\(\Delta\)-critical graphs with a vertex of degree 2

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Abstract

Let \(G\) be a simple graph with maximum degree \(\Delta\). A classic result of Vizing shows that \(\chi'(G)\), the chromatic index of \(G\), is either \(\Delta\) or \(\Delta + 1\). We say \(G\) is of Class 1 if \(\chi'(G) = \Delta\), and is of Class 2 otherwise. A graph \(G\) is \(\Delta\)-critical if \(\chi'(G) = \Delta + 1\) and \(\chi'(H) < \Delta + 1\) for every proper subgraph \(H\) of \(G\), and is overfull if \(|E(G)| > \Delta([|V(G)| - 1]/2)\). Clearly, overfull graphs are Class 2. Hilton and Zhao in 1997 conjectured that if \(G\) is obtained from an \(n\)-vertex \(\Delta\)-regular Class 1 graph with maximum degree greater than \(n/3\) by splitting a vertex, then being overfull is the only reason for \(G\) to be Class 2. This conjecture was only confirmed when \(\Delta \geq \frac{n}{2}(\sqrt{7} - 1) \approx 0.82n\). In this paper, we improve the bound on \(\Delta\) from \(\frac{n}{2}(\sqrt{7} - 1)\) to \(0.75n\). Considering the structure of \(\Delta\)-critical graphs with a vertex of degree 2, we also show that for an \(n\)-vertex \(\Delta\)-critical graph with \(\Delta \geq \frac{2n}{3}\), if it contains a vertex of degree 2, then it is overfull. We actually obtain a more general form of this result, which partially supports the overfull conjecture of Chetwynd and Hilton from 1986, which states that if \(G\) is an \(n\)-vertex \(\Delta\)-critical graph with \(\Delta > n/3\), then \(G\) contains an overfull subgraph \(H\) with \(\Delta(H) = \Delta\). Our proof techniques are new and might shed some light on attacking both of the conjectures when \(\Delta\) is large.

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1 Introduction

For two integer $p, q$ with $q \geq p$, we use $[p, q]$ to denote the set of all integer between $p$ and $q$, inclusively. We consider only simple graphs. Let $G$ be a graph with maximum degree $\Delta(G) = \Delta$. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. Let $v \in V(G)$ with $d_G(v) = i \geq 2$ and $N_G(v) = \{u_1, \ldots, u_t\}$. A vertex-splitting in $G$ at $v$ into two vertices $v_1$ and $v_2$ gives a new graph $G'$ such that $V(G') = (V(G) \setminus \{v\}) \cup \{v_1, v_2\}$ and $E(G') = E(G - v) \cup \{v_1v_2\} \cup \{v_1u_i : i \in [1, s]\} \cup \{v_2u_i : i \in [s + 1, t]\}$, where $s \in [1, t - 1]$ is any integer. We say $G'$ is obtained from $G$ by a vertex-splitting.

An edge $k$-coloring or simply a $k$-coloring of $G$ is a mapping $\varphi$ from $E(G)$ to the set of integers $[1, k]$, called colors, such that no two adjacent edges receive the same color with respect to $\varphi$. The chromatic index of $G$, denoted $\chi'(G)$, is defined to be the smallest integer $k$ so that $G$ has an edge $k$-coloring. We denote by $\mathcal{C}_k(G)$ the set of all edge $k$-colorings of $G$.

A graph $G$ is $\Delta$-critical if $\chi'(G) = \Delta + 1$ and $\chi'(H) < \Delta + 1$ for every proper subgraph $H$ of $G$. In 1960’s, Vizing [14] showed that a graph of maximum degree $\Delta$ has chromatic index either $\Delta$ or $\Delta + 1$. If $\chi'(G) = \Delta$, then $G$ is said to be of Class 1; otherwise, it is said to be of Class 2. Holyer [10] showed that it is NP-complete to determine whether an arbitrary graph is of Class 1. Nevertheless, if a graph $G$ has too many edges, i.e., $|E(G)| > \Delta|V(G)|/2$, then we have to color $E(G)$ using exactly $(\Delta + 1)$ colors. Such graphs are called overfull. Clearly, overfull graphs have an odd order, all regular graphs of an odd order are overfull, and all graphs obtained from a regular Class 1 graph by a vertex-splitting is overfull.

Being overfull is definitely a cause for a graph to be Class 2, but is that the only cause? Hilton and Zhao [8] in 1997 conjectured the following: Let $G$ be an $n$-vertex Class 1 $\Delta$-regular graph with $\Delta > \frac{n}{2}$. If $G^*$ is obtained from $G$ by a vertex-splitting, then $G^*$ is $\Delta$-critical (vertex-splitting conjecture). Clearly, $G^*$ is overfull. The conjecture asserts that for every $e \in E(G^*)$, $G^* - e$ is no longer Class 2. In other words, being overfull is the only cause for $G^*$ to be Class 2. This conjecture was verified when $\Delta \geq \frac{n}{2} (\sqrt{7} - 1) \approx 0.82n$ by Hilton and Zhao [8] in 1997. Song [12] in 2002 showed that the conjecture holds for a particular class of $n$-vertex Class 1 $\Delta$-regular graphs with $\Delta \geq \frac{n}{2}$. No other progress on this conjecture has been achieved since then. We support this conjecture as below.

**Theorem 1.1.** Let $n$ and $\Delta$ be positive integers such that $\Delta \geq \frac{3(n - 1)}{4}$. If $G$ is obtained from an $(n - 1)$-vertex $\Delta$-regular Class 1 graph by a vertex-splitting, then $G$ is $\Delta$-critical.

If $G$ is an $n$-vertex overfull graph, then $|E(G)| \geq \Delta(n - 1)/2 + 1$. Thus $\sum_{v \in V(G)} (\Delta - d_G(v)) \leq \Delta - 2$. Therefore, if $G$ has a vertex of degree 2, then all other vertices of $G$ are of maximum degree. Is the converse of this statement true? That is, when will be a Class 2 graph with a degree 2 vertex overfull? We investigate this question and show that this happens when $\Delta$ is large. In general, for two adjacent vertices $u, v \in V(G)$, we call $(u, v)$ a full-deficiency pair of $G$ if $d(u) + d(v) = \Delta(G) + 2$. In particular, if $v$ is of degree 2 in a $\Delta$-critical graph $G$, then each neighbor $u$ of $v$ has degree $\Delta$ by Vizing’s Adjacency Lemma.
(this lemma will be introduced in Section 2). Therefore, \((u, v)\) is a full-deficiency pair of \(G\). We obtain the following result.

**Theorem 1.2.** Let \(n\) and \(\Delta\) be positive integers such that \(\Delta \geq \frac{3(n-1)}{4}\), and \(G\) be an \(n\)-vertex \(\Delta\)-critical graph. If \(G\) has a full-deficiency pair, then \(G\) is overfull. Consequently, \(G\) is obtained from an \((n-1)\)-vertex \(\Delta\)-regular Class 1 multigraph by a vertex-splitting.

Theorem 1.2 partially supports a conjecture of Chetwynd and Hilton from 1986 [4, 5]. The conjecture states the following: Let \(G\) be a simple graph with \(\Delta(G) > \frac{1}{3}|V(G)|\). Then \(G\) is Class 2 implies that \(G\) contains an overfull subgraph \(H\) with \(\Delta(H) = \Delta(G)\) (overfull conjecture). The overfull conjecture was confirmed only for some special classes of graphs. Chetwynd and Hilton [5] in 1989 verified the conjecture for \(n\)-vertex graphs with \(\Delta \geq n - 3\). Hoffman and Rodger [9] in 1992 confirmed the conjecture for complete multipartite graphs. Plantholt [11] in 2004 showed that the overfull conjecture is affirmative for graphs \(G\) with an even order \(n\), maximum degree \(\Delta\) and minimum degree \(\delta\) satisfying \((3\delta - \Delta)/2 \geq cn\) for any \(c \geq \frac{3}{4}\). The overfull conjecture was also confirmed for large regular graphs in 2013 [6, 7].

Both the overfull conjecture and the vertex-splitting conjecture are best possible in terms of the condition on the maximum degree, by considering the critical Class 2 graph \(P^\ast\), which is obtained from the Petersen graph by deleting a vertex. Hilton and Zhao [8] proved that the overfull conjecture implies the vertex-splitting conjecture.

The results in Theorem 1.1 and Theorem 1.2 together imply that all \(n\)-vertex \(\Delta\)-critical graphs with a vertex of degree 2 can be obtained from an \((n-1)\)-vertex \(\Delta\)-regular Class 1 multigraph by a vertex splitting, when \(\Delta \geq \frac{3(n-1)}{4}\). Thereby, these results provide a way of constructing dense \(\Delta\)-critical graphs, which are known to be hard. The reminder of this paper is organized as follows. We introduce some definitions and preliminary results in Section 2. In Section 3, we prove Theorem 1.1 and Theorem 1.2 by assuming the truth of several lemmas. These lemmas will be proved in the last section.

## 2 Definitions and Preliminary Results

Let \(G\) be a graph. For \(e \in E(G)\), \(G - e\) denotes the graph obtained from \(G\) by deleting the edge \(e\). The symbol \(\Delta\) is reserved for \(\Delta(G)\), the maximum degree of \(G\) throughout this paper. A k-vertex in \(G\) is a vertex of degree exactly \(k\) in \(G\), and a k-neighbor of a vertex \(v\) is a neighbor of \(v\) that is a k-vertex in \(G\). For \(u, v \in V(G)\), we use \(\text{dist}_G(u, v)\) to denote the distance between \(u\) and \(v\), which is the length of a shortest path connecting \(u\) and \(v\) in \(G\). For \(S \subseteq V(G)\), define \(\text{dist}_G(u, S) = \min_{v \in S} \text{dist}_G(u, v)\).

An edge \(e \in E(G)\) is a critical edge of \(G\) if \(\chi'(G - e) < \chi'(G)\). It is not hard to see that if \(G\) is connected, \(\chi'(G) = \Delta + 1\) and every edge of \(G\) is critical, then \(G\) is \(\Delta\)-critical. Critical graphs are useful since they provide more information about the structure around a vertex than general Class 2 graphs. For example, Vizing’s Adjacency Lemma (VAL) from...
1965 [14] is a useful tool that reveals certain structure at a vertex by assuming the criticality of an edge.

**Lemma 2.1 (Vizing’s Adjacency Lemma (VAL)).** Let $G$ be a Class 2 graph with maximum degree $\Delta$. If $e = xy$ is a critical edge of $G$, then $x$ has at least $\Delta - d_G(y) + 1$ $\Delta$-neighbors in $V(G) \setminus \{y\}$.

Let $G$ be a graph and $\varphi \in \mathcal{C}^k(G - e)$ for some edge $e \in E(G)$ and some integer $k \geq 0$. For any $v \in V(G)$, the set of colors present at $v$ is $\varphi(v) = \{\varphi(f) : f$ is incident to $v\}$, and the set of colors missing at $v$ is $\varphi_C(v) = [1, k] \setminus \varphi(v)$. For a vertex set $X \subseteq V(G)$, define

$$\varphi(X) = \bigcup_{v \in X} \varphi(v).$$

The set $X$ is called elementary with respect to $\varphi$ or simply $\varphi$-elementary if $\varphi(u) \cap \varphi(v) = \emptyset$ for every two distinct vertices $u, v \in X$. Sometimes, we just say that $X$ is elementary if the edge coloring is understood.

For two distinct colors $\alpha, \beta \in [1, k]$, let $H$ be the subgraph of $G$ with $V(H) = V(G)$ and $E(H)$ consisting of edges from $E(G)$ that are colored by $\alpha$ or $\beta$ with respect to $\varphi$. Each component of $H$ is either an even cycle or a path, which is called an $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$. If we interchange the colors $\alpha$ and $\beta$ on an $(\alpha, \beta)$-chain $C$ of $G$, we get a new edge $k$-coloring of $G$, and we write

$$\varphi' = \varphi/C.$$  

This operation is called a Kempe change. For a color $\alpha$, a sequence of Kempe $(\alpha, \ast)$-changes is a sequence of Kempe changes that each involves the exchanging of the color $\alpha$ and another color from $[1, k]$.

Let $x, y \in V(G)$, and $\alpha, \beta, \gamma \in [1, k]$ be three colors. If $x$ and $y$ are contained in a same $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$, we say $x$ and $y$ are $(\alpha, \beta)$-linked with respect to $\varphi$. Otherwise, $x$ and $y$ are $(\alpha, \beta)$-unlinked with respect to $\varphi$. Without specifying $\varphi$, when we just say $x$ and $y$ are $(\alpha, \beta)$-linked or $x$ and $y$ are $(\alpha, \beta)$-unlinked, we mean they are linked or unlinked with respect to the current edge coloring. Let $P$ be an $(\alpha, \beta)$-chain of $G$ with respect to $\varphi$ that contains both $x$ and $y$. If $P$ is a path, denote by $P_{[x,y]}(\alpha, \beta, \varphi)$ the subchain of $P$ that has endvertices $x$ and $y$. By swapping colors along $P_{[x,y]}(\alpha, \beta, \varphi)$, we mean exchanging the two colors $\alpha$ and $\beta$ on the path $P_{[x,y]}(\alpha, \beta, \varphi)$. The notion $P_{[x,y]}(\alpha, \beta)$ always represents the $(\alpha, \beta)$-chain with respect to the current edge coloring. Define $P_x(\alpha, \beta, \varphi)$ to be an $(\alpha, \beta)$-chain or an $(\alpha, \beta)$-subchain of $G$ with respect to $\varphi$ that starts at $x$ and ends at a different vertex missing exactly one of $\alpha$ and $\beta$. (If $x$ is an endvertex of the $(\alpha, \beta)$-chain that contains $x$, then $P_x(\alpha, \beta, \varphi)$ is unique. Otherwise, we take one segment of the whole chain to be $P_x(\alpha, \beta, \varphi)$. We will specify the segment when it is used.) If $u$ is a vertex on $P_x(\alpha, \beta, \varphi)$, we write $u \in P_x(\alpha, \beta, \varphi)$; and if $uv$ is an edge on $P_x(\alpha, \beta, \varphi)$, we write $uv \in P_x(\alpha, \beta, \varphi)$. Similarly, the notion $P_x(\alpha, \beta)$ always represents
the \((\alpha, \beta)\)-chain with respect to the current edge coloring. If \(u, v \in P_x(\alpha, \beta)\) such that \(u\) lies between \(x\) and \(v\), then we say that \(P_x(\alpha, \beta)\) meets \(u\) before \(v\). Suppose that \(u, v \in P_x(\alpha, \beta)\) such that \(u\) lies between \(x\) and \(v\), then we say that \(P_x(\alpha, \beta)\) meets \(u\) before \(v\). Suppose that \(\alpha \in \varphi(x)\) and \(\beta, \gamma \in \varphi(x)\). An \((\alpha, \beta)\)–\((\beta, \gamma)\) swap at \(x\) consists of two operations: first swaps colors on \(P_x(\alpha, \beta, \varphi)\) to get an edge \(k\)-coloring \(\varphi'\), and then swaps colors on \(P_x(\beta, \gamma, \varphi')\). By convention, an \((\alpha, \alpha)\)-swap at \(x\) does nothing at \(x\). Suppose the current color of an edge \(uv\) of \(G\) is \(\alpha\), the notation \(uv : \alpha \to \beta\) means to recolor the edge \(uv\) using the color \(\beta\). Recall that \(\varphi(x)\) is the set of colors not present at \(x\). If \(|\varphi(x)| = 1\), we will also use \(\varphi(x)\) to denote the color that is missing at \(x\).

Let \(\alpha, \beta, \gamma, \tau, \eta \in [1, k]\). We will use a matrix with two rows to denote a sequence of operations taken on \(\varphi\). Each entry in the first row represents a path or a sequence of vertices. Each entry in the second row, indicates the action taken on the object above this entry. We require the operations to be taken to follow the “left to right” order as they appear in the matrix. For example, the matrix below indicates three sequential operations taken on the graph based on the coloring from the previous step:

\[
\begin{bmatrix}
P_{[a,b]}(\alpha, \beta) & rs & ab \\
\alpha/\beta & \gamma \to \tau & \eta
\end{bmatrix}
\]

Step 1 Swap colors on the \((\alpha, \beta)\)-subchain \(P_{[a,b]}(\alpha, \beta, \varphi)\).

Step 2 Do \(rs : \gamma \to \tau\).

Step 3 Color the edge \(ab\) using color \(\eta\).

Let \(T\) be a sequence of vertices and edges of \(G\). We denote by \(V(T)\) the set of vertices from \(V(G)\) that are contained in \(T\), and by \(E(T)\) the set of edges from \(E(G)\) that are contained in \(T\).

### 2.1 Multifan

Let \(G\) be a graph, \(e = rs_1 \in E(G)\) and \(\varphi \in \mathcal{C}^k(G-e)\) for some integer \(k \geq 0\). A multifan centered at \(r\) with respect to \(e\) and \(\varphi\) is a sequence \(F_\varphi(r, s_1 : s_p) := (r, rs_1, s_1, rs_2, s_2, \ldots, rs_p, s_p)\) with \(p \geq 1\) consisting of distinct vertices \(r, s_1, s_2, \ldots, s_p\) and distinct edges \(rs_1, rs_2, \ldots, rs_p\) satisfying the following condition:

(F1) For every edge \(rs_i\) with \(i \in [2, p]\), there exists \(j \in [1, i - 1]\) such that \(\varphi(rs_i) \in \varphi(s_j)\).

We will simply denote a multifan \(F_\varphi(r, s_1 : s_p)\) by \(F\) if \(\varphi\) and the vertices and edges in \(F_\varphi(r, s_1 : s_p)\) are clear. Let \(F_\varphi(r, s_1 : s_p)\) be a multifan. By its definition, for any \(p^* \in [1, p]\), \(F_\varphi(r, s_1 : s_{p^*})\) is a multifan. The following result regarding a multifan can be found in [13, Theorem 2.1].

**Lemma 2.2.** Let \(G\) be a Class 2 graph and \(F_\varphi(r, s_1 : s_p)\) be a multifan with respect to a critical edge \(e = rs_1\) and a coloring \(\varphi \in \mathcal{C}^\Delta(G-e)\). Then the following statements hold.
(a) $V(F)$ is $\varphi$-elementary.

(b) Let $\alpha \in \overline{\varphi}(r)$. Then for every $i \in [1, p]$ and $\beta \in \overline{\varphi}(s_i)$, $r$ and $s_i$ are $(\alpha, \beta)$-linked with respect to $\varphi$.

Let $F_{\varphi}(r, s_1 : s_p)$ be a multifan. We call $s_{\ell_1}, s_{\ell_2}, \ldots, s_{\ell_k}$, a subsequence of $s_1 : s_p$, an $\alpha$-sequence with respect to $\varphi$ and $F$ if the following holds:

$$\varphi(rs_{\ell_i}) = \alpha \in \overline{\varphi}(s_i), \quad \varphi(rs_{\ell_i}) \in \overline{\varphi}(s_{\ell_{i-1}}), \quad i \in [2, k].$$

A vertex in an $\alpha$-sequence is called an $\alpha$-inducing vertex with respect to $\varphi$ and $F$, and a missing color at an $\alpha$-inducing vertex is called an $\alpha$-inducing color. For convenience, $\alpha$ itself is also an $\alpha$-inducing color. We say $\beta$ is induced by $\alpha$ if $\beta$ is $\alpha$-inducing. By Lemma 2.2 (a) and the definition of multifan, each color in $\overline{\varphi}(V(F))$ is induced by a unique color in $\overline{\varphi}(s_1)$. Also if $\alpha_1, \alpha_2$ are two distinct colors in $\overline{\varphi}(s_1)$, then an $\alpha_1$-sequence is disjoint with an $\alpha_2$-sequence. For two distinct $\alpha$-inducing colors $\beta$ and $\delta$, we write $\delta \prec \beta$ if there exists an $\alpha$-sequence $s_{\ell_1}, s_{\ell_2}, \ldots, s_{\ell_k}$ such that $\delta \in \overline{\varphi}(s_{\ell_i}), \beta \in \overline{\varphi}(s_{\ell_j})$ and $i < j$. For convenience, $\alpha \prec \beta$ for any $\alpha$-inducing color $\beta \neq \alpha$. As a consequence of Lemma 2.2 (a), we have the following properties for a multifan. A proof of the result can be found in [3, Lemma 3.2].

**Lemma 2.3.** Let $G$ be a Class 2 graph and $F_{\varphi}(r, s_1 : s_p)$ be a multifan with respect to a critical edge $e = rs_1$ and a coloring $\varphi \in C^\Delta(G - e)$. For two colors $\delta \in \overline{\varphi}(s_i)$ and $\lambda \in \overline{\varphi}(s_j)$ with $i, j \in [1, p]$ and $i \neq j$, the following statements hold.

(a) If $\delta$ and $\lambda$ are induced by different colors, then $s_i$ and $s_j$ are $(\delta, \lambda)$-linked with respect to $\varphi$.

(b) If $\delta$ and $\lambda$ are induced by the same color, $\delta \prec \lambda$, and $s_i$ and $s_j$ are $(\delta, \lambda)$-unlinked with respect to $\varphi$, then $r \in P_{s_j}(\lambda, \delta, \varphi)$.

### 2.2 Kierstead path

Let $G$ be a graph, $e = v_0v_1 \in E(G)$, and $\varphi \in C^k(G - e)$ for some integer $k \geq 0$. A Kierstead path with respect to $e$ and $\varphi$ is a sequence $K = (v_0, v_0v_1, v_1v_2, v_2, \ldots, v_{p-1}, v_{p-1}v_p, v_p)$ with $p \geq 1$ consisting of distinct vertices $v_0, v_1, \ldots, v_p$ and distinct edges $v_0v_1, v_1v_2, \ldots, v_{p-1}v_p$ satisfying the following condition:

(K1) For every edge $v_{i-1}v_i$ with $i \in [2, p]$, there exists $j \in [1, i - 1]$ such that $\varphi(v_{i-1}v_i) \in \overline{\varphi}(v_j)$.

Clearly a Kierstead path with at most 3 vertices is a multifan. We consider Kierstead paths with 4 vertices. The result below was proved in Theorem 3.3 from [13].

**Lemma 2.4.** Let $G$ be a Class 2 graph, $e = v_0v_1 \in E(G)$ be a critical edge, and $\varphi \in C^\Delta(G - e)$. If $K = (v_0, v_0v_1, v_1v_2, v_2v_3, v_3)$ is a Kierstead path with respect to $e$ and $\varphi$, then the following statements hold.
(a) If $\min\{d_G(v_2), d_G(v_3)\} < \Delta$, then $V(K)$ is $\varphi$-elementary.

(b) $|\varphi(v_3) \cap (\varphi(v_0) \cup \varphi(v_1))| \leq 1$.

3 Proof of Theorems 1.1 and 1.2

We will prove Theorems 1.1 and 1.2 based on the following lemmas, whose proof will be presented in the last section.

General properties on Kierstead paths with 5 vertices was proved by the first author of this paper \cite{1}. Here we stress only one of the cases.

Lemma 3.1. Let $G$ be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in C^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ be a Kierstead path with respect to $ab$ and $\varphi$. If $|\varphi(t) \cap (\varphi(a) \cup \varphi(b))| \geq 3$, then the following hold:

(a) There exists $\varphi^* \in C^\Delta(G - ab)$ satisfies the following properties:

   (i) $\varphi^*(bu) \in \varphi^*(a) \cap \varphi^*(t)$,
   (ii) $\varphi^*(us) \in \varphi^*(b) \cap \varphi^*(t)$, and
   (iii) $\varphi^*(st) \in \varphi^*(a)$.

(b) $d_G(b) = d_G(u) = \Delta$.

Figure 1 shows a Kierstead path with the properties described in (a).

![Figure 1: Colors on a Kierstead path of 5 vertices](image)

Lemma 3.2. Let $G$ be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in C^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ and $K^* = (a, ab, b, bu, u, ux, x)$ be two Kierstead paths with respect to $ab$ and $\varphi$, where $x \notin V(K)$. If $|\varphi(t) \cap (\varphi(a) \cup \varphi(b))| \geq 4$ and $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, then $d_G(x) = \Delta$.

A short-kite $H$ is a graph with

$$V(H) = \{a, b, c, u, x, y\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, ux, uy\}.$$ 

The lemma below reveals some properties of a short-kite with specified colors on its edges.
Lemma 3.3. Let $G$ be a Class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in C^\Delta(G - ab)$. Suppose

$$K = (a, ab, b, bu, u, ux, x) \quad \text{and} \quad K^* = (b, ab, a, ac, cu, u, uy)$$

are two Kierstead paths with respect to $ab$ and $\varphi$. If $\overline{\varphi}(x) \cup \overline{\varphi}(y) \subseteq \overline{\varphi}(a) \cup \overline{\varphi}(b)$, then

$$\max\{d_G(x), d_G(y)\} = \Delta.$$

A kite $H$ is a graph with

$$V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\} \quad \text{and} \quad E(H) = \{ab, ac, bu, cu, us_1, us_2, s_1t_1, s_2t_2\}.$$

The lemma below reveals some properties of a kite with specified colors on its edges.

Lemma 3.4. Let $G$ be a Class 2 graph, $H \subseteq G$ be a kite with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$, and let $\varphi \in C^\Delta(G - ab)$. Suppose

$$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \text{and} \quad K^* = (b, ab, a, ac, cu, u, us_2, s_2t_2, t_2)$$

are two Kierstead paths with respect to $ab$ and $\varphi$. If $\varphi(s_1t_1) = \varphi(s_2t_2)$, then $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 4$.

Let $G$ be a $\Delta$-critical graph, $ab \in E(G)$, and $\varphi \in C^\Delta(G - ab)$. A fork $H$ with respect to $\varphi$ is a graph with

$$V(H) = \{a, b, u, s_1, s_2, t_1, t_2\} \quad \text{and} \quad E(H) = \{ab, bu, us_1, us_2, s_1t_1, s_2t_2\}$$

such that $\varphi(bu) \in \overline{\varphi}(a), \varphi(us_1), \varphi(us_2) \in \overline{\varphi}(a) \cup \overline{\varphi}(b)$, and $\varphi(s_1t_1) \in (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \cap \overline{\varphi}(t_2)$ and $\varphi(s_2t_2) \in (\overline{\varphi}(a) \cup \overline{\varphi}(b)) \cap \overline{\varphi}(t_1)$. Fork was defined in [2] and it was shown in [2, Proposition B] that a fork can not exist in a $\Delta$-critical graph if the degree sum of $a$, $t_1$ and $t_1$ is small.

Lemma 3.5. Let $G$ be a $\Delta$-critical graph, $ab \in E(G)$, and $\{u, s_1, s_2, t_1, t_2\} \subseteq V(G)$. If $\Delta \geq d_G(a) + d_G(t_1) + d_G(t_2) + 1$, then for any $\varphi \in C^\Delta(G - ab)$, $G$ does not contain a fork on $\{a, b, u, s_1, s_2, t_1, t_2\}$ with respect to $\varphi$.

We need the following two additional results to prove Theorem 1.2. Since all vertices not missing a given color $\alpha$ are saturated by the matching that consists of all edges colored by $\alpha$ in $G$, we have the following result.

Lemma 3.6 (Parity Lemma). Let $G$ be an $n$-vertex graph and $\varphi \in C^\Delta(G)$. Then for any color $\alpha \in [1, \Delta]$, $|\{v \in V(G) : \alpha \in \overline{\varphi}(v)\}| \equiv n \pmod{2}$.

Lemma 3.7. If $G$ is an $n$-vertex Class 2 graph with a full-deficiency pair $(a, b)$ such that $ab$ is a critical edge of $G$, then $G$ satisfies the following properties.

(i) For every $x \in (N_G(a) \cup N_G(b)) \setminus \{a, b\}$, $d_G(x) = \Delta$.
(ii) For every \(x \in V(G) \setminus \{a, b\}\), if \(\text{dist}_G(x, \{a, b\}) = 2\), then \(d_G(x) \geq \Delta - 1\). Furthermore, if \(d_G(a) < \Delta\) and \(d_G(b) < \Delta\), then \(d_G(x) = \Delta\);

(iii) For every \(x \in V(G) \setminus \{a, b\}\), if \(d_G(x) \geq n - |N_G(b) \cup N_G(a)|\), then \(d_G(x) \geq \Delta - 1\). Furthermore, if \(d_G(a) < \Delta\) and \(d_G(b) < \Delta\), then \(d_G(x) = \Delta\);

(iv) If there exists \(x \in V(G) \setminus \{a, b\}\) such that \(d_G(x) < \Delta\), then there exists \(y \in V(G) \setminus \{a, b, x\}\) such that \(d_G(y) < \Delta\).

**Proof.** We let \(\varphi \in C^\Delta(G - ab)\) and \(F = (b, ba, a)\) be the multifan with respect to \(ab\) and \(\varphi\). By Lemma 2.2 (a),

\[
|\varphi(F)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = 2\Delta + 2 - (\Delta + 2) = \Delta. \tag{1}
\]

By Lemma 2.2, for every \(\varphi' \in C^\Delta(G - ab)\), \(\{a, b\}\) is \(\varphi'\)-elementary and for every \(i \in \varphi'(a)\) and \(j \in \varphi'(b)\), \(a\) and \(b\) are \((i, j)\)-linked with respect to \(\varphi'\). We will use this fact very often.

Since all the \(\Delta\) colors appear in \(\varphi(F)\), each of \(N_G(a) \cup \{b\}\) and \(N_G(b) \cup \{a\}\) is the vertex set of a multifan with respect to \(ab\) and \(\varphi\). By Lemma 2.2 (a) and (1), we know that for every \(x \in (N_G(a) \cup N_G(b)) \setminus \{a, b\}\), \(d_G(x) = \Delta\). This proves (i).

For (ii), let \(x \in V(G) \setminus \{a, b\}\) such that \(\text{dist}_G(x, \{a, b\}) = 2\). We assume that \(\text{dist}_G(x, b) = 2\) and let \(u \in (N_G(b) \setminus \{a\}) \cap N_G(x)\). Then by (1), \(K = (a, ab, b, bu, u, xu, x)\) is a Kierstead path with respect to \(ab\) and \(\varphi\). By (1) and Lemma 2.4 (b), it follows that \(d_G(x) \geq \Delta - 1\). If \(d_G(a) < \Delta\) and \(d_G(b) < \Delta\), by (1) and Lemma 2.4 (a), we get \(d_G(x) = \Delta\).

For (iii), let \(x \in V(G) \setminus \{a, b\}\) such that \(d_G(x) \geq n - |N_G(b) \cup N_G(a)|\). By (i), we may assume that \(x \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\}\). Thus \(d_G(x) \geq n - |N_G(b) \cup N_G(a)|\) implies that there exists \(u \in ((N_G(a) \cup N_G(b)) \cap N_G(x))\). Therefore, \(\text{dist}_G(x, \{a, b\}) = 2\). Now Statement (ii) yields the conclusion. Statement (iv) is a consequence of (1) and the Parity Lemma. \(\square\)

**Corollary 3.8.** Let \(G\) be an \(n\)-vertex Class 2 graph with a full-deficiency pair \((a, b)\) such that \(ab\) is a critical edge of \(G\). If \(\Delta \geq \frac{3(n-1)}{4}\), then there exists at most one vertex \(x \in V(G) \setminus \{a, b\}\) such that \(d_G(x) = \Delta - 1\).

**Proof.** Assume to the contrary that there exist distinct \(x, y \in V(G) \setminus \{a, b\}\) such that \(d_G(x) = d_G(y) = \Delta - 1\). By Lemma 3.7 (i), \(x, y \notin (N_G(a) \cup N_G(b)) \setminus \{a, b\}\). By Lemma 3.7 (iii), we may assume that \(d_G(b) = \Delta\). Thus \(d_G(a) = 2\) as \(d_G(a) + d_G(b) = \Delta + 2\). Let \(c\) be the other neighbor of \(a\) in \(G\). Since \((a, c)\) is a full-deficiency pair of \(G\) as well, we may assume \(x, y \notin N_G(c)\).

Since \(d_G(b) = d_G(c) = \Delta\) and \(d_G(x) = d_G(y) = \Delta - 1\), we get \(|N_G(b) \cap N_G(c)| \geq \frac{n}{2} - 1\) and \(|N_G(x) \cap N_G(y)| \geq \frac{n}{2} - 2\). Since \(b, c, x, y \notin N_G(b) \cap N_G(c)\) and \(b, c, x, y \notin N_G(x) \cap N_G(y)\), we get \(|N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)| \geq 1\). Let \(u \in N_G(b) \cap N_G(c) \cap N_G(x) \cap N_G(y)\),
$H$ be the short-kite with $V(H) = \{a, b, c, u, x, y\}$, and let $\varphi \in \mathcal{C}\Delta(G - ab)$. As $\{a, b\}$ is $\varphi$-elementary, $|\varphi(a) \cup \varphi(b)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = \Delta$ and so $\varphi(a) \cup \varphi(b) = [1, \Delta]$. Thus $K = (a, ab, b, bu, u, u, x, x)$ and $K^\ast = (b, ab, ac, c, cu, u, uy)$ are two Kierstead paths with respect to $ab$ and $\varphi$, and $\varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b)$. However, $d_G(x) = d_G(y) = \Delta - 1$, contradicting Lemma 3.3.

\textbf{Proof of Theorem 1.1.} Since $G$ is overfull, $G$ is Class 2. We show that every edge of $G$ is critical. Suppose to the contrary that there exists $xy \in E(G)$ such that $xy$ is not a critical edge of $G$. Let

$$G^\ast = G - xy.$$ 

Then $\chi'(G^\ast) = \Delta + 1$.

Since $ab$ is a critical edge of $G$, $ab \neq xy$. Also, since $ab$ is a critical edge of $G$, and any $\Delta$-coloring of $G - ab$ gives a $\Delta$-coloring of $G^\ast - ab$, $ab$ is also a critical edge of $G^\ast$. Since $d_{G^\ast}(x) = d_{G^\ast}(y) = \Delta - 1$, we reach a contradiction to Corollary 3.8. □

\textbf{Proof of Theorem 1.2.} Let $(a, b)$ be a full-deficiency pair of $G$. It suffices to only show that for every $v \in V(G) \setminus \{a, b\}$, $d_G(v) = \Delta$. To see this, let $G$ be a $\Delta$-critical graph with a full-deficiency pair $(a, b)$ and for every $v \in V(G) \setminus \{a, b\}$, $d_G(v) = \Delta$. Let $\varphi \in \mathcal{C}\Delta(G - ab)$.

Since $\varphi(a) \cap \varphi(b) = \emptyset$ and $d_G(a) + d_G(b) = \Delta + 2$, $\varphi(a) \cap \varphi(b) = \emptyset$. Thus, identifying $a$ and $b$ in $G$ gives a $\Delta$-coloring of a $\Delta$-regular multigraph $G^\ast$. This implies that $|V(G^\ast)| = n - 1$ is even. So $n$ is odd. Consequently, $G$ is overfull.

Thus, for the sake of contradiction, we assume that there exists $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) < \Delta$. By Lemma 3.7 (iv), there exists $y \in V(G) \setminus \{a, b, x\}$ such that $d_G(y) < \Delta$. Furthermore, by Lemma 3.7 (iii) and Corollary 3.8, there exists at most one vertex $x \in V(G) \setminus \{a, b\}$ such that $d_G(x) = \Delta - 1$, and for all other vertex $y \in V(G) \setminus \{a, b, x\}$, if $d_G(y) < \Delta$, then $d_G(y) < n - |N_G(b) \cup N_G(a)|$. This gives a vertex $t \in V(G) \setminus \{a, b\}$ such that $d_G(t) < n - |N_G(b) \cup N_G(a)|$. Let $\varphi \in \mathcal{C}\Delta(G - ab)$ and

$$F = (b, ba, a)$$

be the multifan with respect to $ab$ and $\varphi$. By Lemma 2.2 (a),

$$|\varphi(F)| = 2\Delta + 2 - (d_G(a) + d_G(b)) = \Delta. \quad (2)$$

Assume, without loss of generality, that $d_G(b) \geq d_G(a)$. Then $d_G(b) \geq \frac{3(n-1)}{8} + 1$ as $d_G(a) + d_G(b) = \Delta + 2 \geq \frac{3(n-1)}{4} + 2$. By Lemma 3.7 (i) and (ii), we assume that $\text{dist}_G(t, \{a, b\}) \geq 3$. Since $\Delta \geq \frac{3(n-1)}{4}$, for any $s \in N_G(t)$ with $d_G(s) \geq \Delta - 1$ (such $s$ exists as $t$ is adjacent to at least two $\Delta$-neighbors by VAL), we conclude that there exists $u \in N_G(b) \cap N_G(s)$. Now by (2), $K = (a, ab, b, bu, u, us, s, st)$ is a Kierstead path with respect to $ab$ and $\varphi$. This implies that $d_G(b) = d_G(u) = \Delta$ by Lemma 3.1 (b). Thus $d_G(a) = 2$. We let $c$ be the other $\Delta$-neighbor of $a$. 

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As $d_G(a) = 2$ and $ab \in E(G)$, $|N_G(b) \cup N_G(a)| \geq \Delta + 1 > \frac{3n}{4}$. Since $G$ is $\Delta$-critical, VAL implies that for every $s \in N_G(t)$, $d_G(s) \geq \Delta + 2 - d_G(t) \geq \Delta + 2 + |N_G(b) \cup N_G(a)| - n \geq n - |N_G(b) \cup N_G(a)|$. Thus, by Lemma 3.7 (iii) and Corollary 3.8, there exists at most one vertex $s \in N_G(t)$ such that $d_G(s) < \Delta$. In this case, $d_G(s) = \Delta - 1$.

Next, we claim that for any $x \in V(G) \setminus \{a, b\}$, $d_G(x) < \Delta \Rightarrow d_G(x) < n - |N_G(b) \cup N_G(a)| \leq n - \Delta - 1$. (3)

Assume to the contrary that $d_G(x) \geq n - |N_G(b) \cup N_G(a)|$. By Lemma 3.7 (iii), we have $d_G(x) = \Delta - 1$. Again, as $\Delta \geq \frac{3(n-1)}{4}$ and every vertex in $(N_G(a) \cup N_G(b) \cup N_G(t)) \setminus \{a, b\}$ has degree at least $\Delta - 1$, for any $s \in N_G(t)$, we conclude that there exists $u \in N_G(b) \cap N_G(s) \cap N_G(x)$. Now by (2), $K = (a, ab, b, bu, u, us, s, st)$ and $K^* = (a, ab, b, bu, u, ux, x)$ are two Kiersted paths with respect to $ab$ and $\varphi$. Clearly, $\varphi(t) \subseteq \varphi(a) \cup \varphi(b)$ and $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, and $|\varphi(t)| \geq \Delta - (n - \Delta - 1) = 2\Delta - n + 1 \geq \frac{n-1}{2}$. Since $n \geq |V(K) \cup \{x\}| = 6$, $\Delta \geq \frac{3(n-1)}{4}$ implies that $\Delta \geq 4$. As $\{b, s, t, x\} \cap N_G(b) = \emptyset$, we see that $n \geq 4 + 4 = 8$. Hence, $|\varphi(t)| \geq \lceil \frac{n}{4} \rceil \geq 4$, achieving a contradiction to Lemma 3.2.

By Lemma 3.7 (iii) and (iv) and the conclusion in (3), we let $t_1, t_2 \in V(G) \setminus \{a, b\}$ such that both of them have degree less than $n - \Delta$. Let $s_1 \in N_G(t_1)$ and $s_2 \in N_G(t_2)$ be any two distinct vertices. Since $G$ is $\Delta$-critical, VAL implies that for every $s_i \in N_G(t_i)$, $d_G(s_i) \geq 2\Delta - n > n - \Delta$. Thus, By Lemma 3.7 (iii) and (iv) and the conclusion in (3), $d_G(s_1) = d_G(s_2) = \Delta$. Thus, since $b, c, s_1, s_2 \notin N_G(b) \cap N_G(c)$ and $b, c, s_1, s_2 \notin N_G(s_1) \cap N_G(s_2)$,

$$|N_G(b) \cap N_G(c) \cap N_G(s_1) \cap N_G(s_2)| \geq |N_G(s_1) \cap N_G(s_2)| - (n - |N_G(b) \cap N_G(c)| - 4),$$

$$\geq |N_G(s_1) \cap N_G(s_2)| + |N_G(b) \cap N_G(c)| + 4 - n$$

$$\geq 2\Delta - n + 2\Delta - n + 4 - n \geq 1,$$

as $\Delta \geq \frac{3(n-1)}{4}$. Let $u \in N_G(b) \cap N_G(c) \cap N_G(s_1) \cap N_G(s_2)$. Then $H$ with $V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\}$ is kite. By (2), both

$K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1)$ and $K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)$

are Kiersted paths with respect to $ab$ and $\varphi$. Let

$$A = \{t \in V(G) \setminus \{a, b\} : d_G(t) < n - \Delta\}.$$

We consider two cases below.

**Case 1: there exist two distinct $t_1, t_2 \in A$ such that $\varphi(t_1) \cap \varphi(t_2) \neq \emptyset$.**

In this case, we choose $s_1 \in N_G(t_1)$ and $s_2 \in N_G(t_2)$ such that $\varphi(s_1t_1) = \varphi(s_2t_2)$. Let $\Gamma = \varphi(t_1) \cap \varphi(t_2)$. Since $\varphi(a) \cup \varphi(b) = [1, \Delta]$, $\Gamma \subseteq \varphi(a) \cup \varphi(b)$. By (2) and the assumption of this case, $|\Gamma| \geq \Delta - 2(n - \Delta - 2) = 3\Delta - 2n + 4 \geq \frac{n-1}{4} + 2$. Since $\Delta \geq \frac{3(n-1)}{4}$, 

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3\(|V(H)| - 1\), \(\Delta \geq 6\). Since also \(s_1, s_2, t_1, t_2 \not\in N_G(b)\), we have \(n \geq |V(H)| + 3 \geq 11\). Thus, \(|\Gamma| \geq \lceil \frac{n-1}{4} \rceil + 2 \geq 5\), contradicting Lemma 3.4.

**Case 2:** for each two distinct \(t_1, t_2 \in A\), it holds that \(\varphi(t_1) \cap \varphi(t_2) = \emptyset\).

By (2) and the assumption of this case, we see that \(H^*\) with \(V(H^*) = \{a, b, u, s_1, s_2, t_1, t_2\}\) is fork. However, by (3), \(d_G(a) + d_G(t_1) + d_G(t_2) \leq 2 + 2(n - \Delta - 2) = 2n - 2\Delta - 2 < \Delta\), as \(\Delta \geq \frac{3(n-1)}{4}\), contradicting Lemma 3.5. The proof is now completed.

\(\square\)

## 4 Proof of Lemmas 3.1 to 3.4

**Lemma 3.1.** Let \(G\) be a Class 2 graph, \(ab \in E(G)\) be a critical edge, \(\varphi \in C^\Delta(G - ab)\), and \(K = (a, ab, b, bu, u, us, s, st, t)\) be a Kierstead path with respect to \(ab\) and \(\varphi\). If \(|\varphi(t) \cap (\varphi(a) \cup \varphi(b))| \geq 3\), then the following hold:

(a) There exists \(\varphi^* \in C^\Delta(G - ab)\) satisfies the following properties:

(i) \(\varphi^*(bu) \in \varphi^*(a) \cap \varphi^*(t)\),

(ii) \(\varphi^*(us) \in \varphi^*(b) \cap \varphi^*(t)\), and

(iii) \(\varphi^*(st) \in \varphi^*(a)\).

(b) \(d_G(b) = d_G(u) = \Delta\).

**Proof.** By Lemma 2.2, for every \(\varphi' \in C^\Delta(G - ab)\), \(\{a, b\}\) is \(\varphi'\)-elementary and for every \(i \in \varphi(a)\) and \(j \in \varphi(b)\), \(a, b\) are \((i, j)\)-linked with respect to \(\varphi'\).

Let \(\Gamma = \varphi(t) \cap (\varphi(a) \cup \varphi(b))\), and \(\alpha, \beta \in \Gamma\). If \(\alpha, \beta \in \varphi(a)\), then we let \(\lambda \in \varphi(b)\), and do a \((\beta, \lambda)\)-swap at \(b\). If \(\alpha, \beta \in \varphi(b)\), then we let \(\lambda \in \varphi(a)\), and do a \((\beta, \lambda)\)-swap at \(a\). Therefore, we may assume that

\[\alpha \in \varphi(a) \quad \text{and} \quad \beta \in \varphi(b).\]

If \(\varphi(bu) = \delta \not= \alpha\), then we do an \((\alpha, \delta)\)-swap at \(t\), and rename the color \(\delta\) as \(\alpha\) and vice versa. Thus we may assume

\[\varphi(bu) = \alpha.\]

Assume first that \(\varphi(us) \in \varphi(b)\). We do a \((\beta, \varphi(us))\)-swap at \(t\) and still call the resulting coloring by \(\varphi\), we see that \(\varphi(us) \in \varphi(b) \cap \varphi(t)\). By permuting the name of the colors, we let \(\varphi(us) = \beta\). Let \(\varphi(st) = \gamma\). Since \(\alpha, \beta \in \varphi(t)\), \(\gamma \not= \alpha, \beta\). If \(\gamma \in \varphi(a)\), we are done. So we assume \(\gamma \in \varphi(b) \cup \varphi(u)\). We color \(ab\) by \(\alpha\) and uncolor \(bu\). Denote this resulting coloring by \(\varphi'\). Then \(K' = (b, bu, u, us, s, st, t)\) is a Kierstead path with respect to \(bu\) and \(\varphi'\). However, \(\alpha, \beta \in \varphi'(t) \cap (\varphi'(b) \cup \varphi'(u))\), showing a contradiction to Lemma 2.4 (b).

Thus we let \(\varphi(us) = \delta \in \varphi(a)\). Then \(\delta \not= \beta\) by Lemma 2.2 (a). Let \(\varphi(st) = \gamma\). Clearly, \(\gamma \not= \alpha, \beta, \delta\). We have either \(\gamma \in \varphi(a)\) or \(\gamma \in \varphi(b) \cup \varphi(u)\). We consider three cases below.
Case 1. $\gamma \in \mathcal{F}(b)$.

If $u \in P_a(\beta, \delta) = P_b(\beta, \delta)$, we do a $(\beta, \delta)$-swap at $t$. Since $a$ and $b$ are $(\delta, \gamma)$-linked and $u \in P_t(\delta, \gamma)$, we do a $(\delta, \gamma)$-swap at $a$. This gives a desired coloring $\varphi^*$.

If $u \notin P_a(\beta, \delta) = P_b(\beta, \delta)$, we first do a $(\beta, \delta)$-swap at $a$ and then a $(\beta, \gamma)$-swap at $a$. Again this gives a desired coloring $\varphi^*$.

**Case 2.** $\gamma \in \mathcal{F}(u)$.

If $\delta \in \Gamma$, since $b$ and $u$ are $(\beta, \gamma)$-linked by Lemma 2.2 (b), we do $(\beta, \gamma)$-swap at $t$. Now $u \in P_t(\delta, \beta)$, we do a $(\beta, \delta)$-swap at $a$. This gives a desired coloring $\varphi^*$. Thus we assume $\delta \notin \Gamma$. Since $b$ and $u$ are $(\beta, \gamma)$-linked by Lemma 2.2 (b), and $a$ and $u$ are $(\delta, \gamma)$-linked by Lemma 2.3 (a), we do $(\beta, \gamma) - (\gamma, \delta)$-swaps at $t$. Finally, since $u \in P_t(\beta, \delta)$, we do a $(\beta, \delta)$-swap at $a$. This gives a desired coloring $\varphi^*$.

**Case 3.** $\gamma \in \mathcal{F}(a)$.

If $\delta \in \Gamma$, we do a $(\beta, \gamma)$-swap at $t$ and then a $(\beta, \delta)$-swap at $a$ to get a desired coloring $\varphi^*$. Thus we assume $\delta \notin \Gamma$. Let $\tau \in \Gamma \setminus \{\alpha, \beta\}$. If $\tau \in \mathcal{F}(u)$, since $a$ and $u$ are $(\delta, \gamma)$-linked by Lemma 2.3 (a), we do a $(\tau, \delta)$-swap at $t$. This gives back to the previous case that $\delta \in \Gamma$. Next we assume $\tau \in \mathcal{F}(b)$. It is clear that $u \in P_a(\tau, \delta) = P_b(\tau, \delta)$, as otherwise, a $(\tau, \delta)$-swap at $a$ gives a desired coloring. Thus we do a $(\tau, \delta)$-swap at $t$, giving back to the previous case that $\delta \in \Gamma$.

Now we assume $\tau \in \mathcal{F}(u)$. If $u \notin P_a(\beta, \delta)$, we do a $(\beta, \delta)$-swap at $a$. Since $a$ and $b$ are $(\alpha, \delta)$-linked and $u \in P_a(\alpha, \delta)$, we do an $(\alpha, \delta)$ swap at $t$. Now since $u \in P_t(\gamma, \delta)$, we do a $(\gamma, \delta)$-swap at $a$, and do $(\beta, \gamma) - (\gamma, \alpha)$-swaps at $t$. Since $a$ and $b$ are $(\tau, \gamma)$-linked, we do a $(\tau, \gamma)$-swap at $t$, and then a $(\beta, \gamma)$-swap at $a$. Now since $u \in P_t(\beta, \delta)$, we do a $(\beta, \delta)$-swap at $a$. This gives a desired coloring. Thus, we assume $u \notin P_a(\beta, \delta)$. We do a $(\beta, \delta)$-swap at $t$, and then a $(\beta, \gamma)$-swap at $t$. Next we do a $(\beta, \gamma)$-swap at $a$ and then a $(\gamma, \delta)$-swap at $a$. This gives a desired coloring for (b).

For statement (b), let $\varphi^* \in C^\Delta(G - ab)$ satisfying (i)–(iii). Let $\alpha, \gamma \in \mathcal{F}^*(a)$, $\beta \in \mathcal{F}^*(b)$ with $\alpha, \beta \in \mathcal{F}(t)$ such that

$$\varphi^*(bu) = \alpha, \quad \varphi^*(us) = \beta, \quad \text{and} \quad \varphi^*(st) = \gamma.$$ 

Let $\tau \in \mathcal{F}(t) \setminus \{\alpha, \beta\}$. Suppose to the contrary first that $d_G(b) \leq \Delta - 1$. Let $\lambda \in \mathcal{F}(b) \setminus \{\beta\}$. We do $(\tau, \lambda) - (\lambda, \gamma)$-swaps at $t$. Now we color $ab$ by $\alpha$ and uncolor $bu$ to get a coloring $\varphi'$. Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to $bu$ and $\varphi'$. However, $\alpha, \beta \in \mathcal{F}'(t) \cap (\mathcal{F}(b) \cup \mathcal{F}(u))$, contradicting Lemma 2.4 (b).

Assume then that $d_G(b) = \Delta$ and $d_G(u) \leq \Delta - 1$. Let $\lambda \in \mathcal{F}(u)$. Since $(a, ab, b, bu, u)$ is a multifan, $\lambda \notin \{\alpha, \beta, \gamma\}$. Since $u$ and $b$ are $(\beta, \lambda)$-linked and $u$ and $a$ are $(\gamma, \lambda)$-linked by Lemma 2.3 (b), we do $(\beta, \lambda) - (\lambda, \gamma)$-swap(s) at $t$. Now we color $ab$ by $\alpha$ and uncolor $bu$ to get a coloring $\varphi'$. Then $K' = (b, bu, u, us, s, st, t)$ is a Kierstead path with respect to $bu$ and $\varphi'$. However, $\alpha \in \mathcal{F}(t) \cap \mathcal{F}'(u)$, contradicting Lemma 2.4 (a), since $d_G(u) < \Delta$. \hfill $\square$
Lemma 3.2. Let $G$ be a Class 2 graph, $ab \in E(G)$ be a critical edge, $\varphi \in \mathcal{C}^\Delta(G - ab)$, and $K = (a, ab, b, bu, u, us, s, st, t)$ and $K^* = (a, ab, b, bu, u, ux, x)$ be two Kierstead paths with respect to $ab$ and $\varphi$, where $x \notin V(K)$. If $|\varphi(t) \cap (\varphi(a) \cup \varphi(b))| \geq 4$ and $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, then $d_G(x) = \Delta$.

**Proof.** Assume to the contrary that $d_G(x) \leq \Delta - 1$. Since $\varphi(x) \subseteq \varphi(a) \cup \varphi(b)$, Lemma 2.4 (b) gives that $d_G(x) = \Delta - 1$. By Lemma 3.1, $d_G(b) = d_G(u) = \Delta$ and we assume that $\varphi(b) = \beta, \varphi(bu) = \alpha, \varphi(us) = \beta, \varphi(st) = \gamma, \alpha, \gamma \in \varphi(a)$, and $\alpha, \beta \in \varphi(t)$. In the following, when we swap colors, we always make sure that the colors on the edges $bu$ and $us$ are unchanged. The color on the edge $st$ might be changed, but the new color will still be a color from $\varphi(a)$. This is guaranteed by using the elementary fact that for every coloring $\varphi' \in \mathcal{C}^\Delta(G - ab)$, $a$ and $b$ are $(i, j)$-linked for every $i \in \varphi(a)$ and every $j \in \varphi(b)$.

We use this fact every often without even mentioning it.

Let $\varphi(ux) = \delta$ and $\varphi(x) = \tau$. We first claim that if $\delta \neq \gamma$, then we may assume $\delta \in \varphi(t)$. Clearly, $\delta \neq \alpha, \beta$, and since $K^*$ is a Kierstead path and $\varphi(b) = \beta$, we have $\varphi(ux) = \delta \in \varphi(a)$. Let $\Gamma = \varphi(t) \cap (\varphi(a) \cup \varphi(b))$, and let $\{\alpha, \beta, \eta, \lambda\} \subseteq \Gamma$. Suppose that $\delta \notin \varphi(t)$. We do $(\beta, \gamma) - (\gamma, \eta)$-swaps at $b$. Denote the new coloring by $\varphi'$.

If $ux \notin P_b(\eta, \delta)$, based on $\varphi'$, we do an $(\eta, \delta)$-swap at $b$ and then do an $(\alpha, \delta)$-swap at $t$. If $\tau = \gamma$, we do $(\delta, \gamma) - (\gamma, \eta)$-swaps at $b$ and then do an $(\alpha, \eta)$-swap at $t$. Finally we do $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at $b$. Thus we assume $\tau \neq \gamma$. Clearly, $\tau \neq \alpha$. If $\tau = \beta$, we simply do a $(\beta, \delta)$-swap at $b$, then $(\beta, \gamma) - (\gamma, \eta)$-swaps at $b$, an $(\alpha, \eta)$-swap at $t$, and finally $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at $b$. Thus, $\tau \neq \alpha, \beta$. We do $(\delta, \tau) - (\tau, \eta)$-swaps at $b$, an $(\alpha, \eta)$-swap at $t$, and finally do $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at $b$.

Thus, we assume that $ux \notin P_b(\eta, \delta)$. Based on $\varphi'$, we do an $(\eta, \delta)$-swap at $t$ and then $(\eta, \lambda) - (\lambda, \gamma) - (\gamma, \beta)$-swaps at $b$.

After the operations above, we have $\varphi(bu) = \alpha, \varphi(us) = \beta, \varphi(st) = \gamma, \varphi(ux) = \delta$ and $\varphi(x) = \tau$, and $\alpha, \beta, \delta, \lambda \in \Gamma$.

**Case 1:** $\varphi(x) = \gamma$.

Recall that $\alpha, \beta, \delta \in \Gamma$. We color $ab$ by $\alpha$, recolor $bu$ by $\beta$, and uncolor $us$. Note that $u$ and $t$ are $(\alpha, \gamma)$-linked, as otherwise an $(\alpha, \gamma)$-swap at $u$ and a $(\beta, \gamma)$-swap at $s$ gives a coloring $\varphi'$ such that $\gamma \in \varphi(u) \cap \varphi(s)$. Thus we do an $(\alpha, \gamma)$-swap at both $a$ and $x$, recolor $ux$ by $\alpha$, and then a $(\beta, \delta)$-swap at both $x$ and $a$. It is clear that $ux \in P_t(\alpha, \gamma)$ and $P_t(\alpha, \gamma)$ meets $u$ before $x$. We now do the following operations:

$$
\begin{bmatrix}
P_{[t,u]}(\alpha, \gamma) & ux & bu & ab \\
\alpha/\gamma & \alpha \rightarrow \beta & \beta \rightarrow \gamma & \gamma \rightarrow \beta
\end{bmatrix}
$$

Based on the coloring above, we do $(\alpha, \beta) - (\beta, \delta)$-swaps at both $x$ and $a$, and then an $(\alpha, \delta)$-swap at $x$. Denote the new coloring by $\varphi'$. Now we do an $(\alpha, \beta)$-swap at $t$ and color $us$ by $\alpha$, giving a $\Delta$-coloring of $G$.
Case 2: $\overline{\varphi}(x) \neq \gamma$ and $\varphi(ux) \neq \varphi(st) = \gamma.$

Recall that $\alpha, \beta, \delta \in \Gamma$ and $|\Gamma| \geq 4$. Let $\{\alpha, \beta, \delta, \lambda\} \subseteq \Gamma$. We show that there is a coloring $\varphi'$ such that $\{\alpha, \beta\} \subseteq \overline{\varphi}(t)$ and $\varphi'(x) = \beta$. Since we already have $\{\alpha, \delta\} \subseteq \overline{\varphi}(t)$, we assume that $\overline{\varphi}(x) = \tau \neq \beta$. Thus $\tau \in \overline{\varphi}(a)$. If $\tau = \alpha$, we simply do a $(\alpha, \beta)$-swap at $x$. Therefore, $\tau \neq \alpha$. It is possible that $\tau = \lambda$, but we deal with this together with the case that $\tau \neq \lambda$. We first do $(\beta, \gamma) - (\gamma, \lambda) - (\lambda, \tau)$-swaps at $b$, then we do an $(\alpha, \tau)$-swap at both $x$ and $t$. Now we do $(\tau, \gamma) - (\gamma, \beta)$-swaps at $b$, and an $(\alpha, \beta)$-swap at both $x$ and $t$. We now derive a contradiction based on the coloring of $E(K) \cup E(K^*)$, as shown in Figure 2.

Figure 2: Colors on the edges of $K$ and $K^*$

We color $ab$ by $\alpha$, recolor $bu$ by $\beta$, and uncolor $us$. We then do an $(\alpha, \beta)$-swap at both $x$ and $t$, and an $(\alpha, \delta)$-swap at $x$. Now since $u$ and $s$ are $(\beta, \delta)$-linked, we do a $(\beta, \delta)$-swap at both $x$ and $a$. Since $u$ and $t$ are $(\delta, \gamma)$-linked, we do a $(\delta, \gamma)$-swap at $a$. Finally, we do $(\beta, \delta) - (\delta, \alpha)$-swaps at $x$. Now $P_u(\alpha, \beta) = uba$, so $u$ and $s$ are $(\alpha, \beta)$-unlined. We do an $(\alpha, \beta)$-swap at $u$ and color $us$ by $\beta$. This gives a $\Delta$-coloring of $G$, showing a contradiction.

Case 3: $\overline{\varphi}(x) = \tau \neq \gamma$ and $\varphi(ux) = \varphi(st) = \gamma.$

We again let $\Gamma = \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$, and let $\{\alpha, \beta, \lambda\} \subseteq \Gamma$. We show that this case can be converted to Case 2. We first claim that $\overline{\varphi}(x) = \tau \neq \beta$. As otherwise, we first do an $(\alpha, \beta)$-swap at $x$, and then a $(\beta, \gamma)$-swap at $b$. Now, $P_t(\gamma, \alpha) = bux$, showing a contradiction to the fact that $a$ and $b$ are $(\alpha, \gamma)$-linked. Next, we claim that $\tau \neq \alpha$. As otherwise, we simply do a $(\beta, \gamma)$-swap at $b$ and achieve a same contradiction as above. Thus, $\tau \neq \alpha, \beta$.

We do a $(\beta, \gamma)$-swap at $b$ and an $(\alpha, \gamma)$-swap at $t$. Now do a $(\tau, \gamma)$-swap at both $x$ and $t$, a $(\gamma, \lambda)$-swap at $b$, an $(\lambda, \alpha)$-swap at $t$, and finally a $(\beta, \lambda)$-swap at $b$. Let the new coloring be $\varphi'$. We see that $\varphi'(st) = \lambda \neq \varphi'(ux) = \tau$. We verify that it still holds that $|\overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \geq 4$. If $\tau \in \Gamma$, we now have $\alpha, \beta, \gamma, \tau \in \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$. If $\tau \notin \Gamma$, then $(\Gamma \setminus \{\lambda\}) \cup \{\tau\} \subseteq \overline{\varphi}(t) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b)).$  

**Lemma 3.3.** Let $G$ be a Class 2 graph, $H \subseteq G$ be a short-kite with $V(H) = \{a,b,c,u,x,y\}$, and let $\varphi \in C^\Delta(G - ab)$. Suppose

$$K = \{a,ab,b,ba,ux,x\} \text{ and } K^* = \{b,ab,a,ac,c,xb,ub,uy\}$$
are two Kierstead path with respect to \( ab \) and \( \varphi \). If \( \varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \), then 
\[
\max\{d_G(x), d_G(y)\} = \Delta.
\]

**Proof.** Assume to the contrary that \( \max\{d_G(x), d_G(y)\} \leq \Delta - 1 \). Since both \( K \) and \( K^* \) are Kierstead paths and \( \varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \), Lemma 2.4 (a) and (b) implies that 
\[
d_G(b) = d_G(u) = \Delta \quad \text{and} \quad d_G(x) = d_G(y) = \Delta - 1.
\]

Let \( \varphi(b) = \{1\} \). Then \( \varphi(ac) = 1 \). We may assume \( \varphi(uy) = 1 \). The reasoning is below. Since \( a \) and \( b \) are \((1, \alpha)\)-linked for every \( \alpha \in \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \), we may assume 
\[
\varphi(y) = 1.
\]

Then a \((1, \varphi(uy))\)-swap at \( y \) gives a coloring, call it still \( \varphi \), such that \( \varphi(uy) = 1 \). We consider now two cases.

**Case 1:** \( \varphi(x) = \varphi(y) \).

Let \( \varphi(wx) = \gamma \), and \( \varphi(x) = \varphi(y) = \eta \). As \( \varphi(uy) = \varphi(b) = 1, 1 \not\in \{\gamma, \eta\} \). As both \( K \) and \( K^* \) are Kierstead paths and \( \varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \), \( \gamma, \eta \in \varphi(a) \). Denote by \( P_u(1, \gamma) \) the \((1, \gamma)\)-subchain starting at \( u \) that does not include the edge \( ux \).

**Claim 4.1.** We may assume that \( P_u(1, \gamma) \) ends at \( x \), some vertex \( z \in V(G) \setminus \{a, b, c, u, x, y\} \), or passing \( c \) ends at \( a \).

**Proof.** Note that \( P_u(1, \gamma) = P_b(1, \gamma) \). If \( u \notin P_a(1, \gamma) \), then the \((1, \gamma)\)-chain containing \( u \) is a cycle or a path with endvertices contained in \( V(G) \setminus \{a, b, c, u, x, y\} \). Thus \( P_u(1, \gamma) \) ends at \( x \) or some \( z \in V(G) \setminus \{a, b, c, u, x, y\} \). Hence we assume \( u \in P_a(1, \gamma) \). As a consequence, \( P_u(1, \gamma) \) ends at either \( b \) or \( a \). If \( P_x(1, \gamma) \) ends at \( b \), we color \( ab \) by 1, uncolor \( ac \), and exchange the vertex labels \( b \) and \( c \). This gives an edge \( \Delta \)-coloring of \( G - ab \) such that \( P_u(1, \gamma) \) ends at \( a \). Thus, if \( u \in P_u(1, \gamma) \), we may always assume that \( P_u(1, \gamma) \) ends at \( a \).

Let \( \varphi(bu) = \delta \). Again, \( \delta \in \varphi(a) \). Figure 3 depicts the colors and missing colors on these specified edges and vertices, respectively. Clearly, \( \delta \neq 1, \gamma \). Since \( a \) and \( b \) are \((1, \delta)\)-linked with respect to \( \varphi \), \( \eta \neq \delta \). Thus, \( \gamma, \delta \) and \( \eta \) are pairwise distinct.

**Claim 4.2.** It holds that \( ub \in P_y(\eta, \delta) \) and \( P_y(\eta, \delta) \) meets \( u \) before \( b \).

**Proof.** Let \( \varphi' \) be obtained from \( \varphi \) by coloring \( ab \) by \( \delta \) and uncoloring \( bu \). Note that \( \varphi'(b) = 1, \varphi'(u) = \delta \) and \( \varphi'(uy) = 1 \). Thus \( F^* = (u, ub, b, uy, y) \) is a multifan and so \( u \) and \( y \) are \((\eta, \delta)\)-linked by Lemma 2.2 (b). By uncoloring \( ab \) and coloring \( bu \) by \( \delta \), we get back the original coloring \( \varphi \). Therefore, under the coloring \( \varphi \), \( u \in P_y(\eta, \delta) \) and \( P_y(\eta, \delta) \) meets \( u \) before \( b \).

We apply the following operations based on \( \varphi \):

\[
\begin{bmatrix}
ux & P_{u,y}(\eta, \delta) & ub & P_u(1, \gamma) & ab \\
\gamma \to \eta & \delta/\eta & \delta \to 1 & 1/\gamma & \delta
\end{bmatrix}.
\]

By Claim 4.1, \( P_u(1, \gamma) \) does not end at \( b \). In any case, the above operations give an edge \( \Delta \)-coloring of \( G \). This contradicts the earlier assumption that \( \chi'(G) = \Delta + 1 \).
Figure 3: Colors on the edges connecting \( x \) and \( y \) to \( b \)

**Case 2:** \( \varphi(x) \neq \varphi(y) \).

Let
\[
\varphi(bu) = \alpha, \quad \varphi(ux) = \beta, \quad \varphi(x) = \tau, \quad \text{and} \quad \varphi(y) = \gamma.
\]

As \( \varphi(uy) = \varphi(b) = 1 \), \( 1 \not\in \{ \alpha, \beta, \gamma \} \). Also, since \( a \) and \( b \) are \((1, \alpha)\)-linked, \( \gamma \neq \alpha \). Since both \( K \) and \( K^* \) are Kierstead paths and \( \varphi(x) \cup \varphi(y) \subseteq \varphi(a) \cup \varphi(b) \), we have \( \alpha, \beta, \tau, \gamma \in \varphi(a) \).

**Claim 4.3.** We may assume \( \varphi(x) = \tau = 1 \).

**Proof.** If \( uy \not\in P_x(1, \tau) \), we simply do a \((1, \tau)\)-swap at \( x \). Thus, we assume that \( u \in P_x(1, \tau) \). We first do a \((1, \tau)\)-swap at \( b \), then an \((\alpha, \tau)\)-swap at \( x \). Then we do a \((\gamma, \tau)\)-swap at \( b \) and a \((1, \alpha)\)-swap at \( x \) give the desired coloring. \( \square \)

Since \( ux \in P_x(1, \beta) \), and \( a \) and \( b \) are \((1, \beta)\)-linked, we do a \((1, \beta)\)-swap at \( b \). Now we color \( ab \) by \( \alpha \), recolor \( bu \) by \( \beta \) and uncolor \( ux \), see Figure 4 for a depiction.

Note that
\[
F^* = (u, ux, x, uy, y), \quad K^* = (x, xu, u, ub, b, ba, a)
\]
are, respectively, a multifan and a Kierstead path. By Lemma 2.2 (b), \( u \) and \( y \) are \((\alpha, \gamma)\)-linked, and \( u \) and \( x \) are \((\alpha, \beta)\)-linked and \((1, \alpha)\)-linked. Thus, we do an \((\alpha, \gamma)\)-swap at \( a \), an \((\alpha, \beta)\)-swap at \( a \), a \((1, \alpha)\)-swap at \( a \), and then an \((\alpha, \gamma)\)-swap at \( a \). Now \( P_u(\alpha, \beta) = uba \), contradicting Lemma 2.2 (b) that \( u \) and \( x \) are \((\alpha, \beta)\)-linked. The proof is now completed. \( \square \)

**Lemma 3.4.** Let \( G \) be a Class 2 graph, \( H \subseteq G \) be a kite with \( V(H) = \{a, b, c, u, s_1, s_2, t_1, t_2\} \), and let \( \varphi \in C^\Delta(G - ab) \). Suppose
\[
K = (a, ab, b, bu, u, us_1, s_1, s_1t_1, t_1) \quad \text{and} \quad K^* = (b, ab, a, ac, c, cu, u, us_2, s_2, s_2t_2, t_2)
\]
are two Kierstead paths with respect to $ab$ and $\varphi$. If $\varphi(s_1t_1) = \varphi(s_2t_2)$, then $|\overline{\varphi}(t_1) \cap \overline{\varphi}(t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))| \leq 4$.

**Proof.** Let $\Gamma = \overline{\varphi}(s_1t_1) \cap \overline{\varphi}(s_2t_2) \cap (\overline{\varphi}(a) \cup \overline{\varphi}(b))$. Assume to the contrary that $|\Gamma| \geq 5$. By considering $K$ and applying Lemma 3.1, we conclude that $d_G(b) = d_G(u) = \Delta$. We show that there exists $\varphi^* \in C^\Delta(G - ab)$ satisfying the following properties:

(i) $\varphi^*(bu), \varphi^*(cu), \varphi^*(us_2) \in \overline{\varphi^*}(a) \cap \overline{\varphi^*}(t_1) \cap \varphi^*(t_2),$
(ii) $\varphi^*(us_1) \in \overline{\varphi^*}(b) \cap \overline{\varphi^*}(t_1) \cap \varphi^*(t_2)$, and
(iii) $\varphi^*(s_1t_1) = \varphi^*(s_2t_2) \in \overline{\varphi^*}(a)$.

See Figure 5 for a depiction of the colors described above.

Let $\alpha, \beta, \tau, \delta \in \Gamma$, and let $\varphi(s_1t_1) = \varphi(s_2t_2) = \gamma$. We may assume that $\alpha \in \overline{\varphi}(a)$ and $\beta \in \overline{\varphi}(b)$. Otherwise, since $d_G(b) = \Delta$, we have $\alpha, \beta \in \overline{\varphi}(a)$. Let $\lambda \in \overline{\varphi}(b)$. As $a$ and $b$ are $(\beta, \lambda)$-linked, we do a $(\beta, \lambda)$-swap at $b$. Note that this operation may change some colors of the edges of $K$ and $K^*$, but they are still Kierstead paths with respect to $ab$ and the current coloring.

Since $d_G(b) = d_G(u) = \Delta$, and $\beta \in \overline{\varphi}(b) \cap \overline{\varphi}(t_1)$, we know that $\gamma \in \overline{\varphi}(a)$, as $K_1$ is a Kierstead path. Next, we may assume that $\varphi(bu) = \alpha$. If not, let $\varphi(bu) = \alpha'$. Since $a$ and $b$ are $(\alpha, \beta)$-linked, we do an $(\alpha, \beta)$-swap at $b$. Now $a$ and $b$ are $(\alpha, \alpha')$-linked, we do an $(\alpha, \alpha')$-swap at $b$. Finally, we do an $(\alpha', \beta)$-swap at $b$. All these swaps do not change the colors in $\Gamma$, so now we get the color on $bu$ to be $\alpha$.

We may now assume that $\varphi(cu) = \tau$. If not, let $\varphi(cu) = \tau'$. Since $a$ and $b$ are $(\beta, \tau)$-linked, we do a $(\beta, \tau)$-swap at $b$. Then do $(\tau, \tau') - (\tau', \beta)$-swaps at $b$. 

![Figure 4: Colors on the edges connecting $x$ and $y$ to $b$](image)
Finally, we show that we can modify $\varphi$ to get $\varphi'$ such that $\varphi'(us_1) = \beta$ and $\varphi'(us_2) = \delta$. Assume firstly that $\varphi(us_1) = \beta' \neq \beta$. If $\beta' \in \Gamma$, we do $(\beta, \gamma) - (\gamma, \beta')$-swaps at $b$. Thus, we assume $\beta' \not\in \Gamma$. Let $\lambda \in \Gamma \setminus \{\alpha, \beta, \tau, \delta\}$. If $u \not\in P_a(\beta, \beta') = P_b(\beta, \beta')$, we simply do a $(\beta, \beta')$-swap at $b$. Thus, we assume $u \in P_a(\beta, \beta') = P_b(\beta, \beta')$. We do a $(\beta, \beta')$-swap at both $t_1$ and $t_2$. Since $a$ and $b$ are $(\beta, \lambda)$-linked, we do a $(\beta, \lambda)$-swap at both $t_1$ and $t_2$. Now we do $(\beta, \gamma) - (\gamma, \beta')$-swaps at $b$. By switching the role of $\beta$ and $\beta'$, we have $\varphi(us_1) = \beta$. Lastly, we show that $\varphi(us_2) = \delta$.

Note that $bu \in P_{t_1}(\alpha, \gamma)$. Otherwise, let $\varphi' = \varphi/P_{t_1}(\alpha, \gamma)$. Then $P_b(\alpha, \beta) = bus_1t_1$, showing a contradiction to the fact that $a$ and $b$ are $(\alpha, \beta)$-linked with respect to $\varphi'$. Thus, $bu \in P_{t_1}(\alpha, \gamma)$. Next, we claim that $P_{t_1}(\alpha, \gamma)$ meets $u$ before $b$. As otherwise, we do the following operations to get a $\Delta$-coloring of $G$:

$$\begin{bmatrix} s_1t_1 & P_{[s_1\beta][\alpha, \gamma]}(u, \gamma) & us_1 & bu & ab \\ \gamma \rightarrow \alpha & \alpha/\gamma & \beta \rightarrow \alpha & \alpha \rightarrow \beta & \gamma \end{bmatrix}.$$

This gives a contradiction to the assumption that $G$ is $\Delta$-critical. Thus, we have that $P_{t_1}(\alpha, \gamma)$ meets $u$ before $b$. This implies that it is not the case that $P_{t_2}(\alpha, \gamma)$ meets $u$ before $b$. In turn, this implies that $u \in P_{t_1}(\beta, \delta') = P_b(\beta, \delta')$. As otherwise, we get a $\Delta$-coloring of $G$ by doing a $(\beta, \delta)$-swap along the $(\beta, \delta)$-chain containing $u$, and then doing the same operation as above with $t_2$ playing the role of $t_1$.

Since $u \in P_{t_1}(\beta, \delta') = P_b(\beta, \delta')$, we do a $(\beta, \delta')$-swap at both $t_1$ and $t_2$. As $u \in P_{t_1}(\beta, \tau) = P_b(\beta, \tau)$, we do a $(\beta, \tau)$-swap at both $t_1$ and $t_2$. Since $us_1 \in P_{t_1}(\beta, \gamma)$, we do a $(\beta, \gamma)$-swap at $b$, then a $(\gamma, \lambda)$-swap at $b$. Since $a$ and $b$ are $(\tau, \lambda)$-linked, we do a $(\tau, \lambda)$-swap at both $t_1$.
and $t_2$. Now $(\lambda, \delta) - (\delta, \gamma) - (\gamma, \beta)$-swaps at $b$ give a desired coloring.

Still, by the same arguments as above, we have that $P_t(\alpha, \gamma)$ meets $u$ before $b$, and $u \in P_u(\beta, \delta) = P_b(\beta, \delta)$. Let $P_u(\beta, \delta)$ be the $(\beta, \delta)$-chain starting at $u$ not including the edge $us_2$. It is clear that $P_u(\beta, \delta)$ ends at either $a$ or $b$. We may assume that $P_u(\beta, \delta)$ ends at $a$. Otherwise, we color $ab$ by $\beta$, uncolor $ac$, and let $\tau$ play the role of $\alpha$. Let $P_u(\alpha, \gamma)$ be the $(\alpha, \gamma)$-chain starting at $u$ not including the edge $bu$, which ends at $t_1$ by our earlier argument. We do the following operations to get a $\Delta$-coloring of $G$:

$$\begin{bmatrix}
P_u(\alpha, \gamma) & bu & P_u(\beta, \delta) & us_2 & t_2 & ab \\
\alpha/\gamma & \alpha \rightarrow \beta & \beta/\delta & \delta/\gamma & \alpha
\end{bmatrix}.$$ 

This gives a contradiction to the assumption that $G$ is $\Delta$-critical. The proof is now finished.

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