AMPLITUDE AND PHASE FLUCTUATIONS FOR GRAVITATIONAL WAVES PROPAGATING THROUGH INHOMOGENEOUS MASS DISTRIBUTION IN THE UNIVERSE

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ABSTRACT

When a gravitational wave (GW) from a distant source propagates through the universe, its amplitude and phase change due to gravitational lensing by the inhomogeneous mass distribution. We derive the amplitude and phase fluctuations and calculate these variances in the limit of a weak gravitational field of density perturbation. If the scale of the perturbation is smaller than the Fresnel scale \( \sim 100 \, \text{pc} (f / \text{mHz})^{-1/2} \) (\( f \) is the GW frequency), the GW is not magnified due to the diffraction effect. The rms amplitude fluctuation is \( 1\% - 10\% \) for \( f > 10^{-10} \) Hz, but it is reduced to less than \( 5\% \) for a very low frequency of \( f < 10^{-12} \) Hz. The rms phase fluctuation in the chirp signal is \( \sim 10^{-3} \) rad at the LISA frequency band (\( 10^{-3} \) to \( 10^{-1} \) Hz). Measurements of these fluctuations will provide information about the matter power spectrum on the Fresnel scale \( \sim 100 \, \text{pc} \).

Subject headings: gravitational lensing — gravitational waves — large-scale structure of universe

1. INTRODUCTION

The inspiral and merger of supermassive black holes (SMBHs; \( 10^4 - 10^7 \, M_\odot \)) is one of the most promising candidates for LISA (Laser Interferometer Space Antenna), which will be launched around 2014. This detector is sensitive in a frequency range of \( 10^{-3} \) to \( 10^{-1} \) Hz and can measure SMBH mergers at cosmological distances with a high signal-to-noise ratio (S/N). The binary SMBH systems have recently been called "cosmological standard sirens" (Holz & Hughes 2005). This is because the distance to the source \( r_s \) can be directly measured by using the relation \( A \propto f \cdot (f^3 r_s) \), where \( f \) is the gravitational wave (GW) frequency, \( f \) is its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined to an accuracy of within less than \( 1\% \) if the direction is determined by identifying its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined to an accuracy of within less than \( 1\% \) if the direction is determined by identifying its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined to an accuracy of within less than \( 1\% \) if the direction is determined by identifying its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined to an accuracy of within less than \( 1\% \) if the direction is determined by identifying its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986). The distance can be determined to an accuracy of within less than \( 1\% \) if the direction is determined by identifying its time derivative, and \( A \) is the amplitude at the detector (Schutz 1986).

However, in practice, the distance cannot be determined with such high accuracy because of the gravitational lensing caused by inhomogeneous mass distribution in the universe. Recently, Holz & Hughes (2005) and Koosik et al. (2006) discussed the effects of lensing magnification (or demagnification) on determining the distance to SMBH mergers. They concluded that lensing errors are \( 5\% - 10\% \), which is greater than the intrinsic distance error.

In weak gravitational lensing, the magnification is \( \mu \simeq 1 + 2\kappa \), where \( \kappa \) is the convergence. The rms convergence fluctuation was derived by Blandford et al. (1991), Miralda-Escudé (1991), and Kaiser (1992) on the basis of linear perturbation theory. In this previous work, geometrical optics was assumed.

However, for the lensing of GWs, since the wavelength is much longer than that of light, geometrical optics is not valid in some cases. Recently, Macquart (2004) and Takahashi et al. (2005) have suggested that if the scale of the density perturbation \( k^{-1} \) is smaller than the Fresnel scale \( r_F \sim (\lambda r)_{1/2} \) (\( \lambda \) is the wavelength), the wave effects become important. This condition is rewritten as \( k^{-1} < 100 \, \text{pc} (f / \text{mHz})^{-1/2} (r_s / 10 \, \text{Gpc})^{1/2} \). In such a case, the incident wave does not experience perturbation and its amplitude is not magnified.

In this paper, we consider a situation in which GWs propagate through the density perturbations of cold dark matter (CDM) and baryons. In the wave optics, the lensing affects not only the amplitude but also the phase; hence, we discuss the lensing effects on both of them. We use units of \( c = G = 1 \).

2. GRAVITATIONALLY LENSED WAVEFORM

The background metric is the Friedmann-Robertson-Walker (FRW) model with a gravitational potential of \( U \) (\( \ll 1 \)). The perturbed FRW metric \( \tilde{g}^B_{\mu\nu} \) for a flat universe is given by

\[
d\tilde{s}^2 = \tilde{g}^B_{\mu\nu} \, dx^\mu \, dx^\nu = a^2(\eta)g^B_{\mu\nu} \, dx^\mu \, dx^\nu = a^2(\eta)\left[ -(1 + 2U) \, d\tilde{r}^2 + (1 - 2U) \, dx^2 \right],
\]

where \( \eta \) is the conformal time, the scale factor is normalized to the electromagnetic counterpart (e.g., Cutler 1998; Seto 2002; Hughes 2002; Vecchio 2004).

Since the propagation equation of GWs is (1) conformally invariant if the wavelength is much smaller than the Hubble radius (see Appendix) and (2) the same as the scalar field wave equation (Peters 1974), we use the scalar field \( \phi \) propagating under the Minkowski background \( g^B_{\mu\nu} \). The basic equation is

\[
(\nabla^2 + \omega^2)\tilde{\phi} = 4\omega^2 U\tilde{\phi},
\]

where \( \omega \) is the frequency at the observer and \( \tilde{\phi}(\omega, x) \) is the Fourier transform of \( \phi(\eta, x) \).

We take \( \tilde{\phi}^0 \) as the incident wave emitted from the source, which is the solution of equation (2) in the unlensed case \( U = 0 \). We use the spherical wave as \( \tilde{\phi}^0(\omega, x) \propto \exp(\pm i\omega|x - x_0|/|x - x_0|) \), including the effect of \( U \) on the first order (Born approximation),

\[1\] The quantity \( \omega \) is the same as the comoving frequency \( \omega_s \), since \( \omega_s = \omega c \) and \( a = 1 \) at present.
the gravitationally lensed wave at the observer is given by Takahashi et al. (2005),

$$\tilde{\phi}_{\text{obs}}^L(\omega) = \tilde{\phi}_{\text{obs}}^0(\omega) + \tilde{\phi}_{\text{obs}}^L(\omega),$$

with

$$\tilde{\phi}_{\text{obs}}^L(\omega) = -\frac{\omega^2}{\pi} \int d^3x \frac{\bar{e}^{i|\omega|\hat{n}|}}{|\hat{n}|} U(x) \tilde{\phi}^0(\omega, x),$$

where $$\tilde{\phi}^0$$ represents the effect of lensing ($$|\tilde{\phi}^1| \ll |\tilde{\phi}^0|$$). The incident wave $$\tilde{\phi}^0$$ is gravitationally lensed at $$x = (r, r_s)$$ and changed into the lensed wave $$\tilde{\phi}^0 + \tilde{\phi}^1$$ (see Fig. 1).

Let us define $$K$$ and $$S$$ as (Ishimaru 1978, p. 338)

$$K(\omega) = \text{Re} \left[ \frac{\tilde{\phi}_{\text{obs}}^1(\omega)}{\tilde{\phi}_{\text{obs}}^0(\omega)} \right], \quad S(\omega) = \text{Im} \left[ \frac{\tilde{\phi}_{\text{obs}}^1(\omega)}{\tilde{\phi}_{\text{obs}}^0(\omega)} \right].$$

Then we have

$$\tilde{\phi}_{\text{obs}}^L(\omega) = [1 + K(\omega)]\tilde{\phi}_{\text{obs}}^0(\omega)e^{iS(\omega)},$$

from equation (3). Hence, $$K$$ means the magnification ($$K > 0$$) or demagnification ($$K < 0$$) of the wave amplitude, while $$S$$ means the phase shift due to the lensing. Hereafter, we call $$K$$ “amplitude fluctuation” and $$S$$ “phase fluctuation.”

Using the Fourier transform of the potential, $$\tilde{U}(r, k) = \int d^3x U(x)e^{i\hat{k} \cdot x}$$, with $$|x_\perp| \ll r$$ and $$|x_\parallel| \ll r_s$$, the result in equation (4) is reduced to

$$\frac{\tilde{\phi}_{\text{obs}}^L(\omega)}{\tilde{\phi}_{\text{obs}}^0(\omega)} = -2i\omega \int_0^{r_s} dr \int \frac{d^3k}{(2\pi)^3} \tilde{U}(r, k)$$

$$\times \exp \left[ -ik \cdot r - i\frac{r}{r_s} k_\perp \cdot x^\perp - i\frac{r(r_s - r)}{2\omega r_s} |k_\perp|^2 \right],$$

where $$k_r$$ and $$k_\perp$$ are the radial and perpendicular components of $$k$$. In particular, for the high-frequency limit $$\omega \to \infty$$, $$K$$ and $$S$$ are rewritten as

$$K(\omega) \to \kappa = \int_0^{r_s} dr \frac{r(r_s - r)}{r_s^2} \nabla^2 \tilde{U}(r, \frac{r}{r_s} x^\perp),$$

$$S(\omega) \to \omega t_d = -2\omega \int_0^{r_s} dr \tilde{U} r \left( \frac{r}{r_s} x^\perp \right).$$

Here $$\kappa$$ is the convergence field along the line of sight to the source and $$t_d$$ is the gravitational time delay. The above results are consistent with that in weak gravitational lensing (Bartelmann & Schneider 2001).

3. AMPLITUDE FLUCTUATION

3.1. Variance in the Amplitude Fluctuation

In this section, we derive the variance in the amplitude fluctuation $$K$$. The gravitational potential satisfies Poisson’s equation (Peebles 1980)

$$\tilde{U}(r, k) = -\frac{3H_0^2\Omega_0}{2a(r)} k^{-2}\tilde{\delta}(r, k),$$

where $$\tilde{\delta}$$ is the density perturbation. The fluctuation of $$\tilde{\delta}$$ is characterized by the power spectrum: $$\langle \tilde{\delta}(r, k)\tilde{\delta}(r', k') \rangle = (2\pi)^4P_{\delta}(r, k)\delta^3(k - k')\delta(r - r').$$ Then the correlation in the potentials $$\tilde{U}(r, k)$$ and $$\tilde{U}(r', k')$$ is

$$\langle \tilde{U}(r, k)\tilde{U}(r', k') \rangle = \frac{3H_0^2\Omega_0}{2a(r)} k^{-4}(2\pi)^4P_{\delta}(r, k)\delta^3(k - k')\delta(r - r').$$

To calculate $$P_{\delta}$$, we use the linear power spectrum (Eisenstein & Hu 1999) with the nonlinear correction of Peacock & Dodds (1996). We adopt a Cosmic Background Explorer (COBE)-normalized, scale-invariant ($$n = 1$$) power spectrum in a flat $$\Lambda$$CDM cosmology with $$\Omega_b = 0.04$$, $$\Omega_\Lambda = 0.3$$, $$\Omega_M = 0.7$$, and $$H_0 = 70$$ km s$$^{-1}$$ Mpc$$^{-1}$$.

The variance in the amplitude fluctuation is given from equations (5), (7), and (10) as

$$\Delta^2_K(\omega) = \langle \tilde{K}^2(\omega) \rangle = \left( \frac{3H_0^2\Omega_0}{4\pi} \right)^2 \int_0^{r_s} dr \frac{1}{a(r)^2} \frac{r^2(r_s - r)^2}{r_s^2}$$

$$\times \int d^2k_\perp P_{\delta}(r, k_\perp)F_K(\omega, r, k_\perp),$$

where the filter function $$F_K$$ is defined as

$$F_K = \left[ \frac{\sin(r_F^2k_\perp^2/2)}{r_F^2k_\perp^2/2} \right]^2; \quad r_F^2 = \frac{r(r_s - r)}{\omega r_s}. $$

Here $$r_F$$ is called the Fresnel scale (Macquart 2004). This is roughly given by

$$r_F \simeq 120 \text{ pc} \left( \frac{f}{\text{mHz}} \right)^{-1/2} \left[ \frac{r(r_s - r)/r_s}{10 \text{ Gpc}} \right]^{1/2},$$

where $$f$$ is the frequency in mHz and $$r_s$$ is the comoving distance from the observer, $$r(r) = \int_0^r ds/h(s)$$, and $$x_\perp$$ is a two-dimensional vector perpendicular to the line of sight. The source position is $$x_s = (r_s, x_\perp^s)$$ with $$|x_\perp^s| \ll \kappa$$; $$\tilde{\phi}^0$$ is the incident wave, and $$\tilde{\phi}^0 + \tilde{\phi}^1$$ is the lensed wave.
where \( f = \omega / 2\pi \). We show the filter function \( F_K \) as a function of \( r_f k \) in Figure 2. From this figure, \( F_K = 1 \) if the scale of the density perturbation \( \sim k^{-1} \) is larger than the Fresnel scale \( r_F \), while \( F_K \) decreases rapidly in proportion to \( (r_f k_f)^{-4} \) if \( k_f^{-1} < r_f \). Hence, the amplitude fluctuation is affected by a density perturbation of scale larger than the Fresnel scale. In the geometrical optics limit \( F_K \rightarrow 1 \), the result (11) is the same as \( \langle \kappa^2 \rangle^{1/2} \).

The reason why the critical scale is \( r_f \) can be explained as follows: For the lensing by a compact object with mass \( M \), if the wavelength is larger than the Schwarzschild radius, \( i > M \), the diffraction effect becomes important (see Takahashi & Nakamura 2003 and references therein). Inserting this condition to the Einstein radius \( r_E = [4M(r_s - r)/r_s]^{1/2} \), we have the Fresnel scale in equation (12).

### 3.2. Results

We show the rms amplitude fluctuation \( \Delta K \) in equation (11) as a function of the frequency in Figure 3. The source redshift is \( z_s = 3 \). The solid line represents \( \Delta K \), and the dashed line represents the geometrical optics limit \( \langle \kappa^2 \rangle^{1/2} \). The difference between these two lines is small for the LISA frequency band (10^{-5} to 10^{-1} Hz). But for very low frequency \( f < 10^{-10} \) Hz, \( \Delta K \) is clearly smaller than \( \langle \kappa^2 \rangle^{1/2} \). Thus, in the frequency band for the pulsar timing array \( f \approx 10^{-9} \) Hz (e.g., Jenet et al. 2005), \( \Delta K \) should be used instead of \( \langle \kappa^2 \rangle^{1/2} \). The wave amplitude is magnified due to lensing by the density perturbation on a scale larger than the Fresnel scale (see § 3.1). The integration of equation (11) is mainly contributed on the scale of 0.1–1 Mpc (since \( P_f k^2 \) has a peak there). For \( f < 10^{-10} \) Hz, the Fresnel scale is \( r_F > 0.4 \) Mpc( \( f/10^{-10} \) Hz)^{-1/2}, which is larger than the contributed scale (0.1–1 Mpc), and hence the amplitude is not magnified.

Figure 4 is the same as Figure 3, but as a function of \( z_s \). The dashed line represents \( \langle \kappa^2 \rangle^{1/2} \), and the solid (dotted) line is for the frequency of 10^{-10} (10^{-12}) Hz. For \( z_s = 1–10 \) the rms amplitude fluctuation is 1%–10% for \( f > 10^{-10} \) Hz, while it decreases to less than 5% for \( f < 10^{-12} \) Hz.

### 3.3. Diffraction Effect in the Amplitude Fluctuation

To investigate the diffraction effect in the amplitude fluctuation, we expand \( K(\omega) \) in terms of \( 1/\omega \). From equations (5) and (7), we have

\[
K(\omega) = 2\omega \int_0^{r_s} dr \sin \left( \frac{1}{2} \frac{r^2 \nabla^2}{r_F} \right) U \left( \frac{r}{r_s} \frac{r}{r_s} \right),
\]

(14)

where \( \sin (r^2 \nabla^2 / 2) = \sum_{n=1}^\infty (-1)^{n+1} (r^2 \nabla^2 / 2)^{2n-1} / (2n - 1)! \). The leading term \( (n = 1) \) is the convergence \( \kappa \) in equation (8).

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**Fig. 2.** Filter function \( F_K \) as a function of \( r_f k \).

**Fig. 3.** The rms amplitude fluctuation \( \Delta K \) as a function of the frequency. The source redshift is \( z_s = 3 \). The solid line represents \( \Delta K \), and the dashed line represents the result in the geometrical optics limit \( \langle \kappa^2 \rangle^{1/2} \).

**Fig. 4.** Same as Fig. 3, but as a function of \( z_s \). The dashed line represents \( \langle \kappa^2 \rangle^{1/2} \), and the solid (dotted) line represents \( \Delta K \) for the frequency of 10^{-10} (10^{-12}) Hz.
4. PHASE FLUCTUATION

4.1. Effects of the Phase Fluctuation in Chirp Signal

Next, we discuss the effects of the phase fluctuation. We consider the inspiraling BH binaries as the sources. As the binary system loses its energy due to gravitational radiation, the orbital separation decreases and the orbital frequency increases. Hence, the frequency of the gravitational waves increases with time ($df/dt > 0$). This is called a chirp signal. We consider the frequency to be swept from $f_1$ to $f_2$. For the binary masses $M_1$ and $M_2$ at redshift $z_s$, the frequency of 1 yr before the final merging is

$$f_1 = 4.1 \times 10^{-5} \left( \frac{M_c}{10^6 M_\odot} \right)^{-5/8} \text{Hz},$$

where $M_c = (M_1 M_2)^{3/5} (M_1 + M_2)^{-1/5} (1 + z_s)$ is the redshifted chirp mass. The frequency at the final merging is

$$f_2 = 4.4 \times 10^{-3} \left( \frac{M_c}{10^6 M_\odot} \right)^{-1} \text{Hz},$$

where $M_c = (M_1 + M_2)(1 + z_s)$ is the redshifted total mass. The difference in the phase fluctuation between $f_1$ and $f_2$ in the chirp signal is important.

The phase fluctuation $S$ is reduced to the time delay $\omega t_d$ in the geometrical optics limit from equation (8). However, we note that the time delay is physically unimportant, since it means an arrival time shift and hence it does not change the waveform. Hence, we use $S - \omega t_d$ instead of $S$ as the phase fluctuation. This quantity $S - \omega t_d$ has larger value for smaller frequency. We define $\Delta_S$ as the rms phase difference between the two frequencies, $\omega_1$ and $\omega_2$:

$$\Delta_S^2(\omega_1, \omega_2) \equiv \left\langle \left[ S(\omega_1) - \omega_1 t_d \right] - \left[ S(\omega_2) - \omega_2 t_d \right] \right\rangle^2.$$ (17)

This is the same as $\Delta^2$ in equation (11), but the filter function is replaced with

$$F_S = \frac{\left[ \cos \left( \frac{r_{F_1} k^2}{2} \right) - 1 \right] - \left[ \cos \left( \frac{r_{F_2} k^2}{2} \right) - 1 \right]^2}{r_{F_1} k^2 / 2},$$ (18)

where $r_{F_1}$ and $r_{F_2}$ are the Fresnel scales in equation (12) for $\omega_1$ and $\omega_2$, respectively.

We show the behavior of the first and second terms in equation (18), $\left[ \cos \left( \frac{r_{F_1} k^2}{2} \right) - 1 \right]/(r_{F_1} k^2 / 2)$, as a function of $r_{F_1} k$ in Figure 5. From this figure, the function peaks at $r_{F_1} k \approx 1$. Hence, the phase fluctuation $S - \omega t_d$ is affected by the density perturbation of the Fresnel scale. In the chirp signal, as the frequency increases, the GW feels the perturbation of the smaller scale.

4.2. Results

In Table 1, we show the rms phase differences $\Delta_S$ in equation (17) for the LISA frequency band ($10^{-5}$ to $10^{-1}$ Hz) with $z_s = 1$, 3, and 10. We consider the frequency to be swept from $f_1$ to $f_2$ in the chirp signal. The values are in units of radians. This table shows that the typical values of $\Delta_S$ are $\approx 10^{-3}$ rad. The results weakly depend on $f_2$ if $f_1 \ll f_2$. This is because $S - \omega t_d$ in equation (17) has larger (smaller) value for lower (higher) frequency.

In Figure 6, $\Delta_S$ is shown as a function of $f_1$ in the limit of $f_2 \to \infty$. The source redshifts are $z_s = 1$ (dotted line), 3 (solid line), and 10 (dashed line). In order to study the behavior of $\Delta_S$, we assume a single power law for the power spectrum, $P(r, k) \propto k^n$. The index is $n \sim 2.7$ at $k \sim 100$ pc. Inserting this $P(r, k)$ into equations (11) and (18), we have $\Delta_S \propto \omega_1^{1/2} \omega_2^{1/2}$. With this argument and the results in Table 1, $\Delta_S$ in $f_1 \ll f_2$ is roughly fitted by

$$\Delta_S \approx 3 \times 10^{-3} \frac{f_1}{10^{-4} \text{Hz}} \text{ rad}$$

for $z_s = 3$. (19)

The above value, $3 \times 10^{-3}$ rad, is replaced by $1(5) \times 10^{-3}$ rad for $z_s = 1(10)$.

4.3. Implications for GW Observations

In the matched filtering analysis, the phase of a waveform can be measured to an accuracy approximately equal to the inverse of the signal-to-noise ratio, $(S/N)^{-1}$. Here the $S/N$ is typically $\approx 10^4$ for the SMBH mergers detected by LISA. However, including the effect of the phase fluctuation, the phase cannot be determined with an accuracy of less than $\Delta_S$. Hence, if the $S/N$ is larger than $\Delta_S^{-1} = 10^{1/3}(\Delta_S/10^{-3})$ rad, the phase fluctuation becomes important and the phase of the waveform can be determined with an accuracy of $\sim \Delta_S^{-1}$. 

| $f_1 - f_2$ (Hz) | $\Delta_s$ (rad) rms |
|------------------|----------------------|
| $10^{-5}$ to $10^{-4}$ | $1.4 \times 10^{-3}$ |
| $10^{-5}$ to $10^{-3}$ | $3.7 \times 10^{-3}$ |
| $10^{-4}$ to $10^{-3}$ | $6.5 \times 10^{-3}$ |
| $10^{-3}$ to $10^{-2}$ | $4.4 \times 10^{-3}$ |
| $10^{-2}$ to $10^{-1}$ | $3.0 \times 10^{-3}$ |

TABLE 1

rms Phase Differences in Chirp Signal $f_1 \to f_2$ Hz with $z_s = 1$, 3, 10
5. VALIDITY OF THE BORN APPROXIMATION

Throughout this paper we assume weak density fluctuations and employ the Born approximation to discuss the lensing effects on the waveform. Since the variances of the amplitude and the phase fluctuations are much smaller than 1, this approximation is valid in almost all cases. However, in a few cases, the GW may pass through a strong density fluctuation or pass near massive compact objects (i.e., strong lensing). Hence, we have some comments about the validity of the Born approximation.

For the amplitude fluctuation \( K \), the maximum of \( K \) is the convergence \( \kappa \) from Figure 3. For the thin lens plane at \( r_l \), \( \kappa \) is the surface density of the lens \( \Sigma \) divided by the critical density \( \Sigma_{cr} = (1/4\pi)\rho_c r_l (\Sigma/\Sigma_{cr}) \approx 2 \times 10^5 M_{\odot} \text{pc}^{-2} (r|r_r) \text{Gpc}^{-1} \) (here \( r_l = r_c - r_r \)). Hence, if the GWs pass through a high-density region \( \Sigma > \Sigma_{cr} \) such as the core of a galaxy or cluster, the Born approximation breaks down and one should use the Kirchhoff diffraction integral to obtain the exact waveform (Schneider et al. 1992, \S\S 4.7 and 7).

If the gravitational potential is a Gaussian random field, one can obtain exact solutions of the correlation functions of the lensed waveform (Macquart 2004; see also Ishihara 1978, p. 399). If there are many compact lens objects and the GWs are scattered several times, the exact lensed waveform can be obtained by using the multiple lens–plane theory in wave optics (Yamamoto 2003).

6. CONCLUSION AND DISCUSSION

We have discussed the lensing effects on the amplitude and phase of a waveform. The rms amplitude fluctuation is 1%–10% for \( f > 10^{-10} \text{Hz} \), which is the same as the result in weak lensing. However, for a very low frequency of \( f < 10^{-12} \text{Hz} \), it decreases to less than 5%. In the chirp signal, the phase fluctuation is typically \( 10^{-3} \text{rad} \) at the LISA frequency band. The phase cannot be measured with an accuracy less than this value.

The power spectrum \( P_f \) has been measured down to a small scale \( k^{-2} \) around several \( \times 0.1 \text{Mpc} \) from the Ly\( \alpha \) forest (e.g., Zaroubi et al. 2005). In this paper, we assume that the formula of \( P_f \) is valid down to the Fresnel scale \( k^{-1} \sim 100 \text{pc} \). If the amplitude or the phase fluctuation is measured in the future, the constraints for \( P_f \) at \( \sim 100 \text{pc} \) could be obtained.

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APPENDIX

CONFORMAL TRANSFORMATION

The relation between the Einstein tensor \( G_{\mu\nu} \) for the metric \( g_{\mu\nu} \) and \( \tilde{G}_{\mu\nu} \) for \( \tilde{g}_{\mu\nu} (= a^2 g_{\mu\nu}) \) is given by Wald (1984),

\[
\tilde{G}_{\mu\nu} = G_{\mu\nu} - 2 \nabla_{\mu} \nabla_{\nu} \ln a + 2(\nabla_{\mu} \ln a)(\nabla_{\nu} \ln a) + g_{\mu\nu}(\nabla^\sigma \ln a)(\nabla_{\sigma} \ln a) + 2g_{\mu\nu} \nabla^\sigma \nabla_{\sigma} \ln a. \tag{A1}
\]

Let us consider the linear perturbation \( \tilde{h}_{\mu\nu} (= a^2 h_{\mu\nu}) \) in the background metric \( \tilde{g}_{\mu\nu} (= a^2 \tilde{g}_{\mu\nu}) \) given in equation (1):

\[
\tilde{g}^R_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{h}_{\mu\nu} = a^2 \left( g_{\mu\nu} + h_{\mu\nu} \right). \tag{A2}
\]

Inserting equation (A2) into (A1), we obtain the linearizing Einstein equation

\[
\delta \tilde{G}_{\mu\nu} = \delta G_{\mu\nu} + 2a \frac{a^\prime}{a} \delta G^0_{\mu\nu} + \left( 2a^\prime a^\prime a - 2a^2 \right) \delta (g_{\mu\nu} g^{00}) - 2a^3 \delta \left( g_{\mu\nu} g^0 g^0 \Gamma^a_{\mu\nu} \right), \tag{A3}
\]
where $a' = da/d\eta$, $\Gamma$ is the Christoffel symbol, and $\delta$ means the perturbed component. The first term $\delta G_{\mu\nu}$ is of the order of $|h_{\mu\nu}|/\lambda^2$ ($\lambda$ is the wavelength of the gravitational wave). The other terms on the right-hand side are roughly $|h_{\mu\nu}|/(\lambda \lambda_H)$ or $|h_{\mu\nu}|/\lambda_H^2$, where $\lambda_H$ is the horizon scale, since $a' = a^2 H \sim a^2 / \lambda_H$. Hence, if $\lambda \ll \lambda_H$, the propagation equation for the gravitational wave is conformally invariant, i.e., $\delta G_{\mu\nu} = 0$.