TRAVELING WAVE SOLUTIONS TO THE
ALLEN-CAHN EQUATION

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Abstract. For the Allen-Cahn equation, it is well known that there is a monotone standing wave joining with the balanced wells of the potential. In this paper we study the existence of traveling wave solutions for the Allen-Cahn equation on an infinite channel. Such traveling wave solutions possess a large number of oscillation and they are obtained with the aid of variational arguments.

1. Introduction

Let \((\xi, y) \in \Omega := \mathbb{R}^1 \times \Omega_y\), a cylinder with cross section \(\Omega_y\). Here \(\Omega_y\) is a bounded open set in \(\mathbb{R}^{N-1}\) with \(C^{2, \alpha}\) boundary \(\partial \Omega\), \(\alpha \in (0, 1)\). We are concerned with the traveling wave solutions of

\[
\begin{aligned}
&u_t = u_{\xi\xi} + \Delta_y u + u(1 - u^2), \\
&u|_{\partial \Omega} = 0.
\end{aligned}
\]

The Allen-Cahn equation and related problems have been investigated for several decades, because not only this nonlinear PDE serves as a model \([1, 4, 31]\) in studying phase transition theory but it has attracted a great attention from different fields in mathematics (see e.g. \([12, 23, 27]\) and the references therein).

Traveling wave played an important role in understanding dynamics of evolution systems. There are many interesting results \([3, 8, 13, 14, 15, 20, 22, 21, 28, 29, 30, 32, 33]\) for the traveling wave solutions of

\[
\begin{aligned}
&u_t = u_{\xi\xi} + \Delta_y u + g(u), \\
&(\xi, y) \in \Omega, \quad t > 0.
\end{aligned}
\]

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Particular examples include $g(u) = u(1 - u)$ in the KPP equation and $g(u) = u(1 - u)(u - \beta)$ with $\beta \in (0, 1/2)$ in the Nagumo equation.

A simple traveling wave can be represented by a function $u(\xi - ct)$ which satisfies (1.2) with $c$ being a constant. Along the moving co-ordinates with speed $c$, this planar traveling wave is a solution of an ordinary differential equation. Such wave solutions have been successfully investigated by shooting method [3, 13]. A wave with zero speed is referred to as a standing wave. It is known that (1.1) possesses a planar standing wave joining 1 and $-1$, the double wells of the potential. Our aim in this paper is to investigate the traveling wave solutions of (1.1).

Following the ansatz proposed in [20], if $u(c(\xi - ct), y)$ satisfies (1.1) then

\begin{equation}
\begin{aligned}
&c^2(u_{xx} + u_x) + \Delta_y u + u(1 - u^2) = 0, \\
&u|_{\partial\Omega} = 0,
\end{aligned}
\end{equation}

where $x = c(\xi - ct)$. Denote by $L^p_w$ the Banach space of functions in $L^p_{\text{loc}}(\Omega)$ equipped with the norm

$$
\|u\|^p_{L^p_w} = \int_{\Omega} e^x |u|^p \, dx \, dy.
$$

The appearance of weight function $e^x$ is due to the first order derivative term $u_x$ in (1.3). Choosing the ansatz $u(c(\xi - ct), y)$ instead of $u(\xi - ct, y)$ seems to be more convenient in dealing with $\Phi_c$ on function spaces with a fixed weight; for instance, in studying the continuous dependence on $c$. Given $u \in H^1_{\text{loc}}(\Omega)$ let $\|u\|^2_E = \|u\|^2_{L^2_w} + \|u\|^2_{L^4_w} + \|u_x\|^2_{L^2_w} + \|
abla_y u\|^2_{L^2_w}$. The set of functions with $\|u\|^2_E < \infty$ is denoted by $E$, while $E_0$ is the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_E$.

Let $F(w) := -\frac{w^2}{2} + \frac{w^4}{4}$ and

\begin{equation}
\Phi_c[w] := \frac{c^2}{2} \int_{\Omega} e^x w_x^2 \, dx \, dy + \int_{\Omega} e^x \left( \frac{1}{2} |\nabla_y w|^2 + F(w) \right) \, dx \, dy.
\end{equation}

By standard theory of calculus of variations, a critical point of $\Phi_c$ in $E_0$ is a solution of (1.3).

Consider a cross section $\Omega_y$ and the boundary value problem

\begin{equation}
\begin{aligned}
&\Delta_y u + u(1 - u^2) = 0, \\
&u|_{\partial\Omega_y} = 0.
\end{aligned}
\end{equation}

The existence of multiple solutions to (1.3) has been established by Variational methods. These solutions are the critical points of the
functional \( J : H^1_0(\Omega_y) \to \mathbb{R} \) defined by
\[
(1.6) \quad J[v] := \int_{\Omega_y} \left( \frac{1}{2} |\nabla_y v|^2 + F(v) \right) \, dy.
\]
Denote by \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \) the eigenvalues of
\[
(1.7) \quad \begin{cases}
\Delta_y \psi + \lambda \psi = 0, \\
\psi|_{\partial \Omega_y} = 0.
\end{cases}
\]
Clearly \( u \equiv 0 \) is a trivial solution of (1.5) and \( J[0] = 0 \). If \( \lambda_1 < 1 \), it is known [9, 18, 25] that there is a unique positive solution \( u^+ \) for (1.5) and \( J[u^+] = \inf_{v \in H^1_0(\Omega_y)} J[v] < 0 \) (see proposition 2.2).

Our goal is to show the following result.

**Theorem 1.1.** Let \( \Omega_y \) be a \( C^{2,\alpha} \) bounded domain and \( \lambda_j \) be the eigenvalues of (1.7).

(i) Assume that \( \lambda_1 < 1 \leq \lambda_2 \). Then for every \( c \in (0, 2\sqrt{1 - \lambda_1}) \), there is a traveling wave solution \( u \) of (1.1) with wave speed \( c \) such that \( u(c(\xi - ct), y) \to 0 \) as \( t \to -\infty \) and \( u(c(\xi - ct), y) \to u_+(y) \) as \( t \to +\infty \).

(ii) If \( J \) has finite number of critical points in \( H^1_0(\Omega_y) \), then for every \( c \in (0, 2\sqrt{1 - \lambda_1}) \), there is a traveling wave solution \( u \) of (1.1) with wave speed \( c \) such that \( u(c(\xi - ct), y) \to u^* \) as \( t \to -\infty \) and \( u(c(\xi - ct), y) \to u_+(y) \) as \( t \to +\infty \), where \( u_+, u^* \) are two critical points of \( J \) such that \( J[u_+] < J[u^*] \).

(iii) \( -u \) is also a traveling wave solution of (1.1).

In particular \( J \) has finite number of critical points in \( H^1_0(\Omega_y) \) if \( \Omega_y \) is an interval. Indeed [10, 11] there are \( 2k + 1 \) critical points if \( \lambda_k < 1 \leq \lambda_{k+1} \).

For the scalar reaction-diffusion equation (1.2), the ordered method has been developed to show the existence of traveling waves on the cylindrical domain [8, 33]. As a consequence of the maximum principle [10], such traveling front solutions possess certain monotonicity properties. For instance, let \( v_+(y) \) and \( v_-(y) \) be the stable solutions of
\[
(1.8) \quad \begin{cases}
v_t = \Delta_y v + g(v), \\
v|_{\partial \Omega_y} = 0.
\end{cases}
\]
If \( v_+(y) > v_-(y) \) for all \( y \in \Omega_y \), Vega [33] proved for (1.2) the existence of a traveling front solution \( w(\xi - ct, y) \) which satisfies
\[
(1.9) \quad v_+(y) > w(x, y) > v_-(y) \quad \text{for all} \quad (x, y) \in \mathbb{R} \times \Omega_y.
\]
Moreover the method of moving planes and the sliding method [7] show that such a wave is strictly monotone in the $x$-direction. Based on the extension of comparison technique, the ordered method has been generalized to studying traveling front solutions in monotone systems [34]. In an earlier work [15], Gardner considered a discretization of (1.2) and applied the Conley index to establish an existence result similar to [33]. The monotonicity properties for traveling waves in combustion models have been studied in [3].

More recently variational methods have been employed to investigate the traveling wave solutions for reaction-diffusion equations with Ginzburg-Landau or bistable type nonlinearities. In [20, 21, 22], the authors proved existence of traveling waves via constrained minimization in a weighted Sobolev space like $E_0$. This constrained minimization requires the traveling front solution stay in the weighted Sobolev space and leads the solution to have certain monotonicity properties.

The remainder of this paper is organized as follows. Section 2 begins with some known results as a preliminary. For the traveling wave solution with a given speed $c$, a variational formulation is introduced in Section 3 to establish a sequence of approximated solutions through a mini-max scheme based on the Krasnoselski genus. As the number of genus is increasing to infinity along this sequence of approximated solutions, it is expected [17, 18, 19] that such a traveling wave solution should possess a large number of oscillation as a limit of the approximated solutions.

In using variational approach to study traveling wave solution, a commonly used weighted Sobolev space $E_0$ is the Hilbert space $H$ or $H_0$ equipped with the norm $||·||_H$, where $||u||^2_H = ||u||^2_{L^2_w} + ||u_x||^2_{L^2_w} + ||\nabla_y u||^2_{L^2_w}$. Choosing the space $E_0$ enables us to work out the boundedness of the solutions and the compactness of Palais-Smale sequences in dealing with the mini-max argument. Then in Section 4 utilizing a suitable limit procedure, we establish the traveling wave solutions. Moreover, as stated in Theorem 1.1, there are infinite number of traveling wave solutions which are distinguished by their speed. To the best of our knowledge, using mini-max method to establish traveling wave solution seems to be new and such a class of traveling front solutions have not been studied before. It should be interesting to point out that all the traveling front solutions stated in Theorem 1.1 do not belong to $E_0$ or $H_0$. Because of this fact, we need a delicate procedure in passing to the limit form the approximated solutions; however this procedure does not keep tracking the number of oscillation. It is not clear to us if
a direct argument is available for the proof. We remark that the Allen-Cahn equation has infinite number of planar traveling front solutions which satisfy \( \lim_{t \to -\infty} u(\xi - ct) = 0 \) and \( \lim_{t \to \infty} u(\xi - ct) = 1 \). These solutions change sign infinitely many times.

2. Preliminary

We state some useful inequalities whose proofs can be found in [21, 24].

**Lemma 2.1.** If \( w(x, y) \) is such that \( \|w\|_{L^2_w}^2 + \|w_x\|_{L^2_w}^2 + \|\nabla_y w\|_{L^2_w}^2 < +\infty \),

\[
(2.1) \quad \int_{r}^{+\infty} \int_{\Omega_y} e^x w^2 \, dx \, dy 
\leq 4 \int_{r}^{+\infty} \int_{\Omega_y} e^x w_x^2 \, dx \, dy,
\]

\[
(2.2) \quad \int_{\Omega_y} w^2(r, y) \, dy 
\leq e^{-r} \int_{r}^{+\infty} \int_{\Omega_y} e^x w_x^2 \, dx \, dy,
\]

for any \( r \in \mathbb{R} \); in particular

\[
(2.3) \quad \int_{\Omega} e^x w^2 \, dx \, dy 
\leq 4 \int_{\Omega} e^x w_x^2 \, dx \, dy.
\]

For the non-trivial solutions of (1.5), some existence and uniqueness results can be found in [10, 11, 17, 19, 25]. Part of such results are stated in the next proposition.

**Proposition 2.2.** Let \( \Omega_y \) be a bounded open set in \( \mathbb{R}^n \) and \( \lambda_i \) the eigenvalues of (1.7).

(a) If \( \lambda_1 < 1 \) there is a unique positive solution \( u_+ \) for (1.5) and \( J[u_+] = \inf_{v \in H^1_0(\Omega_y)} J[v] < 0 \).

(b) If \( \lambda_1 < 1 \leq \lambda_2 \) then \( u_+ \) and \( -u_+ \) are the only non-trivial solutions of (1.5).

3. Variational framework

In this section, a variational framework will be used to construct approximation solutions to a traveling wave solution of (1.1). Let \( \Omega_* := (-\infty, 0) \times \Omega_y \) and consider the following boundary value problem:

\[
(3.1) \quad \begin{cases} 
   c^2 (u_{xx} + u_x) + \Delta_y u + u(1 - u^2) = 0, \\
   u|_{\partial \Omega_*} = 0.
\end{cases}
\]

Let \( E_* \) be the closure of \( C^0_0(\Omega_*) \) in \( E \) and, for all \( u \in E_* \)

\[
(3.2) \quad I_c[u] := \int_{-\infty}^{0} \int_{\Omega_y} \left( \frac{c^2}{2} u_x^2 + \frac{1}{2} \nabla_y u^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4 \right) e^x \, dx \, dy.
\]
Proposition 3.1. The functional \( I_c \in C^1(E_*; \mathbb{R}) \) and is bounded from below.

The proof is standard. We omit it.

Proposition 3.2. Suppose that \( u \in E_* \) is a critical point of \( I_c \), with \( \| u \|_\infty < \infty \). Then \( u \in C^{2,\alpha}(\Omega_*) \cap C^{1,\alpha}(\bar{\Omega}_*) \) and satisfies (3.1). Moreover \( \| u \|_{C^{1,\alpha}((-\infty,0) \times \bar{\Omega}_*)} \) is bounded; in particular, \( u_x \) and \( \nabla_y u \) are uniformly continuous in \( \Omega \).

Proof. A critical point \( u \) satisfies

\[
0 = I'_c[u]\phi = \int_{\infty}^{0} \int_{\bar{\Omega}_y} \left( e^2 u_x \phi_x + \nabla_y u \cdot \nabla_y \phi - u \phi + u^3 \phi \right) e^x dxdy
\]

for \( \phi \in E_*^\dagger \) (the dual of \( E_* \)), in particular for all \( \phi \in C_0^\infty(\Omega_*) \). Since \( u \) is bounded by assumption and \( e^x \) is bounded on bounded subsets of \( \Omega_* \), it immediately follows that \( u \in H^2_{loc}(\Omega_*) \). Then standard regularity theory \([16]\) shows that \( u \in C^{2,\alpha}(\Omega_*) \cap C^{1,\alpha}(\bar{\Omega}_*) \), and thus it is a classical solution of (3.1). \( \square \)

Lemma 3.3. If \( u \in E_* \) and \( I_c[u] \leq 0 \) then

\[
\int_{\Omega_*} e^x u^4 dxdy \leq 4 \int_{\Omega_*} e^x dxdy, \tag{3.3}
\]

\[
\int_{\Omega_*} e^x u^2 dxdy \leq 2 \int_{\Omega_*} e^x dxdy, \tag{3.4}
\]

\[
\int_{\Omega_*} e^x u_x^2 dxdy \leq \frac{2}{c^2} \int_{\Omega_*} e^x dxdy \tag{3.5}
\]

and

\[
\int_{\Omega_*} e^x |\nabla_y u|^2 dxdy \leq 2 \int_{\Omega_*} e^x dxdy. \tag{3.6}
\]

In particular

\[
\| u \|_{L^4} \leq \sqrt{2} |\Omega_*|^{1/4} \quad \text{for all } u \in E_* \text{ such that } I_c[u] \leq 0. \tag{3.7}
\]

Proof. By Holder inequality

\[
\int_{\Omega_*} e^x u^2 dxdy \leq \left( \int_{\Omega_*} e^x dxdy \right)^{1/2} \left( \int_{\Omega_*} e^x u^4 dxdy \right)^{1/2}. \tag{3.8}
\]

Clearly \( I_c[u] \leq 0 \) implies that

\[
\int_{\Omega_*} \left( \frac{e^2}{2} u_x^2 dxdy + \frac{1}{2} |\nabla_y u|^2 + \frac{1}{4} u^4 \right) e^x dxdy \leq \int_{\Omega_*} \frac{1}{2} e^x u^2 dxdy. \tag{3.9}
\]
This together with (3.8) yields (3.3). Substituting (3.3) into (3.8) gives (3.4). Then (3.5) and (3.6) easily follow from (3.9).

\[ \square \]

**Lemma 3.4.** Assume that \( u_n \in E_* \) is a sequence such that \( I_c[u_n] \leq 0 \) and \( \| u_n \|_{E_w(\Omega_*)} \leq C \).

Then there exists a subsequence \( u_{n_k} \) which converges weakly in \( E_* \) and strongly in \( L^p(\Omega_*) \) for all \( p \in [2, 4] \) to a function \( \bar{u} \in E_* \).

**Proof.** It immediately follows from Lemma 3.3 that \( u_n \) is bounded in \( E_* \). From the boundedness of \( u_n \), there exists a subsequence \( u_{n_k} \) which converges weakly to some \( \bar{u} \in E_* \) and strongly in \( L^\infty([-L, 0] \times \Omega_y) \) for all \( L > 0 \).

We next show that \( u_{n_k} \to \bar{u} \) in \( L^p_w(\Omega_*) \) if \( p \in [2, 4] \). Let us first remark that Lemma 3.3 implies that

\[ \int_{\Omega_*} e^x |u_{n_k} - \bar{u}|^p \, dx \leq C \]

for \( 2 \leq p \leq 4 \), and the same inequality holds for \( \bar{u} \), with a constant \( C \) not depending on \( k \).

Given \( \epsilon > 0 \), since

\[ \int_{-L}^{-L} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx \]

\[ \leq \left( \int_{-L}^{-L} \int_{\Omega_y} e^x \, dy \, dx \right)^{1/2} \left( \int_{-L}^{-L} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4 \, dy \, dx \right)^{1/2} \]

\[ \leq |\Omega_y|^{1/2} e^{-L/2} \left( \int_{-L}^{0} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4 \, dy \, dx \right)^{1/2} \]

\[ \leq \bar{C} e^{-L/2}, \]

we take \( L > 0 \) such that

\[ \int_{-L}^{-L} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx \leq \bar{C} e^{-L/2} < \frac{\epsilon^2}{2} \]

and then \( k_0 \in \mathbb{N} \) such that

\[ \int_{-L}^{0} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx \leq \frac{\epsilon^2}{2} \quad \text{for all } k \geq k_0. \]
Then for all $k \geq k_0$
\[
\|u_{n_k} - \bar{u}\|^2_{L^2_w(\Omega_\ast)} = \int_{-\infty}^0 \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx
\]
\[
= \int_{-\infty}^{-L} \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx + \int_{\Omega} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx < \epsilon^2
\]
and $\|u_{n_k} - \bar{u}\|^2_{L^2_w(\Omega_\ast)} \to 0$.

Observe that
\[
\|u_{n_k} - \bar{u}\|^4_{L^4_w(\Omega_\ast)} = \int_{-\infty}^0 \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^4 \, dy \, dx
\]
\[
\leq \left( \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^2 \, dy \, dx \right)^{1/2} \left( \int_{\Omega_y} e^x |u_{n_k} - \bar{u}|^6 \, dy \, dx \right)^{1/2}
\]
\[
\leq C \|u_{n_k} - \bar{u}\|_{L^2_w(\Omega_\ast)}
\]
by the boundedness of $u_{n_k}$ in $L^6_w(\Omega_\ast)$. Now the lemma follows. \qed

**Lemma 3.5.** Assume that $u_n \in \mathbf{E}_\ast$ is a sequence such that $\|u_n\|_{L^6_w} \leq C$, $I_c[u_n] \to b \leq 0$ and $I'_c[u_n] \to 0$.

Then there exists a subsequence $u_{n_k}$ such that $u_{n_k} \to \bar{u} \in \mathbf{E}_\ast$, $I_c[\bar{u}] = b$ and $I'_c[\bar{u}] = 0$.

**Proof.** Lemma 3.4 implies that there is a subsequence $u_{n_k}$ which converges weakly in $\mathbf{E}_\ast$ and strongly in $L^2(\Omega_\ast)$ to a function $\bar{u} \in \mathbf{E}_\ast$. Then
\[
\int_{\Omega_y} \left[ c^2 (u_{n_k})_x \cdot \bar{u}_x + |(u_{n_k})_y \cdot \bar{u}_y|^2 \right] e^x \, dx \, dy
\]
\[
= \int_{\Omega_y} \left[ c^2 (u_{n_k})_x ((u_{n_k})_x - \bar{u}_x) + \nabla_y u_{n_k} \cdot \nabla_y (u_{n_k} - \bar{u}) \right] e^x \, dx \, dy
\]
\[
- \int_{\Omega_y} \left[ c^2 \bar{u}_x ((u_{n_k})_x - \bar{u}_x) + \nabla_y \bar{u} \cdot \nabla_y (u_{n_k} - \bar{u}) \right] e^x \, dx \, dy
\]
\[
= \left< I'_c[u_{n_k}], u_{n_k} - \bar{u} \right> + \int_{\Omega_y} \left[ u_{n_k} (u_{n_k} - \bar{u}) - u_{n_k}^3 (u_{n_k} - \bar{u}) \right] e^x \, dx \, dy
\]
\[
- \int_{\Omega_y} \left[ c^2 \bar{u}_x ((u_{n_k})_x - \bar{u}_x) + \nabla_y \bar{u} \cdot \nabla_y (u_{n_k} - \bar{u}) \right] e^x \, dx \, dy,
\]
which converges to zero since $u_{n_k} - \bar{u}$ is bounded, converges weakly in $\mathbf{E}_\ast$, strongly in $L^2(\Omega_\ast)$ to 0 while $u_{n_k}$ is bounded in $L^6_w(\Omega_\ast)$. This immediately deduces that $I'_c[\bar{u}] = 0$ and $I_c[\bar{u}] = b$. \qed

**Lemma 3.6.** There exists $L_\ast > 0$ such that
\[
\|I'_c[u] - I'_c[v]\|_{\mathbf{E}_\ast} \leq L_\ast \|u - v\|_{\mathbf{E}_\ast}
\]
if \( u, v \in E_* \), \( I_c[u] \leq 0 \) and \( I_c[v] \leq 0 \).

**Proof.** Suppose that \( u, v \in E_* \) and \( I_c[u] \leq 0 \), \( I_c[v] \leq 0 \). Set \( h = u - v \), then for all \( \phi \in E_* \)

\[
(I'_c[u] - I'_c[v])[\phi] = \int_{\Omega_*} (c^2 h_x \phi_x + \nabla_y h \cdot \nabla_y \phi - h \phi + (u^3 - v^3) \phi) e^x \, dx \, dy.
\]

By (3.7)

\[
\int_{\Omega_*} ((u^3 - v^3) \phi) e^x \, dx \, dy = \int_{\Omega_*} ((u - v)(u^2 + uv + v^2) \phi) e^x \, dx \, dy
\]

\[
\leq \|u - v\|_{L^4_*} (\|u\|_{L^4_*}^2 + \|u\|_{L^4_*} \|v\|_{L^4_*} + \|v\|_{L^4_*}^2) \|\phi\|_{L^4_*}
\]

\[
\leq C \|u - v\|_{L^4_*} \|\phi\|_{L^4_*},
\]

with the constant \( C \) not depending on \( u \) or \( v \). Thus the proof is complete. \( \square \)

We now introduce the min-max classes:

\[
\Gamma_k = \{ A \subset E_* \setminus \{0\} \mid A \text{ closed}, A = -A \text{ and } \gamma(A) \geq k \}
\]

where \( \gamma(A) \) is the Krasnoselski genus, and

\[
\hat{\Gamma}_k = \{ A \in \Gamma_k \mid |u(x, y)| \leq 1, \text{ for all } u \in A \text{ and } (x, y) \in \Omega_* \}.
\]

For fixed \( c \), we denote \( I^a = \{ u \in E_* \mid I_c[u] < a \} \), the sublevels of \( I_c \).

**Proposition 3.7.** For all \( k \in \mathbb{N} \), there is a set \( A \) of genus \( k \) such that

\[
(3.10) \quad c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I_c[u] = \inf_{A \in \hat{\Gamma}_k} \sup_{u \in A} I_c[u] < 0
\]

**Proof.** Take \( L > 0 \) (to be fixed later) and consider the linear problem

\[
\begin{cases}
-c^2(u_{xx} + u_x) - \Delta_y u - u = \lambda_1 u & \text{in } \Omega_L = [-L, 0] \times \Omega_y, \\
u = 0 & \text{on } \partial \Omega_L,
\end{cases}
\]

where \( \lambda_1 \) is the first eigenvalue of (1.7) and \( \varphi_+ \) the corresponding eigenfunction. Set

\[
\phi_k(x, y) = e^{-x/2} \sin \left( \frac{k \pi}{L} x \right) \varphi_+(y)
\]

Notice that \( \phi_k(x, y) = 0 \) for \( (x, y) \in \partial \Omega_L \) and it is a solution of

\[
(3.11) \quad -c^2(\phi_{xx} + \phi_x) - \Delta_y \phi - \phi = \left[ c^2 \left( \frac{1}{4} + \frac{k^2 \pi^2}{L^2} \right) + (\lambda_1 - 1) \right] \phi.
\]
Multiplying (3.11) by $\phi e^x$ and integrating it, we obtain
\[\int_{-L}^{0} \int_{\Omega_y} \left[ -c^2(\phi_{xx} + \phi_x) - \Delta_y \phi - \phi \right] \phi e^x \, dydx \]
\[= \left[ c^2 \left( \frac{1}{4} + \frac{k^2 \pi^2}{L^2} \right) + (\lambda_1 - 1) \right] \int_{-L}^{0} \int_{\Omega_y} \phi^2 e^x \, dydx. \]
Set
\[Q_L[\phi] = \frac{1}{2} \int_{-L}^{0} \int_{\Omega_y} \left[ c^2 \phi_x^2 + |\nabla_y \phi|^2 - \phi^2 \right] e^x \, dydx. \]
Since
\[2Q_L[\phi] = \int_{-L}^{0} \int_{\Omega_y} \left[ -c^2 ((\phi_x \phi e^x)_x - \phi_x^2 e^x) + |\nabla_y \phi|^2 e^x - \phi^2 e^x \right] dydx \]
\[= \int_{-L}^{0} \int_{\Omega_y} \left[ -c^2(\phi_{xx} + \phi_x) - \Delta_y \phi - \phi \right] \phi e^x \, dydx, \]
it follows that $Q_L[\phi_k] < 0$ if
\[c^2 \left( \frac{1}{4} + \frac{k^2 \pi^2}{L^2} \right) + (\lambda_1 - 1) < 0. \]
In fact, for every given $k \in \mathbb{N}$, if $L$ is large enough then $Q_L[\phi_k] < 0$, because $\lambda_1 < 1$ and $c < 2\sqrt{(1 - \lambda_1)}$. Clearly
\[Q_L[\alpha_i \phi_i + \alpha_j \phi_j] = \alpha_i^2 Q_L[\phi_i] + \alpha_j^2 Q_L[\phi_j]. \]
We now extend the functions $\phi_i(x, y)$ to be defined on $\Omega_*$ by setting $\phi_i(x, y) = 0$ for all $x < -L$.
Consider an odd map
\[\xi: S^k \to E_*, \quad \xi(\alpha_1, \alpha_2, \ldots, \alpha_k) = \epsilon \sum_{i=1}^{k} \alpha_i \phi_i. \]
By direct calculation
\[I_c[\epsilon \xi(\alpha_1, \ldots, \alpha_k)] = \epsilon^2 \sum_{i=1}^{k} \alpha_i^2 Q_L[\phi_i] \]
\[+ \epsilon^4 \int_{-L}^{0} \int_{\Omega_y} \frac{1}{4} |\sum_{i=1}^{k} \alpha_i \phi_i|^4 e^x \, dydx, \]
which is negative for all $(\alpha_1, \ldots, \alpha_k) \in S^k$, provided that we pick $L$ to be large enough and $\epsilon$ small enough. Since
\[A = \xi(S^k) \in \Gamma_k, \]
it follows that
\[ c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} I_c[u] < 0 \]
for all \( k \in \mathbb{N} \).

Let
\[ \hat{c}_k = \inf_{A \in \tilde{\Gamma}_k} \sup_{u \in A} I_c[u]. \]
Clearly \( c_k \leq \hat{c}_k \), while setting a truncation \( \hat{u} \) from \( u \) by
\[ \hat{u}(x, y) = \begin{cases} 
  u(x, y) & \text{if } |u(x, y)| \leq 1 \\
  \frac{u(x, y)}{|u(x, y)|} & \text{if } |u(x, y)| > 1
\end{cases} \]
yields \( I_c[\hat{u}] \leq I_c[u] \) for all \( u \in E_* \). Given any \( A \in \Gamma_k \), define
\[ \hat{A} = \{ \hat{u} \mid u \in A \}. \]
Since the map \( u \mapsto \hat{u} \) is continuous in \( E_* \), we conclude that \( \hat{A} \in \tilde{\Gamma}_k \), \( c_k = \hat{c}_k \) and (3.10) holds.

**Proposition 3.8.** For \( k \in \mathbb{N} \), let \( \hat{A}_n \in \tilde{\Gamma}_k \) be such that
\[ c_k \leq \sup_{u \in \hat{A}_n} I_c[u] \leq c_k + \frac{1}{n} < 0. \]
Then there is a \( u_n \in \hat{A}_n \) such that
\[ c_k - \frac{2}{n} \leq I_c[u_n] \leq c_k + \frac{1}{n}, \quad \| I'_c[u_n] \|_{E_*^!} \leq 8 \sqrt{\frac{2L_*}{n}}. \]

**Proof.** The proof is based on the deformation theory. Suppose the assertion is false, then
\[ \| I'_c[v] \|_{E_*^!} > 8 \sqrt{\frac{L_*}{n}} \]
for all \( v \in \hat{A}_n \) such that \( c_k - \frac{2}{n} \leq I_c[v] \leq c_k + \frac{1}{n} \). Let \( \delta = \frac{2}{\sqrt{nL_*}} \) with \( L_* \) given by Lemma 3.6. If \( u \in I_c^{-1}([c_k - \frac{2}{n}, c_k + \frac{1}{n}]) \) and \( \| u - v \|_{E_*} < 2\delta \), invoking Lemma 3.6 yields
\[ \| I'_c[u] \|_{E_*^!} \geq \| I'_c[v] \|_{E_*^!} - \| I'_c[u] - I'_c[v] \|_{E_*^!} \geq 8 \sqrt{\frac{L_*}{n}} - 2L_*\delta = 4 \sqrt{\frac{L_*}{n}}. \]
We can then apply Lemma 2.3 of [35] with \( S = \hat{A}_k \) and \( \epsilon = \frac{1}{n} \) since
\[ \frac{8\epsilon}{\delta} = \frac{8 \sqrt{nL_*}}{2} = 4 \sqrt{\frac{L_*}{n}}. \]
As a consequence
\[ \eta(1, \hat{A}_n) \subset I_{c_k}^{1/n} \]
and we have reached a contradiction since \( \eta(1, \hat{A}_n) \in \Gamma_k \). □

The following result follows from an application of the Ljusternik-Schnirelman theory. We refer to [19, 26] for related applications to differential equations.

**Proposition 3.9.** Let \( \lambda_1 < 1 \) and \( c < 2\sqrt{(1 - \lambda_1)} \). Then there exist a sequence of critical points \( \{\hat{u}_k\} \) of \( I_c \) such that \( I_c[\hat{u}_k] \leq I_c[\hat{u}_{k+1}] < 0 \), \( |u(x, y)| \leq 1 \) for all \( (x, y) \in \Omega_s \),

\[
(3.12) \lim_{k \to +\infty} I_c[\hat{u}_k] = 0
\]

and

\[
(3.13) \int_{-\infty}^{0} \int_{\Omega_y} e^{x}(\hat{u}_k)^2 dxdy > 0.
\]

**Proof.** It has been shown that \( \hat{c}_k = c_k \) is a critical level. Following from Proposition 3.8 we can find a Palais-Smale sequence \( v_n \) at level \( c_k \) such that \( |v_n(x, y)| \leq 1 \) for all \( (x, y) \in \Omega_s \). By Lemma 3.5 we deduce that \( v_n \) converge to a critical point \( \hat{u}_k \) at level \( c_k \) such that \( |\hat{u}_k(x, y)| \leq 1 \) for all \( (x, y) \in \Omega_s \). Thus we get a sequence of critical points \( \{\hat{u}_k\} \) such that \( I_c[\hat{u}_k] = c_k \).

If \( \int_{-\infty}^{0} \int_{\Omega_y} e^{x}(\hat{u}_k)^2 dxdy = 0 \) for some \( k \in \mathbb{N} \), then \( \hat{u}_k \equiv 0 \); however this would violate \( I_c[\hat{u}_k] < 0 \), thus (3.13) must hold.

To show (3.12), we can follow a variant of a rather standard procedure (see e.g. [2, Theorem 10.10]). Let

\[ \mathcal{B}_C = \{ u \in E_* \mid \|u\|_{L_w^6} \leq C \} \]

Notice that if \( u \in E_* \) and \( \|u\|_{L^\infty(\Omega_s)} \leq 1 \) then \( \|u\|_{L_w^6} \leq |\Omega_y|^{1/6} \). Since \( |u(x, y)| \leq 1 \) implies \( u \in \mathcal{B}_C \) for all \( C \geq \bar{C} = |\Omega_y|^{1/6} \), we can find sets with genus \( k \) in \( I^0 \cap \mathcal{B}_C \) for all \( k \in \mathbb{N} \).

Suppose that

\[
\lim_{k \to +\infty} I_c[\hat{u}_k] = \lim_{k \to +\infty} c_k = \chi < 0.
\]

Then \( \gamma(I_c^{x+\epsilon} \cap \mathcal{B}_C) = +\infty \) for all \( \epsilon > 0 \) such that \( \chi + \epsilon < 0 \). Since the set

\[ \hat{Z}_\chi = \{ u \in E_* \mid I_c[u] = \chi \text{ and } I_c'[u] = 0 \text{ and } \|u\|_{L_w^6(\Omega_s)} \leq \bar{C} \} \]

is compact in \( E_* \), using a property of genus, we can find a neighborhood \( U \) of \( \hat{Z}_\chi \) which has finite genus, say \( \gamma(U) = k_0 < +\infty \). Let \( \mathcal{A} = I_c^{x+\epsilon} \cap \mathcal{B}_C \). As in proving Proposition 3.8 and Lemma 3.6, since the
Palais-Smale property holds in $\mathcal{A}$, when $\epsilon$ is small enough we can find a map $\eta$ such that

$$I_c[u] < \chi - \epsilon \quad \text{for all } u \in \eta(\mathcal{A} \setminus U).$$

This implies that $\gamma(\mathcal{A} \setminus U) \leq \gamma(\eta(\mathcal{A} \setminus U)) = k_1 < +\infty$. Then $\mathcal{A} = (\mathcal{A} \setminus U) \cup (\mathcal{A} \cap U)$ gives

$$\gamma(\mathcal{A}) \leq \gamma(\mathcal{A} \setminus U) + \gamma(\mathcal{A} \cap U) \leq k_1 + k_0 < +\infty,$$

which leads to a contradiction. □

**Lemma 3.10.** If $\{\hat{u}_k\}$ is the sequence of critical points obtained by Proposition 3.9, then

$$\lim_{k \to +\infty} \int_{\Omega^*} e^x \hat{u}_k^4 \, dx dy = 0,$$

$$\lim_{k \to +\infty} \int_{\Omega^*} e^x \hat{u}_k^2 \, dx dy = 0,$$

$$\lim_{k \to +\infty} \int_{\Omega^*} e^x (\hat{u}_k)^2 \, dx dy = 0,$$

$$\lim_{k \to +\infty} \int_{\Omega^*} e^x |\nabla y \hat{u}_k|^2 \, dx dy = 0.$$

**Proof.** Since

$$\int_{\Omega^*} \left( c^2 (\hat{u}_k)^2 \, dx dy + |\nabla y \hat{u}_k|^2 - F'(\hat{u}_k) u_k \right) e^x \, dx dy = \langle I_c'[\hat{u}_k], \hat{u}_k \rangle = 0,$$

it follows that

$$0 = \lim_{k \to +\infty} I_c[\hat{u}_k] = \lim_{k \to +\infty} \int_{\Omega^*} \left( \frac{c^2}{2} (\hat{u}_k)^2 \, dx dy + \frac{1}{2} |\nabla y \hat{u}_k|^2 + F(\hat{u}_k) \right) e^x \, dx dy
\quad = \lim_{k \to +\infty} -\frac{1}{4} \int_{\Omega^*} e^x \hat{u}_k^4 \, dx dy.$$

This together with (3.18) yields

$$\lim_{k \to +\infty} \int_{\Omega^*} e^x \hat{u}_k^2 \, dx dy = 0.$$

Combining with (3.18) completes the proof. □

**Lemma 3.11.** If $u \in E_*$ is a bounded critical point of $I_c$ then $u_x \in L^2(\Omega_*)$ and

$$\lim_{x \to -\infty} u_x(x, y) = 0.$$
uniformly in $y$.

Proof. Multiplying (1.3) by $u_x$ and integrating by parts, we get

\begin{align}
(3.20) \quad c^2 \int_{x_n}^{0} \int_{\Omega_y} u_x^2 \, dy \, dx \\
= - \int_{\Omega_y} \left( \frac{c^2}{2} u_x^2 - \frac{1}{2} |\nabla_y u|^2 - F(u) \right) \, dy \bigg|_{x=x_n} \\
- \int_{\partial\Omega_y} \int_{x_n}^{0} u_x \frac{\partial u}{\partial y} \, dx \, d\sigma_y,
\end{align}

where $\nu_y$ is a normal vector to $\partial\Omega_y$ on which $d\sigma_y$ is an integral element. The last term of (3.20) vanishes since $u_x \equiv 0$ on $\partial\Omega_y$ due to the boundary conditions and hence

\begin{align}
(3.21) \quad c^2 \int_{x_n}^{0} \int_{\Omega_y} u_x^2 \, dy \, dx \\
= - \int_{\Omega_y} \left( \frac{1}{2} |\nabla_y u(x_n, y)|^2 + F(u(x_n, y)) \right) \, dy \\
+ \frac{c^2}{2} \int_{\Omega_y} \left( u_x^2(x_n, y) - u_x^2(0, y) \right) \, dy.
\end{align}

Using the facts that $u$ and $\nabla u$ are uniformly bounded, we arrive at

\begin{align}
(3.22) \quad \int_{x_n}^{0} \int_{\Omega_y} u_x^2 \, dx \, dy \leq C
\end{align}

with $C$ being a constant independent of $n$. Passing to the limit as $n \to \infty$ yields $u_x \in L^2(\Omega_y)$. Then (3.19) follows, since $u_x$ is uniformly continuous in $\Omega_*$. \hfill \Box

Lemma 3.12. Suppose that $J$ has only isolated critical points in $H^1_0(\Omega_y)$ and $u$ is a nonconstant critical point of $I_c$ obtained by Proposition 3.9. Then

\begin{align}
(3.23) \quad \lim_{x \to -\infty} u(x, y) = v(y) \text{ uniformly in } y
\end{align}

and $v$ is a critical point of $J$ with $J[v] < 0$. Furthermore if $\lambda_1 < 1 \leq \lambda_2$ then $v = u_+$ or $-u_+$.

Proof. We first show that for any sequence $x_n \to -\infty$ there exist a subsequence $x_{nk}$ and a critical point $v(y)$ of $J$ such that

\[ u(x_{nk}, y) \to v(y) \text{ in } C^1(\bar{\Omega}_0), \]

where $\Omega_0 = (-1, 0) \times \Omega_y$. 


Take any sequence \( x_n \to -\infty \). By Proposition 3.2 for all \( n \in \mathbb{N} \), \( \| u(x + x_n, y) \|_{C^{1,\alpha}(\bar{\Omega}_0)} \) are uniformly bounded. Hence along a subsequence \( x_{n_k} \) (3.24) \[ u(x + x_{n_k}, y) \to v(x, y) \] in \( C^1(\bar{\Omega}_0) \).

It follows from (3.19) that \( v_x(x, y) \equiv 0 \); thus \( v \in C^1(\bar{\Omega}_y) \), a function which depends on \( y \) only.

Let \( \phi \in H^1_0(\Omega_y) \). Multiplying (1.3) by \( \phi \) and integrating over \( \Omega_0 \), we get

\[
\begin{align*}
 c^2 \int_{\Omega_y} (u_x(x + x_{n_k}, y) + u(x + x_{n_k}, y)) \cdot \phi(y) dy \big|_{x=1}^{x=0} \\
 - \int_{\Omega_0} \left[ \nabla_y u(x + x_{n_k}, y) \cdot \nabla_y \phi(y) - f(u(x + x_{n_k}, y)) \phi(y) \right] dy dx = 0.
\end{align*}
\]

Passing to the limit as \( k \to \infty \), we use (3.19) and (3.24) to obtain

\[
\int_{\Omega_0} \left[ \nabla_y \phi(y) \right. \left. \cdot \nabla_y \phi(y) \right. \right. \left. \left. - f(v(y)) \phi(y) \right] dx dy = 0.
\]

Then

\[
\int_{\Omega_y} \left[ \nabla_y \cdot \nabla_y \phi - f(v) \phi \right] dy = 0,
\]

which shows \( v \) is a critical point of \( J \) and our claim holds. Invoking (3.19) and letting \( k \to \infty \) in (3.20), we also get

(3.25) \[ J[v] = \int_{\Omega_y} \left[ \frac{1}{2} |\nabla_y v|^2 + F(v) \right] dy \]

\[ = -c^2 \int_{\Omega_y} u_x^2 dy dx - \frac{c^2}{2} \int_{\Omega_y} u_x^2(0, y) dy < 0. \]

From the above equality we deduce that, while \( v \) in principle depends on the sequence \( \{x_n\} \) and its subsequence \( n_k \), the critical value \( J[v] \) does not.

To show (3.23), we claim

\[ u(x + x_n, y) \to v(y) \] in \( C^1(\bar{\Omega}_y) \) along any sequence \( x_n \to -\infty \).

For otherwise, there exists a decreasing sequence \( x_n \to -\infty \) such that (3.26) \[ u(x + x_n, y) \to \tilde{v}(y) \] if \( n \) is odd, \( u(x + x_n, y) \to v(y) \) if \( n \) is even, and

\[ \kappa := \| \tilde{v} - v \|_{C(\bar{\Omega}_y)} > 0. \]

It follows from (3.25) that

\[ E(v) = E(\tilde{v}). \]
(i) Suppose that there exists a decreasing sequence \( x_n \to -\infty \) such that (3.26) holds and \( |x_{n+1} - x_n| \leq M \) for all \( n \in \mathbb{N} \). If \( n \) is large then there exist \( y \in \Omega_y \) and \( \xi_n \in (x_{n+1}, x_n) \) such that
\[
|u_x(\xi_n, y)| = \frac{|u(x_n, y) - u(x_{n+1}, y)|}{|x_n - x_{n+1}|} \geq \frac{\kappa}{3M}.
\]
This contradicts (3.19).

(ii) It remains to treat the case in which \( |x_{n+1} - x_n| \to \infty \), \( \|u(x + x_n, y) - \tilde{v}(y)\|_{C^1(\bar{\Omega}_y)} \to 0 \) and \( \|u(x + x_n, y) - v(y)\|_{C(\bar{\Omega}_y)} \to 0 \) as \( n \to \infty \).

From Lemma 3.11 we know \( u_x \in L^2(\Omega_x) \). Hence there exists a sequence \( \{\eta_n\} \) with \( \lim_{n \to \infty} \eta_n = 0 \) such that \( \|u(x + x_n, y) - \tilde{v}(y)\|_{C(\bar{\Omega}_y)} < \eta_n \),
\( \|u(x + x_n, y) - v(y)\|_{C(\bar{\Omega}_y)} < \eta_n \) and
\[
\int_{x_n}^{x_{n+1}} \int_{\Omega_y} |u_x(x, y)|^2 \, dy \, dx \leq \eta_n.
\]

Since \( J \) has only isolated critical points in \( H_0^1(\Omega_y) \), there exist \( \kappa_1, \kappa_2 \in (0, \kappa) \) such that \( w \) is not a critical point of \( J \) if \( \kappa_1 \leq \|w - v\|_{C(\bar{\Omega}_y)} \leq \kappa_2 \).

Since \( \|u(x + \xi, y)\|_{C(\bar{\Omega}_y)} \) is continuous with respect to \( \xi \), there exists \( \tilde{\xi}_n \in (x_{n+1}, x_n) \) such that \( \|u(x + \tilde{\xi}_n, y) - v\|_{C(\bar{\Omega}_y)} = \frac{\kappa_1 + \kappa_2}{2} \). Arguing like in (i), we see that \( |\tilde{\xi}_n - x_n| \to \infty \) and \( |\tilde{\xi}_n - x_{n+1}| \to \infty \) as \( n \to \infty \). Set \( V_n(x, y) = u(x + \tilde{\xi}_n, y) \). We then know that, along a subsequence, still denoted by \( \{V_n\} \), we have that \( V_n(x, y) \to V(y) \) in \( C^1(\bar{\Omega}_y) \) with \( V(y) \) a critical point of \( J \). This is not possible, since
\[
\|V(y) - v(y)\|_{C(\bar{\Omega}_y)} = \lim_{n \to \infty} \|u(x + \tilde{\xi}_n, y) - v(y)\|_{C(\bar{\Omega}_y)} = \frac{\kappa_1 + \kappa_2}{2}.
\]

The last assertion follows from Proposition 2.2. Now the proof is complete.

\[\square\]

4. PASSING TO LIMIT FROM APPROXIMATE SOLUTIONS

Let \( \{\hat{u}_k\} \) be a sequence of solutions obtained in section 3. First we consider the case that \( \lambda_1 < 1 \leq \lambda_2 \). Then along a subsequence
\[
\lim_{x \to -\infty} \hat{u}_k(x, y) = u_+(y)
\]
or
\[
\lim_{x \to -\infty} \hat{u}_k(x, y) = -u_+(y).
\]
We may assume (4.1) holds, for otherwise taking \( -\hat{u}_k \) will do.
By (3.16) and Proposition 3.9

\( I_c[\hat{u}_k] \leq I_c[\hat{u}_{k+1}] < 0, \)

\( \lim_{{k \to +\infty}} I_c[\hat{u}_k] = 0, \)

\( \lim_{{k \to +\infty}} \int_{\Omega^*} e^x (\hat{u}_k)_x^2 \, dx \, dy = 0, \)

while from (3.5) and (2.3) we deduce

\( 0 < \int_{\Omega^*} e^x (\hat{u}_k)_x^2 \, dx \, dy \leq 2 \int_{\Omega^*} e^x \hat{u}_k^2 \, dx \, dy, \)

\( \int_{\Omega^*} e^x \hat{u}_k^2 \, dx \, dy \leq 4 \int_{\Omega^*} e^x (\hat{u}_k)_x^2 \, dx \, dy. \)

Furthermore using (3.9) yields

\( \int_{\Omega^*} e^x \hat{u}_k^4 \, dx \, dy \leq 2 \int_{\Omega^*} e^x \hat{u}_k^2 \, dx \, dy, \)

\( \int_{\Omega^*} e^x |\nabla_y \hat{u}_k|^2 \, dx \, dy \leq \int_{\Omega^*} e^x \hat{u}_k^2 \, dx \, dy. \)

Proof of theorem 1.1. We prove (i) first. Let \( \mu = \|v_+\|_{H^1_0(\Omega_y)} \) and \( x_k \) be the largest real number \( \bar{x} \) satisfying

\( \int_{\bar{x}-1}^{\bar{x}} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y) - \nabla_y v_+(y)|^2 + |\hat{u}_k(x,y) - v_+(y)|^2) \, dy \, dx = \frac{\mu}{8}, \)

and

\( \int_{z-1}^{z} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y) - \nabla_y v_+(y)|^2 + |\hat{u}_k(x,y) - v_+(y)|^2) \, dy \, dx < \frac{\mu}{8}, \)

if \( z < \bar{x} \). From (4.4), (4.6), (4.7) and (4.8), we deduce that for all \( z < 0 \)

\( \int_{z-1}^{z} \int_{\Omega_y} (|\nabla_y \hat{u}_k(x,y)|^2 + |\hat{u}_k(x,y)|^2) \, dy \, dx \to 0 \)

as \( k \to +\infty \). This implies \( x_k \to -\infty \). Define

\( w_k(x,y) = \begin{cases} \hat{u}_k(x+x_k, y) & \text{if } x \leq -x_k, \\ 0 & \text{if } x > -x_k. \end{cases} \)

It is clear that \( w_k(x,y) \to v_+(y) \) as \( x \to -\infty \) and \( w_k(x,y) \to 0 \) as \( x \to +\infty \). Along a subsequence \( w_k(x,y) \to U(x,y) \) in \( C^2_{\text{loc}} \), and \( U \) is a bounded solution of (1.3). By (4.9)

\( \int_{-1}^{0} \int_{\Omega_y} (|\nabla_y U(x,y) - \nabla_y v_+(y)|^2 + |U(x,y) - v_+(y)|^2) \, dy \, dx = \frac{\mu}{8}. \)
which ensures that $U(x, y)$ is a nontrivial solution of (1.3). We remark that for all $a, b \in \mathbb{R}$ and $a < b$,

$$\int_a^b \int_{\Omega_y} U_x^2(x, y) \, dx \, dy \leq \lim_{k \to +\infty} \int_a^b \int_{\Omega_y} (w_k)_x^2(x, y) \, dx \, dy.$$ 

From the proof of (3.22), we know

$$\int_{-\infty}^0 \int_{\Omega_y} (\hat{u}_k)_x^2 \, dx \, dy$$

is bounded. Hence $U_x \in L^2(\Omega)$ and $U_x(x, y) \to 0$ as $x \to \pm \infty$. Arguing like Lemma 3.12, we deduce that

$$v_{-\infty}(y) = \lim_{x \to -\infty} U(x, y), \quad v_{\infty}(y) = \lim_{x \to +\infty} U(x, y),$$

where $v_{-\infty}$ and $v_{\infty}$ are the solutions of (1.5).

As in the proof of (3.21), we have

$$c^2 \int_a^b \int_{\Omega_y} U_x^2 \, dy \, dx = -\int_{\Omega_y} \left(\frac{1}{2} |\nabla_y U(b, y)|^2 + F(U(b, y))\right) dy$$

$$+ \int_{\Omega_y} \left(\frac{1}{2} |\nabla_y U(a, y)|^2 + F(U(a, y))\right) dy$$

$$+ \frac{c^2}{2} \int_{\Omega_y} (U_x^2(a, y) - U_x^2(b, y)) \, dy. \quad (4.11)$$

Letting $a \to -\infty$ and $b \to +\infty$ gives

$$- J[v_{-\infty}] + J[v_{\infty}] = c^2 \int_{-\infty}^\infty \int_{\Omega_y} U_x^2 \, dy \, dx > 0. \quad (4.12)$$

This implies $v_{-\infty} = u_+$ and $v_{\infty} = 0$, which completes the proof of (i).

The proof of (iii) is trivial. It remains to show (ii). Since $J$ has finite number of critical points in $H_0^1(\Omega_y)$, there is a subsequence of $\{\hat{u}_k\}$, still denoted by $\{\hat{u}_k\}$, such that

$$\lim_{x \to -\infty} \hat{u}_k(x, y) = u_* (y)$$

and $u_*$ is a solution of (1.5). With a slight modification, we obtain a bounded nontrivial solution $U(x, y)$ of (1.3) and (4.12) holds. Clearly $v_{-\infty} = u_*$. Then setting $v_{\infty} = u^*$ completes the proof.
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