Kepler's Laws without Calculus

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Kepler’s Laws are derived from the inverse squared force law without the use of calculus and are simplified over previous such derivations. It uses just elementary algebra and trigonometry, and does not even use any advanced geometry.
I.

Newton in the Principia used the inverse squared force law (and Galileo’s idea of compound motion) to derive Kepler’s laws. As usual for him, the proof is a purely geometric proof, using no calculus, although he might well have used calculus in private. Whether or not his proof that inverse-square implied conic section as the only solutions has been the topic of some controversy but that is not the topic of this paper. Maxwell\textsuperscript{5}, using the hodographic technique of Hamilton\textsuperscript{3,4} gave a very different proof. In the 1960’s, Feynman\textsuperscript{6} gave a geometric proof very similar to Maxwell’s. Finally, Vogt\textsuperscript{7} in the American Journal of Physics also carried out a derivation which started out with the energy conservation equation and the angular momentum conservation to again present a geometric proof.

In all of these cases, the derivation that the orbit actually is an ellipse was, to me, somewhat torturous and difficult to follow. In the following, starting off using the Maxwell-Feynman technique of solving for the velocities as a function of the angle about the center of attraction, the derivation that the orbit is an ellipse is simplified about as much as possible. I do so both using calculus and, following Newton’s proof of Kepler’s second law, a discrete geometric argument to arrive at the equation for a conic section.

II. KEPLER’S SECOND LAW

In keeping with all of the above, I will start with Newton’s discretized proof of Kepler’s second law, showing that a central force produces orbits which obey the equal areas in equal times law. I will repeat it because I will use the same ideas in the proof of the first law.

Consider first a straight line along which an object travels inertially. This means it continues in the same direction with constant speed. Consider the motion in two adjacent very short time intervals, of length $dt$ where this is assumed small. Join the points at times $0, dt, 2dt$ to what will be the centre of force. We now have two triangles, each with base of length $|v|dt$ and common vertex. They must therefore have the same area. Now look at these triangles and the common base joining the the point at time $dt$ to the vertex. Since the areas are common, the heights to the tips at 0 an $2dt$ must be equal. Draw the perpendicular to the line from the vertex to the point $dt$ to the two lines which go through those two tips and are parallel to line from the vertex to $dt$. The two line segments from $dt$ to the parallel
FIG. 1. These three diagrams show the two triangular areas swept out in equal times by a polygonal orbit with a central force acting between two situations in which the object travels inertially. The first figure shows that if there is no force acting at all, the areas of the two triangles are equal since the bases are of equal length since the speeds are the same. The second figure shows that the perpendicular to the common side to the height of the end points of the line must therefor also be the same. In the third figure that is an acceleration acting along the common line, and perpendicular to the orthogonal line to this common line.

through 0 and from $dt$ to the parallel through $2dt$ are equal as these are just the heights of the triangles from the common base. Ie, the particle travels with uniform speed along this straight line. Galileo stated that the if there was an acceleration perpendicular to this line, that uniform motion along the line would not be disturbed. Now accelerate the body suddenly along the radius through $dt$. The second triangle will be distorted, but its common base will not change, and their heights will also not change. Thus the areas of the triangles will not change. If we now take the limits as $dt$ goes to zero, we find that the areas swept in equal times will be constant no matter what the magnitude is of the radial force. This is just the expression that a radial force has a conservation law, the conservation of angular momentum.

We can write the above in terms of differentiation In the limit as $\Delta \theta$ goes to 0, the area is $r^2(d\theta/dt)/2 = \ell/2$ where $\ell$ is thus twice the rate of change of area with time and is equal to the angular momentum.

\[
\frac{d\theta}{dt} = \frac{\ell}{r^2}
\]
III. INVERSE SQUARE LAW AND ELLIPSE–CALCULUS

Let me first go through the derivation using calculus, but not solving any differential equations. Since the geometric proof will follow the procedure in detail, I will first solve the equations using calculus.

Consider that the force law is an inverse square force law. i.e., the acceleration of the object has the form \( \frac{\mu}{r^2} \) where \( r \) is the distance from the object to the above vertex. Now consider the change in velocity, but not in a unit time but in a unit angle. We can write the equation of motion in a small time interval as

In differential form, Newton’s second law becomes,

\[
\frac{d\vec{v}}{d\theta} = \frac{2r^2}{\ell} \left( -\frac{\mu}{r^2} \vec{n}_r \right)
\]

from which

\[
\vec{v} = \vec{v}_0 + \frac{\mu}{\ell} \vec{n}_\theta
\]

Where \( r \vec{n}_r = \vec{r} \) and \( \vec{n}_\theta \) is the unit vector orthogonal to \( \vec{n}_r \) and points in the direction of increasing \( \theta \).

\[
\frac{1}{r} \frac{d(r \vec{n}_r)}{d\theta} = \frac{r}{\ell} \left( \vec{v}_0 + \frac{\mu}{\ell} \vec{n}_\theta \right)
\]

Taking only the tangential component, we have

\[
1 = \frac{r}{\ell} (\vec{v}_0 \cdot \vec{n}_\theta) + 2\frac{\mu}{\ell}
\]

\[
= \frac{|v_0|}{\ell} r \cos(\theta) + \frac{\mu}{\ell^2} r
\]

or

\[
r + \frac{|v_0|}{\mu} x = \frac{\ell^2}{\mu}
\]

with \( x = r \cos(\theta) \), or

\[
r + \epsilon x = R_0
\]

where \( \epsilon = |v_0|\ell/\mu \) and \( R_0 = \ell^2/\mu \).

This is the equation for an ellipse with the semi-major axis

\[
a = R_0/(1 - \epsilon^2) = (\ell^2/\mu) - (|v_0|\ell)^2
\]
IV. ELLIPSE WITH GEOMETRY

Now, let us try doing the above using geometry rather than calculus. (It is suspected that Newton solved many problems using calculus that he had invented, and translated them into geometry for publication.) This provides a way of proving that that orbit is elliptical without calculus.

We follow the spirit of the procedure used to prove the second law, but now look at a larger section of the orbit. We again draw the orbit as a polygon but now with the various lines defining triangles from the vertex $P$ being separated by angles $\Delta \theta$. Every second radial line goes to a vertex where the velocity changes discontinuously as in the first law proof. The other radial lines go from the vertex to a point near the midpoints of the lines between the discontinuity vertexes. Each pair of adjacent radial lines are separated by the above angle. The area of the triangle from the centre $P$ to point $b$ to point $1$ is given by

$$A_1 = \frac{1}{2} r_b r_1 \sin(\Delta \theta)$$  \hspace{1cm} (10)

while the second triangle is

$$A_2 = \frac{1}{2} r_b r_2 \sin(\Delta \theta)$$  \hspace{1cm} (11)
The change in velocity at the vertex \( b \) is \( \Delta \vec{v} = \vec{v}_2 - \vec{v}_1 \). It is assumed that there is a sudden acceleration \( \vec{a} \) along \( r_b \). Newton’s equation gives

\[
\delta \vec{v} = \vec{a} T_{12}
\]

where \( T_{12} \) is the time of travel between points 1 and 2 and \( \delta Q = Q_2 - Q_1 \) for any quantity \( Q \). But by Kepler’s second law,

\[
T_{12} = \frac{2}{\ell} (A_1 + A_2)
\]

where \( \ell \) is again is twice the rate at which the area is swept out by the radius vector. Thus we have

\[
\delta \vec{v} = \frac{1}{\ell} (r_1 + r_2) r_b \sin(\Delta \theta)
\]

where \( \vec{a} \) is the acceleration vector. Again use the unit vectors \( \vec{n}_r, \vec{n}_\theta \).

Assume that

\[
\frac{1}{2} (r_1 + r_2) r_b \vec{a} = -\mu \vec{n}_{rb}
\]

where \( \mu \) is a constant. This is the assumption that the acceleration falls off as the inverse of the distance squared (Newton’s law of gravitation) adapted to the discretized version. \( (r_1 + r_2) r_b \) is of order \( \Delta \theta^2 \) away from \( r^2 \).

Using that the vectors \( \vec{n}_{r1} \) and \( \vec{n}_{\theta 1} \) are just rotations through the angle \(-\Delta \theta\) from those at the point \( b \), and similarly for the vectors at point 2 through the angle \( \Delta \theta \), one has

\[
\vec{n}_{r1} + \vec{n}_{r2} = 2 \cos(\Delta \theta) \vec{n}_{rb}
\]

\[
\vec{n}_{\theta 1} + \vec{n}_{\theta 2} = 2 \cos(\Delta \theta) \vec{n}_{\theta b}
\]

\[
\vec{n}_{r2} - \vec{n}_{r1} = 2 \sin(\Delta \theta) \vec{n}_{\theta b}
\]

\[
\vec{n}_{\theta 2} - \vec{n}_{\theta 1} = 2 \sin(\Delta \theta) \vec{n}_{rb}
\]

We can write

\[
\vec{v} = v_r \vec{n}_r + v_\theta \vec{n}_\theta
\]

and the equation for the discontinuity of \( \vec{v} \) caused by the acceleration at \( b \) is

\[
\frac{1}{2} \left[ \delta v_r \vec{n}_r + v_r \delta \vec{n}_r + \delta v_\theta \vec{n}_\theta + v_\theta \delta \vec{n}_\theta \right] = -\frac{\mu}{\ell} \vec{n}_{rb} \sin(\Delta \theta)
\]
where $\delta Q = Q_2 - Q_1$ and $\overrightarrow{Q} = (Q_2 + Q_1)$ for any quantity $Q$.

Using the equations for the sums and differences and separating out the components of $\vec{n}_{rb}$ and $\vec{n}_{\theta b}$, we have the two equations for the components

$$2\delta v_r \cos(\Delta \theta) - 2\overline{v}_\theta \sin(\Delta \theta) = \frac{2\mu \sin(\Delta \theta)}{\ell}$$

(22)

$$2\delta v_\theta \cos(\Delta \theta) + 2\overline{v}_r \sin(\Delta \theta) = 0$$

(23)

which are a pair of difference equations for $v_{r2}$ and $v_{\theta2}$ in terms of $v_{r1}$, $v_{\theta1}$ and $\mu$.

The solution of these linear equations can always be written as a sum of a particular solution of the inhomogeneous equations plus an arbitrary solution of the homogeneous equations ($\mu = 0$). The simplest solution to the former is to take $v_{ri} = 0$ for all numerical $i$. Then $v_{\theta i}$ are all equal, $\hat{v}_\theta$. One has

$$2\hat{v}_\theta \sin(\Delta \theta) = \frac{2\mu \sin(\Delta \theta)}{\ell}$$

(24)

(25)

or

$$\hat{v}_\theta = \frac{\mu}{\ell}$$

(26)

The simplest way to solve the homogeneous equation is to go back to eqn [14] setting the rhs (proportional to $\mu$) to zero, which gives

$$\vec{v}_2 = \vec{v}_1$$

(27)

The homogeneous solution is an arbitrary constant vector $\vec{v}_0$ independent of which vertex one is looking at.

The generic solution is exactly the solution obtained above for the calculus solution, namely a constant vector plus a “rotating” vector orthogonal to the radius vector.

Now let us solve for the equation of the polygon. The vectors $\vec{v}_0 + \hat{v}_\theta \vec{n}_{\theta i}$ point along the sides of the polygon and the vector difference of the two location vectors $\vec{r}_b$ and $\vec{r}_b'$ is just the velocity $\vec{v}_2$ times the time $T_{bb'}$ it takes the particle to travel between the two points $b$ and $b'$.

$$r'_b \vec{n}_{rb} - r_2 \vec{n}_{rb} = (\vec{v}_0 + \frac{\mu}{\ell} \vec{n}_{\theta 2}) T_{2b'}$$

(28)
or, using Kepler’s second law with \( T_{2b'} = [r'_b \sin(\Delta \theta)]/2 \ell \) as the time to go from the point 2 to b'. The equation for \( r_b \mathbf{n}_{r_b} - r_2 \mathbf{n}_{r_b} \) is the same with \( \Delta \theta \to -\Delta \theta \) We now take the dot product of this with \( \mathbf{n}_{\theta_2} \) Taking the dot product with \( \mathbf{n}_{\theta_2} \) and dividing by \( r_b \) we get

\[
1 = \left( \mathbf{n}_{\theta_2} \cdot \mathbf{v}_0 + \frac{\mu}{\ell} \right) \frac{r_2}{\ell}
\]

which, after defining \( n_{\theta_2} \cdot \mathbf{v}_0 = |v_0| \cos(\theta_2) \) and \( r_2 \cos(\theta_2) = x_2 \), gives

\[
r_2 + \frac{|v_0| \ell}{\mu} x_2 = \frac{\ell^2}{\mu}
\]

which is again exactly the equation for an ellipse passing through the endpoints of lines from the vertex P to the straight segments of the polygon, near the centres of those lines. This is true even in the limit as \( \Delta \theta \) goes to zero giving exactly the same solution as above done with calculus. This analysis is identical to that using \( r_b \) since the only difference was taking \( \Delta \theta \to -\delta \theta \).

We now take the dot product with \( \mathbf{n}_{\theta_2} \) and get

\[
r'_b \cos(\Delta \theta) - \frac{R_0}{1 + \epsilon \cos(\theta_2)}(\frac{v_0}{\ell} \sin(\theta_2))r_b r_2 \sin(\Delta \theta) = \frac{\epsilon \sin(\Delta \theta)}{1 + \epsilon \cos(\theta_2)} r_b
\]

or solving for \( r'_b \),

\[
r'_b \frac{(\cos(\Delta \theta) + \epsilon(\cos(\theta_2) \cos(\Delta \theta) - \sin(\theta_2) \sin(\Delta \theta))}{1 + \epsilon \cos(\theta_2)} = r_2
\]

or

\[
r'_b = \frac{R_0}{\cos(\Delta \theta) + \epsilon \cos(\theta_2 + \Delta \theta)}
\]

where \( \cos(\theta_2 + \Delta \theta) = \cos(\theta_{b'}) \).

One gets exactly the same equation for \( r_b \) but with \( \Delta \theta \to -\Delta \theta \) and thus in terms of \( \theta_b \).

Thus all of the \( b \) vertexes lie on the ellipse

\[
r + \frac{\epsilon}{\cos(\Delta \theta)} x = \frac{R_0}{\cos(\Delta \theta)}
\]

Thus we can exactly solve the equations of motion for a discretization of the orbit, a la Newton’s proof of the second law without doing any calculus whatsoever (but with some algebra and trigonometry). In particular, there is no need to solve any differential equation. It also does not require any tricky geometric proofs, since geometry has, unfortunately, been removed from many high school and college curricula. This makes this proof that Kepler’s first law is obeyed accessible to even good high school students.
FIG. 3. This is the polygonal approximation to the Kepler orbit with $\Delta \theta = \pi/8$. The eccentricity of the inner ellipse is .3 and of the outer $.3/\cos(\pi/8) = .325$

V. ELLIPSE

To show that the expression

$$r + \epsilon x = R_0$$

where $r^2 = x^2 + y^2$, is an ellipse we can define

$$x' = x + \frac{2\epsilon R_0}{1 - \epsilon^2}$$

$$r' = x'^2 + y'^2$$

(37) (38)

to get

$$0 = r^2 - (-\epsilon x + R_0)^2 = (1 - \epsilon^2)x^2 + y^2 + 2\epsilon x R_0 - R_0^2$$

(39)

$$= r'^2 - (\epsilon x' + R_0)^2$$

(40)

or

$$r' - \epsilon x' = R_0$$

(41)

This gives

$$r + r' + \epsilon(x - x') = 2R_0$$

(42)
or

\[ r + r' = \frac{R_0}{1 - \epsilon^2} \]  

(43)

\( r \) and \( r' \) are the two distances from the foci of the figure to a point on the ellipse. That the sum of the distances of a point on the figure to the two foci is constant is one of the definitions of the ellipse.

If \( v_0 \) is sufficiently large, then the orbit is no longer an ellipse but rather a parabola or a hyperbola \((\epsilon = 1 \text{ or } \epsilon > 1.)\) Thus this proof is valid for all conic sections. Furthermore if \( \mu < 0 \) (repulsive force law), \( R_0 \) and \( \epsilon \) are both less than 0, and the figure is again a hyperbola, but with the centre of force at the outside focus of the hyperbola for any value of \( \epsilon < 0. \)

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