TOPOLOGICAL AND DITOPLOGICAL UNOSEMIGROUPS

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Abstract. In this paper we introduce and study a new topologo-algebraic structure called a (di)topological unosemigroup. This is a topological semigroup endowed with continuous unary operations of left and right units (which have certain continuous division property called the dicontinuity). We show that the class of ditopological unosemigroups contains all topological groups, all topological semilattices, all uniformizable topological unosemigroups, all compact topological unosemigroups, and is closed under the operations of taking subunosemigroups, Tychonoff product, reduced product, semidirect product, and the Hartman-Mycielski extension.

1. Introduction

In this paper we shall introduce and study a new topologo-algebraic structure called a [di]topological (left, right) unosemigroup. This is a topological semigroup endowed with continuous unary operations of (left, right) units [having certain continuous division property called the dicontinuity]. Introducing topological unosemigroups was motivated by the problem of generalization of Hryniv’s Embedding Theorems [7] (on embeddings of Clifford compact topological inverse semigroups into products of cones over compact topological groups) beyond the class of compact topological inverse semigroups.

Topological and ditopological (left, right) unosemigroups will be introduced in Sections 2 and 3. In Sections 4 and 5 we shall study the class of ditopological unosemigroups and shall show that it contains all topological groups, all topological semilattices, and all compact Hausdorff topological unosemigroups. In Section 6 we shall introduce and study some operations over topological (left, right) unosemigroups.

2. Topological unosemigroups

In this section we shall introduce the notion of a topological (left, right) unosemigroup.

Let \( S \) be a semigroup (i.e., a non-empty set \( S \) endowed with an associative binary operation \( \cdot : S \times S \to S \)). An unary operation \( \lambda : S \to S \) is called a left unit operation on \( S \) if \( \lambda(x) \cdot x = x \) for all \( x \in S \). A semigroup \( S \) endowed with a left unit operation \( \lambda : S \to S \) is called a left unosemigroup.

By analogy we can introduce right unosemigroups. Namely, an unary operation \( \rho : S \to S \) on a semigroup \( S \) is called a right unit operation on \( S \) if \( x \cdot \rho(x) = x \) for all \( x \in S \). A right unosemigroup is a semigroup \( S \) endowed with a right unit operation \( \rho : S \to S \).

A unosemigroup is a semigroup \( S \) endowed with a left unit operation \( \lambda : S \to S \) and a right unit operation \( \rho : S \to S \).

In should be mentioned that the class of left (resp. right) unosemigroups contains the class of domain (resp. range) semigroups, considered in [5].

Now we introduce topological versions of these notions.

A topological semigroup is a topological space \( S \) endowed with a continuous associative binary operation \( \cdot : S \times S \to S \). A topological left unosemigroup is a topological semigroup \( S \) endowed with a continuous left unit operation \( \lambda : S \to S \). A topological right unosemigroup is a topological semigroup \( S \) endowed with a continuous right unit operation \( \rho : S \to S \). A topological unosemigroup is a topological semigroup \( S \) endowed with a continuous left unit operation \( \lambda : S \to S \) and a continuous right unit operation \( \rho : S \to S \).

Topological (left, right) unosemigroups are objects of the category whose morphisms are continuous (left, right) unosemigroup homomorphisms. A function \( h : X \to Y \) between two topological left unosemigroups \( (X, \lambda_X) \) and \( (Y, \lambda_Y) \) is called a left unosemigroup homomorphism if \( h \) is a semigroup homomorphism and \( h(\lambda_X(x)) = \lambda_Y(h(x)) \) for all \( x \in X \).
\(\lambda_x(h(x))\) for all \(x \in X\). By analogy we can define right unosemigroup homomorphisms between topological right unosemigroups and unosemigroup homomorphisms between topological unosemigroups.

3. Ditopological unosemigroups

To introduce ditopological (left, right) unosemigroups, we first introduce two kinds of division operations on each semigroup. Namely, for two points \(a, b\) of a semigroup \(S\) consider the sets

\[ a \succ b = \{ x \in S : ax = b \} \quad \text{and} \quad b \prec a = \{ x \in S : xa = b \} \]

which can be thought as the results of left and right division of \(b\) by \(a\). Respectively, for two subsets \(A, B \subseteq S\) the sets

\[ A \succ B = \bigcup_{(a,b) \in A \times B} a \succ b \quad \text{and} \quad B \prec A = \bigcup_{(a,b) \in A \times B} b \prec a \]

can be thought as the results of left and right division of \(B\) by \(A\).

A continuous left unit operation \(\lambda : S \to S\) on a topological semigroup \(S\) is called dicontinuous at a point \(x \in S\) if for every neighborhood \(O_x \subseteq S\) of \(x\) there are neighborhoods \(U_x \subseteq S\) and \(W_{\lambda(x)} \subseteq \lambda(S)\) of the points \(x\) and \(\lambda(x)\) in \(S\) and \(\lambda(S)\) respectively, such that

\[ (W_{\lambda(x)} \times U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subseteq O_x. \]

A left unit operation \(\lambda : S \to S\) is defined to be dicontinuous if it is dicontinuous at each point \(x \in X\).

A ditopological left unosemigroup is a topological semigroup \(S\) endowed with a dicontinuous left unit operation \(\lambda : S \to S\).

A trivial (but important) example of a ditopological left unosemigroup is an idempotent topological left unosemigroup. A (topological) left unosemigroup \((S, \lambda)\) is called idempotent if \(\lambda(x) = x\) for all \(x \in S\). In this case \(x = \lambda(x) \cdot x = xx\), which means that \(S\) is an idempotent semigroup.

**Proposition 1.** Each idempotent topological left unosemigroup \((S, \lambda)\) is ditopological.

**Proof.** Given a point \(x \in S\) and a neighborhood \(O_x \subseteq S\) of \(x\) put \(U_x = W_{\lambda(x)} = O_x\) and observe that

\[ (W_{\lambda(x)} \times U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subseteq W_{\lambda(x)} = O_x, \]

which means that the left unit operation \(\lambda\) is dicontinuous at \(x\) and hence the topological left unosemigroup \((S, \lambda)\) is dicontinuous.

By analogy we can introduce the right and two-sided versions of these notions.

Namely, a continuous right unit operation \(\rho : S \to S\) on a topological semigroup \(S\) is called dicontinuous if for every point \(x \in S\) and every neighborhood \(O_x \subseteq S\) of \(x\) there are neighborhoods \(U_x \subseteq S\) and \(W_{\rho(x)} \subseteq \rho(S)\) of \(x\) and \(\rho(x)\) such that

\[ (U_x \times W_{\rho(x)}) \cap \rho^{-1}(W_{\rho(x)}) \subseteq O_x. \]

A ditopological right unosemigroup is a topological semigroup \(S\) endowed with a dicontinuous right unit operation \(\rho : S \to S\).

A (topological) right unosemigroup \((S, \rho)\) is called idempotent if \(\rho(x) = x\) for all \(x \in X\). By analogy with Proposition 1 we can prove:

**Proposition 2.** Each idempotent topological right unosemigroup \((S, \rho)\) is ditopological.
4. Uniformizable topological unosemigroups

In this section we shall prove that the discontinuity of a continuous left (right) unit operation on a topological semigroup automatically follows from the right (left) uniformizability of the topological semigroup.

We define a topological semigroup $S$ of inversion $(\_\cdot \_)$ implies:

Corollary 1. Each continuous right unit operation $\lambda : S \to S$ on a right-uniformizable topological semigroup $S$ is dicontinuous. Consequently, each right-uniformizable topological left unosemigroup $(S, \lambda)$ is a ditopological left unosemigroup.

Proof. Let $U$ be a uniformity on $S$ witnessing that the semigroup $S$ is right-uniformizable. To show that the left unit operation $\lambda : S \to S$ is dicontinuous, fix a point $x \in S$ and a neighborhood $O_x \subset S$ of $x$. Since $U$ generates the topology of $S$, the neighborhood $O_x$ contains the ball $B(x, U) = \{y \in S : (x, y) \in U\}$ for some entourage $U \in U$. Find an entourage $V \in U$ such that $V \cdot U \cdot V \subset V$ for all points $x, y \in S$. Here $B(x, V) = \{z \in S : (x, z) \in V\}$ stands for the $V$-ball centered at the point $x \in S$.

Each topological group $G$ is left (resp. right) uniformizable by its left (resp. right) uniformity generated by the base consisting of the entourages $\{(x, y) \in G \times G : y \in x U\}$ (resp. $\{(x, y) \in G \times G : y \in U x\}$) where $U = U^{-1}$ runs over symmetric neighborhoods of the idempotent in $G$.

Theorem 1. Each continuous left unit operation $\lambda : S \to S$ on a right-uniformizable topological semigroup $S$ is dicontinuous. Consequently, each right-uniformizable topological left unosemigroup $(S, \lambda)$ is a ditopological left unosemigroup.

Proof. Let $U$ be a uniformity on $S$ witnessing that the semigroup $S$ is right-uniformizable. To show that the left unit operation $\lambda : S \to S$ is dicontinuous, fix a point $x \in S$ and a neighborhood $O_x \subset S$ of $x$. Since $U$ generates the topology of $S$, the neighborhood $O_x$ contains the ball $B(x, U) = \{y \in S : (x, y) \in U\}$ for some entourage $U \in U$. Find an entourage $V \in U$ such that $V \cdot U \cdot V \subset V$ for all points $x, y \in S$. Here $B(x, V) = \{z \in S : (x, z) \in V\}$ stands for the $V$-ball centered at the point $x \in S$.

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Theorem 2. Each continuous right unit operation $\lambda : S \to S$ on a left-uniformizable topological semigroup $S$ is dicontinuous. Consequently, each left-uniformizable topological right unosemigroup $(S, \lambda)$ is a ditopological right unosemigroup.

Theorems 1 and 2 imply:

Theorem 3. A topological unosemigroup $(S, \lambda, \rho)$ is ditopological if the topological semigroup $S$ is both left-uniformizable and right-uniformizable.

Each compact Hausdorff space $X$ carries a unique uniformity generating its topology. The uniform continuity of continuous maps defined on a compact Hausdorff space implies:

Corollary 1. Each compact Hausdorff topological (left, right) unosemigroup is ditopological.

Since discrete topological semigroups are (left and right) uniformizable by the discrete uniformity, Theorem 3 implies:

Corollary 2. Each discrete topological (left, right) unosemigroup is ditopological.

5. Ditopological inverse semigroups

Continuous left and right unit operations appear naturally in topological inverse semigroups. Let us recall that a semigroup $S$ is called inverse if for each element $x \in S$ there is a unique element $x^{-1} \in S$ (called the inverse to $x$) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. By a topological inverse semigroup we understand an inverse semigroup $S$ endowed with a topology such that the semigroup operation $\cdot : S \times S \to S$ and the operation of inversion $(\_)^{-1} : S \to S$ are continuous.
Each topological inverse semigroup $S$ possesses the canonical continuous left unit operation
\[ \lambda : S \rightarrow S, \quad \lambda : x \mapsto xx^{-1}, \]
and the canonical continuous right unit operation
\[ \rho : S \rightarrow S, \quad \rho : x \mapsto x^{-1}x, \]
which turn $S$ into a topological unosemigroup.

Observe that the sets $\lambda(S) = \{ xx^{-1} : x \in S \}$ and $\rho(S) = \{ x^{-1}x : x \in S \}$ coincide with the set $E(S) = \{ x \in S : xx = x \}$ of idempotents of $S$. It is well-known that the set $E(S)$ is a commutative subsemigroup of the inverse semigroup $S$.

A topological inverse semigroup $S$ is called a **ditopological inverse semigroup** if the topological unosemigroup $(S, \lambda, \rho)$ is ditopological.

The following proposition shows that the discontinuity of the left unit operation on a topological inverse semigroup is equivalent to the discontinuity of the right unit operation.

**Proposition 3.** Let $S$ be a topological inverse semigroup. The left unit operation $\lambda : S \rightarrow S, \lambda : x \mapsto xx^{-1}$, is discontinuous at a point $x \in S$ if and only if the right unit operation $\rho : S \rightarrow S, \rho : x \mapsto x^{-1}x$, is discontinuous at the point $x^{-1} \in S$.

**Proof.** Assume that the left unit operation $\lambda$ is discontinuous at a point $x \in X$. To show that the right unit operation $\rho$ is discontinuous at the point $x^{-1}$, fix a neighborhood $O_{x^{-1}} \subset S$ of $x^{-1}$. Then $O_{x^{-1}} = \{ y^{-1} : y \in O_{x^{-1}} \}$ is a neighborhood of the point $x$ in $S$. By the dicontinuity of the left unit operation $\lambda$ on $S$, the points $x$ and $\lambda(x) = xx^{-1}$ have neighborhoods $U_x \subset S$ and $W_{xx^{-1}} \subset E(S)$ such that $(W_{xx^{-1}} \setminus U_x) \cap \lambda^{-1}(W_{xx^{-1}}) \subset O_{x^{-1}}$.

After the inversion the latter inclusion turns into $(U_x \setminus W_{xx^{-1}}) \cap \rho^{-1}(W_{xx^{-1}}) \subset O_{x^{-1}}$. So, $U_{x^{-1}} = U_x^{-1}$ and $W_{\rho(x^{-1})} = W_{xx^{-1}}^{-1} \cap W_{xx^{-1}}$ are neighborhoods of the points $x^{-1}$ and $\rho(x^{-1}) = xx^{-1}$ witnessing the discontinuity of the right unit operation $\rho$ at $x^{-1}$.

By analogy we can prove that the discontinuity of the right unit operation $\rho$ at $x^{-1} \in S$ implies the discontinuity of the left unit operation $\lambda$ at the point $x$.

By a **topological semilattice** we understand a commutative idempotent topological semigroup $S$. Each topological semilattice is a topological inverse semigroup with $x^{-1} = x$ for all $x \in S$.

**Theorem 4.** The class of ditopological inverse semigroups includes all topological groups, all topological semilattices, all compact Hausdorff topological inverse semigroups, and all discrete topological inverse semigroups.

**Proof.** Let $G$ be a topological group. In this case $\lambda$ and $\rho$ are constant operations assigning to each $x \in G$ the unique idempotent $e$ of the group $G$, so $\lambda(G) = \rho(G) = \{ e \}$ is a singleton. By Proposition 3, to prove that $G$ is a ditopological inverse semigroup, it suffices to check that the left unit operation $\lambda$ is discontinuous at each point $x \in G$. Given any neighborhood $O_x \subset G$ of $x$, put $U_x = O_x$ and $W_{\lambda(x)} = \{ e \}$ and observe that
\[ (W_{\lambda(x)} \setminus U_x) \cap \lambda^{-1}(W_{\lambda(x)}) = (\{ e \} \setminus U_x) \cap G = U_x = O_x, \]
which means that the left unit operation $\lambda$ is discontinuous at $x$. So, the topological group $G$ is ditopological. The same conclusion can be also derived from the right-uniformizability of topological groups and Theorem 4.

Each topological semilattice, being an idempotent topological unosemigroup, is a ditopological unosemigroup according to Propositions 3 and 4. Corollaries 3 and 4 imply that the class of ditopological inverse semigroups contains all compact Hausdorff topological inverse semigroups and all discrete topological inverse semigroups.

Now we present a simple example of a locally compact commutative topological inverse semigroup $S$ which is not ditopological.

**Example 1.** There exists a commutative topological inverse semigroup $S$ such that
(1) $S$ is countable, metrizable, and locally compact;
(2) the idempotent semilattice $E(S)$ of $S$ is compact and each subgroup of $S$ has cardinality $\leq 2$;
(3) the left unit operation $\lambda : S \rightarrow S, \lambda : x \mapsto xx^{-1}$, is not discontinuous, which implies that the topological Clifford semigroup $S$ is not ditopological.
Proof. Let $H$ be a two-element group and $T = \{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a convergent sequence endowed with the semilattice operation

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then the product $T \times H$ is a commutative inverse semigroup whose idempotent semilattice coincides with the set $E = T \times \{e\}$ where $e$ is the idempotent of the group $H$. Let $h$ be the non-identity element of the two-element group $H$.

Endow the inverse semigroup $S = T \times H$ with the topology $\tau$ which induces the original (compact metrizable) topology on the set $T \times \{e\}$ and the discrete topology on $T \times \{h\}$. It is easy to see that the topology $\tau$ is metrizable, locally compact, and turns $S$ into a topological inverse semigroup.

We claim that the left unit operation $\lambda : S \to S$, $\lambda : x \mapsto xx^{-1}$, is not dicontinuous at the point $x = (0, h) \in T \times H$. Assuming the opposite, for the neighborhood $O_x = \{x\}$ we would find neighborhoods $U_x \subset S$ of $x$ and $W_{\lambda(x)} \subset \lambda(S)$ of the idempotent $\lambda(x) = (0, e)$ such that

$$(W_{\lambda(x)} \wedge U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset O_x = \{x\}.$$  

By the definition of the topology on $S$, the neighborhood $W_{\lambda(x)}$ contains a point $(\frac{1}{n}, e)$ for some $n \in \mathbb{N}$. It follows from $(0, e) \cdot (\frac{1}{n}, h) = (0, h) = x \in U_x$ and $\lambda((\frac{1}{n}, h))) = (\frac{1}{n}, e) \in W_{\lambda(x)}$ that

$$(\frac{1}{n}, h) \in (W_{\lambda(x)} \wedge U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset \{x\} = \{(0, h)\}$$

and hence $\frac{1}{n} = 0$, which is a desired contradiction. $\Box$

6. Operations over ditopological unosemigroups

As we know from Theorem 4 the class of ditopological unosemigroups contains all topological groups, all topological semilattices, and all compact Hausdorff topological inverse semigroups. In this section we shall show that this class is stable under many natural operations over topological unosemigroups.

6.1. Subunosemigroups of topological (left, right) unosemigroups. Let $(S, \lambda)$ be a topological left unosemigroup and $X \subset S$ be a subsemigroup such that $\lambda(X) \subset X$. Then $\lambda|X : X \to X$ is a continuous left unit operation on $X$, which implies that $(X, \lambda|X)$ is a topological left unosemigroup. Such left unosemigroup will be called a left subunosemigroup of $(S, \lambda)$.

By analogy we can define right subunosemigroups of topological right unosemigroups and subunosemigroups of topological unosemigroups.

Since the dicontinuity of a left or right unit operation $u : S \to S$ on a topological semigroup $S$ implies the dicontinuity of the restriction $u|X$ to any subsemigroup $X \subset S$ with $u(X) \subset X$, we get:

Proposition 4. If a topological (left, right) unosemigroup $S$ is ditopological, then so is any its subunosemigroup $X \subset S$.

6.2. Tychonoff products of topological (left, right) unosemigroups. For any family of topological left unosemigroups $(S_\alpha, \lambda_\alpha)$, $\alpha \in A$, the Tychonoff product $S = \prod_{\alpha \in A} S_\alpha$ carries a natural structure of a topological left unosemigroup endowed with the left unit operation $\lambda : S \to S$, $\lambda : (x_\alpha)_{\alpha \in A} \mapsto (\lambda_\alpha(x_\alpha))_{\alpha \in A}$. By analogy we can define the operation of Tychonoff product of (right) unosemigroups.

The dicontinuity of left (right) unit operations on the topological semigroups $S_\alpha$, $\alpha \in A$, implies the dicontinuity of the left (right) unit operation on their Tychonoff product. This proves our next simple:

Proposition 5. The Tychonoff product of ditopological (left, right) unosemigroups is a ditopological (left, right) unosemigroup.

6.3. The reduced product of topological (left, right) unosemigroups. Let $X, Y$ be two topological semigroups and $I \subset X$ be a closed two-sided ideal in $X$. By the reduced product $X \times_I Y$ of the semigroups $X$ and $Y$ over the ideal $I$ we mean the set $I \cup (X \setminus I) \times Y$ endowed with the smallest topology such that

- the map $(X \setminus I) \times Y \to X \times_I Y$ is a topological embedding,
- the projection $\pi : X \times_I Y \to X$ is continuous.

Proof. Let $H$ be a two-element group and $T = \{0\} \cup \{\frac{1}{n}\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a convergent sequence endowed with the semilattice operation

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then the product $T \times H$ is a commutative inverse semigroup whose idempotent semilattice coincides with the set $E = T \times \{e\}$ where $e$ is the idempotent of the group $H$. Let $h$ be the non-identity element of the two-element group $H$.

Endow the inverse semigroup $S = T \times H$ with the topology $\tau$ which induces the original (compact metrizable) topology on the set $T \times \{e\}$ and the discrete topology on $T \times \{h\}$. It is easy to see that the topology $\tau$ is metrizable, locally compact, and turns $S$ into a topological inverse semigroup.

We claim that the left unit operation $\lambda : S \to S$, $\lambda : x \mapsto xx^{-1}$, is not dicontinuous at the point $x = (0, h) \in T \times H$. Assuming the opposite, for the neighborhood $O_x = \{x\}$ we would find neighborhoods $U_x \subset S$ of $x$ and $W_{\lambda(x)} \subset \lambda(S)$ of the idempotent $\lambda(x) = (0, e)$ such that

$$(W_{\lambda(x)} \wedge U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset O_x = \{x\}.$$  

By the definition of the topology on $S$, the neighborhood $W_{\lambda(x)}$ contains a point $(\frac{1}{n}, e)$ for some $n \in \mathbb{N}$. It follows from $(0, e) \cdot (\frac{1}{n}, h) = (0, h) = x \in U_x$ and $\lambda((\frac{1}{n}, h))) = (\frac{1}{n}, e) \in W_{\lambda(x)}$ that

$$(\frac{1}{n}, h) \in (W_{\lambda(x)} \wedge U_x) \cap \lambda^{-1}(W_{\lambda(x)}) \subset \{x\} = \{(0, h)\}$$

and hence $\frac{1}{n} = 0$, which is a desired contradiction. $\Box$
This formula is well-defined since for every $z \in (X \setminus I) \times Y$, we need to show that

$$q(x, y) = \begin{cases} x & \text{if } x \in I; \\ (x, y) & \text{otherwise} \end{cases}$$

is a semigroup homomorphism.

If the semigroups $X$ and $Y$ carry continuous left unit operations $\lambda_X : X \to X$ and $\lambda_Y : Y \to Y$ such that $\lambda_X(I) \subset I$, then the reduced product $X \times_I Y$ carries a natural left unit operation $\lambda : X \times_I Y \to X \times_I Y$ defined by the formula:

$$\lambda(z) = \begin{cases} \lambda_X(z) & \text{if } z \in I \\ (\lambda_X(x), \lambda_Y(y)) & \text{if } z = (x, y) \in (X \setminus I) \times Y. \end{cases}$$

This formula is well-defined since for every $x \in X \setminus I$ the equality $x = \lambda_X(x) \cdot x \notin I$ implies that $\lambda_X(x) \notin I$.

The semigroup $X \times_I Y$ endowed with the left unit operation $\lambda$ is a topological left unosemigroup called the reduced product of the topological left unosemigroups $(X, \lambda_X)$ and $(Y, \lambda_Y)$.

**Theorem 5.** If the topological left unosemigroups $X = (X, \lambda_X)$ and $Y = (Y, \lambda_Y)$ are ditopological, then so is their reduced product $X \times_I Y = (X \times_I Y, \lambda)$.

**Proof.** Assume that the topological left unosemigroups $X = (X, \lambda_X)$ and $Y = (Y, \lambda_Y)$ are ditopological. To show that $X \times_I Y$ is a ditopological left unosemigroup, fix a point $z \in X \times_I Y$ and a neighborhood $O_z \subset X \times_I Y$ of $z$. We divide the proof into two parts.

First assume that $z = (x, y) \in (X \setminus I) \times Y$. In this case we can assume that $O_z = O_x \times O_y$ for some open neighborhoods $O_x \subset X \setminus I$ and $O_y \subset Y$ of the points $x \in X \setminus I$ and $y \in Y$, respectively. Since $X$ and $Y$ are left ditopological unosemigroups, there are open neighborhoods $U_x \subset X$, $W_{\lambda_X}(x) \subset \lambda_X(X)$ of the points $x$, $\lambda_X(x)$ and $U_y \subset Y$, $W_{\lambda_Y}(y) \subset \lambda_Y(Y)$ of $y$, $\lambda_Y(y)$ such that $(W_{\lambda_X}(x) \times U_x) \cap \lambda_X^{-1}(W_{\lambda_X}(x)) \subset O_x$ and $(W_{\lambda_Y}(y) \times U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y}(y)) \subset O_y$.

It follows from $\lambda(x) \cdot x = x \notin I$ that $\lambda(x) \notin I$. So, we can assume that the sets $U_x$ and $W_{\lambda(x)}$ are contained in $X \setminus I$.

We claim that the open sets $U_z = U_x \times U_y$ and $W_{\lambda(z)} = W_{\lambda_X(z)} \times W_{\lambda_Y(y)}$ witness that the left unit operation $\lambda$ on $X \times_I Y$ is discontinuous at the point $z$. Given any point $u \in X \times_I Y$ such that $u \in (W_{\lambda(z)} \times U_z) \cap \lambda^{-1}(W_{\lambda(z)})$, we need to show that $u \notin O_z$. It follows that $\lambda(u) \in W_{\lambda(z)}$ and $\mu u \in U_z$ for some $\mu \in W_{\lambda(z)}$, which implies that $w, u \notin I$ and hence $w = (w_{1w}, w_{2u})$ and $u = (x_u, y_u)$ for some points $x, x_u \in X \setminus I$ and $y, y_u \in Y$. Since $w = (x_{1w}, w_{2u}) = (x_{1w}, y_{2u}), u = (x_u, y_u) \in U_z$, $x_{1w} x_u \in U_z \times U_y$ and $\lambda(u) = (\lambda_X(x_u), \lambda_Y(y_u)) \in W_{\lambda(z)} = W_{\lambda_X(z)} \times W_{\lambda_Y(y)}$, we obtain $x_{1w} x_u \in U_z \times U_y \subset U_z$ and $\lambda_X(x_u) \in W_{\lambda_X(z)}$. Hence $x_{1w} \in (W_{\lambda_X(z)} \times U_z) \cap \lambda_X^{-1}(W_{\lambda_X(z)}) \subset O_x$ and $y_u \in (W_{\lambda_Y(y)} \times U_y) \cap \lambda_Y^{-1}(W_{\lambda_Y(y)}) \subset O_y$ and thus $u \notin O_x \times O_y = O_z$.

In case $z \in I$, we can assume that $O_z$ is of the form $O_z = \pi^{-1}(O_1')$, where $O_1'$ is an open neighborhood of the point $z \in I$ in the left unosemigroup $X$. Since $\lambda_X$ left continuous at $z$, there are open neighborhoods $U_z' \subset X$ and $W_{\lambda_X(z)}' \subset \lambda_X(X)$ of the points $z \in X$ and $\lambda_X(z) \in \lambda_X(X)$ such that $(W_{\lambda_X(z)} \times U_z') \cap \lambda_X^{-1}(W_{\lambda_X(z)}) \subset O_z'$.

The latter inclusion implies that for the open neighborhoods $U_z = \pi^{-1}(O_1') \subset X \times_I Y$ of $z$ and $W_{\lambda(z)} = \lambda(X \times_I Y) \cap \pi^{-1}(W_{\lambda_X(z)}) \subset \lambda(X \times_I Y)$ of $\lambda(z)$ we get

$$(W_{\lambda(z)} \times U_z) \cap \lambda^{-1}(W_{\lambda(z)}) \subset \pi^{-1}((W_{\lambda_X(z)} \times U_z') \cap \lambda_X^{-1}(W_{\lambda_X(z)}')) \cap \pi^{-1}(O_1') = O_z,$$

which witnesses that the left unit operation $\lambda$ on $X \times_I Y$ is left discontinuous at $z$. Thus, the reduced product $X \times_I Y = (X \times_I Y, \lambda)$ is a ditopological left unosemigroup.

By analogy we can introduce the reduced product of topological right unosemigroups. Namely, if $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ are two topological right unosemigroups and $I \subset X$ is a closed two-sided ideal with $\rho_X(I) \subset I$,
then the reduced product \( X \times_I Y \) carries an induced right unit operation \( \rho : X \times_I Y \to X \times_I Y \) defined by

\[
\rho(z) = \begin{cases} 
\rho_X(z) & \text{if } z \in I \\
(\rho_X(x), \rho_Y(y)) & \text{if } z = (x, y) \in (X \setminus I) \times Y.
\end{cases}
\]

The reduced product \( X \times_I Y \) endowed with the right unit operation \( \rho \) is a topological right unosemigroup called the reduced product of the topological right unosemigroups \( (X, \lambda_X) \) and \( (Y, \lambda_Y) \).

By analogy with Theorem 5, we can prove:

**Theorem 6.** If the topological right unosemigroups \( X = (X, \lambda_X) \) and \( Y = (Y, \rho_Y) \) are ditopological, then so is their reduced product \( X \times_I Y = (X \times_I Y, \rho) \).

The above discussion implies that for topological unosemigroups \( X = (X, \lambda_X, \rho_X) \) and \( Y = (Y, \lambda_Y, \rho_Y) \) and a closed two-sided ideal \( I \subseteq X \) with \( \lambda_X(I) \cup \rho_X(I) \subseteq I \), the triple \( X \times_I Y = (X \times_I Y, \lambda, \rho) \) is a topological unosemigroup. This topological unosemigroups will be called reduced product of the topological unosemigroups \( X \) and \( Y \). Theorems 5 and 6 imply:

**Corollary 3.** If topological unosemigroups \( X \) and \( Y \) are ditopological, then so is their reduced product \( X \times_I Y \).

Now we present some important examples of reduced products.

**Example 2.** Let \( G \) be a topological group and let \( 2 = ([0, 1], \min) \) be a two-element semilattice endowed with the discrete topology. By Theorem 4, the semigroups \( G \) and \( 2 \) endowed with the canonical left and right unit operations are ditopological inverse semigroups and by Corollary 5 so is their reduced product \( G = 2 \times_{\{0\}} G \) called the 0-extension of \( G \).

**Example 3.** Let \( G \) be a topological group and let \( I \) be the unit interval \([0, 1]\) endowed with the semilattice operation of minimum. By Theorem 4 the semigroups \( G \) and \( I \) endowed with the canonical left and right unit operations are ditopological inverse semigroups and by Corollary 5 so is their reduced product \( G = I \times_{\{0\}} G \) called the cone over \( G \).

The 0-extensions and cones of topological groups will be essentially used in the paper [2] devoted to constructing embeddings of Clifford ditopological inverse semigroups into Tychonoff products of topological semilattices and cones over topological groups.

### 6.4. Semidirect products of topological unosemigroups.

In this subsection we shall consider the operation of a semidirect product of topological (left, right) unosemigroups. Let us mention that semidirect products of semigroups were studied in Chapter 2 of [4].

By a continuous action of a topological semigroup \( F \) on a topological semigroup \( S \) we understand a continuous function \( \alpha : F \times S \to S \) having the following two properties:

- for each \( f \in F \) the function \( \alpha_f : S \to S, \alpha_f : x \mapsto \alpha(f, x) \), is a semigroup homomorphism of \( S \);
- \( \alpha_{fg} = \alpha_f \circ \alpha_g \) for each \( f, g \in F \).

The action \( \alpha : F \times S \to S \) induces a continuous associative binary operation

\[
(s, f) \cdot (t, g) = (s \cdot \alpha_f(t), f \cdot g)
\]

on the product \( S \times F \). The product \( S \times F \) endowed with this binary operation is denoted by \( S \times_\alpha F \) and called the semidirect product of the topological semigroups \( S \) and \( F \).

We shall say that the action \( \alpha : F \times S \to S \)

- respects a (left, right) unit operation \( u : F \to F \) on \( F \) if \( \alpha(u(f), s) = s \) for all \( (f, s) \in F \times S \);
- respects a (left, right) unit operation \( u : S \to S \) on \( S \) if \( \alpha(f, u(s)) = u(s) \) for all \( (f, s) \in F \times S \).

If \((F, \lambda_F)\) and \((S, \lambda_S)\) are topological left unosemigroups and a continuous action \( \alpha : F \times S \to S \) of \( F \) on \( S \) respects the left unit operation \( \lambda_F \), then the unary operation

\[
\lambda : S \times_\alpha F \to S \times_\alpha F, \quad \lambda : (s, f) \mapsto (\lambda_S(s), \lambda_F(f)),
\]

is a continuous left unit operation on the semidirect product \( S \times_\alpha F \) as

\[
(\lambda_S(s), \lambda_F(f)) \cdot (s, f) = (\lambda_S(s) \cdot \alpha(\lambda_F(f), s), \lambda_F(f) \cdot f) = (\lambda_S(s) \cdot s, f) = (s, f)
\]

for all \((s, f) \in S \times F\).
Therefore, $S \times_\alpha F = (S \times_\alpha F, \lambda)$ is a topological left unosemigroup, called the semidirect product of the topological left unosemigroups $S = (S, \lambda_S)$ and $F = (F, \lambda_F)$.

**Theorem 7.** Let $S = (S, \lambda_S)$ and $F = (F, \lambda_F)$ be topological left unosemigroups and $\alpha : F \times S \to F$ be a continuous action of $F$ on $S$, which respects the left operation $\lambda_F$ of the left unosemigroup $F$. The topological left unosemigroup $S \times_\alpha F$ is ditopological if and only if the topological left unosemigroups $S$ and $F$ are ditopological.

**Proof.** To prove the “only if” part, assume that $S \times_\alpha F$ is a ditopological left unosemigroup. We need to prove that the topological left unosemigroups $S$ and $F$ are ditopological.

To prove the discontinuity of the left unit operation $\lambda_S$, fix any point $s \in S$ and a neighborhood $O_s$ of $s$ in $S$. Fix any point $f \in F$ and consider the neighborhood $O_{s,f} = O_s \times F$ of $(s, f)$ in the topological semigroup $S \times_\alpha F$. This implies that $\lambda_S(s,f)$ is discontinuous at $(s, f)$ in $S \times_\alpha F$, where $\lambda_S(s,f)$ yields neighborhoods $U(s,f) \subset S \times F$ and $W(s,f) \subset \lambda(S \times F)$ of $(s, f)$ and $(\lambda_S(s), \lambda_F(f))$ such that $(W(s,f), U(s,f)) \cap \lambda^{-1}(W(s,f)) \subset O(s,f)$. Clearly, we can assume that these neighborhoods are of the form $U(s,f) = U_s \times U_f$ and $W(s,f) = W(s) \times W(f)$ for some open sets $U_s \subset S$, $W(s) \subset \lambda(S)$, $U_f \subset F$, and $W(f) \subset \lambda(F)$.

We claim that the neighborhoods $U_s$ and $W(s)$ witness that $\lambda_S$ is ditopological at $s$. Let $t \in (W(s) \setminus U_s) \cap \lambda^{-1}(W(s))$. This implies $\lambda_S(t) \in W(s)$ and $w \in U_s$ for some $w \in W(s)$. Then for the elements $(t, f) \in S \times_\alpha F$ and $(w, \lambda_F(f)) \in W(s)$ we get

$$(w, \lambda_F(f)) \cdot (t, f) = (w \cdot \alpha \lambda_F(f))(t), \lambda_F(f) \cdot f = (w t, f) \in U_s \times U_f = U(s,f)$$

and

$$\lambda(s) = (\lambda_S(s), \lambda_F(f)) \in W(s) \times W(f) = W(s,f).$$

Here we used the fact that the action $\alpha$ respects the left unit operation $\lambda_F$. The choice of the neighborhoods $U(s,f)$ and $W(s,f)$ guarantees that $(t, f) \in O(s,f)$ and hence $t \in O_s$.

To check the discontinuity of the left unit operation $\lambda_F$, take any point $f \in F$ and a neighborhood $O_f \subset F$ of $f$. Fix any element $s \in S$ and consider the neighborhood $O_{s,f} = S \times O_f$ of $(s, f)$ in $S \times_\alpha F$. The discontinuity of the left unit operation $\lambda$ on $\lambda(S \times F)$ yields neighborhoods $\lambda(U(s,f)) \subset S \times F$ and $\lambda(W(s,f)) \subset \lambda(S \times F)$ of $(s, f)$ and $(\lambda_S(s), \lambda_F(f))$ such that $(\lambda(U(s,f)), W(s,f)) \cap \lambda^{-1}(\lambda(U(s,f))) \subset O(s,f)$. We lose no generality assuming that $U(s,f) = U_s \times U_f$ and $W(s,f) = W(s) \times W(f)$ for some open sets $U_s \subset S$, $W(s) \subset \lambda(S)$, $U_f \subset F$, and $W(f) \subset \lambda(F)$.

We claim that the neighborhoods $U_f$ and $W(s)$ witness the discontinuity of $\lambda_F$ at the point $f$. Given any point $g \in (W(s), f) \cap \lambda^{-1}(W(s))$, observe that $\lambda_F(g) \in W(s)$ and $w \in U_f$ for some $w \in W(s) \subset \lambda(F)$. Taking into account that the action $\alpha$ respects the left unit operation $\lambda_F$ and $w \in \lambda(F)$, we conclude that $\alpha_w(s) = s$. Then for the elements $(s, g)$ and $(\lambda(s), w) \in W(s) \times W(f) = W(s,f)$ we get

$$(\lambda_S(s), s) \cdot (s, g) = (\lambda_S(s) \cdot \alpha_w(s), s w) = (s, s, w) = (s, g) \in U(s,f)$$

and $\lambda(s) = (\lambda_S(s), \lambda_F(g)) \in W(s) \times W(f) = W(s,f)$, which means that

$$\lambda(s, g) \in (\lambda(U(s,f)) \setminus U(s, f)) \cap \lambda^{-1}(\lambda(U(s,f))) \subset O(s,f) = S \times O_f$$

and hence $g \in O_f$. This completes the proof of the “only if” part of the theorem.

To prove the “if” part, assume that the topological left unosemigroups $S$ and $F$ are ditopological. We need to check that the left unit operation $\lambda : (s, f) \mapsto (\lambda_S(s), \lambda_F(f))$ on $S \times_\alpha F$ is discontinuous at every point $(s, f) \in S \times_\alpha F$. Fix any open neighborhood $O_{s,f} \times O_f$ of $(s, f)$ in $S \times_\alpha F$. We lose no generality assuming that it is of basic form: $O_{s,f} = O_s \times O_f$, where $O_s$ and $O_f$ are open neighborhoods of $s$ and $f$ in $S$ and $F$, respectively.

By the discontinuity of the left unit operation $\lambda_F$ at $f$, there are neighborhoods $U_f \subset F$ and $W(s,f) \subset \lambda(F)$ of $f$ and $\lambda_F(f)$ such that $(W(s,f) \setminus U_f) \cap \lambda^{-1}(W(s,f)) \subset O_f$. By the discontinuity of the left unit operation $\lambda_S$ at $s$, there are neighborhoods $U_s \subset S$ and $W(s) \subset \lambda(S)$ of $s$ and $\lambda_S(s)$ such that $(W(s) \setminus U_s) \cap \lambda^{-1}(W(s)) \subset O_s$.

We claim that the neighborhoods $U_{s,f} = U_s \times U_f$ and $W(s,f) = W(s) \times W(f)$ of $(s, f)$ and $w \in W(s)$ witness that the left unit operation $\lambda$ is ditopological at $(s, f)$. Given any pair $(t, g) \in (W(s) \setminus U(s,f)) \cap \lambda^{-1}(W(s,f))$, we need to show that $(t, g) \in O(s,f)$. It follows that $(w, h) \cdot (t, g) \in U(s,g)$ for some pair $(w, h) \in W(s,f)$.
into account that the action $\alpha$ respects the left unit operation $\lambda_F$ and $h \in W_{\lambda_F(f)} \subset \lambda_F(F)$, we conclude that

$$(w, h) \cdot (t, g) = (w \cdot \alpha_h(t), hg) = (wt, hg)$$

and hence $(wt, hg) \in U_{s, f} = U_s \times U_f$ and

$$(t, g) \in \left( (W_{\lambda_S(s)} \times U_s) \cap \lambda_S^{-1}(W_{\lambda_S(s)}) \right) \times \left( (W_{\lambda_F(f)} \times U_f) \cap \lambda_F^{-1}(W_{\lambda_F(f)}) \right) \subset O_s \times O_f = O_{s,f}.$$

□

Now, given two topological right unosemigroups $S = (S, \rho_S)$ and $F = (F, \rho_F)$ and a continuous action $\alpha : F \times S \to S$ of $F$ on $S$, we shall define a right unit operation on the semidirect product $S \rtimes_{\alpha} F$. This can be done under an additional assumption that the action $\alpha$ is $\rho_S$-invertible in the sense that for every $f \in F$ the restriction $\tilde{\alpha}_f = \alpha_f|\rho_S(S)$ is a bijective map of $\rho_S(S)$ and the map

$$\alpha^- : F \times \rho_S(S) \to \rho_S(S), \quad \alpha^- : (f, s) \mapsto \tilde{\alpha}_f^{-1}(s),$$

is continuous.

In this case the map $\rho : S \times_{\alpha} F \to S \times_{\alpha} F$ defined by

$$\rho(s, f) = (\tilde{\alpha}_f^{-1}(\rho_S(s)), \rho_F(f)) = (\alpha^- (f, \rho_S(s)), \rho_F(f))$$

is continuous.

Since

$$(s, f) \cdot \rho(s, f) = (s, f) \cdot (\tilde{\alpha}_f^{-1}(\rho_S(s)), \rho_F(f)) = (s \cdot \alpha_f \circ \tilde{\alpha}_f^{-1}(\rho_S(s)), f \cdot \rho_F(f)) =

= (s \cdot \tilde{\alpha}_f \circ \tilde{\alpha}_f^{-1}(\rho_S(s)), f) = (s \cdot \rho_S(s), f) = (s, f),$$

the map $\rho$ is a continuous right unit operation on $S \times_{\alpha} F$. Therefore, $S \times_{\alpha} F = (S \times_{\alpha} F, \rho)$ is a topological right unosemigroup, called the semidirect product of the topological right unosemigroups $S = (S, \rho_S)$ and $F = (F, \rho_F)$.

Let us observe that if the action $\alpha$ respects the right unit operation $\rho_S$, then for every $f \in F$ the restriction $\tilde{\alpha}_f = \alpha_f|\rho_S(S)$ is an identity map of $\rho_S(S)$ and hence the action $\alpha$ is $\rho_S$-invertible. Moreover, in this case $\rho(s, f) = (\rho_S(s), \rho_F(f))$ for all $(s, f) \in S \times F$.

The following propositions will help us to detect $\rho_S$-invertible actions.

Proposition 6. A continuous action $\alpha : F \times S \to S$ of a topological right unosemigroup $(F, \rho_F)$ on a topological right unosemigroup $(S, \rho_S)$ is $\rho_S$-invertible if

1. $\alpha$ respects the right unit operation $\rho_F$;
2. $\alpha_f|\rho_S(S) = \rho_S(S)$ for all $f \in F$;
3. there is a continuous unary operation $(\cdot)^{-1} : F \to F$ such that $\rho_F(f) = f^{-1}f$ for all $f \in F$.

Proof. Taking into account that the action $\alpha$ preserves the right unit operation $\rho_F : F \to F, \rho_F : f \mapsto f^{-1}f$, we conclude that for every $f \in F$ and $s \in S$ we get

$$s = \alpha(\rho_F(f), s) = \alpha(f^{-1}f, s) = \alpha_f^{-1}(s) = \alpha_f^{-1} \circ \alpha_f(s),$$

which implies that the homomorphism $\alpha_f : S \to S$ is injective and hence has the inverse $\alpha_f^{-1} : \alpha_f(S) \to S$. It follows from $\alpha_f|\rho_S(S) = \rho_S(S)$ that the restriction $\tilde{\alpha}_f = \alpha_f|\rho_S(S)$ is a bijective map of $\rho_S(S)$.

It remains to check that the map $\alpha^- : F \times \rho_S(S) \to \rho_S(S), \alpha^- : (f, s) \mapsto \tilde{\alpha}_f^{-1}(s)$, is continuous. For this observe that the function $\alpha^-$ coincides with the continuous function $\beta : F \times \rho_S(S) \to \rho_S(S), \beta(f, s) \mapsto \alpha(f^{-1}, s)$. Indeed, given any $f \in F$ and $s \in \rho_S(S)$, we can find a unique point $x \in \rho_S(S)$ with $s = \tilde{\alpha}_f(x) = \alpha_f(x)$ and conclude that

$$\beta(f, s) = \alpha(f^{-1}, s) = \alpha_f^{-1}(s) = \alpha_f^{-1} \circ \alpha_f(x) = \alpha_f^{-1}(x) = \alpha_{\rho_F(f)}(x) = x = \tilde{\alpha}_f^{-1}(s) = \alpha^-(f, s).$$

□

Now we study the semidirect products of ditopological right unosemigroups.

Theorem 8. Let $S = (S, \rho_S)$ and $F = (F, \rho_F)$ be topological right unosemigroups and $\alpha : F \times S \to S$ be a $\rho_S$-invertible continuous action of $F$ on $S$. The semidirect product $S \rtimes_{\alpha} F$ is a ditopological right unosemigroup if and only if the topological right unosemigroups $S$ and $F$ are ditopological.
Proof. To prove the “only if” part, assume that the topological right unosemigroup $S \times_\alpha F = (S \times_\alpha F, \rho)$ is ditopological. We need to show that the right unit operations $\rho_S$ and $\rho_F$ are discontinuous.

To prove the discontinuity of the right unit operation $\rho_S$, fix any point $s \in S$ and a neighborhood $O_s \subset S$ of $s$. Fix any point $f \in \rho_F(F)$ and consider the continuous function $\rho_S : S \to S$, whose restriction $\rho_S^f$ is a bijection of the set $\rho_S(S)$. We claim that $\rho_S^f$ is the identity map of $\rho_S(S)$. It follows from $f \in \rho_F(F)$ that $f = \rho_F(g)$ for some $g \in F$. The equality $g = g \cdot \rho_F(g) = gf$ implies that $\rho_S^f = \rho_S^g \circ \rho_S^f$, which is possible only in case of identity map $\rho_S^f$.

Now consider the point $(s, f) \in S \times_\alpha F$ and its neighborhood $O_{(s, f)} = O_s \times F$. It follows that $\rho_S(s, f) = (\rho_S^{-1}(\rho_S(s)), \rho_F(f))$. The discontinuity of the right unit operation $\rho$ on $S \times_\alpha F$ yields neighborhoods $U_{(s, f)} \subset S \times_\alpha F$ and $W_{\rho(s, f)} \subset \rho(S \times_\alpha F) = \rho_S(S) \times \rho_F(F)$ of the elements $(s, f)$ and $\rho(s, f) = (\rho_S(s), \rho_F(f))$ such that $(U_{(s, f)} \times W_{\rho(s, f)}) \cap \rho^{-1}(W_{\rho(s, f)}) \subset O_{(s, f)}$. We lose no generality assuming that $U_{(s, f)} = U_s \times U_f$ and $W_{\rho(s, f)} = W_{\rho_S(s)} \times W_{\rho_F(f)}$ for some open sets $U_s \subset S$, $W_{\rho_S(s)} \subset \rho_S(S)$, $U_f \subset F$, and $W_{\rho_F(f)} \subset \rho_F(F)$.

We claim that the neighborhoods $U_s$ and $W_{\rho_S(s)}$ have the required property: $(U_s \times W_{\rho_S(s)}) \cap \rho^{-1}(W_{\rho_S(s)}) \subset O_s$.

Given any point $t \in (U_s \times W_{\rho_S(s)}) \cap \rho^{-1}(W_{\rho_S(s)})$, find a point $w \in W_{\rho_S(s)} \subset \rho_S(S)$ such that $tw \in U_s$. Consider the point $(t, f) \in S \times_\alpha F$ and $(w, \rho_F(f)) \in W_{\rho_S(s)} \times W_{\rho_F(f)}$ and observe that $$(t, f) \cdot (w, \rho_F(f)) = (t \cdot \rho_S(w), f \cdot \rho_S(f)) = (tw, f) \in U_s \times U_f.$$ Since $\rho_S(t, f) = (\rho_S^{-1}(\rho_S(t)), \rho_F(f)) = (\rho_S(t), \rho_F(f)) \in W_{\rho_S(s)} \times W_{\rho_F(f)} = W_{\rho(s, f)},$ we get the desired inclusion $$(t, f) \in (U_s \times W_{\rho_S(s)}) \cap \rho^{-1}(W_{\rho_S(s)}) \subset O_{(s, f)} = O_s \times F,$$ which implies $t \in O_s$.

Next, we show that the right unit operation $\rho_F$ on $F$ is discontinuous at every point $f \in F$. Fix any neighborhood $O_f$ of $f$ in $F$. Fix any point $s \in S$ and consider the pair $(s, f)$ and its neighborhood $O_{(s, f)} = S \times O_f$ in $S \times_\alpha F$. The discontinuity of the right unit operation $\rho$ on $S \times_\alpha F$ yields neighborhoods $U_{(s, f)} \subset S \times_\alpha F$ and $W_{\rho(s, f)} \subset \rho(S \times_\alpha F) = \rho_S(S) \times \rho_F(F)$ of the elements $(s, f)$ and $\rho(s, f)$ such that $(U_{(s, f)} \times W_{\rho(s, f)}) \cap \rho^{-1}(W_{\rho(s, f)}) \subset O_{(s, f)}$. Consider the point $r = \rho_S^{-1}(\rho_S(s))$ and observe that $\rho^{-1}(W_{\rho(s, f)}) \subset (r, \rho_F(f))$. Without lose of generality we can assume that $W_{\rho(s, f)} = W_r \times W_{\rho_F(f)}$ and $U_{(s, f)} = U_s \times U_f$ for some open sets $W_r \subset \rho_S(S)$, $W_{\rho_F(f)} \subset \rho_F(F)$, $U_s \subset S$ and $U_f \subset F$.

Consider the continuous functions $\beta : F \to \rho_S(S)$, $\beta : g \to \rho_S^{-1}(\rho_S(s)) = \rho_S^{-1}(\rho_S(s))$, and $\gamma : F \to S$, $\gamma : g \to s \cdot \alpha(g, r)$, and observe that $\beta(f) = r$ and $\gamma(f) = s \cdot \alpha(f, r) = s \cdot \rho_S(r) = s \in O_f$.

Using the continuity of the functions $\beta$ and $\gamma$, find a neighborhood $U_f' \subset U_f$ of $f$ such that $\beta(U_f') \subset W_r$ and $\gamma(U_f') \subset U_s$.

We claim that the neighborhoods $U_f'$ and $W_{\rho_F(f)}$ have the required property: $(U_f' \times W_{\rho_F(f)}) \cap \rho^{-1}(W_{\rho_F(f)}) \subset O_f$. Given any point $g \in (U_f' \times W_{\rho_F(f)}) \cap \rho^{-1}(W_{\rho_F(f)})$, find a point $h \in W_{\rho_F(f)} \subset \rho_F(F)$ with $gh \in U_f'$. Consider the points $(s, g) \in S \times_\alpha F$ and $(r, h) \in W_r \times W_{\rho_F(f)} = W_{\rho(s, f)}$. Since $h \in \rho_F(F)$, the map $\rho_h$ is an identity homeomorphism of $\rho_S(S)$. Hence, $\alpha(h)(r) = r$ and $s \cdot \alpha(g)(r) = s \cdot \alpha(g)(\alpha(h)(r)) = s \cdot \alpha(g)(r) = \gamma(g) \in \gamma(U_f') \subset U_s$.

Also $\alpha_{gh} = \alpha_h \circ \alpha_g = \alpha_g$ implies that $\rho(s, g) = (\rho_S^{-1}(\rho_S(s)), \rho_F(g)) = (\rho_S^{-1}(\rho_S(s)), \rho_F(g)) = (\beta(gh), \rho_F(g)) \in \beta(U_f') \times W_{\rho_F(f)} \subset W_r \times W_{\rho_F(f)} = W_{\rho(s, f)}$. Since $$(s, g) \cdot (r, h) = (s \cdot \alpha(g)(r), gh) \in U_s \times U_f' \subset U_{(s, f)},$$ we get $$(s, g) \in (U_{(s, f)} \times W_{\rho(s, f)}) \cap \rho^{-1}(W_{\rho(s, f)}) \subset O_{(s, f)} = S \times O_f$$ and hence $g \in O_f$, which completes the proof of the “only if” part of the theorem.
To prove the “if” part, assume that the topological right unosemigroups \((S, \rho_S)\) and \((F, \rho_F)\) are ditopological. We need to check that the right unit operation \(\rho : S \times F \to S \times F\), \(\rho : (s, f) \mapsto (\bar{\alpha}_f^{-1}(\rho_S(s)), \rho_F(f))\), is diconintuous at each point \((s, f) \in S \times F\).

Fix any neighborhood \(O_{(s, f)}\) of the point \((s, f)\) in the topological semigroup \(S \times F\). We lose no generality assuming that this neighborhood is of basic form: \(O_{(s, f)} = O_s \times O_f\) where \(O_s\) and \(O_f\) are open neighborhoods of \(s\) and \(f\) in \(S\) and \(F\), respectively. The diconintuity of the right unit operation \(\rho_S\) yields neighborhoods \(U_s \subset S\) and \(W_{\rho_S(s)} \subset \rho_S(S)\) of \(s\) and \(\rho_S(S)\) such that \((U_s \times W_{\rho_S(s)}) \cap \rho_S^{-1}(W_{\rho_S(s)}) \subset O_s\).

Now consider the point \(t = \bar{\alpha}_f^{-1}(\rho_S(s)) \in \rho_S(S)\) and observe that
\[
\alpha(f, r) = \bar{\alpha}_f(r) = \bar{\alpha}_f \circ \bar{\alpha}_f^{-1}(\rho_S(s)) = \rho_S(s) \in W_{\rho_S(s)}.
\]
The continuity of the action \(\alpha(F \times \rho_S(S))\) yields a neighborhood \(O_f' \subset O_f\) of \(f\) and a neighborhood \(W_r \subset \rho_S(S)\) of \(r\) such that \(\alpha(O_f' \times W_r) \subset W_{\rho_S(s)}\).

Since the right unit operation \(\rho_F\) is diconintuous at \(f\), there are neighborhoods \(U_f \subset F\) and \(W_{\rho_F(f)} \subset \rho_F(F)\) of \(f\) and \(\rho_F(f)\) such that \((U_f \times W_{\rho_F(f)}) \cap \rho_F^{-1}(W_{\rho_F(f)}) \subset O_f'\).

The \(\rho_S\)-invertibility of the action \(\alpha\) guarantees that \(\rho(S \times \alpha F) = \{ (\bar{\alpha}_f^{-1}(\rho_S(s)), \rho_F(f)) : (s, f) \in S \times F \} = \rho_S(S) \times \rho_F(F)\), which implies that the product \(W_{\rho_S(s)} \times \rho_F(F)\) is a neighborhood of the point \((\rho_S(s), \rho_F(f)) = (r, \rho_F(f))\) in \(\rho(S \times \alpha F) = \rho(S) \times \rho_F(F)\). We claim that the neighborhoods \(U_{(s, f)} = U_s \times U_f\) and \(W_{\rho_S(s)} = W_r \times \rho_F(F)\) have the required property: \((U_{(s, f)} \times W_{\rho_S(s)}) \cap \rho^{-1}(W_{\rho_S(s)}) \subset O_{(s, f)}\). Fix any point \((t, g) \in (U_{(s, f)} \times W_{\rho_S(s)}) \cap \rho^{-1}(W_{\rho_S(s)})\). It follows that \((t, g)(w, h) \in U_s \times U_f\) for some \((w, h) \in W_{\rho_S(s)}\).

Observe that
\[
W_r \times W_{\rho_F(f)} = W_{\rho_S(s)} \cap \rho^{-1}(W_{\rho_S(s)}) \subset O_f'.
\]
implies \(\rho_F(g) \in W_{\rho_F(f)}\) and \(\bar{\alpha}_g^{-1}(\rho_S(t)) \in W_r\). Also \((w, h) \in W_{\rho_S(s)} \cap \rho^{-1}(W_{\rho_S(s)})\) yields \(w \in W_r\) and \(\rho_S(t) = \bar{\alpha}_g \circ \bar{\alpha}_g^{-1}(\rho_S(t)) = \alpha(g, \bar{\alpha}_g^{-1}(\rho_S(t))) \in \alpha(O_f' \times W_r) \subset W_{\rho_S(s)}\).

By the same reason,
\[
\alpha_g(w) = \alpha(g, w) \in \alpha(O_f' \times W_r) \subset W_{\rho_S(s)}.
\]
Observe also that
\[
t \cdot \alpha_g(w), gh \in U_{(s, t)} = U_s \times U_f
\]
yields \(gh \in U_f\) and \(t \cdot \alpha_g(w) \in U_s\). The choice of the neighborhoods \(U_s\) and \(W_{\rho_S(s)}\) guarantees that \(t \in O_s\). So, \((t, g) \in O_s \times O_f = O_{(s, f)}\).

The above discussion implies that for topological unosemigroups \(F = (F, \lambda_F, \rho_F), S = (S, \lambda_S, \rho_S)\) and a \(\lambda_F\)-respecting \(\rho_S\)-invertible topological unosemigroups \(S \times \alpha F = (S \times \alpha F, \lambda, \rho)\) is a topological unosemigroup. This topological unosemigroup will be called the semidirect product of the topological unosemigroups \(F\) and \(S\). By Theorems 7 and 8 the topological unosemigroup \(S \times \alpha F\) is ditopological if and only if so are the topological unosemigroups \(S\) and \(F\).

6.5. Hartman-Mycielski Extension. Given a topological space \(X\), for every \(n \in \mathbb{N}\) by \(HM_n(X)\) we denote the set of all functions \(f : [0,1) \to X\) for which there exists a sequence \(0 = a_0 < a_1 < \cdots < a_n = 1\) such that \(f\) is constant on each interval \([a_i, a_{i+1}), 0 \leq i < n\). The union \(HM(X) = \bigcup_{n \in \mathbb{N}} HM_n(X)\) is called the Hartman-Mycielski extension of the space \(X\), see [3, 4].

A neighborhood sub-base of the topology of \(HM(X)\) at \(f \in HM(X)\) consists of sets \(N(a, b, V, \varepsilon)\), where
\[
\begin{align*}
(1) & \ 0 \leq a < b \leq 1, \ f \text{ is constant on } [a, b], \ V \text{ is a neighborhood of } f(a) \text{ in } X, \text{ and } \varepsilon > 0, \\
(2) & \ g \in N(a, b, V, \varepsilon) \text{ means that } |t \in [a, b] : g(t) \notin V| < \varepsilon, \text{ where } |\cdot| \text{ denotes the Lebesgue measure.}
\end{align*}
\]
If \(X\) is a topological semigroup, then \(HM(X)\) also is a topological semigroup with respect to the pointwise operations of multiplication of functions. Moreover, for any continuous left (right) unit operation \(u_X : X \to X\) on \(X\), the unary operation \(u_{HM(X)} : HM(X) \to HM(X), u_{HM(X)} : f \mapsto u \circ f\), is a continuous left (right) unit operation on \(HM(X)\), see [1] Proposition 2.

Theorem 9. If \(X = (X, \lambda_X)\) is a ditopological left unosemigroup, then so is its Hartman-Mycielski extension \(HM(X) = (HM(X), \lambda_{HM(X)})\).
Proof. For simplicity of notation, we write $\lambda$ instead of $\lambda_{HM(X)}$. Assume that $X$ is a ditopological left unosemigroup. To show that the topological left unosemigroup $HM(X)$ is ditopological, fix any element $f \in HM(X)$ and its sub-basic neighborhood $O_f = N(a, b, O_{f(a)}, \varepsilon)$ such that $f$ is constant on the half-interval $[a, b] \subset [0, 1)$ and $O_{f(a)}$ is an open neighborhood of $f(a)$ in $X$. Since the topological left unosemigroup $X$ is ditopological, there are neighborhoods $U_x \subset S$ and $W_{\lambda x} \subset \lambda X$ of the points $x = f(a)$ and $\lambda x = \lambda f(a)$ such that $(W_{\lambda x} \setminus U_x) \cap \lambda^{-1}(W_{\lambda x}(x)) \subset O_x$.

Consider the open neighborhoods $U_f = N(a, b, U_x, \frac{\varepsilon}{3}) \subset HM(X)$ of $f$ and

$$W_{\lambda(f)} = \lambda(HM(X)) \cap N(a, b, W_{\lambda X(x)}, \frac{\varepsilon}{3}) \subset \lambda(HM(X))$$

of $\lambda(f)$. We claim that each function $g \in (W_{\lambda(f)} \setminus U_f) \cap \lambda^{-1}((W_{\lambda(f)}) \subset HM(X)$ belongs to the neighborhood $O_f$. The inclusion $\lambda(g) \in W_{\lambda(f)} = N(a, b, W_{\lambda X(x)}, \frac{\varepsilon}{3})$ guarantees that the set $A = \{ t \in [a, b] : \lambda X(f(t)) \notin W_{\lambda X(x)} \}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. On the other hand, the inclusion $wg \in U_f = N(a, b, U_x, \frac{\varepsilon}{3})$ implies that the set $B = \{ t \in [a, b] : w(t) \cdot g(t) \notin U_x \}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. Finally, the inclusion $w \in W_{\lambda(f)}$ implies that the set $C = \{ t \in [a, b] : w(t) \notin W_{\lambda X(x)} \}$ has Lebesgue measure $< \frac{\varepsilon}{3}$. Then the union $A \cup B \cup C$ has Lebesgue measure $< \varepsilon$ and for each $t \in [a, b] \setminus (A \cup B \cup C)$ we get $\lambda X(g(t)) \in W_{\lambda X(x)}, w(t) \in W_{\lambda X(x)}$ and $w(t)g(t) \in U_x$. Then the choice of the neighborhoods $U_x$ and $W_{\lambda X(x)}$ guarantees that $g(t) \in O_x = O_{f(a)}$ and hence $g \in N(a, b, O_{f(a)}, \varepsilon) = O_f$. □

By analogy we can prove:

Theorem 10. If $X = (X, \rho_X)$ is a ditopological right unosemigroup, then so is its Hartman-Mycielski extension $HM(X) = (HM(X), \rho_{HM(X)})$.

Theorems 9 and 10 imply:

Theorem 11. If $X = (X, \lambda X, \rho_X)$ is a ditopological unosemigroup, then so is its Hartman-Mycielski extension $HM(X) = (HM(X), \lambda_{HM(X)}, \rho_{HM(X)})$.

Since the Hartman-Mycielski space $HM(X)$ is contractible [9], Theorems 9, 10, 11 imply:

Corollary 4. Each topological (left, right) unosemigroup is topologically isomorphic to (left, right) subunosemigroup of a contractible topological (left, right) unosemigroup.

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