The Diophantine equation $aX^4 - bY^2 = 1$

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Abstract. As an application of the method of Thue-Siegel, we will resolve a conjecture of Walsh to the effect that the Diophantine equation $aX^4 - bY^2 = 1$, for fixed positive integers $a$ and $b$, possesses at most two solutions in positive integers $X$ and $Y$. Since there are infinitely many pairs $(a,b)$ for which two such solutions exist, this result is sharp.

1. Introduction

In a series of papers over nearly forty years, Ljunggren (see e.g. [11], [12], [13], [14] and [15]) derived remarkable sharp bounds for the number of solutions to various quartic Diophantine equations, particularly those of the shape
\[ aX^4 - bY^2 = \pm 1, \]
typically via sophisticated application of Skolem’s $p$-adic method. More recently, there has been a resurgence of interest in Ljunggren’s work; results along these lines are well surveyed in the paper of Walsh [26]. By way of example, using lower bounds for linear forms in logarithms, together with an assortment of elementary arguments, Bennett and Walsh [2] showed that the equation
\[ aX^4 - bY^2 = 1 \]
has at most one solution in positive integers $X$ and $Y$, when $a$ is an integral square and $b$ is a positive integer. For general $a$ and $b$, however, there is no absolute upper bound for the number of integral solutions to (2) available in the literature, unless one makes strong additional assumptions (see e.g. [2], [3], [5], [7], [14], [15] and [22]). This lies in sharp contrast to the situation for the apparently similar equation
\[ aX^4 - bY^2 = -1 \]
where Ljunggren [12] was able to bound the number of positive integral solutions by 2 for arbitrary fixed $a$ and $b$. Moreover, it appears that the techniques employed to treat equation (3) and, in special cases, (2), do not lead to results for (2) in general.

It is our goal in this paper to rectify this situation. To be precise, we will prove the following

**Theorem 1.1.** Let $a$ and $b$ be positive integers. Then equation (2) has at most two solutions in positive integers $(X,Y)$.

This resolves a conjecture of Walsh (see [2], [3], [7] and [26]), which had been suggested by computations and assorted heuristics. Since there are infinitely many pairs $(a,b)$ for which two such solutions exist (see Section 2), this result is best possible.

To prove this, we will appeal to classical results of Thue [21] from the theory of Diophantine approximation, together with modern refinements, particularly those of Evertse [8]. Such an approach, based on Padé approximation to binomial functions, has been used in a number of previous works to explicitly solve Thue inequalities and equations (see e.g. [3], [5], [9], [22], [23], [24]) or to bound the
number of such solutions (see e.g. [1], [8], [10]). We will apply similar techniques to a certain family of quartic inequalities.

2. An Equivalent Problem

Let \(a\) denote a non-square positive integer, and \(b\) a positive integer for which the quadratic equation

\[
aX^2 - bY^2 = 1
\]

is solvable in positive integers \(X\) and \(Y\). Let \((v, w)\) be a pair of positive solutions to (4) so that

\[
\tau = v\sqrt{a} + w\sqrt{b} > 1,
\]

and \(\tau\) is minimal with this property. All solutions in positive integers of (4) are given by

\[
(v^{2k+1}, w^{2k+1}),
\]

where

\[
\tau^{2k+1} = v_{2k+1}\sqrt{a} + w_{2k+1}\sqrt{b} \quad (k \geq 0)
\]

(see [25] for a proof). Solving the quartic equation (2) is thus equivalent to the problem of determining all squares in the sequence \(\{v_{2k+1}\}\). One can find a proof of the following result in [16].

Proposition 2.1. If \(v_{2k+1}\) is a square for some \(k \geq 0\), then \(v_1\) is also a square.

Let us assume that equation (2) is solvable. Proposition 2.1 implies that \(\tau = \tau(a,b)\) is of the form

\[
\tau = x\sqrt{a} + w\sqrt{b},
\]

where \(t = ax^4 - 1\). Thus, for \(k \geq 0\)

\[
\tau^{2k+1} = V_{2k+1}\sqrt{t} + W_{2k+1}\sqrt{t},
\]

where \(V_{2k+1} = \frac{v_{2k+1}}{v_1}\). Hence, by Proposition 2.1, \(v_{2k+1}\) is a square if and only if \(V_{2k+1}\) is a square. In other words, in order to bound the number of positive integer solutions to an equation of the form \(aX^4 - bY^2 = 1\), it is sufficient to determine an upper bound for the number of integer solutions to Diophantine equations of the shape

\[
(t + 1)X^4 - tY^2 = 1.
\]

The main result of [3] is the following.

Proposition 2.2. Let \(m\) be a positive integer. Then the only positive integral solutions to the equation

\[
(m^2 + m + 1)X^4 - (m^2 + m)Y^2 = 1
\]

are given by \((X, Y) = (1, 1)\) and \((X, Y) = (2m + 1, 4m^2 + 4m + 3)\).

In fact, these are the only values of \(t\) for which equation (5) is known to have as many as two positive solutions (suggesting a stronger version of Theorem [11]).

Note that if \(V_3 = z^2\), where \(z\) is a positive integer, then since \(V_3 = 1 + 4t\), we have

\[
4t = z^2 - 1 = (z - 1)(z + 1)
\]

and therefore there exist positive integers \(m\) and \(n\) such that \(t = mn\), \(2m = z - 1\) and \(2n = z + 1\). We conclude, therefore, that \(n = m + 1\) and \(t = m^2 + m\). Proposition 2.2 thus implies the following.
Corollary 2.3. If $V_3$ is a square then for any $k > 1$, $V_{2k+1}$ is not a square and there are only 2 solutions to equation (7) in positive integers $X$ and $Y$.

As it transpires, we will need to account for the possibility of $V_{2k+1}$ being square, for odd values of $k$. The preceding result handles the case $k = 1$. For $k = 3$ and $k = 5$, we will appeal to

Lemma 2.4. If $t > 204$, then neither $V_7$ nor $V_{11}$ is an integral square.

Proof. The equation $z^2 = V_7 = 64t^3 + 80t^2 + 24t + 1$ was treated independently using the function faintp on SIMATH and IntegralPoints on MAGMA, and found to have only the solutions corresponding to $t = 0$ and $t = 1$. For the case $z^2 = V_{11}$, we first put $x = 4t$, and see that the desired result will follow by determining the set of rational points on the curve $z^2 = x^5 + 9x^4 + 28x^3 + 35x^2 + 15x + 1$. The proof now follows exactly as the proof for the case $M^2 = U_{11}$ on pages 8–10 of [4], but with $x = P^2$ and $Q = -1$, as the proof therein does not take into account the fact that $P^2$ is a square..

3. Reduction To A Family Of Thue Equations

We will begin by applying an argument of Togbe, Voutier and Walsh [22] to reduce (5) to a family of Thue equations. We subsequently apply the method of Thue-Siegel to find an upper bound for the number of solutions to this family. Let $P(x, y) = x^4 + 4tx^3y - 6tx^2y^2 - 4t^2xy^3 + t^2y^4$.

The following is a modified version of Proposition 2.1 of [22]. We will include a proof primarily for completeness (and since we will have need of one of the inequalities derived therein).

Proposition 3.1. Let $t$ be a positive integer such that $t \neq m^2 + m$ for all $m \in \mathbb{Z}$. If $(X, Y) \neq (1, 1)$ is a positive integer solution to equation (7), then there is a solution in coprime positive integers $(x, y)$ to the equation

$$P(x, y) = t_1^2,$$

where $t_1$ divides $t$, $t_1 \leq \sqrt{t}$ and $xy > 64t^3$.

Proof. For $k \geq 0$, let us define $\tau$, $V_{2k+1}$ and $W_{2k+1}$ as in Section 2 and choose $T_k$ and $U_k$ to satisfy

$$\tau^{2k} = T_k + U_k \sqrt{t(t+1)}.$$

Assume that $V_{2k+1} = z^2$ for some integer $z > 1$. We will suppose that $k$ is odd, $k = 2n + 1$ say, as the case that $k$ is even is similar and discussed in [22]. When $k = 2n + 1$,

$$V_{4n+1} = z^2 = V_{2n+2}^2 + V_{2n+1}^2 = tU_{n+1}^2 + V_{2n+1}^2,$$

with, via Corollary 2.3, $n > 0$. Thus

$$tU_n^2 - V_{2n+1}^2 = tU_{n+1}^2 = z^2 - (T_n + tU_n)^2.$$

Since $U_{n+1} = 2T_n + (2t+1)U_n$ and $\gcd(U_{n+1}, U_n) = 1$, we have

$$\gcd(U_{n+1}, T_n + tU_n) = 1$$

and hence there exist positive integers $G$, $H$, $t_1$, $t_2$, with $U_{n+1} = 2GH$ and $t = t_1t_2$, such that

$$z - (T_n + tU_n) = 2t_1G^2$$

and

$$z + (T_n + tU_n) = 2t_2H^2.$$
Therefore, \(T_n + tU_n = t_2H^2 - t_1G^2\), and since
\[
2GH = U_{n+1} = 2T_n + (2t + 1)U_n,
\]
we deduce that
\[
U_n = 2GH - 2t_2H^2 + 2t_1G^2
\]
and
\[
T_n = t_2H^2 - t_1G^2 - t(2GH - 2t_2H^2 + 2t_1G^2).
\]
Substituting for \(T_n\) and \(U_n\) in the equation \(T_n^2 - t(t + 1)U_n^2 = 1\), we obtain the equation
\[
t_2^2G^4 - 4tt_1G^3H - 6tG^2H^2 + 4tt_2G^3H + t_2^2H^4 = 1.
\]
Multiplying both sides by \(t_1^2\) and taking \(x = -t_1G, y = H\), we find that \(x\) and \(y\) are coprime positive integers satisfying \(P(x, y) = t_1^2\). To complete the proof, we observe that, since Lemma 2.4 and Corollary 2.3 imply that \(n \geq 3\),
\[
xy = t_1GH = t_1 \frac{1}{2} U_{n+1} \geq \frac{t_1}{2} U_4 > 64t^3.
\]
□

Our focus for the remainder of the paper will be to find, for fixed \(t\), an upper bound upon the number of coprime positive integral solutions to the constrained inequality
\[
0 < P(x, y) \leq t^2, \quad xy > 64t^3.
\]
We should note that for \(t \leq 204\), Theorem 1.1 with \((a, b) = (t + 1, t)\) has been verified in [22]. Here and henceforth, therefore, we will assume that \(t > 204\). To proceed, let \(\xi = \xi(x, y)\) and \(\eta = \eta(x, y)\) be linear functions of \((x, y)\) so that
\[
\xi^4 = 4(\sqrt{-t} + 1)(x - \sqrt{-ty})^4 \quad \text{and} \quad \eta^4 = 4(\sqrt{-t} - 1)(x + \sqrt{-ty})^4.
\]
We call \((\xi, \eta)\), a pair of resolvent forms. Note that
\[
P(x, y) = \frac{1}{8}(\xi^4 - \eta^4)
\]
and if \((\xi, \eta)\) is a pair of resolvent forms then there are precisely three others with distinct ratios, say \((-\xi, \eta), (i\xi, \eta)\) and \((-i\xi, \eta)\). Let \(\omega\) be a fourth root of unity, \((\xi, \eta)\) a fixed pair of resolvent forms and set
\[
z = 1 - \left(\frac{\eta(x, y)}{\xi(x, y)}\right)^4.
\]
We say that the integer pair \((x, y)\) is related to \(\omega\) if
\[
|\omega - \frac{\eta(x, y)}{\xi(x, y)}| < \frac{\pi}{12}|z|.
\]
It turns out that each nontrivial solution \((x, y)\) to (7) is related to a fourth root of unity:

**Lemma 3.2.** Suppose that \((x, y)\) is a positive integral solution to inequality (7), with
\[
|\omega_j - \frac{\eta(x, y)}{\xi(x, y)}| = \min_{0 \leq k \leq 3} |e^{k\pi i/2} - \frac{\eta(x, y)}{\xi(x, y)}|.
\]
Then
\[
|\omega_j - \frac{\eta(x, y)}{\xi(x, y)}| < \frac{\pi}{12}|z(x, y)|.
\]
Proof. We begin by noting that
\[
|z| = \left| \frac{\xi^4 - \eta^4}{\xi^4} \right| = \frac{8P(x, y)}{|\xi^4|},
\]
and, from \(xy \neq 0\),
\[
|\xi^4(x, y)| \geq 4(\sqrt{1 + t})^5,
\]
whereby
\[
|z| \leq \frac{2t^2}{(\sqrt{1 + t})^5} < 1.
\]
Since \(\eta = -\xi\), it follows that
\[
\left| \frac{\eta}{\xi} \right| = 1, \quad |1 - z| = 1.
\]
Now let \(4\theta = \arg\left(\frac{\eta(x, y)^4}{\xi(x, y)^4}\right)\). We have
\[
\sqrt{2 - 2\cos(4\theta)} = |z| < 1,
\]
and so \(|\theta| < \frac{\pi}{12}\). Since
\[
\left| \frac{\omega_j - \eta(x, y)}{\xi(x, y)} \right| \leq |\theta|,
\]
it follows that
\[
\left| \frac{\omega_j - \eta(x, y)}{\xi(x, y)} \right| \leq \frac{1}{4} \frac{|4\theta|}{\sqrt{2 - 2\cos(4\theta)}} |1 - \frac{\eta(x, y)^4}{\xi(x, y)^4}|.
\]
From the fact that \(\frac{|4\theta|}{\sqrt{2 - 2\cos(4\theta)}} < \frac{\pi}{3}\) whenever \(0 < |\theta| < \frac{\pi}{12}\), we obtain inequality (8), as desired. \(\square\)

This lemma shows that each integer pair \((x, y)\) is related to precisely one fourth root of unity. Let us fix such a fourth root, say \(\omega\), and suppose that we have distinct coprime positive solutions \((x_1, y_1)\) and \((x_2, y_2)\) to inequality (7), each related to \(\omega\). We will assume, as we may, that \(|\xi(x_2, y_2)| \geq |\xi(x_1, y_1)|\). For concision, we will write \(\eta_1 = \eta(x_1, y_1)\) and \(\xi_1 = \xi(x_1, y_1)\). Before we move into the heart of our proof, we will mention a pair of results that will be the starting point for our later proving that \((x_1, y_1)\) and \((x_2, y_2)\) are far apart in height.

Since
\[
|z| = \frac{8P(x, y)}{|\xi^4|} \leq \frac{8t^2}{|\xi^4|},
\]
it follows from (8) that
\[
|\xi_1\eta_2 - \xi_2\eta_1| = |\xi_1(\eta_2 - \omega\xi_2) - \xi_2(\eta_1 - \omega\xi_1)| \leq \frac{2\pi t^2}{3} \left( \frac{|\xi_1|}{|\xi_2|} + \frac{|\xi_2|}{|\xi_1|} \right) \leq \frac{4\pi t^2}{3|\xi_1^4|} |\xi_2|.
\]
On the other hand, choosing our fourth root appropriately, we have
\[
\begin{pmatrix}
\sqrt{2}/(\sqrt{t} - 1)^{1/4} & -\sqrt{2}/(\sqrt{t} + 1)^{1/4} \\
\sqrt{2}/(\sqrt{t} + 1)^{1/4} & \sqrt{2}/(\sqrt{t} - 1)^{1/4}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
y_1
\end{pmatrix}
= \begin{pmatrix}
x_2 \\
y_2
\end{pmatrix}
\]
and so
\[
|\xi_1\eta_2 - \xi_2\eta_1| = 4(t + 1)^{1/4} \sqrt{t} |x_1y_2 - x_2y_1|.
\]
Since \( x_1y_2 - x_2y_1 \) is a nonzero integer (recall that we assumed \( \gcd(x_1, y_1) = 1 \)), we have
\[
(11) \quad |\xi_1\eta_2 - \xi_2\eta_1| \geq 4\sqrt{t} (t + 1)^{1/4}
\]
and thus, combining (10) and (11), we conclude that if \((x_1, y_1)\) and \((x_2, y_2)\) are distinct solutions to (4), related to \( \omega \), with \( |\xi(x_2, y_2)| \geq |\xi(x_1, y_1)| \) then
\[
(12) \quad |\xi| > \frac{3}{\pi} t^{-5/4} |\xi_1|^3.
\]
As a final preliminary result, we have the following lemma, whose proof is an immediate consequence of the definition of resolvent forms:

**Lemma 3.3.** If \((x_1, y_1)\) and \((x_2, y_2)\) are two pairs of rational integers then
\[
\frac{\xi(x_1, y_1)\eta(x_2, y_2)}{(-t - 1)^{1/4}}, \quad \xi(x_1, y_1)^3\xi(x_2, y_2) \quad \text{and} \quad \eta(x_1, y_1)^3\eta(x_2, y_2)
\]
are integers in \( \mathbb{Q}(\sqrt{-t}) \).

### 4. Padé Approximation

The main focus of this section is to construct a family of dense approximations to \( \xi/\eta \) from rational function approximations to the binomial function \((1 - z)^{1/4}\).

Consider the system of linear forms
\[
R_r(z) = -Q_r(z) + (1 - z)^{1/4}P_r(z),
\]
where \( R_r(z) = z^{2r+1}R_r(z) \), \( R_r(z) \) is regular at \( z = 0 \), and \( P_r(z) \) and \( Q_r(z) \) are polynomials of degree \( r \). Thue [19], [20] explicitly found polynomials \( P_r(z) \) and \( Q_r(z) \) that satisfy such a relationship, and Siegel [17] identified them in terms of hypergeometric polynomials. Refining the work of Thue and Siegel, Evertse [8] used the theory of hypergeometric functions to sharpen Siegel’s upper bound for the number of solutions to the equation \( f(x, y) = 1 \), where \( f \) is a cubic binary form with positive discriminant. In this paper, we will apply similar arguments to certain quartic forms.

We begin with some preliminaries on hypergeometric functions. A *hypergeometric function* is a power series of the shape
\[
F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha + 1) \cdots (\alpha + n - 1)\beta(\beta + 1) \cdots (\beta + n - 1)}{\gamma(\gamma + 1) \cdots (\gamma + n - 1)n!} z^n.
\]

Here \( z \) is a complex variable and \( \alpha, \beta \) and \( \gamma \) are complex constants. If \( \alpha \) or \( \beta \) is a non-positive integer and \( m \) is the smallest integer such that
\[
\alpha(\alpha + 1) \cdots (\alpha + m)\beta(\beta + 1) \cdots (\beta + m) = 0,
\]
then \( F(\alpha, \beta, \gamma, z) \) is a polynomial in \( z \) of degree \( m \). Furthermore, if \( \gamma \) is a non-negative integer, we will assume that at least one of \( \alpha \) and \( \beta \) is also a non-positive integer, smaller than \( \gamma \).

We note that \( F(\alpha, \beta, \gamma, z) \) converges for \( |z| < 1 \). By a result of Gauss, if \( \alpha, \beta \) and \( \gamma \) are real with \( \gamma > \alpha + \beta \) and \( \gamma, \gamma - \alpha \) and \( \gamma - \beta \) are not non-positive integers, then \( F(\alpha, \beta, \gamma, z) \) converges for \( z = 1 \) and we have
\[
F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.
\]
For future use, it is worth noting that the hypergeometric function $F(\alpha, \beta, \gamma, z)$ satisfies the differential equation

\begin{equation}
(14) \quad z(1-z)\frac{d^2 F}{dz^2} + (\gamma - (1+\alpha +\beta)z)\frac{dF}{dz} - \alpha\beta F = 0.
\end{equation}

Our family of dense approximations to $\xi/\eta$ are as given in the following lemma; their connection to hypergeometric functions will be made apparent later.

**Lemma 4.1.** Let $r$ be a positive integer and $g \in \{0,1\}$. Put

\begin{align*}
A_{r,g}(z) &= \sum_{m=0}^{r} \binom{r - g + \frac{1}{4}}{m} \binom{2r - g - m}{r - g}(-z)^m, \\
B_{r,g}(z) &= \sum_{m=0}^{r-g} \binom{r - \frac{1}{4}}{m} \binom{2r - g - m}{r}(-z)^m.
\end{align*}

(i) There exists a power series $F_{r,g}(z)$ such that for all complex numbers $z$ with $|z| < 1$

\begin{equation}
(16) \quad A_{r,g}(z) - (1-z)^{1/4}B_{r,g}(z) = z^{2r+1-g}F_{r,g}(z)
\end{equation}

and

\begin{equation}
(17) \quad |F_{r,g}(z)| \leq \frac{(r-g+1/4)(r-1/4)}{(2r+1-g)} (1-|z|)^{-\frac{1}{2}(2r+1-g)}.
\end{equation}

(ii) For all complex numbers $z$ with $|1-z| \leq 1$ we have

\begin{equation}
(18) \quad |A_{r,g}(z)| \leq \binom{2r-g}{r}.
\end{equation}

(iii) For all complex numbers $z \neq 0$ and for $h \in \{1,0\}$ we have

\begin{equation}
(19) \quad A_{r,0}(z)B_{r+h,1,1}(z) \neq A_{r+h,1}(z)B_{r,0}(z).
\end{equation}

**Proof.** Put

\begin{align*}
C_{r,g}(z) &= \sum_{m=0}^{r} \binom{r - 1/4}{r-m} \binom{r-g+1/4}{m} z^m, \\
D_{r,g}(z) &= \sum_{m=0}^{r-g} \binom{r - 1/4}{m} \binom{r-g+1/4}{r-g+m} z^m.
\end{align*}

Note that, in terms of hypergeometric functions,

\begin{align*}
A_{r,g}(z) &= \binom{2r-g}{r} F(-1/4-r+g,-r,-2r+g,z), \\
B_{r,g}(z) &= \binom{2r-g}{r-g} F(1/4-r,-r+g,-2r+g,z), \\
C_{r,g}(z) &= \binom{r-1/4}{r} F(-1/4-r+g,-r,3/4,z) \\
\text{and} \\
D_{r,g}(z) &= \binom{r-g+1/4}{r-g} F(1/4-r,-r+g,5/4,z).
\end{align*}

We will begin by proving that

\begin{equation*}
C_{r,g}(z) = A_{r,g}(1 - z), \quad D_{r,g}(z) = B_{r,g}(1 - z).
\end{equation*}
The power series \( F(z) = \sum_{m=0}^{\infty} a_m z^m \) is a solution to the differential equation (14) precisely when
\[
(n + 1)(\gamma + n)a_{n+1} = (\alpha + n)(\beta + n)a_n \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]
Both \( A_{r,g}(1 - z) \) and \( C_{r,g}(z) \) satisfy (14) with \( \alpha = -1/4 - r + g \), \( \beta = -r \), \( \gamma = 3/4 \). Since \( \gamma \) is not a non-positive integer, all coefficients \( a_i \) of power series \( g(z) \) are determined by \( a_0 \). Hence the solution space of (14) is one-dimensional. Therefore, \( A_{r,g}(1 - z) \) and \( C_{r,g}(z) \) are linearly dependent. On equating the coefficients of \( z^r \) in
\[
(1 + z)^{2r+g} = (1 + z)^{-1/4}(1 + z)^{r-g+1/4},
\]
we find that
\[
C_{r,g}(1) = \sum_{m=0}^{\infty} \binom{r - 1/4}{r - m} \binom{r - g + 1/4}{m} = \binom{2r - g}{r} = A_{r,g}(0),
\]
and hence \( C_{r,g}(z) = A_{r,g}(1 - z) \). Similarly, \( D_{r,g}(z) = B_{r,g}(1 - z) \). One can easily observe that \( C_{r,g}(z) \) has positive coefficients. Hence when \( |1 - z| \leq 1 \),
\[
|A_{r,g}(z)| = |C_{r,g}(1 - z)| \leq C_{r,g}(1) = A_{r,g}(0) = \binom{2r - g}{r}.
\]
This proves part (ii) of our lemma.

To prove (15), we define
\[
G_{r,g}(z) = F(r + 1 - g, r + 3/4, 2r + 2 - g, z)
\]
and notice that, for \( |z| < 1 \), the functions \( A_{r,g}(z), (1 - z)^{1/4}B_{r,g}(z) \) and \( z^{2r+1-g}G_{r,g}(z) \) satisfy (14) with \( \alpha = -1/4 - r + g \), \( \beta = -r \), \( \gamma = -2r + g \). Suppose
\[
G_{r,g}(z) = \sum_{m=0}^{\infty} g_m z^m.
\]
We have \( g_0 = 1 \) and, for \( m \geq 0 \),
\[
g_{m+1}g_m = \frac{(r + 1 - g + m)(r + 3/4 + m)}{(m + 1)(2r + 2 - g + m)} \leq \frac{r + 1/2 - g/2 + m}{m + 1} = \frac{(-1)^{m+1}(-r-1/2+g/2)}{(-1)^m(-r-1/2+g/2)m}.
\]
Therefore,
\[
|G_{r,g}(z)| \leq \sum_{m=0}^{r} \binom{-r-1/2+g/2}{m}(-|z|)^m = (1 - |z|)^{-\frac{1}{2}(2r+1-g)}.
\]
Since \( r \geq 1 \) and \( g \in \{0, 1\} \), \( \gamma = -2r + g \) is a negative integer. By (20), If \( F(z) = \sum_{m=0}^{\infty} a_m z^m \) is a solution to (14), then since \( a_0 \) and \( a_{2r-g+1} \) may vary independently, the solution space of (14) is two-dimensional. Therefore, there are constants \( c_1, c_2 \) and \( c_3 \), not all zero, such that
\[
c_1A_{r,g}(z) + c_2(1 - z)^{1/4}B_{r,g}(z) + c_3z^{2r+1-g}G_{r,g}(z) = 0.
\]
Letting \( z = 0 \), since \( A_{r,g}(0) = B_{r,g}(0) \neq 0 \), we find that \( c_1 = -c_2 \neq 0 \). We may thus assume \( c_1 = 1 \). Substituting \( z = 1 \) in above identity thus yields \( c_3 = -\frac{A_{r,g}(z)}{G_{r,g}(z)} \), whence
\[
F_{r,g}(z) = A_{r,g}(1)G_{r,g}(1)^{-1}G_{r,g}(z).
\]
Lemma 5.1. It is clear that

$$A_{r,g}(1)G_{r,g}(1)^{-1} = \binom{r-1/4}{r} \frac{\Gamma(r+1)\Gamma(r+5/4-g)}{\Gamma(2r+2-g)\Gamma(1/4)} = \frac{(r-1/4)(r-g+1/4)}{(2r+1-g)}.$$  

Proof. In order to complete the proof of part (i), note that, by (13), we have

$$P_{r,g}(z) = \frac{\Lambda(z)}{z^{2r+g}},$$

where $$P_{r,g}(z)$$ is a power series. However, the left hand side of the above identity is a polynomial of degree at most $$2r + g$$, and so $$P_{r,g}(z)$$ must be a constant. Letting $$z = 1$$, we obtain that $$P_{r,g}(z)$$ is not 0. Therefore,

$$A_{r,0}(z)B_{r+1,z}(z) - A_{r+1,z}(z)B_{r,0}(z) = 0$$

if and only if $$z = 0$$. □

5. SOME ALGEBRAIC NUMBERS

Combining our polynomials of the previous section with the resolvent forms defined in Section 3 we will consider the complex sequences $$\Sigma_{r,g}$$ given by

$$\Sigma_{r,g} = \frac{\eta_2}{\xi_2}A_{r,g}(z_1) - (-1)^r \eta_1 \xi_1 B_{r,g}(z_1)$$

where $$z_1 = 1 - \eta_1^4/\xi_1^4$$. Define

$$\Lambda_{r,g} = \frac{\xi_1^{4r+1-g}\xi_2^{4r+1-2g}}{(-t-1)^{1/4}} \Sigma_{r,g}.$$

We will show that $$\Lambda_{r,g}$$ is either an integer in $$\mathbb{Q}$$ or a fourth root of such an integer. If $$\Lambda_{r,g} \neq 0$$, this provides a lower bound upon $$|\Lambda_{r,g}|$$. In conjunction with the inequalities derived in Lemma 4.1 this will induce a strong “gap principle”, guaranteeing that solutions to inequality (7) must, in a certain sense, increase rapidly in height.

For a polynomial $$P(z)$$ of degree $$n$$, we will denote by $$P^*(x,y) = x^nP(y/x)$$ an associated binary form. Let $$A_{r,g}$$ and $$B_{r,g}$$ be as in (15) and, as in the proof of Lemma 4.1 set

$$C_{r,g}(z) = A_{r,g}(1-z)$$ and $$D_{r,g}(z) = B_{r,g}(1-z).$$

For $$z \neq 0$$, we have $$D_{r,0}(z) = z^rC_{r,0}(z^{-1})$$, hence

$$A^*_r (z, z + \bar{z}) = z^r A_{r,0} \left(1 + \frac{\bar{z}}{z}\right) = z^r C_{r,0} \left(\frac{-\bar{z}}{z}\right)$$

(21)

$$= (-1)^r z^r D_{r,0} \left(\frac{-\bar{z}}{z}\right) = (-1)^r z^r B_{r,0} \left(1 + \frac{\bar{z}}{z}\right)$$

$$= (-1)^r B^*_{r,0} (\bar{z}, z + \bar{z}) = (-1)^r B^*_{r,0} (z, z + \bar{z}).$$

Lemma 5.1. For any pair of integers $$(x, y)$$, both $$A^*_r (\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))$$ and $$B^*_r (\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))$$ are algebraic integers in $$\mathbb{Q}(\sqrt{-1})$$.

Proof. It is clear that $$A^*_r (\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))$$ and $$B^*_r (\xi^4(x, y), \xi^4(x, y) - \eta^4(x, y))$$ belong to $$\mathbb{Q}(\sqrt{-1})$$; we need only show that they are algebraic integers. From the definitions of $$A^*_r (x, y)$$, $$B^*_r (x, y)$$, $$\xi(x, y)$$ and $$\eta(x, y)$$ (in particular, since
\[\xi^4(x, y) - \eta^4(x, y) = 8P(x, y),\] this is an immediate consequence of Lemma 4.1 of [6], which, in this case, implies that
\[
\binom{a/4}{n}8^n
\]
is, for fixed nonnegative integers \(a\) and \(n\), a rational integer. \(\square\)

We now proceed to show that \(\Lambda_{r,g}\) has the desired property. We have
\[
\Lambda_{r,g} = \frac{\xi_1^{-g}\eta_2}{(-t - 1)^{1/4}} A_{r,g}(\xi^4, \xi^4 - \eta^4) - \frac{(-1)^g\xi_2^2\eta_1}{(-t - 1)^{1/4}} B_{r,g}(\xi^4, \xi^4 - \eta^4).
\]
By Lemmas 3.3, 5.1 and (21), \(\Lambda_{r,g} \in \mathbb{Z}\sqrt{-t}\). Similarly, Lemmas 3.3 and 5.1 imply that \(\Lambda_{r,1}\) is an algebraic integer in \(\mathbb{Q}(\sqrt{-t})\). We claim that it is not a rational integer. To see this, let us start by noting that
\[
\frac{\eta_2/\eta_2}{\xi_2/\xi_2} A_{r,g}(z_1) - (-1)^g\frac{\eta_2/\eta_2}{\xi_2/\xi_2} B_{r,g}(z_1) = \frac{\eta_2/\eta_2}{\xi_2/\xi_2} A_{r,g}(z_1) - (-1)^g\frac{\eta_2/\eta_2}{\xi_2/\xi_2} B_{r,g}(z_1),
\]
where \(\eta = (\sqrt{-t} - 1)^{1/4}\) and \(\xi = (\sqrt{-t} + 1)^{1/4}\). By Lemma 5.1
\[
\frac{\eta_2/\eta_2}{\xi_2/\xi_2} A_{r,g}(z_1) - (-1)^g\frac{\eta_2/\eta_2}{\xi_2/\xi_2} B_{r,g}(z_1) \in \mathbb{Q}(\sqrt{-t})
\]
and so
\[
(22) \quad \mathfrak{f} = \mathbb{Q}(\sqrt{-t}, \Sigma_{r,g}) = \mathbb{Q}(\sqrt{-t}, (-t - 1)^{1/4}\eta_2/\xi_2)
\]
\[
= \mathbb{Q}(\sqrt{-t}, (-t + 1 - 2\sqrt{-t})^{1/4}).
\]
If we choose a complex number \(X\) so that \(\xi(X, 1) = \eta(X, 1)\) then \(X \in \mathfrak{f}\) and
\[
P(X, 1) = \frac{1}{8}(\xi^4(X, 1) - \eta^4(X, 1)) = 0.
\]
Since we have assumed that \(P\) is irreducible, \(X\) and \(\Sigma_{r,g}\) both have degree 4 over \(\mathbb{Q}(\sqrt{-t})\).

Suppose that \(\Lambda_{r,1} \in \mathbb{Z}\). Then we have for some \(\rho, \rho_1 \in \{\pm 1, \pm i\}\), that \(\Lambda_{r,1} = \rho\Lambda_{r,1}\) and \((-t - 1)^{1/4} = \rho_1(-t - 1)^{1/4}\), whence, from Lemma 3.3
\[
\Sigma_{r,1} = (-t - 1)^{1/4}\xi_1^{-4r}\xi_2^{-1}\rho\Lambda_{r,1}
\]
\[
= \xi_1^{-4r}\xi_2^{-1}\eta_2\rho_1\left(\frac{\xi_2}{\eta_2} A_{r,1} \left(1 - \xi_1^4/\eta_1^4\right) - (-1)^g\frac{\xi_2}{\eta_2} B_{r,1} \left(1 - \xi_1^4/\eta_1^4\right)\right)
\]
\[
= \rho\rho_1\xi_1^{4r}\xi_2^{1}\left(A_{r,1} \left(1 - \xi_1^4/\eta_1^4\right) - (-1)^g\frac{\xi_1\eta_2}{\xi_2\eta_1} B_{r,1} \left(1 - \xi_1^4/\eta_1^4\right)\right).
\]
This together with Lemmas 3.3 and 5.1 imply that \(\Sigma_{r,1} \in \mathbb{Q}(\sqrt{-t}, \rho_1)\), which contradicts the fact that \(\Sigma_{r,1}\) has degree 4 over \(\mathbb{Q}(\sqrt{-t})\). We conclude that \(\Lambda_{r,1}\) cannot be a rational integer.

From the well-known characterization of algebraic integers in quadratic fields, we may therefore conclude that, if \(\Lambda_{r,g} \neq 0, g \in \{0, 1\}\), then
\[
(23) \quad |\Lambda_{r,g}| \geq 2^{4r + 1 - \frac{1}{2}}.
\]
We will now combine inequality (23) with upper bounds from Lemma 4.1 to show that solutions to (7) are widely spaced:

**Lemma 6.1.** If \( \Sigma_{r,g} \neq 0 \), then

\[
c_1(r, g) |\xi_1|^{4r+1-g} |\xi_2|^{-3} + c_2(r, g) |\xi_1|^{-4r-3(1-g)} |\xi_2| > 1,
\]

where we may take

\[
c_1(r, g) = \frac{2^{2r+1+g/4}}{\sqrt{\pi r}} t^{5/4+3g/8}
\]

and

\[
c_2(r, g) = \frac{2^{1/2+g/4-2r} \pi^{4r+2-2g}}{\pi \sqrt{r}} t^{4r+5/4-13g/8}.
\]

**Proof.** By (10), we can write

\[
| (t+1)^{1/4} \Lambda_{r,g} | = |\xi_1|^{4r+1-g} |\xi_2| \left( \frac{\eta}{\xi_2} - \omega \right) \Lambda_{r,g}(\xi_1) + \omega \xi_1^{2r+1-g} F_{r,g}(\xi_1).
\]

Since \(|1 - \xi_1| = 1\) and \(|\xi_1| \leq 1\), from (3), (9), (17), (18), and the inequality

\[
|\xi_1|^4 > 4(1+t)^{5/2},
\]

we have

\[
| (t+1)^{1/4} \Lambda_{r,g} | \leq |\xi_1|^{4r+1-g} |\xi_2| \left( \frac{2r-g}{r} |\xi_2| + \frac{(r-g+1/4)}{(r+1-g)} \left( \frac{4}{r} \right) \left( \frac{9}{\xi_1} \right)^{2r+1-g} \right).
\]

Comparing this with (23), we obtain

\[
c_1(r, g) |\xi_1|^{4r+1-g} |\xi_2|^{-3} + c_2(r, g) |\xi_1|^{-4r-3(1-g)} |\xi_2| > 1,
\]

where we may take \(c_1\) and \(c_2\) so that

\[
c_1(r, g) \geq 2^{1+g/4} t^{5/4+3g/8} \left( \frac{2r}{r} \right)
\]

and

\[
c_2(r, g) \geq 2^{g/4} \pi^{4r+2-2g} t^{4r+5/4-13g/8} \left( \frac{r-g+1/4}{r+1-g} \right) \left( \frac{r-1/4}{r} \right) \left( \frac{9}{\xi_1} \right)^{2r+1-g}.
\]

Applying the following version of Stirling’s formula (see Theorem (5.44) of [18])

\[
\frac{1}{2\sqrt{k}} 4^k < \left( \frac{2k}{k} \right) < \frac{1}{\sqrt{\pi k}} 4^k,
\]

(valid for \(k \in \mathbb{N}\)) leads immediately to the stated choice of \(c_1\).

To evaluate \(c_2(r, g)\), we begin by noting that

\[
\left( \frac{2r+1-g}{r} \right) \geq \left( \frac{2r}{r} \right) \geq \frac{4^r}{2\sqrt{r}}.
\]

Next we will show that

\[
\left( \frac{r-g+1/4}{r+1-g} \right) \left( \frac{r-1/4}{r} \right) < \frac{1}{\sqrt{2\pi r}}.
\]
for \( r \in \mathbb{N} \) and \( g \in \{0, 1\} \), whence we may conclude that
\[
\frac{(r-g+1/4)(r-1/4)}{r+1-g} < \frac{\sqrt{2}}{\sqrt{\pi} 4^r}.
\]
This gives the desired value for \( c_2(r, g) \). To bound \((r-g+1/4)(r-1/4)\), first we note that
\[
\left(\frac{r-3/4}{r}\right) > \left(\frac{r+1/4}{r+1}\right).
\]
for \( r \in \mathbb{N} \). Put
\[
X_r = \left(\frac{r-3/4}{r}\right)\left(\frac{r-1/4}{r}\right) = \frac{y_r}{r},
\]
whereby
\[
X_{r+1} = \left(\frac{r+1/4}{r}\right)\left(\frac{r+3/4}{r+1}\right) = \left(\frac{r^2 + r + 2/9}{r^2 + r}\right) \frac{y_r}{r+1}.
\]
Hence,
\[
y_1 = 3/16, \quad y_r = 3/16 \prod_{k=1}^{r-1} \frac{k^2 + k + 3/16}{k^2 + k}.
\]
Since
\[
\prod_{k=1}^{\infty} \frac{k^2 + k + 3/16}{k^2 + k} = \frac{16}{3\Gamma(1/4)\Gamma(3/4)} = \frac{16}{3\sqrt{2\pi}},
\]
we obtain
\[
X_r < \frac{1}{\sqrt{2\pi} r},
\]
which completes the proof.

We will also have need of the following:

**Lemma 6.2.** If \( r \in \mathbb{N} \) and \( h \in \{0, 1\} \), then at most one of \( \{\Sigma_{r,0}, \Sigma_{r+h,1}\} \) can vanish.

**Proof.** Let \( r \) be a positive integer and \( h \in \{0, 1\} \). Following an argument of Bennett [1], we define the matrix \( M \):
\[
M = \begin{pmatrix}
A_{r,0}(z_1) & A_{r+h,1}(z_1) & (-1)^{r+1}\frac{\eta_1}{\xi_1} \\
A_{r,0}(z_1) & A_{r+h,1}(z_1) & (-1)^r\frac{\eta_1}{\xi_1} \\
B_{r,0}(z_1) & B_{r+h,1}(z_1) & \frac{\eta_2}{\xi_2}
\end{pmatrix}.
\]
The determinant of \( M \) is zero because it has two identical rows. Expanding along the first row, we find that
\[
A_{r,0}(z_1)\Sigma_{r+h,1} - A_{r+h,1}(z_1)\Sigma_{r,0} + (-1)^r\frac{\eta_1}{\xi_1}(A_{r,0}(z_1)B_{r+h,1}(z_1) - A_{r+h,1}(z_1)B_{r,0}(z_1))
\]
vanishes and hence if \( \Sigma_{r,0} = \Sigma_{r+h,1} = 0 \), then
\[
A_{r,0}(z_1)B_{r+h,1}(z_1) - A_{r+h,1}(z_1)B_{r,0}(z_1) = 0,
\]
contradicting part (iii) of Lemma [4.1].

Our final result of this section follows similar lines to an argument of Evertse [8]. We show:
Lemma 6.3. Suppose that $t > 204$. For $r \in \{1, 2, 3, 4, 5\}$, we have

$$\Sigma_{r,0} \neq 0.$$ 

Proof. Let $r \in \{1, 2, 3, 4, 5\}$ and suppose that $\Sigma_{r,0} = 0$. From [16], for each $r$, the polynomial

$$A_{r,0}(z) - (1 - z)B_{r,0}^4$$

has a zero at 0 of order at least 2. We can thus find polynomials $A_r(z), B_r(z)$ and $F_r(z) \in \mathbb{Z}[z]$, satisfying

$$A_r(z)^4 - (1 - z)B_r^4 = z^{2r+1}F_r(z).$$

In fact, we have

$$A_1(z) = \frac{4}{3}A_{1,0}(z) = 8 - 5z,$$

$$B_1(z) = \frac{4}{3}B_{1,0}(z) = 8 - 3z,$$

$$F_1(z) = 320 - 320z + 81z^2,$$

$$A_2(z) = \frac{32}{3}A_{2,0}(z) = 64 - 72z + 15z^2,$$

$$B_2(z) = \frac{32}{3}B_{2,0}(z) = 64 - 56z + 7z^2,$$

$$F_2(z) = 86016 - 172032z^2 + 114624z^2 - 28608z^3 + 2401z^4,$$

$$A_3(z) = 128A_{3,0}(z) = 2560 - 4160z + 1872z^2 - 195z^3,$$

$$B_3(z) = 128B_{3,0}(z) = 2560 - 3520z + 1232z^2 - 77z^3,$$

$$F_3(z) = 14057472000 - 4217241600z + 52948365200z^2 - 26679910400z^3$$

$$+ 7150266240z^4 - 389047040z^5 + 35153041z^6,$$

$$A_4(z) = \frac{2048}{5}A_{4,0}(z) = 28672 - 60928z + 42432z^2 - 10608z^3 + 663z^4,$$

$$B_4(z) = \frac{2048}{5}B_{4,0}(z) = 28672 - 53760z + 31680z^2 - 6160z^3 + 231z^4,$$

$$F_4(z) = 13989396348928 - 55957585395712z + 191916125077504z^2$$

$$- 79896826347520z^3 + 39463764078592z^4 - 11050000539648z^5$$

$$+ 1648475542656z^6 - 113348764800z^7 + 2847396321z^8,$$

$$A_5(z) = \frac{8192}{21}A_{5,0}(z) = 98304 - 258048z + 243712z^2 - 99008z^3 + 15912z^4 - 663z^5,$$

$$B_5(z) = \frac{8192}{21}B_{5,0}(z) = 98304 - 233472z + 194560z^2 - 66880z^3 + 8360z^4 - 209z^5.$$ 

and

$$F_5(z) = 12173331812352 - 60866659061760z + 1301756554248192z^2$$

$$- 155502626222208z^3 + 1136607561252864z^4 - 523630732640256z^5$$

$$+ 15102916176512z^6 - 26204424888320z^7 + 2515441608384z^8$$

$$- 113971885760z^9 + 1908029761z^{10}.$$ 

We also define $A_r^*$ and $B_r^*$ via

$$A_r^*(x,y) = x^r A_r(y/x) \quad \text{and} \quad B_r^*(x,y) = x^r B_r(y/x).$$
Since $\Sigma_{r,0}$ is assumed to be zero,

$$\frac{\eta_1^2}{\xi_2^2} = \frac{\eta_1^4(B_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4}{\xi_1^2(A_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4}.$$

Let $\mathcal{I}$ be the ideal integral in $\mathbb{Q}(\sqrt{-r})$ generated by $\xi_1^4(A_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4$ and $\eta_1^4(B_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4$, and $N(\mathcal{I})$ be the absolute norm of $\mathcal{I}$. Since the ideal generated by $\xi_1^4(A_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4 - \eta_1^4(B_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4$ divides $(\xi_2^2 - \eta_2^2)\mathcal{I}$, we obtain

$$|\xi_1|^{4(4r+1)}|A_r^4(z_1) - (1 - z_1)B_r^4(z_1)| = |\xi_1^4(A_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4 - \eta_1^4(B_r^*(\xi_1^4, \xi_1^4 - \eta_1^4))^4|.$$

From the fact that $\mathcal{I}$ is an imaginary quadratic field,

$$|\xi_1|^{4(4r+1)}|A_r^4(z_1) - (1 - z_1)B_r^4(z_1)| \leq N(\mathcal{I})^{1/2}|\xi_2^4 - \eta_2^4|.$$ 

By (16),

$$A_r^4(z_1) - (1 - z_1)B_r^4(z_1) = z_2^{2r+1}F_r(z_1),$$

so we may conclude

$$|z_1|^{2r+1}|F_r(z_1)| \leq N(\mathcal{I})^{1/2}|\xi_2^4 - \eta_2^4||\xi_1|^{-4(4r+1)},$$

i.e.

$$1 \leq \frac{N(\mathcal{I})^{1/2}|\xi_2^4 - \eta_2^4||\xi_1|^{-4(4r+1)}}{|z_1|^{2r+1}|F_r(z_1)|}.$$

Since $|z_1| = |\xi_1^{-4}||\xi_4^4 - \eta_4^4|$ and $|\xi_4^4 - \eta_4^4| = 8P(x, y)$, it follows after a little work that

$$|\xi_1|^{4r} \leq N(\mathcal{I})^{1/2}|\xi_4^4 - \eta_4^4|^{-4r+1}(8P(x, y))^{2r+1}|F_r(z_1)|^{-1}.$$

To estimate $N(\mathcal{I})^{1/2}$, we choose a finite extension $\mathbb{M}$ of $\mathbb{Q}(\sqrt{-r})$ so that the ideal generated by $\xi_4^4$ and $\xi_4^4 - \eta_4^4$ in $\mathbb{M}$ is a principal ideal, with generator $p$. We denote the extension of $\mathcal{I}$ to $\mathbb{M}$, by $\mathcal{I}'$. Let $\mathcal{I}$ be the ideal in $\mathbb{M}$ generated by $A_r^*(u, v)$ and $B_r^*(u, v)$, where $u = \xi_4^4/p$ and $v = \xi_4^4 - \eta_4^4$. Since $A_r^*(x, y) = B_r^*(y, x - y)$,

$$p^{4r+1}\xi_4^4B_r^*(0, 1)^4 \subset p^{4r+1}\xi_4^4(u, B_r^*(0, v)^4) = p^{4r+1}\xi_4^4(u, v)B_r^*(u, v)^4 \subset p^{4r+1}\xi_4^4(u - v)B_r^*(u, v)^4 \subset p^{4r+1}\xi_4^4(uA^*(u, v)^4, (u - v)B_r^*(v, u)^4) = \mathcal{I}'_1,$$

where $(m_1, \ldots, m_n)$ denote the ideal in $\mathbb{M}$ generated by $m_1, \ldots, m_n$.

We have

$$A_r^1(x, y) - B_r^1(x, y) = -2y.$$

Therefore,

$$2(v) \subset (A_r^1(u, v), B_r^1(u, v)) \subset r_1,$$

where $(v)$ is the ideal generated by $v$ in $\mathbb{M}$. Since $B^1(0, 1) = -3$, it follows from (25) that

$$1296(\xi_4^4 - \eta_4^4)^5 \subset 1296p(\xi_4^4 - \eta_4^4)^4 = p^516v^4B_r^1(0, 1)^4 \subset \mathcal{I}'_1.$$

For $r = 2$, we first observe that

$$B^1(x, y)A^2_2(x, y) - A^1_2(x, y)B^2_2(x, y) = -10y^3$$

and

$$(-32x + 7y)A^2_2(x, y) - (-32x + 15y)B^2_2(x, y) = 80xy^2.$$
Therefore, by (25) we have

$$80(v^2) \subset (-10v^2, 80uv^2) \subset (A^*_v(u, v), B^*_v(u, v)) \subset r_2.$$ 

Since $B^*_v(0, 1) = 7$, we have

$$80^4 \times 7^4(\xi^4_1 - \eta^4_1)^9 \subset 80^4 \times 7^4 p(\xi^4_1 - \eta^4_1)^8 = 80^4 p^9 v^8 B^*_v(0, 1)^4 \subset r_2.$$ 

When $r = 3$, we have

$$B^*_v(x, y) A^*_v(x, y) - A^*_v(x, y) B^*_v(x, y) = -210y^5$$

$$(1616x^2 - 1078xy + 77y^2) A^*_v(x, y) - (1616x^2 - 1482xy + 195y^2) B^*_v(x, y) = -16800x^2y^3.$$ 

Substituting 77 for $B^*_v(0, 1)$, we conclude

$$16800^4 \times 7^4(\xi^4_1 - \eta^4_1)^{13} \subset 16800^4 \times 7^4 p(\xi^4_1 - \eta^4_1)^{12} = 16800^4 p^{13} v^{12} B^*_v(0, 1)^4 \subset r_2.$$ 

For $r = 4$, setting

$$G_4(x, y) = 14178304x^3 - 15889280x^2y + 4071760xy^2 - 162393y^3,$$

$$H_4(x, y) = 14178304x^3 - 19433856x^2y + 6714864xy^2 - 466089y^3,$$

we may verify that

$$B^*_v(x, y) A^*_v(x, y) - A^*_v(x, y) B^*_v(x, y) = -6006y^7$$

and

$$G_4(x, y) A^*_v(x, y) - H_4(x, y) B^*_v(x, y) = -150678528y^4x^3.$$ 

This implies that

$$150678528^4 \times 231^4(\xi^4_1 - \eta^4_1)^{17} \subset 150678528^4 \times 231^4 p(\xi^4_1 - \eta^4_1)^{16}.$$ 

Since this latter quantity is equal to $150678528^4 p^{17} v^{16} B^*_v(0, 1)^4$, it follows that

$$150678528^4 \times 231^4(\xi^4_1 - \eta^4_1)^{17} \subset r_4.$$ 

Finally, for $r = 5$, we have

$$B^*_v(x, y) A^*_v(x, y) - A^*_v(x, y) B^*_v(x, y) = -14586y^7$$

and

$$G_5(x, y) A^*_v(x, y) - H_5(x, y) B^*_v(x, y) = -134424576y^5x^4.$$ 

where

$$G_5(x, y) = 43706368x^4 - 69346048x^3y + 32767856x^2y^2 - 4764782xy^3 + 123519y^4,$$

$$H_5(x, y) = 43706368x^4 - 80272640x^3y + 46006896x^2y^2 - 8845746xy^3 + 391833y^4.$$ 

This implies that

$$134424576^4 \times 209^4(\xi^4_1 - \eta^4_1)^{21} \subset 134424576^4 \times 209^4 p(\xi^4_1 - \eta^4_1)^{20}$$

wherby

$$134424576^4 \times 209^4(\xi^4_1 - \eta^4_1)^{21} \subset 134424576^4 p^{21} v^{20} B^*_v(0, 1)^4 \subset r_5.$$ 

From the preceding arguments, we are thus able to deduce the following series of inequalities:

$$N(3_1)^{1/2} |\xi_1^4 - \eta_1^4|^{-5} \leq 1296,$$

$$N(3_2)^{1/2} |\xi_1^4 - \eta_1^4|^{-9} \leq 560^4,$$

$$N(3_3)^{1/2} |\xi_1^4 - \eta_1^4|^{-13} \leq (77 \times 16800)^4,$$

$$N(3_4)^{1/2} |\xi_1^4 - \eta_1^4|^{-17} \leq (231 \times 150678528)^4.$$
and
\[ N(3^r)\frac{1}{2}|\xi_1^4 - \eta_1^4|^{-21} \leq (209 \times 134424576)^4. \]
These will enable us to contradict inequality (24) for \( r \leq 5 \), provided we can find a suitably strong lower bound for \( |\xi_1| \). Since \( \xi_1^4 = 4(\sqrt{-t} + 1)(x_i - \sqrt{-ty_i})^4 \) and \( x_1y_1 > 64t^3 \), via calculus we have that
\[ |\xi_1|^4 > 2^{16}t^{15/2}, \]
whence (24) and the assumption that \( P(x, y) \leq t^2 \) imply
\[ 2^{26r-3}t^{11r-2} < N(3^r)^{1/2}|\xi_1^4 - \eta_1^4|^{-4r-1}|F_r(z_1)|^{-1}. \]
From (20), we have
\[ |z_1| = \left| \frac{8P(x, y)}{\xi_1^4} \right| < \left( 2^{13}t^{11/2} \right)^{-1} < 0.001, \]
and consequently,
\[ F_1(z_1) > 10^2, F_2(z_1) > 10^4, F_3(z_1) > 10^{10}, F_4(z_1) > 10^{13} \quad \text{and} \quad F_5(z_1) > 10^{14}. \]
In case \( r = 1 \), inequality (27) thus implies that
\[ 2^{2349} < 6635.52 \times t^6, \]
a contradiction for all \( t \). Arguing similarly for \( r = 2, 3, 4 \) and 5, and noting that \( t > 204 \), completes the proof of Lemma 6.3. \( \square \)

7. The Proof of Theorem 1.1

Assume that there are two distinct coprime solutions \((x_1, y_1)\) and \((x_2, y_2)\) to inequality (17) with \( |\xi_2| > |\xi_1| \). We will show that \( |\xi_2| \) is arbitrary large in relation to \( |\xi_1| \). In particular, we will demonstrate via induction that
\[ |\xi_2| > \frac{\sqrt{r}}{5^{14r+7/4}} \left( \frac{4}{81} \right)^r |\xi_1|^{4r+3}, \]
for each positive integer \( r \). Since inequality (20) thus implies that
\[ |\xi_2| > t^{7r/2+31/8}, \]
for arbitrary \( r \), we deduce an immediate contradiction.

We first prove inequality (28) for \( r = 1 \). By (12) and (20),
\[ c_1(1, 0)|\xi_1|^5|\xi_2|^{-3} < 2^{-13} \pi^{-1/2} t^{-5/2} < 0.1, \]
and hence, since \( \Sigma_{1, 0} \neq 0 \), Lemma 6.1 yields
\[ c_2(1, 0)|\xi_1|^{-7}|\xi_2| > 0.9, \]
which, after a little work, implies (28).

We now proceed by induction. Suppose that (28) holds for some \( r \geq 1 \). Then
\[ c_1(r + 1, 0)|\xi_1|^{4r+5}|\xi_2|^{-3} < \frac{2000}{9\sqrt{\pi}r^2} t^{12r+13/2} \left( \frac{312}{24} \right)^r |\xi_1|^{-8r-4}, \]
and hence, from (26),
\[ c_1(r + 1, 0)|\xi_1|^{4r+5}|\xi_2|^{-3} < \frac{125}{2^{12} \sqrt{\pi}r^2} t^{-3r-1} \left( \frac{312}{236} \right)^r < 0.1. \]
If $\Sigma_{r+1,0} \neq 0$, then by Lemma 6.1

$$c_2(r + 1, 0)|\xi_1|^{-4(r+1)-3}|\xi_2| > 0.9,$$

which leads to inequality (28) with $r$ replaced by $r+1$. If, however, $\Sigma_{r+1,0} = 0$ then by Lemmas 6.2 and 6.3 both $\Sigma_{r+1,1}$ and $\Sigma_{r+2,1}$ are nonzero, and $r \geq 5$. Using the induction hypothesis, we find as previously that

$$c_1(r + 1, 1)|\xi_1|^{4r+4}|\xi_2|^{-3} < 0.1$$

and thus by Lemma 6.1 conclude that

$$c_2(r + 1, 1)|\xi_1|^{-4r-4}|\xi_2| > 0.9.$$ 

It follows that

$$|\xi_2| > 0.08 \times \sqrt{\frac{r+1}{t^{4r+28/8}}} \left(\frac{4}{81}\right)^r |\xi_1|^{4r+4}.$$ 

Consequently,

$$c_1(r + 2, 1)|\xi_1|^{4r+8}|\xi_2|^{-3} < \frac{24000}{(r + 1)^2} \left(\frac{3^{12}}{24}\right)^r t^{4r+28/8} |\xi_1|^{-8r-4},$$

whereby, from (26) and the fact that $r \geq 5$,

$$c_1(r + 2, 1)|\xi_1|^{4r+8}|\xi_2|^{-3} < \frac{1}{2(r+1)^2} \left(\frac{3^{12}}{24}\right)^r t^{4r+25/2} |\xi_1|^{-8r-4} < 0.1.$$ 

Lemma 6.1 thus implies the inequality

$$c_2(r + 2, 1)|\xi_1|^{-4r-8}|\xi_2| > 0.9$$

and so

$$|\xi_2| > 0.08 \sqrt{\frac{r+1}{5t^{4r+4+7/4}}} \left(\frac{4}{81}\right)^r |\xi_1|^{4r+7}.$$ 

From (26), it follows that

$$|\xi_2| > \sqrt{\frac{r+1}{5t^{4r+4+7/4}}} \left(\frac{4}{81}\right)^r |\xi_1|^{4r+7},$$

as desired. This completes the proof of inequality (28) and hence we conclude that there is at most one solution to (7) related to each fourth root of unity.

To finish the proof of Theorem 1.1, it is enough to show that three of the roots of unity under consideration do not have solutions to (7) associated to them. We recall that the polynomial

$$P(x, 1) = x^4 + 4tx^3 - 6tx^2 - 4t^2x + t^2$$

has 4 real roots $\beta_1, \beta_2, \beta_3, \beta_4$, say, where

$$\sqrt{\frac{1}{2}} + \frac{1}{8\sqrt{t}} - \frac{2}{8t} < \beta_1 < \sqrt{\frac{1}{2}} + \frac{1}{8\sqrt{t}} - \frac{1}{8t}$$

$$-\sqrt{\frac{1}{2}} + \frac{1}{8\sqrt{t}} - \frac{1}{8t} < \beta_2 < -\sqrt{\frac{1}{2}} + \frac{1}{8\sqrt{t}} - \frac{1}{8t}$$

$$\frac{1}{4} - \frac{5}{64t} + \frac{22}{512t^2} < \beta_3 < \frac{1}{4} - \frac{5}{64t} + \frac{23}{512t^2}$$

$$-4t - \frac{5}{4} + \frac{21}{64t} - \frac{87}{512t^2} < \beta_4 < -4t - \frac{5}{4} + \frac{21}{64t} - \frac{84}{512t^2}.$$
(since \( t \geq 18 \), the polynomial \( P(x, 1) \) changes sign between the given bounds). Since
\[
P(\beta, 1) = \frac{1}{8} (\xi^4(\beta, 1) - \eta^4(\beta, 1)) = 0,
\]
it follows that, for each \( 1 \leq i \leq 4 \), \( \frac{\eta(\beta, 1)}{\xi(\beta, 1)} \) is a fourth root of unity. Noting that
\[
\frac{\eta(\beta, 1)}{\xi(\beta, 1)} - \frac{\eta(\beta_j, 1)}{\xi(\beta_j, 1)} = \left( \frac{\sqrt{-t} - 1}{\sqrt{-t} + 1} \right)^{1/4} \frac{2\sqrt{-t}(\beta_j - \beta)}{(\beta - \sqrt{-t})(\beta_j - \sqrt{-t})},
\]
they are in fact distinct. We now proceed to show that solutions to (7) necessarily correspond to fourth roots of unity related to \( \beta_2 \).

In [22], it is shown that for \( \{V_{2n+1}\} \) defined in Section 2, the equation \( z^2 = V_{4n+1} \) has no solution. Supposing that \( z^2 = V_{4n+3} \), as in the proof of Proposition 3.1 there exist integers \( t_1, t_2, G \) and \( H \), so that the integers \( x, y \) arising from Proposition 3.1 satisfy \( x = -t_1 G \) and \( y = H \). We have
\[
\frac{x}{y} = \frac{-t_1 G}{H} = \frac{-2t_1 G^2}{2GH} = \frac{-\sqrt{V_{4n+3}} - V_{2n+1}}{U_{n+1}}
\]
\[
= \frac{-\sqrt{V_{2n+2}^2 + V_{2n+1}^2} - V_{2n+1}}{V_{2n+2}^2 \sqrt{t}}
\]
\[
= -\sqrt{t} \left( \sqrt{1 + \frac{V_{2n+1}^2}{V_{2n+2}^2}} - \frac{V_{2n+1}}{V_{2n+2}} \right),
\]
using the fact that \( V_{2n+2} = \sqrt{t} U_{n+1} \). Thus
\[
\left| \frac{x}{y} + \sqrt{t} \right| = \sqrt{t} \left| 1 + \frac{V_{2n+1}^2}{V_{2n+2}^2} - \frac{V_{2n+1}}{V_{2n+2}} - 1 \right|
\]
A crude application of the Mean Value Theorem therefore implies that
\[
\left| \frac{x}{y} + \sqrt{t} \right| < \sqrt{t}
\]
and consequently, \( x/y \in (-2\sqrt{t}, 0) \), whereby the inequalities for \( \beta_i \) yield
\[
\left| \frac{x}{y} - \beta_1 \right| \geq \left| \sqrt{t} + \beta_1 \right| - \frac{x}{y} + \sqrt{t} > 2\sqrt{t} - \sqrt{t} = \sqrt{t},
\]
\[
\left| \frac{x}{y} - \beta_3 \right| \geq \left| \sqrt{t} + \beta_3 \right| - \frac{x}{y} + \sqrt{t} > \sqrt{t} + \frac{1}{2} - \sqrt{t} = \frac{1}{2}
\]
and
\[
\left| \frac{x}{y} - \beta_4 \right| \geq \left| \sqrt{t} + \beta_4 \right| - \left| \frac{x}{y} + \sqrt{t} \right| > 3t - \sqrt{t} > 2t.
\]
Let \( \beta \in \{\beta_1, \beta_3, \beta_4\} \). We have just shown that if \( (x, y) \) is a solution to inequality (7), then
\[
\left| \frac{x}{y} - \beta \right| > \frac{1}{5}.
\]
If we suppose that \( \omega = \frac{\eta(\beta, 1)}{\xi(\beta, 1)} \), then

\[
\left| \omega - \frac{\xi(x, y)}{\eta(x, y)} \right| = \left| \frac{\eta(\beta, 1)}{\xi(\beta, 1)} - \frac{\eta(z, 1)}{\xi(z, 1)} \right| = \left| \frac{2\sqrt{-t}(\beta - \sqrt{-t})}{(\beta - \sqrt{-t})(\beta - \sqrt{-t})} \right|
\]

whence the inequalities

\[
|\beta - \sqrt{-t}| < \sqrt{16t^2 + 17t}
\]

and

\[
\left| \frac{x}{y} - \sqrt{-t} \right| < \sqrt{5t}
\]

(recall that \(|x/y| < 2\sqrt{t}\)) imply

\[
\left| \omega - \frac{\xi(x, y)}{\eta(x, y)} \right| > \frac{2}{5\sqrt{80t^2 + 85t}}
\]

Since

\[
|z| = \frac{8P(x, y)}{|\xi^4(x, y)|} \leq \frac{8t^2}{|\xi^4(x, y)|},
\]

this, together with (26), contradicts Lemma 3.2.

This shows that there is no solution related to three of the fourth roots of unity (those corresponding to \(\beta_1, \beta_3\) and \(\beta_4\)). Therefore, there is at most a single solution to inequality (7). Together with Propositions 2.2 and 3.1 this completes the proof of Theorem 1.1.

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