Non-renormalisation of extremal correlators in $\mathcal{N}=4$ SYM theory

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Abstract

We show that extremal correlators of chiral primary operators in $\mathcal{N}=4$ supersymmetric Yang–Mills theory with $SU(N)$ gauge group are neither renormalised at first ($g^2$) order in perturbation theory nor receive contribution from any instanton sector at leading order in the semiclassical expansion. This lends support to the strongest version of a new prediction recently put forward on the basis of the AdS/SCFT correspondence.

1 Introduction

A new prediction of the correspondence between type IIB superstring on $AdS_5 \times S^5$ and $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) theory with $SU(N)$ gauge group [1, 2] is that a certain class of $n$-point correlation functions, represented by “extremal correlators” of chiral primary operators (CPO’s), satisfy some non-renormalisation theorem. In [3], D’Hoker, Freedman, Mathur, Matusis and Rastelli have argued that type IIB supergravity requires the above $n$-point functions to decompose into products of $n-1$ free-field two-point functions. Moreover the overall coefficient was argued not to be renormalised with respect to its free field value. Something similar is known to happen for two- and three-point functions of CPO’s [4, 5, 6] as well as for the chiral [7] and Weyl [8] anomalies.

The extremal correlation functions we consider are of the form

$$G_{\text{ext}}(x, x_1, \ldots, x_n) = \langle Q^{(\ell)}(x) Q^{(\ell_1)}(x_1) \ldots Q^{(\ell_n)}(x_n) \rangle,$$  \hspace{1cm} (1)

1The $1/N^2$ corrections to the chiral anomaly are reproducible in terms of one-loop corrections to the lowest order supergravity approximation due to the absence in the bulk of the would-be singleton fields [9].
with $\ell = \ell_1 + \ell_2 + \ldots + \ell_n$. The operators $Q^{(l)}$ in (1) are CPO’s, i.e. scalar composite operators of protected dimension $\Delta = \ell$ belonging to the representation with Dynkin labels $[0, \ell, 0]$ of the $SU(4)$ R-symmetry group. They are lowest components of short $SU(2,2|4)$ supermultiplets. Other shortening conditions are possible, see [10, 11] for a detailed discussion.

For single-trace CPO’s one has

$$Q^{i_1 i_2 \ldots i_\ell}(x) = \sum_{\text{perms } \sigma} \text{tr} \left[ \varphi^{(i_1)}\varphi^{(i_2)} \ldots \varphi^{(i_\ell)} \right] - \text{flavour contractions} .$$

For multi-trace CPO’s one has an obvious generalisation of (2). The correlator $[\ ]$ is “extremal” in that there is only one $SU(4)$ invariant contraction of the (implicit) “flavour” indices, i.e. there is only one $SU(4)$ singlet in the product of the representations with Dynkin labels $[0, \ell, 0]$, $[0, \ell_1, 0]$, $\ldots$, $[0, \ell_n, 0]$.

It is the purpose of this letter to show that extremal correlators of CPO’s are neither renormalised at first order in perturbation theory nor receive contribution from any instanton sector at leading order in the semiclassical expansion.

The non-renormalisation properties displayed by extremal correlators of CPO’s at first order in perturbation theory suggest that the strong version of the argument proposed in [3] on the basis of the AdS/SCFT correspondence should be valid. One might then expect that any extremal correlators, either involving single-trace operators or multi-trace operators, should be independent of the coupling constant $g$, hence tree-level exact, for any finite $N$. If this were the case no higher derivative term in the type IIB superstring effective action should be capable of giving a non-vanishing amplitude of this kind. $SL(2,Z)$ invariance of type IIB superstring requires that string loop corrections to higher derivative terms be accompanied by non-perturbative D-instanton corrections. It is by now widely appreciated that the counterpart of type IIB D-instantons in the AdS/SCFT correspondence are SYM instantons [12, 13, 14]. These are responsible for interesting $U(1)_B$ violating processes such as a 16-dilatino amplitude, that has been used for a quantitative test of the AdS/SCFT correspondence [13, 14], as well as for some $U(1)_B$ preserving processes such as the higher derivative corrections to the four stress-tensor/graviton amplitude. Motivated by these considerations we have extended our analysis to the non-perturbative level.

The plan of the letter is as follows. After briefly reviewing the description of $\mathcal{N}=4$ supersymmetric Yang–Mills theory in terms of $\mathcal{N}=1$ superfields, we demonstrate the vanishing of the lowest perturbative correction to the extremal correlators. We then pass to describe the fermion zero-mode counting in instanton backgrounds and show that non-perturbative corrections are absent as well. Finally we comment on the bearing and extension of our results in view of the AdS/SCFT correspondence.

We would like to stress that the results we are going to show are valid both for single- and for multi-trace operators in the relevant $SU(4)$ representation and for any number of colours $N$, suggesting the validity of the non-renormalisation theorem well beyond the reach of the lowest supergravity approximation, valid at large $N$ and large ’t Hooft

\footnote{Dan Freedman has informed us that a similar result has been independently obtained by Witold Skiba.}
coupling. We suspect, but we do not explicitly show, that the result should hold for extremal correlators in $\mathcal{N}=4$ SYM theories with other gauge groups.

\section{$\mathcal{N}=4$ SYM theory in the $\mathcal{N}=1$ formulation}

The field content of $\mathcal{N}=4$ SYM \cite{[15]} is realised combining one $\mathcal{N}=1$ vector superfield, $V$, with three $\mathcal{N}=1$ chiral superfields, $\Phi^I$ ($I=1,2,3$), all in the adjoint representation of the gauge group. The six real scalars, $\varphi^i$ ($i=1,2,\ldots,6$), of the theory are assembled into three complex fields, namely

$$
\phi^I = \frac{1}{\sqrt{2}} (\varphi^I + i\varphi^{I+3}) , \quad \phi_I = \frac{1}{\sqrt{2}} (\varphi^I - i\varphi^{I+3}) ,
$$

that are scalar components of the superfields $\Phi^I$. Three of the Weyl fermions are the spinors of the chiral multiplets, denoted by $\lambda^I$, and the fourth spinor, $\lambda = \lambda^0$, together with the vector, $A_{\mu}$, form the vector multiplet. In this formulation only a $SU(3) \times U(1)$ subgroup of the original $SU(4)$ R-symmetry group is manifest, with $\Phi^I$ and $\Phi_I$ transforming in the representations $3$ and $\overline{3}$ of $SU(3)$ respectively, while $V$ is a singlet.

The action in the $\mathcal{N}=1$ superfield formulation reads

$$
S = \frac{1}{l_r g^2} \text{tr} \left\{ \int d^4x \left[ \left( \int d^4\theta \Phi^I e^V \Phi_I \right) + \left( \int d^2\theta \frac{1}{16} W^a W_a + \text{h.c.} \right) + \frac{i\sqrt{2}}{3!} \left( \int d^2\theta \varepsilon_{IJK} \Phi^I [\Phi^J, \Phi^K] + \int d^2\bar{\theta} \varepsilon^{IJK} \Phi_I^\dagger [\Phi^J_\dagger, \Phi^K_\dagger] \right) \right] \right\} ,
$$

where $l_r$ denotes the Dynkin index of the representation. In what follows we will mostly restrict our attention to the case of an $SU(N)$ gauge group. All the (super)fields belong to the adjoint representation, \textit{viz.}

$$
V(x, \theta, \bar{\theta}) = V^a(x, \theta, \bar{\theta}) T_a , \quad \Phi^I(x, \theta, \bar{\theta}) = \Phi^{ai}(x, \theta, \bar{\theta}) T_a , \quad \Phi_I(x, \theta, \bar{\theta}) = \Phi^a_I(x, \theta, \bar{\theta}) T_a .
$$

In equation (5) the standard dependence on the coupling constant $g$ is recovered by substituting $V \rightarrow 2gV$, $\Phi^I \rightarrow g\Phi^I$ and $\Phi_I \rightarrow g\Phi_I$. Notice that the full action also contains the ghosts that are not displayed here since they do not contribute to the Green functions that we will consider at the order at which we compute them.

Expanding the exponential $e^{2gV}$ in (6) gives

$$
S = \int d^4 x \ [V^a [-\Box + (1-\xi)(P_1 + P_2)\Box] V_a + \Phi^{aI}_I \Phi^{I}_I + ig f_{abc} \Phi^a_I V^b \Phi^c_I + \ldots + \frac{\sqrt{2}}{3!} g f_{abc} \left[ \varepsilon_{IJK} \Phi^a_I \Phi^b_J \Phi^c_K \delta(\theta) + \varepsilon^{IJK} \Phi_I^a \Phi_J^b \Phi_K^c \delta(\theta) \right] + \ldots ,
$$

where $f_{abc}$ are the $SU(N)$ structure constants and terms that are not relevant for first order perturbative calculations have been neglected. The projection operators $P_1$ and $P_2$ are defined as

$$
P_1 = \frac{1}{16} \overline{\Box} \Box , \quad P_2 = \frac{1}{16} \Box \overline{\Box} .
$$
where $\mathcal{D}$ and $\mathcal{D}'$ are the standard super-covariant derivatives [16]. In (1) a gauge fixing term ($\xi = 1/\alpha$ is a gauge parameter)

$$S_{g.f.} = \frac{1}{16g^2} \text{tr} \int d^4x \int d^2\theta d^2\bar{\theta} \left[ -\frac{\xi}{32} (\mathcal{D}^2 V) (\mathcal{D}'^2 V) \right]$$

has been introduced as well.

We choose to work in components and without fixing the Wess-Zumino gauge. By expanding chiral and vector superfields according to

$$\Phi^I(x, \theta, \bar{\theta}) = \phi^I(x) + \sqrt{2} \theta \lambda^I(x) + \theta \Phi^I(x) + i \theta \sigma^\mu \partial_\mu \phi^I(x) + \frac{1}{\sqrt{2}} \theta \bar{\theta} \sigma^\mu \partial_\mu \lambda^I(x) +$$

$$+ \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \phi^I(x),$$

$$V(x, \theta, \bar{\theta}) = C(x) + i \theta \chi(x) - i \bar{\theta} \chi(x) + \frac{i}{\sqrt{2}} \theta \theta S(x) - \frac{i}{\sqrt{2}} \bar{\theta} \bar{\theta} \bar{\theta} \bar{\theta} S^I(x) - \theta \sigma^\mu \bar{\theta} A_\mu(x) +$$

$$+ i \theta \bar{\theta} \left[ \lambda(x) + \frac{i}{2} \sigma^\mu \partial_\mu \chi(x) \right] - i \bar{\theta} \theta \left[ \tilde{\lambda}(x) + \frac{i}{2} \sigma^\mu \partial_\mu \tilde{\chi}(x) \right] + \frac{1}{2} \theta \theta \bar{\theta} \bar{\theta} \left[ D(x) + \frac{1}{2} \Box C(x) \right],$$

in the action, one obtains the equivalent component-field formulation. In the Fermi-Feynman gauge, $\alpha = 1$, the kinetic part of the action in components reads [17]

$$S_0 = \int d^4x \left[ \phi^I_{Ia} \Box \phi^I_a - \lambda^I_a \sigma^\mu \partial_\mu \lambda^I_a + F_{Ia}^{ab} F_a^{bc} - S_a \Box S^a + \frac{1}{2} A_\mu^a \Box A_\mu^a + \right.$$  

$$\left. - \frac{1}{2} C^a_a \square C_a + C^a \Box D_a + D^a \Box C_a \right] + \frac{1}{2} \left( \lambda^a \square \sigma^\mu \partial_\mu \lambda^a + \lambda^a \sigma^\mu \partial_\mu \lambda^a + \right.$$  

$$\left. - \lambda^a \Box \lambda^a - \lambda^a \Box \lambda^a \right) .$$

The trilinear couplings, that are needed to compute the first order, i.e. $O(g^2)$, perturbative corrections to $n$-point functions of scalar composite operators, such as the extremal correlators of CPO’s, can be obtained from the action (10) by an analogous expansion. The result is [17]

$$S_{\text{int}} = \int d^4x \left\{ ig f_{abc} \left[ \frac{1}{2} \phi^c_a \Box \phi^b_{Ia} + \frac{1}{2} \phi^a_{Ia} \Box \phi^c_I - \frac{1}{2} (\partial_\mu \phi^a_{I}) C^b (\partial_\mu \phi^c_I) + \right. \right.$$  

$$\left. + \frac{i}{2} (\phi^a_{Ia} A^b_\mu (\partial_\mu \phi^c_I) - (\partial_\mu \phi^a_{I}) A^b_\mu \phi^c_I) + \frac{i}{\sqrt{2}} (\phi^a_{Ia} S^b_{I} F^{cI} - F_{Ia}^{bc} S^b_{I} \phi^c_I) + \right.$$  

$$\left. - F_{Ia}^{bc} F^{cI} + \frac{i}{\sqrt{2}} (\lambda^I_a \lambda^b_{Ia} \phi^c_I - \phi^a_{Ia} \lambda^b \lambda^c) + \frac{i}{\sqrt{2}} (F_{Ia}^{bc} \chi^b \lambda^c_I - \lambda^I_a \chi^b \lambda^c_I) + \right.$$  

$$\left. \frac{i}{\sqrt{2}} (\phi^{aI} (\partial_\mu \lambda^c_I) \sigma^\mu \lambda^c_I + \lambda^I_a \sigma^\mu (\partial_\mu \lambda^c_I) \phi^c_I) + \right.$$  

$$\left. \frac{1}{2} (\sigma^\mu C^a_I \sigma^\mu (\partial_\mu \lambda^c_I) - C^b (\partial_\mu \lambda^c_I) \sigma^\mu \lambda^c_I) + \frac{i}{2} \lambda^a \sigma^\mu \lambda^c_I A^b_\mu - \frac{\sqrt{2}}{3!} g f_{abc} \left[ \varepsilon^{IJK} (\phi^a_{Ia} \phi^b_{JK} F^{cI} - \phi^a_{Ia} \lambda^b_{JK}) + \varepsilon^{IJK} (\phi^a_{Ia} \phi^b_{JK} F^{cJ} + \right. \right.$$  

$$\left. \phi^a_{Ia} \lambda^b_{IK} \lambda^c_I) \right\} .$$

(12)
3 Perturbative non-renormalisation

In the following we will concentrate on extremal correlators of the specific form

\[
G(x, x_1, \ldots, x_n) = \langle \text{tr} \left[(\phi^1)^\ell_1(x)\right] \text{tr} \left[(\phi^1)^\ell_2(x_2)\right] \ldots \text{tr} \left[(\phi^1)^\ell_n(x_n)\right]\rangle. \tag{13}
\]

Up to an overall non-vanishing Clebsch–Gordan coefficient, computing correlators of this kind is equivalent to computing the generic correlator (I).

The tree-level contribution to (13) corresponds to a diagram with \(\ell\) lines exiting from the point \(x\), which form \(n\) different “rainbows” connecting \(x\) to the points \(x_i\), the \(i\)th rainbow containing \(\ell_i\) lines

The free scalar propagator is

\[
\langle \phi^I_a(x_0) \phi^J_b(y) \rangle_{(0)} = \frac{\delta_{ab}}{(2\pi)^2} \frac{\delta^I_J}{(x-y)^2}, \tag{14}
\]

hence at tree-level the Green function (13) reads

\[
G_{(0)}(x, x_1, \ldots, x_n) = \frac{c_N}{(2\pi)^{2\ell}} \frac{1}{(x-x_1)^2 \ldots (x-x_n)^2} \sum_{\text{perms } \sigma} \left[ \text{tr} \left(T^{a_{\sigma(1)}} T^{a_{\sigma(2)}} \ldots T^{a_{\sigma(\ell)}} \right) \right] \cdot \text{tr} \left(T^{a_{1}} \ldots T^{a_{\ell_1}}\right) \text{tr} \left(T^{a_{\ell_1+1}} \ldots T^{a_{\ell_1+\ell_2}}\right) \ldots \text{tr} \left(T^{a_{\ell_1+\ldots+\ell_{n-1}+1}} \ldots T^{a_{\ell}}\right). \tag{15}
\]

Using the \(\mathcal{N}=1\) description introduced in the previous section, we will momentarily show that the first order perturbative correction to (13) and hence to (I) is zero. Let us preliminarily notice that if one keeps all the component fields in the vector supermultiplets, i.e. if one does not employ the WZ gauge, and works in the Fermi-Feynman gauge \(\alpha = 1\) there is no first order correction to the propagators of the elementary fields. The off-shell self-energy corrections due to vector exchange, including a very peculiar \(C - D\) exchange, cancel the contributions due to the three chiral multiplets (I). In this approach, we do not need bother with UV and IR problematic corrections to two-point functions of elementary fields.

It is easier to first analyse the possible corrections to the relevant diagrams in superfield language, in which (I) is obtained as the \(\theta = 0\) component of a correlator of (anti)chiral superfields. The choice of flavour indices made in (I) is crucial in all subsequent computations. In particular, one can easily check that it prevents the insertion of
(anti)chiral trilinear vertices at order $g^2$. The only relevant diagrams are thus obtained by the insertion of vector superfield lines into the tree diagram. It is very convenient to regularise the diagrams by point-splitting. According to the form of the action in components, see equations (11) and (12), vector exchange between a pair of chiral lines

$$\begin{align*}
\Phi &\rightarrow V & \Phi^\dagger
\end{align*}$$

corresponds to three different diagrams for the lowest scalar components

$$\begin{align*}
A_\mu &\rightarrow \phi & A^\mu &\rightarrow \phi^\dagger \\
D &\rightarrow \phi & CD &\rightarrow \phi^\dagger
\end{align*}$$

In the following we will refer to the sum of these three terms as vector exchange unless otherwise stated. In (16) the internal propagators are respectively

$$\begin{align*}
\langle A^a(x)A^b(y)\rangle_{(0)} &= \frac{\delta_{\mu\nu}\delta^{ab}}{(2\pi)^2(x-y)^2} , \\
\langle C^a(x)D^b(y)\rangle_{(0)} &= -\frac{\delta^{ab}}{(2\pi)^2(x-y)^2} , \\
\langle D^a(x)D^b(y)\rangle_{(0)} &= \delta^{ab}\delta(x-y).
\end{align*}$$

The resulting first order corrections to the correlator (13) are of the form

$$G^{(A)}_{(1)}(x, x_1, \ldots, x_n) = g^2 c(n, N) \mathcal{G}(x, x_1, x_2, \ldots, x_n) ,$$

where the coefficient $c(n, N)$ comes from colour contractions and the spatial dependence is encoded in the function $\mathcal{G}(x, x_1, x_2, \ldots, x_n)$. In order to proceed, it is convenient to distinguish two types of corrections, those in which there is a vector exchange within a single rainbow and those in which the vector lines are inserted between two different rainbows. Contributions of the first kind will be denoted by $G^{(A)}_{(1)}$ and the others by $G^{(B)}_{(1)}$.

Corrections of the first kind are exactly those that appear in the two-point functions of CPO’s and have been argued to vanish for a variety of reasons. Nevertheless we feel worth showing their vanishing by an explicit computation. Each of the diagrams in $G^{(A)}_{(1)}$ is zero due to the vanishing of the corresponding contribution to the function $\mathcal{G}(x, x_1, x_2, \ldots, x_n)$. More precisely, for a diagram with the insertion of the vector in the rainbow connecting $x$ and $x_i$
one obtains

\[ G_{(1)}^{(A)}(x, x_1, \ldots, x_n) = c^{(A)}(n, N) \frac{g^2}{(2\pi)^{2\ell-8}} \prod_{j=1}^{n} \frac{1}{(x - x_j)^{2\ell_j}} \cdot \frac{1}{(x - x_j)^{2\ell_j}} \lim_{u \to x \atop v \to x_i} \left[ \frac{-1}{2(2\pi)^2} \ln((\partial_x + \partial_u)^2 f(x, x_i, u, v) \right], \]

where the function \( f \) is defined as

\[ f(x_1, x_2, x_3, x_4) = \int dx_5 dx_6 \left[ \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 x_7^2 x_8^2} \right], \]

with \( x_{ij} = x_i - x_j \). In (21) a regularisation by point-splitting on the pair of lines interested by the interaction has been introduced. The limit \( v \to x_i \) in (21) can be taken without subtleties, it gives a finite result \[6\]

\[ \lim_{v \to x_i} \left[ (\partial_x + \partial_u)^2 f(x, x_i, u, v) \right] = -(2\pi)^2 \frac{(x - u)^2}{(x - x_i)^2(u - x_i)^2} g(x, x_i, u). \] (23)

The function \( g \) is defined as

\[ g(x_1, x_2, x_3) = \frac{\pi^2}{(x_{12})^2} B(\hat{r}, \hat{s}), \]

with \( \hat{r} = \frac{x_{12}^2}{x_{12}^2}, \hat{s} = \frac{x_{12}^2}{x_{12}^2} \) and

\[ B(\hat{r}, \hat{s}) = \frac{1}{\sqrt{p}} \left\{ \ln(\hat{r}) \ln(\hat{s}) - \left[ \ln \left( \frac{\hat{r} + \hat{s} - 1 - \sqrt{p}}{2} \right) \right]^2 \right\} + 2 \text{Li}_2 \left( \frac{2}{1 + \hat{r} - \hat{s} + \sqrt{p}} \right) - 2 \text{Li}_2 \left( \frac{2}{1 - \hat{r} + \hat{s} + \sqrt{p}} \right), \]

where \( p(\hat{r}, \hat{s}) = 1 + \hat{r}^2 + \hat{s}^2 - 2\hat{r} - 2\hat{s} - 2\hat{r}\hat{s} \). \( g(x, x_i, u) \) is only logarithmically divergent in the limit \( u \to x \) \[3, 19\], so that (23) and thus (21) go to zero due to the prefactor \( (x - u)^2 \).

In conclusion

\[ G_{(1)}^{(A)}(x, x_1, \ldots, x_n) = 0. \] (26)
For the second kind of first order perturbative corrections the insertion of vector lines between the $i$th and $j$th rainbows corresponds to the diagram

which gives

$$G^{(B)}_{(1)}(x, x_1, \ldots, x_n) = c^{(B)}(n, N) \frac{g^2}{(2\pi)^{2l-s}} \prod_{k=1}^{n} \frac{1}{(x-x_k)^{2k}} \cdot$$

$$\frac{1}{(x-x_i)^{2(l_i-1)}}(x-x_j)^{2(l_j-1)} \lim_{u \to x} \left[ -\frac{1}{2(2\pi)^{10}} (\partial_x + \partial_u)^2 f(x, x_i, u, x_j) \right]. \quad (27)$$

In this case the limit $u \to x$ (removal of point-splitting) is finite but non-vanishing, however the group-theory coefficient $c^{(B)}(n, N)$ turns out to be zero. This can be proven as follows. For compactness of notation and without loss of generality we consider the case in which the interaction is between the rainbows $i=1$ and $j=2$. One then obtains

$$c^{(B)}(n, N) = C_N \sum_{i,r=1}^{\ell_1} \sum_{\substack{j,s=\ell_1+1 \atop j \neq s}}^{\ell_1+\ell_2} \left\{ \sum_{\text{perms } \sigma} \left[ \text{tr} \left( T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(i)}} \ldots T^{a_{\sigma(j)}} \ldots T^{a_{\sigma(l)}} \right) \cdot \right. \right.$$  

$$\cdot \text{tr} \left( T^{a_1} \ldots T^{a_{\ell_1}} \ldots T^{a_{\ell_1+i_1}} \right) \text{tr} \left( T^{a_{\ell_1+i+1}} \ldots T^{a_{\ell_1+\ell_2}} \right) \ldots$$  

$$\cdot \text{tr} \left( T^{a_{\ell_1+i+\ell_{n-1}+1}} \ldots T^{a_{\ell_1+\ell_2}} \right) f_{a_1b_1} f_{a_2b_2} \left] \right\} =$$  

$$= -C_N \sum_{i=1}^{\ell_1} \sum_{j=\ell_1+1}^{\ell_1+\ell_2} \left\{ \sum_{\text{perms } \sigma} \left[ \text{tr} \left( T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(i)}} \ldots T^{a_{\sigma(j)}} \ldots T^{a_{\sigma(l)}} \right) \cdot \right. \right.$$  

$$\cdot \text{tr} \left( T^{a_1} \ldots [T^{a_i}, T^{a_j}] \ldots T^{a_{\ell_1+i_1}} \right) \text{tr} \left( T^{a_{\ell_1+i+1}} \ldots [T^{a_j}, T^{a_i}] \ldots T^{a_{\ell_1+\ell_2}} \right) \ldots$$  

$$\cdot \text{tr} \left( T^{a_{\ell_1+i+\ell_{n-1}+1}} \ldots T^{a_{\ell_1+\ell_2}} \right) \right\}, \quad (28)$$

where the minus sign in the last equality comes from the factor of $i$ in the definition of the structure constants $f_{abc}$. Using cyclicity of the trace and the relation

$$\text{tr} \left( [M, T^1] T^2 T^3 \ldots T^n \right) = \sum_{i=2}^{n} \text{tr} \left( M T^2 \ldots [T^1, T^i] \ldots T^n \right), \quad (29)$$

8
valid for any matrix $M$, yields
\[
\begin{align*}
c^{(B)}(n, N) &= -C_N \sum_{i=1}^{\ell_1} \sum_{j=\ell_1+1}^{\ell_1+\ell_2} \left\{ \sum_{\text{perms } \sigma} \left[ \text{tr} \left( T^{a_{\sigma(1)}} T^{a_{\sigma(i)}} \ldots T^{a_{\sigma(j)}} \ldots T^{a_{\sigma(\ell)}} \right) \right] \right. \\
&\quad \times \sum_{p=1}^{\ell_1} \text{tr} \left( T^{b} T^{a_1} \ldots [T^{a_i}, T^{a_p}] \ldots T^{a_{\ell_1}} \right) \sum_{q=\ell_1+1}^{\ell_1+\ell_2} \text{tr} \left( T^{b} T^{a_{\ell_1+1}} \ldots [T^{a_j}, T^{a_q}] \ldots T^{a_{\ell_1+\ell_2}} \right) \ldots \text{tr} \left( T^{a_{\ell_1+\ldots+\ell_n-1+1}} \ldots T^{a_\ell} \right) \right\} = 0,
\end{align*}
\]
(30)

since the first factor is completely symmetric in the indices $a_m$ and in particular under the exchange of the pairs $a_i, a_p$ and $a_j, a_q$, that enter the commutators in the second line.

In conclusion extremal correlators of the form (13) have zero first order perturbative corrections
\[
G^{(1)}(x, x_1, \ldots, x_n) = 0.
\]
(31)

Notice that the same analysis can be repeated step by step in the case of extremal correlators involving multi-trace operators in short multiplets [10, 11, 19, 20]. The vanishing of the corrections of the first kind, previously denoted by $G^{(A)}$, is still valid in this case since it does not depend on the colour structure. For what concerns the second type of contributions, $G^{(B)}$, the argument described above still applies since it only relies on the symmetry of the sum over permutations, that is a consequence of Wick contractions.

4 Non-perturbative non-renormalisation

The power of supersymmetric instanton calculus [21], that has allowed non-perturbative tests of the AdS/SCFT correspondence [13, 14] will enable us to show that extremal correlators receive no instanton contribution from any topological sector (labelled by $K$) and for any number of colours (labelled by $N$) at leading semiclassical order. The following argument heavily relies on the systematic of the gaugino zero-modes in the multi-instanton background. An index theorem for fermions in the adjoint of $SU(N)$ tells us that the number of gaugino zero-modes is $2NKN$, where the factor of $N$ takes into account the number of supersymmetries. In the $N=4$ case under consideration, however, only 16 of these gaugino zero-modes are exact zero-modes [22, 23, 24], i.e. those corresponding to the 8 supersymmetry and 8 superconformal transformations broken in the YM instanton background. The remaining $8KN - 16$ are lifted by the Yukawa interactions and appear in quadrilinear terms in the “classical action”, i.e. in the action obtained after expanding the fields around the instanton configuration.

The gaugino zero-modes may be written as
\[
\lambda^A = \frac{1}{2} F_{\mu\nu} \sigma^{\mu\nu} \zeta^A + \tilde{\lambda}^A,
\]
(32)

where $\zeta^A_\alpha = n^A_\alpha + x_{\mu}^{\alpha\dot{\alpha}} \tilde{\zeta}_{A\dot{\alpha}}$ parameterise the 16 exact “geometric” zero-modes and $\tilde{\lambda}^A$ only involve the lifted zero-modes ($8KN - 16$ of them) whose explicit expression will not concern us here. $\tilde{\lambda}^A$ can be written in terms of the ADHM data and their superpartners.
that satisfy some complicated non-linear constraints, collectively denoted by $\Gamma^{\text{super}}_{\text{ADHM}} = 0$ in the following. The only crucial point is that the “true” zero-modes neither enter the non-linear “super-ADHM constraints” $\Gamma^{\text{super}}_{\text{ADHM}}$ nor the fermion-quadrilinear term in the classical action.

Taking into account the Yukawa couplings (in $N=1$ notation)

$$L_Y = g \text{tr} \left( \phi^I_1 [\lambda^0, \lambda^I] + \frac{1}{2} \varepsilon_{IJK} \phi^J [\lambda^I, \lambda^K] + \text{h.c.} \right)$$

the equations of motion of the scalar fields in the instanton background and in the presence of the gaugino zero-modes read

$$D^2 \phi^I = g [\lambda^0, \lambda^I]$$

$$D^2 \phi^I_1 = \frac{1}{2} g \varepsilon_{IJK} [\lambda^I, \lambda^K],$$

where $D$ denotes the covariant derivative in the instanton background and the trilinear terms due to the variation of the scalar potential have been neglected being of higher order in $g$.

The scalar-field solutions induced by the presence of the fermionic zero-modes in (34) are

$$\phi^1 = \zeta^0 F_{\mu\nu} \sigma^{\mu\nu} \zeta^1 + \hat{\phi}^1,$$  \hspace{1cm} (35)

where $\hat{\phi}^1$ is a bilinear in the lifted gaugino zero-modes of “flavour” 0 and 1 only, and

$$\phi^\dagger_1 = \zeta^2 F_{\mu\nu} \sigma^{\mu\nu} \zeta^3 + \hat{\phi}^\dagger_1,$$  \hspace{1cm} (36)

where $\hat{\phi}^\dagger_1$ is a bilinear in the lifted gaugino zero-modes of “flavour” 2 and 3 only. Substituting the instanton-induced expressions for the scalar fields in the extremal correlators (13) yields

$$G^{(K)}_{\text{inst}}(x, x_1, \ldots, x_n) = c^{(K)}(n, N, g) e^{-\left( \frac{\pi^2}{\sigma^2} + i\theta \right) K} \int d\mu^{(K)} \delta(\Gamma^{\text{super}}_{\text{ADHM}}) \cdot (\zeta^0 F \sigma \zeta^1 + \hat{\phi}^1)^{\ell_1 + \ldots + \ell_n} (x) \ldots (\zeta^2 F \sigma \zeta^3 + \hat{\phi}^\dagger_1)^{\ell_n} (x_n),$$  \hspace{1cm} (37)

where, except for an obvious $d^{16} \zeta = d^8 \eta d^8 \bar{\xi}$ factor, the measure of integration $d\mu^{(K)}$ and the explicit form of the “super”-ADHM constraints $\Gamma^{\text{super}}_{\text{ADHM}}$ are independent of $\zeta$’s and play no rôle in the following. Since only the first operator could possibly absorb the zero-modes of type 0 and 1, the integrand necessarily contains a factor of the form $(\zeta^0 F \sigma \zeta^1)^{\ell - 4} (\hat{\phi}^1)^{\ell - 4}$. Then, using the fact that $(\zeta)^3 = 0$ for any two-component Grassmann variable,

$$G_{\text{inst}}(x, x_1, \ldots, x_n) = 0$$  \hspace{1cm} (38)

immediately follows for any extremal correlator. In fact in the case of three- and higher-point functions extremality requires $\ell \geq 4$. For two-point functions of CPO’s, that are always extremal, the values $\ell = 2, 3$ are allowed. The corresponding correlators have vanishing instanton correction at lowest order because the 16 geometric zero-modes cannot be saturated in these cases.
This proves the non-perturbative non-renormalisation of extremal correlators of CPO’s with any number \( n \) of insertion points and for all \( N \) and \( K \) to leading order in the semiclassical expansion around the instanton background. We will not dwell into the study of perturbative corrections to the above instanton result, but we expect them to vanish as much as perturbative corrections around the trivial vacuum configuration vanish. The extension to any gauge group, though technically involved, should be straightforward.

5 Discussion

The results presented in this letter lend firm support to a new prediction of the AdS/SCFT correspondence [3], well beyond the regime of validity of the supergravity approximation.

It would be extremely useful to fully characterise the class of correlators that are (expected to be) tree-level exact in \( \mathcal{N}=4 \) SYM theory. From a preliminary analysis of CPO’s correlators, it seems that this interesting class could be restricted to those correlators that involve only one \( SU(4) \) singlet. In addition to the extremal correlators that we have considered above, two- and three-point functions of CPO’s belong to this class. As observed above, all two-point functions and some three-point functions are extremal too. Tests of the predictions arising for these cases from the AdS/SCFT correspondence [4] have been performed from the SYM perspective both at one-loop [5, 19, 20, 22] and at the non-perturbative level [19]. Judiciously using the “bonus” \( U(1)_B \) symmetry [18] in the context of \( \mathcal{N}=4 \) analytic superspace [3] and disposing of some potentially troublesome contact terms [23] lead to a demonstration of the non-renormalisation of two- and three-point functions of CPO’s [24]. The identification of \( U(1)_B \)-violating nilpotent super-invariants that began at five points and above [24] and were not listed in [23] prevents one from generically extending the same argument to higher-point functions. Proving the absence of the relevant nilpotent super-invariants should allow one to prove the absence of quantum corrections to the extremal correlators of CPO’s by simple algebraic means.

A related issue is whether the knowledge of correlators of CPO’s completely determines those of their super-descendants or there are other nilpotent super-invariants that can spoil this naive expectation.

It would be interesting to see whether the \( \mathcal{N}=4 \) analytic superspace analysis could also shed some light onto the non-renormalisation properties of protected multi-trace operators that satisfy more general shortening conditions [10, 11]. In particular, the vanishing of their anomalous dimensions is supported by explicit perturbative and non-perturbative computations (e.g. for a dimension four double-trace operator in the \( 84 \) representation of the \( SU(4) \) R-symmetry) at weak coupling [13], but it is still waiting for an AdS confirmation.

Acknowledgements

We would like to acknowledge Dan Freedman for early collaboration on the project, stimulating discussions and a careful reading of the manuscript. We have also benefitted from discussions with Ya.S. Stanev, E. Sokatchev, P. West and G.C. Rossi. Preliminary results were presented by one of us (M.B.) at the 3rd Annual TMR Conference held at SISSA, Trieste, Italy, September 20- 24 1999. M.B. would like to thank the Aspen Center for
Physics, where this work was initiated, for the kind hospitality.

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