RECTIFIABILITY OF A CLASS OF INTEGRALGEOMETRIC MEASURES AND APPLICATIONS

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Abstract. In this paper we introduce a new class of integralgeometric measures in \( \mathbb{R}^n \), built upon the idea of slicing, and depending on the dimension \( 0 \leq m \leq n \) and on the exponent \( p \in [1, \infty] \). Among this class we find general conditions which guarantee rectifiability. Two main consequences follow. The solution to a long standing open problem proposed by Federer and concerning the rectifiability of the Integralgeometric measure with exponent \( p \in (1, \infty) \) ([17, 3.3.16], [25, 1], and [26, 5.5]), as well as a novel criterion of rectifiability via slicing for arbitrary Radon measures. The latter is reminiscent of the rectifiable slices theorem originally discovered by White for flat chains and by Ambrosio and Kircheim for metric currents. As far as we know, such a criterion is the first result that sheds light in the understanding of rectifiability of Radon measures by slicing. For this reason its proof requires a completely new technique. Eventually, an alternative proof of the closure theorem for flat chains with discrete-group coefficients is provided.

1. Introduction

In geometric measure theory the notion of rectifiability is of central interest. Rectifiable sets inherit in a measure theoretic sense several properties of manifolds, among which the concepts of dimension and tangent space. Such a notion is fundamental in a multitude of cases. Ranging from pure mathematical research fields, like the study of minimal surfaces, to more applied mathematical fields, like the modelling of fracture in materials. For this reason, it is of fundamental importance to develop criteria capable of recognising when a set is rectifiable, or more in general when a measure \( \mu \) is rectifiable.

In order to better describe the problem let us fix some notation. We begin with the notion of countably \( m \)-rectifiable set \( R \) in \( \mathbb{R}^n \). Precisely, given an integer \( m \) with \( 0 \leq m \leq n \), the set \( R \) is countably \( m \)-rectifiable if and only if

\[
R = \bigcup_i f_i(E_i),
\]

where \( E_i \subset \mathbb{R}^m \) are bounded sets and \( f_i : E_i \rightarrow \mathbb{R}^n \) is Lipschitz for every \( i = 1, 2, \ldots \). Since the geometric properties of rectifiable sets are of measure theoretic nature, the notion of rectifiability extends to measures in the following sense. We say that a positive measure \( \mu \) of \( \mathbb{R}^n \) is \( m \)-rectifiable, whenever

\[
\mu = \theta \mathcal{H}^m \upharpoonright R
\]

for some countably \( m \)-rectifiable set \( R \) and some real-valued measurable function \( \theta \), where \( \mathcal{H}^m \upharpoonright R \) denotes the restriction of the \( m \)-dimensional Hausdorff measure to \( R \).

An effective way to study the structure of a measure \( \mu \) in \( \mathbb{R}^n \), is to perform the so called slicing technique. The idea of slicing is to decompose \( \mu \) into more elementary measures, which are loosely speaking the restriction of \( \mu \) to level sets of a sufficiently regular function. To render the idea, given \( V \subset \text{Gr}(n, m) \), namely, the Grassmannian of all \( m \)-planes of \( \mathbb{R}^n \), denoting by \( \pi_V : \mathbb{R}^n \rightarrow V \) the orthogonal projection, under suitable conditions on \( \mu \), it is possible to consider for a.e. \( y \in V \) a measure \( \mu_y \) which is supported on \( \pi_V^{-1}(y) \) and whose structure reflects that of \( \mu \). For this reason, properties of \( \mu \) can be derived by suitably averaging its slices with respect to \( \gamma_{n,m} \), namely, the unique probability measure on \( \text{Gr}(n, m) \) which is invariant under the action of the orthogonal group \( O(n) \) [25, 3.5, Section 3.9]. In mathematical analysis, slicing techniques have been used several times to reduce the study of the structure of measures to the one of its slices, which are usually simpler and lower dimensional. We refer to [3] for the case of functions with bounded variation, to [2, 8] for...
the case of vector fields with bounded deformation, and to \cite{16, 34, 6, 11} for the case of currents or flat chains. The aim of this work is to provide a rectifiability criterion which is based on the idea of slicing and whose applicability could possibly find applications in different areas of mathematical analysis. Before stating it we present two significant outcomes of such a criterion. The first contains the solution of a long standing open problem proposed by Federer’s \cite{17, 3.3.16} and regarding the structure of the Integralgeometric measure $T^p_m$ of $\mathbb{R}^n$. We refer to Subsection 1.2 for a detailed discussion of this result. We split it into two theorems whose proofs can be found in Section 5.

**Theorem 1.1** (Structure of Integralgeometric measures ($p > 1$)). Let $E \subset \mathbb{R}^n$ satisfy $T^p_m(E) < \infty$ for some $p \in (1, \infty]$. Then $T^p_m \upharpoonright E$ is $m$-rectifiable.

Here we limit ourselfs to observe that the case $p = \infty$ was addressed and solved by Federer \cite{17, Theorem 3.3.14} while Mattila provided a counterexample for $p = 1$ \cite{23}. Theorem 1.1 closes the problem for all the remaining exponents $p > 1$.

Our second contribution for the structure of the Integralgeometric measure reads as follows.

**Theorem 1.2** (Structure of Integralgeometric measure ($p = 1$)). Let $E \subset \mathbb{R}^n$ satisfy $T^1_m(E) < \infty$. Then there exists a unique $m$-rectifiable measure $\mu$, such that

$$\pi_{V^1}(T^m_1 \upharpoonright E - \mu_r) \perp \mathcal{H}^m, \quad \text{for } \gamma_{n,m}\text{-a.e. } V \in \text{Gr}(n, m).$$

(1.3)

The novelty of Theorem (1.2) is the characterization of the singular part $\mu_s := T^m_1 \upharpoonright E - \mu_r$ by means of the singular-pushforward property (1.3). This latter is derived from a decomposition result \cite{Proposition 4.12} which is somehow reminiscent of Besicovitch-Federer's structure theorem.

The second outcome is a novel rectifiability criterion via slicing for Radon measure in $\mathbb{R}^n$. For a correct statement let us recall that given a finite Radon measure $\mu$ in $\mathbb{R}^n$, and given an $m$-plane $V \subset \mathbb{R}^n$, there exists a $\mathcal{H}^m$-a.e. uniquely determined family of probability measures $(\mu^V_y)_{y \in V}$ which disintegrates $\mu$ by means of the map $\pi_V$ (Theorem 2.3). We call $\mu^V_y$ slicing measure.

**Theorem 1.3** (Rectifiability via slicing). Let $\mu$ be a Radon measure in $\mathbb{R}^n$ and let $0 \leq m \leq n$. If there exists a measurable set $\Lambda \subset \text{Gr}(n, m)$ of strictly positive $\gamma_{n,m}$-measure such that for every $V \in \Lambda$

$$\pi_{V^1} \mu \ll \mathcal{H}^m \quad \text{and } \mu^V_y \text{ is supported on a finite set for } \pi_{V^1} \mu \text{-a.e. } y \in V,$$

(1.4)

then $\mu$ is $m$-rectifiable.

A detailed discussion can be found in Subsection 1.3. Here we point out that even if Theorem 1.3 is stated for orthogonal projections, it is possible to consider disintegration of $\mu$ with respect to more general transversal families of maps which are not necessarily Lipschitz in the space variable (Theorem 6.3). This can be done at the expense of losing the absolute continuity of $\mu$ with respect to $\mathcal{H}^m$, but still preserving the condition $\mu(\mathbb{R}^n \setminus R) = 0$ for a countably $m$-rectifiable set $R \subset \mathbb{R}^n$. Moreover the requirement (1.4) for a positive-set of planes is rather delicate; it is indeed possible that a measure $\mu$ satisfies (1.4) on a dense subset of $m$-planes even though it is supported on a purely $(\mu, m)$-unrectifiable set (Subsection 1.3).

At the best of our knowledge, Theorem 1.3 is the first result that sheds light in the understanding of rectifiability of Radon measures by slicing. For this reason its proof requires a completely new technique. Eventually, as an application of the rectifiability via slicing, we present an alternative proof of the well-known closure theorem for flat chains with coefficients in a discrete group (Theorem 6.6).

1.1. **The central result.** We assume that $l$ is an integer satisfying $l \geq m$ and that $p \in [1, \infty]$. Given $\Lambda \subset \mathbb{R}^l$ open and bounded we consider a continuous map $P: \mathbb{R}^n \times \Lambda \to \mathbb{R}^m$. Letting $P_\lambda: \mathbb{R}^n \to \mathbb{R}^m$ be defined as $P_\lambda(x) := P(x, \lambda)$, we assume that the family $(P_\lambda)$ is “smoothly” parametrized by $\lambda \in \Lambda$ in a sense which is made precise in Definition 3.3 and which is referred to as transversality condition. Informally, one can think of $(P_\lambda)$ as a family of generalized projections. Eventually, letting $(\mu_\lambda)$ be a measurable family of Borel regular measures of $\mathbb{R}^n$, for every $p \in [1, \infty]$ and every Borel set $B \subset \mathbb{R}^n$ we let $\zeta_p(B) := (\int_{\mathbb{R}^n} \mu_\lambda(B^p) \, d\lambda)^{1/p}$ (with the obvious extension for $p = \infty$).

Via the classical Carathéodory’s construction we define the measure $\mathcal{F}_p^m$ on $\mathbb{R}^n$ as

$$\mathcal{F}_p^m(E) := \sup_{\delta > 0} \inf_{G_\delta} \sum_{B \in G_\delta} \zeta_p(B), \quad E \subset \mathbb{R}^n$$

(1.5)
where $G_{\delta}$ is the family of all countable Borel coverings of $E$ made of sets having diameter less than or equal to $\delta$. Then we say that $\mathcal{I}_{p}^{m}$ is integralgeometric if and only if there exists a Borel set $E \subset \mathbb{R}^{n}$ such that for $\mathcal{L}^{1}$-a.e. $\lambda \in \Lambda$ the following conditions hold true (see Definition 4.1)

$$P_{\mathcal{L}^{1}} \ll \mathcal{L}^{m} \quad \text{and} \quad P_{\mathcal{L}^{1}}(y) \text{ is finite for } P_{\mathcal{L}^{1}} \text{-a.e. } y \in \mathbb{R}^{m}. \quad (1.6)$$

The following rectifiability criterion for integralgeometric measures is the core of the paper. The proof can be found in Section 4.

**Theorem 1.4.** Let $\mathcal{I}_{p}^{m}$ be a finite integralgeometric measure in $\mathbb{R}^{n}$. If $p \in [1, \infty)$ then $\mathcal{I}_{p}^{m}(\mathbb{R}^{n} \setminus R) = 0$ for a countably $m$-rectifiable set $R \subset \mathbb{R}^{n}$. If in addition there exists $\alpha \in (0,1]$ such that $P_{\lambda} : \mathbb{R}^{n} \to \mathbb{R}^{m}$ is $\alpha$-Hölder for $\mathcal{L}^{1}$-a.e. $\lambda \in \Lambda$, then we have also $\mathcal{I}_{p}^{m} \ll \mathcal{H}^{m} \setminus R$.

Informally, Theorem 1.4 investigates conditions on the family $(\mu_{\lambda})$ which guarantee that the average measure $\int_{\lambda} \mu_{\lambda} d\lambda$ is rectifiable. We observe that for $p = 1$ the same result does not hold true since it is possible to construct a counterexample (see Subsection 1.2). Indeed, the peculiarity in the choice $p > 1$ relies on the reflexivity of the $L^{p}$-space which prevents certain concentrations of measures in the space of parameters $\Lambda$. We further observe that in the Lipschitz case ($\alpha = 1$), the rectifiability of $\mathcal{I}_{p}^{m}$ follows immediately from the $\sigma$-finiteness of $R$ with respect to $\mathcal{H}^{m}$ and hence from a straightforward application of Radon-Nikodym’s theorem. Moreover, as explained in Remark 4.11, the advantage of working with transversal families of maps relies on the possibility to extend our result to non-euclidean settings.

We refer to Subsection 1.4 for a detailed discussion and for an outline of the proof of Theorem 1.4.

**1.2. Structure of Federer’s Integralgeometric measure.** The $m$-dimensional Integralgeometric measure $\mathcal{I}_{m}^{n}$ of $\mathbb{R}^{n}$ was first defined by Fàvarel for $p = 1$ in [15], then for $p = \infty$ by Mickle and Nemitz in [27] and [28], respectively, and for $p \in (1, \infty)$ by Federer in [17]. Following [25] we recall here its definition. For every $V \in \text{Gr}(n,m)$ and every Borel set $B \subset \mathbb{R}^{n}$ we let $f_{B}(V) := \mathcal{H}^{m}(\pi_{V}(B))$ and $\eta_{p}(B) := \|f_{B}\|_{L^{p}(\text{Gr}(n,m))}$ where the $L^{p}$-space in $\text{Gr}(n,m)$ is considered with respect to the invariant measure $\gamma_{m,m}$. The $m$-dimensional Integralgeometric measure $\mathcal{I}_{m}^{n}$ is thus defined as in (1.5) via Caratheodory’s construction.

For fixed $m$ the measures $\mathcal{I}_{m}^{n}$ vanish on the same sets: for $p \in [1, \infty]$, $\mathcal{I}_{m}^{n}(E) = 0$ if and only if $E$ is contained in a Borel set $B$ and $\mathcal{H}^{m}(\pi_{V}(B)) = 0$ for $\gamma_{m,m}$-a.e. $V \in \text{Gr}(n,m)$. Nevertheless, in contrast with the behaviour of Hausdorff null-sets, the complex nature of the Integralgeometric measure is highlighted by the fact that null-sets are in general not stable under smooth maps. In fact, in order to disprove Vitushkin’s conjecture [32, p. 147], suggesting that the compact null-sets of $\mathcal{I}_{1}$ in the plane might be exactly the compact null-sets for analytic capacity, Mattila proved in [24] that a $C^{2}$ diffeomorphism $f : \mathbb{R}^{2} \to \mathbb{R}^{2}$ preserves negligible sets if and only if it is affine.

In this paper we are interested in the structure of Integralgeometric finite-sets. Precisely, we address the following two questions.

**(Q.1)** Given $p \in [1, \infty]$ does $\mathcal{I}_{p}^{m}(E) < \infty$ imply the $m$-rectifiability of $\mathcal{I}_{p}^{m} \setminus E$?

**(Q.2)** If (Q.1) is false, how can be characterized the non-rectifiable part of $\mathcal{I}_{p}^{m} \setminus E$?

When $p = \infty$ (Q.1) received a positive answer in [17, Theorem 3.3.14]. Federer’s proof is essentially based on the structure theorem together with an ad hoc argument. The same author posed the question whether the measure $\mathcal{I}_{p}^{m}$ could be replaced by any $\mathcal{I}_{p}^{m}$ with $p \in [1, \infty)$. He suggested that a negative answer to (Q.1) is equivalent to the construction of a compact set $E \subset \mathbb{R}^{2}$ with $0 < \mathcal{I}_{p}^{m}(E) < \infty$ and satisfying [17, 3.3.16]

$$\Theta^{1}(\mathcal{I}_{p}^{m} \setminus E, x) = 0, \quad \text{for } x \in E. \quad (1.7)$$

The case $p = 1$ was addressed by Mattila in [23]. Based on an idea of Talagrand [31] the author iterates simple operations on a parallelogram to construct a compact set $E \subset \mathbb{R}^{2}$ with $0 < \mathcal{I}_{1}^{1}(E) < \infty$ but for which $\mathcal{I}_{1}^{1} \setminus E$ is not 1-rectifiable. This gives at the same time a negative answer to (Q.1) and the existence of a set fulfilling (1.7). Nevertheless the answer to (Q.1) for $p > 1$ still remained unknown. Notice that in the questions above, only sets having non $\sigma$-finite $\mathcal{H}^{m}$-measure are problematic. This is because $\mathcal{H}^{m}(E) < \infty$ implies $\mathcal{I}_{p}^{m}(E) < \infty$, and thus the $m$-rectifiability of $\mathcal{I}_{p}^{m} \setminus E$ for every $p \geq 1$ follows from the case $p = \infty$.

Theorem 1.1 closes the problem by providing an affirmative answer to (Q.1) for $p \in (1, \infty)$. We obtain this result by rewriting $\mathcal{I}_{p}^{m}$ as an integralgeometric measure in the sense of Definition 4.1 and appealing to Theorem 1.4. As a direct consequence we can infer a precise relation between $\mathcal{I}_{p}^{m}$ for
different values of $p$. Indeed, it is not difficult to show that the measures $\mathcal{I}^m_p$, for fixed $m$ and different values of $p$, coincide up to constant multiples on every countably $m$-rectifiable sets [17, subsection 3.3.16]. We have thus the validity of the following corollary.

**Corollary 1.5.** Given $1 < p \leq q \leq \infty$ the measures $\mathcal{I}^m_p$ and $\mathcal{I}^m_q$ coincide up to a constant multiple.

Eventually, in order to answer to question (Q.2), we apply the decomposition provided by Proposition 4.12 to obtain a characterization of the singular part of $\mathcal{I}^m_1 \downarrow E$ by means of the singular-pushforward property. This is exactly the content of Theorem 1.2. In the case $E$ is the set of Mattila’s counterexample, then the decomposition of $\mathcal{I}^m_1 \downarrow E$ contains only singular part (Remark 5.2).

### 1.3. Rectifiability of measures via slicing.

The result contained in Theorem 1.3 is reminiscent of the so called **rectifiable slices theorem**, originally discovered by White for flat chains [34] and independently by Ambrosio and Kircheim for metric currents [6]. The latter precisely asserts that a finite-mass flat $m$-chain $T$ is rectifiable if and only if its slices $(T, V, y)$ are rectifiable 0-chains for every projection onto a coordinate $m$-plane $V$ and for $\mathcal{H}^m$-a.e. $y \in V$. The structure of flat chain or flat current is essential in both their proofs: the existence of a boundary operator, the deformation theorem for flat chains [35], or, in the case of Ambrosio and Kircheim, the fact that $y \mapsto (T, V, y)$ can be seen as a function of metric bounded variation ([6, Section 7]).

In Theorem 1.3 we investigate rectifiability of measures under suitable assumptions on their disintegrations. Precisely, if for a positive-set of $m$-planes $V$ we have $\pi_{V \upharpoonright \mu} \ll \mathcal{H}^m$ and the slicing measure $\mu^V$ consists of finitely many atoms for $\mathcal{H}^m$-a.e. $y \in V$, then $\mu$ is $m$-rectifiable. The hypothesis of the rectifiable slices theorem instead, only asks for a control on finitely many planes. The reason of our stronger assumption is inherent in the nature of our problem and can be well explained by means of the purely ($\mathcal{H}^1, 1$)-unrectifiable Sierpiński gasket $S \subset \mathbb{R}^2$ [21]. To this purpose we recall that $S$ is a self-similar 1-set satisfying the following condition: the projection of $S$ has positive $\mathcal{H}^1$-measure if and only if the projection contains an interval. This occurs exactly for all lines whose slope can be written as $p/q$ in lowest term for integers solving $p + q = 0 \mod 3$ ([21, Lemma 5]). In addition we recall the following general result ([14, Theorem 1.9]): if a self-similar 1-set $K \subset \mathbb{R}^2$ satisfies $\mathcal{H}^1(\pi(K)) > 0$ for an orthogonal projections $\pi$ of $\mathbb{R}^2$ onto a line $t$ through the origin, then

$$\pi_{\ell}(\mathcal{H}^1 \downarrow K) = \frac{\mathcal{H}^1(\pi(K))}{\mathcal{H}^1(\pi_{\ell}(K))} \mathcal{H}^1 \downarrow \pi_t(K).$$

Therefore, by virtue of the previously mentioned projection property of $S$, letting $\mu := \mathcal{H}^1 \downarrow S$, then $\mu$ fulfills the left hand-side of (1.4) for a countable dense set of lines. By applying the general argument in [13] we also know $0 < \mathcal{H}^1(S) < \infty$. For this reason the right hand-side of (1.4) is verified for every lines by virtue of the general formula [17, Corollary 2.10.11].

The previous example shows that exceptional lines may even be dense, excluding thus the possibility to infer rectifiability of $\mu$ by only looking at its behaviour on finitely many projections. How large can be the set of such exceptional lines has not been investigated in this paper. This issue is equivalent to understand whether in Theorem 1.3, $\gamma_{n,m}$ can be replaced by some lower dimensional measure. For instance, one may ask if in the planar case $\gamma_{2,1}$ can be substituted by some $\mathcal{H}^\alpha$ for $\alpha \in (0, 1)$. In view of [21, 22], when the measure $\mu$ coincides with the restriction of $\mathcal{H}^1$ to some self similar planar 1-set, it is reasonable to expect that exceptional lines can be at most countably many.

In this direction, asking for the optimal dimension $\alpha > 0$ which realizes the rectifiability criterion of Theorem 1.3 for general Radon measures $\mu$ seems to be a challenging question.

The proof of Theorem 1.3 is based on a combination of our central result (Theorem 1.4) together with Theorem 6.2 giving sufficient conditions which guarantee that a family of Radon measures $(\mu_\lambda)$ is admissible in (4.5)-(4.6).

### 1.4. Outline of the proof of Theorem 1.4.

The proof of Theorem 1.4 is obtained by proving that a finite integralgeometric measure $\mathcal{I}^m_\phi$ ($p > 1$) cannot give mass to purely $(\mathcal{I}^m_\phi, m)$-unrectifiable sets. The first ingredient is represented by a novel structure theorem for subsets of $\mathbb{R}^n$ having finite $\phi$-measure (Theorem 3.8). Precisely, letting $\phi$ be a Borel regular measure in $\mathbb{R}^n$ and assuming (4.1)-(4.2), we consider a family of measures $(\sigma_\lambda)$ in $\mathbb{R}^m$ absolutely continuous with respect to $\mathcal{L}^m$ and a set $E \subset \mathbb{R}^n$ satisfying

$$\mathcal{H}^0(E \cap P_\lambda^{-1}(y)) < \infty, \quad \text{for } \sigma_\lambda{\text{-a.e. }} y \in \mathbb{R}^m.$$

(1.8)
We find conditions on \( \phi \) ensuring that \( E = (E \setminus R) \cup R \) for a countably \( m \)-rectifiable set \( R \subset \mathbb{R}^n \) such that \( \sigma_\lambda(P_\lambda(E \setminus R)) = 0 \) for \( L^\lambda \)-a.e. \( \lambda \in \Lambda \):

\[
S \subset \Lambda \times \mathbb{R}^n \text{ and } (L^\lambda \otimes \phi \restriction E)(S) = 0 \text{ implies } \sigma_\lambda(P_\lambda(S \cap E)) = 0 \text{ for a.e. } \lambda \in \Lambda,
\]

where \( L^\lambda \otimes \phi \restriction E \) denotes the product measure and \( S_\lambda := \{ x \in \mathbb{R}^n \mid (\lambda, x) \in S \} \). The main novelties in our decomposition result are condition (1.9) (compare this with condition (I) in [17, Theorem 3.3.12]) and the possibility to consider slices of \( E \) with respect to transversal maps which are only continuous (compare this with [19, Theorem 1.2] in which the authors assume lipschitzianity in the spatial variable).

With this result at our disposal, a further step consists in showing that \( \mathcal{I}_p^m \) is admissible in (1.9). To this regard, the key remark consists in noticing that, at least formally, the measure \( \mathcal{I}_p^m \) can be extended to a measure on the product \( \Lambda \times \mathbb{R}^n \) as

\[
\mathcal{I}_m(A) := \int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda, \text{ for } A \subset \Lambda \times \mathbb{R}^n \text{ Borel.}
\]

The fundamental feature of the measure in (1.10) relies on the following implication: if the family \( (\mu_\lambda) \) is such that the corresponding integralgeometric measure \( \mathcal{I}_p^m \) is finite for some \( p \in (1, \infty) \), then a disintegration of the form

\[
\mathcal{I}_m = \eta_x \otimes \mathcal{I}_p^m, \quad (\eta_x)_{x \in \mathbb{R}^n} \text{ family of probability measures in } \Lambda,
\]

must satisfy \( \eta_x \ll L^x \) for \( \mathcal{I}_p^m \)-a.e. \( x \in \mathbb{R}^n \) (Proposition 4.8). This implication is obtained by showing that the sequence \( (f_\lambda) \) is relatively weakly sequentially compact in \( L^p(\Lambda) \) whenever \( f_\lambda(\lambda) := \mu_\lambda(B_r(\lambda)) / \mathcal{I}_p^m(B_r(\lambda)) \). In particular, this allows us to infer the admissibility of \( \mathcal{I}_p^m \) in (1.9) (Proposition 4.10).

Eventually, by putting together all these ingredients in the proper way we obtain Theorem 1.4.

1.5. Plan of the paper. In Section 2 we recall some notation as well as some classical results which are relevant for our analysis. In Section 3 we introduce the notion of transversal family of maps and prove the structure of Theorem 3.8 for Borel regular measures \( \phi \). In Section 4 we rigorously define the class of integralgeometric measures and give the proof of our main result (Theorem 1.4); the last part of the same section is dedicated to a decomposition result for Radon measures (Proposition 4.12) which will serve as the main ingredient in deriving the structure of Integralgeometric measure for \( p = 1 \). Section 5 is devoted to the structure of the Integralgeometric measure (Theorem 1.1 and Theorem 1.2). Section 6 contains a rectifiability criterion via slicing (Theorem 6.3) which includes in particular Theorem 1.3, and an alternative proof of the closure theorem for flat chains with coefficients in a discrete group. We conclude with Appendix A by verifying that the family of orthogonal projections can be represented as a union of transversal family of maps.

2. Some preliminary tools

Let us briefly fix some notation and state some standard facts which we will systematically use throughout the paper.

Let \( X \) denote a set and \( 2^X \) the family of all subsets of \( X \). A measure on \( X \) is as set function \( \mu : 2^X \to [0, \infty] \) satisfying

\[
\mu(\emptyset) = 0 \quad \text{and} \quad \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \text{ whenever } E \subset \bigcup_{i=1}^{\infty} E_i.
\]

If \( X \) is a topological space and every Borel set \( B \subset X \) is \( \mu \)-measurable in the sense of Caratheodory then \( \mu \) is Borel. We say that \( \mu \) is Borel regular whenever given \( E \subset X \) we find a Borel set \( B \supset E \) such that \( \mu(B) = \mu(E) \), while \( \mu \) is Radon if it is Borel regular and finite on compact sets. Given another topological space \( Y \) and a Borel map \( P : X \to Y \), we define the pushforward measure \( P_* \mu \) on \( Y \) as

\[
P_* \mu(E) := \inf_{E \subset B \subset Y \text{ Borel}} \mu(P^{-1}(B)), \text{ for every } E \subset Y,
\]

and we notice that \( P_* \mu(B) = \mu(P^{-1}(B)) \) for \( B \subset Y \) Borel and \( P_* \mu \) is Borel regular whenever \( \mu \) is Borel. In addition, if \( \mu \) is a \( \sigma \)-additive set function defined on Borel subsets of \( X \) and satisfying \( \mu(\emptyset) = 0 \), we always consider its extension to a Borel regular measure in \( X \) via the following formula

\[
\mu(E) := \inf_{E \subset B \subset X \text{ Borel}} \mu(B), \text{ for every } E \subset X.
\]
Here we present three standard results for Radon measures. We start with a density estimate (see [25, Theorem 6.9]). We denote by $\Theta^m(\mu,x)$ the upper $m$-density of $\mu$ at $x$.

**Theorem 2.1 (Density estimates).** Let $\mu$ be a Radon measure in $\mathbb{R}^n$, $E \subset \mathbb{R}^n$, and $0 < t < \infty$. Then

1. if $\Theta^m(\mu,x) \leq t$ \( \forall x \in A \) then $\mu(A) \leq 2^n t \mathcal{H}^m(A)$
2. if $\Theta^m(\mu,x) \geq t$ \( \forall x \in A \) then $\mu(A) \geq t \mathcal{H}^m(A)$

We continue with a version of the Radon-Nikodym’s theorem (see [25, Theorem 2.12, Theorem 2.17]).

**Theorem 2.2 (Radon-Nikodym’s Theorem).** Let $\mu$ and $\eta$ be Radon measures in $\mathbb{R}^n$. Then

1. if $f(x) := \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\mu(B_r(2x))}$ exists finite for $\eta$-a.e. $x \in \mathbb{R}^n$
2. if $\mu \ll \eta$ then $\mu(B) = \int_B f \, d\eta$ for every $B \subset \mathbb{R}^n$ Borel.

We need the following version of the disintegration theorem (see [5, Theorem 5.3.1]).

**Theorem 2.3 (Disintegration Theorem).** Let $P : \mathbb{R}^n \to \mathbb{R}^m$ be a Borel map and let $\mu$ be a positive Radon measure in $\mathbb{R}^n$. Then there exists a $P^\mu$-a.e. uniquely determined family of positive Radon measures in $\mathbb{R}^n$, say $(\eta_y)_{y \in \mathbb{R}^m}$, such that

\[ \eta_y(\mathbb{R}^n \setminus P^{-1}(y)) = 0, \quad \text{for } P^\mu \text{-a.e. } y \in \mathbb{R}^m \]
\[ y \mapsto \eta_y(B) \text{ is Borel measurable for every } B \subset \mathbb{R}^n \text{ Borel} \]
\[ \int_{\mathbb{R}^n} f(x) \, d\mu(x) = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(x) \, d\eta_y(x) \right) \, dP^\mu(y), \]

for every $f : \mathbb{R}^n \to [0,\infty]$ Borel.

We say that a family of measures $(\eta_y)$ is measurable whenever condition (2.4) is satisfied. Eventually, given a non-negative function $f : X \to [0,\infty]$ its upper integral is denoted by

\[ \int_X^* f \, d\mu. \]

It is defined as the infimum of the integrals of all step functions lying $\mu$-a.e. above $f$ [17, 2.4.2]. The following properties are satisfied

- $\int_X f_1 \, d\mu + \int_X f_2 \, d\mu$ whenever $f_1 \leq f_2$ $\mu$-a.e.
- $\int_X f_1 + f_2 \, d\mu \leq \int_X f_1 \, d\mu + \int_X f_2 \, d\mu$.

### 3. A Structure Theorem

Throughout this section $n, m, l$ satisfy (4.1) and $\Lambda \subset \mathbb{R}^l$ is open and bounded. We recall two basic definitions in geometric measure theory.

**Definition 3.1 (Countably $(\phi, m)$-rectifiability).** Let $\phi$ be a measure in $\mathbb{R}^n$, then we say that a set $E \subset \mathbb{R}^n$ is countably $(\phi,m)$-rectifiable if and only if $\phi(E \setminus R) = 0$ for some countably $m$-rectifiable set $R \subset \mathbb{R}^n$.

**Definition 3.2 (Purely $(\phi,m)$-unrectifiability).** Let $\phi$ be a measure in $\mathbb{R}^n$, then we say that a set $E \subset \mathbb{R}^n$ is purely $(\phi,m)$-unrectifiable if and only if for every $C^1$-regular submanifolds $\Sigma \subset \mathbb{R}^n$ it holds true $\phi(\Sigma \cap E) = 0$.

The notion of *transversal family of maps* will play a fundamental role along this section. It was originally introduced in [29, Definition 7.2] for families of maps $P_\lambda : X \to \mathbb{R}^m$ parametrized by $\lambda$ and defined on a compact metric space $(X,d)$. For our purposes it will be sufficient to consider the case when $X$ coincides with $\mathbb{R}^n$. Our definition of transversality can be found in [19] except for the fact that we do not assume lipschitzianity with respect to the spatial variable $x \in \mathbb{R}^n$.

**Definition 3.3.** Let $P : \mathbb{R}^n \times \Lambda \to \mathbb{R}^m$ be continuous. Setting $P_\lambda(x) := P(x,\lambda)$, we say that the family $(P_\lambda)_{\lambda \in \Lambda}$ is transversal if and only if the following conditions hold true.

(H.1) For every $x \in \mathbb{R}^n$ the map $\lambda \mapsto P_\lambda(x)$ belongs to $C^2(\Lambda;\mathbb{R}^m)$ and

\[ \sup_{(\lambda,x) \in \Lambda \times \mathbb{R}^n} \|D^2_{\lambda} P_\lambda(x)\| < \infty, \quad \text{for } j = 1, 2. \] (3.1)
(H.2) For \( \lambda \in \Lambda \) and \( x, x' \in \mathbb{R}^n \) with \( x \neq x' \) define
\[
T_{xx'}(\lambda) := \frac{P_\lambda(x) - P_\lambda(x')}{|x - x'|};
\] (3.2)
then there exists a constant \( C' > 0 \) such that the property
\[
|T_{xx'}(\lambda)| \leq C' \quad \text{implies} \quad |J_{\lambda}T_{xx'}(\lambda)| \geq C'.
\] (3.3)

(H.3) There exists a constant \( C'' > 0 \) such that
\[
\|D_\lambda T_{xx'}(\lambda)\|, \|D_{\lambda^2} T_{xx'}(\lambda)\| \leq C'',
\] (3.4)
for \( \lambda \in \Lambda \) and \( x, x' \in \mathbb{R}^n \) with \( x \neq x' \).

Here we need also to define cones around the preimages of points with respect to the maps \((P_\lambda)\).

**Definition 3.4** (Cone 1). Let \( \lambda \in \Lambda, a \in \mathbb{R}^n, 0 < s < 1, \) and \( r > 0, \) we define
\[
X(a, \lambda, s) := \{x \in \mathbb{R}^n \mid |P_\lambda(x) - P_\lambda(a)| < s |x - a|\},
\] (3.5)
\[
X(a, r, \lambda, s) := X(a, \lambda, s) \cap \overline{\mathcal{T}}(a).
\] (3.6)

**Definition 3.5** (Cone 2). Let \( \lambda \in \Lambda, a \in \mathbb{R}^n, 0 < s < 1, \) and let \( V \subset \mathbb{R}^l \) be an \( m \)-dimensional plane, we define
\[
L_V(a, \lambda, s) := \{x \in \mathbb{R}^n \mid P_\lambda(x) - P_\lambda(a) = 0, \ |\lambda' - \lambda| < s, \ \pi_{V^\perp}(\lambda' - \lambda) = 0\},
\] (3.7)
where \( V^\perp \) is the orthogonal to \( V \).

In the study of rectifiability, the key property of transversality relies on the equivalence between the two definitions of cones introduced above. This is the content of the following proposition.

**Proposition 3.6.** Let \((P_\lambda)\) be a family of transversal maps and let \( a \in \mathbb{R}^n, \lambda_0 \in \Lambda, \) and \( \delta_0 > 0 \) such that \( \overline{\mathcal{T}}(\lambda_0) \subset \Lambda. \) If we denote by \( B := \{e_1, \ldots, e_{\ell}\} \) an orthonormal basis of \( \mathbb{R}^l \) and by \( V_m \) the family of all \( m \)-dimensional coordinate planes, there exists \( c > 0 \) and \( s_0 > 0 \) such that, for every \( \lambda \in \overline{\mathcal{T}}(\lambda_0), s < s_0, \) and \( r > 0, \) the following inclusions hold true
\[
\bigcup_{V \in V_m} \mathcal{T}_r(a) \cap L_V(a, \lambda, s/c) \setminus \{a\} \subset X(a, r, \lambda, s) \subset \bigcup_{V \in V_m} \mathcal{T}_r(a) \cap L_V(a, \lambda, s/c) \setminus \{a\}. \quad (3.8)
\]

**Proof.** The first inclusion follows straightforwardly from the lipschitzianity of \( \lambda \mapsto T_{xx'}(\lambda). \) The second inclusion can be deduced from [20, Lemma 3.3]. \( \square \)

**Remark 3.7.** Inclusions (3.8) are the only properties of transversality that we use throughout the paper. Since condition (3.8) is local in \( \Lambda, \) it is possible to work with family of maps \( P_\lambda : \mathbb{R}^n \to \mathbb{R}^m \) parametrized on some \( \ell \)-dimensional manifolds \( M \) and by working locally with charts \((U, \psi). \) In order to extend all the results presented in this paper to this more general case, it will be then sufficient to verify that \((P_\psi^{-1}(\lambda))_{\lambda \in \Psi(U)}\) satisfies Definition 3.3 or, more directly, inclusions (3.8) with \( \Lambda = \Psi(U), \) and to replace the Lebesgue measure \( \mathcal{L}^l \) with (some multiple of) the volume measure of \( M. \) With this procedure the family of orthogonal projections \((\pi_V)_{V \in \text{Gr}(n,m)}\) can be represented as a finite union of transversal families (see Appendix A), and by exploiting the transitive action of \( O(n) \) on \( \text{Gr}(n,m), \) the same applies to the family \((\pi_{\sigma(V)})_{V \in O(n)}\) for any fixed \( V \in \text{Gr}(n,m). \) In these two cases the Lebesgue measure \( \mathcal{L}^l \) will be replaced by the related Haar measures \( \gamma_{n,m} \) and \( \theta_n, \) respectively.

The remaining part of this section is devoted to the proof of the following structure theorem.

**Theorem 3.8** (Structure). Let \((P_\lambda)\) be a family of transversal maps, let \((\sigma_\lambda)\) be measures on \( \mathbb{R}^m \) absolutely continuous with respect to \( \mathcal{L}^m, \) and let \( \phi \) be a Borel regular measure on \( \mathbb{R}^n. \) Given \( E \subset \mathbb{R}^n \) \( \phi \)-measurable with \( \phi(E) < \infty \) we have
\[
E = R \cup (E \setminus R),
\] (3.9)
where \( R \) is countably \((\phi, m)\)-rectifiable and \( E \setminus R \) is purely \((\phi, m)\)-unrectifiable and \( \sigma \)-compact. Moreover, if for \( \mathcal{L}^l \)-a.e. \( \lambda \in \Lambda \) we have
\[
\sigma_\lambda(P_\lambda(S \cap E)) = 0 \quad \text{whenever} \quad S \subset \Lambda \times \mathbb{R}^n \quad \text{and} \quad (\mathcal{L}^l \otimes \phi)(E)(S) = 0, \quad (3.10)
\]
\[
\mathcal{H}^0(E \cap P_\lambda^{-1}(y)) < \infty, \quad \text{for} \ \sigma_\lambda \text{-a.e.} \ y \in \mathbb{R}^m, \quad (3.11)
\]
the following property holds true
\[
\sigma_\lambda(P_\lambda(E \setminus R)) = 0, \quad \text{for} \ \mathcal{L}^l \text{-a.e.} \ \lambda \in \Lambda. \quad (3.12)
\]
The proof of this theorem is achieved through several intermediate propositions. In what follows we implicitly assume that \((P_{\lambda})\) and \(\phi\) satisfy the hypothesis of Theorem 3.8.

**Proposition 3.9.** Let \(E \subset \mathbb{R}^n\) be a purely \((\phi, m)\)-unrectifiable set with \(\phi(E) < \infty\) and let \(\delta > 0\) and \(\lambda \in \Lambda\). Let the set \(E_{1, \delta}(\lambda)\) be defined as

\[
E_{1, \delta}(\lambda) := \{ a \in \mathbb{R}^n \mid \limsup_{s \to 0^+} \sup_{0 < r < \delta} (rs)^{-m}\phi(E \cap X(a, r, \lambda, s)) = 0 \}. \tag{3.13}
\]

Then \(\phi(E_{1, \delta}(\lambda) \cap E) = 0\).

In order to prove the previous proposition we pass through the following lemma.

**Lemma 3.10.** Let \(E \subset \mathbb{R}^n\) be a purely \((\phi, m)\)-unrectifiable set, let \(0 < s < 1\), \(\alpha > 0\), \(\delta > 0\), \(\lambda \in \Lambda\), and

\[
\phi(E \cap X(x, r, \lambda, s)) \leq \alpha \omega(m)(rs)^m, \tag{3.14}
\]

whenever \(x \in E\) and \(0 < r \leq \delta\), then

\[
\phi(E \cap \overline{B}_{4p/\delta}(a) \cap P_{\lambda}^{-1}(\overline{B}_\rho(P_{\lambda}(a)))) \leq 2(84)^m \alpha \omega(m)\rho^m, \tag{3.15}
\]

whenever \(a \in \mathbb{R}^n\) and \(0 < \rho \leq s\delta/24\). In particular

\[
\Theta^m(\phi \circ E, a) \leq 2(84)^m \alpha. \tag{3.16}
\]

**Proof.** We claim that given \(E \subset \mathbb{R}^n\) satisfying

\[
E \cap X(a, \infty, \lambda, s) = \emptyset \text{ whenever } a \in E, \tag{3.17}
\]

then \(E\) is countably \(m\)-rectifiable. To show this, notice that (3.17) implies for every \(x, z \in E\)

\[
|P_{\lambda}(x) - P_{\lambda}(z)| \geq |x - z|,
\]

hence \(P_{\lambda} \cap E\) is injective and \(f: \text{Im}(P_{\lambda} \cap E) \to \mathbb{R}^n\) satisfying \(P_{\lambda} \cap E \circ f(y) = y\) is a well defined map with Lipschitz constant equal to \(s^{-1}\). Now the proof follows straightforwardly the one of [17, Lemma 3.3.6], where the orthogonal projection \(P\) is replaced by \(P_{\lambda}\). \(\square\)

**Proof of Proposition 3.9.** Consider first a set \(E' \subset E\) with \(\text{diam}(E') \leq \delta/6\). Given \(\alpha > 0\) and \(a \in E_{1, \delta}(\lambda) \cap E'\) by hypothesis we know that there exists \(0 < \overline{s}_a \leq 1\) (depending also on \(\alpha\)) such that (3.14) holds true whenever \(0 < r \leq \delta\) and \(0 < s \leq \overline{s}_a\). If we choose \(\rho\) and \(s\) such that \(0 < \rho := s\delta/24 \leq \overline{s}_a\delta/24\), we can make use of (3.15) to write

\[
\phi(E' \cap P_{\lambda}^{-1}(\overline{B}_\rho(P_{\lambda}(a)))) \leq 2(84)^m \alpha \omega(m)\rho^m, \tag{3.18}
\]

where we have used the fact that, thanks to the choice of \(\rho\), \(B \subset \overline{B}_{4p/\delta}(a) = \overline{B}_{\delta/6}(a)\) for every \(a \in E'\) since \(\text{diam}(E') \leq \delta/6\). This means that for every \(y \in P_{\lambda}(E_{1, \delta}(\lambda) \cap E')\)

\[
\limsup_{\rho \to 0^+} P_{\lambda}\phi \circ E'(\overline{B}_\rho(y))\rho^{-m} \leq 2(84)^m \alpha \omega(m). \tag{3.19}
\]

We can apply the density estimates for Radon measures to infer

\[
P_{\lambda}\phi \circ E'(P_{\lambda}(E_{1, \delta}(\lambda) \cap E')) \leq 2^{2m+1}(84)^m \alpha \omega(m)\mathcal{L}^m(P_{\lambda}(E_{1, \delta}(\lambda) \cap E')).
\]

The arbitrariness of \(\alpha\) implies \(P_{\lambda}\phi \circ B(P_{\lambda}(E_{1, \delta}(\lambda) \cap E')) = 0\). From the definition of pushforward (see formula (2.1)) there exists \(B\) Borel such that \(P_{\lambda}(E_{1, \delta}(\lambda) \cap E') \subset B\) and \(0 = P_{\lambda}\phi \circ E'(B) = \phi(P_{\lambda}^{-1}(B) \cap E')\). Finally, since \(E_{1, \delta}(\lambda) \cap E' \subset P_{\lambda}^{-1}(B) \cap E'\) we conclude \(\phi(E_{1, \delta}(\lambda) \cap E') = 0\). In the general case we can simply write \(E\) as the union of at most countably many sets \(E_i\) \(i = 1, 2, \ldots\) such that \(E_i \subset E\) and \(\text{diam}(E_i) \leq \delta/6\) for \(i = 1, 2, \ldots\). The previous result applied to each \(E_i\) tells us \(\phi(E_{1, \delta}(\lambda) \cap E_i) = 0\) which immediately gives the desired conclusion. \(\square\)

**Proposition 3.11.** Let \(E \subset \mathbb{R}^n\) be \(\phi\)-measurable with \(\phi(E) < \infty\) and let \(\delta > 0\), \(\lambda \in \Lambda\). Let the set \(E_{2, \delta}(\lambda)\) be defined as

\[
E_{2, \delta}(\lambda) := \{ a \in \mathbb{R}^n \mid \limsup_{s \to 0^+} \sup_{0 < r < \delta} (rs)^{-m}\phi(E \cap X(a, r, \lambda, s)) = \infty \}, \tag{3.20}
\]

then \(\mathcal{L}^m(P_{\lambda}(E_{2, \delta}(\lambda))) = 0\).
Proof. By definition we have $P_{\lambda}(X(x, r, \lambda, s)) \subset \mathcal{T}_{\lambda}(P_{\lambda}(x))$ whenever $x \in \mathbb{R}^n$ and $r, s$ are positive numbers. Our hypothesis implies that, given $x \in E_{2, \delta}(\lambda)$, we can find sequences $(s_i)$ and $(r_i)$ with $s_i \to 0$ as $i \to \infty$ and $0 < r_i < \delta$ for $i = 1, 2, \ldots$ for which
\[
\lim_{i \to \infty} P_{\lambda} \phi \cap E(\overline{B}_{r_i s_i}(P_{\lambda}(x)))(r_i s_i)^{-m} = \infty.
\]
But this means that
\[
\limsup_{\rho \to 0^+} P_{\lambda} \phi \cap E(\overline{B}_\rho(y))\rho^{-m} = \infty,
\]
whenever $y \in P_{\lambda}(E_{2, \delta}(\lambda))$. Since $\phi(E) < \infty$, by using [25, Theorem 6.9], we finally infer $\mathcal{L}^m\left(P_{\lambda}(E_{2, \delta}(\lambda))\right) = 0$.

The three alternatives contained in the following proposition are crucial for the structure theorem. They were originally proved in [17, Theorem 3.3.4] in a slightly different form. The proof presented is mainly inspired by the one in [10, Lemma 2.5].

**Proposition 3.12.** Let $E \subset \mathbb{R}^n$ be a $\phi$-measurable set with $\phi(E) < \infty$ and let $\delta > 0$. For every $a \in \mathbb{R}^n$, for $\mathcal{L}^1$-a.e. $\lambda \in \Lambda$ one of the following conditions holds true
\[
\limsup_{s \to 0^+} \sup_{0 < r < \delta} \phi(E \cap X(a, r, \lambda, s))(rs)^{-m} = 0,
\]
\[
\limsup_{s \to 0^+} \sup_{0 < r < \delta} \phi(E \cap X(a, r, \lambda, s))(rs)^{-m} = \infty,
\]
\[
(E \setminus \{a\}) \cap P_{\lambda}^{-1}(P_{\lambda}(a)) \cap \overline{B}_\delta(\lambda) \neq \emptyset.
\]

Proof. We can assume without loss of generality that $E$ is bounded. By exploiting the fact that $\phi \cap E$ is Radon a measure and hence inner regular, we find a $\sigma$-compact set $E'$ with $E \subset E'$ satisfying $\phi(E \cap E') = 0$. Notice that, if the conclusion of the proposition holds true with $E$ replaced by $E'$, then the same must be true for the original set $E$. We may thus suppose that $E$ is a $\sigma$-compact set. Fix $a \in \mathbb{R}^n$, $\lambda_0 \in \Lambda$ and $0 < \delta < \theta_0$ such that $\overline{B}_{2\delta}(\lambda_0) \subset \Lambda$. Let $V \subset \mathbb{R}^l$ be a $m$-dimensional linear subspace and let $V_\sigma := V + \sigma$ for all $\sigma \in V^\perp$. For all $\sigma \in V^\perp$, define a measure $\psi_{\sigma}$ on $\Lambda$ and supported on $V_\sigma \cap \overline{B}_{\delta}(\lambda_0)$ by
\[
\psi_{\sigma}(\Sigma) := \sup_{0 < r < \delta} r^{-m} \phi((E \setminus \{a\}) \cap \overline{B}_r(\lambda) \cap L_{V_\sigma}(\Sigma)),
\]
for all $\Sigma \subset \Lambda$, where
\[
L_{V_\sigma}(\Sigma) := \bigcup_{\lambda \in \Sigma \cap V_\sigma \cap \overline{B}_{\delta}(\lambda_0)} P_{\lambda}^{-1}(P_{\lambda}(a)).
\]
We claim that the set
\[
C_{V_\sigma} := \{\lambda \in \overline{B}_{\delta}(\lambda_0) \cap V_\sigma \mid (E \setminus \{a\}) \cap L_{V_\sigma}(\lambda) \cap \overline{B}_\delta(\lambda) \neq \emptyset\}
\]
is $\mathcal{H}^m$-measurable. This follows from the fact that it is $\sigma$-compact which can be seen as follows. Define the $\sigma$-compact sets
\[
S_1 := \{(\lambda, x) \in (\overline{B}_{\delta}(\lambda_0) \cap V_\sigma) \times \mathbb{R}^n \mid P_{\lambda}(x) = P_{\lambda}(a) = 0\}
\]
and
\[
S_2 := S_1 \cap [\Lambda \times ((E \setminus \{a\}) \cap \overline{B}_\delta(\lambda))],
\]
then we have $C_{V_\sigma} = \pi_1(S_2)$, where $\pi_1 : \Lambda \times \mathbb{R}^n \to \Lambda$ denotes the orthogonal projection onto the first component.

Let $D_{V_\sigma} := (\overline{B}_{\delta}(\lambda_0) \cap V_\sigma) \setminus C_{V_\sigma}$. From the definitions of $\psi_{\sigma}$ and $C_{V_\sigma}$ we deduce that $\psi_{\sigma}(D_{V_\sigma}) = 0$. Now [25, Theorem 18.5] implies that for $\mathcal{H}^m$-a.e. $\lambda \in \overline{B}_{\delta}(\lambda_0) \cap V_\sigma$ one of the following conditions holds true
\[
\limsup_{s \to 0^+} \psi_{\sigma}(\overline{B}_s(\lambda))s^{-m} = 0,
\]
\[
\limsup_{s \to 0^+} \psi_{\sigma}(\overline{B}_s(\lambda))s^{-m} = \infty,
\]
\[
\lambda \in C_{V_\sigma}.
\]
We have thus proved that for every $\sigma \in V^\perp$ and for $\mathcal{H}^m$-a.e. $\lambda \in \overline{B}_{\delta}(\lambda_0) \cap V_\sigma$ one among (3.27)-(3.29) holds true. Denote by $\pi_{\sigma, V} : \mathbb{R}^l \to V^\perp$ the orthogonal projection of $\mathbb{R}^l$ onto $V^\perp$. Notice that once we prove the $\mathcal{L}^l$-measurability of the set of points $\lambda \in \overline{B}_{\delta}(\lambda_0)$ for which one between (3.27)-(3.29)
holds true with $\sigma$ replaced by $\pi_{\nu,\lambda}(\cdot)$, then by applying Tonelli’s theorem we see that for $L^1$-a.e. $\lambda \in \overline{B}_{\delta_0}(\lambda_0)$ one of the following conditions holds true
\begin{align}
\limsup_{s \to 0^+} \psi_{\pi_{\nu,\lambda}}(\lambda) (\overline{B}(s)) s^{-m} &= 0, \\
\limsup_{s \to 0^+} \psi_{\pi_{\nu,\lambda}}(\lambda) (\overline{B}(s)) s^{-m} &= \infty, \\
\lambda &\in C_{\nu,\lambda}(\lambda).
\end{align}
(3.30) (3.31) (3.32)
However, notice that the exceptional set of $L^1$-measure zero depends on the $m$-plane $V$ and this fact does not allow to deduce (3.21)-(3.23) immediately from (3.30)-(3.32). Nevertheless this issue can be solved by combining Proposition 3.6 with properties (3.30)-(3.32). This allows us to infer the validity of (3.21)-(3.23) on $B_{\delta_0}(\lambda_0)$ and thus the proposition follows by a simple covering argument.

It remains to prove the measurability. In order to simplify the notation let us denote $\pi_{\nu,\lambda}(\cdot)$ by $\sigma_\lambda$. First we prove that the map $\varphi_{a,r,s}: \overline{B}_{\delta_0}(\lambda_0) \to [0, \infty]$ defined by
\begin{equation}
\varphi_{a,r,s}(\lambda) := \phi(E \cap \overline{B}(a) \cap L_{V_{\lambda}}(\overline{B}(\lambda)))
\end{equation}
is $L^1$-measurable. For every $a \in E$, $r > 0$, and $0 < s < 1$ define the $\sigma$-compact sets
\begin{equation}
S_1(a,r,s) := \{(\lambda', \lambda, x) \in \Lambda^2 \times \mathbb{R}^n \mid P_\lambda(x) = P_{\lambda'}(a), |\lambda' - \lambda| \leq s, \pi_{\nu,\lambda}(\lambda' - \lambda) = 0\},
\end{equation}
\begin{equation}
S_2(a,r,s) := S_1(a,r,s) \cap \overline{B}_{\delta_0}(\lambda_0) \times \overline{B}_{\delta_0}(\lambda_0) \times ((E \setminus \{a\}) \cap L_{V_{\lambda}}(E)).
\end{equation}
Notice that
\[E \cap \overline{B}(a) \cap L_{V_{\lambda}}(\overline{B}(\lambda)) = \pi_{2,3}(S_2(a,r,s)),\]
for every $\lambda \in \overline{B}_{\delta_0}(\lambda_0)$ where $\pi_{2,3}: \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1 \times \mathbb{R}^n$ denotes the orthogonal projection onto the second and third components. Moreover $S_2(a,r,s)$ is $\sigma$-compact because $S_1(a,r,s)$ is closed and hence $S_2(a,r,s)$ is the intersection of a closed set and a $\sigma$-compact set. This means that also $\pi_{2,3}(S_2(a,r,s))$ is $\sigma$-compact and we can apply Tonelli’s theorem [9, Proposition 5.2.1] on the product space $\Lambda \times \mathbb{R}^n$ with product measure $L^1 \otimes \phi \restriction E$ to deduce that $\varphi_{a,r,s}(\cdot)$ is a $L^1$-measurable map. For every $j = 1, 2, \ldots$ notice that
\begin{equation}
\chi_{a,j}(\lambda) := \sup_{s < 1/j, r < \delta} \sup_{0 < r < s} \varphi_{a,r,s}(\lambda)(rs)^{-m} = \sup_{s \in (0, 1/j) \cap \mathbb{Q}} \sup_{r \in (0, \delta) \cap \mathbb{Q}} \varphi_{a,r,s}(\lambda)(rs)^{-m},
\end{equation}
whenever $\lambda \in \overline{B}_{\delta_0}(\lambda_0)$. Indeed we claim that for every $a \in E$ we have $\lim_{k \to \infty} \varphi_{a, s_k, s_k}(\lambda) = \varphi_{a,r,s}(\lambda)$ whenever $s_k \searrow s$ and $r_k \searrow r$ for every $0 < s < 1$ and $r > 0$. This last fact is a consequence of the following pointwise convergence
\begin{equation}
\mathbb{P}_{\pi_{\nu,\lambda}(\lambda)}(\overline{B}(\lambda))(x) \to \mathbb{P}_{\pi_{\nu,\lambda}(\lambda)}(\overline{B}(\lambda))(x)
\end{equation}
for every $x \in \mathbb{R}^n$ whenever $s_k \searrow s$ and $r_k \searrow r$. Our claim follows thus from Lebesgue’s dominated convergence theorem applied to the sequence of Borel functions in (3.34) and with measure $\phi \restriction E$.

But this means that $\chi_{a,j}(\cdot)$ is a $L^1$-measurable map. Finally, we notice that the points $\lambda \in \overline{B}_{\delta_0}(\lambda_0)$ for which (3.30)-(3.31) hold true are exactly
\begin{equation}
\{\lambda \in \overline{B}_{\delta_0}(\lambda_0) \mid \lim_{j \to \infty} \chi_{a,j}(\lambda) = 0\} \quad \text{and} \quad \{\lambda \in \overline{B}_{\delta_0}(\lambda_0) \mid \lim_{j \to \infty} \chi_{a,j}(\lambda) = \infty\},
\end{equation}
respectively. It remains therefore to prove the measurability of points satisfying (3.32). For this purpose define the $\sigma$-compact sets
\begin{equation}
S_1 := \{(\lambda, x) \in \overline{B}_{\delta_0}(\lambda_0) \times \mathbb{R}^n \mid P_{\lambda}(x) = P_{\lambda}(a) = 0\}
\end{equation}
and
\begin{equation}
S_2 := S_1 \cap (\Lambda \times (E \setminus \{a\}) \cap \overline{B}(a)),
\end{equation}
then $C_{\nu,\lambda} = \pi_1(S_2)$ where $\pi_1: \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^1$ denotes the orthogonal projection onto the first component. Therefore $C_{\nu,\lambda}$ is $\sigma$-compact and this concludes the proof. □

**Lemma 3.13.** Let $E \subset \mathbb{R}^n$ be a $\phi$-measurable set with $\phi(E) < \infty$ and let $\delta > 0$. For $L^1$-a.e. $\lambda \in \Lambda$, for $\phi \restriction E$-a.e. $a \in \mathbb{R}^n$, one of the following conditions holds true
\begin{equation}
\limsup_{s \to 0^+} \sup_{0 < r < \delta} \sup_{0 < r < \delta} \phi(E \cap X(a, r, \lambda, s))(rs)^{-m} = 0, \text{ for some } \delta > 0,
\end{equation}
(3.35)
\[
\limsup_{s \to 0^+} \sup_{0 < r < \delta} \phi(E \cap X(a, r, \lambda, s))(rs)^{-m} = \infty, \quad \text{for every } \delta > 0, \tag{3.36}
\]

\[
(E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap \overline{B}_\delta(a) \neq \emptyset, \quad \text{for every } \delta > 0. \tag{3.37}
\]

**Proof.** First we notice that for every \(a \in \mathbb{R}^n\) we have that \(L^1\)-a.e. \(\lambda \in \Lambda\) satisfies one among (3.35)-(3.37). This follows by using the fact that the map

\[
(0, \infty) \ni \delta \mapsto \limsup_{s \to 0^+} \sup_{0 < r < \delta} \phi(E \cap X(a, r, \lambda, s))(rs)^{-m}
\]

is increasing together with Proposition 3.12. Since \(\phi \restriction E\) is a Radon measure, the same argument at the beginning of the proof of Proposition 3.12 tells us that we may equivalently prove the lemma supposing that \(E\) is \(\sigma\) compact. In this situation we claim that for \((L^j \otimes \phi \restriction E)\text{-a.e. } (\lambda, a) \in \Lambda \times E\) one between (3.35)-(3.37) holds true. Clearly our claim follows from Tonelli’s theorem [9, Proposition 5.2.1] once that we prove the \((L^j \otimes \phi \restriction E)\text{-measurability of the set of points } (\lambda, a) \in \Lambda \times E\) satisfying (3.35) or (3.36) or (3.37). This last fact can be obtained as in the last part of the proof of Theorem 3.8 below.

We are now in position to prove Theorem 3.8.

**Proof of Theorem 3.8.** Let us denote by \(R\) the family of all countably \((\phi, m)\)-rectifiable subsets of \(\mathbb{R}^n\) and consider the minimum problem

\[
\min_{R \in R} \phi(E \setminus R). \tag{3.38}
\]

Decomposition (3.9) follows immediately if we show that (3.38) admits a solution. In addition we claim that we can find a solution to (3.38) which guarantees also that \(E \setminus R\) is \(\sigma\) compact. To show this consider a minimizing sequence \((R_j)\) in \(R\) and define \(R := \bigcup_j R_j\). Clearly \(R \in R\) and \(\phi(E \setminus R) \leq \phi(E \setminus R_j)\) for \(j = 1, 2, \ldots\), so that \(R\) solves (3.38). Moreover being \(E \setminus R\) \(\phi\text{-measurable and being } \phi \restriction E\) a Radon measure, we can find \(a\)-compact set \(S \subset E \setminus R\) with \(\phi(E \setminus R) \cap S = 0\). Redefining \(R\) with abuse of notation as \(R \cup [(E \setminus R) \setminus S]\) we finally obtain the desired decomposition.

To prove the remaining part we may assume without loss of generality that \(E\) is a purely \((\phi, m)\)-unrectifiable \(\sigma\) compact set. Proposition 3.9 applied to the purely \((\phi, m)\)-unrectifiable set \(E\) tells us that for every \(\lambda \in \Lambda\)

\[
\phi(E_{1, \delta})(\lambda) \cap E = 0. \tag{3.39}
\]

Define

\[
S_\delta := \{(\lambda, a) \in \Lambda \times E \mid \limsup_{s \to 0^+} \sup_{0 < r < \delta} (rs)^{-m} \phi(E \cap X(a, r, \lambda, s)) = 0 \}
\]

and suppose for a moment that \(S_\delta\) is \((L^j \otimes \phi \restriction E)\text{-measurable for every } \delta > 0\). Since \((S_\delta)_{\lambda} = E_{1, \delta}(\lambda) \cap E\) for every \(\lambda \in \Lambda\), property (3.39) together with Tonelli’s theorem [9, Proposition 5.2.1] allow us to deduce

\[
(L^j \otimes \phi \restriction E)(S_\delta) = 0, \quad \delta > 0 \tag{3.41}
\]

and hence by setting \(S := \bigcup_{j=1}^{\infty} S_{1/j} \) also

\[
(L^j \otimes \phi \restriction E)(S) = 0. \tag{3.42}
\]

Condition (3.10) implies thus that for \(L^j\text{-a.e. } \lambda \in \Lambda\)

\[
\sigma_\lambda(P_\lambda(S_\lambda \cap E)) = 0. \tag{3.43}
\]

Combining this with condition (3.11) we infer that for \(L^j\text{-a.e. } \lambda \in \Lambda\)

\[
\sigma_\lambda(P_\lambda(S_\lambda \cap E)) = 0 \quad \text{and } E \cap P_\lambda^{-1}(y) \text{ is finite for } \sigma_\lambda\text{-a.e. } y \in \mathbb{R}^m. \tag{3.44}
\]

Define

\[
S' := \{(\lambda, a) \in \Lambda \times E \mid (3.35)-(3.37) \text{ do not hold true}\}, \tag{3.45}
\]

and suppose for a moment that \(S'\) is \((L^j \otimes \phi \restriction E)\text{-measurable. Lemma 3.13 in combination with Tonelli’s theorem tell us that } (L^j \otimes \phi \restriction E)(S') = 0\). As a consequence, applying again property (3.10) we deduce that for \(L^j\text{-a.e. } \lambda \in \Lambda\)

\[
\sigma_\lambda(P_\lambda(S'_\lambda \cap E)) = 0. \tag{3.46}
\]

In addition notice that for every \(\lambda \in \Lambda\)

\[
E = \{a \in E \mid (3.35) \text{ or (3.36) holds true}\} \cup (S'_\lambda \cap E) \tag{3.47}
\]

\[
= \{a \in E \mid (3.36) \text{ or (3.37) holds true}\} \cup (S_\lambda \cap E) \cup (S'_\lambda \cap E).
\]
Putting together (3.44)-(3.45), Proposition 3.11, and $\sigma_\lambda \ll L^m$ (a.e. $\lambda$), we are finally in position to infer that for $L^l$-a.e. $\lambda \in \Lambda$

$$\sigma_\lambda(P_\lambda(\{a \in E \mid (3.36) \text{ or } (3.37) \text{ holds true}\})) = 0,$$

$$\sigma_\lambda(P_\lambda(S_\lambda \cap E)) = 0,$$

$$\sigma_\lambda(P_\lambda(S_\lambda' \cap E)) = 0.$$  

(3.48) (3.49) (3.50)

This together with (3.47) gives the desired conclusion.

It remains to prove the measurability of $S_3$ and $S'$. To this purpose define the $(\phi \circ E \otimes L^l \circ \phi \circ E)$-measurable set $Z$ by

$$Z := \{(a, \lambda, x) \in E \times \Lambda \times E \mid |P_\lambda(x) - P_\lambda(a)| < s|a - x|, |x - a| \leq r\},$$

and notice that $E \cap X(a, r, \lambda, s) = Z_{(a, \lambda)}$. Hence by Tonelli’s theorem the map $(a, \lambda) \mapsto \phi(E \cap X(a, r, \lambda, s))$ is $(\phi \circ E \otimes L^n)$-measurable for every $r > 0$ and $0 < s < 1$. Since for every $a \in E$ and for every $\lambda \in \Lambda$ we can make use of Lebesgue’s dominated convergence theorem to infer

$$\lim_{k \to \infty} \phi(E \cap X(a, r_k, \lambda, s_k)) = \phi(E \cap X(a, r, \lambda, s)), \quad r > 0, \quad 0 < s < 1,$$

whenever $r_k \downarrow r$ and $s_k \nearrow s$, we can write for $(L^l \circ \phi \circ E)$-a.e. $(\lambda, a)$, for every $j = 1, 2, \ldots$, and every $\delta > 0$

$$\chi_j(a, \lambda) := \sup_{0<s<1/j} \sup_{0<r<\delta} (rs)^{-m} \phi(E \cap X(a, r, \lambda, s))$$

$$= \sup_{s \in (0,1/j) \cap Q} \sup_{r \in (0,\delta) \cap Q} (rs)^{-m} \phi(E \cap X(a, r, \lambda, s)).$$

But this means that $\chi_j(a, \lambda)$ is $(L^l \circ \phi \circ E)$-measurable. Since

$$S_3 = \{(a, \lambda) \in E \times \Lambda \mid \lim_{j \to \infty} \chi_j(a, \lambda) = 0\},$$

we deduce also the $(L^l \circ \phi \circ E)$-measurability of $S_3$ for every $\delta > 0$.

Now we pass to the measurability of $S'$. Notice that

$$S_4 := \{(a, \lambda) \in E \times \Lambda \mid (3.35) \text{ holds true}\} = \bigcup_{j=1}^{\infty} S_{1/j}$$

and this gives the $(L^l \circ \phi \circ E)$-measurability of $S_4$. Analogously one can prove the $(L^l \circ \phi \circ E)$-measurability of $S_2 := \{(a, \lambda) \in E \times \Lambda \mid (3.36) \text{ holds true}\}$. Define $S_3 := \{(a, \lambda) \in E \times \Lambda \mid (3.37) \text{ holds true}\}$

and notice that

$$\pi_{1,2}(\{(a, \lambda, x) \in E \times \Lambda \times E \mid P_\lambda(x) = P_\lambda(a), \quad 0 < |x - a| \leq \delta\})$$

$$= \{(a, \lambda) \in E \times \Lambda \mid (E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap B_\delta(a) \neq \emptyset\}.$$  

(3.51)

Since $E$ is Borel by assumption, we can apply the measurable projection theorem [9, Proposition 8.4.4] to infer that the set in (3.51) is $(L^l \circ \phi \circ E)$-measurable and since

$$S_3 = \bigcap_{j=1}^{\infty} \{(a, \lambda) \in E \times \Lambda \mid (E \setminus \{a\}) \cap P_\lambda^{-1}(P_\lambda(a)) \cap B_{1/j}(a) \neq \emptyset\},$$

we deduce that also $S_3$ is $(L^l \circ \phi \circ E)$-measurable. Finally, the $(L^l \circ \phi \circ E)$-measurability of $S_1, S_2, S_3$ immediately implies the $(L^l \circ \phi \circ E)$-measurability of $S'$.

4. RECTIFIABILITY IN THE CLASS OF INTEGRALGEOMETRIC MEASURES

For convenience of the reader we list here the basic conditions that we assume throughout this section.

$$\Lambda \subset \mathbb{R}^l \text{ open and bounded and } n \geq 1, \quad 0 \leq m \leq n, \quad m \leq l$$

(4.1)

$$(P_\lambda)_{\lambda \in \Lambda} \text{ transversal family of maps in the sense of Definition 3.3}$$

(4.2)

$$(\mu_\lambda)_{\lambda \in \Lambda} \text{ measurable family of Borel regular measures on } \mathbb{R}^n$$

(4.3)

We present the definition of integralgeometric measure.
Definition 4.1. The measure $\mathcal{J}^m_p$ defined in (1.5) is integralgeometric if and only if

$$P_M \mu_\lambda \ll \mathcal{L}^m$$

and there exists a Borel set $E \subset \mathbb{R}^n$ satisfying the following two conditions

$$\mu_\lambda(\mathbb{R}^n \setminus E) = 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } \lambda \in \Lambda$$

and

$$\mathcal{H}^i(E \cap \mathcal{P}_\Lambda^{-1}(y)) < \infty \quad \text{for } \mathcal{L}^1\text{-a.e. } \lambda \in \Lambda \text{ and } P_M \mu_\lambda\text{-a.e. } y \in \mathbb{R}^m.$$  

(4.4)  
(4.5)  
(4.6)

We are also interested in finding a suitable extension of the measure $\mathcal{J}^m_p$ given by (1.5) to the product space $\mathbb{R}^n \times \Lambda$. To this regard, under assumptions (4.1)-(4.3) we define for every Borel set $B' \subset \mathbb{R}^n \times \Lambda$ the set function $\hat{\zeta}(B')$ by

$$\hat{\zeta}(B') := \int_\Lambda \mu_\lambda(B'_\lambda) \, d\lambda,$$

and for every $A \subset \mathbb{R}^n \times \Lambda$ the measure $\hat{\mathcal{J}}_m$ by

$$\hat{\mathcal{J}}_m(A) := \sup_{\delta > 0} \inf_{G'_\delta \subseteq \mathcal{G}'_\delta} \sum_{B' \in G'_\delta} \hat{\zeta}(B'),$$

where $G'_\delta$ is the family of all countable Borel coverings of $A$ made of sets having diameter less or equal than $\delta$. We assume for the rest of the section that $\mathcal{J}^m_p$ is the measure defined in (1.5) ($p \in [1, \infty]$).

Proposition 4.2. The measures $\hat{\mathcal{J}}_m$ and $\mathcal{J}^m_p$ ($p \in [1, \infty]$) are Borel regular.

Proof. We first prove that $\hat{\mathcal{J}}_m$ is Borel regular. As customary, in order to check that it is Borel we verify Carathéodory’s criterion [25, Theorem 1.7], namely given $A_1, A_2 \subset \mathbb{R}^n \times \Lambda$ with dist$(A_1, A_2) > 0$ it holds true $\hat{\mathcal{J}}_m(A_1 \cup A_2) = \hat{\mathcal{J}}_m(A_1) + \hat{\mathcal{J}}_m(A_2)$. Indeed let $t := \text{dist}(A_1, A_2) > 0$, then if we choose $\delta < t/2$ in (4.8) any $G'_\delta$ and $G'_\delta$ countable Borel cover of $A_1$ and $A_2$, respectively, admits a subcovering $G'_\delta$ and $G'_\delta$ satisfying

$$B_1 \subseteq G'_\delta \text{ and } B_2 \subseteq G'_\delta \text{ implies } B_1 \cap B_2 = \emptyset.$$  

(4.9)

Let $\delta < t/2$ and $G_\delta$ be a countable Borel covering of $A_1 \cup A_2$. If we define $G'_\delta := \{B \in G_\delta \mid B \cap A_1 \neq \emptyset\}$ and $G'_\delta := \{B \in G_\delta \mid B \cap A_2 \neq \emptyset\}$ then $G'_\delta$ and $G'_\delta$ are countable Borel coverings of $A_1$ and $A_2$, respectively. Using (4.9), we can consider subcoverings $G'_\delta$ and $G'_\delta$ of $A_1$ and $A_2$, respectively, such that $G'_\delta$ and $G'_\delta$ are disjoint. This gives

$$\inf \left\{ \sum_{B \in G_\delta} \hat{\zeta}(B) \mid G_\delta \text{ countable Borel cover of } A_1 \cup A_2, \text{ diam}(B) \leq \delta \right\}$$

$$\geq \inf \left\{ \sum_{B \in G_\delta} \hat{\zeta}(B) \mid G_\delta \text{ countable Borel cover of } A_1, \text{ diam}(B) \leq \delta \right\}$$

$$\quad + \inf \left\{ \sum_{B \in G_\delta} \hat{\zeta}(B) \mid G_\delta \text{ countable Borel cover of } A_2, \text{ diam}(B) \leq \delta \right\}.$$  

This immediately implies $\hat{\mathcal{J}}_m(A_1 \cup A_2) \geq \hat{\mathcal{J}}_m(A_1) + \hat{\mathcal{J}}_m(A_2)$. The opposite inequality comes from the fact that $\hat{\mathcal{J}}_m$ is a measure.

To prove the regularity we fix $A \subset \mathbb{R}^n \times \Lambda$. From the definition of $\hat{\mathcal{J}}_m$ and the monotonicity of $\hat{\zeta}(\cdot)$ we can consider for $i = 1, 2, \ldots$ countable Borel coverings of $A$, say ($G_{i/1}$), made of sets having diameter less or equal than $i^{-1}$ and such that

$$\bigcup_{B \in G_{i/1}} B \subset \bigcup_{B \in G_{i/1}} B \quad \text{and} \quad \lim_{i \to \infty} \sum_{B \in G_{i/1}} \hat{\zeta}(B) = \hat{\mathcal{J}}_m(A).$$

Define the Borel set $B' := \bigcap_{i=1}^{\infty} \bigcup_{B \in G_{i/1}} B$. Then $B' \subset \bigcup_{B \in G_{i/1}} B$ for $i = 1, 2 \ldots$ and $A \subset B'$. But this means that

$$\hat{\mathcal{J}}_m(B') = \sup_{\delta > 0} \left\{ \sum_{B \in G_\delta} \hat{\zeta}(B) \mid G_\delta \text{ countable Borel cover of } B', \text{ diam}(B) \leq \delta \right\}$$

$$\leq \lim_{i \to \infty} \sum_{B \in G_{i/1}} \hat{\zeta}(B) = \hat{\mathcal{J}}_m(A),$$

where we have used the monotonicity with respect to $\delta$ of the infimum above. This immediately gives $\hat{\mathcal{J}}_m(A) = \hat{\mathcal{J}}_m(B')$ and hence also the Borel regularity of $\hat{\mathcal{J}}_m$. 


With the same argument we obtain that also $\mathcal{I}^m_p$ is Borel regular. \hfill $\square$

**Proposition 4.3.** The following relations hold true

$$\mathcal{I}^m_1(E) = \inf_{E \subset B \text{ Borel}} \int_{\Lambda} \mu_\lambda(B) \, d\lambda, \quad \text{for every } E \subset \mathbb{R}^n,$$  

and in particular

$$\mathcal{J}_m(B \times U) \leq \int_U \mu_\lambda(B) \, d\lambda, \quad \text{for every } B \subset \mathbb{R}^n, \ U \subset \Lambda \text{ Borel} \quad (4.11)$$

$$\mathcal{J}^m_1(B) = \zeta_1(B) = \int_{\Lambda} \mu_\lambda(B) \, d\lambda, \quad \text{for every } B \subset \mathbb{R}^n \text{ Borel}. \quad (4.12)$$

**Proof.** We notice that the measure on the right hand-side of (4.10) is Borel *regular* by construction while $\mathcal{J}_m$ is Borel regular thanks to Proposition 4.2. Hence (4.10) follows from (4.12). We prove (4.12). Given $B \subset \mathbb{R}^n$ Borel and $\epsilon > 0$, from the definition of $\mathcal{J}^m_1$ we find a countable Borel covering $(B_i)$ of $B$ satisfying $\sum_i \zeta_1(B_i) \leq \mathcal{J}^m_1(B) + \epsilon$. Therefore

$$\int_{\Lambda} \mu_\lambda(B) \, d\lambda = \zeta_1(B) \leq \sum_i \zeta_1(B_i) \leq \mathcal{J}^m_1(B) + \epsilon,$$

where we have used the $\sigma$-subadditivity of $\zeta_1(\cdot)$. The arbitrariness of $\epsilon$ gives

$$\zeta_1(B) \leq \mathcal{J}^m_1(B), \quad B \subset \mathbb{R}^n \text{ Borel}. \quad (4.13)$$

It remains to prove

$$\zeta_1(B) \geq \mathcal{J}^m_1(B), \quad B \subset \mathbb{R}^n \text{ Borel}. \quad (4.14)$$

Without loss of generality we may assume $\zeta_1(B) < \infty$. We notice that the measure $\zeta_1$ is Borel simply because for $\mathcal{L}^1$-a.e. $\lambda \in \Lambda$ the measure $\mu_\lambda$ is Borel. Indeed, for every pairwise disjoint sequence $(B_k)$ of Borel sets

$$\zeta_1(\bigcup_k B_k) = \int_{\Lambda} \mu_\lambda(\bigcup_k B_k) \, d\lambda = \sum_k \int_{\Lambda} \mu_\lambda(B_k) \, d\lambda = \sum_k \zeta_1(B_k),$$

where we have used Beppo Levi's monotone convergence theorem. This means that the $\sigma$-algebra of all $\zeta_1$-measurable subsets of $\mathbb{R}^n$ contains the Borel sets, namely $\zeta_1$ is a Borel measure. Notice that, in order to prove (4.14), it is enough to show that for every $\delta > 0$ we can find a Borel countable covering of $B$, say $G_\delta$, made of sets having diameter less or equal than $\delta$ and such that

$$\zeta_1(B) = \sum_{\tilde{B} \in G_\delta} \zeta_1(\tilde{B}). \quad \zeta_1(B) \leq \zeta_1(\bigcup_k B_k)$$

But this can be easily obtained thanks to the fact that $\zeta_1$ is $\sigma$-additive on Borel sets. This concludes the proof of (4.12).

To prove (4.11) we define the measure $\eta$ in $\mathbb{R}^n$ as

$$\eta(E) := \sup_{\delta > 0} \inf \sum_B \int_U \mu_\lambda(B) \, d\lambda, \quad E \subset \mathbb{R}^n,$$

where the infimum is considered among the family of all countable Borel covering of $E$ made of sets having diameter less or equal than $\delta$. Since by definition the map $\lambda \mapsto \mu_\lambda(B)$ is $\mathcal{L}^1$-measurable for every $B \subset \mathbb{R}^n$ Borel, we have

$$\int_U \mu_\lambda(B) \, d\lambda = \int_U \mu_\lambda(B) \, d\lambda, \quad \text{for every } B \subset \mathbb{R}^n \text{ Borel.}$$

This immediately implies that

$$\mathcal{J}_m(E \times U) \leq \eta(E), \quad \text{for every } E \subset \mathbb{R}^n. \quad (4.16)$$

To conclude, we notice that the previous argument applies to $\eta$ and allows us to obtain

$$\eta(B) = \int_U \mu_\lambda(B) \, d\lambda, \quad \text{for every } B \subset \mathbb{R}^n \text{ Borel.}$$

Putting together this last information with (4.16) we infer the validity of (4.11). \hfill $\square$
Remark 4.4. If \( \lambda \mapsto \mu_\lambda(A_\lambda) \) is \( \mathcal{L}' \)-measurable for every \( A \subset \mathbb{R}^n \times \Lambda \) Borel, the same argument in the proof of (4.12) gives

\[
\hat{\mathcal{H}}_m(A) = \hat{\zeta}(A) = \int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda, \quad \text{for every } A \subset \mathbb{R}^n \times \Lambda \text{ Borel.} 
\]  

(4.17)

In this situation, notice that thanks to (4.12) we have also the equality

\[
\pi_I \hat{\mathcal{H}}_m(E) = \mathcal{J}^m_I(E), \quad E \subset \mathbb{R}^n,
\]

where \( \pi_1: \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \) is the orthogonal projection.

**Proposition 4.5.** For every \( p > 1 \) the measures \( \mathcal{J}^m_p \) satisfy

\[
\left( \int_{\Lambda} \mu_\lambda(B)^p \, d\lambda \right)^{\frac{1}{p}} \leq \mathcal{J}^m_p(B), \quad B \subset \mathbb{R}^n \text{ Borel.}
\]

(4.19)

**Proof.** It is enough to use the \( \sigma \)-subadditivity of \( \mu_\lambda \) with Minkowski’s inequality. \( \square \)

**Remark 4.6.** We have \( \mathcal{J}^m_q \leq \mathcal{L}'(\Lambda)^{\frac{1}{q}} \mathcal{J}^m_q \) for every \( 1 \leq q \leq p \).

**Proposition 4.7.** Given \( A \subset \mathbb{R}^n \times \Lambda \) Borel and \( B \subset \mathbb{R}^n \) Borel the following equivalences hold true.

1. \( \hat{\mathcal{H}}_m(A) = 0 \) if and only if \( \mu_\lambda(A_\lambda) = 0 \) for \( \mathcal{L}' \)-a.e. \( \lambda \in \Lambda \).
2. \( \mathcal{J}^m_p(B) = 0 \) if and only if \( \mu_\lambda(B) = 0 \) for \( \mathcal{L}' \)-a.e. \( \lambda \in \Lambda \).
3. For every \( p \geq 1 \) we have \( \mathcal{J}^m_p(B) = 0 \) if and only if \( \mathcal{J}^m_p(B) = 0 \).

**Proof.** Suppose that \( \hat{\mathcal{H}}_m(A) = 0 \). From the definition of \( \hat{\mathcal{H}}_m \) we find finite Borel coverings of \( A \), say \( (A^m_i)_{i=1}^M \), such that \( \sum_i \hat{\zeta}(A^m_i) \to 0 \) as \( m \to \infty \). Using the monotonicity and subadditivity of the upper-integral we can write for every \( m = 1, 2, \ldots \)

\[
\int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda \leq \sum_{i=1}^M \int_{\Lambda} \mu_\lambda(A_\lambda \cap A^m_i) \, d\lambda \leq \sum_{i=1}^M \hat{\zeta}(A^m_i).
\]

Letting \( m \to \infty \) on last term of the previous inequalities we infer

\[
\int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda = \int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda = 0,
\]

and therefore \( \mu_\lambda(A_\lambda) = 0 \) for \( \mathcal{L}' \)-a.e. \( \lambda \in \Lambda \). Now suppose \( \mu_\lambda(A_\lambda) = 0 \) for \( \mathcal{L}' \)-a.e. \( \lambda \in \Lambda \). Fix \( \delta > 0 \) and notice that given any Borel covering \( (A_i) \) of \( A \) such that \( \text{diam}(A_i) \leq \delta \) for \( i = 1, 2, \ldots \), also \( (A_i \cap A_j) \) is a Borel covering of \( A \) satisfying \( \text{diam}(A_i \cap A_j) \leq \delta \) for \( i = 1, 2, \ldots \). From the arbitrariness of \( \delta \) together with the following inequality

\[
\sum_i \hat{\zeta}(A \cap A_i) = \sum_i \int_{\Lambda} \mu_\lambda((A \cap A_i)_\lambda) \, d\lambda \leq \sum_i \int_{\Lambda} \mu_\lambda(A_\lambda) \, d\lambda = 0,
\]

we immediately conclude \( \hat{\mathcal{H}}_m(A) = 0 \).

Equivalence (2) can be proved with the same argument. So let us prove (3). Suppose \( \mathcal{J}^m_p(B) = 0 \) then from (2) we know \( \mu_\lambda(B) = 0 \) for \( \mathcal{L}' \)-a.e. \( \lambda \in \Lambda \). Fix \( \delta > 0 \) and notice that given any Borel covering \( (B_i) \) of \( B \) such that \( \text{diam}(B_i) \leq \delta \) for \( i = 1, 2, \ldots \), also \( (B_i \cap B_j) \) is a Borel covering of \( B \) satisfying \( \text{diam}(B_i \cap B_j) \leq \delta \) for \( i = 1, 2, \ldots \). From the arbitrariness of \( \delta \) together with the following inequality

\[
\sum_i \hat{\zeta}(B \cap B_i) = \sum_i \left( \int_{\Lambda} \mu_\lambda(B \cap B_i)^p \, d\lambda \right)^{\frac{1}{p}} \leq \sum_i \left( \int_{\Lambda} \mu_\lambda(B)^p \, d\lambda \right)^{\frac{1}{p}} = 0,
\]

we immediately conclude \( \mathcal{J}^m_p(B) = 0 \). The opposite implication comes from Remark 4.6. \( \square \)

Here we present a key property of integralgeometric measures with \( p > 1 \).

**Proposition 4.8.** Let \( \mathcal{J}^m_p \) be given by formula (1.5) and finite for some \( p > 1 \). Suppose that \( (\eta_x)_{x \in \mathbb{R}^n} \) is a family of Radon measure on \( \Lambda \) such that

\[
\hat{\mathcal{H}}_m = \eta_x \otimes \pi_1 \hat{\mathcal{H}}_m,
\]

(4.20)

in the sense of disintegration of measures. We have that \( \eta_x \ll \mathcal{L}' \) for \( \pi_1 \hat{\mathcal{H}}_m \)-a.e. \( x \in \mathbb{R}^n \).
Proof. In order to simplify the notation we set
\[ \mathcal{J}^+_m := \tau_{12} \mathcal{J}_m. \]
If \( x_0 \in \mathbb{R}^n \) we can consider for every \( r > 0 \) such that \( \overline{B}_r(x_0) \subset \mathbb{R}^n \) the \( \mathcal{L}^r \)-measurable maps \( f_r(\lambda) := \mu_\lambda(\overline{B}_r(x_0))/\mathcal{J}^+_m(\overline{B}_r(x_0)) \). Notice that by (4.19)
\[
\|f_r\|_{L^p(\Lambda)} = \frac{1}{\mathcal{J}^+_m(\overline{B}_r(x_0))} \left( \int_\Lambda \mu_\lambda(\overline{B}_r(x_0))^p d\lambda \right)^{\frac{1}{p}} \leq \frac{\mathcal{J}^+_m(\overline{B}_r(x_0))}{\mathcal{J}^+_m(\overline{B}_r(x_0))}.
\]
Applying Radon-Nikodym’s theorem for the two Radon measures \( \mathcal{J}^+_m, \mathcal{J}^+_m \) (notice that being \( \mathcal{J}^+_m \) finite also \( \mathcal{J}^+_m \) is finite) we obtain
\[
\lim_{r \to 0^+} \frac{\mathcal{J}^+_m(\overline{B}_r(x_0))}{\mathcal{J}^+_m(\overline{B}_r(x_0))} \text{ exists and is finite for } \mathcal{J}^+_m \text{-a.e. } x_0 \in \mathbb{R}^n.
\]
Now let \( \varphi \in C^0_c(\Lambda) \) be non negative and notice that by (4.11)
\[
\frac{1}{\mathcal{J}^+_m(\overline{B}_r(x_0))} \int_{\mathbb{R}^n \times \Lambda} \mathbb{1}_{\overline{B}_r(x_0)}(y) \varphi d\mathcal{J}_m \leq \int_\Lambda \varphi \frac{\mu_\lambda(\overline{B}_r(x_0))}{\mathcal{J}^+_m(\overline{B}_r(x_0))} d\lambda = \int_\Lambda \varphi f_r d\lambda.
\]
On the other hand, by using (4.26) we know also
\[
\frac{1}{\mathcal{J}^+_m(\overline{B}_r(x_0))} \int_{\mathbb{R}^n \times \Lambda} \mathbb{1}_{\overline{B}_r(x_0)}(y) \varphi d\mathcal{J}_m = \int_\Lambda \varphi d\mu_\lambda.
\]
Now consider \( D \) a countable dense subset of all non-negative functions in \( C^0_c(\Lambda) \). By using (4.24) together with Lebesgue’s differentiation theorem, for \( \mathcal{J}^+_m \text{-a.e. } x_0 \in \mathbb{R}^n \) and for every \( \varphi \in D \) we have
\[
\lim_{r \to 0^+} \frac{1}{\mathcal{J}^+_m(\overline{B}_r(x_0))} \int_{\mathbb{R}^n \times \Lambda} \mathbb{1}_{\overline{B}_r(x_0)}(y) \varphi d\mathcal{J}_m = \int_\Lambda \varphi d\mu_\lambda.
\]
If we look to \( x_0 \in \mathbb{R}^n \) for which (4.25) holds true, we can make use of (4.21)-(4.23) to deduce that up to pass through a subsequence on \( r \), still denoted by \( r \), we have \( f_r \to f \) weakly in \( L^p(\Lambda) \), and hence
\[
\int_\Lambda \varphi d\mu_\lambda \leq \int_\Lambda \varphi f d\lambda, \quad \varphi \in D.
\]
Since \( D \) is dense we deduce from (4.26) that for \( \mathcal{J}^+_m \text{-a.e. } x_0 \in \mathbb{R}^n \) we have \( \eta_{x_0} \leq f \mathcal{L}^1 \) as measures. This gives the desired assertion. \( \square \)

Remark 4.9. If \( \lambda \mapsto \mu_\lambda(A_\lambda) \) is \( \mathcal{L}^r \)-measurable for every \( A \subset \mathbb{R}^n \times \Lambda \) Borel, using (4.17), we deduce that actually (4.23) becomes an equality and hence, \( \eta_{x_0} = f \mathcal{L}^1 \) for \( \mathcal{J}^+_m \text{-a.e. } x_0 \in \mathbb{R}^n \).

We observe that if \( \mathcal{J}^+_m \) is given by (1.5) and is finite for some \( p \in [1, \infty) \), then in view of Remark 4.6 also \( \mathcal{J}^+_m \) is finite. Applying formula (4.12) we deduce that \( \mu_\lambda \) is a finite and hence a Radon measure for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \). We can therefore find a disintegration of the form
\[
\mu_\lambda = \eta^0 \otimes P_\lambda \mu_\lambda, \quad \text{for } \mathcal{L}^1 \text{-a.e. } \lambda \text{ and for } \mathcal{L}^m \text{-a.e. } y \in \mathbb{R}^m.
\]
for a uniquely determined family of probability measures \( \{\eta^0_y\}_{y \in \mathbb{R}^m} \) each of which (for a.e. \( y \)) concentrated on \( P_\lambda^{-1}(y) \). Having this in mind we can state the next proposition.

Proposition 4.10. Let \( \mathcal{J}^+_m \) be given by (1.5) and be finite for some \( p > 1 \). Assume that the measure \( \eta^0 \) given by disintegration (4.27) is 0-rectifiable for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \) and for \( \mathcal{L}^m \)-a.e. \( y \in \mathbb{R}^m \). If \( S \subset \mathbb{R}^n \times \Lambda \) is \( (\mathcal{J}^+_m \otimes \mathcal{L}^1) \)-negligible, then for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \)
\[
P_\lambda \mu_\lambda(P_\lambda(S_\lambda)) = 0.
\]

Proof. Since \( \mathcal{J}^+_m \otimes \mathcal{L}^1 \) is Borel regular we can reduce ourselves to prove the corollary in the case \( S \) is Borel. By our hypothesis we can apply Tonelli’s theorem to deduce that
\[
\mathcal{L}^1(S_\lambda) = 0, \quad \mathcal{J}^+_m \text{-a.e. } x \in \mathbb{R}^n.
\]
Since \( \mathcal{J}^+_m(\mathbb{R}^n) < \infty \) implies \( \mathcal{J}_m(\mathbb{R}^n \times \Lambda) < \infty \) we can make use of disintegration theorem to find a family of Radon measures on \( \Lambda \), say \( \{\eta_x\}_{x \in \mathbb{R}^n} \), such that
\[
\mathcal{J}_m = \eta_x \otimes \tau_{12} \mathcal{J}_m.
\]
From Proposition 4.8 we already know that \( \eta_y \ll L^1 \) for \( \pi_{1, \mathcal{I}_m} \)-a.e. \( x \in \mathbb{R}^n \). Hence, (4.29) gives \( \mathcal{I}_m(S) = 0 \). Proposition 4.7 together with the Borel regularity of \( \mathcal{I}_m \) allow us write
\[
\mu_\lambda(S_\lambda) = 0, \quad \text{for } L^1 \text{-a.e. } \lambda \in \Lambda. \tag{4.31}
\]
We achieve therefore the desired conclusion combining (4.31) with disintegration (4.27) and the assumption on the \( \eta_y^\lambda \).

We are now in position to prove the main result of this paper.

**Proof of Theorem 1.4.** We claim that the Borel set \( E \) and the measure \( \mathcal{I}_p^m \) satisfy the hypothesis of Theorem 3.8 with \( (\sigma_\lambda) \) replaced by \( (P_\lambda \mu_\lambda) \). We need only to verify (3.10). But this will follow from Proposition 4.10 once that we verify that \( \eta_y^\lambda \) is atomic for \( L^1 \)-a.e. \( \lambda \in \Lambda \) and for \( L^m \)-a.e. \( y \in \mathbb{R}^m \). To this regard we observe that, since \( \mathcal{I}_p^m \) is finite, applying Remark 4.6 together with (4.12), we deduce that also \( \mu_\lambda \) is finite and hence Radon for a.e. \( \lambda \in \Lambda \). We have thus the validity of the disintegration (4.27). Now using (4.5)-(4.6) we know that for \( L^1 \)-a.e. \( \lambda \in \Lambda \) it holds true \( \mu_\lambda = \mu_\lambda \mid E \). This last information can be used in (4.27) to infer that for \( L^1 \)-a.e. \( \lambda \in \Lambda \)
\[
\eta_y^\lambda \mid E \otimes P_\lambda \mu_\lambda = \mu_\lambda \mid E = \mu_\lambda = \eta_y \otimes P_\lambda \mu_\lambda,
\]
from which we deduce that
\[
\eta_y^\lambda \mid E = \eta_y^0, \quad \text{for } L^1 \text{-a.e. } \lambda \in \Lambda \text{ and for } P_\lambda \mu_\lambda \text{-a.e. } y \in \mathbb{R}^m.
\]
But now remember that \( \eta_y^\lambda \) are non null measures concentrated on \( P_\lambda^{-1}(y) \) for \( P_\lambda \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \) and that, by assumption, \( E \cap P_\lambda^{-1}(y) \) is finite for \( P_\lambda \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \). We obtain thus
\[
\eta_y^\lambda = \sum_{x_\lambda \in E \cap P_\lambda^{-1}(y)} m_i \delta_{x_\lambda}, \quad \text{for } P_\lambda \mu_\lambda \text{-a.e. } y \in \mathbb{R}^m,
\]
where \( m_i \) are strictly positive real multiplicities and \( \delta_{x_\lambda} \) is the probability measure supported on \( \{x_\lambda\} \). This allows us to make use of Proposition 4.10 and infer the validity of condition (3.10).

We find therefore a countably \( (\mathcal{I}_p^m, m) \)-rectifiable set \( R' \) such that \( \mathbb{R}^n \setminus R' \) is a \( \sigma \)-compact purely \( (\mathcal{I}_p^m, m) \)-unrectifiable set and satisfies (3.12), namely for \( L^1 \)-a.e. \( \lambda \in \Lambda \) we have
\[
P_\lambda \mu_\lambda(P_\lambda(E \setminus R')) = 0. \tag{4.33}
\]
This last information implies
\[
\int \mu_\lambda(E \setminus R') \, d\lambda = \int \left( \int_{\mathbb{R}^m} \eta_y^\lambda(E \setminus R') \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda = 0.
\]
Hence from (4.12) we immediately deduce \( \mathcal{I}_p^m(E \setminus R') = 0 \) and from Proposition 4.7 also \( \mathcal{I}_m^m(E \setminus R') = 0 \). From this we immediately find a countably \( m \)-rectifiable set \( R \subset R' \) such that \( \mathcal{I}_p^m(E \setminus R) = \mathcal{I}_p^m(E \setminus R) = 0 \).

Now using the obvious estimate
\[
\mathcal{H}^m(P_\lambda(B)) \leq \sup_{x \neq x'} \left( \frac{P_\lambda(x) - P_\lambda(x')}{|x - x'|^m} \right) \mathcal{H}^m(B), \quad \text{for every } B \subset \mathbb{R}^n \text{ Borel},
\]
and arguing as in [17, Theorem 2.10.10, Corollary 2.10.11] we infer
\[
\int_{\mathbb{R}^m} \mathcal{H}^m(R \cap B \cap P_\lambda^{-1}(y)) \, dy \leq \sup_{x \neq x'} \left( \frac{P_\lambda(x) - P_\lambda(x')}{|x - x'|^m} \right) \mathcal{H}^m(R \cap B), \quad B \subset \mathbb{R}^n \text{ Borel}. \tag{4.34}
\]
Since \( \eta_y^\lambda \) is concentrated on \( P_\lambda^{-1}(y) \) for \( P_\lambda \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \), we deduce that \( \eta_y^\lambda \mid R \) is concentrated on \( R \cap P_\lambda^{-1}(y) \) for \( P_\lambda \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \). From (4.34) we can thus write for \( L^1 \)-a.e. \( \lambda \in \Lambda \)
\[
\mu_\lambda \mid R = \eta_y^\lambda \mid (R \cap P_\lambda^{-1}(y)) \otimes P_\lambda \mu_\lambda \ll \mathcal{H}^m \mid (R \cap P_\lambda^{-1}(y)) \otimes P_\lambda \mu_\lambda \ll \mathcal{H}^m \mid R
\]
where we used the hypothesis \( P_\lambda \mu_\lambda \ll L^m \). Finally, since \( \mathcal{I}_m^m(E \setminus R) = 0 \), we can make use of formula (4.12) together with \( \mu_\lambda \mid R \ll \mathcal{H}^m \mid R \) to infer \( \mathcal{I}_p^m \ll \mathcal{I}_m^m \ll \mathcal{H}^m \mid R \).

**Remark 4.11 (Transversality in a metric setting).** In this remark we briefly discuss how transversality allows for a significant adaptability to non-euclidean settings. The notion of transversality was originally introduced as a sort of unifying criterion in studying the set of singular projections of a measure. More precisely, given a measure \( \mu \) on a compact metric space \( (X, d) \) and a transversal family of maps \( P_\lambda \colon X \to \mathbb{R}^m (\lambda \in \Lambda \subset \mathbb{R}^1) \), in [29] is estimated the Hausdorff dimension of parameters \( \lambda \) for
which $P_{\lambda}^{\mu}$ is singular with respect to the $m$-dimensional Hausdorff measure. Their approach covers in particular the case of orthogonal projections in relation with Falconer’s Theorem \cite{falconer_1974} as well as the case of Bernoulli convolutions. For instance, the latter case involves the study of parameters $\lambda$ for which the random series $\sum_{i=1}^{\infty} \pm \lambda^i$ has distribution $\nu_\lambda$ which is singular. In this situation, where $X = \{-1, 1\}^\mathbb{N}$, $m = l = 1$, and $P_{\lambda}(x) = \sum_{i=1}^{\infty} x_i \lambda^i$, transversality means exactly that the power series $P_{\lambda}(x) - P_{\lambda}(x')$ has no double zeros.

In a more geometric framework, the concept of transversality has been used for manifolds as well as for Riemann surfaces. Precisely, in \cite{tasso_2021} suitable 2-dimensional measures invariant under the geodesic flow are constructed on Riemann surfaces with constant negative curvature and with boundary. When $X$ is a generic Riemannian manifold, the codimension-one case of Theorem 1.4 has found application in \cite{tasso_2019}. In that paper it was shown the local existence of transversal families of maps whose level sets are supported on geodesics. This allows the authors to make use of the technique developed here and to infer the rectifiability of a certain class of measures related to the jump sets of one-dimensional slices of a measurable function.

More in general, the machinery developed in this paper can be adapted with minor changes to rather general metric spaces. For example the argument in Section 3 still works for measures which are Borel regular and gives finite mass to every bounded sets, while the technique of Section 4 are based on the disintegration theorem (Theorem 2.3) which works well for finite Borel measures on any separable Radon space \cite[Definition 5.14]{tasso_2020}. When trying to apply Theorem 1.4 in a non-euclidean framework, the most delicate part essentially reduces to the study of transversality. For this reason our approach may be potentially interesting also in metric settings.

4.1. A decomposition result for Radon measures.

As a consequence of our main result, we have the following decomposition result for general Radon measures of $\mathbb{R}^n$.

**Proposition 4.12.** Assume (4.1)-(4.2) and let $\mu$ be a finite Radon measure in $\mathbb{R}^n$. Then there exist $\mu_r, \mu_s$ Radon measures in $\mathbb{R}^n$ such that, for any $E \subset \mathbb{R}^n$ Borel satisfying

\[ \mathcal{H}^m(E \cap P^{-1}_{\lambda}(y)) < \infty, \] for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda$ and for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, \hspace{1cm} (4.35)

we have that $\mu_r \perp E$ is $m$-rectifiable, while

\[ P_{\lambda}^{\mu_r} \perp \mathcal{L}^m \text{ for } \mathcal{L}^l$-a.e. $\lambda \in \Lambda, \] \hspace{1cm} (4.36)

and $\mu$ is decomposed as $\mu(B) = \mu_r(B) + \mu_s(B)$ for every $B \subset \mathbb{R}^n$ Borel.

**Proof.** Denote by $S$ the family of (positive) Radon measures $\eta$ in $\mathbb{R}^n$ such that $|\mu - \eta|(\mathbb{R}^n) + \eta(\mathbb{R}^n) = \mu(\mathbb{R}^n)$, where $|\mu - \eta|$ denotes the total variation measure, and $\eta \perp E$ is $m$-rectifiable whenever $E \subset \mathbb{R}^n$ is Borel and satisfies hypothesis (4.35). Consider the minimum problem

\[ \min_{\eta \in S} |\mu - \eta|(\mathbb{R}^n). \] \hspace{1cm} (4.37)

Given a generic (positive) Radon measure $\eta$ notice that, since for every $B \subset \mathbb{R}^n$ Borel it holds true $|\mu - \eta|(B) + \eta(B) \geq \mu(B)$, condition $\eta \in S$ implies $|\mu - \eta|(B) + \eta(B) = \mu(B)$ for every $B \subset \mathbb{R}^n$ Borel. In particular, we deduce that $\eta \in S$ implies $|\mu - \eta| \leq \mu_s$, $\eta \leq \mu$ and from Radon-Nikodým’s Theorem there exists $f_\eta : \mathbb{R}^n \to [0,1]$ Borel such that $|\mu - \eta| = (1 - f_\eta) \mu$ and $\eta = f_\eta \mu$. Now let $(\eta_i)$ be a minimizing sequence for (4.37) and let $f_\eta_i : \mathbb{R}^n \to [0,1]$ be such that $\eta_i = f_\eta_i \mu$ for $i = 1, 2, \ldots$. Define $f_1 : \mathbb{R}^n \to [0,1]$ as $f_1 := \sup_{1 \leq j \leq i} f_{\eta_j}$ and notice that the Radon measure $\mu_i := f_i \mu$ satisfies

\[ |\mu - \mu_i|(B) + \mu_i(B) = \mu(B) \quad \text{and} \quad |\mu - \mu_i|(\mathbb{R}^n) \leq |\mu - \eta_i|(\mathbb{R}^n), \]

whenever $B \subset \mathbb{R}^n$ is Borel and $i = 1, 2, \ldots$. Since $\mu_i \leq \sum_{1 \leq j \leq i} \eta_j$ and by assumption each $\eta_j \perp E$ is $m$-rectifiable, Radon-Nikodým’s theorem tells us that the same holds true for $\mu_i \perp E$. Hence $\mu_i$ is still a minimizing sequence for (4.37). Moreover, since $f_1 \not\sim f$ for some $f : \mathbb{R}^n \to [0,1]$ Borel, we can make use of Lebesgue’s dominated convergence theorem to infer $|\mu_i - \mu_i|(\mathbb{R}^n) \to 0$ as $i \to \infty$, where $\mu_r := f \mu$. Thanks to the strong convergence we easily deduce

\[ |\mu - \mu_r|(B) + \mu_r(B) = \mu(B) \quad \text{and} \quad |\mu - \mu_r|(\mathbb{R}^n) \leq |\mu - \eta|(\mathbb{R}^n), \]

and also the $m$-rectifiability $\mu_r \perp E$. We deduce that $\mu_r$ is a minimizer for (4.37). Therefore, if we define $\mu_s := \mu - \mu_r$ since $\mu_s \in S$ implies $\mu(B) = \mu_s(B) + \mu_r(B)$ for every $B \subset \mathbb{R}^n$ Borel, it remains only to prove that $\mu_s$ satisfies condition (4.36). We argue by contradiction and assume that $P_{\lambda}^{\mu_s}$ is not singular with respect to $\mathcal{L}^m$ for every $\lambda$ belonging to a set $\Lambda' \subset \Lambda$ with strictly positive
\(L^1\)-measure. Disintegration theorem gives for every \(\lambda \in \Lambda\) a family of Radon measures \((\eta^\lambda_y)_{y \in \mathbb{R}^m}\) such that
\[
\mu_s(B) = \int_{\mathbb{R}^m} \eta^\lambda_y(B) d[P_{M^s}](y) + \int_{\mathbb{R}^m} \eta^\lambda_y(B) d[P_{M^s}](y), \quad B \in \mathbb{R}^n \text{ Borel},
\]
where \([P_{M^s}](y)\) and \([P_{M^s}](y)\) denote the absolutely continuous part and the singular part of \(P_{M^s}\) with respect to \(\mathbb{L}^m\), respectively. Notice that, by exploiting that the two measures in the right-hand side of the previous equality are mutually orthogonal, it can be proved that the following maps are \(L^1\)-measurable whenever \(B \subset \mathbb{R}^n\) is Borel
\[
g^\lambda_y(\lambda) := \int_{\mathbb{R}^m} \eta^\lambda_y(B) d[P_{M^s}](y) \quad \text{and} \quad g^\lambda_y(\lambda) := \int_{\mathbb{R}^m} \eta^\lambda_y(B) d[P_{V^s}](y).
\]
Integrating both sides of (4.38) with respect to \(L^1\) we get two finite Radon measures in \(\mathbb{R}^n\), say \(\eta^a, \eta^s\), such that
\[
\mu_s(B) = \eta^a(B) + \eta^s(B), \quad B \subset \mathbb{R}^n \text{ Borel.}
\]

Thanks to our contradiction argument and the fact that disintegration satisfies \(\eta^\lambda_y \neq 0\) for \(P_{M^s}\)-a.e. \(y \in \mathbb{R}^m\), we deduce that \(\eta^a \neq 0\). By virtue of (4.39), if we prove that \(\eta^a \cap E\) is \(m\)-rectifiable for every set \(E \subset \mathbb{R}^n\) Borel and satisfying (4.39), the minimality of \(\mu_s\) would give a contradiction with \(\eta^a \neq 0\). To this regard, we consider the family of Borel measures \((\mu^\lambda)_{\lambda \in \Lambda}\), defined as
\[
\mu^\lambda(B) := g^\lambda_y(B), \quad B \subset \mathbb{R}^n \text{ Borel},
\]
and for \(p \geq 1\) the Borel regular measures \(\mathcal{H}^m_p\) according to (1.5). We claim that \(\mathcal{H}^m\) is a finite integralgeometric measure. Properties (4.4)-(4.6) follow by construction. The finiteness is a consequence of the following inequality
\[
\|g^\lambda_y(B)\|_{L^\infty(\Lambda)} = \text{ess sup} \sup_{\Lambda} \int_{\mathbb{R}^m} \eta^\lambda_y(E \cap B) d[P_{M^s}](y) \leq \mu_s(B) \leq \mu(B),
\]
whenever \(B \subset \mathbb{R}^n\) is Borel. Indeed, given \(\delta > 0\), we can consider a countable Borel covering \((B_i)\) of \(\mathbb{R}^n\) made of sets having diameter less than \(\delta\) and such that for every \(i = 1, 2, \ldots\) it holds true \(#\{B_j \mid B_j \cap B_i \neq \emptyset\} \leq N\) for some constant \(N\) (for example the closed cubes belonging to a \(n\)-dimensional \(\delta\)-grid with size parallel to the coordinate axis). Using (4.40) we have thus
\[
\sum_{i=1}^{\infty} \|g^\lambda_y(B)\|_{L^\infty(\Lambda)} \leq N \mu(\mathbb{R}^n).
\]
This last inequality together with the arbitrariness of \(\delta\) give the finiteness of \(\mathcal{H}^m\). Theorem 4.4 tells us that \(\mathcal{H}^m\) is \(m\)-rectifiable and we already know from Proposition 4.7 that also \(\mathcal{H}^m_1\) is \(m\)-rectifiable. Finally, we use formula (4.12) to infer for every \(B \subset \mathbb{R}^n\) Borel
\[
\mathcal{H}^m_1(B) = \int_{\Lambda} \mu^\lambda(E \cap B) d\lambda = \int_{\Lambda} \left(\int_{\mathbb{R}^m} \eta^\lambda_y(E \cap B) d[P_{M^s}](y)\right) d\lambda
\]
This gives that \(\eta^a \cap E\) is \(m\)-rectifiable and the desired contradiction follows.

**Remark 4.13.** The decomposition contained in Proposition 4.12 can be seen as a relaxed version for finite Radon measures of Besicovitch-Federer’s structure theorem. In order to make a parallel, we restrict our attention to the case of \(\mu\) concentrated on a Borel set \(E\) satisfying (4.35). In this situation, the result contained in [17, Theorem 3.3.12] asserts the existence of a decomposition of \(E\) of the form \(E = R \cup E', \) where \(R \subset \mathbb{R}^n\) is countably \(m\)-rectifiable and the (purely unrectifiable) remaining part \(E' \setminus R\) has the null-projection property
\[
\mathcal{H}^m(\pi_V(E \setminus R)) = 0, \quad E = r_{m,n}, \text{a.e.} \quad V \in \text{Gr}(n, m),
\]
provided the measure \(\mu\) satisfies the additional condition: for \(r_{m,n}, \text{a.e.} \quad V \in \text{Gr}(n, m)
\]
Under the same hypothesis on \(E\), Proposition 4.12 provides a unique decomposition of \(\mu\) of the form \(\mu = \mu_s + \mu_r\), where \(\mu_s\) is \(m\)-rectifiable and the remaining singular part \(\mu_r = \mu - \mu_s\) satisfies the singular-pushforward property with no additional requirements on \(\mu\). For this reason, the null pushforward property can be seen as a relaxation of the null-projection property. For instance, in
Remark 5.2 we point out an example of a compact set $E \subset \mathbb{R}^2$ having finite and strictly positive $I_1^1$-measure, hence not satisfying (4.41), but such that $I_1^1 \setminus E = \mu_e$.

In the case of a generic Radon measure the above decomposition may not be unique. The rectifiable restriction property of $\mu_e$ is thus meaningful whenever $\mu$ gives mass to some set $E$ satisfying condition (4.35). For instance, when $\mu$ is the total variation of the gradient of a BV-function [3] or the symmetric gradient of a BD-vectorfield [2] (see also [8] for the general case of functions having bounded $A$-variation) the set $E = \Theta_{\mu}^m := \{ x \in \mathbb{R}^n \mid \Theta_{\mu}^m(\mu, x) > 0 \}$ with $m = n - 1$ is of significance in connection with the study of jump type discontinuity of a function. For general codimension, we refer to the case of $\mu$ coinciding with the mass measure of a real flat chain [16] and $E = \Theta_{\mu}^m$. In fact, the common feature underlying the above mentioned spaces is that slicing techniques can be applied. As a consequence, it can be proved that the pushforward of $\mu$ on a.e. $m$-plane $V$ is absolutely continuous with respect to $\mathcal{H}^m$ which implies $\mu_e = 0$. Property (4.35) for $\Theta_{\mu}^m$ is in general achieved from the density estimate of Radon measures. In this case $\Theta_{\mu}^m$ is actually $\sigma$-finite with respect to $\mathcal{H}^m$ and Besicovitch-Federer’s structure theorem applies. Nevertheless, we notice that the rectifiable restriction property of $\mu_e$ is possibly stronger, since sets satisfying (4.35) are not necessarily $\sigma$-finite with respect to $\mathcal{H}^m$.

5. Structure of Federer’s Integralgeometric measure

For convenience of the reader we recall here the relevant notation. We denote by $I_p^m$ the $m$-dimensional Integralgeometric measure in $\mathbb{R}^n$ with exponent $p \geq 1$. Letting for every $V \in \text{Gr}(n, m)$ and every Borel set $B \subset \mathbb{R}^n$

$$g_B(V) := \mathcal{H}^m(\pi_V(B)),$$

we define the set function

$$\eta_p(B) := \|g_B\|_{L^p(\text{Gr}(n, m))}.$$  

The measure $I_p^m$ is thus defined via Carathéodory’s construction as in (1.5).

We want to rewrite $I_p^m$ as an integralgeometric measure $J_p^m$. This does not come directly from its definition because of the fact that $B \mapsto g_B(V)$ is not a Borel measure. In order to overcome this issue we proceed as follows. Define for every $V \in \text{Gr}(n, m)$ and for every $B \subset \mathbb{R}^n$ Borel, the Borel measure $\mu_V$ on $\mathbb{R}^n$ by

$$\mu_V(B) := \int_{\mathbb{R}^n} \mathcal{H}^0(B \cap \pi_V^{-1}(y)) \, dy.$$  

Define for every $B' \subset \text{Gr}(n, m) \times \mathbb{R}^n$ Borel, the set function $\tilde{\zeta}$ by

$$\tilde{\zeta}(B') := \int_{\text{Gr}(n, m)} \mu_V(B' \cap \pi_{B'}^{-1}(y)) \, d\gamma_{n,m},$$

where $B' := \{ x \in \mathbb{R}^n \mid (V, x) \in B' \}$. Letting $f_B(V) := \mu_V(B)$ for every $V \in \text{Gr}(n, m)$ and $B \subset \mathbb{R}^n$ Borel we define the set function

$$\zeta_p(B) := \|f_B\|_{L^p(\text{Gr}(n, m))}.$$  

The required measurability to define the integral in (5.3) can be obtained as follows. We consider for every $k = 1, 2, \ldots$ a countable Borel partition $D_k$ of $B$ satisfying $\text{diam}(D) \leq k^{-1}$ for every $D \in D_k$. Then we observe the pointwise convergences

$$\lim_k \# \{ D \in D_k \mid \pi_V^{-1}(y) \cap D \neq \emptyset \} = \mathcal{H}^0(B \cap \pi_V^{-1}(y)), \quad \text{for } y \in V$$

$$\lim_k \sum_{D \in D_k} \mathcal{H}^m(\pi_V(D)) = \int_V \# \{ D \in D \mid \pi_V^{-1}(y) \cap D \neq \emptyset \} \, d\mathcal{H}^m(y), \quad \text{for } V \in \text{Gr}(n, m),$$

where in the second limit we made use of the measurable projection theorem [9, Proposition 8.4.4] to deduce the $\mathcal{H}^m$-measurability of $\pi_V(D)$. By virtue of these two convergences and by applying Fatou’s lemma we can write for $V \in \text{Gr}(n, m)$

$$\int_V \mathcal{H}^0(B \cap \pi_V^{-1}(y)) \, d\mathcal{H}^m(y) = \sum_{D \in D_k} \int_V \mathcal{H}^0(D \cap \pi_V^{-1}(y)) \, d\mathcal{H}^m(y)$$

$$\geq \limsup_k \sum_{D \in D_k} \mathcal{H}^m(\pi_V(D)) \geq \liminf_k \sum_{D \in D_k} \mathcal{H}^m(\pi_V(D))$$

$$= \liminf_k \int_V \mathcal{H}^0(\{ D \in D_k \mid \pi_V^{-1}(y) \cap D \neq \emptyset \}) \, d\mathcal{H}^m(y)$$

$$\geq \liminf_k \int_V \mathcal{H}^0(D \cap \pi_V^{-1}(y)) \, d\mathcal{H}^m(y)$$
\[ \geq \int_V \liminf_k \mathcal{H}^0(\{D \in D_k \mid \pi^{-1}_V(y) \cap D \neq \emptyset\}) \, d\mathcal{H}^m(y) \]
\[ = \int_V \mathcal{H}^0(B \cap \pi^{-1}_V(y)) \, d\mathcal{H}^m(y), \]

which immediately implies
\[ \lim_k \sum_{D \in D_k} \mathcal{H}^m(\pi_V(D)) = \int_V \mathcal{H}^0(B \cap \pi^{-1}_V(y)) \, d\mathcal{H}^m(y), \quad \text{for } V \in \mathrm{Gr}(n,m). \]

Hence, the \( \gamma_{n,m} \)-measurability of \( \mathrm{Gr}(n,m) \supset V \mapsto \mathcal{H}^m(\pi_V(D)) \) for every \( D \subset \mathbb{R}^n \) Borel \( [17, \, 2.10.5] \) implies that of \( \mathrm{Gr}(n,m) \supset V \mapsto \int_V \mathcal{H}^0(B \cap \pi^{-1}_V(y)) \, d\mathcal{H}^m(y) \).

Consider now the Borel regular measure (see Proposition 4.2) \( \hat{\mathcal{S}}_m \) on \( \mathrm{Gr}(n,m) \times \mathbb{R}^n \) defined as in \( (4.8) \) for every \( A \subset \mathrm{Gr}(n,m) \times \mathbb{R}^n \) by
\[ \hat{\mathcal{S}}_m(A) := \sup_{\delta > 0} \left\{ \sum_{B' \in G'_\delta} \hat{\zeta}(B') \mid G'_\delta \text{ countable Borel cover of } A, \, \text{diam}(B') \leq \delta \right\}, \quad (5.6) \]
and the Borel regular measure \( \mathcal{S}_p^m \) on \( \mathbb{R}^n \) (see Proposition 4.2) defined as in \( (1.5) \) for every \( E \subset \mathbb{R}^n \) by
\[ \mathcal{S}_p^m(E) := \sup_{\delta > 0} \left\{ \sum_{B \in G_\delta} \zeta_p(B) \mid G_\delta \text{ countable Borel cover of } A, \, \text{diam}(B) \leq \delta \right\}. \quad (5.7) \]

We have the following lemma.

**Lemma 5.1.** For every \( p \geq 1 \) we have \( T^m_p = \mathcal{S}_p^m \).

**Proof.** Being both measures \( T^m_p \) and \( \mathcal{S}_p^m \) Borel regular it is enough to prove that they coincide on Borel sets. In addition, since \( \eta_p(B) \leq \zeta_p(B) \) for every \( B \subset \mathbb{R}^n \) Borel we have only to prove that
\[ \mathcal{T}^m_p(B) \geq \mathcal{S}_p^m(B), \quad B \subset \mathbb{R}^n \text{ Borel.} \quad (5.8) \]

For this purpose fix \( B \subset \mathbb{R}^n \) Borel and assume without loss of generality that \( \mathcal{T}^m_p(B) < \infty \). Consider for \( k = 1, 2, \ldots \) a sequence of Borel countable coverings of \( B \), say \( (D_k) \), such that \( D \in D_k \) implies \( \text{diam}(D) \leq 1/k \) and \( \lim_{k \to \infty} \sum_{D \in D_k} \eta_p(D) = \mathcal{T}^m_p(B) \). Notice that for every \( k \)
\[ \sum_{D \in D_k} \eta_p(D) = \sum_{D \in D_k} \left( \int_{\mathrm{Gr}(n,m)} \mathcal{H}^m(\pi_V(D)) \, d\gamma_{n,m} \right)^{\frac{1}{p}} \]
\[ \geq \left( \int_{\mathrm{Gr}(n,m)} \left\{ \sum_{D \in D_k} \mathcal{H}^m(\pi_V(D)) \right\}^p \, d\gamma_{n,m} \right)^{\frac{1}{p}} \]
\[ = \left( \int_{\mathrm{Gr}(n,m)} \left( \int_V \mathcal{H}^0(\{D \in D_k \mid \pi^{-1}_V(y) \cap D \neq \emptyset\}) \, d\mathcal{H}^m(y) \right)^p \, d\gamma_{n,m} \right)^{\frac{1}{p}}. \]

Since every \( D_k \) is made of sets whose diameter is shrinking to zero as \( k \to \infty \), we have
\[ \liminf_{k \to \infty} \mathcal{H}^0(\{D \in D_k \mid \pi^{-1}_V(y) \cap D \neq \emptyset\}) \geq \mathcal{H}^0(\pi^{-1}_V(y) \cap B), \]
for every \( V \in \mathrm{Gr}(n,m) \) and \( y \in V \). Therefore we can apply Fatou’s lemma to deduce
\[ \mathcal{T}^m_p(B) = \liminf_{k \to \infty} \sum_{D \in D_k} \eta_p(D) \]
\[ \geq \left( \int_{\mathrm{Gr}(n,m)} \left( \int_V \liminf_{k \to \infty} \mathcal{H}^0(\{D \in D_k \mid \pi^{-1}_V(y) \cap D \neq \emptyset\}) \, d\mathcal{H}^m(y) \right)^p \, d\gamma_{n,m} \right)^{\frac{1}{p}} \]
\[ \geq \left( \int_{\mathbb{R}^n} \mathcal{H}^0(\pi^{-1}_V(y) \cap B) \, d\mathcal{H}^m(y) \right)^p \, d\gamma_{n,m} \right)^{\frac{1}{p}} = \zeta_p(B). \quad (5.9) \]

For every \( \delta > 0 \), since \( \mathcal{T}^m_p \) is a Borel measure, we easily find a countable Borel covering of \( B \), say \( G_\delta \), with \( \text{diam}(B) \leq \delta \) for \( \hat{B} \in G_\delta \) and
\[ \sum_{B \in G_\delta} \mathcal{T}^m_p(B) = \mathcal{T}^m_p(B) \]. Thanks to \( (5.9) \) we can write for every \( \delta > 0 \)
\[ \mathcal{T}^m_p(B) = \sum_{B \in G_\delta} \mathcal{T}^m_p(B) \geq \sum_{B \in G_\delta} \zeta_p(\hat{B}) \geq \inf_{G_\delta} \sum_{B \in G_\delta} \zeta_p(\hat{B}) \geq \mathcal{S}_p^m(B) - o(1). \]
where $G_\delta$ denotes any countable Borel coverings of $B$ made of sets having diameter less or equal than $\delta$ and $o(1) \to 0$ as $\delta \to 0^+$. Taking the limit as $\delta \to 0^+$ on both sides of the previous inequality we obtain (5.8).

We are now in position to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Thanks to Lemma 5.1 we know that $T^m_\rho \cap E$ coincides with $S^m_\rho \cap E$ where $S^m_\rho$ is the measure defined in (5.7). Moreover, from the Borel regularity of $S^m_\rho$, we may assume that $S^m_\rho$ is Borel. Finally, since $S^m_\rho \cap E$ is a finite integral geometric measure by construction we obtain the desired conclusion by applying Theorem 1.4 together with Remark 3.7.

**Proof of Theorem 1.2.** Since $T^m_\rho$ is Borel regular we may assume that $E$ is Borel. We apply the decomposition result of Theorem 4.12 to obtain a decomposition of the form $T^m_\rho \cap E = \mu_\rho + \mu_s$ for some Radon measures $\mu_\rho, \mu_s$ such that $\mu_\rho \cap B$ is $m$-rectifiable whenever $B$ has the geometric property (4.35) and $\mu_s$ satisfies (1.3). Since $T^m_\rho(E) < \infty$ we know that the set $E$ enjoys property (4.35) and hence we can infer the $m$-rectifiability of $\mu_\rho = \mu_\rho \cap E$ and the validity of (1.3) immediately follows.

In addition suppose by contradiction that there exist $\mu_\rho^1 \neq \mu_\rho^2$ both satisfying (5.2). Since the singular-pushforward is a linear condition on the vector space of signed Radon measures, we deduce that $\mu_\rho^1 - \mu_\rho^2$ is non-null and satisfies (5.2). But this gives a contradiction with the fact that both measures $\mu_\rho^1, \mu_\rho^2$ are $m$-rectifiable. The uniqueness follows.

We conclude this section with a remark.

**Remark 5.2 (Null-projection vs singular-pushforward property).** Let $E \subset \mathbb{R}^2$ be the compact set given in [23]. Setting $\mu := T^1_\rho \cap E$, we notice that the zero density property (1.7) forces $\mu = 0$. This provides therefore an example of a compact set $E$ having finite Integral geometric measure and not satisfying the null-projection property, but fulfilling the singular-pushforward property.

## 6. Rectifiability via slicing

For convenience of the reader we list the basic conditions that we tacitly assume throughout this section.

\begin{align}
\Lambda &\subset \mathbb{R}^l \text{ open and bounded and } n \geq 1, \ 0 \leq m \leq n, \ m \leq l \quad (6.1) \\
(P_\lambda)_{\lambda \in \Lambda} &\text{ transversal family of maps in the sense of Definition 3.3} \quad (6.2) \\
\lambda \mapsto \mu_\lambda(A_\lambda) &\text{ is } L^1 \text{-measurable for every } A \subset \mathbb{R}^n \times \Lambda \text{ Borel} \quad (6.3) \\
P_{\lambda} \mu_\lambda &\ll L^m \text{ for } L^1 \text{-a.e. } \lambda \in \Lambda. \quad (6.4)
\end{align}

Notice that condition (6.3) tells us that the further properties contained in Remarks 4.4 and 4.9 hold true.

The finiteness of $S^m_\rho$ for some $p > 1$ tells us that, when studying structure property of $S^m_\rho$, we can reduce ourselves to work on a particular subset of $\mathbb{R}^n$. This is the content of the next lemma.

**Lemma 6.1.** Assume (6.1)-(6.4). Let $S^m_\rho$ be given by (1.5) and be finite for some $p > 1$, and let $(\eta_x)_{x \in \mathbb{R}^n}$ and $(\eta^y_x)_{y \in \mathbb{R}^m}$ be the family of probability measures satisfying (4.30) and (4.27), respectively. Assuming that

1. $\eta^y_x$ is $0$-rectifiable for $L^1$-a.e. $\lambda \in \Lambda$ and for $P_\lambda \mu_\lambda$-a.e. $y \in \mathbb{R}^m$,

then there exists a Borel function $f : \mathbb{R}^n \times \Lambda \to \mathbb{R}$ such that for $L^1$-a.e. $\lambda \in \Lambda$

\begin{equation}
J^m_\rho((x \in \mathbb{R}^n \mid f_x(\lambda) > 0 \text{ and } \eta^x_{P_\lambda y}(|\{x\}|) = 0)) = 0. \quad (6.5)
\end{equation}

**Proof.** In order to simplify the notation we let $f_x(\lambda) := f(x, \lambda)$. Using (4.17) and the disintegration $J^m_\rho = \eta_x \otimes P_\lambda \mu_\lambda$, we can write for every $\lambda \in \Lambda$, every $x > 0$, and every non-negative $\varphi \in C_0^\infty(\mathbb{R}^n)$

\begin{equation}
\frac{1}{L^1(B_r(\lambda))} \int_{\mathbb{R}^n} \eta_x(B_r(\lambda)) \varphi(x) dJ^m_\rho(x) = \frac{1}{L^1(B_r(\lambda))} \int_{B_r(\lambda)} \left( \int_{\mathbb{R}^n} \varphi(x) d\mu_N(x) \right) d\lambda. \quad (6.6)
\end{equation}

From the following two properties

\begin{align}
\eta_x(B_r(\lambda)) &\text{ is Borel for every } \lambda \in \Lambda \text{ (see (2.4))} \\
\lambda \mapsto \eta_x(B_r(\lambda)) &\text{ is continuous for every } x \in \mathbb{R}^n,
\end{align}

...
we infer that \((x, \lambda) \mapsto \eta_x(B_r(\lambda))\) is Borel on the product space \(\mathbb{R}^n \times \Lambda\). By using also the continuity of \(r \mapsto \eta_x(B_r(\lambda)) / \mathcal{L}^n(B_r(\lambda))\) \((r > 0)\) we deduce that the following set is Borel

\[
\left\{(x, \lambda) \in \mathbb{R}^n \times \Lambda \mid \limsup_{r \to 0^+} \frac{\eta_x(B_r(\lambda))}{\mathcal{L}^n(B_r(\lambda))} = \liminf_{r \to 0^+} \frac{\eta_x(B_r(\lambda))}{\mathcal{L}^n(B_r(\lambda))}\right\}. \tag{6.7}
\]

Moreover, since from Lebesgue’s differentiation theorem we know that for every \(x \in \mathbb{R}^n\) the equality in (6.7) is satisfied for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda\), we apply Tonelli’s theorem with product measure \(\mathcal{F}^m \otimes \mathcal{L}^1\) to infer that for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda\) the equality in (6.7) is satisfied for \(\mathcal{F}^m\)-a.e. \(x \in \mathbb{R}^n\). The same argument gives also a Borel function \(f : \mathbb{R}^n \times \Lambda \to \mathbb{R}\) such that

\[
f_x(\lambda) := f(x, \lambda) = \lim_{r \to 0^+} \frac{\eta_x(B_r(\lambda))}{\mathcal{L}^n(B_r(\lambda))}, \quad \text{for } (\mathcal{F}^m \otimes \mathcal{L}^1)\text{-a.e. } (x, \lambda) \in \Lambda \times \mathbb{R}^n.
\]

Therefore, if we consider a countable dense subset \(D\) of the set of all non-negative functions in \(C_0^\infty(\mathbb{R}^n)\), we can take both sides of (6.6) the limit as \(r \to 0^+\) and apply Fatou’s lemma in the left-hand side to infer for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda\)

\[
\int_{\mathbb{R}^n} f_x(\lambda) \varphi(x) \, d\mathcal{F}^m(x) \leq \int_{\mathbb{R}^n} \varphi(x) d\mu_\lambda(x), \quad \text{for } \varphi \in D,
\]

from which we easily obtain for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda\)

\[
\int_B f_x(\lambda) \, d\mathcal{F}^m(x) \leq \mu^B_\lambda(B) = \int_{\mathbb{R}^m} \eta_y^B(B) \, dP_{\mathcal{M}^\lambda}(y) \quad \text{for } B \subset \mathbb{R}^n \text{ Borel}. \tag{6.8}
\]

If we assume that (6.5) does not hold true for a subset of \(\Lambda\) with strictly positive measure, then we find \(\lambda_0 \in \Lambda\) such that (6.5) is false, \(\eta^\lambda_0\) is atomic for \(P_{\mathcal{M}^\lambda}\)-a.e. \(y \in \mathbb{R}^m\), \(f(\cdot, \lambda_0)\) is Borel, and (6.8) holds true. By replacing the set \(B\) in (6.8) with the Borel set defined in (6.5) for \(\lambda = \lambda_0\) we reach a contradiction thanks to the fact that \(\eta^\lambda_0\) are atomic for \(P_{\mathcal{M}^\lambda}\)-a.e. \(y \in \mathbb{R}^m\). Condition (6.5) is thus proved.

It is a well-known fact that \(\sigma\)-finiteness of measures with respect to \(\mathcal{H}^m\) can be deduced by studying their \(m\)-dimensional upper densities. In analogy, the next theorem provides a criterion for a family of measures \((\mu_\lambda)_{\lambda}\) giving sufficient conditions for which \(\int_{\Lambda} \mu_\lambda \, d\lambda\) concentrates on a \(\sigma\)-finite set with respect to condition (4.6).

**Theorem 6.2.** Assume (6.1)-(6.4). Let \(\Lambda' \subset \Lambda\) be Borel and let \(\mathcal{F}^m_p\) be given by (1.5) and be finite for some \(p \in (1, \infty]\). Consider \((\eta_\lambda)\) and \((\eta^\lambda_p)\) the families of probability measures satisfying (4.30) and (4.27), respectively, and suppose in addition that

1. \(\mathcal{F}^m \ll \mu_\lambda\) for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda'\) and \(\mu_\lambda \gg 0\) for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda \setminus \Lambda'\).
2. \(\eta^\lambda_p\) is 0-rectifiable for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda\) and \(P_{\mathcal{M}^\lambda}\)-a.e. \(y \in \mathbb{R}^m\).

Then there exists a Borel set \(E \subset \mathbb{R}^n\) satisfying (4.5)-(4.6).

**Proof.** Define the Borel set \(A \subset \mathbb{R}^n \times \Lambda\) as \(A := \{(x, \lambda) \in \mathbb{R}^n \times \Lambda \mid f(x, \lambda) > 0\}\), where \(f : \mathbb{R}^n \times \Lambda \to [0, \infty)\) is the Borel function of Lemma 6.1. Thanks to hypothesis (1), we can apply Radon-Nikodym’s theorem to infer that for \(\mathcal{F}^m\)-a.e. \(\lambda \in \Lambda'\)

\[
\lim_{r \to 0^+} \frac{\mu_\lambda(\overline{B_r(x)})}{\mathcal{F}^m(\overline{B_r(x)})} > 0, \quad \text{for } \mathcal{F}^m\text{-a.e. } x \in \mathbb{R}^n,
\]

and for \(\mathcal{L}^1\)-a.e. \(\lambda \in \Lambda \setminus \Lambda'\)

\[
\lim_{r \to 0^+} \frac{\mu_\lambda(\overline{B_r(x)})}{\mathcal{F}^m(\overline{B_r(x)})} = 0, \quad \text{for } \mathcal{F}^m\text{-a.e. } x \in \mathbb{R}^n. \tag{6.9}
\]

Suppose for a moment that we already know that the following sets are Borel

\[
\{(x, \lambda) \in \mathbb{R}^n \times \Lambda \mid \lim_{r \to 0^+} \frac{\mu_\lambda(\overline{B_r(x)})}{\mathcal{F}^m(\overline{B_r(x)})} > 0\}\tag{6.11}
\]

\[
\{(x, \lambda) \in \mathbb{R}^n \times \Lambda \mid \lim_{r \to 0^+} \frac{\mu_\lambda(\overline{B_r(x)})}{\mathcal{F}^m(\overline{B_r(x)})} = 0\}. \tag{6.12}
\]
In this case we can apply Tonelli’s theorem with product measure $\mathcal{F}_m^\Lambda \otimes \mathcal{L}^l$ to deduce from (6.9) and (6.10) that for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$

$$g(x, \lambda) := \lim_{r \to 0^+} \frac{\mu_L(B_r(x))}{\mathcal{F}_m(B_r(x))} > 0, \text{ for } \mathcal{L}^l$$.a.e. $\lambda \in \Lambda',$$

and

$$g(x, \lambda) := \lim_{r \to 0^+} \frac{\mu_L(B_r(x))}{\mathcal{F}_m(B_r(x))} = 0, \text{ for } \mathcal{L}^l$$.a.e. $\lambda \in \Lambda \setminus \Lambda'.$

If we set $f_r(x, \lambda) := \frac{\mu_L(B_r(x))}{\mathcal{F}_m(B_r(x))}$ we know from Proposition 4.8 (see also Remark 4.9) that for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$ it holds true

$$f_r(x, \cdot) \to f(x, \lambda), \text{ weakly in } L^p(\Lambda) \text{ as } i \to \infty,$$

for a suitable subsequence (depending on $x$) $r_i \to 0^+$ as $i \to \infty$. The pointwise convergences (6.13)-(6.14) together with the weak convergence (6.15) imply that for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$ we have

$$f(x, \lambda) = g(x, \lambda) \text{ for } \mathcal{L}^l$$.a.e. $\lambda \in \Lambda$.

In particular we deduce that for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$

$$f(x, \lambda) > 0, \text{ for } \mathcal{L}^l$$.a.e. $\lambda \in \Lambda'$

$$f(x, \lambda) = 0, \text{ for } \mathcal{L}^l$$.a.e. $\lambda \in \Lambda \setminus \Lambda'.$

But this means that the Borel set $A$ defined above is vertical. Precisely, defining the Borel set $E := \{x \in \mathbb{R}^n | \mathcal{L}^l(A_x \Delta \Lambda') = 0\}$, we have

$$\mathcal{F}_m^\Lambda \otimes \mathcal{L}^l(A \Delta (E \times \Lambda')) = 0.$$  

Moreover, by disintegrating $\hat{\mathcal{F}}_m = \eta_x \otimes \mathcal{F}_m^\Lambda$ we know from Proposition 4.8 that $\eta_x \leq f(x, \cdot)$ for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$. Therefore the measure $\mathcal{F}_m$ is concentrated on $\Lambda$. In addition, since condition (6.16) gives $\mathcal{L}^l(A_x \Delta \Lambda') = 0$ for $\mathcal{F}_m^\Lambda$-a.e. $x \in \mathbb{R}^n$, we deduce from $\eta_x \ll \mathcal{L}^l$ that $\hat{\mathcal{F}}_m(A \Delta (E \times \Lambda')) = 0$. We can thus infer

$$\hat{\mathcal{F}}_m((\mathbb{R}^n \setminus E) \times \Lambda') = \mathcal{F}_m \downarrow A((\mathbb{R}^n \setminus E) \times \Lambda') \leq \hat{\mathcal{F}}_m(A \setminus (E \times \Lambda')) = 0.$$  

By virtue of Proposition 4.7 we have thus $\mu_L(\mathbb{R}^n \setminus E) = 0$ for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda'$. Since $\mu_L = 0$ for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda \setminus \Lambda'$, we have as well $\mu_L(\mathbb{R}^n \setminus E) = 0$ for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda$. This gives (4.5).

To prove (4.6) we first notice that (6.16) together with Tonelli’s theorem tell us that $\mathcal{F}_m^\Lambda(A \Delta E) = 0$ for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda'$. Therefore, by virtue of Lemma 6.1 and Proposition 4.7, we deduce that for $\mathcal{L}^l$-a.e. $\lambda \in \Lambda'$

$$\mathcal{F}_m^\Lambda(\{x \in E | \eta_{P_x}(\{x\}) = 0\}) = 0.$$  

Now suppose for a moment that we can find a family of measures $(\tilde{\eta}^\Lambda_\beta)$ satisfying

$$(y, \lambda) \mapsto \tilde{\eta}^\Lambda_\beta(B) \text{ is Borel, for every } B \subset \mathbb{R}^n \text{ Borel}.$$  

Define the $S := \{(x, \lambda) \in \mathbb{R}^n \times \Lambda' | \tilde{\eta}^\Lambda_\beta(\{x\}) = 0\}$ and suppose for a moment that it is Borel measurable. Since (6.18) together with the Borel measurability of $S$ allows us to make use of Tonelli’s theorem to infer $\mathcal{F}_m^\Lambda \otimes \mathcal{L}^l(S) = 0$, we apply Proposition 4.10 to obtain

$$P_{\Omega \setminus \Lambda}(S \cap \Lambda) = 0,$$  

On the other hand, from the definition of $S$ we have for every $\lambda \in \Lambda'$ the inclusion

$$E \setminus S_{\lambda} \subset \{x \in \mathbb{R}^n | \tilde{\eta}^\Lambda_\beta(\{x\}) > 0\}.$$  

Using hypothesis (2) and (6.19) we deduce from (6.22)

$$\mathcal{H}^d((E \setminus S_{\lambda}) \cap P^{-1} \lambda(y)) < \infty, \text{ for } \mathcal{L}^d \text{-a.e. } \lambda \in \Lambda' \text{ and } P_{\Omega \setminus \Lambda} \text{-a.e. } y \in \mathbb{R}^m.$$  

Since $\mu_L = 0$ for $\mathcal{L}^d$-a.e. $\lambda \in \Lambda \setminus \Lambda'$ we can trivially extend property (6.23) to the whole of $\Lambda$, namely,

$$\mathcal{H}^d((E \setminus S_{\lambda}) \cap P^{-1} \lambda(y)) < \infty, \text{ for } \mathcal{L}^d \text{-a.e. } \lambda \in \Lambda \text{ and } P_{\Omega \setminus \Lambda} \text{-a.e. } y \in \mathbb{R}^m.$$  

Putting together (6.21) with (6.24) we finally obtain (4.6).

It remains to prove that the sets in (6.11)-(6.12) are Borel, the existence of a family $(\tilde{\eta}^\Lambda_\beta)$ satisfying (6.19)-(6.20), and that the set $S$ is Borel.
We start by showing that (6.11)-(6.12) are Borel. By Exploiting the right continuity of both maps \( r \mapsto \mu_\lambda(\overline{B}_r(x)) \) and \( r \mapsto \mathcal{J}_r^m(\overline{B}_r(x)) \) we have that the couple \((x, \lambda)\) satisfies condition (6.11) if and only if
\[
\lim_{i \to \infty} \sup_{r \in (0,1/r) \cap \mathbb{Q}} \frac{\mu_\lambda(\overline{B}_r(x))}{\mathcal{J}_r^m(\overline{B}_r(x))} = \lim_{i \to \infty} \inf_{r \in (0,1/r) \cap \mathbb{Q}} \frac{\mu_\lambda(\overline{B}_r(x))}{\mathcal{J}_r^m(\overline{B}_r(x))} > 0.
\]
Therefore, the Borel measurability of the sets in (6.11)-(6.12) follows from the Borel measurability of the map \((x, \lambda) \mapsto \mu_\lambda(\overline{B}_r(x))/\mathcal{J}_r^m(\overline{B}_r(x))\).

In order to prove the existence of \((\tilde{\eta}_\lambda^m)\) satisfying (6.19)-(6.20) we consider \( \hat{P} : \mathbb{R}^n \times \Lambda \to \mathbb{R}^m \times \Lambda \) defined as \( \hat{P}(x, \lambda) := (P_\lambda(x), \lambda) \). From Remark 4.4 we have for every \( S \subset \mathbb{R}^m \times \Lambda \) Borel
\[
\hat{P}_* \tilde{\mathcal{J}}_m(S) = \int_{\mathbb{R}^m} \mu_\lambda((\hat{P}^{-1}(S))_\lambda) \, d\lambda = \int_{\Lambda} \left( \int_{\mathbb{R}^m} \eta^m_\lambda((\hat{P}^{-1}(S))_\lambda) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda
\]
\[
= \int_{\Lambda} \left( \int_{\mathbb{R}^m} \eta^m_\lambda(P_\lambda^{-1}(S)) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda
\]
\[
= \int_{\Lambda} \left( \int_{\mathbb{R}^m} \eta^m_\lambda(\mathbb{R}^m) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda
\]
\[
= \int_{\Lambda} P_\lambda \mu_\lambda(S) \, d\lambda,
\]
where we used that disintegration satisfies \( \eta^m_\lambda(\mathbb{R}^m \setminus P_\lambda^{-1}(y)) = 0 \) and \( \eta^m_\lambda(\mathbb{R}^n) = 1 \) for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \) and \( P_\lambda \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \). In particular, for every \( \varphi : \mathbb{R}^m \times \Lambda \to \mathbb{R} \) Borel, we have
\[
\int_{\mathbb{R}^m \times \Lambda} \varphi \, d\hat{P}_* \tilde{\mathcal{J}}_m = \int_{\Lambda} \left( \int_{\mathbb{R}^m} \varphi(y, \lambda) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda.
\]
On the other hand, applying disintegration theorem with the map \( \hat{P} \), we can write for every \( S \subset \mathbb{R}^n \times \Lambda \) Borel
\[
\tilde{\mathcal{J}}_m(S) = \int_{\mathbb{R}^m \times \Lambda} \eta(y, \lambda)(S) \, d\hat{P}_* \tilde{\mathcal{J}}_m(y, \lambda).
\]
for a \((\tilde{\mathcal{J}}_m \eta_{y, \lambda})\)-a.e. uniquely determined family of probability measures \((\eta_{y, \lambda})\) satisfying
\[
(y, \lambda) \mapsto \eta_{y, \lambda}(S) \text{ is Borel, for every } S \subset \mathbb{R}^n \times \Lambda \text{ Borel.}
\]
Without loss of generality we may also assume that
\[
\eta_{y, \lambda} \text{ is a probability measure, for every } (y, \lambda) \in \mathbb{R}^m \times \Lambda.
\]
Letting \( \pi_1 : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n \) be the orthogonal projection we set
\[
\tilde{\eta}_\lambda^m := \pi_1 \eta_{y, \lambda}, \text{ for } (y, \lambda) \in \mathbb{R}^m \times \Lambda.
\]
Clearly (6.20) is satisfied. In addition, putting together (4.17) with assumption (6.3) we have for every \( B \subset \mathbb{R}^n \) and \( \Sigma \subset \Lambda \) Borel sets
\[
\tilde{\mathcal{J}}_m(B \times \Sigma) = \int_{\Lambda} \left( \int_{\mathbb{R}^n} \eta^m_\lambda(B) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda,
\]
while by exploiting (6.25)-(6.26) we have also
\[
\tilde{\mathcal{J}}_m(B \times \Sigma) = \int_{\mathbb{R}^m \times \Lambda} \eta_{y, \lambda}(B \times \Sigma) \, d\hat{P}_* \tilde{\mathcal{J}}_m(y, \lambda)
\]
\[
= \int_{\mathbb{R}^m \times \Sigma} \eta_{y, \lambda}(B \times \Lambda) \, d\hat{P}_* \tilde{\mathcal{J}}_m(y, \lambda)
\]
\[
= \int_{\Sigma} \left( \int_{\mathbb{R}^m} \eta_{y, \lambda}(B \times \Lambda) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda
\]
\[
= \int_{\Sigma} \left( \int_{\mathbb{R}^m} \pi_1 \tilde{\eta}_{y, \lambda}(B) \, dP_\lambda \mu_\lambda(y) \right) \, d\lambda
\]
Now consider a countable dense subset \( D \subset C^0(\mathbb{R}^n) \). The arbitrariness of the sets \( B \) and \( \Sigma \) in the above equalities allow us to infer that for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \) and for every \( \varphi \in D \) we have

\[
\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \varphi(x) \, d\eta_\mu^\lambda(x) \right) \, dP_{\mathcal{M}} \mu_\lambda(y) = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \varphi(x) \, d\pi_{\lambda y} \tilde{\eta}_n(y, \lambda)(x) \right) \, dP_{\mathcal{M}} \mu_\lambda(y). \tag{6.29}
\]

We can use the density of \( D \) to deduce that actually (6.29) holds true for every \( \varphi \in C_c^0(\mathbb{R}^n) \). Finally, the uniqueness in the disintegration theorem allows us to infer \( \eta_\mu^\lambda = \tilde{\eta}_n^\lambda \) for \( \mathcal{P}_{\mathcal{M}} \mu_\lambda \)-a.e. \( y \in \mathbb{R}^m \) from which the validity of (6.19) follows.

Eventually, we conclude by showing that \( S \) is Borel. To this purpose we denote for every \( z \in \mathbb{R}^n \), \( \tau_z : \mathbb{R}^n \to \mathbb{R}^n \) the translation map \( \tau_z(x) := x - z \). From (6.20) we notice that for every \( \varphi \in C_c^0(\mathbb{R}^n) \) the map \( K_{\varphi} : \mathbb{R}^m \times \mathbb{R}^n \times \Lambda \to \mathbb{R} \) defined as

\[
K_{\varphi}(y, z, \lambda) := \int_{\mathbb{R}^n} \varphi(x) \, d\tau_z \tilde{\eta}_n^\lambda(x)
\]

is Borel measurable because it satisfies the following two conditions

\[
(y, \lambda) \mapsto K_{\varphi}(y, z, \lambda) \text{ is Borel measurable for every } z \in \mathbb{R}^n
\]

\[
z \mapsto K_{\varphi}(y, z, \lambda) \text{ is continuous for every } (y, \lambda) \in \mathbb{R}^m \times \Lambda \text{ (see (6.28)).}
\]

From the Borel measurability of \( K_{\varphi} \) for every \( \varphi \in C_c^0(\mathbb{R}^n) \) and by exploiting the outer regularity of \( \tilde{\eta}_n^\lambda \) it easily follows the Borel measurability of \( K_{I_0} \) for every \( B \subset \mathbb{R}^n \) Borel, where \( I_B \) denotes the characteristic function of \( B \). \( S' := \{(y, \lambda) \in \mathbb{R}^m \times \Lambda \mid \eta_\mu^\lambda(x) = 0\} \) is Borel. Finally, by defining the continuous map \( T : \mathbb{R}^n \times \Lambda \to \mathbb{R}^m \times \mathbb{R}^n \times \Lambda \) as \( T(x, \lambda) := (P_\lambda(x), x, \lambda \Lambda) \), we notice that \( \tilde{\eta}_n^\lambda \) is Borel measurable.

We infer thus that \( S \) is Borel from the Borel measurability of the composition \( K_{I_0} \circ T \).

We are now in position to prove the following rectifiability criterion for Radon measures.

**Theorem 6.3.** Assume (6.1)-(6.2) and let \( \mu \) be a Radon measure in \( \mathbb{R}^n \). If there exists a measurable set \( N' \subset \Lambda \) of strictly positive \( \mathcal{L}^1 \)-measure such that for every \( \lambda \in N' \)

\[
P_{\mathcal{M}} \mu \ll \mathcal{L}^m \quad \text{and} \quad \mu_\lambda^\lambda \text{ is supported on a finite set for } P_{\mathcal{M}} \mu \text{-a.e. } y \in \mathbb{R}^m,
\]

then \( \mu(\mathbb{R}^n \setminus R) = 0 \) for a countably \( m \)-rectifiable set \( R \subset \mathbb{R}^n \). In addition, if there exists \( \lambda_0 \in \Lambda \) and \( \alpha \in (0, 1] \) such that \( P_{\lambda_0} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n \times \Lambda \) as \( T(x, \lambda) := (P_\lambda(x), x, \lambda \Lambda) \), we have also \( \mu \ll \mathcal{H}^m \setminus R \).

**Proof.** Suppose first that \( \mu \) is finite. By using the regularity of \( \mathcal{L}^1 \) we may assume that \( \Lambda' \) is Borel. Define for every \( \lambda \in \Lambda \) the Radon measure \( \mu_\lambda = \mu \) if \( \lambda \in \Lambda' \) and \( \mu_\lambda = 0 \) if \( \lambda \in \Lambda \setminus \Lambda' \). With this definition of the family \( \mu_\lambda \), in order to apply Theorem 6.2, we need only to verify condition (1). But this is trivially satisfied because of the following equalities

\[
\mathcal{F}^m_1(B) = \int_{\Lambda} \mu_\lambda(B) \, d\lambda = \mathcal{L}^1(\Lambda') \mu(B), \quad \text{for } B \subset \mathbb{R}^n \text{ Borel.} \tag{6.31}
\]

From (6.31) we see also that we can reduce ourselves to prove that \( \mathcal{F}^m_1(\mathbb{R}^n \setminus R) = 0 \) for a countably \( m \)-rectifiable set \( R \subset \mathbb{R}^n \). This can be achieved by applying Theorem 6.2. To this purpose we need to verify that \( P_{\mathcal{M}} \mu_\lambda \ll \mathcal{L}^m \) for \( \mathcal{L}^1 \)-a.e. \( \lambda \in \Lambda \) and that \( \mathcal{F}^m_1 \) is finite for some \( p > 1 \). The absolute continuity of the pushforward measures follows by hypothesis and by definition of \( (\mu_\lambda) \), while a simple covering argument as in the proof of Theorem 4.12 gives the finiteness of \( \mathcal{F}^m_1 \).

For a general Radon measure \( \mu \), we consider compact sets \( K_i \subset K_{i+1} \) for \( i = 1, 2, \ldots \), whose union is the whole of \( \mathbb{R}^n \). Applying the previous rectifiability result to each measure \( \mu \setminus K_i \), we conclude that \( \mu \) is \( m \)-rectifiable because it is the monotone limit of a sequence of \( m \)-rectifiable measures \( \mu \setminus K_i \setminus \mu \) as \( i \to \infty \).

Eventually, we make use of formula (4.34) to write for every \( K \subset \mathbb{R}^n \) compact

\[
\int_{\mathbb{R}^m} H^0(R \cap B \cap P_{\lambda_0}^{-1}(y)) \, dy \leq \sup_{x, x' \in K \setminus \lambda \neq x'} \left( \frac{P_{\lambda_0}(x) - P_{\lambda_0}(x')}{|x - x'|^m} \right)^m H^0(R \cap B), \quad B \subset K \text{ Borel.} \tag{6.32}
\]

By virtue of (6.32) and of \( \mu(\mathbb{R}^n \setminus R) = 0 \), we can argue as in the last part of the proof of Theorem 1.4 to infer \( \mu = \mu_{\lambda_0} \ll \mathcal{H}^m \setminus R \). This concludes the proof. \( \square \)
6.1. Closure theorem for flat chains. The closure theorem asserts that, under suitable conditions on the coefficients group, any finite-mass flat chain is rectifiable. It was originally proved by Federer and Fleming for integral flat chains in their pioneering work [16]. Later on, many authors contribute to similar results by providing different techniques and by generalizing the space of coefficients [30, 33, 36, 18, 34] (see also [6, 10, 11, 7, 4] in a metric framework). Here we propose an alternative proof in the case of coefficients in a discrete group, which is based on Theorem 1.3.

Before entering into details we present a brief account of a few useful tools for our analysis. In [18] Fleming developed a theory of flat chains with coefficients in a normed abelian group G which is a complete metric space. In case G is the group of integers or real numbers, with the usual norm, the flat norm and mass norm are related by

\[ F(T) = \inf \{ M(T - \partial S) + M(S) \mid S \in \mathcal{F}_{m+1} \}. \]

Mass is lower-semicontinuous on \( \mathcal{F}_k(K; G) \). Also, given any flat chain \( T \), there exist polyhedral chains \( T_i \) converging to \( T \) with \( M(T_i) = \lim_i M(T_i) \). A polyhedral chain \( T \) can be expressed as a G-linear combination of a finite collection of non-overlapping oriented \( m \)-dimensional polyhedra:

\[ T = \sum_i m_i \sigma_i. \]

The mass of \( T \) is then \( \sum_i |m_i|H^m(\sigma_i) \).

Given a finite-mass \( m \)-chain \( T \) and a Borel set \( B \subset \mathbb{R}^n \), we can consider the finite mass \( m \)-chain \( T \cap B \) which is, roughly speaking, the portion of \( T \) in \( B \). Then \( B \mapsto M(T \cap B) \) defines a Radon measure \( \mu_T \).

A Lipschitz map \( P: \mathbb{R}^n \to \mathbb{R}^m \) induces a homomorphism \( P_T: \mathcal{F}(\mathbb{R}^n; G) \to \mathcal{F}(\mathbb{R}^m; G) \). If \( P \) is affine and \( T \) is polyhedral, it is given by

\[ P_T \sum_i m_i \sigma_i = \sum_i m_i [P(\sigma_i)]. \]

We will make use of the homomorphism \( g_2 \) for \( g \in O(n) \).

In addition a flat \( m \)-chain \( T \) can also be sliced by affine \((n - m)\)-planes. Precisely, following [34], given \( V \in \text{Gr}(n, m) \) and a polyhedral chain \( T \) we can associate for \( H^m \)-a.e. \( y \in V \) the slices \( \langle T, V, y \rangle \) defined as the flat 0-chains satisfying

\[ \langle T, V, y \rangle = \sum_i m_i [\sigma_i \cap \pi_V^{-1}(y)]. \] (6.33)

For a general flat \( m \)-chains \( T \) it is possible to prove [34, Proposition 3.1] that its slices \( \langle T, V, y \rangle \) are 0-flat chains in \( \pi_V^{-1}(y) \cong \mathbb{R}^{n-m} \) and satisfy

\[ \int_V F(\langle T, V, y \rangle) dH^m(y) \leq F(T) \] (6.34)

\[ \int_V M(\langle T, V, y \rangle) dH^m(y) \leq M(T), \] (6.35)

and given a Borel set \( B \subset \mathbb{R}^n \) also

\[ \langle T, V, y \rangle \cap B = \langle T \cap B, V, y \rangle, \quad \text{for } H^m \text{-a.e. } y \in V. \] (6.37)

In particular, if \( T \) has finite mass, for each \( V \in \text{Gr}(n, m) \) we can consider a measure in \( \mathbb{R}^n \) defined as

\[ B \mapsto \int_V \mu_{\langle T, V, y \rangle}(B) dH^m(y), \quad \text{for every } B \subset \mathbb{R}^n \text{ Borel.} \] (6.38)
From the identity \( g \circ \pi_{g^{-1}(V)} \circ g^{-1} = \pi_V \) for \( g \in O(n) \) and \( V \in Gr(n, m) \) the following formula holds true for every flat \( m \)-chain
\[
g_y(T, g^{-1}(V), g^{-1}(y)) = \langle q_T, V, y \rangle, \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in V. \quad (6.39)
\]

Following [34, 1.2] we define rectifiable chains as follows.

**Definition 6.4** (Rectifiability). Let \( T \in \mathcal{M}_m(\mathbb{R}^n; G) \) and let us denote by \( \mu_T \) its mass measure. We say that \( T \) is rectifiable if there exists an \( m \)-rectifiable set \( R \) such that \( \mu_T(\mathbb{R}^n \setminus R) = 0 \).

Before proving the closure theorem we still need an intermediate proposition.

**Proposition 6.5.** Let \( G \) be a discrete and normed abelian group. Then every \( T \in \mathcal{M}_m(\mathbb{R}^n; G) \) satisfies
\[
\mu_{(T,V,y)} \text{ is supported on a finite set for every } V \in Gr(n, m) \text{ and for } \mathcal{H}^m\text{-a.e. } y \in V. \quad (6.40)
\]

**Proof.** Given \( T \in \mathcal{M}_m(\mathbb{R}^n; G) \), by definition we can consider a sequence of polyhedral chains \( T_i \) converging rapidly to \( T \), namely, \( \sum_i \mathcal{F}(T_i - T) < \infty \), and such that \( \mathcal{M}(T_i) \to \mathcal{M}(T) \). In particular, from (6.34)-(6.35) we have
\[
(1) \quad \mathcal{F}(\langle T_i, V, y \rangle - \langle T, V, y \rangle) \to 0, \quad V \in Gr(n, m) \text{ and } \mathcal{H}^m\text{-a.e. } y \in V
\]
\[
(2) \quad \lim \inf_i \mathcal{M}(\langle T_i, V, y \rangle) < \infty, \quad V \in Gr(n, m) \text{ and } \mathcal{H}^m\text{-a.e. } y \in V.
\]

Using a combination of (1) and (2), for every \( V \) and for \( \mathcal{H}^m\text{-a.e. } y \in V \) we can pass through a subsequence (depending on \( V \) and \( y \)), still denoted by \( T_i \), such that (1) holds true together with \( \lim_i \mathcal{M}(\langle T_i, V, y \rangle) < \infty \). Therefore, thanks to our hypothesis on \( G \), it is not difficult to show that any weak* limit in the sense of measures of \( \mu_{(T,V,y)} \) is a sum of finitely many atoms endowed with real and positive multiplicities. From the lower semicontinuity of the mass with respect to the flat-convergence, up to pass to a further subsequence which is weak* convergent, still denoted by \( T_i \), we infer that for every continuous and compactly supported function \( \varphi : \mathbb{R}^n \to [0, \infty) \) it holds true
\[
\sum_{\ell} m_{\ell} \varphi(x_{\ell}) = \lim \inf_i \int_{\mathbb{R}^n} \varphi \, d\mu_{(T,V,y)} \geq \int_{\mathbb{R}^n} \varphi \, d\mu_{(T,V,y)},
\]
for suitable positive real numbers \( m_{\ell} \) and points \( x_{\ell} \) in \( \mathbb{R}^n \). Thanks to the arbitrariness of \( V \) this concludes the proof. \( \square \)

In the next theorem we show how a combination of Theorems 1.3 and 6.2 leads to the closure theorem for finite mass flat chains with coefficients in a discrete group. The most involved result for flat chains that we are going to use is that a slice null flat chain is zero. Its proof relies on the duality pairing with differential forms, the slice null property is obvious for real and hence also integer flat chains.

**Theorem 6.6.** Let \( G \) be a discrete and normed abelian group. Then every \( T \in \mathcal{M}_m(\mathbb{R}^n; G) \) is rectifiable.

**Proof.** Let \( (V_k) \) denote the coordinate \( m \)-planes of \( \mathbb{R}^n \). Consider for each \( k \) an element \( g_k \in O(n) \) such that \( g_k(V_i) = V_k \) and for every \( g \in O(n) \) let \( \pi_g : \mathbb{R}^n \to g(V_i) \) denotes the orthogonal projection. Setting \( h_k := gg_k^{-1}g^{-1} \) we define the Radon measures \( \mu_g^k \) for every \( g \in O(n) \) and \( k = 1, \ldots, (m)_n \) as
\[
\mu_g^k := \int_{g(V_i)} |\mu_{(h_k T, g(V_i), y)}| \, d\mathcal{H}^m(y). \quad (6.41)
\]
From (6.39) we obtain for every \( k = 1, \ldots, (m)_n \) and every \( g \in O(n) \)
\[
\langle h_k T, g(V_i), y \rangle = h_k \langle T, g(V_i), h_k^{-1}(y) \rangle, \quad \text{for } \mathcal{H}^m\text{-a.e. } y \in g(V_i). \quad (6.42)
\]
In particular, we deduce from (6.40) and (6.41)-(6.42) that for every \( k = 1, \ldots, (m)_n \)
\[
\pi_{g^k} \mu_g^k \ll \mathcal{H}^m \text{ for } \theta, \text{-a.e. } g \in O(n),
\]
\[
|\mu_{(h_k T, g(V_i), y)}| \text{ is supported on a finite set for every } g \in O(n) \text{ and for } \mathcal{H}^m\text{-a.e. } y \in g(V_i) \quad (6.43)
\]
Defining the Radon measure \( \mu_g \) for every \( g \in O(n) \) as \( \mu_g := \sum_k \mu_g^k \), by applying (6.43)-(6.44) we deduce the validity of the following conditions
\[
\pi_{g^k} \mu_g \ll \mathcal{H}^m \text{ for } \theta, \text{-a.e. } g \in O(n) \quad (6.45)
\]
\( \eta_0 \) is supported on a finite set for every \( g \in O(n) \) and for \( \pi_{g*}\mu_g \)-a.e. \( y \in g(V) \)
\( (6.46) \)
where \( (\eta_0) \) is the uniquely determined family of probability measures satisfying \( \mu_g = \eta_0 \otimes \pi_{g*}\mu_g \). Moreover, if \( \mu_0^k(B) = 0 \) for every \( k \), then (6.42) tells us that the flat \( m \)-chain \( T \setminus B \) satisfies the hypothesis of [34, Theorem 3.2] and hence \( T \setminus B = 0 \). This implies
\[ \mu_T \ll \mu_g, \text{ for every } g \in O(n). \] (6.47)
Finally, by virtue of (6.45)-(6.47), we can combine Remark 3.7 and Theorem 6.3 with \( \alpha = 1 \) to infer the \( m \)-rectifiability of the Radon measure \( \mu_T \). This concludes the proof. \( \square \)

Appendix A. Orthogonal projections as transversal maps

In this appendix we verify that the family of orthogonal projections \( (\pi_v)_{V \in Gr(n,m)} \) can be represented as a finite union of transversal families in the sense of Definition 3.3. Notice that the same property immediately applies for the family \( (\pi_g(V))_{g \in O(n)} \) for any fixed \( V \in Gr(n,m) \).

We equip \( Gr(n,m) \) with the distance
\[ d(V_1, V_2) := \sup_{x \in \mathbb{R}^n} |\pi_{V_1}(x) - \pi_{V_2}(x)|. \] (A.1)
and notice that \( d(\cdot, \cdot) \) is invariant under the action of \( O(n) \). For this reason, Hausdorff measures constructed with respect to this metric is invariant under such a action. As a consequence, the structure of \( m(n-m) \)-dimensional manifold of \( Gr(n,m) \) allows us to identify \( \gamma_{n,m} \) with a constant multiple of \( \mathcal{H}^m(n-m) \). We further observe that, denoting \( (V_l) \) the collection of all coordinate \( m \)-planes of \( \mathbb{R}^n \) and defining
\[ U_i := \{ V \in Gr(n,m) \mid \det(\pi_{V_l} \cap V) > (2\sqrt{e_{n,m}})^{-1} \}, \]
where \( e_{n,m} \) is the cardinality of the family \( (V_l) \), then the sets \( (U_i) \) form an open covering of \( Gr(n,m) \).

We observe that, in order to obtain the desired property, it is sufficient to find a finite open refinement of \( (U_i) \), say \( (W_j) \), open sets \( \Lambda_j \) in \( \mathbb{R}^{m(n-m)} \), and bi-Lipschitz diffeomorphisms \( \varphi_j \) where \( \varphi_j : \Lambda_j \rightarrow W_j \) such that
\[ |(\pi_{V_{i(j)}} \circ \pi_x)(\varphi_j(\lambda))| \leq C' \implies J_\lambda(\pi_{V_{i(j)}} \circ \pi_x)(\varphi_j(\lambda)) \geq C', \quad \lambda \in \Lambda_j, \quad x \in \mathbb{S}^{n-1}, \] (A.2)
for some constant \( C' > 0 \), where \( \pi_x : Gr(n,m) \rightarrow \mathbb{R}^n \) is defined as \( \pi_x(V) := \pi_V(x) \) \((x \in \mathbb{R}^n)\), and where \( W_j \subset U_{i(j)} \) for every \( j \). Indeed, defining \( P_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^m \) as
\[ P_\lambda(x) := (\pi_{V_{i(j)}} \circ \pi_x)(\varphi_j(\lambda)), \quad \lambda \in \Lambda_j, \] (A.3)
by linearity we have
\[ \frac{P_\lambda(x) - P_\lambda(x')}{|x - x'|} = P_\lambda \left( \frac{x - x'}{|x - x'|} \right), \]
and hence, from (A.2), the families \( (P_\lambda) \) with \( \lambda \in \Lambda_j \) satisfy hypothesis (H.2) of transversality. Since the remaining hypothesis (H.1) and (H.3) are trivially satisfied we deduce that \( (P_\lambda) \) is transversal for every \( j \).

It remains to discuss the validity of (A.2). First we check that
\[ |(\pi_V \circ \pi_x)(V)| \leq C' \implies \text{Rank}(\pi_V \circ \pi_x)(V) = m, \quad V \in U_i, \quad x \in \mathbb{S}^{n-1} \] (A.4)
for some \( C' > 0 \). If not, we use the definition of \( U_i \) together with a compactness argument to find \( V_0 \in U_i \) and \( x \in V_0^\perp \cap \mathbb{S}^{n-1} \) such that
\[ \text{Rank}(\pi_V \circ \pi_x)(V_0) < m. \] (A.5)
By exploiting the (transitive) action of \( O(n) \) on \( Gr(n,m) \), condition (A.5) implies that, defining \( G : O(n) \rightarrow \mathbb{R}^m \) as \( G(y) := (\pi_V \circ \pi_x)(y V_0) \), then
\[ \text{Rank}(G(Id)) < m, \] (A.6)
where \( Id \) denotes the identity \((n \times n)\) matrix. Using that the tangent space of \( O(n) \) at the identity is isomorphic to \( M_{skw}^{n \times n} \), namely, the space of \((n \times n)\) skew symmetric matrices, it is not difficult to show that the differential of \( G \) computed at \( Id \) has the following form
\[ - (\pi_V \circ \pi_{V_0})(Ax), \quad A \in M_{skw}^{n \times n}. \] (A.7)
By denoting \(\{e_1, \ldots, e_m\}\) an orthonormal basis of \(V_0\) we can consider its completion to an orthonormal basis of \(\mathbb{R}^n\), say \(\{e_1, \ldots, e_n\}\), in such a way that \(x = e_n\). Writing \((A.7)\) in this coordinates we get
\[
[Ae_n \cdot e_1] \pi_{V_e}(e_1) + \ldots + (Ae_n \cdot e_m) \pi_{V_e}(e_m) = A \in \mathbb{M}^{n \times n}_{\text{skw}}.
\]  
(A.8)
Since \(V_0 \in \mathcal{T}_i\) we have that \(\pi_{V_e}(e_1), \ldots, \pi_{V_e}(e_m)\) are linearly independent vectors of \(V_e\). Hence the expression in \((A.8)\) vanishes if and only if \((Ae_n \cdot e_i) = 0\) for every \(i = 1, \ldots, m\). But this means that the dimension of \(\text{im}(\pi_{V_e} \circ \pi_{V_0}(Ax))\) for \(A \in \mathbb{M}^{n \times n}_{\text{skw}}\) equals \(m\). Finally, this is in contradiction with \((A.5)\) and we deduce the validity of \((A.4)\). From the compactness of \(\text{Gr}(n,m)\) we easily find an open refinement of \((U_i)\), say \((W_i)\), open subsets of \(\mathbb{R}^{m(n-m)}\), say \((\Lambda_j)\), and bi-Lipschitz diffeomorphisms \((\varphi_j)\) such that \(\varphi_j: \Lambda_j \to W_j\). To conclude, we combine \((A.4)\) with the bi-lipschitzianity of \(\varphi_j\) to infer the validity of \((A.2)\)

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