We give an elementary geometric re-proof of a formula discovered by Brion as well as two variants thereof. A subset $K$ of $\mathbb{R}^n$ gives rise to a formal Laurent series with monomials corresponding to lattice points in $K$. Under suitable hypotheses, these series represent rational functions $\sigma(K)$. We will prove formul\ae relating the rational function $\sigma(P)$ of a lattice polytope $P$ to the sum of rational functions corresponding to the supporting cones subtended at the vertices of $P$. The exposition should be suitable for everyone with a little background in topology.

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1 Brion’s formula

The goal of this note is to exhibit a geometric proof of an astonishing formula discovered by Brion, relating the lattice point enumerator of a rational polytope to the lattice points enumerators of supporting cones subtended at its vertices. Roughly speaking, the theorem is about the surprising fact that in a certain sum of rational functions which are all given by infinite Laurent series, there is enough cancellation so that only finitely many terms survive: The sum collapses to a Laurent polynomial.

The argument is based on systematic usage of Euler characteristics of visibility complexes. The method of proof will readily yield two variants of Brion’s formula as well. These notes are intended as an easily accessible introduction for non-experts with some topological background.

We start with a 1-dimensional example to demonstrate the cancellation. The first series we consider is the well-known geometric series $\sum_{j=0}^{\infty} x^j$. For $|x| < 1$ this series converges to the rational function $f_1(x) = 1/(1-x)$. The second series is a variant of the geometric series in $x^{-1}$, namely $\sum_{j=-\infty}^{2} x^j = x^2 \sum_{j=-\infty}^{0} x^j$. For $|x^{-1}| < 1$ this series converges to $f_2(x) = x^2/(1-x^{-1})$. The two series have no common domain of convergence; we can, however, add the rational functions...
they represent and obtain

\[
f_1(x) + f_2(x) = \frac{1}{1-x} + \frac{x^2}{1-x^{-1}} = \frac{1}{1-x} + \frac{x^3}{x-1} = \frac{1-x^3}{1-x} = 1 + x + x^2,
\]

a polynomial with three terms only. Note that this happens only on the level of rational functions; adding the power series yields a (non-convergent, formal) power series with infinitely many terms. The interested reader might want to check the paper [BHS] which contains a careful exposition of a 2-dimensional example.

To formulate the main theorem, and to link the example to geometry, we have to introduce some notation first. Given a subset \( K \subseteq \mathbb{R}^n \) and a vector \( b \in \mathbb{R}^n \) we define \( b + K \) as the set of points of \( K \) shifted by the vector \( b \):

\[
b + K = \{ b + x \mid x \in K \}.
\]

We associate to each subset \( K \) of \( \mathbb{R}^n \) a formal Laurent series \( S(K) \) with complex coefficients in \( n \) indeterminates as follows. We write \( \mathbb{C}[[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]] \) for the set of Laurent series; it is a module over the ring \( \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \) of Laurent polynomials.

For a given vector \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \) we write \( x^a \) for the product \( x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \).

**1.1 Definition.** For a subset \( K \subseteq \mathbb{R}^n \) we define the formal Laurent series

\[
S(K) = \sum_{a \in \mathbb{Z}^n \cap K} x^a \in \mathbb{C}[[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]].
\]

A straightforward calculation shows \( S(b + K) = x^b S(K) \) for any \( b \in \mathbb{Z}^n \).

In favourable cases the series \( S(K) \) represents a rational function which we will denote \( \sigma(K) \in \mathbb{C}(x_1, x_2, \ldots, x_n) \). For example, if \( K = (-\infty, 2] \subset \mathbb{R} \), then \( S(K) = \sum_{j=-\infty}^2 x^j \), so \( \sigma(K) = x^2/(1-x^{-1}) \).

Let \( P \) denote a polytope (the convex hull of a finite set of points) in \( \mathbb{R}^n \). We assume throughout that \( P \) has non-empty interior, i.e., that \( P \) is of dimension \( n \). Given a vertex \( v \) of \( P \) we define the barrier cone \( C_v \) of \( P \) at \( v \) as the set of finite linear combination with non-negative real coefficients spanned by the set \(-v + P\). This is a cone based at the origin of the coordinate system, having the origin as a vertex. It is the smallest such cone containing the translate \(-v + P\) of \( P \).

Since \( C_v \) is a pointed cone, the associated Laurent series represents a rational function. We can thus formulate the following result (where \(-P = \{-x \mid x \in P\} \) in Equation (4)):
1.2 Theorem. Suppose $P$ is a polytope such that all its faces admit rational normal vectors (this happens, for example, if $P$ has vertices in $\mathbb{Z}^n$). Then there are equalities of rational functions

$$\sum_{v \text{ vertex of } P} \sigma(v + C_v) = \sigma(P), \quad (1)$$

$$\sum_{v \text{ vertex of } P} \sigma(C_v) = 1, \quad (2)$$

$$\sum_{v \text{ vertex of } P} \sigma(-v + C_v) = (-1)^n \sigma(\text{int}(-P)). \quad (3)$$

A version of this theorem appears as Proposition 3.1 in [BV97]. We give a more geometric proof, working out how visibility subcomplexes of polytopes enter the picture.

The example above shows how Equation (1) works for $P = [0, 2] \subset \mathbb{R}$. The vertices of $P$ are $v = 0$ and $v = 2$, the respective barrier cones are

$$C_0 = [0, \infty) \quad \text{and} \quad C_2 = (-\infty, 0],$$

so $0 + C_0 = C_0$ and $2 + C_2 = (-\infty, 2]$. For Equation (2), since $S(C_0) = \sum_{j=0}^{\infty} x^j$ we have $\sigma(C_0) = 1/(1 - x)$, and similarly $\sigma(C_2) = 1/(1 - x^{-1})$, so the Theorem predicts correctly that

$$\sigma(C_0) + \sigma(C_2) = \frac{1}{1 - x} + \frac{1}{1 - x^{-1}} = \frac{1 - x}{1 - x} = 1.$$

Finally, we consider Equation (3). We have $-P = [-2, 0]$, so the only integral point in the interior of $-P$ is $-1$, and the right-hand side of (3) is the single term $-x^{-1}$. On the left, we have $\sigma(-0 + C_0) = 1/(1 - x)$ as before, and $\sigma(-2 + C_2) = x^{-2}/(1 - x^{-2})$, and indeed

$$\sigma(-0 + C_0) + \sigma(-2 + C_2) = \frac{1}{1 - x} + \frac{x^{-2}}{1 - x^{-1}} = \frac{1 - x^{-1}}{1 - x} = x^{-1} \cdot \frac{x - 1}{1 - x} = -x^{-1}.$$

We will prove Theorem 1.2 in §4–5. Equation (1) of the Theorem is the original version of Brion’s formula [Bri88, Theorem 2.2] [Bri96, Theorem 2.1 (ii)].

The paper is inspired by Beck, Haase and Sottile [BHS] who gave a new, elementary proof of Brion’s formula. The approach taken in this note is a rather straightforward elaboration: It replaces the elegant but delicate combinatorics of [BHS] with a geometric analysis of visibility subcomplexes (§3) of a polytope. From a topologist’s point of view this makes the proof more transparent, while still avoiding the elaborate machinery of toric algebraic geometry used in the original proof [Bri88].

The basic strategy of proof is to establish an identity of formal Laurent power series first (Theorem 1.1), then pass to rational functions (Theorem 1.2). This is explained in detail in [BHS], but we will recall the relevant arguments for the convenience of the reader.
2 Polytopal complexes

A polytope $P$ is the convex hull of a non-empty finite set of points in $\mathbb{R}^n$. A face of $P$ is the intersection of $P$ with some supporting hyperplane; as a matter of convention, we also have the improper faces $F = P$ and $F = \emptyset$. See [Ewa96] and [Zie95] for more on polytopes and their faces.

2.1 Definition. A non-empty finite collection $K$ of non-empty polytopes in some $\mathbb{R}^n$ is called a polytopal complex if the following conditions are satisfied:

1. If $F \in K$ and $G$ is a non-empty face of $F$, then $G \in K$.
2. For all $F, G \in K$, the intersection $F \cap G$ is a (possibly empty) face of both $F$ and $G$.

A subset $L \subseteq K$ of a polytopal complex is called an order filter if for all $F \in L$ and $G \in K$ with $F$ a face of $G$, we have $G \in L$. A subset $L \subseteq K$ of a polytopal complex is called a subcomplex of $K$ if $L$ is a polytopal complex.

Important examples of polytopal complexes are the complex $F(P)_0$ of non-empty faces of a polytope $P$, and its subcomplex $F(P)_0^1$ of non-empty proper faces of $P$ (sometimes called boundary complex of $P$).

The intersection of two subcomplexes, if non-empty, is a subcomplex. The (set-theoretic) complement of a subcomplex is an order filter.

2.2 Definition. Suppose $K$ is a polytopal complex, and $L$ is a non-empty subset of $K$. We call $|L| := \bigcup_{F \in L} F$ the realisation or the underlying space of $L$.

If $P$ is an $n$-dimensional polytope, we have homeomorphisms $|F(P)_0| = P \cong B^n$ and $|F(P)_0^1| = \partial P \cong S^{n-1}$.

2.3 Definition. Let $L$ be a non-empty subset of the polytopal complex $K$. The Euler characteristic $\chi(L)$ of $L$ is defined by

$$\chi(L) = \sum_{A \in L} (-1)^{\dim(A)} .$$

If $L$ is a subcomplex of $K$, then $\chi(L)$ agrees with the Euler characteristic of $|L|$ as defined in algebraic topology. In particular, $\chi(F(P)_0) = \chi(P) = 1$ and $\chi(F(P)_0^1) = \chi(\partial P) = 1 + (-1)^{\dim(P)}$ for any polytope $P$.

2.4 Lemma. The Euler characteristic is additive: For a polytopal complex $K$ and a non-empty proper subset $L \subset K$ we have

$$\chi(K) = \chi(L) + \chi(K \setminus L) .$$
3 Visibility subcomplexes of a polytope

Understanding visibility subcomplexes of polytopes is the key to our approach to Brion’s theorem. The notions of visible, back and lower faces are defined, and we indicate a proof that these subcomplexes are balls in the boundary sphere of $P$. In particular, these complexes are contractible and have Euler characteristic 1.—We assume throughout that $P \subset \mathbb{R}^n$ is a polytope with int($P$) $\neq \emptyset$.

Visible and invisible faces

3.1 Definition. A face $F \in F(P)_0^1$ is called visible from the point $x \in \mathbb{R}^n \setminus P$ if $[p, x] \cap P = \{p\}$ for all $p \in F$. (Here $[p, x]$ denotes the line segment between $p$ and $x$. ) Equivalently, $F$ is visible if $p + \lambda(x - p) \notin P$ for all points $p \in F$ and real numbers $\lambda > 0$. We denote the set of visible faces by $\text{Vis}(x)$; its complement $\text{Inv}(x) := F(P)_0^1 \setminus \text{Vis}(x)$ is the set of invisible faces.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Visible faces}
\end{figure}

3.2 Lemma. A facet $F$ of $P$ is visible from $x$ if and only if $x$ and int $P$ are on different sides of the affine hyperplane spanned by $F$. A proper non-empty face of $P$ is visible if and only if it is contained in a visible facet of $P$.

In particular, the sets $\text{Vis}(x)$ and $\text{Inv}(x)$ are non-empty. Since a face of a visible face is visible itself, $\text{Vis}(x)$ is a subcomplex while $\text{Inv}(x)$ is an order filter.

3.3 Proposition. The space $|\text{Vis}(x)|$ is homeomorphic to an $(n - 1)$-ball. In particular, $\chi(\text{Vis}(x)) = 1$.

Proof. Applying a translation if necessary we may assume $x = 0$. Let $H$ be any hyperplane separating 0 and $P$ (Fig. 1). Let $C$ denote the cone (with apex 0)
on \( P \). Then \( C \) is a pointed polyhedral cone, hence \( C \cap H \) is a ball \cite{Ewa96}, Theorem V.1.1. Projection along \( C \) provides a homeomorphism \(|\text{Vis}(x)| \cong C \cap H\). \( \square \)

**Front and back faces**

3.4 Definition. A face \( F \in F(P)_{10}^1 \) is called a back face with respect to the point \( x \in \mathbb{R}^n \setminus \text{int} \ P \) if for all points \( p \in F \) and all real numbers \( \lambda > 0 \) we have \( p + \lambda(p - x) \notin P \). The set of back faces is denoted by \( \text{Back}(x) \); its complement \( \text{Front}(x) := F(P)_{10}^1 \setminus \text{Back}(x) \) is the set of front faces.

![Back faces](image)

**Figure 2: Back faces**

3.5 Lemma. Suppose \( F \) is a facet of \( P \). Then \( F \) is a back face with respect to \( x \) if and only if \( x \) and \( \text{int} \ P \) are on the same side of the affine hyperplane spanned by \( F \). A proper non-empty face \( F \) of \( P \) is a back face if and only if it is contained in a facet of \( P \) which is a back face. \( \square \)

In particular, the sets \( \text{Back}(x) \) and \( \text{Front}(x) \) are non-empty. Since a face of a back face is a back face itself, \( \text{Back}(x) \) is a subcomplex while \( \text{Front}(x) \) is an order filter.

By arguments similar to the ones used for the case of visible faces, we can show:

3.6 Proposition. The space \(|\text{Back}(x)|\) is homeomorphic to an \((n - 1)\)-ball. In particular, \( \chi(\text{Back}(x)) = 1 \). \( \square \)

**Upper and lower faces**

3.7 Definition. A face \( F \in F(P)_{10}^1 \) is called a lower face with respect to the direction \( x \in \mathbb{R}^n \setminus \{0\} \) if for all points \( p \in F \) and all real numbers \( \lambda > 0 \) we
have $p - \lambda x \notin P$. The set of lower faces is denoted by $\text{Low}(x)$; its complement $\text{Up}(x) := F(P)_0 \setminus \text{Low}(x)$ is the set of upper faces.

![Figure 3: Lower faces](image)

### 3.8 Lemma. Suppose $F$ is a facet of $P$ with inward pointing normal vector $v$. Then $F$ is a lower face with respect to $x$ if and only if $\langle x, v \rangle > 0$. A proper non-empty face of $P$ is a lower face if and only if it is contained in a facet of $P$ which is a lower face. \(\blacksquare\)

In particular, the sets $\text{Low}(x)$ and $\text{Up}(x)$ are non-empty. Since a face of a lower face is a lower face itself, $\text{Low}(x)$ is a subcomplex while $\text{Up}(x)$ is an order filter.

By arguments similar to the ones used for the case of visible faces, we can show:

### 3.9 Proposition. The space $|\text{Low}(x)|$ is homeomorphic to an $(n - 1)$-ball. In particular, $\chi(\text{Low}(x)) = 1$. \(\blacksquare\)

## 4 Barrier cones, tangent cones, and the Brianchon-Gram theorem

Let $P \subset \mathbb{R}^n$ be a polytope with non-empty interior. Given a non-empty face $F$ of $P$ we define the barrier cone $C_F$ of $P$ at $F$ as the set of finite linear combination with non-negative real coefficients spanned by the set

$$P - F := \{p - f | p \in P \text{ and } f \in F\}.$$ 

Clearly $C_F$ contains the vector space spanned by $F - F$ which is the vector space associated to the affine span of $F$. This definition generalises the previous one if $F$ is a vertex of $P$. 
Let $F$ be a non-empty face of $P$. One should think of the translated cone $F + C_F = \{ f + x \mid x \in C_F, f \in F \}$ as the cone $C_F$ attached to the face $F$.

For a non-empty proper face $F$ of $P$ let $T_F$ denote the supporting cone (or tangent cone) of $F$; it is the intersection of all supporting half-spaces containing $F$ in their boundary. (Of course it is enough to restrict to facet-defining half-spaces.) By convention $T_P = \mathbb{R}^n$. Using Farkas’ lemma ([Zie95 §1.4] or [Ewa96, Lemma I.3.5]) it can be shown that $F + C_F = T_F$. Moreover, every polytope is the intersection of all its supporting half-spaces, thus $P = \bigcap_{F \in F(P)} T_F$.

Also of interest are the cones $-F + C_F = \{ -f + x \mid x \in C_F, f \in F \}$. Up to a reflection at the origin, they can be thought of as the negatives of the barrier cones, attached to the corresponding face (i.e., with cones pointing towards the outside of $P$).

The following theorem is the heart of this paper; expressed in combinatorial terms, it uses the Euler characteristic to give specific inclusion-exclusion formulae for lattice point in (the interior of) $P$. Part (4) is known as the BRIANCHON-GRAM theorem, the remaining two equations are variations of the theme.

4.1 Theorem. Let $P \subset \mathbb{R}^n$ be an arbitrary $n$-dimensional polytope. There are equalities of formal LAURENT series

\[
\sum_{F \in F(P)_0} (-1)^{\dim F} S(F + C_F) = S(P),
\]

(4)

\[
\sum_{F \in F(P)_0} (-1)^{\dim F} S(C_F) = 1,
\]

(5)

\[
\sum_{F \in F(P)_0} (-1)^{\dim F} S(-F + C_F) = (-1)^n \cdot S(\text{int } -P).
\]

(6)

Proof. We verify Equation (4) first. Fix a vector $a \in \mathbb{Z}^n$. We have to show that the coefficient of $x^a$ is the same on both sides of the equation.

If $a \in P$ then the monomial $x^a$ occurs with coefficient 1 in all the LAURENT series $S(F + C_F)$ on the left. Thus the coefficient of $x^a$ in the sum is the Euler characteristic of $P$, which is known to be 1. Hence the coefficients of $x^a$ agree on the left and right side in this case.

Now assume $a \notin P$. Let $F$ denote a proper non-empty face of $P$. From Lemma 3.2 and the definition of supporting cones we conclude that $a \in T_F = F + C_F$ if and only if $F$ is invisible from $a$. In particular, the coefficient of $x^a$ in $S(F + C_F)$ is 1 if $F \in \text{Inv}(a)$, and it is 0 if $F \notin \text{Inv}(a)$. In total, the coefficient of $x^a$ on the left is

$$\ell = (-1)^n + \sum_{F \in \text{Inv}(a)} (-1)^{\dim(F)} = (-1)^n + \chi(\text{Inv}(a)),$$
the extra \((-1)^n\) corresponding to the contribution coming from \(P\). Now by
definition of the Euler characteristic, we have

\[ 1 = \chi(P) = \chi(\text{Vis}(a)) + \ell. \]

Since \(|\text{Vis}(a)|\) is a ball by Proposition 3.3, we infer that \(\ell = 0\). Consequently,
the monomial \(x^a\) does not occur on either side of Equation (1), as required.

Next we deal with Equation (5). Observe first that \(0 \in C_F\) for all \(F \in F(P)_0\),
so the coefficient of \(1 = x^0\) is \(\chi(P) = 1\).

Now fix any non-zero vector \(a \in \mathbb{Z}^n\). We have to show that the coefficient
of \(x^a\) is trivial. For a given face \(F \in F(P)_0\), let

\[ N_F := \{ \forall p \in C_F: (p, v) \geq 0 \} \]

be the dual cone \(C_F\) [Ewa96, §I.4 and §V.2].

Let \(U(a)\) denote the poset of all non-empty proper faces \(F\) of \(P\) satisfying
\(a \in C_F\). By the above we have equivalences

\[ F \in U(a) \iff a \in C_F = N_F^\vee \iff \forall v \in N_F: (-a, v) \leq 0. \]

This means that \(U(a) = \text{Up}(-a)\) is the set of upper faces of \(P\) with respect
to \(-a\) in the sense of Definition 3.7. Hence the coefficient of \(x^a\) in the left-hand
side of Equation (5) can be rewritten as

\[ (-1)^n + \sum_{F \in U(a)} (-1)^{\dim(F)} = (-1)^n + \chi(\text{Up}(-a)) = \chi(P) - \chi(\text{Low}(-a)) = 0 \]

where we have used additivity of Euler characteristic and Proposition 3.9 as well.

Finally we discuss Equation (6). Fix a point \(a \in \mathbb{Z}^n\) and a face \(F \in F(P)_1\).
Then \(a \notin -F + C_F\) if and only if there is a facet \(G \supseteq F\) of \(P\) such that \(a\) and
\(\text{int}(-P)\) are on the same side of the affine hyperplane spanned by \(-G\). Such
a facet certainly exists if \(a \in \text{int}(-P)\). Hence the only summand on the left
contributing to \(x^a\) is the one corresponding to \(P\), giving a coefficient \((-1)^n\) as
required.

If \(a\) is not in the interior of \(-P\), Lemma 3.5 applied to the polytope \(-P\),
shows that \(a \in -F + C_F\) if and only if \(-F\) is a front face of \(-P\) in the sense
of Definition 3.4. It follows from Proposition 3.6 and additivity of Euler
characteristics that the contribution to \(x^a\) is

\[ (-1)^n + \sum_{-F \in \text{Front}(-P)} (-1)^{\dim(F)} = \chi(P) - \chi(\text{Back}(-P)) = 0. \]
5 Brion’s formula

From the Brianchon-Gram theorem we deduce Brion type formulæ by passing to rational functions. We follow the treatment as exemplified in [BHS]. Write \( \Pi \) for the \( \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \)-module of formal power series \( S(w + C) \) where \( w \in \mathbb{R}^n \) is an arbitrary vector, and \( C \) is a polyhedral rational cone

\[
C = \{ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r \mid \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0} \}
\]  

(7)

where \( v_1, \ldots, v_r \) are vectors in \( \mathbb{Z}^n \). For \( r = 0 \) the cone \( C \) degenerates to the single point 0, so \( \Pi \) contains the series \( S(P) \) for any polytope \( P \). In general, if \( C \) does not contain an affine subspace of positive dimension (i.e., if \( C \) is pointed), the series \( S(w + C) \) represents a rational function, denoted \( \sigma(w + C) \in \mathbb{C}(x_1, x_2, \ldots, x_n) \).

The following Lemma has been attributed to Brion, see [BHS, Theorem 2.4] and [Bri96, Theorem 2.1 (i)].

5.1 Lemma. There is a unique \( \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \)-module homomorphism

\[
\phi: \Pi \to \mathbb{C}(x_1, x_2, \ldots, x_n)
\]

such that \( \phi(S(w + C)) = \sigma(w + C) \) for each pointed cone of the form (7) where \( v_1, \ldots, v_r \in \mathbb{Z}^n \) and \( w \in \mathbb{R}^n \).

Moreover, if \( C \) is of the form (7), and \( C \) contains an affine subspace of positive dimension, then \( \phi(S(w + C)) = 0 \).

We now come to the proof of Theorem 1.2. We treat Equation (2) only, the other cases being similar. Note that for all \( F \in F(P)_0 \) the barrier cone \( C_F \) is a rational polyhedral cone since the facets of \( P \) admit rational normal vectors. We can thus apply the homomorphism \( \phi \) from Lemma 5.1 to Equation (5), Theorem 4.1. The results follows immediately if one recalls that \( C_F \) contains an affine subspace of positive dimension if and only if \( \dim F \geq 1 \), so all summands coming from faces of positive dimension disappear upon application of \( \phi \).

Concluding remarks

Visibility subcomplexes can be used to compute higher sheaf cohomology of certain line bundles on projective toric varieties; the reader will easily recognise the similarity between the present paper and the exposition in [Hütb], Appendix of §2.5. Brion’s theorem can be generalised substantially to include the case of arbitrary torus-invariant line bundles on complete toric varieties or, formulated in more combinatorial terms, arbitrary support functions on complete fans [Hütta].
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