The proximal augmented Lagrangian method for nonsmooth composite optimization

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Abstract—We study a class of optimization problems in which the objective function is given by the sum of a differentiable but possibly nonconvex component and a nondifferentiable convex regularization term. We introduce an auxiliary variable to separate the objective function components and utilize the Moreau envelope of the regularization term to derive the proximal augmented Lagrangian—a continuously differentiable function obtained by constraining the augmented Lagrangian to the manifold that corresponds to the explicit minimization over the variable in the nonsmooth term. This function is used to develop customized algorithms based on the method of multipliers (MM) and a primal-descent dual-ascent gradient method in order to compute optimal primal-dual pairs. Our customized MM algorithm is applicable to a broader class of problems than proximal gradient methods and it has stronger convergence guarantees and a more refined step-size update rules than the alternating direction method of multipliers. These features make it an attractive option for solving structured optimal control problems. When the differentiable component of the objective function is (strongly) convex and the regularization term is convex, we prove (exponential) asymptotic convergence of the primal-descent dual-ascent algorithm. Finally, we solve the edge addition in directed consensus networks and optimal placement problems to demonstrate the merits and the effectiveness of our approach.

I. INTRODUCTION

We study a class of composite optimization problems in which the objective function is a sum of a differentiable but possibly nonconvex component and a convex nondifferentiable component. Problems of this form are encountered in diverse fields including compressive sensing [1], machine learning [2], statistics [3], image processing [4], and control [5]. In feedback synthesis, they typically arise when a traditional performance metric (such as the $H_2$ or $H_{\infty}$ norm) is augmented with a regularization function to promote certain structural properties in the optimal controller. For example, the $\ell_1$ norm and the nuclear norm are commonly used nonsmooth convex regularizers that encourage sparse and low-rank optimal solutions, respectively.

The lack of a differentiable objective function precludes the use of standard descent methods for smooth optimization. Proximal gradient methods [6] and their accelerated variants [7] generalize gradient descent, but typically require the nonsmooth term to be separable over the optimization variable. Furthermore, standard acceleration techniques are not well-suited for problems with constraint sets that do not admit an easy projection (e.g., closed-loop stability).

An alternative approach is to split the smooth and nonsmooth components in the objective function over separate variables which are coupled via an equality constraint. Such a reformulation facilitates the use of the alternating direction method of multipliers (ADMM) [8]. This augmented-Lagrangian-based method splits the optimization problem into subproblems which are either smooth or easy to solve. It also allows for a broader class of regularizers than proximal gradient and it is convenient for distributed implementation. However, there are limited convergence guarantees for nonsmooth convex problems and parameter tuning greatly affects its convergence rate.

The method of multipliers (MM) is the most widely used algorithm for solving constrained nonlinear programing problems [9]–[11]. In contrast to ADMM, it is guaranteed to converge for nonconvex problems and there are systematic ways to adjust algorithmic parameters. However, MM is not a splitting method and it requires joint minimization of the augmented Lagrangian with respect to all primal optimization variables. This subproblem is typically nonsmooth and as difficult to solve as the original optimization problem.

In this paper, we transform the augmented Lagrangian into a continuously differentiable form by exploiting the structure of proximal operators associated with nonsmooth regularizers. This new form is obtained by constraining the augmented Lagrangian to the manifold that corresponds to the explicit minimization over the variable in the nonsmooth term. The resulting expression, that we call the proximal augmented Lagrangian, is given in terms of the Moreau envelope of the nonsmooth regularizer and it is continuously differentiable. This allows us to take advantage of standard optimization tools, including gradient descent and quasi-Newton methods, and enjoy the convergence guarantees of the standard method of multipliers.

We also examine Arrow-Hurwicz-Uzawa gradient flow dynamics for the proximal augmented Lagrangian. Such dynamics can be used to identify saddle points of the Lagrangian [12] and have enjoyed recent renewed interest in the context of networked optimization because, in many cases, the gradient can be computed in a distributed manner [13]. Our approach yields a dynamical system with a continuous right-hand side for a broad class of nonsmooth optimization problems. This is in contrast to existing techniques which employ subgradient methods [14] or use discontinuous projected dynamics [15]–[17] to handle inequality constraints. Furthermore, since the proximal augmented Lagrangian is not strictly convex-concave we make additional developments relative to [18] to show asymptotic convergence. Finally, inspired by recent advances [19], [20], we employ the theory of integral quadratic constraints (IQCs) [21] to prove exponential convergence when the differentiable component of the objective function is strongly convex with a Lipschitz continuous gradient.

The rest of the paper is structured as follows. In Section II we formulate the nonsmooth composite optimization problem and provide a brief background on proximal operators. In Section III we exploit the structure of proximal operators to introduce the proximal augmented Lagrangian. In Section IV-A we provide an efficient algorithmic implementation of the method of multipliers using the proximal augmented Lagrangian. In Section IV-B we prove global (exponential) asymptotic convergence of primal-descent dual-ascent gradient flow dynamics under a (strong) convexity assumption. In Section V we use the problem of edge addition in directed consensus networks and optimal placement to illustrate the effectiveness of our approach. We close the paper in Section VI with concluding remarks.

II. PROBLEM FORMULATION AND BACKGROUND

We consider a composite optimization problem,

$$\min_x f(x) + g(T(x))$$

(1)

where the optimization variable $x$ belongs to a finite-dimensional Hilbert space (e.g., $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$) equipped with an inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. The function $f$ is continuously differentiable but possibly nonconvex, the function $g$ is convex but
potentially nondifferentiable, and $T$ is a bounded linear operator. We further assume that $g$ is proper and lower semicontinuous, that $f$ is feasible, and that its minimum is finite.

Problem (1) is often encountered in structured controller design [23–24], where $f$ is a measure of closed-loop performance, e.g., the $H_2$ norm, and the regularization term $g$ is introduced to promote certain structural properties of $T(x)$. For example, in wide-area control of power systems, $f$ measures the quality of synchronization between different generators and $g$ penalizes the amount of communication between them [25–27].

In particular, for $z := T(x) \in \mathbb{R}^m$, the $\ell_1$ norm, $\|z\|_1 := \sum |z_i|$, is a commonly used convex proxy for promoting sparsity of $z$. For $z \in \mathbb{R}^{m \times n}$, the nuclear norm, $\|z\|_* := \sum \sigma_i(z)$, can be used to obtain low-rank solutions to (1), where $\sigma_i(z)$ is the $i$th singular value.

The indicator function, $I_C(z) := \{0, z \in C; \infty, z \notin C\}$ associated with the convex set $C$ is the proper regularizer for enforcing $z \in C$.

Regularity of $T(x)$ instead of $x$ is important in the situations where the desired structure has a simple characterization in the co-domain of $T$. Such problems arise in spatially-invariant systems, where it is convenient to perform standard control design in the spatial frequency domain [28] but necessary to promote structure in the physical space, and in consensus/synchronization networks, where the objective function is expressed in terms of the deviation of node values from the network average but it is desired to impose structure on the network edge weights [23, 24].

A. Background on proximal operators

Problem (1) is difficult to solve directly because $f$ is, in general, a nonconvex function and $g$ is typically not differentiable. Since the existing approaches and our method utilize proximal operators, we first provide a brief overview; for additional information, see [6].

The proximal operator of the function $g$ is given by

$$\text{prox}_{\mu g}(v) := \arg\min g(x) + \frac{1}{2\mu} \|x - v\|^2 \quad (2a)$$

and the associated optimal value specifies its Moreau envelope,

$$M_{\mu g}(v) := \inf g(x) + \frac{1}{2\mu} \|x - v\|^2 \quad (2b)$$

where $\mu > 0$. The Moreau envelope is a continuously differentiable function, even when $g$ is not, and its gradient is
differentiable, namely

$$\nabla M_{\mu g}(v) = \frac{1}{\mu} (v - \text{prox}_{\mu g}(v)). \quad (2c)$$

For example, when $g$ is the $\ell_1$ norm, the proximal operator is determined by soft-thresholding,

$$\text{prox}_{\mu g}(v_i) = S_\mu(v_i) := \text{sign}(v_i) \max\{|v_i| - \mu, 0\} \quad (3a)$$

and the associated Moreau envelope is given by the Huber function,

$$M_{\mu g}(v_i) = \begin{cases} \frac{1}{2\mu} v_i^2 & |v_i| \leq \mu \\ |v_i| - \frac{\mu}{2} & |v_i| \geq \mu \end{cases} \quad (3b)$$

and the gradient of this Moreau envelope is the saturation function,

$$\nabla M_{\mu g}(v_i) = \text{sign}(v_i) \min(|v_i|/\mu, 1), \quad (3c)$$

Similarly, the proximal operator associated with the nuclear norm is obtained by soft-thresholding the singular values of a matrix.

B. Existing algorithms

1) Proximal gradient: The proximal gradient method generalizes standard gradient descent to certain classes of nonsmooth optimization problems. This method can be used to solve (1) when $g(T)$ has an easily computable proximal operator. When $T = I$, the proximal gradient method for problem (1) with step-size $\alpha_1$ is given by,

$$x^{k+1} = \text{prox}_{\alpha_1 g}(x^k - \alpha_1 \nabla f(x^k)).$$

When $g = 0$, the proximal gradient method simplifies to standard gradient descent, and when $g$ is indicator function of a convex set, it simplifies to projected gradient descent. The proximal gradient algorithm applied to the $\ell_1$-regularized least-squares problem (LASSO)

$$\min_{x} \frac{1}{2} \|Ax - b\|^2 + \gamma \|x\|_1 \quad (4)$$

where $\gamma$ is a positive regularization parameter, yields the Iterative Soft-Thresholding Algorithm (ISTA) [7].

$$x^{k+1} = S_{\alpha_1}(x^k - \alpha_1 A^T(Ax^k - b)).$$

This method is effective only when the proximal operator of $g(T)$ is easy to evaluate. Except in special cases, e.g., when $T$ is diagonal, efficient computation of $\text{prox}_{\mu g(T)}$ does not necessarily follow from an efficiently computable $\text{prox}_{\mu g}$. This makes the use of proximal gradient method challenging for many applications and its convergence can be slow. Acceleration techniques improve the convergence rate [7], [29], but they do not perform well in the face of constraints such as closed-loop stability.

1) Proximal gradient: A common approach for dealing with a nondiagonal linear operator $T$ in (1) is to introduce an additional optimization variable $z$

$$\min_{x,z} f(x) + g(z)$$

subject to $T(x) - z = 0. \quad (5)$

The augmented Lagrangian is obtained by adding a quadratic penalty on the violation of the linear constraint to the regular Lagrangian associated with (5),

$$L_{\mu}(x,z,y) = f(x) + g(z) + \langle y, T(x) - z \rangle + \frac{1}{2\mu} \|T(x) - z\|^2$$

where $y$ is the Lagrange multiplier and $\mu$ is a positive parameter.

ADMM solves (5) via an iteration which involves minimization of $L_{\mu}(x,z,y)$ separately over $x$ and $z$ and a gradient ascent update (with step-size $1/\mu$) of $y$ [8].

$$x^{k+1} = \arg\min_{x} L_{\mu}(x, z^k; y^k) \quad (6a)$$

$$z^{k+1} = \arg\min_{z} L_{\mu}(x^{k+1}, z; y^k) \quad (6b)$$

$$y^{k+1} = y^k + \frac{1}{\mu} (T(x^{k+1}) - z^{k+1}) \quad (6c).$$

ADMM is appealing because, even when $T$ is nondiagonal, the $z$-minimization step amounts to evaluating $\text{prox}_{\mu g(z)}$, and the $x$-minimization step amounts to solving a smooth (but possibly nonconvex) optimization problem. Although it was recently shown that ADMM is guaranteed to converge to a stationary point of (5) for some classes of nonconvex problems [30], its rate of convergence is strongly influenced by the choice of $\mu$.

The method of multipliers (MM) is the most widely used algorithm for solving constrained nonconvex optimization problems [9, 10] and it guarantees convergence to a local minimum. In contrast to ADMM, each MM iteration requires joint minimization of the augmented Lagrangian with respect to the primal variables $x$ and $z$,

$$(x^{k+1}, z^{k+1}) = \arg\min_{x,z} L_{\mu}(x, z; y^k)$$

$$y^{k+1} = y^k + \frac{1}{\mu} (T(x^{k+1}) - z^{k+1}). \quad (7a)$$

It is possible to refine MM to allow for inexact solutions to the $(x,z)$-minimization subproblem and adaptive updates of the penalty parameter $\mu$. However, until now, MM has not been a feasible choice for solving (5) because the nonconvex and nondifferentiable $(x,z)$-minimization subproblem is as difficult as the original problem (1).

III. THE PROXIMAL AUGMENTED LAGRANGIAN METHOD

In this section, we derive the proximal augmented Lagrangian, a continuously differentiable function resulting from explicit minimization of the augmented Lagrangian over the auxiliary variable $z$. This
brings the \((x, z)\)-minimization problem in (7) into a form that is continuously differentiable with respect to \(x\) and facilitates the use of a wide suite of standard optimization tools, including the method of multipliers and the Arrow-Hurwicz-Uzawa method.

A. Derivation of the proximal augmented Lagrangian

The first main result of the paper is provided in Theorem 1. We use the proximal operator of the function \(g\) to eliminate the auxiliary optimization variable \(z\) from the augmented Lagrangian and transform (7a) into a tractable continuously differentiable problem.

**Theorem 1:** For a proper, lower semicontinuous, and convex function function \(g\), minimization of the augmented Lagrangian \(\mathcal{L}_\mu(x; z; y)\) associated with problem (5) over \((x, z)\) is equivalent to minimization of the proximal augmented Lagrangian

\[
\mathcal{L}_\mu(x; y) := f(x) + M_{y\mu}(T(x) + \mu y) - \frac{\mu}{2} \|y\|^2
\]

over \(x\). Moreover, if \(f\) is continuously differentiable then \(\mathcal{L}_\mu(x; y)\) is continuously differentiable over \(x\) and \(y\), and if \(f\) has a Lipschitz continuous gradient \(\nabla \mathcal{L}_\mu(x; y)\) is Lipschitz continuous.

**Proof:** Through the completion of squares, the augmented Lagrangian \(\mathcal{L}_\mu\), associated with (5) can be equivalently written as

\[
\mathcal{L}_\mu(x; z; y) = f(x) + g(z) + \frac{1}{2\mu} \|z - (T(x) + \mu y)\|^2 - \frac{\mu}{2} \|y\|^2.
\]

Minimization with respect to \(z\) yields an explicit expression,

\[
z^*_{\mu}(x, y) = \text{prox}_{\mu g}(T(x) + \mu y)
\]

and substitution of \(z^*_{\mu}\) into the augmented Lagrangian provides (8). i.e., \(\mathcal{L}_\mu(x; y) = \mathcal{L}_\mu(x, z^*_{\mu}(x, y); y)\). Continuous differentiability of \(\mathcal{L}_\mu(x; y)\) follows from continuous differentiability of \(M_{y\mu}\) and Lipschitz continuity of \(\nabla \mathcal{L}_\mu(x; y)\) follows from Lipschitz continuity of \(\text{prox}_{\mu g}\) and boundedness of the linear operator \(T\); see [23].

Expression (8), that we refer to as the proximal augmented Lagrangian, characterizes \(\mathcal{L}_\mu(x; z; y)\) on the manifold corresponding to explicit minimization over the auxiliary variable \(z\). Theorem 1 allows joint minimization of the augmented Lagrangian with respect to \(x\) and \(z\), which is in general a nondifferentiable problem, to be achieved by minimizing differentiable function (8) over \(x\). It thus facilitates the use of the method of multipliers in Section III-B and the Arrow-Hurwicz-Uzawa gradient flow dynamics in Section IV.

B. MM based on the proximal augmented Lagrangian

Theorem 1 allows us to solve nondifferentiable subproblem (7a) by minimizing the continuously differentiable proximal augmented Lagrangian \(\mathcal{L}_\mu(x; y^k)\) over \(x\). Relative to ADMM, our customized MM algorithm guarantees convergence to a local minimum and offers systematic update rules for the parameter \(\mu\). Relative to proximal gradient, we can solve (11) with a general bounded linear operator \(T\) and can incorporate second order information about \(f\).

Using reformulated expression (8) for the augmented Lagrangian, MM minimizes \(\mathcal{L}_\mu(x; y^k)\) over the primal variable \(x\) and updates the dual variable \(y\) using gradient ascent with step-size \(1/\mu\),

\[
x^{k+1} = \text{argmin}_{x} \mathcal{L}_\mu(x; y^k) \quad \text{(MMa)}
\]

\[
y^{k+1} = y^k + \frac{1}{\mu} \nabla y \mathcal{L}_\mu(x^{k+1}; y^k) \quad \text{(MMb)}
\]

where

\[
\nabla y \mathcal{L}_\mu(x^{k+1}; y^k) = \nabla (T(x^{k+1}) - \text{prox}_{\mu g}(T(x^{k+1}) + \mu y^k))
\]

denotes the primal residual, i.e., the difference between \(T(x^{k+1})\) and \(z^*_{\mu}(x^{k+1}, y^k)\).

In contrast to ADMM, our approach does not attempt to avoid the lack of differentiability of \(g\) by fixing \(z\) to minimize over \(x\). By constraining \(\mathcal{L}_\mu(x; z; y)\) to the manifold resulting from explicit minimization over \(z\), we guarantee continuous differentiability of the proximal augmented Lagrangian \(\mathcal{L}_\mu(x; y)\). MM is a gradient ascent algorithm on the Lagrangian dual of problem (5) strengthened by a quadratic penalty on primal infeasibility. Since a closed-form expression of the dual is typically unavailable, MM uses the \((x, z)\)-minimization subproblem (7a) to evaluate it computationally and then take a gradient ascent step (7b) in \(y\). ADMM avoids this issue by solving simpler, separate subproblems over \(x\) and \(z\). However, the \(x\) and \(z\) minimization steps (6a) and (6b) do not solve (7a) and thus unlike the \(y\)-update (7b) in MM, the \(y\)-update (6c) in ADMM is not a gradient ascent step on the strengthened dual. MM thus has stronger convergence results [8, 9] and may lead to fewer \(y\)-update steps.

Remark 1: The proximal augmented Lagrangian enables MM because the \(x\)-minimization subproblem in MM (MMa) is not more difficult than in ADMM (6a). For LASSO problem (4), the \(x\)-update in ADMM (6a) is given by soft-thresholding \(z^{k+1} = S_{\gamma\mu}(x^{k+1} + \mu y^k)\), and the \(x\)-update (6a) requires minimization of the quadratic function (8). In contrast, the \(x\)-update (MMa) in MM requires minimization of \((1/2)\|Ax - b\|^2 + \mu_{\nu\mu}(x + \mu y^k)\), where \(\mu_{\nu\mu}(y)\) is the Moreau envelope associated with the \(\ell_1\) norm; i.e., the Huber function [35]. Although in this case the solution to (6a) cannot be characterized explicitly by a matrix inversion, this is not true in general. The computational cost associated with solving either (6a) or (MMa) using first-order methods scales at the same rate.

1) Algorithm: The procedure outlined in [11, Algorithm 17.4] allows minimization subproblem (MMa) to be inexact, provides a method for adaptively adjusting \(\mu_k\), and describes a more refined update of the Lagrange multiplier \(\mu\). We incorporate these refinements into our proximal augmented Lagrangian algorithm for solving (5).

In Algorithm 1 \(\mu\) and \(\omega\) are convergence tolerances, and \(\mu_{\min}\) is a minimum value of the parameter \(\mu\). Because of the equivalence established in Theorem 1 convergence to a local minimum follows from the convergence results for the standard method of multipliers [11].

**Algorithm 1 Proximal augmented Lagrangian algorithm for (5).**

input: Initial point \(x^0\) and Lagrange multiplier \(\mu^0\)
initialize: \(\mu_0 = 10^{-1}, \mu_{\min} = 10^{-5}, \omega_0 = \mu_0, \eta_0 = \mu_0^{-1}\)
for \(k = 0, 1, 2, \ldots\)

Solve (MMa) such that

\[
\|\nabla y \mathcal{L}_\mu(x^{k+1}; y^k)\| \leq \omega_k
\]

if \(\|\nabla y \mathcal{L}_\mu(x^{k+1}; y^k)\| \leq \eta_k\)

else if \(\|\nabla y \mathcal{L}_\mu(x^{k+1}; y^k)\| \leq \eta\) and \(\|\nabla x \mathcal{L}_\mu(x^{k+1}; y^k)\| \leq \omega\)

stop with approximate solution \(x^{k+1}\)

else

\[
y^{k+1} = y^k + \frac{1}{\mu_k} \nabla y \mathcal{L}_\mu(x^{k+1}; y^k), \quad \mu_{k+1} = \mu_k
\]

\[
\eta_{k+1} = \eta_k \mu_0^{-1}, \quad \omega_{k+1} = \omega_k \mu_{k+1}^{-1}
\]

endif

else

\[
y^{k+1} = y^k, \quad \mu_{k+1} = \max\{\mu_k/5, \mu_{\min}\} \quad \omega_{k+1} = \omega_k \mu_{k+1}^{-1}
\]

endif
endfor

2) Minimization of \(\mathcal{L}_\mu(x; y)\) over \(x\): MM based on the proximal augmented Lagrangian alternates between minimization of \(\mathcal{L}_\mu(x; y)\) with respect to \(x\) (for fixed values of \(\mu\) and \(y\)) and the update of \(\mu\) and \(y\). Since \(\mathcal{L}_\mu(x; y)\) is once continuously differentiable, many techniques can be used to find a solution to subproblem (MMa). We next summarize three of them.
Gradient descent: The gradient with respect to $x$ is given by,
$$\nabla_x L_\mu(x; y) = \nabla f(x) + \frac{1}{\mu} T^\dagger(T(x) + \mu y - \text{prox}_{\mu g}(T(x) + \mu y))$$
where $T^\dagger$ denotes the adjoint of $T$. Backtracking conditions such as the Armijo rule can be used to select a step-size.

Proximal gradient: Gradient descent does not exploit the structure of the Moreau envelope of the function $g$; in some cases, it may be advantageous to use proximal operator associated with the Moreau envelope to solve [1]. In particular, when $T = I$, [1] and [2] imply that $\text{prox}_{\alpha L_\mu g}(v) = x^*$ where $x^*$ satisfies,
$$x^* = \frac{1}{\mu + \alpha} \left( \alpha \text{prox}_{\mu g}(x^*) + \mu v \right).$$
If $g$ is separable and has an easily computable proximal operator, its Moreau envelope also has an easily computable proximal operator. For example, the $i$th element of the proximal operator of the Huber function [5] (the Moreau envelope associated with the $\ell_i$ norm), is
$$\text{prox}_{\mu L_{\text{hub}}}(v_i) = \begin{cases} \frac{v_i}{\mu + \alpha}, & |v_i| \leq \mu + \alpha \\ v_i - \alpha \text{sign}(v_i), & |v_i| \geq \mu + \alpha. \end{cases}$$

In [32], proximal gradient methods were used to solve subproblem [MMa] to solve a sparse feedback synthesis problem introduced in [3]. Computational savings were shown relative to standard proximal gradient method and ADMM.

Quasi-Newton method: Although $\text{prox}_{\mu g}$ is typically not differentiable, it is Lipschitz continuous and therefore differentiable almost everywhere [33]. To improve computational efficiency, we employ the limited-memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) method [41] to estimate the Hessian $\nabla^2_x L_\mu(x; y^*)$. While BFGS is only guaranteed to converge for twice-differentiable convex functions, good practical performance has been observed for nonsmooth and nonconvex problems [43]. Since $L_\mu(x; y^*)$ is at least once continuously differentiable and BFGS approximates the Hessian by a positive definite operator, the BFGS update for subproblem [MMa] is well-defined and it is a descent direction for $L_\mu(x; y^*)$.

Remark 2: For regularization functions that do not admit simply computable proximal operators, $\text{prox}_{\mu g}$ has to be evaluated numerically by solving [6]. If this is expensive, the primal-descent dual-ascent algorithm of Section IV offers an appealing alternative because it requires one evaluation of $\text{prox}_{\mu g}$ per iteration. When the regularization function $g$ is nonconvex, the proximal operator may not be single-valued and the Moreau envelope may not be continuously differentiable. In spite of this, the convergence of proximal algorithms has been established for nonconvex, proper, lower semicontinuous regularizers that obey the Kurdyka-Lojasiewicz inequality [35].

IV. ARROW-HURWICZ-UZAWA GRADIENT FLOW

Instead of minimizing over the primal variable and performing gradient ascent in the dual, an alternative approach is to simultaneously update the primal and dual variables to find the saddle point of the augmented Lagrangian. The continuous differentiability of $L_\mu(x; y)$ established in Theorem [3] enables the use of Arrow-Hurwicz-Uzawa gradient flow dynamics [12],
$$\begin{cases} \dot{x} = -\nabla_x L_\mu(x; y) \\ \dot{y} = -\nabla_y L_\mu(x; y) \end{cases}.$$ (GF)

In Section IV-A we show that the continuous-time gradient flow dynamics [41] globally converge to the set of saddle points of the proximal augmented Lagrangian $L_\mu(x; y)$ for a convex $f$. In Section IV-B we employ the theory of IQCs to establish exponential convergence for a strongly convex $f$ with a Lipschitz continuous gradient. Finally, in Section IV-C we identify classes of problems for which dynamics [41] are convenient for distributed implementation and compare/contrast our framework to the existing approaches.

A. Asymptotic convergence for convex $f$

We first characterize the optimal primal-dual pairs of optimization problem [5] with the Lagrangian, $f(x) + g(z) + (y, T(x) - z)$. The associated first-order optimality conditions are given by,
$$0 = \nabla f(x^*) + T^\dagger(y^*)$$
$$0 \in \partial g(z^*) - y^*$$
$$0 = T(x^*) - z^*$$

where $\partial g$ is the subgradient of $g$. Clearly, these are equivalent to the optimality condition for [1], i.e., $0 \in \nabla f(x^*) + T^\dagger(\partial g(T(x^*)))$.

To show convergence, we introduce a lemma about proximal operators which follows almost directly from their definition. Even though we state the result for $x \in \mathbb{R}^n, g: \mathbb{R}^m \to \mathbb{R}$, and $T(x) = Tx$ where $T \in \mathbb{R}^{m \times n}$ is a given matrix, the proof for $x$ in a Hilbert space and a bounded linear operator $T$ follows from similar arguments.

Lemma 2: Let $g: \mathbb{R}^m \to \mathbb{R}$ be a proper, lower semicontinuous, convex function and let $\text{prox}_{\mu g}: \mathbb{R}^m \to \mathbb{R}^m$ be the corresponding proximal operator. Then, for any $a, b \in \mathbb{R}^m$, we can write
$$\text{prox}_{\mu g}(a) - \text{prox}_{\mu g}(b) = (a - b)$$ (11)

where $D$ is a symmetric matrix satisfying $I \preceq D \preceq I$.

Proof: Let $\tilde{c} := a - b$, $\tilde{p} := \text{prox}_{\mu g}(a) - \text{prox}_{\mu g}(b)$, and $D := (\tilde{p} \tilde{p}^T / (\tilde{p}^T \tilde{c}), \rho \not\equiv 0; 0, \text{otherwise})$. Then, by construction, $D = D^T \succeq 0$ and (11) can be written as $\tilde{p} = D^{-1} \tilde{c}$. Since $\text{prox}_{\mu g}$ is firmly nonexpansive [6], $\tilde{p}^T \tilde{c} \geq 2\tilde{p}^T \tilde{c}$, or, equivalently, $\tilde{c}^T (I - D) \tilde{c} \geq 0$ for every $\tilde{c} \in \mathbb{R}^m$. Positive semi-definiteness of $I - D$ thus follows from $D \succeq 0$ and commutativity of $D$ and $I - D$.

Theorem 3: Let $f$ be a continuously differentiable convex function, and let $g$ be a proper, lower semicontinuous, convex function. Then, for the primal-descent dual-ascent gradient flow dynamics [41]
$$\begin{cases} \dot{x} = -\nabla f(x) + T^\dagger \nabla M_\mu(T(x + \mu y)) \\ \dot{y} = -\mu \nabla M_\mu(T(x + \mu y) - y) \end{cases}$$ (GF)

the set of optimal primal-dual pairs $x^*, y^*$ of (5) is globally asymptotically stable, and $x^*$ is an optimal solution of (1).

Proof: We introduce a change of variables $\tilde{x} := x - x^*$, $\tilde{y} := y - y^*$ and a Lyapunov function candidate,
$$V(\tilde{x}, \tilde{y}) = \frac{1}{2} (\tilde{x}, \tilde{x}) + \frac{1}{2} (\tilde{y}, \tilde{y})$$
where $x^*, y^*$ is an optimal solution to (5) that satisfies (10). The dynamics in the $(\tilde{x}, \tilde{y})$-coordinates are given by,
$$\begin{cases} \dot{\tilde{x}} = -\nabla f(x) - \nabla f(x^*) + (1/\mu) T^\dagger \tilde{m} \\ \dot{\tilde{m}} = -\mu \tilde{y} \end{cases}$$ (12)

where
$$\tilde{m} := \mu (\nabla M_\mu(T(x + \mu y)) - \nabla M_\mu(T(x^* + \mu y^*)).$$ (13a)

Using expression (11) to construct $D$ such that,
$$D(T \tilde{x} + \mu \tilde{y}) = \text{prox}_{\mu g}(T \tilde{x} + \mu \tilde{y}) - \text{prox}_{\mu g}(T \tilde{x}^* + \mu \tilde{y}^*)$$ (13b)
and definition (22) of $\nabla M_\mu$, we can write
$$\tilde{m} = (I - D)(T \tilde{x} + \mu \tilde{y}).$$ (13c)

Thus, the derivative of $V$ along the solutions of (12) is given by
$$\dot{V}(\tilde{x}, \tilde{y}) = -\langle \tilde{x}, \nabla f(x) - \nabla f(x^*) \rangle$$
$$- (1/\mu) \langle T \tilde{x}, (I - D) T \tilde{x} \rangle - \mu \langle \tilde{y}, D \tilde{y} \rangle.$$ (14)

Since $f$ is convex, $\langle \tilde{x}, \nabla f(x) - \nabla f(x^*) \rangle \geq 0$ and Lemma 2 imply $\dot{V} \leq 0$.

When $\nabla f(x) = \nabla f(x^*)$ and when matrices $D$ and $T^\dagger (I - D) T$ have nontrivial kernels, it is possible that $\dot{V} = 0$ for a nonzero $\tilde{y}$ in ker$(D)$ and $\tilde{x}$ such that $T \tilde{x} \in \ker\{(I - D)\}$. From (13c), these
conditions imply \( \hat{\nu} = \mu \hat{y} \) and \( \|x\| \) simplifies to,
\[
\dot{x} = -T^T \hat{y}, \quad \dot{\hat{y}} = 0.
\]
Thus, the only invariant set for dynamics \( \hat{\nu} = \nabla f(x) \), \( \hat{y} \in \ker(T^T) \cap \ker(D) \), and \( \tilde{T} \in \ker(I - D) \). Global asymptotic stability of these points follows from LaSalle’s invariance principle. To complete the proof, we show that any \( x \) and \( y \) that lie in this invariant set also satisfy the optimality conditions \( \|x\| \) of problem \( \|u\| \) with \( z = z^*(x, y) \) and thus that \( x \) solves problem \( \|u\| \).

Since \( x^* \) and \( y^* \) are optimal, \( \|x\| \) implies \( \nabla f(x^*) + T^T y^* = 0 \).

For any \( x \) and \( y \) that lie in the invariant set of \( \|x\| \), we can replace \( \nabla f(x^*) \) with \( \nabla f(x) \) and add \( T^T \hat{y} = 0 \) to obtain
\[
\nabla f(x) + T^T (y^* + \hat{y}) = \nabla f(x) + T^T y = 0
\]
which implies that \( x \) and \( y \) also satisfy \( \|x\| \). Furthermore, \((I - D) T \hat{x} = 0 \) and \( D \hat{y} = 0 \) can be combined to yield \( T \hat{x} = D(T \hat{x} + \mu y) = 0 \), which, by \( \|x\| \), leads to
\[
T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = \text{prox}_{\mu y}(T \hat{x} + \mu y^*) = 0.
\]

By optimality condition \( \|x\| \), \( x^* = z^*(x^*, y^*) = \text{prox}_{\mu y}(T \hat{x} + \mu y^*) \). It thereby follows that \( T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = 0 \) and thus
\[
T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = z^*(x, y) = z
\]
which implies that \( x \) and \( z \) satisfy \( \|x\| \).

Finally, the optimality condition of the minimization problem \( \|x\| \) that defines \( \text{prox}_{\mu y}(x) \) is \( \partial g(z) + \frac{1}{2} (z - v) > 0 \). Setting \( v = T \hat{x} + \mu y \) from the expression \( \|x\| \) that characterizes the \( z^* \)-manifold and \( T \hat{x} = z \) by \( \|x\| \) yields the final optimality condition \( \|x\| \). \( \|x\| \)

B. Convergence rates for strongly convex conditions imply \( \tilde{\nu} = \mu \tilde{y} \), implying \( \|x\| \) with \( z = z^*(x, y) \) and thus that \( x \) solves problem \( \|u\| \).

For any \( x \) and \( y \) that lie in the invariant set of \( \|x\| \), we can replace \( \nabla f(x^*) \) with \( \nabla f(x) \) and add \( T^T \hat{y} = 0 \) to obtain
\[
\nabla f(x) + T^T (y^* + \hat{y}) = \nabla f(x) + T^T y = 0
\]
which implies that \( x \) and \( y \) also satisfy \( \|x\| \). Furthermore, \((I - D) T \hat{x} = 0 \) and \( D \hat{y} = 0 \) can be combined to yield \( T \hat{x} = D(T \hat{x} + \mu y) = 0 \), which, by \( \|x\| \), leads to
\[
T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = \text{prox}_{\mu y}(T \hat{x} + \mu y^*) = 0.
\]

By optimality condition \( \|x\| \), \( x^* = z^*(x^*, y^*) = \text{prox}_{\mu y}(T \hat{x} + \mu y^*) \). It thereby follows that \( T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = 0 \) and thus
\[
T \hat{x} = \text{prox}_{\mu y}(T \hat{x} + \mu y) = z^*(x, y) = z
\]
which implies that \( x \) and \( z \) satisfy \( \|x\| \).

Finally, the optimality condition of the minimization problem \( \|x\| \) that defines \( \text{prox}_{\mu y}(x) \) is \( \partial g(z) + \frac{1}{2} (z - v) > 0 \). Setting \( v = T \hat{x} + \mu y \) from the expression \( \|x\| \) that characterizes the \( z^* \)-manifold and \( T \hat{x} = z \) by \( \|x\| \) yields the final optimality condition \( \|x\| \). \( \|x\| \)

W.
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\[
\begin{align*}
\frac{\partial g(z)}{\partial z} + \frac{1}{2} (z - v) > 0 \quad \text{setting} \quad v = T \hat{x} + \mu y \quad \text{from the expression} \quad \|x\| \quad \text{that characterizes the} \quad z^* \text{-manifold} \quad \text{and} \quad T \hat{x} = z \quad \text{by} \quad \|x\| \quad \text{yields the final optimality condition} \quad \|x\|. \\
\end{align*}
\]

\[
\text{B. Convergence rates for strongly convex } f
\]

We express \( \text{GF} \) or equivalently \( \text{GF1} \) as a linear system \( G \) connected in feedback with nonlinearities that correspond to the gradients of \( f \) and of the Moreau envelope of \( g \); see Fig. \( | \| \) These nonlinearities can be conservatively characterized by IQCs. Exponential stability of \( G \) connected in feedback with any nonlinearity that satisfies these IQCs implies exponential convergence of \( \text{GF} \) to the primal-dual optimal solution of \( \|u\| \). In what follows, we adjust the tools of \( \|u\| \) to our setup and establish exponential convergence by evaluating the feasibility of an LMI. We assume that the function \( f \) is \( 1 \)-strongly convex with an \( 1 \)-Lipschitz continuous gradient. Characterizing additional structural restrictions on \( f \) and \( g \) with IQCs may lead to tighter bounds on the rate of convergence.

As illustrated in Fig. \( | \| \) \( \text{GF} \) can be expressed as a linear system \( G \) connected via feedback to a nonlinear block \( \Delta \).

\[
\begin{align*}
\dot{w} &= Aw + Bu, \quad \xi = Cw, \quad u = \Delta(\xi) \\
A &= \begin{bmatrix} -mI & \\
-L & -\mu I \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 \\
0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \\
0 & T \mu \end{bmatrix} \\
\end{align*}
\]

\[
\text{where} \quad \begin{bmatrix} x^T & y^T \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1^T & \xi_2^T \end{bmatrix}, \quad \text{and} \quad u = \begin{bmatrix} u_1^T & u_2^T \end{bmatrix}^T. \\
\end{align*}
\]

Nonlinearity \( \Delta \) maps the system outputs \( \xi_1 = x \) and \( \xi_2 = T \hat{x} + \mu y \) to the inputs \( u_1 \) and \( u_2 \) via \( u_1 = \Delta_1(\xi_1) = \nabla f(\xi_1) - \mu \xi_1 \) and \( u_2 = \Delta_2(\xi_2) = \nabla g(\xi_2) = \nabla \text{prox}_{\mu y}(\xi_2). \)

When the mapping \( u_1 = \Delta_1(\xi_1) \) is the \( 1 \)-Lipschitz continuous gradient of a convex function, it satisfies the IQC \( \|u\| \) in Lemma \( \|u\| \) where \( \xi_0 \) is some reference point and \( u_0 = \Delta(\xi_0) \). Since \( f \) is strongly convex with parameter \( m \), the mapping \( \Delta_1(\xi_1) \) is the gradient of the convex function \( f(\xi_1) - (m/2)\|\xi_1\|^2 \). Lipschitz continuity of \( \nabla f \) with parameter \( L \) implies Lipschitz continuity of \( \Delta_1(\xi_1) \) with \( L := L_1 - m; \) thus, \( \Delta_1 \) satisfies \( \|u\| \).

End of Proof.
Condition \([17d]\) is satisfied for all \(\omega \in \mathbb{R}\) if there are no \(\omega^2 \geq 0\) for which the left-hand side is nonpositive. When \(\rho = 1\), both the constant term and the coefficient of \(\omega^2\) are strictly positive, which implies that the roots of \([17d]\) as a function of \(\omega^2\) are either not real or lie in the domain \(\omega^2 < 0\), which cannot occur for \(\omega \in \mathbb{R}\). Finally, continuity of \([17d]\) with respect to \(\rho\) implies the existence a positive \(\rho\) that satisfies \([17d]\) for all \(\omega \in \mathbb{R}\).

Remark 3: Each eigenvalue \(\lambda_i\) of a full rank matrix \(TT^T\) is positive and hence to estimate the exponential convergence rate it suffices to check \([17d]\) only for the smallest \(\lambda_i\). A sufficient condition for \([17d]\) to hold for each \(\omega \in \mathbb{R}\) is positivity of the constant term and the coefficient multiplying \(\omega^2\). These can be expressed as quadratic inequalities in \(\rho\) that admit explicit solutions, thereby providing an estimate of the rate of exponential convergence.

Remark 4: A similar convergence rate result can be obtained by applying [19, Theorem 4] to a discrete-time implementation of the primal-descent dual-ascent dynamics that results from a forward Euler discretization of \([GF]\).

C. Distributed implementation

Gradient flow dynamics \([GF]\) are convenient for distributed implementation. If the state vector \(x\) corresponds to the concatenated states of individual agents, \(x_i\), the sparsity pattern of \(T\) and the structure of the gradient map \(\nabla f \colon \mathbb{R}^n \rightarrow \mathbb{R}^n\) dictate the communication topology required to form \(\nabla L_{x_i}\) in \([GF]\). For example, if \(f(x) = \sum f_i(x_i)\) is separable over the agents, then \(\nabla f_i(x_i)\) can be formed locally. If in addition \(TT^T\) is an incidence matrix of an undirected network with the graph Laplacian \(TT^T\), each agent need only share its state \(x_i\) with its neighbors and maintain dual variables \(y_i\) that correspond to its edges. A distributed implementation is also natural when the mapping \(\nabla f\colon \mathbb{R}^n \rightarrow \mathbb{R}^n\) is sparse.

Our approach provides several advantages over existing distributed optimization algorithms. Even for problems \([1]\) with non-differentiable regularizers \(g\), a formulation based on the proximal augmented Lagrangian yields gradient flow dynamics \([GF]\) with a continuous right-hand side. This is in contrast to existing approaches which employ subgradient methods \([14]\) or use discontinuous projected dynamics \([15]-[18]\). Furthermore, when \(T\) is not diagonal, a distributed proximal gradient cannot be implemented because the proximal operator of \(g(Tx)\) may not be separable. Finally, ADMM has been used for distributed implementation in the situations where \(f\) is separable and \(T\) is an incidence matrix. Relative to such a scheme, our method does not require solving an \(x\)-minimization subproblem in each iteration and provides a guaranteed rate of convergence.

Remark 5: A special instance of our framework has strong connections with the existing methods for distributed optimization on graphs; e.g., \([13], [14], [18]\). The networked optimization problem of minimizing \(f(\bar{x}) = \sum f_i(x_i)\) over a single variable \(x\) can be reformulated as \(\sum f_i(x_i)\) + \(g(Tx)\) where the components \(f_i\) of the objective function are distributed over independent agents \(x_i\), \(x\) is the aggregate state, \(TT^T\) is the incidence matrix of a strongly connected and balanced graph, and \(g\) is the indicator function associated with the set \(Tx = 0\). The \(g(Tx)\) term ensures that at feasible points, \(x_i = x_j = \bar{x}\) for all \(i\) and \(j\). It is easy to show that \(\nabla M_{\bar{x}}(\bar{v}) = (1/\mu) \bar{v}\) and that the primal-descent dual-ascent dynamics \([GF]\) are given by,

\[
\dot{x} = -\nabla f(x) - (1/\mu) L x - \dot{y} \\
\dot{y} = L x
\]

where \(\dot{y} := TT^y \) and \(L := TT^T\) is the graph Laplacian. The only difference relative to \([18]\, Equation 11\) is that \(-\dot{y}\) appears instead of \(-L\dot{y}\) in the equation for the dynamics of the primal variable \(x\).

V. EXAMPLES

We solve the problems of edge addition in directed consensus networks and optimal placement to illustrate the effectiveness of the proximal augmented Lagrangian method.

A. Edge addition in directed consensus networks

A consensus network with \(N\) nodes converges to the average of the initial node values \(\bar{v} = (1/N) \sum_{i=1}^{N} v_i(0)\) if and only if it is strongly connected and balanced \([39]\). Unlike for undirected networks \([23]\), the problem of edge addition in directed consensus networks is not known to be convex. The steady-state variance of the deviations from average is given by the square of the \(H_2\) norm of,

\[
\dot{\bar{v}} = -(L_p + L_x) \bar{v} + d, \quad \xi = \left[ Q^{1/2} - R^{1/2} L_x \right] \psi
\]

where \(d\) is a disturbance, \(L_p\) is a weighted directed graph Laplacian of a plant network, \(Q := I - (1/\mu) L^T\) penalizes the deviation from average, and \(R \succ 0\) is the control weight. The objective is to optimize the \(H_2\) norm (from \(d\) to \(\xi\)) by adding a few additional edges, specified by the graph Laplacian \(L_x\) of a controller network.

To ensure convergence of \(\bar{v}\) to the average of the initial node values, we require that the closed-loop graph Laplacian, \(L = L_p + L_x\), is balanced. This condition amounts to the linear constraint, \(1^T L = 0\). We express the directed graph Laplacian of the controller network as,

\[
L_x = \sum_{i,j} \ell_{ij} \delta_{ij} =: \sum_{i} L_i z_i \quad \text{where} \quad z_i,j \geq 0 \quad \text{is the added edge weight that connects node} \ j \ \text{to node} \ i, \ \ell_{ij} := e_i e_j^T - e_i^T e_j, \ e_i \ \text{is the} \ i\text{th basis vector in} \ \mathbb{R}^n, \ \text{and the integer} \ l \ \text{indexes the edges such that} \ z_i = z_i,j \ \text{and} \ L_i = L_i,j. \ \text{For simplicity, we assume that the plant network} \ L_p \ \text{is balanced and connected. Thus, enforcing that} \ L \ \text{is balanced amounts to enforcing the linear constraint} \ 1^T L_x = 1^T \left( \sum \delta_{ij} \right) = \left( E z \right)^T = 0 \ \text{on} \ z, \ \text{where} \ E \ \text{is the incidence matrix} \ [39]\ \text{of the edges that may be added. Any vector of edge weights} \ z \ \text{that satisfies this constraint can be written as} \ z = T x \ \text{where the columns of} \ T \ \text{span the nullspace of the matrix} \ E \ \text{and provide a basis for the space of balanced graphs, i.e., the cycle space} \ [39].

Each feasible controller Laplacian can thus be written as,

\[
L_x = \sum_i L_i [T x]_i = \sum_i \left[ \sum_k (T e_k) x_k \right]_i =: \sum_k \hat{L}_k x_k
\]

where the matrices \(\hat{L}_k\) are given by \(\hat{L}_k = \sum_i L_i \left[ T e_k \right]_i\).

Since the mode corresponding to \(1\) is marginally stable, unobservable, and uncontrollable, we introduce a change of coordinates to the deviations from average \(\phi = V^T \bar{v}\) where \(V^T V = 0\) and discard the average mode \(\bar{v} = 1^T \psi\). The energy of the deviations from average is given by the \(H_2\) norm squared of the reduced system,

\[
f(x) = \left( V^T (Q + L_p^T R L_x) V \right) X, \quad \hat{A} X + X \hat{A}^T + B \hat{B}^T = 0
\]

where \(X\) is the controllability gramian of the reduced system with \(\hat{A} := -V^T (L_p + L_x) V\) and \(\hat{B} := V^T\).

To balance the closed-loop \(H_2\) performance with the number of added edges, we introduce a regularized optimization problem

\[
x_\gamma = \arg\min_x f(x) + \gamma 1^T T x + I_+ (T x)
\]

(19)

Here, the regularization parameter \(\gamma > 0\) specifies the emphasis on sparsity relative to the closed-loop performance \(f\) and \(I_+\) is the indicator function associated with the nonnegative orthant \(\mathbb{R}^n_+\). When the desired level of sparsity for the vector of the added edge weights \(z_\gamma = T x_\gamma\) has been attained, optimal weights for the identified set of edges are obtained by solving,

\[
\min z \quad f(x) + I_{Z_0}(T x) + I_+ (T x)
\]

(20)

where \(Z_0\) is the set of vectors with the same sparsity pattern as \(z_\gamma\) and \(I_{Z_0}\) is the indicator function associated with this set.
1) Implementation: We next provide implementation details for solving (19) and (20). The proof of next lemma is omitted for brevity.

Lemma 5: Let a graph Laplacian of a directed plant network $L_p$ be balanced and connected and let $A, B, L_s, \text{ and } V$ be as defined in (18a)–(18b). The gradient of $f(x)$ defined in (18b) is given by,

$$\nabla f(x) = 2 \text{vec} \left( \left( (RL_sV - VP) XV^T, \hat{L}_k \right) \right)$$

where $X$ and $P$ are the controllability and observability gramians determined by (18b) and $A^T P + P A + V^T (Q + L_s^T RL_s) V = 0$.

The proximal operator associated with the regularization function $g_{\mu}(z) := \gamma \hat{I}^2 x + I_k(z)$ in (19) is given by $\text{prox}_{g_{\mu}}(v) = \max \{ 0, v - \gamma \mu \}$. The associated Moreau envelope is,

$$M_{g_{\mu}}(v) = \sum_i \left( \frac{1}{2\mu} v_i^2 \right) \quad v_i \leq \gamma \mu$$

and its gradient is $\nabla M_{g_{\mu}}(v) = \max \left\{ \frac{1}{\gamma} v, \gamma \right\}$. The proximal operator associated with the regularization function in (20), $g_{\mu}(z) := I_s(z) + I_k(z)$, is a projection onto the intersection of the set $Z_\gamma$ and the nonnegative orthant, $\text{prox}_{g_{\mu}}(v) = P_{E}(v)$ where $E := Z_\gamma \cap \mathbb{R}_{+}^n$. The associated Moreau envelope is the distance to $E$, $M_{g_{\mu}}(v) = \frac{1}{2} \| v - P_{E}(v) \|_2^2$ and its gradient is determined by a vector pointing from $E$ to $v$, $\nabla M_{g_{\mu}}(v) = \frac{1}{2} (v - P_{E}(v))$.

2) Computational experiments: We solve (19) and (20) using Algorithm 1, where L-BFGS is employed in the primal-descent dual-ascent gradient dynamics we propose are suitable for solving difficult combinatorial optimization problems.

To illustrate the utility of our primal-descent dual-ascent approach, we consider an example in which mobile agents aim to minimize their Euclidean distances relative to a set of targets $\{b_i\}$ while staying within a desired distance from their neighbors in a network with the incidence matrix $T$.

$$\text{minimize } x \sum (x_i - b_i)^2 + I_{[1-1,1]}(Tx) \quad \text{(21)}$$

Here, $Tx$ is a vector of inter-agent distances which must be kept within an interval $[-1, 1]$. Applying primal-descent dual-ascent update rules to (21) achieves path planning for first-order agents $\dot{x} = u$

$$u = -\nabla_x L(x; y)$$

The proximal operator is projection onto a box, $\text{prox}_{\mu I_{[1-1,1]}}(z) = \max(\min(z, 1), -1)$, the Moreau envelope is the distance squared to that set, $M_{\mu I_{[1-1,1]}}(z) = \frac{1}{2} \sum S_i^2(z)$, and $\nabla M_{\mu I_{[1-1,1]}}(z) = \frac{1}{2} S_i(z)$. To update its state, each agent $x_i$ needs information from its neighbors in a network with a Laplacian $T^T T$.

Methods based on the subderivative are not applicable because the indicator function is not subdifferentiable. Proximal methods are hindered because the proximal operator of $I_{[1-1,1]}(Tx)$ is difficult to compute due to $T$. Since $f(x) = \sum (x_i - b_i)^2$ is separable, a distributed ADMM implementation can be applied; however, it may require large discrete jumps in agent positions, which could be unsuitable for vehicles. Moreover, when $f$ is not separable a distributed implementation of the $\varepsilon$-minimization step (2a) in ADMM would not be possible.

VI. CONCLUDING REMARKS

For a class of nonsmooth composite optimization problems that arise in structured optimal control, we have introduced continuously differentiable proximal augmented Lagrangian function. This function is obtained by collapsing the associated augmented Lagrangian onto the manifold resulting from explicit minimization over the variable in the nonsmooth part of the objective function. Our approach facilitates development of customized algorithms based on the method of multipliers and the primal-descent dual-ascent method.

MM based on the proximal augmented Lagrangian is applicable to a broader class of problems than proximal gradient methods, and it has more robust convergence guarantees, more rigorous parameter update rules, and better practical performance than ADMM. The primal-descent dual-ascent gradient dynamics we propose are suitable for distributed implementation and have a continuous right-hand side. When the differentiable component of the objective function is (strongly) convex, we establish (exponential) asymptotic convergence. Finally, we illustrate the efficacy of our algorithms using the edge addition in consensus networks and optimal placement problems. Future work will focus on developing second-order updates for the primal and dual variables and on providing an extension to nonconvex regularizers.

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Fig. 3: (a) Total time; (b) number of outer iterations; and (c) average time per outer iteration required to solve (19) with $\gamma = 0.01, 0.1, 0.2$ for a cycle graph with $N = 5$ to 50 nodes and $m = 20$ to 2450 potential added edges using PAL (---), ADMM (- - - - -), and ADMM with the adaptive $\mu$-update heuristic [8] (· · · · · · · · · · · · · · ·). PAL requires fewer outer iterations and thus a smaller total solve time.

Fig. 4: Set of 5 distributed agents tracking targets (black ⋄) whose optimal positions are determined by the solution to [21] (red ×).