FINITE SPEED OF PROPAGATION IN DEGENERATE EINSTEIN BROWNIAN MOTION MODEL

ISANKA GARLI HEVAGE\textsuperscript{1} AND AKIF IBRAGIMOV\textsuperscript{1}

\textsuperscript{1}DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS TECH UNIVERSITY, TX 79409–1042, U. S. A.
Email address: \textsuperscript{1}isankaupul.garlihevage@ttu.edu, akif.ibragimov@ttu.edu

ABSTRACT. We considered qualitative behaviour of the generalization of Einstein’s model of Brownian motion when the key parameter of the time interval of \textit{free jump} degenerates. Fluids will be characterised by number of particles per unit volume (density of fluid) at point of observation. Degeneration of the phenomenon manifests in two scenarios: a) flow of the fluid, which is highly dispersing like a non-dense gas and b) flow of fluid far away from the source of flow, when the velocity of the flow is incomparably smaller than the gradient of the density. First, we will show that both types of flows can be modeled using the Einstein paradigm. We will investigate the question: What features will particle flow exhibit if the time interval of the \textit{free jump} is inverse proportional to the density and its gradient ? We will show that in this scenario, the flow exhibits localization property, namely: if at some moment of time \( t_0 \) in the region, the gradient of the density or density itself is equal to zero, then for some \( T \) during time interval \([t_0, t_0 + T]\) there is no flow in the region. This directly links to Barenblatt’s finite speed of propagation property for the degenerate equation. The method of the proof is very different from Barenblatt’s method and based on the application of Ladyzhenskaya - De Giorgi iterative scheme and Vespri - Tedeev technique. From PDE point of view it assumed that solution exists in appropriate Sobolev type of space.

1. INTRODUCTION

In his celebrated work from 1905, Einstein introduced frame work of the random motion of particles suspended in media. Einstein’s paradigm took a pivotal role in this study as we are considering an extension of his thought experiment to explain localization properties which define many physical problems. Einstein introduced three key parameters which characterize random movement of the particles , i.e., non-colliding time interval \((\tau)\), value of changes in the length of particle displacement \((\Delta)\) which are “free of collision” with other particles and media during time interval \([t, t + \tau]\) and frequency \((\varphi(\Delta))\) of the occurrence of the “free of collision” displacements \(\Delta\). In our work we interpret ”non-collision displacements”, which we call hereafter as a \textit{free jump}. In order to be specific, we state all above in the form of Einstein’s Brownian motion axioms as a definition of \textit{free jump}.

Definition 1.1.

(1) There exists a time interval \(\tau\), which is very small compared to the time interval over which the system is observed, but large enough that the motions performed by a particle during two consecutive time intervals \(\tau\) can be considered as mutually independent events.

(2) Length of non-colliding jumps corresponding to time interval \(\tau\), which we call “free jumps” is \(\Delta\).

Traditionally physicists use free path to determine pass of non-collision of particles, as in \cite{1}. In this sense the term \textit{free jump} has the same meaning as free path. To make the definition of \textit{free jump} transparent, we cite the Einstein’s statement as follows: “\textit{Evidently it must be assumed that each single particle executes a movement which is independent of the movement of all other particles ; the movements of one and the same particle after different intervals of time must be considered as mutually independent processes, so long as we think of these intervals of time as being chosen not too small. We will introduce a time-interval \(\tau\) in our discussion, which is to be very small compared with the observed interval of time, but, nevertheless, of such a magnitude that the movements executed by a particle in two consecutive
intervals of time $r$ are to be considered as mutually independent phenomena. In order not to designate this two definition we use term free jump instead of free path. Next, Einstein postulates that

**Axiom 1.1.** At any point of observation $x$ at time $t + \tau$ total number of particles $u(x, t)$ in the unit volume $dv$ containing point $x$ is equal to accumulated total number of particles which has free jumps $\Delta$ weighted by frequency $\varphi(\Delta)$ from the same point $x$ at time $t$

$$u(x, t + \tau) \cdot dv = \left( \int_{\mathbb{R}} u(x + \Delta, t) \varphi(\Delta) d\Delta \right) \cdot dv.$$  

Regarding frequency Einstein also postulates the following axioms.

**Axiom 1.2.** whole universes axiom:

$$\int_{\mathbb{R}} \varphi(\Delta) d\Delta = 1.$$

**Axiom 1.3.** Evenness of the frequency:

$$\varphi(-\Delta) = \varphi(\Delta).$$

Therefore expected value of length in free jump $\Delta_e = 0$.

Assuming in the Axiom 1.1, evenness of the $\varphi(\Delta)$ and smoothness of the function $u(x, t)$ Einstein approximates function $u(x, t)$ with the solution of diffusivity equation, which is often cited as the Brownian motion equation

$$u_t = D u_{xx}. \quad (1.1)$$

where $D \triangleq \frac{1}{\tau} \left( \int_{\mathbb{R}} \frac{\Delta^2}{2} \varphi(\Delta) d\Delta \right)$ is called Einstein’s diffusion coefficient. If $D(x, t)$ is space and time dependent function and $1/c_0 > D > c_0 > 0$ for some constant $c_0$ then, due to strong maximum principle, the solutions of Eq.(1.1) exhibit the so called important feature: " infinite speed of propagation". In many cases this can lead to non-accurate and non physical interpretations of the observations. In this article we will show that there exists an extended Einstein paradigm of random motion such that solution of the corresponding equation will exhibit finite speed of propagation. Namely, in our thought experiment (see Section 2), we consider the case when duration of the time interval $\tau$ of free jump is inverse proportional to density $u$ (number of particles in unit volume) and its gradient $|\nabla u|$. We will show rigorously that this will lead to localisation property which in many sources is called finite speed of propagation:

**Remark 1.1.** It is worth to mention that the number of particles per unit volume is a monotone function with respect to the density of the fluid, and correspondingly is monotone function with respect to the pressure for isothermic fluids.

In the present work, we reconsider the above phenomena, and their governing nonlinear parabolic equations from the point of view of generalized Einstein’s random walk model in a continuous medium with diffusion, drift and absorption or reaction. Starting from basic stochastic principles, we derive a generic degenerating via solution and its gradient parabolic equation. This solution models concentration of the particles in a unit volume. We then prove the strong localization property by hypothesising the process such that in the Einstein framework time interval of a "free" movement of particles depends on density particles and its gradient. This approach can be interpreted as a complementary conceptual derivation of the governing equation for the
flows, but without appealing to Fick’s, Darcy’s or Fourier’s type laws, the continuity (or conservation of mass) equation, and thermodynamic closures that binds together the various variables of the system. We assume that the number of molecules in a liquid is proportional to the concentration of compound of interest, characterized by a scalar function \( u(x,t) \), which depends on the spatial and time coordinates \( x \) and \( t \), respectively.

This Article is organized as follows: In the Section 2 we formally introduce definition of random motion which includes: vector \( \Delta = (\Delta_1, \cdots, \Delta_N) \) of free jumps in \( N \) dimensional space, its expected vector \( \Delta_e \) and some probability distribution \( \phi(\Delta) \). To derive our nonlinear model we use the scalar \( \tau \) as length of the time interval \([t, t+\tau]\) during which particles are not colliding. Generalizing the Einstein Axioms of mass conservation for the free jump process in \( N \) dimension , we introduce ”conservation of the mass” in (2.4) with absorption or reaction and drift. Next, assuming that density function \( u(x,t) \) is sufficiently smooth, we derive the PDE inequality (2.12) for function \( u(x,t) \) using Taylor’s expansion and Caratheodory theorems. Assuming the co-variances and the expected vector are \( u \) independent, we model the process of ”non-linear \( \alpha - \beta \) jumps” by choosing \( \tau \) to be inverse proportional to density and its gradient in (2.13). This axiom , alongside with assumption on regularity of density function allows reduction of conservation of mass equation into a nonlinear IBVP for differential inequalities. Then , by proposed method of Tedeev-Vespri we use De - Giorgi-Ladyzhenskaya construction to map original problem for non-divergent differential inequality to iterative integral inequalities in Lemma 3.2 and Lemma 3.3. Based on Lemmas 3.2 and Lemma 3.3 in Theorem 3.1 , we transform obtained inequalities into one iterative inequality for specific functional . This Theorem enables us to implement Ladyzhenskaya Lemma to prove one of the main result in Theorem 3.2 . Then , by Corollary 3.2.1 we show that if initial data has compact support, the solution of the differential inequality 2.15 will exhibit finite speed of propagation property.

2. Generalized Einstein paradigm

In this section we will extend model for Brownian Motion for dynamical process of transport, diffusion and absorption with parameters, depending on number of particles and its gradient function.

2.1. Mass conservation law. Let \( x \in \mathbb{R}^N \) and \( u(x,t) \) be the function which represent the number of particles per unit volume at point \( x \) and at time \( t \). Consider a particle \( (P) \) of particular type suspended in the medium of interest. Denote \( P(\tau) \) to be the set of vectors with non-colliding jumps of \( P \) corresponding to time interval \( \tau \). We call \( \Delta = (\Delta_1, \cdots, \Delta_N)^T \) to be a “ vector of free jump of particles \( P \)” if \( \Delta \in P(\tau) \). We assume the following extension of the definition 1.1:

Assumption 2.1.

(1) All possible interactions between particles during time interval \( \tau \) are via absorbing thorough surrounding media which may include media itself, other particles, and all possible boundaries. This key parameter \( \tau \) in general can depend on the concentration of the particles \( (P) \) and its gradient and also space coordinate and time itself.

(2) Time interval of free jumps \( \tau \), expected vector \( \Delta_e \), of a free jump \( \Delta \) and probability density function of free jump \( \phi(\Delta) \) are the only parameters which characterise process of free jumps. Note that in a view of the definition of the set \( P(\tau) \), if \( \Delta \notin P(\tau) \) then \( \phi(\Delta) = 0 \).

(3) During time interval \([t, t+\tau]\) in the unit volume around the observation point \( x \) , there are possible bonding and/or absorption with other particles or with the media which approximated by integral

\[
\int_t^{t+\tau} A(u(x,s)) \, ds.
\]

It is important to state that if the non-linear function \( A(u) > 0 \), \( A(0) = 0 \) then it has the growth constrain which will be introduced in Lemma 3.3.

Axiom 2.1. Whole universe axiom:

\[
\int_{P(\tau)} \phi(\Delta) d\Delta = 1. \quad (2.1)
\]
Let us define an expected vector of the "jumps" and corresponding co-variance matrix.

**Definition 2.1.**

1. Expected vector of free jumps
   \[
   \bar{\Delta}_e \equiv (\Delta_1^e, \Delta_2^e, \ldots, \Delta_N^e)^T \quad \text{where} \quad \Delta_i^e \equiv \int_{\Omega(t)} \Delta_i \varphi(\bar{\Delta}) d\bar{\Delta}.
   \] (2.2)

2. Standard Co-variance matrix of a free jump
   \[
   \sigma_{ij}^2 \equiv \int_{\Omega(t)} (\Delta_i - \Delta_i^e) (\Delta_j - \Delta_j^e) \varphi(\bar{\Delta}) d\bar{\Delta}.
   \] (2.3)

Evidently \( \bar{\Delta}_e(x,t) \) and \( \sigma_{ij}(x,t) \) depend on space \( x \) and time \( t \). We postulate generalized Einsteins Axiom for the number of particles found at time \( t + \tau \) in the control volume \( d\bar{\Delta} \) contained point \( x \) by

**Axiom 2.2.**

\[ u(x, t + \tau) \cdot dv = \left( \int_{\Omega(x)} u(x + \bar{\Delta}, t) \varphi(\bar{\Delta}) d\bar{\Delta} + \int_{t}^{t+\tau} A(u(x, s), t) ds \right) \cdot dv. \] (2.4)

Mass conservation law (2.4) intuitively is easy to interpret, and it says that at any given point in space \( x \) at time \( t + \tau \) we will observe the number of particles per unit volume all particles with free jumps from the point \( x \) at time \( t \), + density of particles which "produced" and − density of particles which "consumed" during time interval \( [t, t + \tau] \). For comparison see [9] (pages 14) first formula with integral.

**Remark 2.1.** Einstein definition of density of particles as a number of particles per unit volume differ from the fundamental definition of the density of fluid: denser the fluid, denser the number of particles in unit volume in the Einstein Paradigm.

### 2.2. Derivation of diffusion and absorption model with drift.

Let \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N) \) multi-index and \( x^\zeta \equiv x_{i_1}^{\zeta_1} \cdot x_{i_2}^{\zeta_2} \cdot \ldots \cdot x_{i_N}^{\zeta_N} \). Assuming that \( u(x,t) \in C^{2,1}_{x,t} \) we apply Taylor’s Expansion, and using (2.1) - (2.3) we get

\[
\int_{\Omega(x)} u(x + \bar{\Delta}, t) \varphi(\bar{\Delta}) d\bar{\Delta} = u(x + \bar{\Delta}_e, t) + \sum_{i,j} \sigma_{ij}^2 u_{x_i x_j} (x + \bar{\Delta}_e, t) + \frac{1}{2} \sum_{i=1} \sigma_{ii}^2 u_{x_i x_i} (x + \bar{\Delta}_e, t) + R_\zeta
\] (2.5)

where

\[
R_\zeta \equiv \int_{\Omega(x)} \sum_{|\zeta| = 2} H_\zeta(x, \bar{\Delta}, t)(\bar{\Delta} - \bar{\Delta}_e)^\zeta \varphi(\bar{\Delta}) d\bar{\Delta}.
\] (2.6)

Here \( \lim_{\Delta \rightarrow \Delta_e} H_\zeta(x, \bar{\Delta}, t) = 0 \). Using (2.5) in (2.4) we get

\[
u(x, t + \tau) - u(x + \bar{\Delta}_e, t) - \sum_{i,j} a_{ij}(x,t) u_{x_i x_j} (x + \bar{\Delta}_e, t) + R_\zeta + \int_{t}^{t+\tau} A(u(x, s), t) ds.
\] (2.7)

Here \( a_{ij}(x,t) = \sigma_{ij}^2(x,t) \) if \( i = j \), and \( a_{ij}(x, t) = \sigma_{ij}^2(x, t) \) if \( i \neq j \). Moreover, using Holder inequality for (2.6) with \( 0 < l < 1 \) one can estimate

\[
R_\zeta \leq \sum_{|\zeta| = 2} M_\zeta \cdot \left( \int_{\Omega(x)} |H_\zeta(x, \bar{\Delta}, t)|^l d\bar{\Delta} \right)^{1-l}
\] (2.8)

where \( M_\zeta \equiv \left( \int_{\Omega(x)} |(\bar{\Delta} - \bar{\Delta}_e)^\zeta \varphi(\bar{\Delta})|^{1-l} d\bar{\Delta} \right)^{1-l-l} \). Observe that LHS and RHS in the Eq.(2.7) are defined in different points. In order to eliminate this ambiguity and derive the equation at the same point
we will assume that \( u(x, t) \in C^{3,2}_{x,t} \). Then by Carathéodory’s criterion there exists a function \( \tilde{u}_{ij}^{\text{xx}} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N \) such that
\[
\sum_{i,j=1}^{N} a_{ij}(x, t)u_{x_i x_j}(x + \Delta_x, t) = \sum_{i,j=1}^{N} [\tilde{u}_{ij}^{\text{xx}}(x, \Delta_x, t) \cdot \Delta_x]a_{ij}(x, t) + \sum_{i,j=1}^{N} a_{ij}(x, t)u_{x_i x_j}(x, t). \tag{2.9}
\]

Similarly \( \exists \) functions \( \psi^t, \psi^\tau, \psi^\tau_1 \in \mathbb{R} \) and \( \psi^\tau_1^\tau \in \mathbb{R}^N \) such that
\[
u(x, t + \tau) - u(x + \Delta_x, t) =
[x^2 \psi^{tt}(x, t, \tau)] - \sum_{i=1}^{N} [\tilde{u}_{ij}^{\text{xx}}(x, \Delta_x, t) \cdot \Delta_x] \Delta^i_t + \tau \psi^t(x, t, 0) - \sum_{i=1}^{N} \psi^\tau_i(x, 0, t) \Delta^i_{\tau}, \tag{2.10}
\]
where \( \psi^t \) and \( \psi^\tau_1 \) are such that \( \lim_{\nu \to 0} \psi^t(x, t, \nu) = u_t(x, t) \) and \( \lim_{\Delta \to 0} \psi^\tau_1(x, \Delta, t) = u_x(x, t) \).

Moreover, in this article we will assume the term \( |\tau \psi^t| \) negligible with respect to \( |\psi^\tau_1| \). Using (2.10) in LHS of (2.7) and (2.8),(2.9) in RHS of (2.7) yields
\[
u_t - \frac{1}{\tau} \sum_{i=1}^{N} \Delta^i_t u_{x_i} - \frac{1}{\tau} \sum_{i,j=1}^{N} a_{ij}(x, t)u_{x_i x_j} \leq \frac{B}{\tau} + |\psi^\tau|,
\]
where
\[
B = \sum_{i=1}^{N} [\tilde{u}_{ij}^{\text{xx}}(x, \Delta_x, t) \cdot \Delta_x] \Delta^i_t + \sum_{i,j=1}^{N} [\tilde{u}_{ij}^{\text{xx}}(x, \Delta_x, t) \cdot \Delta_x]a_{ij}(x, t) + R_{xx}.
\]

Here \( \Delta_x \) and all functions are subject of growth condition with respect to \( |\nabla u| \). Further we will consider

**Assumption 2.2.** \( \exists \) constant \( C_1 \geq 0 \) such that in R.H.S of inequality (2.11) function
\[
|\nabla u| \leq C_1 |\nabla u(x, t)|,
\]
where \( |\nabla u| = \sqrt{\sum_{i=1}^{N} \left( \frac{\partial u}{\partial x_i} \right)^2} \). This was introduced from general, mathematical point of view. For interpretation see [10]. From above it follows that the function \( u(x, t) \) can be approximated by the differential inequality
\[
u_t - \frac{1}{\tau} \sum_{i=1}^{N} b_i(x, t)u_{x_i} - \frac{1}{\tau} \sum_{i,j=1}^{N} a_{ij}(x, t)u_{x_i x_j} \leq \frac{1}{\tau} C_1 |\nabla u| + |\psi^\tau|,
\]
where \( b_i(x, t) = \Delta^i_t \). In forthcoming sections we will study a qualitative properties of the function \( u(x, t) \) which solves inequality (2.12), without refereeing to the origin of the this process which led to (2.12). Obtained results then will provide interpretation for for physical processes which can lay under Einstein paradigm.

**Remark 2.2.** Partial differential inequality (PDI) (2.12) has no unique solution for any initial data and boundary conditions but any solution which satisfies this inequality exhibits finite speed of propagation under the condition on \( \tau \) in (2.13) with appropriately chosen \( \alpha \) and \( \beta \) in Lemma 3.1 and Lemma 3.3. In contrast to PDEs, PDI s has much wider application and corresponding dynamical systems will also exhibit localization property (see [11, 12]).

The PDI (2.12) has no unique solution but in spite of that any solution of this inequality with initial data having compact support will exhibit finite speed of propagation under our conditions on Non-linearity. RHS of (2.12) has two terms. The second term associated to the reaction term of Einstein type dynamic processes, whether first term is modeling non-linear drift, which bounded by \( \nabla u \). Therefore inequality (2.12) has the generalization of the Einstein Brownian Motion model not only due to \( \tau \), but also because of the non-linear drift and reaction- absorption in the system.
2.3. Non-linear degenerate partial differential inequality. In our model $\tau$, $\Delta_x$ and $\varphi$ are key characteristics of the process dynamics. Not only can they be functions of spatial and time variables $x$ and $t$, properties of the medium and its boundary but also functions of dependent variables and their derivatives. In this article we assume that the process of the "free" jumps is characterised only by the time interval of free jump ($\tau$). In our sought experiment we claim the process is such the time interval of free jumps ($\tau$) is inverse proportional to number of particles $u(x,t)$, and its gradient. This constrain can be intuitively justified by following arguments:

a) Time interval of the free jumps of non-colliding particles increases as number of particles decreases.

Remark 2.3. Assuming the number of particles per unit volume to be proportional to density, then time interval of free jump will be reciprocal to density. In this way Einstein’s paradigm can be used for characterization of highly "disperse" gases. It is known that the density of the disperse gases in porous media is proportional to pressure (see [13]). This lead to so called porous media equation for pressure function, with degeneration in diffusion coefficient (see [14]). Equation obtained from Einstein’s thought experiment can be mapped to a Barenblatt type equation for density function. This provides one more justification for assumption a) above. It is worth to mentioned that we also used Einstein’s model with "free jumps" to interpret the multi-component flow in the 1-dimensional tube, and provided method of the evaluation of the parameter $\tau$ based on spectrometer data (see [15]).

b) The Smaller the gradient of the density of particle lesser the "mobility" of the fluid particles and, consequently the bigger the time interval of free jumps of non-colliding particles.

Remark 2.4. To justify this assumption we correlate the number of particles per unit volume through density to fluid pressure in porous media which will relate Einstein’s non-linear equation to so called pre-Darcy flow (see [7, 16]).

The major question which we address in this article is following: Can the degeneration of the $\tau$, with respect to solution or/and its gradient lead to localisation property of the particle-transport due to diffusion? To prove this localisation property we postulate the following dependence of the parameter $\tau$ which stated in form of definition.

Definition 2.2. Let $\alpha > 0$ and $\beta \geq 0$. The dynamical process, governs by partial differential inequality (2.12) is called the process of $\alpha - \beta$ jumps, if $\tau$ in (2.12) has the property

$$\tau \approx \frac{1}{u^\alpha \cdot |\nabla u|^\beta}. \quad (2.13)$$

In the recent paper [17] in 1-D case, it was shown that if $\tau$ does not degenerate then qualitative property of Brownian motion is similar to linear case. In this article we consider the process of the jumps of particles when $\tau$ degenerates and subject to constrain (2.13). We assume that variance matrix is homogeneous, positively defined and isotropic: $\sigma_{ij}(x,t) = 2k_2\delta_{ij}$, where $k_2 > 0$, $\delta_{ij}$ is the Kronecker symbol. Let

$$Lu \triangleq u_t - u^\alpha|\nabla u|^\beta \sum_{i=1}^N b_i(x,t)u_{x_i} - k_2 \cdot u^\alpha|\nabla u|^\beta \Delta u \quad (2.14)$$

Under the Definition 2.2 and extended assumptions in assumption 2.1, the differential inequality (2.12) with the corresponding initial and boundary condition in bounded domain $\Omega \subset \mathbb{R}^N$, we defined $u(x,t)$ as a non-negative classical solution of the following IBVP

$$\begin{cases}
Lu \leq C_1u^\alpha|\nabla u|^{\beta+1} + |A(u)| & \text{in } \Omega \times (0,T) \\
u(x,0) = u_0(x) & \text{in } \Omega \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0,T). \quad (2.15)
\end{cases}$$

Remark 2.5. Note that the operator $L$ degenerates when $u = 0$ or $|\nabla u| = 0$, therefore its solution $u(x,t)$ does not belong to $C^0_{x,t}(\Omega \times (0,T))$, strictly speaking. Our result is qualitative and does not address the existence of the solutions, but the obtained property of the solution will be applicable for the
weak viscous solution which one can define as follows. Let $\epsilon$ to be positive, and

$$L_{\epsilon}u_{\epsilon} \triangleq (u_{\epsilon})_t - (u_{\epsilon}^{\alpha}|\nabla u_{\epsilon}|^{\beta} + \epsilon) \sum_{i=1}^{N} b_i(x,t)(u_{\epsilon})_{x_i} - k_2 \cdot (u_{\epsilon}^{\alpha}|\nabla u_{\epsilon}|^{\beta} + \epsilon) \Delta u_{\epsilon}$$

then we define $u_{\epsilon}$ as a solution of the following IBVP

$$
\begin{align*}
L_{\epsilon}u_{\epsilon} & \leq C_1 u_{\epsilon}^{\alpha}|\nabla u_{\epsilon}|^{\beta+1} + |A(u_{\epsilon})| & \text{in } \Omega \times (0, T) \\
u_{\epsilon}(x,0) & = u_0(x) & \text{in } \Omega \\
u_{\epsilon}(x,t) & = 0 & \text{on } \partial \Omega \times (0, T).
\end{align*}
$$

In this article we showed that all estimates for the solution of the IBVP with $L_{\epsilon} u_{\epsilon} = 0$, do not depend on $\epsilon$. All dependence on the regularization parameter $\epsilon$ appears as separate terms in the respective estimates, and they do not depend on the solution $u_{\epsilon}$ or its derivatives. This observation allow us to pass to the limit in the final estimates, and conclude the localization property for the limiting function

$$u(x, t) = \lim_{\epsilon \to 0} u_{\epsilon}(x, t),$$

which is considered as a weak passage to the limit (see [18]). The obtained function $u(x, t)$ is called a weak viscosity solution of the IBVP, which will exhibit localisation property.

3. LOCALIZATION PROPERTY OF THE SOLUTION IBVP

The main goal of this article is to prove the Localization Property of the solution of IBVP (2.15) with $\sup u_0 \subset B_{R_0}(0) = \{|x| < R_0\}$. In order to prove in this section let us assume that the expected value of free jumps in each direction is uniformly bounded:

$$|\Delta^{i}_{\epsilon}| \leq k_1 < \infty. \quad (3.1)$$

Then we will proceed with De Giorgi-Ladyzhenskaya iteration procedure as in [19, 20]. Consider the sequence of $r_n = 2r \left(1 - \frac{1}{2^{n+1}}\right)$ for $n = 0, 1, 2, \ldots$ with $r > 2R_0$. Let $\bar{r}_n = \left(\frac{r_n + \bar{r}_n + 1}{2}\right)$, $\Omega_n = \Omega \setminus B_{r_n}(0)$ and $\bar{\Omega}_n = \bar{\Omega} \setminus B_{r_n}(0)$. Note that $\Omega_{n+1} \subset \bar{\Omega}_n \subset \Omega_n \subset \Omega$. Let $0 \leq \eta_n \leq 1$ be a sequence of cut off functions satisfying

$$\eta_n(x) = 0 \text{ for } x \in B_{r_n}(0), \eta_n(x) = 1 \text{ for } x \in \bar{\Omega}_n \text{ and } |\nabla \eta_n| \leq \frac{c^{2n}}{r} \text{ otherwise}. \quad (3.2)$$

3.1. Preliminary Lemmas.

**Lemma 3.1.** Let $u(x, t) \geq 0$ be a classical solution of IBVP (2.15). Let $\theta \geq 1$ and $p \geq 2$ be such that

$$\beta + 2 \leq p < \frac{\theta + \alpha}{\beta + 1} - \frac{k_1}{k_2} - C_1. \quad (3.3)$$

Then

$$\sup_{0 < \tau < t} \int_{\Omega_n} u^{\theta + 1} dx + C \int_{0}^{t} \int_{\Omega_{n+1}} u^{\theta + \alpha - 1}|\nabla u|^{\beta + 2} dx d\tau$$

$$\leq D_n \int_{0}^{t} \int_{\Omega_n} u^{\theta + \alpha + 1} dx d\tau + (\theta + 1) \int_{0}^{t} \int_{\Omega_n} |A(u)| u^{\theta} dx d\tau. \quad (3.4)$$

for $0 < t \leq T$. Here

$$C = (\theta + 1) \left( k_2 \left( \frac{\theta + \alpha}{1 + \beta} \right) - k_1 - C_1 - k_2 p \right), D_n = (\theta + 1) \left( k_2 p \left( \frac{c^{2n}}{r} \right)^{\beta + 2} + k_1 + C_1 \right), \quad (3.5)$$

and $C_1$ is from Assumption 2.2.
Proof. Let $\Omega_t \triangleq \Omega \times (0, t)$. Multiply both side of the inequality in (2.15) by $\eta_m^p u^\theta$, integrating by parts over $\Omega_t$ and using (3.1) we find

$$
\frac{1}{\theta + 1} \int_\Omega \eta_m^p u^{\theta + \alpha} \frac{d}{dx} u \frac{\partial}{\partial x} u dx - k_1 \int_\Omega \eta_m^p u^{\theta + \alpha} |\nabla u| u \, dx \leq C_1 \int_\Omega \eta_m^p u^{\theta + \alpha} \nabla u \cdot u \, dx + \int_\Omega A(\nabla u) \eta_m^p u^\theta \, dx.
$$

We compute $\nabla (\eta_m^p u^{\theta + \alpha})$ and (3.6) yields

$$
\frac{1}{\theta + 1} \int_\Omega \eta_m^p u^{\theta + \alpha} \frac{d}{dx} u \frac{\partial}{\partial x} u \, dx - k_1 \int_\Omega \eta_m^p u^{\theta + \alpha} |\nabla u| u \, dx \leq C_1 \int_\Omega \eta_m^p u^{\theta + \alpha} \nabla u \cdot u \, dx + \int_\Omega A(\nabla u) \eta_m^p u^\theta \, dx.
$$

Apply Young’s Inequality

$$
\eta_m^{p-1} |\nabla u| |\nabla u|^{\theta + \alpha} \leq \eta_m^{p-1} |\nabla u|^{\theta + \alpha} + |\nabla u|^{\theta + \alpha + 1},
$$

and

$$
|\nabla u|^{\theta + \alpha} \leq |\nabla u|^{\theta + \alpha + 1} + u^{\theta + \alpha + 1}.
$$

Then estimate (3.7) becomes

$$
\frac{1}{\theta + 1} \int_\Omega \eta_m^p u^{\theta + \alpha} \frac{d}{dx} u \frac{\partial}{\partial x} u \, dx - k_1 \int_\Omega \eta_m^p u^{\theta + \alpha} |\nabla u| u \, dx \leq C_1 \int_\Omega \eta_m^p u^{\theta + \alpha} \nabla u \cdot u \, dx + \int_\Omega A(\nabla u) \eta_m^p u^\theta \, dx.
$$

Note that \((k_2^{(\theta + \alpha)} - k_1 - C_1) \eta_m^p > k_2 \eta_m^{(p-1)\frac{\beta + 2}{\beta + \alpha}}\) by (3.3). Using (3.2), one can rearrange (3.8) to get

$$
\frac{1}{\theta + 1} \int_{\Omega_{n+1}} u^{\theta + \alpha} \, dx + \int_0^t \int_{\Omega_{n+1}} \left( k_2^{(\theta + \alpha)} - k_1 - C_1 - k_2 \right) u^{\theta + \alpha - 1} u \frac{\partial}{\partial x} u \, dx \, dt
$$

$$
\leq k_2 \int_0^t \int_{\Omega_n \setminus \Omega_n} \left( \frac{c_1}{r} \right)^{\theta + 1} \nabla u \cdot u \, dx \, dt + (k_1 + C_1) \int_0^t \int_{\Omega_n} u^{\theta + \alpha + 1} \, dx \, dt + \int_0^t \int_{\Omega_n} A(\nabla u) u^\theta \, dx \, dt.
$$

So we will have the inequality

$$
L_n[u](t) \triangleq \sup_{0 < r < t} \int_{\Omega_{n+1}} u^{\theta + 1} \, dx + C \int_0^t \int_{\Omega_{n+1}} u^{\theta + \alpha - 1} u \frac{\partial}{\partial x} u \, dx \, dt
$$

$$
\leq D_n \int_0^t \int_{\Omega_n} u^{\theta + \alpha + 1} \, dx \, dt + (\theta + 1) \int_0^t \int_{\Omega_n} A(\nabla u) u^\theta \, dx \, dt
$$

$$
\triangleq J_n[u](t) + K_n[u](t). \quad (3.9)
$$

Next we introduce the following mapping
Definition 3.1. Let
\[ z \triangleq u^{(\theta+\alpha+\beta+1)/(\beta+2)} \quad \text{and} \quad \Lambda \triangleq (\theta + 1)(\beta + 2)/(\theta + \alpha + \beta + 1). \] (3.10)

Define
\[ I_n[z](t) \triangleq \sup_{0 < \tau < t} \int_{\Omega_{n+1}} z^\lambda \, dx + C \int_0^t \int_{\Omega_{n+1}} |\nabla z|^{\beta+2} \, dx \, d\tau, \quad n = 0, 1, 2 \ldots . \]

Using (3.10) one can get
\[ L_n[u](t) = \sup_{0 < \tau < t} \int_{\Omega_{n+1}} z^\lambda \, dx + C \int_0^t \int_{\Omega_{n+1}} \left( \frac{\lambda}{\theta + 1} \right)^{\beta+2} |\nabla z|^{\beta+2} \, dx \, d\tau \triangleq \tilde{L}_n[z](t). \]

Therefore
\[ \tilde{L}_n[z](t) \geq G \cdot I_n[z](t), \] (3.11)

where \( G \triangleq \min \{1, C[\lambda/(\theta + 1)]^{\beta+2}\} \). Next in Lemma 3.2 and Lemma 3.3, we will provide the relations for \( \tilde{K}_n[z](t) \) and \( \tilde{J}_n[z](t) \) using \( I_n[z](t) \).

Lemma 3.2. Let \( u(x,t) \geq 0 \) be a classical solution of IBVP (2.15) and \( u \) and \( z \) be as in (3.10). Then \( \exists C_L, \epsilon_0 > 0 \) and \( b_L > 1 \) such that
\[ \tilde{J}_n[z](t) \leq t^{1-\Lambda} C_L b_L^{-n-1} I_{n-1}^{1+\epsilon_0}[z](t), \] (3.12)

for any \( n = 1, 2, \ldots \). Here \( 0 < \Lambda < 1 \) and \( \tilde{J}_n[z](t) \) defined below in (3.13).

Proof. By the substitution (3.10)
\[ J_n[u](t) \triangleq D_n \int_0^t \int_{\Omega_n} u^{\theta+\alpha+\beta+1} \, dx \, d\tau = D_n \int_0^t \int_{\Omega_n} z^{\beta+2} \, dx \, d\tau \triangleq \tilde{J}_n[z](t). \] (3.13)

By Gagliardo-Nirenberg-Sobolev inequality (see [21]) we obtain
\[ \int_{\Omega_n} z^{\beta+2} \, dx \leq c_G \left( \int_{\Omega_n} |\nabla z|^{\beta+2} \, dx \right)^{\Lambda} \left( \int_{\Omega_n} z^\lambda \, dx \right)^{(1-\Lambda)(\beta+2)/\lambda}, \]

with
\[ \Lambda \triangleq \frac{\alpha + \beta}{\alpha + \beta + N(\beta + 2)(\theta + 1)}. \] (3.14)

Integrating above inequality over time and using Holder inequality, we get
\[ \int_0^t \int_{\Omega_n} z^{\beta+2} \, dx \, d\tau \leq c_G t^{1-\Lambda} \left( \int_0^t \int_{\Omega_n} |\nabla z|^{\beta+2} \, dx \, d\tau \right)^{\Lambda} \left( \sup_{0 < \tau < t} \int_{\Omega_n} z^\lambda \, dx \right)^{(1-\Lambda)(\beta+2)/\lambda}. \] (3.15)

Using (3.15) in (3.13) provides estimate
\[ \tilde{J}_n[z](t) \leq D_n c_G t^{1-\Lambda} I_{n-1}^{1+\epsilon_0} [z](t). \] (3.16)

By (3.14), let
\[ \epsilon_0 \triangleq N\Lambda(\beta + 2), \] (3.17)

then \( \Lambda + \frac{(1-\Lambda)(\beta+2)}{\lambda} = 1 + \epsilon_0 \). Moreover by (3.3) and (3.5) one can get
\[ D_n \leq (\theta + 1) \left( k_1 + \left( \frac{c}{2R_0} \right) k_2(\theta + \alpha) \right) 2^{n\beta + 2n}. \]

Therefore (3.16) will lead to
\[ \tilde{J}_n[z](t) \leq t^{1-\Lambda} C_L b_L^{-n-1} I_{n-1}^{1+\epsilon_0}[z](t). \] (3.18)

Here \( C_L = (\theta + 1) \left( k_1 + \left( \frac{c}{2R_0} \right) k_2(\theta + \alpha) \right) c_G 2^{\beta+2} \) and \( b_L = 2^{\beta+2} \). \( \Box \)
Remarque 3.1. Note that by (3.17)
\[ \epsilon_0 = \frac{N(\alpha + \beta)(\beta + 2)}{\alpha + \beta + N(\beta + 2)(\theta + 1)}. \] (3.19)

Let \( \alpha = 0 \) and \( \beta = 0 \) in \( Lu = 0 \) in (2.14) which provides the non-Degenerate parabolic equation without absorption. Then \( \epsilon_0 \) in (3.18) will vanish. As we will see later, in this case Lemma 3.4 will not provide Localization property as it intended. This reflects an important feature of the solution of parabolic equation which is called infinite speed of propagation. Indeed due to strong maximum principle if \( u(x, t) \) is non-negative solutions of second order-linear parabolic and \( u(x_0, t_0) = 0 \) then it will vanish to zero on all subordinates to this point sub domain (see [10]).

Lemma 3.3. Let \( u \) and \( z \) be as in (3.10).

(P1) Let \( \alpha > 0, \beta \geq 0 \) and \( N > 1 \) be such that \( \exists \theta \geq 1 \) and \( s \geq 1 \), satisfy
\[ \max \{ 1 + \epsilon_0, N \epsilon_0 \} \leq s < \min \{ \beta + 2, N(1 + \epsilon_0) \}, \quad \text{where \( \epsilon_0 \) is from (3.19)}. \] (3.20)
Define
\[ \gamma \triangleq \left( \frac{1 + \epsilon_0}{s} - \frac{1}{\beta + 2} \right) \lambda + 1, \quad \text{where \( \lambda \) is from (3.10)}. \] (3.21)

For given \( A \), assume that \( \exists \) a transformation \( F \) satisfying the following properties:

(P2) \( |A(u)|u^\theta = F^s(z^\gamma) \)

(P3) \( \frac{F'}{F} \leq M_0 \) in \( \Omega \). Here \( \frac{F'}{F} \triangleq \frac{dF}{dx}(\cdot) \).

(P4) \( F(z^\gamma(x)) \in W_0^{1,m}(\Omega_n, \partial \Omega) \) where \( 1 \leq m < N \).

Then \( \exists M_L > 0 \) such that
\[ \tilde{K}_n(z)(t) \leq t^{-\frac{\beta+2-s}{s}} M_L k_L^{n-1} I_{n-1}^{1+\epsilon_0}(t). \] (3.22)

for any \( n = 1, 2, \ldots \). Here \( \tilde{K}_n(z)(t) \) defined below in (3.23).

Proof. Recall that
\[ K_n[u](t) \triangleq (\theta + 1) \int_0^t \int_{\Omega_n} |A(u)|u^\theta dx d\tau = (\theta + 1) \int_0^t \int_{\Omega_n} |F^s(z^\gamma)| \ dx d\tau \triangleq \tilde{K}_n[z](t). \] (3.23)

Let \( m \triangleq \frac{s}{1 + \epsilon_0} \). By property (3.20) one has \( 1 \leq s \leq \frac{Nm}{N-m} \). Consequently by Poincare-Sobolev inequality in (see [2])
\[ \int_{\Omega_n} |F(z^\gamma)|^s \ dx \leq \left( c_p \gamma \sup_{x \in \Omega_n} |F'| \right)^s \left[ \int_{\Omega_n} z^{(\gamma-1)m} \ | \nabla z |^m \ dx \right]^{1+\epsilon_0}. \] (3.24)

Using Holder Inequality we get
\[ \left[ \int_{\Omega_n} z^{(\gamma-1)m} \ | \nabla z |^m \ dx \right]^{1+\epsilon_0} \leq \left[ \int_{\Omega_n} | \nabla z |^{\beta+2} \ dx \right]^{(1+\epsilon_0)H} \cdot \left[ \int_{\Omega_n} z^{\lambda} \ dx \right]^{(1+\epsilon_0)(1-H)}, \] (3.25)
with \( H \triangleq \frac{s}{1 + \epsilon_0} \frac{\beta+2}{\beta+2} \). Note that \( \frac{\gamma-1}{1-H} = \frac{\lambda}{m} \), by (3.21) in Lemma 3.3. Integrating (3.24) over time and using (3.25) we obtain
\[ \int_0^t \int_{\Omega_n} |F(z^\gamma)|^s \ dx d\tau \leq M \int_0^t \left( \int_{\Omega_n} | \nabla z |^{\beta+2} \ dx \right)^{(1+\epsilon_0)H} d\tau \cdot \left[ \sup_{0 < \tau < t} \int_{\Omega_n} z^{\lambda} \ dx \right]^{(1+\epsilon_0)(1-H)}, \]
where \( M^{1/s} \triangleq c_p \gamma \sup_{(x,t) \in \Omega_n \times (0,t)} |F'| \). Applying Holder inequality in time
\[ \int_0^t \int_{\Omega_n} |F(z^\gamma)| \ dx d\tau \leq Mt^{\frac{\beta+2-s}{\beta+2}} \left[ \int_0^t \int_{\Omega_n} | \nabla z |^{\beta+2} \ dx d\tau \right]^{\frac{\beta+2}{\beta+2}} \left[ \sup_{0 < \tau < t} \int_{\Omega_n} z^{\lambda} \ dx \right]^{(1+\epsilon_0)(1-H)}. \] (3.26)
Observe that \([1+\epsilon_0][1-H] + \frac{s}{\beta+2} = 1 + \epsilon_0\). Then using (3.26) in (3.23) one has

\[K_n(z)(t) \leq t^{\frac{\theta+2-s}{2+s}} M_L b_L^{n-1} I_{n-1}^{1+\epsilon_0}[z](t),\]

where \(M_L = (\theta + 1)c_n^s \gamma^s \sup_{(x,t) \in \Omega_n \times (0,t)} |F| \gamma^s\).

Combining the obtained inequality (3.11), (3.12) and (3.22) then (3.9) yields to generate the following iterative inequality.

### 3.2. Localization property.

**Theorem 3.1.** Assume that all conditions in Lemmas 3.1, 3.2 and 3.3 are satisfied. Let \(u(x, t) \geq 0\) be a classical solution of IBVP (2.15) and functions \(u(x, t)\) and \(z(x, t)\) be related as (3.10). Then

\[I_n[z](t) \leq t^q \left( \frac{C_L + M_L}{G} \right) b_L^{n-1} I_{n-1}^{1+\epsilon_0}[z](t).\]  

(3.27)

for \(0 < t \leq T\) and \(n = 1, 2, \ldots\). Here \(t^q \triangleq \max \{t^{1-\lambda}, t^{\frac{\theta+2-s}{2+s}}\}\), \(\lambda\) is from (3.14) and \(s\) is from (3.20). Next we prove localization property using the following generic Lemma by Ladyzhenskaya in [21].

**Lemma 3.4.** Let sequence \(y_n\) for \(n = 0, 1, 2, \ldots\) be non-negative, satisfying the recursion inequality

\[y_{n+1} \leq c b^n y_{n+\epsilon},\]

for \(n = 0, 1, 2, \ldots\) with some positive constants \(c, \epsilon > 0\) and \(b \geq 1\). Then

\[y_n \leq c \frac{(1+\epsilon)^n-1}{\epsilon} b^{\frac{n}{b}} y_0^{(1+\epsilon)^n}.\]

In particular if \(y_0 \leq \theta_L = c^{-\epsilon} b^{-\epsilon}\) and \(b > 1\) then \(y_n \leq \theta_L b^{-\epsilon n}\). Consequently

\[y_n \rightarrow 0 \text{ when } n \rightarrow \infty.\]

From inequality (3.27) and Lemma 3.4 it follows the main theorem on localization.

**Theorem 3.2.** Assume that all conditions in Theorem 3.1 are satisfied. Let

\[I_0[z](T) \leq 2 - \left( \frac{\theta+2}{\theta} \right) \left( \frac{G}{C_L + M_L} \right)^{\frac{1}{\theta_0}} T^{-\frac{\theta}{\theta_0}}.\]

(3.28)

then

\[I_n[z](T) \rightarrow 0 \text{ as } n \rightarrow \infty.\]

By the De- Giorgi construction we have \(\Omega_{n+1} \subset \Omega_n\). From above it follows that:

**Corollary 3.2.1.** Let \(u(x, t) \geq 0\) be a classical solution of IBVP (2.15) and \(u\) and \(z\) be related as in (3.10). Assume as before condition (3.28). Then \(z(x, t) = 0\) a.e. in \(\Omega_\infty = \bigcap_{n=1}^\infty \Omega_n\) for any \(t \leq T\). Consequently

\[u(x, t) = 0 \text{ a.e. in } \Omega \setminus B_{2r}(0) \text{ for any } t \leq T.\]

Next, we can control value of the integral in (3.28) explicitly via initial data by assuming that in IBVP (2.15) reaction term to be equal zero. In this case

\[I_0[z](t) \leq \frac{D_0}{G} \int_0^t \int_{\Omega_0} z^{\beta+2} dx d\tau \leq \frac{D_0 |\Omega|}{G} \|z\|_{L^\infty(\Omega_0 \times (0,t))} \cdot t.\]

This estimate leads to the following

**Theorem 3.3.** Let \(A(u) = 0\) in IBVP (2.15) and assume that all conditions in Lemma 3.1 are satisfied.

Let \(z(x, t)\) be such that in (3.10) and \(\|z\|_{L^\infty(\Omega_0 \times (0,t))} \leq 2^{-\frac{1}{\theta_0}} B_0^{-\frac{1}{\theta+2}} t^{-(\alpha+\beta+\theta+1)/(\alpha+\beta)(\beta+2)}\) for \(0 < t \leq T\). Then

\[I_0[z](t) \leq 2^{-\frac{\theta+2}{\theta_0}} \left( \frac{G}{C_L} \right)^{\frac{1}{\theta_0}} t^{-\frac{\theta+1}{\alpha+\beta}} \triangleq \theta_L.\]

Here \(\epsilon_0\) from (3.17), \(B_0 \triangleq \frac{C_L^{-\frac{1}{\theta_0}} G^{1+\frac{1}{\sigma_0}}}{D_0 |\Omega_0|} \) and \(|\cdot|\) is the size of domain.
Corollary 3.3.1. Let $u(x, t) \geq 0$ be a classical solution of IBVP (2.15). Let the initial function $u_0(x)$ be such that $\text{supp } u_0(x) \in B_{R_0}(0)$ and

$$
\|u_0\|_{L^\infty(\Omega)} \leq 2^{-\mu} \left( \frac{C_L}{D_0|\Omega|} \right)^{\frac{1}{\alpha+\beta+1}} \cdot T^{-\frac{1}{\alpha+\beta}},
$$

(3.29)

where $\mu = \frac{[\alpha + \beta + N(\beta + 2)(\theta + 1)]^2}{(\beta + 2)N^2(\alpha + \beta)^2(\alpha + \beta + \theta + 1)}$. Then by the maximum principle function $z(x, t)$ which relate with $u(x, t)$ in (3.10) satisfies $I_0[z](t) \leq \theta_L$. Then due to Theorem 3.2 if (3.29) holds then

$$
u(x, t) = 0 \quad \text{a.e. in } \Omega \setminus B_{2r}(0),$$

for $r > 2R_0$ and $t \leq T$. Observe that the estimate (3.29) depends on volume $|\Omega|$. Smaller the volume “bigger” initial data are allowed. Note that Vespri and Tedeeve (see [19]) obtained boundedness of the solution of degenerate parabolic equation without drift and right hand side (R.H.S.) for equation in divergence form without use of the maximum principle.

4. DISCUSSION

Localization was first proposed by Zeldovich in 1950 (see [14]) and then proved by Barenblatt by constructing “Barenblatt” type of barrier for corresponding degenerate non-linear PDE (see [22]). Localisation property closely relates to the property which in some sources is called a “dead zone”. This property for solution of the steady state elliptic equation with strong absorption was investigated in the work by Landis [10] using his methods of lemma of growth for narrow domain. In the recent work by Vespri and Tedev method based on De- Giorgi (see [23]) and Ladyzhenskaya iterative process (see [21]) was employed to prove this essential feature for the class of degenerate parabolic equation in divergent form in [19]. We use this as our groundwork and provide proof of the Localization property for degenerate parabolic equation in non-divergent form derived from generalized Einstein paradigm, which has a clear physical interpretation. It is appropriate to mention that Landis used his method to provide alternative prove of the De-Giorgi celebrated theorem on Holder continuity of the solution of elliptic equation of second order (see [24]). We believe that by employing Landis method we can significantly widen class of the equations for which localization property holds by including absorption term.

5. CONCLUSION

Einstein paradigm was implemented for generalized Brownian motion process with drift and absorption or reaction. We consider processes which allow implementation of Einstein paradigm for dynamic process with time interval of free jump to be inverse proportional to density and norm of its gradient. The problem was reduced to a nonlinear partial differential inequality with drift and absorption or reaction. We proved that this type of processes of $\alpha - \beta$ jumps, exhibits Localization property. To prove this Ladyzhenskaya- De Giorgi iterative procedure was used. Obtained result has very clear physical interpretation.

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