The boundedness of intrinsic square functions on the weighted Herz-type Hardy spaces

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Abstract

In this paper, the boundedness properties are obtained for intrinsic square functions including the Lusin area function, Littlewood-Paley $g$-function and $g^*_A$-function on the weighted Herz-type Hardy spaces.

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1. Introduction and preliminaries

First, let’s recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [15]. A weight $w$ is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. $B = B(x_0, R)$ denotes the ball with the center $x_0$ and radius $R$. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C$$

for every ball $B \subseteq \mathbb{R}^n$, where $C$ is a positive constant which is independent of the choice of $B$.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) \, dx \leq C \inf_{x \in B} w(x)$$

for every ball $B \subseteq \mathbb{R}^n$.

A weight function $w$ is said to belong to the reverse Hölder class $RH_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse

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Hölder inequality holds

\[
\left( \frac{1}{|B|} \int_B |w(x)|^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)
\]

for every ball \( B \subseteq \mathbb{R}^n \).

It is well known that if \( w \in A_p \) with \( 1 < p < \infty \), then \( w \in A_r \) for all \( r > p \), and \( w \in A_q \) for some \( 1 < q < p \). If \( w \in A_p \) with \( 1 \leq p < \infty \), then there exists \( r > 1 \) such that \( w \in RH_r \).

Given a ball \( B \) and \( \lambda > 0 \), \( \lambda B \) denotes the ball with the same center as \( B \) whose radius is \( \lambda \) times that of \( B \). For a given weight function \( w \), we denote the Lebesgue measure of \( B \) by \( |B| \) and the weighted measure of \( B \) by \( w(B) \), where \( w(B) = \int_B w(x) \, dx \).

We state the following results that we will use frequently in the sequel.

**Lemma A** ([4]). Let \( w \in A_p, p \geq 1 \). Then, for any ball \( B \), there exists an absolute constant \( C \) such that

\[
w(2B) \leq C w(B).
\]

In general, for any \( \lambda > 1 \), we have

\[
w(\lambda B) \leq C \lambda^p w(B),
\]

where \( C \) does not depend on \( B \) nor on \( \lambda \).

**Lemma B** ([4,5]). Let \( w \in A_p \cap RH_r, p \geq 1 \) and \( r > 1 \). Then there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 \left( \frac{|E|}{|B|} \right)^p \leq \frac{w(E)}{w(B)} \leq C_2 \left( \frac{|E|}{|B|} \right)^{(r-1)/r}
\]

for any measurable subset \( E \) of a ball \( B \).

Next we shall give the definitions of the weighted Herz space and weighted Herz-type Hardy space. In 1964, Beurling [2] first introduced some fundamental form of Herz spaces to study convolution algebras. Later Herz [6] gave versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in harmonic analysis. For instance, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the classical Hardy spaces, and used by Lu and Yang [12] in the study of partial differential equations.
On the other hand, a theory of Hardy spaces associated with Herz spaces has been developed in [3,10]. These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces \( H^p(\mathbb{R}^n) \) and are good substitutes for \( H^p(\mathbb{R}^n) \) when we study the boundedness of non-translation invariant operators (see [11]). For the weighted case, in 1995, Lu and Yang introduced the following weighted Herz-type Hardy spaces \( H_{q}^{\alpha,p} (w_1, w_2) \) \( \left( H_{q}^{\alpha,p} (w_1, w_2) \right) \) and established their atomic decompositions. In 2006, Lee gave the molecular characterizations of these spaces, he also obtained the boundedness of the Hilbert transform and the Riesz transforms on \( H_{q}^{\alpha,p} (1/p-1/q,p)(w, w) \) and \( H_{q}^{\alpha,p} (1/p-1/q,p)(w, w) \) for \( 0 < p \leq 1 \). For the results mentioned above, we refer the readers to the book [14] and the papers [7,8,9,13] for further details.

Let \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \) and \( C_k = B_k \setminus B_{k-1} \) for \( k \in \mathbb{Z} \). Denote \( \chi_k = \chi_{C_k} \) for \( k \in \mathbb{Z} \), \( \tilde{\chi}_k = \chi_k \) if \( k \in \mathbb{N} \) and \( \tilde{\chi}_0 = \chi_B \), where \( \chi_{C_k} \) is the characteristic function of \( C_k \). Given a weight function \( w \) on \( \mathbb{R}^n \), for \( 1 \leq p < \infty \), we denote by \( L_w^p {\left( \mathbb{R}^n \right)} \) the space of all functions satisfying

\[
\| f \|_{L_w^p (\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
\]

**Definition 1.** Let \( \alpha \in \mathbb{R} \), \( 0 < p, q < \infty \) and \( w_1, w_2 \) be two weight functions on \( \mathbb{R}^n \).

(i) The homogeneous weighted Herz space \( \hat{K}_{q}^{\alpha,p} (w_1, w_2) \) is defined by

\[
\hat{K}_{q}^{\alpha,p} (w_1, w_2) = \{ f \in L_w^q {\left( \mathbb{R}^n \setminus \{ 0 \} \right)}, w_2) : \| f \|_{\hat{K}_{q}^{\alpha,p} (w_1, w_2)} < \infty \},
\]

where

\[
\| f \|_{\hat{K}_{q}^{\alpha,p} (w_1, w_2)} = \left( \sum_{k \in \mathbb{Z}} \left( w_1(B_k) \right)^{\alpha p/n} \| \chi_k \|_{L_w^p}^p \right)^{1/p}.
\]

(ii) The non-homogeneous weighted Herz space \( K_{q}^{\alpha,p} (w_1, w_2) \) is defined by

\[
K_{q}^{\alpha,p} (w_1, w_2) = \{ f \in L_w^q {\left( \mathbb{R}^n \right)}, w_2) : \| f \|_{K_{q}^{\alpha,p} (w_1, w_2)} < \infty \},
\]

where

\[
\| f \|_{K_{q}^{\alpha,p} (w_1, w_2)} = \left( \sum_{k=0}^{\infty} \left( w_1(B_k) \right)^{\alpha p/n} \| \tilde{\chi}_k \|_{L_w^p}^p \right)^{1/p}.
\]

Let \( \mathcal{S} (\mathbb{R}^n) \) be the class of Schwartz functions and let \( \mathcal{S}'(\mathbb{R}^n) \) be its dual space. For \( f \in \mathcal{S}'(\mathbb{R}^n) \), the grand maximal function of \( f \) is defined by

\[
G(f)(x) = \sup_{\varphi \in \mathcal{S}_N} \sup_{|y-x| < t} |\varphi_t * f(y)|,
\]
where $N > n + 1$, \$N = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta| \leq N} |x^\alpha D^\beta \varphi(x)| \leq 1 \}$ and \$\varphi_t(x) = t^{-n} \varphi(x/t) \$.

**Definition 2.** Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 < q < \infty$ and $w_1$, $w_2$ be two weight functions on $\mathbb{R}^n$.

(i) The homogeneous weighted Herz-type Hardy space $HK_q^{\alpha,p}(w_1,w_2)$ associated with the space $\tilde{K}_q^{\alpha,p}(w_1,w_2)$ is defined by

$$HK_q^{\alpha,p}(w_1,w_2) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G(f) \in \tilde{K}_q^{\alpha,p}(w_1,w_2) \}$$

and we define $\| f \|_{HK_q^{\alpha,p}(w_1,w_2)} = \| G(f) \|_{\tilde{K}_q^{\alpha,p}(w_1,w_2)}$.

(ii) The non-homogeneous weighted Herz-type Hardy space $HK_q^{\alpha,p}(w_1,w_2)$ associated with the space $\tilde{K}_q^{\alpha,p}(w_1,w_2)$ is defined by

$$HK_q^{\alpha,p}(w_1,w_2) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : G(f) \in K_q^{\alpha,p}(w_1,w_2) \}$$

and we define $\| f \|_{HK_q^{\alpha,p}(w_1,w_2)} = \| G(f) \|_{K_q^{\alpha,p}(w_1,w_2)}$.

**2. The atomic decomposition**

In this article, we will use Lu and Yang’s atomic decomposition theory for weighted Herz-type Hardy spaces in [8,9]. We characterize weighted Herz-type Hardy spaces in terms of atoms in the following way.

**Definition 3.** Let $1 < q < \infty$, $n(1-1/q) \leq \alpha < \infty$ and $s \geq [\alpha+n(1/q-1)]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha,q,s)$-atom with respect to $(w_1,w_2)$ (or a central $(\alpha,q,s;w_1,w_2)$-atom), if it satisfies

(a) $\text{supp} \ a \subseteq B(0,R) = \{ x \in \mathbb{R}^n : |x| < R \}$,

(b) $\| a \|_{L^q_{\alpha/2}} \leq w_1(B(0,R))^{-\alpha/n}$,

(c) $\int_{\mathbb{R}^n} a(x) x^\beta \, dx = 0$ for every multi-index $\beta$ with $|\beta| \leq s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha,q,s)$-atom of restricted type with respect to $(w_1,w_2)$ (or a central $(\alpha,q,s;w_1,w_2)$-atom of restricted type), if it satisfies the conditions (b), (c) above and

(a') $\text{supp} \ a \subseteq B(0,R)$ for some $R > 1$.

**Theorem C.** Let $w_1,w_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$ and $n(1-1/q) \leq \alpha < \infty$. Then we have

(i) $f \in HK_q^{\alpha,p}(w_1,w_2)$ if and only if

$$f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x), \quad \text{in the sense of} \ \mathcal{S}'(\mathbb{R}^n),$$

where $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < \infty$, each $a_k$ is a central $(\alpha,q,s;w_1,w_2)$-atom. Moreover,

$$\| f \|_{HK_q^{\alpha,p}(w_1,w_2)} \approx \inf \left( \sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p},$$

...
where the infimum is taken over all the above decompositions of \( f \).

(ii) \( f \in HK_q^{\alpha,p}(w_1,w_2) \) if and only if

\[
f(x) = \sum_{k=0}^{\infty} \lambda_k a_k(x), \quad \text{in the sense of } \mathscr{S}'(\mathbb{R}^n),
\]

where \( \sum_{k=0}^{\infty} |\lambda_k|^p < \infty \), each \( a_k \) is a central \((\alpha,q,s;w_1,w_2)\)-atom of restricted type. Moreover,

\[
\|f\|_{HK_q^{\alpha,p}(w_1,w_2)} \approx \inf \left( \sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p},
\]

where the infimum is taken over all the above decompositions of \( f \).

3. The intrinsic square functions and our main results

The intrinsic square functions were first defined by Wilson in [17] and [18]. For \( 0 < \beta \leq 1 \), let \( C_\beta \) be the family of functions \( \varphi \) defined on \( \mathbb{R}^n \), such that \( \varphi \) has support containing in \( \{ x : |x| \leq 1 \} \), \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 0 \) and for all \( x, x' \in \mathbb{R}^n \),

\[
|\varphi(x) - \varphi(x')| \leq |x - x'|^\beta.
\]

For \( (y,t) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty) \) and \( f \in L^1_{loc}(\mathbb{R}^n) \), we set

\[
A_\beta(f)(y,t) = \sup_{\varphi \in C_\beta} |f \ast \varphi_t(y)|.
\]

Then we define the intrinsic square function of \( f \) (of order \( \beta \)) by the formula

\[
S_\beta(f)(x) = \left( \iint_{\Gamma(x)} \left( A_\beta(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]

where \( \Gamma(x) \) denotes the usual cone of aperture one:

\[
\Gamma(x) = \{ (y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t \}.
\]

We can also define varying-aperture version of \( S_\beta(f) \) by the formula

\[
S_{\beta,\gamma}(f)(x) = \left( \iint_{\Gamma_\gamma(x)} \left( A_\beta(f)(y,t) \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2},
\]

where \( \Gamma_\gamma(x) \) is the usual cone of aperture \( \gamma > 0 \):

\[
\Gamma_\gamma(x) = \{ (y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \gamma t \}.
\]
The intrinsic Littlewood-Paley $g$-function (could be viewed as “zero-aperture” version of $S_\beta (f)$) and the intrinsic $g^*_\lambda$-function (could be viewed as “infinite aperture” version of $S_\beta (f)$) will be defined respectively by

\[ g_\beta (f)(x) = \left( \int_0^\infty \left( A_\beta (f)(x,t) \frac{2 \, dt}{t} \right) \right) ^{1/2} \]

and

\[ g^*_\lambda,\beta (f)(x) = \left( \iiint_{\mathbb{R}^{n+1}_+} \left( \frac{t}{t + |x-y|} \right) \lambda^n \left( A_\beta (f)(y,t) \frac{2 \, dydt}{t^{n+1}} \right) \right) ^{1/2} \].

In [18], Wilson showed the following weighted $L^p$ boundedness of the intrinsic square functions.

**Theorem D.** Let $w \in A_p$, $1 < p < \infty$ and $0 < \beta \leq 1$. Then there exists a constant $C > 0$ such that

\[ \| S_\beta (f) \|_{L^p_w} \leq C \| f \|_{L^p_w}. \]

The main purpose of this paper is to discuss the boundedness properties of intrinsic square functions on the homogeneous and non-homogeneous weighted Herz-type Hardy spaces. Our main results are stated as follows.

**Theorem 1.** Let $w_1, w_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$, $0 < \beta \leq 1$ and $n(1-1/q) \leq \alpha < n(1-1/q) + \beta$. Then there exists a constant $C$ independent of $f$ such that

\[ \| g_\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)} \]

\[ \| g^*_\lambda,\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)}. \]

**Theorem 2.** Let $w_1, w_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$, $0 < \beta \leq 1$ and $n(1-1/q) \leq \alpha < n(1-1/q) + \beta$. Then there exists a constant $C$ independent of $f$ such that

\[ \| S_\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)} \]

\[ \| S_\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)}. \]

**Theorem 3.** Let $w_1, w_2 \in A_1$, $0 < p < \infty$, $1 < q < \infty$, $0 < \beta \leq 1$ and $n(1-1/q) \leq \alpha < n(1-1/q) + \beta$. If $\lambda > 3 + (2\beta)/n$, then there exists a constant $C$ independent of $f$ such that

\[ \| g^*_\lambda,\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)} \]

\[ \| g^*_\lambda,\beta (f) \|_{K_q^{\alpha,p}(w_1,w_2)} \leq C \| f \|_{HK_q^{\alpha,p}(w_1,w_2)}. \]
Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence.

4. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. First we note that the assumption $n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \beta$ implies that $N = \lceil \alpha + n(1/q - 1) \rceil = 0$. For any central $(\alpha, q, 0; w_1, w_2)$-atom $a$ with supp $a \subseteq B(0, R)$, we are going to show that $\|g_\beta(a)\|_{K_\alpha}^{\alpha,p(w_1,w_2)} \leq C$, where $C > 0$ is independent of the choice of $a$. For given $R > 0$, we are able to choose $k_0 \in \mathbb{Z}$ satisfying $2^{k_0} - 2 < R \leq 2^{k_0} - 1$. Write

$$
\|g_\beta(a)\|_{K_\alpha}^{\alpha,p(w_1,w_2)} = \sum_{k \in \mathbb{Z}} w_1(B_k) \|g_\beta(a)\|_{L_{w_2}^p}^p
$$

$$
= \sum_{k=-\infty}^{k_0} w_1(B_k) \|g_\beta(a)\|_{L_{w_2}^p}^p + \sum_{k=k_0+1}^{\infty} w_1(B_k) \|g_\beta(a)\|_{L_{w_2}^p}^p
$$

$$
= I_1 + I_2.
$$

Since $w_2 \in A_1$, then $w_2 \in A_q$ for any $1 < q < \infty$. It follows from Theorem D that

$$
I_1 \leq \sum_{k=-\infty}^{k_0} w_1(B_k) \|g_\beta(a)\|_{L_{w_2}^p}^p \leq C \sum_{k=-\infty}^{k_0} w_1(B_k) \|a\|_{L_{w_2}^p}^p.
$$

Since $w_1 \in A_1$, then we know $w \in RH_r$ for some $r > 1$. When $k \leq k_0$, then $B_k \subseteq B_{k_0}$. By Lemma B, we have

$$
w_1(B_k) \leq C w_1(B_{k_0}) |B_k|^\delta |B_{k_0}|^{-\delta}, \tag{1}
$$

where $\delta = (r - 1)/r > 0$. Using the size condition of central atom $a$ and (1), we obtain

$$
I_1 \leq C \sum_{k=-\infty}^{k_0} \left( 2^{(k-k_0)\alpha \delta p} w_1(B_{k_0}) \|a\|_{L_{w_2}^p}^p \right)
$$

$$
\leq C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)\alpha \delta p}
$$
Substituting the above inequality (6) into the term $I \in y$, the following inequality holds
\[ 1_R < \infty. \]

To estimate the other term $I_2$, we first claim that for any $(x, t) \in \mathbb{R}_{+}^{n+1}$, the following inequality holds
\[ A_\beta(a)(x, t) \leq C \cdot \frac{R^{n+\beta}}{t^{n+\beta}} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{-1/q}. \] (3)

In fact, for any $\varphi \in C_\beta$, by the vanishing moment condition of $a$, we have
\[ |a * \varphi_t(x)| = \left| \int_{B(0, R)} (\varphi_t(x - y) - \varphi_t(x)) a(y) \, dy \right| \leq \int_{B(0, R)} \frac{|y|^{\beta}}{t^{n+\beta}} |a(y)| \, dy \quad (4) \]
\[ \leq \frac{R^{\beta}}{t^{n+\beta}} \int_{B(0, R)} |a(y)| \, dy. \]

Denote the conjugate exponent of $q > 1$ by $q' = q/(q - 1)$. Using Hölder’s inequality, $A_q$ condition and the size condition of $a$, we can get
\[ \int_{B(0, R)} |a(y)| \, dy \leq \left( \int_{B(0, R)} |a(y)|^q w_2(y) \, dy \right)^{1/q} \left( \int_{B(0, R)} \left( w_2^{-1/q}' d'y \right)^{1/q'} \right) \leq C \cdot \|a\|_{L^q_{w_2}} |B(0, R)| w_2(B(0, R))^{-1/q} \leq C \cdot |B(0, R)| w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{-1/q}. \] (5)

Substituting the above inequality (5) into (4) and taking the supremum over all functions $\varphi \in C_\beta$, we obtain (3).

Observe that when $x \in C_k = B_k \setminus B_{k-1}, k > k_0$, the choice of $k_0$ implies $R < \frac{1}{2} |x|$. We also note that $\text{supp} \varphi \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$, then for any $y \in B(0, R)$, we have $t \geq |x - y| \geq \frac{1}{2} |x|$. Hence, by inequality (3), we deduce
\[ (g_\beta(a)(x))^2 \leq C \left( R^{n+\beta} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{-1/q} \right)^2 \int_{|x|}^\infty \frac{dt}{t^{2n+2\beta+1}} \leq C \left( R^{n+\beta} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{-1/q} \right)^2 \frac{1}{|x|^{2n+2\beta}}. \] (6)

Substituting the above inequality (6) into the term $I_2$, we obtain
\[ I_2 = \sum_{k=k_0+1}^\infty w_1(B_k)^{\alpha p/n} \left( \int_{2^{k-1} \leq |x| \leq 2^k} |g_\beta(a)(x)|^q w_2(x) \, dx \right)^{p/q} \]
\[ \leq C \sum_{k=k_0+1}^{\infty} w_1(B_k)^{\alpha p/n} \left( R^{n+\beta} w_1(B(0,R))^{-\alpha/n} w_2(B(0,R))^{-1/q} \right)^p \]
\[ \cdot \left( \int_{2^{k-1} < |x| \leq 2^k} \frac{w_2(x)}{|x|^{(n+\beta)q}} dx \right)^{p/q} \]
\[ \leq C \sum_{k=k_0+1}^{\infty} \left( \frac{p^{n+\beta}}{2^{k p(n+\beta)}} \right) \left( \frac{w_1(B_k)}{w_1(B(0,R))} \right)^{\alpha p/n} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{p/q}. \]

When \( k > k_0 \), then \( B_k \supseteq B_{k_0} \). Using Lemma B again, we can get
\[ w_i(B_k) \leq C w_i(B_{k_0}) |B_k| |B_{k_0}|^{-1} \text{ for } i = 1 \text{ or } 2. \]

Hence
\[ I_2 \leq C \sum_{k=k_0+1}^{\infty} \left( \frac{p^{k p(n+\beta)}}{2^{k p(n+\beta)}} \right) \left( \frac{2^k n}{2^{k_0 n}} \right)^{\alpha p/n} \left( \frac{2^k n}{2^{k_0 n}} \right)^{p/q} \]
\[ = C \sum_{k=k_0+1}^{\infty} \frac{1}{2^{(k-k_0)(np+\beta p-\alpha p-np/q)}} \]
\[ = C \sum_{k=1}^{\infty} \frac{1}{2^{k(np+\beta p-\alpha p-np/q)}} \]
\[ \leq C, \quad (7) \]
where the last series is convergent since \( \alpha < n(1 - 1/q) + \beta \). Combining the above estimate (7) with (2), we get the desired result.

We are now in a position to give the proof of Theorem 1 for the case \( 0 < p \leq 1 \). For every \( f \in \dot{H}_{q}^{p,\alpha,p}(w_1, w_2) \), then by Theorem C, we have the decomposition \( f = \sum_{j \in \mathbb{Z}} \lambda_j a_j \), where \( \sum_{j \in \mathbb{Z}} |\lambda_j|^p < \infty \) and each \( a_j \) is a central \((\alpha, q, 0; w_1, w_2)\)-atom. Therefore
\[ \|g_\beta(f)\|_{\dot{H}_{q}^{\alpha,p}(w_1, w_2)}^p \leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j \in \mathbb{Z}} |\lambda_j| \|g_\beta(a_j)\|_{L^q_{\nu_2}} \right)^p \]
\[ \leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j \in \mathbb{Z}} |\lambda_j|^p \|g_\beta(a_j)\|_{L^q_{\nu_2}} \right)^p \]
\[ \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \]
\[ \leq C \|f\|_{\dot{H}_{q}^{\alpha,p}(w_1, w_2)}^p. \]
We now consider the case $1 < p < \infty$. Without loss of generality, we may assume that $\text{supp } a_j \subseteq B(0, R_j)$ and $R_j = 2^j$. Write

$$\|g_\beta(f)\|_{K^\alpha,q'}(w_1,w_2) \leq C \sum_{k \in \mathbb{Z}} w_1(B_k) \sum_{j=k-1}^{\infty} |\lambda_j| \|g_\beta(a_j)\chi_k\|_{L^q_{w_2}}^p + C \sum_{k \in \mathbb{Z}} w_1(B_k) \sum_{j=-\infty}^{k-2} |\lambda_j| \|g_\beta(a_j)\chi_k\|_{L^q_{w_2}}^p = I_1' + I_2'. $$

We first deal with $I_1'$. Applying Hölder’s inequality and Theorem D, we have

$$I_1' \leq C \sum_{k \in \mathbb{Z}} w_1(B_k) \sum_{j=k-1}^{\infty} |\lambda_j| \|a_j\|_{L^q_{w_2}}^p \leq C \sum_{k \in \mathbb{Z}} w_1(B_k) \sum_{j=k-1}^{\infty} |\lambda_j|^p w_1(B_j)^{-\alpha p/2n} \left( \sum_{j=k-1}^{\infty} w_1(B_j)^{-\alpha p'/2n} \right)^{p'/p'}.$$

By Lemma B, we thus obtain

$$\left( \sum_{j=k-1}^{\infty} w_1(B_j)^{-\alpha p'/2n} \right)^{p'/p'} \leq C w_1(B_{k-1})^{-\alpha p/2n},$$

which gives

$$I_1' \leq C \sum_{k \in \mathbb{Z}} w_1(B_{k-1})^{\alpha p/2n} \left( \sum_{j=k-1}^{\infty} |\lambda_j|^p w_1(B_j)^{-\alpha p/2n} \right) \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \left( \sum_{k=-\infty}^{j+1} w_1(B_{k-1})^{\alpha p/2n} w_1(B_j)^{-\alpha p/2n} \right).$$

By using Lemma B again, it is easy to show that the above series in the bracket is bounded by a constant which is independent of $j \in \mathbb{Z}$. Hence

$$I_1' \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq C \|f\|_{HK^\alpha,q'(w_1,w_2)}^p. \quad (8)$$

We now turn to estimate $I_2'$. Observe that if $y \in B(0, R_j)$, $j \leq k - 2$ and $x \in C_k = B_k \setminus B_{k-1}$, then we have $|y| \leq \frac{1}{2} |x|$. As before, it follows
immediately from the inequality (6) that

\[ I'_2 \leq C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \left( \sum_{j=\infty}^{k-2} |\lambda_j| \frac{R_j^{n+\beta}}{2^{k(n+\beta)}} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} w_2(B_k)^{1/q} \right)^p \]

\[ = C \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2(B_k)^{p/q} \left( \sum_{j=\infty}^{k-2} |\lambda_j| \frac{R_j^{n+\beta}}{2^{k(n+\beta)}} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q} \right)^p. \]

By using Hölder’s inequality, we obtain that the above expression in the bracket is bounded by

\[ \left( \sum_{j=\infty}^{k-2} |\lambda_j|^p \left( \frac{R_j}{2^k} \right)^{p(n+\beta)/2} w_1(B_j)^{-\alpha p/2n} w_2(B_j)^{-p/2q} \right)^p \times \left( \sum_{j=\infty}^{k-2} \left( \frac{R_j}{2^k} \right)^{p'(n+\beta)/2} w_1(B_j)^{-\alpha p'/2n} w_2(B_j)^{-p'/2q} \right)^{p/p'}. \]

It follows from Lemma B that

\[ \left( \sum_{j=\infty}^{k-2} \left( \frac{R_j}{2^k} \right)^{p'(n+\beta)/2} w_1(B_j)^{-\alpha p'/2n} w_2(B_j)^{-p'/2q} \right)^{p/p'} \leq C \cdot w_1(B_{k-2})^{-\alpha p/2n} w_2(B_{k-2})^{-p/2q}, \]

which implies

\[ I'_2 \leq C \sum_{k \in \mathbb{Z}} w_1(B_{k-2})^{\alpha p/2n} w_2(B_{k-2})^{p/2q} \]

\[ \cdot \left( \sum_{j=\infty}^{k-2} |\lambda_j|^p \left( \frac{R_j}{2^k} \right)^{p(n+\beta)/2} w_1(B_j)^{-\alpha p/2n} w_2(B_j)^{-p/2q} \right) \]

\[ \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \left( \sum_{k=j+2}^{\infty} \left( \frac{R_j}{2^k} \right)^{p(n+\beta)/2} \frac{w_1(B_{k-2})}{w_1(B_j)} \alpha p/2n \left( \frac{w_2(B_{k-2})}{w_2(B_j)} \right)^{p/2q} \right). \]

By using Lemma B again, it is easy to check that the above series in the bracket is bounded by a constant which is independent of \( j \in \mathbb{Z} \). Therefore

\[ I'_2 \leq C \sum_{j \in \mathbb{Z}} |\lambda_j|^p \leq C \|f\|^p_{\mathcal{H}K_\alpha^p(w_1,w_2)}. \quad (9) \]

Summarizing the estimates (8) and (9) derived above, we get the desired result. This completes the proof of Theorem 1. \qed
Proof of Theorem 2. The proof is similar. We only point out the minor differences. Write

\[ \| S_\beta(a) \|_{K^{\alpha,p}(w_1,w_2)}^p = \sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \| S_\beta(a) \chi_k \|_{L^q_{w_2}}^p = \sum_{k=0}^{k_0} w_1(B_k)^{\alpha p/n} \| S_\beta(a) \chi_k \|_{L^q_{w_2}}^p + \sum_{k=k_0+1}^\infty w_1(B_k)^{\alpha p/n} \| S_\beta(a) \chi_k \|_{L^q_{w_2}}^p = J_1 + J_2. \]

Using the same arguments as in the proof of Theorem 1, we can get

\[ J_1 \leq C. \]

Let \( k_0 \) and \( R \) be the same as in Theorem 1. Observe that if \( x \in B_k \setminus B_{k-1} \), \( k > k_0 \) and \( z \in B(0, R) \), then we have \( |z| \leq \frac{1}{2} |x| \). Furthermore, when \( |x - y| < t \) and \( |y - z| < t \), then we can deduce

\[ 2t > |x - z| \geq |x| - |z| \geq \frac{1}{2} |x|. \]

By using the estimate (3), we thus obtain

\[ (S_\beta(a)(x))^2 \leq C \left( R^{n+\beta} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{1/q} \right)^2 \int_{|x|}^\infty \int_{|y-x|<t} \frac{dy dt}{t^{2n+2\beta+1}} \leq C \left( R^{n+\beta} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{1/q} \right)^2 \int_{|x|}^\infty \frac{dt}{t^{2n+2\beta+1}} \leq C \left( R^{n+\beta} w_1(B(0, R))^{-\alpha/n} w_2(B(0, R))^{1/q} \right)^2 \frac{1}{|x|^{2n+2\beta}}. \tag{10} \]

Following the same lines as that of Theorem 1, we can prove

\[ J_2 \leq C. \]

This completes the proof of Theorem 2 for the case \( 0 < p \leq 1 \). As for the case \( 1 < p < \infty \), by the above observation and the estimate (10), we are able to establish our desired result. \( \square \)
5. Proof of Theorem 3

We first establish the following three propositions which will be used in the proof of our theorem.

**Proposition 5.1.** Let \( w \in A_1, 0 < \beta \leq 1 \). Then for any \( j \in \mathbb{Z}_+ \), we have

\[
\| S_{\beta,2j}(a) \|_{L_w^2} \leq C \cdot 2^{jn/2} \| S_{\beta}(a) \|_{L_w^q}.
\]

**Proof.** Since \( w \in A_1 \), then by Lemma A, we have

\[
w(B(y, 2^j t)) = w(2^j B(y, t)) \leq C \cdot 2^j w(B(y, t)) \quad j = 1, 2, \ldots.
\]

Therefore

\[
\| S_{\beta,2j}(a) \|_{L_w^2}^2 = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_{+}^{n+1}} (A_{\beta}(a)(y, t))^2 \chi_{|x-y|<2^j t} \frac{dydt}{t^{n+1}} \right) w(x) dx
\]

\[
= \int_{\mathbb{R}_{+}^{n+1}} \left( \int_{|x-y|<2^j t} w(x) dx \right) (A_{\beta}(a)(y, t))^2 \frac{dydt}{t^{n+1}}
\]

\[
\leq C \cdot 2^{jn} \int_{\mathbb{R}_{+}^{n+1}} \left( \int_{|x-y|<t} w(x) dx \right) (A_{\beta}(a)(y, t))^2 \frac{dydt}{t^{n+1}}
\]

\[
= C \cdot 2^{jn} \| S_{\beta}(a) \|_{L_w^q}^2.
\]

We are done. \( \square \)

**Proposition 5.2.** Let \( w \in A_1, 0 < \beta \leq 1 \) and \( 2 < q < \infty \). Then for any \( j \in \mathbb{Z}_+ \), we have

\[
\| S_{\beta,2j}(a) \|_{L_w^q} \leq C \cdot 2^{jn/2} \| S_{\beta}(a) \|_{L_w^q}.
\]

**Proof.** It is easy to see that

\[
\| S_{\beta,2j}(a) \|_{L_w^q}^2 = \| S_{\beta,2j}(a) \|_{L_w^{q/2}}^2.
\]

Since \( q/2 > 1 \), then we have

\[
\| S_{\beta,2j}(a) \|_{L_w^{q/2}}^2 = \sup_{\| b \|_{L_w^{q/2}} \leq 1} \left| \int_{\mathbb{R}^n} S_{\beta,2j}(a)(x)^2 b(x) w(x) dx \right|
\]

\[
= \sup_{\| b \|_{L_w^{q/2}} \leq 1} \left| \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_{+}^{n+1}} (A_{\beta}(a)(y, t))^{2} \chi_{|x-y|<2^j t} \frac{dydt}{t^{n+1}} \right) b(x) w(x) dx \right| 
\]

\[
= \sup_{\| b \|_{L_w^{q/2}} \leq 1} \left| \int_{\mathbb{R}_{+}^{n+1}} \left( \int_{|x-y|<2^j t} b(x) w(x) dx \right) (A_{\beta}(a)(y, t))^2 \frac{dydt}{t^{n+1}} \right|. 
\]
Denote the weighted maximal operator by $M_w$; that is
$$M_w(f)(x) = \sup_{x \in B \wedge w(B)} \frac{1}{|B|} \int_B |f(y)| w(y) \, dy.$$ 

By Lemma A, we can get
$$\int_{|x-y|<2^{j}t} b(x)w(x) \, dx \leq C \cdot 2^{jn} \int_{B(y,2^{j}t)} b(x)w(x) \, dx \
\leq C \cdot 2^{jn} \inf_{x \in B(y,2^{j}t)} M_w(b)(x) \leq C \cdot 2^{jn} \sup_{x \in B(y,2^{j}t)} M_w(b)(x).$$

Substituting the above inequality (12) into (11) and using Hölder’s inequality and the $L_{q/(2)'}$ boundedness of $M_w$, we obtain
$$\|S_{\beta,2^{j}}(a)^2\|_{L_{q/2}} \leq C \cdot 2^{jn} \sup_{\|b\|_{L_{q/(2)'}^r} \leq 1} \int_{\mathbb{R}^n} S_{\beta}(a)(x)^2 M_w(b)(x)w(x) \, dx \leq C \cdot 2^{jn} \|S_{\beta}(a)^2\|_{L_{q/2}} \|M_w(b)\|_{L_{q/(2)'}^r} \leq C \cdot 2^{jn} \|S_{\beta}(a)^2\|_{L_{q/2}}^2 = C \cdot 2^{jn} \|S_{\beta}(a)\|_{L_{q}^r}^2.$$ 

This implies the desired estimate. 

**Proposition 5.3.** Let $w \in A_1$, $0 < \beta < 1$ and $1 < q < 2$. Then for any $j \in \mathbb{Z}_+$, we have
$$\|S_{\beta,2^{j}}(a)^2\|_{L_{q/2}} \leq C \cdot 2^{jn/q} \|S_{\beta}(a)\|_{L_{q}^r}.$$ 

**Proof.** We will adopt the method given in [16, page 315-316]. Set $\Omega_{\lambda} = \{x \in \mathbb{R}^n : S_{\beta}(a)(x) > \lambda\}$ and $\Omega_{\lambda,j} = \{x \in \mathbb{R}^n : S_{\beta,2^{j}}(a)(x) > \lambda\}$. We also set $\Omega_{\lambda,j}^* = \{x \in \mathbb{R}^n : M_w(\chi_{\lambda,j})(x) > 2^{j+1}\}$. Observe that $w(\Omega_{\lambda,j}) \leq w(\Omega_{\lambda,j}^*) + w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda,j}^*))$.

Thus
$$\|S_{\beta,2^{j}}(a)^2\|_{L_{q/2}} = \int_{\mathbb{R}^n} q\lambda^{q-1}w(\Omega_{\lambda,j}) \, d\lambda \leq \int_{\mathbb{R}^n} q\lambda^{q-1}w(\Omega_{\lambda,j}^*) \, d\lambda + \int_{\mathbb{R}^n} q\lambda^{q-1}w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_{\lambda,j}^*)) \, d\lambda = \text{I} + \text{II}.$$
The weighted weak type estimate of $M_w$ implies
\[ I \leq C \cdot 2^{jn} \int_0^\infty q\lambda^{q-1} w(\Omega_\lambda) \, d\lambda \leq C \cdot 2^{jn} \| S_\beta(a) \|_{L^q_w}^q. \] (13)

To estimate $II$, we now claim that
\[ \int_{\mathbb{R}^n \setminus \Omega_\lambda^*} S_{\beta,2j}(a)(x)^2 w(x) \, dx \leq C \cdot 2^{jn} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_{\beta}(a)(x)^2 w(x) \, dx. \] (14)

It follows from Chebyshev’s inequality and the inequality (14) that
\[ w(\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) \leq \lambda^{-2} \int_{\Omega_{\lambda,j} \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} S_{\beta,2j}(a)(x)^2 w(x) \, dx \]
\[ \leq \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_{\beta,2j}(a)(x)^2 w(x) \, dx \]
\[ \leq C \cdot 2^{jn} \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_{\beta}(a)(x)^2 w(x) \, dx. \]

Hence
\[ II \leq C \cdot 2^{jn} \int_0^\infty q\lambda^{q-1} \left( \lambda^{-2} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_{\beta}(a)(x)^2 w(x) \, dx \right) \, d\lambda. \]

Changing the order of integration yields
\[ II \leq C \cdot 2^{jn} \int_{\mathbb{R}^n} S_{\beta}(a)(x)^2 \left( \int_{|S_{\beta}(a)(x)|} q\lambda^{q-3} \, d\lambda \right) w(x) \, dx \]
\[ \leq C \cdot 2^{jn} \frac{q}{2-q} \| S_{\beta}(a) \|_{L^q_w}^q. \] (15)

Combining the above estimate (15) with (13), we complete the proof of Proposition 5.3.

So it remains to prove the inequality (14).

Set $\Gamma_{2j}(\mathbb{R}^n \setminus \Omega_\lambda^*) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda^*} \Gamma_{2j}(x)$ and $\Gamma(\mathbb{R}^n \setminus \Omega_\lambda) = \bigcup_{x \in \mathbb{R}^n \setminus \Omega_\lambda} \Gamma(x)$. For each given $(y, t) \in \Gamma_{2j}(\mathbb{R}^n \setminus \Omega_\lambda^*)$, by Lemma A, we thus have
\[ w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)) \leq C \cdot 2^{jn} w(B(y, t)). \]

It is easy to check that $w(B(y, t) \cap \Omega_\lambda) \leq \frac{1}{2} w(B(y, t))$ and $\Gamma_{2j}(\mathbb{R}^n \setminus \Omega_\lambda^*) \subseteq \Gamma(\mathbb{R}^n \setminus \Omega_\lambda)$. Hence
\[ w(B(y, t)) = w(B(y, t) \cap \Omega_\lambda) + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)) \]
\[ \leq \frac{1}{2} w(B(y, t)) + w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)), \]

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which implies
\[ w(B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*) \leq C \cdot 2^{jn} w(B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda)). \]

Therefore
\[
\int_{\mathbb{R}^n \setminus \Omega_\lambda^*} S_{\beta, 2^j}(a)(x)^2 w(x) \, dx \\
= \int_{\Gamma_{2^j}(\mathbb{R}^n \setminus \Omega_\lambda^*)} \left( \int_{B(y, 2^j t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} w(x) \, dx \right) \left( A_\beta(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\
\leq C \cdot 2^{jn} \int_{\Gamma(\mathbb{R}^n \setminus \Omega_\lambda)} \left( \int_{B(y, t) \cap (\mathbb{R}^n \setminus \Omega_\lambda^*)} w(x) \, dx \right) \left( A_\beta(y, t) \right)^2 \frac{dy dt}{t^{n+1}} \\
\leq C \cdot 2^{jn} \int_{\mathbb{R}^n \setminus \Omega_\lambda} S_\beta(a)(x)^2 w(x) \, dx.
\]

We are done. \( \square \)

We are now in a position to give the proof of Theorem 3.

**Proof of Theorem 3.** From the definition, we readily see that
\[
g_{\lambda, \beta}^*(a)(x)^2 = \int_{\mathbb{R}^n_{+1}} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(a)(y, t))^2 \frac{dy dt}{t^{n+1}} \\
= \int_0^\infty \int_{|x - y| < t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(a)(y, t))^2 \frac{dy dt}{t^{n+1}} \\
\quad + \sum_{j=1}^\infty \int_0^\infty \int_{2^{j-1} t \leq |x - y| < 2^j t} \left( \frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(a)(y, t))^2 \frac{dy dt}{t^{n+1}} \\
\leq C \left[ S_\beta(a)(x)^2 + \sum_{j=1}^\infty 2^{-j\lambda n} S_{\beta, 2^j}(a)(x)^2 \right]. \tag{16}
\]

We write
\[
\|g_{\lambda, \beta}^*(a)\|_{K_q^{p,\alpha,n}(w_1,w_2)}^p = \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \|g_{\lambda, \beta}^*(a) \chi_k\|_{L_{w_2}^p}^p \\
\quad + \sum_{k=k_0+1}^\infty w_1(B_k)^{\alpha p/n} \|g_{\lambda, \beta}^*(a) \chi_k\|_{L_{w_2}^p}^p \\
= K_1 + K_2.
\]
Applying Proposition 5.1–5.3 and the inequality (16), we obtain

\[ K_1 \leq C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \left\{ \|S_\beta(a)\chi_k\|_{L_{q,w_2}^q}^p + \left( \sum_{j=1}^{\infty} 2^{-\frac{j\alpha n}{2}} \|S_{\beta,2j}(a)\chi_k\|_{L_{q,w_2}^q} \right)^p \right\} \]

\[ \leq C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \left\{ \|S_\beta(a)\|_{L_{q,w_2}^q}^p + \left( \sum_{j=1}^{\infty} 2^{-\frac{j\alpha n}{2}} 2^{jn} \|S_\beta(a)\|_{L_{q,w_2}^q} \right)^p \right\} \]

\[ \leq C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \|S_\beta(a)\|_{L_{q,w_2}^q}^p \left( 1 + \left( \sum_{j=1}^{\infty} 2^{-\frac{j\alpha n}{2}} 2^{jn} \right)^p \right). \]

Since \( \lambda > 3 \), then the last series in the bracket is convergent. Hence, by Theorem D, we have

\[ K_1 \leq C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \|a\|_{L_{k,w_2}^q}^p. \]

Following the same lines as in Theorem 1, we can prove

\[ K_1 \leq C. \] (17)

Let \( k_0 \) and \( R \) be the same as in Theorem 1. By a simple calculation, we know that for any given \((y,t) \in \Gamma_{2j}(x)\), \( x \in B_k \setminus B_{k-1} \), \( k > k_0 \), then \( t \geq \frac{1}{2^{j+2} |x|} \). It follows from the inequality (3) that

\[ \left| S_{\beta,2j}(a)(x) \right|^2 \]

\[ \leq C \left( R^{n+\beta} \|a\|_{L_{q,w_2}^q} \|w_2(B(0,R))^{-1/q} \right)^2 \int_{|y-x|<2^{j+2}t} \int_{|x|/2^{j+2}} \frac{dy dt}{t^{2n+2\beta+n+1}} \]

\[ \leq C \left( R^{n+\beta} \|a\|_{L_{q,w_2}^q} \|w_2(B(0,R))^{-1/q} \right)^2 2^{jn} \cdot \int_{|x|/2^{j+2}} \int_{|y-x|<2^{j+2}t} dt \]

\[ \leq C \left( R^{n+\beta} \|a\|_{L_{q,w_2}^q} \|w_2(B(0,R))^{-1/q} \right)^2 2^{j(3n+2\beta)} |x|^{-2n-2\beta}. \]

Consequently

\[ \|S_{\beta,2j}(a)\chi_k\|_{L_{q,w_2}^q} \leq C \cdot R^{n+\beta} \|a\|_{L_{q,w_2}^q} \|w_2(B(0,R))^{-1/q} \cdot 2^{j(3n+2\beta)} \]

\[ \cdot \left( \int_{2^{k-1} \leq |x| \leq 2^k} \frac{w_2(x)}{|x|^{(n+\beta)q}} dx \right)^{1/q} \]

\[ \leq C \cdot 2^{j(3n+2\beta)/2} 2^{-k(n+\beta)} R^{n+\beta} \|a\|_{L_{q,w_2}^q} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{1/q}. \]
Hence
\[
\sum_{j=1}^{\infty} 2^{-\frac{j\lambda n}{2}} \|S_{\beta,j}(a)\chi_k\|_{L^q_w}
\leq C \sum_{j=1}^{\infty} 2^{-\frac{j\lambda n}{2} j(3n+2\beta)} 2^{-k(n+\beta)} R^{n+\beta} \|a\|_{L^q_w} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{1/q}
= C 2^{-k(n+\beta)} R^{n+\beta} \|a\|_{L^q_w} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{1/q} \sum_{j=1}^{\infty} 2^{-\frac{j(\lambda n - 3n - 2\beta)}{2}}
\leq C 2^{-k(n+\beta)} R^{n+\beta} \|a\|_{L^q_w} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{1/q},
\]
(18)
where the last series is convergent since \(\lambda > 3 + (2\beta)/n\). Substituting the above inequality (18) into the term \(K_2\), we thus obtain
\[
K_2 \leq C \sum_{k=k_0+1}^{\infty} w_1(B_k)^{\alpha p/n} \left\{ \|S_\beta(a)\chi_k\|_{L^p_w}^p + \left( \sum_{j=1}^{\infty} 2^{-\frac{j\lambda n}{2}} \|S_{\beta,j}(a)\chi_k\|_{L^q_w} \right)^p \right\}
\leq C \sum_{k=k_0+1}^{\infty} w_1(B_k)^{\alpha p/n} \left\{ 2^{-k p(n+\beta)} R^{p(n+\beta)} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{p/q} \|a\|_{L^q_w}^p \right\}
+ 2^{-k p(n+\beta)} R^{p(n+\beta)} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{p/q} \|a\|_{L^q_w}^p \left\{ 2^{-\frac{kp(n+\beta)}{2}} \left( \frac{w_1(B_k)}{w_1(B(0,R))} \right)^{\alpha p/n} \left( \frac{w_2(B_k)}{w_2(B(0,R))} \right)^{p/q} \right\}.
\]
(19)
The rest of the proof is exactly the same as that of Theorem 1, we can get
\[
K_2 \leq C.
\]
Therefore, we conclude the proof of Theorem 3 for the case \(0 < p \leq 1\) by combining the above estimates (17) and (19). Finally, by using the same arguments as in Theorem 1, we can also prove the case of \(1 < p < \infty\). We leave the details to the readers.

Remark. The corresponding results for non-homogeneous weighted Herz-type Hardy spaces can also be proved by atomic decomposition theory. The arguments are similar, so the details are omitted here.
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