ABSTRACT. We study the decreasing rearrangement of functions in VMO, and show that for rearrangeable functions, the mapping \( f \mapsto f^* \) preserves vanishing mean oscillation. Moreover, as a map on BMO, while bounded, it is not continuous, but continuity holds at points in VMO (under certain conditions). This also applies to the symmetric decreasing rearrangement. Many examples are included to illustrate the results.

1. Introduction

For equimeasurable rearrangements, boundedness and continuity do not always go hand-in-hand. On Lebesgue spaces, both the decreasing rearrangement and the symmetric decreasing rearrangement are non-expansive, but the situation is more complicated for the Sobolev spaces \( W^{1,p} \), \( 1 \leq p < \infty \). The Pólya–Szegő inequality \( \| \nabla Sf \|_p \leq \| \nabla f \|_p \) ensures that symmetrization decreases the norm in these spaces. Furthermore, Coron [12] proved that this rearrangement is continuous on \( W^{1,p}(\mathbb{R}) \). However, Almgren and Lieb [2] discovered that continuity fails in \( W^{1,p}(\mathbb{R}^n) \) in all higher dimensions \( n > 1 \). In essence, convergence can fail because the symmetric decreasing rearrangement can reduce the measure of the critical set where \( \nabla f \) vanishes. In one dimension, this is precluded by Sard’s lemma; for the same reason, Steiner symmetrization is continuous in any dimension [7]. Similar questions have been studied regarding the boundedness and continuity of maximal functions on Sobolev spaces and BV [1, 10, 18, 19].

Turning to the space BMO of functions of bounded mean oscillation, it is well known that the decreasing rearrangement (that is, the map \( f \mapsto f^* \) ) is bounded on BMO: there are constants \( C_n \), depending only on dimension, such that

\[
\| f^* \|_{\text{BMO}} \leq C_n \| f \|_{\text{BMO}}
\]

whenever the decreasing rearrangement is defined. The sharp dependence of the constants \( C_n \) or, indeed, if they do depend on dimension at all is still an open question for dimensions \( n > 1 \). See [8] for a discussion of this inequality and for a proof that \( C_n \) exhibits at most square-root-dependence.
on \( n \). It was proven in [8] that the symmetric decreasing rearrangement (that is, the map \( f \mapsto Sf \)) is also bounded on BMO. The sharp constants and their dependence on dimension remains unknown for \( n \geq 1 \).

In this paper, we address the question of continuity of these rearrangements on BMO. This question has not previously been studied in the literature, even in dimension \( n = 1 \). We show by means of an example (see Example 3.2) that the decreasing rearrangement is discontinuous on \( \text{BMO}(\mathbb{R}^n) \) and it follows that the same is true for the symmetric decreasing rearrangement. The phenomenon in Example 3.2 is somewhat similar to that described above for the Sobolev spaces: the sequence \( f_k \) and the limit \( f = f^* \) have jumps of height 1, but the decreasing rearrangement erases the jumps, and \( f_k^* \) is continuous. Unlike the situation in \( W^{1,p} \), this can happen even in one dimension.

To eliminate the possibility of jump discontinuities, we consider the subspace VMO of functions of vanishing mean oscillation, which often plays the role of the continuous functions within BMO. The definition of VMO originates with Sarason [23], who identified the closure of the uniformly continuous functions in BMO with those functions whose mean oscillation over any cube converges to zero uniformly in the diameter of the cube. VMO can be viewed as the 0-endpoint on the smoothness scale: vanishing mean oscillation is a common minimal regularity condition on the coefficients of PDE [11, 17] and on the normal to the boundary of non-smooth domains [20].

Our first result shows that the decreasing rearrangement \( f^* \) of a function \( f \in \text{VMO} \) is in VMO. As a decreasing function of a single variable, \( f^* \) has vanishing mean oscillation if and only if it satisfies a sub-logarithmic growth condition (i.e., a vanishing John-Nirenberg inequality) at the origin. Note that for functions in the critical Sobolev spaces \( W^{s,n/s}(\mathbb{R}^n) \), which embed in VMO, stronger vanishing at the origin was proved by Hansson [15] and Brezis-Wainger [6].

**Theorem 1 (Boundedness).** Let \( Q_0 \subset \mathbb{R}^n \) be a cube. If \( f \in \text{VMO}(Q_0) \), then \( f^* \in \text{VMO}(\mathbb{R}^n) \).

It turns out that functions in VMO on an unbounded domain are not automatically rearrangeable (see Example 3.1). Nonetheless, on \( \mathbb{R}^n \), we have the following analogue of the above theorem:

\[
f \in \text{VMO}(\mathbb{R}^n) \text{ rearrangeable} \implies f^* \in \text{VMO}(\mathbb{R}_+) \nonumber.
\]

See Theorem 4.1 for this result and in a more general context.

Our second result shows that the decreasing rearrangement is continuous at all points in VMO, in the following sense.

**Theorem 2 (Continuity).** Let \( Q_0 \subset \mathbb{R}^n \) be a cube. If \( f_k, k \in \mathbb{N}, \) are in \( \text{BMO}(Q_0) \) and \( f \) is in \( \text{VMO}(Q_0) \) with \( f_k \to f \) in \( \text{BMO}(Q_0) \) such that the means \( \int_{Q_0} f_k \) converge to \( \int_{Q_0} f \), then \( f_k^* \to f^* \) in \( \text{BMO}(0, |Q_0|) \).

Similar continuity results hold on \( \mathbb{R}^n \) and in more general settings – see Theorem 4.10. The corresponding conclusions (see Corollaries 4.4 and 4.11) also hold for the symmetric decreasing rearrangement.
Both theorems will be proved more generally for functions on a domain $\Omega \subset \mathbb{R}^n$ that have vanishing mean oscillation with respect to a suitable basis $\mathcal{S}$ in $\Omega$. The VMO space in this setting is introduced in Section 2.3.

2. Preliminaries

2.1. Rearrangements. To define the decreasing rearrangement (also called the nonincreasing rearrangement) and the symmetric decreasing rearrangement, we will restrict ourselves to functions which we call rearrangeable. For a measurable function $f$ on a domain $\Omega \subset \mathbb{R}^n$, this means that $\mu_f(\alpha) \to 0$ as $\alpha \to \infty$, where $\mu_f$ is the distribution function of $f$. Recall that for $\alpha \geq 0$, $\mu_f(\alpha) = |E_\alpha(f)|$, the Lebesgue measure of the level set

$$E_\alpha(f) := \{x \in \Omega : |f(x)| > \alpha\}.$$

The decreasing rearrangement is the right-continuous generalized inverse of the distribution function, given by

$$f^*(s) = \mu(\mu_f)(s) = |\{\alpha \geq 0 : \mu_f(\alpha) > s\}|,$$

see Figure 2.1. If the domain has finite measure $|\Omega|$, then $f^*(s) = 0$ for all $s \geq |\Omega|$, so we consider $f^*$ as a function on $(0, |\Omega|)$.

The following standard properties of the decreasing rearrangement will be used throughout this paper.

Property R1. (Equimeasurability.) For all $\alpha \geq 0$, $\mu_f(\alpha) = \mu_{f^*}(\alpha)$.

This property uniquely characterizes $f^*$ among the right-continuous decreasing functions on $(0, |\Omega|)$.

Property R2. The decreasing rearrangement $f \mapsto f^*$ is norm-preserving from $L^p(\Omega)$ to $L^p(0, |\Omega|)$ for all $1 \leq p \leq \infty$. Furthermore, it is non-expansive and, therefore, continuous.

Property R3. If $f_k, k \in \mathbb{N}$, and $f$ are nonnegative rearrangeable functions on $\Omega$ satisfying $f_k \uparrow f$ pointwise, then $f_k^* \uparrow f^*$ pointwise on $(0, |\Omega|)$. 

![Figure 1. A nonnegative function $f \in \text{BMO}(\mathbb{R})$ and its decreasing rearrangement $f^* \in \text{BMO}(\mathbb{R}^+)$.](image)
To see this, note first that for any $\alpha > 0$, the level sets of $f$ satisfy $E_\alpha(f) = \bigcup_{k \geq 1} E_\alpha(f_k)$. By continuity of the measure, $\mu_{f_k}(\alpha) \uparrow \mu_f(\alpha)$ for all $\alpha > 0$. By Eq. (2.1), $f^*$ and $f_k^*$ can be represented as the distribution functions of $\mu_f$ and $\mu_{f_k}$, respectively. Therefore, by the same argument as above, $f_k^*(s) \uparrow f^*(s)$ for all $s > 0$.

**Property R4. (Truncation.)** For any nonnegative rearrangeable function $f$ and any $0 \leq \alpha < \beta \leq \infty$,

$$\left(\min \{\max \{f, \alpha\}, \beta\}\right)^* = \min \{\max \{f^*, \alpha\}, \beta\}.$$  

For a rearrangeable function $f$ on $\mathbb{R}^n$, we define its symmetric decreasing rearrangement $Sf$ by

$$Sf(x) = f^*(\omega_n|x|^n), \quad x \in \mathbb{R}^n \setminus \{0\},$$  

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. The symmetric decreasing rearrangement, as a map from functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$, inherits Properties [R1]-[R4].

For later use, we record the behaviour of distribution functions and rearrangements under scaling, dilation, and translation. If $\tilde{f}(x) = af(b^{-1}(x - x_0))$ for some positive constants $a, b$ and some $x_0 \in \mathbb{R}^n$, then

$$(\tilde{f})^*(s) = af^*(b^{-n}s) \quad \text{and} \quad S\tilde{f}(x) = a(Sf)(b^{-1}x).$$

More details on the decreasing rearrangement can be found in [24]; see [3] for the symmetric decreasing rearrangement.

### 2.2. Mean oscillation.

John and Nirenberg [14] introduced functions of bounded mean oscillation over cubes in $\mathbb{R}^n$ with sides parallel to the axes. We follow the terminology used in [13] in order to define mean oscillation over more general sets than cubes or balls. A basis of shapes in a domain $\Omega \subset \mathbb{R}^n$ is a collection $\mathcal{S}$ of open sets $S \subset \Omega$, $0 < |S| < \infty$, forming a cover of $\Omega$. We will use $\mathcal{I}$ to denote the basis of finite open intervals in $\mathbb{R}$; in $\mathbb{R}^n$, we denote by $\mathcal{B}$ the basis of Euclidean balls, by $\mathcal{Q}$ the basis of cubes with sides parallel to the axes, and by $\mathcal{R}$ the rectangles with sides parallel to the axes.

Let $f$ be a real-valued function with $f \in L^1(S)$ for every $S \in \mathcal{S}$. Define the mean oscillation of $f$ on a shape $S \in \mathcal{S}$ by

$$\mathcal{O}(f, S) := \int_S |f - f_S|,$$

where $|S|$ denotes the measure of $S$ and $f_S := \frac{1}{|S|} \int_S f$ is the average of $f$ over $S$.

We will use the following properties of mean oscillation of an integrable function over a given shape $S \in \mathcal{S}$. These shapewise identities and inequalities are proved in [13].

**Property O1.** For any constant $\alpha$, $\mathcal{O}(f + \alpha, S) = \mathcal{O}(f, S)$. 


Property O2. Denoting \( y_+ = \max(y, 0) \),
\[
\mathcal{O}(f, S) = \frac{2}{|S|} \int_S (f - f_S)_+.
\]
When \( f = X_E \) for a measurable set \( E \), then
\[
(2.4) \quad \mathcal{O}(X_E, S) = 2\rho(E, S)(1 - \rho(E, S)), \quad \text{where} \quad \rho(E, S) := \frac{|E \cap S|}{|S|}.
\]

Property O3. \( \mathcal{O}(|f|, S) \leq 2\mathcal{O}(f, S) \).

Property O4.
\[
\mathcal{O}(f, S) \leq 2\inf_{\alpha} \int_S |f - \alpha| = 2\int_S |f - m|,
\]
where the infimum is taken over all constants \( \alpha \), and \( m \) is a median of \( f \) on \( S \), defined by the property that \( |\{ x \in S : f(x) > m \}| \leq \frac{1}{2}|S| \) and \( |\{ x \in S : f(x) < m \}| \leq \frac{1}{2}|S| \).

Property O5.
\[
\mathcal{O}(f, S) \leq \int_S \int_S |f(x) - f(y)| \, dx \, dy \leq 2\mathcal{O}(f, S).
\]

Property O6. For \( -\infty \leq \alpha < \beta \leq \infty \), the truncation \( \tilde{f} = \min\{\max\{f, \alpha\}, \beta\} \) satisfies
\[
\mathcal{O}(\tilde{f}, S) \leq \mathcal{O}(f, S).
\]

We will also frequently use the following comparison principle, which can be obtained by applying Property O2 to both sides.

Property O7. For any pair of shapes \( S \subset \tilde{S} \),
\[
\mathcal{O}(f, S) \leq \frac{|\tilde{S}|}{|S|} \mathcal{O}(f, \tilde{S}).
\]

If \( \mathcal{I} \) and \( \tilde{\mathcal{I}} \) are two bases of shapes in \( \Omega \) such that for every \( S \in \mathcal{I} \) there exists \( \tilde{S} \in \tilde{\mathcal{I}} \) with \( S \subset \tilde{S} \) and \( |\tilde{S}| \leq c|S| \), and for every \( \tilde{S} \in \tilde{\mathcal{I}} \) there exists \( S \in \mathcal{I} \) with \( \tilde{S} \subset S \) and \( |S| \leq \bar{c}|\tilde{S}| \), for some constants \( c, \bar{c} > 0 \), then we say that \( \mathcal{I} \) is equivalent to \( \tilde{\mathcal{I}} \), written \( \mathcal{I} \approx \tilde{\mathcal{I}} \).

Definition 2.1. A function \( f \) has bounded mean oscillation with respect to a basis \( \mathcal{I} \), denoted \( f \in \text{BMO}_{\mathcal{I}}(\Omega) \), if \( f \in L^1(S) \) for all \( S \in \mathcal{I} \) and
\[
(2.5) \quad \|f\|_{\text{BMO}_{\mathcal{I}}} := \sup_{S \in \mathcal{I}} \mathcal{O}(f, S) < \infty.
\]

When \( \mathcal{I} = \mathcal{Q} \), the basis of cubes, \( \text{BMO}_{\mathcal{Q}}(\Omega) \) will be simply denoted by \( \text{BMO}(\Omega) \). Many interesting properties of \( \text{BMO}_{\mathcal{I}} \) for \( \mathcal{I} = \mathcal{Q} \) and \( \mathcal{R} \) can be found in [16].

Remark 2.2. Functions in \( \text{BMO}_{\mathcal{I}}(\Omega) \) are locally integrable in \( \Omega \). Note, however, that shapes need not be compactly contained in \( \Omega \). In particular, functions in \( \text{BMO}(\mathbb{R}_+) \) are also integrable at the origin in the sense of being integrable on \((0, b)\) for any \( b < \infty \).
Property [O1] tells us that Eq. (2.5) defines a seminorm that vanishes on constant functions. It is therefore natural to consider \( \text{BMO}_{\mathcal{S}}(\Omega) \) modulo constants, and it was shown in [13] that this gives a Banach space. When dealing with rearrangements, however, working modulo constants will not serve our purpose, and we will just consider \( \text{BMO}_{\mathcal{S}}(\Omega) \) as a linear space with a seminorm. Convergence in \( \text{BMO}_{\mathcal{S}}(\Omega) \) will always mean convergence with respect to this seminorm; i.e.

\[
f_k \to f \text{ in } \text{BMO}_{\mathcal{S}}(\Omega) \iff \lim_{\delta \to 0^+} \| f_k - f \|_{\text{BMO}_{\mathcal{S}}} = 0.
\]

We collect here some properties of BMO that will be used subsequently.

**Property B1.** If \( \mathcal{S} \approx \tilde{\mathcal{S}} \), then \( f \in \text{BMO}_{\mathcal{S}}(\mathbb{R}^n) \) if and only if \( f \in \text{BMO}_{\tilde{\mathcal{S}}}(\mathbb{R}^n) \), with

\[
\tilde{c}^{-1} \| f \|_{\text{BMO}_{\mathcal{S}}} \leq \| f \|_{\text{BMO}_{\tilde{\mathcal{S}}}} \leq c \| f \|_{\text{BMO}_{\mathcal{S}}}.
\]

This follows from Property [O7].

**Property B2.** For any shape \( S \in \mathcal{S} \),

\[
\| f - f_S \|_{L^1(S)} \leq |S| \| f \|_{\text{BMO}_{\mathcal{S}}}.
\]

**Property B3.** On \( \Omega = \mathbb{R}^n \), if \( \tilde{f}(x) = f(b^{-1}(x - x_0)) \) for some \( b > 0 \) and \( x_0 \in \mathbb{R}^n \), then

\[
\| \tilde{f} \|_{\text{BMO}} = \| f \|_{\text{BMO}}.
\]

**Property B4.** Suppose \( f \) is nonnegative and \( S \in \mathcal{S} \). Then

\[
O(f, S) \leq 2 \left( \inf \frac{|S'|}{|S|} \right) \| f \|_{\text{BMO}_{\mathcal{S}}},
\]

where the infimum is taken over all shapes \( S' \) with \( S' \supset E_0(f) \cap S \) and \( |S'| \geq 2|E_0(f) \cap S'| \). On such a shape \( S' \), we have that \( m = 0 \) is a median of \( f \), and by Properties [O2] and [O4],

\[
(2.6) \quad O(f, S) \leq \frac{2}{|S|} \int_S f \leq \frac{2}{|S|} \int_{S'} |f - m| = 2 \frac{|S'|}{|S|} \int_{S'} |f - m| \leq 2 \frac{|S'|}{|S|} O(f, S').
\]

In general, this is not sharp. For intervals on \( \mathbb{R} \), if \( E_0(f) \) is an interval \( I \) then for any interval \( S \) with \( |I \cap S| < |S| \), the minimum is attained by taking an interval \( S' \) of length \( 2|I \cap S| \), and we get the estimate

\[
(2.7) \quad O(f, S) \leq \frac{4|I \cap S|}{|S|} \| f \|_{\text{BMO}}.
\]

This is sharp: for indicator functions, the norm of \( \| X_E \|_{\text{BMO}(\mathbb{R})} = \frac{1}{2} \) and \( \rho(E, S) \) can be taken arbitrarily close to zero in Eq. (2.4).

## 2.3. Vanishing mean oscillation.

An important subspace of BMO is the space of functions of vanishing mean oscillation, VMO, originally defined by Sarason on \( \mathbb{R} \). We generalize the definition here to a basis \( \mathcal{S} \) of shapes in a domain \( \Omega \subset \mathbb{R}^n \).

**Definition 2.3.** We say that a function \( f \in \text{BMO}_{\mathcal{S}}(\Omega) \) is in \( \text{VMO}_{\mathcal{S}}(\Omega) \) if

\[
(2.8) \quad \lim_{\delta \to 0^+} \sup_{|S| \leq \delta} O(f, S) = 0,
\]

where the supremum is taken over all shapes \( S \in \mathcal{S} \) of measure at most \( \delta \).
In what follows, we want to exclude the possibility that Eq. (2.8) holds vacuously because there are no shapes of arbitrarily small measure (an example is the basis of cubes with sidelength bounded below by some constant). This is implicit in the density condition (4.1).

By Properties $O^7$ and $B^1$, equivalent bases define the same VMO-space, with equivalent BMO-seminorms. Again, the notation VMO$(\Omega)$ will be reserved for the case $\mathcal{S} = \mathcal{Q}$. For bases equivalent to cubes, having vanishing diameter is the same as having vanishing measure, hence the supremum in Definition 2.3 can instead be taken over all shapes $S$ of diameter at most $\delta$. For general bases, however, vanishing diameter is strictly stronger than vanishing measure. Consider the basis of rectangles, $\mathcal{R}$, in which a sequence of rectangles of constant diameter can have measure tending to zero.

For nice domains $\Omega$, VMO$(\Omega)$ is the closure, in the BMO-seminorm, of the set of uniformly continuous functions in BMO$(\Omega)$ (see [9]). General functions in VMO, though, need be neither continuous nor bounded; an example is $(-\log|x|)^+$ for $0 < p < 1$. On $\Omega = \mathbb{R}^n$, VMO can also be characterized as the subset of BMO$(\mathbb{R}^n)$ on which translation is continuous. In the case when $\Omega$ is unbounded, note that there is a strictly smaller VMO-space, sometimes denoted CMO (see [4, 9, 25]), in which additional vanishing mean oscillation conditions are required as the cube or its sidelength go to infinity.

For any choice of basis, we have that VMO$\mathcal{S}(\Omega)$ is a closed subspace of BMO$\mathcal{S}(\Omega)$. To see this, let $f_k \to f$ in BMO$\mathcal{S}(\Omega)$, where $f_k \in$ VMO$\mathcal{S}(\Omega)$ for all $k$. Then $f \in$ VMO$\mathcal{S}(\Omega)$ since

$$\sup_{|S| \leq \delta} O(f, S) \leq \sup_{|S| \leq \delta} O(f_k, S) + \|f - f_k\|_{\text{BMO}\mathcal{S}(\Omega)}$$

can be made arbitrarily small by choosing $k$ sufficiently large and $\delta$ sufficiently small.

2.4. Rearrangement meets mean oscillation. To deal with the decreasing rearrangement for functions in BMO, we need to establish several conventions. First, as noted above, while mean oscillation is invariant under the addition of constants, this is not true for the rearrangement. Therefore, the mapping from $f$ to $f^*$ is not between equivalence classes modulo constants, but between individual functions.

A second issue relates to rearrangeability. When defining the decreasing rearrangement for functions in spaces like $L^p$ (see Property $R^2$) or weak-$L^p$, the rearrangeability condition is automatically satisfied. This is not true for functions in BMO, which may fail to be rearrangeable. Nevertheless, as functions in BMO are locally integrable, hence finite almost everywhere, we will have that $f \in$ BMO is rearrangeable provided that $\mu_f(\alpha) < \infty$ for some $\alpha \geq 0$. This property is preserved when we add constants.

If $f$ were not rearrangeable, then defining $f^*$ would lead to $f^* \equiv \infty$. This is the case, for instance, for $-\log|x|$, the prototypical unbounded function in BMO$(\mathbb{R}^n)$. On the other hand, the positive part $(-\log|x|)^+$ is rearrangeable, as is any other BMO-function of compact support, since such functions are integrable.
3. Some examples

3.1. Rearrangeable functions. To go beyond the case of bounded functions, functions of compact support, and integrable functions, we consider some examples in BMO($\mathbb{R}$) defined, pointwise, as series. Fix a nonnegative, nonconstant integrable function $g$ in BMO($\mathbb{R}$) vanishing outside $I := (-1, 1)$. Define

\begin{equation}
(3.1) \quad f = \sum_{k=1}^{\infty} g_k ,
\end{equation}

where each $g_k$ is obtained from $g$ by scaling, dilation and translation; that is,

\begin{equation}
(3.2) \quad g_k(x) = a_k g(b_k^{-1}(x-n_k)) ,
\end{equation}

for some sequences $\{a_k\}, \{b_k\}, \{n_k\}$ of positive numbers. Here $\{a_k\}$ is assumed to be bounded, $n_1 \geq b_1$, and $n_k \uparrow \infty$. From Property [B3] it follows that $\|g_k\|_{BMO} = a_k \|g\|_{BMO}$. Note that $g_k$ vanishes outside $I_k := (n_k - b_k, n_k + b_k)$, and we further assume that consecutive intervals are well-spaced, namely

\begin{equation}
(3.3) \quad n_{k+1} - n_k \geq 9(b_k + b_{k+1}) ,
\end{equation}

so that the larger intervals $\tilde{I}_k := (n_k - 9b_k, n_k + 9b_k)$ are disjoint. This implies that if an interval $J$ intersects $I_k$ and an adjacent interval $I_{k\pm 1}$, then $|J \cap \tilde{I}_k| \geq 8b_k \geq 4|J \cap I_k|$. The series in Eq. (3.1) converges pointwise and in $L^1_{loc}(\mathbb{R})$, and

\begin{equation*}
\int_J f \leq \|g\|_{L^1} \sum_{k: J \cap I_k \neq \emptyset} a_k b_k ,
\end{equation*}

It converges in $L^1(\mathbb{R})$ if and only if the sequence $\{a_kb_k\}$ is summable; in that case,

\begin{equation*}
\|f\|_{L^1} = \|g\|_{L^1} \sum_{k \geq 1} a_k b_k .
\end{equation*}

We claim that $f \in \text{BMO}$, with

\begin{equation}
(3.4) \quad \|f\|_{\text{BMO}} = a \|g\|_{\text{BMO}} ,
\end{equation}

where $a := \sup a_k$. In particular, applying this to the tail of the series, we see that the convergence of the series in the BMO-seminorm is equivalent to $a_k \to 0$.

To prove Eq. (3.4), recalling that $\|g_k\|_{\text{BMO}} = a_k \|g\|_{\text{BMO}}$ and the definition of $a$, one direction reduces to showing $\|f\|_{\text{BMO}} \geq \|g_k\|_{\text{BMO}}$ for each $k$. We fix $k$ and estimate the oscillation of $g_k$ on an interval $J$. If $J$ intersects only $I_k$, then $O(g_k, J) = O(f, J)$. On the other hand, as was already pointed out, by the well-spacing assumption Eq. (3.3), if $J$ is sufficiently long to intersect $I_k$ and one of its neighbors, it must satisfy $|J| \geq 8b_k \geq 4|J \cap I_k|$. We now apply the calculation in
Eq. (2.6) to the function \( g_k \) with \( S = J \) and an interval \( S' \) whose half consists of \( J \cap I_k \), noting that \(|S'| = 2|J \cap I_k| \leq 4b_k \) implies \( S' \) does not intersect any neighbor of \( I_k \). Thus

\[
O(g_k, J) \leq \frac{2|S'|}{|J|} O(g_k, S') \leq O(g_k, S') = O(f, S')
\]
and we have shown that \( O(g_k, J) \leq \|f\|_{BMO} \).

The other direction follows by similar arguments applied this time to the mean oscillation of \( f \) on an interval \( J \). Again we have that if \( J \) intersects exactly one interval \( I_k \), then \( O(f, J) = O(g_k, J) \leq a_k \|g\|_{BMO} \). Otherwise, by the subadditivity of oscillation and Eq. (2.7), noting once more that \(|J| \geq 4|J \cap I_k| \) for each \( k \) for which \( I_k \cap J \neq \emptyset \), we write

\[
O(f, J) \leq \sum_{k: I_k \cap J \neq \emptyset} O(g_k, J) \leq \sum_{k: I_k \cap J \neq \emptyset} 4 \frac{|I_k \cap J|}{|J|} \|g_k\|_{BMO} \leq \frac{a \|g\|_{BMO}}{|J|} \sum_{k: I_k \cap J \neq \emptyset} 4 |I_k \cap J|.
\]

A final application of the consequences of well-spacing, namely that \( \tilde{I}_k := (n_k - 9b_k, n_k + 9b_k) \) are disjoint, gives

\[
\sum_{k: I_k \cap J \neq \emptyset} 4 |I_k \cap J| \leq \sum_{k: I_k \cap J \neq \emptyset} |\tilde{I}_k \cap J| \leq |J|.
\]

To check whether \( f \) is rearrangeable, we compute its distribution function. By disjointness of the supports of the \( \{g_k\} \) and Eq. (2.3),

\[
\mu_f(\alpha) = \sum_{k=1}^{\infty} \mu_{g_k}(\alpha) = \sum_{k=1}^{\infty} b_k \mu_g(a_k^{-1} \alpha)
\]
for any \( \alpha > 0 \). In particular, recalling that \( a := \sup a_k \) and assuming \( b := \sum_{k \geq 1} b_k < \infty \), we have \( \mu_f(\alpha) \leq b \mu_g(a^{-1} \alpha) \) and \( f^*(s) \leq ag(b^{-1}s) \). Even when \( \{b_k\} \) is not summable, \( f \) may be rearrangeable provided that \( \{a_k\} \) decays sufficiently quickly.

To understand this phenomenon, we specialize to the function \( g(x) = (-\log |x|)_+ \), which satisfies \( g_1 = 1 \), and \( \|g\|_{BMO} = 2/e \). Its mean oscillation is maximized on any symmetric subinterval of \( I = (-1, 1) \). Its distribution function and decreasing rearrangement are given by

\[
\mu_g(\alpha) = 2e^{-\alpha}, \quad g^*(s) = (-\log s + \log 2)_+.
\]

**Example 3.1.** Define \( f \) by Eqs. (3.1) and (3.2), where the factors \( a_k \) and \( b_k \) will be further specified below. For \( \{n_k\} \) we choose any sequence with \( n_1 \geq b_1 \) and satisfying Eq. (3.3). We consider three scenarios.

(a) \( f \) is rearrangeable, but \( \sum g_k \) diverges in BMO. Taking \( a_k = 1 \) and \( b_k = e^{-k} \) in Eq. (3.1), we obtain for the distribution function and decreasing rearrangement of \( f \):

\[
\mu_f(\alpha) = 2e^{-\alpha} \sum_{k=1}^{\infty} e^{-k} = 2be^{-\alpha}, \quad f^*(s) = (-\log s + \log(2b))_+,
\]
where \( b = \frac{1}{e-1} \). Then \( \|f^*\|_{BMO} = \|g\|_{BMO} = \|f\|_{BMO} \). Since \( a_k \to 0 \), the series from Eq. (3.1) diverges in BMO. In fact, \( f \) cannot be approximated in BMO by compactly supported functions because \( \|f\|_{BMO([|x| > R])} \to 0 \) as \( R \to \infty \) (see Theorem 6 in [4]).
(b) \( f \) is not rearrangeable, but \( \sum g_k \) converges in BMO. Taking \( a_k = k^{-1/2} \) and \( b_k = e^k \), the series from Eq. (3.1) converges in BMO. However, \( f \) fails to be rearrangeable:

\[
\mu_f(\alpha) = 2 \sum_{k=1}^{\infty} e^{k - \sqrt{k}} = \infty, \quad f^* \equiv +\infty.
\]

Since the partial sums are integrable functions of compact support, this demonstrates that rearrangeability is not preserved under limits in BMO. Note also that the series diverges in \( L^1(\mathbb{R}) \), because \( \{a_k b_k\} \) is not summable.

(c) \( f \) is rearrangeable, \( \sum g_k \) converges in BMO, but \( \inf f^* > \inf f \). Taking \( a_k = k^{-1} \) and \( b_k = e^k \) yields

\[
\mu_f(\alpha) = 2 \sum_{k=1}^{\infty} e^{-k(\alpha-1)} = \frac{2}{e^{\alpha-1} - 1} \quad (\alpha > 1), \quad f^*(s) = 1 + \log(1 + \frac{2}{s}) .
\]

Although neither \( f \) nor \( f^* \) is integrable, \( f^* \) is integrable at the origin and \( \|f^*\|_{\text{BMO}} \leq 2\|f\|_{\text{BMO}} = \frac{4}{e} \). In this example, \( \inf f = 0 \) but \( \inf f^* = 1 \).

Similar series can be constructed in VMO, by taking \( g(x) = (-\log |2x|)^p_+ \) with \( p \in (0, 1) \), and adjusting \( a_k, b_k, n_k \) accordingly.

3.2. Convergence of rearrangements. The next two examples show that the decreasing rearrangement is discontinuous on BMO and VMO. We consider sequences of the form

\[(5.5) \quad f_k = f + g_k ,\]

where \( f \) and \( g \) are fixed functions of compact support, and \( g_k \) is obtained from \( g \) by scaling, dilation, and translation, as in Eq. (3.2).

Example 3.2. \( f, f_k \) rearrangeable on a finite interval, \( f_k \to f \) in BMO, but \( f_k^* \not\to f^* \) in BMO. Choose \( f = \chi_{(0,1)} \) and \( g = (-\log |2x|)_+ \), and let \( a_k = \frac{1}{k}, b_k = 1, \) and \( n_k = -\frac{1}{2} \), see Figure 2.

Since \( a_k \to 0 \), the sequence \( f_k \) converges to \( f \) pointwise (except at \( x = -\frac{1}{2} \)) and in BMO.

The rearrangements are given by

\[
f_k^*(s) = \begin{cases} 
-\frac{1}{k} \log s, & 0 \leq s < e^{-k} \\
1, & e^{-k} \leq s < 1 + e^{-k} \\
-\frac{1}{k} \log(s - 1), & s \geq 1 + e^{-k} .
\end{cases}
\]

By monotone convergence, \( f_k^* \to f = f^* \) pointwise (except at \( x = -\frac{1}{2} \)) and in \( L^1 \). However, since \( f_k^* \) is constant on a short interval \( J \) centred at 1, we find that

\[
\|f_k^* - f^*\|_{\text{BMO}} \geq \mathcal{O}(f_k^* - f^*, J) = \mathcal{O}(f^*, J) = \frac{1}{2}
\]

for all \( k \).
Example 3.2 can easily be modified to show that the decreasing rearrangement is not continuous from $\text{BMO}(\Omega)$ to $\text{BMO}(0, |\Omega|)$ for other domains. In particular, $f_k$ and $f$ can be considered as functions on $\mathbb{R}^n$ that happen to depend only on the first component. By scaling, translation, and restriction, one obtains examples on any bounded domain $\Omega \subset \mathbb{R}^n$.

If we replace the function $g(x) = (-\log |x|)_+$ in Example 3.2 with $g(x) = (-\log |x|)^p_+$ for $p \in (0, 1)$, we obtain examples of functions in $\text{BMO} \setminus \text{VMO}$ with rearrangements in VMO: Since $f_k = f + g_k \notin \text{VMO}(\mathbb{R})$ as they have jump discontinuities at $x = 0$, while $f_k^* \in \text{VMO}(\mathbb{R}_+)$.

Even on the subspace VMO, the decreasing rearrangement is not continuous unless additional conditions are imposed either on the sequence of functions, or on the domain and the collection of shapes, see Theorems 4.8 and 4.10.

Example 3.3. $f$, $f_k$ rearrangeable, $f_k \to f$ in VMO, but $f_k^* \not\to f^*$ in BMO. We again consider a sequence $f_k$ given by Eq. (3.5), with $g_k$ given by Eq. (3.2), but this time with

$$f(x) = \sqrt{(-\log(\frac{1}{2} |x + 6|))_+}, \quad g(x) = \sqrt{(-\log |x|)_+},$$

and $a_k = k^{-\frac{1}{2}}$, $b_k = e^k$, $n_k = b_k + k + 1$, see Figure 3. The distribution function and decreasing rearrangement of $f$ are given by

$$\mu_f(\alpha) = 4e^{-\alpha^2}, \quad f^*(s) = \sqrt{(-\log s + \log 4)_+}.$$ 

Using the properties of scaling and dilation (see Property B3), we have

$$\|g_k\|_{L^1} = k^{-\frac{1}{2}}e^k\|g\|_{L^1}, \quad \|g_k\|_{\text{BMO}} = k^{-\frac{1}{2}}\|g\|_{\text{BMO}},$$

hence $f_k \to f$ in $L^1_{\text{loc}}(\mathbb{R})$ and in $\text{BMO}(\mathbb{R})$, but not in $L^1(\mathbb{R})$.

Since the supports of $f$ and $g_k$ are disjoint, the distribution function of $f_k$ is given by

$$\mu_{f_k}(\alpha) = \mu_f(\alpha) + \mu_{g_k}(\alpha) = \mu_f(\alpha) + 2e^k(1-\alpha^2),$$

![Figure 2. A convergent sequence $\{f_k\}$ in BMO(−1, 1) whose decreasing rearrangements $\{f_k^*\}$ do not converge in BMO(0, 2).](image)
see Eq. (2.3). The right-hand side increases with $k$ for $0 \leq \alpha < 1$, decreases for $\alpha > 1$, and

$$\lim_{k \to \infty} \mu_{f_k}(\alpha) = \begin{cases} +\infty, & 0 \leq \alpha < 1, \\ 4e^{-1} + 2, & \alpha = 1, \\ \mu_f(\alpha), & \alpha > 1. \end{cases}$$

We decompose $f_k = \min\{f_k, 1\} + (f_k - 1)_+$. The distribution function of the first summand is given by

$$\mu_{\min\{f_k, 1\}}(\alpha) = \mu_f(\alpha) + 2e^{k(1-\alpha^2)}$$

for $0 \leq \alpha < 1$ and vanishes for $\alpha \geq 1$. Since this increases with $k$, Eq. (2.1) yields

$$\lim_{k \to \infty} \min\{f_k^*(s), 1\} = \left| \bigcup_{k \geq 1} \{\alpha \in (0, 1] : \mu_f(\alpha) + 2e^{k(1-\alpha^2)} > s\} \right| = 1,$$

where we have used continuity of the measure in the first step.

The second summand has distribution function $\mu_{(f_k - 1)_+}(\alpha) = \mu_f(\alpha + 1) + 2e^{-k(2\alpha + \alpha^2)}$, which decreases with $k$. Eq. (2.1) yields

$$\lim_{k \to \infty} (f_k^* - 1)_+(s) = \left| \bigcap_{k \geq 1} \{\alpha > 0 : \mu_f(\alpha + 1) + 2e^{-k(2\alpha + \alpha^2)} > s\} \right|
= \left| \{\alpha > 0 : \mu_f(\alpha + 1) \geq s\} \right|
= (f^* - 1)_+(s).$$

We have used that the level sets of $f_k$ have finite measure to apply continuity from above. Since $\mu_f$ is strictly decreasing, the set where $\mu_f(\alpha + 1) = s$ consists of a single point. It follows that

$$f_k^* = \min\{f_k^*, 1\} + (f_k^* - 1)_+ \longrightarrow \max\{f^*, 1\}$$

pointwise on $\mathbb{R}$ as $k \to \infty$. By dominated convergence,

$$\lim_{k \to \infty} \|f_k^* - f^*\|_{\text{BMO}} \geq \lim_{k \to \infty} \mathcal{O}(f_k^* - f^*, (0, 2)) = \mathcal{O}((1 - f^*)_+, (0, 2)) > 0.$$

This failure to converge is accompanied by a loss of mass reminiscent of Fatou’s lemma:

$$\| \lim f_k^* \|_{\text{BMO}} = \|\max\{f^*, 1\}\|_{\text{BMO}} < \|f^*\|_{\text{BMO}} = \|\lim f_k^*\|_{\text{BMO}},$$

see Property O6.
For the next two examples, we consider sequences of the form

(3.6) \[ f_k = f - g_k, \]

where \( f \) and \( g \) are fixed functions defined on \( \mathbb{R}^+ \), and \( g_k \) is obtained from \( g \) in the following way:

(3.7) \[ g_k(x) = \min\left\{ \frac{1}{k} g(x - n_k), 1 \right\} \]

for some choice of positive increasing sequence \( \{n_k\} \).

The following example shows that on domains of infinite measure, the decreasing rearrangement of a convergent sequence in VMO need not converge globally on \( \mathbb{R}^+ \).

**Example 3.4.** \( f, f_k \) rearrangeable in VMO\((\mathbb{R}^+)\), \( 0 \leq f_k \uparrow f \) in BMO\((\mathbb{R}^+)\), but \( f^*_k \not\to f^* \) in BMO\((\mathbb{R}^+)\). Choose \( f \) to be the periodic function given by \( f(x) = \frac{1}{2} \cos(\frac{\pi}{2} x) + \frac{3}{2} \) on \( \mathbb{R}^+ \) and \( g(x) = (\ln(x))^+ \) on \( \mathbb{R}^+ \), and let \( n_k \) the smallest integer divisible by 4 with \( n_k \geq ke^k \), see Figure 4. Since \( g_k \leq g_{k+1} \) and \( \|g_k\|_{\text{BMO}} \leq \frac{1}{k} \|g\|_{\text{BMO}} \to 0 \), the sequence \( f_k \) converges to \( f \) monotonically, pointwise, and in BMO.

To see that \( f^*_k \not\to f^* \), we derive a lower bound on the oscillation of \( f^*_k - f^* \) over the interval \( J = (0, n_k + e^k) \). The oscillation of \( f^*_k \) on \( J \) equals that of \( \max\{f_k, 1\} \) on \( J \), and so

\[
\mathcal{O}(f^*_k, J) = \mathcal{O}(\max\{f_k, 1\}, J)
\]

\[
\geq \frac{n_k + e^k}{n_k} \mathcal{O}(\max\{f_k, 1\}, (0, n_k))
\]

\[
\geq \frac{1}{2} \mathcal{O}(\cos x, (0, \pi)) = \frac{1}{\pi}.
\]

Since \( f^* \equiv 2 \), it follows with Property \( O1 \) that \( \|f^*_k - f^*\|_{\text{BMO}} \geq \mathcal{O}(f^*_k, J) \geq \frac{1}{\pi} \).

Our final example shows that on domains of infinite measure, \( L_k = \inf f^*_k \) need not converge to \( L := \inf f^* \), even if \( f_k \uparrow f \) pointwise and \( (L - f^*_k)_+ \to 0 \) in BMO\((\mathbb{R}^+)\).

**Example 3.5.** \( f, f_k \) rearrangeable, \( (L - f^*_k)_+ \to 0 \) in BMO\((\mathbb{R}^+)\), but \( L_k \not\to L \). Still considering sequences given by Eq. (3.6), with \( g_k \) given by Eq. (3.7), take \( f \equiv 2 \) on \( \mathbb{R}^+ \) and \( g(x) = (\ln(x))^+ \) on \( \mathbb{R}^+ \), and let \( \{n_k\} \) be any increasing sequence of positive numbers. Since \( f \) and \( f_k \) are continuous
and decreasing, they coincide with their decreasing rearrangements. Moreover, \( L = 2 \) and \( \| (L - f^*_k) + \|_{\text{BMO}} = \| g_k \|_{\text{BMO}} \leq \frac{1}{k} \| g \|_{\text{BMO}} \to 0. \) On the other hand, \( L_k = 0 \) for every \( k \), and so \( L_k \not\to L. \)

4. Rearrangements on VMO

In this section, we consider rearrangements on VMO and prove the main results. Recall, as explained in the introduction, that rearrangements are fundamentally nonlinear, and so boundedness does not imply continuity. We have already seen in Example 3.2 that the decreasing rearrangement fails to be continuous on \( \text{BMO}(\Omega) \).

4.1. Boundedness. We first show that under suitable assumptions on the basis \( \mathcal{S} \), the decreasing rearrangement of any rearrangeable function \( f \in \text{VMO}_\mathcal{S}(\Omega) \) lies in \( \text{VMO}(0, |\Omega|) \). In addition to the boundedness of the decreasing rearrangement on \( \text{BMO}_\mathcal{S} \), this requires the following assumption on \( \mathcal{S} \).

Density condition: There exists \( q \in (0, \frac{1}{4}] \) such that for every measurable \( E \subset \Omega \) with \( |E||E^c| > 0 \),

\[
\limsup_{S \in \mathcal{S}, |S| \to 0} \rho(E, S)(1 - \rho(E, S)) \geq q, \quad \text{where} \quad \rho(E, S) := \frac{|E \cap S|}{|S|}.
\]

Note that implicit in this is the existence of shapes of arbitrarily small measure. By the Lebesgue density theorem and continuity of the integral, this condition holds for the standard bases \( \mathcal{B}, \mathcal{Q}, \) and \( \mathcal{R} \).

**Theorem 4.1.** Assume that \( \mathcal{S} \) satisfies the density condition \((4.1)\) and that \( f^* \in \text{BMO}(0, |\Omega|) \) whenever \( f \in \text{BMO}_\mathcal{S}(\Omega) \) is rearrangeable, with

\[
\|f^*\|_{\text{BMO}} \leq c\|f\|_{\text{BMO}_\mathcal{S}}.
\]

Then, \( f^* \in \text{VMO}(0, |\Omega|) \) whenever \( f \in \text{VMO}_\mathcal{S}(\Omega) \).

We need two technical lemmas. The first provides a sufficient condition for a nonnegative decreasing function of a single variable to be in VMO.

**Lemma 4.2.** Let \( I \) be an open interval (possibly infinite) with left endpoint at the origin, and let \( g \in L^1_{\text{loc}}(I) \) be nonnegative and decreasing. Then \( g \in \text{VMO}(I) \) if and only if \( g \) is continuous on \( I \) and

\[
\lim_{\delta \to 0} \sup_{J \subset [0, \delta] \cap I} \mathcal{O}(g, J) = 0.
\]

**Proof:** If \( g \in \text{VMO}(I) \), then Eq. \((4.3)\) holds by definition. Furthermore, \( g \) cannot have jump discontinuities hence, being monotone, is continuous on \( I \).

Conversely, suppose that \( g \) is continuous and Eq. \((4.3)\) holds. Given \( \varepsilon > 0 \), take \( \delta > 0 \) so that \( [0, \delta) \subset I \) and \( \sup_{J \subset [0, \delta]} \mathcal{O}(g, J) < \varepsilon. \)

Since \( g \) is continuous, decreasing, and bounded below, it is uniformly continuous on \( [\delta/2, \infty) \cap I. \) Thus there exists \( \eta > 0 \) such that \( |g(x) - g(y)| < \varepsilon \) for every pair of points \( x, y \in I \) with \( x, y \geq \delta/2 \)
and $|x - y| < \eta$. By Property [O5] this implies that $O(g, J) < \varepsilon$ for every interval $J \subset [\delta/2, \infty) \cap I$ with $|J| < \eta$.

Thus for an interval $J \subset I$ with $|J| < \min\{\eta, \delta/2\}$, either $J \subset [0, \delta)$ or $J \subset [\delta/2, \infty) \cap I$, hence $O(g, J) < \varepsilon$.

The next lemma shows that if $f^*$ has a jump discontinuity then the oscillation of $f$ must be large on shapes of arbitrarily small measure, and this can be quantified in terms of the size of the jump.

**Lemma 4.3.** Suppose $\mathcal{S}$ satisfies the density assumption (4.1). Then the decreasing rearrangement $f^*$ of any function $f \in \text{VMO}_{\mathcal{S}}(\Omega)$ is continuous on $(0, |\Omega|)$.

**Proof.** Let $f \in \text{BMO}_{\mathcal{S}}(\Omega)$ be a rearrangeable function. Replacing $f$ with $|f|$, we assume without loss of generality that $f$ is nonnegative.

Since $f^*$ is monotone decreasing and right-continuous, its only possible discontinuities are jumps of the form

$$\beta := \lim_{s \to t^-} f^*(s) > f^*(t) =: \alpha,$$

at some $t \in (0, |\Omega|)$. We will estimate the size of the jump, $\beta - \alpha$, in terms of the modulus of oscillation of $f$.

Consider the truncation $\tilde{f} = \min(\max(f, \alpha), \beta)$. By Property [R4],

$$(\tilde{f})^* = \min(\max(f^*, \alpha), \beta) = \alpha + (\beta - \alpha)\chi_{(0, t)}.$$

This implies, since $\tilde{f} \geq 0$, that $\tilde{f}$ agrees with $\alpha + (\beta - \alpha)\chi_E$ almost everywhere, where $E := E_\gamma(f)$ is the level set of $f$ at any $\gamma \in (\alpha, \beta)$. By equimeasurability (see Property [R1]), $|E| = t > 0$ and $|E^c| > 0$.

Given $\delta > 0$, the density assumption in Eq. (4.1) gives us the existence of a shape $S \in \mathcal{S}$ with $|S| \leq \delta$ such that $\rho(E, S)(1 - \rho(E, S)) \geq q > 0$. We estimate

$$O(f, S) \geq O(\tilde{f}, S) = (\beta - \alpha)O(\chi_E, S) \geq 2q(\beta - \alpha),$$

where we have used Property [O6] in the middle step, then applied Property [O1] and finally Eq. (2.4). As $\delta > 0$ was arbitrary, we have

$$\beta - \alpha \leq (2q)^{-1}\limsup_{|S| \to 0} O(f, S).$$

Thus for $f \in \text{VMO}_{\mathcal{S}}(\Omega)$, $f^*$ cannot have any jumps.

**Proof of Theorem 4.1.** Suppose $f \in \text{VMO}_{\mathcal{S}}(\Omega)$ is rearrangeable. We need to show $f^* \in \text{VMO}(0, |\Omega|)$. By Property [O3] it suffices to consider nonnegative $f$, and by Lemma 4.3 we may further assume that $f^*$ is continuous.

Therefore, by Lemma 4.2 we only need to show that $f^*$ has vanishing mean oscillation at the origin. To do so, we will bound $O(f^*, J)$ for $J \subset [0, \delta)$ with $\delta < |\Omega|$.
Writing $J = (a, b)$ and $\beta = f^*(b)$, consider the function $g = (f^* - \beta)_+ = (f^* - \beta)^*$. Since $f^* \in \text{BMO}(\Omega)$, and $f^* \geq \beta$ on $J$, we have by Eq. (4.2) that

$$\mathcal{O}(f^*, J) = \mathcal{O}((f^* - \beta)_+, J) \leq \| (f^* - \beta)_+\|_{\text{BMO}} \leq c\| (f^* - \beta)_+\|_{\text{BMO}}.$$  

Let $S$ be a shape with $\mathcal{O}((f - \beta)_+, S) \geq \frac{1}{2}\| (f - \beta)_+\|_{\text{BMO}}$. It follows from the above, together with Property 4 and the equimeasurability of $(f - \beta)_+$ and $g$, that

$$\mathcal{O}(f^*, J) \leq 2c \mathcal{O}((f - \beta)_+, S) \leq \frac{4c}{|S|} \int_{\Omega} (f - \beta)_+ = \frac{4c}{|S|} \int_{0}^{b} g \leq \frac{4c}{|S|} \int_{0}^{b} f^*.$$  

For any $\eta > 0$, we therefore get, using Property 6, that

$$\mathcal{O}(f^*, J) \leq \max \left\{ 2c \sup_{|S| \leq \eta} \mathcal{O}(f, S), \frac{4c}{\eta} \int_{0}^{\delta} f^* \right\}.$$  

Since $f \in \text{VMO}_f(\Omega)$ and $f^*$ is in $\text{BMO}(\Omega)$, hence integrable at the origin (see Remark 2.2), we can choose $\eta$ and then $\delta$ to make the right-hand-side arbitrary small.

As a consequence of this bound on the decreasing rearrangement, we are able to obtain an analogous result for the symmetric decreasing rearrangement defined by Eq. (2.2).

**Corollary 4.4.** If $f \in \text{VMO}(\mathbb{R}^n)$ is rearrangeable then $Sf \in \text{VMO}(\mathbb{R}^n)$.

To prove this corollary, we make use of the following technical lemma from [3] that allows for the transfer of mean oscillation estimates for the decreasing rearrangement to the symmetric decreasing rearrangement.

**Lemma 4.5 ([3]).** Let $R > 0$ and $Q \subset B(0, R)$ be a cube of diameter $d$, centred at a point $x$ with $|x| \leq R - d/2$. There is an interval $I \subset (0, \omega_n R^{n-1})$ of length $|I| \leq n\omega_n R^{n-1}d$, such that if $f_1, f_2$ are rearrangeable, then

$$\mathcal{O}(Sf_1 - Sf_2, Q) \leq n^2 \omega_n \mathcal{O}(f^*_1 - f^*_2, I).$$

**Proof of Corollary 4.4.** Let $f \in \text{VMO}(\mathbb{R}^n)$ be rearrangeable. By the boundedness of the decreasing rearrangement on $\text{BMO}(\mathbb{R}^n)$ and Theorem 4.1, $f^* \in \text{VMO}(\mathbb{R}_+)$. Since by definition $Sf(x) = f^*(\omega_n|x|^n)$, it is continuous on $\mathbb{R}^n \setminus \{0\}$ by Lemma 4.3. Moreover, as a radially decreasing, nonnegative function, it is uniformly continuous on the complement of any centred ball of finite radius. By the same argument as in Lemma 4.2, it therefore suffices to show that the modulus of oscillation on $B(0, R)$ vanishes as $R \to 0$.

For a cube $Q$ contained in $B(0, R)$, Lemma 4.5 applied to $f_1 = f$ and $f_2 = 0$, yields that

$$\mathcal{O}(Sf, Q) \leq n^2 \omega_n \mathcal{O}(f^*, I)$$

for an interval $I$ with $|I| \leq c_n R^n$. Since $f^* \in \text{VMO}(\mathbb{R}_+)$, we have

$$\lim_{R \to 0} \sup_{Q \subset B(0, R)} \mathcal{O}(Sf, Q) = 0.$$  

$\square$
4.2. **Continuity.** In this section, we derive conditions on a sequence of functions $f_k$ in BMO converging to a function $f$ in VMO that ensure that the sequence of rearrangements $f_k^*$ converges in BMO to $f^*$ in VMO.

For VMO, there exists an analogue of the Arzelà-Ascoli theorem that can be used to characterize relative compactness [5]. In our case, we take advantage of the monotonicity of rearrangements and make use of a theorem of Pólya (see [21] and [22, page 270]), given here under slightly weakened assumptions.

**Lemma 4.6.** Let $f_k, k \in \mathbb{N}$, be monotone decreasing functions on $(0, b)$ for some $0 < b \leq \infty$ converging almost everywhere to a continuous function $f$. Then, the convergence is uniform on any compact subinterval of $(0, b)$. Furthermore, if $b = \infty$, $f_k, k \in \mathbb{N}$, and $f$ are bounded below and $\inf t f_k(t) \to \inf t f(t)$, then the convergence is uniform on $[a, \infty)$ for any $a > 0$.

**Proof.** Given $\varepsilon > 0$, select a partition $a < x_0 < \ldots < x_n < b$ such that for each $i = 1, \ldots, n$, $|f(y) - f(z)| < \varepsilon/2$ for all $y, z \in [x_{i-1}, x_i]$, and there exists $K_i$ such that $|f_k(x_i) - f(x_i)| < \varepsilon/2$ whenever $k \geq K_i$. Fix $x \in [x_0, x_n]$ and select $i$ such that $x \in [x_{i-1}, x_i]$. Then for $k \geq \max K_i$,

$$
|f(x) - \varepsilon| < f(x_i) - \varepsilon/2 \leq f_k(x_i) \leq f_k(x) \leq f_k(x_{i-1}) < f(x_{i-1}) + \varepsilon/2 < f(x) + \varepsilon.
$$

Thus, for a compact subinterval $I \subset (0, b)$, if $\{f_k\}$ converges at the endpoints of $I$ this shows that $\{f_k\}$ converges uniformly to $f$ on $I$. If $\{f_k\}$ does not converge at either of the endpoints, $I$ can always be extended to a larger compact subinterval $\tilde{I} \subset (0, b)$ on which the convergence is uniform, implying uniform convergence on $I$.

In the case $b = \infty$, then the assumption that $\inf t f_k(t) \to \inf t f(t)$ means that one may choose $x_n = \infty$ in the previous argument, giving the result. \hfill $\square$

The next lemma provides a sufficient condition for the decreasing rearrangements of a convergent sequence in VMO to be relatively compact.

**Lemma 4.7.** Let $\mathcal{S}$ be a basis of shapes in a domain $\Omega \subset \mathbb{R}^n$, satisfying the hypotheses of Theorem 4.1. Let $f_k, k \in \mathbb{N}$, and $f$ be rearrangeable functions in $\text{BMO}_\mathcal{S}(\Omega)$.

If $f \in \text{VMO}_\mathcal{S}(\Omega)$, $f_k \to f$ in $\text{BMO}_\mathcal{S}(\Omega)$, and $f_k^* \to f^*$ in $L^1(0, b)$ for some $0 < b \leq |\Omega|$, then $f_k^* \to f^*$ in $\text{BMO}(0, b)$.

**Proof.** Fix $b \in (0, |\Omega|]$ such that $\{f_k^*\}$ converges to $f^*$ in $L^1(0, b)$. Any subsequence of $\{f_k^*\}$ will then also converge to $f^*$ in $L^1(0, b)$ and so have a further subsequence that converges pointwise almost everywhere to $f^*$ on $(0, b)$. It suffices to show that this subsequence, which we continue to denote by $\{f_k^*\}$, converges to $f^*$ in $\text{BMO}(0, b)$.

As $f^*$ is continuous by Lemma 4.3 and the functions $f_k^*$ are monotone decreasing and converge pointwise almost everywhere to $f^*$, Lemma 4.6 tells us that the convergence is uniform on $[\delta, b)$ for any $\delta, 0 < \delta < b$. Note that if $b = \infty$, then the fact that $f_k^*, f^*$ are in $L^1(\mathbb{R}_+)$ means that $\inf t f_k^*(t) = 0 = \inf t f^*(t)$ for all $k$. 

Given such a $\delta$, if $J \subset (0, \delta)$, then we have, as in the proof of Theorem 4.1,
\[
\mathcal{O}(f^*, J) \leq \max \left\{ 2c \sup_{|S| < \eta} \mathcal{O}(f, S), \frac{4c}{\eta} \int_0^{\delta} f^* \right\},
\]
and correspondingly for each $f_k^*$. Given $\epsilon > 0$, since $f \in \text{VMO}_\mathcal{S}(\Omega)$, we can choose $\eta > 0$ to make $2c \sup_{|S| < \eta} \mathcal{O}(f, S) < \epsilon/2$, and the convergence of $f_k$ to $f$ in $\text{BMO}_\mathcal{S}(\Omega)$ means that for this $\eta$ and all sufficiently large $k$, $2c \sup_{|S| < \eta} \mathcal{O}(f_k, S) < \epsilon$. For this $\eta$, we can choose $\delta > 0$ such that $\frac{4c}{\eta} \int_0^{\delta} f^* < \epsilon$, and also, since $\{f_k^*\}$ is convergent in $L^1(0, b)$, hence uniformly integrable, $\frac{4c}{\eta} \int_0^{\delta} f_k^* < \epsilon$ for all $k$.

Combining, we get that for $\delta$ sufficiently small and $k$ sufficiently large, $\mathcal{O}(f_k^* - f^*, J) < 2\epsilon$ for $J \subset (0, \delta)$. By uniform convergence, the stronger estimate $\sup_J |f_k^* - f^*| < 2\epsilon$ holds when $J \subset [\delta/2, b)$.

If $J \subset (0, b)$ is not in one of these cases, then $J \supset (\delta/2, \delta)$. Let $g = f_k^* - f^*$, $I = J \cap (0, \delta)$, $I' = J \cap (\delta/2, \delta)$. Noting that $|I'| \geq |I|/2$, we can estimate
\[
\mathcal{O}(g, J) \leq 2 \int_I |g - g_r| \leq \frac{1}{|J|} \int_J |g - g_I| + |g_I - g_r| + \frac{1}{|J|} \int_{J \setminus I} |g - g_r| \leq 3 \mathcal{O}(g, I) + 2 \sup_{[\delta/2, b]} |g|.
\]
Thus we have shown that for $k$ sufficiently large, $\|f_k^* - f^*\|_{\text{BMO}(0, b)} \leq 10\epsilon$. \hfill \Box

**Theorem 4.8.** Let $\mathcal{S}$ be a basis of shapes in a domain $\Omega \subset \mathbb{R}^n$, satisfying the hypotheses of Theorem 4.1. Let $f_k, k \in \mathbb{N}$, and $f$ be rearrangeable functions in $\text{BMO}_\mathcal{S}(\Omega)$.

If $f \in \text{VMO}_\mathcal{S}(\Omega)$, and $f_k \rightarrow f$ in $\text{BMO}_\mathcal{S}(\Omega)$ and in $L^1(\Omega)$, then $f_k^* \rightarrow f^*$ in $\text{BMO}(0, |\Omega|)$.

**Proof.** By Property R2, $\{f_k\}$ to $f$ in $L^1(\Omega)$ implies the convergence of $\{f_k^*\}$ to $f^*$ in $L^1(0, |\Omega|)$. The convergence in $\text{BMO}(0, |\Omega|)$ follows then by taking $b = |\Omega|$ in Lemma 4.7. \hfill \Box

We are now ready to prove the results given in the introduction. Note that in the case of $\Omega = Q_0$, the assumption of $L^1$ convergence of $\{f_k\}$ follows from BMO convergence upon normalization of the means — see Property B2.

**Proof of Theorems 1 and 2.** The basis $Q$ is well known to satisfy the hypotheses of Theorem 4.1, see the remark after the density condition (4.1).

Let $Q_0$ be a finite cube and $f \in \text{VMO}(Q_0)$. Since $Q_0$ has finite measure, $f$ is rearrangeable, and Theorem 4.1 yields that $f^* \in \text{VMO}(Q_0)$. If, moreover, $f_k \rightarrow f$ in $\text{BMO}(Q_0)$ and $\int_{Q_0} f_k \rightarrow \int_{Q_0} f$, then $f_k \rightarrow f$ also in $L^1(Q_0)$. It follows from Theorem 4.8 that $f_k^* \rightarrow f^*$ in $\text{BMO}(Q_0)$. \hfill \Box

It remains to consider the case of infinite domains. While under the condition of the previous theorem, we have convergence of the rearrangements in BMO on any finite interval, Example 3.4 shows that convergence on all of $\mathbb{R}_+$ requires further assumptions at infinity.

**Lemma 4.9.** Let $\mathcal{S}$ be a basis of shapes in a domain $\Omega \subset \mathbb{R}^n$ of infinite measure, satisfying the hypotheses of Theorem 4.1. Let $f_k, k \in \mathbb{N}$, and $f$ be rearrangeable functions in $\text{BMO}_\mathcal{S}(\Omega)$, and write $L := \inf f^*$. 


If \( f \in \text{VMO}_\mathcal{F}(\Omega) \), \( f_k \to f \) in \( \text{BMO}_\mathcal{F}(\Omega) \), \( f_k^* \to f^* \) in \( L^1(0, b) \) for every \( 0 < b < \infty \), and
\[
\|(L - f_k^*)_+\|_{\text{BMO}} \to 0 \quad \text{as} \quad k \to \infty,
\]
then \( f_k^* \to f^* \) in \( \text{BMO}(\mathbb{R}_+) \).

The hypothesis that \( (L - f_k^*)_+ \to 0 \) can be replaced by the convenient assumption that \( \inf f_k \to L \) as \( k \to \infty \), i.e., \( \mu_{f_k}(\alpha) \to \infty \) for all \( \alpha < L \). However, this assumption is strictly stronger, see Example 3.5.

Proof of Lemma 4.9 Consider \( f_k, k \in \mathbb{N} \), and \( f \) satisfying the hypotheses. By Theorem 4.1, \( f^* \in \text{VMO}(0, \lvert \Omega \rvert) \).

Exhausting \( \mathbb{R}_+ \) by intervals of the form \( (0, b) \) for \( 0 < b < \infty \), and using the convergence of \( \{f_k^*\} \) to \( f^* \) in \( L^1(0, b) \), we see that any subsequence of \( \{f_k^*\} \) has a further subsequence that converges pointwise almost everywhere to \( f^* \) on \( \mathbb{R}_+ \). Denote by \( E \subset \mathbb{R}_+ \) the set on which convergence holds. It suffices to show that this subsequence, which we continue to denote by \( \{f_k^*\} \), converges to \( f^* \) in \( \text{BMO}(\mathbb{R}_+) \). We will show that for any \( a \in E \),
\[
\limsup_{k \to \infty} \|f^* - f_k^*\|_{\text{BMO}(\mathbb{R}_+)} \leq 4(f^*(a) - L).
\]
The result follows by taking \( a \to \infty \) from the monotonicity of \( f^* \) and the definition of \( L \).

Note that for any \( 0 < \lambda < \infty \), we can write \( f_k^* \) as
\[
\max\{f_k^*, \lambda\} - (\lambda - f_k^*)_+.
\]
Taking \( \lambda = L \), we have that
\[
\|f^* - f_k^*\|_{\text{BMO}(\mathbb{R}_+)} \leq \|f^* - \max\{f_k^*, L\}\|_{\text{BMO}(\mathbb{R}_+)} + \|(L - f_k^*)_+\|_{\text{BMO}(\mathbb{R}_+)}.
\]
By assumption, the last term converges to zero as \( k \to \infty \).

For the first term on the right hand side of Eq. (4.5), let \( a \in E \), take \( b \geq 2a \), and consider an arbitrary interval \( J \subset \mathbb{R}_+ \). For \( J \subset (0, b) \) we use that \( f_k^*(s) - \max\{f_k^*(s), L\} = (f_k^*(s) - L)_+ \) decreases with \( s \) to estimate
\[
\sup_{J \subset (0, b)} \mathcal{O}(f^* - \max\{f_k^*, L\}, J) \leq \sup_{J \subset (0, b)} \mathcal{O}(f^* - f_k^*, J) + \sup_{J \subset (0, b)} \mathcal{O}(f_k^* - \max\{f_k^*, L\}, J)
\]
\[
\leq \|f^* - f_k^*\|_{\text{BMO}(0, b)} + (L - f_k^*(a))_.
\]
As \( k \to \infty \), the first term converges to zero by Lemma 4.7 and the second converges to \( (L - f^*(a))_+ = 0 \) since \( a \in E \). For \( J \subset (a, \infty) \) we have
\[
\sup_{J \subset (a, \infty)} \mathcal{O}(f^* - \max\{f_k^*, L\}, J) \leq \sup_{J \subset (a, \infty)} \frac{1}{|J|} \int_J |f^* - L| + \sup_{J \subset (a, \infty)} \int_J |L - \max\{f_k^*, L\}|
\]
\[
\leq \sup_{s \geq a} |f^*(s) - L| + \sup_{s \geq a} (f_k^*(s) - L)_+
\]
\[
= (f^*(a) - L) + (f_k^*(a) - L)_+,
\]
which converges to \( 2(f^*(a) - L) \) as \( k \to \infty \) since \( a \in E \). Finally, if \( J \supset (a, b) \), then
\[
\frac{1}{|J|} \int_{J \cap (0, b)} |f^* - \max\{f_k^*, L\}| \leq \frac{1}{b - a} \|f^* - \max\{f_k^*, L\}\|_{L^1(0, b)} \leq \frac{1}{a} \|f^* - f_k^*\|_{L^1(0, b)}.
\]
and
\[ \frac{1}{|J|} \int_{J \cap (b, \infty)} |f^* - \max \{ f^*_k, L \}| \leq (f^*(a) - L) + (f^*_k(a) - L)_+, \]
where we have again used monotonicity of $f^*$ and $f^*_k$ in the last step. It follows that
\[ \sup_{J \supset (a, b)} \mathcal{O}(f^* - \max \{ f^*_k, L \}, J) \leq \frac{2}{d} \| f^* - f^*_k \|_{L^1(0,b)} + 2 (f^*(a) - L) + 2 (f^*_k(a) - L)_+. \]
As $k \to \infty$, the first term on the right hand side converges to zero by assumption, and the last term converges to $2(f^*(a) - L)$ since $a \in E$. This completes the proof of Eq. (4.4).

**Theorem 4.10.** Let $\mathcal{S}$ be a basis of shapes in a domain $\Omega \subset \mathbb{R}^n$ of infinite measure, satisfying the hypotheses of Theorem [4.1]. Let $f_k, k \in \mathbb{N}$, and $f$ be rearrangeable functions in $\text{BMO}_\mathcal{S}(\Omega)$, and write $L := \inf f^*$.

If $f \in \text{VMO}_\mathcal{S}(\Omega)$, $f_k \to f$ in $\text{BMO}_\mathcal{S}(\Omega)$, $0 \leq f_k \uparrow f$ pointwise, and $(L - f^*_k)_+ \to 0$ in $\text{BMO}(\mathbb{R}_+)$, then $f^*_k \to f^*$ in $\text{BMO}(\mathbb{R}_+)$.  

**Proof.** By Property [R3], if $f_k \uparrow f$ on $\Omega$, then $f^*_k \uparrow f^*$ on $\mathbb{R}_+$. By monotone convergence, $f^*_k \to f^*$ in $L^1(0, b)$ for any $b < \infty$, and we can apply Lemma [4.9].

By Lemma [4.5], the conclusion of Theorem [4.10] directly extends to the symmetric decreasing rearrangement.

**Corollary 4.11.** Under the hypotheses of Theorem [4.10] $Sf_k \to Sf$ in $\text{BMO}(\mathbb{R}^n)$.

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