VECTOR BUNDLES OVER ITERATED SUSPENSIONS OF STUNTED REAL PROJECTIVE SPACES

ANIRUDDHA C. NAOLEKAR AND AJAY SINGH THAKUR

Abstract. Let $X_{m,n}^k = \Sigma^k(\mathbb{RP}^m/\mathbb{RP}^n)$. In this note we completely determine the values of $k, m, n$ for which the total Stiefel-Whitney class $w(\xi) = 1$ for any vector bundle $\xi$ over $X_{m,n}^k$.

1. Introduction

Recall (see [6]) that a CW-complex $X$ is said to be $W$-trivial if for every vector bundle $\xi$ over $X$ the total Stiefel-Whitney class $w(\xi) = 1$. A theorem of Atiyah-Hirzebruch ([1], Theorem 2) says that for any finite CW-complex $X$, the 9-fold suspension $\Sigma^9X$ is $W$-trivial. In the same paper Atiyah-Hirzebruch have shown ([1], Theorem 1) that the sphere $S^k$ is $W$-trivial if and only if $k \neq 1, 2, 4, 8$ (see also, [5], Theorem 1).

In view of the Atiyah-Hirzebruch theorem it is interesting to understand whether or not the iterated suspension $\Sigma^kX$ of a finite CW-complex $X$ is $W$-trivial with $0 \leq k \leq 8$. In recent times there has been some interest in understanding $W$-triviality of iterated suspensions of spaces (see, [6], [7], [8] and the references therein).

In [7], the author has completely determined the values of $k$ and $n$ for which the iterated suspension $\Sigma^k\mathbb{FP}^n$ is $W$-trivial. Here $\mathbb{FP}^n$ denotes the projective space of 1-dimensional subspaces of $\mathbb{F}^{n+1}$ where $\mathbb{F}$ is the field $\mathbb{R}$ of reals, the field $\mathbb{C}$ of complex numbers or the skew-field $\mathbb{H}$ of quaternions. In [6], the author has completely described the cases under which the stunted projective space $\mathbb{RP}^m/\mathbb{RP}^n$ is $W$-trivial. In [8], the second author has almost complete results concerning $W$-triviality of the iterated suspension $\Sigma^kD(m,n)$ of the Dold manifold $D(m,n)$.

Let $X_{m,n}$ denote the stunted projective space $\mathbb{RP}^m/\mathbb{RP}^n$ and let $X_{m,n}^k$ denote the $k$-fold suspension $\Sigma^kX_{m,n}$ of $X_{m,n}$. In this note we completely determine the values of $k, m, n$ for which $X_{m,n}^k$ is $W$-trivial.

In view of the Atiyah-Hirzebruch theorem, we assume that $0 \leq k \leq 8$. Also note that the cases $X_{m,0}^k = \Sigma^k\mathbb{RP}^m$ and $X_{m-1}^k = \Sigma^kS^n = S^{n+k}$ are completely understood. In the sequel we assume that $0 < n < m$ and hence, in particular, $m \geq 2$. Since the case $X_{m,n}$ is completely understood, we state our results for $X_{m,n}^k$ with $1 \leq k \leq 8$. The following statements completely describe the cases when $X_{m,n}^k$ is $W$-trivial.

Theorem 1.1. Let $X_{m,n}^k$ be as above with $0 < n < m$.

1. If $k = 3, 5$, then $X_{m,n}^k$ is $W$-trivial if and only if $m + k \neq 8$.
2. $X_{m,n}^6$ is $W$-trivial if and only if $(m, n) \neq (3, 1), (2, 1)$.
3. $X_{m,n}^7$ is $W$-trivial.

2010 Mathematics Subject Classification. 57R20 (55R40, 57R22).
Key words and phrases. Stiefel-Whitney class, stunted projective spaces, $W$-triviality.
Theorem 1.2. Let 0 < n < m. Then $X_{m,n}^1$ is W-trivial if and only if $m \neq 3, 7$.

Theorem 1.3. Let 0 < n < m. Then

1. $X_{m,n}^2$ is not W-trivial if $n = 1$.
2. $X_{m,n}^2$ is W-trivial if $m \not\equiv 0, 6, 7 \pmod{8}$ and $n \geq 2$.
3. $X_{8t+6,n}^2$ is W-trivial if $t \geq 1$ and $n \geq 2$. $X_{6,n}^2$ is not W-trivial.
4. $X_{8t+7,n}^2$ is W-trivial if $t \geq 1$ and $n \geq 2$. $X_{6,n}^2$ is W-trivial if and only if $n = 6$.
5. $X_{8t,n}^2$ is W-trivial if $n \geq 2$.

Theorem 1.4. Let 0 < n < m. Then

1. $X_{m,n}^4$ is W-trivial if $m = 2, 3$ and not W-trivial if $m = 4$.
2. $X_{m,n}^4$ is not W-trivial if $m > 4$ and $n = 1, 2, 3$. $X_{m,n}^4$ is W-trivial if $m > 4$ and $n \geq 4$.

Theorem 1.5. Let 0 < n < m. Then $X_{m,n}^8$ is W-trivial.

The proofs of the above theorems crucially make use of the computations of the $\widetilde{KO}$-groups of the real projective spaces and the stunted real projective spaces. In the next section we first state some easy to verify general observations and prove our main results.

Conventions. All references to cohomology groups will mean singular cohomology with $\mathbb{Z}_2$ coefficients. Given a map $\alpha : X \to Y$, the induced homomorphism in cohomology and $\widetilde{KO}$-groups will again be denoted by $\alpha$.

2. Proof of the theorems

We begin by recording some important observations which will be crucial in the proofs of the main theorems.

Proposition 2.1. (1) $X_{m,n}^k$ is W-trivial if there does not exist an integer $s$ such that $n + 1 + k \leq 2^s \leq m + k$.
(2) Suppose $0 \leq n' < n < m$. If $X_{m,n'}^k$ is W-trivial, then so is $X_{m,n}^k$. In particular, if $\Sigma^k \mathbb{R}P^n$ is W-trivial, then so is $X_{m,n}^k$.

Proof. (1) follows from the well-known fact that the first non-zero Stiefel-Whitney class of a vector bundle is in degree a power of 2. (2) follows from the fact that the obvious map $X_{m,n'}^k \to X_{m,n}^k$ induces isomorphism in cohomology in degree $i$ with $n + 1 + k \leq i \leq m + k$.

We note that if $m$ is odd, then we have a splitting
$$X_{m,m-2} = S^m \vee S^{m-1}$$
and if $m$ is even, then we have
$$X_{m-2} = M(\mathbb{Z}_2, m-1) = \Sigma^{m-2} \mathbb{R}P^2$$
where $M(\mathbb{Z}_2, m-1)$ denotes the Moore space of type $(\mathbb{Z}_2, m-1)$. Also note that if $\widetilde{KO}^{-k}(X) = 0$, then $\Sigma^k X$ is W-trivial.

Given a sequence of integers $0 \leq p < n < m$, the cofiber sequence
$$X_{n,p} \to X_{m,p} \to X_{m,n}$$
gives rise to an exact sequence
\[ \cdots \rightarrow \widetilde{KO}^{-i}(X_{m,n}) \rightarrow \widetilde{KO}^{-i}(X_{m,p}) \rightarrow \widetilde{KO}^{-i}(X_{n,p}) \rightarrow \widetilde{KO}^{-i+1}(X_{m,n}) \rightarrow \cdots. \]

Our proofs, in many cases, involve analyzing the above exact sequence of \( \widetilde{KO} \)-groups corresponding to a suitable choice of a cofiber sequence as above. We shall use the above observations implicitly in the sequel.

We record here what is known about the \( W \)-triviality of \( \Sigma^k \mathbb{RP}^m \) and the stunted projective spaces \( X_{m,n} \) for easy reference.

**Theorem 2.2.** ([7], Theorem 1.4) (1) If \( k = 1, 2, 4, 8 \), then \( \Sigma^k \mathbb{RP}^m \) is not \( W \)-trivial if and only if \( m \geq k \). (2) If \( k = 3, 5, 7 \), then \( \Sigma^k \mathbb{RP}^m \) is not \( W \)-trivial if and only if \( m + k = 4, 8 \).

(3) \( \Sigma^6 \mathbb{RP}^m \) is not \( W \)-trivial if and only if \( m = 2, 3 \). \( \square \)

**Theorem 2.3.** ([6], Theorem 4.1) Suppose \( 1 \leq n \leq m - 2 \). Then \( X_{m,n} \) is \( W \)-trivial if and only if \( m < 2^\varphi(n) \) where \( \varphi(n) \) denotes the number of integers \( i \) such that \( 0 < i \leq n \) and \( i \equiv 0, 1, 2, 4 \pmod{8} \). \( \square \)

*Proof of Theorem 2.2.* Assume first that \( k = 3, 5 \). If \( m + k \neq 8 \) then as \( \Sigma^k \mathbb{RP}^m \) is \( W \)-trivial it follows from Proposition [2.1](2) that \( X^k_{m,n} \) is also \( W \)-trivial. Next we look at \( X^5_{3,n} \). The obvious map \( X^3_{5,n} \rightarrow S^8 \) induces isomorphism in the top cohomology and hence the Hopf bundle on \( S^8 \) pulls back to a bundle \( \xi \) over \( X^5_{3,n} \) with \( w(\xi) \neq 1 \). A similar argument works for \( X^5_{3,n} \). This completes the proof of (1).

The case (2) when \( m \neq 2, 3 \) follows from arguments similar to the above case. That \( X^6_{3,n} \) is not \( W \)-trivial for \( n = 1, 2 \) follows as from the facts that \( X^6_{2,1} = S^9 \vee S^8 \) and \( X^6_{3,2} = S^8 \).

Clearly, \( X^6_{2,1} = S^8 \) is not \( W \)-trivial. This completes the proof of (2).

Finally as \( m \geq 2 \), \( W \)-triviality of \( \Sigma^7 \mathbb{RP}^m \) implies that \( X^7_{m,n} \) is \( W \)-trivial. This completes the proof of (3) and the theorem.

*Proof of Theorem 2.3.* There are obvious maps \( X^1_{3,n} \rightarrow S^4 \) and \( X^1_{7,n} \rightarrow S^8 \) that induce isomorphisms in cohomology in the top dimension. The Hopf bundles on \( S^4 \) and \( S^8 \) then pull back to give bundles with total Stiefel-Whitney class not equal to 1. This shows that \( X^1_{m,n} \) is not \( W \)-trivial if \( m = 3, 7 \).

Next assume that \( m \neq 3, 7 \pmod{8} \). Then by ([11], Table 3) we have \( \widetilde{KO}^{-1}(X_{m,1}) = 0 \). Hence when \( m \neq 3, 7 \pmod{8} \), we have that \( X^1_{m,1} \) and hence \( X^1_{m,n} \) is \( W \)-trivial for all \( n \geq 1 \).

Finally we look at the case when \( m = 8t + 3, 8t + 7 \) with \( t \geq 1 \). First consider \( X^1_{m,n} \) with \( m = 8t + 3 \) and \( t \geq 1 \). Consider the exact sequence
\[ \cdots \rightarrow \widetilde{KO}^{-1}(X_{8t+3,8t+2}) \rightarrow \widetilde{KO}^{-1}(X_{8t+3,1}) \rightarrow \widetilde{KO}^{-1}(X_{8t+2,1}) \rightarrow \cdots. \]

As \( 8t + 2 \neq 3, 7 \pmod{8} \), the last group in the above sequence is zero ([11], Table 3) and hence \( \alpha \) is an epimorphism. Hence \( W \)-triviality of the sphere \( X^1_{8t+3,8t+2} \) implies \( W \)-triviality of \( X^1_{8t+3,1} \). Hence \( X^1_{m,n} \) is \( W \)-trivial for all \( n \geq 1 \) when \( m = 8t + 3 \) and \( t \geq 1 \). The case \( X^1_{8t+7,1} \) is dealt with similarly by looking at the cofiber sequence
\[ X_{8t+6,1} \rightarrow X_{8t+7,1} \rightarrow X_{8t+7,8t+6}. \]

Thus \( X^1_{m,n} \) is \( W \)-trivial if and only if \( m \neq 3, 7 \). This completes the proof of the theorem.

*Proof of Theorem 2.3.* We prove each of the claims in the theorem.
Proof of (1). We consider the exact sequence
\[ \cdots \to KO^{-2}(X_{m,2}) \to KO^{-2}(X_{m,1}) \xrightarrow{\alpha} KO^{-2}(S^2) \to KO^{-1}(X_{m,2}) \to \cdots. \]
Assume that \( m \not\equiv 3, 7 \pmod{8} \). It then follows from ([4], Table 4) that the last group in the above sequence is zero. Hence the homomorphism \( \alpha \) is an epimorphism. Thus there exists a vector bundle \( \xi \) over \( X_{m,1}^{7} \) that pulls back to the Hopf bundle over \( S^4 \). Hence \( w(\xi) \neq 1 \) showing that if \( m \not\equiv 3, 7 \pmod{8} \), then \( X_{m,1}^{2} \) is not \( W \)-trivial.

We now deal with the cases \( m \equiv 3, 7 \pmod{8} \). Consider the exact sequence
\[ \cdots \to KO^{-2}(X_{8t+3,8t+2}) \to KO^{-2}(X_{8t+3,1}) \xrightarrow{\alpha} KO^{-2}(X_{8t+2,1}) \to KO^{-1}(X_{8t+3,8t+2}) \xrightarrow{\beta} \]
\[ \to KO^{-1}(X_{8t+3,1}) \to KO^{-1}(X_{8t+2,1}) \to \cdots. \]
The last group in the above sequence is zero ([4], Table 3). Now \( KO^{-1}(X_{8t+3,8t+2}) = \mathbb{Z} \) and \( KO^{-1}(X_{8t+3,1}) = \mathbb{Z} \) by ([4], Table 3). Thus \( \beta \) is an isomorphism and hence \( \alpha \) is an epimorphism. As \( 8t + 2 \not\equiv 3, 7 \pmod{8} \) it follows that \( X_{8t+2,1}^{2} \) is not \( W \)-trivial and hence \( X_{8t+3,1}^{2} \) is not \( W \)-trivial. The case \( X_{8t+7,1}^{2} \) is dealt with similarly by looking at the cofiber sequence \( X_{8t+6,1} \to X_{8t+7,1} \to X_{8t+7,8t+6} \). This completes the proof of (1).

Proof of (2). Assume that \( m \not\equiv 0, 6, 7 \pmod{8} \). By ([4], Table 4), \( KO^{-2}(X_{m,2}) = 0 \). Thus \( X_{m,n}^{2} \) is \( W \)-trivial for all \( n \geq 2 \) if \( m \not\equiv 0, 6, 7 \pmod{8} \). This completes the proof of (2).

Proof of (3). We consider \( X_{8t+6,6}^{2} \). If \( t = 0 \), the obvious map \( X_{6,6}^{2} \to S^8 \) induces isomorphism in cohomology in the top dimension and hence the Hopf bundle on \( S^8 \) pulls back to a bundle \( \xi \) with \( w(\xi) \neq 1 \).

Assume now that \( t \geq 1 \). In the exact sequence
\[ \cdots \to KO^{-2}(X_{8t+6,8t+5}) \xrightarrow{\alpha} KO^{-2}(X_{8t+6,2}) \xrightarrow{\alpha} KO^{-2}(X_{8t+5,2}) \to \cdots \]
\( \alpha \) is an epimorphism as the last group is zero ([4], Table 4). Clearly \( X_{8t+6,2}^{2} \) is \( W \)-trivial showing that \( X_{8t+6,2}^{2} \) is \( W \)-trivial. Hence \( X_{8t+6,6}^{2} \) is \( W \)-trivial for all \( n \geq 2 \). This completes the proof of (3).

Proof of (4). First note that as \( X_{7,5}^{2} = S^9 \vee S^8 \) we have that \( X_{7,5}^{2} \) is not \( W \)-trivial. Thus \( X_{7,n}^{2} \) is not \( W \)-trivial for \( n \leq 5 \). Clearly, \( X_{7,6}^{2} = S^9 \) is \( W \)-trivial.

Next we consider \( X_{8t+7,n}^{2} \) with \( t \geq 1 \). In the exact sequence
\[ \to KO^{-2}(X_{8t+7,8t+6}) \to KO^{-2}(X_{8t+7,2}) \xrightarrow{\alpha} KO^{-2}(X_{8t+6,2}) \xrightarrow{\beta} \]
\[ KO^{-1}(X_{8t+7,8t+6}) \to \]
as \( KO^{-2}(X_{8t+6,2}) = \mathbb{Z}_2 \) ([4], Table 4) and the last group is infinite cyclic we have that \( \alpha \) is an epimorphism. By (3) above, \( X_{8t+6,2}^{2} \) is \( W \)-trivial and hence \( X_{8t+7,2}^{2} \) is \( W \)-trivial. Hence the proof of (4) is complete.

Proof of (5). We first concentrate on the case \( t = 1 \). Note that \( X_{8,7}^{2} = S^{10} \) and hence is \( W \)-trivial. As
\[ X_{8,6}^{2} = \Sigma^2 M(\mathbb{Z}_2, 7) = \Sigma^8 \mathbb{R} P^2, \]
we have, by Theorem 2.2, that \( X_{8,6}^{2} \) is \( W \)-trivial. Next consider the exact sequence
\[ \to KO^{-2}(X_{8,6}) \to KO^{-2}(X_{8,5}) \xrightarrow{\beta} KO^{-2}(S^6) \xrightarrow{\alpha} KO^{-1}(X_{8,6}) \to KO^{-1}(X_{8,1}) \to \]
We note that \( KO^{-2}(S^6) = \mathbb{Z} \) generated by the Hopf bundle \( \nu \), \( KO^{-1}(X_{8,6}) = \mathbb{Z}_2 \) ([4], Table 4), \( KO^{-1}(X_{8,1}) = 0 \) ([4], Table 3) and \( KO^{-2}(X_{8,5}) = \mathbb{Z} \oplus \mathbb{Z}_2 \) ([4], Table 3). Hence \( \alpha \) is an
epimorphism. Since \(\beta\) induces isomorphism in 8th cohomology, the equality \(w(\beta(\xi)) = 1\) implies that \(w(\xi) = 1\). It follows from the exactness of the above sequence that there is a generator \(\xi\) of the torsion free part of \(\widetilde{KO}^{-2}(X_{8,5})\) with \(\beta(\xi) = 2\nu\). Since \(w(2\nu) = 1\), we have that \(w(\xi) = 1\). If \(\eta\) is the generator of the torsion part, \(w(\beta(\eta)) = 1\) as \(\beta(\eta)\) is (stably) trivial. Hence \(w(\eta) = 1\). This shows that for any \(\theta \in \widetilde{KO}^{-2}(X_{8,5})\), \(w(\theta) = 1\) and hence \(X_{8,5}^4\) is W-trivial. W-triviality of \(X_{8,4}^2\) follows from that of \(X_{8,5}^4\) by considering the exact sequence

\[
\cdots \to \widetilde{KO}^{-2}(X_{8,5}) \to \widetilde{KO}^{-2}(X_{8,4}) \to \widetilde{KO}^{-2}(S^8) \to \cdots
\]

and noting that the last group in the above sequence is zero. Similar considerations shows that \(X_{8,3}^2, X_{8,2}^2\) are W-trivial. This completes the proof in the case \(t = 1\).

Next assume that \(t > 1\) and consider the exact sequence

\[
\cdots \to \widetilde{KO}^{-2}(X_{8t,8t-3}) \to \widetilde{KO}^{-2}(X_{8t,8}) \to \widetilde{KO}^{-2}(X_{8t-3,2}) \to \cdots
\]

The last group in the above sequence is zero \((\mathbb{I}, \text{Table 4})\). Hence \(\alpha\) is an epimorphism. We claim that \(X_{8t,8t-3}^2\) is W-trivial. The inclusion map \(S^{8t} \to X_{8t,8t-3}^2\) induces isomorphism in cohomology in degree 8t. If \(\xi\) is a vector bundle over \(X_{8t,8t-3}^2\) with \(w(\xi) \neq 1\), then \(w_{8t}(\xi) \neq 0\). This bundle then pulls back to a bundle \(\eta\) over \(S^{8t}\) with \(w(\eta) \neq 1\). This is a contradiction as \(t > 1\). Hence \(X_{8t,8t-3}^2\) is W-trivial. The surjectivity of \(\alpha\) now implies that \(X_{8t,2}^2\) is W-trivial. This completes the proof of (5) and the theorem.

**Proof of Theorem 1.4** First note that if \(m = 2, 3\), then \(X_{m,n}^4\) is always W-trivial. Since the obvious map \(X_{4,n}^4 \to S^8\) induces isomorphism in cohomology in top dimension, we have that \(X_{4,n}^4\) is not W-trivial. This completes the proof of (1).

Now assume that \(m > 4\). Consider the exact sequence

\[
\cdots \to \widetilde{KO}^{-4}(X_{m,3}) \to \widetilde{KO}^{-4}(\mathbb{R}P^{m}) \to \widetilde{KO}^{-4}(\mathbb{R}P^{3}) \to \cdots
\]

The last group in the above sequence is well known to be zero \((\mathbb{I})\) and hence \(\alpha\) is an epimorphism. By Theorem 2.2, \(\Sigma^4\mathbb{R}P^m\) is not W-trivial. The surjectivity of \(\alpha\) now implies that \(X_{m,3}^4\) is not W-trivial if \(m > 4\). In view of Proposition 2.1 (3), \(X_{m,n}^4\) is not W-trivial for \(m > 4\) and \(n = 1, 2, 3\). To complete the proof of the theorem we now show that if \(m > 4\), then \(X_{m,4}^4\) is W-trivial. To see this consider the exact sequence (see \(\mathbb{I}\) Section 3)

\[
0 \to \widetilde{KO}^{-4}(X_{m,4}) \to \widetilde{KO}^{-4}(\mathbb{R}P^{m}) \to \widetilde{KO}^{-4}(\mathbb{R}P^{4}) \to 0.
\]

The last two groups in the above exact sequence are finite cyclic \(\mathbb{I}\). By Theorem 2.2, the spaces \(\Sigma^4\mathbb{R}P^m\) and \(\Sigma^4\mathbb{R}P^{4}\) are not W-trivial. Thus if \(\theta, \eta\) are generators of the second and the third group respectively, then we must have \(w(\xi) \neq 1\) and \(w(\eta) \neq 1\). Assume that \(\beta(\theta) = \eta\). Now let \(\xi \in \widetilde{KO}^{-4}(X_{m,4})\) be such that \(w(\xi) \neq 1\). Then \(w(\alpha(\xi)) \neq 1\) as \(\alpha\) induces isomorphism in cohomology in degrees \(j, j \geq 9\). Let \(\alpha(\xi) = s\theta\). Then \(s\) is odd as the cup products in \(\tilde{H}^*(\Sigma^4\mathbb{R}P^m)\) are all zero. The calculation

\[
w(\beta(\alpha(\xi))) = w(s\beta(\theta)) = w(s\eta) = w(\eta) \neq 1
\]

contradicts the exactness of the above sequence. Thus \(w(\xi) = 1\) for every \(\xi \in \widetilde{KO}^{-4}(X_{m,4})\) proving that \(X_{m,4}^4\) is W-trivial. This shows that \(X_{m,n}^4\) is W-trivial for all \(n \geq 4\) if \(m > 4\). This completes the proof of the theorem.

**Proof of Theorem 1.5** Consider the exact sequence

\[
\widetilde{KO}(X_{m,n}) \to \widetilde{KO}(\mathbb{R}P^{m}) \to \widetilde{KO}(\mathbb{R}P^{n}) \to 0.
\]
It is known (see [2]) that the image of $\alpha$ is generated by $2^{\varphi(n)}\xi$ where $\xi$ is the canonical line bundle over $\mathbb{RP}^m$ and $\varphi(n)$ is as in Theorem 2.3. It follows that in the exact sequence

$$\tilde{KO}^{-8}(X_{m,n}) \xrightarrow{\beta} \tilde{KO}^{-8}(\mathbb{RP}^m) \rightarrow \tilde{KO}^{-8}(\mathbb{RP}^n) \rightarrow 0$$

the image of $\beta$ is generated by $2^{\varphi(n)}\eta$ where $\eta$ corresponds to $\xi$ under the Bott periodicity isomorphism. Since $2^{\varphi(n)}$ is even and the cup products in $\tilde{H}^*(\Sigma^8\mathbb{RP}^m)$ are zero, it follows that $w(2^{\varphi(n)}\eta) = 1$. Hence if $\theta$ is in the image of $\beta$, then $w(\theta) = 1$. As $\alpha$ induces isomorphism in cohomology in degrees $j, j \geq n + 9$, it follows that $X_{m,n}^8$ is $W$-trivial. This completes the proof of the theorem.

Acknowledgement. The second author acknowledges the hospitality of Indian Statistical Institute, Bangalore, where part of this work was done.

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Indian Statistical Institute, 8th Mile, Mysore Road, RVCE Post, Bangalore 560059, INDIA.

E-mail address: ani@isibang.ac.in

School of Maths, TIFR, Homi Bhabha Road, Colaba, Mumbai 400005, INDIA.

E-mail address: athakur@math.tifr.res.in