Abstract

A new iterative technique is presented for solving of initial value problem for certain classes of multidimensional linear and nonlinear partial differential equations. Proposed iterative scheme does not require any discretization, linearization or small perturbations and therefore significantly reduces numerical computations. Rigorous convergence analysis of presented technique and an error estimate are included as well. Several numerical examples for high dimensional initial value problem for heat and wave type partial differential equations are presented to demonstrate reliability and performance of proposed iterative scheme.

Keywords: Partial differential equation, iterative technique, initial value problem, heat equation, wave equation, convergence analysis, error estimate

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1. Introduction

Many physical phenomena can be described by mathematical models that involve partial differential equations. Therefore, in recent years, researchers look for new numerical methods which are more cost effective and simple in implementation to solve partial differential equations. Investigation of exact and approximate solution helps us to understand meaning and relevance of these mathematical models. Several techniques including scattering method [1], sine-cosine method [2], homotopy analysis method [3, 4], homotopy perturbation method [5-9], differential transform method [10], variational iteration method [11, 12], or decomposition methods [13-20] have been used for solving these problems, but mostly for two dimensional partial differential equations.

Inspired and motivated by ongoing research in this area, we apply new iterative scheme for solving heat- and wave-type equations. Several examples are given to verify reliability and efficiency of proposed technique.

2. Preliminaries

Let $\Omega$ be a compact subset of $\mathbb{R}^k$. Denote $J = [-\delta, \delta] \times \Omega$, where $\delta > 0$ will be specified later, then $J$ is a compact subset of $\mathbb{R}^{k+1}$. Let $u(t,x) = u(t,x_1,\ldots,x_k)$ be a real function of $k+1$ variables defined on $J$. We introduce the following operators: $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$ and $D = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k})$. We deal with partial differential equations

$$\frac{\partial^m}{\partial t^m} u(t,x) = F(t,x,u,\nabla u,\ldots,\nabla^m u) \quad \text{for } m < n \quad (1)$$

and

$$\frac{\partial^n}{\partial t^n} u(t,x) = F(t,x,u,\nabla u,\ldots,\nabla^{n-1} u, D\nabla^{n-1} u, D^2\nabla^{n-1} u, \ldots, D^{m-(n-1)}\nabla^{n-1} u) \quad \text{for } m \geq n \quad (2)$$

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In both cases, left-hand side of the equation contains only the highest derivative with respect to $t$. We do not consider equations where the order of partial derivatives with respect to $t$ is $n$ or higher on the righthand side, including mixed derivatives.

When convenient, we will use multiindex notation as well:

$$\frac{\partial|\alpha|}{\partial x^\alpha} = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}},$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k$.

Denote $N = \max\{m, n\}$. We consider equation (1) or (2) with the set of initial conditions

$$u(0, x) = c_1(x),$$

$$\frac{\partial}{\partial t} u(0, x) = c_2(x),$$

$$\vdots$$

$$\frac{\partial^{n-1}}{\partial t^{n-1}} u(0, x) = c_n(x),$$

where initial functions $c_i(x), \ i = 1, \ldots, n$ are taken from space $C^N(\Omega, \mathbb{R})$. It means that we are looking for classical solutions.

For the purpose of clarity, we emphasize that our formulation covers for instance heat, wave, Burger, Boussinesq or Korteweg-de Vries (KdV) equations.

Obviously, $F: J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{(k+1)^2} \times \ldots \times \mathbb{R}^{(k+1)^2} \times \ldots \times \mathbb{R}^{(k+1)^2} \times \mathbb{R} \to \mathbb{R}$ if $m < n$, and $F: J \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^{(k+1)^2} \times \ldots \times \mathbb{R}^{(k+1)^2} \times \ldots \times \mathbb{R}^{(k+1)^2} \times \mathbb{R} \to \mathbb{R}$ if $m \geq n$. Denote

$$K_1 = \frac{(k + 1)^{n+1} - 1}{k} \quad \text{for } m < n$$

and

$$K_2 = \frac{(k + 1)^{n+1} - 1}{k} + (k + 1)^{n+1} - 1 \quad \text{for } m \geq n.$$ (5)

Then, if we consider $u$ as dependent variable, we see that $F$ is a function of $k + 1 + K_1$ variables in case $m < n$ or $k + 1 + K_2$ variables in case $m \geq n$.

Denote

$$u_0(t, x) = \sum_{i=1}^n c_i(x) \frac{t^{i-1}}{(i-1)!} = \sum_{i=1}^n \left( \frac{\partial^{i-1}}{\partial t^{i-1}} u(0, x) \right) \frac{t^{i-1}}{(i-1)!}.$$ (6)

Then $u_0 \in C^N(J, \mathbb{R})$.

We suppose that $F$ is Lipschitz continuous in last $K_1$ ($m < n$) or $K_2$ ($m \geq n$) variables, i.e. $F$ satisfies condition

$$|F(t, x, y_1, \ldots, y_K) - F(t, x, z_1, \ldots, z_K)| \leq L \left( \sum_{i=1}^K |y_i - z_i| \right), \quad l = 1 \text{ or } 2,$$ (7)

on a compact set which is defined as follows: There is $R \in \mathbb{R}, R > 0$ such that (7) holds on

$$J \times \prod_{a_0 + |\alpha| \geq m} [c_{a_0, \alpha}, d_{a_0, \alpha}],$$ (8)

where

$$c_{a_0, \alpha} = \min_{t, \alpha \in J} \left( \frac{\partial^{a_0 + |\alpha|}}{\partial t^{a_0} \partial x^\alpha} u_0(t, x) \right) - R, \quad d_{a_0, \alpha} = \max_{t, \alpha \in J} \left( \frac{\partial^{a_0 + |\alpha|}}{\partial t^{a_0} \partial x^\alpha} u_0(t, x) \right) + R,$$ (9)

and $a_0 < n$ in all cases.

Since $F$ is continuous on compact set, $|F|$ attains its maximal value on this set, denote it $M$. Then we put

$$\delta = \left( \frac{R \cdot (n-1)!}{M} \right)^{1/n}.$$ (10)
3. Main results

**Theorem 1.** Let the condition \((7)\) hold. Then problem consisting of equation \((1)\) and \((2)\) and initial conditions \((3)\) has a unique local solution on \((-\delta, \delta) \times \Omega\), where \(\delta\) is defined by \((10)\).

**Proof.** First define the following operator

\[
Tu(t, x) = u_0(t, x) + \int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} F(\xi, x, u(\xi, x), \nabla u(\xi, x), \ldots) d\xi,
\]

where function \(F\) has either \(k + 1 + K_1\) or \(k + 1 + K_2\) arguments and the last \(K_1\) or \(K_2\) arguments involve dependent variable \(u\).

Starting with equation \((1)\), respective \((2)\), and using repeated integration by parts, it can be easily proved that if \(u\) is a solution of equation \(u = Tu\), i.e., if \(u\) is a fixed point of operator \(T\), then it is a solution of problem \((1)\), \((3)\), respective \((2)\), \((3)\).

Denote \(J_1 = [-\delta_1, \delta_1] \times \Omega\), where \(0 < \delta_1 < \delta\). Then \(J_1\) is compact. Let \(C^N(J_1, \mathbb{R})\) be the space of functions from \(J_1\) to \(\mathbb{R}\) with continuous partial derivatives up to order \(N\). This space is a Banach space with respect to the norm

\[
\|u\|_{C^N} = \sum_{a_0 + |\alpha| \leq N} \max_{(t, x) \in J_1} \left| \frac{\partial^{\alpha} u}{\partial^a t^a x^a} (t, x) \right|.
\]

It is obvious that, considering space \(C^N\), the order of partial derivatives with respect to \(t\) is allowed to be greater than or equal to \(n\) in case \(m \geq n\) when calculating the norm. Further, we define closed ball \(B_R(u_0) \subseteq C^N(J_1, \mathbb{R})\) as follows:

\[
B_R(u_0) = \left\{ y \in C^N(J_1, \mathbb{R}) : \|y - u_0\|_{C^N} \leq R \right\}.
\]

It is not difficult to verify that \(F\) composed with any \(y \in B_R(u_0)\) and its appropriate derivatives satisfies Lipschitz condition \((7)\) and that upper bound \(|F| \leq M\) remains valid as well. Indeed, \(J_1 \subseteq J\) and if \(\|y - u_0\|_{C^N} \leq R\), then

\[
\left| \frac{\partial^{\alpha} u}{\partial^a t^a x^a} (y(t, x) - u_0(t, x)) \right| \leq R
\]

for all \((t, x) \in J_1\) and for all \(a_0 + |\alpha| \leq N\). Hence point \((t, x, y(t, x), \nabla y(t, x), \ldots \nabla^m y(t, x))\) or \((t, x, y(t, x), \nabla y(t, x), \ldots, D^{m-(n-1)} \nabla y(t, x))\) respectively lies in the set \(J \times \prod_{a_0 + |\alpha| \leq M} [c_0, a, \partial c_0, \alpha] \) defined by \((8)\) and \((9)\).

We need to show that \(T\) is a contraction on \(B_R(u_0)\) for sufficiently small \(\delta_1\).

In the first step, we prove that \(T\) maps \(B_R(u_0)\) into itself. Take any \(y \in B_R(u_0)\). Then

\[
\|Ty - u_0\|_{C^N} = \|a_0 + \int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} F(\xi, x, y(\xi, x), \nabla y(\xi, x), \ldots) d\xi - u_0\|_{C^N}
\]

\[
\leq \int_0^t \frac{M}{(n-1)!} F(\xi, x, y(\xi, x), \nabla y(\xi, x), \ldots) d\xi \|_{C^N} \leq \frac{\delta_1^{n-1}}{(n-1)!} M \|a_0 + \int_0^t F(\xi, x, y(\xi, x), \nabla y(\xi, x), \ldots) d\xi\|_{C^N}
\]

\[
\leq \frac{\delta_1^n}{(n-1)!} M < R \text{ for } 0 < \delta_1 < \left( \frac{R}{M} \right)^\frac{1}{n}.
\]

Thus \(Ty \in B_R(u_0)\) for \(0 < \delta_1 < \left( \frac{R}{M} \right)^\frac{1}{n}\).
The second step is to show that $T$ is a contraction. Choose arbitrary $y, z \in B_p(u_0)$. Then we have
\[
\|Ty - Tz\|_{C^N} = \left\| \int_0^\gamma \frac{(t - \xi)^{p-1}}{(n-1)!} \left[ F(\xi, x, y(\xi, x), \nabla y(\xi, x), \ldots) - F(\xi, x, z(\xi, x), \nabla z(\xi, x), \ldots) \right] d\xi \right\|_{C^N}
\leq \left\| \frac{\rho^{p-1}}{(n-1)!} \int_0^\gamma \left[ F(\xi, x, y(\xi, x), \nabla y(\xi, x), \ldots) - F(\xi, x, z(\xi, x), \nabla z(\xi, x), \ldots) \right] d\xi \right\|_{C^N}
\leq \frac{\delta_1^{p-1}}{(n-1)!} L \left( \sum_{\alpha + \beta < \infty} \left\| \frac{\partial^\alpha x(\xi, x) - \partial^\alpha \xi(\xi, x)}{(\nabla^\alpha \xi) (\nabla^\beta \xi)} \right\|_{C^N} \right)
\leq \frac{\delta_1^{p-1}}{(n-1)!} L \left( \sum_{\alpha + \beta < \infty} \left\| \frac{\partial^\alpha x(\xi, x) - \partial^\alpha \xi(\xi, x)}{(\nabla^\alpha \xi) (\nabla^\beta \xi)} \right\|_{C^N} \right) \int_0^\gamma d\xi \leq \frac{\delta_1^{p}}{(n-1)!} L \|y - z\|_{C^N},
\]
where $\alpha_0 < n$ for equation (3) and $L$ is a Lipschitz constant for $F$ introduced in (7). It follows that $T$ is a contraction for $0 < \delta_1 < \left( \frac{(p-1)\delta_0}{2L} \right)^{1/n}$.

Combining all results, we obtain that if $0 < \delta_1 \leq \min \left\{ \frac{\delta_1}{p} \left( \frac{L}{\sqrt{2}} \right)^{1/n}, \left( \frac{p-1}{2L} \right)^{1/n} \right\}$, then operator $T$ is a contraction on $B_p(u_0)$. Applying Banach contraction principle, we can conclude that $T$ has a unique fixed point in $B_p(u_0)$ which is a unique solution of problem (1), (3), respective (2), (3).

Since $\delta_1$ depends only on the Lipschitz constant $L$ and on the distance $R$ from initial data to the boundaries of the intervals $[c_{u_0}, d_{u_0}]$ wherein the estimate $M$ holds, we can apply our result repeatedly to get a unique local solution defined for $(t, x) \in (-\delta, \delta) \times \Omega$.

**Theorem 2.** Assume that condition (7) holds. Then iterative scheme $u_p = Tu_{p-1}$, $p \geq 1$ with initial approximation $u_0$ defined by (6), where $T$ is defined by (11), converges to unique local solution $u(t, x)$ of problem (1), (3), respective (2), (3). Moreover, we have the following error estimate for this scheme:
\[
\|u(t, x) - u_p(t, x)\|_{C^N} \leq \frac{R \cdot \gamma^p}{1 - \gamma}
\]  
(13)
on $(-\delta_1, \delta_1) \times \Omega$, where $\delta_1$ is chosen such that operator $T$ is a contraction,
\[
\gamma = \frac{L \cdot \delta_1}{(n-1)!},
\]  
(14)and constants $L$ and $R$ are defined by (7) and (8).

**Proof.** First we need to show that sequence $(u_p)_{p=0}^\infty$ is convergent. We prove it by showing that it is a Cauchy sequence. Take any $p, q \in \mathbb{N}$, $q \geq p$. Then
\[
\|u_q - u_p\|_{C^N} = \left\| \int_0^\gamma \frac{(t - \xi)^{p-1}}{(n-1)!} \left[ F(\xi, x, u_{q-1}(\xi, x), \nabla u_{q-1}(\xi, x), \ldots) - F(\xi, x, u_{p-1}(\xi, x), \nabla u_{p-1}(\xi, x), \ldots) \right] d\xi \right\|_{C^N}
\leq \frac{\delta_1^{p-1}}{(n-1)!} L \|u_{q-1} - u_{p-1}\|_{C^N} \leq \gamma \|u_{q-1} - u_{p-1}\|_{C^N},
\]
where $0 \leq \gamma \leq \frac{1}{q} < 1$. Put $q = p + 1$. We obtain
\[
\|u_{p+1} - u_p\|_{C^N} \leq \gamma \|u_p - u_{p-1}\|_{C^N} \leq \gamma^2 \|u_{p-1} - u_{p-2}\|_{C^N} \leq \ldots \leq \gamma^p \|u_1 - u_0\|_{C^N}.
\]
Now, using the triangle inequality, we get
\[
\|u_q - u_p\|_{C^N} = \|u_q - u_{q-1}\|_{C^N} + \ldots + \|u_{p+2} - u_{p+1}\|_{C^N} + \|u_{p+1} - u_p\|_{C^N} \leq (\gamma^p + 1 + \gamma^2 \gamma \gamma^p + \ldots + \gamma^p) \|u_1 - u_0\|_{C^N}
\leq \gamma^p \left( 1 + \gamma + \gamma^2 + \ldots + \gamma^{p-1} \right) \|u_1 - u_0\|_{C^N} \leq \gamma^p \frac{1 - \gamma^{p-1}}{1 - \gamma} \|u_1 - u_0\|_{C^N}.
\]
Since $\gamma < 1$, then $1 - \gamma^{n+p} < 1$ as well, and we estimate
\[
\|u_q - u_p\|_{C^\gamma} \leq \frac{\gamma^p}{1 - \gamma}\|u_1 - u_0\|_{C^\gamma} \leq \frac{\gamma^p}{1 - \gamma}\|Tu_0 - u_0\|_{C^\gamma} = \frac{\gamma^p}{1 - \gamma}\int_0^\infty (t - \xi)^{n-1} F(\xi, x, u_0(\xi, x), \nabla u_0(\xi, x), \ldots) d\xi \|_{C^\gamma}
\]
\[
\leq \frac{\gamma^p}{1 - \gamma} \cdot \frac{\delta_0^p}{(n-1)!} \cdot M \leq \frac{\gamma^p}{1 - \gamma} \cdot \frac{(\gamma^p)^2}{2} \cdot R.
\]
(15)

It follows that for arbitrary $\varepsilon > 0$ there is a $P \in \mathbb{N}$, $P > 1 - \log_2 \varepsilon + \log_2 R$ such that if $p, q \geq P$, then $\|u_q - u_p\|_{C^\gamma} < \varepsilon$.

Thus sequence $(u_p)_{p=0}^\infty$ is a Cauchy sequence and consequently a convergent sequence.

Then there is a limit $u = \lim_{p \to \infty} u_p$ such that $u = Tu$. Hence $u$ is a fixed point of operator $T$. Applying Theorem 1, we conclude that this fixed point is unique and it is a unique local solution of problem (1), (3), respective (2), (3).

Error estimate (13) follows immediately from (15) by taking a limit for $q \to \infty$.

\[ \square \]

**Corollary 1.** Let condition (7) be valid and suppose that $F$ can be written as $G + g$:
\[
F(t, x, z(t, x), \nabla z(t, x), \ldots) = G(t, x, z(t, x), \nabla z(t, x), \ldots) + g(t, x).
\]

Then we may choose initial approximation
\[
\bar{u}_0 = u_0 + \int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} g(\xi, x) d\xi = \sum_{i=1}^n \frac{\partial^{i-1}}{\partial t^{i-1}} u(0, x) \frac{t^{i-1}}{(i-1)!} + \int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} g(\xi, x) d\xi.
\]

(16)

**Proof.** Denote $M_2 = \max[|G| + |g|]$. According to the proof of Theorem 1, we only need to show that $\bar{u}_0 \in B_{g}(u_0)$, i.e., $\|\bar{u}_0 - u_0\|_{C^\gamma} \leq R$. Indeed, we have
\[
\|\bar{u}_0 - u_0\|_{C^\gamma} = \left\|\int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} g(\xi, x) d\xi - u_0\right\|_{C^\gamma} = \left\|\int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} g(\xi, x) d\xi\right\|_{C^\gamma}
\]
\[
\leq \left\|\int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} |g(\xi, x)| d\xi\right\|_{C^\gamma} \leq \frac{\delta_2^p}{(n-1)!} M_2 < R \text{ for } 0 < \delta_2 < \left(\frac{R \cdot (n-1)!}{M_2}\right)^{1/n}.
\]

Consequently, integral
\[
\int_0^t \frac{(t - \xi)^{n-1}}{(n-1)!} g(\xi, x) d\xi
\]
is small enough for sufficiently small $\delta_2$ and thus $\bar{u}_0 \in B_{g}(u_0)$ for this $\delta_2$.

\[ \square \]

**4. Applications**

We demonstrate potentiality of our approach on several initial value problems (IVP’s).

**Example 1.** Consider the following two-dimensional heat-type equation
\[
\frac{\partial u(x, y, t)}{\partial t} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + u(x, y, t) = (1 + t) \sinh (x + y)
\]
with initial condition
\[
u(x, y, 0) = \sinh (x + y).
\]

Then
\[
u_0(x, y, t) = \sum_{k=0}^{n-1} \frac{\partial^k u(x, y, 0)}{\partial x^k} \frac{t^k}{k!} + \int_0^\infty g(x, y, \xi) d\xi = \sinh (x + y) \left(1 + t + \frac{t^2}{2}\right)
\]
and

\[ u_p(x, y, t) = u_0(x, y, t) + \int_0^t \left( \frac{\partial^2 u_{p-1}(x, y, \xi)}{\partial x^2} - \frac{\partial^2 u(x, y, \xi)}{\partial y^2} - u(x, y, \xi) \right) d\xi, \quad p \geq 1. \]

Hence

\[ u_1(x, y, t) = \sinh(x + y) \left( 1 + t + \frac{t^2}{2} \right) - \int_0^t \sinh(x + y) \left( 1 + \xi + \frac{\xi^2}{2} \right) d\xi, \]

\[ u_2(x, y, t) = \sinh(x + y) \left( 1 + t + \frac{t^2}{2} \right) - \int_0^t \sinh(x + y) \left( 1 - \frac{\xi^3}{3!} \right) d\xi, \]

\[ u_3(x, y, t) = \sinh(x + y) \left( 1 + t + \frac{t^2}{2} \right) - \int_0^t \sinh(x + y) \left( 1 + \frac{\xi^2}{2} + \frac{\xi^4}{4!} \right) d\xi, \]

\[ u_4(x, y, t) = \sinh(x + y) \left( 1 + t + \frac{t^2}{2} \right) - \int_0^t \sinh(x + y) \left( 1 + \frac{\xi^2}{2} - \frac{\xi^3}{3!} - \frac{\xi^5}{5!} \right) d\xi, \]

\[ u_5(x, y, t) = \sinh(x + y) \left( 1 + t + \frac{t^2}{2} \right) - \int_0^t \sinh(x + y) \left( 1 + \frac{\xi^2}{2} - \frac{\xi^3}{3!} + \frac{\xi^4}{4!} + \frac{\xi^6}{6!} \right) d\xi, \]

\[ \vdots \]

We can see that so-called self-canceling terms appear between various components (see, for example, \( u_1, u_2, u_3 \)). Keeping the remaining non-cancelled terms, we have

\[ u(x, y, t) = \sinh(x + y) \left( 1 + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots \right) = (t + e^{-y}) \sinh(x + y), \]

which is unique exact solution of IVP [18], [19].

**Example 2.** Consider the following initial value problem for two-dimensional heat-type equation with variable coefficients

\[
\frac{\partial^2 u(x, y, t)}{\partial t^2} = \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \tag{20}
\]

with initial condition

\[ u(x, y, 0) = y^2. \tag{21} \]

Then

\[ u_0(x, y, t) = y^2 \]

and

\[ u_p(x, y, t) = y^2 + \int_0^t \left( \frac{\partial^2 u_{p-1}(x, y, \xi)}{\partial x^2} + \frac{\partial^2 u_{p-1}(x, y, \xi)}{\partial y^2} \right) d\xi, \quad p \geq 1. \]
Thus

\[ u_1(x, y, t) = y^2 + \int_0^t x^2 d\xi = y^2 + x^2 t, \]

\[ u_2(x, y, t) = y^2 + \int_0^t (y^2 + x^2) d\xi = y^2 \left( 1 + \frac{t^2}{2!} \right) + x^2 t, \]

\[ u_3(x, y, t) = y^2 + \int_0^t \left( y^2 + x^2 + \frac{x^2}{2} \right) d\xi = y^2 \left( 1 + \frac{t^2}{2!} \right) + x^2 \left( 1 + \frac{t^3}{3!} \right), \]

\[ u_4(x, y, t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \right) + x^2 \left( 1 + \frac{t^3}{3!} + \frac{t^5}{5!} \right), \]

\[ u_5(x, y, t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \right) + x^2 \left( 1 + \frac{t^3}{3!} + \frac{t^5}{5!} \right), \]

\[ \vdots \]

\[ u_{2l-1}(x, y, t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2l-2}}{(2l-2)!} \right) + x^2 \left( 1 + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2l-1}}{(2l-1)!} \right), \]

\[ u_2(x, y, t) = y^2 \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots + \frac{t^{2l-1}}{(2l-1)!} + \cdots \right) + x^2 \left( 1 + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2l}}{(2l)!} + \cdots \right) \]

\[ = x^2 \cosh t + y^2 \sinh t. \]

**Example 3.** Consider nonlinear wave-type equation

\[ \frac{\partial^2 u(x, y, t)}{\partial t^2} = 2x^2 + 2y^2 + \frac{15}{2} \left( \frac{\partial^2 u(x, y, t)}{\partial x^2} \right)^2 + 15 \left( \frac{\partial^2 u(x, y, t)}{\partial y^2} \right)^2 \]

with initial conditions

\[ u(x, y, 0) = \frac{\partial u(x, y, 0)}{\partial t} = 0. \]  

Then

\[ u_0(x, y, t) = \int_0^t (t - \xi)(2x^2 + 2y^2) d\xi = \dot{r}^2(x^2 + y^2) \]

and

\[ u_p(x, y, t) = \dot{r}^2(x^2 + y^2) + \frac{15}{2} \int_0^t (t - \xi) \left( x \left( \frac{\partial^2 u_{p-1}(x, y, \xi)}{\partial x^2} \right)^2 + y \left( \frac{\partial^2 u_{p-1}(x, y, \xi)}{\partial y^2} \right)^2 \right) d\xi, \]

\[ p \geq 1. \] From here we obtain the following iterations:

\[ u_1(x, y, t) = \dot{r}^2(x^2 + y^2) + 30 \int_0^t (t - \xi)(x^4 + y^4) d\xi = \dot{r}^2(x^2 + y^2) + \dot{r}^6(x + y), \]

\[ u_2(x, y, t) = \dot{r}^2(x^2 + y^2) + 30 \int_0^t (t - \xi)(x^4 + y^4) d\xi = \dot{r}^2(x^2 + y^2) + \dot{r}^6(x + y), \]

\[ \vdots \]

\[ u_l(x, y, t) = \dot{r}^2(x^2 + y^2) + \dot{r}^6(x + y), \quad l \geq 2. \]  

\[ (24) \]
Then we obtain required unique solution of \( u(x, y, t) \) as
\[
u(x, y, t) = t^2(x^2 + y^2) + \ell(x + y).
\]

**Example 4.** Consider three-dimensional wave-type equation with variable coefficients
\[
\frac{\partial^2 u}{\partial t^2} - \frac{1}{2} \left( \frac{x^2 \partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} \right) = x^2 + y^2 + z^2
\]
(25)
with initial conditions
\[
u(x, y, z, t) = 0, \quad \frac{\partial u(x, y, z, t)}{\partial t} = x^2 + y^2 - z^2.
\]
(26)

Then
\[
u_0(x, y, z, t) = t(x^2 + y^2 - z^2) + \int_0^t (t - \xi)(x^2 + y^2 + z^2) d\xi = t(x^2 + y^2 - z^2) + \frac{t^2}{2!}(x^2 + y^2 + z^2)
\]
and
\[
u_p(x, y, z, t) = t(x^2 + y^2 - z^2) + \frac{t^2}{2!}(x^2 + y^2 + z^2)
\]
\[+ \frac{1}{2} \int_0^t (t - \xi) \left( x^2 \frac{\partial^2 u_{p-1}}{\partial x^2} + y^2 \frac{\partial^2 u_{p-1}}{\partial y^2} + z^2 \frac{\partial^2 u_{p-1}}{\partial z^2} \right) d\xi, \quad p \geq 1.
\]

From here we get
\[
u_1(x, y, z, t) = t(x^2 + y^2 - z^2) + \frac{t^2}{2!}(x^2 + y^2 + z^2)
\]
\[+ \frac{1}{2} \int_0^t (t - \xi) \left( x^2 \frac{\partial^2 u_{p-1}}{\partial x^2} + y^2 \frac{\partial^2 u_{p-1}}{\partial y^2} + z^2 \frac{\partial^2 u_{p-1}}{\partial z^2} \right) d\xi
\]
\[= t(x^2 + y^2 - z^2) + \frac{t^2}{2!}(x^2 + y^2 + z^2) + \frac{t^3}{3!}(x^2 + y^2 - z^2) + \frac{t^4}{4!}(x^2 + y^2 + z^2)
\]
\[= (x^2 + y^2) \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \right) + z^2 \left( -t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} \right),
\]
\[
u_2(x, y, z, t) = (x^2 + y^2) \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} \right) + z^2 \left( -t + \frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^6}{6!} \right),
\]
\[\vdots
\]
\[
u_k(x, y, z, t) = (x^2 + y^2) \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^{2k+2}}{(2k+2)!} \right) + z^2 \left( -t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots - \frac{t^{2k+1}}{(2k+1)!} \right),
\]
\[\vdots
\]
Thus
\[
u(x, y, z, t) = \lim_{k \to \infty} \left[ (x^2 + y^2) \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{t^{2k+1}}{(2k+1)!} \right)
\]
\[+ z^2 \left( -t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots - \frac{t^{2k+1}}{(2k+1)!} \right) \right]
\[= (x^2 + y^2)e^t + z^2e^{-t} - (x^2 + y^2 + z^2)
\]
which is unique exact solution of IVP (25),(26).

**Remark 1.** Some above mentioned initial value problems have been solved using homotopy analysis method, homotopy perturbation method or Adomian decomposition method respectively (see [4],[8],[17],[19],[20]). However, in contrast to iterative technique proposed in this paper, these methods require complicated calculations of multidimensional integrals, high order derivatives or Adomian’s or He’s polynomials as it can be seen in cited papers.
5. Conclusion

- We conclude that iterative algorithm presented in this paper is a powerful and efficient analytical technique suitable for numerical approximation of a solution of initial problem for wide class of partial differential equations of arbitrary order.

- There is no need for calculating multiple integrals or derivatives, only one integration in each step is performed. Less computational work is demanded compared to other methods (Adomian decomposition method, variational iteration method, homotopy perturbation method, homotopy analysis method).

- Expected solution is a limit of a sequence of functions, in contrast to other frequently used methods where a sum of a functional series is considered. Consequently, the form of a solution can be immediately controlled in each step.

- All notations are carefully described and proofs are treated rigorously, compared to many recently presented algorithms and methods.

- Region and rate of convergence depend on Lipschitz constant for righthand side $F$.

- Using presented approach, we are able not only to obtain approximate solution, but even there is a possibility to identify unique solution of initial problem in closed form.

- A specific advantage of this technique over any purely numerical method is that it offers a smooth, functional form of the solution in each step.

- Another advantage is that using our approach we avoided discretization, linearization or perturbation of the problem.

- There is a possibility to reduce computational effort by combining presented algorithm with Laplace transform since there is a convolution integral inside the iterative formula.

- Finally, a subject of further investigation is to develop the presented technique for systems of PDE’s, to find modifications for solving equations with deviating arguments and for other types of problems (e.g. BVP’s).

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References

[1] V.O. Vakhnenko, E.J. Parkes, A.J. Morrison, A Backlund transformation and the inverse scattering transform method for the generalized Vakhnenko equation, Chaos Solitons Fractals 17 (4), 683–692, 2003.
[2] A.M. Wazwaz, Handbook of Differential Equations: Evolutionary Equations 4, Elsevier, 485–568, 2008.
[3] S.J. Liao, On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computation 147 (2), 499–513, 2004.
[4] H. Jafari, M. Saeidy, M. A. Firoozjaee, The Homotopy Analysis Method for Solving Higher Dimensional Initial Boundary Value Problems of Variable Coefficients, Numerical Methods for Partial Differential Equations, doi10.1002/nme.20471.
[5] J.H. He, Homotopy perturbation method: A new nonlinear technique, Applied Mathematics and Computation 135, 73–79, 2003.
[6] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, International Journal of Nonlinear Sciences and Numerical Simulation 6 (2), 207–208, 2005.
[7] S.Gupta, D.Kumar, J.Singh, Application of He’s homotopy perturbation method for solving nonlinear wave-like equations with variable coefficients, Journal of Advances in Applied Mathematics and Mechanics, Vol.1(2), 65–79, 2013.
[8] L. Jin, Homotopy perturbation method for solving partial differential equations with variable coefficients, Int. J. Contemp. Math. Sciences Vol. 3(28), 1395–1407, 2008.
[9] J. Singh, D. Kumar, A. Kilicman, Application of Homotopy Perturbation Sumudu Transform Method for Solving Heat and Wave-Like Equations, Malaysian Journal of Mathematical Science 7(1), 79–95, 2013.

[10] B. Soltanalizadeh, Application of differential transformation method for solving a fourth-order parabolic partial differential equations, International Journal of Pure and Applied Mathematics, vol.78(3), 299–308, 2012.

[11] J.H. He, Variational iteration method-a kind of non-linear analytical technique: Some examples, International Journal of Non-Linear Mechanics 34 (4), 699–708, 1999.

[12] J. Biazar, H. Ghazvini, He's variational iteration method for solving hyperbolic differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (3), 311–314, 2007.

[13] D.D. Ganji, A. Sadighi, I. Khatami, Assessment of two analytical approaches in some nonlinear problems arising in engineering sciences, Physics Letters A 372, 4399–4406, 2008.

[14] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994.

[15] D. Kaya, M. Aassila, An application for a generalized KdV equation by the decomposition method, Physics Letters A 299, 201–206, 2002.

[16] D.D. Ganji, E.M.M. Sadeghi, M.G. Rahmat, Modified Camassa-Holm and Degasperis-Procesi Equations Solved by Adomian’s Decomposition Method and Comparison with HPM and Exact Solutions, Acta Appl Math 104, 303–311, 2008.

[17] A.M. Wazwaz, Exact Solutions for Heat-Like and Wave-Like Equations with Variable Coefficients. Appl. Math. Comput. 149, 15–29, 2013.

[18] W. Abdul-Majid, The decomposition method for solving higher dimensional initial boundary value problems of variable coefficients, Int J Comput Math 76, 159–172, 2000.

[19] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994.

[20] S.T. Mohyud-Din, Solving heat and wave-like equations using He’s polynomials, Mathematical Problems in Engineering, Vol.2009, Article ID 427516, 1–12, 2009.

[21] Zhu, H., Shu, H., Ding, M., Numerical solutions of two-dimensional Burgers’ equations by discrete Adomian decomposition method, Comput. Math. Appl. 60 (3), 840–848, 2010.

[22] Biazar, J., Ghazvini, H., Homotopy perturbation method for solving hyperbolic partial differential equation, Comput. Math. Appl. 56 (2), 453–458, 2008.

[23] Chun, Ch., Jafari, H., Kim, Y., Numerical method for the wave and nonlinear diffusion equations with the homotopy perturbation method, Comput. Math. Appl. 57, 1226–1231, 2009.