A GRAPH TQFT FOR HAT HEEGAARD FLOER HOMOLOGY

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Abstract. In this paper we introduce an extension of the hat Heegaard Floer TQFT which allows cobordisms with disconnected ends. Our construction goes by way of sutured Floer homology, and uses some elementary results from contact geometry. We provide some model computations, which allow us to realize the $H_1(Y;\mathbb{Z})/\text{Tors}$ action and the first order term, $\partial_1$, of the differential of $CF^\infty$ as cobordism maps. As an application we prove a conjectured formula for the action of $\pi_1(Y,p)$ on $\hat{HF}(Y,p)$. We provide enough model computations to completely determine the new cobordism maps without the use of any contact geometric constructions.

1. Introduction

For a connected, closed and oriented $Y^3$ and a Spin$^c$ structure $s$, the various flavors of Heegaard Floer homology associate useful algebraic objects to $Y$. The simplest is the hat flavor of Heegaard Floer homology, denoted $\hat{HF}(Y,p,s)$, where $p \subseteq Y$ is a finite collection of basepoints. As shown in [OS06], over a single basepoint $p \in Y$, the groups

$$\hat{HF}(Y,p) = \bigoplus_{s \in \text{Spin}^c(Y)} \hat{HF}(Y,p,s),$$

form a sort of 3+1 dimensional TQFT. To each cobordism $(W,\gamma): (Y_1,p_1) \rightarrow (Y_2,p_2)$, with $W,Y_1$ and $Y_2$ connected and $\gamma$ a path connecting the basepoint $p_1$ to $p_2$, and each $s \in \text{Spin}^c(W)$, we get a map

$$\hat{F}_{W,s}: \Lambda^*(H_1(W;\mathbb{Z})/\text{Tors}) \otimes_{\mathbb{Z}_2} \hat{HF}(Y_1,p_1,s|_{Y_1}) \rightarrow \hat{HF}(Y_2,p_2,s|_{Y_2}).$$

The map

$$\hat{F}_W = \sum_{s \in \text{Spin}^c(W)} \hat{F}_{W,s}$$

is functorial with respect to composition of cobordisms. The original references are [OS04b], [OS06], though in those papers the dependence on paths was not fully understood. There are two main limitations of this theory: the dependence on basepoints and paths, and the requirement that $W,Y_1$ and $Y_2$ be connected manifolds. In this paper we show that the dependence on paths and basepoints is an essential feature of the $\hat{HF}$ TQFT, while the connectedness requirement is not necessary.

In [Juh09] András Juhász develops a TQFT for sutured manifolds, which allows for disconnected manifolds. By specializing the cobordism maps in sutured Floer homology to graph complements, and using some basic facts from contact geometry, we get the following:

**Theorem A.** Suppose that $(W,\Gamma)$ is a pair with $W^4$ a compact cobordism from the closed, multipointed $3$-manifolds $(Y_1,p_1)$ to $(Y_2,p_2)$, and $\Gamma \subseteq W$ is a properly embedded graph such that $\Gamma \cap Y_i = p_i$. Also suppose that the $Y_i$ have at least one basepoint per component. Then there is a map

$$\hat{F}_{W,\Gamma}: \hat{HF}(Y_1,p_1) \rightarrow \hat{HF}(Y_2,p_2)$$

with the following properties:

1. if $(Y_1,p_1),(Y_2,p_2)$ and $W$ are all connected, and $\Gamma$ is a single arc from $p_1$ to $p_2$, then $\hat{F}_{W,\Gamma}$ is the cobordism map defined in [OS06].
2. $\hat{F}_{W,\Gamma}$ is functorial, i.e. if $(W,\Gamma) = (W_2 \cup W_1, \Gamma_1 \cup \Gamma_2)$ then
   $$\hat{F}_{W,\Gamma} = \hat{F}_{W_2,\Gamma_2} \circ \hat{F}_{W_1,\Gamma_1}.$$ 

**Theorem B.** There is a refinement of the new graph TQFT for $\hat{HF}$ over Spin$^c$ structures:
(a) For 3-manifolds the groups \( \widehat{HF}(Y, p) \) decompose naturally over \( \text{Spin}^c(Y) \) structures as

\[
\widehat{HF}(Y, p) = \bigoplus_{s \in \text{Spin}^c(Y)} \widehat{HF}(Y, p, s);
\]

(b) For cobordisms \((W, \Gamma)\), the maps decompose over \( \text{Spin}^c(W) \) structures as

\[
\widehat{F}_{W, \Gamma} = \sum_{s \in \text{Spin}^c(W)} \widehat{F}_{W, \Gamma, s};
\]

(c) If \((W, \Gamma) = (W_2, \Gamma_2) \circ (W_1, \Gamma_1) = (W_2 \cup W_1, \Gamma_2 \cup \Gamma_1)\) then

\[
\widehat{F}_{W_2, \Gamma_2, s_2} \circ \widehat{F}_{W_1, \Gamma_1, s_1} = \sum_{s \in \text{Spin}^c(W)} \widehat{F}_{W, \Gamma, s}.
\]

The reader might notice that the cobordism maps in [OS06] had an action of \( \Lambda^*(H_1(W; \mathbb{Z})/\text{Tors}) \) while our new ones do not explicitly have one. Our new TQFT contains all of the previous information since the \( H_1 \) action is now encoded directly into the graph data in our TQFT, as exhibited by the following theorem:

**Theorem C.** Suppose that \((W, \gamma) : (Y_1, p_1) \to (Y_2, p_2)\) is cobordism so that \( \gamma \) is a path, \( W_1, Y_1 \) and \( Y_2 \) are connected, and suppose that \( \xi \) is a closed simple curve in \( W \) which intersects \( \gamma \) at a single point. Then

\[
\widehat{F}_{W, \gamma \cup \xi} (\cdot) = \widehat{F}_W ([\xi] \otimes \cdot)
\]

where \( \widehat{F}_W ([\xi] \otimes \cdot) \) denotes the twisted coefficients map defined in [OS06].

Using this we recover the formula described in [Juh13] for the action of \( \pi_1(Y, p) \) on \( \widehat{HF}(Y, p) \). For a diagram \((\Sigma, \alpha, \beta, p)\) which is strongly admissible for an \( s \in \text{Spin}^c(Y) \), the infinity Heegaard Floer complex, denoted \((CF^\infty(\Sigma, \alpha, \beta, s, p), \partial^\infty)\), is a complex of \( \mathbb{Z}_2[U, U^{-1}]\)-modules. We can write the differential \( \partial^\infty = \partial_0 + \partial_1U + \partial_2U^2 + \cdots \), where \( \partial_0 \) can be naturally identified with \( \partial \). We will write \( \partial \) for the map on the chain complex \( \overline{CF}(\Sigma, \alpha, \beta, p) \), and \((\partial_1)_*\) for the induced map on \( \widehat{HF}(Y, p) \). We have the following:

**Theorem D.** The first order term, \( \partial_1 \), of the differential \( \partial^\infty \) descends to a map \( C = (\partial_1)_* : \widehat{HF}(Y, p) \to \widehat{HF}(Y, p) \). If \( \gamma \in \pi_1(Y, p) \) is an embedded curve, letting \( [\gamma] \) denote the \( (H_1(Y, \mathbb{Z})/\text{Tors})\)-action and \( \gamma_* \) the \( \pi_1(Y, p)\)-action, we have the following relations:

1. \( \gamma_* = 1 + (\partial_1)_*[\gamma] \);
2. \( (\partial_1)_*[\gamma] = [\gamma](\partial_1)_* \);
3. \( (\partial_1)^2_* = 0 \).

Using only graph cobordisms, we also define a relative \( H_1 \) action:

**Theorem E.** Using the graph cobordism maps we can construct an action of \( \Lambda^*(H_1(Y, p; \mathbb{Z})/\text{Tors}) \) on \( \widehat{HF}(Y, p) \). For singly pointed diagrams this agrees with the standard \( \Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors}) \) action defined in [OS04b].

We provide enough computations so that the new \( \widehat{F}_{W, \Gamma} \) cobordism maps can be computed without the use of sutured Floer homology:

**Theorem F.** The \( \widehat{F}_{W, \Gamma} \) cobordism maps can be defined without the use of sutured Floer homology. In the case of connected cobordisms between nonempty, connected 3-manifolds with a single basepoint, they are completely determined by the cobordism maps with twisted \( \Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors}) \) coefficients defined in [OS06], as well as a new map corresponding to \( (\partial_1)_* : \widehat{HF}(Y, p) \to \widehat{HF}(Y, p) \). With arbitrary graphs inside of possibly disconnected manifolds, the cobordism maps can be described by maps for i-handle attachments, maps corresponding to four simple graphs (the splitting maps, and termination/creation maps), as well as an additional map which acts by \( \partial_1 \).

Explicit descriptions of the fundamental graphs we provide computations for can be found in Lemmas 5.3, 5.4, 5.5 and 5.6. The cobordism corresponding to \( (\partial_1)_* \) is described in Lemma 9.2.
2.1. The category of sutured manifolds and cobordisms. We define the graph cobordism category and prove Theorem A, i.e. that there exists a graph TQFT for hat Heegaard Floer homology.

2.2. We provide a brief introduction to sutured manifolds, sutured Floer homology, and contact geometry and provide the necessary background which will be used to construct the graph TQFT.

2.3. We prove some technical lemmas about the gluing map from [HKM08], which we use to provide model computations for relatively simple graph cobordisms. We describe most of the new maps in the graph TQFT in these sections.

2.4. We prove Theorem C and describe how the $\Lambda^*(H_1(Y;\mathbb{Z})/\text{Tors})$ action appears in the graph TQFT maps.

2.5. We introduce a new map $C : \widehat{HF}(Y,p) \to \widehat{HF}(Y,p)$ which first appears as a component of a transition map. We show that $C$ is actually a graph cobordism map, and then identify $C$ as $(\partial_1)_*$, the map induced by $\partial_1$. This allows us to finish the proof of Theorem D, namely that we can provide a complete description of the graph cobordism maps without using contact geometry. We use these observations to prove Theorem E and give a formula for the $\pi_1(Y,p)$ action on $\widehat{HF}(Y,p)$.

2.6. We consider the effect of having disconnected components of the graph and make some useful observations about the $H_1(Y;\mathbb{Z})/\text{Tors}$-action on multipointed diagrams.

2.7. We continue the theme of constructing endomorphisms of $\widehat{HF}(Y,p)$ as graph cobordism maps by constructing a full $\Lambda^*(H_1(Y,p;\mathbb{Z})/\text{Tors})$ action on $\widehat{HF}(Y,p)$ using only graph cobordism maps.

2.8. We prove that the graph TQFT maps are naturally graded over $\text{Spin}^c$ structures, establishing Theorem B.

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2. Background material about sutured manifolds and contact geometry

2.1. The category of sutured manifolds and cobordisms. We first summarize and review the definitions of sutured manifolds and cobordisms as presented in [Juhi06a] and [Juhi09].

Definition 2.1. A sutured manifold $(M, \gamma)$ is a compact oriented 3-manifold $M$ together with a set $\gamma \subseteq \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$, together with a set $s(\gamma)$ of homologically nontrivial, oriented, simple closed curves (the set of sutures) such that each component of $A(\gamma)$ contains exactly one suture. Also each component of $R(\gamma) = \partial M \setminus \text{int}(\gamma)$ is oriented. We let $R_+(\gamma)$ (resp. $R_-(\gamma)$) denote the components of $R(\gamma)$ whose normal vectors point out of (resp. into) $M$. The orientation on $R(\gamma)$ must also be coherent with respect to the sutures.

For our purposes, we will only consider sutured manifolds which have no toroidal sutures. The reader should also note that we will often ignore the distinction between $\gamma$ and $s(\gamma)$.

Definition 2.2. A balanced sutured manifold is a sutured manifold $(M, \gamma)$ such that $M$ has no closed components, the map $\pi_0(\gamma) \to \pi_0(\partial M)$ is surjective, and $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Example 2.3. Suppose $(Y^3, p)$ is a manifold with basepoints $p = \{p_1, \ldots, p_k\}$ (with $p_1, \ldots, p_k$ all distinct). Picking pairwise disjoint open balls $B_1, \ldots, B_k$ such that $p_i \in B_i$, simple closed curves $s_i \subseteq \partial B_i$, and regular neighborhoods $\gamma_i = N(s_i) \subseteq \partial B_i$, we set let $M = Y \setminus \bigcup_{i=1}^k \text{int}(B_i)$ and $\gamma = \bigcup_{i=1}^k \gamma_i$. Then $(M, \gamma)$ is a balanced sutured manifold, which we denote by $Y(p)$.

Following [Juhi09] we now define cobordisms of sutured manifolds.

Definition 2.4. If $(M_1, \gamma_1)$ and $(M_2, \gamma_2)$ are sutured manifolds, a we call a triple $(W, Z, \xi)$ a cobordism of sutured manifolds between $(M_1, \gamma_1)$ and $(M_2, \gamma_2)$ if

1. $W$ is a compact oriented 4-manifold with boundary;
2. $Z \subseteq \partial W$ is a compact codimension-0 submanifold with boundary, and $\partial W \setminus \text{int}(Z) = -M_1 \sqcup M_2$;
(3) $\xi$ is a positive contact structure on $Z$ such that $\partial Z$ is a convex surface with dividing set $s_i(\gamma_i)$ on $\partial M_i$ for $i = 1, 2$.

Example 2.5. If $(M, \gamma)$ is a balanced sutured manifold, we define the identity cobordism $\text{id}_{(M, \gamma)} : (M, \gamma) \to (M, \gamma)$ as the sutured cobordism $(W, Z, \xi)$ where $W = M \times I$ and $Z = \partial M \times I$. The sutures $\gamma$ determine an $I$-invariant, tight contact structure $\xi$ on $\partial M \times I$ with dividing set $s(\gamma)$. A picture is shown in Figure 1.

Figure 1. A sutured manifold $(M, \gamma)$ and the identity cobordism $\text{id}_{(M, \gamma)}$. The sutures $\gamma$ are shown with a dotted line in the picture to the left.

2.2. Sutured Floer homology. An important feature of the sutured category is that sutured manifolds and cobordisms fit into the framework of a 3+1 dimensional TQFT. This TQFT was constructed by András Juhász in [Juh06a] and [Juh09].

If $(M, \gamma)$ is a balanced sutured manifold, we can construct a sutured Heegaard splitting $(\Sigma, \alpha, \beta)$ where $\Sigma \subseteq M$ is an oriented surface which divides $M$ into two disjoint open subsets (the positive side and the negative side), such that the following are true:

1. $\partial \Sigma = s(\gamma)$ as oriented manifolds;
2. $|\alpha| = |\beta|$;
3. the $\alpha$ curves bound compression disks on the negative side, and compressing $\Sigma$ along $\alpha$ yields a surface isotopic to $R_-(\gamma)$ relative to $\gamma$;
4. the $\beta$ curves bound compression disks on the positive side and compressing $\Sigma$ along $\beta$ yields a surface isotopic to $R_+(\gamma)$ relative to $\gamma$.

Assuming admissibility of the diagram $(\Sigma, \alpha, \beta)$ one then picks a path of almost complex structures $J_t$ on $\text{Sym}^k(\Sigma)$, where $k = |\alpha| = |\beta|$. The chain complex $CF(\Sigma, \alpha, \beta)$ is then defined as the free $\mathbb{Z}_2$-module with generators $T_\alpha \cap T_\beta$ where $T_\alpha = \alpha_1 \times \cdots \times \alpha_n$ and $T_\beta = \beta_1 \times \cdots \times \beta_n$. The differential on $CF(\Sigma, \alpha, \beta)$ is defined by the formula

$$\partial(x) = \sum_y \sum_{\phi \in \pi_2(x,y), \mu(\phi) = 1} \#(\hat{M}(\phi))y,$$

obtained by counting moduli spaces of Whitney disks between intersection points. We denote the homology of $CF(\Sigma, \alpha, \beta)$ by $SFH(\Sigma, \alpha, \beta)$. Most of the details can be found in [Juh06a], though some questions of naturality were resolved later in [JT12]. To define this as an invariant of a sutured manifold $(M, \gamma)$, one must consider different choices of diagrams $(\Sigma, \alpha, \beta)$ and $(\Sigma', \alpha', \beta')$ and define transition functions between $SFH(\Sigma, \alpha, \beta)$ and $SFH(\Sigma', \alpha', \beta')$, and show the transition functions commute in an appropriate fashion.

Theorem 2.6 ([JT12 Theorem 2.34]). The $\mathbb{Z}_2$-modules $SFH(\Sigma, \alpha, \beta)$ fit into a transitive system and hence yield an invariant of $(M, \gamma)$, which we denote by $SFH(M, \gamma)$.
In particular if \((Y^3, p)\) is a closed, oriented, connected 3-manifold and \(p\) is a single point, we have that \(SFH(Y(p))\) is isomorphic to \(\widehat{HF}(Y, p)\), as constructed by Ozsváth and Szabó in [OS04b].

Sutured Floer homology also fits into a TQFT:

**Theorem 2.7** ([Juh09]). To a sutured cobordism \((W, Z, \xi)\) we can associate a map \(F_W : SFH(M_1, \gamma_1) \to SFH(M_2, \gamma_2)\) which is functorial under composition.

We will give a very brief outline of the construction here since we will go into further detail later when it is necessary for some model computations. We need some notation introduced in [Juh09].

**Definition 2.8.** A sutured cobordism \((W, Z, \xi) : (M_1, \gamma_1) \to (M_2, \gamma_2)\) is called a **boundary cobordism** if \((W, Z, \xi)\) is equivalent to a cobordism with \(W = (M_1 \cup Z) \times I\) with \(M_2 = M_1 \cup Z\).

A sutured cobordism \((W, Z, \xi) : (M_1, \gamma_1) \to (M_2, \gamma_2)\) is called a **special cobordism** if \(W\) is equivalent to a cobordism formed by only handle attachments to \((\text{int } M_1) \times I\), with \(\xi\) an \(I\) invariant contact structure on \(\partial M_1 \times I\) divided by the sutures \(\gamma_i\).

To define the cobordism maps in [Juh09], one takes a sutured cobordism \((W, Z, \xi)\) and first writes it as a composition of a boundary cobordism followed by a special cobordism \((\partial M, \gamma)\). Juhász uses the gluing map in [HKM08] to define the cobordism map for boundary cobordisms. Juhász then defines the sutured cobordism map for special cobordisms. The special cobordism maps are defined for each index of handle attachments, similar to how the maps are defined in [OS06] in Heegaard Floer homology. The total cobordism map \(F_W\) is the composition of these maps. We will go into more detail when we need to do model computations.

**2.3. Characteristic foliations and dividing sets.** We will review some basic definitions and results from contact geometry. We review the basic fact that \((M, \gamma) = (S^1 \times S^2)_{\#n}(\mathcal{P})\) has an essentially unique tight contact structure relative \(\partial M\), and this contact structure can be taken to be divided by \(\gamma\).

Suppose \((M, \xi)\) is a contact manifold, co-oriented by a global contact form \(\alpha\) such that \(\alpha \wedge d\alpha > 0\). We say an embedded \(\Sigma \subseteq M\) is **convex** if there is a vector field \(v\) on \(M\) (referred to as a contact vector field) which is transverse to \(\Sigma\) such that the flow of \(v\) preserves \(\xi\). The characteristic foliation \(\mathcal{F}_\xi\) on \(\Sigma\) is defined as \(\mathcal{F}_\xi(p) = T_p \Sigma \cap \xi(p)\). The **dividing set** \(\Gamma_\xi \subseteq \Sigma\) are the points where \(v(x) \in \xi(x)\). It is a basic fact from contact geometry that the isotopy class of \(\Gamma_\xi\) is independent of \(v\), so we will often refer to \(\Gamma_\xi\) without reference to \(v\).

![Figure 2. The characteristic foliation and a dividing set \(\Gamma_\xi\) for the unit sphere in \((\mathbb{R}^3, \xi_{\text{std}})\).](image)

A key idea in contact geometry is that the dividing set on \(\Sigma\) contains most of the contact theoretic information in a neighborhood of \(\Sigma\). If \(\mathcal{F}\) is a singular foliation on \(\Sigma\), we say that a collection of closed embedded curves \(\Gamma\) divides \(\mathcal{F}\) if there is an \(I\)-invariant contact structure on \(\Sigma \times I\) with \(\mathcal{F} = \xi|_{\Sigma \times \{0\}}\) with \(\Gamma\) the dividing set of \(\Sigma \times \{0\}\).

**Theorem 2.9** (Giroux’s Flexibility Theorem [Gir91]). Suppose that \(\Sigma\) is a convex surface with characteristic foliation \(\xi|_{\Sigma}\), contact vector field \(v\) and dividing set \(\Gamma\). If \(\mathcal{F}\) is another singular foliation on \(\Sigma\) divided by \(\Gamma\), then there is an isotopy \(\phi_t \in [0, 1]\) such that \(\phi_0(\Sigma) = \Sigma, \xi|_{\phi_t(\Sigma)} = \mathcal{F}\), the isotopy is fixed on \(\Gamma\), and \(\phi_t(\Sigma)\) is transverse to \(v\) for all \(t\).
If $M$ is an oriented, compact 3-manifold with nonempty boundary, and $\Gamma_M$ is a dividing set and $F$ is a singular foliation which is divided by $\Gamma_M$, we define $T(M, F)$ to be the set of isotopy classes of tight contact structures whose characteristic foliation on $\partial M$ is $F$.

**Lemma 2.10 ([Hon00 Proposition 4.2]).** Let $M$ be a compact, oriented 3-manifold with nonempty boundary. Let $F_1$ and $F_2$ be two characteristic foliations on $\partial M$ which are divided by $\Gamma_M$. There exists a bijection

$$\phi_{12} : T(M, F_1) \sim T(M, F_2).$$

Thus we write $T(M, \Gamma_M)$ for any $T(M, F)$ where $F$ is divided by $\Gamma$.

We will now discuss tight contact structures on connected sums of $S^1 \times S^2$ because they are important in defining the $\hat{F}_{W, \Gamma}$ cobordism maps.

**Lemma 2.11 ([Gei08 Theorem 4.10.1]).** Up to isotopy, there are unique tight contact structures on $S^3$ and $S^1 \times S^2$, i.e. $|T(S^3)| = |T(S^1 \times S^2)| = 1$.

If $\Sigma \subseteq M$ is a properly embedded surface in a compact 3-manifold, we let $M \setminus \Sigma$ denote the metric completion of the complement of $\Sigma$ in $M$. Equivalently we can consider $M \setminus \Sigma$ to be the complement of a small, open regular neighborhood of $\Sigma$ in $M$.

Using [Col97 Theorem 3], one can prove the following:

**Theorem 2.12 ([Col97]).** Suppose $\Sigma = S^2$. If $(M \setminus \Sigma, \xi|_{M \setminus \Sigma})$ is tight, then $(M, \xi)$ is tight.

As a corollary to the above theorem, one can show the following, commonly known as Colin’s gluing theorem:

**Corollary 2.13** (Colin’s Gluing Theorem). Suppose that

$$M = M_1 \# \cdots \# M_n$$

where each $M_i$ is a closed 3-manifold. Then there is a bijection

$$T(M) \sim T(M_1) \times \cdots \times T(M_n).$$

One can also use the theorem of Colin to show the following:

**Corollary 2.14.** If $(M, \gamma)$ is a balanced sutured manifold, and $(M', \gamma')$ denotes the sutured manifold obtained by removing $k$ disjoint open balls from $M$ and adding one simple closed curve to each new boundary component, then there is a bijection

$$T(M, \gamma) \sim T(M', \gamma').$$

As a consequence of the previous Corollary and the theorem of Colin, we have the following:

**Corollary 2.15.** If $p \subseteq (S^1 \times S^2)^\#n$ is a finite collection of points, then

$$|T((S^1 \times S^2)^\#n)| = |T((S^1 \times S^2)^\#n(p))| = 1.$$

Similarly, as a consequence of the theorems of Colin, we have the following:

**Corollary 2.16.** Suppose $Z_1 = (S^1 \times S^2)^\#n_1(p_1)$ and $Z_2 = (S^1 \times S^2)^\#n_2(p_2)$ and that $Z$ is obtained by gluing $Z_1$ and $Z_2$ along some number of their boundary components. Then $Z$ is diffeomorphic to $(S^1 \times S^2)^\#n(q)$ for some $q$. The unique tight contact structure (rel $\partial Z$) restricts to the unique tight contact structures (rel $\partial Z_i$) on each $Z_i$.

**Proof.** The family of manifolds $(S^1 \times S^2)^\#n(p)$ is characterized by the property of having a finite collection of disjoint embedded 2-spheres, such that blowing up along those spheres yields $S^3(x)$ for some finite $x \subseteq S^3$. Gluing two manifolds in this family along components of their boundaries still preserves this property. The statement about contact structures is obvious. \qed
2.4. Cobordism maps and contact structures. The following propositions are implicitly used in sutured Floer homology. The main point is that on sutured cobordisms, we can safely put the “standard” contact structure on manifolds like \((S^1 \times S^2)^\#n(p)\) and \(S^3\) without worrying about minor ambiguities in the choice of contact structure.

**Proposition 2.17.** Suppose that \((W, Z, \xi), (W, Z, \xi') : (M_1, \Gamma_1) \rightarrow (M_2, \Gamma_2)\) are balanced sutured cobordisms. If \(\xi\) and \(\xi'\) are isotopic rel \(\partial Z\), then \(F_{W, \xi} = F_{W, \xi'}\).

**Proof.** Extend the isotopy of \(Z(\text{rel} \partial Z)\) to an isotopy of \(W(\text{rel} - M_1 \sqcup M_2)\). This yields an equivalence of cobordisms and hence the induced maps are the same. \(\square\)

**Proposition 2.18.** Suppose that \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are two characteristic foliations which are divided by the sutures \(\Gamma\) on \(\partial Z\), and that \(\phi_{12}\) denotes the map \(\mathcal{T}(Z, \mathcal{F}_1) \rightarrow \mathcal{T}(Z, \mathcal{F}_2)\) in Lemma 2.16 then \(F_{W, \xi} = F_{W, \phi_{12} \xi}\).

**Proof.** By examining the map \(\phi_{12}\) described in [Hon00, Proposition 4.2], it is not hard to describe a diffeomorphism between \((W, Z, \xi)\) and \((W, Z, \phi_{12} \xi)\) which is isotopic on \(- M_1 \sqcup M_2 \subseteq \partial W\) to the identity map. Hence the maps on sutured Floer homology are identical. \(\square\)

3. The graph TQFT for \(\widehat{HF}\)

Let \(\text{Cob}_{3+1}^\Gamma\) be the category such that \(\text{Ob}(\text{Cob}_{3+1}^\Gamma)\) consists of pairs \((Y, p)\) such that \(Y\) is a closed oriented 3-manifold and \(p\) is a tuple of basepoints such that each component of \(Y\) contains at least one basepoint. The empty set with no basepoint is such a manifold.

The morphisms, \(\text{Mor}(\text{Cob}_{3+1}^\Gamma)\), are equivalence classes of tuples 

\[(W, h_1, h_2, (Y_1, p_1), (Y_2, p_2), \Gamma)\]

such that

1. \(W\) is a closed oriented manifold with oriented boundary \(- V_1 \sqcup V_2\);
2. \((Y_i, p_i)\) are objects of \(\text{Cob}_{3+1}^\Gamma\);
3. \(h_i : Y_i \rightarrow V_i\) is a diffeomorphism;
4. \(\Gamma \subseteq W\) is a nonempty embedded graph (with finitely many edges and vertices) such that \(\Gamma \cap V_i = h_i(p_i)\);
5. \(\Gamma\) intersects \(V_i\) transversely;
6. two tuples \((W, h_1, h_2, (Y_1, p_1), (Y_2, p_2), \Gamma)\) and \((W, \tilde{h}_1, \tilde{h}_2, (Y_1, p_1), (Y_2, p_2), \Gamma)\) are equivalent if there is a diffeomorphism \(\phi : W \rightarrow W\) which makes the obvious diagrams involving \(h_i\) and \(\tilde{h}_i\) commute, and which maps \(\Gamma\) homeomorphically onto \(\Gamma\).

The equivalence condition is necessary since otherwise the cobordism category does not even have identity morphisms. Given two cobordisms \((W_1, \Gamma_1) : (Y_1, p_1) \rightarrow (Y_2, p_2)\) and \((W_2, \Gamma_2) : (Y_2, p_2) \rightarrow (Y_3, p_3)\) we form their composition

\[(W_2, \Gamma_2) \circ (W_1, \Gamma_1) = (W_2 \cup W_1, \Gamma_2 \cup \Gamma_1)\].

**Remark 3.1.** One could alternatively define the category \(\text{Cob}_{3+1}^\Gamma\) with an additional equivalence relation where two graph cobordisms are equivalent if they become diffeomorphic after removing a regular neighborhood of the graph. In any event, the cobordism maps we describe in the more restrictive category described above are invariant under the equivalent in this looser category. In Figure [3] we show three graphs which would become equivalent under this looser notion of equivalence. The cobordism maps we construct will be equal for all three graphs.

**Definition 3.2.** We construct the category \(\text{Cob}_{3+1}^\Omega\) by defining \(\text{Ob}(\text{Cob}_{3+1}^\Omega)\) to be the collection of connected, oriented, nonempty 3-manifolds with one basepoint. The morphisms are equivalence classes of tuples

\[(W, h_1, h_2, (Y_1, p_1), (Y_2, p_2), \gamma)\],

exactly as in \(\text{Cob}_{3+1}^\Gamma\), except \(\gamma\) is a properly embedded path in \(W\) such that \(\gamma \cap V_i = p_i\).
Figure 3. Three graphs which have isotopic regular neighborhoods and which induce the same cobordism maps. We distinguish these in $\text{Cob}^\Gamma_{3+1}$, though it is not strictly necessary for us to do so.

**Theorem 3.3** ([OS06]). Hat Heegaard Floer homology yields a functor

$$\widehat{HF} : \text{Cob}_0^{3+1} \to \text{Vect}(\mathbb{Z}_2).$$

We now prove Theorem A, restated as follows:

**Theorem A.** There is functor $\widehat{HF}_\Gamma : \text{Cob}_\Gamma^{3+1} \to \text{Vect}(\mathbb{Z}_2)$ extending the $\widehat{HF}$ functor defined in [OS06].

**Proof.** For each 3-manifold $(Y, p) \in \text{Ob}(\text{Cob}_\Gamma^{3+1})$ define

$$\widehat{HF}_\Gamma(Y, p) = SFH(Y(p))$$

where $Y(p)$ denotes the 3-manifold obtained by removing open balls centered at each $p_i \in p$ and adding trivial sutures to the boundary components.

For each cobordism $(W, h_1, h_2, (Y_1, p_1), (Y_2, p_2), \Gamma)$ (which we will refer to as $(W, \Gamma)$), define the sutured cobordism $(W_0, Z, \xi)$ where $W_0$ is obtained by removing a regular neighborhood of $\Gamma$ from $W_0$. The manifold $W_0$ has boundary

$$\partial W_0 = (-Y_1(p_1)) \cup Z \cup Y_2(p_2)$$

where $Z$ is the boundary of the regular neighborhood of $\Gamma$. As a 3-manifold $Z$ is diffeomorphic to a disjoint union of manifolds of the form $(S^1 \times S^2) \# n(x)$. There is a unique tight contact structure rel $\partial Z$ with these trivial sutures (more precisely for a fixed characteristic foliation divided by the sutures, there is a unique tight contact structure). Define $\widehat{HF}_\Gamma(W, \Gamma) = SFH(W_0, Z, \xi)$.

To any cobordism $(W, \Gamma)$ we have a well defined map, though we need to show that these maps are functorial under composition. For this, it is sufficient to show that if $(W, \Gamma) = (W_2, \Gamma_2) \circ (W_1, \Gamma_1)$ then

$$\widehat{HF}_\Gamma(W, \Gamma) = \widehat{HF}_\Gamma(W_2, \Gamma_2) \circ \widehat{HF}_\Gamma(W_1, \Gamma_1).$$

To this end it is sufficient to show that the contact structures glue appropriately. This was shown in Corollary 2.16.

We let $\widehat{F}_{W, \Gamma}$ denote the map induced by a graph cobordism $(W, \Gamma)$.

### 4. Gluing maps and product disk decompositions

We would like to compute the new graph cobordisms without the use of contact geometry. In order to do so we need some model computations of the gluing map in sutured Floer homology, defined in [HKM08]. The main result of the following section is Lemma 4.10, which will be necessary to perform model computations of graph cobordisms.

#### 4.1. Product disk decompositions and connected sums of 3-manifolds

We start by discussing connected sums. In Heegaard Floer homology, one can show that $\widehat{HF}(Y_1 \# Y_2)$ and $\widehat{HF}(Y_1) \otimes \widehat{HF}(Y_2)$ are isomorphic, though one must be somewhat careful in describing the isomorphism. This is reflected in sutured Floer homology in that the analogous isomorphism (cf. [Juh06a, Proposition 9.15]) relies on maps induced by “product disk decompositions” instead of being defined only in terms of a dividing sphere in $Y_1 \# Y_2$.

We now proceed with some definitions, following [Juh06a].

**Definition 4.1.** Let $(M, \gamma)$ be a sutured manifold, and $(S, \partial S) \subseteq (M, \partial M)$ a properly embedded surface such that for each component $\lambda$ of $S \cap \gamma$ one of the following holds:

1. $\lambda$ is a properly embedded nonseparating arc in $\gamma$;
2. $\lambda$ is a simple closed curve in an annular component $A$ of $\gamma$ in the same homology class as $A \cap s(\gamma)$;
(3) \( \lambda \) is a homotopically nontrivial curve in a torus component \( T \) of \( \gamma \), and if \( \delta \) is another component of \( T \cap S \), then \( \lambda \) and \( \delta \) represent the same homology class in \( H_1(Y) \).

We say \( S \) defines a sutured manifold decomposition and write

\[
(M, \gamma) \rightarrow^S (M', \gamma'),
\]

where

\[
M' = M \setminus \text{int}(N(S))
\]

and

\[
\gamma' = (\gamma \cap M') \cup N(S'_+ \cap R_-(\gamma)) \cup N(S'_- \cap R_+(\gamma)),
\]

\[
R_+\gamma' = ((R_+\gamma) \cap M') \cup S'_+ \setminus \text{int}(\gamma'),
\]

and

\[
R_-\gamma' = ((R_-\gamma) \cap M') \cup S'_- \setminus \text{int}(\gamma'),
\]

where \( S'_+ \) (resp. \( S'_- \)) is the component of \( \partial N(S) \cap M' \) whose normal vector points out of (resp. into) \( M' \).

Figure 4. An example of a sutured manifold decomposition. Here \( (M, \gamma) \) has \( \partial M = S^2 \) with \( \gamma \) the trivial sutures. The surface \( S \) intersects \( \gamma \) at two points. The manifold \( (M', \gamma') \) has two components. All sutures are shown in bold red. Notice that \( S \) satisfies condition (1) of Definition 4.1.

The only case that we will need is the case that \( S \) is actually a disk. We have a special name for such a sutured manifold decomposition:

**Definition 4.2.** A sutured manifold decomposition \( (M, \gamma) \rightarrow^D (M', \gamma') \) is called a **product decomposition** if \( D \) is a properly embedded disk in \( M \) and \( |D \cap s(\gamma)| = 2 \).

Notice that in the case of a product decomposition, each component \( \lambda \) of \( S \cap \gamma \) satisfies condition (1) of Definition 4.1.

In [Juh06b], given a surface decomposition \( (M, \gamma) \rightarrow^S (M', \gamma') \), Juhász defines a map \( SFH(M', \gamma') \rightarrow SFH(M, \gamma) \), called the inclusion map, which yields an isomorphism

\[
SFH(M', \gamma') = \bigoplus_{s \in O_S} SFH(M, \gamma, s) \subseteq SFH(M, \gamma),
\]

where \( O_S \subseteq \text{Spin}^c(M, \gamma) \) is a subset of so called “outer” \( \text{Spin}^c \) structures. For a product decomposition, the inclusion map is particularly simple:

**Lemma 4.3** ([Juh06a Lemma 9.13]). Suppose \( (M, \gamma) \) is a balanced sutured manifold and \( (M, \gamma) \rightarrow^D (M', \gamma') \) is a product decomposition. Then \( (M', \gamma') \) is also balanced, and the inclusion map

\[
SFH(M, \gamma) \rightarrow SFH(M', \gamma')
\]

is an isomorphism.
Remark 4.4. Suppose \((Y^3, p, q)\) is a doubly based three-manifold, and \(S\) is an embedded 2-sphere in \(Y\) dividing \(Y\) into two pieces, with one basepoint in each component of \(Y \setminus S\). The sphere \(S\) gives a topological connected sum decomposition of \(Y\). By choosing of path from \(S\) to one of the basepoints in \(Y\), we can push \(S\) to one of the boundary components of the sutured manifold \(Y(p, q)\) to get a product disk decomposition. The fact that the inclusion map is an isomorphism should be thought of as a K"unneth theorem.

4.2. Gluing maps in sutured Floer homology. There are several gluing maps in sutured Floer homology, which correspond to cutting sutured manifolds into pieces.

4.2.1. A first version of the gluing map. One component of the construction of the cobordism maps in sutured Floer homology is the gluing map in sutured Floer homology. Introduced in \([HKM08]\), there is a map on sutured Floer homology of a sutured submanifold to the ambient sutured manifold, assuming the presence of a contact structure.

Definition 4.5. We say a sutured manifold \((M', \gamma')\) is a sutured submanifold of \((M, \gamma)\) if \(M' \subseteq M\) is a codimension 0 submanifold with boundary and \(M' \subseteq \text{int}(M)\). A connected component \(C \subseteq \partial M \setminus \text{int}(M')\) is isolated if \(C \cap \partial M = \emptyset\).

![Figure 5](image)

**Figure 5.** A sutured submanifold \((M', \gamma')\) of \((M, \gamma)\). The manifold \(M \setminus \text{int} M'\) has the contact structure \(\xi\). The component \(C\) of \(M \setminus \text{int}(M')\) is isolated.

Theorem 4.6 (\([HKM08]\)). Suppose \((M', \gamma') \subseteq (M, \gamma)\) is a sutured submanifold with \(m\) isolated components, and \(\xi\) is a contact structure on \(M \setminus \text{int}(M')\) with convex boundary and dividing set \(\gamma\) on \(\partial M\) and \(\gamma'\) on \(\partial M'\). Then there is a map

\[
\Phi_\xi : SFH(-M', -\gamma') \to SFH(-M, -\gamma) \otimes V^\otimes m
\]

where \(V = \widehat{HF}(S^1 \times S^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Over \(\mathbb{Z}\) this is well defined up to a necessary \(\pm 1\) ambiguity.

We now summarize the construction of the gluing map, as described in \([HKM08]\). Let \(T = \partial M'\). We consider a bicolored neighborhood \(N = T \times [-1, 1] \subseteq M\) with a \([-1, 1]\)-invariant contact structure \(\xi\) such that

- \(T_t = T \times \{t\}\) are convex surfaces with dividing set \(\gamma' \times \{t\}\);
- \(T \times [-1, 0] \subseteq M'\) and \(T' \times [0, 1] \subseteq M \setminus \text{int}(M')\);
- \(\xi|_{T \times [0, 1]} = \xi|_{T \times [0, 1]}\).

The authors of \([HKM08]\) then construct suitable “contact compatible” Heegaard splittings for \(M'\) and \(M\). Roughly speaking they pick an appropriate \((\Sigma', \alpha', \beta')\) Heegaard splitting for \((-M', -\Gamma')\) and an appropriate extension \((\Sigma, \alpha, \beta) = (\Sigma' \cup \Sigma'', \alpha' \cup \alpha'', \beta' \cup \beta'')\) for \((-M, -\Gamma)\). The gluing map is then defined as

\[
\Phi_\xi : CF(\Sigma', \beta', \alpha') \to CF(\Sigma, \beta, \alpha)
\]

\[y \mapsto y \otimes x_0,\]

where \(x_0\) is a specially defined collection of intersection points arising from the contact structure \(\xi\).
4.2.2. Another version of the gluing map. Following [HKM08], one can specialize the above gluing map to a situation where a particular contact structure is determined by sutures, thus giving a gluing map which isn’t defined with respect to a particular contact structure.

More precisely, suppose that \((M, \gamma)\) is a sutured manifold. The sutures are equivalent to defining a translation invariant contact structure \(\zeta_{\partial M}\) in a product neighborhood \(N = \partial M \times I\) of \(\partial M\) with dividing set \(\gamma \times \{i\}\) for \(i = 0, 1\). Suppose \((U, \partial U)\) is a properly embedded surface in \((M, \gamma)\) satisfying the following:

1. There exists an invariant contact structure \(\zeta_U\), defined in a neighborhood of \(U\) which agrees with \(\zeta_{\partial M}\) near \(\partial U\);
2. \(U\) is convex with possibly empty Legendrian boundary and has dividing set \(\gamma_U\) with respect to \(\zeta_U\).

Let \((M', \gamma')\) be the sutured manifold obtained from cutting \((M, \gamma)\) along \(U\) and edge rounding. By shrinking \(M\) we obtain the tight contact structure \(\zeta = \zeta_{\partial M} \cup \zeta_U\) on \(M \setminus \text{int}(M')\). The contact structure \(\zeta\) induces the gluing map as in Theorem 4.6

\[
\Phi \coloneqq \Phi_\zeta : SFH(-M', -\gamma') \to SFH(-M, -\gamma) \otimes V^\otimes m.
\]

Since the contact structure \(\zeta\) only depends on the sutures, the map \(\Phi\) is defined on sutured Floer homology without reference to contact structures:

**Theorem 4.7 ([HKM08] Theorem 1.3).** Let \((M', \gamma')\) be a sutured manifold and let \(U_+, U_-\) be disjoint subsurfaces of \(\partial M\) which satisfy the following:

1. each component of \(\partial U_\pm\) transversely and nontrivially intersects \(\gamma'\);
2. there is an orientation reversing diffeomorphism \(\phi : U_+ \to U_-\) which takes \(\gamma'|_{U_+}\) to \(\gamma'|_{U_-}\) and takes \(R_{\pm}(U_+)\) to \(R_{\mp}(U_-)\).

Let \((M, \gamma)\) be the sutured manifold obtained by gluing \(U_+\) and \(U_-\) via \(\phi\) and smoothing. Then there is a natural gluing map

\[
\Phi : SFH(-M', -\gamma') \to SFH(-M, -\gamma) \otimes V^\otimes m
\]

where \(m\) is the number of components of \(U_\pm\) which are closed surfaces.

**Theorem 4.8 ([HKM08] Proposition 6.4).** If \(\gamma_U\) is \(\partial\)-parallel, i.e. each component of \(\gamma_U\) cuts off a half disk which intersects no other component of \(\gamma_U\), then the convex decomposition \((M, \gamma) \leadsto (U, \gamma_U)\) \((M', \gamma')\) corresponds to a sutured manifold decomposition, and the gluing map

\[
\Phi : SFH(-M', -\gamma') \to SFH(-M, -\gamma)
\]

corresponds to the direct summand map constructed in [HKM09] and [Juh06b] which we described in Theorem 4.3.

4.3. Model gluing map computations. We now need to perform some model calculations of the contact gluing map defined in [HKM08]. Suppose that \((Y^3, p)\) is a closed 3-manifold with basepoints \(p\). Recall that we let \(Y(p)\) denote the sutured manifold obtained by removing neighborhoods of each of the points \(p\) and adding trivial sutures.

We now do the easiest case: when \(Z\) consists of a trivial extension of \(\partial(Y(p))\) together with the disjoint union of a contact manifold:

**Lemma 4.9.** Suppose that \(Z = (\partial Y(p) \times I) \cup Z_0\) where \((Z_0, \gamma_0, \xi_0)\) is a balanced sutured manifold with sutures induced by contact structure \(\xi_0\). Suppose that \(\xi\) is a contact structure on \(Z\) such that \(\xi|_{Z_0} = \xi_0\) and \(\xi|_{\partial Y(p) \times I}\) is an I-invariant contact structure inducing the sutures on \(\partial Y(p)\). Then the gluing map

\[
\Phi_\xi : SFH(Y(p)) \to SFH(-Y(p)) \otimes SFH(-Z_0, -\gamma_0)
\]

is the map

\[
x \mapsto x \otimes EH(Z_0, \xi_0, \gamma_0),
\]

under the identification of \(Y(p)\) as \(Y(p) \cup (\partial Y(p) \times I)\).

**Proof.** This is an immediate consequence of [HKM08] Theorem 6.1 and the fact that “contact compatible” along \(Z_0\) in this case can be replaced by just taking a diagram induced by a partial open book decomposition of \(Z_0\).
We now bootstrap our way to a somewhat more complicated case. The following is a key Lemma which
we will use repeatedly to compute the graph cobordism maps. For notational simplicity, we phrase this for
the case when \( p \) contains a single point \( p \), though the obvious analog of the statement holds true for more
basepoints.

**Lemma 4.10.** Suppose \( (Y, p) \) is a singly basepointed manifold. Suppose that \( (Z, \xi, \gamma) \) is a balanced sutured
manifold with tight contact structure \( \xi \) inducing the sutures \( \gamma \), and an identification of a component of \( \partial Z \)
with \( \partial Y(p) \). Suppose further that \( \partial Y(p) \cap \partial Z \) consists of a single copy of \( S^2 \), and that \( (M, \gamma) = (Y(p) \cup Z, \gamma) \),
where \( \gamma \) are the remaining sutures of \( Z \), is a balanced sutured manifold. Let \( S \) denote \( \partial Y(p) \). Then a choice
of path \( \lambda \) from \( S \) to \( \partial M \) induces a product disk decomposition

\[
M \rightsquigarrow D \cup Z,
\]

with \( Z_0 \subseteq Z \) and an identification

\[
\phi : M_0 \to Y(p).
\]

Writing \( \xi_{Z_0} = \xi|_{Z_0} \) and \( \gamma_{Z_0} \) for the induced sutures on \( Z_0 \), and \( \gamma_{M_0} \) for the sutures on \( M_0 \), we have the
following:

(a) The map associated to the product decomposition gives an identification of \( SFH(-M, -\gamma) \) as

\[
SFH(-M_0, -\gamma_{M_0}) \otimes SFH(-Z_0, -\gamma_{Z_0});
\]

(b) The gluing map

\[
\Phi_{\xi} : SFH(-Y(p)) \to SFH(-M, -\gamma)
\]

is the map

\[
\Phi_{\xi} = x \mapsto x \otimes EH(Z_0, \gamma_{Z_0}, \xi_{Z_0})
\]

under these identifications.

**Proof.** Our strategy is to decompose the \( M \) into several pieces and use the functoriality of the gluing map
(cf. [HKM08, Proposition 6.2]). Let \( S \subseteq M = Y(p) \cup Z \) denote the sphere \( \partial Y(p) \). Let \( C \) be a component of
\( \partial Z \setminus \partial Y(p) \) and let \( \lambda \) be a path from a point on \( S \) to a point on the sutures on \( C \). Let \( N = N(S \cup \gamma) \subseteq Z \)
be a regular neighborhood of \( S \cup \gamma \) in \( Z \). We have the picture in Figure 6.

![Figure 6](image)

**Figure 6.** The manifolds \( M = Y(p) \cup Z \) and the path \( \gamma \). In the picture, \( Y(p) \) is displayed
as the region “inside” of \( S \) and \( Z \) is “outside”.

Let \( N' = N'(S \cup \gamma) \subseteq Z \) be a slightly larger regular neighborhood of \( S \cup \gamma \) in \( Z \) so that

1. \( \overline{N} \subseteq N' \);
2. \( \overline{N} \setminus N \subseteq Z \) is diffeomorphic to \( D^2 \times I \),
3. \( \partial N \) and \( \partial N' \) are convex surfaces;
Let $Z_0 = Z \setminus N'$, and let $\gamma_{Z_0}^0$ be the sutures induced by $\xi$. Let $M_0 = Y(p) \cup N$ and let $\gamma_{M_0}^0$ be the sutures induced by $\xi$. Let $\xi_{Z_0}^0 = \xi|_{Z_0}$.

By construction, we had $N' \setminus N$ diffeomorphic to $D^2 \times I$. Let $D$ be the disk $D^2 \times \{\frac{1}{2}\} \subseteq N' \setminus N$. Notice that $N' \setminus N$ is a regular neighborhood of $D$. By construction there is a product disk decomposition $M \to D^2 M_0 \sqcup Z_0$. Part (a) thus follows from Theorem 4.3.

We can glue the pieces in any order we want by [HKM08, Proposition 6.2] since the gluing map is functorial. Hence we have the commutative diagram:

$$\begin{array}{c}
SFH(-Y(p)) \quad \xrightarrow{\Phi_{\xi}} \quad SFH(-M, -\gamma_M) \\
\Phi_{\xi|_{N' \setminus N}} \quad \downarrow \quad \Phi_{\xi|_{N' \setminus N}} \\
SFH(-M_0 \sqcup -Z_0, -\gamma_{M_0} \sqcup -\gamma_{Z_0}^0) \quad \rightarrow \\
\end{array}$$

Since $\xi$ was assumed to be tight, we know that the sutures on $N' \setminus N$ satisfy the requirements of Theorem 4.8. In particular the map $\Phi_{\xi|_{N' \setminus N}}$ is the product decomposition map defined by Juhasz in [Juh06b]. On the other hand since $\xi$ is tight we know that $\xi|_{N}$ is the standard contact structure on $S^2 \times I$, and hence by [HKM08, Theorem 6.1], the map $\Phi_{\xi|_{N' \sqcup Z_0}}$ is the isomorphism induced by an identification of $Y(p)$ with $Y(p) \cup N$ which is the identity outside of a small neighborhood of $S \subseteq Y(p)$. Part (b) now follows from an application of Lemma 4.9. For clarity we include a picture in Figure 7.

Figure 7. The sets $N, N'$ and $Z_0$ in the proof of Lemma 4.10

5. Model graph cobordism computations

We are now in position to begin providing explicit descriptions of the graph cobordism maps. We would like to describe our new graph TQFT without the use of contact geometry. As it stands, the existence of our new cobordism maps relies on the cobordism maps defined in [Juh09], which in turn uses several constructions in sutured Floer homology, such as the gluing map defined in [HKM08]. In this section, we compute the cobordism maps associated to cobordisms of the form $(W, \Gamma)$ with $W = Y \times I$ and $\Gamma$ a relatively simple graph in $W$.

We first introduce some important notation. The model computations for graph cobordism maps will be done with respect to certain special diagrams. Given a based 3-manifold $(Y, p, q)$, we will often pick a basepoint, say $p$, and mark it with an asterisk to indicate that whatever map we write down is defined with respect to a diagram for $(Y, q)$ which is $(0,3)$-stabilized at $p$. This is analogous to viewing the sutured manifold $Y(p, q)$ as $(Y(q))(p)$. 

\[\]
5.1. **Nullhomotopic loops.** Suppose that \( W = Y \times I \) and \( \Gamma = p \times I \) with a small interval removed and a nullhomotopic loop inserted, as in Figure 8. Note that since we are in dimension 4, being nullhomotopic is equivalent to nullisotopic.

![Figure 8](image)

**Figure 8.** A cobordism \((W, \Gamma)\) which is a trivial cobordism with a nullhomotopic loop spliced in.

**Lemma 5.1.** For \((W, \Gamma) : (Y, p) \to (Y, p)\) as above, with \( \Gamma \) the graph with a nullhomotopic loop spliced in, the cobordism map is zero.

**Proof.** The induced sutured cobordism \((W, Z, \xi)\) can be computed as a boundary cobordism of gluing \(-Y(1)\) and \(-S^1 \times S^2(1)\), followed by special cobordism which is attaching a cancelling two handle. The gluing map is \(x \mapsto x \otimes EH(\xi) = x \otimes \theta^-\) by Lemma 4.10 and the fact that the contact class of the standard contact structure on \(S^1 \times S^2(1)\) is \(\theta^-\), the generator of lower relative degree. The two handle map is \(x \otimes \theta^+ \mapsto x\) and \(x \otimes \theta^- \mapsto 0\), by a model calculation in, e.g., [OS06]. Hence the composition is zero. \(\square\)

More generally, the above model computation can be carried out for arbitrarily many basepoints by the same argument. In general we have the following:

**Corollary 5.2.** If \((W, \Gamma) : (Y_1, p_1) \to (Y_2, p_2)\) is a graph cobordism and \(\Gamma\) contains a nullhomotopic loop spliced into another arc, then \(\hat{F}_{W, \Gamma} = 0\).

**Proof.** Isotope the graph and decompose the cobordism as a composition of cobordisms so that an intermediate cobordism is diffeomorphic to a cobordism of the form shown in Figure 8 with additional trivial strands between additional basepoints. The result follows as in the previous lemma. \(\square\)

5.2. **Path termination/creation cobordism.** Suppose that \((\Sigma_0, \alpha, \beta)\) is a Heegaard surface for the sutured manifold \(Y(q)\). Suppose further that \(\Sigma_0\) contains the point \(p \in Y\). Then by removing a small ball in \(Y(q)\) and \(\Sigma_0\) at the point \(p\) we get a Heegaard surface \((\Sigma, \alpha \cup \alpha_0, \beta \cup \beta_0)\) for the sutured manifold \(Y(p, q)\) where \(\alpha_0\) and \(\beta_0\) are parallel curves to the new boundary component of \(\Sigma\) which intersect each other twice. Suppose further that all of the \(\alpha\) and \(\beta\) curves are chosen so that \(p\) is in a domain of \(\Sigma_0\) which intersects the boundary \(\partial \Sigma_0\) nontrivially. This is shown in Figure 9.

![Figure 9](image)

**Figure 9.** The curves \(\alpha_0\) and \(\beta_0\).

By [Juh06a, Proposition 9.14] we know that \(SFH(Y(p, q)) \cong SFH(Y(q)) \otimes V\) where \(V = \mathbb{Z}_2^2\) is the two dimensional \(\mathbb{Z}_2\)-vector space generated by generators \(\theta^-\) and \(\theta^+\) corresponding to the intersection points of \(\alpha_0\) and \(\beta_0\).

Consider the cobordism drawn in Figure 10. The underlying manifold on both ends is \(Y\). As a 4-manifold the cobordism is \(W = Y \times I\). On the top end \(Y\) has two basepoints, \(p\) and \(q\), and on the bottom it has one. The asterisk indicates where the top end is stabilized (for the purposes of computing the cobordism maps).
**Figure 10.** The path termination/creation cobordism. The asterisk identifies which point is the stabilized point for the purposes of computing the map. Equivalently, the asterisk corresponds to a choice of disk yielding a product decomposition of $SFH(Y(p, q))$.

**Lemma 5.3.** If $(W, \Gamma) : (Y, p, q) \to (Y, q)$ is the graph cobordism shown in Figure 10, viewed as a cobordism from top to bottom, then the associated map is as in Figure 10, i.e.

$$x \otimes \theta^- \mapsto x \quad \text{and} \quad x \otimes \theta^+ \mapsto 0.$$ 

**Proof.** Consider the associated sutured cobordism $(W, Z, \xi) : Y(p, q) \to Y(q)$. This cobordism is drawn in Figure 11.

**Figure 11.** The sutured cobordism corresponding to the $\hat{HF}_\Gamma$ termination cobordism.

In the above sutured cobordism $(W, Z, \xi)$ we have that $Z = S^2 \times I \sqcup S^3(1)$ and $\xi$ is the standard tight contact structure. Using the terminology of [Juh09] the component $S^2 \times I$ is not isolated since it intersects $Y(q)$ nontrivially, while the $S^3(1)$ component is isolated, since it doesn’t intersect $Y(q)$. Following the construction of the cobordism map in [Juh09], we first remove a standard contact ball from $S^3(1)$ and view the cobordism $(W, Z, \xi)$ instead as $(W, Z', \xi') : Y(p, q) \to Y(q) \sqcup B$ where $Z' = Z \setminus B$ and $\xi' = \xi|_{Z'}$. Now $Z = S^2 \times I \sqcup S^2 \times I$ with the standard tight contact structure, so the gluing map $SFH(Y(p, q)) \to SFH(Y(p, q) \cup Z) = SFH(Y(p, q))$ is the identity map by [Juh09, Theorem 6.7].

To compute $F_W$, we then compute the map associated to the special cobordism $Y(p, q) \to Y(q) \sqcup S^3(1)$. This cobordism is obtained by attaching a 3-handle to a sphere containing the boundary sphere associated to $p$. Hence the cobordism map is obtained via the three handle map, which is exactly the map described in the lemma statement.

A schematic is shown in Figure 12.

**Figure 12.** A schematic of the computation of the termination cobordism.

\[\Phi_{\xi'} = \text{Id} \]

\[
\begin{align*}
SFH(Y(p, q)) & \xrightarrow{\Phi_{\xi'}} = \text{Id} \\
SFH(Y(p, q)) & \xrightarrow{\text{id}} SFH(Y(q)) \\
SFH(Y(q)) & \xrightarrow{\text{id}} SFH(Y(q) \sqcup S^3(1))
\end{align*}
\]
Lemma 5.4. If \((W, \Gamma) : (Y, q) \to (Y, p, q)\) is the graph cobordism shown in Figure 10, viewed as a cobordism from bottom to top, then the associated map is as in Figure 10, i.e. 
\[ x \mapsto x \otimes \theta^+. \]

Proof. We proceed analogously for the computation of the cobordism. We note that in this case, \(Z = (S^2 \times I) \cup S^3(1)\), though in this case the component \(S^3(1)\) is no longer isolated because it intersects \(\partial(Y(q) \cup Z)\) nontrivially. The gluing map is thus \(\Phi_\xi : SFH(Y(q)) \to SFH(Y(q) \cup S^3(1))\). By Lemma 4.9 the gluing map \(\Phi_\xi : SFH(Y(q)) \to SFH(Y(q) \cup S^3(1))\) is the map \(x \mapsto x \otimes 1\) where 1 is the generator of \(SFH(S^3(1))\). This computes the boundary cobordism part of the cobordism map. The special cobordism is obtained by attaching a 1-handle with one foot at \(p \in Y(q)\) and the other foot in at a point in \(S^3(1)\). After destabilizing the resulting diagram to get rid of the torus connected sum of the Heegaard surface coming from \(S^3(1)\), we get exactly the map in the Lemma statement. A schematic is shown in Figure 13.

Figure 13. A schematic of the computation of the edge create cobordism

5.3. Splitting Cobordisms. We now wish to compute the maps associated to another cobordism. The cobordism is from \(Y(p')\) to \(Y(p, q)\). The cobordism and the associated map is shown in Figure 14. For the purpose of computing the map in terms of a particular diagram, we need to pick one of the basepoints \(p\) or \(q\) to identify as the stabilized point. The asterisk denotes which point is the stabilized point.

To define the cobordism precisely, we need to make some choices. For one thing we need to pick a path \(\lambda\) from the points \(p\) and \(q\) (or equivalently pick a ball containing both \(p\) and \(q\)). The choice of \(\lambda\) does effect the actual map on \(\widehat{HF}(Y)\), as a different choice of \(\lambda\) corresponds to post composition by the action of an element in \(\pi_1(p)\).

\[ x \otimes \theta^+ x \otimes \theta^- \]

Figure 14. The effect of the splitting cobordism.

Lemma 5.5. The splitting cobordism map 
\[ F_W : SFH(Y(p')) \to SFH(Y(p, q)) \]
is the map 
\[ x \mapsto x \otimes \theta^- . \]

Proof. The cobordism is a boundary cobordism, and hence can be computed with only a gluing map. By Lemma 4.10 we know that the map is of the form \(x \mapsto x \otimes EH(\xi)\) where \(\xi\) is the standard contact structure on \(S^3(2)\), but this is the lower degree generator, and so the map is as described in the lemma statement. □
In the other direction, we have the following:

**Lemma 5.6.** The splitting cobordism map  

\[ F_W : SFH(Y(p,q)) \rightarrow SFH(Y(p')) \]

is  

\[ x \otimes \theta^+ \mapsto x, \quad \text{and} \quad x \otimes \theta^- \mapsto 0. \]

**Proof.** To see that \( x \otimes \theta^+ \mapsto x \), we can just use TQFT properties. Compose the cobordism with a path termination/creation cobordism as shown in Figure 15.

![Figure 15. Composing the termination/creation cobordism with the splitting cobordism](image)

The composition is the identity, so considering the composition in the downward direction, we see that \( x \otimes \theta^- \mapsto 0 \).

We now need to show that \( x \otimes \theta^- \mapsto 0 \). To see this, we compose two splitting cobordisms to get a cobordism \( Y(p) \rightarrow Y(p) \). The graph has a nullhomotopic loop spliced in, so the induced map is zero. We already know one splitting map sends \( x \mapsto x \otimes \theta^- \), and hence the other splitting map must compose with this to get zero, so \( x \otimes \theta^- \mapsto 0 \).

\[ \square \]

**Remark 5.7.** Using the proofs as above, one can prove identically that if we add additional basepoints \( p_i \) and additional components of the graph of the form \( \{p_i\} \times I \), then the above formulas still hold true for the termination, creation and splitting cobordisms.

6. **Disconnected cobordisms and empty three-manifolds**

It is now worthwhile to provide a short discussion of how empty and disconnected manifolds fit into the \( \hat{HF} \) and sutured cobordism categories. Firstly, we define \( SFH(\emptyset) \) by  

\[ SFH(\emptyset) = SFH(S^3(p)) = \mathbb{Z}_2. \]

Since we are working over \( \mathbb{Z}_2 \), the situation is also easy for disconnected 3-manifolds: if \( (Y_1, p_1) \) and \( (Y_2, p_2) \) are multi-based three manifolds, then following the construction in [Juh06a], we have  

\[ \hat{HF}(Y_1 \sqcup Y_2, p_1 \sqcup p_2) = \hat{HF}(Y_1, p_1) \otimes_{\mathbb{Z}_2} \hat{HF}(Y_2, p_2). \]

If we were to work over \( \mathbb{Z} \), we would have to define \( \hat{HF}(Y_1 \sqcup Y_2, p_1 \sqcup p_2) \) to be \( H_*(CF(Y_1(p_1) \otimes CF(Y_2(p_2))) \) because of the additional Tor terms.

It’s perhaps worthwhile mentioning the algebraic possibility that over \( \mathbb{Z}_2 \) if \( (W_1, \Gamma_1) \) and \( (W_2, \Gamma_2) \) were two cobordisms, then \( \hat{F}_{(W_1, \Gamma_1), (W_2, \Gamma_2)} \) may not vanish even though both \( \hat{F}_{W_1, \Gamma_1} \) and \( \hat{F}_{W_2, \Gamma_2} \) both vanish because of the additional Tor terms. Over \( \mathbb{Z}_2 \) the situation simplifies and we just have that  

\[ \hat{F}_{(W_1, \Gamma_1), (W_2, \Gamma_2)} = \hat{F}_{W_1, \Gamma_1} \otimes \hat{F}_{W_2, \Gamma_2}. \]

We now discuss cobordisms with one or both ends equal to the empty set. In our TQFT we allow \( (W, \Gamma) \) to have \( \partial W = \emptyset \), but we require \( \Gamma \) to be nonempty.

In the case that \( (W, \Gamma) : (Y_1, p_1) \rightarrow (Y_2, p_2) \) is a cobordism and one of \( Y_i \) is empty, the construction of the cobordism map in [Juh09] is to remove a ball \( B^4 \) from \( W \), and connect the resulting boundary \( S^3 \) with a single arc to a component of \( \Gamma \) arbitrarily. By [Juh09] Lemma 8.4, this is independent of the path in \( W \) or the component of \( \Gamma \) which we use to connect the sphere to \( \Gamma \), though it’s also enlightening to the see a proof of the independence of the choice of path explicitly.
Lemma 6.1. Suppose that $W, Y : (Y_1, p_1) \rightarrow (Y_2, p_2)$ is a graph cobordism with $Y_1 \neq \emptyset$. Remove a ball $B$ from $W$ and let $\lambda$ be an arc from a point $p \in \partial B$ to a point on $\Gamma$. Then

$\widehat{F}_{W, \Gamma} : \widehat{HF}(Y_1, p_1) \rightarrow \widehat{HF}(Y_2, p_2)$

and

$\widehat{F}_{W \setminus B, \Gamma \cup \lambda} : \widehat{HF}(Y_1 \sqcup S^3, p_1 \cup p) \rightarrow \widehat{HF}(Y_2, p_2)$

are equal under the identification of

$\widehat{HF}(Y_1 \sqcup S^3, p_1 \cup p) = \widehat{HF}(Y_1, p_1) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2$.

Proof. Removing a ball $B$ near $Y_1 \subseteq \partial W$ and adding a path from the the new copy of $S^2$ to a component of the graph has the effect of introducing a splitting cobordism and a 1- or 3-handle map. This has the same effect as not removing a ball, but adding a trivial strand under the identification of $SFH(Y(p) \sqcup S^3(p))$ with $SFH(Y(p))$. Adding a trivial strand does not change the cobordism map though, as can be explicitly seen by composing the splitting cobordism map with a path continuation cobordism map (or alternatively noting that a regular neighborhood of the graph with a trivial strand added is isotopic to a regular neighborhood of the graph with no strand added). □

Figure 16. Three cobordisms which all induce the same map after identifying $\widehat{HF}(Y_1, p_1)$ with $\widehat{HF}(Y_1 \sqcup S^3, p_1 \cup \{p\})$.

7. Relation to the homology action in Heegaard Floer homology

In [OS04b] Ozsváth and Szabó define a $\Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors})$ module structure on $HF^\circ(Y, p)$. We now wish to describe a relation between the cobordism maps in $\text{Cob}_{\text{S}}^{s, \text{1}}$ and the $\Lambda^*H_1(Y)/\text{Tors}$ action. In [OS06] they define a cobordism map associated to a cobordism between connected three-manifolds $W : Y_1 \rightarrow Y_2$ and a Spin$^c$ structure $s$ on $W$. This map has turned out to be somewhat more complicated than their original definition, instead relying on a choice of basepoints $p_i \in Y_i$ and a path in $W$ between the basepoints.

Theorem 7.1 (Essentially [OS06]). Given a connected cobordism $(W, \gamma) : (Y_1, p_1) \rightarrow (Y_2, p_2)$ with $\gamma$ a path between $p_1$ and $p_2$ there is a map

$\widehat{F}_{W, s} : \Lambda^*(H_1(W)/\text{Tors}) \otimes \widehat{HF}(Y_1, p_1, s|_{Y_1}) \rightarrow \widehat{HF}(Y_2, p_2, s|_{Y_2})$.

Summing over Spin$^c(W)$ structures there is a map

$\widehat{F}_W : \Lambda^*(H_1(W)/\text{Tors}) \otimes \widehat{HF}(Y_1, p_1) \rightarrow \widehat{HF}(Y_2, p_2)$

such that if $W = W_1 \circ W_2$ as path cobordisms and $\xi_i \in H_1(W_i; \mathbb{Z})$ then

$\widehat{F}_W((\iota_*\xi_2 \wedge \iota_*\xi_1) \otimes x) = \widehat{F}_{W_2}(\xi_2 \otimes \widehat{F}_{W_1}(\xi_1 \otimes x))$,

where $\iota_*$ is the map on $H_1$ induced by inclusion.

Theorem C. Suppose that $(W, \gamma) : (Y_1, p_1) \rightarrow (Y_2, p_2)$ is a path cobordism and suppose that $\xi$ is a closed simple curve in $W$ which intersects a single path of $\gamma$ at a single point and no other paths. Let $\gamma \xi = \gamma \cup \xi$. Then

$\widehat{F}_{W, \gamma \xi} = \hat{F}_W([\xi] \otimes \cdot)$

where $\hat{F}_W([\xi] \otimes \cdot)$ denotes the map with twisted $\Lambda^*H_1/\text{Tors}$ coefficients defined in [OS06].
The rest of this section will be devoted to the proof of Theorem C. The reader should note that we are only phrasing the above theorem for cobordisms between singly pointed manifolds. We will consider the analogous question in Sections 12 and 13, though there is some subtlety in that the maps in [OS06] were only defined for cobordisms between singly pointed 3-manifolds.

First suppose that \((W, \gamma) : (Y, p) \rightarrow (Y, p)\) is a trivial cobordism \((W, \gamma) = (Y \times I, p \times I)\). Our first goal will be to show the claim is true for such a cobordism.

Let \(s \in \text{int} \, I\). We can construct a map
\[
F : \pi_1(Y, p) \rightarrow \text{End}_{\mathbb{Z}_2}(\widehat{HF}(Y, p))
\]
to be defined as \(\alpha \mapsto \widehat{F}_{W, \gamma_\alpha}\), where \(\gamma_\alpha\) is the graph formed by taking a simple closed curve \(\alpha\) in \(Y\) passing through \(p\) at exactly one point and forming
\[
\gamma_\alpha = \gamma \cup (\alpha \times \{s\}).
\]

As constructed this is only well defined as a map from the set of simple closed curves in \(Y\) which meet \(p\) at exactly one point. To see that this is well defined as a function on \(\pi_1(Y, p)\), note that if \(\alpha\) and \(\alpha'\) are simple closed curves in \(Y\) passing through \(p\) at a single point which are homotopic relative \(p\), then in \(Y \times I\) they are isotopic, and by transversality, the isotopy can be taken to not intersect \(\gamma\) except at the point where the isotopy is fixed. Hence the graphs \(\gamma_\alpha\) and \(\gamma_{\alpha'}\) are isotopic in \(Y \times I\) relative the endpoints \(p \times \{0\}\) and \(p \times \{1\}\) and hence determine the same cobordism map. Thus the map \(F\) is well defined on \(\pi_1(Y, p)\).

**Lemma 7.2.** The map \(F\) satisfies \(F(\xi) \circ F(\xi) = 0\).

**Proof.** The graph cobordism maps \(\widehat{F}_{W, \Gamma}\) are invariant under isotopy of vertices along edges or across other vertices. By performing such an operation to the graph in the cobordism representing the map \(F(\xi) \circ F(\xi)\), one can obtain a nullhomotopic loop, and hence \(F(\xi) \circ F(\xi) = 0\) by Corollary 5.2. This is shown in Figure 17.

![Figure 17. Passing the bottom \(\xi\) along the top \(\xi\) on the left yields a nullhomotopic loop (on the right).](image)

**Lemma 7.3.** If \(\xi, \zeta \in \pi_1(Y, w)\) then \(F(\xi \circ \zeta) = F(\xi) + F(\zeta)\) where \(\circ\) is concatenation.

**Proof.** If \(\widehat{\xi}\) denotes the curve in \(Y \times I\) given by \(\widehat{\xi}(t) = (\xi(t), t)\), then by isotoping the graph, we can replace the graph \((p \times I) \cup \xi\) with \((p \times I) \cup \widehat{\xi}\). Similarly we can replace \((p \times I) \cup \zeta\) with \((p \times I) \cup \widehat{\zeta}\). The statement of the equality we want to show is shown pictorially in Figure 18.

Note that if \(e \subseteq \text{int} \, W\) is an embedded arc in a cobordism \(W\) and \(\Gamma \subseteq W\) is a graph such that \(e \cap W\) is a single point, then \(\widehat{F}_{W, \Gamma} = \widehat{F}_{W, \Gamma \cup e}\) because the regular neighborhoods of \(\Gamma\) and \(\Gamma \cup e\) are isotopic rel \(\partial W\) inside of \(W\). Add trivial edges to the graphs representing \(F(\xi)\) and \(F(\zeta)\) so the desired equality is pictorially stated in Figure 19.

By decomposing the cobordisms in Figure 19 into compositions of cobordisms, it is sufficient to show the equality of cobordism maps shown in Figure 20.

We can verify the equality in Figure 20 explicitly. The map on the right is the identity. The two maps on the left are composition of splitting and termination cobordisms. Viewing the cobordisms as maps from the top to the bottom, the map on the left is
\[
\widehat{F}_{W, \Gamma_1}(x \otimes \theta^-) = x \otimes \theta^-,
\]
and
\[
\widehat{F}_{W, \Gamma_1}(x \otimes \theta^+) = 0.
\]
Figure 18. A pictorial representation of the statement that $F(\zeta \ast \xi) = F(\zeta) + F(\xi)$.

Figure 19. Adding edges to the cobordisms in Figure 18 which don’t affect $F(\xi), F(\zeta)$ or $F(\xi \ast \zeta)$.

Figure 20. An equality which if true is sufficient to show $F(\xi \ast \zeta) = F(\xi) + F(\zeta)$. The asterisks denote the basepoints which we will view as stabilized, i.e. we will pick a diagram for $\hat{HF}(Y, p)$ where $p$ is the basepoint not marked with an asterisk, and then stabilize it at the basepoint marked with a *.

The map in the middle is

$$\hat{F}_{W,\Gamma_2}(x \otimes \theta^-) = 0, \quad \text{and} \quad \hat{F}_{W,\Gamma_2}(x \otimes \theta^+) = x \otimes \theta^+.$$

Clearly $\hat{F}_{W,\Gamma_1} + \hat{F}_{W,\Gamma_2} = \text{id}$, completing the proof.

Corollary 7.4. The map $F$ descends from $\pi_1(Y, p)$ to a map on $H_1(Y; \mathbb{Z})$.

Suppose that $W = (Y \times I) \cup h_1$ where $h_1$ is a 1-handle. Say that $W$ is a cobordism between $Y$ and some $Y'$ (where $Y'$ is diffeomorphic to $Y\#(S^1 \times S^2)$). Pick an embedded arc $\lambda$ in $W \times I$ between the feet of the 1-handle in $Y$, such that $\lambda$ intersects $p$ exactly once and $\lambda \cap \partial W$ consists of just the endpoints of $\lambda$. Concatenate $\lambda$ with the core curve of the 1-handle to form a closed loop $\xi$. Let $\Gamma = (p \times I) \cup *$.

We wish to compute $F_W^\Gamma$, and to do so we need to describe the diagrams which we can compute $F_W^\Gamma$ with respect to. Pick a diagram $(\Sigma, \alpha, \beta)$ for $Y$ such that both feet of $h$ intersect $\Sigma$, the curve $\lambda$ is embedded in $\Sigma$, such that $\lambda$ does not pass through any $\alpha$ or $\beta$ curves, and such that $\lambda$ is in a domain which contains the basepoint $p$. Let $(E, \alpha_0, \beta_0)$ be a standard diagram for $S^1 \times S^2$ with $E$ a torus and $\alpha_0$ and $\beta_0$ two isotopic
closed curves which intersect at exactly two points. Then \((\Sigma\#E, \alpha \cup \alpha_0, \beta \cup \beta_0)\) is a diagram for \(Y'\), where the connect sum is taken at the domain in \(\Sigma\) containing \(\lambda\) and the domain in \(E\) containing the basepoint.

**Lemma 7.5.** Using the diagrams \((\Sigma, \alpha, \beta)\) and \((\Sigma\#E, \alpha \cup \alpha_0, \beta \cup \beta_0)\) described in the previous paragraph, the map \(F^\Gamma_W\) is

\[
x \mapsto x \otimes \theta^-.
\]

**Proof.** After removing a regular neighborhood of \(\Gamma\), the sutured cobordism \(W(\Gamma)\) is a boundary cobordism in the nomenclature of [Juh09], and hence the cobordism map is just the gluing map. The path \(\lambda\) determines a diffeomorphism of \(Y'\) with \(Y\#(S^1 \times S^2)\), under which we can use Lemma [4.10] to compute that the map \(SFH(Y(p)) \to SFH(Y'(p))\) is

\[
x \mapsto x \otimes \text{EH}(S^1 \times S^2) = x \otimes \theta^-,
\]

since \(\theta^-\) is the contact class of the standard contact structure on \(S^1 \times S^2\). \Box

**Lemma 7.6.** For \(W = Y \times I\) and the graph \(\gamma_\xi\) as described above, the cobordism maps satisfy \(F^\Gamma_W = F_{W, [\xi]}\).

**Proof.** We just need to use naturality of the standard Heegaard Floer cobordism maps. Note that in Lemma 7.5 we showed that if \(W_1 = (Y \times I) \cup h\) where \(h\) is a 1-handle and \(\xi_1\) is a path as in the lemma, and \(\Gamma_1 = \gamma \cup \xi_1\) then

\[
\tilde{F}_{W_1, \Gamma_1}(x) = x \otimes \theta^- = [\xi] \cdot (x \otimes \theta^+) = \tilde{F}_{W_1, \gamma}(\xi \otimes x).
\]

Pick a curve \(\ell\) which has the same endpoints as \(\lambda\) (the curve selected in the previous paragraph), such that concatenating \(\ell\) with \(\lambda\) yields the embedded curve \(\xi\).

Consider the cobordism \((W', \Gamma')\) formed by attaching a two handle along the closed curve formed by concatenating \(\ell\) and the core curve of the 1-handle. This cancels the 1 handle \(h\), so \(W'\) is diffeomorphic to \(W = Y \times I\) under an identification which fixes \(Y \times \{0, 1\}\) except in a neighborhood of the handle attachments. Let \(\phi : W' \to W\) be such a diffeomorphism. Let \(Y''\) denote the outgoing end of \(W'\), so that \(\phi|_{Y''} : Y'' \to Y\) is a diffeomorphism. The graph \(\Gamma_0\) becomes \(\Gamma = \gamma \cup \xi\). The situation is shown in Figure 21.

![Figure 21](image-url)

**Figure 21.** The arcs \(\lambda\) and \(\ell\), as well as the core of the 1-handle, which are used to form the closed curves \(\xi\) and \(\xi_0\).

We will now compute \(\tilde{F}_{W, \Gamma}\) as a composition. For clarity we will use the symbol \(F^\text{Graph}_{W, \Gamma}\) for the graph cobordism map and \(F^\text{Stand}_{W, \gamma}\) for the standard \(HF\) cobordism map. We compute as follows:
\[
F^\text{Graph}_{W,(p \times I) \cup \xi}(x) = \phi_* F^\text{Graph}_{W',(p \times I) \cup \xi}(x)
\]

\[
= \phi_* F^\text{2-handle}_{p \times I}(\Gamma_1)(\xi_0(x))
\]

\[
= \phi_* F^\text{Stand}_{p \times I}(\xi_0(x) \otimes x)
\]

\[
= F^\text{Stand}_{W,p \times I}(\xi_0(x) \otimes x)
\]

\[
= F^\text{Stand}_{W,p \times I}(\xi) \otimes x
\]

\[
= (\phi : W' \rightarrow W \text{ induces an equivalence in } \widehat{HF}_1)
\]

\[
(\text{Functoriality of } \widehat{HF}_1 \text{ maps})
\]

\[
(\text{Lemma 7.5})
\]

\[
(\text{Theorem 7.1})
\]

\[
(\phi : W' \rightarrow W \text{ induces an equivalence in } \widehat{HF}_{\text{Stand}})
\]

\[
(\phi_* \xi_0(x) = \xi, \text{ by construction})
\]

exactly as we wanted. □

We are now in position to prove Theorem C.

**Proof of Theorem C.** Write \((W, \gamma)\) as a composition of handle attachments

\[W = W_3 \circ W_2 \circ W_1\]

and write \(W_i : Y_0 \rightarrow Y_1, W_2 : Y_1 \rightarrow Y_2\) and \(W_3 : Y_2 \rightarrow Y_3\), where \(W_i\) is obtained by attaching \(i\)-handles. Notice that \(\pi_1(Y_1) \hookrightarrow \pi_1(W)\) is surjective, so we can isotope \(\xi\) so that it lies completely in \(Y_1\). We now just compute

\[\hat{F}_{W,\gamma}(x) = (\hat{F}_{W_3,\gamma}|_{W_3} \circ \hat{F}_{W_2,\gamma}|_{W_2} \circ \hat{F}_{Y_1 \times I,\gamma}|_{(p \times I) \cup \xi} \circ \hat{F}_{W_1,\gamma}|_{W_1})(x),\]

where \(p \in Y_1\) is the basepoint of \(Y_1\). After applying Lemma 7.6, we see immediately that this is \(\hat{F}_W(\xi) \otimes x\).

\[\square\]

**Remark 7.7.** We will later see that if \(\Gamma\) is formed by taking the disjoint union of closed loop and a collection of paths \(\gamma\) then the cobordism map vanishes. In fact we will show in Theorem 12.7 that if \(\partial W\) is not empty and \(\Gamma\) has any components which don’t intersect \(\partial W\), then the cobordism map \(\hat{F}_{W,\Gamma}\) vanishes.

### 8. An Important Transition Map Computation

In this section, we will partially compute a transition map which appears frequently in the graph TQFT. In the splitting cobordism, given a diagram of the singly pointed end, there are two natural diagrams which we could consider on the doubly pointed end. Let \(H_0\) denotes the diagram \((\Sigma, \alpha, \beta, p')\) of \((Y, p')\). Suppose that \(p\) and \(q\) are points in \(\Sigma\) such that \(p\) and \(q\) are in the same domain as \(p'\). Let \(H_1\) denote the diagram formed by performing \((0, 3)\)-stabilization of \((\Sigma, \alpha, \beta, q)\) at \(p\). Let \(H_2\) denote the diagram formed by performing \((0, 3)\) stabilization of \((\Sigma, \alpha, \beta, p)\) at \(q\). These are shown in Figure 22.

As \(\mathbb{Z}_2\)-modules, we can view \(\widehat{CF}(H_1)\) as \(\widehat{CF}(H_0) \otimes \mathbb{Z}_2 = \widehat{CF}(H_0) \oplus \widehat{CF}(H_0)\). The differential acts diagonally, in the sense that

\[
\partial H_i = \begin{pmatrix}
\partial H_0 & 0 \\
0 & \partial H_0
\end{pmatrix}.
\]

There is a canonical isomorphism \(\Phi_{H_1,H_2} : \widehat{HF}(H_1) \rightarrow \widehat{HF}(H_2)\) which is well defined on homology, corresponding to changing diagrams. The map \(\Phi_{H_1,H_2}\) is defined on chain complexes by taking a sequence of Heegaard moves which transform from one diagram to the other and composing special maps for each Heegaard move. According to [JT12 Theorem 2.39], the resulting composition is independent on homology of the choice of intermediate Heegaard moves.

**Lemma 8.1.** With respect to the above decompositions of \(\widehat{HF}(H_i) = \widehat{HF}(H_0) \oplus \widehat{HF}(H_0)\), the map \(\Phi_{H_1,H_2}\) is of the form

\[
\Phi_{H_1,H_2} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},
\]

for some map \(C\).
The diagrams $H_1, H_2$ and the change of diagram map $\Phi = \Phi_{H_1, H_2}$.

The map $\Phi_{H_1, H_2}$ can be computed as a composition of two triangle maps. Let $(\Sigma, \alpha, \beta)$ be the unpointed diagram used to construct $H_1$ and $H_2$, and let $\beta'$ be small Hamiltonian isotopies of the $\beta$. Let $\alpha'$ be small Hamiltonian isotopies of the $\alpha$ curves. The curves $\alpha_0, \beta_0, \alpha_1, \beta_1$ can be arranged so that the transition map is the composition of the two triangle maps in Figure 23.

**Figure 23.** The two Heegaard triples used to compute the transition map $\Phi_{H_1, H_2}$.

**Lemma 8.2.** On the complexes $\widehat{CF}$, the map corresponding to the first handleslide has the following form

$$
\tilde{\Phi}_{\alpha_0 \cup \beta_0}^{\beta_0 \cup \beta_1} = \begin{pmatrix}
\tilde{\Phi}_\alpha^{\beta_0 \rightarrow \beta'} & 0 \\
C_1 & \tilde{\Phi}_\alpha^{\beta \rightarrow \beta'}
\end{pmatrix}
$$

for sufficiently stretched complex structure. Similarly on complexes $\widehat{CF}$ the map corresponding to the second handleslide has the form

$$
\tilde{\Phi}_{\alpha_0 \cup \alpha}^{\alpha_0 \cup \alpha_1} = \begin{pmatrix}
\tilde{\Phi}_\alpha^{\alpha_0 \rightarrow \alpha'} & 0 \\
C_2 & \tilde{\Phi}_\alpha^{\alpha_0 \rightarrow \alpha'}
\end{pmatrix}
$$

for sufficiently stretched complex structure.

We will only provide a computation of the first map, since the second one can be computed by the same procedure. Our approach is similar to the analysis of quasistabilizations in [MO10]. Write

$$
\tilde{\Phi}_{\alpha_0 \cup \beta_0}^{\beta_0 \cup \beta_1} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

Our goal will be to compute $A, B$ and $D$. 

The transition maps are computed using the triangle map, which counts pseudoholomorphic triangles of Maslov index 0.

Let \((S^2, \alpha_0, \beta_0, \beta_1)\) denote the Heegaard triple in Figure 24.

**Lemma 8.3.** If \(\theta^+\) denotes the positive generator of \(\widehat{CF}(S^2, \beta_0, \beta_1, p, q)\), then for any \(\phi \in \pi_2(a, \theta^+, b)\) for \(a, b\) intersection points in \(\alpha_0 \cap \beta_0\) or \(\alpha_0 \cap \beta_1\), we have that

\[
\mu(\phi) = m_1 + m_2 + m_3 + m_4,
\]

where the \(m_i\) are the multiplicities shown in Figure 24.

**Proof.** The proof is identical to the proof of [MO10, Lemma 5.16]. One just checks that the claim is true for a particular index 0 triangle with vertex at \(\theta^+\). Any other triangle in \(\pi_2(a, \theta^+, b)\) can be obtained by a juxtaposing index 1 disks without \(\theta^+, \theta^-\) as a vertex, or index 2 boundary degenerations, and the formula respects juxtaposition by any such homology class.

Let \(x^+\) (resp. \(x^-\)) denote the points of \(\alpha_0 \cap \beta_0\) of greater (resp. lower) relative grading. Let \(y^+\) (resp. \(y^-\)) denote the points of \(\alpha_0 \cap \beta_1\) of greater (resp. lower) relative grading. Let \(\theta^+\) denote the point of \(\beta_0 \cap \beta_1\) of higher relative grading. For a homology class in the triple \((S^2, \alpha_0, \beta_0, \beta_1, p, q)\) with \(m_1 = 0\), the remaining multiplicities are determined by three coefficients, labelled \(a, b, c\) in the diagrams in Figure 25. In Figure 25, the three diagrams correspond to homology triangles which will be relevant in computing \(A, B\) and \(D\). By simply adding \(m_1\) to all of the multiplicities determined in one of the diagrams with \(m_1 = 0\), we can achieve any homology triangle.

**Figure 24.** The Heegaard triple \((S^2, \alpha_0, \beta_0, \beta_1, p, q)\)

**Figure 25.** Homology triangles with \(m_1 = 0\) are determined by \(a, b, c\) though the multiplicities in the other regions depends on what the other vertices of the homology triangle are.
We are now in position to prove Lemma 8.2.

**Proof of Lemma 8.2.** We essentially just need to pay attention to the multiplicities of triangles which are counted in the hat triangle maps.

**Computing the map $A$.** Suppose that $\phi \in \pi_2(x \otimes x^+, \theta^+ \otimes \theta^+, y \otimes y^+)$. Write $\phi = \phi_1 \# \phi_2$ where $\phi_1 \in \pi_2(x, \theta^+, y)$ and $\phi_2 \in \pi_2(x^+, \theta^+, y^+)$. Following the proof of [OS08 Theorem 5.1], if $\phi$ has holomorphic representatives for arbitrarily long neck length then by a Gromov compactness argument we need that $\phi_1$ and $\phi_2$ admit holomorphic representatives for some almost complex structures. We conclude that $\mu(\phi_1), \mu(\phi_2) \geq 0$.

On the other hand, from the combinatorial description of the Maslov index given in [Sar11], one can compute that

$$\mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2m_1.$$  

Let $a, b, c$ be the multiplicities of $\phi_2$ as in Figure 25 after subtracting $m_1$.

We compute using Lemma 8.3

$$\mu(\phi) = \mu(\phi_1) + 2(m_1 + a + b + c - 1).$$

Since $m_1 + a + b + c - 1$ is a multiplicity in the diagram and $\phi$ has holomorphic representatives, we conclude that

$$\mu(\phi_1) = 2(m_1 + a + b + c - 1) = 0.$$  

On the other hand, we are computing over $HF$ so the multiplicity of the domains containing $p$ and $q$ is zero. Hence

$$m_1 + a + c - 1 = 0$$

and

$$m_1 + b = 0,$$

as well. Solving for various quantities yields

$$m_1 = 0, \quad a = 0, \quad b = 0, \quad c = 1.$$  

On the other hand, given a $\phi_1$ with $\mu(\phi_1) = 0$ and $m_1 = 0$, we can construct a triangle in $\pi_2(x \otimes x^+, \theta^+ \otimes \theta^+, y \otimes y^+)$ whose moduli space has the same cardinality as $\phi_1$. It follows that $A = \Phi_{\alpha} \circ \Phi_{\beta}$.

**Computing the map $D$.** Consider a homology triangle $\phi = \phi_1 \# \phi_2 \in \pi_2(x \otimes x^-, \theta^+ \otimes \theta^+, y \otimes y^+)$. This corresponds to the middle diagram in Figure 25. We have

$$\mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2m_1$$

$$= \mu(\phi_1) + 2(m_1 + a + b + c).$$

Since the multiplicity of the middle region must be nonnegative, we have

$$m_1 + a + b + c \geq -1.$$  

By stretching the complex structure, we can assume that the only $\phi = \phi_1 \# \phi_2$ which would be counted in the triangle maps satisfy $\mu(\phi_1) \geq 0$. Hence there are two possibilities

$$\mu(\phi_1) = 2 \quad \text{and} \quad m_1 + a + b + c = -1,$$

or

$$\mu(\phi_1) = 0 \quad \text{and} \quad m_1 + a + b + c = 0.$$  

Consider the first case. Assuming that no triangles cross the basepoints, we have that

$$m_1 + a + c = 0 \quad \text{and} \quad m_1 + b = 0$$

from which we conclude that

$$0 = m_1 + (m_1 + a + b + c) = m_1 - 1$$

so $m_1 = 1$. Since $m_1 + b = 0$ we have $b = -1$. Similarly $m_1 + a + c = 1 + a + c = 0$ so we have $a + c = -1$. Since $m_1 + a = 1 + a \geq 0$ and $m_1 + c = 1 + c \geq 0$ we thus know that one of $a$ and $c$ must be $-1$ and the other must be $0$. But we have regions with multiplicity $m_1 + a = b = a$ and $m_1 + b + c = c$, so this forces a region to have negative multiplicity. Hence for sufficiently stretched almost complex structure these homology triangles don’t contribute to the triangle map.
We now consider the second case, i.e. triangles with

\[ \mu(\phi_1) = 0 \quad \text{and} \quad m_1 + a + b + c = 0. \]

Since we have \( m_1 + a + c = m_1 + b = 0 \) we get \( m_1 = b = 0 \). Hence \( a + c = 0 \) and \( a \geq 0 \) and \( c \geq 0 \), from which we conclude that \( a = c = 0 \). From this we see as before that

\[ D = \hat{\Phi}_{\alpha}^{\beta \rightarrow \beta'}. \]

**Computing the map** \( B \). Consider a homology triangle \( \phi \in \pi_2(x \otimes x^+, \theta^+ \otimes \theta^+, y \otimes y^+) \) which we write as \( \phi = \phi_1 \# \phi_2 \). We use the same strategy as before. By stretching the complex structure sufficiently, we can assume that any homology triangle \( \phi = \phi_1 \# \phi_2 \) which has complex representative satisfies \( \mu(\phi_1) \geq 0 \) and \( \mu(\phi_2) \geq 0 \). Suppose that \( \mu(\phi) = 0 \). We have

\[ \mu(\phi) = \mu(\phi_1) + \mu(\phi_2) - 2m_1, \]

and considering the rightmost diagram in Figure 26 we have

\[ \mu(\phi) = \mu(\phi_1) + 2(m_1 + a + b + c) - 1. \]

The integer \( m_1 + a + b + c \) must be nonnegative, since there is a region of \( \phi \) with that multiplicity. The only possibility is that

\[ \mu(\phi_1) = 1 \quad \text{and} \quad m_1 + a + b + c = 0. \]

If we assume that \( \phi \) is counted in the hat triangle map, then the multiplicities of the regions containing \( p \) and \( q \) must be zero, and hence we have that

\[ m_1 + a + c = 0 \quad \text{and} \quad m_1 + b = 0. \]

Since we also have that \( m_1 + a + b + c = 0 \), one finds that \( m_1 = b = 0 \) and \( a + c = 0 \). Since both \( a \) and \( c \) are multiplicities of domains of a holomorphic triangle, they must be nonnegative, so \( a = c = 0 \). Finally we note that there is a region with multiplicity \( m_1 + b + c - 1 = -1 \). Hence there can be no holomorphic triangle and the map \( B \) is zero, as claimed.

**Proof of 8.1** By multiplying the two matrices, we get

\[ \Phi_{\alpha \cup \alpha \cup \alpha_0 \rightarrow \alpha' \cup \alpha' \cup \alpha_0}^{\beta \cup \beta \cup \beta_1} = \begin{pmatrix} \Phi_{\alpha \rightarrow \alpha'}^{\beta \rightarrow \beta'} & 0 \\ C & \Phi_{\alpha \rightarrow \alpha'}^{\beta \rightarrow \beta'} \end{pmatrix}. \]

It’s easy to see (using the above techniques for instance) that

\[ \Phi_{\alpha \cup \alpha \cup \alpha_0 \rightarrow \alpha' \cup \alpha' \cup \alpha_0}^{\beta' \cup \beta' \cup \beta_1} = \begin{pmatrix} \Phi_{\alpha' \rightarrow \alpha}^{\beta' \rightarrow \beta} & 0 \\ 0 & \Phi_{\alpha' \rightarrow \alpha}^{\beta' \rightarrow \beta} \end{pmatrix}. \]

Hence we have that

\[ \Phi_{\alpha \cup \alpha \cup \alpha_0 \rightarrow \alpha' \cup \alpha' \cup \alpha_0}^{\beta \cup \beta \cup \beta_1} \simeq \begin{pmatrix} \Phi_{\alpha \rightarrow \alpha}^{\beta \rightarrow \beta} & 0 \\ C & \Phi_{\alpha \rightarrow \alpha}^{\beta \rightarrow \beta} \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}, \]

as claimed.

From this point forward, \( C \) will denote the lower left component of the transition map \( \Phi_{H_1 H_2} \).

**9. A Cobordism Map for \( C \).**

We will show in Section 10 that \( C \) can be identified with \( \partial_1 \cdot \), the map which counts disks going over the basepoint exactly once. Before we show that, we will first show that \( C \) induces a well defined map \( C : \hat{H} \hat{F}(Y, p) \rightarrow \hat{H} \hat{F}(Y, p) \) instead of just being a component of a transition map on certain diagrams.

Recall that \( C \) appears in the computation of a “swap” cobordism. That is, if we pick a ball containing two basepoints \( p, q \in Y \) (or equivalently a distinguished path \( \lambda \) between \( p \) and \( q \)), the effect of the cobordism in Figure 26

**Lemma 9.1.** The swap cobordism shown in Figure 26 has block matrix form

\[ \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}. \]
Proof. The effect of the swap cobordism is that of $\phi_* : \widehat{HF}(Y, p) \to \widehat{HF}(Y, q, p)$ where $\phi : Y \to Y$ is a diffeomorphism swapping $p$ and $q$ which is equal to the identity outside of the chosen ball containing $p$ and $q$. We computed in Lemma 8.1 that the effect of $\phi_*$ with the chosen diagrams was that of the block matrix $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$.

□

Lemma 9.2. The map $C : \widehat{HF}(Y, p) \to \widehat{HF}(Y, p)$ is the map induced by the cobordism in Figure 27.

Proof. Note that the cobordism in Figure 27 can be decomposed as in Figure 28.

By Lemmas 5.3, 5.4 and 9.1 the map on the left is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = (C)$, and hence so is the one on the right.

□

Remark 9.3. One can compute without enormous difficulty the cobordism map explicitly instead of just using TQFT properties, and see that it corresponds exactly to the lower left block of the map induced by the swap cobordism.
In the next section we will see an identification of $C$ as $(\partial_1)_*$. Note that our geometric description of $C$ as a cobordism map recovers the fact that $\partial_2^C$ is chain homotopic to 0 on $\widehat{HF}(Y,p)$.

This can expressed by a sequence of pictures and then an easy computation, as expressed in Figure 29.

![Figure 29. A composition useful for computing $C^2$.](image)

The composition on the right (as a cobordism from top to bottom) is

$$C \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 0.$$

### 10. An Identification of $C$ as $(\partial_1)_*$

In this section, we give an identification of $C$ as $(\partial_1)_*$. The results of this section of course only apply to connected $(Y,p)$ since the differential $\partial_1$ isn’t defined for disconnected manifolds. Consider the complex $CF^\infty(Y,p,s)$. In general we can write the differential $\partial^\infty$ as a sum based on the number of times a disk crosses over the basepoint, i.e.

$$\partial^\infty = \partial_0 + \partial_1 U + \partial_2 U^2 + \cdots.$$

Pick a diagram which is strongly admissible for a fixed $s \in \text{Spin}^c(Y)$ structure $s$. Such a diagram exists by [OS04b, Section 5]. By [OS04b, Remark 4.11] a diagram which is strongly admissible for a given $s$ is weakly admissible for all $\text{Spin}^c(Y)$ structures and hence can be used to compute $\widehat{HF}(Y,p) = \bigoplus_{s \in \text{Spin}^c(Y)} \widehat{HF}(Y,p,s)$.

The following is a key observation from the definitions:

**Lemma 10.1.** The differential $\partial_1$ induces a map on $\widehat{HF}(\Sigma, \alpha, \beta, J, p, s)$.

**Proof.** Simply note

$$0 = \partial^\infty \partial^\infty = \partial_0^2 + (\partial_0 \partial_1 + \partial_1 \partial_0) U + (\partial_1^2 + \partial_2 \partial_0 + \partial_0 \partial_2) U^2 + \cdots.$$ 

The zeroth order term shows $\partial_1$ is a differential on $\widehat{CF}(\Sigma, \alpha, \beta, J, p, s)$, the first order term shows that $\partial_1$ is a chain map on $\widehat{CF}(Y,p)$, and the second order term shows that $\partial_1^2$ is chain homotopic to 0 on $\widehat{CF}(Y,p,s)$. \qed

**Lemma 10.2.** For a certain choice of complex structure $J$ on $W_\partial = \Sigma \times \mathbb{R} \times [0,1]$, we have that $C$ and $\partial_1$ are $\partial_0$-chain homotopic on $\widehat{CF}(\Sigma, \alpha, \beta, J, p, s)$.

**Proof.** We recall how $C$ was defined. Most simply it’s defined as a cobordism map, but it’s also defined in terms of certain double pointed diagrams for $\widehat{HF}(Y,p,q,s)$. Let $H_p$ denote the diagram $(\Sigma, \alpha, \beta, J, p)$ and let $H_{p,q}$ denote the diagram with (0,3) stabilization performed at $q \in \Sigma$ which is in the same region as $p$ on $\Sigma$. This is shown in Figure 22. By Lemma 8.1 the transition map $\widehat{\Phi}_{H_{p,q}H_{q,p}}$ takes the form

$$\widehat{\Phi}_{H_{p,q}H_{q,p}} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$$

when viewed as a map from $\widehat{HF}(H_{p,q}) \cong \widehat{HF}(H_q) \oplus \widehat{HF}(H_p) \to \widehat{HF}(H_q) \oplus \widehat{HF}(H_q) \cong \widehat{HF}(H_{q,p})$.

We will consider the transition map $\Phi^\infty = \widehat{\Phi}_{H_{p,q}H_{q,p}}$ over $CF^\infty(\Sigma, \alpha, \beta, J, p, s)$, which we can regard as the $\mathbb{Z}_2[U, U^{-1}, V, V^{-1}]$-module

$$\widehat{CF}(\Sigma, \alpha, \beta, J, p, s)[U, U^{-1}, V, V^{-1}].$$

Let $U$ correspond to $p$ and $V$ correspond to $q$. By [OS08, Proposition 6.5], we know that if we view $H_{p,q}$ and $H_{q,p}$ as connected sums of diagrams of the form $(\Sigma, \alpha, \beta)\#(S^2, \alpha_0, \beta_0, p, q)$ then for sufficiently stretched...
complex structure on each diagram, and for connected sum point on \(S^2\) sufficiently close to \(\beta_0\), we can realize the differentials as
\[
\partial_{H_1} = \left( \sum_{i=0}^{\infty} \partial_i V^i \begin{array}{c} V \quad U \end{array} \right)
\]
and
\[
\partial_{H_2} = \left( \sum_{i=0}^{\infty} \partial_i U^i \begin{array}{c} U \quad V \end{array} \right).
\]
An easy argument shows that we can realize both diagrams \(H_{p,q}\) and \(H_{q,p}\) as having the above differentials for the same complex structure \(J\) (make the \(\alpha_0\) and \(\beta_0\) curves on \(H_{p,q}\) very small perturbations of the \(\alpha_0\) and \(\beta_0\) curves on \(H_{q,p}\) to ensure the connected sum point is sufficiently close to \(\beta_0\) in both diagrams, then stretch the neck sufficiently).

Write
\[
\Phi^\infty = \Phi_0 + \Phi_{10} U + \Phi_{01} V + \cdots.
\]
Note that
\[
\Phi_0 = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},
\]
by definition. We have that \(\Phi^\infty\) is a chain map, so we have
\[
\Phi^\infty \partial_{H_1} = \partial_{H_2} \Phi^\infty.
\]
Collecting \(U^1\) terms we get
\[
\Phi_{10} \begin{pmatrix} \partial_0 & 0 \\ 0 & \partial_0 \end{pmatrix} U + \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial_0 & 0 \\ 0 & \partial_0 \end{pmatrix} \Phi_{10} U + \begin{pmatrix} \partial_1 U & U \\ 0 & \partial_1 U \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}
\]
and hence
\[
\begin{pmatrix} \partial_1 U + UC \\ \partial_1 UC \\ UC + \partial_1 U \end{pmatrix} = \begin{pmatrix} \partial_0 & 0 \\ 0 & \partial_0 \end{pmatrix} \Phi_{10} U + \Phi_{10} \begin{pmatrix} \partial_0 & 0 \\ 0 & \partial_0 \end{pmatrix} U.
\]
Hence all of the terms in the left hand side are chain homotopic to zero in \(\tilde{CF}\). Thus we have shown for a particular complex structure \(J\) we have an identification of \(C\) with \((\partial_1)_\ast\). \(\square\)

**Theorem 10.3.** For a generic choice of \(J\) on \(W_D\) we have that \(\partial_1 \simeq C\) on \(\tilde{CF}(\Sigma, \alpha, \beta, J, p, s)\).

**Proof.** Let \(J_1\) be a complex structure for which \(\partial_1\) is chain homotopic to \(C\) on \(\tilde{CF}\) and let \(J_2\) be an arbitrary complex structure. The map \(C\), being the component of a transition map between two diagrams, must commute with the change of almost complex structure maps. Similarly \(\partial^\infty\) must also commute with the change of almost complex structure maps of \(CF^\infty\). Let \(\Psi^\infty = \Psi_{1,J_2}^\infty\) and \(\Psi_0 = \tilde{\Psi}_{J_1,J_2}\), denote the change of almost complex structure maps over \(CF^\infty\) and \(\tilde{CF}\) respectively. We have that
\[
C_{J_2} \Psi_0 = \Psi_0 C_{J_1}.
\]
We also have that
\[
\partial_{J_2}^\infty \Phi^\infty = \Phi^\infty \partial_{J_2}^\infty.
\]
Writing \(\Psi^\infty = \Psi_0 + \Psi_1 U + \cdots\) and collecting the \(U^1\) terms of the above commutation relation, we see that
\[
\partial_{1,J_2} \Psi_0 = \Psi_0 \partial_{1,J_1} + \Psi_1 \partial_{0,J_1} + \partial_{0,J_2} \Psi_1.
\]
In particular
\[
\partial_{1,J_2} \Psi_0 \simeq \Psi_0 \partial_{1,J_1}.
\]
Combining the fact that \(\partial_{1,J_1} \simeq C_{J_1}\) with equations \([1]\) and \([2]\), we get
\[
C_{J_2} \Psi_0 \simeq \partial_{1,J_2} \Psi_0.
\]
Precomposing with \(\Psi_0' = \tilde{\Psi}_{J_2,J_1}\) we thus get
\[
C_{J_2} \Psi_0 \Psi_0' \simeq \partial_{1,J_2} \Psi_0 \Psi_0'.
\]
Since \(\Psi_0 \Psi_0' \simeq \text{id}\) we thus have that
\[
C_{J_2} \simeq \partial_{1,J_2},
\]
concluding the proof. \(\square\)
As a consequence, we can now compute any $\widehat{H}_F \gamma$ cobordism map without any contact geometry:

**Theorem F.** The $\widehat{H}_F \gamma$ cobordism maps can be defined without the use of contact geometry. The cobordism maps can be computed via the action of the mapping class group, explicit formulas for 0-, 1-, 3-, and 4- handles, the triangle map for 2-handles, as well as the splitting, termination, creation, and $(\partial_1)_*$ maps for certain simple graphs.

**Proof.** Given a graph cobordism $(W, \Gamma) : (Y_1, p_1) \to (Y_2, p_2)$, pick a Morse pair $(f, g)$ on $W$ with distinct critical values, and no critical points on the boundary. This gives a sequence of handle attachments which build $W$ from $Y_1 \times I$. Modify the graph $\Gamma$ by performing isotopies, vertex slides, and other modifications which preserve an isotopy class of a regular neighborhood of $\Gamma$ so that each component of each level set containing a critical value of $f$ intersects $\Gamma$ nontrivially and transversely. By perturbing the graph slightly, without changing a regular neighborhood of the graph we can ensure that all vertices have valence 1 or 3. We can perturb the embedding of the graph so that on each edge, the function $f|_{\Gamma}$ is Morse and so that no critical points of $f|_{\Gamma}$ occur on the endpoints of an edge. At any critical point of $f|_{\Gamma}$ along the interior of an edge, we can add a small trivial strand in the direction of $\nabla f$, and then modify the graph slightly so that the old edge and the new edge form a trivalent vertex with no critical points of $f|_{\Gamma}$ anywhere. Thus we can assume that there are no critical points of $f|_{\Gamma}$ on the graph.

As we pass through the level sets, when we reach a univalent vertex, we either apply a creation/termination cobordism map (Lemmas 5.5 and 5.6). If we reach a critical point of $f$ in the 4-manifold $W$, we apply the 4-manifold cobordism maps. If we reach no critical points or vertices, then the effect is simply by the effect of the mapping class group induced by the paths. □

11. The $\pi_1(Y, p)$ action on $\widehat{H}_F \gamma(Y, p)$.

In this section we apply the $\widehat{H}_F \gamma$-TQFT to the action of $\pi_1(Y, p)$ on $\widehat{H}_F \gamma(Y, p)$. In [JT12], Juhász and Thurston show that the based mapping class group $\text{MCG}_\partial(Y, p) = \pi_0 \text{Diff}(Y, p)$ acts on $\widehat{H}_F \gamma(Y, p)$. There is a fibration

$$\text{Diff}(Y, p) \to \text{Diff}(Y) \to Y.$$ 

The long exact sequence of homotopy groups associated to the fibration gives a map

$$\pi_1(Y, p) \to \pi_0 (\text{Diff}(Y, p)) = \text{MCG}(Y, p),$$

and hence induces an action of $\pi_1(Y, p)$ on $\widehat{H}_F \gamma(Y, p)$.

The following result was conjectured in [Juh13]: the $\pi_1(Y, p)$ action on $\widehat{H}_F \gamma(Y, p)$ has the formula $\gamma_* = 1 + [\gamma](\pi \circ \iota)$, where $\pi$ and $\iota$ are the maps in the long exact sequence

$$\cdots \to \widehat{H}_F \gamma(Y, p) \xrightarrow{\cdot} \text{HF}^+(Y, p) \xrightarrow{\text{U}} \text{HF}^+(Y, p) \xrightarrow{\text{Z}} \widehat{H}_F \gamma(Y, p) \to \cdots.$$ 

We will show how the $\widehat{H}_F \gamma$ TQFT recovers this formula.

**Theorem D.** The action of $\pi_1(Y, p)$ on $\widehat{H}_F \gamma(Y, p)$ descends to an action of $H_1(Y; \mathbb{Z})$. In fact, the map $(\partial_1)_* : \widehat{H}_F \gamma(Y, p) \to \widehat{H}_F \gamma(Y, p)$ induced by the first differential $\partial_1$ satisfies the following for any $\gamma \in \pi_1(Y, p)$:

1. $\gamma_* = 1 + (\partial_1)_* [\gamma]$;
2. $(\partial_1)_* [\gamma] = [\gamma](\partial_1)_*$;
3. $\partial_1 |^2 = 0$.

This computation will be the result of composing cobordisms and using TQFT properties. Pick an auxiliary basepoint $q$ such that $\gamma$ does not intersect the point $q$. Consider the cobordism $(W, \Gamma) : (Y, p, q) \to (Y, p, q)$ where $W = Y \times I$ and $\Gamma$ is the union of the two paths $\widehat{\gamma}(t) = (\gamma(t), t)$ and $\widehat{q}(t) = (q, t)$. This is shown in Figure 30.

We wish to compute explicitly the map $\widehat{F}_{W, \Gamma}$. For this, we use functoriality of composition in our TQFT. Given an unpointed Heegaard splitting $(\Sigma, \alpha, \beta)$ such that $p, q \in \Sigma$, and such that $p$ and $q$ are in the same component of $\Sigma \setminus (\alpha \cup \beta)$. Let $H_1$ be the diagram formed by taking a diagram $(\Sigma, \alpha, \beta, q)$ and performing $(0,3)$-stabilization at $p$. Let $H_2$ be the diagram formed by taking $(\Sigma, \alpha, \beta, p)$ and performing $(0,3)$-stabilization at $q$. These diagrams are shown in Figure 31.
As $\mathbb{Z}_2$-modules, there is an obvious isomorphism between $\widehat{HF}(H_1)$ and $\widehat{HF}(H_2)$, since if $H_0$ denotes the unstabilized isotopy diagram with a single basepoint (either $p$ or $q$) but all of the same other $\alpha$ and $\beta$ curves, then both $\widehat{HF}(H_1)$ and $\widehat{HF}(H_2)$ are both isomorphic to $\widehat{HF}(H_0) \otimes \mathbb{Z}_2^2 = \widehat{HF}(H_0) \oplus \widehat{HF}(H_0)$. We let $\Phi_{H_1,H_2} : \widehat{HF}(H_1) \to \widehat{HF}(H_2)$ denote the transition map. Let $M_1$ be the block matrix for $\widetilde{F}_{W,\Gamma}$ with respect to the diagram $H_1$, and let $M_2$ be the block matrix for $\widetilde{F}_{W,\Gamma}$ with respect to $H_2$.

**Lemma 11.1.** The matrices $M_i$ can be computed to be

$$M_1 = \begin{pmatrix} 1 & [\gamma] \\ 0 & 1 \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} \gamma_* & [\gamma] \\ 0 & \gamma_* \end{pmatrix}$$

**Proof.** We now use TQFT properties to compute $M_i$. For this we will use the computation of the splitting cobordisms and the termination/creation cobordisms from Section 5.

We compute the effect of composing these cobordisms with $(W, \Gamma)$ on both sides appropriately. Write

$$M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$ 

In Figure 32 we provide two useful equivalences of cobordisms which allow us to compute the components $a_1, c_1,$ and $d_1$ of $M_1$. The remaining component, $b_1$, can be computed by considering the third cobordism in Figure 32. The composition is equivalent to the identity cobordism with $\gamma$ spliced in, so by Theorem C we know that the result is the homology action $[\gamma]$. To compute $M_2$, we can do the same trick, but with slightly different cobordism maps. These are shown in Figure 33. The desired form of $M_1$ and $M_2$ follow easily from the relations shown in Figures 32 and 33. 

---

**Proof of Theorem D** This is now just an easy computation. We must have that $M_2 \Phi_{H_1,H_2} = \Phi_{H_1,H_2} M_1$ and hence

$$\begin{pmatrix} \gamma_* & [\gamma] \\ 0 & \gamma_* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (\partial_1)_* & 1 \end{pmatrix} \begin{pmatrix} 1 & [\gamma] \\ 0 & 1 \end{pmatrix}.$$ 


Multiplying this out we see
\[
\left( \gamma_* + \left[ \gamma \right]\left( \partial_1 \right)_* \right) \left( \left[ \gamma \right] \right) = \left( \left( \left( \partial_1 \right)_* \right) \left( \left[ \gamma \right] \right) + 1 \right),
\]
from which all of the identities readily follow.

\[\square\]

**Remark 11.2.** If \([\gamma]\) vanishes then so does \(\gamma_*\). Since \([2\gamma]\) always vanishes, we conclude that \(\gamma_* \gamma_* = \text{id}\), so the \(\pi_1\) action acts by involutions, at least when we work over \(\mathbb{Z}_2\).

**Remark 11.3.** It is straightforward to compute explicitly that \(\pi \circ \iota = (\partial_1)_*\) where \(\iota\) and \(\pi\) are the maps in the long exact sequence in equation (3), as one would expect given the description of the \(\pi_1(Y,p)\) action in [Juh13].

12. **Graphs with isolated components and the \(H_1(Y;\mathbb{Z})/\text{Tor}\) action on multipointed diagrams**

In Theorem [C] we computed the cobordism map associated to splicing in a loop \(\alpha\) into a graph \(\Gamma\), assuming that \(\alpha\) intersected \(\Gamma\) in a single point. By splicing in a loop, we recovered the \(H_1(Y;\mathbb{Z})/\text{Tor}\) action. A natural question is what the effect of \(\Gamma \cup \alpha\) is if \(\alpha \cap \Gamma = \emptyset\). The answer is that it acts by zero. In fact we have the following.

**Theorem 12.1.** Suppose that \((W,\Gamma)\) is a graph cobordism and \(\Gamma_0\) is a graph in \(W\) such that \(\Gamma_0 \cap \Gamma = \emptyset\) and \(\Gamma \cap \partial W = \emptyset\). Then \(\hat{F}_W,\Gamma\cup\Gamma_0 = 0\).

The main point of the proof is a so called “loop swapping lemma”, essentially saying that we can move a closed loop from one strand to another without changing the cobordism map. An isolated component of the graph can be modified by sliding vertices across edges and can be assumed to be an arc with loops attached.
Figure 33. Three compositions of cobordisms which fully compute $M_2$. Each cobordism is viewed as cobordism from top to bottom. Next to each cobordism we indicate the implied equality of cobordism maps.

at a single point. The “loop swap lemma” allows us to move all of the loops onto a different component without changing the cobordism map. Hence we will be able to show that the cobordism map is the same as one with an arc as a component of the graph, but such a cobordism map can be explicitly computed to be zero.

In principle we could just directly apply our result claiming that splicing in a loop results in the homology action, but a-priori there may be some ambiguity about what is meant with the $H_1$ action on multipointed diagrams, so instead we approach the problem from more basic considerations.

Lemma 12.2. Suppose that $Y^3$ is connected, and $(W, \Gamma) : (Y, p) \rightarrow (Y, p)$ is a graph cobordism with $W = Y \times I$ and $p = \{p_1, p_2\}$. Suppose that $\zeta$ is a loop in $Y \times I$ which intersects $p_1 \times I$ at a single point and doesn’t intersect $p_2 \times I$. If we pick a path from $p_1$ to $p_2$ and take a diagram for $(Y, p_1)$ and stabilize it at $p_2$ using this path, then with respect to the decomposition of $\widehat{HF}(Y, p)$ induced by this diagram the cobordism map takes the form

$$\widehat{F}_{W, \Gamma} = \begin{pmatrix} [\zeta] & 0 \\ 0 & [\zeta] \end{pmatrix}. $$

Proof. We use functoriality to compute $\widehat{F}_{W, \Gamma}$ by pre-and postcomposing with convenient cobordisms. Write

$$\widehat{F}_{W, \Gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. $$

Then three useful compositions and the inequalities that one can read off from them are shown in Figure 34. The equations shown in the figure allow one determine all of $a, b, c, $ and $d$. 

\[\square\]
Figure 34. Compositions which allow us to compute $\tilde{F}_{W,\Gamma}$ where $\Gamma$ is a $\{p_1, p_2\} \times I \cup \zeta$ where $\zeta$ intersects $p_1 \times I$ in a single point.

We now prove another lemma, essentially letting us take a curve spliced into a graph and swap it onto another component.

**Lemma 12.3.** Suppose that $\zeta$ and $\zeta'$ are curves in $W = Y \times I$, and that $\zeta$ intersects $p_1 \times I$ at one point, but not $p_2 \times I$, and $\zeta'$ intersects $p_2 \times I$ at one point but not $p_1 \times I$. Suppose $p = \{p_1, p_2\}$. Suppose that $\lambda$ is a path from $p_1$ to $p_2$, and that the concatenation $\lambda \ast \zeta \ast \lambda^{-1}$ is homotopic to $\zeta'$ as curves in $\pi_1(Y, p_2)$. Writing $\gamma$ for $p \times I$, we have

$$\tilde{F}_{W, \gamma \cup \zeta} = \tilde{F}_{W, \gamma \cup \zeta'}.$$  

*Proof.* We can directly compute the cobordism maps. A picture of the two cobordism maps is shown in Figure 35.

We now claim that the cobordisms $(W, \gamma \cup \zeta)$ and $(W, \gamma \cup \zeta')$ are related by

$$(W, \gamma \cup \zeta) = S \circ (W, \gamma \cup \zeta') \circ S,$$

where $S$ is the “swap” cobordism along the distinguished path $\lambda$ from $p_1$ to $p_2$, obtained by swapping the basepoints along a prescribed path between them. This is displayed in Figure 36.

Using Lemma 10.1 and the computations of the maps $\tilde{F}_{W, \gamma \cup \zeta}$ in Lemma 12.2, we see that

$$\tilde{F}_{W, \gamma \cup \zeta'} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} [\gamma] & 0 \\ 0 & [\gamma] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}.$$
Figure 36. A relation between the cobordisms. The cobordism $S$ is the map on the top and bottom on the left. Note that the cobordism is a four manifold, so over and under crossings are meaningless.

Hence

$$\hat{F}_{W; \gamma \cup \zeta'} = \left( \begin{array}{c} \gamma \ 0 \\ 0 \ [\gamma] \end{array} \right),$$

which is equal to $\hat{F}_{W; \gamma \cup \zeta}$ by Lemma 12.2.

We now bootstrap our way to higher numbers of strands to a full “loop swap lemma”:

**Lemma 12.4 (Loop Swap Lemma).** Suppose that $\zeta$ and $\zeta'$ are two homotopic curves in $W = Y \times I$, suppose that $\zeta$ intersects $p_1 \times I$ at one point, but not $p_2 \times I$, and $\zeta'$ intersects $p_2 \times I$ at one point but not $p_1 \times I$. Suppose additionally that neither $\zeta$ nor $\zeta'$ intersect any of the $p_i$ for $i > 2$. We have

$$\hat{F}_{W; (p \times I) \cup \zeta} = \hat{F}_{W; (p \times I) \cup \zeta'}. $$

**Proof.** This proceeds by induction. The base case is Lemma 12.3. Suppose that the claim is true for $|p| = n \geq 2$. Let $p_{n+1}$ be a new basepoint. Pick a ball containing all of the basepoints. Let $H_0$ be a diagram for $(Y, p)$ and let $H$ be the diagram stabilized at $p_{n+1}$ with respect to the chosen ball. Let define the maps

$$F_1 = \hat{F}_{W; (p \times I) \cup \zeta},$$

$$F_2 = \hat{F}_{W; (p \times I) \cup \zeta'},$$

and

$$\hat{F}_1 = \hat{F}_{W; (p \times I) \cup \zeta},$$

$$\hat{F}_2 = \hat{F}_{W; (p \times I) \cup \zeta'}.$$

Note that $F_1 = F_2$ by induction. By following our time tested strategy of pre- and post-composing with path termination, path creation, or splicing cobordisms, and then rearranging, we see that

$$\hat{F}_i = \left( \begin{array}{c} F_i \\ 0 \ F_i \end{array} \right).$$

Hence $\hat{F}_1 = \hat{F}_2$ by induction.

We are now in position to prove the main theorem of this section:

**Proof of Theorem 12.1.** First suppose that $(W, \Gamma) : (Y, p) \to (Y', p')$ is a cobordism with $W$ connected and $\partial W$ nonempty. Suppose that $\Gamma \cap \partial Y$ and $\Gamma \cap \partial W = \emptyset$. As smooth manifolds, write $W = W_3 \cup W_2 \cup W_1$, where $W_i$ is obtained by attaching $i$-handles. Say that $W_i$ is a cobordism from $Y$ to $Y_1$. Isotope $\Gamma$ so $\Gamma \cap Y_1 \neq \emptyset$ and so that $\Gamma$ is transverse to $Y_1$. Let $p_1 = Y_1 \cap \Gamma$. Insert a trivial cobordism $(Y_1 \times I, p_1 \times I)$ and write $W = W_3 \circ W_2 \circ (Y_1 \times I) \circ W_1$. The map $\pi_1(Y_1) \to W$ is a surjection, and hence we can isotope the graph $\Gamma_0$ so that $\Gamma_0$ is completely contained in $Y_1 \times I$. By isotoping edges across each other in $\Gamma_0$ and by deleting or adding trivial strands to $\Gamma_0$ we can assume that $\Gamma_0$ is a bouquet of closed paths which mutually intersect at a single point $q \times t_i$, as well as the interval $q \times I'$ for some proper subinterval $I' \subseteq I$. Using the loop-swapping lemma (Lemma 12.4), the cobordism map is unchanged after we swap all of the loops onto other strands. A picture is shown in Figure 37.
A cobordism map with an isolated arc which doesn’t intersect the boundary is zero, as can be computed using Lemmas 5.3 and 5.4 and functoriality, since the the composition of a path creation and path termination cobordism is zero.

We now consider the case where $W$ is connected but $\partial W = \emptyset$. By construction, the cobordism map $\tilde{F}_{W,F,\Gamma_0} : SFH(\emptyset) \to SFH(\emptyset)$ (where $SFH(\emptyset) = \mathbb{Z}_2$) is defined by removing two balls from $W$, and connecting them with a path to $\Gamma$. Hence in the case that $W$ is connected but $\partial W = \emptyset$, we simply need to show that if $\Gamma$ has two components, then $\tilde{F}_{W,\Gamma} = 0$. By construction, we remove two balls from $W$, and connect the new copies of $S^3$ to $\Gamma$ by arbitrary paths. Pick paths which go to the same component of $\Gamma$, and then apply the result to the case that $\partial W \neq \emptyset$ to see that $\tilde{F}_{W,\Gamma} = 0$.

We finally consider the case where $W$ is disconnected. Writing $(W,\Gamma) = (W_1,\Gamma_1) \sqcup (W_2,\Gamma_2)$, we have $\tilde{F}_{W,\Gamma} = \tilde{F}_{W_1,\Gamma_1} \circ \tilde{F}_{W_2,\Gamma_2}$ since we are working over $\mathbb{Z}_2$. By assumption one of the maps $\tilde{F}_{W_i,\Gamma_i}$ or $\tilde{F}_{W_2,\Gamma_2}$ vanishes, and hence the $\mathbb{Z}_2$ tensor product of the maps must as well. \hfill $\square$

13. The action of $\Lambda^*(H_1(Y;p,Z)/\text{Tors})$ on $\tilde{HF}(Y,p)$

In this section, we prove Theorem 13, namely that the using the graph cobordism maps we can construct an action of $\Lambda^*(H_1(Y;p,Z)/\text{Tors})$ on $\tilde{HF}(Y,p)$.

Suppose that $(Y,p)$ is a multibased 3-manifold. We now describe an action of $H_1(Y,p;\mathbb{Z})/\text{Tors}$ on $\tilde{HF}(Y,p)$. One approach to describing the $H_1(Y,p;\mathbb{Z})$ action would be to use an approach similar to the one taken in [Ni10] and try to describe an action by counting holomorphic disks. We will not take this approach, and instead will show that an $H_1(Y,p;\mathbb{Z})/\text{Tors}$ action can be described by graph cobordism maps.

For convenience we will assume that $Y$ is connected, though the same construction yields an action of $H_1(Y,p;\mathbb{Z})/\text{Tors}$ on $\tilde{HF}(Y,p)$ even if $Y$ is disconnected. The following argument requires slightly more bookkeeping, so we assume that $Y$ is connected for convenience.

Writing $Y/p$ for the quotient space obtained by identifying all of the basepoints $p$ to a point, there is an isomorphism $H_1(Y,p;\mathbb{Z}) \to H_1(Y/p;\mathbb{Z})$, so we will exhibit a $\Lambda^*(H_1(Y/p;\mathbb{Z})/\text{Tors})$ action on $\tilde{HF}(Y,p)$.

Write $p = \{p_1, \ldots, p_\ell\}$. Pick a distinguished basepoint $p_1$ and a set of paths $\lambda_{ij}$ from $p_1$ to $p_j$ for $2 \leq j \leq \ell$. Assume that these paths are disjoint and embedded. Let $T$ denote the collection of paths $\lambda_{ij}$. Let $N$ be a regular neighborhood of the union of these paths. Notice that $\partial N = S^2$ and hence by Mayer-Vietoris, inclusion yields an isomorphism

$$H_1(Y \setminus N) \oplus H_1(N/p) \to H_1(Y/p).$$

Now $H_1(N/p)$ is freely generated by the images of the paths $\lambda_{ij}$. Also the inclusion map $H_1(Y \setminus N) \to H_1(Y)$ is an isomorphism by Mayer-Vietoris. Hence given such a $T$, to define a map

$$\mathcal{F}_T : H_1(Y/p) \to \text{End}_{\mathbb{Z}_2}(\tilde{HF}(Y,p))$$

it is sufficient to define a map

$$\mathcal{F}_0 : H_1(Y) \to \text{End}_{\mathbb{Z}_2}(\tilde{HF}(Y,p))$$

and a map

$$\mathcal{F}_T^\ell : H_1(N/p) \to \text{End}_{\mathbb{Z}_2}(\tilde{HF}(Y,p)).$$

Since $H_1(N/p)$ is freely generated by the paths $\lambda_{ij}$, we know that such a homomorphism $\mathcal{F}_T^\ell$ is equivalent to a choice of elements $\mathcal{F}_T^\ell(\lambda_{ij}) \in \text{End}_{\mathbb{Z}_2}(\tilde{HF}(Y,p))$ for each $2 \leq j \leq \ell$. The map $\mathcal{F}_T$ then can be constructed as

$$\mathcal{F}_T(w + \lambda) = \mathcal{F}_0(w) + \mathcal{F}_T^\ell(\lambda)$$

where $w \in H_1(Y)$ and $\lambda \in H_1(N/p)$. 36
We define $F_0(w)$ to be equal to the cobordism on the left in Figure 38 and $F_T^1(\lambda)$ to be equal to the cobordism map on the right in Figure 38.

**Figure 38.** The cobordisms defining the maps $F_0(w)$ and $F_T^1(\lambda)$. Note that by Lemma 12.4 the map $F_0$ is independent of the choice of vertical path which the loop $w$ is spliced into.

**Lemma 13.1.** The map $F_T : H_1(Y/p) \to \text{End}_{\mathbb{Z}_2}(\hat{HF}(Y,p))$ is well defined for a particular $T$.

**Proof.** The map $F_0$ is well defined on $H_1(Y)$ by the same argument as in the proof Theorem C. Also $F_0(w)$ is independent of the vertical path which the curve $w$ is spliced into by Theorem 12.4. □

**Lemma 13.2.** The map $F_T : H_1(Y/p) \to \text{End}_{\mathbb{Z}_2}(\hat{HF}(Y,p))$ is independent of the choice of $T$.

**Proof.** Any $T'$ which is a collection of embedded paths from a distinguished basepoint to all the other basepoints differs from $T$ by a sequence of the following moves:

1. homotoping one of the $\lambda_{1j}$;
2. concatenating a $\lambda_{1j}$ with a closed loop in $Y$;
3. changing the distinguished basepoint from $p_1$ to $p_k$ by replacing each $\lambda_{1j}$ for $j \neq k$ with a path which is homotopic to $\lambda_{1k}^{-1} \ast \lambda_{1j}$. The path $\lambda_{1k}$, the original path from $p_1$ to $p_k$, is then replaced with $\lambda_{k1} = \lambda_{1k}^{-1}$.

We need to show that $F_T$ is independent of the above moves. A homotopy of the $\lambda_{1j}$ results in an isotopy of the corresponding graph in the cobordism since the cobordism is 4-dimensional, so homotoping a $\lambda_{1j}$ curve has no effect on the map $F_T$.

If $T'$ results from $T$ by concatenating $\lambda_{1j}$ with a closed loop $w$, then to show invariance we need to show that

$$F_0(w) + F_T^1(\lambda_{1j}) = F_T^1(\lambda_{1j} \ast w).$$

In Figure 39 we display the above relation in terms of cobordism maps.

**Figure 39.** A relation which is sufficient to show invariance under (2).
For the doubly pointed diagrams shown in Figure 39, if we put an asterisk at either basepoint, the map on the left takes the form 
\[
\left( \begin{array}{cc}
[\lambda] & 0 \\
0 & [\lambda]
\end{array} \right) + \left( \begin{array}{cc}
0 & 0 \\
1 & 0
\end{array} \right),
\]
by Lemmas 12.2, 5.5, and 5.6. The map on the right takes the form 
\[
\left( \begin{array}{cc}
1 & [\gamma] \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
0 & 0 \\
1 & 0
\end{array} \right) \left( \begin{array}{cc}
1 & -[\gamma] \\
0 & 1
\end{array} \right) = \left( \begin{array}{cc}
[\gamma] & 0 \\
1 & [\gamma]
\end{array} \right).
\]
Hence the maps on the left and right are equal. For diagrams with more than two basepoints, the standard technique of pre- and post-composing with convenient maps as in Lemma 12.4 shows that we have the desired equality for any number of basepoints.

We now need to show invariance of \( F_T \) under move (3). Suppose \( T' \) is obtained from \( T \) by the move in (3). First note that 
\[
F_T(\lambda_{1k}) = F_T(\lambda_{1k}) = -F_T'(1) \lambda_{k1} = -F_T'(\lambda_{k1}).
\]
since the cobordisms representing \( F_T(\lambda_{1k}) \) and \( F_T'(\lambda_{k1}) \) are equal and we are working over \( \mathbb{Z}_2 \). Hence 
\[
F_T(\lambda_{1k}) = F_T'(\lambda_{k1}).
\]
The other equality needed to show (3) is 
\[
F_T(\lambda_{1k} + \lambda_{1j}) = F_T'(\lambda_{1k} + \lambda_{1j}).
\]
Since \( F_T \) is a homomorphism and we are working over \( \mathbb{Z}_2 \), the desired equality becomes 
\[
F_T(\lambda_{1k}) + F_T(\lambda_{1j}) = F_T'(\lambda^{-1}_{1k} \lambda_{1j}).
\]
The desired equality of cobordism maps is demonstrated in Figure 40.

![Figure 40](image1.png)

**Figure 40.** A relation of cobordisms which is sufficient to show that relation (3).

We first perform a manipulation of the graphs as in Figure 41.

![Figure 41](image2.png)

**Figure 41.** Two graphs in \( Y \times I \) which have isotopic regular neighborhoods.

Hence the desired equality can be reduced to the equality shown in Figure 42.

We now add trivial strands and manipulate the graphs in such a way that doesn’t change the isotopy class of a regular neighborhood, as in Figure 43.

![Figure 43](image3.png)

**Figure 43.** The relation in Figure 43 can be verified as in Lemma 7.3 since the equality in Figure 20 is easily verified. Hence \( F_T \) is invariant under the move described in (3).
Using the equality in Figure 41, we can reduce the equality in Figure 40 to this equality. Here $\lambda$ is a curve which is isotopic in $T$ to $\lambda^{-1}_k * \lambda_j$ but doesn’t pass through $p_1$.

We manipulate the graphs in Figure 42 to get an equality which, if true, implies invariance under (3). This equality follows from the equality in Figure 20, which is easily verified.

We now define $F : H_1(Y, p; \mathbb{Z}) \to \text{End}_{\mathbb{Z}_2}(\widetilde{HF}(Y, p))$ to be $F_T$ for any $T$. We now restate Theorem E as follows:

**Theorem E** The above map $F$ induces a map

\[ \Lambda^*(H_1(Y, p; \mathbb{Z})/\text{Tors}) \to \text{End}_{\mathbb{Z}_2}(\widetilde{HF}(Y, p)). \]

For singly pointed diagrams this is the standard $\Lambda^*(H_1(Y; \mathbb{Z})/\text{Tors})$ action defined in [OS04b].

**Proof.** First note that fact that this is the standard action for singly pointed diagrams follows from Theorem C.

We need to show that $F(\eta) = 0$ for any $\eta \in H_1(Y, p; \mathbb{Z})$ which is torsion. In light of the exact sequence

\[ 0 \to H_1(Y) \to H_1(Y, p) \to \tilde{H}_0(p) \to 0, \]

we know that if $\eta \in H_1(Y, p)$ is torsion, then $\eta$ is in the image of $H_1(Y)$. Hence

\[ F(\eta) = F_0(\eta). \]

Using Theorem C for singly pointed diagrams and Lemma 12.2 with the proof of Lemma 12.4 for multipointed diagrams, we know that $F_0(\eta) = 0$ since the standard $H_1(Y; \mathbb{Z})$ defined in [OS04b] for singly pointed diagrams vanishes on torsion elements.

We now need to show that if $\eta$ is an arbitrary element of $H_1(Y, p; \mathbb{Z})$, then $F(\eta) \circ F(\eta) = 0$. Write $\eta = \sum_i t_i$ where each $t_i$ is either a single closed loop or a path between distinct basepoints. We then have

\[ F \left( \sum_i t_i \right) \circ F \left( \sum_i t_i \right) = \sum_{i,j} F(t_i) \circ F(t_j). \]

We first note that $F(t_i) \circ F(t_i) = 0$. To see this in the case that $t_i$ is a closed loop, the graph in the cobordism associated to $F(t_i) \circ F(t_i)$ can be manipulated to have a nullisotopic loop, as in Figure 20. To see that $F(t_i) \circ F(t_i)$ in the case that $t_i$ is a single path between distinct basepoints, the composition is immediately seen to have a nullisotopic loop. Hence to show that $\sum_{i,j} F(t_i) \circ F(t_j) = 0$, since we are working over $\mathbb{Z}_2$, it is sufficient to show that $F(t_i) \circ F(t_i) = F(t_j) \circ F(t_i)$ for all $i$ and $j$. This is can now be seen by manipulating the appropriate graphs, depending on what types of curves $t_i$ and $t_j$ are.
Figure 44. A manipulation of graphs showing that $F(t_i) \circ F(t_j) = F(t_j) \circ F(t_i)$ in the case that $t_i$ and $t_j$ are both closed loops.

If $t_i$ and $t_j$ are two closed loops, then the equality is demonstrated in Figure 44. If $t_i$ is a closed loop and $t_j$ is an arc between distinct basepoints, then the desired equality is seen much the same as in the previous case. If $t_i$ and $t_j$ are both arcs between distinct basepoints, we have two sub-cases. Suppose that $t_i$ is an arc between $p_i^1$ and $p_i^2$ and $t_j$ is an arc between points $p_j^2$ and $p_j^1$. In the case that $|\{p_i^1, p_i^2, p_j^1, p_j^2\}| = 2$ or 4, the desired equality is immediate from the graphs. In the case that $|\{p_i^1, p_i^2, p_j^1, p_j^2\}| = 3$, the desired equality is demonstrated in Figure 38.

Figure 45. A manipulation of graphs showing that $F(t_i) \circ F(t_j) = F(t_j) \circ F(t_i)$ in the case that $t_i$ and $t_j$ are arcs from $p_i^1$ to $p_i^2$ and $p_j^1$ to $p_j^2$ respectively and $|\{p_i^1, p_i^2, p_j^1, p_j^2\}| = 3$.

Remark 13.3. If $(Y, p) = (Y_1, p_1) \sqcup (Y_2, p_2)$ is disconnected, the above construction also yields an action $\Lambda^*(H_1(Y, p; \mathbb{Z})/\text{Tors})$ on $\widehat{HF}(Y, p)$ and the above proof is easily adapted.

14. A refinement over Spin$^c(W)$

In this section, we prove Theorem B, i.e., the $\widehat{HF}_\Gamma$ TQFT is naturally graded over 3- and 4-dimensional Spin$^c$ structures on closed manifolds (as is the case for the $\mathbb{Z}_2$-modules and maps defined in [OS04b] and [OS06]).

14.1. 3-manifolds and Spin$^c$ structures.

Definition 14.1. Suppose $Y^3$ is a closed 3-manifold. A Spin$^c$ structure on $Y$ is a homology class of nonvanishing vector field. Here two vector fields $v, w$ are homologous iff there is a set $B$ which is a disjoint union of balls and $v|_{Y\setminus B} \simeq w|_{Y\setminus B}$ where $\simeq$ denotes homotopic through nonzero vector fields.

For a closed $Y^3$, the set $\text{Spin}^c(Y)$ is nonempty and has a free and transitive $H^2(Y; \mathbb{Z})$ action. The Heegaard Floer Homology groups $\widehat{HF}(Y, p)$ are graded over $\text{Spin}^c(Y)$ structures, i.e. we can write

$$\widehat{HF}(Y, p) = \bigoplus_{s \in \text{Spin}^c(Y)} \widehat{HF}(Y, p, s).$$

For sutured manifolds the situation is similar. Suppose that $v_0$ is a nonzero vector field along $\partial M$ that points into $M$ along $R_-(\gamma)$, points out of $M$ along $R_+(\gamma)$ and on $\gamma$ is the gradient of a height function $s(\gamma) \times I \to I$. 

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Definition 14.2. We define \( \text{Spin}^c(M, \gamma) \) to be the set of homology classes of nonvanishing vector fields \( v \) on \( M \) such that \( v|_{\partial M} = 0 \). Two vector fields are homologous if there is \( B \subseteq \text{int}(M) \) such that \( B \) is a disjoint union of balls and \( v|_{M \setminus B} \cong w|_{M \setminus B} \) rel \( \partial M \).

For a sutured manifold \((M, \gamma)\), the set \( \text{Spin}^c(M, \gamma) \) is nonempty and has a free and transitive action of \( H^2(M, \partial M; \mathbb{Z}) \). The sutured Floer homology groups \( \text{SFH}(M, \gamma) \) are graded over relative \( \text{Spin}^c(M, \gamma) \) structures.

Since \( H^2(B^3, S^2; \mathbb{Z}) = 0 \), we know that there is a unique relative \( \text{Spin}^c \) structure on the sutured manifold \((B^3, \gamma_t)\) where \( \gamma_t \) are the trivial sutures on \( \partial B^3 = S^2 \). Let \( v_t \) be a vector field corresponding to this \( \text{Spin}^c \) structure. If \((Y, p)\) is a multi-based 3-manifold and \( s \in \text{Spin}^c(Y(p)) \), then by gluing balls into each boundary component and gluing \( s \) to copies of \( v_t \) along each boundary component, we get a \( s \in \text{Spin}^c(Y) \). By doing this procedure, we thus get a well defined map

\[
f : \text{Spin}^c(Y(p)) \rightarrow \text{Spin}^c(Y),
\]

which we call the filling map.

We now prove the 3-manifold portion of Theorem \([B]\)

Theorem \([B](a)\). For the sutured manifold \( Y(p) \), the map

\[
f : \text{Spin}^c(Y(p)) \rightarrow \text{Spin}^c(Y)
\]

is an isomorphism.

Proof. We first show that \( f \) is injective. Pick a trivialization of \( TY \), so a vector field \( v \) on \( Y \) determines a map \( Y \rightarrow S^2 \). Similarly a relative \( \text{Spin}^c \) structure on \((Y, p)\) determines a map \( Y(p) \rightarrow S^2 \) which is fixed on \( \partial Y(p) \). Suppose that \( v \) and \( v' \) are nonvanishing vector fields on \( Y(p) \) which when extended over \( Y \) become homotopic on \( Y \setminus B \) for some set \( B \) which is a disjoint union of balls. An easy argument shows that if two vector fields are homotopic on \( Y \setminus B \) then they are also homotopic over \( Y \setminus B' \) where \( B' \) is the image of an isotopy of \( B \). Hence we can assume that if \( Y(p) = Y \setminus \sqcup_i B_i \), then \( B \cap B_i = \emptyset \) for all \( i \). It thus becomes sufficient to show that given a homotopy \( h_t \) between vector fields \( v \) and \( w \) on \( B^3 \) with \( v = w \), the homotopy \( h_t \) can be homotoped relative \( t = 1 \) and \( t = 0 \) to the constant homotopy. The homotopy \( h_t \) can be thought of as a loop in \( \text{Map}(B^3, S^2) \) which starts and ends at \( v = w \). The question of homotoping \( h_t \) to the constant path is equivalent to asking whether \( \pi_1(\text{Map}(B^3, S^2), v) = 1 \). Since \( B^3 \) is contractible, we know that \( \pi_1(\text{Map}(B^3, S^2), v) = 1 \), which is true. Hence \( f \) is injective.

We now need to show that \( f \) is surjective. Suppose that \( v \) is a nonvanishing vector field on \( Y \). Let \( B_i \) denote the ball containing \( p_i \) so that \( Y(p) = Y \setminus \sqcup_i B_i \). We need to show that \( v \) is homologous on \( Y \) to a standard vector field \( v_t \) on each \( B_i \). Note that \( v|_{\partial B_i} \) and \( v_t|_{\partial B_i} \) are both nullhomotopic since they extend over all of \( B_i \), and hence we can homotope \( v \) so that \( v|_{\partial B_i} = v_t|_{\partial B_i} \). By restriction to \( B_i \), we thus get an element of \( \text{Spin}^c(B_i, \gamma_t) \) (where \( \gamma_t \) denotes the trivial sutures). Since \( \text{Spin}^c(B_i, \gamma_t) \) has a unique element, we know that \( v|_{B_i} \) and \( v_t \) are homologous on \( B_i \). This clearly implies that \( v \) is homologous on \( Y \) to a vector field \( w \) which is equal to \( v_t \) on each \( B_i \). Hence \( v \sim f(w|_{Y(p)}) \) so \( f \) is surjective.

\[ \square \]

14.2. 4-manifolds and \( \text{Spin}^c \) structures. The cobordism maps in sutured Floer homology are graded over equivalence classes of relative 4-dimensional \( \text{Spin}^c \) structures, whereas the ones in \([OS06]\) are graded over absolute 4-dimensional \( \text{Spin}^c \) structures. In this section, we show that grading in sutured Floer homology over equivalence classes of relative \( \text{Spin}^c \) structures induces a grading by absolute \( \text{Spin}^c \) structures.

Definition 14.3. Suppose that \( W^4 \) is a compact oriented 4-manifold. We define \( \text{Spin}^c(W) \) to be the homology classes of pairs \( (J, P) \) where

- \( P \subseteq W \) is a finite collection of points;
- \( J \) is an almost complex structure defined over \( W \setminus P \).

Two pairs \( (J, P) \) and \( (J', P') \) are said to be homologous if there exists a compact 1-manifold \( C \subseteq W \) such that \( P, P' \subseteq C \) and \( J|_{W \setminus C} \) is isotopic to \( J'|_{W \setminus C} \).
Definition 14.4 ([Juh09]). Suppose that \( W = (W, Z, \xi) \) is a sutured cobordism. Let \( J(\xi) \) denote the space of almost-comple structures on \( TW|_Z \) such that \( \xi \) consists of almost-complex lines. Fix a \( J_0 \in J(\xi) \). A relative Spin\(^c\) structure on \( W \) is a pair \((J, P)\) where

- \( P \subseteq W \setminus Z \) is a finite collection of points;
- \( J \) is an almost complex structure defined over \( W \setminus P \);
- \( J|_Z = J_0 \).

We say that \((J, P)\) and \((J', P')\) are homologous if there exists a compact 1-manifold \( C \subseteq W \setminus Z \) such that \( P, P' \subseteq C \) and \( J|_{W \setminus C} \) and \( J'|_{W \setminus C} \) are homotopic through almost complex structures relative to \( Z \).

As described in [Juh09] the space \( J(\xi) \) is contractible, and \( \text{Spin}^c(W) \) is an affine space over \( H^2(W, Z) \).

Definition 14.5. If \( (W, Z, \xi) \) is a sutured cobordism from \((M_0, \gamma_0)\) to \((M_1, \gamma_1)\), we say that a component \( C \) of \( Z \) is isolated if \( Z \cap M_1 = \emptyset \).

Recall that if \((W, \Gamma)\) is a graph cobordism, we let \( N(\Gamma) \) denote a regular neighborhood of \( \Gamma \) in \( W \), and \( Z(\Gamma) \) denote the boundary of \( N(\Gamma) \) in \( W \). Putting the standard contact structure on \( Z(\Gamma) \), we get the sutured cobordism \((W(\Gamma), Z(\Gamma), \xi_{Z(\Gamma)})\). We let \((W(\Gamma), Z'(\Gamma))\) be the sutured cobordism obtained by taking the standard sutured graph cobordism \((W(\Gamma), Z(\Gamma), \xi_{Z(\Gamma)})\) and removing a standard contact ball from each isolated component of \( Z(\Gamma) \).

Definition 14.6 ([Juh09]). Suppose that \( W = (W, Z, \xi) \) is a balanced cobordism such that \( Z \) has no isolated components. Write \( W = W_s \circ W_0 \) where \( W_s \) is a special cobordism and \( W_0 \) is a boundary cobordism. Two relative Spin\(^c\) structures \( s, s' \in \text{Spin}^c(W, Z, \xi) \) are said to be equivalent if \( s|_{W_s} = s'|_{W_s} \) and \( s|_{W_0} = s'|_{W_0} \). We define \( \text{Spin}^c(W)/_{\sim} \) to be the set of equivalence classes of \( \text{Spin}^c(W, Z, \xi) \) structures. If \((W, Z, \xi)\) is a balanced sutured cobordism such that \( Z \) has isolated components, we set \( \text{Spin}^c(W)/_{\sim} \) to be \( \text{Spin}^c(W')/_{\sim} \) where \( W' \) is the sutured cobordism obtained by removing a standard contact ball from each isolated component of \( Z \).

The cobordism maps in sutured Floer homology are graded over equivalence classes of relative Spin\(^c\) structures:

Theorem 14.7 ([Juh09] Proposition 7.10]). Given a balanced cobordism \( W = (W, Z, \xi) \), we have

\[
F_W = \bigoplus_{s \in \text{Spin}^c(W)/_{\sim}} F_{W, s}.
\]

We will also need the following fact from [Juh09] about Spin\(^c\) structures and boundary cobordisms:

Lemma 14.8 ([Juh09] Lemma 6.10). If \( W_0 : (M_0, \gamma_0) \to (M_1, \gamma_1) \) is a boundary cobordism (or equivalently if \(-M_0 \) is a sutured submanifold of \(-M_1 \)), then \( \text{Spin}^c(W_0) \cong \text{Spin}^c(M_0, \gamma_0) \) and there is a map \( f_\xi : \text{Spin}^c(M_0, \gamma_0) \to \text{Spin}^c(M_1, \gamma_1) \), which satisfies

\[
f_\xi(s_1) - f_\xi(s_2) = e_*(s_1 - s_2)
\]

where \( e_* : H_1(M_0) \to H_1(M_1) \) is the map induced by inclusion.

We now need a brief digression on Stein fillings.

Definition 14.9 ([CE12]). Suppose that \( W^4 \) is a smooth compact manifold with boundary. A Stein domain structure on \( W \) is a pair \((J, \phi)\) where \( J \) is a complex structure and \( \phi : W \to \mathbb{R} \) is a \( J \)-convex generalized Morse function.

One should consult [CE12] for the definition of a \( J \)-convex generalized Morse function, since it is not important for our purposes.

Definition 14.10. If \((Y^3, \xi)\) is a closed, oriented contact manifold, then a Stein filling of \((Y, \xi)\) is a Stein domain \((W, J, \phi)\) such that there is an orientation preserving contactomorphism between \( \partial W \) with the field of complex tangencies and \((Y, \xi)\). Two Stein fillings \((W, J, \phi)\) and \((W', J', \phi')\) are said to be deformation equivalent if there is a diffeomorphism \( h : W \to W' \) such that \((J, \phi)\) and \((h^* J', h^* \phi)\) are homotopic.

Theorem 14.11 ([CE12] Theorem 16.9). Any Stein filling of a \( k \)-fold connected sum \( \#^k \left(S^2 \times S^1\right) \) (with the standard tight contact structure) is deformation equivalent to the canonical Stein structure on the 4-ball with \( k \) 1-handles attached.
We now apply the above theorem to graph cobordisms. If \((W(\Gamma), Z'(\Gamma))\) is the sutured cobordism as above, then the triple \((N(\Gamma), Z'(\Gamma), \xi_t)\) embeds properly into \((B^4 \cup k 1\text{-handles}, \#^k S^1 \times S^2, \xi_{\text{std}})\). In particular there is a Stein filling of \((\#^k S^1 \times S^2, \xi_{\text{std}})\) by \(B^4 \cup (k 1\text{-handles})\) which is unique up to deformation. In particular, given an almost complex structure \(J_0\) on \(TW|Z'(\Gamma)\) which has \(\xi_{Z'(\Gamma)}\) as complex lines, the Stein structure determines a unique homotopy class of almost complex structures extending \(J_0\), which is given by the homotopy class induced by the Stein structure on \(B^4 \cup (k 1\text{-handles})\). Hence we have a well defined map
\[
f : \text{Spin}^c(W(\Gamma), Z'(\Gamma)) \to \text{Spin}^c(W),
\]
which we call the \textit{filling map}.

We now show that the filling map induces an isomorphism between equivalence classes of relative Spin\(^c\) structures on the sutured cobordisms \((W(\Gamma), Z'(\Gamma))\) and the absolute Spin\(^c\) structures on \(W\).

**Lemma 14.12.** If \((W, \Gamma)\) is a graph cobordism, then the filling map \(f : \text{Spin}^c(W(\Gamma), Z'(\Gamma)) \to \text{Spin}^c(W)\) descends to an isomorphism
\[
f : \text{Spin}^c(W(\Gamma), Z'(\Gamma))/\sim \to \text{Spin}^c(W).
\]

**Proof.** We first show that \(f : \text{Spin}^c(W(\Gamma), Z'(\Gamma)) \to \text{Spin}^c(W)\) is surjective. Trivialize \(TW|_{N(\Gamma)}\). The space of almost complex structures on a 4-dimensional real vector space is homotopy equivalent to \(S^2\) (cf. [Juha99, Lemma 3.3]). To show surjectivity, it’s sufficient to show that we can homotope a given almost complex structure on \(W\) so that on \(N(\Gamma)\) it is equal to the map induced by the Stein structure. All maps from \(N(\Gamma)\) to \(S^2\) are homotopic since \(N(\Gamma)\) has the homotopy type of \(\Gamma\), which is a 1-dimensional CW complex. Hence \(f\) is surjective.

We now show that \(f : \text{Spin}^c(W(\Gamma), Z'(\Gamma)) \to \text{Spin}^c(W)\) satisfies
\[
f(s) = f(s') \quad \text{iff} \quad s \sim s',
\]
for \(s, s' \in \text{Spin}^c(W(\Gamma), Z'(\Gamma))\), which will show that \(f\) induces a well defined map on \(\text{Spin}^c(W(\Gamma), Z'(\Gamma))/\sim\) and that the induced map is injective.

Decompose \((W(\Gamma), Z'(\Gamma))\) as \(\mathcal{W}_s \circ \mathcal{W}_\partial\) where \(\mathcal{W}_s\) is a special cobordism and \(\mathcal{W}_\partial\) is a boundary cobordism. Fill in \(\mathcal{W}_\partial\) and \(\mathcal{W}_s\) separately to get (absolute) cobordisms \(W_2\) and \(W_1\). If \((W, \Gamma) : (Y_1, p) \to (Y_2, p)\), note that as undecorated cobordisms, we have
\[
W = W_2 \circ W_1
\]
and \(W_1\) is obtained from \(Y_1\) by adding 1-handles. Given \(s, s' \in \text{Spin}^c(W)\), we thus have \(s = s'\) iff \(s|_{W_2} = s'|_{W_2}\) since \(W_1\) is obtained by adding only 1-handles.

On the other hand, if \(s, s' \in (W(\Gamma), Z'(\Gamma))\) are relative Spin\(^c\) structures, then by definition \(s \sim s'\) iff \(s|_{W_2} = s'|_{W_2}\) and \(s|_{W_1} = s'|_{W_1}\). On the other hand, the inclusion map
\[
e_s : H_1(Y(p)) \to H_1(Y(p) \cup Z(\Gamma))
\]
is injective. If we write \(W_\partial : Y(p) \to M\) for some sutured manifold \(M\), then by Lemma [14.8] we have \(s|_{W_\partial} = s'|_{W_\partial}\) iff \(s|_{Y(p)} = s'|_{Y(p)}\) iff \(s|_{M} = s'|_{M}\). But since \(M\) is the incoming boundary of \(W_s\), we know that \(s|_{M} = s'|_{M}\) if \(s|_{W_s} = s'|_{W_s}\). Hence we know that \(s \sim s'\) iff \(s|_{W_s} = s'|_{W_s}\).

Combining these observations, it is thus sufficient to show that if \(s, s' \in \text{Spin}^c(W(\Gamma), Z'(\Gamma))\) then
\[
f(s)|_{W_2} = f(s')|_{W_2} \quad \text{iff} \quad s|_{W_s} = s'|_{W_s}.
\]

Notice however that \(W_s\) is the sutured graph cobordism for \((W_2, \Gamma_2)\) where \(\Gamma_2\) is a collection of paths from the incoming end to the outgoing end. It is thus sufficient to show that \(f : \text{Spin}^c(\widehat{W}(\gamma), Z(\gamma)) \to \text{Spin}^c(\widehat{W})\) is injective if \((\widehat{W}_\gamma, \gamma)\) is a graph cobordism with \(\gamma\) a collection of paths from the incoming end of \(\widehat{W}\) to the outgoing end of \(\widehat{W}\).

To show injectivity of \(f\) in this case, we wish to show that two relative Spin\(^c\) structures on \(W(\gamma)\) which become homologous when extended over \(N(\gamma)\) were originally homologous over \(W(\gamma)\). Given two almost complex \(J_1, J_2\) structures on \(W \setminus P\) (where \(P\) is a set of points) which are homotopic through almost complex structures over \(W \setminus C\) (where \(C\) is a compact 1-manifold with \(\partial C = P\)), an easy argument shows that \(J_1\) and \(J_2\) are also homotopic through almost complex structures on \(W \setminus C'\) where \(C'\) is another 1-submanifold with \(\partial C = P\) which is isotopic to \(C\) relative \(P\). Hence we can assume that \(C \cap N(\gamma) = \emptyset\). Note also that \(N(\gamma)\) is a disjoint union of \(\sqcup_i B^3 \times I\). In light of this, it is sufficient to show that if \(J_t\) is a path of
almost complex structures on $B^3 \times I$, such that $J_0 = J_1$, then it can be homotoped through paths of almost complex structures fixing $t = 0, 1$ to the constant path. Since the space a almost complex structures on a 4-dimensional vector space is homotopy equivalent to $S^2$, by picking a trivialization of $T(B^2 \times I)$, it is sufficient to show that $\pi_1(\text{Map}(B^3 \times I, S^2)) = 1$. Note that by contracting $B^3 \times I$ to a point, we get that $\text{Map}(B^3 \times I, S^2) \simeq S^2$, so the statement becomes equivalent to $\pi_1(S^2) = 1$, which is clearly true. Hence $\mathfrak{f}$ is an injection. 

We thus can define the refinement of the graph TQFT maps over $\text{Spin}^c(W)$ structures as

$$F_{W, \Gamma, s} = F_{W(\Gamma), \mathfrak{f}^{-1}(s)}.$$ 

**Theorem B(b).** If $(W, \Gamma)$ is a graph cobordism, then

$$F_{W, \Gamma} = \bigoplus_{s \in \text{Spin}^c(W)} F_{W, \Gamma, s}.$$ 

**Proof.** This follows from [Juh09 Proposition 7.10] and Lemma 14.12.

**Theorem B(c).** Let $(W_1, \Gamma_1) : (Y_0, p_0) \rightarrow (Y_1, p_2)$ and $(W_2, \Gamma_2) : (Y_1, p_1) \rightarrow (Y_2, p_2)$ be graph cobordisms. Let $(W, \Gamma) = (W_2, \Gamma_2) \circ (W_1, \Gamma_1)$. Then the refinements satisfy

$$F_{W_2, \Gamma_2, s_2} \circ F_{W_1, \Gamma_1, s_1} = \sum_{s \in \text{Spin}^c(W) : s_{|W_i} = s_i} F_{W, \Gamma, s}.$$ 

**Proof.** This follows from [Juh09 Theorem 8.3], and the observation that if $s \in \text{Spin}^c(W(\Gamma), Z(\Gamma))$ then $f(s)|_{W_i} = f(s)|_{W_i(\Gamma_i)}$.

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