Virasoro 3-algebra from scalar densities

T. A. Larsson
Vanadisvägen 29, S-113 23 Stockholm, Sweden
e-mail: thomas.larsson@hdd.se

June 25, 2008

Abstract

It is shown that the ternary Virasoro-Witt algebra of Curtright, Fairlie and Zachos can be constructed by applying the Nambu commutator to the vect(1) realization on scalar densities. This construction is generalized to vect(d), but the corresponding 3-algebra fails to close.

There has recently been a surge of interest in 3-algebras in M-theory [1, 2, 3, 4], which is closely related to the ternary brackets introduced long ago by Nambu [5] and developed by Filippov [6], Curtright and Zachos [7] and many others. In particular, Lin [8] considered a Kac-Moody 3-algebra and Curtright, Fairlie and Zachos [9] considered very recently a 3-algebra related to the Witt (centerless Virasoro) algebra. In this note I observe that their Virasoro-Witt 3-algebra (eqn (22) in [9]) can be constructed by applying the Nambu commutator

\[
[A, B, C] = ABC + BCA + CAB - BAC - CBA - ACB
\]

(1)

to the Virasoro representation acting on scalar densities, i.e. primary fields. Consider the operators

\[
E_m = e^{imx}, \\
L_m = e^{imx}(-i \frac{d}{dx} + \lambda m), \\
S_m = e^{imx}(-i \frac{d}{dx} + \lambda m)^2.
\]

1
They satisfy

\[\begin{align*}
E_m E_n &= E_{m+n}, \\
E_m L_n &= L_{m+n} - \lambda m E_{m+n}, \\
L_m E_n &= L_{m+n} + (1 - \lambda) n E_{m+n}, \\
L_m L_n &= S_{m+n} + (n - \lambda (m + n)) L_{m+n} + (\lambda^2 - \lambda) m n E_{m+n}.
\end{align*}\]  

(3)

The products involving \(S\)'s can also be readily computed but are not needed here. The commutators \([A, B] = AB - BA\) are

\[\begin{align*}
[E_m, E_n] &= 0, \\
[L_m, E_n] &= n E_{m+n}, \\
[L_m, L_n] &= (n - m) L_{m+n}.
\end{align*}\]  

(4)

This is recognized as the semidirect product of the Witt algebra \(\text{ vect}(1)\) (centerless Virasoro algebra) with an abelian current algebra. The corresponding Nambu commutators (1) become

\[\begin{align*}
[E_m, E_n, E_r] &= 0, \\
[L_m, E_n, E_r] &= (n - r) E_{m+n+r}, \\
[L_m, L_n, E_r] &= (n - m) L_{m+n+r} + (1 - 2\lambda)(n - m) r E_{m+n+r}, \\
[L_m, L_n, L_r] &= (\lambda - \lambda^2)(m - n)(n - r)(r - m) E_{m+n+r}.
\end{align*}\]  

(5)

This is the Virasoro-Witt 3-algebra [9], eqn (22). To recover their conventions, we substitute \(L_m \rightarrow -L_m\), \(E_m \rightarrow M_m\), \(\lambda \rightarrow \beta\), and note that their constant \(C = \beta(1 - \beta)\), cf their eqn (10). We have thus shown that the Virasoro-Witt 3-algebra reported by Curtright, Fairlie and Zachos can be obtained by inserting the \(\text{ vect}(1)\) representation on scalar densities into the Nambu commutator.

It was noted by those authors that the 3-algebra [5] does not satisfy the fundamental identity of type \(LLLLL\), except when \(\lambda = \pm 2i\). However, it was emphasized earlier [7] that there is nothing fundamental about the fundamental identity as far as associativity is concerned. Indeed, since we started from an infinite-dimensional matrix representation [2] of \(\text{ vect}(1)\), the 3-algebra [5] does possess an associative matrix representation by construction.

Let us generalize this construction to higher dimensions; instead of the algebra \(\text{ vect}(1)\) of vector fields on the circle, consider the algebra \(\text{ vect}(d)\) of vector fields on the \(d\)-dimensional torus. Let \(x = (x^\mu)\) be the coordinates and
\( \partial_\mu = \partial / \partial x^\mu \) be the corresponding derivatives. The Fourier modes \( e^{im \cdot x} \) are labelled by momenta \( m = (m_\mu) \in \mathbb{Z}^d \). For simplicity, we restrict ourselves to scalar fields, i.e. densities with weight \( \lambda = 0 \). Consider in analogy with (2)

\[
E(m) = e^{im \cdot x}, \\
L_\mu(m) = e^{im \cdot x}(-i\partial_\mu), \\
S_{\mu\nu}(m) = S_{\nu\mu}(m) = e^{im \cdot x}(-i\partial_\mu)(-i\partial_\nu).
\]

These operators satisfy the products

\[
E(m)E(n) = E(m+n), \\
E(m)L_\nu(n) = L_\nu(m+n), \\
L_\mu(m)E(n) = L_\mu(m+n) + n_\mu E(m+n), \\
L_\mu(m)L_\nu(n) = S_{\mu\nu}(m+n) + n_\mu L_\nu(m+n).
\]

The commutators

\[
[E(m), E(n)] = 0, \\
[L_\mu(m), E(n)] = n_\mu E(m+n), \\
[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n),
\]

satisfy a semidirect product of \( \text{vect}(d) \) with an abelian current algebra in \( d \) dimensions. The Nambu commutators become

\[
[E(m), E(n), E(r)] = 0, \\
[L_\mu(m), E(n), E(r)] = (n_\mu - r_\mu)E, \\
[L_\mu(m), L_\nu(n), E(r)] = (m_\nu + r_\nu)L_\mu - (n_\mu + r_\mu)L_\nu + (n_\mu r_\nu - m_\nu r_\mu)E, \\
[L_\mu(m), L_\nu(n), L_\rho(r)] = (m_\rho - n_\rho)S_{\mu\nu} + (n_\mu - r_\mu)S_{\nu\rho} + (r_\nu - m_\nu)S_{\rho\mu} \\
+ (n_\mu r_\nu - m_\nu r_\mu)L_\rho + (r_\nu m_\rho - n_\rho m_\nu)L_\mu + (m_\rho n_\mu - r_\mu n_\rho)L_\nu,
\]

where the common argument \( (m+n+r) \) on the RHS has been suppressed. Unlike the Witt 3-algebra (5), these brackets do not close, due to the appearence of \( S \) terms in the \( LLL \) bracket. Hence we must also consider brackets with \( S \)'s in the LHS, which leads to infinite hierarchy of new generators.

In the one-dimensional case we started from the realization (2) on scalar densities, but in \( d \) dimensions we only considered the case \( \lambda = 0 \). The \( \text{vect}(d) \) realization (6) can readily be generalized to scalar densities:

\[
L_\mu(m) = e^{im \cdot x}(-i\partial_\mu + \lambda m_\mu).
\]
More generally, the classical irreps of \( \text{vect}(d) \) are closely related to tensor densities, which correspond to the \( \text{vect}(d) \) realization

\[
L_\mu(m) = e^{im \cdot x}(-i\partial_\mu + m_\nu T^\nu_\mu),
\]

(11)

where \( T^\mu_\nu \) are some matrices which satisfy \( gl(d) \):

\[
[T^\mu_\nu, T^\rho_\sigma] = \delta^\rho_\nu T^\mu_\sigma - \delta^\mu_\sigma T^\rho_\nu.
\]

(12)

For each \( gl(d) \) representation \( R \), these formulas yield a \( \text{vect}(d) \) representation, which describes how infinitesimal diffeomorphisms act on tensor densities of type \( R \). In particular, we recover scalar densities by setting

\[
T^\mu_\nu = \lambda \delta^\mu_\nu.
\]

(13)

The tensor representation (11) is irreducible, except when \( R \) is totally antisymmetric with weight zero; in that case, the module of differential forms has the submodule of closed forms. There are essentially no other irreducible \( \text{vect}(d) \) representations [10].

We can now construct new \( \text{vect}(d) \) 3-algebras by replacing the definition of \( L_\mu(m) \) in (6) by (10) or (11), and change the definition of \( S^\mu_\mu(m) \) accordingly. However, since the \( \text{vect}(d) \) 3-algebra (9) fails to close already when we start from scalar fields, we must introduce an infinite hierarchy of new generators and relations. The situation becomes very complex, and the physical relevance of such an exercise is unclear to me.

References

[1] J Bagger and N Lambert, Phys Rev D77, 065008 (2008) arXiv:0711.0955 [hep-th];
J Bagger and N Lambert, JHEP 0802, 105 (2008) arXiv:0712.3738 [hep-th].

[2] J Gomis, G Milanesi, J G Russo, arXiv:0805.1012v2[hep-th]

[3] A Gustavsson, arXiv:0709.1260[hep-th];
A Gustavsson, arXiv:0802.3456[hep-th].

[4] P-M Ho, R-C Hou, and Y Matsuo, arXiv:0804.2110v2 [hep-th];
P-M Ho, Y Imamura, and Y Matsuo, arXiv:0805.1202v2 [hep-th].

[5] Y Nambu, Phys Rev D7 (1973) 2405-2412.
[6] V T Filippov, Sib Mat Zh 26 (1985) 126-140 (Sib Math Journal 26 (1986) 879-891).

[7] T L Curtright and C K Zachos, Phys. Rev. D68, 085001 (2003) [hep-th/0212267].

[8] H Lin, arXiv:0805.4003 [hep-th].

[9] T L Curtright, D B Fairlie and C K Zachos, arXiv:0806.3515v1 [hep-th].

[10] A N Rudakov, Math. USSR Izv. 8, 836866 (1974)