A DYNAMICAL-TOPOLOGICAL OBSTRUCTION FOR SMOOTH
ISOMETRIC EMBEDDINGS OF RIEMANNIAN MANIFOLDS VIA
INCOMPRESSIBLE EULER EQUATIONS

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Abstract. We obtain a dynamical-topological obstruction for the existence of isometric embedding of a Riemannian manifold-with-boundary \((M, g)\): if the first real homology of \(M\) is nontrivial, if the centre of the fundamental group is trivial, and if \(M\) is isometrically embedded into a Euclidean space of dimension at least 3, then the isometric embedding must violate a certain dynamical, kinetic energy-related condition (the “rigid isotopy extension property” in Definition 1.1). The arguments are motivated by the incompressible Euler equations with prescribed initial and terminal configurations in hydrodynamics.

0. Introduction

0.1. The Euler Equation. In his seminal paper [2] in 1966, Arnold interpreted the Euler equation describing the motion of incompressible inviscid flows on a Riemannian manifold-with-boundary \((M, g)\) as the geodesic equation on \(SDiff(M)\), the group of orientation-, volume-, and boundary-preserving diffeomorphisms on \(M\). The Euler equation on \((M, g)\) reads as follows:

\[
\begin{aligned}
\partial_t v + \text{div}(v \otimes v) + \nabla p &= 0 \quad \text{in } [0, T] \times M, \\
\text{div} v &= 0 \quad \text{in } [0, T] \times M, \\
\langle v, \nu \rangle &= 0 \quad \text{on } [0, T] \times \partial M.
\end{aligned}
\]

Here \(v : [0, T] \times M \to TM\) is the velocity, \(p : [0, T] \times M \to \mathbb{R}\) is the pressure, \(\nu\) is the outward unit normal vectorfield on the boundary \(\partial M\), and \(\langle \bullet, \bullet \rangle\) is the inner product on \(\partial M\) induced by the metric \(g\). The infinite-dimensional Lie group \(SDiff(M)\) is equipped with the \(L^2\)-metric. In this note we shall normalise \(T = 1\).

0.2. Two problems on \(SDiff(M)\). Arnold’s viewpoint initiated fruitful researches on the Euler equation via global-geometric and variational methods. One important question is the following

**Two-Point Problem:** Given \(h_0\) and \(h_1 \in SDiff(M)\). Find a smooth curve \(\{\xi_t\}_{t \in [0,1]}\) such that \(\xi_0 = h_0, \xi_1 = h_1\), and that the action

\[
J(\{\xi_t\}) := \frac{1}{2} \int_0^1 \left\| \frac{\partial \xi_t(\bullet)}{\partial t} \right\|_{L^2}^2 \, dt
\]

is finite.

Without loss of generality, we shall take \(h_0 = \text{id}\) throughout the paper. We call the diffeomorphism \(h_1 \in SDiff(M)\) **attainable** if and only if the two-point problem is soluble for \(h_1\). In this case, it is natural to furthermore consider:
**Classical Variational Problem:** Given \( h_0 = \text{id}, h_1 \in \text{SDiff}(M) \). Find a smooth curve \( \{ \xi_t \}_{t \in [0,1]} \) such that \( \xi_0 = h_0, \xi_1 = h_1 \), and that \( \{ \xi_t \} \) minimises the action \( J(\{ \xi_t \}) \).

The above problems are essentially different from the usual Cauchy problem. They are concerned with whether two prescribed configurations of an incompressible fluid on \( M \) can be nicely deformed into each other inside \( \text{SDiff}(M) \). A solution for the classical variational problem will give rise to a smooth solution for the Euler Eq. (1).

### 0.3. Attainable and unattainable diffeomorphisms.

In a series of original works [13, 15, 16] around 1990s, Shnirelman showed that answers to both the two-point problem and the classical variational problem are, in general, negative. More precisely, denote by \( I^n \) the \( n \)-dimensional unit cube \([0,1]^n\). The following result was was proved in [13, 15, 16]:

**Theorem 0.1.** There are unattainable diffeomorphisms on \( I^2 \); in contrast, any diffeomorphism \( h \in \text{SDiff}(I^n) \) is attainable when \( n \geq 3 \). Also, there exists \( h_{\text{BAD}} \in \text{SDiff}(I^{n \geq 3}) \) such that there are no action-minimising curves of diffeomorphisms connecting \( \text{id} \) and \( h_{\text{BAD}} \).

The existence of unattainable diffeomorphisms on \( I^2 \) is closely related to the fact that the diameter of \( \text{SDiff}(I^2) \) with respect to the \( L^2 \)-metric is infinite. This can be proved for general surfaces using symplectic geometric techniques; see Eliashberg–Ratiu [9].

On the other hand, if \( h = h_1 \) is sufficiently close to \( \text{id} = h_0 \) in the Sobolev \( H^s \)-norm for \( s > [n/2] + 1 \), then the two-point and classical variational problems are both soluble. See Ebin–Marsden [8].

### 0.4. Some subsequent developments.

This paper relies crucially on Theorem 0.1. Nevertheless, to put our work into perspectives, we digress briefly to discuss some developments further to the resolutions of the two-point and classical variational problems.

The non-existence of action-minimising paths in \( \text{SDiff}(M) \) suggests that the putative fluid flows which “minimise the kinetic energy” may allow collisions, merges, and splits of particle trajectories. Thus, one may need to forgo the smoothness of \( \text{SDiff}(M) \) and work in measure-theoretic settings. In 1989 Brenier [5] introduced the notion of generalised flows (GF). They are the probability measures on the path space \( C^0(I, M) \), where \( I = [0,1] \). A GF is incompressible if and only if its marginals for all times \( t \) are equal to the normalised Riemannian volume measure on \( M \). Roughly speaking, GFs correspond to measure-valued solutions for the Euler Eq. (1).

Pioneered by [5], variational models for incompressible GFs remain up to date a crucial topic of mathematical hydrodynamics. Many studies have been devoted to understanding the approximations by diffeomorphisms, the rôle of pressure, the optimal transportational aspects, and the numerical algorithms for GFs. We refer the readers to Bernot–Figalli–Santambrogio [4], Ambrosio–Figalli [1], Brenier–Gangbo [6], Eyink [10], and Benamou–Carlier–Nenna [3] among many others papers, as well as the exposition [7] by Daneri–Figalli.

In addition, Arnold’s paper [2] triggered active research activities on the global geometry of diffeomorphism groups. See Misiołek–Preston [12] and the many references cited therein.

### 0.5. Fluid domains other than \( I^n \) or \( T^n \)?

The works [3, 4, 5, 6, 10, 13, 15, 16] cited above pertaining to unattainable diffeomorphisms or generalised flows are all working essentially with \( I^n \), the unit cube, or \( T^n \), the flat torus. (Ambrosio–Figalli [1] obtained results for GF on measure-preserving Lipschitz preimages of \( I^n \) or \( T^n \).) It seems difficult to generalise many of the
results in the above works to general manifolds, as they make crucial use of explicit constructions of fluid flows on \(I^n\) or \(\mathbb{T}^n\).

On the other hand, considerable developments concerning measure-preserving homeomorphisms and diffeomorphisms on manifolds-with-boundary have taken place in topology and dynamics. See, e.g., Fathi [11], Schwartzman [14], and many subsequent works.

1. The main result

Motivated by the discussions in the Introduction [11] we set out to combine various ideas from (A), the topological properties of Lie groups of measure-preserving (also, boundary- and orientation-preserving) homeomorphisms/diffeomorphisms on manifolds-with-boundaries; and (B), the two-point problem for the Euler equation.

Indeed, making use of results on unattainable diffeomorphisms via the mass flow homomorphism as in §5 of Fathi [11] and on symplectomorphism groups by Eliashberg–Ratiu [9], we arrive at a somewhat surprising topological-geometrical result. The surprise mainly comes from the method of the proof.

Let us introduce a dynamical notion as follows. For a compact Riemannian manifold-with-boundary \((M^n, g)\), let

\[ \iota : (M^n, g) \hookrightarrow \mathbb{E}^n \]

be an isometric embedding into the Euclidean space \(\mathbb{E}^n\).

**Definition 1.1.** We say that \(\iota\) has the “rigid isotopy extension property” if for every \(\{\xi_t\}_{t \in I} \subset \text{SDiff}(M, g)\) and for any \(\epsilon > 0\), the following two conditions hold:

- There exists \(\{\hat{\xi}_t\}_{t \in I} \subset \text{SDiff}(\mathbb{E}^n)\) such that \(J(\{\xi_t\}) + \epsilon > J(\{\hat{\xi}_t\})\) and \(\hat{\xi}_t \circ \iota = \iota \circ \xi_t\) for all \(t \in I\).
- For any \(\{\hat{\gamma}_t\}_{t \in I} \subset \text{SDiff}(\mathbb{E}^n)\) such that \(\hat{\gamma}_t \circ \iota = \iota \circ \xi_t\) for all \(t \in I\), one must have \(J(\{\xi_t\}) < J(\{\hat{\gamma}_t\}) + \epsilon\).

**Remark 1.2.** Strictly speaking, in the above definition \(J(\{\xi_t\})\) is computed with respect to \((M, g)\), while \(J(\{\hat{\xi}_t\})\) and \(J(\{\hat{\gamma}_t\})\) are computed with respect to \(\mathbb{E}^n\). But this difference is immaterial here as \(\iota\) is an isometric embedding.

The heuristic idea can be summarised as follows: an isometric embedding \((M, g) \hookrightarrow \mathbb{E}^n\) has the rigid isotopy extension property if and only if any smooth, volume-preserving isotopy on \(M\) can be extended to an ambient isotopy on \(\mathbb{E}^n\) at the expense of arbitrarily little extra kinetic energy; meanwhile, any such extension cannot effectively save any kinetic energy either.

**Example 1.3.** As will be proved in the second item in §5, a handlebody of genus 2 does not possess the rigid isotropy extension property in \(\mathbb{E}^3\).

Recall also that the centre of \(\pi_1(M)\) is the subgroup

\[ Z(\pi_1(M)) = \left\{ \sigma \in \pi_1(M) : \sigma \gamma = \gamma \sigma \text{ for all } \gamma \in \pi_1(M) \right\}. \]

The main result of this note is as follows:

**Theorem 1.4.** Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold-with-boundary. Assume that \((M, g)\) admits an isometric embedding into \(\mathbb{E}^n\) with the rigid isotopy extension property; \(n \geq 3\). Then, if \(H_1(M; \mathbb{R}) \neq \{0\}\), then \(\pi_1(M)\) has nontrivial centre.
Theorem 1.4 provides a characterisation for isometric embeddings of “topologically complex” manifolds-with-boundary from the dynamical perspectives. It can be understood via the following formulation — If \((M, g)\) has nontrivial first real homology and if its fundamental group has trivial centre, then any of its isometric embeddings into Euclidean spaces must satisfy the following condition: one can find an isotopy of \((M, g)\), such that either some of its extensions (to an isotopy of the ambient Euclidean space) costs a lot of extra energy, or some of its extensions saves a lot of energy. In other words, we can always find in the above case an isotopy of \((M, g)\) that is flexible rather than rigid from the energetic point of view. This will be an isotopy that “rotates a lot”. It arises from some generator of the nontrivial real homology of \((M, g)\).

2. Notations and Nomenclatures

The following notations shall be used throughout this note:

- \(I = [0, 1]\) and \(I^n = [0, 1]^n\).
- For a set \(S\), denote by \(\mathring{S}\) its interior (with respect to an obvious topology in context).
- For sets \(S\) and \(R\), \(S \sim R\) denotes the set difference.
- For a group \(G\), \(\mathcal{Z}(G)\) denotes its centre.
- \((M, g)\) is a Riemannian manifold-with-boundary if \(M\) is locally diffeomorphic to \(\mathbb{R}^n\) or \(\mathbb{R}^n_+ = \{x = (x', x^n) : x' \in \mathbb{R}^{n-1}, x^n \geq 0\}\), and \(g\) is a Riemannian metric on \(M\).
- \(\partial M\) is the boundary of the manifold \(M\).
- \(\text{id}\) denotes the identity map that is clear from the context.
- For a topological space \(X\) containing \(\text{id}\), \(X_0\) denotes the identity component of \(X\).
- \(\mathbb{E}^n\) is \(\mathbb{R}^n\) equipped with the Euclidean metric.
- \(\text{Homeo}(M)\) is the group of homeomorphisms on \(M\).
- Let \(\mu\) be a measure on \(M\). \(\text{Homeo}(M; \mu)\) is the \(\mu\)-preserving subgroup of \(\text{Homeo}(M)\), namely that \(\text{Homeo}(M; \mu) := \{\Phi \in \text{Homeo}(M) : \Phi_\# \mu = \mu\}\).

As in [11], we always consider \(\mu\) that is “good” in the sense of Oxtoby–Ulam [17], i.e., \(\mu\) is atomless, its support is the whole of \(M\), and \(\mu(\partial M) = 0\).

- \(\text{Homeo}_0(M; \mu)\) is the identity component of the universal cover of \(\text{Homeo}(M; \mu)\).
- \(\text{Diff}(M)\) is the group of diffeomorphisms on \(M\). When \(\partial M \neq \emptyset\), an element of \(\text{Diff}(M)\) also fixes the boundary (hence the corresponding vectorfields are perpendicular to \(\partial M\)).
- \(\Phi_\#\) and \(\Phi^\#\) denote the pushforward and pullback under \(\Phi\), respectively. They can be defined for suitable maps, maps, tensorfields, etc.
- \(\Omega\) is the Riemannian volume measure on \(M\).
- \(\text{SDiff}(M) := \{\Phi \in \text{Diff}(M) : \Phi_\# \Omega = \Omega\}\).
- A path \(\{\xi_t\}_{t \in I}\) in \(\text{SDiff}(M)\) with \(\xi_0 = \text{id}\) is known as a smooth isotopy connecting \(\text{id}\) to \(\xi_1\). We also write it as \(\{\xi_t\}\). In Fathi [11], Appendix A.5 it is written as \(\{\xi_t\}\), and the group of isotopies is denoted by \(3\mathcal{S}_\infty(M)\). Moreover, \(3\mathcal{S}(M; \mu)\) denotes the subgroup of \(\mu\)-preserving smooth isotopies. Therefore, the two-point problem in §0.2 concerns the existence of smooth isotopies with volume measure-preserving prescribed endpoints.
- \(J\{\xi_t\}\) denotes the action (total kinetic energy) of the isotopy \(\{\xi_t\}\), defined in Eq. (2).
- \(H_k(M; \mathbb{R})\) and \(H^k(M; \mathbb{R})\) denote the \(k\)-th \(\mathbb{R}\)-coefficient homology and cohomology groups of \(M\), respectively.

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\(\pi_1(M)\) is the fundamental group of \(M\).

- For a vectorfield \(X\) and a differential form \(\beta\) on \(M\), \(X \cdot \beta\) denotes the interior multiplication of \(X\) to \(\beta\).
- Unless otherwise specified, \([\bullet]\) designates an equivalence class. For example, for a closed \(k\)-form \(\beta\) on \(M\), \([\beta]\) denotes the corresponding cohomology class in \(H^k(M; \mathbb{R})\).
- \(T^1 = \mathbb{R}/2\pi\mathbb{Z} = S^1\). We write \(T^1\) to emphasise that the group operation is additive.
- For a map \(f : X \to T^1\) with \(f(0) = 0\), we write \(\widetilde{f}\) for its lift to the universal \(\mathbb{R}\) with \(\widetilde{f}(0) = 0\).
- We reserve the symbol \(\theta : \text{Homeo}_0(M, \mu) \to H_1(M; \mathbb{R})\) for the mass flow homomorphism on \(M\); see \(\S 3\).
- \(\tilde{\theta}\) is the lift of \(\theta\); see \(\S 3\) too.

It is important to point out the following.

**Remark 2.1.** In the definition of the diffeomorphism group \(\text{Diff}(M)\) and its subgroup \(\text{SDiff}(M)\), for a manifold-with-boundary \(M\), we do not require an element to be smooth up to the boundary. In fact, we only require interior smoothness and continuity up to the boundary. This is the setting of Shnirelman’s work \([16]\), which will justify the construction of \(S_\infty\) in Eq. \((8)\).

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3. Mass flow

An essential ingredient of our developments is the mass flow homomorphism defined by Fathi in \(\S 5\), \([11]\). The idea had also appeared in \([14]\) by Schwartzman, under the name of “asymptotic cycles”. We summarise here the definition and some properties of the mass flow, which will be needed for the proof of Theorem \([14]\) in \(\S 4\).

Let \(M\) be a compact metric space. Given any good measure \(\mu\) on \(M\) (in the sense of Oxtoby–Ulam \([17]\)), one can define a natural map

\[\tilde{\theta} : \text{Homeo}_0(M, \mu) \to \text{Hom} \left(\left[ M, T^1 \right] ; \mathbb{R} \right),\]

where \(\left[ M, T^1 \right] \) is the homotopy class of maps from \(M\) to \(T^1\). When \(M\) is a compact manifold-with-boundary, we have

\[\text{Hom} \left(\left[ M, T^1 \right] ; \mathbb{R} \right) \cong H_1(M; \mathbb{R}),\]

as well as

\[\text{Homeo}_0(M, \mu) = \mathcal{S}(M, \mu)/\sim,\]

with the equivalence relation \(\{h_t\} \sim \{k_t\}\) if and only if \(h_0 = k_0 = \text{id}, h_1 = k_1 = e,\) and there is a continuous map \(H : I \times I \to \text{Homeo}(M, \mu)\) such that \(H_{0,s} = \text{id}, H_{1,s} = e, H_{t,0} = h_t,\) and \(H_{t,1} = k_t.\) The space \(\mathcal{S}(M, \mu)/\sim\) is equipped with the quotient topology derived from the compact-open topology on the isotopy group.

To define \(\tilde{\theta}\) in Eq. \((3)\), take any \(\{h_t\} \in \mathcal{S}(M, \mu);\) then for any \(f \in C^0 \left( M, T^1 \right)\) we have a homotopy \((f \circ h_t - f) : M \to T^1\) which equals 0 at \(t = 0.\) So one may lift it in a unique way to a function \(\tilde{f} \circ h_t - \tilde{f} : M \to \mathbb{R},\) with \(\tilde{f} \circ h_0 = \tilde{f} = 0.\) We set

\[\tilde{\theta}(\{h_t\})(f) := \int_M \tilde{f} \circ h_1 - \tilde{f} \, d\mu.\]
One can easily check that $\tilde{\theta}$ induces a homomorphism as in Eq. (3), and that $\tilde{\theta}$ is continuous when $\text{Hom}\left([M, T^1]; \mathbb{R}\right)$ is endowed with the weak topology.

Write $\mathcal{N}(M; \mu)$ for the kernel of the universal covering $\text{Homeo}_0(M; \mu) \to \text{Homeo}_0(M; \mu)$. It is identified with the loop space:

$$\mathcal{N}(M; \mu) \cong \left\{\{h_t\} \in \mathcal{I}(M; \mu)/\sim; h_1 = \text{id}\right\} \cong \left\{h_t \in \text{Homeo}_0(M; \mu): h_1 = \text{id}\right\}.$$ 

For $M$ being a compact manifold-with-boundary, we set

$$\Gamma := \tilde{\theta}(\mathcal{N}(M; \mu)) \leq H_1(M; \mathbb{R}).$$ (5)

Then $\tilde{\theta}$ declines to a homomorphism

$$\theta : \text{Homeo}_0(M; \mu) \to H_1(M; \mathbb{R}).$$ (6)

We shall call either $\tilde{\theta}$ in Eq. (3) or $\theta$ in Eq. (6) the mass flow homomorphism.

**Remark 3.1.** The integrand $f \circ h_1 - f$ on the right-hand side of Eq. (4) should be understood as $\lim_{t \to 1} (f \circ h_t - f)$. For $f : M \to T^1$ with $M$ being connected, $f \circ h_1 - f$ takes a constant integer value when $h_1 = \text{id}$. Thus, $\Gamma$ is Eq. (5) is nontrivial in general.

When $M$ is a compact Riemannian manifold-with-boundary equipped with a normalised volume form $\Omega$ ($\int_M \Omega = 1$), one may further characterise $\tilde{\theta}$ via smooth isotopies. Indeed, $\tilde{\theta}$ is equivalent via the Poincaré duality to

$$\tilde{V} : \mathcal{I}(M; \Omega) \to H^{n-1}(M; \mathbb{R}),$$

where for $\{h_t\} \in \mathcal{I}(M; \Omega)$ we define $\tilde{V}(\{h_t\})$ as follows: let $X_t \in \Gamma(TM)$ be the vectorfield whose integral flow is $h_t$. Then

$$\tilde{V}(\{h_t\}) := \int_0^1 X_t \cdot \Omega \, dt,$$

where $\cdot \Omega$ denotes the interior multiplication.

Finally, let us present here Proposition 5.1 in [11], attributed therein to D. Sullivan. Consider the homomorphisms:

$$a_M : [M, T^1] \to H^1(M; \mathbb{Z}), \quad f \mapsto f^\# \sigma;$$

$$b_M : H_1(M; \mathbb{Z}) \to \text{Hom}(H^1(X; \mathbb{Z}); \mathbb{Z}), \quad \text{the natural duality};$$

$$h_M : \pi_1(M) \to H_1(M; \mathbb{Z}), \quad \text{the Hurewicz map},$$

and define

$$\alpha_M := a_M^* \circ b_M \circ h_M.$$ 

Then, whenever $M$ is a connected compact metric space, it holds that

$$\Gamma \subset \alpha_M\left(\mathcal{Z}(\pi_1(M))\right).$$ (7)

4. Proof of the main Theorem 1.4

Our proof has three key components:

1. the mass flow homomorphism (3);
(2) Shnirelman’s proof for the existence of unattainable diffeomorphisms on $I^2$ (cf. Theorem 0.1 and 16); 
(3) control of the action $J$ by the mass flow (cf. Appendix A, Eliashberg–Ratiu 9).

Proof of Theorem 1.4. Let $(M, g)$ be a compact Riemannian manifold-with-boundary of dimension $n$. Without loss of generality, we may assume that $M$ is connected.

Suppose for contradiction that $\pi_1(M)$ has trivial centre, $H_1(M; \mathbb{R}) \neq \{0\}$, and that there is an isometric embedding $\iota : (M, g) \hookrightarrow \mathbb{R}^n$.

Let $\{B_j = B(x_j, \rho_j)\}_{j=1}^\infty$ be a countable collection of disjoint Euclidean $n$-balls inside $\mathbb{R}^n$, such that $x_j$ has an accumulation point on the boundary $\partial I^n$.

Consider the mass flow homomorphism in Eq. (6):
$$\theta : \text{Homeo}_0(M, \mu) \to H_1(M; \mathbb{R})/\Gamma,$$

where $\mu$ is the normalised volume measure on $(M, g)$, and $\text{Homeo}_0(M, \mu)$ is the identity component of the group of $\mu$-preserving homeomorphisms. In this case, $\mu$ is the Lebesgue measure restricted on $M$, which is clearly good in the Oxtoby–Ulam sense [17]. Under the assumption that $\pi_1(M)$ has trivial centre, $\Gamma$ is trivial in view of Eq. (7). In fact, $\theta$ is a group epimorphism; see Fathi [11], §5.

Moreover, as $M$ is a smooth manifold-with-boundary, we can choose the generators of $H_1(M; \mathbb{R})$ to be smooth loops with tubular neighbourhoods in $M$. For a generator $[\sigma] \in H_1(M; \mathbb{R})$ and any $j \in \mathbb{N}$, let us select $h_j \in \text{Homeo}_0(M, \mu)$ such that $\theta(h_j) = N(j)[\sigma]$,

where $N(j)$ is a large number to be specified later. Thus $\|\theta(h_j)\| = c_1 N(j)$ with $c_1$ depending only on $[\sigma]$.

Here and hereafter, the norm $\|\cdot\|$ is the quotient norm on cohomology/homology groups induced by the usual $L^1$-norm on differential forms. We shall fix $\sigma$ once and for all.

Let $\iota$ be an isometric embedding of $(M, g)$ into the open Euclidean ball $\mathbb{B}(0, R) \subset \mathbb{R}^n$. Let $O_j$ be suitable compositions of homotheties and translations on $\mathbb{R}^n$ such that

$$O_j \circ \iota(M) \subset B_j$$

for each $j$. Note that the Jacobian of $O_j$ is $(\rho_j/R)^n$.

Then we set

$$S_\infty(x) := \begin{cases} x & \text{if } x \in I^n \sim \bigcup_{j=1}^\infty B_j, \\ O_j \# h_j(x) & \text{if } x \in B_j. \end{cases}$$

(8)

It is a volume-preserving and orientation-preserving diffeomorphism on the cube $I^n$ that fixes the boundary $\partial I^n$.

Since the dimension $n \geq 3$ and $S_\infty \in \text{SDiff}(I^n)$, $S_\infty$ is an attainable diffeomorphism by Theorem 0.1 (cf. Shnirelman [15]). That is, there exists an isotopy

$$\{k_t\}_{t \in I} \in J_\infty(I^n; \mu)$$

connecting $\text{id}$ to $S_\infty$ with $J(\{k_t\}) < \infty$; $J$ is the action/total kinetic energy in Eq. (2).
By the construction of $S_{\infty}$ and the assumption that $\iota : (M, g) \hookrightarrow \mathbb{M}^\mathbb{P}$ has the rigid isotopy extension property, for arbitrarily small $\epsilon_0 > 0$ we have

$$J(\{k_t\}) + \epsilon_0 \geq \sum_{j=1}^{\infty} J_j, \quad (9)$$

where

$$J_j = \frac{1}{2} \int_0^1 \int_{B_j} \left| \frac{\partial k_t(x)}{\partial t} \right|^2 \mu \left( \partial \{O_j \# \mu(x)\} \right) \, dt. \quad (10)$$

Note that $k_0 = \text{id}$ and $k_1 = O_j \# h_j$ on each $B_j$. The rigidity assumption on $\iota$ is crucial here; otherwise Eq. (9) fails in general, as one can find on each $B_j$ an ambient isotopy that effectively saves action/total kinetic energy to reach the rescaled configuration compared to $\{k_t\}$. Moreover, again by the rigidity assumption on $\iota$, such $\{k_t\}$ can be obtained via extending an isotopy on $(M, g)$ at the cost of arbitrarily little extra action (not relabelled).

To proceed, we relate the integral of $\frac{\partial k_t}{\partial t}$ to the mass flow homomorphism. The arguments essentially follow from Appendix A in Eliashberg–Ratiu [4]. First note that

$$\int_{B_j} \left| \frac{\partial k_t(x)}{\partial t} \right|^2 \mu \left( \partial \{O_j \# \mu(x)\} \right) = \int_{M} \left| \frac{\partial (O_j \# k_t)}{\partial t} \right| J_{O_j}(y) \, d\mu(y), \quad (11)$$

where $J_{O_j}$ is the Jacobian of $O_j$. Since $O_j$ is a homothety that sends $\iota(M)$ into an $\mathbb{P}$-ball of radius $\rho_j$ (modulo rigid motions in $\mathbb{M}^\mathbb{P}$), it is clear that

$$J_{O_j} \geq c_2(\rho_j)^\mathbb{P} \quad (12)$$

with some universal constant $c_2$ depending only on the volume of $(M, g)$.

Thanks to Eq. (12), we have

$$\int_{M} \left| \frac{\partial (O_j \# k_t)}{\partial t} \right| J_{O_j}(y) \, d\mu(y) \geq c_3 (\rho_j)^\mathbb{P} \int_{M} \left| \frac{\partial (O_j \# k_t)}{\partial t} \right| \, d\mu(y), \quad (13)$$

where $c_3$ again depends only on the volume of $(M, g)$. This follows immediately from the Cauchy–Schwarz inequality.

Now, notice that for any vectorfield $X \in \Gamma(TM)$, it holds that

$$\int_{M} |X| \Omega \geq c_4 \|X \llcorner \Omega\| \quad (14)$$

for a dimensional constant $c_4$, where $\llcorner$ is the interior multiplication and $\Omega$ is the normalised volume form corresponding to $\mu$. We can see this, for example, by computing in local co-ordinates. Let $\{\partial_1, \ldots, \partial_n\} \subset \Gamma(TM)$ be the canonical frame induced from the isometric embedding $\iota$ into Euclidean space, and let $\{dx^1, \ldots, dx^n\}$ be the dual coframe. Then write $X = \sum X^i \partial_i$ and $\Omega = \rho \, dx^1 \wedge \cdots \wedge dx^n$ for scalarfields $X^1, \ldots, X^n$, and $\rho$. It follows that

$$\int_{M} |X| \Omega = \int_{M} \sqrt{\sum_i |X^i|^2 \rho} \, d\mu$$

and

$$X \llcorner \Omega = \sum_i (\rho X^i) \, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$
On the other hand, $\|X \mathcal{L} \Omega\|$ is less than or equal to the $L^1$-norm of $X \mathcal{L} \Omega$. Hence
\[
\|X \mathcal{L} \Omega\| \leq \int_M \rho \left( \sum_i |X_i| \right) \, d\mu,
\]
from which Eq. (14) follows.

Let us take
\[
X = \frac{\partial (O_j^#k_t)}{\partial t}.
\]
An application of the Fubini theorem and Eq. (14) yields that
\[
\int_0^1 \int_M \left| \frac{\partial (O_j^#k_t)}{\partial t}(y) \right| \, d\mu(y) \, dt \geq c_4 \left\| \int_0^1 \left\{ \frac{\partial (O_j^#k_t)}{\partial t} \mathcal{L} \Omega \right\} \, dt \right\|.
\]
(15)

However, recall from §3 that the group homomorphism sending a smooth isotopy to a representitive of the $(n-1)$-cohomology:
\[
\{O_j^#k_t\}_{t \in I} \mapsto \left[ \int_0^1 \left\{ \frac{\partial (O_j^#k_t)}{\partial t} \mathcal{L} \Omega \right\} \, dt \right] \in H^{n-1}(M; \mathbb{R})
\]
is the Poincaré dual of $\tilde{\theta}(\{O_j^#k_t\}_{t \in I})$, where $\tilde{\theta}$ is the lift of the mass flow homomorphism $\theta$. The square bracket over $\int_0^1 \{ \cdots \} \, dt$ denotes an equivalence class in the cohomology group.

Since $\pi_1(M)$ has trivial centre, we can work with $\theta$ and $\tilde{\theta}$ interchangeably, thanks to Proposition 5.1 in [11] (attributed to D. Sullivan).

We also recall that $\theta$-image of the volume-preserving diffeomorphism $h_j \equiv O_j^#k_1$ is $N(j)[\sigma]$, where $[\sigma]$ is the nontrivial cycle in $H_1(M; \mathbb{R})$ chosen before. By the definition of mass flow homomorphism (see Eqs. (3) and (4)), it holds that
\[
\tilde{\theta}(\{O_j^#k_t\}_{t \in I})(\varphi) = \int_M \varphi \circ h_j - \varphi \, d\mu
\]
for each test function $\varphi \in C^0(M, \mathbb{T}^1)$, where $\varphi \circ h_j - \varphi$ is the lift of $\varphi \circ k_0 - \varphi$ from $\mathbb{T}^1$ to the universal cover $\mathbb{R}$ with $\varphi \circ k_0 - \varphi = 0$ (recall that $k_0 = \text{id}$). So
\[
\left\| \tilde{\theta}(\{O_j^#k_t\}_{t \in I}) \right\| \geq c_5 N(j)
\]
with another universal constant $c_5 > 0$, which may depend on $c_1$.

Thus, combining Eqs. (17) and (16), we obtain
\[
\left\| \int_0^1 \left\{ \frac{\partial (O_j^#k_t)}{\partial t} \mathcal{L} \Omega \right\} \, dt \right\| \geq c_5 N(j).
\]
This together with Eqs. (9), (10), (11), (12), (13), and (15) leads to
\[
J(\{k_t\}) + \epsilon_0 \geq \frac{c_3 c_4 c_5}{2} \left( \sum_{j=1}^{\infty} N(j) \cdot (\rho_j)^{1-\frac{1}{\pi}} \right),
\]
where $\epsilon_0 > 0$ is as small as we would like. By choosing
\[
N(j) := (\rho_j)^{-\frac{1}{\pi}} j^{-1}
\]
we get
\[
J(\{k_t\}) = \infty.
\]
which contradicts the attainability of $S_\infty \in \text{SDiff}(I^m)$. The proof is now complete. □

5. Remarks

(1) Consider a rope in $E^3$, i.e., the thickening of a knot by making a small tubular neighbourhood of it solid. It has nontrivial first real homology and its fundamental group has nontrivial centre.

(2) Consider a handlebody of genus 2 in $E^3$ which has nontrivial first real homology. The fundamental group is free on 2-generators and hence has trivial centre. Theorem 1.4 implies that the natural inclusion of the handlebody in $E^3$ does not have the rigid isotopy extension property.

(3) Shnirelman’s “binary star” construction ([16], §2.4, Theorem 2.6) of unattainable diffeomorphisms on the 2-dimensional cube has the rigid isotopy extension property. This is implicit in [16], Lemma 2.5.

(4) The rigid isotopy extension property in Definition 1.1 is a dynamical condition on the isometric embedding $\iota : (M, g) \rightarrow E^n$. To some extent, it is pertaining more to the extrinsic geometry for $\iota$ than to the intrinsic geometry of $(M, g)$.

(5) All the results in this paper apply (as a special case) to closed — namely, compact and without boundary — Riemannian manifolds $(M, g)$.

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References

[1] L. Ambrosio and A. Figalli, Geodesics in the space of measure-preserving maps and plans, Arch. Ration. Mech. Anal. 194 (2009), 421–462
[2] V. I. Arnold, Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966), 319–361
[3] J.-D. Benamou, G. Carlier, and L. Nenna, Generalized incompressible flows, multi-marginal transport and Sinkhorn algorithm, Numer. Math. 142 (2019), 33–54
[4] M. Bernot, A. Figalli, and F. Santambrogio, Generalized solutions for the Euler equations in one and two dimensions, J. Math. Pures Appl. 91 (2009), 137–155
[5] Y. Brenier, The least action principle and the related concept of generalized flows for incompressible perfect fluids, J. Amer. Math. Soc. 2 (1989), 225–255
[6] Y. Brenier and W. Gangbo, $L^p$ approximation of maps by diffeomorphisms, textitCalc. Var. Partial Differential Equations 16 (2003), 147–164
[7] S. Daneri and A. Figalli, Variational models for the incompressible Euler equations, HCDTE lecture notes. Part II. Nonlinear hyperbolic PDEs, dispersive and transport equations, 51 pp., AIMS Ser. Appl. Math., 7, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2013
[8] D. G. Ebin and J. E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. of Math. 92 (1970), 102–163
[9] Y. Eliashberg and T. Ratiu, The diameter of the symplectomorphism group is infinite, *Invent. Math.* **103** (1991), 327–340
[10] G. L. Eyink, Stochastic least-action principle for the incompressible Navier–Stokes equation, *Phys. D* **239** (2010), 1236–1240
[11] A. Fathi, Structure of the group of homeomorphisms preserving a good measure on a compact manifold, *Ann. Sci. École Norm. Sup.* **13** (1980), 45–93
[12] G. Misiołek and S. C. Preston, Fredholm properties of Riemannian exponential maps on diffeomorphism groups, *Invent. Math.* **179** (2010), 191–227
[13] A. I. Shnirelman, The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid (in Russian), *Mat. Sb. (N.S.)* **128**(1985), 82–109
[14] S. Schwartzman, Asymptotic cycles, *Ann. of Math.* **66**(1957), 270–284
[15] A. I. Shnirelman, Attainable diffeomorphisms, *Geom. Funct. Anal.* **3** (1993), 279–294
[16] A. I. Shnirelman, Generalized fluid flows, their approximation and applications, *Geom. Funct. Anal.* **4** (1994), 586–620
[17] J. C. Oxtoby and S. M. Ulam, Measure-preserving homeomorphisms and metrical transitivity, *Ann. of Math.* **42** (1941), 874–920

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