TWISTED EHRESMANN SCHAUENBURG BIALGEBROIDS

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Abstract. We construct an invertible normalised 2 cocycle on the Ehresmann Schauenburg bialgebroid of a cleft Hopf Galois extension under the condition that the corresponding Hopf algebra is cocommutative and the image of the unital cocycle corresponding to this cleft Hopf Galois extension belongs to the centre of the coinvariant subalgebra. Moreover, we show that any Ehresmann Schauenburg bialgebroid of this kind is isomorphic to a 2-cocycle twist of the Ehresmann Schauenburg bialgebroid corresponding to a Hopf Galois extension without cocycle, where comodule algebra is an ordinary smash product of the coinvariant subalgebra and the Hopf algebra (i.e. \( C(B\#_\sigma H, H) \cong C(B\# H, H)^{\sigma} \)). We also study the theory in the case of a Galois object where the base is trivial but without requiring the Hopf algebra to be cocommutative.

Contents

1. Introduction 1
2. Algebraic preliminaries 2
  2.1. Cleft Hopf Galois extensions 3
  2.2. Bialgebroids 5
3. Ehresmann-Schauenburg bialgebroids 7
  3.1. 2-cocycles on Ehresmann-Schauenburg bialgebroids 7
References 14

1. Introduction

The study of principal bundles and groupoids are important in different areas of mathematics and physics. In the area of noncommutative geometry, we are always interested in Hopf Galois extensions, which can be viewed as quantisation of principal bundles. For any principal bundle, we can construct a gauge groupoid. In the ‘quantum’ case, the ‘quantum’ gauge groupoid can also be constructed for any Hopf Galois extension \( B = A^{coH} \subseteq A \) (quantum principal bundles). This kind of ‘quantum’ gauge groupoids \( C(A, H) \) are called Ehresmann Schauenburg bialgebroids.

In this paper, we will study cleft Hopf Galois extensions, which can be viewed as the quantisation of trivial principal bundles. Since any cleft Hopf Galois extension is isomorphic to the crossed product \( B\#_\sigma H \) of the coinvariant subalgebra \( B = A^{coH} \) and the Hopf algebra \( H \) for a unital cocycle \( \sigma : H \otimes H \to B \), so instead of studying the comodule algebra \( A \) directly, we can work on the crossed product algebra \( B\#_\sigma H \).

As it was shown in [6] that given a cocommutative Hopf algebra \( H \), there is a bijective correspondence between the equivalence classes of \( H \)-cleft Hopf Galois extensions and the second cohomology group \( \mathcal{H}^2(H, Z(B)) \), where \( Z(B) \) is the centre of the coinvariant subalgebra \( B \). So in this paper we will mainly consider cleft Hopf Galois extensions with cocommutative Hopf algebra and its second cohomology group \( \mathcal{H}^2(H, Z(A)) \). Under this
special assumption, we can show that there is an invertible normalised 2-cocycle $\tilde{\sigma}$ on the Ehresmann Schauenburg bialgebroid associated to the cleft Hopf Galois extension. Moreover, the Ehresmann Schauenburg bialgebroid is isomorphic to the 2 cocycle twisted algebra $C(B\#H, H)^\tilde{\sigma}$, where $B\#H$ is the smash product of $B$ and $H$. We can also show that if the action of $H$ on $B$ is trivial, then the corresponding Ehresmann Schauenburg bialgebroid associated to the cleft Hopf Galois extension is isomorphic to $C(B\#H, H)^\tilde{\sigma}$ even if the Hopf algebra $H$ is not cocommutative. In particular, we will study the case of Galois objects for any Hopf algebras.

In Section §2, we will give a brief introduction to Hopf Galois extension, and in particular the cleft Hopf Galois extension with its properties. Then we will also recall bialgebroids and 2-cocycles of them. In Section §3, we will first study Ehresmann Schauenburg bialgebroid, then we show under some conditions there is an invertible norm ailsed 2 cocycle on $C(B\#H, H)$, such that $C(A, H) \simeq C(B\#_{\sigma}H, H) \simeq C(B\#H, H)^{\tilde{\sigma}}$. Finally, we will apply the general theory on Galois objects.

2. Algebraic preliminaries

In this section we will first recall the definitions of comodule algebras and module algebras of Hopf algebras, then we will study Hopf Galois extensions which can be viewed as the quantisation of principal bundles. In particular, we will recall some properties of cleft Hopf Galois extensions, which can be viewed as trivial noncommutative principal bundles. We also recall the more general notions of rings and corings over an algebra as well as the associated notion of bialgebroids. In this paper, we will assume all the algebras, comodules and modules are vector spaces over $\mathbb{C}$.

Let $H$ be a Hopf algebra with coproduct $\Delta$, counit $\epsilon$ and antipode $S$. We use the sumless Sweedler notation to denote the image of the coproduct, i.e. $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for all $h \in H$. The convolution algebra of the dual space $H' := \text{Hom}(H, \mathbb{C})$ is a unital associative algebra with the product given by $\phi \star \psi(h) := \phi(h_{(1)})\psi(h_{(2)})$, for all $\phi, \psi \in H'$.

Given a Hopf algebra $H$, a left $H$ module algebra is an algebra $A$, such that it is a left $H$ module. Moreover, it satisfies:

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad h \triangleright 1 = \epsilon(h)1, \quad (2.1)$$

for all $a, b \in A$ and $h, g \in H$, where $\triangleright : H \otimes B \to B$ is the action of the left module $B$.

Given a left $H$ module algebra $A$, we can define a new algebra $A\#H$, which as a vector space is equal to $A \otimes H$, and the product is given by

$$(a\#h)(a'\#g) = a(h_{(1)} \triangleright a')\#h_{(2)}g, \quad (2.2)$$

for all $a, a' \in A$ and $g, h \in H$, where we write $a\#h$ for the tensor product $a \otimes h$. We call this algebra the smash product of $A$ and $H$.

Dually, given a Hopf algebra $H$, the right $H$-comodule is vector space $V$, together with a coaction, which is a linear map $\delta : V \to V \otimes H$ such that

$$(\text{id} \otimes \Delta) \circ \delta = (\delta \otimes \text{id}) \circ \delta, \quad (\text{id} \otimes \epsilon) \circ \delta = \text{id}. \quad (2.3)$$

We also use the sumless Sweedler notation to denote the image of coaction, i.e. $\delta(v) = v_{(0)} \otimes v_{(1)}$ for all $v \in V$. The morphism between two right $H$-comodules $V$ and $W$ is a
Given a Hopf algebra $H$, a right $H$-comodule algebra is an algebra $A$, such that $A$ is a right $H$-comodule, and the comodule map $\delta : A \to A \otimes H$ is an algebra map. The morphism between two comodule algebras is both a comodule map and an algebra map, where $A \otimes H$ has the usual tensor product algebra structure.

2.1. Cleft Hopf Galois extensions. In this section, we will recall the definition of Hopf Galois extensions. In particular, we will study cleft Hopf Galois extensions, which could be thought of as trivial noncommutative principal bundles.

Definition 2.1. Let $A$ be a comodule algebra of a Hopf algebra $H$, with the coinvariant subalgebra $B := \{ b \in A \mid \delta(b) = b \otimes 1_H \} \subseteq A$, the extension $B = A^{co H} \subseteq A$ is called a Hopf Galois extension, if the canonical map
\[ \chi : A \otimes_B A \to A \otimes H, \quad a \otimes_B a' \mapsto aa'_{(0)} \otimes a'_{(1)} \] (2.6)
is bijective, where $\otimes_B$ is the balanced tensor product over $B$ (i.e. $ab \otimes_B a' = a \otimes_B ba'$ for all $a,a' \in A$ and $b \in B$).

In the following, we will always assume that for any Hopf Galois extension $B = A^{co H} \subseteq A$, $A$ is a faithful flat left $B$-module.

Given a Hopf Galois extension $B = A^{co H} \subseteq A$, the translation map is
\[ \tau := \chi^{-1}|_{1 \otimes H} : H \to A \otimes_B A, \quad h \mapsto h^{<1> \otimes_B h^{<2>}}. \] (2.7)

Since the canonical map $\chi$ is left $B$ linear, then the inverse of the canonical map can be determined by the translation map. It is shown in [3 Prop. 3.6] and [4 Lemma 34.4] that the translation map of a Hopf Galois extension satisfy the following properties:

\begin{align*}
h^{<1> \otimes_B h^{<2>}}_{(0)} \otimes h^{<2>}_{(1)} &= h^{<1>}_{(1)} \otimes_B h^{<2>}_{(1)} \otimes h_{(2)}^{<2>}, \quad (2.8) \\
h^{<1>}_{(2)} \otimes_B h^{<2>}_{(2)} \otimes S(h_{(3)}) &= h^{<1>}_{(0)} \otimes_B h^{<2>} \otimes h^{<1>}_{(1)}, \quad (2.9) \\
h^{<1>h^{<2>}}_{(0)} \otimes h^{<2>}_{(1)} &= 1_A \otimes h, \quad (2.10) \\
a_{(0)}^{<1> \otimes_B a_{(1)}^{<2>}} &= 1_A \otimes_B a, \quad (2.11)
\end{align*}

for all $h \in H$ and $a \in A$.

Now we recall cleft Hopf Galois extensions.

Definition 2.2. A Hopf–Galois extension $B = A^{co H} \subseteq A$ is cleft if there is a convolution invertible right $H$ comodule map $\gamma : H \to A$.

A Hopf Galois extension has normal basis property, if $A \simeq B \otimes H$ as left $B$-modules and right $H$-comodules.
Definition 2.3. Given a Hopf algebra $H$ and an algebra $B$, we call $H$ measures $B$, if there is a linear map $H \otimes B \rightarrow B$, given by $h \otimes b \mapsto h \triangleright b$, such that

1. $h \triangleright 1 = \epsilon(h)1$,
2. $h \triangleright (bb') = (h_{(1)} \triangleright b)(h_{(2)} \triangleright b')$,

for all $h \in H$ and $b, b' \in A$.

It was shown in [7] that we can construct an algebra by $H$ and $A$, if $H$ measures $A$.

Lemma 2.4. Let $H$ be a Hopf algebra, $B$ be an algebra and $H$ measures $B$. If there is a convolution invertible linear map $\sigma : H \otimes H \rightarrow B$ (i.e. there is a linear map $\sigma^{-1} : H \otimes H \rightarrow B$, such that $\sigma(h_{(1)}, h'_{(1)})\sigma^{-1}(h_{(2)}, h'_{(2)}) = \epsilon(h)\epsilon(h')1 = \epsilon^{-1}(h_{(1)}, h'_{(1)})\sigma(h_{(2)}, h'_{(2)})$ for all $h, h' \in H$), such that

1. $1 \triangleright b = b$,
2. $h \triangleright (k \triangleright b) = \sigma(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \triangleright b)\sigma^{-1}(h_{(3)}, k_{(3)})$,
3. $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$,
4. $(h_{(1)} \triangleright \sigma(k_{(1)}, m_{(1)}))\sigma(h_{(2)}, k_{(2)}m_{(2)}) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m)$,

for all $h, k, m \in H$ and $b \in B$. Then there is an algebra structure on $B\#_\sigma H$, which is equal to $B \otimes H$ as vector space, with the product given by

\[ (b\#_\sigma h)(b'\#_\sigma h') = b(h_{(1)} \triangleright b')\sigma(h_{(2)}, h'_{(1)})\#_\sigma h_{(3)}h'_{(2)}, \]  

(2.12)

for all $h, h' \in H$ and $b, b' \in B$. Here we have written $b\#_\sigma h$ for the tensor product $b \otimes h$. Conversely, assume $H$ measures $B$ and $\sigma : H \otimes H \rightarrow B$ is a convolution invertible linear map, if the vector space $B\#_\sigma H$ (equal to $B \otimes H$ as vector space) with a product defined by (2.12) is an associative algebra with identity element $1\#\sigma 1$, then we have the four conditions above.

The algebra $B\#_\sigma H$ given above is called the crossed product of $B$ and $H$. Let $H$ measures $B$, we call $\sigma$ an unital cocycle, if $\sigma(h, 1) = \sigma(1, h) = \epsilon(h)1$ and

\[ (h_{(1)} \triangleright \sigma(k_{(1)}, m_{(1)}))\sigma(h_{(2)}, k_{(2)}m_{(2)}) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m), \]  

(2.13)

for any all $h, k, m \in H$.

If $\sigma$ is trivial (i.e. $\sigma(h, h') = \epsilon(hh')1$), then from Lemma 2.4 we know that $B$ is a left module algebra of $H$ and $B\# H$ is the smash product of $B$ and $H$ given by (2.2).

It is known (cf. [14], Proposition 7.2.7) that:

Proposition 2.5. Let $H$ measures $B$ with a convolution invertible linear map $\sigma : H \otimes H \rightarrow B$, such that $B\#_\sigma H$ is an unital associative algebra, then we have the following properties of $\sigma$:

1. $\sigma^{-1}(h_{(1)}, k_{(1)}m_{(1)})(h_{(2)} \triangleright \sigma^{-1}(k_{(2)}, m_{(2)})) = \sigma^{-1}(h_{(1)}k_{(1)}, m)\sigma^{-1}(h_{(2)}, k_{(2)})$,
2. $h \triangleright \sigma(k, m) = \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m_{(1)})\sigma^{-1}(h_{(3)}, k_{(3)}m_{(2)})$,
3. $h \triangleright \sigma^{-1}(k, m) = \sigma(h_{(1)}, k_{(1)}m_{(1)})\sigma^{-1}(h_{(2)}k_{(2)}, m_{(2)})\sigma^{-1}(h_{(3)}, k_{(3)})$,
4. $(h_{(3)} \triangleright \sigma^{-1}(S(h_{(4)}), h_{(5)}))\sigma(h_{(2)}, S(h_{(3)})) = \epsilon(h)1$. 


Proof. We can see that (1) and (2) can be derived from Lemma \[2.4\] and (3) can be derived from (1). Now let’s check (4):
\[
\begin{align*}
(h_{(1)} \triangleright \sigma^{-1}(S(h_{(4)}), h_{(5)}))\sigma(h_{(2)}, h_{(3)}) &= \sigma(h_{(1)}, S(h_{(8)})h_{(9)})\sigma^{-1}(h_{(2)}S(h_{(7)}), h_{(10)})\sigma^{-1}(h_{(3)}, S(h_{(6)})\sigma(h_{(4)}, S(h_{(5)}))) \\
&= \sigma(h_{(1)}, S(h_{(4)}h_{(5)})\sigma^{-1}(h_{(2)}S(h_{(3)}), h_{(6)}) \\
&= \epsilon(h)1,
\end{align*}
\]
for any \(h \in H\). \(\square\)

We can see that \(B = (B\#_A H)^{co H} \subseteq B\#_A H\) is a Hopf Galois extension with the coaction \(\delta(b\#_A h) := b\#_A h_{(1)} \otimes h_{(2)}\) for all \(b\#_A h \in B\#_A H\). The inverse of its canonical map is given by \[14\], Theorem 8.2.4) that:
\[
\chi^{-1}(b\#_A g \otimes h) = (b\#_A g)(\sigma^{-1}(S(h_{(2)}), h_{(3)})\#_A S(h_{(1)})) \otimes_B 1\#_A h_{(4)} , \tag{2.14}
\]
for all \(b \in B\) and \(g, h \in H\). Indeed, we can check that
\[
\begin{align*}
\chi((b\#_A g)(\sigma^{-1}(S(h_{(2)}), h_{(3)})\#_A S(h_{(1)})) \otimes_B 1\#_A h_{(4)}) &= (b\#_A g)(\sigma^{-1}(S(h_{(3)}), h_{(4)})\sigma(S(h_{(2)}), h_{(5)})\#_A S(h_{(1)})h_{(6)}) \otimes h_{(7)} \\
&= b\#_A g \otimes h.
\end{align*}
\]
We also have
\[
\chi^{-1}(\chi(b\#_A g \otimes_B b'\#_A h)) \\
&= (b\#_A g)(b'\#_A h_{(1)})((\sigma^{-1}(S(h_{(3)}), h_{(4)})\#_A S(h_{(2)})) \otimes_B 1\#_A h_{(5)} \\
&= (b\#_A g)(b'(h_{(1)} \triangleright \sigma^{-1}(S(h_{(6)}), h_{(7)}))(\sigma(h_{(2)}, S(h_{(5)}))\#_A h_{(3)}h_{(4)}) \otimes_B 1\#_A h_{(8)} \\
&= (b\#_A g)(b'\#_A 1) \otimes_B 1\#_A h \\
&= b\#_A g \otimes_B b'\#_A h,
\]
for all \(b, b' \in B\) and \(g, h \in H\), where the third step uses Proposition \[2.5\] It is known (cf. \[14\], Theorem 8.2.4) that:

**Theorem 2.6.** Let \(B = A^{co H} \subseteq A\) be a Hopf Galois extension. Then the following are equivalent:

- \(B = A^{co H} \subseteq A\) is cleft.
- \(B = A^{co H} \subseteq A\) has normal basis property.
- \(A \simeq B\#_A H\) as left \(B\)-modules and right \(H\)-comodule algebras.

2.2. **Bialgebroids.** Here we also give an introduction of bialgebroids (cf. \[4, 5\]). For an algebra \(B\), a \(B\)-ring is a triple \((A, \mu, \eta)\). Here \(A\) is a \(B\)-bimodule and \(\mu : A \otimes_B A \to A\) and \(\eta : B \to A\) are \(B\)-bimodule maps, satisfying the associativity and unit conditions
\[
\mu \circ (\mu \otimes B) \id_A = \mu \circ (\id_A \otimes B \mu) \quad \text{and} \quad \mu \circ (\eta \otimes B) \id_A = \id_A = \mu \circ (\id_A \otimes B \eta) \tag{2.15}
\]
A morphism of \(B\)-rings \(f : (A, \mu, \eta) \to (A', \mu', \eta')\) is an \(B\)-bimodule map \(f : A \to A'\), such that \(f \circ \mu = \mu' \circ (f \otimes_B f)\) and \(f \circ \eta = \eta'\).

**Remark 2.7.** Let \(B\) and \(A\) be algebras, if there is an algebra map \(\eta : B \to A\), then \(A\) is a \(B\)-bimodule with \(b \triangleright a \triangleleft b' = \eta(b)a\eta(b')\). Moreover, \(A\) is a \(B\)-ring with the product obtained from the universality of the coequaliser \(A \otimes A \to A \otimes_B A\) which identifies an element \(ab \otimes a'\) with \(a \otimes ba'\).
For an algebra $B$, a $B$-coring is a triple $(C, \Delta, \epsilon)$. Here $C$ is an $B$-bimodule and $\Delta : C \to C \otimes_B C$ and $\epsilon : C \to B$ are $B$-bimodule maps, satisfying the coassociativity and counit conditions.

$$(\Delta \otimes_B \text{id}_C) \circ \Delta = (\text{id}_C \otimes_B \Delta) \circ \Delta \quad \text{and} \quad (\epsilon \otimes_B \text{id}_C) \circ \Delta = \text{id}_C = (\text{id}_C \otimes_B \epsilon) \circ \Delta.$$  

(2.16)

A morphism of $B$-corings $f : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ is a $B$-bimodule map $f : C \to C'$, such that $\Delta' \circ f = (f \otimes_B f) \circ \Delta$ and $\epsilon' \circ f = \epsilon$.

**Definition 2.8.** Let $B$ be an algebra. A left $B$-bialgebroid $\mathcal{L}$ consists of a $B \otimes B^{\text{op}}$-ring $\mathcal{L}$ with the unit $\eta : B \otimes B^{\text{op}} \to \mathcal{L}$. The restrictions of $\eta$

$$s := \eta(\cdot \otimes_B 1_B) : B \to \mathcal{L} \quad \text{and} \quad t := \eta(1_B \otimes_B \cdot) : B^{\text{op}} \to \mathcal{L}$$

are called source and target map, with their ranges commute in $B$.

Moreover, $\mathcal{L}$ is a $B$-coring $(\mathcal{L}, \Delta, \epsilon)$ on the same vector space $\mathcal{L}$. They are subject to the following compatibility axioms.

(i) The bimodule structure in the $B$-coring $(\mathcal{L}, \Delta, \epsilon)$ is related to the $B \otimes B^{\text{op}}$-ring $\mathcal{L}$ via

$$b \triangleright a \triangleleft b' := s(b)t(b')a, \quad \text{for } b, b' \in B, a \in \mathcal{L}. \quad (2.17)$$

(ii) Considering $\mathcal{L}$ as an $B$-bimodule as in (2.17), the coproduct $\Delta$ corestricts to an algebra map from $\mathcal{L}$ to

$$\mathcal{L} \times_B \mathcal{L} := \{ \sum_i a_i \otimes_B a_i' \mid \sum_i a_i t(b) \otimes_B a_i' = \sum_i a_i \otimes_B a_i' s(b), \quad \forall b \in B \}, \quad (2.18)$$

where $\mathcal{L} \times_B \mathcal{L}$ is an algebra via factorwise multiplication.

(iii) The counit $\epsilon$ is a left character on the $B$-ring $(\mathcal{L}, s)$:

1. $\epsilon(1_\mathcal{L}) = 1_B$.
2. $\epsilon(s(b)a) = b\epsilon(a)$,
3. $\epsilon(\epsilon(s(a)')\epsilon(a')) = \epsilon(\epsilon(\epsilon(a')\epsilon(a')))$,

for all $a, a' \in \mathcal{L}$ and $b \in B$.

Morphisms between left $B$ bialgebroids are $B$-coring maps which are also algebra maps. Given a left $B$-bialgebroid $\mathcal{L}$, then there is an algebra structure on $B \text{Hom}_B(\mathcal{L} \otimes_B \mathcal{L} \otimes_B B^{\text{op}} \mathcal{L}, B)$ with the (convolution) product given by

$$f \ast g(a, a') := f(a_{(1)}, a'_{(1)})g(a_{(2)}, a'_{(2)}), 
$$

(2.19)

for all $a, a' \in \mathcal{L}$ and $f, g \in B \text{Hom}_B(\mathcal{L} \otimes_B \mathcal{L} \otimes_B B^{\text{op}} \mathcal{L}, B)$. The unit of this algebra is $\epsilon : a \otimes a' \mapsto \epsilon(aa')$. Moreover, the $B$-bimodule structure on $\mathcal{L} \otimes_B \mathcal{L} \otimes_B B^{\text{op}} \mathcal{L}$ is given by $b \triangleright (a \otimes a') \triangleleft b' = s(b)t(b')a \otimes a'$, for all $b, b' \in B$, and the balanced tensor product $\mathcal{L} \otimes_B \mathcal{L} \otimes_B B^{\text{op}} \mathcal{L}$ is induced by the algebra map $\epsilon : B \otimes B^{\text{op}} \to \mathcal{L}$.

**Definition 2.9.** Let $\mathcal{L}$ be a left $B$-bialgebroid, an invertible normalised $2$-cocycle on $\mathcal{L}$ is a convolution normalised $2$-cocycle $\tilde{\sigma} \in B \text{Hom}_B(\mathcal{L} \otimes_B \mathcal{L} \otimes_B B^{\text{op}} \mathcal{L}, B)$, such that

1. $\tilde{\sigma}(s(b)t(b')a, a') = s\tilde{\sigma}(a, a')b'$ (bilinearity),
2. $\tilde{\sigma}(a, s(\tilde{\sigma}(a_{(1)}, a'_{(1)}))a'_{(2)}a''_{(2)}) = \tilde{\sigma}(s(\tilde{\sigma}(a_{(1)}, a'_{(1)}))a_{(2)}a'_{(2)}, a'')$ (cocycle condition),
3. $\sigma(1_\mathcal{L}, a) = \epsilon(a) = \sigma(a, 1_\mathcal{L})$ (normalisation),

for all $a, a', a'' \in \mathcal{L}$ and $b, b' \in B$.

It is known in [3], we can define a new left bialgebroid by an invertible normalised $2$-cocycle.
Proposition 2.10. Let $L$ be a left $B$-bialgebroid and $\tilde{\sigma} \in B \text{Hom}_B(L \otimes B \otimes B, L \otimes B)$ be an invertible normalised 2-cocycle, with inverse $\tilde{\sigma}^{-1}$, then the $B$-coring $L$ with the twisted product

$$a \cdot \tilde{\sigma} a' := s(\tilde{\sigma}(a_{(1)}, a'_{(1)}))t(\tilde{\sigma}^{-1}(a_{(3)}, a'_{(3)}))a_{(2)}a'_{(2)},$$

for all $a, a' \in L$, constitute a left $B$-bialgebroid $L_{\tilde{\sigma}}$.

The proposition above can also apply to Hopf algebras, which are left bialgebroid over $\mathbb{C}$.

3. Ehresmann-Schauenburg bialgebroids

Ehresmann Schauenburg Bialgebroids can be viewed as the quantisation of Gauge groupoids associated to principal bundles. Here we will first give the definition of Ehresmann Schauenburg Bialgebroids [4, §34.13].

Definition 3.1. Let $B = A^{coH} \subseteq A$ be a Hopf Galois extension such that $A$ is a faithful flat left $B$-module, the $B$-bimodule

$$C(A, H) := \{a \otimes \hat{a} \in A \otimes A \mid a_{(0)} \otimes \tau(a_{(1)})\hat{a} = a \otimes \hat{a} \otimes_B 1\},$$

is a $B$-coring with the coring product and counit given by

$$\Delta(a \otimes \hat{a}) = a_{(0)} \otimes \tau(a_{(1)}) \otimes \hat{a},$$

$$\epsilon(a \otimes \hat{a}) = a\hat{a}.$$

Moreover, $C(A, H)$ is a $B \otimes B^{op}$-ring with the product given by

$$(a \otimes \hat{a}) \bullet_C (a' \otimes \hat{a}') = aa' \otimes \hat{a}'\hat{a},$$

for all $a \otimes \hat{a}, a' \otimes \hat{a}' \in C$. The source and target maps are

$$s(b) = b \otimes 1,$$

$$t(b) = 1 \otimes b.$$

All of the structures given above form a left $B$-bialgebroid, which is called Ehresmann Schauenburg bialgebroid.

It was shown in [1], [8] that the $B$-bimodule $C(A, H)$ is isomorphic to the $B$-bimodule of coinvariant elements, that is

$$C(A, H) \cong (A \otimes A)^{coH} := \{a \otimes \hat{a} \in A \otimes A \mid a_{(0)} \otimes \hat{a}_{(0)} \otimes a_{(1)}\hat{a}_{(1)} = a \otimes \hat{a} \otimes 1 \otimes H\}.$$

3.1. 2-cocycles on Ehresmann-Schauenburg bialgebroids. Let $B = A^{coH} \subseteq A$ be a cleft Hopf Galois extension, then by Theorem 2.6 we know $C(A, H) \cong C(B\#^*H, H)$. In particular, $B = (B\#H)^{coH} \subseteq B\#H$ is a cleft Hopf Galois extension. From (3.7) we can see

$$C(B\#H, H) = \{b\#h_{(1)} \otimes b'\#S(h_{(2)}) \mid \forall b, b' \in B \text{ and } \forall h \in H\}.$$  

We can also see the coproduct is given by

$$\Delta(b\#h_{(1)} \otimes b'\#S(h_{(2)})) = b\#h_{(1)} \otimes 1\#S(h_{(2)}) \otimes_B 1\#h_{(3)} \otimes b'\#S(h_{(4)}),$$

for all $b, b' \in B$ and $h, h' \in H$. 

7
since from (2.14) we know the translation map $\tau : H \to B\#H \otimes_B B\#H$ is
\[
\tau(h) = 1\#S(h_{(1)}) \otimes_B 1\#h_{(2)},
\] (3.10)
for all $h \in H$.

**Lemma 3.2.** For a crossed product $B\#_{\sigma}H$, with $H$ being cocommutative and the image of $\sigma$ belonging to the centre of $B$, then the linear map $\tilde{\sigma} \in B\text{Hom}_B(C(B\#H,H) \otimes_B B^{opp} C(B\#H,H), B)$ given by
\[
\tilde{\sigma}(b\#h_{(1)} \otimes b'\#S(h_{(2)}), c\#g_{(1)} \otimes c'\#S(g_{(2)})) := b(h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)})((h_{(3)}g_{(2)}) \triangleright c')(h_{(4)} \triangleright b'),
\] (3.11)
for all $b\#h_{(1)} \otimes b'\#S(h_{(2)}), c\#g_{(1)} \otimes c'\#S(g_{(2)}) \in C(B\#H,H)$ is an invertible normalised 2-cocycle.

**Proof.** Since $H$ is cocommutative and the image of $\sigma$ belongs to the centre of $B$, then by Lemma 2.3 $B$ is a left $H$-module algebra, which ensures the smash product $B\#H$ is well defined. Let $X = b\#h_{(1)} \otimes b'\#S(h_{(2)}), Y = c\#g_{(1)} \otimes c'\#S(g_{(2)})$ and $Z = d\#h_{(1)} \otimes d'\#S(h_{(2)})$ be arbitrary three elements in $C(B\#H,H)$. First we can show this cocycle is well defined over the balanced tensor over $B \otimes B^{opp}$: On the one hand we have
\[
\tilde{\sigma}(X\eta(d \otimes d'), Y) = \tilde{\sigma}((b\#h_{(1)} \otimes b'\#S(h_{(2)}))\eta(d \otimes d'), c\#g_{(1)} \otimes c'\#S(g_{(2)}))
= b(h_{(1)} \triangleright d)(h_{(2)} \triangleright c)\sigma(h_{(3)}, g_{(1)})((h_{(4)}g_{(2)}) \triangleright c')(h_{(5)} \triangleright (d'b')),
\]
for all $d, d' \in B$; On the other hand
\[
\tilde{\sigma}(X, \eta(d \otimes d'))Y) = \tilde{\sigma}((b\#h_{(1)} \otimes b'\#S(h_{(2)})), \eta(d \otimes d')c\#g_{(1)} \otimes c'\#S(g_{(2)}))
= \tilde{\sigma}((b\#h_{(1)} \otimes b'\#S(h_{(2)})), dc\#g_{(1)} \otimes c'(S(g_{(3)} \triangleright d')\#S(g_{(2)}))
= b(h_{(1)} \triangleright (dc))\sigma(h_{(2)}, g_{(1)})((h_{(3)}g_{(2)}) \triangleright (c'(S(g_{(3)} \triangleright d')))(h_{(4)} \triangleright b')
= b(h_{(1)} \triangleright d)(h_{(2)} \triangleright c)\sigma(h_{(3)}, g_{(1)})((h_{(4)}g_{(2)}) \triangleright c')(h_{(5)} \triangleright (d'b')).
\]
The inverse $\tilde{\sigma}^{-1}$ is given by
\[
\tilde{\sigma}^{-1}(X, Y) := b(h_{(1)} \triangleright c)\sigma^{-1}(h_{(2)}, g_{(1)})((h_{(3)}g_{(2)}) \triangleright c')(h_{(4)} \triangleright b').
\] (3.12)

We can see that
\[
\tilde{\sigma}^{-1} \star \tilde{\sigma}(X, Y) = \epsilon(X, Y),
\]
where $\epsilon$ is the unit in the algebra $B\text{Hom}_B(L \otimes_B B^{opp} L, B)$. Similarly, we can also see $\tilde{\sigma} \star \tilde{\sigma}^{-1} = \epsilon$.

It is clear that $\tilde{\sigma}$ is left $B$-linear, we can also show that $\tilde{\sigma}$ is also right $B$-linear:
\[
\tilde{\sigma}(t(b'')X, Y) = \tilde{\sigma}(t(b'')(b\#h_{(1)} \otimes b'\#S(h_{(2)}), c\#g_{(1)} \otimes c'\#S(g_{(2)}))
= \tilde{\sigma}(b\#h_{(1)} \otimes b'(S(h_{(3)}) \triangleright b'')\#S(h_{(2)}), c\#g_{(1)} \otimes c'\#S(g_{(2)}))
= b(h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)})((h_{(3)}g_{(2)}) \triangleright c')(h_{(4)} \triangleright (b'(S(h_{(5)} \triangleright b'')))
= b(h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)})((h_{(3)}g_{(2)}) \triangleright c')(h_{(4)} \triangleright b'b''),
\]
for all $b'' \in B$. Now let's show the cocycle condition of $\tilde{\sigma}$.
On the one hand we have:
\[
\tilde{\sigma}(X, s(\tilde{\sigma}(Y_1, Z_1)Y_{23}) = \tilde{\sigma}(b\#h_{11} \otimes b'\#S(h_{22}), c(g_{11}) \triangleright d)\sigma(g_{22}, k_{11})\#g_{33}k_{22} \otimes d'(S(k_{44}) \triangleright c')\#S(k_{55})S(g_{66}))
\]
\[
= b\left(h_{11} \triangleright (c(g_{11}) \triangleright d)\sigma(g_{22}, k_{11})\right)\sigma(h_{22}, g_{33}k_{22})((h_{33}g_{44}k_{33}) \triangleright (d'(S(k_{44}) \triangleright c')))\left(h_{44} \triangleright b'\right).
\]

On the other hand,
\[
\tilde{\sigma}(s(\tilde{\sigma}(X_1, Y_1)X_{23}, Z) = \tilde{\sigma}(b(h_{11} \triangleright c)\sigma(h_{22}, g_{11})\#h_{33}g_{22} \otimes c'(S(g_{44}) \triangleright b')\#S(g_{55})S(h_{66}), d\#k_{11} \otimes d'\#S(k_{22}))
\]
\[
= b(h_{11} \triangleright c)\sigma(h_{22}, g_{11})((h_{33}g_{22}) \triangleright d)\sigma(h_{44}g_{33}, k_{11})\sigma(h_{55}g_{44}k_{33} \triangleright d')(h_{66}g_{55}) \triangleright (c'(S(g_{66}) \triangleright b'))).
\]

Compare the results on both hand sides, it is sufficient to show
\[
h_{11} \triangleright ((g_{11} \triangleright d)\sigma(g_{22}, k_{11}))\sigma(h_{22}, g_{33}k_{22}) = \sigma(h_{11}, g_{11})((h_{22}g_{23}) \triangleright d)\sigma(h_{33}g_{33}, k).
\]

We can see the left hand side is
\[
h_{11} \triangleright ((g_{11} \triangleright d)\sigma(g_{22}, k_{11}))\sigma(h_{22}, g_{33}k_{22})
\]
\[
= (h_{11} \triangleright (g_{11} \triangleright d))(h_{22} \triangleright \sigma(g_{22}, k_{11}))\sigma(h_{33}g_{33}, k_{22})
\]
\[
= (h_{11} \triangleright (g_{11} \triangleright d))\sigma(h_{22}, g_{33})\sigma(h_{33}g_{33}, k)
\]
\[
= \sigma(h_{11}, g_{11})((h_{22}g_{23}) \triangleright d)\sigma(h_{33}g_{33}, k),
\]

where the second step uses \[(2,13)\], and the last step uses Lemma \[2,4\]. Finally, we can show the normalisation condition:
\[
\tilde{\sigma}(X, 1) = b(h_{11} \triangleright b') = \epsilon(X)
\]
\[
\tilde{\sigma}(1, X) = b(h_{11} \triangleright b') = \epsilon(X).
\]

\[\square\]

**Lemma 3.3.** For a crossed product \(B\#_\sigma H\), with \(H\) being cocommutative and the image of \(\sigma\) belonging to the centre of \(B\), then the map \(\phi : \mathcal{C}(B\#H, H) \to \mathcal{C}(B\#_\sigma H, H)\) is a \(B\)-coring map, which is given by
\[
\phi(b\#h_{11} \otimes b'\#S(h_{22})) := b\#h_{11} \otimes b'\sigma^{-1}(S(h_{33}), h_{44})\#_\sigma S(h_{22}) \quad (3.13)
\]
for any \(b\#h_{11} \otimes b'\#S(h_{22}) \in \mathcal{C}(B\#H, H)\).

**Proof.** First we check \(\epsilon = \epsilon^\sigma \circ \phi\), where \(\epsilon^\sigma\) is the counit of \(\mathcal{C}(B\#_\sigma H, H)\). Let \(X = b\#h_{11} \otimes b\#_\sigma S(h_{22}) \in \mathcal{C}(B\#H, H)\), then
\[
\epsilon^\sigma(\phi(X)) = (b\#_\sigma h_{11})(b'\sigma^{-1}(S(h_{33}), h_{44})\#_\sigma S(h_{22}))
\]
\[
= b(h_{11} \triangleright b')(h_{22} \triangleright \sigma^{-1}(S(h_{33}), h_{44}))(h_{55}, S(h_{44})\#_\sigma 1)
\]
\[
= b(h \triangleright b')\#_\sigma 1
\]
\[
= \epsilon(b\#h_{11} \otimes b'\#S(h_{22})),
\]

where in the third step we use Proposition \[2,3\]. Here we always identify \(B\) with its image in \(B\#H\) and \(B\#_\sigma H\) by \(b \mapsto b\#1\) and \(b \mapsto b\#_\sigma 1\) respectively. We can see \(\phi\) is left
B-module map. We can also check it is right B-linear:

\[
\phi(X \triangleright b'') = \phi(b\# h_{(1)} \otimes b'(S(h_{(3)}) \triangleright b'') \# S(h_{(2)})) \\
= b\# h_{(1)} \otimes b'(S(h_{(3)}) \triangleright b'')\sigma^{-1}(S(h_{(3)}), h_{(4)}) \#_\sigma S(h_{(2)}) \\
= b\# h_{(1)} \otimes b'\sigma^{-1}(S(h_{(4)}), h_{(5)}) (S(h_{(3)}) \triangleright b'') \#_\sigma S(h_{(2)}) \\
= \phi(X) \triangleright b''
\]

for all \( b'' \in B \), where the third step uses the fact that \( H \) is cocommutative and the image of \( \sigma \) belongs to the centre of \( B \). Recall that the translation map of the Hopf Galois extension \( B = (B\#_\sigma H)^{coH} \subseteq B\#_\sigma H \) is given by

\[
\tau(h) = \sigma^{-1}(S(h_{(2)}), h_{(3)}) \#_\sigma S(h_{(1)}) \otimes B 1\#_\sigma h_{(4)},
\]

for all \( h \in H \). So we have

\[
(\phi \otimes_B \phi)(\Delta(X)) \\
= (\phi \otimes_B \phi)(b\# h_{(1)} \otimes 1\# S(h_{(2)}) \otimes_B 1\# h_{(3)} \otimes b'\# S(h_{(4)})) \\
= b\# h_{(1)} \otimes \sigma^{-1}(S(h_{(3)}), h_{(4)}) \#_\sigma S(h_{(2)}) \otimes_B 1\# h_{(5)} \otimes b'\sigma^{-1}(S(h_{(7)}), h_{(8)}) \#_\sigma S(h_{(6)}) \\
= \Delta^\sigma(b\# h_{(1)} \otimes b'\sigma^{-1}(S(h_{(3)}), h_{(4)}) \#_\sigma S(h_{(2)})) \\
= \Delta^\sigma(\phi(X)),
\]

where \( \Delta^\sigma \) is the coproduct of \( C(B\#_\sigma H, H) \).

\[\square\]

**Theorem 3.4.** For a crossed product \( B\#_\sigma H \), with \( H \) being cocommutative and the image of \( \sigma \) belonging to the centre of \( B \), then there is an invertible normalized 2-cocycle \( \tilde{\sigma} \) on \( C(B\# H, H) \), such that \( \phi : C(B\#_\sigma H, H)^{\tilde{\sigma}} \rightarrow C(B\#_\sigma H, H) \) is an isomorphism of left \( B \)-bialgebroids, where \( \tilde{\sigma} \) is given by \((3.11)\) and \( \phi \) is given by \((3.13)\).

**Proof.** Since \( \phi \) is a coring map, so we only need to show \( \phi \) is an algebra map. Let \( X = b\# h_{(1)} \otimes b'\# S(h_{(2)}) \), \( Y = c\# g_{(1)} \otimes c'\# S(g_{(2)}) \in C(B\# H, H) \). On the one hand,

\[
\phi(X \triangleright Y) \\
= \phi(\tilde{\sigma}(b\# h_{(1)} \otimes 1\# S(h_{(2)}), c\# g_{(1)} \otimes 1\# S(g_{(2)})) \# h_{(3)} g_{(3)} \otimes \\
((S(g_{(3)})S(h_{(4)})) \triangleright \sigma^{-1}(1\# h_{(5)} \otimes b'\# S(h_{(7)}), 1\# g_{(6)} \otimes c'\# S(g_{(7)}))) \# S(g_{(4)})S(h_{(4)})) \\
= \phi(b\# h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)}) \# h_{(3)} g_{(2)} \otimes \\
(S(g_{(3)})S(h_{(5)})) \triangleright (\sigma^{-1}(h_{(6)}, g_{(3)})((h_{(7)}g_{(6)}) \triangleright c')(h_{(8)} \triangleright b'))) \# S(g_{(3)})S(h_{(4)})) \\
= b\# h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes \\
(S(g_{(3)})S(h_{(7)})) \triangleright \sigma^{-1}(h_{(8)}, g_{(6)}c'(S(g_{(8)}) \triangleright b')\sigma^{-1}(S(g_{(4)})S(h_{(5)}), h_{(6)} g_{(5)})) \#_\sigma S(g_{(3)})S(h_{(4)}) \\
= b\# h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes \\
(S(g_{(3)})S(h_{(7)})) \triangleright \sigma^{-1}(h_{(8)}, g_{(6)}c'(S(g_{(8)}) \triangleright b')\sigma^{-1}(S(g_{(4)})S(h_{(5)}), h_{(6)} g_{(5)})) \#_\sigma S(g_{(3)})S(h_{(4)}) \\
= b\# h_{(1)} \triangleright c)\sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes \\
\sigma^{-1}(S(g_{(4)}), g_{(5)}) \sigma^{-1}(S(g_{(6)})S(h_{(5)}), h_{(6)} c'(S(g_{(7)}) \triangleright b') \#_\sigma S(g_{(3)})S(h_{(4)}), \)
where in the 4th, 5th and 6th step we use that $H$ is cocommutative and the image of $\sigma$ belongs to the centre of $B$, and the 5th step also uses Proposition 2.5. On the other hand,

$$
\phi(X) \hat{\phi}(Y) = (b \#_\sigma h_{(1)} \otimes b' \sigma^{-1}(S(h_{(2)}), h_{(4)}) \#_\sigma S(h_{(2)})) (c \#_\sigma g_{(1)} \otimes c' \sigma^{-1}(S(g_{(3)}), g_{(4)}) \#_\sigma S(g_{(2)})) = b(h_{(1)} \triangleright c) \sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes c' \sigma^{-1}(S(g_{(5)}), g_{(6)}) (S(g_{(5)}), h_{(7)}) \sigma^{-1}(S(h_{(6)}, h_{(7)})) \sigma(S(g_{(5)}), S(h_{(5)})) \#_\sigma S(g_{(3)}) S(h_{(4)})) = b(h_{(1)} \triangleright c) \sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes c'(S(g_{(5)}), g_{(6)}) \sigma^{-1}(S(h_{(6)}), S(h_{(6)})) \sigma^{-1}(S(g_{(5)}), S(g_{(5)})) \#_\sigma S(g_{(3)}) S(h_{(4)})) = b(h_{(1)} \triangleright c) \sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes c'(S(g_{(5)}), g_{(6)}) \sigma^{-1}(S(h_{(6)}), S(h_{(6)})) \sigma^{-1}(S(g_{(5)}), S(g_{(5)})) \#_\sigma S(g_{(3)}) S(h_{(4)}))
$$

where in the 3rd, 4th and 5th steps we use that $H$ is cocommutative and the image of $\sigma$ belongs to the centre of $B$, and the 4th step also uses Proposition 2.5. Since $H$ is cocommutative, we have $\phi(X \triangleright Y) = \phi(X) \hat{\phi}(Y)$.

Similarly, we can also show the following theorem:

**Theorem 3.5.** For a crossed product $B \#_\sigma H$, with the image of $\sigma$ belonging to the centre of $B$, if the action of $H$ on $B$ is trivial (i.e. $h \triangleright b = e(h)b$, for all $h \in H$ and $b \in B$), then there is an invertible normalised 2-cocycle $\tilde{\sigma}$ on $C(B \#_\sigma H, H)$, such that $\phi : C(B \#_\sigma H, H)^\sigma \to C(B \#_\sigma H, H)$ is an isomorphism of left $B$-bialgebroids, where $\tilde{\sigma}$ is given by (3.11) and $\phi$ is given by (3.13).

**Proof.** If $H$ measures $B$ with trivial action, and $B \#_\sigma H$ is an unital associative algebra with the image of $\sigma$ belongs to the centre of $B$, then we can also get Lemma 3.2. Lemma 3.3 and Theorem 3.4 without asking $H$ being cocommutative. Indeed, in this case $B \# H$ is equal to $B \otimes H$ with factorwise multiplication, so by Lemma 3.2

$$
\tilde{\sigma}(b \#_\sigma h_{(1)} \otimes b' \#_\sigma S(h_{(2)}), c \#_\sigma g_{(1)} \otimes c' \#_\sigma S(g_{(2)})) := b c \sigma(h, g) c' b', \tag{3.14}
$$

for all $b \#_\sigma h_{(1)} \otimes b' \#_\sigma S(h_{(2)}), c \#_\sigma g_{(1)} \otimes c' \#_\sigma S(g_{(2)}) \in C(B \#_\sigma H, H)$. In this case, Lemma 3.3 can be also shown, since even without the cocommutativity of $H$ we can show that $\phi$ is right $B$-linear. Moreover, in the proof of Theorem 3.4 we can see on the one hand

$$
\phi(X \triangleright Y) = b(h_{(1)} \triangleright c) \sigma(h_{(2)}, g_{(1)}) \#_\sigma h_{(3)} g_{(2)} \otimes (S(g_{(5)}), S(g_{(6)})) \sigma^{-1}(S(h_{(6)}, h_{(7)})) (S(g_{(5)}), h_{(6)} g_{(5)}) \#_\sigma S(g_{(3)}) S(h_{(4)}) = b c \sigma(h_{(1)}, g_{(1)}) \#_\sigma h_{(2)} g_{(5)} \otimes \sigma^{-1}(h_{(7)}, g_{(6)}) c' b' \sigma^{-1}(S(h_{(5)}, g_{(6)})) \sigma^{-1}(S(h_{(6)}, h_{(7)})) \sigma(S(g_{(5)}), S(h_{(4)})) \#_\sigma S(g_{(3)}) S(h_{(3)}) = b c \sigma(h_{(1)}, g_{(1)}) \#_\sigma h_{(2)} g_{(5)} \otimes c' b' \sigma^{-1}(S(g_{(5)}), g_{(6)}) \sigma^{-1}(S(h_{(5)}, h_{(6)}) \sigma(S(g_{(5)}), S(h_{(4)})) \#_\sigma S(g_{(3)}) S(h_{(4)}))
$$



where in the 2nd and 3rd steps we use the fact that action is trivial, and the image of \( \sigma \) belongs to the centre of \( B \), and the 2nd step also uses

\[
\sigma^{-1}(S(g_{(1)}), h_{(2)}g_{(2)}) = \sigma^{-1}(S(g_{(2)}), g_{(3)})\sigma^{-1}(S(h_{(2)}), h_{(3)}g_{(4)})\sigma(S(g_{(1)}), S(h_{(1)})),
\]

which can be derived from Proposition 2.5. On the other hand, Proposition 2.5, the 3rd step also uses Proposition 2.5. On the other hand,

\[
\phi(X)\phi(Y) = b(h_{(1)}c\sigma(h_{(2)}, g_{(1)})\#h_{(3)}g_{(2)} \otimes \\
\theta^{-1}(S(g_{(6)}), (g_{(7)})S(g_{(5)}) \otimes (b'\sigma^{-1}(S(h_{(6)}), h_{(7)}))\sigma(S(g_{(4)}), S(h_{(5)}))\#h_{(3)}S(h_{(4)})
\]

where in the last step we use the fact that image of \( \sigma \) belongs to the centre of \( B \) and the action is trivial. \( \square \)

Recall that a Galois object of a Hopf algebra \( H \) is a comodule algebra \( A \), such that the canonical Galois map is bijective, and \( A^{coH} = \mathbb{C} \). As a result of Theorem 2.6 and Theorem 3.5 we have:

**Corollary 3.6.** Let \( A \) be a cleft Galois object of \( H \), then \( C(A, H) \) is isomorphic to \( H^\gamma \) as Hopf algebra, where \( \gamma : H \otimes H \to \mathbb{C} \) is an invertible normalised 2-cocycle on \( H \).

**Proof.** By Theorem 2.6, we know \( A \simeq \mathbb{C}#_A H \), then by Theorem 3.5, we know \( C(\mathbb{C}#_A H, H) \simeq C(\mathbb{C}#H, H)^\gamma \). Since in [8] we know \( H \simeq C(H, H) \simeq C(\mathbb{C}#H, H) \), where the isomorphic map \( f \) is given by \( f : h \mapsto h_{(1)} \otimes S(h_{(2)}) \), and its inverse is given by \( f^{-1} : g \otimes h \mapsto ge(h) \). Then there is a 2-cocycle \( \gamma \) on \( H \) such that \( C(A, H) \simeq H^\gamma \). As we know in [15] that \( C(A, H) \) is a Hopf algebra, then we get the corollary. \( \square \)

Corollary 3.6 is also shown in [15], here we can view Theorem 3.5 as a generalisation of it.

Let \( \gamma : H \otimes H \to \mathbb{C} \) be an invertible normalised 2-cocycle on a Hopf algebra \( H \), and \( B = A^{coH} \subseteq A \) be a cleft Hopf Galois extension. It is shown in [11] that we can define a \( H^\gamma \)-comodule algebra \( A_\gamma \) on the same underlying \( H \)-comodule \( A \) (i.e. \( \delta^A =: \delta^{A_\gamma} : A_\gamma \to A_\gamma \otimes H^\gamma \)) with a new product given by

\[
a_\gamma a' := a_{(0)}a'_{(0)}\gamma^{-1}(a_{(1)}, a'_{(1)}), \quad (3.15)
\]

for all \( a, a' \in A \). Moreover, \( B = A_\gamma^{coH^\gamma} \subseteq A_\gamma \) is a Hopf Galois extension, with the translation map given by

\[
\tau_\gamma(h) := h_{(3)}^{<1>} \otimes_B h_{(3)}^{<2>}_2 \gamma(h_{(1)}), \quad (3.16)
\]

for all \( h \in H \). Indeed, since the canonical map \( \chi_\gamma \) is a left \( A_\gamma \)-module map, it is sufficient to show

\[
(\chi_\gamma \circ \tau_\gamma)(h) = h_{(3)}^{<1>}(h_{(3)}^{<2>}_0) \otimes \gamma^{-1}(h_{(3)}^{<1>}_1, h_{(3)}^{<2>}_1, h_{(3)}^{<2>}_2) \gamma(h_{(1)}), S(h_{(2)})
\]

\[
= h_{(4)}^{<1>}_1 h_{(4)}^{<2>}_2 \otimes \gamma^{-1}(S(h_{(3)}), h_{(4)}^{<2>}_1, h_{(4)}^{<2>}_2) \gamma(h_{(1)}), S(h_{(2)})
\]

\[
= 1 \otimes \gamma^{-1}(S(h_{(3)}), h_{(4)} S(h_{(2)}))
\]

\[
= 1 \otimes h,
\]

12
for all $h \in H$, where the 2nd step uses (2.9), the 3rd step uses (2.10), and the last step uses the fact that $\gamma$ is a 2-cocycle on the Hopf algebra $H$. We can also see

\[
(\chi^{-1}_\gamma \circ \chi_\gamma)(a' \otimes_B a) = a' \cdot \gamma a'_0 \cdot \gamma a_{(3)}^{1>} \otimes_B a_{(3)}^{2>} \gamma(a_{(1)}, S(a_{(2)})) = a' \cdot \gamma(a_{(0)}a_{(4)}^{1>}) \otimes_B a_{(4)}^{2>} \gamma^{-1}(a_{(1)}, a_{(4)}^{1>}) \gamma(a_{(2)}, S(a_{(3)})) = a' \cdot \gamma(a_{(0)}a_{(5)}^{1>}) \otimes_B a_{(5)}^{2>} \gamma^{-1}(a_{(1)}, S(a_{(4)})) \gamma(a_{(2)}, S(a_{(3)})) = a' \cdot \gamma(a_{(0)}a_{(1)}^{1>}) \otimes_B a_{(1)}^{2>} = a' \otimes_B a,
\]

for all $a, a' \in A_\gamma$, where the 3rd step use (2.9), and the last step uses (2.11). Therefore, $B = A^\gamma_{A^H} \subseteq A_\gamma$ is a Hopf Galois extension.

**Lemma 3.7.** Let $B = A^\gamma_{A^H} \subseteq A$ be a cleft Hopf Galois extension, then $B = A^\gamma_{A^H} \subseteq A_\gamma$ is also a cleft Hopf Galois extension.

**Proof.** Since $B = A^\gamma_{A^H} \subseteq A$ is a cleft Hopf Galois extension, from Theorem 2.6 there is an isomorphic map $F : A \to B \otimes H$ between left $B$-modules and right $H$-comodules. Define $F_\gamma = F$ on the underlying vector space of $A$. Then we can see $F_\gamma$ is left $B$-linear and right $H$-colinear:

\[
F_\gamma(b \cdot \gamma a) = F(ba) = bF(a) = b \cdot \gamma F_\gamma(a),
\]

for all $b \in B$ and $a \in A$. Thus $F_\gamma$ is left $B$ linear. We can also see

\[
F_\gamma(a_{(0)}) \otimes F_\gamma(a_{(1)}) = F(a_{(0)}) \otimes F(a_{(1)}) = F(a_{(0)}) \otimes a_{(1)} = F_{\gamma}(a_{(0)}) \otimes a_{(1)},
\]

thus $F_\gamma$ is a $H$ comodule map. Therefore, $B = A^\gamma_{A^H} \subseteq A_\gamma$ is a cleft Hopf Galois extension.

**Corollary 3.8.** Let $A$ be a cleft Galois object of $H$, and $\gamma : H \otimes H \to C$ be an invertible normalised 2-cocycle on $H$, then $C(A_\gamma, H^\gamma)$ is isomorphic to $C(A, H)^\omega$, where $\omega$ is an invertible normalised 2-cocycle on $C(A, H)$.

**Proof.** Since $A$ is a cleft Galois object, from the Lemma 3.7 we know $A_\gamma$ is a cleft Galois object of $H^\gamma$.

Therefore, by Corollary 3.6 there is an invertible normalised 2-cocycle $\rho$ on $H$, such that

\[
C(A_\gamma, H^\gamma) \simeq (H^\gamma)^\rho,
\]

Since $C(A, H) \simeq H^\sigma$, we have

\[
C(A_\gamma, H^\gamma) \simeq C(A, H)^{\sigma^{-1} \gamma \star \rho}, \quad (3.17)
\]

and $\omega = \sigma^{-1} \star \gamma \star \rho$. \qed

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