Point-splitting regularization of composite operators and anomalies

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Abstract

The point-splitting regularization technique for composite operators is discussed in connection with anomaly calculation. We present a pedagogical and self-contained review of the topic with an emphasis on the technical details. We also develop simple algebraic tools to handle the path ordered exponential insertions used within the covariant and non-covariant version of the point-splitting method. The method is then applied to the calculation of the chiral, vector, trace, translation and Lorentz anomalies within diverse versions of the point-splitting regularization and a connection between the results is described. As an alternative to the standard approach we use the idea of deformed point-split transformation and corresponding Ward-Takahashi identities rather than an application of the equation of motion, which seems to save the complexity of the calculations.

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1 Introduction

The point-splitting regularization of the composite operators is a method which has a long history. The early works \cite{1} date back more than fifty years ago. The idea of the method is very simple: the slightly different space time points are assigned to the elementary fields from which a composite operator is built. As a result, the short distance singularities of the composite operator which appear in the limit of the coinciding points are regulated. E.g. the fermionic electromagnetic current

\[ J_{\mu}^{em}(x) = \overline{\psi}(x)\gamma_{\mu}\psi(x) \]  \hspace{1cm} (1)

can be regulated according to the prescription

\[ J_{\mu}^{em}(x)^{\text{reg}} = \overline{\psi}(x + \varepsilon)\gamma_{\mu}\psi(x - \varepsilon) \]  \hspace{1cm} (2)

and the short distance (UV) singularities can be isolated as the divergent terms in the expansion of the regularized expression for \( \varepsilon \to 0 \). However, the naive form of the point-splitting \( \text{(2)} \) suffers from the violation of the \( U(1)_{em} \) gauge invariance. This loophole can be cured introducing a suitable compensating factor, the gauge invariant regulated current can be constructed as

\[ J_{\mu}^{em}(x)^{\text{reg}} = \overline{\psi}(x + \varepsilon)\gamma_{\mu}\psi(x - \varepsilon) \exp\left(i\varepsilon \int_{x-\varepsilon}^{x+\varepsilon} dy \cdot A(y)\right). \]  \hspace{1cm} (3)

Various modifications of this approach were described in the literature \cite{2}, and the point-splitting method still attracts interest in various contexts \cite{3}.

One of the most important applications of this regularization scheme is the calculation of the quantum anomalies. The first calculation of such a type was done already in the pioneering work \cite{4} on the Adler-Bell-Jackiw anomaly and was also systematically used in the fundamental work \cite{5} to get the general form of the nonabelian anomaly. It also proved to be a suitable tool for the calculation of the anomalies connected with the trace and divergence of the energy-momentum tensor \cite{6} in the gravitational background. Since the time of the first works on this topic, the point-splitting regularization method has become a well-understood standard routine.

In this short review we attempt to present a self-contained pedagogical introduction to the method and its application to the anomaly calculation with an emphasis on the technical details and the language of the contemporary field theory. We also offer a rather nonstandard approach to the derivation of the anomalies in terms of the Ward-Takahashi identities for deformed point-split transformations rather than using the equations of motion, which seems to save the complexity of the calculations.

The paper is organized as follows. In Section 2 we concentrate on the problem of the short distance singularities of the fermion propagator in the background of a nonabelian gauge field and the heat kernel method is briefly reviewed. In Section 3 we outline the general strategy of the anomalies calculation. In Section 4, 5, 6, 7 and 8 we illustrate this general strategy using the concrete examples of the chiral, trace, translational and Lorentz anomalies respectively. The non-covariant point-splitting and the corresponding modifications of the anomalies are described in Section 9. The properties of the vector current in various versions of the point-splitting and integrability of the vector current are discussed in Section 10. Some of the technical details are postponed to Appendices A and B.
2 The singularities of the fermion propagator at short distances, the heat kernel method

In this section we give a pedagogical overview of one of the most efficient methods for obtaining the short distance properties of the euclidean propagator of the fermions in the background nonabelian gauge field. This method is based on the properties of the so called heat kernel, associated with a suitable elliptic differential operator of the second order. We also introduce some notation which will be useful in the rest of the article.

Let us first consider the following elliptic operator, operating on the sections of some Hermitian vector bundle over four-dimensional flat Euclidean space

$$ \Delta = -D^2 + V, $$

(4)

where the covariant derivative is given as

$$ D = \partial + A $$

(5)

and the gauge field $A$ and the (positive) potential $V$ satisfy the following hermiticity properties

$$ A^+ = -A, \quad V^+ = V, $$

(6)

i.e. the operator $\Delta$ is positive and Hermitian. Let us define an Euclidean “scalar propagator with mass $m$” as the $x$-representation matrix element of the inverse of the operator $\Delta + m^2$

$$ G(x, y; m^2) = (x| (\Delta + m^2)^{-1} |y). $$

(7)

First we will investigate the $|x - y| \to 0$ asymptotics of this propagator with a general potential $V$. The result (with specific choice of the potential $V$) will be then used to obtain the short distance behavior of the fermion propagator defined as the inverse of the Dirac operator $\gamma \cdot D + i m$

$$ S(x, y; m) = (x| (\gamma \cdot D + im)^{-1} |y) $$

(8)

where $\gamma_\mu$ are the antihermitian Euclidean $\gamma$- matrices, which satisfy the anticommutation relations

$$ \{\gamma_\alpha, \gamma_\beta\} = -2 \delta_{\alpha\beta}. $$

(9)

Let us introduce now the key object of our further considerations. The heat kernel $G(x, y; \tau)$ of the elliptic operator $\Delta$ given by the formula (4) is defined as the kernel (i.e. the $x$-representation matrix element) of the operator $e^{-\tau \Delta}$, explicitly

$$ G(x, y; \tau) = (x| e^{-\tau \Delta} |y). $$

(10)

The heat kernel satisfies the following partial differential equation,

$$ -\frac{\partial}{\partial \tau} G(x, y; \tau) = \Delta G(x, y; \tau), $$

(11)

with the initial condition

$$ \lim_{\tau \to 0} G(x, y; \tau) = \delta^{(4)}(x - y). $$

(12)
Let us summarize some of its properties. If we find the solution of the equation (11) in the form

\[ G(x, y; \tau) = \frac{1}{(4\pi \tau)^2} e^{-\frac{|x-y|^2}{4\tau}} F(x, y; \tau), \]  

(13)

the initial condition (12) requires then

\[ \lim_{\tau \to 0} F(x, y; \tau) = 1. \]  

(14)

Assuming for the function \( F(x, y; \tau) \) the following asymptotic expansion \([7]\) for \( \tau \to 0 \)

\[ F(x, y; \tau) = \sum_{n=0}^{\infty} a_n(x, y) \tau^n, \]  

(15)

the coefficients of this expansion satisfy then the following set of recursion relations, which can be obtained by inserting the asymptotics (15) to the equation (11) with the initial condition (12)

\[ (x - y) \cdot D_x a_0(x, y) = 0, \quad a_0(x, x) = 1, \]

\[ n a_n(x, y) + (x - y) \cdot D_x a_n(x, y) = -\Delta x a_{n-1}(x, y). \]  

(16)

The coefficients \( a_n(x, y) \) (known as the Seeley-DeWitt coefficients\([8]\)) are smooth for \( |x - y| \to 0 \). Let us indicate, how these relations can be explicitly solved. Introducing (with \( x \) and \( y \) fixed)

\[ x_t = y + t(x - y) \]  

(17)

and using the first relation (16), we get for \( a_0(x_t, y) \) the following ordinary differential equation

\[ \frac{d}{dt} a_0(x_t, y) = -(x - y) \cdot A(x_t) a_0(x_t, y), \]  

(18)

with the initial condition \( a_0(x_0, y) = 1 \), which can be easily solved in terms of the \( T \)-ordered exponential

\[ a_0(x_t, y) = T \exp \left( -\int_0^t dt (x - y) \cdot A(x_t) \right). \]  

(19)

\( a_0(x, y) \) is then given by the following formula

\[ a_0(x, y) = a_0(x_1, y) = T \exp \left( -\int_0^1 dt (x - y) \cdot A(x_t) \right) = \Omega(x, y) \]  

(20)

as the parallel transporter \( \Omega(x, y) \) along the straight line connecting the points \( x \) and \( y \). In the same way we get the following differential equation for \( a_1(x_t, y) \)

\[ \frac{d}{dt} a_1(x_t, y) = -(x - y) \cdot A(x_t) a_1(x_t, y) - \Delta x a_0(x_t, y), \]  

(21)

\[ ^3 \text{Let us note, that } \Delta^{(4)}(x, 0) = \int_0^{\infty} \frac{d\tau}{(4\pi \tau)^2} e^{-\frac{|x-y|^2}{4\tau}} = \frac{1}{4\pi^2|x-y|^2}, \text{ corresponds to the free scalar Euclidean propagator with zero mass.} \]

\[ ^4 \text{For a review and complete list of references see also [8].} \]
the solution of which can be expressed in terms of the known coefficient $a_0(x_t, y)$

$$ta_1(x_t, y) = -a_0(x_t, y) \int_0^t d\tau a_0(x_\tau, y)^{-1} \Delta_x a_0(x_\tau, y),$$

therefore

$$a_1(x, y) = -a_0(x, y) \int_0^1 dt a_0(x_t, y)^{-1} \Delta_x a_0(x_t, y)$$

$$= -\int_0^1 dt a_0(x_t, y) \Delta_x a_0(x_t, y)$$

(here we used simple properties of the coefficient $a_0(x, y)$, namely $a_0(x, y)^{-1} = a_0(y, x)$ and $a_0(x, y)a_0(y, z) = a_0(x, z)$; valid for $x, y, z$ on the same line). The generalization of this procedure for general $n$ is straightforward, the general formula reads

$$a_n(x, y) = -\int_0^1 dt t^{n-1} a_0(x_t, y) \Delta_x a_{n-1}(x_t, y).$$

Let us now use the properties of the heat kernel to get the short distance asymptotics of the scalar propagator $G(x, y; m^2)$. This can be achieved by expressing it in terms of the heat kernel as

$$G(x, y; m^2) = \int_0^\infty d\tau e^{-\tau m^2} G(x, y; \tau);$$

this formula is known as the Schwinger proper-time representation of the propagator. The short distance asymptotics of $G(x, y; m^2)$ can be then obtained as a consequence of the $\tau \to 0$ asymptotics of the heat kernel $G(x, y; \tau)$. Indeed, from the Schwinger representation of the scalar propagator (25) we have

$$G(x, y; m^2) = \int_0^\infty \frac{d\tau}{(4\pi \tau)^2} e^{-\tau m^2 - \frac{|x-y|^2}{4\tau}} F(x, y, \tau),$$

i.e. for $|x-y| \neq 0$ the integrand is well-behaved both for $\tau \to 0$ and $\tau \to \infty$. For $|x-y| = 0$ we have however a nonintegrable singularity at $\tau \to 0$. In order to isolate the short distance singularities, let us divide the integrand into two parts by means of adding and subtracting the leading asymptotics of the integrand for $\tau \to 0$:

$$G(x, y; m^2) = \int_0^\infty \frac{d\tau}{(4\pi \tau)^2} e^{-\tau m^2 - \frac{|x-y|^2}{4\tau}} \sum_{i=0}^n a_i(x, y) \tau^i$$

$$+ \int_0^\infty \frac{d\tau}{(4\pi \tau)^2} e^{-\tau m^2 - \frac{|x-y|^2}{4\tau}} (F(x, y; \tau) - \sum_{i=0}^n a_i(x, y) \tau^i).$$

(27)

The second term on the right hand side has smooth limit for $|x-y| \to 0$, if we take $n$ large enough, (the corresponding integrand behaves as $\tau^{n-1}$ for $\tau \to 0$ and $\tau^{n-2} e^{-\tau m^2}$ for $\tau \to \infty$).

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5Let us recall the limit $F(x, y, \tau) \to 1$ for $\tau \to 0$ and, in the finite volume, the asymptotic behaviour of the heat kernel $G(x, y, \tau) \approx \exp(-\lambda_0 \tau) \Phi_0(y) \Phi_0(x)$ for $\tau \to \infty$, where $\lambda_0 > 0$ is the lowest eigenvalue and $\Phi_0(x)$ the corresponding eigenvector of the operator $\Delta$. 
The source of the singular behavior of the scalar propagator for $|x-y| \to 0$ corresponds to the first term.

Let us describe this in more detail. We have

$$G(x, y; m^2) = \sum_{i=0}^{n} a_i(x, y) I_n(x - y) + R(x, y),$$

(28)

where $R(x, y)$ is finite in the limit $|x - y| \to 0$ and $I_n(x)$ are the following integrals

$$I_n(x) = \int_0^\infty \frac{d\tau}{(4\pi^2)^2} \tau^n e^{-\tau m^2 - \frac{|x|^2}{4\tau}} = \frac{m^{-n+1}}{8\pi^2} \left( \frac{|x|}{2} \right)^{n-1} K_{-n+1}(m|x|).$$

(29)

Here $K_n(z)$ are the MacDonald functions:

$$K_n(z) = \frac{1}{2} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k-n} + \frac{(-1)^{n+1}}{2} \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left( \frac{z}{2} \right)^{2(k+n)} (2\ln \left( \frac{z}{2} \right) - \psi(k+1) - \psi(k+n+1)),$$

(30)

satisfying

$$K_n(z) = K_{-n}(z).$$

(31)

Explicitly

$$I_0(x) = \Delta_E^{(4)}(x; m^2) = \frac{1}{4\pi^2 x^2} m|x| K_1(m|x|)$$

$$= \frac{1}{4\pi^2} \left( \frac{1}{x^2} + \frac{m^2}{4} \ln m^2 |x|^2 - \frac{m^2}{4} (2\ln 2 + 2\gamma_E - 1) \right)$$

$$+ O(m^2|x|^2, m^2|x|^{2\ln m^2|x|^2})$$

(32)

and

$$I_1(x) = \Delta_E^{(2)}(x; m^2) = \frac{1}{8\pi^2} K_0(m|x|)$$

$$= - \frac{1}{16\pi^2} (\ln m^2 |x|^2 - 2\ln 2 + 2\gamma_E - 2) + O(m^2|x|^2, m^2|x|^{2\ln m^2|x|^2})$$

(33)

where $\Delta_E^{(d)}(x - y; m^2)$ is the free Euclidean scalar propagator (i.e. for $A = 0$ and $V = 0$) in $d$ dimensional euclidean space. Note also, that for $n \geq 2$, the $I_n(x)$ (and also $\partial_\mu I_n(x)$) are

$^6$Here

$$\psi(k+1) = -\gamma_E + \sum_{j=1}^{k} \frac{1}{j}.$$
regular for \( x \to 0 \), i.e. we can keep only the first two terms in the sum on the right hand side of (28) in order to pick up explicitly the potentially singular part.

Therefore we have

\[
G(x, y; m^2) = \frac{1}{4\pi^2} \left( \frac{a_0(x, y)}{|x - y|^2} + \frac{1}{4} \ln m^2 |x - y|^2 (a_0(x, y)m^2 - a_1(x, y)) \right) + O(1).
\]  

The fermion propagator, defined as

\[
S(x, y; m) = (x | \frac{1}{\gamma \cdot D + im} | y),
\]  

can be rewritten in the form

\[
S(x, y; m) = (x | (\gamma \cdot D - im) \frac{1}{(\gamma \cdot D)^2 + m^2} | y)
\]  
\[
= (\gamma \cdot D - im)(x | -D^2 - \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu} + m^2 | y)
\]  
\[
= (\gamma \cdot D - im)(x | \frac{1}{\Delta + m^2} | y)
\]  

with positive operator

\[
\Delta = -D^2 - \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}.
\]

The matrices \( \sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta] \) are the generators of the \( SO(4) \) euclidean rotations and

\[
F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
\]

is the gauge field strength. I.e., expressing the fermion propagator in terms of the “scalar propagator with mass \( m \)” corresponding to the operator (37), we have for the potentially singular part of the fermion propagator

\[
S(x, y; m) = (\gamma \cdot D - im)G(x, y; m^2)
\]  
\[
= \frac{1}{4\pi^2} \left( \frac{(\gamma \cdot D - im)a_0(x, y)}{|x - y|^2} - \frac{2\gamma_\mu(x - y_\mu a_0(x, y)}{|x - y|^4}ight.
\]
\[
+ \frac{1}{4} \ln m^2 |x - y|^2 (\gamma \cdot D - im)(a_0(x, y)m^2 - a_1(x, y))
\]
\[
+ \frac{1}{2} \frac{2\gamma_\mu(x - y_\mu}{|x - y|^2} (a_0(x, y)m^2 - a_1(x, y)) + O(1) \right).
\]  

To isolate the singular part of the propagators (34) and (39), it remains now to expand the Seeley- de Witt coefficients in power series in \( |x - y| \). Before doing this, we add one important note. The above defined scalar and fermion propagators have one inconvenient property - they are not gauge covariant. E.g., for the fermion propagator we have the following transformation relation with respect to the gauge transformation \( U(x) \):

\[
S(x, y; m) \to U(x)S(x, y; m)U^+(y).
\]
As a consequence of this, if we introduce

\[ \mathfrak{P} = \frac{1}{2}(x + y), \quad \varepsilon = \frac{1}{2}(y - x), \]  

(41)

the coefficients of the expansion of the propagators \( G(x, y) \) and \( S(x, y; m) \) in powers and logarithms of \( \varepsilon \) are not gauge covariant functions of \( \mathfrak{P} \).

For the calculations presented in the following sections we need rather objects, which transform covariantly. For this purpose, let us note, that the parallel transporter along the straight line (which is nothing but the Seeley-de Witt coefficient \( a_0(x, y) \))

\[ \Omega(x, y) = P \exp \left( - \int_0^1 dt(x - y) \cdot A(y + t(x - y)) \right) = a_0(x, y). \]  

(42)

transforms according to the prescription

\[ \Omega(x, y) \rightarrow U(x)\Omega(x, y)U^+(y). \]  

(43)

Let us therefore define the “covariant propagator”

\[ S^{\text{cov}}(x, \varepsilon) = \Omega(x, x - \varepsilon)S(x - \varepsilon, x + \varepsilon; m)\Omega(x + \varepsilon, x). \]  

(44)

The transformation properties of the gauge exponential ensure the following transformation of the covariant propagator:

\[ S^{\text{cov}}(x, \varepsilon) \rightarrow U(x)S^{\text{cov}}(x, \varepsilon)U^+(x). \]  

(45)

In the same way we can “improve” the scalar propagator, which transforms noncovariantly

\[ G(x, y; m^2) \rightarrow U(x)G(x, y; m^2)U^+(y), \]  

(46)

and define the covariant propagator

\[ G^{\text{cov}}(x, \varepsilon) = \Omega(x, x - \varepsilon)G(x - \varepsilon, x + \varepsilon; m^2)\Omega(x + \varepsilon, x) \]  

(47)

with covariant transformation law

\[ G^{\text{cov}}(x, \varepsilon) \rightarrow U(x)G^{\text{cov}}(x, \varepsilon)U^+(x). \]  

(48)

The main advantage of such covariant propagators is, that the \( x \)-dependent coefficients of the expansion in the powers and logarithms of \( \varepsilon \) are now gauge covariant functions.

From the expression (54), we get then

\[ G^{\text{cov}}(x, \varepsilon) = \frac{1}{4\pi^2} \left( \frac{1}{|\varepsilon|^2} + \frac{1}{4}\text{Im}m^2|\varepsilon|^2(m^2 - a_0(x, x - \varepsilon)a_1(x - \varepsilon, x + \varepsilon)a_0(x + \varepsilon, x)) \right) + O(1). \]  

(49)

To make use of this formula, we need the expansion of the expression \( a_0(x, x - \varepsilon)a_1(x - \varepsilon, x + \varepsilon)a_0(x + \varepsilon, x) \) in the powers of \( \varepsilon \). We only quote the result of the calculation here, the details can be found in Appendix A:

\[ a_0(x, x - \varepsilon)a_1(x - \varepsilon, x + \varepsilon)a_0(x + \varepsilon, x) = \frac{i}{2}\sigma_{\alpha\beta}F_{\alpha\beta}(x) + \frac{1}{3}\varepsilon_{\alpha}[D_\beta, F_{\beta\alpha}](x) + O(\varepsilon^2). \]  

(50)
With this formula at hand, we have finally
\[
G^{\text{cov}}(x, \varepsilon) = \frac{1}{16\pi^2} \left( \frac{1}{|\varepsilon|^2} + \ln m^2 |\varepsilon|^2 (m^2 - \frac{i}{2} \sigma_{\alpha\beta} F_{\alpha\beta}(x)) \right) + \mathcal{O}(1). \tag{51}
\]

Before we proceed to the analogous expression for the fermion propagator, let us now make a useful observation. For any (sufficiently smooth) section \(\phi(x)\), the operation
\[
\phi(x) \rightarrow \Omega(x, x - \varepsilon) \phi(x - \varepsilon) \tag{52}
\]
can be expressed in the form
\[
\Omega(x, x - \varepsilon) \phi(x - \varepsilon) = e^{-\varepsilon \cdot \vec{D} \phi(x)}. \tag{53}
\]
Indeed, the functions \(\phi_1(x, t) = \Omega(x, x - t\varepsilon) \phi(x - t\varepsilon)\) and \(\phi_2(x, t) = e^{-t\varepsilon \cdot \vec{D} \phi(x)}\) both satisfy the following differential equation
\[
\frac{d}{dt} \phi_i(x, t) = -\varepsilon \cdot \vec{D} \phi_i(x, t) \tag{54}
\]
with the initial condition
\[
\phi_i(x, 0) = \phi(x). \tag{55}
\]
This statement is clear for \(i = 2\), let us prove it for \(i = 1\). We have
\[
\frac{d}{dt} \phi_1(x, t) = (\varepsilon \cdot \delta y \Omega(x, y)) \big|_{y=x-t\varepsilon} \phi(x - t\varepsilon) - \Omega(x, x - t\varepsilon) \varepsilon \cdot \partial_x \phi(x - t\varepsilon) \\
= -\varepsilon \cdot \partial_x (\Omega(x, x - t\varepsilon) \phi(x - t\varepsilon)) + (\varepsilon \cdot \partial_y \Omega(x, y)) \big|_{y=x-t\varepsilon} \phi(x - t\varepsilon) \\
= -\varepsilon \cdot \partial_x (\Omega(x, x - t\varepsilon) \phi(x - t\varepsilon)) - \varepsilon \cdot A(x) \Omega(x, x - t\varepsilon) \phi(x - t\varepsilon) \\
= -\varepsilon \cdot \vec{D} \phi_1(x, t), \tag{56}
\]
where we used the relation \((x - y) \cdot D_2 \Omega(x, y) = 0\) in the third line.

Using now this result and the formula for the fermion propagator
\[
S(x, y; m) = -imG(x, y; m^2) + \gamma \cdot \vec{D} G(x, y; m^2) \tag{57}
\]
we have
\[
S^{\text{cov}}(x, \varepsilon) = -imG^{\text{cov}}(x, \varepsilon) + e^{-\varepsilon \cdot \vec{D} \gamma} \vec{D} G(x, y; m^2) e^{\varepsilon \cdot \vec{D} y} \big|_{x=y},
\]
where
\[
\vec{D} = \partial + A, \tag{58}
\]
\[
\vec{D} = \overleftarrow{\partial} - A. \tag{59}
\]
It is not difficult to show (the details are postponed to Appendix A), that
\[
e^{-\varepsilon \cdot \vec{D} \mu} \vec{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D} y} \big|_{x=y} = \frac{1}{2} \left( [\vec{D}_\mu, G^{\text{cov}}(x, \varepsilon)] - \partial^\mu G^{\text{cov}}(x, \varepsilon) \right) \bigg|_{x=y} + G_\mu(x, \varepsilon) G^{\text{cov}}(x, \varepsilon) - G^{\text{cov}}(x, \varepsilon) G_\mu(x, -\varepsilon) + H_\mu(x, \varepsilon) G^{\text{cov}}(x, \varepsilon) + G^{\text{cov}}(x, \varepsilon) H_\mu(x, -\varepsilon)), \tag{60}
\]
where the functions $G_\mu(x,\varepsilon)$ and $H_\mu(x,\varepsilon)$ have the following expansion (the complete formulae are given in Appendix A)

\begin{align}
G_\mu(x,\varepsilon) &= \varepsilon_\nu F^\nu_\mu(x) - \frac{1}{2}[\varepsilon \cdot D, \varepsilon_\nu F^\nu_\mu(x)] + O(\varepsilon^3), \\
H_\mu(x,\varepsilon) &= -\frac{1}{2}\varepsilon_\nu F^\nu_\mu(x) + \frac{1}{3}[\varepsilon \cdot D, \varepsilon_\nu F^\nu_\mu(x)] + O(\varepsilon^3).
\end{align}

As a result we have

\begin{align}
S^{\text{cov}}(x,\varepsilon) &= -imG^{\text{cov}}(x,\varepsilon) + \frac{1}{2}\gamma_\mu([D_\mu, G^{\text{cov}}(x,\varepsilon)] - \partial_\mu G^{\text{cov}}(x,\varepsilon)) \\
&+ G_\mu(x,\varepsilon)G^{\text{cov}}(x,\varepsilon) - G^{\text{cov}}(x,\varepsilon)G_\mu(x,-\varepsilon) \\
&+ H_\mu(x,\varepsilon)G^{\text{cov}}(x,\varepsilon) + G^{\text{cov}}(x,\varepsilon)H_\mu(x,-\varepsilon)).
\end{align}

and, putting the formulas together,

\begin{align}
S^{\text{cov}}(x,\varepsilon) &= \frac{1}{16\pi^2} \left( \frac{\gamma \cdot \varepsilon - im}{|\varepsilon|^2} - \frac{m^2 \gamma \cdot \varepsilon - \varepsilon_\mu F^\mu_\nu(x)\gamma_\nu \gamma_5}{|\varepsilon|^2} \\
&+ \text{Im} \frac{m^2}{|\varepsilon|^2}(-im^2 - \frac{1}{2}m\sigma_{\alpha\beta}F_{\alpha\beta}(x) - \frac{1}{3}[D_\mu, F^\mu_\nu]\gamma_\nu + O(\varepsilon)) \\
&+ \frac{1}{3} \frac{\gamma \cdot \varepsilon[D_\mu, F^\mu_\nu]_\nu}{|\varepsilon|^2} - \frac{1}{3} \frac{\gamma_\mu[\varepsilon \cdot D, F^\mu_\nu]_\nu|\varepsilon|^2}{|\varepsilon|^2} + O_{\text{reg}}(1) \right).
\end{align}

Here we keep for the sake of further convenience also the terms, which are formally $O(1)$, but the $\varepsilon \to 0$ limit of which does not exist, but rather depends on the direction in which $\varepsilon$ approaches zero. As we shall see in the following, this terms are responsible for the anomalous divergence of the energy-momentum tensor.

3 The anomalies via point-splitting

The quantum anomalies generally mean a violation of the classical symmetry due to the quantum corrections, which cannot be avoided by suitable renormalization of the quantities involved. Historically, the first example of such a phenomenon was the famous Adler-Bell-Jackiw chiral anomaly \[10\]. It is connected with the so-called chiral $U(1)$ transformation

\begin{align}
\delta \psi &= i\alpha \gamma_5 \psi, \\
\delta \psi^+ &= i\psi^+ \alpha \gamma_5,
\end{align}

where $\alpha$ is an infinitesimal real parameter. The corresponding Noether current

\begin{equation}
j_{5\mu}(x) = \psi^+ \gamma_\mu \gamma_5 \psi
\end{equation}

obeys classically the following equation

\begin{equation}
\partial \cdot j_5(x) = 2im\psi^+ \gamma_5 \psi
\end{equation}

and it is conserved in the chiral limit $m \to 0$. In the quantum case, however, the anomalous term $\frac{1}{8\pi^2}F^*_\mu\nu F^\mu_\nu$, (where $F^*_\mu\nu$ is the dual tensor to $F^\mu_\nu$) appears on the right hand side of the previous equation, spoiling the chiral invariance of the theory. Though this term could be
removed by a suitable counterterm added to the chiral current \( j_{\mu 5} (x) \), this counterterm spoils
gauge invariance and is therefore not admissible in any reasonable gauge theory. This situation
is typical; the anomalous symmetries appear in pairs and saving one of them necessarily brings
about spoiling of the other. Further example of the anomalous pair is the conflict between
the scale and translational symmetries leading to the so-called trace anomaly\footnote{The anomalous trace of the energy-momentum tensor.}. In the
gravitational background there is also the so-called Lorentz anomaly, the consequence of which
is the anomalous antisymmetric part of the energy-momentum tensor. For a comprehensive
discussion of this and related topics see the books\footnote{The anomalous trace of the energy-momentum tensor.}, where also an extensive list of references
can be found.

In this section we give rather nonstandard derivation of the axial and trace anomalies
via point-splitting regularization. We will show, that the anomalies in this formalism can
be understood as the result of the non-invariance of the classical action with respect to the
regularized nonlocal “point-split” transformations, which replace the naive form of the chiral
rotation, scale transformation and (covariant) translation. The remnant of this noninvariance
survives in the quantum case the procedure of removing the cut-off. This gives a partial ex-
planation of why the point-splitting regularization produces anomaly in the divergence rather
than in the trace of the energy-momentum tensor. The idea of the deformed transformations
was used in similar context of the Fujikawa-like regularization procedures in the paper\footnote{The anomalous trace of the energy-momentum tensor.},
where the anomalous pair of the chiral and abelian gauge symmetry was investigated.

The main object of our interest is the Euclidean generating functional of the correlators
of the gauge fermionic currents given formally as the fermionic functional integral

\[
Z_E[A] = \int \mathcal{D} \psi^+ \mathcal{D} \psi \exp(-S_E),
\]

where

\[
S_E = \int d^4 x \frac{1}{2} \gamma \cdot \overrightarrow{D} + i m \psi
\]
is the Euclidean action. We shall tacitly assume some intermediate regularization (e.g. Pauli-Villars) with the cut-off parameter \( \Lambda \), which makes the functional integral well defined and
which justifies the formal operations we are going to perform. This intermediate regularization
will be removed at the end of the calculation. The only requirements on this regularization
are that it does not deform the four-dimensional algebra of the \( \gamma \)-matrices (in order to
preserve the four-dimensional \( \gamma_5 \) ) and also the property which we call “naturalness” - i.e.
that it reproduces the true value of the convergent integrals after the cut-off is removed. The
Ward-Takahashi identity, corresponding to the general infinitesimal change of the integration
variables

\[
\begin{align*}
\psi & \rightarrow \psi + \delta \psi \\
\psi^+ & \rightarrow \psi^+ + \delta \psi^+
\end{align*}
\]

has the form

\[
\int \mathcal{D} \psi^+ \mathcal{D} \psi \left( \delta S_E + \int d^4 x \left( \frac{\delta \psi(x)}{\delta \psi^+(x)} + \frac{\delta \psi^+(x)}{\delta \psi(x)} \right) \right) \exp(-S_E) = 0,
\]

where \( \delta S_E = \int d^4 x \delta \mathcal{L}_E (x) \) is the variation of the action with respect to the transformation
\footnote{The anomalous trace of the energy-momentum tensor.}. Here we pick up the formal remnant of the determinant of the change of the integration

\[
\int \mathcal{D} \psi^+ \mathcal{D} \psi \left( \delta S_E + \int d^4 x \left( \frac{\delta \psi(x)}{\delta \psi^+(x)} + \frac{\delta \psi^+(x)}{\delta \psi(x)} \right) \right) \exp(-S_E) = 0,
\]
variables, which we assume to be consistently regularized by the above mentioned intermediate regularization.\(^8\) Within the naive form of the Ward-Takahashi identity this term is omitted, on the other hand it can be the source of the quantum anomaly provided it survives after the cut-off is removed.\(^9\) Note that the local form of the Ward-Takahashi identity can be obtained by means of the standard technique of localizing the transformation (70) and can be written in the form

\[
\int D\psi^+ D\psi \left( \int d^4x \theta(x) \left( \delta L_E(x) - \partial \cdot j(x) + \left( \frac{\delta \delta \psi(x)}{\delta \psi(x)} + \frac{\delta \delta \psi^+(x)}{\delta \psi^+(x)} \right) \right) \right) \exp(-S_E) = 0,
\]

(72)

where \(\theta(x)\) is an infinitesimal parameter of the localized transformation

\[
\psi \rightarrow \psi + \theta \delta \psi
\]

\[
\psi^+ \rightarrow \psi^+ + \theta \delta \psi^+
\]

(73)

and \(j_\mu(x)\) is the Noether current. Another subject of our investigation will be therefore the generating functional of the connected correlators of the gauge currents with one insertion of the divergence of the Noether current \(j_\mu(x)\) or one insertion of the \(\delta L_E(x)\) corresponding to the transformation (70), assuming these operators to be regularized by point-splitting, i.e.

\[
\partial \cdot j(x)^{\text{reg}} \cdot Z_E[A]_c = \langle \partial \cdot j(x)^{\text{reg}} \rangle_c = \int D\psi^+ D\psi \ j_\mu(x)^{\text{reg}} \exp(-S_E)|_c
\]

\[
= Z_E[A]^{-1} \int D\psi^+ D\psi \ j_\mu(x)^{\text{reg}} \exp(-S_E)
\]

(74)

and

\[
\delta L_E(x)^{\text{reg}} \cdot Z_E[A]_c = \langle \delta L_E(x)^{\text{reg}} \rangle_c = \int D\psi^+ D\psi \delta L_E(x)^{\text{reg}} \exp(-S_E)|_c
\]

\[
= Z_E[A]^{-1} \int D\psi^+ D\psi \delta L_E(x)^{\text{reg}} \exp(-S_E).
\]

(75)

Here we need not assume any other intermediate regularization, since the loops with one insertion of the regularized operator are already finite and the loops without the operator insertion are formally cancelled. Alternatively we can think about this point-split functionals as being produced by removing the intermediate cut-off, which is possible because all the relevant integrals are finite and thanks to the required naturalness property of the intermediate regularization are not deformed by this procedure. This is the point of view we will adopt in the following.

In the point-splitting method we introduce the anomaly as the insertion defined by the difference

\[
\mathcal{A}(x) \cdot Z_E[A]_c = \lim_{\text{removed point-splitting}} \left( \partial \cdot j(x)^{\text{reg}} \cdot Z_E[A]_c - \delta L_E(x)^{\text{reg}} \cdot Z_E[A]_c \right),
\]

(76)

\(^8\)E.g. for the Pauli-Villars regularization this formally divergent expression is cancelled by the analogous term coming from the transformation of the regulator fields with opposite statistics. Within the framework of the dimensional regularization it vanishes by definition.

\(^9\)However, for the Pauli-Villars regularization, the source of the axial anomaly is the additional contribution of the regulator fields to \(\delta S_E\). Within the consistent dimensional regularization in this case also the additional terms appear in \(\delta S_E\) due to the commutation of \(\gamma_5\) with the \(\gamma_\mu\) with \(\mu > 4\).
i.e. as the anomalous divergence of the Noether current in the limit of the point-splitting cut-off removed. Our strategy for calculation of the anomalies will be as follows. We first deform the localized transformation prescription (73) in such a way, that the corresponding Noether current is identical with the point-split Noether current of the original transformation. As a result, the local Ward-Takahashi identity (72) for the deformed transformation will produce the point-split version of the original identity, i.e. with the relevant composite operators regularized by the point-splitting. As we shall see, the deformation of the transformation brings about the presence of additional terms in $\delta S_E$ which prove to be the source of the anomaly.

On the other hand, the determinant of the deformed transformation will be identically equal to one; this property also survives the process of removing of the intermediate cut-off. In the next sections we give explicit examples of this general strategy.

4 The chiral anomaly

Let us give the first illustration of the above described strategy. For the nonabelian chiral transformation

$$\delta \psi = \alpha \gamma_5 \psi$$
$$\delta \psi^+ = \psi^+ \alpha \gamma_5$$

(77)

where $\alpha = \alpha_a T^a$ is an antihermitian infinitesimal matrix from the Lie algebra of the gauge group, the Noether currents are identical with the axial currents given by

$$j_{\mu}^a = \psi^+ T^a \gamma_\mu \gamma_5 \psi$$

(78)

and on the classical level we have

$$D_\mu j_{\mu}^a = \delta \mathcal{L}_E(x)^a = 2i m \psi^+ T^a \gamma_5 \psi,$$

(79)

where $\delta \mathcal{L}_E(x)^a$ is the variation of the Lagrangian under the chiral rotations. Our choice for the point-split version of these currents is

$$j_{\mu}^{a, \text{reg}}(x) = \psi^+(x + \varepsilon) \Omega(x + \varepsilon, x) T^a \gamma_\mu \gamma_5 \Omega(x, x - \varepsilon) \psi(x - \varepsilon),$$

(80)

where $\Omega(x, y)$ is the parallel transporter along the straight line connecting the points $x$ and $y$. Note, that this definition gives gauge covariant expression; the regularized current multiplet transforms according to the adjoint representation of the gauge group. For the further convenience, let us introduce also the point-split fields (cf. also Section 2)

$$\psi_\varepsilon(x) = e^{-\varepsilon \cdot D} \psi(x) = \Omega(x, x - \varepsilon) \psi(x - \varepsilon),$$
$$\psi^+_\varepsilon(x) = \psi^+(x + \varepsilon) \Omega(x + \varepsilon, x)$$

(81)

in terms of which

$$j_{\mu}^{a, \text{reg}} = \psi^+_\varepsilon T^a \gamma_\mu \gamma_5 \psi_\varepsilon.$$
Let us note, that the fields $\psi_\varepsilon(x)$ and $\psi_\varepsilon^+(x)$ have the same transformation properties with respect to the gauge transformations as the original fields $\psi(x)$ and $\psi^+(x)$. As a result, the regularized current $J_5^{\mu,\text{reg}}$ is gauge covariant. Note also, that the covariant propagator introduced in the previous section can be expressed in terms of the correlator of the point-split fields

$$\langle \psi_\varepsilon(x) \psi_\varepsilon^+(x) \rangle_c = Z_E[A]^{-1} \int D\psi^+ D\psi \psi_\varepsilon(x) \psi_\varepsilon^+(x) \exp(-S_E) = S^{\text{cov}}(x, \varepsilon).$$  \hspace{1cm} (83)

The variation of the Lagrangian under the chiral rotations, which represent the naive divergence of the axial currents

$$\delta\mathcal{L}_E(x)^a = 2i m^{\psi^+} T^a \gamma_5 \psi$$  \hspace{1cm} (84)

can be regulated analogously as

$$\delta\mathcal{L}_E(x)^{a,\text{reg}} = 2i m^{\psi^+}(x + \varepsilon) \Omega(x + \varepsilon, x) T^a \gamma_5 \Omega(x, x - \varepsilon) \psi(x - \varepsilon) = 2i m^{\psi^+} T^a \gamma_5 \psi_\varepsilon.$$

Let us now introduce the deformed local chiral rotation with infinitesimal parameter $\alpha(x) = \alpha_a(x) T^a$ as follows

$$\delta\psi = e^{-\varepsilon D} \alpha \gamma_5 e^{-\varepsilon D} \psi,$$
$$\delta\psi^+ = \psi^+ e^{\varepsilon D} \alpha \gamma_5 e^{\varepsilon D}.  \hspace{1cm} (85)$$

Because of the shift of the argument in the expression for $\delta\psi$ and $\delta\psi^+$, we have

$$\int d^4x \left( \frac{\delta\delta\psi(x)}{\delta\psi(x)} + \frac{\delta\delta\psi^+(x)}{\delta\psi^+(x)} \right) = 0  \hspace{1cm} (86)$$

independently of the intermediate regularization. The change of the Euclidean action under the deformed local chiral rotation \(S^3\) is

$$\delta S_E = \int d^4x \left( \psi_\varepsilon^+ [\gamma \cdot D, \alpha] \gamma_5 \psi_\varepsilon + 2i m^{\psi^+} \alpha \gamma_5 \psi_\varepsilon \right)$$
$$+ \int d^4x \left( \psi_\varepsilon^+ \left( \alpha \gamma_5 [e^{-\varepsilon D}, \gamma \cdot D] \right) e^{\varepsilon D} - e^{-\varepsilon D} [\gamma \cdot D, e^{\varepsilon D}] \gamma_5 \alpha \right) \psi_\varepsilon).$$  \hspace{1cm} (87)

Inserting this to the Ward-Takahashi identity \((71)\) and removing the intermediate cut-off we get for the covariant divergence of the point-split axial current

$$D_\mu \langle J_5^{a,\text{reg}} \rangle_c = 2i m \langle \psi_\varepsilon^+ T^a \gamma_5 \psi_\varepsilon \rangle_c + A_5^\alpha,$$  \hspace{1cm} (88)

where the point-split axial anomaly is given by the formula (cf. \((82)\) and \((83)\))

$$A_5^\alpha = \langle \psi_\varepsilon^+ (T^a \gamma_5 e^{-\varepsilon D}, \gamma \cdot D) e^{\varepsilon D} - e^{-\varepsilon D} [\gamma \cdot D, e^{\varepsilon D}] T^a \gamma_5 \psi_\varepsilon \rangle_c$$
$$= -\text{Tr} S^{\text{cov}} \gamma_5 (T^a [e^{-\varepsilon D}, \gamma \cdot D] e^{\varepsilon D} + e^{-\varepsilon D} [\gamma \cdot D, e^{\varepsilon D}] T^a);$$  \hspace{1cm} (89)

here the trace is taken over both the Dirac and color indices. Remembering the formulae

$$[e^{-\varepsilon D}, D_\mu] e^{\varepsilon D} = G_\mu(x, \varepsilon)  \hspace{1cm} (90)$$
and
\[ e^{-\varepsilon \cdot \vec{D}[D_\mu, e^\varepsilon \cdot \vec{D}]} = -G_\mu(x, -\varepsilon) \]  
(91)
we get for the anomaly
\[ A_5^a = -\text{Tr} S^{\text{cov}} \gamma_5 ([T^a, \gamma \cdot G^+] + \{T^a, \gamma \cdot G^- \}) \]  
(92)
where
\[ G^\pm_\mu(x, \varepsilon) = \frac{1}{2} (G_\mu(x, \varepsilon) \pm G_\mu(x, -\varepsilon)) \]  
(93)
i.e. (cf. also Appendix A)
\[ G^+_\mu(x, \varepsilon) = -\frac{1}{2} [\varepsilon \cdot D, F_{\mu\nu}] \varepsilon_{\nu} + O(|\varepsilon|^4), \]  
(94)
\[ G^-_\mu(x, \varepsilon) = \varepsilon_{\nu} F_{\mu\nu}(x) + O(|\varepsilon|^3). \]  
(95)
From the expression for the short distance behavior of the covariant propagator we see that only the second term on the right hand side of (92) contributes in the limit \( \varepsilon \to 0 \)
\[ A_5^a = -\text{Tr} \left( -\frac{1}{16\pi^2} \left( \frac{\varepsilon_\mu}{|\varepsilon|^2} F^{\ast\mu\nu} \gamma_\nu \gamma_\alpha + O(1) \right) \{T^a, F_{\alpha\sigma} \varepsilon_\sigma \} + O(\varepsilon^2) \right). \]  
(96)
Taking the average over the directions of the four-vector \( \varepsilon_\mu \) and performing the trace of the Dirac matrices we get finally the covariant form of the axial anomaly
\[ A_5^a = \frac{1}{8\pi^2} \text{Tr}_C T^a F^{\ast}_{\mu\nu} F_{\mu\nu} + O(\varepsilon), \]  
(97)
where \( \text{Tr}_C \) means a trace over the color indices only.

5 The vector anomaly

As another simple application of the above described method let us consider the nonabelian gauge transformation
\[ \delta \psi = \alpha \psi \]
\[ \delta \psi^+ = -\psi^+ \alpha \]  
(98)
where again \( \alpha = \alpha_a T^a \). The corresponding Noether currents
\[ j^a_\mu = \psi^+ T^a \gamma_\mu \psi \]  
(99)
are covariantly conserved at the classical level, i.e.
\[ D_\mu j^a_\mu = 0. \]  
(100)
Let us choose the following point-split version of these vector currents\[ j^a_{\mu, \text{reg}} = \psi^\varepsilon_\varepsilon T^a \gamma_\mu \psi^\varepsilon \]  
(101)
\footnote{Here again the regularized current \( j^a_{\mu, \text{reg}} \) is gauge covariant because of the transformation properties of the point-split fields \( \psi_\varepsilon \) and \( \psi^\varepsilon_\varepsilon \).}
Because the gauge covariance is manifest within our point-splitting regularization scheme, we do not expect any anomalous divergence. This can be easily proved using the technique described above. Let us consider the point-split transformation with infinitesimal parameter \( \alpha(x) = \alpha_a(x)T^a \)

\[
\begin{align*}
\delta \psi &= e^{-\epsilon \cdot D} \alpha e^{-\epsilon \cdot D} \psi, \\
\delta \psi^+ &= -\psi^+ e^{\epsilon \cdot \hat{D}} \alpha e^{\epsilon \cdot \hat{D}}.
\end{align*}
\] (102)

The variation of the action is then

\[
\delta S_E = \int d^4x \left( \psi^+ [\gamma \cdot D, \alpha] \psi + \psi^+ (\alpha [e^{-\epsilon \cdot \hat{D}}, \gamma \cdot \hat{D}] e^{\epsilon \cdot \hat{D}} + e^{-\epsilon \cdot \hat{D}} [\gamma \cdot \hat{D}, e^{\epsilon \cdot \hat{D}}] \alpha) \psi \right),
\] (103)

We have therefore the following point-split Ward-Takahashi identity

\[
D_\mu \langle j^a, \text{reg} \rangle = A^a
\] (104)

where the possible vector anomaly is

\[
A^a = \langle \psi^+ (T^a [e^{-\epsilon \cdot \hat{D}}, \gamma \cdot \hat{D}] e^{\epsilon \cdot \hat{D}} + e^{-\epsilon \cdot \hat{D}} [\gamma \cdot \hat{D}, e^{\epsilon \cdot \hat{D}}] T^a) \psi \rangle.
\] (105)

This can be rewritten in the form

\[
A^a = -\text{Tr} S_{\text{cov}}^{\gamma \mu} (T^a G(x, \epsilon) \cdot \gamma - \gamma \cdot G(x, -\epsilon) T^a)
\] (106)

where \( G_\mu(x, \epsilon) \) was introduced in the previous section. Note, that \( \epsilon \cdot G(x, \epsilon) = 0 \) and \( G_\mu(x, \epsilon) = O(\epsilon) \). Using the formula for the covariant propagator, which implies

\[
\text{Tr}_D S_{\text{cov}}^{\gamma \mu} = \frac{1}{4\pi^2} \left\{ \frac{\epsilon_\mu}{|\epsilon|^4} + m^2 \frac{\epsilon_\mu}{|\epsilon|^2} + \frac{1}{3} \ln m^2 |\epsilon|^2 [D_\sigma, F_{\sigma \mu}] - \frac{1}{3} \frac{\epsilon_\mu \epsilon_\sigma}{|\epsilon|^2} [D_\rho, F_{\rho \sigma}] + \frac{1}{3} \frac{\epsilon_\rho \epsilon_\sigma}{|\epsilon|^2} [D_\mu, F_{\mu \rho}] + O_{\text{reg}}(1) \right\},
\] (107)

we get

\[
A^a = -\frac{1}{4\pi^2} \left( -\frac{\epsilon_\mu}{|\epsilon|^4} + m^2 \frac{\epsilon_\mu}{|\epsilon|^2} + O(\ln |\epsilon|^2) \right) \text{Tr}_C (T^a G_\mu(x, \epsilon) - G_\mu(x, -\epsilon) T^a) = O(\epsilon \ln |\epsilon|^2)
\] (108)

and therefore the vector current is anomaly free in the limit of the removed point-splitting.

6 The trace anomaly

Another example of the strategy of deformed transformation is the trace anomaly of the canonical gauge invariant energy-momentum tensor\[12\]

\[
\theta_{\mu \nu} = \frac{1}{2} \psi^+ \gamma_\mu \hat{\nabla}_\nu \psi.
\] (109)

\[12\] This tensor is not symmetric; the construction of the symmetric energy-momentum tensor will be discussed in the next section.
Let us consider the scale transformation

\[ \delta \psi = \lambda \psi, \]
\[ \delta \psi^+ = \psi^+ \lambda, \]  

(110)

where \( \lambda \) is a real infinitesimal parameter. The corresponding Noether current is identically zero and the change of the action under localized scale transformation with parameter \( \lambda(x) \) is

\[ \delta S_E = \int d^4x 2\lambda(x) \left( \frac{1}{2} \psi^+ \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu \psi + im \psi^+ \psi \right) = \int d^4x 2\lambda(x) (\theta_{\mu\mu} + im \psi^+ \psi). \]  

(111)

On the classical level we have therefore

\[ \theta_{\mu\mu} = -im \psi^+ \psi. \]  

(112)

The regularized form of the relevant composite operators is\[13\]

\[ \theta^\text{reg}_{\mu\nu} = \frac{1}{2} \psi^+ \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\nu \psi \]  

and

\[ \psi^+ \psi^\text{reg} = \psi^\text{reg}_\psi. \]  

(114)

these regularized expressions are gauge invariant. The point-split version of the transformation (110) is then

\[ \delta \psi = e^{-\varepsilon \cdot \nabla} \lambda e^{-\varepsilon \cdot \nabla} \psi, \]
\[ \delta \psi^+ = \psi^+ e^{\varepsilon \cdot \nabla} \lambda e^{\varepsilon \cdot \nabla}. \]  

(115)

The Jacobian of this deformed scaling is identically equal to one for the same reason as before. The variation of the action reproduces the point-split form of (111) and acquires an additional term, which is the potential source of the trace anomaly

\[ \delta S_E = \int d^4x 2\lambda(x) \left( \frac{1}{2} \psi^+ \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu \psi + im \psi^+ \psi \right) + \int d^4x \lambda \left( \psi^+ \left( \left( e^{-\varepsilon \cdot \nabla} \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu \right) e^{\varepsilon \cdot \nabla} - e^{-\varepsilon \cdot \nabla} \left[ \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu, e^{\varepsilon \cdot \nabla} \right] \right) \psi \right), \]  

(116)

i.e. we have

\[ \langle \theta^\text{reg}_{\mu\nu} \rangle_c = -i \langle m \psi^+ \psi \rangle_c + A^\text{trace} \]  

(117)

where

\[ A^\text{trace} = -\frac{1}{2} \langle \psi^+ \left( \left( e^{-\varepsilon \cdot \nabla} \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu \right) e^{\varepsilon \cdot \nabla} - e^{-\varepsilon \cdot \nabla} \left[ \gamma_\mu \stackrel{\leftrightarrow}{\nabla}_\mu, e^{\varepsilon \cdot \nabla} \right] \right) \psi \rangle_c \]
\[ = \text{Tr} S^\text{cov} \gamma \cdot G^+, \]  

(118)

\[ ^{13}\text{Other forms of the point-split tensor } \theta^\text{reg}_{\mu\nu} \text{ are used in the literature as well. Our choice is again motivated by simplicity.} \]
where $G^+_{\mu}$ was introduced via the formula (93). Using the formulae (107) and (94) we get

$$A^{\text{trace}} = \text{Tr} \left( \frac{1}{16\pi^2} \left( -4\frac{\varepsilon_{\mu}}{|\varepsilon|^2} + O\left( \frac{\varepsilon_{\mu}}{|\varepsilon|^4} \right) \right) \left( -\frac{1}{2} |\varepsilon \cdot D, F_{\mu\nu}| \varepsilon_{\nu} + O(|\varepsilon|^4) \right) \right),$$

and, after taking the average over the direction of the four-vector $\varepsilon_{\mu}$,

$$A^{\text{trace}} = O(\varepsilon).$$

It means that there is no trace anomaly within the point-splitting regularization.

7 The translation anomaly

Because there is no anomaly in the trace of the point-split energy-momentum tensor, we expect the appearance of an anomaly in its divergence. Assume therefore the covariant translation of the fields, parametrized by an infinitesimal four-vector $a$

$$\delta \psi = a \cdot D \psi,$$
$$\delta \psi^+ = \psi^+ \cdot \vec{D} \cdot a.$$

The canonical Noether current associated with this transformation is identical with the energy-momentum tensor (109) introduced above. The variation of the action with respect to the localized version of this transformation, which gives the classical divergence of the energy-momentum tensor, is

$$\delta S_E = \int d^4x \left[ -a_\mu \partial_{\mu} \left( \frac{1}{2} \psi^+ \gamma_{\mu} \vec{D}_{\nu} \psi \right) + a \cdot \partial \left( \frac{1}{2} \psi^+ \gamma \cdot \vec{D} \psi + i m \psi^+ \psi \right) + a_\nu \psi^+ \gamma_\mu F_{\mu\nu} \psi \right]$$

on the classical level we have therefore

$$\partial_{\mu} \theta_{\mu\nu} = \psi^+ \gamma_{\mu} F_{\mu\nu} \psi.$$ (123)

We again regularize the operators involved according to the (113), (114) and to the prescription

$$\psi^+ \gamma_{\mu} F_{\mu\nu} \psi_{\text{reg}} = \psi^+_{\varepsilon} \gamma_{\mu} F_{\mu\nu} \psi_{\varepsilon}.$$ (124)

In order to get the Ward-Takahashi identity for the point-split operators, we deform the (local) covariant translation as follows

$$\delta \psi = e^{-\varepsilon \cdot D} a \cdot D e^{-\varepsilon \cdot D} \psi$$
$$\delta \psi^+ = \psi^+ e^{\varepsilon \cdot \vec{D}} \cdot \vec{D} \cdot e^{\varepsilon \cdot \vec{D}}$$

We have then for the variation of the action

$$\delta S_E = \int d^4x \left[ -a_\mu \partial_{\mu} \left( \frac{1}{2} \psi^+_{\varepsilon} \gamma_{\mu} \vec{D}_{\nu} \psi_{\varepsilon} \right) + a \cdot \partial \left( \frac{1}{2} \psi^+_{\varepsilon} \gamma \cdot \vec{D} \psi_{\varepsilon} + i m \psi^+_{\varepsilon} \psi_{\varepsilon} \right) + a_\nu \psi^+_{\varepsilon} \gamma_\mu F_{\mu\nu} \psi_{\varepsilon} \right]$$

$$+ \int d^4x a_\nu \left( \psi^+_{\varepsilon} (\vec{D}_{\nu} [e^{-\varepsilon \cdot \vec{D}}, \gamma \cdot \vec{D}] e^{\varepsilon \cdot \vec{D}} - e^{-\varepsilon \cdot \vec{D}} [\gamma \cdot \vec{D}, e^{\varepsilon \cdot \vec{D}}] \vec{D}_{\nu}) \psi_{\varepsilon} \right)$$ (126)
and therefore

\[ \partial_\mu ( \theta^{\text{reg}}_{\mu \nu} )_c = \partial_\nu ( \frac{1}{2} \psi_\xi^+ \gamma \cdot \vec{D} \psi_\xi + \text{imm} \psi_\xi^+ \psi_\xi)_c + ( \psi_\xi^+ \gamma_\mu F_\mu \nu \psi_\xi )_c + A_\nu, \]

where the anomaly is given by the formula

\[ A_\nu = ( \psi_\xi^+ ( \vec{D}_\nu [ e^{-\varepsilon \cdot \vec{D}} \gamma \cdot \vec{D} ] e^{\varepsilon \cdot \vec{D}} - e^{-\varepsilon \cdot \vec{D}} [ \gamma \cdot \vec{D}, e^{\varepsilon \cdot \vec{D}} ] \vec{D}_\nu ) \psi_\xi )_c. \]

Expressing this in terms of the above introduced \( G \) functions we have

\[ A_\nu = \langle \psi_\xi^+ ( \vec{D}_\nu [ \gamma \cdot G^+ + \gamma \cdot G^+ \vec{D}_\nu ] ) \psi_\xi \rangle + \langle \psi_\xi^+ ( \vec{D}_\nu [ \gamma \cdot G^- - \gamma \cdot G^- \vec{D}_\nu ] ) \psi_\xi \rangle
\]
\[ \quad + \langle \psi_\xi^+ ( \vec{D}_\nu [ \gamma \cdot G^- - \gamma \cdot G^- \vec{D}_\nu ] ) \psi_\xi \rangle. \]

Let us work out also the last term. For the derivatives of the point-split fields with respect to the parameter \( \varepsilon \) we have the following useful formulae

\[ \partial^\xi \psi_\xi = - \int_0^1 d\varepsilon e^{-\varepsilon \cdot \vec{D}} D_\nu e^{\varepsilon \cdot \vec{D}} \psi_\xi = -D_\nu \psi_\xi + \Gamma_\nu ( x, \varepsilon ) \psi_\xi \]

and

\[ \partial^\xi \psi_\xi^+ = \psi_\xi^+ \int_0^1 d\varepsilon e^{-\varepsilon \cdot \vec{D}} \vec{D}_\nu e^{\varepsilon \cdot \vec{D}} \]
\[ \quad = \psi_\xi^+ \vec{D}_\nu + \psi_\xi^+ \Gamma_\nu ( x, -\varepsilon ), \]

where \( \partial^\xi \) means the partial derivative with respect to \( \varepsilon \) and where \( \Gamma_\mu ( x, \varepsilon ) = H_\mu ( x, \varepsilon ) - G_\mu ( x, \varepsilon ) \) (the functions \( G_\mu ( x, \varepsilon ) \) and \( H_\mu ( x, \varepsilon ) \) were already introduced in the previous sections, their explicit form\(^{14}\) is given in Appendix A.) I.e.

\[ A_\nu = \partial_\nu ( \psi_\xi^+ ( \gamma \cdot G^+ ) \psi_\xi )_c - \langle \psi_\xi^+ [ D_\nu, \gamma \cdot G^+ ] \psi_\xi \rangle_c
\]
\[ \quad + \langle \psi_\xi^+ ( \vec{D}_\nu \gamma \cdot G^- + \gamma \cdot G^- \vec{D}_\nu ) \psi_\xi \rangle_c
\]
\[ \quad - \langle \psi_\xi^+ ( \Gamma_\nu ( x, -\varepsilon ) \psi_\xi ) \rangle_c, \]

\(^{14}\)Note also, that we can also write

\[ \Gamma_\mu ( x, \varepsilon ) = - \int_0^1 d\varepsilon [ e^{-\varepsilon \cdot \vec{D}} D_\mu ] e^{\varepsilon \cdot \vec{D}} = - \int_0^1 d\varepsilon G_\mu ( x, \varepsilon )
\]
\[ \quad = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+2)!} [ \varepsilon \cdot D, \ldots, [ \varepsilon \cdot D, \varepsilon \nu F_\mu \nu ( x ) ] ] ]

Here the second line is a consequence of the explicit formula for \( G_\mu ( x, \varepsilon ) \), which is given in Appendix A.
or, in terms of the covariant propagator,

\[ A_\nu = -\text{Tr}(\partial_\nu S^{\text{cov}})\gamma \cdot G^- - \partial_\nu \text{Tr} S^{\text{cov}} \gamma \cdot G^+ + \text{Tr} S^{\text{cov}} ([D_\nu, \gamma \cdot G^+] + \Gamma_\nu(x, -\varepsilon)\gamma \cdot G^- + \gamma \cdot G^- \Gamma_\nu(x, \varepsilon)). \] (133)

Let us recall, that \( G^+_\mu(x, \varepsilon) = \mathcal{O}(\varepsilon^2) \) and \( G^-_\mu(x, \varepsilon) = \mathcal{O}(\varepsilon) \), \( \Gamma_\mu(x, \varepsilon) = \mathcal{O}(\varepsilon) \), therefore in the second and third term on the right hand side of (133) only the first two terms of the expansion of \( \text{Tr} S^{\text{cov}} \gamma_\mu \) can contribute. However, these terms are proportional to \( \varepsilon_\mu \) and because \( \varepsilon \cdot G^\pm(x, \varepsilon) = 0 \), their contribution vanishes. So that only the first term on the right hand side of (133) can survive and we get for the anomaly the following simple formula

\[ A_\nu = -\text{Tr}(\partial_\nu S^{\text{cov}})\gamma \cdot G^- + \mathcal{O}(\varepsilon). \] (134)

This gives, after some algebra\(^{[15]}\)

\[ A_\nu = -\frac{1}{24\pi^2} \text{Tr}_C \left( 5[D_\alpha, F_{\alpha\beta}]F_{\beta\nu} + [D_\nu, F_{\alpha\beta}]F_{\alpha\beta} + [D_\alpha, F_{\beta\nu}]F_{\beta\alpha} \right). \] (135)

Inserting here the identity \([D_\alpha, F_{\beta\nu}]F_{\beta\alpha} = \frac{1}{2}[D_\nu, F_{\alpha\beta}]F_{\alpha\beta}\), we get finally

\[ A_\nu = -\frac{1}{24\pi^2} \theta_\nu \text{Tr}_C \left( \frac{5}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} + \frac{5}{3} \theta_\lambda \right) + \mathcal{O}(\varepsilon) \] (136)

where

\[ \theta_\mu^\lambda = F_{\mu\rho} F_{\rho\sigma} + \frac{1}{4} \delta_{\mu\rho} F_{\alpha\beta} F_{\alpha\beta}. \] (137)

As we have expected, the anomaly is shifted to the divergence of the energy-momentum tensor. We can recover the trace anomaly by redefinition (finite subtraction) of this tensor according to the prescription

\[ \theta_{\mu\nu}^{\text{reg}} = \theta_{\mu\nu}^{\text{reg}} + \frac{1}{24\pi^2} \text{Tr}_C \left( \frac{5}{4} \delta_{\mu\nu} F_{\alpha\beta} F_{\alpha\beta} + \frac{5}{3} \theta_\lambda \right). \] (138)

The redefined energy-momentum tensor has the standard trace anomaly

\[ \langle \theta_{\mu\nu}^{\text{reg}} \rangle_c = -i \langle m \psi^+_\varepsilon \psi^c \rangle_c + \frac{1}{24\pi^2} \text{Tr}_C F_{\alpha\beta} F_{\alpha\beta} + \mathcal{O}(\varepsilon) \] (139)

\(^{[15]}\)Note, that \( G^\mu_\nu = \varepsilon_\mu F_{\mu\sigma} + \mathcal{O}(\varepsilon^3) \) and

\[ \text{Tr}_D S^{\text{cov}} \gamma_\mu = \frac{1}{4\pi^2} \left\{ \begin{array}{c}
\frac{\varepsilon_\mu}{|\varepsilon|^4} + m^2 \frac{\varepsilon_\mu}{|\varepsilon|^2} + \frac{1}{3} \text{ln} m^2 |\varepsilon|^2 [D_\nu, F_{\mu\rho}] \\
- \frac{1}{3} \frac{\varepsilon_\mu \varepsilon_\rho}{|\varepsilon|^2} [D_\rho, F_{\mu\sigma}] + \frac{1}{3} \frac{\varepsilon_\mu \varepsilon_\rho}{|\varepsilon|^2} [D_\rho, F_{\mu\sigma}] + \mathcal{O}_{\text{reg}}(1)
\end{array} \right\}, \]

and therefore

\[ \text{Tr}_D \partial_\nu S^{\text{cov}} \gamma_\mu = \frac{1}{4\pi^2} \left\{ \begin{array}{c}
4 \varepsilon_\mu \varepsilon_\nu - \delta_{\mu\nu} |\varepsilon|^2 + m^2 \delta_{\mu\nu} |\varepsilon|^2 - \frac{2}{3} \varepsilon_\mu |\varepsilon|^2 [D_\nu, F_{\mu\rho}] \\
- \frac{1}{3} \delta_{\mu\rho} [D_\rho, F_{\mu\sigma}] + \frac{1}{3} \delta_{\mu\rho} [D_\rho, F_{\mu\sigma}] + \mathcal{O}_{\text{reg}}(1)
\end{array} \right\}. \]
and its divergence is anomaly free. This is an illustration of the anomalous symmetry pair phenomenon.

Let us briefly comment on this result. As we have seen, the nonstandard terms in the variation of the action under corresponding point-split transformations are present both for the scale transformation and translation. The anomalies are therefore not a priori excluded either in the trace or in the divergence of the energy-momentum tensor. Although our point of view does not explain why it appears in the divergence and not in the trace (this result is hidden in the short distance asymptotics of the covariant propagator), it at least sheds a new light on the well known fact that point-splitting regularization is in a conflict with (covariant) translations.

8 The symmetric energy-momentum tensor and the Lorentz anomaly

As we have mentioned above, the canonical energy-momentum tensor is not symmetric. In this section we give construction of the modified energy-momentum tensor, which does not suffer from this inconvenient property.

Let us start with the local $SO(4)$ rotation of the fermion fields

$$
\delta \psi = \frac{i}{2} \omega_{\mu \nu} \sigma_{\mu \nu} \psi
$$

$$
\delta \psi^+ = -\frac{i}{2} \psi^+ \sigma_{\mu \nu} \omega_{\mu \nu}
$$

with the parameter $\omega_{\mu \nu}(x)$. The variation of the action with respect to this transformation can be written as

$$
\delta S_E = \frac{i}{4} \int d^4x \omega_{\mu \nu} \left[ -\partial_\alpha (\psi^+ \{\gamma_\alpha, \sigma_{\mu \nu}\} \psi) + 4i(\theta_{\mu \nu} - \theta_{\nu \mu}) \right].
$$

with $\theta_{\mu \nu}$ being the canonical energy-momentum tensor (109). Classically, using the identities

$$
\gamma_\alpha \sigma_{\mu \nu} = i(\delta_{\nu \alpha} \gamma_\mu - \delta_{\mu \alpha} \gamma_\nu + \epsilon_{\mu \nu \alpha \lambda} \gamma_\lambda \gamma_5)
$$

$$
\sigma_{\mu \nu} \gamma_\alpha = i(\delta_{\mu \alpha} \gamma_\nu - \delta_{\nu \alpha} \gamma_\mu + \epsilon_{\mu \nu \alpha \lambda} \gamma_\lambda \gamma_5)
$$

we get then the following on-shell relation,

$$
\theta_{\mu \nu} - \theta_{\nu \mu} = -\frac{i}{4} \partial_\alpha (\psi^+ \{\gamma_\alpha, \sigma_{\mu \nu}\} \psi) = \frac{1}{2} \epsilon_{\mu \nu \alpha \lambda} \partial_\alpha (\psi^+ \gamma_\lambda \gamma_5 \psi).
$$

Thus, the classical symmetric energy-momentum tensor $T_{\mu \nu}$

$$
T_{\mu \nu} = \frac{1}{2}(\theta_{\mu \nu} + \theta_{\nu \mu}) = \frac{1}{4} (\psi^+ \gamma_\mu \bar{D}_\nu \psi + \psi^+ \gamma_\nu \bar{D}_\mu \psi)
$$

can be expressed on shell as

$$
T_{\mu \nu} = \theta_{\mu \nu} - \frac{1}{4} \epsilon_{\mu \nu \alpha \lambda} \partial_\alpha (\psi^+ \gamma_\lambda \gamma_5 \psi)
$$

and has therefore the same trace and divergence as the canonical tensor $\theta_{\mu \nu}$.
Let us now derive the quantum analog of the identity (147), valid for the point-split energy-momentum tensor $T^\text{reg}_{\mu\nu}$ defined as

$$T^\text{reg}_{\mu\nu} = \frac{1}{2} (\theta^\text{reg}_{\mu\nu} + \theta^\text{reg}_{\nu\mu}) = \frac{1}{4} (\psi^+_\varepsilon \gamma_\mu \overset{\leftrightarrow}{D}_\nu \psi_\varepsilon + \psi^+_\varepsilon \gamma_\nu \overset{\leftrightarrow}{D}_\mu \psi_\varepsilon)$$  \hspace{1cm} (146)

We shall use here again the idea of the point-split transformation. The suitable deformation of the local $SO(4)$ rotation (140) is

$$\delta \psi = \frac{1}{2} \epsilon^{-\varepsilon \cdot D} \omega_{\mu\nu} e^{-\varepsilon \cdot D} \sigma_{\mu\nu} \psi$$

$$\delta \psi^+ = -\frac{1}{2} \psi^+ \sigma_{\mu\nu} e^{\varepsilon \cdot D} \omega_{\mu\nu} e^{\varepsilon \cdot D}$$  \hspace{1cm} (147)

The variation of the action under this transformation gives the following regularized form of the formula (141) and

$$\delta S_E = \frac{1}{4} \int d^4x \omega_{\mu\nu} \left[ -\partial_\alpha (\psi^+_\varepsilon \left\{ \gamma_\alpha, \sigma_{\mu\nu} \right\} \psi_\varepsilon) + 4i (\theta^\text{reg}_{\mu\nu} - \theta^\text{reg}_{\nu\mu}) 
- 2\psi^+_\varepsilon (\sigma_{\mu\nu} [e^{-\varepsilon \cdot D}, \gamma \cdot \overset{\leftrightarrow}{D}] e^{\varepsilon \cdot D} + e^{-\varepsilon \cdot D} [\gamma \cdot \overset{\leftrightarrow}{D}, e^{\varepsilon \cdot D}] \sigma_{\mu\nu} \psi_\varepsilon) \right].$$  \hspace{1cm} (148)

The Ward-Takahashi identity corresponding to the transformation (147) can be therefore rewritten in the form

$$(\theta^\text{reg}_{\mu\nu} - \theta^\text{reg}_{\nu\mu})_c = \frac{1}{2} \varepsilon_{\mu\nu\alpha\lambda} \partial_\alpha (\psi^+_\varepsilon \gamma_\alpha \gamma_5 \psi_\varepsilon)_c + A_{\mu\nu}$$  \hspace{1cm} (149)

where

$$A_{\mu\nu} = -A_{\nu\mu} = \frac{1}{2} (\psi^+_\varepsilon (\sigma_{\mu\nu} [e^{-\varepsilon \cdot D}, \gamma \cdot \overset{\leftrightarrow}{D}] e^{\varepsilon \cdot D} + e^{-\varepsilon \cdot D} [\gamma \cdot \overset{\leftrightarrow}{D}, e^{\varepsilon \cdot D}] \sigma_{\mu\nu} \psi_\varepsilon)_c$$  \hspace{1cm} (150)

is a possible anomaly term corresponding to the violation of the naive identity (143). Such an anomaly is called Lorentz anomaly. The regularized form of the symmetric energy-momentum tensor $T^\text{reg}_{\mu\nu}$ is therefore\[16]

$$T^\text{reg}_{\mu\nu} = \theta^\text{reg}_{\mu\nu} - \frac{1}{4} \varepsilon_{\mu\nu\alpha\lambda} \partial_\alpha (\psi^+_\varepsilon \gamma_\alpha \gamma_5 \psi_\varepsilon) - \frac{1}{2} A_{\mu\nu}.$$  \hspace{1cm} (151)

Provided the anomaly does not vanish, we may get modification of the divergence of the quantum symmetric energy-momentum $T^\text{reg}_{\mu\nu}$ tensor, which, unlike in the classical case, would not be the same as the divergence of the canonical tensor $\theta^\text{reg}_{\mu\nu}$. For the anomaly we have, using the identities (142) and the definition of the functions $G(x, \varepsilon)$ and $G^\pm(x, \varepsilon)$

$$A_{\mu\nu} = \frac{i}{2} \text{Tr} S^\text{cov}_{\mu\nu} (\sigma_{\mu\nu} \gamma \cdot G(x, \varepsilon) - G(x, -\varepsilon) \cdot \gamma \sigma_{\mu\nu})$$

$$= -\text{Tr} S^\text{cov}_{\mu\nu} (\delta_{\mu\alpha} \gamma_\nu - \delta_{\nu\alpha} \gamma_\mu) G^+_\alpha(x, \varepsilon) + \varepsilon_{\mu\nu\alpha\lambda} \gamma_\alpha \gamma_5 G^\alpha_{\alpha}(x, \varepsilon)).$$  \hspace{1cm} (152)

\[16\]This operator identity should be understood as the statement for the corresponding generating functionals with operator insertions, i.e.

$$T^\text{reg}_{\mu\nu} \cdot Z_E [A]_c = \theta^\text{reg}_{\mu\nu} \cdot Z_E [A]_c - \frac{1}{4} \varepsilon_{\mu\nu\alpha\lambda} \partial_\alpha (\psi^+_\varepsilon \gamma_\alpha \gamma_5 \psi_\varepsilon) \cdot Z_E [A]_c - \frac{1}{2} A_{\mu\nu}. $$

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Using now the explicit formulae for $S^{\text{cov}}$ and the functions $G(x, \varepsilon)$ and $G^\pm(x, \varepsilon)$, it is easy to show, that in the symmetric limit $\varepsilon \to 0$ we get

$$\mathcal{A}_{\mu\nu} = O(\varepsilon) \quad (153)$$

i.e., there is no SO(4) anomaly and the regularized symmetric energy-momentum tensor (151) has the same trace and divergence as the canonical one in the limit of the removed cut-off.

9 Non-covariant point-splitting

In the previous sections we have introduced a gauge covariant version of the point-splitting regularization for various Noether currents. In this section we relax this restriction and try to show how the violation of the gauge covariance effects the structure of the anomalies discussed above.

The simplest type of the non-covariant point-splitting can be achieved by replacing the parallel transporter $\Omega(x, y)$ (which ensured the gauge covariance of the regularized currents) by a deformed one according to the prescription

$$\tilde{\Omega}(x, y) = P \exp \left( - \int_0^1 dt (x - y) \cdot A(y + t(x - y)) \right) \to P \exp \left( -a \int_0^1 dt (x - y) \cdot A(y + t(x - y)) \right) = \tilde{\Omega}(x, y) \quad (154)$$

where $a$ is a real parameter\(^{17}\). Gauge covariance is then recovered putting $a = 1$. For further convenience let us introduce the following notation

$$\tilde{\psi}_\varepsilon(x) = \tilde{\Omega}(x, x - \varepsilon) \psi(x - \varepsilon) = e^{-\varepsilon \tilde{D}} \psi(x) = \Omega_- \psi(x),$$

$$\tilde{\psi}_\varepsilon^+(x) = \psi^+(x + \varepsilon) \tilde{\Omega}(x + \varepsilon, x) = \psi^+(x) e^{\varepsilon \tilde{D}} = \psi^+(x) \Omega_+. \quad (155)$$

Here $\tilde{D} = \partial + aA$ and, using the results of the previous sections,

$$\Omega_- = \tilde{\Omega}(x, x - \varepsilon) \Omega_+^+(x, x - \varepsilon) = e^{-\varepsilon \tilde{D}} e^{\varepsilon \tilde{D}},$$

$$\Omega_+ = \Omega_+^+(x + \varepsilon, x) \tilde{\Omega}(x + \varepsilon, x) = e^{-\varepsilon \tilde{D}} e^{\varepsilon \tilde{D}}. \quad (156)$$

From the operator expressions the $\varepsilon$-expansion of these matrix functions can be easily obtained, the explicit formulae are given in Appendix B. The non-covariantly regularized currents are now built from the modified point-split fields $\tilde{\psi}_\varepsilon(x)$ and $\tilde{\psi}_\varepsilon^+(x)$ and (in analogy with the previous sections) they can be understood as the Noether currents corresponding to the modified point-split transformations (85), (102), (102) and (125). Such modification is given by replacing the covariant derivative with the deformed one $D \to \tilde{D}$ e.g. the modified point-split chiral rotation reads

$$\tilde{\delta}\psi = e^{-\varepsilon \tilde{D}} \alpha_5 e^{-\varepsilon \tilde{D}} \psi,$$

$$\tilde{\delta}\psi^+ = \psi^+ e^{\varepsilon \tilde{D}} \alpha_5 e^{\varepsilon \tilde{D}}. \quad (157)$$

\(^{17}\)That means $\tilde{\Omega}(x, y)$ corresponds to the parallel transporter along the straight line connecting the points $x$ and $y$ in the gauge field $aA(x)$ instead of $A(x)$. 

22
The above described change of the regularization scheme modifies generally the Noether currents \( j^a_\mu, j^5_\mu, \theta_\mu \) and the composite operators \( s^a = \psi^+ T^a \psi \) and \( p^a = \psi^+ T^a \gamma_5 \psi \) entering the Ward-Takahashi identities by means of adding non-covariant counterterms \( \Delta \tilde j^a_\mu, \Delta \tilde j^a_5, \Delta \tilde \theta^a_\mu, \Delta \tilde s^a \) and \( \Delta \tilde p^a \) respectively. In this section we give a list of such counterterms and deduce from it the additional spurious contributions to the anomalies. These contributions can be expressed by the following formulae, which follow from the regularized form of the Ward-Takahashi identities\(^\text{18}\)

\[
\begin{align*}
\Delta \tilde A^a &= D_\mu \Delta \tilde j^a_\mu \\
\Delta \tilde A^a_5 &= D_\mu \Delta \tilde j^a_5 - 2i m \Delta \tilde p^a \\
\Delta \tilde A^\text{trace} &= \Delta \tilde \theta^a_\mu + i m \Delta \tilde s^a \\
\Delta \tilde A^a_\nu &= \partial_\mu \Delta \tilde \theta^a_{\mu \nu} - \partial_\nu (\Delta \tilde \theta^a_\mu + i m \Delta \tilde s^a) - F^a_{\mu \nu} \Delta \tilde j^a_\mu \\
\Delta \tilde A^a_{\mu \nu} &= \Delta \tilde \theta^a_{\mu \nu} - \Delta \tilde \theta^a_{\nu \mu} - \frac{1}{2} \varepsilon_{\mu \nu \alpha \lambda} \partial_\alpha \Delta \tilde j^a_\lambda \\
\end{align*}
\]  

(158)

Let us start with the axial currents. The non-covariant point-split regularization of this operator is

\[
\tilde j^a_5^\text{reg}(x) = \psi^+(x + \varepsilon) \tilde \Omega(x + \varepsilon, x) T^a \gamma_\mu \gamma_5 \tilde \Omega(x, x - \varepsilon) \psi(x - \varepsilon)
\]

\[
= \tilde \psi^+ T^a \gamma_\mu \gamma_5 \tilde \psi \varepsilon = \psi^+ \Omega T^a \gamma_\mu \gamma_5 \Omega^{-1} \psi \varepsilon.
\]  

(159)

and it differs from the covariant one by the following counterterm

\[
\Delta \tilde j^a_5^\text{reg} = \langle \tilde \psi^+ T^a \gamma_\mu \gamma_5 \tilde \psi \varepsilon - \psi^+ T^a \gamma_\mu \gamma_5 \psi \varepsilon \rangle
\]

\[
= -\text{Tr} S^\text{cov} \gamma_\mu \gamma_5 (\Omega T^a \Omega - T^a).
\]  

(160)

Using the expansion

\[
\Omega \pm = 1 + (1 - a) \varepsilon \cdot A(x) + \mathcal{O}(\varepsilon^2)
\]

we get

\[
\Delta \tilde j^a_5^\text{reg} = \frac{1 - a}{4 \pi^2} \text{Tr} C \left( \left( \frac{\varepsilon_\mu}{|\varepsilon|^2} F^{*}_\nu (x) + \mathcal{O}(1) \right) \left\{ T^a, \varepsilon \cdot A(x) \right\} + \mathcal{O}(\varepsilon^2) \right)
\]

(162)

and after the symmetrization over the direction of \( \varepsilon \)

\[
\Delta \tilde j^a_5^\text{reg} = \frac{1 - a}{16 \pi^2} \text{Tr} C T^a \left\{ A_\nu (x), F^{*}_\nu (x) \right\} + \mathcal{O}(\varepsilon).
\]

(163)

In the same way we can prove, that for the pseudoscalar density \( p = \psi^+ \gamma_5 \psi \) we get counterterm which vanishes in the limit of the removed point-splitting

\[
\Delta \tilde p^a_\text{reg} = \langle \tilde \psi^+ T^a \gamma_5 \tilde \psi \varepsilon \rangle = \text{Tr} S^\text{cov} \gamma_5 (\Omega T^a \Omega - T^a)
\]

\[
= \mathcal{O}(\varepsilon).
\]  

(164)

The chiral Ward-Takahashi identity has now the following form

\[
D_\mu \langle \tilde j^a_5^\text{reg} \rangle = 2i m \langle \tilde \psi^+ T^a \gamma_5 \tilde \psi \varepsilon \rangle + \tilde A^a_5
\]

(165)

\(^{18}\)Here \( j_\mu = \psi^\dagger \gamma_\mu \psi \) and \( j^5_\mu = \psi^\dagger \gamma_\mu \gamma_5 \psi \) are the singlet currents and \( \Delta \tilde j_\mu \) and \( \Delta \tilde j_5 \) are corresponding counterterms. The same notation without superscript is used for the singlet scalar and pseudoscalar densities.
After the symmetrization over the direction of $\varepsilon$ we get
\[
\tilde{A}_5^a = A_5^a + D_\mu \Delta \tilde{J}_{5\mu}^{a,\text{reg}} = \frac{1}{8\pi^2} \text{Tr}_C T^a F_{\mu\nu}^* (x) F_{\mu\nu} (x) + \frac{1-a}{16\pi^2} \text{Tr}_C T^a \{ [D_\mu, A_\nu (x)], F_{\nu\mu}^* (x) \} + \mathcal{O} (\varepsilon).
\]  
(166)

Note, that for the abelian case we have
\[
\tilde{A}_5 = \frac{1+a}{16\pi^2} F_{\mu\nu}^* (x) F_{\mu\nu} (x)
\]  
(167)

and the anomaly can be eliminated taking $a = -1$, cf. [14].

Because the gauge symmetry is violated in this regularization scheme, we expect also anomalous divergence of the gauge current
\[
\tilde{J}_{\mu}^{a,\text{reg}} = \psi^A_\xi T^a \gamma_\mu \psi_\xi = \psi^A_\xi \Omega_+ T^a \gamma_\mu \Omega_- \psi_\xi.
\]  
(168)

For the counterterm we have
\[
\Delta \tilde{J}_{\mu}^{a,\text{reg}} = \langle \psi^A_\xi T^a \gamma_\mu \psi_\xi - \psi^A_\xi T^a \gamma_\mu \psi_\xi \rangle
= -\text{Tr}_S \text{cov}_\gamma \mu (\Omega_+ T^a \Omega_- - T^a)
= \frac{1}{4\pi^2} \left( \frac{\varepsilon_\mu}{|\varepsilon|^2} - m^2 \frac{\varepsilon_\mu}{|\varepsilon|^2} + \mathcal{O} (\ln |\varepsilon|) \right) \text{Tr}_C (\Omega_+ T^a \Omega_- - T^a)
\]  
(169)

Using the formula (see Appendix B)
\[
\Omega_- \Omega_+ = 1 + 2(1-a)(A(x) \cdot \varepsilon) + 2(1-a)^2 (A(x) \cdot \varepsilon)^2
+ \frac{4}{3} (1-a)^3 (A(x) \cdot \varepsilon)^3
+ \frac{1}{3} (1-a)(2-a)[(A(x) \cdot \varepsilon), (\varepsilon \cdot \partial)(A(x) \cdot \varepsilon)]
+ \frac{1}{3} (a-\varepsilon \cdot \partial)^2 (A(x) \cdot \varepsilon) + \mathcal{O} (\varepsilon^4)
\]  
(170)

we get after the symmetrization over the direction of $\varepsilon$
\[
\Delta \tilde{J}_{\mu}^{a,\text{reg}} = \frac{1-a}{8\pi^2} \text{Tr}_C T^a \left( \left( \frac{1}{|\varepsilon|^2} - m^2 \right) A_\mu 
+ \frac{(1-a)^2}{9} (A_\mu A \cdot A + A \cdot AA_\mu + A_\nu A_\mu A_\nu)
+ \frac{(2-a)}{36} ([A_\mu, \partial \cdot A] + [A_\nu, \partial_\mu A_\nu] + [A_\nu, \partial_\nu A_\mu])
+ \frac{1}{36} (2\partial_\mu \partial \cdot A + \partial^2 A_\mu) \right) + \mathcal{O} (\varepsilon).
\]  
(171)

The regularized current $\tilde{J}_{\mu}^{a,\text{reg}}$ has therefore anomalous divergence
\[
D_\mu (\tilde{J}_{\mu}^{a,\text{reg}}) = \tilde{A}^a
\]  
(172)

and the gauge anomaly is
\[
\tilde{A}^a = \frac{1-a}{8\pi^2} \text{Tr}_C T^a [D_\mu, (\frac{1}{|\varepsilon|^2} - m^2) A_\mu 
+ \frac{(1-a)^2}{9} (A_\mu A \cdot A + A \cdot AA_\mu + A_\nu A_\mu A_\nu)
+ \frac{(2-a)}{36} ([A_\mu, \partial \cdot A] + [A_\nu, \partial_\mu A_\nu] + [A_\nu, \partial_\nu A_\mu])
+ \frac{1}{36} (2\partial_\mu \partial \cdot A + \partial^2 A_\mu)] + \mathcal{O} (\varepsilon).
\]  
(173)
In the abelian case (cf. [14]) we get

$$\tilde{A} = \frac{1 - a}{8\pi^2} \left( \frac{1}{|\varepsilon|^2} - m^2 \right) \partial \cdot A + \frac{1}{12} \partial^2 \partial \cdot A + \frac{(1 - a)^2}{3} (A^2 \partial \cdot A + 2A_\mu A \cdot \partial A_\mu)$$ (174)

which together with (167) reproduces the well-known one-parameter family of gauge and chiral anomalies [14].

Let us now calculate the non-covariantly regularized energy-momentum tensor. We have

$$\tilde{\theta}^{\text{reg}}_{\mu\nu} = \frac{1}{2} \tilde{\psi}^+ \gamma_\mu \tilde{D}_\nu \tilde{\psi} = \frac{1}{2} \psi^+ \gamma_\mu \tilde{D}_\nu \psi$$ (175)

and the non-covariant counterterm can be rewritten after simple manipulation in the form

$$\Delta \tilde{\theta}^{\text{reg}}_{\mu\nu} = \frac{1}{2} \langle \psi^+ \gamma_\mu (\Omega_+ [\tilde{D}_\nu, \Omega_-] - [\Omega_+, \tilde{D}_\nu] \Omega_- + (\Omega_+ \Omega_- - 1) \tilde{D}_\nu - \tilde{D}_\nu (\Omega_+ \Omega_- - 1)) \psi \rangle.$$ (176)

Using the identities (130) and (131), we have then

$$\Delta \tilde{\theta}^{\text{reg}}_{\mu\nu} = - \frac{1}{2} \text{Tr} S^{\text{cov}} \gamma_\mu (\Omega_+ [\tilde{D}_\nu, \Omega_-] - [\Omega_+, \tilde{D}_\nu] \Omega_- + (\Omega_+ \Omega_- - 1) \tilde{D}_\nu - \tilde{D}_\nu (\Omega_+ \Omega_- - 1))$$

$$+ \frac{1}{2} \text{Tr} (\tilde{D}_\nu S^{\text{cov}} \gamma_\mu) (\Omega_+ \Omega_- - 1).$$ (177)

After some algebra (the details are given in Appendix B) we get

$$\Delta \tilde{\theta}^{\text{reg}}_{\mu\nu} = \frac{(1 - a)^2}{48\pi^2} \text{Tr}_C \left( \delta_{\mu\nu} A^2 \left( 2m^2 - \frac{1}{|\varepsilon|^2} \right) - 2A_\mu A_\nu \left( m^2 - \frac{2}{|\varepsilon|^2} \right) \right)$$

$$- \frac{(1 - a)^2}{576\pi^2} \text{Tr}_C (8\delta_\mu\nu A_\alpha [D_\beta, F_{\beta\alpha}] - 11 (A_\mu [D_\alpha, F_{\alpha\nu}] + A_\nu [D_\alpha, F_{\alpha\mu}])$$

$$- 5 (A_\mu [D_\nu, F_{\mu\alpha}] + A_\nu [D_\mu, F_{\mu\alpha}])$$

$$+ (1 - a)^2 (2A_\mu \partial \partial \cdot A + A_\alpha \partial^2 A_\alpha) + (A_\mu \partial_\nu + A_\nu \partial_\mu) \partial \cdot A$$

$$+ A_\alpha \partial (\partial_\alpha A_\mu + \partial_\mu A_\alpha) + 4A_\alpha \partial_\nu \partial_\mu A_\alpha - A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu$$

$$+ 3 ([A_\alpha, A_\mu] (\partial_\nu A_\alpha + \partial_\alpha A_\nu) + [A_\alpha, A_\nu] (\partial_\mu A_\alpha + \partial_\alpha A_\mu))$$

$$+ (1 - a)^2 (2A^4 + A_\alpha A_\beta A_\alpha A_\beta - 4 ([A_\mu, A_\nu] A^2 + A_\mu A_\alpha A_\alpha A_\nu))$$

$$- \frac{(1 - a)^2}{192\pi^2} \text{Tr}_C ((a + 6) (A_\mu [D_\alpha, F_{\alpha\nu}] - A_\nu [D_\alpha, F_{\alpha\mu}]) + (a - 4) A \cdot \partial F_{\mu\nu}$$

$$+ (1 - a) (A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2 A \cdot \partial [A_\mu, A_\nu]))$$ (178)

where we picked up the divergent, symmetric and antisymmetric part.

Let us calculate also the non-covariant counterterm for the scalar density $s = \psi^+ \psi$, we have using (170)

$$\Delta \tilde{s} = \langle \tilde{\psi}^+ \tilde{\psi} - \psi^+ \psi \rangle = - \text{Tr} S^{\text{cov}} (\Omega_+ \Omega_- - 1)$$

$$= \frac{(1 - a)^2}{8\pi^2} \text{ImTr}_C A^2.$$ (179)

For the trace anomaly in the non-covariant regularization scheme we get then

$$\tilde{A}^{\text{Trace}} = \Delta \tilde{\theta}^{\text{reg}}_{\mu\nu} + \text{Im} \Delta \tilde{s} = - \frac{(1 - a)^2}{96\pi^2} \text{Tr}_C (2A \cdot \partial \partial \cdot A + A_\alpha \partial^2 A_\alpha),$$ (180)
The spurious Lorentz anomaly can be also easily found as

\[ \Delta \tilde{A}_{\mu} = \Delta \tilde{\theta}_{\mu}^{\text{reg}} - \Delta \tilde{\theta}_{\mu}^{\text{reg}} - \frac{1}{2} \varepsilon_{\mu \nu \alpha \lambda} \partial_\alpha \Delta \tilde{J}_{\nu \lambda}^{\text{reg}} \]

\[ = -\frac{(1-a)}{96\pi^2} \text{Tr}_C \left( (a + 6)(A_\mu [D_\alpha, F_{\alpha \nu}] - A_\nu [D_\alpha, F_{\alpha \mu}]) + (a - 4) A \cdot \partial F_{\mu \nu} \right. \]

\[ + (1-a)(A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2A \cdot \partial [A_\mu, A_\nu]) \]

\[ \left. - \frac{1-a}{16\pi^2} \partial_\alpha \text{Tr}_C (A_\alpha F_{\mu \nu} + A_\mu F_{\nu \alpha} - A_\nu F_{\mu \alpha}) \right), \quad (181) \]

or, after some algebra,

\[ \Delta \tilde{A}_{\mu} = -\frac{1-a}{16\pi^2} \left( a (A_\mu [D_\alpha, F_{\alpha \nu}] - A_\nu [D_\alpha, F_{\alpha \mu}]) + (a + 2) A \cdot \partial F_{\mu \nu} \right. \]

\[ + 6[D_\alpha, A_\mu] \partial_\nu A_\alpha - [D_\alpha, A_\nu] \partial_\mu A_\alpha + F_{\mu \nu} \partial_\cdot A \]

\[ \left. + (1-a)(A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2A \cdot \partial [A_\mu, A_\nu]) \right). \quad (182) \]

Especially in the abelian case we have

\[ \Delta \tilde{A}_{\mu} = \left( \frac{a}{6} (A_\mu \partial_\nu - A_\nu \partial_\mu) \partial \cdot A - \frac{1}{6} (2 + a) A \cdot \partial F_{\mu \nu} \right. \]

\[ - \frac{1}{6} (A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu) \]

\[ - F_{\mu \nu} \partial \cdot A - \partial_\alpha A_\mu (\partial_\nu A_\alpha) + (\partial_\alpha A_\mu) (\partial_\nu A_\alpha) \right). \quad (183) \]

For the translation anomaly we get

\[ \Delta \tilde{A}_{\mu} = \partial_\nu \Delta \tilde{\theta}_{\mu \nu} - \partial_\nu \Delta \tilde{A}_{\mu}^{\text{trace}} - F_{\mu \nu} \Delta \tilde{a}_{\mu} \]

\[ = \frac{(1-a)^2}{48\pi^2} \text{Tr}_C \left( \partial_\nu A_\alpha \left( \frac{2m^2 - \frac{1}{|\varepsilon|^2}}{2} \right) - 2 \partial_\mu (A_\mu A_\nu) \left( m^2 - \frac{2}{|\varepsilon|^2} \right) \right. \]

\[ - 6F_{\mu \nu} A_\mu \left( \frac{1}{|\varepsilon|^2} - m^2 \right) \]

\[ - \frac{(1-a)}{576\pi^2} \partial_\mu \text{Tr}_C (8\delta_{\mu \alpha} A_{\alpha} [D_\beta, F_{\beta \alpha}] - 11(A_\mu [D_\alpha, F_{\alpha \nu}] + A_\nu [D_\alpha, F_{\alpha \mu}]) \]

\[ - 5(A_\alpha [D_\beta, F_{\beta \alpha}] + A_\alpha [D_\beta, F_{\alpha \mu}]) \]

\[ + (1-a) (-5 \delta_{\mu \nu} (2A \cdot \partial \cdot A + A_\alpha \partial^2 A_\alpha) + (A_\alpha \partial_\mu + A_\mu \partial_\alpha) \partial \cdot A \]

\[ + A \cdot \partial (\partial_\mu A_\nu + \partial_\nu A_\mu) + 4A_\nu \partial_\mu \partial_\alpha A_\alpha - A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu \]

\[ + 3([A_\alpha, A_\mu] (\partial_\nu A_\alpha + \partial_\alpha A_\nu) + [A_\alpha, A_\nu] (\partial_\nu A_\alpha + \partial_\alpha A_\mu)) \]

\[ + (1-a)^2 (\delta_{\mu \nu} (2A^2 + A_{\alpha \beta} A_{\alpha \beta} A_{\alpha \beta} - 4([A_\mu, A_\nu] A^2 + A_{\mu \alpha} A_{\nu \beta} A_{\beta \alpha}))) \]

\[ - \frac{(1-a)}{192\pi^2} \partial_\mu \text{Tr}_C ((a + 6)(A_\mu [D_\alpha, F_{\alpha \nu}] - A_\nu [D_\alpha, F_{\alpha \mu}]) + (a - 4) A \cdot \partial F_{\mu \nu} \]

\[ + (1-a)(A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2A \cdot \partial [A_\mu, A_\nu]) \]

\[ - \frac{(1-a)}{288\pi^2} \text{Tr}_C F_{\mu \nu} (4(1-a)^2 (A_\mu A \cdot A + A \cdot AA_\mu + A_\nu A_\mu A_\nu) \]

\[ + (2-a)([A_\mu, \partial \cdot A] + [A_\nu, \partial \mu A_\nu] + [A_\nu, \partial_\mu A_\nu]) \]

\[ + (2 \partial_\mu \partial \cdot A + \partial^2 A_\mu)) \right). \quad (184) \]
Let us note, that the formulae (166), (173), (180), (182) and (184) for the anomalies can be also obtained by means of the corresponding modification of the Ward-Takahashi identities approach used in the previous sections. E.g. we have the following Ward-Takahashi identity for the non-covariantly point-split chiral transformation (157)

\[
\delta S_E = \int d^4x \left( \tilde{\psi}^+_\varepsilon \gamma \cdot D, \alpha \gamma_5 \tilde{\psi}_\varepsilon + 2im\tilde{\psi}^+_\varepsilon \alpha \gamma_5 \tilde{\psi}_\varepsilon \right)
\]

\[
+ \int d^4x \left( \tilde{\psi}^+_\varepsilon (\alpha \gamma_5 [e^{-\varepsilon \cdot D}, \gamma \cdot \tilde{D}] e^{-\varepsilon \cdot D} - e^{-\varepsilon \cdot D} [\gamma \cdot \tilde{D}, e^{-\varepsilon \cdot D}] \gamma_5 \alpha) \tilde{\psi}_\varepsilon \right),
\]

that means we have

\[
\tilde{A}_0^a = \langle \tilde{\psi}^+_\varepsilon (\alpha \gamma_5 [e^{-\varepsilon \cdot D}, \gamma \cdot \tilde{D}] e^{-\varepsilon \cdot D} - e^{-\varepsilon \cdot D} [\gamma \cdot \tilde{D}, e^{-\varepsilon \cdot D}] \gamma_5 \alpha) \tilde{\psi}_\varepsilon \rangle.
\]

It is an easy exercise to prove, that this expression reproduces the formula (166).

10 Properties of the vector current

In the previous section we have defined the covariant vector current by means of the formula

\[
j_{\mu, \text{reg}}^a = \psi^+_\varepsilon T^a \gamma_\mu \tilde{\psi}_\varepsilon.
\]

From this formula it follows immediately that the regularized current transforms covariantly under the gauge transformations. There exist also other definitions of the covariant point-split currents, which differ from this by the order of the parallel transporter and the gauge generator \( T^a \). E.g. in ref. 2 the authors use the prescription

\[
j_{\mu, (1)}^{a, \text{reg}} = \psi^+_\varepsilon (x + \varepsilon) T^a \gamma_\mu \Omega(x + \varepsilon, x - \varepsilon) \psi(x - \varepsilon)
\]

or

\[
j_{\mu, (2)}^{a, \text{reg}} = \frac{1}{2} \psi^+_\varepsilon (x + \varepsilon) \{ T^a, \Omega(x + \varepsilon, x - \varepsilon) \} \gamma_\mu \psi(x - \varepsilon).
\]

Let us now clarify the relations between these currents. We can write

\[
j_{\mu, (1)}^{a, \text{reg}} = \psi^+_\varepsilon e^{-\varepsilon \cdot \tilde{D}} e^\varepsilon \cdot \partial T^a \gamma_\mu \Omega(x + \varepsilon, x) \psi_\varepsilon,
\]

or, using the identity

\[
\Omega(x + \varepsilon, x) = e^{-\varepsilon \cdot \partial} e^\varepsilon \cdot \partial,
\]

we have

\[
j_{\mu, (1)}^{a, \text{reg}} = \psi^+_\varepsilon e^{-\varepsilon \cdot \tilde{D}} e^\varepsilon \cdot \partial T^a \gamma_\mu e^{-\varepsilon \cdot \partial} e^\varepsilon \cdot \partial \psi_\varepsilon
\]

\[
= \psi^+_\varepsilon e^{-\varepsilon \cdot \tilde{D}} T^a \gamma_\mu e^\varepsilon \cdot \partial \psi_\varepsilon.
\]

That means the current (187) and (188) differ by the counterterm

\[
\Delta_{(1)} j_{\mu}^a = \langle \psi^+_\varepsilon (e^{-\varepsilon \cdot \tilde{D}} T^a e^\varepsilon \cdot \partial - T^a) \gamma_\mu \psi_\varepsilon \rangle
\]

\[
= -\text{Tr} S^\text{cov} \gamma_\mu (e^{-\varepsilon \cdot \tilde{D}} T^a e^\varepsilon \cdot \partial - T^a).
\]
Because
\[ e^{-\varepsilon\hat{D}T^a\varepsilon\hat{D} - T^a} = \sum_{n=1}^{\infty} \frac{1}{n!} [\varepsilon \cdot D, [\varepsilon \cdot D, \ldots [\varepsilon \cdot D, T^a] \ldots]], \tag{194} \]
we have
\[
\Delta(1)j_\mu^a = \frac{1}{4\pi^2} \text{Tr}c \left( \left( \frac{\varepsilon \mu}{|\varepsilon|^2} - m^2 \frac{\varepsilon \mu}{|\varepsilon|^2} + \mathcal{O}(\ln |\varepsilon|) \right) \right)
\times \left( [\varepsilon \cdot D, T^a] + \frac{1}{2} [\varepsilon \cdot D, [\varepsilon \cdot D, T^a]] + \frac{1}{6} [\varepsilon \cdot D, [\varepsilon \cdot D, [\varepsilon \cdot D, T^a]]] + \mathcal{O}(\varepsilon^4) \right)
\]
\[ = \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|), \tag{195} \]
in the same way we can easily prove
\[
\Delta(2)j_\mu^a = j_{\mu(2)}^a - j_{\mu}^{a,\text{reg}} = \mathcal{O}(\varepsilon). \tag{196} \]
That means that all the above introduced currents are equivalent in the limit of the removed point-splitting (cf. [2]). But only one of them, namely \(j_{\mu}^{a,\text{reg}}(x)\), transforms covariantly according to the naive prescription
\[ j_\mu(x) \rightarrow U(x)j_\mu(x)U^+(x) \tag{197} \]
already on the regularized level, the other currents have shifted the space time argument of the transformation matrix \(U\).

Let us investigate also another property of the general regularized current \(j_{\mu}^{a,\text{reg}}(x)\). Provided the integrability conditions
\[ I_{\mu\nu}(x,y)[A] = \frac{\delta}{\delta A^a_\nu(y)}(j_{\mu}^{a,\text{reg}}(x))_c - \frac{\delta}{\delta A^a_\mu(x)}(j_{\mu}^{b,\text{reg}}(y))_c = 0, \tag{198} \]
are satisfied, the current can be expressed as a functional derivative of the effective action functional \(\Gamma_E[A] = \ln Z_E[A]\), i.e.
\[ j_{\mu}^{a,\text{reg}}(x) \cdot Z_E[A] = (j_{\mu}^{a,\text{reg}}(x))_c = -\frac{\delta}{\delta A^a_\mu(x)} \Gamma_E[A]_c \tag{199} \]
or, equivalently, it can be integrated to define the consistent effective action consistent with the relation (199)
\[ \Gamma_E[A] = -\int_0^1 dt \int d^4x A^a_\mu(x) j_{\mu}^{a,\text{reg}}(x) \cdot Z_E[tA]_c. \tag{200} \]

\(j_{\mu}^{a,\text{reg}}(x) \cdot Z_E[A]_c\) can be then interpreted as a consistent generator of the correlators of the vector currents (i.e. which obeys Bose symmetry) corresponding to the naive formula
\[ (j_{\mu_1}^{a_1}(x_1)j_{\mu_2}^{a_2}(x_2) \ldots j_{\mu_n}^{a_n}(x_n))_c = (-1)^{n-1} \frac{\delta^{n-1}}{\delta A^a_{\mu_2}(x_2) \ldots \delta A^a_{\mu_n}(x_n)} j_{\mu_1}^{a_1,\text{reg}}(x_1) \cdot Z_E[A]_c. \tag{201} \]
On the other hand, provided the right hand side of (198) is nonzero, the effective action defined by means of the integral (200) does not yield the original current, but the integrable one (cf. also [2] and [13])
\[ j_{\mu}^{a,\text{reg}}(x) \cdot Z_E[A] = -\frac{\delta}{\delta A^a_\mu(x)} \Gamma_E[A]_c = j_{\mu}^{a,\text{reg}}(x) \cdot Z_E[A] - \int_0^1 dt \int d^4 y \hat{r}^{ab}(x,y)[tA]_c \tag{202} \]
the covariant derivative of which gives the consistent anomaly satisfying the Wess-Zumino consistency conditions [16].

Another characteristics of the general point-split current is the violation of the covariance

$$C_{\mu}^{ab}(x,y)[A] = D_{\nu}^{ac}(y)\frac{\delta}{\delta A_{\nu}^{a}(x)} j_{\mu}^{b,\text{reg}}(x) \cdot Z_{E}[A]_{c} - \delta^{(4)}(x-y) f^{abc} j_{\mu}^{c,\text{reg}}(x) \cdot Z_{E}[A]_{c}.$$  \hspace{1cm} (203)

where $D_{\nu}^{ac}(y) = \partial_{\nu}\delta^{ac} + f^{dca} A_{\nu}^{d}(y)$ is the covariant derivative in the adjoint representation and $f^{dca}$ are the totally antisymmetric structure constants of the gauge group. $C_{\mu}^{ab}(x,y)[A]$ can be rewritten using (198) to the form (cf. ref. [15])

$$C_{\mu}^{ab}(x,y)[A] = D_{\nu}^{ac}(y) J_{\mu \nu}^{b,c}(x,y)[A] + \frac{\delta}{\delta A_{\mu}^{a}(x)} D_{\nu}^{ac}(y) j_{\nu}^{c,\text{reg}}(y) \cdot Z_{E}[A]_{c}$$

$$= -D_{\nu}^{ac}(y) T_{\nu \mu}^{b}(y,x) [A] + \frac{\delta}{\delta A_{\mu}^{a}(x)} A^{a}(y)[A]$$\hspace{1cm} (204)

where $A^{a}(y)[A] = D_{\nu}^{ac}(y) j_{\nu}^{c,\text{reg}}(y)_{c}$ is the anomaly. From this equation it is easily seen, that anomaly free integrable current is necessary covariant and covariant and integrable current must be anomaly free.

As an illustration, we prove in the following that for the covariant point-splitting described in the previous sections the integrability conditions (198) are not violated (cf. also [2]). Let us decompose the covariantly regularized current (187) into two parts, namely

$$J_{\mu}^{a,\text{reg}} = J_{\mu}^{a,\text{reg}} - \Delta J_{\mu}^{a,\text{reg}},$$  \hspace{1cm} (205)

where $J_{\mu}^{a,\text{reg}}$ is the non-covariantly regularized vector current [168] described in the previous section and $\Delta J_{\mu}^{a,\text{reg}}$ is the corresponding counterterm and choose appropriately the value of the regularization parameter $a$ in order to simplify the calculations of $T_{\mu \nu}^{b}(x,y)$ as much as possible. The contribution of $\Delta T_{\mu \nu}^{ab}(x,y)$ of the counterterm $\Delta J_{\mu}^{a,\text{reg}}$ to the left hand side of the integrability condition (198) can be easily obtained from the formula (168) as

$$\Delta T_{\mu \nu}^{ab}(x,y) = \frac{\delta}{\delta A_{\nu}^{b}(y)} \langle \tilde{J}_{\mu}^{a,\text{reg}}(x) \rangle_{c} - \frac{\delta}{\delta A_{\mu}^{a}(x)} \langle \tilde{J}_{\nu}^{b,\text{reg}}(y) \rangle_{c}$$

$$= \frac{1}{96\pi^{2}} (1 - a)(2 - a) \delta^{(4)}(x-y) Tr_{C}[T^{a}, T^{b}] \langle \delta_{\mu \nu} \partial \cdot A + \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} \rangle$$

$$+ O(\varepsilon, \varepsilon \ln |\varepsilon|).$$  \hspace{1cm} (206)

The contribution of the non-covariantly regularized vector current, $\tilde{T}_{\mu \nu}^{ab}(x,y)$, can be rewritten in the form

$$\tilde{T}_{\mu \nu}^{ab}(x,y) = \frac{\delta}{\delta A_{\nu}^{b}(y)} \langle \tilde{J}_{\mu}^{a,\text{reg}}(x) \rangle_{c} - \frac{\delta}{\delta A_{\mu}^{a}(x)} \langle \tilde{J}_{\nu}^{b,\text{reg}}(y) \rangle_{c}$$

$$= -\langle \tilde{\psi}_{\varepsilon}^{+}(x) T^{a} \gamma_{\mu} \tilde{\psi}_{\varepsilon}(y) \rangle_{c} + \langle \psi_{\varepsilon}^{+}(x) T^{b} \gamma_{\mu} \tilde{\psi}_{\varepsilon}(y) \rangle_{c}$$

$$+ \langle \psi_{\varepsilon}^{+}(x) \frac{\delta}{\delta A_{\mu}^{a}(x)} e^{D} T^{a} \gamma_{\mu} \tilde{\psi}_{\varepsilon}(y) \rangle_{c} + \langle \psi_{\varepsilon}^{+}(x) T^{a} \gamma_{\mu} \frac{\delta}{\delta A_{\mu}^{a}(x)} e^{-D} \tilde{\psi}_{\varepsilon}(y) \rangle_{c}$$

$$- \langle \psi_{\varepsilon}^{+}(y) T^{b} \gamma_{\mu} \tilde{\psi}_{\varepsilon}(y) \rangle_{c} + \langle \tilde{\psi}_{\varepsilon}^{+}(y) T^{b} \gamma_{\mu} \frac{\delta}{\delta A_{\mu}^{a}(x)} e^{-D} \tilde{\psi}_{\varepsilon}(y) \rangle_{c}.$$  \hspace{1cm} (207)
Putting in this formula the value of the regularization parameter \( a = 0 \), we get much less complicated expression

\[
\begin{align*}
\bar{T}^{ab}_{\mu \nu}(x, y) |_{a=0} &= -\langle \psi^+(x + \varepsilon) T^a \gamma_\mu \psi(x - \varepsilon) \psi^+(y) T^b \gamma_\nu \psi(y) \rangle_c \\
&\quad + \langle \psi^+(x) T^a \gamma_\mu \psi(x) \psi^+(y + \varepsilon) T^b \gamma_\nu \psi(y - \varepsilon) \rangle_c \\
&= \text{Tr} T^a \gamma_\mu S(x - \varepsilon, y) T^b \gamma_\nu S(y, x + \varepsilon) \\
&\quad - \text{Tr} T^a \gamma_\mu S(x, y + \varepsilon) T^b \gamma_\nu S(y - \varepsilon, x).
\end{align*}
\]

(208)

For \( x \neq y \) the right hand side is easily seen to vanish in the limit \( \varepsilon \to 0 \), however for \( x = y \) we get singular expression; i.e. \( \bar{T}^{ab}_{\mu \nu}(x, y) |_{a=0} \) as a function of \( y \) is a distribution concentrated at \( y = x \). Let us therefore calculate the expression

\[
\int d^4 y B^b_\nu(y) \bar{T}^{ab}_{\mu \nu}(x, y) |_{a=0} = \int d^4 y (\text{Tr} T^a \gamma_\mu S(x - \varepsilon, y) B(y) \cdot \gamma S(y, x + \varepsilon) \\
- \text{Tr} T^a \gamma_\mu S(x, y + \varepsilon) B(y) \cdot \gamma S(y - \varepsilon, x)),
\]

(209)

where \( B^b_\nu(y) \) are appropriately smooth test functions with compact support. Inserting here the expansion of the propagator

\[
S(x, y) = \frac{1}{4\pi^2} \left( -2 \gamma \cdot (x - y) \frac{1}{|x|} \right) + 2 \frac{\gamma \cdot (x - y)(y - x)}{|x - y|^2} \text{A}(\mathbf{x}) \bigg) - \text{im} \frac{1}{|x - y|^2} \bigg( \\
- \frac{\gamma \cdot (x - y)(y - x)}{|x - y|^2} \text{A}(\mathbf{x}) \bigg) + \frac{1}{2} m^2 \gamma \cdot (x - y) \frac{1}{|x - y|^2} \bigg) + O(\ln|\varepsilon|)
\]

(210)

where \( \mathbf{x} = (x + y)/2 \), we find after the substitution \( y = x + |\varepsilon| z \), that the potentially most singular term stemming from the product of the leading singularity in the expansion of \( S(x, y) \) cancel and the only terms, which survives the limit \( \varepsilon \to 0 \) comes from the product of the first two terms of (210), namely

\[
\int d^4 y B^b_\nu(y) \bar{T}^{ab}_{\mu \nu}(x, y) |_{a=0} = - \frac{1}{4\pi^4} \text{Tr} D(\gamma \mu \alpha \nu \gamma_\beta) \frac{1}{|\varepsilon|} \int d^4 z B^b_\nu(x + |\varepsilon| z) \frac{\Delta_{+\gamma} \Delta_{-\beta}}{\Delta_{+} |\Delta_{+}|^2} \bigg( \\
\times (\text{Tr} C(T^a T^b (A_\gamma(\mathbf{x}_+) - A_\gamma(\mathbf{x}_-)) \Delta_{-\gamma}) \\
+ \text{Tr} C(T^b T^a (A_\gamma(\mathbf{x}_+) - A_\gamma(\mathbf{x}_-)) \Delta_{+\gamma})) + O(\varepsilon \ln|\varepsilon|),
\]

(211)

here we denote

\[
\Delta_{\pm} = \frac{x - y \mp \varepsilon}{|\varepsilon|} = \mp(z \pm n)
\]

(212)

\[
\mathbf{x}_{\pm} = \frac{1}{2} (x + y \pm \varepsilon) = x + \frac{1}{2} |\varepsilon|(z \pm n)
\]

(213)

and \( n = \varepsilon/|\varepsilon| \). We have therefore, after the substitution \( z \to -z \) in the second term

\[
\int d^4 y B^b_\nu(y) \bar{T}^{ab}_{\mu \nu}(x, y) |_{a=0} = - \frac{1}{4\pi^4} \text{Tr} D(\gamma \mu \alpha \nu \gamma_\beta) B^b_\nu(x) \text{Tr} C([T^a, T^b] \partial_\alpha A_\gamma) n_\delta J_{\alpha \beta \gamma}(n) \\
+ O(\varepsilon \ln|\varepsilon|),
\]

(214)
Using the same method as in the main text, we can easily prove that

\[ J_{\alpha\beta}(n) = \int d^4x \frac{\Delta_+^{\alpha} \Delta_-^{\beta} \Delta_+^{\gamma} \Delta_-^{\delta}}{|\Delta_+|^4 |\Delta_-|^4} \]

\[ = \frac{\pi^2}{8} (\delta_{\alpha\beta} n_\gamma - \delta_{\beta\gamma} n_\alpha + \delta_{\alpha\gamma} n_\beta - 2n_\alpha n_\beta n_\gamma). \tag{215} \]

I.e. after taking the average over the directions of \( n \) and some algebra we get

\[ \tilde{I}_{\mu\nu}(x, y)_{|a=0} = \frac{1}{48\pi^2} \delta^{(4)}(x - y) \text{Tr}_C([T^a, T^b](\delta_{\mu\nu} \partial \cdot A + \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu})) \]

\[ + \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|) \tag{216} \]

and\footnote{We could also obtain this result in a more direct way and get independent check of the calculation. Let us write

\[ I_{\mu\nu}^{ab}(x, y) = I_{\mu\nu}^{ab}(x, y)_{|a=0} - \Delta I_{\mu\nu}^{ab}(x, y)_{|a=0} = \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|). \]

That means the covariantly regularized vector current is integrable and, because its divergence is zero, the effective action is gauge invariant.}

\[ I_{\mu\nu}^{ab}(x, y) = \tilde{I}_{\mu\nu}(x, y)_{|a=0} - \Delta I_{\mu\nu}^{ab}(x, y)_{|a=0} = \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|). \]

\[ \Delta^b_\nu(x, y, \varepsilon) = \left( \frac{\delta}{\delta A^b_\nu(y)} \right) e^{-\varepsilon D} \right) e^{\varepsilon D} = -\int_0^1 d\varepsilon e^{-\varepsilon D} \varepsilon, \delta^{(4)}(x - y) T^b e^{\varepsilon D}\]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} [\varepsilon \cdot D, [\varepsilon \cdot D, [\ldots [\varepsilon \cdot D, \varepsilon, \delta^{(4)}(x - y) T^b] \ldots]]. \]

Using the same method as in the main text, we can easily prove that \( I_{\mu\nu}^{ab}(x, y)_{|a=0} = 0 \). We have therefore, dropping terms which vanish after symmetrization over the directions of the four-vector \( \varepsilon \),

\[ I_{\mu\nu}^{ab}(x, y) = \frac{1}{4\pi^2} \left( \frac{\varepsilon_{\mu \varepsilon_{\nu}}}{|\varepsilon|^4} - m^2 \frac{\varepsilon_{\mu \varepsilon_{\nu}}}{|\varepsilon|^2} + \mathcal{O}(\ln |\varepsilon|) \right) \]

\[ \times \text{Tr}_C \left( \frac{2}{3} \varepsilon \cdot \partial \delta^{(4)}(x - y) |\varepsilon \cdot A(x), T^a||T^a \right) \\
+ \frac{1}{3} \delta^{(4)}(x - y) [\varepsilon \cdot D, [\varepsilon \cdot A(x), T^b]|T^a \right) \\
+ \frac{1}{3} (\varepsilon \cdot \partial)^2 \delta^{(4)}(x - y) T^b T^a + \mathcal{O}(\varepsilon^3) \right) - (\varepsilon, a) \leftrightarrow (\varepsilon, b) \]

\[ = \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|), \]

i.e.

\[ I_{\mu\nu}^{ab}(x, y) = \mathcal{O}(\varepsilon, \varepsilon \ln |\varepsilon|). \]
As it is seen from formula (206), for the non-covariant point-splitting
\[
\tilde{T}^{ab}_{\mu}(x,y) = \Delta \tilde{T}^{ab}_{\mu}(x,y) = \frac{(1-a)(2-a)}{96\pi^2} \delta^{(4)}(x-y) \text{Tr}_{C}[T^a,T^b](\partial_{\mu} \partial \cdot A + \partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu}) + O(\epsilon, \epsilon \ln |\epsilon|)
\]
and therefore we can recover integrability also for \(a = 2\). We can also enlarge the class of regularizations and construct one parametric set of integrable non-covariant vector currents within such an enlarged point-splitting scheme in the following way. Let us define the regularized current to be
\[
j_{\mu}^{a,\text{reg}}(x;a,b) = \frac{1}{2}(\bar{\psi}_{\epsilon}^{-}(x,a)T^a \gamma_{\mu} \psi_{\epsilon}(x,b) + \bar{\psi}_{\epsilon}^{+}(x,b)T^a \gamma_{\mu} \psi_{\epsilon}(x,a))
\]
where
\[
\bar{\psi}_{\epsilon}(x,a) = e^{-\epsilon \cdot D(a)} \psi(x)
\]
\[
\bar{\psi}_{\epsilon}^{+}(x,a) = \psi(x) e^{\epsilon \cdot \bar{D}(a)}
\]
and \(D(a) = \partial \cdot a A\). This broad class of regularizations naturally incorporates the previously introduced currents for special choice of the parameters \(a\) and \(b\), namely \(j_{\mu}^{a,\text{reg}}(x) = j_{\mu}^{a,\text{reg}}(x;a,a)\) and \(j_{\mu}^{a,\text{reg}}(x) = j_{\mu}^{a,\text{reg}}(x;1,1)\). It is not difficult to calculate the counterterm
\[
\Delta j_{\mu}^{a,\text{reg}}(x;a,b) = j_{\mu}^{a,\text{reg}}(x;a,b) - j_{\mu}^{a,\text{reg}}(x),
\]
using the methods described in the previous section. We have
\[
\Delta j_{\mu}^{a,\text{reg}}(x;a,b) = -\frac{1}{2} \text{Tr} S^{\text{cov}} \gamma_{\mu} (\Omega_{+}(a) T^a \Omega_{-}(b) + \Omega_{+}(b) T^a \Omega_{-}(a) - 2T^a)
\]
where we explicitly picked up the value of the parameters in the functions \(\Omega_{\pm}\). Using the formulae of Appendix B we get
\[
\frac{1}{2}(\Omega_{-}(a) \Omega_{+}(b) + \Omega_{-}(b) \Omega_{+}(a)) = 1 + 2 \left(1 - \frac{a + b}{2}\right) \epsilon \cdot A + 2 \left(1 - \frac{a + b}{2}\right)^2 (\epsilon \cdot A)^2 + \frac{4}{3} \left(1 - \frac{a + b}{2}\right)^3 (\epsilon \cdot A)^3 + \frac{1}{3} \left(1 - \frac{a + b}{2}\right) (\epsilon \cdot \partial)^2 \epsilon \cdot A + \frac{1}{12}((1-a)(a-3b+4) + (1-b)(b-3a+4)) \times [\epsilon \cdot A, \epsilon \cdot \partial A \cdot \epsilon] + O(\epsilon^4)
\]
and therefore
\[
\Delta j_{\mu}^{a,\text{reg}}(x;a,b) = \frac{1}{8\pi^2} \text{Tr}_{C} T^a \left( \left(1 - \frac{a + b}{2}\right) \left(\frac{1}{|\epsilon|^2} - m^2\right) A_{\mu} + \frac{1}{9} \left(1 - \frac{a + b}{2}\right)^3 (A_{\mu} A \cdot A + A A_{\mu} + A_{\mu} A_{\nu} + A_{\nu} A A_{\mu}) + \frac{1}{144}((1-a)(a-3b+4) + (1-b)(b-3a+4)) \times ([A_{\mu}, \partial \cdot A] + [A_{\nu}, \partial_{\nu} A_{\mu}] + [A_{\nu}, A_{\nu} A_{\mu}]) + \frac{1}{36} \left(1 - \frac{a + b}{2}\right) (2\partial_{\mu} \partial \cdot A + \partial^2 A_{\mu})\right) + O(\epsilon).
\]

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This generally nonintegrable counterterm is “minimal” for \(a + b = 2\), in this case

\[
\Delta j_{\mu, \text{reg}}^a(x; a, 2 - a) = -\frac{(1 - a)^2}{144\pi^2} \text{Tr} C T^a ([A_\mu, \partial \cdot A] + [A_\nu, \partial_\mu A_\nu] + [A_\nu, \partial_\nu A_\mu]),
\]

This minimal choice is exceptional also for another reason. Let us note, that provided we regularize the axial current in the analogous way, i.e. if we put

\[
j_{\mu, \text{reg}}^a(x; a, b) = \frac{1}{2} (\bar{\psi}_\epsilon^+(x, a) T^a \gamma_5 \psi_\epsilon(x, b) + \bar{\psi}_\epsilon^+(x, b) T^a \gamma_5 \psi_\epsilon(x, a)),
\]

we get the following non-covariant counterterm, which modifies the covariantly regularized current

\[
\Delta j_{\mu, \text{reg}}^a(x; a, b) = j_{\mu, \text{reg}}^a(x; a, b) - j_{\mu, \text{reg}}^b,
\]

\[
= -\frac{1}{2} \text{Tr} S^{\text{cov}} \gamma_5 \gamma_5 (\Omega_+ (a) T^a \Omega_- (b) + \Omega_+ (b) T^a \Omega_- (a) - 2 T^a)
\]

\[
= \frac{1}{16 \pi^2} \left(1 - \frac{a + b}{2}\right) \text{Tr} C T^a \{A_\nu, F^a_{\nu \mu}\} + \mathcal{O}(\epsilon).
\]

This counterterm vanishes and the gauge covariance is recovered in the limit of the removed cut-off again for \(a\) and \(b\) satisfying \(a + b = 2\).

The non-covariant current \(j_{\mu, \text{reg}}^a(x; a, b)\) can be made integrable for \(a\) and \(b\) for which

\[
(1 - a)(a - 3b + 4) + (1 - b)(b - 3a + 4) = 0;
\]

especially for the “minimal” choice \(a = 1\) (however then \(b = 2 - a = 1\) and the gauge covariance is recovered). The gauge invariance of the effective action \(\bar{\Gamma}[A]\) integrated from such a current is generally lost, its variation under gauge transformation is determined by the non-covariant anomaly,

\[
\bar{\Gamma}[A - [D, \alpha]] = \frac{1}{8\pi^2} \left(1 - \frac{a + b}{2}\right) \int d^4 x \text{Tr} C \alpha(x)[D_\mu, ((\frac{1}{|\alpha|^2} - m^2)A_\mu
\]

\[
+ \frac{1}{9} \left(1 - \frac{a + b}{2}\right)^2 (A_\mu A_\mu A + A \cdot AA_\mu + A_\nu A_\mu A_\nu)
\]

\[
+ \frac{1}{36}(2\partial_\mu \partial \cdot A + \partial^2 A_\mu)] + \mathcal{O}(\epsilon).
\]

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Appendix A

In this appendix we give some technical details of the previous calculation. We present
here the explicit form of the functions \( a_0(x, x - \varepsilon) a_1(x - \varepsilon, x + \varepsilon) a_0(x + \varepsilon, x) \), \( G_\mu(x, \varepsilon) \) and \( H_\mu(x, \varepsilon) \) used in the main text.

Let us first derive the formula (50). As we have seen in Section 2, for \( a_1(x, y) \) we have the following expression

\[
a_1(x, y) = - \int_0^1 \! dt a_0(x, t\bar{x}) \Delta_{x_t} a_0(t\bar{x}, y),
\]

(229)

where

\[
\bar{x}_t = y + t(x - y)
\]

and

\[
\Delta = -D^2 - \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}.
\]

(230)

(231)

I.e. we have

\[
a_0(x, x - \varepsilon) a_1(x - \varepsilon, x + \varepsilon) a_0(x + \varepsilon, x) = - \int_0^1 \! dt a_0(x, t) \Delta_x a_0(x, x + \varepsilon) a_0(x + \varepsilon, x)
\]

\[
= \frac{i}{2} \int_0^1 \! dt a_0(x, t) \sigma_{\mu\nu} F_{\mu\nu}(x) a_0(x, x) + \int_0^1 \! dt a_0(x, x) (D^2_{x_t} a_0(x, x + \varepsilon) a_0(x + \varepsilon, x)
\]

(232)

where now

\[
x_t = x + \varepsilon(1 - 2t).
\]

Let us remind that

\[
a_0(x, y) = T \exp \left( - \int_0^1 \! dt (x - y) \cdot A(y + t(x - y)) \right),
\]

(233)

(234)

i.e.

\[
a_0(x, x_t) = T \exp \left( - \int_0^1 \! d\tau (x - x_t) \cdot A(x + \tau(x - x_t)) \right)
\]

(235)

and

\[
a_0(x_t, x) = T \exp \left( \int_0^1 \! d\tau (x - x_t) \cdot A(x + \tau(x - x)) \right).
\]

(236)

Expanding this to the first order in \( \varepsilon \) we get

\[
a_0(x, x_t) = 1 - (2t - 1)\varepsilon \cdot A(x) + \mathcal{O}(\varepsilon^2)
\]

(237)

and

\[
a_0(x_t, x) = 1 + (2t - 1)\varepsilon \cdot A(x) + \mathcal{O}(\varepsilon^2).
\]

(238)

We have therefore immediately

\[
\frac{i}{2} \int_0^1 \! dt a_0(x, t) \sigma_{\mu\nu} F_{\mu\nu}(x) a_0(x, x)
\]

\[
= \frac{i}{2} \int_0^1 \! dt a_0(x, t) \sigma_{\mu\nu} (F_{\mu\nu}(x) + (1 - 2t)\varepsilon \cdot \partial F_{\mu\nu}(x)) a_0(x, x) + \mathcal{O}(\varepsilon^2)
\]

\[
= \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) + \frac{i}{2} \int_0^1 \! dt (1 - 2t)\varepsilon \cdot (\partial F_{\mu\nu}(x) + [A(x), F_{\mu\nu}(x)]) \sigma_{\mu\nu} + \mathcal{O}(\varepsilon^2)
\]

\[
= \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) + \mathcal{O}(\varepsilon^2).
\]

(239)
Let us calculate also two other quantities, namely $D_{x,\mu}a_0(x, y)$ and $D^2_{x}a_0(x, y)$. From the equation

$$(x - y) \cdot D_{x}a_0(x, y) = 0$$

we get

$$D_{x,\mu}a_0(x, y) + (x - y)\alpha D_{x,\mu}D_{x,\alpha}a_0(x, y) = (1 + (x - y) \cdot D_{x})D_{x,\mu}a_0(x, y) - (x - y)\alpha F_{\mu\alpha}a_0(x, y) = 0$$

and, using the left hand side of the previous equation and the identity $[D^2, D_{\alpha}] = [D_{\mu}, F_{\mu\alpha}] + 2F_{\mu\alpha}D_{\mu}$, we have

$$2D^2_{x}a_0(x, y) + (x - y)\alpha D^2_{x,\alpha}a_0(x, y)
= (2 + (x - y) \cdot D_{x})D^2_{x}a_0(x, y) + (x - y)\alpha([D_{\mu}, F_{\mu\alpha}] + 2F_{\mu\alpha}D_{\mu})a_0(x, y) = 0.$$ 

From this equations we get for $tD_{x_t,\mu}a_0(\bar{x}_t, y)$ and $tD^2_{x_t}a_0(\bar{x}_t, y)$ the following ordinary differential equations

$$\frac{d}{dt}tD_{x_t,\mu}a_0(\bar{x}_t, y) = -(x - y) \cdot A(\bar{x}_t)tD_{x_t,\mu}a_0(\bar{x}_t, y) + t(x - y)\alpha F_{\alpha\mu}(\bar{x}_t)a_0(\bar{x}_t, y)$$

and

$$\frac{d}{dt}tD^2_{x_t}a_0(\bar{x}_t, y) = -(x - y) \cdot A(\bar{x}_t)tD^2_{x_t}a_0(\bar{x}_t, y) - t^2(x - y)\alpha([D_{x_t,\mu}, F_{\mu\alpha}(\bar{x}_t)] + 2F_{\mu\alpha}(\bar{x}_t)D_{x_t,\mu})a_0(\bar{x}_t, y)$$

with the solutions

$$tD_{x_t,\mu}a_0(\bar{x}_t, y) = (x - y)\alpha \int_0^t d\tau \tau a_0(\bar{x}_t, \bar{x}_\tau)F_{\alpha\mu}(\bar{x}_\tau)a_0(\bar{x}_\tau, y)$$

and

$$tD^2_{x_t}a_0(\bar{x}_t, y) = -(x - y)\alpha \int_0^t d\tau \tau^2 a_0(\bar{x}_t, \bar{x}_\tau)[D_{x_t,\mu}, F_{\mu\alpha}(\bar{x}_\tau)]a_0(\bar{x}_\tau, y)
- 2(x - y)\alpha \int_0^t d\tau \tau a_0(\bar{x}_t, \bar{x}_\tau)F_{\mu\alpha}(\bar{x}_\tau)(\tau D_{x_t,\mu}a_0(\bar{x}_\tau, \bar{x}_\tau)).$$

Now, setting $x \to x - \varepsilon$, $y \to x + \varepsilon$, we can easily expand in powers of $\varepsilon$. From (243) we see that up to the order $O(\varepsilon)$ only the first term on the right hand side of (244) contributes, i.e.

$$t^2D^2_{x_t}a_0(x_t, y) = -2\varepsilon\alpha \int_0^t d\tau \tau^2 [D_\mu, F_{\mu\alpha}(x)] + O(\varepsilon^2)
= -2\varepsilon\alpha \frac{t^3}{3} [D_\mu, F_{\mu\alpha}(x)] + O(\varepsilon^2).$$

As a result we have

$$\int_0^1 dt a_0(x, t)(D^2_{x_t}a_0(x_t, x + \varepsilon)a_0(x + \varepsilon, x) = -2\varepsilon\alpha \int_0^1 dt \frac{t}{3} [D_\mu, F_{\mu\alpha}(x)] + O(\varepsilon^2)
= -\frac{1}{3} \varepsilon\alpha [D_\mu, F_{\mu\alpha}(x)] + O(\varepsilon^2).$$
Finally we get

\[ a_0(x, x - \varepsilon)a_1(x - \varepsilon, x + \varepsilon)a_0(x + \varepsilon, x) = \frac{i}{2} \sigma_{\mu \nu} F_{\mu \nu}(x) + \frac{1}{3} \varepsilon_\alpha [D_{\mu}, F_{\mu \alpha}(x)] + O(\varepsilon^2). \] (249)

Now we present the derivation of the right hand side of the formula (249). Let us evaluate the following expression

\[
\begin{align*}
\partial_\mu (e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y}) &= \int_0^1 dt e^{-\varepsilon \cdot \vec{D}_x} \vec{D}_{\mu,x} e^{\varepsilon \cdot \vec{D}_x} e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} \\
&+ \int_0^1 dt e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} e^{-\varepsilon \cdot \vec{D}_y} \vec{D}_{\mu,y} e^{\varepsilon \cdot \vec{D}_y} \\
&= e^{-\varepsilon \cdot \vec{D}_x} \vec{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} + e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) \vec{D}_{\mu,y} e^{\varepsilon \cdot \vec{D}_y} \\
&- \int_0^1 dt t e^{-\varepsilon \cdot \vec{D}_x} [\varepsilon \cdot \vec{D}_y, \vec{D}_{\mu,x}] e^{\varepsilon \cdot \vec{D}_x} e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} \\
&+ \int_0^1 dt t e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} e^{-\varepsilon \cdot \vec{D}_y} [\varepsilon \cdot \vec{D}_y, \vec{D}_{\mu,y}] e^{\varepsilon \cdot \vec{D}_y},
\end{align*}
\] (250)

where we have used integration by parts to get the last lines. Introducing the function

\[
H_\mu(x, \varepsilon) = -\int_0^1 dt t e^{-\varepsilon \cdot \vec{D}_x} [\varepsilon \cdot \vec{D}_y, \vec{D}_{\mu,x}] e^{\varepsilon \cdot \vec{D}_x} \\
= -\int_0^1 dt t e^{-\varepsilon \cdot \vec{D}_x} \varepsilon_\mu F_{\nu \mu}(x) e^{\varepsilon \cdot \vec{D}_x} \\
= \sum_{n=0}^{\infty \frac{(-1)^{n+1}}{n!(n+2)}} [\varepsilon \cdot D, [\varepsilon \cdot D, [... [\varepsilon \cdot D, \varepsilon_\mu F_{\nu \mu}(x)]]]].
\] (251)

we can rewrite it in the form

\[
\begin{align*}
\partial_\mu (e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y}) &= e^{-\varepsilon \cdot \vec{D}_x} \vec{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} + e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) \vec{D}_{\mu,y} e^{\varepsilon \cdot \vec{D}_y} \\
&+ H_\mu(x, \varepsilon) e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} \\
&+ e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} H_\mu(x, -\varepsilon).
\end{align*}
\] (252)

Analogously we have

\[
[D_\mu, G^{G^{\text{cov}}}(x, \varepsilon)] = \partial_\mu G^{G^{\text{cov}}}(x, \varepsilon) + A_\mu(x) G^{G^{\text{cov}}}(x, \varepsilon) - G^{G^{\text{cov}}}(x, \varepsilon) A_\mu(x) \\
= \partial_\mu e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} + e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} \partial_\mu \\
+ A_\mu(x) e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} - e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} A_\mu(y) |_{x=y} \\
= D_{x, \mu} e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} + e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} \partial_{y, \mu} |_{x=y} \\
= e^{-\varepsilon \cdot \vec{D}_x} \vec{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \cdot \vec{D}_y} + e^{-\varepsilon \cdot \vec{D}_x} G(x, y; m^2) \vec{D}_{\mu,y} e^{\varepsilon \cdot \vec{D}_y} |_{x=y} \\
+ [D_{\mu,x}, e^{-\varepsilon \cdot \vec{D}_x}] e^{-\varepsilon \cdot \vec{D}_x} G^{G^{\text{cov}}}(x, \varepsilon) + G^{G^{\text{cov}}}(x, \varepsilon) e^{-\varepsilon \cdot \vec{D}_x} [e^{-\varepsilon \cdot \vec{D}_x}, \vec{D}_{\mu,x}].
\] (253)
and introducing the function

\[ G_\mu(x, \varepsilon) = -[\tilde{D}_{\mu,x}, e^{-\varepsilon \tilde{D}_x}] e^{\varepsilon \tilde{D}_x} = \int_0^1 dt e^{-t\varepsilon \tilde{D}_x} F_{\mu\nu} \varepsilon \nu e^{t\varepsilon \tilde{D}_x} \]

we have

\[ [D_\mu, G_\text{cov}(x, \varepsilon)] = e^{-\varepsilon \tilde{D}_x} \tilde{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \tilde{D}_y} + e^{-\varepsilon \tilde{D}_x} G(x, y; m^2) \tilde{D}_{\mu,y} e^{\varepsilon \tilde{D}_y}|_{x=y} \]

As a result we have

\[ e^{-\varepsilon \tilde{D}_x} \tilde{D}_{\mu,x} G(x, y; m^2) e^{\varepsilon \tilde{D}_y}|_{x=y} = \frac{1}{2} \left( [D_\mu, G_\text{cov}(x, \varepsilon)] - \partial_\varepsilon G_\text{cov}(x, \varepsilon) \right) + G_\mu(x, \varepsilon) G_\text{cov}(x, \varepsilon) - G_\text{cov}(x, \varepsilon) G_\mu(x, -\varepsilon) + H_\mu(x, \varepsilon) G_\text{cov}(x, \varepsilon) + G_\text{cov}(x, \varepsilon) H_\mu(x, -\varepsilon); \]

this is the formula (60).

**Appendix B**

In this appendix we give some details of the calculation within the non-covariant point-splitting. Let us first derive the \( \varepsilon \)-expansion of the matrix functions \( \Omega_\pm \). We have

\[ \Omega_- = \tilde{\Omega}(x, x - \varepsilon) \Omega^+(x, x - \varepsilon) = e^{-\varepsilon \tilde{D}} e^{\varepsilon \tilde{D}}, \]

\[ \Omega_+ = \Omega^+(x + \varepsilon, x) \tilde{\Omega}(x + \varepsilon, x) = e^{-\varepsilon \tilde{D}} e^{\varepsilon \tilde{D}}. \]

where \( \tilde{D} = \partial + aA \) and \( \tilde{\partial} = \partial - aA \) are the non-covariant modification of the covariant derivatives. Using the Baker-Campbell-Hausdorff formula we get

\[ \Omega_- = \tilde{\Omega}(x, x - \varepsilon) \Omega^+(x, x - \varepsilon) = e^{-\varepsilon \tilde{D}} e^{\varepsilon \tilde{D}} \]

\[ = e^{-\varepsilon \tilde{D} + \varepsilon \tilde{D} - \frac{1}{2}[\varepsilon \tilde{D}, \varepsilon \tilde{D}] + \frac{1}{12}[\varepsilon \tilde{D}, \varepsilon \tilde{D}, \varepsilon \tilde{D}] + \mathcal{O}(\varepsilon^4)} \]

\[ = 1 + \varepsilon \cdot (D - \tilde{D}) + \frac{1}{2} \left( [\varepsilon \cdot \tilde{D}, (\varepsilon \cdot \tilde{D} - \varepsilon \cdot D)] + (\varepsilon \cdot (D - \tilde{D}))^2 \right) \]

\[ + \frac{1}{6} (\varepsilon \cdot (D - \tilde{D}))^3 - \frac{1}{4} [\varepsilon \cdot (D - \tilde{D}), [\varepsilon \cdot \tilde{D}, \varepsilon \cdot D]] + \frac{1}{12} [\varepsilon \cdot (D + \tilde{D}), [\varepsilon \cdot \tilde{D}, \varepsilon \cdot D]] \]

\[ + \mathcal{O}(\varepsilon^4). \]

Using now \( \varepsilon \cdot D - \varepsilon \cdot \tilde{D} = (1-a)\varepsilon \cdot A \), we rewrite this to the final form

\[ \Omega_- = 1 + (1-a)\varepsilon \cdot A + \frac{1}{2} (1-a)(1-a)((1-a)(\varepsilon \cdot A)^2 - (\varepsilon \cdot \partial \varepsilon \cdot A) \]

\[ + \frac{1}{6} (1-a)((\varepsilon \cdot 0)^2 + (1-a)^2 (\varepsilon \cdot A)^3 \]

\[ + (2a - 1)\varepsilon \cdot A \varepsilon \cdot \partial \varepsilon \cdot A + (a-2)(\varepsilon \cdot \partial \varepsilon \cdot A \varepsilon \cdot A) + \mathcal{O}(\varepsilon^4). \]

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If we realize, that \( \Omega_+ = e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}} = e^{\varepsilon \cdot \vec{D} e^{-\varepsilon \cdot \vec{D}}} = (e^{\varepsilon \cdot \vec{D} e^{-\varepsilon \cdot \vec{D}}})^+ \), we can immediately obtain \( \Omega_+ \) by means of Hermitian conjugation and replacing then \( \varepsilon \rightarrow -\varepsilon \), i.e.

\[
\Omega_+ = 1 + (1 - a)\varepsilon \cdot A + \frac{1}{2} (1 - a)((1 - a)(\varepsilon \cdot A)^2 + (\varepsilon \cdot \partial \varepsilon \cdot A) \\
+ \frac{1}{6} (1 - a)((\varepsilon \cdot \partial)^2 \varepsilon \cdot A + (1 - a)^2 (\varepsilon \cdot A)^3 \\
- (2a - 1)(\varepsilon \cdot \partial \varepsilon \cdot A) \varepsilon \cdot A - (a - 2)\varepsilon \cdot A \varepsilon \cdot \partial \varepsilon \cdot A + O(\varepsilon^4). \tag{260}
\]

It is now straightforward to get the formula

\[
\Omega_- \Omega_+ = 1 + 2(1 - a)\varepsilon \cdot A + 2(1 - a)^2 (\varepsilon \cdot A)^2 \\
+ \frac{4}{3}(1 - a)^3 (\varepsilon \cdot A)^3 + \frac{1}{3}(1 - a)(\varepsilon \cdot \partial)^2 \varepsilon \cdot A \\
+ \frac{1}{3}(1 - a)(2 - a)[\varepsilon \cdot A, \varepsilon \cdot \partial \varepsilon \cdot A] + O(\varepsilon^4), \tag{261}
\]

which was used in the main text. We need also the expansion of the trace \( \text{Tr}_C(\Omega_+ \Omega_- - 1) \) to the fourth order in \( \varepsilon \). We have, using the operator expression for \( \Omega_{\pm} \), for the fourth term of the \( \varepsilon \) expansion

\[
\text{Tr}_C(\Omega_+ \Omega_-)^{(4)} = \frac{1}{2} \text{Tr}_C(\Omega_+ \Omega_- + \Omega_- \Omega_+)^{(4)} \\
= \frac{1}{2} \text{Tr}_C(\frac{4}{3}(\varepsilon \cdot A)^2 + \frac{4}{3}(\varepsilon \cdot \partial)^2 - \frac{4}{3}(\varepsilon \cdot \partial D, (\varepsilon \cdot \partial)^2) - \frac{4}{3}(\varepsilon \cdot \partial, (\varepsilon \cdot D)^2) \\
- \frac{1}{3}(\varepsilon \cdot D, \{\varepsilon \cdot D, (\varepsilon \cdot \partial)^2\}) - \frac{1}{3}(\varepsilon \cdot \partial, \{\varepsilon \cdot \partial, (\varepsilon \cdot D)^2\}) \\
- \frac{1}{3}(\varepsilon \cdot D, \{\varepsilon \cdot D, \{\varepsilon \cdot D, (\varepsilon \cdot \partial)^2\})} \\
- \frac{1}{3}(\varepsilon \cdot \partial, \{\varepsilon \cdot \partial, \{\varepsilon \cdot D, (\varepsilon \cdot \partial)^2\})} \}. \tag{262}
\]

After some algebra we get

\[
\text{Tr}_C(\Omega_+ \Omega_-)^{(4)} = \frac{1}{6} \text{Tr}_C((4\varepsilon \cdot (D - \vec{D})) + \{\varepsilon \cdot (D - \vec{D})}, [\varepsilon \cdot (D + \vec{D}), [\varepsilon \cdot \vec{D}, \varepsilon \cdot (D - \vec{D})]\}] \\
= \frac{2}{3} \text{Tr}_C((1 - a)^4 (\varepsilon \cdot A)^2 + (1 - a)^2 \varepsilon \cdot A (\varepsilon \cdot \partial)^2 \varepsilon \cdot A). \tag{263}
\]

Let us now work out the expression \( \Omega_+[\vec{D}_\nu, \Omega_-] - [\Omega_+, \vec{D}_\nu] \Omega_- \). We have

\[
\Omega_+[\vec{D}_\nu, \Omega_-] - [\Omega_+, \vec{D}_\nu] \Omega_- = \Omega_+[\vec{D}_\nu, e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}] - [e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}, \vec{D}_\nu] \Omega_- \\
= \Omega_+[\vec{D}_\nu, e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}]e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}} + \Omega_+e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}[\vec{D}_\nu, e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}] - \\
e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}[\vec{D}_\nu, e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}]e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}[\vec{D}_\nu, e^{-\varepsilon \cdot \vec{D} e^\varepsilon \cdot \vec{D}}] \\
= -\Omega_+\vec{G}_\nu(x, \varepsilon) \Omega_- + \Omega_+\Omega_- \vec{G}_\nu(x, \varepsilon) \\
- \Omega_+\vec{G}_\nu(x, -\varepsilon) \Omega_- + \vec{G}_\nu(x, -\varepsilon) \Omega_+ \Omega_- \tag{264}
\]

where

\[
\vec{G}_\nu(x, \varepsilon) = [e^{-\varepsilon \cdot \vec{D}} \vec{D}_\nu]e^{\varepsilon \cdot \vec{D}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + 1)!}[\varepsilon \cdot \vec{D}, [\varepsilon \cdot \vec{D}, \ldots, [\varepsilon \cdot \vec{D}, \vec{D}_\nu] \ldots]] \tag{265}
\]
is a non-covariant modification of the function $G_{\nu}(x, \varepsilon)$ introduced above. For the calculation of the counterterm $\Delta \tilde{g}_{\mu \nu}$ we have derived

$$
\Delta \tilde{g}_{\mu \nu}^{\text{reg}} = \frac{1}{2} \text{Tr} S^{\text{cov}} \gamma_\mu (\Omega_+ \hat{D}_\nu, \Omega_-) - (\Omega_+ \hat{D}_\nu, \Omega_-) \Omega_+ \Omega_- \\
+ (\Omega_+ \Omega_- - 1) \Gamma_\nu (x, \varepsilon) + \Gamma_\nu (x, -\varepsilon) (\Omega_+ \Omega_- - 1) \\
+ \frac{1}{2} \text{Tr} (\mathcal{O} S^{\text{cov}} \gamma_\mu) (\Omega_+ \Omega_- - 1). \tag{266}
$$

Remembering the formula for the $\text{Tr}_D S^{\text{cov}} \gamma_\mu$ and noting that

$$
\Omega_+ \hat{D}_\nu, \Omega_- - (\Omega_+ \hat{D}_\nu, \Omega_-) + (\Omega_+ \Omega_- - 1) \Gamma_\nu (x, \varepsilon) + \Gamma_\nu (x, -\varepsilon) (\Omega_+ \Omega_- - 1) = \mathcal{O}(\varepsilon^2) \tag{267}
$$

it is easily seen that we need only the trace over the color indices of the expression expanded to the order $\mathcal{O}(\varepsilon^3)$. We have then up to the terms which do not contribute to the final expression after the symmetrization over the direction of $\varepsilon$

$$
\text{Tr}_C ((\Omega_+ \Omega_- - 1) (\Gamma_\nu (x, \varepsilon) + \Gamma_\nu (x, -\varepsilon)) + \Omega_+ \hat{D}_\nu, \Omega_-) = - \frac{2}{3} (1 - a) \text{Tr}_C ((2 - 3a) (\varepsilon \cdot A) [D_\nu, (\varepsilon \cdot \partial) (\varepsilon \cdot A)] + (\varepsilon \cdot A) (\varepsilon \cdot \partial)^2 A_\nu) + \mathcal{O}(\varepsilon^4). \tag{268}
$$

Putting all the ingredients together we get

$$
\Delta \tilde{g}_{\mu \nu}^{\text{reg}} = \frac{1 - a}{8 \pi^2} \text{Tr}_C \left( \frac{1}{36} ((2 - 3a) (A_\mu \partial_\nu \partial \cdot A + A_\alpha \partial_\mu \partial_\nu A_\alpha + A \cdot \partial \partial_\nu A_\mu \right. \\
+ A_\mu [A_\nu, \partial \cdot A] + A_\alpha [A_\nu, \partial_\mu A_\alpha] + A_\alpha [A_\nu, \partial_\alpha A_\mu]) \\
+ A_\mu \partial^2 A_\nu + 2 A \cdot \partial_\mu A_\nu) \\
- \frac{(1 - a)^3}{72} (\delta_{\mu \nu} (2 A^4 + A_\alpha A_\beta A_\alpha A_\beta) - 4 (\{ A_\mu, A_\nu \} A^2 + A_\mu A_\alpha A_\nu A_\alpha)) \\
- \frac{(1 - a)^3}{72} (\delta_{\mu \nu} (2 A \cdot \partial \cdot A + A_\alpha \partial^2 A_\alpha) - 2 (A_\mu \partial_\nu + A_\nu \partial_\mu) \partial \cdot A) \\
- 2 A \cdot \partial_\nu A_\mu + \partial_\mu A_\nu) - A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2 A_\alpha \partial_\mu \partial_\nu A_\alpha \\
+ \frac{1 - a}{6} \left( \delta_{\mu \nu} A^2 \left( 2 m^2 - \frac{1}{|\varepsilon|^2} \right) - 2 A_\mu A_\nu \left( m^2 - \frac{2}{|\varepsilon|^2} \right) \right) \\
- \frac{1}{9} (\delta_{\mu \nu} A_\alpha [D_\beta, F_{\beta \alpha}] + A_\mu [D_\alpha, F_{\alpha \mu}] - 4 A_\nu [D_\alpha, F_{\alpha \mu}] \\
- A_\alpha [D_\alpha, F_{\mu \nu}] - A_\alpha [D_\nu, F_{\mu \alpha}] \right). \tag{269}
$$

Using the following identities

$$
\text{Tr}_C (A_\mu [D_\alpha, F_{\alpha \nu}] + A_\nu [D_\alpha, F_{\alpha \mu}]) = \text{Tr}_C (A_\mu \partial^2 A_\nu + A_\nu \partial^2 A_\mu - (A_\mu \partial_\nu + A_\nu \partial_\mu) \partial \cdot A \\
+ 2 A_\alpha [\partial_\alpha A_\nu, A_\mu] + 2 A_\alpha [\partial_\alpha A_\mu, A_\nu] \\
- A_\alpha [\partial_\nu A_\alpha, A_\mu] - A_\alpha [\partial_\mu A_\alpha, A_\nu] \\
+ 2 \{ A_\mu, A_\nu \} A^2 - 4 A_\alpha A_\mu A_\alpha A_\nu) \tag{270}
$$
\[
\text{Tr}_C(A_\alpha[D_\nu, F_{\mu\alpha}] + A_\alpha[D_\mu, F_{\nu\alpha}]) = \text{Tr}_C(2A_\alpha \partial_\mu \partial_\nu A_\alpha - A \cdot \partial(\partial_\nu A_\mu + \partial_\mu A_\nu)
- 2A_\alpha [\partial_\nu A_\alpha, A_\mu] - 2A_\alpha [\partial_\mu A_\alpha, A_\nu] + A_\alpha [\partial_\alpha A_\nu, A_\mu] + A_\alpha [\partial_\alpha A_\mu, A_\nu] + 2[A_\mu, A_\nu] A^2 - 4A_\alpha A_\mu A_\alpha A_\nu)
\]

we get

\[
\Delta \tilde{\theta}^{\text{reg}}_{\mu\nu} = \frac{(1-a)^2}{48\pi^2} \text{Tr}_C \left( \delta_{\mu\nu} A^2 \left(2m^2 - \frac{1}{|\epsilon|^2}\right) - 2A_\mu A_\nu \left(m^2 - \frac{2}{|\epsilon|^2}\right) \right)
- \frac{1-a}{576\pi^2} \text{Tr}_C(8\delta_{\mu\nu} A_\alpha [D_\beta, F_{\beta\alpha}] - 11(A_\mu [D_\alpha, F_{\nu\alpha}] + A_\nu [D_\alpha, F_{\mu\alpha}])
- 5(A_\alpha [D_\nu, F_{\mu\alpha}] + A_\alpha [D_\mu, F_{\nu\alpha}])
+ (1-a)(\delta_{\mu\nu}(2A \cdot \partial \partial A + A_\alpha \partial^2 A_\alpha) + (A_\mu \partial_\nu + A_\nu \partial_\mu) \partial \cdot A
+ A \cdot \partial(\partial_\nu A_\mu + \partial_\mu A_\nu) + 4A_\alpha \partial_\mu \partial_\nu A_\alpha - A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu
+ 3([A_\alpha, A_\mu])(\partial_\nu A_\alpha + \partial_\alpha A_\nu) + [A_\alpha, A_\nu](\partial_\mu A_\alpha + \partial_\alpha A_\mu))
+ (1-a)^2(\delta_{\mu\nu}(2A^2 + A_\alpha A_\beta A_\alpha A_\beta) - 4([A_\mu, A_\nu] A^2 + A_\mu A_\alpha A_\alpha A_\nu))
- \frac{(1-a)}{192\pi^2} \text{Tr}_C((a + 6)(A_\mu [D_\nu, F_{\alpha\nu}] - A_\nu [D_\alpha, F_{\mu\alpha}]) + (a - 4)A \cdot \partial F_{\mu\nu}
+ (1-a)(A_\mu \partial^2 A_\nu - A_\nu \partial^2 A_\mu - 2A \cdot \partial[A_\mu, A_\nu]));
\]

this is the formula (178).

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