NEIGHBORLY INScribed POLYTOPES AND DELAUNAY TRIANGULATIONS

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Abstract. We prove that there are superexponentially many combinatorially distinct $d$-dimensional neighborly Delaunay triangulations on $n$ points. These are the first examples of neighborly Delaunay triangulations that cannot be obtained via a stereographic projection of an inscribed cyclic polytope, and provide the current best lower bound for the number of combinatorial types of Delaunay triangulations. To prove this bound we combine recent results on constructions for neighborly and inscribable polytopes to obtain a very simple explicit technique to generate a rich family of inscribable neighborly polytopes, and hence of point configurations with neighborly Delaunay triangulations.

1. Introduction

Delaunay triangulations and Voronoi diagrams are one of the most important objects in computational geometry. For example, they are used for nearest-neighbors search, pattern matching, clustering, mesh generation for the finite-element method and surface reconstruction. Understanding their combinatorial complexity (in terms of the number of faces) becomes crucial for these applications. This subject has received a lot of recent attention both for low dimensional configurations [4, 5, 12, 13, 14] and for those with arbitrary dimension [2, 3, 7, 11].

Seidel [22, 23] proved an upper bound theorem for the complexity of Delaunay triangulations in terms of neighborly polytopes. Indeed, Brown observed in 1979 [8] that using a stereographic projection one can identify the combinatorial types of Delaunay triangulations in $\mathbb{R}^d$ with those of inscribable $(d + 1)$-dimensional polytopes. Therefore, by McMullen’s Upper Bound Theorem [19], the complexity of Voronoi diagrams is bounded by that of neighborly polytopes.

The existence of inscribed neighborly polytopes was already known by Carathéodory in 1911, when he presented an inscribed realization of the cyclic polytope [9]. Every instance of inscribed neighborly polytope found since then has been combinatorially equivalent to the cyclic polytope.

In this paper we provide a very simple construction (Theorem 5.6 and Construction 6.1) for higher dimensional inscribable neighborly polytopes E-mail address: GonskaB@gmail.com, arnau.padrol@fu-berlin.de.

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and hence Delaunay triangulations and Voronoi diagrams). In short, the vertex set of an initial neighborly polytope in $\mathbb{R}^d$ is lifted to a neighborly Delaunay triangulation in $\mathbb{R}^{d+1}$, which can then be lifted to an inscribable neighborly polytope in $\mathbb{R}^{d+2}$. This technique generates a surprisingly rich family of combinatorial types of neighborly inscribable polytopes/Delaunay triangulations, and shows that, in combinatorial terms, inscribable neighborly polytopes are more frequent than what one might have thought.

Indeed, we show in Theorem 6.6 that $\text{inp}(n, d)$, the number of different labeled combinatorial types of inscribable neighborly $d$-polytopes with $n$ vertices, is at least $\text{inp}(n, d) \geq \frac{n^2}{2} n^{(1+o(1))}$ (as $n \to \infty$ with $d$ fixed). This contrasts with the fact that $\text{pol}(n, d)$, the number of different labeled combinatorial types of $d$-polytopes with $n$ vertices, is not larger than $\text{pol}(n, d) \leq \left(\frac{n}{2}\right)^{d^2 n^{(1+o(1))}}$ when $\frac{n}{d} \to \infty$ (see [1] and [17]). This means that $\text{inp}(n, d)$ is at least of the same order of magnitude as $\sqrt[d]{\text{pol}(n, d)}$ when $d$ is fixed and $n \to \infty$.

1.1. Previous work. A polytope is called inscribable if it is combinatorially equivalent to a polytope that has all its vertices on a sphere. The question of inscribability was raised for the first time by Steiner in 1832 [25], who asked whether all 3-dimensional polytopes are inscribable. A negative answer was given by Steinitz in 1928 [26]. For example, the Triakis tetrahedron is not inscribable.

The existence of inscribable neighborly polytopes has been known since their discovery by Carathéodory [9], who found a realization of cyclic polytopes with all their vertices on a sphere. While many inscribed realizations of the cyclic polytope have been found (c.f. [15], [16], [18] and [23]), a proof for the existence of other combinatorial types of neighborly polytopes inscribed on the sphere has been elusive.

Without the constraint of inscribability, Grünbaum [18] found the first examples of non-cyclic neighborly polytopes. Even more, Shemer used a “sewing construction” to prove in 1982 that the number of combinatorial types of neighborly $d$-polytopes with $n$ vertices is of order $n^{\frac{n}{2}(1+o(1))}$ [24]. His bound was recently improved by the second author in [20, 21], by proposing a new construction for neighborly polytopes that contains Shemer’s family. When $d$ is even, the number of different combinatorial types of labeled neighborly $d$-polytopes with $n$ vertices obtained with this construction is at least

$$\frac{(n - 1)(\frac{n-1}{2})^2}{(n - d - 1)(\frac{n-d-1}{2})^d 2^d c^{\frac{n-d-1}{2} d^2}}$$

which implies that, regardless of the dimension, the number of labeled neighborly $d$-polytopes with $n$ vertices is greater than

$$n^{\frac{d}{2} n^{(1+o(1))}} \text{ as } n \to \infty.$$ (1)

Even if this method cannot generate all neighborly polytopes, [11] is currently also the best lower bound for the number of combinatorial types of labeled $d$-polytopes with $n$ vertices.

Our main contribution in this paper is to show that all these neighborly polytopes are inscribable. To this end, we revisit the construction in [20]
providing a new direct proof that avoids the use of Gale duality and oriented matroid theory needed in the original proof. Thanks to this new proof we are able to show that these polytopes are inscribable by using a technique developed by the first author in [15, 16] to construct inscribable cyclic polytopes.

This implies that the lower bound (1) is also valid for the number of neighborly Delaunay triangulations, and begs the question of whether every neighborly polytope is inscribable.

2. Preliminaries

2.1. Polytopes and point configurations. Let $A = \{a_1, \ldots, a_n\}$ be a point configuration in $\mathbb{R}^d$ whose points are labeled by $\{1, \ldots, n\}$. We will say that $A$ is in general position if no $d+1$ points lie in a common hyperplane and no $d+2$ points lie on a common sphere.

The convex hull of $A$ is a polytope $P := \text{conv}(A) \subset \mathbb{R}^d$ and the intersection of $P$ with a supporting hyperplane is a face of $P$. Faces of dimensions 0 and $d-1$ are called vertices and facets, respectively. If all facets of $P$ are simplices, then $P$ is called simplicial and if $A$ is in general position, then $\text{conv}(A)$ is always simplicial.

A face $F$ of a polytope $P$ is called an equatorial face if the projection $\pi$ that forgets the last coordinate maps $F$ onto a face of $\pi(P)$. If $A$ is in sufficient general position, then the dimensions of $F$ and $\pi(F)$ coincide for every equatorial face $F$. In particular, in this case there cannot be equatorial facets.

A face $F$ of a polytope $P$ is visible from a point $p \notin P$ if there is a point $x \in \text{relint}(F)$ such that the segment $[x,p]$ intersects $P$ only at $x$.

We will usually assume that $A$ is in convex position, i.e. it coincides with $\text{vert}(P)$, the set of vertices of $P$. In particular, each face $F$ of $P$ can be identified with the set of labels of the points $a_i \in F$. The face lattice of $P$ is then a poset of subsets of $\{1, \ldots, n\}$. In this context, two vertex-labeled polytopes are combinatorially equivalent if their face lattices coincide.

A polytope $P$ is called $k$-neighborly if every subset of $k$ vertices of $P$ forms a face of $P$. No $d$-dimensional polytope other than the simplex can be $k$-neighborly for any $k > \lceil \frac{d}{2} \rceil$, which motivates the definition of neighborly polytopes as those that are $\lfloor \frac{d}{2} \rfloor$-neighborly.

McMullen’s Upper Bound Theorem [19] states that the number of $i$-dimensional faces of a $d$-polytope $P$ with $n$ vertices is maximized by simplicial neighborly polytopes, for all $i$.

A canonical example of neighborly polytope is the cyclic polytope, $C_n(d)$, obtained as the convex hull of $n$ points in any $d$-order curve in $\mathbb{R}^d$ [27]. For example, the moment curve $\gamma : t \mapsto (t, t^2, \ldots, t^d)$ is a $d$-order curve. In even dimensions, the trigonometric moment curve

$$
\tau : t \mapsto (\sin(t), \cos(t), \sin(2t), \cos(2t), \ldots, \sin(\frac{d}{2}t), \cos(\frac{d}{2}t))
$$

is a $d$-order curve, providing an inscribed realization of the cyclic polytope.

2.2. Triangulations. A triangulation of a point configuration $A$ is a collection $\mathcal{T}$ of simplices called cells such that

- $\text{vert}(S) \subseteq A$ for every $S \in \mathcal{T}$.
- If $S \in \mathcal{T}$ and $F$ is a face of $S$, then $F \in \mathcal{T}$.
• \( \bigcup_{S \in T} S = \text{conv}(A) \).

• If \( S, S' \in T \) and \( S \neq S' \) then \( \text{relint}(S) \cap \text{relint}(S') = \emptyset \).

Just as with polytopes, we can define the \textit{face lattice} of a triangulation and consider their \textit{combinatorial equivalence}.

\textbf{Definition 2.1.} The \textit{Delaunay triangulation} \( \mathcal{D}(A) \) of a point configuration \( A \subset \mathbb{R}^d \) in general position is the triangulation that consists of all cells defined by the \textit{empty circumsphere condition}: \( S \in \mathcal{D}(A) \) if and only if there exists a \((d-1)\)-sphere that passes through all the vertices of \( S \) and all other points of \( A \) lie outside this sphere.

If \( A \) is in general position, this condition always defines a unique triangulation of \( A \).

We will be particularly interested in neighborly triangulations, which will provide us with extreme examples in terms of number of faces.

\textbf{Definition 2.2.} A triangulation \( T \) of a point configuration \( A \subset \mathbb{R}^d \) is \textit{neighborly} if \( \text{conv}(S) \) is a cell of \( T \) for each subset \( S \subset A \) of size \(|S| = \lfloor \frac{d+1}{2} \rfloor \).

\textbf{Theorem 2.3 (Upper Bound Theorem for balls [10, Corollary 2.6.5])}. A triangulation \( T \) of a point configuration \( A \subset \mathbb{R}^d \) has the maximal number of \( d \)-cells among all triangulations of \( n \) points in \( \mathbb{R}^d \) if and only if \( T \) is neighborly and \( \text{conv}(A) \) is a \( d \)-simplex.

\section{Liftings and Triangulations}

\subsection{Lexicographic liftings.}

The main tool for our construction are lexicographic liftings, which are a way to derive \((d+1)\)-dimensional point configurations from \( d \)-dimensional point configurations.

\textbf{Definition 3.1.} Let \( A = \{a_1, \ldots, a_n\} \) be a configuration of \( n \) labeled points in \( \mathbb{R}^d \).

We say that a configuration \( \hat{A} = \{\hat{a}_1, \ldots, \hat{a}_n\} \) of \( n \) labeled points in \( \mathbb{R}^{d+1} \) is a \textit{Delaunay lexicographic lifting} (or just a \textit{D-lifting}) of \( A \) (with respect to the order induced by the labeling) if \( \hat{a}_i = (a_i, h_i) \in \mathbb{R}^{d+1} \) for each \( 1 \leq i \leq n \), for some collection of heights \( h_i \in \mathbb{R} \) that fulfill:

(i) \( h_1 = \cdots = h_{d+1} = 0 \),

(ii) \( |h_{d+2}| > 0 \), and

(iii) for each \( i > d + 2 \), \( |h_i| > 0 \) is large enough so that if \( h_i > 0 \) (resp. \( h_i < 0 \)) then \( \hat{a}_i \) is above (resp. below) \( H \) for every hyperplane \( H \) spanned by \( d + 1 \) points of \( \{\hat{a}_1, \ldots, \hat{a}_{i-1}\} \).

(iv) for each \( i > d + 2 \), \( \hat{a}_i \) is not contained in any of the circumspheres of any simplex spanned by \( d + 2 \) points of \( \{\hat{a}_1, \ldots, \hat{a}_{i-1}\} \).

If \( h_i \geq 0 \) for every \( 1 \leq i \leq n \), the lexicographic lifting is called \textit{positive}.

\textbf{Remark 3.2.} If \( A \) is in general position, then any D-lifting \( \hat{A} \) of \( A \) is also in general position. Furthermore, if \( A \) is in convex position, so is \( \hat{A} \).

We omit the proof of this lemma, which follows from the definition.

\textbf{Lemma 3.3.} For any point configuration \( A \), and for any D-lifting \( \hat{A} \), every equatorial face of \( \text{conv}(\hat{A} \setminus \hat{a}_n) \) is visible from \( \hat{a}_n \).
Figure 1. A D-lifting \( \{\hat{a}_1, \ldots, \hat{a}_5\} \) of the point configuration \( \{a_1, \ldots, a_5\} \).

3.2. Lexicographic triangulations. The combinatorics of D-liftings are easily explained in terms of lexicographic triangulations. We refer to [10, Section 4.3] for a detailed presentation, and we will only present here the parts that will be directly useful for us.

**Lemma 3.4** ([10, Lemma 4.3.2]). Let \( A = \{a_1, \ldots, a_n\} \) be a point configuration in convex position and let \( T \) be a triangulation of the point configuration \( A \setminus a_i \). Then there is a triangulation \( T' \) of \( A \) whose cells are

\[
T' := T \cup \{ \text{conv}(B \cup a_i) \mid B \in T \text{ and is visible from } a_i \}.
\]

Moreover, \( T' \) is the only triangulation of \( A \) that contains \( T \).

**Definition 3.5.** The triangulation \( T' \) in the previous lemma is called a refinement of \( T \) obtained by placing \( a_i \).

The **placing triangulation** of \( A \) (with respect to the order induced by the labels) is then the triangulation \( T := T_n \) obtained iteratively as follows: \( T_1 \) is the trivial triangulation of \( \{a_1\} \) and \( T_i \) is the triangulation of \( \{a_1, \ldots, a_i\} \) obtained by placing \( a_i \) on \( T_{i-1} \).

4. The construction

Our construction is based on the following theorems, which show how certain triangulations of certain D-liftings are always Delaunay (Theorem 4.1) and neighborly (Theorem 4.2).

**Theorem 4.1.** Let \( A = \{a_1, \ldots, a_n\} \) be a configuration of \( n \geq d + 2 \) labeled points in general position in \( \mathbb{R}^d \) and let \( \hat{A} \) be a D-lifting of \( A \). Then the Delaunay triangulation \( T := D(\hat{A}) \) coincides with the placing triangulation of \( \hat{A} \).

**Proof.** The proof is by induction on \( n \). If \( n = d + 2 \) then both triangulations consist on the single simplex spanned by \( \hat{A} \). Otherwise, let \( T' \) be the Delaunay triangulation of \( \hat{A} \setminus \hat{a}_n \). By induction hypothesis, this is the placing triangulation of \( \hat{A} \setminus \hat{a}_n \). Moreover, since the lifting fulfills condition [iv], every Delaunay cell of \( T' \) is still a Delaunay cell of \( T \). In particular, \( T \)
refines $T'$. Since by Lemma 3.4 the triangulation that refines $T'$ is unique, $T$ must be the placing triangulation of $\hat{A}$. \hfill \square

Theorem 4.2 can be deduced from the “Gale sewing” technique presented in [20] (cf. [21, Theorem 4.5]). However, the original proof of the theorem exploits Gale duality and oriented matroid theory, while this primal proof is elementary. Moreover, this is the setting that will eventually allow us to prove inscribability in Theorem 5.6.

**Theorem 4.2.** Let $A = \{a_1, \ldots, a_n\}$ be a configuration of $n \geq d + 2$ labeled points in convex general position in $\mathbb{R}^d$ and let $\hat{A}$ be a D-lifting of $A$.

If $\text{conv}(A)$ is $k$-neighborly (as a polytope), then $\text{conv}(\hat{A})$ is $k$-neighborly (as a polytope) and the placing triangulation of $\hat{A}$ is $(k+1)$-neighborly (as a triangulation).

**Proof.** The proof of the first claim ($\text{conv}(\hat{A})$ is $k$-neighborly) is straightforward. Indeed, since $A$ is a projection of $\hat{A}$, every face of $\text{conv}(A)$ is a projection of an equatorial face of $\text{conv}(\hat{A})$. In particular, since every subset of $k$-points of $A$ is a face of $\text{conv}(A)$, the corresponding points must also form an equatorial face of $\text{conv}(\hat{A})$.

The second claim is proved by induction on $n$, and it is trivial when $n = d + 2$. For $n > d + 2$, let $T$ be the placing triangulation of $\hat{A}$ and $T'$ the corresponding placing triangulation of $\hat{A} \setminus \hat{a}_n$. Now fix a subset $S$ of $\hat{A}$ of size $k+1$. If $\hat{a}_n \notin S$, then $S$ forms a cell of $T'$ by induction hypothesis, and hence of $T$. Otherwise, if $\hat{a}_n \in S$, then $S = S \setminus \hat{a}_n$ must be an equatorial face of $\hat{A} \setminus \hat{a}_n$. By Lemma 3.3, $S$ is visible from $\hat{a}_n$ and hence by the definition of the placing triangulation, $S$ must be a cell of $T$. \hfill \square

The combination of these two theorems directly proves our main result (see also Remark 6.4).

**Theorem 4.3.** Let $A = \{a_1, \ldots, a_n\}$ be a configuration of $n \geq d + 2$ labeled points in convex general position in $\mathbb{R}^d$ such that $\text{conv}(A)$ is $k$-neighborly, and let $\hat{A}$ be a D-lifting of $A$. Then $\text{conv}(\hat{A})$ is a $k$-neighborly polytope with vertex set $\hat{A}$ and its Delaunay triangulation is a $(k+1)$-neighborly triangulation.

5. Polytopes

We can easily adapt Theorem 4.2 to obtain a statement in terms of neighborly polytopes instead of neighborly triangulations. It suffices to do another suitable D-lifting. The construction will start on a neighborly polytope and increase its dimension by 2.

A first approach would be to start with a $k$-neighborly configuration $A \subset \mathbb{R}^d$ of $n$ points, then make a D-lifting of $A$ to obtain $\hat{A}$, and finally a positive D-lifting of $\hat{A}$ to obtain $\hat{\hat{A}}$. Then $\hat{\hat{A}}$ is easily seen to be a $(d+2)$-dimensional $(k+1)$-neighborly configuration of $n$ points.

However, we will do a slight variation of this method to obtain a $(d+2)$-dimensional configuration of $n+2$ points. The reason is that, in a D-lifting $\hat{A}$ of $A$, the position of the point $a_n$ in $A$ does not affect the combinatorics
of $\text{conv}(\hat{A})$. Hence, we can always add an “extra point” to play the role of this last point. To this end, we will extend the concept of D-lifting to that of pointed D-lifting, which could be understood as doing a D-lifting and then placing the extra point so high that plays the role of a point at infinity. This will become convenient to translate from triangulations to polytopes, and to estimate with more precision the number of combinatorial types obtained.

5.1. Pointed lexicographic liftings. As usual, let $A = \{a_1, \ldots, a_n\}$ be a configuration of $n$ labeled points in $\mathbb{R}^d$. A configuration $\hat{A}[\hat{p}] = \{\hat{a}_1, \ldots, \hat{a}_n, \hat{p}\}$ of $n+1$ labeled points in $\mathbb{R}^{d+1}$ is a pointed Delaunay lexicographic lifting of $A$ (with respect to the order induced by the labels) if $\hat{p}$ has label $n+1$ and there is some $p \in \mathbb{R}^d$ such that $\hat{A}[\hat{p}]$ is a D-lifting of $A' = A \cup p$ and such that the height for $\hat{p}$ is positive. We call $\hat{p}$ the apex of the D-lifting.

In short, to do a pointed lifting, a point $p$ is added to $A$ and is lifted in the positive direction as the last element of the configuration.

Observation 5.1. Let $\hat{A}[\hat{p}]$ be a pointed D-lifting of $A$. Then the faces of $\text{conv}(\hat{A}[\hat{p}])$ that contain the apex $\hat{p}$ are the join of $\hat{p}$ with every face of $\text{conv}(A)$.

Observe that Theorems 4.1, 4.2 and 4.3 still hold if we replace the D-lifting by a pointed D-lifting with apex $\hat{p}$. Indeed, the proof of Theorem 4.2 only uses the neighborliness of $A \setminus a_n$, and those of Theorems 4.1 and 4.3 follow directly.

Remark 5.2. The pointed D-lifting of $A$ with heights $h_i$ corresponds to the lexicographic extension by $[a_{n-\text{sign}(h_n)}, \ldots, a_{d+2-\text{sign}(h_{d+2})}]$ of the Gale dual of $A$ (see [9 Section 7.2]). This provides the link with the results presented in [20] and [21].

5.2. Polytopes from triangulations. To recover inscribed polytopes from Delaunay triangulations, we need the following classical result from Brown [8] (cf. [15, Proposition 0.3.13]).

Lemma 5.3. Let $A = \{a_1, \ldots, a_n\}$ be a configuration of $n$ points in $\mathbb{R}^d$ in general position, and let $D(A)$ be its Delaunay triangulation. Then there is an inscribable simplicial $(d+1)$-polytope $P_A$ with $n+1$ vertices $\{\hat{a}_1, \ldots, \hat{a}_n, \hat{p}\}$ whose faces are:

(i) $\{\hat{a}_{i_1}, \ldots, \hat{a}_{i_k}\}$ if $\{a_{i_1}, \ldots, a_{i_k}\}$ is a cell of $D(A)$ and
(ii) $\{\hat{p}, \hat{a}_{i_1}, \ldots, \hat{a}_{i_k}\}$ if $\{a_{i_1}, \ldots, a_{i_k}\}$ is a face of $\text{conv}(A)$.

An inscribed realization of $P_A$ can be found by inverting a stereographic projection.

Observation 5.4. This lemma implies a bijection between labeled combinatorial types of inscribable $d$-polytopes with $n$ vertices and $(d-1)$-dimensional Delaunay triangulations of $n-1$ points. To go from the face lattice of an inscribable polytope to the face lattice of a Delaunay triangulation, simply delete all faces containing the point with the largest label. This is clearly a bijection.

We omit the proof of the next lemma, which only needs a combinatorial description of the face lattice of a positive pointed lifting (cf. [10, Lemma 4.3.4]).
Lemma 5.5. If the Delaunay triangulation of $A$ coincides with its placing triangulation, then the polytope $P_A$ of Lemma 5.3 is combinatorially equivalent to a positive pointed $D$-lifting of $A$.

As a direct consequence of the combination of Lemma 5.3 and Theorem 4.3, we obtain the following result. Observe how the strategy is to start with a $k$-neighborly $d$-polytope, lift it to a $(k+1)$-neighborly $(d+1)$-triangulation, and lift it again (with a positive lifting with the same order) to a $(k+1)$-neighborly $(d+1)$-polytope.

Theorem 5.6. Let $A$ be a $d$-dimensional configuration of $n \geq d+2$ points in convex general position such that $\text{conv}(A)$ is $k$-neighborly, let $\hat{A}[\hat{p}]$ be a pointed $D$-lifting of $A$ and let $\hat{A}[\hat{p}][\hat{q}]$ be a positive pointed $D$-lifting of $\hat{A}[\hat{p}]$.

Then $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$ is an inscribable $(k+1)$-neighborly $(d+2)$-polytope. In particular, if $\text{conv}(A)$ is neighborly, so is $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$.

Proof. Inscribability follows from Lemma 5.5, which says that $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$ is combinatorially equivalent to the inscribable polytope $P_A$ of Lemma 5.3.

For $k$-neighborliness, observe that every subset of $k+1$ points of $A[\hat{p}]$ is a face of its placing triangulation, and hence a face of $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$ by point (ii) of Lemma 5.3. And every set of $k+1$ points of $\hat{A}[\hat{p}][\hat{q}]$ containing $\hat{q}$ is a face of $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$, by point (iii) of Lemma 5.3 because $\text{conv}(\hat{A}[\hat{p}])$ is $k$-neighborly.

6. LOWER BOUNDS

Theorem 5.6 provides the following method to construct many inscribable neighborly $(d \geq 2)$-polytopes with $n$ vertices starting with an arbitrary polygon or 3-polytope.

Construction 6.1. To construct an inscribable neighborly $d$-polytope with $n$ vertices $P$ and a $(d-1)$-dimensional set of $n-1$ points with a neighborly Delaunay triangulation $T$:

1. Set $n_0 := n - 2 \left\lfloor \frac{d-2}{2} \right\rfloor$ and $d_0 := d - 2 \left\lfloor \frac{d-2}{2} \right\rfloor$.
2. Let $P_0$ be any simplicial $d_0$-polytope with $n_0$ vertices.
3. $P_0$ is neighborly because $d_0 \in \{2, 3\}$.
4. Set $A_0 := \text{vert}(P_0)$.
5. For $i$ from 1 to $\left\lfloor \frac{d-2}{2} \right\rfloor$ do:
   1.1. Choose a permutation $\sigma \in S_{n_{i-1}}$ and relabel the points of $A_{i-1}$ with $a_j \mapsto a_{\sigma(j)}$.
   2. Let $\hat{A}_{i-1}[\hat{p}]$ be a pointed $D$-lifting of $A_{i-1}$.
   3. Set $A_i := \hat{A}_{i-1}[\hat{p}][\hat{q}]$ be a pointed positive $D$-lifting of $\hat{A}_{i-1}[\hat{p}]$.
   4. Set $n_i := n_{i-1} + 2$ and $d_i := d_{i-1} + 2$.
   5. By Theorem 5.6, $P_i := \text{conv}(A_i)$ is a neighborly $d_i$-polytope with $n_i$ vertices.
6. $P := P\left\lfloor \frac{d-2}{2} \right\rfloor$ is an inscribable neighborly $d$-polytope with $n$ vertices by Theorem 5.6.
7. $T := \mathcal{D}(\hat{A}\left\lfloor \frac{d-1}{2} \right\rfloor[\hat{p}])$ is a neighborly Delaunay triangulation of a $(d-1)$-dimensional set of $n-1$ points by Theorem 4.3.
Remark 6.2. We deferred the discussion about the order the points for the D-lifting until now. However, the relabeling step in Construction 6.1 is crucial, since it is the choice of these permutations what produces the variety of combinatorial types. It is also important to remark that the choice of the permutation is only done once every two dimensions, since the lifting with apex $\hat{p}$ and the lifting with apex $\hat{q}$ need to follow the same order or otherwise Theorem 5.6 does not hold.

Remark 6.3. Construction 6.1 relies on Delaunay lexicographic liftings, which in their definition depend on some $h_i$'s being “large enough”. One might wonder how feasible, computationally, is to find these $h_i$'s. On the one hand, it is not hard to find an upper bound for the minimal valid $h_i$ that depends only on $d$ and $n$ if we assume, for example, that the $a_i$'s have integer entries whose absolute value is bounded by $K$. Indeed, the conditions required in Definition 3.1 only depend on certain determinants being positive. However, if we apply this method to construct our point configurations from scratch, we will end up with points having extremely large coordinates. It remains open which are the minimal coordinates needed to realize these configurations. Can they be realized with polynomially large coordinates? In lower dimensions this kind of questions has already been considered (see [12] and [13]).

Remark 6.4. Theorem 4.3 provides neighborly Delaunay triangulations that are stereographic projections of neighborly polytopes from vertices. However, they do not attain the upper bound of Theorem 2.3, because their convex hull is not a simplex. To obtain such triangulations, we need neighborly polytopes with an inscribed realization that can be stacked on a facet such that the result is still inscribed (see [23] and [15]).

Construction 6.1 can be modified to get them, following [15] Remark 1.3.5. In the last D-lifting step (from $A_{\lfloor \frac{d-4}{2} \rfloor}$ to $\hat{A}_{\lfloor \frac{d-4}{2} \rfloor}([\hat{p}])$), apply a stellar subdivision to the first simplex directly after it appears. What we then get is the vertex projection of an inscribed neighborly polytope that has been stacked with a point on its circumsphere. We can then apply Brown’s projection from the stacked point to get the desired triangulation with a maximal number of faces.

6.1. The bounds. It remains to discuss how many different labeled combinatorial types of inscribable neighborly polytopes (and of neighborly Delaunay triangulations) can be obtained with Construction 6.1. These bounds were obtained in [20] using a construction that can be seen to be equivalent. We will only sketch the main ideas for the original proof, which is based on oriented matroids, and refer to [20] and [21, Chapter 5] for details.

These bounds concentrate on the case of even-dimensional neighborly polytopes (see Remark 6.7). The main ingredient for them is the following lemma, which ignores the variability provided by the signs of the $h_i$'s and only focuses on positive liftings. Stronger bounds are discussed in [21, Chapter 5].

Lemma 6.5 ([21 Proposition 5.4], [21 Lemma 5.9]). Let $A$ be a configuration of $n > d + 2$ labeled points in convex general position in $\mathbb{R}^d$, for $d \geq 2$ even, such that $\text{conv}(A)$ is neighborly.
Denote by $\mathcal{P}$ the set of different labeled combinatorial types of polytopes $P$ that fulfill:

- $P = \text{conv}(\hat{A}[\hat{p}][\hat{q}])$ is obtained by doing a couple of positive D-liftings (as in Theorem 5.6) from an initial relabeling of $A$, and
- the labeled combinatorial type of $\text{conv}(A)$ can be recovered from that of $\text{conv}(\hat{A}[\hat{p}][\hat{q}])$.

Then there are at least $\frac{(n+1)!}{(d+2)!}$ different elements in $\mathcal{P}$.

**Proof idea.** The idea is that to go from $A$ to $\hat{A}[\hat{p}][\hat{q}]$, first a permutation $\sigma$ is applied to the labels of $A$. Then $\hat{A}[\hat{p}]$ is a pointed positive D-lifting of $A$ and $\hat{A}[\hat{p}][\hat{q}]$ is a pointed positive D-lifting of $\hat{A}[\hat{p}]$ (with respect to the same ordering). Finally the permutation $\sigma^{-1}$ is applied on the labels of $\hat{A}$ so that the original labeling of $A$ is preserved.

The next step in the proof of this lemma in [21, Proposition 5.4] is to show that, if we restrict to the case when both D-liftings are positive, then the $n - d - 2$ first elements of $\sigma$ can be recovered from the combinatorics of $P$, the convex hull of $\hat{A}[\hat{p}][\hat{q}]$, and that 2 different cases can be distinguished for the last $d + 2$ elements.

A final step is that, when $\text{conv}(A)$ is neighborly, if $\hat{q}$ and the face lattice of $P$ are fixed, then there are at most two points of $\hat{A}[\hat{p}]$ that could be $\hat{p}$. Hence the label of $\hat{p}$ can also be chosen almost arbitrarily, adding a factor of $\frac{n+1}{2}$ to the number of combinatorial types.

Applying recursively this lemma one obtains the following bound, which is estimated using Euler-Maclaurin approximation.

**Theorem 6.6.** For even $d$, $\text{inp}(n, d)$, the number of different labeled combinatorial types of inscribable neighborly $d$-polytopes with $n$ vertices, fulfills

$$\text{inp}(n, d) \geq \prod_{i=1}^{\frac{d}{2}} \frac{(n - d - 1 + 2i)!}{(2i)!} \geq \frac{(n - 1)(\frac{n+1}{2})^2}{(n - d - 1)(\frac{n+1}{2})^3 d^{\frac{3}{4}}} e^{\frac{3d(n-d-1)}{4}} \geq \left(\frac{n - 1}{e^{3/2}}\right)^{\frac{1}{2}(n-d-1)} d^d.$$  

By Observation 5.4 the same bound holds for the number of labeled combinatorial types of neighborly Delaunay triangulations of configurations of $n - 1$ points in $\mathbb{R}^{d-1}$.

**Remark 6.7.** For the odd-dimensional case, observe that any pyramid over an even dimensional inscribable neighborly polytope is again an inscribable neighborly polytope, which shows that the bound (1) also applies for odd dimensional configurations.

**Remark 6.8.** Although the use of labeled types is needed to prove these bounds, observe that we can easily recover bounds for non-labeled combinatorial types just by dividing by $n!$. The bounds obtained this way are still superexponential and comparable to (1).
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