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Instantons in field strength formulated Yang-Mills theories\textsuperscript{1}

H. Reinhardt, K. Langfeld, L. v. Smekal
Institut für theoretische Physik, Universität Tübingen
D–7400 Tübingen, Germany

Abstract

It is shown that the field strength formulated Yang-Mills theory yields the same semiclassics as the standard formulation in terms of the gauge potential. This concerns the classical instanton solutions as well as the quantum fluctuations around the instanton.

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1. Introduction

Yang-Mills (YM) theories can be reformulated entirely in terms of field strength thereby eliminating the gauge potential 1, 2, 3. The field strength formulation allows for a non-perturbative treatment of the gluon self-interaction and offers a simple description of the non-perturbative vacuum where the gauge bosons are presumed condensed. In the so-called FSA an effective action for the field strength is obtained which contains a term of order \( \hbar \) with an explicit energy scale. Hence in the FSA the anomalous breaking of scale invariance is described already at tree level 2. At this level the YM vacuum is determined by homogeneous field configurations with constant \( \langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle \neq 0 \) and instantons do no longer exist as stationary points of the effective action. This is already dictated by the fact that the effective action contains an energy scale. This energy scale cannot tolerate instantons which have a free scale (size) parameter.

In this paper we show that if the FSA is consistently formulated in powers of \( \hbar \) one recovers the same semiclassics as in the standard formulation of YM theory in terms of the gauge potential. All instantons which extremize the standard YM action are also stationary points of the action to \( O(\hbar) \) of the FSA. Furthermore the leading \( \hbar \) corrections originating from the integral over quantum fluctuations around the instanton are also the same in both approaches. Although one might have expected this result on general grounds it is completely non-trivial how this result emerges in the field strength formulation. Furthermore the equivalence proof will also shed some new light on the FSA.

2. Semiclassical approximation to the Yang-Mills theory

We start from the generating functional of Euclidean YM theory

\[
Z[j] = \int \mathcal{D}A \delta(f^a(A)) \text{Det} M_f \exp\{-S_{YM}[A] + \int d^4x j A\},
\]

where

\[
S_{YM} = \frac{1}{4g^2} \int d^4x F^a_{\mu\nu}(A) F^a_{\mu\nu}(A), \quad F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu,
\]

is the classical YM action. Furthermore \( \delta(f^a(A)) \) is the gauge fixing constraint and \( \text{Det} M_f \) denotes the Faddeev-Popov determinant.

It is well known that all finite action self-dual or anti-self-dual field configurations extremize the YM action 3 i.e. solve the classical YM equation of motion

\[
\partial_\mu F^a_{\mu\nu} = -f^{abc} A^b_\mu F^c_{\mu\nu}.
\]
These classical solutions are referred to as instantons. Expanding the fluctuating
gauge field around the classical instanton solution \( A_{\text{inst}}^\mu \) up to second order in the
quantum fluctuations and performing the integral in semiclassical approximation
one finds

\[
Z[j = 0] = Q (\text{Det}' D_{YM}^{-1}(x_1, x_2))^{-1/2} e^{-S_{YM}[A_{\text{inst}]}},
\]

where

\[
D_{YM}^{-1}(x_1, x_2)^{ab}_{\mu\nu} = \frac{\delta^2 S_{YM}[A]}{\delta A^a_\mu(x_1) \delta A^b_\nu(x_2)} |_{A_\mu = A^\mu_{\text{inst}}}
\]

and the prime indicates that the zero modes of \( D_{YM}^{-1} \) have to be excluded from the
determinant. This can be done in the standard fashion and yields the factor \( Q \) in (4).
Furthermore even when the zero modes are excluded the determinant is still singular
and needs regularization. As will become clear later, for our purpose a regularization
scheme that only depends on eigenvalues is convenient, e.g. Schwinger’s proper time
regularization or \( \zeta \)-function regularization. The second variation of the YM action reads

\[
g^2 D_{YM}^{-1}(x_1, x_2)^{ab}_{\mu\nu} = \hat{F}^{ab}_{\mu\nu}(x_1) \delta(x_1, x_2) + \frac{1}{2} \int d^4x \frac{\delta F^c_{\kappa\lambda}(x)}{\delta A^a_\mu(x_1)} \frac{\delta F^c_{\lambda\kappa}(x)}{\delta A^b_\nu(x_2)} ,
\]

where

\[
\hat{F}^{ab}_{\mu\nu} = f^{abc} F^c_{\mu\nu}
\]
denotes the field strength in the adjoint representation and

\[
\frac{\delta F^c_{\kappa\lambda}(x)}{\delta A^a_\mu(x_1)} = [\hat{D}^a_{\kappa}(x) \delta_{\mu\lambda} - \hat{D}^a_{\lambda}(x) \delta_{\mu\kappa}] \delta(x - x_1) ,
\]

with

\[
\hat{D}^{ab}_{\mu}(x) = \partial_\mu \delta^{ab} - \hat{A}^{ab}_{\mu}(x)
\]

being the covariant derivative. The functional determinant \( \text{Det} D_{YM}^{-1} \) in an instanton
background has been explicitly evaluated by t’Hooft [3].

3. Field strength formulated Yang-Mills theory

Non-Abelian YM theory can be equivalently formulated in terms of field strength
[4, 3, 5]. Inserting the identity

\[
\exp\{-\frac{1}{4g^2} \int d^4x F^2(A)\} = \int D\chi \ exp \left(-\int d^4x \left\{ \frac{1}{4} \chi^a_{\mu\nu} \chi^a_{\mu\nu} + \frac{i}{2g} \chi^a_{\mu\nu} F^{a}_{\mu\nu}(A) \right\} \right)
\]

(10)
into (1) we obtain

$$Z = \int D\chi \, DA \, \delta(f^a(A)) \, \text{Det} M_f \, \exp\left\{-S[\chi, A] + \int d^4 x \, j A\right\} \quad (11)$$

$$S[\chi, A] = \frac{1}{g^2} \int d^4 x \left( \frac{1}{4} \chi_{\mu\nu}^{a} \chi_{\mu\nu}^{a} \pm \frac{i}{2} \chi_{\mu\nu}^{a} F_{\mu\nu}^{a}(A) \right) , \quad (12)$$

If there were no gauge fixing the integral over the gauge field would be Gaussian. At first sight it seems that with the presence of the gauge fixing constraint the $A_{\mu}$-integration can no longer be performed explicitly. However, one can transfer the gauge fixing from the gauge potential $A_{\mu}$ to the field strengths. For this purpose we insert the following identity into (12)

$$1 = \text{Det} M_g(\chi) \int d(\theta) \delta(g^a(\chi^\theta)) , \quad (13)$$

where $d(\theta)$ is the invariant measure of the functional integration over the group space, $\chi^\theta$ denotes the gauge transformed of $\chi$, and $\text{Det} M_g(\chi)$ does not depend on $\theta$. The key observation now is that the action $S[\chi, A]$ in (12) is only invariant under simultaneous gauge transformations of $A$ and $\chi$. Therefore a change of the integration variable $\chi^\theta \rightarrow \chi$ implies also a change in the gauge potential $A_{\mu} \rightarrow A_{\mu}^{\theta}$ to leave the exponent in (12) invariant. Because of the gauge invariance of the measure and the determinants one then obtains

$$Z = \int d(\theta) \int D\chi \, DA \, \delta(f^a(A^{\theta-1})) \, \delta(g^b(\chi)) \, \text{Det} M_f \, \text{Det} M_g \, e^{-S[\chi, A]} \quad . \quad (14)$$

Now the integration over the gauge group can be performed again yielding

$$Z = \int D\chi \, DA \, \delta(g^b(\chi)) \, \text{Det} M_g \, \exp\left\{-S[\chi, A] + \int d^4 x \, j A\right\} \quad . \quad (15)$$

Fixing the gauge in terms of field strengths leaves a residual invariance with respect to transformations, which leave the field strengths invariant. In the case of YM theories these transformations belong to the discrete invariant subgroup of the gauge group. Therefore (in contrast to the Abelian case) this residual invariance is harmless.

Once the field strength $\chi$ is gauge fixed there is no invariance left in the potentials (up to the irrelevant residual invariance mentioned above) and the integration $\int DA$ becomes unconstraint. For non-singular $\hat{\chi}_{\mu\nu}^{a}$ it yields

$$Z = \int D\chi \, \delta(g^a(\chi)) \, \text{Det} M_g \, (\text{Det} \frac{i}{2g} \hat{\chi})^{-1/2} \, \exp\left\{-S_{FS}[\chi, j]\right\} \quad . \quad (16)$$
\[ S_{FS}[\chi, j] = \frac{1}{g^2} \int d^4 x \left( \frac{1}{4} \chi^a_{\mu\nu} \chi^a_{\mu\nu} + \frac{i}{2} \chi^a_{\mu\nu} F^a_{\mu\nu}(J) + j^a_{\mu} J^a_{\mu} - i g^2 j^a_{\mu} (\hat{\chi}^{-1})^{ab}_{\mu\nu} j^b_{\nu} \right), \]

where

\[ J^a_{\mu} = \left( \hat{\chi}^{-1} \right)^{ab}_{\mu\nu} \partial_{\rho} \chi^b_{\rho\nu} \]

is an induced gauge potential. For singular \( \hat{\chi} \), integration over the gauge potential \( A^a_{\mu}(x) \) yields an expression similar to (16), where the matrix \( \hat{\chi} \) is, however, replaced by its projection onto the non-singular subspace. But in addition constraints for the \( \chi \)-integration result. These constraints indicate that singular field configurations \( \hat{\chi} \) are statistically suppressed. Since \( \chi^a_{\mu\nu} \) behaves under gauge transformations as the field strength \( F^a_{\mu\nu}(A) \) the induced potential \( J^a_{\mu} \) transforms precisely like the original gauge field \( A^a_{\mu} \). In practice, gauge fixing of the \( \chi \) can be done by using the familiar gauges for the induced gauge potential \( J^a_{\mu} \) (18).

The presence of the external source \( j^a_{\mu} \) in the exponent of (16) ensures that Green’s functions of the original gauge potential \( A^a_{\mu} \) are still accessible in the field strength formulation.

Finally in the field strength formulation a current current interaction is induced which dominates the fermion dynamics at low energies [7, 8, 9, 10].

4. Instantons in the field strength formulation

We are interested in a semiclassical analysis of the field strength formulated YM theory. For simplicity we discard the external gluon source \( (j^a_{\mu} = 0) \). The extrema of the action \( S_{FS}[\chi] = S_{FS}[\chi, j = 0] \) occur for

\[ g \chi^a_{\mu\nu} = -i F^a_{\mu\nu}(J). \]

It was observed by Halpern [1] that the effective action of the \( \chi \) field is extremized by the standard Polyakov t’Hooft instantons. This fact is, however, not only true for the standard SU(2) instanton but holds for any classical solution extremizing the Yang-Mills action (2). For a proof we rewrite the classical YM equation of motion (3) with the definition (18) as

\[ J^a_{\mu}(F(A)) = A^a_{\mu}(x) \]

where we have for simplicity assumed that \( F^a_{\mu\nu}(A) \) is not singular. Now let \( A^{inst}_{\mu} \) denote an instanton solution to (20). Since \( J^a_{\mu}(\chi = -i F/g) = J(F) \) it follows from (20) that the equation of motion of the field strength formulation (19) is solved indeed for

\[ \chi^a_{\mu\nu} = -i g F^a_{\mu\nu}(A^{inst}_{\mu}). \]
Furthermore, it follows then that also the classical action of the instanton in the field strength formulation is the same as in the standard approach

\[ S_{FS}[\chi = -\frac{i}{g}F(A_{\mu}^{\text{inst}})] = S_{YM}[A_{\mu}^{\text{inst}}]. \] (22)

The equivalence between both approaches holds, however, not only at the classical level but also the quantum fluctuations give identical contributions as we will explicitly prove in the following for the leading order in \( \hbar \) corrections.

In the field strength formulation quantum fluctuations around a background field were considered in [3]. In the semiclassical approximation the background field is chosen as the instanton (21). If we expand the tensor field \( \chi_{a \mu \nu} \) in terms of the t’Hooft symbols \([6]\)

\[ \chi_{a \mu \nu} = \chi_{i \mu \nu} Z_{i}^{a} , \quad Z_{i \mu \nu} = \{ \eta_{i \mu \nu} , \bar{\eta}_{i \mu \nu} \} \] (23)

the generating functional (16) becomes then

\[ Z[j = 0] = Q_{FS} (\text{Det} \frac{i}{g} \tilde{\chi}^{\text{inst}})^{-1/2} (\text{Det} \hat{D}_{FS}^{-1}[\chi^{\text{inst}}])^{-1/2} e^{-S_{FS}[\chi^{\text{inst}}]} \] (24)

where we have used (22) and [3]

\[ \hat{D}_{FS}^{-1}[\chi](x_{1}, x_{2})_{ij}^{ab} = \frac{\delta^{2} S_{FS}[A]}{\delta \chi_{i \mu}^{a}(x_{1}) \delta \chi_{j \nu}^{b}(x_{2})} \bigg|_{\chi^{\text{inst}}} \] (25)

is the second variation of the field strength action taken at a background field \( \chi^{\text{inst}} = -iF(A_{\mu}^{\text{inst}})/g \). The prime indicates again that zero modes are excluded. Their contribution is included in the factor \( Q_{FS} \). For space-time dependent dependent background fields \( \chi_{a \mu \nu} = -iF_{a \mu \nu}(x)/g \) one finds

\[ \hat{D}_{FS}^{-1}[F](x_{1}, x_{2})_{ij}^{ab} = 2\delta^{ab} \delta_{ij} \delta(x_{1}, x_{2}) + K_{ij}^{ab}[F](x_{1}, x_{2}) \] (26)

\[ K_{ij}^{ab}[F] = -\hat{D}_{\kappa}^{ac}(x_{1})Z_{\kappa \mu}^{i}(\bar{F}^{-1})_{\mu \nu}^{cd}\hat{D}_{\lambda}^{\phi}(x_{1})Z_{\lambda \nu}^{j}(x_{2}) \] (27)

where \( \hat{D}_{\mu}^{ab} \) denotes here the covariant derivative (3) with respect to the induced gauge potential \( J_{\mu}^{a}(\chi = -iF) = J_{\mu}^{a}(F) \)

\[ \hat{D}_{\mu}^{ab}(x) = \partial_{\mu} \delta^{ab} - \hat{j}_{\mu}^{ab}(x). \] (28)

For a constant background field the above expressions for the fluctuations reduce to the expressions given in [3]. Comparison of (3) and (24) shows if both approaches give the same semiclassical result we should have the relation

\[ Q (\text{Det} g^{2} \hat{D}_{YM}^{-1}[A_{\text{inst}}](x_{1}, x_{2}))^{-\frac{1}{2}} = C Q_{FS} (\text{Det} \tilde{F}_{\text{inst}} \text{Det} \hat{D}_{FS}^{-1}[F_{\text{inst}}])^{-\frac{1}{2}}, \] (29)
where $C$ is an irrelevant (but non-vanishing) constant, which does not depend on the instanton solution. We will now explicitly prove this relation.

5. Equivalence proof

In order to establish the validity of (29) we cast the functional matrix (3) of the standard approach YM into the form of its field strength formulated counterpart (20) by writing

$$2g^2 D_{YM}^{-1}[F](x_1,x_2)_{ab}^{\mu \nu} = (\hat{F}^{1/2})_{\rho \sigma}^{ac}(x_1) M_{\rho \sigma}^{bc}[F](x_1,x_2) (\hat{F}^{1/2})_{\rho \sigma}^{eb}(x_2)$$

$$M_{\mu \nu}^{ab}[F] = 2 \delta_{ab} \delta_{\mu \nu} \delta(x_1,x_2) + (\hat{F}^{-1/2})_{\mu \sigma}^{ad} \int d^4x \frac{\delta F_{\kappa \lambda}^c(x)}{\delta A_{\rho}^d(x_1)} \frac{\delta F_{\kappa \lambda}^c(x)}{\delta A_{\rho}^e(x_2)} (\hat{F}^{-1/2})_{\rho \nu}^{eb}.$$  

We first prove that $M[F]$ has the same eigenvalues (including the zero modes) as $D_{FS}^{-1}[F]$. Let $\phi_{\mu}^a(x)$ and $\lambda$ denote the eigenvectors and eigenvalues of $M$:

$$\int d^4x_2 \ M_{\mu \nu}^{ab}[F](x_1,x_2) \phi_{\nu}^b(x_2) = \lambda \phi_{\mu}^a(x_1)$$

Defining

$$\psi_{\kappa \lambda}^c(x) := \int d^4x_2 \frac{\delta F_{\kappa \lambda}^c(x)}{\delta A_{\rho}^e(x_2)} (\hat{F}^{-1/2})_{\rho \nu}^{eb}(x_2) \phi_{\nu}^b(x_2)$$

the eigenvalue equation becomes

$$\int d^4x \ K_{d c}^{\rho \sigma,\kappa \lambda}[F](y,x) \psi_{\kappa \lambda}^c(x) = (\lambda - 2) \psi_{\rho \sigma}^d(y)$$

where

$$K_{d c}^{\rho \sigma,\kappa \lambda}[F](y,x) = \int d^4x_1 \frac{\delta F_{\rho \sigma}^d(y)}{\delta A_{\mu}^a(x_1)} [\hat{F}^{-1}(x_1)]_{\mu \nu}^{ab} \frac{\delta F_{\kappa \lambda}^c(x)}{\delta A_{\rho}^e(x_1)}. $$

By construction the amplitudes $\psi_{\mu \nu}^a$ are antisymmetric in $(\mu, \nu)$ and can hence be expanded in terms of t’Hooft symbols (c.f. (23))

$$\psi_{\mu \nu}^a = \psi_{i}^a Z_{\mu \nu}^i .$$

The eigenvalue equation (34) then reads

$$\int d^4x \ K_{ij}[F](y,x) \psi_{j}^i(x) = (\lambda - 2) \psi_{i}^d(y)$$

$$K_{ij}^{d b}(y,x) = \frac{1}{4} Z_{i}^a K[F](y,x)_{\mu \nu,\kappa \lambda}^{a b} Z_{j}^b .$$

Inserting the explicit form of $\delta F_{\kappa \lambda}^c(x)/\delta A_{\mu}^a(y)$ (3) into $K[F]$ (35) the integration over the intermediate coordinate $x_1$ can be carried out upon using
\[ \hat{D}_\mu^{ab}(x)\delta(x-y) = -\hat{D}_\mu^{ba}(y)\delta(x-y). \]

Exploiting also the antisymmetry of the \( Z_{\mu\nu}^i \)
the kernel \( \bar{K}[F] \) takes the form

\[
\bar{K}_{ij}^{ab}(x,y) = -\hat{D}_\mu^{ac}(x)Z_{\mu\kappa}^i(\hat{F}^{-1}(x))^{cd}\hat{D}_\nu^{db}(x)Z_{\nu\lambda}^j \delta(x-y). \tag{39}
\]

For an instanton background field \( A_\mu \) we have in view of (20)
the covariant derivatives in (9) and (28) are the same so that the kernels \( K \) [27] and \( \bar{K} \) [39] are identical. We thus proved that all eigenvalues of \( M \) [31] are also eigenvalues of \( \mathcal{D}_{FS}^{-1} \), and the corresponding eigenvectors are related by (33).

On the other hand, not all eigenvalues of \( \mathcal{D}_{FS}^{-1} \) [26] are also eigenvalues of \( M \) [31]. This is because the transformation [33] maps the Hilbert space of eigenstates of \( M \) of dimension \( n = D(N^2 - 1) \) onto a subspace of the \( m = \left( \begin{array}{c} D \\ 2 \end{array} \right) (N^2 - 1) \) dimensional space of eigenvectors of \( \bar{K} \). This implies that the \( m-n \) additional eigenvectors of \( \bar{K} \), denoted by \( \psi_{\mu\nu}^{(0)} \), are zero modes

\[
\int d^4x \bar{K}_{\rho\epsilon,\kappa\lambda}(y,x) \psi^{(0)}_{\kappa\lambda}(x) = 0 \tag{40}
\]
satisfying

\[
\int d^4x_1 \frac{\delta F_{\epsilon\lambda}(x)}{A_\mu^c(x_1)} \psi^{(0)}_{\kappa\lambda}(x_1) = 0. \tag{41}
\]

These additional zero eigenvalues of \( \bar{K} \) give rise to additional eigenvalues 2 of \( \mathcal{D}_{FS}^{-1} \) in the field strength formulation. The latter contribute only an irrelevant constant to the functional determinant of \( \mathcal{D}_{FS}^{-1} \), which can be absorbed into the constant \( C \) in (29).

If there were no zero modes the proof of (29) would be completed. This is because in the absence of zero modes \( (Q = Q_{FS} = 1) \) from (30) would follow

\[
\text{Det} \mathcal{D}_{YM}^{-1} = \text{Det} \hat{F} \text{ Det} M \tag{42}
\]

and we have shown above that

\[
\text{Det}' M[F] = C \text{ Det}' \mathcal{D}_{FS}^{-1} \tag{43}
\]

provided the same regularization is used.

In the presence of zero modes some more care is required. Their contribution [4] to the functional integrals is represented by the preexponential factors in (4) and (24)

\[
Q = \text{Det}^{1/2}\langle \delta_i A_{cl} \mid \delta_k A_{cl} \rangle \quad \text{and} \quad Q_{FS} = \text{Det}^{1/2}\langle \delta_i \chi_{cl} \mid \delta_k \chi_{cl} \rangle \tag{44}
\]
respectively. Here $\delta_i A_{cl}$ denotes the variation of the classical (instanton) solution with respect to its $i$th symmetry parameter, which is the (unnormalized) zero mode. Its counterpart $\delta_i \chi_{cl}$ in the field strength formulation is related to $\delta_i A_{cl}$ by

$$\delta_i (\chi_{cl})^a_{\kappa\lambda}(x) = -\frac{i}{g} \delta_i F^a_{\kappa\lambda}[A_{cl}] = -\frac{i}{g} \int d^4x \frac{\delta F^a_{\kappa\lambda}}{\delta A^b_{\mu}(x)} \delta_i A_{cl,\mu}(x). \quad (45)$$

Exploiting the fact $D_{YM}^{-1} \delta_i A_{cl} = 0$ one readily verifies that

$$Q^2_{FS} = \det(\delta_i A_{cl} | \hat{F} | \delta_k A_{cl}) = Q^2 \det(\hat{F}), \quad (46)$$

where $\det(\hat{F})$ is the determinant of the matrix arising from projection $\hat{F}$ onto the space of zero modes of $D_{YM}^{-1}$. Inserting $(46)$ into $(29)$ it remains to be proven that

$$\frac{1}{\det(D_{YM}^{-1})} = \frac{\det(\hat{F})}{\det F \det(D_{FS}^{-1})}. \quad (47)$$

In order to extract $\det(D_{FS}^{-1})$ from $\det(D_{YM}^{-1})$ it is convenient to introduce the complete set of orthonormal eigenvectors $\varphi_i$ and $\hat{\varphi}_i$ of the symmetric matrices $D_{YM}^{-1} = \hat{F}^{1/2} M \hat{F}^{-1/2}$ and $M$. We denote the normalized zero and non-zero modes of $D_{YM}^{-1}$ ($M$) by $\varphi_i^{(0)}$, $(\varphi_i^{(0)})$ and $\varphi_i$, $(\varphi_i)$, respectively. Accordingly $\det(\hat{F})$ and $\det(D_{FS}^{-1})$ denote the respective subspace determinants of $\hat{F}$. From the defining equation $(30)$ follows that the vectors $\{\hat{F}^{1/2} \varphi_k^{(0)}\}$ span the space of zero modes $\{\varphi_k^{(0)}\}$ of $M$ and conversely, the $\{\hat{F}^{-1/2} \varphi_k^{(0)}\}$ span the space of zero modes $\{\varphi_k^{(0)}\}$ of $\det(D_{YM}^{-1})$. From the orthogonality of the $\varphi_k$ and the $\varphi_k$ then follows

$$\varphi_i^{(0)T} \hat{F}^{1/2} \varphi_k^{(0)} = 0 \quad \text{and} \quad \varphi_i^{(0)T} \hat{F}^{-1/2} \varphi_k^{(0)} = 0 \quad (48)$$

respectively, implying that we may write

$$\hat{F}^{1/2} \varphi_i^{(0)} = U_{lm} \varphi_m^{(0)} \quad \text{and} \quad \hat{F}^{-1/2} \varphi_i^{(0)} = O_{lm} \varphi_m^{(0)}. \quad (49)$$

For later use we calculate the determinant of the matrices $U$ and $O$. Due to the orthonormality of the eigenvectors $\varphi_i$ we have

$$\varphi_i^{(0)T} \varphi_k^{(0)} = U_{il} U_{km} \varphi_l^{(0)T} \hat{F}^{-1} \varphi_m^{(0)} = \delta_{ik}, \quad \varphi_i^{(0)T} \varphi_k^{(0)} = O_{il} O_{km} \varphi_l^{(0)T} \hat{F} \varphi_m^{(0)} = \delta_{ik} \quad (50)$$

and thus

$$1 = \det U \det(\hat{F}) \det U^T \quad \text{and} \quad 1 = \det O \det(\hat{F}) \det O^T, \quad (51)$$

implying

$$\det(\hat{F}) = (\det O)^{-2}, \quad \text{and} \quad \det(\hat{F}) = (\det U)^2. \quad (52)$$
In order to show that
\[ \text{Det} \hat{F} = \text{Det}' \hat{F} \text{ Det}^{(0)} \hat{F} , \] (53)
we note that the orthonormal sets \( \varphi_i \) and \( \phi_k \) are related by an orthogonal transformation. Therefore we may write
\[ \text{Det} \hat{F}^{1/2} = \text{Det} \begin{pmatrix} \varphi_i^T, \varphi_i^{(0)T} \end{pmatrix} \hat{F}^{1/2} \begin{pmatrix} \phi_k^T \\ \phi_k \end{pmatrix} \] . (54)

In view of (48) the matrix on the right hand side is of triangular block form.
\[ \text{Det} \hat{F}^{1/2} = \text{Det} \begin{pmatrix} \phi_i^T, \phi_i^{(0)T} \end{pmatrix} \hat{F}^{1/2} \begin{pmatrix} \phi_k^T \\ \phi_k \end{pmatrix} \]
\[ = \text{Det} \begin{pmatrix} O_m \varphi_m^{(0)T} \end{pmatrix} \text{ Det} \begin{pmatrix} \phi_i^{(0)T} \end{pmatrix} \]
\[ = \text{Det} O \text{ Det}' \hat{F} \text{ Det} U \]
\[ = \left( \text{Det}' \hat{F} \right)^{1/2} \left( \text{Det}^{(0)} \hat{F} \right)^{1/2} \] , (55)

An analogous manipulation, again using (48) and (49), shows that the determinant \( \text{Det}' D^{-1}_{YM} \) in (47) factorizes
\[ \text{Det}' D^{-1}_{YM} = \text{Det} \begin{pmatrix} \varphi_k^T \hat{F}^{1/2} \phi_m \phi_m^{(0)T} M \phi_n \phi_n^{(0)T} \hat{F}^{1/2} \varphi_i \end{pmatrix} \]
\[ = \text{Det} O \text{ Det}' \hat{F} \text{ Det}' M \text{ Det}' \hat{F} \text{ Det} O T = \text{Det}' \hat{F} \text{ Det}' M \] . (56)

This completes the proof of (47), which establishes explicitly the equivalence of the field strength formulation and the standard formulation at the semiclassical level.

In the so called field strength approach of [2] the \( (\text{Det} \hat{\chi})^{-1/2} \) arising from the integration over the gauge field is included into an effective action
\[ S_{FSA} = \frac{1}{4} \int d^4x \left\{ \chi^2 + \frac{\mu^4}{2} \text{tr} \ln \frac{i}{g} \hat{\chi} / \mu^2 + \frac{i}{2g} \chi F(J) \right\} \] (57)
where the scale \( \mu \) arises from the regularizations of \( \text{Tr} \ln \frac{i}{g} \hat{\chi} \). Due to the appearance of the scale this effective action does no longer tolerate instantons as stationary points [11] as one might have expected since instantons have a free scale (size) parameter. This was explicitly shown already in [2] for t’Hooft-Polyakov instantons. Instead of instantons the effective action (57) has (up to gauge transformations) constant solutions \( \chi = -iG \), which can be interpreted as instanton solids [11]. If one considers fluctuations around these constant solutions the propagator of the fluctuations is given by [3]
\[ D^{-1}_{FSA} = 2 (1 + C) + K \] (58)
where $K$ is defined by (27) and the extra term

$$C_{ij}^{ab} = \frac{1}{2} \text{tr} \left[ \hat{G}^{-1} T^a Z^i \hat{G}^{-1} T^b Z^j \right] (T^a)^{bc} = f^{abc}$$

arises from the $\text{Tr} \ln \frac{1}{g} \hat{\chi}$ term in the effective action (57). For a constant background field the additional zero modes $\psi^{(0)}$ of $\hat{K}$ found above in the instanton background correspond to non-propagating (non-dynamical) modes in the field strength formulation. This also reflects the fact that the field strength formulation, although using the larger number of field variables $\chi^a_{\mu\nu}$, contains the same number of propagating (dynamical) modes as the standard formulation as was already observed in [3].

For large momenta $p^2$ the term $C_{ij}^{ab}$ is however negligible compared to the $p$-dependent term $K(p)$ and the propagator of the fluctuations in the field strength approach $D_{FSA}$ (58) reduces to $D_{FS}$ (26), which we have shown to yield the same quantum effects as the standard propagator. This implies that the FSA yields the same asymptotic ($p^2 \to \infty$) gluon propagator ($\sim 1/p^2$) as the standard formulation as will be explicitly demonstrated elsewhere [12].

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