On Reconfigurability of Target Sets

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Abstract

We study the problem of deciding reconfigurability of target sets of a graph. Given a graph $G$ with vertex thresholds $\tau$, consider a dynamic process in which vertex $v$ becomes activated once at least $\tau(v)$ of its neighbors are activated. A vertex set $S$ is called a target set if all vertices of $G$ would be activated when initially activating vertices of $S$. In the Target Set Reconfiguration problem, given two target sets $X$ and $Y$ of the same size, we are required to determine whether $X$ can be transformed into $Y$ by repeatedly swapping one vertex in the current set with another vertex not in the current set preserving every intermediate set as a target set. In this paper, we investigate the complexity of Target Set Reconfiguration in restricted cases. On the hardness side, we prove that Target Set Reconfiguration is PSPACE-complete on bipartite planar graphs of degree 3 and 4 and of threshold 2, bipartite 3-regular graphs and planar 3-regular graphs of threshold 1 and 2, and split graphs, which is in contrast to the fact that a special case called Vertex Cover Reconfiguration is in $P$ for the last graph class. On the positive side, we present a polynomial-time algorithm for Target Set Reconfiguration on graphs of maximum degree 2 and trees. The latter result can be thought of as a generalization of that for Vertex Cover Reconfiguration.

1 Introduction

Combinatorial reconfiguration is a research field studying the following problem: Given a pair of feasible solutions for a search problem, can we find a step-by-step transformation between them while keeping the feasibility? Studying such problems may help understand the structure of the solution space of a search problem and have applications in dynamic and changing environments [Mou15, HIM+16]. Countless reconfiguration problems are derived from classical search problems, e.g., Boolean Satisfiability [GKMP09, MTY11, MNPR17], Clique, Matching [IDH+11], Coloring [CvdHJ08, BC09, CvdHJ11], Subset Sum [ID14], and Shortest Path [KMM11, Bon13]. See also the survey of Nishimura [Nis18] and van den Heuvel [vdH13]. One of the most well-studied reconfiguration problems is based on Vertex Cover [HD05, IDH+11, Bon16, BKW14, KMM12, LM19, Wro18]. Given a graph $G$ and two vertex covers $X$ and $Y$ of $G$,\footnote{A vertex cover of $G$ is a vertex set that includes at least one endpoint of every edge.} Vertex Cover Reconfiguration requests to decide if $X$ can be transformed into $Y$ by applying a sequence

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of prespecified transformation rules preserving every intermediate set as a vertex cover. Such a sequence is called a reconfiguration sequence. Under a token jumping rule [KMM12], we can add one vertex and remove another vertex simultaneously by a single step (see Section 2 for a formal definition), where Vertex Cover Reconfiguration was shown to be PSPACE-complete [HD05, IDH+11, KMM12].

In this paper, we present an initial study on reconfigurability of target sets (to the best of our knowledge). Target Set Selection is a combinatorial optimization problem on a graph that finds applications in social network analysis [Che09, KKT03] and distributed computing [Pel02, Pel98]. Given a graph $G$ with vertex thresholds $\tau$, we consider a dynamic process where vertex $v$ becomes activated once at least $\tau(v)$ of $v$’s neighbors are activated, which models the spread of influence, information, and opinion over a network. A vertex set $S$ is called a target set if all vertices of $G$ would be activated when initially activating vertices of $S$. The objective of Target Set Selection is to identify the minimum target set of $G$. This problem generalizes well-studied Vertex Cover and Feedback Vertex Set problems: It is known [Dre00, DR09] that a target set is a vertex cover (resp. a feedback vertex set) if $\tau(v)$ is the degree of $v$ (resp. the degree of $v$ minus 1) for every vertex $v$. In a reconfiguration version of Target Set Selection, namely, Target Set Reconfiguration, we are asked to decide if there exists a reconfiguration sequence between two particular target sets.

1.1 Related Work and Known Results

Reconfiguration Problems. We review known results for Vertex Cover Reconfiguration (VC-R for short) and Feedback Vertex Set Reconfiguration (FVS-R for short), which are included as a special case of Target Set Reconfiguration. Hearn and Demaine [HD05] are the first to prove that VC-R is PSPACE-complete on planar graphs of maximum degree 3 by reducing from Nondeterministic Constraint Logic (see also [HD09]). Since the unified framework of reconfiguration has been established by Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, and Uno [IDH+11], great effort has been devoted to analyzing the restricted-case hardness and solvability of VC-R, summarized in Table 1. Typically, a search problem in P induces a reconfiguration counterpart that belongs to P; e.g., polynomial-time algorithms are known for chordal graphs, split graphs, trees [KMM12, INZ16, MNRS18], claw-free graphs [BKW14], and cacti [MNRS18]. Some exceptions, however, are known: VC-R is PSPACE-complete on perfect graphs [KMM12] and bounded-treewidth graphs [Wro18], and it is NP-complete on bipartite graphs [LM19], for which Vertex Cover is in P. Since any hardness result of VC-R directly applies to Target Set Reconfiguration, it is interesting to explore the complexity of Target Set Reconfiguration for cases where VC-R is in P.

Some parameterized complexity results [DF12, CFK+15] for FVS-R are known: FVS-R is fixed-parameter tractable when parameterized by the size of a feedback vertex set, but it is W[1]-hard when parameterized by the length of reconfiguration sequences [MNRW14]. On the other hand, parameterization by the length of reconfiguration sequences and the treewidth of a graph is fixed-parameter tractable [MNR+17]. Ito and Otachi [IO19] showed that FVS-R is polynomial-time solvable on split graphs and interval graphs.
Table 1: Complexity of Target Set Selection (TSS), Target Set Reconfiguration (TS-R), Vertex Cover (VC), and Vertex Cover Reconfiguration (VC-R) on restricted graph classes.

| graph class         | TSS          | TS-R          | VC            | VC-R          |
|---------------------|--------------|---------------|---------------|---------------|
| planar              | NP-c (from VC) | PSPACE-c (from VC-R) | NP-c          | PSPACE-c [HD05, IDH+11, KMM12] |
| even-hole-free      | NP-c (from split) | PSPACE-c (from split) | open          | P [KMM12, INZ16, MNRS18]         |
| perfect             | NP-c (from split) | PSPACE-c (from VC-R) | P             | PSPACE-c [KMM12]                  |
| chordal             | NP-c (from split) | PSPACE-c (from split) | P             | P (since even-hole-free)          |
| split               | NP-c [NNUW13] | PSPACE-c (Theorem 4.4) | P             | P (since even-hole-free)          |
| claw-free           | NP-c [Mun17] | open          | P             | P [BK14]                           |
| tree                | P [Che09]    | P (Theorem 4.1) | P             | P (since even-hole-free)          |
| bipartite           | NP-c [Che09] | PSPACE-c (Theorem 3.7) | P             | NP-c [LM19]                        |
| bounded treewidth   | P [BHLN11]   | PSPACE-c (from VC-R) | P             | PSPACE-c [Wro18]                   |
| cactus              | P [BHLN11, CHL+13] | open          | P             | P [MNRS18]                        |

**Target Set Selection.** Since Target Set Selection (TSS for short) naturally arises in many different fields, it is also known by various names such as irreversible k-conversion sets [DR09, CDP+11] and dynamic monopolies [Pel98, Pel02]. We review the complexity results of TSS in restricted cases. One direction is to investigate the case of bounded degree and/or bounded threshold. The following settings of vertex thresholds are well established: (1) majority thresholds, where a vertex becomes activated if at least half of its neighbors are already activated; (2) constant thresholds, that is, all thresholds are some constant $t$, e.g., $t = 2$. Peleg [Pel02] showed that it is NP-hard to find a minimum target set for majority thresholds. Dreyer [Dre00] and Dreyer and Roberts [DR09] proved that a target set is a vertex cover if $\tau(v)$ is the degree of $v$ for every vertex $v$, and it is a feedback vertex set if $\tau(v)$ is the degree of $v$ minus 1. So, TSS turns out to be NP-hard even when the threshold $\tau(v)$ of every vertex $v$ is a constant $t$ for any $t \geq 3$. Chen [Che09] provided the first NP-hardness result for the case of $t = 2$, which is tight in the sense that the case of $t = 1$ is trivially solved. In fact, Chen [Che09] gave NP-hardness of approximating TSS within a polylogarithmic factor. Subsequently, NP-hardness under $t = 2$ was established for graphs of maximum degree 11 [CDP+11] and graphs of maximum degree 4 [PPRS14, KLV17]. On cubic (i.e., 3-regular) graphs of threshold 2, TSS is equivalent to Feedback Vertex Set, which is solvable in polynomial time [UKG88]. Further, polynomial-time algorithms for TSS on subcubic graphs of threshold 2 are known [TU15, KLV17]. Feige and Kogan [FK19] reported hardness-of-approximation results of TSS for several classes of bounded-degree graphs, including 3-regular graphs of threshold 1 and 2, 4-regular graphs of threshold 2, and 4-regular graphs of threshold 3. Note that the case of maximum degree 2 is trivial [DR09].

A different direction is to consider restricted classes of graphs. Chen [Che09] gave a linear-time algorithm for trees. Ben-Zwi, Hermelin, Lokshtanov, and Newman [BHLN11] developed an $n^{O(tw)}$-time algorithm, where $n$ is the number of vertices, and $tw$ is the treewidth of a graph, as a generalization of Chen [Che09]’s algorithm, and they ruled out the existence of an $n^{o(tw)}$-time algorithm under some plausible complexity-theoretic assumption. Other graph classes rendering TSS tractable include block-cactus graphs [CHL+13], cliques [NNUW13], chordal graphs with bounded thresholds [CHL+13], and interval graphs with bounded thresholds [BEPR19]. Con-
versely, NP-hardness was shown for split graphs [NNUW13], claw-free graphs [Mun17], planar graphs [DR09], and bipartite graphs [DR09]. Parameterized complexity of TSS is examined for numerous parameters [NNUW13,CNNW14,DKT18,BCNS14a].

1.2 Our Results

In this paper, we study the complexity of Target Set Reconfiguration (TS-R for short) in restricted cases, aiming to reveal a dividing line between easy and hard instances. Section 3 examines small-degree graphs, the results for which are outlined in Table 2. One of the simplest cases is when all thresholds are 1, which ensures reconfigurability between any pair of target sets (Observation 3.1). Graphs of maximum degree 2 are seemingly easy to handle since they consist only of paths and cycles. However, there exist a nontrivial pair of reconfigurable target sets, requiring a kind of “detour” (see Section 3.1). We devise a characterization of reconfigurable target sets by careful analysis, yielding a polynomial-time algorithm (Theorem 3.3). Once a graph can include degree-3 vertices, TS-R becomes computationally challenging. We first show PSPACE-completeness on bipartite planar graphs of degree 3 and 4 and of threshold 2 (Theorem 3.7). This restricted-case result is of particular interest because it satisfies constant and majority thresholds simultaneously.

On cubic graphs, the case of $t = 3$ is identical to VC-R, which is known to be PSPACE-complete (see [HD05, IDH+11, KMM12] and Observation 3.6). Besides, we derive PSPACE-completeness on bipartite cubic graphs and planar cubic graphs even if thresholds are taken from $\{1, 2\}$ (Theorem 3.12). Our proofs involve several gadgets that are constructed so carefully that they preserve reconfigurability.

Section 4 explores restricted graph classes, the results for which are summarized in Table 1. On the algorithmic side, we develop a polynomial-time algorithm for TS-R on trees (Theorem 4.1), which can be thought of as a generalization of that for VC-R [KMM12, INZ16, MNRS18]. Similar to the case of other reconfiguration problems on trees [HIM+16, DDF+15], we demonstrate that any pair of target sets is reconfigurable, and that an actual reconfiguration sequence can be found in polynomial time. On the hardness side, we prove that TS-R is PSPACE-complete on split graphs (Theorem 4.4), on which VC-R belongs to P. This result relies on a technique for reducing from Hitting Set by Nichterlein, Niedermeier, Uhlmann, and Weller [NNUW13].

Proofs of the statements marked with “★” are deferred to Appendix A.
2 Preliminaries

2.1 Notations and Definitions

For nonnegative integers \( m \) and \( n \) with \( m \leq n \), we define \( [n] \triangleq \{1, 2, \ldots, n\} \) and \( [m..n] \triangleq \{m, m+1, \ldots, n-1, n\} \). A sequence \( S \) consisting of sets \( S^{(0)}, S^{(1)}, \ldots, S^{(t)} \) is denoted as \( \langle S^{(0)}, S^{(1)}, \ldots, S^{(t)} \rangle \), and we write \( S^{(t)} \in S \) to mean that \( S^{(t)} \) appears in \( S \) (at least once). The symbol \( \cup \) is used to emphasize that the union is taken over two disjoint sets. For a graph \( G = (V, E) \), let \( V(G) \) and \( E(G) \) denote the vertex set \( V \) and the edge set \( E \) of \( G \), respectively. We assume that graphs are simple; i.e., they have no self-loops or multi-edges. For a vertex \( v \) of \( G \), we denote the neighborhood of \( v \) by \( N_G(v) \triangleq \{u : (u, v) \in E\} \) and the degree of \( v \) by \( d_G(v) \triangleq |N_G(v)| \). We omit the subscript when \( G \) is clear from the context. For a vertex set \( S \subseteq V \), we write \( G[S] \) for denoting the subgraph of \( G \) induced by \( S \), and we write \( G - S \) for denoting the induced subgraph \( G[V \setminus S] \). In this paper, a threshold function \( \tau : V \to \mathbb{N} \) is often associated with graph \( G = (V, E) \). Hence, we also refer to a triplet \( G = (V, E, \tau) \) as a graph. A vertex \( v \) of \( G \) is referred to as a \( (d', \tau') \)-vertex if \( d(v) = d' \) and \( \tau(v) = \tau' \). A graph \( G = (V, E, \tau) \) is referred to as a \( (D, T) \)-graph for two integer sets \( D \) and \( T \) if \( d(v) \in D \) and \( \tau(v) \in T \) for all \( v \in V \).

We define the activation process over a graph \( G = (V, E, \tau) \). Each vertex takes either of two states: active or inactive. For a seed set \( S \subseteq V \), we define \( A_G^{(t)}(S) \) as the set of already activated vertices at discrete-time step \( t \). Initially, the vertices of \( S \) are active, and the others are inactive; i.e., \( A_G^{(0)}(S) \triangleq S \). Given \( A_G^{(t-1)}(S) \) at step \( t-1 \), we verify whether each inactive vertex \( v \) has at least \( \tau(v) \) active neighbors. If this is the case, then \( v \) becomes active at step \( t \); i.e., \( v \) is added into \( A_G^{(t)}(S) \). This process is irreversible; i.e., an active vertex may not become inactive. Formally, the set of active vertices at step \( t \geq 1 \) is recursively defined as:

\[
A_G^{(t)}(S) \triangleq A_G^{(t-1)}(S) \cup \left\{ v \in V : |N_G(v) \cap A_G^{(t-1)}(S)| \geq \tau(v) \right\}.
\]

Observe that \( A_G^{(n)}(S) = A_G^{(n+1)}(S) \) for \( n \geq |V| \) by the fact that \( A_G^{(t-1)}(S) \subseteq A_G^{(t)}(S) \) for all \( t \geq 1 \). Therefore, we define the active vertex set of \( S \) as \( A_G(S) \triangleq A_G^{(\infty)}(S) \), and we say that \( S \) activates a vertex \( v \) or \( v \) is activated by \( S \) in \( G \) if \( v \in A_G(S) \). In particular, if \( S \) activates the whole graph; i.e., \( A_G(S) = V \), \( S \) is called a target set of \( G \). The Target Set Selection problem is defined as follows.

**Problem 2.1 (Target Set Selection).** Given a graph \( G = (V, E, \tau) \), find a minimum target set of \( G \).

Throughout this paper, we assume that \( 1 \leq \tau(v) \leq d(v) \) for all \( v \in V(G) \). This is because if \( \tau(v) > d(v) \), then any target set of \( G \) must include \( v \); if \( \tau(v) = 0 \), then \( v \) is not included in any minimum target set of \( G \) [NNUW13, Observation 1].

We then formulate a reconfiguration version of Target Set Selection according to the reconfiguration framework of Ito, Demaine, Harvey, Papadimitriou, Sideri, Uehara, and Uno [IDH*11]. We consider the following two types of reconfiguration steps, which specify how a target set can be transformed.

\[\text{Note that this assumption forces a graph to have no isolated vertex.}\]
**Token jumping (TJ)** [KMM12]: Given a target set, a TJ-step can remove one vertex from it and add another vertex not in it simultaneously, which does not change the set size.

**Token addition or removal (TAR)** [IDH’11]: Given a target set, a TAR-step can remove a vertex from it or add a vertex not in it.

For two target sets $X$ and $Y$, a reconfiguration sequence from $X$ to $Y$ is a sequence of target sets $S = (S^{(0)}, S^{(1)}, \ldots, S^{(t)})$ starting from $X$ (i.e., $S^{(0)} = X$) and ending with $Y$ (i.e., $S^{(t)} = Y$) such that $S^{(i)}$ is obtained from $S^{(i-1)}$ by a single reconfiguration step for $i \in [t]$. The length $t$ of $S$ is defined as the number of sets in it minus 1. If $S$ consists only of TJ-steps, then it is called a TJ-sequence; if $S$ consists only of TAR-steps and every set in $S$ is of size at most $k + 1$, then it is called a $k$-TAR-sequence. Moreover, we say that $X$ and $Y$ are TJ-reconfigurable on $G$ if there exists a TJ-sequence of target sets of $G$ from $X$ to $Y$; we say that $X$ and $Y$ are $k$-TAR-reconfigurable on $G$ if there exists a $k$-TAR-sequence from $X$ to $Y$. We define the Target Set Reconfiguration problem as follows.

**Problem 2.2 (Target Set Reconfiguration).** Given a graph $G = (V, E, \tau)$ and two target sets $X$ and $Y$ of the same size, decide if $X$ and $Y$ are TJ-reconfigurable or not.

Observe easily that this problem is in PSPACE [vdH13]. Note that the present problem definition does not request an actual TJ-sequence. We concern TJ-reconfigurability only because it is essentially equivalent to $k$-TAR-reconfigurability. We define Minimum Target Set Reconfiguration as a special case where $X$, $Y$, and all intermediate sets are promised to be minimum.

### 2.2 Useful Lemmas

Here, we introduce some lemmas, which are convenient for proving our results in the subsequent sections. We first define TJN-sequences that we use for a technical reason. Given a target set, a TJN-step can perform either a TJ-step or do nothing. A TJN-sequence is a reconfiguration sequence consisting only of TJN-steps. We say that $X$ and $Y$ are TJN-reconfigurable if there exists a TJN-sequence from $X$ to $Y$. The following is trivial by definition.

**Observation 2.3.** Let $X$ and $Y$ be two target sets of a graph $G$. Then, $X$ and $Y$ are TJ-reconfigurable on $G$ if and only if they are TJN-reconfigurable on $G$.

We then show the equivalence between TJ-reconfigurability and TAR-reconfigurability, whose proof is an adaptation of Kamiński, Medvedev, and Milanič [KMM12, Theorem 1].

**Observation 2.4.** Let $G$ be a graph and $X$ and $Y$ be two size-$k$ target sets of $G$. Then, $X$ and $Y$ are TJ-reconfigurable if and only if they are $k$-TAR-reconfigurable.

**Proof.** Suppose that we are given a TJ-sequence $(S^{(i)})_{i \in [0..t]}$ from $X$ to $Y$. Observe that a single TJ-step from $S^{(i-1)}$ to $S^{(i)}$ can be converted into two TAR-steps: add a vertex $S^{(i)} \setminus S^{(i-1)}$ to $S^{(i-1)}$ to obtain $S^{(i-1)} \cup S^{(i)}$, and remove a vertex $S^{(i-1)} \setminus S^{(i)}$ from $S^{(i-1)} \cup S^{(i)}$ to obtain $S^{(i)}$. The resulting sequence is a $k$-TAR-sequence from $X$ to $Y$. This completes the only-if direction.
Suppose then that we are given a \( k\)-TAR-sequence \( (S^{(i)})_{i \in [0..\ell]} \) from \( X \) to \( Y \). Until we obtain a \( k\)-TAR-sequence consisting of target sets of size \( k \) or \( k+1 \), we modify the current \( k\)-TAR-sequence according to the following procedure: Let \( \sigma \) be a length-2 subsequence of the current sequence consisting of removal of vertex \( x \) and addition of vertex \( y \) such that the middle set is of size less than \( k \), say, \( \sigma = (S, S \setminus \{x\}, S \setminus \{x\} \cup \{y\}) \), where \( |S \setminus \{x\}| \leq k - 1 \). If \( x = y \), then we can remove the two consecutive sets \( S \setminus \{x\} \) and \( S \setminus \{x\} \cup \{y\} \) to shorten the \( k\)-TAR-sequence. Otherwise, we can replace \( \sigma \) with the subsequence \( \sigma' \) in which we first add \( y \) and then remove \( x \), say, \( \sigma' = (S, S \cup \{y\}, S \cup \{y\} \setminus \{x\}) \), which is still a \( k\)-TAR-sequence. One can continue this procedure until the resulting \( k\)-TAR-sequence consists of target sets of size \( k \) or \( k+1 \), which turns out to be a TJ-sequence, completing the proof. \( \square \)

We introduce a combinatorial characterization of target sets due to Ackerman, Ben-Zwi, and Wolfovitz [ABW10].

**Theorem 2.5** (Adaptation of Lemma 2.1 of [ABW10]). For a graph \( G = (V, E, \tau) \), a seed set \( S \subseteq V \) is a target set of \( G \) (i.e., \( \mathcal{A}_G(S) = V \)) if and only if there exists an acyclic orientation \( D \) of \( G \) such that \( d_D^-(v) \geq \tau(v) \) for every vertex \( v \in V \setminus S \), where \( d_D^-(v) \) is the number of edges entering into \( v \) in \( D \).

For a graph \( G = (V, E, \tau) \) and a seed set \( S \subseteq V \), the residual is defined as \( G_S = (V_S, E_S, \tau_S) \), where \( V_S \) is the set of vertices that would not have been activated by \( S \) on \( G \); i.e., \( V_S = V \setminus \mathcal{A}_G(S) \), \( E_S = \{(u, v) \in E : u \in V_S, v \in V_S\} \), and \( \tau_S(v) \) for each \( v \in V_S \) is defined as \( \tau_S(v) \) minus the number of \( v \)'s active neighbors; i.e., \( \tau_S(v) = \tau(v) - |N_G(v) \cap \mathcal{A}_G(S)| \).

**Lemma 2.6.** For a graph \( G = (V, E, \tau) \) and two disjoint vertex subsets \( S \) and \( T \) of \( V \), let \( G_S = (V_S, E_S, \tau_S) \) be the residual. Then, \( S \cup T \) is a target set of \( G \) if and only if \( T \cap V_S \) is a target set of \( G_S \). Moreover, if \( S \cup T \) is a minimum target set of \( G \), then \( T \cap V_S \) is a minimum target set of \( G_S \).

**Proof.** We first prove the only-if direction. Suppose that \( S \cup T \) is a target set of \( G \). By **Theorem 2.5**, there exists an acyclic orientation \( D \) of \( G \) such that \( d_D^-(v) \geq \tau(v) \) for every vertex \( v \in V \setminus (S \cup T) \). Define \( D_S = D[V_S] \), which is an acyclic orientation of \( G_S \). By definition, it holds that \( \tau_S(v) = \tau(v) - |N_G(v) \cap \mathcal{A}_G(S)| \) and \( d_{D_S}^-(v) = d_D^-(v) - |N_D^-(v) \cap \mathcal{A}_G(S)| \) for every vertex \( v \in V_S \). Observing that \( |N_G(v)| \geq |N_D^-(v)| \), we have that \( d_{D_S}^-(v) \geq \tau_S(v) \) for any \( v \in V_S \setminus T \); i.e., \( T \cap V_S \) is a target set of \( G_S \) by **Theorem 2.5**.

We then prove the if direction. Suppose that \( T \cap V_S \) is a target set of \( G_S \). By **Theorem 2.5**, there exists an acyclic orientation \( D_S \) of \( G_S \) such that \( d_{D_S}^-(v) \geq \tau_S(v) \) for every vertex \( v \in V_S \setminus T \). Since \( S \) is a target set of \( G[\mathcal{A}_G(S)] \), there exists an acyclic orientation \( D' \) of \( G[\mathcal{A}_G(S)] \) such that \( d_{D'}^-(v) \geq \tau(v) \) for every vertex \( v \in \mathcal{A}_G(S) \setminus S \). Recall that \( \mathcal{A}_G(S) \cup V_S = V \). We define an orientation \( D \) of \( G \) as follows: for each (undirected) edge \( (u, v) \) of \( G \),

1. if \( u, v \in \mathcal{A}_G(S) \), then its direction coincides with that of \( D' \);
2. if \( u, v \in V_S \), then its direction coincides with that of \( D_S \);
3. if \( u \in \mathcal{A}_G(S) \) and \( v \in V_S \), it is directed from \( u \) to \( v \).
Theorem 2.5. We have that the sequence 

\[ \forall v \in A_G(S) \setminus S, \text{ and we have that } d_D^-(v) = d_D^+(v) + |A_G(S) \cap N_G(v)| \geq \tau_S(v) + |A_G(S) \cap N_G(v)| = \tau(v) \]

for each vertex \( v \in V_S \setminus T \). Accordingly, \( S \cup T \) is a target set of \( G \) by Theorem 2.5. The argument regarding minimality is obvious.

The disjoint union of two graphs \( G_1 \) and \( G_2 \) is defined as a graph \( G_1 \oplus G_2 \) with vertex set \( V(G_1 \oplus G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \).

**Lemma 2.7.** Let \( G_1 \) and \( G_2 \) be two graphs, and let \( (X_1, Y_1) \) and \( (X_2, Y_2) \) be pairs of two minimum target sets of \( G_1 \) and \( G_2 \), respectively. Then, \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) are \( \text{TJ} \)-reconfigurable on the disjoint union \( G_1 \oplus G_2 \) if and only if \( X_1 \) and \( Y_1 \) are \( \text{TJ} \)-reconfigurable on \( G_1 \) and \( X_2 \) and \( Y_2 \) are \( \text{TJ} \)-reconfigurable on \( G_2 \).

**Proof.** Observe easily that a seed set \( S \subseteq V(G_1 \oplus G_2) \) is a minimum target set of \( G_1 \oplus G_2 \) if and only if \( S \cap V(G_1) \) is a minimum target set of \( G_1 \) and \( S \cap V(G_2) \) is a minimum target set of \( G_2 \). Given a \( \text{TJ} \)-sequence \( S_1 = \langle S_1(t) \rangle_{t \in [0..\ell]} \) from \( X_1 \) to \( Y_1 \) for \( G_1 \) and a \( \text{TJ} \)-sequence \( S_2 = \langle S_2(t) \rangle_{t \in [0..\ell]} \) from \( X_2 \) to \( Y_2 \) for \( G_2 \), we have that the sequence \( \langle X_1 \cup X_2, S_1^{(1)} \cup X_2, \ldots, S_1^{(\ell-1)} \cup X_2, S_1^{(\ell)} \cup Y_2 \rangle \) is a \( \text{TJ} \)-sequence from \( X_1 \cup X_2 \) to \( Y_1 \cup Y_2 \) for \( G_1 \oplus G_2 \); i.e., \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) are \( \text{TJ} \)-reconfigurable on \( G_1 \oplus G_2 \). On the other hand, given a \( \text{TJ} \)-sequence \( S = \langle S(t) \rangle_{t \in [0..\ell]} \), we have that the sequence \( S_1 \triangleq \langle S(t) \cap V(G_1) \rangle_{t \in [0..\ell]} \) must be a \( \text{TJN} \)-sequence from \( X_1 \) to \( Y_1 \) for \( G_1 \); i.e., \( X_1 \) and \( Y_1 \) are \( \text{TJ} \)-reconfigurable on \( G_1 \), and the sequence \( S_2 \triangleq \langle S(t) \cap V(G_2) \rangle_{t \in [0..\ell]} \) must be a \( \text{TJN} \)-sequence from \( X_2 \) to \( Y_2 \) for \( G_2 \); i.e., \( X_2 \) and \( Y_2 \) are \( \text{TJ} \)-reconfigurable on \( G_2 \), as desired.

Note that minimality is necessary for ensuring the only-if direction. More precisely, we show by an example in Figure 1 that if either \( (X_1, Y_1) \) or \( (X_2, Y_2) \) is not minimum, \( \text{TJ} \)-reconfigurability between \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) in \( G_1 \oplus G_2 \) may not guarantee \( \text{TJ} \)-reconfigurability between \( X_1 \) and \( Y_1 \) in \( G_1 \) or \( X_2 \) and \( Y_2 \) in \( G_2 \).

**Example 2.8.** Let \( G_1 \) be a cycle graph on four vertices \( c_1, c_2, c_3, c_4 \), and let \( G_2 \) be a path graph on two vertices \( p_1, p_2 \). Define vertex thresholds \( \tau \) as \( \tau(c_i) = 2 \) for all \( c_i \in V(G_1) \) and \( \tau(p_i) = 1 \) for all \( p_i \in V(G_2) \).

Let \( X_1 = \{ c_1, c_3 \} \), \( Y_1 = \{ c_2, c_4 \} \), \( X_2 = \{ p_1, p_2 \} \), and \( Y_2 = \{ p_1, p_2 \} \). Here, \( X_1 \) and \( Y_1 \) are minimum target sets of \( G_1 \), while \( X_2 \) and \( Y_2 \) are not minimum; i.e., \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) are not minimum for \( G_1 \oplus G_2 \). Observe that \( X_1 \cup X_2 \) and \( Y_1 \cup Y_2 \) are \( \text{TJ} \)-reconfigurable in \( G_1 \oplus G_2 \): add \( w_2 \) and remove \( v_1 \); add \( w_4 \) and remove \( w_1 \); add \( v_1 \) and remove \( w_3 \). However, \( X_1 \) and \( Y_1 \) are not \( \text{TJ} \)-reconfigurable in \( G_1 \).
We then refer to [NNUW13] to show that a target set does not need to include threshold-1 vertices.

**Observation 2.9** (Nichterlein et al. [NNUW13]). For a graph $G = (V, E, \tau)$, let $S$ be a target set of $G$. Let $v$ be a vertex in $S$ and $w$ be a neighbor of $v$ in $G$, which may or may not be in $S$. If $\tau(v) = 1$, then $S \setminus \{v\} \cup \{w\}$ is a target set of $G$.

The subdivision of an edge $(u, v)$ of graph $G$ consists of the removal of $(u, v)$ from $G$ and the addition of a new vertex $w$ and two edges $(u, w)$ and $(w, v)$. Let $G'$ be a graph obtained from $G$ by subdividing an edge $(u, v) \in E(G)$ by a new vertex $w$ whose threshold is 1. For a seed set $S' \subseteq V'$ of $G'$, we define $\phi_{sd}(S') \subseteq V$ as

$$
\phi_{sd}(S') \triangleq \begin{cases} 
S' & \text{if } w \notin S', \\
S' \setminus \{w\} \cup \{u\} & \text{if } w \in S'.
\end{cases}
$$

(2)

**Lemma 2.10.** A seed set $S' \subseteq V'$ is a minimum target set of $G'$ if and only if $\phi_{sd}(S')$ is a minimum target set of $G$. Moreover, two minimum target sets $X$ and $Y$ of $G$ are TJ-reconfigurable on $G$ if and only if they are TJ-reconfigurable on $G'$.

**Proof.** One can verify that if $S \subseteq V$ is a target set of $G$, then $S$ is also a target set of $G'$; if $S' \subseteq V'$ is a target set of $G'$, then $\phi_{sd}(S')$ is a target set of $G$. Since a minimum target set of $G'$ does not include $u$ and $w$ at the same time owing to **Observation 2.9**, we have $|\phi_{sd}(S')| = |S'|$ for a minimum target set $S'$ of $G'$, i.e., $\phi_{sd}(S')$ is a minimum target set of $G$.

We then demonstrate that two minimum target sets $X$ and $Y$ of $G$ are TJ-reconfigurable on $G$ if and only if they are TJ-reconfigurable on $G'$. Given a TJ-sequence $S$ of minimum target sets of $G$ from $X$ to $Y$, we find $S$ to be a TJ-sequence of minimum target sets of $G'$ from $X$ to $Y$, completing the only-if direction. On the other hand, given a TJ-sequence $S'$ of minimum target sets of $G'$ from $X$ to $Y$, we find the sequence $S = (\phi_{sd}(S'))_{S' \in S'}$ to be a TJN-sequence of minimum target sets of $G$ from $\phi_{sd}(X) = X$ to $\phi_{sd}(Y) = Y$; i.e., $X$ and $Y$ are TJ-reconfigurable on $G$, completing the if direction. □

We finally introduce a one-way gadget [KLV17,CNW16,BCNS14b], which is defined as a graph $D = (V, E, \tau)$ such that $V = \{t, h, b_1, b_2\}$, $E = \{(t, b_1), (t, b_2), (h, b_1), (h, b_2)\}$, and $\tau(t) = \tau(b_1) = \tau(b_2) = 1, \tau(h) = 2$. We say that $D$ connects from vertex $v$ to vertex $w$ if there exist two edges $(v, t)$ and $(w, h)$. The vertices of $V$ are called the internal vertices of a one-way gadget $D$. Observe that $v$ activates vertices $t, h, b_1, b_2$, but $w$ does not.

### 3 Small Degree Graphs

In this section, we study the complexity of **Target Set Reconfiguration** on small degree graphs. As a warm-up, we show that threshold-1 graphs are amenable.

**Observation 3.1.** Let $G$ be a graph in which every vertex has threshold 1. Then, any two target sets of the same size are TJ-reconfigurable.
Figure 2: A graph $G$ of maximum degree 2 and four target sets $X_1, Y_1, X_2, Y_2$. $\bigcirc$ and $\circ$ represent threshold-2 and threshold-1 vertices, respectively. Seed vertices are colored black $\bullet$. $X_1$ and $Y_1$ are not $\text{TJ}$-reconfigurable while $X_2$ and $Y_2$ are $\text{TJ}$-reconfigurable.

Proof. Suppose that $G$ consists of $c$ connected components, denoted $C_1, \ldots, C_c$. Then, a minimum target set $S^*$ of $G$ includes exactly one vertex, say, $v_i$, of each connected component $C_i$. Hence, for any size-$k$ target set $S$, we can construct a $k$-$\text{TAR}$-sequence from $S$ to $S^*$ as follows: for each $i \in [c]$, add a vertex $v_i$ if $v_i \notin S$ and remove the vertices of $C_i \setminus \{v_i\}$ one by one. By Observation 2.4, any two target sets of the same size are $\text{TJ}$-reconfigurable. $\square$

3.1 Polynomial-time on Maximum Degree Two Graphs

We address a graph of maximum degree 2. Since such a graph consists of paths and cycles, finding a minimum target set is an easy problem [DR09]. On the other hand, the reconfiguration problem becomes somewhat intricate, as shown below.

Example 3.2. Take Figure 2 as an example of two similar but different instances of $\text{Target Set Reconfiguration}$. Here, a graph $G$ is made up of a cycle having four threshold-2 vertices $w_1, w_2, w_3, w_4$ and a path consisting of two threshold-1 vertices $v_1, v_2$. Two target sets $X_1$ and $Y_1$ respectively in Figures 2a and 2b are not $\text{TJ}$-reconfigurable because any seed in $X_1$ cannot be moved. On the other hand, in Figures 2c and 2d, we have the following $\text{TJ}$-sequence from $X_2$ to $Y_2$: add $c_2$ and remove $p_1$; add $c_4$ and remove $c_1$; add $p_1$ and remove $c_3$. Note that this $\text{TJ}$-sequence requires a kind of detour ($v_1$ appears in two $\text{TJ}$-steps).

We show that $\text{Target Set Reconfiguration}$ on a graph of maximum degree 2 is polynomial-time solvable, as stated below.

Theorem 3.3. $\text{Target Set Reconfiguration}$ can be solved in polynomial time for graphs of maximum degree 2. Moreover, if the answer is “yes,” an actual $\text{TJ}$-sequence can be found in polynomial time.

To prove Theorem 3.3, we characterize reconfigurable target sets on path and cycle graphs respectively in Lemmas 3.4 and 3.5.
Lemma 3.4. Let $G$ be a path graph including $m$ threshold-2 vertices. Then, the size of the minimum target set is $\left\lceil \frac{m}{2} \right\rceil + 1$. Any two target sets $X$ and $Y$ are $\max\{|X|, |Y|\}$-TAR-reconfigurable. In particular, when $|X| = |Y|$, they are TJ-reconfigurable. Moreover, an actual reconfiguration sequence can be found in polynomial time.

Proof. Of a path graph $G$, let $w_1, \ldots, w_m$ denote $m$ threshold-2 vertices in a path order. If $m = 0$, the statement is obvious by Observation 3.1. Hereafter, suppose $m > 0$. Let $S$ be a minimum target set of $G$. We can assume that $S$ only includes threshold-2 vertices, because if $S$ includes threshold-1 vertices, we can replace them by some threshold-2 vertices owing to Observation 2.9. Observe then that $S$ must include $w_i$ or $w_{i+1}$ for each $i \in \{m-1\}$; thus, it holds that $|S| \geq \left\lceil \frac{m}{2} \right\rceil + 1$. On the other hand, the set $\{w_1, w_3, w_5, \ldots, w_{m-2}, w_m\}$ is a target set of $G$ if $m$ is positive odd, and the set $\{w_1, w_3, w_5, \ldots, w_{m-1}, w_m\}$ if $m$ is positive even. Therefore, the size of the minimum target set of $G$ is $\left\lceil \frac{m}{2} \right\rceil + 1$.

Since a path graph is a tree, we use Lemma 4.3, which will be proved later on and states that any target set $S$ of a tree is $|S|$-TAR-reconfigurable to some minimum target set. Thus, any two target sets $X$ and $Y$ are $\max\{|X|, |Y|\}$-TAR-reconfigurable, which completes the proof. □

Lemma 3.5. For a cycle graph $G$ including $m$ threshold-2 vertices, we have the following:

- If $m = 0$: Any two target sets $X$ and $Y$ of $G$ are $\max\{|X|, |Y|\}$-TAR-reconfigurable.

- If $m$ is positive even: The size of the minimum target set is $\frac{m}{2}$. For a size-$k$ target set $S$ such that $k \geq \frac{m}{2} + 1$ (i.e., $S$ is not minimum), there exists a $k$-TAR-sequence from $S$ to some minimum target set. There are exactly two minimum target sets; they are TJ-reconfigurable when $m = 2$, and they are not TJ-reconfigurable but $(\frac{m}{2} + 1)$-TAR-reconfigurable when $m \geq 4$.

- If $m$ is positive odd: The size of the minimum target set is $\frac{m+1}{2}$. For a size-$k$ target set $S$ such that $k \geq \frac{m+1}{2}$, there exists a $k$-TAR-sequence from $S$ to a special minimum target set consisting of threshold-2 vertices. Moreover, such special minimum target sets are TJ-reconfigurable to each other. In particular, for $k \geq \frac{m+1}{2}$, there exists a $k$-TAR-sequence from a size-$k$ target set to some minimum target set.

Moreover, an actual reconfiguration sequence can be found in polynomial time.

Proof. Of a cycle graph $G$, let $w_1, \ldots, w_m$ denote $m$ threshold-2 vertices in a cyclic order. For notational convenience, we assume that arithmetic operations regarding the subscript of variables are performed over modulo $m$; e.g., $w_{m+1} \equiv w_1$ and $w_{2m} \equiv w_m$. In the case of $m = 0$, the statement is obvious by Observation 3.1.

Suppose that $m$ is positive even. We show that there exist two minimum target sets of size $\frac{m}{2}$. We first claim that any minimum target set $S$ does not include threshold-1 vertices. This is because if $S$ includes a threshold-1 vertex $v$, the residual graph $G_{\{v\}}$ turns out to be a path graph having $m - 2$ threshold-2 vertices, whose minimum target set has size $\frac{m}{2}$ due to Lemma 3.4; thus, $|S| \geq \frac{m}{2} + 1$. Observing further that any minimum target set must include $w_i$ or $w_{i+1}$ for all $i \in \{m\}$, we come up with the following two minimum target sets: $S_1^* \equiv \{w_1, w_3, \ldots, w_{m-1}\}$ and $S_2^* \equiv \{w_2, w_4, \ldots, w_m\}$. 

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Figure 3: Illustration of TAR-reconfigurability from an initial target set $S^{(0)} = S$ to $S^{(4)} = S^{(2)}$ in a cycle graph including six threshold-2 vertices (Lemma 3.5). $\bigcirc$ and $\circ$ represent threshold-2 and threshold-1 vertices, respectively. Seed vertices are colored black $\bullet$. 

(a) An initial target set $S^{(0)} = S$. 
(b) Add $w_2$ and remove the other vertices of $I_1$ to obtain a target set $S^{(1)}$. 
(c) Add $w_4$ and remove the other vertices in $I_2$ to obtain a target set $S^{(2)}$. 
(d) Remove all vertices but $w_6$ in $I_3$ to obtain a target set $S^{(3)}$. 
(e) Remove finally the vertices not in $S^{(2)}$ to obtain the target set $S^{(4)} = S^{(2)} = \{w_2, w_4, w_6\}$.
Let \( S \) be a target set of size \( k \geq \frac{m}{2} + 1 \). We construct a \( k\text{-TAR} \)-sequence from \( S \) to either \( S'_1 \) or \( S'_2 \). For each \( i \in \left[ \frac{m}{2} \right] \), let \( I_i \) denote the unique path from \( w_{2i-1} \) to \( w_{2i} \) (including end vertices) passing only through threshold-1 vertices. Note that \( \{I_1, \ldots, I_{\frac{m}{2}}\} \) forms a packing of \( V(G) \), and any target set of \( G \) includes at least one vertex of \( I_i \) for all \( i \in \left[ \frac{m}{2} \right] \). See Figure 3 for an example. Once \( S \) contains \( S'_1 \), we just need to remove the vertices of \( S \setminus S'_1 \) to obtain \( S'_1 \); thus, \( S \) and \( S'_1 \) are \( k\text{-TAR} \)-reconfigurable. Consider then that \( S \) does not include some \( w_i \) in \( S'_1 \). Here, we can safely assume that \( w_i \not\in S \) without loss of generality: Otherwise, we can reorder threshold-2 vertices to obtain \( w'_1, \ldots, w'_m \) so that \( w'_j = w_{j+1} \) for each \( j \in [m] \). Starting from \( S^{(0)} = S \), we transform \( S^{(i-1)} \) into \( S^{(i)} \) for each \( i \in \left[ \frac{m}{2} \right] \) by the following \( \text{TAR} \)-steps (see Figure 3):

**Step 1.** Add the vertex \( w_{2i} \) to \( S^{(i-1)} \) if \( w_{2i} \not\in S^{(i-1)} \).

**Step 2.** Remove the vertices of \( S^{(i-1)} \cap (I_i \setminus \{w_{2i}\}) \) one by one.

**Step 3.** Let \( S^{(i)} \) be the resulting set. Note that \( S^{(i)} \cap (\bigcup_{j \in [i]} I_j) = S'_2 \cap (\bigcup_{j \in [i]} I_j) \).

Since \( S'_2 \subseteq S^{(\frac{m}{2})} \), we finally remove the vertices of \( S^{(\frac{m}{2})} \setminus S'_2 \) to obtain \( S^{(\frac{m}{2}+1)} = S'_2 \). Let \( S \) be the resulting \( \text{TAR} \)-sequence from \( S \) to \( S'_2 \).

We show that \( S^{(i)} \) is a target set for each \( i \in [0 \ldots \frac{m}{2} + 1] \) by induction on \( i \). The base case of \( i = 0 \) is obvious as \( S^{(0)} = S \). Suppose that \( S^{(i-1)} \) is a target set for \( i \in \left[ \frac{m}{2} \right] \). Consider the residual \( G' = (V', E', E') \bowtie G_{S^{(i-1)}} \). Note that \( S^{(i-1)} \setminus I_i = S^{(i)} \setminus I_i \). Since \( w_{2i-2} \in S^{(i-1)} \) by construction, \( E'(w_{2i-1}) = 1 \); since \( S^{(i-1)} \) (which is a target set by induction hypothesis) includes a vertex of \( I_{i+1} \), \( E'(w_{2i+1}) = 1 \). Thus, \( w_{2i} \) is a unique vertex that may have threshold 2 in \( G' \). Since \( w_{2i} \in S^{(i)} \) by construction, \( (S^{(i)} \setminus I_i) \cup \{w_{2i}\} = S^{(i)} \) must be a target set of \( G \). Of course, \( S^{(\frac{m}{2}+1)} = S'_2 \) is a target set.

We then claim that \( |S^{(i-1)}| \geq |S^{(i)}| \) for all \( i \in \left[ \frac{m}{2} \right] \). If \( w_{2i} \in S^{(i-1)} \), then the claim is obvious because we only remove vertices in Step 2. Otherwise \( (w_{2i} \notin S^{(i-1)}) \), we have \( S^{(i-1)} \cap I_i \neq \emptyset \); thus, we remove at least one vertex in Step 2, implying that \( |S^{(i-1)}| \geq |S^{(i)}| \). It is easy to see that \( |S^{(\frac{m}{2})}| \geq |S^{(\frac{m}{2}+1)}| \). Since any target set in the subsequence of \( S \) from \( S^{(i-1)} \) to \( S^{(i)} \) has a size of at most \( |S^{(i-1)}| + 1 \), the maximum size of any target set in \( S \) from \( S^{(0)} = S \) to \( S^{(\frac{m}{2}+1)} = S'_2 \) is at most \( k + 1 \); i.e., \( S \) and \( S'_2 \) are \( k\text{-TAR} \)-reconfigurable.

We then consider \( \text{TAR} \)-reconfigurability between \( S'_1 \) and \( S'_2 \). Consider transforming \( S'_1 \) into \( S'_2 \) by the following \( \text{TAR} \)-steps:

**Step 1.** Add the vertex \( w_m \) to \( S'_1 \).

**Step 2.** For each \( i \in [\frac{m}{2} - 1] \), add the vertex \( w_{2i} \) and remove the vertex \( w_{2i-1} \).

**Step 3.** Remove the vertex \( w_m \).

Observe easily that every intermediate vertex set is a target set of size at most \( \frac{m}{2} + 2 \); i.e., \( S'_1 \) and \( S'_2 \) are \( (\frac{m}{2} + 1)\text{-TAR} \)-reconfigurable.

Finally, when \( m = 2 \), we have that \( S'_1 = \{w_1\} \) and \( S'_2 = \{w_2\} \), which are clearly \( \text{TJ} \)-reconfigurable. On the other hand, when \( m \geq 4 \), \( S'_1 \) and \( S'_2 \) are not \( \text{TJ} \)-reconfigurable as the symmetric difference between \( S'_1 \) and \( S'_2 \) has at least four vertices, as desired.
(a) An initial target set $S$.

(b) Add $w_1$ and remove the other vertices of $I_1$ to obtain a target set $S^{(0)}$.

(c) Add $w_3$ and remove the other vertices of $I_2 \cup I_3$ to obtain a target set $S^{(1)}$.

(d) Add $w_5$ and remove the other vertices of $I_4 \cup I_5$ to obtain a minimum target set $S^{(2)}$ consisting only of threshold-2 vertices.

Figure 4: Illustration of TAR-reconfigurability from an initial target set $S$ to a minimum target set in a cycle graph including five threshold-2 vertices (Lemma 3.5). $\bigcirc$ and $\bigcirc$ represent threshold-2 and threshold-1 vertices, respectively. Seed vertices are colored black $\bullet$. 
Suppose that \( m \) is positive odd. For each \( i \in [m] \), let \( I_i \) denote the unique path from \( w_i \) to one vertex before \( w_{i+1} \). Note that \( \{I_1, \ldots, I_m\} \) forms a partition of \( V(G) \). See Figure 4 for an example. Observe easily that for any target set \( S \), \( |S \cap I_i| \geq 1 \) or \( |S \cap I_{i+1}| \geq 1 \) for all \( i \in [m] \). It turns out that every target set \( S \) satisfies that \( |S| \geq \frac{m+1}{2} \). On the other hand, we can easily verify that the set \( \{w_1, w_3, w_5, \ldots, w_{m-2}, w_m\} \) of size \( \frac{m+1}{2} \) is a target set; i.e., the size of the minimum target set is \( \frac{m+1}{2} \).

For a target set \( S \) of size \( k \), we construct a \( k\)-TAR-sequence \( S \) from \( S \) to a minimum target set consisting only of threshold-2 vertices as follows (see Figure 4):

**Step 1.** Let \( i \in [m] \) be an integer such that \( S \cap I_i \neq \emptyset \). Add the vertex \( w_i \) to \( S \) if \( w_i \notin S \) and remove the vertices of \( S \cap (I_i \setminus \{w_i\}) \) one by one. Let \( S^{(0)} \) be the resulting set.

**Step 2.** For each \( j \in \left[ \frac{m-1}{2} \right] \), do the following:

- **Step 2-1.** Add \( w_{i+2j} \) if \( w_{i+2j} \notin S^{(j-1)} \).
- **Step 2-2.** Remove the vertices of \( S^{(j-1)} \cap (I_i \cap \{w_{i+2j}\}) \) one by one.
- **Step 2-3.** Let \( S^{(j)} \) be the resulting set.

Similarly to the case of positive even \( m \), we can prove that \( S^{(i)} \) is a target set for all \( i \in [0..\frac{m-1}{2}] \), and prove that the maximum size of any target set in \( S \) from \( S \) to \( S^{(m-1)} \) is at most \( k + 1 \). Note that \( S^{(m-1)} \) consists of exactly \( \frac{m+1}{2} \) threshold-2 vertices, which is a minimum target set.

We finally prove that any two minimum target sets consisting only of threshold-2 vertices are TJ-reconfigurable. For each \( i \in [m] \), we define \( S_i^* \triangleq \{w_{i+2j} : j \in [0..\frac{m-1}{2}]\} \). It is not hard to see that any minimum target set consisting only of threshold-2 vertices is identical to \( S_i^* \) for some \( i \in [m] \). Observe further that \( S_i^* \) and \( S_{i+2}^* \) for \( i \in [m] \) are TJ-reconfigurable: it suffices to add \( w_{i+2} \) and remove \( w_{i+1} \) by a single TJ-step. Since \( m \) is an odd integer, we eventually have that \( S_i^* \) and \( S_j^* \) are TJ-reconfigurable for any pair of \( i, j \in [m] \), as desired.

We are now ready to prove Theorem 3.3.

**Proof of Theorem 3.3.** We say that a cycle graph is terrible if the number of threshold-2 vertices in it is four or more and an even number. Given a graph \( G \) of maximum degree 2 and two size-\( k \) target sets \( X \) and \( Y \), we demonstrate by case analysis that \( X \) and \( Y \) are not TJ-reconfigurable if and only if the following conditions hold:

(C1) \( X \) and \( Y \) are minimum;

(C2) \( G \) contains a terrible cycle \( C \) such that \( X \cap V(C) = Y \cap V(C) \).

**Case 1.** \( G \) contains no terrible cycles: For each path and cycle \( C \) of \( G \), \( X \cap V(C) \) and \( Y \cap V(C) \) are \( \max(|X \cap V(C)|, |Y \cap V(C)|)\)-TAR-reconfigurable by Lemmas 3.4 and 3.5, respectively. Concatenating such TAR-sequences (where components \( C \) with \( |X \cap V(C)| \geq |Y \cap V(C)| \) are processed before those with \( |X \cap V(C)| < |Y \cap V(C)| \)), we obtain a \( k\)-TAR-sequence from \( X \) to \( Y \); i.e., \( X \) and \( Y \) are TJ-reconfigurable due to Observation 2.4.
Case 2. $G$ contains terrible cycles, but $X$ and $Y$ are not minimum: By assumption, we can find two elements $x \in X$ and $y \in Y$ such that $X' \triangleq X \setminus \{x\}$ and $Y' \triangleq Y \setminus \{y\}$ are target sets of size $k' \triangleq k - 1$ in polynomial time by a brute-force search. By Lemmas 3.4 and 3.5, for each component $C$ of $G$, $X' \cap V(C)$ and $Y' \cap V(C)$ are $(\max\{|X' \cap V(C)|, |Y' \cap V(C)|\} + 1)$-TAR-reconfigurable. Concatenating such TAR-sequences (in a similar manner to Case 1), we obtain a $(k' + 1)$-TAR-sequence from $X'$ to $Y'$, implying that $X$ and $Y$ are $k$-TAR-reconfigurable; i.e., $X$ and $Y$ are TJ-reconfigurable due to Observation 2.4.

Case 3. $G$ contains terrible cycles, and $X$ and $Y$ are minimum, but it holds that $X \cap V(C) = Y \cap V(C)$ for every terrible cycle $C$: Observe that there is no need to modify the vertices of terrible cycles. For each path and nonterrible cycle $C$ of $G$, $X \cap V(C)$ and $Y \cap V(C)$ are $\max\{|X \cap V(C)|, |Y \cap V(C)|\}$-TAR-reconfigurable by Lemmas 3.4 and 3.5, respectively. Concatenating such TAR-sequences (in a similar manner to Case 1), we obtain a $k$-TAR-sequence from $X$ to $Y$; i.e., $X$ and $Y$ are TJ-reconfigurable due to Observation 2.4.

Case 4. Otherwise (i.e., (C1) and (C2) hold): For a terrible cycle $C$ such that $X \cap V(C) \neq Y \cap V(C)$, $X \cap V(C)$ and $Y \cap V(C)$ are not TJ-reconfigurable on $C$ by Lemma 3.5. By Lemma 2.7, $X$ and $Y$ are not TJ-reconfigurable on $G$, as desired.

Since we can verify if $G$ satisfies (C1) and (C2) in polynomial time, the above analysis completes the proof.  

3

3.2 PSPACE-completeness on Planar $(3, 3)$-Graphs [HD05, IDH11, KMM12]

Hearn and Demain [HD05] proved that Minimum Vertex Cover Reconfiguration is PSPACE-complete on planar graphs of degree 2 and 3, which implies that it is also PSPACE-complete on planar 3-regular graphs (see, e.g., [Moh01]). For the sake of completeness, we give an explicit proof of the following statement.

Observation 3.6 (⋆ [HD05,IDH11,KMM12]). Minimum Vertex Cover Reconfiguration is PSPACE-complete on planar 3-regular graphs; i.e., Minimum Target Set Reconfiguration is PSPACE-complete on planar $(3, 3)$-graphs.

3.3 PSPACE-completeness on Bipartite Planar $\{3, 4\}$-Graphs

We prove the PSPACE-completeness result on bipartite planar $\{3, 4\}$-graphs.

Theorem 3.7. Target Set Reconfiguration is PSPACE-complete on bipartite planar $\{3, 4\}$-graphs.

The proof of Theorem 3.7 is based on a series of reductions starting from a planar $(3, 3)$-graph. Suppose that $G = (V, E, \tau)$ is a planar $(3, 3)$-graph and $w$ is a $(3, 3)$-vertex, whose neighbors are denoted $N_G(w) = \{x, y, z\}$. We then modify the subgraph induced by $\{w, x, y, z\}$ according to the following procedure (see Figure 5).

3For example, Figures 2a and 2b fall into Case 4; Figures 2c and 2d fall into Case 2.
Construction of $\Upsilon$-gadget (Figure 5).

Step 1. Remove $(x, w)$, $(y, w)$, and $(z, w)$, and set $\tau(w) = 2$.

Step 2. Create vertices $v_x, v_y, v_{xy}$ with $\tau(v_x) = \tau(v_y) = 1$ and $\tau(v_{xy}) = 2$, and edges $(x, v_x)$, $(y, v_y)$, $(v_x, v_{xy})$, and $(v_y, v_{xy})$.

Step 3. Create a cycle graph $C_4$ on four vertices $w, v_z, \overline{w}, v_w$ such that $\tau(\overline{w}) = 2$ and $\tau(v_z) = \tau(v_w) = 1$, and edges $(v_w, v_{xy})$ and $(v_z, z)$.

Step 4. Create a one-way gadget $D_x$ on vertex set $\{t_x, h_x, b_{x,1}, b_{x,2}\}$ connecting from $w$ to $v_x$, and create an edge $(b_{x,1}, b_{x,2})$; create a one-way gadget $D_y$ on vertex set $\{t_y, h_y, b_{y,1}, b_{y,2}\}$ connecting from $\overline{w}$ to $v_y$ and create an edge $(b_{y,1}, b_{y,2})$.

We call the resulting subgraph a $\Upsilon$-gadget, which plays a role in removing a $(3,3)$-vertex using $(3,1)$- and $(3,2)$-vertices without sacrificing planarity. After uploading an early draft of this paper on arXiv, Ryuhei Uehara discovered this gadget, which is designed to preserve planarity, improving upon the old $\Upsilon$-gadget. Gratefully, Uehara allowed us to include it here. Vertices of \{v_x, v_y, v_{xy}\} $\cup V(C_4) \cup V(D_x) \cup V(D_y)$ are referred to as internal vertices of a $\Upsilon$-gadget. Let $G'$ be a graph obtained from $G$ by the above procedure. The most crucial property of $\Upsilon$-gadgets is that $G'_{[w]} = G_{[w]}$. For a seed set $S' \subseteq V'$ of $G'$, we define $\phi_\Upsilon(S') \subseteq V$ as

$$\phi_\Upsilon(S') = \begin{cases} S' & \text{if } S' \subseteq V \setminus \{w\}, \\ (S' \cap V) \cup \{w\} & \text{otherwise.} \end{cases}$$

Lemma 3.8. Let $G = (V, E, \tau)$ be a graph including a $(3,3)$-vertex $w$ and $G' = (V', E', \tau')$ be a graph obtained from $G$ by replacing $w$ and its incident edges with a $\Upsilon$-gadget. Then, a seed set $S' \subseteq V'$ is a
minimum target set of $G'$ if and only if $\phi_T(S')$ is a minimum target set of $G$. Moreover, two minimum target sets are TJ-reconfigurable on $G$ if and only if they are TJ-reconfigurable on $G'$.

**Proof.** We give a relation between minimum target sets of $G$ and $G'$. We first claim that if a seed set $S \subseteq V$ is a target set of $G$, then it is also a target set of $G'$ according to the following case analysis.

- If $w \notin S$: Since it holds that $d(w) = \tau(w) = 3$, $S$ must be a target set of $G - \{w\}$. Observing that $G - \{w\} = G'[V \setminus \{w\}]$, we find $S$ to activate $x, y, z$ in $G'$, eventually activating the internal vertices of the $\Upsilon$-gadget: $x$ and $y$ respectively activate $v_x$ and $v_y$, which then activates $v_{xy}$ and $v_w$; $z$ activates $v_z$; $v_z$ and $v_w$ activate $w$ and $\overline{w}$, which further activates the vertices of $D_x$ and $D_y$. Therefore, $S$ is a target set of $G'$.

- If $w \in S$: By applying Lemma 2.6 on the residual $G_{\{w\}}$ and $S$, we find $(S \setminus \{w\}) \cap V(G_{\{w\}}) = S \setminus \{w\}$ to be a target set of $G_{\{w\}}$. Since $G_{\{w\}} = G'_{\{w\}}$, $(S \setminus \{w\}) \cup \{w\} = S$ is a target set of $G'$ by Lemma 2.6.

We then claim that if $S' \subseteq V'$ is a target set of $G'$, then $\phi_T(S')$ is a target set of $G$. To see this, the following case analysis is sufficient.

- If $S' \subseteq V \setminus \{w\}$: Since $S'$ activates $x, y, z$ before $w$ on $G'$, $S'$ is a target set of $G'[V \setminus \{w\}] = G - \{w\}$. Thus, $S'$ activates the vertices of $V \setminus \{w\}$ on $G$, finally activating $w$ on $G$; i.e., $S' = \phi_T(S')$ is a target set of $G$.

- Otherwise: Observe that $(S' \cap V) \cup \{w\}$ is also a target set of $G'$. By applying Lemma 2.6 on the residual $G'_{\{w\}}$ and $(S' \cap V) \cup \{w\}$, we find $S' \cap V$ to be a target set of $G'_{\{w\}}$. Since $G'_{\{w\}} = G_{\{w\}}$, $(S' \cap V) \cup \{w\} = \phi_T(S')$ is a target set of $G$ by Lemma 2.6.

Note that a minimum target set $S' \subseteq V'$ of $G'$ includes at most one internal vertex of the $\Upsilon$-gadget because otherwise $(S' \cap V) \cup \{w\}$ is a target set of $G$, which contradicts the minimality of $S'$. Therefore, we have $|\phi_T(S')| = |S'|$ for a minimum target set $S'$ of $G'$, namely, $\phi_T(S')$ is a minimum target set of $G$.

We finally demonstrate that two minimum target sets $X$ and $Y$ of $G$ are TJ-reconfigurable on $G$ if and only if they are TJ-reconfigurable on $G'$. Given a TJ-sequence $S$ of minimum target sets of $G$ from $X$ to $Y$, we find $S$ to be a TJ-sequence of minimum target sets of $G'$ from $X$ to $Y$, completing the only-if direction. On the other hand, given a TJ-sequence $S'$ of minimum target sets of $G'$ from $X$ to $Y$, we find the sequence $(\phi_T(S'))_{S' \in S'}$ to be a Tjn-sequence of minimum target sets of $G$ from $\phi_T(X) = X$ to $\phi_T(Y) = Y$; i.e., $X$ and $Y$ are TJ-reconfigurable on $G$ by Observation 2.3, completing the if direction.

After replacing each $(3, 3)$-vertex and its incident edges with a $\Upsilon$-gadget, we obtain a planar graph $H$ in which each vertex is a $(2, 1)$-, $(3, 1)$-, $(3, 2)$-, or $(4, 2)$-vertex. We then subdivide every edge according to Lemma 2.10 to obtain a bipartite planar graph $I$. We further introduce the following gadget (see Figure 6).
Construction of $\Theta$-gadget (Figure 6).

**Step 1.** Create a hexagonal prism graph $Y_6$ on 12 vertices such that

$$V(Y_6) \triangleq \{t_{i,j} : i \in [2], j \in [6]\},$$
$$E(Y_6) \triangleq \{(t_{i,j}, t_{i,j+1 \mod 6}) : i \in [2], j \in [6]\} \cup \{(t_{1,j}, t_{2,j}) : j \in [6]\},$$
$$\tau(t_{i,j}) \triangleq 2$$

for all $i \in [2], j \in [6]$.

**Step 2.** Create an edge $(t_{2,3}, t_{2,6})$.

**Step 3.** Create a vertex $r$ with $\tau(r) = 2$ and two edges $(r, t_{1,1})$ and $(r, t_{1,5})$.

We call this gadget a $\Theta$-gadget. Observe that a $\Theta$-gadget is bipartite and planar. We say that a $\Theta$-gadget is connected to a vertex $v$ if there exists an edge $(r, v)$. We analyze the minimum target set of $\Theta$-gadgets, followed by TJ-reconfigurability.

**Lemma 3.9 (•).** Let $R$ be a $\Theta$-gadget and $R'$ be a graph obtained from $R$ by redefining the threshold of $r$ as 1. Then, the size of the minimum target set of $R$ and $R'$ is 3. In particular, the seed set $M \triangleq \{r, t_{1,2}, t_{2,3}\}$ is a minimum target set of $R$ and $R'$.

**Lemma 3.10.** Let $G = (V, E, \tau)$ be a graph and $G' = (V', E', \tau')$ be a graph obtained from $G$ by connecting a $\Theta$-gadget $R$ to a vertex $v$ of $G$ and defining $\tau'(v) \triangleq \tau(v) + 1$ and $\tau'(w) \triangleq \tau(w)$ for the other vertices $w \in V \setminus \{v\}$. Then, a seed set $S \subseteq V$ is a minimum target set of $G$ if and only if $S \cup M$ is a minimum target set of $G'$, where $M$ is the minimum target set of the $\Theta$-gadget given in Lemma 3.9. Moreover, two minimum target sets $X$ and $Y$ of $G$ are TJ-reconfigurable on $G$ if and only if $X \cup M$ and $Y \cup M$ are TJ-reconfigurable on $G'$.

**Remark 3.11.** The crux of the proof of Lemma 3.10 is that $R$ and $R'$ have the same-sized minimum target set as promised by Lemma 3.10. Suppose we have constructed a different $\Theta$-gadget such that the minimum target set of $R$ has size 3 and that of $R'$ has size 2. Then, let $S'$ be any minimum target set of above defined $G'$, and consider the residual $G'_{S' \cap V}$. On one hand, if $v \in S'$, the threshold of $r$ must be 1 in $G'_{S' \cap V}$; i.e., $|S' \cap V(R)| = 2$. On the other hand, if $v \notin S'$, the threshold of $r$ must be 2; i.e., $|S' \cap V(R)| = 3$. Therefore, $S' \cap V$ in the former case must not be a minimum target set of $G$. This is undesirable.
Proof of Lemma 3.10. We give a relation between minimum target sets of \( G \) and \( G' \). Let \( R \) be a \( \Theta \)-gadget connected to \( v \) and \( R' \) be a graph obtained from \( R \) by decreasing the threshold of \( r \) by 1. Define \( k \) as the size of the minimum target set of \( G \). Suppose that a seed set \( S' \subseteq V' \) is a target set of \( G' \). Then, \( S' \) satisfies the following:

\((C1)\) \( S' \cap V \) is a target set of \( G \). In particular, it holds that \( |S' \cap V| \geq k \).

\((C2)\) \( S' \cap V(R) \) is a target set of \( R' \). In particular, it holds that \( |S' \cap V(R)| \geq 3 \) due to Lemma 3.9.

The reason \((C1)\) holds is that the residual \( G'_V(R) \) is identical to \( G \); the reason \((C2)\) holds is that the residual \( G'_V(R) \) is identical to \( R' \). Indeed, for any minimum target set \( S \subseteq V \) of \( G \) and the minimum target set \( M \) of the \( \Theta \)-gadget given in Lemma 3.9, the union \( S \cup M \) is a size-(\( k + 3 \)) target set of \( G' \) from the fact that \( G'_M = G \). By \((C1)\) and \((C2)\), \( S \cup M \) turns out to be minimum. Consequently, whenever \( S' \) is a minimum target set of \( G' \), we have that \( S' \cap V \) is a minimum target set of \( G \) due to \((C1)\) and \( |S' \cap V(R)| = 3 \) due to \((C2)\).

We finally demonstrate that two minimum target sets \( X \) and \( Y \) of \( G \) are TJ-reconfigurable if and only if \( X \cup M \) and \( Y \cup M \) are TJ-reconfigurable on \( G' \). Given a TJ-sequence \( S \) of minimum target sets of \( G \) from \( X \) to \( Y \), we find the sequence \( (S \cup M)_{S \in S} \) to be a TJ-sequence of minimum target sets of \( G' \) from \( X \cup M \) to \( Y \cup M \) thanks to the above discussion. On the other hand, given a TJ-sequence \( S' \) of minimum target sets of \( G' \) from \( X \cup M \) to \( Y \cup M \), we remove the vertices of \( V(R) \) from every set in \( S' \) to obtain a new sequence \( S = (S' \cap V)_{S \in S'} \). Since each seed set in \( S' \) is a union of a minimum target set of \( G \) and a minimum target set of \( R' \), \( S \) is a TJN-sequence of minimum target sets of \( G \) from \( X \) to \( Y \); i.e., \( X \) and \( Y \) are TJ-reconfigurable on \( G \) by Observation 2.3.

We are now ready to prove Theorem 3.7 using Lemmas 2.10, 3.8 and 3.10.

Proof of Theorem 3.7. The reduction from a planar \((3,3)\)-graph to a bipartite planar \((\{3,4\},2)\)-graph is presented below.

| Reduction from planar \((3,3)\)-graph \( G \) to bipartite planar \((\{3,4\},2)\)-graph \( J \). |
|---------------------------------------------------------------|
| **Step 1.** Replace each vertex \( w \in V(G) \) and its incident edges with a \( \Upsilon \)-gadget to obtain a graph \( H \). Note that \( H \) is planar and that each vertex of \( H \) is a \((2,1)\)-, \((3,1)\)-, \((3,2)\)-, or \((4,2)\)-vertex. |
| **Step 2.** Subdivide each edge of \( H \) by a new threshold-1 vertex to obtain a graph \( I \). Note that \( I \) is bipartite and planar. |
| **Step 3.** Create a \( \Theta \)-gadget connecting to each \((2,1)\)- or \((3,1)\)- vertex \( v \) of \( I \), and increase the threshold of \( v \) by 1 to obtain a graph \( J \). Note that \( J \) is a bipartite planar \((\{3,4\},2)\)-graph. Let \( M_v \) denote a minimum target set of the \( \Theta \)-gadget connected to vertex \( v \) defined in Lemma 3.9. |

Obviously, the reduction completes in polynomial time, and \( J \) is a bipartite planar \((\{3,4\},2)\)-graph. We now show the correctness of the reduction. Let \( X \) and \( Y \) be two minimum target sets
of $G$. By applying Lemmas 2.10 and 3.8 repeatedly, we have that $X$ and $Y$ are $\text{TJ}$-reconfigurable on $G$ if and only if $X$ and $Y$ are $\text{TJ}$-reconfigurable on $I$. By applying Lemma 3.10 repeatedly, we have that $X$ and $Y$ are $\text{TJ}$-reconfigurable on $I$ if and only if $X_j \approx X \cup \bigcup_v M_v$ and $Y_j \approx Y \cup \bigcup_v M_v$ are $\text{TJ}$-reconfigurable on $J$. Consequently, it turns out that $X$ and $Y$ are $\text{TJ}$-reconfigurable on $G$ if and only if $X_j$ and $Y_j$ are $\text{TJ}$-reconfigurable on $J$. By Observation 3.6, Target Set Reconfiguration on bipartite planar $(\{3, 4\}, 2)$-graphs is PSPACE-hard, as desired. □

3.4 PSPACE-completeness on Bipartite $(3, \{1, 2\})$-Graphs and Planar $(3, \{1, 2\})$-Graphs

We next prove the PSPACE-completeness result on bipartite $(3, \{1, 2\})$-graphs and planar $(3, \{1, 2\})$-graphs.

Theorem 3.12. Target Set Reconfiguration is PSPACE-complete on bipartite $(3, \{1, 2\})$-graphs and planar $(3, \{1, 2\})$-graphs.

Though Theorem 3.12 is a more-or-less similar statement to Theorem 3.7, its proof involves a different gadget. Our reduction begins from a planar $(3, 3)$-graph. Suppose that $G = (V, E, \tau)$ is a planar $(3, 3)$-graph and $w$ is a $(3, 3)$-vertex, whose neighbors are denoted $N_G(w) = \{x, y, z\}$. We then modify the subgraph induced by $\{w, x, y, z\}$ by replacing it with a $\Upsilon$-gadget (see Figure 5). We obtain a planar graph $H$ that consists only of $(3, 1)$- or $(3, 2)$-vertices. We then make $H$ bipartite by subdividing every edge according to Lemma 2.10, which, however, produces $(2, 1)$-vertices. We thus introduce the following gadget (see Figure 7):

Construction of $\Xi$-gadget (Figure 7).

**Step 1.** Create two cycle graphs $C_1$ and $C_2$ on vertex sets $\{a_1, b_1, c_1, d_1\}$ and $\{a_2, b_2, c_2, d_2\}$, respectively, such that $a_1$ and $a_2$ have a threshold of 2 and the other vertices have a threshold of 1.

**Step 2.** Create three edges $(b_1, b_2)$, $(c_1, c_2)$, and $(d_1, d_2)$.

We call this gadget a $\Xi$-gadget. Observe that a $\Xi$-gadget is bipartite. We say that a $\Xi$-gadget connects between two distinct vertices $v_1$ and $v_2$ if there exist two edges $(v_1, a_1)$ and $(v_2, a_2)$. Vertices of $V(C_1) \cup V(C_2)$ are referred to as internal vertices of a $\Xi$-gadget. We show the following lemma on $\text{TJ}$-reconfigurability.
Lemma 3.13. Let $G = (V, E, \tau)$ be a graph and $G' = (V', E', \tau')$ be a graph obtained from $G$ by connecting a $\Xi$-gadget $R$ between two distinct vertices $v_1$ and $v_2$ of $G$ and defining $\tau'(v_1) = \tau(v_1) + 1$, $\tau'(v_2) = \tau(v_2) + 1$, and $\tau'(w) = \tau(w)$ for the other vertices $w \in V \setminus \{v_1, v_2\}$. Then, $S' \subseteq V'$ is a minimum target set of $G'$ if and only if $S' \cap V$ is a minimum target set of $G$ and $S' \cap V(R)$ is a minimum target set of $R$ consisting of a single internal vertex. Moreover, two minimum target sets $X$ and $Y$ of $G$ are $TJ$-reconfigurable on $G$ if and only if $X \cup \{a_1\}$ and $Y \cup \{a_1\}$ are $TJ$-reconfigurable on $G'$.

Proof. We give a relation between minimum target sets of $G$ and $G'$. Let $S' \subseteq V'$ be a minimum target set of $G'$. Since $V$ is not a target set of $G'$, $S'$ includes at least one internal vertex of the $\Xi$-gadget. Further, if $S'$ includes two or more internal vertices, we can remove all but one of them to obtain a smaller target set. Therefore, $S'$ must include exactly one internal vertex, say, $v$. Observing that $G'_{\{v\}} = G$ for any internal vertex $v$ of the $\Xi$-gadget, we apply Lemma 2.6 on the residual $G'_{\{v\}}$ and $S'$, and find $S' \cap V$ to be a minimum target set of $G$. On the other hand, if $S \subseteq V$ is a minimum target set of $G$, then $S \cup \{a_1\}$ is a minimum target set of $G'$.

We finally demonstrate that two minimum target sets $X$ and $Y$ of $G$ are $TJ$-reconfigurable if and only if $X \cup \{a_1\}$ and $Y \cup \{a_1\}$ are $TJ$-reconfigurable on $G'$. Given a $TJ$-sequence $S$ of minimum target sets of $G$ from $X$ to $Y$, we find the sequence $\langle S \cup \{a_1\}\rangle_{S \in S}$ to be a $TJ$-sequence of minimum target sets of $G'$ from $X \cup \{a_1\}$ to $Y \cup \{a_1\}$, completing the only-if direction. On the other hand, given a $TJ$-sequence $S'$ of minimum target sets of $G'$ from $X \cup \{a_1\}$ to $Y \cup \{a_1\}$, we find the sequence $\langle S \cap V\rangle_{S \in S}$ to be a $TJN$-sequence of minimum target sets of $G$ from $X$ to $Y$, i.e., $X$ and $Y$ are $TJ$-reconfigurable on $G$, completing the if direction.

We are now ready to prove Theorem 3.12 using Lemmas 2.7, 2.10, 3.8 and 3.13.

Proof of Theorem 3.12. The proof for planar $(3, \{1, 2\})$-graphs is immediate from Lemma 3.8. We present the reduction from a $(3, 3)$-graph to a bipartite $(3, \{1, 2\})$-graph below.

| Reduction from $(3, 3)$-graph $G$ to bipartite $(3, \{1, 2\})$-graph $I$. |
|---|
| **Step 1.** For each vertex $w$ of $G$, replace $w$ and its incident edges with a $Y$-gadget to obtain a graph $H$. Note that $H$ is a $(3, \{1, 2\})$-graph. |
| **Step 2.** Subdivide each edge of $H$ by a new threshold-1 vertex to obtain a graph $I$. Note that $I$ is bipartite. |
| **Step 3.** Create two copies of $I$, denoted $I_1$ and $I_2$. For each vertex $v$ of $I$, the two vertices of $I_1$ and $I_2$ corresponding to $v$ are denoted by $v_1$ and $v_2$, respectively. |
| **Step 4.** For each $(2, 1)$-vertex $v$ of $I$, connect a $\Xi$-gadget between $v_1$ and $v_2$, and increase the threshold of $v_1$ and $v_2$ by 1 to obtain a graph $J$. Denote by $a_{1,v}$ the internal vertex $a_1$ of the $\Xi$-gadget connected between $v_1$ and $v_2$. |

Obviously, the reduction finishes in polynomial time, and $J$ is a bipartite $(3, \{1, 2\})$-graph. Let $X$ and $Y$ be two minimum target sets of $G$. By applying Lemmas 2.7, 2.10 and 3.8 repeatedly, we have that $X$ and $Y$ are $TJ$-reconfigurable on $G$ if and only if $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are $TJ$-reconfigurable on $I_1 \oplus I_2$, where $X_1 = \{v_1 : v \in X\}$, $Y_1 = \{v_1 : v \in Y\}$, $X_2 = \{v_2 : v \in X\}$, and $Y_2 = \{v_2 : v \in Y\}$.
Algorithm 4.1 Chen’s polynomial-time algorithm [Che09] for Target Set Selection on a tree.

Input: tree $G = (V,E,\tau)$.
1. let $T$ be a rooted-tree representation of $G$ with root $r \in V$; initialize $S^* \leftarrow \emptyset$.
2. while $\exists$ vertex $v$ s.t. $\tau(v)$ is not defined, but $\tau'(w)$ is determined for all $v$’s children $w$ do
3. let $\tau'(v) \leftarrow \tau(v)$—(# $v$’s children $w$ such that $\tau'(w) = 0$ or $w \in S^*$).
4. if $v \neq r$ and $\tau'(v) \geq 2$ then $S^* \leftarrow S^* \cup \{v\}$.
5. if $v = r$ and $\tau'(v) \geq 1$ then $S^* \leftarrow S^* \cup \{v\}$.
6. return $S^*$.

By applying Lemma 3.13 repeatedly, we have that $X_1 \cup X_2$ and $Y_1 \cup Y_2$ are TJ-reconfigurable on $I_1 \oplus I_2$ if and only if $X_I \cong X_1 \cup X_2 \cup \{a_{1,v} : v \in V(I)\}$ and $Y_I \cong Y_1 \cup Y_2 \cup \{a_{1,v} : v \in V(I)\}$ are TJ-reconfigurable on $I$. Consequently, $X$ and $Y$ are TJ-reconfigurable on $G$ if and only if $X_I$ and $Y_I$ are TJ-reconfigurable on $I$. By Observation 3.6, Target Set Reconfiguration on bipartite $(3, (1, 2))$-graphs is PSPACE-hard, as desired. □

4 Restricted Graph Classes

This section investigates the tractability of Target Set Reconfiguration on restricted graph classes: trees and split graphs.

4.1 Polynomial Time on Trees

Vertex Cover Reconfiguration is known to be solvable in polynomial time on trees [KMM12, INZ16, MNRS18]. We show that Target Set Reconfiguration is also tractable on trees.

Theorem 4.1. Target Set Reconfiguration is polynomial-time solvable on trees.

As will be shown, any pair of same-size target sets of a tree is TJ-reconfigurable; i.e., we just answer “yes.” Our idea for proving Theorem 4.1 is to construct a “canonical” target set that is TAR-reconfigurable to any target set, which is reminiscent of the idea for Dominating Set Reconfiguration by Haddadan, Ito, Mouawad, Nishimura, Ono, Suzuki, and Tebbal [HIM+16].

We first recapitulate Chen’s polynomial-time algorithm [Che09] for Target Set Selection on a tree, presented in Algorithm 4.1. Let $G = (V,E,\tau)$ be a tree. For an arbitrary vertex $r \in V$, let $T$ denote a tree representation of $(V,E)$ rooted at $r$, which naturally introduces parents, children, and leaves. Starting from an empty set $S^* = \emptyset$, we determine whether or not to include each vertex of $G$ into $S^*$ in a bottom-up fashion. Suppose that there exists a vertex $v$ that has not been scanned yet but its children have already been examined.\footnote{Note that we first select a leaf, which has no children.} We then define $\tau'(v)$ as follows:

$$\tau'(v) \triangleq \tau(v) - \left| \{ \text{child } w \text{ of } v : \tau'(w) = 0 \text{ or } w \in S^* \} \right|,$$

which indicates the number of $v$’s children that would have been activated by the current $S^*$. If $v$ is not the root $r$, then we include $v$ into $S^*$ only if $\tau'(v) \geq 2$. On the other hand, if $v = r$, then
we include \( v \) into \( S^* \) only if \( \tau'(v) \geq 1 \). Running through every vertex of \( G \), we finally return \( S^* \) as an output. See Figure 8 for a running example of Algorithm 4.1. Chen [Che09] proves that \( S^* \) is a minimum target set of \( G \). We will present a simple characterization of target sets of \( G \) using \( S^* \). Define \( k \triangleq |S^*| \). Let \( s_1, s_2, \ldots, s_k \) denote the vertices of \( S^* \) ordered by a postorder depth-first traversal starting from the root \( r \) of \( T \). For each \( i \in [k] \), let \( T_i \) denote the subtree of \( T \) rooted at \( s_i \), and define the vertex set \( P_i \) as:

\[
P_i = V(T_i) \setminus \bigcup_{j \in [i-1]} V(T_j). \tag{8}
\]

\( \{P_1, \ldots, P_k\} \) forms a packing of \( V(T) \), and it holds that \( S^* \cap P_i = \{s_i\} \) for all \( i \in [k] \); see also Figure 8.

**Lemma 4.2.** For any target set \( S \) of a tree \( G \), it holds that \( |S \cap P_i| \geq 1 \) for all \( i \in [k] \).

**Proof.** We show that for each \( i \in [k] \), the seed set \( S_i \triangleq V(G) \setminus P_i \) is not a target set, which is sufficient because \( S_i \) is the maximum set disjoint to \( P_i \). In fact, we show that \( S'_i \triangleq S_i \cup \{v \in P_i : \tau'(v) = 0\} \) is not a target set, where the values of \( \tau' \) are obtained by running Algorithm 4.1. Fix \( i \) and consider first the case of \( s_i \neq r \). By definition of \( \tau' \) and \( S'_i \), we have the following relation for each \( v \in P_i \):

- If \( \tau'(v) = 0 \): \( v \in S'_i \);
If $\tau'(v) = 1$: $\tau(v)$ is equal to the number of $v$’s children in $S'_i$ plus 1;

If $\tau'(v) \geq 2$: $\tau(v)$ is at least the number of $v$’s children in $S'_i$ plus 2.

Define $Q_i$ as the set of vertices that are reachable from $s_i$ in $T$ without touching any vertices of $S'_i$. Observe that $s_i \in Q_i \subseteq P_i$, and the subtree of $T$ induced by $Q_i$ is connected. See Figure 8 for an example.

For each vertex $v \in Q_i \setminus \{s_i\}$, $v$’s parent is not in $S'_i$ and $\tau'(v)$ must be 1 because in Line 4 of Algorithm 4.1 we have not added $v$ into $S'_i$; thus, we have $\tau(v) = |N_G(v) \cap S'_i| + 1$. Similarly, $s_i$’s parent is in $S'_i$ and $\tau'(s_i)$ must be at least 2 because in Line 4 of Algorithm 4.1 we have added $s_i$ into $S'_i$; hence, $\tau(s_i) \geq |N_G(s_i) \cap S'_i| + 1$.

Consequently, $\tau(v) \geq |N_G(v) \cap S'_i| + 1$ for all $v \in Q_i$, implying that no vertices of $Q_i$ would be activated; i.e., $S'_i$ is not a target set, as desired. The case of $s_i = r$ can be shown in the same manner except that $\tau'(r)$ is at least the number of $r$’s children in $S'_i$ plus 1, which eventually results in that $\tau(r) \geq |N_G(r) \cap S'_i| + 1$ because $r$ has no parent.

\[ \square \]

Lemma 4.2 gives a different (perhaps simple) proof of the optimality of Algorithm 4.1 from Chen’s proof. Indeed, we claim that any target set $S$ and $S^*$ are $|S|$-TAR-reconfigurable.

**Lemma 4.3.** For any target set $S$ of $G$, $S$ and $S^*$ are $|S|$-TAR-reconfigurable. Moreover, an actual $|S|$-TAR-sequence can be found in polynomial time.

**Proof.** We construct a TAR-sequence $S$ of target sets from a target set $S$ to $S^*$. Starting from $S^{(0)} = S$, for each $i \in [k]$, we transform $S^{(i-1)}$ into $S^{(i)}$ by the following TAR-steps:

**Step 1.** Add the vertex $s_i \in S^* \cap P_i$ if $s_i \notin S^{(i-1)}$.

**Step 2.** Remove the vertices of $S^{(i-1)} \cap (P_i \setminus \{s_i\})$ one by one.

**Step 3.** Let $S^{(i)}$ be the resulting set. Note that $S^{(i)} \cap V(T_i) = S^* \cap V(T_i)$.

Since it holds that $S^* \subseteq S^{(k)}$, we finally remove the vertices of $S^{(k)} \setminus \bigcup_{i \in [k]} P_i$ to obtain $S^{(k+1)} = S^*$.

We here show that $S^{(0)}$ is a target set for each $i \in [k]$. The proof is done by induction on $i$. The base case of $i = 0$ is obvious since $S^{(0)} = S$. Suppose that $S^{(i-1)}$ is a target set for $i \in [k]$. Since $G$ is a tree, the residual $G_{\{s_i\}}$ can be decomposed into the subtree $G_1$ of $G_{\{s_i\}}$ induced by $V(T_i) \setminus \{s_i\}$ and the subtree $G_2$ of $G_{\{s_i\}}$ induced by $V(G) \setminus V(T_i)$. Note that $S^{(i)} \cap V(G_1) = S^* \cap V(G_1)$ and $S^{(i)} \cap V(G_2) = S^{(i-1)} \cap V(G_2)$. Since $S^*$ is a target set of $G$, $S^* \setminus \{s_i\}$ is a target set of $G_{\{s_i\}}$ by Lemma 2.6, which implies that $(S^* \setminus \{s_i\}) \cap V(G_1) = S^* \cap V(G_1) = S^{(i)} \cap V(G_1)$ is a target set of $G_1$. By the induction hypothesis, $S^{(i-1)} \cup \{s_i\}$ is a target set of $G$, implying that $(S^{(i-1)} \cup \{s_i\}) \cap V(G_2)$ is a target set of $G_2$ by Lemma 2.6; i.e., $(S^{(i-1)} \cup \{s_i\}) \cap V(G_2) = S^{(i-1)} \cap V(G_2) = S^{(i)} \cap V(G_2)$ is a target set of $G_2$. Eventually, we have that $S^{(i)} \cap (V(G_1) \cup V(G_2)) = S^{(i)} \setminus \{s_i\}$ is a target set of $G_1 \oplus G_2 = G_{\{s_i\}}$; i.e., $S^{(i)}$ is a target set of $G$ due to Lemma 2.6. Obviously, $S^{(k+1)} = S^*$ is a target set. Since every seed set appearing in $S$ is a superset of $S^{(i)}$ for some $i \in [k+1]$, $S$ is a TAR-sequence of target sets from $S$ to $S^*$.

We then claim that $|S^{(i-1)}| \geq |S^{(i)}|$ for all $i \in [k]$. If $s_i \in S^{(i-1)}$, then the claim is obvious because we only remove vertices in Step 2 without adding $s_i$ in Step 1. On the other hand, if $s_i \notin S_{i-1}$,
Lemma 4.2 tells that $S^{(i-1)} \cap (P_i \setminus \{s_i\}) \neq \emptyset$. Hence, we remove at least one vertex in Step 2; i.e., it must hold that $|S^{(i-1)}| \geq |S^{(i)}|$. It is easy to observe that $|S^{(k)}| \geq |S^{(k+1)}|$. Since every target set in the subsequence of $S$ from $S^{(i-1)}$ to $S^{(i)}$ has a size of at most $|S^{(i-1)}| + 1$, the maximum size of any target set in $S$ from $S^{(0)} = S$ to $S^{(k+1)} = S^*$ is at most $|S| + 1$; i.e., $S$ is a $|S|$-TAR-sequence from $S$ to $S^*$, completing the proof. □

Proof of Theorem 4.1. By Lemma 4.3, two target sets $X$ and $Y$ are max$(|X|, |Y|)$-TAR-reconfigurable. In particular, when $|X| = |Y|$, they are TJ-reconfigurable, as desired. □

4.2 PSPACE-completeness on Split Graphs

A graph is called a split graph if the vertex set can be partitioned into a clique and an independent set. On split graphs, VC-R is solvable in polynomial time [KMM12, INZ16, MNRS18]. But, Target Set Reconfiguration is PSPACE-complete on split graphs.

Theorem 4.4. Target Set Reconfiguration is PSPACE-complete on split graphs.

We adapt a reduction from Hitting Set to Target Set Selection due to Nichterlein, Niedermeier, Uhlmann, and Weller [NNUW13]. Given a set family $\mathcal{F} = \{F_1, \ldots, F_m\}$ over a universe $U = \{u_1, \ldots, u_n\}$, a subset $S$ of $U$ is called a hitting set if $S \cap F_j \neq \emptyset$ for all $j \in [m]$. The Hitting Set problem requires deciding if there exists a hitting set of size $k$ for a parameter $k \in [n]$, which is known to be NP-complete [JG79]. In Hitting Set Reconfiguration, given $\mathcal{F}$, $U$, and two hitting sets $S$ and $T$ of size $k$, we are requested to determine the existence of a TJ-sequence of hitting sets from $S$ to $T$. Due to the equivalence between SET COVER and HITTING SET, HITTING SET RECONFIGURATION can be shown to be PSPACE-complete [IDH11]. Given $U$, $\mathcal{F}$, and a set size $k \in [n]$, we construct a graph $G = (V, E, \tau)$ according to the following procedure [NNUW13]:

Construction of $G = (V, E, \tau)$ from $U, \mathcal{F}, k$ [NNUW13].

Step 1. Create vertex sets $V_U \doteq \{v_u : u \in U\}$, $W_F \doteq \{w_F : F \in \mathcal{F}\}$, and an isolated vertex $x$.
Define $V \doteq V_U \cup W_F \cup \{x\}$.

Step 2. Create an edge $(v_u, w_F)$ for each $v_u \in V_U$ and $w_F \in W_F$ such that $u \in F$.

Step 3. Connect $x$ to all vertices in $W_F$; i.e., create edges $(x, w_F)$ for each $w_F \in W_F$.

Step 4. Render $V_U \cup \{x\}$ a clique; i.e., create edges between every pair of vertices of $V_U \cup \{x\}$.

Step 5. Set $\tau(v_u) \doteq |\{F \in \mathcal{F} : u \in F\}| + k + 1$ for each $v_u \in V_U$, $\tau(w_F) \doteq 1$ for each $w_F \in W_F$, and $\tau(x) \doteq |W_F| + k$.

Since $V_U \cup \{x\}$ forms a clique and $W_F$ forms an independent set, $G$ is a split graph. Moreover, the diameter of $G$ is 2 as $x$ is adjacent to every other vertex. Nichterlein, Niedermeier, Uhlmann, and Weller [NNUW13] proved that there exists a size-$k$ hitting set if and only if there exists a size-$k$ target set of $G$, implying the NP-hardness of Target Set Selection on split graphs of diameter 2. In particular, we use the following fact to prove Theorem 4.4.
Lemma 4.5 (Nichterlein et al. [NNUW13]). For a set family $\mathcal{F} = \{F_1, \ldots, F_m\}$ of a universe $U = \{u_1, \ldots, u_n\}$ and a positive integer $k \in [n]$, let $G = (V, E, \tau)$ be a graph constructed from $U, \mathcal{F}, k$ according to the procedure described above. Then, any size-$k$ target set $S$ of $G$ is a subset of $V_U$; i.e., it does not intersect $W_F$ or include $x$. Moreover, $S \subseteq U$ is a size-$k$ hitting set if and only if $V_S \doteq \{v_u : u \in S\}$ is a size-$k$ target set. 

Proof. In [NNUW13, Proof of Theorem 1], it is shown that a size-$k$ target set $S$ never includes the vertex $x$. Suppose then that $S$ includes a vertex $w_F$ of $W_F$. By Observation 2.9 and the fact that $\tau(w_F) = 1$, $S \setminus \{w\} \cup \{x\}$ must be a target set, which is a contradiction. See [NNUW13, Proof of Theorem 1] for the proof of the equivalence between a size-$k$ hitting set and a size-$k$ target set. $\square$

Proof of Theorem 4.4. We present a polynomial-time reduction from Hitting Set Reconfiguration, which is a PSPACE-complete problem [IDH+11]. Let $\mathcal{F}$ be a set family of a universe $U$ and $X$ and $Y$ be two hitting sets of size $k$. Let $G$ be a graph constructed from $U, \mathcal{F}, k$ according to the procedure described above in polynomial time. Define $V_X \doteq \{v_x \in V_U : x \in X\}$ and $V_Y \doteq \{v_y \in V_U : y \in Y\}$. Given a $\mathcal{TJ}$-sequence $S$ of size-$k$ hitting sets from $X$ to $Y$, we have that the sequence $\langle \{v_u : u \in S\} \rangle_{S \in S}$ is a $\mathcal{TJ}$-sequence of size-$k$ target sets from $V_X$ to $V_Y$ by Lemma 4.5. On the other hand, given a $\mathcal{TJ}$-sequence $\mathcal{T}$ from $V_X$ to $V_Y$, we have that the sequence $\langle \{u : v_u \in T\} \rangle_{T \in \mathcal{T}}$ is a $\mathcal{TJ}$-sequence of size-$k$ hitting sets from $X$ to $Y$ by Lemma 4.5. Consequently, $X$ and $Y$ are $\mathcal{TJ}$-reconfigurable on $\mathcal{F}$ if and only if $V_X$ and $V_Y$ are $\mathcal{TJ}$-reconfigurable on $G$, which completes the proof. $\square$

5 Discussion

Our results follow a typical pattern, that is, that an NP-complete (resp. P) search problem induces a PSPACE-complete (resp. P) reconfiguration problem, which however left some open questions. One of the important unsettled cases is cubic graphs of threshold 2, which is equivalent to FVS-R on cubic graphs and was mentioned by Suzuki in the open problem session of the 3rd International Workshop on Combinatorial Reconfiguration.5 Since the respective search problem can be solved in polynomial time using a graphic matroid parity algorithm [TU15, KLV17], a polynomial-time algorithm might be expected. We stress that PSPACE-completeness has been shown for cubic graphs of threshold 1 and 2 in this paper. A superclass of trees is another case whose complexity remains open; e.g., VC-R is known to be polynomial-time solvable on cacti [MNRS18] but is PSPACE-complete on $O(1)$-treewidth graphs [Wro18]. The complexity status for claw-free graphs [BKW14, Mun17] is also left unanswered.

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5https://pagesperso.g-scop.grenoble-inp.fr/~bousquen/CoRe_2019/CoRe_2019_Open_Problems.pdf
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A Missing Proofs

Proof of Observation 3.6. We reduce from Minimum Vertex Cover Reconfiguration on graphs of degree 2 and 3. To this end, we introduce the following gadget (see Figure 9):

**Construction of Σ-gadget (Figure 9).**

**Step 1.** Create a vertex set \( V \equiv \{r, t_1, t_2, t_3, t_4\} \).

**Step 2.** Create an edge set \( E \equiv \{(r, t_1), (r, t_3), (t_1, t_2), (t_1, t_4), (t_2, t_3), (t_2, t_4), (t_3, t_4)\} \).

Similar gadgets can be found in [HD05, HD09]. We here call this gadget a Σ-gadget. Observe that a Σ-gadget is planar. We say that a Σ-gadget is connected to vertex \( v \) if there exists an edge \((r, v)\).

Figure 10 lists minimum vertex covers of a Σ-gadget. We define \( M \equiv \{r, t_2, t_4\} \), which is the minimum vertex cover drawn in Figure 10a. We then have the following.

**Lemma A.1.** Let \( G = (V, E) \) be a graph and \( G' = (V', E') \) be a graph obtained from \( G \) by connecting a Σ-gadget \( R \) to a vertex \( v \) of \( G \). Then, a vertex set \( S \subseteq V \) is a minimum vertex cover of \( G \) if and only if \( S \uplus M \) is a minimum vertex cover of \( G' \). Moreover, two minimum vertex covers \( X \) and \( Y \) of \( G \) are TJ-reconfigurable on \( G \) if and only if \( X \uplus M \) and \( Y \uplus M \) are TJ-reconfigurable on \( G' \).

**Proof.** Observing the following facts suffices to ensure the statement:
(1) if $S \subseteq V$ is a minimum vertex cover of $G$, then $S \cup M$ is a minimum vertex cover of $G'$;

(2) if $S' \subseteq V'$ is a minimum vertex cover of $G'$, then $S' \cap V$ is a minimum vertex cover of $G$ and $S' \cap V(R)$ is a minimum vertex cover of $R$;

(3) $M$ is not TJ-reconfigurable to any other minimum vertex cover of $R$ (see Figure 10).

The reduction from a planar graph of degree 2 and 3 to a planar 3-regular graph is presented below.

**Reduction from planar graph $G$ of degree 2 and 3 to planar 3-regular graph $H$.**

**Step 1.** Connect a $\Sigma$-gadget to each degree-2 vertex $v$ of $G$ to obtain a graph $H$. Let $M_v$ denote the minimum vertex cover of the $\Sigma$-gadget connected to vertex $v$ defined above.

Obviously, the reduction completes in polynomial time, and $H$ is a planar 3-regular graph. Let $X$ and $Y$ be two minimum vertex covers of $G$. By applying Lemma A.1 repeatedly, we have that $X$ and $Y$ are TJ-reconfigurable on $G$ if and only if $X \cup \bigcup_v M_v$ and $Y \cup \bigcup_v M_v$ are TJ-reconfigurable on $H$. Since Minimum Vertex Cover Reconfiguration on planar graphs of degree 2 and 3 is PSPACE-complete [HD05], we obtain the desired result.

**Proof of Lemma 3.9.** One can verify that the seed set $\{r, t_{1,2}, t_{2,3}\}$ is a target set of $R$, which is drawn in Figure 11: $r$, $t_{1,2}$, and $t_{2,3}$ become activated initially; $t_{1,1}$, $t_{1,3}$, and $t_{2,2}$ become activated at step 1; $t_{2,1}$ becomes activated at step 2; $t_{2,6}$ becomes activated at step 3; $t_{1,6}$ becomes activated at step 4; $t_{1,5}$ becomes activated at step 5; $t_{2,5}$ and $t_{1,4}$ become activated at step 6; $t_{2,4}$ becomes activated at step 7. Showing that any size-2 seed set is not a target set of $R'$ is sufficient to prove the statement. To this end, we exhaustively enumerate all possible size-2 seed sets in Figures 12 and 13, where a black circle • denotes a seed, and a cross-hatched circle denotes an activated vertex. Note that we have omitted some seed sets that are identical due to the symmetry of $R'$; e.g., $\{t_{1,1}, t_{2,6}\}$ and $\{t_{1,5}, t_{2,6}\}$ are identical.
Figure 11: A minimum target set of Θ-gadget.
Figure 12: Size-2 seed sets for $\Theta$-gadget. Here, $\bullet$ denotes a seed, and a cross-hatched circle denotes an activated vertex.
Figure 13: Size-2 seed sets for \( \Theta \)-gadget (cont'd).