Galilean and Carrollian invariant Hodge star operators

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Standard Hodge star operator is naturally associated with metric tensor (and orientation). It is therefore invariant, on Minkowski spacetime, with respect to Poincaré transformations and is routinely used to succintly write down Poincaré invariant physics equations. In Galilean (Carrollian) spacetime, there is no Galilean (Carrollian) invariant metric tensor available. So usual construction of Galilean (Carrollian) invariant Hodge star does not work. Here we propose analogs of the Hodge star operator which are Galilean (or Carrollian) invariant. They may be used to write down Galilean (Carrollian) invariant physics equations, e.g. equations of Galilean (Carrollian) electrodynamics.

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1 Introduction - standard Hodge star operator

The Hodge star $\ast \equiv \ast_{g,o}$ is a notorious linear operator (see e.g. [1] - [7]) acting on differential forms on (pseudo)Riemannian oriented manifold $(M, g, o)$

$$\ast : \Omega^p \rightarrow \Omega^{n-p} \quad \Omega^p \equiv \Omega^p(M, g, o)$$

($n$ being the dimension of $M$). In components (w.r.t. a local frame field $e_a$), it is defined as

$$(\ast \alpha)_{a\ldots b} := \frac{1}{p!} \alpha_{c\ldots d} \omega^{c\ldots d}_{a\ldots b} \quad \omega^{c\ldots d}_{a\ldots b} := g^{ce} \ldots g^{df} \omega_{e\ldots f a\ldots b}$$

where $\omega \equiv \omega_{g,o}$ stands for the metric volume form on $(M, g, o)$.

The Hodge star operator has a lot of nice properties. Among the most important is that it is natural w.r.t. diffeomorphisms of $M$, meaning that if $f : M \rightarrow M$ is a diffeomorphism and $\alpha$ is a form on $M$, then

$$f^* (\ast_{g,o} \alpha) = \ast_{f^* g, f^*(o)} (f^* \alpha) \quad \text{or, in short,} \quad f^* \ast_{g,o} = \ast_{f^* g, f(o)} f^*$$

(see more details in Appendix A.1 here $f(o)$ denotes the opposite orientation, if $f$ does not preserve orientation $o$).

Then, clearly, the Hodge star operator is invariant w.r.t. (orientation preserving) isometries of $(M, g)$, i.e.

$$f^* g = g \quad \Rightarrow \quad f^* \ast_g = \ast_g f^*$$

In particular, if $g \equiv \eta$ corresponds to Minkowski spacetime $(M, g) = (\mathbb{R}^4, \eta)$, the corresponding Hodge star $\ast_\eta$ is invariant w.r.t. Poincaré transformations

$$f : x \mapsto Ax + a \quad A^T \eta A = \eta \quad \Rightarrow \quad f^* \ast_\eta = \ast_\eta f^*$$
This property promotes Minkowski Hodge star $\ast_\eta$ to become especially effective tool in writing important Poincaré invariant equations of physics extremely neatly. For example, the Maxwell equations may be written as

$$dF = 0 \quad d\ast_\eta F = -J$$

(here $J$ stands for 3-form of current). Then, just applying $f^*$ on both sides of (6) we see, that if $F$ is a solution generated by $J$, then, for any Poincaré transformation $f$, $f^*F$ is a solution as well, namely generated by $f^*J$. (Here we also used the fact that the exterior derivative $d$ is natural, too, $f^*d = df^*$.)

A lot of research is recently devoted to Galilean and Carrollian spacetimes, see e.g. [8] - [14]. Now if one wanted to find similarly effective instruments in these spacetimes, a serious problem would arise immediately: There is no Galilean invariant metric tensor in Galilean spacetime and, similarly, no Carrollian invariant metric tensor in Carrollian spacetime. And since invariance of $g$ turns out to be the key element in invariance of the resulting $\ast_g$, routine procedure fails, there.

## 2 Galilean and Carrollian invariant Hodge stars

What is used for construction of the (standard) Hodge star is

- the metric volume form $\omega_g$
- the cometric $g^{-1}$ (for raising of indices).

Since both objects happen to be natural (see Appendix A.1), they become invariant for isometries, leading to invariance (w.r.t. isometries) of the resulting Hodge star.

So, effectively, in the Minkowski case,

- the volume form is invariant w.r.t. Poincaré transformations,
- the raising of indices procedure is invariant w.r.t. Poincaré transformations, and therefore
- the Minkowski Hodge star is invariant w.r.t. Poincaré transformations, too.

Then, necessarily, if

- a volume form invariant w.r.t. Galilean (Carrollian) transformations and
- raising of indices invariant w.r.t. Galilean (Carrollian) transformations were available, then

- the corresponding “Hodge star” would be invariant w.r.t. Galilean (Carrollian) transformations as well.

### 2.1 Construction of Galilean invariant Hodge star

Do the two needed Galilean-invariant objects (in Galilean spacetime) exist?
As for the volume form, it indeed does exist (see Appendix A.2) and it is, when expressed in (arbitrary) Galilean coordinates \( x^\mu = (x^0 = t, x^i) \), simply
\[
\omega = dt \wedge dx \wedge dy \wedge dz \tag{7}
\]
(See a nice discussion of the concepts of Lorentzian and Galilean coordinates in § 12.3 in [15].) So it looks exactly as we are accustomed to the Minkowski metric volume form expressed in standard flat coordinates. (Recall, however, that the Galilean spacetime volume form, in spite of the formal coincidence, is not metric, since there is no (distinguished) metric tensor, there.)

As for the raising of indices, one needs a \((\frac{d}{2})\)-type tensor for it, in general. On a (pseudo)Riemannian manifold \((M, g)\), the cometric \(g^{-1}\) is used to perform the job. (Then, the metric \(g\) itself is used for the reverse procedure, lowering of indices.) On Galilean spacetime, this cannot be done (since no relevant metric is available). However, it is well-known that there is a distinguished (namely Galilean invariant) \((\frac{d}{2})\)-type tensor, there (see Appendix A.2 again). When expressed in Galilean coordinates, it reads
\[
h \equiv h^{\mu\nu} \partial_\mu \otimes \partial_\nu = - \delta^{ij} \partial_i \otimes \partial_j \tag{8}
\]
(minus sign is just a useful convention, see (101)) so that the component matrix looks as follows
\[
h^{\mu\nu} \leftrightarrow \begin{pmatrix} h^{00} & h^{0i} \\ h^{i0} & h^{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\delta^{ij} \end{pmatrix} \tag{9}
\]
Notice that the tensor happens to be degenerate, so it cannot be regarded as a true (Galilean) spacetime cometric. Nevertheless, this is not sufficient argument against using it for raising of indices, albeit with no way back (no reverse lowering of indices is possible, no undo button exists for the raising of indices operation, here).

So, being equipped with all necessary elements, \(\omega\) and \(h\) satisfying, for any Galilean transformation \(f\),
\[
f^* \omega = \omega \quad f^* h = h \tag{10}
\]
we can define a new Hodge star operator as
\[
(*\alpha)_{\mu...\nu} := \frac{1}{p!} \alpha_{\rho...\tau} \omega^{\rho...\tau}_{\mu...\nu} \quad \omega^{\rho...\tau}_{\mu...\nu} := h^{\rho\lambda} \ldots h^{\tau\sigma} \omega_{\lambda...\sigma\mu...\nu} \tag{11}
\]
(cf. (2)) or, in component-free way, as
\[
\alpha \mapsto *_h \alpha \sim C \ldots C(h \otimes \cdots \otimes h \otimes \omega \otimes \alpha) \tag{12}
\]
(cf. [71]). And exactly because, for any Galilean transformation \(f\), conditions (10) do hold, this particular modification of Hodge star turns out to be Galilean-invariant
\[
f^* *_h = *_h f^* \tag{13}
\]
2.2 Construction of Carrollian invariant Hodge star

Recall that, in the Galilean spacetime, the crucial tensor field \( \omega^{\rho\ldots\tau}_{\mu\ldots\nu} \), needed for construction of the new Hodge star operator (defined in (11) or (12)), was based on existence of two canonical (Galilean invariant) tensor fields,

\[
\omega \leftrightarrow \omega^{\rho\ldots\nu} \\
h \leftrightarrow h_{\mu\nu}
\]

Namely we used \( h \) to raise (part of) indices of \( \omega \).

In the Carrollian spacetime, this cannot be repeated as it stands, since \( \omega \) and \( h \) are guaranteed to be Galilean (rather than Carrollian) invariant. Fortunately, there are other useful canonical (Carrollian invariant) tensors available, namely

\[
\tilde{\omega} \leftrightarrow \tilde{\omega}^{\rho\ldots\nu} \\
\tilde{h} \leftrightarrow \tilde{h}^{}_{\mu\nu}
\]

In detail,

\[
\tilde{\omega} \equiv \partial_t \wedge \partial_x \wedge \partial_y \wedge \partial_z \quad \text{and} \quad \tilde{h} = \delta_{ij}dx^i \otimes dx^j
\]

i.e.

\[
\tilde{\omega}^{0123} = 1 \quad \tilde{h}^{}_{\mu\nu} \leftrightarrow \begin{pmatrix} \tilde{h}_{00} & \tilde{h}_{0i} \\ \tilde{h}_{i0} & \tilde{h}_{ij} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} \end{pmatrix}
\]

(see Appendix A.2 again) and one can clearly reach the same goal (construction of the mixed tensor field \( \omega^{\rho\ldots\tau}_{\mu\ldots\nu} \)) by lowering (part of) indices of \( \tilde{\omega} \) using \( \tilde{h} \).

So, being equipped with \( \tilde{\omega} \) and \( \tilde{h} \) which satisfy, for any Carrollian transformation \( f \),

\[
f^*\tilde{\omega} = \tilde{\omega} \quad f^*\tilde{h} = \tilde{h}
\]

we can define a new Hodge star operator again as

\[
(*\alpha)^{}_{\mu\ldots\nu} := \frac{1}{p!} \alpha^{\rho\ldots\tau} \omega^{\rho\ldots\tau}_{\mu\ldots\nu} \quad \omega^{\rho\ldots\tau}_{\mu\ldots\nu} := \tilde{\omega}^{\rho\ldots\tau\lambda\ldots\sigma} \tilde{h}^{\lambda\mu} \cdots \tilde{h}^{\sigma\nu}
\]

(cf. (11)) or, in component-free way, as

\[
\alpha \mapsto *_{\tilde{h}}\alpha \sim C \cdots C(\tilde{h} \otimes \cdots \tilde{h} \otimes \tilde{\omega} \otimes \alpha)
\]

(cf. (12)). And exactly because, for any Carrollian transformation \( f \), conditions (13) do hold, this particular modification of Hodge star turns out to be Carrollian-invariant

\[
f^* *_{\tilde{h}} = *_{\tilde{h}} f^*
\]

For further possibilities, see Appendix A.4.

3 Explicit formulas and basic properties

3.1 General explicit formulas

In both Galilean and Carrollian spacetimes, exactly as is the case in Minkowski spacetime (see e.g. Section 16.1 in [7]), any \( p \)-form \( \alpha \), may be uniquely written as follows:

\[
\alpha = dt \wedge \hat{s} + \hat{r}
\]
where the two hatted forms, \((p - 1)\)-form \(\hat{s}\) and \(p\)-form \(\hat{r}\), respectively, are spatial, meaning that they do not contain (“temporal”) differential \(dt\) in its (adapted) coordinate expression. (The coordinate \(t\) itself may be present, however, in components within the expression.)

When applying all three versions of Hodge star operator \(*\) on the decomposed version (22) of \(\alpha\), we get (in any dimension \(n = 1 + d\)) the following results (see Appendix A.3):

\[
\begin{align*}
*(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{*r} + \hat{\eta}\hat{s} & \text{Minkowski Hodge star} \\
*(dt \wedge \hat{s} + \hat{r}) &= dt \wedge \hat{r} & \text{Galilean Hodge star} \\
*(dt \wedge \hat{s} + \hat{r}) &= \hat{\eta}\hat{s} & \text{Carrollian Hodge star}
\end{align*}
\]

Here \(\hat{*}\) stands for standard “Euclidean” \((d\)-dimensional\) Hodge star (it only operates on spatial forms) and \(\hat{\eta}\) is the “main automorhism” on forms (just a multiple by \(\pm 1\); see e.g. Sec.5.3 in [7]).

We see that
- the first term on the r.h.s. of (23) coincides with
- the only term on the r.h.s. of (24).

Similarly,
- the second term on the r.h.s. of (23) coincides with
- the only term on the r.h.s. of (25).

(The sign conventions from (8) and (16) play a role, by the way, in this coincidences, see Appendix A.3.)

The missing terms on the r.h.s. of (24) and (25), in comparison with (23), correspond to non-zero kernels

\[
\begin{align*}
*(dt \wedge \hat{s}) &= 0 & \text{Galilean Hodge star} \\
* \hat{r} &= 0 & \text{Carrollian Hodge star}
\end{align*}
\]

For more explicit versions of (24) and (25) for the usual \(1 + 3\) case, see Appendix A.5.

3.2 Basic properties

Exactly as is the case for the standard Hodge star, both new Hodge stars map linearly \(p\)-forms on \((n - p)\)-forms and vice versa

\[
\Omega^p \overset{*}{\leftrightarrow} \Omega^{n-p}
\]

Further, as already mentioned above (see (3), (13) and (21)), from construction they commute with (pull-back of) distinguished transformations \(f\) in “their” space-
times, i.e.

\[ f^* *_M = *_M f^* \quad f = \text{Poincaré transformation} \quad (29) \]
\[ f^* *_G = *_G f^* \quad f = \text{Galilean transformation} \quad (30) \]
\[ f^* *_C = *_C f^* \quad f = \text{Carrollian transformation} \quad (31) \]

Put it differently, the new operators happen to be invariant w.r.t. Galilean and Carrollian transformations, respectively. This property makes them potentially interesting from the point of view of writing Galilean or Carrollian *invariant* equations in terms of differential forms on corresponding spacetimes (see Section 4.2).

A straightforward consequence of (24) and (25) is a striking feature of the new Hodge stars - they both square to zero

\[ ** = 0 \quad ** : \Omega^p \rightarrow \Omega^p \quad (32) \]

This is in sharp contrast with the standard Hodge operator \( ^* \) from (2) which, as is well-known, squares to (plus or minus) the *unit* operator

\[ ^* *^_G = \pm \mathbf{1} \quad ^* *^_G : \Omega^p \rightarrow \Omega^p \quad (33) \]

So, contrary to the standard case, neither the *Galilean-Hodge* star nor the *Carrollian-Hodge* star are *isomorphisms* between the two spaces of equal dimension (spaces of \( p \)-forms and \( (n-p) \)-forms, respectively). Their (non-zero) kernels are displayed in (26) and (27).

One can also rephrase the statement as that we can no longer speak of
- Galilean-Hodge *duality* as well as
- Carrollian-Hodge *duality*.

This is the tax to be paid in order to switch from *Poincaré* invariance (29) of (23) to
- *Galilean invariance* (30) of (24) and
- *Carrollian invariance* (31) of (25).

4 An application: Electrodynamics

As an application of Galilean/Carrollian Hodge star operator in physics, let us have just a brief look at Galilean and Carrollian *electrodynamics*. In these two versions of electrodynamics (particular limits of the standard Lorentzian one, see e.g. [16] and [8]), the corresponding equations of motion for the fields \( E \) and \( B \) are Galilean/Carrollian (rather than Poincaré) invariant.
4.1 Standard (Lorentzian) electrodynamics

A folklore knowledge says that, in Minkowski spacetime, the (source-less, for simplicity) Maxwell equations

\[
\begin{align*}
\text{div } E &= 0 \quad (34) \\
\text{curl } B - \partial_t E &= 0 \quad (35) \\
\text{curl } E + \partial_t B &= 0 \quad (36) \\
\text{div } B &= 0 \quad (37)
\end{align*}
\]

may be neatly written in terms of 2-form of electromagnetic field

\[ F = dt \wedge E \cdot dr - B \cdot dS \quad (38) \]

as follows:

\[
\begin{align*}
d \ast F &= 0 \quad (39) \\
dF &= 0 \quad (40)
\end{align*}
\]

As already mentioned in Section 1, just because of properties of \(d\) and \(*_\eta\), this way of presentation of the Maxwell equations makes their Poincaré invariance evident.

In more detail, the assignment reads

\[
\begin{align*}
d \ast F &= 0 \quad \leftrightarrow \quad \text{div } E = 0 \quad (41) \\
dF &= 0 \quad \leftrightarrow \quad \text{curl } E + \partial_t B = 0 \quad (43) \\
\text{div } B &= 0 \quad (44)
\end{align*}
\]

(see, e.g., Section 16.1 in [7]). Notice that each spacetime equation on the l.h.s. corresponds to as many as two spatial equations on the r.h.s.

4.2 Galilean and Carrollian electrodynamics

Now, it looks natural, in an effort to switch to Galilean (Carrollian) electrodynamics, to just repeat the complete story with Galilean (Carrollian) Hodge star operator. So again, introduce (see (109)) the 2-form of electromagnetic field exactly as in (38) and write down equations of motion à la (39) and (40), the only change being replacement of Minkowski Hodge star with the Galilean (Carrollian) one (so with \(*_h\) from (24) or (25); more explicitly from (124) or (129)): 

\[
\begin{align*}
d \ast F &= 0 \quad * = *_h \text{ or } *_{\tilde{h}} , \text{ here} \quad (45) \\
dF &= 0 \quad (46)
\end{align*}
\]

Then it is clear that, again just because of the properties of \(d\) and \(*_h\), this is
- a system of 1-st order partial differential equations
- for the fields $E$ and $B$
- which is *Galilean or Carrollian* (rather than Poincaré) *invariant*.

All the items listed above provide a promising signal that equations (45) - (46) are probably closely associated with the desired Galilean (Carrollian) electrodynamics.

How the system (45) - (46) actually looks like in terms of the fields $E$ and $B$?

First, it is clear that (46) looks *the same* for all three cases (standard Lorentzian, Galilean as well as Carrollian), namely it always corresponds to (43) and (44). So, both the Faraday’s law and non-existence of magnetic monopoles hold in all three versions of electrodynamics.

What really makes a difference is equation (45). When (124) and (129) - rather than (119) - is used for computation of $\star$, we get

\begin{align*}
  d \star F &= 0 \quad \Leftrightarrow \quad \text{curl } B = 0 \quad \text{Galilean case} \quad (47) \\
  d \star F &= 0 \quad \Leftrightarrow \quad \partial_t E = 0 \ , \ \text{div } E = 0 \quad \text{Carrollian case} \quad (48)
\end{align*}

So what we get when *all* spatial equations, resulting from

\[ d \star F = 0 \quad dF = 0 \quad (49) \]

are displayed side by side for *all three* cases is

| Lorentzian | Galilean | Carrollian |
|------------|----------|------------|
| div $E = 0$ | div $E = 0$ | div $E = 0$ | (51) |
| curl $B - \partial_t E = 0$ | curl $B = 0$ | $\partial_t E = 0$ | (52) |
| curl $E + \partial_t B = 0$ | curl $E + \partial_t B = 0$ | curl $E + \partial_t B = 0$ | (53) |
| div $B = 0$ | div $B = 0$ | div $B = 0$ | (54) |

Checking equations in Galilean and Carrollian columns versus standard Maxwell equations (left column) shows that there are some objects *missing* in Galilean as well as Carrollian versions of “Maxwell equations” (49), namely

- *displacement current* $\partial_t E$ in Ampère’s law in *Galilean case*
- “*Ampère term*” curl $B$ in Ampère’s law in *Carrollian case*
- complete equation (51), i.e. *Gauss’s law*, in *Galilean case*.

Now, while the absence of

- the displacement current in Galilean electrodynamics and
- the “basic Ampère term” in Carrollian electrodynamics

are well-known (so expected and desired) facts (namely in so-called “magnetic limit” version of Galilean electrodynamics, see [16], and “electric limit” version of Carrollian electrodynamics, see [8]), the Gauss’ law *should* be present in all three versions of electrodynamics, i.e. *the* Galilean system actually should contain the equations from the central column plus the equation (41).
So, do represent correct equations in all three versions of electrodynamics, albeit they do not provide, in the Galilean case, the whole story.

[Technically, the reason why just a single equation (47) corresponds to (45) (so that, at the end of the day, one equation is missing), contrary to two equations (41) and (42) in Minkowski case, looks to be simple: The Galilean star is no longer an isomorphism, it kills the “electric” part of $F$ and then action of $d$ only produces a single term. Notice, however, that neither the Carrollian star is an isomorphism (it kills the “magnetic” part of $F$). And here action of $d$ produces two terms. So the number of resulting equations alone actually depends on more subtle details of actions of the two star operators, rather than on their general common property of not being isomorphism.]

5 Summary and conclusions

Hodge star operator belongs to essential tools in applications of differential forms on (pseudo)Riemannian manifolds. One of its key features is naturalness w.r.t. diffeomorphisms and, consequently, invariance w.r.t. isometries. For example, it is invariant w.r.t. Poincaré transformations on Minkowski spacetime.

However, Galilean and Carrollian spacetimes are not (pseudo)Riemannian manifolds. There is no (distinguished, non-degenerate) metric tensor available on them. Consequently, there is no “full-fledged” Hodge star operator on them.

In spite of that, it turns out that one can mimic the usual construction of Hodge star using specific (well-known) invariant tensors available. Since algebra of invariant tensors depends on particular spacetime, particular constructions and the two resulting “Hodge stars” differ as well.

Both of them retain some important properties of the standard Hodge star, but lose some other properties.

In particular, they remain to be invariant (w.r.t. Galilean or Carrollian transformations, respectively). Put it differently, the operators commute with (pull-back of) distinguished transformations $f$ in “their” spacetimes, i.e.

\[
\begin{align*}
  f^{**} = f^{*} & \quad f = \text{Poincaré transformation} \\
  f^{*} & \quad f = \text{Galilean transformation} \\
  f^{*} & \quad f = \text{Carrollian transformation}
\end{align*}
\]

(see (5), (13) and (21)). This makes them potentially interesting from the point of view of writing Galilean or Carrollian invariant equations in terms of differential forms on corresponding spacetimes.

Both Hodge stars, however, cease to be isomorphisms (they are no longer Hodge dualities).
In all three spacetimes (Minkowski, Galilean and Carrollian) one can express
differential forms in terms of a pair of spatial forms. Then the three Hodge star
operators act as follows:

\[ *_M (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{*r} + \hat{*\eta s} \]  
\[ *_G (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{*r} \]  
\[ *_C (dt \wedge \hat{s} + \hat{r}) = \hat{\ast} \hat{\eta s} \]  

(see (23) - (25); the fact that \( *_G \) and \( *_C \) are not isomorphisms is clear).

For example, action on the 2-form \( F \) of electromagnetic field reads

\[ *_M (dt \wedge E \cdot dr - B \cdot dS) = dt \wedge (-B) \cdot dr - E \cdot dS \]  
\[ *_G (dt \wedge E \cdot dr - B \cdot dS) = dt \wedge (-B) \cdot dr \]  
\[ *_C (dt \wedge E \cdot dr - B \cdot dS) = -E \cdot dS \]

Consequently, single (universal) manifestly invariant equation on spacetime

\[ d \ast F = 0 \]  

has substantially different 3-dimensional content (expression in terms of fields \( E \) and \( B \)) depending on which particular star \( (\ast_M, \ast_G \text{ or } \ast_C) \) is used (i.e. whether we speak of invariance w.r.t. Lorentzian, Galilean or Carrollian boosts). In (51) - (54) the corresponding (source-free) “Maxwell equations” are displayed. From this we see that what we get in this way is

- complete set of (“electric type” of) Carrollian “Maxwell equations”
- incomplete set of (“magnetic type” of) Galilean “Maxwell equations”.

### A Appendices

#### A.1 Hodge star is natural

Let’s check that the standard Hodge star operator indeed does behave naturally w.r.t. diffeomorphisms (i.e. that (3) holds). In order to do that, it might be useful to see, in more detail, how the operator works.

As we can see from (2), one first raises (some) indices of the (metric) volume form \( \omega_g \) with the help of the cometric \( g^{-1} \):

\[ \omega_{e...d...a...b} \mapsto \omega_{e...d...a...b} := g^{ce} \cdots g^{df} \omega_{e...f...a...b} \]  

(here we use the notation \( g \leftrightarrow g_{ab} \) and \( g^{-1} \leftrightarrow g^{ab} \)). This may be written, in a component-free way, as

\[ \omega_g \mapsto C \cdots C (g^{-1} \otimes \cdots \otimes g^{-1} \otimes \omega_g) \]

where \( C \) stands for (appropriate) contraction.
The second step then consists in contraction with the input form \( \alpha \). The complete procedure (modulo the multiplicative constant \( 1/p! \)) then looks, in component-free way, as follows:

\[
\alpha \mapsto \ast g \alpha \sim C \ldots C(g^{-1} \otimes \ldots \otimes g^{-1} \otimes \omega_g \otimes \alpha)
\]

(67)

Now, it is a folklore knowledge that (any) pull-back

1. *commutes* with contractions
2. *preserves* tensor product

i.e.

\[
f^*C = C f^* \quad f^*(a \otimes b) = f^*a \otimes f^*b
\]

(68)

In addition,

3. \( g^{-1} \) as well as \( \omega_g \) are natural w.r.t. diffeomorphisms:

\[
f^*g^{-1} = (f^*g)^{-1} \quad f^*\omega_g = \omega_{f^*g}
\]

(69)

(see below). Then, action of \( f^* \) on the r.h.s. of (67) may be summarized as mere replacement

- \( g \mapsto f^*g \) for each \( g \) present there
- \( \alpha \mapsto f^*\alpha \).

And this is exactly

\[
f^* \ast g = \ast f^* g f^*
\]

(70)

as (69) claims.

Concerning statements in (69), let us denote (just in this Appendix) the co-metric w.r.t. the metric \( g \) as \( G \), i.e.

\[
G^{ac}g_{cb} = \delta^a_b \quad \text{or} \quad C(G \otimes g) = \hat{1}
\]

(71)

Then we get

\[
f^*(C(G \otimes g)) = C((f^*G) \otimes (f^*g)) = f^*\hat{1} = \hat{1}
\]

(72)

This says that \( f^*G \) is the cometric w.r.t. the metric \( f^*g \), i.e. the first statement of (69) holds.

Now let \( e^a \) be an *orthonormal* (and right-handed) coframe w.r.t. \( g \), i.e. (locally)

\[
g = \eta_{ab} e^a \otimes e^b
\]

(73)

Then the metric volume form \( \omega_g \) reads

\[
\omega_g = e^1 \wedge \cdots \wedge e^n
\]

(74)

Now

\[
f^*g = \eta_{ab}(f^*e^a) \otimes (f^*e^b) = \eta_{ab}E^a \otimes E^b \quad E^a := f^*e^a
\]

(75)
so that $E^a$ is an ON coframe w.r.t. $f^*g$. Therefore, if $f$ also preserves orientation (otherwise additional minus occurs),

$$\omega_{f^*g} = E^1 \wedge \cdots \wedge E^n$$

But then

$$\omega_{f^*g} = f^*e^1 \wedge \cdots \wedge f^*e^n = f^*(e^1 \wedge \cdots \wedge e^n) = f^*\omega_g$$

i.e. the second statement of (69) holds.

### A.2 Galilean and Carrollian invariant tensors

In Galilean coordinates, both tensor fields

$$\omega \equiv dt \wedge dx \wedge dy \wedge dz \equiv dt \wedge dV$$

and

$$h = -\delta^{ij} \partial_i \otimes \partial_j$$

from Section 2.1 are clearly translation and rotation invariant. That is, they are Lie invariant w.r.t. vector fields

$$\partial_t \equiv \partial_0$$

(time translation) \hspace{1em} (79)

$$\partial_i$$

(space translations) \hspace{1em} (80)

$$x^i \partial_j - x^j \partial_i$$

(rotations) \hspace{1em} (81)

Three generators of Galilean boosts read

$$\xi(i) = x^0 \partial_i \equiv t \partial_i \hspace{1em} i = 1, 2, 3$$

(see (157) - (158)). It is an elementary check that

$$\mathcal{L}_{\xi(i)} \omega = 0 \quad \text{and} \quad \mathcal{L}_{\xi(i)} h = 0 \hspace{1em} i = 1, 2, 3$$

so (any constant multiples of) both $\omega$ and $h$ are invariant w.r.t. Galilean boosts. (As for $\omega$, see also (160) and (163).) Moreover, one can similarly check that, modulo constant multiple, $h$ is the only $\binom{3}{0}$-type tensor field with these properties.

The geometrical role of the volume form is clear.

Concerning the role of the tensor field $h$:

1. It defines, in each point of the Galilean spacetime, spatial vectors via the image space of $h$ w.r.t. the map $\alpha \mapsto h(\alpha, \cdot)$, $\alpha$ being a spacetime covector.

2. It defines, on the (3-dimensional) sub-space of spatial vectors, the spatial metric $\hat{h}$ via

$$\hat{h}(h(\alpha, \cdot), h(\beta, \cdot)) := -h(\alpha, \beta) \hspace{1em} \alpha, \beta \text{ spacetime covectors}$$

(The minus on the r.h.s. just makes $\hat{h}$ positive (rather than negative) definite for our choice of sign in (78).) Formally, the trick from 1. and 2. is exactly the same as
- **cometric** is used in sub-Riemannian geometry
  (in order to define the distribution and then the **metric** on the subspaces)
- **Poisson** tensor is used in Poisson geometry
  (in order to define symplectic leaves and then **symplectic form** on them).

Similarly in *Carrollian* case, both tensor fields

\[
\tilde{\omega} \equiv \partial_t \wedge \partial_x \wedge \partial_y \wedge \partial_z \quad \text{and} \quad \tilde{h} = \delta_{ij} dx^i \otimes dx^j \quad (85)
\]

i.e.

\[
\tilde{\omega}_{0123} = 1 \quad \tilde{h}_{\mu
u} \leftrightarrow \left( \begin{array}{cc} \tilde{h}_{00} & \tilde{h}_{0i} \\ \tilde{h}_{i0} & \tilde{h}_{ij} \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & \delta_{ij} \end{array} \right) \quad (86)
\]

are clearly translation and rotation invariant. That is, they are Lie invariant w.r.t. vector fields (79) - (81). Three generators of *Carrollian boosts* read

\[
\tilde{\xi}_{(i)} = x^i \partial_t \quad i = 1, 2, 3 \quad (87)
\]

(see (109) - (110)). It is an elementary check that

\[
\mathcal{L}_{\tilde{\xi}_{(i)}} \tilde{\omega} = 0 \quad \text{and} \quad \mathcal{L}_{\tilde{\xi}_{(i)}} \tilde{h} = 0 \quad i = 1, 2, 3 \quad (88)
\]

so (any constant multiples of) both \( \tilde{\omega} \) and \( \tilde{h} \) are invariant w.r.t. Carrollian boosts as well. As for the signs in (78) and (85), see (101).

It is also worth mentioning (elementary check; it is used in Appendix A.4) that

- Galilean invariant \( \omega \) from (78) is also *Carrollian* invariant
- Carrollian invariant \( \tilde{\omega} \) from (85) is also *Galilean* invariant
- \( k = dt \otimes dt \) is *Galilean* invariant and \( \tilde{k} = \partial_t \otimes \partial_t \) is *Carrollian* invariant.

Knowledge of tensor field \( h \) is notorious in the literature on Galilean spacetimes or Galilean structure. Starting, at least, in [17] and [18], up to, say, [19]. The volume form \( \omega \) is already mentioned explicitly in [20].

As for \( k \) and \( \tilde{k} \), they are just tensor products of \( dt \) (which is Galilean invariant) and \( \partial_t \) (which is Carrollian invariant), respectively. So they are invariant as well. All of them are ubiquitous in the literature.

### A.3 Proof of (24) and (25)

Definition formula (11) in the language of components leads directly, on coordinate basis of \( p \)-forms (w.r.t. which the components are defined), to

\[
\star (dx^\mu \wedge \cdots \wedge dx^\nu) = \frac{1}{(n-p)!} \omega^{\mu...\nu}_{\rho...\tau} dx^\rho \wedge \cdots \wedge dx^\tau \quad (89)
\]

where

\[
\omega^{\mu...\nu}_{\rho...\tau} := h^{\mu\alpha} \cdots h^{\nu\beta} \omega_{\alpha...\beta\rho...\tau} \quad \text{Galilean spacetime} \quad (90)
\]

\[
\tilde{\omega}^{\mu...\nu}_{\rho...\tau} := \tilde{h}_{\rho\alpha} \cdots \tilde{h}_{\tau\beta} \tilde{\omega}^{\mu...\nu\alpha...\beta} \quad \text{Carrollian spacetime} \quad (91)
\]
\[
\begin{align*}
&h^{0\mu} = h^{\mu 0} = 0 \quad \text{Galilean} \quad (92) \\
&\tilde{h}_{0\mu} = \tilde{h}_{\mu 0} = 0 \quad \text{Carrollian} \quad (93)
\end{align*}
\]
(here \(\lambda_1\) and \(\lambda_2\) are free parameters to be fixed later). From this we easily see vanishing components:
\[
\begin{align*}
&\omega^{0...i}_{\quad j...k} = 0 \quad \text{i.e.} \quad * (dt \wedge \hat{s}) = 0 \quad \text{Galilean} \quad (94) \\
&\omega^{i...j}_{\quad 0...k} = 0 \quad \text{i.e.} \quad * \hat{r} = 0 \quad \text{Carrollian} \quad (95)
\end{align*}
\]
For nonvanishing components we get, similarly,
\[
\begin{align*}
&\omega^{i...j}_{\quad 0k...l} = (-\lambda_1)^p \epsilon_{i...jk...l} \quad \text{Galilean} \quad (96) \\
&\omega^{0i...j}_{\quad k...l} = \lambda_2^{n-p} \epsilon_{i...jk...l} \quad \text{Carrollian} \quad (97)
\end{align*}
\]
and, consequently,
\[
\begin{align*}
&* \hat{r} = dt \wedge (-\lambda_1)^p \hat{\epsilon} \hat{r} \quad \text{Galilean} \quad (98) \\
&* (dt \wedge \hat{s}) = \lambda_2^{n-1} \hat{\epsilon} \hat{s} \quad \text{Carrollian} \quad (99)
\end{align*}
\]
Now in Minkowski spacetime (convention \((+,-,\ldots, -)\)) we have
\[
* (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{\epsilon} \hat{r} + \hat{\epsilon} \hat{s} \quad (100)
\]
Then we see that if we want to get as similar expressions as possible to the Minkowski case, we are to choose
\[
\lambda_1 = -1 \quad \lambda_2 = 1 \quad \text{i.e.} \quad h^{ij} = -\delta^{ij} \quad \tilde{h}_{ij} = \delta_{ij} \quad (101)
\]
For this particular choice of free parameters we end with the following formulas:
\[
\begin{align*}
&* (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{\epsilon} \hat{r} + \hat{\epsilon} \hat{s} \quad \text{Minkowski Hodge star} \quad (102) \\
&* (dt \wedge \hat{s} + \hat{r}) = dt \wedge \hat{\epsilon} \hat{r} \quad \text{Galilean Hodge star} \quad (103) \\
&* (dt \wedge \hat{s} + \hat{r}) = \hat{\epsilon} \hat{s} \quad \text{Carrollian Hodge star} \quad (104)
\end{align*}
\]
Finally, notice that we can see from the proof that results in both (103) and (104) hold in Galilean and Carrollian spacetimes of any dimension (one time and any number of spatial dimensions). The formulas (117) - (131) are, on the contrary, already tailored for the usual 1 + 3 case.
A.4 Alternative versions of star operators

Recall that, in all three cases (standard, Galilean and Carrollian), the Hodge star works according to the common formula

\[(\ast \alpha)_{\mu...\nu} := \frac{1}{p!} \alpha_{\rho...\tau} \omega^{\rho...\tau}_{\mu...\nu} \tag{105}\]

and where the construction differs is how the necessary mixed tensor field \(\omega^{\rho...\tau}_{\mu...\nu}\) is obtained:

1a On a general \((M, g)\) (including Minkowski spacetime) we have at our disposal distinguished - volume \(n\)-form \(\omega \leftrightarrow \omega_{\rho...\tau\mu...\nu}\) for any orthonormal coframe \(e^a\)
- cometric \(g^{-1} \leftrightarrow g^{\mu\nu}\)
and the needed tensor \(\omega^{\rho...\tau}_{\mu...\nu}\) is produced by raising of indices on \(\omega\) with the help of \(g^{-1}\).

1b Still on a general \((M, g)\) we also have (distinguished)
- \(n\)-vector \(\tilde{\omega} \leftrightarrow \tilde{\omega}_{\rho...\tau\mu...\nu}\)
- metric \(g \leftrightarrow g_{\mu\nu}\)
- \(\omega^{\rho...\tau}_{\mu...\nu}\) by lowering of (complementary) indices on \(\tilde{\omega}\) with the help of \(g\)
- and we clearly get the same tensor \(\omega^{\rho...\tau}_{\mu...\nu}\) in this way.

2a On Galilean spacetime, we have at our disposal invariant
- volume \(n\)-form \(\omega \leftrightarrow \omega_{\rho...\tau\mu...\nu}\)
- tensor field \(h \leftrightarrow h^{\mu\nu}\)
- \(\omega^{\rho...\tau}_{\mu...\nu}\) produced by raising of indices on \(\omega\) with the help of \(h\).

2b On Carrollian spacetime, we have at our disposal invariant
- \(n\)-vector \(\tilde{\omega} \leftrightarrow \tilde{\omega}_{\rho...\tau\mu...\nu}\)
- tensor field \(\tilde{h} \leftrightarrow \tilde{h}_{\mu\nu}\)
- \(\omega^{\rho...\tau}_{\mu...\nu}\) produced by lowering of indices on \(\tilde{\omega}\) with the help of \(\tilde{h}\).

However, as is mentioned at the end of Appendix A.2 (and readily checked),
- Galilean invariant \(\omega\) from (78) is also Carrollian invariant
- Carrollian invariant \(\tilde{\omega}\) from (85) is also Galilean invariant
- \(k = dt \otimes dt \leftrightarrow k^{\mu\nu}\) is Galilean invariant (folklore knowledge)
- \(\tilde{k} = \partial_t \otimes \partial_t \leftrightarrow \tilde{k}^{\mu\nu}\) is Carrollian invariant (folklore knowledge).

This means that we can actually add \(b\)-versions to both Galilean and Carrollian constructions:

3a On Galilean spacetime, we have at our disposal invariant
- \(n\)-vector \(\tilde{\omega} \leftrightarrow \tilde{\omega}_{\rho...\tau\mu...\nu}\)
- tensor field \(k \leftrightarrow k^{\mu\nu}\)
- \(\omega^{\rho...\tau}_{\mu...\nu}\) is produced by lowering of indices on \(\tilde{\omega}\) with the help of \(k\).

3b On Carrollian spacetime, we have at our disposal invariant
- volume n-form \( \omega \leftrightarrow \omega_{\rho\ldots\tau\ldots\nu} \)
- tensor field \( \tilde{k} \leftrightarrow \tilde{k}^{\mu}_{\nu} \)
- \( \omega^{\rho\ldots\tau\ldots\nu} \) is produced by raising of indices on \( \omega \) with the help of \( \tilde{k} \).

For explicit expressions (as well as some discussion) see Appendix A.5.

### A.5 Explicit formulas for all degrees of forms

In the usual 1+3 dimensional Minkowski, Galilean as well as Carrollian spacetimes, the general decomposition formula

\[
\alpha = dt \wedge \hat{s} + \hat{r}
\]  

(106)

from [22] may be specified, for the five relevant degrees of differential forms, as follows:

\[
\Omega^0: \quad \alpha = f
\]  

(107)

\[
\Omega^1: \quad \alpha = f dt + \mathbf{a} \cdot d\mathbf{r}
\]  

(108)

\[
\Omega^2: \quad \alpha = dt \wedge \mathbf{a} \cdot d\mathbf{r} + b \cdot d\mathbf{S}
\]  

(109)

\[
\Omega^3: \quad \alpha = dt \wedge \mathbf{a} \cdot d\mathbf{S} + fdV
\]  

(110)

\[
\Omega^4: \quad \alpha = f dt \wedge dV
\]  

(111)

where

\[
\mathbf{a} \cdot d\mathbf{r} = a_x dx + a_y dy + a_z dz
\]  

(112)

\[
\mathbf{a} \cdot d\mathbf{S} = a_x dS_x + a_y dS_y + a_z dS_z
\]  

(113)

\[
\equiv a_x dy \wedge dz + a_y dz \wedge dx + a_z dx \wedge dy
\]  

(114)

\[
dV = dx \wedge dy \wedge dz
\]  

(115)

So, concerning this presentation of forms alone, there is no difference between the three spacetimes (for the Minkowski case, see e.g. Sec.16.1 in [7]).

Due to (102) - (104) and well-known Euclidean 3D-results

\[
\hat{*}f = fdV \quad \hat{*}(\mathbf{a} \cdot d\mathbf{r}) = \mathbf{a} \cdot d\mathbf{S} \quad \hat{*}(\mathbf{a} \cdot d\mathbf{S}) = \mathbf{a} \cdot d\mathbf{r} \quad \hat{*}(fdV) = f
\]  

(116)

(see e.g. Sec.8.5 in [7]) we get, in this language, the following explicit results for action of the three Hodge star operators:

**Poincaré** invariant Hodge star:

\[
\hat{*}f = f dt \wedge dV
\]  

(117)

\[
\hat{*}(dt + \mathbf{a} \cdot d\mathbf{r}) = dt \wedge \mathbf{a} \cdot d\mathbf{S} + fdV
\]  

(118)

\[
\hat{*}(dt \wedge \mathbf{a} \cdot d\mathbf{r} + b \cdot d\mathbf{S}) = dt \wedge b \cdot d\mathbf{r} - \mathbf{a} \cdot d\mathbf{S}
\]  

(119)

\[
\hat{*}(dt \wedge \mathbf{a} \cdot d\mathbf{S} + fdV) = f dt + \mathbf{a} \cdot d\mathbf{r}
\]  

(120)

\[
\hat{*}(dt \wedge dV) = -f
\]  

(121)
Galilean invariant Hodge star:

\[ *f = fdt \wedge dV \]  
\[ *(fdt + a \cdot dr) = dt \wedge a \cdot dS \]  
\[ *(dt \wedge a \cdot dr + b \cdot dS) = dt \wedge b \cdot dr \]  
\[ *(dt \wedge a \cdot dS + fdV) = fdt \]  
\[ *(fdt \wedge dV) = 0 \]

Carrollian invariant Hodge star:

\[ *f = 0 \]  
\[ *(fdt + a \cdot dr) = fdV \]  
\[ *(dt \wedge a \cdot dr + b \cdot dS) = -a \cdot dS \]  
\[ *(dt \wedge a \cdot dS + fdV) = a \cdot dr \]  
\[ *(fdt \wedge dV) = -f \]

When alternative versions (namely cases 2b and 3b from Appendix A.4) of Galilean and Carrollian Hodge stars are computed, we get:

**Alternative Galilean invariant Hodge star:**

\[ *f = 0 \]  
\[ *(fdt + a \cdot dr) = 0 \]  
\[ *(dt \wedge a \cdot dr + b \cdot dS) = 0 \]  
\[ *(dt \wedge a \cdot dS + fdV) = fdt \]  
\[ *(fdt \wedge dV) = -f \]

(signs hold for the convention \( \tilde{\omega}^{0123} = -1, k_{00} = +1 \)).

**Alternative Carrollian invariant Hodge star:**

\[ *f = fdt \wedge dV \]  
\[ *(fdt + a \cdot dr) = fdV \]  
\[ *(dt \wedge a \cdot dr + b \cdot dS) = 0 \]  
\[ *(dt \wedge a \cdot dS + fdV) = 0 \]  
\[ *(fdt \wedge dV) = 0 \]

(signs hold for the convention \( \omega_{0123} = +1, k_{00} = +1 \)).

Recall that, as was explained in Appendix A.4, the two “alternative” versions are also invariant w.r.t. corresponding transformations, i.e.

- both (122) - (126) and (132) - (136) are Galilean invariant
- both (127) - (131) and (137) - (141) are Carrollian invariant.
However, when explicit expressions (namely (122) - (126) versus (132) - (136) and, similarly, (127) - (131) versus (137) - (141)) are compared, what we see is that the “b-versions”
- often lead just to zero operator (where “a-versions” provide non-zero result)
- exceptionally it is contrariwise, “b-versions” give more (see (136), (137)).

In general, it is our free choice which version is more useful for us. Since, however,
zero operator is always (trivially) invariant, non-zero result makes more sense, from practical point of view.

For a completely different approach (and more results), see the forthcoming paper [21].

A.6 Action of rotations, Galilean and Carrollian boosts on forms

Consider two observers who are rotated with respect to one another. So they use coordinates related by (in all three spacetimes)
\[
\begin{align*}
  t' &= t \\
  r' &= Rr \
\end{align*}
\]
i.e.
\[
  x'_i = R_{ij} x_j
\]
and therefore, their coordinate basis forms are related as
\[
\begin{align*}
  dt' &= dt \\
  dr' &= Rdr \\
  dS' &= RdS \\
  dV' &= dV
\end{align*}
\]
Then, from
\[
\begin{align*}
  f' &= f \\
  f'dt' + a' \cdot dr' &= f dt + a \cdot dr \\
  dt' \wedge a' \cdot dr' + b' \cdot dS' &= dt \wedge a \cdot dr + b \cdot dS \\
  dt' \wedge a' \cdot dS' + f' dV' &= dt \wedge a \cdot dS + f dV \\
  f'dt' \wedge dV' &= f dt \wedge dV
\end{align*}
\]
we easily get primed components in terms of unprimed to be
\[
\begin{align*}
  \Omega^0 : & \quad f \quad \mapsto \quad f' \quad = \quad f
\\
  \Omega^1 : & \quad (f, a) \quad \mapsto \quad (f', a') \quad = \quad (f, Ra) \\
  \Omega^2 : & \quad (a, b) \quad \mapsto \quad (a', b') \quad = \quad (Ra, Rb) \\
  \Omega^3 : & \quad (a, f) \quad \mapsto \quad (a', f') \quad = \quad (Ra, f) \\
  \Omega^4 : & \quad f \quad \mapsto \quad f' \quad = \quad f
\end{align*}
\]
[What we actually compute in this way are effective changes of (properly chosen) components when pull-back of the (complete) original forms w.r.t. to coordinate change \((t,r) \mapsto (t,R^{-1}r)\). So, in all primed components (i.e. in \(f',a',b'\)), corresponding change of the argument \(r \mapsto R^{-1}r\) is also meant implicitly. As an example, what \([154]\) says in detail is that \(a(t,r) \mapsto a'(t,r) = (Ra)(t,R^{-1}r)\). The same comment holds for boost formulas presented below.]

Similarly, for observers related by Galilean boosts with respect to one another, we have

\[
\begin{align*}
t' &= t \\
r' &= r + vt
\end{align*}
\]

and therefore, on basis coordinate forms,

\[
\begin{align*}
dt' &= dt \\
dr' &= dr + vt \\
dS' &= dS + dt \wedge (v \times dr) \\
dV' &= dV + dt \wedge v \cdot dS
\end{align*}
\]

Then, writing again \([148] - [152]\), we get

\[
\begin{align*}
\Omega^0 : & \quad f \quad \mapsto \quad f' \quad = \quad f \\
\Omega^1 : & \quad (f,a) \quad \mapsto \quad (f',a') \quad = \quad (f - v \cdot a,a) \\
\Omega^2 : & \quad (a,b) \quad \mapsto \quad (a',b') \quad = \quad (a + v \times b,b) \\
\Omega^3 : & \quad (a,f) \quad \mapsto \quad (a',f') \quad = \quad (a - vf,f) \\
\Omega^4 : & \quad f \quad \mapsto \quad f' \quad = \quad f
\end{align*}
\]

Finally, for Carrollian boosts we have (see (9) in \([22]\), II.10 in \([8]\))

\[
\begin{align*}
t' &= t + v \cdot r \\
r' &= r
\end{align*}
\]

and therefore, on basis coordinate forms,

\[
\begin{align*}
dt' &= dt + v \cdot dr \\
dr' &= dr \\
dS' &= dS \\
dV' &= dV
\end{align*}
\]
Then, writing \(148\) - \(152\) still again, we get

\[
\Omega^0: \quad f \mapsto f' = f \quad (175)
\]

\[
\Omega^1: \quad (f, a) \mapsto (f', a') = (f, a - f v) \quad (176)
\]

\[
\Omega^2: \quad (a, b) \mapsto (a', b') = (a, b - v \times a) \quad (177)
\]

\[
\Omega^3: \quad (a, f) \mapsto (a', f') = (a, f - v \cdot a) \quad (178)
\]

\[
\Omega^4: \quad f \mapsto f' = f \quad (179)
\]

One can check, using these formulas (as well as explicit expressions for Hodge stars from Appendix A.5) that the Hodge indeed commutes with rotations and boosts.

As an example, for rotations, two-forms and Galilean spacetime, we get (using \(124\) and \(155\) and the fact, that Hodge does not change the argument \((t, r)\))

\[
\Omega^2: \quad (a, b) \mapsto (Ra, Rb) \mapsto (Rb, 0) \quad (180)
\]

\[
\ast \mapsto (b, 0) \mapsto (Rb, 0) \quad (181)
\]

so that, for the specified case,

\[
R \ast = \ast R \quad (182)
\]

Similarly, for boosts, two-forms and Galilean spacetime, we get (using \(124\) and \(166\))

\[
\Omega^2: \quad (a, b) \mapsto (a + v \times b, b) \mapsto (b, 0) \quad (183)
\]

\[
\ast \mapsto (b, 0) \mapsto (b, 0) \quad (184)
\]

so that, for the specified case,

\[
B \ast = \ast B \quad (185)
\]

Finally, for boosts, two-forms and Carrollian spacetime, we get (using \(129\) and \(177\))

\[
\Omega^2: \quad (a, b) \mapsto (a, b - v \times a) \mapsto (0, -a) \quad (186)
\]

\[
\ast \mapsto (0, -a) \mapsto (0, -a) \quad (187)
\]

so that, for the specified case,

\[
B \ast = \ast B \quad (188)
\]

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References

[1] P.Bamberg, S.Sternberg: *A Course in Mathematics for Students of Physics*, Cambridge University Press, Cambridge, 1990

[2] B.F.Schutz: *Geometrical Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1982

[3] M.Crampin, F.A.E.Pirani: *Applicable Differential Geometry*, Cambridge University Press, Cambridge, 1986

[4] M.Göckeler, T.Schücker: *Differential Geometry, Gauge Theories and Gravity*, Cambridge University Press, Cambridge, 1987

[5] T.Frankel: *The Geometry of Physics, An Introduction* 
Cambridge University Press 1997 (1-st ed), 2004 (2-nd ed), 2012 (3-rd ed)

[6] A.Trautman: *Differential Geometry for Physicists*, Bibliopolis, Napoli, 1984

[7] M.Fecko: *Differential Geometry and Lie Groups for Physicists*, Cambridge University Press 2006 (paperback 2011)

[8] Ch.Duval, G.W.Gibbons, P.A.Horvathy, P.M.Zhang: 
Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time, 
Classical and Quantum Gravity, Volume 31, Number 8 (2014) 
[arXiv:1402.0657 [gr-qc]]

[9] D.Van den Bleeken, and C.Yunus: 
Newton-Cartan, Galileo-Maxwell and Kaluza-Klein, 
Classical and Quantum Gravity, 33, 13 (2016) 
[arXiv:1512.03799v2 [math-ph]]

[10] J.Figueroa-O’Farrill, R.Grassie, S.Prohazka: Geometry and BMS Lie Algebras of Spatially Isotropic Homogeneous Spacetimes, 
Journal of High Energy Physics 119 (2019); [arXiv:2009.01948 [hep-th]]

[11] L.Ciambelli, R.G.Leigh, Ch.Marteau and P.M.Petropoulos: 
Carroll structures, null geometry, and conformal isometries, 
Physical Review D 100, 046010 (2019); [arXiv:1905.02221 [hep-th]]

[12] K.Morand: 
Embedding Galilean and Carrollian geometries I. Gravitational waves, 
Journal of Mathematical Physics 61, 082502 (2020) 
[arXiv:1811.12681-2 [hep-th]]

[13] K.Banerjee, R.Basu, A.Mehra, A.Mohan, A.Sharma: 
Interacting Conformal Carrollian Theories: Cues from Electrodynamics, 
Physical Review D 103, 105001 (2021); [arXiv:2008.02829v4 [hep-th]]
[14] M. Henneaux, P. Salgado-Rebolledo: Carroll contractions of Lorentz-invariant theories,
Journal of High Energy Physics 180 (2021); [arXiv:2109.06708 [hep-th]]

[15] Ch. W. Misner, K. S. Thorne, J. A. Wheeler: *Gravitation*,
W. H. Freeman and Company, 1973

[16] M. Le Bellac, J.-M. Lévy Leblond: Galilean Electromagnetism,
Il Nuovo Cimento, Vol. 14 B, N. 2, 217 - 234 (1973)

[17] A. Trautman: Sur la théorie newtonienne de la gravitation,
C. R. Acad. Sci. Paris, t. 257, p. 617-720 (1963)

[18] A. Trautman:
Comparison of Newtonian and Relativistic Theories of Space-Time,
pp. 413–425 in: *Perspectives in Geometry and Relativity*, Essays in honor of V. Hlavatý, ed. by B. Hoffmann, Indiana Univ. Press, Bloomington, 1966.

[19] J. Figueroa-O’Farrill: On the intrinsic torsion of spacetime structures,
Report number EMPG-20-14; [arXiv:2009.01948 [hep-th]] (2020)

[20] H. P. Künzle: Galilei and Lorentz structures on spacetime: Comparison of the corresponding geometry and physics,
Ann. Inst. H. Poincaré. Phys. Théor. 17, 4, 337-362. (1972)

[21] M. Fecko: Some useful operators on differential forms in Galilean and Carrollian spacetimes,
in preparation (2022)

[22] J.-M. Lévy Leblond: Une nouvelle limite non-relativiste du groupe de Poincaré,
Annales de l’ I.H.P. Physique théorique vol. 3, no. 1, pp. 1-12, (1965)