QCD Phase Shifts and Rising Total Cross-Sections

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Abstract

An attempt is made in QCD to explain the growth of total cross-sections with energy, without violating the Froissart bound. This is achieved by computing the phase shifts of elastic scatterings of partons rather than their amplitudes. To render that possible a general formalism of phase-shifts in QCD is developed. Computed to two-loop order, agreements with hadronic and virtual-photon total cross-sections are quite satisfactory. Predictions for the slower rate of growth at higher energies are also presented.
The recent observation [1,2] of the rapid rise of parton density at small $x$ has generated much theoretical interest [3]. This rise is equivalent to an increase of the total photon-proton cross-section $\sigma_T(\gamma^*p)$ with its c.m. energy $\sqrt{s}$. For photons with virtuality $Q^2$ in the range of the HERA data, $\sigma_T(\gamma^*p) \sim s^a$ with the power $a$ substantially larger than the power $0.08$ [4] observed in hadronic and real-photon-hadron total cross-sections, but substantially less than the power $\sim 0.5$ given by the BFKL Pomeron [5]. See Fig. 2. A power growth violates the Froissart bound $\sigma_T \sim (\ln s)^2$ so eventually all these increases have to slow down. In terms of parton density, this means saturation at small $x$ when the partons begin to overlap in transverse dimensions [6].

A considerable amount of work exists in computing parton densities at small $x$ for large nuclei, and in understanding its saturation [7,8]. We shall approach this problem by looking at the total cross-section $\sigma_T(\gamma^*p)$, which at least for $Q \gg \Lambda_{QCD}$ is also amenable to perturbative QCD calculations. At sufficiently high energies, the energy variation of total cross sections is expected to be universal, so for simplicity we shall only study the energy dependence of quark-quark interactions.

Optical theorem relates total cross-section to the imaginary part of forward elastic scattering amplitude. If the latter is computed via the exchange of a BFKL Pomeron [5], then $\sigma_T(s)$ violates the Froissart bound [3,4]. Multi-reggeon exchanges are necessary to restore unitarity but that calculation is very difficult [9]. We take a different approach by looking at the phase shift, which will almost certainly guarantee the Froissart bound. The idea of using phase shift is not new [10], though it remains a challenge to calculate it within the framework of QCD. We shall discuss how this can be carried out, then use it to compute the quark-quark scattering phase-shift to two-loop order, in the leading-log approximation. The result compares well with the energy increase of the HERA data at different $Q^2$ (Fig. 2), and it also agrees with the hadronic data up to laboratory energy of $10^9$ GeV (Fig. 3).

Let $A(s, \Delta)$ be the elastic scattering amplitude at momentum transfer $\Delta$. At high energies it is well described by the impact-parameter representation
where $\Delta$ and $\vec{b}$ are two-dimensional vectors in the transverse plane. At large impact parameters the phase shift $\delta(s,b)$ is small and it goes approximately like $a \exp(-\mu b)$ if the interaction range is $\mu^{-1}$. Only for $b$ less than the effective radius $R(s)$ does the phase shift contribute substantially to the amplitude, so $R$ can be estimated from the condition $\delta(s,R) \sim 1$. If $a$ is an increasing function of $s$, then $R(s)$ and hence $\sigma_T \sim \pi R(s)^2$ also increase with $s$. In particular, if $a \sim s^\ell$ for $\ell > 0$, then $R(s) \sim (\ell/\mu) \ln s$, and $\sigma_T \sim (\ln s)^2$ reaches the Froissart bound. This conclusion is stable and robust, and is qualitatively independent of the magnitude of $\ell$ and $\mu$. Unless $a$ grows faster than a power of $s$, the Froissart bound is almost certainly guaranteed when computed via the phase shift.

Feynman diagrams tell us how to calculate perturbative amplitudes but not directly the phase shifts. This is because the impact-space amplitude $A(s,b) = e^{2i\delta(s,b)} - 1 = -\sum_{n=1}^{\infty} [2i\delta(s,b)]^n / n!$ is given by an infinite sum of all powers of the phase shift. Even if $\delta(s,b)$ is computed to the lowest order, the resulting $A(s,\Delta)$ already contains terms of all orders. Thus a proper understanding of phase shifts cannot be obtained unless an infinite sum of Feynman diagrams is considered. More generally, if the phase shift is expanded in powers of the coupling constant $g^2$, $\delta(s,b) \simeq \sum_m g^{2m} \delta^{(m)}$, then from (1) the $(2n)$th order contribution to $A(s,b)$ is given by a sum of products of the phase shifts $\delta^{(m)}$. Individual Feynman diagrams certainly do not factorize in this manner so even at a given order we have to sum over many Feynman diagrams. A hint for what to sum is given by the formula $\delta(s,b) \sim \int V(s,b,z)dz$ for potential scattering, where $\delta(s,b)$ is given by an accumulation of interactions (vertices) along the path without involving any energy denominators or propagators. Recall that in perturbation theory the energy denominator $1/\Delta E$ comes from the uncertainty relation, and the fact that the time $\Delta t \sim 1/\Delta E$ allowed for each interaction in a given Feynman diagram is constrained by the times of its two neighbouring interactions. To get rid of the energy denominator we must relax the time constraint by summing over diagrams with all possible orderings of the interactions. When this is carried out with the help of the
The eikonal formula \[1\] for potential scattering and for QED, indeed the energy denominators disappear, factorization occurs, and the phase shift is given by the first Born approximation.

For QCD, the presence of non-commuting colour matrices makes factorization considerably more difficult to achieve, and representations in terms of phase shifts harder to obtain. In what follows, we shall first discuss how this is done for the simpler case of scattering from an external colour source, before tackling the more difficult problem of quark-quark scattering where the leading-log approximation has to be invoked.

The desired factorization for a fast parton of momentum \( p \) interacting with external colour sources, in the tree approximation, comes from the decomposition theorem \[12,13\]

\[
A_n(\vec{b}) = \sum_{\{m\}} C_{m_1}(\vec{b})C_{m_2}(\vec{b}) \cdots C_{m_k}(\vec{b}) / \prod_{i=1}^{k-1} \sum_{j=i}^{k} m_j, \tag{2}
\]

with the first sum taken over all \( m_i \geq 1 \) so that \( \sum_{i=1}^{k} m_i = n \), and over all \( k \). For \( n > 1 \), the irreducible amplitude \( C_n(\vec{b}) \) is identical to the Feynman amplitude \( A_n(\vec{b}) \), except the product of colour matrices \( \lambda_{a_1}, \lambda_{a_2}, \cdots \lambda_{a_{n-1}}, \lambda_{a_n} \) in \( A_n \) is replaced by its nested commutators \([\lambda_{a_1}, [\lambda_{a_2}, [\cdots, [\lambda_{a_{n-1}}, \lambda_{a_n}]]]]\) in \( C_n \). For \( n = 1 \), by definition \( C_1 = A_1 \). Note that \( A_n \) is given by a product of colour matrices so it contains many colours, but \( C_n \) is given by their nested commutators so it carries colour only in the adjoint representation, same as the colour of a single gluon.

With this factorization it is possible to sum up \( A_n(\vec{b}) \) to obtain an impact-parameter representation \[13\], from which the formula for the phase shift can be extracted. The phase shift defined in \(1\), with \( s \) replaced by \( 2p \), is given by

\[
2\delta(\vec{b}) = \sum_{k=1}^{\infty} \frac{C_k}{k} + \frac{1}{12} [C_2, C_1] + \frac{1}{12} [C_3, C_1] + \cdots, \tag{3}
\]

where the ellipses consists of commutators of \( C_m \) of order \( g^{10} \) and above. The \( O(g^{10}) \) expression is explicitly known \[13\] and the higher order expressions can be computed when they are needed.

For quark-quark scattering the presence of loops and colour matrices of the other quark makes the problem so much more difficult that we find it necessary to invoke the leading-log approximation. It is crucial to recognize however that the leading-log approximation
used here may not be the same as what is found in the literature. For example, it is known that the BFKL Pomeron computed in the leading-log approximation violates the Froissart bound; subleading-logs supplied by multiple-reggeon exchanges are needed to restore unitarity. This however refers to leading and subleading logs with respect to a fixed $t$-channel colour (singlet in this case). The leading-log approximation to be used in the phase shift calculation keeps leading-log terms with respect to fixed colour structures. Leading logs in fixed colour structures may give rise to subleading logs in fixed $t$-channel colours, like those supplied by multiple reggeons, which is why leading-log for fixed colour structures may give rise to phase shifts that preserves the Froissart bound. Indeed, there is a very close relationship between colour structures and reggeon structures that will be discussed further below.

The colour structure of a diagram is defined by its colour factor when quarks carry arbitrary $SU(N_c)$ colours. Colour factors are linearly dependent only if they can be related through $SU(N_c)$ commutation relations alone, with coefficients independent of $N_c$ and the specific quark colours. The six diagrams in Fig. 1 have linearly independent colour structures.

The colour structure of a diagram is said to be primitive if all the lines in the diagram remain connected after the top and the bottom quark lines are removed. Using $G$ to denote the colour matrices of a colour structure, $G_a$ and $G_d$ in Fig. 1 are primitive and all the others are not. Decomposing gluons attached to one of the quark lines using (4), it can be shown in the leading-log approximation [14,15] that (i). Non-primitive colour factors may be considered as commutative products of primitive colour factors. For example, using $G$ to denote the colour structure of a diagram, then $G_b = G_a^2$, $G_c = G_a^3$, $G_e = G_f = G_d^2$; (ii). This colour factorization also leads to an amplitude factorization in the impact-parameter space. If the impact-parameter amplitude for a primitive colour structure $G_i$ is $d_i$, then the spacetime amplitude for the colour factor $M(G_i)$, where $M$ is a monomial function of the $G_i$’s, can be factorized into $M(d_i)$; (iii). As a result of this factorization, diagrams of all orders can be summed up to an impact-parameter representation with a phase shift given by $2\delta(s,b) =$
\[ \sum_i d_i G_i, \text{ where the sum is taken over all linearly independent primitive colour factors } G_i. \]

To two-loop order in the leading-log approximation, the quark-quark scattering amplitude \( A(s, \Delta) \) is known to be [16,17]

\[
A(s, \Delta)/s = \left[ -g^2 I_1(\Delta) + (g^4 N_c \ln s/4\pi) I_2(\Delta) \right. \\
\left. - (g^6 N_c^2 \ln^2 s/32\pi^2) \Delta^2 I_2^2(\Delta) \right] G_a \\
+i \left[ g^4 I_2(\Delta)/2 - g^6 N_c \ln s I_3(\Delta)/4\pi \right] G_b \\
+ \left[ g^6 I_3(\Delta)/6 \right] G_c \\
+i(g^6 \ln s/2\pi) \left[ I_3(\Delta) - \Delta^2 I_2^2(\Delta)/2 \right] G_d,
\]

\[
I_n(\Delta) \equiv \int \prod_{i=1}^n \left( \frac{d^2 k_{i\perp}}{(2\pi)^2(k_{i\perp}^2 + \mu^2)} \right) (2\pi)^2. \\
\delta^2 \left( \Delta - \sum_{i=1}^n k_{i\perp} \right) = \int d^2 b \ e^{i\vec{\Delta} \cdot \vec{b}} \left[ K_0(\mu b)/2\pi \right]^n. \quad (4)
\]

The infrared cutoff \( \mu \) is introduced as a parameter to simulate hadronic size and confinement [18]. The factorization property mentioned above can be explicitly verified in eq. (4). The resulting phase shift to two-loop order is then given by

\[
2\delta(s, b) = 2\delta_a(s, b)G_a + 2\delta_d(s, b)G_d, \\
2\delta_a(s, b) = -g^2 K_0(x)/2\pi + g^4 N_c \ln s K_0^2(x)/16\pi^3 \\
- g^6 N_c^2 \ln^2 s V(x)/32\pi^2, \\
2\delta_d(s, b) = ig^6 \ln s \left[ K_0^3(x)/16\pi^4 - V(x)/4\pi \right], \quad (x \equiv \mu b) \\
\Delta^2 I_2^2(\Delta) \equiv \int d^2 b \ e^{i\vec{\Delta} \cdot \vec{b}} V(\mu b). \quad (5)
\]

The colour factors are respectively \( G_a = \lambda_a \times \lambda_a \) and \( G_d = \lambda_a \lambda_b \times \lambda_c \lambda_d (if_{ace})(if_{bed}) \), where \( \times \) indicates tensor product of colour matrices associated with the two quark lines. To obtain the elastic amplitude and hence the total cross section, we must extract from (5) the contribution of the colour-singlet. The resulting formula for the total cross section at \( N_c = 3 \) is
\[ \sigma_T(s) = \frac{1}{\mu^2} \int d^2x \left[ 1 - \frac{2}{3} e^{-D} \cos \left( \frac{2\delta_a}{3} \right) - \frac{1}{3} e^{-2D} \cos \left( \frac{4\delta_a}{3} \right) \right], \] (6)

where the damping exponent \( D = -i\delta_d \) is real and so is \( \delta_d \). Note that these two functions depend on the impact parameter only through the combination \( x = \mu b \), so that all the \( \mu \) dependences are factored out into the \( 1/\mu^2 \) factor in front of the integral. In this way the uncalculable parameter \( \mu \) affects only the overall magnitude of the cross-section, but not its energy dependence which is expected to be universal (as is experimentally the case for hadronic and real photon total cross-sections). We may therefore use this formula for quark-quark total cross-section on the energy dependence of experimentally accessible beams and targets.

To compare the prediction of (6) with experiment we have to realize that perturbative QCD contains no energy scale per se, so an energy scale \( \Lambda(Q) \) must be introduced externally to replace all \( \ln s \) factors by \( \ln(\frac{s}{\Lambda^2(Q)}) \). Recall that (5) is obtained in the leading-log approximation, where \( \ln s \) comes from an integral of the form \( \ln(\frac{s}{\Lambda^2}) = \int_{\Lambda^2/s} d\omega/\omega \), with \( \Lambda \) being a combination of the other energy scales present (masses, momentum transfers, virtualities). For massless quarks and forward amplitude, on dimensional grounds it is therefore reasonable to assume \( \Lambda(Q) = \Lambda_0 + cQ \), with \( \Lambda_0 \) to be of the order of \( \Lambda_{QCD} \), which we shall take it to be 0.2 GeV, and a parameter \( c \) which must be determined phenomenologically. We shall also take the coupling constant to be \( \alpha_s = \frac{g^2}{4\pi} = 0.26 \), which is its value at \( Q^2 = 0.65 \) (GeV/c)^2, the largest \( Q^2 \) involved in these data. The dependence on this parameter is very weak so its actual value is really not that important. In particular, we will not let it vary with \( Q^2 \) though there is no problem to include such a variation. As mentioned before, the parameter \( \mu \) simulates confinement and hadronic-size effects so it is not computable within the framework of perturbative QCD so it must be fitted for each \( Q \).

Total cross sections for deep-inelastic [1] and hadronic [19] data are shown in Figs. 2 and 3 respectively. The energy variations predicted by (5) and (6) are given by the solid curves, with \( c = 4 \). No attempt has been made to obtain a best fit. Shown for comparison
in Fig. 2 are also the dotted lines representing a power variation \( s^{0.08} \) and a dashed line giving \( s^{0.5} \). In Fig. 3 the dashed line represents the Donnachie-Landshoff fit [4,19] \( \sigma_T = 22s^{0.079} + 56.1s^{-0.46} \) mb.

At \( s \) values above a few tens of \((\text{GeV})^2\) for the deep-inelastic data, and perhaps a bit higher for the hadronic data, the energy variation of this theory, with essentially one free parameter \( c \), agrees well with the data. The \( s \)-variation of this theory is slower than a power, as it must be to obey the Froissart bound, but at this energy range it is really not very different from a power as can be seen in the plots. This theory with its leading-log approximation is not designed to work at low energies, but it is amusing to note from Fig. 3 that the dotted line with the low-energy correction term \( s^{-0.46} \) neglected will also come to very similar values at \( s \sim 10^2 \) (GeV\(^2\)).

To summarize, we have suggested that the energy dependence of total cross sections should be understood through the phase shifts of the elastic amplitudes, because the Froissart bound is almost certainly guaranteed in such an approach. The problem of how to calculate the phase shifts in QCD at high energies is discussed, exactly for partons scattered from external sources, and in the leading-log approximation for parton-parton scatterings. The latter is also explicitly calculated to two-loop order and compared with experiments. With an energy scale \( \Lambda(Q) = \Lambda_0 + cQ \) this theory compares well with the energy variations of both the deep inelastic and the hadronic data.

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FIG. 1. Colour structure diagrams.
FIG. 2. $\gamma^*$-proton total cross-sections as a function of $s$. Data are taken from Ref. [1] where references to the original experiments can be found. $Q^2$ of the photon, in (GeV)$^2$, are given on the right. Where two numbers are listed, the second one refers to the low-energy data of E665. The solid curve is the prediction of the present theory. The dotted curve and the dashed curve depict respectively energy variations of $s^{0.08}$ and $s^{0.5}$.
FIG. 3. $pp$ and $\bar{p}p$ total cross sections as a function of the laboratory momentum $p_{\text{lab}}$. Data are taken from Ref. [19], dashed line is the Donnachie-Landshoff fit, and the solid line is the prediction of the present theory. All parameters are identical to those used in Fig. 2.
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