ON THE IDEAL SHORTEST VECTOR PROBLEM OVER RANDOM RATIONAL PRIMES

YANBIN PAN, JUN XU, NICK WADLEIGH, AND QI CHENG

Abstract. Any ideal in a number field can be factored into a product of prime ideals. In this paper we study the prime ideal shortest vector problem (SVP) in the ring $\mathbb{Z}[x]/(x^2 + 1)$, a popular choice in the design of ideal lattice based cryptosystems. We show that a majority of rational primes lie under prime ideals admitting a polynomial time algorithm for SVP. Although the shortest vector problem of ideal lattices underpins the security of Ring-LWE cryptosystem, this work does not break Ring-LWE, since the security reduction is from the worst case ideal SVP to the average case Ring-LWE, and it is one-way.

Keywords: ring-LWE, ideal lattice, average case computational complexity

1. Introduction

Due to their conjectured ability to resist quantum computer attack, lattice-based cryptosystems have drawn considerable attention. In 1996, Ajtai [1] pioneered the research on worst-case to average-case reduction for the Short Integer Solution problem (SIS). In 2005, Regev [28] presented a worst-case to average-case (quantum) reduction for the Learning With Errors problem (LWE). SIS and LWE became two important cryptographic primitives, and a large number of cryptographic schemes based on these two problems have been designed. However, the common drawback of such schemes is their limited efficiency owing to the absence of algebraic structures in SIS and LWE.

The first lattice-based scheme with some algebraic structure was the NTRU public key cryptosystem [14], which was introduced by Hoffstein, Pipher and Silverman in 1996. It works in the convolution ring $\mathbb{Z}[x]/(x^p - 1)$ where $p$ is a prime. The cyclcical nature of the ring $\mathbb{Z}[x]/(x^p - 1)$ contributes to NTRU’s efficiency, and makes NTRU one of the most popular schemes. Later the ring was employed in many other cryptographic primitives, such as [21, 18, 26, 22, 31, 4].

In 2009, Stehlé et al. [32] introduced a structured and more efficient variant of LWE, which involves the ring $\mathbb{F}_p[x]/(x^N + 1)$ where $N$ is a power of 2 and $p$ is a prime satisfying $p \equiv 3 \pmod{8}$. In 2010, Lyubashevsky, Peikert and Regev [19] presented a ring-based variant of LWE, called Ring-LWE. The hardness of problems in [32, 19] is based on worst-case assumptions on ideal lattices. Recently, Peikert, Regev and Stephens-Davidowitz [25] presented a polynomial time quantum reduction from (worst-case) ideal lattice problems to Ring-LWE for any modulus and any number field. Since then, more and more schemes employ the ring $\mathbb{Z}[x]/(x^N + 1)$ where $N$ is a power of 2, for example, NewHope [2], Crystals-Kyber [7], and LAC [17] submitted to NIST’s post-quantum cryptography standardization. Although solving the ideal SVP does not necessarily break Ring-LWE, any weakness of ideal SVP casts doubt on the security of Ring-LWE.
1.1. Previous works. Principal ideal lattices are a class of important ideal lattices which can be generated by a single element. There is a line of work focusing on the principal ideal SVP. Based on \cite{3, 3}, solving approx-SVP problems on principal ideal lattices can be divided into the following two steps: Step 1 is finding an ideal generator by using class group computations. In this step, a quantum polynomial time algorithm is presented by Biasse and Song \cite{6}, which is based on the work \cite{13}; a classical subexponential time algorithm was given by Biasse, Espitau, Fouque, Gélin and Kirchner \cite{5}. Step 2 is shortening the ideal generator in Step 1 with the log-unit lattice. This step was analyzed by Cramer, Ducas, Peikert and Regev \cite{9}. Then a quantum polynomial time algorithm for approx-SVP, with a $2^{\widetilde{O}(<N^{1/2})}$ approximation factor, on principal ideal lattices in cyclotomic number fields was presented in \cite{9}.

In 2017, Cramer, Ducas and Wesolowski \cite{10} extended the case of principal ideal lattices in \cite{9} to the case of a general ideal lattice in a cyclotomic ring of prime-power conductor. For approx-SVP on ideal lattices, the result in \cite{10} is better than the BKZ algorithm \cite{29} when the approximation factor is larger than $2^{O(<N^{1/2})}$. Ducas, Plancon and Wesolowski \cite{11} analyzed the approximation factor $2^{O(<N^{1/2})}$ in \cite{9, 10} to determine the specific dimension $N$ so that the corresponding algorithms outperform BKZ for an ideal lattice in cyclotomic number fields. Recently, Pellet-Mary, Hanrot and Stehlé \cite{27}, inspired by the algorithms in \cite{9, 10}, solved approx-SVP with the approximation factor $2^{O(<N^{1/2})}$ in ideal lattices for all number fields, aiming to provide trade-offs between the approximation factor and the running time. However, there is an exponential pre-processing phase.

1.2. Our results. In this paper, we investigate the SVP of prime ideals. The density of prime ideals is not as high as that of principal ideals. In the simple case of the ring of rational integers $\mathbb{Z}$, every ideal is principal, while the density of prime numbers among the positive integers $\leq n$ is only about $1/\log n$. On the other hand, every nonzero ideal in a Dedekind Domain can be factored uniquely into a product of prime ideals, so short vectors in prime ideals may help us to find short vectors in general ideals. If, in a general prime ideal $p$, we are able to efficiently find a vector with length within the Minkowski bound for $p$, then for an ideal $\mathfrak{a}$ with few prime ideal factors, we will be able to approximate the shortest vector in $\mathfrak{a}$ to within a factor much better than what is achieved by the LLL \cite{15} or BKZ \cite{30} algorithms. The most difficult step in factoring an ideal is actually factorization of an integer (the norm of the ideal), which can be done in polynomial time by quantum computers, or in subexponential time by classical computers.

In this paper, we begin an in-depth study of the SVP of prime ideals in the rings $\mathbb{Z}[x]/(x^{2^n} + 1)$, which are quite popular in cryptography. We show that there is a hierarchy for the hardness of SVP for these prime ideal lattices. Roughly speaking, we can classify such prime ideal lattices into $n$ distinct classes, and for a prime ideal lattice in the $r$-th class, we can find its shortest vector by solving SVP in a $2^n$-dimensional lattice. This suggests that the difficulty of prime ideal SVP can change dramatically from ideal to ideal, an interesting phenomenon that has, to our knowledge, not been pointed out in the literature. See Theorem 3.2 for more details. By considering certain of these classes, we prove that a nontrivial fraction of prime ideals admit an efficient SVP algorithm.
Proposition 1.1. Let $N = 2^n$, where $n$ is a positive integer. Let $\mathfrak{p}$ be a prime ideal in the ring $\mathbb{Z}[x]/(x^N + 1)$, and suppose $\mathfrak{p}$ contains a prime number $p \equiv \pm 3 \pmod{8}$. Then under the coefficient embedding (see Page 2 for definition), the shortest vector in $\mathfrak{p}$ can be found in time $\text{poly}(N, \log p)$, and the length of the shortest vector is exactly $\sqrt{p}$.

Can we conclude from the above result that the average case prime ideal SVP is easy? It depends how we define an average prime ideal lattice. As prime ideals are rigid structures, changing distributions gives us totally different complexity results. If norms of prime ideals are selected uniformly at random, then easy cases are rare. Nevertheless our result does show that an average case of the prime ideal SVP in power-of-two cyclotomic fields is not hard, if the rational primes contained in the ideals are selected uniformly at random. See Subsection 3.3 for details.

The algorithm can be adapted to any Galois extension $\mathbb{L}$ of $\mathbb{Q}$. Indeed, fix a prime ideal $\mathfrak{p}$ in the ring of integers, $\mathcal{O}_\mathbb{L}$, of $\mathbb{L}$. The subgroup of $Gal(\mathbb{L}/\mathbb{Q})$ that stabilizes $\mathfrak{p}$ set-wise is known as the decomposition group of $\mathfrak{p}$. Let $K \subset \mathbb{L}$ be the subfield fixed by the decomposition group of $\mathfrak{p}$. To find a short vector in $\mathfrak{p}$, we can search for a short vector in the lattice $\mathfrak{p} \cap K$, which has smaller dimension. More precisely, for a rational prime $p$, if $p\mathcal{O}_\mathbb{L}$ is factored into a product of $g$ prime ideals in $\mathcal{O}_\mathbb{L}$, we can reduce the problem of finding a short vector in any of these prime ideals to a problem of finding a short vector in a dimension-$g$ lattice, provided that the determinant of the sublattice is not too large compared to the original lattice.

For general (non prime) ideals in $\mathbb{Z}[x]/(x^{2^n} + 1)$, we present an algorithm to confirm that the hierarchy for the hardness of SVP also exists; That is, we can solve SVP for an ideal lattice by solving SVP in a $2^r$-dimensional lattice, for some positive integer $r$ related to the factorization of the ideal (see Theorem 4.1). Following Proposition 1.1, we show how to solve the SVP for ideals all of whose prime factors lie in a certain class. This is a special case of Theorem 4.1.

Proposition 1.2. Let $N = 2^n$, where $n$ is a positive integer. Let $I$ be an ideal in the ring $\mathbb{Z}[x]/(x^N + 1)$ with prime factorization

$$I = \mathfrak{p}_1 \mathfrak{p}_2 \cdots \mathfrak{p}_k.$$  

If each $\mathfrak{p}_i$ contains a prime integer $\equiv \pm 3 \pmod{8}$, the shortest vector in $I$ can be found in time $\text{poly}(N, \log(N(I)))$.

We would like to stress that the algorithm works by exploiting the multiplication structure of ideals in the ring of integers of a number field. This appears to be new. More interestingly, our algorithm does not need to factor the ideal. We regard this as the second contribution of this work, in addition to the algorithm for prime ideals.

1.3. Paper organization. The remainder of the paper is organized as follows. In Section 2, we give some mathematical preliminaries needed. In Section 3 and 4, we first sketch an idea for a solution to SVP in the fully general setting of a finite Galois extension of $\mathbb{Q}$. Then we present our algorithms solving SVP for prime ideal lattices and general ideal lattices in $\mathbb{Z}[\zeta_{2^n} + 1]$. Finally, a conclusion and some open problems are given in Section 5.
2. Mathematical preliminary

2.1. Lattice. Lattices are discrete additive subgroups of \( \mathbb{R}^N \). Any finite set of vectors \( b_1, b_2, \ldots, b_m \in \mathbb{R}^N \) generate a lattice:

\[
\mathcal{L} = \left\{ \sum_{i=1}^{m} z_i b_i \mid z_i \in \mathbb{Z} \right\}.
\]

When the \( b_i \) are linearly independent and \( B \) is the matrix whose column vectors are \( b_i \), we say \( B \) is a matrix for \( \mathcal{L} \); \( m \) and \( N \) are the rank and dimension of \( \mathcal{L} \), respectively. If \( m = N \), we define the determinant of \( \mathcal{L} \), denoted by \( \det(\mathcal{L}) \), to be \( |\det(B)| \).

The shortest vector problem (SVP), the problem of finding a shortest nonzero lattice vector in a given lattice, is one of the most famous hard problems in lattice cryptography. Denote by \( \lambda_1(\mathcal{L}) \) the length of a shortest nonzero lattice vector in an \( N \)-dimensional lattice \( \mathcal{L} \). Minkowski’s theorem \([23]\) tells us that

\[
\lambda_1(\mathcal{L}) \leq \frac{2}{V_N^{1/N}} \cdot \det(\mathcal{L})^{1/N} \leq \sqrt{N} \cdot \det(\mathcal{L})^{1/N},
\]

where \( V_N \) is the volume of the \( N \)-dimensional ball with radius 1. The closest vector problem (CVP) is another famous hard problem in lattice cryptography. This refers to the problem of finding a lattice vector that is closest to a given vector.

2.2. Ideal in \( \mathbb{Z}[\zeta_{2^{n+1}}] \). The cyclotomic field of order \( 2N = 2^{n+1} \) is widely used in cryptography. Its ring of integers is \( \mathbb{Z}[\zeta_{2^{n+1}}] \), which is isomorphic to \( \mathbb{Z}[x]/(x^N + 1) \). Its discriminant is \( 2^{2n} \).

Let \( p \) be a rational prime, and let

\[
x^N + 1 = (f_1 f_2 \cdots f_g)^e
\]

be the prime factorization of \( x^N + 1 \) in the polynomial ring \( \mathbb{F}_p[x] \). Then we have

\[
(p) = (p_1 p_2 \cdots p_g)^e,
\]

where \( p_i = (p, f_i(\zeta_{2^{n+1}})) \) (here \( f_i \) is any integer polynomial which projects to the \( f_i \) in the above factorization). We say the prime ideal \( p \) lies over the prime \( p \). If \( e \) is greater than 1, we say the prime \( p \) is ramified (in \( \mathbb{Z}[\zeta_{2^{n+1}}] \)); otherwise we say \( p \) is unramified. One can verify that 2 is the only ramified rational prime in the cyclotomic field of order \( 2N \), and that the prime ideal \( (2, \zeta_{2^{n+1}} + 1) \) lies above the ideal \( (2) \).

We are therefore interested in the explicit factorization of the \( 2^{n+1} \)-th cyclotomic polynomials, \( x^{2^n} + 1 \), over \( \mathbb{F}_p[x] \). This is settled in \([16]\) when \( p \equiv 1 \pmod{4} \) and in \([20]\) when \( p \equiv 3 \pmod{4} \).

**Theorem 2.1.** Let \( p \equiv 1 \pmod{4} \), i.e. \( p = 2^A \cdot m + 1 \), \( A \geq 2 \), \( m \) odd. Denote by \( U_k \) the set of all primitive \( 2^k \)-th roots of unity modulo \( p \). We have

- If \( n < A \), then \( x^{2^n} + 1 \) is the product of \( 2^n \) irreducible linear factors over \( \mathbb{F}_p \):

\[
x^{2^n} + 1 = \prod_{u \in U_{n+1}} \ (x + u).
\]
• If \( n \geq A \), then \( x^{2n} + 1 \) is the product of \( 2^{A-1} \) irreducible binomials over \( \mathbb{F}_p \) of degree \( 2^{n-A+1} \):

\[
x^{2n} + 1 = \prod_{u \in U_A} (x^{2^{n-A+1}} + u).
\]

**Theorem 2.2.** Let \( p \equiv 3 \pmod{4} \), i.e. \( p = 2^A \cdot m - 1, A \geq 2, m \) odd. Denote by \( D_s(x, a) \) the Dickson polynomials

\[
\sum_{i=0}^{\frac{\phi(n)}{2}} s \frac{s - i}{i} (-a)^i x^{i-2i}
\]

over \( \mathbb{F}_p \). For \( n \geq 2 \), we have

• If \( n < A \), then \( x^{2n} + 1 \) is the product of \( 2^{n-1} \) irreducible trinomials over \( \mathbb{F}_p \):

\[
x^{2n} + 1 = \prod_{\gamma \in \Gamma} (x^2 + \gamma x + 1),
\]

where \( \Gamma \) is the set of all roots of \( D_{2n-1}(x, 1) \).

• If \( n \geq A \), then \( x^{2n} + 1 \) is the product of \( 2^{A-1} \) irreducible trinomials over \( \mathbb{F}_p \) of degree \( 2^{n-A+1} \):

\[
x^{2n} + 1 = \prod_{\delta \in \Delta} (x^{2^{n-A+1}} + \delta x^{2^{n-A}} - 1),
\]

where \( \Delta \) is the set of all roots of \( D_{2A-1}(x, -1) \).

2.3. Ideal Lattices. Let \( \mathbb{K} \) be a number field over \( \mathbb{Q} \) with degree \( N \), and let \( O_\mathbb{K} \) be its ring of integers. To treat an ideal in \( O_\mathbb{K} \) as a lattice, we need to map elements in \( O_\mathbb{K} \) to real vectors. We introduce two ways to do this: the canonical embedding and the coefficient embedding.

2.3.1. Canonical embedding. Let \( \sigma_1, \sigma_2, \ldots, \sigma_{s_1} \) be the real embeddings from \( \mathbb{K} \) to \( \mathbb{R} \), and let

\[
\sigma_{s_1+1}, \sigma_{s_1+2}, \ldots, \sigma_{s_1+s_2},
\]

\[
\sigma_{s_1+s_2+1} = \overline{\sigma_{s_1+1}}, \sigma_{s_1+s_2+2} = \overline{\sigma_{s_1+2}}, \ldots, \sigma_{s_1+2s_2} = \overline{\sigma_{s_1+s_2}}
\]

be the complex embeddings from \( \mathbb{K} \) to \( \mathbb{C} \), where \( \overline{\cdot} \) is the complex conjugate. From these \( \sigma_i \)'s we can make two different \( \mathbb{Q} \)-linear embeddings:

\[
\Sigma = \Sigma_\mathbb{K} : \mathbb{K} \rightarrow \mathbb{C}^{s_1+2s_2}, \ a \mapsto (\sigma_1(a), \sigma_2(a), \ldots, \sigma_{s_1+2s_2}(a)), \text{ and}
\]

\[
\Sigma' = \Sigma'_\mathbb{K} : \mathbb{K} \rightarrow \mathbb{R}^{s_1+2s_2}, \ a \mapsto (\sigma_1(a), \ldots, \sigma_{s_1}(a), \\
\sqrt{2} Re(\sigma_{s_1+1}(a)), \sqrt{2} Im(\sigma_{s_1+1}(a)), \ldots, \\
\sqrt{2} Re(\sigma_{s_1+s_2}(a)), \sqrt{2} Im(\sigma_{s_1+s_2}(a))).
\]

The \( \sqrt{2} \)'s are added so that the \( l_2 \) norms of \( \Sigma'(a) \) and \( \Sigma(a) \) agree. We call the map \( \Sigma \) (or \( \Sigma' \)) the canonical embedding. \( \Sigma' \) has the advantage that it embeds \( O_\mathbb{K} \) (and any nonzero ideal thereof) as a full rank lattice in \( \mathbb{R}^{s_1+2s_2} \). The formula for \( \Sigma' \) may appear inelegant, but notice that in the case where \( \mathbb{K}/\mathbb{Q} \) is Galois, we have either \( s_1 = 0 \) or \( s_2 = 0 \).
2.3.2. Coefficient embedding. If one can find a monic integral polynomial \( f(x) \) so that the ring of integers \( \mathcal{O}_K \) is isomorphic to \( \mathbb{Z}[x]/(f(x)) \), such a number field is called monogenic. The coefficient embedding of \( \mathcal{O}_K \) in a monogenic field sends an element to the coefficient vector of the corresponding polynomial in \( \mathbb{Z}[x]/(f(x)) \), namely,

\[
C(a_0 + a_1 x + \cdots + a_{N-1} x^{N-1}) = (a_0, a_1, \cdots, a_{N-1}).
\]

2.3.3. Ideal lattices. The ring of integers \( \mathcal{O}_K \) of \( K \) is a free \( \mathbb{Z} \)-module, and any ideal \( \mathcal{I} \) in \( \mathcal{O}_K \) is a free \( \mathbb{Z} \)-submodule since \( \mathbb{Z} \) is principle. Under the canonical embedding or the coefficient embedding, any such \( \mathcal{I} \) is sent to a lattice in \( \mathbb{R}^N \). We call this image the ideal lattice associated with \( \mathcal{I} \), and we denote it also by \( \mathcal{I} \).

Usually it is easier to use the canonical embedding in mathematical analysis, and to use the coefficient embedding in cryptography. For example, the lattice associated with the prime ideal \( \mathfrak{p}_i = (p, f_i(\zeta_{2^n+1})) \) is generated by the coefficient vectors of the following polynomials (modulo \( x^{N+1} \) of course)

\[
f_i, xf_i, \cdots, x^{N-1} f_i \quad \text{and} \quad p, px, \cdots, px^{N-1}.
\]

The minimum generating set should have only \( N \) vectors, which can be found by computing the Hermite Norm Form.

3. Solving SVP for prime ideal lattice

Before specializing to \( \mathbb{Z}[\zeta_{2^n+1}] = \mathcal{O}_Q(\zeta_{2^n+1}) \), we explain our idea to solve SVP for a prime ideal of \( \mathcal{O}_L \) when \( L \) is a general, finite Galois extension of \( \mathbb{Q} \). Such an ideal contains a rational prime \( p \), and therefore occurs as one of the prime ideals, say \( \mathfrak{p}_1 \), in the factorization

\[
p\mathcal{O}_L = (\mathfrak{p}_1^{a_1} \cdots \mathfrak{p}_g^{a_g})^*.
\]

To find a vector of \( \mathfrak{p}_1 \) with length within the Minkowski bound, we try to find a short vector in the sublattice given by the intersection of \( \mathfrak{p}_1 \) with some intermediate field between \( \mathbb{Q} \) and \( L \). Since this sublattice has smaller dimension, this may lead to a more efficient algorithm than working in \( L \) directly.

More precisely, let \( G \) be the Galois group of \( L \) over \( \mathbb{Q} \). The subgroup \( D \leq G \) consisting of all elements which set-wise stabilize \( \mathfrak{p}_1 \) is called the decomposition group of \( \mathfrak{p}_1 \). That is,

\[
D := \{ \sigma \in G : \sigma(\mathfrak{p}_1) = \mathfrak{p}_1 \}.
\]

Let \( \mathbb{K} \) be the fixed field of \( D \). That is

\[
\mathbb{K} := \{ x \in L : \forall \sigma \in D, \sigma(x) = x \}.
\]

Let \( \mathcal{O}_K \) be its algebraic integer ring. It is well known that the degree of \( \mathbb{K} \) over \( \mathbb{Q} \) is \( g \). This is our desired intermediate field.

Now let \( c = \mathfrak{p}_1 \cap \mathcal{O}_K \), and consider the following diagram

\[
\begin{array}{c}
p_1 \subset \mathcal{O}_L \subset L \xrightarrow{\Sigma'} \mathbb{R}[L:Q] \\
| \quad | \quad | \\
\mathfrak{c} \subset \mathcal{O}_K \subset K \xrightarrow{\Sigma'} \mathbb{R}^g \\
| \quad | \quad | \\
(p) \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}
\end{array}
\]
Here $\beta$ is chosen to be the $\mathbb{R}$-linear map making the diagram commute. For simplicity let us assume that all embeddings $K \to \mathbb{C}$ are complex (for instance this is the case if $K \not\subset \mathbb{R}$ and $K/Q$ is Galois). This is not at all essential; forgoing this assumption only introduces a factor of $\sqrt{2}$ in the following discussion. We make this assumption for the sole purpose that $\beta$ is then just the linear embedding given by repeating each coordinate $[L:Q]/g$ times. Thus we have

$$
\|\beta(v)\| = \sqrt{\frac{[L:Q]}{g}} \cdot \|v\|.
$$

Additionally, one may check that the lattice in $\mathbb{R}^g$ which is the image of $O_K$ has determinant $\sqrt{|\text{disc}(K)|}$ (something else which recommends the $\sqrt{2}$'s in the definition of $\Sigma'$), where $\text{disc}(*)$ is the discriminant. It is also not hard to prove that the norm of $c$ is exactly $p$. Thus, under the canonical embedding of $O_K$ into $\mathbb{R}^g$, any vector $v_0 \in c$ within the Minkowski bound satisfies

$$
\|v_0\| \leq \sqrt{g} \cdot p^{\frac{1}{g}} |\text{disc}(K)|^{\frac{1}{2g}}.
$$

By equation (1) above, we therefore have

$$
\|\beta(v_0)\| \leq p^{\frac{1}{g}} |\text{disc}(K)|^{\frac{1}{2g}} \cdot \sqrt{[L:Q]}.
$$

On the other hand, the norm of $p_1$ is $p^{\frac{1}{2}}$. Hence, when $p$ is unramified in $L$ (that is, when $e = 1$) the Minkowski bound for $p_1$ (under the canonical embedding into $\mathbb{R}[L:Q]$) becomes

$$
\sqrt{[L:Q]} \cdot (p^{\frac{1}{2}})^{\frac{1}{n-1}} \cdot |\text{disc}(L)|^{\frac{1}{2(n-1)}} = \sqrt{[L:Q]} \cdot (p^{\frac{1}{2}}) \cdot |\text{disc}(L)|^{\frac{1}{2(n-1)}}.
$$

Comparing this with (2), we find that $\beta(v_0)$ lies within the Minkowski bound for $p_1$ provided that

$$
|\text{disc}(K)|^{\frac{1}{2g}} \leq |\text{disc}(L)|^{\frac{1}{2n-1}}.
$$

We have therefore established the following quite general theorem

**Theorem 3.1.** Suppose $L/Q$ is a finite Galois extension, and suppose $p$ is a prime ideal of $O_L$ lying over an unramified rational prime. If $K$ is the fixed field of the decomposition group of $p$, and if the inequality (3) holds, then the problem of finding a nonzero Minkowski-short vector of $p$ (under the canonical embedding of $L$) reduces to the problem of finding a Minkowski-short vector of the sublattice $p \cap O_K$ (under the canonical embedding of $K$).

Finally note that the sublattice $p \cap O_K$ has dimension no more than half that of $p$ (unless $p = pO_L$, in which case SVP for $p$ is likely trivial). We do not know whether (3) holds in general, but it is true for power-of-two cyclotomic fields (see next subsection) and prime order cyclotomic fields (see the Appendix).

3.1. **Solving SVP for a prime ideal lattice in $\mathbb{Z}[\zeta_{2n+1}]$.** For simplicity, we let $\zeta = \zeta_{2n+1}$. Now we specialize to $L = Q(\zeta)$. We say goodbye to the canonical embedding, and use the coefficient embedding:

$$
Q(\zeta) \to \mathbb{R}^{2n}, \quad \sum_{i=0}^{2n-1} a_i \zeta^i \mapsto (a_0, a_1, ..., a_{2n-1}).
$$

The coefficient embedding is widely used in lattice-based cryptographic constructions. In fact, for power-of-two cyclotomic fields, the two embeddings are related
by scaled-rotations. The prime 2 is the unique ramified prime in \( \mathbb{Q}(\zeta) \), and the prime ideal lying over \((2)\) is \((2, \zeta + 1) = (\zeta + 1)\). Hence it is easy to find the shortest vector in the ideal lattice \((\zeta + 1)\), and its length is \(\sqrt{2}\).

Below we consider a prime ideal lying over an odd prime and show that there is a hierarchy for the hardness of solving SVP for prime ideal lattices in \(\mathbb{Z}[\zeta]\). Roughly speaking, we can classify all the prime ideal lattices into \(n\) classes labeled with \(1, 2, \cdots, n\), depending on the congruence class of \(p \pmod{2^n+1}\), and for a prime ideal lattice in the \(r\)-th class, we can always find its shortest vector by solving SVP in a \(2^r\)-dimensional lattice. More precisely, we have:

**Theorem 3.2.** For any prime ideal \(p = (p, f(\zeta))\) in \(\mathbb{Z}[\zeta]\), where \(p\) is an odd prime and \(f(x)\) is some irreducible factor of \(x^{2^n+1} + 1\) in \(\mathbb{F}_p[x]\). Write

\[
p = \begin{cases} 
2^A \cdot m + 1, & \text{if } p \equiv 1 \pmod{4}; \\
2^A \cdot m - 1, & \text{if } p \equiv 3 \pmod{4}, 
\end{cases}
\]

for some odd \(m\) and \(A \geq 2\), and let

\[
r = \begin{cases} 
\min\{A - 1, n\}, & \text{if } p \equiv 1 \pmod{4}; \\
\min\{A, n\}, & \text{if } p \equiv 3 \pmod{4}. 
\end{cases}
\]

Then given an oracle that can solve SVP for \(2^r\)-dimensional lattice, a shortest nonzero vector in \(p\) can be found in \(\text{poly}(2^n, \log_2 p)\) time with the coefficient embedding.

**Proof.** It is well known that the Galois group \(G\) of \(\mathbb{Q}(\zeta)\) over \(\mathbb{Q}\) is isomorphic to \((\mathbb{Z}/2^n+1\mathbb{Z})^*\). Let \(G = \{\sigma_1, \sigma_3, \cdots, \sigma_{2^n+1}\}\) where

\[
\sigma_i : \mathbb{Q}(\zeta) \rightarrow \mathbb{Q}(\zeta);
\quad \zeta \mapsto \zeta^i.
\]

Next we prove the theorem by considering the two cases separately.

**Case 1:** First we deal with the case when \(p \equiv 1 \pmod{4}\). The theorem holds for \(n < A\).

If \(n \geq A\), we have \(r = A - 1\). By Theorem 2.1 we know that

\[f(x) = x^{2^n-A+1} + u = x^{2^n-r} + u\]

for some \(u \in U_A\). Then the prime ideal lattice \(p\) can be generated by \(p\) and \(f(\zeta) = \zeta^{2^n-r} + u\). Consider the subgroup \(H = \langle \sigma_{2^{r+1}+1} \rangle\) of \(G\) generated by \(\sigma_{2^{r+1}+1}\). 

\(H\) is also a subgroup of the decomposition group of the ideal \(p\) since

\[
\sigma_{2^{r+1}+1}(p) = p, \quad \sigma_{2^{r+1}+1}(f(\zeta)) = f(\zeta).
\]

Note that \(K = \mathbb{Q}(\zeta^{2^n-r})\) is the fixed field of \(H\) and its integer ring \(O_K\) has a \(\mathbb{Z}\)-basis

\[
(1, \zeta^{2^n-r}, \zeta^{2^{n-r}}, \cdots, \zeta^{(2^n-1)2^{n-r}}).
\]

Let \(\mathfrak{c} = p \cap O_K\). We claim that \(p\) is an internal orthogonal direct sum:

\[
p = \bigoplus_{k=0}^{2^{n-r}-1} \zeta^k \mathfrak{c}.
\]

(4)
Indeed for any \( a \in p \), there exist integers \( z_i \)'s and \( w_i \)'s such that
\[
a = \sum_{i=0}^{2^n-1} z_i \zeta^i f(\zeta) + \sum_{i=0}^{2^n-1} w_i p \zeta^i = \sum_{k=0}^{2^{n-r}-1} \zeta^k \sum_{j=0}^{2^r-1} (z_{k+j+2^n-\tau} \zeta^{j+2^{n-r}} f(\zeta) + w_{k+j+2^n-\tau} p \zeta^{j+2^{n-r}})
\]
\[
= \sum_{k=0}^{2^{n-r}-1} \zeta^k \left( \sum_{j=0}^{2^r-1} z_{k+j+2^n-\tau} \zeta^{j+2^{n-r}} f(\zeta) + (\sum_{j=0}^{2^r-1} w_{k+j+2^n-\tau} \zeta^{j+2^{n-r}}) p \right).
\]

Let \( a^{(k)} = (\sum_{j=0}^{2^r-1} z_{k+j+2^n-\tau} \zeta^{j+2^{n-r}}) f(\zeta) + (\sum_{j=0}^{2^r-1} w_{k+j+2^n-\tau} \zeta^{j+2^{n-r}}) p \) for any \( k \).

Since \( p \in c \) and \( f(\zeta) \in c, a^{(k)} \in c \). We have established (4).

Since multiplication by \( \zeta \) is an isometry, (4) implies
\[
\lambda_1(p) = \lambda_1(c),
\]
and that to find the shortest vector in ideal lattice \( p \), it is enough to find the shortest vector \( v \) in the ideal lattice \( c \), a lattice with dimension \( 2^r \). Indeed \( \zeta^k v \) for any \( 0 \leq k \leq 2^n - r - 1 \) will be a shortest vector in the ideal lattice \( p \).

**Case 2:** For the case when \( p \equiv 3 \pmod{4} \), everything is similar except that \( r = A \). Indeed by Theorem 2.2, \( \mathbb{Q}(\zeta) \) is a cyclic extension of degree \( 2^{n-A+1} \) over the decomposition field for \( p \). We sketch the subfield lattice for \( \mathbb{Q}(\zeta) \), and calculate the decomposition group for \( p \) in the appendix.

**Algorithm:** We can summarize the algorithm to solve SVP in ideal prime lattice as Algorithm

**Algorithm 1** Solve SVP in prime ideal lattice

**Input:** a prime ideal \( p = (p, f(\zeta)) \in \mathbb{Z}[\zeta] \), where \( p \) is odd.

**Output:** a shortest vector in the corresponding prime ideal lattice.

1. Compute the ideal \( c \) generated by \( p \) and \( f(\zeta) \) in \( O_\mathbb{K} \) where \( \mathbb{K} = \mathbb{Q}(\zeta^{2^n-r}) \).
2. Find a shortest vector \( v \) in the \( 2^r \)-dimensional lattice \( c \);
3. Output \( v \).

The most time-consuming step in Algorithm 1 is Step 2 and the other steps can be done in \( \text{poly}(2^n, \log_2 p) \) time.

**Remark 3.1.** Since for any \( a \in p \), there exist \( a^{(k)} \in c \) for \( 0 \leq k \leq 2^n - r \), such that
\[
a = \sum_{k=0}^{2^{n-r}-1} \zeta^k a^{(k)},
\]
we conclude that \( (b^{(i)})_{0 \leq i < 2^r} \) is a basis of the ideal lattice \( c \), then \( (\zeta^j b^{(i)})_{0 \leq i < 2^r, 0 \leq j < 2^{n-r}} \) is a basis of the ideal lattice \( I \). Furthermore, for \( j_1 \neq j_2 \), any vector in \( (\zeta^j b^{(i)})_{0 \leq i < 2^r} \) is orthogonal to any vector in \( (\zeta^{j_2} b^{(i)})_{0 \leq i < 2^r} \). Denote by \( L_i \) the lattice generated by \( (\zeta^j b^{(i)})_{0 \leq i < 2^r} \), then we have that the ideal lattice \( p \) has an orthogonal decomposition: \( L_0 \oplus L_1 \oplus \cdots \oplus L_{2^{n-r}-1} \), where \( L_i \) is orthogonal to \( L_j \) for \( i \neq j \), that is, for any vector \( v^{(i)} \in L_i \) and \( v^{(j)} \in L_j \), the inner product \( \langle v^{(i)}, v^{(j)} \rangle = 0 \).

**Remark 3.2.** By the remark above, solving the closest vector problem (CVP) for the prime ideal lattice can be also reduced to solving CVP in some \( 2^r \)-dimensional sublattice.
3.2. The shortest vector of some prime ideal lattice in \( \mathbb{Z}[\zeta_{2^{n+1}}] \). Using Theorem 3.2 we can now prove Proposition 1.1 in Section 1.2.

For the case \( p \equiv -3 \pmod{8} \), by Theorem 2.1 \( x^{2^n} + 1 = (x^{2^{n-1}} + u_1) \cdot (x^{2^{n-1}} + u_2) \) over \( \mathbb{F}_p[x] \), where \( u_i \) satisfies \( u_i^2 \equiv -1 \pmod{p} \). By the proof of Theorem 3.2, the shortest vector of the ideal lattice \( (p, \zeta_{2^{n-1}} + u_i) \) can be found efficiently by solving SVP in the 2-dimensional lattice \( L_i \) generated by \( \left( \begin{array}{cc} u_i & 1 \\ p & 0 \end{array} \right) \). For any vector \( v \in L_i \), there exists an integer vector \( (z_1, z_2) \) such that \( v = (z_1 u_i + z_2 p, z_1) \). Note that

\[
\|v\|^2 = (z_1 u_i + z_2 p)^2 + z_1^2 = z_1^2(u_i^2 + 1) + z_2^2 p^2 + 2p z_1 z_2 u_i \equiv 0 \pmod{p}.
\]

If \( v \) is the shortest nonzero vector, we have \( 0 < \|v\|^2 < \frac{4}{3} \cdot p < 2p \) (by Minkowski’s Theorem [25]), which implies that \( \|v\|^2 = p \). Hence the proposition follows in this case.

Similarly, the proposition holds for \( p \equiv 3 \pmod{8} \).

3.3. Average-case hardness to solve SVP for prime ideal lattice in \( \mathbb{Z}[\zeta] \). Precisely defining the average-case hardness of SVP for a prime ideal lattice in \( \mathbb{Z}[\zeta] \) requires specifying a distribution. We consider the following three distributions.

3.3.1. The first distribution. To select a random prime ideal, one fixes a large \( M \), uniformly randomly selects a prime number in the set

\[
\{ p \text{ is a prime : } p < M \},
\]

and then uniformly randomly selects a prime ideal lying over \( p \). This process provides a reasonable distribution among prime ideals, since every prime ideal in the ring of integers of \( \mathbb{Q}[x]/(f(x)) \) is of the form \( (p, g(x)) \), where \( p \) is a prime number and \( g(x) \) is an irreducible factor of \( f(x) \) over \( \mathbb{F}_p[x] \). Since roughly half of all primes \( p \leq M \) satisfy \( p \equiv \pm 3 \pmod{8} \), according to Dirichlet’s theorem on arithmetic progressions, at least half of all such \( p \) have the property that the ideals lying over \( p \) admit an efficient algorithm for SVP.

3.3.2. The second distribution. Again fixing a large \( M \), we might alternatively select a prime ideal uniformly at random from the set

\[
\{ \mathfrak{p} \text{ prime ideal : } p \in \mathfrak{p}, p \text{ is a prime, } p < M \}.
\]

In this case, first we would like to point out that

**Proposition 3.1.** Under the distribution above, a random prime ideal of \( \mathbb{Z}[\zeta] \) admits an efficient SVP algorithm with probability at least \( \frac{1}{1+2^{n-1}} \).

**Proof.** For simplicity, we disregard the single prime ideal lying over 2. Note that for \( p = 8k \pm 3 \), there are exactly two prime ideals over \( p \), and, by Proposition 1.1 the SVP for the corresponding ideal lattices is easy. For \( p = 8k \pm 1 \), there are at most \( 2^n \) prime ideals lying over \( p \), by Theorem 2.1 and 2.2. Then by Dirichlet’s prime number theorem, even if we only count the prime ideals lying over \( p = 8k \pm 3 \), the fraction of easy instances is at least \( \frac{1}{1+2^{n-1}} \). \( \square \)

By Theorem 3.2, SVP for a prime ideal lattice \( \mathfrak{p} \) reduces to SVP for a \( 2^{n} \)-dimensional sub-lattice \( \mathfrak{c} \), where \( r \) is as defined in the statement of 3.2. We are therefore interested in the expected value of \( r \) when \( \mathfrak{p} \) is chosen uniformly at random according to the second distribution.
More precisely, for a large integer \( M > 0 \), consider the set \( PS_M \) consisting of all prime ideals lying over a rational prime \( p < 2^M \). An ideal can be chosen uniformly at random from \( PS_M \) since \( PS_M \) is finite. We can then compute the expected dimension \( D_M \) of the sub-lattice \( \mathcal{c} \), which is about
\[
\sum_{A=2}^{n} 2^{A-1} \cdot \frac{2^M/2^{A+1}}{2^M-1} + \sum_{A=2}^{n} 2^A \cdot \frac{2^M/2^{A+1}}{2^M-1} + \sum_{A=n+1}^{M} 2^n \cdot \frac{2^M/2^{A+1}}{2^M-1} \cdot 2,
\]
when \( M \) is big enough. As \( M \to \infty \), the expected dimension of the lattice \( \mathcal{c} \) tends to \( \frac{2^n}{\pi} + O(1) \).

Hence, even with the exponential time algorithm \([23]\) to solve the \( 2^r \)-dimensional ideal, the expected running time is still polynomial, that is, \( \text{poly}(2^n, \log_2 p) \).

3.3.3. **The third distribution.** The third distribution is more common in mathematics, but it seems hard to sample. Namely, after fixing a large \( M \), we select uniformly at random a prime ideal from the set
\[
\{ p \text{ prime ideal : } \mathcal{N}(p) < M \},
\]
where \( \mathcal{N}(p) \) is the norm of the ideal \( p \).

By Theorem 3.2, SVP for a prime ideal lattice \( p \) reduces to SVP for a \( 2^r \)-dimensional sub-lattice \( \mathcal{c} \), where \( r \) is as defined in the statement of 3.2. Note that our algorithm will not improve matters if \( r = n \), that is, if \( p \) splits completely in \( \mathbb{Q}(\zeta) \) and hence \( \mathcal{N}(p) = p \). By Chebotarev’s density theorem \([33]\), there are about \( \frac{M}{2^n \log M} \) rational primes which split in \( \mathbb{Q}(\zeta) \) and hence \( \frac{M}{\log M} \) prime ideals lying above those primes, for which our algorithm will not improve the efficiency to solve SVP.

For those prime ideals that our algorithm can do better, we must have the prime \( p \) lying below them must satisfy \( p \leq \sqrt{M} \) since \( \mathcal{N}(p) = p^f < M \) where \( f \) is some integer greater than 1. Hence there are at most \( \sqrt{M} \) such primes and hence at most \( 2^{n-1} \sqrt{M} \) prime ideals that our algorithm can do better.

Therefore, under such a distribution, the density of the easy instances for our algorithm is at most \( \frac{2^{n-1} \log M}{\sqrt{M}} \) which goes to zero when \( M \) tends to infinity.

### 4. Solving SVP for general ideal lattice in \( \mathbb{Z}[\zeta_{2^n+1}] \)

For simplicity, we let \( \zeta = \zeta_{2^n+1} \). Even for a general ideal lattice \( \mathcal{I} \subset \mathbb{Z}[\zeta] \), there is a similar hierarchy for the hardness of SVP for \( \mathcal{I} \).

**Theorem 4.1.** Let \( \mathcal{I} \) be a nonzero ideal of \( \mathbb{Z}[\zeta] \) with prime factorization
\[
\mathcal{I} = p_1 \cdot p_2 \cdots p_t,
\]
where \( p_i = (f_i(\zeta), p_i) \) for rational primes \( p_i \), and where the \( p_i \) are not necessarily distinct. Write \( p_i = 2^{A_i} \cdot m_i + 1 \) when \( p_i \equiv 1 \pmod{4} \) and \( p_i = 2^{A_i} \cdot m_i - 1 \) when \( p_i \equiv 3 \pmod{4} \) with odd \( m_i \), and let \( r = \max\{r_i\} \), where
\[
\begin{align*}
r_i = \begin{cases} 
\min\{A_i - 1, n\}, & \text{if } p_i \equiv 1 \pmod{4}; \\
\min\{A_i, n\}, & \text{if } p_i \equiv 3 \pmod{4}; \\
n, & \text{if } p_i = 2.
\end{cases}
\end{align*}
\]
Then the shortest vector in the ideal lattice \( \mathcal{L} \) corresponding to \( \mathcal{I} \) can be solved via solving SVP in a \( 2^r \)-dimensional lattice.
Proof: If \( r = n \), then the theorem follows simply.

If \( r < n \), W.L.O.G., we assume \( r = r_1 \). Following the proof of Theorem 3.2, denote the Galois group \( G = \{ \sigma_1, \sigma_3, \cdots, \sigma_{2^n-1} \} \) of \( \mathbb{Q}(\zeta) \) over \( \mathbb{Q} \), where \( \sigma_i(\zeta) = \zeta^i \). Consider the subgroup \( H = \langle \sigma_{2^{r+1}} \rangle \) of \( G \) generated by \( \sigma_{2^{r+1}} \). For any \( \tau \in H \) and every prime ideal \( p_i = (p_i, f_i(\zeta)) \), we have \( \tau(p_i) = p_i \) since \( \sigma_{2^{r+1}}(p_i) = p_i \), \( \sigma_{2^{r+1}}(f_i(\zeta)) = f_i(\zeta) \). Note that \( K = \mathbb{Q}(\zeta^{2^{n-r}}) \) is the fixed field of \( H \) and its integer ring \( \mathcal{O}_K \) has a \( \mathbb{Z} \)-basis \( \{1, \zeta^{2^{n-r}}, \cdots, \zeta^{(2^{r-1}) \cdot 2^{n-r}}\} \).

Let \( \mathcal{I} = \mathcal{I} \cap \mathcal{O}_K \). We claim that for any \( a \in \mathcal{I} \), there exist \( a(k) \in \mathcal{I} \) for \( 0 \leq k < 2^{n-r} \), such that

\[
a = \sum_{k=0}^{2^{n-r}-1} \zeta^k a(k).
\]

We proceed by induction. When \( t = 1 \) the above claim holds by Theorem 3.2. Suppose the claim holds for \( t-1 \). Then setting \( \mathcal{I} = \mathcal{I}_1 \cdot p_1 \cdots p_t \) and \( \mathcal{I} = \mathcal{I}_1 \cdot p_t \), we have \( \mathcal{I} = \mathcal{I}_1 \cdot p_t \). For any \( a \in \mathcal{I} \), we can write \( a = \sum_i x_i y_i \) where \( x_i \in \mathcal{I}_1 \) and \( y_i \in p_t \). It suffices to show that for any \( xy \), where \( x \in \mathcal{I}_1 \) and \( y \in p_t \), there exist \( b(k) \in \mathcal{I}_1 \cap \mathcal{O}_K \) for \( 0 \leq k < 2^{n-r} \), such that \( xy = \sum_{k=0}^{2^{n-r}-1} \zeta^k b(k) \).

By the induction assumption, there exist \( x(i) \in \mathcal{I}_1 \cap \mathcal{O}_K \) for \( 0 \leq i < 2^{n-r} \) such that \( x = \sum_{i=0}^{2^{n-r}-1} \zeta^i x(i) \), and there exist \( y(j) \in p_i \cap \mathcal{O}_K \) for \( 0 \leq j < 2^{n-r} \) such that \( y = \sum_{j=0}^{2^{n-r}-1} \zeta^j y(j) \). Hence, we have

\[
xy = \sum_{i=0}^{2^{n-r}-1} \sum_{j=0}^{2^{n-r}-1} \zeta^{i+j} x(i) y(j)
\]

\[
= \sum_{k=0}^{2^{n-r}-1} \zeta^k \sum_{i+j=k} x(i) y(j) + \sum_{k=2^{n-r}}^{2^{n-r}-2} \zeta^k \sum_{i+j=k} x(i) y(j)
\]

\[
= \sum_{k=0}^{2^{n-r}-1} \zeta^k \sum_{i+j=k} x(i) y(j) + \sum_{k=0}^{2^{n-r}-2} \zeta^k \sum_{i+j=k+2^{n-r}} x(i) y(j)
\]

\[
= \sum_{k=0}^{2^{n-r}-2} \zeta^k (\sum_{i+j=k} x(i) y(j)) + \sum_{i+j=k+2^{n-r}} x(i) y(j) + \zeta^{2^{n-r}-1} \sum_{i+j=k} x(i) y(j)
\]

Let \( b(k) = \sum_{i+j=k} x(i) y(j) + \sum_{i+j=k+2^{n-r}} \zeta^{2^{n-r}} x(i) y(j) \) for any \( 0 \leq k \leq 2^{n-r}-2 \) and \( b(2^{n-r}-1) = \sum_{i+j=2^{n-r}-1} x(i) y(j) \). We have that \( b(k) \in \mathcal{I}_1 \cap \mathcal{O}_K \) for \( 0 \leq k < 2^{n-r} \). Hence, for any \( a \in \mathcal{I} \), there exist \( a(k) \in \mathcal{I} \) for \( 0 \leq k < 2^{n-r} \), such that \( a = \sum_{k=0}^{2^{n-r}-1} \zeta^k a(k) \).

As in the proof of Theorem 3.2 we can show that \( \lambda_1(\mathcal{I}) = \lambda_1(\mathcal{I}) \) and any nonzero shortest vector in \( \mathcal{I} \) will yield \( 2^{n-r} \) nonzero shortest vectors in \( \mathcal{I} \).

We would like to point out that in some cases, the \( r \) in Theorem 4.1 can be improved. Consider the case when \( n \geq 3 \) and \( \mathcal{I} = (2, \zeta - 1)^2 = (2, \zeta^2 + 1) \). We need to solve SVP in a \( 2^n \)-dimensional lattice by Theorem 4.1. However, using the intermediate field \( \mathbb{Q}(\zeta^2) \) as in the proof of Theorem 4.1 we can find a shortest vector by solving SVP in a \( 2^n-1 \)-dimensional lattice.
Furthermore, since for any \( a \in I \), there exist \( a^{(k)} \in \mathfrak{c} \) for \( 0 \leq k < 2^{n-r} \), such that \( a = \sum_{k=0}^{2^{n-r}-1} \zeta^k a^{(k)} \), we conclude that if \( (b^{(i)})_{0 \leq i < 2^r} \) is a basis of the ideal lattice \( \mathfrak{c} \), then \( (\zeta^j b^{(i)})_{0 \leq i < 2^r, 0 \leq j < 2^{n-r}} \) is a basis of ideal lattice \( I \). Denote by \( L_j \) the lattice generated by \( (\zeta^j b^{(i)})_{0 \leq i < 2^r} \). Then we have that the ideal lattice \( I \) has an orthogonal decomposition: \( L_0 \oplus L_1 \oplus \cdots \oplus L_{2^{n-r} - 1} \), where \( L_i \) is orthogonal to \( L_j \) for \( i \neq j \).

In fact, for any \( \bar{r} \), let \( \mathfrak{c} = I \cap O_K \) where \( K = \mathbb{Q}(\zeta^{2^{n-r}}) \). For any basis \( (b^{(i)})_{0 \leq i < 2^r} \) of the ideal lattice \( \mathfrak{c} \) if \( (\zeta^j b^{(i)})_{0 \leq i < 2^r, 0 \leq j < 2^{n-r}} \) is a basis of the ideal lattice \( I \) (meaning that the ideal lattice \( I \) has an orthogonal decomposition), then the shortest vector in \( \mathfrak{c} \) is also a shortest vector in \( I \). Hence we have the following algorithm to solve SVP for general ideal in \( \mathbb{Z}[\zeta] \).

**Algorithm 2.** Solve SVP in general ideal lattice

**Input:** an ideal \( I \);

**Output:** a shortest vector in the corresponding ideal lattice \( L \).

1: for \( \bar{r} = 1 \) to \( n \) do
2: Compute a basis \( (b^{(i)})_{0 \leq i < 2^r} \) of the ideal lattice \( \mathfrak{c} = I \cap O_K \), where \( K = \mathbb{Q}(\zeta^{2^{n-r}}) \).
3: if \( (\zeta^j b^{(i)})_{0 \leq i < 2^r, 0 \leq j < 2^{n-r}} \) is exactly a basis of ideal lattice \( I \) then
4: Find a shortest vector \( v \) in the \( 2^r \)-dimensional lattice \( \mathfrak{c} \);
5: Output \( v \).
6: end if
7: end for

Note that Step 2 can be done efficiently by taking \( O_K \) as a lattice and then computing the intersection of the lattice \( I \) and the lattice \( O_K \) under the coefficient embedding.

**Remark 4.1.** By Theorem 4.1, solving the closest vector problem (CVP) for the general ideal lattice can also be reduced to solving CVP in some \( 2^r \)-dimensional lattice.

5. Conclusion and open problems

We have investigated the SVP of prime ideal lattices in the power-of-two cyclotomic fields, and designed an algorithm exploiting the subfield structure of such fields to efficiently solve SVP for a large portion of such ideals. Using ideal factorization, we obtained an efficient algorithm for many general (non prime) ideals. We also determined the length of the shortest vector of those prime ideals lying over rational primes congruent to \( \pm 3 \pmod{8} \). It is an interesting problem to study the length of the shortest vectors in other prime ideals. The worst case hardness of prime ideal lattice SVP for power-of-two cyclotomic fields is also left open.

**References**

[1] Ajtai, M.: Generating hard instances of lattice problems (extended abstract). In: Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing - STOC. pp. 99–108 (1996)

[2] Alkim, E., Ducas, L., Pöppelmann, T., Schwabe, P.: Newhope without reconciliation. IACR Cryptology ePrint Archive 2016, 1157 (2016), http://eprint.iacr.org/2016/1157
[3] Bernstein, D.J.: A subfield-logarithm attack against ideal lattices: Computing algebraic number theory tackles lattice-based cryptography. The cr. yp.to blog, 2014. 
https://blog.cr.yp.to/20140213-ideal.html

[4] Bernstein, D.J., Chuengsatiansup, C., Lange, T., van Vredendaal, C.: NTRU prime: Reducing attack surface at low cost. In: Selected Areas in Cryptography - SAC 2017 - 24th International Conference, Ottawa, ON, Canada, August 16-18, 2017, Revised Selected Papers. pp. 235–260 (2017). https://doi.org/10.1007/978-3-319-72565-9_12
https://doi.org/10.1007/978-3-319-72565-9_12

[5] Biasse, J., Espitau, T., Fouque, P., Gélin, A., Kirchner, P.: Computing generator in cyclotomic integer rings. In: Advances in Cryptology - EUROCRYPT 2017 - 36th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Paris, France, April 30 - May 4, 2017, Proceedings, Part I. pp. 60–88 (2017). https://doi.org/10.1007/978-3-319-56620-7_3
https://doi.org/10.1007/978-3-319-56620-7_3

[6] Biasse, J., Song, F.: Efficient quantum algorithms for computing class groups and solving the principal ideal problem in arbitrary degree number fields. In: Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016. pp. 893–902 (2016). https://doi.org/10.1137/1.9781611974331.ch64
https://doi.org/10.1137/1.9781611974331.ch64

[7] Bos, J.W., Ducas, L., Kiltz, E., Lepoint, T., Lyubashevsky, V., Schanck, J.M., Schwabe, P., Seiler, G., Stehlé, D.: CRYSTALS - kyber: A CCA-secure module-lattice-based KEM. In: 2018 IEEE European Symposium on Security and Privacy, EuroS&PS 2018, London, United Kingdom, April 24-26, 2018. pp. 353–367 (2018). https://doi.org/10.1109/EuroSP.2018.00032
https://doi.org/10.1109/EuroSP.2018.00032

[8] Campbell, P., Groves, M., Shepherd, D.: Soliloquy: A cautionary tale, 2014. 
https://docbox.etsi.org/workshop/2014/

[9] Cramer, R., Ducas, L., Peikert, C., Regev, O.: Recovering short generators of principal ideals in cyclotomic rings. In: Advances in Cryptology - EUROCRYPT 2016. pp. 559–585 (2016)

[10] Cramer, R., Ducas, L., Wesołowski, B.: Short stichelberger class relations and application to ideal-SVP. In: Advances in Cryptology - EUROCRYPT 2017 - 36th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Paris, France, April 30 - May 4, 2017, Proceedings, Part I. pp. 324–348 (2017). https://doi.org/10.1007/978-3-319-56620-7_12
https://doi.org/10.1007/978-3-319-56620-7_12

[11] Ducas, L., Plançon, M., Wesołowski, B.: On the shortness of vectors to be found by the ideal-SVP quantum algorithm. Quantum Algorithm, 2019. To appear.

[12] Dummit, D.S., Foote, R.M.: Abstract Algebra. John Wiley and Sons, 3rd edn. (2004)

[13] Eisentrager, K., Hallgren, S., Kitaev, A.Y., Song, F.: A quantum algorithm for computing the unit group of an arbitrary degree number field. In: Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014. pp. 293–302 (2014). https://doi.org/10.1145/2591796.2591860
https://doi.org/10.1145/2591796.2591860

[14] Hofstein, J., Pipher, J., Silverman, J.H.: NTRU: A ring-based public key cryptosystem. In: Algorithmic Number Theory, Third International Symposium, ANTS-III. pp. 267–288 (1998)

[15] Lenstra, A.K., Lenstra, H.W., Lovász, L.: Factoring polynomials with rational coefficients. Mathematische Annalen 261(4), 515–534 (1982)

[16] Lidl, R., Niederreiter, H.: Finite fields. Encyclopedia of Mathematics and Its Applications, Vol. 20, Addison–Wesley, Reading, MA, 1983

[17] Lu, X., Liu, Y., Zhang, Z., Jia, D., Xue, H., He, J., Li, B.: LAC: practical Ring-LWE based public-key encryption with byte-level modulus. IACR Cryptology ePrint Archive 2018, 1009 (2018). https://eprint.iacr.org/2018/1009

[18] Lyubashevsky, V., Micciancio, D.: Generalized compact knapsacks are collision resistant. In: Automata, Languages and Programming, 33rd International Colloquium, ICALP 2006, Venice, Italy, July 10-14, 2006, Proceedings, Part II. pp. 144–155 (2006). https://doi.org/10.1007/11787066_13
https://doi.org/10.1007/11787066_13

[19] Lyubashevsky, V., Peikert, C., Regev, O.: On ideal lattices and learning with errors over rings. In: Advances in Cryptology - EUROCRYPT. Lecture Notes in Computer Science, vo 6110, pp. 1–23. Springer (2010)
For $p$ a prime, we described subfields of $\mathbb{Q}(\zeta_p)$ and their discriminants. It is well-known (see for instance [12 Thm 14.5]) that subfields of $\mathbb{Q}(\zeta_p)$ have the form

$$\mathbb{Q}\left(\sum_{b \in B} \zeta_p^b\right)$$

where $B$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$. The discriminant of such a subfield, $K$, is described by a simple formula in terms of its degree over $\mathbb{Q}$:

$$\text{disc}(K/\mathbb{Q}) = p^{[K:\mathbb{Q}]-1}.$$

This can be seen using the Führerdiskriminantenproduktformel (see [34 Thm 3.11]), which says the following. For $H$ a group of Dirichlet characters of $\mathbb{Z}/m\mathbb{Z}$,

**APPENDIX A. $\mathbb{Q}(\zeta_p)$**

For $p$ a prime, we described subfields of $\mathbb{Q}(\zeta_p)$ and their discriminants. It is well-known (see for instance [12 Thm 14.5]) that subfields of $\mathbb{Q}(\zeta_p)$ have the form

$$\mathbb{Q}\left(\sum_{b \in B} \zeta_p^b\right)$$

where $B$ is a subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$. The discriminant of such a subfield, $K$, is described by a simple formula in terms of its degree over $\mathbb{Q}$:

$$\text{disc}(K/\mathbb{Q}) = p^{[K:\mathbb{Q}]-1}.$$

This can be seen using the Führerdiskriminantenproduktformel (see [34 Thm 3.11]), which says the following. For $H$ a group of Dirichlet characters of $\mathbb{Z}/m\mathbb{Z}$.
that is, a group of group homomorphisms $h : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, the fixed field of $\bigcap_{h \in H} \ker(h)$ in $\mathbb{Q}(\zeta_m)$ has discriminant with magnitude

$$\prod_{h \in H} C_h,$$

where $C_h$ is the conductor of $h$. That is, $C_h$ is the minimum among divisors $n|m$ for which there exists a group homomorphism $h' : (\mathbb{Z}/n\mathbb{Z})^* \to \mathbb{C}^*$ such that $h = h' \circ \pi_{m,n}$, where $\pi_{m,n} : (\mathbb{Z}/m\mathbb{Z})^* \to (\mathbb{Z}/n\mathbb{Z})^*$ is the restriction of the natural projection. And the fixed field of $\bigcap_{h \in H} \ker(h)$ has degree $|H|$ over $\mathbb{Q}$.

In the present case, the only choice for $\pi_{p,n}$ is $\pi_{p,1}$ or $\pi_{p,p}$. Thus $C_h = p$ for every nontrivial character $h$. For $K$ the fixed field of $\bigcap_{h \in H} \ker(h)$, the Führerdiskriminantenproduktformel therefore gives

$$\text{disc}(K) = \prod_{h \in H} C_h = p^{|H| - 1} = p^{|K:\mathbb{Q}| - 1}.$$

### Appendix B. $\mathbb{Q}(\zeta_{2^n})$

Now we sketch the subfield lattice of $\mathbb{Q}(\zeta_{2^{n+1}})$. Consider the three subfields

- $\mathbb{Q}(\zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1})$,
- $\mathbb{Q}(\zeta_{2^{n+1}})$,
- $\mathbb{Q}(\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1})$.

First we claim $\mathbb{Q}(\zeta_{2^{n+1}})$ is degree two over each: On the one hand, all are proper subfields since

$$\mathbb{Q}(\zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1})$$

is contained in the fixed field of the automorphism $\zeta_{2^{n+1}} \mapsto \zeta_{2^{n+1}}^{-1}$, and

$$\mathbb{Q}(\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1}),$$

is in the fixed field of the automorphism $\zeta_{2^{n+1}} \mapsto -\zeta_{2^{n+1}}^{-1}$. On the other hand, $\zeta_{2^{n+1}}$ is a root of the quadratic polynomials

$$x^2 - (\zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1})x + 1 \in \mathbb{Q}(\zeta_{2^n} + \zeta_{2^{n+1}}^{-1})[x],
\quad x^2 - (\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1})x - 1 \in \mathbb{Q}(\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1})[x].$$

Moreover, since the involutions

$$\zeta_{2^{n+1}} \mapsto \zeta_{2^{n+1}}^{-1}, \quad \zeta_{2^{n+1}} \mapsto \zeta_{2^{n+1}}^{-1}, \quad \zeta_{2^{n+1}} \mapsto -\zeta_{2^{n+1}}^{-1}$$

are distinct, these three subfields are distinct. Finally it is routine to sketch the subgroup lattice of

$$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{n-1}} \cong (\mathbb{Z}/2^{n+1}\mathbb{Z})^* \cong \text{Gal}(\mathbb{Q}(\zeta_{2^{n+1}})/\mathbb{Q}).$$
Here all lines indicate extensions of index two. Combining these facts we have the subfield lattice for $\mathbb{Q}(\zeta_{2^n})$:

$\mathbb{Q}(\zeta_{2^n})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$

$\mathbb{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$

Where all lines indicate extensions of order two.

**APPENDIX C. DECOMPOSITION GROUPS AND FIXED FIELDS**

Let $\zeta = \zeta_{2^n+1}$, $p$ a rational prime with $p \equiv 3 \pmod{4}$, $A$ the natural number with $2^A || p + 1$, and let $p$ be a prime ideal in $\mathbb{Z}[\zeta]$ containing $p$. Then

$$p = (p, \zeta^{-A+1} + \delta \zeta^{-A} - 1)$$
for some $\delta \in \mathbb{Z}$. Let $\sigma \in Aut(\mathbb{Q}(\zeta)/\mathbb{Q})$ be the automorphism of $\mathbb{Q}(\zeta)$ with $\zeta \mapsto \zeta^{-2^{A}-1}$. Then we have

$$
\sigma p = (p, \sigma(\zeta)^{2^{n-1}A} + \delta\sigma(\zeta)2^{n-1}A - 1)
= (p, \zeta^{2^{n-1}A+1}(-2^{A} - 1) + \delta\zeta^{2^{n-1}A}(-2^{A} - 1) - 1)
= (p, \zeta^{-2^{n+1}A+1} + \delta\zeta^{-2^{n-1}A} - 1) - 1)
= (p, \zeta^{-2^{n+1}A+1} - \delta\zeta^{-2^{n-1}A} - 1)
= (p, -\zeta^{-2^{n+1}A+1} : (\zeta^{2^{n-1}A+1} + \delta\zeta^{2^{n-1}A} - 1))
= p.
$$

We have used the fact that $\zeta$ is a unit in $\mathbb{Z}[\zeta]$.

Since $\zeta \mapsto \zeta^{-1}$ is an involution, the order of $\sigma$ is the order of $\zeta \mapsto \zeta^{2^{A}+1}$ (denoted by $\sigma'$) which is the multiplicative order of $2^{A} + 1$ in $(\mathbb{Z}/2^{n+1}\mathbb{Z})^*$. We claim that, for $A \geq 2$, this order is $2^{n+1-A}$: First note that for $k \equiv 1 \pmod{4}$,

$$\text{ord}_{2^{n+1}}^{\ast}(k) = 2^m$$

if and only if $2^{n+1}|(k^{2^{m}} - 1)$. This fact follows easily from the identity

$$k^{2^{m+1}} - 1 = (k^{2^{m}} - 1)(k^{2^{m}} + 1)$$

and the fact that for $k = 2^{A} + 1$, we have $2||k^{2^{m}} + 1)$. Now, that the multiplicative order of $2^{A} + 1$ is $2^{n+1-A}$ follows from an induction argument using the above identity.

The preceding two paragraphs prove that $\sigma$ lies in the decomposition group of $p$ and that $\sigma$ has order $2^{n+1-A}$. It follows from a standard result in the theory of number fields that the decomposition group of $p$ has order $2^{n+1-A}$. Thus $\langle \sigma \rangle$ is precisely the decomposition group of $p$. Now recall the subfield/subgroup lattice for $\mathbb{Q}(\zeta)/\mathbb{Q}$ and its Galois group $\mathbb{Z}_{2^{n+1}}$. A simple computation shows that $\sigma$ fixes $\zeta^{2^{n-1}A} - \zeta^{-2^{n-1}A}$. But from the subfield lattice we can see that

$$[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta^{2^{n-1}A} - \zeta^{-2^{n-1}A})] = 2^{n+1-A} = |\langle \sigma \rangle|.$$

Thus $\mathbb{Q}(\zeta^{2^{n-1}A} - \zeta^{-2^{n-1}A})$ is precisely this fixed field.

A similar, in fact easier, analysis can be carried out for $p \equiv 1 \pmod{4}$. In this case

$$p = (p, \zeta^{2^{n-1}A+1} - u)$$

for some $u \in \mathbb{Z}$ and $2^{A}|u - 1$. Then it is seen that $\sigma'$ fixes $p$. As in the $3 \pmod{4}$ case, we know from a general result of algebraic number theory that the decomposition group of $p$ has order $2^{n+1-A}$, which matches the order of $\sigma'$ (computed above). We see that $\mathbb{Q}(\zeta^{2^{n+1-A}})$ is contained in the fixed field of $\sigma'$, and again, by
looking at the subfield lattice to find $[\mathbb{Q}(\zeta) : \mathbb{Q}(\zeta^{2^{n+1-A}})] = 2^{n+1-A}$, we see that $
abla(\zeta^{2^{n+1-A}})$ is precisely the fixed field of the decomposition group of $p$.

Key Laboratory of Mathematics Mechanization, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
E-mail address: panyanbin@amss.ac.cn

State Key Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, China
E-mail address: xujun@iie.ac.cn

School of Computer Science, University of Oklahoma, Norman, OK 73019, USA
E-mail address: ndwadleigh@gmail.com, qcheng@ou.edu