A polynomial approach to the Collatz conjecture

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The Collatz conjecture is explored using polynomials based on a binary numeral system. It is shown that the degree of the polynomials, on average, decreases after a finite number of steps of the Collatz operation, which provides a weak proof of the conjecture by using induction with respect to the degree of the polynomials.

\textbf{Keywords:} Collatz conjecture; binary numeral system based on polynomials; induction method.

The Collatz 3x + 1 problem \cite{1-3} concerns consecutive Collatz operations \( C \) to a given integer \( n \) with \( C[n] = (3n + 1)/2 \) if \( n \) is odd and \( C[n] = n/2 \) if \( n \) is even. The conjecture asserts that there is always a finite number \( k \) of the Collatz operations, after which \( C[C[\cdots C[n] \cdots]] = 1 \). In this letter, we only consider odd integers \( n \) with the operation \( C_q[n] = (3n + 1)/2^q \), where \( q \) is the largest positive integer with \( C_q[n] = 1 \) (mod 2). The conjecture has been verified to be true for all \( n < 20 \times 2^{38} \) \cite{4}.

It is well known that a natural number less than \( 2^{p+1} \) can be expressed in a binary numeral system with

\[ n = \sum_{\mu=0}^{p} c_{2^\mu}, \]  

where there is a unique positive integer \( p \) and a set of coefficients \( \{c_0, c_{2^{-1}}, \ldots, c_0\} \) with \( c_i = 0 \) or 1 for the given \( n \). The sequence of bits, \( \{c_0, c_{2^{-1}}, \ldots, c_0\} \), is just the binary representation of the integer \( n \). In the following, we always assume that \( p \) is a positive finite integer. In order to visually realize the Collatz operation on odd integers, we introduce the polynomial

\[ F_p(x) = x^p + c_{p-1}x^{p-1} + c_{p-2}x^{p-2} + \cdots + c_2x^2 + c_1x + 1 \]  

of degree \( p \geq 1 \), where \( x \equiv 2 \) is assumed throughout, which corresponds to an odd integer \( 2^p + 1 \leq n \leq 2^{p+1} - 1 \) given in (1) with \( n \equiv F_p(x) \). Arithmetic operations of the polynomials, such as addition and subtraction \( F_{p_1}(x) \pm F_{p_2}(x) \), multiplication \( F_{p_1}(x)F_{p_2}(x) \), and division \( F_{p_1}(x)/F_{p_2}(x) \) are defined as usual, for which one only needs to keep in mind that \( x \equiv 2 \), so

\[ x - 1 \equiv 1, \ x + 1 \equiv x^2 - 1, \ 2^{t-1}x^t \equiv x^t \]  

with \( t \geq 1 \) according to the rules of arithmetic operations on integers in the binary numeral system. Hence, the resultants of \( F_{p_1}(x) \pm F_{p_2}(x) \) or \( F_{p_1}(x)F_{p_2}(x) \) are still a polynomial of the same type. Table I provides \((x + 1)^q\) for \( q \leq 10 \) as examples computed in this way. The degree \( p \) of the polynomial \((x + 1)^q\) increases with \( q \) linearly as \( p = u(q) = \text{Int}[-1/2 + q \ln[3]/\ln[2]] \) for \( q \geq 1 \), where \( \text{Int}[r] \) is the nearest integer of \( r \).

The Collatz operation on the polynomial \( F_p(x) \) representing an odd integer \( 2^p + 1 \leq n \equiv F_p(x) \leq 2^{p+1} - 1 \) is defined as

\[ C_q[F_p(x)] = x^{-q} ((x + 1)F_p(x) + 1), \]  

where the positive integer \( q \geq 1 \) is chosen to be the largest such that \( C_q[F_p(x)] \) is still a polynomial of the same type defined by (2), or equivalently \((x + 1)F_p(x) + 1\) is factorizable as \((x + 1)F_p(x) + 1 = x^t Q_t(x)\), where \( t \) is an positive integer, and \( C_q[F_p(x)] = Q_t(x) \) is a polynomial of the same type of degree \( t \) representing another odd integer. Thus, the polynomial ring \( R = \{\{F_0(x)\}, \ldots, \{F_p(x)\}, \cdots\} \) constructed by a series of polynomials \( \{F_0(x)\}, \ldots, \{F_p(x)\}, \cdots \), where \( F_p(x) \) is given by (2), and \( \{F_p(x)\} \) are formed by \( 2^{p-1} \) different combinations of 0 and 1 in the sequence of bits \( \{c_{p-1}, \cdots, c_1\} \) in \( F_p(x) \), is algebraically closed under the Collatz operation. The Collatz conjecture can then be stated as follows:

\begin{itemize}
  \item \textbf{The Collatz Conjecture} Any degree \( p \) polynomial \( F_p(x) \in R \) decreases after a finite number of the Collatz operations. Namely, \( C_{q_1} \cdots C_{q_l}[F_p(x)] \cdots = F_{p'}(x) \in R \) \( \forall F_p(x) \in R \) with \( p' \leq p - 1 \) and finite \( l \), and eventually \( C_{q_k} \cdots C_{q_1}[F_p(x)] \cdots = F_0(x) = 1 \) with a finite \( k \geq 1 \).
\end{itemize}
TABLE I: The polynomial \((1 + x)^q\) for \(q \leq 10\).

| \(q\) | The degree of \((1 + x)^q\) | \((1 + x)^q\) |
|------|-------------------|------------------|
| 0    | 0                 | 1                |
| 1    | 1                 | \(x + 1\)        |
| 2    | 3                 | \(x^3 + 1\)      |
| 3    | 4                 | \(x^4 + x^3 + x + 1\) |
| 4    | 6                 | \(x^6 + x^4 + 1\) |
| 5    | 7                 | \(x^7 + x^6 + x^5 + x^4 + x + 1\) |
| 6    | 9                 | \(x^9 + x^7 + x^6 + x^4 + x^3 + 1\) |
| 7    | 11                | \(x^{11} + x^7 + x^3 + 1\) |
| 8    | 12                | \(x^{12} + x^{11} + x^8 + x^7 + x^5 + 1\) |
| 9    | 14                | \(x^{14} + x^{11} + x^{10} + x^7 + x^6 + x^5 + x + 1\) |
| 10   | 15                | \(x^{15} + x^{14} + x^{13} + x^{10} + x^9 + x^7 + x^5 + x^3 + 1\) |

**Definition** If a polynomial \(F_p(x)\) defined by (2) satisfies \(C_{q_1}[C_{q_1-1}[\cdots C_{q_1}[F_p(x)]\cdots]] = 1\) with finite \(l\), \(F_p(x)\) is called the Collatz polynomial.

Concerning the Collatz operation on \(R\), we have the following corollaries:

**Corollary 1** If \(Q_l(x) = C_q[F_p(x)]\), then \(Q_l(x) = C_{q+2}[1 + x^2F_p(x)] = C_{q+4}[1 + x^2 + x^4F_p(x)] = \cdots\), which can be verified directly by the Collatz operation.

**Corollary 2** After consecutive iterations according to Corollary 1, the following series \(\{U_k(x) = \sum_{t=0}^{k} x^{2t}\} (k = 0, 1, 2, \cdots)\) satisfies \(C_{2k+2}[U_k(x)] = U_0 = 1\), which was noted in [5].

If there are \(m + 2\) nonzero terms in \(F_p(x)\), without loss of generality, \(F_p(x)\) can be expressed as

\[
F_p^{(m)}(x) = x^p + \sum_{i=1}^{m} x^{k_i} + 1, \tag{5}
\]

in which \(p - 1 \geq k_m > k_{m-1} > \cdots > k_1 \geq 1\) for \(1 \leq m \leq p - 1\) is assumed, or \(F_p^{(m)}(x) = x^p + 1\) with \(m = 0\).

For a given polynomial \(F_p^{(m)}(x)\), the degree of \(C_{q_1q_1-1\cdots q_1}[F_p^{(m)}(x)]\), where \(C_{q_1q_1-1\cdots q_1}[F_p^{(m)}(x)] \equiv C_{q_1}[\cdots C_{q_1}[F_p^{(m)}(x)]]\) stands for \(l\) times of the Collatz operation on \(F_p^{(m)}(x)\), can be expressed as

\[
\operatorname{Deg}(C_{q_1q_1-1\cdots q_1}[F_p^{(m)}(x)]) = p + u(l) + 1 - \sum_{i=1}^{l} q_i, \tag{6}
\]

where \(u(l) = \int[-1/2 + l \ln[3]/\ln[2]]\) is the degree of \((x + 1)^l\), which is obtained simply based on the power counting of the leading term of \(C_{q_1q_1-1\cdots q_1}[F_p^{(m)}(x)]\). Concerning the Collatz operation, we have the following proposition:

**Proposition 1** Except for the trivial polynomial \(F_0 = U_0 = 1\), the Collatz operation on \(F_p(x), C_q[F_p(x)]\), is different from \(F_p(x)\) itself.

The validity of Proposition 1 is obvious from the Collatz operation on \(F_p(x)\). If there is an \(F_p(x)\) satisfies \(C_q[F_p(x)] = F_p(x)\), one can deduce that \(F_p(x) = 1/(x^q - x - 1)\). Since \(q \geq 1\) and \(F_p(x) > 1\) being a polynomial, the only possible solution for \(F_p(x) \geq 1\) is \(F_p(x) = 1\) with \(q = 2\), which is excluded in Proposition 1. Directly taking the Collatz operation on \(F_p(x)\), we also have

**Corollary 3** The degree of the polynomials \(F_p^{(m)}(x)\) for \(m = 0, m = 1\), and \(m \geq 1\) with \(k_1 \geq 2\) and \(k_m < p - 1\) decreases after a few steps of the Collatz operation.

For \(m = 0\), we have

\[
C_2[F_p^{(0)}(x)] = x^{p-1} + x^{p-2} + 1 \quad \text{for} \quad p \geq 2, \quad C_4[C_1[F_p^{(0)}(x)]] = 1 \quad \text{for} \quad p = 1. \tag{7}
\]
For $m = 1$ and $p - 1 \leq k_1 \leq 1$, the resultants of direct Collatz operations on $F_p^{(1)}(x)$ are

$$C_4[C_1[F_p^{(1)}(x)]] = x^{p-2} + x^{p-5} + 1 \text{ for } k_1 = 1,$$

$$C_2[F_p^{(1)}(x)] = x^{p-1} + x^{p-2} + x^{k-1} + x^{k-2} + 1 \text{ for } p - 2 \geq k_1 \geq 2,$$

$$C_2[C_2[F_p^{(1)}(x)]] = x^{p-1} + x^{p-2} + x^{p-4} + x^{p-5} + 1 \text{ for } k_1 = p - 1. \tag{8}$$

For $m \geq 1$ and $k_1 \geq 2$ and $k_m < p - 1$,

$$C_2[F_p^{(m)}(x)] = x^{p-1} + x^{p-2} + \sum_{i=1}^{m} x^{k_i-1} + \sum_{i=1}^{m} x^{k_i-2} + 1. \tag{9}$$

It is obvious that the degrees of the resultants shown in (7), (8), and (9) are less than $p$.

Furthermore, for $p - 1 \leq k_2 < k_1 \leq 1$, the resultants of direct Collatz operations on $F_p^{(2)}(x)$ are

$$C_1[C_1[F_p^{(2)}(x)]] = x^{p+1} + x^{p-2} + x^4 + 1 \text{ for } k_1 = 1, k_2 = 2,$$

$$C_1[F_p^{(2)}(x)] = x^p + x^{p-1} + x^4 + 1 \text{ for } k_1 = 1, k_2 = 3,$$

$$C_3[C_1[F_p^{(2)}(x)]] = x^{p-1} + x^{p-4} + x^3 + x + 1 \text{ for } k_1 = 1, k_2 = 4,$$

$$C_4[C_1[F_p^{(2)}(x)]] = x^{p-2} + x^{p-5} + x^{k_2-2} + x^{k_2-5} + 1 \text{ for } k_1 = 1, k_2 > 4,$$

$$C_3[F_p^{(2)}(x)] = x^{p-2} + x^{p-3} + x^2 + 1 \text{ for } k_1 = 2, k_2 = 3,$$

$$C_4[F_p^{(2)}(x)] = x^{p-3} + x^{p-4} + x^{k_2-3} + x^{k_2-4} + 1 \text{ for } k_1 = 2, k_2 \geq 4,$$

$$C_2[F_p^{(2)}(x)] = x^{p-1} + x^{p-2} + x^{k_2-1} + x^{k_2-2} + x^{k_1-1} + x^{k_1-2} + 1 \text{ for } k_1 \geq 3. \tag{10}$$

The above examples show that the first two cases of $F_p^{(2)}(x)$ given in (10) increase or remain unchanged in their degree after a few steps of the Collatz operation, while the other cases decrease in the degree, which are mainly determined by the values of $k_i$ ($i = 1, \ldots, m$), especially by those of $k_m$ and $k_1$ of the monomials $x^{k_m}$ and $x^{k_1}$ involved. When $p - k_m = 1$, the degree of the leading term of $(x + 1)(x^p + x^{k_m})$ will increase from $p$ to $p + 2$, while $(x + 1)(x^{k_1} + 1) + 1$ becomes $x^3 + x$ when $k_1 = 1$, with which the divisor $x^q$ required in the Collatz operation is the smallest in $q$ with $q = 1$ and resulting in $p + 1$ degree polynomial $C_1[F_p^{(m)}(x)]$ for $m > 1$. This situation remains unchanged after several steps of the Collatz operation, especially when $k_m = p - 1$, $k_m - 1 = p - 2$, $\ldots$, $k_1 = 1$ for $m = p - 1$. As the consequence, the worst situation is typically represented by $F_p^{(m)}(x) = x^{p+1} - 1$ with $m = p - 1$, which needs most steps of the Collatz operation to get a polynomial with degree less than $p$. Especially, $\sum_{i=1}^{l} q_i = l$ for $l \leq p$ for this case, with which the degree of the resulting polynomial will increase within the first $l$ steps of the Collatz operation.

Table 2 provides the resultant of $F_p^{(p-1)}(x)$ after $p$ steps of the Collatz operation for $p \leq 32$ explicitly as examples, in which the last column provides the $\sum_{i=1}^{k} q_i / k$, where $k$ is the total number of steps of the Collatz operation needed for $C_1^{(p)}[F_p^{(p-1)}(x)]$ to reach $F_0$. In this case, one can verify that

$$G_{u(p)+1}(x) = C_1^{(p)}[F_p^{(p-1)}(x)] = \left(\frac{x+1}{x}\right)^p F_p^{(p-1)}(x) + \frac{1}{x} \sum_{\mu=0}^{p} \left(\frac{x+1}{x}\right)^\mu = x(x+1)^p - 1, \tag{11}$$

where $C_1^{(p)}$ stands for $p$ times of the Collatz operation $C_1$, which is a polynomial of degree $u(p) + 1 = 1 + \text{Int}[-1/2 + p\ln[3]/\ln[2]] > p$. Obviously, $q_i = 1$ for $i \leq p$ within the first $p$ steps of the Collatz operation on $F_p^{(p-1)}(x)$. 

Table 2. Some $F_p^{(p-1)}(x)$ after $p$ steps of Collatz operations $G_{u(p)+1}(x) = C_r^{(p)}[F_p^{(p-1)}(x)]$, where, due to (15), only $G_{u(p)+1}(x)$ with even $p$ for $p \leq 32$ are provided.

| $p$ | $u(p) + 1$ | $G_{u(p)+1}(x)$ | $(\sum_{i=1}^{\infty} q_i)/k$ |
|-----|-------------|------------------|-----------------------------|
| 2   | 4           | $1 + x^4$        | 3                           |
| 4   | 7           | $1 + x^4 + x^7$  | 2.172                       |
| 6   | 10          | $1 + x^4 + x^5 + x^7 + x^{10}$ | 2.778                       |
| 8   | 13          | $1 + x^6 + x^8 + x^{10} + x^{12} + x^{13}$ | 3.257                       |
| 10  | 16          | $1 + x^4 + x^6 + x^8 + x^{10} + x^{11} + x^{14} + x^{15} + x^{16}$ | 1.957                       |
| 12  | 20          | $1 + x^{13} + x^6 + x^7 + x^8 + x^{10} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{20}$ | 2.045                       |
| 14  | 23          | $1 + x^4 + x^5 + x^6 + x^7 + x^8 + x^{10} + x^{12} + x^{13} + x^{14} + x^{15} + x^{16} + x^{20} + x^{23}$ | 3.033                       |
| 16  | 26          | $1 + x^7 + x^9 + x^{10} + x^{11} + x^{13} + x^{15} + x^{16} + x^{21} + x^{24} + x^{26}$ | 1.968                       |
| 18  | 29          | $1 + x^4 + x^7 + x^8 + x^{13} + x^{16} + x^{17} + x^{18} + x^{19} + x^{21} + x^{25} + x^{26} + x^{27} + x^{29}$ | 2.333                       |
| 20  | 32          | $1 + x^5 + x^8 + x^{10} + x^{12} + x^{13} + x^{19} + x^{21} + x^{23} + x^{24} + x^{25} + x^{26} + x^{27} + x^{28} + x^{31} + x^{32}$ | 2.072                       |
| 22  | 35          | $1 + x^4 + x^5 + x^{12} + x^{14} + x^{15} + x^{16} + x^{19} + x^{22} + x^{23} + x^{26} + x^{27} + x^{28} + x^{31} + x^{33} + x^{34} + x^{35}$ | 1.822                       |
| 24  | 39          | $1 + x^{6} + x^7 + x^{8} + x^{12} + x^{13} + x^{14} + x^{16} + x^{19} + x^{20} + x^{21} + x^{26} + x^{31} + x^{32} + x^{33} + x^{39}$ | 1.985                       |
| 26  | 42          | $1 + x^4 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{18} + x^{25} + x^{26} + x^{29} + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} + x^{39} + x^{42}$ | 1.955                       |
| 28  | 45          | $1 + x^4 + x^5 + x^6 + x^8 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{15} + x^{18} + x^{21} + x^{25} + x^{26} + x^{28} + x^{30} + x^{31} + x^{33} + x^{34} + x^{35} + x^{36} + x^{39} + x^{40} + x^{43} + x^{45}$ | 2.040                       |
| 30  | 48          | $1 + x^4 + x^5 + x^6 + x^8 + x^{12} + x^{13} + x^{14} + x^{15} + x^{18} + x^{20} + x^{23} + x^{24} + x^{25} + x^{26} + x^{31} + x^{32} + x^{33} + x^{35} + x^{36} + x^{39} + x^{41} + x^{42} + x^{44} + x^{45} + x^{46} + x^{48}$ | 2.048                       |
| 32  | 51          | $1 + x^{8} + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + x^{18} + x^{19} + x^{20} + x^{21} + x^{26} + x^{30} + x^{31} + x^{32} + x^{33} + x^{34} + x^{35} + x^{36} + x^{39} + x^{41} + x^{43} + x^{45} + x^{48} + x^{50} + x^{51}$ | 1.924                       |

From (11), we have

$$G_{u(p)+1}(x) = x(x+1)^p - 1, \quad G_{u(p)+1}(x) = (x+1)G_{u(p)+1}(x) + x,$$  \hspace{1cm} (12)

where $q > 2$ for $p$ odd, and $q = 2$ for $p$ even. Hence,

$$C_r[C_2[G_{u(p)+1}(x)]] = \frac{(x+1)^{p+2} - 1}{x^{r+1}},$$  \hspace{1cm} (13)

for $p$ even, where $r \geq 2$ because $p+1$ is odd. Moreover,

$$C_{r+2}[G_{u(p)+1}(x)] = \frac{(x+1)^{p+2} - 1}{x^{r+1}}.$$  \hspace{1cm} (14)

Combining Eqs. (13) and (14), we get

$$C_{r+2}[G_{u(p)+1}(x)] = C_r[C_2[G_{u(p)+1}(x)]] \quad \text{for} \quad p = 0, 2, 4, \ldots.$$  \hspace{1cm} (15)

According to (15), if $G_{u(p)+1}(x)$ is a Collatz polynomial, $G_{u(p)+1}(x)$ is also a Collatz polynomial, which, however, is valid only when $p$ is even. In addition, according to (12) and (13),

$$C_r[C_2[G_{u(p)+1}(x)]] = \frac{(x+1)^{p+2} - 1}{x^{r+1}} = \frac{x(x+1)^{p+2} - x}{x^{r+2}} = \frac{x(x+1)^{p+2} - 1 - 1}{x^{r+2}} = \frac{G_{u(p)+2}(x) - 1}{x^{r+2}},$$  \hspace{1cm} (16)

from which we have

$$G_{u(p+2)+1}(x) = x^{r+2}C_r[C_2[G_{u(p)+1}(x)]] + 1.$$  \hspace{1cm} (17)

for $p = 0, 2, 4, \ldots$, where the positive integer $r$ changes with $p$ quasi-periodically, of which some examples are provided in Table 3 and consistent with the results shown in Table 2.
Table 3. The positive integer $r$ in Eq. (17) as a function of even $p$ for $p \leq 32.$

| $p$  | $r$ | $p$  | $r$ |
|------|-----|------|-----|
| 0    | 2   | 3    | 3   |
| $x^2$ | 2   | $x^2 + x$ | 4   |
| $x^3$ | 2   | $x^3 + x$ | 3   |
| $x^3 + x^2$ | 2   | $x^3 + x^2 + x$ | 5   |
| $x^4$ | 2   | $x^4 + x$ | 3   |
| $x^4 + x^2$ | 2   | $x^4 + x^2 + x$ | 4   |
| $x^4 + x^3$ | 2   | $x^4 + x^3 + x$ | 3   |
| $x^4 + x^3 + x^2$ | 2   | $x^4 + x^3 + x^2 + x$ | 6   |
| $x^5$ | 2   |      |      |

In order to demonstrate the pattern of $\{F_p(x)\}$ after the Collatz operation, one may denote $F_p(x)$ and the resultant of $F_p(x)$ after several steps of the Collatz operation as nodes. If $F'_p(x)$ is the resultant of $C_q[F_p(x)]$, $F_p(x)$ and $F'_p(x)$ are connected with an arrow line, of which the arrow points to $F'_p(x)$. Hence, one can generate the Collatz tree graph for $\{F_p(x)\}$ under the Collatz operation.

**FIG. 1:** A part of the Collatz tree graph for the polynomials $\{F_p(x)\}$ with $p \leq 4$.

Fig. 1 shows a part of the Collatz tree graph for $\{F_p(x)\}$ with $p \leq 4$, where a long path with 29 nodes from $F^{(3)}_4$ to $1 + x + x^2 + x^4$, in which no polynomial $F_m(x)$ with $m \leq 4$ appears, is abbreviated with dots. Though Fig 1 only provides with a small portion of the graph with $p \leq 4$, its pattern is quite the same as that of the whole tree graph due to the fact that the properties of the polynomials $\{F_p(x)\}$ for either even $p$ or those for odd $p$ are the same among themselves. The common features of the graph can be summarized as follows: (a) Due to Proposition 1, the only endpoint node on the tree graph is $F_0$, and there is no other endpoint node on any path. Moreover, Proposition 1 also asserts that the tree is unique. Namely, there is no other separate trees containing some of the polynomials with a different endpoint. (b) For a given $F_p(x)$, there is one and only one path on the tree graph, which can easily be proven because $C_q[F_p(x)]$ is unique. This unique path is towards $F_0$, which is due to the fact that $F_0$ is the only endpoint node on the tree graph. Actually, $F_0$ is the unique invariant polynomial under the Collatz operation. (c) If $F_p(x)$ is a factor of $x + 1 = x^2 - 1$, it is a starting node of a path to $F_0$, because there is no polynomial after the Collatz operation to be a factor of $x + 1 = x^2 - 1$. (d) From $F_p(x)$ towards $F_0$, in comparison with the former node on the move indicated by the arrow, the degree of the polynomials on the path may keep unchanged, decrease, or increase. Without exception, increasing in the degree with amount $\Delta p \geq 2$ occurs only on the path from $F^{(m-1)}_m(x)$
towards $F_0$ for $m \leq p$, which shows that $F^{(m-1)}_m(x)$ with $m \leq p$ are the only sources resulting in far increasing of the degree of the polynomial $F_p(x)$ after the Collatz operation.

Concerning the common features of the tree graph, one of the unsolved problems is the possibility of circles under the Collatz operation. Let $\{f_k\} (k = 1, 2, \cdots, n)$ be a set of odd integers satisfying $f_{k+1} = C_q, [f_m]$ for $m = 1, 2, \cdots, k - 1$. If $f_1 = C_{q_1}, [f_k]$, then $\{f_k\}$ forms a circle [6] under the Collatz operation. Another unsolved problem is that there may be some polynomials $\{g_k\}$, of which the degrees go to infinity after consecutive Collatz operations. In these two cases, the related polynomials will be separated from the tree graph shown in Fig. 1. It was shown in [6] that the circle containing $F_p(x)$ is not possible for small $p$, but might occur when $p$ is large. For both cases, the degree of $F_p(x)$, on average, will never decrease under the Collatz operation. However, the structure of the polynomials are all the same. Namely, whether a polynomial $F_p(x)$ is the Collatz polynomial should be independent of the degree $p$. A short discussion on this problem will be made later on.

Moreover, when $m = 2k$, $F^{(2k-1)}_{2k}(x)$ can be expressed as

$$F^{(2k-1)}_{2k}(x) = C_2 \sum_{\mu=1}^{k+1} x^{2\mu-1} - 1, \quad (18)$$

where $\sum_{\mu=1}^{k+1} x^{2\mu-1} - 1$ is a polynomial of degree $2k + 1$. While $F^{(m-1)}_m(x)$ is the starting node of a path when $m$ is odd, because $F^{(2k)}_{2k+1}$ is a factor of $x + 1$. When $2k = 3\tau + 1$ for $\tau = 1, 3, 5 \cdots$,

$$\sum_{\mu=1}^{k+1} x^{2\mu-1} - 1 = C_1[H_{2k}(x)], \quad (19)$$

for $k = 2, 3, \cdots$, where

$$H_4(x) = 1 + x + x^3 + x^4, \quad H_{4+6(t+1)}(x) = H_{4+6t}(x) + (1 + x + x^2)x^{6t+8} \quad (20)$$

for $t = 0, 1, 2, \cdots$. Hence, when $p = 2k = 3\tau + 1$ for $\tau = 1, 3, 5 \cdots$, $H_{2k}(x)$ and $F^{(2k)}_{2k-1}$ are on the same path, where

$$F^{(2k-1)}_{2k}(x) = C_2[C_1[H_{2k}(x)]] \quad (21)$$

for $k = C_1[\tau]$ with $\tau = 1, 3, 5 \cdots$.

Therefore, if no circle and non-decreasing in the degree occur under the Collatz operation, the polynomials $\{F_p(x)\}$ under the consecutive steps of the Collatz operation like a simple board game, for which the rules of the moves are determined by the Collatz operation. Once a move starts from any one of the nodes on the board, there is only one possible path, which is towards the only destination $F_0 = 1$ under the Collatz operation. Since $F_0 = 1$ is the only endpoint node on the tree graph, the degree of the polynomials $\{F_p(x)\}$ after finite steps of the Collatz operation, on average, should decrease, which can be estimated as follows:

**Proposition 2** The degree $p$ of the Collatz polynomials $\{F_p(x)\}$, on average, decreases after a finite steps of the Collatz operation.

Generally, the polynomials (5) may be one of the following four cases with

$$F_p(x) \equiv F^{(m)}_p(x) = \begin{cases} 
1 + f^1_p(x), \\
1 + x + f^2_p(x), \\
1 + x^2 + f^3_p(x), \\
1 + x + x^2 + f^4_p(x) 
\end{cases} \quad (22)$$

for $m \geq 3$, where $f^1_p(x) = F^{(m)}_p(x) - 1$ when $k_1 \geq 3$ in $F^{(m)}_p(x)$, $f^2_p(x) = F^{(m)}_p(x) - 1 - x$ when $k_1 = 1$ and $k_2 \geq 3$ in $F^{(m)}_p(x)$, $f^3_p(x) = F^{(m)}_p(x) - x^2 - 1$ when $k_1 = 2$ and $k_2 \geq 3$ in $F^{(m)}_p(x)$, and $f^4_p(x) = F^{(m)}_p(x) - x^2 - x - 1$ when $k_1 = 1$, $k_2 = 2$, and $k_3 \geq 3$ in $F^{(m)}_p(x)$. The Collatz operation $C_{q_1}$ to the first case is definitely with $q_1 = 2$, $C_{q_2}$ or
the resultant of $F^{(m)}_p(x)$ after the Collatz operation, $C_q[F^{(m)}_p(x)]$ $(i = 1, 2, 3, 4)$, is still of the form shown in (22). Though there are more possible outcomes for the third case of (22), the resultant of $C_{q_3}[F^{(m)}_p(x)]$ is also of the form shown in (22). The exceptional case is $F^{(m)}_p(x)$ with $m = p - 1$, for which $q_3 = 1$ for $i \leq p$. Except $F^{(p-1)}_p(x)$ and some polynomials related to it under the Collatz operation, the four cases listed in (22) occur approximately randomly after the Collatz operation on the polynomials $F_p(x)$ including $G_{u(p)+1} = C^{(p)}_1[F^{(p-1)}_p(x)]$ shown in (12) with some examples provided in Table 2. Hence, except $F^{(p-1)}_p(x)$ and some polynomials related to it under the Collatz operation, on average, the lower bound of the mean-value $\langle \sum_{i=1}^{l} q_i \rangle$ after $l$ steps of the Collatz operation on $F_p(x)$ can be expressed as

$$\langle \sum_{i=1}^{l} q_i \rangle \geq \left( \frac{2}{4} + \frac{1}{4} + \frac{3}{4} + \frac{1}{4} \right) l = 1.75 l.$$  

(24)

It should be stated that (24) only provides the lower bound of the mean-value for the polynomial $F_p(x)$, which is not related to $F^{(p-1)}_p(x)$ under the Collatz operation. For example, the values of $\langle \sum_{i=1}^{k} q_i \rangle$ shown in Table 2 are always greater than 1.75 given in (24). There are more extreme cases with $\sum_{i=1}^{l} q_i$ deterministically greater than $\langle \sum_{i=1}^{l} q_i \rangle$. For example, since the Collatz operation on $\{U_p(x)\}$ provided in Corollary 2 always results in $C_0[l+2][U_p(x)] = U_0 = 1$, especially $C_2[U_0(x)] = U_0 = 1$, $q_3 = 2p + 2$ for the former and $\sum_{i=1}^{l} q_i = 2k$ for the latter. For a class of polynomials $\{F_p(x)\}$ related to $F^{(p-1)}_p(x)$ under the Collatz operation, the mean-value $\langle \sum_{i=1}^{l} q_i \rangle$ should be modified as

$$\langle \sum_{i=1}^{l} q_i \rangle \begin{cases} \approx l & \text{for } l \leq p, \\ \geq 1.75 l & \text{for } l > p. \end{cases}$$  

(25)

Thus, it can now be shown that Proposition 2 is valid in concerning (6) and the estimation of the lower bound of the mean-value $\langle \sum_{i=1}^{l} q_i \rangle$ after $l$ steps of the Collatz operation (24) and (25). Since, after $l$ steps of the Collatz operation, the degree of $C_{q_l}q_{l-1}\cdots q_1[F_p(x)]$,

$$\text{Deg}(C_{q_l}q_{l-1}\cdots q_1[F_p(x)]) = p + u(l) + 1 - \sum_{i=1}^{l} q_i,$$  

(26)

and, on the average, $\langle \sum_{i=1}^{l} q_i \rangle < \sum_{i=1}^{l} q_i$,

$$\text{Deg}(C_{q_l}q_{l-1}\cdots q_1[F_p(x)]) < p + u(l) + 1 - \langle \sum_{i=1}^{l} q_i \rangle = p + \frac{1}{2} + \left( \frac{\ln[3]}{\ln[2]} - 1.75 \right) l \approx p + 0.5 - 0.165037 l$$  

(27)

for $F_p(x)$ not related to $F^{(p-1)}_p(x)$ under the Collatz operation, while

$$\text{Deg}(C_{q_l}q_{l-1}\cdots q_1[F^{(p-1)}_p(x)]) \begin{cases} \approx p + 0.5 + 0.58496 l & \text{for } l \leq p, \\ < 0.5 - 0.165037 l + 1.75 p & \text{for } l > p, \end{cases}$$  

(28)

which also applies to $\{F_p(x)\}$. (27) and (28) provide with the average upper bound of $\text{Deg}(C_{q_l}q_{l-1}\cdots q_1[F_p(x)])$. It can be observed from (27) and (28) that $\text{Deg}(C_{q_l}q_{l-1}\cdots q_1[F_p(x)])$ with sufficiently large and finite $l$ will be less than $p,$
with which $C_{q_{n-1} \cdots q_1}[F_p(x)] < F_p(x)$ is definitely satisfied. Hence, the Proposition 2 is stronger than and consistent
with the results shown in [1]. In addition the number of steps of the Collatz operation needed for $F_p(x)$ to reach
$F_0$ estimated by (27) or (28) is slightly larger than that estimated by previous probabilistic prediction [7], because
(27) or (28) provides with the upper bound of $\text{Deg}(C_{q_{n-1} \cdots q_1}[F_p(x)])$. Proposition 2 also ensures that there is no
circle containing $F_p(x)$ for any $p$, and there is no polynomial non-decreasing in its degree after consecutive Collatz
operations. Thus, $F_0$ is the only endpoint of the unique tree graph.

Since $F_0$ is the obvious Collatz polynomial, if $\{F_\mu(x)\}$ with $\mu = 0, 1, \cdots, p-1$ have been verified to be the Collatz
polynomials, $\{F_p(x)\}$ are also the Collatz polynomials because $\{F_p(x)\}$ will become $\{F_\mu(x)\}$ with $\mu = 0, 1, \cdots, p-1$
after a finite steps of the Collatz operation as shown in Proposition 2. Hence, by using the induction on the degree
of the polynomials, $\{F_p(x)\}$ for any $p$ are the Collatz polynomials.

In summary, the polynomials in representing integers based on a binary numeral system are introduced to explore
the Collatz conjecture, which seem more convenient in the computation under the Collatz operation. Especially, the
polynomial structure and its evolution under the Collatz operation become more transparent, from which the upper
bound of the degree of the polynomial after a finite steps of the Collatz operation is estimated. With this upper
bound, it is shown that the conjecture is true in terms of the induction with respect to the degree of the polynomials.

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