Universal Finite Subgroup of the Tate Curve

Zhen Huan

Abstract. In [10] Katz and Mazur discuss the moduli problem of the subgroup-schemes of elliptic curves. We give the classification of the finite subgroups of the Tate curve in [2]. Moreover, Katz and Mazur define the universal finite subgroup of an elliptic curve. In this paper we give an explicit construction of the universal finite subgroup of the Tate curve via isogenies and the stringy power operation of Tate K-theory.

1. Introduction

An elliptic curve $E \to S$ over a base $S$ is an smooth and proper abelian group $S$-scheme $E$ whose fiber at every geometric point is an elliptic curve. In [10] Chapter 6 Katz and Mazur discussed the moduli problem $[N\text{-Isog}]$, where the symbol $[N\text{-Isog}]$ means isogenies with finite kernel of order $N$. The $S$-scheme $[N\text{-Isog}](E/S)$ is defined to be the set of finite locally free commutative $S$-subgroup-schemes $G < E[N]$ which are of rank $N$ over $S$. In other words, $[N\text{-Isog}](E/S)$ is the set of subgroup schemes of rank $N$ in $E$. In [10] Proposition 6.5.1, Theorem 6.8.1] Katz and Mazur prove that this moduli problem is relatively representable and is finite and flat over (Ell).

Let $E$ be an elliptic curve over a commutative ring $A$. Among the subgroup-schemes of rank $N$ in $E[N]$, based on the idea in [10], we define the universal finite subgroup of $E$. Fix an integer $N \geq 0$. There exists a ring homomorphism $A \to B$ and a subgroup $G < E_B$ of rank $N$ in $E_B$, where $E_B$ is the pullback of $E$ over Spec $(B)$, with the following property. The pair $(B,G < E_B)$ is the universal subgroup of rank $N$ of $E \to \text{Spec } (A)$ in the sense that: given a ring homomorphism $A \to C$ and a subgroup $H$ of rank $N$ in $E_C$, there exists a unique ring homomorphism $g : B \to C$ compatible with the maps from $A$ such that $H = g^*(G)$, i.e. there is a pullback diagram in schemes of the form

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In this paper we show the universal finite subgroup of the Tate curve exists and provide an explicit construction of it.

The Tate curve $\text{Tate}(q)$ is an elliptic curve over $\text{Spec } \mathbb{Z}(q)$. The formal group of it is the multiplicative group. Thus, it is a form of $K$-theory. We call the theory $K_{\text{Tate}}$. It is the elliptic cohomology theory that is first discovered; See [11]. In [7] we discuss the moduli problem [N-Isog] for the Tate curve and prove in [7, Theorem 7.4] that the finite subgroups of order $N$ of the Tate curve $\text{Tate}(q)$ can be classified by the Tate $K$-theory of the symmetric group $\Sigma_N$ modulo a certain transfer ideal. The application of an intermediate theory, quasi-elliptic cohomology, is essential in the proof of this classification theorem, as shown in Section 6.3 in [7].

In this paper we construct the universal finite subgroup of order $N$ of the Tate curve $\text{Tate}(q)$. The finite subgroups of the Tate curve are the kernels of isogenies. In light of this fact, we construct the universal finite subgroup of $\text{Tate}(q)$ as the kernel of an isogeny. Furthermore, as shown in [1], there is some correspondence between isogenies and operations among complex oriented cohomology theories under some condition. By [1, Theorem A, Section 6.3], the isogenies of certain form give rise to operations of Tate $K$-theory with proper coefficient rings. The construction of the isogenies will be recalled in Section 2. Especially, these operations have relations with the additive power operation of Tate $K$-theory, which is constructed from the stringy power operation of Tate $K$-theory [5].

In [7] we present a construction of the stringy power operation. We first construct a power operation $\{\mathbb{P}_N\}_N$ of quasi-elliptic cohomology, which can be viewed as the Tate $K$-theory with coefficient ring $\mathbb{Z}[q^\pm]$ and is introduced in Section 3.1. This power operation is related to the level structure of the Tate curve. Its construction mixes power operations in $K$-theory with the natural operations of dilating and rotating loops, and can be generalized to other equivariant cohomology theories. $\{\mathbb{P}_N\}_N$ can extend uniquely to the stringy power operation of Tate $K$-theory. Moreover, from the power operation, we construct in [7, Proposition 6.5] the additive power operation $\mathbb{P}_N : \text{QEll}(X/G) \to \text{QEll}(X/G) \otimes_{\mathbb{Z}[p^\pm]} \text{QEll}(\text{pt}//\Sigma_N)/I_{tr}^\Sigma_N$, which is a ring homomorphism. In [3] Ando, Hopkins and Strickland discuss the additive power operation of Morava $E$-theories $E^0 \to E^0(B\Sigma_p)/I_{tr}$. Applying Strickland’s theorem in [16] they show that it has a nice algebro-geometric interpretation in terms of the formal group and it takes the quotient by the universal subgroup. The additive operation $\mathbb{P}_N$ plays an essential part in the construction of the isogeny whose kernel is the universal subgroup of order $N$ of the Tate curve.
In Section 5 we prove the main theorem that the finite subgroup constructed in Section 4 is indeed the universal finite subgroup of the Tate curve.

**Theorem 1.1.** The universal finite subgroup of order $N$ of $\text{Tate}(q)$ is the pair $G_{\text{univ}} := (D_N, \text{Ker}(\psi) < i_N^* \text{Tate}(q)[N])$, where the $\mathbb{Z}((q))$–algebra $D_N$ and the isogeny $\psi$ are defined in Section 4, in the sense that for any $\mathbb{Z}((q))$–algebra $R$, there is a 1-1 correspondence which is natural

\[
\{\mathbb{Z}((q)) – \text{algebra maps } D_N \longrightarrow R\} \\
\downarrow \\
\{\text{finite subgroup schemes } G \leq \text{Tate}(q)[N]_R \text{ of order } N\}
\]

where $\text{Tate}(q)[N]_R$ is the pullback

\[
\begin{array}{ccc}
\text{Tate}(q)[N]_R & \longrightarrow & \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec (R)} & \longrightarrow & \text{Spec (Z((q)))}
\end{array}
\]

It is a group scheme over $\text{Spec (R)}$. In other words, given a subgroup scheme $G \leq \text{Tate}(q)[N]_R$ of degree $N$, there exists a unique pullback square

\[
\begin{array}{ccc}
G & \longrightarrow & \text{Ker}(\psi) \\
\downarrow & & \downarrow \\
\text{Tate}(q)[N]_R & \longrightarrow & i_N^* \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec (R)} & \longrightarrow & \text{Spec (D_N)}.
\end{array}
\]

In Section 2 we recall the Tate curve and its torsion points. We discuss the finite subgroups of the Tate curve and their relation with isogenies. In Section 3 we recall Tate K-theory. Some constructions are made via quasi-elliptic cohomology. We also recall the classification theorems of the Tate curve and the power operations of the theories that we need later in Section 4. The main references are [7] and [13]. In Section 4 we construct the universal finite subgroup of $\text{Tate}(q)$ explicitly as the kernel of an isogeny constructed from the additive power operation. In Section 5 we show that the construction in Section 4 is indeed the universal finite subgroup and prove the main theorem.

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2. The Tate curve and its finite subgroups

In this section we introduce the Tate curve and its torsion points. We discuss the isogenies between the Tate curve in Example 2.1, 2.2 and 2.4, and the finite subgroups of it as kernel of isogenies. The main references are [2] Section 2.6 and [10], Section 8.7-8.8.

The Tate curve \(\text{Tate}(q)\) is the elliptic curve

\[
E_q : y^2 + xy = x^3 + a_4x + a_6
\]

whose coefficients are given by the formal power series in \(\mathbb{Z}((q))\):

\[
a_4(q) = -\frac{5}{2} \sum_{n \geq 1} n^3 q^n/(1 - q^n) \quad a_6(q) = -\frac{1}{12} \sum_{n \geq 1} (7n^5 + 5n^3) q^n/(1 - q^n).
\]

The Tate curve \(\text{Tate}(q)\), defined by (2.1) and (2.2), is a pointed curve of genus 1 over \(\mathbb{Z}((q))\). It is a one-dimensional abelian group. There is an isomorphism

\[
\hat{\mathbb{G}}_m \cong \text{Tate}(q)
\]

of formal groups; see [4], VII, 1.16. Since its formal group is multiplicative, the cohomology theory associated to the Tate curve is a form of \(K\)-theory; we call it Tate \(K\)-theory \(K_{\text{Tate}}\).

It is shown in [14], Theorem 3.1 (b)] that the Weierstrass cubic (2.1) has discriminant

\[
\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}.
\]

This implies that over \(\mathbb{Z}((q))\) the Tate curve is an elliptic curve. It is modeled on the multiplicative parameterization of elliptic curves over \(\mathbb{C}\): for any complex number \(q\) with \(0 < |q| < 1\), \(\mathbb{C}^*/q^2\) is an elliptic curve which fits into the exact sequence

\[
1 \longrightarrow q^2 \longrightarrow \mathbb{C}^* \longrightarrow \mathbb{C}^*/q^2 \longrightarrow 1.
\]

Thus, we obtain a family of elliptic curves

\[
\mathbb{C}^*/q^2 \rightarrow D^*(\mathbb{C})
\]

where \(D^*(\mathbb{C})\) is the punctured open unit disk

\[
D^*(\mathbb{C}) = \{ q \in \mathbb{C} \mid 0 < |q| < 1 \}.
\]

For \(q \in D^*(\mathbb{C})\), \(a_4(q)\) and \(a_6(q)\) converge to complex numbers. Let \(E_q\) denote the resulting elliptic curve over \(\mathbb{C}\). As shown in [14], p. 410], we have an analytic isomorphism

\[
\mathbb{C}^*/q^2 \cong E_q.
\]

If \(F\) is a non-archimedean field, let

\[
D^*(F) = \{ q \in F \mid 0 < |q| < 1 \}.
\]

We have the isomorphism of sets

\[
\text{hom}^{\text{cts}}(\mathbb{Z}((q)), F) \rightarrow D^*(F), \quad g \mapsto g(q).
\]

For any continuous homomorphism

\[
\mathbb{Z}((q)) \xrightarrow{\phi} F,
\]
there is an isomorphism of groups
\[ F^* / g(q)^\mathbb{Z} \cong g^* \text{Tate}(F). \]

See [14, p. 423] for more details.

**Example 2.1.** In this example we describe the finite subgroups and the isogenies for \( E_q \), the analytic Tate curve over \( \mathbb{C} \).

Let \( N \) be any positive integer. To give a subgroup of order \( N \) of \( E_q \), we pick a pair of integers \((d, e)\) such that \( N = de \) and \( d, e \geq 1 \) and let \( q' \) be a nonzero complex number such that \( q^d = q'^e \). Consider the isogeny
\[
(2.6) \quad G_d : \mathbb{C}^* \to \mathbb{C}^*/q^d \mathbb{Z}, \quad x \mapsto x^d.
\]

It is well-defined since \( G_d(q^d \mathbb{Z}) \subseteq q'^d \mathbb{Z} \).

We can check that \( \ker(G_d) \) has order \( N \). Explicitly, it is \( \{ \mu_d^m q^n \mathbb{Z} \mid n, m \in \mathbb{Z} \} \) where \( \mu_d \) is a \( d \)'th primitive root of 1 and \( q^{1/2} \) is an \( e \)'th primitive root of \( q \). In fact
\[
\{ \ker(G_d) \mid d \text{ divides } N \text{ and } d \geq 1 \}
\]
gives all the subgroups of \( \mathbb{C}^*/q^d \mathbb{Z} \) of order \( N \).

**Example 2.2.** It is shown in [14, Exercise 5.10] a generalization of Example 2.1. For a \( p \)-adic field \( K \), if \( q, q' \in K, 0 < |q|, |q'| < 1 \), and \( q^d = q'^e \), then the function
\[
(2.7) \quad \mathbb{K}^* / q^d \mathbb{Z} \to \mathbb{K}^*/q'^e \mathbb{Z}, \quad u \mapsto u^d
\]
lifts to an isogeny \( E_q \to E_{q'} \) of elliptic curves over \( K \), where \( E_q \) and \( E_{q'} \) are defined by the Tate curve equations (2.1) (2.2).

Moreover, as shown in [1 Section 6.3], if \( F \) is a non-archimedean field, there is an isogeny
\[ F^* / q^d \mathbb{Z} \to F^*/q^e \mathbb{Z}, \quad z \mapsto z^d \]
for any positive integer \( d \).

We recall a model for the torsion points of the Tate curve from [10 Section 8.7] and [1 Section 2.3]. The \( N \)-torsion points \( T[N] \) of it is the disjoint union of
\[ T_0[N], \ldots, T_{N-1}[N], \]
where, for each integer \( 0 \leq i < N \),
\[
(2.8) \quad T_i[N] = \text{Spec } (\mathbb{Z}[q^\pm][x]/(x^N - q^i)).
\]

It fits into a short exact sequence of group schemes [10 (8.7.1.4)]
\[
(2.9) \quad 0 \longrightarrow \mu_N \cong T_0(N) \xrightarrow{a_N} T[N] \xrightarrow{b_N} \mathbb{Z}[\frac{1}{N}] / \mathbb{Z} \longrightarrow 0
\]
where \( a_N \) is the inclusion sending \( \zeta \in \mu_N \) to \((\zeta, 0) \in T[N]\), and \( b_N \) sends \((X, \frac{1}{N}) \in T[N]\) to \( \frac{X}{N} \mod \mathbb{Z} \).

Then we recall a smooth one-dimensional commutative group scheme \( T \) over \( \mathbb{Z}[q^\pm] \), which is defined in [10 Section 8.7]. As a scheme, it is the disjoint union of
schemes \( T_\alpha \), indexed by \( \alpha \)’s running over all rational numbers in the interval \([0, 1)\), where each \( T_\alpha \) is the scheme \( \mathbb{G}_m \).

It sits in a short exact sequence of group-schemes over \( \mathbb{Z}[q^\pm] \):
\[
0 \longrightarrow \mathbb{G}_m \longrightarrow T \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.
\]

For any \( \mathbb{Z}[q^\pm] \)-algebra \( R \) with connected spectrum,
\[
T(R) = \frac{R^* \times \mathbb{Q}}{(q, -1)}
\]
is still a group with the group structure defined in [10] (8.7.2.3), Section 8.7, i.e. for any \((X, \alpha), (Y, \beta)\) in \( T(R) \),
\[
(2.10) \quad (X, \alpha) \cdot (Y, \beta) := \begin{cases} 
(XY, \alpha + \beta) & \text{if } \alpha + \beta < 1; \\
\left(\frac{XY}{q^\alpha}, \alpha + \beta - 1\right) & \text{if } \alpha + \beta \geq 1.
\end{cases}
\]

Thus, \( T \) is a functor from the category of \( \mathbb{Z}[q^\pm] \)-algebras to the category of abelian groups.

Over a faithfully flat \( \mathbb{Z}[q^\pm] \)-algebra \( R \) containing a compatible system of \( N \)’th roots \( q^{\frac{1}{N}} \) of \( q \) for every \( N \), i.e. for every integer \( N \geq 1 \), we are given \( Y_N \in R^* \) such that \( Y_1 = q \) and \( (Y_{NM})^M = Y_N \) for every \( M, N \geq 1 \), \( T \) is isomorphic to the product
\[
\mathbb{G}_m \times \mathbb{Q}/\mathbb{Z}
\]
and the torsion points \( T_{\text{torsion}} \) is therefore isomorphic to
\[
\mu_\infty \times \mathbb{Q}/\mathbb{Z}.
\]

We have the conclusion below, which is Theorem 8.7.5 in [10].

**Theorem 2.3.** There exists a faithfully flat \( \mathbb{Z}[q^\pm] \)-algebra \( R \), an elliptic curve \( E/R \), and an isomorphism of ind-group-schemes over \( R \)
\[
T_{\text{torsion}} \otimes \mathbb{Z}[q^\pm] R \xrightarrow{\sim} E_{\text{tors}},
\]
such that for every \( N \geq 1 \), the isomorphism on \( N \)-torsion points \( T[N] \otimes R \xrightarrow{\sim} E[N] \) is compatible with \( e_N \)-pairings.

Thus, we have the unique isomorphism of ind-group-schemes on \( \mathbb{Z}(\!(q)\!) \):
\[
T_{\text{torsion}} \otimes \mathbb{Z}[q^\pm] \mathbb{Z}(\!(q)\!) \xrightarrow{\sim} Tate(\!(q)\!)_{\text{tors}}.
\]

**Example 2.4.** As shown in [12] and [1], over \( \mathbb{Z}(\!(q)\!) \) there is an isogeny
\[
(d) : Tate(q) \rightarrow Tate(q^d)
\]
with kernel \( \mu_d \). And over \( \mathbb{Z}(\!(q^{\frac{1}{e}})\!) \) there is an isogeny
\[
\pi_e: Tate(q) \rightarrow Tate(q^{\frac{1}{e}}),
\]
with kernel \( \mathbb{Z}(\!(\frac{1}{e})\!)/\mathbb{Z} \). Especially, the restriction
\[
\pi_e: Tate(q^e) \rightarrow Tate(q)
\]
is the dual isogeny of \( [e] : Tate(q) \rightarrow Tate(q^e) \).

Let \( D(d, e) \) denote the \( \mathbb{Z}(\!(q)\!) \)-algebra
\[
(2.12) \quad D(d, e) := \mathbb{Z}(\!(q)\!)[q_{d,e}]/(q^d - q_{d,e}^e).
\]
There are two ring homomorphisms
\[ \mathbb{Z}((q))^i_{d,e} \to D(d,e), \quad q \mapsto q \]
\[ \mathbb{Z}((q))^j_{d,e} \to D(d,e), \quad q \mapsto q_{d,e}^{i} \]
From them we can define the group schemes \( i_{d,e}^{*}Tate(q) \) and \( j_{d,e}^{*}Tate(q) \) as the pullbacks below.
\[
\begin{array}{ccc}
Tate(q) & \xrightarrow{i_{d,e}^{*}} & Tate(q) \\
\downarrow & & \downarrow \\
Spec(D(d,e)) & \xrightarrow{i_{d,e}} & Spec(\mathbb{Z}((q)))
\end{array}
\]
\[
\begin{array}{ccc}
Tate(q) & \xrightarrow{j_{d,e}^{*}} & Tate(q) \\
\downarrow & & \downarrow \\
Spec(D(d,e)) & \xrightarrow{j_{d,e}} & Spec(\mathbb{Z}((q)))
\end{array}
\]
We can define an isogeny over \( D(d,e) \) by the composition
\[
(2.14) \quad \phi_{d,e}^{i}: i_{d,e}^{*}Tate(q) = Tate(q) \to Tate(q) \to Tate(q_{d,e}^{i}) = j_{d,e}^{*}Tate(q).
\]
Let \( N = de \). The kernel of \( \phi_{d,e}^{i} \) is the \( D(d,e) \)-scheme
\[
(2.15) \quad T[d,e] := \{ z \in T[N] \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) | z^{d} = q_{d,e}^{i} \text{ for some } i \}.
\]
It fits into a short exact sequence
\[
0 \to \mu_{d} \to T[d,e] \to \mathbb{Z}[\frac{1}{e}] / \mathbb{Z} \to 0.
\]
And \( T[d,e] \) is a finite subgroup of \( Tate(q) \) of order \( N \).

3. Tate K-theory

3.1. Quasi-elliptic cohomology and Tate K-theory. In this section we give an explicit description of Tate K-theory via quasi-elliptic cohomology, which can be defined in terms of equivariant K-theories. For more details on quasi-elliptic cohomology, please refer to [7] and [13].

Let \( X \) be a \( G \)-space. Let \( G^{tors} \subseteq G \) be the set of torsion elements of \( G \). Let \( \sigma \in G^{tors} \). And let
\[ C_{G}(\sigma) = \{ g \in G \mid \sigma g = g \sigma \} \]
denote the centralizer. The fixed point space
\[ X^{\sigma} := \{ x \in X \mid x \cdot \sigma = x \} \]
is a \( C_{G}(\sigma) \)-space. Let
\[ \Lambda_{G}(\sigma) := C_{G}(\sigma) \times \mathbb{R} / (\sigma, -1) \]
We can define a \( \Lambda_{G}(\sigma) \)-action on \( X^{\sigma} \) by
\[ [g, t] \cdot x := g \cdot x. \]
Then quasi-elliptic cohomology of the orbifold \( X/G \) is defined by

**Definition 3.1.**
\[
(3.1) \quad QEll^{*}(X/G) := \prod_{\sigma \in G^{tors}{\text{conj}}} K^{*}_{\Lambda_{G}(\sigma)}(X^{\sigma}) = \left( \prod_{\sigma \in G^{tors}{\text{conj}}} K^{*}_{\Lambda_{G}(\sigma)}(X^{\sigma}) \right)^{G},
\]
where \( G^{tors}{\text{conj}} \) is a set of representatives of \( G \)-conjugacy classes in \( G^{tors} \).
We have the ring homomorphism

\[ \mathbb{Z}[q^\pm] = K^0_{T}(pt) \xrightarrow{\pi^*} K^0_{\Lambda G}(pt) \xrightarrow{\pi^*} K^0_{\Lambda G}(X) \]

where \( \pi : \Lambda G(g) \rightarrow T \) is the projection \([a, t] \mapsto e^{2\pi it}\) and the second is via the collapsing map \( X \rightarrow pt \). So \( QEll^*(X/G) \) is naturally a \( \mathbb{Z}[q^\pm] \)-algebra.

**Proposition 3.2.** The relation between quasi-elliptic cohomology and Tate K-theory is

\[ QEll^*(X/G) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = K^*_\text{Tate}(X/G). \]

**3.2. Moduli Problems.** In this section we discuss several moduli problems for the Tate curve involving the classification of \( N \)-torsion points and that of finite subgroups. The computation of quasi-elliptic cohomology via representation theory plays a role in the study of the moduli problems.

We have the computation \([9, Example 3.3]\) that

\[ QEll^*(pt/(\mathbb{Z}/N\mathbb{Z})) = \prod_{k=0}^{N-1} \mathbb{Z}[q^\pm] [x_k]/(x_k^N - q^k), \]

where each \( x_k \) is the representation of \( \Lambda_{\mathbb{Z}/N\mathbb{Z}}(k) \) defined by

\[ \Lambda_{\mathbb{Z}/N\mathbb{Z}}(k) = (\mathbb{Z} \times \mathbb{R})/(\mathbb{Z}(N, 0) + \mathbb{Z}(1, k)) \xrightarrow{[a,t] \mapsto (kt-a)/N} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1). \]

By \([28]\) and \([33]\), we have the isomorphism below.

**Proposition 3.3.** \( T[N] \cong \text{Spec} (QEll^*(pt/(\mathbb{Z}/N\mathbb{Z}))). \)

In \([7, Section 4]\), we construct a power operation \( P_N \) for quasi-elliptic cohomology, which extends uniquely to the stringy power operation \( P^\text{string}_N \) for Tate K-theory \([6, Definition 3.15]\). Via \( P_N \) we show by computing representation rings the following conclusion, which is part of \([7, Theorem 7.4]\).

**Proposition 3.4.**

\[ QEll^*(pt/\Sigma N)/I_{tr}^{\Sigma N} \cong \prod_{N=de} \mathbb{Z}[q^\pm] [q'_{d,e}]/\langle q^d - q'^d_{d,e} \rangle, \]

where

\[ q' = \prod_{N=de} q'_{d,e} \]

is the image of \( q \) under the power operation \( P_N \) and

\[ I_{tr}^{\Sigma N} := \sum_{N>0} \text{Image}[\mathbb{Z}_{\Sigma_i \times \Sigma_j} : QEll^*(pt/\Sigma_i \times \Sigma_j) \rightarrow QEll^*(pt/\Sigma N)] \]

is the transfer ideal for quasi-elliptic cohomology. The product goes over all the ordered pairs of positive integers \((d, e)\) such that \( N = de \).

Applying the relation \([32]\), we can get the conclusion below as a corollary of Proposition \([34]\) which is part of \([7, Theorem 7.4]\).

**Theorem 3.5.** The Tate K-theory of symmetric groups modulo the transfer ideal \( I_{tr}^{\Sigma N} \) classifies the finite subgroups of the Tate curve. Explicitly,

\[ K^*_{\text{Tate}}(pt/\Sigma N)/I_{tr}^{\Sigma N} \cong \prod_{N=de} D(d, e) \]
where $D(d,e)$ is the ring defined in (2.12). And
\[ q' := \prod_{N=de} q'_{d,e} \]
is the image of $q$ under the power operation $P^\text{string}_N$ constructed in [3] Definition 3.15] and
\[ I^\Sigma_N := \sum_{i+j=N, N>0} \text{Image}[I^\Sigma_{i,x\Sigma_j} : K_{\text{Tate}}(pt//\Sigma_i \times \Sigma_j) \to K_{\text{Tate}}(pt//\Sigma_N)] \]
is the transfer ideal of Tate K-theory. The product goes over all the ordered pairs of positive integers $(d,e)$ such that $N = de$.

### 3.3. Power operation.

Moreover, via the power operation $P^N$ we will construct a new operation
\[ T^N : Q\text{Ell}(X//G) \to Q\text{Ell}(X//G) \otimes_{\mathbb{Z}[q^\pm]} Q\text{Ell}(\Sigma_N)/I^\Sigma_N \]
of quasi-elliptic cohomology. It is essential in the construction of the universal finite subgroup of order $N$ of $Tate(q)$.

**Proposition 3.6.** The composition
\[
T^N : Q\text{Ell}(X//G) \xrightarrow{P^N} Q\text{Ell}(X//G \otimes_{\mathbb{Z}[q^\pm]} Q\text{Ell}(\Sigma_N)) \xrightarrow{\text{res}} Q\text{Ell}(X//G) \otimes_{\mathbb{Z}[q^\pm]} Q\text{Ell}(\Sigma_N)/I^\Sigma_N
\]
\[ \xrightarrow{\text{diag}^*} Q\text{Ell}(X//G \times \Sigma_N) \cong Q\text{Ell}(X//G) \otimes_{\mathbb{Z}[q^\pm]} Q\text{Ell}(\Sigma_N)/I^\Sigma_N \]
\[ \cong Q\text{Ell}(X//G) \otimes_{\mathbb{Z}[q^\pm]} \prod_{N=de} \mathbb{Z}[q^\pm]/[q^d - q'_{d,e}] \]
defines a ring homomorphism, where $\text{res}$ is the restriction map given by the inclusion
\[ G \times \Sigma_N \hookrightarrow G \wr \Sigma_N, \ (g, \sigma) \mapsto (g, \cdot \cdot \cdot g; \sigma), \]
and $\text{diag}$ is the diagonal map
\[ X \to X \times N, \ x \mapsto (x, \cdot \cdot \cdot x). \]
In addition, we refer the reader to [7] (4.1), (4.2) for the construction of the power operation $\{T^N\}_N$ and [7] (4.17) for the explicit formula of it.

The operation $T^N$ sends $q$ to $q'$. In addition, it extends uniquely to a ring homomorphism
\[ P^\text{string}_N : K_{\text{Tate}}(X//G) \to K_{\text{Tate}}(X//G) \otimes_{\mathbb{Z}[q]} \prod_{N=de} \mathbb{Z}(d,e) \cong \prod_{N=de} (d,e) \otimes_{id,e} K_{\text{Tate}}(X//G) \]
constructed in [5 Section 5.4]. The operations gives an isogeny of formal groups
\[ (3.9) \quad P^\text{string}_N : \prod_{N=de} i^*_d eTate(q) \to Tate(q) \]
4. The universal finite subgroup of the Tate curve

In this section we construct a finite subgroup over $D_N$ of order $N$ of $i_N^*Tate(q)$ as the kernel of an isogeny $\psi$, which is defined in (4.13). An explicit description of $\text{Ker}(\psi)$ is given in (4.15). We prove in Section 5 that $\text{Ker}(\psi)$ is the universal finite subgroup of $Tate(q)$ of order $N$.

Let $D_N$ denote the product

$$\prod_{N=de} D(d,e).$$

And let $i_N$ denote the map

$$i_N := \prod_{N=de} i_{d,e} : \mathbb{Z}((q)) \to D_N, \quad q \mapsto \prod_{N=de} q.$$

And let $j_N$ denote the map

$$j_N := \prod_{N=de} j_{d,e} : \mathbb{Z}((q)) \to D_N, \quad q \mapsto \prod_{N=de} q'_{d,e}.$$

We have the isomorphisms

$$i_N^*Tate(q) \cong \prod_{N=de} i_{d,e}^*Tate(q);$$

$$j_N^*Tate(q) \cong \prod_{N=de} j_{d,e}^*Tate(q),$$

and the isogeny over $D_N$

$$(4.1) \quad \phi_N' := \prod_{N=de} \phi_{d,e}' : i_N^*Tate(q) \longrightarrow j_N^*Tate(q)$$

where each $\phi_{d,e}'$ is defined in (2.14).

The kernel of $\phi_N'$ is the $D_N$-scheme

$$\prod_{N=de} T[d,e].$$

From the pullback squares (2.13) we obtain the pullback squares below.

$$\begin{array}{ccc}
\text{Spec } (D_N) & \longrightarrow & \text{Spec } (\mathbb{Z}((q))) \\
\downarrow \quad i_N & & \quad \downarrow \\
\text{Spec } (D_N) & \longrightarrow & \text{Spec } (\mathbb{Z}((q))) \\
\end{array}$$

To study the finite subgroups of $Tate(q)$ of order $N$, it suffices to work inside the world of $N$-torsion points of $Tate(q)$. By Theorem 2.3 the $N$-torsion points $Tate(q)[N]$ of $Tate(q)$ is isomorphic to

$$(4.3) \quad Tate(q)[N] \cong T[N] \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong \prod_{i=0}^{N-1} \text{Spec } (\mathbb{Z}((q))[x_k]/(x_k^N - q^i)).$$

By Proposition 3.2 and Proposition 3.3 we have the isomorphism of rings of functions

$$(4.4) \quad O_{Tate(q)[N]} \cong K^*_\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z})).$$
The $N$-torsion points of $i_N^*\text{Tate}(q)$ is
\[(i_N^*\text{Tate}(q))[N] = i_N^*(\text{Tate}(q)[N]).\]
And the $N$-torsion points of $j_N^*\text{Tate}(q)$ is
\[(j_N^*\text{Tate}(q))[N] = j_N^*(\text{Tate}(q)[N]).\]
Thus, we have the pullback squares below.

\[\begin{array}{ccc}
  i_N^*\text{Tate}(q)[N] & \rightarrow & j_N^*\text{Tate}(q)[N] \\
  \downarrow & & \downarrow \\
  \text{Spec } (D_N) & \rightarrow & \text{Spec } (\mathbb{Z}((q)))
\end{array}\]

And we have the pushout squares of the induced maps on the rings of functions

\[\begin{array}{ccc}
  \mathbb{Z}((q)) & \rightarrow & D_N \\
  \downarrow & & \downarrow \\
  K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))} & \rightarrow & D_N \otimes_{i_N} K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))}
\end{array}\]

By the universal property of the pushout, there is a unique map
\[\psi^* : D_N \otimes_{j_N} K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))} \rightarrow D_N \otimes_{i_N} K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))}\]
making the diagrams below commute.

\[\begin{array}{ccc}
  \mathbb{Z}((q)) & \rightarrow & D_N \\
  \downarrow & & \downarrow \\
  K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))} & \rightarrow & D_N \otimes_{j_N} K_{\text{Tate}(\text{pt}//(\mathbb{Z}/N\mathbb{Z}))}
\end{array}\]

We apply Spec to the diagram (4.5) and obtain the commutative diagrams of group schemes below.

\[\begin{array}{ccc}
  i_N^*\text{Tate}(q)[N] & \rightarrow & j_N^*\text{Tate}(q)[N] \\
  \downarrow & & \downarrow \\
  \text{Spec } D_N & \rightarrow & \text{Spec } (\mathbb{Z}((q)))
\end{array}\]
Next we show the explicit formula for $\psi^*$. We define an element
\[ x_k \in K_{\text{Tate}}(pt/(\mathbb{Z}/N\mathbb{Z})) \]
in (3.3). For any $\mathbb{Z}((q))$-algebra $R$ with connected spectrum, the formula of the map
\[ x_k : \text{Tate}(q)[N](R) \to R \]
is
\[ x_k([a, t]) = \begin{cases} a, & \text{if } t = \frac{k}{N} \text{ with } k = 0, 1, \ldots, N-1; \\ 0, & \text{if } [a, t] \neq [a', \frac{k}{N}] \text{ for any } a'. \end{cases} \]

Note that, in $\text{Tate}(q)[N](R)$, $[a, t + 1] = [a, t]$. By the formula for the operation $\overline{\text{string}}_N : K_{\text{Tate}}(pt/(\mathbb{Z}/N\mathbb{Z})) \to D_N \otimes_{i_N} K_{\text{Tate}}(pt/(\mathbb{Z}/N\mathbb{Z}))$ given in Proposition 3.6, it sends $\prod_{m=0}^{N-1} q$ to
\[ \prod_{m=0}^{N-1} q'_{d,e} \otimes 1 = q', \]
and sends each $x_k$ to
\[ \prod_{N=de \alpha_m = 0}^{e-1} \prod_{e|k} q'_{d,e}^{-\alpha_m} \otimes x_m^d \]
where $m = \frac{k}{e} + \alpha_m d$.

Therefore, by the commutativity of the diagram (4.9), $\psi^*$ sends $1 \otimes (\prod_{m=0}^{N-1} q)$ and $q' \otimes 1$ to the image of
\[ \prod_{m=0}^{N-1} q \in K_{\text{Tate}}(pt/(\mathbb{Z}/N\mathbb{Z})) \]
under $\overline{\text{string}}_N$, i.e.
\[ \prod_{m=0}^{N-1} q'_{d,e} \otimes 1 = q', \]
and it sends $x_k \otimes 1$ to the image of $x_k$ under $\overline{\text{string}}_N$, i.e.
\[ \prod_{N=de \alpha_m = 0}^{e-1} \prod_{e|k} q'_{d,e}^{-\alpha_m} \otimes x_m^d \]
with each $m = \frac{k}{e} + \alpha_m d$, and $\psi^*$ sends $q \otimes 1$ to the image of $q \in D_N$ under $p$, i.e.
\[ q \otimes 1 \in D_N \otimes_{i_N} K_{\text{Tate}}(pt/(\mathbb{Z}/N\mathbb{Z})). \]

Next, we give an explicit formula for $\psi$. 
For any $D(d,e)$-algebra $A$ with connected spectrum, we have the equivalences for the $N$-torsion points
\[(4.11) \quad i_{d,e}^* \text{Tate}(q)[N](A) = \frac{A^* \times (\mathbb{Z}/N\mathbb{Z})}{(q,-1)} \quad \text{and} \quad j_{d,e}^* \text{Tate}(q)[N](A) = \frac{A^* \times (\mathbb{Z}/N\mathbb{Z})}{(q_{d,e}',-1)}.
\]
Under this identification, the map
\[
\psi_{d,e}(A) : i_{d,e}^* \text{Tate}(q)[N](A) \longrightarrow j_{d,e}^* \text{Tate}(q)[N](A)
\]
is defined by
\[(4.12) \quad [a,x] \mapsto [a^d, ex].
\]
It is well-defined since $[q^d,-e] = [q^e,-e] = 0$ in $j_{d,e}^* \text{Tate}(q)[N](A)$.

The map
\[(4.13) \quad \psi : i^* \text{Tate}(q)[N] \longrightarrow j^* \text{Tate}(q)[N]
\]
defined in the diagram (4.9) can be constructed as the coproduct of the maps
\[
\psi_{d,e} : i_{d,e}^* \text{Tate}(q)[N] \longrightarrow j_{d,e}^* \text{Tate}(q)[N].
\]
For each $(d,e)$,
\[
(\psi_{d,e}^* x_k)[a,x] = x_k(\psi_{d,e}[a,x]) = x_k([a^d, ex]).
\]
Note that $\psi_{d,e}^* x_k$ is in $D_N \otimes_{i_N} K_{\text{Tate}}(\text{pt} / (\mathbb{Z}/N\mathbb{Z}))$.

With the formula given in (4.12), we have an explicit formula for $(\text{Ker}(\psi))(A)$.
\[(4.14) \quad (\text{Ker}(\psi))(A) = \prod_{N=de} \text{Ker}(\psi_{d,e})(A)
\]
\[(4.15) \quad = \prod_{N=de} \{ [a,x] \in i_{d,e}^* \text{Tate}(q)[N](A) \mid [a^d, ex] = [1,0] \}.
\]
\text{Ker}(\psi) is isomorphic to
\[
\text{Ker}(\psi_N') = \prod_{N=de} T[d,e]
\]
where each $T[d,e]$ is the kernel of the isogeny
\[
\phi_{d,e} : i_{d,e}^* \text{Tate}(q) \longrightarrow j_{d,e}^* \text{Tate}(q)
\]
deфинирован в (2.14).

**Remark 4.1.** The isogeny $\psi$ is the same as the restriction of the isogeny
\[
\phi_N' : i_N^* \text{Tate}(q) \longrightarrow j_N^* \text{Tate}(q)
\]
deфинирован в (4.1) to the $N$-torsion part $i_N^* \text{Tate}(q)[N]$. By [1] Theorem A, Corollary 6.7, there is a ring operation
\[
\Psi_N : D_N \otimes_{i_N} K_{\text{Tate}}(-) \longrightarrow D_N \otimes_{j_N} K_{\text{Tate}}(-)
\]
such that $\Psi_N(\mathbb{C}P^\infty) = \phi_N'$. And $\Psi_N(\mathbb{B}(\mathbb{Z}/N\mathbb{Z}))$ is the homomorphism $\psi^*$ defined in (4.8).
5. The main theorem

Now we are ready to state the main conclusion of this paper.

In this section, by a \( \mathbb{Z}((q)) \)-algebra \( R \) containing a compatible system of \( N \)’th roots \( q^{1/N} \) of \( q \) for every \( N \), i.e. for every integer \( N \geq 1 \), we are given \( Y_N \in R^* \) such that \( Y_1 = q \) and \((Y_N^M)_M = Y_N \) for every \( M, N \geq 1 \).

**Theorem 5.1.** The universal finite subgroup of order \( N \) of Tate\((q)\) is the pair \( G_{univ} : = (D_N, \text{Ker}(\psi) < i_N^* \text{Tate}(q)[N]) \) in the sense that for any \( \mathbb{Z}((q)) \)-algebra \( R \), there is a 1-1 correspondence which is natural

\[
\{ \mathbb{Z}((q)) - \text{algebra maps } D_N \to R \} \\
\downarrow \\
\{ \text{finite subgroup schemes } G \leq \text{Tate}(q)[N]_R \text{ of order } N \}
\]

where \( \text{Tate}(q)[N]_R \) is the pullback

\[
\begin{array}{ccc}
\text{Tate}(q)[N]_R & \longrightarrow & \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R) & \longrightarrow & \text{Spec } (\mathbb{Z}((q)))
\end{array}
\]

It is a group scheme over \( \text{Spec } (R) \). In other words, given a subgroup scheme \( G \leq \text{Tate}(q)[N]_R \) of degree \( N \), there exists a unique pullback square

\[
\begin{array}{ccc}
G & \longrightarrow & \text{Ker}(\psi) \\
\downarrow & & \downarrow \\
\text{Tate}(q)[N]_R & \longrightarrow & i_N^* \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R) & \longrightarrow & \text{Spec } (D_N).
\end{array}
\]

**Remark 5.2.** Given the subgroup \( G \leq \text{Tate}(q)[N]_R \) in the construction, there is an isogeny

\[
\psi_R : i_N^* \text{Tate}(q)[N]_R \to j_N^* \text{Tate}(q)[N]_R
\]

over \( R \) such that \( G \) is a subgroup of the kernel of \( \psi_R \), where \( j_N^* \text{Tate}(q)[N]_R \) and \( i_N^* \text{Tate}(q)[N]_R \) are the pullbacks

\[
\begin{array}{ccc}
j_N^* \text{Tate}(q)[N]_R & \longrightarrow & j_N^* \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R) & \longrightarrow & \text{Spec } (D_N)
\end{array}
\]

and \( \psi_R \) is constructed as the base change of the isogeny

\[
\psi : i_N^* \text{Tate}(q)[N] \longrightarrow j_N^* \text{Tate}(q)[N].
\]
Proof of Theorem 5.1. We prove the conclusion by three steps.

Step I: Since Spec \((R)\) is connected, the image of each map

\[
\text{Spec } (R) \to \text{Spec } (D_N) = \bigsqcup_{N=de} \text{Spec } (D(d,e))
\]

lies in some component Spec \((D(d,e))\) of Spec \((D_N)\).

We show, given a finite subgroup \(G < \text{Tate}(q)[N]_R\) of order \(N\), we can construct a map

\[F_{d,e}^*: \text{Spec } (R) \to \text{Spec } (D(d,e))\]

for some pair of integers \((d,e)\) with \(N = de\).

For the subgroup \(G < \text{Tate}(q)[N]_R\), we have the exact sequences of group schemes over \(R\) and the commutative diagrams

\[
\begin{array}{ccccccccc}
0 & \to & \mathbb{G}_m & \to & \text{Tate}(q)[N]_R & \to & \mathbb{Q}/\mathbb{Z} & \to & 0 \\
0 & \to & \mathbb{G}_m \cap G & \to & G & \to & \mathbb{Z}[\frac{1}{e}] / \mathbb{Z} & \to & 0
\end{array}
\]

for some positive integer \(e \geq 1\) dividing \(N\). The group \(\mathbb{G}_m \cap G\) is the kernel of the projection \(G \to \mathbb{Q}/\mathbb{Z}\). Let \(d\) denote the order of \(\mathbb{G}_m \cap G\). We have \(N = de\).

In addition, we have the exact sequences of group schemes over \(R\)

\[
\begin{array}{ccccccccc}
0 & \to & \mu_d & \to & G & \to & \mathbb{Z}[\frac{1}{e}] / \mathbb{Z} & \to & 0 \\
0 & \to & \mu_d & \to & \text{Tate}(q)_R & \to & \text{Tate}(q^d)_R & \to & 0
\end{array}
\]

where \(\text{Tate}(q)_R\) is the pullback

\[
\begin{array}{cccc}
\text{Tate}(q)_R & \to & \text{Tate}(q) \\
\downarrow & & \downarrow \\
\text{Spec } (R) & \to & \text{Spec } (\mathbb{Z}(q))
\end{array}
\]

and \([d]_R\) is the isogeny over \(R\)

\([d]_R([x, \lambda]) = [x^d, \lambda]\).

There exists a unique homomorphism

\[r : \mathbb{Z}[\frac{1}{e}] / \mathbb{Z} \to \text{Tate}(q^d)_R\]

making the diagram (5.4) commute.

Let \(a \in \mathbb{Z}[\frac{1}{e}] / \mathbb{Z}\) be a generator.

\[
r(a) = [q^e_{R}, \frac{1}{e}]
\]

for some \(q^e_{R} \in R^*\). Note that, in \(\text{Tate}(q^d)_R\), \(0 = e \cdot [q^e_{R}, \frac{1}{e}] = [q^e_{R}, 1] = [q^e_{R}q^{-d}, 0]\)

and

\[q^e_{R} = q^d.
\]

Then we define a ring map

\[F_{d,e} : D(d,e) \to R\]
by sending $q$ to $q$ and sending $q'_{d,e}$ to $q'_R$.

Then the map $F_{d,e}^*$ defined by the composition

$$\text{Spec } (\mathcal{R}) \to \text{Spec } (\mathcal{D}(d,e)) \to \text{Spec } (\mathcal{D}_N)$$

is the one we want.

**Step II:** We show that, given a $\mathbb{Z}((q))$-algebra map $\mathcal{D}_N \to \mathcal{R}$, we can construct a finite subgroup $G < \text{Tate}(\mathbb{Z}((q))[N]_R$ of order $N$.

Since $\text{Tate}(\mathbb{Z}((q))[N]$ is a finite flat group scheme over $\mathbb{Z}((q))$ for any integer $N \geq 1$, and $\mathcal{R}$ is a faithful flat $\mathbb{Z}((q))$-algebra, thus the group scheme $\text{Tate}(\mathbb{Z}((q))[N]_R$ is finite flat. The order of it is defined by the locally constant map

$$\delta : \text{Spec } (\mathcal{R}) \to \mathbb{Z}_{\geq 0}$$

$$p \mapsto \text{rank}_{\mathcal{R}_p}(\text{O}_{\text{Tate}(\mathbb{Z}((q))[N]_R}_p)$$

Then there exists a $\mathbb{Z}((q))$-algebras $R_{d,e}$ for each pair of integers $(d,e)$ with $N = de$ such that

$$\text{Spec } (R_{d,e}) = \{ x \in \text{Spec } (\mathcal{R}) \mid \delta(x) = d \}$$

And $\mathcal{R}$ is the product

$$\prod_{N = de} R_{d,e}$$

Thus the ring homomorphism

$$\alpha : \mathcal{D}_N \to \mathcal{R}$$

is the product of the maps

$$\prod_{N = de} (\alpha_{d,e} : \mathcal{D}(d,e) \to R_{d,e}).$$

If there is no element $x$ in $\mathcal{R}^*$ such that $x^e = q^d$, then $\alpha_{d,e} = 0$. In this case the contribution of $\alpha_{d,e}$ to the subgroup $G$ is 0. There is only one factor $\alpha_{d_0,e_0}$ that is non-trivial. We have the pull-back diagrams

$$\begin{array}{ccc}
\mathcal{G} & \to & \text{Ker}(\psi_{d_0,e_0}) \\
\downarrow & & \downarrow \\
\text{Tate}(\mathbb{Z}((q))[N]_{R_{d_0,e_0}} & \to & \text{I}_{d_0,e_0}^* \text{Tate}(\mathbb{Z}((q))[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R_{d_0,e_0}) & \xrightarrow{\alpha_{d_0,e_0}} & \text{Spec } (\mathcal{D}(d_0,e_0))
\end{array}$$

And, for any $\mathcal{D}(d,e)$-algebra $A$ with connected spectrum, $G(A)$ is the factor

$$\{ [t,x] \in \text{I}_{d_0,e_0}^* \text{Tate}(\mathbb{Z}((q))[N](A) \mid [t^d,e, x] = [1,0] \}.$$ 

Thus, the order of $G$ is $N$.

**Step III:** In this step we check the two maps in Step I and Step II are the inverse of each other. Then, we have the 1-1 correspondence.

Let $H < \text{Tate}(\mathbb{Z}((q))[N]_R$ denote a finite group of order $N$. By Step I, we have a $\mathbb{Z}((q))$-algebra map

$$F_{d,e} : \mathcal{D}(d,e) \to \mathcal{R}$$
for some pair \((d, e)\) of positive integers with \(N = de\). By the construction of \(F_{d,e}\) in Step I, \(H\) is the pullback in the diagram

\[
\begin{array}{ccc}
H & \longrightarrow & \text{Ker}(\psi_{d,e}) \\
\downarrow & & \downarrow \\
\text{Tate}(q)[N]_{R_{d,e}} & \longrightarrow & i_{d,e}^* \text{Tate}(q)[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R_{d,e}) & \longrightarrow & \text{Spec } (D(d,e))
\end{array}
\]

By the uniqueness of pullback, we know the subgroup \(G\) of \(\text{Tate}(q)[N]_R\) obtained from \(F_{d,e}: D(d, e) \longrightarrow R\) in Step II is isomorphic to \(H\).

For the other direction, given a \(\mathbb{Z}((q))\)-algebra map \(D_N \xrightarrow{\alpha} R\), we can construct a finite subgroup \(G < \text{Tate}(q)[N]_R\) as the pullback of the diagram

\[
\begin{array}{ccc}
G & \longrightarrow & \text{Ker}(\psi_{d,e}) \\
\downarrow & & \downarrow \\
\text{Tate}(q)[N]_{R_{d,e}} & \longrightarrow & i_{d,e}^* \text{Tate}[N] \\
\downarrow & & \downarrow \\
\text{Spec } (R_{d,e}) & \longrightarrow & \text{Spec } (D(d,e))
\end{array}
\]

Especially, there is only one pair \((d, e)\) that is relevant to the construction of \(G\). Then, the construction in Step I recovers the map \(\alpha\).

\[\square\]
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ZHEN HUAN, CENTER FOR MATHEMATICAL SCIENCES, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HUBEI 430074, CHINA
Email address: 2019010151@hust.edu.cn