Meiotic Drive in *D. albomicans* and *D. nasuta*

Supplementary Mathematica Notebook for Wang et al. “Neosex chromosome evolution shapes sex-dependent asymmetrical introgression barrier”

The General Model

Definitions

Define an ordering of Karyotype alleles:

\[ \text{Nx}=1, \text{Ax}=2, \text{Ny}=3, \text{Ay}=4 \]

```
In[1]= order = {{1, 1}, {1, 2}, {1, 3}, {1, 4}, {2, 2}, {2, 3}, {2, 4}};
order // MatrixForm
```

```
\begin{array}{cccc}
1 & 1 \\
1 & 2 \\
1 & 3 \\
1 & 4 \\
2 & 2 \\
2 & 3 \\
2 & 4 \\
\end{array}
```

```
In[2]= \text{hexToRGB} = \text{RGBColor} @@
    (\text{IntegerDigits}[\text{ToExpression}@\text{StringReplace}[\#, "\#" \rightarrow \text{"16^^"}], 256, 3]/255.) &;
\text{Col1} = \text{hexToRGB}["\#DC8665"];
\text{Col2} = \text{hexToRGB}["\#138086"];
\text{Col3} = \text{hexToRGB}["\#534666"];
\text{Col4} = \text{hexToRGB}["\#CD7672"];
\text{Col5} = \text{hexToRGB}["\#EEB462"];
```

```
In[3]= \text{AxNyCol} = \text{hexToRGB}["\#FFD700"];
\text{NxNyCol} = \text{hexToRGB}["\#FFFFD4"];
\text{AxAyCol} = \text{hexToRGB}["\#00bfff"];
```
Differential Equations

The Birth Matrix
Birth = Block[(RM, S, HI, MD, NXW, out),
(*Random Mating (White)*)
RM = {F[1, 1, t] * F[1, 3, t], F[1, 1, t] * F[1, 4, t],
    F[1, 1, t] * F[2, 3, t], F[1, 1, t] * F[2, 4, t], F[1, 2, t] * F[1, 3, t],
    F[1, 2, t] * F[1, 4, t], F[1, 2, t] * F[2, 3, t], F[1, 2, t] * F[2, 4, t],
    F[2, 2, t] * F[1, 3, t], F[2, 2, t] * F[1, 4, t],
    F[2, 2, t] * F[2, 3, t], F[2, 2, t] * F[2, 4, t]};
(*Segregation (Orange)*)
S = {};
(*NxNx * NxNy*) AppendTo[S, \{\frac{1}{2}, 0, \frac{1}{2} (1-\mu N), 0, 0, 0, 0\}];
(*NxNx * NxAy*) AppendTo[S, \{\frac{1}{2}, 0, 0, \frac{1}{2}, 0, 0, 0\}];
(*NxNx * AxNy*) AppendTo[S, \{0, \frac{1}{2}, \frac{1}{2} (1-\mu H), 0, 0, 0, 0\}];
(*NxNx * AxAy*) AppendTo[S, \{0, \frac{1}{2}, 0, \frac{1}{2}, (1-\mu A), 0, 0, 0\}];
(*NxAx * NxNy*) AppendTo[S, \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, (1-\mu N), 0, 0\}];
(*NxAx * NxAy*) AppendTo[S, \{\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}, 0, 0, \frac{1}{4}\}];
(*NxAx * AxNy*) AppendTo[S, \{0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, (1-\mu H), 0\}];
(*NxAx * AxAy*) AppendTo[S, \{0, \frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \frac{1}{4}\}];
(*AxAx * NxNy*) AppendTo[S, \{0, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, (1-\mu N), 0\}];
(*AxAx * NxAy*) AppendTo[S, \{0, \frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}\}];
(*AxAx * AxNy*) AppendTo[S, \{0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, (1-\mu H), 0\}];
(*AxAx * AxAy*) AppendTo[S, \{0, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, (1-\mu A)\}];
(*Hybrid Incompatability*)
HI = \{1, 1-\rho, 1, 1-\rho, 1, 1-\rho, 1\};
(*Miotic Drive*)
(*MD=(1,1,1-\mu N,1,1,1-\mu H,1-\mu A);*)
(*Selection for Ax (sX) Selection against sY*)
NXW = \{1, 1+sX, 1, 1, 1+2 sX, 1+sX, 1+sX-sY\};
Transpose[Transpose[RM * S] * HI * MD * NXW]
];

The total birth rate for normalization
The differential equations for numerical integration

\[
\texttt{fOdes = \{D[F[1, 1, t], t] = wVec[[1]] - F[1, 1, t],} \\
D[F[1, 2, t], t] = wVec[[2]] - F[1, 2, t], D[F[1, 3, t], t] = wVec[[3]] - F[1, 3, t],} \\
D[F[1, 4, t], t] = wVec[[4]] - F[1, 4, t], D[F[2, 2, t], t] = wVec[[5]] - F[2, 2, t],} \\
D[F[2, 3, t], t] = wVec[[6]] - F[2, 3, t], D[F[2, 4, t], t] = wVec[[7]] - F[2, 4, t]\};
\]

The differential equations for equilibria analysis

\[
\texttt{equList = (wVec[[1]] - F[1, 1, t],} \\
wVec[[2]] - F[1, 2, t], wVec[[3]] - F[1, 3, t], wVec[[4]] - F[1, 4, t],} \\
wVec[[5]] - F[2, 2, t], wVec[[6]] - F[2, 3, t], wVec[[7]] - F[2, 4, t]\};
\]

The differential equations for the Jacobian Matrix

\[
\texttt{VarList = \{F[1, 1, t], F[1, 2, t], F[1, 3, t], F[1, 4, t], F[2, 2, t], F[2, 3, t], F[2, 4, t]\};}
\]
\[ dF[1, 1, t_] := wVec[[1]] - F[1, 1, t] \\
\[ dF[1, 2, t_] := wVec[[2]] - F[1, 2, t] \\
\[ dF[1, 3, t_] := wVec[[3]] - F[1, 3, t] \\
\[ dF[1, 4, t_] := wVec[[4]] - F[1, 4, t] \\
\[ dF[2, 2, t_] := wVec[[5]] - F[2, 2, t] \\
\[ dF[2, 3, t_] := wVec[[6]] - F[2, 3, t] \\
\[ dF[2, 4, t_] := wVec[[7]] - F[2, 4, t] \\
\]

**Numerical Dynamics**

\[ testPars = \{\rho \to 0.2, \mu A \to 0.3, \mu H \to 0.2, \mu N \to 0.2, sX \to 0.1, sY \to 0.1\}; \]

The initial conditions given the experimental design

\[ Inits = \{F[1, 1, 0] = 1/2 * 1/4, F[1, 2, 0] = 2 * 1/2 * 1/4, \]
\[ F[1, 3, 0] = 1/2 * 1/4, F[1, 4, 0] = 1/2 * 1/4, \]
\[ F[2, 2, 0] = 1/2 * 1/4, F[2, 3, 0] = 1/2 * 1/4, F[2, 4, 0] = 1/2 * 1/4 \}

Solving the ODEs numerically

\[ Clear[FNSol] \\
\[ FNSol[pars_] := Flatten[NDSolve[Join[fOdes /. pars, Inits], fVars, \{t, 0, 100\}]] \]

\[ FN[k1_, k2_, t1_, pars_] := F[k1, k2, t] /. FNSol[pars] /. t \to t1 \]

\[ FN[1, 1, 0.2, testPars] \]

\[ 0.12805 \]

**Plot By Parental Type**

Plotting the karyotype frequencies by Parental type
PlotF[pars_] := GraphicsRow[
    Plot[{FN[1, 1, t1, pars], FN[1, 3, t1, pars]}, (t1, 0, 30),
        Frame -> True, PlotLabel -> "N Parental", PlotRange -> {0, Automatic},
        Epilog -> Inset[LineLegend[({ColorData[97][1], ColorData[97][2],
            "NxNx", "NxNy"}), Scaled[{0.8, 0.5}]],
        Plot[{FN[1, 2, t1, pars], FN[1, 4, t1, pars], FN[2, 3, t1, pars]},
        (t1, 0, 30), Frame -> True, PlotLabel -> "Hybrid", PlotRange -> {0, All},
        PlotStyle -> {Thickness[0.01], Automatic, Dotted}, Epilog ->
        Inset[LineLegend[({ColorData[97][1], ColorData[97][2], ColorData[97][3]},
            "(1/2)NxAx", "NxAy", "AxNy"), Scaled[{0.8, 0.5}]]],
        Plot[{FN[2, 2, t1, pars], FN[2, 4, t1, pars]}, (t1, 0, 30), Frame -> True,
        PlotLabel -> "A Parental", PlotRange -> {0, Automatic},
        Epilog -> Inset[LineLegend[({ColorData[97][1], ColorData[97][2]},
            "AxAx", "AxAy"), Scaled[{0.8, 0.25}])}, ImageSize -> Full]
]

Manipulate[PlotF[{ρ → r, μA → uA, μH → uH, μN → uN, sX → Sx, sY → Sy}],
    {{r, 0.2}, 0, 1, 0.1}, {{uA, 0.3}, 0, 1, 0.4}, {{uH, 0.2}, 0, 1, 0.4},
    {{uN, 0.2}, 0, 1, 0.4}, {{Sx, 0.1}, 0, 0.2, 0.01}, {{Sy, 0.1}, 0, 0.2, 0.01}]

Plot of Karyotype and Allele Frequencies by Sex

FemaleTot[pars_, t1_] := FN[1, 1, t1, pars] + FN[1, 2, t1, pars] + FN[2, 2, t1, pars]
MaleTot[pars_, t1_] :=
    FN[1, 3, t1, pars] + FN[2, 3, t1, pars] + FN[1, 4, t1, pars] + FN[2, 4, t1, pars]
Plot2[pars_] := GraphicsRow[
  Plot[{FN[1, 1, t1, pars] / FemaleTot[pars, t1], FN[1, 2, t1, pars] / FemaleTot[pars, t1], FN[2, 2, t1, pars] / FemaleTot[pars, t1], 
    FN[1, 1, t1, pars] + FN[1.2, t1, pars] / 2, FemaleTot[pars, t1], 
    FN[1.2, t1, pars] / 2 + FN[2, 2, t1, pars] / 2, FemaleTot[pars, t1], 
    {t1, 0, 100}, Frame -> True, PlotLabel -> "Female", PlotRange -> (0, All), 
    PlotStyle -> {ColorData[97][1], ColorData[97][2], ColorData[97][3], 
      {ColorData[97][4], Dashed}, {ColorData[97][5], Dashed}}, 
    Epilog -> Inset[LineLegend[{{ColorData[97][1], ColorData[97][2], 
      ColorData[97][3], (ColorData[97][4], Dashed), (ColorData[97][5], Dashed), 
      "NXNX", "(0.5)NXAX", "AXAX", "NX", "AX"}], Scaled[(0, 0.5)]], 
      Plot[{FN[1, 4, t1, pars] / MaleTot[pars, t1], FN[1, 3, t1, pars] / MaleTot[pars, t1], 
        FN[2, 3, t1, pars] / MaleTot[pars, t1], 
        FN[1, 4, t1, pars] + FN[1, 3, t1, pars], FN[2, 3, t1, pars] + FN[2, 4, t1, pars], 
        MaleTot[pars, t1], 
        FN[1, 3, t1, pars] + FN[2, 3, t1, pars], FN[1, 4, t1, pars] + FN[2, 4, t1, pars], 
        MaleTot[pars, t1], 
        {t1, 0, 100}, Frame -> True, PlotLabel -> "Male", PlotRange -> (0, Automatic), 
        PlotStyle -> {ColorData[97][1], ColorData[97][2], ColorData[97][3], 
          ColorData[97][4], {ColorData[97][5], Dashed}, {Dashed, ColorData[97][6]}, 
          Directive[ColorData[97][7], Dashed], (Dashed, ColorData[97][8])}, Epilog -> 
          Inset[LineLegend[{{ColorData[97][1], ColorData[97][2], ColorData[97][3], 
            ColorData[97][4], {ColorData[97][5], Dashed}, {Dashed, ColorData[97][6]}, 
            Directive[ColorData[97][7], Dashed], (Dashed, ColorData[97][8])}, 
            "NXAY", "NXNY", "AXAY", "AXY", "NX", "AX", "NY", "AY"], 
            Scaled[(0.8, 0.5)]], ImageSize -> Full}]}]
Manipulate[PlotF2[{\(\rho \to r\), \(\mu_A \to u_A\), \(\mu_H \to u_H\), \(\mu_N \to u_N\), \(s_X \to S_x\), \(s_Y \to S_y\)}],
{{r, 0.2}, 0, 1, 0.1}, {{u_A, 0.3}, 0, 1, 0.4}, {{u_H, 0.2}, 0, 1, 0.4},
{{u_N, 0.2}, 0, 1, 0.4}, {{S_x, 0.1}, 0, 0.2, 0.01}, {{S_y, 0.1}, 0, 0.2, 0.01}]

Equilibria-New

Numerical Equilibrium

Define an ordering of Karyotype alleles:
\[N_x=1, A_x=2, N_y=3, A_y=4\]

\[\text{order} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\};\]

\[\text{testPars} = \{\rho \to 0.2, \mu_A \to 0.3, \mu_H \to 0.2, \mu_N \to 0.15, s_X \to 0.1, s_Y \to 0.05\};\]
Finding AxAx/AxAy equilibrium (Equ 1)

\[
\text{AxAxAx} = \{F[1, 1, t] \to 0, F[1, 2, t] \to 0, F[1, 3, t] \to 0, F[1, 4, t] \to 0, F[2, 3, t] \to 0\};
\]

\[
\text{equ1sol} = \text{Solve}[\{\text{equList} /. \text{AxAxAx}\}[[\{5, 7\}]] = \{0, 0\},
\{F[2, 2, t], F[2, 4, t]\}] // \text{Simplify} // \text{Flatten}
\]

\[
\text{Equ1} = \text{Join}[\text{AxAxAx}, \text{equ1sol}];
\]

Finding AxAx/AxNy equilibrium (Equ 2)

\[
\text{AxAxAxNy} = \{F[1, 1, t] \to 0, F[1, 2, t] \to 0, F[1, 3, t] \to 0, F[1, 4, t] \to 0, F[2, 4, t] \to 0\};
\]

\[
\text{equ2sol} = \text{Solve}[\{\text{equList} /. \text{AxAxAxNy}\}[[\{5, 6\}]] = \{0, 0\},
\{F[2, 2, t], F[2, 3, t]\}] // \text{Simplify} // \text{Flatten}
\]

\[
\text{Equ2} = \text{Join}[\text{AxAxAxNy}, \text{equ2sol}];
\]

Finding NxNx/NxAy equilibrium (Equ 3)

\[
\text{NxNxNxAy} = \{F[1, 2, t] \to 0, F[1, 3, t] \to 0, F[2, 2, t] \to 0, F[2, 3, t] \to 0, F[2, 4, t] \to 0\};
\]
equ3sol = SolveAlways[(equList /. NxNxNxAy)[[{
{1, 4}]]] == \{0, 0\}, \{F[1, 1], F[1, 4]\}] // Simplify // Flatten
Out[66] = \{F[1, 1, t] \rightarrow \frac{1}{2 - \rho}, F[1, 4, t] \rightarrow \frac{-1 + \rho}{-2 + \rho}\}

Equ3 = Join[NxNxNxAy, equ3sol];

Finding NxNx/NxNy equilibrium (Equ 4)

NxNxNxNy = \{F[1, 2, t] \rightarrow 0, F[1, 4, t] \rightarrow 0, F[2, 2, t] \rightarrow 0, F[2, 3, t] \rightarrow 0, F[2, 4, t] \rightarrow 0\};
equ4sol = Solve[(equList /. NxNxNxNy)[[\{1, 3\}]] == \{0, 0\}, \{F[1, 1, t], F[1, 3, t]\}] // Flatten
Out[9] = \{F[1, 1, t] \rightarrow \frac{1}{2 - \mu N}, F[1, 3, t] \rightarrow \frac{-1 + \mu N}{-2 + \mu N}\}

Equ4 = Join[NxNxNxNy, equ4sol];

Stability-New

JMtrx = Block[{out, a1, a2, b1, b2}, out = \{
 (*Loop through equations*)
 For[a1 = 1, a1 \leq 2, a1++,
  For[b1 = 1, b1 \leq 4, b1++,
   If[b1 \geq a1,
     AppendTo[out, {}];
   (*Loop through variables*)
   For[a2 = 1, a2 \leq 2, a2++,
    For[b2 = 1, b2 \leq 4, b2++,
     If[b2 \geq a2,
      AppendTo[out[[\{-1\}], D[dF[a1, b1, t], F[a2, b2, t]]];
      ]
     ]
    ]
   ]
  ]
 ]
out];
Equilibrium 1: AxAx/AxAy

Case A: sY=0, μH=0, μN=0

\[ J M t r x 1 = J M t r x / . A x A x A x A y / . E q u l / / S i m p l i f y ; \]

\[ C a s e A S u b = \{ s Y \rightarrow 0, \mu H \rightarrow 0, \mu N \rightarrow 0 \}; \]

\[ \lambda L i s t 1 A = E i g e n v a l u e s [ J M t r x 1 / . C a s e A S u b ] / / S i m p l i f y \]

\[ \{ -1, -1, -1, -1, \frac{\mu A + \rho}{1 + \mu A}, -\frac{3 - \sqrt{9 + 18 s X + s X^2 + \rho + \sqrt{9 + 18 s X + s X^2} \rho + s X (7 + \rho)}}{4 + 8 s X}, \]

\[ -\frac{3 + \sqrt{9 + 18 s X + s X^2 + \rho - \sqrt{9 + 18 s X + s X^2} \rho + s X (7 + \rho)}}{4 + 8 s X} \} \]

\[ \text{Reduce} \]

\[ \{ \lambda L i s t 1 A[[5]] < 0, \lambda L i s t 1 A[[6]] < 0, \lambda L i s t 1 A[[7]] < 0, 0 < s X, 0 < \rho < 1, 0 < \mu A < 1 \} \]

\[ 0 < \rho < 1 && s X > 0 && 0 < \mu A < \rho \]

Equilibrium 1A: is stable whenever \( \mu A < \rho \)

Case B: sY=0, μA=0

\[ C a s e B S u b = \{ s Y \rightarrow 0, \mu A \rightarrow 0 \}; \]

\[ \lambda L i s t 1 B = E i g e n v a l u e s [ J M t r x 1 / . C a s e B S u b ] / / S i m p l i f y \]

\[ \{ -1, -1, -1, -1, -\frac{3 - \sqrt{9 + 18 s X + s X^2 + \rho + \sqrt{9 + 18 s X + s X^2} \rho + s X (7 + \rho)}}{4 + 8 s X}, \]

\[ -\frac{3 + \sqrt{9 + 18 s X + s X^2 + \rho - \sqrt{9 + 18 s X + s X^2} \rho + s X (7 + \rho)}}{4 + 8 s X}, \mu H (-1 + \rho) - \rho \} \]

\[ \text{Reduce} \]

\[ \{ \lambda L i s t 1 B[[5]] < 0, \lambda L i s t 1 B[[6]] < 0, \lambda L i s t 1 B[[7]] < 0, 0 < s X, 0 < \rho < 1, 0 < \mu H < 1 \} \]

\[ 0 < \rho < 1 && s X > 0 && 0 < \mu H < 1 \]

Equilibrium 1B: is always stable

Case C: μH=0, μA=0, μN=0

\[ C a s e C S u b = \{ \mu H \rightarrow 0, \mu A \rightarrow 0, \mu N \rightarrow 0 \}; \]
\( \lambda_{\text{List1C}} = \text{Eigenvalues}[\text{JMtrx1} /. \text{CaseDSub}] // \text{Simplify} \)

\[
\{-1, -1, -1, -1, \frac{sY - (1 + sX) \rho}{1 + sX - sY}, -\frac{1}{4 (1 + 2 sX) (1 + sX - sY)} \left\{ 3 - 3 sY - \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho - sY \rho + \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho + 2 sX (5 + \rho) + sX^2 (7 + \rho) - sX sY (7 + \rho) \right\}, -\frac{1}{4 (1 + 2 sX) (1 + sX - sY)} \left\{ 3 - 3 sY + \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho - sY \rho - \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho + 2 sX (5 + \rho) + sX^2 (7 + \rho) - sX sY (7 + \rho) \right\} \}
\]

\( \text{Reduce} \[
\{ \lambda_{\text{List1C}}[[5]] < 0, \lambda_{\text{List1C}}[[6]] < 0, \lambda_{\text{List1C}}[[7]] < 0, 0 < sX, 0 < \rho < 1, 0 < sY < 1 \}
\]

Equilibrium 1C: is stable whenever \( \frac{sY}{1 + sX} < \rho < 1 \)

Case D: \( \mu N = 0 \)

\( \text{CaseDSub} = \{ \mu A \rightarrow 0, \mu N \rightarrow 0 (*, \mu H \rightarrow 0.12, sY \rightarrow 0.015*) \} \)

\( \lambda_{\text{List1D}} = \text{Eigenvalues}[\text{JMtrx1} /. \text{CaseDSub}] // \text{Simplify} \)

\[
\{-1, -1, -1, -1, -\frac{1}{4 (1 + 2 sX) (1 + sX - sY)} \left\{ 3 - 3 sY - \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho - sY \rho + \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho + 2 sX (5 + \rho) + sX^2 (7 + \rho) - sX sY (7 + \rho) \right\}, -\frac{1}{4 (1 + 2 sX) (1 + sX - sY)} \left\{ 3 - 3 sY + \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho - sY \rho - \sqrt{1 + sX} \sqrt{9 + sX^3 + sX^2 (19 - 2 sY) - 10 sY + sY^2 + sX (27 - 20 sY + sY^2)} + \rho + 2 sX (5 + \rho) + sX^2 (7 + \rho) - sX sY (7 + \rho) \right\} \}
\]

\( \text{Simplify} \frac{\lambda_{\text{List1D}}[1]}{\lambda_{\text{List1D}}[5]} \)

\( \frac{sY}{1 + sX} < \rho < 1 \)
\textbf{Equilibrium 1D: Complicated}

\textbf{Equilibrium 2: AxAx/AxNy}

\textbf{Case A: sY=0, \mu H=0, \mu N=0}

\textbf{Case ASub} = \{sY \to 0, \mu H \to 0, \mu N \to 0\};

\textbf{\lambda List2A} = \text{Eigenvalues}[JMtrx2 /. CaseASub] // Simplify

\textbf{Out[77]} = \{
  \frac{\mu A - \rho}{-1 + \rho}, -1, -1, -1, -1, -1
\}

Reduce[\{\lambda List1D[[5]] < 0, \lambda List1D[[6]] < 0, 
\lambda List1D[[7]] < 0, 0 < sX < 1, 0 < \rho < 1, 0 < \mu H < 1, 0 < sY < 1\}] // Simplify

Out[74] = \begin{align*}
0 < sX < \frac{1}{\sqrt{3}} & \quad \land \quad 0 < sY < \frac{3 sX (1 + sX)}{1 + 3 sX} \quad \land \quad \mu H < 1 \land \\
\left(0 < \mu H \land \frac{sY}{1 + sX} < \rho \land \rho < 1\right) & \quad \land \quad \left(0 < \rho \land \frac{-sY + \rho + sX \rho}{1 + sX} < \mu H \land \frac{sY}{1 + sX} < 0 \right) \land \\
\left(0 < \rho \land \frac{sY}{1 + sX} < \mu H \land \frac{-sY + \rho + sX \rho}{1 + sX} < 0 \right) & \quad \land \quad \left(0 < \rho \land \frac{-sY + \rho + sX \rho}{1 + sX} < 0 \right) \land \\
\left(0 < \rho \land \frac{sY}{1 + sX} < 0 \right) & \quad \land \quad \left(0 < \rho \land \frac{sY}{1 + sX} < 0 \right) \land \\
3 sX (1 + sX) < sY < 1 \land \mu H < 1 \land \\
\left(0 < \mu H \land \frac{sY}{1 + sX} < 0 \right) & \quad \land \quad \left(0 < \mu H \land \frac{sY}{1 + sX} < 0 \right)
\end{align*}
\begin{verbatim}
In[78]:= Reduce[
{\lambda List2A[[1]] < 0, \lambda List2A[[6]] < 0, \lambda List2A[[7]] < 0, 0 < sX, 0 < \rho < 1, 0 < \mu A < 1}]
Out[78]= 0 < \rho < 1 && sX > 0 && \rho < \mu A < 1

Equilibrium 2A: is stable whenever \rho < \mu A

Case B: sY=0, \mu A=0

In[82]:= CaseBSub = {sY -> 0, \mu A -> 0};
In[82]:= \lambda List2B = Eigenvalues[JMtrx2 /. CaseBSub] // Simplify
Out[82]= \{-1, -1, -1, -1, \frac{\mu H + \rho - \mu H \rho}{(-1 + \mu H) (-1 + \rho)},
-3 + \rho + sX (7 + \rho) + \sqrt{9 + sX^2 (-1 + \rho)^2 - 2 \rho + \rho^2 + 2 sX (9 - 2 \rho + \rho^2)} - \frac{4 + 8 sX}{4 + 8 sX},
-3 + \rho + sX (7 + \rho) - \sqrt{9 + sX^2 (-1 + \rho)^2 - 2 \rho + \rho^2 + 2 sX (9 - 2 \rho + \rho^2)} \}

In[81]:= Reduce[
{\lambda List2B[[5]] < 0, \lambda List2B[[6]] < 0, \lambda List2B[[7]] < 0, 0 < sX, 0 < \rho < 1, 0 < \mu H < 1}]
Out[81]= False

Equilibrium 2B: is NEVER stable

Case C: \mu H=0, \mu A=0,\mu N=0

In[82]:= CaseCSub = {\mu H -> 0, \mu A -> 0, \mu N -> 0};
In[83]:= \lambda List2C = Eigenvalues[JMtrx2 /. CaseCSub] // Simplify
Out[83]= \{-1, -1, -1, -1, \frac{sY - (1 + sX) \rho}{(1 + sX) (-1 + \rho)},
-3 + \rho + sX (7 + \rho) + \sqrt{9 + sX^2 (-1 + \rho)^2 - 2 \rho + \rho^2 + 2 sX (9 - 2 \rho + \rho^2)} - \frac{4 + 8 sX}{4 + 8 sX},
-3 + \rho + sX (7 + \rho) - \sqrt{9 + sX^2 (-1 + \rho)^2 - 2 \rho + \rho^2 + 2 sX (9 - 2 \rho + \rho^2)} \}

In[84]:= Reduce[{\lambda List2C[[5]] < 0, \lambda List2C[[6]] < 0, \lambda List2C[[7]] < 0, 0 < sX, 0 < \rho < 1, 0 < sY < 1}, \rho]
Out[84]= 0 < sY < 1 && sX > 0 && \rho < \frac{sY}{1 + sX}

Equilibrium 2C: is stable whenever \rho < \frac{sY}{1 + sX}
\end{verbatim}
Case D: $\mu N=0$

\text{In[85]} = \text{CaseDSub} = \{\mu A \to 0, \mu N \to 0*, \mu H+0.12, sY \to 0.015*\};

\text{In[86]} = \lambda\text{List2D} = \text{Eigenvalues}[\text{JMtrx2} /. \text{CaseDSub}] // \text{Simplify}

\text{Out[86]} = \{-1, -1, -1, -1, -\frac{sY + (1 + sX) \left[\mu H (-1 + \rho) - \rho\right]}{(1 + sX) \left(-1 + \mu H\right)} - 2 \mu H \rho^3 + \mu H^2 \rho^3 + sX^3 \left(-1 + \mu H\right) \left(3 + \mu H (-1 + \rho) - \rho\right) \left(-7 + 6 \rho + \rho^2\right) + sX^2 \left(-1 + \mu H\right) \left(-1 + \rho\right) \left(29 - 6 \rho - 3 \rho^2 + \mu H (-13 + 10 \rho + 3 \rho^2)\right) + sX^2 \left(-1 + \mu H\right) \left(-1 + \rho\right) \left(44 - 9 \rho - 3 \rho^2 + \mu H (-17 + 14 \rho + 3 \rho^2)\right)\}

\text{In[87]} = \text{Reduce}\left[\left\{\lambda\text{List2D}[5] < 0, \lambda\text{List2D}[6] < 0, \lambda\text{List2D}[7] < 0, 0 < sX < 1, 0 < \rho < \frac{1}{2}, 0 < \mu H < 1, 0 < sY < 1\right\}\right] // \text{Simplify}

\text{Out[87]} = \rho > 0 \&\& sX > 0 \&\& \mu H > 0 \&\& 2 \rho < 1 \&\& sX < 1 \&\& \rho < 1 \&\& (1 + sX) \left(\mu H + \rho\right) < 1 + (1 + sX) \mu H \rho \&\& (1 + sX) \left(\mu H + \rho\right) < sY + (1 + sX) \mu H \rho \&\& sY < 1

Equilibrium 3: N$x$$N$x/N$x$Ay

\text{In[88]} = \text{JMtrx3} = \text{JMtrx} /. \text{N$x$$N$xN$x$Ay} /. \text{Equ3} // \text{Simplify};

Case A: $sY=0, \mu H=0, \mu N=0$

\text{In[89]} = \text{CaseASub} = \{sY \to 0, \mu H \to 0, \mu N \to 0\};

\text{In[90]} = \lambda\text{List3A} = \text{Eigenvalues}[\text{JMtrx3} /. \text{CaseASub}] // \text{Simplify}

\text{Out[90]} = \{-1, -1, -1, -1, -\frac{\rho}{1 + \rho}, \frac{1}{4} \left(-3 - \rho - \sqrt{9 - 2 \rho + \rho^2} - sX \left(-1 + \rho + \sqrt{9 - 2 \rho + \rho^2}\right)\right), \frac{1}{4} \left(-3 - \rho + \sqrt{9 - 2 \rho + \rho^2} + sX \left(1 + \rho + \sqrt{9 - 2 \rho + \rho^2}\right)\right)\}
Reduce[
{λList3A[[5]] < 0, λList3A[[6]] < 0, λList3A[[7]] < 0, 0 < sX, 0 < ρ < 1, 0 < μ < 1}]

Out[91]= False

Equilibrium 3A: is never stable

Case B: sY=0, μA=0

In[92]= CaseBSub = {sY → 0, μA → 0};

In[93]= λList3B = Eigenvalues[JMtrx3 /. CaseBSub] // Simplify

Out[93]= \[\{-3 - \rho + \sqrt{9 - 2 \rho + \rho^2} - sX \left(1 - \rho + \sqrt{9 - 2 \rho + \rho^2}\right), \\
\frac{1}{4} \left(-3 - \rho + \sqrt{9 - 2 \rho + \rho^2} + sX \left(1 - \rho + \sqrt{9 - 2 \rho + \rho^2}\right)\right}\}\]

In[94]= Reduce[{λList3B[[1]] < 0, λList3B[[6]] < 0, λList3B[[7]] < 0, 0 < sX < 1, 0 < ρ < 1, 0 < μ < 1}]

Out[94]= 0 < sX < -1 + 2 && \frac{3 sX + sX^2}{1 + sX} < ρ < 1 && μN > ρ && μ < μH < 1

Equilibrium 3B: is stable when sX < -1 + 2 && \frac{3 sX + sX^2}{1 + sX} μN>ρ

Case C: μH=0, μA=0, μN=0

In[95]= CaseCSub = {μH → 0, μA → 0, μN → 0};

In[96]= λList3C = Eigenvalues[JMtrx3 /. CaseCSub] // Simplify

Out[96]= \{-1 - 1, -1 - 1, -1 - \frac{\rho}{1 + \rho}, \\
\frac{1}{4} \left(-3 + sX - sX ρ - \sqrt{1 + sX} \sqrt{9 - 8 sY - 2 \rho + \rho^2 + sX \left(9 - 2 \rho + \rho^2\right)}\right), \\
\frac{1}{4} \left(-3 + sX - sX ρ + \sqrt{1 + sX} \sqrt{9 - 8 sY - 2 \rho + \rho^2 + sX \left(9 - 2 \rho + \rho^2\right)}\right)\}

In[97]= Reduce[{λList3C[[5]] < 0, λList3C[[6]] < 0, λList3C[[7]] < 0, 0 < sX, 0 < ρ < 1, 0 < sY < 1}, ρ]

Out[97]= False

Equilibrium 3C: is never stable

Case D: μN=0

In[98]= CaseDSub = {μA → 0, μN → 0 (*μH+0.12,sY+0.015*)};
\( \lambda_{\text{List3D}} = \text{Eigenvalues}[\text{JMtrx3} /. \text{CaseDSub}] \) // Simplify

\[
\left\{ -1, -1, -1, -1, -\frac{\rho}{1 + \rho}, \frac{1}{4} \left( -3 + sX - \rho - sX \rho - \sqrt{1 + sX} \cdot \sqrt{9 - 8 sY - 2 \rho + \rho^2 + sX (9 - 2 \rho + \rho^2)} \right), \frac{1}{4} \left( -3 + sX - \rho - sX \rho + \sqrt{1 + sX} \cdot \sqrt{9 - 8 sY - 2 \rho + \rho^2 + sX (9 - 2 \rho + \rho^2)} \right) \right\}
\]

\( \text{Reduce}[(\lambda_{\text{List3D}}[[5]] < 0, \lambda_{\text{List3D}}[[6]] < 0, \lambda_{\text{List3D}}[[7]] < 0, 0 < sX < 1, 0 < \rho < 1, 0 < \muH < 1, 0 < sY < 1)] \) // Simplify

\( \text{Out}[100] = \text{False} \)

Equilibrium 3D: is never stable

Finding \( \text{NxNx/NxNy} \) equilibrium (Equ 4)

\( \text{JMtrx4} = \text{JMtrx} /. \text{NxNxNxNy} /. \text{Equ4} \) // Simplify;

Case A: \( sY=0, \muH=0, \muN=0 \)

\( \text{CaseASub} = \{ sY \to 0, \muH \to 0, \muN \to 0 \} \);

\( \lambda_{\text{List4A}} = \text{Eigenvalues}[\text{JMtrx4} /. \text{CaseASub}] \) // Simplify

\( \lambda_{\text{List4A}}[[5]] < 0, \lambda_{\text{List4A}}[[6]] < 0, \lambda_{\text{List4A}}[[7]] < 0, 0 < sX < 1, 0 < \rho < 1, 0 < \muA < 1 \), \( \rho \)

\( \text{Out}[104] = sX > 0 \) \&\& \( 0 < \muA < 1 \) \&\& \( \frac{sX}{1 + sX} < \rho < 1 \)

Equilibrium 4A: is stable when \( \frac{sX}{1 + sX} < \rho \)

Case B: \( sY=0, \muA=0 \)

\( \text{CaseBSub} = \{ sY \to 0, \muA \to 0 \} \);

\( \lambda_{\text{List4B}} = \text{Eigenvalues}[\text{JMtrx4} /. \text{CaseBSub}] \) // Simplify

\( \lambda_{\text{List4B}}[[5]] < 0, \lambda_{\text{List4B}}[[6]] < 0, \lambda_{\text{List4B}}[[7]] < 0, 0 < sX < 1, 0 < \rho < 1 \) \&\& \( 0 < \muH < 1, 0 < \muN < \frac{1}{2} \)

\( \text{Out}[107] = 0 < \muH < 1 \) \&\& \( 0 < \muN < \frac{1}{2} \) \&\& \( 0 < \muA < \frac{1}{2} \) \&\& \( 0 < \muH < 1, 0 < \muN < \frac{1}{2} \)

\( \text{Out}[107] = 0 < \muH < 1 \) \&\& \( 0 < \muA < \frac{1}{2} \) \&\& \( 0 < \muN < \frac{1}{2} \) \&\& \( 0 < \muH < 1, 0 < \muN < \frac{1}{2} \)
Equilibrium 4B: is stable when \( sX < -\frac{\rho}{1+\rho} \)  

**Case C: \( \mu H=0, \mu A=0, \mu N=0 \)**

```math
\text{In[108]=} \quad \text{CaseCSub} = \{\mu H \to 0, \mu A \to 0, \mu N \to 0\};

\text{In[109]=} \quad \lambda List4C = \text{Eigenvalues}[\text{JMtrx4 /. CaseCSub}] // \text{Simplify}

\text{Out[109]=} \quad \{-1, -1, -1, -1, -\rho, \frac{1}{2} (-3 + sX (-1 + \rho) + \rho), sX - \rho - sX \rho\}

\text{In[110]=} \quad \text{Reduce[}
\quad \{\lambda List4C[[5]] < 0, \lambda List4C[[6]] < 0, \lambda List4C[[7]] < 0, 0 < sX, 0 < \rho < 1, 0 < sY < 1\}
\quad \text{Out[110]=} \quad sX > 0 && \frac{sX}{1+sX} < \rho < 1 && 0 < sY < 1

Equilibrium 4C: is stable when \( sX < \frac{\rho}{1+\rho} \)

**Case D: \( \mu N=0 \)**

```math
\text{In[111]=} \quad \text{CaseDSub} = \{\mu A \to 0, \mu N \to 0 (*, \mu H \to 0.12, sY \to 0.015*)\};

\text{In[112]=} \quad \lambda List4D = \text{Eigenvalues}[\text{JMtrx4 /. CaseDSub}] // \text{Simplify}

\text{Out[112]=} \quad \{-1, -1, -1, -1, -\rho, \frac{1}{2} (-3 + sX (-1 + \rho) + \rho), sX - \rho - sX \rho\}

\text{In[113]=} \quad \text{Reduce[}
\quad \{\lambda List4D[[5]] < 0, \lambda List4D[[6]] < 0, \lambda List4D[[7]] < 0, 0 < sX < 1, 0 < \rho < \frac{1}{2}, 0 < \mu H < 1, 0 < sY < 1\}, sX\}
\quad \text{Out[113]=} \quad 0 < \rho < \frac{1}{2} && 0 < \mu H < 1 && 0 < sY < 1 && 0 < sX < \frac{\rho}{-1+\rho}

Equilibrium 3D: is stable when \( sX < \frac{\rho}{1-\rho} \)

**Case I: \( sY=0, \mu A=0 \)**

**Phase Plane**

```math
\text{In[132]=} \quad \text{Solve[}\quad sX = -\frac{\rho}{-1+\rho}, \rho\] // \text{Simplify}
\quad \text{Out[132]=} \quad \{(\rho \to \frac{sX}{1+sX})\}
```
\[ \text{Show[Plot}\left[\left\{\frac{sX}{1+sX}, \frac{3sX + sX^2}{1+sX}\right\}, \{sX, 0, 0.5\}, \text{PlotRange} \rightarrow \{0, 0.5\}, \text{PlotStyle} \rightarrow \{\text{Black}, \{\text{Red}, \text{Dashed}\}\}], \text{ListLinePlot}\left[\left\{-1+\sqrt{2}, 0\right\}, \{0, 0.5\}\right], \text{PlotStyle} \rightarrow \text{Red}\], \text{Plot}\left[\{0.2, 0.3, 0.4\}, \{sX, 0, 0.5\}, \text{PlotStyle} \rightarrow \text{Dashed}\], \text{Frame} \rightarrow \text{True}, \text{FrameLabel} \rightarrow \{"sX", "\rho"\}\]

AxAx/AxAy is potentially stable anywhere
AxAx/AxNy is never stable
NxNxNxAy is stable above the red-dashed line and to the left of the solid red line and below the coloured dashed lines
NxNxNxNy is stable above the black solid line

**Numerical Equilibrium**

Define an ordering of Karyotype alleles:
Nx=1, Ax=2, Ny=3, Ay=4

\[ \text{order} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}; \text{order} // \text{MatrixForm}\]

\[ \text{testParsB} = \{\rho \rightarrow 0.1, \mu A \rightarrow 0.0, sX \rightarrow 0.1, sY \rightarrow 0, \mu H \rightarrow 0.12, \mu N \rightarrow 0.08\}; \]
\begin{verbatim}
ln[137]= Clear[DynTestB]
DynTestB[pars_, Ax_, Ay_] := DynTestB[pars, Ax, Ay] = Block[{
    F2NSol, F2N, test},
    F2NSol[pars, Ax, Ay] = Flatten[NDSolve[Join[fodes /. pars,
        \[F1, 1, 0] = \frac{(1 - Ax)^2}{2}, \[F1, 2, 0] = \frac{2 \times (1 - Ax) \cdot Ax}{2},
        \[F1, 3, 0] = \frac{(1 - Ax) \cdot (1 - Ay)}{2}, \[F1, 4, 0] = \frac{(1 - Ax) \cdot Ay}{2}, \[F2, 2, 0] = \frac{Ax^2}{2},
        \[F2, 3, 0] = \frac{Ax \cdot (1 - Ay)}{2}, \[F2, 4, 0] = \frac{Ax \cdot Ay}{2}]}, fVars, {t, 0, 5000}]];
F2N[k1_, k2_, t1_, pars, Ax0, Ay0_] := F2NSol[pars, Ax0, Ay0] /. t \rightarrow t1;
test[t1_] := Round[
    (*AxAx*) F2N[2, 2, t1, pars, Ax0, Ay0], (*AxAy*)
    F2N[2, 4, t1, pars, Ax0, Ay0] - {F[2, 2, t], F[2, 4, t]} /. Equ1 /. pars,
    (*NxNx*) F2N[1, 1, t1, pars, Ax0, Ay0], (*NxNy*)
    F2N[1, 4, t1, pars, Ax0, Ay0] - {F[1, 1, t], F[1, 4, t]} /. Equ3 /. pars,
    (*AxAx*) F2N[1, 3, t1, pars, Ax0, Ay0], (*AxAy*)
    F2N[1, 1, t1, pars, Ax0, Ay0] - {F[1, 1, t], F[1, 3, t]} /. Equ4 /. pars, 0.001];
Position[test[5000], {0., 0.}]][[1, 1]]

ln[138]= (*Purple*)
temp0 = Flatten[Table[{
    DynTestB[\{
        \rho \rightarrow r, \mu A \rightarrow 0.0, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0.12, \mu N \rightarrow 0.2\}, 0.5, 0.5]},
    {ssX, 0.005, 0.2, 0.005}, {r, 0.01, 0.42, 0.01}], 1];
(*Blue*)
temp1 = Flatten[Table[{
    DynTestB[\{
        \rho \rightarrow r, \mu A \rightarrow 0.0, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0.12, \mu N \rightarrow 0.3\}, 0.5, 0.5]},
    {ssX, 0.005, 0.2, 0.005}, {r, 0.01, 0.42, 0.01}], 1];
(*Red*)
temp2 = Flatten[Table[{
    DynTestB[\{
        \rho \rightarrow r, \mu A \rightarrow 0.0, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0.12, \mu N \rightarrow 0.4\}, 0.5, 0.5]},
    {ssX, 0.005, 0.2, 0.005}, {r, 0.01, 0.42, 0.01}], 1];
\end{verbatim}

**Case II: sY=0, \mu H=0, \mu N=0**

Define an ordering of Karyotype alleles:

\begin{align*}
    Nx &= 1, Ax &= 2, Ny &= 3, Ay &= 4
\end{align*}
In[142] := order = {{1, 1}, {1, 2}, {1, 3}, {1, 4}, {2, 2}, {2, 3}, {2, 4}};

order // MatrixForm

Out[143]=\[
\begin{pmatrix}
1 & 1 \\
1 & 2 \\
1 & 3 \\
1 & 4 \\
2 & 2 \\
2 & 3 \\
2 & 4 \\
\end{pmatrix}
\]

In[144] := testParsA = {ρ -> 0.1, μA -> 0.3, sX -> 0.1, sY -> 0, μH -> 0, μN -> 0};

In[145] := Clear[DynTestA]

DynTestA[pars_, Ax0_, Ay0_] := DynTestA[pars, Ax0, Ay0] = Block[{F2NSol, F2N, test},

F2NSol[pars, Ax0, Ay0] = Flatten[NDSolve[Join[fOdes /. pars,

{F[1, 1, 0] = \(\frac{(1 - Ax0)^2}{2}\), F[1, 2, 0] = \(\frac{2 \times (1 - Ax0) Ax0}{2}\),

F[1, 3, 0] = \(\frac{(1 - Ax0)(1 - Ay0)}{2}\), F[1, 4, 0] = \(\frac{(1 - Ax0) Ay0}{2}\), F[2, 2, 0] = \(\frac{Ax0^2}{2}\),

F[2, 3, 0] = \(\frac{Ax0 (1 - Ay0)}{2}\), F[2, 4, 0] = \(\frac{Ax0 Ay0}{2}\)}, fVars, {t, 0, 5000})];

F2N[k1_, k2_, t1_, pars, Ax0, Ay0] := F[k1, k2, t] /. F2NSol[pars, Ax0, Ay0] /. t -> t1;

test[t1_] := Round[{{{(*AxAx*)} F2N[2, 2, t1, pars, Ax0, Ay0], (*AxAy*)

F2N[2, 4, t1, pars, Ax0, Ay0]} - {F[2, 2, t], F[2, 4, t]}} /. Equ1 /. pars,

{(*AxAx*)} F2N[2, 2, t1, pars, Ax0, Ay0], (*AxNy*)

F2N[2, 3, t1, pars, Ax0, Ay0]} - {F[2, 2, t], F[2, 3, t]} /. Equ2 /. pars,

{(*(*NxNx*)} F2N[1, 1, t1, pars, Ax0, Ay0], (*NxAy*)} F2N[1, 4, t1, pars, Ax0, Ay0] -

{F[1, 1, t], F[1, 4, t]}/.Equ3/.pars,*

{(*(*AxAx*)} F2N[1, 1, t1, pars, Ax0, Ay0], (*AxAy*)} F2N[1, 3, t1, pars, Ax0, Ay0] -

{F[1, 1, t], F[1, 3, t]} /. Equ4 /. pars), 0.001];

Position[test[5000], {0., 0.}][[1, 1]]
(*Purple*)

\[temp0 = \text{Flatten}[\text{Table}[
\{ssX, r, \text{DynTestA}[\{\rho \rightarrow r, \mu A \rightarrow 0.15, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0, \mu N \rightarrow 0\}, 0.5, 0.5]\}, sX, 0.005, 0.1, 0.0025], \{r, 0.01, 0.42, 0.01\}], 1] ;\]

(*Blue*)

\[temp1 = \text{Flatten}[\text{Table}[
\{ssX, r, \text{DynTestA}[\{\rho \rightarrow r, \mu A \rightarrow 0.3, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0, \mu N \rightarrow 0\}, 0.5, 0.5]\}, sX, 0.005, 0.1, 0.0025], \{r, 0.01, 0.42, 0.01\}], 1] ;\]

(*Red*)

\[temp2 = \text{Flatten}[\text{Table}[
\{ssX, r, \text{DynTestA}[\{\rho \rightarrow r, \mu A \rightarrow 0.2, sX \rightarrow ssX, sY \rightarrow 0, \mu H \rightarrow 0, \mu N \rightarrow 0\}, 0.5, 0.5]\}, sX, 0.005, 0.1, 0.0025], \{r, 0.01, 0.42, 0.01\}], 1] ;\]
Figure for paper

```
In[150]= (*Extracting boundary between Equ 1 and equ 2*)
points0A = Cases[Normal@ListContourPlot[temp0, Contours -> {1}],
   Line[pts_] -> pts, Infinity][[1]];
points1A = Cases[Normal@ListContourPlot[temp1, Contours -> {1}],
   Line[pts_] -> pts, Infinity][[1]];
points2A = Cases[Normal@ListContourPlot[temp2, Contours -> {1}],
   Line[pts_] -> pts, Infinity][[1]];
(*Extracting boundary between Equ 2 and equ 3*)
points0B = Cases[Normal@ListContourPlot[temp0, Contours -> {3}],
   Line[pts_] -> pts, Infinity][[1]];
points1B = Cases[Normal@ListContourPlot[temp1, Contours -> {3}],
   Line[pts_] -> pts, Infinity][[1]];
points2B = Cases[Normal@ListContourPlot[temp2, Contours -> {3}],
   Line[pts_] -> pts, Infinity][[1]];
int0 = points0B[[Flatten[Position[Abs[points0B[[ ;; , 2]] - 0.15],
   Min[Abs[points0B[[ ;; , 2]] - 0.15]]]]]] // Mean;
int1 = points1B[[Flatten[Position[Abs[points1B[[ ;; , 2]] - 0.3],
   Min[Abs[points1B[[ ;; , 2]] - 0.3]]]]]] // Mean;
int2 = points2B[[Flatten[Position[Abs[points2B[[ ;; , 2]] - 0.2],
   Min[Abs[points2B[[ ;; , 2]] - 0.2]]]]]] // Mean;
```
PhasePlane =
Show[
(*Yellow Region*)
Show[ListLinePlot[Join[Select[points0B, #[[2]] < 0.15 &], {int0, {0.1, 0.15}}],
Filling -> Axis, FillingStyle -> Directive[{AxNyCol, Opacity[0.3]}],
PlotStyle -> Directive[{Col3, Thickness[0.01]}],
ListLinePlot[Join[Select[points1B, #[[2]] < 0.3 &], {int1, {0.1, 0.3}}],
Filling -> Axis, FillingStyle -> Directive[{AxNyCol, Opacity[0.3]}],
PlotStyle -> Directive[{Col2, Thickness[0.01]}],
ListLinePlot[Join[Select[points2B, #[[2]] < 0.2 &], {int2, {0.1, 0.2}}],
Filling -> Axis, FillingStyle -> Directive[{AxNyCol, Opacity[0.3]}],
PlotStyle -> Directive[{Col5, Thickness[0.01]}],
PlotRange -> {{0.005, 0.1}, {0, 0.42}}, Frame -> True,
FrameStyle -> Directive[Black, 30, Thick, Bold],
PlotRange -> {{0.005, 0.1}, {0, 0.42}}, AspectRatio -> 0.75],
(*Dotted Region*)
Show[
ListLinePlot[Select[temp0, #[[3]] == 1 &]], PlotStyle ->
{Directive[{Col3, Opacity[0.3]}], PlotMarkers -> {Automatic, Medium}],
ListLinePlot[Select[temp2, #[[3]] == 1 &]], PlotStyle ->
{Directive[{Col5, Opacity[0.75]}], PlotMarkers -> {Automatic, Medium}],
ListLinePlot[Select[temp1, #[[3]] == 1 &]], PlotStyle ->
{Directive[{Col2, Opacity[0.75]}], PlotMarkers -> {Automatic, Medium}],
ListLinePlot[points0B, PlotStyle ->
{Directive[{Col3, Thickness[0.01], Opacity[1]}], ListLinePlot[
points1B, PlotStyle -> {Directive[{Col2, Thickness[0.01], Opacity[1]}],
ListLinePlot[points2B, PlotStyle ->
{Directive[{Col5, Thickness[0.01], Opacity[1]}]}],
(*Horizontal Line*)
ListLinePlot[Select[int0, {0.1, 0.15}]], {int1, {0.1, 0.3}}, {int2, {0.1, 0.2}}],
PlotStyle -> {Directive[{Col3, Thickness[0.01], Opacity[1]}],
Directive[{Col2, Thickness[0.01], Opacity[1]}],
Directive[{Col5, Thickness[0.01], Opacity[1]}]},
Epilog -> Inset[Leg, Scaled[{0.15, 0.75}]], ImageSize -> {1000, 800},
FrameLabel -> {"sX", "p"}]
Export[NotebookDirectory[] <> "CaseII.jpeg", %, ImageResolution -> 300];
Case III: $\mu_H=0, \mu_A=0, \mu_N=0$

Phase Plane

```math
\text{Phase} [sY, \text{sty}_] := \text{Plot} \left[ \left\{ \frac{sY}{1 + sX}, \frac{sX}{1 + sX} \right\}, \{sX, 0, 0.2\}, \text{PlotStyle} \rightarrow \text{sty} \right]
```

```math
\text{Show} [\text{Phase} [0.15, \text{Automatic}], \text{Phase} [0.1, \text{Dashed}], \text{Phase} [0.05, \text{Dotted}]]
```

Given $sY$
Equilibrium 1 is stable above the blue line
Equilibrium 2 is stable below the blue line
Equilibrium 4 is stable above the yellow line

Numerical Equilibrium

Define an ordering of Karyotype alleles:
$N_x=1, A_x=2, N_y=3, A_y=4$

```math
\text{order} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\};
\text{order} // \text{MatrixForm}
```

```math
\begin{pmatrix}
1 & 1 \\
1 & 2 \\
1 & 3 \\
1 & 4 \\
2 & 2 \\
2 & 3 \\
2 & 4 \\
\end{pmatrix}
```

```math
\text{testParsC} = \{\rho \rightarrow 0.1, \mu_A \rightarrow 0.0, sX \rightarrow 0.1, sY \rightarrow 0.3, \mu_H \rightarrow 0.0, \mu_N \rightarrow 0.0\};
```
\[\text{Clear[DynTestC]}\]

\[
\text{DynTestC[pars\_, Ax\_, Ay\_]} := \text{DynTestC[pars, Ax\_, Ay\_]} = \text{Block[}\{\text{F2NSol, F2N, test}\},
\]

\[
\text{F2NSol[pars, Ax\_, Ay\_]} = \text{Flatten[}\text{NDSolve[}\text{Join}[\text{fodes} /. \text{pars},
\]

\[
\left\{ \begin{array}{l}
F[1, 1, 0] = \frac{(1 - Ax\_0)^2}{2}, F[1, 2, 0] = 2 \times \frac{(1 - Ax\_0) Ax\_0}{2}, \\
F[1, 3, 0] = \frac{(1 - Ax\_0)(1 - Ay\_0)}{2}, F[1, 4, 0] = \frac{(1 - Ax\_0) Ay\_0}{2}, F[2, 2, 0] = \frac{Ax\_0^2}{2}, \\
F[2, 3, 0] = \frac{Ax\_0(1 - Ay\_0)}{2}, F[2, 4, 0] = \frac{Ax\_0 Ay\_0}{2}\end{array} \right\}], \text{fVars, }\{t, 0, 5000}\};
\]

\[\text{F2N[k1\_, k2\_, t1\_, pars, Ax\_, Ay\_]} := F[k1, k2, t] /.
\]

\[\text{F2NSol[pars, Ax\_, Ay\_]} /. t \rightarrow t1;\]

\[\text{test[t1\_]} := \text{Round[}\{\]
Figure for paper

(*Extracting boundary between Equ 1 and equ 2*)
points0A = Cases[Normal@ListContourPlot[temp0, Contours -> {1}],
    Line[pts_] -> pts, Infinity][[1]];
points1A = Cases[Normal@ListContourPlot[temp1, Contours -> {1}],
    Line[pts_] -> pts, Infinity][[1]];
points2A = Cases[Normal@ListContourPlot[temp2, Contours -> {1}],
    Line[pts_] -> pts, Infinity][[1]];
(*Extracting boundary between Equ 2 and equ 3*)
points0B = Cases[Normal@ListContourPlot[temp0, Contours -> {3}],
    Line[pts_] -> pts, Infinity][[1]];
points1B = Cases[Normal@ListContourPlot[temp1, Contours -> {3}],
    Line[pts_] -> pts, Infinity][[1]];
points2B = Cases[Normal@ListContourPlot[temp2, Contours -> {3}],
    Line[pts_] -> pts, Infinity][[1]];
(*int0=points0B[[Flatten[Position[Abs[points0B[[;;,2]]-0.15],
        Min[Abs[points0B[[;;,2]]-0.15]]]]/Mean;*
    int1=points1B[[Flatten[Position[Abs[points1B[[;;,2]]-0.3],
        Min[Abs[points1B[[;;,2]]-0.3]]]]/Mean;
    int2=points2B[[Flatten[Position[Abs[points2B[[;;,2]]-0.2],
        Min[Abs[points2B[[;;,2]]-0.2]]]]/Mean;*)
\begin{verbatim}
Leg = LineLegend[
   
   {Directive[{Col2, Thickness[0.01]}], Directive[{Col5, Thickness[0.01]}],
    Directive[{Col3, Thickness[0.01]}]}, 
   "s0=0.15", "s0=0.1", "s0=0.05"],
   LabelStyle \to \{Bold, 18, Black\}, ImageSize \to \{200, 170\}, Background
   \to Directive[White, Opacity[0.9]]];

Show[
   Show[ListLinePlot[Table[{{ssX, \frac{0.15}{1+ssX}}, {ssX, 0, 0.1, 0.01}],
          PlotStyle \to Directive[Col2, Thickness[0.01]], Filling \to \emptyset,
          FillingStyle \to Directive[{AxNyCol, Opacity[0.3]}]],
          ListLinePlot[points0B, PlotStyle \to Directive[Col2, Thickness[0.01]],
          Filling \to \emptyset, FillingStyle \to White],
          PlotRange \to \{(0.01, 0.1), \{0, 0.4\}\}, Frame \to True, Axes \to False],
   Show[ListLinePlot[Table[{{ssX, \frac{0.1}{1+ssX}}, {ssX, 0, 0.1, 0.01}],
          PlotStyle \to Directive[Col5, Thickness[0.01]], Filling \to \emptyset,
          FillingStyle \to Directive[{AxNyCol, Opacity[0.3]}]],
          ListLinePlot[points1B, PlotStyle \to Directive[Col5, Thickness[0.01]],
          Filling \to \emptyset, FillingStyle \to White],
          PlotRange \to \{(0.01, 0.1), \{0, 0.4\}\}, Frame \to True, Axes \to False],
   Show[ListLinePlot[Table[{{ssX, \frac{0.05}{1+ssX}}, {ssX, 0, 0.1, 0.01}],
          PlotStyle \to Directive[Col3, Thickness[0.01]], Filling \to \emptyset,
          FillingStyle \to Directive[{AxNyCol, Opacity[0.3]}]],
          ListLinePlot[points2B, PlotStyle \to Directive[Col3, Thickness[0.01]],
          Filling \to \emptyset, FillingStyle \to White],
          ListPlot[Select[temp2, #[[3]] = 1 \& ; ; , {1, 2}]],
          PlotStyle \to \{Directive[Col3, Opacity[0.5]\}], PlotMarkers \to \{Automatic, Medium\}],
          ListPlot[Select[temp1, #[[3]] = 1 \& ; ; , {1, 2}]],
          PlotStyle \to \{Directive[Col5, Opacity[0.5]\}], PlotMarkers \to \{Automatic, Medium\}],
          ListPlot[Select[temp0, #[[3]] = 1 \& ; ; , {1, 2}]],
          PlotStyle \to \{Directive[Col2, Opacity[0.5]\}], PlotMarkers \to \{Automatic, Medium\}],
          Frame \to True, FrameStyle \to Directive[Black, 30, Thick, Bold], AspectRatio \to 1,
          Axes \to \emptyset, PlotRange \to \{(0.01, 0.1), \{0.01, 0.4\}\}, ImageSize \to \{1000, 1000\},
          Epilog \to Inset[Leg, Scaled[\{0.75, 0.75\}\}], FrameLabel \to \{"sX", "p\}"
   ];

Export[NotebookDirectory[] <> "CaseIII.jpeg", \%, ImageResolution \to 300];
\end{verbatim}
Case IV: $\mu N=0$

Numerical Equilibrium

Define an ordering of Karyotype alleles:

$N_1, A_2, N_3, A_4$

In[168]:= order = {{1, 1}, {1, 2}, {1, 3}, {1, 4}, {2, 2}, {2, 3}, {2, 4}};

Out[168]= MatrixForm[
    1 1
    1 2
    1 3
    1 4
    2 2
    2 3
    2 4
]

In[169]:= testParsD = {\rho \to 0.1, \mu A \to 0.0, sX \to 0.1, sY \to 0.015, \mu H \to 0.12, \mu N \to 0.0};

In[170]:= Clear[DynTestD]

DynTestD[pars_, Ax0_, Ay0_] := DynTestD[pars, Ax0, Ay0] = Block[{F2NSol, F2N, test}, F2NSol[pars, Ax0, Ay0] = Flatten[NDSolve[Join[fOdes /. pars,

    \{F[1, 1, 0] = \left(1 - Ax0\right)^2, \quad F[1, 2, 0] = \frac{2 \left(1 - Ax0\right) Ax0}{2},
    \quad F[1, 3, 0] = \frac{2 \left(1 - Ax0\right) \left(1 - Ay0\right)}{2}, \quad F[1, 4, 0] = \frac{\left(1 - Ax0\right) Ay0}{2}, \quad F[2, 2, 0] = \frac{Ax0^2}{2},
    \quad F[2, 3, 0] = \frac{Ax0 \left(1 - Ay0\right)}{2}, \quad F[2, 4, 0] = \frac{Ax0 Ay0}{2}\}]], fVars, {t, 0, 5000}]];

F2N[k1_, k2_, t1_, pars, Ax0, Ay0_] := F[k1, k2, t] /. F2NSol[pars, Ax0, Ay0] /. t -> t1;

test[t1_] := Round[{
\{(*AxAx*) F2N[2, 2, t1, pars, Ax0, Ay0], (*AxAy*) F2N[2, 4, t1, pars, Ax0, Ay0]\} - \{F[2, 2, t], F[2, 4, t]\} /. Equ1 /. pars,
\{(*AxAx*) F2N[2, 2, t1, pars, Ax0, Ay0], (*AxNy*) F2N[2, 3, t1, pars, Ax0, Ay0]\} - \{F[2, 2, t], F[2, 3, t]\} /. Equ2 /. pars,
\{(*NxNx*) F2N[1, 1, t1, pars, Ax0, Ay0], (*NxNy*) F2N[1, 4, t1, pars, Ax0, Ay0]\} - \{F[1, 1, t], F[1, 4, t]\} /. Equ3 /. pars,]
\{(*AxAx*) F2N[1, 1, t1, pars, Ax0, Ay0], (*AxAy*) F2N[1, 3, t1, pars, Ax0, Ay0]\} - \{F[1, 1, t], F[1, 3, t]\} /. Equ4 /. pars, 0.001];

Position[test[5000], {0., 0.}][[1, 1]]
```
(*Black*)

temp0 = Flatten[Table[{
  ssX, r,
  DynTestC[{
    ρ → r, μA → 0.0, sX → ssX, sY → 0.15, μH → 0.12, μN → 0.0}, 0.5, 0.5]},
  {ssX, 0.005, 0.5, 0.01}, {r, 0.01, 0.5, 0.01}], 1];
temp1 = Flatten[Table[{
  ssX, r, DynTestC[{
    ρ → r, μA → 0.1, sX → ssX, sY → 0.015, μH → 0.12, μN → 0.0}, 0.5, 0.5]},
  {ssX, 0.005, 0.5, 0.01}, {r, 0.01, 0.5, 0.01}], 1];
temp2 = Flatten[Table[{
  ssX, r, DynTestC[{
    ρ → r, μA → 0.1, sX → ssX, sY → 0.15, μH → 0.12, μN → 0.0}, 0.5, 0.5]},
  {ssX, 0.005, 0.5, 0.01}, {r, 0.01, 0.5, 0.01}], 1];

points0A = Cases[
  Normal@ListContourPlot[temp0, Contours -> {1, 2}], Line[pts_] -> pts, Infinity];
points1A = Cases[
  Normal@ListContourPlot[temp1, Contours -> {1, 2}],
  Line[pts_] -> pts, Infinity];
points2A = Cases[
  Normal@ListContourPlot[temp2, Contours -> {1, 2}],
  Line[pts_] -> pts, Infinity];

Leg = LineLegend[{
  Directive[{
    Col2, Thickness[0.01]}],
  Directive[{
    Col5, Thickness[0.01]}], Directive[{
    Col3, Thickness[0.01]}]},
  {"sY=0.15, μA=0.1", "sY=0.015, μA=0.1", "sY=0.15, μA=0"},
  LabelStyle -> {Bold, 18, Black}, ImageSize -> {350, 180}, Background
  -> Directive[White, Opacity[0.9]]]
```

```
- sY=0.15, μA=0.1
- sY=0.015, μA=0.1
- sY=0.15, μA=0
```
Show[
  (*Turquoise*)
  Show[ListLinePlot[points2A[[1]], PlotRange -> {{0, 0.5}, {0, 0.5}},
    PlotStyle -> Directive[Col2, Thickness[0.01]], Filling -> Bottom,
    FillingStyle -> Directive[AxNyCol, Opacity[0.5]], PlotRange -> All],
  ListLinePlot[points2A[[2]], PlotRange -> {{0, 0.5}, {0, 0.5}},
    PlotStyle -> Directive[Col2, Thickness[0.01]]], PlotRange -> All],
  (*Purple*)
  Show[ListLinePlot[points0A[[1]], PlotRange -> {{0, 0.5}, {0, 0.5}},
    PlotStyle -> Directive[Col3, Thickness[0.01]], Filling -> Bottom,
    FillingStyle -> Directive[AxNyCol, Opacity[0.5]]],
  ListLinePlot[points0A[[2]], PlotRange -> {{0, 0.5}, {0, 0.5}},
    PlotStyle -> Directive[Col3, Thickness[0.01]]],
  (*Yellow*)
  ListLinePlot[points1A[[1]], PlotStyle -> Directive[Col5, Thickness[0.01]]],
  ListPlot[Select[temp1, #[[3]] == 1 &][;; , {1, 2}],
    PlotStyle -> {Directive[Col5, Opacity[0.5]]}, PlotMarkers -> {Automatic, Medium}],
  ListPlot[Select[temp0, #[[3]] == 1 &][;; , {1, 2}],
    PlotStyle -> {Directive[Col3, Opacity[0.5]]}, PlotMarkers -> {Automatic, Medium}],
  ListPlot[Select[temp2, #[[3]] == 1 &][;; , {1, 2}],
    PlotStyle -> {Directive[Col2, Opacity[0.5]]}, PlotMarkers -> {Automatic, Medium}],
  Frame -> True, FrameStyle -> Directive[Black, 30, Thick, Bold], AspectRatio -> 1,
  Axes -> False, PlotRange -> {{0.0, 0.5}, {0.0, 0.5}}, ImageSize -> {1000, 1000},
  FrameLabel -> {"sX", "ρ"}, Epilog -> Inset[Leg, Scaled[[0.75, 0.75]]]
]
Export[NotebookDirectory[] <> "CaseIV.jpeg", %, ImageResolution -> 300];
