Flory-like Statistics of Fracture in Fiber Bundle Model as obtained via Kolmogorov Dispersion for Turbulence: A conjecture

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It has long been conjectured that (rapid) fracture propagation dynamics in materials and turbulent motion of fluids are two manifestations of the same physical process. The universality class of turbulence (Kolmogorov dispersion in particular) had been conjectured to be identifiable with the Flory statistics for linear polymers (self-avoiding walks on lattices). These help us to relate fracture statistics to those of linear polymers (Flory statistics). The statistics of fracture in the Fiber Bundle Model (FBM) are now well studied and many exact results are now available for the equal load-sharing (ELS) scheme. Yet, the correlation length exponent in this model was missing and we show here how the correspondence between fracture statistics and the Flory mapping of Kolmogorov statistics for turbulence helps us to make a conjecture about the value of the correlation length exponent for fracture in the ELS limit of FBM, and also about the upper critical dimension. Besides, the fracture avalanche size exponent values in lower dimensions (as estimated from such mapping to Flory statistics) also compare well with the observations.

I. INTRODUCTION

The dynamics of fracture propagation and the resulting multi-fractal nature of the roughness of the fractured surfaces cannot be captured directly from the linear elastic theory of fracture. The same is true for the intermittent nature of the energy avalanches that are frequently observed in experiments with a scale free size distribution, from the tectonic scale earthquakes (Gutenberg-Richter law) to the laboratory scale fracture (power law statistics of acoustic emission experiments).

Nevertheless, a well developed literature exist for the past three decades or so that explores fracture as a stochastic critical phenomenon through various experiments and approximate models $^{[1–3]}$. Particularly, the existence of a universal roughness exponent that is largely independent of the details of the material, a diverging relaxation time, signifying critical slowing down near the global failure point and the above mentioned scale free size distribution of the energy avalanches, support this dynamical critical phenomena picture of fracture $^{[4,5]}$.

Among the several attempts to develop a consistent theory for the critical behavior in fracture, a relatively less explored path is the hydrodynamic analogy of fracture propagation $^{[6,7]}$. The long wave-length limit of a hydrodynamic description will average over the details of the sample in question and could come up with a universality class, as is often the case for known static critical phenomena, such as magnetism. In particular, the analogy between fracture and homogeneous, isotropic turbulence has been explored before in several cases, in order to relate the velocity fluctuation in turbulence with the roughness of the fractured surfaces (see e.g., $^{[8]}$). Here we show that in so far as the vortex lines of turbulence in a non-viscous fluid can be mapped $^{[10]}$ to the Self Avoiding Walks (SAWs) $^{[12]}$, the Kolmogorov energy cascade statistics $^{[13]}$ help obtaining a consistent failure statistics of the Fiber Bundle Model (FBM) in one and two dimensions with Local Load Sharing and its convergence to the Equal Load Sharing (ELS) or mean field limit $^{[2,5]}$ beyond the upper critical dimension ($d_c$). Exploiting the same analogy between SAW statistics and of turbulence and, in turn, of fracture, we could also conjecture about the correlation length exponent ($\nu$) for the fiber bundle model in the mean field (ELS) limit. In what follows we will obtain $\nu = 1/4$ and $d_c = 6$. It may be mentioned that these estimates were missing earlier $^{[1,2,5]}$.

II. ENERGY DISPERSION IN FRACTURE AND TURBULENCE

As mentioned before, the analogy between a fully developed turbulence and the “frozen” undulation of a fractured surfaces in terms of it’s roughness exponent, have been explored before. Specifically an analogy between the hydrodynamics description of turbulence velocities and the roughness of fractured surface in mode-I cracks predicts a multi fractal surface, consistent with many experimental observations of fracture $^{[4]}$. The correspondence between the Kolmogorov energy dispersion and that observed in the intermittent acoustic energy emissions for fracture is, however, not explored.

In pursuing this line of argument, let us recall that turbulent fluids can be thought of as an ensemble of vortex lines that do not cross each other as long as the fluids are non-viscous $^{[10]}$. Therefore, the vortex lines can be modeled as SAWs on lattices, and it is straight forward to relate the Fourier transform of the correlation function...
of vortex density with the wave number as
\[ g_k \sim k^{-\alpha} \tag{1} \]
with \( \alpha = 1/\nu_F \), where \( \nu_F \) is the so called compactness exponent of the SAW and is estimated from the Flory’s theory of polymers (see e.g., [12, 14]) as:
\[ \nu_F = \frac{3}{2 + d}, \text{ for } d \leq 4 \]
\[ = \frac{1}{2}, \text{ for } d > 4 \tag{2} \]
where \( d \) denotes the spatial dimension. Now, the energy density in the real space is proportional to the correlation function, implying that the energy spectrum in the Fourier space is proportional to \( g_k \) i.e.
\[ E_k \sim k^{-\alpha}, \tag{3} \]
with \( \alpha = (2 + d)/3 \), which is known to recover (see e.g., [12, 14]) the Kolmogorov exponent 5/3 for \( d = 3 \).

Let us now turn to the equivalent dispersion relation in the case of fractures. The probability distribution functions of the avalanches of acoustic energies are known to have a scale-free variation of the form [1]
\[ P(E) \sim E^{-\delta}, \tag{4} \]
which is verified in numerous experiments and theoretical models [1, 2], including the observation of the Gutenberg-Richter scaling of earthquake statistics. If we take that the probability scales as the inverse of the correlated volume \( \xi^d \), where \( \xi \) is the correlation length, then considering a wave number \( k \) that scales as the inverse of the length scale \( \xi \), we end up with
\[ k^d \sim E_k^{-\delta}, \tag{5} \]
or
\[ E_k \sim k^{-d/\delta}. \tag{6} \]

Drawing the parallel now with Eq. (3), we get
\[ \delta = \frac{3d}{d + 2}, \text{ for } d \leq 4 \]
\[ = \frac{d}{2}, \text{ for } d > 4 \tag{7} \]
This is now giving an estimate of the exponent value seen in the size distribution of the energy avalanches, based on the Flory statistics. Given the upper critical dimension for the Flory statistics is 4, we should have
\[ E_k \sim k^{-2} \tag{8} \]
for the mean field limit of the Kolmogorov dispersion (cf. [15]; see also the Appendix A). It is also possible to arrive at this dispersion form for \( d = 4 \) from a purely dimensional analysis (see Appendix A). In comparing the corresponding limit in fracture, we note that in the fiber bundle model of fracture, the upper critical dimension is known to be higher than 5. It is consistent, therefore, to take \( d = 6 \) as the upper critical dimension (\( d_c \)) for the fiber bundle model of fracture and then use the exact value for the avalanche size exponent \( \delta = 3 \) for fixed load increase protocol [11] (see also [2], pp. 57-61; notwithstanding the difference in the \( \delta \) value in the quasistatic limit [2]) and Eq. (3) then gives \( E \sim k^{-2} \), consistent with Eq. (5) estimated above.

To probe the estimate in Eq. (4) in lower dimensions, we recall that in numerical estimates of the avalanche size distribution for a two-dimensional interface propagation in the fiber bundle model, \( \delta \approx 1.5 [12, 14] \), which is consistent with Eq. (7) with \( d = 2 \) (considering a fixed rate of load increase, rather than following a quasi-static increase).

In the \( d = 1 \) limit, obviously the “compactness” exponent becomes 1. But it is possible to calculate the energy avalanches in the fiber bundle model in one dimension, when the loading is done locally (see Appendix B) and the result matches with the prediction in Eq. (7).

Therefore, we see that the energy cascades shown in the Kolmogorov dispersion for turbulence are also captured in the fracture propagation models, to the extent that a SAW statistics for turbulence is considered.

### III. LENGTH AND TIME SCALE DIVERGENCES IN FBM IN MEAN FIELD (ELS) LIMIT

Given that the scale-free statistics of energy avalanches in fracture can be viewed as a result of the growing correlation length and critical slowing down, a well developed literature of fracture models explored the corresponding critical exponent values [3]. These are supported by the experimental observations [4]. Specifically, the critical slowing down, represented by a diverging relaxation time
\[ \tau \sim (\sigma - \sigma_c)^{-\zeta} \tag{9} \]
can be exactly calculated for the fiber bundle model in the mean field limit, with \( \zeta = 1/2 \) [4]. However, given the lack of an explicit notion of length scale in the mean field, a correlation length exponent in the form \( \xi \sim (\sigma - \sigma_c)^{-\nu} \) is yet unknown. Here we exploit the correspondence with the turbulence described above to estimate the exponent \( \nu \) for the mean field fiber bundle model.

Writing the wave number \( k \) as an inverse length scale as before, we end up with
\[ k \sim (\sigma - \sigma_c)^{\nu'}, \tag{10} \]
and using the Kolmogorov dispersion, assuming \( E \sim \tau^\eta \), we get
\[ E_k \sim \tau^{\eta} \sim (\sigma - \sigma_c)^{-\eta\zeta} \sim k^{-\frac{6\eta}{\nu}}, \tag{11} \]
giving
\[ \alpha = \frac{1}{\nu_F} = \frac{\eta\zeta}{\nu} \tag{12} \]
and in the mean field limit, the exponent values of $\zeta$ and $\nu_F$ are $1/2$ \cite{12} and $2$ \cite{14} respectively, giving $\nu = \eta/4$ in the mean field limit.

Although an explicit estimate of the correlation length exponent was not made for the FBM in the mean field limit, there are some related results along this line, which we demonstrate to be consistent with the estimate we obtained above. Utilizing the Fisher finite size scaling argument (see e.g., \cite{17}), where the correlation length $\xi \sim (\sigma - \sigma_c)^{\nu}$, becomes of the order of system size $L$, giving the effective value of the breaking load $\sigma_c(L)$ by $L \sim (\sigma_c(L) - \sigma_c(\infty))^{-\nu}$, where the global ($L \to \infty$) failure load is denoted by $\sigma_c(\infty)$. This implies,

$$\sigma_c(L) - \sigma_c(\infty) \sim L^{-1/\nu}. \tag{13}$$

Now, for a system in $d$ dimensions, the total number of fibers is $N = L^d$. Then the above equation in terms of the total number of fibers becomes

$$\sigma_c(N) - \sigma_c(\infty) \sim N^{-1/\nu d}. \tag{14}$$

This relation, however, was derived exactly \cite{18} and was extensively checked numerically to find $1/\nu d = 2/3$ \cite{19}, in the mean field limit. Therefore, we need to insert the upper critical dimension for the fiber bundle model in order to get the correlation length exponent. According to our conjecture discussed earlier, the upper critical dimension of FBM is 6 (it is numerically known to be higher than 5 \cite{20}). In some other numerical studies (see for example \cite{21}), the value of $d_c$ seems to be higher than 8, though there occurs a crossover to brittle behavior, unlike the case of continuous transition considered here. We therefore get $\nu = 1/4$ for the fiber bundle model in the mean field limit, which is consistent with our estimate (indicated above) from the Kolmogorov dispersion and SAW description of turbulence, if $\eta = 1$.

IV. SUMMARY AND DISCUSSIONS

We have utilized here the SAW mapping \cite{10} of the vortex lines in turbulence together with the Flory statistics of SAWs \cite{12,13} and Kolmogorov dispersion of energy in turbulence \cite{12}, which in turn, helps us to relate the statistics of fracture \cite{6,7} in the ELS (mean field) limit of fracture statistics (in FBM) with the Flory statistics. The exponent values of various quantities that have been obtained independently for these fields, seem to agree with each other well in various dimensions, and such mappings also helps some reasonable scaling conjectures. It may be noted however (see discussions in sec. III), as in some other cases (see e.g., \cite{22}), the upper critical dimension (4) of Flory statistics differs from that (6) of FBM.

Specifically, for the FBM in the ELS limit, as discussed in sec. III, the exact result $\zeta = 1/2$ \cite{6} for the growth of relaxation time $\tau$ (or critical slowing down; Eq. \cite{9}) can be recast as Eq. \cite{4} with $\tau^\eta$ as $E_k$ and $\xi^{-1}$ as $k$, giving $\alpha \nu = \eta/2$ (with $\eta = 1$) as suggested in the discussion following Eq. \cite{13}. It may be noted, though the energy $E$ or the avalanche size $\Delta$ (e.g., in Eq. \cite{11}) and the relaxation time $\tau$ (in Eq. \cite{9}) are shown to be linearly related ($\eta = 1$), the obtained power laws can be also be argued on the basis of a dimensional analysis. Also, the exact result \cite{18} for finite size scaling behavior (Eq. \cite{13}) gives $d\nu = 3/2$. Earlier, in sec. II, we got the relationship $\alpha = d/\delta$ (from Eq. \cite{6}). These three relationships, when rewritten, give $\alpha = 1/\delta = 1$ and $\nu = 1/\delta = 1$. or $n = \delta$ and $\alpha = d/\delta$. These, in turn, give $d = d_c = 6$ for the upper critical dimension for FBM, using Flory formula (Eq. \cite{2} giving $\alpha = 1/\nu_F = 2$ for $d \geq 4$) and consequently $\nu = 1/4$ for the correlation length exponent in FBM (with $d = d_c = 6$ for ELS limit).

In conclusion, the analogy between fracture propagation in fiber bundle model and turbulence in fluids, and the use of Self-Avoiding Walk map of the vortex lines in a fully developed turbulence, have helped us here to relate the fracture statistics in Fiber Bundle Model with those of the Flory statistics of the linear polymers and provide with some consistent relationships. In particular, we obtain the new results for the correlation length exponent $\nu = 1/4$ and the upper critical dimension $d_c = 6$ in the Equal Load Sharing limit of the Fiber Bundle Model.

APPENDIX A: KOLMOGOROV EXPONENT FOR $d = 4$ FROM DIMENSIONAL ANALYSIS

It is worth recalling the dimensional analysis that led to the estimate of $\alpha$ in Eq. \cite{3} ($\alpha = 5/3$ for $d = 3$) can be extended for $d = 4$.

In three dimensions, noting that relevant dimensions are the following: $k \sim 1/L$, $E_k \sim \text{force} \times \text{area} \sim L^2 \sim T^2/\mathcal{L}$ and the dissipation scale $\psi$ as the rate of energy per unit length of flow should have the dimension $E_k/\mathcal{L} \sim L^2/\tau^2$. So, if $E_k \sim k^{-\alpha} \psi^\beta$, then

$$\frac{L^3}{\tau^2} \sim \frac{L^{3+\alpha}}{T^{3\beta}}. \tag{15}$$

Collecting the exponents for $T$ and $L$, we arrive at $\beta = 2/3$ and $\alpha = 5/3$ for $d = 3$.

Extending now for the four dimensions, $E_k \sim \text{force} \times \text{volume} \sim \frac{L^3}{\mathcal{L}} L^3 \sim L^6$ and $\psi$ is, as before, the rate of energy per unit length of flow $\sim \frac{E_k}{\mathcal{L}^2} \sim L^4$, therefore, if $E_k \sim k^{-\alpha} \psi^\beta$, then

$$\frac{L^4}{\tau^2} \sim \frac{L^{4+\alpha}}{T^{4\beta}}. \tag{16}$$

which on separating the powers of $T$ and $L$, gives $\beta = 2/3$ and $\alpha = 2$. This wonderful matching with the estimate from the Flory theory mentioned in the main text (at the upper critical dimension 4) is noteworthy.
APPENDIX B: ENERGY AVALANCHE SIZE DISTRIBUTION IN FBM IN ONE DIMENSION

If we consider a threshold ($\sigma_{th}$) distribution in (0:1), then the probability that an avalanche of size $\Delta$ occurs following an application of load $\sigma$, is the probability that $\Delta$ number of fibers have their failure thresholds less than $\sigma$ i.e. $P(\Delta, \sigma) \sim \sigma^\Delta$. Integrating over $\sigma$, e.g., for uniformly distributed fiber strengths ($\sigma_{th}$), we get $P(\Delta) \sim \frac{1}{1 + \Delta}$. So, for large $\Delta$, we must have $P(\Delta) \sim \frac{1}{\Delta}$, giving back the exponent $\delta = 1$ predicted from Eq. (12), which also matches well with simulations (not shown) for uniform as well as other threshold distributions.

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[1] S. Biswas, P. Ray, B. K. Chakrabarti, *Statistical physics of fracture, breakdown and earthquake: effects of disorder and heterogeneity*, Wiley-VCH, Singapore (2015).
[2] A. Hansen, P. C. Hemmer, S. Pradhan, *The fiber bundle model: Modeling failure in materials*, Wiley-VCH, Singapore (2015).
[3] H. Kawamura, T. Hatano, N. Kato, S. Biswas, B. K. Chakrabarti, *Statistical physics of fracture, friction, and earthquakes*, Rev. Mod. Phys. 84, 839 (2012).
[4] D. Bonnay, E. Bouchaud, *Failure of heterogeneous materials: A dynamics phase transition?*, Phys. Rep. 498, 1 (2011).
[5] S. Pradhan, A. Hansen, B. K. Chakrabarti, *Failure processes in elastic fiber bundles*, Rev. Mod. Phys. 82, 499 (2010).
[6] A. Basu, B. K. Chakrabarti, *Hydrodynamic descriptions for surface roughness in fracture front propagation*, Phil. Trans. R. Soc. A 377, 20170387 (2018).
[7] B. K. Chakrabarti, *Kolmogorov dispersion for turbulence in porous media: A conjecture*, Physica A 384, 25 (2007).
[8] B. K. Chakrabarti, *Turbulence in porous media & scaling behavior of fractured surfaces: A conjecture*, in proceedings of Continuum Models and Discrete Systems CMD-11, D. Jeulin, S. Forest (eds), (2005).
[9] E. Bouchbinder, I. Procaccia, S. Sela, *Disentangling scaling properties in anisotropic fracture*, Phys. Rev. Lett. 95, 255503 (2005).
[10] K. Huang, *Lectures on Statistical Physics and Protein Folding*, World Scientific, Singapore (2005).
[11] S. Pradhan, P. Bhattacharyya, B. K. Chakrabarti, *Dynamic critical behavior of failure and plastic deformation in the random fiber bundle model*, Phys. Rev. E 66, 016116 (2002).
[12] P. J. Flory, *Statistical Mechanics of Chain Molecules*, Wiley (1969).
[13] A. N. Kolmogorov, *Local structure of turbulence in an incompressible fluid at very high Reynolds number*, Dokl. Acad. Nauk SSSR 30, 299 (1941).
[14] P. G. de Gennes, *Scaling concepts in Polymer Physics*, Cornell University Press, Ithaca (1979).
[15] S. Biswas, B. K. Chakrabarti, *Self-organized dynamics in local load sharing fiber bundle models*, Phys. Rev. E 88, 042112 (2013).
[16] S. Biswas, L. Goehring, *Interface propagation in fiber bundles: local, mean-field and intermediate range-dependent statistics*, New J. Phys. 18, 103048 (2016).
[17] M. E. Fisher, M. N. Barber, *Scaling theory for finite-size effects in the critical region*, Phys. Rev. Lett. 28, 1516 (1972).
[18] R. L. Smith, *The asymptotic distribution of the strength of a series-parallel system with equal load sharing*, Ann. Prob. 10, 137 (1982).
[19] C. Roy, S. Kundu, S. S. Manna, *Scaling forms for relaxation times of the fiber bundle model*, Phys. Rev. E 87, 062137 (2013).
[20] S. Sinha, J. T. Kjellstadli, A. Hansen, *Local load-sharing fiber bundle model in higher dimensions*, Phys. Rev. E 92, 020401(R) (2015).
[21] Z. Danku, G. Odor, F. Kun, *Avalanche dynamics in higher-dimentional fiber bundle models*, Phys. Rev. E 98, 042126 (2018).
[22] P. L. Krapivsky, S. Redner, E. Ben-Naim, *A Kinetic view of statistical physics*, Cambridge University Press, Cambridge (2010).