On the existence and analyticity of solitary waves solutions to a two-dimensional Benjamin-Ono equation.

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Abstract

We show the existence, regularity and analyticity of solitary waves associated to the following equation

\[(u_t + u^p u_x + \mathcal{H} \partial_x^2 u + \lambda \mathcal{H} \partial_y^2 u)_x + \mu u_{yy} = 0,\]

where \(\mathcal{H}\) is the Hilbert transform with respect to \(x\) and \(\lambda\) and \(\mu\) are nonnegative real numbers, not simultaneously zero.

In this paper we will use the following notations:

- \(L^2 = L^2(\mathbb{R}^2)\)
- \(H^s = H^s(\mathbb{R}^2)\)
- \(S = S(\mathbb{R}^2)\)
- \(X^{\frac{1}{2}} = \{f \in L^2 | D_x^{\frac{1}{2}} f \in L^2 \text{ and } D_x^{-\frac{1}{2}} f_y \in L^2\}\)
- \(\dot{X}^{\frac{1}{2}} = \{f \in L^2 | D_x^{\frac{1}{2}} f \in L^2 \text{ and } \partial_x^{-1} f_y \in L^2\}\)
- \(Y = \{f \in H^1 | \partial_x^{-1} f_y \in L^2\}\)

1 Introduction

In this paper we are interested in the proof of the existence and analyticity of solitary waves associated to the following equation

\[(u_t + u^p u_x + \mathcal{H} \partial_x^2 u + \lambda \mathcal{H} \partial_y^2 u)_x + \mu u_{yy} = 0,\] (1)

where \(\mathcal{H}\) is the Hilbert transform with respect to \(x\) and \(\lambda\) and \(\mu\) are nonnegative real numbers, not simultaneously zero. We emphasize 2 striking cases commonly appearing in the mathematical literature, when \(\lambda = 1\) and \(\mu = 0\) and when \(\lambda = 0\)
and \( \mu = 1 \). Using Kato’s theory, for instance, it can be proved that the equation (1) is local well-posed in \( H^s \cap X^{+} \) and \( H^s \cap Y \), for \( s > 2 \).

Observe that
\[
E_1(u) = \int_{\mathbb{R}^2} \frac{1}{2} \left( (D_x^2 u)^2 + \lambda(D_x^{-1} \partial_y u)^2 + \mu(\partial_x^{-1} \partial_y u)^2 \right) + \frac{u^{p+1}}{p(p+1)} \, dx
\]
and
\[
Q(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \, dx,
\]
are conserved by the flow of (1).

This is a two-dimensional case of the Benjamin-Ono equation
\[
\partial_t u + 3(\partial_x^2 u + u\partial_x u) = 0,
\]
which describes certain models in physics about wave propagation in a stratified thin regions (see [3] and [15]). This last equation shares with the equation KdV
\[
u_t + u_x + uu_x + u_{xxx} = 0
\]
many interesting properties. For example, they both have infinite conservation laws, they have solitary waves as solutions which are stable and behave like soliton (this last is evidenced by the existence of multisoliton type solutions) (see [1] and [14]). Also, the local and global well-posedness was proven in the Sobolev spaces context (in low regularity spaces inclusive, see, e.g., [7], [16], [9], [11] and [17].

We should note that the equation (1) is the model of dispersive long wave motion in a weakly nonlinear two-fluid system, where the interface is subject to capillarity and bottom fluid is infinitely deep (see [1], [2] and [10]). For this equation, with \( \alpha = 0 \), the local well-posedness was proven in [15] and the existence of solitary wave solution was claimed in [6], however their proof is not complete, they do not present a satisfactory proof of Lemma 3.4 there (Lemma 2.12 here). We use interpolation spaces techniques for this.

This paper is organized as follows. In Section 2 we present theorem about interpolation of the spaces involved. In Section 3 we present the proof of the existence of solitary waves solution to the equation (1), to this we use minimax theory techniques. Finally, in Section 4 we shall show that these solitary waves are analytic (in the real sense) using the Lizorkin theorem (see [13]) and ideas developed in [8].

## 2 Preliminaries

In this section we examine some properties that we shall use later. It is easy to see that \( X^{+} \) is a Hilbert space with the inner product defined by
\[
\langle f, g \rangle_\frac{1}{2} = \int_{\mathbb{R}^2} fg + D_x^\frac{1}{2} f D_x^\frac{1}{2} g + \lambda D_x^{-1} \partial_y g_y + \mu \partial_x^{-1} f_y \partial_x^{-1} g_y dx dy.
\]

**Proposition 2.1.** \( \partial_x S \subset X^{\frac{1}{2}} \).
Let us recall two important results whose proofs can be found in [12]

**Lemma 2.2.** Let $s \in (0, 1)$, then $H^s(\mathbb{R}^2)$ is continuously embedded in $L^p(\mathbb{R}^2)$ with $p = \frac{2}{1-s}$. Moreover, for $f \in H^s(\mathbb{R}^2)$, $s \in (0, 1)$,

$$
\|f\|_{L^p} \leq C_s \|D^s_x f\|_{L^2} \leq c\|f\|
$$

where

$$
D^s = (-\Delta)^{s^2} f = ((2\pi|\xi|)^{s} \hat{f})^\vee.
$$

**Lemma 2.3.** Suppose that $D^s_1 f \in L^2$ and $D^s_2 f \in L^2$. Then, for $s \in [s_1, s_2]$, $D^s_x f \in L^2$ and

$$
\|D^s_x f\| \leq C_s \|D^{s_1}_x f\|^\theta \|D^{s_2}_x f\|^{1-\theta}
$$

where $\theta = \frac{s_2 - s}{s_2 - s_1}$.

As consequence of these two lemmas we have the following useful embedding lemma.

**Lemma 2.4.** (a) For $\lambda > 0$ and $p \leq 2$, there exists $C > 0$, such that, for each $f \in X^\frac{1}{2}$,

$$
\|f\|_L^{p+2} \leq C \|f\|_L^{2-p} \|D^{1/2}_x \partial_y f\|_L^{p/2} \|D^{1/2}_x f\|_L^{3p/2}
$$

(b) For $\lambda = 0$, $\mu > 0$ and $p \leq \frac{4}{3}$, there exists $C > 0$, such that, for each $f \in X^\frac{1}{2}$,

$$
\|f\|_L^{p+2} \leq C \|f\|_L^{\frac{4}{3-p}} \|D^{1/2}_x f\|_L^{\frac{2p+4}{3-p}} \|\partial_x^{-1} \partial_y f\|_L^{\frac{3p}{2}}
$$

In particular, for each $f \in X^\frac{1}{2}$,

$$
\|f\|_L^{p+2} \leq C \|f\|_X^{p+2},
$$

where

$$
\begin{cases}
0 \leq p \leq 2 & \text{if } \lambda > 0, \\
0 \leq p \leq \frac{4}{3} & \text{if } \lambda = 0 \text{ and } \mu > 0.
\end{cases}
$$

**Proof.** By Proposition 2.1, it is enough show (7) for $f = \partial_y \phi$, $\phi \in S$. First, let us suppose $\lambda > 0$. Lemmas 2.2, 2.3 and Hölder inequality allow us to show that

$$
\|f\|_L^{p+2} = \int \|f(x, y)|^{p+2} dxdy \leq C \int \|D^{p/2(p+2)}_x f(., y)\|_L^{p+2} dy
$$

$$
\leq C \int \|D^{1/2}_x f(., y)\|_L^p \|f(., y)\|_L^{2} dy
$$

$$
\leq C \|D^{1/2}_x f\|_L^p \left( \sup_{y \in \mathbb{R}} \|f(., y)\|_L^{4/(2-p)} \right)^{(2-p)/2}
$$

$$
\leq C \|D^{1/2}_x f\|_L^p \|f\|_L^{2-p} \sup_{y \in \mathbb{R}} \|f(., y)\|_L^p
$$
On the other hand, for each $y \in \mathbb{R}$,
\[
\|f(\cdot, y)\|^2 = \int_{\mathbb{R}} f^2(x, y) dx = 2 \int_{\mathbb{R}} \int_{-\infty}^{y} f(x, y_1) \partial_y f(\cdot, y_1) dy_1 dx \\
= 2 \int_{-\infty}^{y} \int_{\mathbb{R}} D^{1/2}_x f(x, y_1) D^{-1/2}_x \partial_y f(x, y_1) dx dy_1 \\
\leq 2\|D^{1/2}_x f\| \|D^{-1/2}_x \partial_y f\|
\]
Hence, we obtain the inequality $\text{(8)}$.

Now, let us suppose $\lambda = 0$. Proceeding as above, we get
\[
\int_{\mathbb{R}^2} |f|^{p+2} dxdy \leq c \int_{\mathbb{R}} \|D^{p/[2(p+2)]}_x f(\cdot, y)\|^{p+2} dy \\
\leq c \int_{\mathbb{R}} \|D^{1/2}_x f(\cdot, y)\|^{(3p+2)/3} \|D^{-1/4}_x f(\cdot, y)\|^{4/3} dy \\
\leq c\|D^{1/2}_x f\|^{(3p+2)/3} \|f\|^{(4-3p)/3}_{0} \left( \sup_{y \in \mathbb{R}} \|D^{-1/4}_x f(\cdot, y)\|_{L^2(\mathbb{R})} \right)^6 \hspace{1cm} (9)
\]
In this case, for each $y \in \mathbb{R}$,
\[
\|D^{-1/4}_x f(\cdot, y)\|^{2}_{L^2} = \int_{\mathbb{R}} \left( D^{-1/4}_x f \right)^2 (x, y) dx \\
= 2 \int_{\mathbb{R}} \int_{-\infty}^{y} D^{-1/4}_x f(x, y_1) D^{-1/4}_x f_y(x, y_1) y_1 dy_1 dx \\
= -2 \int_{-\infty}^{y} \int_{\mathbb{R}} D^{1/2}_x f(x, y_1) D^{-1}_x f_y(x, y_1) dx dy_1 \\
\leq 2 \int_{\mathbb{R}} \|D^{1/2}_x f\| \|D^{-1}_x f_y\| dy \\
\leq 2\|D^{1/2}_x f\|_0 \|D^{-1}_x f_y\|_0.
\hspace{1cm} (10)
\]
Hence it follows $\text{(8)}$.

Let $X_0 = \{ f \in L^2 \mid \partial_x f, \partial_x^{-1} f_{yy} \text{ and } \partial_x^{-2} f_{yy} \in L^2 \}$. $X_0$ is a Hilbert space with the inner product
\[
\langle f, g \rangle_0 = \int_{\mathbb{R}^2} fg + \partial_x f \partial_x g + \lambda \partial_x^{-1} f_{yy} \partial_x^{-1} g_{yy} + \mu \partial_x^{-2} f_{yy} \partial_x^{-2} g_{yy} dxdy.
\]
It is obvious $(X_0, L^2)$ is a compatible couple in the interpolation theory sense (see $\text{[4]}$).
Theorem 2.5. \( X^\frac{1}{2} = (X_0, L_2)_{\cdot\cdot\cdot} \).

Proof. Let \( \phi \in X^\frac{1}{2} \) and

\[
f(z) = e^{-\lambda(z - \frac{1}{2})^2} \left[ 1 + |\xi| \left( 1 + \lambda \left| \frac{\eta^2}{\xi^2} \right| + \mu \left| \frac{\eta^2}{\xi^3} \right| \right)^{\frac{1}{2}} \right] ^{\frac{1}{2}}.
\]

It is easy to see that

\[
\begin{cases}
f(z) \in L^2, & \text{for all } 0 \leq \text{Im}(z) \leq 1 \\
f \text{ is analytic on } 0 < \text{Im}(z) < 1 & \\
f(it) \in X_0, & \text{for all } t \in \mathbb{R} \\
f(1 + it) \in L^2, & \text{for all } t \in \mathbb{R} \\
f(z) \to 0, & \text{as } |\text{Im}(z)| \to \infty, \text{ for } \text{Re}(z) = 0 \\
f\left( \frac{1}{2} \right) = \phi.
\end{cases}
\]  

(11)

Then \( \phi \in (X_0, L_2)_{\cdot\cdot\cdot} \) and \( \|\phi\|_{\cdot\cdot\cdot} \leq c\|\phi\|_{X^\frac{1}{2}} \). Now let \( \phi \in (X_0, L_2)_{\cdot\cdot\cdot} \), \( \hat{\phi}_n = \chi_{\left( \xi, \eta \right)}{\leq n} \), \( \hat{\phi}_n = \chi_{\left( \xi, \eta \right)}{\leq n} \), \( \hat{\phi}_n = \chi_{\left( \xi, \eta \right)}{\leq n} \) is the characteristic function of \( \|\xi, \eta\| \leq n \), \( \Phi_n(z) = \left( 1 + |\xi| + \lambda |\eta| \right) + \mu |\eta|^2 \left( \frac{1}{2} - z \hat{\phi}_n \right) \) and \( f \) a function on \( 0 \leq \text{Im}(z) \leq 1 \) into \( L^2 \) that satisfies (11). It is clear that \( \Phi_n \) is analytic on \( \mathbb{C} \) with values in \( L^2 \). Therefore \( (f(z), \Phi_n(z))_{L^2} \) is a continuous function on \( 0 \leq \text{Im}(z) \leq 1 \) and analytic on \( 0 < \text{Im}(z) < 1 \). Furthermore, \( \|f(it)\|_{L^2} \leq \|f(it)\|_{X_0} \|\phi_n\|_{X^\frac{1}{2}} \) and \( \|f(1 + it)\|_{L^2} \leq \|f(1 + it)\|_{L^2} \|\phi_n\|_{X^\frac{1}{2}} \). By three lines lemma, we have that

\[
\|\langle f(z), \Phi_n(z) \rangle_{L^2} \| \leq \max(\sup \|f(it)\|_{X_0}, \sup \|f(1 + it)\|_{L^2}) \|\phi_n\|_{X^\frac{1}{2}}.
\]

Taking \( z = \frac{1}{2} \), we have that \( \|\phi_n\|_{X^\frac{1}{2}} \leq \max(\sup \|f(it)\|_{X_0}, \sup \|f(1 + it)\|_{L^2}) \), for all \( n \). So, by Lebesgue monotone convergence theorem we have that \( \Phi \in X_0 \). \( \square \)

Definition 2.1. Let \( \Omega \) an open connected set in \( \mathbb{R}^2 \) and \( X \) any \( X_0 \) or \( X^\frac{1}{2} \). We denote by \( X(\Omega) \) the set \( \{f \in L^2(\Omega) \mid f = g \text{ for some } g \in X \} \). With the norm

\[
\|f\|_{X(\Omega)} = \inf_{g(\omega) = f} \|g\|_{X^\frac{1}{2}}
\]

\( X(\Omega) \) is a Banach space.

Lemma 2.6. Suppose that \( \Omega = (a, b) \times (c, d) \) and \( \phi \) is a non-negative function \( C^\infty \) on \( \mathbb{R} \) such that \( \text{supp } \phi \subseteq [a, b] \) and \( \int_{\mathbb{R}} \phi = 1 \). Then, there exists a constant \( C \), depending only on \( \Omega \) and \( \phi \), such that for all \( f \in L^2_{\text{loc}} \) with \( (\partial^2_x f, \partial^2_y f) \in L^2_{\text{loc}} \),

\[
\left\| f - \frac{1}{b-a} \int_a^b f dx - \left( x - \frac{a+b}{2} \right) \int_a^b f x dx \right\|_{L^2(\Omega)} \leq C \|\partial^2_x f\|_{L^2(\Omega)},
\]

(12)
\[
\left\| f_x - \int_a^b f_x \phi \, dx \right\|_{L^2(\Omega)} \leq C \| \partial_x^2 f \|_{L^2(\Omega)} \quad (13)
\]
and
\[
\left\| f_{yy} - \frac{1}{b-a} \int_a^b f_{yy} \, dx - \left( x - \frac{a+b}{2} \right) \int_a^b f_{yxy} \phi \, dx \right\|_{L^2(\Omega)} \leq C \| \partial_y^2 f \|_{L^2(\Omega)} \quad (14)
\]

**Proof.** First, let us see the following obvious Poincaré inequality generalization.

**Lemma 2.7.** Let \(a < b\) and \(\phi\) be a non-negative continuous function on \([a, b]\) such that \(\int \phi = 1\). Then, for all \(f \in L^p[a, b]\) with \(f' \in L^p[a, b]\),
\[
\left\| f - \int_a^b f \phi \, dx \right\|_{L^p[a, b]} \leq C \| f' \|_{L^p[a, b]},
\]
where \(C\) depends only on \([a, b]\) and \(p\).

**Proof.** For \(x \in [a, b]\),
\[
\left| f(x) - \int_a^b f(\xi) \phi(\xi) \, d\xi \right| = \left| \int_a^x \int_{\xi}^b f'(s) \, ds \phi(\xi) \, d\xi \right| \leq \| f' \|_{L^1[a, b]}.
\]

The lemma follows immediately from this inequality.

By Poincaré’s inequality and lemma above,
\[
\int_a^b \left| f(x, y) - \frac{1}{b-a} \int_a^b f - \left( x - \frac{a+b}{2} \right) \int_a^b f_x \phi \, dx \right|^2 \, dx \leq C^2 \int_a^b \left| f_x(x, y) - \int_a^b f_x \phi \, dx \right|^2 \, dx \leq C^2 \int_a^b |\partial_x^2 f(x, y)|^2 \, dx.
\]

This inequality shows \(12\).

\(13\) is an immediate consequence of Lemma \(2.7\). Now we shall prove \(14\). By the Cauchy-Schwarz inequality we have that
\[
\left| \frac{1}{b-a} \int_a^b f_{yy} \, dx \right|^2 \leq \frac{1}{b-a} \int_a^b |f_{yy}|^2 \, dx.
\]
Therefore
\[
\left\| \frac{1}{b-a} \int_a^b f_{yy} \, dx \right\|_{L^2(\Omega)} \leq \| \partial_y^2 f \|_{L^2(\Omega)}. \quad (15)
\]
Additionally, observe that
\[
\left| \int_a^b f_{xyy}(x, y)\phi(x) \, dx \right| = \left| \int_a^b f_{yy}(x, y)\phi_x(x) \, dx \right| \leq \|f_{yy}(\cdot, y)\|_{L^2[a, b]}\|\phi_x\|_{L^2[a, b]}.
\] (16)

[15], [16] and the triangle inequality imply (14).

Lemma 2.8. Let \( \Omega = (a, b) \times (c, d) \). There exists an extension operator \( E : X^0(\Omega) \rightarrow X^0 \), i.e., there exists a bounded linear operator \( E \) from \( X^0(\Omega) \) to \( X^0 \) such that, for any \( u \in X^0(\Omega) \), \( Eu = u \) in \( \Omega \), \( \|Eu\|_{L^2} \leq C\|u\|_{L^2(\Omega)} \) and \( \|Eu\|_{X^0} \leq C\|u\|_{X^0(\Omega)} \), where \( C \) depends only on \( \Omega \).

Proof. Let \( u \in X_0(\Omega) \). Without loss generality, we can suppose that \( u = \partial_x^2 f \) in \( \Omega \) for some \( f \in S(\mathbb{R}^2) \) with \( \|\partial_x^2 f\|_{X^0} \leq 2\|u\|_{X^0(\Omega)} \). Let us take \( f_0 = f - \frac{1}{b-a} \int_a^b f \, dx \) and \( (x - \frac{a+b}{2}) \int_a^b f \phi \, dx \). It is obvious that \( u = \partial_x^2 f_0 \) in \( \Omega \). Now consider \( f_1 \) defined on \([2a - b, 2b - a] \times [c, d]\) by
\[
f_1(x, y) = \begin{cases} 
  f_0(x, y) & \text{if } x \in [a, b] \\
  \sum_{i=1}^4 a_i f_0 \left( \frac{i+1}{3}a - \frac{1}{3}x, y \right) & \text{if } x \in [b, 2b - a] \\
  \sum_{i=1}^4 a_i f_0 \left( \frac{i+1}{3}a - \frac{1}{3}x, y \right) & \text{if } x \in [2a - b, a],
\end{cases}
\]
where
\[
\begin{align*}
  a_1 + a_2 + a_3 + a_4 &= 1 \\
  a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \frac{a_4}{4} &= -1 \\
  a_1 + \frac{a_2}{4} + \frac{a_3}{9} + \frac{a_4}{16} &= 1 \\
  a_1 + \frac{a_2}{8} + \frac{a_3}{27} + \frac{a_4}{64} &= -1.
\end{align*}
\]

Clearly \( f_1 \) is a \( C^3 \) function on \([2a - b, 2b - a] \times [c, d]\) and satisfy
\[
\|\partial^\alpha f_1\|_{L^2([2a-b,2b-a]\times[c,d])} \leq C\|\partial^\alpha f_0\|_{L^2(\Omega)},
\] (17)
for all \( \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq 3 \). In the same way, from \( f_1 \), we can define a \( C^3 \) function \( f_2 \) on \( \tilde{\Omega} = [2a - b, 2b - a] \times [2c - d, 2d - c] \) such that
\[
\|\partial^\alpha f_2\|_{L^2(\Omega)} \leq 9\|\partial^\alpha f_0\|_{L^2(\Omega)},
\] (18)
for all \( \alpha \in \mathbb{N}^2 \) with \( |\alpha| \leq 3 \). Now, let \( \eta \) a \( C^\infty \) function in \( \mathbb{R}^2 \) such that \( \eta \equiv 1 \) in \( \Omega \) and 0 out of \( \Omega \), and let \( Eu = \partial_x^2(\eta f_2) \) in \( \tilde{\Omega} \) and 0 in \( \mathbb{R}^2 - \tilde{\Omega} \). From [18] and Lemma 2.6 follows that \( Eu = u \) in \( \Omega \), \( \|Eu\|_{L^2} \leq C\|u\|_{L^2(\Omega)} \) and \( \|Eu\|_{X^0} \leq C\|u\|_{X^0(\Omega)} \), where \( C \) depends only on \( \Omega \) and \( \phi \).

Corollary 2.9. If \( \Omega = (a, b) \times (c, d) \) then \( X^\frac{1}{2}(\Omega) = [L^2(\Omega), X^0(\Omega)]_{\frac{1}{2}} \).

Proof. It is enough to observe that \( E \) defined in Lemma 2.8 can be see as a coretract of the restriction operator from \( (X^0, L^2) \) to \( (X^0(\Omega), L^2(\Omega)) \). Then, the corollary follows from Theorem 1.2.4 in [18].
Theorem 2.10. Suppose that \( \{\Omega_i\}_{i \in \mathbb{N}} \) is a cover of \( \mathbb{R}^2 \), where each \( \Omega_i \) is an open cube with edges parallel to the coordinate axis and side-length \( R \), and such that each point in \( \mathbb{R}^2 \) is contained in at most 3 \( \Omega_i \)'s. Then

\[
\sum_{i=0}^{\infty} \|u\|_{X(\Omega_i)}^2 \leq C\|u\|_{X}^2, \tag{19}
\]

for all \( u \in X^{\frac{1}{2}} \).

Proof. Proceeding as in the proof of Lemma 2.8 we can show that

\[
\|E_iu\|_{X^0}^2 \leq C \int_{\Omega_i} u^2 + \partial_x u^2 + \lambda \partial_x^{-1} \partial_y^2 u^2 + \mu \partial_x^{-2} \partial_y^2 u^2 \, dx,
\]

where \( E_i \) is the extension operator from \( X^0(\Omega_i) \) to \( X^0 \). It is easy to check that \( C \) depends only on length of \( x \)-side of \( \Omega_i \). Then \( C \) is independent of \( i \). Since \( \|u\|_{X^0(\Omega_i)} \leq \|E_iu\|_{X^0} \), for all \( i \), we get

\[
\sum_{i=0}^{\infty} \|u\|_{X^0(\Omega_i)}^2 \leq C \sum_{i=0}^{\infty} \int_{\Omega_i} u^2 + \partial_x u^2 + \lambda \partial_x^{-1} \partial_y^2 u^2 + \mu \partial_x^{-2} \partial_y^2 u^2 \, dx \, dy \leq 3C\|u\|_{X^0}^2.
\]

Also, it is obvious that

\[
\sum_{i=0}^{\infty} \|u\|_{L^2(\Omega_i)}^2 \leq 3\|u\|_{L^2}^2.
\]

Then the operator \( u \mapsto (u_{\Omega_i})_{i \in \mathbb{N}} \) (\( u_{\Omega_i} \) is the restriction of \( u \) to \( \Omega_i \)) is continuous from \( L^2 \) to \( \ell^2(L^2(\Omega_i)) \) and from \( X^0 \) to \( \ell^2(X^0(\Omega_i)) \). By Theorem 1.18.1 in [18], we have that the operator \( u \mapsto (u_{\Omega_i})_{i \in \mathbb{N}} \) is continuous from \( X^{\frac{1}{2}} \) to \( \ell^2(X^{\frac{1}{2}}(\Omega_i)) \). Thence we obtain (19) for \( X = X^{\frac{1}{2}} \). The proof of (19) with \( X = \tilde{X}^{\frac{1}{2}} \) is completely analogous. \( \square \)

Lemma 2.11. The embedding \( X^{\frac{1}{2}} \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^2) \) is compact, if

\[
\begin{align*}
0 \leq p < 4 & \quad \text{if } \lambda > 0, \\
0 \leq p < \frac{4}{3} & \quad \text{if } \lambda = 0.
\end{align*}
\]

In other words, if \( (u_n) \) is a bounded sequence in \( X^{\frac{1}{2}} \) and \( R > 0 \), there exists a subsequence \( (u_{n_k}) \) which converges strongly to \( u \) in \( L^p(B_R) \).

Proof. We prove the lemma when \( \lambda > 0 \), the proof when \( \lambda = 0 \) is just to make some obvious modifications. Suppose that \( (u_n)_{k=1}^{\infty} \) is a bounded sequence in \( X^{\frac{1}{2}} \). Let \( \Omega_R \) be the cube with center at the origin and edges parallel to the coordinate axis and length \( R \), and let \( E_R \) the extension operator from \( L^2(\Omega_R) \) to \( L^2 \) as in proof Lemma 2.8. By interpolation, \( E_R \) is a continuous operator from \( X^s(\Omega) \) to \( X^{\frac{s}{2}} \). Also, it is easy to observe that \( E_R(u) \) is 0 out of \( \Omega_{3R} \), for all \( u \in X^{\frac{1}{2}} \), where \( \Omega_{3R} \) is the cube with
center at the origin and edges parallel to the coordinate axis and length \(3R\). Because \(u = E_R(u)\) in \(\Omega\), without loss of generality, we can assume that \(u_n = E_R(u_n)\), for all \(n\). Now, since \(u_n\) is bounded in \(X^1\), we can also suppose that \(u_n \to u\) in \(X^1\), and replacing, if necessary, \(u_n\) by \(u_n - u\), we can assume that \(u = 0\) too.

Let

\[
Q_1 = \{ (\xi, \eta) \in \mathbb{R}^2 / |\xi| \leq \rho, |\eta| \leq \rho \}
\]

\[
Q_2 = \{ (\xi, \eta) \in \mathbb{R}^2 / |\xi| > \rho \}
\]

\[
Q_3 = \{ (\xi, \eta) \in \mathbb{R}^2 / |\xi| < \rho, |\eta| > \rho \}
\]

Then \(\mathbb{R}^2 = \bigcup_{i=1}^{3} Q_i\) and \(Q_i \cap Q_j = \emptyset, i \neq j\). For \(\rho > 0\), there holds

\[
\int_{\Omega \setminus R} |u_n(x, y)|^2 dxdy = \int_{\mathbb{R}^2} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta = \sum_{i=1}^{3} \int_{Q_i} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta
\]

It is clear that

\[
\int_{Q_2} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta = \int_{Q_2} \frac{1}{|\xi|^2} |D_{\xi}^{1/2} u_n(\xi, \eta)|^2 d\xi d\eta \leq \frac{C}{\rho} \| D_{\xi}^{1/2} u_n \|^2_0,
\]

and

\[
\int_{Q_3} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta = \int_{Q_3} \frac{|\xi|^2}{|\eta|^2} |D_{\xi} \partial_y u_n(\xi, \eta)|^2 d\xi d\eta.
\]

Therefore, for any \(\epsilon\), there exists \(\rho > 0\) large enough such that

\[
\int_{Q_2} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta + \int_{Q_3} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta \leq \epsilon/2.
\]

Since, by the fact that \(u_n \to 0\) in \(L^2(\mathbb{R}^2)\),

\[
\lim_{n \to 0} \hat{u}_n(\xi, \eta) = \lim_{n \to 0} \int_{\Omega \setminus R} u_n(x, y)e^{-i(\xi x + \eta y)} dxdy = 0,
\]

and \(|\hat{u}(\xi, \eta)| \leq \| u_n \|_1\), the Lebesgue dominated convergence theorem guarantees that

\[
\int_{Q_1} |\hat{u}_n(\xi, \eta)|^2 d\xi d\eta = 0
\]

as \(n \to \infty\). Hence \(u_n \to 0\) in \(L^2_{loc}(\mathbb{R}^2)\). By Lemma 2.4, \(u_n \to 0\) in \(L^p_{loc}(\mathbb{R}^2)\) if \(2 \leq p < 4\).

Lemma 2.12. If \((u_n)\) is bounded in \(X^1\) and

\[
\lim_{n \to \infty} \sup_{(x, y) \in \mathbb{R}^2} \int_{B(x, y; R)} |u_n|^2 dxdy = 0,
\]

(20)

as \(n \to \infty\), then \(u_n \to 0\) in \(L^p(\mathbb{R}^2)\) for

\[
\begin{cases}
2 < p < 4 & \text{if } \lambda > 0 \\
2 < p < 4/3 & \text{if } \lambda = 0
\end{cases}
\]
Proof. Suppose $\lambda > 0$ ($\lambda = 0$ follows in the same way). Let $2 < s < 4$ and let $\Omega_R$ be the cube with center at the origin, edges parallel to the coordinate axis and side-length $R$. Then, by Hölder inequality and Lemma 2.4, we have that

$$\|u_n\|_{L^s((x,y)+\Omega_R)} \leq \|u_n\|_{L^2((x,y)+\Omega_R)}^{1-\frac{\vartheta}{2}} \|u_n\|_{X_{(x,y)\Omega_R}^\frac{1}{2}}^\vartheta,$$

where $\vartheta = \frac{2(s-2)}{s}$. Choosing $s$ such that $\vartheta = 1$, i.e., $s = 3$, there holds

$$\int_{(x,y)+\Omega_R} |u_n|^3 dx dy \leq C \|u_n\|_{L^2((x,y)+\Omega_R)} \|u_n\|_{X_{(x,y)\Omega_R}^\frac{1}{2}}^2,$$

Now, covering $\mathbb{R}^2$ by cubes with edges parallel to the coordinate axis and side-length $R$ in such a way that each point of $\mathbb{R}^2$ is contained in at most 3 of these cubes, by Theorem 2.10 we get

$$\int_{\mathbb{R}^2} |u_n|^3 dx dy \leq C \sup_{(x,y)\in\mathbb{R}^2} \|u_n\|_{L^2((x,y)+\Omega_R)} \|u_n\|_{X_{(x,y)\Omega_R}^\frac{1}{2}}^2.$$ 

Since $u_n$ is bounded in $X_{\frac{1}{2}}$ and satisfies (20), $u_n \to 0$ in $L^3(\mathbb{R}^2)$. Because $2 < 3 < 4$, the Hölder inequality implies that $u_n \to 0$ in $L^p(\mathbb{R}^2)$, for all $2 < p < 4$. \hfill \square

The following lemma gives us a minimax principle and is an immediate consequence of Theorem 2.8 in [19, pg. 41]

**Lemma 2.13.** Suppose $X$ is a Banach space and $\Phi \in C^1(X, \mathbb{R})$ satisfies the following properties:

1. $\Phi(0) = 0$, and there exists $\rho > 0$, such that $\Phi|_{\partial B_\rho(0)} \geq \alpha > 0$.
2. There exists $\beta \in X \setminus \overline{B}_\rho(0)$ such that $\Phi(\beta) \leq 0$.

Let $\Gamma$ be the set of all paths which connects 0 and $\beta$, i.e.,

$$\Gamma = \{g \in C([0,1], X) \mid g(0) = 0, g(1) = \beta\},$$

and

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} \Phi(g(t)). \quad (21)$$

Then $c \geq \alpha$ and $\Phi$ possesses a Palais-Smale sequence at level $c$, i.e., there exists a sequence $(u_n)$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$. 

10
3 Existence of Solitary Waves

If \( \phi(x - ct, y) \) is a solitary wave solution to (1), then

\[
(-c \partial_x \phi + \phi \partial_x \phi + \mathcal{H}(\partial_x^2 \phi + \lambda \partial_y^2 \phi))_x + \mu \partial_y^2 \phi = 0.
\]

(22)

If \( \phi \in X^\perp \), we can write (22) as

\[
-c \phi + \mathcal{H} \partial_x \phi + \lambda \mathcal{H} \partial_x^{-1} \partial_y \phi + \mu \partial_x^{-1} \partial_y \phi + \frac{1}{p+1} \phi^{p+1} = 0.
\]

(23)

where the term on the right hand is in \( (X^\perp)^* \), the topological dual of \( X^\perp \). Then \( \phi \) is a critical point of the functional \( \Phi \) on \( X^\perp \) defined as

\[
\Phi(\phi) = \int_{\mathbb{R}^2} \left( c\phi^2 + (D_x^2 \phi)^2 + \lambda(D_x^{-1} \partial_y \phi)^2 + \mu(\partial_x^{-1} \partial_y \phi)^2 \right) - \frac{\phi^{p+2}}{(p+1)(p+2)} \, dxdy.
\]

Let us see that \( \Phi \) satisfies the conditions of the Lemma 2.13. It is obvious that \( \Phi \) is a \( C^1 \) functional for \( 0 < p \leq 2 \). \( \Phi(0) = 0 \) and, since

\[
\Phi(\phi) \geq \frac{\min\{c, 1\}}{2} \|\phi\|_{X^\perp} - \frac{|\phi|^{p+2}}{(p+1)(p+2)},
\]

by Lemma 2.4 there exist a \( \rho \) such that

\[
\inf_{\partial B_\rho(0)} \Phi = \alpha > 0,
\]

which shows 1). Now, for \( \vartheta \in \mathbb{R} \) and \( u \in X^\perp \),

\[
\Phi(\vartheta u) = \vartheta^2 \left( \Phi(u) + \int_{\mathbb{R}^2} \frac{u^{p+2}}{(p+1)(p+2)} \, dxdy \right) - \vartheta^{p+2} \int_{\mathbb{R}^2} \frac{u^{p+2}}{(p+1)(p+2)} \, dxdy.
\]

Then, taking \( u \) fixed and \( \vartheta \) large enough, we have 2) with \( \beta = \vartheta u \). So, we have shown the following lemma.

**Lemma 3.1.** Let \( \Phi, \alpha \) and \( \beta \) be defined as above and let \( \Gamma \) and \( c \) be defined as Lemma 2.13. Then, there exists a sequence \( (\phi_n) \) such that \( \Phi(\phi_n) \to c \) and \( \Phi'(\phi_n) \to 0 \).

Now, we can prove the following theorem.

**Theorem 3.2.** (22) has nontrivial solutions in \( X^\perp \).

**Proof.** It is enough to show that \( \Phi \) have non-zero critical points in \( X^\perp \). By Lemma 3.1 there exists a Palais-Smale sequence \( (\phi_n) \) at level \( c \) of \( \Phi \). Therefore,

\[
c + 1 \geq \Phi(\phi_n) - \frac{\langle \Phi'(\phi_n), \phi_n \rangle_{X^\perp}}{p+2} \geq (\frac{1}{2} - \frac{1}{p+2}) \min\{1, c\} \|\phi_n\|^2_{X^\perp},
\]

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for $n$ big enough. Hence $(\phi_n)$ is bounded in $X^{1/2}$. Considering that

$$0 < c = \lim_{n \to \infty} \Phi(\phi_n) - \frac{1}{2} \langle \Phi'(\phi_n), \phi_n \rangle_{X^{1/2}} = \lim_{n \to \infty} \frac{p}{2(p+2)(p+1)} \int_{\mathbb{R}^2} \phi_n^{p+2} \, dx \, dy,$$

the Lemma 2.12 implies that

$$\delta = \limsup_{n \to \infty} \sup_{(x,y) \in \mathbb{R}^2} \int_{(x,y)+\Omega_1} \phi_n^2 \, dx \, dy > 0.$$  

Then, passing to a subsequence if necessary, we can assume that there exists a sequence $(x_n, y_n)$ in $\mathbb{R}$ such that

$$\int_{(x_n,y_n)+\Omega_1} \phi_n^2 \, dx \, dy > \delta/2,$$  

(24)

for $n$ big enough. Let $\tilde{\phi}_n = \phi_n(\cdot + (x_n, y_n))$. Then, again passing to a subsequence if necessary, we can assume that, for some $\phi \in X^{1/2}$, $\tilde{\phi}_n \to \phi$ in $X^{1/2}$. In view of (24), for $n$ large enough, and Lemma 2.11, $\phi \neq 0$. The Lemma 2.11 and the continuity of the function $u \to u^{p+1}$ from $L^{p+2}$ to $L^{p+1}$, in any measure space, imply that

$$\langle \Phi'(\phi), w \rangle_{X^{1/2}} = \lim_{n \to \infty} \langle \Phi'(\tilde{\phi}_n), w \rangle_{X^{1/2}} = 0.$$  

This shows this theorem.

4 Smoothness of solitary wave

In this section we shall prove that the solitary wave solution of (1) is $C^\infty$.

Theorem 4.1. Let $p = 1$. If $\phi \in X^{1/2}$ is solution to (22), $\phi \in H^\infty = \bigcap_{n=0}^\infty H^n$. Moreover, $\phi$ is analytic.

Proof. Suppose first that $\mu = 0$. In this case, without loss of generality, we can suppose that $\alpha = 1$. By Lemma 2.4, $\phi \in L^4$. In particular, $\frac{1}{2} \phi^2 + c \phi \in L^2$. Now, from (22), we have

$$\Delta \phi = \mathcal{H} \partial_x \left( \frac{\phi^2}{2} - c \phi \right).$$  

(25)

Then, the Plancherel theorem implies that $\phi \in H^1$. So, by Sobolev embedding theorem, $\frac{1}{2} \phi^2 + c \phi \in L^p$, $2 \leq p < \infty$. Since the Hilbert transform is bounded from $L^p \to L^p$ and, by Lizorkin theorem (see [13]), $\frac{\xi^2}{\xi^2 + \eta^2}$ and $\frac{\eta^2}{\xi^2 + \eta^2}$ are $L^p$ multipliers, from (22), we have that $\phi_x$ and $\phi_y \in L^p$. Whence, again (22) implies that $\phi \in H^2$. The theorem follows once we have observed that if $\phi \in H^n$ then $\phi \in H^{n+1}$, for $n \geq 2$. This last affirmation follows from (22), the fact that $H^n$ is a Banach algebra, for $n \geq 2$, and Plancherel theorem.

Suppose now $\gamma \mu \neq 0$. Without loss of generality, we can suppose also that $\gamma = \mu = 1$. So, (22) becomes in

$$\mathcal{H} \partial_y^2 \phi + \mathcal{H} \partial_x \delta_y \phi - \partial_y^2 \phi = -\partial_x^2 \left( \frac{\phi^2}{2} - c \phi \right).$$  

(26)

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From here, thanks to Lizorkin theorem, we have that \( \xi^3 \) and \( \xi^2 \eta^2 \) are multipliers in \( L^p \), \( 1 < p < \infty \). From now on it is just follow the steps in the previous case. The case \( \gamma = 0 \) is was done in [6].

To see the analyticity of \( \phi \) it is enough to prove that
\[
\| \partial^\alpha \phi \|_{H^2} \leq C|\alpha|!(\frac{R}{2})^{|\alpha|},
\]
for some \( R > 0 \) and for all \( \alpha \in \mathbb{N}^2 \). We shall show that there exists \( R > 0 \) such that
\[
\| \partial^\alpha \phi \|_{H^2} \leq C\frac{|\alpha| - 1)!}{(|\alpha| + 1)^s}(\frac{R}{2})^{\frac{|\alpha|}{s}},
\]
where \( s > 1 \). We see this by induction. For \( |\alpha| = 1 \) the inequality (28) is obvious; it is sufficient to choose \( C \) large enough. Suppose now that (28) is valid for \( |\alpha| = 1, \cdots, n \) and \( R \) (that we shall conveniently choose later). From equation (22) we have that
\[
\partial^2_x \phi + \alpha \partial^2_y \phi - \gamma \mathcal{H}_x^{-1} \partial^2_y \phi = \mathcal{H}_x \left( \frac{\phi^2}{2} - c\phi \right).
\]
Applying \( \partial^\alpha \) on both sides of the equation and making the inner product in \( H^2 \) with \( \partial^\alpha \phi \) in the last equation, we can show that
\[
\| \nabla \partial^\alpha \phi \|_{H^2} \leq C \left( |\partial^\alpha \left( \frac{\phi^2}{2} - c\phi \right) | \right)_{H^2}.
\]
For finishing the theorem’s proof we need the following lemma.

**Lemma 4.2.** (a) If \( f \) and \( \phi \in C^\infty(\mathbb{R}) \), then
\[
\partial^\alpha \left( f(\phi) \right) = \sum_{j=1}^{\vert \alpha \vert} \frac{f^{(j)}(\phi)}{j!} \sum_{\alpha_1 + \cdots + \alpha_j = \alpha \atop |\alpha_i| \geq 1, \forall 1 \leq i \leq j} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \partial^{\alpha_i} \phi \cdots \partial^{\alpha_j} \phi.
\]
(b) For each \( (n_1, \ldots, n_j) \in \mathbb{N}^j \) we have
\[
|\alpha|! = \sum_{\alpha_1 + \cdots + \alpha_j = \alpha \atop |\alpha_i| = n_i, \forall 1 \leq i \leq j} \frac{\alpha_1! \cdots \alpha_j!}{\alpha_1! \cdots \alpha_j!}.
\]
(c) For \( s > 1 \) there exists \( C_2 \) such that for all \( j \in \mathbb{N} \)
\[
\sum_{k_1 + \cdots + k_j = k} \frac{1}{(k_1 + 1)^s \cdots (k_j + 1)^s} \leq \frac{C_2^{j-1}}{(k + 1)^s}
\]
Now we return to the proof of the theorem. Part (a) of Lemma 4.2, inequality (30) and the fact \( H^2 \) is a Banach algebra imply that
\[
\| \nabla \partial^\alpha \phi \|_{H^2} \leq C_1 \sum_{j=1}^2 \sum_{\alpha_1 + \cdots + \alpha_j = \alpha \atop |\alpha_i| \geq 1, \forall 1 \leq i \leq j} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \| \partial^{\alpha_1} \phi \|_{H^2} \cdots \| \partial^{\alpha_j} \phi \|_{H^2}.
\]
By the induction hypothesis and part (b) of the same lemma, we have

\[
\| \nabla \partial^\alpha \phi \|_{H^2} \leq C_1 \sum_{j=1}^{2} C_j A^{[\alpha]-j} \sum_{n_1+\cdots+n_j=|\alpha|} \sum_{n_i \geq 1, \forall 1 \leq i \leq j} \frac{\alpha!}{\alpha_1! \cdots \alpha_j!} \frac{(|\alpha_1|-1)! \cdots (|\alpha_j|-1)!}{(|\alpha_1|+1)^s \cdots (|\alpha_j|+1)^s}
\]

where \( A = \frac{R}{2} \), and from this inequality and part (c) of Lemma 4.2, we obtain that

\[
\| \nabla \partial^\alpha \phi \|_{H^2} \leq C_1 \left( \frac{|\alpha|!}{(|\alpha|+2)^s} A^{[\alpha]} \sum_{j=1}^{2} (\tilde{C} C_2)^j A^{-j} \right). \]

Now we can choose \( R \). We take \( A \) large enough such that \( C_1 \sum_{j=1}^{2} (\tilde{C} C_2)^j A^{-j} \leq C \).

It is clear that this choice does not depend on \( \alpha \). Therefore, with \( R = 2A \),

\[
\| \nabla \partial^\alpha \phi \|_{H^2} \leq C \left( \frac{|\alpha|!}{(|\alpha|+2)^s} \left( \frac{R}{2} \right)^{|\alpha|} \right),
\]

that shows (28). This completes the proof.

\[\square\]

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