Equation of State, Radial Flow and Freeze-out in High Energy Heavy Ion Collisions

C.M. Hung and E. Shuryak
Physics Department, State University of New York at Stony Brook, NY 11794-3800
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Abstract

We have shown that recent experimental data on radial flow, both from AGS and SPS energies, are in agreement with the Equation of State (EOS) including the QCD phase transition. New hydro-kinetic model (HKM) is developed, which incorporates hydrodynamical treatment of expansion and proper kinetics of the freeze-out. We show that the freeze-out surfaces for different secondaries and different collisions are very different, and they are not at all isotherms $T = \text{const}$ (as was assumed in most previous hydro works). Comparison of HKM results with cascade-based event generator RQMD is also made in some details: we found that both EOS and flow are in rather good agreement, while the space-time picture is still somewhat different.

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I. INTRODUCTION

One of the main physics goals of high energy nuclear collisions includes a test of whether for heavy enough ions at the AGS/SPS energy range (10-200 GeV/A) production of (locally) equilibrated hot/dense hadronic matter really takes place. We do know that at the early stages of those collisions (few fm/c after the first impact) a very large energy density of the order of several $GeV/fm^3$ is actually reached. How rapidly it is equilibrated and whether new phase of matter - Quark-Gluon Plasma (QGP) - is indeed produced remains unclear. One possible strategy to answer those questions is relying on special rare processes happening at earlier stages, the e/m probes \[1\], or $J/\psi$ suppression \[2\]. In both directions we have recent exciting experimental findings \[3,4\].

This work is however devoted to hadronic observables related with production of the usual secondaries, $\pi, N, K$ etc. It is widely believed that their spectra are not actually sensitive to questions mentioned above: and indeed, as the produced multi-particle system expands and cools, the re-scattering erases most traces of the dense stage. Nevertheless, those which are accumulated during the expansion remain, and thus provide valuable information about the state of matter through its evolution.

The central phenomenon of such kind discussed in this paper is a collective flow. Its multiple studies at Bevalac/SIS energies ($E/A \sim 1 GeV$) have shown a number of interesting effects. However, it was concluded that nuclear matter do not really reach equilibration under such conditions.

In contrast to that, in high energy pp (or even $e^+e^-$) collisions, the thermal description for particle spectra and composition works surprisingly well \[5\]. At the same time, (except maybe at very high energies) there is no observed collective radial flow in these cases (see \[6\] and the next section): this alone shows that the system is not truly macroscopic\[1\].

In contrast to that, data for heavy ion collisions show very strong flow. therefore suggesting that the excited system created do indeed behave as a truly macroscopic system. To test whether it is indeed so is the main physics objective of this paper. More specifically, we study whether available experimental data on heavy ion collisions in AGS/SPS energy domain are consistent with (so far semi-qualitative) information about the Equation of State (below EOS) of hot/dense hadronic matter as obtained from current lattice QCD.

The central phenomenon studied in this work is the so called radial (or axially symmetric) flow, observed in central collisions. Current data are now rich enough to allow systematic study of its collision energy and rapidity dependence, as well as dependence on the nuclear size (A-dependence) and the particular secondary particle involved. All those dependencies are discussed below, and to a large extent reproduced by our model.

Another, more practical objective of this paper is to create a next generation model for heavy ion collisions, to be called Hydro-Kinetic Model (HKM). It incorporates three

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1 An explanation suggested in \[5\] is that in pp/$e^+e^-$ collisions matter excitation is not strong enough to overcome the “bag pressure”, and small systems created have stabilised transverse size at some equilibrium value, and thus zero pressure. Modern models explaining these data use strings: those are precisely such objects. Large systems created in nuclear collisions must have positive pressure, and thus expand.
basic elements of the macroscopic approach – (i) thermodynamics of hadronic matter, (ii) hydrodynamics of its expansion, and (iii) realistic hadronic kinetics at the freeze-out. Most elements of the model have in fact been worked out in literature, some are new, but we think they are taken together for the first time.

The hydro-based works available in literature aimed more at proper parameterization of the initial conditions, which would then lead to \( y, p_t \) spectra comparable with data. Among recent papers let us mention \([11]\) which study the first few \( \text{fm}/c \) and attempt to derive the initial conditions from 3-liquid model and \([13]\) which has studied some specific EOS-related observables. Probably the closest in spirit to our work is recent paper \([14]\), in which the same freeze-out conditions is used. Unfortunately, its physical consequences are not studied in any details, and their method (referred to as “global” hydrodynamics) include unnecessary averaging, which significantly obscure them. To the extent we could trace them, our findings actually qualitatively agree with the results of \([14]\). In particular, we also found that the resonance gas EOS leads to too strong flow at SPS, while the softer EOS including the phase transition gives it about right.

Let us now comment on the relations between our approach and widely used cascade “event generators” (Fritjof, Venus, RQMD, ARC etc). Hydrodynamics and cascades were often treated as alternatives, and many people trust cascades much more, as those are “based on known physics”. In reality, both rather should be used as complementary tools.

The very fact that all event generators approximately work, in spite of huge differences between them (their tables of cross sections, lists of resonances included, etc, are different, some have strings or even color “ropes”) indicate that bulk results are insensitive to those differences. For example, particle composition appears to be rather well equilibrated, explaining insensitivity to details of the model in some observables. The simplest way to test which of those parameters are relevant is to vary the input parameters: unfortunately, very little work was so far done along this line. Considering flow, one should obviously have a look components of the stress tensor, the pressure \( p \) and energy density \( \epsilon \). Hadronic cascades (see \([28]\) for RQMD) have a very simple EOS, \( p/\epsilon \approx \text{const} \), typical to thermal resonance gas (see below).

Obviously cascades have a lot to say about later stages of the collisions, at the so called freeze-out stage where interactions stop, resonances decay etc. They also provide more detailed information (e.g. the degree of local chemical/thermal equilibration) which in principle help us understand the validity limits of the macroscopic variables/approaches. We will discuss many of these issues below.

At the same time, description based on hadronic cascade of the earlier stages of the collisions obviously has little theoretical justification, and fails in practice for sufficiently

\[2\] After a very long break, there was a workshop in Trento ECT, May 1997 devoted entirely to this subject. Its proceedings (which will appear as a series of papers in Heavy Ion Physics, 5 (1997).) should give rather complete description of recent activities.

\[3\] In practice, to our knowledge it was not even demonstrated that any of these cascade codes satisfy the detailed balance (say there are resonance decays and other 2 to many hadronic processes without their inverse), and that, even if given time, they do lead to correct thermal equilibrium.
high energies (SPS). “Event generators” therefore rely on specific models (color strings and their breaking, etc), introducing plenty of unknown parameters or even concepts (e.g. “color ropes”). What is even worst, these models have so far no connection to developments in non-perturbative QCD, say to lattice studies of QCD thermodynamics. They disregard such issues as chiral restoration and deconfinement, leading to disappearance the very objects they work with, hadrons and strings\(^4\). Hydro description, on the other hand, is much simpler and operates directly with EOS, so in this framework and can easily incorporate different scenarios (e.g., with or without the QCD phase transition).

Our last comment is practical: with experiments proceeding from light ions to heavy ones, and from the AGS/SPS to the RHIC/LHC energies we have to deal with many thousands of secondaries. Direct simulation of all their re-scattering are neither practical not necessary: as soon as the system are much larger than the interaction range, the system can be cut into parts (or “cells”) which evolve independently from each other. Furthermore, one may separate internal evolution (thermodynamics and kinetics) from cell’s motion (hydrodynamics), enormously simplifying the problem. As multiplicities grow cascades become more and more expensive, while the macroscopic approach becomes only more accurate: at some point going from one language to another becomes inevitable.

The paper is structured as follows. In section 2 we start with some phenomenological introduction into properties of the radial flow, setting the problem to be discussed below. In section 3 we consider thermodynamics of hadronic matter, using a rather standard model of resonance gas plus QGP with bag-model EOS. The important step is determination of the particular paths the volume elements of matter make in the phase diagram (e.g. temperature \(T\) - baryonic chemical potential \(\mu_b\)) during expansion. Then we determine effective EOS on these paths, to be used section 4 in hydro calculations. In this paper we would not discuss the non-equilibrium phenomena neither at formation stage nor during the passage of the phase transition. It is more important however to address kinetic phenomena at the end of hydro expansion, the so called freeze-out stage: this we do in considerable details in section 5. Here we separately discuss chemical and thermal freeze-out and discuss how the final spectra of secondaries are generated. Then we go to comparison of observables, and especially the radial flow, with experiment and cascades (RQMD), see section 6. Summary of the paper is contained in section 7.

\section{Flow: The Phenomenological Introduction}

First of all, in order to put things into proper perspective and introduce terminology, we remind that the collective flow can be observed as follows. (i) Axially symmetric radial and (ii) longitudinal flow exist even for central collisions. For non-zero impact parameter experiments have also seen clear signals for at least two non-zero harmonics in the angle \(\phi\), known as (iii) dipole and (iv) elliptic flow.

In this paper we study only the first of them, the radial flow, so let us now comment on

\footnote{Therefore their phenomenological success is even used as an argument against reality of the QCD phase transition itself: needless to say, we are strongly opposed to this point of view.}
others. The longitudinal flow was studied a lot in other hydro-based works [9]: we decided not to discuss it here in details. It is a parameterization rather than a real prediction: the issue is obscured by an uncertainty in initial conditions.

Asymmetric flow (iii) and (iv) is potentially very interesting, especially the elliptic one [15,16]. The difference between the elliptic and radial flow should mostly appear due to earlier stages: it is obviously an exciting subject for further work. (At the moment we however feel that it is too early, one should be able to get more details from experiment first. For recent summary see [17]).

The existence of radial flow in nuclear collisions was widely debated in literature for a decade. Phenomenological fits of the $p_t$ spectra of various secondaries by some (ad hoc) velocity profile (or even a single velocity value $v_t$) and the same decoupling temperature $T_f$ is possible, see [10]. Unfortunately, the data allow for multiple fits, with wide margin for the trade-off between $v_t$ and $T_f$. In particular, for heavy ions one can obtain equally good fits with $(T_f = 140 MeV, v_t = 0.4)$ and $(T_f = 120 MeV, v_t = 0.6)$. However, as we will show below, the model used was oversimplified. Even the main assumption that one should expect the same $v_t$ and $T_f$ for all secondaries is in obvious contradiction to elementary kinetics of the freeze-out.

More important is that rather rich experimental systematics is now emerging. In Fig.4 we show a collection of slopes from NA44 [5] and NA49 experiments at SPS, for $\pi, K, N, \phi, \Lambda, d$. The definition is
\[
E dN \over d^3p = C(y) \exp(-{m_t \over T(y)}); \quad m_t^2 = p_t^2 + m^2
\]

One major observation is a very strong mass dependence: the slopes show consistent growth with the particle mass. It is clear how collective flow may explain it: for heavier secondaries, the thermal motion is smaller and collective velocity $v_t$ starts to show up. (Or, alternatively, collective motion generates larger momentum $m v_t$ for larger $m$.) Note however, that there are no lines on this plot: as it will be clear from what follows, we do not believe in any simple parameterizations, because participation in flow depends also on particle ability to interact with others.

An excellent example is the obvious exception from the general trend, the $\phi$ meson: with about the same mass as the nucleon, it has small slope. This is because of much smaller cross section of its re-scattering, leading to earlier decoupling, so that $\phi$ does not participate in flow. (If, on the contrary, the increased slopes are due to initial state scattering, as advocated e.g. in [22], one should instead get larger slope for $\phi$, since it is not stopped by later “friction force” in matter.)

Another excellent test for existence of the flow is provided by deuterons. The shape of their spectrum, its slope $\tilde{T}_d$ and even yield are all very sensitive to the magnitude of flow. For

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5A disclaimer: NA49 data we use are preliminary [21], and is presented here for qualitative comparison only. Also NA49 has rapidity coverage wider than NA44, and therefore their slopes should be somewhat smaller.

6The tilde should remind that slopes are not temperatures, as they also include effect of the flow and resonance decays.
example, if flow is absent and both protons and neutrons be produced independently, with a distribution $\sim \exp(-p_t^2/2m_N T_N)$, their coalescence into $d$ would generate distribution with the same $\tilde{T}_d = \tilde{T}_N$. The observed value is much larger. The flow imply a specific correlation between position and momentum, which helps to produce larger $p_t$. If this correlation is artificially removed (see [26], where in RQMD output the nucleon’s positions or momenta were interchanged) the deuteron spectra change shape and their yield drops.

The next point is strong $A$-dependence, also quite evident from Fig.1. While the pp data show perfect thermal-looking spectra without a slightest trace of the radial flow [8], for SS collision the slopes start growing with the mass of the secondary particle, and for PbPb the effect is about twice larger. So, the larger the nuclei, the stronger is the flow.

This point is very important, because such trend qualitatively contradicts to what most of the hydro models in literature would obtain. The initial longitudinal size is usually taken to be either (i) the same for all, or (ii) it scales as $A^{1/3}$. With such assumptions and $A$-independent freeze-out temperature one gets either (i) a system which looks more and more one-dimensional, with the radial flow decreasing with $A$, or (ii) a system with a geometric scaling, with $A$-independent flow. Correct flow (increasing with $A$) naturally follows from improved freeze-out conditions to be discussed below.

One more aspect of the systematics of the radial flow is their rapidity dependence. The nucleon slopes (taken from [29]) from E877 and E866 experiments at AGS are compiled in Fig.2. One can clearly see from it that strong flow (and presumably its $A$-dependence) comes preferentially from the central region, $y \sim 0$ in CM.

Compared to these strong trends, the observed dependence of the flow on the collision energy appears to be weak. Unlike for Bevalac/SIS energies, in which $v_t$ steadily grows, for AGS (10-15 GeV/A) and SPS (160-200 GeV/A) domain the radial flow velocity is (inside uncertainties) about the same. It must be a mere coincidence, since both the meson/baryon ratio, the EOS and even the general picture of space-time development of the collisions are radically different. Furthermore, as we have shown previously [47], hydrodynamics predicts a rather non-monotonous dependence of the lifetime of the excited matter as a function of the collision energy, with a sharp maximum between AGS and SPS energies. This long time is related with a rather specific “burning pancake” regime: and although no detailed calculation of radial flow was made, it is hard to see how it can avoid having some kind of discontinuity as well. The simplicity [17] of the model used in [17] somewhat limited its predictive power, we believe the main conclusion about the peak of the lifetime should persist. One can look for this effect experimentally by scanning to lower energies at SPS, or by scanning various impact parameters. (Although we have not studied non-central collisions in this work, it is probably worth mentioning that such scan for $J/\psi$ suppression have shown discontinuous behavior at about the same energy density.)

Another interesting manifestation of the “softness” of EOS is stabilization of the radial flow at much higher RHIC energies. In this case hydro predicts a “burning log” picture [18], leading to mixed phase surviving for 25-30 fm/c. As we will show shortly, this regime actually appears at SPS energies already.

7 The EOS had no baryon number and full stopping was assumed, which may not to be the case even for heaviest nuclei.
III. THERMODYNAMICS OF HADRONIC MATTER

A. Quark-gluon plasma

Unlike real experiments, numerical ones performed on the lattice are easier to do at higher T. As a result, current lattice data have significantly clarified the QCD thermodynamics of the quark-gluon plasma phase. Above the phase transition region (see below) the thermodynamics was found to be close to that of the ideal quark-gluon gas. Deviations are typically 10-15 percent downward shift in pressure and energy density \( p, \epsilon \)\(^2\), which are roughly reproduced by the lowest order \( O(g^2) \) perturbative corrections\(^3\). Since for hydro only the \( p/\epsilon \) ratio matters, this common factor can safely be ignored. The non-perturbative corrections are more important: they are well seen in lattice data for \( T = (1 - 2)T_c \). Following tradition, we parameterize it simply by addition of the bag-type term \( B \) to the EOS of ideal quark gluon plasma

\[
\epsilon = \frac{\pi^2 T^4}{15} (16 + \frac{7}{8} 6N_f) + \frac{3N_f}{2} (T^2 \mu_b^2 + \frac{\mu^4}{2\pi^2}) + B
\]

\[
p = \frac{\pi^2 T^4}{45} (16 + \frac{7}{8} 6N_f) + \frac{N_f}{2} (T^2 \mu_b^2 + \frac{\mu^4}{2\pi^2}) - B
\]

(2)

The value of \( B \) is tuned to get \( T_c = 160\, \text{MeV} \) for zero baryon density (see below) resulting in the value\(^4\) \( B = 320\, \text{MeV/fm}^3 \).

Lattice data are also displaying a very spectacular phase transition in the vicinity of \( T_c \), in which \( \epsilon \) grows by a large factor. Although the exact dependence of order of the transition on the theory parameters (such as quark masses, number of colors and flavors) is still far from being completely clarified (see \([23,24]\) for recent review), it already quite clear that in practical sense the transition is close to the 1-st order one with large latent heat. Whether there is real jump, or just rapid rise inside few MeV range of T can hardly be practically relevant: high accuracy of T cannot be reached for finite-size systems we work with.

The actually relevant variable is not T but \( \epsilon \): and below we would refer to matter in wide range of energy densities \( \epsilon \sim .3 - 1.5\, \text{GeV/fm}^3 \) as a “mixed phase” domain. Its precise structure remains unknown: but fortunately it should not matter for hydro, provided the inhomogeneous domains (known also as “bubbles” of QGP) do not become too large. Fortunately, the hint we have from lattice data is a predicted tiny value (about 1 percent of \( T_c^3 \)) for the surface tension. If it is true, the boundaries between the two phases cost little energy, and so this phase should be very well mixed indeed.

\(^8\) All higher orders which can be perturbatively calculable have been now calculated, but those show divergent (or at least non-convergent) series, with large and alternating sign terms.

\(^9\) Note that it is about 6 times the original constant of the MIT bag model, and also only about a 1/2-1/3 of what one would get if all gluon condensate would be eliminated.
B. Hadronic matter as a resonance gas

Ironically enough, properties of the hadronic matter at $T < T_c$ are theoretically understood much less than QGP. In many applications people usually simply used the ideal pion gas as the simplest model: it leads than to a huge latent heat in the transition. However this approach is clearly inadequate, and many more hadronic degrees of freedom are actually excited.

We use instead the resonance gas approach, suggested very early by Landau and Belenky [18]. They have shown, using the lowest order virial expansion, that resonances seen in scattering phases in fact contribute to thermodynamical parameters exactly as stable particles. It was later used by Hagedorn in his statistical bootstrap studies of 60’s: his main point was that the exponential mass spectrum leads to the upper possible temperature of the hadronic gas. However, it was noticed by one of us long ago [43] that the observed resonance mass spectrum can better be fitted by power of the mass than the exponent. It leads to rather simple EOS, for zero baryon number $p, \epsilon \sim T^6$ [43] or $p \approx 0.2 \epsilon$. Later much more detailed calculations with actual scattering phases have confirmed it.

In this work we also include non-zero baryon density, and so our thermodynamics have two variables, $T$ and $\mu_b$. Except at low energies (when we are close to nuclear matter), we know very little about the role of non-zero baryon density in the EOS. As it is well known, lattice calculations are so far impossible in this case, due to complex weight function for the non-zero chemical potential.

Simple generalization of the resonance gas to the non-zero chemical potential is of course natural, but it is known to have a problem at low T/high density. Naive fermi-gas for nucleons clearly overestimates the pressure of nuclear matter. The QGP with reasonable bag constant cannot compete with it, and therefore a phase transition line has a pathological behavior at $\mu > 0.8$ GeV (see dotted line at fig3(a)). Following many others (e.g. [39]) we have solved this problem by the excluded volume correction, which effectively reduces the baryonic pressure at high $\mu$. Specifically we adopted the excluded volume model in [40], which is thermodynamically consistent, and is characterized by the canonical partition function

$$Z_{excl}(T, \{N_i\}, V) = \sum_i Z(T, N_i, V - V_0 N_i) \theta(V - V_0 N_i)$$

from which

$$p_{excl}(T, \{\mu_i\}) = \sum_i p_{i}^{ideal}(T, \mu_i - V_0 p_{excl}(T, \{\mu_i\})) = \sum_i p_{i}^{ideal}(T, \tilde{\mu}_i)$$

$V_0$ is the excluded volume, which we assume to be the same for all fermions, while $V_0 = 0$ for bosons.

10Those should be narrow enough: $\Gamma << T$.

11The chemical potential for strangeness $\mu_s$ is a dependent variable, with its value always fixed from the total strangeness $S=0$ condition.
\[ n_i^{\text{excl}}(T, \{\mu_i\}) = \left( \frac{\partial P^{\text{excl}}}{\partial \mu_i} \right)_{T,\{\mu_i\} \{\mu_i\}} = \frac{n_i^{\text{ideal}}(T, \tilde{\mu}_i)}{1 + V_0 \sum_j n_j^{\text{ideal}}(T, \tilde{\mu}_j)} \]

\[ s^{\text{excl}}(T, \{\mu_i\}) = \left( \frac{\partial P^{\text{excl}}}{\partial T} \right)_{\{\mu_i\}} = \frac{\sum_j s_j^{\text{ideal}}(T, \tilde{\mu}_j)}{1 + V_0 \sum_j n_j^{\text{ideal}}(T, \tilde{\mu}_j)} \]

\[ \epsilon^{\text{excl}}(T, \{\mu_i\}) = T s^{\text{excl}}(T, \{\mu_i\}) - P^{\text{excl}}(T, \{\mu_i\}) + \sum_j \mu_j n_j^{\text{excl}}(T, \{\mu_i\}) \]

\[ = \frac{\epsilon^{\text{ideal}}(T, \tilde{\mu}_i)}{1 + V_0 \sum_j n_j^{\text{ideal}}(T, \tilde{\mu}_j)} \]

and the excluded volume radius \( r_0 = 0.7 \text{fm} \). We do it just for completeness of the phase diagram: however we have to stress that all our results are completely independent on what is happening in this corner of the \( T, \mu_b \) phase diagram, since all the paths we discuss (see below) are far from it.

Apart from the excluded volume factor in \( p \) (which is also \( \log Z \)), we use standard thermodynamical formulae for the ideal gas of hadrons, stable and resonances. We use all resonances till \( m=2 \text{ GeV} \). All other variables are obtained from the pressure \( p(T, \mu) \) by standard thermodynamical relations.

### C. Adiabatic paths in the phase diagram and the resulting EOS

Although the \( T - \mu \) plane is rather convenient for the determination of the thermodynamical parameters in both phases, the mixed phase domain is hidden behind the transition line. As it is well known, in the mixed phase new thermodynamical variable is the fraction of the volume occupied by the QGP phase. Besides, as we will show shortly, the cooling trajectory in the \( T - \mu \) plane is rather complicated.

If expansion of matter\(^{12}\) is slow enough, the entropy is conserved. We assume it in what follows. If so, the conjugates to \( (T - \mu_b) \) pair – the entropy \( s \) and the baryonic density \( n_b \) – provide more natural description. If those variables are used, the cooling paths would be just straight lines, going from initial point toward the origin. As the entropy per baryon ratio stays constant, the paths can be marked by this ratio.

For the EOS described above (the resonance gas for hadronic phase is supplemented by a simple bag-type quark-gluon plasma) we have calculated those paths in all variables. In Fig.3(a) we show how these paths look like on the original phase diagram. The lines are marked by the \( n_b/s \) ratio. Those for \( n_b/s = 0.02, 0.1 \) correspond approximately to SPS (160 GeV A) and AGS (11 GeV A) heavy ion collisions, respectively. Note that the trajectory has a non-trivial zigzag shape\(^{13}\) with re-heating in the mixed phase. The endpoint of the QGP branch was named \(^{[47]} \) the “softest point”, while the beginning of the hadronic one

\(^{12}\)Note that we do not discuss compression stage here: it is not slow and therefore entropy is in fact produced here.

\(^{13}\)As far as we found, such shape first appeared in literature in \(^{[12]}\).
can be called the “hottest point”\cite{hagedorn}.

The next step is to define the effective EOS in the form $p(\epsilon)$ (needed for hydro) on these lines: that is shown in Fig.\ref{fig:fig}(b). Note that the QCD resonance gas in fact has a very simple EOS\cite{res_gas}, $p/\epsilon \approx \text{const}$, while displaying strong dive toward the minimum of $p/\epsilon$ (the “softest point”). The contrast between “softness” of matter at dense stages and relative “stiffness” at the dilute ones is strongly enhanced for the SPS case: it is the main physical phenomenon we study below.

For comparison, one should also look at the (effective) EOS corresponding to popular cascade event generators. For RQMD (with repulsive potential between baryons) it was studied in recent work \cite{rqmd_es} for AGS energies. Rather simple EOS was found, about the same for compression and expansion stages. For (transverse) pressure and energy density it is approximately $p/\epsilon \approx 0.14$. It is very close to what our resonance gas gives for the corresponding $s/n_b$ ratio. (It would be nice to have similar results for other cascades, and in wider energy range.)

In summary: resonance gas (even with baryons) has a very simple EOS, $p/\epsilon \approx \text{const}(\epsilon)$. However lattice results (modeled via bag-type model for QGP) indicate that EOS of hadronic matter is much softer, with small $p/\epsilon$ in the interval of the energy densities near the end of the mixed phase.

**IV. HYDRODYNAMICS**

The equations of relativistic hydrodynamics are standard

\[ \partial_\mu T_{\mu\nu} = 0; \quad \partial_\mu n_b u_\mu = 0 \]  

In absence of any dissipative terms, they imply conservation of the entropy $\partial_\mu s u_\mu = 0$ and baryon number $N_b$. The ratio of their local densities $n_b/s$ is not changing, and that is why in our discussion of the thermodynamics above we have parameterized by it the paths on the phase diagram (e.g. $T,\mu_b$). Furthermore, we have shown that hydro-relevant form of EOS, namely $p(\epsilon)/\epsilon$, depends smoothly on this ratio.

It was shown many times (see e.g. a recent review \cite{review}) that\cite{review}, for PbPb/AuAu collisions at SPS/AGS energies the rapidity spectra of $\pi, K, N, d$ can be described by by some common collective motion, convoluted with thermal one (and this is certainly different for $\pi, K, N, d$). It suggests that all matter elements have about the same composition ($n_b/s$).  

\footnote{Of course, in the “Hagedorn sense”, as the hottest point of the hadronic phase.}

\footnote{The main difference between the curves with various $n_b/s$ is at the low energy density side: obviously adding baryons one contribute much more to the energy density than to pressure. as we will show below, it will have a significant impact on the mean radial flow.}

\footnote{For clarity: this statements holds for central collisions of heavy enough ions, which have very small “corona” of punched through nucleons. For medium/light ions it is obviously more visible, and asymmetric systems have many spectator nucleons. Certainly those are not part of the hydrodynamic fireball.}
As it was explained in previous section, $n_b/s$ ratio is conserved for each matter elements. However, if initial conditions have different $n_b/s$ in different places, it becomes space and time-dependent due to flow. Phenomenological observations mention in the previous paragraph imply that we may in fact significantly simplify the problem, assuming “well mixed” initial conditions which have constant $n_b/s$ everywhere. If so, equations for baryon flow and entropy flow become the same, and $n_b/s$ ratio is space-time independent. In practice, one can determine $n_b/s$ from the baryon/meson ratio at the freeze-out stage. We use the values $n_b/s = 0.02, 0.085$ as representative for SPS (160 GeV A) and AGS (11 GeV A) heavy ion collisions, respectively. These paths corresponding to them on the $T − \mu$ plot are also shown in fig3(a).

The initial geometry of the fireball was chosen to be Saxon-like with natural Lorentz contraction in the longitudinal direction. In this work we have not even attempted to discuss kinetics at the formation stage, and simply adopt a phenomenological approach, introducing initial longitudinal size $z_0$ and velocity $v_z = v_0 \tanh(z/z_0)$ as a phenomenological parameters. As a result, we do not have a predictive power as far as rapidity distribution is concerned: but we can just fit it (as it was many times done before, see [9]). We concentrate below at central region of rapidity, and do not intend to describe well spectra in the target or projectile fragmentation region. Although we describe most of the secondaries, the total energy of hydro subsystem is only a fraction of the total one. For 160A GeV Pb+Pb the total initial energy of the hydrodynamical system is about 0.4 of the total center of mass collision energy (which corresponds to initial central energy density of 4GeV/fm$^3$), while for 11.6A GeV Au+Au this ratio (the inelasticity coefficient) is about 0.7 (which corresponds to initial central energy density of 1GeV/fm$^3$).

The uncertain initial conditions are not important for transverse flow, because it is accumulated over long time. We will return to discussion of hydro and radial flow results later, after we study kinetics of the freeze-out in more details.

Typical solution for 11.6A GeV Au+Au is shown in Fig.4 while for 160A GeV Pb+Pb it is shown in Fig.5. Let us make few comments about them. First of all, they are qualitatively different. While at the former (AGS) energy the longitudinal and transverse expansion are not that different, at SPS ones the longitudinal flow has already distict ultrarelativistic (Bjorken-like) features, with most isotherms being close to hyperbolae, the lines of constant proper time $\tau = \sqrt{(t^2 - z^2)}$. What is less obvious (and follows from particular EOS including the QCD phase transition) is also a dramatic difference in the transverse flow at AGS and SPS as well. The former can be described as “burning in”, the lines of constant energy density moves inward with some small constant speed. At SPS the mixed phase matter burns into the low density hadron gas at a “burning log”, which is nearly time-independent and positioned at transverse radius 6-8 fm. With time, as more matter flow from the center, there is even tendency to get by the end of the expansion a hole at r=0, with less density there than in the “burning log” region. Such behaviour is a result of overshooting the “softest point” in the intitial conditions, and it is even more dramatic at higher (RHIC/LHC) energies, see [48].

(It is interesting to note, that our hydro solutions in many cases show late implosion, with subsequent secondary explosion from the center r=0. However, it is happening well after freeze-out, and thus such hydro solution can only become physical if colliding nuclei are much larger than the heaviest existing ones.)
V. KINETICS OF THE FREEZE-OUT

Although there is rather substantial theoretical literature related with kinetics of freeze-out, and all major concept (and most of the details) to be used below has been developed before, in most of previous hydrodynamical models the freeze-out is formulated in a very crude, oversimplified form. Most of them simply assume that all reactions stop when the system reaches some universal “final temperature” $T_f \approx 140\text{MeV}$. However this approximation is clearly inadequate since both (i) different processes have different rates, say inelastic and elastic ones; (ii) different secondaries have different rates; (iii) the expansion rates are very different for different colliding nuclei, and even for the same nuclei for different matter elements.

To learn more about freeze-out conditions and resolve the issue phenomenologically, one can study various observables, such as HBT radii, deuteron production, Coulomb effects [20], event-per-event fluctuations [42], the pion chemical potential etc. Except for the last one, we do not so far have HKM predictions for them, and live those for studies elsewhere.

A. Local freeze-out condition

The central point we would like to make is as follows. Although each individual matter element follow roughly the same path on the phase diagram, the relevant kinetics is not the same because they move along these paths with different speed. In particular, the freeze-out happens at higher $T, \mu$ when time evolution is faster (smaller initial system, or closer to the edge of the system) relative to matter elements for which the evolution is slower. Accounting for it turns out to be crucial for applications we have in mind in this work.

Fortunately, after the global collective motion of matter is already determined from hydro calculation, we know the expansion rate of any matter element at any time. With this information at hand, plus known kinetics of various hadronic processes, we can formulate realistic conditions under which subsequent freeze-outs (decoupling of a particular reaction) take place.

The principle idea of a freeze-out goes back to the famous 1951 Pomeranchuck paper (which initiated Landau to suggest a hydro-based approach for the first time). The condition Pomeranchuck had in mind is a relation between the mean free path and the system dimensions. The particular form we use (as far as we know, mentioned first in [30]) is based on the similar condition, which is however local (or differential) value of the ratio

$$\xi = \tau_{\text{exp}}/\tau_{\text{coll}}$$

of the where $1/\tau_{\text{coll}}$ is a collision rate per particle considered per unit proper time. The invariant expression for the expansion time can be given in terms of the 4-velocity $u_\mu$ of the flow

\[17\] The non-local condition in line with original Pomeranchuck idea is worked out in [46], but we feel it is still way too complicated to use it in practice in hydro context, because of the integrals toward future propagation involved.
\[ \frac{1}{\tau_{\text{exp}}} = \partial_{\mu} u_{\mu} \] (5)

while \( 1/\tau_{\text{coll}} \) is a collision rate per particle considered, calculated in the cell proper time. Hydrodynamics is applicable (the dissipative terms are small) when \( \xi \gg 1 \), while if \( \xi \ll 1 \) the reactions in question can be ignored. The boundary at which \( \xi \sim 1 \) exist both at the formation and expansion stages, forming some 3-surface around the 4-volume in which hydro is applicable. Furthermore, in principle the situation is more complicated, with \( \xi \) large and small for different variables.

First of all, let us distinguish two classes of reactions: (i) the inelastic reactions leading to creation or annihilation of a certain species of particles; (ii) elastic rescatterings leading to simple momentum exchange. It is well known that the former need higher collision energies than the latter. (For example, in the gas of massless pions one can use chiral perturbation theory to evaluate reaction rates, and pion production depends on temperature as \( 1/\tau_{\text{production}} \sim T^9 \) while elastic re-scattering is \( 1/\tau_{\text{rescattering}} \sim T^5 \). Clearly, as the expansion cools the gas, their decoupling happens at different points.) Separating those two classes, one usually defines chemical and thermal freeze-out, for these two classes of reactions. The second important point: both freeze-outs should be determined for each species separately.

For chemical freeze-out this distinction however is not very important in practice, since in fact all reactions changing the particle composition can bee seen to be rather ineffective during the hadronic phase, for all AGS/SPS collisions\(^{18}\). In QGP (most) hadrons do not exist at all, and thus the natural place for “hadronization” is what we call the mixed phase. How it happens remains unknown: but there are rather convincing arguments that it happens rapidly enough. Those are based on quite extensive work on thermal description of many hadronic species\(^{34,39}\), mainly in connection with the so called “strangeness enhancement” phenomenon. It was found that (within the existing experimental uncertainties, not always small) one can describe most of particle ratios in a thermal model. (Moreover, for heavy ions one can do it even without any “strangeness suppression” factors.) The resulting values for \( T, \mu_b \) are shown in Fig.3 as two crosses, for the AGS and SPS energies respectively. Both are close to the “hottest points” of the corresponding paths: this is consistent with the idea that chemical equilibration cannot indeed be kept in the hadronic gas phase.

We have built in this idea into HKM: any “hadronic chemistry” in the hadronic phase is ignored. It is assumed that it ends together with hadronization, and when the path departs from the phase transition line \( T_c, \mu_c \) no more changes in particle composition (apart of resonance decays) are included. Therefore our particle composition is exactly the same as in the thermal model\(^{39}\) (which has thermodynamics of exactly the same resonance gas with excluded volume). We therefore do not duplicate the tables for particle ratios here, referring the interested reader to this work.

\(^{18}\)For example, for strangeness production reactions this statement was well documented long ago, see\(^{33}\).
B. Between chemical and thermal freeze-out

We do not provide extensive introduction for this section: for a good overview and references see [41]. Switching off all reactions changing the particle composition, we have made any particle number $N_i$ to be a conserved quantity. The point is simply that at this stage of the evolution one has to introduce chemical potentials for all particle species, $\mu_i$. Their values are then determined by those pre-determined values of $N_i$ in the usual way. This is in contrast to chemical equilibrium, in which most of them are zero, and only chemical potentials conjugated to conserved quantities (baryon charge and strangeness) were needed.

It is instructive to see how, as one starts with chemically equilibrated hadron gas with $\mu_i = 0$, the non-zero values appear as the system cools further. Thermodynamical relation written in the form

$$(\epsilon + p)/nT - s/n = \mu/T.$$  \hspace{1cm} (6)

is especially useful. For slow (adiabatic) expansion the $s/n$ ratio is not changing, while in the first term in l.h.s. the chemical potential nearly cancels (it does provided Boltzmann approximation is used). So, one can read T-dependence of $\mu$ directly from the r.h.s. The notorious exceptional case worth mentioning is that for massless particles, for which the whole l.h.s is just a constant. Therefore $\mu/T = \text{const}$, and so if $\mu = 0$ at the beginning it remains so for any T.

Accounting for the non-zero pion mass and Bose statistics one finds $\mu_\pi(T)$, see [41]. For example, if one assumes that $\mu_\pi(T_c = 160\text{MeV}) = 0$, one finds that by a thermal freeze-out (which happens for PbPb collisions at CERN at T=110-120 MeV) the pion chemical potential $\mu_\pi = 60 - 80\text{MeV}$.

In order to see whether such effect really occurs in experiment, we have plotted in Fig.6 the ratio of $p_t$ spectra for PbPb collisions (in which we expect thermal freeze-out at T=100-120 MeV, and thus formation of significant pion chemical potential) to our reference point, central SS collisions (for which the effect should be much smaller). The data sets are both for positive pions from NA44 experiment [5], in the same experimental settings (and thus...

19For clarity: those potentials are conjugated to total number of particles, so say for pions they enter distributions of $\pi^+,\pi^-,\pi^0$ with the same sign.

20To our knowledge, it was first pointed out in the context of the pion gas by G.Baym. Further discussion of this idea and of kinetics of the pion gas can be found in [35], mostly in relation with the question of the possible evolution of the non-zero chemical potential for the pions. For discussion of the opposite scenario, suggesting overpopulation and large positive chemical potential for pions already at this point, see [36].

21This is what happens in the case of background radiation in expanding Universe: photons do not collide after the Big Bang, but they still have the Plank spectrum, with $\mu = 0$.

22The $\pi^-/\pi^+$ ratio show larger enhancement, which is known to be due to Coulomb effects, see e.g. [20].
systematic errors should somewhat cancel). One finds, that there is significant enhancement of this ratio at small $p_t$, which agrees with formation of the non-zero pion chemical potential. Moreover, as one can see, the magnitude of the effect is in approximate agreement with our estimates.

(Additional comments: PbPb and SS collisions have somewhat different stopping of baryons. For positive pions extra stopped charge for PbPb would decrease low $p_t$ pion production due to the Coulomb field, contrary to observations. Another effect contributes in the opposite direction is feeding to low $p_t$ pions from extra $\Delta$ decays coming from extra baryons in PbPb as compared to SS. Magnitude of those effects is comparable, and thus they may cancel out to some extent.)

The secondaries other then pions can be to a good accuracy treated as Boltzmann non-relativistic gas, and so one can easily derive the following relation between the chemical potentials at chemical and thermal freeze-out

$$\mu_{th} = \mu_{ch} \frac{T_{th}}{T_{ch}} + m(1 - \frac{T_{th}}{T_{ch}})$$  \hspace{1cm} (7)

(In particular, for very large systems $T_{th} \rightarrow 0$, the chemical potential $\mu_{th} \rightarrow m$ as it should, and one can then proceed to normal non-relativistic notations.) When implemented in HKM, this relation ensure that particle ratios are independent on any details of thermal decoupling we discuss below.

C. Thermal freeze-out for different species

Now we are in the position to discuss particular reactions in the resonance gas. Rather extensive studies have been made in the past, see [44]. Let us start with qualitative comments first.

Out of many reactions which include pions the major processes are the low energy elastic $\pi\pi$, $\pi K$ and $\pi N$ scattering. Those have especially large cross section due to existence of the low energy resonances, $\rho, K^*, \Delta$ respectively.

The estimates of $\pi\pi$ collision rate using the chiral Lagrangian was made by one of us [37] and, in more details, in ref. [38]. The result

$$1/\tau_{\pi\pi} = T^5/(12F_{\pi}^4)$$  \hspace{1cm} (8)

display very strong T-dependence. This feature remains true when one includes the resonances [11]: basically in the interval we deal with ($T=120-150$ MeV) the pion-pion scattering rate increases by the factor 223. These rates are increased further by the inclusion of the non-zero value of pion chemical potential discussed in the preceding subsection.

23Furthermore, the inclusion of resonances changes the dependence on the pion momentum $p$: in contrast to chiral result the rate becomes basically flat for $p < 700 MeV$ we need, and decrease for larger $p$ (now, in contrast to the lowest order chiral result which predicts the unphysical rise with $p$).
Strong T-dependence leads to the following qualitative feature of freeze-out: relatively modest changes in the freeze-out temperature correspond to quite significant changes in duration of the collision-dominated (hydro) expansion. As we will see below, this will translates to significantly stronger flow.

The $\pi N$ cross section is very large, reaching about 200 mb at the $\Delta$ resonance peak. Naive radius of the interaction $R = \sqrt{(\sigma/\pi)} \approx 2.6$ fm is so large that one may question simple cascades and think about collective effects ("pi-sobars"). Absolute scattering rates depends on the density of nucleons at the decoupling stage. At AGS the (isospin averaged) rate is of the order of $1/\tau_{\pi N} \approx 100$ MeV, which is larger than $1/\tau_{\pi \pi}$. Since nucleon to pion ratio is about one, the rates are very close also. At SPS energies the situation is quite different: the nucleon/pion ratio is about 1/5. It makes the $\pi N$ scattering less important for pions, but nucleons have very large collision rate and thus should freeze-out very late.

Kaon and other strange secondaries have smaller collision rates. We have already mention a special case of $\phi$ with the scattering and absorption cross sections in few mb range. Clearly one can completely ignore their re-scattering in hadronic phase: we assume therefore that their thermal freeze-out (as well as chemical one) coincides with the end of the mixed phase.

Let us now provide more quantitative information about the rates we use (see also [44]). The general formula for the averaged collision rate of particle $a$ resulting from binary collision with particle $b$ is given by

$$\Gamma_{ab}^a(T) = \frac{1}{\int \frac{d^3p_a}{(2\pi)^3} e^{E_a/T} \pm 1} \int \frac{d^3p_b}{(2\pi)^3} \frac{1}{e^{E_b/T} + 1} \sigma_{ab} (p_a + p_b)^2 \left| \frac{p_a}{E_a} - \frac{p_b}{E_b} \right|$$

where $E_a$ is the energy of $a$ (minus any chemical potential for $E_a$ within the thermal exponent), and similarly for $b$. $g_b$ is the multiplicity of $b$ and the sign in the denominator of the thermal weights are chosen based on whether $a$, $b$ is a fermion or boson. For example, for the $\pi N$ rate we take the $\pi^+ p$ total cross-section from Particle Data Group [45] and notice that by isospin arguments, the averaged $\pi N$ cross-section is

$$\sigma_{\pi N}^\pi \approx \frac{2}{3} \sigma_{\pi^+ p}$$

also noting that $g_N = 2$, we get for the $\pi N$ pion collision rate

$$\Gamma_{\pi N}^\pi(T) = \frac{1}{\int \frac{d^3p_\pi}{(2\pi)^3} e^{E_\pi/T} \pm 1} \int \frac{d^3p_N}{(2\pi)^3} \frac{1}{e^{E_N/T} - 1} 4/3 \left| \frac{p_\pi}{E_\pi} - \frac{p_N}{E_N} \right|$$

Using $\mu_b(T)$ from the previous section we can evaluate the above integral numerically. The results are shown in Fig.7, for the $\pi \pi$, $\pi K$ rates combined (dots and fitted curve) and for its $\pi N$ component at AGS and SPS. Total pion collision time is then given by

$$\tau_{\text{collision}}(T) = (\Gamma_{\pi \pi}^\pi(T) + \Gamma_{\pi N}^\pi(T))^{-1}$$

16
For kaons we simply take the $\pi K$ rate from [44] which we show in Fig.8. We have ignored smaller $KN$ collision rates: therefore (as also noted in [44]) we do not distinguish the rates for kaons and anti-kaons.

For nucleons, $\pi N$ interaction is the dominant process [44] and we have the expression:

$$
\Gamma_{\pi N}^N(T) = \frac{1}{\int \frac{d^3 \mathbf{p}_N}{(2\pi)^3} \frac{1}{e^{\mu_b} + 1}} \int \frac{d^3 \mathbf{p}_\pi}{(2\pi)^3} \frac{2}{e^{E_{\pi}/T} - 1} \frac{e^{(E_N - \mu_b)/T} + 1}{\sigma_{\pi + p} [(p_\pi + p_N)^2]} \left| \frac{p_\pi}{E_{\pi}} - \frac{p_N}{E_N} \right| \tag{13}
$$

noting that $g_\pi = 3$. It turns out that due to the almost factorizable nucleon density inside the main integral which almost cancels the denominator in front, the effect of $\mu_b$ is almost negligible. In Fig.9 we present the common rate for the AGS and SPS and compare with the SPS nucleon rate from [44].

The remaining issue is what value of the ratio in the condition (4) one should use in order to optimize the surface. Consider for example the simplest case in which expansion/reactions proceed with the same rates for some time, the densities are $n \sim \exp(-\tau/\tau_{exp})$. Plugging it into the rate and integrating from freeze-out to infinity one gets the number of collisions left over to be $\sim (1 - \exp(-\xi))$. As we want to cut roughly in the middle of the last collision, one may think the optimal point is close to $\xi = 1/2$. Our checks with cascades (see below) confirm this choice, although one may in future improve on this point.

Using these rates, and the condition (4), we determine the freeze-out (3-d) surface. Several representative cases are shown in Figs.10 to 13, shown as a section by time t-longitudinal coordinate z plane (at transverse coordinate r=0) and the t-r plane (z=0). We have already commented about dependence on the particle kind above. Note also a significant difference between heavy and light ions. As expected, one finds that the larger is the system, the lower is $T_{th}$ on this surface.

Furthermore, the shape of the freeze-out surface is very different from simple isotherms. It means that there is a significant variation of this temperature over the surface itself: in order to find the coolest pion gas, one should look at the very center of central collisions of heaviest nuclei at highest available energy!

Finally, after elements of the the freeze-out surface are determined (from hydro solution plus kinetic condition discussed above) by (3-d) triangularization (see appendix), the HKM program generates secondaries using the Cooper-Frye formula$^{24}$

$$
E dN = \frac{1}{(2\pi)^3} \int_{\sigma_f} \frac{p^\mu d\sigma_\mu}{e^p u/T} \pm 1 \tag{14}
$$

where the integral is taken over freeze-out surface, with $T_{th}, \mu_{th}, u_\mu$ changing from point to point.

$^{24}$Although this formula is conserving energy and is widely used, there is still a well known problem with it when applied to the space-like part of the freeze-out surface: it includes also particles which move toward the excited system, which would be re-absorbed. Possible improvements are discussed e.g. in recent work [19]: we have not included those in HKM.

17
The last step of the HKM is the decay of all resonances (and stable particles, if needed) into the final secondaries. The standard output, like from other event generators, includes information about particle momenta, the time and place of their production, and the parent resonance (if they come from a decay).

VI. FURTHER HYDRO RESULTS: THE RADIAL FLOW

In the previous section we have shown that the improved thermal freeze-out condition lead to huge difference compared with simple isotherms. For the same fixed value of the $\tau_{\exp}/\tau_{col}$ one gets very different conditions for different $A$, also for large and small $y$ (the central region cools further). These observations provide natural resolutions to the puzzling observations related to strong $A$- and $y$-dependence of the flow discussed above.

The key point here is as follows: although these modifications does not significantly prolong total lifetime, it significantly increases the lifetime of hadronic phase. It is important for flow, because it is the part of the evolution path at which the matter is most “stiff” (have larger $p/\epsilon$). Thus improved freeze-out leads to a significant “extra push”, and explains strong flow.

The typical $p_t$ spectra for $\pi, K, N$ we obtain (after resonance decays) are shown in Fig.14 (a), together with their fixed-slope fits.

In Figs.21 and 22 we show the distribution over transverse velocities calculated over all matter element on the freeze-out surfaces. We show only heavy ions, for AGS and SPS energies. The distributions always have sharp peak at their right end, which is more pronounced at SPS. Its position depend significantly on the particle type, reaching as high peak as $v_t = 0.6$ for N at SPS. Note dramatic difference with the isotherms $T = 14 GeV$ which were used in many previous works: for them there is also a peak, but for much smaller $v_t \approx 0.17$, plus a shoulder toward larger values. This difference is much smaller for medium ions (not shown).

In Figs.15 to 20 we show how this translates into the observable quantity, the $m_t$ slopes $\tilde{T}(y)$. Recall that they include effect of freeze-out temperature, flow plus resonance decays, and we show them as a function of rapidity $y$. We show 4 cases: AuAu at 11 GeV/N, PbPb at 158 GeV/N SiAl at 14.6 GeV/N and S S at 200 GeV/N. In all cases we compare our results with the experimental data available, as well as with the RQMD (which was obtained from standard output files and fitted in the same way as the HKM ones).

For AuAu data at AGS Fig.15 one can see, that RQMD reproduces slopes very accurately, while our results slightly under-predict the flow. However, it is precisely how it should be, because this version of RQMD has been tuned with a repulsive baryon-induced potential, on the top of the pure cascade. We have checked that the version without potential gives smaller flow, and agrees with our results very well. At the same time, the results following from “naive” freeze-out with $T_f = 140 MeV$ is way below.

Fig.18, showing PbPb at 158 GeV/N, look very similar to Fig.15. This feature however must be a mere coincidence, since both the EOS and the space-time picture are quite

\[\text{Due to completely different matter composition: the } \pi/N \text{ ratio different by factor 5.}\]
different. Apart from obviously rather different longitudinal motion, at AGS the transverse velocity is gained gradually in time (due to about constant $p/\epsilon$, or acceleration, while at SPS our hydro solution clearly display appearance of a “burning wall” regime, at which most of acceleration occurs. Note that nevertheless our results agree with data and RQMD in this case as well, for the same $\tau_{\text{exp}}/\tau_{\text{coll}} = .5$. This agreement is very non-trivial.

For comparison, let us now discuss lighter ions. An example is shown in Fig.19, for S at 200 GeV/N, and one can see from it that our results over-predict flow in the central region $y \approx 0$. Although for light ions HKM predicts shorter lifetime of the hadronic phase and smaller flow ($\tilde{T}_N(y)$ about 30 percent lower), the data (and RQMD) show that this drop should in fact be larger. It is hardly surprising to see that for medium ions the HKM (and probably hydro-based models in general) are less accurate.

Let us finally stress that we have not attempted any fine tuning of the parameters used. The main ingredients, EOS and freeze-out parameter $\xi = 1/2$ were fixed rather early and not modified when hadron spectra/slopes were calculated. Clearly one can do it and get better agreement. In this work our main objective was to test crudely the systematics of the flow discussed in the introduction (and, of course, its magnitude).

Finally, a comment on agreement with RQMD is in order here. We emphasized above that its EOS is similar to ours for AGS domain, in which both represent the resonance gas: but how can both agree at SPS energies, where our EOS has the notorious softness due to the QCD phase transition? In fact, RQMD has its own reasons for changing its EOS to larger “softness": at SPS conditions at early times the energy is stored no longer in resonances, but in (longitudinally stretched) strings. Naturally, those make little pressure in the transverse direction.

By no means we want to create an impression that our model and RQMD are to a large extent identical. The magnitude and various dependences of the flow we discussed in this work are important observables, but even those give only partial information on the space-time picture of the collision. Looking at these results more closely, one however finds significant differences here, which should affect the “freeze-out sizes” extracted by pion interferometry analysis. This statement is illustrated in Figs.23 and 24, comparing distributions in the points of the last interaction in our model and RQMD. One can see from it that although the average sizes generally agree (and thus flow velocities), their dispersions (relevant for interferometry) are rather different. With better data coming, one would be soon able to address this aspect as well.

VII. SUMMARY AND DISCUSSION

In this work we have developed next-generation hydro-kinetic model for heavy ion collisions. Although most of the ideas in it are not new, we believe they are now brought together in an economic and practical way.

Compared to previous hydro-based models in literature we have included a number of improvements: (i) realistic EOS including the QCD phase transition together with the effect of baryons; (ii) more realistic “local” freeze-out condition, which is based directly on kinetics of re-scattering; (iii) decay of all resonances in the final state, etc.

Our main focus was on new data on radial flow, including its magnitude, $y$, $A$ and $s$-dependence. We have found that our model in general reproduces it well enough.
shows that the lattice-based EOS (which is very soft in the transition region, as we repeatedly emphasized) is in fact consistent with flow data. This is our main result.

The crucial observation which was important for this success is point (ii). It leads to a very simple property of the freeze-out: the larger is the size of the system, the cooler the matter at the end becomes. (We have found it surprising that this effect was overlooked before.) Clearly, deeper cooling for larger A should be seen in many different ways, and we look forward other ways of testing it.

One may further ask, whether data can restrict the EOS. We have not attempt to quantify this in the present work, and only note that for EOS without the QCD phase transition (e.g. a resonance gas, with \( p/\epsilon \approx \text{constant} \)) discussed above) the magnitude of the flow for SPS is indeed too large, and the expansion time too short. Clearly further studies are needed to clarify these issues. Natural extension is discussing 3+1 hydro at non-zero impact parameters, leading to dipole/elliptic components of the flow.

Another obvious way to proceed is to calculate the HBT radii and compared it with data. We have already mention that dispersions of the emission time/positions in our model are quite different from those resulting from the RQMD.

Clearly only a small fraction of data is considered in this work. To facilitate further use of the model, we plan to deviate from the usual scenario in which only the basics formulae plus some results are presented in the paper, and plan to provide the source code/output files, in the same form as event generators do. We hope it will prompt the experimentalist to use it widely, revealing in wider scope its agreement and disagreement with particular data.

**APPENDIX A: TRIANGULATION OF THE FREEZE-OUT SURFACE**

The surface element on a 3-surface of space-time is given by [49]

\[
d^3s_\mu = \epsilon_{\mu\alpha\beta\gamma} \frac{\partial x^\alpha}{\partial a} \frac{\partial x^\beta}{\partial b} \frac{\partial x^\gamma}{\partial c} dadbdc
\]  

(A1)

where \( a, b, c \) are the coordinates on the 3-surface and \( \epsilon_{\mu\alpha\beta\gamma} \) is the totally antisymmetric Levi-Civita tensor. For our purposes, due to the cylindrical symmetry of our model, we need to express a finite 3-surface element of a freeze-out surface in terms of its corner points. More specifically we want to find \( d^3s_\mu \) for a triangle defined in \( (z,r,t) \) space with corner points \( (z_i, r_i, t_i) \) for \( i = 1, 2, 3 \). It can be shown that up to a sign,

\[
d^3s_\mu = \frac{1}{2} rd\theta \begin{vmatrix} z_1 & r_1 & 1 & t_1 & z_1 & 1 & t_1 & z_1 & 1 & r_1 & t_1 & 1 \\ z_2 & r_2 & 1 & t_2 & z_2 & 1 & t_2 & z_2 & 1 & r_2 & t_2 & 1 \\ z_3 & r_3 & 1 & t_3 & z_3 & 1 & t_3 & z_3 & 1 & r_3 & t_3 & 1 \end{vmatrix}
\]  

(A2)

The sign would have to be determined by choosing a direction for the normal of the surface which points outward from the hotter interior of the surface. The output file of the hydro program gives \( (t, z, r, \epsilon, \xi = \tau_{\text{exp}}/\tau_{\text{col}}, ...) \) at each point of the output grid (which is typically of size 25 \( \times \) 25 \( \times \) 25 in \( (z,r,t) \)). To triangulate the freeze-out surface, we pick a cell and check to see whether \( \xi \) (or \( \epsilon \) if we want a freeze-out surface of constant temperature)
on its vertices are above or below the freeze-out value $\xi_f$ (or $\epsilon_f$). By interpolation we can determine the intersections (if any) between the freeze-out surface and the edges of the cell. Once the intersections are found, we find the center point of these intersection points and connect it to two adjacent intersection points to form a triangle. Continuing this process for all the cells we obtain the desired triangulation of the freeze-out surface.
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FIG. 1. Experimentally measured slopes of $m_t$ distributions as a function of particle mass (MeV), for $\pi, K, N$ (NA44) and $\phi, \Lambda, d$ (NA49), in acceptance of these experiments. Three types of points correspond to pp, SS and PbPb collisions.

FIG. 2. Experimentally measured proton slopes of $m_t$ distributions at AGS as a function of rapidity $y$ (counted from CM).
FIG. 3. (a) Paths in the $T - \mu$ plane for different baryon admixture, for resonance gas plus the QGP; (b) the ratio of pressure to energy density $p/\epsilon$ versus $\epsilon$, for different baryon admixture.
FIG. 4. Hydrodynamical solution for 11.6A GeV Au+Au. The solid contours are energy density contours, with the bold contour being the boundary between the mixed and hadronic phase ($\epsilon = 0.35\text{GeV/fm}^3$). The dotted contours are the longitudinal (left) and radial (right) velocity contours, with values starting from the left of 0.01, 0.05, 0.1, 0.2,...

FIG. 5. Hydrodynamical solution for 160A GeV Pb+Pb. The solid contours are energy density contours, with the bold contours being the mixed-hadronic and quark-mixed boundaries, with energy densities $\epsilon = 0.18, 1.4\text{GeV/fm}^3$) respectively. The dotted contours are the longitudinal (left) and radial (right) velocity contours, with values starting from the left of 0.01, 0.05, 0.1, 0.2,...
FIG. 6. The ratio of $\pi^+$ $p_t$ spectra for PbPb to SS collisions. Points are experimental data from NA44 experiment, three curves correspond to pion chemical potential $\mu_\pi = 60, 80$ and 100 MeV (from bottom up).

FIG. 7. Pion collision rates $\nu = \tau^{-1}$ [GeV] in a pion-kaon-nucleon gas versus temperature $T$ [GeV].
FIG. 8. Kaon collision rates

FIG. 9. Nucleon collision rates. “Data” refers to our numerical results, and “Fit” refers to a fit to our results using the data points below 0.15 GeV
FIG. 10. Freeze-out surfaces for 11.6A GeV Au+Au

FIG. 11. Freeze-out surfaces for 14.6A GeV Si+Al
FIG. 12. Freeze-out surfaces for 160A GeV Pb+Pb

FIG. 13. Freeze-out surfaces for 200A GeV S+S
FIG. 14. Typical hydro output and fit of the $m_t$ distributions for pion, kaon and protons, for central PbPb collisions at 158 GeV A, at central rapidity $|y| < .5$.

FIG. 15. Nucleon slope parameters for 11.6A GeV Au+Au
FIG. 16. Pion slope parameters for 11.6A GeV Au+Au
Kaon Inverse Slope as a Function of Rapidity

out210.dat, 11.6A GeV Au+Au

τ_{exp}/τ_{col} = 0.5
T_f = 0.14 GeV
RQMD, auau12ab_1.mf
E866 data

FIG. 17. Kaon slope parameters for 11.6A GeV Au+Au

Nucleon Inverse Slope as a Function of Rapidity

out213.dat, 160A GeV Pb+Pb

τ_{exp}/τ_{col} = 0.5
T_f = 0.14 GeV
RQMD, pbpb160a.dat
NA44 data
NA49 data

FIG. 18. Nucleon slope parameters for 158A GeV Pb+Pb
FIG. 19. Nucleon slope parameters for 200A GeV S+S

FIG. 20. Nucleon slope parameters for 14.6A GeV Si+Al
FIG. 21. Transverse velocity distribution over the various freeze-out surfaces for 160A GeV Pb+Pb collision

FIG. 22. Transverse velocity distribution over the various freeze-out surfaces for 11.6A GeV Au+Au collision
FIG. 23. Space-time distributions for 11.6A GeV Au+Au, hydro. Eight pictures are projections of the emission points on the z,t and r,t planes, for all secondaries, pions, nucleons and kaons, subsequently. Resonance decays are included.
FIG. 24. Same distributions as the previous figure (also 11.6 A GeV Au+Au) but for RQMD output file.

FIG. 25. Triangulation of the freeze-out surface within a single cell. Here $P_1$ is the only vertex with $\xi > \xi_f$ while all other vertices have $\xi < \xi_f$. This cell will yield 3 triangles upon triangulation.