CONVERGENCE OF DIAGONAL PADÉ APPROXIMANTS FOR A CLASS OF DEFINITIZABLE FUNCTIONS

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Abstract. Convergence of diagonal Padé approximants is studied for a class of functions which admit the integral representation
\[ F(\lambda) = r_1(\lambda) \int_{-1}^{1} \frac{t \sigma(t)}{t - \lambda} \mathop{dt} - r_2(\lambda), \]
where \( \sigma \) is a finite nonnegative measure on \([-1,1]\), \( r_1 \) and \( r_2 \) are real rational functions bounded at \( \infty \), and \( r_1 \) is nonnegative for real \( \lambda \). Sufficient conditions for the convergence of a subsequence of diagonal Padé approximants of \( F \) on \( \mathbb{R} \setminus [-1,1] \) are found. Moreover, in the case when \( r_1 \equiv 1, r_2 \equiv 0 \) and \( \sigma \) has a gap \((\alpha, \beta)\) containing 0, it turns out that this subsequence converges in the gap. The proofs are based on the operator representation of diagonal Padé approximants of \( F \) in terms of the so-called generalized Jacobi matrix associated with the asymptotic expansion of \( F \) at infinity.

1. Introduction

Let \( F(\lambda) = -\sum_{j=0}^{\infty} s_j \lambda^{j+1} \) be a formal power series with \( s_j \in \mathbb{R} \), and let \( L, M \) be positive integers. An \([L/M]\) Padé approximant for \( F \) is defined as a ratio
\[ F^{[L/M]}(\lambda) = \frac{A^{[L/M]}(\lambda)}{B^{[L/M]}(\lambda)} \]
of polynomials \( A^{[L/M]} \), \( B^{[L/M]} \) of formal degree \( L \) and \( M \), respectively, such that \( B^{[L/M]}(0) \neq 0 \) and
\[ \sum_{j=0}^{L+M-1} \frac{s_j}{\lambda^{j+1}} + F^{[L/M]}(\lambda) = O \left( \frac{1}{\lambda^{L+M+1}} \right), \quad \lambda \to \infty. \]

The classical Markov theorem \[31\] states that for every nonnegative measure \( \sigma \) on the interval \([-1,1]\) and the function
\[ F(\lambda) = \int_{-1}^{1} \frac{t \sigma(t)}{t - \lambda} \]
with the Laurent expansion \( F(\lambda) = -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}} \) at \( \infty \) the diagonal Padé approximants \( F^{[n/n]} \) exists for every \( n \in \mathbb{N} \) and converge to \( F \) locally uniformly on

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\(\mathbb{C} \setminus [−1, 1]\). However, it should be noted that in the case when \(\sigma\) has a gap \((\alpha, \beta)\) in its support, the diagonal \(\text{Padé}^\star\) approximants \(F[n/n]\) do not usually converge inside the gap (see \([31]\)).

In \([6]\) it was conjectured that for every function \(F\) holomorphic in a neighborhood of \(\infty\) there is a subsequence of diagonal \([n/n]\) \(\text{Padé}^\star\) approximants which converges to \(F\) locally uniformly in the neighborhood of \(\infty\) (Padé hypothesis). In general, as was shown by D. Lubinsky \([28]\) (see also \([9]\)), this conjecture fails to hold, but for some classes of functions the Padé hypothesis is still true. For example, if \(F\) has the form

\[
F(\lambda) = \int_{-1}^{1} \frac{d\sigma(t)}{t - \lambda} + r(\lambda),
\]

where \(r\) is a rational function with poles outside of \([-1, 1]\), the convergence of \(\text{Padé}^\star\) approximants was proved by A. Gonchar \([16]\) and E. Rakhmanov \([33]\).

In \([11, 12]\) we studied the Padé hypothesis in the class of generalized Nevanlinna functions introduced in \([23]\) (see the definition at the beginning of Section 2), which contains, in particular, functions of the form

\[
F(\lambda) = r_1(\lambda) \int_{-1}^{1} \frac{d\sigma(t)}{t - \lambda} + r_2(\lambda),
\]

where:

- (A1) \(\sigma\) is a finite nonnegative measure on \([-1, 1]\);
- (A2) \(r_1 = q_1/w_1\) is a rational function, nonnegative for real \(\lambda\) (\(\deg q_1 \leq \deg w_1\));
- (A3) \(r_2 = q_2/w_2\) is a real rational function such that \(\deg q_2 < \deg w_2\).

Let \(F\) have the Laurent expansion \(F(\lambda) = -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}\) at \(\infty\), and let \(\mathcal{N}(s)\) be the set of all normal indices of the sequence \(s = \{s_i\}_{i=0}^{\infty}\), i.e. natural numbers \(n_1 < n_2 < \cdots < n_j < \cdots\), for which

\[
\det(s_{i+k})_{i,k=0}^{n_j-1} \neq 0, \quad j = 1, 2, \ldots.
\]

As is known the sequence \(\{n_j\}_{j=1}^{\infty}\) contains all natural \(n\) big enough. The Padé approximants for \(F\) were considered in \([12]\) in connection with the theory of difference equations

\[
\bar{b}_{j-1}u_{j-1} - p_j(\lambda)u_j + b_ju_{j+1} = 0, \quad j \in \mathbb{N},
\]

naturally related to the function \(F\), where \(p_j\) are monic polynomials of degree \(k_j = n_{j+1} - n_j\), \(b_j > 0\), \(\bar{b}_j\) are real numbers, such that \(|\bar{b}_j| = b_j, j \in \mathbb{Z}_+: = \mathbb{N} \cup \{0\}\).

It turned out that the diagonal Padé approximants for \(F\) exist for all \(n = n_j\) and are calculated by the formula

\[
F[n_j/n_j](\lambda) = -\frac{Q_j(\lambda)}{P_j(\lambda)},
\]

where \(P_j, Q_j\) are polynomials of the first and the second type associated with the difference equation \((1.5)\) (see \([11]\)). In \([12]\) it was shown that the sequence of diagonal Padé approximants for \(F\) converges to \(F\) locally uniformly in \(\mathbb{C} \setminus ([−1, 1] \setminus \mathcal{P}(F))\), where \(\mathcal{P}(F)\) is the set of poles of \(F\). Subdiagonal Padé approximants \(F[n_j/n_j-1]\) for \(F\) exist if and only if

\[
n_j \in \mathcal{N}_F := \{n_j \in \mathcal{N}(s) : P_{j-1}(0) \neq 0\}.
\]
The convergence of the sequence \( \{ F_{(n/j-1)/n_{j-1}} \}_{n_j \in \mathbb{N}_F} \) of subdiagonal Padé approximants for \( F \) on \( \mathbb{C} \setminus \mathbb{R} \) was also proved.

In Theorem 3.5 we improve the result of [12, Theorem 4.16] by pointing out sufficient condition for convergence of the sequence of subdiagonal Padé approximants \( F_{(n/n-1)} \) to \( F \) in a neighborhood of \( \infty \). In the previous notations this condition takes the form

(B) The sequence \( \left\{ b_{j-1} P_{j-1}(0) / P_j(0) \right\}_{n_j \in \mathbb{N}_F} \) is bounded.

The main part of the present paper is dedicated to the study of convergence of diagonal Padé approximants of definitizable functions with one "turning point". This class was introduced by P. Jonas in [18]. We postpone the exact definitions until Section 4 and we mention only that the typical representative of this class is the function

\[
(1.6) \quad \mathfrak{F}(\lambda) = r_1(\lambda) \int_{-1}^{1} \frac{t \sigma(t)}{t - \lambda} + r_2(\lambda),
\]

where \( \sigma, r_1, r_2 \) satisfy the assumptions (A1), (A2) and

(A3') \( r_2 = q_2/w_2 \) is a real rational function such that \( \deg q_2 \leq \deg w_2 \).

We prove that for every \( \mathfrak{F} \) of the form (1.6) satisfying condition (B) the Padé hypothesis is still true. The idea is that the diagonal Padé approximants for \( \mathfrak{F} \) are proportional to the subdiagonal Padé approximants for \( F \). This fact was observed by A. Magnus in a formal setting [29] and its operator interpretation for the Nevanlinna class was given by B. Simon [34]. This observation and [12, Theorem 4.16] enable us to prove in Theorem 4.6 that the sequence of diagonal Padé approximants for \( F \) converges to \( F \) locally uniformly in \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(\mathfrak{F})) \). Moreover, using the result of Theorem 3.5 we show that condition (B) is sufficient for the convergence of diagonal Padé approximants of \( \mathfrak{F} \) in a neighborhood of \( \infty \) (see Theorem 5.1).

In Theorem 5.5 we specify this result to the case when the function \( \mathfrak{F} \) in (1.3) takes the form

\[
(1.7) \quad \mathfrak{F}(\lambda) = \int_{E} \frac{t \sigma(t)}{t - \lambda}, \quad E = [-1, \alpha] \cup [\beta, 1],
\]

where the measure \( \sigma \) has a gap \((\alpha, \beta)\) with \( \alpha < 0 < \beta \). For this function one can observe a new effect, that the sequence of diagonal Padé approximants \( \{ \mathfrak{F}_{(n_j-1)/n_{j-1}} \}_{n_j \in \mathbb{N}_F} \) converges to \( \mathfrak{F} \) in the gap \((\alpha, \beta)\). The proof of this result is based on the theory of generalized Jacobi matrices associated with generalized Nevanlinna functions and on the operator representation of the subdiagonal Padé approximants for the generalized Nevanlinna function \( F(\lambda) := \mathfrak{F}(\lambda)/\lambda \). Moreover, in Theorem 5.5 we prove that for such a function condition (B) is also necessary and sufficient for the convergence of the sequence \( \{ \mathfrak{F}_{(n_j-1)/n_{j-1}} \}_{n_j \in \mathbb{N}_F} \) to \( \mathfrak{F} \) in a neighborhood of \( \infty \).

This theorem makes a bridge to the theory of classical orthogonal polynomials. In Proposition 5.7 we show that the condition (B) is in force, if 0 is not an accumulation point of zeros of polynomials \( P_n \) orthogonal with respect to \( \sigma \). In the case when the measure \( \sigma \) in (1.7) satisfies the Szegő condition on each of the intervals \([-1, \alpha]\) and \([\beta, 1]\) we inspect the question: under what conditions 0 is not an accumulation point of zeros of polynomials \( P_n \)? In Proposition 5.9 we show that the results of E. Rakhmanov [33] can be applied to give a partial answer to this question and, hence, to find some sufficient conditions on \( \alpha, \beta \) and \( \sigma \) for the existence of
a subsequence of diagonal Padé approximants \( \tilde{\mathcal{F}}^{[n/n]} \) which converges to \( F \) locally uniformly in a neighborhood of \( \infty \).

The paper is organized as follows. In Section 2 the basic facts concerning generalized Nevanlinna functions and their operator representations in terms of generalized Jacobi matrices are given. In Section 3 we state and improve some results from [12] on the locally uniform convergence of subdiagonal Padé approximants for generalized Nevanlinna functions. In Section 4 we introduce the class \( \mathcal{D}_{\kappa,-\infty} \) of definitizable functions with one "turning point", and find the formula connecting diagonal Padé approximants for \( \mathcal{F} \in \mathcal{D}_{\kappa,-\infty} \) with subdiagonal Padé approximants for generalized Nevanlinna function \( F(\lambda) = \mathcal{F}(\lambda)/\lambda \). In Section 5 we apply our results to subclasses of definitizable functions of the form (1.6) and (1.7).

This paper is dedicated to the memory of Peter Jonas. Discussions with him during several of our visits to Berlin have had a significant influence on the development of this paper.

2. Preliminaries

2.1. Moment problem in the class of generalized Nevanlinna functions.

Let \( \kappa \) be a nonnegative integer. Recall that a function \( F \), meromorphic in \( \mathbb{C}_+ \cup \mathbb{C}_- \), is said to belong to the class \( \mathcal{N}_{\kappa} \) if the domain of holomorphy \( \rho(F) \) of the function \( F \) is symmetric with respect to \( \mathbb{R} \), \( F(\bar{\lambda}) = F(\lambda) \) for \( \lambda \in \rho(F) \), and the kernel

\[
\begin{align*}
N_F(\lambda, \omega) &= \frac{F(\lambda) - F(\omega)}{\lambda - \omega}, & \lambda, \omega &\in \rho(F); \\
N_F(\lambda, \lambda) &= F'(\lambda), & \lambda &\in \rho(F)
\end{align*}
\]

has \( \kappa \) negative squares on \( \rho(F) \). The last statement means that for every \( n \in \mathbb{N} \) and \( \lambda_1, \lambda_2, \ldots, \lambda_n \in \rho(F) \), \( n \times n \) matrix \( (N_F(\lambda_i, \lambda_j))_{i,j=1}^n \) has at most \( \kappa \) negative eigenvalues (with account of multiplicities) and for some choice of \( n, \lambda_1, \lambda_2, \ldots, \lambda_n \) it has exactly \( \kappa \) negative eigenvalues (see [23]).

We will say (cf. [15]) that a generalized Nevanlinna function \( F \) belongs to the class \( \mathcal{N}_{\kappa,-2n} \) if \( F \in \mathcal{N}_{\kappa} \) and for some numbers \( s_0, \ldots, s_{2n} \in \mathbb{R} \) the following asymptotic expansion holds true

\[
F(\lambda) = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} + o\left( \frac{1}{\lambda^{2n+1}} \right), \quad \lambda \xrightarrow[\hat{\lambda}]{} \infty,
\]

where \( \lambda \xrightarrow[\hat{\lambda}]{} \infty \) means that \( \lambda \) tends to \( \infty \) nontangentially, that is inside the sector \( \varepsilon < \arg \lambda < \pi - \varepsilon \) for some \( \varepsilon > 0 \). Let us set

\[
\mathcal{N}_{\kappa,-\infty} := \bigcap_{n \geq 0} \mathcal{N}_{\kappa,-2n}.
\]

In particular, every function of the form (1.3) where \( r_1, r_2, \sigma \) are subject to the assumptions (A1)–(A3), belongs to the class \( \mathcal{N}_{\kappa,-\infty} \) for some \( \kappa \in \mathbb{Z}_+ \) (see [23]). Moreover, every generalized Nevanlinna function \( F \in \mathcal{N}_{\kappa,-\infty} \) holomorphic at \( \infty \) admits the representation (1.3) for some \( r_1, r_2, \sigma \) satisfying (A1)–(A3).

It will be sometimes convenient to use the following notation

\[
F(\lambda) \sim -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}, \quad \lambda \xrightarrow[\hat{\lambda}]{} \infty
\]

to denote the validity of (2.1) for all \( n \in \mathbb{N} \).
Problem $\text{M}_n(s)$. Given are a nonnegative integer $\kappa$ and a sequence $s = \{s_j\}_{j=0}^{\infty}$ of real numbers, such that the matrices $S_n := (s_{i+j})_{i,j=0}^{n}$ are nondegenerate for all $n$ big enough. Find a function $F \in \mathbb{N}_\kappa$, which has the asymptotic expansion \((2.2)\).

We say that the problem $\text{M}_n(s)$ is determinate if $\text{M}_n(s)$ has a unique solution. The problem $\text{M}_n(s)$ was considered in \([24]\), where it was shown that the problem $\text{M}_n(s)$ is solvable if and only if the number of negative eigenvalues of the matrix $S_n$ does not exceed $\kappa$ for all $n \in \mathbb{N}$. The Schur algorithm for solving the problem $\text{M}_n(s)$ considered in \([10]\) proceeds as follows.

Let $\mathcal{N}(s)$ be the set of all normal indices of the sequence $s$, i.e., natural numbers $n_j$, for which

\begin{equation}
\label{eq:m_n(s)_3}
det S_{n_j-1} \neq 0, \quad j = 1, 2, \ldots.
\end{equation}

If $F_0 := F$ is a generalized Nevanlinna function with the asymptotic expansion \((2.2)\), then the function $-1/F_0$ can be represented as

\begin{equation}
\label{eq:m_n(s)_4}
-\frac{1}{F_0(\lambda)} = \varepsilon_0 p_0(\lambda) + b_0^2 F_1(\lambda),
\end{equation}

where $\varepsilon_0 = \pm 1$, $b_0 > 0$, $p_0$ is a monic polynomial of degree $k_0 = n_1$ and $F_1$ is a generalized Nevanlinna function. Continuing this process one gets sequences $\varepsilon_j = \pm 1$, $b_j > 0$, $j \in \mathbb{Z}_+$ and a sequence of real monic polynomials $p_j$ of degree $k_j = n_{j+1} - n_j$, such that $F$ admits the following expansion into a $P$-fraction

\begin{equation}
\label{eq:m_n(s)_5}
-\frac{\varepsilon_0}{p_0(\lambda)} - \frac{\varepsilon_0 \varepsilon_1 b_1^2}{p_1(\lambda)} - \cdots - \frac{\varepsilon_{N-1} \varepsilon_N b_N^2}{p_N(\lambda)} \cdots .
\end{equation}

A similar algorithm for a continued fraction expansion of a formal power series was proposed by A. Magnus in \([29]\). The objects $\varepsilon_j$, $b_j$, $p_j$ are uniquely defined by the sequence $s = \{s_j\}_{j=0}^{\infty}$ (see \([10]\), \([12]\)). The function $F_1$ in \((2.4)\) is called the Schur transform of $F_0 \in \mathbb{N}_\kappa$ (cf. \([5]\)).

2.2. Generalized Jacobi matrices. Let $p(\lambda) = p_k \lambda^k + \cdots + p_1 \lambda + p_0$, $p_k = 1$ be a monic scalar real polynomial of degree $k$. Let us associate with the polynomial $p$ its symmetrizer $E_p$ and the companion matrix $C_p$, given by

\begin{equation}
\label{eq:m_n(s)_6}
E_p = \begin{pmatrix} p_1 & \cdots & p_k \\ \vdots & \ddots & \vdots \\ p_k \end{pmatrix}, \quad C_p = \begin{pmatrix} 0 & \cdots & 0 & -p_0 \\ 1 & \cdots & 0 & -p_1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 1 & -p_{k-1} \end{pmatrix},
\end{equation}

where all the non-specified elements are supposed to be equal to zero. As is known (see \([17]\)), $\det(\lambda - C_p) = p(\lambda)$ and the spectrum $\sigma(C_p)$ of the companion matrix $C_p$ is simple. The matrices $E_p$ and $C_p$ are related by

\begin{equation}
\label{eq:m_n(s)_7}
C_p E_p = E_p C_p^T.
\end{equation}

Definition 2.1. Let $p_j$ be real monic polynomials of degree $k_j$

\begin{equation}
\label{eq:m_n(s)_8}
p_j(\lambda) = \lambda^{k_j} + p_{k_j-1}^{(j)} \lambda^{k_j-1} + \cdots + p_1^{(j)} \lambda + p_0^{(j)},
\end{equation}
and let \( \varepsilon_j = \pm 1, b_j > 0, j \in \mathbb{Z}_+ \). The tridiagonal block matrix

\[
(2.8) \quad \mathcal{J} = \begin{pmatrix} A_0 & \bar{B}_0 \\ B_0 & A_1 & \bar{B}_1 \\ \vdots & \ddots & \ddots & \ddots \\ \end{pmatrix}
\]

where \( A_j = C_{p_j} \) and \( k_j+1 \times k_j \) matrices \( B_j \) and \( k_j \times k_{j+1} \) matrices \( \bar{B}_j \) are given by

\[
(2.9) \quad B_j = \begin{pmatrix} 0 & \ldots & b_j \\ \cdots & \ddots & \cdots \\ 0 & \ldots & 0 \end{pmatrix}, \quad \bar{B}_j = \begin{pmatrix} 0 & \cdots & \bar{b}_j \\ \cdots & \ddots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}, \quad \bar{b}_j = \varepsilon_j \varepsilon_{j+1} b_j, j \in \mathbb{Z}_+.
\]

will be called a \textit{generalized Jacobi matrix} associated with the sequences of polynomials \( \{\varepsilon_j p_j\}_{j=0}^{\infty} \) and numbers \( \{b_j\}_{j=0}^{\infty} \).

\textbf{Remark 2.2.} Define an infinite matrix \( G \) by the equality

\[
(2.10) \quad G = \text{diag}(G_0, \ldots, G_N, \ldots), \quad G_j = \varepsilon_j E_{p_j}^{-1}, \quad j = 0, \ldots, N
\]

and let \( \ell^2_{[0,\infty)}(G) \) be the space of \( \ell^2 \)-vectors with the inner product

\[
(2.11) \quad [x, y] = (Gx, y)_{\ell^2_{[0,\infty)}}, \quad x, y \in \ell^2_{[0,\infty)}.
\]

The inner product \( (2.11) \) is indefinite, if either \( k_j > 1 \) for some \( j \in \mathbb{Z}_+ \), or at least one \( \varepsilon_j \) is equal to \(-1\). The space \( \ell^2_{[0,\infty)}(G) \) is equivalent to a Krein space (see [4]) if both \( G \) and \( G^{-1} \) are bounded in \( \ell^2_{[0,\infty)} \). If \( k_j = \varepsilon_j = 1 \) for all \( j \) big enough, then \( \ell^2_{[0,\infty)}(G) \) is a Pontryagin space. In these cases it follows from \( (2.7) \) that the generalized Jacobi matrix \( \mathcal{J} \) determines a symmetric operator \( S \) in the space \( \ell^2_{[0,\infty)}(G) \) (see details in [11]). More general definition of generalized Jacobi matrix which is not connected with the Schur algorithm has been considered in [23].

Setting \( \bar{b}_{-1} = \varepsilon_0 \), define polynomials of the first kind (cf. [23]) \( P_j(\lambda) \), \( j \in \mathbb{Z}_+ \), as solutions \( u_j = P_j(\lambda) \) of the following system:

\[
(2.12) \quad \bar{b}_{j-1} u_{j-1} - p_j(\lambda) u_j + b_j u_{j+1} = 0, \quad j \in \mathbb{Z}_+,
\]

with the initial conditions

\[
(2.13) \quad u_{-1} = 0, \quad u_0 = 1.
\]

Similarly, the polynomials of the second kind \( Q_j(\lambda) \), \( j \in \mathbb{Z}_+ \), are defined as solutions \( u_j = Q_j(\lambda) \) of the system \( (2.12) \) subject to the following initial conditions

\[
(2.14) \quad u_{-1} = -1, \quad u_0 = 0.
\]

It follows from \( (2.12) \) that \( P_j \) is a polynomial of degree \( n_j = \sum_{i=0}^{j-1} k_i \) with the leading coefficient \( (b_0 \ldots b_{j-1})^{-1} \) and \( Q_j \) is a polynomial of degree \( n_j - k_0 \) with the leading coefficient \( \varepsilon_0 (b_0 \ldots b_{j-1})^{-1} \). The equations \( (2.12) \) coincide with the three-term recurrence relations associated with \( P \)-fractions (see also [19] Section 5.2). The following statement is immediate from \( (2.12) \).

\textbf{Proposition 2.3.} ([12]). \textit{Polynomials} \( P_j \) and \( P_{j+1} \) \( (Q_j \) and \( Q_{j+1}) \) \textit{have no common zeros.}
The following connection between the polynomials of the first and second kinds $P_j, Q_j$ and the shortened Jacobi matrices $J_{[0,j]}$ can be found in [12 Proposition 3.3] (in the classical case see [8 Section 7.1.2]).

**Proposition 2.4.** Polynomials $P_j$ and $Q_j$ can be found by the formulas

\begin{align}
P_j(\lambda) &= (b_0 \ldots b_{j-1})^{-1} \det(\lambda - J_{[0,j-1]}), \\
Q_j(\lambda) &= \varepsilon_0 (b_0 \ldots b_{j-1})^{-1} \det(\lambda - J_{[1,j-1]}).
\end{align}

Clearly, $J_{[0,j-1]}$ is a symmetric operator in the subspace $l^2_{[0,j-1]}(G)$ of the indefinite inner product space $l^2_{[0,\infty]}(G)$, which consists of vectors $u = \{u_{i,k}\}_{i=0,\ldots,\infty}$ such that $u_{i,k} = 0$ for $i \geq j$. The m-function of the shortened matrix $J_{[0,j-1]}$ is defined by

\begin{equation}
m_{[0,j-1]}(\lambda) = [(J_{[0,j-1]} - \lambda)^{-1}e, e].
\end{equation}

Due to formulas (2.15), (2.16) it is calculated by

\begin{equation}
m_{[0,j-1]}(\lambda) = -\varepsilon_0 \frac{\det(\lambda - J_{[1,j-1]})}{\det(\lambda - J_{[0,j-1]})} = -\frac{Q_j(\lambda)}{P_j(\lambda)}.
\end{equation}

**Remark 2.5.** Let us emphasize that the polynomials $P_j$ and $Q_j$ have no common zeros (see [11 Proposition 2.7]) and due to (2.18) the set of holomorphy of $m_{[0,j-1]}$ coincides with the resolvent set of $J_{[0,j-1]}$.

**Theorem 2.6** ([12]). Let $F \in N_{\kappa,-\infty}$ and the corresponding indefinite moment problem $M_\kappa(s)$ be determinate. Then:

(i) the generalized Jacobi matrix corresponding to $F$ via (2.6) and (2.8) generates a selfadjoint operator $J$ in $l^2_{[0,\infty]}(G)$ and

\begin{equation}
F(\lambda) = [(J - \lambda)^{-1}e, e].
\end{equation}

(ii) the diagonal $[n_j/n_j]$ Padé approximants of $F(\lambda)$ coincide with $m_{[0,j-1]}(\lambda)$ and converge to $F(\lambda)$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}$.

The proof of this result is based on the fact that the compressed resolvents of $J_{[0,j-1]}$ converge to the compressed resolvent of $J$ (see [12 Theorem 4.8]). Theorem 2.6 contains as partial cases some results of A. Gonchar [10] and E. Rakhmanov [33], mentioned in Introduction, as well as the results of G.L. Lopes [27] concerning convergence of diagonal Padé approximants for rational perturbations of Stieltjes functions.

3. The Convergence of Subdiagonal Padé Approximants.

Let us consider the following finite generalized Jacobi matrix

\begin{equation}
J_{[0,j]}(\tau) = \begin{pmatrix}
A_0 & \tilde{B}_0 \\
B_0 & \ddots & \ddots \\
& \ddots & A_{j-1} & \tilde{B}_{j-1} \\
& & B_{j-1} & A_j(\tau)
\end{pmatrix},
A_j(\tau) = \begin{pmatrix}
0 & \ldots & 0 & -p_0^{(j)} + \tau \\
1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & 1 & -p_{k-1}^{(j)}
\end{pmatrix}.
\end{equation}

A vector $u = (u_{i,k})_{i=0,\ldots,j}^{k=0,\ldots,n-1} \in \mathbb{C}^n$ is a left eigenvector of the matrix $J_{[0,j]}(\tau)$, corresponding to the eigenvalue 0 if and only if $u_i = u_{i0}, i = 0, \ldots, j,$ satisfy the
Next, applying the determinant decomposition theorem one can obtain
\[ \tau = \tau_j := \frac{b_{j-1}P_{j-1}(0) - p_{j}(0)P_{j}(0)}{P_{j}(0)} = b_{j}P_{j+1}(0) P_{j}(0). \]

**Proposition 3.1.** If \( P_{j}(0) \neq 0 \) then there exists a number \( \tau_j \in \mathbb{R} \) such that
\[ 0 \in \sigma_p(\mathcal{J}_{[0,j]}(\tau_j)). \]
Moreover, \( \tau_j \) can be found by the formula (3.3).

**Remark 3.2.** We will use the notation \( \mathcal{J}^{(K)}_{[0,j]} \) for the matrix \( \mathcal{J}_{[0,j]}(\tau_{j}) \) with the property (3.4). In the case when the corresponding indefinite moment problem \( M_{\kappa}(s) \) is determinate for all \( \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(F)) \) we have
\[ m^{(K)}_{[0,j-1]}(\lambda) := [(\mathcal{J}^{(K)}_{[0,j-1]} - \lambda)^{-1}e, e] \rightarrow F(\lambda) = [(\mathcal{J}^{(K)} - \lambda)^{-1}e, e] \]
as \( j \rightarrow \infty \) (see [12, Proposition 4.4]).

**Theorem 3.3 ([12]).** Let a function \( F \in \mathbb{N}_\kappa \) have the expansion (2.2) for every \( n \in \mathbb{N} \), let the corresponding moment problem \( M_{\kappa}(s) \) be determinate, and let \( \{n_j\}_{j=1}^{\infty} = N(s) \) be the set of normal indices of the sequence \( s = \{s_j\}_{j=0}^{\infty} \). Then:

(i) The \( \lfloor n_j/n_j-1 \rfloor \) Padé approximant \( F^{(n_j/n_j-1)} \) exists if and only if
\[ n_j \in \mathcal{N}_F := \{ n_j \in N(s) : P_{j-1}(0) \neq 0 \}; \]

(ii) The sequence
\[ F^{(n_j/n_j-1)} = \frac{Q_j(\lambda)P_{j-1}(0) - Q_{j-1}(\lambda)P_{j}(0)}{P_{j}(\lambda)P_{j-1}(0) - P_{j-1}(\lambda)P_{j}(0)}, \quad n_j \in \mathcal{N}_F, \]
converges to \( F \) locally uniformly in \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(F)) \).

**Proof.** We will sketch the proof of this theorem presented in [12].

Let \( P_{j-1}(0) \neq 0 \) and let \( P^{(K)}_{j}, Q^{(K)}_{j} \) be the polynomials of the first and second kinds, respectively, associated with the matrix \( \mathcal{J}^{(K)}_{[0,j-1]} \), that is
\[ P^{(K)}_{j}(\lambda) = (b_0 \ldots b_{j-1})^{-1} \det(\lambda - \mathcal{J}^{(K)}_{[0,j-1]}), \]
\[ Q^{(K)}_{j}(\lambda) = (b_0 \ldots b_{j-1})^{-1} \det(\lambda - \mathcal{J}^{(K)}_{[1,j-1]}). \]

Next, applying the determinant decomposition theorem one can obtain
\[ P^{(K)}_{j}(\lambda) = P_{j}(\lambda) - \frac{\tau_{j-1} P_{j-1}(0)}{b_{j-1}} = \frac{P_{j}(\lambda)P_{j-1}(0) - P_{j-1}(\lambda)P_{j}(0)}{P_{j-1}(0)}, \]
\[ Q^{(K)}_{j}(\lambda) = Q_{j}(\lambda) - \frac{\tau_{j-1} Q_{j-1}(0)}{b_{j-1}} = \frac{Q_{j}(\lambda)P_{j-1}(0) - Q_{j-1}(\lambda)P_{j}(0)}{P_{j-1}(0)}. \]

It obviously follows from (3.3) that
\[ m^{(K)}_{[0,j-1]}(\lambda) := [(\mathcal{J}^{(K)}_{[0,j-1]} - \lambda)^{-1}e, e] \sim - \sum_{i=0}^{\infty} \frac{s_i^{(K)}(\lambda)}{\lambda^{i+1}}, \quad \lambda \searrow \infty, \]
where \( s_i^{(K)} = \left( J^{(K)}_{[0,j-1]} \right)^i e, e \). Using the form of the matrix \( J^{(K)}_{[0,j-1]} \) one gets

\[
\begin{align*}
  s_i^{(K)} &= s_i \text{ if } i \leq 2n_j - 2, \\
  s_{2n_j-1}^{(K)} &= s_{2n_j-1} + (b_0 \ldots b_{j-1})^2 \tau_{j-1}.
\end{align*}
\]

So, the function \( m^{(K)}_{[0,j-1]}(\lambda) \) has the following asymptotic expansion

\[
m^{(K)}_{[0,j-1]}(\lambda) = - \sum_{i=0}^{2n_j-2} \frac{s_i}{\lambda^{i+1}} + O\left( \frac{1}{\lambda^{2n_j}} \right), \quad \lambda \to \infty,
\]

where the sequence \( \{s_j\}_{j=0}^\infty \) corresponds to the generalized Jacobi matrix \( J \). On the other hand, due to \((2.13)\)

\[
m^{(K)}_{[0,j-1]}(\lambda) = - \frac{Q^{(K)}_j(\lambda)}{P^{(K)}_j(\lambda)}.
\]

Further, setting

\[
A^{[n_j/n_j-1]} \left( \frac{1}{\lambda} \right) = \left( \frac{1}{\lambda} \right)^{n_j} Q^{(K)}_i(\lambda), \quad B^{[n_j/n_j-1]} \left( \frac{1}{\lambda} \right) = \left( \frac{1}{\lambda} \right)^{n_j} P^{(K)}_i(\lambda),
\]

for \( i = 0, 1 \ldots, j \) and taking into account the equality \( P^{(K)}_j(0) = 0 \), one obtains

\[
m^{(K)}_{[0,j-1]}(\lambda) = \frac{A^{[n_j/n_j-1]}(1/\lambda)}{B^{[n_j/n_j-1]}(1/\lambda)},
\]

where

\[
\deg A^{[n_j/n_j-1]} = n_j, \quad \deg B^{[n_j/n_j-1]} = n_j - 1, \quad B^{[n_j/n_j-1]}(0) = \frac{1}{b_0 \ldots b_{j-1}} \neq 0.
\]

Therefore, \( m^{(K)}_{[0,j-1]}(\lambda) \) is the \([n_j/n_j-1]\) Padé approximant for the corresponding Hamburger series. This proves the first part of the theorem. The second statement rests on the fact mentioned in Remark 3.2.

**Remark 3.4.** Condition \( P_{j-1}(0) \neq 0 \) is equivalent to

\[
\begin{vmatrix}
  s_1 & \cdots & s_{n_j-1} \\
  \cdots & \cdots & \cdots \\
  s_{n_j-1} & \cdots & s_{2n_j-1}
\end{vmatrix} \neq 0.
\]

It follows from Proposition 2.3 that the set \( \mathcal{N}_F \) is infinite.

**Example 1.** Consider the following classical 2-periodic Jacobi matrix

\[
J = \begin{pmatrix}
  a_0 & b_0 & b_1 & \cdots \\
  b_0 & a_1 & b_1 & \cdots \\
  b_1 & a_2 & \cdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}, \quad a_n = (-1)^n + \frac{1}{2}, \quad b_n = 1, \quad n \in \mathbb{Z}_+.
\]

The m-function corresponding to this Jacobi matrix can be found by using standard methods (see [32])

\[
\varphi(\lambda) = ((J - \lambda)^{-1} e, e)\mathbb{E}_{[a, \infty)} = \frac{\lambda - \lambda^2 + \sqrt{(\lambda^2 - \lambda - 2)^2 - 4}}{2(\lambda - 1)}
\]
Theorem 3.5. Let sequence f be holomorphic at infinity. (This example was given in \[12\] with several misprints.)

\[ \varphi(\lambda) = \int_E \frac{d\sigma(t)}{t - \lambda}, \]

where the support \( E := \text{supp } \sigma \) of the measure \( \sigma \) is contained in \([-2, 3]\). Since the \([n/n-1]\) Padé approximant is equal to \( f^{[n/n-1]}(\lambda) = m^{(K)}_{[0,n-1]}(\lambda) \), its poles coincide with eigenvalues of the matrix \( J_{[0,n-1]}^{(K)} \). Let us show that the eigenvalue of the matrix \( J_{[0,2k]}^{(K)} \) with the largest absolute value tends to infinity as \( k \to +\infty \). First we compute \( \tau_n \). Since \( \tau_n = b_n p_{n+1}(0)/p_n(0) = -b_{n-1} \bar{b}_{n-1}/\tau_{n-1} + p_n(0) \), we have

\[ \tau_n = -1/\tau_{n-1} - ((-1)^n + 1)/2. \]

Clearly, \( \tau_0 = -1 \). By induction, we have the following formulas

\[ \tau_{2k} = -(k + 1), \quad \tau_{2k+1} = 1/(k + 1). \]

Taking into account that \( J_{[0,2k]}^{(K)} \) is a self-adjoint matrix, one obtains

\[ |\lambda_{\text{max}}(J_{[0,2k]}^{(K)})| = \|J_{[0,2k]}^{(K)}\| \geq |(J_{[0,2k]}^{(K)} e_{2k}, e_{2k})| = k, \]

where \( \lambda_{\text{max}}(J_{[0,2k]}^{(K)}) \) is the eigenvalue of the matrix \( J_{[0,2k]}^{(K)} \) with the largest absolute value. Therefore, \( |\lambda_{\text{max}}(J_{[0,2k]}^{(K)})| \to +\infty \) as \( k \to +\infty \). So, infinity is an accumulation point of the set of poles of the Padé approximants \( f^{[n/n-1]} \) of the function \( \varphi \) holomorphic at infinity. (This example was given in \[12\] with several misprints.)

Under certain conditions, it is possible to say more about the convergence of the sequence \( F^{[n/n-1]} \) on the real line.

Theorem 3.5. Let \( F \) have the form

\[ F(\lambda) = r_1(\lambda) \int_1^\lambda \frac{d\sigma(t)}{t - \lambda} + r_2(\lambda) = -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}, \quad |\lambda| > R, \]

where \( \sigma, r_1, r_2 \) satisfy the assumptions (A1)-(A3). If the sequence \( \{\tau_{j-1}\}_{n_j \in \mathcal{N}_F} \) is bounded, i.e.

\[ \sup_{n_j \in \mathcal{N}_F} \left| b_j^{-1} \frac{P_j(0)}{P_{j-1}(0)} \right| < \infty, \]

then there exists a constant \( \varepsilon > 0 \) such that the sequence \( \{F^{[n_j/n_j-1]}\}_{n_j \in \mathcal{N}_F} \) converges to \( F \) locally uniformly in \( \mathbb{C} \setminus ([-1 - \varepsilon, 1 + \varepsilon] \cup \mathcal{P}(F)) \).

Proof. It is obvious that \( F \) corresponds to the determinate moment problem \( M_\varepsilon(s) \). Moreover, the corresponding generalized Jacobi matrix \( J \) is a bounded linear operator. According to Theorem 3.3, the sequence \( \{F^{[n_j/n_j-1]}\}_{n_j \in \mathcal{N}_F} \) converges to \( F \) locally uniformly in \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(F)) \). Due to (3.3) we have

\[ F^{[n_j/n_j-1]}(\lambda) = m^{(K)}_{[0,j-1]}(\lambda) = \left[(J_{[0,j-1]}^{(K)} - \lambda)^{-1}, e, e \right]. \]

Since the sequence \( \{\tau_{j-1}\}_{n_j \in \mathcal{N}_F} \) is bounded, one obtains

\[ \|J_{[0,j-1]}^{(K)}\| \leq \|J_{[0,j-1]}\| + |\tau_{j-1}|, \quad n_j \in \mathcal{N}_F \]
for some $\varepsilon > 0$. It follows from the inequality
\[
\| (\mathcal{J}^{(K)}_{[0,j-1]} - \lambda)^{-1} \| \leq \frac{1}{|\lambda| - \| \mathcal{J}^{(K)}_{[0,j-1]} \|} \quad (|\lambda| > \| \mathcal{J}^{(K)}_{[0,j-1]} \|)
\]
and (3.12) that
\[
| m^{(K)}_{[0,j-1]}(\lambda) \| \leq \left| (\mathcal{J}^{(K)}_{[0,j-1]} - \lambda)^{-1} e_{\varepsilon} \right| \leq \frac{\| G_e \|_{l^2}}{|\lambda| - 1 - \varepsilon}.
\]
for $|\lambda| > 1 + \varepsilon$. To complete the proof, it is sufficient to apply the Vitali theorem. □

**Corollary 3.6.** If the sequence $\{ \tau_{j-1} \}_{n_j \in \mathbb{N}_F}$ tends to 0 then the sequence $\{ F_{[0,j-1]} \}_{n_j \in \mathbb{N}_F}$ converges to $F$ locally uniformly in $\mathbb{C} \setminus ([-1,1] \cup \mathcal{P}(F))$.

**Proof.** The statement is implied by (2.17), (3.11), the relation
\[
\| \mathcal{J}_{[0,j-1]} - \mathcal{J}^{(K)}_{[0,j-1]} \| \to 0, \quad j \to \infty,
\]
and Theorem 2.6. □

**Remark 3.7.** The condition (3.10) can be reformulated in terms of the monic orthogonal polynomials
\[
\hat{P}_j(\lambda) = \frac{1}{\det S_{n_j-1}} \begin{vmatrix} s_0 & s_1 & \cdots & s_{n_j} \\ \vdots & \cdots & \cdots & \vdots \\ s_{n_j-1} & s_{n_j} & \cdots & s_{2n_j-1} \\ 1 & \lambda & \cdots & \lambda^{n_j} \end{vmatrix},
\]
which are connected with $P_j(\lambda)$ by the formulas $\hat{P}_j(\lambda) = (b_0 \cdots b_{j-1}) P_j(\lambda)$, $j \in \mathbb{N}$. Therefore, the condition (3.10) takes the form
\[
(3.13) \sup_{n_j \in \mathbb{N}_F} \left| \frac{\hat{P}_j(0)}{\hat{P}_{j-1}(0)} \right| < \infty.
\]

**Remark 3.8.** It is clear from the proof of Theorem 3.5 that the existence of a converging subsequence of the $[n/n-1]$ Padé approximants follows from the existence of a bounded subsequence of $\{ \tau_{j-1} \}_{n_j \in \mathbb{N}_F}$.

### 4. A CLASS OF DEFINITIZABLE FUNCTIONS AND Pade APPROXIMANTS.

#### 4.1. Classes $D_{\kappa,-\infty}$ and $D^*_{\kappa,-\infty}$.

**Definition 4.1.** Let us say that a function $\mathfrak{F}$ meromorphic in $\mathbb{C}+$ belongs to the class $D_{\kappa,-\infty}$ if
\[
F(\lambda) := \frac{\mathfrak{F}(\lambda)}{\lambda} \in N_{\kappa,-\infty} \quad \text{and} \quad \mathfrak{F}(\lambda) = O(1), \quad \lambda \to \infty.
\]

Clearly, every function $\mathfrak{F} \in D_{\kappa,-\infty}$ is definitizable in the sense of [18]. Indeed, consider the factorization
\[
\mathfrak{F}(\lambda) = r^{-1}(\lambda)(r^2)^{-1}(\lambda)F_0(\lambda),
\]
where $r$ is a real rational function, $r^\sharp(\lambda) = r(\lambda)$, and $F_0 \in \mathbb{N}_0$. Then
\[
\frac{r(\lambda) r^\sharp(\lambda)}{\lambda} \hat{\mathfrak{F}}(\lambda) = F_0(\lambda) \in \mathbb{N}_0
\]
and, hence, $\hat{\mathfrak{F}}$ is definitizable, $\frac{r(\lambda) r^\sharp(\lambda)}{\lambda}$ is definitizing multiplier.

It follows from (2.1) that every function $\mathfrak{F} \in D^{\kappa,-\infty}$ admits the asymptotic expansion
\[
\mathfrak{F}(\lambda) \sim -s_{-1} - \frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} - \cdots, \quad \lambda \to \infty
\]
which is connected with the asymptotic expansion (2.1) of $F(\lambda) = \frac{\hat{\mathfrak{F}}(\lambda)}{\lambda}$ via the formulas
\[
s_{j-1} = s_j, \quad j \in \mathbb{Z}.
\]
In what follows we use the Gothic script for all the notations associated with the $D^{\kappa,-\infty}$ function and the Roman script for the $N^{\kappa,-\infty}$ function to avoid confusion.

We also say that a function $F$ meromorphic in $\mathbb{C}^+$ belongs to the class $D^{\circ,\kappa,-\infty}$ if
\[
F(\lambda) := \frac{\hat{\mathfrak{F}}(\lambda)}{\lambda} \in N^{\kappa,-\infty}, \quad \hat{\mathfrak{F}}(\lambda) = o(1), \quad \lambda \to \infty,
\]
and the asymptotic expansion of the function $\hat{\mathfrak{F}}$
\[
\hat{\mathfrak{F}}(\lambda) \sim -s_{-1} - \frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n}}{\lambda^{2n+1}} - \cdots, \quad \lambda \to \infty
\]
is normalized in a sense that the first nontrivial coefficient in (4.3) has modulus 1, $|s_{n_1-1}| = 1$.

Let the set $\mathcal{N}(s)$ of normal indices of the sequence $s = \{s_i\}_{i=0}^{\infty}$ corresponding to a function $\hat{\mathfrak{F}} \in D^{\kappa,-\infty}$ be defined by (2.3), that is
\[
\mathcal{N}(s) = \{n_j : \det(s_{i+j+k})_{i,k=0}^{n_j-1} \neq 0, \quad j = 1, 2, \ldots\}.
\]

4.2. Normal indices of the $D^{\circ,\kappa,-\infty}$ functions. Remind that the point $\infty$ is called a generalized pole of nonpositive type of $F \in N^{\kappa}$ with multiplicity $\kappa_{\infty}(F)$, if
\[
0 \leq \lim_{\lambda \to \infty} \frac{F(\lambda)}{\lambda^{2\kappa_{\infty}+1}} < \infty, \quad -\infty < \lim_{\lambda \to \infty} \frac{F(\lambda)}{\lambda^{2\kappa_{\infty}+1}} < 0.
\]
Similarly, the point $\infty$ is called a generalized zero of nonpositive type of $F$ with multiplicity $\pi_{\infty}(F)$, if
\[
-\infty \leq \lim_{\lambda \to \infty} \lambda^{2\pi_{\infty}+1} F(\lambda) < 0, \quad 0 \leq \lim_{\lambda \to \infty} \lambda^{2\pi_{\infty}-1} F(\lambda) < \infty.
\]

It was shown in [24] that the multiplicity of $\infty$ as a generalized pole (zero) of nonpositive type of $F \in N^{\kappa}$ does not exceed $\kappa$.

Lemma 4.2. Let $\hat{\mathfrak{F}} \in D^{\circ,\kappa,-\infty}$, let the sequence $s = \{s_j\}_{j=0}^{\infty}$ be defined by the asymptotic expansion (4.3), and let $\mathcal{N}(s) = \{n_j\}_{j=1}^{\infty}$ be the set of normal indices of $s$. Then
\[
n_1 \leq 2\kappa.
\]
Moreover, if $n_1 = 2\kappa$, then
\[
s_{n_1-1} > 0.
\]
Proof. Since \( \pi_\infty(F) \leq \kappa \) it follows from (4.4), that

\[
\lim_{\lambda \to \infty} \lambda^{2\pi_\infty(F)+1} F(\lambda) < 0.
\]

The normal index \( n_1 \) can be characterized by the relations

\[
g_0 = \cdots = g_{n_1-2} = 0, \quad g_{n_1-1} \neq 0.
\]

Hence \( F(\lambda) = g(\lambda)/\lambda \) has the asymptotic expansion

\[
F(\lambda) \sim \frac{g_{n_1-1}}{\lambda^{n_1+1}} - \cdots - \frac{g_{2n}}{\lambda^{2n+2}} - \cdots, \quad \lambda \to \infty
\]

and (4.9) implies the inequality (4.7).

If equality prevails in (4.7) then \( \pi_\infty(F) = \kappa \), the limit in (4.9) is finite and coincides with \(-s_{n_1-1}\). This implies the inequality (4.8). \( \square \)

Proposition 4.3. Let \( \mathfrak{F} \in \mathcal{D}_{\kappa,-\infty}^o \) and \( F(\lambda) = \mathfrak{F}(\lambda)/\lambda \) have asymptotic expansions (4.3) and (2.2), and let \( \mathcal{N}_F, \mathcal{N}(s) \), \( \mathcal{N}(s) \) be defined by (4.6), (2.3), (4.4). Then

\[
\mathcal{N}_F = \mathcal{N}(s) \cap \mathcal{N}(s).
\]

Proof. Let \( \mathcal{N}(s) = \{ n_j \}_{j=1}^\infty \). The statement is implied by (3.6) and the equality

\[
\begin{bmatrix}
  g_0 & \cdots & g_{n_j-1} \\
  \cdots & \cdots & \cdots \\
  g_{n_j-2} & \cdots & g_{2n_j-2}
\end{bmatrix} = \begin{bmatrix}
  s_1 & \cdots & s_{n_j} \\
  \cdots & \cdots & \cdots \\
  s_{2n_j-2} & \cdots & s_{2n_j-2}
\end{bmatrix} \neq 0,
\]

which is implied from (4.2) and (3.9). \( \square \)

4.3. The Schur transform of the \( \mathcal{D}_{\kappa,-\infty}^o \) functions. Let a function \( \mathfrak{F} \in \mathcal{D}_{\kappa,-\infty}^o \) have the asymptotic expansion (4.3), let \( \{ n_j \}_{j=1}^\infty \) be the set of normal indices for \( s = \{ s_j \}_{j=0}^\infty \) and let \( \mathfrak{S}_n = (s_{i+j})_{i,j=0}^n \). Let us set \( \epsilon_0 = \text{sign} s_{n_1-1} \),

\[
p_0(\lambda) = \frac{1}{\det \mathfrak{S}_{n_1-1}} \det \left( \begin{array}{ccc}
  0 & \cdots & g_{n_1-1} & g_{n_1} \\
  \vdots & \ddots & \vdots & \vdots \\
  g_{n_1-1} & g_{n_1} & \cdots & s_{n_1-1} \\
  1 & \cdots & \cdots & \lambda^{n_1}
\end{array} \right).
\]

The Schur transform of the function \( \mathfrak{F} \in \mathcal{D}_{\kappa,-\infty}^o \) is defined by the equality

\[
-\frac{1}{\mathfrak{F}(\lambda)} = \epsilon_0 p_0(\lambda) + \epsilon_0 b_0^2 \mathfrak{F}(\lambda),
\]

where \( b_0 \) is chosen in such a way that \( \mathfrak{F} \) has a normalized expansion at \( \infty \).

Theorem 4.4. Let \( \mathfrak{F} \in \mathcal{D}_{\kappa,-\infty}^o \) and let \( \hat{\mathfrak{F}} \) be the Schur transform of \( \mathfrak{F} \). Then:

\begin{enumerate}
  \item \( \hat{\mathfrak{F}} \in \mathcal{D}_{\kappa',-\infty}^o \) for some \( \kappa' \leq \kappa \);
  \item If \( \hat{\mathfrak{F}} \in \mathcal{D}_1^o \), then \( \hat{\mathfrak{F}} \in \mathcal{D}_1^o \);
  \item The inverse Schur transform is given by
\end{enumerate}

\[
\mathfrak{F}(\lambda) = -\frac{\epsilon_0}{p_0(\lambda) + \epsilon_0 b_0^2 \hat{\mathfrak{F}}(\lambda)}.
\]
Proof. (i) Direct calculations presented in \cite{10} Lemmas 2.1, 2.4 show that \(\hat{\mathcal{F}}\) admits the asymptotic expansion

\[ \hat{\mathcal{F}}(\lambda) \sim -\frac{s_0^{(1)}}{\lambda} - \frac{s_1^{(1)}}{\lambda^2} - \cdots - \frac{s_{2n}^{(1)}}{\lambda^{2n+1}} - \cdots, \quad \lambda \to \infty \]

with some \(s_j^{(1)} \in \mathbb{R}, j \in \mathbb{Z}_+\). Setting

\[ G_1(\lambda) := \lambda \hat{\mathcal{F}}(\lambda) \]

one obtains from (4.12)

\[ b_0 G_1(\lambda) + \epsilon_0 \lambda \mathcal{P}_0(\lambda) = -\frac{1}{F(\lambda)} \in N_{\kappa, -\infty}. \]

Since \(\deg \epsilon_0 \lambda \mathcal{P}_0(\lambda) = n_1 + 1 \geq 2\) then \(\infty\) is a generalized pole of nonpositive type of the polynomial \(\epsilon_0 \lambda \mathcal{P}_0(\lambda)\) with multiplicity

\[ \kappa(\epsilon_0 \lambda \mathcal{P}_0(\lambda)) \geq 1. \]

It follows from (4.14) and (4.15) that

\[ \lim_{\lambda \to \infty} G_1(\lambda) = -s_0^{(1)} \]

and hence \(\infty\) is not a generalized pole of nonpositive type of \(G_1\). By \cite{22} Satz 1.13 one obtains

\[ \kappa(G_1) + \kappa(\epsilon_0 \lambda \mathcal{P}_0(\lambda)) = \kappa(-1/F) = \kappa, \]

and hence \(G_1 \in N_{\kappa''}, -\infty\) for some \(\kappa'' \leq \kappa - 1\).

Consider the function

\[ F_1(\lambda) := \frac{\hat{\mathcal{F}}(\lambda)}{\lambda} = \frac{G_1(\lambda)}{\lambda^2}. \]

It follows from (4.6) that the multiplicities of generalized zeros at \(\infty\) of \(F_1\) and \(G_1\) are related as follows

\[ \pi_\infty(F_1) = \pi_\infty(G_1) + 1. \]

So, by a theorem of M.G. Krein and H. Langer \cite{24} Theorem 3.5] \(F_1 \in N_{\kappa', -\infty}\), where \(\kappa' = \kappa'' + 1 \leq \kappa\).

(ii) By Proposition 4.2 \(n_1 \leq 2\) in the case \(\kappa = 1\).

Assume first that \(n_1 = 1\). Then \(\deg \lambda \mathcal{P}_0(\lambda) = 2, \kappa(\lambda \mathcal{P}_0(\lambda)) = 1, \) and hence \(G_1 \in N_{0, -\infty}\). Then it follows from (4.18) that \(F_1 \in N_{1, -\infty}\).

Let now \(n_1 = 2\). Then \(\deg \lambda \mathcal{P}_0(\lambda) = 3\) and in view of (4.6) the leading coefficient of \(\mathcal{P}_0\) is positive. Therefore \(\kappa(\lambda \mathcal{P}_0(\lambda)) = 1, \) and hence \(G_1 \in N_{0, -\infty}\) and \(F_1 \in N_{1, -\infty}\).

(iii) The last statement is checked by straightforward calculations. \(\square\)

4.4. Diagonal Padé approximants of the function \(\hat{\mathcal{F}} \in D_{\kappa, -\infty}\). To prove the uniform convergence of diagonal Padé approximants for a function belonging to \(D_{\kappa, -\infty}\), we need the following lemma.

Lemma 4.5 (cf. \cite{12}, \cite{29}). Let \(\hat{\mathcal{F}} \in D_{\kappa, -\infty}\) and let \(F(\lambda) := \hat{\mathcal{F}}(\lambda)/\lambda\). Then

\[ \hat{\mathcal{F}}^{[n-1/n-1]}(\lambda) = \lambda F^{[n/n-1]}(\lambda) \] for every \(n \in N_F\).
**Proof.** Suppose that \( n \in \mathcal{N}_F \). Then by Theorem 3.3 the Padé approximant \( F_{[n/n-1]} \) exists and

\[
F_{[n/n-1]}(\lambda) + \sum_{j=0}^{2n-2} \frac{s_j}{\lambda^{j+1}} = O(\lambda^{-2n}), \quad \lambda \rightarrow \infty.
\]

Multiplying by \( \lambda \) one obtains

\[
\lambda F_{[n/n-1]}(\lambda) + \sum_{j=0}^{2n-2} \frac{s_j}{\lambda^{j}} = O(\lambda^{-(2n-1)}), \quad \lambda \rightarrow \infty.
\]

Now the first term in (4.21) can be represented as

\[
\lambda F_{[n/n-1]}(\lambda) = \lambda A_{[n/n-1]}(1/\lambda) B_{[n/n-1]}(1/\lambda),
\]

where \( \deg A_{[n/n-1]} \leq n \), \( \deg B_{[n/n-1]} \leq n-1 \), and \( B_{[n/n-1]}(0) \neq 0 \). Hence, \( A_1(1/\lambda) = \lambda A_{[n/n-1]}(1/\lambda) \) is a polynomial in \( 1/\lambda \) of degree \( \leq n-1 \). This proves that

\[
\lambda F_{[n/n-1]}(\lambda) = A_1(1/\lambda) B_{[n/n-1]}(1/\lambda),
\]

where \( \deg A_1 \leq n-1 \), \( \deg B_{[n/n-1]} \leq n-1 \), and \( B_{[n/n-1]}(0) \neq 0 \). So, it follows from (4.21) that \( \lambda F_{[n/n-1]}(\lambda) \) is the \([n-1/n-1]\) Padé approximant for \( \mathcal{F} \). \hfill \( \square \)

**Theorem 4.6.** Let \( \mathcal{F} \in \mathcal{D}_{\kappa,-\infty} \) and let

\[
F(\lambda) := \frac{1}{\lambda} \mathcal{F}(\lambda) - \sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}}, \quad \lambda \rightarrow \infty,
\]

generate the determinate moment problem \( M_{\kappa}(\mathcal{F}) \). Then the sequence of diagonal Padé approximants \( \{\mathcal{F}_{[n/n-1]}\}_{n \in \mathcal{N}_F} \) converges to \( \mathcal{F} \) locally uniformly on \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(\mathcal{F})) \).

Moreover, if the condition (3.10) is fulfilled for the function \( F \) of the form (1.3) then the sequence of diagonal Padé approximants converges to \( \mathcal{F} \) locally uniformly on \( \mathbb{C} \setminus ([-1-\varepsilon, 1+\varepsilon] \cup \mathcal{P}(\varphi)) \) for some \( \varepsilon > 0 \).

**Proof.** It follows from Theorem 3.3 that the sequence \( \{F_{[n/n-1]}\}_{n \in \mathcal{N}_F} \) converges to \( F \) locally uniformly on \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(F)) \). Since \( \mathbb{R} \cup \mathcal{P}(\mathcal{F}) = \mathbb{R} \cup \mathcal{P}(F) \) the statement on the convergence on \( \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{P}(\mathcal{F})) \) is implied by Lemma 4.5.

Under the condition (3.10) the convergence on \( \mathbb{C} \setminus ([-1-\varepsilon, 1+\varepsilon] \cup \mathcal{P}(\varphi)) \) for some \( \varepsilon > 0 \) is a consequence of Theorem 3.4. \hfill \( \square \)

**Remark 4.7.** It should be noted that the above theorem and the appropriate variation of \( \mathcal{F} \) give us the possibility to make conclusions on the locally uniform convergence of Padé approximants for the function \( \mathcal{F} \) such that

\[
\frac{\mathcal{F}(\lambda)}{\lambda + \zeta} \in \mathcal{N}_{\kappa,-\infty}
\]

for some \( \zeta \in \mathbb{R} \) and \( \mathcal{F}(\lambda) = O(1) \) as \( \lambda \rightarrow \infty. \)
4.5. Generalized Jacobi matrix associated with the function $\mathfrak{F} \in D_{\kappa, -\infty}^\circ$.

**Theorem 4.8.** Let $\mathfrak{F} \in D_{\kappa, -\infty}^\circ$, let the sequence $s = \{s_j\}_{j=0}^\infty$ be defined by the asymptotic expansion (\ref{4.3}), and let $N(s) = \{n_j\}_{j=1}^\infty$ be the set of normal indices of $s$. Then:

(i) $\mathfrak{F}$ admits the expansion into the $P$-fraction

$$
(4.22) \quad -\frac{\epsilon_0}{p_0(\lambda)} - \frac{\epsilon_0 \epsilon_1 b_0^2}{p_1(\lambda)} - \cdots - \frac{\epsilon_{N-1} \epsilon_N b_{N-1}^2}{p_N(\lambda)} - \cdots,
$$

where $p_j$ are polynomials of degree $t_j := n_{j+1} - n_j$ ($\leq 2\kappa$), $\epsilon_i = \pm 1$, $b_i > 0$, $i \in \mathbb{Z}_+$;

(ii) If $\mathbf{J}$ is the generalized Jacobi matrix associated with the $P$-fraction (\ref{4.22}), and $\mathbf{P}_i(\lambda), \mathbf{Q}_i(\lambda)$ are given by

$$
P_i(\lambda) = (b_0 \cdots b_{i-1})^{-1} \det(\lambda - \mathbf{J}_{[0,i-1]}),
$$

then the $i$-th convergent to (\ref{4.22}) coincides with $-\mathbf{Q}_i(\lambda)/\mathbf{P}_i(\lambda)$ and is the $[n_i/n_i]$ Padé approximant for $\mathfrak{F}(\lambda)$.

**Proof.** (i) It follows from Theorem 4.4 that any function $\mathfrak{F} \in D_{\kappa, -\infty}^\circ$ can be represented as follows

$$
\mathfrak{F}(\lambda) = -\frac{\epsilon_0}{p_0(\lambda) + \epsilon_0 b_0^2 \mathfrak{F}_1(\lambda)},
$$

where $p_0$ is a monic polynomial of degree $\deg p_0 = n_1 \leq 2\kappa$ (see formula (4.17)), $\epsilon_0 = \pm 1$, $b_0 \in \mathbb{R}_+$, and $\mathfrak{F}_1 \in D_{\kappa_1, -\infty}$ with $\kappa_1 \leq \kappa$. Further, one can apply Theorem 4.4 to $\mathfrak{F}_1$ and so on. Thus, the Schur algorithm leads to (4.22). To complete the proof, note that the relation $\deg p_i = n_{i+1} - n_i$ follows from [10 Corollary 3.6].

(ii) This part is proved in line with [11 Proposition 2.3] (see also [12]).

**Remark 4.9.** Let $\hat{\mathbf{P}}_i$ be monic polynomials associated with $\mathbf{P}_i$ by the equalities

$$
\hat{\mathbf{P}}_i(\lambda) := (b_0 \cdots b_{i-1}) \mathbf{P}_i(\lambda), \quad \deg \mathbf{P}_i = n_i, \quad i \in \mathbb{N}.
$$

Then it follows from Theorem 4.5, Lemma 4.5, and Theorem 4.8 that $\hat{\mathbf{P}}_i$ are the Christoffel transformations of the polynomials $\hat{P}_j$ corresponding to $F(\lambda) = \mathfrak{F}(\lambda)/\lambda$

$$
\hat{\mathbf{P}}_i(\lambda) = (b_0 \cdots b_{j-1}) \left( P_j(\lambda) - \frac{P_j(0)}{P_{j-1}(0)} P_{j-1}(\lambda) \right) \frac{1}{\lambda}
$$

$$
= \frac{\hat{P}_j(\lambda) \hat{P}_{j-1}(0) - \hat{P}_{j-1}(\lambda) \hat{P}_j(0)}{\hat{P}_{j-1}(0) \lambda},
$$

such that $n_i = \deg \hat{\mathbf{P}}_i = \deg \hat{P}_j = n_j - 1$, $n_j \in N(s)$. Since $F \in N_\kappa$, the Christoffel transformation is defined for every natural $n(= n_j)$ large enough.

In the case when $\mathfrak{F} \in D_{1, -\infty}^\circ$ one can simplify the form of the generalized Jacobi matrix $\mathfrak{J}$.

**Proposition 4.10.** Let $\mathfrak{F} \in D_{1, -\infty}^\circ$ satisfy the assumptions of Theorem 4.8 and let $\mathfrak{J}$ be the generalized Jacobi matrix associated with the $P$-fraction (4.22). Then
\( \mathfrak{t}_i := n_{i+1} - n_i \) is either 1 or 2 and the block matrix \( \mathfrak{A}_i \) in \( \mathfrak{J} \) takes the form

\[
\mathfrak{A}_i = \begin{cases} 
-p_0^{(i)}, & \text{if } \mathfrak{t}_i = 1; \\
\begin{pmatrix} 0 & -p_0^{(i)} \\
1 & -p_1^{(i)} \end{pmatrix}, & \text{if } \mathfrak{t}_i = 2,
\end{cases}
\]

where \( p_0^{(i)}, p_1^{(i)} \) are coefficients of the polynomials \( p_i \) in (4.22).

It may happen that the generalized Jacobi matrix \( \mathfrak{J} \) is unbounded even in the case when the support of \( \mathfrak{J} \) is bounded (see Example 2). It should be noted that bounded generalized Jacobi matrices associated to (4.22) were considered in [13].

5. PARTICULAR CASES

5.1. The case when \( \mathfrak{J} \) is holomorphic at \( \infty \). Consider the function \( \mathfrak{F} \) of the form

\[
\mathfrak{F}(\lambda) = r_1(\lambda) \int_{-1}^{1} \frac{t \, d\sigma(t)}{t - \lambda} + r_2(\lambda),
\]

where \( \sigma, r_1 \) and \( r_2 \) satisfy the assumptions (A1), (A2) and (A3').

**Theorem 5.1.** Let the function \( \mathfrak{F} \) be of the form (5.1) and let \( F(\lambda) = \frac{\mathfrak{F}(\lambda)}{\lambda} \). Then \( \mathfrak{F} \in D_{\kappa, -\infty} \) for some \( \kappa \in \mathbb{Z}_+ \) and the sequence of \([n-1/n-1]\) Padé approximants \( \{\mathfrak{F}^{[n-1/n-1]}\}_{n \in \mathbb{N}} \) converges to \( \mathfrak{F} \) locally uniformly in \( \mathbb{C} \setminus ([-1, -\varepsilon] \cup \mathbb{P}(\varphi)) \).

Moreover, if the condition (4.19) is fulfilled for the function \( F \) then the sequence \( \{\mathfrak{F}^{[n-1/n-1]}\}_{n \in \mathbb{N}} \) converges to \( \mathfrak{F} \) locally uniformly in \( \mathbb{C} \setminus ([-1 - \varepsilon, 1 + \varepsilon] \cup \mathbb{P}(\varphi)) \) for some \( \varepsilon > 0 \).

**Proof.** The function \( \frac{\mathfrak{F}(\lambda)}{\lambda} \) admits the representation

\[
F(\lambda) = \frac{\mathfrak{F}(\lambda)}{\lambda} = r_1(\lambda) \int_{\alpha}^{\beta} \frac{d\sigma(t)}{t - \lambda} + r_2(\lambda),
\]

where

\[
\bar{r}_2(\lambda) = \frac{r_1(\lambda)}{\lambda} \int_{\alpha}^{\beta} d\sigma(t) + \frac{r_2(\lambda)}{\lambda}.
\]

Therefore \( F \in N_{\kappa, -\infty} \) (see [22]), and hence \( \mathfrak{F} \in D_{\kappa, -\infty} \).

The statements concerning convergence of the sequence of diagonal Padé approximants of \( \mathfrak{F} \) are implied by Theorem 4.6.

**Remark 5.2.** In fact, as is easily seen from [22], every function \( \mathfrak{F} \in D_{\kappa, -\infty} \) holomorphic at infinity admits the representation (5.1).

**Example 2.** Let \( \theta \in \mathbb{R} \) be an irrational number and consider the function

\[
\mathfrak{F}(\lambda) = \int_{-1}^{1} \frac{td\sigma(t)}{t - \lambda}, \text{ where } d\sigma(t) = \frac{dt}{\sqrt{1 - (t - \cos \pi \theta)^2}}.
\]

Substitution \( x = t - \cos \pi \theta \) leads to the equality

\[
\mathfrak{F}(\lambda + \cos \pi \theta) = \int_{-1}^{1} \frac{(x + \cos \pi \theta) \, d\omega(x)}{x - \lambda}, \text{ where } d\omega(x) = \frac{dx}{\sqrt{1 - x^2}}.
\]

As was shown in [35] every point of \( \mathbb{R} \) is an accumulation point of the set of poles of the diagonal Padé approximants for \( \mathfrak{F}(\cdots + \cos \pi \theta) \). As a consequence, the diagonal
Padé approximants for \( \mathfrak{F} \) do not converge on \( \mathbb{R} \setminus [-1 + \cos \pi \theta, 1 + \cos \pi \theta] \). Therefore, the corresponding generalized Jacobi matrix \( J \) is unbounded.

However, there exists a subsequence of \( \mathfrak{F}^{[n-1/n-1]} \) converging in a neighborhood of \( \infty \). Indeed, applying Lemma 4.5 to the function \( \mathfrak{F} - \gamma \) with \( \gamma = \frac{-1 + \cos \pi \theta}{1 + \cos \pi \theta} \int_{-1 + \cos \pi \theta}^{1 + \cos \pi \theta} d\sigma(t) \), one obtains

\[
\mathfrak{F}^{[n-1/n-1]}(\lambda) = \lambda F^{[n/n-1]}(\lambda) + \gamma, \quad \text{where} \quad F(\lambda) = \int_{-1 + \cos \pi \theta}^{1 + \cos \pi \theta} \frac{d\sigma(t)}{t - \lambda}.
\]

Clearly, the shifted Chebyshev polynomials \( T_n(\cdot - \cos \pi \theta) \) are orthonormal with respect to \( \sigma \). Consequently, we can calculate explicitly the coefficient

\[
\tau_n = \frac{T_{n+1}(\cos \pi \theta)}{2T_n(\cos \pi \theta)} = -\frac{\cos(n + 1)\pi \{\theta\}}{2 \cos n\pi \{\theta\}} = \frac{1}{2}(\cos \pi \{\theta\} - \sin \pi \{\theta\} \tan n\pi \{\theta\}),
\]

where \( n \in \mathbb{N} \) and \( \{x\} \) denotes the fractional part of \( x \in \mathbb{R} \). Since the set \( \{n\theta\}_{n=0}^{\infty} \) is dense in \((0, 1)\), there is a bounded subsequence of \( \{\tau_n\}_{n=0}^{\infty} \) and thus, by Remark 3.8, there exists a subsequence of diagonal Padé approximants converging in a neighborhood of \( \infty \).

**Remark 5.3.** Let us consider a function \( \mathfrak{F} \) of the following form

\[
(5.2) \quad \mathfrak{F}(\lambda) = \int_{-1}^{1} \frac{td\sigma(t)}{t - \lambda},
\]

where \( \sigma \) is a nonnegative probability measure on \([-1, 1]\). It is clear that

\[
\mathfrak{F}(\lambda) = (J(J - \lambda)^{-1}e, e)_{\ell^2_{[0, \infty)}} = 1 + \lambda((J - \lambda)^{-1}e, e)_{\ell^2_{[0, \infty)}},
\]

where \( J \) is the classical Jacobi matrix constructed by the measure \( \sigma \) via the usual procedure [1]. Now, let us consider the following modified Padé approximant

\[
(5.3) \quad \mathfrak{F}^{[n/n]}(\lambda) = (J_{[0,n-1]}(J_{[0,n-1]} - \lambda)^{-1}e, e) = \frac{P_n(\lambda) - \lambda Q_n(\lambda)}{P_n(\lambda)}
\]

where \( P_n, Q_n \) are polynomials of the first and second kinds corresponding to the measure \( \sigma \). It follows from the Markov theorem (as well as from the spectral decomposition theorem) that

\[
\mathfrak{F}^{[n/n]} \to \mathfrak{F}
\]

locally uniformly in \( \mathbb{C} \setminus [-1, 1] \). So, to avoid the phenomenon described in the above example one can use the modified Padé approximants (5.3) for the function \( \mathfrak{F} \) of the form (5.2).

**5.2. The case when \( \text{supp} \ \sigma \text{ has a gap.} \)** Assume now that \( r_1(\lambda) \equiv r_2(\lambda) \equiv 1 \) in (5.1) and the support \( E \) of the finite nonnegative measure \( \sigma \) is contained in the union of two intervals

\[
E = [-1, \alpha] \cup [\beta, 1], \quad \alpha < 0 < \beta.
\]

First, we will show that in this case the diagonal Padé approximants for \( \mathfrak{F} \) have no poles inside the gap \((\alpha, \beta)\).
**Proposition 5.4.** Let $\sigma$ be a finite nonnegative measure on $E = [-1, \alpha] \cup [\beta, 1]$ and let

$$\mathfrak{F}(\lambda) = \int_E \frac{td\sigma(t)}{t - \lambda}. \quad (5.4)$$

Then:

(i) $\mathfrak{F} \in D_{1,-\infty}$;

(ii) The polynomials $\mathfrak{P}_j$, $j \in \mathbb{N}$ have no zeros inside the gap $(\alpha, \beta)$;

(iii) The function

$$\mathfrak{F}_0(\lambda) = \int_E \frac{td\sigma(t)}{t - \lambda} - \gamma, \quad \gamma = \int_E d\sigma(t). \quad (5.5)$$

belongs to the class $D_{0,-\infty}$.

**Proof.** (i) The first statement is implied by the equality

$$F(\lambda) = \frac{\mathfrak{F}(\lambda)}{\lambda} = \int_E \frac{d\sigma(t)}{t - \lambda} + \frac{1}{\lambda} \int_E d\sigma(t), \quad (5.6)$$

since $F \in N_{1,-\infty}$.

(ii) Next, $\mathfrak{P}_j(0)$ coincides with the Hankel determinants

$$\mathfrak{P}_j(0) = \begin{vmatrix} s_1 & \cdots & s_j \\ \vdots & \ddots & \vdots \\ s_j & \cdots & s_{2j-1} \end{vmatrix}$$

which are positive since $s_j = s_{j+1}$, $j = 0, 1, \ldots$ are moments of the positive measure $t^2d\sigma(t)$ with infinite support $E$.

Similarly, $\mathfrak{P}_j(\theta)$ coincides with the Hankel determinants

$$\mathfrak{P}_j(\theta) = \begin{vmatrix} s_1 - \theta s_0 & \cdots & s_j - \theta s_{j-1} \\ \vdots & \ddots & \vdots \\ s_j - \theta s_{j-1} & \cdots & s_{2j-1} - \theta s_{2j-2} \end{vmatrix}$$

which are positive for $\theta \in (\alpha, \beta)$, since $s_j := s_{j+1} - \theta s_j$ are moments of the positive measure $t(t - \theta)d\sigma(t)$.

(iii) The third statement follows from the equality

$$F_0(\lambda) = \frac{\mathfrak{F}_0(\lambda)}{\lambda} = \int_E \frac{d\sigma(t)}{t - \lambda}. \quad (5.7)$$

Next, one can apply Theorem 3.5 to prove the convergence of diagonal Padé approximants for $\mathfrak{F}$ on the real line.

**Theorem 5.5.** Let $\sigma$ be a finite nonnegative measure on $E = [-1, \alpha] \cup [\beta, 1]$, let $\mathfrak{F}$ have the form $\mathfrak{F}(\lambda) = \int_E \frac{td\sigma(t)}{t - \lambda}$ and let $\{\mathfrak{P}_j\}_{j=0}^\infty$ be the set of normalized polynomials orthogonal with respect to $\sigma$. Then:

(i) The sequence of diagonal Padé approximants $\{\mathfrak{F}^{[n-1/n-1]}\}_{n \in \mathbb{N}}$ converges to $\mathfrak{F}$ locally uniformly in $\mathbb{C} \setminus \{(-\infty, \alpha] \cup [\beta, \infty)\}$;

(ii) The sequence $\{\mathfrak{F}^{[n-1/n-1]}\}_{n \in \mathbb{N}}$ converges to $\mathfrak{F}$ locally uniformly in $\mathbb{C} \setminus \{[-1 - \varepsilon, \alpha] \cup [\beta, 1 + \varepsilon]\}$ for some $\varepsilon > 0$ if and only if the condition $3.13$ is fulfilled.
Proof. (i) As follows from Lemma 4.5 and [12, Theorem 4.16] the Padé approximant $\tilde{F}_{0}^{[n-1/n-1]}$ takes the form

\[ (5.8) \quad \tilde{F}_{0}^{[n-1/n-1]}(\lambda) = \tau_{n-1/n-1}(\lambda) = \lambda((\mathcal{J}_{[0,n-1]}^{(K)})^{-1} - \lambda)^{-1}e, \quad n = n_j \in \mathcal{N}_{F_0}, \]

where $\mathcal{J}$ is a classical Jacobi matrix corresponding to the measure $\sigma$, and $\mathcal{J}_{[0,n-1]}^{(K)}$ is defined in Remark 5.2. Let us emphasize that $\mathcal{J}_{[0,n-1]}^{(K)}$ is a classical Jacobi matrix since $F_0 \in \mathbb{N}_0$. By Proposition 5.4 and Theorem 4.8, $\tilde{F}_{0}^{[n-1/n-1]}$ is holomorphic on $(\alpha, 0) \cup (0, \beta)$ and hence by (5.8) and Remark 2.5, the set $(\alpha, 0) \cup (0, \beta)$ is contained in $\rho(\mathcal{J}_{[0,n-1]}^{(K)})$.

It follows from the spectral theorem that for arbitrary $\varepsilon > 0$

\[ \| (\mathcal{J}_{[0,n-1]}^{(K)} - \lambda)^{-1} \| \leq \frac{1}{\varepsilon} \quad \text{for all } \lambda \in (\alpha + \varepsilon, -\varepsilon) \cup (\varepsilon, \beta - \varepsilon). \]

Then by the Vitali theorem the sequence $\{\tilde{F}_{0}^{[n-1/n-1]}(\lambda)\}_{n \in \mathcal{N}_{F}}$ converges to $\tilde{F}_{0}(\lambda) = \lambda((\mathcal{J} - \lambda)^{-1}e, e)$ locally uniformly on $\mathbb{C} \setminus ((-\infty, \alpha] \cup \{0\} \cup [\beta, \infty))$. Moreover, for $\varepsilon > 0$ small enough $\tilde{F}_{0}^{[n-1/n-1]}(\lambda)$ converges to $\tilde{F}_{0}(\lambda)$ uniformly on the circle $|\lambda| = \varepsilon$. Then by the mean value theorem

\[ \tilde{F}_{0}^{[n-1/n-1]}(0) = \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{F}_{0}^{[n-1/n-1]}(\varepsilon e^{it}) dt \to \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{F}_{0}(\varepsilon e^{it}) dt = \tilde{F}_{0}(0) \]

as $n \to \infty$. To complete the proof it remains to mention that the $[n - 1/n - 1]$ Padé approximants $\tilde{F}_{0}^{[n-1/n-1]}$, $n \in \mathcal{N}_{F}$, are connected with the $[n - 1/n - 1]$ Padé approximants $\tilde{F}_{0}^{[n-1/n-1]}$ by the equality

\[ \tilde{F}_{0}^{[n-1/n-1]} = \tilde{F}_{0}^{[n-1/n-1]} + \gamma. \]

(ii) The necessity of the second statement is contained in Theorem 4.6. Let us prove the sufficiency by proving the inverse statement. So, suppose that

\[ (5.9) \quad \sup_{n_j \in \mathcal{N}_{F}} \left| \frac{\tilde{P}_j(0)}{\tilde{P}_{j-1}(0)} \right| = \infty. \]

Due to (5.8) the poles of the Padé approximant $\tilde{F}_{0}^{[n-1/n-1]}$ coincide with eigenvalues of the matrix $\mathcal{J}_{[0,n-1]}^{(K)}$. Taking into account that $\mathcal{J}_{[0,n-1]}^{(K)}$ is a self-adjoint matrix, one obtains

\[ |\lambda_{\max}(\mathcal{J}_{[0,n-1]}^{(K)})| = \|\mathcal{J}_{[0,n-1]}^{(K)}\| \geq |(\mathcal{J}_{[0,n-1]}^{(K)}e_{n-1}, e_{n-1})| = p_{n-1}^{(n-1)} - \frac{\tilde{P}_n(0)}{\tilde{P}_{n-1}(0)}, \]

where $\lambda_{\max}(\mathcal{J}_{[0,n-1]}^{(K)})$ is the eigenvalue of the matrix $\mathcal{J}_{[0,n-1]}^{(K)}$ with the largest absolute value. Since the sequence $\{p_{n-1}^{(n-1)}\}_{n=1}^{\infty}$ is bounded, we have that infinity is an accumulation point of the set of all poles of Padé approximants $\tilde{F}_{0}^{[n-1/n-1]}$.

Remark 5.6. Let $\mathcal{F}$ be a function having the following form

\[ \mathcal{F}(\lambda) = \int_{-1}^{1} \frac{\rho(t) dt}{t - \lambda}, \]

where $\rho$ is a nonvanishing on $[-1, 1]$ complex-valued function. Under some assumptions on $\rho$, the locally uniform convergence of the diagonal Padé approximants for
\( \mathcal{F} \) was proved by A. Magnus [30] (see also [35]). Using the technique of Riemann-Hilbert problems, the result was reproved by A.I. Aptekarev and W. van Assche [3]. Example 3. Let \( \mathcal{F} \) have the form (5.4) with an absolutely continuous measure \( d\sigma(t) = \rho(t)dt \) such that \( \rho(t) \) is an even function on \( E = [-1, -\beta] \cup [\beta, 1] \) and \( \rho(t) = 0 \) for \( t \in \mathbb{R} \setminus E \). Then the polynomials \( P_{2j+1} \) are odd (for instance, see [34] formula (5.90)) and, therefore, \( P_{2j+1}(0) = 0 \) for all \( j \in \mathbb{N} \). Hence the condition (5.10) is fulfilled and by Theorem 5.5 the Padé approximants \( \mathcal{F}^{[2j/2j]} \) converge to \( \mathcal{F} \) on \( C \setminus E \). This fact can be shown directly, since \( \mathcal{F}(\lambda) \) admits the representation

\[ \mathcal{F}(\lambda) = \varphi(\lambda^2), \quad \text{where } \varphi(\mu) = \int_{\beta^2}^1 \frac{\sqrt{s\rho(\sqrt{s})} \, ds}{s - \mu} \in \mathbb{N}_0 \]

and, hence, \( \mathcal{F}^{[2j/2j]}(\lambda) = \varphi[j/j](\lambda^2) \) converge to \( \mathcal{F}(\lambda) = \varphi(\lambda^2) \) for all \( \lambda^2 \in \mathbb{C} \setminus [\beta^2, 1] \), or, equivalently, for all \( \lambda \in \mathbb{C} \setminus E \).

Due to Remark 3.3 it is enough to find a bounded subsequence of \( \{\tau_{j-1}\}_{n_j \in N} \) to say that there exists a subsequence of diagonal Padé approximants of \( \mathcal{F} \) which converges locally uniformly in a neighborhood of \( \infty \). In the following proposition we find a sufficient condition for the boundedness of a subsequence of \( \{\tau_{j-1}\}_{n_j \in N} \).

**Proposition 5.7.** Let \( \sigma \) be a finite nonnegative measure on \( E = [-1, \alpha] \cup [\beta, 1] \) and let \( \{P_j\}_{j=0}^\infty \) be the set of monic polynomials orthogonal with respect to \( \sigma \). Assume that 0 is not an accumulation point of zeros of a subsequence \( \{P_{j_k}\}_{k=1}^\infty \). Then the sequence \( \left\{\frac{P_{j_k+1}(0)}{P_{j_k}(0)}\right\}_{k=1}^\infty \) is bounded.

**Proof.** The orthogonal polynomials \( \hat{P}_j \) satisfy the following recurrence relations

\[ \lambda \hat{P}_j(\lambda) = b_j^2 \hat{P}_{j-1}(\lambda) + a_j \hat{P}_j(\lambda) + \hat{P}_{j+1}(\lambda), \quad b_j > 0, \quad a_j \in \mathbb{R}, \]

which implies the following equality

\[ \frac{\hat{P}_{j+1}(\lambda)}{\hat{P}_j(\lambda)} = \lambda - \alpha_j + b_j^2 \left( \frac{\hat{P}_{j-1}(\lambda)}{\hat{P}_j(\lambda)} \right). \]

(5.10)

It is well known (see [2]) that \( -\frac{\hat{P}_{j-1}(\lambda)}{\hat{P}_j(\lambda)} \) belongs to \( N_0 \) and hence there is a nonnegative measure \( \sigma^{(j)} \) such that

\[ \frac{\hat{P}_{j-1}(\lambda)}{\hat{P}_j(\lambda)} = \int_{-1}^1 \frac{d\sigma^{(j)}(t)}{t - \lambda}. \]

Moreover \( \sigma^{(j)} \) satisfies the condition

\[ \int_{-1}^1 d\sigma^{(j)}(t) = 1 \]

because of the asymptotic relation \( -\frac{\hat{P}_{j-1}(\lambda)}{\hat{P}_j(\lambda)} = -\frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right) \) as \( \lambda \to \infty \). Since the zeros of \( \{\hat{P}_{j_k}\}_{k=1}^\infty \) do not accumulate to 0, there exists \( \delta > 0 \) such that

\[ (-\delta, \delta) \cap \text{supp } \sigma^{(j_k)} = \emptyset, \quad k = 1, 2, \ldots \]

So, we have the following estimate

\[ \left| \frac{\hat{P}_{j-1}(0)}{\hat{P}_j(0)} \right| = \left| \int_{-1}^1 \frac{d\sigma^{(j_k)}(t)}{t} \right| \leq \frac{1}{\delta}. \]

(5.11)
Finally, the boundedness of \( \left\{ \frac{P_{j+1}(0)}{P_j(0)} \right\}_{k=1}^{\infty} \) follows from the boundedness of the sequences \( \{a_j\}_{j=0}^{\infty} \), \( \{b_j\}_{j=0}^{\infty} \), the equality (5.11), and the equality (5.10). \( \square \)

5.3. The case when \( \sigma \) satisfies the Szegő conditions. Now, a natural question arises: under what conditions 0 is not an accumulation point of zeros of a subsequence \( \{P_{j_k}\}_{j=0}^{\infty} \)? In this subsection the answer to this question is given for functions of the form (5.3) under an additional assumption that the measure \( d\sigma(t) = \rho(t)dt \) satisfies the Szegő condition on each of the intervals \([-1, \alpha]\) and \([\beta, 1]\)

\[
\int_{-1}^{\alpha} \frac{\log \rho(t)}{\sqrt{(\alpha - t)(t + 1)}} dt > -\infty, \quad \int_{\beta}^{1} \frac{\log \rho(t)}{\sqrt{(1 - t)(t - \beta)}} dt > -\infty.
\]

As is known the polynomials \( P_j \) have at most one zero in the interval \((\alpha, \beta)\). The information about accumulation points of these zeros can be formulated in terms of the harmonic measure \( \omega(\lambda) \) of \([-1, \alpha]\) with respect to \( \mathbb{C} \setminus E \), i.e. harmonic function on \( \mathbb{C} \setminus E \) whose boundary values are equal 1 on \([-1, \alpha]\) and 0 on \([\beta, 1]\).

**Remark 5.8.** For more detailed and deep analysis of the behavior of zeros of orthogonal polynomials see [36] (see also [37]).

Assume first that \( \omega(\infty) \) is an irrational number. Then by a theorem of E. Rakhmanov ([33 Theorem 0.2]) every point of \((\alpha, \beta)\) and, in particular, 0 is an accumulation point of zeros of a sequence \( \{P_{j_k}\}_{j=0}^{\infty} \). However, since there is only one zero of \( P_j \) in the gap \((\alpha, \beta)\) it is possible to choose a subsequence of \( \{P_{j_k}\}_{j=0}^{\infty} \) which zeros do not accumulate to 0. Further, as follows from Proposition 5.7 there is a subsequence of \([n/n]\) Padé approximants of \( \mathcal{F} \) which converges to \( \mathcal{F} \) locally uniformly on \( \mathbb{C} \setminus ([1-\varepsilon, \alpha] \cup [\beta, 1+\varepsilon]) \) for some \( \varepsilon > 0 \).

Assume now that \( \omega(\infty) \) is a rational number \( m/n \), where \( m, n \in \mathbb{N} \) and \( \gcd(m, n) = 1 \). Then it follows from [33 formula (57)] that every accumulation point of zeros of polynomials \( \{P_j\}_{j=1}^{\infty} \) in the interval \((\alpha, \beta)\) satisfies one of the equation

\[
\omega_1(z) \equiv \frac{k}{n} (\text{mod } 2), \quad k \in \mathbb{Z}, |k| \leq n.
\]

In the case when 0 is not a solution of the equation (5.13) it follows from Proposition 5.7 and Theorem 5.5 that the sequence \( \{\mathcal{F}^{n_j-1/n_j-1}\}_{n_j \in N'} \) of Padé approximants of \( \mathcal{F} \) converges to \( \mathcal{F} \) locally uniformly on \( \mathbb{C} \setminus ([1-\varepsilon, \alpha] \cup [\beta, 1+\varepsilon]) \) for some \( \varepsilon > 0 \).

The harmonic measure of \([-1, \alpha]\) with respect to \( \mathbb{C} \setminus E \) can be calculated explicitly (see [2]). Let real \( k \) be defined by

\[
k^2 = \frac{2(\beta - \alpha)}{(1 - \alpha)(1 + \beta)}.
\]

Consider the function \( x = \text{sn} \ w \) with the modulus \( k \), and with the primitive periods \( 4K, 2iK' \). As is known (see [2 p.190]) the mapping

\[
z = \alpha + \frac{1 - \alpha^2}{2\text{sn}^2 K'/\text{sn}^2 k} \frac{\text{sn} \ w}{\sqrt{\pi}} + \alpha - 1
\]

maps conformally the ring

\[
r := e^{-\frac{2K}{K'}} < |w| < 1
\]
onto the plane $\mathbb{C}$ with the cuts $[-1, \alpha] \cup [\beta, 1]$, moreover, the semicircle $|w| = 1$ ($\text{Im } w \geq 0$) is mapped onto the upper shore of the cut $[-1, \alpha]$.

As is well known (see [25]) the harmonic measure $\omega_R$ of the ring (5.15) has the form

$$\omega_R(w) = \frac{\ln |w| - \ln r}{\ln 1 - \ln r} = \frac{\ln |w| + \frac{2K}{\pi K'}}{\pi K'}.$$ 

So, the harmonic measure of $[-1, \alpha]$ with respect to $\mathbb{C} \setminus E$ can be found by

$$\omega(z) = \frac{K'}{\pi K} \ln |w| + 1. \tag{5.16}$$

Let us choose $w_\infty \in (r, 1)$ and $w_0 \in (-1, r)$ such that

$$1 - 2\sin^2 \frac{K'}{\pi} \ln w_\infty = \alpha, \quad 1 - 2\sin^2 \frac{K'}{\pi} \ln w_0 = \frac{1}{\alpha}. \tag{5.17}$$

Then the numbers $w_\infty, w_0$ correspond to $z = \infty$ and $z = 0$ via (5.14). It follows from (5.14) that $\omega(\infty)$ is a rational number $m/\pi$ if and only if $\frac{K'}{\pi} \ln w_\infty = \frac{m-n}{n}$, which in view of (5.16) is equivalent to

$$1 - 2\sin^2 K' r = \alpha, \quad r \in \mathbb{Q} \tag{5.18}$$

with $r = \frac{m-n}{n}$. Since $w_0 = -|w_0|$ one obtains from (5.17) and the reduction formula (see [2] Table XII)

$$1 - 2\sin^2 \frac{K'}{\pi} \ln w_0 = 1 - \frac{2}{k^2\sin^2(K'r)} = 1 - \frac{2}{k^2\sin^2(K'r)} \tag{5.19}$$

Hence one obtains that $\omega(0)$ is a rational number $\frac{m}{\pi}$ if and only if $\frac{K'}{\pi} \ln |w_0|$ is a rational number, or, equivalently,

$$1 - \frac{2}{k^2\sin^2(K'r')} = \frac{1}{\alpha}. \tag{5.20}$$

for $r' = \frac{m-n}{n}$.

These calculations lead to the following

**Proposition 5.9.** Let a finite nonnegative measure $\sigma$ on $E = [-1, \alpha] \cup [\beta, 1]$ be absolutely continuous ($d\sigma(t) = \rho(t) dt$) and satisfies the Szegő conditions (5.12). Then:

(i) If $\alpha$ cannot be represented in the form (5.18) for some $r \in \mathbb{Q}$ then there is a subsequence of $[n/n]$ Padé approximants of $\mathcal{F}$ which converges to $\mathcal{F}$ locally uniformly on $\mathbb{C} \setminus (-1 - \varepsilon, \alpha] \cup [\beta, 1 + \varepsilon)$ for some $\varepsilon > 0$.

(ii) If $\alpha$ satisfies (5.18) for some $r = \frac{m}{n}$ with $m, n \in \mathbb{N}$ ($\text{gcd}(m, n) = 1$) and does not satisfies (5.18) for any $r' = \frac{k}{n}$ with $k \in \mathbb{N}$ $|k| \leq n$, then the sequence $\{\mathcal{F}[n^{-1/n-1}]\}_{n \in \mathbb{N}}$ of Padé approximants of $\mathcal{F}$ converges to $\mathcal{F}$ locally uniformly on $\mathbb{C} \setminus (-1 - \varepsilon, \alpha] \cup [\beta, 1 + \varepsilon)$ for some $\varepsilon > 0$.

In this paper we considered the case of one ”turning point”. The following example shows that in the case of 2 ”turning points” the behaviour of diagonal Padé approximants seems to be more complicated.

**Example 4 ([33]).** Let $\theta_1, \theta_2, 1 (0 < \theta_1 < \theta_2 < 1)$ be rationally independent real, and let

$$F(\lambda) = \int_{-1}^{1} \frac{1}{t - \lambda} \frac{(t - \cos \pi \theta_1)(t - \cos \pi \theta_2)}{\sqrt{1 - t^2}} dt.$$
Then all the diagonal Padé approximants $F^{[k/k]}$ exist, but do not converge locally uniformly on $\mathbb{C} \setminus \mathbb{R}$ since

$$
\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} P(F^{[k/k]}) = \mathbb{C}.
$$

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