Faithful Action on the Space of Global Differentials of an Algebraic Curve

Bernhard Köck

Abstract. Given a faithful action of a finite group on an algebraic curve of genus at least 2, we prove that the induced action on the space of global holomorphic differentials is faithful as well, except in the following very special case: the given action is not tame, the genus of the quotient curve is 0 and the characteristic of the base field is 2.

Mathematics Subject Classification 2000. 14H30; 14F10; 11R32.

Let \( X \) be a connected smooth projective algebraic curve over an algebraically closed field \( k \) equipped with a faithful action of a finite group \( G \) of order \( n \). Then \( G \) also acts on the vector space \( H^0(X, \Omega_X) \) of global holomorphic differentials on \( X \). A widely studied problem is to determine the structure of \( H^0(X, \Omega_X) \) as module over the group ring \( k[G] \). It goes back to the Chevalley-Weil when \( k = \mathbb{C} \), see \([CW]\). If the canonical projection \( \pi : X \to Y \) from \( X \) to the quotient curve \( Y = X/G \) is tamely ramified, a fairly explicit answer to this problem has been given in 1986 by Kani in \([Ka]\). For more recent answers to (related) questions in more general situations the reader is referred to the papers \([Bo]\) and \([FWK]\). In the case of arbitrary wild ramification the explicit calculation of the \( k[G] \)-isomorphism class of \( H^0(X, \Omega_X) \) is still an open problem.

This note is concerned with the weaker question whether \( G \) acts faithfully on the space \( H^0(X, \Omega_X) \). We give the following answer to this question. Let \( g_X \) and \( g_Y \) denote the genus of \( X \) and \( Y \), respectively, and let \( p \) denote the characteristic of \( k \).

Theorem. Suppose that \( G \) does not act faithfully on \( H^0(X, \Omega_X) \). Then \( g_X \in \{0, 1\} \) or all of the following three conditions hold:

(i) The projection \( \pi \) is not tamely ramified.
(ii) \( g_Y = 0 \).
(iii) \( p = 2 \).

The proof of this theorem will be given after the proof of Proposition 1 below.

For trivial reasons the group \( G \) does normally not act faithfully on \( H^0(X, \Omega_X) \) if \( g_X \in \{0, 1\} \), see parts (a) and (b) of the following example. That we also cannot expect the action of \( G \) on \( H^0(X, \Omega_X) \) to be faithful in the (very special) situation described by conditions (i), (ii) and (iii) of the above theorem, is explained in part (c) of the following example and in Proposition 2 below.

Example.

(a) If \( g_X = 0 \) and \( G \) is not the trivial group then \( G \) does obviously not act faithfully on \( H^0(X, \Omega_X) = \{0\} \).
(b) If \( g_X = 1 \) (that is if \( X \) is an elliptic curve) and if \( G \) is a finite subgroup of
Let $X(k)$ acting on $X$ by translation then $G$ leaves invariant any global non-vanishing holomorphic differential and hence $G$ acts trivially on $H^0(X, \Omega_X)$.

(c) Let $p = 2$ and let $r$ be an odd natural number. Let $k(x, y)$ be the cyclic field extension of the rational function field $k(x)$ of degree 2 given by the Artin-Schreier equation $y^2 - y = x^r$. Let $\pi : X \to \mathbb{P}^1_k$ be the corresponding cover of nonsingular curves over $k$. Then $\pi$ is not tamely ramified (see Example 2.5 on p. 1095 in [Kö]) and the Galois group $G = \mathbb{Z}/2\mathbb{Z}$ acts trivially on $H^0(X, \Omega_X)$. This follows from Proposition 2 below.

The next lemma is crucial for the proof of Proposition 1 which in turn is the main idea for the proof of our theorem. We begin by introducing some notations. For any $G$-invariant divisor $D$ on $X$ let $\mathcal{O}_X(D)$ denote the corresponding equivariant invertible $\mathcal{O}_X$-module, as usual. Furthermore let $\pi^G_*(\mathcal{O}_X(D))$ denote the subsheaf of the direct image $\pi_*\mathcal{O}_X(D)$ fixed by the obvious action of $G$ on $\pi_*\mathcal{O}_X(D)$ and let $\left\lfloor\frac{\pi_*D}{n}\right\rfloor$ denote the divisor on $Y$ obtained from the push-forward $\pi_*D$ by replacing the coefficient $m_Q$ of $Q$ in $\pi_*D$ with the integral part $\left\lfloor\frac{m_Q}{n}\right\rfloor$ of $\frac{m_Q}{n}$ for every $Q \in Y$. The function fields of $X$ and $Y$ are denoted by $K(X)$ and $K(Y)$, respectively. Finally, for any $P \in X$, let $e_P$ denote the ramification index of $\pi$ at $P$ and let $\text{ord}_P$ and $\text{ord}_Q$ denote the respective valuations of $K(X)$ and $K(Y)$ at $P$ and $Q := \pi(P)$.

**Lemma.** Let $D$ be a $G$-invariant divisor on $X$. Then the sheaves $\pi^G_*(\mathcal{O}_X(D))$ and $\mathcal{O}_Y\left(\left\lfloor\frac{\pi_*D}{n}\right\rfloor\right)$ are equal as subsheaves of the constant sheaf $K(Y)$ on $Y$. In particular the sheaf $\pi^G_*(\mathcal{O}_X(D))$ is an invertible $\mathcal{O}_Y$-module.

For the reader’s convenience we include a proof of this lemma although it may already exist in the literature.

**Proof.** For every open subset $V$ of $Y$ we have

$$\pi^G_*(\mathcal{O}_X(D))(V) = \mathcal{O}_X(D)(\pi^{-1}(V))^G \subseteq K(X)^G = K(Y).$$

In particular both sheaves are subsheaves of the constant sheaf $K(Y)$ as stated. It therefore suffices to check that their stalks are equal. Let $Q \in Y$, let $P \in \pi^{-1}(Q)$ and let $n_P$ denote the coefficient of $D$ at $P$. Then we have

$$\pi^G_*(\mathcal{O}_X(D))_Q = \mathcal{O}_X(D)_P \cap K(Y)$$

$$= \left\{ f \in K(Y) : \text{ord}_P(f) \geq -n_P \right\}$$

$$= \left\{ f \in K(Y) : \text{ord}_Q(f) \geq -\left\lfloor\frac{n_P}{e_P}\right\rfloor \right\}$$

$$= \mathcal{O}_Y\left(\left\lfloor\frac{\pi_*D}{n}\right\rfloor\right)_Q,$$

as desired. \qed

Let $R := \sum_{P \in X} \dim_k(\Omega_{X/Y}[P])$ denote the ramification divisor of $\pi$. The following proposition computes the dimension of the subspace of $H^0(X, \Omega_X)$ fixed by $G$. 

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Proposition 1.

\[ \text{dim}_k \left( H^0(X, \Omega_X)^G \right) = \begin{cases} 
    g_Y & \text{if } \deg \left( \frac{\pi_*(R)}{n} \right) = 0 \\
    g_Y - 1 + \deg \left( \frac{\pi_*(R)}{n} \right) & \text{if } \deg \left( \frac{\pi_*(R)}{n} \right) > 0.
\end{cases} \]

Proof. Let \( K_X \) be a \( G \)-invariant canonical divisor on \( X \), that is we have an equivariant isomorphism \( \mathcal{O}_X(K_X) \cong \Omega_X \). Let the divisor \( K_Y \) on \( Y \) be defined by the equality \( \pi^*(\Omega_Y) = \mathcal{O}_X(\pi^*(K_Y)) \) of subsheaves of \( \mathcal{O}_X(K_X) \). Note that we consider \( \pi^*(\Omega_Y) \) as a subsheaf of \( \Omega_X \cong \mathcal{O}_X(K_X) \) and that we have a short exact sequence

\[ 0 \to \pi^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0. \]

In particular we have \( K_X = \pi^*K_Y + R \) and hence

\[ \left\lceil \frac{\pi_*(K_X)}{n} \right\rceil = \left\lceil \frac{\pi_*\pi^*(K_Y) + \pi_*(R)}{n} \right\rceil = K_Y + \left\lceil \frac{\pi_*(R)}{n} \right\rceil. \]

Using the previous lemma we conclude that

\[ \pi_*^G(\Omega_X) \cong \mathcal{O}_Y \left( K_Y + \left\lceil \frac{\pi_*(R)}{n} \right\rceil \right) \]

and finally

\[ \text{dim}_k \left( H^0(X, \Omega_X)^G \right) = \text{dim}_k \left( H^0(Y, \pi_*^G(\Omega_X)) \right) = \text{dim}_k \left( H^0 \left( Y, \mathcal{O}_Y \left( K_Y + \left\lceil \frac{\pi_*(R)}{n} \right\rceil \right) \right) \right). \]

If \( \deg \left( \frac{\pi_*(R)}{n} \right) = 0 \) then \( \left\lceil \frac{\pi_*(R)}{n} \right\rceil \) is the zero divisor and we conclude that

\[ \text{dim}_k \left( H^0(X, \Omega_X)^G \right) = \text{dim}_k \left( H^0(Y, \Omega_Y) \right) = g_Y, \]

as desired. If \( \deg \left( \frac{\pi_*(R)}{n} \right) > 0 \) the divisor \( K_Y + \left\lceil \frac{\pi_*(R)}{n} \right\rceil \) is non-special and using the Riemann-Roch theorem (see Theorem 1.3 on p. 295 and Example 1.3.4 on p. 296 in [Ha]) we obtain

\[ \text{dim}_k \left( H^0(X, \Omega_X)^G \right) = \deg \left( K_Y + \left\lceil \frac{\pi_*(R)}{n} \right\rceil \right) + 1 - g_Y = g_Y - 1 + \deg \left( \frac{\pi_*(R)}{n} \right), \]

as stated. \( \square \)
Proof of Theorem. We assume that \( g_X \geq 2 \) and prove conditions (i), (ii) and (iii). By replacing \( G \) with the (non-trivial) kernel \( H \) of the action of \( G \) on \( H^0(X, \Omega_X) \) we may assume that \( G \) is non-trivial and that \( G \) acts trivially on \( H^0(X, \Omega_X) \): for condition (i) we note that if the projection \( X \to X/H \) is not tamely ramified then the projection \( \pi : X \to Y \) cannot be tamely ramified either; for part (ii) we note that if the genus of \( X/H \) is 0 also the genus of \( Y = X/G \) is zero by the Hurwitz formula (see Corollary 2.4 on p. 301 in [Ha]) applied to the canonical morphism \( X/H \to Y \); there isn’t anything to note for part (iii) in this reduction.

We first suppose that \( \pi \) is tamely ramified. Then we have \( R = \sum_{P \in X} (e_P - 1)[P] \) by Proposition 2.2(c) on p. 300 in [Ha]; hence \( \lfloor \pi_*(R) \rfloor/n \) is the zero divisor. Therefore we obtain

\[
g_X = \dim_k (H^0(X, \Omega_X)) = \dim (H^0(X, \Omega_X)^G) = g_Y
\]

by Proposition 1. Substituting this equality into the Hurwitz formula

\[
2(g_X - 1) = 2n(g_Y - 1) + \deg(R)
\]

yields the desired contradiction because \( n \geq 2, \ g_X \geq 2 \) and \( \deg(R) \geq 0 \).

We arrive at this contradiction whenever \( \lfloor \pi_*(R) \rfloor/n \) is the zero divisor. To prove condition (ii) we can therefore assume that \( \lfloor \pi_*(R) \rfloor/n \) is not the zero divisor. Then Proposition 1 tells us that

\[
g_X = g_Y - 1 + \deg \left( \frac{\pi_*(R)}{n} \right).
\]

Substituting this equality into the Hurwitz formula we obtain

\[
2 \left( g_Y - 1 + \deg \left( \frac{\pi_*(R)}{n} \right) - 1 \right) = 2n(g_Y - 1) + \deg(R).
\]

For any \( Q \in Y \) let \( n_Q \) denote the coefficient of the ramification divisor \( R \) at any \( P \in \pi^{-1}(Q) \) and let \( e_Q := e_P \) for any \( P \in \pi^{-1}(Q) \). Rewriting the previous equation yields

\[
(2n - 2)g_Y = 2n - 4 + 2 \deg \left( \frac{\pi_*(R)}{n} \right) - \deg(R)
\]

\[
= 2 \left( n - 2 + \sum_{Q \in Y} \left( \left\lfloor \frac{n \ n_Q}{e_Q} \right\rfloor - \frac{n \ n_Q}{e_Q} \right) \right)
\]

\[
= 2 \left( n - 2 + \sum_{Q \in Y} \left( \frac{n_Q}{e_Q} \right) - \frac{n_Q \ n}{e_Q} \right)
\]

\[
\leq 2(n - 2)
\]

because \( \frac{n}{2} \geq 1 \) and \( \left\lfloor \frac{n_Q}{e_Q} \right\rfloor \leq \frac{n_Q}{e_Q} \) for all \( Q \in Y \). Hence we obtain \( g_Y \leq \frac{n - 2}{n - 1} < 1 \) and therefore \( g_Y = 0 \), as stated in condition (ii).

By condition (i) the characteristic of \( k \) is positive and we may furthermore replace \( G \) by
a cyclic subgroup of $G$ of order $p$ in order to prove condition (iii). Then condition (iii) is the conclusion of one direction in the following proposition. \hfill\Box

**Proposition 2.** Let $p > 0$ and let $G$ be cyclic of order $p$. We furthermore assume that $g_Y = 0$. Then $G$ acts trivially on $H^0(X, \Omega_X)$ if and only if one of the following three conditions holds:

(i) $p = 2$.

(ii) $g_X = 0$.

(iii) $p = 3$ and $g_X = 1$.

**Proof.** Let $P_1, \ldots, P_r \in X$ be the ramified points of $\pi : X \to Y$ and, for $i = 1, \ldots, r$, define $N_i \in \mathbb{N}$ by $\text{ord}_P(\sigma(\pi_i) - \pi_i) = N_i + 1$ where $\pi_i$ is a local parameter at $P_i$ and $\sigma$ is a generator of $G$. From Lemma 1 on p. 87 in \cite{Na} we know that $p$ does not divide $N_i$, a fact we will use several times below. The ramification divisor $R$ of $\pi$ is equal to $\sum_{i=1}^{r}(N_i + 1)(p - 1)[P_i]$ by Hilbert’s formula for the order of the different (see Prop. 4, §1, Ch. IV on p. 72 in \cite{Se}). Let $N := \sum_{i=1}^{r}N_i$. Using the Hurwitz formula we obtain

$$2g_X - 2 = -2p + (N + r)(p - 1)$$

and hence

$$\dim_k(H^0(X, \Omega_X)) = g_X = \frac{(N + r - 2)(p - 1)}{2}.$$ 

Since $g_X \geq 0$ we obtain $r \geq 1$; that is, $\pi$ is not unramified. Therefore we have

$$\deg \left[ \frac{\pi_*(R)}{p} \right] = \sum_{i=1}^{r} \left[ \frac{(N_i + 1)(p - 1)}{p} \right] \geq \sum_{i=1}^{r} \left[ \frac{2(p - 1)}{p} \right] = r > 0.$$ 

From Proposition 1 we then conclude that

$$\dim_k(H^0(X, \Omega_X)^G) = g_Y - 1 + \deg \left[ \frac{\pi_*(R)}{p} \right] = -1 + \sum_{i=1}^{r} \left[ \frac{(N_i + 1)(p - 1)}{p} \right] = -1 + N + r + \sum_{i=1}^{r} \left[ \frac{-N_i + 1}{p} \right].$$

If $p = 2$ the dimension of both $H^0(X, \Omega_X)$ and $H^0(X, \Omega_X)^G$ is therefore equal to $\frac{N + r - 2}{2}$. If $g_X = 0$ both dimensions are obviously equal to 0. If $p = 3$ and $g_X = 1$ we obtain $N + r = 3$ and hence $r = 1$ and $N = 2$; thus both dimensions are equal to 1. Therefore in all three of these cases $G$ acts trivially on $H^0(X, \Omega_X)$. This finishes the proof of one direction in Proposition 2.

To prove the other direction we now assume that $G$ acts trivially on $H^0(X, \Omega_X)$ and that $p \geq 3$ and prove that condition (ii) or condition (iii) holds. For each $i = 1, \ldots, r$, we write $N_i = s_ip + t_i$ with $s_i \in \mathbb{N}$ and $t_i \in \{1, \ldots, p - 1\}$. We furthermore put $S := \sum_{i=1}^{r}s_i$ and $T := \sum_{i=1}^{r}t_i \geq r$. Then we have

$$\frac{(N + r - 2)(p - 1)}{2} = \dim_k(H^0(X, \Omega_X)) = \dim_k(H^0(X, \Omega_X)^G) = N - S - 1.$$
Rearranging this equation we obtain

$$(3-p)N - 2S = (r-2)(p-1) + 2$$

and hence

$$(-p^2 + 3p - 2)S = (r-2)(p-1) + 2 - (3-p)T.$$ 

Since $-p^2 + 3p - 2 = -(p-1)(p-2)$ and $p \geq 3$ this equation implies that

$$S = \frac{(r-2)(1-p) - 2 + T(3-p)}{(p-1)(p-2)}.$$ 

Because $S \geq 0$ the numerator of this fraction is non-negative, that is

$$0 \leq (r-2)(1-p) - 2 + T(3-p) \leq (r-2)(1-p) - 2 + r(3-p) = 2(r-1)(2-p).$$

Hence we have $r = 1$ and that numerator is 0. We conclude that $S = 0$ and hence that $T = 1$ or $p = 3$. If $T = 1$ we also have $N = 1$ and finally

$$g_X = \frac{(N + r - 2)(p - 1)}{2} = 0,$$

i.e. condition (ii) holds. If $T \neq 1$ and $p = 3$ we obtain $N = T = 2$ and finally

$$g_X = \frac{(N + r - 2)(p - 1)}{2} = 1,$$

i.e. condition (iii) holds. $\square$

Acknowledgments. The question underlying this paper goes back to Michel Matignon. I would like to thank Niels Borne for communicating this question to me, for sketching a proof of condition (i) of the above theorem and in particular for pointing me to the above lemma.

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