ON SYMMETRIC FUSION CATEGORIES IN POSITIVE CHARACTERISTIC

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Abstract. We propose a conjectural extension to positive characteristic case of a well known Deligne’s theorem on the existence of super fiber functors. We prove our conjecture in the special case of semisimple categories with finitely many isomorphism classes of simple objects.

1. Introduction

1.1. Let $k$ be an algebraically closed field of characteristic $p \geq 0$. We recall that a symmetric tensor category $C$ is a category endowed with the functor $\otimes : C \times C \to C$ of tensor product, and with associativity and commutativity isomorphisms and unit object $1$ satisfying suitable axioms, see e.g. [SR] or [EGNO]. In this paper we consider symmetric tensor categories $C$ satisfying the following assumptions:

1) $C$ is an essentially small $k$–linear abelian category such that any morphism space is finite dimensional and each object has finite length;
2) the functor $\otimes$ is $k$–linear and the natural morphism $k \to \text{End}(1)$ to the endomorphism ring of the unit object is an isomorphism;
3) $C$ is rigid (this implies that the functor $\otimes$ is exact in each variable, see [DM, Proposition 1.16]).

Such categories are precisely tensor categories satisfying finiteness assumptions of [D90, 2.12.1]; they were called pre-Tannakian in [CO, 2.1].

Example 1.1. (i) The category Vec of finite dimensional vector spaces is pre-Tannakian. Now let $p \neq 2$. Then the category sVec of finite dimensional super vector spaces over $k$ is pre-Tannakian.

(ii) Let $G$ be an affine group scheme over $k$. Then the category $\text{Rep}_k(G)$ of finite dimensional representations of $G$ over $k$ is pre-Tannakian.

(iii) (see [D02, 0.3]) Let $G$ be an affine super group scheme over $k$ and let $\varepsilon \in G(k)$ be an element of order $\leq 2$ such that its action by conjugation on $G$ coincides with the parity automorphism of $G$. Let $\text{Rep}_k(G, \varepsilon)$ be the full subcategory of super representations of $G$ such that $\varepsilon$ acts by parity automorphism. Then $\text{Rep}_k(G, \varepsilon)$ is pre-Tannakian. A special case of this construction is when $G$ is a finite group and $\varepsilon \in G$ is a central element of order $\leq 2$, see [D02] 0.4 (i)].

(iv) (see [D90, Section 8]) Let $\mathcal{C}$ be a pre-Tannakian category and let $\pi \in \mathcal{C}$ be its fundamental group as defined in [D90] 8.13]. Thus $\pi$ is an affine groups scheme in the category $\mathcal{C}$ and it acts on any object of $\mathcal{C}$ in a canonical way. Let $G$ be an affine group scheme in the category $\mathcal{C}$ and let $\varepsilon : \pi \to G$ be a homomorphism such that the action of $\pi$ on $G$ by conjugations coincides with the canonical action. Let $\text{Rep}_{\mathcal{C}}(G)$ be the category of representations of $G$ in category $\mathcal{C}$ and let $\text{Rep}_{\mathcal{C}}(G, \varepsilon)$ be the full
subcategory of $\text{Rep}_C(G)$ consisting of representations such that the action of $\pi$ via homomorphism $\varepsilon$ coincides with the canonical action of $\pi$. Then both $\text{Rep}_C(G)$ and $\text{Rep}_C(G, \varepsilon)$ are pre-Tannakian. Example (iii) is a special case of this with $C = s\text{Vec}$ since the fundamental group of $s\text{Vec}$ is the finite group of order 2 and its canonical action is given by the parity automorphism.

In this paper a symmetric tensor functor between symmetric tensor categories is a monoidal functor compatible with the commutativity isomorphism. Recall (see [SR]) that a fiber functor for a pre-Tannakian category $C$ is a $k$–linear exact symmetric tensor functor $C \to \text{Vec}$. Similarly, a super fiber functor is a $k$–linear exact symmetric tensor functor $C \to s\text{Vec}$, see [D02].

For example the forgetful functor $\text{Rep}_k(G) \to \text{Vec}$ assigning to a representation its underlying vector space is a fiber functor. Similarly, the forgetful functor $\text{Rep}_k(G, \varepsilon) \to s\text{Vec}$ is a super fiber functor. Conversely, in the theory of Tannakian categories (see [SR, DM, D90]) one shows that any pre-Tannakian category $C$ with a fiber functor is tensor equivalent to $\text{Rep}_k(G)$ endowed with the forgetful functor. Similarly and more generally, any pre-Tannakian category $C$ with a super fiber functor is tensor equivalent to $\text{Rep}_k(G, \varepsilon)$ endowed with the forgetful functor, see [D90, 8.19]. These results reduce many questions about pre-Tannakian categories to the theory of affine group schemes and super schemes.

Furthermore, Deligne showed that for $p = 0$ many pre-Tannakian categories admit a super fiber functor. Namely, we say that $C$ is of subexponential growth if for any object $X \in C$ the length of the objects $X \otimes^n$ is bounded by the function $a_X^n$ for a suitable $a_X \in \mathbb{R}$, see [D02], [EGNO, 9.11].

**Theorem 1.2** ([D02] Théorème 0.6). Assume that $p = 0$. A pre-Tannakian category $C$ admits a super fiber functor if and only if it is of subexponential growth.

1.2. The main goal of this paper is to propose a conjectural extension of Theorem 1.2 to the case $p \neq 0$. The counterexamples constructed by Gelfand and Kazhdan in [GK] (see also [A, GM]) show that a direct counterpart of Theorem 1.2 fails for $p > 0$. For instance for $p = 5$ there exists a semisimple pre-Tannakian category $C$, called Yang-Lee category or Fibonacci category, with two isomorphism classes $\mathbf{1}$ and $X$ of simple objects and such that $X \otimes X = \mathbf{1} \oplus X$, see [GK, GM] and Example 3.2 below. It is clear that this category has no super fiber functors since for any monoidal functor $F : C \to s\text{Vec}$ the dimension $d$ of vector space $F(X)$ would be a root of the equation $d^2 = 1 + d$ which is impossible.

Thus in Section 3 for each prime $p$ we introduce the universal Verlinde category $\text{Ver}_p$ which is a semisimple pre-Tannakian category with $p - 1$ isomorphism classes of simple objects (this category is equivalent to one of the categories constructed in [GK, GM], see Section 3.2).

**Conjecture 1.3.** Assume that $p > 0$. A pre-Tannakian category $C$ of subexponential growth admits a (unique up to isomorphism) $k$–linear exact symmetric tensor functor $C \to \text{Ver}_p$.

**Remark 1.4.** We refer the reader to [D07] for examples of pre-Tannakian categories which are not of subexponential growth. We note that no such examples are currently known in the case $p > 0$.

In view of Example 3.2 Conjecture 1.3 states that in the case $p = 2$ any pre-Tannakian category of subexponential growth admits a fiber functor and in the case
$p = 3$ any pre-Tannakian category of subexponential growth admits a super fiber functor. Thus Conjecture 1.3 predicts that for $p = 2$ any pre-Tannakian category of subexponential growth is of the form $\text{Rep}_k(G)$ for a suitable affine group scheme $G$, and for $p = 3$ any pre-Tannakian category of subexponential growth is of the form $\text{Rep}_k(G, \varepsilon)$ for a suitable affine super group scheme $G$.

1.3. We recall (see [ENO]) that a fusion category is a $k$-linear semisimple rigid monoidal category with finite dimensional Hom-spaces, finitely many isomorphism classes of simple objects, and simple unit object. In particular, a symmetric fusion category (that is a fusion category equipped with a symmetric braiding) is the same as semisimple pre-Tannakian category with finitely many isomorphism classes of simple objects. It is not difficult to see that a fusion category is of subexponential growth, see [D02, Lemme 4.8]. Thus the following statement which is the main result of this paper is a special case of Conjecture 1.3:

**Theorem 1.5.** Let $p > 0$. A symmetric fusion category $\mathcal{C}$ admits a $k$-linear symmetric tensor functor $\mathcal{C} \to \text{Ver}_p$.

We note that Theorem 1.5 holds true also for $p = 0$ if we set $\text{Ver}_0 = \text{sVec}$ by Theorem 1.2. Using [D90, Théorème 8.17] we get the following

**Corollary 1.6.** A symmetric fusion category is of the form $\text{Rep}_{\text{Ver}_p}(G, \varepsilon)$ where $G$ is a finite group scheme in the category $\text{Ver}_p$.

A well known Nagata’s theorem ([DG, IV, 3.6]) gives a classification of finite group schemes $G$ such that $\text{Rep}_k(G)$ is semisimple; thus Corollary 1.6 yields a classification of symmetric fusion categories in the case $p = 2$. Namely, any such category is an equivariantization (see e.g. [EGNO, 8.4]) of a pointed category associated with a 2-group by the action of a group of odd order. It is natural to ask

**Question 1.7.** What is classification of finite group schemes $G$ in $\text{Ver}_p$ such that $\text{Rep}_{\text{Ver}_p}(G)$ or $\text{Rep}_{\text{Ver}_p}(G, \varepsilon)$ is semisimple?

1.4. The main ingredient in the proof of Theorem 1.5 is the notion of Frobenius functor which is an abstract version of the pullback functor under the Frobenius morphism from a group scheme to itself. The definition of this functor is given in Section 4 and it works only in the case of semisimple pre-Tannakian categories. We expect that a similar definition can be given more generally and hope to address this issue in future publications.

Another essential tool in the proof of Theorem 1.5 is the theory of non-degenerate fusion categories developed in [ENO] Section 9. A crucial property of such categories is that they can be lifted to characteristic zero, see loc. cit.

1.5. **Acknowledgements.** It is my great pleasure to thank Pierre Deligne, Pavel Etingof, Michael Finkelberg, Shlomo Gelaki, Alexander Kleshchev, Dmitri Nikshych, Julia Pevtsova, Alexander Polishchuk, and Vadim Vologodsky for very useful conversations.

2. Preliminaries

For a tensor category $\mathcal{C}$ we will denote by $1$ its unit object. For a braided (in particular, symmetric) tensor category $\mathcal{C}$ we will denote by $c_{X,Y}$ the braiding
morphism \(X \otimes Y \to Y \otimes X\). For an abelian category \(C\) we will denote by \(\cal O(C)\) the set of isomorphism classes of simple objects of \(C\).

2.1. Fusion categories. The definition of fusion category was given in introduction. A fusion subcategory of a fusion category \(C\) is a full tensor subcategory \(C' \subset C\) such that if \(X \in C\) is isomorphic to a direct summand of an object of \(C'\) then \(X \in C'\), see [DGNO] 2.1. For a collection \(S\) of objects of \(C\) there is a smallest fusion subcategory containing \(S\); it is called a fusion subcategory generated by \(S\).

A tensor functor \(F\) between fusion categories \(C\) and \(D\) is called injective if it is fully faithful; such a functor is called surjective if any object of \(D\) is isomorphic to a direct summand of \(F(X), X \in C\), see [ENO] 5.7. Thus a tensor functor is an equivalence if and only if it is both injective and surjective.

For a tensor functor \(F : C \to D\) its image \(F(C)\) is the fusion subcategory of \(D\) generated by objects \(F(X), X \in C\).

Let \(G\) be a finite group. A \(G\)-grading on a fusion category \(C\) is a function \(\phi : \cal O(C) \to G\) such that for \(X, Y \in \cal O(C)\) the tensor product \(X \otimes Y\) contains only simple summands \(Z\) with \(\phi(Z) = \phi(X)\phi(Y)\), see e.g. [DGNO] 2.3; such a grading is faithful if the function \(\phi\) is surjective. It is clear that direct sums of simple objects \(X\) with \(\phi(X) = 1 \in G\) form a fusion subcategory of \(C\); this is neutral component of the grading.

2.2. External tensor product. Let \(C\) and \(D\) be two \(k\)-linear tensor categories. We define category \(C \times_k D\) as follows: objects are pairs \((X, Y)\) where \(X \in C\) and \(Y \in D\) and morphisms are \(\text{Hom}(X_1 Y_1, X_2 Y_2) = \text{Hom}_c(X_1, X_2) \otimes_k \text{Hom}(Y_1, Y_2)\). The category \(C \times_k D\) has an obvious structure of \(k\)-linear tensor category with tensor product given by \((X_1 Y_1, X_2 Y_2) := (X_1 \otimes X_2, Y_1 \otimes Y_2)\). If \(C\) and \(D\) are symmetric tensor categories the so is \(C \times_k D\).

We define external tensor product \(C \boxtimes D\) to be the Karoubian envelope (see e.g. [D77] 1.8) of \(C \times_k D\); the image of pair \((X, Y) \in C \times_k D\) in \(C \boxtimes D\) will be denoted \(X \boxtimes Y\). We have an obvious tensor functors \(C \to C \boxtimes D, X \mapsto X \boxtimes 1\) and \(D \to C \boxtimes D, Y \mapsto 1 \boxtimes Y\). If \(C, D, A\) are symmetric \(k\)-linear Karoubian tensor categories we have the following universal property of the category \(C \boxtimes D\):

(a) the functor assigning to \(F : C \boxtimes D \to A\) its composition with functors \(C \to C \boxtimes D\) and \(D \to C \boxtimes D\) is an equivalence of categories:

\[\{ \text{\(k\)-linear symmetric tensor functors } C \boxtimes D \to A\} \to \{ \text{pairs of } k\text{-linear symmetric tensor functors } C \to A\text{ and } D \to A\}\]

In general the category \(C \boxtimes D\) is not abelian even if \(C\) and \(D\) are. However if \(C\) and \(D\) are abelian and one of these categories is semisimple then \(C \boxtimes D\) is also abelian (say if \(C\) is semisimple then \(C \boxtimes D\) is equivalent to direct sum of copies of \(D\) indexed by the isomorphism classes of simple objects in \(C\) as an additive category).

**Example 2.1.** Let \(C\) be a semisimple pre-Tannakian category and let \(G\) be a finite group. Let \(C_G\) be the equivariantization of \(C\) with respect to the trivial action of \(G\) on \(C\), see [DGNO] 4.1.3. In other words the objects of \(C_G\) are objects of \(C\) equipped with \(G\)-action; the morphisms are morphisms in \(C\) commuting with \(G\)-action, and the tensor product is obvious. We have the following symmetric tensor functors:

\[C \to C_G, X \mapsto (X, \text{trivial action of } G),\]

\[\text{Rep}_k(G) \to C_G, V \mapsto (V \otimes 1, G \text{ acts on } \text{first factor})\].
Thus by the universal property (a) we have a symmetric tensor functor $\mathbb{C}\boxtimes \text{Rep}_K(G) \to \mathcal{C}_G$. We leave it to the reader to check that this functor is an equivalence.

For the future use we will record the following result:

**Lemma 2.2.** Let $\mathcal{C}$ be a symmetric fusion category and let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{C}$ be two fusion subcategories. Assume that the only simple object $X \in \mathcal{C}$ satisfying $X \in \mathcal{A}_1$ and $X \in \mathcal{A}_2$ is $X = 1$. Then fusion subcategory $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ of $\mathcal{C}$ generated by (the objects of) $\mathcal{A}_1$ and $\mathcal{A}_2$ is equivalent to $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ as a symmetric tensor category.

**Proof.** We have obvious symmetric tensor functors $\mathcal{A}_1, \mathcal{A}_2 \to \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$; thus by the universal property (a) we have a symmetric tensor functor $\mathcal{A}_1 \boxtimes \mathcal{A}_2 \to \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ sending $X \boxtimes Y$ to $X \otimes Y$. This functor is clearly surjective, so we just need to show that it is injective. Any simple object of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$ is of the form $X \boxtimes Y$ where $X \in \mathcal{O}(\mathcal{A}_1)$ and $Y \in \mathcal{O}(\mathcal{A}_2)$. For two such objects $X_1 \boxtimes Y_1$ and $X_2 \boxtimes Y_2$ we have
\[
\text{Hom}(X_1 \otimes Y_1, X_2 \otimes Y_2) = \text{Hom}(X_1 \otimes X_2^*, Y_1^* \otimes Y_2) = \begin{cases} k & \text{if } X_1 = X_2 \text{ and } Y_1 = Y_2 \\ 0 & \text{otherwise} \end{cases}
\]
since $X_1 \otimes X_2^* \in \mathcal{A}_1$ and $Y_1^* \otimes Y_2 \in \mathcal{A}_2$ which proves the injectivity. \qed

**Remark 2.3.** Lemma 2.2 extends trivially to the case when $\mathcal{C}$ is a semisimple pre-Tannakian category. On the other hand the condition that $\mathcal{C}$ is symmetric can not be dropped. Namely if $\mathcal{C}$ is braided we still have an equivalence $\mathcal{A}_1 \boxtimes \mathcal{A}_2 \simeq \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ of monoidal categories but it is not necessarily braided. If $\mathcal{C}$ is not braided then even a functor $\mathcal{A}_1 \boxtimes \mathcal{A}_2 \to \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ can not be defined in general.

2.3. **Frobenius-Perron dimension.** For an abelian tensor category $\mathcal{C}$ with exact tensor product we will denote by $K(\mathcal{C})$ its Grothendieck ring, see e.g. [EGNO, 4.5]. Class of an object $X \in \mathcal{C}$ in $K(\mathcal{C})$ will be denoted $[X]$. We recall that for a fusion category $\mathcal{C}$ there is a unique ring homomorphism $\text{FPdim} : K(\mathcal{C}) \to \mathbb{R}$ called **Frobenius-Perron dimension** such that $\text{FPdim}(X) := \text{FPdim}([X]) > 0$ for any $0 \neq X \in \mathcal{C}$, see [EGNO, 4.5]. This definition implies that $\text{FPdim}(X) \geq 1$ for any $X \neq 0$, see [EGNO, Proposition 3.3.4].

Recall that **Frobenius-Perron dimension** $\text{FPdim}(\mathcal{C})$ of a fusion category $\mathcal{C}$ is defined as
\[
\text{FPdim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X)^2.
\]

It is easy to see that for any $M \in \mathbb{R}$ the set
\[
\{ x \in \mathbb{R} | x < M \text{ and there exists a fusion category } \mathcal{C} \text{ with } x = \text{FPdim}(\mathcal{C}) \}
\]
is finite. In particular any nonempty set of fusion categories has an element $\mathcal{C}$ with minimal possible $\text{FPdim}(\mathcal{C})$.

We have the following result:

**Lemma 2.4 (EGNO, Propositions 6.3.3 and 6.3.4).** Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between fusion categories.

(i) If $F$ is injective then $\text{FPdim}(\mathcal{C}) \leq \text{FPdim}(\mathcal{D})$ and we have equality if and only if $F$ is an equivalence;

(ii) If $F$ is surjective then $\text{FPdim}(\mathcal{C}) \geq \text{FPdim}(\mathcal{D})$ and we have equality if and only if $F$ is an equivalence.

Since the functor $F : \mathcal{C} \to F(\mathcal{C})$ is surjective we have the following
Corollary 2.5. For a tensor functor \( F : \mathcal{C} \to \mathcal{D} \) between fusion categories we have \( \text{FPdim}(F(C)) \leq \text{FPdim}(C) \) and we have equality if and only if \( F \) is injective.

2.4. Non-degenerate fusion categories. Let \( \mathcal{C} \) be a \( k \)-linear rigid tensor category such that \( k \to \text{End}(1) \) is an isomorphism. We recall that a pivotal structure on \( \mathcal{C} \) is a functorial tensor isomorphism \( X \simeq X^{**} \) for any \( X \in \mathcal{C} \), see [BW] or [EGNO] 4.7]. Such a structure allows to define the left and right traces of any morphism \( a : X \to X \), see loc. cit. A pivotal structure is called spherical if for any morphism \( a : X \to X \) its left trace equals right trace, so the notion of trace is unambiguous. In particular we can defined dimension \( \dim(X) \in k \) of any object \( X \) as a trace of the identity morphism. If \( \mathcal{C} \) is abelian the dimension determines a ring homomorphism \( \dim : \mathcal{K}(\mathcal{C}) \to k \) sending \([X]\) to \( \dim(X) \). In particular this discussion applies in the case when \( \mathcal{C} \) is symmetric, since for such categories we have a canonical choice of spherical structure given by

\[
X \xrightarrow{id_X \otimes \text{coev}_X^{**}} X \otimes X^{**} \xrightarrow{\text{ev}_X \otimes id_{X^{**}}} X^{**},
\]

see e.g. [EGNO] Section 9.9]. This is the only spherical structure that is used in this paper. Recall that (see e.g. [EGNO] Proposition 4.8.4):

(i) if \( \mathcal{C} \) is semisimple and \( X \in \mathcal{O}(\mathcal{C}) \) then \( \dim(X) \neq 0 \).

(ii) If \( \mathcal{C} \) is a fusion category equipped with a spherical structure. For such category \( \mathcal{C} \) one defines its global dimension \( \dim(\mathcal{C}) \in k \) via

\[
\dim(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \dim(X)^2.
\]

Definition 2.6 ([ENO] Definition 9.1). A spherical fusion category \( \mathcal{C} \) is called non-degenerate if \( \dim(\mathcal{C}) \neq 0 \).

Remark 2.7. (i) In fact \( \dim(\mathcal{C}) \) is independent of the choice of spherical structure. Moreover, \( \dim(\mathcal{C}) \) and the notion of non-degeneracy can be defined for a fusion category without a reference to the spherical structures, see [ENO] Definition 2.2].

(ii) It is known that for \( p = 0 \) any fusion category is non-degenerate, see [ENO] Theorem 2.3]. Thus this notion is of interest only for \( p > 0 \).

A crucial property of non-degenerate fusion categories is that they can be lifted to characteristic zero, see [ENO] Section 9]. In particular we have the following

Proposition 2.8 ([EG] Theorem 5.4). Let \( \mathcal{C} \) be a non-degenerate symmetric fusion category.

(i) If \( p = 2 \) then there exists a fiber functor \( \mathcal{C} \to \text{Vec} \);

(ii) If \( p > 2 \) then there exists a super fiber functor \( \mathcal{C} \to s\text{Vec} \).

Proof. Let \( W(k) \) be the ring of Witt vectors of \( k \) and let \( \mathbb{F} \) be its field of quotients.

Thus we have ring homomorphisms \( W(k) \to k \) and \( W(k) \to \mathbb{F} \).

By [ENO] Corollary 9.4] the category \( \mathcal{C} \) has a lifting \( \mathcal{C}_{W(k)} \) to characteristic zero. Thus \( \mathcal{C}_{W(k)} \) is a symmetric tensor category over \( W(k) \); its objects are the same as objects of \( \mathcal{C} \) and its morphisms are free \( W(k) \)-modules and we have that \( \mathcal{C}_{W(k)} \otimes_{W(k)} k \simeq \mathcal{C} \) and \( \mathcal{C}_{W(k)} \otimes_{W(k)} \mathbb{F} \) is a symmetric fusion category over \( \mathbb{F} \). It is easy to see that \( \dim(\mathcal{C}_{W(k)} \otimes_{W(k)} \mathbb{F}) \in W(k) \subset \mathbb{F} \) and its image in \( k \) equals \( \dim(\mathcal{C}) \).

Thus by [D02] Corollaire 0.8] we have an equivalence \( \mathcal{C}_{W(k)} \otimes_{W(k)} \mathbb{F} \simeq \text{Rep}_\mathbb{F}(G, \varepsilon) \) for a suitable finite group \( G \) and central element \( \varepsilon \in G \) of order \( \leq 2 \), see Example [(ii)]. Since \( \dim(\text{Rep}_\mathbb{F}(G, \varepsilon)) = |G| \) the non-degeneracy of \( \mathcal{C} \) forces that \( |G| \) is not
Proof. Let \( C \) be a spherical fusion category such that the ring \( K(C) \otimes k \) is semisimple. Then \( C \) is non-degenerate.

**Proposition 2.9.** Let \( C \) be a spherical fusion category such that the ring \( K(C) \otimes k \) is semisimple. Then \( C \) is non-degenerate.

**Proof.** Let \( \text{Tr}(x) \in k \) be the trace of the operator of left multiplication by \( x \in K(C) \otimes k \). Since \( K(C) \otimes k \) is semisimple the trace form \( x, y \mapsto \text{Tr}(xy) \) on \( K(C) \otimes k \) is non-degenerate. For \( X, Y \in C \) we have a congruence modulo \( p \):

\[
\text{Tr}([X][Y]) \equiv \sum_{Z \in \mathcal{O}(C)} \dim \text{Hom}(X \otimes Y \otimes Z, Z) = \dim \text{Hom}(X \otimes Y, \oplus_{Z \in \mathcal{O}(C)} Z^* \otimes Z).
\]

Now consider an element \( R = [\oplus_{Z \in \mathcal{O}(C)} Z^* \otimes Z] \in K(C) \otimes k \). In the basis \([X], X \in \mathcal{O}(C)\) the operator of left multiplication by \( R \) has matrix entries

\[
\dim \text{Hom}(\oplus_{Z \in \mathcal{O}(C)} Z^* \otimes Z, X, Y) = \dim \text{Hom}(X \otimes Y^*, \oplus_{Z \in \mathcal{O}(C)} Z^* \otimes Z).
\]

Thus the matrix of this operator differs from the matrix of the trace form only by permutations of columns. Thus under the assumptions of the Proposition this matrix is non-degenerate and the element \( R \in K(C) \otimes k \) is invertible. Thus its image under the homomorphism \( \dim : K(C) \otimes k \to k \) is nonzero. The result follows since

\[
\dim(R) = \dim(\oplus_{Z \in \mathcal{O}(C)} Z^* \otimes Z) = \sum_{Z \in \mathcal{O}(C)} \dim(Z)^2 = \dim(C).
\]

\( \square \)

**Remark 2.10.** It seems reasonable to expect that conversely for a non-degenerate fusion category \( C \) the ring \( K(C) \otimes k \) is semisimple.

The following result is well known. However we did not find a reference, so a proof is included for reader’s convenience.

**Lemma 2.11.** Let \( C \) be a faithfully \( G \)-graded spherical fusion category with neutral component \( C_1 \). Then

\[
\dim(C) = |G| \dim(C_1).
\]

**Proof.** Let \( \mathcal{O}_g(C) \subset \mathcal{O}(C) \) consists of \( X \) with \( \phi(X) = g \in G \). Let

\[
D_g = \sum_{X \in \mathcal{O}_g(C)} \dim(X)[X] \in K(C) \otimes k.
\]

Note that by \( 2.3 \) (a) we have \( D_g \neq 0 \) for any \( g \in G \). We claim that for \( X \in \mathcal{O}_g(C) \) we have

\[
[X]D_h = \dim(X)D_{gh} \quad \text{and} \quad D_h[X] = \dim(X)D_{hg}.
\]

Here is the proof of the first formula (and the second one is similar):

\[
[X]D_h = \sum_{Y \in \mathcal{O}_h(C)} \dim(Y)[X \otimes Y] = \sum_{Y \in \mathcal{O}_h(C), Z \in \mathcal{O}_{gh}(C)} \dim(Y) \dim \text{Hom}(X \otimes Y, Z)[Z] =
\]

\[
\sum_{Z \in \mathcal{O}_{gh}(C)} \dim(Z) \dim \text{Hom}(Y, Z) = \dim(Y) \sum_{Z \in \mathcal{O}_{gh}(C)} \dim(Z)^2 = \dim(X) \dim(C).
\]

\( \square \)
Thus Proposition 2.13 combined with Lemma 2.2 imply that \( \overline{X} \) is negligible and \( \overline{\text{Hom}}(X, X^\ast \otimes Z)[Z] = \sum_{Z \in O_{gh}(C)} \dim(X^\ast) \dim(Z)[Z] = \sum_{Z \in O_{gh}(C)} \dim(X^\ast) \dim(Z)[Z] = \dim(X)D_{gh}. \)

It follows that \( D_gD_h = \dim(D_g)D_{gh} = \dim(D_h)D_{gh}, \) so \( \dim(D_g) = \dim(D_h) \) for all \( g, h \in G. \) The result follows since
\[
\dim(\mathcal{C}) = \sum_{g \in G} \dim(D_g) \quad \text{and} \quad \dim(\mathcal{C}_1) = \dim(D_1).
\]

\[\square\]

**Remark 2.12.** (i) Argument in the proof of Lemma 2.11 is fairly standard, see e.g. [EGNO, Theorem 3.5.2].

(ii) Using construction of pivotalization (see [EGNO, Definition 7.21.9]) one can extend Lemma 2.11 to fusion categories which are not necessarily spherical.

(iii) In the special case \( p = 0 \) Lemma 2.11 is [DGNO, Corollary 4.28].

### 2.5. Negligible morphisms.

Let \( C \) be as in the beginning of Section 2.4 and assume that \( C \) is equipped with a spherical structure. We recall that a morphism \( f : X \to Y \) in \( C \) is called negligible if for any morphism \( u : Y \to X \) the trace of the composition \( fu \) equals zero, see e.g. [AAITV, BW, DD7]. For \( X, Y \in C \) let \( N(X, Y) \subset \text{Hom}(X, Y) \) denote the subspace of negligible morphisms. It is well known that negligible morphisms form a tensor ideal in \( C \). This means that a composition \( fg \) and tensor product \( f \otimes g \) is negligible whenever at least one of \( f \) and \( g \) is negligible. Thus one defines a new category \( \overline{C} \) called quotient of \( C \) by negligible morphisms as follows: objects of \( \overline{C} \) are the same as objects of \( C \) and \( \text{Hom}_{\overline{C}}(X, Y) = \text{Hom}_C(X, Y)/N(X, Y) \) and the composition of morphisms in \( \overline{C} \) is induced by composition in \( C \). We will denote by \( \overline{\text{Hom}} \) an object of \( \overline{C} \) corresponding to \( X \in C \).

The tensor product in \( C \) descends to a tensor product in \( \overline{C} \); thus \( \overline{C} \) is a tensor category endowed with a \( k \)-linear quotient tensor functor \( C \to \overline{C} \) sending \( X \in C \) to \( \overline{X} \in \overline{C} \). The category \( \overline{C} \) is equipped with spherical structure and the quotient functor is compatible with the spherical structures. In addition the category \( \overline{C} \) is braided or symmetric if \( C \) is.

We will use the following result:

**Proposition 2.13** ([BW] Proposition 3.8, see also [AAITV, Theorem 2.7 and Exercise 8.18.9]). Assume that \( C \) is abelian and that all morphism spaces in \( C \) are finite dimensional. Then \( \overline{C} \) is semisimple and its simple objects are precisely \( \overline{X} \) where \( X \) is an indecomposable object of \( C \) with \( \dim(X) \neq 0 \).

Note that if \( X \) is an indecomposable object with \( \dim(X) = 0 \) then \( \text{id} \in \text{Hom}(X, X) \) is negligible and \( \overline{X} = 0 \).

**Example 2.14.** In the setup of Example 2.1 consider the quotient \( \overline{C}_G \) of \( C_G \) by the negligible morphisms. The indecomposable objects of \( C_G = C \otimes \text{Rep}_k(G) \) are of the form \( X \boxtimes V \) where \( X \in O(C) \) and \( V \) is an indecomposable object of \( \text{Rep}_k(G) \). We have \( \dim(X \boxtimes V) = \dim(X) \dim(V) = 0 \) if and only if \( \dim(V) = 0 \), see 2.4(a).

Thus Proposition 2.13 combined with Lemma 2.2 imply that \( \overline{C}_G = C \boxtimes \overline{\text{Rep}_k(G)} \) where \( \overline{\text{Rep}_k(G)} \) is the quotient of \( \text{Rep}_k(G) \) by the negligible morphisms.
3. Frobenius functor

3.1. Representations of the cyclic group. Assume that \( p > 0 \). Let \( C_p \) be the cyclic group of order \( p \) with generator \( \sigma \). Let \( k[C_p] \) be the group algebra of \( C_p \); clearly \( k[C_p] = k[\sigma]/(\sigma^p - 1) = k[\sigma]/(\sigma - 1)^p \). For any \( s \in \mathbb{Z} \) satisfying \( 1 \leq s \leq p \) let \( L_s \) be \( C_p \)-module \( k[\sigma]/(1 - \sigma)^s \). Clearly \( \dim(L_s) = s \). By the Jordan normal form theory we have:

(a) \( L_s \) exhaust all the isomorphism classes of indecomposable objects in the category \( \text{Rep}_k(C_p) \). Moreover, \( L_s^s \simeq L_s \).

The decompositions of the tensor products of the modules \( L_s \) were described by Green in [4]. We record here some of his results:

(b) \( L_1 = 1 \) is the unit object of \( \text{Rep}_k(C_p) \):

\[
L_2 \otimes L_s = \begin{cases} 
L_2 & \text{if } s = 1 \\
L_{s-1} \oplus L_{s+1} & \text{if } s = 2, \ldots, p - 1 \\
L_p \oplus L_p & \text{if } s = p 
\end{cases}
\]

(1)

\[
L_2 \otimes L_s = L_{p-s} \oplus (s-1)L_p
\]

(2)

3.2. Universal Verlinde category.

Definition 3.1. We define the universal Verlinde category \( \text{Ver}_p \) to be the quotient of the category \( \text{Rep}_k(C_p) \) by the negligible morphisms.

By the results of section 2.5, \( \text{Ver}_p \) is a symmetric fusion category. The simple objects of \( \text{Ver}_p \) are precisely \( L_s, s = 1, \ldots, p - 1 \) (note that \( L_p = 0 \)). Obviously \( L_1 \) is the unit object of \( \text{Ver}_p \). The results of section 3.1 imply the following relations in the Grothendieck ring \( K(\text{Ver}_p) \):

\[
[L_2][L_s] = [L_{s-1}] + [L_{s+1}], s = 1, \ldots, p - 1,
\]

(3)

where we define \( [L_0] = [L_p] = 0 \). Using relation (3) one determines the multiplication in \( K(\text{Ver}_p) \):

\[
[L_r][L_s] = \sum_{i=1}^c |L_{r-s+2i-1}|, \text{ where } c = \min(r, s, p-r, p-s)
\]

(4)

We record the following consequence of (4):

\[
\hat{L}_3 \text{ is a summand of } L_s \otimes L_s^* \text{ if } s \neq 1, p - 1.
\]

Note that the Grothendieck ring of \( \text{Ver}_p \) as a ring with basis coincides with so called Verlinde ring associated with the quantum group \( SL_2 \) at \( 2p \)-th root of unity or affine Lie algebra \( \hat{sl}_2 \) at the level \( p - 2 \), see e.g. [EGNO] 4.10.6. This is our motivation for the choice of the name.

Example 3.2. (i) The category \( \text{Ver}_2 \) has just one simple object \( \hat{L}_1 = 1 \) up to isomorphism; thus we have \( \text{Ver}_2 \simeq \text{Vec} \).

(ii) The category \( \text{Ver}_3 \) has two simple objects \( \hat{L}_1 = 1 \) and \( \hat{L}_2 \) up to isomorphism; by (2) we have \( L_2 \otimes L_2 \simeq L_1 \); since \( \dim(L_2) = -1 \) we get that \( \text{Ver}_2 \simeq s\text{Vec} \).

(iii) The category \( \text{Ver}_7 \) has four simple objects \( \hat{L}_1 = 1, \hat{L}_2, \hat{L}_3, \hat{L}_4 \) up to isomorphism; one determines from (4) that \( \hat{L}_4 \otimes \hat{L}_4 \simeq \hat{L}_1 = 1 \) and \( \hat{L}_3 \otimes \hat{L}_3 \simeq \hat{L}_1 \oplus \hat{L}_3 \). Thus the fusion subcategory \( (\hat{L}_1, \hat{L}_3) \) is an example of Yang-Lee category from Section 1.3.
It follows from (2) that we have
\[ \bar{L}_{p-1} \otimes \bar{L}_s = \bar{L}_{p-s} \]
In particular we have \( \bar{L}_{p-1} \otimes \bar{L}_{p-1} \cong \bar{L}_1 = 1 \). Thus for \( p > 2 \) the direct sums of \( \bar{L}_1 \) and \( \bar{L}_{p-1} \) form a fusion subcategory of \( \text{Ver}_p \); it is easy to see that this subcategory is tensor equivalent to \( \text{sVec} \) and we will refer to this subcategory as \( \text{sVec} \subset \text{Ver}_p \). On the other hand it follows from (4) that for \( p > 3 \) the direct sums of \( \bar{L}_1, \bar{L}_3, \ldots, \bar{L}_{p-2} \) also for a fusion subcategory \( \text{Ver}_p^+ \subset \text{Ver}_p \).

**Proposition 3.3.** Assume \( p > 3 \).

(i) The subcategory \( \text{Ver}_p^+ \subset \text{Ver}_p \) is generated by \( \bar{L}_3 \);

(ii) The category \( \text{Ver}_p \) has precisely four fusion subcategories: \( \text{Vec}, \text{sVec}, \text{Ver}_p^+, \text{Ver}_p \);

(iii) We have an equivalence of symmetric fusion categories \( \text{Ver}_p \cong \text{Ver}_p^+ \boxtimes \text{sVec} \).

**Proof.** (i) is immediate from (4). If a fusion subcategory of \( \text{Ver}_p \) contains \( \bar{L}_s \) with \( s \neq 1, p - 1 \) then by (5) and (i) it contains \( \text{Ver}_p^+ \). This implies (ii). Finally (iii) is immediate from Lemma 2.2. \( \square \)

3.3. Let \( \mathcal{C} \) be a semisimple pre-Tannakian category. Let \( \mathcal{C}^{(1)} \) be the Frobenius twist of \( \mathcal{C} \), that is \( \mathcal{C}^{(1)} = \mathcal{C} \) as an additive symmetric tensor category with \( k \)-linear structure changed as follows: for \( \lambda \in k \) and a \( \mathcal{C} \)-morphism \( f \) we set \( \lambda \cdot f := \lambda^p f \).

Thus the Grothendieck ring \( K(\mathcal{C}^{(1)}) \) is canonically isomorphic to the Grothendieck ring of \( \mathcal{C} \). Since \( \lambda \mapsto \lambda^p \) is an automorphism of \( k \), \( \mathcal{C}^{(1)} \) is a Galois conjugate of \( \mathcal{C} \). In particular we have \( \text{Ver}_p^{(1)} \cong \text{Ver}_p \) since \( \text{Ver}_p \) is defined over the prime subfield of \( k \).

Consider a functor \( F_0 : \mathcal{C} \to \mathcal{C} \) sending an object \( X \) to \( X^{\otimes p} \) and a morphism \( f : X \to Y \) to \( f \otimes \ldots \otimes f \). This functor is not additive but it has an obvious \( p \) factors structure of symmetric tensor functor. Moreover, the commutativity isomorphisms determine an action of the symmetric group \( S_p \) on an object \( X^{\otimes p} \) (see e.g. [EGNO, 9.9]), so we can upgrade the functor \( F_0 \) to the functor taking values in the category of equivariant objects. We will use only a part of this structure as follows. Let \( C_p \subset S_p \) be the cyclic subgroup generated by \( p \)-cycle \( \sigma = (1, 2, \ldots, p) \). Restricting the action of \( S_p \) above to \( C_p \) we get a symmetric tensor functor \( P_1 : \mathcal{C} \to C_{C_p} \) where \( \mathcal{C}_p \) is the equivariantization of \( \mathcal{C} \) as in Example 2.1.

Let \( \mathcal{C}_{C_p} \) be the quotient of \( \mathcal{C}_p \) by the negligible morphisms, see Section 2.5. Recall that we have an identification \( \mathcal{C}_{C_p} = \mathcal{C} \boxtimes \text{Rep}_k(C_p) = \mathcal{C} \boxtimes \text{Ver}_p \), see Example 2.14. Let \( Q \) be the quotient functor \( \mathcal{C}_{C_p} \to \mathcal{C}_{C_p} \). We define a symmetric tensor functor \( F_{0q} \) as a composition
\[
\mathcal{C} \xrightarrow{P_1} \mathcal{C}_{C_p} \xrightarrow{Q} \mathcal{C}_{C_p} = \mathcal{C} \boxtimes \text{Ver}_p.
\]

We have the following

**Lemma 3.4.** The functor \( F_{0q} \) is additive.

**Proof.** We have
\[
P_1(f + g) = (f + g) \otimes \ldots \otimes (f + g) = P_1(f) + P_1(g) + \text{other terms}
\]
\[\text{p factors}\]
where the other terms are monomials \( h_1 \otimes \ldots \otimes h_p \) where each \( h_i \) is \( f \) or \( g \) and not all \( h_i \) are the same. The group \( C_p \) acts on such monomials by permuting tensorands cyclically; clearly such an action has no fixed points. Thus \( P_1(f + g) - P_1(f) - P_1(g) \) splits into summands of the form

\[
(7) \quad h_1 \otimes h_2 \otimes \ldots \otimes h_p + h_2 \otimes h_3 \otimes \ldots \otimes h_1 + \ldots + h_p \otimes h_1 \otimes \ldots \otimes h_{p-1}.
\]

Sum \( (7) \) is a morphism in the category \( C_{C_p} \). Let us show that this morphism is negligible. Thus we need to show that the trace of the composition of \( (7) \) with a suitable morphism \( u \) is zero.

Observe that

\[
h_2 \otimes h_3 \otimes \ldots \otimes h_1 = \sigma(h_1 \otimes h_2 \otimes \ldots \otimes h_p)\sigma^{-1}
\]

whence

\[
\text{Tr}((h_2 \otimes h_3 \otimes \ldots \otimes h_1)u) = \text{Tr}((h_1 \otimes h_2 \otimes \ldots \otimes h_p)\sigma^{-1}u\sigma) = \text{Tr}((h_1 \otimes h_2 \otimes \ldots \otimes h_p)u)
\]

since by the definition of morphisms in \( C_{C_p} \) the morphism \( u \) commutes with \( \sigma \). We see that the contribution of each summand in \( (7) \) to the total trace is the same, which shows that the total trace is zero since we have \( p \) summands. Hence \( \text{Fr}_0(f + g) = \text{Fr}_0(f) + \text{Fr}_0(g) \) as desired. \( \square \)

The functor \( \text{Fr}_0 \) is not \( k \)-linear since obviously \( \text{Fr}_0(\lambda f) = \lambda^p\text{Fr}_0(f) \) for a morphism \( f \) and \( \lambda \in k \).

**Definition 3.5.** The Frobenius functor \( \text{Fr} : C \to C^{(1)} \boxtimes \text{Ver}_p \) is a \( k \)-linear symmetric tensor functor derived from \( \text{Fr}_0 \) using the identifications

\[
C \boxtimes \text{Ver}_p = (C \boxtimes \text{Ver}_p)^{(1)} = C^{(1)} \boxtimes C^{(1)} = C^{(1)} \boxtimes \text{Ver}_p.
\]

**Example 3.6.** Let \( p = 5 \) and let \( C \) be the Yang-Lee category from Section 1.2. Let us compute \( \text{Fr}(X) \). We have \( X^{\otimes 5} = 3 \cdot 1 \oplus 5X \) whence \( \text{Fr}(X) = 1 \boxtimes V_1 \oplus X \boxtimes V_X \) where \( V_1, V_X \in \text{Rep}_k(C_p) \) and \( \dim(V_1) = 3, \dim(V_X) = 5 \). The only possibility compatible with \( \text{FPdim}(\text{Fr}(X)) = \text{FPdim}(X) \) is \( \text{Fr}(X) = 1 \boxtimes L_3 \). Thus \( \sigma \) acts on \( V_1 \) as a Jordan cell of size 3 and on \( V_X \) as a Jordan cell of size 5.

The following result is follows directly from definitions:

**Lemma 3.7.** Let \( \text{id} \otimes \dim \) be the ring homomorphism \( K(C) \otimes K(\text{Ver}_p) \to K(C) \otimes k \) sending \( x \otimes y \) to \( x \otimes \dim(y) \). Then \( \text{id} \otimes \dim([\text{Fr}(X)]) = [X]^p \in K(C) \otimes k \). \( \square \)

**Example 3.8.** Let \( C = \text{Ver}_p \) and \( p > 2 \). Using \( (4) \) one computes \( [L_2]^p = -2[L_{p-1}] \) (mod \( p \)) in \( K(C) \). Using Lemma 3.7 and the Frobenius-Perron dimension we deduce that \( \text{Fr}(L_2) = L_{p-1} \boxtimes L_{p-2} \in C^{(1)} \boxtimes \text{Ver}_p \). Using (4) again we obtain

\[
\text{Fr}(L_s) = \begin{cases} 1 \boxtimes L_s & \text{if } s \text{ is odd}, \\ L_{p-1} \boxtimes L_{p-s} & \text{if } s \text{ is even}. \end{cases}
\]

We will say that \( C \) is of Frobenius type \( A \) if \( A \subset \text{Ver}_p \) is the smallest fusion subcategory such that the image \( \text{Fr}(C) \) is contained in \( C^{(1)} \boxtimes A \subset C^{(1)} \boxtimes \text{Ver}_p \). Thus by Proposition 3.3 (ii) for \( p > 3 \) there are just four possibilities for the Frobenius type of \( C \). For example the categories \( \text{Vec}, s\text{Vec} \) are of Frobenius type \( \text{Vec} \) and the category \( \text{Ver}_p \) is of Frobenius type \( \text{Ver}_p^+ \) by Example 3.8.
Remark 3.9. Observe that the formation of the Frobenius functor is compatible with \( k \)-linear symmetric tensor functors, that is for such a functor \( F : \mathcal{C} \to \mathcal{D} \) we have \( \text{Fr}(F(X)) = (F^{(1)} \boxtimes \text{id})(\text{Fr}(X)) \in \mathcal{D}^{(1)} \boxtimes \text{Ver}_p \). It follows that any category that admits a fiber functor or a super fiber functor is of Frobenius type \( \text{Vec} \).

Moreover, Conjecture 1.3 implies that a semisimple pre-Tannakian category \( \mathcal{C} \) of subexponential growth is of Frobenius type \( \text{Vec} \) or \( \text{Ver}_p \).

In the special case of category \( \mathcal{C} \) of Frobenius type \( \text{Vec} \) the image of Frobenius functor is contained in \( \mathcal{C}^{(1)} \boxtimes \text{Vec} = \mathcal{C}^{(1)} \).

We have the following immediate consequence of Lemma 3.7:

Corollary 3.10. Let \( \mathcal{C} \) be of Frobenius type \( \text{Vec} \). Then we have the following equality in \( K(\mathcal{C}) \otimes k \):

\[
[X]^p = [	ext{Fr}(X)].
\]

4. Proof of Theorem 1.5

4.1. Frobenius injective categories. We say that a symmetric fusion category is Frobenius injective if the Frobenius functor is injective. The following result is crucial.

Lemma 4.1. Let \( \mathcal{C} \) be a symmetric fusion category which is Frobenius injective of Frobenius type \( \text{Vec} \). Then \( \mathcal{C} \) is non-degenerate.

Proof. By the assumptions the Frobenius functor \( \mathcal{C} \to \mathcal{C}^{(1)} \boxtimes \text{Ver}_p \) lands to \( \mathcal{C}^{(1)} \boxtimes \text{Vec} = \mathcal{C}^{(1)} \) and is an equivalence. By Corollary 3.10 we have \( [X]^p = [\text{Fr}(X)] \) in \( K(\mathcal{C}) \otimes k \). It follows that the linear map \( x \mapsto x^p \) on the ring \( K(\mathcal{C}) \otimes k \) is surjective and hence injective. Therefore the commutative ring \( K(\mathcal{C}) \otimes k \) has no nilpotent elements and therefore it is semisimple. The result follows by Proposition 2.9.

Remark 4.2. We give here an easy alternative argument in the special case \( p = 2 \). We claim that in this case a Frobenius injective category \( \mathcal{C} \) has no nontrivial self-dual simple objects. Indeed, if \( X \) is self-dual then \( 1 \) appears in \( X \otimes 2 \) with multiplicity 1 and Corollary 3.10 implies that \( 1 \) appears as a direct summand in \( \text{Fr}(X) \) which contradicts Frobenius injectivity. This implies Lemma 4.1 in this case since contribution of each pair \((X, X^*)\) of non self-dual simple objects to \( \dim(\mathcal{C}) \) equals \( 2 \dim(X)^2 = 0 \) and hence \( \dim(\mathcal{C}) = 1 \).

Corollary 4.3. Let \( p > 2 \) and let \( \mathcal{C} \) be a symmetric fusion category which is Frobenius injective of Frobenius type \( s\text{Vec} \). Then \( \mathcal{C} \) is non-degenerate.

Proof. For a simple object \( X \in \mathcal{C} \) we have either \( \text{Fr}(X) = Y \boxtimes 1 \) or \( \text{Fr}(X) = Y \boxtimes \bar{L}^{p-1} \). Let \( \phi : \mathcal{O}(\mathcal{C}) \to \mathbb{Z}/2\mathbb{Z} \) be the function sending the objects of the first type to \( 0 \in \mathbb{Z}/2\mathbb{Z} \) and the objects of the second type to \( 1 \in \mathbb{Z}/2\mathbb{Z} \). Then \( \phi \) is a faithful \( \mathbb{Z}/2\mathbb{Z} \)-grading (see Section 2.11) of the category \( \mathcal{C} \). Moreover the neutral component \( \mathcal{C}_0 \) is Frobenius injective of Frobenius type \( \text{Vec} \). Thus \( \mathcal{C}_0 \) is non-degenerate by Lemma 2.11 The result follows since by Lemma 2.11 \( \dim(\mathcal{C}) = 2 \dim(\mathcal{C}_0) \).

Lemma 4.4. Let \( \mathcal{C} \) be of Frobenius type \( \text{Ver}_p \) or \( \text{Ver}_p^+ \) (so \( p > 3 \)). Then the image of Frobenius functor contains \( 1 \boxtimes \text{Ver}_p^+ \subset \mathcal{C}^{(1)} \boxtimes \text{Ver}_p \).
Proof. By assumption there is a simple object $X \in \mathcal{C}$ such that $\text{Fr}(X)$ contains a summand of the form $Y \boxtimes \bar{L}_s$ with $s \neq 0, p - 1$. Then $\text{Fr}(X \otimes X^*)$ contains a summand $(Y \otimes Y^*) \boxtimes (\bar{L}_s \otimes \bar{L}_s^*)$. Since $1 \subset Y \otimes Y^*$ and $\bar{L}_3 \subset \bar{L}_s \otimes \bar{L}_s^*$ we see that the image of Frobenius functor contains $1 \boxtimes \bar{L}_3$. The result follows since $\bar{L}_3$ generates $\text{Ver}_p$ by Proposition 3.4 (i).

4.2. Completion of the proof. For a sake of contradiction let us assume that Theorem 1.7 does not hold. Then there exists a counterexample $\mathcal{C}$ with minimal possible $\text{FPdim}(\mathcal{C})$, see Section 2.3. Then any $k$–linear symmetric tensor functor from $\mathcal{C}$ to another symmetric fusion category is injective by Corollary 2.5. In particular, the category $\mathcal{C}$ is Frobenius injective.

If $\mathcal{C}$ is of Frobenius type $\text{Vec}$ then by Lemma 4.1 and Proposition 2.8 there exists (necessarily injective) $k$–linear symmetric tensor functor $\mathcal{C} \to s\text{Vec}$ and we have a contradiction. Similarly, if $\mathcal{C}$ is of Frobenius type $s\text{Vec}$ then by Corollary 1.3 and Proposition 2.8 there exists $k$–linear symmetric tensor functor $\mathcal{C} \to s\text{Vec}$ and we also have a contradiction. Note that this completes the proof in the cases $p = 2$ and $p = 3$.

Thus $\mathcal{C}$ is forced to be of Frobenius type $\text{Ver}_p^+$ or $\text{Ver}_p$. Recall that $\mathcal{C}$ is Frobenius injective. Let $\tilde{\mathcal{C}} \subset \mathcal{C}$ be the subcategory generated by simple objects $X$ such that $\text{Fr}(X) = Y \boxtimes 1$ or $\text{Fr}(X) = Y \boxtimes \bar{L}_{p-1}$. Clearly $\tilde{\mathcal{C}}$ is a fusion subcategory of $\mathcal{C}$ of Frobenius type $\text{Vec}$ or $s\text{Vec}$. Thus by Lemma 4.1 and Corollary 4.3 $\tilde{\mathcal{C}}$ is non-degenerate.

Lemma 4.5. (i) The image of $\mathcal{C}$ under the Frobenius functor is generated by $\text{Fr}(\tilde{\mathcal{C}})$ and $1 \boxtimes \text{Ver}_p^+$. (ii) The fusion subcategory generated by $\text{Fr}(\tilde{\mathcal{C}})$ and $\text{Ver}_p^+$ is equivalent to $\tilde{\mathcal{C}} \boxtimes \text{Ver}_p^+$.

Proof. (i) We have $\text{Fr}(\mathcal{C}) \supset \text{Fr}(\tilde{\mathcal{C}})$; also the image $\text{Fr}(\mathcal{C})$ contains $1 \boxtimes \text{Ver}_p^+$ by Lemma 4.2. Thus it remains to show that $\text{Fr}(\mathcal{C})$ is contained in the fusion subcategory generated by $\text{Fr}(\tilde{\mathcal{C}})$ and $1 \boxtimes \text{Ver}_p^+$.

Recall that $\mathcal{C}$ is Frobenius injective. Let $X \in \mathcal{O}(\mathcal{C})$ with $\text{Fr}(X) = T \boxtimes L_s$. Let $\delta = 1$ if $s$ is odd and $\delta = \bar{L}_{p-1}$ if $s$ is even. Then $L_s \otimes \delta \in \text{Ver}_p^+$ (see (3)) and there exists $Y \in \mathcal{O}(\mathcal{C})$ such that $\text{Fr}(Y) = 1 \boxtimes (\bar{L}_s \otimes \delta)$. Then $\text{Fr}(X \otimes Y) = T \boxtimes (L_s \otimes L_s \otimes \delta)$ contains a summand $T \boxtimes \delta_s$. Hence $X \otimes Y$ contains a summand $Z \in \mathcal{O}(\mathcal{C})$ such that $\text{Fr}(Z) = T \boxtimes \delta_s$. Thus $Z \in \tilde{\mathcal{C}}$ and $\text{Fr}(X) \neq T \boxtimes L_s$ is isomorphic to $T \boxtimes (\delta_s \otimes L_s \otimes \delta_s)$ which shows that $\text{Fr}(X)$ is contained in the subcategory generated by $\text{Fr}(\tilde{\mathcal{C}})$ and $1 \boxtimes \text{Ver}_p^+$ as desired.

(ii) The simple objects of $\text{Fr}(\tilde{\mathcal{C}})$ are of the form $T \boxtimes \delta$ where $\delta = 1$ or $\delta = \bar{L}_{p-1}$, and the simple objects of $1 \boxtimes \text{Ver}_p^+$ are $1 \boxtimes \bar{L}_s$ with odd $s$. Thus the only simple object which belongs to both subcategories is $1 \boxtimes 1$ and the result follows from Lemma 2.2.

Thus by Lemma 4.5 the Frobenius functor induces a $k$–linear symmetric tensor functor $\mathcal{C} \to \tilde{\mathcal{C}} \boxtimes \text{Ver}_p^+$. Since $\tilde{\mathcal{C}}$ is non-degenerate by Proposition 2.8 there exists a $k$–linear symmetric tensor functor $\tilde{\mathcal{C}} \to s\text{Vec}$. Taking the composition we get a functor $\mathcal{C} \to s\text{Vec} \boxtimes \text{Ver}_p^+ = \text{Ver}_p$ (see Proposition 3.3 (iii)). Thus $\mathcal{C}$ is not a counterexample to Theorem 1.7; so no such counterexample exists.

4.3. Examples and complements.
4.3.1. Let $p > 0$ and let $G_{a,1}$ be the Frobenius kernel of the additive group $G_a$, see e.g. [J. II.2.2]. Then representations of $G_{a,1}$ are the same as representations of the Hopf algebra $k[x]/x^p$ where $x$ is primitive element (that is $\Delta(x) = x \otimes 1 + 1 \otimes x$). The indecomposable objects of $\text{Rep}_k(G_{a,1})$ are Jordan cells of sizes $1, \ldots, p$; moreover the decompositions of tensor products are precisely the same as in Section 3.1, see e.g. [G] p. 611. In particular, the Grothendieck ring of the quotient category $\text{Rep}_k(G_{a,1})$ is isomorphic to the Verlinde ring $K(\text{Ver}_p)$ as a based ring; moreover the isomorphism respects the dimensions of objects. We claim that the functor $F : \text{Rep}_k(G_{a,1}) \rightarrow \text{Ver}_p$ existing by Theorem 1.5 is an equivalence. Indeed, it is easy to see from explicit formula [EGNO, Exercise 4.10.7] that for any $s \neq 1, 2, p-2, p-1$ we have $\text{FPdim}(L_s) > \text{FPdim}(L_2) > \text{FPdim}(L_2) \notin \mathbb{Z}$ for $p > 3$. Hence $F$ should send the two dimensional Jordan cell to either $L_2$ or $L_{p-2}$; however the second case is impossible since $\dim(L_{p-2}) = p - 2 \neq 2$. Therefore (8) implies that $F$ sends the Jordan cell of size $s$ to $L_s$ and thus $F$ is an equivalence $\text{Rep}_k(G_{a,1}) \simeq \text{Ver}_p$.

4.3.2. In [GM] the authors construct examples of symmetric fusion categories over $k$ as follows. Let $G$ be a semisimple algebraic group such that its Coxeter number (see e.g. [J. II.6.2]) is smaller than $p$. The category $\text{Rep}_k(G)$ contains a Karoubian (but not abelian) tensor subcategory $\mathcal{T}(G)$ of tilting modules, see [A] or [J. II.E]. The quotient $\mathcal{T}(G)$ of this category by the negligible morphisms (see Section 3.5) is an example of symmetric fusion category. In the special case $G = SL_2$ it is known that $K(\mathcal{T}(SL_2)) \simeq K(\text{Ver}_p)$ as a based ring and the isomorphism respects the dimensions, see [GM]. As in the preceding paragraph it follows that we have an equivalence of symmetric fusion categories $\mathcal{T}(SL_2) \simeq K(\text{Ver}_p)$ as it was promised in Section 1.2.

4.3.3. Finally, we sketch a direct construction of the functor $\mathcal{T}(G) \rightarrow \text{Ver}_p$ guaranteed by Theorem 1.5. Let $G_{a,1} \subseteq G$ be an embedding associated with a regular nilpotent element of the Lie algebra of $G$. The restriction gives a $k$-linear symmetric tensor functor $\mathcal{T}(G) \rightarrow \text{Rep}_k(G_{a,1})$. The theory of support varieties shows that an indecomposable object of $\mathcal{T}(G)$ of dimension zero is sent by this functor to a projective object of $\text{Rep}_k(G_{a,1})$, see [J. E13]. It follows that the restriction functor descends to a functor $\mathcal{T}(G) \rightarrow \text{Rep}_k(G_{a,1})$. Combining this with equivalence $\text{Rep}_k(G_{a,1}) \simeq \text{Ver}_p$ we get a desired functor $\mathcal{T}(G) \rightarrow \text{Ver}_p$.

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