Total Variation Bound for Kac’s Random Walk

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Abstract

We show that the classical Kac’s random walk on $S^{n−1}$ starting from the point mass at $e_1$ mixes in $O(n^5 \log n)$ steps in total variation distance. This improves a previous bound by Diaconis and Saloff-Coste of $O(n^{2n})$.

1 Introduction

Mark Kac proposed the following simplified model of one-dimensional Boltzmann gas dynamics (for historical details, see [3], [4]): For $n$ particles on $\mathbb{R}$, we can represent their velocities $(v_1, \ldots, v_n)$ as a point on the unit sphere $S^{n−1}$, after normalization so that

$$\sum_{i=1}^{n} v_i^2 = 1.$$ 

Conservation of kinetic energy (assuming 0 potential energy) in the gas dynamics is equivalent to $(v_1(t), \ldots, v_n(t))$ staying on $S^{n−1}$ for all $t \geq 0$. We will not introduce momentum conservation in our model, because that will force the collision to be inelastic (see 2nd paragraph below).

Each time there is a collision, it occurs with probability 1 between two particles, which corresponds to choosing two coordinate directions $x_i, x_j$ and rotating $S^{n−1}$ along the 2-plane $x_i \wedge x_j$ by some angle $\theta$.

Notice that $v_i^2 + v_j^2$ before and after the collision stays the same, since the velocities of the other particles are not affected by the collision.

By disregarding the position information of the particles (which have to be confined in some compact domain, for example $S^1$, else they will eventually run off to infinity), each collision occurs between any pair of the particles with equal probability $\frac{1}{\binom{n}{2}}$. The rotation angle $\theta$ can be chosen from some distribution on $[0, 2\pi)$, which physically is a measure of the elasticity of the collision; for example, inelastic collision in $\mathbb{R}$ will correspond to a distribution of $\theta$ that’s a delta measure concentrated at $\pi$. In this paper, we will assume that $\theta$ is uniformly distributed on $[0, 2\pi)$.

Thus we obtain a discrete time Markov chain on $S^{n−1}$ with transition kernel given by, for $f : S^{n−1} \to \mathbb{R}$ continuous, and $x \in S^{n−1}$,

$$Kf = \frac{1}{\binom{n}{2}} \sum_{i \neq j}^{n} \int_{0}^{2\pi} f(R(i, j; \theta)x) \frac{1}{2\pi} d\theta$$

where $R(i, j; \theta)$ denotes the rotation along the oriented $i \wedge j$ plane by the angle $\theta$. By transposing, $K$ defines a map from the set of probability measures on $S^{n−1}$ to itself, since $K(1) = 1$.

It is not too hard to see that the $U_{n−1}$, the uniform distribution on $S^{n−1}$, is a stationary distribution for $K$: for each summand $K_{i,j}$ (without $\frac{1}{\binom{n}{2}}$ in (1)), we have

$$U_{n−1}(K_{i,j}f) = \int_{S^{n−1}} (K_{i,j}f)(x)U_{n−1}(dx)$$

$$= \int_{S^{n−1}} \left( \int_{0}^{2\pi} f(R(i, j; \theta)x) \frac{1}{2\pi} d\theta \right)U_{n−1}(dx)$$

$$= \int_{S^{n−1}} \frac{1}{2\pi} \int_{0}^{2\pi} f(x)d\theta U_{n−1}(R(i, j; \theta)dx)$$

$$= \int_{S^{n−1}} f(x)U_{n−1}(dx)$$
using a change of variable formula and the fact that $U_{n-1}$ is invariant under rotations.

This establishes that $U_{n-1}K_{i,j} = U_{n-1}$ for all $i \neq j$. Thus their average $U_{n-1}K = U_{n-1}$ as well.

We further claim that Markov chain is aperiodic because once a point is reached, it can be reached in the next step with positive probability density for any rotation. It is also irreducible since along a sequence of rotations $(i_1 \wedge i_2, \ldots, i_k \wedge i_{k+1})$ that form a connected spanning graph in $K_n$, the complete graph on $n$ vertices, one can transport any point on $S^{n-1}$ to any other point with positive probability density; such sequence of rotations certainly occur with positive probability. Thus by general theory of Markov chains, we know that with any initial distribution $\mu$ on $S^{n-1}$,

$$\lim_{i \to \infty} \mu K^i(A) - U_{n-1}(A) = 0$$

uniformly in $A \subset S$. This implies total variation convergence.

Using the $L^2$ theory of discrete time Markov chains, it can be shown that if the starting distribution $\mu$ is in $L^2(S^{n-1}, U_{n-1})$, then we get the following convergence bound

$$||\mu K^i - U_{n-1}||_{TV} < ||\mu - 1||_{L^2}(1 - \frac{1}{2^n})^i$$

by the result in [4], that shows the spectral gap of $K$ is bounded below by $\frac{1}{2n}$. In fact it’s given exactly by $\frac{n+2}{2n(n-1)}$ for $n \geq 2$.

If the initial distribution $\mu$ does not have an $L^2$ density with respect to $U_{n-1}$, then the $L^2$ theory above fails. The best result for the rate of convergence when the initial distribution is say concentrated $\mathbb{P}(\mu = \delta_i) > 0$, we have the following two claims:

1. the density $g := \frac{d\mu'}{d\mu}$ of the resulting distribution $\mu'$ of the conditioned random walk with respect to the uniform distribution on $S^{n-1}$ satisfies the following bound

$$g(x) \leq \frac{\min_i x_i}{n}^{n} \left(\sum_{i=1}^{n} (-\log |x_i|)^k\right) C^k \prod_{m=1}^{k} m!$$

(2)

$$\leq C^k k^2 \left(\min_i x_i\right)^{-n} \left(\sum_{i=1}^{n} (-\log \min_i x_i)^k\right)$$

(3)

for some fixed absolute constant $C$.

2. For $k > -n^2 \log n \log \epsilon$, and $\epsilon < \frac{n^{-3/2}}{2}$ the set $H_\epsilon := \{x \in S^{n-1} : |x_i| < \epsilon \text{ for some } i\}$ satisfies the following bound on its probability

$$\lim_{i \to \infty} \mu K^i(A) - U_{n-1}(A) = 0$$

2 Bounding the Total Variation Distance

Next we want to bound the convergence rate of the Kac’s random walk on $S^{n-1}$, in total variation distance.

Recall the total variation distance between two probability measures $\mu$ and $\nu$ on the same probability space $(S, \mathcal{S})$ is defined by the following variational quantity:

$$||\mu - \nu||_{TV} = \sup_{A \in \mathcal{S}} |\mu(A) - \nu(A)|$$

where $\mathcal{S}$ is the $\sigma$-algebra on $S$.

Let $A_k$ be the event that at the $k$th step of the walk, every pair of coordinates has been used. Then we have

$$P(A_k) := \eta_k < \left(\frac{n}{2}\right) \left(1 - \frac{1}{(n/2)^k}\right)$$

Conditioning on this event, we have the following two claims:

1. the density $g := \frac{d\mu'}{d\mu}$ of the resulting distribution $\mu'$ of the conditioned random walk with respect to the uniform distribution on $S^{n-1}$ satisfies the following bound

$$g(x) \leq \frac{\min_i x_i}{n}^{n} \left(\sum_{i=1}^{n} (-\log |x_i|)^k\right) C^k \prod_{m=1}^{k} m!$$

(2)

$$\leq C^k k^2 \left(\min_i x_i\right)^{-n} \left(\sum_{i=1}^{n} (-\log \min_i x_i)^k\right)$$

(3)

for some fixed absolute constant $C$.

2. For $k > -n^2 \log n \log \epsilon$, and $\epsilon < \frac{n^{-3/2}}{2}$ the set $H_\epsilon := \{x \in S^{n-1} : |x_i| < \epsilon \text{ for some } i\}$ satisfies the following bound on its probability
\[ \mu_k'(H_v) \leq e^t \] (4)

Let us first show how claims 1 and 2 lead to a polynomial time convergence rate for the Kac walk under total variation norm. Let \( \mu_k \) be the distribution on \( S^{n-1} \) after \( k \) steps of the random walk, and let \( \mu_k' \) be \( \mu_k \) conditional on \( A_k \), i.e., for \( B \subset S^{n-1} \),

\[ \mu_k'(B) = P(\delta e_1 R^k \in B | A_k) \]

where \( R \) is the one step transition kernel of the Kac random walk.

Then we have

\[ ||\mu_k' - \mu_k||_{TV} < \eta_k < \left( \frac{n}{2} \right) (1 - \frac{1}{n})^k \] (5)

To check this, let \( B \subset S^{n-1} \) be Lebesgue measurable. Then we have

\[ \mu_k(B) = P(\delta e_1 R^k \in B | A_k) P(A_k) + P(\delta e_1 R^k \in B | A_k^c) P(A_k^c) \]

\[ \leq \mu_k'(B) + \eta_k \]

This implies

\[ \mu_k(B) - \mu_k'(B) \leq \eta_k \]

On the other hand, since

\[ \mu_k'(B) = \frac{P(\{\delta e_1 R^k \in B \} \cap A_k)}{P(A_k)} \]

we also get

\[ \frac{\mu_k(B)}{1 - \eta_k} > \mu_k'(B) \]

which gives

\[ \mu_k(B) > \mu_k'(B) - \eta_k \mu_k'(B) \]

hence

\[ \mu_k'(B) - \mu_k(B) < \eta_k \]

which shows (5).

Next recall that a Markov kernel is weakly contracting in total variation norm because if \( f \) is a bounded continuous function on the state space with

\[ ||f||_{\infty} \leq 1 \]

then \( Rf(x) = \int R(x, dy) f(y) \) satisfies the same \( L^\infty \) bound, hence

\[ (\mu R - \nu R)(f) = (\mu - \nu)(Rf) \leq ||\mu - \nu||_{TV} \]

Thus by the triangle inequality we just need to bound \( ||\mu_k' R^k - U_{n-1}||_{TV} \) from now on, where \( U_{n-1} \) denotes the uniform distribution on \( S^{n-1} \) and at the end add \( \eta_k \) to the resulting bound.

Next we modify \( \mu_k' \) to a different distribution \( \nu_k \) as following. We define \( \nu_k \) in terms of its density with respect to \( U_{n-1} \).
On the set $H^c\varepsilon$,
\[
\frac{d\nu_k}{dU_{n-1}} = \frac{d\mu'_k}{dU_{n-1}}
\]

On the set $H_{\varepsilon}$, we let its density be a constant equal to the mass of $H_{\varepsilon}$ under $\mu'_k$ divided by its mass under $U_{n-1}$, which is what’s needed for $\nu_k$ to be a probability distribution on $S^{n-1}$; we invoke claim 1 above to get an upper bound on this constant:
\[
\frac{d\nu_k}{dU_{n-1}} = \frac{\mu'_k(H_{\varepsilon})}{U_{n-1}(H_{\varepsilon})} < \frac{1}{\epsilon^{1/4}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \epsilon^{1/4} < \frac{1}{\epsilon^{1/4}} \frac{2\pi}{n-2}
\]
Here we used the fact that
\[
\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} > \sqrt{\frac{n-2}{2}}
\]
which follows from log convexity of the $\Gamma$ function.

The total variation distance between $\mu'_k$ and $\nu_k$ is given simply by their total variation distance over the region $H_{\varepsilon}$, hence we have
\[
||\mu'_k - \nu_k||_{TV} \leq \mu'_k(H_{\varepsilon}) + \frac{n\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \epsilon^{1/4} \leq n^{3/2} \epsilon + \epsilon^{1/4}
\]

Thus by choosing $\epsilon$ sufficiently small, whose exact value we will determine in the end, we can make sure that $\mu'_k$ and $\nu_k$ are very close in total variation distance. And again by weak contractivity of Markov kernel, we now simply need to focus on bounding $||\nu_kR^c - U_{n-1}||_{TV}$. Since $\nu_k$ has an $L^2$ density with respect to $U_{n-1}$, we can use the spectral gap to bound the rate of convergence. First we bound the $L^2(dU_{n-1})$ distance between $\nu_k$ and $U_{n-1}$:
\[
||\nu_k - U_{n-1}||_{L^2(U_{n-1})} = \left(\int_{H_{c\varepsilon}} |\frac{d\nu_k}{dU_{n-1}} - 1|^2 dU_{n-1} + \int_{H^{c\varepsilon}} |\frac{d\nu_k}{dU_{n-1}} - 1|^2 dU_{n-1}\right)^{1/2}
\]
Let’s bound the two integrals separately.

For the first integral on the right hand side of (8), we have
\[
\int_{H_{c\varepsilon}} |\frac{d\nu_k}{dU_{n-1}} - 1|^2 dU_{n-1} \leq \int_{H_{c\varepsilon}} \left(\frac{d\nu_k}{dU_{n-1}} - 1\right)^2 dU_{n-1} + U_{n-1}(H_{\varepsilon}) < e^{-3/2} \frac{8\pi}{n-2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \leq 4\epsilon^{-1/2} \sqrt{\frac{2\pi}{n-2}}
\]

For the second integral, notice first that $H^c_{\varepsilon}$ is the set of points on $S^{n-1}$ for which all the coordinates are greater than $\epsilon$. So claim 2 tells us that the density $\frac{d\nu_k}{dU_{n-1}}$ over this region is bounded above by $\epsilon^{-n}$, from which we immediately get the following bound
\[
\int_{H^c_{\varepsilon}} |\frac{d\nu_k}{dU_{n-1}} - 1|^2 dU_{n-1} < \epsilon^{-2n} + 1
\]
Combining (9) and (10), we get, for $\epsilon < \frac{1}{\sqrt{2}}$ and $n > 2$ say, that
\[
||\nu_k - U_{n-1}||_{L^2(U_{n-1})} \leq 2\epsilon^{-n}
\]
By the results in \[4\], we know that the spectral gap of the Kac kernel is \(1/n\), so we get

\[
\|\nu_k R^t - U_{n-1}\|_{\text{TV}} \leq \left\| \frac{d\nu_k}{dU_{n-1}} - 1 \right\|_{L^2(dU_{n-1})} (1 - \frac{1}{n})^m
\]

\[
\leq 2\epsilon^{-n} (1 - \frac{1}{n})^m
\]  \hspace{1cm} (11)

Finally combining (5) (6) and (11), we get

\[
\|\delta\epsilon_1 R^{k+l} - U_{n-1}\|_{\text{TV}} \leq \left(\frac{n}{2}\right) (1 - \frac{1}{(\frac{n}{2})})^k + n^{3/2}\epsilon + \epsilon^{1/4} + C^k k^2 |\epsilon|^{-n} (-\log \epsilon)^k (1 - \frac{1}{n})^l
\]  \hspace{1cm} (12)

So it remains to minimize the right hand side of (12) with respect to \(k\) and \(l\). Suppose our target total variation distance is \(3\delta\). Then we can simply divide \(3\delta\) into three equal parts and bound each summand in (12) by \(\delta\). We look at each summand below:

Bounding the first summand yields

\[
\left(\frac{n}{2}\right) (1 - \frac{1}{(\frac{n}{2})})^k < \delta
\]

\[
\Rightarrow k > (-\log \delta + 2 \log n) \left(\frac{n}{2}\right) \hspace{1cm} \text{(13)}
\]

So it suffices to take

\[
k > n^2 \log n \log \frac{1}{\delta}
\]

Bounding the second summand \(\epsilon^{1/4} + n^{3/2}\epsilon < \delta\), it suffices to have

\[
\epsilon^{1/4} < \delta/2
\]

\[
n^{3/2}\epsilon < \delta/2
\]

which gives

\[
\epsilon < \frac{1}{2} \delta^4 n^{-3/2}
\]

This will be used to bound \(l\) from above in the third summand:

\[
C^k k^2 |\epsilon|^{-n} (-\log \epsilon)^k (1 - \frac{1}{n})^l < \delta
\]

implies we need \(l\) greater than

\[
n(-\log \delta + k \log C + k^2 \log k - n \log \epsilon + k \log \log(\epsilon^{-1}))
\]

\[
< n(-\log \delta + k \log C + n^4 (\log n)^2 (\log \delta)^2 (2 \log n + \log \log n + \log \log(\delta^{-1})) + n(-4 \log \delta + \frac{3}{2} \log n) + k \log \log(\epsilon^{-1}))
\]

\[
< C' n^5 (\log n)^3 (\log \delta)^3
\]

for some constant \(C'\).

Clearly \(l\) dominates \(k\), so it requires a total of \(C' n^5 (\log n)^2 (\log \delta)^2\) steps to bring the running distribution of the Kac random walk to be \(3\delta\) close to its stationary distribution on the unit sphere \(S^{n-1}\).

Lastly we prove the two claims introduced in the beginning.
3 Proof of Claim 1

Starting at the delta mass at \( c_1 \), an admissible sequence of rotations in \( A_k \) will distribute it over the entire \( S^{n-1} \) with positive probability everywhere provided that \( P(A_k) > 0 \), i.e., for \( k \) sufficiently large. This will certainly be the case if \( k \geq -n^2 \log n \log \delta \) for \( - \log \delta > 2 \).

Observe that at step \( j - 1 \), \( j \leq k \), the support of the running distribution is a subsphere of \( S^{n-1} \). Without loss of generality, we call this subsphere \( S^m \). Denote by \( u_j, v_j \) the axes that span the plane of rotation \( \gamma_j \).

The way \( \gamma_j \) affects the previous running distribution can be classified into three cases:
1. \( u_j, v_j \notin S^m \), in which case the running distribution remains unchanged.
2. \( u_j, v_j \in S^m \), in which case the support at step \( j \) is still on \( S^m \).
3. \( u_j \in S_m, v_j \notin S_m \), in which case the support of the running distribution grows to be a sphere with 1 dimension higher than \( S^m \), denoted without loss of generality \( S^{m+1} \), and the density with respect to the uniform distribution on \( S^{m+1} \) is bounded by \((u_j^2 + v_j^2)^{-1/2}\) times the previous density bound with respect to \( S^m \).

Case 1 is clear because the rotation does not take \( S^m \) outside itself and for \( \theta \in [0, 2\pi] \), the density at 

\[(x_1, \ldots, x_{m+1}, (u_j^2 + v_j^2)^{1/2} \cos \theta, \ldots, (u_j^2 + v_j^2)^{1/2} \sin \theta, \ldots, x_n)\]

with respect to \( U_m \) only depends on the first \( m + 1 \) coordinates, which means that averaging over \( \theta \) uniformly in \([0, 2\pi]\) remains the same.

To understand case 3, we have the following

**Lemma 3.1.** Assuming the running density \( h(x_1, \ldots, x_{m+1}) \) with respect to \( U_m \) after step \( j - 1 \) is bounded by \( g(x_1, \ldots, x_{m+1}) \), and that without loss of generality \( u_j = x_{m+1}, v_j = x_{m+2} \), then the new density with respect to \( U_{m+1} \) after step \( j \) is bounded by

\[
\frac{1}{2\pi} g(x_1, \ldots, (x_{m+1} + x_{m+2})^{1/2})(x_{m+1}^2 + x_{m+2}^2)^{-1/2}
\]

*Proof.* Denote the new density with respect to \( U_{m+1} \) by \( h(x_1, \ldots, x_{m+2}) \) with a slight abuse of notation. Then we have

\[
h(x_1, \ldots, (x_{m+1}^2 + x_{m+2}^2)^{1/2} \cos \theta, (x_{m+1}^2 + x_{m+2}^2)^{1/2} \sin \theta)
\]

is independent of \( \theta \) and in particular equals

\[h(x_1, \ldots, (x_{m+1}^2 + x_{m+2}^2)^{1/2}, 0)\]

Furthermore the total contribution of density from \((x_1, \ldots, (x_{m+1}^2 + x_{m+2}^2)^{1/2} \cos \theta, (x_{m+1}^2 + x_{m+2}^2)^{1/2} \sin \theta)\) for all \( \theta \) should add up to the previous density at the point \((x_1, \ldots, (x_{m+1}^2 + x_{m+2}^2)^{1/2})\). In other words,

\[
(x_{m+1}^2 + x_{m+2}^2)^{1/2} \int_{\theta=0}^{2\pi} h(x_1, \ldots, (x_{m+1}^2 + x_{m+2})^{1/2} \cos \theta, (x_{m+1}^2 + x_{m+2})^{1/2} \sin \theta)
\]

\[= h(x_1, \ldots, (x_{m+1}^2 + x_{m+2})^{1/2})\]

Notice that the factor \((x_{m+1}^2 + x_{m+2}^2)^{1/2}\) accounts for the measure of the circle over which we aggregate. Thus we get

\[
h(x_1, \ldots, (x_{m+1}^2 + x_{m+2})^{1/2} \cos \theta, (x_{m+1}^2 + x_{m+2})^{1/2} \sin \theta)
\]

\[= \frac{1}{2\pi} (x_{m+1}^2 + x_{m+2}^2)^{-1/2} h(x_1, \ldots, (x_{m+1}^2 + x_{m+2})^{1/2})
\]

\[\leq \frac{1}{2\pi} (x_{m+1}^2 + x_{m+2}^2)^{-1/2} g(x_1, \ldots, (x_{m+1}^2 + x_{m+2})^{1/2})\]

\[\square\]

To study case 2, we make the assumption that after step \( j - 1 \), the density with respect to \( U_m \) is bounded by an expression of the form

\[
C(a_1^2 + b_1^2)^{-1/2} \ldots (a_{m-1}^2 + b_{m-1}^2)^{-1/2} \left[ (-\log |x_1|)^{j-1} + \ldots + (-\log |x_{m+1}|)^{j-1} \right]
\]

where \( C \) is a constant that varies with \( j \) and \( m \). Here \( a_i \neq b_i \) for each \( i \) and \((a_1, b_1), \ldots, (a_{m-1}, b_{m-1})\) are pairs in \((x_1, \ldots, x_{m+1})^2\) with the property that no two pairs are the same and each coordinate appears at most twice.
Lemma 3.2. Under the assumption above, if \(j\)th rotation is in case 2, then the new density bound takes the form

\[
2C(j+1)(\frac{1}{a_1^2} + \frac{1}{b_1^2})^{-1} \ldots (\frac{1}{a_{m-1}^2} + \frac{1}{b_{m-1}^2})^{-1}[-\log |x_1|] + \ldots + (-\log |x_{m+1}|)^2
\]

with possibly a different sequence of \((a_i, b_i)\) satisfying the same property as before.

Proof. Without loss of generality assume \((u_j, v_j) = (1, 2)\).

The new density \(h'\) is obtained from the old density \(h\) by averaging over \(\theta \in [0, 2\pi]\) of \(h(R(1, 2, \theta)x)\), where \(R(1, 2, \theta)x\) denotes the rotation of the vector \(x \in S^m\) by angle \(\theta\) along \(x_1 \wedge x_2\). In formula we have,

\[
h'(x) = \frac{1}{2\pi} \int_0^{2\pi} h(R(1, 2, \theta)x)\,d\theta
\]

We write the bound \((\ref{14})\) as a sum of \(m + 1\) terms and consider one of the terms

\[
g_i(x) = C(\frac{1}{a_1^2} + \frac{1}{b_1^2})^{-1} \ldots (\frac{1}{a_{m-1}^2} + \frac{1}{b_{m-1}^2})^{-1}(-\log |x_i|)^{j-1}
\]

By assumption, at most two of \(a_1, b_1, \ldots, a_{m-1}, b_{m-1}\) equals \(x_i\) at most two equals \(x_2\).

By the circle averaging formula \((\ref{15})\), we have

\[
g_i'(x) = \frac{1}{2\pi} \int_0^{2\pi} g((x_1^2 + x_2^2)^{1/2} \cos \theta, (x_1^2 + x_2^2)^{1/2} \sin \theta, x_3, \ldots, x_{m+1})\,d\theta
\]

We shall break the integral into two parts, where the range of integration is over \(I_c = [0, \pi/4] \cup [3\pi/4, 5\pi/4] \cup [7\pi/4, 2\pi]\) and its complement \(I_i = [0, 2\pi]\) respectively, i.e., the ranges are where \(\cos \theta\) is close to 1 and \(\sin \theta\) is close to 1 respectively.

For \(\theta \in I_c\), all the factors in \(g_i(x)\) of the form \((x_1^2 + x_2^2)^{-1/2}\), that involves \(x_2\) but not \(x_1\) upon the rotation \(R(1, 2, \theta)\) becomes \((x_1^2 + x_2^2)^{1/2} \sin \theta + x_2^2)^{-1/2}\), which can be bounded above by \(\sqrt{2}(x_1^2 + x_2^2 + x_2^2)^{-1/2}\).

As of the factors that involve both \(x_1\) and \(x_2\), i.e., \((x_1^2 + x_2^2)^{-1/2}\), there can be at most one of such. And it remains the same under the rotation \(R(1, 2, \theta)\) since \((x_1^2 + x_2^2)^{1/2} \cos \theta + (x_1^2 + x_2^2)^{1/2} \sin \theta = x_1^2 + x_2^2\).

The factors that involve \(x_1\) and \(x_i, l \neq 2\), becomes \((x_1^2 + x_2^2)^{1/2} \cos \theta + (x_2^2)^{-1/2}\), which we can bound as follows:

using the fact that \(\frac{1}{2}(|a| + |b|) \leq (a^2 + b^2)^{1/2} \leq |a| + |b|\), we get

\[
(x_1^2 + x_2^2) \cos^2 \theta + x_2^2)^{-1/2} \sim (|x_1| + |x_2|)(|\cos \theta| + |x_2|)^{-1}\]

where \(a \sim b\) means \(b/C \leq a \leq bC\) for some constant \(C\). Here the constant can be taken to be 2.

Finally it’s also possible that \(i \in \{1, 2\}\), in which case we also have to deal with a \((-\log((x_1^2 + x_2^2)^{-1/2} \cos \theta))^{-1}\) factor that goes to infinite for \(\theta \in I_c\).

In fact when \(i = 1\), the only factors that have singularities for \(\theta \in I_c\) and for the coordinates bounded away from \(0\) take the following form:

\[
((-|x_1| + |x_2|)|\cos \theta| + |x_1|)^{-1}((-|x_1| + |x_2|)|\cos \theta| + |x_1|)^{-1}(-\log((x_1^2 + x_2^2)^{-1/2} \cos \theta))^t
\]

where \(s \neq t\), or without the \(x_t\) factor. In the former case we show that the following integral

\[
\frac{1}{2\pi} \int_{\theta \in I_c} ((|x_1| + |x_2|)|\cos \theta| + |x_1|)^{-1}((-|x_1| + |x_2|)|\cos \theta| + |x_1|)^{-1}(-\log((x_1^2 + x_2^2)^{1/2} \cos \theta))^t\,d\theta
\]

is bounded by

\[
j((x_1^2 + x_2^2)^{-1/2}(x_1^2 + x_2^2)^{-1/2}((-\log |x_1|)^t + (-\log |x_2|)^t + (-\log |x_1|)^t)^t
\]

whereas in the case \(x_t\) is not present, the same bound applies to the expression.
\[
\frac{1}{2\pi} \int_{\theta \in I_c} \left( (|x_1| + |x_2|) \cos \theta + |x_s| \right)^{-1} \left( -\log \left( (x_1^2 + x_2^2 + x_s^2)^{1/2} \cos \theta \right) \right)^{i-1} d\theta
\]

(17)

clearly because

\[
\left( (|x_1| + |x_2|) \cos \theta + |x_s| \right)^{-1} \geq 1
\]

When \( i \neq 1 \), the logarithmic singularity will not arise when integrating over \( \theta \in I_c \), so it will trail off as a remaining factor of the form \( (-\log |x_s|)^{i-1} \leq 1 + (-\log |x_s|)^i \).

Recall also that we have factors of the form

\[
2(x_1^2 + x_2^2 + x_s^2)^{-1/2} (x_1^2 + x_2^2 + x_s^2)^{-1/2}
\]

(18)

coming from the uniform bound on the factors involving \( x_2 \) but not \( x_3 \); here \( s, t \) are possibly different indices than those appearing in the singular factors. (13) can be trivially bounded above by \( 2(x_1^2 + x_2^2)^{-1/2}(x_2^2 + x_3^2)^{-1/2} \). The remaining inverse factors in \( g(R(1, 2, \theta)x) \) do not contain \( x_1 \) or \( x_2 \), so one can easily check that the inductive hypothesis is satisfied.

The best way to visualize this branching inductive argument is to consider a simple graph on \( m + 1 \) vertices with degrees bounded above by 2. The edges between \( i \) and \( j \) represent a factor of the form \( (x_i^2 + x_j^2)^{-1/2} \) in the bound on the density. A rotation in the \( x_1 \wedge x_2 \) plane has the effect of producing two graphs on \( m + 1 \) vertices. Without loss of generality let’s describe one of those two descendant graphs, the one associated with \( x_1 \):

there will be edges \( (1, 2), (3, 4), (1, 3), \) and \( (2, 4) \) if \( x_3 \) and \( x_4 \) were incident to \( x_1 \) in the previous graph, or simply \((1, 2)\) when \( x_1 \) only has degree 1. If \( x_1 \) had degree 0, then it remains isolated in the \( x_1 \) component of the descendant graph. In the process of this rewiring, some logarithmic factors are also introduced, namely, if \( (-\log |x_s|)^{i-1} \) or \( (-\log |x_t|)^{i-1} \) was a factor in the bound for the previous step running distribution, then the new bound will have \( j!(-\log |x_i|)^j \). If it’s a log factor of other coordinates, then the exponent remains the same.

It remains to prove the bound (16), which is given by the following technical lemma.

**Lemma 3.3.** For \( 0 \leq x_t, x_s, 0 \leq x_1, x_2 \),

\[
\int_0^1 (x_1 + x_2)\theta + x_s)^{-1} (x_1 + x_2)\theta + x_t)^{-1} (-\log (x_1 + x_2)\theta)^{i-1} d\theta
\]

\[
\leq 4(j + 1)! (x_1 + x_2)^{-1} (x_1 + x_2)^{-1} (\log x_1)^j + (\log x_2)^j + (\log x_s)^j + (\log x_t)^j
\]

**Proof.** Without loss of generality, we can assume \( x_t \leq x_s \).

First of all the factor \((x_1 + x_2)\theta + x_s)^{-1}\) can be bounded above by \( 2(x_1 + x_t)^{-1} \) for \( \theta \in [0, 1] \). So it remains to bound the integral of the remaining factors

\[
\int_0^1 (x_1 + x_2)\theta + x_t)^{-1} (-\log (x_1 + x_2)\theta)^{i-1} d\theta
\]

\[
\leq x^{-1} \int_0^1 (-\log (x_1 + x_2)\theta)^{i-1} d\theta + (-\log (x_1 + x_2)(x_1 + x_2)^{-1})^{i-1} \int_0^1 (x_1 + x_2)\theta + x_1)^{-1} \theta d\theta
\]

\[
\leq x^{-1} j! x_1 (\log (x_1 + x_2)\theta)^j + (\log (x_1 + x_2)\theta)^{i-1} (x_1 + x_2)^{-1} \log (x_1 + x_2 + x_s) / (x_1 + x_2) + x_1)
\]

\[
\leq x^{-1} j! x_1 (\log (x_1 + x_2)\theta)^j + (\log (x_1 + x_2)\theta)^{i-1} (x_1 + x_2)^{-1} \log (x_1 + x_2 + x_s)
\]

Taking \( \epsilon = x_t \), we obtain the result.

\boxcheck

### 4 Proof of Claim 2

Here we use the result from [5] that after \( k = n^2 \log n = loge \) steps the \( L^2 \) transportation distance between the running distribution of the Kac random walk on \( S^{n-1} \) and the uniform distribution \( U_{n-1} \) is less than \( \epsilon \). So by Holder’s inequality, the \( L^1 \) transportation distance is also less than \( \epsilon \). Now suppose \( \mu_k(H_s) > \epsilon^{1/4} \). We know that the uniform measure \( U_{n-1}(H_s) \) is of the order \( \epsilon \); in fact using the marginal
density formula for a single coordinate on the unit sphere, one sees that it is bounded above by $n^{3/2} \epsilon$, and similarly $U_{n-1}(H_{2\epsilon}) \leq 2n^{3/2} \epsilon$. So for $\epsilon$ sufficiently smaller than $n^{-3/2}$, we can make sure that

$$(n^{3/2} \epsilon)^{1/4} - 2cn^{3/2} > \epsilon^{1/2}$$

then in order to transport the excessive mass in $H$, under the $k$th running distribution of the random walk to other places on $S^{n-1}$, any transport strategy has to take all the mass outside $H_{2\epsilon}$, which means a distance more than $\epsilon$ has to be traversed for each particle mass. This shows the $L^1$ transportation cost for smoothing out the region $H_\epsilon$ is at least $c \epsilon^{1/2}$, which is greater than the total $L^1$ transportation distance, a contradiction.

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