METASTABILITY FOR THE DILUTE CURIE–WEISS MODEL WITH GLAUBER DYNAMICS

ANTON BOVIER, SAEDA MARELLO, AND ELENA PULVIRENTI

ABSTRACT. We analyse the metastable behaviour of the dilute Curie–Weiss model subject to a Glauber dynamics. The model is a random version of a mean-field Ising model, where the coupling coefficients are Bernoulli random variables with mean $p \in (0, 1)$. This model can be also viewed as an Ising model on the Erdős–Rényi random graph with edge probability $p$. The system is a Markov chain where spins flip according to a Metropolis dynamics at inverse temperature $\beta$. We compute the average time the system takes to reach the stable phase when it starts from a certain probability distribution on the metastable state (called the last-exit biased distribution), in the regime where $N \to \infty$, $\beta > \beta_c = 1$ and $h$ is positive and small enough. We obtain asymptotic bounds on the probability of the event that the mean metastable hitting time is approximated by that of the Curie–Weiss model. The proof uses the potential theoretic approach to metastability and concentration of measure inequalities.

1. INTRODUCTION AND MAIN RESULTS

The randomly dilute Curie–Weiss model (RDCW) is a classical model of a disordered ferromagnet and was studied, e.g. in Bovier and Gayrard in [6]. It generalises the standard Curie–Weiss model (CW) in that the fixed interactions between each pair of spins is replaced by independent, identically distributed, random ferromagnetic couplings between any pair of spins. In [6] it is proven that the RDCW free energy converges, in the thermodynamic limit, to that of the CW model, under some assumptions on the coupling distribution. Their result relies on the fact that the RDCW Hamiltonian can be approximated by that of the CW model up to a small perturbation which can be uniformly bounded in high probability. In the last decade the RDCW model have gained again some attention and various results at equilibrium have been proven, both in the annealed and quenched case. De Sanctis and Guerra [9] give an exact expression of the free energy first in the high temperature and low connectivity regime, and then at zero temperature. The control of the fluctuations of the magnetisation in the high temperature limit is addressed by De Sanctis [8], while recently Kabluchko, Löwe and Schubert [15] prove a quenched Central Limit Theorem for the magnetisation in the high temperature regime.

One of the features which make these random systems with “bond disorder” very appealing is their deep connection with the theory of random graphs, which attracted great interest in the last years due to their application to real-world networks. Indeed, if the
random couplings are chosen as i.i.d. Bernoulli random variables with mean $p$, one can view the model as a spin system on an Erdős–Rényi random graph with edge probability $p$. There has been an extensive study of the Ising model on different kinds of random graphs, e.g. in Dembo, Montanari [10] and Dommers, Giardinà, van der Hofstad [14], where several thermodynamic quantities where analysed. We refer to van der Hofstad [17] for a general overview of these results.

In contrast to the substantial body of literature on the equilibrium properties of the RDCW model, much less is known about its dynamical properties. The present paper focuses on the phenomenon of metastability for the RDCW model where, for simplicity, the couplings are Bernoulli distributed with fixed parameter $p \in (0, 1)$, independent of the number of vertices $N$, and the system evolves according to a Glauber dynamics. In particular, we give a precise estimate of the mean transition time from a certain probability distribution on the metastable state (called the last-exit biased distribution) to the stable state, when the external magnetic field is small enough and positive and when $N$ tends to infinity. We obtain asymptotic bounds on the probability of the event that the average time is close to the CW one times some constants of order 1 which depend on the parameters of the system.

In the context of metastability for interacting particle systems on random graphs, progress has been made for the case of the random regular graph, analysed by Dommers [12] and for the configuration model, studied by Dommers, den Hollander, Jovanovski, and Nardi [13], both subject to Glauber dynamics, in the limit as the temperature tends to zero and the number of vertices is fixed. In [11] den Hollander and Jovanovski investigate the same model considered in the present paper and obtain estimates on the average crossover time for fixed temperature in the thermodynamic limit. They show that, with high probability, the exponential term is the same as in the CW model, while the multiplicative term is polynomial in $N$. Their analysis relies on coupling arguments and uses the pathwise approach to metastability.

In contrast, in the present paper, we use the potential theoretic approach initiated by Bovier, Eckhoff, Gayrard and Klein in a series of papers [3, 4, 5]. (see the monograph of Bovier and den Hollander [2] for an in-depth review). This method has also successfully applied to the random field CW model, where the external magnetic field is given by i.i.d. random variables, by Bianchi, Bovier and Ioffe in [1]. Furthermore, inspired by the results of Bovier and Gayrard [6], namely that the equilibrium properties of the RDCW model are very close to those of the CW model, we observe that, using Talagrand’s concentration inequality, the mesoscopic measure can be expressed in terms of that of CW.

Before stating our results we give a precise definition of the model.

1.1. Glauber dynamics for the RDCW model. Let $[N] = \{1, \ldots, N\}, N \in \mathbb{N}$, be a set of vertices. To each vertex $i \in [N]$ an Ising spin $\sigma_i$ with values in $\{-1, +1\}$ is associated. We denote by $\sigma = \{\sigma_i : i \in [N], \sigma_i \in \{-1, +1\}\}$ a spin configuration and we define the state space $S_N = \{-1, +1\}^{[N]}$ to be the set of all such configurations $\sigma$. We fix a probability $p \in (0, 1)$. Then the randomly dilute Curie–Weiss model (RDCW) has the following random Hamiltonian $H_N : S_N \to \mathbb{R}$

$$H_N(\sigma) = -\frac{1}{Np} \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j - h \sum_{i \in [N]} \sigma_i, \quad (1.1)$$
where $h \in \mathbb{R}$ represents an external constant magnetic field, while $J_{ij}/Np$ is a ferromagnetic random coupling. In particular, \{$J_{ij}$\}, \(i,j \in [N]\) is a sequence of i.i.d. random variables with $J_{ij} \sim \text{Ber}(p)$ and $J_{ij} = J_{ji}$.

The RDCW model can be seen as the Ising model on the Erdős–Rényi random graph with vertex set \([N]\), edge set $E$ and edge probability $p \in (0,1)$ (see van der Hofstad \[16\] for a general overview on random graphs). In this picture the Hamiltonian can also be written as

$$H_N(\sigma) = -\frac{1}{Np} \sum_{\langle i,j \rangle \in E} \sigma_i \sigma_j - h \sum_{i \in [N]} \sigma_i.$$

The Gibbs measure associated to the random Hamiltonian $H_N$ is

$$\mu_{\beta,N}(\sigma) = \frac{e^{-\beta H_N(\sigma)}}{Z_{\beta,N}}, \quad \sigma \in S_N,$$

where $\beta \in (0,\infty)$ is the inverse temperature and the partition function is defined as

$$Z_{\beta,N} = \sum_{\sigma \in S_N} e^{-\beta H_N(\sigma)}.$$

The Gibbs measure $\mu_{\beta,N}$ is the unique invariant (and reversible) measure for the (discrete time) Glauber dynamics on $S_N$ with Metropolis transition probabilities

$$p_N(\sigma,\sigma') = \begin{cases} \frac{1}{N} \exp(-\beta[H_N(\sigma') - H_N(\sigma)]_+), & \text{if } \sigma \sim \sigma', \\ 1 - \sum_{\eta \neq \sigma} p(\sigma,\eta), & \text{if } \sigma = \sigma', \\ 0, & \text{else}, \end{cases}$$

where $\sigma \sim \sigma'$ means $||\sigma - \sigma'|| = 2$ with $|| \cdot ||$ the $\ell_1$-norm on $S_N$. We denote this Markov chain by $\{\sigma(t)\}_{t \geq 0}$ the Markov chain and write $\mathbb{P}_\nu$, for the law of the process $\sigma(t)$ with initial distribution $\nu$ conditioned on the realisation of the random couplings. Analogously, $\mathbb{E}_\nu$ is the quenched expectation w.r.t. the Markov chain with initial distribution $\nu$. Moreover, we set $\mathbb{P}_\sigma = \mathbb{P}_{\delta_\sigma}$. For any subset $A \subset S_N$ we define the hitting time of $A$ as

$$\tau_A = \inf\{t > 0 : \sigma_t \in A\}.$$  

### 1.2. The Curie–Weiss model

Before stating the main results, we recall some results for the mean-field Curie–Weiss (CW) model (see Bovier and den Hollander \[2, Section 13\]). The CW Hamiltonian can be obtained taking the mean value of (1.1). An simplifying feature of the CW model is that its Hamiltonian depends on $\sigma$ only through the empirical magnetisation $m_N : S_N \to \Gamma_N$ defined as

$$m_N(\sigma) = \frac{1}{N} \sum_{i=1}^N \sigma_i \in \Gamma_N = \{-1, -1 + \frac{2}{N}, ..., 1 - \frac{2}{N}, 1\}.$$

From now on we will drop the dependency on $N$ from the magnetisation. Then

$$\tilde{H}_N(\sigma) = -N \left(\frac{1}{2}m(\sigma)^2 + hm(\sigma) \right) \equiv NE(m(\sigma))$$

and the associated Gibbs measure is

$$\tilde{\mu}_{\beta,N}(\sigma) = \frac{e^{-\beta NE(m(\sigma))}}{\tilde{Z}_{\beta,N}}, \quad \sigma \in S_N,$$

where $\tilde{Z}_{\beta,N}$ is the normalising partition function. We denote the law of $m(\sigma)$ under the Gibbs measure by

$$\tilde{Q}_{\beta,N} = \tilde{\mu}_{\beta,N} \circ m^{-1}.$$
Then
\[ \tilde{Q}_{\beta,N}(m) = \frac{e^{-\beta N E(m)}}{Z_{\beta,N}} \sum_{\sigma \in S_N} 1_{m(\sigma) = m} = \frac{e^{-\beta N E(m)}}{Z_{\beta,N}} \left( \frac{N!}{2^{m N}} \right) \frac{e^{-\beta N f_{\beta,N}(m)}}{Z_{\beta,N}}, \] (1.11)
where
\[ f_{\beta,N}(m) = -\frac{m^2}{2} - hm + \beta^{-1} I_N(m) \] (1.12)
is the finite volume free energy, while the entropy of the system is given by the following combinatorial coefficient
\[ I_N(m) = -\frac{1}{N} \log \left( \frac{N}{2^{m N}} \right). \] (1.13)
As \( N \to \infty \),
\[ I_N(m) \to I(m) \equiv \frac{1 - m}{2} \log \frac{1 - m}{2} + \frac{1 + m}{2} \log \frac{1 + m}{2}. \] (1.14)
More precisely,
\[ I_N(m) - I(m) = \frac{1}{2N} \ln \frac{1 - m^2}{4} + \frac{\ln N + \ln(2\pi)}{2N} + O\left( \frac{1}{N^2} \right). \] (1.15)

We use the notation \( f_{\beta}(m) = \lim_{N \to \infty} f_{\beta,N}(m) \).

We consider the Glauber dynamics associated to the CW Hamiltonian in analogy with (1.5) and with transition probabilities \( \bar{p}_y(\sigma,\sigma') \). A particular feature of this model is that the image process \( m(t) \equiv m(\sigma(t)) \) of the Markov process \( \sigma(t) \) under the map \( m \) is again a Markov process on \( \Gamma_N \), with transition probabilities
\[ \bar{r}_N(m,m') = \begin{cases} \exp(-\beta N[E(m') - E(m)]) \frac{(1 - m)}{2} & \text{if } m' = m + \frac{2}{N}, \\ \exp(-\beta N[E(m') - E(m)]) \frac{(1 + m)}{2} & \text{if } m' = m - \frac{2}{N}. \end{cases} \] (1.16)

The equilibrium CW model displays a phase transition. Namely, there is a critical value of the inverse temperature \( \beta_c = 1 \) such that, in the regime \( \beta > \beta_c, h > 0 \) and small, the free energy \( f_{\beta}(m) \) is a double-well function with local minimisers \( m_-, m_+ \) and saddle point \( m^* \). They are the solutions of equation \( m = \tanh(\beta(m + h)) \). Since \( f_{\beta}(m_-) > f_{\beta}(m_+) \), the phase with \( m_- \) represents the metastable state, while \( m_+ \) represents the stable state for the system. Defining \( m_-(N), m^*(N), m_+(N) \) as the closest points in \( \Gamma_N \) to \( m_-, m^*, m_+ \) respectively, then \( (m_-(N), m_+(N)) \) form a metastable set in the sense of Definition 8.2 of Bovier and den Hollander [2]. Let \( \mathbb{E}^{\text{CW}}_{m_-(N)} \) be the expectation w.r.t. the Markov process \( m(t) \) with transition probabilities \( \bar{r}_N \) and starting at \( m_-(N) \). Then the following theorem holds.

**Theorem 1.1.** For \( \beta > 1 \) and \( h > 0 \) small enough, as \( N \to \infty \),
\[ \mathbb{E}^{\text{CW}}_{m_-(N)}[\tau_{m_+(N)}] = \exp \left( \beta N \left[ f_{\beta}(m^*) - f_{\beta}(m_-) \right] \right) \times \frac{\pi}{1 - m^2} \sqrt{\frac{1 - m^2}{1 - m^2}} \frac{N(1 + o(1))}{\beta \sqrt{f''_{\beta}(m_-)(-f''_{\beta}(m^*))}}. \] (1.17)

We conclude this section by giving the explicit formula of the capacity for the CW model. The definition of capacity is given in (1.27), while its relation with the mean hitting time is given by the key relation (1.26). Let us denote, for any subset \( U \) of \( \Gamma_N \), the set of configurations with magnetisation in \( U \) by
\[ S_N[U] = \{ \sigma \in S_N : m(\sigma) \in U \}. \] (1.18)
and for simplicity, for any \( m \in \Gamma_N \), the set of configurations with given magnetisation \( m \) by \( S_N[m] \).

Then, the following formula,
\[
\text{cap}^\text{CW}(S_N[m_-(N)], S_N[m_+(N)]) = \frac{1}{Z_{\beta,N}} e^{-BN_{\beta}(m^*)} \sqrt{\beta(-f''_B(m^*))} \sqrt{\frac{1-m^*}{1+m^*}(1 + o(1))},
\]
follows from standard arguments (see e.g. Bovier and den Hollander \[2\] Section 13).

1.3. Main results. For any \( A, B \subset S_N \) disjoint, we define the so-called last-exit biased distribution on \( A \) for the transition from \( A \) to \( B \) as
\[
\nu_{A,B}(\sigma) = \frac{\mu_{\beta,N}(\sigma)\mathbb{P}_{\sigma}(\tau_B < \tau_A)}{\sum_{\sigma \in A} \mu_{\beta,N}(\sigma)\mathbb{P}_{\sigma}(\tau_B < \tau_A)}, \quad \sigma \in A.
\]
Since we are going to use \( \nu_{A,B} \) on the sets \( S_N[m_-(N)], S_N[m_+(N)] \) defined above, we introduce the following simplified notation
\[
\nu^N_{m_-, m_+} = \nu_{S_N[m_-(N)], S_N[m_+(N)]}.
\]

The following theorem gives a description of the dynamical properties of the RDCW model in the metastable regime where \( h \) is positive and small enough, \( \beta > \beta_c = 1 \) (\( \beta_c \) is the critical inverse temperature for the CW model) and \( N \) is going to infinity. We provide an estimate on the mean time it takes to the system, starting with initial distribution \( \nu^N_{m_-, m_+} \), to reach \( S_N[m_+(N)] \). More precisely, we estimate, in the limit as \( N \to \infty \), its ratio with the mean metastable exit time for the CW model to go from \( m_-(N) \) to \( m_+(N) \), providing constant upper and lower bounds independent of \( N \). Because of the random interaction, the result is given in the form of tail bounds.

We are now ready to formulate our main theorem.

**Theorem 1.2** (Mean metastable exit time). For \( \beta > 1, h > 0 \) small enough and for \( s > 0 \), there exist absolute constants \( k_1, k_2 > 0 \) and \( C_1(p, \beta) < C_2(p, \beta, h) \) independent of \( N \), such that
\[
\lim_{N \to \infty} \mathbb{P}_J \left( C_1 e^{-s} \frac{\mathbb{E}^N_{S_N[m_-(N)]} [\tau_{S_N[m_+(N)]}]}{\mathbb{E}^N_{m_-(N)} [\tau_{m_+(N)}]} \right) \leq C_2 e^s \geq 1 - k_1 e^{-k_2 s^2}.
\]

The quantities \( C_1 \) and \( C_2 \) in the previous theorem can be explicitly written. Set
\[
\alpha = \frac{\beta^2 (1-p)}{4p}, \quad \kappa = \alpha + \max_{\eta \in (0,1)} \left\{ \log \eta - \frac{\beta \sqrt{2 \alpha + \log \left( \frac{c_1}{(1-\eta)^p} \right)}}{p \sqrt{2c_2}} \right\},
\]
where \( c_1, c_2 > 0 \) are absolute constants coming from Theorem 2.7. It is easy to see that \( \kappa < \alpha \). With this notation
\[
C_1 = C_1(\beta, p) = e^{-2\beta - \alpha + \kappa (1 + o(1))}, \quad (1.24)
\]
\[
C_2 = C_2(\beta, h, p) = e^{2\beta(1+h)+2\alpha (1 + o(1))}. \quad (1.25)
\]
1.4. **Proof of the main theorem.** The proof of Theorem 1.2 is based on the potential theoretic approach to metastability, which turns out to be a rather powerful tool to analyse the main object we are interested in, i.e. the mean hitting time of $S_N[m, (N)]$ for the system with initial distribution $\nu^N_{m, m}$. The general ideas of this approach were first introduced in a series of papers by Bovier, Eckhoff, Gayrard and Klein [3, 4, 5]. We refer to Bovier and den Hollander [2] for an overview on this method.

The crucial formula in the study of metastability is given by the following relation linking mean hitting time and capacity of two sets $A, B \in S_N$.

$$E_{\nu, \lambda}[\tau_B] = \sum_{\omega \in A} \mu_{\beta, N}(\omega)\nu_{A, B}(\omega)E_{\nu, \lambda}[\tau_B] = \frac{1}{\text{cap}(A, B)} \sum_{\omega \in S_N} \mu_{\beta, N}(\omega)h_{AB}(\omega),$$

(1.26)

where

$$\text{cap}(A, B) = \sum_{\omega \in A} \mu_{\beta, N}(\omega)\mathbb{P}_\omega(\tau_B < \tau_A).$$

(1.27)

The function $h_{AB}$ is called harmonic function and has the following probabilistic interpretation

$$h_{AB}(\omega) = \begin{cases} \mathbb{P}_\omega(\tau_A < \tau_B), & \omega \in S_N \setminus (A \cup B), \\ 1, & \omega \in A \cup B. \end{cases}$$

(1.28)

By (1.26), in order to estimate mean hitting times one needs estimates both on the capacity and on the harmonic function.

We prove bounds on the capacity of two sets $S_N[m_1], S_N[m_2]$, with $m_1 < m_2$, stated in the two following theorems.

**Theorem 1.3.** For any $m_1 < m_2 \in \Gamma_N$ and any $s > 0$, there exist absolute constants $k_1, k_2 > 0$ such that

$$\mathbb{P}_{\beta} \left( \frac{Z_{\beta, N} \cap (S_N[m_1], S_N[m_2])}{Z_{\beta, N} \cap CW(S_N[m_1], S_N[m_2])} \leq e^{s+2\beta+\sigma}(1 + o(1)) \right) \geq 1 - k_1 e^{-k_2 s^2},$$

(1.29)

asymptotically as $N \to \infty$, where $\sigma$ is defined in (1.23).

**Theorem 1.4.** For any $m_1 < m_2 \in \Gamma_N$ and any $s > 0$, there exist absolute constants $k_1, k_2 > 0$ such that

$$\mathbb{P}_{\beta} \left( \frac{Z_{\beta, N} \cap (S_N[m_1], S_N[m_2])}{Z_{\beta, N} \cap CW(S_N[m_1], S_N[m_2])} \geq e^{-(s+2\beta+\sigma+1)(1 + o(1))} \right) \geq 1 - k_1 e^{-k_2 s^2},$$

(1.30)

asymptotically as $N \to \infty$, where $\sigma$ is defined in (1.23).

We state asymptotic upper and lower bounds on the sum over the harmonic function in the numerator of (1.26) in the following proposition. We used the simplified notation

$$h^N_{m, m} = h_{S_N[m, (N)], S_N[m, (N)]}.$$  

(1.31)

**Theorem 1.5.** For any $s > 0$, there exist absolute constants $k_1, k_2 > 0$ such that

$$\mathbb{P}_{\beta} \left( \sum_{\omega \in S_N} \mu_{\beta, N}(\omega)h^N_{m, m}(\omega) \leq e^{s+\sigma} \exp \left( -\beta N f_\beta(m) \right) \frac{Z_{\beta, N}}{(1 - m^2)\beta f''_\beta(m)} (1 + o(1)) \right) \geq 1 - k_1 e^{-k_2 s^2},$$

(1.32)

and

$$\mathbb{P}_{\beta} \left( \sum_{\omega \in S_N} \mu_{\beta, N}(\omega)h^N_{m, m}(\omega) \geq e^{-s} \exp \left( -\beta N f_\beta(m) \right) \frac{Z_{\beta, N}}{(1 - m^2)\beta f''_\beta(m)} (1 + o(1)) \right) \geq 1 - k_1 e^{-k_2 s^2},$$

(1.33)
asymptotically as \( N \to \infty \), and where \( \alpha \) and \( \kappa \) are defined in (1.23).

We conclude this section using Theorems 1.3-1.5 to prove the main theorem. First, we introduce the following notation which will be extensively used:

\[
P(\sigma) \quad \text{is equivalent to} \quad \mathbb{P}_{\beta}(A \geq B) \geq 1 - k_1 e^{-k_2 s^2},
\]

for all \( s > 0 \) and for some absolute constants \( k_1, k_2 > 0 \), whose values might change along the paper.

**Proof of Theorem 1.2.** We prove here only the upper bound, as the lower bound follows similarly. More precisely, we prove

\[
\mathbb{E}^{\nu}_{\mathcal{E}_{m_{\sigma}}(N)} \tau_{S_{N}[m_{\sigma}]} \leq C_2 e^{s}.
\]

We start from (1.26), which in our case reads

\[
\mathbb{E}^{\nu}_{\mathcal{E}_{m_{\sigma}}(N)} \tau_{S_{N}[m_{\sigma}]} \leq \sum_{\sigma \in S_{N}} \mu_{\beta} \left( h_{m_{\sigma}}^{N}(\sigma) \right) \frac{\beta}{\text{cap}(S_{N}[m_{\sigma}], S_{N}[m_{\sigma}])}.
\]

From (1.32) we obtain

\[
\mathbb{E}^{\nu}_{\mathcal{E}_{m_{\sigma}}(N)} \tau_{S_{N}[m_{\sigma}]} \leq \frac{e^{s}}{Z_{\beta,N} \text{cap}(S_{N}[m_{\sigma}], S_{N}[m_{\sigma}])(1 - m_{\sigma})^{\beta f''(m_{\sigma})}}.
\]

Via the lower bound on the capacity from Theorem 1.4, we obtain

\[
\mathbb{E}^{\nu}_{\mathcal{E}_{m_{\sigma}}(N)} \tau_{S_{N}[m_{\sigma}]} \leq e^{2s + 2\beta(1 + h) + 2\alpha} \frac{\beta N}{1 - m_{\sigma}} \exp \left( \frac{f_{\beta}(m_{\sigma}) \beta f''(m_{\sigma})}{1 - m_{\sigma}^2} \right) (1 + o(1))
\]

where we used (1.19) and Theorem 1.1.

\[\square\]

**1.5. Outline.** The remainder of this paper is organised as follows. In Section 2 we use the powerful Talagrand’s concentration inequality to obtain bounds on the equilibrium measure of the RDCW model. These bounds allow us to write the RDCW mesoscopic measure in terms of the deterministic CW one, times a random factor which is the exponential of a sub-Gaussian random variable. In Section 3 we give the proof of Theorems 1.3 and 1.4 via two dual variational principles, the Dirichlet and the Thomson principles, which are the building blocks of the potential theoretic approach to metastability. In obtaining upper and lower bounds on the capacity, the main strategy is to use the results of Section 2 in order to recover the capacity of the CW model. In Section 4 we prove Theorem 1.5, i.e. we compute the asymptotics of the numerator in the formula for the mean hitting time using estimates on the harmonic function.
2. Equilibrium analysis via Talagrand’s concentration inequality

In this section we prove that the equilibrium mesoscopic measure of the RDCW model is in fact very close to that of the CW model. This is done in two steps. First, we prove that the difference between the random free energy at fixed magnetisation and its average can be controlled via Talagrand’s concentration inequality. Second, we find upper and lower bounds on the aforementioned average by estimating first and second moments of the partition function of the RDCW model at fixed magnetisation.

2.1. Mesoscopic measure and closeness to the CW model. We start by analysing the equilibrium measure of the RDCW model. The aim is to express the equilibrium measure \( Q_{\beta,N} \) in terms of the empirical magnetisation in order to obtain a mesoscopic description, as we did for the CW model in Section 1.2. Let us define the measure \( Q_{\beta,N} \) on \( \Gamma_N \), and let the partition function be its normalisation

\[
Q_{\beta,N}(\cdot) = \mu_{\beta,N} \circ m^{-1}(\cdot) = \sum_{\sigma \in S_N} \mu_{\beta,N}(\sigma), \quad Z_{\beta,N} = \sum_{m \in \mathbb{R}} Q_{\beta,N}(m). \tag{2.1}
\]

A priori the Hamiltonian of the RDCW model is not only depending on \( m \), but it depends of course on the whole spin configuration. Nonetheless, we will see later in this section that the mesoscopic measure \( Q_{\beta,N} \) can be written in terms of the mesoscopic measure \( \tilde{Q}_{\beta,N} \) of the standard CW model.

We first notice that the expectation of the RDCW Hamiltonian is the CW one, i.e.

\[
\mathbb{E}[H_N(\sigma)] = -\frac{1}{Np} \sum_{i<j} \mathbb{E}[J_{ij}]\sigma_i\sigma_j - h \sum_i \sigma_i = -\frac{p}{Np} \sum_{i<j} \sigma_i\sigma_j - h \sum_i \sigma_i = H_N(\sigma). \tag{2.2}
\]

Therefore, we can split the Hamiltonian into the mean-field part and the remaining random part obtaining

\[
H_N(\sigma) = \mathbb{E}[H_N(\sigma)] + \Delta_{N,p}(\sigma), \tag{2.3}
\]

where

\[
\Delta_{N,p}(\sigma) = H_N(\sigma) - H_N(\sigma) = -\frac{1}{Np} \sum_{i<j} J_{ij}\sigma_i\sigma_j. \tag{2.4}
\]

Note that \( \Delta_{N,p} \) is a random variable with zero mean. In order to simplify the notation, we drop from now on the dependence on \( N \) and \( p \), from \( \Delta_{N,p} \). Next, we write the mesoscopic measure as

\[
Q_{\beta,N}(m) = e^{-\beta E(m)} \sum_{\sigma \in S_N} e^{-\Delta(\sigma)} , \tag{2.5}
\]

where \( E(m) \) is defined in (1.8). We introduce the following notation, where we drop the dependence on \( \beta \) for simplicity

\[
Z_{N,m} = \sum_{\sigma \in S_N} e^{-\beta \Delta(\sigma)} = \exp(Np_{N,m}) \exp(N [F_{N,m} - p_{N,m}]), \tag{2.6}
\]

\[
F_{N,m} = \frac{1}{N} \log Z_{N,m}, \tag{2.7}
\]

\[
p_{N,m} = \mathbb{E}(F_{N,m}), \tag{2.8}
\]

where \( Z_{N,m} \) can be interpreted as a partition function for a system of spins with fixed magnetisation, \( F_{N,m} \) as the associated random free energy and \( p_{N,m} \) as its average.

We are interested in finding precise estimates on \( Z_{N,m} \) by writing it in terms of the entropic exponential term \( e^{-N \mathbb{E}(m)} \) times some random factor which takes into account the
randomness of the couplings. We notice that \( Z_{N,m} \) is the product of a deterministic factor \( e^{Np_{N,m}} \) and a random factor \( e^{N(F_{N,m} - p_{N,m})} \).

We first characterise the random variable \( N(F_{N,m} - p_{N,m}) \) in the following Proposition.

**Proposition 2.1.** For any \( \beta \), any \( t > 0 \),

\[
\mathbb{P}(N(F_{N,m} - p_{N,m}) \geq t) \leq c_1 \exp \left(-\gamma t^2\right).
\] (2.9)

where \( \gamma \propto \frac{\beta^2}{p^2} \).

The previous result intuitively means that the random free energy \( F_{N,m} \) is in fact very well concentrated around its mean \( p_{N,m} \).

As a second step we provide asymptotic bounds on the average of \( F_{N,m} \), i.e. the deterministic term \( p_{N,m} \).

**Lemma 2.2.** Asymptotically, as \( N \to \infty \),

\[
p_{N,m} \leq \frac{\alpha}{N} - I_N(m) + o \left(\frac{1}{N}\right),
\] (2.10)

where \( \alpha \) is defined in (1.23).

**Lemma 2.3.** Asymptotically, as \( N \to \infty \),

\[
p_{N,m} \geq \frac{\kappa}{N} - I_N(m) + o \left(\frac{1}{N}\right),
\] (2.11)

where \( \kappa \) is defined in (1.23).

Proposition 2.1 together with Lemmas 2.2 and 2.3 imply the following result.

**Proposition 2.4.** Asymptotically, as \( N \to \infty \), we have

\[
Z_{N,m} \leq e^\alpha \exp \left[N(F_{N,m} - p_{N,m}) - NI_N(m)\right] (1 + o(1)),
\] (2.12)

and

\[
Z_{N,m} \geq e^\kappa \exp \left[N(F_{N,m} - p_{N,m}) - NI_N(m)\right] (1 + o(1)),
\] (2.13)

where \( \alpha \) and \( \kappa \) are defined in (1.23). Moreover, \( N(F_{N,m} - p_{N,m}) \) is a sub-Gaussian random variable with variance

\[
\text{Var}[N(F_{N,m} - p_{N,m})] \leq c \beta^2 \frac{p^2}{p^2},
\] (2.14)

where \( c \) is a positive constant.

We prove Proposition 2.1 in Section 2.2 and the Lemmas 2.2 and 2.3 in Section 2.3.

We proceed now by giving the main result of this section, as a corollary of Proposition 2.4.

**Corollary 2.5.** For every \( m \in \Gamma_N \),

\[
Z_{\beta,N}Q_{\beta,N}(m) \leq e^\alpha \tilde{Z}_{\beta,N}(m) \exp \left(N \left[F_{\beta,N,m} - p_{\beta,N,m}\right]\right) (1 + o(1)),
\] (2.15)

and

\[
Z_{\beta,N}Q_{\beta,N}(m) \geq e^\kappa \tilde{Z}_{\beta,N}(m) \exp \left(N \left[F_{\beta,N,m} - p_{\beta,N,m}\right]\right) (1 + o(1)),
\] (2.16)

where \( \alpha \) and \( \kappa \) are defined in (1.23).
Proof. Using the decomposition \( (2.3) \) and the upper bound in Proposition \( 2.4 \) we have
\[
Z_{\beta,N} Q_{\beta,N}(m) = \sum_{\sigma \in \mathcal{S}_N[m]} e^{-\beta H_N(\sigma)} = e^{-\beta N E(m)} \mathcal{Z}_{N,m}
\]
\[
\leq e^\sigma \exp \left( -\beta N f_{\beta}(m) + N [F_{N,m} - p_{N,m}] \right) (1 + o(1))
\]
\[
= e^\sigma \tilde{Z}_{\beta,N} \mathcal{Q}_{\beta,N}(m) \exp \left( N [F_{N,m} - p_{N,m}] \right) (1 + o(1)).
\]
The lower bound is proven similarly. \( \square \)

We conclude this section by introducing some notation which will be widely used later.

**Property 2.6.** Let \( \mathcal{Y} \) be a sub-Gaussian random variable such that
\[
\mathbb{P}_f (|\mathcal{Y}| \geq s) \leq k_1 e^{k_2 s^2},
\]
where \( k_1, k_2 > 0 \) are absolute constants, and consider the random variable \( X = \exp(\mathcal{Y}) \). For all \( s > 0 \), it is trivial to see that
\[
\mathbb{P}_f (X \leq e^s) \geq 1 - k_1 e^{-k_2 s^2} \quad \text{and} \quad \mathbb{P}_f (X \geq e^{-s}) \geq 1 - k_1 e^{-k_2 s^2}.
\]

### 2.2. Sub-Gaussian bounds on the random term

**Proposition 2.7** follows from Talagrand’s concentration inequality, which we cite for completeness in the version of Tao [19].

**Theorem 2.7** (Talagrand concentration inequality). Let \( G : \mathbb{R}^M \rightarrow \mathbb{R} \) be a 1-Lipschitz and convex function. Let \( M \in \mathbb{N} \), \( g = (g_1, \ldots, g_M) \), with \( g_i \) be independent r.v., uniformly bounded by \( K > 0 \), i.e. \( |g_i| \leq K \), for every \( 1 \leq i \leq M \). Then, for any \( t \geq 0 \),
\[
\mathbb{P}(|G(g) - \mathbb{E}(G(g))| \geq tK) \leq c_1 \exp \left( -c_2 t^2 \right),
\]
with positive absolute constants \( c_1, c_2 \).

**Proof of Proposition 2.7** We can apply Theorem 2.7 to the free energies \( F_{N,m} \) as a function of the \( N^2 \) coupling constants \( \hat{J}_{ij} \). It is standard to see the \( F_{N,m} \) is convex and Lipschitz continuous with constant \( \tfrac{2}{Np \sqrt{2}} \) (see e.g. Talagrand [18, Corollary 2.2.5]). Thus, for some positive constants \( c_1, c_2 \),
\[
\mathbb{P}_f (|F_{N,m} - p_{N,m}| \geq t) \leq c_1 \exp \left( -c_2 \tfrac{2p^2}{\beta^2} t^2 \right),
\]
concluding the proof of \( (2.9) \) and hence Proposition 2.7. \( \square \)

### 2.3. Asymptotic bounds on the deterministic term

In this section we prove first the upper bound on \( p_{N,m} \) (Lemma 2.2) and then the lower bound (Lemma 2.3). The upper bound is obtained by estimates on the first moment of the random partition function \( \tilde{Z}_{N,m} \), while the lower bound is in the spirit of Talagrand [18, Theorem 2.2.1] and is more delicate. We will see that it involves also estimates on the second moment of the random partition function.

**Proof of Lemma 2.2** Observing that \( \{\hat{J}_{ij}\}_{i,j \in [N]} \) defined in \( (2.4) \) are i.i.d. random variables such that \( \mathbb{E} \hat{J}_{ij} = 0 \), we easily obtain
\[
\mathbb{E} [Z_{N,m}] = \sum_{\sigma \in \mathcal{S}_N[m]} \mathbb{E} \left( \exp \left[ \frac{\beta}{Np} \sum_{i<j} \hat{J}_{ij} \sigma_i \sigma_j \right] \right) = \sum_{\sigma \in \mathcal{S}_N[m]} \prod_{i<j} \mathbb{E} \left( \exp \left[ \frac{\beta}{Np} \hat{J}_{ij} \sigma_i \sigma_j \right] \right).
\]
In order to find estimates for \((2.22)\), we first define
\[
\Phi(x) := \mathbb{E}[\exp(x\hat{\mathcal{J}}_{ij})],
\]  
which is a function independent of \(i, j\), being \(\{\hat{\mathcal{J}}_{ij}\}_{i,j}\) i.i.d., with first and second derivatives
\[
\Phi'(0) = \mathbb{E}\hat{\mathcal{J}}_{ij} = 0,
\]
\[
\Phi''(0) = \mathbb{E}\hat{\mathcal{J}}_{ij}^2 = p(1 - p).
\]

Performing a Taylor expansion of \(\Phi\) we get
\[
\Phi(x) = \Phi(0) + x\Phi'(0) + \frac{x^2}{2}\Phi''(0) + o_0(x^2) = 1 + \frac{x^2}{2}p(1 - p) + o(x^2).
\]  
Thus, we can exponentiate \(\Phi(x)\) to obtain
\[
\Phi(x) = \exp \left( \log (\Phi(x)) \right) = \exp \left( \frac{x^2}{2}p(1 - p) + o(x^2) \right),
\]  
where we used the expansion \(\log(1 + x) = x + o(x)\). Therefore, for any sequence of coefficients \(x_{ij}^2\) which are independent of \(i, j\) and \(\sigma\), we have the following
\[
\sum_{\sigma \in \mathcal{S}_N[m]} \prod_{i < j} \mathbb{E}\left[\exp(x_{ij}\hat{\mathcal{J}}_{ij})\right] = \sum_{\sigma \in \mathcal{S}_N[m]} \prod_{i < j} \Phi(x_{ij})
\]
\[
= \sum_{\sigma \in \mathcal{S}_N[m]} \prod_{i < j} \exp \left( \frac{x_{ij}^2}{2}p(1 - p) + o(x_{ij}^2) \right)
\]
\[
= e^{-NI_{N}(m)} \exp \left( \frac{x_{ij}^2}{2}p(1 - p) + o(x_{ij}^2) \right)^{N(N - 1)/2}
\]
\[
= e^{-NI_{N}(m)} \exp \left( x_{ij}^2p(1 - p) \frac{N(N - 1)}{4} + o(x_{ij}^2N(N - 1)) \right),
\]  
asymptotically, for \(x_{ij} \to 0\), where the second equality holds only if \(x_{ij}^2\) is independent of \(i, j\) and the last equality holds only if \(x_{ij}^2\) is independent of \(\sigma\). Applying \((2.28)\) to \(x_{ij} = \frac{\theta}{\sqrt{4p}}\sigma_i \sigma_j\) we get, asymptotically as \(N \to \infty\),
\[
\mathbb{E}\left[\mathcal{Z}_{N,m}\right] = e^{-NI_{N}(m)} \exp \left( \frac{\theta^2(1 - p)}{4p} + o(1) \right) = \exp \left( \alpha - NI_{N}(m) + o(1) \right).
\]  
Therefore, by Jensen’s inequality and \((2.29)\), we have
\[
\mathbb{E}\left[\log \mathcal{Z}_{N,m}\right] \leq \log \left( \mathbb{E}\left[\mathcal{Z}_{N,m}\right] \right) = \alpha - NI_{N}(m) + o(1),
\]  
which proves the upper bound. \(\square\)

**Proof of Lemma** \((2.3)\) A key ingredient in the proof is to control the upper bound on the second moment of \(\mathcal{Z}_{N,m}\), i.e. prove that the following bound holds
\[
\mathbb{E}\left[\mathcal{Z}_{N,m}^2\right] \leq e^{2\alpha} \mathbb{E}\left[\mathcal{Z}_{N,m}\right]^2 (1 + o(1)),
\]  
\((2.31)\)
where \( \alpha \) is defined in (1.23).

\[
\mathbb{E} \left[ \mathcal{Z}_{N,m}^2 \right] = \mathbb{E} \left[ \sum_{\sigma^{(1)}, \sigma^{(2)} \in S_N[m]} \exp \left( \sum_{i<j} \frac{\beta}{Np} \delta_{ij} \left( \sigma_i^{(1)} \sigma_j^{(1)} + \sigma_i^{(2)} \sigma_j^{(2)} \right) \right) \right]
\]

\[
= \sum_{\sigma^{(1)}, \sigma^{(2)} \in S_N[m]} \prod_{i<j} \exp \left( \frac{\beta^2}{2N^2p^2} \left( \sigma_i^{(1)} \sigma_j^{(1)} + \sigma_i^{(2)} \sigma_j^{(2)} \right)^2 p(1-p) + o \left( \frac{\beta^2}{N^2} \right) \right)
\]

\[
\leq e^{-2N \beta_m} \exp \left( \frac{\beta^2}{p} (1-p) + o(1) \right) = e^{-2N \beta_m} \exp (4\alpha + o(1))
\]

where in the second line we used again (2.28) with \( x_{ij} = \frac{\beta}{Np} \left( \sigma_i^{(1)} \sigma_j^{(1)} + \sigma_i^{(2)} \sigma_j^{(2)} \right) \), while in the last line we used (2.29).

We recall the Paley–Zygmund inequality, which states that

\[
P(X \geq \eta \mathbb{E}X) \geq (1 - \eta)^2 \frac{\mathbb{E}X^2}{\mathbb{E}X^2},
\]

for any non negative random variable \( X \) and any \( \eta \in (0, 1) \). By (2.29), (2.32) and (2.33) we get, asymptotically as \( N \to \infty \),

\[
P \left( \frac{1}{N} \log \mathcal{Z}_{N,m} \geq \frac{1}{N} \log \left( \eta \mathbb{E} \mathcal{Z}_{N,m} \right) \right)
\]

\[
= P \left( \frac{1}{N} \log \mathcal{Z}_{N,m} \geq -I_N(m) + \frac{1}{N}(\alpha + \log \eta) + o \left( \frac{1}{N} \right) \right) \geq \frac{(1 - \eta)^2}{\exp (2\alpha + o(1))}.
\]

Moreover, after a change of variables in (2.21), we obtain \( \forall \, t > 0 \),

\[
P \left( \frac{1}{N} \log \mathcal{Z}_{N,m} - p_{N,m} \right) \geq t \leq c_1 \exp \left(- \frac{2c_2 N^2 p t^2}{\beta^2} \right),
\]

which implies

\[
P \left( \frac{1}{N} \log \mathcal{Z}_{N,m} \leq p_{N,m} + \frac{\eta \beta}{Np \sqrt{2c_2}} \right) \geq 1 - c_1 \exp(-t^2).
\]

Next we prove that the intersection of the events in (2.34) and (2.36) is non empty. Assuming, for \( \eta \in (0, 1) \), that

\[
P \left( \frac{1}{N} \log \mathcal{Z}_{N,m} \leq p_{N,m} + \frac{\eta \beta}{Np \sqrt{2c_2}} \right) > 1 - \frac{(1 - \eta)^2}{\exp (2\alpha + o(1))}
\]

and comparing (2.34) and (2.37), we notice that the sum of the probabilities of the two events

\[
\left\{ \frac{1}{N} \log \mathcal{Z}_{N,m} \leq p_{N,m} + \frac{\eta \beta}{Np \sqrt{2c_2}} \right\},
\]

and

\[
\left\{ \frac{1}{N} \log \mathcal{Z}_{N,m} \geq -I_N(m) + \frac{1}{N}(\alpha + \log \eta) + o \left( \frac{1}{N} \right) \right\}
\]
is strictly greater than 1. Therefore, they intersect in the not empty event
\[ -I_N(m) + \frac{1}{N}[\alpha + \log \eta] + o\left(\frac{1}{N}\right) \leq \frac{1}{N} \log Z_{N,m} \leq p_{N,m} + \frac{t\beta}{Np \sqrt{2c_2}} \]  
which is contained in the deterministic set
\[ -I_N(m) + \frac{1}{N}[\alpha + \log \eta] + o\left(\frac{1}{N}\right) \leq p_{N,m} + \frac{t\beta}{Np \sqrt{2c_2}} \]  
As a consequence, the latter set is non empty and, being deterministic,
\[ p_{N,m} \geq -I_N(m) + \frac{1}{N}[\alpha + \log \eta] - \frac{t\beta}{Np \sqrt{2c_2}} + o\left(\frac{1}{N}\right) \]  
holds with probability 1.

It remains to choose a suitable \( t > 0 \) for assumption (2.37) to hold. A sufficient condition is, for every \( \eta \in (0,1) \),
\[ c_1 \exp(-t^2) < \frac{(1 - \eta)^2}{\exp(2\alpha + o(1))}, \]  
namely
\[ t^2 > 2\alpha + \log\left(\frac{c_1}{(1 - \eta)^2}\right) + o(1). \]  
Therefore, we obtain, for every \( \eta \in (0,1) \),
\[ p_{N,m} \geq -I_N(m) + \frac{\alpha + \log \eta}{N} - \frac{\beta}{Np \sqrt{2c_2}} + o\left(\frac{1}{N}\right) \]  
where
\[ \kappa_\eta = \alpha + \log \eta - \frac{\beta}{p \sqrt{2c_2}} \]  
Notice that \( \kappa_\eta < \alpha \). In order to obtain the best lower bound, namely the closer to the upper bound proven in Lemma 2.2 we choose \( \eta \in (0,1) \) s.t. \( \alpha - \kappa_\eta \) is minimised and we conclude the proof. This choice motivates the maximum in the definition of \( \kappa \), in (1.23). \( \square \)

3. Capacity estimates

This section is entirely devoted to obtain upper and lower bounds on capacities between sets with a fixed magnetisation. These bounds are obtained via two dual variational principles, i.e. the *Dirichlet* and *Thomson principles* which are extensively discussed in Bovier and den Hollander [2]. The result will be expressed in terms of the capacity for the Curie–Weiss model, see (1.19). In particular, we prove Theorem 1.3 in Section 3.1 and Theorem 1.4 in Section 3.2.
3.1. **Asymptotics on capacity: upper bound.** In this section we prove Theorem 1.3 obtaining the upper bound on the capacity of the RDCW model in terms of the capacity of the CW model.

**Proof of Theorem 1.3** The main idea of the proof is to find an upper bound on the capacity via the following Dirichlet principle (see Bovier and den Hollander [2] Section 7.3.1) for details)

\[ \text{cap}(S_N[m_1], S_N[m_2]) = \min_{f \in \mathcal{H}} \sum_{\sigma, \sigma' \in S_N} \mu_{\beta, N}(\sigma) \rho_N(\sigma, \sigma')[f(\sigma) - f(\sigma')]^2, \]  

where

\[ \mathcal{H} = \{ f : S_N \to [0, 1] \text{ s.t. } f|_{S_N[m_1]} = 0, f|_{S_N[m_2]} = 1 \}. \]

Later it will be clear that we can restrict the previous variational principle over the functions on the space \( \Gamma_N \), hence it is useful to define

\[ \tilde{\mathcal{H}} = \{ g : \Gamma_N \to [0, 1] \text{ s.t. } g(m_1) = 0, g(m_2) = 1 \}. \]

In order to simplify the notation we will often neglect the dependency on \( m_1, m_2 \) when this will not generate confusion. From (3.1), we have

\[ Z_{\beta, N} \text{cap}(S_N[m_1], S_N[m_2]) \]

\[ = \min_{f \in \mathcal{H}} \frac{1}{N} \sum_{\sigma, \sigma' \in S_N} 1_{\sigma=\sigma'} \exp(-\beta H_N(\sigma)) \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) [f(\sigma) - f(\sigma')]^2 \]

\[ = \min_{f \in \tilde{\mathcal{H}}} \frac{1}{Z_{\beta, N}^N} \sum_{m, m' \in \Gamma_N} \sum_{\sigma \in S_N[m], \sigma' \in S_N[m']} \exp(-\beta N \mu_{\sigma, \sigma'}) \exp(-\beta N [E(\sigma') - E(\sigma)])_+ \]

\[ \times [f(\sigma) - f(\sigma')]^2 \exp(-\beta \Delta(\sigma)) \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) \]

\[ \leq \min_{g \in \tilde{\mathcal{H}}} \frac{1}{Z_{\beta, N}^N} \sum_{m, m' \in \Gamma_N} \exp(-\beta N \mu_{\sigma, \sigma'}) \exp(-\beta N [E(\sigma') - E(\sigma)])_+ \]

\[ \times \sum_{\sigma \in S_N[m]} \sum_{\sigma' \in S_N[m']} 1_{\sigma=\sigma'} \exp(-\beta \Delta(\sigma)) \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) \]

\[ \times \exp(-\beta \Delta(\sigma)) \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) \exp(-\beta N [E(\sigma') - E(\sigma)])_+. \]  

(3.4)

We turn now to the last sum in (3.4) and call this quantity \( G(\sigma, m') \). If \( \sigma \sim \sigma' \), then \( \sigma \) and \( \sigma' \) differ on a single state, say \( \ell \in [N] \), i.e. \( \forall i \in [N] \setminus \{\ell\}, \sigma_i = \sigma'_i \) and \( \sigma_\ell = -\sigma'_\ell \). Thus, setting \( m = m(\sigma) \), we can write

\[ \Delta(\sigma') - \Delta(\sigma) = -\frac{2}{Np} \sum_{i \neq \ell} \hat{J}_{\ell}(\sigma') \hat{J}_{\ell}(\sigma'_\ell) = \sigma_\ell \frac{2}{N} \sum_{i \neq \ell} \hat{J}_{\ell}(\sigma_i) \hat{J}_{\ell}(\sigma'_\ell), \]

(3.5)

\[ \hat{H}_N(\sigma') - \hat{H}_N(\sigma) = \sigma_\ell \frac{2}{N} \sum_{i \neq \ell} \sigma_i + 2h = \sigma_\ell \left[ \frac{2}{N}(Nm - \sigma_\ell) + 2h \right], \]

(3.6)

\[ H_N(\sigma') - H_N(\sigma) = \hat{H}_N(\sigma') - \hat{H}_N(\sigma) + \Delta(\sigma') - \Delta(\sigma) = \sigma_\ell \left[ \frac{2}{N} \sum_{i \neq \ell} \hat{J}_{\ell}(\sigma_i) + 2h \right]. \]

(3.7)

Due to the presence of the indicator function \( 1_{\sigma=\sigma'} \), \( G(\sigma, m') \) vanishes if \( m' \notin \{m \pm \frac{2}{N} \} \). Moreover, we can rewrite the sum \( \sum_{\sigma' \in S_N[m']} 1_{\sigma=\sigma'} \) in terms of the single index \( \ell \in [N] \) on which \( \sigma \) and \( \sigma' \) differ. Notice that if \( m(\sigma') = m + \frac{2}{N} \) then \( \sigma_\ell = -1 = -\sigma'_\ell \) and if
$m(\sigma^r) = m - \frac{2}{N}$ then $\sigma_\ell = 1 = -\sigma_\ell^r$. Therefore, calling $i^s(\sigma) := \{ j \in [N] : \sigma_j = \pm 1 \}$, we obtain

$$G(\sigma, m + \frac{2}{N}) = \sum_{i \in \iota(\sigma)} \exp\left( -\beta \left[ -\frac{2}{Np} \sum_{i \in \iota(\sigma)} J_{li}(\sigma_i - 2h_i) \right] \right) \leq N \frac{1 - m}{2} e^{2\beta},$$  \hspace{1cm} (3.8)

$$G(\sigma, m - \frac{2}{N}) = \sum_{i \in \iota^*(\sigma)} \exp(-\beta \left[ \frac{2}{N} \sum_{i \in \iota(\sigma)} J_{li}(\sigma_i + 2h_i) \right]) \exp(-\beta \left[ \frac{2}{N} (Nm + 1) - 2h \right]) \leq N \frac{1 - m}{2} e^{2\beta},$$  \hspace{1cm} (3.9)

where in (3.8) we have used the following elementary facts holding asymptotically in $N$,

$$\exp\left( -\beta \left[ -\frac{2}{Np} \sum_{i \in \iota(\sigma)} J_{li}(\sigma_i - 2h_i) \right] \right) \leq 1,$$  \hspace{1cm} (3.10)

$$\exp\left( \beta \left[ -\frac{2}{N} (Nm + 1) - 2h \right] \right) \leq \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right) \leq e^{2\beta}.$$  \hspace{1cm} (3.11)

Similar inequalities hold for the terms in (3.9). Thus, by (3.8), (3.9) and Proposition 2.4 we obtain

$$Z_{\beta,N} \text{cap} (S_N[m_1], S_N[m_2])$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta NE(m)) \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right) \leq e^{2\beta} \sum_{\sigma \in S_N[m]} \exp(-\beta \Delta(\sigma)) \left[ N^{1 - m} \mathbb{1}_{m - \frac{2}{N}}(m') + N^{1 - m} \mathbb{1}_{m + \frac{2}{N}}(m') \right]$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta NE(m)) \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta NE(m)) \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq \min_{g \in \mathcal{G}} \bar{Z}_{\beta,N} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

We conclude the upper bound by applying Property 2.6 obtaining

$$Z_{\beta,N} \text{cap} (S_N[m_1], S_N[m_2])$$

$$\leq e^{+2\beta + \alpha} \bar{Z}_{\beta,N} \min_{g \in \mathcal{G}} \sum_{m,m' \in \Gamma_N} \exp(-\beta \left[ E(m') - E(m) \right]_+) \exp\left( \beta \left[ -2m - \frac{2}{N} - 2h \right] \right)$$

$$\leq e^{+2\beta + \alpha} \bar{Z}_{\beta,N} \text{cap}^{\text{GW}} (S_N[m_1], S_N[m_2]) (1 + o(1)),$$  \hspace{1cm} (3.13)

where we used the notation (3.4). Furthermore, we noticed that the variational form appearing in the previous inequality is given by the Dirichlet principle applied to the CW model and therefore it is equal to the capacity of the CW model.

### 3.2. Asymptotics on capacity: lower bound

In this section we prove Theorem 1.4, obtaining the lower bound on the capacity of the RDCW model in terms of the capacity of the CW model.

The main idea of the proof is to find a lower bound in terms of the capacity of CW via the Thomson principle using a suitably defined unitary flow (see Bovier and den Hollander).
Section 7.3.2]. Let us denote by $\mathcal{U}_{S_N[m_1], S_N[m_2]}$ the space of all unit flows from $S_N[m_1]$ to $S_N[m_2]$. For all $\sigma, \sigma' \in S_N$, we define the candidate flow $\Psi_N$ as follows

$$
\Psi_N(\sigma, \sigma') = \phi_N(m(\sigma), m(\sigma')),
$$

where, for all $m \in \Gamma_N$,

$$
\phi_N(m, m') = \begin{cases} 
\left[ \frac{(1-m)N}{2} \exp(-NI(m)) \right]^{-1} & \text{if } m_1 \leq m \leq m_2 - \frac{2}{N} \text{ and } m' = m + \frac{2}{N}, \\
0 & \text{otherwise.}
\end{cases}
$$

(3.15)

The proof of Theorem 1.4 is postponed after three technical intermediate results which are essential for it. The following lemma allows us to use $\Psi_N$ in the Thomson principle.

**Lemma 3.1.** The flow $\Psi_N$ on $S_N$, defined in (3.14) is a unitary flow from $S_N[m_1]$ to $S_N[m_2]$, i.e. $\Psi_N \in \mathcal{U}_{S_N[m_1], S_N[m_2]}$.

**Proof.** Let us first prove that the Kirchhoff law holds, i.e., for all $\bar{\sigma} \in S_N \setminus (S_N[m_1] \cup S_N[m_2])$

$$
\sum_{\sigma \in S_N : \sigma \sim \bar{\sigma}} \Psi_N(\sigma, \bar{\sigma}) = \sum_{\sigma' \in S_N : \sigma' \sim \bar{\sigma}} \Psi_N(\bar{\sigma}, \sigma').
$$

(3.16)

For all $\bar{\sigma} \in S_N \setminus (S_N[m_1] \cup S_N[m_2])$, letting $\bar{m} = m(\bar{\sigma})$, the left hand side of (3.16) is

$$
\sum_{\sigma \in S_N : \sigma \sim \bar{\sigma}} \phi_N(m(\sigma), \bar{m}) = \sum_{\sigma \in S_N : \sigma \sim \bar{\sigma}, m(\sigma) = \bar{m}} \phi_N(m(\sigma), \bar{m}) + \sum_{\sigma \in S_N : \sigma \sim \bar{\sigma}, m(\sigma) = \bar{m} - \frac{2}{N}} \left[ \frac{(1-m)N}{2} \exp(-NI(m)) \right]^{-1}.
$$

(3.17)

After a similar computation for the right hand side of (3.16), we get that $\Psi_N$ satisfies (3.16) if and only if, $\forall \bar{m} \in \Gamma_N \setminus \{m_1, m_2\}$,

$$
\frac{(1+\bar{m})N}{2} \left[ \frac{(1-(\bar{m}-\frac{2}{N}))N}{2} \exp(-NI(\bar{m} - \frac{2}{N})) \right]^{-1} = \frac{(1+\bar{m})N}{2} \left[ \frac{(1-\bar{m})N}{2} \exp(-NI(\bar{m})) \right]^{-1},
$$

(3.18)

which indeed holds. We are left to show that $\Psi_N$ is unitary from $S_N[m_1]$ to $S_N[m_2]$, namely

$$
\sum_{a \in S_N[m_1]} \sum_{\sigma' \in S_N : a \sim \sigma'} \Psi_N(a, \sigma') = \sum_{b \in S_N[m_2]} \sum_{\sigma \in S_N : \sigma \sim b} \Psi_N(\sigma, b).
$$

(3.19)

The left hand side of (3.19) equals

$$
\sum_{a \in S_N[m_1]} \sum_{\sigma' \in S_N : a \sim \sigma'} \phi_N(m(a), m(\sigma')) = \sum_{a \in S_N[m_1]} \sum_{\sigma' \in S_N : a \sim \sigma', m(\sigma') = a + \frac{2}{N}} \left[ \frac{(1-m_1)N}{2} \exp(-NI(m_1)) \right]^{-1}
$$

(3.20)

$$
= \frac{(1-m_1)N}{2} \exp(-NI(m_1)) \left[ \frac{(1-m_1)N}{2} \exp(-NI(m_1)) \right]^{-1} = 1.
$$

After a similar computation for the right hand side, we obtain that (3.19) is satisfied. □
Lemma 3.2. For all $\sigma \in S_N$, the following holds
\[
\sum_{\sigma' \in S_N[m]} \frac{\exp(-\beta \tilde{H}_N(\sigma') - \tilde{H}_N(\sigma))}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]) + 1} \leq e^{2\beta(1+h)} \left[N \frac{1 + m(\sigma)}{2} \mathbb{1}_{m(\sigma)\neq \hat{m}}(m') + N \frac{1 - m(\sigma)}{2} \mathbb{1}_{m(\sigma)\neq \hat{m}}(m') \right].
\] (3.21)

Proof. Let $m = m(\sigma)$. The left hand side is non-zero only if $m' \in \{m + \frac{2}{N}, m - \frac{2}{N}\}$.

Recalling the definition $i^+(\sigma) = \{j \in [N]: \sigma_j = \pm 1\}$, if $m' = m + \frac{2}{N}$, we have
\[
\sum_{\sigma' \in S_N[m + \frac{2}{N}]} \frac{\exp(-\beta \tilde{H}_N(\sigma') - \tilde{H}_N(\sigma))}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]) + 1} = \sum_{\ell \in i^+} \frac{\exp(-\beta \tilde{H}_N(\sigma') - \tilde{H}_N(\sigma))}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]) + 1} 
\leq \sum_{\ell \in i^+} \exp \left( -\beta \frac{2}{N} \sum_{i \in \ell} J_{il} \sigma_i - 2h \right) \leq \sum_{\ell \in i^+} e^{2\beta} = N \frac{1 - m}{2} e^{2\beta},
\] (3.22)

where we have used that, since $h > 0$,
\[
-\frac{2}{N} \sum_{i \in \ell} J_{il} \sigma_i - 2h \leq \frac{2}{N} \sum_{i \in \ell} J_{il} \sigma_i \mid \leq \frac{2(N - 1)}{N} \leq 2.
\] (3.23)

Similarly, if $m' = m - \frac{2}{N}$, we get
\[
\sum_{\sigma' \in S_N[m - \frac{2}{N}]} \frac{\exp(-\beta \tilde{H}_N(\sigma') - \tilde{H}_N(\sigma))}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]) + 1} = \sum_{\ell \in i^+} \frac{\exp(-\beta \tilde{H}_N(\sigma') - \tilde{H}_N(\sigma))}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]) + 1} \leq \sum_{\ell \in i^+} e^{2\beta} = N \frac{1 + m}{2} e^{2\beta}. \quad (3.24)
\]

In the proof of Theorem 1.4 we will need an upper bound on $\sum_{\sigma \in S_N[m]} \exp(\beta \Delta(\sigma))$. Noticing the analogy with (2.6) one proves the following Lemma, which is very similar to Proposition 2.4. We introduce the following overlined notation, in analogy to (2.6)-(2.8),
\[
\overline{Z}_{N,m} = \sum_{\sigma \in S_N[m]} e^{\beta \Delta(\sigma)} = 1 \frac{N}{N} \log \overline{Z}_{N,m}, \quad \overline{F}_{N,m} = \mathbb{E}(\overline{F}_{N,m}).
\] (3.25)

Lemma 3.3. Asymptotically as $N \to \infty$,
\[
\sum_{\sigma \in S_N[m]} e^{\beta \Delta(\sigma)} \leq e^\alpha \exp(N(\overline{F}_{N,m} - \overline{F}_{N,m}) + NI_N(m))(1 + o(1)),
\] (3.26)

where $\alpha$ is defined in (1.23). Moreover, $N(\overline{F}_{N,m} - \overline{F}_{N,m})$ is a sub-Gaussian random variable.

Proof. It is sufficient to go along the whole proof of Proposition 2.4 replacing carefully $\beta$ with $-\beta$. Notice that all results concerning bounds on $\overline{F}_{N,m}$ and properties of $\overline{F}_{N,m} - \overline{F}_{N,m}$ have to be proven as well, but the proofs carry trivially out. □
With the previous lemmas at hand, we are now ready to prove the lower bound on the capacity.

**Proof of Theorem 7.4.** By the Thomson principle (see Bovier and den Hollander [2, Section 7.3.2]) we have

$$\text{cap} (S_N[m_1], S_N[m_2]) \geq \sup \left\{ \frac{1}{D(\Psi)} : \Psi \in \mathcal{U}_{S_N[m_1], S_N[m_2]} \right\} \geq \frac{1}{D(\Psi_N)},$$

(3.27)

where $\Psi_N$ is the test flow we defined in (3.14), which by Lemma 3.1 is in $\mathcal{U}_{S_N[m_1], S_N[m_2]}$. Thus, we are interested in upper bounds on

$$D(\Psi_N) = \frac{1}{2} \sum_{\sigma, \sigma' \in S_N} \mathbb{I}_{\sigma' < \sigma} \mu_{\text{GL}}(\sigma, \sigma')^2 \Psi_N(\sigma, \sigma')^2 = \frac{N}{2} \sum_{\sigma, \sigma' \in S_N} \mathbb{I}_{\sigma' < \sigma} \phi_N(m(\sigma), m(\sigma'))^2 \exp(-\beta H_N(\sigma')) \exp(-\beta \Delta(\sigma)) \exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+).$$

(3.28)

By multiplying and dividing by $\exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+) \tilde{Z}_\beta$, we get

$$D(\Psi_N) = \frac{N}{2} \tilde{Z}_\beta \sum_{m, m' \in \mathcal{E}^N} \tilde{\mu}_{\text{GL}}(m) \exp(-\beta N [E(m') - E(m)]_+) \sum_{\sigma \in S_N[m]} \exp(\beta \Delta(\sigma)) \times \sum_{\sigma' \in S_N[m']} \mathbb{I}_{\sigma' < \sigma} \frac{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+)}{\exp(-\beta [H_N(\sigma') - H_N(\sigma)]_+)}. \quad (3.29)$$

We use Lemma 3.3 to bound the sum over $\sigma$ and Lemma 3.2 to bound the sum over $\sigma'$, obtaining

$$D(\Psi_N) = \frac{1}{2} \tilde{Z}_\beta \sum_{m, m' \in \mathcal{E}^N} \frac{\phi_N(m, m')^2}{\tilde{Q}_\beta(m, m')} \exp(N (\tilde{F}_{N,m} - \tilde{F}_{N,m'} - 2N I_N(m))) \times \left[N \frac{1 + m}{2} 1_{m \leq \frac{N}{2}} (m') + N \frac{1 - m}{2} 1_{m \geq \frac{N}{2}} (m') \right]^2 (1 + o(1)).$$

(3.30)

Substituting into (3.28) the flow $\Psi_N$ defined in (3.14) - (3.15) and using Property 2.6, we obtain

$$D(\Psi_N) = \frac{Z_{\beta}}{2 \tilde{Z}_\beta} \sum_{m_1 \leq m \leq m_2} \frac{1}{\tilde{Q}_\beta(m, m + \frac{N}{2})} \exp(N (\tilde{F}_{N,m} - \tilde{F}_{N,m'}))(1 + o(1)) \leq \frac{Z_{\beta,N}}{2 \tilde{Z}_\beta} \sum_{m_1 \leq m \leq m_2} \frac{1}{\tilde{Q}_\beta(m, m + \frac{N}{2})} (1 + o(1), \quad (3.31)$$

where we used the notation (1.34). Therefore, by (3.27) and (3.31), we obtain

$$Z_{\beta,N} \text{cap} (S_N[m_1], S_N[m_2]) \geq \frac{Z_{\beta,N}}{D(\Psi_N)} \geq \tilde{Z}_{\beta,N} e^{2(1+h) - \alpha} \left[ \frac{1}{2} \sum_{m_1 \leq m \leq m_2} \frac{1}{\tilde{Q}_\beta(m, m + \frac{N}{2})} \right]^{-1} (1 + o(1)) \quad (3.32)$$

$$= \tilde{Z}_{\beta,N} e^{-2(1+h) - \alpha} \text{cap}^\text{CW} (S_N[m_1], S_N[m_2]) (1 + o(1), \quad (3.33)$$
where we used the notation \((1.34)\) and we noticed that the inverse of the expression appearing in brackets in \((3.32)\) gives exactly the capacity for the CW model. Indeed, to compute the capacity for the CW model with Glauber dynamics, one can simply notice that it is equivalent to a one-dimensional random walk in \(\Gamma_N\) and use the formula for the capacity in Bovier and den Hollander [2, Section 7.1.4].

\[\square\]

4. Estimates on the harmonic function

As pointed out in Section [1.4], the proof of Theorem [1.2] relies on sharp estimates on capacities, carried out in Section [3] and estimates on the harmonic function. We entirely devote this section to obtain asymptotic upper and lower bounds on the numerator in \((1.36)\), which is given by the following sum

\[
\sum_{\sigma \in S_N} \mu_{\beta,N}(\sigma) h_{m,m_\ast}(\sigma),
\]

that is to give the proof of Theorem [1.5].

In order to control the sum \((4.1)\), one generally uses a renewal argument which relies again on estimates over capacities. However, in our case this is not possible, due to the fact that capacities of single spins are too small.

We first prove the upper bound and then give some details about how to prove the lower bound, which is very similar and more straightforward. Our proof follows Bianchi, Bovier and Ioffe [1, Section 6].

4.1. Notation and decomposition of the space. Before starting with the proof, we introduce some notation. We refer to Figure [1] below for a better visual understanding of the objects we are defining.

Recall that we denote by \(m_+\) the global minimum, by \(m_-\) the local minimum, and by \(m^*\) the local maximum of \(f_\beta(\cdot)\) in \([-1, 1]\), where \(f_\beta(\cdot) = \lim_{N \to \infty} f_{\beta,N}(\cdot)\), defined in \((1.12)\).

We want to decompose the space \(\Gamma_N\) (and eventually the set of spin configurations \(S_N\)) according to the values of \(f_\beta\). The notation and the decomposition is organised in 4 steps.

Step 1. First, let \(\delta > 0\) be small in a way which will become clear later, and define the set

\[
U_\delta = \{m \in [-1, 1] : f_\beta(m) \leq f_\beta(m_-) + \delta\}.
\]

We write \(U_\delta^c = [-1, 1] \setminus U_\delta\) and we denote by \(U_\delta(m)\) the connected component of \(U_\delta\) containing \(m\). Note that \([m_-, m_+] \in U_\delta\). In general, \(U_\delta(m_-)\) and \(U_\delta(m_+)\) may have non empty intersection, but we choose \(\delta\) such that \(m^* \notin U_\delta\), implying that \(U_\delta\) is partitioned by the disjoint sets \(U_\delta(m_-)\) and \(U_\delta(m_+)\). For this to hold, it suffices to take \(\delta < f_\beta(m^*) - f_\beta(m_-)\).

Furthermore, let us denote by \(m_\delta\) the unique point in \((m_-, m_+)\) such that

\[
f_\beta(m_\delta) = f_\beta(m_-) + \delta.
\]

Step 2. With \(\delta\) chosen as above, we define a sequence \((\delta_N)_{N \in \mathbb{N}}\), converging to \(\delta\) from below, such that the left extreme of \(U_{\delta_N}(m_+)\) is in \(\Gamma_N\). Specifically, we define \(\delta_N\) as follows:

\[
\delta_N = \max \{\delta \in (0, \delta] : \exists m \in U_{\delta}(m_+) \cap \Gamma_N \setminus [m_-, 1] \text{ s.t. } f_\beta(m) = f_\beta(m_-) + \delta\},
\]

for \(N\) sufficiently large. Moreover, set

\[
U_{\delta,N} = U_{\delta,N} \cap \Gamma_N, \quad U_{\delta,N}^c = \Gamma_N \setminus U_{\delta,N} \quad \text{and} \quad U_{\delta,N}(m) = U_{\delta,N}(m) \cap \Gamma_N,
\]

for all \(m \in [-1, 1]\). Thus, we have the partition

\[
\Gamma_N = U_{\delta,N}(m_-) \cup U_{\delta,N}(m_+) \cup U_{\delta,N}^c.
\]
Remark. Notice that, for $N$ sufficiently large, $U_{\delta,N}(m_-(N)) = U_{\delta,N}(m_-(N))$ and $U_{\delta,N}(m_+(N)) = U_{\delta,N}(m_+(N))$. Furthermore, with these definitions, $m_{\delta'} \in U_{\delta,N}$ and it is the left extreme of $U_{\delta,N}(m_+)$. 

Step 3. Let $\varepsilon > 0$ be arbitrarily small (the choice of $\varepsilon$ will be relevant in Section 4.2). We denote by $m_{\varepsilon}$ the only point in a small left neighbourhood of $m_+$, more precisely in $U_{\delta}(m_+) \setminus [m_+, 1]$, such that

$$f_\beta(m_\varepsilon) = f_\beta(m_+) + \varepsilon.$$  

Let us define an $\varepsilon$-dependent parameter $\theta > 0$ by

$$\theta = m_\varepsilon - m_+.$$  

(4.8)

Step 4. Similarly to Step 2, fixed $\varepsilon > 0$, we want to define a sequence $(\varepsilon_N)_{N \in \mathbb{N}}$ converging to $\varepsilon$ from below such that $m_{\varepsilon_N}$ is in $\Gamma_N$. More precisely, we define $\varepsilon_N$ as follows

$$\varepsilon_N = \max \left\{ \varepsilon \in (0, \varepsilon] : \exists m \in U_{\delta,N}(m_+) \setminus [m_+, 1] \text{ s.t. } f_\beta(m) = f_\beta(m_+) + \varepsilon \right\}. \quad (4.9)$$

We will use later that $m_{\varepsilon_N} \in U_{\delta,N}(m_+)$ and it satisfies $f_\beta(m_{\varepsilon_N}) = f_\beta(m_+) + \varepsilon_N$.

Moreover, given $\varepsilon > 0$, we define the sequence $(\theta_N)_{N \in \mathbb{N}}$, analogously to (4.8), by setting $\theta_N = m_+(N) - m_{\varepsilon_N}$. $\theta_N$ plays an important role in Lemma 4.4 below. Notice that $\lim_{N \to \infty} \theta_N = \theta$ and, if $m_+ \neq m_+(N)$, then $f(m_{\varepsilon_N}) - f(m_+(N)) \neq \varepsilon_N$.

4.2. Upper bound on the harmonic sum. In this section we prove the first part of Theorem [1,5] by giving an upper bound on the harmonic sum in (4.1).

With the notation introduced in Steps 1–4 in Section 4.1 we partition $S_N$ as follows

$$S_N = S_N \left[ U_{\delta,N}(m_-) \right] \cup S_N \left[ m_+(N) \right] \cup S_N \left[ U_{\delta,N}(m_+) \right] \cup S_N \left[ U_{\delta,N}(m_+) \setminus \{m_+(N)\} \right]. \quad (4.10)$$

We will estimate the contribution of each of these sets to the sum in (4.1). As one expects, the only relevant contribution will be given by the terms in $S_N[U_{\delta,N}(m_-)]$. Indeed, $\mu_{\delta,N}$ is very small in $S_N[U_{\delta,N}(m_+)]$ while $h_{m_- m_+}^N$ is very small in $S_N[U_{\delta,N}(m_+)]$ and we will see the two contributions on these two sets turn out to be irrelevant.
The main ingredients in the proof of the upper bound are Lemma 4.1 and Lemma 4.3. The proof of the latter result is quite technical and it is postponed to Section 4.3. We now state Lemma 4.1 which allows us to find bounds on the random mesoscopic measure $Q_{\beta,N}$ in terms of the infinite volume CW free energy $f_{\beta}$. 

**Lemma 4.1.** For all $m \in (-1, 1)$, for all $s > 0$, 

\[
Q_{\beta,N}(m) \leq \frac{\mathcal{P}(s)}{Z_{\beta,N}} \exp \left(-\beta N f_{\beta}(m)\right) \sqrt{\frac{2}{\pi N(1 - m^2)}} (1 + o(1))
\]  

and, for $m \in \{1, -1\}$,

\[
Q_{\beta,N}(m) \leq \frac{\mathcal{P}(s)}{Z_{\beta,N}} \exp \left(-\beta N f_{\beta}(m)\right) (1 + o(1)),
\]

where we used the notation (1.34).

The proof of Lemma 4.1 is straightforward and uses Corollary 2.5, Property 2.6 and the following useful expansion.

**Lemma 4.2.** For $m \in (-1, 1)$,

\[
e^{-\beta N f_{\beta,N}(m)} = e^{-\beta N f_{\beta}(m)} (1 + o(1)) \sqrt{\frac{2}{\pi N(1 - m^2)}}
\]

and for $m \in \{1, -1\}$, $f_{\beta,N}(m) = f_{\beta}(m)$.

**Proof of Theorem 1.5** Upper bound. We are ready to start estimating the contributions on each disjoint set of the partition.

**Part 1. Sum on $S_N[U_{\delta,N}(m_{-})]$.** This will be the relevant part. Using that $h^N_{m_{-},m_{-}}(\sigma) = 1$ when $m(\sigma) = m_{-}(N)$, the fact that $\mathbb{P}_\sigma (\tau_{S_N[m_{-}(N)]} < \tau_{S_N[m_{+}(N)]}) = 0$ for $\sigma \in S_N$ with $m(\sigma) < m_{-}(N)$, and Lemma 4.1, we obtain

\[
\sum_{\sigma \in S_N[U_{\delta,N}(m_{-})]} \mu_{\beta,N}(\sigma) h^N_{m_{-},m_{-}}(\sigma) \leq \sum_{m \in U_{\delta,N}(m_{-})[1-m_{-}(N)]} Q_{\beta,N}(m)
\]

\[
\leq \frac{\mathcal{P}(s)}{Z_{\beta,N}} \exp \left(-\beta N f_{\beta}(m)\right) \sqrt{\frac{2}{\pi N(1 - m^2)}}
\]

\[
= \frac{e^{\epsilon+\alpha}(1+o(1))}{Z_{\beta,N}} \sqrt{\frac{1}{1-m^2}} \beta f'_{\beta}(m_{-}).
\]

In the last step we first approximated, for $N$ sufficiently large, the sum with an integral and then applied the saddle point method (see, for instance de Bruijn [7] Chp 5.7), where $m_{-}$ is the maximum point of $-\beta f_{\beta}$ on the considered domain. More precisely,

\[
\sum_{m \in U_{\delta,N}(m_{-})[1-m_{-}(N)]} \exp \left(-\beta N f_{\beta}(m)\right) \frac{1}{\sqrt{1-m^2}}
\]

\[
\approx \frac{N}{2} \int_{m_{-}(N)}^{b} \exp \left(-\beta N f_{\beta}(x)\right) \frac{1}{\sqrt{1-x^2}} dx
\]

\[
= \exp \left(-\beta N f_{\beta}(m_{-})\right) \frac{1}{\sqrt{1-m^2}} \sqrt{\frac{\pi N}{2 \beta f'_{\beta}(m_{-})}} (1 + o(1)),
\]
where \( b \) is the right extreme of \( U_{\delta,N}(m_-) \).

**Part 2. Sum on** \( S_N[m_+(N)] \). Being by definition \( h_{m_-,m_+}^N(\sigma) = 0 \) for all \( \sigma \in S_N[m_+(N)] \), we trivially have

\[
\sum_{\sigma \in S_N[m_+(N)]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) = 0. \tag{4.16}
\]

**Part 3. Sum on** \( S_N[U_{\delta,N}^{c}] \).

Being \( h_{m_-,m_+}^N \leq 1 \), we have

\[
\sum_{\sigma \in S_N[U_{\delta,N}^{c}]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \leq \sum_{\sigma \in S_N[U_{\delta,N}^{c}]} \mu_{\beta,N}(\sigma) = \sum_{m \in U_{\delta,N}^{c}} Q_{\beta,N}(m). \tag{4.17}
\]

In order to bound \( Q_{\beta,N}(m) \), we use Lemma 4.1 and the following fact. Let \( j \in \{1, -1\} \). If \( j \in U_{\delta,N}^{c} \) then \( e^{-\beta N f_\delta(j)} \leq e^{-\beta N [\ell_{m_+(\delta_N^0)}] - \delta_N] \), by definition of \( U_{\delta,N}^{c} \). If \( \{1, -1\} \not\subset U_{\delta,N}^{c} \), then adding a positive term to the sum in the bracket yields an upper bound. Therefore,

\[
\sum_{\sigma \in S_N[U_{\delta,N}^{c}]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \leq \frac{e^{s+\tau} (1 + o(1))}{Z_{\beta,N}} \sum_{m \in U_{\delta,N}^{c} \setminus \{1, -1\}} \exp \left( - \beta N f_\delta(m) \right) \frac{2}{\pi N (1 - m^2)} \sum_{m \in U_{\delta,N}^{c} \setminus \{1, -1\}} \frac{1}{\sqrt{1 - m^2}} + 2 \bigg), \tag{4.18}
\]

where the last inequality holds by definition of \( U_{\delta,N}^{c} \).

**Part 4. Sum on** \( S_N[U_{\delta,N}(m_-) \setminus m_+(N)] \). Using (1.28) and the fact that, for any \( \sigma \in S_N \) such that \( m(\sigma) > m_+(N) \), \( \mathbb{P}_\sigma(\tau_{S_N[m_-(N)]} < \tau_{S_N[m_+(N)]}) = 0 \), we get

\[
\sum_{\sigma \in S_N[U_{\delta,N}(m_-) \setminus m_+(N)]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) = \sum_{\sigma \in S_N[m_-(N)]} \mu_{\beta,N}(\sigma) \mathbb{P}_\sigma (\tau_{S_N[m_+(N)]} < \tau_{S_N[m_-(N)]}) \tag{4.19}.
\]

Thus, applying Lemma 4.3 below, we have

\[
\sum_{\sigma \in S_N[U_{\delta,N}(m_-) \setminus m_+(N)]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \leq \exp \left( - \beta N (1 - \gamma) f_\delta(m_-) \right) \sum_{m \in[m_-(m_+),m_+(m_-)]} Q_{\beta,N}(m) \left[ \exp \left( \beta N (1 - \gamma) f_\delta(m) \right) \right] e^{-\beta N (1 - \gamma) \delta_N (1 + o(1)). \tag{4.20}}
\]
From the bound in Lemma 4.11 we obtain

\[
\sum_{\sigma \in S_N \{U_N(m_\sigma) \mid (m_\sigma(N))\}} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \leq \frac{e^{+\alpha} (1 + o(1))}{Z_{\beta,N}} \exp \left( -\beta N (1 - \gamma) f_\beta(m_-) \right) \exp \left( -\beta N (1 - \gamma) f_\beta(m_+) \right) \sum_{m \in \{m_{N_0}, m_\sigma(N)\}} \exp \left( -\beta N f_\beta(m) \right) \times \frac{2}{\pi N (1 - m^2)} \left\{ \exp \left( \beta N (1 - \gamma) f_\beta(m) \right) + e^{\beta N (1 - \gamma) 2\varepsilon_N + \ell_N(\theta_N)} \exp \left( \beta N (1 - \gamma) f_\beta(m_+) \right) \right\}
\]

\[
\leq \frac{e^{+\alpha} (1 + o(1))}{Z_{\beta,N}} \exp \left( -\beta N (1 - \gamma) f_\beta(m_-) \right) e^{-\beta N (1 - \gamma) \delta_N} N \sqrt{\frac{2}{\pi N (1 - m_+^2)}} \times \left\{ \exp \left( -\gamma \beta N f_\beta(m_+) \right) + e^{\beta N (1 - \gamma) 2\varepsilon_N + \ell_N(\theta_N)} \exp \left( -\beta N f_\beta(m_+) \right) \exp \left( \beta N (1 - \gamma) f_\beta(m_+) \right) \right\}
\]

\[
= \frac{e^{+\alpha} (1 + o(1))}{Z_{\beta,N}} \exp \left( -\beta N f_\beta(m_-) \right) \sqrt{\frac{2}{\pi (1 - m_+^2)}} \exp \left( -\gamma \beta N \left[ f_\beta(m_+) - f_\beta(m_-) \right] \right) \exp \left( -\gamma \beta N \left[ f_\beta(m_+) - f_\beta(m_-) \right] \right) \times \left\{ \exp \left[ -\beta N \left( \gamma \left[ f_\beta(m_+) - f_\beta(m_-) \right] + (1 - \gamma) (\delta_N - 2\varepsilon_N) - \ell_N(\theta_N) - \frac{\log(N)}{\beta N} \right) \right] \right\}.
\]

(4.21)

Now we prove that this part is not relevant compared to the right hand side of (4.14). In particular, we show that, for a certain choice of \( \gamma \),

\[
c_N = \gamma (f_\beta(m_+) - f_\beta(m_-)) + (1 - \gamma) (\delta_N - 2\varepsilon_N) - \ell_N(\theta_N) - \varepsilon_N
\]

is positive and its limit,

\[
\lim_{N \to \infty} c_N = \gamma (f_\beta(m_+) - f_\beta(m_-)) + (1 - \gamma) (\delta - 2\varepsilon) - \frac{\theta}{2} (\log(2) + 3 - \log(1 - m_+)) - \varepsilon,
\]

(4.23)

is positive and finite. In order to achieve this, we choose \( \gamma \in (0, 1) \) small enough, such that \( c_N \) and its limit are positive, definitely in \( N \). In particular, we want to impose

\[
0 < \gamma < \frac{\delta_N - 3\varepsilon_N - \ell_N(\theta_N)}{f_\beta(m_+) - f_\beta(m_-) + \delta_N - 2\varepsilon_N} < 1,
\]

(4.24)

definitely in \( N \), and

\[
0 < \gamma < \frac{\delta - 3\varepsilon - \lim_{N \to \infty} \ell_N(\theta)}{f_\beta(m_+) - f_\beta(m_-) + \delta - 2\varepsilon} < 1.
\]

(4.25)

First, we notice that it is easy to check that the previous quantities are strictly smaller than 1. Second, we want to show that a strictly positive \( \gamma \) satisfying (4.24), (4.25) exists. Note that \( \ell_N(\theta_N) \), defined in (4.39), has the following trivial upper bound for every \( N \),

\[
\ell_N(\theta_N) \leq \theta_N (\beta + \log 2 + O(\theta_N)).
\]

(4.26)

Thus, a sufficient condition is to choose, for \( N \) large enough, \( \gamma \geq \gamma_0 \), where

\[
\gamma_0 = \frac{\delta - 3\varepsilon - \theta (\beta + \log 2 + O(\theta))}{f_\beta(m_+) - f_\beta(m_-) + \delta},
\]

(4.27)
is clearly strictly positive. Indeed, we can choose $\varepsilon > 0$ sufficiently small for the numerator on the left hand side of (4.27) to be positive, while $\theta$ is small accordingly to $\varepsilon$. We conclude by obtaining, for $N$ sufficiently large,

\[
\sum_{\sigma \in S_N[U_{\delta N}(m_{\gamma})]} \mu_{\beta,N}(\sigma) h^N_{m_{\gamma}, m_\theta}(\sigma) \leq \frac{e^{\varepsilon + \alpha(1)}}{Z_{\beta,N}} \exp\left(-\beta N f_\beta(m_-) + c_N\right) \sqrt{\frac{2}{N(1-m_-^2)}},
\]

(4.28)

where $0 < c_N = O(1)$.

Conclusion.

With the previous bounds at hand, we are now ready to conclude the proof of the upper bound. Decomposing the sum over $S_N$ using (4.10), and inserting the estimates we computed above into (4.1), we obtain

\[
\sum_{\sigma \in S_N} \mu_{\beta,N}(\sigma) h^N_{m_{\gamma}, m_\theta}(\sigma)
\]

\[
\leq \frac{e^{\varepsilon + \alpha}}{Z_{\beta,N}} \exp\left(-\beta N f_\beta(m_-)\right) \left[ e^{-\beta N o_N} \left( \sum_{m \in U_{\delta N}(1,-1)} \frac{1}{\sqrt{1-m^2}} + 2 \right) + \sqrt{\frac{2}{\pi (1-m_-^2)}} e^{-\beta N o_N} + \frac{1}{\sqrt{(1-m_-^2) \beta f''(m_-)}(1+o(1))} \right]
\]

(4.29)

concluding the proof. \hfill \Box

4.3. Some technical results. In this section we prove Lemma 4.3 which is pivotal in obtaining the upper bound in Theorem 1.5 (see (4.20)). The proof is quite involved, therefore we split it into subsequent technical results. Before starting the proof, we give a brief outline of this section. First, we state Lemma 4.3 and prove it via Lemmas 4.4, 4.5 and 4.6 which follow later on. Second, we give the proof of Lemmas 4.4 and 4.5. The latter relies on Lemma 4.6 which we subsequently prove using Lemma 4.7. We conclude the section proving Lemma 4.7. Throughout this section we will use the notation introduced in Section 4.1.

Lemma 4.3. For all $\sigma \in S_N[[m_{\delta_N}, m_\gamma(N)]]$, for all $\gamma \in (0, 1)$ and $\varepsilon > 0$,

\[
P_{\sigma}\left(\tau_{S_N[m_{\gamma}(N)]} < \tau_{S_N[m_{\gamma}(N)]}\right) \leq \exp\left(-\beta N(1-\gamma)[f_\beta(m_-) + \delta_N]\right)(1+o(1)) \times \exp\left(\beta N(1-\gamma)[f_\beta(m_\gamma) + 2\varepsilon_N + \ell_N(\theta_N)]\right),
\]

(4.30)

where $\ell_N(\cdot)$ is defined in (4.39).
Proof. For all $\sigma \in S_N \left[ [m_{\delta_N}, m_+] \right]$, we have
\[
\P_{\sigma} \left( \tau_{S_N[\{ \sigma \}]} < \tau_{S_N[m_{\delta_N}]}, \tau_{S_N[m_{\delta_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+]}, \tau_{m_{\epsilon_N}, m_+] < \tau_{S_N[m_{\epsilon_N}, m_+]} \right) \\
+ \sum_{\eta \in S_N[m_{\epsilon_N}]} \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right)
\]
\[
= \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \\
+ \sum_{\eta \in S_N[m_{\epsilon_N}]} \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \\
\times \P_{\sigma} \left( \tau_{m_{\epsilon_N} + \{ \sigma \} + \{ \eta \} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \} + \{ \eta \}] + \{ \sigma \}} \right),
\]
where we notice that,
\[
\P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \tau_{m_{\epsilon_N} + \{ \eta \}} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \} + \{ \eta \}] + \{ \sigma \}} \right) = \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right). \tag{4.32}
\]
Using the Markov property and taking the maximum of the first factor out of the sum, we have that, for all $\sigma \in S_N \left[ [m_{\delta_N}, m_+] \right]$,
\[
\P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \leq \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+]}, \tau_{m_{\epsilon_N}, m_+] < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right) \\
+ \left( \max_{\eta \in S_N[m_{\epsilon_N}]} \P_{\eta} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \eta \}]} \right) \right) \sum_{\eta \in S_N[m_{\epsilon_N}]} \P_{\sigma} \left( \tau_{m_{\epsilon_N} + \{ \eta \}} < \tau_{S_N[m_{\epsilon_N} + \{ \eta \} + \{ \sigma \}]} \right)
\]
\[
= \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+]}, \tau_{m_{\epsilon_N}, m_+] < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right) \\
+ \left( \max_{\eta \in S_N[m_{\epsilon_N}]} \P_{\eta} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \eta \}]} \right) \right) \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right)
\]
\[
\leq \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+]}, \tau_{m_{\epsilon_N}, m_+] < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right) \\
+ \left( \max_{\eta \in S_N[m_{\epsilon_N}]} \P_{\eta} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \eta \}]} \right) \right) \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right).
\]
We first consider the case $\sigma \in S_N[m_{\epsilon_N}]$. By Lemma 4.4, we get
\[
\P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \leq \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right) \\
+ \left( \max_{\eta \in S_N[m_{\epsilon_N}]} \P_{\eta} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \eta \}]} \right) \right) \left( 1 - e^{-B N f_\theta(\gamma)} (1 + o(1)) \right).
\tag{4.34}
\]
Taking the maximum over $\sigma$ and noticing that the same term appears in both right and left hand side of the inequality, we obtain, for all $\sigma \in S_N[m_{\epsilon_N}]$,
\[
\max_{\sigma \in S_N[m_{\epsilon_N}]} \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N} + \{ \sigma \}]} \right) \\
\leq \max_{\sigma \in S_N[m_{\epsilon_N}]} \P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}, m_+] + \{ \sigma \}} \right) e^{-B N f_\theta(\gamma)} (1 + o(1)) \\
\leq \max_{\sigma \in S_N[m_{\epsilon_N}]} \exp \left( -B N \left( 1 - \gamma \right) f_\beta \left( m_- \right) + \delta_N - f_\beta \left( m(\sigma) - \frac{\gamma}{\beta} \right) - \ell_N(\theta_N) \right) (1 + o(1)),
\tag{4.35}
\]
where we used Lemma 4.5.
By Taylor expansion of $f_\beta\left(m_{\text{ex}} - \frac{2}{N}\right)$ and definition of $m_{\text{ex}}$, we get
\[
\max_{\sigma \in S_N[m_{\text{ex}}]} \mathbb{P}_\sigma \left(\tau_{S_N[m_{\text{ex}}]} < \tau_{S_N[\{m_{\text{ex}}\}] = \tau_{S_N[m_{\text{ex}}]} \wedge \tau_{S_N[m_{\text{ex}}]} \right)
\]
\[
= \exp \left[-\beta N \left((1 - \gamma) \left[f_\beta(m_\sigma) + \delta_N - f_\beta\left(m_{\text{ex}} - \frac{2}{N}\right)\right] - \ell_N(\theta_N)\right] \right](1 + o(1)) \quad (4.36)
\]
\[
\leq \exp \left[-\beta N \left((1 - \gamma) \left[f_\beta(m_\sigma) + \delta_N - 2\epsilon_N - f_\beta(m_\sigma)\right] - \ell_N(\theta_N)\right] \right](1 + o(1)),
\]
where the last inequality holds for $N$ sufficiently large.

Now we consider the case where $\sigma \in S_N[[m_{\text{ex}}, m_+(N)) \setminus \{m_{\text{ex}}\}]$. Going back to \(4.33\) and using again \(4.36\) we obtain
\[
\mathbb{P}_\sigma \left(\tau_{S_N[m_{\text{ex}}]} < \tau_{S_N[m_{\text{ex}}]} \right) \leq \mathbb{P}_\sigma \left(\tau_{S_N[m_{\text{ex}}]} < \tau_{S_N[m_{\text{ex}}]} \right) + \exp \left[-\beta N \left((1 - \gamma) \left[f_\beta(m_{\text{ex}}) + \delta_N - 2\epsilon_N - f_\beta(m_{\text{ex}})\right] - \ell_N(\theta_N)\right] \right](1 + o(1))
\]
\[
\leq \exp \left[-\beta N \left((1 - \gamma) \left[f_\beta(m_{\text{ex}}) + \delta_N\right] - \ell_N(\theta_N)\right] \right](1 + o(1))
\]
\[
\times \left[\exp \left[\beta N \left(1 - \gamma\right) f_\beta(m(\sigma))\right] + \exp \left[\beta N \left(1 - \gamma\right) f_\beta(m_\sigma) + 2\epsilon_N + \ell_N(\theta_N)\right] \right].
\]
(4.37)

In the last inequality we used Lemma 4.6, which holds for $\sigma \in S_N[m_{\text{ex}}, m_+(N)]$, and that $\mathbb{P}_\sigma(\tau_{S_N[m_{\text{ex}}]} < \tau_{S_N[m_{\text{ex}}]}\right) = 0$ for all $\sigma \in S_N[m_{\text{ex}}, m_+(N)]$.

Remark. In Lemma 4.3 one might try to further bound the r.h.s. of \(4.30\) using that $f_\beta(m(\sigma))$ is bounded by $f_\beta(m_{\text{ex}}) + \delta_N$. This would yield to the trivial upper bound 1 on $\mathbb{P}_\sigma(\tau_{S_N[\{m_{\text{ex}}\}] < \tau_{S_N[\{m_{\text{ex}}\}]})$, which is not sufficient for our purpose of proving that the second term in \(4.29\) is negligible with respect to the last one. The way to go is, therefore, to keep the dependence on $m(\sigma)$ in order to obtain later a more suitable bound, uniform in $m$, by exploiting the smallness of $Q_{\beta,N}(m(\sigma))$ in \(4.20\) and \(4.21\).

In order for \(4.34\) to be true, we have to prove the following result.

**Lemma 4.4.** For all $\sigma \in S_N[m_{\text{ex}}]$, for $\epsilon$ sufficiently small and for $N$ sufficiently large,
\[
\mathbb{P}_\sigma(\tau_{S_N[\{m_{\text{ex}}\}] < \tau_{S_N[\{m_{\text{ex}}\}]}) \geq e^{-\beta N(\epsilon N)}(1 + o(1)),
\]
(4.38)
where $\ell_N : \mathbb{R} \to \mathbb{R}$ is defined by
\[
\ell_N(x) = \frac{1}{2} \left[x \left( \log 2 + \beta |2 - 2h| + 1 \right) - (1 - m_+(N) + x) \log(1 - m_+(N) + x) + (1 - m_+(N)) \log(1 - m_+(N)) \right].
\]
(4.39)

**Proof.** Recall that $\{\sigma(t)\}_{t \geq 0}$ is the Markov chain with transition probabilities \(1.5\) and, for $\sigma \in S_N$ with $m(\sigma) < m_+(N)$, let
\[
A_N(\sigma) = \left\{(\sigma(0), \sigma(1), \sigma(2), \ldots) : \sigma(0) = \sigma, \forall i \in \mathbb{N}, \sigma(i) \in S_N, \sigma(i) \sim \sigma(i + 1), \exists k \in \mathbb{N} \text{ s.t. } \sigma(k) \in S_N[m_+(N), \text{ and } \forall i \leq k - 1, m(\sigma(i + 1)) = m(\sigma(i)) + \frac{2}{N} \right\}
\]
(4.40)
be the set of infinite paths starting in $\sigma$ and having increasing magnetisation until the set $S_N[m_+(N)$ is reached.

Notice that, for fixed $\sigma$ and $N$, the number $k$ of steps of increasing magnetisation to reach $S_N[m_+(N)$ is fixed, namely $k = \frac{N}{2}(m_+(N) - m(\sigma))$.\]
We want to partition \( A_N(\sigma) \) according to the values of the first \( k + 1 \) elements of its paths. Given a sequence \( \pi \in S_N^{k+1} \), let us denote by \( \{\pi\} \) the set of all paths in \( A_N(\sigma) \) in which the first \( k + 1 \) elements are exactly given by \( \pi \), namely

\[
\{\pi\} = \{(\sigma(0), \sigma(1), \ldots, \sigma(k), \sigma(k+1), \ldots) \in A_N(\sigma) : (\sigma(0), \ldots, \sigma(k)) = \pi\}. 
\] (4.41)

Notice that, by definition of \( A_N(\sigma) \), \( \{\pi\} \) is empty for many \( \pi \in S_N^{k+1} \). We denote by \( B_N(\sigma) \) the set of all the sequences \( \pi \in S_N^{k+1} \) such that \( \{\pi\} \) is not empty. Thus, we obtain the following partition of \( A_N(\sigma) \)

\[
A_N(\sigma) = \bigsqcup_{\pi \in B_N(\sigma)} \{\pi\}. 
\] (4.42)

Fix \( \sigma \in S_N[m_{eN}] \), then one simply notices that

\[
\mathbb{P}_\sigma \left( \tau_{S_N[m_e(N)]} < \tau_{S_N[m_{eN}]} \right) \geq \mathbb{P}_\sigma (A_N(\sigma)) = \sum_{\pi \in B_N(\sigma)} \mathbb{P}_\sigma (\{\pi\}). 
\] (4.43)

Thus, we first find a lower bound on \( \mathbb{P}_\sigma (\{\pi\}) \) independent of \( \pi \) in \( B_N(\sigma) \) and later we compute the cardinality of \( B_N(\sigma) \). Fix \( \pi = (\sigma(0), \sigma(1), \sigma(2), \ldots, \sigma(k)) \in B_N(\sigma) \), then we have

\[
\mathbb{P}_\sigma (\{\pi\}) = \prod_{i=1}^{k} p_N(\sigma(i-1), \sigma(i)) = \frac{1}{N^k} \prod_{i=1}^{k} \exp\left(-\beta \left[ H(\sigma(i)) - H(\sigma(i-1)) \right] \right), 
\] (4.44)

where \( m_i = m(\sigma(i)) \), \( C = \exp(-\beta[2 - 2h]) \) and we used the following fact

\[
\frac{\exp\left(-\beta \left[ H(\sigma(i)) - H(\sigma(i-1)) \right] \right)}{\exp\left(-\beta N[E(m_i) - E(m_{i-1})] \right)} = \left[ \exp\left(-\beta \left[ -2m_{i-1} - \frac{2}{N} - 2h \right] \right) \right]^{\frac{1}{2}} 
\] (4.45)

\[
\geq \exp\left(-\beta \left[ 2 - 2h - \frac{2}{N} \right] \right) \geq \exp(-\beta[2 - 2h]),
\] where \( r \) is the index of the spin to be flipped to go from \( \sigma(i-1) \) to \( \sigma(i) \). Therefore, recalling that \( m_i \in [m_{eN}, m_e(N)] \), we obtain the following lower bound independent of \( \pi \)

\[
\mathbb{P}_\sigma (\{\pi\}) \geq \frac{C^k}{N^k} \prod_{i=1}^{k} \exp\left(-\beta \left[ -2m_{eN} - \frac{2}{N} - 2h \right] \right) = \frac{C^k}{N^k}. 
\] (4.46)

Indeed, for \( \varepsilon_N \) sufficiently small, \( m_{eN} \) is close to \( m_e(N) \) > 0, allowing us to assume \( m_{eN} > 0 \). Therefore, \(-2m_{eN} - \frac{2}{N} - 2h < 0\), which implies the last equality in (4.46).

We are left to compute the cardinality of \( B_N(\sigma) \), with \( \sigma \in S_N[m_{eN}] \), namely we have to count all paths from \( \sigma \) to \( S_N[m_{eN}(N)] \) with increasing magnetisation and length \( k + 1 \). Any of these paths is characterised by a final spin \( \tilde{\sigma} \in S_N[m_{eN}(N)] \) and a sequence of negative spins which are flipped. Notice that \( \tilde{\sigma} \) is reachable by \( \sigma \) through a path with increasing magnetisation if and only if the two following properties are satisfied: \( \tilde{\sigma} \) has \( k \) positive spins more than \( \sigma \) and, for all \( i \in [N] \), \( \sigma_i = +1 \) implies \( \tilde{\sigma}_i = +1 \). Thus, a configuration \( \tilde{\sigma} \in S_N[m_{eN}(N)] \) reachable by \( \sigma \) through a path with increasing magnetisation
Lemma 4.5. For $\sigma \in S_N[\varepsilon N]$, for $N$ sufficiently large and any $\gamma \in (0,1)$,
\[
\mathbb{P}_\sigma \left( \tau_{S_N[\varepsilon N]} < \tau_{S_N[\varepsilon N, m_N]} \right) \leq \exp \left( -\beta N(1 - \gamma) \left[ f_\beta(m_-) - \delta_N - f_\beta \left( m_{\varepsilon N} - \frac{2}{N} \right) \right] \right). \tag{4.53}
\]
Proof. Let us denote by $W_N(m)$ the event of making the first flip in $S_N[m]$. For $\sigma \in S_N[m_{\epsilon_N}]$, conditioning on the first step, we obtain

$$
P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) = P_{\sigma} \left( W_N(m_{\epsilon_N} + \frac{2}{N}) P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \mid W_N(m_{\epsilon_N} + \frac{2}{N}) \right) + P_{\sigma} \left( W_N(m_{\epsilon_N} - \frac{2}{N}) P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \mid W_N(m_{\epsilon_N} - \frac{2}{N}) \right) \right)
$$

$$= P_{\sigma} \left( W_N(m_{\epsilon_N} + \frac{2}{N}) \right) \sum_{\sigma' \in S_N[m_{\epsilon_N} + \frac{2}{N}, \sigma, \sigma']} P_{\sigma'} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) + P_{\sigma} \left( W_N(m_{\epsilon_N} - \frac{2}{N}) \right) \sum_{\sigma' \in S_N[m_{\epsilon_N} - \frac{2}{N}, \sigma, \sigma']} P_{\sigma'} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right).$$

(4.54)

The first term vanishes because all the probabilities in the sum are zero. Thus, we get the upper bound

$$P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) \leq \sum_{\sigma' \in S_N[m_{\epsilon_N} - \frac{2}{N}, \sigma, \sigma']} P_{\sigma'} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right).$$

(4.55)

Using first Lemma 4.6 and then Lemma 4.2, we obtain

$$P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) \leq \sum_{\sigma' \in S_N[m_{\epsilon_N} - \frac{2}{N}, \sigma, \sigma']} \exp \left( -\beta N (1 - \gamma) \left[f_{\beta, N}(m_{\epsilon}) + \delta_N - f_{\beta, N}(m(\sigma')) \right] \right)
$$

$$= N \frac{1 + m_{\epsilon}}{2} \sqrt{1 - \frac{\left(m_{\epsilon} - \frac{2}{N}\right)^2}{1 - m^2}} \exp \left( -\beta N (1 - \gamma) \left[f_{\beta}(m_{\epsilon}) + \delta_N - f_{\beta}\left(m_{\epsilon} - \frac{2}{N}\right) \right] \right)
$$

$$= \exp \left( -\beta N (1 - \gamma) \left[f_{\beta}(m_{\epsilon}) + \delta_N - f_{\beta}\left(m_{\epsilon} - \frac{2}{N}\right) \right] \right) \leq \exp \left( -\beta N (1 - \gamma) \left[f_{\beta}(m_{\epsilon}) + \delta_N - f_{\beta}\left(m_{\epsilon} - \frac{2}{N}\right) \right] \right).$$

(4.56)

In the proofs of Lemmas 4.3 and 4.5, we use the following fact.

**Lemma 4.6.** For $\sigma \in S_N[m_{\epsilon_N}, m_{\epsilon_N}]$, for $N$ sufficiently large and any $\gamma \in (0, 1),$

$$P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) \leq \exp \left( -\beta N (1 - \gamma) \left[f_{\beta}(m_{\epsilon}) + \delta_N - f_{\beta}(m(\sigma)) \right] \right).$$

(4.57)

**Proof.** For $\sigma \in S_N[m_{\delta_N}, m_{\epsilon_N}]$, we obtain

$$P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right) \leq P_{\sigma} \left( \tau_{S_N[m_{\epsilon_N}]} < \tau_{S_N[m_{\epsilon_N}]} \right),$$

(4.58)

being $m_{\epsilon}(N) < m^* < m_{\delta_N} < m_{\epsilon_N} < m_{\epsilon}(N)$. Therefore, we focus on finding an upper bound on the right hand side of (4.58). Assume that there exists a function $\psi$ superharmonic in $S_N[m_{\delta_N}, m_{\epsilon_N}]$. As a consequence, $0 > L\psi(\sigma) = \frac{\partial}{\partial t} \mathbb{E}_\sigma \left[ \psi(\sigma(t)) \right]$. This implies $\mathbb{E}_\sigma \left[ \psi(\sigma(s)) \right] \leq \mathbb{E}_\sigma \left[ \psi(\sigma(t)) \right]$, for all $s < t$. Take $s = 0$, and $\sigma(0) = \sigma$, therefore $\mathbb{E}_\sigma \left[ \psi(\sigma(t)) \right] \leq \psi(\sigma)$, for all $t > 0$. Thus, $\psi(\sigma(t))$ is a super-martingale. For the integrable stopping time $T = \tau_{S_N[m_{\epsilon_N}]} \wedge \tau_{S_N[m_{\epsilon_N}]}$, we use Doob’s optional stopping
theorem for super-martingales to show that, for all \( \sigma \) in the domain \( S_N[[m_{\delta_N}, m_{\epsilon_N}]] \) of \( \psi \),
\[ E_\sigma [\psi(\sigma(T))] \leq \psi(\sigma). \]
Therefore,
\[
\psi(\sigma) \geq E_\sigma [\psi(\sigma(T))] \\
\geq \min_{\sigma' \in S_N[m_{\delta_N}]} \psi(\sigma') \mathbb{P}_\sigma (\tau_{S_N[m_{\delta_N}] < \tau_{S_N[[m_\epsilon(N),m_{\epsilon_N}]]}}) \\
+ \min_{\sigma' \in S_N[[m_\epsilon(N),m_{\epsilon_N}]]} \psi(\sigma') \left[ 1 - \mathbb{P}_\sigma (\tau_{S_N[m_{\delta_N}] < \tau_{S_N[[m_\epsilon(N),m_{\epsilon_N}]]}}) \right] \tag{4.59}
\]
which implies that
\[
\mathbb{P}_\sigma (\tau_{S_N[m_{\delta_N}] < \tau_{S_N[[m_\epsilon(N),m_{\epsilon_N}]]}}) \leq \frac{\psi(\sigma)}{\min_{\sigma' \in S_N[m_{\delta_N}]} \psi(\sigma')} \tag{4.60}
\]
For a suitably chosen \( \psi \) the latter inequality will yield the desired upper bound. Now we are left with the choice of a suitable \( \psi : S_N \to \mathbb{R} \) such that \( L\psi(x) < 0 \), for all \( x \in S_N[[m_{\delta_N}, m_{\epsilon_N}]]. \) We define a function \( \psi \) which depends on a parameter \( \gamma \in (0, 1) \) and is constant on fixed magnetisation sets, i.e., for all \( \sigma \in S_N, \)
\[
\psi(\sigma) = \phi(m(\sigma)), \tag{4.61}
\]
where \( \phi : [-1, 1] \to \mathbb{R} \) is defined by
\[
\phi(m) = \exp \left( \beta N (1 - \gamma) f_\beta(m) \right). \tag{4.62}
\]
Our choice of \( \psi \) is similar to the one used in [1] Proposition 6.4, while the choice of \( \gamma \) is relevant in [4,24].

We claim and prove later in Lemma 4.7 that \( \psi \) is super-harmonic in \( S_N[[m_{\delta_N}, m_{\epsilon_N}]]. \)
Therefore, we conclude the proof by inserting \( \psi \) in (4.60) and obtaining
\[
\mathbb{P}_\sigma (\tau_{S_N[m_{\delta_N}] < \tau_{S_N[[m_\epsilon(N),m_{\epsilon_N}]]}}) \leq \frac{\exp \left( \beta N (1 - \gamma) f_\beta(m(\sigma)) \right)}{\min_{\sigma' \in S_N[m_{\delta_N}]} \exp \left( \beta N (1 - \gamma) f_\beta(m(\sigma')) \right)} \phan{4.63}
\]
where we used the definition of \( m_{\delta_N}. \)
\[ \square \]

We are now left with the proof of the super-harmonicity of \( \psi \), which is used in the proof of Lemma 4.6.

**Lemma 4.7.** \( \psi \) defined in (4.61) is super-harmonic in \( S_N[[m_{\delta_N}, m_{\epsilon_N}]]. \)

**Proof.** We have to prove that \( L\psi(x) < 0 \), for all \( x \in S_N[[m_{\delta_N}, m_{\epsilon_N}]]. \) Fix \( x \) in \( S_N[[m_{\delta_N}, m_{\epsilon_N}]] \) and use the notation \( \tilde{m} = m(x). \) As usual, we try to rewrite the terms appearing in the expression for \( L\psi(x) \) in terms of their mean-field version.
Therefore, we can rewrite (4.64) as

\[
L\psi(x) = \sum_{y \in \Sigma_N} p(x, y)[\psi(y) - \psi(x)]
\]

\[
= \frac{1}{N} \sum_{y \in \Sigma_N} \mathbb{I}_{y-x} \exp \left( -\beta[H(y) - H(x)]_+ \right)
\times \left[ \exp \left( \beta(1 - \gamma)N f_\beta(m(y)) \right) - \exp \left( \beta(1 - \gamma)N f_\beta(m(x)) \right) \right]
\]

\[
= \frac{1}{N} \sum_{m \in \Gamma_N} \exp \left( -\beta N[E(m) - E(\bar{m})]_+ \right)
\times \left[ \exp \left( \beta N(1 - \gamma)f_\beta(m) \right) - \exp \left( \beta N(1 - \gamma)f_\beta(\bar{m}) \right) \right]
\times \frac{1}{N} \sum_{y: m(y) = m} \mathbb{I}_{x-y} \exp \left( -\beta[H(\gamma) - H(x)]_+ \right)
\exp \left( -\beta N[E(m) - E(\bar{m})]_+ \right)
\]

\[
\leq \sum_{m \in \Gamma_N} \exp \left( -\beta N[E(m) - E(\bar{m})]_+ \right) \phi(\bar{m}) \left[ \exp \left( \beta N(1 - \gamma)[f_\beta(m) - f_\beta(\bar{m})] \right) - 1 \right]
\times e^{2\beta} \left[ \frac{1 + \bar{m}}{2} \mathbb{I}_{\bar{m}-\frac{N}{2}}(m) + \frac{1 - \bar{m}}{2} \mathbb{I}_{\bar{m}+\frac{N}{2}}(m) \right],
\]

(4.64)

where we used the upper bound \(\exp(2\beta)\) on \(G(\sigma, m')\) as in the proof of the upper bound on capacity (see (3.3), (3.9)).

Now, recalling definition (1.16), we use the following notation

\[
r_+ = \bar{r}_N \left( \bar{m}, \bar{m} + \frac{N}{2} \right) = \exp \left( -2\beta \left[ -\frac{p}{N} - (p\bar{m} + h) \right]_+ \right) \frac{1 - \bar{m}}{2},
\]

(4.65)

\[
r_- = \bar{r}_N \left( \bar{m}, \bar{m} - \frac{N}{2} \right) = \exp \left( -2\beta \left[ -\frac{p}{N} + p\bar{m} + h \right]_+ \right) \frac{1 + \bar{m}}{2},
\]

(4.66)

and, for all \(m \in \Gamma_N \setminus \{1\},\)

\[
g(m) = \frac{N}{2} \left[ f_\beta \left( m + \frac{N}{2} \right) - f_\beta(m) \right].
\]

(4.67)

Therefore, we can rewrite (4.64) as

\[
L\psi(x) \leq e^{2\beta} \phi(\bar{m}) r_+ \left[ \exp \left( 2\beta(1 - \gamma)g(\bar{m}) \right) - 1 \right]
+ e^{2\beta} \phi(\bar{m}) r_- \left[ \exp \left( -2\beta(1 - \gamma)g \left( \bar{m} - \frac{N}{2} \right) \right) - 1 \right]
\]

(4.68)

\[
= e^{2\beta} \phi(\bar{m}) r_+ G_+,
\]

where

\[
G_+ = \left( e^{2\beta(1 - \gamma)g(\bar{m})} - 1 \right) + \frac{r_-}{r_+} \left( e^{-2\beta(1 - \gamma)g \left( \bar{m} - \frac{N}{2} \right)} - 1 \right).
\]

(4.69)

Being \(e^{2\beta}, \phi(\bar{m})\) and \(r_+\) positive, we have only to show that \(G_+ < 0\). First we notice that

\[
g(m) = -pm - h + \frac{1}{\beta} f'(m) + O \left( \frac{1}{N} \right)
\]

(4.70)

and that a similar expansion holds for \(g \left( m - \frac{N}{2} \right)\). Therefore,

\[
g(m) - g \left( m - \frac{N}{2} \right) = -2h + \frac{1}{\beta} \frac{2}{N} f''(m) + O \left( \frac{1}{N} \right) = -2h + O \left( \frac{1}{N} \right).
\]

(4.71)
Then, recalling that $I'(m) = \frac{1}{T} \log \left( \frac{m + 1}{m - 1}\right)$, and using (4.70) we have
\[
\frac{r_-}{r_+} = \frac{1 + \tilde{m} \exp(2\beta \left[-\frac{p}{N} - (p\tilde{m} + h)\right])}{1 - \tilde{m} \exp(2\beta \left[-\frac{p}{N} + p\tilde{m} + h\right])} \\
= \frac{1 + \tilde{m} \exp(-2\beta(p\tilde{m} + h))(1 + O\left(\frac{1}{N}\right))}{1 - \tilde{m} \exp(-2\beta(p\tilde{m} + h))(1 + O\left(\frac{1}{N}\right))} \\
= \exp(2I'(\tilde{m}) - 2\beta(p\tilde{m} + h))(1 + O\left(\frac{1}{N}\right)) \\
= \exp(2\beta \left[g\left(\tilde{m} - \frac{2}{N}\right) + p\tilde{m} + O\left(\frac{1}{N}\right) - h\right] - 2\beta(p\tilde{m} + h))(1 + O\left(\frac{1}{N}\right)) \\
= \exp(2\beta g(\tilde{m} - \frac{2}{N}) - 4\beta h)(1 + O\left(\frac{1}{N}\right)).
\]
(4.72)

Therefore, rearranging (4.69) and using (4.71) and (4.72), we obtain
\[
G_* = \left[ \exp(2\beta(1 - \gamma)g(\tilde{m})) - 1 \right] \left[ 1 - \frac{r_-}{r_+} \exp(-2\beta(1 - \gamma)g(\tilde{m} - \frac{2}{N})) \right] \\
+ \frac{r_-}{r_+} \left[ \exp(2\beta(1 - \gamma)g(\tilde{m}) - g(\tilde{m} - \frac{2}{N})) \right] - 1 \\
= \left[ \exp(2\beta(1 - \gamma)g(\tilde{m})) - 1 \right] \left[ 1 - \exp(2\beta \gamma g(\tilde{m} - \frac{2}{N}) - 4\beta h)(1 + O\left(\frac{1}{N}\right)) \right] \\
+ \frac{r_-}{r_+} \left[ \exp(-4\beta h(1 - \gamma))(1 + O\left(\frac{1}{N}\right)) \right] - 1
\]
(4.73)

Notice that, for every $m \in [m_{mN}, m_{EN}) \subset [m^*, m_+]$, $g(m)$ is negative, being $f_\beta$ strictly decreasing in $[m^*, m_+]$. As a consequence, being $h > 0$, $e^{2\beta(1 - \gamma)g(\tilde{m})} - 1 < 0$ and, for $N$ sufficiently large, $1 - e^{2\beta \gamma g(\tilde{m} - \frac{2}{N}) - 4\beta h}(1 + O\left(\frac{1}{N}\right)) > 0$, the first term is negative. Moreover, being $\frac{r_-}{r_+} \geq 0$ and $e^{-4\beta h(1 - \gamma)}(1 + O\left(\frac{1}{N}\right)) - 1 < 0$, the second term is negative. Therefore, for $N$ sufficiently large, $G_*$ is negative, concluding the proof.

\[\square\]

4.4. Lower bound on the harmonic sum. In this section we provide the main ideas to prove the second part of Theorem 1.5, namely the lower bound on the harmonic sum in (4.1).

Proof of Theorem 1.5. Lower bound. The proof is very similar to the proof of the upper bound we gave in Section 4.2 therefore we omit the details. The main contribution is given once again by the sum on $S_N[U_{\delta,N}(m_-)]$.

We have,
\[
\sum_{\sigma \in S_N} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \geq \sum_{\sigma \in S_N[U_{\delta,N}(m_-)]} \mu_{\beta,N}(\sigma) h_{m_-,m_+}^N(\sigma) \\
= \sum_{\sigma \in S_N[U_{\delta,N}(m_-)]} \mu_{\beta,N}(\sigma) - \sum_{\sigma \in S_N[U_{\delta,N}(m_-)]} \mu_{\beta,N}(\sigma)(1 - h_{m_-,m_+}^N(\sigma)) \\
= \sum_{m \in U_{\delta,N}(m_-)(-1,m_+)} Q_{\beta,N}(m) \\
- \sum_{\sigma \in S_N[U_{\delta,N}(m_-)(-1,m_+)]} \mu_{\beta,N}(\sigma) P_{\sigma}(\tau_{S_N[m_+]} < \tau_{S_N[m_-]}).
\]
(4.74)

The first term, i.e. the sum on the mesoscopic measure $Q_{\beta,N}$, gives the main contribution. This sum can be estimated from below by using the second bound in Corollary 2.5 and
then obtaining a lower bound similar to the one in Lemma 4.1. More precisely, using the notation (1.34), we have the following lower bound for $s > 0$:

$$\sum_{m \in U_{\delta,N}(-1,m_-(N))} Q_{\beta,N}(m) P(s) \exp\left(\frac{-\beta N f_\beta(m_-)}{Z_{\beta,N} \sqrt{(1 - m^2) f''_\beta(m_-)}}\right) (1 + o(1)) \geq e^{\kappa - s \exp\left(-\beta N f_\beta(m_-)\right)} Z_{\beta,N} \sqrt{2 \pi (1 - m^2) f''_\beta(m_-)} e^{-\beta Nc} (1 + o(1)), \tag{4.75}$$

The second term in (4.74), appearing with a negative sign in front, is estimated via an upper bound, obtaining

$$\sum_{\sigma \in S_N[U_{\delta,N}(m_-)[-1,m_-(N)]]} \mu_{\beta,N}(\sigma) \mathbb{P}_{\sigma} \left( \tau_{S_N[m_+(N)]} < \tau_{S_N[m_-(N)]} \right) \leq \frac{e^{s + \sigma} \exp\left(-\beta N f_\beta(m_-)\right)}{Z_{\beta,N}} \sqrt{\frac{2}{\pi(1 - m^2)}} e^{-\beta Nc} (1 + o(1)), \tag{4.76}$$

which is negligible compared to the right hand side of (4.75), concluding the proof.

We omit the proof of (4.76) being it again technical and very similar to the proof of the upper bound (4.28) in Part 4 of Section 4.2. An analogue construction to the one given in Section 4.1 and similar proofs to those in Section 4.3 are needed. The main difference consists in restricting the analysis on a right neighbourhood of $m_-(N)$ instead of a left neighbourhood of $m_+(N)$.

□

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A. BOVIER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, RHEINISCHE FRIEDRICH-WILHELM-UNIVERSITÄT, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

_E-mail address:_ bovier@uni-bonn.de

S. MARELLO, INSTITUT FÜR ANGEWANDTE MATHEMATIK, RHEINISCHE FRIEDRICH-WILHELM-UNIVERSITÄT, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

_E-mail address:_ marello@iam.uni-bonn.de

E. PULVIRENTI, INSTITUT FÜR ANGEWANDTE MATHEMATIK, RHEINISCHE FRIEDRICH-WILHELM-UNIVERSITÄT, ENDENICHER ALLEE 60, 53115 BONN, GERMANY

_E-mail address:_ pulvirenti@iam.uni-bonn.de