ON THE THEORY OF MATRIX-VALUED FUNCTIONS
BELONGING TO THE SMIRNOV CLASS

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A theory of matrix-valued functions from the matricial Smirnov class \( \mathcal{N}_n^+ (\mathbb{D}) \) is systematically developed. In particular, the maximum principle of V.I. Smirnov, inner-outer factorization, the Smirnov-Beurling characterization of outer functions and an analogue of Frostman’s theorem are presented for matrix-valued functions from the Smirnov class \( \mathcal{N}_n^+ (\mathbb{D}) \). We also consider a family \( F_\lambda = F - \lambda \mathbf{I} \) of functions belonging to the matricial Smirnov class which is indexed by a complex parameter \( \lambda \). We show that with the exception of a ”very small” set of such \( \lambda \) the corresponding inner factor in the inner-outer factorization of the function \( F_\lambda \) is a Blaschke-Potapov product.

The main goal of this paper is to provide users of analytic matrix-function theory with a standard source for references related to the matricial Smirnov class.

NOTATIONS :

\( \mathbb{C} \) - the complex plane.

\( \mathbb{T} := \{ t \in \mathbb{C} : |t| = 1 \} \) - the unit circle.

\( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) - the unit disc.

\( \mathfrak{B}_T \) - the \( \sigma \) - algebra of Borel subsets of \( \mathbb{T} \).

\( m \) - normalized Lebesgue measure on the measurable space \( (\mathbb{T}, \mathfrak{B}_T) \).

\( \mathbb{C}^n \) - the \( n \)-dimensional complex space equipped with the usual Euclidean norm, i.e.,

\[
\| x \|_{\mathbb{C}^n} := \left\{ \sum_{k=1}^{n} |\xi_k|^2 \right\}^{1/2}.
\]

\( \mathcal{M}_n \) - the set of all complex \( n \times n \) matrices equipped with the standard matrix norm, namely if \( M \in \mathcal{M}_n \) then \( \| M \| := \sup_{x \in \mathbb{C}^n \setminus \{0\}} \| Mx \|_{\mathbb{C}^n} / \| x \|_{\mathbb{C}^n} \).

\( \mathfrak{C}_n := \{ M \in \mathcal{M}_n : \| M \| \leq 1 \} \) - the subset of all contractive matrices in \( \mathcal{M}_n \).

\( I_n \) - the \( n \times n \) unit matrix.

As usual for \( r \in \mathbb{R} \) we set \( r^+ := \max\{r,0\} \) and \( r^- := \max\{-r,0\} \).

Hence, \( r = r^+ - r^- \) and \( |r| = r^+ + r^- \). In particular, if \( a \in (0, \infty) \), then

\[
\ln^+ a = \max\{\ln a, 0\} \quad \text{and} \quad \ln^- a = \max\left\{ \ln \frac{1}{a}, 0 \right\},
\]

\[
\ln a = \ln^+ a - \ln^- a \quad \text{and} \quad |\ln a| = \ln^+ a + \ln^- a.
\]

If \( A \in \mathbb{C}^{p \times q} \), then the symbol \( A^\top \) stands for the transposed matrix.
In this paper, we discuss various aspects of a class of matrix-valued functions which is named after V.I. Smirnov who introduced it for the scalar case in his famous paper [Sm]. It should be mentioned that the scalar Smirnov class also appeared in early papers of Doob (see e.g. [Doo1], [Doo2] and the bibliographies in the monographs Collingwood and Lohwater [CoLo] and Noshiro [No] which contain references to many other related works of Doob). For a collection of basic facts on the Smirnov class and the intimately related function spaces named after Nevanlinna and Hardy we refer the reader to the monographs of P.L. Duren [Dur], J.B. Garnett [G], K. Hoffman [Hoff], P. Koosis [Koo], I.I. Privalov [Pri] and M. Rosenblum and J. Rovnyak [RoRo2]. These books concentrate more or less on function-theoretic properties of functions belonging to some of the mentioned classes.

In the last two decades much progress has been made in clearing up topological and functional-analytic questions connected with the structure of the Smirnov class (see e.g. Yanagihara [Y1] - [Y10], Yanagihara and Kawase [YK], Yanagihara and Nakamura [YN], Stoll [St1], [St2], Roberts [Rob], Roberts and Stoll [RoSt1], [RoSt2], Mochizuki [Mo1], [Mo2], Helson [Hel2] - [Hel4], McCarthy [McC], Camera [Cam]).

A systematic study of the matricial Smirnov class was mainly promoted by the work of D.Z. Arov. In his paper [Ar1] on Darlington synthesis matricial generalization of V.I. Smirnov's important maximum principle was used in an essential way, namely with its aid a powerful criterion for proving the $J$-contractivity of a meromorphic matrix function was established. Moreover, D.Z. Arov's description of all Darlington representations of a given (pseudocontinuable) Schur function is based on the concept of denominators. A pair $[b_1, b_2]$ of inner matrix-valued functions of appropriate sizes is called a denominator of a given meromorphic matrix-valued function $f$ of bounded characteristic if $b_1 f b_2$ belongs to the matricial Smirnov class.

Nehari interpolation and generalized bitangential Schur - Nevanlinna - Pick interpolation are other important problems which turned out to be closely related with the matricial Smirnov class. This is an immediate consequence of D.Z. Arov's work [Ar3] - [Ar9] (see also Nicolau [Nic1], [Nic2]). In his investigations on the corresponding inverse problem D.Z. Arov introduced particular subclasses of $J$-inner functions which are now called the classes of Arov-regular and Arov-singular $J$-inner functions. Here a $J$-inner function $V$ is called Arov-singular if $V$ and $V^{-1}$ belong to the matricial Smirnov class. Furthermore, a $J$-inner function $W$ is called left Arov-regular (resp. right Arov-regular) if it does not contain any nonconstant Arov-singular left (resp. right) divisors. D.Z. Arov (see [Ar3] - [Ar7]) proved that each $J$-inner function $W$ admits (an essentially unique) factorizations

$$W = W_{l,r} \cdot W_{l,s} = W_{r,s} \cdot W_{r,r}$$

where the $J$-inner functions $W_{l,s}$ and $W_{r,s}$ are Arov-singular whereas the $J$-inner functions $W_{l,r}$ and $W_{r,r}$ are left Arov-regular and right Arov-regular, respectively. Furthermore, D.Z. Arov proved that a $J$-inner function is a left (resp. right) resolvent matrix of a completely indeterminate bitangential Schur - Nevanlinna - Pick interpolation problem if and only if it is left Arov-regular (resp. right Arov-regular). For several connections between left and right Arov-regularity we refer the reader to the papers [Kats1], [Kats2] where essential
connections between left and right Blaschke-Potapov products were established. In this way the first author (see [Kats3], [Kats4]) was led to a weighted approximation problem for pseudocontinuable functions belonging to the Smirnov class. The papers [Kats1]-[Kats3] laid the basis for the study of an inverse problem for Arov-singular $J$-inner functions which was considered in [AFK7]. The papers [Ar2], [AFK1]-[AFK6] deal with several completion problems for $J$-inner functions with particular emphasis on various subclasses of $J$-inner functions (Smirnov type, inverse Smirnov type, Arov-singular type). Using the concept of Arov-singularity and Arov-regularity of $J$-inner functions and the approximation method created in [Kats3], A. J. Kheifets [Kh] answered a question of D. Sarason [Sar1] (see also [Sar2]) on exposed points in the Hardy space $H^1(\mathbb{D})$. Prediction theory for multivariate stationary sequences formed an important source for the development of the theory of matrix-valued holomorphic functions (see Wiener and Masani [WM1], [WM2], Helson and Lowdenslager [HL1], [HL2], Rozanov [Roz1], [Roz2], Masani [Ma1]-[Ma4]). In particular, the matricial Hardy class $H_n^2(\mathbb{D})$ (see Definition 5.1 below) became an essential tool. It turned out that the basic problems of prediction theory could be reformulated as analytic problems for appropriate functions belonging to the Hardy class $H_n^2(\mathbb{D})$. Using functional-analytic methods, Beurling’s inner-outer factorization was generalized to $H_n^2(\mathbb{D})$ (see Masani [Ma2], Rozanov [Roz1]). Moreover classical results due to Szegö [Sz1]-[Sz3], Kolmogorov [Kol] and Krein [Kr] were extended to the multivariate case. Here, it turned out (see Devinatz [De]) that the matrix version of Szegö’s factorization theorem and other results due to Wiener and Masani [WM1], [WM2] and Helson and Lowdenslager [HL1], [HL2] are not so much generalizations of Szegö’s classical results as consequences of it. An algebraic treatment of this theory was given by Helson [Hel1].

Carrying on from the theory of matrix-valued functions belonging to the Hardy class $H_n^2(\mathbb{D})$, we will study various aspects of outer functions from the matricial Smirnov class in this paper. In particular, we will extend the theory of inner-outer factorization to the matricial Smirnov class. A central topic in our investigations is to describe the situation where the inner factor in the inner-outer factorization of a matrix-valued Smirnov class function is a Blaschke-Potapov product. Moreover, we will consider a family of functions belonging to the matricial Smirnov class which is indexed by a complex parameter $\lambda$. Then it will be shown that with exception of a ”very small” set of such parameters $\lambda$ the corresponding inner factor in the inner-outer factorization of the function $F_\lambda$ is a Blaschke-Potapov product. Our methods to prove this use a matrix generalization of logarithmic potentials. In this way, we obtain a generalization of a classical theorem of Frostman [Fr] (see also Heins [Hei] and Rudin [Ru1], [Ru2]). It should be mentioned that it was Yu. P. Ginzburg who was a pioneer in matrix (and in operator) generalizations of Frostman’s results (see [Gi6] and [GiTa1]-[GiTa3]).

1. ON THE MATRICIAL NEVANLINNA AND SMIRNOV CLASSES

For $F : \mathbb{D} \to \mathcal{M}_n$ and $r \in [0, 1)$, we define the function $F_{[r]} : T \to \mathcal{M}_n$ via $t \to F(rt)$.

**Definition 1.1.** A matrix-valued function $F : \mathbb{D} \to \mathcal{M}_n$ is said to belong to the matricial Nevanlinna class $\mathcal{N}_n(\mathbb{D})$ if $F$ is holomorphic in $\mathbb{D}$ and if the family $(\ln^+ \|F_{[r]}\|)_{r \in [0, 1)}$
is bounded in $L^1(m)$, or more precisely, if
\[
\sup_{r \in [0,1]} \int_T \ln^+ \|F_r(t)\| \, m(dt) < +\infty. \tag{1.1}
\]

**Remark 1.1.** Let $F : \mathbb{D} \to \mathcal{M}_n$ be a matrix-valued function which is holomorphic in $\mathbb{D}$. Then $F$ belongs to $\mathcal{N}_n(\mathbb{D})$ if and only if the (subharmonic) function $\ln^+ \|F\|$ has a harmonic majorant in $\mathbb{D}$.

The definition of the Smirnov class $\mathcal{N}_n(\mathbb{D})$ and of its matricial analogue $\mathcal{N}_n^+(\mathbb{D})$ are connected with the notion of uniform integrability. Since this notion is not used very often we give the definition.

**Definition of Uniform Integrability.** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then the family $(f_\alpha)_{\alpha \in \mathcal{A}}$ belonging to $L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})$ is called uniformly integrable with respect to $\mu$ if the following conditions are satisfied:

(i) \[
\sup_{\alpha \in \mathcal{A}} \int_{\Omega} |f_\alpha(t)| \, \mu(dt) < +\infty.
\]

(ii) For every $\epsilon \in (0, \infty)$ there exists a $\delta \in (0, \infty)$ (which depends only on $\epsilon$) such that for all $\alpha \in \mathcal{A}$ and for all $\Delta \in \mathcal{A}$, with $\mu(\Delta) < \delta$, the inequality
\[
\int_{\Delta} |f_\alpha(t)| \, \mu(dt) < \epsilon
\]

is fulfilled.

**Remark 1.2.** If $\mu(\Omega) < +\infty$ and if for each fixed $\delta \in (0, \infty)$ there exist an $N(\delta) \in \mathbb{N}$ and a sequence $(X_{k,\delta})_{k=1}^{N(\delta)}$ from $\Delta$ such that $\Omega = \bigcup_{k=1}^{N(\delta)} \Omega_{k,\delta}$ and $\mu(\Omega_{k,\delta}) \leq \delta$ for all $k \in \{1, 2, \ldots, N(\delta)\}$, then a family of functions for which condition (ii) in the preceding definition is fulfilled, automatically satisfies condition (i). Consequently, in the case of a finite measure space $(\Omega, \mathcal{A}, \mu)$ condition (i) can be omitted in the definition of uniform integrability. A special case of such a measure space is the Lebesgue space on $\mathbb{T}$, where $\mathcal{A}$ is the $\sigma$-algebra of Borel subsets of $\mathbb{T}$ and $m$ is the normalized Lebesgue measure on $\mathbb{T}$.

In the sequel we will repeatedly use the following theorem from measure theory which goes back to G. Vitali [Vit] (see also [Ru3, p.133, Exercise 10]).

**Vitali's Convergence Theorem.** Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space (i.e., $\mu(\Omega) < \infty$). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence from $L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})$ which is uniformly integrable with respect to $\mu$ and converges $\mu$-a.e. to a Borel measurable function $f : \Omega \to \mathbb{C}$.

Then $f \in L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})$,

\[
\lim_{n \to \infty} \int_{\Omega} |f_n - f| \, d\mu = 0
\]

and

\[
\lim_{n \to \infty} \int_{\Omega} f_n \, d\mu = \int_{\Omega} f \, d\mu.
\]
PROOF. Let \( \epsilon \in (0, \infty) \). In view of the uniform \( \mu \)-integrability of \((f_n)_{n \in \mathbb{N}}\) there exists a number \( \delta \in (0, \infty) \) such that for all \( n \in \mathbb{N} \) and for all \( \Delta \in \mathfrak{A} \), which satisfy \( \mu(\Delta) \leq \delta \), the inequality

\[
\int_{\Delta} |f_n| \, d\mu < \frac{\epsilon}{3} \tag{1.2}
\]

is satisfied. Since \( \mu(\Omega) < \infty \), Egorov’s Theorem guarantees the existence of a set \( B_\delta \in \mathfrak{A} \) such that

\[
\mu(B_\delta) < \delta \tag{1.3}
\]

and

\[
\lim_{n \to \infty} \sup_{w \in \Omega \setminus B_\delta} |f_n(w) - f(w)| = 0. \tag{1.4}
\]

Thus, there exists an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and all \( \omega \in \Omega \setminus B_\delta \) the inequality

\[
|f_n(\omega) - f(\omega)| < \frac{\epsilon}{3[1 + \mu(\Omega)]} \tag{1.5}
\]

is satisfied. In view of (1.2) and (1.3) for \( n \in \mathbb{N} \) we have

\[
\int_{B_\delta} |f_n| \, d\mu < \frac{\epsilon}{3}. \tag{1.6}
\]

From Fatou’s Theorem and (1.6) we obtain

\[
\int |f| \, d\mu \leq \lim_{n \to \infty} \int_{B_\delta} |f_n| \, d\mu \leq \frac{\epsilon}{3}. \tag{1.7}
\]

Combining (1.5) - (1.7) we obtain the estimate

\[
\int_{\Omega} |f_n - f| \, d\mu = \int_{\Omega \setminus B_\delta} |f_n - f| \, d\mu + \int_{B_\delta} |f_n - f| \, d\mu \leq \frac{\epsilon}{3[1 + \mu(\Omega)]} \mu(\Omega \setminus B_\delta) + \int_{B_\delta} |f| \, d\mu + \int_{B_\delta} |f_n| \, d\mu \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\]

for \( n \geq n_0 \). Thus,

\[
\lim_{n \to \infty} \int_{\Omega} |f_n - f| \, d\mu = 0.
\]

From this, all the remaining assertions follow immediately.

DEFINITION 1.2. A function \( \varphi : \mathbb{R} \to \mathbb{R} \) is called strongly convex if it has the following properties:
(i) $\varphi$ is convex.
(ii) $\varphi$ is monotonically nondecreasing.
(iii) $\varphi$ takes its values in $[0, \infty)$.
(iv) $\lim_{x \to +\infty} \frac{\varphi(x)}{x} = +\infty$.
(v) For some $c \in (0, \infty)$ there exist constants $M \in [0, \infty)$ and $a \in \mathbb{R}$ such that $\varphi(t + c) \leq M \cdot \varphi(t)$ for all $t \in [a, \infty)$.

If (v) holds for just one value of $c \in (0, \infty)$, say $c = c_0$, then by (ii) it holds for all $c \in (0, c_0)$. By iteration it holds for $c = nc_0$, $n \in \mathbb{N}$ and hence it holds for all $c \in (0, \infty)$.

**THEOREM 1.1.** (de la Vallée Poussin [LVP1], Nagumo [Na].) Let $(\Omega, \mathcal{A}, \mu)$ be a (finite or infinite) measure space, and let $(f_\alpha)_{\alpha \in A}$ be a family of functions belonging to $L^1(\Omega, \mathcal{A}, \mu; \mathbb{C})$. In case $\mu(\Omega) = +\infty$, we assume also that

$$
\sup_{\alpha \in A} \int_{\Omega} |f_\alpha| \, d\mu < \infty.
$$

(i) Suppose that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying

$$
\lim_{x \to +\infty} \frac{\varphi(x)}{x} = +\infty
$$

and

$$
\sup_{\alpha \in A} \int_{\Omega} \varphi(|f_\alpha|) \, d\mu < +\infty.
$$

Then the family $(f_\alpha)_{\alpha \in A}$ is uniformly integrable with respect to $\mu$.

(ii) Suppose that the family $(f_\alpha)_{\alpha \in A}$ is uniformly integrable with respect to $\mu$. Then there exists a strongly convex function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$
\sup_{\alpha \in A} \int_{\Omega} \varphi(|f_\alpha|) \, d\mu < +\infty.
$$

For a modern proof of Theorem 1.1 we refer to [RoRo2, Theorem 3.10] (see also Theorem 3.1.2 in [Ru2]). This modern proof based on Vitali’s Convergence Theorem.

**DEFINITION 1.3.** A matrix-valued function $F : \mathbb{D} \to \mathcal{M}_n$ is said to belong to the matricial Smirnov class $\mathcal{N}_n^+(\mathbb{D})$ if $F$ is holomorphic in $\mathbb{D}$ and if the family $(\ln^+ \|F_r\|)_{r \in [0,1)}$ is uniformly integrable with respect to the normalized Lebesgue measure $m$, i.e., if for each $\epsilon \in (0, \infty)$ there exists a $\delta \in (0, \infty)$ (which depends only on $\epsilon$) such that for all $r \in [0,1)$ and for all Borel subsets $\Delta$ of $\mathbb{T}$ satisfying $m(\Delta) < \delta$ the inequality

$$
\int_{\Delta} \ln^+ \|F_r(t)\| \, m(dt) < \epsilon
$$

(1.8)
is fulfilled.

REMARK 1.3. In view of Remark 1.2, each matrix-valued function $F \in \mathfrak{N}_n^+(D)$ satisfies condition (1.1). Hence, the matricial Smirnov class $\mathfrak{N}_n^+(D)$ is a subclass of the matricial Nevanlinna class $\mathfrak{N}_n(D)$:

$$\mathfrak{N}_n^+(D) \subseteq \mathfrak{N}_n(D).$$

(1.9)

For a matrix-valued function $F$ belonging to $\mathfrak{N}_n(D)$ we denote by $\overline{F} : \mathbb{T} \to \mathfrak{M}_n$ a boundary limit function associated with $F$, i.e., $\overline{F}$ is a Borel measurable function and there exists a Borel subset $\Delta_0$ of $\mathbb{T}$ satisfying $m(\Delta_0) = 0$ such that for all $t \in \mathbb{T} \setminus \Delta_0$ we have

$$\lim_{r \to 1^-} F(rt) = \overline{F}(t).$$

Observe that in view of Vitali’s theorem a function $F \in \mathfrak{N}_n^+(D)$ satisfies

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ \|F(rt)\| \ m(dt) = \int_{\mathbb{T}} \ln^+ \|\overline{F(t)}\| \ m(dt).$$

(1.10)

According to Fatou’s theorem,

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \ln^- \|F(rt)\| \ m(dt) \geq \int_{\mathbb{T}} \ln^- \|\overline{F(t)}\| \ m(dt).$$

(1.11)

(where equality does not hold in general). Hence,

$$\overline{\lim}_{r \to 1^-} \int_{\mathbb{T}} \ln \|F(rt)\| \ m(dt) \leq \int_{\mathbb{T}} \ln \|\overline{F(t)}\| \ m(dt).$$

(1.12)

LEMMA 1.1. A matrix-valued function $F : \mathbb{D} \to \mathfrak{M}_n$ belongs to the matricial class $\mathfrak{N}_n(D)$ (resp. $\mathfrak{N}_n^+(D)$) if and only if each of its entries belongs to the scalar class $\mathfrak{N}(D)$ (resp. $\mathfrak{N}^+(D)$).

As the determinant of a matrix is a polynomial of its elements and because each of the classes $\mathfrak{N}(D)$ and $\mathfrak{N}^+(D)$ is an algebra over $\mathbb{C}$ the following result holds true.

LEMMA 1.2. (i) If $F \in \mathfrak{N}_n(D)$, then $\det F \in \mathfrak{N}(D)$.

(ii) If $F \in \mathfrak{N}_n^+(D)$, then $\det F \in \mathfrak{N}^+(D)$.

As a special case of (1.10), (1.11) and (1.12) (corresponding to the scalar case) we obtain for a function $F \in \mathfrak{N}_n^+(D)$ from part (ii) of Lemma 1.2 that

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ |\det[F(rt)]| \ m(dt) = \int_{\mathbb{T}} \ln^+ |\det[\overline{F(t)}]| \ m(dt),$$

(1.13)

$$\lim_{r \to 1^-} \int_{\mathbb{T}} \ln^- |\det[F(rt)]| \ m(dt) \geq \int_{\mathbb{T}} \ln^- |\det[\overline{F(t)}]| \ m(dt),$$

(1.14)
and, finally, that
\[
\lim_{r \to 1^{-}} \int_{\mathbb{T}} \ln |\det[F(rt)]| \ m(dt) \leq \int_{\mathbb{T}} \ln |\det[F(t)]| \ m(dt). \tag{1.15}
\]

In the following we will use the Poisson kernel \(P : \mathbb{D} \times \mathbb{T} \to (0, \infty)\) which is defined by the formula
\[
P(z, t) := \frac{1 - |z|^2}{|t - z|^2}.
\]

**THEOREM 1.2.** Let \(F \in \mathcal{H}_n(\mathbb{D})\) with \(F \not\equiv 0\) and let \(u_F\) denote the least harmonic majorant of \(\log \|F\|\). Then the following statements are equivalent:

(i) \(F \in \mathcal{H}_n^+(\mathbb{D})\).

(ii) \(u_F(z) \leq \int_{\mathbb{T}} \ln \|F(t)\| \frac{1 - |z|^2}{|t - z|^2} m(dt)\) for every \(z \in \mathbb{D}\).

(iii) \(\ln \|F(z)\| \leq \int_{\mathbb{T}} \ln \|F(t)\| \frac{1 - |z|^2}{|t - z|^2} m(dt)\) for every \(z \in \mathbb{D}\).

(iv) There exist a strongly convex function \(\varphi : \mathbb{R} \to \mathbb{R}\) and a number \(r_0 \in (0, 1)\) such that
\[
\sup_{r \in [r_0, 1]} \int_{\mathbb{T}} \varphi \left( \ln \|F_r(t)\| \right) m(dt) < +\infty.
\]

(v) There exists a strongly convex function \(\psi : \mathbb{R} \to \mathbb{R}\) such that
\[
\sup_{r \in [0, 1]} \int_{\mathbb{T}} \psi \left( \ln^+ \|F_r(t)\| \right) m(dt) < +\infty.
\]

**PROOF.** Theorem 1.2 can be proved by a slight modification of the proof of Theorem 3.3.5 in [Ru2]. Here, Theorem 1.1 plays an essential role.

For further results on matrix-valued functions belonging to one of the classes named after Nevanlinna, Smirnov and Hardy we refer the reader to chapter 4 in [RoRo1].

**2. MATRIX FUNCTIONS OF THE SMIRNOV CLASS AS MULTIPLES OF CONTRACTIVE MATRIX FUNCTIONS**

Recall that a scalar function \(e : \mathbb{D} \to \mathbb{C}\) is said to be outer (in the sense of V.I. Smirnov) if there exist a unimodular constant \(C \in \mathbb{T}\) and a function \(w : \mathbb{T} \to [0, \infty)\) for which \(\log w\) is \(m\)-integrable such that for \(z \in \mathbb{D}\) the relation
\[
e(z) = C \cdot \exp \left\{ \int_{\mathbb{T}} \frac{t + z}{t - z} \ln [w(t)] \ m(dt) \right\} \tag{2.1}
\]
holds true. Let $\mathcal{E}(\mathbb{D})$ denote the class of all outer functions. From its definition it is obvious, that the class $\mathcal{E}(\mathbb{D})$ is multiplicative: If $e_1, e_2 \in \mathcal{E}(\mathbb{D})$, then $e_1 \cdot e_2 \in \mathcal{E}(\mathbb{D})$.

The following statement is well-known (see e.g. Theorem 4.29 in [RoRo2]).

**LEMMA 2.1.** Let $e : \mathbb{D} \to \mathbb{C}$ be some function. Then the following statements are equivalent:

- (i) $e \in \mathcal{E}(\mathbb{D})$.
- (ii) $e \in \mathcal{N}^+(\mathbb{D})$, $e \not\equiv 0$ and $e^{-1} \in \mathcal{N}^+(\mathbb{D})$.

In particular, a function $e$ of type (2.1) belongs to the class $\mathcal{N}(\mathbb{D})$ and, consequently, it possesses a boundary function $e : \mathbb{T} \to \mathbb{C}$. It is known that for almost all $t \in \mathbb{T}$ with respect to $m$,

$$|e(t)| = w(t). \quad (2.2)$$

As the function $\ln |e|$ is harmonic in $\mathbb{D}$ we obtain

$$\int_{\mathbb{T}} \ln |e(rt)| \ m(dt) = \ln |e(0)| = \int_{\mathbb{T}} \ln |w(t)| \ m(dt).$$

for $r \in [0, 1)$. Consequently, if $e \in \mathcal{E}(\mathbb{D})$, then for $r \in [0, 1)$ we obtain

$$\int_{\mathbb{T}} \ln |e(rt)| \ m(dt) = \int_{\mathbb{T}} \ln |e(t)| \ m(dt). \quad (2.3)$$

Let us recall the following useful characterization of outer functions (see e.g. Corollaries 4.16 and 4.17 in [RoRo2]).

**LEMMA 2.2.** Let $e \in \mathcal{N}(\mathbb{D})$ but $e \not\equiv 0$. Then the following statements are equivalent:

- (i) $e$ is outer.
- (ii) For all $z \in \mathbb{D}$, $\ln |e(z)| = \int_{\mathbb{T}} \Re \frac{t + z}{t - z} \ln |e(t)| \ m(dt)$.
- (iii) There is a $z_0 \in \mathbb{D}$ such that $\ln |e(z_0)| = \int_{\mathbb{T}} \Re \frac{t + z_0}{t - z_0} \ln |e(t)| \ m(dt)$.
- (iv) If $h \in \mathcal{N}^+(\mathbb{D})$ satisfies $|h(t)| \leq |e(t)|$ for almost all $t \in \mathbb{T}$ with respect to $m$, then for all $z \in \mathbb{D}$ the inequality $|h(z)| \leq |e(z)|$ holds true.
- (v) If $z_0 \in \mathbb{D}$ and if $h \in \mathcal{N}^+(\mathbb{D})$ satisfies $|h(t)| \leq |e(t)|$ for almost all $t \in \mathbb{T}$ with respect to $m$, then the inequality $|h(z_0)| \leq |e(z_0)|$ holds true.

Observe that conditions (iii) and (v) of Lemma 2.2 are usually used with the choice $z_0 = 0$. 

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In the proof of Lemma 2.4 and also in further considerations we will use the following result which goes back to V.I. Smirnov [Sm].

THE MAXIMUM PRINCIPLE OF V.I. SMIRNOV. Let $f \in \mathcal{N}^+(\mathbb{D})$ be such that its boundary function $f$ is $m$-essentially bounded. Then $f$ is bounded in the unit disc and satisfies

$$\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{t \in \mathbb{T}} |f(t)|$$

This result can be generalized to the matrix case.

THE MAXIMUM PRINCIPLE OF V.I. SMIRNOV FOR MATRIX FUNCTIONS. Let $F \in \mathcal{N}^{+}(\mathbb{D})$ be such that its boundary function $F$ satisfies $\text{ess sup}_{t \in \mathbb{T}} \|F(t)\| < \infty$. Then $F$ is bounded in the unit disc and satisfies

$$\sup_{z \in \mathbb{D}} \|F(z)\| = \text{ess sup}_{t \in \mathbb{T}} \|F(t)\|.$$ 

PROOF. Let $F = (F_{j,k})_{j,k=1}^{n}$, and fix the indices $j,k \in \{1, \ldots, n\}$. In view of the inequality $|F_{j,k}(z)| \leq \|F(z)\| \ (z \in \mathbb{D})$, then $F_{j,k} \in \mathcal{N}^+(\mathbb{D})$ and

$$\text{ess sup}_{t \in \mathbb{T}} \|F_{j,k}(t)\| \leq \text{ess sup}_{t \in \mathbb{T}} \|F(t)\|.$$ 

According to the maximum principle for scalar functions we then have

$$\sup_{z \in \mathbb{D}} |F_{j,k}(z)| < +\infty.$$ 

Hence,

$$\sup_{z \in \mathbb{D}} \|F(z)\| < +\infty.$$ 

The bounded holomorphic matrix-valued function $F$ admits the Poisson integral representation

$$F(z) = \int_{\mathbb{T}} F(t) \cdot P(z, t) \ m(dt), \ z \in \mathbb{D}.$$ 

Therefore, by the integral version of the triangle inequality, we obtain

$$\|F(z)\| \leq \int_{\mathbb{T}} \|F(t)\| \cdot P(z, t) \ m(dt), \ z \in \mathbb{D}.$$ 

But this in turn implies the inequality $\|F(z)\| \leq \text{ess sup}_{t \in \mathbb{T}} \|F(t)\| \ (z \in \mathbb{D})$. 

Since the function $\ln^+ \|F\|$ is subharmonic for an analytic matrix-valued function $F$ the following result is true.

LEMMA 2.3. Let $F \in \mathcal{N}^+_n(\mathbb{D})$. Then for all $z \in \mathbb{D}$ the inequality

$$\|F(z)\| \leq \exp \left\{ \int_{\mathbb{T}} P(z, t) \ln \|F(t)\| \ m(dt) \right\} \quad (2.4)$$
holds true.

For a proof of Lemma 2.3 we refer to Theorem 3.13 in [RoRo2].

Clearly, the maximum principle of V.I. Smirnov is a consequence of inequality (2.4).

DEFINITION 2.1. The set $S_{n\times n}(D)$ of all holomorphic matrix-valued functions $S : D \to \mathbb{C}_n$ is called the $n \times n$ Schur class.

LEMMA 2.4. A matrix-valued function $F : D \to \mathcal{M}_n$ belongs to the Smirnov class $\mathcal{N}_n^+(D)$ if and only if it admits a representation of the form

$$F = \frac{1}{d} \cdot \Phi,$$

where $\Phi \in S_{n\times n}(D)$ and $d$ is an outer function which belongs to $S(D)$.

PROOF. I. Suppose that $F$ admits a representation of the form (2.5). Then $\Phi \in \mathcal{N}_n^+(D)$ and, as $d$ is outer, we have $d^{-1} \in \mathcal{N}^+(D)$. Thus, as $\mathcal{N}^+(D)$ is an algebra over $\mathbb{C}$, we get $\Phi \cdot d^{-1} \in \mathcal{N}_n^+(D)$.

II. Suppose that $F \in \mathcal{N}_n^+(D)$. We can assume that $F$ is not the null function in $D$. Then $\ln \|F\|$ is $m$-integrable. We define $d : D \to \mathbb{C}$ via

$$d(z) := \exp \left\{ - \int_T \frac{t + z}{t - z} \ln \|F\| \ m(dt) \right\}.$$  \(\text{(2.6)}\)

Then, from our earlier considerations (see (2.1) - (2.4)), it is clear that $d$ is a scalar outer function and that the corresponding boundary function $\underline{d}$ satisfies

$$|\underline{d}(t)| = \|F(t)\|^{-1} \quad \text{(2.6)}$$

for almost all $t \in \mathbb{T}$ with respect to $m$. Now define $\Phi : D \to \mathcal{M}_n$ via

$$\Phi(z) := d(z) \cdot F(z). \quad \text{(2.7)}$$

Then, since $F \in \mathcal{N}_n^+(D), d \in \mathcal{N}^+(D)$ and $\mathcal{N}^+(D)$ is an algebra over $\mathbb{C}$, we see that

$$\Phi \in \mathcal{N}_n^+(D). \quad \text{(2.8)}$$

From (2.6) and (2.7) we get

$$\|\Phi(t)\| = 1 \quad \text{(2.9)}$$

for almost all $t \in \mathbb{T}$ with respect to $m$. Finally, in view of (2.8) and (2.9), the maximum principle of V.I. Smirnov implies that for $z \in D$ we obtain $\|\Phi(z)\| \leq 1$. Thus, $\Phi \in S_{n\times n}(D)$.

3. OUTER MATRIX-VALUED FUNCTIONS

The main goal of this section is to discuss outer matrix-valued functions which belong to the Smirnov class $\mathcal{N}_n^+(D)$. The needs of prediction theory of multivariate stationary
stochastic processes initiated an intensive study of matrix-valued outer functions belonging to the Hardy class $H^2_n(D)$ (see Definition 5.1 below) which is a subclass of $\mathcal{M}^+_n(D)$. The formula for the best predictor of a multivariate stationary stochastic process of a given time in terms of its past depends in an essential manner on a particular outer matrix-valued function belonging to $H^2_n(D)$ (see Wiener and Masani [WM1], [WM2], Helson and Lowdenslager [HL1], [HL2], Rozanov [Roz1], [Roz2], Masani [Ma1] - [Ma4] and for operator-valued generalizations also Devinatz [De], Helson [Hel1], Sz.-Nagy and Foias [SZNF], Nikolskii [Nik2]).

DEFINITION 3.1. A matrix-valued function $E : D \rightarrow \mathbb{M}_n$ is called outer (in the sense of V.I. Smirnov) if $E \in \mathcal{M}^+_n(D)$ and det $E$ is outer. The class of all $n \times n$ matrix-valued outer functions will be denoted by $E_n(D)$.

If $E \in \mathcal{E}_n(D)$ then, in particular, for all $z \in D$ we have

$$\det [E(z)] \neq 0.$$  

Definition 3.1 is clearly an immediate generalization of the notion of a scalar outer function. This definition of an outer matrix-valued function enables us to avoid the study of the question of a matricial analogue of formula (2.1).

REMARK 3.1. The class $\mathcal{E}_n(D)$ is multiplicative: If $E_1, E_2 \in \mathcal{E}_n(D)$ then $E_1 \cdot E_2 \in \mathcal{E}_n()$.

REMARK 3.2. Let $E \in \mathcal{E}_n(D)$. Then $E^\top \in \mathcal{E}_n(D)$.

THEOREM 3.1. (Determinant characterization of outer matrix-valued functions)

(i) Let $E \in \mathcal{E}_n(D)$. Then $\det [E(z)] \neq 0$ for all $z \in D$ and $E^{-1} \in \mathcal{M}^+_n(D)$.

(ii) Let $E$ be a function from $\mathcal{M}^+_n(D)$ for which $\det E$ never vanishes in $D$ and $E^{-1}$ belongs to $\mathcal{M}^+_n(D)$. Then $E \in \mathcal{E}_n(D)$.

PROOF. (i) According to the rule for computing the inverse matrix we have the representation

$$E^{-1} = \frac{1}{\det E} \cdot A$$  

where $A : D \rightarrow \mathbb{M}_n$ is a matrix-valued function the entries of which are polynomials of the elements of matrix $E$ (namely, the cofactors of the corresponding elements). Since the class $\mathcal{M}^+_n(D)$ is an algebra over $\mathbb{C}$, each entry of $A$ belongs to $\mathcal{M}^+_n(D)$. Hence, $A \in \mathcal{M}^+_n(D)$. From the fact that $E \in \mathcal{E}_n(D)$ and Lemma 2.1 it then follows that $(\det E)^{-1} \in \mathcal{M}^+_n(D)$, and thus in view of (3.1), $E^{-1} \in \mathcal{M}^+_n(D)$. Hence, (i) is proved.

(ii) By Lemma 1.2, $\det E \in \mathcal{M}^+_n(D)$ and $\det (E^{-1}) \in \mathcal{M}^+_n(D)$. Therefore, the function $\det E$ satisfies condition (ii) in Lemma 2.1. Thus, $\det E \in \mathcal{E}(D)$, and so, in view of Definition 3.1, $E \in \mathcal{E}_n(D)$. Hence (ii) is proved.

The following result supplements the statement of Lemma 2.4.

LEMMA 3.1. Let $E \in \mathcal{E}_n(D)$. Then $E$ has a representation of the form

$$E = \frac{1}{d} \cdot C,$$  

where $C \in \mathcal{S}_{n \times n}(D) \cap \mathcal{E}_n(D)$ and $d \in \mathcal{E}(D)$. 

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PROOF. In view of Lemma 2.4, the function $E$ has a representation of the form

$$E = \frac{1}{d} \cdot C,$$

where $C \in \mathcal{S}_{n \times n}(\mathbb{D})$ and $d \in \mathcal{C}(\mathbb{D})$. Lemma 2.1 guarantees that $d^{-1} \in \mathcal{H}^+(\mathbb{D})$. Since $E \in \mathcal{E}_n(\mathbb{D})$, it follows from Theorem 3.1 that $E^{-1} \in \mathcal{H}_n^+(\mathbb{D})$. Therefore, as $\mathcal{H}^+ (\mathbb{D})$ is an algebra over $\mathbb{C}$, from $C^{-1} = d^{-1} E^{-1}$ we see that $C^{-1} \in \mathcal{H}_n^+(\mathbb{D})$. Thus, as $C \in \mathcal{S}_{n \times n}(\mathbb{D}) \subseteq \mathcal{H}_n^+(\mathbb{D})$ it follows from Theorem 3.1 that $C \in \mathcal{E}_n(\mathbb{D})$.

Let us recall the following notion.

DEFINITION 3.2. The class $H_n^\infty(\mathbb{D})$ consists of all matrix-valued functions $F : \mathbb{D} \to \mathcal{M}_n$ which are holomorphic and bounded in $\mathbb{D}$, i.e.,

$$\sup_{z \in \mathbb{D}} \|F(z)\| < \infty. \quad (3.3)$$

THEOREM 3.2. (i) Let $E \in \mathcal{E}_n(\mathbb{D})$. Then there exists a sequence $(F_k)_{k \in \mathbb{N}}$ from $H_n^\infty(\mathbb{D})$ with the following properties:

$(\alpha)$ For almost all $t \in \mathbb{T}$ with respect to $m$, $\lim_{k \to \infty} F_k(t) = I_n$.

$(\beta)$ The family $\{\ln \|F_k\|\}_{k \in \mathbb{N}}$ is uniformly integrable with respect to $m$.

$(\gamma)$ There exists a Borel subset $B_0$ of $\mathbb{T}$ with $m(B_0) = 0$ such that for all $k \in \mathbb{N}$ and all $t \in \mathbb{T} \setminus B_0$ the inequality $\|F_k(t)\| < 1$ holds true.

(ii) Let $E \in \mathcal{H}_n^+(\mathbb{D})$ be such that there exists a sequence $(F_k)_{k \in \mathbb{N}}$ belonging to $H_n^\infty(\mathbb{D})$ satisfying the above conditions $(\alpha)$ and $(\beta)$. Then $E \in \mathcal{E}_n(\mathbb{D})$.

REMARK 3.3. Theorem 3.2 expresses in some sense a Smirnov class generalization of that characterization of the property that a function is outer which is formulated in terms of the shift-invariant subspace generated by this function. Sometimes the approximation property contained in Theorem 3.2 is called weak invertibility of the function $E$ (see [Sh] or [Nik1,Ch.2]). For the spaces $H_n^\infty(\mathbb{D})$ or $H_n^2(\mathbb{D})$ this approximation property (weak invertibility) will be often used for defining the notion ”outer function”. Observe that in the scalar case ($n = 1$) it was already shown by V.I. Smirnov [Sm] that for an outer function $e$ the linear subspace $e \cdot H^2(\mathbb{D})$ is dense in $H^2(\mathbb{D})$. Concerning several generalizations of this result of V.I. Smirnov we refer the reader to chapter 2 in [Nik1] (in particular, see Theorem 3 in Section 2.2.).

PROOF OF THEOREM 3.2. (i) Since $E$ is a matrix-valued outer function, Theorem 3.1 guarantees that $E^{-1} \in \mathcal{H}_n^+(\mathbb{D})$. We fix a boundary function $F$ of $E$ such $\det [\underline{F}(t)] \neq 0$ for $t \in \mathbb{T}$. Then for $k \in \mathbb{N}$ we define $w_k : \mathbb{T} \to (0, \infty)$ via

$$w_k(t) := \begin{cases} 1, & \text{if } \|E^{-1}(t)\| < k \\ \frac{1}{\|E^{-1}(t)\|}, & \text{if } \|E^{-1}(t)\| \geq k. \end{cases} \quad (3.4)$$

Clearly

$$0 < w_1(t) \leq w_2(t) \leq w_3(t) \leq \ldots \leq 1 \quad (3.5)$$

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for \( t \in \mathbb{T} \) and
\[
\lim_{k \to \infty} w_k(t) = 1. \tag{3.6}
\]
From (3.5) we see that the inequality
\[
w_1(t) \geq \|E^{-1}(t)\|^{-1} \tag{3.7}
\]
holds for \( t \in \mathbb{T} \). Since \( E^{-1} \in \mathfrak{N}_{\mathbb{M}}^+ (\mathbb{D}) \), we infer that
\[
\ln [\|E^{-1}\|^{-1}] \in L^1(\mathbb{T}, \mathfrak{B}_\mathbb{T}, m; \mathbb{C}). \tag{3.8}
\]
From (3.5) - (3.7) we obtain
\[
\int_{\mathbb{T}} \ln [w_k(t)] m(dt) > -\infty. \tag{3.9}
\]
Hence, for \( k \in \mathbb{N} \) the function \( \varphi_k : \mathbb{D} \to \mathbb{C} \) which is given by
\[
\varphi_k(z) := \exp \left\{ \int_{\mathbb{T}} \frac{t + z}{t - z} \ln [w_k(t)] m(dt) \right\}
\]
is well-defined. Moreover from its definition it is clear that \( \varphi_k \in \mathfrak{N}^+ (\mathbb{D}) \) (or more precisely, that \( \varphi_k \) is even outer). In view of (3.5) and (3.6) the monotone convergence theorem guarantees that
\[
\lim_{k \to \infty} \varphi_k(z) = 1, \quad z \in \mathbb{D}. \tag{3.10}
\]
Since \( |\varphi_k(t)| = w_k(t) \) for almost all \( t \in \mathbb{T} \) with respect to \( m \), formula (3.6) yields
\[
\lim_{k \to \infty} |\varphi_k(t)| = 1
\]
for almost all \( t \in \mathbb{T} \) with respect to \( m \). In view of (3.5) and (3.6), another application of the monotone convergence theorem gives us
\[
\lim_{k \to \infty} \int_{\mathbb{T}} |\varphi_k(t)|^2 m(dt) = \lim_{k \to \infty} \int_{\mathbb{T}} [w_k(t)]^2 m(dt) = \int_{\mathbb{T}} 1 \, dm = 1. \tag{3.11}
\]
For \( k \in \mathbb{N} \), we have
\[
\int_{\mathbb{T}} |\varphi_k(t) - 1|^2 m(dt) = \int_{\mathbb{T}} |\varphi_k(t)|^2 m(dt) - 2\Re[\varphi_k(0)] + 1. \tag{3.12}
\]
Combining (3.10) - (3.12) it follows that
\[
\lim_{k \to \infty} \int_{\mathbb{T}} |\varphi_k(t) - 1|^2 m(dt) = 0. \tag{3.13}
\]
In view of (3.13), the F. Riesz - Fischer theorem yields a subsequence \( (\varphi_{l_k})_{k \in \mathbb{N}} \) of \( (\varphi_k)_{k \in \mathbb{N}} \) such that
\[
\lim_{k \to \infty} \varphi_{l_k}(t) = 1 \tag{3.14}
\]
for almost all $t \in \mathbb{T}$ with respect to $m$. Let $k \in \mathbb{N}$ and set

$$F_k := E^{-1} \cdot \varphi_{l_k}.$$  

(3.15)

Then, since $E^{-1} \in \mathcal{M}_n^+(\mathbb{D})$ and $\varphi_{l_k} \in \mathcal{M}^+(\mathbb{D})$, we get $F_k \in \mathcal{M}_n^+(\mathbb{D})$. Thus as $|\varphi_{l_k}| = w_{l_k}$ almost everywhere with respect to $m$ it follows from (3.15) and (3.4) that

$$\|F_k(t)\| = w_{l_k}(t) \cdot \|E^{-1}(t)\| \leq l_k$$  

for almost all $t \in \mathbb{T}$ with respect to $m$. Thus, the maximum principle of V.I. Smirnov implies that $\|F_k(z)\| \leq l_k$ for all $z \in \mathbb{D}$. Consequently, $F_k \in H^\infty_n(\mathbb{D})$. From (3.15) it follows that

$$E \cdot F_k = \varphi_{l_k} \cdot I_n.$$  

(3.16)

From (3.5) we obtain

$$|\varphi_{l_k}(t)| = w_{l_k}(t) \leq 1$$  

and hence since $\varphi_{l_k} \in \mathcal{M}^+(\mathbb{D})$, the maximum principle of V.I. Smirnov guarantees that that

$$|\varphi_{l_k}(z)| \leq 1, \quad z \in \mathbb{D}.$$  

(3.17)

Thus, combining (3.16) and (3.17) we see that $(\gamma)$ is fulfilled. Moreover, from (3.16) and (3.14) we get that $(\alpha)$ is satisfied.

For almost all $t \in \mathbb{T}$ with respect to $m$ we have $|\varphi_{l_k}(t)| \leq 1$ and, consequently, in view of (3.15), the inequality

$$\ln^+ \|F_k(t)\| \leq \ln^+ \|E^{-1}(t)\|$$

holds for almost all $t \in \mathbb{T}$ with respect to $m$. Hence, the family $(\ln^+ \|F_k(t)\|)_{k \in \mathbb{N}}$ has an $m$-integrable majorant. This implies that $(\beta)$ is fulfilled.

Part (i) of Theorem 3.2 is now proved.

Before proving part (ii) of Theorem 3.2 we recall the following result (see [WM1, Lemma 3.12]).

**THE GENERALIZED MINKOWSKI INEQUALITY.** Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $M : \Omega \to \mathcal{M}_n$ be a $P$-integrable matrix function with nonnegative Hermitian values. Then

$$\ln \left[ \det \left( \int_\Omega M \ dP \right) \right] \geq \int_\Omega \ln [\det M] \ dP.$$  

(3.18)

**PROOF OF PART (ii) OF THEOREM 3.2.** For $k \in \mathbb{N}$ we define $v_k : \mathbb{T} \to [1, \infty)$ via the rule

$$v_k(t) := \begin{cases} \|E(t) \cdot F_k(t)\|, & \text{if } \|E(t) \cdot F_k(t)\| \geq 1 \\ \|E(t) \cdot F_k(t)\|^2, & \text{if } \|E(t) \cdot F_k(t)\| < 1. \end{cases}$$  

(3.19)

For $k \in \mathbb{N}$ and $t \in \mathbb{T}$ we then have

$$\ln [v_k(t)] \in [0, \infty).$$  

(3.20)
Combining \((\alpha)\) and (3.19) we infer that for almost all \(t \in \mathbb{T}\) with respect to \(m\),

\[
\lim_{k \to \infty} \ln [v_k(t)] = 0. \tag{3.21}
\]

For \(k \in \mathbb{N}\) and \(t \in \mathbb{T}\) we get the inequality

\[
\ln [v_k(t)] \leq \ln^+ \|E(t)\| + \ln^+ \|F_k(t)\|
\]

from (3.19), which together with \((\beta)\) implies that the family \((\ln v_k)_{k \in \mathbb{N}}\) is uniformly \(m\)-integrable. Combining this fact with (3.20) and (3.21), an application of Vitali’s Theorem provides

\[
\lim_{k \to \infty} \int_{\mathbb{T}} \ln [v_k(t)] \, m(dt) = 0. \tag{3.22}
\]

For \(k \in \mathbb{N}\) we define \(\Psi_k : \mathbb{D} \to \mathbb{C}\) via the formula

\[
\Psi_k(z) := \exp \left\{ - \int_{\mathbb{T}} \ln [v_k(t)] \frac{t + z}{t - z} \, m(dt) \right\}. \tag{3.23}
\]

Therefore, in view of (3.20), we obtain the inequality

\[
|\Psi_k(z)| = \exp \left\{ \Re \left[ - \int_{\mathbb{T}} \ln [v_k(t)] \frac{t + z}{t - z} \, m(dt) \right] \right\}
\]

\[
= \exp \left\{ - \int_{\mathbb{T}} \ln [v_k(t)] \frac{1 - |z|^2}{|t - z|^2} \, m(dt) \right\} \leq \exp \{0\} = 1 \tag{3.24}
\]

for \(z \in \mathbb{D}\). In view of (3.21), an application of Lebesgue's dominated convergence theorem yields

\[
\lim_{k \to \infty} \Psi_k(z) = 1 \tag{3.25}
\]

for all \(z \in \mathbb{D}\). For almost all \(t \in \mathbb{T}\) with respect to \(m\) we get from (3.24)

\[
|\Psi_k(t)| \leq 1 \tag{3.26}
\]

and hence, upon taking into account that formula (3.23) implies that

\[
|\Psi_k(t)| = [v_k(t)]^{-1}, \tag{3.27}
\]

we see from (3.19) and \((\alpha)\) that

\[
\lim_{k \to \infty} |\Psi_k(t)| = 1. \tag{3.28}
\]

For \(k \in \mathbb{N}\),

\[
\int_{\mathbb{T}} |\Psi_k(t) - 1|^2 \, m(dt) = \int_{\mathbb{T}} |\Psi_k(t)|^2 \, m(dt) - 2 \Re[\Psi_k(t)] + 1. \tag{3.29}
\]
In view of (3.26) and (3.28), Lebesgue’s dominated convergence theorem yields
\[
\lim_{k \to \infty} \int_T |\Psi_k(t)|^2 \, m(dt) = m(\mathbb{T}) = 1. \tag{3.30}
\]

Combining (3.25), (3.29) and (3.30) we obtain
\[
\lim_{k \to \infty} \int_T |\Psi_k(t) - 1|^2 \, m(dt) = 0. \tag{3.31}
\]

In view of (3.31), the F. Riesz - Fischer theorem provides a subsequence \((\Psi_{l_k})_{k \in \mathbb{N}}\) of \((\Psi_k)_{k \in \mathbb{N}}\) such that
\[
\lim_{k \to \infty} \Psi_{l_k}(t) = 1 \tag{3.32}
\]
for almost all \(t \in \mathbb{T}\) with respect to \(m\). Suppose that \(k \in \mathbb{N}\) and define
\[
\Phi_k := E \cdot F_k \cdot \Psi_k. \tag{3.33}
\]

Then, since \(E \in \mathfrak{N}_n^+(\mathbb{D})\), \(F_k \in H_n^\infty(\mathbb{D})\) and (3.24) holds, we get
\[
\Phi_k \in \mathfrak{N}_n^+(\mathbb{D}). \tag{3.34}
\]

For almost all \(t \in \mathbb{T}\) with respect to \(m\) it follows from (3.33), (3.19) and (3.27) that
\[
\|\Phi_k(t)\| = |\Psi_k(t)| \cdot \|E(t) \cdot F_k(t)\| \leq |\Psi_k(t)| \cdot v_k(t) = 1. \tag{3.35}
\]

Therefore the maximum principle of V.I. Smirnov implies that
\[
\|\Phi_k(z)\| \leq 1 \tag{3.36}
\]
for all \(z \in \mathbb{D}\). In particular,
\[
\Phi_k \in H_n^\infty(\mathbb{D}). \tag{3.37}
\]

From (3.34) and (3.35) it follows that
\[
\Phi_k^*(t) \cdot \Phi_k(t) \leq I_n \tag{3.38}
\]
for almost all \(t \in \mathbb{T}\) with respect to \(m\). Combining (3.33), (\(\alpha\)) and (3.28) we get
\[
\lim_{k \to \infty} \Phi_k^*(t) \cdot \Phi_k(t) = \lim_{k \to \infty} |\Psi_k(t)|^2 |(E(t)F_k(t))| \cdot |E(t)F_k(t)| = I_n. \tag{3.39}
\]

From (3.32), (3.33) and (\(\alpha\)) we now obtain
\[
\lim_{k \to \infty} \Phi_{l_k}(t) = \lim_{k \to \infty} E(t) \cdot F_{l_k}(t) \cdot \Psi_{l_k}(t) = I_n. \tag{3.40}
\]

Using (3.37), (3.38), (3.40) and Lebesgue’s dominated convergence theorem we get
\[
\lim_{k \to \infty} \Phi_{l_k}(0) = \lim_{k \to \infty} \int_T \Phi_{l_k}(t) \, m(dt) = I_n. \tag{3.41}
\]
Suppose that \( k \in \mathbb{N} \). We define \( M_k : \mathbb{T} \to \mathfrak{M}_n \) via the rule
\[
M_k(t) := \Phi_k^*(t) \cdot \Phi_k(t).
\] (3.42)

Then (3.42) and (3.38) imply that the inequality \( 0 \leq M_k(t) \leq I_n \) holds true for almost all \( t \in \mathbb{T} \) with respect to \( m \). Hence,
\[
0 \leq \int_{\mathbb{T}} M_k(t) \, m(dt) \leq I_n.
\] (3.43)

Now we apply the Generalized Minkowski inequality to the \( M_k \). (Note that Lebesgue measure \( m \) is a probability measure.) From (3.43) we infer first that
\[
\ln \left[ \det \left( \int_{\mathbb{T}} M_k(t) \, m(dt) \right) \right] \leq \ln \det I_n = 0.
\] (3.44)

Hence, (3.44) and the Generalized Minkowski inequality guarantee that
\[
\int_{\mathbb{T}} \ln \left( \det [M_k(t)] \right) \, m(dt) \leq 0.
\] (3.45)

Using (3.42) and (3.33) it follows that
\[
\frac{1}{2} \ln \left( \det [M_k(t)] \right) = \ln \left| \det [\Phi_k(t)] \right| = \ln \left| \det [E(t)] \right| + \ln \left| \det \{[F_k(t)] \cdot [\Psi_k(t)]\} \right|
\] (3.46)

for almost all \( t \in \mathbb{T} \) with respect to \( m \). Thus, from (3.45) and (3.46) we see that
\[
\int_{\mathbb{T}} \ln \left| \det [E(t)] \right| \, m(dt) \leq - \int_{\mathbb{T}} \ln \left| \det \{[F_k(t)] \cdot [\Psi_k(t)]\} \right| \, m(dt).
\] (3.47)

By assumption, \( F_k \in H_n^\infty(\mathbb{D}) \). Using (3.23) and (3.24) we see that \( \Psi_k \in H_n^\infty(\mathbb{D}) \). Thus, \( F_k \cdot \Psi_k \in H_n^\infty(\mathbb{D}) \) and, consequently, \( \det [F_k \cdot \Psi_k] \in H_\infty(\mathbb{D}) \). Now Jensen’s inequality gives
\[
- \int_{\mathbb{T}} \ln \left| \det \{[F_k(t)] \cdot [\Psi_k(t)]\} \right| \, m(dt) \leq - \ln \left| \det \{[F_k(0)] \cdot [\Psi_k(0)]\} \right|.
\] (3.48)

From (3.47) and (3.48) it now follows that
\[
\int_{\mathbb{T}} \ln \left| \det [E(t)] \right| \, m(dt) \leq - \ln \left| \det \{[F_k(0)] \cdot [\Psi_k(0)]\} \right|.
\] (3.49)

From (3.33) and (3.41) we obtain
\[
\lim_{k \to \infty} \ln \left| \det \{[F_k(0)] \cdot [\Psi_k(0)]\} \right| = - \ln \left| \det [E(0)] \right|.
\] (3.50)
Combining (3.49) and (3.50) we obtain
\[
\int_{\mathbb{T}} \ln |\det [\mathbf{E}(t)]| \, m(dt) \leq \ln |\det [\mathbf{E}(0)]|.
\]
By assumption, \( E \in \mathfrak{M}^+(\mathbb{D}) \). Thus, \( \det E \in \mathfrak{M}^+(\mathbb{D}) \) and Jensen’s inequality yields
\[
\ln |\det [\mathbf{E}(0)]| \leq \int_{\mathbb{T}} \ln |\det [\mathbf{E}(t)]| \, m(dt).
\]
Hence,
\[
\int_{\mathbb{T}} \ln |\det [\mathbf{E}(t)]| \, m(dt) = \ln |\det [\mathbf{E}(0)]|.
\] (3.51)
From (3.51) and Lemma 2.1 we see that \( \det E \in \mathfrak{E}(\mathbb{D}) \). Therefore, by definition 3.1, \( E \in \mathfrak{E}_n(\mathbb{D}) \). Part (ii) of Theorem 3.2 is now proved. \( \square \)

**THEOREM 3.3.** (i) Let \( E \in \mathfrak{E}_n(\mathbb{D}) \). Then there exists a sequence \( (F_k)_{k \in \mathbb{N}} \) from \( H^\infty_n(\mathbb{D}) \) with the following properties:

(a) For almost all \( t \in \mathbb{T} \) with respect to \( m \), \( \lim_{k \to \infty} F_k(t) \cdot E(t) = I_n \).

(b) The family \( (\ln^+ \|F_k\|)_{k \in \mathbb{N}} \) is uniformly integrable with respect to \( m \).

(c) There exists a Borel subset \( B_0 \) of \( \mathbb{T} \) with \( m(B_0) = 0 \) such that for all \( k \in \mathbb{N} \) and all \( t \in \mathbb{T} \setminus B_0 \) the inequality \( \|F_k(t) \cdot E(t)\| \leq 1 \) holds.

(ii) Let \( E \in \mathfrak{M}^+_n(\mathbb{D}) \) be such that there exists a sequence \( (F_k)_{k \in \mathbb{N}} \) which belongs to \( H^\infty_n(\mathbb{D}) \) and satisfies the above conditions (a) and (b). Then \( E \in \mathfrak{E}_n(\mathbb{D}) \).

**PROOF.** Combine Theorem 3.2 and Remark 3.2. \( \square \)

It should be mentioned that Ginzburg [Gi1] obtained a multiplicative integral representation for outer functions which belong to \( \mathfrak{E}_n(\mathbb{D}) \).

**4. MATRIX-VALUED INNER FUNCTIONS**

In this section, we draw our attention to a distinguished subclass of the Schur class \( \mathfrak{S}_{n \times n} \) (compare Definition 2.1).

**DEFINITION 4.1.** Let \( \Theta \in \mathfrak{S}_{n \times n}(\mathbb{D}) \). Then \( \Theta \) is called inner if
\[
I_n - \Theta^*(t) \cdot \Theta(t) = O_{n \times n}
\] (4.1)
for almost all \( t \in \mathbb{T} \) with respect to \( m \). The class of all \( n \times n \) matrix-valued inner functions will be denoted by \( \mathfrak{I}_n(\mathbb{D}) \).

**REMARK 4.1.** Let \( \Theta \in \mathfrak{I}_n(\mathbb{D}) \). Then obviously \( \det \Theta \neq 0 \).
REMARK 4.2. Let $\Theta \in \mathcal{I}_n(\mathbb{D})$. Then $\Theta^T \in \mathcal{I}_n(\mathbb{D})$.

The class $\mathcal{I}_n(\mathbb{D})$ contains two important subclasses, namely the so-called singular inner functions and the Blaschke-Potapov products. Now we will formulate the corresponding definitions.

DEFINITION 4.2. Let $S \in \mathcal{I}_n(\mathbb{D})$. Then $S$ is called singular, if $\det [S(z)] \neq 0$ for all $z \in \mathbb{D}$ (or in other words if $S^{-1}$ is holomorphic in $\mathbb{D}$). The class of all $n \times n$ matrix-valued singular inner functions will be denoted by $\mathcal{I}_{n,s}(\mathbb{D})$.

REMARK 4.3. If $S \in \mathcal{I}_{n,s}(\mathbb{D})$, then $S^{-1} \in \mathcal{N}_{n}(\mathbb{D})$, because $S^{-1}$ admits the representation $S^{-1} = L \cdot (\det S)^{-1}$ with bounded holomorphic functions $L$ and $\det S$.

LEMMA 4.1. Let $S \in \mathcal{I}_{n,s}(\mathbb{D})$ be such that $S^{-1} \in \mathcal{N}_{n}(\mathbb{D})$. Then $S$ is constant.

PROOF. Since $S(t)$ is unitary for a.e. $t \in \mathbb{T}$ it follows that

$$\|S^{-1}(t)\| = 1.$$ 

Therefore, by the maximum principle of V.I. Smirnov, $\|S^{-1}(z)\| \leq 1$ for all $z \in \mathbb{D}$. Since $\|S(z)\| \leq 1$ then it follows that $S(z)$ is a unitary matrix for all $z \in \mathbb{D}$. However a holomorphic matrix function with unitary values is necessarily constant (see e.g. Corollary 2.3.2 in [DFK]).

Now we are going to define Blaschke-Potapov products. For this reason, we recall first the notion of a scalar elementary Blaschke factor. Let $a \in \mathbb{D}$. Then we define $b_a : \mathbb{D} \to \mathbb{C}$ via the rule

$$b_a(z) := \begin{cases} \frac{|a|}{a} \cdot \frac{a - z}{1 - \overline{a}z}, & \text{if } a \in \mathbb{D} \setminus \{0\} \\ z, & \text{if } a = 0 \end{cases}.$$ 

(4.2)

Assume that $P \in \mathcal{M}_n$ is a non-zero orthoprojection matrix, i.e., that the conditions

$$P^2 = P \quad P = P^*$$ 

(4.3)

are satisfied. Then the matrix-valued function $B_{a,P} : \mathbb{D} \to \mathcal{M}_n$ which is defined by

$$B_{a,P}(z) := I_n + [b_a(z) - 1] \cdot P$$ 

(4.4)

is called the Blaschke-Potapov elementary factor associated with $a$ and $P$.

From (4.3) and (4.4) it is clear that

$$\det [B_{a,P}] = (b_a)^{\text{rank } P}.$$ 

(4.5)

Suppose that $(z_k)_{k \in I}$ is a sequence from $\mathbb{D}$ and that $(P_k)_{k \in I}$ is a sequence of orthoprojection matrices for which the condition

$$\sum_{k \in I} (1 - |z_k|) \cdot \text{tr } P_k < +\infty$$ 

(4.6)
is fulfilled. (The index set $I$ can be finite or infinite.) Then, according to a result due to V.P. Potapov [Pot], the product

$$
\prod_{k \in I} B_{z_k,p_k}(z) \quad \text{resp.} \quad \prod_{k \in I} B_{z_k,p_k}(z)
$$

converges for all $z \in \mathbb{D}$. (V.P. Potapov has also shown that condition (4.6) is necessary for the convergence of the product in (4.7)).

**DEFINITION 4.3.** Let $B : \mathbb{D} \to \mathcal{M}_n$. Then $B$ is called a left (resp. right) Blaschke-Potapov product if $B$ is a constant function with unitary value or if there exist a unitary matrix $V$, a set of orthoprojection matrices $(P_k)_{k \in I}$ and sequences $(z_k)_{k \in I}$ which belong to $\mathbb{D}$ such that (4.6) is satisfied and moreover the representation

$$
B = \left( \prod_{k \in I} B_{z_k,p_k}(z) \right) \cdot V \quad \text{resp.} \quad B = V \cdot \left( \prod_{k \in I} B_{z_k,p_k}(z) \right)
$$

is valid. The set of left (resp. right) Blaschke-Potapov products will be denoted by $\mathcal{I}_{n,B,l}(\mathbb{D})$ (resp. $\mathcal{I}_{n,B,r}(\mathbb{D})$).

We will see below that each left Blaschke-Potapov product is also a right Blaschke-Potapov product and vice versa. Moreover, it will turn out that $\mathcal{I}_{n,B,l}(\mathbb{D}) \subseteq \mathcal{I}_n(\mathbb{D})$.

**LEMMA 4.2.** Let $A, B \in \mathcal{C}_n$ be such that $A \cdot B$ is unitary. Then $A$ and $B$ are unitary too.

**PROOF.** Since $A, B \in \mathcal{C}_n$ we have $I_n - AA^* \geq \mathbb{O}_{n \times n}$ and $I_n - BB^* \geq \mathbb{O}_{n \times n}$. Hence, $A(I_n - BB^*)A^* \geq \mathbb{O}_{n \times n}$. Therefore, the identity

$$
\mathbb{O}_{n \times n} = I_n - (AB)(AB)^* = (I_n - AA^*) + A(I_n - BB^*)A^*
$$

implies that $I_n - AA^* = \mathbb{O}_{n \times n}$ and $A(I_n - BB^*)A^* = \mathbb{O}_{n \times n}$. Thus, $A$ is unitary. In particular, we have $\det A \neq 0$. This implies that $I_n - BB^* = \mathbb{O}_{n \times n}$ and hence that $B$ is unitary too.

**THEOREM 4.1.** Suppose that $\Theta \in \mathcal{I}_n(\mathbb{D})$.

(a) There exist functions $B \in \mathcal{I}_{n,B,l}(\mathbb{D})$ (resp. $C \in \mathcal{I}_{n,B,r}(\mathbb{D})$) and $S \in \mathcal{I}_{n,s}(\mathbb{D})$ (resp. $T \in \mathcal{I}_{n,s}(\mathbb{D})$) such that the multiplicative representation

$$
\Theta = B \cdot S \quad \text{resp.} \quad \Theta = T \cdot C
$$

holds true.

(b) Suppose that the functions $B_1, B_2 \in \mathcal{I}_{n,B,l}(\mathbb{D})$ (resp. $C_1, C_2 \in \mathcal{I}_{n,B,r}(\mathbb{D})$) and $S_1, S_2 \in \mathcal{I}_{n,s}(\mathbb{D})$ (resp. $T_1, T_2 \in \mathcal{I}_{n,s}(\mathbb{D})$) satisfy $B_1S_1 = B_2S_2 = \Theta$ (resp. $T_1C_1 = T_2C_2 = \Theta$). Then there exist a unitary matrix $U \in \mathcal{M}_n$ (resp. $V \in \mathcal{M}_n$) such that $B_2 = B_1U$ and $S_2 = U^*S_1$ (resp. $C_2 = VC_1$ and $T_2 = T_1V^*$) are fulfilled.

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PROOF. Theorem 4.1 is a special case of a much more general result due to V.P. Potapov [Pot]. The Potapov theory handles the case of meromorphic matrix-valued functions in \( D \) which have a nonidentically vanishing determinant and which are \( J \)-contractive where \( J \) is a signature matrix (i.e. \( J = J^* \) and \( J^2 = I_n \)). In the special case that \( J = I \), V.P. Potapov’s result (see [Pot] and also a series of papers by Ginzburg [Gi1] - [Gi5], [GiSh]) provides the existence of functions \( B \in \mathcal{H}_{n,B,l}(D) \) and \( S \in \mathcal{S}_{n \times n}(D) \) such that

\[
\Theta = B \cdot S \tag{4.9}
\]

and for all \( z \in D \),

\[
\det [S(z)] \neq 0. \tag{4.10}
\]

Since the boundary function \( \Theta \) has unitary values almost everywhere with respect to \( m \) we infer from Lemma 4.2 that the boundary functions \( B \) and \( S \) also have unitary values almost everywhere with respect to \( m \). Taking into account (4.10) we obtain \( S \in I_{n,s}(D) \).

The uniqueness part goes back to V.P. Potapov [Pot] too.

LEMMA 4.3. Let \( M \in C_n \). Then

(a) \(| \det M | \leq 1 \)

(b) \(| \det M | = 1 \) if and only if \( M \) is unitary.

PROOF. Let \( (l_k(M^*M))_{k=1}^n \) denote the system of eigenvalues of \( M^*M \). Then, since \( M \in C_n \), \( 0 \leq l_k(M^*M) \leq 1 \) for all \( k \in \{1, \ldots, n\} \). Thus, as

\[
| \det M |^2 = | \det (M^*M) | = \prod_{k=1}^n l_k(M^*M), \tag{4.11}
\]

we see that \( | \det M | \leq 1 \) with equality if and only if \( l_k(M^*M) = 1 \) for all \( k \in \{1, \ldots, n\} \).

But \( l_k(M^*M) = 1 \) for all \( k \in \{1, \ldots, n\} \) if and only if \( M^*M = I_n \).

Now we recall a well-known characterization of Blaschke products (see e.g., Privalov [Pri, Ch.I, Sec.7.1]).

LEMMA 4.4. Let \( \Theta \in \mathcal{S}_{1 \times 1}(D) \). Then \( \Theta \) is a Blaschke product if and only if

\[
\lim_{r \to 1-} \int_{\mathbb{T}} \ln | \det [\Theta(rt)] | \ m(dt) = 0.
\]

THEOREM 4.2. Let \( f \in \mathcal{S}_{n \times n}(D) \). Then:

(a) The function \( \det f \) belongs to \( \mathcal{S}_{1 \times 1}(D) \).

(b) \( f \in \mathcal{I}_n(D) \) if and only if \( \det f \in \mathcal{I}_1(D) \). If \( f \in \mathcal{I}_n(D) \) then \( \det f \neq 0 \).

(c) \( f \in \mathcal{I}_{n,s}(D) \) if and only if \( \det f \in \mathcal{I}_{1,s}(D) \).

(d) The following statements are equivalent:

(i) \( f \in \mathcal{I}_{n,B,l}(D) \),

(ii) \( f \in \mathcal{I}_{n,B,r}(D) \),

(iii) \( \det f \) is a Blaschke product.

(iv) The limit relation \( \lim_{r \to 1-} \int_{\mathbb{T}} \ln | \det [f(rt)] | \ m(dt) = 0 \) holds true.
PROOF. The assertions stated in part (a) and (b) are immediate consequences of Lemma 4.3. Part (c) follows from part (a) and the definition of a singular inner function. It remains to prove part (d). From (a) and Lemma 4.4 we can immediately conclude the equivalence of statements (iii) and (iv). In view of (4.5), it is readily checked that each of the conditions (i) and (ii) implies (iii). Now suppose that (iii) holds. By virtue of part (b) we see that \( f \) is an inner function. From Theorem 4.1 we infer that there exist functions \( B \in \mathcal{S}_{n,B,I}(\mathbb{D}) \) and \( S \in \mathcal{S}_{n,s}(\mathbb{D}) \) satisfying the multiplicative decomposition \( f = B \cdot S \). Hence, \( \det f = \det B \cdot \det S \). The implication “(i) \( \Rightarrow \) (iii)” which is already verified shows that \( \det B \) is a Blaschke product. Part (c) yields that \( \det S \) is a singular inner function. Therefore, the uniqueness part of Theorem 4.1 yields that \( \det S \) is a constant inner function with unimodular value. Hence, we obtain from part (b) of Lemma 4.3 that the matrix \( S(z) \) is unitary for each \( z \in \mathbb{D} \). Since \( S \) belongs to \( \mathcal{S}_{n \times n}(\mathbb{D}) \), the maximum modulus principle for matrix-valued Schur functions (see e.g. [DFK, Corollary 2.3.2]) implies that \( S \) is a constant function. From \( f = B \cdot S \) we infer that (i) holds. The implication “(iii) \( \Rightarrow \) (ii)” can be shown analogously. The theorem is proved.

For further results on matrix-valued and operator-valued inner functions we refer the reader to the monographs Helson [Hel1], Sz.-Nagy and Foias [SZNF] and Nikolskii [Nik2].

5. INNER-OUTER FACTORIZATION

This section is aimed at a Smirnov class generalization of the inner-outer factorization of matrix-valued functions belonging to the Hardy class \( H^2_n(\mathbb{D}) \).

Let us recall the following notions:

DEFINITION 5.1. The Hardy class \( H^2_n(\mathbb{D}) \) is the set of all matrix-valued functions \( F : \mathbb{D} \to \mathcal{M}_n \) which are holomorphic in \( \mathbb{D} \) and satisfy

\[
\sup_{r \in [0,1]} \int_{\mathbb{T}} \| F(rt) \|^2 m(dt) < \infty.
\]

REMARK 5.1. Obviously, \( H^\infty_n(\mathbb{D}) \subseteq H^2_n(\mathbb{D}) \subseteq \mathcal{N}^+_n(\mathbb{D}) \).

REMARK 5.2. Define \( \| \cdot \|_{H^2_n} : H^2_n(\mathbb{D}) \to [0, \infty) \) via

\[
F \to \sqrt{\sup_{r \in [0,1]} \int_{\mathbb{T}} \| F(rt) \|^2 m(dt)}.
\]

Then \( (H^2_n(\mathbb{D}), \| \cdot \|_{H^2}) \) is a complex Hilbert space.

REMARK 5.3. Let \( S \in \mathcal{S}_{n \times n}(\mathbb{D}) \) be such that \( \det (I_n + S) \) does not identically vanish in \( \mathbb{D} \). Then \( (I_n + S) \in \mathcal{E}_n(\mathbb{D}) \cap H^\infty_n(\mathbb{D}) \) (see Arov [Ar1], Lemma 3.1).

The definition of a matrix-valued outer function given above (see Definition 3.1) is too rough for the purposes of prediction theory of stationary sequences. For this reason, P.R. Masani [Ma1, Ma2] introduced the following notion for the space \( H^2_n(\mathbb{D}) \) (compare Lemma 2.2).
DEFINITION 5.2. Let $E \in H_n^2(\mathbb{D})$. Then $E$ is said to be left optimal (resp. right optimal) if $E$ has the following property: If $F \in H_n^2(\mathbb{D})$ satisfies $[F(t)] \cdot [E(t)]^* = [E(t)] \cdot [E(t)]^*$ (resp. $[E(t)]^* \cdot [E(t)] = [E(t)]^* \cdot [F(t)]$) then $[F(0)] \cdot [F(0)]^* \leq [E(0)] \cdot [E(0)]^*$ (resp. $[F(0)]^* \cdot [F(0)] \leq [E(0)]^* \cdot [E(0)]$).

REMARK 5.4. Let $E \in H_n^2(\mathbb{D})$. Then $E$ is left optimal if and only if $E^\top$ is right optimal.

This notion of optimality is closely related to the following definition which in the scalar case goes back to Beurling [Be].

DEFINITION 5.3. Let $E \in H_n^2(\mathbb{D})$. Then $E$ is called left Beurling-outer (resp. right Beurling outer) if there exists a sequence $(f_k)_{k \in \mathbb{N}}$ from $H_n^\infty(\mathbb{D})$ which satisfies

$$\lim_{k \to \infty} \int_T ||F_k(t) \cdot E(t) - I_n||^2 \, m(dt) = 0 \quad \text{(resp. } \lim_{k \to \infty} \int_T ||E(t) \cdot F_k(t) - I_n||^2 \, m(dt) = 0).$$

The class of all $n \times n$ matrix-valued left Beurling-outer (resp. right Beurling outer) functions will be denoted by $\mathcal{E}_{n,B,l}(\mathbb{D})$ (resp. $\mathcal{E}_{n,B,r}(\mathbb{D})$).

REMARK 5.5. Let $E \in H_n^2(\mathbb{D})$. Then $E \in \mathcal{E}_{n,B,l}(\mathbb{D})$ if and only if $E^\top \in \mathcal{E}_{n,B,r}(\mathbb{D})$.

REMARK 5.6. Let $E \in H_n^2(\mathbb{D})$. Then it is readily checked that $E$ is left Beurling outer (resp. right Beurling outer) if and only if the subspace $H_n^2(\mathbb{D}) \cdot E$ (resp. $E \cdot H_n^2(\mathbb{D})$) is dense in $(H_n^2(\mathbb{D}), || \cdot ||_{H^2})$.

REMARK 5.7. Let $E$ be a function belonging to $\mathcal{E}_{n,B,l}(\mathbb{D})$ or $\mathcal{E}_{n,B,r}(\mathbb{D})$. Then for all $z \in \mathbb{D}$ the relation $\det [E(z)] \neq 0$ holds true.

PROOF : Let us consider the case $E \in \mathcal{E}_{n,B,l}(\mathbb{D})$. Then there exists a sequence $(F_k)_{k \in \mathbb{N}}$ from $H^\infty(\mathbb{D})$ such that

$$\lim_{k \to \infty} \int_T ||E(t) \cdot F_k(t) - I_n||^2 \, m(dt) = 0.$$ 

From this it follows by the Poisson integral representation for $H_n^2(\mathbb{D})$ functions that

$$\lim_{k \to \infty} E(z) \cdot F_k(z) = I_n$$

for $z \in \mathbb{D}$ and hence that

$$\lim_{k \to \infty} \det [E(z)] \cdot \det [F_k(z)] = 1.$$ 

Thus, $\det [E(z)] \neq 0$. If $E \in \mathcal{E}_{n,B,l}(\mathbb{D})$, then the assertion follows from Remark 5.5 and the preceding analysis.

The following result due to Masani [Ma2, Corollary 4.6] clarifies the relation between optimality and Beurling-outerness.
THEOREM 5.1. Let $E \in H_n^2(\mathbb{D})$. Then:

(a) If $\det E \not\equiv 0$ and $E$ is left optimal (resp. right optimal), then $E \in \mathcal{E}_{n,B,l}(\mathbb{D})$ (resp. $E \in \mathcal{E}_{n,B,r}(\mathbb{D})$).

(b) If $E \in \mathcal{E}_{n,B,l}(\mathbb{D})$ (resp. $E \in \mathcal{E}_{n,B,r}(\mathbb{D})$), then $E$ is left optimal (resp. right optimal).

The notion of optimality is more general than the notion of Beurling - outer because it allows the functions in question to have identically vanishing determinants. In the theory of multivariate stationary stochastic processes this corresponds to the case of a singular prediction error matrix.

The following result plays a key role in the theory of holomorphic matrix-valued functions.

THEOREM 5.2. Let $F \in H_n^2(\mathbb{D})$ be such that $\det F \not\equiv 0$. Then:

(i) There exist functions $\Theta_r \in \mathcal{I}_n(\mathbb{D})$ and $E_r \in \mathcal{E}_{n,B,r}(\mathbb{D})$ such that the multiplicative decomposition

$$F = \Theta_r \cdot E_r$$

is satisfied.

(ii) Suppose that the functions $\Theta_{r_1}, \Theta_{r_2} \in \mathcal{I}_n(\mathbb{D})$ and $E_{r_1}, E_{r_2} \in \mathcal{E}_{n,B,r}(\mathbb{D})$ satisfy

$$\Theta_{r_1} \cdot E_{r_1} = \Theta_{r_2} \cdot E_{r_2} = F.$$

Then there exists a unitary matrix $V \in \mathbb{M}_n$ such that $\Theta_{r_2} = \Theta_{r_1} \cdot V$ and $E_{r_2} = V^* \cdot E_{r_1}$ are fulfilled.

(iii) There exist functions $\Theta_l \in \mathcal{I}_n(\mathbb{D})$ and $E_l \in \mathcal{E}_{n,B,l}(\mathbb{D})$ such that the multiplicative decomposition

$$F = E_l \cdot \Theta_l$$

is satisfied.

(iv) Suppose that the functions $\Theta_{l_1}, \Theta_{l_2} \in \mathcal{I}_n(\mathbb{D})$ and $E_{l_1}, E_{l_2} \in \mathcal{E}_{n,B,l}(\mathbb{D})$ satisfy

$$E_{l_1} \cdot \Theta_{l_1} = E_{l_2} \cdot \Theta_{l_2} = F.$$

Then there exists a unitary matrix $U \in \mathbb{M}_n$ such that $\Theta_{l_2} = U \cdot \Theta_{l_1}$ and $E_{l_2} = E_{l_1} \cdot U^*$ are fulfilled.

Theorem 5.2 was proved independently by several authors (see Masani [Ma2, 4.3, 4.4], Helson and Lowdenslager [HL2, Theorem 15], Rozanov [Roz1, Theorem 5]). The Beurling-Lax-Halmos Theorem (see Beurling [Be], Lax [La], Halmos [Hal] and also Masani [Ma2, Theorem 3.8.]) which describes the structure of shift invariant left (resp. right) submodules of $H_n^2(\mathbb{D})$ lies at the heart of the proof.

THEOREM 5.3. The identities

$$\mathcal{E}_{n,B,l}(\mathbb{D}) = \mathcal{E}_{n,B,r}(\mathbb{D}) = \mathcal{E}_n(\mathbb{D}) \cap H_n^2(\mathbb{D})$$

are valid.
PROOF. First we show that

\[ \mathcal{E}_{n,B,r}(\mathbb{D}) = \mathcal{E}_n(\mathbb{D}) \cap H_n^2(\mathbb{D}). \]

Our proof is based mainly on Theorem 3.2.

First assume that \( E \in \mathcal{E}_n(\mathbb{D}) \cap H_n^2(\mathbb{D}) \). Then part (i) of Theorem 3.2 guarantees the existence of a sequence \((F_k)_{k \in \mathbb{N}}\) from \( H_n^\infty(\mathbb{D}) \) with the properties (\( \alpha \)), (\( \beta \)) and (\( \gamma \)) formulated there. In view of property (\( \gamma \)), there exists a Borel subset \( B_0 \) of \( \mathbb{T} \) with \( m(B_0) = 0 \) such that for all \( k \in \mathbb{N} \) and all \( t \in \mathbb{T} \setminus B_0 \) the inequality

\[ \| F(t) \cdot F_k(t) - I_n \| \leq \| F(t) \cdot F_k(t) \| + \| I_n \| \leq 2 \quad \text{(5.1)} \]

holds. In view of (\( \alpha \)) and (5.1), an application of Lebesgue’s dominated convergence theorem yields

\[ \lim_{k \to \infty} \int_{\mathbb{T}} \| F(t) \cdot F_k(t) - I_n \|^2 \ m(dt) = 0. \]

Thus, \( E \in \mathcal{E}_{n,B}(\mathbb{D}) \). Hence, the inclusion

\[ \mathcal{E}_n(\mathbb{D}) \cap H_n^2(\mathbb{D}) \subseteq \mathcal{E}_{n,B}(\mathbb{D}) \quad \text{(5.2)} \]

holds true.

Now assume that \( E \in \mathcal{E}_{n,B}(\mathbb{D}) \). Then Definition 5.3 implies that

\[ E \in H_n^2(\mathbb{D}). \quad \text{(5.3)} \]

We will show that \( E \) satisfies the conditions (\( \alpha \)) and (\( \beta \)) in Theorem 3.2. In view of Definition 5.2 there exists a sequence \((F_k)_{k \in \mathbb{N}}\) from \( H_n^\infty(\mathbb{D}) \) for which

\[ \lim_{k \to \infty} \int_{\mathbb{T}} \| F(t) \cdot F_k(t) - I_n \|^2 \ m(dt) = 0. \quad \text{(5.4)} \]

Obviously, for \( k \in \mathbb{N} \) and \( t \in \mathbb{T} \) the inequality

\[ 0 \leq \ln^+ \| F(t) \cdot F_k(t) \| \leq \| F(t) \cdot F_k(t) - I_n \| \quad \text{(5.5)} \]

holds true. From (5.4) and (5.5) it then follows that

\[ \lim_{k \to \infty} \int_{\mathbb{T}} \ln^+ \| F(t) \cdot F_k(t) \| \ m(dt) = 0. \]

Hence, the family \((\ln^+ \| F \cdot F_k \|)_{k \in \mathbb{N}}\) is uniformly \( m \) - integrable. In view of Remark 5.7 we see that \( \det [E(z)] \neq 0 \) for all \( z \in \mathbb{D} \). Since \( E \in H_n^2(\mathbb{D}) \subseteq \mathfrak{N}_n(\mathbb{D}) \) we now obtain \( E^{-1} \in \mathfrak{N}_n(\mathbb{D}) \). Hence, \( \ln \| E^{-1} \| = \ln \| E^{-1} \| \) is \( m \)-integrable. Clearly, for \( k \in \mathbb{N} \) and \( t \in \mathbb{T} \) the inequality

\[ \ln^+ \| F_k(t) \| \leq \ln^+ \| F(t) \cdot F_k(t) \| + \ln^+ \| [F(t)]^{-1} \| \quad \text{(5.6)} \]
holds true. Since the family \((\ln^+ \| E \cdot F_k \|)_{k \in \mathbb{N}}\) is uniformly \(m\)-integrable and since \(\ln \| E^{-1} \|\) is \(m\)-integrable it follows from (5.6) that the family \((\ln^+ \| F_k \|)_{k \in \mathbb{N}}\) is uniformly \(m\)-integrable. Taking into account (5.4), the Theorem of F. Riesz - Fischer provides the existence of a subsequence \((F_k)_{k \in \mathbb{N}}\) of \((F_k)_{k \in \mathbb{N}}\) such that

\[
\lim_{k \to \infty} E(t) \cdot F_k(t) = I_n
\] for \(m\)-almost all \(t \in \mathbb{T}\). Since the family \((\ln^+ \| F_k \|)_{k \in \mathbb{N}}\) is also uniformly \(m\)-integrable the conditions \((\alpha)\) and \((\beta)\) in Theorem 3.2 are satisfied for the sequence \((F_k)_{k \in \mathbb{N}}\). Thus, part (ii) of Theorem 3.2 implies that

\[
E_n \in E_n(D).
\] (5.7)

From (5.3) and (5.7) we obtain

\[
E_{n,B}(D) \subseteq E_n(D) \cap H^2_n(D).
\] (5.8)

From (5.8) and Remarks 3.2 and 5.5 we then get

\[
E_{n,B,l}(D) = E_n(D) \cap H^2_n(D).
\] Thus, the theorem is proved.

THEOREM 5.4. (Inner - outer factorization in the Smirnov class \(\mathcal{M}^+_n(D)\)). Let \(F \in \mathcal{M}^+_n(D)\) be such that \(\det F \neq 0\). Then:

(i) There exist functions \(\Theta_r \in I_n(D)\) and \(E_r \in E_n(D)\) such that

\[
F = \Theta_r \cdot E_r.
\]

(ii) Suppose that the functions \(\Theta_{r1}, \Theta_{r2} \in I_n(D)\) and \(E_{r1}, E_{r2} \in E_n(D)\) satisfy

\[
\Theta_{r1} \cdot E_{r1} = \Theta_{r2} \cdot E_{r2} = F.
\]

Then there exists a unitary matrix \(V \in \mathcal{M}_n\) such that \(\Theta_{r2} = \Theta_{r1} \cdot V\) and \(E_{r2} = V^* \cdot E_{r1}\).

(iii) There exist functions \(\Theta_l \in I_n(D)\) and \(E_l \in E_n(D)\) such that

\[
F = E_l \cdot \Theta_l.
\]

(iv) Suppose that the functions \(\Theta_{l1}, \Theta_{l2} \in I_n(D)\) and \(E_{l1}, E_{l2} \in E_n(D)\) satisfy

\[
E_{l1} \cdot \Theta_{l1} = E_{l2} \cdot \Theta_{l2} = F.
\]

Then there exists a unitary matrix \(U \in \mathcal{M}_n\) such that \(\Theta_{l2} = U \cdot \Theta_{l1}\) and \(E_{l2} = E_{l1} \cdot U^*\).

PROOF. We derive these results from Theorem 5.2.

(i) In view of Lemma 2.4 there exist functions \(d \in \mathcal{E}(D)\) and \(\Phi \in \mathcal{S}_{n \times n}(D)\) such that

\[
F = \frac{1}{d} \cdot \Phi.
\] (5.9)
Since \( \det F \neq 0 \), it follows from (5.9) that \( \det \Phi \neq 0 \). Thus as \( \mathcal{G}_{n \times n}(\mathbb{D}) \subseteq H^2_\alpha(\mathbb{D}) \), Theorem 5.2 ensures the existence of functions \( \Theta_r \in \mathcal{I}_n(\mathbb{D}) \) and \( E_{r,B} \in \mathcal{E}_{n,B}(\mathbb{D}) \) such that
\[
\Phi = \Theta_r \cdot E_{r,B}.
\] (5.10)

We set
\[
E := d \cdot E_{r,B}.
\] (5.11)

According to Theorem 5.3 it follows that \( E_{r,B} \in \mathcal{E}_n(\mathbb{D}) \). Since \( d \in \mathcal{E}(\mathbb{D}) \) we get \( E \in \mathcal{E}_n(\mathbb{D}) \) from (5.11). Thus (i) is proved.

(ii) The factorizations \( F = \Theta_{r_1} \cdot E_{r_1} = \Theta_{r_2} \cdot E_{r_2} \) yield the factorizations
\[
\Theta_{r_1} \cdot E_{r_1,B} = \Theta_{r_2} \cdot E_{r_2,B} = \Phi,
\] (5.12)
upon setting \( E_{r_1,B} := d \cdot E_{r_1} \) and \( E_{r_2,B} := d \cdot E_{r_2} \) and invoking (5.9).

From \( \Phi \in \mathcal{G}_{n \times n}(\mathbb{D}) \), (5.12) and its definition it is clear that
\[
E_{r_1,B}, E_{r_2,B} \in \mathcal{E}_n(\mathbb{D}) \cap \mathcal{G}_{n \times n}(\mathbb{D}).
\]
Thus, from Theorem 3.2 we get \( E_{r_1,B}, E_{r_2,B} \in \mathcal{E}_{n,B}(\mathbb{D}) \). Now part (ii) of Theorem 5.2 provides the existence of a unitary matrix satisfying \( \Theta_{r_2} = \Theta_{r_1} \cdot V \) and \( E_{r_2,B} = V^* \cdot E_{r_1,B} \).

Hence,
\[
E_{r_2} = \frac{1}{d} \cdot E_{r_2,B} = \frac{1}{d} \cdot V^* \cdot E_{r_1,B} = V^* \cdot E_{r_1}.
\]
Thus, (ii) is proved.

Assertions (iii) and (iv) can be established analogously.

COROLLARY 5.1. Let \( F \in \mathcal{N}_n^+(\mathbb{D}) \) be such that \( \det F \neq 0 \). Then there exist functions \( B_1 \in \mathcal{I}_{n,B}(\mathbb{D}), S_1 \in \mathcal{I}_{n,s}(\mathbb{D}) \) and \( E_1 \in \mathcal{E}_n(\mathbb{D}) \) (resp. \( B_2 \in \mathcal{I}_{n,B,r}(\mathbb{D}), S_2 \in \mathcal{I}_{n,s}(\mathbb{D}) \) and \( E_2 \in \mathcal{E}_n(\mathbb{D}) \)) such that
\[
F = B_1 \cdot S_1 \cdot E_1 \quad \text{(resp. } F = E_2 \cdot S_2 \cdot B_2). \]

PROOF. The assertion follows immediately by combining Theorem 4.1 and Theorem 5.4.

It should be mentioned that using deep results and methods of V. Potapov [Pot] an alternate approach to Theorem 5.4 and Corollary 5.1 was presented by J.P. Ginzburg [Gi1]. His result contains also a multiplicative integral representation for the outer factor and the singular inner component.

The following theorem provides a useful characterization of the case that the inner component in the inner-outer factorization of a given function from \( \mathcal{N}_n^+(\mathbb{D}) \) is a Blaschke-Potapov product.

THEOREM 5.5. Let \( F \in \mathcal{N}_n^+(\mathbb{D}) \) be such that \( \det F \neq 0 \). Suppose that the functions \( \Theta_r, \Theta_l \in \mathcal{I}_n(\mathbb{D}) \) and \( E_r, E_l \in \mathcal{E}_n(\mathbb{D}) \) satisfy \( \Theta_r \cdot E_r = E_l \cdot \Theta_l = F \).

Then the following statements are equivalent:

(i) \( \Theta_r \in \mathcal{I}_{n,B,r}(\mathbb{D}) \).
PROOF. In view of the fact that $E_r, E_l \in \mathfrak{C}_n(\mathbb{D})$, the functions $\det E_r$ and $\det E_l$ are outer. Moreover, since $\Theta_r, \Theta_l \in \mathfrak{I}_n(\mathbb{D})$, part (b) of Theorem 4.2 implies that the functions $\det \Theta_r, \det \Theta_l$ are inner. From part (d) of Theorem 4.2 it follows that (i) (resp. (ii)) holds if and only if $\det \Theta_r$ (resp. $\det \Theta_l$) is a Blaschke product. According to Lemma 4.4 this is equivalent to

$$\lim_{s \to 1^{-}} \int_{\mathbb{T}} \ln |\det \Theta_r(t)| \ m(dt) = 0$$  \hspace{1cm} (5.13)

(resp.

$$\lim_{s \to 1^{-}} \int_{\mathbb{T}} \ln |\det \Theta_l(t)| \ m(dt) = 0.$$  \hspace{1cm} (5.14)

From the multiplicative decomposition $F = \Theta_r \cdot E_r$ (resp. $F = E_l \cdot \Theta_l$) it follows immediately that (5.13) (resp. (5.14)) is equivalent to (iii).

Thus, the statements (i) - (iii) are equivalent.

**REMARK 5.8.** It is instructive to compare statement (iii) in Theorem 5.5 with the inequality (1.15) which is fulfilled for an arbitrary function $F$ from $\mathfrak{N}^+_{n}(\mathbb{D})$.

### 6. AN ANALOGUE OF FROSTMAN’S THEOREM FOR MATRIX FUNCTIONS OF THE SMIRNOV CLASS

Let $f$ be a nonconstant function from the Smirnov class $\mathfrak{N}^+(\mathbb{D})$. For $\lambda \in \mathbb{C}$ the function

$$f_{\lambda} := f - \lambda$$  \hspace{1cm} (6.1)

clearly belongs to $\mathfrak{N}^+(\mathbb{D})$ too. Thus, there exists an inner function $\theta_{\lambda}$ and an outer function $e_{\lambda}$ such that

$$f_{\lambda} = \theta_{\lambda} \cdot e_{\lambda}.$$  \hspace{1cm} (6.2)

It will turn out that in some sense ”the typical situation” corresponds to the case that the inner function $\theta_{\lambda}$ in (6.2) is a Blaschke product. The set of all $\lambda \in \mathbb{C}$ for which $\theta_{\lambda}$ is not a Blaschke product is very thin. (A remarkable result of this type goes back to Frostman [Fr].) The corresponding notion of thinness can be formulated in terms of potential theory. For this reason, now we recall some notions of potential theory.

Suppose that $\nu$ is a nonnegative Borel measure with compact support. For all $\xi \in \mathbb{C}$ the integral

$$U^{(\nu)}(\xi) := \int_{\mathbb{C}} \ln |\xi - \lambda| \ \nu(d\lambda)$$  \hspace{1cm} (6.3)

is then well-defined and takes its values in $[-\infty, \infty)$. The function $U^{(\nu)} : \mathbb{C} \to [-\infty, \infty)$ is called the logarithmic potential of $\nu$. A Borel measure $\nu$ on $\mathbb{C}$ is said to be nontrivial if it is not the zero measure. If $K$ is a Borel subset of $\mathbb{C}$, the Borel measure $\nu$ is said to be
concentrated on $K$ if $\nu(C \setminus K) = 0$. By definition, a Borel subset $K$ on $C$ is called thin if for each nontrivial Borel measure $\nu$ which is concentrated on $K$ the associated logarithmic potential $U^{(\nu)}$ is not bounded from below, or in other words if

$$\inf_{\xi \in C} U^{(\nu)}(\xi) = -\infty.$$ 

If $K$ is not thin, then there exists a nontrivial Borel measure $\nu$ which is concentrated on $K$ and satisfies

$$\inf_{\xi \in C} U^{(\nu)}(\xi) > -\infty. \quad (6.4)$$

The notion of logarithmic capacity is introduced in potential theory. More precisely, this means that with each Borel subset $K$ of $C$ there is associated a nonnegative number $\text{cap}K$ which is called the logarithmic capacity of $K$. It turns out that a Borel subset $K$ of $C$ is thin if and only if $\text{cap}K = 0$. In other words, if $\text{cap}K > 0$, then there exists a nontrivial Borel measure $\nu$ which is concentrated on $K$ and satisfies condition (6.4). If $\text{cap}K > 0$, then amongst all the nontrivial Borel measures $\nu$ which are concentrated on $K$ and satisfy (6.4) there is a distinguished probability measure $\nu_K$, the so-called equilibrium measure of $K$. This measure $\nu_K$ is a solution of several natural extremal problems. (If $\text{cap}K = 0$ the equilibrium measure is not defined.)

The logarithmic potential is not always continuous on $C$ but only upper semicontinuous on $C$. More precisely, for all $\xi \in C$,

$$\liminf_{\xi' \to \xi} U^{(\nu)}(\xi') \leq U^{(\nu)}(\xi).$$

Although it is bounded below, the logarithmic potential of the equilibrium measure need not be continuous on $C$. If the set $K$ is “bad” there are so-called irregular points. Nevertheless it can be proved (see de la Vallée Poussin [LVP2], [LVP3]) that if $\text{cap}K > 0$, then there exists a nontrivial nonnegative measure which is concentrated on $K$ and for which the associated logarithmic potential is continuous on $C$ (as already mentioned, the equilibrium measure $\nu_K$ can not generally be used for this purpose). We will not enter into such detailed and rather delicate potential - theoretical considerations. To avoid them we give the following definition.

**DEFINITION 6.1.** A bounded Borel subset $K$ of $C$ is said to have positive logarithmic capacity if there exists a nontrivial Borel measure $\nu$ which is concentrated on $K$ and for which the associated logarithmic potential is continuous on $C$.

Clearly, if $K_1 \subseteq K_2$ and $K_1$ is a set of positive logarithmic capacity, then $K_2$ is also a set of positive logarithmic capacity.

**LEMMA ON THE CAPACITY OF AN INTERVAL.** Every interval of the complex plane is a set of positive logarithmic capacity.

**PROOF.** Without loss of generality we can assume that the considered interval is a subinterval $(\alpha, \beta)$ of the real axis where $-\infty < \alpha < \beta < \infty$. Now we take for $\nu$ the restriction of one dimensional Lebesgue measure to this interval $(\alpha, \beta)$. The function $U^{(\nu)} : C \to [-\infty, \infty)$ which is defined by the rule

$$\xi \to \int_{(\alpha, \beta)} \ln|\xi - \lambda| \nu(d\lambda).$$

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is continuous in $C$. This can be checked in several ways, e.g. one can compute explicitly and then obtain the continuity of $U^{(\nu)}$ by direct estimates.

W. Rudin [Ru1] (see also section 3.6 of the monograph [Ru2]) proved the following fact which generalizes Frostman’s original result:

Let $f \in \mathfrak{N}^+(\mathbb{D})$ with $f \not\equiv 0$ and let $K$ be some bounded Borel subset of $\mathbb{C}$ with positive logarithmic capacity. Then there exist a $\lambda \in K$ such that the inner factor in the multiplicative decomposition (6.2) is a Blaschke product. (Indeed, W. Rudin obtained a more general result which is formulated for the Smirnov class $\mathfrak{N}^+(\mathbb{D}^p)$ in the polydisc $\mathbb{D}^p$. This class is a natural analogue of $\mathfrak{N}^+(\mathbb{D})$ and coincides with it in the case $p = 1$.) It should be mentioned that S.A. Vinogradov [Vin] independently obtained such a generalization of Frostman’s theorem too.

REMARK 6.1. Let $F \in \mathfrak{N}_n^+(\mathbb{D})$ and define $F_\lambda := F - \lambda \cdot I_n$ for $\lambda \in \mathbb{C}$. Then the set $M_F := \{ \lambda \in \mathbb{C} : \det(F_\lambda) \equiv 0 \}$ is finite.

Now we formulate our main result.

**THEOREM 6.1.** Let $F \in \mathfrak{N}_n^+(\mathbb{D})$. Assume that for $\lambda \in \mathbb{C} \setminus M_F$ the functions $\Theta_{\lambda,r} \in \mathcal{J}_n(\mathbb{D})$ and $E_{\lambda,r} \in \mathcal{E}_n(\mathbb{D})$ are factors in the multiplicative decomposition

$$F_\lambda = \Theta_{\lambda,r} \cdot E_{\lambda,r}.$$ 

Suppose that $K$ is a bounded Borel subset of $\mathbb{C}$ with positive logarithmic capacity. Then there exists a point $\lambda \in K \cap (\mathbb{C} \setminus M_F)$ for which $\Theta_{\lambda,r}$ is a Blaschke-Potapov product.

**COROLLARY 6.1.** The set of all $\lambda \in \mathbb{C} \setminus M_F$ for which $\Theta_{\lambda,r}$ is a Blaschke-Potapov product is dense in $\mathbb{C}$.

**PROOF:** Combine Theorem 6.1 and the Lemma on the capacity of an interval.

In order to follow the strategy of W. Rudin’s proof we shall need to introduce a number of classes of scalar functions of several variables.

**DEFINITION 6.2.** A function $\sigma : \mathbb{C}^n \to \mathbb{R}$ is called symmetric if for all permutations

$$\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$$

and all $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ the relation

$$\sigma((x_{i_1}, \ldots, x_{i_n})^T) = \sigma((x_1, \ldots, x_n)^T)$$

is valid.

In view of Definition 6.2 the following object is well-defined.

**DEFINITION 6.3.** Let $\sigma : \mathbb{C}^n \to \mathbb{R}$ be a symmetric function. Then the function $\varphi_\sigma : \mathcal{M}_n \to \mathbb{R}$ which is defined by the rule

$$A \to \sigma((l_1(A), \ldots, l_n(A))^T),$$

where $(l_j(A))_{j=1}^n$ are the roots of the characteristic polynomial of $A$ (taking into account their algebraic multiplicities), is called the function of matrix argument which is generated
LEMMA 6.1. Suppose that $\sigma : \mathbb{C}^n \rightarrow \mathbb{R}$ is a continuous symmetric function. Then $\varphi_\sigma$ is a continuous function.

PROOF. The lemma is an immediate consequence of Theorem 5.1 from Chapter II in Kato’s monograph [Ka]. (See there especially formula (5.3) and the text following it.)

If the symmetric function $\sigma : \mathbb{C}^n \rightarrow \mathbb{R}$ is a polynomial or a rational function in $n$ variables $x_1, \ldots, x_n$, then it can be expressed as a polynomial or a rational function of the elementary symmetric functions. In this case the function $\varphi_\sigma$ is a polynomial or a rational function of the elements of the matrix variable.

We introduce now a potential of the matrix argument. Roughly speaking, we insert a matrix argument in formula (6.3) instead of the complex variable.

REMARK 6.2. Suppose that $A \in \mathbb{M}_n$. Then the function $h_A : \mathbb{C} \rightarrow \mathbb{R}$ which is defined by $\lambda \rightarrow |\det (A - \lambda I_n)|$ is continuous. Hence, the function $\ln h_A$ is continuous and locally bounded above. If $\nu$ is a finite Borel measure on $\mathbb{C}$ with compact support, then the function $\Phi^{(\nu)} : \mathbb{M}_n \rightarrow [-\infty, \infty)$ with

$$
\Phi^{(\nu)}(A) := \int_{\mathbb{C}} \ln |\det (A - \lambda I_n)| \, \nu(d\lambda)
$$

(6.5)

is well-defined.

DEFINITION 6.4. Suppose that $\nu$ is a finite Borel measure on $\mathbb{C}$ with compact support. Then the function $\Phi^{(\nu)} : \mathbb{M}_n \rightarrow [-\infty, \infty)$ which is defined by (6.5) is called the potential of the matrix argument associated with $\nu$.

Assume that $\nu$ is a finite Borel measure on $\mathbb{C}$ with compact support. Let $U^{(\nu)}$ denote the logarithmic potential of $\nu$. Let $A \in \mathbb{M}_n$ and let $(l_k(A))_{k=1}^n$ be roots of the characteristic polynomial of $A$. For $\lambda \in \mathbb{C}$ we then have

$$
\ln |\det (A - \lambda I_n)| = \sum_{k=1}^n \ln |l_k(A) - \lambda|.
$$

(6.6)

Hence, upon taking (6.3) into account we get

$$
\Phi^{(\nu)}(A) = \sum_{k=1}^n U^{(\nu)}(l_k(A)).
$$

(6.7)

We define $\sigma^{(\nu)} : \mathbb{C}_n \rightarrow [-\infty, \infty)$ via

$$
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\rightarrow \sum_{k=1}^n U^{(\nu)}(x_k).
$$

(6.8)

Obviously, the function $\sigma^{(\nu)}$ is symmetric. From Definition 6.3, (6.7) and (6.8) we infer that

$$
\Phi^{(\nu)}(A) = \varphi_{\sigma^{(\nu)}}(A).
$$

(6.9)
LEMMA 6.2. Suppose that \( \nu \) is a finite Borel measure on \( \mathbb{C} \) with compact support such that the associated logarithmic potential \( U^{(\nu)} \) is continuous on \( \mathbb{C} \). Then the function \( \Phi^{(\nu)} : \mathcal{M}_n \to [-\infty, \infty) \) which is defined by (6.5) is continuous on \( \mathcal{M}_n \).

PROOF. Indeed, from (6.8) it follows that \( \sigma^{(\nu)} \) is a continuous function on \( \mathbb{C} \). Then in view of (6.9) and Lemma 6.1 the assertion follows.

DEFINITION 6.5. Let \( \nu \) be a finite Borel measure on \( \mathbb{C} \) with compact support. Then the functions \( \Phi^{(\nu)}_+ : \mathcal{M}_n \to [0, \infty) \) and \( \Phi^{(\nu)}_- : \mathcal{M}_n \to [0, \infty) \) are defined via the formulas

\[
\Phi^{(\nu)}_+(A) := \int_{\mathbb{C}} \ln^+ |\det (A - \lambda I_n)| \nu(d\lambda) \tag{6.10}
\]

and

\[
\Phi^{(\nu)}_-(A) := \int_{\mathbb{C}} \ln^- |\det (A - \lambda I_n)| \nu(d\lambda), \tag{6.11}
\]

respectively.

LEMMA 6.3. Suppose that \( \nu \) is a finite Borel measure on \( \mathbb{C} \) with compact support. Then the function \( \Phi^{(\nu)}_+ \) defined by (6.10) is continuous on \( \mathcal{M}_n \).

PROOF. The function \( f : \mathcal{M}_n \times \mathbb{C} \to [0, \infty) \) which is defined by

\[
f(A, \lambda) := |\det (A - \lambda I_n)|
\]

is continuous on \( \mathcal{M}_n \times \mathbb{C} \). Since the function \( \ln^+ := \max\{\ln, 0\} \) is continuous on \([0, \infty)\) the composition mapping \( \ln^+ f \) is continuous on \( \mathcal{M}_n \times \mathbb{C} \). From this we infer that the function \( \Phi^{(\nu)}_+ \) is continuous on \( \mathcal{M}_n \).

LEMMA 6.4. Suppose that \( \nu \) is a finite Borel measure on \( \mathbb{C} \) with compact support such that the associated logarithmic potential \( U^{(\nu)} \) is continuous on \( \mathbb{C} \). Then the function \( \Phi^{(\nu)}_- \) which is defined by (6.11) is continuous on \( \mathcal{M}_n \); it is also bounded:

\[
\sup_{A \in \mathcal{M}_n} \Phi^{(\nu)}_-(A) < +\infty. \tag{6.12}
\]

PROOF. From Definitions 6.4 and 6.5 we get the identity

\[
\Phi^{(\nu)} = \Phi^{(\nu)}_+ - \Phi^{(\nu)}_. \tag{6.13}
\]

In view of Lemma 6.2 the function \( \Phi^{(\nu)} \) is continuous whereas Lemma 6.3 provides the continuity of \( \Phi^{(\nu)}_+ \). Thus, (6.13) shows the continuity of \( \Phi^{(\nu)}_- \). We define the functions \( U^{(\nu)}_+ : \mathbb{C} \to [0, \infty) \) and \( U^{(\nu)}_- : \mathbb{C} \to [0, \infty) \) by

\[
U^{(\nu)}_+(\xi) := \int_{\mathbb{C}} \ln^+ |\xi - \lambda| \nu(d\lambda) \tag{6.14}
\]

and

\[
U^{(\nu)}_-(\xi) := \int_{\mathbb{C}} \ln^- |\xi - \lambda| \nu(d\lambda). \tag{6.15}
\]
Combining (6.3), (6.14) and (6.15) we see that
\[ U^{(\nu)} = U_+^{(\nu)} - U_-^{(\nu)}. \] (6.16)

Since the function \( U^{(\nu)} \) is continuous by assumption and since the function \( U_+^{(\nu)} \) is always continuous (by Lemma 6.3 with \( n = 1 \)) the continuity of \( U_-^{(\nu)} \) follows from (6.16). If \( (r_k)_{k=1}^n \) is a sequence from \([0, \infty)\), then clearly
\[ \ln^- \left( \prod_{k=1}^n r_k \right) \leq \sum_{k=1}^n \ln^- r_k. \] (6.17)

Let \( A \in \mathcal{M}_n \) and let \((l_k(A))_{k=1}^n\) be the roots of the characteristic polynomial of \( A \). In view of (6.6) we get
\[ \ln^- |\det (A - \lambda I_n)| = \ln^- \left( \prod_{k=1}^n |l_k(A) - \lambda| \right). \] (6.18)

From (6.17), (6.18) and (6.15) we infer that
\[ \Phi_+^{(\nu)}(A) \leq \sum_{k=1}^n U_-^{(\nu)}(l_k(A)). \]

Hence,
\[ \sup_{A \in \mathcal{M}_n} \Phi_-^{(\nu)}(A) \leq n \cdot \sup_{\xi \in \mathbb{C}} U_-^{(\nu)}(\xi). \] (6.19)

Now it remains to prove that our assumptions ensure that
\[ \sup_{\xi \in \mathbb{C}} U_-^{(\nu)}(\xi) < \infty \] (6.20)
is fulfilled. If \( \xi \in \mathbb{C} \) satisfies
\[ |\xi| \geq 1 + \sup_{\lambda \in \text{supp } \nu} |\lambda|, \] (6.21)
then using (6.15) we see that
\[ U_-^{(\nu)}(\xi) = 0. \] (6.22)

Now the continuity of \( U_-^{(\nu)} \), (6.21), (6.22) and a classical theorem due to Weierstrass yield (6.20). The lemma is proved.

**REMARK 6.3.** If \( a, b \in [0, \infty) \), then
\[ \ln^+ (a + b) \leq \ln^+ a + \ln^+ b + \ln 2. \]

**REMARK 6.4.** Let \( A \in \mathcal{M}_n \). Then \( |\det A| \leq \|A\|^n \).

**REMARK 6.5.** Let \( A \in \mathcal{M}_n \) and \( \lambda \in \mathbb{C} \). Then
\[ \ln^+ |\det [A - \lambda I_n]| \leq n \cdot [\ln^+ \|A\| + \ln^+ |\lambda| + \ln 2]. \]
Indeed, using remarks 6.4 and 6.3 we obtain
\[ \ln^+ | \det [A - \lambda I_n] | \leq \ln^+ \| A - \lambda I_n \|^n \]
\[ \leq n \cdot \ln^+ \| A - \lambda I_n \| \leq n \cdot \ln^+ \| A \| + | \lambda | \leq n \cdot \ln^+ \| A \| + \ln 2. \]

PROOF OF THEOREM 6.1. Let \( \lambda \in \mathbb{C} \). For \( r \in [0, 1) \) we define
\[ v_r(\lambda) := \int_{\mathbb{T}} \ln | \det [F(rt) - \lambda I_n] | \ m(dt). \] (6.23)
Assume that \( r_1, r_2 \in [0, 1) \) satisfy \( r_1 \leq r_2 \). Since the function \( \det [F - \lambda I_n] \) is holomorphic we get \( v_{r_1}(\lambda) \leq v_{r_2}(\lambda) \). Thus, the limit
\[ v_{1-0}(\lambda) := \lim_{r \to 1-0} v_r(\lambda) \] (6.24)
exists. Define
\[ v(\lambda) := \int_{\mathbb{T}} \ln | \det [F(t) - \lambda I_n] | \ m(dt). \] (6.25)
If we apply inequality (1.15) to the function \( F - \lambda I_n \), then using (6.23) - (6.25) we obtain
\[ v_{1-0}(\lambda) \leq v(\lambda). \] (6.26)
According to Theorem 5.5, equality holds in (6.26) for those and only those \( \lambda \in \mathbb{C} \setminus M_F \) for which the inner factor \( \Theta_{\lambda,r} \) is a Blaschke-Potapov product. Consequently, Theorem 5.5 reduces the question which is discussed in Theorem 6.1 to the study of the structure of the set of all \( \lambda \in \mathbb{C} \setminus M_F \) for which the inequality in (6.26) is strict. More formally, we will show that if \( K \) is a bounded Borel subset of positive logarithmic capacity then there exists a point \( \lambda \in K \cap (\mathbb{C} \setminus M_F) \) such that equality holds true in (6.26). Furthermore, we will show that if \( K \) is such a set and if \( \nu \) is a finite Borel measure on \( \mathbb{C} \) which is concentrated on \( K \), i.e.,
\[ \nu(\mathbb{C} \setminus K) = 0, \]
and if the associated logarithmic potential \( U^{(\nu)} \) (see (6.3)) is continuous in \( \mathbb{C} \), then the identity
\[ \int_{\mathbb{C}} \left[ v(\lambda) - v_{1-0}(\lambda) \right] \nu(d\lambda) = 0 \] (6.27)
is valid. Clearly, from (6.26) and (6.27) it will follow that \( v(\lambda) = v_{1-0}(\lambda) \) for almost all \( \lambda \) with respect to \( \nu \). In particular, there exists a \( \lambda \in K \cap (\mathbb{C} \setminus M_F) \) for which \( v(\lambda) = v_{1-0}(\lambda) \) is satisfied. Now we are going to prove (6.27). According to (6.23) for \( r \in [0, 1) \) and \( \lambda \in \mathbb{C} \) we have
\[ \int_{\mathbb{T}} \ln^+ | \det [F(rt) - \lambda I_n] | \ m(dt) - \int_{\mathbb{T}} \ln^- | \det [F(rt) - \lambda I_n] | \ m(dt) = v_r(\lambda). \] (6.28)
In view of Remark 6.5, the inequality

\[ \ln^+ |\det [F(rt) - \lambda I_n]| \leq n \cdot [\ln^+ \|F(rt)\| + \ln^+ |\lambda| + \ln 2] \quad (6.29) \]

holds for \( r \in [0, 1), \lambda \in \mathbb{C} \) and \( t \in \mathbb{T} \). For \( \lambda \in \mathbb{C} \) and \( r \in [0, 1) \) the function \( G_{\lambda,r} : \mathbb{T} \to [0, \infty) \) is defined by

\[ G_{\lambda,r}(t) := \det [F(rt) - \lambda I_n]. \quad (6.30) \]

Suppose that \( \lambda \in \mathbb{C} \) is fixed. Then from (6.29) and (6.30) we infer that the family \( (\ln^+ |G_{\lambda,r}|)_{r \in [0,1)} \) is uniformly \( m \)-integrable. Clearly, for almost all \( t \in \mathbb{T} \) with respect to \( m \) we have

\[ \lim_{r \to 1^-} \ln^+ |\det [F(rt) - \lambda I_n]| = \ln^+ |\det [F(t) - \lambda I_n]|. \]

Thus, using Vitali’s convergence theorem again, we get

\[ \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ |\det [F(rt) - \lambda I_n]| \, m(dt) = \int_{\mathbb{T}} \ln^+ |\det [F(t) - \lambda I_n]| \, m(dt). \quad (6.31) \]

Taking into account (6.31) we obtain the formula

\[ \int_{\mathbb{T}} \ln^+ |\det [F(t) - \lambda I_n]| \, m(dt) - \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ |\det [F(rt) - \lambda I_n]| \, m(dt) \]

\[ = v_1(\lambda) \quad (6.32) \]

by letting \( r \to 1^- \) in (6.28), where the limit of the second term on the left hand side of (6.32) necessarily exists. From (6.25) and (6.32) it follows that

\[ v(\lambda) - v_1(\lambda) = \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ |\det [F(rt) - \lambda I_n]| \, m(dt) \]

\[ - \int_{\mathbb{T}} \ln^+ |\det [F(t) - \lambda I_n]| \, m(dt). \quad (6.33) \]

In general, the family \( (\ln^- |G_{\lambda,r}|)_{r \in [0,1)} \) is not uniformly \( m \)-integrable. For this reason, the right hand side in (6.33) is not necessarily zero. (However, according to Fatou’s theorem this difference is nonnegative.) Nevertheless, it will turn out that after applying the following averaging procedure the right hand side of (6.33) vanishes. Suppose that \( \nu \) is a finite nonnegative measure with compact support for which the associated logarithmic potential \( U^{(\nu)} \) is continuous. We will prove that

\[ \int_{\mathbb{C}} \left( \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^+ |\det [F(rt) - \lambda I_n]| \, m(dt) - \int_{\mathbb{T}} \ln^- |\det [F(t) - \lambda I_n]| \, m(dt) \right) \nu(d\lambda) = 0. \quad (6.34) \]

Using Fubini’s theorem and (6.11) we get

\[ \int_{\mathbb{C}} \left( \int_{\mathbb{T}} \ln^- |\det [F(t) - \lambda I_n]| \, m(dt) \right) \nu(d\lambda) = \int_{\mathbb{T}} \left( \int_{\mathbb{C}} \ln^- |\det [F(t) - \lambda I_n]| \nu(d\lambda) \right) m(dt) \]
\[
= \int_{\mathbb{T}} \Phi_\nu(F(t)) \, m(dt).
\] (6.35)

In view of (6.12) it follows that
\[
\int_{C} \left( \int_{\mathbb{T}} \ln^{-} \left| \det [F(t) - \lambda I_n] \right| \, m(dt) \right) \nu(d\lambda) < \infty.
\] (6.36)

Now we integrate identity (6.33) with respect to \(\nu\) and use (6.36) to rewrite the integral of the difference as the difference of integrals. Then we rewrite the second term using (6.35) and apply Fatou's theorem to the first one. Finally, we use Fubini's theorem and (6.11) to rewrite the first term. This leads us to the following estimate
\[
\int_{C} \left[ v(\lambda) - v_{1-0}(\lambda) \right] \nu(d\lambda)
= \int_{C} \left[ \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^{-} \left| \det [F(rt) - \lambda I_n] \right| \, m(dt) \right] \nu(d\lambda)
= \int_{C} \left[ \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^{-} \left| \det [F(rt) - \lambda I_n] \right| \, m(dt) \right] \nu(d\lambda)
\]
\[
- \int_{C} \left[ \int_{\mathbb{T}} \ln^{-} \left| \det [F(t) - \lambda I_n] \right| \, m(dt) \right] \nu(d\lambda)
\]
\[
= \int_{C} \left[ \lim_{r \to 1^-} \int_{\mathbb{T}} \ln^{-} \left| \det [F(rt) - \lambda I_n] \right| \, m(dt) \right] \nu(d\lambda)
\]
\[
= \lim_{r \to 1^-} \int_{\mathbb{T}} \Phi_\nu(F(rt)) \, m(dt) - \int_{\mathbb{T}} \Phi_\nu(F(t)) \, m(dt).
\] (6.37)

According to Lemma 6.4 and our choice of \(\nu\), the function \(\Phi_\nu\) is continuous. Thus, for almost all \(t \in \mathbb{T}\) with respect to \(m\) we get
\[
\lim_{r \to 1^-} \Phi_\nu(F(rt)) = \Phi_\nu(F(t)).
\] (6.38)

Since the function \(\Phi_\nu\) is also bounded (see Lemma 6.4), Lebesgue's theorem on dominated convergence and (6.38) guarantee
\[
\lim_{r \to 1^-} \int_{\mathbb{T}} \Phi_\nu(F(rt)) \, m(dt) = \int_{\mathbb{T}} \Phi_\nu(F(t)) \, m(dt).
\] (6.39)
Thus, combining (6.37) and (6.39) we obtain (6.27).

As explained above this completes the proof. □

**COMMENTS ON THEOREM 6.1.** These comments are intended to clarify the function-theoretic content of Theorem 6.1. Let \( t \in T \). Then the function \( G : C \to [-\infty, \infty) \) defined by

\[
G(\lambda) := \ln |\det [F(t) - \lambda I_n]|
\]

is subharmonic. Let \( r \in [0, 1) \) and let the function \( G_r : C \to [-\infty, \infty) \) be defined by

\[
G_r(\lambda) := G_{\lambda,r}(t), \\
G_{\lambda,r} \text{ given in (6.30)}.
\]

Then \( G_r \) is subharmonic too. From standard theorems on integrating parametric families of subharmonic functions (see e.g. Ronkin [Ron, Ch.I, §5] or Lelong and Gruman [LG, Appendix I, Proposition I.14]) it follows that the function \( v \) defined in (6.25) is subharmonic and that for each \( r \in [0, 1) \) the function \( v_r \) defined in (6.23) is also subharmonic. Since the family \((v_r)_{r \in [0,1)}\) increases monotonically with \( r \), the function \( v_{1-0} \) defined in (6.24) is the upper envelope of this family. The function \( v_{1-0} \) is not necessarily subharmonic but its regularization \( v^*_1 \) is defined by

\[
v^*_1(\lambda) := \lim_{\lambda' \to \lambda} v_{1-0}(\lambda').
\]

The identity \( v^*_1(\lambda) = v(\lambda) \) for all \( \lambda \) belonging to some dense subset of \( C \) clearly follows from the identity

\[
\int_I [v(\lambda) - v_{1-0}(\lambda)] \mu(d\lambda) = 0,
\]

where \( I \) is an arbitrary one dimensional interval of \( C \) and \( \mu \) is the one dimensional Lebesgue measure. The use of H. Cartan’s theorem on upper envelopes of families of subharmonic functions.
functions for proving the smallness (in the sense of capacity) of exceptional sets has many traditions in the theory of functions of one or several complex variables. The application of the recently created complex potential theory, in particular the analogue of H. Cartan’s theorem for the upper envelope of a family of plurisubharmonic functions (see Bedford and Taylor [BT, Section 7] and Sadullaev’s survey paper [Sad]) enables one to derive results on families of matrix-valued functions of a more general type, namely on families which depend holomorphically on \( p \) variables where \( p \in \mathbb{N} \).

Finally, we turn our attention to the left version of our main result.

**THEOREM 6.2.** Let \( F \in \mathcal{H}^+_n(D) \). Assume that for \( \lambda \in \mathbb{C} \setminus M_F \) the functions \( \Theta_{\lambda,l} \in \mathcal{J}_n(D) \) and \( E_{\lambda,l} \in \mathcal{E}_n(D) \) are factors in the multiplicative decomposition

\[
F_\lambda = E_{\lambda,l} \cdot \Theta_{\lambda,l}.
\]

Suppose that \( K \) is a bounded Borel subset of \( \mathbb{C} \) with positive logarithmic capacity. Then there exists a \( \lambda \in K \cap (\mathbb{C} \setminus M_F) \) for which \( \Theta_{\lambda,l} \) is a Blaschke-Potapov product.

**PROOF.** Use Theorem 6.1, Remark 3.2 and Remark 4.2. \( \square \)

**COROLLARY 6.2.** The set of all \( \lambda \in \mathbb{C} \setminus M_F \) for which \( \Theta_{\lambda,l} \) is a Blaschke-Potapov product is dense in \( \mathbb{C} \).

For further matricial generalizations of the classical theorems of Frostman [Fr], Heins [Hei] and Rudin [Ru1] we refer the reader to the papers [Gi6] and [GiTa1] - [GiTa3].

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