Weight Enumerators and Cardinalities for Number-Theoretic Codes

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Abstract—The number-theoretic code is a class of codes defined by single or multiple congruences. These codes are mainly used for correcting insertion and deletion errors, and for correcting asymmetric errors. This paper presents a formula for a generalization of the complete weight enumerator for the number-theoretic codes. This formula allows us to derive the weight enumerators and cardinalities for the number-theoretic codes. As a special case, this paper provides the Hamming weight enumerators and cardinalities of the non-binary Tenengolts’ codes, correcting single insertion or deletion. Moreover, we show that the formula deduces the MacWilliams identity for the linear codes over the ring of integers modulo \( r \).

Index Terms—Cardinality, Insertion/deletion correcting code, Non-binary Tenengolts’ code, Number-theoretic code, Weight enumerator

I. INTRODUCTION

The number-theoretic code [4] is a class of codes defined by single or multiple congruences. These codes are mainly used for correcting insertion and deletion errors [4–6], and for correcting asymmetric errors [7], [8]. In general, the number-theoretic codes are non-linear.

The code rate characterizes the performance of the code and is derived from the number of codewords or cardinality of the code. The cardinalities of the linear codes are easily derived from the size and rank of the generator or parity-check matrices. On the other hand, in the case of number-theoretic codes, derivation of the cardinalities is not an easy problem. Once we can explicitly derive the cardinalities of the number-theoretic codes, we can choose a code with the largest cardinality. Although it is also non-trivial problem to encode the number-theoretic codes [9], [10], there is a prospect that a low redundant encoding algorithm will be given for the code with the largest cardinality.

The Varshamov-Tenengolts (VT) codes [3] are binary number-theoretic single insertion/deletion correcting codes. Its cardinality is given by Ginzburg [11], Stanley and Yoder [12]. More precisely, they [11], [12] defined an \( r \)-ary VT code, which is a natural generalization of the binary VT codes, and derived the cardinality of this code. To derive cardinalities of the VT codes, Stanley and Yoder [12] derived the Hamming weight enumerators of the codes. In other words, the Hamming weight enumerators are used for deriving the cardinalities.

Bibak and Milenkovic [5] defined the binary linear congruence (BLC) code, which is a general class of number-theoretic codes, and derived its Hamming weight enumerator. The BLC code includes the binary codes defined by a single linear congruence, e.g., the binary VT codes [3], the Levenshtein codes [4], the Helberg codes [2], and the odd weight codes [6]. Sakurai [13] generalized this result, namely, defined the \( r \)-ary linear congruence code and derived its Hamming weight enumerator. Moreover, Sakurai [13] provided a simple derivation for its Hamming weight enumerator. However, those classes of codes do not include several useful number-theoretic codes, e.g., the non-binary Tenengolts’ codes [14], [15], the shifted VT (SVT) codes [15], and the non-binary SVT codes [16].

This paper investigates the simultaneous congruences (SC) code, a general class of non-binary codes defined by multiple non-linear congruences. This paper provides the definition of the SC code, and to my best knowledge, the SC code is first formulated in this paper. The SC code is a generalization of the \( r \)-ary linear congruence code and includes the non-binary Tenengolts’ codes, the SVT codes, and the non-binary SVT codes. Furthermore, the SC code includes codes over the ring of integers modulo \( r \) and codes defined by the finite Abelian group. Unfortunately, it is difficult directly to derive the Hamming weight enumerator for the SC codes. Hence, we define the extended weight enumerator, a generalization of the Hamming weight enumerator, and present a formula for the extended weight enumerator of the SC code. This formula allows us to derive the weight enumerators and cardinalities for the SC codes. Moreover, using this result, the paper derives the Hamming weight enumerators and cardinalities of the non-binary Tenengolts’ codes. From this, we clarify the parameters which give maximum cardinality of the non-binary Tenengolts’ code. Furthermore, we show that the formula deduces the MacWilliams identity for the linear codes over the ring of integers modulo \( r \).

Summarizing above, the paper contributions are (i) showing a formula for the extended weight enumerators of the SC codes by generalizing the results in [5], [13], (ii) deriving the Hamming weight enumerators and cardinalities of the non-binary Tenengolts’ codes, (iii) clarifying the parameters which give the maximum cardinality of the non-binary Tenengolts’ code, and (iv) deriving the MacWilliams identity for the linear codes over the ring of integers modulo \( r \) by the formula for the extended weight enumerator of SC codes.

This paper is an extended version of a conference proceeding [11] and introduces an example of a conference code, codes defined by the finite Abelian group. Moreover, we additionally present...
the fourth contribution above.

The rest of the paper is organized as follows. Section II introduces the notations and definitions used throughout the paper. Section III derives the formula for the extended weight enumerators of the SC codes. Section IV presents the Hamming weight enumerators and cardinalities of the non-binary Tenengolts’ codes. Section V derives the MacWilliams identity for the linear codes over the ring of integers modulo r. Section VI concludes the paper.

II. PRELIMINARIES

This section gives notations used throughout the paper. This section also introduces several classes of number-theoretic codes and the weight enumerators.

A. Notations and Definitions

Let $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{C}$ be the set of integers, positive integers, and complex numbers, respectively. Define $[a, b] := \{i \in \mathbb{Z} | a \leq i \leq b\}$ for $a, b \in \mathbb{Z}$, and $[a] := [0, a - 1]$ for $a \in \mathbb{Z}^+$. Let $\mathbb{I}(P)$ be the indicator function, which equals 1 if the proposition $P$ is true and equals 0 otherwise. Denote the cardinality of a set $T$, by $|T|$. Denote a vector of length $n$, by $(x_1, x_2, \ldots, x_n)$.

For $a, b \in \mathbb{Z}$, we write $a | b$ if $a$ divides $b$. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, denote $a \equiv b \pmod{n}$ if $(a - b) | n$. Denote the ring of integers modulo $r$, by $\mathbb{Z}_r$.

Let $i$ be the imaginary unit. Define $e(x) := \exp(2\pi i x)$.

B. Number-Theoretic Codes

Bibak and Milenkovic [5] defined the binary linear congruence (BLC) codes as follows:

**Definition 1:** Suppose $n, m \in \mathbb{Z}^+$, $h = \langle h_1, h_2, \ldots, h_n \rangle \in \mathbb{Z}^n$, and $a \in [m]$. Then, the BLC code of length $n$ with parameters $a, m, h$ is defined by

$$\text{BLC}_a(n, m, h) := \{ (x_1, x_2, \ldots, x_n) \in [0, 1]^n | \sum_{i=1}^{n} h_i x_i \equiv a \pmod{m} \}.$$  

Sakurai [13] extended this definition to the $r$-ary case and defined the linear congruence (LC) codes.

**Definition 2:** Suppose $n, m \in \mathbb{Z}^+$, $h = \langle h_1, h_2, \ldots, h_n \rangle \in \mathbb{Z}^n$, and $a \in [m]$. Then, the LC code of length $n$ with parameters $a, m, h$ is defined by

$$\text{LC}_a(n, m, r, h) := \{ (x_1, x_2, \ldots, x_n) \in [r]^n | \sum_{i=1}^{n} h_i x_i \equiv a \pmod{m} \}.$$  

In general, the number-theoretic insertion/deletion correcting codes are defined by multiple nonlinear congruences over non-binary elements. We refer such codes as *non-binary multiple nonlinear congruence codes* or simply *simultaneous congruences (SC) codes*.

**Definition 3:** Suppose $n, r, s \in \mathbb{Z}^+$ and $m := \langle m_1, m_2, \ldots, m_s \rangle \in (\mathbb{Z}^+)^s$. Let $\rho_i : [r]^n \rightarrow \mathbb{Z}$ and $a_i \in [m_i]$ for $i \in [1, s]$. Define $\rho := (\rho_1, \rho_2, \ldots, \rho_s)$. Then, the r-ary SC code of length $n$ with parameters $s, \rho, a, m$ is defined as

$$C_{p, a, m}(n, r, s) := \{ x \in [r]^n | \forall i \in [1, s] \rho_i(x) \equiv a_i \pmod{m_i} \}.$$  

### TABLE I

**CODEWORDS OF NON-BINARY TENENGOLOTS’ CODES $T_{a_1, a_2}(3, 3)$**

| $a_1$ | $a_2 = 0$ | $a_2 = 1$ | $a_2 = 2$ |
|-------|-----------|-----------|-----------|
| $a_1 = 0$ | $\{000, 012, 111, 210, 222\}$ | $\{001, 022, 112\}$ | $\{002, 011, 122\}$ |
| $a_1 = 1$ | $\{102, 201\}$ | $\{100, 202, 211\}$ | $\{101, 200, 212\}$ |
| $a_1 = 2$ | $\{021, 120\}$ | $\{010, 121, 220\}$ | $\{020, 110, 221\}$ |

**Remark 1:** For $h = \langle h_1, h_2, \ldots, h_n \rangle \in \mathbb{Z}^n$, define a linear mapping $\ell_h(x) := \sum_{i=1}^{n} h_i x_i$. Then, we get

$$C_{\ell_h, a, m}(n, 2, 1) = \text{BLC}_a(n, m, h),$$

$$C_{\ell_h, a, m}(n, r, 1) = \text{LC}_a(n, m, r, h).$$

In words, the SC codes are generalization of the BLC codes and LC codes.

The non-binary Tenengolts’ codes [14] are single insertion/deletion correcting codes and defined as follows:

**Definition 4:** Let $n, r \in \mathbb{Z}^+$, $a_1 \in [n]$ and $a_2 \in [r]$. Define

$$\gamma(x) := \sum_{i=1}^{n-1} \mathbb{I}\{x_i > x_{i+1}\}, \quad \sigma(x) := \sum_{i=1}^{n} x_i.$$  

Then, the $r$-ary Tenengolts’ code of length $n$ with parameters $a_1, a_2$ is

$$T_{a_1, a_2}(n, r) := \{ x \in [r]^n | \gamma(x) \equiv a_1 \pmod{n}, \sigma(x) \equiv a_2 \pmod{r} \}.$$  

Define

$$\gamma(\geq)(x) := \sum_{i=1}^{n-1} \mathbb{I}\{x_i \geq x_{i+1}\},$$

$$\lambda(\leq)(x) := \sum_{i=1}^{n-1} \mathbb{I}\{x_i < x_{i+1}\},$$

Then, we have the following variants of the $r$-ary Tenengolts’ code.

$$T_{a_1, a_2}(\geq)(n, r) := \gamma(\geq), \gamma(\leq), a_1, a_2, (n, r, 2),$$

$$T_{a_1, a_2}(\leq)(n, r) := \gamma(\leq), \gamma(\geq), a_1, a_2, (n, r, 2),$$

Note that $\gamma(x), \gamma(\geq)(x), \lambda(\leq)(x),$ and $\lambda(\leq)(x)$ are non-linear mappings.

**Example 1:** Table I displays the codewords of $T_{a_1, a_2}(3, 3)$ for $a_1, a_2 \in [3]$. From this table, we confirm that the non-binary Tenengolts’ codes are generally non-linear. Note that the cardinalities of the non-binary Tenengolts’ codes depend on the parameters $a_1, a_2$. We will derive the cardinalities of the non-binary Tenengolts’ codes in Section IV.

**Example 2:** Table II shows the codewords of non-binary Tenengolts’ codes $T_{a_1, a_2}(2, 3)$ and ones for its variants $T_{a_1, a_2}(\geq)(2, 3), T_{a_1, a_2}(\leq)(2, 3),$ and $T_{a_1, a_2}(\leq)(2, 3)$. We will derive the
TABLE II

| (a₁, a₂) | T_{a₁, a₂} | T_{a₁, a₂}^{(3, 2)} | T_{a₁, a₂}^{(4, 2)} | T_{a₁, a₂}^{(5, 2)} |
|----------|------------|-----------------|-----------------|-----------------|
| (0, 0)   | {00, 12}   | {12}            | {00, 21}        | {00, 21}        |
| (0, 1)   | {01, 22}   | {01, 22}        | {00, 22}        | {00, 22}        |
| (0, 2)   | {02, 11}   | {02, 20}        | {00, 12}        | {00, 12}        |
| (1, 0)   | {21}       | {00, 21}        | {12}            | {00, 21}        |
| (1, 1)   | {10}       | {10, 22}        | {01}            | {01, 22}        |
| (1, 2)   | {20}       | {11, 20}        | {02}            | {02, 11}        |

cardinality and property for the variants of Tenegolts’ code in Section IV-E.

Example 3: In this example, we list some codes included in the SC code. Define

ω(x) := η_{1,2,\ldots,n}(x) = \sum_{i=1}^{n} i x_i,

δ(x) := \sum_{i=1}^{n-1} 1\{x_i > x_{i+1}\}.

For fixed t, r ∈ Z^{+}, an integer sequence \{g^{(t, r)}\} is defined recursively as

\begin{align*}
g^{(t, r)}_i &= 1 + (r - 1) \sum_{j=1}^{i} g^{(t, r)}_j 1\{i - j \geq 1\}, \quad \text{for } i \in Z^{+},
\end{align*}

Table III shows some special cases of the SC code. Note that the BLC (resp. LC) code is also special cases of the SC code with r = 2 and s = 1 (resp. s = 1).

The codes over Z_{r} are also special cases of the SC code. In particular, every linear code \mathcal{L} over Z_{r} is defined by a full-rank parity check matrix \mathbf{H} as \mathcal{L} = \{x ∈ Z_{r}^n | \mathbf{H}x^T = 0^T\}.

Hence, \mathcal{L} is rewritten as

\mathcal{L} = \{x ∈ [r]^n | ∀i ∈ [1, s], \ e_i(x) \equiv 0 \pmod{r}\},

where \mathbf{H}_{i} represents the \text{i}-th row of \mathbf{H}.

The binary codes defined by the finite Abelian group G are also in the SC code as shown in below. For any finite Abelian group G, there exists e_1, e_2, \ldots, e_s such that G is isomorphic to Z_{e_1} × Z_{e_2} × \cdots × Z_{e_s} and e_i | e_{i+1} for i ∈ [1, s - 1].

Hence, the binary codes defined by G is also defined by s congruences. An example of the binary code defined by G is the Constantin-Rao code [7].

C. Hamming and Extended Weight Enumerator

Let wt_H(x) be the Hamming weight for x ∈ [r]^n, i.e., wt_H(x) := |\{i | x_i \neq 0\}|. We define the Hamming weight enumerator for a code T ⊆ [r]^n by

\begin{align*}
\mathcal{H}(T; w) &= \sum_{x \in T} w^{wt_H(x)}.
\end{align*}

This paper investigates the extended weight enumerator, which is a generalization of the Hamming weight enumerator. To my best knowledge, it is first defined in the paper. We will explain how we define the extended weight enumerator in Remark 3.

Definition 5: Let n, r, s ∈ Z^{+}. Let \rho_i : [r]^n → Z for i ∈ [1, s]. We denote the number of components of \mathbf{x} ∈ [r]^n that equal j, by τ_j(\mathbf{x}), i.e., τ_j(\mathbf{x}) := |\{i | x_i = j\}|. We define the extended weight enumerator associated to \rho := (\rho_1, \rho_2, \ldots, \rho_s) for a code T ⊆ [r]^n as

\begin{align*}
\mathcal{W}(T; \rho; z, w) &= \sum_{x \in T} \prod_{j \in [r]} z^{\nu_j(x)} \prod_{j \in [r]} w^{\tau_j(x)}.
\end{align*}

where z = (z_1, z_2, \ldots, z_r) and w = (w_0, w_1, \ldots, w_{r-1}).

Remark 2: Denote all one vector of length s, by 1^s. Define a vector \mathbf{w}^* := (1, w, w, \ldots, w) of length r. Then, the complete weight enumerator \mathcal{W}(T; w) for a code T is

\begin{align*}
\mathcal{W}(T; w) &= \mathcal{W}(T; \rho; 1^s, w^*) \sum_{x \in T} w^{\tau_j(x)}.
\end{align*}

Furthermore, the cardinality of T satisfies

\begin{align*}
|T| &= \mathcal{H}(T; 1) = \mathcal{W}(T; \rho; 1^s, 1^r).
\end{align*}

The following lemma is easily derived from the definition of extended weight enumerator.

Lemma 1: Let T_1, T_2 ⊆ [r]^n. If T_1 ∩ T_2 = ∅ holds, then

\begin{align*}
\mathcal{W}(T_1 \cup T_2; \rho; z, w) &= \mathcal{W}(T_1; \rho; z, w) + \mathcal{W}(T_2; \rho; z, w), \quad (2)
\end{align*}

Example 4: Similar to Example 1 consider the ternary Tenegolts’ code of length 3. For each codeword \mathbf{x} in T_{0,0}(3, 3), Table IV summarizes the values of (γ(x), σ(x)) and τ_i(x) for i = 0, 1, 2. From this table, the following gives the extended weight enumerator associated to (γ, σ) for T_{0,0}(3, 3):

\begin{align*}
\mathcal{W}(T_{0,0}(3, 3); \gamma(x), \sigma(x)) &= w_0^3 + w_1^3 + w_2^3 + w_3^3.
\end{align*}

Similarly, the complete and Hamming weight enumerators for T_{0,0}(3, 3) are

\begin{align*}
\mathcal{W}(T_{0,0}(3, 3); w) &= w_0^3 + w_0 w_1 w_2 + w_1^3 + w_0 w_1 w_2 + w_2^3;
\mathcal{H}(T_{0,0}(3, 3); w) &= 1 + 2w^2 + 2w^3.
\end{align*}

III. EXTENDED WEIGHT ENUMERATORS FOR SC CODES

This section presents a formula for the extended weight enumerators of the SC codes. Section III-B gives the main results of this section. Section III-C proves them.

A. Main Result and Corollary

The following main theorem presents an important property to derive the extended weight enumerators for the SC codes.

Theorem 1: Define the SC code (resp. extended weight enumerator) as in Definition 3 (resp. 5). Denote
TABLE III
CODENBS Included in the SC Code

| Code          | r | s | ρ | m       |
|---------------|---|---|---|---------|
| Binary VT [3]| 2 | 1 | ω | n + 1   |
| Levenshtein [4]| 2 | 1 | ω | m       |
| Ternary integer [13]| 3 | 1 | ω | 2m + 1  |
| Helberg [2]   | 2 | 1 | ω | 2m + 1  |
| Le-Nguyen [13]| 2 | 1 | ω | 2m + 1  |
| Odd coefficient [6]| 2 | 1 | ω | 2m + 1  |
| Exponential coefficient [6]| 2 | 1 | ω | 2m + 1  |
| Shifted VT [15]| 2 | 2 | ω | (m, 2)  |
| Han Vinck-Morita [19]| 2 | 2 | ω | (n + 1, 3)|
| Non-binary Tenengolts [14]| 2 | 2 | γ | (n, r)  |
| Non-binary SVT [16]| 2 | 3 | γ | (m, 2, r)|
| Linear code over ℤr | r | s | γ | (ℓ1, ℓ2, . . . , ℓk) |

TABLE IV
CODENBS in T0.0,0(3, 3) and VALUES ASSOCIATED TO EXTENDED WEIGHT ENUMERATOR

| x   | γ(x) | σ(x) | τ0(x) | τ1(x) | τ2(x) | z1γ(x) z2τ0(x) τ2(x) | W(C, p, a, m)(n, r, s, ρ; z, w) |
|-----|------|------|-------|-------|-------|-----------------------|--------------------------------|
| 000 | 0    | 0    | 3     | 0     | 0     | w1       | 1  |
| 012 | 0    | 3    | 1     | 1     | 1     | z2w1       | 1  |
| 111 | 0    | 3    | 0     | 3     | 0     | z2w1       | 1  |
| 210 | 3    | 3    | 1     | 1     | 1     | z2w1       | 1  |
| 222 | 0    | 6    | 0     | 0     | 3     | z2w1       | 1  |

ze( \frac{u}{m} ) := \langle z1e( \frac{u}{m_1} ), z2e( \frac{u}{m_2} ), . . . , zs e( \frac{u}{m_s} ) \rangle.

Then, the following equation holds:

W(C, p, a, m)(n, r, s, ρ; z, w) = \sum_{u \in [m_1] \times [m_2] \times . . . \times [m_s]} W([r]^n, \rho; e( \frac{u}{m} ), w) \prod_{i=1}^{s} \frac{1}{m_i} e( -\frac{a_i u_i}{m_i} ).

Corollary 2 shows that the extended weight enumerator for [r] is also used for deriving the complete/Hamming weight enumerator of an SC code. On the other hand, when we derive the cardinality of an SC code, Corollary 2 only uses W([r]^n, \rho; e( \frac{u}{m} ), 1^r).

From Corollary 2 when we have an explicit formula of W([r]^n, \rho; z, 1^r), we can obtain the cardinality of an SC code in O(\prod_{i=1}^{s} m_i). Because m_i is a polynomial of n in most SC codes, the complexity of the derivation is also a polynomial of n. It is much smaller than the complexity O(r^n) of enumerating all codewords of the SC code.

B. Proof of Main Result and Corollary

The following well-known identity is used in the proof.

Lemma 2: For any A ∈ ℤ, m ∈ ℤ^+, the following holds:

\begin{align*}
\mathbb{I}\{ A \equiv 0 \pmod{m} \} = \mathbb{I}\{ m \mid A \} = \frac{1}{m} \sum_{j \in [m]} e\left( \frac{Aj}{m} \right).
\end{align*}

The following lemma gives the key technique of the proof. The technique is well-known in the graph-based codes (e.g., see [20]).
Lemma 3: The code membership function \( \mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} \) is factorized as follows:

\[
\mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} = \prod_{i=1}^{s} \mathcal{I}\{\rho_i(x) - a_i \equiv 0 \pmod{m_i}\}
\]

By combining those lemmas, the code membership function is written as in the following lemma.

Lemma 4: The code membership function \( \mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} \) is written as

\[
\mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} = \prod_{i=1}^{s} \sum_{u \in [m_i]} \frac{1}{m_i} \left( -\frac{a_i u_i}{m_i} \right) \left( e\left( \frac{u_i}{m_i} \right) \right) \rho_i(x).
\]

Proof: Note that \( e(x + y) = e(x)e(y) \) and \( e(xy) = e(x)^y \). Definition 3 and Lemmas 2 and 3 lead

\[
\mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} = \prod_{i=1}^{s} \sum_{u \in [m_i]} \frac{1}{m_i} \left( -\frac{a_i u_i}{m_i} \right) \left( e\left( \frac{u_i}{m_i} \right) \right) \rho_i(x).
\]

1) Proof of Theorem 4 By combining Lemma 4 and Definition 5 we obtain the theorem as follows:

\[
\mathcal{W}(C_{p,a,m}(n, r, s), \rho; z, w) = \sum_{x \in [r]^n} \prod_{j=1}^{s} w_{j}^{r_j(x)} \mathcal{I}\{x \in C_{p,a,m}(n, r, s)\}
\]

\[
= \sum_{x \in [r]^n} \prod_{j=1}^{s} w_{j}^{r_j(x)} \prod_{i=1}^{s} \sum_{u \in [m_i]} \frac{1}{m_i} \left( -\frac{a_i u_i}{m_i} \right) \left( e\left( \frac{u_i}{m_i} \right) \right) \rho_i(x)
\]

\[
= \sum_{u \in [m_1] \times [m_2] \times \cdots \times [m_s]} \left( \prod_{i=1}^{s} \frac{1}{m_i} \left( -\frac{a_i u_i}{m_i} \right) \right) \times \sum_{x \in [r]^n} \prod_{j=1}^{s} \left\{ \sum_{i=1}^{s} \left( e\left( \frac{u_i}{m_i} \right) \right) \right\} \rho_i(x)
\]

\[
= \sum_{u \in [m_1] \times [m_2] \times \cdots \times [m_s]} \prod_{i=1}^{s} \frac{1}{m_i} \left( -\frac{a_i u_i}{m_i} \right) \times \mathcal{W}([r]^n, \rho; z e(u/m), w).
\]

2) Proof of Corollary 7 The following identity holds

\[
\mathcal{W}([r]^n, \ell_h; z, w) = \prod_{j=1}^{s} \sum_{k \in [r]} w_k z^{h_k}.
\]

This equation and Theorem 4 lead

\[
\mathcal{W}(\text{LC}_a(n, m, r, h), \ell_h; z, w) = \sum_{u \in [m]} \prod_{j=1}^{s} \sum_{k \in [r]} w_k z^{h_k} e\left( \frac{h_k u}{m} \right).
\]

By combining this equation and Remark 2 we get the Hamming weight enumerator for the LC codes:

\[
\mathcal{H}(\text{LC}_a(n, m, r, h); w) = \mathcal{W}(\text{LC}_a(n, m, r, h), \ell_h; 1, w^*)
\]

\[
= \sum_{u \in [m]} \prod_{j=1}^{s} \sum_{k \in [r]} \left( 1 + w \sum_{i=1}^{r-1} e\left( \frac{h_k u}{m} \right) \right).
\]

Since \( \text{BLC}_a(n, m, h) \) = \( \text{LC}_a(n, m, 2, h) \), we also get the Hamming weight enumerator for the BLC codes by substituting \( r = 2 \) into the equation above.

Remark 4: Now, we explain how we define the extended weight enumerator based on the proof of Theorem 4 The code membership function \( \mathcal{I}\{x \in C_{p,a,m}(n, r, s)\} \) generates the factor \( \prod_{i=1}^{s} e(u_i/m) \rho_i(x) \). The extended weight enumerator is defined by adding parameters \( z \) to the complete weight enumerator for including this factor.

IV. CARDINALITIES AND HAMMING WEIGHT ENUMERATORS FOR NON-BINARY TENENGOLETS’ CODES

This section derives the Hamming weight enumerators and cardinalities for non-binary Tenengolts’ codes by using Theorem 4 Moreover, we show the parameters which give maximum cardinality of the \( r \)-ary Tenengolts’ code of length \( n \). Section IV-A gives the notations used in this section. Section IV-B presents the results above and Section IV-C proves them. Section IV-D shows a numerical example.

A. Notations Used in This Section

For \( a, b \in \mathbb{Z} \), let \((a, b)\) be the greatest common divisor of \( a \) and \( b \). For \( n \in \mathbb{Z}^+ \), let \( \phi(n) \) and \( \mu(n) \) be Euler’s totient function and Möbius function, respectively, i.e., when \( n \) has a prime factorization \( n = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k} \),

\[
\phi(n) = n \prod_{j=1}^{k} \left( 1 - \frac{1}{p_j} \right),
\]

\[
\mu(n) = \begin{cases} (-1)^k, & \text{if } i_1 = i_2 = \cdots = i_k = 1, \\ 0, & \text{otherwise.} \end{cases}
\]

The value \( \eta_d \in \mathbb{C} \) is called a primitive \( d \)-th root of unity if \( \eta_d \neq 1 \) for \( i \in [1, d - 1] \) and \( \eta_d^d = 1 \) hold. For \( d \in \mathbb{Z}^+ \) and \( a \in \mathbb{Z} \), denote Ramanujan’s sum, by \( c_d(a) \), i.e.,

\[
c_d(a) := \sum_{j=[1, d], (j, d)=1} e\left( \frac{aj}{d} \right) = \frac{\phi(d) \mu(d)(a, d)}{\phi(d(a, d))}.
\]

Notice that \( c_d(a) = c_d(b) \) if \( a \equiv b \pmod{d} \).
For non-negative integers $a, b, t_0, t_1, \ldots, t_{r-1}$, denote $t := \langle t_0, t_1, \ldots, t_{r-1} \rangle$ and we define the binomial coefficient and multinomial coefficient as

\[
\binom{a + b}{a} := \frac{(a + b)!}{a!b!},
\]

\[
\binom{\sum_{i=0}^{r-1} t_i}{t} := \binom{\sum_{i=0}^{r-1} t_i}{t_0, t_1, \ldots, t_{r-1}} = \frac{(\sum_{i=0}^{r-1} t_i)!}{t_0!t_1! \cdots t_{r-1}!}.
\]

For $n \in \mathbb{Z}^+$, the $q$-integer is defined by

\[
[n]_q := 1 + q + q^2 + \cdots + q^n - 1.
\]

Similarly, for non-negative integers $a, b, t_0, t_1, \ldots, t_{r-1}$, we define the $q$-factorial, the $q$-binomial coefficient, and the $q$-multinomial coefficient as follows

\[
[a]_q! := \prod_{i=1}^{a} [i]_q = [a]_q[a - 1]_q \cdots [1]_q,
\]

\[
\binom{a + b}{a}_q := \frac{[a + b]_q!}{[a]_q!b!_q},
\]

\[
\binom{\sum_{i=0}^{r-1} t_i}{t}_q := \binom{\sum_{i=0}^{r-1} t_i}{t_0, t_1, \ldots, t_{r-1}}_q = \frac{[\sum_{i=0}^{r-1} t_i]!_q}{[t_0]_q! [t_1]_q! \cdots [t_{r-1}]_q!},
\]

where $t = \langle t_0, t_1, \ldots, t_{r-1} \rangle$. Hereafter, we drop the subscript $q$ if it is clear from the context.

**B. Main Result**

The following theorem presents the Hamming weight enumerators and cardinalities of the non-binary Tenengolts’ codes.

**Theorem 2:** Define the non-binary Tenengolts’ code as in Definition 4. For any $a_1 \in [n], a_2 \in [r]$, the following equations give the Hamming weight enumerator and cardinality of the $r$-ary non-binary Tenengolts’ code of length $n$ with parameters $a_1, a_2$:

\[
\mathcal{H}(T_{a_1, a_2}(n, r); w) = \frac{1}{nr} \sum_{d \in \mathbb{Z}^+, d \mid n} \sum_{a \in \mathbb{Z}^+, e \mid r} c_d(a_1)c_e(a_2)
\]

\[
\quad \times \{1 - w^d + rw^d|e|d\}^\frac{1}{q},
\]

\[
|T_{a_1, a_2}(n, r)| = \frac{1}{nr} \sum_{d=1}^{\lfloor n/r \rfloor} c_d(a_1)r^\frac{1}{q} (r, d) \prod_{j=1}^{\lfloor n/r \rfloor} \left(\frac{1}{q} \right) \prod_{d=1}^{\lfloor n/r \rfloor} \left(\frac{1}{q} \right)
\]

where $a = \langle a_0, a_1, \ldots, a_{r-1} \rangle$. For all non-negative integer $a$ and $j \in [d]$, the following holds

\[
\lim_{q \to q_d} [\sum_{i=0}^{r-1} t_i]_q = \prod_{k=1}^{r-1} \left(\frac{1}{q} \right) \prod_{k=1}^{r-1} \left(\frac{1}{q} \right) \left(\frac{1}{q} \right).
\]

The following lemma is shown by Hagiwara and Kong [22, Lem. 2.4].

**Lemma 6:** Let $\eta_d \in \mathbb{C}$ be a primitive $d$-th root of unity. Let $a, b$ be non-negative integers. Assume $d \mid (a + b)$. Then, we get

\[
\lim_{q \to q_d} \left(\frac{a + b}{d} \right) = \left(\frac{a + b}{d} \right) (d \mid a, b).
\]

This lemma is easily generalized as the following.

**Lemma 7:** Let $t_0, t_1, \ldots, t_{r-1}$ be non-negative integers. Let $\eta_d \in \mathbb{C}$ be a primitive $d$-th root of unity. Assume $d \mid \sum_{i=0}^{r-1} t_i$. Then, we get

\[
\lim_{q \to q_d} \left(\frac{\sum_{j \in [r]} t_j}{t/d} \right) = \left(\frac{\sum_{j \in [r]} t_j}{t/d} \right) \prod_{i=0}^{t_i} \left(\frac{1}{q} \right).
\]

where $t = \langle t_0, t_1, \ldots, t_{r-1} \rangle$.

**Proof:** Firstly, we suppose $d \mid t_i$ for all $i \in [r]$. The $q$-multinomial coefficient is factorized as follows:

\[
\binom{\sum_{i=0}^{r-1} t_i}{t}_q = \binom{\sum_{i=0}^{r-1} t_i}{t_0, t_1, \ldots, t_{r-1}}_q = \prod_{k=1}^{r-1} \left(\frac{1}{q} \right) \prod_{j=1}^{r} \left(\frac{1}{q} \right)
\]

Note that $d \mid \sum_{i=0}^{r-1} t_i$ holds for all $k \in [1, r-1]$. By applying Lemma 6 to this equation, we get

\[
\lim_{q \to q_d} \left(\frac{\sum_{i=0}^{r-1} t_i}{t/d} \right) = \prod_{k=1}^{r-1} \left(\frac{1}{q} \right) \prod_{j=1}^{r} \left(\frac{1}{q} \right) \left(\frac{1}{q} \right).
\]

Next, we suppose that there exists $i \in [r]$ such that $d \nmid t_i$. Since $t_i$ is a non-negative integer, it is expressed as $t_i = a_id + b_i$, where $b_i \in [d]$. From the above, there exists $i$ such that $b_i \neq 0$. Since $d \mid \sum_{i=0}^{r-1} t_i$, we get $d \mid \sum_{i=0}^{r-1} b_i$. Hence, we have $\sum_{i=0}^{r-1} b_i \geq d$. Note that $q$-multinomial coefficient is rewritten as

\[
\binom{\sum_{i=0}^{r-1} t_i}{t}_q = \binom{\sum_{i=0}^{r-1} a_id}{t} \prod_{j=1}^{r-1} \left(\frac{1}{q} \right) \prod_{j=1}^{r-1} \left(\frac{1}{q} \right) \left(\frac{1}{q} \right),
\]

where $a = \langle a_0, a_1, \ldots, a_{r-1} \rangle$. For all non-negative integer $a$ and $j \in [d]$, the following holds

\[
\lim_{q \to q_d} [\sum_{i=0}^{r-1} t_i]_q = 0,
\]

if $j \equiv 0 \pmod{d}$, otherwise.

From this, the denominator of the second factor in (4) does not contain zero factor. On the other hand, the numerator of the second factor in (4) contain zero factor, since $\sum_{i=0}^{r-1} b_i \geq d$. Hence, $\lim_{q \to q_d} [\sum_{i=0}^{r-1} t_i] = 0$.

**1) Proof of Theorem 2** We divide the proof into several lemmas.

**Lemma 8:** Let $n, r \in \mathbb{Z}^+$. Define $T_n := \{\langle t_0, t_1, \ldots, t_{r-1} \rangle \in [n]^r \mid \sum_{i \in [r]} t_i = n\}$. The following holds:

\[
W([r]^n, \langle r, \sigma \rangle; \langle q, z \rangle, w) = \sum_{t \in T_n} \prod_{j \in [r]} w_j t_j z^{t_j}.
\]
Proof: Lemma 5 gives
\[ W(S(t), \langle \gamma, \sigma \rangle; \langle q, z \rangle, w) = \left[ \sum_{t \in [r]} u_j^t z^{jt} \right]_{j=1}^n, \]

For \( t, t' \in T_n \) (\( t \neq t' \)), \( S(t) \cap S(t') = \emptyset \) holds. Moreover, \( \bigcup_{t \in T_n} S(t) = [r]^n \). Equation (2) yields
\[ W([r]^n, \langle \gamma, \sigma \rangle; \langle q, z \rangle, w) = \sum_{t \in T_n} W(S(t), \langle \gamma, \sigma \rangle; \langle q, z \rangle, w) = \sum_{t \in T_n} \left[ \sum_{j \in [r]} u_j^t z^{jt} \right]_{j=1}^n, \]

which concludes the proof.

Lemma 9: For any \( u_1 \in [n], u_2 \in [r] \), we get
\[ W([r]^n, \langle \gamma, \sigma \rangle; \langle e(nu_1), e(nu_2) \rangle, w) = \left( \sum_{i \in [r]} w_i u_2^{i(nu_2)} e \left( \frac{i(nu_2)}{n(u_1)} \right) \right)_{(n,u_1)}. \]

Proof: For a fixed \( u_1 \), define \( d := \frac{nu_1}{n(u_1)} \). Then, \( e(u_1/n) \) is a primitive \( d \)-th root of unity. Combining Lemmas 7 and 8 gives
\[ W([r]^n, \langle \gamma, \sigma \rangle; \langle e(nu_1), e(nu_2) \rangle, w) = \left( \sum_{i \in [r]} w_i u_2^{i(nu_2)} e \left( \frac{i(nu_2)}{n(u_1)} \right) \right)_{(n,u_1)}. \]

Proof of Theorem 2: The Hamming weight enumerator is derived as follows:
\[ \mathcal{H}(T_{a_1, a_2}(n, r); w) = W(T_{a_1, a_2}(n, r), \langle \gamma, \sigma \rangle; \langle 1, 1 \rangle, w^*) = \frac{1}{nr} \sum_{a_1 \in [n]} \sum_{u_2 \in [r]} e \left( -\frac{a_1 u_1}{n} \right) e \left( -\frac{a_2 u_2}{r} \right) \times W \left( \left[ r \right]^n, \langle \gamma, \sigma \rangle; \langle \frac{a_1 u_1}{n}, \frac{a_2 u_2}{r} \rangle, w^* \right) = \frac{1}{nr} \sum_{d \in \mathbb{Z}^+} \sum_{u_1 \in [n], u_2 \in [r]} \sum_{d_{nr} \in \{e, e_{d_{nr}}\}} \sum_{(u_1, n/d) = d} e \left( -\frac{a_1 u_1}{n} \right) e \left( -\frac{a_2 u_2}{r} \right) \times \left( 1 - w^{n/d} + rw^{n/d} \right)^d \]
Note that $c_1(0) = 1, c_3(0) = 2, c_3(1) = c_3(2) = -1$. We get

$$|T_{a_1, a_2}(3, 3)| = \begin{cases} 
5, & \text{if } a_1 = 0 \text{ and } a_2 = 0, \\
2, & \text{if } a_1 = 1, 2 \text{ and } a_2 = 0, \\
3, & \text{if } a_2 = 1, 2.
\end{cases}$$

This result is confirmed by Table XI.

E. Property for Variants of Non-binary Tenengoltz’ code

For $x = (x_1, x_2, \ldots, x_n)$, denote the vector reversing the order of $x$, by $\overline{x}$, i.e., $\overline{x} := \langle x_n, \ldots, x_2, x_1 \rangle$. The following theorem shows that the variants of the non-binary Tenengoltz’ code are essentially equivalent.

**Theorem 3:** Let $n, r \in \mathbb{Z}^+$. For $a_1 \in [n]$, define $\overline{a_1} := (n-a_1)\{a_1 \neq 0\}$ and

$$a'_1 := \begin{cases} 
(n-a_1)\{a_1 \neq 0\}, & \text{odd } n, \\
\frac{n}{2} - a_1 + n\{a_1 > n/2\}, & \text{even } n.
\end{cases}$$

Then, for any $a_1 \in [n]$ and $a_2 \in [r]$,

$$T_{a_1, a_2}^<(n, r) = \{x | x \in T_{a_1, a_2}(n, r)\},$$

$$T_{a_1, a_2}^≤(n, r) = \{x | x \in T_{a_1, a_2}^<(n, r)\},$$

$$T_{a_1, a_2}^≥(n, r) = T_{a'_1, a_2}(n, r),$$

$$T_{a_1, a_2}^>(n, r) = \overline{T_{a'_1, a_2}}(n, r).$$

**Proof:** ObVIOUSLY, $\sigma(\overline{x}) = \sigma(x)$ for any $x \in [r]^n$. Moreover, for any $x \in [r]^n$, we get

$$\lambda_{\leq}(x) = \frac{n(n-1)}{2}.$$

Hence, $\lambda_{\leq}(x) \equiv a_1 \pmod{\text{odd } n}$ if and only if $\sigma(x) \equiv \overline{a_1} \pmod{n}$, thus, we get (7). Similarly, we get (8).

For any $x \in [r]^n$, we have

$$\gamma(x) + \lambda_{\leq}(x) = \frac{n(n-1)}{2}.$$ 

Hence, we get

$$\gamma(x) + \lambda_{\leq}(x) = \begin{cases} 
0 \pmod{\text{odd } n}, & \text{odd } n, \\
\frac{n}{2} \pmod{\text{even } n}.
\end{cases}$$

Hence, $\lambda_{\leq}(x) \equiv a_1 \pmod{\text{odd } n}$ if and only if $\gamma(x) \equiv a'_1 \pmod{n}$, thus, we get (9). Similarly, we get (10).

Theorem 3 leads the cardinalities and Hamming weight enumerators for the variants of non-binary Tenengoltz’ codes.

**Theorem 4:** Let $n, r \in \mathbb{Z}^+$. For $a_1 \in [n]$, define $\overline{a_1}$ and $a'_1$ as in Theorem 3. Define

$$\overline{a}' := \begin{cases} 
a_1, & \text{odd } n, \\
\frac{n}{2} + a_1 - n\{a_1 \geq n/2\}, & \text{even } n.
\end{cases}$$

Then, for any $a_1 \in [n]$ and $a_2 \in [r],$

$$H(T_{a_1, a_2}^<(n, r); z) = H(T_{\overline{a}_1, a_2}(n, r); z),$$

$$H(T_{a_1, a_2}^≤(n, r); z) = H(T_{a'_1, a_2}(n, r); z),$$

$$H(T_{a_1, a_2}^≥(n, r); z) = H(T_{\overline{a}'_1, a_2}(n, r); z),$$

$$H(T_{a_1, a_2}^>(n, r); z) = H(T_{a'_1, a_2}(n, r); z),$$

$$H(T_{a_1, a_2}^≤(n, r); z) = H(T_{\overline{a}'_1, a_2}(n, r); z),$$

$$H(T_{a_1, a_2}^>(n, r); z) = H(T_{a'_1, a_2}(n, r); z).$$

**Proof:** We get (11) and (9) from (7) and (12), respectively. Equation (10) leads $H(T_{a_1, a_2}^>(n, r); z) = H(T_{a'_1, a_2}(n, r); z)$. Combining this and (12), we get (13). By substituting $z = 1$ into (11), (12), and (13), we obtain (14), (15), and (16), respectively.

The following corollary gives the parameters $\langle a_1, a_2 \rangle$ which attain the maximum cardinality.

**Corollary 4:** For any $n, r \in \mathbb{Z}^+$, the following holds

$$\arg\max_{a_1, a_2} H(T_{a_1, a_2}^<(n, r); z) \equiv (0, 0),$$

$$\arg\max_{a_1, a_2} H(T_{a_1, a_2}^≤(n, r); z) \equiv \begin{cases} 
(0, 0), & \text{odd } n, \\
\langle n/2, 0 \rangle, & \text{even } n.
\end{cases}$$

$$\arg\max_{a_1, a_2} H(T_{a_1, a_2}^≥(n, r); z) \equiv \begin{cases} 
(0, 0), & \text{odd } n, \\
\langle n/2, 0 \rangle, & \text{even } n.
\end{cases}$$

**Proof:** Combining Corollary 3 and Theorem 4 we obtain the corollary.

As shown in Example 2 and Corollary 4 when $n$ is even, the variants $T_{a_1, a_2}^<(n, r), T_{a_1, a_2}^≤(n, r)$ of the non-binary Tenengoltz’ code do not attain the maximum cardinality for $a_1 = a_2 = 0$.

V. MacWilliams Identity for Complete Weight Enumerators of Linear Codes

Recall that $L$ is a linear code over $\mathbb{Z}_r$. Let $L_\perp$ be the dual code of $L$, i.e.,

$$L_\perp := \{y \in \mathbb{Z}_r | y x^T = 0 \ \forall x \in L\}.$$

Note that $L_\perp$ is represented by a full-rank parity-check matrix $H$ for $L$, as

$$L_\perp = \{u H | u \in \mathbb{Z}_r^*\}.$$

The MacWilliams identity [23] gives the complete weight enumerator of a code from one of its dual code. This identity is widely investigated in the coding theory, e.g., [24], [25]. Now, we will derive the MacWilliams identity for the complete weight enumerator of the linear code over $\mathbb{Z}_r$ (e.g., see [26]) from Theorem 4.

**Corollary 5:** Let $L$ be the linear code over $\mathbb{Z}_r$, and $L_\perp$ its dual code. For a fixed $w = \langle w_0, w_1, \ldots, w_r-1 \rangle$, define $v_i := \sum_{k=0}^{r-1} w_k e(i k/r)$ for $i \in [r]$. Then, the following identity holds:

$$\overline{W}(L; v) = \frac{r^3}{r} \overline{W}(L_\perp; v).$$

**Proof:** Denote $h_i := \langle h_{i, 1}, h_{i, 2}, \ldots, h_{i, n} \rangle$. Note that

$$W([r]^n; \langle \ell_{h_1}, \ell_{h_2}, \ldots, \ell_{h_n} \rangle; z, w) = \prod_{j=1}^{n-r+1} \sum_{k=0}^{s-1} z^{h_{j, k}}.$$
Denote \( e(u/r) = (e(u_1/r), e(u_2/r), \ldots, e(u_s/r)) \). From Theorem 1 we get
\[
\overline{W}(L; w) = \frac{1}{r^s} \sum_{u \in [r]^s} W([r]^n, \langle \ell_{h_1}, \ell_{h_2}, \ldots, \ell_{h_s}; e(u/r), w \rangle) \\
= \frac{1}{r^s} \sum_{u \in [r]^s} \prod_{j=1}^{r-1} \sum_{i=0}^{r-1} w_k e \left( \frac{k \sum_{i=1}^{n} h_{i,j} u_i}{r} \right) \\
= \frac{1}{r^s} \sum_{y \in L\perp} \prod_{j=1}^{r-1} \sum_{i=0}^{r-1} w_k e \left( \frac{ky_i}{r} \right)^{\tau_i(y)} \\
= \frac{1}{r^s} \sum_{y \in L\perp} \prod_{i=0}^{r-1} \left( \sum_{k=0}^{r-1} w_k e \left( \frac{ik}{r} \right) \right)^{\tau_i(y)}.
\]
Recall that \( \overline{W}(L\perp; w) = \sum_{y \in L\perp} \prod_{i=0}^{r-1} w_i^{\tau_i(y)} \). Combining these, we complete the proof.

VI. Conclusion

This paper has derived a formula for the extended weight enumerator of the SC code. As a special case, this paper has also provided the Hamming weight enumerators and cardinalities of the non-binary Tenengolts' codes. Moreover, we have shown that the formula deduces the MacWilliams identity for the complete weight enumerator of the linear codes over \( \mathbb{Z}_r \).

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References

[1] T. Nozaki, “Weight enumerators for number-theoretic codes and cardinalities of Tenengolts’ non-binary codes,” in 2020 IEEE International Symposium on Information Theory (ISIT), 2020, pp. 729–733.
[2] A. S. J. Helberg, “Coding for the correction of synchronization errors,” Ph.D. dissertation, University of Johannesburg, 1993.
[3] R. Varshamov and G. Tenenholz, “Codes which correct single asymmetric errors,” Automatica i Telemekhanika, vol. 26, no. 2, pp. 288–292, 1965.
[4] V. I. Levenshtein, “Binary codes capable of correcting deletions, insertions, and reversals,” in Soviet physics doklady, vol. 10, no. 8, 1966, pp. 707–710.
[5] K. Bibok and O. Milenkovic, “Weight enumerators of some classes of deletion correcting codes,” in 2018 IEEE International Symposium on Information Theory (ISIT), 2018, pp. 431–435.
[6] T. Nozaki, “Bounded single insertion/deletion correcting codes,” in 2019 IEEE International Symposium on Information Theory (ISIT), June 2019, pp. 2379–2383.
[7] S. D. Constantin and T. Rao, “On the theory of binary asymmetric error correcting codes,” Information and Control, vol. 40, no. 1, pp. 20–36, 1979.
[8] A. Shiozaki, “Single asymmetric error-correcting cyclic AN codes,” IEEE Computer Architecture Letters, vol. 31, no. 06, pp. 554–555, 1982.
[9] K. A. Abdel-Ghaffar and H. C. Ferreira, “Systematic encoding of the Varshamov-Tenengolts codes and the Constantin-Rao codes,” IEEE Transactions on Information Theory, vol. 44, no. 1, pp. 340–345, 1998.
[10] M. Abroshan, R. Venkataramanan, and A. G. I. Fabregas, “Efficient systematic encoding of non-binary VT codes,” in 2018 IEEE International Symposium on Information Theory (ISIT), 2018, pp. 91–95.
[11] B. Ginzburg, “A certain number-theoretic function which has an application in coding theory,” Problemy Kibern., vol. 19, pp. 249–252, 1967.