PARTITION-THEORETIC FROBENIUS-TYPE LIMIT FORMULAS

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Abstract. Using partition generating function techniques, we prove q-series analogues of a formula of Frobenius generalizing Abel’s convergence theorem for complex power series. Frobenius’ result states that for |q| < 1, \( \lim_{q \to 1} \frac{1}{N} \sum_{k=1}^{N} f(k) \) is equal to the average value \( \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) of the sequence \( \{ f(n) \} \) as \( n \to \infty \), if the average value exists.

1. Introduction and Statement of Results

In [1], Abel proves a foundational theorem on the convergence of complex power series.

Proposition 1.1 (Abel’s Convergence Theorem). Let \( f : \mathbb{N} \to \mathbb{C} \) be an arithmetic function. For \( q \in \mathbb{C}, |q| < 1 \), if the limit \( L = \lim_{N \to \infty} \sum_{1 \leq k \leq N} f(k) \) exists, then

\[
\lim_{q \to 1} \sum_{n \geq 1} f(n)q^n = L
\]
as \( q \to 1 \) radially from within the unit disk.

Another “Abel type” theorem giving limiting values as \( q \to 1 \) for certain classes of complex power series, is proved by Frobenius in [4].

Proposition 1.2 (Frobenius’ Theorem). Let \( f : \mathbb{N} \to \mathbb{C} \) be an arithmetic function. For \( q \in \mathbb{C}, |q| < 1 \), if the average value \( f_{\text{avg}} := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
\lim_{q \to 1} (1 - q) \sum_{n \geq 1} f(n)q^n = f_{\text{avg}}
\]
as \( q \to 1 \) radially from within the unit disk.

In this paper, we prove theorems analogous to Proposition 1.2 using methods from \( q \)-series, that we will refer to as Frobenius-type limit formulas. We note that the limit in Proposition 1.2 holds if \( q \to 1 \) through any path in a Stolz sector of the unit disk, a region with vertex at \( z = 1 \) such that \( \frac{1-|q|}{1-|q|} \leq M \) for some \( M > 0 \) (see [12]).

Let \( P \) denote the integer partitions [3]. For \( \lambda \in P \), let \(|\lambda|\) denote the size of \( \lambda \) (sum of parts), \( \ell(\lambda) \) denote the length (number of parts), and let \( \text{sm}(\lambda) \) and \( \text{lg}(\lambda) \) denote the smallest part and largest part of \( \lambda \), respectively, noting \(|\emptyset| = \ell(\emptyset) = \text{sm}(\emptyset) = \text{lg}(\emptyset) := 0 \) for \( \lambda = \emptyset \) the empty partition. For \( z, q \in \mathbb{C}, |zq| < 1 \), let \( (z;q)_n := \prod_{0 \leq k < n} (1 - zq^k) \) denote the \( q \)-Pochhammer symbol, with \( (z;q)_\infty := \lim_{n \to \infty} (z;q)_n \). Let \( p(n) = \sum_{|\lambda|=n} 1 \) denote the partition function (number of partitions of size \( n \geq 0 \)), with the initial value \( p(0) := 1 \).

Note that if \( f(n) \) is the indicator function for a subset \( S \subseteq \mathbb{N} \) with arithmetic density \( d_S \), then Proposition 1.2 gives the limiting value \( f_{\text{avg}} = d_S \) as \( q \to 1 \). Inspired by work of Alladi [2], in [7, 8, 11], the author and my collaborators exploited this idea to prove

\footnote{Prop. 1.2 is an equivalent statement to the second equation of [6], replacing \( a_n \) by \( f(n) \), and \( A \) by \( f_{\text{avg}} \); the condition that \( f_{\text{avg}} \) exists is equivalent to the Tauberian condition \( \sum_{k \leq n} a_k \sim An \).}
partition-theoretic and $q$-series formulas for $d_S$ with $q \to 1$, as well as at other roots of unity $\zeta$. The present note is a complement to the papers [7, 8]; we give a general setting in which such partition-theoretic density computations arise naturally. Throughout this paper, we take $q \to 1$ in a Stolz sector of the unit disk.

It is not hard to write down partition-theoretic analogues of Proposition 1.2. Noting that

$$f(n)q^n = \frac{f(n)}{p(n)} q^n \cdot \sum_{|\lambda|=n} 1 = \sum_{|\lambda|=n} \frac{f(|\lambda|)}{p(|\lambda|)} q^{|\lambda|},$$

then Proposition 1.2 can be rewritten as a sum over partitions:

$$\lim_{q \to 1} (1 - q) \sum_{\lambda \in P} f(|\lambda|) q^{|\lambda|} = f_{\text{avg}}.$$ 

This resembles Proposition 1.2 somewhat in form, but writing down the coefficients explicitly requires one to repeatedly compute the partition function, a nontrivial task. Alternatively, replacing $f(n)$ in Proposition 1.2 with $(f \cdot p)(n) := f(n)p(n)$ gives by the same argument

$$\lim_{q \to 1} (1 - q) \sum_{\lambda \in P} f(|\lambda|) q^{|\lambda|} = (f \cdot p)_{\text{avg}},$$

if $(f \cdot p)_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k)p(k)$ exists. However, neither (1) nor (2) strongly resembles Proposition 1.2 in that the limits on the right-hand sides do not equal the average values of the coefficients $f(|\lambda|)$ on the left, but versions weighted by $p(n)$.

Below we prove a number of Frobenius-type limit formulas that represent more faithful analogues of Proposition 1.2. The proofs of these formulas hold for arithmetic functions $f(n)$ that we will refer to as having the property of “$q$-summability” \footnote{We do not prove general $q$-summability theorems here. The property must be checked for a given $f(n)$; general proofs of $q$-summability would be useful. We note here, as remarks, examples from previous works [7, 8, 10] proved by less general methods, as demonstrations that our general limit theorems are not vacuous.}

**Definition 1.3.** Suppose for arithmetic function $f: \mathbb{N} \to \mathbb{C}$ that the limit $f_{\text{avg}} := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k)$ exists. For $|q| < 1$, write

$$\lim_{q \to 1} (1 - q) \sum_{n \geq 1} f(n)q^n = f_{\text{avg}}q + \epsilon_f(q)q,$$

noting by Proposition 1.2 that as $q \to 1$, the error function $\epsilon_f(q) \to 0$.

We define $f(n)$ to be a $q$-summable function of type (Q, 1) if $\sum_{k \geq 1} f(k)q^k(q; q_k)^{-1}$ is absolutely convergent, and the following condition holds:

$$\lim_{q \to 1} \sum_{k \geq 1} \frac{\epsilon_f(q^k)q^k}{(q; q_k)} = 0.$$

We define $f(n)$ to be a $q$-summable function of type (Q, 2) if $\sum_{k \geq 1} f(k)q^k(q; q_k)^{-1}$ is absolutely convergent, and the following condition holds:

$$\lim_{q \to 1} \sum_{k \geq 1} \frac{(-1)^{k+1} \epsilon_f(q^k)q^{\frac{k(k+1)}{2}}}{(q; q_k)} = 0.$$

**Remark.** The property of $q$-summability generalizes the idea of $q$-commensurate subsets of $\mathbb{N}$ in [8]: $S \subseteq \mathbb{N}$ is $q$-commensurate if and only if the indicator function of $S$ is $q$-summable.

**Remark.** We loosely imitate the notations for summation methods (C, 1), (C, 2), etc. in [5].
**Theorem 1.4.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 1)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
\lim_{q \to 1} (q; q)_\infty \sum_{\lambda \in \mathcal{P}} f(\text{sm}(\lambda)) q^{\lambda} = f_{\text{avg}},
\]

where the sum is taken over all partitions, and \( \text{sm}(\lambda) \) denotes the smallest part of \( \lambda \in \mathcal{P} \).

We prove this theorem and all other results in Section 2 below. Theorem 1.4 is a true partition-theoretic analogue of Proposition 1.2.

Partition generating function methods yield further formulas to compute the limit.

**Corollary 1.5.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 1)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
- \lim_{q \to 1} \sum_{\lambda \in \mathcal{P}} \mu_{\mathcal{P}}(\lambda) f(\text{lg}(\lambda)) q^{\lambda} = f_{\text{avg}}.
\]

**Corollary 1.6.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 1)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
\lim_{q \to 1} \sum_{n \geq 1} f(n) q^n (q; q)_{n-1} = \lim_{q \to 1} \sum_{n \geq 1} \sum_{k \geq 1} \frac{f(n) q^{nk}}{(q; q)_{k-1}} = f_{\text{avg}}.
\]

**Remark.** Setting \( f(n) \) equal to the indicator function for \( S \subseteq \mathbb{N} \), then that the first limit in Corollary 1.6 is equal to \( f_{\text{avg}} = ds \), re-proves Theorem 3.6 of [8].

Somewhat surprisingly, if one replaces “sm” with “lg” in Theorem 1.4, the limit still holds.

**Theorem 1.7.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 2)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
\lim_{q \to 1} (q; q)_\infty \sum_{\lambda \in \mathcal{P}} f(\text{lg}(\lambda)) q^{\lambda} = f_{\text{avg}},
\]

where the sum is taken over all partitions, and \( \text{lg}(\lambda) \) denotes the largest part of \( \lambda \in \mathcal{P} \).

Theorem 1.7 is a second partition analogue of Proposition 1.2. As with Theorem 1.4, generating function methods yield further formulas to compute \( f_{\text{avg}} \).

**Corollary 1.8.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 2)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
- \lim_{q \to 1} \sum_{\lambda \in \mathcal{P}} \mu_{\mathcal{P}}(\lambda) f(\text{sm}(\lambda)) q^{\lambda} = f_{\text{avg}}.
\]

**Corollary 1.9.** For \( f(n) \) a \( q \)-summable arithmetic function of type \((Q, 2)\), if the limit \( f_{\text{avg}} = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \) exists, then

\[
\lim_{q \to 1} (q; q)_\infty \sum_{n \geq 1} f(n) q^n (q; q)_{n} = - \lim_{q \to 1} \sum_{n \geq 1} \sum_{k \geq 1} \frac{(-1)^k f(n) q^{nk + \frac{k(k-1)}{2}}}{(q; q)_{k-1}} = f_{\text{avg}}.
\]
Remark. Setting \( f(n) = \frac{\varphi(n)}{N} \) in Corollary 1.8 with \( \varphi(n) \) the Euler phi function; it is a well-known result that \( \frac{1}{N} \sum_{k=1}^{N} f(k) \sim 6/\pi^2 \) as \( N \to \infty \). Noting \( f(n) \) is \( q \)-summable of type \((Q, 2)\) (see remark below), then

\[
\lim_{q \to 1} (q; q)_\infty \sum_{n \geq 1} \frac{\varphi(n)q^n}{n \cdot (q; q)_n} = \frac{6}{\pi^2}.
\]

Remark. This re-proves Example E.1.1 of [10] for the case \( \zeta = 1 \).

Example 1. Set \( f(n) = \frac{\varphi(n)}{N} \) in Corollary 1.9 with \( \varphi(n) \) the Euler phi function; it is a well-known result that \( \frac{1}{N} \sum_{k=1}^{N} f(k) \sim 6/\pi^2 \) as \( N \to \infty \). Noting \( f(n) \) is \( q \)-summable of type \((Q, 2)\) (see remark below), then

\[
\lim_{q \to 1} (q; q)_\infty \sum_{n \geq 1} \frac{\varphi(n)q^n}{n \cdot (q; q)_n} = \frac{6}{\pi^2}.
\]

Remark. One anticipates similar limiting formulas hold as \( q \) approaches other roots of unity.

2. Proofs of results

The proofs in this section begin with a rewriting of equation (3) from Definition (1.3):

\[
(7) \quad \sum_{n \geq 1} f(n)q^n = f_{\text{avg}} \cdot \frac{q}{1-q} + \frac{\varepsilon f(q)q}{1-q},
\]

with \( f_{\text{avg}} := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(k) \), as above, and \( \varepsilon f(q) \) as defined by (3).

We use equation (7) as a building block to produce further \( q \)-series formulas to compute the limit \( f_{\text{avg}} \). The proof of Theorem 1.4 below generalizes the proof of Theorem 3.6 in [8].

Proof of Theorem 1.4. Take \( q \mapsto q^k \) in (7). Multiply both sides by \( (q; q)_{k-1} \), sum over \( k \geq 1 \), then swap order of summation of the double summation, and make the change of indices \( k \mapsto k+1 \) on the left, to give

\[
(8) \quad \sum_{n \geq 1} \sum_{k \geq 1} \frac{f(n)q^{nk}}{(q; q)_{k-1}} = \sum_{n \geq 1} f(n)q^n \sum_{k \geq 0} \frac{q^{nk}}{(q; q)_k} = f_{\text{avg}} \cdot \sum_{k \geq 1} \frac{q^k}{(q; q)_k} + \sum_{k \geq 1} \varepsilon f(q^k)q^k (q; q)_k \sim f_{\text{avg}} \cdot \sum_{k \geq 1} \frac{q^k}{(q; q)_k}
\]

as \( q \to 1 \). We note that both the asymptotic, and the order-of-summation swap by absolute convergence, are justified by the hypothesis that \( f \) is \( q \)-summable of type \((Q, 1)\) (see (11)).

By the \( q \)-binomial theorem [3], the inner sum over \( k \geq 0 \) in the second double series is equal to \( (q^n; q)_{-1}^{-1} \), and the summation on the right is \( (q; q)_{-1}^{-1} - 1 \). Multiplying both sides of (8) by \( (q; q)_\infty \) gives, from standard generating function arguments, as \( q \to 1 \):

\[
(9) \quad (q; q)_\infty \sum_{\lambda \in P} f(\text{sm}(\lambda)) q^{\lambda} = (q; q)_\infty \sum_{n \geq 1} \frac{f(n)q^n}{(q^n; q)_\infty} = \sum_{n \geq 1} f(n)q^n (q; q)_{n-1} = \sum_{\lambda \in P} \mu_P(\lambda) f(\text{lg}(\lambda)) q^{\lambda} \sim f_{\text{avg}}.
\]

That the left-hand side is asymptotically equal to \( f_{\text{avg}} \) as \( q \to 1 \), is equivalent to the statement of the theorem. \( \square \)
Proof of Corollaries 1.5 and 1.6. These corollaries record alternative expressions for the limit \( f_{\text{avg}} \) derived during the proof of Theorem 1.4 above.

The following proof of Theorem 1.7 generalizes the proof Theorem 3.5 in [8].

Proof of Theorem 1.7. Take \( q \mapsto q^k \) in (7). Multiply through by \((-1)^k q^{\frac{k(k-1)}{2}} (q; q)_{k-1}^{-1}\) and sum both sides over \( k \geq 1 \), swapping order of summation on the left-hand side, to give

\[
\sum_{n \geq 1} \sum_{k \geq 1} \frac{(-1)^k f(n) q^{nk + \frac{k(k-1)}{2}}}{(q; q)_{k-1}}.
\]

For each \( k \geq 1 \), the factor \((q; q)_{k-1}^{-1}\) generates partitions with largest part strictly less than \( k \). The factor \( q^{nk} \) adjoins a largest part \( k \) with multiplicity \( n \) to each partition, for every \( n \geq 1 \). The \( q^{k(k-1)/2} = q^{1+2+3+\ldots+(k-1)} \) factor guarantees at least one part of each size less than \( k \). Thus (10) is the generating function for partitions \( \gamma \) with every natural number \( \lg(\gamma) \) appearing as a part, weighted by \((-1)^{\lg(\gamma)} f(m_{\lg}(\gamma)) = (-1)^k f(n)\), where \( m_{\lg}(\gamma) = n \) denotes the multiplicity of the largest part of each partition \( \gamma \).

Under conjugation, this set of partitions \( \gamma \) maps to partitions \( \lambda \) into distinct parts weighted by \( \mu_P(\lambda) f(\text{sm}(\lambda)) = (-1)^{\ell(\lambda)} f(\text{sm}(\lambda)) = (-1)^k f(n)\), which is nonzero since \( \lambda \) has no repeated part. Thus, noting \( f(0) := 0 \), and multiplying through by a factor of \(-1\) to produce the desired end result, we have

\[
\sum_{n \geq 1} \sum_{k \geq 1} \frac{(-1)^k f(n) q^{nk + \frac{k(k-1)}{2}}}{(q; q)_{k-1}} = -\sum_{\lambda \in P} \mu_P(\lambda) f(\text{sm}(\lambda)) q^{\lfloor \lambda \rfloor} = \sum_{n \geq 1} f(n) q^{n(q^n+1); q}_{\infty}
\]

\[
= (q; q)_{\infty} \sum_{n \geq 1} \frac{f(n) q^n}{(q; q)_n} = (q; q)_{\infty} \sum_{\lambda \in P} f(\lg(\lambda)) q^{\lfloor \lambda \rfloor},
\]

using standard partition generating function methods. Manipulating the right-hand side of (7) accordingly gives, as \( q \to 1 \),

\[
\sum_{n \geq 1} \sum_{k \geq 1} \frac{(-1)^k f(n) q^{nk + \frac{k(k-1)}{2}}}{(q; q)_{k-1}} = -f_{\text{avg}} \cdot \sum_{k \geq 1} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q; q)_k} + \sum_{k \geq 1} \frac{(-1)^{k+1} \varepsilon f(q^k) q^{\frac{k(k+1)}{2}}}{(q; q)_k}
\]

\[
\sim -f_{\text{avg}} \cdot \sum_{k \geq 1} \frac{(-1)^k q^{\frac{k(k+1)}{2}}}{(q; q)_k} = -f_{\text{avg}} \cdot ((q; q)_{\infty} - 1).
\]

The asymptotic and the order-of-summation swaps (by absolute convergence) are justified by the hypothesis that \( f \) is \( q \)-summable of type \((Q, 2)\) (see [5]); and we use an identity of Euler for the final equality (see [3]), noting the right-hand side approaches \( f_{\text{avg}} \) as \( q \to 1 \). Comparing the right-hand sides of (11) and (12) as \( q \to 1 \) completes the proof.

Proof of Corollaries 1.8 and 1.9. These two corollaries record alternative expressions for the limit \( f_{\text{avg}} \) derived during the proof of Theorem 1.7 above.

The identity (7) could be involved in series manipulations in diverse ways. Using techniques from \( q \)-series, modular forms, Lambert series, et al., it seems likely one can produce other classes of limit formulas analogous to Proposition 1.2.
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