A note on the location of polynomial roots

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Abstract

We review some known inclusion results for the roots of a polynomial, and adapt them to a conjecture recently presented by S. A. Vavasis. In particular, we provide strict upper and lower bounds to the distance of the closest root of a polynomial $p(z)$ from a given $\zeta \in \mathbb{C}$ such that $p'(\zeta) = 0$.

1 Introduction

Recently S.A. Vavasis [2] has presented the following conjecture.

Conjecture There exist two universal constants $0 < \iota_1 \leq 1 \leq \iota_2$ with the following property. Let $\xi_1, \ldots, \xi_n$ be the roots of a degree-$n$ univariate polynomial $p(z)$. Let $\zeta_1, \ldots, \zeta_{n-1}$ be the roots of its derivative. Define

$$\rho_j = \min_{k=2, \ldots, n} \left| \frac{k! p(\zeta_j)}{p^{(k)}(\zeta_j)} \right|^{1/k}, \quad j = 1, \ldots, n-1$$

(1)

and the annuli

$$A_j = \{ z : \iota_1 \rho_j \leq |z - \zeta_j| \leq \iota_2 \rho_j \}, \quad j = 1, \ldots, n-1.$$  

Then for each $i = 1, \ldots, n$

$$\xi_i \in A_1 \cup \cdots \cup A_{n-1}.$$  

The author also refers to an unpublished communication by Giusti et Al., where it is shown that $\iota_1$ exists and can be taken $(\sqrt{5} - 1)/2$ and where a sequence of $n$-degree polynomials is given such that $\lim_n |z - \zeta_j|/\rho_j = +\infty$ so that $\iota_2$ does not exist.

In this note we revisit some known general bounds to the roots of a polynomial from [4], in particular Theorem 6.4b on pages 451,452, and Theorem 6.4e on page 454, and adapt them to the conditions of the Vavasis conjecture.

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More specifically, we show that for any polynomial \( p(z) \), and for any \( \zeta \) such that \( p'(\zeta) = 0 \), there exists a root \( \xi \) of \( p(z) \) satisfying

\[
|\xi - \zeta| \leq \rho \sqrt{n/2}, \quad \rho = \min_{k=2,\ldots,n} \left( k!p(\zeta) \frac{1}{p^{(k)}(\zeta)} \right)^{1/k},
\]

and that the bound is sharp since it is attained by a suitable polynomial.

We provide also some sharp lower bound to \( |\xi - \zeta| \) under the condition that \( p^{(k)}(\zeta) = 0 \) for \( k \in \Omega \), where \( \Omega \) is a nonempty subset of \( \{1, 2, \ldots, n-1\} \).

Moreover, we also show that \( \nu_2 \) does not exist by providing an example of a sequence \( \{p_n(z)\}_n \) of polynomials of degree \( n+1 \) having a common root \( \xi \), where the ratio \( |\xi - \zeta_j^{(n)}|/\rho_j^{(n)} \) is independent of \( j \) and tends to infinity as \( n^{1-\epsilon} \) for any \( i = 1, \ldots, n \) and for any \( 0 < \epsilon < 1 \), where \( \zeta_j^{(n)} \) are the roots of \( p_n'(z) \).

2 Main results

In this section, after providing a counterexample of the Vavasis conjecture, we review some inclusion theorems of [1], which give lower bounds and upper bounds to the distance of the roots of a polynomial from a given complex number \( \zeta \).

2.1 Counterexample

Consider the monic polynomial of degree \( n+1 \)

\[
p_n(z) = z^{n+1} - (n+1)z.
\]

Clearly \( z = 0 \) is one of its roots, and we have \( p_n'(z) = (n+1)(z^n - 1) \), so that the roots \( \zeta_i \) of \( p_n' \) are the complex \( n \)-th roots of the unity. Define

\[
\rho^{(n,k)} = \left| \frac{k!p_n(\zeta)}{p^{(k)}(\zeta)} \right|^{1/k}, \quad \rho^{(n)} = \min_{k=2,\ldots,n+1} \rho^{(n,k)},
\]

where \( \zeta \) stands for any \( n \)-th root \( \zeta_i \) of 1, and observe that \( p_n(\zeta) = -n\zeta, p_n^{(2)}(\zeta) = n(n+1)\zeta^{-1} \). Therefore, for \( k = 2 \) one has

\[
\rho^{(n)} \leq \rho^{(n,2)} = \left| \frac{2!p_n(\zeta)}{p_n^{(2)}(\zeta)} \right|^{1/2} = \left| \frac{2n}{n(n+1)} \right|^{1/2} = \sqrt{\frac{2}{n+1}}
\]

hence \( \rho_n \to 0 \) as \( n \to \infty \). Observe that this bound is independent of the root \( \zeta_i \). The annuli \( A_i \) have their centers on the unit circle and for \( \nu_2 \) constant, their external radii tend to 0 as \( n \to \infty \). Thus, for sufficiently large values of \( n \) they cannot contain the origin, and this contradicts the conjecture as \( z = 0 \) is a common root to all the polynomials \( p_n(z) \).

Moreover, for \( z = 0 \) one has

\[
\frac{|z - \zeta_i|}{\rho^{(n)}} \geq \frac{|z - \zeta_i|}{\rho^{(n,2)}} = (n+1)^{1/(n+1)} \sqrt{\frac{n+1}{2}} \geq \sqrt{\frac{n+1}{2}}.
\]
That is, the ratio \( \frac{|z - \zeta|}{\rho(n,k)} \) can grow as much as \( \sqrt{n/2} \). For general \( k \) one can easily get

\[
\frac{|z - \zeta|}{\rho(n,k)} = \left[ \frac{1}{n} \binom{n+1}{k} \right]^{1/k} (n+1)^{n/k} \geq \left[ \frac{1}{n} \binom{n+1}{k} \right]^{1/k}.
\]

(2)

Thus, for a fixed \( k \) the ratio \( |z - \zeta|/\rho(n,k) \) can grow as much as \( n^{1-\frac{1}{k}} \).

### 2.2 Lower bounds

Let us recall the following result (see [1], Theorem 6.4b).

**Theorem 1** Let \( p(z) = \sum_{i=0}^{n} a_i z^i \) be a monic polynomial of degree \( n \) and \( \zeta \) any complex number. Assume \( a_0 \neq 0 \). Then any root \( \xi \) of \( p(x) \) is such that

\[
\gamma \rho < |\xi - \zeta|, \quad \rho = \rho(\zeta) = \min_{k=2, \ldots, n} \left| k! \frac{p(\zeta)}{p^{(k)}(\zeta)} \right|^{1/k}
\]

(3)

where \( \gamma = 1/2 \).

The following proof of the above theorem can be easily adjusted to the case where \( \zeta \) is a (numerical) root of some derivative of \( p(z) \).

Without loss of generality we may assume \( \zeta = 0 \). In fact, if \( \zeta \neq 0 \) consider \( \tilde{p}(z) = p(z - \zeta) \) so that \( \tilde{p}'(z) = p'(z - \zeta) \) and \( \tilde{p}(0) = \rho(\zeta) \), and reduce the case to \( \zeta = 0 \).

From the definition of \( \rho \) one has

\[
\rho^k \leq k! \left| \frac{p(0)}{p^{(k)}(0)} \right| = \left| \frac{a_0}{a_k} \right|.
\]

(4)

Then taking the moduli in both sides of the equation \( -a_0 = a_1 \xi + a_2 \xi^2 + \ldots + a_n \xi^n \) yields

\[
1 \leq \sum_{i=1}^{n} \left| \frac{a_i}{a_0} \xi^i \right|
\]

which, in view of (4) provides the bound

\[
1 \leq \sum_{i=1}^{n} t^i, \quad t = \frac{|\xi|}{\rho},
\]

whence

\[
1 \leq \frac{t - t^{n+1}}{1 - t}.
\]

If \( t < 1 \) then we have \( 1 - t \leq t - t^{n+1} < t \) which implies \( t > 1/2 \). This proves the bound \( |\xi| > \frac{\rho}{4} \) for any root \( \xi \) of \( p(z) \).

Observe that the bound is strict since the polynomial \( p_n(z) = \sum_{i=1}^{n} z^i - 1 \) has a root in the interval \( (1/2, 1/2(1 + 1/n)) \) for \( n \geq 2 \).

The proof of Theorem 1 can be adjusted to the case where \( \zeta \) satisfies some additional condition. We have the following result:
Proposition 1 Assume that $\zeta$ satisfies the following condition
\[
\theta^i \left| \frac{p(i)(\zeta)}{i! p(\zeta)} \right| \leq \epsilon, \quad i \in \Omega = \{i_1, \ldots, i_h\} \subset \{1, 2, \ldots, n - 1\}
\]
where $0 \leq \epsilon < 1/h$, $1 \leq h < n$ and $\theta$ is an upper bound to $|\zeta - \xi_i|$ for $i = 1, \ldots, n$. Then $\bullet$ holds where $\gamma$ is the only solution in $(1/2, 1)$ of the equation
\[
(t - 1) \sum_{i \in \Omega} t^i + 2t - 1 + (1-t)h\epsilon = 0.
\]

Proof. By following the same arguments of the proof of Theorem 1 with $\zeta = 0$ one obtains
\[
1 \leq \sum_{i=1}^{n} \left| \frac{a_i}{a_0} \xi^i \right| \leq \sum_{i=1, n; i \in \Omega} \left| \frac{a_i}{a_0} \xi^i \right| + h\epsilon \leq \sum_{i=1, n; i \in \Omega} t^i + h\epsilon.
\]
If $t < 1$, replacing $\sum_{i=1, n; i \notin \Omega} t^i = (t - t^{n+1})/(1-t) - \sum_{i \in \Omega} t^i$ in the latter inequality yields $1 - t \leq t - t^{n+1} - (1-t) \sum_{i \in \Omega} t^i + (1-t)h\epsilon \leq t + (1-t) \sum_{i \in \Omega} t^i + (1-t)h\epsilon$. Whence, $t > \gamma$ where $\gamma$ is the only solution of $\bullet$ in $(1/2, 1)$. $\square$

Let us look at some specific instances of the above result. For $\epsilon = 0$ the condition of the proposition turns into $p(i)(\zeta) = 0$ for $i \in \Omega$. If in addition $\Omega = \{1\}$ one finds the condition $p'(\zeta) = 0$ of the Vavasis conjecture and $\bullet$ turns into $t^2 + t - 1 = 0$ that implies $\gamma = (\sqrt{5} - 1)/2 = 0.618\ldots$. Weaker bounds are obtained assuming $\epsilon = 0$ and $\Omega = \{k\}$ for some $k > 1$ since the only root of the polynomial $t^{k+1} - t^k + 2t - 1$ in $(1/2, 1)$ is lower than $(\sqrt{5} - 1)/2$.

Better bounds are obtained if $\zeta$ is a root of multiplicity $h$ of $p'(z)$; in fact, $\gamma$ is the only positive root of the polynomial $t^{h+1} + t - 1$. In particular, if $h = 2$ then $\gamma = 0.682\ldots$ if $h = 3$, $\gamma = 0.724\ldots$.

If $\zeta$ is close to a root of $p'(z)$, so that the condition $\theta |p'(\zeta)/p(\zeta)| < \epsilon$ for some “small” $\epsilon$ is satisfied, then $\gamma = (\sqrt{5} - 1)/2 - \epsilon(1 + 3/\sqrt{5}) + O(\epsilon^2)$.

For $\epsilon = 0$ the bound in the above proposition is strict since it is asymptotically attained by the polynomial $t^n - (t - 1) \sum_{i \in \Omega} t^i - 2t + 1$. The advantage of this bound is that it allows to compute sharper values for $\gamma$ just by solving a low degree equation if $\Omega$ is made up by small integers.

Slightly better lower bounds can be obtained from the following known result of $\bullet$ which requires to compute a positive root of a polynomial of degree $n$.

Theorem 2 Any root $\xi$ of $p(z)$ is such that $|\xi| \geq \sigma$, where $\sigma$ is the only positive solution to the equation $|a_0| = \sum_{i=1}^{n} t^i |a_i|$. 

2.3 Upper bounds

Throughout this section we denote
\[
\rho^{(k)} = \left( k! p(\zeta)/p^{(k)}(\zeta) \right)^{1/k}, \quad \rho = \min_k \rho^{(k)}
\]
for a given $\zeta \in \mathbb{C}$. Concerning upper bounds to the distance of a root from $\zeta$ we recall the following result of [1] (Theorem 6.4e, page 454).

**Theorem 3** For any $\zeta \in \mathbb{C}$ there exists a root $\xi$ of $p(z)$ such that

$$|\xi - \zeta| \leq \rho^{(k)} \left( \frac{n}{k} \right)^{1/k}, \quad k = 1, \ldots, n.$$  \hspace{1cm} (6)

Observe that, for $k = 2$ one has

$$|\xi - \zeta| \leq \rho^{(2)} \sqrt{n(n-1)/2},$$  \hspace{1cm} (7)

while

$$|\xi - \zeta| \leq \min_k \left( \frac{n}{k} \right)^{1/k} \rho^{(k)} \leq \max_k \left( \frac{n}{k} \right)^{1/k} \rho \leq n \rho.$$  \hspace{1cm} (8)

The bound (8) is sharp since it is attained by the polynomial $p(z) = (z - n)^n$ with $\zeta = 0$. In fact, it holds $\rho = \rho^{(1)} = 1$ and $p(z)$ has roots of modulus $n$.

Under the condition $p'(\zeta) = 0$ the bounds (6), (7) and (8) can be substantially improved. In fact we may prove the following result

**Proposition 2** For any $\zeta \in \mathbb{C}$ such that $p'(\zeta) = 0$ there exists a root $\xi$ of $p(z)$ such that

$$|\xi - \zeta| \leq \begin{cases} \\
\rho^{(2)} \sqrt{n/2} \\
\rho^{(3)} \sqrt[3]{n/3} \\
\rho^{(k)} \sqrt{n} \left( \frac{1}{k} \prod_{i=2}^{k/2} \left( \frac{1}{n^{i-1}} + \frac{1}{n^{i-2}} \right) \right)^{1/k} & \text{for } 4 \leq k \leq n
\end{cases}$$  \hspace{1cm} (9)

Moreover,

$$|\xi - \zeta| \leq \rho \sqrt{\frac{n}{2}}$$  \hspace{1cm} (10)

**Proof.** Without loss of generality we may assume $\zeta = 0$ and $a_0 = 1$ so that the polynomial can be written as $p(z) = 1 + a_2 z^2 + \ldots + a_n z^n$. Recall the Newton identities [1], page 455:

$$ka_k = -s_k - \sum_{i=1}^{k-1} a_i s_{k-i}, \quad k = 1, 2, \ldots,$$

where $s_k = \sum_{i=1}^{n} \xi_i^{-k}$ are the power sums of the reciprocal of the roots $\xi$ of $p(z)$. Clearly, $a_1 = s_1 = 0$ so that for $k \geq 4$ the Newton identities turn into

$$ka_k = -s_k - \sum_{i=2}^{k-2} a_i s_{k-i}, \quad k = 4, 5, \ldots.$$  \hspace{1cm} (11)

5
Let $\Delta = \min_i |\xi_i|$ so that $|s_k| \leq n\Delta^{-k}$. It holds $|2a_2| = |s_2| \leq n\Delta^{-2}$, $|3a_3| = |s_3| \leq n\Delta^{-3}$ and
\[
k|a_k| \leq \Delta^{-k}n(1 + \sum_{i=2}^{k-2} |a_i|\Delta^i), \; k \geq 4.
\]
Denoting $\gamma_k = n(1 + \sum_{i=2}^{k-2} |a_i|\Delta^i)$, for $k \geq 4$ and $\gamma_2 = \gamma_3 = n$, by using the induction argument one easily finds that
\[
k|a_k| \leq \Delta^{-k}\gamma_k
\]
\[
\gamma_k \leq \gamma_{k-1} + \frac{1}{k-2}\gamma_{k-2}, \; k \geq 4
\]
\[
\gamma_2 = \gamma_3 = n.
\]
(12)
The above expression provides the bound
\[
\Delta \leq \rho^{(k)} \left( \frac{\gamma_k}{k} \right)^{1/k}
\]
so that it remains to give upper bounds to $\gamma_k$. Since $\gamma_2 = \gamma_3 = n$, from (13) we deduce (9) for $k = 2, 3$. For the general case $k \geq 4$, we express the recurrence (12) in matrix form as
\[
\begin{bmatrix}
\gamma_{k+1} \\
\gamma_k
\end{bmatrix} \leq
\begin{bmatrix}
1 & \frac{n}{k-1} \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\gamma_k \\
\gamma_{k-1}
\end{bmatrix},
\]
where the inequality holds component-wise. Applying twice the above bound yields
\[
\begin{bmatrix}
\gamma_{k+1} \\
\gamma_k
\end{bmatrix} \leq
\begin{bmatrix}
1 + \frac{n}{k-1} & \frac{n}{k-2} \\
1 & \frac{n}{k-2}
\end{bmatrix}
\begin{bmatrix}
\gamma_{k-1} \\
\gamma_{k-2}
\end{bmatrix}.
\]
(14)
Whence, since $\gamma_2 = \gamma_3 = n$, one finds that $\gamma_{2i}$ and $\gamma_{2i+1}$ are polynomials in $n$ of degree $i$. Denoting
\[
\gamma_{2i} = n^i\delta_{2i}, \quad \gamma_{2i+1} = n^i\delta_{2i+1},
\]
we may give upper bounds to $\delta_k$. In fact, from (14) with $k = 2i$ it holds
\[
\begin{bmatrix}
\delta_{2i+1} \\
\delta_{2i}
\end{bmatrix} \leq
\begin{bmatrix}
\frac{1}{n} + \frac{1}{k-1} & \frac{1}{k-2} \\
\frac{1}{n} & \frac{1}{k-2}
\end{bmatrix}
\begin{bmatrix}
\delta_{2i-1} \\
\delta_{2i-2}
\end{bmatrix}.
\]
(16)
Let us denote $W_k$ the matrix in the right-hand side of (16), so that for $n \geq 4$ we have
\[
\begin{bmatrix}
\delta_{2i+1} \\
\delta_{2i}
\end{bmatrix} = W_{2i}W_{2(i-1)} \cdots W_4 \begin{bmatrix}
\delta_3 \\
\delta_2
\end{bmatrix}.
\]
(17)
Since for $n \geq 4$ we have $||W_k||_{\infty} = \frac{1}{n} + \frac{1}{k-1} + \frac{1}{k-2}$, taking norms in (17) yields
\[
||(\delta_{2i+1}, \delta_{2i})||_{\infty} \leq \prod_{j=2}^{i} ||W_{2j}||_{\infty}||(\delta_{3}, \delta_{2})||_{\infty} \leq \prod_{j=2}^{i} \left( \frac{1}{n} + \frac{1}{2j-1} + \frac{1}{2j-2} \right),
\]
(18)
since \( ||(δ_1, δ_2)||_∞ = ||(1, 1)||_∞ = 1\). In view of (13) and (15) this proves (9).

In order to prove the bound (10), from (13) it is sufficient to prove that

\[
\gamma_k \leq k(\sqrt{\frac{n}{2}})^k.
\]

We prove the latter bound by induction on \( k \) for \( 2 \leq k \leq n \). For \( k = 2, 3 \), the inequality (18) is true since \( \gamma_2 = \gamma_3 = n \). Moreover, from (12) one has

\[
\gamma_4 \leq \gamma_3 + \frac{n}{2} \gamma_2 = n(n + 2)/2
\]

so that (18) is satisfied also for \( k = 4 \). Now we assume that the bound (18) is true for \( k \) and \( k - 1 \), where \( k \geq 4 \) and we prove it for \( k + 1 \leq n \), i.e., \( \gamma_{k+1} \leq (k + 1)(\sqrt{n/2})^{k+1} \). From (12) and from the inductive assumption one has

\[
\gamma_{k+1} = \left(\sqrt{\frac{n}{2}}\right)^{k+1} \left(k\sqrt{\frac{n}{2}} + 2\right)
\]

Therefore it is sufficient to prove that

\[
k\sqrt{\frac{n}{2}} + 2 \leq k + 1,
\]

that is, \( \sqrt{\frac{n}{2}} \geq \frac{k}{k+1} \) which is satisfied for \( n \geq k \geq 4 \). This completes the proof. \(\square\)

Observe that the bound of Theorem 2 is sharp since it is attained by the polynomial \( p(z) = (z^2 - m)^n \) with \( \zeta = 0 \), where \( n = 2m \). In fact, \( p'(0) = 0 \), \( \rho = \rho^{(2)} = 1 \) and the roots of \( p(z) \) have moduli \( \sqrt{n/2} \).

If \( \zeta \) is such that \( p^{(j)}(\zeta) = 0, \ j = 1, \ldots, h \), then from the Newton identities one finds that \( s_i = a_i = 0, \ i = 1, \ldots, h \) so that equation (11) turns into

\[
ka_k = -s_k - \sum_{i=h+1}^{k-h-1} a_is_{k-i}, \ k \geq 2(h+1).
\]

By following the same argument used in the proof of Proposition 2 we can prove that there exists a root \( \xi \) of \( p(z) \) such that

\[
|\xi - \zeta| \leq \rho^{(h+i)} \sqrt{\frac{n}{h+1}}, \ i = 1, \ldots, h + 1.
\]

References

[1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 1, Wiley, 1974.

[2] S. A. Vavasis, A conjecture that the roots of a univariate polynomial lie in a union of annuli (Interim Revised Version), arXiv:math.CV/0606194 v3, 28 Jul 2006.