IMPROVEMENT OF $A$-NUMERICAL RADIUS INEQUALITIES OF SEMI-HILBERTIAN SPACE OPERATORS

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Abstract. Let $H$ be a complex Hilbert space and let $A$ be a positive operator on $H$. We obtain new bounds for the $A$-numerical radius of operators in semi-Hilbertian space $B_A(H)$ that generalize and improve on the existing ones. Further, we estimate bounds for the $B$-operator seminorm and $B$-numerical radius of $2 \times 2$ operator matrices, where $B = \text{diag}(A, A)$. The bounds obtained here improve on the existing bounds.

1. Introduction

The purpose of the present article is to study the numerical radius inequalities of semi-Hilbertian space operators and operator matrices, which generalize the classical numerical radius inequalities of complex Hilbert space operators and operator matrices. The motivation comes from the recent papers [2, 8]. Let us first introduce the following notations and terminologies.

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. Let $T \in B(H)$. As usual the Range of $T$ and the Kernel of $T$ are denoted by $R(T)$ and $N(T)$, respectively. By $\overline{R(T)}$ we denote the norm closure of $R(T)$. Let $T^*$ be the adjoint of $T$. The letters $I$ and $O$ are reserved for the identity operator and the zero operator on $H$, respectively. Throughout the article, $A \in B(H)$ is a positive operator and $B = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$. Clearly, $A$ induces a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle_A : H \times H \to \mathbb{C}$, defined by $\langle x, y \rangle_A = \langle Ax, y \rangle$ for all $x, y \in H$. This sesquilinear form induces a seminorm $\| \cdot \|_A : H \to \mathbb{R}^+$, defined by $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for all $x \in H$. Clearly, $\| \cdot \|_A$ is a norm if and only if $A$ is injective and $(H, \| \cdot \|_A)$ is complete if and only if $R(A)$ is closed in $H$. An operator $R \in B(H)$ is called an $A$-adjoint of $T$ if $\langle Tx, y \rangle_A = \langle x, Ry \rangle_A$ for all $x, y \in H$. The existence of an $A$-adjoint of $T$ is not guaranteed. Let $B_A(H)$ denote the collection of all operators in $B(H)$, which admit $A$-adjoints. By Douglas theorem [9], it follows that

$$B_A(H) = \{ T \in B(H) : R(T^*A) \subseteq R(A) \}.$$
If \( T \in \mathcal{B}_A(\mathcal{H}) \) then the operator equation \( AX = T^*A \) has a unique solution, denoted by \( T^{z_A} \), satisfying \( \mathcal{R}(T^{z_A}) \subseteq \mathcal{R}(A) \). Note that \( T^{z_A} = A^\dagger T^*A \), where \( A^\dagger \) is the Moore-Penrose inverse of \( A \). Clearly, \( A T^{z_A} = A \). For \( T \in \mathcal{B}_A(\mathcal{H}) \), we have \( AT^{z_A} = T^*A \) and \( N(T^{z_A}) = N(T^*A) \). Note that, if \( T \in \mathcal{B}_A(\mathcal{H}) \) then \( T^{z_A} \in \mathcal{B}_A(\mathcal{H}) \) and \( (T^{z_A})^{z_A} = PTP \), where \( P \) is the orthogonal projection onto \( \overline{\mathcal{R}(A)} \). For further study on the A-adjoint of an operator, we refer the interested readers to [1]. Let us now define the A-operator seminorms on \( \mathcal{B}_A(\mathcal{H}) \). Let \( T \in \mathcal{B}_A(\mathcal{H}) \). The A-operator seminorm of \( T \), denoted by \( \|T\|_A \), is defined as

\[
\|T\|_A = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\|_A = 1\}.
\]

Clearly, \( \|TT^{z_A}\|_A = \|T^{z_A}T\|_A = \|T^{z_A}\|_A^2 = \|T\|_A^2 \). The A-minimum norm of \( T \), denoted by \( m_A(T) \), is defined as

\[
m_A(T) = \inf\{\|Tx\| : x \in \mathcal{H}, \|x\|_A = 1\}.
\]

The A-numerical range and the A-numerical radius of \( T \), denoted by \( W_A(T) \) and \( w_A(T) \), respectively, are defined as

\[
W_A(T) = \{\langle Tx, x \rangle_A : x \in \mathcal{H}, \|x\|_A = 1\},
\]

\[
w_A(T) = \sup\{\|\langle Tx, x \rangle_A\| : x \in \mathcal{H}, \|x\|_A = 1\}.
\]

For \( T \in \mathcal{B}_A(\mathcal{H}) \), we also have

\[
\|T\|_A = \sup\{\|\langle Tx, y \rangle_A\| : x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1\}.
\]

In particular, if we consider \( A = I \) in the definitions of A-operator seminorm, A-minimum norm and A-numerical radius then we have the classical operator norm, minimum norm and numerical radius, respectively, i.e., \( \|T\|_A = \|T\|_1, m_A(T) = m(T) \) and \( w_A(T) = w(T) \). It is well-known that \( w_A(\cdot) \) and \( \|\|_A \) are equivalent seminorms on \( \mathcal{B}_A(\mathcal{H}) \), satisfying the following inequality

\[
\frac{1}{2} \|T\|_A \leq w_A(T) \leq \|T\|_A.
\]

Recently many eminent mathematicians have studied A-numerical radius inequalities, we refer the interested readers to [4, 5, 12, 15] and references therein.

This article is separated into three sections, including the introductory one. In the second section, we develop inequalities for the A-numerical radius of operators in \( \mathcal{B}_A(\mathcal{H}) \). The inequalities obtained here generalize and improve on the inequalities in [16]. In particular, we show that if \( T \in \mathcal{B}_A(\mathcal{H}) \) then the following inequalities hold

\[
w_A^2(T) \leq \min_{0 \leq \alpha \leq 1} \|\alpha T^{z_A} + (1 - \alpha)TT^{z_A}\|_A^2,
\]

\[
w_A^2(T) = \min_{0 \leq \alpha \leq 1} \left\{ \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4} TT^{z_A} + \left(1 - \frac{3\alpha}{4} \right) T^{z_A}T \right\| \right\}
\]

and

\[
w_A^2(T) = \min_{0 \leq \alpha \leq 1} \left\{ \frac{\alpha}{2} w_A(T^2) + \left\| \left(1 - \frac{3\alpha}{4} \right) TT^{z_A} + \frac{\alpha}{4} T^{z_A}T \right\| \right\}.
\]

In third section we study the inequalities on operator matrices. We obtain bounds for the B-numerical radius of \( 2 \times 2 \) operator matrices of the form \( \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \).
where $X, Y, Z, W \in B_A(\mathcal{H})$. Also, we obtain an upper bound for the $B$-operator seminorm of $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$. We show that the bounds obtained here improve on the existing ones.

2. Inequalities of operators

We begin with the following theorem that gives an upper bound for the $A$-numerical radius of bounded linear operators on $\mathcal{H}$ that admits $A$-adjoint.

**Theorem 2.1.** Let $T \in B_A(\mathcal{H})$. Then
\[
 w^2_A(T) \leq \min_{0 \leq \alpha \leq 1} \|\alpha T^{z_A}T + (1 - \alpha)TT^{z_A}\|_A.
\]

**Proof.** Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Then for all $\alpha \in [0, 1]$ we get,
\[
 |\langle Tx, x \rangle_A| = \alpha |\langle Tx, x \rangle_A| + (1 - \alpha) |\langle x, T^{z_A}x \rangle_A| \\
 \Rightarrow |\langle Tx, x \rangle_A| \leq \alpha \|Tx\|_A + (1 - \alpha) \|T^{z_A}x\|_A \\
 \Rightarrow |\langle Tx, x \rangle_A|^2 \leq \alpha \|Tx\|^2_A + (1 - \alpha) \|T^{z_A}x\|^2_A, \text{ by convexity of } t^2 \\
 \Rightarrow |\langle Tx, x \rangle_A|^2 \leq \alpha \langle Tx, Tx \rangle_A + (1 - \alpha) \langle T^{z_A}x, T^{z_A}x \rangle_A \\
 \Rightarrow |\langle Tx, x \rangle_A|^2 \leq \alpha \langle T^{z_A}Tx, x \rangle_A + (1 - \alpha) \langle TT^{z_A}x, x \rangle_A \\
 \Rightarrow |\langle Tx, x \rangle_A|^2 \leq \langle \{\alpha T^{z_A}T + (1 - \alpha) TT^{z_A}\}x, x \rangle_A \\
 \Rightarrow |\langle Tx, x \rangle_A|^2 \leq \|\alpha T^{z_A}T + (1 - \alpha) TT^{z_A}\|_A.
\]

Taking supremum over $\|x\|_A = 1$, we get
\[
 w^2_A(T) \leq \|\alpha T^{z_A}T + (1 - \alpha) TT^{z_A}\|_A, \forall \alpha \in [0, 1].
\]

Taking minimum over all $\alpha \in [0, 1]$ we get the desired inequality. \hfill \square

**Remark 2.2.** In [16, Th. 2.10], Zamani proved that
\[
 w^2_A(T) \leq \frac{1}{2} \|T^{z_A}T + TT^{z_A}\|_A, \quad (2.1)
\]
which follows clearly from Theorem 2.1, in fact, we have
\[
 w^2_A(T) \leq \min_{0 \leq \alpha \leq 1} \|\alpha T^{z_A}T + (1 - \alpha) TT^{z_A}\|_A \leq \frac{1}{2} \|T^{z_A}T + TT^{z_A}\|_A.
\]

Considering
\[
 T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\]
we get,
\[
 \min_{0 \leq \alpha \leq 1} \|\alpha T^{z_A}T + (1 - \alpha) TT^{z_A}\|_A = \frac{6}{5} \\
 \text{and} \quad \frac{1}{2} \|T^{z_A}T + TT^{z_A}\|_A = \frac{4}{3}.
\]

Thus, we observe that Theorem 2.1 is a non-trivial improvement of [16, Th. 2.10].

To develop the next inequality we need the following lemma.
Lemma 2.3. [7] Let $x, y, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then
\[
\|\langle a, e \rangle_A \langle b, e \rangle_A \| \leq \frac{1}{2} (\|\langle a, b \rangle_A \| + \|a\|_A \|b\|_A).
\]

We now obtain the following inequality for the $A$-numerical radius of operators in $B_A(\mathcal{H})$.

Theorem 2.4. Let $T \in B_A(\mathcal{H})$ and let $r \geq 1$. Then
\[
w_A^{2r}(T) \leq \frac{1}{2} w_A(T^2) + \frac{1}{2^{r+1}} \|TT^{t_A} + T^{t_A}T\|_A^r.
\]

Proof. Let $x \in \mathcal{H}$ with $\|x\|_A = 1$. Taking $a = Tx$, $b = T^{t_A}x$ and $e = x$ in Lemma 2.3, we get
\[
\|\langle Tx, x \rangle_A \|^2 \leq \frac{1}{2} (\|\langle T^2 x, x \rangle_A \| + \|Tx\|_A \|T^{t_A}x\|_A)
\]
\[
\Rightarrow \|\langle Tx, x \rangle_A \| \leq \frac{1}{2} \|\langle T^2 x, x \rangle_A \| + \frac{1}{4} (\|Tx\|_A^2 + \|T^{t_A}x\|_A^2), \text{ by AM-GM inequality}
\]
\[
\Rightarrow \|\langle Tx, x \rangle_A \| \leq \frac{1}{2} \|\langle T^2 x, x \rangle_A \| + \frac{1}{4} \langle (TT^{t_A} + T^{t_A}T) x, x \rangle_A
\]
\[
\Rightarrow \|\langle Tx, x \rangle_A \| \leq \frac{1}{2} w_A(T^2) + \frac{1}{4} \|TT^{t_A} + T^{t_A}T\|_A
\]
\[
\Rightarrow \|\langle Tx, x \rangle_A \|^{2r} \leq \frac{1}{2} w_A(T^2) + \frac{1}{2^{r+1}} \|TT^{t_A} + T^{t_A}T\|_A^r, \text{ by convexity of } t^r.
\]

Taking supremum over $\|x\|_A = 1$, we get the required inequality. \hfill \Box

Remark 2.5. 1. In particular, if we consider $r = 1$ in Theorem 2.4 then we get the inequality in [16, Th. 2.11].
2. In [13, Th. 2.4], Sattari et. al. proved that the following numerical radius inequality
\[
w_A^{2r}(T) \leq \frac{1}{2} (w(T^2) + \|T\|^{2r}),
\]
for $T \in B(\mathcal{H})$ and $r \geq 1$. Clearly, if we consider $A = I$ then Theorem 2.4 gives better bound than that in [13, Th. 2.4].
3. It is pertinent to mention here that there was a mathematical mistake in the proof of a similar inequality developed in [3, Th. 2.16], the mistake was in the consideration of $TT^{t_A}$ and $T^{t_A}T$ as positive operators, which is not necessarily true.

In our next theorem we obtain upper bounds for the $A$-numerical radius of operators in $B_A(\mathcal{H})$ which generalize the inequality in [16, Th. 2.11].

Theorem 2.6. Let $T \in B_A(\mathcal{H})$. Then for all $\alpha \in [0, 1],
\[
w_A^{2}(T) \leq \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4} TT^{t_A} + \left(1 - \frac{3\alpha}{4}\right) T^{t_A}T \right\|_A
\]
and
\[
w_A^{2}(T) \leq \frac{\alpha}{2} w_A(T^2) + \left\| \left(1 - \frac{3\alpha}{4}\right) TT^{t_A} + \frac{\alpha}{4} T^{t_A}T \right\|_A.
\]
Proof. Let \( x \in \mathcal{H} \) with \( \|x\|_A = 1 \). Considering \( a = Tx, b = T^{\sharp A}x, e = x \) in Lemma 2.3 and then by using AM-GM inequality, we get
\[
|\langle Tx, x \rangle_A|^2 \leq \frac{1}{2} |\langle T^2 x, x \rangle_A| + \frac{1}{4} |\langle TT^{\sharp A} + T^{\sharp A}T \rangle x, x \rangle_A.
\]

Now,
\[
|\langle Tx, x \rangle_A| = \alpha |\langle Tx, x \rangle_A| + (1 - \alpha) |\langle Tx, x \rangle_A|
\]

\[\Rightarrow |\langle Tx, x \rangle_A| \leq |\langle Tx, x \rangle_A| + (1 - \alpha) \|Tx\|_A, \text{ by Cauchy-Schwarz inequality} \]

\[\Rightarrow |\langle Tx, x \rangle_A|^2 \leq |\langle Tx, x \rangle_A|^2 + (1 - \alpha) \|Tx\|^2_A, \text{ by convexity of } t^2 \]

\[\Rightarrow |\langle Tx, x \rangle_A|^2 \leq \frac{\alpha}{2} |\langle T^2 x, x \rangle_A| + \frac{\alpha}{4} \left( (TT^{\sharp A} + T^{\sharp A}T)x, x \rangle_A + (1 - \alpha) \langle T^{\sharp A}Tx, x \rangle_A \right)
\]

\[\Rightarrow |\langle Tx, x \rangle_A|^2 \leq \frac{\alpha}{2} w_A(T^2) + \left\{ \left\{ \frac{\alpha}{4} TT^{\sharp A} + \left( 1 - \frac{3\alpha}{4} \right) T^{\sharp A}T \right\} \right\} \]

Taking supremum over \( \|x\|_A = 1 \), we get the inequality (2.2). Similarly as in (2.2), we can prove the inequality (2.3).

As a consequence of Theorem 2.6, we get the following corollary.

Corollary 2.7. Let \( T \in \mathcal{B}_A(\mathcal{H}) \). Then
\[
w^2(T) \leq \min \{ \beta_1, \beta_2 \},
\]

where
\[
\beta_1 = \min_{0 \leq \alpha \leq 1} \left\{ \frac{\alpha}{2} w_A(T^2) + \left\| \frac{\alpha}{4} TT^{\sharp A} + \left( 1 - \frac{3\alpha}{4} \right) T^{\sharp A}T \right\| \right\}
\]

and
\[
\beta_2 = \min_{0 \leq \alpha \leq 1} \left\{ \frac{\alpha}{2} w_A(T^2) + \left\| \left( 1 - \frac{3\alpha}{4} \right) TT^{\sharp A} + \frac{\alpha}{4} T^{\sharp A}T \right\| \right\}.
\]

Remark 2.8. In particular, if we consider \( \alpha = 1 \) in Corollary 2.7, then we get the inequality in [16, Th. 2.11], i.e.,
\[
w^2_A(T) \leq \frac{1}{2} w_A(T^2) + \frac{1}{4} \left\| TT^{\sharp A} + T^{\sharp A}T \right\|_A.
\]

Now we consider an example. Let \( T \) and \( A \) be the same as described in Remark 2.2. Then by elementary calculations we have, \( w_A(T^2) = \frac{1}{\sqrt{3}} \). Also,

\[
\beta_1 = \min_{0 \leq \alpha \leq 1} \max \left\{ \frac{\alpha}{2} + \frac{\alpha}{2\sqrt{3}}, \frac{2 - 4\alpha}{3} + \frac{\alpha}{2\sqrt{3}}, \frac{2}{3} \left( 1 - \frac{3\alpha}{4} \right) + \frac{\alpha}{2\sqrt{3}} \right\} = \frac{2}{3} + \frac{1}{2\sqrt{3}}
\]

and

\[
\beta_2 = \min_{0 \leq \alpha \leq 1} \max \left\{ 2 - \frac{3\alpha}{2} + \frac{\alpha}{2\sqrt{3}}, \frac{2}{3} + \frac{\alpha}{2\sqrt{3}}, \frac{\alpha}{6} + \frac{\alpha}{2\sqrt{3}} \right\} = \frac{2}{3} + \frac{4}{9\sqrt{3}}.
\]

Therefore,
\[
\min \{ \beta_1, \beta_2 \} = \frac{2}{3} + \frac{4}{9\sqrt{3}} < \frac{2}{3} + \frac{1}{2\sqrt{3}} = \frac{1}{2} w_A(T^2) + \frac{1}{4} \left\| TT^{\sharp A} + T^{\sharp A}T \right\|_A.
\]
Thus, we conclude that the inequality (2.4) is a non-trivial improvement of (2.5).

Finally, we obtain an inequality that involves A-operator seminorm and A-minimum norm.

**Theorem 2.9.** Let $T \in B_A(H)$. Then we have,

$$||T||^2_A + \max \left\{ m^2_A(T), m^2_A(T^{T_A}) \right\} \leq ||T^{T_A} T + T T^{T_A}||_A.$$

**Proof.** Let $x \in H$ with $||x||_A = 1$. Then by Cauchy-Schwarz inequality, we get

$$||T x||^2_A + ||T^{T_A} x||^2_A = \langle (T^{T_A} T + T T^{T_A}) x, x \rangle_A \leq ||T^{T_A} T + T T^{T_A}||_A.$$  

Therefore, $||T x||^2_A + m^2_A(T^{T_A}) \leq ||T^{T_A} T + T T^{T_A}||_A$. Taking supremum over $||x||_A = 1$, we get,

$$||T||^2_A + m^2_A(T^{T_A}) \leq ||T^{T_A} T + T T^{T_A}||_A. \tag{2.6}$$

Similarly, $m^2_A(T) + ||T^{T_A} x||^2_A \leq ||T^{T_A} T + T T^{T_A}||_A$. Taking supremum over $||x||_A = 1$, we get,

$$||T||^2_A + m^2_A(T) \leq ||T^{T_A} T + T T^{T_A}||_A. \tag{2.7}$$

Combining (2.6) and (2.7), we get the desired inequality. \hfill \Box

**Remark 2.10.** The inequality in [10, Th. 1] follows from Theorem 2.9 and by using the fact (see in [16]) that $||T^{T_A} T + T T^{T_A}||_A \leq 4w^2_A(T)$.

3. Inequalities of operator matrices

We begin this section with the following known results, the proof of which can be found in [3, 6, 11].

**Lemma 3.1.** Let $X, Y, Z, W \in B_A(H)$. Then the following results hold:

(i) $w_B \left( \begin{array}{cc} X & O \\ O & Y \end{array} \right) = \max \{ w_A(X), w_A(Y) \}$.

(ii) $w_B \left( \begin{array}{cc} O & X \\ Y & O \end{array} \right) = w_B \left( \begin{array}{cc} O & Y \\ X & O \end{array} \right)$.

(iii) $w_B \left( \begin{array}{cc} O & X \\ e^{i\theta}Y & O \end{array} \right) = w_B \left( \begin{array}{cc} O & Y \\ X & O \end{array} \right)$, For any $\theta \in \mathbb{R}$.

(iv) $w_B \left( \begin{array}{cc} X & Y \\ Y & X \end{array} \right) = \max \{ w_A(X + Y), w_A(X - Y) \}$.

In particular, $w_B \left( \begin{array}{cc} O & Y \\ Y & O \end{array} \right) = w_A(Y)$.

(v) $\left\| \left( \begin{array}{cc} X & O \\ O & Y \end{array} \right) \right\|_B = \left\| \left( \begin{array}{cc} O & X \\ Y & O \end{array} \right) \right\|_B = \max \{ ||X||_A, ||Y||_A \}$.

(vi) $\left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right)^{T_B} = \left( \begin{array}{cc} X^{T_A} & Z^{T_A} \\ Y^{T_A} & W^{T_A} \end{array} \right)$.

First we obtain an upper bound for the $B$-operator seminorm of $2 \times 2$ operator matrices of the form $\left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right)$, where $X, Y, Z, W \in B_A(H)$. 
Theorem 3.2. Let $X, Y, Z, W \in \mathcal{B}_A(\mathcal{H})$. Then
\[
\left\| \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \right\|_B^2 \leq \max\{\|X\|_A^2, \|W\|_A^2\} + \max\{\|X\|_A, \|W\|_A\} \max\{\|Y\|_A, \|Z\|_A\} + \max\{\|Y\|_A^2, \|Z\|_A^2\} + \max_x \left( \begin{pmatrix} O & \tilde{Z}_A^X \end{pmatrix} Z \begin{pmatrix} O \\ W \end{pmatrix} \right).
\]

Proof. Let $x, y$ be two B-unit vectors in $\mathcal{H} \oplus \mathcal{H}$. Then, we get
\[
\left\| \left\langle \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} x, y \right\rangle_B \right\|_B^2
= \left\| \left\langle \left[ \begin{pmatrix} X & O \\ Z & O \end{pmatrix} + \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} \right] x, y \right\rangle_B \right\|_B^2
= \left\| \left\langle \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x, y \right\rangle_B + \left\langle \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2
\leq \left\| \left\langle \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2 + \left\| \left\langle \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2
+ 2 \left\| \left\langle \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B \left\| \left\langle \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x \right\rangle_B \right\|_B,
\]
\[
= \left\| \left\langle \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2 + \left\| \left\langle \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2
+ \left\| \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x \right\|_B \left\| \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x \right\|_B + \left\| \begin{pmatrix} O & \tilde{Z}_A^X \end{pmatrix} Z \begin{pmatrix} O \end{pmatrix} x, y \right\|_B \right\|_B + \left\| \begin{pmatrix} O & \tilde{Z}_A^X \end{pmatrix} Z \begin{pmatrix} O \end{pmatrix} x, x \right\|_B.
\]

by Lemma 2.3
\[
= \left\| \left\langle \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2 + \left\| \left\langle \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x, y \right\rangle_B \right\|_B^2
+ \left\| \begin{pmatrix} X & O \\ Z & O \end{pmatrix} x \right\|_B \left\| \begin{pmatrix} O & Y \\ Z & O \end{pmatrix} x \right\|_B + \left\| \begin{pmatrix} O & \tilde{Z}_A^X \end{pmatrix} Z \begin{pmatrix} O \end{pmatrix} x, x \right\|_B.
\]

Taking supremum over $x, y$ with $\|x\|_B = \|y\|_B = 1$ and using Lemma 3.1, we get the required inequality. \qed

Remark 3.3. We would like to note that the inequality in [2, Th. 2.1] follows from Theorem 3.2 by considering $A = I$.

Next we obtain an upper bound for the $B$-numerical radius of $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where $X, Y, Z, W \in \mathcal{B}_A(\mathcal{H})$. 
Theorem 3.4. Let $X, Y, Z, W \in \mathcal{B}_A(\mathcal{H})$. Then

$$w_B^2 \left( \begin{array}{c} X \\ Z \\ W \end{array} \right) \leq \max \{w_A^2(X), w_A^2(W)\} + w_B^2 \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) + w_B \left( \begin{array}{c} O \\ Z^A W \\ O \end{array} \right) + \frac{1}{2} \max \{\|X^A + Z^A Z\|_A, \|W^A W + Y^A Y\|_A\}.$$ 

Proof. Let $x$ be a B-unit vector in $\mathcal{H} \oplus \mathcal{H}$. Then, we get

$$\left| \left\langle \left( \begin{array}{c} X \\ Z \\ W \end{array} \right), x \right\rangle_B \right|^2 = \left| \left\langle \left[ \left( \begin{array}{c} X \\ O \\ W \end{array} \right) + \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) \right) x, x \right\rangle_B \right|^2$$

$$= \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B + \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|$$

$$\leq \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B \right|^2 + \left| \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|^2 + 2 \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|$$

$$= \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B \right|^2 + \left| \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|^2 + 2 \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x \right\rangle_B \right|$$

by Lemma 2.3

$$= \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B \right|^2 + \left| \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|^2 + \left| \left\langle \left( X^A X \begin{array}{c} O \\ O \end{array} W^A W \right) x, x \right\rangle_B \right|^{\frac{1}{2}} \left| \left\langle \left( Z^A Z \begin{array}{c} O \\ O \end{array} Y^A Y \right) x, x \right\rangle_B \right|^{\frac{1}{2}}$$

$$+ \left| \left\langle \left( \begin{array}{c} O \\ Y^A X \end{array} Z^A W \right) x, x \right\rangle_B \right|$$

$$\leq \left| \left\langle \left( \begin{array}{c} X \\ O \\ W \end{array} \right) x, x \right\rangle_B \right|^2 + \left| \left\langle \left( \begin{array}{c} O \\ Z \\ O \end{array} \right) x, x \right\rangle_B \right|^2 + \frac{1}{2} \left| \left\langle \left( X^A X + Z^A Z \begin{array}{c} O \\ O \end{array} Y^A Y + W^A W \right) x, x \right\rangle_B \right|$$

$$+ \left| \left\langle \left( \begin{array}{c} O \\ Y^A X \end{array} Z^A W \right) x, x \right\rangle_B \right|,$$ by AM-GM inequality.
Taking supremum over \(\|x\|_B = 1\), and using Lemma 3.1, we get the required inequality of the theorem.

\[ \square \]

In particular, considering \(W = X, Z = Y\) in Theorem 3.4 and then using Lemma 3.1, we get the following corollary.

**Corollary 3.5.** Let \(X, Y \in \mathcal{B}_A(\mathcal{H})\). Then

\[
\max\{w_A^2(X+Y), w_A^2(X-Y)\} \leq w_A^2(X) + w_A^2(Y) + \frac{1}{2}\|X^{\sharp A}X + Y^{\sharp A}Y\|_A + w_A(Y^{\sharp A}X).
\]

**Remark 3.6.** Consider \(T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\), where \(Y = Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\), \(X = W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\). If we take \(A = I\) then the bound obtained by Rout et. al. in [14, Th. 3.5] gives \(w_B^2(T) \leq 4\), whereas the bound obtained in Theorem 3.4 gives \(w_B^2(T) \leq \frac{3}{4}\). Therefore, for this operator matrix \(T\), the bound obtained in Theorem 3.4 is better than that in [14, Th. 3.5].

Finally, we obtain another upper bound for the \(B\)-numerical radius of \(2 \times 2\) operator matrices of the form \(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\), where \(X, Y, Z, W \in \mathcal{B}_A(\mathcal{H})\).

**Theorem 3.7.** Let \(X, Y, Z, W \in \mathcal{B}_A(\mathcal{H})\). Then

\[
w_B^2\left(\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\right) \leq \max\{w_A^2(X), w_A^2(W)\} + \frac{1}{2}\max\{w_A(YZ), w_A(ZY)\}
\]

\[
+ w_B\left(\begin{pmatrix} O & YW \\ ZX & O \end{pmatrix}\right)
\]

\[
+ \frac{1}{4}\max\\{\|YY^{\sharp A} + Z^{\sharp A}Z\|_A, \|Y^{\sharp A}Y + ZZ^{\sharp A}\|_A\}
\]

\[
+ \frac{1}{2}\max\\{\|X^{\sharp A}X + YY^{\sharp A}\|_A, \|W^{\sharp A}W + ZZ^{\sharp A}\|_A\}.
\]
Proof. Let $x$ be a B-unit vector in $\mathcal{H} \oplus \mathcal{H}$. Then

$$|\langle \left( \begin{array}{cc} X & Y \\ Z & W \end{array} \right) x, x \rangle_B|^2 = |\langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) + \left( \begin{array}{cc} O & Y \\ Z & O \end{array} \right) \rangle x, x \rangle_B|^2$$

$$= |\langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) x, x \rangle_B|^2 + |\langle \left( \begin{array}{cc} O & Y \\ Z & O \end{array} \right) x, x \rangle_B|^2$$

$$\leq |\langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) x, x \rangle_B|^2 + |\langle \left( \begin{array}{cc} O & Y \\ Z & O \end{array} \right) x, x \rangle_B|^2 + 2|\langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) x, x \rangle_B \langle \left( \begin{array}{cc} O & Y \\ Z & O \end{array} \right) x, x \rangle_B|$$

$$= |\langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) x, x \rangle_B|^2$$

$$+ \frac{1}{2} \left| \langle \left( \begin{array}{cc} O & Y \\ Z & O \end{array} \right) \rangle x, x \rangle_B \left| \langle \left( \begin{array}{cc} O & Z \end{array} \right) x, x \rangle_B \right| \right| \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \left| \langle \left( \begin{array}{cc} O & Z \end{array} \right) x, x \rangle_B \right|$$

$$+ \left| \langle \left( \begin{array}{cc} X & O \\ O & W \end{array} \right) \rangle x, x \rangle_B \left| \langle \left( \begin{array}{cc} O & Z \end{array} \right) x, x \rangle_B \right| \right| \left( \begin{array}{cc} O & Y \end{array} \right) x, x \rangle_B \left| \langle \left( \begin{array}{cc} O & Y \end{array} \right) x, x \rangle_B \right|$$

by Lemma 2.3

$$= \frac{1}{2} \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right| \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right|$$

$$+ \frac{1}{2} \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right| \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right|$$

$$+ \frac{1}{2} \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right| \left| \langle \left( \begin{array}{cc} Y \end{array} \right) x, x \rangle_B \right|$$

by AM-GM inequality.
Taking supremum over $\|x\|_B = 1$ and then using Lemma 3.1, we get the desired result.

Considering $W = X$, $Z = Y$ in Theorem 3.7 and using Lemma 3.1, we get the following corollary.

**Corollary 3.8.** Let $X, Y \in \mathcal{B}_A(\mathcal{H})$. Then

\[
\max \left\{ w_A^2(X + Y), w_A^2(X - Y) \right\} \leq w_A^2(X) + \frac{1}{4}\|YY^2A + Y^2A Y\|_A
\]

\[
+ \frac{1}{2}w_A(Y^2) + \frac{1}{2}\|X^2A X + YY^2A\|_A + w_A(YX).
\]

**Remark 3.9.** Consider $T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where $X = Z = W = O, Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = I$, then the bound obtained in [14, Th. 3.7] gives $w_B^2(T) \leq 1$, whereas the bound in Theorem 3.7 gives $w_B^2(T) \leq \frac{3}{2}$. Therefore, for this operator matrix $T$, Theorem 3.7 gives a better bound than that in [14, Th. 3.7].

**Incomparability of Theorem 3.4 and Theorem 3.7.** If we consider $X = W = O$ and $Y = Z$ then it follows from (2.5) that Theorem 3.4 gives better bound than that in Theorem 3.7. Again, if we consider $A = I$ and $X = (\frac{1}{2}), Y = (1), Z = W = (0)$ then for the operator matrix $T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ Theorem 3.4 gives $w_B^2(T) \leq \frac{10}{3}$, whereas Theorem 3.7 gives $w_B^2(T) \leq \frac{3}{2}$. Therefore, for this operator matrix, Theorem 3.7 gives better bound than that in Theorem 3.4. Thus, bounds obtained in Theorem 3.4 and Theorem 3.7 are not comparable, in general.

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