Parametric Resonance in a Vibrating Cavity

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We present the study of parametric resonance in a one-dimensional cavity based on the analysis of classical optical paths. The recursive formulas for field energy are given. We separate the mechanism of particle production and the resonance amplification of radiation. The production of photons is a purely quantum effect described in terms of quantum anomalies in recursive formulas. The resonance enhancement is a classical phenomenon of focusing and amplifying beams of photons due to Döppler effect.

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I. INTRODUCTION

The phenomenon of parametric resonance is related to the instability of open systems under the action of an external periodic force. In quantum field theory, we expect the resonant amplification of quantum fluctuations [1, 2, 3]. The resonance enhancement of vacuum fluctuations is usually referred as the dynamical Casimir effect. The standard model to investigate this effect is the system of electromagnetic fields confined inside a vibrating one-dimensional cavity [4, 5, 6, 7].

Meplan and Gignoux [8] found the correspondence between the wave propagation in a vibrating cavity and the motion of massless particles in a two-dimensional space-time billiard. They used the generalized Korringa-Kohn-Rostoker method. The method presented in this paper offers both a better insight in resonance mechanism and a more useful tool for detailed calculations. Moreover, the brief discussion of the quantum version in [8] does not trace properly the impact of conformal anomaly [9].

In this Letter, we present a new approach to study resonance solutions. We show that the resonance amplification together with the formation of narrow packets in energy densities [10] is a purely classical phenomenon. It can be explained as a consequence of a cumulative Döppler effect [11]. In this way, we are allowed to interpret a wave packet as a beam of massless and non-interacting particles. Hence, we introduce the billiard function which contains full information about possible reflections from a cavity wall. This function is helpful to establish recursive formulas for physical quantities in a vibrating cavity. Production of particles in quantum case is represented by conformal anomaly contribution in such formulas for energy densities. Our technique is illustrated with solutions of several examples of vibrating cavity systems discussed in literature.

II. CLASSICAL BILLIARDS

Before studying the field-theoretic models, we consider briefly the classical billiards. Let us start with the simplest example, namely a head-on collision of two non-relativistic objects. A particle moves with velocities \( v \) and \( v' \) before and after the collision, while respective velocities of a target are denoted by \( u \) and \( u' \). This elementary physical problem is fully described by the reflection law in one dimension (on a line),

\[
v + v' = u + u'.
\]  

(1)

If we assume a target to be very heavy and skip its recoil, then the velocity of a particle after the collision is just \( v' = -v + 2u \). We see that if a particle were bounced with a regular frequency, then its energy would grow quadratically with time. Let us look now at a head-on collision of relativistic objects. This two-body problem is described by the relativistic reflection law which says that the sums of rapidities are equal,

\[
\text{artanh } v + \text{artanh } v' = \text{artanh } u + \text{artanh } u'.
\]  

(2)

In this case, if a particle bounces rhythmically on a heavy target, then its rapidity grows linearly with time. It follows that the energy grows exponentially with time. In particular, we can consider a left-moving massless particle (photon)
hitting a target with a velocity of light. The exact formula for the change of the particle energy after a single collision yields:

\[ E' = E \frac{1 - u}{1 + u} \left( 1 + \frac{2E}{M} \sqrt{\frac{1 - u}{1 + u}} \right)^{-1}. \]  

(3)

Two consecutive factors on the right-hand side are obviously the Döppler factor and the Compton factor. For a large target mass \( M \), the Compton factor is neglected. Therefore, during each head-on collision the energy of a massless particle increases (or decreases if the target moves in the same direction) by the Döppler factor.

Now, we insert a heavy target (mirror) which moves with some prescribed trajectory \( L(t) \). Let us then introduce the following billiard function\[12\]:

\[ f(t + L(t)) = t - L(t). \]  

(4)

We define the retarded time \( t^* \) using the retardation relation \( t = t^* + L(t^*) \) and recognize that the derivative of the billiard function is just the retarded Döppler factor:

\[ \dot{f}(t) = \frac{1 - \dot{L}(t^*)}{1 + L(t^*)}. \]  

(5)

Since the derivative is additive, the billiard function is increasing and its inverse is well defined. The Döppler factor is greater than one (amplifying) if the mirror is moving left, i.e. towards an arriving test massless particle. For any physical trajectory of the mirror, its corresponding billiard function is defined unambiguously. On the other hand, any proper billiard function may be used to reconstruct its genuine mirror’s trajectory. More information can be found in\[12\]. Although rather cumbersome in general, the billiard function is the most useful characteristic to analyze the parametric resonance within a particular cavity model. We can make use of the correspondence between waves and massless particles and relate the field theory problem with the analysis of trajectories in a space-time billiard\[8\].

### III. OPTICAL CAVITIES

Under the guidance of the above simple examples from classical billiards, we consider the case of field theory. As usual, we will discuss the ideal resonator model\[4\] formed by two perfect mirrors. The electromagnetic potential \( A(x, t) \) obeys the one-dimensional wave equation and any classical solution is composed of left- and right-moving wave packets,

\[ A(t, x) = A_L(t + x) + A_R(t - x). \]  

(6)

For the sake of simplicity, we assume the left mirror to be fixed at \( x = 0 \), whereas the right mirror is vibrating according to a prescribed trajectory \( L(t) \). Then, the field \( A(x, t) \) is subject to Dirichlet’s boundary conditions, namely:

\[ A_L(\tau) = -A_R(\tau) \equiv \varphi(\tau), \quad \varphi(\tau) = \varphi(f(\tau)). \]  

(7)

We follow the solution of the Cauchy problem with the help of the billiard function. From initial data for \( A \) and \( \partial_t A \) at \( t = 0 \), we read the function \( \varphi \) for arguments in the interval \([-L(0), L(0)]\) (up to some irrelevant constant). The
solution is obtained if we extend \( \varphi \) into the whole real domain. We can do it using Eq. (7) provided that the billiard function is known. Usually, if we predetermine some trajectory of the mirror, then the corresponding billiard function is to be found only in the way of numerical iterations based on Eq. (4). On the other hand, to specify the mirror motion we can predetermine the billiard function instead.

The energy density of the classical wave packet is given by:

\[
T_{00}(t, x) = 1/2(\partial_t A)^2 + 1/2(\partial_x A)^2 = \varrho(t + x) + \varrho(t - x) ,
\]

where \( \varrho(\tau) = \varphi^2(\tau) \). The formula for the total energy can be presented in the following form:

\[
E(t) = \int_0^{L(t)} dx \ T_{00}(t, x) = \int_{\tau - L(t)}^{t + L(t)} d\tau \ \varrho(\tau) .
\]

The billiard function determines all possible trajectories of a massless particle or all classical optical paths inside the cavity. Each trajectory can be represented by the sequence \( \{T_n(\tau)\} \equiv \{(f^{-1})^n(\tau)\}_{n=0}^{\infty} \), where \( (f^{-1})^n(\tau) \) denotes here \( n \)-fold composition \( f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1} \). A light-like particle starts at time \( \tau \) moving right from \( x = 0 \) and next elements of the sequence read next times of collisions with the static mirror. We define also retarded times \( T_n^*(\tau) \) by demanding \( T_n^*(\tau) + L(T_n^*(\tau)) = T_n(\tau) \). It is easy to find the relation \( T_n^*(\tau) = (T_n(\tau) + T_{n-1}(\tau))/2 \).

The appearance of the parametric resonance is related to the existence of periodic particle trajectories [8]. Such trajectories obey the following periodicity condition (for any non-negative integer \( n \)):

\[
T_n(\tau_0) = \tau_0 + nT ,
\]

where \( \tau_0 \) is a starting point in time and \( T \) is a period. The retarded times are respectively \( T_n^*(\tau_0) = T_n(\tau_0) - T/2 \).

Looking at Fig.1, we notice immediately that any periodic particle trajectory appears on condition that the trajectory of the mirror has return points [13]:

\[
L(\tau_0^* + nT) = T/2 .
\]

It is usually assumed that the position of the mirror \( T/2 \) refers to the length of the static (unperturbed) cavity, so that \( T/2 = L = L(0) \) in this paper.

Let us define the characteristic for particle trajectories which will play the most important role in our analysis. It is the cumulative Döppler factor:

\[
D_n(\tau) = \frac{1}{T_n(\tau)} = \prod_{k=1}^{n} f(T_n(\tau)) = \prod_{k=1}^{n} \frac{1 - \dot{L}(T_n^*(\tau))}{1 + \dot{L}(T_n^*(\tau))} .
\]

We will call a trajectory to be positive (stable, attractive) if its cumulative Döppler factor tends to infinity for large \( n \), and to be negative (unstable, chaotic) if its cumulative Döppler factor goes to zero with increasing \( n \).

The existence of periodic particle trajectories inside the cavity is equivalent to the existence of suitable return points in the trajectory of the cavity wall. In particular, a periodic particle trajectory is positive (negative) if any return point of the mirror trajectory gives a Döppler factor greater (less) than one. It happens if the right mirror is always moving toward (outward) the left mirror at all return points.

Denote by \( \tau_+ \) and \( \tau_- \) starting points for positive and negative periodic trajectories respectively. For small perturbations \( \varepsilon \), it is straightforward to derive the following formulas:

\[
T_n(\tau_+ + \varepsilon) \cong \tau_+ + nT + \varepsilon D_n^{-1}(\tau_+) ,
\]

\[
T_n(\tau_- + \varepsilon D_n(\tau_-)) \cong \tau_- + nT + \varepsilon .
\]

Now, we can show how the existence of periodic particle trajectories inside the cavity triggers the resonance instability of the system. For the sake of simplicity, we assume that there are only two periodic particle trajectories, a positive one and a negative one, with starting points \( \tau_+ \) and \( \tau_- \) lying in the initial interval \([- L(0), L(0)]\). These simplifications are only for clarity, and the method can be adapted for more sophisticated examples. As it was said, to solve the Cauchy problem we need to extend the function \( \varphi \) given in Eq. (7) from the initial interval to the whole real domain. If we are interested only in the evolution forward in time, then it is enough to obtain data only for arguments to the right of the initial interval \( \tau > L(0) \). To build such extension of \( \varphi \), we pick points from the initial
interval, run along particle trajectories and read extended values of \( \varphi \) from Eq. (7). To analyze our solution obtained in this way, let us look first at the profile function of the energy density. If we take the following iteration formula:

\[
\varrho(T_n(\tau)) = \varrho(\tau)D_n^2(\tau),
\]

(15)

then it is straightforward to derive the following asymptotic formulas for large arguments \( n \gg 1 \):

\[
\varrho(nT + \tau_+ + \varepsilon D_n^{-1}(\tau_+)) \approx \varrho(\tau_+ + \varepsilon)D_n^2(\tau_+ + \varepsilon),
\]

(16)

\[
\varrho(nT + \tau_- + \varepsilon) \approx \varrho(\tau_- + \varepsilon D_n(\tau_-))D_n^2(\tau_- + \varepsilon D_n(\tau_-)).
\]

(17)

The above formulas explain the formation of travelling narrow packets in the energy density \( T_{i0}(t,x) \). The profile function \( \varrho(\tau) \) is concentrated around spots of the positive periodic trajectory. From Eqs. (16,17) one can estimate easily the width and the height of peaks. The height grows like \( D_n^2(\tau_+) \), and the width diminishes like \( D_n^2(\tau_-) \). The values of the energy density far off the peaks decreases according to \( D_n^2(\tau_-) \).

The total energy can be analyzed using the following formula:

\[
E(T_n(\tau_0)) = \int_{t_{n-1}(\tau_0)}^{t_n(\tau_0)} d\tau \varrho(\tau) = \int_{f(\tau_0)}^{\tau_0} d\tau \varrho(\tau)D_n(\tau).
\]

(18)

We can calculate the evolution of the total accumulated energy using only initial data and the cumulative Döppler factor. It is apparent that an initial shape of the classical field inside the cavity is of minor importance. Let us test our method with an example of sinusoidal cavity wall motion:

\[
L(t) = L + \Delta L \sin(\omega t).
\]

(19)

Obviously, we are to assume that \( \Delta L < L \) and \( \omega \Delta L < 1 \). The parametric resonance frequencies are \( \omega_N = N \pi/L \), where \( N \) is the order of the resonance. It is easy to find periodic particle trajectories. First, we read all return points of the mirror from Eq. (11): \( \sin(\omega_N \tau^*) = 0 \). They correspond to the following starting points of positive and negative particle trajectories:

\[
\tau_{+m} = \frac{(-N + 2m + 1)L}{N}, \quad \tau_{-m} = \frac{(-N + 2m)L}{N},
\]

(20)

where \( m = 0, 1, ..., N - 1 \). The corresponding values of the cumulative Döppler factors can be calculated:

\[
D_n(\tau_{\pm m}) = \left( \frac{1 \pm \omega_N \Delta L}{1 \mp \omega_N \Delta L} \right)^n \approx \exp(\pm 2n\omega_N \Delta L) \quad \text{for} \quad \Delta L \ll L.
\]

(21)

There are \( N \) travelling peaks in the energy density. All the peaks have the same height and the same width. The corresponding time evolution of peaks agrees with the results of quantized version [6]. The formation of peaks is basically a classical phenomenon.

The resonance instability in the field theory appears not only for finely tuned frequencies. We obtain usually some band structure. Our method enable us to get insight into off resonant behavior of the cavity system as well. Therefore, we assume some perturbation of the resonant frequency: \( \omega = \omega_N + \Delta \omega \). The equation for return points is now:

\[
\sin(\omega_N \tau^*) = -\frac{L \Delta \omega}{\omega \Delta L}.
\]

(22)

The solutions for return points exist provided that:

\[
\frac{\Delta \omega}{\omega} < \frac{\Delta L}{L}.
\]

(23)

This condition has been obtained numerically for quantized version in [14]. It defines the band structure of the parametric resonance [14]. For small perturbations, the structure defined by Eq. (23) is typical and resembles those obtained in classical mechanics. The cumulative Döppler factor at starting points yields:

\[
D_n(\tau_{\pm m}) = \left( \frac{1 \pm \sqrt{(\omega \Delta L)^2 - (L \Delta \omega)^2}}{1 \mp \sqrt{(\omega \Delta L)^2 - (L \Delta \omega)^2}} \right)^n.
\]

(24)
For off resonant vibrations of the cavity, the formation of narrow peaks and the energy growth take a longer time. Let us conclude our classical analysis. Within the framework of our method the calculations are straightforward for any type of cavity motion. The parametric resonance is related to the existence of periodic trajectories in a two-dimensional cavity billiard. As it was already noticed in [13], there is no need for periodicity of cavity motion. We need only periodic returns of the cavity to the unperturbed position. For typical cavity motions, the growth of the total energy is exponential with time. The cumulative Doppler factor describes all details of the formation of narrow wave packets and the energy growth. Essentially, many details of the resonant enhancement of classical electromagnetic radiation inside a vibrating cavity match the results of quantum theory.

Finally, we discuss vibrating optical cavities in the quantum field theory. Apart from the interference effects which we have already demonstrated in the classical theory of the cavity, we are to account for instability of the quantum vacuum and possibility for particle production. The vacuum expectation value of the energy density is given by [4]:

$$\langle T_{00}(t, x) \rangle = \varrho(t + x) + \varrho(t - x) \ ,$$  (25)

where

$$\varrho(\tau) = -\frac{\pi}{48}\dot{R}^{2}(\tau) - \frac{1}{24\pi}S[R](\tau) \ .$$  (26)

The second term is responsible for particle production. It is defined by the Schwartz derivative:

$$S[R] = \frac{\ddot{R}}{R} - \frac{3}{2} \left( \frac{\dot{R}}{R} \right)^{2} \ .$$  (27)

The crucial information is contained in the phase function, which obeys the following Moore’s equation (being a counterpart of kinematical laws Eqs.(1,2,7)):

$$R(\tau) - R(f(\tau)) = 2 \ .$$  (28)

We see that our billiard function is useful for the quantum case as well. It is straightforward to derive the following recursive relation:

$$\varrho(T_{n}(\tau)) = \varrho(\tau)D_{n}(\tau) + A_{n}(\tau)D_{n}^{2}(\tau) \ ,$$  (29)

where the cumulative Doppler factor is given again by Eq. (12) and the cumulative conformal anomaly contribution can be calculated from the following formulae:

$$A_{n}(\tau) = \frac{1}{24\pi}S[T_{n}](\tau) = -\frac{1}{24\pi}\sum_{k=1}^{n}D_{k}^{2}f(\tau)S[f](T_{k}(\tau)) \ .$$  (30)

The total energy can be calculated now from:

$$E(T_{n}^{\tau_{0}}(\tau_{0})) = \int_{T_{n-1}^{\tau_{0}}(\tau_{0})}^{T_{n}^{\tau_{0}}(\tau_{0})} d\tau \ \varrho(\tau) = \int_{T_{n}^{\tau_{0}}(\tau_{0})}^{\tau_{0}} d\tau \ \left[ \varrho(\tau) + A_{n}(\tau) \right] D_{n}(\tau) \ .$$  (31)

In comparison with the classical formula Eq. (15), aside from the Doppler factor there appears a new additive term on the right hand side. It represents the conformal anomaly of the theory. Most of classical results, like the band structure around resonance frequencies, the formation and the shape of travelling packets in the energy density, the exponential growth of the total accumulated energy, can be reproduced in the quantum case as well. These features are to be calculated from the cumulative Doppler factor. The novelty of the quantum description is the conformal anomaly. We show that for the lowest resonance channel, the anomalous mechanism of energy growth clashes with the resonance enhancement of the initial vacuum fluctuations.

Let us refer again to the sinusoidal motion of the cavity [10]. We consider motions with resonance frequencies \(\omega_{N}\). At the beginning, the cavity is static and empty. It corresponds to the following condition:

$$\varrho(\tau) = -\frac{\pi}{48L^{2}} = -\frac{\omega_{N}^{2}}{48\pi} \quad \text{for} \quad \tau \in [-L, L] \ .$$  (32)

The above value is just the static Casimir energy density which is present even the cavity is at rest. We can compute the cumulated anomaly contribution at starting points of positive trajectories directly from Eq. (30):

$$A_{n}(\tau_{+m}) = \frac{\omega_{N}^{2}}{48\pi L} \frac{1}{1 - \omega_{N}^{2}\Delta L^{2}} \left[ 1 - D_{n}^{2}(\tau_{+m}) \right] \ .$$  (33)
We see that for a long time limit (here large $n$) and small amplitudes, the initial density contribution $\varrho(\tau)$ and the anomaly contribution $A_n(\tau)$ cancel each other out for $N = 1$. It explains why the lowest resonance (called sometimes semi-resonance [6]) is suppressed. No particles are produced inside a cavity. For higher resonance frequencies $N > 1$, the anomalous mechanism surpasses resonant enhancement of the negative energy of the initial state. We obtain again the exponential growth of the total energy and the formation of travelling narrow wave packets.

The first exact analytical solution corresponding to a vibrating cavity system was presented in [10] for the resonance channel $N = 2$ and generalized in [17] for higher resonances. The solutions correspond to a family of mirror trajectories given by:

$$L(t) = L + \frac{1}{\omega_N} \left\{ \arcsin \left[ \sin \frac{\omega_N \Delta L}{2} \cos (\omega_N t) \right] - \frac{\omega_N \Delta L}{2} \right\}. \tag{34}$$

The picture of resonantly enhanced radiation is similar there except of the fact that the energy happens to grow quadratically with time. Such a power-like growth of the total energy is observed at the boundary of the frequency band Eq. (23). For trajectories Eq. (34), the billiard function can be derived exactly:

$$f(\tau) = \frac{2}{\omega_N} \arccot \left[ \cot \left( \frac{\omega(\tau - L)}{2} - 2 \tan \frac{\omega_N \Delta L}{2} \right) - L \right], \tag{35}$$

where the branch of the multivalued function arccot should be properly chosen to have the billiard function increasing. It is straightforward to calculate the cumulative Döppler factor and the cumulative anomaly contribution:

$$D_n(\tau) = 1 + 2n^2 \tan^2 \frac{\omega_N \Delta L}{2} \left[ 1 - \cos (\omega_N (\tau + L)) \right] + 2n \tan \frac{\omega_N \Delta L}{2} \sin (\omega_N (\tau + L)), \tag{36}$$

$$A_n(\tau) = \frac{\omega_N^2}{48\pi} \left[ 1 - D_n(\tau) \right]. \tag{37}$$

There are $N$ periodic particle trajectories corresponding to starting points $\tau_{0m} = (-N + 2m + 1)L/N$ where $m = 0, 1, ..., N - 1$. But all $D_n(\tau_{0m}) = 1$. The periodic trajectories are neither positive nor negative. There is no Döppler factor since the reflections occur at the turning points where the mirror stops. We obtain the corresponding energy density:

$$\varrho(\tau) = -\frac{N^2 \pi}{48L^2} + \frac{(N^2 - 1) \pi}{48L^2} \left\{ 1 + 2n^2 \tan^2 \frac{\omega_N \Delta L}{2} \left[ 1 - (-1)^N \cos (\omega_N \tau) \right] - 2n(-1)^N \tan \frac{\omega_N \Delta L}{2} \sin (\omega_N \tau) \right\}^{-2}. \tag{38}$$

In this case, it is easy to show that the total energy grows quadratically with time, and the heights and widths of travelling wave packets are proportional to $\tau^4$ and $\tau^{-2}$ respectively.

IV. SUMMARY

In summary, we have studied the resonance behavior of the electromagnetic field inside a vibrating one-dimensional (linear) cavity. Our approach exploits fully the analogy with the classical billiard of massless particles. The parametric resonance is determined by the existence of periodic particle trajectories. The billiard function can be used to establish recursive relations for all physical quantities. The resonant exponential enhancement and the concentration of energy into narrow wave packets is described in both classical and quantum field theories of cavities referring to the same aggregated Döppler factor. It is enough to calculate this factor, even numerically, in some finite interval of the length related to the period of cavity oscillations. A purely quantum phenomenon of particle production from the vacuum is described by the anomalous contribution in recursive relations. For higher resonances, this contribution dominates and the evolution of the system is insensitive to the initial state of the quantum field inside a cavity. However, for higher resonances the band structure defined by Eq. (23) is squeezed. From experimental point of view, the inequality Eq. (23) challenges someone to secure either fine tuning or big enough amplitude with gigahertz frequencies. The resonance amplification and photon production from thermal field fluctuations [17], being practically more feasible, can be studied in the same way as well.

The opto-mechanical resonance in vibrating cavities (dynamical Casimir effect) is the subject of numerous studies. The analysis of real, three-dimensional models shows that each photon mode can be truly described by some one-dimensional model. We have presented a new and simple method to analyze resonance solutions in such fundamental
models. Our new results: the mechanism of particle production and the mechanism of resonance amplification of radiation are separated and clarified, exact formulas for off-resonant solutions are given, the band structure is described, the puzzle with the lack of the amplification for the lowest resonant frequency is solved, it is also explained when the instability of the system (the energy growth) is either exponential or power-like. Moreover, the correspondence with classical optics and classical mechanics is presented here from a new point of view as far as the quantum parametric resonance is concerned.

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