DYNAMICS OF RAREFACTION WAVES IN GENERAL-RELATIVISTIC COLLAPSING CLOUDS

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Abstract

Formation and evolution of general relativistic collapse inhomogeneity is investigated. It is shown that a rarefaction wave forms at initial collapse stages and propagates inside from the cloud boundary to its center. The focusing time of the rarefaction wave is found. In massive clouds, the rarefaction wave focusing time equals the free-fall time. The collapse of such clouds leads to black hole formation. In low-mass clouds, the rarefaction wave focusing time is less than the free-fall time. After focusing, the collapse of such clouds becomes fully inhomogeneous and can be sufficiently decelerated by the pressure gradient.

1 Introduction

The study of general relativistic gravitational collapse is a complex problem because of mathematical difficulties appearing when solving the corresponding non-linear partial differential equations even in spherical symmetry [1]. Only a few exact solutions are known for any special cases [2]. More realistic models for gravitational collapse can be investigated using numerical simulations (see, e.g., [3]).

An initially homologous collapsing cloud with zero pressure (dust cloud) remains homologous during the collapse. But contraction of an initially homologous cloud with non-zero pressure develops with time in a sufficiently non-homologous way because a pressure gradient must necessarily develop at the cloud surface [4]. Even if the cloud is initially surrounded by an ambient gas of the same pressure as the cloud, the further contraction breaks this equilibrium [5]. A rarefaction wave propagates in from the cloud surface to its center with sound speed. The rarefaction front divides the cloud into two parts [6, 7, 4]. The pressure gradient becomes zero in the internal region. However, some inhomogeneity forms in the external region. This dynamical inhomogeneity can evolve in a self-similar manner at last stages of contraction [8, 9].

The dynamics of rarefaction wave has been studied for non-relativistic (protostellar) collapse of an isothermal cloud [10]. The effects of rotation [11] and magnetic field [7] have been taken into account. In this paper, a general-relativistic analogue of this problem is considered under the condition of spherical symmetry.

In the next section, the basic equations are described. In Sec. 3, the flow structure in the inner homogeneously collapsing core is considered. The equation of rarefaction front propagation is solved in Sec. 4. In Sec. 5, the radial null geodesics are investigated. Finally, Sec. 6 presents the main results and conclusions.

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2 Basic equations

Let us work in a system of spherical coordinates \((t, R, \theta, \varphi)\) moving at each point with the gas located at that point (comoving or Lagrangian coordinates). The interior space-time geometry for a collapsing cloud in spherical symmetry can be described by the metric of a “collapsing star” (see, e.g., [12]):

\[
\text{d}s^2 = e^{2\Phi}c^2 \text{d}t^2 - e^{2\Lambda} \text{d}R^2 - r^2 \text{d}\Omega^2,
\]

where \(\text{d}\Omega^2 = \text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2\). Here \(\Phi, \Lambda, r\) are some functions of the radial coordinate \(R\) and the time \(t\) to be determined by the Einstein equations. Note that the form of the line element (1) is invariant under the transformation:

\[
R' = R'(R), \quad t' = t'(t).
\]

In the comoving reference frame, the components of the four-velocity of gas are \(u^0 = e^{-\Phi}, u^1 = u^2 = u^3 = 0\). Therefore, for a perfect fluid, the stress-energy tensor has following non-zero components:

\[
T^{00} = e^{-2\Lambda}, \quad T^{11} = T^{22} = T^{33} = -P,
\]

where \(e\) is the gas energy density (gas energy per unit proper rest volume) and \(P\) is the gas pressure.

Using these relations, we can obtain the following Einstein field equations:

\[
\frac{8\pi G}{c^4}P = e^{-2\Lambda} \left( \frac{\dddot{r}^2}{r^2} + 2\dddot{\Phi} \frac{r' \dot{r}'}{r} \right) - \frac{1}{c^2} e^{-2\Phi} \left( \frac{2}{r} \frac{\dot{r}}{r^2} - 2\dddot{\Phi} \frac{\dot{r}}{r} \right) - \frac{1}{r^2}, \tag{2}
\]

\[
\frac{8\pi G}{c^4}e = -e^{-2\Lambda} \left( 2 \frac{r''}{r} + \frac{\dddot{r}^2}{r^2} - 2\Lambda \frac{r' \dot{r}'}{r} \right) + \frac{2}{c^2} e^{-2\Phi} \left( \frac{\dot{\Lambda}}{r} + \frac{1}{2} \frac{\dot{r}^2}{r^2} \right) + \frac{1}{r^2}, \tag{3}
\]

\[
0 = \frac{\dddot{r}}{r} - \dddot{\Lambda} \frac{r'}{r} - \dddot{\Phi} \frac{\dot{r}'}{r}, \tag{4}
\]

where the prime means a derivative with respect to the radial coordinate \(R\) and the overdot means a derivative with respect to the time \(t\). The equations of gas motion give

\[
\frac{\dot{e}}{e + P} = -\dddot{\Lambda} - 2 \frac{\dddot{\Phi}}{r}, \quad \frac{P'}{e + P} = -\dddot{\Phi}.
\]

The system (2-5) is still not complete. We must add an equation of state to close this system. In this paper, we assume the following equation of state:

\[
P = (\kappa - 1)e, \tag{6}
\]

where \(\kappa = \text{const} \ (1 \leq \kappa \leq 2)\). From this equation it follows that the sound speed in the gas is \(a = \sqrt{\kappa - 1}c\).

Eq. (6) can correspond to two reasonable astrophysical situations [8]. In the first case (protocluster and protogalactic clouds) this equation can be considered as the equation of state of a non-relativistic isothermal ideal gas when adiabatic heating and cooling is neglected and therefore the internal energy is negligible compared to the rest energy. In the second case (presupernova cores), Eq. (6) can be considered as an extreme relativistic limit of an adiabatic (or polytropic) equation of state when the rest energy density is negligible relative to the internal energy.

3 Homogeneously collapsing core

Inside the internal homogeneous core of a collapsing cloud, the pressure gradient is zero. From the second equation in (5) it follows that \(\Phi' = 0\) and therefore \(\Phi = \Phi(t)\). Thus we can define
another time variable \( T = T(t) \) such that \( \Phi = 0 \). It is clear that the time \( T \) coincides with the proper time of the gas. In addition, suppose that the radial coordinate \( R \) satisfies the relation \( r(T, R) = R(1 - \eta(T)) \), where \( \eta(T) \) is a function of the proper time \( T \) such that \( 0 \leq \eta \leq 1 \). The value \( \eta = 1 \) corresponds to free-fall time (without the pressure gradient) for the considered homogeneous region.

From Eq. (4) we get the following solution:

\[
\Lambda = \ln \frac{1 - \eta}{\sqrt{1 + f(R)}},
\]

where \( f(R) \) is some function. Combining (7) and (5), we find

\[
e = \frac{e_0}{(1 - \eta)^{3\kappa}},
\]

where \( e_0 \) is initial energy density.

Now we can use Eqs. (7), (8) to transform Eq. (3) to the following form:

\[
8\pi G \frac{e}{c^4} = 3 \frac{\dot{\eta}^2}{c^2 (1 - \eta)^2} - \frac{f}{R^2 (1 - \eta)^2} - \frac{f'}{R (1 - \eta)^2}.
\]

Since the left-hand side of this equation is a function of the proper time \( T \), we see that \( f = f_0 R^2 \), where \( f_0 \) is some constant. Taking into account the initial values \( \eta(0) = 1 \) and \( \dot{\eta}(0) = 0 \), we can find: \( f_0 = -8\pi G e_0 / (3c^4) \).

The invariant line element (11) can be written as

\[
d^2 = c^2 dT^2 - (1 - \eta)^2 \left( \frac{dR^2}{8\pi G e_0 R^2} + \frac{1}{3c^2} \frac{d\Omega^2}{R^2} \right).
\]

Note that this metric corresponds to the well-known Friedmann solution for a closed homogeneous isotropic universe with uniform pressure.

Now we can combine the expression for energy density (8) with (9) to obtain an equation for the function \( \eta(T) \):

\[
\frac{d\eta}{dT} = \sqrt{8\pi G e_0 \frac{1}{3c^2} \left( \frac{1}{(1 - \eta)^{3\kappa - 2}} - 1 \right)^{1/2}}.
\]

This equation can be solved analytically in the special case \( \kappa = 4/3 \). For an arbitrary parameter \( \kappa \), Eq. (11) can be solved numerically.

### 4 Propagation of a rarefaction wave

The rarefaction wave front propagates in collapsing gas with a sound speed. The location \( R_{rf}(T) \) of the rarefaction wave front satisfies the equation:

\[
e^\Lambda \frac{dR_{rf}}{dT} = -a.
\]
Figure 1: Functions $R_{rf}(\eta)$ in the case $\kappa = 5/3$ for various values of parameter $\alpha$.

After some manipulations, we can transform Eq. (12) to the form:

$$\frac{dr_{rf}}{d\eta} = -\frac{\sqrt{\kappa - 1}(1 - \eta)^{3\kappa - 2} \sqrt{1 - \alpha^2 r_{rf}^2}}{\alpha \sqrt{1 - (1 - \eta)^{3\kappa - 2}}}.$$  \hspace{1cm} (13)

where $r_{rf} = R_{rf}/R_0$, $R_0$ is the initial cloud radius,

$$\alpha = \sqrt{\frac{R_g}{R_0}}; \quad R_g = \frac{2GM_0}{c^2} = \frac{8\pi G\epsilon_0 R_0^3}{3c^4}. \hspace{1cm} (14)$$

The parameter $\alpha$ controls the general relativity effect in the problem. In the case of very small $\alpha$, the space-time curvature is negligible, and the dynamics of the rarefaction wave can be considered as non-relativistic. In the opposite limit, as $\alpha \to 1$, the problem is highly general-relativistic.

The solution of Eq. (13) can be written as:

$$r_{rf} = \frac{1}{\alpha} \sin \left( \arcsin \alpha + \frac{2\sqrt{\kappa - 1}}{3\kappa - 2} \arccos(1 - \eta)^{\frac{3\kappa - 2}{2}} \right). \hspace{1cm} (15)$$

The solutions obtained, for the case $\kappa = 5/3$ and various $\alpha$, are shown in Fig. 1.

The rarefaction wave front reaches the center of the cloud at a focusing time $T_*$. The value of $T_*$ can be determined by an equivalent value $\eta_* = \eta(T_*)$. There are two possible regimes for propagation of rarefaction waves in general relativistic collapsing clouds.

For sufficiently large $\alpha \geq \alpha_*$, where

$$\alpha_* = \sin \frac{\pi \sqrt{\kappa - 1}}{3\kappa - 2}, \hspace{1cm} (16)$$

we have $\eta_* = 1$. This result means that the rarefaction wave front reaches the cloud center at a moment of big crunch when the central homogeneous core collapses to space-time singularity.
For relatively small values of parameter $\alpha < \alpha^*$ the rarefaction wave front reaches the cloud center before the formation of a central singularity. The focusing time is determined by the value

$$
\eta^* = 1 - \cos^2 \left( \frac{3\kappa - 2}{2\sqrt{\kappa - 1}} \arcsin \alpha \right).
$$

(17)

In such a cloud, after focusing of the rarefaction wave, a strong pressure gradient can be formed. Hence the further collapse can be sufficiently decelerated and can even pass to slower quasi-hydrostatic contraction with formation of some stable object at final stages of cloud evolution (e.g., a neutron star). Note that, for a fixed initial radius $R_0$, a small value of the parameter $\alpha$ corresponds to a small initial mass of the cloud. The simplest expression for the focusing time can be obtained in case $\kappa = 4/3$:

$$
T_* = \sqrt{\frac{3c^2}{8\pi G\epsilon_0}} \sin \left( \sqrt{3} \arcsin \alpha \right).
$$

(18)

The critical value $\alpha^*$ is a function of the parameter $\kappa$. As $\kappa \to 0$ we have the asymptotic relation $\alpha^* = \pi \sqrt{\kappa - 1}$. In particular, in the case of a non-relativistic isothermal gas, the obtained criterion coincides with the non-relativistic one (see [10, 11, 17]). The maximum value of $\alpha^*$ is reached at $\kappa = 4/3$. At this point, $\alpha^* = \sin \pi \sqrt{3/6} \approx 0.79$, and therefore the initial cloud radius is $R_0 \approx 1.61R_g$.

## 5 Null radial geodesics

We can probe the space-time geometry in the internal homogeneous region by emitting outgoing light rays. The equation for outgoing radial null geodesics in the inner homogeneous region is

$$
e^\Lambda \frac{dR}{dT} = c.
$$

(19)

After some simple manipulations we can transform this equation to the following form:

$$
\frac{dR}{d\eta} = \sqrt{\frac{3c^4}{8\pi G\epsilon_0} \left( 1 - \eta \right)^{\frac{3\kappa - 4}{2}} \sqrt{1 - \frac{8\pi G\epsilon_0}{3c^4} R^2}} \frac{R^2}{1 - (1 - \eta)^{3\kappa - 2}}.
$$

(20)

For null geodesics that are outgoing from the cloud center, we have the following solution:

$$
R = \frac{R_0}{\alpha} \sin \left\{ \frac{1}{3\kappa - 2} \left[ \arcsin \left( 1 - \eta_0 \right)^{\frac{3\kappa - 2}{2}} - \arcsin \left( 1 - \eta \right)^{\frac{3\kappa - 2}{2}} \right] \right\}.
$$

(21)

Here $\eta_0$ is an initial value of the parameter $\eta$ for a given outgoing light ray.

Fig. 2 shows the space-time geometry in a collapsing cloud with the parameters $\kappa = 5/3$ and $\alpha = 0.85$ in terms of the coordinates $\eta$ and $R$. The heavy solid line corresponds to the rarefaction wave front. The arrowed lines denote outgoing radial light rays. Each arrow points the direction of light ray propagation. Some outgoing rays can reach the rarefaction wave front and then get out to the outer inhomogeneous region of the collapsing cloud. Other rays cannot reach the rarefaction wave front because they fall back to the cloud center at $\eta = 1$. Every light ray emitted at any point of the shaded area cannot escape the internal homogeneous region.
Figure 2: Space-time geometry of a collapsing cloud in case $\kappa = 5/3$ and $\alpha = 0.85$. Lines with arrows denote outgoing radial light rays with various values of the initial parameter $\eta_0$.

In other words, we can conclude that an event horizon forms in the considered cloud at final stages of its collapse.

Therefore, for clouds with $\alpha < \alpha_*$, there are no effects that can retard the collapse, and some part of matter necessarily falls under an event horizon with the formation of a black hole.

The obtained criterion for black hole formation in collapsing clouds can be reformulated in another form: $M_0 > M_*$, where the critical mass $M_* = 0.5\alpha_2^2 c^2 R_0/G$ depends on the initial cloud radius $R_0$. Note that in the case $\kappa \approx 1$ the value of the critical mass $M_*$ can be very small because in such a cloud the critical parameter $\alpha_*$ is very close to zero. On the other hand, an event horizon can appear in a collapsing cloud after the rarefaction wave focusing. Hence the real critical mass can even be less than $M_*$.

6 Conclusion

The basic results of this work are:

1. The problem of inhomogeneity of general relativistic collapse of an initially homogeneous cloud surrounded by an ambient gas with the same pressure is investigated. It is shown that, at initial collapse stages, a rarefaction wave forms at the cloud boundary and then propagates to the cloud center. The front of this rarefaction wave divides the cloud into two parts. In the inner homogeneous region, the space-time metric is like the Friedmann one. In the outer region behind the rarefaction front, a strong inhomogeneity forms.

2. The rarefaction wave front propagates in the collapsing gas with the sound speed. The characteristic equation of the rarefaction front propagation has been solved. The focusing time of the rarefaction wave has been found.
3. An analysis reveals two possible regimes of rarefaction wave dynamics. In very massive clouds, the rarefaction wave focusing occurs at the central singularity formation time (big crunch). The collapse of such clouds leads inevitably to black hole formation. In a cloud with a relatively small mass, rarefaction wave focusing takes place before the big crunch. After the rarefaction wave focusing, the collapse of such a cloud can pass to quasi-static contraction with further formation of some stable object.

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