New results for electromagnetic quasinormal and quasibound modes of Kerr black holes

Denitsa Staicova
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Bulgaria

Plamen Fiziev
Sofia University Foundation for Theoretical and Computational Physics and Astrophysics and JINR, Dubna, 141980 Moscow Region, Russia

The perturbations of Kerr metric and the miracle of their exact solutions play a critical role in the comparison of predictions of GR with astrophysics of compact objects, see the recent review article by Teukolsky [1]. The differential equations governing the late-time ring-down of the perturbations of the Kerr metric, the Teukolsky Angular Equation and the Teukolsky Radial Equation, can be solved analytically in terms of confluent Heun functions. In this article, we use those exact solutions to obtain the electromagnetic (EM) quasinormal and quasibound spectra of the Kerr black hole. This is done by imposing the appropriate boundary conditions on the solutions and solving numerically the so obtained two-dimensional transcendental system.

The EM quasinormal modes (QNM) spectra are found to match the already published results. Additionally, one obtains a symmetric with respect to the real axis spectrum corresponding to quasibound boundary conditions and also a spurious spectrum which can be shown to be numerically unstable. This new result demonstrates the importance of understanding the peculiarities of the numerical integration in understanding the physics of the problem. This may become particularly important considering the recent interest in the spectra of the electromagnetic counterparts of events producing gravitational waves.

QUASI-NORMAL MODES OF BLACK HOLES

During the long history of the study of the quasinormal modes (QNMs) of a black hole (BH) ([1–29]), the case of electromagnetic (EM) perturbations has been often ignored in favor to the gravitational one. It is considered that the gravitational output should be significantly more luminous than the electromagnetic one ([30]), which combined with the strong absorption of the EM spectrum by the interstellar medium, makes the detection very difficult at the predicted low frequencies for the electromagnetic QNMs. On the other hand, the gravitational waves (GW) interact very weakly with matter and thus they can travel big distances without getting absorbed or scattered, i.e. without obscuring the signature of the body that emitted them. It is, therefore, reasonable to expect that the GW should be much better suited for studying the central engines of astrophysical events, such as gamma-ray bursts (GRBs).

For now, however, there are no gravitational waves detected, even though both LIGO and VIRGO detectors already work on the design sensitivity ([31–37]). Particularly puzzling is the lack of GW detection from short GRBs ([38, 39]) whose progenitors are expected to emit GWs in the range of sensitivity of the detectors. A simple explanation of those negative results may be a new mechanism of generation of short GRBs which in good approximation preserves the spherical symmetry of the central engine and thus admits only significant dipole radiation (electromagnetic waves) and no quadrupole one (gravitational waves). If so, we may expect that most of the energy release from GRBs is in the form of electromagnetic radiation. A more traditional point of view is a physical process which yields both electromagnetic and gravitational radiation with unknown, model-dependent ratio. While hopes are laid on the Advanced LIGO and Advanced VIRGO, which should start operating in the next years, this situation offers a good motivation for optimizing the GW search strategy and understanding better the physics of the GW sources. Particularly, this points to the advantages of studying the EM counterpart of the GW emission, the so-called multimessenger approach, which can help the localization of the source (improving on the big error box of the GW detector, [30, 40, 43, 44]) and also it may give additional clues to the physics of the event.

The discrete spectrum of complex frequencies called QNMs describe only the linearized perturbations of the metric. For this reason, they cannot describe the dynamics of the process during the early, highly intensive period of those events, when the linearized theory is not applicable. However, it is known from full numerical simulations that it is the QNMs that dominate the late-time evolution of the object’s response to perturbations ([20, 45]). Since with our EM observations, we “see” only the tails of the corresponding events, being far from them, the QNM’s are important from observational point of view.

Furthermore, the QNMs correspond to particular boundary conditions characteristic of the object in question. In the case of BHs, the no-hair theorem states that
they should depend only on the parameters of the metric, which means that measuring those frequencies observationally can be used to test the nature of the object – a black hole or other compact massive object like super-spinars (naked singularities), neutron stars, black hole mimickers etc. [46–51]. It also can constrain additionally the no-hair theorem, which has recently been put into question for the case of black holes formed as a result of the collapse of rotating neutron stars [52].

An interesting possibility is to find a way to use the damping times of the EM quasinormal modes for comparison with observations. While the frequencies are a subject of interaction with the surrounding matter which can significantly change the spectra, their damping times should be much less prone to deviation. A suggestion for such use can be found in simulations of jet propagation, which imply that the short-scale variability of the light-curve should be due to the central engine and not to the interaction of the jet with the surrounding medium (see [53] and references therein). This could be particularly interesting in the case of the central engines of GRBs, whose extreme luminosity (~10^{51}–10^{55} erg/s) and peculiar time-variability cannot yet be fully explained in the frames of current models. Even though numerical simulations proved to be capable of describing some of the features of the GRBs light curves (for a recent review on GRBs, see [54]), the biggest stumbling stone seems to be the lack of proper understanding of the central engine of the GRBs.

Common ingredients of the existing GRB models include a compact massive object (black hole or a millisecond magnetar) and extreme magnetic field (~10^{15} G), which accelerate and collimate the matter via different processes. Although those processes are still an open question for both theory and numerical simulations, the very central engine can be studied approximately by the linearized EM (and also GW if data are available) perturbations of the Kerr black hole. The QNM spectra do not depend on the origin of the perturbation, but only on the parameters of the compact massive object (mass and spin). In the idealized EM case, the perturbation is described by free EM waves in vacuum. While the astrophysical black holes are thought to be not charged, they are immersed into EM waves with different energy and origin. The black hole response to such EM perturbations in linear approximation will be, then, the QNM spectrum defined by the appropriate boundary conditions.1

Studying the so obtained electromagnetic spectrum can give important insights into the key parameters of the physics occurring during high-energy events as GRBs. In particular, the electromagnetic QNMs are subject to resonant amplification (the idea of the black hole bomb, [9, 27, 29]) and additionally, it is known from previous evaluations of the spectrum that they exhibit very low damping in the limit \( \alpha \to M \). For the moment, there are no observations of the rotations of the GRB progenitors, but the theoretical expectations are that they should be highly rotating in order to produce jets with such luminosity and collimation. Available observationally measured rotations of astrophysical compact massive objects show that there are many cases of near extremal values (as a recent example, in [55], the rotation rates of two astrophysical black holes, Sw J1644+57 and Sw J2058+05, have most probable values \( \alpha = 0.9M \) and \( \alpha = 0.99M \) respectively), thus studying the extremal limit could be relevant to such objects. Clearly, the theoretical study of QNMs is relevant to the physics of such objects, especially if combined with identifying the QNM component in the observed EM spectra.

Theoretical calculations of the QNMs, however, is not simple. The linear perturbations of the rotating BHs are described by two second-order linear differential equations: the Teukolsky radial equation (TRE) and the Teukolsky angular equation (TAE) on which specific boundary conditions have been imposed ([11, 16]). Until recently, solving those equations analytically was considered impossible in terms of known functions, so approximations with more simple wave functions were used instead. The resulting system of spectral equations – a connected problem with two complex spectral parameters: the frequency \( \omega \) and the separation constant \( E \) – has been solved using different methods ([21, 23, 24, 28]) with notably the most often used of them the method of continued fractions adapted by Leaver from the problem of the hydrogen molecule ion in quantum mechanics [17, 18]. This method, while being successful in obtaining the QNMs spectra, has the disadvantage of not being directly connected with the physics of the problem, thus making it harder to further explore the spectra – for example studying its dependence on the choice of the branch cuts of the exact solutions of the radial equation. In addition one has some specific numerical problems in calculation of particular modes, for example, in calculation of the 9th one in the gravitational case [19, 21, 24].

The analytical solutions of the TRE and the TAE can be written in terms of the confluent Heun function (for \( \alpha \neq M \)) as done for the first time in [22, 25, 26, 56]. Those functions are the unique local Frobenius solutions of a second-order linear ordinary differential equation of the Fuchsian type [57–60] with 2 regular singularities (\( z = 0,1 \)) and one irregular (\( z = \infty \)) (for details see [26]) and in MAPLE notation, they are denoted as: HeunC(\( \alpha, \beta, \gamma, \delta, \eta, z \)) (normalized to HeunC(\( \alpha, \beta, \gamma, \delta, \eta, 0 \)) = 1). While the theory of the Heun functions is still far from being complete, they are implemented in the software package MAPLE and despite the problems in that numerical realization (see the dis-

---

1 Other conditions more suitable for describing a primary jet were studied in [51].
cussion in [61]), the confluent Heun function was used successfully in our previous works [22, 51, 61, 62]. The advantage of using the analytical solutions is that one can impose the boundary conditions on them directly (see [22, 51]) and thus to be able to control all the details of the physics of the problem. Furthermore, it is thought that the Teukolsky-Starobinsky identities stem from the fact that both the TRE and TAE can be solved in terms of the confluent Heun function [1].

In a series of articles, we developed a method for solving numerically two-dimensional systems featuring the Heun functions (the two-dimensional generalization of the Miller method described in [63]) and we used it successfully in the case of gravitational perturbation ($s = -2$) of the Schwarzschild metric [61]. The so-obtained frequencies repeat with high precision the results, already published by other authors. Additionally, we used the epsilon-method (see below) to study the branch cuts of the solutions, which are particularly important in the case of the $9^\text{th}$ mode for $s = -2$, often wrongly considered to represent the purely imaginary algebraically special mode. While the analysis of the potentials of the Regge-Wheeler equation (EWE) and the Sterilize equation (ZRE) showed that there is a branch cut on the imaginary axis for this mode [64, 65] which leads to its interesting properties, this result is directly obtainable from the actual solutions of the EWE and ZRE in terms of the confluent Heun functions. Furthermore, the numerical stability 2 of the solutions with respect to the position of those branch cuts was studied in the whole interval of applicability of the method. Such study cannot be carried out with the continued fractions method, where the radial variable does not explicitly enter into the equations being solved and which cannot be used for purely imaginary frequencies ([18], p.8). If one looks at the equations used by this method in detail, it turns out that the angular equation [17] in the continued fractions method coincides with the the three-term recurrence defining the confluent Heun function, solution of the TAE, in the neighborhood of two regular singular points, $u = -1, 1$, where $u = \cos(\theta)$ ([69] Eq. (1.9-1.10). The radial equation in the continued fractions method, however, differs from the solution of the TRE in terms of confluent Heun functions. This is because in the Leaver’s paper, the series from which the continued fractions are obtained, are developed for the powers of $\frac{r-r_+}{r-r_-}$ (due to switching the places of the singular points, see [18], p.7), while the asymptotic three-term recurrence of the confluent Heun function at infinity is developed for $\frac{1}{r-r_-}$. Note that in MAPLE, for $r > r_+$, the evaluation of the confluent Heun functions at infinity is obtained by numerical integration from the second singularity $r = r_+$.

In this article, we continue the exploration of the application of the confluent Heun functions by studying the QNMs of the Kerr BH. Previous results, obtained in [66] showed that using the confluent Heun function one can obtain the QNMs to a very good precision for the lower modes and within the $\epsilon$-method. For the first time, there the dependence of the so obtained frequencies on the position in the complex radial plane and its non-trivial evolution with $n$ and $a$ was studied. It also demonstrated some weaknesses of the Heun routines in MAPLE, which led to their significant improvement in recent versions. Using the new routines, in the current work, we are able to obtain the QNMs for a wide range of modes and rotational parameters and we show that there is a very good agreement between our results and those obtained within other methods. Taking advantage of direct way of imposing the boundary conditions on our system one obtains not only the QNM spectrum but also the quasibound one (QBM). The QBMs form a discrete spectrum of frequencies with negative imaginary parts, obtained by imposing boundary conditions inverse to those of the QNMs. This is to say that the equations no longer describe a black hole. The QBMs are resonant in their nature and an example of the study of the quasibound states in the case of massive vector propagating on the Schwarzschild space-time can be found in [67]. Here, we obtain them numerically under the appropriate boundary conditions and study their dependence on the rotation parameter $a$.

THE TEUKOLSKY ANGULAR EQUATIONS

In Chandrasekhar’s notation, the Teukolsky Master Equation ([9]), for $|s| = 1$, is separable under the substitution $\Psi = e^{i(\omega t + m\phi)} g(\theta) R(r)$, where $m = 0, \pm 1, \pm 2$ for integer spins and $\omega$ is the complex frequency. Due to the choice of this form of $\Psi$, the sign of $\omega$ differs from the one Teukolsky used, and the stability condition, guaranteeing that the perturbations will damp with time reads $\Im(\omega) > 0$.

The TAE for EM perturbations ($s = -1$) has 16 classes of exact solutions $S(\theta)$ in terms of the confluent Heun functions (for full details see [26]). To fix the spectrum approximately, one requires an additional regularity condition for the angular part of the perturbation, which means that if we choose one solution, $S_1(\theta)$ is regular around the one pole of the sphere ($\theta = 0$) and another, $S_2(\theta)$, which is regular around the other pole ($\theta = \pi$), then in order to ensure a simultaneous regularity, the Wronskian of the two solutions should become equal to zero, $W[S_1(\theta), S_2(\theta)] = 0$. This gives us one of the equations for the two-dimensional system that need to be solved to obtain the QNMs of the Kerr BH.

In [26], there are four pairs of Wronskians, each pair

\[ s > 0 \text{ for } a > 0, s < 0 \text{ for } a < 0. \]

2 Here by numerical stability of a mode we will understand that small deviations in the parameters of the radial variable do not change the mode up to certain significant digits.
being valid in a sector of the plane \( \{ s, m \} \). Ideally, using any of them should lead to the same spectrum. Numerically, the results obtained with the different Wronskians coincide within 11-13 digits of precision. The Wronskians used to obtain the spectrum are:

\[
W[S_1, S_2] = \frac{\text{HeunC}'(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, \cos(\pi/6)^2)}{\text{HeunC}(\alpha_1, \beta_1, \gamma_1, \delta_1, \eta_1, \cos(\pi/6)^2)} + \frac{\text{HeunC}'(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, \sin(\pi/6)^2)}{\text{HeunC}(\alpha_2, \beta_2, \gamma_2, \delta_2, \eta_2, \sin(\pi/6)^2)} + p = 0
\]

where the derivatives are with respect to \( z \) and the values of the parameters for the two confluent Heun functions for each \( m \) are as follows:

For the case \( m = 0 \):
\[
\alpha_1 = 4 \omega, \beta_1 = 1, \gamma_1 = -1, \delta_1 = 4 \omega, \eta_1 = 1/2 - E - 2 \omega - a^2 \omega^2 \quad \text{and} \quad \alpha_2 = -4 \omega, \beta_2 = 1, \gamma_2 = 1, \delta_2 = -4 \omega, \eta_2 = 1/2 - E + 2 \omega - a^2 \omega^2, \quad p = \frac{1}{(\sin(\pi/6))^2}
\]

For the case \( m = 1 \):
\[
\alpha_1 = -4 \omega, \beta_1 = 2, \gamma_1 = 0, \delta_1 = 4 \omega, \eta_1 = 1 - E - 2 \omega - a^2 \omega^2 \quad \text{and} \quad \alpha_2 = -4 \omega, \beta_2 = 0, \gamma_2 = 2, \delta_2 = -4 \omega, \eta_2 = 1 - E + 2 \omega - a^2 \omega^2, \quad p = -4 \omega
\]

For the case \( m = 2 \):
\[
\alpha_1 = -4 \omega, \beta_1 = 3, \gamma_1 = -1, \delta_1 = 4 \omega, \eta_1 = 5/2 - E - 2 \omega - a^2 \omega^2 \quad \text{and} \quad \alpha_2 = -4 \omega, \beta_2 = 1, \gamma_2 = -3, \delta_2 = -4 \omega, \eta_2 = 5/2 - E + 2 \omega - a^2 \omega^2, \quad p = 8 - 4 \omega
\]

where we use \( \theta = \pi/3 \) (the QNMs should be independent of the choice of \( \theta \) in the spectral conditions).

These Wronskians differ from those in [26], most notably by the presence of the term \( p \). The reason for this is that they were constructed using two different solutions \( \{ S_1(\theta), S_2(\theta) \} \) of the TAE (note that the sign convention in this paper differs from the one in [26]), each of which being still regular on one of the poles. That was done to improve the numerical convergence of the root-finding algorithm and to avoid MAPLE’s problems with the evaluation of the confluent Heun function and its derivative for certain values of the parameters.

### THE TEUKOLSKY RADIAL EQUATION

The TRE differential equation is of the confluent Heun type, with \( r = r_+ \) regular singular points and \( r = \infty \) –irregular one. As it was noted in [51], the point \( r = 0, \theta = \pi/2 \) is not a singularity for this equation and, therefore, it needs not to be considered when imposing the boundary conditions. The solutions of the TRE for \( r > r_+ \), are:

\[
R(r) = C_1 R_1(r) + C_2 R_2(r), \quad \text{for} \quad R_1(r) = e^{\frac{r}{\omega}} (r-r_+) \frac{\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, \zeta)}{\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, \zeta)} \quad \text{and} \quad R_2(r) = e^{\frac{r}{2\omega}} (r-r_+) \frac{\text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, \zeta)}{\text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, \zeta)},
\]

where \( z = \frac{r-r_+}{r-r_-} \) and the parameters are:

\[
\alpha = -2 i (r_+ - r_-) \omega, \beta = -2 i (\omega (a^2 + r_+^2 + am) - 1), \gamma = 2 i (\omega (a^2 + r_+^2 + am) - 1), \delta = -2 i (r_+ - r_-) \omega (1 - i (r_+ + r_-) \mu), \eta = \frac{1}{2} \frac{1}{(r_+ - r_-)} \left[ 4 \omega^2 r_+^4 + 4 (i \omega - 2 a^2 r_-) r_+^3 + (1 - 4 a \omega m - 2 a^2 a^2 - 2 E) \times (r_+^2 + r_-^2) + 4 (i \omega r_- - 2 i \omega r_+ + E - \omega^2 a^2 - \frac{1}{2}) r_- r_+ - 4 a^2 (m + \omega a)^2 \right].
\]

Here we have followed MAPLE’s internal rules when constructing the general solution of the differential equation from the confluent Heun type. Accounting for the symmetries of the confluent Heun function, the solutions (3) coincide with those in [26] (for \( \omega \) replaced with \( -\omega \)).

The TRE has 3 singular points \( r_-, r_+, \infty \) and in order to fix the spectrum, one needs to impose specific boundary conditions on two of those singularities (i.e. to solve the central two-point connection problem [57]). Different boundary conditions on different pairs of singular points will mean specifying of different physics of the problem. In our case, we impose the black hole boundary conditions (BHBC) – waves going simultaneously into the event horizon \( (r_+) \) and into infinity – following the same reasoning as in [51] where additional details can be found.

Then, the BHBC read:

1. BHBC on the KBH event horizon \( r_+ \).

   For \( r \rightarrow r_+ \), from \( r(t) = r_+ + e^{-R(t) t + \text{const}} \rightarrow r_+ \), where \( n_{1,2} \) are the powers of the factors \( (r-r_-)^{n_{1,2}} \) in \( R_{1,2} \), it follows that for \( m = 0 \), the only valid solution in the whole interval \( (\infty, \infty) \) is \( R_2 \), while for \( m \neq 0 \), the solution \( R_2 \) is valid for frequencies for which \( R(\omega) \notin (-\infty, 0) \). This means that the rotation splits on two parts the area of validity of \( R_2 \). If this condition is not fulfilled, then the spectrum corresponds to waves going out of the horizon – a white hole case. Similarly, if we work with \( R_1 \) in the same interval, we deal with a white hole.

2. BHBC at infinity.

   At \( r \rightarrow \infty \), the solution is a linear combination of an ingoing \( (R_-) \) and an outgoing \( (R_+) \) wave: \( R = C_- R_- + C_+ R_+ \), where \( C_- \), \( C_+ \) are unknown constants and \( R_- \), \( R_+ \) are found using the asymptotics of the confluent Heun function as defined in [26, 57].

   To ensure only outgoing waves at infinity, one needs to have \( C_- = 0 \).

\footnote{It is important to emphasize that the so obtained solutions cannot be used for extremal KBH (\( a = M \)) since in this case, the differential equation is of the biconfluent type and its treatment differs, so it is outside the scope of this work.}
To achieve this, one first finds the direction of the steepest descent in the complex plane \( \mathbb{C}_r \), for which
\[
\lim_{r \to \infty} \frac{R}{R_n} = r^{-4i \omega M + 2} e^{-2i \omega r} = 0
\]
tends to zero most quickly: \( \sin(\arg(\omega) + \arg(r)) = -1 \). This gives us a relation, \( r = |r| e^{\frac{3}{2} i \pi - i \arg(\omega)} \) ([22]), between \( \omega \) and \( r \), which is exact only if one uses the first term of the asymptotic series for the confluent Heun function (i.e. Heun\( C \)). More details on this approximation can be found in the next section.

Then, it is enough to solve :
\[
C_r = r^{\frac{3}{2} i \omega + \frac{2 i m n}{1 + |r|}} \operatorname{HeunC}(\alpha, -\beta, \gamma, \delta, \eta, z) = 0, \quad (3)
\]
in order to completely specify the spectra \( \{\omega_{n,m}, E_{n,m}\} \). We use \( |r| = 110 \) as the actual numerical infinity and \( M = 1/2 \).

By similar reasoning, one can also obtain the so called quasibound frequencies corresponding to boundary conditions as follow: waves outgoing from the horizon and outgoing from infinity. Although the QBM frequencies are considered unphysical, because they will lead to non-damping waves and thus to a black hole bomb, mathematically the differential system describes those states on equal footing as the QNMs and they can be found by imposing appropriate, non-BH boundary conditions. In order to differentiate between the two states, one must keep the track of 1) which radial equation they solve: \( R_1(r) \) or \( R_2(r) \) 2) which boundary condition at infinity they obey: \( \sin(\arg(\omega) + \arg(r)) \leq 0 \).

**THE EPSILON-METHOD**

Equation (3) relies on the direction of the steepest descent defined by the phase condition \( \sin(\arg(\omega) + \arg(r)) < 0 \) leading to \( \arg(r) + \arg(\omega) = 3/2\pi \). This approximate direction was chosen ignoring the higher terms in the asymptotic expansion of the solution around the infinity point. Therefore, one can expect that the true path in the complex plane may not be a straight line but a curve. In principle, the spectrum should not depend on this curve as long as \( r \) stays in the sector of the complex plane where \( \lim_{r \to \infty} \frac{R}{R_n} = 0 \), i.e. as long as \( \sin(\arg(\omega) + \arg(r)) < 0 \) with only the convergence of the algorithm being affected.

The spectrum obtained numerically in this interval, however, depends in a non-trivial way on this curve. The complications are partially due to the appearance of branch cuts in the numerical realization of the confluent Heun functions in MAPLE. The branching points of the confluent Heun function in the complex \( z \)-plane are found at the singular points \( z = 1 \) and \( z = \infty \). In MAPLE, as a branch cut the semi-infinite interval \( (1, \infty) \) on the real axis is chosen. In the case of QNMs of non-rotating BHs [61, 62], it was observed that when those branch cuts are found near a frequency (after taking into account the direction of the steepest descent, the branch cuts appear also in the complex \( \omega \)-plane), they have serious effect on it.

As a way to find the correct sheet of the multivalued function and to remain on it, we introduced the epsilon-method, which consists in adding a variation \( (|r| < 1) \) in the phase condition:
\[
\arg(r) + \arg(\omega) = \frac{3 + \epsilon}{2} \pi. \quad (4)
\]

Using the \( \epsilon \)-method one can change the location of the branch cut with respect to the eventual roots of the system, and this way to try to minimize the effect of the jump discontinuity of the radial function \(^4\). Using \( \epsilon \), one can also explore the whole sector \( \pi < \arg(r) + \arg(\omega) < 2\pi \), i.e. effectively moving \( r = |r| e^{i \arg(r)} \) in the complex plane, and this way to test the numerical stability of the QNM spectrum.

Using the parameter \( \epsilon \), the observed branch cuts in the realization of the confluent Heun function in MAPLE are as follows:

1. For \( r \)-real, one encounters one of the branch cuts of the confluent Heun function. The equation of the line of this branch cut is: \( \Im(\omega) / \Re(\omega) = \tan(3/2\pi + \epsilon \pi/2) = -\cot(\epsilon \pi/2) \). This line rotates when \( \epsilon \) changes.
2. If \( \Im(\omega) = 0 \), then one encounters the branch cut of the argument-function. In this case the branch cut is defined for \( \Re(\omega) = (-\infty, 0) \). This branch cut, however, affects the solutions only very close to \( a = M \) when the frequencies can become almost real.
3. If \( \Re(\omega) = 0 \) and \( \Im(\omega) = 2n, n = 1, 2, 3, ... \), then one can have \( \Im(r) = 0 \) for certain values of \( \epsilon \) and thus to reach the branch cut of the confluent Heun function on the real axis. This condition can affect modes which are very near the imaginary axis (for example, similar condition holds around the algebraically special mode for a non-rotating BH).

The knowledge of the branch cuts is critical for the successful numerical evaluation and analysis of the results. Another way to use the \( \epsilon \)-method it is to use directly \( r = |r| e^{i \arg(r)} \) and to vary \( \arg(r) \) and afterwards to check whether the QNM or QBM condition is satisfied.

**NUMERICAL ALGORITHMS**

The spectral equations we need to solve to find the spectrum \( \omega_{n,m}(a) \) for \( M = 1/2 \) are Eqs.(1) and (3). This

---

\(^4\) Here, the radial function refers to the solutions of the radial equation and not to the differential equation itself.
system represents a two-dimensional connected problem of two complex variables – the frequency $\omega$ and the separation parameter $E$ – and in both of its equations one encounters the confluent Heun function and in the case of the TAE – their derivatives.

Such a system cannot be easily solved by conventional methods like the Newton method and the Broyden method, as outlined in [61] and [63] since they do not work well with the confluent Heun function in MAPLE. For this reason, our team developed a new method, namely, the two-dimensional Müller algorithm, which proved to be much better adapted to work with those functions. The details on the algorithm can be found in [61–63], but for completeness, we will mention only that it relies on the Müller method, which is a quadratic generalization of the secant method having better convergence than the latter. The new algorithm does not need the evaluation of derivatives, thus avoiding one of the biggest problems when using the confluent Heun function in MAPLE. Clearly, in the system we solve, the angular spectral equation (Eq. (1)) includes derivatives, but in this case, they remain in the domain $|z| < 1$, where they can be evaluated correctly (for most values of the parameters) and with precision comparable to that of the radial function. It is important to note that both $\omega, E$ are found directly from the spectral system (Eqs. (1) and (3)) and with equal precision.\footnote{The algorithm is realized in MAPLE code and the numbers presented below are obtained using MAPLE 17 and 18. The software floating point number is set to 64 (unless stated otherwise), the precision of the algorithm – to 25 digits.} \footnote{An important precaution when working with the confluent Heun function in MAPLE is that its precision or over-all behavior may depend on different factors which are not always under user’s control [61]. In the presented results, one can trust at least 11-12 digits.}

**NUMERICAL RESULTS FOR ELECTROMAGNETIC QNMS**

While the evaluation of QNMs is not new to physics, the actual numbers published for EM perturbations of KBH are scarce. Because of this, for comparison, we use the numbers published by Berti et al. [24, 46], the numerical data can be found on http://www.phy.olemiss.edu/~berti/qnms.html. Those numbers were obtained using the continued fractions method, which is still considered as the most accurate method for obtaining the QNMs from the KBH. The available control frequencies are $n = 0.6$ for $l = 1$ and $n = 0.3$ for $l = 2$. Using those “control” numbers, denoted as $\omega_{n,m}^B, E_{n,m}^B$, one can easily check the precision of the method.

The first 10 modes of the spectrum obtained using the new method in the interval $a = [0, M)$ can be found on http://tcpa.uni-sofia.bg/conf/research. In the Appendix, one can find some of the QNMs for specific values of $a$.

**Non-rotating BH**

It is already well known that when there is no rotation ($a = 0$), the electromagnetic QNMs come in pairs symmetrical to the imaginary axis $\omega_{n,m} = \pm |\Re(\omega_{n,m})| + i\Im(\omega_{n,m})$ ($n = 0, 1..$ numbering the mode). In this case, the system reduces to one equation – the radial function (3) (for $E = l(l + 1), l = 1, 2..$), solved here using the one-dimensional Müller algorithm.

In our studies, however, we chose not to use the direction of the steepest descent or the $\epsilon$-method, but instead we varied $\arg(r)$ directly to find the zeros of the two equations $R_1(r)$ and $R_2(r)$ and then we checked whether they satisfied the quasinormal condition, the quasibound one, or none. This way, we can study not only the QNM modes but also the quasibound modes. This means that now, we get frequencies in the 4 quadrants of the complex plane, i.e.: $\omega_{n,m} = \pm |\Re(\omega_{n,m})| \pm i\Im(\omega_{n,m})$ ($n = 0, 1..$).

![Figure 1](image_url)

Figure 1. (a) QNMs for $a = 0$, for $m = 0, l = 1$. (b) the boundary condition $\sin(\arg(\omega) + \arg(r))$ for the different modes. The red diamonds are obtained from $R_1(r)$ with $\arg(r) = 1/2\pi$, the red crosses – from $R_2(r)$ with $\arg(r) = 3/2\pi$, the blue diamonds – from $R_1(r)$ with $\arg(r) = 1/2\pi$, the blue crosses – from $R_2(r)$ with $\arg(r) = 3/2\pi$. 
The results can be seen on Fig. 1 a). On it, one can see the complex frequencies obtained for \( m = 0 \) and \( l = 1 \). On the plot, the blue symbols correspond to QNM and the red-ones – to QBM. On Fig. 1 b), one can see the boundary condition at infinity for them (\( \sin(\arg(\omega) + \arg(r)) \leq 0 \)) which allows us to differentiate between the two types of modes. The symmetry with respect to both the real axis and the imaginary axis is clearly visible.

A numerical comparison of QNMs (i.e. those from the upper part of the complex plain on Fig. 1 a) with the frequencies obtained by Berti et al. shows that the average deviation is \( |\omega_{n,m}^{E} - \omega_{n,m}^{Q}| \approx 10^{-10} \).

As already discussed for the gravitational perturbation for non-rotating BH ([61]), the dependence \( \omega(\epsilon) \) in the electromagnetic case is not a trivial one. The so-found QNM and QBM are stable in certain intervals of \( \epsilon \), whose width depends on \( n \). Outside these intervals, they start to vary with \( \epsilon \), usually close to \( \sin(\arg(\omega) + \arg(r)) = 0 \). We did not present here an in-depth analysis of this behavior. Additionally, we checked the dependence of those modes on \( |r| \) and, as expected, those modes are stable with respect to an increase in \( |r| \), which means that \( |r| = 110 \) is a valid actual infinity.

Rotating KBH

The results presented here were obtained for \( a = [0,M] \) and can be seen on figures 2 – 8.

For the QNM modes, when \( a \neq 0 \), the symmetry with respect to the imaginary axis \( \omega_{m,n}^{1,2}(0) = \pm |\Re(\omega_{m,n})| + i\Im(\omega_{m,n}) \) breaks down, but it is replaced by the symmetry:

\[
\{ \Re(\omega_{m,n}^{j}), \Im(E_{m,n}^{j}), m \} \rightarrow \{-\Re(\omega_{m,n}^{j}), -\Im(E_{m,n}^{j}), -m\},
\]

where \( j = 1,2 \) coincides with the upper index of \( \omega_{m,n}^{1,2}(0) \).

Thus to study the complete behavior of the modes for \( a \in [0,M] \), it is enough to trace both symmetric frequencies in the pair corresponding to each \( \{m,n\} \) for \( a = 0 \), for only \( m > 0 \) (the index \( l \) here is omitted to simplify the notation, since we consider only \( l = 1 \))

If one considers both the QNM and the QBM, i.e. the roots of the transcendental system in all the 4 quadrants of the complex plane \((I, II, III, IV)\) for \( m = 0,1 \), one observes the symmetry

\[
\Re(\omega_{I}) = \Re(\omega_{II}), \Re(\omega_{II}) = \Re(\omega_{III}),
\]

\[
\Im(\omega_{I}) = -\Im(\omega_{II}), \Im(\omega_{II}) = -\Im(\omega_{IV})
\]

(and analogously for \( E \)). This symmetry is preserved at least up to \( n < 4 \) within the precision of the numerical method. A peculiarity of the numerical routines evaluating HeunC in Maple is that one needs to increase the floating point number in order to calculate higher modes. For example, for \( n = 9 \) one already needs to set the parameter Digits=192. This means that it becomes impractical to calculate the modes with \( n > 9 \) and that evaluating with lower than the needed precision may introduce numerical errors. To avoid this, we have presented here only the modes with \( n < 5 \).

It is important to note that for \( m = 0 \), in the modes \( n \geq 3 \), one observes loops. An example can be seen on Fig. 2. Those loops appear in all the higher modes, and their position depends on \( n \). As those loops require a finer structure of the plot (i.e. smaller step), on some plots, for example Figs. 3, and 6, we plotted only the points before the first loop observed in each curve. On Fig. 6 one can see all the results plotted together.

From the radial boundary conditions, it follows that only frequencies for which \( \Re(\omega) \notin (0, -m\frac{\pi}{2M_{r}}) \) correspond to black hole boundary conditions. From Fig. 7 a), it is clear that the so obtained QNM spectrum obeys this condition. A deviation from this condition was observed in [51], where some of the frequencies describing primary jets crossed the line defined by \( -m\frac{\pi}{2M_{r}} \), thus

---

Figure 2. On the plots the real and imaginary parts of \( \omega_{0,3}(a) \) and \( E_{0,3}(a) \).
Figure 3. Complex plots of $\omega_{0,n}(a)$ and $E_{0,n}(a)$ for $a = [0, M)$, the first 5 modes with both positive and negative real parts.

Figure 4. On the plots the real and the imaginary parts of $\omega_{0,n}(a)$ and $E_{0,n}(a)$ for $a = [0, M)$ for the modes $n = 0..4$.

Figure 5. Complex plots of $\omega_{m,n}(a)$ and $E_{m,n}(a)$ for $a = [0, M)$, $n = 0..3$. With blue is $m = 0$, with green $m = 1$, with black dot is denoted $a = 0$.

Figure 6. A complex plot of all the $\omega_{m,n}(a)$ and $E_{m,n}(a)$ obtained for $a = [0, M)$ for $m = 0, 1, l = 1, n = 0..4$.

The spectrum corresponds to perturbation of a black hole. Note that because of the symmetry in the 4 quadrants, the same will also apply to the solutions in quadrant III. Since for them we are working with the solution $R_1(r)$, $\omega_n < -m_ia^2/M$ means that they correspond to a white hole solution and there are no incoming in the horizon modes. Combined with the boundary condition at infinity, i.e. $\sin(\arg(\omega) + \arg(r)) > 0$, this means that we deal with quasibound modes.

From the same figure, one can see in the negative sector of the plot, that the real parts of the QNMs for increasing $n$ seem to tend to the line $-m_ia^2/2Mr$, which requires
Further investigation. For the positive sector (i.e. the frequencies with positive real parts), we were not able to trace the frequencies with high $n$ near $a \to M$, thus we cannot confirm the relation $\Re(\omega) = m$ for $a \to M$ observed in [20].

Finally, obtaining the modes in the limit $a \approx M$ could be of serious interest, if one is to compare the EM QNMs with the spectra obtained from astrophysical objects but it is also technically challenging. This happens because for $a = M$ the TRE changes its type and near this limit the confluent Heun function becomes numerically unstable since these functions are transformed in the biconfluent Heun ones. Due to this, the examination of the limit $a \to M$ for modes with high $n$ is impossible with current numerical realization of that function in MAPLE. For the lowest modes, however, the function is stable enough in the interval $a \in [0.49, 0.4995]$ and the results of the numerical experiment for $m = 1$ are plotted on Fig. 8. As expected, for $n = 0$, for $a > 0.91M$ the imaginary part of the frequency quickly tends to zero, thus proving that for extremal objects, the perturbations damp very slowly. The other two modes also seem to tend to zero, although a somewhat more slowly than $n = 0$. In physical units, the difference between the 3 modes for $a = 0.4995$ is only 6Hz ($\omega_{1,1} \approx 1.582kHz$), but the damping times of the first mode is approximately 4.86 times bigger than that of the third and is $t_{1,1}^{\text{damp}} \approx 4.2ms$ for KBH with mass $M = 10M_\odot$. The frequencies in physical units, for some other values of the rotational parameter, can be found in the tables I in the Appendix.

Algebraically special modes, branch cuts and spurious modes

The algebraically special (AS) modes are obtained from the condition that the Teukolsky-Starobinsky constant vanishes ([16]) and they correspond to the so called total transmission modes (TTM) – modes moving only in one direction: to the right or to the left. In the case of gravitational perturbations ($s = -2$) from non-rotating BH, because the 9th QNM coincides approximately with the theoretically expected purely imaginary AS mode, there were speculations that the two modes coincide (see [20] for a review, and also [64, 65]). A study of this mode in the case of gravitational perturbations of KBH showed numerical peculiarities as the “doublet” emerging from the “AS mode” for $m > 0$ ([20]). For the non-rotating gravitational case, Maassen van den Brink [64, 65] found that the peculiarities of the 9th mode are due to the branch cut in the asymptotics of the Regge-Wheeler potential, which the method of continued fractions is not adapted to handle. This result was confirmed by the use of the $\epsilon$-method in our previous work [62], where the AS character of the 9th mode was disproved.

For electromagnetic perturbations, the algebraically special modes have not been discussed much, because in the limit $a \to 0$, the Teukolsky-Starobinsky constant do not vanish for purely imaginary modes (in fact, for $a = 0$ the Teukolsky-Starobinsky constant does not depend on $\omega$ at all, see Eq. (60) [16] p.392) and there appear to be no correlation between TTM and QNM modes [68].

As discussed in Section 4, the confluent Heun function has different branch cuts in the complex $r$–plane. Using the $\epsilon$–method, one can change the position of the branch cuts in the complex $\omega$–plane with respect to certain $\omega_{m,n}$. We can switch between the complex $\omega$–plane and the complex $r$–plane because of the relation due to the boundary conditions: arg$(r) + \arg(\omega) = \frac{3\pi}{2}$. Thus by using the $\epsilon$ method, it is possible to examine the effect of the proximity of a branch cut over the modes and to look for the so-called spurious modes.

An example of such a spurious additional spectrum can be seen on Fig. 9. In this case, the modes seem to fulfill the QNM boundary condition for certain $n$ and the QBM boundary condition for other $n$. This makes understanding the nature of those modes difficult; however, an additional test of their numerical stability with respect to changes in $|r|$ show that those modes are unstable and they basically decrease with the increase of the actual infinity. We discard those modes because they contradict
the basic assumption upon which one solves the Teukolsky Master Equation – i.e., that $\omega \approx r$. Such spurious modes were reported in [20] without further explanation.

Additional spurious modes can be observed through the $\epsilon$-methods. For example, for $n = 0$, one has a stable mode with precision of 11 digits in the interval $\epsilon = -0.5..0.35$. For $\epsilon > 0.35$ the mode starts increasing its real part until it reaches a value for which the boundary condition at infinity is no longer satisfied. We consider those modes to be a product of numerical instability due to inadequate choice of the direction of the steepest descent. Another possible reason for such dramatic changes in the numerical stability of the mode with respect to $\epsilon$ are that the branch cut in the radial function is moved by the parameter $\epsilon$, i.e. they are due to the complex character of the used analytical functions (the confluent Heun functions) in the vicinity of the irregular singular point $r = \infty$ in the complex $r$-plane. Understanding better the theory of the confluent Heun function is critical for the complete understanding of the numerical results.

Considering all the numerical peculiarities demonstrated above, the use of the $\epsilon$-method poses a very serious question before astrophysical application of those spectra – if one is to compare the numerical results with some observational frequencies, which $\epsilon$ should be trusted? In our numerical experiments we were able find both the frequencies obtained with the well-established methods, and also other spurious-type frequencies which also evolve with rotation, and they can be found even in the naked singularity case, unlike the QNM and QBM frequencies. Those results show that one needs additional criteria for sifting out the physical modes based on better understanding of the behavior of the radial function in the complex plane of the radial variable. Such study is outside the scope of the current work, which aims to demonstrate the dependence of the method for obtaining the frequencies with respect to changes in the phase-condition and thus to provoke work in this area.

**CONCLUSION**

From the recent developments in the field of gravitational waves detection, it is clear that finding the EM counterpart to those events can prove to be very useful. In this case, it is needed to better understand the fundamental physics of quasinormal ringing. In this paper, our team offered a new approach to finding the QNMs for the KBH, based on directly solving the system obtained by the analytical solutions of the TRE and TAE in terms of the confluent Heun function. This approach has the advantage of being more traditional (i.e. imposing directly the corresponding boundary conditions on the exact analytical solutions of the problem) and hence it should allow better understanding of the peculiar properties of the EM QNMs and the physics they imply.

It was shown that using this approach one can reproduce the frequencies already obtained by other authors, but without relying on approximate methods. Particularly important is the ability to impose the boundary condition directly on the solutions of the differential equations. We require the standard regularity condition on the TAE and explore in detail the radial boundary condition (the BHBC). We then can solve the system for each of the radial equations, solution of the Teukolsky Master Equation and find both quasinormal and quasibound modes, and additional spurious spectra. By tracking the boundary condition at infinity we are able to work with all those modes at the same time and to obtain their spectra with and without rotation. Such a result shows the advantage of our novel method over more traditional methods. It also raises the question of the theory of the Heun functions as a critical part of understanding the numerical results.

Our work demonstrates that the confluent Heun function and their implementation in Maple can be a useful tool in scientific investigations. More precisely, they enabled us to repeat with increased precision the already published results and also to reveal new properties of the numerical stability of the EM QNMs with respect to changes in the phase-condition.

**ACKNOWLEDGEMENTS**

The authors would like to thank Prof. E. Berti for discussion of the numerical values of the EM QNM frequencies obtained within the Leaver method, which was important for the comparison of our method with this already well-established one. The authors would like to thank Dr. Edgardo Cheb-Terrab for useful discussions of the algorithms evaluating the Heun functions in Maple and for continuing the improvement of those algorithms in the latest versions of Maple.

This article was supported by the Foundation "Theoretical and Computational Physics and Astrophysics", by
the Bulgarian National Scientific Fund under contracts DO-1-872, DO-1-895, DO-02-136, and Sofia University Scientific Fund, contract 185/26.04.2010, Grants of the Bulgarian Nuclear Regulatory Agency for 2013 and 2014.

**AUTHOR CONTRIBUTIONS**

P.F. posed the problem of evaluation of the EM QNMs of rotating BHs as a continuation of previous studies of the applications of the confluent Heun functions in astrophysics. He proposed the epsilon method and supervised the project.

D.S. is responsible for the numerical results, their analysis and the plots and tables presented here.

Both authors discussed the results at all stages. The manuscript was prepared by D.S. and edited by P.F..

**Tables of the obtained EM QNMs**

Table I presents some of the values obtained for the EM QNM, converted to physical units using the relations:

\[
\omega^{\text{phys}} = \mathfrak{R}(\omega) \frac{c^3}{2\pi G M}, \quad \tau^{\text{phys}} = \frac{1}{\Im(\omega)} \frac{GM}{c^3}.
\]

Note that in those formulas a factor of 2 is missing because the EM QNMs were obtained for \(M_{\text{KBH}} = \frac{1}{2}\) and not for \(M_{\text{KBH}} = 1\). Then if \(M\) is the mass of the object in physical units (we use \(M = 10M_\odot\)), \(M_\odot\) – the mass of the Sun \((M_\odot = 1.98892 \times 10^{30}[kg])\) and \(G = 6.673 \times 10^{-11}[m^3/kg\cdot s^2], c = 2.99792458 \times 10^8[m/s]\), one obtains

\[
\omega^{\text{phys}} \approx \frac{32310}{M/M_\odot} \mathfrak{R}(\omega)[Hz], \quad \tau^{\text{phys}} \approx \frac{0.4925 \times 10^{-5} M/M_\odot}{\Im(\omega)}[s].
\]

| \(n\) = 0 | \(m = 0\) | \(\omega^{\text{phys}}_{m=0}[Hz]\) | \(\tau^{\text{phys}}_{m=0}[ms]\) | \(m = -1\) | \(\omega^{\text{phys}}_{m=-1}[Hz]\) | \(\tau^{\text{phys}}_{m=-1}[ms]\) | \(m = 1\) | \(\omega^{\text{phys}}_{m=1}[Hz]\) | \(\tau^{\text{phys}}_{m=1}[ms]\) |
|---|---|---|---|---|---|---|---|---|---|
| \(a/M\) | \(\omega^{\text{phys}}_{m=0}[Hz]\) | \(\tau^{\text{phys}}_{m=0}[ms]\) | \(\omega^{\text{phys}}_{m=-1}[Hz]\) | \(\tau^{\text{phys}}_{m=-1}[ms]\) | \(\omega^{\text{phys}}_{m=1}[Hz]\) | \(\tau^{\text{phys}}_{m=1}[ms]\) |
| 0 | 802.1512449166 | 0.5325890917 | 802.1512449166 | 0.5325890917 | 802.1512449167 | 0.5325890917 |
| 0.2 | 804.9393652797 | 0.5343561642 | 849.8315682698 | 0.5388677452 | 763.6902591869 | 0.5299212818 |
| 0.6 | 829.2637578502 | 0.5526525743 | 996.9258848852 | 0.5772488810 | 704.6451920585 | 0.5313970112 |
| 0.98 | 884.6086875757 | 0.6427140687 | 1445.8814670353 | 1.2178343064 | 661.8628389523 | 0.5372375077 |

| \(n\) = 3 | \(m = 0\) | \(\omega^{\text{phys}}_{m=0}[Hz]\) | \(\tau^{\text{phys}}_{m=0}[ms]\) | \(m = -1\) | \(\omega^{\text{phys}}_{m=-1}[Hz]\) | \(\tau^{\text{phys}}_{m=-1}[ms]\) | \(m = 1\) | \(\omega^{\text{phys}}_{m=1}[Hz]\) | \(\tau^{\text{phys}}_{m=1}[ms]\) |
|---|---|---|---|---|---|---|---|---|---|
| \(a/M\) | \(\omega^{\text{phys}}_{m=0}[Hz]\) | \(\tau^{\text{phys}}_{m=0}[ms]\) | \(\omega^{\text{phys}}_{m=-1}[Hz]\) | \(\tau^{\text{phys}}_{m=-1}[ms]\) | \(\omega^{\text{phys}}_{m=1}[Hz]\) | \(\tau^{\text{phys}}_{m=1}[ms]\) |
| 0 | 472.3043572607 | 0.0638131626 | 472.3043572599 | 0.0638131626 | 472.3043572609 | 0.0638131626 |
| 0.2 | 479.7880705182 | 0.0641624606 | 549.5382706431 | 0.0658306413 | 415.5213067645 | 0.0624052214 |
| 0.6 | 530.3546588805 | 0.0677548111 | 792.4033820584 | 0.0741706889 | - | - |
| 0.98 | 481.050806664 | 0.0750272782 | - | - | - | - |

Table I. Table of the frequencies, \(\omega^{\text{phys}}\), in Hz , the damping times, \(\tau^{\text{phys}}\), in milliseconds for \(n = 0, 3, l = 1\) for some chosen values of the rotational parameter \(a\). Here \(M = 10M_\odot\).
Table II. Table of the separation parameter $E$ for $n = 0, 3, l = 1$ for some chosen values of the rotational parameter $a$. Here $M = 10M_\odot$.

|      |      |      |
|------|------|------|
| $a/M$ | $E_{m=0}$ | $E_{m=-1}$ | $E_{m=1}$ |
| 0    | 2.000000000 + 9.410^{-6}i | 2.000000000 + 4.8210^{-6}i | 2.000000000 + 4.8310^{-6}i |
| 0.2  | 1.999142948 - 0.000350608i | 1.946050875 + 0.0193509134i | 2.0642372214 + 0.0176327659i |
| 0.6  | 1.991655231 - 0.0066071264i | 1.797091424 + 0.0620215343i | 1.252126190 + 0.0478086914i |
| 0.98 | 1.973406595 - 0.0162271167i | 1.4506680177 + 0.0650778034i | 1.2533566585 + 0.0705481152i |

$E_{m=0}$ = 0.6
$E_{m=-1}$ = 0.2
$E_{m=1}$ = 0.0

* dstaicova@inrne.bas.bg
† fiziev@phys.uni-sofia.bg

[1] TEUKOLSKY S. A., REVIEW ARTICLE, The Kerr Metric, to appear in Classical and Quantum Gravity for its "Milestones of General Relativity" focus issue to be published during the Centenary Year of GR, arXiv:1410.2130
[2] REGGE, T., WHEELER J. A., Stability of a Schwarzschild Singularity, Phys. Rev. 108:14: 1063-1069 (1957)
[3] ZERILLI, F. J., Effective Potential for Even-Parity Regge-Wheeler Gravitational Perturbation Equations, Phys. Rev. Lett. 24:13: 737-738 (1970)
[4] VISHVESHWARA, C. V., Stability of the Schwarzschild Metric, Phys. Rev. D 1:1.10: 2870-2879 (1970)
[5] BARDEEN, J.M., PRESS, W. H., TEUKOLSKY, S. A., Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation, ApJ 178: 347-370 (1972)
[6] TEUKOLSKY, S. A., Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations, Phys. Rev. Lett. 29: 1114-1118 (1972)
[7] TEUKOLSKY, S. A., Perturbations of a rotating black hole I Fundamental Equations for Gravitational, Electromagnetic and Neutrino-field Perturbations, ApJ 185: 635-648 (1973)
[8] PRESS W.A., TEUKOLSKY S. A., Perturbations of a Rotating Black Hole. II. Dynamical Stability of the Kerr Metric, ApJ 185: 649-674 (1973)
[9] TEUKOLSKY S. A., PRESS W. H., Perturbations of a rotating black hole. III - Interaction of the hole with gravitational and electromagnetic radiation, ApJ 193: 443 (1974)
[10] CHANDRASEKHAR S., On the Equations Governing the Perturbations of the Schwarzschild Black Hole, Proc. Roy. Soc. London A343: 289-298 (1975)
[11] CHANDRASEKHAR S., AND DETWEILER S. L., The quasi-normal modes of the Schwarzschild black hole, Proc. Roy. Soc. London A344: 441-452 (1975)
[12] CHANDRASEKHAR S., On a transformation of Teukolsky’s equation and the electromagnetic perturbations of Kerr black hole, Proc. R. Soc. Lond. A 348, 39-55 (1976)
[13] S. DETWEILER, On the equations governing the electromagnetic perturbations of the Kerr black hole, Proc. R. Soc. London A 349, 217-230 (1976)
[14] CHANDRASEKHAR S., On the equations governing the perturbations of Reissner - Nordstrø m black hole, Proc. R. Soc. London A 3365, 453-465 (1976)
[15] DETWEILER S., Black holes and gravitational waves. III - The resonant frequencies of rotating holes, ApJ:239, 292-295 (1980)
[16] CHANDRASEKHAR S., The mathematical theory of black holes, Clarendon Press/Oxford University Press (International Series of Monographs on Physics. Volume 69), (1983)
[17] LEAVER E. W., An analytic representation for the quasinormal modes of Kerr black holes, Proc. Roy. Soc. London A402: 265-298 (1985)
[18] LEAVER E. W., Solutions to a generalized spheroidal wave equation: Teukolsky’s equations in general relativity, and the two-center problem in molecular quantum mechanics, J.Math. Phys. 27:5:1238 (1986)
[19] ANDERSSON N., A numerically accurate investigation of black-hole normal modes, Proc. Roy. Soc. London A 439 no.1905: 47-58 (1992)
[20] BERITI E., CARDOSO V., KOKKOTAS K. D., ONOZAWA H., Highly damped quasinormal modes of Kerr black holes Phys.Rev. D68 (2003) 124018, arXiv:hep-th/0310032v2 (2004)
[21] BERITI E., Exact Solutions of Regge-Wheeler Equation and Quasi-Normal Modes of Compact Objects, Class. Quant. Grav. 23 2447-2468 (2006), arXiv:0509123v5 [gr-qc]
[22] FERRARI V., GUALTIERI L., Quasi-normal modes and gravitational wave astronomy, Gen.Rel.Grav.40: 945-970 (2008), arXiv:0709.0657v2 [gr-qc] (2007)
[23] BERITI E., CARDOSO V. AND STARINETS A. O., Quasinormal modes of black holes and black branes, Class. Quantum Grav. 26 163001 (108pp) (2009)
[24] FIZIEV P. P., Teukolsky-Starobinsky identities: A novel derivation and generalizations, Phys. Rev. D80, 124001
[26] Fiziev P. P., Classes of exact solutions to the Teukolsky master equation Class. Quantum Grav. 27 135001 (2010), arXiv:0908.4234v4 [gr-qc]

[27] Hod S., Hod O., Analytic treatment of the black-hole bomb, Phys. Rev. D 81, 061502 (2010) Rapid communication, arXiv:0910.0734v1 [gr-qc]

[28] Konoplya, R. A., Zhidenko, A., Quasinormal modes of black holes: from astrophysics to string theory, Reviews of Modern Physics, 83: 793 - 836, issue 3, (2011), arXiv:1102.401v1 [gr-qc] (2011)

[29] Hod S., Quasinormal resonances of a massive scalar field in a near-extremal Kerr black hole spacetime Physical Review D 84, 044046 (2011), arXiv:1109.4080v1 [gr-qc]

[30] Schnittman, J. D., Electromagnetic counterparts to black hole mergers Classical and Quantum Gravity, 28, Issue 9, pp. 094021 (2011) arXiv:1010.3250v1 [astro-ph.HE]

[31] Abbott B. P. et al., LIGO scientific collaboration, Directional limits on persistent gravitational waves using LIGO S5 science data arXiv:1109.1809v2 [astro-ph.CO]

[32] The LIGO Scientific Collaboration, the Virgo Collaboration, Search for gravitational waves from binary black hole inspiral, merger and ringdown Phys.Rev.D83:122005,(2011), arXiv:1102.3781v1 [gr-qc]

[33] The LIGO Scientific Collaboration, Abadie J. et al., Search for Gravitational Wave Bursts from Six Magnetars Astrophys.J.734:(L35,2011), arXiv:1011.4079v2 [astro-ph.HE]

[34] The LIGO Scientific Collaboration, A search for gravitational waves associated with the August 2006 timing glitch of the Vela pulsar Phys.Rev.D83:094030, (2010), arXiv:0912.0360v2 [gr-qc]

[35] LIGO Scientific Collaboration, First search for gravitational waves from the youngest known neutron star, Astrophys.J.722:1504,(2010), arXiv:1006.2535v2 [gr-qc]

[36] The LIGO Scientific Collaboration, the Virgo Collaboration, Search for Gravitational Waves from Compact Binary Coalescence in LIGO and Virgo Data from S5 and VSR1, Phys.Rev.D82:102001,(2010), arXiv:1005.4655v1 [gr-qc]

[37] The LIGO Scientific Collaboration, the Virgo Collaboration, All-sky search for gravitational-wave bursts in the first joint LIGO-GEO-Virgo run, Phys.Rev.D81:102001,2010 arXiv:1002.1036v2 [gr-qc]

[38] Dietz A. (LIGO Scientific Collaboration and the Virgo Collaboration) , Searches for inspiral gravitational waves associated with short gamma-ray bursts in LIGO’s fifth and Virgo’s first science run arXiv:1006.3393v1 [gr-qc]

[39] The LIGO Scientific Collaboration, the Virgo Collaboration, Search for gravitational-wave bursts associated with gamma-ray bursts using data from LIGO Science Run 5 and Virgo Science Run 1, Astophysical Journal 715 (2010) 1438-1452 arXiv:0908.3824v2 [astro-ph.HE]

[40] Coward D. M., Gendre B., Sutton P. J., Howell E. J., Regimbau T., Laas-Bourez M., Klotz A., Boer M., Branchesi M., Toward an optimal search strategy of optical and gravitational wave emissions from binary neutron star coalescence, MNRAS, 415:L26, arXiv:1104.5552v1 [astro-ph.HE]

[41] Bogdanovic T. , Bode T. , Haas R. , Laguna P. , Shoemaker D. , Properties of Accretion Flows Around Coalescing Supermassive Black Holes, Classical and Quantum Gravity, 28:094020 (2011), arXiv:1010.2496v2 [astro-ph.CO]

[42] Moesta P., Alic D., Rezzolla L., Zanotti O., Palenz C., On the detectability of dual jets from binary black holes arXiv:1109.1177v1 [gr-qc]

[43] The LIGO Scientific Collaboration, the Virgo Collaboration, Implementation and testing of the first prompt search for electromagnetic counterparts to gravitational wave transients, arXiv:1109.3498v1 [astro-ph.IM]

[44] Christensen N.L., for the LIGO Scientific Collaboration, the Virgo Collaboration, Multimessenger Astronomy, arXiv:1105.5843v1 [gr-qc]

[45] Jaramillo J. L., Macedo R. P., Moesta P., Rezzolla L., Black-hole horizons as probes of black-hole dynamics i: post-merger recoil in head-on collisions, Submitted to PRD, arXiv:1108.0060v1 [gr-qc]

[46] Berti E., Cardoso V., Will C. M. , On gravitational-wave spectroscopy of massive black holes with the space interferometer LISA, Phys.Rev.D 73:064030, (2006), arXiv:0512160v2 [gr-qc]

[47] Schutz B. F., Centrella J., Cutler C., Hughes S. A., Will Einstein Have the Last Word on Gravity?, astro2010: The Astronomy and Astrophysics Decadal Survey, arXiv:0903.0100v1 [gr-qc]

[48] Chirenti C. B. M. H., Rezzolla L., How to tell gravastar from black hole, Class. Quant. Grav. 24:, 4191-4206, (2007), arXiv:0706.1531v2 [gr-qc]

[49] Chirenti C. B. M. H., Rezzolla L., Ergoregion instability in rotating gravastars, Phys.Rev.D 78:084011, (2008), arXiv:0808.4080v1 [gr-qc]

[50] Pani P., Berti E., Cardoso V., Chen Y., Norte R., Gravitational wave signatures of the absence of an event horizon: Nonradial oscillations of a thin-shell gravastar , Phys.Rev.D 80:124047,(2009) , arXiv:0909.0287v2 [gr-qc]

[51] Staciva D., Fiziev P., The Spectrum of Electromagnetic Jets from Kerr Black Holes and Naked Singularities in the Teukolsky Perturbation Theory, Astrophysics and Space Science, 332, pp.385-401, arXiv:1002.0480v2 [astro-ph.HE], (2010)

[52] Lyutikov M., McKinney J. C., Slowly balding black holes, Phys. Rev. D, 84:084019, arXiv:1109.0584v1 [astro-ph.HE], (2011)

[53] Gao, H., Zhang B. ,B and Zhang B., Evidence Of Superposed Variability Components In GRB Prompt Emission Lightcurves, arXiv:1103.0074v2 [astro-ph.HE], (2011)

[54] Zhang B., Open Questions in GRB Physics Comptes Rendus Physique, 12, 206-225 (2011), arXiv:1104.0932v1 [astro-ph.HE]

[55] Lei W.-H., Zhang B., Black hole Spin in Sw J1644+57 and Sw J2058+05 , ApJ L27: 740, arXiv:1108.3115v2 [astro-ph.HE],(2011)

[56] Fiziev P. P., Novel relations and new properties of confluent Heun’s functions and their derivatives of arbitrary order, J. Phys. A: Math. Theor. 43 (2010) 035203, arXiv:0904.0245 [math-ph]

[57] Slavanov S. Y., Lay W., Special Functions, A Unified Theory Based on Singularities (Oxford: Oxford
[58] HEUN K., Math. Ann. 33 161, (1889)

[59] DECARREAU A., DUMONT-LEPAGE M. CL., MARONI P., ROBERT A. AND RONEAUX A., Ann. Soc. Buxelles 92 53, (1978)

[60] DECARREAU A., MARONI P. AND ROBERT A., 1978 Ann. Soc. Buxelles 92 151. 1995 Heun’s Differential Equations ed Roneaux A. Oxford: Oxford Univ. Press, (1995)

[61] FIZIEV P., STAICOVA D., Solving systems of transcendental equations involving the Heun functions., American Journal of Computational Mathematics Vol. 02 : 02, pp.95 (2012)

[62] FIZIEV P., STAICOVA D., Application of the confluent Heun functions for finding the QNMs of non-rotating black hole, Phys. Rev. D 84, 127502 (2011) , arXiv:1109.1532 [gr-qc]

[63] FIZIEV P., STAICOVA D., Two-dimensional generalization of the Muller root-finding algorithm and its applications (2011), arXiv:1005.5375v2 [cs.NA]

[64] MAASSEN VAN DEN BRINK A, Analytic treatment of black-hole gravitational waves at the algebraically special frequency, Phys. Rev. D 62 064009 (2000), arXiv:gr-qc/0001032v1

[65] LEUNG P. T., MAASSEN VAN DEN BRINK A., MAK K. W., YOUNG K., Unconventional Gravitational Excitation of a Schwarzschild Black Hole, Class. Quant. Grav. 20 L217 (2003), arXiv:gr-qc/0301018v4

[66] STAICOVA D., FIZIEV P., New results for electromagnetic quasinormal modes of black holes, arXiv:1112.0310 [astro-ph.HE]

[67] ROSA J. G. AND DOLAN S. R., Massive vector fields on the Schwarzschild spacetime: Quasinormal modes and bound states, Phys. Rev. D 85, 044043 (2012)

[68] ONOZAWA H., A detailed study of quasinormal frequencies of the Kerr black hole, Phys.Rev. D 55: 3593-3602 (1997), arXiv:gr-qc/9610048v1