Minimization of quotients with variable exponents

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Abstract

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, $p \in C^1(\Omega)$, $q \in C(\Omega)$ and $l, j \in \mathbb{N}$. We describe the asymptotic behavior of the minimizers of the Rayleigh quotient $\frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}}$, first when $j \to \infty$ and after when $l \to \infty$.

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1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, and consider the Rayleigh quotient

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}}, \quad (1)$$

associated with the immersion of the Sobolev space $W^{1,p(x)}_0(\Omega)$ into the Lebesgue space $L^{q(x)}(\Omega)$, where the variable exponents satisfy

$$1 < \inf_{\Omega} p(x) \leq \sup_{\Omega} p(x) < \infty$$

and

$$1 < q(x) < p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

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In this paper we study the behavior of the least Rayleigh quotients when the functions \( p(x) \) and \( q(x) \) become arbitrarily large. Our script is based on the paper [3], where these functions are constants. Thus, in order to overcome the difficulties imposed by the fact that the exponents depend on \( x \), we adapt arguments developed by Franzina and Lindqvist in [18], where \( p(x) = q(x) \). Actually, our results in the present paper generalize those of [8] for variable exponents and complement the approach of [18].

In [8], Ercole and Pereira first studied the behavior, when \( q \to \infty \), of the positive minimizers \( w_q \) corresponding to \( \lambda_q := \min \left\{ \| \nabla u \|_{L^p(\Omega)} : u \in W_0^{1,p}(\Omega) \quad \text{in} \quad \| u \|_{L^q(\Omega)} = 1 \right\} \), for a fixed \( p > N \). An \( L^\infty \)-normalized function \( u_p \in W_0^{1,p}(\Omega) \) is obtained as the uniform limit in \( \Omega \) of a sequence \( w_{q_n} \), with \( q_n \to \infty \). Such a function is positive in \( \Omega \), assumes its maximum only at a point \( x_p \) and satisfies
\[
\begin{cases}
-\Delta_p u = \Lambda_p \delta_{x_p} & \text{in } \Omega \\
0 & \text{on } \partial \Omega,
\end{cases}
\]
where \( \Lambda_p := \min \left\{ \| \nabla u \|_{L^p(\Omega)} : u \in W_0^{1,p}(\Omega) \quad \text{in} \quad \| u \|_{L^\infty(\Omega)} = 1 \right\} \) and \( \delta_{x_p} \) denotes the Dirac delta distribution concentrated at \( x_p \). In the sequence, the behavior of the pair \( (\Lambda_p, u_p) \), as \( p \to \infty \), is determined. In fact, it is proved that
\[
\lim_{p \to \infty} \Lambda_p = \Lambda_\infty := \inf_{0 \neq v \in W_0^{1,\infty}(\Omega)} \frac{\| \nabla v \|_\infty}{\| v \|_\infty}
\]
and that there exist a sequence \( p_n \to \infty \), a point \( x_\ast \in \Omega \) and a function \( u_\infty \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \) such that: \( x_{p_n} \to x_\ast \), \( \| \rho(x_\ast) \|_\infty = \frac{\rho(x_\ast)}{\| \rho \|_\infty} \) and \( u_{p_n} \to u_\infty \), uniformly in \( \Omega \). Moreover, it is shown that: \( u_\infty \) is also a minimizer of \( \Lambda_\infty \), assumes its maximum value 1 only at \( x_\ast \) and satisfies
\[
\begin{cases}
\Delta_\infty u = 0 & \text{in } \Omega \setminus \{ x_\ast \} \\
u = \frac{\rho}{\| \rho \|_\infty} & \text{on } \partial (\Omega \setminus \{ x_\ast \}) = \{ x_\ast \} \cup \partial \Omega
\end{cases}
\]
in the viscosity sense.

In [18], Franzina and Lindqvist determined the exact asymptotic behavior, as \( j \to \infty \), of both the minimum \( \Lambda_{jp(x)} \) of the quotients \( \frac{\| \nabla u \|_{jp(x)}}{\| u \|_{jp(x)}} \) and its respective \( jp(x) \)-normalized minimizer \( u_j \). It is proved that
\[
\lim_{j \to \infty} \Lambda_{jp(x)} = \Lambda_\infty
\]
and that a subsequence of \( (u_j)_{j \in \mathbb{N}} \) converges uniformly in \( \overline{\Omega} \) to a nonnegative function \( 0 \neq u_\infty \in C(\overline{\Omega}) \cap W_0^{1,\infty}(\Omega) \) satisfying, in the viscosity sense, the equation
\[
\max \left\{ \Lambda_\infty - \frac{\| \nabla u \|_\infty}{u}, \quad \Delta_{\infty(x)} \left( \frac{u}{\| \nabla u \|_\infty} \right) \right\} = 0 \quad \text{in } \Omega,
\]
where the operator $\Delta_{\infty(x)}$ is defined by

$$\Delta_{\infty(x)} u := \Delta u + |\nabla u|^2 \ln |\nabla u| \langle \nabla u, \nabla \ln p \rangle.$$ 

In the present paper we assume that $p \in C^1(\Omega)$, $q \in C(\Omega)$ and $1 \leq q(x) < p^*(x)$ in $\Omega$. After presenting, in Section 2, a brief review on the theory of Sobolev-Lebesgue spaces with variable exponents, we show in Section 3 that

$$\Lambda_1 := \inf \left\{ \frac{\|\nabla v\|_{p(x)}}{\|v\|_{q(x)}} : v \in W_{0}^{1,p(x)}(\Omega) \setminus \{0\} \right\} = \frac{\|\nabla u\|_{p(x)}}{\|u\|_{q(x)}} > 0$$

for some $u \in W_{0}^{1,p(x)}(\Omega) \setminus \{0\}$. Moreover, taking [18] and [25] as reference, we derive the following Euler-Lagrange equation corresponding to this minimization problem

$$- \text{div} \left( \frac{\nabla u}{K(u)} \right)^{p(x)-2} \nabla u = \Lambda S(u) \left( \frac{u}{k(u)} \right)^{q(x)-2},$$

where $\Lambda = \Lambda_1$,

$$K(u) := \|\nabla u\|_{p(x)}, \quad k(u) := \|u\|_{q(x)}, \quad \text{and} \quad S(u) := \frac{\int_{\Omega} |\nabla u|^{p(x)} d\Omega}{\int_{\Omega} |u|^{q(x)} d\Omega}.$$

We consider (2)-(3) as an eigenvalue problem. Thus, if a pair $(\Lambda, u) \in \mathbb{R} \times W_{0}^{1,p(x)}(\Omega) \setminus \{0\}$ solves (2)-(3) we say that $\Lambda$ is an eigenvalue and $u$ is an eigenfunction corresponding to $\Lambda$. In this setting, $\Lambda_1$ is the first eigenvalue and any of its corresponding eigenfunctions is a first eigenfunction. We show that any first eigenfunction do not change sign in $\Omega$ and, for the sake of completeness, we apply a minimax scheme based on Kranoselskii genus to obtain an increasing and unbounded sequence of eigenvalues.

Our main results are established in Sections 4 and 5. First we consider a natural $l > N$ and show, in Section 4, that

$$\mu_l := \inf \left\{ \frac{\|\nabla v\|_{p(x)}}{\|v\|_{\infty}} : v \in W_{0}^{1,lp(x)}(\Omega) \setminus \{0\} \right\} = \lim_{j \to \infty} \Lambda_{l,j},$$

where

$$\Lambda_{l,j} := \inf \left\{ \frac{\|\nabla v\|_{lp(x)}}{\|v\|_{jq(x)}} : v \in W_{0}^{1,lp(x)}(\Omega) \setminus \{0\} \right\}.$$ 

Moreover, by using the results of Section 3, we argue that for each fixed $j > 1$ there exists a positive minimizer $u_{l,j} \in W_{0}^{1,lp(x)}(\Omega) \setminus \{0\}$ for $\Lambda_{l,j}$. Hence, the compactness of the embedding $W_{0}^{1,lp(x)}(\Omega) \hookrightarrow C(\Omega)$ implies that $\mu_l$ is achieved at a function $w_l$ which is obtained as the uniform limit of $u_{l,j,m}$ for a subsequence $j_{m} \to \infty$. 

3
We also show in Section 4, by using arguments developed in [19], that \( \mu_l \) is achieved at \( u \) if, and only if,

\[
-\text{div} \left( \frac{\nabla u}{|K_l(u)|} \frac{|\nabla u|}{K_l(u)} \right) = \mu_l \left( \int_{\Omega} |\nabla u|^{\frac{j}{p(x)}} \frac{|\nabla u|}{K_l(u)} \, dx \right) \text{sgn}(u(x_0)) \delta_{x_0},
\]

where \( K_l(u) = \|\nabla u\|_{l^p(x)} \) and \( x_0 \) is the only point where \( u \) reaches its uniform norm.

Finally, in Section 5, we study the asymptotic behavior of \( \mu_l \) and of its normalized extremal function \( w_l \), where \( \|w_l\|_{\infty} = 1 \) and \( \mu_l = \|\nabla w_l\|_{l^p(x)} \), when \( l \to \infty \). We prove that

\[
\lim_{l \to \infty} \mu_l = \Lambda_{\infty}
\]

and that there exist \( l_n \to \infty \), \( x_\star \in \Omega \) and \( w_\infty \in W^{1,\infty}_0(\Omega) \cap C(\overline{\Omega}) \) such that \( w_{l_n} \to w_\infty \) uniformly in \( \overline{\Omega} \) and

\[
0 \leq w_\infty \leq \frac{d}{\|d\|_{\infty}} \quad \text{a.e.} \ \Omega \quad \text{and} \quad w_\infty(x_\star) = \frac{d(x_\star)}{\|d\|_{\infty}},
\]

where \( d(x) = \text{dist}(x, \partial \Omega) \) is the function distance to the boundary. It is well-known that \( d \in W^{1,\infty}_0(\Omega) \) and

\[
\Lambda_{\infty} = \frac{1}{\|d\|_{\infty}}.
\]

Moreover, we prove that \( \Lambda_{\infty} \) is attained at \( w_\infty \) and that this function satisfies

\[
\begin{cases}
\Delta_{\infty}(x) \left( u \over \|\nabla u\|_{\infty} \right) = 0 & \text{in } D = \Omega \setminus \{x_\star\} \\
\frac{u}{\|\nabla u\|_{\infty}} = d & \text{on } \partial D = \partial \Omega \cup \{x_\star\}.
\end{cases}
\]

in the viscosity sense.

Due to the lack of a suitable version of the Harnack’s inequality for the "variable infinity operator" \( \Delta_{\infty}(x) \), one cannot guarantee that the function \( w_\infty \) is strictly positive in \( \Omega \).

At the end of Section 5, by using a uniqueness result proved in [21] for the equation \( \Delta_{\infty}(x) u = 0 \), we provide a sufficient condition on \( \Omega \) for the equality

\[
w_\infty = \frac{d}{\|d\|_{\infty}} \quad \text{in } \Omega
\]

to hold.

After comparing our results with those of [18], it is interesting to remark that the minimum of the quotients \( \|\nabla u\|_{l^p(x)} \) converges to \( \Lambda_{\infty} \) independently of how \( l^p(x) \) and \( j^q(x) \) go to \( \infty \) : if either \( l = j \to \infty \) in the case \( p(x) = q(x) \) or \( j \to \infty \) firstly and then \( l \to \infty \). However, the same do not hold for the corresponding minimizers (or for their respective limit problems). The distinction seems to be due to the Dirac delta that appears in the right-hand term of the Euler-Lagrange equation [2] when \( q(x) \) is replaced by \( j^q(x) \) and \( j \) is taken to infinity. The same distinction appears when \( p \) and \( q \) are constant, as one can check from [8] and [20].
2 Preliminaries

In this section we recall some definitions and results on the Sobolev-Lebesgue spaces with variable exponents.

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) and \( p \in C(\Omega) \) such that \( 1 < p^- := \inf p(x) \leq p^+ := \sup p(x) < \infty \). Let \( L^{p(x)}(\Omega) \) denote the space of the Lebesgue measurable functions \( u : \Omega \to \mathbb{R} \) such that
\[
\int_{\Omega} |u|^{p(x)} \, dx < \infty,
\]
endowed with the Luxemburg norm
\[
\|u\|_{p(x)} = \inf \left\{ \gamma > 0 : \int_{\Omega} \frac{|u(x)|^{p(x)}}{\gamma} \, dx \leq 1 \right\}.
\] (4)

Note that (4) is equivalent to the norm
\[
|u|_{p(x)} = \inf \left\{ \gamma > 0 : \int_{\Omega} \left( \frac{|u(x)|}{\gamma} \right)^{p(x)} \, dx \leq 1 \right\}.
\] (5)

An important concept in the theory of spaces \( L^{p(x)}(\Omega) \) is the modular function.

**Definition 2.1** The function \( \rho : L^{p(x)}(\Omega) \to \mathbb{R} \) defined by
\[
\rho(u) = \int_{\Omega} |u|^{p(x)} \, dx / p(x),
\]
is called the modular function associated to the space \( L^{p(x)}(\Omega) \).

The following proposition lists some properties of the modular function.

**Proposition 2.2** Let \( u \in L^{p(x)}(\Omega) \setminus \{0\} \), then

a) \( \|u\|_{p(x)} = a \) if, and only if, \( \rho(\frac{u}{a}) = 1 \);

b) \( \|u\|_{p(x)} < 1 \) (\( = 1; > 1 \)) if, and only if, \( \rho(u) < 1 \) (\( = 1; > 1 \));

c) If \( \|u\|_{p(x)} > 1 \) then \( \|u\|_{p(x)} < \|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+} \);

d) If \( \|u\|_{p(x)} < 1 \) then \( \|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-} \leq \|u\|_{p(x)} < \|u\|_{p(x)}^{p^+} \).
For a posterior use, we recall the following estimate valid for an arbitrary $u \in L^\infty(\Omega)$:

$$
\|u\|_{p(x)} \leq \alpha \|u\|_\infty, \quad \text{where} \quad \alpha := \begin{cases} 
|\Omega|^{1/p^+} & \text{if} \quad |\Omega| \leq 1, \\
|\Omega|^{1/p^-} & \text{if} \quad |\Omega| > 1.
\end{cases}
$$

(6)

This estimate is easily verified by applying item b) of Proposition 2.2 to the function $\frac{u}{\alpha \|u\|_\infty}$.

We define the Sobolev space

$$
W^{1,p(x)}(\Omega) := \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \},
$$

endowed with the norm

$$
\|u\|_{1,p(x)} := \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.
$$

Both $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)})$ are separable and uniformly convex (therefore, reflexive) Banach spaces.

The Sobolev space $W^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. In this space, $\|\nabla \cdot \|_{p(x)}$ is a norm equivalent to norm $\|\cdot\|_{1,p(x)}$ and this is a consequence of the following proposition.

**Proposition 2.3** (see [10]) Let $p \in C(\overline{\Omega})$ with $p^- > 1$. There exists a positive constant $C$ such that

$$
\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).
$$

Now, we recall some facts involving exponents $q(x) \leq p(x)$.

**Proposition 2.4** (see [10]) Let $p, q \in C(\overline{\Omega})$. Then

$$
L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega)
$$

if, and only if, $q(x) \leq p(x)$ in $\Omega$. Additionally, the embedding is continuous.

From now on, the notation $f \ll g$ will mean that $f(x) \leq g(x)$ for all $x \in \overline{\Omega}$ and

$$
\inf_{\overline{\Omega}} (g(x) - f(x)) > 0.
$$

**Proposition 2.5** ([13], [10]) Let $p, q \in C(\overline{\Omega})$ and $1 \leq q(x) \leq p^*(x)$ in $\overline{\Omega}$. The embedding

$$
W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)
$$

is continuous. Moreover, it is compact whenever $q \ll p^*$.

We define the operator $p(x)$-Laplacian by $\Delta_{p(x)} u := \text{div}(|\nabla u|^{p(x)-2} \nabla u)$ and consider the Dirichlet problem

$$
\begin{aligned}
-\Delta_{p(x)} u &= f(x,u), \quad x \in \Omega \\
u &= 0 \quad x \in \partial \Omega
\end{aligned}
$$

(7)

where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.

We say that a function $u \in W^{1,p(x)}_0(\Omega)$ is a weak solution of (7) if, and only if,

$$
\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \eta \, dx = \int_{\Omega} f(x,u) \eta \, dx, \quad \forall \eta \in W^{1,p(x)}_0(\Omega).
$$
Proposition 2.6  Weak solutions of (7) belong to $L^\infty(\Omega)$ provided that $f$ satisfies the sub-critical growth condition

$$|f(x,t)| \leq c_1 + c_2 |t|^\alpha(x)-1, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\alpha \in C(\overline{\Omega})$ and $1 < \alpha \ll p^*$.

Proposition 2.7  Suppose that $p(x)$ is Hölder continuous on $\overline{\Omega}$. If $u \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (7), then $u \in C^{1,\tau}(\overline{\Omega})$ for some $\tau \in (0,1)$.

The following strong maximum principle for $p(x)$-Laplacian is taken from [11].

Proposition 2.8  Suppose that $p \in C^1(\overline{\Omega})$, $u \in W^{1,p(x)}_0(\Omega) \setminus \{0\}$ and $u \geq 0$ in $\Omega$. If $-\Delta_{p(x)} u \geq 0$ in $\Omega$ then $u > 0$ in $\Omega$.

We recall that the inequality $-\Delta_{p(x)} u \geq 0$, for a function $u \in W^{1,p(x)}_0(\Omega)$, means

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \eta \, dx \geq 0, \quad \forall \eta \in W^{1,p(x)}_0(\Omega), \text{ with } \eta \geq 0.$$  

Theoretical results involving operators with variable exponent can be found among the papers [2, 7, 10–18, 23–27, 29] and in the references therein. For applications in rheology and image restoration we refer the reader to [1, 3, 28] and [5, 6], respectively.

3  The minimization problem

In this section we will consider $p \in C^1(\overline{\Omega})$ and $q \in C(\overline{\Omega})$, with $1 \leq q \ll p^*$. For practical purposes, $X$ will denote the Sobolev space $W^{1,p(x)}_0(\Omega)$ and $k, K : X \to \mathbb{R}$ will denote, respectively, the functionals

$$k(u) := \|u\|_{q(x)} \quad \text{and} \quad K(u) := \|\nabla u\|_{p(x)}, \quad u \in X.$$  

Since $K(u) = \|u\|_X$, the functional $K$ is sequentially weakly lower semicontinuous in $X$.

We will also consider

$$\Lambda_1 := \inf_{u \in X \setminus \{0\}} \frac{K(u)}{k(u)},$$

which is positive number, according to Proposition 2.5.

We say that a function $u \in X \setminus \{0\}$ is an extremal function (or minimizer) of $\Lambda_1$ if

$$\frac{K(u)}{k(u)} = \Lambda_1.$$  

The next proposition shows that such a function always exists.

Proposition 3.1  There exists a nonnegative extremal function of $\Lambda_1$.  

7
Proof. Let \((v_n) \subset X \setminus \{0\}\) be a minimizing sequence of admissible functions such that \(k(v_n) = 1\). Thus,

\[
\Lambda_1 = \lim_{n \to \infty} K(v_n).
\]

Since the sequence \((v_n)\) is bounded in the reflexive space \(X\), there exist a subsequence \((v_{n_j})\) and \(u \in X\) such that \(v_{n_j} \rightharpoonup u\) in \(X\). We can assume, from Proposition 2.3, that \(v_{n_j} \to u\) in \(L^{q(x)}(\Omega)\), so that \(k(v_{n_j}) \to k(u) = 1\). Since \(\|u\|_X \leq \liminf_j \|v_{n_j}\|_X\) we have

\[
K(u) \leq \lim_{j \to \infty} K(v_{n_j}) = \Lambda_1,
\]

showing thus that \(u\) is an extremal function of \(\Lambda_1\). It is simple to see that (the nonnegative) function \(|u|\) is also an extremal function of \(\Lambda_1\). □

Our next goal is to derive the Euler-Lagrange equation associated with the minimizing problem (9), which must be satisfied for the extremal functions of \(\Lambda_1\). For this we need the following lemma.

Lemma 3.2 (Lemma A.1, [18]) Let \(u \in X\) and \(\eta \in C_0^\infty(\Omega)\). Then,

\[
\frac{d}{d\varepsilon} K(u + \varepsilon \eta) \Big|_{\varepsilon=0} = \int_\Omega \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \cdot \nabla \eta dx
\]

and

\[
\frac{d}{d\varepsilon} k(u + \varepsilon \eta) \Big|_{\varepsilon=0} = \int_\Omega \left| \frac{u}{k(u)} \right|^{q(x)-2} \frac{u}{k(u)} \eta dx
\]

We observe that a necessary condition for the inequality

\[
\frac{K(u)}{k(u)} \leq \frac{K(u + \varepsilon \eta)}{k(u + \varepsilon \eta)}
\]

to hold is that

\[
\frac{d}{d\varepsilon} K(u + \varepsilon \eta) \Big|_{\varepsilon=0} = 0,
\]

which can be written as

\[
\frac{1}{K(u)} \frac{d}{d\varepsilon} K(u + \varepsilon \eta) \Big|_{\varepsilon=0} = \frac{1}{k(u)} \frac{d}{d\varepsilon} k(u + \varepsilon \eta) \Big|_{\varepsilon=0}.
\]

Therefore, according to Lemma 3.2 if \(u\) is an extremal function, then one must have

\[
\int_\Omega \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \cdot \nabla \eta dx = \Lambda_1 S(u) \int_\Omega \left| \frac{u}{k(u)} \right|^{q(x)-2} \frac{u}{k(u)} \eta dx, \quad \forall \eta \in C_0^\infty(\Omega)
\]
where

\[ S(u) := \frac{\int_{\Omega} |\nabla u|^p K(u) \, dx}{\int_{\Omega} |u|^q K(u) \, dx}. \tag{10} \]

Hence, since \( X \) is the closure of \( C^\infty_0(\Omega) \) in the norm \( \| \cdot \|_X \), the Euler-Lagrange equation for the extremal functions is

\[- \text{div} \left( \frac{\nabla u}{K(u)} \cdot \frac{p(x)}{u} \right) = \Lambda S(u) \frac{u}{k(u)} \sum_{\Omega} \frac{q(x)}{u}. \tag{11} \]

**Definition 3.3** We say that a real number \( \Lambda \) is an eigenvalue if there exists \( u \in X \setminus \{0\} \) such that

\[ \int_{\Omega} |\nabla u|^p K(u) \cdot \nabla \eta \, dx = \Lambda S(u) \int_{\Omega} |u|^q K(u) \cdot u \eta \, dx, \quad \forall \eta \in X. \tag{12} \]

In this case, we say that \( u \) is an eigenfunction corresponding to \( \Lambda \). 

**Remark 3.4** One can easily verify the following homogeneity property: if \( u \) is an eigenfunction corresponding to \( \Lambda \) the same holds for \( tu \), for any \( t \in \mathbb{R} \setminus \{0\} \).

Taking \( \eta = u \) in (12) and recalling the definition of \( S(u) \) in (10) we obtain

\[ K(u) \int_{\Omega} |\nabla u|^p K(u) \, dx = \Lambda S(u) k(u) \int_{\Omega} |u|^q K(u) \, dx = \Lambda k(u) \int_{\Omega} |\nabla u|^p K(u) \, dx, \]

so that

\[ \Lambda = \frac{K(u)}{k(u)} \geq \Lambda_1. \]

Hence, \( \Lambda_1 \) is called the first eigenvalue and the corresponding eigenfunctions are called first eigenfunctions. Clearly, the extremal functions are precisely the first eigenfunctions.

**Proposition 3.5** There exists a continuous, strictly positive first eigenfunction.

**Proof.** Proposition 3.1 shows that a nonnegative first eigenfunction \( u \in X \) exists, Propositions 2.6 and 2.7 guarantee that \( u \in C(\Omega) \) and the strong maximum principle (Proposition 2.8) yields that \( u > 0 \) in \( \Omega \).

**Remark 3.6** It can be verified that if the norm (5) is taken to define

\[ \tilde{\Lambda}_1 := \inf_{u \in X \setminus \{0\}} \frac{|\nabla u|_p}{|u|_q} \]

then

\[ \frac{1}{p^+} \tilde{\Lambda}_1 \leq \Lambda_1 \leq \frac{q^+}{q^+} \tilde{\Lambda}_1. \]
Moreover, the same results of Propositions 3.1 and 3.5 can be obtained, but associated with an Euler-Lagrange equation a bit more complicated:

\[
\int_{\Omega} p(x) \left| \frac{\nabla u}{K_*} \right|^{p(x)-2} \nabla u \cdot \nabla \eta \, dx = K_* \int_{\Omega} q(x) \left| \frac{u}{k_*} \right|^{q(x)-2} \frac{u}{k_*} \eta \, dx
\]

where,

\[K_* = |\nabla u|_{p(x)}, \quad k_* = |u|_{q(x)}, \quad S_u = \frac{\int_{\Omega} p(x) \left| \frac{\nabla u}{K_*} \right|^{p(x)} \, dx}{\int_{\Omega} q(x) \left| \frac{u}{k_*} \right|^{q(x)} \, dx}.
\]

Noting that \(\frac{p^-}{q^+} \leq S_u \leq \frac{p^+}{q^-}\) we can show the existence of a strictly positive eigenfunction. In fact, if \(u\) is a nonnegative function of (13) then, for any \(\eta \in W^{1,p(x)}_0(\Omega)\), with \(\eta \geq 0\), we have

\[
p^+ \int_{\Omega} \left| \frac{\nabla u}{K_*} \right|^{p(x)-2} \nabla u \cdot \nabla \eta \, dx \geq q^+ \left( \frac{p^-}{q^+} \right) K_* \int_{\Omega} \left| \frac{u}{k_*} \right|^{q(x)-2} \frac{u}{k_*} \eta \, dx \\
\geq \left( \frac{p^-}{q^+} \right) K_* \int_{\Omega} \left| \frac{u}{k_*} \right|^{q(x)-2} \frac{u}{k_*} \eta \, dx.
\]

It follows that \(-\Delta_{p(x)} \left( \frac{u}{k_*} \right) \geq 0\) what implies, by Proposition 2.8, that \(u > 0\) in \(\Omega\). Thus, we can see that the use of \(p(x)^{-1} \, dx\) simplifies the equations a little.

According to Lemma 3.2, the Gateaux derivatives \(K', k'\) are given, respectively, by

\[
\langle K'(u), v \rangle = \frac{\int_{\Omega} \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \cdot \nabla v \, dx}{\int_{\Omega} \left| \frac{\nabla u}{K(u)} \right|^{p(x)} \, dx}, \quad u, v \in X
\]

and

\[
\langle k'(u), v \rangle = \frac{\int_{\Omega} \left| \frac{u}{k(u)} \right|^{q(x)-2} \frac{u}{k(u)} v \, dx}{\int_{\Omega} \left| \frac{u}{k(u)} \right|^{q(x)} \, dx}, \quad u, v \in X.
\]

It is simple to check that \(K, k \in C^1(X, \mathbb{R})\) (see [12,14]). Thus, we define

\[\mathcal{M} := \{u \in X : k(u) = 1\} = k^{-1}(1).
\]

Since \(1\) is a regular value of \(k\), the set \(\mathcal{M}\) is a submanifold of class \(C^1\) in \(X\). The functional

\[\tilde{K} := K |_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}
\]
is of class $C^1$ and bounded from below in $\mathcal{M}$.

We know that $u$ is a critical point of $\bar{K}$ in $\mathcal{M}$ if there exists $\Lambda \in \mathbb{R}$ such that

$$K'(u) = \Lambda k'(u) \quad \text{in} \quad X^*,$$

meaning that

$$\langle K'(u), v \rangle = \Lambda \langle k'(u), v \rangle, \quad \forall v \in X.$$ 

Therefore, if $u$ is critical point of $\bar{K}$ then $u$ is solution of (12) with $\Lambda = K(u)/k(u)$.

Now, by adapting arguments of [25, Lemma 2.3], we show that $\bar{K}$ satisfies the Palais-Smale condition.

**Proposition 3.7** $\bar{K}$ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$, namely, every sequence $(u_n) \subset \mathcal{M}$ such that $\bar{K}(u_n) \to c$ and $\bar{K}'(u_n) \to 0$, has a convergent subsequence.

**Proof.** First, we show that if $u \in X \setminus \{0\}$, then

$$|\langle K'(u), v \rangle| \leq K(v) \quad \text{and} \quad |\langle k'(u), v \rangle| \leq k(v), \quad \forall v \in X. \quad (14)$$

We assume that $v \neq 0$ (otherwise the equality in (14) holds trivially). Then

$$|\langle k'(u), v \rangle| \leq \frac{\int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)-1}{q(x)}} |v| \, dx}{\int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)}{q(x)}} \, dx} \quad (15)$$

and, by using the Young inequality

$$ab \leq \left(1 - \frac{1}{r}\right) a^{r/r-1} + \frac{1}{r} b^r, \quad \forall a, b \geq 0, \ r > 1,$$

with $a = |u/k(u)|^{q(x)-1}$, $b = |v/k(v)|$, $r = q(x)$, and integrating over $\Omega$, we have

$$\int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)-1}{q(x)}} \left| \frac{v}{k(v)} \right| \, dx \leq \int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)}{q(x)}} \, dx - \int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)}{q(x)}} \, dx + \int_\Omega \left| \frac{v}{k(v)} \right|^{\frac{q(x)}{q(x)}} \, dx. \quad (16)$$

Since

$$\|u\|_{q(x)} = k(u) \quad \text{and} \quad \|v\|_{q(x)} = k(v)$$

it follows from Proposition 2.2 (item a)) that

$$\int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)}{q(x)}} \, dx \left| \frac{v}{q(x)} \right| \, dx = 1 = \int_\Omega \left| \frac{v}{k(v)} \right|^{\frac{q(x)}{q(x)}} \, dx \left| \frac{v}{q(x)} \right| \, dx.$$

This implies that (16) can be rewritten as

$$\int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)-1}{q(x)}} |v| \, dx \leq k(v) \int_\Omega \left| \frac{u}{k(u)} \right|^{\frac{q(x)}{q(x)}} \, dx.$$
which, in view of (15), leads to the second inequality in (14).

The first inequality in (14) is obtained by using the same arguments.

Now, let \( c \in \mathbb{R} \) and take a sequence \( (u_n) \subset \mathcal{M} \) such that \( \tilde{K}(u_n) \to c \) and \( \tilde{K}'(u_n) \to 0 \). It follows that \( K(u_n) \to c \) in \( X \) and

\[
K'(u_n) - c_n k'(u_n) \to 0
\]

in \( X^* \), for some sequence \( (c_n) \subset \mathbb{R} \). Since

\[
\langle K'(u_n) - c_n k'(u_n), u_n \rangle = \langle K'(u_n), u_n \rangle - c_n \langle k'(u_n), u_n \rangle = K(u_n) - c_n
\]

and

\[
\|\langle K'(u_n) - c_n k'(u_n), u_n \rangle\|_X \leq \|K'(u_n) - c_n k'(u_n)\|_{X^*} \|u_n\|_X
\]

\[
= \|K'(u_n) - c_n k'(u_n)\|_{X^*} K(u_n) \to 0,
\]

one has \( c_n \to c \).

Taking into account that \( K(u_n) = \|u_n\|_X \) is bounded and that \( X \) is reflexive and compactly embedded into \( L^q(x)(\Omega) \), we can select a subsequence \( (u_{n_j}) \) converging weakly in \( X \) and strongly in \( L^q(x)(\Omega) \) to a function \( u \in X \). The weak convergence guarantees that \( K(u) \leq \liminf K(u_{n_j}) \).

Thus, since \( X \) is uniformly convex, in order to conclude that \( u_{n_j} \) converges to \( u \) strongly, it is enough to verify that

\[
\limsup_{j \to \infty} K(u_{n_j}) \leq K(u).
\]

It follows from (14) that

\[
\left| \langle k'(u_{n_j}), u_{n_j} - u \rangle \right| \leq k(u_{n_j} - u) = \|u_{n_j} - u\|_{q(x)} \to 0.
\]

Combining this fact, (17) and the boundedness of both sequences \( (c_{n_j}) \) and \( (\|u_{n_j} - u\|_X) \) we conclude that \( \langle K'(u_{n_j}), u_{n_j} - u \rangle \to 0 \). Since

\[
\langle K'(u_{n_j}), u_{n_j} - u \rangle = K(u_{n_j}) - \langle K'(u_{n_j}), u \rangle \geq K(u_{n_j}) - K(u)
\]

we have

\[
\limsup_{j \to \infty} K(u_{n_j}) \leq \limsup_{j \to \infty} \langle K'(u_{n_j}), u_{n_j} - u \rangle + K(u) = K(u),
\]

what finishes the proof. \( \blacksquare \)

Since \( \mathcal{M} \) is a closed symmetric submanifold of class \( C^1 \) in \( X \) and \( \tilde{K} \in C^1(\mathcal{M}, \mathbb{R}) \) is even, bounded from below and satisfies the \((PS)_c\) condition, we can define an increasing and unbounded sequence of eigenvalues, by a minimax scheme. For this, we set

\[
\Sigma := \{ A \subset X \setminus \{0\} : A \text{ is compact and } A = -A \}
\]

and

\[
\Sigma_n := \{ A \in \Sigma : A \subset \mathcal{M} \text{ and } \gamma(A) \geq n \}, \quad n = 1, 2, \ldots,
\]
where $\gamma$ is the Krasnoselkii genus.

Let us define
\[
\lambda_n := \inf_{A \in \Sigma_n} \sup_{u \in A} \tilde{K}(u), \quad n \geq 1.
\]

It is known that under the above conditions for $M$ and $\tilde{K}$, we have that $\lambda_n$ is a critical value of $\tilde{K}$ in $M$ (see \[30\], Corollary 4.1). Moreover, since $\Sigma_{k+1} \subset \Sigma_k$, we have $\lambda_{k+1} \geq \lambda_k$, and so
\[
0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \ldots \to \infty.
\]

In particular $\lambda_1 = \inf_{v \in M} K(v) = \Lambda_1$ (this latter equality is consequence of Remark \[34\]).

Let us consider the sets
\[
Q_n = \{u \in M : \tilde{K}'(u) = 0 \text{ and } \tilde{K}(u) = \lambda_n\}
\]
and
\[
A_{p(x),q(x)} = \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue}\}.
\]

If $u \in Q_n$, for a given $n \geq 1$, there exists $\Lambda \in \mathbb{R}$ such that $K'(u) = \Lambda k'(u) = \lambda_n$, so that $(\lambda_n)_{n \geq 1} \subset A_{p(x),q(x)}$. Since $\lambda_n \to \infty$, we conclude the following.

**Proposition 3.8** The set of eigenvalues $A_{p(x),q(x)}$ is non-empty, infinite and $\sup A_{p(x),q(x)} = +\infty$.

**Remark 3.9** When $p$ and $q$ are constants, the equation (11) reduces to
\[
- \text{div} \left( \frac{|\nabla u|^{p-2} \nabla u}{\|\nabla u\|_p^{p-1}} \right) = \Lambda \left( \frac{\sqrt[p]{q}}{\sqrt[q]{p}} \right) \frac{|u|^{q-2} u}{\|u\|_q^{q-1}}
\]
and the first eigenvalue is given by
\[
\Lambda_1 = \frac{\sqrt[q]{q}}{\sqrt[p]{p}} \inf_{W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|\nabla v\|_p}{\|v\|_q}.
\]

4 Extremal functions for $\frac{\|\cdot\|_{L^p(x)}}{\|\cdot\|_{L^\infty}}$

We recall the Morrey inequality, valid for $p > N$ :
\[
\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega),
\]
where $\gamma := 1 - \frac{N}{p}$ and the positive constant $C$ depends only on $N$, $\Omega$ and $p$. An important consequence of this inequality is the compactness of the embedding
\[
W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}).
\]

Combining this fact with Proposition 2.4, we can verify the compactness of the embeddings
\[
W_0^{1,p(x)}(\Omega) \hookrightarrow C(\overline{\Omega}) \quad \text{and} \quad W_0^{1,p(x)}(\Omega) \hookrightarrow L^{jq(x)}(\Omega),
\]
where here, and throughout this section:
\begin{itemize}
  \item $l, j \in \mathbb{N}$, with $l \geq N$;
  \item $p \in C^1(\overline{\Omega})$, with $1 < p^- \leq p^+ < \infty$ (so that $lp(x) \geq lp > N$);
  \item $q \in C(\overline{\Omega})$ with $1 < q^- \leq q^+ < \infty$.
\end{itemize}

The following lemma is proved in [18].

**Lemma 4.1** If $u \in L^\infty(\Omega)$, then

$$
\lim_{j \to \infty} \|u\|_{jq(x)} = \|u\|_\infty.
$$

The previous lemma is also valid if we consider an increasing sequence of functions $(q_j) \subset C(\overline{\Omega})$ such that $q_j \to \infty$ uniformly.

Let us define

$$
\Lambda_{l,j} := \inf \left\{ \frac{\|\nabla v\|_{lp(x)}}{\|v\|_{jq(x)}} : v \in W^{1,lp}_0(\Omega) \setminus \{0\} \right\}
$$

and

$$
\mu_l := \inf \left\{ \frac{\|\nabla v\|_{lp(x)}}{\|v\|_\infty} : v \in W^{1,lp}_0(\Omega) \setminus \{0\} \right\}.
$$

\begin{equation}
(18)
\end{equation}

**Proposition 4.2** One has,

$$
\lim_{j \to \infty} \Lambda_{l,j} = \mu_l.
$$

\begin{equation}
(19)
\end{equation}

**Proof.** It follows from Lemma 4.1 that

$$
\limsup_{j \to \infty} \Lambda_{l,j} \leq \lim_{j \to \infty} \frac{\|\nabla v\|_{lp(x)}}{\|v\|_{jq(x)}} = \frac{\|\nabla v\|_{lp(x)}}{\|v\|_\infty}, \quad \forall v \in W^{1,lp}_0(\Omega) \setminus \{0\}.
$$

Therefore,

$$
\limsup_{j \to \infty} \Lambda_{l,j} \leq \mu_l.
$$

For any $j \geq 1$, let $u_{l,j}$ denote the extremal of $\Lambda_{l,j}$, that is,

$$
\Lambda_{l,j} = \frac{\|\nabla u_{l,j}\|_{lp(x)}}{\|u_{l,j}\|_{jq(x)}}
$$

It follows from (6) that

$$
\|u_{l,j}\|_{jq(x)} \leq |\Omega|^{\alpha_j} \|u_{l,j}\|_\infty,
$$

where

$$
\alpha_j := \begin{cases}
\frac{1}{jq^-} & \text{if } |\Omega| \leq 1, \\
\frac{1}{jq^+} & \text{if } |\Omega| > 1.
\end{cases}
$$

14
Hence,
\[
\frac{\mu_l}{|\Omega|^{1/3}} \leq \frac{||\nabla u_{l,j}\||_{lp(x)}}{|\Omega|^{1/3} ||u_{l,j}\||_{\infty}} \leq \frac{||\nabla u_{l,j}\||_{lp(x)}}{||u_{l,j}\||_{jq(x)}} = \Lambda_{l,j}
\]
and by making \(j \to \infty\) we obtain
\[
\mu_l \leq \liminf \Lambda_{l,j},
\]
concluding thus the proof of (19).

We say that \(u \in W^{1,lp}(\Omega)\) is an extremal function of \(\mu_l\) if
\[
\mu_l = \frac{||\nabla u\||_{lp(x)}}{||u\||_{\infty}}.
\]

**Proposition 4.3** Let \(l \geq N\) be fixed. There exists \(j_m \to \infty\) and a function \(w_l \in W^{1,lp}(\Omega) \cap C(\Omega)\) such that \(u_{l,j_m} \to w_l\) strongly in \(C(\Omega)\) and also in \(W^{1,lp}(\Omega)\). Moreover, \(w_l\) is an extremal function of \(\mu_l\).

**Proof.** Let \(u_{l,j}\) denote the extremal function of \(\Lambda_{l,j}\). Without loss of generality we assume that \(||u_{l,j}\||_{jq(x)} = 1\). Since the sequence \((u_{l,j})_{j \geq 1}\) is uniformly bounded in \(W^{1,lp}(\Omega)\), there exist \(j_m \to \infty\) and \(w_l \in W^{1,lp}(\Omega) \subset C(\Omega)\) such that \(u_{l,j_m} \to w_l\), weakly in \(W^{1,lp}(\Omega)\) and strongly in \(C(\Omega)\). It follows from (20) that
\[
1 = \lim_{m \to \infty} ||u_{l,j_m}\||_{jq(x)} \leq \lim_{m \to \infty} ||u_{l,j_m}\||_{\infty} = ||w_l||_{\infty},
\]
so that
\[
\mu_l \leq \frac{||\nabla w_l||_{lp(x)}}{||w_l||_{\infty}} \leq ||\nabla w_l||_{lp(x)} \leq \lim_{m \to \infty} ||\nabla u_{l,j_m}||_{lp(x)} = \lim_{m \to \infty} \Lambda_{l,j_m} = \mu_l.
\]
Hence,
\[
||w_l||_{\infty} = 1, \quad \mu_l = ||\nabla w_l||_{lp(x)} = \lim_{m \to \infty} ||\nabla u_{l,j_m}||_{lp(x)},
\]
implying that \(w_l\) is an extremal function of \(\mu_l\) and that \(u_{l,j_m} \to w_l\) strongly in \(W^{1,lp}(\Omega)\). ■

Now, by adapting arguments of [19] we characterize of the extremal functions of \(\mu_l\). For this, let us denote by \(\Gamma_u\) the set of the points where a function \(u \in C(\Omega)\) assumes its uniform norm, that is
\[
\Gamma_u := \{x \in \overline{\Omega} : |u(x)| = ||u||_{\infty}\}.
\]

**Lemma 4.4** Let \(u, \eta \in C(\Omega), \text{ with } u \neq 0\). One has
\[
\lim_{\epsilon \to 0^+} \frac{||u + \epsilon \eta||_{\infty} - ||u||_{\infty}}{\epsilon} = \max\{\text{sgn}(u(x))|\eta(x)| : x \in \Gamma_u\}.
\]
Proof. Let \( r > 1, \delta > 0 \) and \( t \in \mathbb{R} \). Since the function \( s \mapsto |s|^{r-2} s \) is increasing we have

\[
\frac{|t + \delta|^{r}}{r} = \int_{0}^{t + \delta} |s|^{r-2} s \, ds = \frac{|t|^{r}}{r} + \int_{t}^{t + \delta} |s|^{r-2} s \, ds \geq \frac{|t|^{r}}{r} + |t|^{r-2} t \delta.
\]

Thus, for \( x_{0} \in \Gamma_{u}, \eta \in C(\overline{\Omega}) \) and \( \epsilon > 0 \), we obtain

\[
\frac{\|u + \epsilon \eta\|_{x_{0}}^{r}}{\epsilon} \geq \frac{|u(x_{0}) + \epsilon \eta(x_{0})|^{r}}{\epsilon} \geq \frac{|u(x_{0})|^{r}}{\epsilon} + |u(x_{0})|^{r-2} u(x_{0}) \epsilon \eta(x_{0}).
\]

Making \( r \to 1^{+} \) (and using that \( |u(x_{0})| = \|u\|_{\infty} \neq 0 \)) we arrive at the inequality

\[
\frac{\|u + \epsilon \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon} \geq \text{sgn}(u(x_{0})) \eta(x_{0}),
\]

which, in view of the arbitrariness of \( x_{0} \in \Gamma_{u} \), implies that

\[
\liminf_{\epsilon \to 0^{+}} \frac{\|u + \epsilon \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon} \geq \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_{u}\}.
\]

In order to conclude this proof we will obtain the reverse inequality for \( \limsup_{\epsilon \to 0^{+}} \). For this, we take \( \epsilon_{m} \to 0^{+} \) such that

\[
\limsup_{\epsilon \to 0^{+}} \frac{\|u + \epsilon \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon_{m}} = \limsup_{m \to \infty} \frac{\|u + \epsilon_{m} \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon_{m}}
\]

and select a sequence \((x_{m}) \subset \overline{\Omega}\) satisfying

\[
|u(x_{m}) + \epsilon_{m} \eta(x_{m})| = \|u + \epsilon_{m} \eta\|_{\infty}.
\]

We can assume (by passing to a subsequence, if necessary) that \( x_{m} \to x_{0} \in \overline{\Omega} \). Of course, \( x_{0} \in \Gamma_{u} \) since \( u + \epsilon_{m} \eta \to u \) in \( C(\overline{\Omega}) \).

Since \( \|u\|_{\infty} \geq |u(x_{m})| \) we have

\[
\limsup_{m \to \infty} \frac{\|u + \epsilon_{m} \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon_{m}} \leq \limsup_{m \to \infty} \frac{|u(x_{m}) + \epsilon_{m} \eta(x_{m})| - |u(x_{m})|}{\epsilon_{m}}
\]

and since \( u(x_{m}) + \epsilon_{m} \eta(x_{m}) \to u(x_{0}) \) we have, for all \( m \) large enough,

\[
\frac{|u(x_{m}) + \epsilon_{m} \eta(x_{m})| - |u(x_{m})|}{\epsilon_{m}} = \begin{cases} 
\eta(x_{m}), & u(x_{0}) > 0 \\
-\eta(x_{m}), & u(x_{0}) < 0 
\end{cases} = \text{sgn}(u(x_{0})) \eta(x_{m}).
\]

It follows that

\[
\limsup_{\epsilon \to 0^{+}} \frac{\|u + \epsilon \eta\|_{\infty} - \|u\|_{\infty}}{\epsilon} \leq \text{sgn}(u(x_{0})) \eta(x_{0}) \leq \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_{u}\}.
\]
Theorem 4.5 Let \( l \geq N \) be fixed. A function \( u \in W^{1,l,p(x)}_0(\Omega) \setminus \{0\} \) is extremal of \( \mu_l \) if, and only if, \( \Gamma_u = \{x_0\} \) for some \( x_0 \in \Omega \) and

\[
\int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx = \mu_l S_l(u) \, \text{sgn}(u(x_0)) \eta(x_0), \quad \forall \eta \in W^{1,l,p(x)}_0(\Omega),
\]

where

\[
K_1(u) := \|\nabla u\|_{l,p(x)} \quad \text{and} \quad S_l(u) := \int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)} \, dx.
\]

Proof. Let \( u \in W^{1,l,p(x)}_0(\Omega) \setminus \{0\} \) be an extremal function of \( \mu_l \) and fix \( \eta \in W^{1,l,p(x)}_0(\Omega) \). Then

\[
\mu_l = \frac{\|\nabla u\|_{l,p(x)}}{\|u\|_\infty} \leq \frac{\|\nabla u + \epsilon \nabla \eta\|_{l,p(x)}}{\|u + \epsilon \eta\|_\infty}, \quad \forall \epsilon > 0.
\]

It follows that

\[
0 \leq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \frac{\|\nabla u + \epsilon \nabla \eta\|_{l,p(x)}}{\|u + \epsilon \eta\|_\infty} - \frac{\|\nabla u\|_{l,p(x)}}{\|u\|_\infty} \right)
\]

\[
= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \frac{\|\nabla u + \epsilon \nabla \eta\|_{l,p(x)}}{\|u + \epsilon \eta\|_\infty} - \frac{\|\nabla u\|_{l,p(x)}}{\|u + \epsilon \eta\|_\infty} + \frac{\|\nabla u\|_{l,p(x)}}{\|u + \epsilon \eta\|_\infty} - \frac{\|\nabla u\|_{l,p(x)}}{\|u\|_\infty} \right)
\]

\[
= \frac{1}{\|u\|_\infty} \left( \frac{1}{S_l(u)} \int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx - \frac{\|\nabla u\|_{l,p(x)}}{\|u\|_\infty} \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\} \right)
\]

\[
= \frac{1}{\|u\|_\infty} \left( \frac{1}{S_l(u)} \int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx - \mu_l \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\} \right),
\]

where we have used Lemma 3.2 and Lemma 4.4.

Therefore,

\[
\int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx \geq \mu_l S_l(u) \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\}. \tag{22}
\]

Now, by replacing \( \eta \) by \(-\eta\) in this inequality we obtain

\[
\int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx \leq \mu_l S_l(u) \min\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\}. \tag{23}
\]

We then conclude from (22) and (23) that

\[
\mu_l S_l(u) \min\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\} = \int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{l,p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx
\]

\[
= \mu_l S_l(u) \max\{\text{sgn}(u(x)) \eta(x) : x \in \Gamma_u\}.
\]
Taking into account the arbitrariness of \( \eta \in W^{1,p(x)}_0(\Omega) \) this implies that \( \Gamma_u = \{ x_0 \} \) for some \( x_0 \in \Omega \). Consequently, \( u \) satisfies (21) for \( x_0 \).

Reciprocally, if \( u \in W^{1,p(x)}_0(\Omega) \setminus \{ 0 \} \) is such that \( \Gamma_u = \{ x_0 \} \) for some \( x_0 \in \Omega \) and, additionally, satisfies (21) for this point, we can choose \( \eta = u \) in (21) to get

\[
\int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx = \mu_l S_l(u) \text{sgn}(u(x_0))u(x_0) = \mu_l S_l(u) \| u \|_\infty,
\]

so that

\[
\mu_l = \frac{\| \nabla u \|_{L^{p(x)}}}{\| u \|_\infty}.
\]

**Corollary 4.6** Extremal functions of \( \mu_l \) do not change sign in \( \Omega \).

**Proof.** Let \( u \in W^{1,p(x)}_0(\Omega) \setminus \{ 0 \} \) be an extremal function of \( \mu_l \) and \( x_0 \in \Omega \) the only point where \( u \) achieves its uniform norm. If \( u(x_0) > 0 \), Theorem 4.5 yields

\[
\int_\Omega \left| \frac{\nabla u}{K_1(u)} \right|^{p(x)-2} \frac{\nabla u}{K_1(u)} \cdot \nabla \eta \, dx = \mu_l S_l(u)\eta(x_0) \geq 0,
\]

for all nonnegative \( \eta \in W^{1,p(x)}_0(\Omega) \). Proposition 2.8 then implies that \( u > 0 \) in \( \Omega \). If \( u(x_0) < 0 \) we repeat the argument for the extremal function \( -u \). \( \square \)

We can say that

\[
- \text{div} \left( \left| \frac{\nabla u}{K_1(u)} \right|^{p(x)-2} \frac{\nabla u}{K_1(u)} \right) = \mu_l S_l(u) \text{sgn}(u(x_0))\delta_{x_0},
\]

is the Euler-Lagrange equation associated with the minimization problem defined by (18), where \( \delta_{x_0} \) is the Dirac delta function concentrated in \( x_0 \). We recall that \( \delta_{x_0} \) is defined by

\[
\delta_{x_0}(\eta) = \eta(x_0), \quad \forall \eta \in W^{1,p(x)}_0(\Omega).
\]

Thus, the extremal functions of \( \mu_l \) are precisely the weak solutions of (24) in the sense of (21).

**Remark 4.7** Consider a function \( v \in W^{1,p(x)}_0(\Omega) \) such that \( |v(x_0)| = \| v \|_\infty \) for some \( x_0 \in \Omega \) and suppose that this function satisfies the equation

\[
\int_\Omega \left| \frac{\nabla v}{K_1(v)} \right|^{p(x)-2} \frac{\nabla v}{K_1(v)} \cdot \nabla \eta \, dx = \mu \left( \int_\Omega \left| \frac{\nabla v}{K_1(v)} \right|^{p(x)} \, dx \right) \text{sgn}(v(x_0))\eta(x_0), \quad \forall \eta \in W^{1,p(x)}_0(\Omega)
\]

where \( \mu \in \mathbb{R} \). By making \( \eta = v \), it follows that

\[
\mu = \frac{K_1(v)}{|v(x_0)|} = \frac{\| \nabla v \|_{L^{p(x)}}}{\| v \|_\infty} \geq \mu_l.
\]
Thus, \( \mu_l \) can be interpreted as the first eigenvalue of (21). Moreover, for a given natural \( j \geq 1 \), we know, from Section 3, that there exists a sequence

\[
0 < \lambda_1^{l,j} \leq \lambda_2^{l,j} \leq \cdots \leq \lambda_n^{l,j} \leq \lambda_{n+1}^{l,j} \leq \cdots
\]

of eigenvalues, where the exponent functions, in this case, are \( lp(x) \) and \( jq(x) \). Proposition 4.2 then says that

\[
\lim_{j \to \infty} \lambda_1^{l,j} = \mu_l.
\]

5 The limit problem as \( l \to \infty \)

In this section we maintain \( p \in C^1(\overline{\Omega}) \), with \( 1 < p^- \leq p^+ < \infty \). For each natural \( l \geq N \) we denote by \( w_l \) a positive, \( L^\infty \)-normalized extremal function of \( \mu_l \). Thus,

\[
w_l \in W^{1,lp}_0(\Omega), \quad \|w_l\|_\infty = 1, \quad w_l > 0 \quad \text{in } \Omega,
\]

and

\[
\mu_l = \|\nabla w_l\|_{lp(\Omega)} \leq \frac{\|\nabla v\|_{lp(\Omega)}}{\|v\|_\infty}, \quad \forall v \in W^{1,lp}_0(\Omega) \setminus \{0\}.
\]

We will also denote by \( x_0^l \) the only maximum point of \( w_l \). According to the previous section, \( w_l \) satisfies

\[
-\Delta_{lp}(x) \left( \frac{w_l}{K_l(w_l)} \right) = \mu_l S_l(w_l) \delta_{x_0^l} \quad \text{in } \Omega,
\]

where \( \delta_{x_0^l} \) is the Dirac delta function concentrated in \( x_0^l \),

\[
K_l(w_l) = \|\nabla w_l\|_{lp(\Omega)} = \mu_l \quad \text{and} \quad S_l(w_l) := \int_{\Omega} \frac{\nabla w_l}{K_l(w_l)} \big|_{lp(\Omega)} d\Omega.
\]

Hence,

\[
\int_{\Omega} \left| \frac{\nabla w_l}{K_l(w_l)} \right|_{lp(\Omega)}^{lp(\Omega)-2} \frac{\nabla w_l}{K_l(w_l)} \cdot \nabla \eta \, d\Omega = \mu_l S_l(w_l) \eta(x_0^l), \quad \forall \eta \in W^{1,lp}_0(\Omega).
\]

Let us define

\[
\Lambda_\infty := \inf \left\{ \frac{\|\nabla v\|_{\infty}}{\|v\|_{\infty}} : v \in W^{1,\infty}_0(\Omega) \setminus \{0\} \right\}.
\]

It is a well-known fact that

\[
\Lambda_\infty = \frac{\|\nabla d\|_{\infty}}{\|d\|_{\infty}} = \frac{1}{\|d\|_{\infty}},
\]

where \( d \) denotes the distance function to the boundary \( \partial \Omega \), defined by

\[
d(x) := \inf_{y \in \partial \Omega} |x - y|, \quad x \in \overline{\Omega}.
\]

We recall that

\[
d \in W^{1,\infty}_0(\Omega) \quad \text{and} \quad |\nabla d| = 1 \text{ a.e. in } \Omega.
\]
Lemma 5.1 Let $\alpha := \int_{\Omega} \frac{dx}{p(x)}$ and $e := \exp(1)$. If $\alpha e < m < l$, then
\[
\|u\|_{mp(x)} \leq \|u\|_{lp(x)} \quad \forall u \in L^{lp(x)}(\Omega).
\]

Proof. When $u \equiv 0$ the equality holds trivially in the above inequality. Thus, we fix $u \in L^{lp(x)}(\Omega) \setminus \{0\}$ and denote the modular functions associated to $L^{mp(x)}(\Omega)$ and $L^{lp(x)}(\Omega)$ by $\rho_m$ and $\rho_l$, respectively.

By Hölder’s inequality
\[
\rho_m(u) = \int_\Omega |u|^{mp(x)} \frac{dx}{mp(x)} = \frac{1}{m} \int_\Omega \left( \frac{|u|^{lp(x)}}{p(x)} \right)^{\frac{m}{l}} \frac{dx}{p(x)} \leq \frac{1}{m} \left( \int_\Omega \left( \frac{|u|^{lp(x)}}{p(x)} \right)^{\frac{m}{l}} \right)^{\frac{1}{\frac{m}{l}}} \frac{\int_\Omega dx}{p(x)}^{\frac{1}{\frac{m}{l}}} = \frac{1}{m} \left( \int_\Omega p(x) dx \right)^{\frac{m}{l}} \alpha^{\frac{m}{l}} = \left( \frac{f(l)}{f(m)} \rho_l(u) \right)^{\frac{m}{l}},
\]
where $f(s) = \left( \frac{s}{\alpha} \right)^{\frac{1}{m}}$. Since $f$ is decreasing in $(\alpha e, \infty)$ and $\alpha e < m < l$ we have
\[
\rho_m(u)^{\frac{1}{l}} \leq \frac{f(l)}{f(m)} \rho_l(u)^{\frac{1}{l}} \leq \rho_l(u)^{\frac{1}{l}}.
\]

Hence, by taking $a = \|u\|_{lp(x)} \neq 0$ and applying item b) of Proposition 2.2 we conclude that
\[
\rho_m \left( \frac{u}{a} \right)^{\frac{1}{m}} \leq \rho_l \left( \frac{u}{a} \right)^{\frac{1}{l}} = 1
\]
and then that $\|u\|_{mp(x)} \leq 1$. This implies that
\[
\|u\|_{mp(x)} \leq a = \|u\|_{lp(x)}.
\]

Proposition 5.2 There exists a subsequence of $(w_i)_{i \in \mathbb{N}}$ converging strongly in $C(\overline{\Omega})$ to a non-negative function $w_{\infty} \in W_0^{1, \infty}(\Omega) \setminus \{0\}$ such that
\[
\lim_{l \to \infty} \mu_l = \Lambda_{\infty} = \|\nabla w_{\infty}\|_{\infty}. \tag{25}
\]
Moreover,
\[
0 \leq w_{\infty}(x) \leq \frac{d(x)}{\|d\|_{\infty}}, \quad \text{for almost every } x \in \Omega. \tag{26}
\]

Proof. Since
\[
\|\nabla w_i\|_{lp(x)} = \mu_i \leq \frac{\|\nabla d\|_{lp(x)}}{\|d\|_{\infty}}, \tag{27}
\]
we can apply Lemma 4.1 to get
\[
\limsup_{l \to \infty} \mu_l \leq \frac{\|\nabla d\|_\infty}{\|d\|_\infty} = \Lambda_\infty. \tag{28}
\]

Let us take a natural \( m > \alpha e \), where \( \alpha \) is given by Lemma 5.1, and a subsequence \( (\mu_{ln})_{n \in \mathbb{N}} \) such that
\[
\lim_{n \to \infty} \mu_{ln} = \liminf_{l \to \infty} \mu_l.
\]
Combining Lemma 5.1 with (27) we conclude that the sequence \( (w_{ln})_{ln > m} \) is bounded in \( W_0^{1,mp(x)}(\Omega) \), since
\[
\limsup_{n \to \infty} \|\nabla w_{ln}\|_{mp(x)} \leq \limsup_{n \to \infty} \|\nabla w_{ln}\|_{lnp(x)} \leq \Lambda_\infty.
\]
Thus, up to a subsequence, we can assume that there exists \( w_\infty \in W_0^{1,mp(x)}(\Omega) \), such that \( w_{ln} \) converges to \( w_\infty \), weakly in \( W_0^{1,mp(x)}(\Omega) \) and uniformly in \( \Omega \).

The uniform convergence, implies that \( \|w_\infty\|_\infty = 1 \) (since \( \|w_{ln}\|_\infty = 1 \)). The weak convergence in \( W_0^{1,mp(x)}(\Omega) \) implies that
\[
\|\nabla w_\infty\|_{mp(x)} \leq \liminf_{n \to \infty} \|\nabla w_{ln}\|_{mp(x)}. \tag{29}
\]
Now, applying Lemma 5.1 again, we conclude that
\[
\liminf_{n \to \infty} \|\nabla w_{ln}\|_{mp(x)} \leq \liminf_{n \to \infty} \|\nabla w_{ln}\|_{lnp(x)} = \lim_{n \to \infty} \mu_{ln} = \liminf_{l \to \infty} \mu_l.
\]
Hence, (29) yields
\[
\|\nabla w_\infty\|_{mp(x)} \leq \liminf_{l \to \infty} \mu_l. \tag{30}
\]
Repeating the above arguments we conclude that \( w_\infty \) is the weak limit of a subsequence of \( (w_{ln})_{n \in \mathbb{N}} \) in \( W_0^{1,sp(x)}(\Omega) \), for any \( s > m \). This fact implies that \( w_\infty \in W_0^{1,\infty}(\Omega) \). Then, by making \( m \to \infty \) in (30), using Lemma 4.1 and (28) we conclude that
\[
\Lambda_\infty \leq \|\nabla w_\infty\|_\infty \leq \liminf_{l \to \infty} \mu_l \leq \limsup_{l \to \infty} \mu_l \leq \Lambda_\infty,
\]
which gives (25).

Since the Lipschitz constant of \( w_\infty \) is \( \|\nabla w_\infty\|_\infty = \Lambda_\infty = \frac{1}{\|d\|_\infty} \), we have
\[
\|d\|_\infty |w_\infty(x) - w_\infty(y)| \leq |x - y|
\]
for almost all \( x \in \Omega \) and \( y \in \partial \Omega \). Since \( w \equiv 0 \) on the boundary \( \partial \Omega \), we obtain
\[
\|d\|_\infty w_\infty(x) \leq \inf_{y \in \partial \Omega} |x - y| = d(x).
\]

We show in the sequel that the functions \( w_\infty \) and \( d \) have a maximum point in common, which is obtained as a cluster point of the sequence \( (x_{l0})_{l \in \mathbb{N}} \).
Corollary 5.3  There exists $x_\star \in \Omega$ such that  

$$w_\infty(x_\star) = \|w_\infty\|_\infty = 1 \quad \text{and} \quad d(x_\star) = \|d\|_\infty.$$  

Proof.  Let $(w_{l_n})_{n \in \mathbb{N}}$ be a sequence converging uniformly to $w_\infty$, which is given by Proposition 5.2. Up to a subsequence, we can assume that $x_0^l \to x_\star \in \Omega$. Since $w_{l_n}(x_0^l) = 1 = \|w_{l_n}\|_\infty$ we have $w_\infty(x_\star) = 1 = \|w_\infty\|_\infty$, showing that $x_\star \in \Omega$. The conclusion stems from (26), since

$$1 = w_\infty(x_\star) \leq \frac{d(x_\star)}{\|d\|_\infty} \leq 1.$$  

In the sequel we recall the concept of viscosity solutions for an equation of the form

$$H(x, u, \nabla u, D^2 u) = 0 \quad \text{in } D$$

where $H$ is a partial differential operator of second order and $D$ denotes a bounded domain of $\mathbb{R}^N$.

Definition 5.4  Let $\phi \in C^2(D)$, $x_0 \in D$ and $u \in C(D)$. We say that $\phi$ touches $u$ from below at $x_0$ if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) < u(x), \quad x \neq x_0$$

Analogously, we say that $\phi$ touches $u$ from above at $x_0$ if

$$\phi(x_0) = u(x_0) \quad \text{and} \quad \phi(x) > u(x), \quad x \neq x_0.$$  

Definition 5.5  We say that $u \in C(D)$ is a viscosity supersolution of the equation (31) if, whenever $\phi \in C^2(D)$ touches $u$ from below at a point $x_0 \in D$, we have

$$H(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) \leq 0.$$  

Analogously, we say that $u$ is a viscosity subsolution if, whenever $\psi \in C^2(D)$ touches $u$ from above at a point $x_0 \in D$, we have

$$H(x_0, \psi(x_0), \nabla \psi(x_0), D^2 \psi(x_0)) \geq 0.$$  

And we say that $u$ is a viscosity solution, if $u$ is both a viscosity supersolution and a viscosity subsolution.

Note that the differential operator $H$ is evaluated for the test functions only at the touching point.

In order to interpret the equation

$$\Delta_p(x) \left( \frac{u}{K(u)} \right) = 0 \quad \text{in } \Omega$$  

(32)
in the viscosity sense, we need to find the expression of the corresponding differential operator $H$. If $\phi$ is a function of class $C^2$, one can verify that the $p(x)$-Laplacian is given by
\[
\Delta_{p(x)} \phi = |\nabla \phi|^{p(x)-4} \left\{ |\nabla \phi|^2 \Delta \phi + (p(x) - 2) \Delta_{\infty} \phi + |\nabla \phi|^2 \ln |\nabla \phi| \langle \nabla \phi, \nabla p \rangle \right\}
\]
where $\langle \nabla \phi, \nabla p \rangle = \nabla \phi \cdot \nabla p$ and $\Delta_{\infty}$ denotes the $\infty$-Laplacian defined by
\[
\Delta_{\infty} v := \frac{1}{2} \langle \nabla v, \nabla |\nabla v|^2 \rangle = \sum_{i,j=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}.
\]

Thus, for a positive constant $t$ one can check that
\[
\Delta_{p(x)} (t \phi) = t^{p(x)-1} |\nabla \phi|^{p(x)-4} \left\{ |\nabla \phi|^2 \Delta \phi + (p(x) - 2) \Delta_{\infty} \phi + |\nabla \phi|^2 \ln(|\nabla (t \phi)|) \langle \nabla \phi, \nabla p \rangle \right\}
\]
and by choosing $t = K(u)^{-1}$ we obtain from (33) that the equation (32) can be rewritten in the form (31) with
\[
H(x, u, \nabla u, D^2 u) := |\nabla u|^{p(x)-4} \left\{ |\nabla u|^2 \Delta \phi + (p(x) - 2) \Delta_{\infty} u + |\nabla u|^2 \ln \left( \frac{|\nabla u|}{K(u)} \right) \langle \nabla u, \nabla p \rangle \right\},
\]
where we are assuming that $p(x) \geq 2$.

**Proposition 5.6** If $u \in C(\Omega) \cap W^{1,p(x)}_0(\Omega)$ is a weak solution of (32) with $K(u) = \|\nabla u\|_{p(x)}$, then $u$ is a viscosity solution of this equation.

**Proof.** We must prove that $u$ is both a viscosity supersolution and a viscosity subsolution of the equation
\[
H(x, u, \nabla u, D^2 u) = 0 \quad \text{in } \Omega
\]
for the differential operator $H$ defined by (31) with $K(u) = \|\nabla u\|_{p(x)}$.

By hypothesis, $u$ satisfies
\[
\int_{D} \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \cdot \nabla \eta \, dx = 0, \quad \forall \eta \in W^{1,p(x)}_0(\Omega).
\]

Let us prove by contradiction that $u$ is a viscosity supersolution. Thus, we suppose that there exist $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ with $\phi$ touching $u$ from below at $x_0$ and satisfying
\[
H(x_0, \phi(x_0), \nabla \phi(x_0), D^2 \phi(x_0)) > 0.
\]
By continuity, there exists a ball $B(x_0, r) \subset \Omega$, with $r$ small enough, such that
\[
H(x, \phi(x), \nabla \phi(x), D^2 \phi(x)) > 0, \quad \forall x \in B(x_0, r).
\]
This means that
\[
\Delta_{p(x)} \left( \frac{\phi}{K(u)} \right) > 0 \quad \text{in } B(x_0, r).
\]
Let \( \varphi := \phi + \frac{m}{2} \), with \( m = \min_{\partial B(x_0, r)} (u - \phi) > 0 \), and take
\[
\eta = (\varphi - u) + \chi_{B(x_0, r)} \in W^{1,p(x)}_0(\Omega).
\]
We have \( \varphi < u \) on \( \partial B(x_0, r) \) and \( \varphi(x_0) > u(x_0) \).

If \( \eta \equiv 0 \), we multiply (36) by \( \eta \) and integrate by parts to obtain
\[
\int_D \left| \frac{\nabla \phi}{K(u)} \right|^{p(x)-2} \frac{\nabla \phi}{K(u)} \cdot \nabla \eta \, dx < 0.
\]
Note that \( \nabla \eta = \nabla \varphi - \nabla u = \nabla \phi - \nabla u \) in the set \( \{ \varphi > u \} \) (and \( \nabla \eta = 0 \) in \( B(x_0, r) \cap \{ \varphi \leq u \} \)). Thus, subtracting (36) from the above inequality we obtain
\[
\int_{\{ \varphi > u \}} \left( \left| \frac{\nabla \phi}{K(u)} \right|^{p(x)-2} \frac{\nabla \phi}{K(u)} - \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \right) \cdot \nabla \eta \, dx < 0
\]
implying thus that
\[
\int_{\{ \varphi > u \}} \left( \left| \frac{\nabla \phi}{K(u)} \right|^{p(x)-2} \frac{\nabla \phi}{K(u)} - \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \right) \, dx < 0 \tag{37}
\]
where the domain of integration is contained in \( B(x_0, r) \). However, it is well known that
\[
\langle |Y|^{p-2} Y - |X|^{p-2} X, Y - X \rangle \geq 0 \quad \forall X, Y \in \mathbb{R}^N \text{ and } p > 1.
\]
Thus, by making \( Y = \frac{\nabla \phi}{K(u)} \), \( X = \frac{\nabla u}{K(u)} \) and \( p = p(x) \) we see that (37) cannot occurs. Therefore, \( \eta \equiv 0 \). But this implies that \( \varphi \leq u \) in \( B(x_0, r) \), contradicting \( \varphi(x_0) > u(x_0) \).

Analogously, we can show that \( u \) is a viscosity subsolution.

Our next result states that the function \( w_\infty \) given by Proposition 5.2 is a viscosity solution of the equation
\[
\Delta_{\infty(x)} \left( \frac{u}{\|\nabla u\|_\infty} \right) = 0
\]
in the punctured domain \( D = \Omega \setminus \{x_*\} \), where \( x_* \) is the maximum point of \( w_\infty \) given by Corollary 5.3 and \( \Delta_{\infty(x)} \) is the differential operator
\[
\Delta_{\infty(x)} u := \Delta_{\infty} u + |\nabla u|^2 \ln |\nabla u| \langle \nabla u, \nabla \ln p \rangle.
\]
Note that if \( t > 0 \), then
\[
\Delta_{\infty(x)} \left( \frac{u}{t} \right) := t^{-3} \left\{ \Delta_{\infty} u + |\nabla u|^2 \ln |\nabla (\frac{u}{t})| \langle \nabla u, \nabla \ln p \rangle \right\} \tag{38}
\]

Lemma 5.7 (22) Suppose that \( f_n \to f \) uniformly in \( \overline{D} \), where \( f_n, f \in C(\overline{D}) \). If \( \phi \in C^2(D) \) touches \( f \) from below at \( x_0 \in D \), then there exists \( x_j \to x_0 \) such that
\[
f_{n_j}(x_j) - \phi(x_j) = \min_{\overline{D}} \{f_{n_j} - \phi\}.
\]
Theorem 5.8 The function \( w_\infty \) is a viscosity solution of

\[
\begin{cases}
\Delta_{\infty(x)} \left( \frac{u}{\| \nabla u \|_\infty} \right) = 0 & \text{in } D = \Omega \setminus \{ x_* \} \\
\frac{u}{\| \nabla u \|_\infty} = d & \text{on } \partial D = \partial \Omega \cup \{ x_* \}.
\end{cases}
\]  

(39)

Proof. Taking into account that \( \| \nabla w_\infty \|_\infty = \Lambda_\infty \) we just need to show that \( w_\infty \) satisfies

\[
\begin{cases}
\Delta_{\infty(x)} \left( \frac{u}{\Lambda_\infty} \right) = 0 & \text{in } D = \Omega \setminus \{ x_* \} \\
\frac{u}{\Lambda_\infty} = d & \text{on } \partial D = \partial \Omega \cup \{ x_* \}
\end{cases}
\]

in the viscosity sense.

Since \( \Lambda_\infty = \frac{1}{\| d \|_\infty} \), it follows from Corollary 5.3 that

\[
\frac{w_\infty(x_*)}{\Lambda_\infty} = w_\infty(x_*) \| d \|_\infty = d(x_*).
\]

Thus, taking into account that \( w_\infty|_{\partial \Omega} = 0 = d|_{\partial \Omega} \), we conclude that \( \frac{w_\infty}{\Lambda_\infty} = d \) on \( \partial \Omega \cup \{ x_* \} \).

In order to show that \( w_\infty \) is a viscosity supersolution, let \( x_0 \in \Omega \setminus \{ x_* \} \) and \( \phi \in C^2(\Omega \setminus \{ x_* \}) \) be such that \( \phi \) touches \( w_\infty \) from below at \( x_0 \), i.e.

\[
\phi(x_0) = w_\infty(x_0) \quad \text{and} \quad \phi(x) < w_\infty(x), \quad \text{for } x \neq x_0.
\]

We claim that

\[
\Delta_{\infty(x)} \left( \frac{\phi(x_0)}{\Lambda_\infty} \right) \leq 0,
\]

where the expression of the differential operator is given by (38). Since the above inequality holds trivially when \( \nabla \phi(x_0) = 0 \), we assume that \( |\nabla \phi(x_0)| \neq 0 \). So, let us take a ball \( B_\epsilon(x_0) \subset \Omega \setminus \{ x_* \} \) such that

\[
|\nabla \phi(x)| \neq 0 \quad \text{in } B_\epsilon(x_0).
\]

Proposition 5.2 and its Corollary 5.3 guarantee the existence of a subsequence of indexes \( (l_n)_{n \in \mathbb{N}} \) such that \( w_{l_n} \to w_\infty \) in \( C(\overline{\Omega}) \) and \( x_0 \to x_* \), where \( x_0 \) denotes a maximum point of \( w_{l_n} \). It follows that \( w_{l_n} \to w_\infty \) uniformly in \( B_\epsilon(x_0) \) and \( x_0 \notin B_\epsilon(x_0) \) for all \( n \) large enough.

Applying Lemma 5.7 to \( B_\epsilon(x_0) \) we can assume that (up to pass to another subsequence) there exists \( y_n \to x_0 \) such that

\[
m_n := \min_{B_\epsilon(x_0)} \{ w_{l_n} - \phi \} = w_{l_n}(y_n) - \phi(y_n).
\]

Thus, the function \( \phi_n(x) := \phi(x) + m_n - |x - y_n|^4 \), which belongs to \( C^2(B_\epsilon(x_0)) \), satisfies

\[
\phi_n(y_n) = \phi(y_n) + m_n = w_{l_n}(y_n) \quad \text{and} \quad \phi_n(x) \leq w_{l_n}(x) - |x - y_n|^4 < w_{l_n}(x)
\]

25
for all $x \in B_r(x_0) \setminus \{y_n\}$. That is, $\phi_n$ touches $w_n$ from below at $y_n$.

Let $H_t$ denote the differential operator associated with the equation $\Delta_{tp(x)}(u/\mu_t) = 0$, that is,

$$H_t(x, u, \nabla u, D^2 u) := |\nabla u|^{l_{p(x)}} - 1 \left\{ |\nabla u|^2 \Delta \phi + (l_p(x) - 2) \Delta_\infty u + |\nabla u|^2 \ln \left( \frac{|\nabla u|}{\mu_t} \right) \right\} \langle \nabla u, l \nabla p \rangle.$$

(Recall that $\mu_t = ||\nabla w_l||_{l_{p(x)}} \to \Lambda_\infty$ as $l \to \infty$)

It follows from Proposition 5.6 that $w_l$ is a viscosity (super)solution of the equation

$$H_l(x, u, \nabla u, D^2 u) = 0.$$ 

Hence, taking $\phi_n$ as a test function for $w_n$, it follows that

$$H_{l_n}(y_n, \phi_n(y_n), \nabla \phi_n(y_n), D^2 \phi_n(y_n)) \leq 0.$$ 

This means that

$$\nabla \phi_n(y_n) \neq \nabla \phi(y_n), \quad \Delta \phi_n(y_n) = \Delta \phi(y_n) \quad \text{and} \quad \Delta_\infty \phi_n(y_n) = \Delta_\infty \phi(y_n)$$

and $\nabla \phi(y_n) \neq 0$ for $n$ large enough, we have

$$\nabla \phi(y_n) \big|_{l_{n,p(x)} - 1} \left\{ |\nabla \phi(y_n)|^2 \Delta \phi(y_n) + (l_n p(y_n) - 2) \Delta_\infty \phi(y_n) \right. \left. + |\nabla \phi(y_n)|^2 \ln \left( \frac{|\nabla \phi(y_n)|}{\mu_n} \right) \right\} \langle \nabla \phi(y_n), l_n \nabla p(y_n) \rangle \leq 0.$$ 

Dividing this inequality by $(l_n p(y_n) - 2) |\nabla \phi(y_n)|_{l_{n,p(x)} - 1}$, we obtain

$$\frac{|\nabla \phi(y_n)|^2 \Delta \phi(y_n)}{l_n p(y_n) - 2} + \Delta_\infty \phi(y_n) + |\nabla \phi(y_n)|^2 \ln \left( \frac{|\nabla \phi(y_n)|}{\mu_n} \right) \left( \frac{\nabla \phi(y_n)}{p(y_n) - 2/l_n} \right) \leq 0.$$ 

Then, by making $n \to \infty$ we arrive at

$$\Delta_\infty \phi(x_0) + |\nabla \phi(x_0)|^2 \ln \left( \frac{|\nabla \phi(x_0)|}{\Lambda_\infty} \right) \left( \frac{\nabla \phi(x_0)}{p(x_0)} \right) \leq 0.$$ 

According to (38) this implies that

$$\Delta_\infty \left( \frac{\phi(x_0)}{\Lambda_\infty} \right) \leq 0$$

and we conclude thus the proof that $w_\infty$ is a viscosity supersolution.

Analogously, we can show that if $\psi \in C^2(\Omega \setminus \{x_\ast\})$ touches $w_\infty$ from above at the point $x_0$, then

$$\Delta_\infty \left( \frac{\psi(x_0)}{\Lambda_\infty} \right) \geq 0.$$ 

Therefore, $w_\infty$ satisfies (39) in the viscosity sense.

The following uniqueness result can be found in [21].
Proposition 5.9 (21, Theorem 1.2) Let $D$ be a bounded domain of $\mathbb{R}^N$ and $f : \partial D \rightarrow \mathbb{R}$ be a Lipschitz continuous function. There exists a unique viscosity solution $u \in C(\overline{D}) \cap W^{1,\infty}(D)$ for the Dirichlet boundary value problem

$$\begin{cases} \Delta_{\infty(x)}u = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{cases}$$

It follows from this result, with $f = d$ and $D = \Omega \setminus \{x_\star\}$, that $w_\infty$ is the only solution of the Dirichlet problem [39].

Following the ideas of [31] and [8] we give a condition on $\Omega$ that leads to the equality $w_\infty = d/\|d\|_\infty$. For this we recall that $d \in C^1(\Omega \setminus \mathcal{R}(\Omega))$ where $\mathcal{R}(\Omega)$ denotes the ridge of $\Omega$, defined as the set of all points in $\Omega$ whose distance to the boundary is reached at least at two points (see [4,9]). Notice that $\mathcal{R}(\Omega)$ contains the maximum points of $d$. Since $d$ is a viscosity solution of the eikonal equation $|\nabla u| = 1$ in $\Omega$, it is simple to check that $\Delta_{\infty(x)}d = 0$ in $\Omega \setminus \mathcal{R}(\Omega)$ in the viscosity sense.

Proposition 5.10 If $\mathcal{R}(\Omega)$ is a singleton set, then $w_l \to \frac{d}{\|d\|_\infty}$ uniformly in $\overline{\Omega}$ and $x_0^l \to x_\star$.

Proof. It follows from Corollary 5.3 that $\mathcal{R}(\Omega) = \{x_\star\}$, since $x_\star$ is a maximum point of $d$. Therefore, $d \in C^1(\Omega \setminus \{x_\star\})$ what implies that $d$ is a viscosity solution of

$$\begin{cases} \Delta_{\infty(x)}u = 0 & \text{in } \Omega \setminus \{x_\star\} \\ u = d & \text{on } \partial (\Omega \setminus \{x_\star\}) = \partial \Omega \cup \{x_\star\}. \end{cases}$$

Therefore, by the uniqueness stated in Proposition 5.9 (with $D := \Omega \setminus \{x_\star\}$) we have $w_\infty = \frac{d}{\|d\|_\infty}$ in $\Omega$.

These arguments imply that $d/\|d\|_\infty$ is the only limit function of any uniformly convergent subsequence of $(w_l)_{l \in \mathbb{N}}$ and also that $x_\star$ is the only cluster point of the numerical sequence $(x_0^l)_{l \in \mathbb{N}}$.

Balls, ellipses and other symmetric sets are examples of domains whose ridge is a singleton set.

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