Ultrasound long-range interactions of solitons observed over astronomical distances

Jae K. Jang*, Miro Erkintalo, Stuart G. Murdoch and Stéphane Coen*

We report what we believe is the weakest interaction between solitons ever observed. Our experiment involves temporal optical cavity solitons recirculating in a coherently driven passive optical-fibre ring resonator. We observe pairs of solitons interacting over a range as large as 8,000 times their width. In the most extreme case, their temporal separation changes as slowly as a fraction of an attosecond per roundtrip of the 100-m-long resonator, or equivalently 1/10,000 of the wavelength of the soliton carrier wave per characteristic dispersive length. The interactions are so weak that, at the speed of light, an effective propagation distance of the order of an astronomical unit can be required to reveal the full dynamical evolution. The interaction is mediated by transverse acoustic waves generated in the optical fibre by the propagating solitons through electrostriction.

Solitons are self-localized wave packets that do not spread, the dispersion of the supporting medium being cancelled by a nonlinear effect1–4. They are universal, and in many respects behave like particles5. They exert forces on one another and can interact in various ways, elastically or inelastically6,7. Here, we report what we believe is by far the weakest form of soliton interaction ever observed. Using recirculating optical cavity solitons, we study interactions so weak that the solitons shift their positions by only $\approx 10^{-7}$ of their width—equivalent to 1/10,000 of the wavelength of the soliton carrier wave—per characteristic dispersive length. At the speed of light, for these interactions to be revealed, effective propagation distances of the order of an astronomical unit (AU) are required, that is, tens of millions of kilometres. The fact that we can actually observe such ultraweak interactions in a noisy laboratory environment highlights the robustness and stability of solitons as never before.

Solitons occur in media as diverse as water, DNA, plasma or ultracold gases8,9,10, but over the past 20 years optics has led the way in promoting our understanding of soliton interactions because of the ease with which such experiments can be studied experimentally11–13. Optical solitons have been shown to attract, repel, break up, merge, orbit one another or even annihilate14–16. As in various other media, soliton interactions can be short-range, occurring when the tails of neighbouring solitons overlap17–19, or long-range, through coupling with a non-local response, be it optical radiation, charge carriers or thermal waves20–23. The weakest soliton interactions reported are long range24–28, but their observation is typically limited by the duration or propagation length over which the solitons can be maintained. In optics, this is often dictated simply by the size of a nonlinear crystal or the length of an optical fibre. To overcome this restriction, our experiment involves solitons recirculating in an optical fibre loop.

More specifically, we consider temporal cavity solitons propagating in a coherently driven passive nonlinear fibre ring resonator29. Not only are these objects genuine solitons, for which nonlinear self-trapping occurs in the longitudinal (temporal) direction, but the losses they suffer each roundtrip in the fibre loop are also compensated, thanks to the continuous-wave (cw) beam coherently driving the system30,31. In this respect, cavity solitons are dissipative solitons32–33, and can persist indefinitely in the driven resonator. In our fibre experiment, we find that temporal cavity solitons interact over long ranges through sound. In this scenario, the electric field of a first cavity soliton deforms the fibre material (in a process known as electrostriction) and excites an acoustic wave that propagates outwards across the fibre core and cladding. The associated density variation leads to a small time-varying change of the refractive index. When passing through this perturbation, a second cavity soliton undergoes a slight shift of its carrier frequency, which, due to dispersion, changes its group velocity40. In this way, the trailing cavity soliton either catches up with the leading one (attraction) or is delayed (repulsion). Such acoustic interactions have been studied in the past in the context of optical-fibre telecommunication systems with the traditional Kerr solitons of the nonlinear Schrödinger (NLS) equation29,40–42. However, with temporal cavity solitons, an important difference emerges, because they are robust attracting states of the nonlinear resonator, tied to the cavity solitons, an important difference emerges, because they are robust attracting states of the nonlinear resonator, tied to the

Department of Physics, The University of Auckland, Private Bag 92019, Auckland 1142, New Zealand. *e-mail: jake.jang.ur.mate@gmail.com; s.coen@auckland.ac.nz
boosted by an erbium-doped fibre amplifier before being launched into the cavity. The part of this beam that is reflected off the cavity is used to actively lock the driving laser frequency to the cavity resonance. Temporal cavity solitons are excited with 1.8 ps addressing pulses at a different wavelength and coupled into the cavity through a WDM. The number of solitons excited and their separation are set with two intensity modulators and a pattern generator. Erbium-doped fibre amplifiers (EDFAs) boost the power of the two beams while optical bandpass filters (BPFs) improve the signal-to-noise ratio of the measurements. The output is analysed with an ultrafast real-time oscilloscope.

Figure 1 | Experimental set-up. A passive fibre cavity is driven externally by a c.w. laser beam. The part of this beam that is reflected off the cavity is used to actively lock the driving laser frequency to the cavity resonance. Temporal cavity solitons are excited with 1.8 ps addressing pulses at a different wavelength and coupled into the cavity through a WDM. The number of solitons excited and their separation are set with two intensity modulators and a pattern generator. Erbium-doped fibre amplifiers (EDFAs) boost the power of the two beams while optical bandpass filters (BPFs) improve the signal-to-noise ratio of the measurements. The output is analysed with an ultrafast real-time oscilloscope.

Figure 2 | Cavity soliton characteristics. a, b, Experimental FROG trace (a) and optical spectrum (b, black) of cavity solitons. c, Simulated FROG trace showing excellent agreement with a. d, Corresponding numerical temporal intensity profile of the cavity solitons, which reveals their 2.6 ps duration. The simulated spectrum is shown in red in b. The horizontal line and the hyperbolic fringes visible at and around the middle of the FROG trace, as well as the peak at the centre of the optical spectrum, are all signatures of the c.w. background on which the cavity solitons are superimposed, and which is clearly visible in d.

boosted by an erbium-doped fibre amplifier before being launched into the cavity. The part of the driving beam that is reflected off the cavity is directed to a servo-controller, which controls the laser frequency. In this way, the driving laser is actively locked on a cavity resonance. In comparison to Leo et al., who were only able to maintain the cavity solitons for a few seconds\textsuperscript{35}, our locking system is significantly more robust and works for a larger excursion of the environmental parameters. Combined with the fact that our cavity is about four times shorter, we can routinely achieve stable locking of the cavity for periods in excess of 30 min. This is a key factor in observing the weak interactions we report here.

The cavity solitons were excited using the same technique as in ref. 35, that is, through cross-phase modulation between the intra-cavity c.w. background field and ultrashort pulses at a different wavelength. These addressing pulses are 1.8 ps in duration and are generated by a pulsed laser with a repetition rate locked to the 10 GHz clock of an electronic pattern generator (see left part of
Fig. 1). A series of two intensity modulators is used to select individual addressing pulses so as to control the number of cavity solitons excited into the cavity, as well as their initial separation (in multiples of the 100 ps period of the 10 GHz pulsed laser). The first modulator imprints the repetitive bit pattern of the generator onto the laser pulse train. The resulting optical signal is then gated by the second modulator to select a single period of that pattern when starting measurements. The rest of the time, the second modulator completely blocks the addressing beam. Note that the cavity solitons will have the same wavelength as that of the driving beam17,35,37. The central (d.c.) peak of the optical spectrum is another signature of the background, as are spectrally resolving, as a function of delay, a signal consisting of the FROG trace. To understand the latter phenomenon, it is highlighted the coherence of the cavity soliton with respect to the optical field under study multiplied by its own delayed copy45. The central structure of the FROG trace signals the presence of short pulses with durations of a few picoseconds circulating in the fibre. The FROG trace is also barred in the centre by a horizontal line. This line signals the presence of a c.w. background on which the pulses are superimposed. Such a c.w. background is a central characteristic of cavity solitons and is expected17,35,37. The central (d.c.) peak of the optical spectrum is another signature of the background, as are the spectral interferometric fringes visible around the main peak of the FROG trace. To understand the latter phenomenon, it is worth pointing out that the FROG measurement is performed by spectrally resolving, as a function of delay, a signal consisting of the optical field under study multiplied by its own delayed copy45. In the presence of a pulse on a c.w. background, and for delays larger than the pulse duration, the FROG signal will be composed of two copies of the pulse, naturally leading to fringes in the spectral domain, and we have successfully verified that the observed fringes follow hyperbolic trajectories in the time–frequency plane, \( \tau, f \), \( f = 1/\tau, 2/\tau, 3/\tau, \ldots \). The presence of these fringes therefore also highlights the coherence of the cavity soliton with respect to the c.w. background, and hence with the driving field. All these features are also clearly visible on a numerically simulated FROG trace (Fig. 2c), with the corresponding temporal intensity profile shown in Fig. 2d. The excellent agreement between the experimental and simulated FROG traces allows us to infer that our cavity solitons are \( \sim 2.6 \) ps (full-width at half-maximum, FWHM) in duration (Fig. 2d). Note that the experimental FROG trace is somewhat blurred in comparison to the numerical one due to the limited spectral resolution of the measuring apparatus.

Results
The interactions between the cavity solitons were studied by writing a single pair of cavity solitons into the cavity and by observing through the output coupler how the temporal separation between the two solitons evolves over many roundtrips. Measurements were performed with an ultrahigh sampling rate, real-time digital oscilloscope triggered on the leading pulse of the pair. In Fig. 3a, we show a colour plot of successive oscilloscope traces when the two cavity solitons are initially separated by 100 ps (about 38 soliton widths). Despite the large separation, the trailing cavity soliton can be seen to be gradually repelled away from the leading one. This long-range repulsion persists for \( \sim 20 \) s until a stationary separation of 420 ps is reached (corresponding to about 160 soliton widths). Note that the oscilloscope has a temporal resolution of \( \sim 50 \) ps, significantly larger than the 2.6 ps cavity soliton duration, so the plot does not do full justice to the large difference of scales between the width of the cavity solitons and their separation. In Fig. 3b, the initial separation has been set to 1,500 ps (or 577 soliton widths). The interaction is now attractive, and again persists until the same stationary separation of 420 ps is reached. We find that beyond an initial separation of 1,500 ps, the interaction becomes repulsive again. Figure 3c illustrates this latter case, where, starting from a separation of 1,800 ps (or 692 soliton widths), the trailing pulse slowly moves away from the leading one, to reach a separation of \( \sim 3,600 \) ps after 30 min. It is clear that the repulsion observed in Fig. 3c is significantly weaker than that in Fig. 3a.

The soliton interactions reported above take place over time-scales that are incredibly slow for an optical system. The differences in orders of magnitude can be appreciated by considering that...
during the 30 min interaction shown in Fig. 3c, the cavity solitons travel around the cavity 3.6 billion times, for a total distance of 360 million kilometres, or 2.4 AU, that is, more than twice the distance between the Earth and the Sun. Yet, over that vast distance, the two cavity solitons only shift their relative position by less than 2 ns, equivalent to a change of 40 cm in spatial separation, 12 orders of magnitude smaller than the distance travelled. Such displacement corresponds to half an attosecond per cavity roundtrip or equivalently to \(2 \times 10^{-17}\) soliton widths, or 1/10,000 of the driving laser wavelength, per linear dispersive length (the distance \(L_D\) over which a Gaussian pulse of the same duration would broaden by a factor of \(\sqrt{2}\) due to linear dispersion only \(44\); in our case \(L_D = 114\) m).

It is worth noting that the interaction strength between solitons propagating in an ideal instantaneous Kerr medium decays exponentially if they are separated by more than a few soliton widths \(19,20,23,26,46\). For temporal cavity solitons, this is related to how fast the cavity soliton wings tend to the c.w. background, which in our experimental conditions occurs with a time constant of \(\sim 2\) ps. To understand the physics of the long-range interactions illustrated in Fig. 3, we have to consider the time-varying refractive index perturbation mediated by an acoustic wave excited by the leading cavity soliton. The impulse response of that perturbation can be calculated using the method of ref. 40 (Supplementary Section SI) and is plotted in Fig. 4a, scaled to the energy of our cavity solitons. We must first point out the smallness of this refractive index perturbation. With a maximum of approximately one part per trillion, it is more than three orders of magnitude smaller than the Kerr-induced index change due to the peak power of the cavity soliton. It is also much smaller than the longitudinal refractive index fluctuations present in even the best quality fibres. However, these fluctuations are sampled and averaged in the same way by the two cavity solitons of a pair, and hence do not affect their relative separation \(^4\). The same is true for thermal and mechanical fluctuations. The second important point to notice from Fig. 4a is that the acoustic impulse response consists of short spikes (1–2 ns in duration) occurring every 21 ns. These correspond to increasing orders of echoes, that is, back-and-forth reflections of the acoustic wave from the fibre cladding–coating boundary back into the fibre core region \(^4\). Given the standard cladding diameter of 125 \(\mu\)m of our fibre, 21 ns perfectly matches with the travel time of sound in silica at 5.996 \(m/s\) \(^{47}\) (ref. 47). Between the spikes, the response remains small due to the negligible overlap between the optical mode trapped inside the fibre core and the acoustic wave travelling in the cladding.

The time-varying nature of the refractive index perturbation \(dn/dt\) is responsible for a shift in the instantaneous frequency of the trailing cavity soliton. The slope of the perturbation at the delay of the trailing cavity soliton determines whether the interaction is repulsive or attractive. If \(dn/dt > 0\) (respectively, \(<0\)), the trailing cavity soliton is redshifted (blueshifted) with respect to the leading one. Because our experiment is performed in the anomalous dispersion regime \(^4\), this frequency shift translates into a smaller (larger) group velocity, resulting in an effective repulsion (attraction) between the cavity solitons. It is only when \(dn/dt = 0\) that the two cavity solitons travel at the same group velocity and do not move with respect to each other. A close-up of the first spike of the acoustic impulse response plotted in Fig. 4a is shown in Fig. 4b where we have highlighted, based on the above analysis, the ranges of repulsion and attraction. Furthermore, from this figure, it is straightforward to verify that the maximum of the refractive index perturbation corresponds to a stable separation, whereas the minimum is unstable. This predicted behaviour is in remarkable qualitative agreement with the observations described in Fig. 3. In particular, the weak but non-vanishing repulsion observed for a separation larger than \(\sim 1,500\) ps simply results from the small

![Figure 4](https://example.com/figure4.png)

**Figure 4 | Acoustic response.** a. Theoretical impulse response of the refractive index acoustic perturbation calculated for our experimental conditions, and scaled to the cavity soliton energy. Subsequent spikes (numbered 1-5) are separated by \(\sim 21\) ns and arise from consecutive reflections of the acoustic wave from the fibre cladding-coating boundary, as schematically illustrated in the inset. b. Close-up of the first peak of the acoustic response. Ranges of repulsion and attraction of a trailing cavity soliton are highlighted. The maximum corresponds to a stable separation.

but non-vanishing (positive) slope of the refractive index perturbation in that region and all the way to the first echo at 21 ns. We must note that the measured stable separation of 420 ps does not quite match with the 510 ps at which the theoretical maximum occurs (Fig. 4b). However, the transverse acoustic model of optical fibres is known to be deficient and has systematically failed in precisely reproducing the experimental measurements of the first acoustic contribution \(^41,48,49\). Incidentally, our results may constitute the most accurate measurement of this primary refractive index extremum to date.

The acoustic origin of the interaction can be confirmed by probing the dynamics of a pair of cavity solitons whose initial separation is adjusted around 21 ns to match the re-entrance of the first acoustic echo into the fibre core. Note that this separation is still much smaller than the roundtrip time of 0.5 ps, so the cavity soliton pair can truly be considered as isolated. A close-up of the impulse response of the refractive index perturbation about the first echo is plotted in Fig. 5a (dashed curve). The response here has two maxima, and for this range of separations we would therefore expect to observe two different stable separations. Experimental measurements are juxtaposed on the same graph in Fig. 5b for four different initial separations sampling the four regions of repulsion and attraction around 21 ns separation. Note that we have omitted to plot the leading cavity soliton (at zero delay) for clarity.
Figure 5 | Interactions of two cavity solitons mediated by the first acoustic echo. a, Close-up of the first echo of the acoustic impulse response (dashed line; see spike labelled 2 in Fig. 4a). The red curve shows the actual index change when taking into account the exact cavity soliton intensity profile and its perturbed background. Two stable separations of cavity solitons exist in the range shown. b, Experimental colour plot of the trailing cavity soliton in each of the regions highlighted in a (as in Fig. 3, using the same colour map, but with the leading cavity soliton omitted for clarity). The plot consists of the juxtaposition of four different independent measurements (separated by vertical lines) obtained for different initial separations between the two cavity solitons. Red curves are numerical simulations.

(at the scale of Fig. 5b, including the leading cavity soliton would make the figure two pages wide). We observe stable separations of 19.79 ns and 22.05 ns. This is in excellent agreement with the maxima of the actual acoustic index change (red curve in Fig. 5a) calculated as the convolution of the impulse response (shown in Fig. 4a) with the exact leading cavity soliton intensity profile and its perturbed background (see Methods). Note how the maxima of the impulse response are shifted towards slightly longer delays when considering the more complete convolution calculation, from 19.80 ns and 21.92 ns, to 19.82 ns and 22.03 ns, respectively. For this case, we have also performed a full modelling of the interactions (see Methods) and the corresponding numerical trajectories of the trailing cavity soliton are superimposed on the experimental data as the red curves in Fig. 5b. Considering the large difference of timescales involved (2.6 ps cavity solitons shifting their position by ~1 ns over 100–200 s) and the absence of any fit parameter, the agreement is spectacularly good. Most of the discrepancies are probably due to incomplete knowledge of the acoustic impulse response or mechanical and thermal fluctuations in the environment or in the driving laser. These observations and simulations confirm the acoustic nature of the interactions we observe. We must point out that 21 ns corresponds to more than 8,000 soliton widths. This is by far the longest interaction range ever reported for solitons.

Discussion

Our study provides clear experimental evidence of record-breaking ultraweak interactions between solitons. In our experiment, interacting solitons shift their relative position by an amount as small as $2 \times 10^{-7}$ of their width, or 1/10,000 of the soliton carrier wavelength, per linear dispersive length. This represents a microscopic displacement (~100 pm) over a macroscopic propagation distance (~100 m), with 12 orders of magnitude difference. The fact that such a weak interaction can accumulate in a well-defined manner over distances of several hundred million kilometres (or billions of dispersive lengths) is also truly remarkable. It highlights the extreme stability, robustness and coherence of the process, and of solitons in general. The long-range nature of the interaction further compounds these feats. Indeed, our observations are performed with solitons separated by up to 8,000 times their width. Translated into spatial units, our solitons localized within 0.5 mm of fibre interact across a 4 m separation. Together, these figures defy imagination.

Our observations make clear that in the context of our study, which is performed with temporal optical cavity solitons, the interactions are mediated by acoustic waves. We believe our findings explain the repulsion of unknown origin described in ref. 35, which reported the first experimental realization of temporal cavity solitons. In that work, the evolution of the separation between pairs of cavity solitons could not be tracked over time because the cavity was only stable for a few seconds. Clearly, the remarkable stability of our experiment, with the cavity interferometrically stabilized over periods in excess of 30 min, was a key factor for the success of our measurements. Our results also open up possibilities for precision sensing applications. On the other hand, the existence of long-range interactions constitute a drawback for the application of temporal cavity solitons to optical buffering and to other telecommunication technologies35. We expect, however, that a shallow modulation of the c.w. driving beam would easily trap the cavity solitons into fixed time slots. Finally, we must stress that temporal cavity solitons have recently been shown to be the probable underlying temporal structure of broadband frequency combs generated in high-Q Kerr microresonators50. As acoustic interactions...
have been shown to play a role in the stabilization of the repetition rate of harmonically mode-locked fibre lasers\textsuperscript{45}, it is interesting to speculate whether they have a similar role in the context of Kerr frequency comb generation and cavity optomechanics\textsuperscript{46}.

Methods

Experimental set-up. The resonator was made from 100 m of standard telecommunications single-mode silica optical fibre (Corning SMF-28). The choice of a single-mode waveguide enabled us to make observations unhampered by the diffraction of the beams\textsuperscript{46}; in other words, the system is purely one-dimensional. At a wavelength of 1,550 nm, the fibre presented a second-order dispersion coefficient \( \beta_2 = 21.4 \text{ ps}^2 \text{ km}^{-1} \) (measured by white-light interferometry) and a c.w. nonlinear coefficient, inclusive of the electrostrictive contribution, of \( n_2 = 1.2 \text{ W}^{-1} \text{ km}^{-1} \) (measured from the nonlinear tilt of the cavity resonance)\textsuperscript{48}. The intracavity isolator preventing stimulated Brillouin scattering was polarization-independent and had a 60 dB isolation factor. The WDM coupler used to inject the addressing pulses was a filter WDM coupler with a 10-nm wide transition band centred at 1,540 nm. The free spectral range and finesse of the resonator were measured to be 2.07 MHz and 22, respectively. The driving beam was generated by a Koheras AdjustiK E15 laser, which is an erbium-doped distributed feedback fibre laser. It presented a linewidth of <1 kHz and was operated at an output power level of 20 mW. After amplification, we used a narrow bandpass filter (BPF, 0.5 nm bandwidth) centered on the driving laser wavelengths to reject most of the amplified spontaneous emission noise of the amplifier before the driving beam was launched into the cavity. The driving power at the input to the resonator was 960 mW. A commercial 100 kHz proportional-integral-derivative (PID) controller (SRS S5960) was used to lock the driving power reflected off the resonator to a set level allowing for precise control of the cavity detuning. The error signal of the PID controller was directly fed to the fast piezo-electric tuning system of the AdjustiK laser. The addressing pulses were derived from a tunable semiconductor mode-locked laser (Alnair LLC LD100) centred at 1,532 nm with a pulse width of 1.8 ps and were subsequently amplified to a peak power of 10 W. The two intensity modulations used to select individual addressing pulses had a bandwidth of 12.5 GHz. For the oscilloscope measurements of Figs 3a–c and 5b, detection was performed with a 12.5 GHz PIN amplified photodiode. The oscilloscope had a bandwidth of 12 GHz and a sampling rate of 40 GSample/s, and traces were acquired every second. Also, to improve the dynamic range of the measurements, the c.w. background on which the cavity solitons were superimposed was filtered out by a narrow BPF (bandwidth, 0.5 nm) centered at 1,551.5 nm, slightly off the driving laser wavelength, and placed in front of the photodiode\textsuperscript{48}.

FROG measurements. The time–frequency representation of Fig. 2a was obtained with a second-harmonic frequency-resolved optical gating (SHG-FROG) apparatus (Southern Photonics HR150) with an 0.09 nm (or 45 GHz) spectral resolution at the second-harmonic wavelength. We must point out that we cannot measure the FROG trace of cavity solitons when there is only a single cavity soliton circulating in the cavity, because the energy contained in the cavity soliton c.w. background is four orders of magnitude larger than the energy of the actual 2.6-ps-wide pulse, which is too large for the dynamic range of the FROG spectrometer. For that reason, the FROG measurement was obtained by filling the resonator with about 1,800 cavity solitons. As all the cavity solitons are identical, the result is the same as that with a single cavity soliton, except for the apparent amplitude of the background signal at d.c. in comparison to the wings of the spectrum. The number of cavity solitons circulating in the cavity was used as a fit parameter to calculate the numerically simulated FROG trace of Fig. 2c. The spectrum shown in Fig. 2b was measured with an optical spectrum analyser presenting an 0.05 nm spectral resolution, in the same conditions as the FROG trace.

Modelling of cavity solitons. Our experimental observations of acoustic–mediated interactions of temporal cavity solitons were modelled with a mean-field Lugiato–Modelling of cavity solitons. Let us consider a cavity with a single cavity soliton (Supplementary Fig. S3), therefore validating our approach. This set-up point between the two different intensity profiles considered in the convolution integral is chosen sufficiently far in front of the trailing cavity soliton, that is, at a location where the trailing cavity soliton is not influencing the field. In practice, this is set as 25 normalized units of fast-time in front of the trailing cavity soliton. Note that, for simplicity, we find numerically that the field of the trailing cavity soliton and the trailing background of the leading cavity soliton (Supplementary Fig. S1), therefore validating our approach. This procedure is repeated for a range of delays (note that the leading cavity soliton is only calculated once). The simulated trajectories shown in Fig. 3b are then obtained by simple integration,

\[
\frac{dt}{d\tau} = \sqrt{\frac{\gamma}{\Delta t}} \frac{1}{\sqrt{(\Delta t)^2 - V_t \Delta t}} 
\]

where \( \Delta t = \int_{t_0}^{t_1} \frac{1}{V(\Delta t)} \) is the time delay between the two cavity solitons and \( V_t \) is the drift velocity of the leading cavity soliton. Finally, we have verified our modelling approach using direct split-step Fourier integration\textsuperscript{48} of equation (1) with smaller cavity soliton separations for which split-step integration does not impose an unmanageable computational load. In addition, the mean-field reference frame is tied to our numerical algorithm and will be explained in the following.

The first four terms of the right-hand side of equation (1) represent, respectively, the total cavity losses, the detuning of the pump from resonance (with \( \Delta \) being the detuning parameter), second-order chromatic dispersion (with \( \eta = 1 \) the sign of the group-velocity dispersion coefficient \( \beta_2 \) of the fibre), and the external driving (with \( S \) the amplitude of the driving field). The second line of the right-hand side accounts for the nonlinearities, which is split into an instantaneous (electronic) Kerr contribution and a delayed response due to the electrostriction-induced acoustic wave. \( \hat{w}(\tau) \) is the acoustic impulse response function \( [\hat{w}(\tau) = 0] \) normalized such that \( \int_0^{\infty} \hat{w}(\tau) d\tau = 1 \) and \( f \) is the fraction of the Kerr nonlinearity due to electrostriction. With our parameters, we find \( f = 15.6\% \), in good agreement with experimental studies\textsuperscript{48}. With these notations, it must be clarified that the nonlinear coefficient \( \gamma \) that appears in the normalization of the field amplitude (see ref. 35) must be understood as being the nonlinearity coefficient as seen by a c.w. wave, that is, inclusive of the electrostriction contribution\textsuperscript{48}. The impulse response of the actual refractive index perturbation \( \hat{n}(\tau) \) plotted in Fig. 4a was obtained by considering that the intensity of the cavity soliton is a Dirac–\( \delta \) function. This leads (in real units) to \( \Delta t = \gamma \hat{w}(\tau)/2(\gamma f) \), where \( \Delta t = 23.3 \) ps is the energy of our cavity soliton (calculated after subtracting the background intensity) and \( \lambda_b = 1.550 \) nm is the driving laser wavelength.

The temporal cavity solitons are the stationary solutions \( (\hat{w} = \hat{w} = 0) \) of equation (1).\textsuperscript{18} We have calculated them by looking for the roots of the right-hand side of the equation with a multidimensional globally convergent Newton–Raphson method. The use of a Newton solver is necessary because the large difference of scales between the duration of the cavity solitons (a few picoseconds) and the timescale of the acoustic response (up to tens of microseconds) makes the propagation simulations very inefficient. In the presence of the acoustic response, however, the cavity solitons are not stationary in the reference frame of the driving field. In fact, the cavity solitons suffer from a self-frequency shift due to the leading edge of the refractive index acoustic perturbation they generate\textsuperscript{48}, resulting in a change in their group velocity. To circumvent this issue, the velocity \( V \) of the fast time reference frame is chosen such that the cavity soliton is stationary in that reference frame. In practice, the drift velocity \( V \) is treated as an additional unknown in the Newton solver, with the extra condition that the sought solution must peak in the centre of the computed temporal window. Additionally, we must point out that the convolution integral of equation (1) is computed explicitly, without using Fourier transforms, so as to account correctly for the infinite unperturbed c.w. background preceding the cavity soliton. With our experimental parameters (corresponding to \( \Delta t = 3.27 \) ps and \( \Delta = 2.89 \), determined as in ref. 35), we find (in real units) \( V = 5.2 \times 10^{-6} \text{ ps}\text{rundrip for the leading cavity soliton} \).

In a pair of interacting cavity solitons, the trailing cavity soliton moves with respect to the leading one, and the overall situation is not stationary (even for particular separations). To calculate the velocity of the trailing cavity soliton at arbitrary delays with the Newton method, we took advantage of the fact that the transverse acoustic wave generated by a cavity soliton only affects trailing cavity solitons and not the other way around. This leads to the following approach. First, we calculate the velocity \( V_t \) of the leading cavity soliton by itself. Because of the generated acoustic wave, we must note that the background trailing that cavity soliton is weakly perturbed from the normal c.w. state. In a second step, the equation is solved for the trailing cavity soliton, again all by itself in the numerical window, but with an important modification: in the convolution term \( \int_0^{\infty} \hat{w}(\tau) E(t - \tau) d\tau \), the c.w. background is replaced by \( E(t) \) at the front of the trailing cavity soliton is substituted for the intensity profile of the leading cavity soliton and its perturbed trailing background, with the delay under consideration. In this way, we ensure that the trailing cavity soliton sits in the correct perturbed refractive index wake and the velocity \( V \) output by the Newton solver for the trailing cavity soliton includes contributions both from the self-frequency shift and from the presence of the leading cavity soliton. The transition point between the two different intensity profiles considered in the convolution integral is chosen sufficiently far in front of the trailing cavity soliton, that is, at a location where the trailing cavity soliton is not influencing the field.

In practice, this is set as 25 normalized units of fast-time in front of the trailing cavity soliton. Note that, for simplicity, we find numerically that the field of the trailing cavity soliton and the trailing background of the leading cavity soliton (Supplementary Fig. S3), therefore validating our approach. This procedure is repeated for a range of delays (note that the leading cavity soliton is only calculated once). The simulated trajectories shown in Fig. 3b are then obtained by simple integration,

\[
\frac{dt}{d\tau} = \sqrt{\frac{\gamma}{\Delta t}} \frac{1}{\sqrt{(\Delta t)^2 - V_t \Delta t}} 
\]
model itself has been validated against simulations of a lumped cavity model of the complete experimental set-up (Supplementary Sections S2V.SV).

Received 20 January 2013; accepted 23 May 2013; published online 7 July 2013

References

1. Russell, J. S. in Report of the Fourteenth Meeting of the British Association for the Advancement of Science, York, September 1844, 311–390, Plates XLVII–LVII (John Murray, 1845).

2. Zabusky, N. J. & Kruskal, M. D. Interaction of ‘solitons’ in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240–243 (1965).

3. Hasegawa, A. & Tappert, F. Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl. Phys. Lett. 23, 142–144 (1973).

4. Akhmediev, N. N. & Ankiewicz, A. Solitons — Nonlinear Pulses and Beams 1st edn (Chapman & Hall, 1997).

5. Stegeman, G. I. & Segov, M. Optical spatial solitons and their interactions: universality and diversity. Science 286, 1518–1523 (1999).

6. Craig, W., Guyenne, P., Hammack, J., Henderson, D. & Sulem, C. Solitary water wave interactions. Phys. Fluids 18, 051706 (2006).

7. Gardiner, C. S., Greene, J. M., Kruskal, M. D. & Miura, R. M. Method for solving the Korteweg–deVries equation. Phys. Rev. Lett. 19, 1095–1097 (1967).

8. Longugren, K. E. Soliton experiments in plasmas. Plasma Phys. 25, 943–983 (1982).

9. Saha, M. & Kofane, T. C. Long-range interactions between adjacent and distant bases in a DNA and their impact on the ribonuclease acid polymerase–DNA dynamics. Chaos 22, 013116 (2012).

10. Polturak, E., deVega, P. G. N., Zeise, E. K. & Lee, D. M. Solitonlike propagation of zero sound in superfluid $^3$He. Phys. Rev. Lett. 46, 1588–1591 (1981).

11. Burger, S. et al. Dark solitons in Bose–Einstein condensates. Phys. Rev. Lett. 83, 5198–5201 (1999).

12. Denischl, J. et al. Generating solitons by phase engineering of a Bose–Einstein condensate. Science 287, 97–100 (2000).

13. Bjorkholm, J. E. & Ashkin, A. A. cw self-focusing and self-trapping of light in sodium vapor. Phys. Rev. Lett. 32, 129–132 (1974).

14. Mollenauer, L. F., Stolen, R. H. & Gordon, J. P. Experimental observation of picosecond pulse narrowing and solitons in optical fibers. Phys. Rev. Lett. 45, 1095–1098 (1980).

15. Barthelemy, A., Maneuf, S. & Froehly, C. Propagation soliton et auto-confinement de faisceaux laser par non lineariteé de Kerr. Opt. Commun. 55, 201–206 (1985).

16. Segov, M. Optical spatial solitons. Opt. Quant. Electron. 30, 503–533 (1998).

17. Barland, S. et al. Cavity solitons as pixels in semiconductor microcavities. Nature 419, 699–702 (2002).

18. Greml, P. & Akhmediev, N. N. Dissipative solitons for mode-locked lasers. Nature Photon. 6, 84–92 (2012).

19. Gordon, J. P. Interaction forces among solitons in optical fibers. Opt. Lett. 8, 598–599 (1983).

20. Reindaur, F. & Barthelemy, A. Optically controlled interaction between two fundamental soliton beams. Europhys. Lett. 12, 401–405 (1990).

21. Tikhonenko, V., Christou, I. & Luther-Davies, B. Three-dimensional bright spatial soliton collision and fusion in a saturable nonlinear medium. Phys. Rev. Lett. 76, 2698–2701 (1996).

22. Shih, M., Segev, M. & Salamo, G. Three-dimensional spiraling of interacting spatial solitons. Phys. Rev. Lett. 78, 2551–2554 (1997).

23. Krolowski, W., Luther-Davies, B., Denz, C. & Tischudi, T. Annihilation of photorefractive solitons. Opt. Lett. 23, 97–99 (1998).

24. Snyder, A. W., Mitchell, D. J., Poladian, L. & Ladouceur, F. Self-induced optical fundamental soliton beams. Phys. Rev. Lett. 71, 827–824 (1993).