MILNOR FIBRATION AT INFINITY FOR MIXED POLYNOMIALS

YING CHEN

Abstract. We study the existence of Milnor fibration on a big enough sphere at infinity for a mixed polynomial \( f : \mathbb{R}^{2n} \to \mathbb{R}^2 \). By using strong non-degeneracy condition, we prove a counterpart of Némethi and Zaharia’s fibration theorem. In particular, we obtain a global version of Oka’s fibration theorem for strongly non-degenerate and convenient mixed polynomials.

1. Introduction

In the local case of germs of holomorphic polynomial functions with isolated singularities, it is well known that there exists a locally trivial fibration \( \varphi := \frac{f}{|f|} : S_r \setminus f^{-1}(0) \to S^1 \) in a sufficiently small sphere which is called Milnor fibration, see [Mi]. Unfortunately, in the global case of holomorphic polynomials, the Milnor fibration \( \frac{f}{|f|} \) at infinity does not exist in general. There are some special cases where \( f : \mathbb{C}^n \to \mathbb{C} \) has no atypical values at infinity, for instance: “convenient polynomials with non-degenerate Newton principal part at infinity” (Kouchnirenko [Ku]), polynomials which are “tame” (Broughton [Br1], [Br2]), “quasi-tame” (Némethi [Ne1], [Ne2]). In these cases, the Milnor fibration \( \frac{f}{|f|} \) at infinity exists in a sufficiently large sphere which is equivalent to the fibration \( f : f^{-1}(S^1_R) \to S^1_R \) for \( R \) sufficiently. In [NZ2], Némethi and Zaharia considered a special class of holomorphic polynomials called “semitame” whose atypical values are contained in \( \{0\} \). It was shown that for semitame polynomials, the Milnor fibration \( \frac{f}{|f|} \) at infinity exists. When \( n = 2 \), A.Bodin in [Bo] proved that the Milnor fibration \( \frac{f}{|f|} \) at infinity exists if and only if \( f \) is semitame.

Recently, Oka introduced the terminology of “mixed polynomials” which is a polynomial function \( \mathbb{C}^n \to \mathbb{C} \) with variables \( z \) and \( \bar{z} \), therefore a real polynomial application \( \mathbb{R}^{2n} \to \mathbb{R}^2 \). By defining non-degeneracy conditions for mixed function germs, Oka showed in [Oka2, Theorem 29, 33, 36] that: for a strongly non-degenerate convenient mixed function germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), there exist positive numbers \( r_0, \delta_0 \) and \( \delta \ll \delta_0 \), such that for any \( r \leq r_0 \),

\[
    f : f^{-1}(D^*_\delta) \cap B^*_{r} \to D^*_\delta
\]

is locally trivial fibrations and the topological isomorphism class does not depend on the choice of \( r \) and \( \delta_0 \). Moreover, \( \varphi := \frac{f}{|f|} : S^{2n-1}_r \setminus K_r \to S^1 \) is also a locally trivial fibration which is equivalent to the above fibration. For mixed polynomials, one can also ask under
which condition does the Milnor fibration \( \frac{f}{|f|} \) at infinity exist? In this paper, we get approach to this problem by using the strong non-degeneracy condition at infinity defined in [CT]. Consider a mixed polynomial \( f : \mathbb{C}^n \to \mathbb{C} \). Inspired by Oka’s construction in the local case, we prove a Némethi and Zaharia type fibration:

**Theorem 1.1.** If \( f \) is a Newton strongly non-degenerate mixed polynomial, then there exists \( \exists \delta_0 > 0 \) and \( R_0 > 0 \) such that for any \( \delta \geq \delta_0 \) and \( R > R_0 \)

\[
\frac{f}{|f|} : S^{2n-1}_R \setminus f^{-1}(D_\delta) \longrightarrow S^1
\]

is a locally trivial fibration for \( R \geq R_0 \) and is equivalent to the global fibration

\[
f_1 : f^{-1}(S^1_\delta) \to S^1_\delta.
\]

Unlike in the semitame setting of holomorphic case, we don’t have to suppose any other condition for atypical values of \( f \) in the above theorem. As a consequence of the above theorem, we get the following global version of [Oka2, Theorem 29, 33, 36]:

**Corollary 1.2.** If \( f \) is a Newton strongly non-degenerate convenient mixed polynomial, then there exists \( R_0 > 0 \) such that for all \( R \geq R_0 \) the Milnor fibration at infinity

\[
\frac{f}{|f|} : S^{2n-1}_R \setminus K \longrightarrow S^1
\]

exists and is equivalent to the global fibration

\[
f_1 : f^{-1}(S^1_\delta) \to S^1_\delta
\]

where \( \delta > 0 \) is sufficient large.

In this paper, we will review some basic definitions and properties of mixed polynomials in Section 2. In order to get an effective estimation of atypical values of \( \frac{f}{|f|} \), we define the \( \rho \)-regularity for \( \frac{f}{|f|} \) in Section 3, which allows us to get a type of formulation like [CT] Theorem 1.1]. The proof of Theorem 1.1 and Corollary 1.2 will be given in Section 4. Our example 4.3 shows that the semitame condition is not sufficient to insure the existence of the Milnor fibration \( \frac{f}{|f|} \) at infinity in the mixed setting.

2. Preliminaries

2.1. Mixed singularity and homogeneous polynomials. Let \( f := (g, h) : \mathbb{R}^{2n} \to \mathbb{R}^2 \) be a polynomial application, where \( g(x_1, \ldots, y_n) \) and \( h(x_1, \ldots, y_n) \) are real polynomials. By writing \( z = x + iy \in \mathbb{C}^n \), where \( z_k = x_k + iy_k \) for \( k = 1, 2, \ldots, n \), we get a polynomial function \( f : \mathbb{C}^n \to \mathbb{C} \) in variables \( z \) and \( \bar{z} \), namely \( f(z, \bar{z}) := g(z + \bar{z}, \frac{z - \bar{z}}{2i}) + ih(z + \bar{z}, \frac{z - \bar{z}}{2i}) \), and reciprocally for a polynomial function \( f : \mathbb{C}^n \to \mathbb{C} \) in variables \( z \) and \( \bar{z} \), we can consider it as a polynomial application (Re\( f \), Im\( f \)). Then \( f \) is called a mixed polynomial, after [Oka2]. We write \( f \) as follows:

\[
f(z, \bar{z}) = \sum_{\nu, \mu} c_{\nu, \mu} z^\nu \bar{z}^\mu
\]
where \( c_{v, \mu} \neq 0 \), \( z'' := z''_1 \cdots z''_n \) and \( \overline{z''} := \overline{z''_1} \cdots \overline{z''_n} \) for n-tuples \( v = (v_1, \ldots, v_n) \), \( \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n \). In the sequel, given a mixed polynomial \( f \), we consider \( f \) as in the form of equation (1).

For a mixed polynomial \( f \), we shall often use derivation with respect to \( z \) and \( \overline{z} \) such as in the following notations:

\[
df := \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right), \quad \overline{\partial} f := \left( \frac{\partial f}{\partial \overline{z}_1}, \ldots, \frac{\partial f}{\partial \overline{z}_n} \right)
\]

**Definition 2.1.** We call \( w \) a mixed singularity of \( f : \mathbb{C}^n \to \mathbb{C} \), if \( w \) is a critical point of the mapping \( f := (g, h) : \mathbb{R}^{2n} \to \mathbb{R}^2 \).

By abuse of notation, we continue to denote the set of mixed singularities for a mixed polynomial \( f \) by \( \text{Sing} f \). The next proposition give us a straight way to calculate the locus of mixed singularities.

**Proposition 2.2.** [Oka2, Proposition 1] Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a mixed polynomial. Then \( w \in \mathbb{C}^n \) is a mixed singularity of \( f \) if and only if there exists a complex number \( \lambda \) with \( |\lambda| = 1 \) such that \( \overline{\partial} f = \lambda \overline{df} \).

For mixed polynomials, we have two notions of homogeneous polynomials introduced in [G] and [Oka1].

**Definition 2.3.** A mixed polynomial \( f : \mathbb{C}^n \to \mathbb{C} \) is called **radial weighted homogeneous** if there exist \( n \) integers \( q_1, \ldots, q_n \) with gcd\((q_1, \ldots, q_n) = 1 \) and a positive integer \( m_r \) such that \( \sum_{j=1}^{n} q_j (v_j + \mu_j) = m_r \) for every n-tuples \( v \) and \( \mu \). We call \((q_1, \ldots, q_n)\) the radial weight of \( f \) and \( m_r \) the radial degree of \( f \). More precisely, \( f \) is radial weighted homogeneous of type \((q_1, \ldots, q_n; m_r)\) if and only if it verifies the following equation for all \( t \in \mathbb{R}^+ = \mathbb{R} \setminus \{0\} \):

\[
f(t \circ z) = (t^{q_1}z_1, \ldots, t^{q_n}z_n, t^{\mu_1}\overline{z}_1, \ldots, t^{\mu_n}\overline{z}_n) = t^{m_r} f(z, \overline{z}).
\]

From Definition 2.3, we see that if \( f := (g, h) : \mathbb{R}^{2n} \to \mathbb{R}^2 \) is a radially weighted homogeneous mixed polynomial, then \( g \) and \( h \) are real weighted homogeneous polynomial with the same weights and degrees as \( f \).

**Definition 2.4.** A mixed polynomial \( f : \mathbb{C}^n \to \mathbb{C} \) is called **polar weighted homogeneous** if there exist \( n \) integers \( p_1, \ldots, p_n \) with gcd\((p_1, \ldots, p_n) = 1 \) and a positive integer \( m_p \) such that \( \sum_{j=1}^{n} p_j (v_j - \mu_j) = m_p \) for every n-tuples \( v \) and \( \mu \). We call \((p_1, \ldots, p_n)\) the polar weight of \( f \) and \( m_p \) the polar degree of \( f \). More precisely, \( f \) is polar weighted homogeneous of type \((p_1, \ldots, p_n; m_p)\) if and only if it verifies the following equation for all \( \lambda \in S^1 \):

\[
f(\lambda \circ z) = f(\lambda^{p_1}z_1, \ldots, \lambda^{p_n}z_n, \lambda^{-p_1}\overline{z}_1, \ldots, \lambda^{-p_n}\overline{z}_n) = \lambda^{m_p} f(z, \overline{z}).
\]

**Example 2.5.** Let \( f, g : \mathbb{C}^2 \to \mathbb{C} \), \( f(x, y) = |x|^2 + |y|^2 \) and \( g(x, y) = x^2 + x^4y^2 + y^2 \). We see that \( f \) is a radial weighted homogeneous polynomial of radial weight \((1, 1)\) and degree \(2 \), but \( f \) is not polar weighted homogeneous. \( g \) is a polar weighted homogeneous polynomial of polar weight \((1, 1)\) and degree \(2 \), but \( g \) is not radial weighted homogeneous.
2.2. Newton non-degeneracy at infinity. In this section, we review the definitions of Newton polyhedron and non-degeneracy conditions introduced in [CT]. Let \( f \) be a mixed polynomial:

**Definition 2.6.** We call \( \text{supp}(f) = \{ \nu + \mu \in \mathbb{N}^n \mid c_{\nu, \mu} \neq 0 \} \) the support of \( f \). We say that \( f \) is convenient if the intersection of \( \text{supp}(f) \) with each coordinate axis is non-empty. We denote by \( \text{supp}(f) \) the convex hull of the set \( \text{supp}(f) \setminus \{0\} \). The Newton polyhedron of a mixed polynomial \( f \), denoted by \( \Gamma_0(f) \), is the convex hull of the set \( \{0\} \cup \text{supp}(f) \). The Newton boundary at infinity, denoted by \( \Gamma^+(f) \), is the union of the faces of the polyhedron \( \Gamma_0(f) \) which do not contain the origin. By “face” we mean face of any dimension.

**Definition 2.7.** For any face \( \Delta \subseteq \text{supp}(f) \), we denote the restriction of \( f \) to \( \Delta \cap \text{supp}(f) \) by \( f_\Delta := \sum_{\nu + \mu \in \Delta \cap \text{supp}(f)} c_{\nu, \mu} z^{\nu} \overline{z}^\mu \). The mixed polynomial \( f \) is called non-degenerate if \( \text{Sing} f_\Delta \cap f_\Delta^{-1}(0) \cap \mathbb{C}^n = \emptyset \), for each face \( \Delta \) of \( \Gamma^+(f) \). We say that \( f \) is Newton strongly non-degenerate if \( \text{Sing} f_\Delta \cap \mathbb{C}^n = \emptyset \) for any face \( \Delta \) of \( \Gamma^+(f) \).

It is easily seen that these two non-degeneracy conditions are not equivalent, but they coincide in the holomorphic setting. Let us recall the definition of bad faces for mixed polynomials. (See also [NZ1], [CT].)

**Definition 2.8.** A face \( \Delta \subseteq \text{supp}(f) \) is called bad if:

(i) there exists a hyperplane \( H \subseteq \mathbb{R}^n \) with equation \( a_1 x_1 + \cdots + a_n x_n = 0 \) (where \( x_1, \ldots, x_n \) are the coordinates of \( \mathbb{R}^n \)) such that:

(a) there exist \( i \) and \( j \) with \( a_i < 0 \) and \( a_j > 0 \),

(b) \( H \cap \text{supp}(f) = \Delta \).

Let \( \mathcal{B} \) denote the set of bad faces of \( \text{supp}(f) \). A face \( \Delta \in \mathcal{B} \) is called strictly bad if it satisfies in addition the following condition and the set of strictly bad faces of \( \text{supp}(f) \) will be denoted by \( \mathcal{S} \mathcal{B} \):

(ii) the affine subspace of the same dimension spanned by \( \Delta \) contains the origin.

Let us review here the notions of Milnor set and asymptotic \( \rho \)-non-regular values.

**Definition 2.9.** The Milnor set of a mixed polynomial \( f \) is

\[
M(f) = \{ z \in \mathbb{C}^n \mid \exists \lambda \in \mathbb{R} \text{ and } \mu \in \mathbb{C}^* \text{ such that } \lambda z = \mu d f(z, \overline{z}) + \overline{\mu} \overline{f}(z, \overline{z}) \}.
\]

**Definition 2.10.** The set of asymptotic \( \rho \)-non-regular values of a mixed polynomial \( f \) is

\[
S(f) = \{ c \in \mathbb{C} \mid \exists \{ z_k \}_{k \in \mathbb{N}} \subset M(f), \lim_{k \to \infty} \| z_k \| = \infty \text{ and } \lim_{k \to \infty} f(z_k, \overline{z}_k) = c \}.
\]

3. Approximation of atypical values of \( \frac{f}{|f|} \)

Let us denote by \( \varphi \) the function \( \frac{f}{|f|} : \mathbb{C}^n \setminus V(f) \to S_1 \) where \( V(f) = f^{-1}(0) \). To simplify notation, we continue to write \( d\varphi \) and \( \overline{d\varphi} \) specifically for the partial derivatives of the variables \( z \) and \( \overline{z} \).
Lemma 3.1. For \( z \in \mathbb{C}^n \setminus V(f) \), the fibre \( \varphi^{-1}(\varphi(z, \overline{z})) \) does not intersect transversely the sphere \( S_{\|z\|}^{2n-1} \) at \( z \in \mathbb{C}^n \), if and only if there exists \( \lambda \in \mathbb{R} \), such that

\[
(2) \quad \lambda z = i \int \overline{d}f(z, \overline{z}) - i f \overline{d}f(z, \overline{z}).
\]

In particular, \( \text{Sing} \varphi = \{ z \in \mathbb{C}^n \setminus V(f) \mid \overline{d}f(z, \overline{z}) = f \overline{d}f(z, \overline{z}) \} \).

Proof. Observe first \( \varphi = -\text{Re}(i \log f) \). By [CT] Lemma 2.1, the non-transversality of the fiber \( \varphi^{-1}(\varphi(z, \overline{z})) \) and the sphere \( S_{\|z\|}^{2n-1} \) implies:

\[
(3) \quad \gamma z = \mu \overline{d}\varphi(z, \overline{z}) + \overline{\mu d}\varphi(z, \overline{z}),
\]

for some \( \gamma \in \mathbb{R} \) and \( \mu \in \mathbb{R}^* \). By definition of \( \overline{d}\varphi \) and \( d\varphi \), we have:

\[
\overline{d}\varphi(z, \overline{z}) = -\overline{\text{dRe}(i \log f)} = i \frac{\overline{d}f(z, \overline{z})}{f},
\]

\[
\overline{d}\varphi(z, \overline{z}) = -\overline{\text{dRe}(i \log f)} = -i \frac{\overline{d}f(z, \overline{z})}{f}.
\]

Multiplying the two sides of (3) by \( |f|^2 \), we conclude (2), where \( \lambda = \frac{\mu}{\overline{\mu}} |f|^2 \in \mathbb{R} \). In particular, taking \( \lambda = 0 \) in (2), we obtain \( \text{Sing} \varphi \). \( \square \)

Combining with the above lemma, we are led to define \( \rho \)-regularity for \( \varphi \). (In general case, this regularity condition is defined in [ACT])

Definition 3.2. We call \( \rho \)-non-regular locus of \( \varphi \) the semi-algebraic set:

\[
M(\varphi) = \{ z \in \mathbb{C}^n \setminus V(f) \mid \exists \lambda \in \mathbb{R}, \text{ such that } \lambda z = i \int \overline{d}f(z, \overline{z}) - i f \overline{d}f(z, \overline{z}) \}.
\]

and we call asymptotic \( \rho \)-non-regular values of \( \int f \overline{f} \) the set:

\[
S(\varphi) = \{ c \in S^1 \mid \exists \{ z_k \}_{k \in \mathbb{N}} \subset M(\varphi), \lim_{k \to \infty} \| z_k \| = \infty \text{ and } \lim_{k \to \infty} \varphi(z_k, \overline{z_k}) = c \}.
\]

Recall the notations of Milnor set \( M(f) \) and the asymptotic \( \rho \)-non-regular set \( S(f) \). The above definition enables us to obtain the following structure result of \( S(\varphi) \).

Lemma 3.3. \( S(\varphi) \) is semi-algebraic and \( M(\varphi) \subset M(f) \setminus V(f) \).

Proof. The inclusion of \( M(\varphi) \subset M(f) \setminus V(f) \) follows from the definitions of \( 2.39 \) and \( 3.2 \). Since \( M(\varphi) \) is a semi-algebraic set, we now proceed analogously to the proof of [CT] Proposition 2.1] and we see that \( S(\varphi) \) is semi-algebraic. \( \square \)

Our next proposition shows that under some homogeneous condition, \( \text{Sing} \varphi \) could be equal to \( M(\varphi) \).

Proposition 3.4. If \( f \) is a mixed radial weighted homogeneous polynomial and not constant, then \( \text{Sing} \varphi = \text{Sing} f \setminus V(f) = M(\varphi) \).

Proof. Let us denote the radial weights of \( f \) by \( q_1, \ldots, q_n \) and the radial degree of \( f \) by \( m_r \), where \( q_1, \ldots, q_n \in \mathbb{Z} \) and \( m_r \neq 0 \). First, we have \( \text{Sing} \varphi \subset \text{Sing} f \setminus V(f) \) and
Singφ ⊂ M(φ). To prove the equality, let a ∈ Sing f and f(a, ¯a) ≠ 0. Therefore ∃ λ ∈ S1 such that for 1 ≤ i ≤ n:

\[
(4) \quad \frac{\partial f}{\partial z_i}(a, \bar{a}) = \lambda \frac{\partial f}{\partial z_i}(a, \bar{a}).
\]

Since f is radial weighted homogeneous, by Euler’s lemma, we have:

\[
(5) \quad \sum_{i=1}^{n} q_i a_i \frac{\partial f}{\partial z_i}(a, \bar{a}) + \sum_{i=1}^{n} q_i \bar{a}_i \frac{\partial f}{\partial z_i}(a, \bar{a}) = m_r f(a, \bar{a}).
\]

Let A = \sum_{i=1}^{n} q_i a_i \frac{\partial f}{\partial z_i}(a, \bar{a}) and B = \sum_{i=1}^{n} q_i \bar{a}_i \frac{\partial f}{\partial z_i}(a, \bar{a}). Multiplying (4) by q_i \bar{a}_i, we obtain:

\[
(6) \quad A = \lambda B
\]

which implies \( A\bar{A} = B\bar{B} \) since \( \lambda \in S^1 \). From (5), (6) and \( f(a, \bar{a}) \neq 0 \), we therefore get \( AB \neq 0 \). Consequently,

\[
(7) \quad \frac{f(a, \bar{a})}{f(a, \bar{a})} = \frac{A + B}{A + B} = \frac{B\bar{A} + B\bar{A}}{B(A + B)} = \frac{B\bar{A} + A\bar{A}}{B(A + B)} = \lambda
\]

which proves that a ∈ Singφ from (4). Thus, we have Singφ = Sing f \( \setminus V(f) \). Using Euler vector field as in the proof of [ACT Proposition 3.2], we have M(φ) ⊂ Sing f \( \setminus V(f) \). This finishes the proof.

For simplicity of notation, we write \( \varphi_\Delta := \frac{f_\Delta}{|f_\Delta|} \) for the restriction of \( \frac{f}{|f|} \), where \( \Delta \) is a face of \( \text{supp}(f) \).

**Theorem 3.5.** If f is Newton strongly non-degenerate at infinity for any face of \( \text{supp}(f) \), then M(φ) is bounded and S(φ) = ∅.

**Proof.** Assume that M(φ) is not bounded, then by Curve selection lemma at infinity ([NZ Lemma 2], [CT Lemma 2.3]), there exists \( z(t) \) of M(φ) a real analytic path defined on a small enough interval \( ]0, \varepsilon[ \) such that

\[
\lim_{t \to 0} ||z(t)|| = \infty.
\]

Since \( z(t) \subset M(\varphi) \), there exists a real analytic curve \( \lambda(t) \), such that for \( t \in ]0, \varepsilon[ \) we have:

\[
(8) \quad \lambda(t)z(t) = if \overline{d}(z(t), z(t)) - if \overline{d}(z(t), z(t)).
\]

Suppose here \( \lambda(t) \neq 0 \) and let \( I = \{i \mid z_i(t) \neq 0\} \). Then I ≠ ∅ since \( \lim_{t \to 0} ||z(t)|| = \infty \).

Assuming that I = \( \{1, \ldots, m\} \), we write the expansions of \( f(z(t), \bar{z}(t)) \), \( z(t) \) and \( \lambda(t) \) explicitly as follows:

\[
(9) \quad z_i(t) = a_i t^{p_i} + \text{h.o.t.}, \quad \text{where } a_i \neq 0, p_i \in \mathbb{Z}, \ 1 \leq i \leq m.
\]

\[
(10) \quad f(z(t), \bar{z}(t)) = \begin{cases} bt^\delta + \text{h.o.t.} & \text{where } b \in \mathbb{C}^*, \delta \neq 0, \text{ if } \lim_{t \to 0} f(z(t), \bar{z}(t)) = 0 \text{ or } \infty \\ c + bt^\delta + \text{h.o.t.} & \text{where } c, b \in \mathbb{C}^*, \delta \neq 0, \text{ if } \lim_{t \to 0} f(z(t), \bar{z}(t)) = c. \end{cases}
\]

\[
(11) \quad \lambda(t) = \lambda_0 t^\gamma + \text{h.o.t.}, \quad \text{where } \lambda_0 \in \mathbb{R}^*, \gamma \in \mathbb{Z}, \lambda(t) \in \mathbb{R}.
\]
Set $a = (a_1, \ldots, a_m) \in \mathbb{C}^+I$, $P = (p_1, \ldots, p_m) \in \mathbb{R}^m$ and consider the linear function $l_P = \sum_{i=1}^m p_ix_i$ defined on $\text{supp}(f^I)$. Let $\Delta$ be the maximal face of $\text{supp}(f^I)$ where $l_P$ takes its minimal value, say this value is $d_P$. We have:

\begin{equation}
 f(z(t), z(t)) = f^I_\Delta(a, \bar{a}) t^{d_P} + \text{o.h.t.}
\end{equation}

Let us discuss the following two cases:

(I). If $\lim_{t \to 0} f(z(t), z(t)) = 0$ or $\infty$, we get $d_P \leq \delta$. Since $\lim_{t \to 0} \|z(t)\| = \infty$, this implies $p := \min_{j \in I} p_j < 0$. Now using (9)-(12) in (8), we get:

\begin{equation}
 \frac{\partial f^I_\Delta}{\partial \bar{z}_i}(a, \bar{a}) - i b \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) = \left\{
 \begin{array}{ll}
 \lambda_0 a_i, & \text{if } d_P - p_i + \delta = p_i + \gamma. \\
 0, & \text{if } d_P - p_i + \delta < p_i + \gamma.
 \end{array}
 \right.
\end{equation}

Let $J = \{ j \mid d_P - p_j + \delta = p_j + \gamma \}$. We suppose $J \neq \emptyset$ which gives $J = \{ j \mid p_j = \min_{1 \leq j \leq m} \{ p_j \} < 0 \}$. Consider the derivative of $f(z(t), z(t))$ with respect to $t$. On one hand, we have:

\begin{equation}
 \frac{df(z(t), z(t))}{dt} = b \delta t^{\delta - 1} + \text{o.h.t.}
\end{equation}

On the other hand, we have:

\begin{equation}
 \frac{df(z(t), z(t))}{dt} = \sum_{i=1}^m \left( \frac{\partial f^I_\Delta}{\partial z_i} \frac{\partial z_i}{\partial t} + \frac{\partial f^I_\Delta}{\partial \bar{z}_i} \frac{\partial \bar{z}_i}{\partial t} \right) = \left[ \frac{\partial f^I_\Delta}{\partial a_i} \frac{\partial f^I_\Delta}{\partial a_i} + \frac{\partial f^I_\Delta}{\partial \bar{a}_i} \frac{\partial f^I_\Delta}{\partial \bar{a}_i} \right] t^{d_P - 1} + \text{o.h.t.}
\end{equation}

where $P = (p_1 a_1, \ldots, p_m a_m)$. From (13), we obtain:

\begin{equation}
 \Re \left( \frac{\partial f^I_\Delta}{\partial a_i} - i b \frac{\partial f^I_\Delta}{\partial \bar{a}_i} \right) = \sum_{i \in J} \lambda_0 \delta \| a_j \|^2 \neq 0.
\end{equation}

If $d_P < \delta$, then comparing the orders of the expansions (14) and (15) with respect to $t$, we have $\Re \left( \frac{\partial f^I_\Delta}{\partial a_i} - i b \frac{\partial f^I_\Delta}{\partial \bar{a}_i} \right) = 0$ and $f^I_\Delta(a, \bar{a}) = 0$. Multiplying (13) by $i b$ and comparing the real parts of the equality, we obtain a contradiction with (16). If $d_P = \delta$, then by (14), we have $\Re \left( \frac{\partial f^I_\Delta}{\partial a_i} - i b \frac{\partial f^I_\Delta}{\partial \bar{a}_i} \right) = \Re(i b^2 \delta) = 0$, which contradicts (16). It follows that $J = \emptyset$. Hence $a \in \mathbb{C}^+I$ is a singularity of $f^I_\Delta$ and $f^I_\Delta(a, \bar{a}) = b$. By [CT, Remark 3.3], this is contrary to the strong non-degeneracy of $f^I$. 

(II). If $\lim_{t \to 0} f(z(t), z(t)) = c \in \mathbb{C}^*$, comparing the orders of the expansions (11) and (13) with respect to $t$, we have $d_P < \delta$. Now using (10)-(13) in (9), we get:

\begin{equation}
 i c \frac{\partial f^I_\Delta}{\partial \bar{z}_i}(a, \bar{a}) - i c \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) = \left\{
 \begin{array}{ll}
 \lambda_0 a_i, & \text{if } d_P - p_i = p_i + \gamma. \\
 0, & \text{if } d_P - p_i < p_i + \gamma.
 \end{array}
 \right.
\end{equation}
Let \( J = \{ j \mid d_p - p_j = p_j + \gamma \} \). We suppose \( J \neq \emptyset \) which implies \( J = \{ j \mid p_j = p = \min_{1 \leq j \leq m} \{ p_j \} < 0 \} \). We derive \( f(z(t), \bar{z}(t)) \) with respect to \( t \). On one hand, we get (14). On the other hand, we have (15). From (17), we obtain:

\[
\text{Re} \left( \langle P_a, \overline{\alpha f^I_\Delta(a, \bar{a})} - ic\bar{f}^I_\Delta(a, \bar{a}) \rangle \right) = \sum_{i \in J} \lambda_0 p \| a_j \| ^2 \neq 0. \tag{18}
\]

Since \( d_p < \delta \), comparing the orders of with respect to \( t \) in (14) and (15), we have

\[
\left< P_a, \partial f^I_\Delta(a, \bar{a}) \right> + \left< P\bar{a}, \partial f^I_\Delta(a, \bar{a}) \right> = 0. \tag{19}
\]

Multiplying (15) by \( \overline{\alpha} \) and comparing the real parts, we obtain a contradiction with (18). It follows that \( J = \emptyset \). Hence \( a \in \mathbb{C}^*I \) is a singularity of \( f^I_\Delta \) and \( f^I_\Delta(a, \bar{a}) = 0 \). By [CT, Remark 3.3], this is contrary to the non-degeneracy of \( f^I \).

In general, if we do not assume the strong non-degeneracy of \( f \) and let \( A = (a, 1, 1, \ldots, 1) \) with the \( i^{th} \) coordinate \( z_i = 1 \) for \( i \notin I \), then we have the following conclusion:

(a) If \( d_p < \delta \), then \( A \) is a singularity of \( V(f^I_\Delta) \).

(b) If \( d_p = \delta \), then \( A \in \text{Sing } \varphi = \text{Sing } f^I_\Delta \setminus V(f^I_\Delta) \) by Proposition 3.4.

When \( \lambda(t) \equiv 0 \), by comparing the orders with respect to \( t \) in (19), we have:

\[
\begin{cases}
ib \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) - ib \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) = 0, & \text{if } \lim_{t \to 0} f(z(t), \bar{z}(t)) = 0 \text{ or } \infty.

ic \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) - ic \frac{\partial f^I_\Delta}{\partial z_i}(a, \bar{a}) = 0, & \text{if } \lim_{t \to 0} f(z(t), \bar{z}(t)) = c \in \mathbb{C}^*.
\end{cases}
\]

It follows that \( a \in \mathbb{C}^*I \) is a singularity of \( f^I_\Delta \). By [CT, Remark 3.3], this is contrary to the non-degeneracy of \( f^I \). Hence \( M(\varphi) \) is bounded and \( S(\varphi) = \emptyset \).

We now proceed to formulate the analogue of [CT, Theorem 1.1]. Recall the notation \( \mathcal{SB} \) the union of strictly bad faces of \( \text{supp}(f) \).

**Theorem 3.6.** Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a mixed polynomial. Suppose that \( f \) is Newton strongly non-degenerate polynomial which depends effectively on all the variables. Let \( f(0) = 0 \) and \( 0 \notin S(f) \). Then:

\[
S(\varphi) \subset \bigcup_{\Delta \in \mathcal{SB}} \varphi_{\Delta}(\text{Sing } \varphi_{\Delta} \cap \mathbb{C}^n).
\]

**Proof.** We use the same notations as in the proof of Theorem 3.5. For any \( c \in S(\varphi) \), by Curve selection Lemma at infinity, there exists \( z(t) \) of \( M(\varphi) \) a real analytic path defined on a small enough interval \( ]0, \varepsilon[ \) such that

\[
\lim_{t \to 0} \| z(t) \| = \infty, \text{ and } \lim_{t \to 0} \varphi(z(t), \bar{z}(t)) = c_0
\]

where either \( f(z(t), \bar{z}(t)) = bt^d + \text{h.o.t. and } d_p \leq \delta < 0, c_0 = \frac{b}{|t|} \), or \( f(z(t), \bar{z}(t)) = c + bt^d + \text{h.o.t. and } d_p \leq 0, c \in \mathbb{C}^*, c_0 = \frac{c}{|t|} \). Consider \( \lambda(t) \neq 0 \), if \( \text{ord}_t(f(z(t), \bar{z}(t))) < 0 \), we are in case (I) as in the proof of Theorem 3.5 then we get that \( a \in \mathbb{C}^*I \) is a singularity of \( f^I_\Delta \). If \( \text{ord}_t(f(z(t), \bar{z}(t))) = 0 \), we are in case (II) as in the proof of Theorem 3.5 then \( a \in \mathbb{C}^*I \) is a singularity of \( f^I_\Delta \).
Set $A = (a, 1, 1, \ldots, 1)$ with the $i^{th}$ coordinate $z_i = 1$ for $i \notin I$. By recalling the definition of Newton boundary at infinity for mixed polynomial, we have the following two cases:

(I). If $d_{p} < 0$, then, from [CT] Lemma 3.1, we conclude that $\triangle$ is a face of $\Gamma^+(f^{l})$. On the other hand since $a$ is a singularity of $f_{A}^{l}$, by Remark this contradicts the Newton strong non degeneracy of $f^{l}$.

(II). If $d_{p} = \delta = 0$, then, from [CT] Lemma 3.1, it follows that either $\triangle$ is a face of $\Gamma^+(f^{l})$ or $\triangle$ satisfies condition (ii) of Definition 2.8 Assume first $\triangle$ is a face of $\Gamma^+(f^{l})$, then we get the same contradiction as that in (I). Thus $\triangle$ verifies condition (ii) of Definition 2.8. We proceed to show that $\triangle$ is strictly bad face of $\text{supp}(f)$. Let us denote by $d$ the minimal value of the restriction of $l_{P}$ to $\text{supp}(f)$. Since $\text{supp}(f^{l}) = \text{supp}(f) \cap \mathbb{R}_+^{l}$, we have $d \leq d_{P} = 0$. Let $H$ be the hyperplane of the equation $\sum_{i=1}^{m} p_{i} x_{i} + q \sum_{i=m+1}^{n} x_{i} = 0$, where $q > -d + 1 > 0$. Hence for any $x = (x_{1}, \ldots, x_{n}) \in \text{supp}(f) \setminus \text{supp}(f^{l})$, the value of $\sum_{i=1}^{m} p_{i} x_{i} + q \sum_{i=m+1}^{n} x_{i}$ is positive. We therefore get $\triangle = \text{supp}(f^{l}) \cap H = \text{supp}(f) \cap H$. On the other hand, note that $p_{1} = p = \min_{1 \leq i \leq m} \{ p_{i} \} < 0$ and $q > 0$. If $\triangle$ does not satisfy condition (i)(a) of Definition 2.8 then we have $m = n$ and $p_{i} \leq 0$ for all $1 \leq i \leq n$. It follows that $f$ can not depend on $z_{1}$ otherwise $d_{P}$ will be negative. This contradicts the effectiveness of $f$. Hence we conclude that $\triangle$ is a strictly bad face of $\text{supp}(f)$. Since $d_{P} = 0$, we obtain $c = f_{A}^{l}(a, \pi) = f_{\triangle}(A, \overline{A}) \neq 0$. By $A \in \text{Sing} \varphi_{\triangle}$ and Proposition 3.4 we get $c_{0} \in \varphi_{\triangle}(\text{Sing} \varphi_{\triangle})$. When $\lambda(t) \equiv 0$, it follows that $a \in \mathbb{C}^{n}$ is a singularity of $f_{A}^{l}$ from (19). In the same manner as above reasoning, we get the desired conclusion. □

Remark 3.7. In particular, if a mixed polynomial $f$ is Newton strongly non-degenerate at infinity and convenient, then by [CT] Corollary 4.1, we have $S(f) = \emptyset$. Combining this conclusion with the above theorem, we get $S(\varphi) = \emptyset$ since $\mathcal{S} \mathcal{B} = \emptyset$.

4. Fibration at infinity

Recall that for a strongly non-degenerate polynomial $f$, we have the monodromy fibration:

$$f_{1} : f^{-1}(S_{1}^{l}) \to S_{1}^{l}$$

over some circle $S_{1}^{l}$ of radius $\delta$ which is sufficiently large. We define two vectors on $\mathbb{C}^{n} \setminus V(f)$:

$$v_{1}(z, \overline{z}) = \overline{\text{d} \log f(z, \overline{z}) + \text{d} \log f(z, \overline{z})}$$
$$v_{2}(z, \overline{z}) = i(\text{d} \log f(z, \overline{z}) - \overline{\text{d} \log f(z, \overline{z})})$$

which have the following geometrical meanings: $v_{1}(z, \overline{z})$ is the normal vector of $\text{log} |f|$ and $v_{2}(z, \overline{z})$ is the normal vector of $-i \text{log} |f|$. In order to prove Theorem 1.1 we shall first prove the following proposition.

Proposition 4.1. Under the same assumption as in Theorem 1.1, there exists $\delta_{2} > 0$ sufficiently large, such that for any $z$ of $\{ z \in \mathbb{C}^{n} \mid |f(z, \overline{z})| \geq \delta_{2} \}$ the there vectors

$$z, \ v_{1}(z, \overline{z}), \ v_{2}(z, \overline{z})$$
are either linearly independent over $\mathbb{R}$ or they are linearly dependent over $\mathbb{R}$ with the following relation

$$z = av_1(z, \overline{z}) + bv_2(z, \overline{z})$$

where $a > 0$.

**Proof.** Since $f$ is strongly non-degenerate at infinity, by [CT] Theorem 1.1, $f(\text{Sing } f) \cup S(f)$ is bounded. Let us suppose that $f(\text{Sing } f) \cup S(f) \subset D_{\delta_1}$. For $|f(z, \overline{z})|$ sufficiently large we shall prove either $z, v_1(z, \overline{z}), v_2(z, \overline{z})$ are linearly independent over $\mathbb{R}$ or $z = av_1(z, \overline{z}) + bv_2(z, \overline{z})$ where $ab \neq 0$. Assume that $z$ and $v_2(z, \overline{z})$ are linearly dependent over $\mathbb{R}$. By Curve selection Lemma at infinity, there exist two analytic paths $z(t) \subset \mathbb{C}^n$ and $\lambda(t) \subset \mathbb{R}$ defined on a small enough interval $]0, \varepsilon[$ such that

$$\lim_{t \to 0} \|z(t)\| = \infty, \lim_{t \to 0} f(z(t), \overline{z}(t)) = \infty.$$  

By Lemma 3.3 we have $z(t) \subset M(f)$. Thus $\lim f(z(t), \overline{z}(t)) = \infty$ contradicts our condition $f(\text{Sing } f) \cup S(f) \subset D_{\delta_1}$. For $|f(z, \overline{z})|$ sufficiently large, we have actually proved that $z$ and $v_2(z, \overline{z})$ are linearly independent over $\mathbb{R}$. Since $d\log f + \overline{d}\log f = \frac{1}{|f|^2}(fd\overline{f} + \overline{f}d\overline{f})$, a slightly change in the proof of the above linear independence shows that for $|f(z, \overline{z})|$ sufficiently large, $z$ and $v_1(z, \overline{z})$ are also linearly independent over $\mathbb{R}$. If $v_1(z, \overline{z}), v_2(z, \overline{z})$ are linearly dependent over $\mathbb{R}$, we have $z \in \text{Sing } f \setminus V(f)$. Hence $f(z(t), \overline{z}(t)) \subset f(\text{Sing } f)$. This contradicts the boundness of $f(\text{Sing } f)$. It follows that $v_1(z, \overline{z}), v_2(z, \overline{z})$ are linearly independent over $\mathbb{R}$ for $|f(z, \overline{z})|$ sufficiently large. We are reduce to proving the proposition for $a > 0$. In the remainder of the proof, we assume $a < 0$. By Curve selection Lemma at infinity, there exist the analytic curves $z(t) \in \mathbb{C}^n$, $a(t) < 0$ and $b(t) \in \mathbb{R}$ defined on a small enough interval $]0, \varepsilon[$ such that

$$\lim_{t \to 0} \|z(t)\| = \infty, \lim_{t \to 0} f(z(t), \overline{z}(t)) = \infty.$$  

(21) $$i(\overline{d}\log f - d\log f)(z(t), \overline{z}(t)) = \lambda(t)z(t).$$

By Lemma 3.3 we have $z(t) \subset M(f)$. Thus $\lim f(z(t), \overline{z}(t)) = \infty$ contradicts our condition $f(\text{Sing } f) \cup S(f) \subset D_{\delta_1}$. For $|f(z, \overline{z})|$ sufficiently large, we have actually proved that $z$ and $v_2(z, \overline{z})$ are linearly independent over $\mathbb{R}$. Since $d\log f + \overline{d}\log f = \frac{1}{|f|^2}(fd\overline{f} + \overline{f}d\overline{f})$, a slightly change in the proof of the above linear independence shows that for $|f(z, \overline{z})|$ sufficiently large, $z$ and $v_1(z, \overline{z})$ are also linearly independent over $\mathbb{R}$. If $v_1(z, \overline{z}), v_2(z, \overline{z})$ are linearly dependent over $\mathbb{R}$, we have $z \in \text{Sing } f \setminus V(f)$. Hence $f(z(t), \overline{z}(t)) \subset f(\text{Sing } f)$. This contradicts the boundness of $f(\text{Sing } f)$. It follows that $v_1(z, \overline{z}), v_2(z, \overline{z})$ are linearly independent over $\mathbb{R}$ for $|f(z, \overline{z})|$ sufficiently large. We are reduce to proving the proposition for $a > 0$. In the remainder of the proof, we assume $a < 0$. By Curve selection Lemma at infinity, there exist the analytic curves $z(t) \in \mathbb{C}^n$, $a(t) < 0$ and $b(t) \in \mathbb{R}$ defined on a small enough interval $]0, \varepsilon[$ such that

$$\lim_{t \to 0} \|z(t)\| = \infty, \lim_{t \to 0} f(z(t), \overline{z}(t)) = \infty.$$  

(23) $$z(t) = a(t)v_1(z, \overline{z})(t) + b(t)v_2(z, \overline{z})(t).$$

Let $I = \{i \mid z_i(t) \neq 0\}$. Without loss of generality we can assume $I = \{1, \ldots, m\}$, then we have:

$$z_i(t) = a_it^{p_i} + \text{h.o.t.}, \quad \text{where } a_i \neq 0, p_i \in \mathbb{Z}, i \in I.$$  

$$f(z(t), \overline{z}(t)) = bt^q + \text{h.o.t.}, \quad \text{where } b \in \mathbb{C}^*, q \in \mathbb{Z}, q < 0$$  

$$a(t) = \lambda_0 t^{v_0} + \text{h.o.t.}, \quad \text{where } \lambda_0 \in \mathbb{R}, v_0 \in \mathbb{Z}$$  

$$b(t) = \beta_0 t^{v_0} + \text{h.o.t.}, \quad \text{where } \beta_0 \in \mathbb{R}, v_0 \in \mathbb{Z}$$

where $|\lambda_0| + |\beta_0| \neq 0$. If $\lambda_0 \in \mathbb{R}^*$, then, by our assumption $a(t) < 0$, we have $\lambda_0 < 0$. To shorten notation, we write $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$, $P = (p_1, \ldots, p_m) \in \mathbb{R}^m$ and consider the linear function $l_P = \sum_{i=1}^m p_ix_i$ defined on supp$(f^1)$. Let $\Delta$ be the maximal face of supp$(f^1)$ where $l_P$ takes its minimal value, say this value is $d_P$. We have $d_P \leq$
By (24), we obtain:

\[
\lambda_0 \left( \frac{\partial f}{\partial z_i}(a, \overline{a}) \right) + \lambda_0 \left( \frac{\partial f}{\partial z_i}(a, \overline{a}) \right) - \frac{\partial f}{\partial z_i}(a, \overline{a}) \right) b = \begin{cases} a_i, & \text{if } d_P - p_i - q + v_0 = p_i, \\ 0, & \text{if } d_P - p_i - q + v_0 < p_i. \end{cases}
\]

Let \( J = \{ j \in I \mid d_P - p_j - q + v_0 = p_j \}. \) We observe \( J = \{ j \in I \mid p_j = \min_{j \in I} \{ p_j \} < 0 \}. \)

If \( J = \emptyset, \) then from (24), we have \( a \in \text{Sing} f_{\Delta}^\dagger. \) Since \( d_P < 0, \) by [CT, Lemma 3.1], we conclude that \( \Delta \) is a face of \( \Gamma^+(f^\dagger). \) This contradicts the Newton strongly non degeneracy of \( f^\dagger. \) Hence \( J \neq \emptyset. \) To deduce the contradiction, consider the following expansion:

\[
\frac{\lambda_0 + i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} + \frac{\lambda_0 - i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} \frac{d t}{d t} = 2\lambda_0 q t^{q-1} + \text{h.o.t.}
\]

We also have:

\[
\frac{d f}{d t}(a, \overline{a}) = \left[ \left\langle P a, \frac{d f}{d t}(a, \overline{a}) \right\rangle + \left\langle P a, \overline{d f}(a, \overline{a}) \right\rangle \right] t^{d_p - 1} + \text{h.o.t.}
\]

By (24), we obtain:

\[
\frac{\lambda_0 + i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} + \frac{\lambda_0 - i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} \frac{d t}{d t} = (2 \sum_{j \in J} p\|a_j\|^2) t^{d_p - 1} + \text{h.o.t.}
\]

Since \( d_P \leq q, \) comparing the two expansions of \( \frac{\lambda_0 + i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} + \frac{\lambda_0 - i \beta_0 \frac{d f}{d t}(a, \overline{a})}{b} \), it follows that \( d_P = q \) and \( \lambda_0 = \sum_{i \in J} \frac{p\|a_j\|^2}{q} > 0 \) from \( p < 0 \) and \( q < 0. \) This contradicts \( \lambda_0 < 0. \)

\[\square\]

**Remark 4.2.** In the holomorphic setting, the parallel results of this proposition are [Mi, Lemma 4.4] and [NZ2, Lemma 4 and Lemma 5]; in the mixed setting, this is a global analogue of [Oka2, Lemma 34].

**Proof of Theorem 1.1** The proof is done as in the case of a holomorphic polynomial. The strong non-degeneracy of \( f \) yields a global fibration:

\[ f_1 : f^{-1}(S^1_\delta) \rightarrow S^1_\delta \]

where \( \delta > 0 \) is sufficiently large. Since \( S^1_\delta \) is compact and \( f(S\text{Sing}(f)) \cup S(f) \) is bounded, there exists \( R_0 > 0 \) sufficiently large such that all the fibers intersect \( S_R \) transversely for any \( R \geq R_0. \) We therefore get the restriction

\[ f_1 : f^{-1}(S^1_\delta) \cap B_R \rightarrow S^1_\delta \]
which is equivalent to the global fibration. By Proposition 4.1 there exists a non-zero vector field \( \omega \) on \( N = \{ z \in B_R \mid |f(z, \bar{z})| \geq \delta \} \) such that
\[
\begin{align*}
\text{Re} \langle w(z), v_2(z, \bar{z}) \rangle &= 0 \\
\text{Re} \langle w(z), v_1(z, \bar{z}) \rangle &> 0 \\
\text{Re} \langle w(z), z \rangle &> 0.
\end{align*}
\]

Along the integral curve \( \gamma(t, z_0) \) of \( w \) with \( \gamma(0, z_0) = z_0 \in N \), it is easily seen that the argument of \( f(\gamma(t, z_0), \overline{\gamma(t, z_0)}) \) is constant and \( |f(\gamma(t, z_0), \overline{\gamma(t, z_0)})|, \|\gamma(t, z_0)\| \) are monotone increasing. Thus for every \( z_0 \in N \), there exists a unique \( h(z_0) \in S^{2n-1}_{\delta} \setminus f^{-1}(D_\delta) \) and \( t_0 \in \mathbb{R}_+ \) such that \( \|\gamma(t_0, h(z_0))\| = R \). Consequently, there is an isomorphism \( \phi : f^{-1}(S^1_\delta) \cap B_R \to S^{2n-1}_{\delta} \setminus f^{-1}(D_\delta) \). We therefore get \( f_1 : S^{2n-1}_{\delta} \setminus f^{-1}(D_\delta) \to S^1 \) a locally trivial fibration which is equivalent to the fibration \( f_1 : f^{-1}(S^1_\delta) \cap B_R \to S^1_\delta \). So \( f_1 : S^{2n-1}_{\delta} \setminus f^{-1}(D_\delta) \to S^1 \) is also equivalent to the global one. This completes our proof.

**Proof of Corollary 1.2.** From Remark 3.7 it follows that \( S(\varphi) = \emptyset \) and \( M(\varphi) \) is bounded. Thus we have \( f_1 : S^{2n-1}_{\delta} \setminus K \to S^1 \) is a locally trivial fibration. Note that the proof of Theorem 1.1 yields that this fibration is equivalent to the global fibration:
\[
f_1 : f^{-1}(S^1_\delta) \to S^1_\delta
\]

where \( \delta > 0 \) is sufficient large.

**Example 4.3.** [Oka2, Example 5 IV] Consider a mixed polynomial
\[
f(z, \bar{z}) = \frac{1}{4} z_1^2 - \frac{1}{4} \overline{z}_1^2 + z_1 \overline{z}_1 - (1 + i)(z_1 + z_2)(\overline{z}_1 + \overline{z}_2).
\]

Then we have:

(a) \( f \) is not Newton strongly non-degenerate at infinity and \( S(f) = \emptyset \).
(b) \( \text{Sing} f = \{ z \in \mathbb{C}^2 \mid z_1 = 0, z_2 \in \mathbb{C} \} \cup \{ z \in \mathbb{C}^2 \mid z_1 + z_2 = 0, z_1 - i \overline{z}_1 = 0 \} \cup \{ z \in \mathbb{C}^2 \mid z_1 + z_2 = 0, z_1 + i \overline{z}_1 = 0 \}
\]

(c) \( M(\varphi) \) is not bounded and \( S(\varphi) = \{ -\frac{1+2i}{\sqrt{2}}, \frac{2i}{\sqrt{3}} \} \).

**Remark 4.4.** The above example is due to Oka. In the holomorphic case, Némethi and Zaharia proved the existence of the Milnor fibration at infinity for semitame polynomials in [NZ2]. The definition of semitame is equivalent to \( S(f) \subset \{ 0 \} \). But this example shows that in the mixed case, the condition \( S(f) \subset \{ 0 \} \) fails to insure the existence of the Milnor fibration \( f_1 \) at infinity. We also observe that the Newton strong non-degeneracy condition at infinity of Theorem 1.1 can not be replaced by Newton non-degeneracy condition at infinity.

**Acknowledgement**

The author wish to thank Professor Mihai Tibăr for his help and support during the preparation for this paper which is also a work contained in the thesis research [Ch].
References

[ACT] R.N.Araújo dos Santos, Ying Chen and Mihai Tibăr, Singular open book structures from real mappings., to appear in Cent. Eur. J. Math.

[Bo] A. Bodin, Milnor fibration and fibred links at infinity, Inter. Math. Res. Not. 11 (1999), 615-621.

[Br1] S.A. Broughton, On the topology of polynomial hypersurface, Singularities, Part 1 (Arcata, Calif., 1981), 167-178, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983.

[Br2] S.A. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, Invent. Math. 92 (1988), no. 2, 217-241.

[Ch] Y.Chen, Bifurcation values of mixed polynomials and Newton polyhedra, PhD. thesis, Université de Lille 1 (2012).

[Cl] J. L. Cisneros-Molina, Join theorem for polar weighted homogeneous singularities, Singularities II, Contemp. Math. 475, Amer. Math. Soc., Providence, RI, 2008, 43-59.

[CT] Y. Chen and M. Tibăr, Bifurcation values of mixed polynomials, Math. Res. Lett. 19 (2012), no.1, 59-79.

[Ku] A. Kushnirenko, Polyèdres de Newton et nombres de Milnor, Invent. Math. 32 (1976), 1-31.

[Mi] J. Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies, Princeton University Press 1968.

[Ne1] A. Némethi, Théorie de Lefschetz pour les variétés algébriques affines, C. R. Acad. Sc. Paris, t.303. Serie I., Nr. 12, 1986.

[Ne2] A. Némethi, Lefschetz theory for complex affine varieties, Rev. Roum. Math. Pures Appl., 33 (1988), 233-260.

[NZ1] A. Némethi, A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, Publ. Res. Inst. Math. Sci. 26 (1990), no. 4, 681-689.

[NZ2] A. Némethi, A. Zaharia, Milnor fibration at infinity, Indag. Math. 3 (1992), 323-335.

[Oka1] M. Oka, Topology of polar weighted homogeneous hypersurfaces, Kodai Math. J. 31, 2 (2008), 163-182.

[Oka2] M. Oka, Non degenerate mixed functions, Kodai Math. J. 33, 1 (2010), 1-62.

Mathématiques, Laboratoire Paul Painlevé, Université Lille 1, 59655 Villeneuve d’Ascq, France.

E-mail address: Ying.Chen@math.univ-lille1.fr