Complexity of multivariate Feynman-Kac path integration in randomized and quantum settings

Marek Kwas

Department of Computer Science, Columbia University, New York, NY10027, USA,
Institute of Applied Mathematics and Mechanics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland.

Abstract

The Feynman-Kac path integration problem was studied in the worst case setting by Plaskota et al. (J. Comp. Phys. 164 (2000) 335) for the univariate case and by Kwas and Li (J. Comp. Phys. 19 (2003) 730) for the multivariate case with $d$ space variables. In this paper we consider the multivariate Feynman-Kac path integration problem in the randomized and quantum settings. For smooth multivariate functions, it was proven in Kwas and Li (2003) that the classical worst case complexity suffers from the curse of dimensionality in $d$. We show that in both the randomized and quantum settings the curse of dimensionality is vanquished, i.e., the number of function evaluations and/or quantum queries required to compute an $\varepsilon$-approximation has a bound independent of $d$ and depending polynomially on $\varepsilon^{-1}$. The exponents of these polynomials are at most 2 in the randomized setting and at most 1 in the quantum setting. Hence we have exponential speedup over the classical worst case setting and quadratic speedup of the quantum setting over the randomized setting. However, both the randomized and quantum algorithms presented here still require extensive precomputing, similar to the algorithms of Plaskota et al. (2000) and Kwas and Li (2003).

* This research was supported in part by the Defense Advanced Research Agency (DARPA) and Air Force Research Laboratory under agreement F30602-01-2-0523. Email address: marek@cs.columbia.edu (Marek Kwas).
1 Introduction

In this paper we study the multivariate Feynman-Kac path integration problem. Multivariate Feynman-Kac path integrals are path integrals over the space of continuous functions from $\mathbb{R}^+ \rightarrow \mathbb{R}^d$, equipped with a Wiener measure. The multivariate Feynman-Kac path integral is the solution of the initial value problem for the multivariate heat equation, see Section 2. This type of path integral plays a significant role in many fields, e.g., quantum physics and chemistry, differential equations, and financial mathematics. A brief survey of research concerning path integrals is contained in the introduction of [15].

In this paper we continue the research initiated in [15], where a new algorithm for computing Feynman-Kac path integrals was proposed. That paper dealt with the univariate case (i.e., with one space variable) and the algorithm presented there was based on $L_2$-approximation. An extension of the approach of [15] to the multivariate case (with many space variables) was presented in [7]. It turns out that algorithms based on $L_2$-approximation are no longer applicable in the multivariate case. The multivariate case can be solved by using uniform approximation as a basic building block. In both papers [7,15] the problem was studied in the worst case setting for input functions belonging to a class $\mathcal{F}$. Assuming that the uniform approximation problem for the class $\mathcal{F}$ has worst case complexity of order $\varepsilon^{-\alpha(\mathcal{F})}$ for some positive $\alpha(\mathcal{F})$ it was proved in [7] that the number of function evaluations required to compute an $\varepsilon$-approximation is roughly of the same order $\varepsilon^{-\alpha(\mathcal{F})}$. We stress that typically $\alpha(\mathcal{F})$ depends on the number $d$ of space variables and tends to infinity with $d$, in which case we have the curse of dimensionality.

In this paper, we consider the multivariate Feynman-Kac path integration problem in the randomized and quantum settings. We present algorithms that compute an $\varepsilon$-approximation and analyze their cost. These algorithms are also based on uniform approximation. However, the power of randomization and quantum computation permits the improvement of the worst case complexity bound $O(\varepsilon^{-\alpha(\mathcal{F})})$. Namely, the number of function evaluations required by the randomized algorithm is roughly of order $\varepsilon^{-2\alpha(\mathcal{F})/\alpha(\mathcal{F}) + 2}$, whereas the number of function evaluations and queries required by the quantum algorithm is roughly of order $\varepsilon^{-\alpha(\mathcal{F})/\alpha(\mathcal{F}) + 1}$, see Section 7.1. We stress that the exponent of $\varepsilon^{-1}$ in the randomized setting is at most 2, and in the quantum setting is at most 1.

In addition to providing the algorithms in the randomized and quantum settings we also study the complexity of multivariate Feynman-Kac path integration in the randomized and quantum settings. As in [7,15], the complexity is bounded from below by the complexity of multivariate weighted integration. The upper bounds are provided by the costs of the algorithms presented in
For the class $F$ of $r$ times continuously differentiable functions we have $\alpha(F) = d/r$, and so the worst case setting suffers from the curse of dimensionality. In the randomized setting, the complexity is roughly of order $O(\varepsilon^{-2/(1+2r/d)})$, whereas in the quantum setting it is roughly of order $O(\varepsilon^{-1/(1+r/d)})$. In both cases the curse of dimensionality is vanquished. We thus have exponential speedup over the worst case setting. For $d \gg r$, we have quadratic speedup of quantum complexity over randomized complexity.

2 Multivariate Feynman-Kac path integration

The multivariate Feynman-Kac formula is the solution of the initial value problem for the heat (diffusion) equation

$$\frac{\partial z}{\partial t}(u, t) = \frac{1}{2} \Delta z(u, t) + V(u)z(u, t) \quad \text{for } (u, t) \in \text{int} \mathbb{R}^d \times [0, \infty), \quad (1)$$

$$z(u, 0) = v(u). \quad (2)$$

Here $v, V : \mathbb{R}^d \to \mathbb{R}$ are the initial value function and the potential function, respectively. As usual, $\Delta$ denotes the Laplacian.

The solution $z$ of (1) and (2) is given by the Feynman-Kac formula

$$z(u, t) = \int_{C} v(x(t) + u) \exp \left( \int_{0}^{t} V(x(s) + u) \, ds \right) w(dx). \quad (3)$$

Here, $C$ is the set of continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^d$ such that $x(0) = 0$. The path integral (3) is with respect to the $d$-dimensional Wiener measure $w$, see [5,16]. Obviously, (3) holds for functions $v$ and $V$ for which the path integral exists. In what follows, we assume that the functions $v$ and $V$ belong to a class $F$ for which (3) exists. This class is defined in Section 4.

3 Problem

For a given fixed point $(u^*, t^*) \in \mathbb{R}^d \times [0, \infty)$ and arbitrary functions $v, V$ from the class $F$, we want to compute an $\varepsilon$-approximation of the exact solution $z_{v,V}(u^*, t^*)$ of (3).

The $\varepsilon$-approximation $a_{v,V}(u^*, t^*)$ is computed by an algorithm $A_n$ that uses $n$ function values of $v$ and $V$, i.e.,

$$a_{v,V}(u^*, t^*) = A_n(u^*, t^*, v(u_1), \ldots, v(u_k), V(u_{k+1}), \ldots, V(u_n)).$$
3.1 Worst case setting

In the worst case setting the error of the algorithm $A_n$ is defined as

$$e^{\text{wor}}(A_n) = \sup_{v,V \in F} |z_{v,V}(u^*, t^*) - a_{v,V}(u^*, t^*)|.$$  

We want to determine the minimal number

$$n^{\text{wor}}(\varepsilon, F) = \min \{ n : \exists A_n \text{ such that } e^{\text{wor}}(A_n) \leq \varepsilon \}$$

of function values that are needed to compute an $\varepsilon$-approximation in the worst case setting. This setting was analyzed in [7].

3.2 Randomized setting

In this setting we use randomized algorithms and replace the worst case error assurance by an expected assurance. A randomized algorithm $A_n$ depends on a random element $\omega$ chosen from some probability space $\Omega$. More precisely, we compute

$$a_{v,V}(u^*, t^*; \omega) = A_{n,\omega}(u^*, t^*, v(u_{\omega,1}), \ldots, v(u_{\omega,k}), V(u_{\omega,k+1}), \ldots, V(u_{\omega,n})),$$

with $n = E_\omega(n_\omega)$. This means that we allow a random choice of a mapping $A_{n,\omega}$ and sample points $u_{\omega,i}$, as well as the number $n_\omega$ of sample points, whose expected value is fixed and equal to $n$.

We measure the randomized error of the algorithm $A_n$ with respect to the $L_2$ norm, i.e.,

$$e^{\text{rand}}(A_n) := \sup_{v,V \in F} \left( E_\omega(z_{v,V}(u^*, t^*) - a_{v,V}(u^*, t^*; \omega))^2 \right)^{1/2}.$$  

As before, we want to determine the minimal expected number of function values

$$n^{\text{rand}}(\varepsilon, F) = \min \{ n : \exists A_n \text{ such that } e^{\text{rand}}(A_n) \leq \varepsilon \}$$

needed to compute an $\varepsilon$-approximation in the randomized setting.

3.3 Quantum setting

In the quantum setting we use quantum algorithms with (deterministic or randomized) quantum queries and assume that we can also perform function evaluations and arithmetic operations on a classical computer. These classical
operations are used to prepare an input for a quantum algorithm and to transform the outcome of a quantum algorithm to an approximation of the exact solution. We will be interested in minimizing the total number of quantum queries and function evaluations needed to compute an \( \varepsilon \)-approximation.

In this section, we give a brief overview of a simplified quantum model of computation for continuous problems and describe deterministic and randomized quantum queries. We refer the reader to [1,2,3,8,14,21] for more detailed information.

We first outline a general framework of the quantum setting. Assume that for a given class \( H \) of input functions \( f : D \to C \) we want to approximate the solution operator
\[
S : H \to G,
\]
with \( G \) being a normed space whose norm is denoted by \( \| \cdot \|_G \). We will approximate \( S(f) \) by a quantum algorithm defined below.

First, we transform a given input function \( f \in H \) by using a classical algorithm \( P \) with \( s \) classical function evaluations and obtain
\[
\bar{f} = P_s(f) : D \to C.
\]
Then we use the transformed function \( \bar{f} \) as an input to a quantum algorithm.

Quantum algorithms are defined as follows. Let \( \mathbb{C}^2 \) be the two dimensional complex vector space. Let \( \mathcal{H}_k = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \) be the \( k \)-fold tensor product of \( \mathbb{C}^2 \), having dimension \( 2^k \). Let \( U_n(\bar{f}) : \mathcal{H}_n \to \mathcal{H}_n \) be a unitary operator of the form
\[
U_n(\bar{f}) = Q_n Q_f Q_{n-1} \cdots Q_1 Q_f Q_0,
\]
with unitary operators \( Q_0, \ldots, Q_n \) and a quantum query \( Q_f \), for some \( \bar{f} \in P_s(H) \).

The deterministic quantum query \( Q_f \) is defined as in [2]. Let \( \mathcal{H}_k = \mathcal{H}_m \otimes \mathcal{H}_{k-m} \) for some \( m \leq k \). Then \( Q_f : \mathcal{H}_m \otimes \mathcal{H}_{k-m} \to \mathcal{H}_m \otimes \mathcal{H}_{k-m} \) is a unitary operator of the form
\[
Q_f|x\rangle|y\rangle = |x\rangle|y \oplus \beta(\bar{f}(\tau(x)))\rangle,
\]
with
\[
\tau : \{0, \ldots, 2^k - 1\} \to D, \quad \beta : C \to \{0, \ldots, 2^{k-m} - 1\}
\]
and \( \oplus \) denoting the addition modulo \( 2^{k-m} \), see again [2] for a more detailed discussion.

The randomized quantum query is defined in [21]. In this case, \( Q_f = Q_{f,\omega} \) depends on a random element \( \omega \) and \( Q_{f,\omega} \) has the form (5) with \( \tau = \tau_\omega \) depending on \( \omega \). This permits the computation of approximate values of \( \bar{f} \) at
randomized points. Hence in this case the unitary operator $U_n$ depends on a random element $\omega$ and has the form

$$U_{n,\omega}(\bar{f}) = Q_{n,\omega} Q_{n-1,\omega} \cdots Q_1 Q_{f,\omega} Q_0$$

with $n = E_n(\omega)$.

As usual we assume that the initial state is $|0\rangle$ and we compute

$$|\psi_{\bar{f}}\rangle = U_{n}(\bar{f})|0\rangle = Q_n Q_{n-1} \cdots Q_1 Q_{f} Q_0 |0\rangle$$

for deterministic quantum queries and

$$|\psi_{f,\omega}\rangle = U_{n,\omega}(\bar{f})|0\rangle = Q_n Q_{n-1} \cdots Q_1 Q_{f,\omega} Q_0 |0\rangle$$

for randomized quantum queries. Then we measure the final state and obtain an outcome $j \in \{0, \ldots, 2^k - 1\}$ with probability

$$p_f(j) = |\langle \psi_{\bar{f}}|j\rangle|^2 \quad \text{or} \quad p_{f,\omega}(j) = |\langle \psi_{f,\omega}|j\rangle|^2.$$  

Knowing the outcome $j$ we compute the final result on a classical computer, and the quantum algorithm $A_n$ yields

$$A_n(\bar{f}, j) = \phi(j) \quad \text{or} \quad A_{n,\omega}(\bar{f}, j) = \phi_{\omega}(j).$$

for some $\phi$ or $\phi_{\omega}$.

In this paper we will be using quantum algorithms with randomized quantum queries. The error of such an algorithm $A_n$ is defined as

$$e_{\text{quant}}(A_n, P_s, S) = \sup_{f \in H} \left( E_{\omega} E_{q} \|S(f) - A_{n,\omega}(P_s(f), j)\|_G^2 \right)^{1/2},$$

where $E_{\omega}$ is the expectation over the probability space $\Omega$, and $E_{q}$ is the expectation with respect to distribution of the quantum algorithm outcomes.

Similarly to the other settings, we want to determine the minimal number of random quantum queries and classical function evaluations

$$n_{\text{quant}}(\varepsilon, H) = \min\{s + n : \exists P_s \exists A_n \text{ such that } e_{\text{quant}}(A_n, P_s) \leq \varepsilon\}$$

needed to guarantee that the error does not exceed $\varepsilon$.

**Remark 1** We now briefly comment on the quantum error setting defined by (6). Let us concentrate for a moment on the randomness introduced by a quantum algorithm, leaving aside randomized queries. So far, the literature dealing with continuous problems in the quantum setting has mainly considered probabilistic error. That is, instead of taking an expectation with respect to all possible outcomes of a quantum algorithm (as $E_{q}$ in (6)), we want an error
estimate such that
\[
\left( \mathbb{E}_\omega \| S(f) - A_{n,P}(f) \|_G^2 \right)^{1/2} \leq \varepsilon
\]
holds with a certain (high) probability, for any \( f \in H \). Obviously these two ways of measuring the error of a quantum algorithm are related. We choose to study the average error for simplicity. Moreover, the average error is probably more natural when we consider randomized queries.

The multivariate Feynman-Kac path integration problem in the quantum setting is defined by taking \( f = (v,V) \) with \( H = F \times F \) and \( S(f) = z_{\nu,V}(u^*,t^*) \).

4 The function class \( F \)

To assure the existence of the path integral (3), we need to choose a proper class of input functions \( F \), see [7]. We assume that
\[
F = \{ (f_1, f_2) \in F \times F : \| f_1 \|_F \leq \beta_1, \| f_2 \|_F \leq \beta_2 \}
\]
is a ball of a linear space \( F \times F \) for some positive \( \beta_1, \beta_2 \).

We make the following assumptions about the linear space \( F \).

1. We assume that for every \( u \in \mathbb{R}^d \), the functional \( L_u : F \to \mathbb{R} \) defined by
   \[
   L_u f = f(u)
   \]
is continuous, and for arbitrary \( a, t \in \mathbb{R}_+ \) we have
   \[
   \int_t^\infty \| L_x(s) \|_F \exp \left( a \int_0^s \| L_x(r) \|_F \, dr \right) \, w(dx) < \infty.
   \]
   By the Fernique theorem, see e.g., [6], condition (8) holds if there exists \( \alpha < 2 \) such that \( \| L_x \|_F = O(\| x \|^\alpha) \) for \( \| x \| \) approaching infinity, see [15] for details. Here and elsewhere, \( \| x \| = \sqrt{\sum_{i=1}^d x_i^2} \) is the Euclidean norm in \( \mathbb{R}^d \).

2. We assume that \( F \) is continuously embedded into \( L_\infty(\mathbb{R}^d) \). That is, \( F \subset L_\infty(\mathbb{R}^d) \) and there exists a positive \( K \) such that
   \[
   \| f \|_{L_\infty(\mathbb{R}^d)} \leq K \| f \|_F \quad \forall f \in F.
   \]
   This assumption permits us to relate the multivariate Feynman-Kac path integration problem to uniform approximation in the worst case setting, see again [7].

By uniform approximation we mean the approximation of the embedding operator \( S : F \to L_\infty(\mathbb{R}^d) \), \( Sf = f \) in the norm of \( L_\infty(\mathbb{R}^d) \). Let \( n_{\text{wor}}(\varepsilon, F) \) denote the minimal number of function values needed to compute an \( \varepsilon \)-approximation in the worst case setting. As we shall see in
Section 6.1, uniform approximation also plays a significant role in the randomized and quantum settings.

(3) We assume that

\[ n_{\text{APP}}^\text{wor}(\varepsilon, \mathcal{F}) = O(\varepsilon^{-\alpha(\mathcal{F})}) \quad \text{as} \quad \varepsilon \to 0, \tag{10} \]

for some positive number \( \alpha(\mathcal{F}) \).

The linear space \( \mathcal{F} \) is characterized by the exponent \( \alpha(\mathcal{F}) \). Usually \( \alpha(\mathcal{F}) \) depends on the smoothness and the number of variables of functions in \( \mathcal{F} \), see Section 9.

5 Feynman-Kac formula as a series of multivariate integrals

In this section we briefly recall some results from [7] which are needed for our analysis.

Without loss of generality we can assume \( u = 0 \) in (3). Then we can express the path integral as a series of multivariate integrals

\[ S(v, V) := z(0, t) = \sum_{k=0}^{\infty} S_{k+1}(v, V), \tag{11} \]

where

\[ S_{k+1}(v, V) = \int_{\mathbb{R}^{(k+1)d}} v(z_{k+1}) \prod_{i=1}^{k} V(z_i) g_{k+1}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) \, dz_1 \ldots dz_{k+1}, \tag{12} \]

with

\[ g_{k+1}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} f_{k+1}(t_1, \ldots, t_k, t, \mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) \, dt_1 \ldots dt_k \tag{13} \]

and

\[ f_{k+1}(t_1, \ldots, t_k, t, \mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = \left( (2\pi)^{k+1} t_1(t_2 - t_1) \cdots (t - t_k) \right)^{-d/2} \]

\[ \times \exp \left( -\frac{1}{2} \left( \frac{||\mathbf{z}_1||^2}{t_1} + \frac{||\mathbf{z}_2 - \mathbf{z}_1||^2}{t_2 - t_1} + \cdots + \frac{||\mathbf{z}_{k+1} - \mathbf{z}_k||^2}{t - t_k} \right) \right). \]

Note that the integral (12) depends on the input functions \( v \) and \( V \) only through the product

\[ h_{k+1}(\mathbf{z}_1, \ldots, \mathbf{z}_{k+1}) = v(z_{k+1}) \prod_{i=1}^{k} V(z_i) \]
and the weight functions $g_{k+1}$ can be computed in advance, albeit with difficulty. Let us recall also that

$$\|g_{k+1}\|_{L_1(\mathbb{R}^{(k+1)d})} = \frac{t^k}{k!} \quad \text{for} \quad k \geq 0. \quad (14)$$

6 Approximation of one term of the series

In this section we present algorithms approximating one term of the series (11). To make the notation more clear we define a weighted integration operator

$$I_{k+1}(f) = \int_{\mathbb{R}^{(k+1)d}} f(z_1, \ldots, z_{k+1}) \ g_{k+1}(z_1, \ldots, z_{k+1}) \ dz_1 \ldots dz_{k+1}$$

where $f : \mathbb{R}^{(k+1)d} \to \mathbb{R}$ is an integrable function. We can then rewrite one term of the series (11) as

$$S_{k+1}(v, V) = I_{k+1}(h_{k+1}).$$

In both the randomized and quantum settings, we shall use deterministic uniform approximation of the function $h_{k+1}$. To utilize the power of randomization and/or quantum computation, we will apply the known technique of variance reduction.

6.1 Variance reduction

Smolyak’s algorithm is a powerful tool for computing an $\varepsilon$-approximation of tensor product problems. For $h_{k+1} \in \bigotimes_{i=0}^{k+1} \mathbb{F}$, Smolyak’s algorithm is of the form

$$U_{\varepsilon,k+1}(h_{k+1}) = \sum_{i=1}^{n(\varepsilon,k)} h_{k+1}(t_{i,\varepsilon,1}, \ldots, t_{i,\varepsilon,k+1}) \zeta_{i,\varepsilon,k+1}, \quad (15)$$

for some $t_{i,\varepsilon,j} \in \mathbb{R}^d$ and $\zeta_{i,\varepsilon,k+1} \in L_\infty(\mathbb{R}^{(k+1)d})$. It is proven in [7, Lemma 2] that

$$\|h_{k+1} - U_{\varepsilon,k+1}(h_{k+1})\|_{L_\infty(\mathbb{R}^{(k+1)d})} \leq \varepsilon \|v\|_\mathbb{F} \|V\|_\mathbb{F}^k, \quad (16)$$

where

$$n(\varepsilon, k+1) \leq c_0 \left( c_1 + c_2 \frac{\ln 1/\varepsilon}{k} \right)^{(\alpha(\mathbb{F})+1)k} \varepsilon^{-\alpha(\mathbb{F})}, \quad (17)$$

for some $c_i \in \mathbb{R}$. Here $a_+$ denotes $\max\{a,0\}$, the right hand side of (17) is defined to be $c_0 \varepsilon^{-\alpha(\mathbb{F})}$ when $k = 0$. 

9
The idea underlying variance reduction idea is as follows. First we compute
\[ \bar{h}_{k+1,\varepsilon} = U_{\varepsilon,k+1}(h_{k+1}) \]
using \( n(\varepsilon, k+1) \) function values. Then we compute
\[ I_{k+1}(\bar{h}_{k+1,\varepsilon}) = \sum_{i=1}^{n(\varepsilon,k+1)} \bar{h}_{k+1}(t_{i,\varepsilon,1}, \ldots, t_{i,\varepsilon,k+1})I_{k+1}(\zeta_{i,\varepsilon,k+1}). \]
Observe that the functions \( \zeta_{i,\varepsilon,k+1} \) do not depend on the input functions \( v \) and \( V \) so the integrals \( I_{k+1}(\zeta_{i,\varepsilon,k+1}) \) can be precomputed.

We stress that \( \bar{h}_{k+1,\varepsilon} \) and \( I_{k+1}(\bar{h}_{k+1,\varepsilon}) \) are deterministic. We will use randomized or quantum algorithm to approximate the multivariate integrals
\[ I_{k+1}(h_{k+1} - \bar{h}_{k+1,\varepsilon}). \]
Since the error depends on the norm \( \|h_{k+1} - \bar{h}_{k+1,\varepsilon}\|_{L_\infty(\mathbb{R}^{(k+1)d})} \), which is small, we can do this efficiently. We present the details in the following two sections.

6.2 Randomized algorithm

To make formulas simpler we define
\[ \tilde{f}_{k+1,\varepsilon} = h_{k+1} - \bar{h}_{k+1,\varepsilon}. \]
We use the randomized algorithm of the form
\[ \phi^{\text{rand}}_{\varepsilon,m,k+1,\omega}(v, V) = I_{k+1}(\bar{h}_{k+1,\varepsilon}) + Q^{\text{rand}}_{m,k+1,\omega}(\tilde{f}_{k+1,\varepsilon}). \quad (18) \]
Here
\[ Q^{\text{rand}}_{m,k+1,\omega}(f) = \frac{1}{m} \sum_{j=1}^{m} f(x_j,\omega) \quad (19) \]
denotes the classical Monte Carlo algorithm with \( m \) randomized sample points.

Randomized sample points are chosen with respect to the density \( g_{k+1}/\|g_{k+1}\|_{L_1(\mathbb{R}^{(k+1)d})} \) which is indicated by the random parameter \( \omega \in \Omega \).

Using the well known error formula for the classical Monte Carlo algorithm, we conclude that
\[
\left( \mathbb{E}_\omega\left( I_{k+1}(h_{k+1}) - \phi^{\text{rand}}_{\varepsilon,m,k+1,\omega}(v, V) \right)^2 \right)^{1/2} = \left( \mathbb{E}_\omega\left( I_{k+1}(\tilde{f}_{k+1,\varepsilon}) - Q^{\text{rand}}_{m,k+1,\omega}(\tilde{f}_{k+1,\varepsilon}) \right)^2 \right)^{1/2} = \frac{1}{\sqrt{m}} \left( \text{Var}(\tilde{f}_{k+1,\varepsilon}) \right)^{1/2}, \quad (20)
\]
with
\[ \text{Var}(\bar{f}_{k+1,\varepsilon}) = I_{k+1}(\bar{f}_{k+1,\varepsilon}^2) - \left( I_{k+1}(\bar{f}_{k+1,\varepsilon}) \right)^2. \]

Clearly, from (16) and then from (7), (14) we get
\[ \left( \text{Var}(\bar{f}_{k+1,\varepsilon}) \right)^{1/2} \leq \frac{t^k}{k!} \|\bar{f}_{k+1,\varepsilon}\|_{L_\infty([k+1]d)} \leq \varepsilon \frac{\|v\|_{F} \|V\|_{V} t^k}{k!} \leq \varepsilon \frac{\beta k^2 t^k}{k!}. \]

This yields the error estimate
\[ \left( \mathbb{E}_\omega(I_{k+1}(h_{k+1}) - \phi_{\varepsilon, k+1, \omega}^\text{rand}(v, V))^2 \right)^{1/2} \leq \varepsilon \frac{\beta k^2 t^k}{k!} \] (21)

and the total number of function evaluations is
\[ n(\varepsilon, k + 1) + m. \] (23)

### 6.3 Quantum algorithm

The structure of our quantum algorithm is similar to randomized one, having the form
\[ \phi_{\varepsilon, m, k, k+1, \omega}^\text{quant}(v, V) = I_{k+1}(\bar{h}_{k+1,\varepsilon}) + Q_{m, k, k+1, \omega}^\text{quant}(\bar{f}_{k+1,\varepsilon}), \] (24)

with, as before, \( \bar{f}_{k+1,\varepsilon} = h_{k+1} - \bar{h}_{k+1,\varepsilon} \). Here, we use a quantum algorithm \( Q_{m, k, k+1, \omega}^\text{quant} \) with \( \kappa \) randomized quantum queries, that approximates the classical Monte Carlo algorithm (19). In [4] the problem of approximating
\[ \frac{1}{m} \sum_{j=1}^{m} f(x_{j,\omega}) \]

was analyzed for Boolean functions \( f \). Using the technique of reducing the summation problem for bounded real functions to the summation problem for Boolean functions as in [2], we see that a result similar to that of [4] holds. From [4] and (16) we conclude that
\[ \left( \mathbb{E}_q \left( \frac{1}{m} \sum_{j=1}^{m} f(x_{j,\omega}) - Q_{m, k, k+1, \omega}^\text{quant}(\bar{f}_{k+1,\varepsilon}) \right)^2 \right)^{1/2} = O \left( \frac{1}{\kappa} \|\bar{f}_{k+1,\varepsilon}\|_{L_\infty([k+1]d)} \right) = O \left( \frac{\varepsilon k^2}{k!} \right). \]

By integrating over \( \Omega \), we obtain
\[ \left( \mathbb{E}_\omega \mathbb{E}_q \left| Q_{m, k, k+1, \omega}^\text{rand}(\bar{f}_{k+1,\varepsilon}) - Q_{m, k, k+1, \omega}^\text{quant}(\bar{f}_{k+1,\varepsilon}) \right|^2 \right)^{1/2} = O \left( \frac{\varepsilon \beta k^2 t^k}{k!} \right). \] (25)
The total number of queries and function evaluations is

\[ n(\varepsilon, k + 1) + \kappa, \]

We stress that this number does not depend on \( m \), which is only used for the definition of the Monte Carlo algorithm.

We now estimate the total error as

\[
\left( \mathbb{E}_\omega \mathbb{E}_q(I_{k+1}(h_{k+1}) - \phi^{\text{quant}}_{\varepsilon,m,k+1,\omega}(v, V))^2 \right)^{1/2}
\leq \left( \mathbb{E}_\omega \left( I_{k+1}(\tilde{f}_{k+1,\varepsilon}) - Q^{\text{rand}}_{m,k+1,\omega}(\tilde{f}_{k+1,\varepsilon}) \right)^2 \right)^{1/2}
+
\left( \mathbb{E}_\omega \mathbb{E}_q \left| Q^{\text{rand}}_{m,k+1,\omega}(\tilde{f}_{k+1,\varepsilon}) - Q^{\text{quant}}_{m,k,k+1,\omega}(\tilde{f}_{k+1,\varepsilon}) \right|^2 \right)^{1/2}.
\]

This, by (22) and (25), yields

\[
\left( \mathbb{E}_\omega \mathbb{E}_q(I_{k+1}(h_{k+1}) - \phi^{\text{quant}}_{\varepsilon,m,k+1,\omega}(v, V))^2 \right)^{1/2} = O \left( \frac{\varepsilon}{\sqrt{m}} \frac{\beta_1 \beta_2^k t^k}{k!} + \frac{\varepsilon}{\kappa} \frac{\beta_1 \beta_2^k t^k}{k!} \right).
\]

Letting \( m = \kappa^2 \) we get the error bound

\[
\left( \mathbb{E}_\omega \mathbb{E}_q(I_{k+1}(h_{k+1}) - \phi^{\text{quant}}_{\varepsilon,m,k+1,\omega}(v, V))^2 \right)^{1/2} = O \left( \frac{\varepsilon}{\kappa} \frac{2\beta_1 \beta_2^k t^k}{k!} \right) \quad (26)
\]

using

\[ n(\varepsilon, k + 1) + \kappa \quad (27) \]

function values and quantum queries. For the sake of convenience we denote

\[ \phi^{\text{quant}}_{\varepsilon,\kappa,k+1,\omega} = \phi^{\text{quant}}_{\varepsilon,m,\kappa,k+1,\omega} \quad \text{with } m = \kappa^2. \]

### 7 Complete algorithms

Based on the previous two sections we are ready to present algorithms computing an \( \varepsilon \)-approximation of multivariate Feynman-Kac path integral. We approximate consecutive terms of the series

\[ S(v, V) = \sum_{k=0}^{\infty} S_{k+1}(v, V) \]

by the algorithms

\[ \phi^{\text{rand}}_{\varepsilon,k+1,m,k+1,\omega,\kappa}, \text{ or } \phi^{\text{quant}}_{\varepsilon,k+1,\kappa,k+1,\omega,\kappa}. \]
with the accuracies $\varepsilon_{k+1}^{\text{rand}}$ and $\varepsilon_{k+1}^{\text{quant}}$ in the corresponding settings being

$$\varepsilon_{k+1}^{\text{rand}} = \frac{k!}{\beta_1 \beta_2 \tau k^2 2^{k+1}}$$

$$\varepsilon_{k+1}^{\text{quant}} = \frac{k!}{\beta_1 \beta_2 \tau k^2 2^{k+2}}$$

and the number of randomized sample points $m_{k+1}$ and quantum queries $\kappa_{k+1}$ being

$$m_{k+1} = \left\lceil \frac{-2\alpha(\mathcal{F})}{\alpha(\mathcal{F})+2} \right\rceil, \quad \kappa_{k+1} = \left\lceil \frac{-\alpha(\mathcal{F})}{\alpha(\mathcal{F})+1} \right\rceil$$

(29)

The final forms of randomized and quantum algorithms approximating $S(v, V)$ are

$$\Phi_{\varepsilon, \omega}^{\text{rand}}(v, V) = \sum_{k=0}^{N_{\varepsilon}^{\text{rand}}} \phi_{\varepsilon_{k+1}^{\text{rand}}, m_{k+1}, k+1, \omega}^{\text{rand}},$$

$$\Phi_{\varepsilon, \omega}^{\text{quant}}(v, V) = \sum_{k=0}^{N_{\varepsilon}^{\text{quant}}} \phi_{\varepsilon_{k+1}^{\text{quant}}, \kappa_{k+1}, k+1, \omega}^{\text{quant}},$$

where the finite integers $N_{\varepsilon}^{\text{rand}}$ and $N_{\varepsilon}^{\text{quant}}$ will be determined in the next section.

### 7.1 Error analysis

From (22), (26) and (28), (29), it is easy to check that the error bounds

$$\left(\mathbb{E}_\omega(I_{k+1}(h_{k+1}) - \phi_{\varepsilon_{k+1}^{\text{rand}}, m_{k+1}, k+1, \omega}^{\text{rand}}(v, V))^2\right)^{1/2} \leq \frac{\varepsilon}{2^k}$$

(30)

and

$$\left(\mathbb{E}_\omega \mathbb{E}_q(I_{k+1}(h_{k+1}) - \phi_{\varepsilon_{k+1}^{\text{quant}}, \kappa_{k+1}, k+1, \omega}^{\text{quant}}(v, V))^2\right)^{1/2} \leq \frac{\varepsilon}{2^k}$$

(31)

hold.

It is also easy to see that we need to approximate only a few terms. Indeed, for $k$ approaching infinity, we have $\varepsilon_{k+1}^{\text{rand}}$ and $\varepsilon_{k+1}^{\text{quant}}$ also tending to infinity. Note that by (9), (22) and (26) we see that for

$$\frac{\varepsilon_{k+1}^{\text{rand}}}{m_{k+1}} \geq K^{k+1} \quad \text{and} \quad \frac{\varepsilon_{k+1}^{\text{quant}}}{\kappa_{k+1}} \geq K^{k+1},$$

with $K$ being the embedding constant in (9), the deterministic zero algorithms provide sufficient accuracy. Thus, we need to use the algorithms $\phi_{\varepsilon_{k+1}^{\text{rand}}, m_{k+1}, k+1, \omega}^{\text{rand}}$ and $\phi_{\varepsilon_{k+1}^{\text{quant}}, \kappa_{k+1}, k+1, \omega}^{\text{quant}}$ only for $k = O(\ln(\varepsilon^{-1}))$. Hence, we get $N_{\varepsilon}^{\text{rand}} = O(\ln(\varepsilon^{-1}))$ and $N_{\varepsilon}^{\text{quant}} = O(\ln(\varepsilon^{-1}))$. 

13
The bounds (30) and (31) yield
\[
\left( \mathbb{E}_{\omega}(S(v, V) - \Phi_{\varepsilon, \omega}^{\text{rand}}(v, V))^2 \right)^{1/2} \leq \varepsilon, \tag{32}
\]
\[
\left( \mathbb{E}_{\omega}\mathbb{E}_{q}(S(v, V) - \Phi_{\varepsilon, \omega}^{\text{quant}}(v, V))^2 \right)^{1/2} \leq \varepsilon. \tag{33}
\]
This means that the algorithms \( \Phi_{\varepsilon}^{\text{rand}} \) and \( \Phi_{\varepsilon}^{\text{quant}} \) compute \( \varepsilon \)-approximations of the multivariate Feynman-Kac path integral in the randomized and quantum settings respectively.

7.2 Number of function values and quantum queries

In this section we derive estimates on the numbers of function values and quantum queries \( n(\Phi_{\varepsilon}^{\text{rand}}) \) and \( n(\Phi_{\varepsilon}^{\text{quant}}) \) of the algorithms \( \Phi_{\varepsilon}^{\text{rand}} \) and \( \Phi_{\varepsilon}^{\text{quant}} \). By the bounds (17), (23) and (27) we get the obvious estimates

\[
n(\Phi_{\varepsilon}^{\text{rand}}) = O \left( \beta_1^{\alpha(F)} + \sum_{k=1}^{\infty} \left( c_1 + c_2 \frac{\ln 1/\varepsilon_{k+1}}{k} \right)^{\alpha(F)+1} \right. + \left. \frac{\beta_1 \beta_2 t^2 k^{k+1}}{k!} \alpha(F) \right) + 1 \varepsilon^{-2\alpha(F)/(\alpha(F)+2)},
\]

and

\[
n(\Phi_{\varepsilon}^{\text{quant}}) = O \left( \beta_1^{\alpha(F)} + \sum_{k=1}^{\infty} \left( c_1 + c_2 \frac{\ln 1/\varepsilon_{k+1}}{k} \right)^{\alpha(F)+1} \right. + \left. \frac{\beta_1 \beta_2 t^2 k^{k+1}}{k!} \alpha(F) \right) + 1 \varepsilon^{-\alpha(F)/(\alpha(F)+1)}.
\]

We can now use an argument similar to that in the proof of [15, Theorem 1] to show that

\[
\sum_{k=1}^{\infty} \left( c_1 + c_2 \frac{\ln 1/\varepsilon_{k+1}}{k} \right)^{\alpha(F)+1} + \left( \frac{\beta_1 \beta_2 t^2 k^{k+1}}{k!} \right)^{\alpha(F)} = O(\varepsilon^{-\delta}),
\]

for all \( \delta > 0 \). Thus we finally get

\[
n(\Phi_{\varepsilon}^{\text{rand}}) = O \left( \varepsilon^{-2\alpha(F)/(\alpha(F)+2)-\delta} \right), \tag{34}
\]
\[
n(\Phi_{\varepsilon}^{\text{quant}}) = O \left( \varepsilon^{-\alpha(F)/(\alpha(F)+1)-\delta} \right) \tag{35}
\]

for all \( \delta > 0 \).
8 Complexity of multivariate Feynman-Kac path integration in randomized and quantum settings

An analysis of the complexity of the multivariate Feynman-Kac path integration in randomized and quantum settings is quite similar to the one presented in [7] and [15]. We only point out essential differences.

8.1 Lower bounds

Lower bounds for our problem complexities are provided by the complexities of multivariate weighted integration problem. By this problem we mean an approximation of the integration operator $I : F \to \mathbb{R}$ define by

$$I(f) = (2\pi t^*)^{-d/2} \int_{\mathbb{R}^d} f(u) \exp(-\|u\|/(2t^*)) \, du \quad \forall \, f \in F.$$ 

Consider a randomized algorithm $A_{n}^{\text{rand}}$ that uses $n$ function values and approximates the integration operator $I$. We say that this algorithm computes an $\varepsilon$-approximation of the weighted integral if

$$\left( \mathbb{E}_{\omega} \left( I(f) - A_{n,\omega}^{\text{rand}} \right)^2 \right)^{1/2} \leq \varepsilon \quad \forall \, f \in F.$$

We denote by $n_{\text{INT}}^{\text{rand}}(\varepsilon, F)$ the minimal number of function values needed to compute an $\varepsilon$-approximation in the randomized setting.

Consider a quantum algorithm $A_{n}^{\text{quant}}$ that uses $n$ randomized quantum queries and approximates the operator $I$. We say that $A_{n}^{\text{quant}}$ computes an $\varepsilon$-approximation of the weighted integral if

$$\left( \mathbb{E}_{\omega} \mathbb{E}_{q} \left( I(f) - A_{n,\omega}^{\text{quant}} \right)^2 \right)^{1/2} \leq \varepsilon \quad \forall \, f \in F. \quad (36)$$

We define $n_{\text{INT}}^{\text{quant}}(\varepsilon, F)$ as the minimal number of quantum queries needed to compute an $\varepsilon$-approximation.

As in [7], we can reduce multivariate Feynman-Kac path integration to multivariate integration with a Gaussian weight by taking $V \equiv 0$, since $S(v, 0) = I(v)$. Moreover, (7) and (9) imply that

$$n_{\text{INT}}^{\text{rand}}(\varepsilon, F) \leq n_{\text{INT}}^{\text{rand}}(\varepsilon, F),$$

$$n_{\text{INT}}^{\text{quant}}(\varepsilon, F) \leq n_{\text{INT}}^{\text{quant}}(\varepsilon, F).$$
8.2 Upper bounds

Obvious estimates on the complexity of the multivariate Feynman-Kac path integration are provided by the cost of the algorithms derived in Section 7. Thus, by (34) and (35) we get

\[ n^{\text{rand}}(\varepsilon, \mathbb{F}) = O\left(\varepsilon^{-2\alpha(\mathbb{F})/((\alpha(\mathbb{F}))+2) - \delta}\right), \]
\[ n^{\text{quant}}(\varepsilon, \mathbb{F}) = O\left(\varepsilon^{-\alpha(\mathbb{F})/((\alpha(\mathbb{F}))+1) - \delta}\right) \]

for all \( \delta > 0 \), where \( \alpha(\mathbb{F}) \) is the exponent of the uniform approximation problem complexity for the space \( \mathbb{F} \) containing the class \( F \), i.e.,

\[ n^{\text{wor}}_{\text{APP}}(\varepsilon, \mathbb{F}) = O(\varepsilon^{-\alpha(\mathbb{F})}), \]

see also Section 4.

From the previous two sections we can see when the randomized and quantum algorithms proposed in this paper are almost optimal. This is the case for the classes of input functions for which randomized and quantum complexities of the integration problem defined in Section 8.1 are of orders \( \varepsilon^{-2\alpha(\mathbb{F})/((\alpha(\mathbb{F}))+2)} \) and \( \varepsilon^{-\alpha(\mathbb{F})/((\alpha(\mathbb{F}))+1)} \) respectively.

9 Examples

In this section we present two examples of function classes \( F \) satisfying the assumptions from Section 4 and compute lower and upper bounds of the complexities of the multivariate Feynman-Kac path integration.

Weighted Sobolev space

We use one of the results from [19], which relates the complexity of the approximation of functions defined over a finite domain to the complexity of the weighted approximation of functions over the whole space \( \mathbb{R}^d \). Let

\[ \mathbb{F} = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \in C^r(\mathbb{R}^d) \text{ and } \|f\|_{\mathbb{F}} := \sum_{0 \leq |\alpha| \leq r} \|f^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)} < \infty \right\}. \]

For simplicity, we consider a weight function \( \rho : \mathbb{R}^d \to \mathbb{R}_+ \), given by

\[ \rho(z) = \exp(-\|z\|^2) \quad \forall z \in \mathbb{R}^d \]
which decays exponentially. By [19] there exists an algorithm

\[ U_ε^ρ f = \sum_{i=1}^{n} f(t_i, ε) \bar{a}_{i, ε} \]

that computes a weighted ε-approximation of the function \( f \in \mathbb{F} \), i.e.,

\[ \| (f - U_ε^ρ f) \rho \|_{L_∞(\mathbb{R}^d)} \leq \varepsilon \| f \|_F, \]

with

\[ n = O \left( \varepsilon^{-d/r} \right). \]

Let

\[ \mathbb{F} = \{ f : \mathbb{R}^d \to \mathbb{R} : f/ρ \in \mathbb{F} \text{ and } \| f \|_F := \| f/ρ \|_F < \infty \}. \]

We can use the algorithm \( U_ε^ρ \) to construct an algorithm \( U_ε \) approximating functions from \( \mathbb{F} \). Indeed, define

\[ U_ε f = \sum_{i=1}^{n} f(t_i, ε) a_{i, ε}, \quad \text{where} \quad a_{i, ε} = \frac{\bar{a}_{i, ε} ρ(\bar{t}_i, ε)}{ρ(t_i, ε)}. \]

Then for \( f_ρ = f/ρ \), we have

\[ \| f - U_ε f \|_{L_∞(\mathbb{R}^d)} = \| (f_ρ - U_ε^ρ f_ρ) \rho \|_{L_∞(\mathbb{R}^d)} \leq \varepsilon \| f_ρ \|_F = \varepsilon \| f \|_F, \]

as claimed. We have to check the three remaining conditions which are to be satisfied by \( \mathbb{F} \), namely, the continuity of function evaluation as well as conditions. It is easy to see that for \( f \in \mathbb{F} \) and \( z \in \mathbb{R}^d \) we have

\[ f(z) \leq \| f \|_{L_∞(\mathbb{R}^d)} \leq \| f/ρ \|_{L_∞(\mathbb{R}^d)} \leq \| f \|_F \]

and so function evaluation is continuous. Conditions (8) and (9) follow immediately from this continuity.

The algorithms \( Φ_ε^{\text{rand}} \) and \( Φ_ε^{\text{quant}} \) compute an ε-approximation of the multivariate Feynman-Kac path integration problem for the class \( \mathbb{F} \) with the number of function evaluations and/or quantum queries roughly \( O(ε^{-2/(1+2r/d)}) \) and \( O(ε^{-1/(1+r/d)}) \), respectively. However, the factors appearing in the big O notation depend on \( d \) and this dependence is exponential, see Sections 7.1 and 7.2. For \( d \gg r \) the exponents \( 2d/(d + 2r) \) for the randomized algorithm and \( d/(d + r) \) for the quantum algorithm are close to \( 2 \) and \( 1 \). In fact, the orders \( 2 \) and \( 1 \) can be obtained by the use of the classical Monte Carlo algorithm (without variance reduction). Then, the factors multiplying \( ε^{-2} \) and \( ε^{-1} \) are independent of \( d \) for the class \( \mathbb{F} \) so the curse of dimensionality present in the worst case setting (see [7]) is indeed broken when we switch to the randomized or quantum settings.
To obtain lower bounds on the multivariate Feynman-Kac path integration problem in the class $F$ we may switch to the integration problem as in Section 8.1. We observe that this integration problem is not easier than the uniform integration over the unit cube

$$I(f) = \int_{[0,1]^d} f(z) \, dz$$

by taking functions with support $[0,1]^d$. It is known that the uniform integration problem has the randomized complexity $\Theta(\varepsilon^{-2/(1+2r/d)})$, see [12,13], and quantum complexity $\Theta(\varepsilon^{-1/(1+r/d)})$, see [21]. This shows that the algorithms $\Phi_{\varepsilon}^{\text{rand}}$ and $\Phi_{\varepsilon}^{\text{quant}}$ are roughly optimal for the class $F$.

**Periodic functions**

This example was considered in [7]. We repeat all details for the reader’s convenience. Following [17] we consider the class $\mathcal{F}$ of $2\pi$-periodic functions $f : [0,2\pi]^d \to \mathbb{R}^d$ satisfying the condition

$$\forall f \in \mathcal{F} \quad \forall j = 1, \ldots, d \quad \exists \varphi_j \in L_{\infty}([-2\pi, 2\pi]^d) \quad f(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_j(x_1, \ldots, x_j - t, \ldots, x_d) F_r(t) \, dt, \quad (37)$$

where $r > 0$ and

$$F_r(t) = 1 + 2 \sum_{k=0}^{\infty} k^{-r} \cos \left( k \pi \frac{t}{2} \right).$$

The norm in the class $\mathcal{F}$ is defined as

$$\|f\|_{\mathcal{F}} = \frac{1}{d} \sum_{j=1}^{d} \|\varphi_j\|_{L_{\infty}([-2\pi, 2\pi]^d)},$$

where the $\varphi_j$ are functions from the representation (37) of the function $f$. In [17], there is a linear algorithm $U_\varepsilon$ that computes a uniform $\varepsilon$-approximation of functions from the class $\mathcal{F}$, i.e., that

$$\|f - U_\varepsilon f\|_{L_{\infty}([0,2\pi]^d)} \leq \varepsilon \|f\|_{\mathcal{F}} \quad \forall f \in \mathcal{F},$$

with the cost of order $\varepsilon^{-d/r}$.

Denote by $\mathcal{F}$ the class of functions $f : \mathbb{R}^d \to \mathbb{R}$ that are periodic extensions of functions from $\mathcal{F}$. Let $\|f\|_{\mathcal{F}} := \|f\|_{0,2\pi)^d}$. Obviously, problem of the uniform approximation for the class $\mathcal{F}$ can be obviously solved using the algorithm mentioned above with the same cost as for the class $\mathcal{F}$. Similarly to the previous
example, we have to check the three conditions of Section 4. It is easy to see that for $f \in F$, $z \in \mathbb{R}^d$, and arbitrary $j \in \{1, 2, \ldots, d\}$ we have

$$f(z) \leq \|f\|_{L^\infty(\mathbb{R}^d)} = \|f\|_{[0,2\pi]^d} \leq \|\varphi_j\|_{L^\infty([-2\pi,2\pi]^d)},$$

with $C = (2\pi)^{-1}\int_0^{2\pi} |F_r(t)| \, dt$. Hence

$$|f(z)| \leq \|f\|_{L^\infty(\mathbb{R}^d)} \leq C \|f\|_F$$

and so function evaluation is continuous. The remaining conditions follow immediately.

Thus the algorithms $\Phi_{\varepsilon}^{\text{rand}}$ and $\Phi_{\varepsilon}^{\text{quant}}$, based on the algorithm $U_{\varepsilon}$ described above, compute an $\varepsilon$-approximation of the multivariate Feynman-Kac path integral with a number of function evaluations and/or quantum queries roughly $O(\varepsilon^{-2/(1+2r/d)})$ and $O(\varepsilon^{-1/(1+r/d)})$, respectively. Using an argument similar to that of the previous example, we conclude that the algorithms $\Phi_{\varepsilon}^{\text{rand}}$ and $\Phi_{\varepsilon}^{\text{quant}}$ are roughly optimal for the class $F$.

Acknowledgements

I wish to thank my advisor H. Woźniakowski for many inspiring discussions. I am also grateful to S. Heinrich, J. F. Traub, A. G. Werschultz for valuable comments and remarks.

References

[1] G. Brassard, P. Høyer, M. Mosca, A. Tapp, Quantum Amplitude Estimation and Amplification, Quantum Computation and Information, S. J. Lomonaco and H. E. Brandt, eds., American Math. Society, 2002, [http://arXiv.org/quant-ph/0005055](http://arXiv.org/quant-ph/0005055), 2000.

[2] S. Heinrich, Quantum Summation with an Application to Integration, J. Complexity, 18, 1-50, 2002. [http://arXiv.org/quant-ph/0105116](http://arXiv.org/quant-ph/0105116), 2001.

[3] S. Heinrich, Quantum Complexity of Numerical Problems, submitted for publication.

[4] S. Heinrich, M. Kwas, H. Woźniakowski, Quantum Boolean Summation with Repetitions in the Worst-Average Setting, to appear in the Proceedings of the 5th International Conference on Monte Carlo and Quasi-Monte Carlo Methods, Singapore 2002.
[5] I. Karatzas, S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, Berlin (1988).

[6] H.-H. Kuo, *Gaussian Measures in Banach Spaces*, Springer-Verlag, Berlin (1995).

[7] M. Kwas, Y. Li, Worst case complexity of multivariate Feynman-Kac path integration, *J. Complexity*, 19, 730-743, 2003.

[8] M. Kwas, H. Woźniakowski, Sharp Error Bounds on Quantum Boolean Summation in Various Settings, to appear in *J. Complexity*.

[9] Y. Li, Applicability of Smolyak’s Algorithm to Certain Banach Spaces of Multivariate Functions, *J. Complexity*, to appear.

[10] Y. Li, G. W. Wasilkowski, Worst Case Complexity of Weighted Integration and Approximation over $\mathbb{R}^d$, *J. Complexity*, 18, 330-345 (2002).

[11] A. Nayak and F. Wu, The quantum query complexity of approximating the median and related statistics, Proceedings of the 31th Annual ACM Symposium on the Theory of Computing (STOC), 384-393, 1999, [http://arXiv.org/quant-ph/9804066](http://arXiv.org/quant-ph/9804066), 1998.

[12] E. Novak, Deterministic and Stochastic Error Bounds in Numerical Analysis, *Lecture Notes in Mathematics* 1349, Springer, 1988.

[13] E. Novak, Quantum Complexity of Integration, *J. Complexity*, 17, 2-16, 2001, [http://arXiv.org/quant-ph/0008124](http://arXiv.org/quant-ph/0008124), 2000.

[14] M. A. Nielsen, I. L. Chuang *Quantum Computation and Quantum Information*, Cambridge University Press, 2001.

[15] L. Plaskota, G. W. Wasilkowski, H. Woźniakowski, A New Algorithm and Worst Case Complexity for Feynman–Kac Path Integration, *J. Comp. Phys.*, 164, 335-353 (2000).

[16] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin (1991).

[17] V. N. Temlyakov, On Approximate Recovery of Functions with Bounded Mixed Derivative, *J. Complexity*, 9, 41-59 (1993).

[18] J. F. Traub, G. W. Wasilkowski, H. Woźniakowski, *Information-Based Complexity*, Academic Press, New York (1988).

[19] G. W. Wasilkowski, H. Woźniakowski, Complexity of Weighted Approximation over $\mathbb{R}^d$, to appear in *J. Complexity*, (2001).

[20] G. W. Wasilkowski, H. Woźniakowski, Explicit Cost Bounds for Multivariate Tensor Product Problems, *J. Complexity*, 11, 1-56 (1995).

[21] H. Woźniakowski, Randomized Quantum Queries, in preparation.