A Note On Cover-Free Families

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Abstract

Let \( N((r, w; d), t) \) denote the minimum number of points in a \((r, w; d)\)-cover-free family having \( t \) blocks. Hajiabolhassan and Moazami (2012) \cite{6} showed that the Hadamard conjecture is equivalent to confirm \( N((1, 1; d), 4d - 1) = 4d - 1 \). Hence, it is a challenging and interesting problem to determine the exact value of \( N((r, w; d), t) \). In this paper, we determine the exact value of \( N((r, w; d), t) \) for every \( r, w \), where \( r + w \leq t \) and some \( d \).

Key words: Cover-free families, Biclique covering number

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1 Introduction

A family of sets is called an \((r, w)\)-cover-free family (or \((r, w)\)-CFF) if no intersection of \( r \) sets of the family are covered by a union of any other \( w \) sets of the family. Cover-free families were first described by Kautz and Singleton (1964) to investigate superimposed binary codes \cite{7}. Erdos et al. \cite{3} introduced the \((1, r)\)-cover-free family as a generalization of Sperner family. Stinson et al. \cite{10} considered cover-free families as group testing. Mitchell and Piper \cite{8} considered a key distribution pattern which appears to be equivalent to the notion of cover-free family. For another application and discussion of cover-free families, (see, for example, \cite{1, 4, 5, 6, 9, 11, 13}). Stinson and Wei \cite{11} have introduced a generalization of cover-free families as follows.

Definition 1. Let \( d, n, t, r, \) and \( w \) be positive integers and \( B = \{B_1, \ldots, B_t\} \) be a collection of subsets of a set \( X \), where \( |X| = n \). Each element of the collection \( B \) is called a block and the elements of \( X \) are called points. The pair \((X, B)\) is called an \((r, w; d)\)-CFF \((n, t)\) if for any two sets of indices \( L, M \subseteq [t] \) such that \( L \cap M = \emptyset, |L| = r, \) and \( |M| = w \), we have

\[
|\left( \bigcap_{l \in L} B_l \right) \setminus \left( \bigcup_{m \in M} B_m \right) | \geq d.
\]

Let \( N((r, w; d), t) \) denote the minimum number of points of \( X \) in an \((r, w; d)\)-CFF having \( t \) blocks.

As was shown by Engel \cite{2}, determining the optimal value for a cover-free family is NP-hard. Also, Hajiabolhassan and Moazami \cite{6} showed that the existence of Hadamard matrices results from the existence of some cover-free families and vice versa. A Hadamard matrix of order \( n \) is an \( n \times n \) matrix \( H \) with entries \(+1\) and \(-1\), such that \( HH^T = nI_n \).
Theorem A. [6] Let $d$ be a positive integer, then $N((1,1;d),4d-1) = 4d-1$ if and only if there exists a Hadamard matrix of order $4d$.

It is proved that if $H$ is a Hadamard matrix of order $n$, then $n = 1$, $n = 2$, or $n = 4d$ whenever $d$ is a positive integer [12]. It was conjectured by Jacques Hadamard (1893) that there exists a Hadamard matrix of every order $4d$ whenever $d$ is a positive integer. Actually Hajiabolhassan and Moazami showed that the Hadamard conjecture is equivalent to confirm $N((1,1;d),4d-1) = 4d-1$. Thus the problem of determining the exact value of the parameter $N((r,w;d),t)$, even for special values of $r$, $w$, $d$, and $t$ is a challenging and interesting problem. In this paper, we determine the exact value of $N((r,w;d),t)$ for every $r$, $w$, where $r + w \leq t$ and some $d$.

2 Cover-Free Family

In this section, we restrict our attention to determine the exact value of $N((r,w;d),t)$ for every value of $r$, $w$, and $t$, where $r + w \leq t$, and some special value of $d$. In this regard, we need to use some notation and theorem as follows. A biclique of $G$ is a complete bipartite subgraph of $G$. The $d$-biclique covering (resp. partition) number $bc_d(G)$ (resp. $bp_d(G)$) of a graph $G$ is the minimum number of bicliques of $G$ such that every edge of $G$ belongs to at least (resp. exactly) $d$ of these bicliques. Hajiabolhassan and Moazami [6] showed that the existence of an $(r,w;d)$-cover-free family is equivalent to the existence of $d$-biclique cover of bi-intersection graph. The bi-intersection graph $I_t(r,w)$ is a bipartite graph whose vertices are all $w$- and $r$-subsets of a $t$-element set, where a $w$-subset is adjacent to an $r$-subset if and only if their intersection is empty.

Theorem B. [6] Let $r$, $w$, $d$ and $t$, be positive integers, where $t \geq r + w$. It holds that $N((r,w;d),t) = bc_d(I_t(r,w))$.

Theorem 1. Let $r$, $w$, and $t$ be positive integers, where $t \geq r + w$. Also, assume that the function $\binom{t}{r} \binom{t-x}{w}$ is maximized for $x = t'$. If $d = \binom{t-r-w}{v-r}$, then

$$N((r,w;d),t) = bc_d(I_t(r,w)) = bp_d(I_t(r,w)) = \binom{t}{t'}.$$

Proof. Set $t'' = \binom{t}{r'}$. First, we show that $I_t(r,w)$ can be covered by $t''$ bicliques such that every edge of $I_t(r,w)$ is covered by exactly $d$ bicliques. Denote the vertex set of $I_t(r,w)$ by bipartition $(X,Y)$ in which $X = \binom{[t]}{r}$ and $Y = \binom{[t]}{w}$. Suppose that $A$ is a $t'$-subset of $[t]$ and $A^c$ is the complement of the set $A$ in $[t]$. Denote the number of these pairs by $t''$. Now, for every $t'$-subset $A_j$ of $[t]$, where $1 \leq j \leq t''$, construct the biclique $G_j$ with the vertex set $(X_j,Y_j)$, where $X_j = \binom{A_j}{r}$ and $Y_j = \binom{A^c_j}{w}$. Let $UV$ be an arbitrary edge of $I_t(r,w)$, where $|U| = r$ and $|V| = w$. In view of the definition of $G_j$, $UV$ is covered by every $G_j$ with vertex set $(X_j,Y_j)$, where $U$ is a vertex of $X_j$ and $V$ is a vertex of $Y_j$. Thus every edge of $I_t(r,w)$ is covered by at least $d$ bicliques. One can see that

$$\sum_{j=1}^{t''} |E(G_j)| = \binom{t}{t'} \binom{t'}{r} \binom{t-t'}{w} \quad \& \quad |E(I_t(r,w))| = \binom{t}{r} \binom{t-r}{w}.$$
Now, it is simple to check that
\[
\sum_{j=1}^{t''} |E(G_j)| = d|E(I_t(r, w))|.
\]
Thus every edge of \( I_t(r, w) \) is covered by exactly \( d \) bicliques. Note that we have actually proved that
\[
bp_d(I_t(r, w)) \leq t''.
\] (1)

Conversely, one can see that
\[
bp_d(I_t(r, w)) \geq bc_d(I_t(r, w)) \geq \frac{d|E(I_t(r, w))|}{B(I_t(r, w))}.
\]

Also, In view of the definition of \( t' \), we have
\[
\frac{d|E(I_t(r, w))|}{B(I_t(r, w))} = \binom{t-r-w}{t-r} \binom{t-w}{t-r} \binom{t-w}{t-r-w} = \binom{t}{t'} = t''.
\]

Hence,
\[
bp_d(I_t(r, w)) \geq bc_d(I_t(r, w)) \geq t''.
\] (2)

From (1) and (2), we conclude
\[
bp_d(I_t(r, w)) = bc_d(I_t(r, w)) = t''.
\]

By Theorem B,
\[
N((r, w; d), t) = bc_d(I_t(r, w)),
\]
this completes the proof.

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