NUMBERS AND THE HEIGHTS OF THEIR HAPPINESS

MAY MEI AND ANDREW READ-MCFARLAND

ABSTRACT. A generalized happy function, $S_{e,b}$ maps a positive integer to the sum of its base $b$ digits raised to the $e^{th}$ power. We say that $x$ is a base $b$, $e$ power, height $h$, $u$ attracted number if $h$ is the smallest positive integer so that $S_{e,b}^h(x) = u$. Happy numbers are then base 10, 2 power, 1 attracted numbers of any height. Let $\sigma_{h,e,b}(u)$ denote the smallest height $h$, $u$ attracted number for a fixed base $b$ and exponent $e$ and let $g(e)$ denote the smallest number so that every integer can be written as $x_1^e + x_2^e + ... + x_{g(e)}^e$ for some nonnegative integers $x_1, x_2, ..., x_{g(e)}$. In this paper we prove that if $p_{e,b}$ is the smallest nonnegative integer such that $b^{p_{e,b}} > g(e)$, $d = \left\lceil \frac{g(e) + 1}{1 - \frac{b}{b-1} e} + e + p_{e,b} \right\rceil$, and $\sigma_{h,e,b}(u) \geq b^d$, then $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

1. Introduction

Let $S_{e,b}$ be the function that maps a positive base $b$ integer to the sum of its digits raised to the $e^{th}$ power, where $e$ is a positive integer. That is for $x = \sum_{i=0}^{n-1} a_i b^i$, with $0 \leq a_i \leq b-1$ for all $i$,

$$S_{e,b} \left( \sum_{i=0}^{n-1} a_i b^i \right) = \sum_{i=0}^{n-1} a_i^e.$$ 

If $S_{e,b}^h(x) = 1$ for some integer $h$, then $x$ is said to be an $e$ power, $b$ happy number.

In [1], Guy gives the smallest 2 power, 10 happy numbers of heights 0 through 6 and asks if 78999 is the smallest height 7 happy number. Grundman and Teeple answer Guy in [2], giving the smallest 2 power, 10 happy numbers of heights 0-10, and 3 power, 10 happy numbers of heights 0-8. From Grundman and Teeple’s work, one can extract an algorithm for finding the smallest happy number of height $h + 1$ if the smallest happy number of height $h$ is known. The main results of this paper are Theorems 3.1 and 3.3 which jointly imply that once the smallest height $h + 1$, $u$ attracted, base $b$ number is sufficiently large, applying $S_{e,b}$ to that number will yield the smallest height $h$, $u$ attracted, base $b$ number. The results of this paper hold not only for happy numbers (i.e. 1 attracted), but more generally for $u$ attracted numbers. Moreover, our results hold for all bases and exponents.

Definition 1.1. For a fixed base $b$, exponent $e$, and positive integer $u$, we say that a positive integer $x$ is $u$ attracted if $S_{e,b}^n(x) = u$ for some nonnegative integer $n$. If $h$ is the smallest nonnegative integer so that $S_{e,b}^h(x) = u$ then $x$ is height $h$, $u$ attracted number. (As a convention, $S_{e,b}^0(x) = x$.)
Definition 1.2. For a fixed base \( b \), exponent \( e \), positive integer \( u \), and nonnegative integer \( h \), let \( \sigma_{h,e,b}(u) \) denote the smallest height \( h \), \( u \) attracted number. That is, the smallest positive integer \( k \) with the property that \( S_{e,b}^h(k) = u \) and \( S_{e,b}^n(k) \neq u \) for \( n < h \). Similarly, for positive \( h \), let \( \tau_{h,e,b}(u) \) denote the second smallest height \( h \), \( u \) attracted number. That is, \( S_{e,b}^h(l) = u \), \( S_{e,b}^n(l) \neq u \) for \( n < h \), and \( \sigma_{h,e,b}(u) < l \).

Some of the following proofs rely upon knowing the smallest integer \( x \) such that for a given \( e \), every integer is expressible as the sum of at most \( x \) many integers raised to the \( e \)th power. We define \( g(e) \) for this purpose.

Definition 1.3. For a fixed positive integer \( e \), let \( g(e) \) denote the smallest integer such that every nonnegative integer is expressible as \( x_1^e + x_2^e + \ldots + x_{g(e)}^e \) where \( x_1, x_2, \ldots, x_{g(e)} \) are all nonnegative integers.

This is the well-known Waring’s problem. Many surveys about the history of this problem exist, see for instance [3].

For the entirety of this paper, we assume that base \( b \geq 2 \) is an integer, exponent \( e \geq 1 \) is an integer, height \( h \) is a nonnegative integer, attractor \( u \) is a positive integer, and that \( x \) denotes a positive integer. Additionally, when we say \( \lfloor x \rfloor = y \) we mean that \( y \) is the smallest integer such that \( y \geq x \), and similarly, if \( \lceil x \rceil = y \), then \( y \) is the largest integer such that \( y \leq x \).

2. Mapping Attracted Numbers

In this section, we establish in Theorem 2.2 a criterion, depending on \( g(e) \) that ensures that \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u) \) for a fixed base \( b \), exponent \( e \), height \( h \), and integer \( u \).

Lemma 2.1. Fix a base \( b \), exponent \( e \), and attractor \( u \). The smallest positive integer \( x \) such that \( S_{e,b}(x) = u \) has \( n \) digits, where \( \frac{u}{b^{e+1}} \leq n \leq \frac{u}{b^e} + g(e) \).

Proof. Since the maximum value of the image of each digit under \( S_{e,b} \) is \( \frac{(b-1)^e}{b^{e+1}} \), \( \frac{u}{b^{e+1}} \) is a lower bound for the number of digits of \( x \). Let \( q \) and \( r \) be the quotient and remainder of \( u \) divided by \( (b-1)^e \), respectively, that is \( q \) is a nonnegative integer, \( 0 \leq r < (b-1)^e \), and \( u = q(b-1)^e + r \). Let \( x_1, \ldots, x_{g(e)} \) be integers such that \( x_1^e + \ldots + x_{g(e)}^e = r \). Since \( r < (b-1)^e \), \( x_1, \ldots, x_{g(e)} < b-1 \) and so are valid digits in base \( b \). Without loss of generality, \( x_1 \leq x_2 \leq \ldots \leq x_{g(e)} \). Let \( y \) be the positive integer formed by the digits \( x_1, x_2, \ldots, x_{g(e)} \) followed by \( q \) digits, each of which is \( b-1 \). Since \( x \) is minimal, it follows that \( x \leq y \). So \( n \), the number of digits of \( x \), must be less than or equal to the number of digits of \( y \), which is \( \lfloor \frac{u}{(b-1)^e} \rfloor + g(e) \). \( \square \)

Theorem 2.2. Fix a base \( b \), exponent \( e \), positive height \( h \), and attractor \( u \). If \( \frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) \leq \frac{\tau_{h,e,b}(u)}{(b-1)^e} \), then \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u) \).

Proof. Let \( x \) be the smallest integer such that \( S_{e,b}(x) = \sigma_{h,e,b}(u) \). Let \( z \) be the height \( h \), \( u \) attracted number that is greater than \( \sigma_{h,e,b}(u) \) (recall that \( \tau_{h,e,b} \) is the smallest such number) and \( y \) any integer such that \( S_{e,b}(y) = z \). That is, \( y \) is a height \( h+1 \), \( u \) attracted number whose image is not \( \sigma_{h,e,b}(u) \). Let \( n \) be the number of digits of \( x \) and \( m \) be the number of digits of \( y \). We will show that \( x < y \). By Lemma 2.1 \( n \leq \frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) \) and \( \frac{\tau_{h,e,b}(u)}{(b-1)^e} \leq m \). By hypothesis, \( \frac{\sigma_{h,e,b}(u)}{(b-1)^e} + g(e) \leq \frac{\tau_{h,e,b}(u)}{(b-1)^e} \), so \( n \leq m \). If \( n < m \), then \( x < y \), so let us suppose that
$n = m$. It must then be the case that $\frac{g(e)}{(b-1)^{\sigma_{u}(n)}} + g(e) = \frac{\sigma_u(n)}{(b-1)^{\sigma_u(n)}}$. Since $S_{e,b}(y) = z$ and $y$ has $m = \frac{\sigma_u(n)}{(b-1)^{\sigma_u(n)}}$ digits, $y$ is the concatenation of $m$ digits, each of which is $b - 1$. Since $x \neq y$ (as they have different images under $S_{e,b}$) and $x$ and $y$ have the same number of digits, at least one digit of $x$ is not $b - 1$. Thus, $x < y$. Hence $x$ is less than every other digit $h + 1$, $w$-attracted number, and so $x = \sigma_{h+1,e,b}(u)$. Since $S_{e,b}(x) = \sigma_{h,e,b}(u)$, $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

From [2], it is known that $\sigma_{7,2,10} = 78999$ and $\sigma_{7,2,10}(1) = 79899$.

**Question 2.3.** Under what conditions is $\tau_{h,e,b}(u)$ a permutation of the digits of $\sigma_{h,e,b}(u)$?

### 3. Large $u$ Attracted Numbers

In this section, we prove Theorems 3.1 and 3.3 which imply that once $\sigma_{h,e,b}(u)$ is sufficiently large, $S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u)$.

**Theorem 3.1.** Fix a base $b$, exponent $e$, positive height $h$, and attractor $u$. Let $\delta$ be a positive integer, and let

$$d = \frac{g(e) + 1}{1 - (\frac{b-1}{b})^\delta} + \delta.$$  

If $\sigma_{h,e,b}(u)$ has at least $d$ digits, then the base $b$ expansion of $\sigma_{h,e,b}$ is of the form

$$\sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i$$

with $a_0, ..., a_\delta = b - 1$. More informally, the rightmost $\delta + 1$ digits of $\sigma_{h,e,b}(u)$ are all $b - 1$.

**Proof.** In this proof, we will show that if $\sigma_{h,e,b}$ has “too many” digits which are not equal to $b - 1$, we can construct a smaller number with the same image as $\sigma_{h,e,b}$. This contradicts the definition of $\sigma_{h,e,b}$.

One can verify $\sigma_{1,e,b}(1) = 10$ (in base $b$) for all $e, b$ and that this is the only number of the form $\sigma_{h,e,b}$ with a 0 digit. However, 10 is a 2 digit number and $d > 2$ for integers $e > 1$. Thus, using the base $b$ expansion from the statement of the theorem, $a_{i+1} \leq a_i$ for $0 \leq i < n - 1$ (its digits must appear in increasing order from left to right) and none of its digits can be 0 since $\sigma_{h,e,b}(u)$ is the least height $h, u$ attracted number.

In the case $a_i = b - 1$ for all $i$, this theorem is trivially true. Otherwise, let us construct $z$, the sum of the image of the digits which are not equal to $b - 1$. In the case that some digits of $\sigma_{h,e,b}(u)$ are $b - 1$ and some are not, define an integer parameter $k \geq 2$ to be such that $a_{k-1} < b - 1$ and for all $i < k - 1$, $a_i = b - 1$. That is, the $k$th place is the first (from the right) in which a digit that is not $b - 1$ appears. Hence,

$$\sigma_{h,e,b}(u) = \sum_{i=k-1}^{n-1} a_i b^i + \sum_{i=0}^{k-2} (b-1)b^i.$$  

Let $y = S_{e,b}(\sigma_{h,e,b}(u))$ and let $z = y - (k-1)(b-1)^\epsilon$, that is,

$$z = \sum_{i=k-1}^{n-1} a_i^\epsilon.$$
In the case that no digits of \( \sigma_{h,e,b} \) are \( b-1 \), set \( k = 1 \) and let \( z = \sum_{i=0}^{n-1} a_i^e \). We proceed to show that if \( k \leq \delta + 1 \), we can construct a number smaller than \( \sigma_{h,e,b} \) with the same image as \( \sigma_{h,e,b} \), a contradiction. Let \( n' = n - (k - 1) \) and let
\[
m = \left\lfloor \frac{z}{(b-1)^{k-1}} \right\rfloor.
\]
Since \( z \) is the sum of \( n' \) many terms of the form \( a_i^e \) where \( a_i \leq b-2 \) for all \( i \), \( n' \geq \frac{z}{(b-1)^{k-1}} \). Thus, \( \frac{(k-2)^{n'}}{(b-1)^{n'}} \geq m \).

So,
\[
\left( \frac{b-2}{b-1} \right)^e n' + g(e) + 1 \geq m + g(e) + 1.
\]
By the definition of \( d \), \( d - \delta = \frac{g(e)+1}{1-(\frac{k-2}{b-1})e} \), and since \( k \leq \delta + 1 \), \( d - (k - 1) \geq \frac{g(e)+1}{1-(\frac{k-2}{b-1})e} \).

Thus,
\[
(d - (k - 1)) \left( 1 - \left( \frac{b-2}{b-1} \right)^e \right) \geq g(e) + 1.
\]
And since \( n' \geq d - (k - 1) \) and \( 1 - (\frac{b-2}{b-1})^e > 0 \), we have that \( n'(1 - (\frac{b-2}{b-1})^e) \geq g(e) + 1 \) and hence
\[
n' \geq g(e) + 1 + n' \left( \frac{b-2}{b-1} \right)^e \geq m + g(e) + 1.
\]

Therefore, \( n' \geq m + g(e) + 1 \).

Let \( r \) be the remainder of \( y \) divided by \( (b-1)^e \), that is \( y = q(b-1)^e + r \) where \( q \geq 0 \) and \( (b-1)^e > r \geq 0 \). From the definition of \( m \), \( q = m + (k - 1) \). Let \( x_1, x_2, \ldots, x_{g(e)} \) be integers less than \( b-1 \) so that \( x_1^e + x_2^e + \ldots + x_{g(e)}^e = r \). There are such \( x_j \) since \( g(e) \) is defined so that such integers exist, and all integers must be less than \( b-1 \) since \( r < (b-1)^e \). Without loss of generality, \( x_1 \leq x_2 \leq \ldots \leq x_{g(e)} \).

Let \( x \) be a base \( b \) number with digits \( x_1, \ldots, x_{g(e)} \) followed by \( m + (k - 1) \) many \( b-1 \) digits.

Hence, \( S_{e,b}(x) = y \), and \( x \) has at most \( g(e) + m + (k - 1) \) digits. Since \( n' = n - (k - 1) \), \( n \geq g(e) + 1 + m + (k - 1) \). However, this means that \( x \) has fewer digits than \( \sigma_{h,e,b}(u) \). This contradicts the fact that \( \sigma_{h,e,b}(u) \) is the smallest height \( h, u \) attracted integer, and hence, \( k > \delta + 1 \).

For ease of notation, we define a constant \( p_{e,b} \).

**Definition 3.2.** For a fixed exponent \( e \) and base \( b \), let \( p_{e,b} \) be the smallest integer such that \( b^{p_{e,b}} > g(e) \).

**Theorem 3.3.** Fix a base \( b \), exponent \( e \), positive height \( h \), and attractor \( u \). If \( \sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i \) where \( a_0, \ldots, a_{e+p_{e,b}} = b-1 \), then \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u) \).

**Proof.** Let \( \sigma_{h,e,b}(u) \) be such that \( a_0, \ldots, a_k = b-1 \) where \( k \geq e + p_{e,b} \). Define \( c_j = \sigma_{h,e,b}(u) + j \) for \( 1 \leq j < g(e)(b-1)^e \). We will show that \( c_1 \) through \( c_{g(e)(b-1)^e-1} \) are not height \( h, u \) attracted numbers.

If \( b > 2 \), using the definition of \( p_{e,b} \) we get
\[
j < g(e)(b-1)^e < b^{p_{e,b}}(b-1)^e < b^{p_{e,b}+e} = b^{e+p_{e,b}}.
\]
Since \( \sigma_{h,e,b} \) has at least \( e + p_{e,b} + 1 \) trailing digits equal to \( b-1 \), \( c_1 \) has at least \( e + p_{e,b} + 1 \) trailing zeros. Since \( j < b^{e+p_{e,b}} \), \( j \) has at most \( e + p_{e,b} \) many digits. Hence \( c_j \) has at least one digits which is zero for \( 1 \leq j < g(e)(b-1)^e \). Let \( c'_j \) be formed by removing the all zero digits of \( c_j \). We claim that \( c'_j < \sigma_{h,e,b}(u) \).

Recall that \( n \) denotes the number of digits of \( \sigma_{h,e,b}(u) \). If \( a_i \neq b-1 \) for some \( i \), then \( n \geq e + p_{e,b} + 2 \) and \( c_j \) has \( n \) digits for all \( j \). Thus, \( c'_j \) has at most \( n - 1 \)
Proof. Since \( S = \sigma_{h,e,b}(u) \) by Theorem 3.3, \( 1, \) and since \( \sigma \) by removing the all zero digits of \( c \), which means that \( c_j < b^{e+p_{e,b}+1} + b^{e+p_{e,b}} \). Thus \( c'_j \) has at most \( n \) digits, while the leading digit of \( \sigma_{h,e,b} \) is \( b-1 \), but the leading digit of \( c'_j \) is 1, and since \( b \neq 2 \), \( c'_j < \sigma_{h,e,b} \).

This leaves only the case that \( b = 2 \). In this case,
\[
j < g(e)(2-1)^c = g(e) < 2^{p_{e,2}}.
\]
Since the only allowable digits are 0 and 1, and we argued in the proof of Theorem 3.1 that \( c_j \) is at least 1, \( c_j \) has at least 2 digits, that are equal to 0. Again, let \( c'_j \) be formed by removing the all zero digits of \( c_j \). Then \( c'_j \) has fewer than \( n \) digits and hence \( c'_j < \sigma_{h,e,b} \).

So, if any \( c_j \) are height \( h \), \( u \)-attracted numbers, then \( c'_j \) is a smaller height \( h \), \( u \) attracted number than \( \sigma_{h,e,b}(u) \), contradicting the definition of \( \sigma_{h,e,b}(u) \). Hence, \( \tau_{h,e,b}(u) \geq g(e)(b-1)^c + \sigma_{h,e,b}(u) \). Therefore, by Theorem 2.2 \( S_{e,b}(\sigma_{h+1,e,b}) = \sigma_{h,e,b} \).

**Corollary 3.4.** Fix a base \( b \) and exponent \( e \). Let \( d = \lceil \frac{g(e)+1}{1-b^{-1}} \rceil + e + p_{e,b} \). If \( \sigma_{h,e,b}(u) \geq b^d \), then \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u) \).

**Proof.** Since \( \sigma_{h,e,b}(u) \geq b^d \), \( \sigma_{h,e,b}(u) \) must have at least \( d - 1 \) digits. Hence, by Theorem 3.1 \( \sigma_{h,e,b}(u) = \sum_{i=0}^{n-1} a_i b^i \) where for \( i \leq e + p_{e,b}, a_i = b - 1 \). Therefore, by Theorem 3.3 \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}(u) \).

**Corollary 3.4** gives a bound \( b^d \) for \( \sigma_{h,e,b}(u) \) (in terms of \( e \) and \( b \)) so that if \( \sigma_{h,e,b}(u) \geq b^d \), then \( S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b} \). This leads to the natural question:

**Question 3.5.** Is there a bound \( \beta \) for \( h \) (in terms of \( e \) and \( b \)) so that if \( h \geq \beta \)

\[
S_{e,b}(\sigma_{h+1,e,b}(u)) = \sigma_{h,e,b}?
\]

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**References**

[1] Richard K. Guy. *Unsolved problems in number theory.* Problem Books in Mathematics. Springer-Verlag, New York, third edition, 2004.

[2] H. G. Grundman and E. A. Teeple. Heights of happy numbers and cubic happy numbers. *Fibonacci Quart.*, 41(4):301–306, 2003.

[3] R. C. Vaughan and T. D. Wooley. Waring’s problem: a survey. In *Number theory for the millennium, III* (Urbana, IL, 2000), pages 301–340. A K Peters, Natick, MA, 2002.

Department of Mathematics & Computer Science, Denison University, 43023
E-mail address: meim@denison.edu

Department of Mathematics & Computer Science, Denison University, 43023
E-mail address: readmc_m1@denison.edu