ELECTRO-MAGNETO-STATIC STUDY OF THE NONLINEAR
SCHRÖDINGER EQUATION COUPLED WITH
BOPP-PODOLSKY ELECTRODYNAMICS IN THE
PROCA SETTING

EMMANUEL HEBEY
Emmanuel Hebey, Université de Cergy-Pontoise
Département de Mathématiques, Site de Saint-Martin
2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
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Abstract. We investigate the system consisting of the nonlinear
Schrödinger equation coupled with Bopp-Podolsky electrodynamics in the
Proca setting in the context of closed 3-dimensional manifolds. We prove
existence of solutions up to the gauge, and compactness of the system both
in the subcritical and in the critical case.

Let \((M, g)\) be a 3-dimensional closed manifold. We look for systems of equations
with unknowns \((u, v, A)\), where \(u\) and \(v\) are functions and \(A\) is a 1-form, which
express like
\[
\begin{aligned}
\frac{\hbar^2}{2m_0} \Delta_g u + \Phi(x, v, A)u &= u^{p-1} \\
\alpha^2 \Delta_g^2 v + \Delta_g v + m_1^2 v &= 4\pi qu^2 \\
\alpha^2 \Delta_g^2 A + \Delta_g A + m_1^2 A &= \frac{4\pi q}{m_0} \Psi(A, S)u^2,
\end{aligned}
\]
(0.1)
where \(\Phi(x, v, A) = \frac{\hbar^2}{2m_0} |\Psi(A, S)|^2 + \omega^2 + qv, \Psi(A, S) = \nabla S - \frac{q}{\hbar} A\), \(a, q, m_0, m_1 > 0\) are positive real numbers, \(\omega \in \mathbb{R}\), \(\Delta_g = -\text{div}_g \nabla\) is the Laplace-Beltrami operator
when acting on functions \(u\) and \(v\), \(\Delta_g = \delta d + d\delta\) is the Hodge-de Rham Laplacian
when acting on 1-forms \(A\), \(u \geq 0\) is a function, and \(p \in (2, 6]\) (\(d\) is the differential,
\(\delta\) is the codifferential, 6 is the critical Sobolev exponent). System of equations
like (0.1) are derived from the full BPSP system when we look for solutions of such
systems in the form \(\Psi(x, t) = u(x, t)e^{iS(x, t)}\) with \(u\) depending only on \(x\) and \(S\) in the
splitted form \(S(x, t) = S(x) + \frac{\omega^2}{\hbar^2} t\) (see Section 1 below). Such type of solutions were
introduced in the paper [3] by Benci and Fortunato for the Klein-Gordon-Maxwell equations in \(\mathbb{R}^3\)
(see also D’Avenia, Mederski and Pomponio [9]). We learned from
the Bopp-Podolsky setting in the paper [10] by D’Avenia and Siciliano where the
existence of electrostatic solutions is established for the Schrödinger equation in the
Bopp-Podolsky electrodynamics in the case of \(\mathbb{R}^3\).

We prove in this paper existence and compactness results for (0.1). We recall
that a coercive operator like \(\Delta_g + \Lambda_g\) has positive mass (resp. nonnegative mass)
if the regular part of its Green’s function is positive (resp. nonnegative) on the

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diagonal. By the positive mass theorem of Schoen and Yau [45] (see also Witten [52]), there exists a function $\Lambda_g$ (with $\Lambda_g > 0$ in $M$) such that $\Delta_g + \Lambda_g$ has positive mass when the scalar curvature $S_g$ of the manifold is positive. We consider the condition

$$\omega^2 + \frac{\hbar^2}{2m_0^2} |\nabla S|^2 < \frac{\hbar^2}{2m_0^2} \Lambda_g$$

where $\Lambda_g > 0$ is smooth and such that $\Delta_g + \Lambda_g$ has nonnegative mass. When $S_g > 0$, one can take $\Lambda_g = \frac{n-2}{4(n-1)} S_g$, and this is sharp in the case of the unit sphere. Another way of expressing (0.2) would be that $\Delta_g + \Phi$, with $\Phi$ given by the left hand side of (0.2), has positive mass. We mainly prove the two following theorems in this paper. More results can be found in the sequel. The subscript $R$ in the notation $C^\infty_R$ refers to the fact that real-valued functions are considered.

Theorem 0.1 (Existence). Let $(M, g)$ be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in (4, 6]$. We assume that $am_1 < \frac{1}{2}$ and when $p$ is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we also assume that (0.2) holds true. Then (0.1) possesses a smooth nontrivial solution $(u, v, A)$ with $u > 0$ and $v > 0$. Also $A \not\equiv 0$ if $\nabla S \not\equiv 0$.

Theorem 0.2 (Compactness). Let $(M, g)$ be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in (4, 6]$. When $p$ is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we assume that (0.2) holds true. Then the set $\mathcal{S}$ of solutions $(u, v, A)$ of (0.1) with $u \geq 0$ is compact in the $C^2$-topology.

The above theorems include both the subcritical case for which $p < 6$ and the critical case for which $p = 6$. Condition (0.2) is required in the sole critical case. It is sharp in the sense described at the very end of the paper. As a remark, adding the Coulomb gauge equation $\delta A = 0$ to (0.1) (see Section 1) obviously does not change anything to Theorem 0.2. More general stability results are established in Sections 9 and 10.

1. A quick origin of the problem. The Bopp-Podolsky theory, developed by Bopp [4], and independently by Podolsky [38], is a second order gauge theory for the electromagnetic field. It refines the Maxwell-Schrödinger theory which aims to describe the evolution of a charged nonrelativistic quantum mechanical particle interacting with the electromagnetic field it generates. We adopt here the Proca formalism as in Hebey and Wei [30]. Then the particle is ruled by the Schrödinger Lagrangian $L_{NLS}$ and interacts via the minimum coupling rule

$$\partial_t \rightarrow \partial_t + i \frac{q}{\hbar} \varphi, \quad \nabla \rightarrow \nabla - i \frac{q}{\hbar} A$$

with an external massive vector field $(\varphi, A)$ which is governed by the Bopp-Podolsky-Proca Lagrangian $L_{BPP}$. A very nice presentation of this (without the Proca contribution) is in D’Avenia and Siciliano [10]. More on the Proca contribution can be found in Goldhaber and Nieto [23, 24], Luo, Gillies and Tu [36], and Ruegg and
Ruiz-Altaba [39]. At this point, we let
\[ \mathcal{L}_{BPP} = \frac{1}{8\pi} \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^{2} - \frac{1}{8\pi} |\nabla \times A|^{2} + \frac{m^{2}}{8\pi} \left( |\varphi|^{2} - |A|^{2} \right) + \frac{\alpha^{2}}{8\pi} \mathcal{L}_{\text{Ad}}(\varphi, A), \]
where \( \nabla \times = \ast d \) is the curl operator (\( \ast \) is the Hodge dual and \( d \) the usual differentiation on forms),
\[ \mathcal{L}_{\text{Ad}}(\varphi, A) = (-\Delta_{g} \varphi + \nabla \cdot \partial_{t} A)^{2} - \left| \Delta_{g} A + \partial_{t} (\nabla \varphi + \partial_{t} A) \right|^{2} \]
and \( \Delta_{g} = \nabla \times \nabla \times = \delta d \) is half the Hodge-de Rham Laplacian for 1-forms (\( \delta \) is the codifferential). Then we define
\[ S = \int \int (\mathcal{L}_{BLS} + \mathcal{L}_{BPP}) \, dv_{g} \, dt \]
to be the total action functional. Writing \( \psi = u e^{iS} \) in polar form, \( u \geq 0 \), and taking the variation of \( S \) with respect to \( u, S, \varphi, \) and \( A \), we get the full BPSP system
\[
\begin{align*}
\frac{\hbar^{2}}{2m_{0}^{2}} \Delta_{g} u + \left( \hbar \frac{\partial S}{\partial \varphi} + q \varphi + \frac{\kappa^{2}}{2m_{0}^{2}} |\Psi(A, S)|^{2} \right) u & = u^{p-1} \\
2u \frac{\partial u}{\partial t} + \frac{\hbar}{m_{0}^{2}} \nabla \cdot (\Psi(A, S) u^{2}) & = 0 \\
- \frac{1}{4\pi} \nabla \cdot (\frac{\partial A}{\partial t} + \nabla \varphi) + \frac{\alpha^{2}}{4\pi} \Delta_{g} M(\varphi, A) - \frac{a^{2}}{4\pi} \frac{\partial \nabla \cdot N(\varphi, A) + m_{1}^{2}}{4\pi} \varphi & = qu^{2} \\
\frac{1}{4\pi} \nabla \Delta_{g} A + \frac{1}{4\pi} (\frac{\partial A}{\partial t} + \nabla \varphi) + \frac{m_{1}^{2}}{4\pi} A + \frac{a^{2}}{4\pi} Q(\varphi, A) & = \frac{b_{0}^{2}}{m_{0}^{5}} \Psi(A, S) u^{2},
\end{align*}
\]  
where
\[ \Psi(A, S) = \nabla S - \frac{q}{\hbar} A, \quad M(\varphi, A) = -\Delta_{g} \varphi + \nabla \partial_{t} A, \]
\[ N(\varphi, A) = \Delta_{g} A + \partial_{t} (\nabla \varphi + \partial_{t} A), \]
\[ Q(\varphi, A) = \Delta_{g} N(\varphi, A) + \frac{\partial^{2} \nabla \cdot N(\varphi, A) - \nabla \frac{\partial}{\partial t} M(\varphi, A)}{4\pi}. \]
When \( a = 0 \) we are back to the Schrödinger-Poisson system in Proca form as investigated in Hebey and Wei [30] and Thizy in the series of papers [46, 47, 48, 49, 50]. We assume from now on that we are in the static case of the system, for which \( \partial_{t} u \equiv 0, \partial_{t} A \equiv 0 \) and \( \partial_{t} \varphi \equiv 0 \), and we look for solutions with
\[ S(x, t) = S(x) + \omega \frac{\hbar}{2} t. \]
Then the system (1.1) can be written under the following form
\[
\begin{align*}
\frac{\hbar^{2}}{2m_{0}^{2}} \Delta_{g} u + \Phi(x, \varphi, A) u & = u^{p-1} \\
\nabla \cdot (\Psi(A, S) u^{2}) & = 0 \\
a^{2} \Delta_{g}^{2} \varphi + \Delta_{g} \varphi + m_{1}^{2} \varphi & = 4\pi q u^{2} \\
a^{2} \Delta_{g} A + \Delta_{g} A + m_{1}^{2} A & = \frac{4\pi G}{m_{0}^{5}} \Psi(A, S) u^{2}.
\end{align*}
\]  
By the fourth equation, since \( \nabla \Delta_{g} = 0 \), the second equation in (1.2) is nothing but the Coulomb gauge condition \( \delta A = 0 \), and the three other equations give rise to (0.1) by noting that when \( \delta A = 0 \) we get that \( \Delta_{g} A = \Delta_{g} A \), where \( \Delta_{g} = \delta d + \delta d \) is the Hodge-de Rham Laplacian on forms.
The auxiliary map problem for $A$. For $q > 0$, let $L^q_R$ be the standard Lebesgue’s space for functions ($R$ for real valued), $H^1_R$ be the Sobolev space of functions in $L^2_R$ with one derivative in $L^2_R$ and $L^q_V$ and $H^1_V$ ($V$ for vector valued) be the corresponding Lebesgue’s spaces and Sobolev space for 1-forms. In the same vein, we let $H^2_V$ and $H^4_V$ be the Sobolev spaces of 1-forms with two and four derivatives in $L^2_V$, and we let $L^\infty_V$ be the space of 1-forms which are bounded. The same with the subscript $C_L$ derivatives in $R$.

Moreover $A(u,v)$ is unique.

Proof of Lemma 2.1. First we prove existence. By the Sobolev embedding theorem, $H^2_V \subset C^0_V$. Let $u,v \in H^1_R$, be given and let $F = F(u,v)$. Then $(F,A)uv \in L^2_R$ and $u^2 |A|^2 \in L^2_R$ for all $A \in H^2_V$. We define $I : H^2_V \rightarrow \mathbb{R}$ by letting

$$I(A) = a^2 \int_M |\Delta_g A|^2 dv_g + \int_M |dA|^2 dv_g + \int_M |\delta A|^2 dv_g + \int_M \left( m_1^2 + \frac{4\pi q^2}{m^2_0} u^2 \right) |A|^2 dv_g ,$$

and we let $\mathcal{H} \subset H^2_V$ be given by

$$\mathcal{H} = \left\{ A \in H^2_V \text{ s.t. } \int_M (F,A) uv dv_g = 1 \right\} .$$

There holds that $\mathcal{H} = \emptyset$ if and only if $B = uvF$ is such that $\int_M (B,A) dv_g = 0$ for all $A \in H^2_V$. By density of $C^0_V$ into $L^2_V$, noting that $B \in L^2_V$ since $F \in C^0_V$ and $H^1_R \subset L^1_R$, we get that $\mathcal{H} = \emptyset$ if and only if $B = 0$. Then $A = 0$ solves the problem.

We assume in what follows that $\mathcal{H} \neq \emptyset$. We let

$$\mu = \inf_{A \in \mathcal{H}} I(A) ,$$

and let $(A_\alpha)_\alpha$ be a minimizing sequence for $\mu$. Obviously $\mu \geq 0$, and by (2.1), $(A_\alpha)_\alpha$ is bounded in $H^2_V$. Then, up to passing to a subsequence, by reflexivity of $H^2_V$ and compact embeddings in the Sobolev inclusions, we may assume that $A_\alpha \rightharpoonup A$ in $H^2_V$, $A_\alpha \rightarrow A$ in $H^1_V$ and $A_\alpha \rightarrow A$ in $C^0_V$ for some $A \in H^2_V$. Then $A \in \mathcal{H}$ and by
weak convergence (using (2.1) but also the above strong convergences) there holds that \( I(A) \leq \mu \). It follows that \( I(A) = \mu \), and since \( I'(A).A = 2I(A) \), we get that
\[
I'(A).B = 2\mu \int_M (F, B) wdv_g
\]
for all \( B \in H^2_p \). There holds that \( A \neq 0 \) since \( A \) belongs to \( \mathcal{H} \). Therefore \( \mu > 0 \) and \( \frac{1}{\mu} A \) solves (2.2) in \( H^2_p \). Now we prove uniqueness. Suppose \( A, \tilde{A} \in H^2_p \) solve (2.2).
Then, subtracting the two equations, contracting by \( A - \tilde{A} \) and integrating over \( M \), we get that \( I(A - \tilde{A}) = 0 \). This implies that \( A = \tilde{A} \). Lemma 2.1 is proved.

In the specific case of (0.1), the following important lemma holds true. Equation (2.4) and the first equation in (2.5) are of the same nature. Equation (2.4) is slightly more subtle than the first equation in (2.5) and improves it when \( \|u\|_{L^2_R} \gg 1 \). They can be combined to get the following single estimate:
\[
\|A(u)\|_{H^2_p} \leq C\|u\|_{L^2_R} \min \left(1, \|u\|_{L^2_R}\right)
\]
for all \( u \in H^1_{R} \), where \( C > 0 \) is independent of \( u \). Uniqueness in Lemma 2.2 actually holds in \( H^2_p \).

**Lemma 2.2.** For any \( u \in H^1_{R} \), there exists a unique \( A(u) \in H^1_{V} \cap C^0_p \) such that
\[
a^2 \Delta^2 gA(u) + \Delta gA(u) + \left(\frac{1}{m_0^2} + \frac{4\pi q^2}{m_0^2} u^2\right) A(u) = \frac{4\pi q h}{m_0^2} (\nabla S)u^2 .
\]
Moreover, there exists \( C > 0 \) such that
\[
\|A(u)\|_{H^2_p} \leq C\|u\|_{L^2_R} \tag{2.4}
\]
for all \( u \in H^1_{R} \), and there also exists \( C', C'' > 0 \) such that
\[
\|A(u)\|_{H^2_p} \leq C'\|u\|_{L^2_R}^2 , \quad \text{and} \quad \|A(u)\|_{H^2_p} \leq C'' \left(1 + \|u\|_{L^2_R}^2\right) \|u\|_{L^2_R} \tag{2.5}
\]
for all \( u \in H^1_{R} \).

**Proof of Lemma 2.2.** By Lemma 2.1, letting \( F \) be the constant map (with respect to \( u \) and \( v \)) given by
\[
F = \frac{4\pi q h}{m_0^2} (\nabla S) ,
\]
we get that for any \( u \in H^1_{R} \) there exists a unique \( A(u) = A(u, u) \) in \( H^2_p \cap C^0_p \) which solves (2.3). It remains to prove the estimates in Lemma 2.2. First we prove (2.4) and the first equation in (2.5). Let \( u \in H^1_{R} \). Contracting (2.3) by \( A(u) \), integrating over \( M \), we get that
\[
a^2 \int_M |\Delta gA(u)|^2dv_g + \int_M |dA(u)|^2dv_g + \int_M |\delta A(u)|^2dv_g + m_1^2 \int_M |A(u)|^2dv_g + \frac{4\pi q h}{m_0^2} \int_M |A(u)|^2u^2dv_g \leq \frac{4\pi q h}{m_0^2} \int_M |(\nabla S, A(u))| u^2dv_g .
\]

The proof follows from similar calculations as (2.6).
Since $H^{2}_{V} \subset C^{0}_{V}$, it follows from (2.6) that there exist $C_1, C_2 > 0$ (depending on $a,S$ and $m_1$, but independent of $u$) such that
\[
\|A(u)\|^2_{H^{2}_{V}} \leq C_1 \|A(u)\|_{C^{0}_{V}} \|u\|^2_{L^{2}_{h}} \leq C_2 \|A(u)\|_{H^{2}_{V}} \|u\|^2_{L^{2}_{h}} \tag{2.7}
\]
and the first equation in (2.5) follows from (2.7). In order to get (2.4) we need to be slightly more subtle. Using the Aubin-Dodziuk norm (2.1), it also follows from (2.6) that there exists $\lambda_0 > 0$ (depending on $a$ and $m_1$, but independent of $u$) such that
\[
\lambda_0 \|A(u)\|^2_{H^{2}_{V}} \leq \frac{4 \pi q \hbar}{m_0^2} \int_{M} \left( |(\nabla S, A(u))| - \frac{q}{\hbar} |A(u)|^2 \right) u^2 dv_g \
\leq \frac{4 \pi q \hbar}{m_0^2} \int_{M} \left( \Lambda_0(S) - \frac{q}{\hbar} |A(u)| \right) |A(u)| u^2 dv_g , \tag{2.8}
\]
where $\Lambda_0(S) = \max_{M} |\nabla S|$. Let $C_0 > 0$ be such that
\[
\Lambda_0(S)^2 - \frac{4 q}{\hbar} C_0 < 0 . \tag{2.9}
\]
Then, by (2.9), we get that
\[
\Lambda_0(S) |A(u)| - \frac{q}{\hbar} |A(u)|^2 \leq C_0 \tag{2.10}
\]
and coming back to (2.8), we get from (2.10) that (2.4) holds true with a positive constant $C$, independent of $u$ (depending only on $\lambda_0, q, m_0$ and $C_0$). It remains to prove that $A(u) \in H^{4}_{V} \cap C^{2}_{V}$ and that the second equation in (2.5) holds true. In order to do so, we rewrite (2.3) in the following form:
\[
(1 + a^2 \Delta_g) \Delta_g A(u) = \Phi(A,u) , \tag{2.11}
\]
where
\[
\Phi(A,u) = \frac{4 \pi q \hbar}{m_0^2}(\nabla S) u^2 - \left( m_1^2 + \frac{4 \pi q^2}{m_0^2} u^2 \right) A(u) .
\]
In particular, by (2.4), we get from (2.11) that there exists $C > 0$ independent of $u$ such that
\[
|\Phi(A,u)| \leq C \left( u^2 + \|u\|_{L^2_{h}} + \|u\|_{L^2_{h}} u^2 \right)
\]
for all $u \in H^{2}_{V}$. By elliptic regularity, and again (2.4), we then get that $\Delta_g A(u) \in H^{4}_{V}$ and that
\[
\|\Delta_g A(u)\|_{H^{4}_{V}} \leq C \left( \|\Delta_g A(u)\|_{L^2_{h}} + \|\Phi(A,u)\|_{L^2_{h}} \right) \tag{2.12}
\]
where $C > 0$ is independent of $u$. Then, again by elliptic regularity, using (2.4) and (2.12), see also Dodziuk [11] for the $\Delta$-norm on $H^{4}_{V}$, we get that $A(u) \in H^{4}_{V}$ and that
\[
\|A(u)\|_{H^{4}_{V}} \leq C \left( \|\Delta_g A(u)\|_{H^{2}_{V}} + \|A(u)\|_{L^2_{h}} \right) \tag{2.13}
\]
where $C > 0$ is independent of $u$. Obviously, we get the second equation in (2.5) with (2.13). Noting that $H^{4}_{V} \subset C^{2}_{V}$ by Sobolev, this ends the proof of Lemma 2.2. □
Remark 2.1. Obviously, we refer to the above proof, the constants \( C, C', C'' \) in Lemma 2.2 can be made independent of \( S \) in any bounded subset of \( C^1_{R} \).

The next lemma establishes local lipschitz estimates for the map \( A \) in Lemma 2.2.

**Lemma 2.3.** Given \( K > 0 \), let \( B_K^2 = \left\{ u \in H^1_R \text{ s.t. } \|u\|_{L^2_R} \leq K \right\} \). For any \( K > 0 \) there exists \( C_K > 0 \) such that

\[
\|A(v) - A(u)\|_{H^1_R} \leq C_K \|v - u\|_{L^2_R} \tag{2.14}
\]

for all \( u, v \in H^1_R \), where \( A \) is as in Lemma 2.2. Moreover, we can take \( C_K = CK(1 + K) \) where \( C > 0 \) is independent of \( K \).

**Proof of Lemma 2.3.** Let \( u, v \in H^1_R \) and \( h = v - u \). There holds that

\[
a^2 \Delta^2 g(\Phi(u + h) + \Delta g(\Phi(u + h) + \left( m_1^2 + \frac{4\pi q^2}{m_0^2}u^2 \right) A(u + h)
\]

\[
= \frac{4\pi q h}{m_0^2} (\nabla S)(u + h)^2 - \frac{4\pi q^2}{m_0^2}h^2 A(u + h) - \frac{8\pi q^2}{m_0^2}uhA(u + h) .
\]

Then,

\[
a^2 \Delta^2 g(\Phi(u + h) - A(u)) + \Delta g(\Phi(u + h) - A(u))
\]

\[
+ \left( m_1^2 + \frac{4\pi q^2}{m_0^2}u^2 \right) (A(u + h) - A(u))
\]

\[
= \frac{4\pi q h}{m_0^2} (\nabla S)h^2 + \frac{8\pi q h}{m_0^2} (\nabla S)uh - \frac{4\pi q^2}{m_0^2}h^2 A(u + h) - \frac{8\pi q^2}{m_0^2}uhA(u + h) .
\]

Contracting the equation by \( A(u + h) - A(u) \) and integrating, using the Aubin-Dodziuk norm (2.1), we get that there exists \( C > 0 \) (depending on \( a, S \) and \( m_1 \), but independent of \( u \) and \( h \)) such that

\[
\|A(u + h) - A(u)\|_{H^1_R}^2 \leq C \left[ \int_M h^2 |A(u + h) - A(u)|dv_g + \int_M |uh||A(u + h) - A(u)|dv_g + \int_M h^2 |A(u + h)||A(u + h) - A(u)|dv_g + \int_M |uh||A(u + h)||A(u + h) - A(u)|dv_g \right],
\]

and if we let \( u, u + h \in B_K^2 \) for some \( K > 0 \), then, by Lemma 2.2,

\[
\|A(u + h) - A(u)\|_{H^1_R}^2 \leq C(1 + K) \left[ \int_M h^2 |A(u + h) - A(u)|dv_g + \int_M |uh||A(u + h) - A(u)|dv_g \right]
\]

\[
\leq C(1 + K)\|A(u + h) - A(u)\|_{C^0} \left[ \int_M (h^2 + |uh|) \, dv_g \right]
\]

\[
\leq CK(1 + K)\|A(u + h) - A(u)\|_{C^0} \|h\|_{L^2_R}
\]
2.1 for all $h$

By Sobolev, $H^1_K \subset C^0_V$ and equation (2.14) with $C_K = CK(1 + K)$ follows from the above equations (with $C > 0$ independent of $K$, $u$ and $v$). This proves Lemma 2.3.

Now we prove the differentiability of the map $A$ in Lemma 2.2.

**Lemma 2.4.** The map $A : H^1_K \to H^2_K$ in Lemma 2.2 is differentiable and its differential $A'_u \in L(H^1_K, H^2_K)$ at $u \in H^1_K$ is given by

$$a^2 \Delta^2 A'_u(h) + \Delta A'_u(h) + \left(m_1^2 + \frac{4\pi q^2}{m_0^2} u^2\right) A'_u(h) = \frac{8\pi q h}{m_0^2} \left(\nabla S - \frac{q}{h} A(u)\right) h$$

(2.16)

for all $h \in H^1_K$.

**Proof of Lemma 2.4.** The existence and uniqueness of $A'_u(h)$ follows from Lemma 2.1 with $F = F(u, v)$, independent of $v$, given by

$$F = \frac{8\pi q h}{m_0^2} \left(\nabla S - \frac{q}{h} A(u)\right).$$

Given $u \in H^1_K$, the linearity of $A'_u$ is obvious. Its continuity is easy to get. We fix $u \in H^1_K$ and for $h \in H^1_K$ we let

$$R(h) = A(u + h) - A(u) - A'_u(h),$$

where $A'_u(h)$ is given by (2.16). Then, by (2.15) and (2.16),

$$a^2 \Delta^2 R(h) + \Delta R(h) + \left(m_1^2 + \frac{4\pi q^2}{m_0^2} u^2\right) R(h) = \frac{4\pi q h}{m_0^2} \left(\nabla S - \frac{q}{h} A(u + h)\right) h^2 - \frac{8\pi q^2}{m_0^2} uh (A(u + h) - A(u)).$$

Contracting the equation by $R(h)$ and integrating, using the Aubin-Dodziuk norm (2.1), we get from Lemma 2.2 and Lemma 2.3 that there exists $C > 0$ such that for all $h \in H^1_K$,

$$\|R(h)\|_{L^2_K} \leq C \left[\int_M h^2 |R(h)| + h \|R\|_{L^2_K} \int_M |uh| |R(h)| dv_g\right]$$

$$\leq C \|R(h)\|_{C^0_V} \|h\|_{L^2_K}^2$$

when $\|h\|_{L^2_K} \leq 1$. Then, by Sobolev, since $H^1_K \subset C^0_V$, we get in particular that

$$\|R(h)\|_{H^2_V} \leq C \|h\|_{L^2_K}^2 \leq C \|h\|_{H^1_K}^2$$

for all $h \in H^1_K$ such that $\|h\|_{L^2_K} \leq 1$. This ends the proof of Lemma 2.4.

At this point we prove the following.
Lemma 2.5. Let $\mathcal{I}_1 : H^1_R \to \mathbb{R}$ be given by

$$\mathcal{I}_1(u) = \int_M \left( \nabla S - \frac{q}{\hbar} A(u), \nabla S \right) u^2 dv_g.$$  \hfill (2.17)

Then $\mathcal{I}_1$ is differentiable and

$$\mathcal{I}_1'(u)(v) = 2 \int_M \left| \nabla S - \frac{q}{\hbar} A(u) \right|^2 uv dv_g$$  \hfill (2.18)

for all $u, v \in H^1_R$.

Proof of Lemma 2.5. We can write that $\mathcal{I}_1$ is given by $\mathcal{I}_1 = \Psi \circ \Phi$, where

$$\Phi : \{ H^1_R \to H^2_V \times H^1_R \}$$

$$u \to (\nabla S - \frac{q}{\hbar} A(u), u, u),$$

and

$$\Psi : \{ H^2_V \times H^1_R \times H^1_R \to \mathbb{R} \}$$

$$(A, v, w) \to \int_M (A, \nabla S) uv dv_g.$$

Clearly $\Psi$ is trilinear continuous, while by Lemma 2.4 there holds that $\Phi$ is differentiable. Then $\mathcal{I}_1$ is differentiable and

$$\mathcal{I}_1'(u)(v) = D\Psi \left( \nabla S - \frac{q}{\hbar} A(u), u, u \right).$$

$$= -\frac{q}{\hbar} \Psi \left( A'_u(v), u, u \right) + 2\Psi \left( \nabla S - \frac{q}{\hbar} A(u), u, u \right)$$  \hfill (2.19)

for all $u, v \in H^1_R$. By (2.3), integrating by parts,

$$\int_M (A'_u(v), \nabla S) u^2 dv_g = \frac{m_1^2}{4\pi q\hbar} \int_M (A'_u(v), \mathcal{L}_g A(u)) dv_g$$

$$= \frac{m_1^2}{4\pi q\hbar} \int_M (A(u), \mathcal{L}_g A'_u(v)) dv_g$$  \hfill (2.20)

where

$$\mathcal{L}_g = a^2 \Delta_g^2 + \Delta_g + \left( m_1^2 + \frac{4\pi q^2}{m_0^2} u^2 \right).$$

Then, by (2.16), we get from (2.20) that

$$\int_M (A'_u(v), \nabla S) u^2 dv_g = 2 \int_M \left( \nabla S - \frac{q}{\hbar} A(u), A(u) \right) uv dv_g$$  \hfill (2.21)

for all $u, v \in H^1_R$. Combining (2.19) and (2.21) we get that (2.18) holds true. This ends the proof of Lemma 2.5.

3. The auxiliary map problem.2. We prove here analogs of the results in the preceding section but now for the more classical second equation in (0.1). The first result we prove is the following. We let $B^2_K$ be as in Lemma 2.3.

Lemma 3.1. For any $u \in H^1_R$, there exists a unique $v(u) \in H^2_R \cap C^2_R$ such that

$$a^2 \Delta^2_g v(u) + \Delta_g v(u) + m_1^2 v(u) = 4\pi q u^2.$$  \hfill (3.1)
There exist $C, C' > 0$ such that

$$
\|v(u)\|_{H^2_R} \leq C\|u\|_{L^2_R}^2, \quad \text{and}
$$

$$
\|v(u)\|_{H^2_R} \leq C'\|u\|_{L^2_R}^2
$$

(3.2)

for all $u \in H^1_R$. Moreover for any $K > 0$ there exists $C_K > 0$ such that

$$
\|v(u_2) - v(u_1)\|_{H^2_R} \leq C_K\|u_2 - u_1\|_{L^2_R}^2
$$

(3.3)

for all $u_1, u_2 \in B_K^2$, and $v : H^1_R \to H^2_R$ is locally lipschitz. There also holds that $v : H^1_R \to H^2_R$ is differentiable and for any $u \in H^1_R$ its differential $v'_u \in L(H^1_R, H^2_R)$ is given by

$$
a^2\Delta^2_v u_\mu(h) + \Delta_v u_\mu(h) + m^2 v_\mu(h) = 8\pi quh
$$

(3.4)

for all $h \in H^1_R$.

Proof of Lemma 3.1. (1) First we prove existence in $H^2_R$. Let $u \in H^1_R$ be given. We let $I : H^2_R \to \mathbb{R}$ be given by

$$
I(v) = a^2 \int_M (\Delta_g v)^2 dv_g + \int_M |\nabla v|^2 dv_g + m_1^2 \int_M v^2 dv_g
$$

and $\mathcal{H} \subset H^2_R$ be given by

$$
\mathcal{H} = \left\{ v \in H^2_R / \int_M u^2 v dv_g = 1 \right\}.
$$

Clearly $\mathcal{H} \neq \emptyset$ unless $u = 0$, but then we can take $v = 0$ as a solution for (3.1). We assume that $u$ is not identically zero and we define

$$
\mu = \inf_{v \in \mathcal{H}} I(v).
$$

Let $(v_\alpha)_\alpha$ be a minimizing sequence for $\mu$. Clearly $(v_\alpha)_\alpha$ is bounded in $H^2_R$. Then, up to passing to a subsequence, by reflexivity of $H^2_R$ and Sobolev embeddings, we may assume that $v_\alpha \to v$ in $H^2_R$ and that $v_\alpha \to v$ in $H^1_R$. Then $v \in \mathcal{H}$ and $I(v) \leq \mu$. In particular, $I(v) = \mu$ and we get that $v$ solves

$$
a^2\Delta^2 v + \Delta v + m^2 v = \mu u^2.
$$

Clearly $\mu > 0$. Letting $v(u) = \frac{4\pi q}{\mu} v$ we get that $v(u)$ solves (3.1).

(2) We prove regularity. We can rewrite (3.1) in the form

$$
(a^2\Delta_g + 1)\Delta_g v(u) = 4\pi qu^2 - m_1^2 v(u).
$$

(3.5)

Since $v(u) \in H^3_R$ and $u \in H^1_R$, the right hand side of this equation belongs to $L^3_R$. By regularity results it follows that $\Delta_g v(u) \in H^2_{R,3}$ (the Sobolev space of functions with two derivatives in $L^3$) and then that $v(u) \in H^3_{R,3}$ (the Sobolev space of functions with four derivatives in $L^3$). In particular, $v(u) \in H^3_R$. By Sobolev, $H^4_R \subset C^2_R$. It follows that $v(u) \in H^4_R \cap C^2_R$.

(3) We prove uniqueness. Let $u \in H^1_R$ be given. Suppose $v, \tilde{v} \in H^4_R \cap C^2_R$ solve (3.1). Then $w = \tilde{v} - v$ solves

$$
a^2\Delta^2 w + \Delta w + m^2 w = 0.
$$

We multiply this equation by $w$ and we integrate. It follows that $w = 0$.

(4) We prove (3.2). We multiply (3.1) by $v(u)$ and integrate. Then, by the Aubin [2] norm for $H^2_R$,

$$
\|v(u)\|_{H^2_R}^2 \leq C\|u\|_{L^2_R}^2 \|v(u)\|_{C^0_R}.
$$
and since $H^2_R \subset C^1_0$ by Sobolev, the first equation in (3.2) follows. In order to prove the second equation in (3.2) we return to (3.5). The right hand side in (3.5) is in $L^2$ (even in $L^4_R$) and if $f$ is this right hand side, by the first equation in (3.2),

$$
\|f\|_{L^2_R} \leq C(\|u\|_{L^4_R} + \|v(u)\|_{H^2_R}) \leq C(\|u\|_{L^4_R}^4 + \|u\|_{L^4_R}^2) \leq C\|u\|_{L^4_R}^4,
$$

where $C > 0$ (which changes from one inequality to another) is independent of $u$. By regularity theory, we then get that

$$
C > \|v\|_{H^2_R}
$$

for all $u \in H^R$, where $C > 0$ (which changes from one line to another) is independent of $u$. Then, again by regularity theory, we get that the second equation in (3.2) holds true.

(4) We prove (3.3). Let $K > 0$ and $u_1, u_2 \in B_K$. By (3.1), if we let $w = v(u_2) - v(u_1)$, then

$$
a^2 \Delta^2 w + \Delta g w + m^2 w = 4\pi q(u_2 - u_1) .
$$

We multiply this equation by $w$ and we integrate. Then, by Sobolev and Hölder,

$$
\|w\|_{H^2_R}^2 \leq C\|u_1 + u_2\|_{L^4_R}\|w\|_{C^1_0}\|u_2 - u_1\|_{L^2_R} \leq C_K \|w\|_{H^2_R}\|u_2 - u_1\|_{L^2_R}
$$

and (3.3) follows.

(4) We end the proof of Lemma 3.1 by proving that $v : H^1_R \to H^2_R$ is differentiable and that its differential is given by (3.4). Let $u \in H^1_R$ be given, and let $u' : H^1_R$ be given by (3.4). It is easily checked that $v'_u(h)$ exists for all $u, h \in H^1_R$, that it is unique and that $v'_u$ is linear continuous. Let $w = v(u + h) - v(u) - v'_u(h)$. Then

$$
a^2 \Delta^2 w + \Delta g w + m^2 w = 4\pi q h^2 .
$$

We multiply this equation by $w$ and we integrate. Then, by Sobolev,

$$
\|w\|_{H^2_R}^2 \leq C\|w\|_{C^1_0}\|h\|_{L^4_R}^2 \leq C\|w\|_{H^2_R}\|h\|_{H^1_R}^2
$$

where $C > 0$ (which changes from one inequality to another) is independent of $h$. Then, $\|w\|_{H^2_R} \leq C\|h\|_{H^1_R}$ and this ends the proof of Lemma 3.1.

At this point we prove the following.

**Lemma 3.2.** Let $I_2 : H^1_R \to \mathbb{R}$ be given by

$$
I_2(u) = \int_M v(u)u^2dv_g .
$$

Then $I_2$ is differentiable and

$$
I_2'(u)(h) = 4\int_M v(u)uhdv_g
$$

for all $u, h \in H^1_R$. 

\[\square\]
Proof of Lemma 3.2. We can write that $I_2$ is given by $$I_2 = \Psi \circ \Phi,$$ where

$$\Phi : \{ H^1_R \times H^1_R \times H^1_R \} \rightarrow (v(u),u,u),$$

and

$$\Psi : \{ H^2_R \times H^1_R \times H^1_R \} \rightarrow \mathbb{R} \quad (v,u,w) \rightarrow \int_M uvwdv_g.$$ 

Clearly $\Psi$ is trilinear continuous, while by Lemma 3.1 there holds that $\Phi$ is differentiable. Then $I_2$ is differentiable and by Lemma 3.1, $I'_2(u).h = \Psi'(v(u),u,u).v'(u).(h,h,h)$

$$= \Psi(v'(u),(h),u,u) + 2\Psi(v(u),h,u)$$

$$= \frac{1}{4\pi q} \int_M v'(u).(h) \left(a^2\Delta_g^2 v(u) + \Delta_g v(u) + m_1^2 v(u)\right) dv_g$$

$$+ 2 \int_M v(u)uhdv_g$$

$$= 4 \int_M v(u)uhdv_g$$

for all $u, h \in H^1_R$. This ends the proof of Lemma 3.2. $\square$

4. Recovering coercivity and regularity. We prove here the following result which gives a sign to $v$ and on which coercivity in Lemma 5.1 is based in case $\omega = 0$.

Lemma 4.1. Let $v : H^1_R \rightarrow H^2_R$ be the map in Lemma 3.1. Assuming that $am_1 < 1$ there holds that $v(u) \geq 0$ for all $u \in H^1_R$ and that there exists $\varepsilon_0 > 0$, independent of $u$, such that

$$\int_M \left(\|\nabla u\|^2 + v(u)u^2\right) dv_g \geq \varepsilon_0 \|u\|_{H^1_R}^2 \min\left(1, \|u\|_{H^1_R}^2\right)$$

for all $u \in H^1_R$.

Proof of Lemma 4.1. We assume that $am_1 < \frac{1}{2}$.

(1) We prove that $v(u) \geq 0$ for all $u \in H^1_R$. First we prove that there exists $a_1, a_2, a_3, a_4 > 0$ such that

$$a^2\Delta_g^2 + \Delta_g + m_1^2 = (a_1\Delta_g + a_2)(a_3\Delta_g + a_4).$$

(4.2)

There holds that

$$(a_1\Delta_g + a_2)(a_3\Delta_g + a_4) = a_1a_3\Delta_g^2 + (a_1a_4 + a_2a_3)\Delta_g + a_2a_4$$

and we want to solve the system

$$\begin{cases}
a_1a_3 = a^2 \\
a_1a_4 + a_2a_3 = 1 \\
a_2a_4 = m_1^2
\end{cases}$$

with $a_i > 0$ for all $i$. Then $a_3 = \frac{a^2}{a_1}$, $a_4 = \frac{m_1^2}{a_2}$ and we want that

$$m_1^2\frac{a_1}{a_2} + a^2\frac{a_2}{a_1} = 1.$$

Letting $\kappa = \frac{a_1}{a_2}$ we need to solve the equation

$$m_1^2\kappa^2 - \kappa + a^2 = 0.$$
whose discriminant is precisely $\Delta = 1 - 4a^2m^2$. By our assumption on $a$ and $m_1$, there holds that $\Delta > 0$. Letting

$$\kappa = \frac{1 + \sqrt{\Delta}}{2m^2}$$

we get (4.2) with $a_2 > 0$ fixed in an arbitrary manner, $a_1 = \kappa a_2$, $a_3 = \frac{a^2}{\kappa a_2}$ and $a_4 = \frac{m_2}{a^2}$. Now that we have (4.2), it follows from (3.1), (4.2) and the maximum principle that $a_3 \Delta u + a_4 u \geq 0$ for all $u \in H^1_R$, and then another application of the maximum principle gives that $v(u) \geq 0$ for all $u \in H^1_R$.

(2) We prove that there exists $\varepsilon_0 > 0$ such that

$$\int_M (|\nabla u|^2 + v(u)|u|^2)\,dv_g \geq \varepsilon_0$$

(4.3)

for all $u \in H^1_R$ satisfying $\|u\|_{H^1_R} = 1$. We prove (4.3) by contradiction and assume that there exists a sequence $(u_\alpha)_\alpha$ in $H^1_R$ such that $\|u_\alpha\|_{H^1_R} = 1$ for all $\alpha$ and such that

$$\int_M (|\nabla u_\alpha|^2 + v(u_\alpha)|u_\alpha|^2)\,dv_g \to \mu$$

as $\alpha \to +\infty$, where $\mu \leq 0$. There holds that $v(u_\alpha) \geq 0$ for all $\alpha$ by the above. Therefore, $\mu = 0$. Up to passing to a subsequence, there exists $u \in H^1_R$ such that $u_\alpha \to u$ in $H^1_R$ and $u_\alpha \to u$ in $L^2_R$ as $\alpha \to +\infty$. Then, by (3.3) and the Sobolev embedding theorem, $v(u_\alpha) \to v(u)$ in $C^0_R$ as $\alpha \to +\infty$. In particular,

$$\lim_{\alpha \to +\infty} \int_M v(u_\alpha)|u_\alpha|^2\,dv_g = \int_M v(u)|u|^2\,dv_g$$

and

$$\int_M |\nabla u|^2\,dv_g \leq \liminf_{\alpha \to +\infty} \int_M |\nabla u_\alpha|^2\,dv_g$$

so that

$$\int_M (|\nabla u|^2 + v(u)|u|^2)\,dv_g \leq \mu.$$ 

Since $\mu = 0$ we get that $u \equiv \lambda$ needs to be constant. Noting that

$$v(\lambda) = \frac{4\pi q\lambda^2}{m^2}$$

for all $\lambda \in \mathbb{R}$, we get that $u = 0$ and $v(u) = 0$. In particular $u_\alpha \to 0$ in $L^2_R$ as $\alpha \to +\infty$. Also, since $\mu = 0$ and $v(u) = 0$, we get that $\|\nabla u_\alpha\|_{L^2_R} \to 0$ as $\alpha \to +\infty$. This is in contradiction with the assumption that $\|u_\alpha\|_{H^1_R} = 1$ for all $\alpha$ and (4.3) is proved.

(3) We prove (4.1). There holds that $v(tu) = t^2v(u)$ for all $t \in \mathbb{R}$ and all $u \in H^1_R$. Then, for any $u \in H^1_R \setminus \{0\}$, letting $\tilde{u} = \|u\|_{H^1_R}^{-1}u$, we get with (4.3) that

$$\int_M (|\nabla u|^2 + v(u)|u|^2)\,dv_g$$

$$= \|u\|^2_{H^1_R} \int_M |\nabla \tilde{u}|^2\,dv_g + \|u\|_{H^1_R}^4 \int_M v(\tilde{u})\tilde{u}^2\,dv_g$$

$$\geq \|u\|^2_{H^1_R} \min \left(1, \|u\|^2_{H^1_R}\right) \int_M (|\nabla \tilde{u}|^2 + v(\tilde{u})\tilde{u}^2)\,dv_g$$
Proof of Lemma 4.3. Uniqueness in Lemmas 2.2 and 3.1 implies that $u_{\alpha} > 0$ everywhere in $M$. Suppose by contradiction that the infimum is zero. Then there exists $u \in H^1_R$ such that $u_{\alpha} \to u$ in $L^2_R$ and $u_{\alpha} \to u$ in $H^1_R$ as $\alpha \to +\infty$. Obviously $\|\nabla u_{\alpha}\|_{L^2_R} \to 0$ as $\alpha \to +\infty$. Thus $\nabla u \equiv 0$ and $u \equiv \lambda$ is a constant. Then
\[
\lambda^2 \int_M |\nabla S|^2 dv_g = 0
\]
and this implies that $\lambda = 0$. However $u_{\alpha} \in S^1_R$ for all $\alpha$, and we get the desired contradiction. Lemma 4.2 is proved.

Another coercivity result we will need is the following.

**Lemma 4.2.** Suppose $S \in C^\infty_R$ is not constant. Then $\Delta_g + |\nabla S|^2$ is coercive.

**Proof of Lemma 4.2.** Let $S \in C^\infty_R$ be such that $S$ is not constant, and thus such that $\nabla S \neq 0$. It suffices to prove that
\[
\inf_{u \in S^1_R} \int_M \left( |\nabla u|^2 + |\nabla S|^2 u^2 \right) dv_g > 0 ,
\]
where $S^1_R$ is the unit sphere with center 0 in $H^1_R$. Let $(u_\alpha)_\alpha$ be a minimizing sequence for the above infimum. Suppose by contradiction that the infimum is zero. Then there exists $u \in H^1_R$ such that $u_\alpha \to u$ in $L^2_R$ and $u_\alpha \to u$ in $H^1_R$ as $\alpha \to +\infty$. Obviously $\|\nabla u_\alpha\|_{L^2_R} \to 0$ as $\alpha \to +\infty$. Thus $\nabla u \equiv 0$ and $u \equiv \lambda$ is a constant. Then
\[
\lambda^2 \int_M |\nabla S|^2 dv_g = 0
\]
and this implies that $\lambda = 0$. However $u_\alpha \in S^1_R$ for all $\alpha$, and we get the desired contradiction. Lemma 4.2 is proved.

The following regularity Lemma will be used in several points in the sequel without necessarily being mentioned.

**Lemma 4.3.** Let $(u, v, A) \in H^1_R \times H^2_R \times H^2_V$ be a solution of (0.1) with $u \geq 0$ in $M$. Then $v = v(u)$, $A = A(u)$, $u, v, A$ are $C^2$ and either $u > 0$ everywhere in $M$ or $u \equiv 0$.

**Proof of Lemma 4.3.** Uniqueness in Lemmas 2.2 and 3.1 implies that $v = v(u)$ and $A = A(u)$, while the estimates in these lemmas together with the Sobolev embedding theorem imply that $v \in C^2_V$, that $A \in C^2_V$ and that $\Phi = \Phi(\cdot, v, A)$ is also $C^2$. By regularity theory as in Gilbarg and Trudinger [22], and the Trudinger regularity argument [51] in the critical case, we then get that $u \in C^2_R$. By the maximum principle, either $u \equiv 0$ or $u > 0$ in $M$. This ends the proof of Lemma 4.3.

5. The variational analysis of the problem. We prove the lemma below which highly simplifies the variational setting for our equation.

**Lemma 5.1.** Let $2 < p \leq 6$. We assume that $am_1 < \frac{1}{2}$. Define $I_p : H^1_R \to \mathbb{R}$ to be the functional
\[
I_p(u) = \frac{\kappa^2}{4m_0^2} \int_M |\nabla u|^2 dv_g + \frac{\omega^2}{2} \int_M u^2 dv_g \\
+ \frac{q}{4} \int_M v(u) u^2 dv_g + \frac{\kappa^2}{4m_0^2} \int_M \left( |\nabla S - \frac{q}{m} A(u), \nabla S| u^2 dv_g \right) \\
- \frac{1}{p} \int_M (u^+)^p dv_g ,
\]
where $A$ and $v$ are as in Lemmas 2.2 and 3.1, and where $u^+ = \max(0, u)$. The functional $I_p$ is differentiable and if $u$ is a critical point of $I_p$, then $(u, v(u), A(u))$ is a smooth solution of (0.1) with $u, v \geq 0$. 

This proves (4.1). In particular, Lemma 4.1 is proved. □
Proof of Lemma 5.1. The functional $I_p$ is differentiable and by Lemmas 2.5 and 3.2 we get that if $u \in H^1_R$ is a critical point of $I_p$ then $u$ is a weak solution of

$$\Delta_g u + \Phi(x,v,A)u = (u^+)^{p-1},$$

(5.2)

where $v = v(u)$ and $A = A(u)$. By the maximum principle, and the decomposition (4.2), $v(u) \geq v(0^-)$. Then, multiplying (5.2) by $u^-$ and integrating over $M$, it follows from the coercivity in Lemma 4.1 that $u^- \equiv 0$. Hence, $u$ solves the first equation in (0.1), while by construction $v = v(u)$ solves the second equation in (0.1) and $A = A(u)$ solves the third equation in (0.1). Still by Lemma 4.1, there holds that $v \geq 0$. By regularity theory as in Gilbarg and Trudinger [22], and the Trudinger regularity argument [51] for stationary critical Schrödinger equations applied to the first equation in (0.1), it is easily seen that $\Phi \in C^1_R$ by Lemmas 2.2 and 3.1, we get that $u$ is in $C^\infty_R$ for all $\theta \in (0,1)$, and then that $u \in C^{2,\theta}_R$ for all $\theta \in (0,1)$. By the maximum principle, either $u \equiv 0$ or $u > 0$ in $M$. If $u \equiv 0$, then $v \equiv 0$ and $A \equiv 0$. In case $u > 0$ in $M$, bootstrapping on the three equations of (0.1), we then get that $u$, $v$ and $A$ are smooth. This ends the proof of Lemma 5.1.

6. Generic convergence. For the sake of clearness, we prove the following generic convergence result in this section.

Lemma 6.1. Let $(M,g)$ be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m, m_1 > 0$ be positive real numbers, $p \in (4,6]$ and $S \in C^\infty(M)$ be a smooth real-valued function. Let $(\omega_\alpha)_{\alpha}$ be sequence of real numbers converging to $\omega$, let $(p_\alpha)_{\alpha}$ be a sequence of powers in (4,6] converging to $p$ and let $(S_\alpha)_{\alpha}$ be a sequence in $C^\infty$ which converges to $S$ in $C^{1,\theta}_R$ as $\alpha \to +\infty$ for some $\theta \in (0,1)$. Let $(u_\alpha, v_\alpha, A_\alpha)_{\alpha}$ be a sequence of solutions of

$$\begin{align*}
\frac{k^2}{2m_0^2} \Delta_g u_\alpha + \Phi_\alpha(x,v_\alpha,A_\alpha)u_\alpha &= u_\alpha^{p_\alpha-1} \\
a^2 \Delta_g^2 v_\alpha + \Delta_g v_\alpha + m_1^2 v_\alpha &= 4\pi q u_\alpha \\
a^2 \Delta_g^2 A_\alpha + \Delta_g A_\alpha + m_1^2 A_\alpha &= \frac{4\pi q}{m_0} \Psi(A_\alpha,S_\alpha) \nu_\alpha^2,
\end{align*}$$

(6.1)

where $\Phi_\alpha(x,v,A) = \frac{k^2}{2m_0^2} |\Psi(A,S)|^2 + u_\alpha^2 + q v_\alpha$, $\Psi(A,S) = \nabla S - \frac{q}{4} A$ and $u_\alpha > 0$ in $M$. Assume $\|u_\alpha\|_{L^\infty_R} = O(1)$. Then, up to passing to a subsequence, $u_\alpha \to u$, $v_\alpha \to v$ in $C^2_R$ and $A_\alpha \to A$ in $C^2_R$ as $\alpha \to +\infty$, for some $u, v \in C^2_R$ and $A \in C^2_R$ which solve (0.1). Moreover, if $am_1 < \frac{1}{2}$, then we also have that $u > 0$ and $v > 0$ in $M$. In addition, when $u > 0$ in $M$, there holds that $A \neq 0$ if $S$ is not constant.

Proof of Lemma 6.1. By Lemmas 2.2 and 3.1 (see also the remark after Lemma 2.2), and by the Sobolev embedding theorem, there holds that, up to passing to a subsequence, $(A_\alpha)_{\alpha}$ and $(v_\alpha)_{\alpha}$ converge in $C^0_V$ and $C^2_R$ as $\alpha \to +\infty$. In particular, $(\Phi_\alpha)_{\alpha}$ converges in $C^0_R$ as $\alpha \to +\infty$ where $\Phi_\alpha = \Phi(x,v_\alpha,A_\alpha)$. Then, by elliptic regularity, $(u_\alpha)_{\alpha}$ is bounded in the Sobolev space $H^{2,s}_R$ for all $s > 1$, where $H^{2,s}_R$ refers to the Sobolev space of functions with two derivatives in $L^s_R$. Again by elliptic regularity we can bound $(u_\alpha)_{\alpha}$ in $C^{2,\theta}_R$ and, up to passing to a subsequence, we may also assume that $(u_\alpha)_{\alpha}$ converges in $C^2_R$ as $\alpha \to +\infty$. If $u$, $v$ and $A$ are the limits of $(u_\alpha)_{\alpha}$, $(v_\alpha)_{\alpha}$ and $(A_\alpha)_{\alpha}$, it is easily seen that $(u,v,A)$ solve (0.1). It remains to prove that $u > 0$ when $am_1 < \frac{1}{2}$. Suppose $am_1 < \frac{1}{2}$. Writing that

$$\int_M \left( \frac{k^2}{2m_0^2} |\nabla u_\alpha|^2 + q v_\alpha u_\alpha^2 \right) dv_g \leq \frac{k^2}{2m_0^2} \int_M |\nabla u_\alpha|^2 dv_g + \int_M \Phi_\alpha u_\alpha^2 dv_g.$$
for some $C > 0$ independent of $\alpha$ and for all $\alpha$, we get with Lemma 4.1 that
\[
\|u_\alpha\|_{H^1_R}^2 \min \left(1, \|u_\alpha\|_{H^1_R}^2 \right) \leq C\|u_\alpha\|_{H^1_R}^{p_0}
\]
for some $C > 0$ independent of $\alpha$ and for all $\alpha$. Since $p_\alpha \to p$ with $p > 4$ this implies
that there exists $\varepsilon_1 > 0$ such that $\|u_\alpha\|_{H^1_R} \geq \varepsilon_1$ for all $\alpha$. In particular $u \not= 0$, and
by the maximum principle we get that $u > 0$ in $M$. Still by Lemma 4.1, there holds that $v \geq 0$ and $v \not= 0$. Let $a_1, \ldots, a_4 > 0$ be as in the proof of Lemma 4.1. Then
\[
(a_1\Delta_g + a_2)(a_3\Delta_g + a_4) v = 4\pi u^2
\]
and, again by the maximum principle, we get that $(a_3\Delta_g + a_4) v \geq 0$ and then that $v > 0$ in $M$. At last, it is easily checked that $A \not= 0$ if $\nabla S \not= 0$ when $u > 0$ in $M$. The lemma follows. \hfill \Box

7. Existence in the subcritical case. First we prove Theorem 0.1 in the subcritical case and when $\omega \not= 0$ or $\nabla S \equiv 0$. We assume in what follows that $p \in (4, 6)$ and that $am_1 < \frac{1}{2}$. We exhibit a solution with a mountain pass structure in that case.

Proof of Theorem 0.1 in the subcritical case when $\omega \not= 0$ or $\nabla S \equiv 0$. Fix $p \in (4, 6)$. Suppose $\omega \not= 0$. We apply the mountain pass lemma in Ambrosetti-Rabinowitz [1] to the functional $I_p$ in Lemma 5.1. There holds that $v(u) \geq 0$ for all $u \in H^1_R$ by Lemma 4.1. By Lemma 2.2 and the Sobolev inequalities, assuming that $\omega \not= 0$, we can then write that
\[
I_p(u) \geq (C_1 - \varepsilon(u))\|u\|_{H^1_R}^2 - C_2\|u\|_{H^1_R}^p
\]
for all $u \in H^1_R$ where $\varepsilon(u) \to 0$ as $\|u\|_{L^2_R} \to 0$ and where the $C_i$’s are positive constants independent of $u$. In particular, there exist $\delta > 0$ sufficiently small and $\rho_0 > 0$ such that
\[
I_p(u) \geq \rho_0 \quad (7.1)
\]
for all $u \in L^1_R$ such that $\|u\|_{H^1_R} = \delta$. In parallel, if $\nabla S \equiv 0$, then $A(u) \equiv 0$ for all $u$, and by Lemma 4.1 we get that
\[
I_p(u) \geq C_1\|u\|_{H^1_R}^4 - C_2\|u\|_{H^1_R}^p
\]
for all $u \in L^1_R$ such that $\|u\|_{H^1_R} < 1$, where the $C_i$’s are positive constants independent of $u$. Here again, since $p > 4$, we get that there exist $\delta > 0$ sufficiently small and $\rho_0 > 0$ such that (7.1) holds true for all $u \in H^1_R$ such that $\|u\|_{H^1_R} = \delta$. There holds that $I_p(0) = 0$. Fix $u_0 > 0$ smooth. By Lemma 2.2, $\|A(tu_0)\|_{L^p_R} \leq C t^2 \|u_0\|_{L^2_R}$. In particular, we can write that $I_p(tu_0) \to -\infty$ as $t \to +\infty$. Then, $I_p(T_0 u_0) < \rho_0$ and $\|T_0 u_0\|_{H^1_R} \geq 1$ for $T_0 \gg 1$. Therefore, we can apply the mountain pass lemma in Ambrosetti-Rabinowitz [1] to $I_p$ and we get that there exists a Palais-Smale sequence $(u_\alpha)_{\alpha}$ for $I_p$. Define
\[
c_p = \inf_{\gamma \in \mathcal{P}} \sup_{u \in \gamma} I_p(u), \quad (7.2)
\]
where $\mathcal{P}$ is the set of all continuous paths joining 0 to $T_0u_0$. By (7.1), $c_p > 0$.

Independently, there holds that $DI_p(u_\alpha), (u_\alpha^-) = o\left(\|u_\alpha^-\|_{H^1}^2\right)$ so that

$$
\frac{h^2}{2m_0^2} \int_M |\nabla u_\alpha^-|^2 dv_g + \omega^2 \int_M (u_\alpha^-)^2 dv_g + q \int_M v(u_\alpha)(u_\alpha^-)^2 dv_g
+ \frac{h^2}{2m_0^2} \int_M \nabla S - \frac{q}{\bar{h}} A(u_\alpha)(u_\alpha^-)^2 dv_g = o\left(\|u_\alpha^-\|_{H^1}^2\right)
$$

(7.3)

for all $\alpha$. If $\omega \neq 0$, we get from (7.3) that $\|u_\alpha^-\|_{H^1}^2 = o\left(\|u_\alpha^-\|_{H^1}^2\right)$. Then

$$
\|u_\alpha^-\|_{H^1} \to 0
$$

as $\alpha \to +\infty$. In case $\omega = 0$, and thus $\nabla S \equiv 0$, we remark that $(u_\alpha^-)^2 \leq u_\alpha^2$ in $M$. Therefore, by (4.2), $v(u_\alpha) \geq v(u_\alpha^-) \geq 0$. Then, by (7.3), we get that

$$
\frac{h^2}{2m_0^2} \int_M |\nabla u_\alpha^-|^2 dv_g + q \int_M v(u_\alpha^-)(u_\alpha^-)^2 dv_g = o\left(\|u_\alpha^-\|_{H^1}^2\right)
$$

(7.5)

and it follows from Lemma 4.1 that, here again, (7.4) holds true. Now, since $(u_\alpha)_\alpha$ is a Palais-Smale sequence for $I_p$, there holds that $I_p(u_\alpha) = c_p + o(1)$ and that $DI_p(u_\alpha), (u_\alpha) = o\left(\|u_\alpha\|_{H^1}^2\right)$. Combining these two equations in the classical Brézis and Nirenberg [8] way, we get that

$$
\left(\frac{1}{2} - \frac{1}{p}\right) \|u_\alpha^+\|_{L^p_H}^p = \frac{q}{4} \int_M v(u_\alpha)u_\alpha^2 dv_g + c_p + o(1) + o\left(\|u_\alpha\|_{H^1}^2\right)
$$

$$
- \frac{qh}{4m_0^2} \int_M \left(\nabla S - \frac{q}{\bar{h}} A(u_\alpha), A(u_\alpha)\right) u_\alpha^2 dv_g.
$$

(6.7)

By the Sobolev embedding theorem, by (2.4) in Lemma 2.2, and by (3.2) in Lemma 3.1, we get from (6.6) that

$$
\frac{p-2}{2p} \|u_\alpha^+\|_{L^p_H}^p \leq C \|u_\alpha\|_{L^2_H}^4 + C \|A(u_\alpha)\|_{L^p_H} \|u_\alpha\|_{L^2_H}^2
$$

$$
+ C \|A(u_\alpha)\|_{L^p_H} \|u_\alpha\|_{L^2_H}^2 + c_p + o(1) + o\left(\|u_\alpha\|_{H^1}^2\right)
$$

(7.7)

where $C > 0$ changes from line to line (but remains independent of $\alpha$). Then, by (7.4), we can write with (7.7) that

$$
\|u_\alpha^+\|_{L^p_H}^p \leq C \left(\|u_\alpha^-\|_{L^2_H}^4 + 1\right) + o\left(\|u_\alpha\|_{H^1}^2\right)
$$

where $C > 0$ is independent of $\alpha$. Since $p > 4$ it follows that

$$
\|u_\alpha^+\|_{L^p_H}^p = O(1) + o\left(\|u_\alpha\|_{H^1}^2\right).
$$

(7.8)

There holds that $DI_p(u_\alpha), (u_\alpha) = o\left(\|u_\alpha\|_{H^1}^2\right)$. Then, by (7.8),

$$
\frac{h^2}{2m_0^2} \int_M |\nabla u_\alpha^+|^2 dv_g + \omega^2 \int_M u_\alpha^2 dv_g + q \int_M v(u_\alpha)u_\alpha^2 dv_g
$$

$$
+ \frac{h^2}{2m_0^2} \int_M |\nabla S - \frac{q}{\bar{h}} A(u_\alpha)|^2 u_\alpha^2 dv_g = O(1) + o\left(\|u_\alpha\|_{H^1}^2\right)
$$

(7.9)
and it follows from (7.9) and Lemma 4.1 that

$$\|u_\alpha\|_{H^1_R} = O(1).$$

Then, up to passing to a subsequence, since $p$ is subcritical, there exists $u \in H^1_R$ such that

$$u_\alpha \rightharpoonup u \text{ in } H^1_R; \quad \text{and } u_\alpha \to u \text{ in } L^2_0 \cap L^p_0 \text{ and } u_\alpha \to u \text{ a.e. as } \alpha \to +\infty.$$ 

By (7.4), $u \geq 0$. For any $h \in H^1_R$, $DIP(u_\alpha).\ (h) = o(1)$. Thus, for any $h \in H^1_R$,

$$\frac{h^2}{2m_0} \int_M (\nabla u_\alpha, \nabla h) \ dv_g + \omega^2 \int_M u_\alpha \cdot h \ dv_g + q \int_M v(u_\alpha)u_\alpha \cdot h \ dv_g$$

$$+ \frac{h^2}{2m_0} \int_M |\nabla S - \frac{q}{h} A(u_\alpha)|^2 u_\alpha \cdot h \ dv_g - \int_M (u_\alpha)_+^{p-1} h \ dv_g = o(1).$$

(7.11)

Since

$$||X|^2 - |Y|^2| \leq |X - Y||(|X| + |Y|),$$

we can write with (7.10), the Sobolev embedding theorem and Lemma 2.2 that

$$\left| \int_M |\nabla S - \frac{q}{h} A(u_\alpha)|^2 u_\alpha \cdot h \ dv_g - \int_M |\nabla S - \frac{q}{h} A(u)|^2 u \cdot h \ dv_g \right|$$

$$\leq \int_M |\nabla S - \frac{q}{h} A(u)|^2 u_\alpha - u \cdot ||h| \ dv_g$$

$$+ \int_M \left| |\nabla S - \frac{q}{h} A(u_\alpha)|^2 - |\nabla S - \frac{q}{h} A(u)|^2 \right| |u_\alpha|||h| \ dv_g$$

$$\leq C_1 \int_M |u_\alpha - u||h| \ dv_g + C_2 \int_M |A(u_\alpha) - A(u)||u_\alpha|||h| \ dv_g$$

(7.12)

for all $\alpha$, where $C_1, C_2 > 0$ are independent of $\alpha$. By Lemma 2.3 we then get from (7.12) that for any $h \in H^1_R$,

$$\lim_{\alpha \to +\infty} \int_M |\nabla S - \frac{q}{h} A(u_\alpha)|^2 u_\alpha \cdot h \ dv_g = \int_M |\nabla S - \frac{q}{h} A(u)|^2 u \cdot h \ dv_g.$$ 

By Lemma 3.1, we also have that

$$\lim_{\alpha \to +\infty} \int_M v(u_\alpha)u_\alpha \cdot h \ dv_g = \int_M v(u) \cdot h \ dv_g.$$ 

Coming back to (7.11), passing to the limit in $\alpha$, we get that $(u, v(u), A(u))$ solves (0.1). By Lemma 4.3, $u, v(u)$ and $A(u)$ are $C^2$ and either $u \equiv 0$ or $u > 0$ in $M$. Obviously $u \not\equiv 0$ by (7.6) since $c_p > 0$. Then $u > 0$. By Lemma 4.1 there also holds that $v \geq 0$. Obviously $v \not\equiv 0$ since $u \not\equiv 0$. Here again, see the proof of Lemma 6.1, we let $a_1, \ldots, a_4 > 0$ be as in the proof of Lemma 4.1. Then

$$(a_1 \Delta u + a_2)(a_3 \Delta u + a_4) v = 4\pi qu^2.$$ 

By the maximum principle, we get that $(a_3 \Delta u + a_4) v \geq 0$ and then that $v > 0$ in $M$. At last, it is easily checked that $A \not\equiv 0$ if $\nabla S \not\equiv 0$. This proves Theorem 0.1 in the subcritical case when $\omega \not\equiv 0$ or $\nabla S \equiv 0$. 

Now we prove Theorem 0.1 in the subcritical case when $\omega = 0$ using the stability in Lemma 9.1 and what we just proved.

**Proof of Theorem 0.1 in the subcritical case when $\omega = 0$.** Let $(\omega_\alpha)_\alpha$ be a sequence in $\mathbb{R}^*$ converging to 0. According to what we just proved we do have a sequence
\((u_{\alpha}, v_{\alpha}, A_{\alpha})\) of solutions to

\[
\begin{align*}
\left\{
\frac{\kappa^2}{2m_0^2} \Delta_g u_{\alpha} + \Phi_\alpha(x, v_{\alpha}, A_{\alpha}) u_{\alpha} &= u_{\alpha}^{p_{\alpha}-1} \\
\alpha^2 \Delta_g^2 v_{\alpha} + \Delta_g v_{\alpha} + m_1^2 v_{\alpha} &= 4\pi q u_{\alpha}^2 \\
\alpha^2 \Delta_g A_{\alpha} + \Delta_g A_{\alpha} + m_1^2 A_{\alpha} &= \frac{4\pi q}{m_0} \Psi(A_{\alpha}, S) u_{\alpha}^2,
\end{align*}
\]

where \(\Phi_\alpha(x, v, A) = \frac{\kappa^2}{2m_0^2} |\Psi(A, S)|^2 + \omega_\alpha^2 qv, \Psi(A, S) = \nabla S - \frac{q}{R} A\) and \(u_{\alpha} > 0\) in \(M\). By Lemma 9.1, up to passing to a subsequence, \(u_{\alpha} \to u, v_{\alpha} \to v\) in \(C^2_R\) and \(\alpha \to +\infty\), for some \(u, v \in C^2_R\) which solve (0.1). Moreover, if \(am_1 < \frac{1}{4}\), then we also have that \(u > 0\) and \(v > 0\) in \(M\). And there holds that \(A \neq 0\) if \(S\) is not constant. This ends the proof of Theorem 0.1 in the subcritical case when \(\omega = 0\).

8. **Existence in the critical case.** As a preliminary remark, we can get existence in the critical case from the existence in the subcritical case coupled with stability results such as Proposition 10.1. Instead we choose here to produce mountain-pass type solutions to (0.1) as in the subcritical case. We assume in what follows that \(p = 6, (0.2)\), and that \(am_1 < \frac{1}{2}\).

By the maximum principle, up to replacing \(\Lambda_g\) by \((1 - \varepsilon)\Lambda_g\), with \(0 < \varepsilon \ll 1\), we can assume that \(\Delta_g + \Lambda_g\) has positive mass. Let \(G\) be the Green’s function of \(\Delta_g + \Lambda_g\). Let \(x_0\) be given in \(M\) and \(G(x) = G(x_0, x)\). In geodesic normal coordinates,

\[
G(x) = \frac{1}{\omega_2 |x|} + A + \alpha(x),
\]

where \(\omega_2\) is the volume of the unit 2-sphere, \(A > 0\) is the mass at \(x_0\) and \(\alpha(x) = O(|x|)\). Following Schoen [40], we let \(\rho_0 > 0\) be a small radius and \(\varepsilon_0 > 0\) to be chosen small relative to \(\rho_0\). Let also \(\psi\) be a piecewise smooth decreasing function of \(|x|\) such that \(\psi(x) = 1\) for \(|x| \leq \rho_0\), \(\psi(x) = 0\) for \(|x| \geq 2\rho_0\), and \(|\nabla \psi| \leq 2\rho_0^{-1}\) for \(\rho_0 \leq |x| \leq 2\rho_0\). We define \(u_\varepsilon, \varepsilon > 0\), by

\[
\left\{
\begin{align*}
u_\varepsilon(x) &= \left(\frac{\varepsilon}{\varepsilon + d_g(x_0, x)}\right)^{1/2} \text{ for } d_g(x_0, x) \leq \rho_0, \\
u_\varepsilon(x) &= \varepsilon_0 (G(x) - \psi(x) \alpha(x)) \text{ for } \rho_0 \leq d_g(x_0, x) \leq 2\rho_0, \\
u_\varepsilon(x) &= \varepsilon_0 G(x) \text{ for } d_g(x_0, x) \geq 2\rho_0,
\end{align*}\right.
\]

where \(d_g\) is the Riemannian distance, and we require that

\[
\varepsilon_0 \left(\frac{1}{\omega_2 \rho_0} + A\right) = \sqrt{\frac{\varepsilon}{\varepsilon^2 + \rho_0^2}}.
\]

Computing as in Schoen [40] we get that

\[
\int_M \left(\frac{1}{2m_0^2} |\nabla u_\varepsilon|^2 + \Lambda_g u_\varepsilon^2\right) dv_g < \frac{1}{K_3^2}
\]

for \(\varepsilon \ll 1\), where \(K_3\) is the sharp constant in the Euclidean Sobolev inequality for \(H^1_R(\mathbb{R}^3)\). Also there holds that

\[
\int_M u_\varepsilon^6 dv_g = \int_{\mathbb{R}^3} \left(\frac{1}{1 + |x|^2}\right)^3 dx + o(1).
\]
In particular, since $u_\varepsilon \to 0$ a.e. as $\varepsilon \to 0$, we get from (8.3), (8.4) and the compactness of the embeddings $H^1_R \subset L^p_R$ for $p < 6$ that $u_\varepsilon \to 0$ in $L^p_R$ for all $p < 6$ as $\varepsilon \to 0$. Given $u_0 \in H^1_R$, we define

$$c_6(u_0) = \inf_{P \in \mathcal{P}} \max_{u \in P} I_6(u),$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to $u_0$ and $I_6$ is the functional in (5.1) in the critical case $p = 6$. The following lemma holds true.

**Lemma 8.1.** Suppose $\omega \neq 0$ or $\nabla S \equiv 0$. There exist $u_0 \in H^1_R$, with $u_0^+ \neq 0$, and some $\delta_0 > 0$, such that $I_6(u_0) < 0$ and

$$\delta_0 \leq c_6(u_0) \leq \frac{\sqrt{2h^3}}{12m_0^3K_3} - \delta_0,$$

where $K_3$ is the sharp constant in the Euclidean Sobolev inequality for $\dot{H}^1_R(\mathbb{R}^3)$.

**Proof of Lemma 8.1.** We let $(u_\varepsilon)_\varepsilon$ be as in (8.2). By Lemma 2.2 there holds that $|A(tu_\varepsilon)| \leq \varepsilon^2 \|u_\varepsilon\|_2^2$ for all $t$ and all $\varepsilon$. Similarly, by Lemma 3.1, $|v(tu_\varepsilon)| \leq \varepsilon^2 \|u_\varepsilon\|_2^2$ for all $t$ and all $\varepsilon$. In particular, thanks to (8.3) and (8.4) there exists $T_0 > 1$ such that $I_6(T_0u_\varepsilon) < 0$ for all $0 < \varepsilon \ll 1$. We fix such a $T_0$ > 1. Since $u_\varepsilon \to 0$ in $L^p_R$ as $\varepsilon \to 0$, and all $u_\varepsilon$, there exists $\delta_0$ > 0, we get

$$\max_{0 \leq t \leq T_0} \|A(tu_\varepsilon)\|_{L^\infty_R} + \max_{0 \leq t \leq T_0} \|v(tu_\varepsilon)\|_{L^\infty_R} \to 0$$

as $\varepsilon \to 0$. Then, by (0.2) and (8.7),

$$\max_{0 \leq t \leq T_0} I_6(tu_\varepsilon) \leq \max_{0 \leq t \leq T_0} J(tu_\varepsilon)$$

for all $0 < \varepsilon \ll 1$, where

$$J(u) = \frac{h^3}{4m_0^3} \int_M \left( |\nabla u|^2 dv_g + \Lambda g u^2 \right) dv_g - \frac{1}{6} \int_M |u|^6 dv_g$$

for $u \in H^1_R$. Differentiating $J(tu_\varepsilon)$ with respect to $t$, we get that

$$\max_{0 \leq t \leq T_0} J(tu_\varepsilon) = \frac{\sqrt{2h^3}}{12m_0^3} \left( \int_M \left( |\nabla u_\varepsilon|^2 + \Lambda g u_\varepsilon^2 \right) dv_g \right)^{3/2}$$

for all $0 < \varepsilon \ll 1$. By (8.3), letting $u_0 = T_0u_\varepsilon$ for $\varepsilon > 0$ sufficiently small, we get that

$$c_6(u_0) \leq \frac{\sqrt{2h^3}}{12m_0^3K_3} - \delta_0$$

for some $\delta_0 > 0$. Concerning the left hand side inequality in (8.6), we proceed in the following way. There holds that $v(u) \geq 0$ for all $u \in H^1_R$ by Lemma 4.1. By Lemma 2.2 and the Sobolev inequalities, if $\omega \neq 0$ we can write that

$$I_6(u) \geq (C_1 - \varepsilon(u)) \|u\|_{H^1_R}^2 - C_2\|u\|_{H^1_R}^6$$

for all $u \in H^1_R$, where $\varepsilon(u) \to 0$ as $\|u\|_{L^2_R} \to 0$ and where the $C_i$’s are positive constants independent of $u$. In case $\nabla S \equiv 0$, and possibly $\omega = 0$, then $A(u) \equiv 0$ for all $u \in H^1_R$. In particular, by Lemma 4.1,

$$I_6(u) \geq C_1\|u\|_{H^1_R}^4 - C_2\|u\|_{H^1_R}^6$$

for all $u \in H^1_R$ such that $\|u\|_{H^1_R} < 1$, where here again the $C_i$’s are positive constants independent of $u$. It follows from (8.9) and (8.10) that there exists $\delta > 0$ (as small
as we want) and $C_\delta > 0$ such that $I_6(u) \geq C_\delta$ for all $u \in H^1_R$ with $\|u\|_{H^1_R} = \delta$. In particular, (8.6) holds true for $\delta_0 > 0$ sufficiently small, and this proves Lemma 8.1.

At this point we can prove Theorem 0.1 when $\omega \neq 0$ or $\nabla S \equiv 0$.

Proof of Theorem 0.1 in the critical case when $\omega \neq 0$ or $\nabla S \equiv 0$. We fix $u_0 \in H^1_R$ as in Lemma 8.1. We let $c_6 = c_6(u_0)$. By (8.9) and (8.10) we can apply the mountain pass lemma in Ambrosetti and Rabinowitz [1] and we get that there exists a Palais-Smale sequence $(u_\alpha)_\alpha$ for $I_6$ at the level $c_6$. By (8.6), $c_6 > 0$. Writing that $DI_6(u_\alpha)(u_\alpha) = o\left(\|u_\alpha\|_{H^1_R}\right)$, we get as in the subcritical case that $\|u_\alpha\|_{H^1_R} = o(1)$.

Proof of Theorem 0.1 in the critical case when $\omega$. By the sharp Sobolev inequality, see Hebey and Vaugon [28, 29], $\|u_\alpha\|_{H^1_R} = o(1)$. Still as in the proof of the subcritical case it follows that $\|u_\alpha\|_{H^1_R} = O(1)$. Then there exists $u \in H^1_R$ such that $u_\alpha \rightharpoonup u$ in $H^1_R$, $u_\alpha \rightarrow u$ in $L^2$ and $u_\alpha \rightarrow u$ a.e as $\alpha \rightarrow +\infty$. There holds that $u \geq 0$, and for any $h \in L^2$, $DI_6(u_\alpha)(h) = o(1)$. By the weak convergence, the strong convergence in $L^2$ and Lemmas 2.3 and 3.1, we get that $(u, v(u), A(u))$ solves (0.1). By Lemma 4.3, $u, v(u)$ and $A(u)$ are $C^2$ and either $u \equiv 0$ or $u > 0$ in $M$. By Lemma 4.1 there also holds that $v \geq 0$. Obviously $v \neq 0$ if $u \neq 0$. Here again, see the proof of Lemma 6.1, we let $a_1, \ldots, a_4 > 0$ be as in the proof of Lemma 4.1. Then

$$(a_1\Delta_g + a_2)(a_3\Delta_g + a_4) v = 4\pi q u^2.$$ 

By the maximum principle, we get that $(a_3\Delta_g + a_4) v \geq 0$ and then that $v > 0$ in $M$ if $u > 0$ in $M$. At last, it is easily checked that $A \neq 0$ if $\nabla S \neq 0$ when $u > 0$ in $M$. In other words, it remains to prove that $u \neq 0$. We assume by contradiction that $u \equiv 0$. By Lemmas 2.2 and 3.1 and by standard manipulations on the Palais-Smale equations, we then get that

$$\begin{cases}
\frac{K^2}{2m_0^2} \int_M |\nabla u_\alpha|^2 \nu_g = \int_M |u_\alpha|^6 \nu_g + o(1) \\
\frac{1}{2} \int_M |u_\alpha|^6 \nu_g = c_6 + o(1).
\end{cases}$$ (8.11)

By the sharp Sobolev inequality, see Hebey and Vaugon [28, 29],

$$\|u_\alpha\|_{L^2_R}^2 \leq K_3^2 \|\nabla u_\alpha\|_{L^2_R}^2 + o(1).$$ (8.12)

Combining (8.11) and (8.12) we get that

$$c_6 \geq \frac{\sqrt{2}h^3}{12m_0^3K_3^3},$$

a contradiction with our choice of $u_0$. Hence $u \neq 0$ and this ends the proof of Theorem 0.1 in the critical case when $\omega \neq 0$ or $\nabla S \equiv 0$.

It remains to prove Theorem 0.1 in the critical case when $\omega = 0$ and $S$ is not constant. The sole condition that $S$ is not constant will be used here.

Proof of Theorem 0.1 in the critical case when $S$ is not constant. Let $(\omega_\alpha)_\alpha$ be a sequence in $\mathbb{R}^*$ converging to 0. According to what we just proved we do have a sequence $(u_\alpha, v_\alpha, A_\alpha)$ of solutions to

$$\begin{cases}
\frac{K^2}{2m_0^2} \Delta_g u_\alpha + \Phi_\alpha(x, v_\alpha, A_\alpha) u_\alpha = u_\alpha^{p-1} \\
a^2 \Delta_g v_\alpha + \Delta_g u_\alpha + m^2 v_\alpha = 4\pi q u_\alpha^2 \\
a^2 \Delta_g A_\alpha + \Delta_g u_\alpha + m^2 A_\alpha = \frac{4\pi q h}{m^6} \Psi(A_\alpha, S) u_\alpha^2,
\end{cases}$$
where \( \Phi_\alpha(x,v,A) = \frac{\hbar^2}{2m_0^2} |\Psi(A,S)|^2 + \omega_\alpha^2 + qv, \) \( \Psi(A,S) = \nabla S - \frac{q}{\hbar} A \) and \( u_\alpha > 0 \) in \( M. \) We assume that \( S \) is not constant. By Proposition 10.1, up to passing to a subsequence, \( u_\alpha \to u, v_\alpha \to v \) in \( C_R^2 \) and \( A_\alpha \to A \) in \( C_V^2 \) as \( \alpha \to +\infty, \) for some \( u,v \in C_R^2 \) and \( A \in C_V^2 \) which solve (0.1). Moreover, since \( qm_1 < \frac{1}{2} \) we also have that \( u > 0 \) and \( v > 0 \) in \( M. \) And there holds that \( A \neq 0 \) since \( S \) is not constant. This ends the proof of Theorem 0.1 in the critical case when \( \omega = 0. \) \( \square \)

9. Stability in the subcritical case. We prove the following stability result in the subcritical case following the Gidas and Spruck [21] scheme. The compactness assertion in Theorem 0.2 in the subcritical case easily follows from Lemma 9.1.

**Lemma 9.1.** Let \((M,g)\) be a smooth closed 3-manifold, \( \omega \in \mathbb{R}, \) \( a,q,m_0,m_1 > 0 \) be positive real numbers, \( p \in (4,6) \) and \( S \in C_R^\infty(M) \) be a smooth real-valued function. Let \( (\omega_\alpha)_{\alpha} \) be sequence of real numbers converging to \( \omega, \) let \((p_\alpha)_{\alpha} \) be a sequence of powers in \((4,6)\) converging to \( p \) and let \((S_\alpha)_{\alpha} \) be a sequence in \( C_R^\infty \) which converges to \( S \) in \( C_R^{\infty,0} \) as \( \alpha \to +\infty \) for some \( \theta \in (0,1). \) Let \((u_\alpha,v_\alpha,A_\alpha)_{\alpha} \) be a sequence of solutions of

\[
\begin{cases}
\frac{\hbar^2}{2m_0^2} \Delta g u_\alpha + \Phi_\alpha(x,v_\alpha,A_\alpha) u_\alpha = u_\alpha^{p_\alpha - 1} \\
\alpha^2 \Delta g v_\alpha + \Delta g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \\
\alpha^2 \Delta g A_\alpha + \Delta g A_\alpha + m_1^2 A_\alpha = \frac{4\pi \hbar^2}{m_0^2} \Psi(A_\alpha,S_\alpha) u_\alpha^2 ,
\end{cases}
\tag{9.1}
\]

where \( \Phi_\alpha(x,v,A) = \frac{\hbar^2}{2m_0^2} |\Psi(A,S)|^2 + \omega_\alpha^2 + qv, \) \( \Psi(A,S) = \nabla S - \frac{q}{\hbar} A \) and \( u_\alpha > 0 \) in \( M. \) Then, up to passing to a subsequence, \( u_\alpha \to u, v_\alpha \to v \) in \( C_R^2 \) and \( A_\alpha \to A \) in \( C_V^2 \) as \( \alpha \to +\infty, \) for some \( u,v \in C_R^2 \) and \( A \in C_V^2 \) which solve (0.1). Moreover, if \( qm_1 < \frac{1}{2} \), then we also have that \( u > 0 \) and \( v > 0 \) in \( M. \) In addition, when \( u > 0 \) in \( M, A \neq 0 \) if \( S \) is not constant.

**Proof of Lemma 9.1.** By Lemmas 2.2 and 3.1, \( v_\alpha = v(u_\alpha) \) and \( A_\alpha = A(u_\alpha) \) for all \( \alpha. \) By the first equation in (9.1),

\[
\frac{\hbar^2}{2m_0^2} \Delta g u_\alpha + \Phi_\alpha(x,v_\alpha,A_\alpha) u_\alpha = u_\alpha^{p_\alpha - 2} \tag{9.2}
\]

for all \( \alpha. \) Integrating over \( M \) we then get from (9.2) that

\[
\int_M u_\alpha^{p_\alpha - 2} dv_g \leq \int_M \Phi_\alpha(x,v_\alpha,A_\alpha) dv_g \tag{9.3}
\]

for all \( \alpha. \) Still by Lemmas 2.2 and 3.1, thanks to (2.4), and by (9.3), we then get that

\[
\int_M u_\alpha^{p_\alpha - 2} dv_g \leq C \left( 1 + \|u_\alpha\|^2_{L_R^2} \right) \tag{9.4}
\]

for all \( \alpha, \) where \( C > 0 \) is independent of \( \alpha. \) Since \( p_\alpha \to p \) and \( p > 4, \) we get from (9.4) that \( \|u_\alpha\|_{L_R^2} = O(1) \), and then that \( \|\Phi_\alpha\|_{L_R^6} = O(1), \) where \( \Phi_\alpha = \Phi_\alpha(x,v_\alpha,A_\alpha). \) By Lemma 6.1 it suffices to prove that \((u_\alpha)_{\alpha} \) is bounded in \( L_R^\infty. \) At this point we assume by contradiction that

\[
\max_M u_\alpha \to +\infty \tag{9.5}
\]

as \( \alpha \to +\infty. \) Let \( x_\alpha \in M \) and \( \mu_\alpha > 0 \) be such that

\[
u_\alpha(x_\alpha) = \max_M u_\alpha = \mu_\alpha^{-2/(p_\alpha - 2)}. \]
By (9.5), $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Define $\tilde{u}_\alpha$ by

$$
\tilde{u}_\alpha(x) = \mu_\alpha^{-\frac{2}{p}} u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))
$$

and $g_\alpha$ by $g_\alpha(x) = (\exp^*_{x_\alpha} g)(\mu_\alpha x)$ for $x \in B_0(\delta \mu_\alpha^{-1})$, where $\delta > 0$ is small. Since $\mu_\alpha \to 0$, we get that $g_\alpha \to \xi$ in $C^2_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$. Moreover, by the first equation in (9.1),

$$
\Delta_{g_\alpha} \tilde{u}_\alpha + \mu_\alpha^2 \hat{\Phi}_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^{p_\alpha-1}
$$

(9.6)

for all $\alpha$, where $\hat{\Phi}_\alpha$ is given by

$$
\hat{\Phi}_\alpha(x) = \Phi_\alpha(\exp_{x_\alpha}(\mu_\alpha x))
$$

There holds that $\tilde{u}_\alpha(0) = 1$ and $0 \leq \tilde{u}_\alpha \leq 1$. By (9.6) and standard elliptic theory arguments, we can write that, after passing to a subsequence, $\tilde{u}_\alpha \to u$ in $C^1_{loc}(\mathbb{R}^3)$ as $\alpha \to +\infty$, where $u$ is such that $u(0) = 1$ and $0 \leq u \leq 1$. Then

$$
\Delta_\xi u = u^{p-1}
$$

in $\mathbb{R}^3$, where $\Delta_\xi$ is the Euclidean Laplacian and, since $p < 6$, we get a contradiction with the Liouville result of Gidas and Spruck [21]. As a conclusion, (9.5) is not possible and we get that $\|u_\alpha\|_{L^\infty} = O(1)$. This ends the proof of Lemma 9.1. 

10. Stability in the critical case. Compactness and stability results for critical Yamabe type equations have a long history. Among possible references, we refer to Brendle [5], Brendle and Marques [6, 7], Druet [12, 13], Druet and Hebey [14], Druet, Hebey and Robert [17], Esposito, Pistoia and Vétois [20], Khuri, Marques and Schoen [31], Li and Zhang [32, 33, 34], Li and Zhu [35], Marques [37] and Schoen [41, 42, 43, 44]. A reference in book form is Hebey [25]. This list is far from being exhaustive. We aim here in proving the following stability result, the analog of Lemma 9.1 in the critical case $p = 6$. The compactness assertion in Theorem 0.2 in the critical case easily follows from Proposition 10.1.

**Proposition 10.1.** Let $(M, g)$ be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $(\omega_\alpha)_{\alpha}$ be sequence of real numbers converging to $\omega$, let $(p_\alpha)_{\alpha}$ be a sequence of powers in $(4, 6]$ converging to 6 and let $(S_\alpha)_{\alpha}$ be a sequence in $C^\infty_R$ which converges to $S$ in $C^2_R$ as $\alpha \to +\infty$. We assume (0.2). Let $(u_\alpha, v_\alpha, A_\alpha)_{\alpha}$ be a sequence of solutions of

$$
\begin{cases}
\frac{k^2}{2m_0} \Delta_g u_\alpha + \Phi_\alpha(x, v_\alpha, A_\alpha) u_\alpha = u_\alpha^{p_\alpha-1} \\
a^2 \Delta_g^2 v_\alpha + \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \\
a^2 \Delta_g^2 A_\alpha + \Delta_g A_\alpha + m_1^2 A_\alpha = \frac{4\pi q}{m_0^2} \Psi(A_\alpha, S_\alpha) u_\alpha^2,
\end{cases}
$$

(10.1)

where $\Phi_\alpha(x, v, A) = \frac{k^2}{2m_0} |\Psi(A, S)|^2 + \omega_\alpha^2 + qv$, $\Psi(A, S) = \nabla S - \frac{q}{2} A$ and $u_\alpha > 0$ in $M$. Then, up to passing to a subsequence, $u_\alpha \to u$, $v_\alpha \to v$ in $C^2_R$ and $A_\alpha \to A$ in $C^2_V$ as $\alpha \to +\infty$, for some $u, v \in C^2_R$ and $A \in C^2_V$ which solve (0.1). Moreover, if $am_1 < \frac{1}{2}$, then we also have that $u > 0$ and $v > 0$ in $M$. In addition, when $u > 0$ in $M$, $A \neq 0$ if $S$ is not constant.

The proof is developed around several distinct results of different importance. The starting point is the following, which we already used in the proof of Lemma 9.1.
Lemma 10.1. Let \((u_\alpha, v_\alpha, A_\alpha)_\alpha\) be a sequence of solutions of (10.1). Then \((u_\alpha)_\alpha\) is bounded in \(L^2_h\). In particular, up to passing to a subsequence, the sequence \((\Phi_\alpha)_\alpha\), where \(\Phi_\alpha = \Phi_\alpha(\cdot, v_\alpha, A_\alpha)\) for all \(\alpha\), converges in \(C^1_{p,\theta}\) for some \(\theta \in (0,1)\).

Proof of Lemma 10.1. By Lemmas 2.2 and 3.1, \(v_\alpha = v(u_\alpha)\) and \(A_\alpha = A(u_\alpha)\) for all \(\alpha\). By the first equation in (10.1),
\[
\frac{\hbar^2}{2m_\alpha^2} \Delta_g u_\alpha + \Phi_\alpha(x, v_\alpha, A_\alpha) = u_\alpha^{p_\alpha-2}
\]
for all \(\alpha\). Integrating over \(M\) we then get from (10.2) that
\[
\int_M u_\alpha^{p_\alpha-2} dv_g \leq \int_M \Phi_\alpha(x, v_\alpha, A_\alpha) dv_g
\]
for all \(\alpha\). Still by Lemmas 2.2 and 3.1, thanks to (2.4), and by (10.3), we then get that
\[
\int_M u_\alpha^{p_\alpha-2} dv_g \leq C \left( 1 + \|u_\alpha\|_{L^2_h}^2 \right)
\]
for all \(\alpha\), where \(C > 0\) is independent of \(\alpha\). Since \(p_\alpha \to 6\) we get from (10.4) that \(\|u_\alpha\|_{L^2_h} = O(1)\). Again by Lemmas 2.2 and 3.1, we get that \(\|A_\alpha\|_{H^3_h} = O(1)\) and that \(\|v_\alpha\|_{H^2_h} = O(1)\). The result then follows from the Sobolev embedding theorem.

Now that we have a convergence on the \(\Phi_\alpha\)'s, we can apply the asymptotic analysis in Hebey and Thizy [26, 27]. We refer also to Hebey [25]. Closely related arguments were first developed by Schoen [41], and then by Druet [12, 13] and Li and Zhu [35] assuming \(C^1\)-convergences of the potentials. We refer also to Druet and Hebey [14, 15, 16], Druet, Hebey and Vétois [18, 19], Hebey [25] and Hebey and Wei [30]. In particular, the following result holds true. Up to the \(C^1\)-convergence of the \(\Phi_\alpha\)'s the result essentially goes back to Li-Zhu [35]. This result, with such a generality, is specific to the case \(n = 3\).

Proposition 10.2 (Li-Zhu [35]). Let \((M, g)\) be a smooth closed 3-manifold and \((p_\alpha)_\alpha\) be a sequence of powers in \(4, 6\) converging to 6. Let \((h_\alpha)_\alpha\) be a sequence of smooth functions. We assume that \(\|h_\alpha\|_{L^\infty_h} = O(1)\). Let \((\varphi_\alpha)_\alpha\) be a sequence of positive solutions of
\[
\Delta_g \varphi_\alpha + h_\alpha \varphi_\alpha = \varphi_\alpha^{p_\alpha-1}.
\]
Then \(\|\varphi_\alpha\|_{H^1_h} = O(1)\).

We refer to Hebey and Thizy [26, 27] and Hebey [25] (where only a \(L^\infty\)-bound is assumed on the \(h_\alpha\)'s) for the proof of Proposition 10.2. The key point consists in proving that blow-up points are isolated. An alternative proof in the specific case of \(S^3\) can be found in Hebey and Wei [30]. By Lemma 10.1 and Proposition 10.2, since
\[
\frac{\hbar^2}{2m_\alpha^2} \Delta_g u_\alpha + \Phi_\alpha u_\alpha = u_\alpha^{p_\alpha-1},
\]
the sequence \((u_\alpha)_\alpha\) is bounded in \(H^1_h\). Hence it is also bounded in \(L^3_h\) and we can go back to Lemmas 2.2 and 3.1 (see also the remark after Lemma 2.2) to get that \(\|A_\alpha\|_{H^3_h} = O(1)\) and \(\|v_\alpha\|_{H^2_h} = O(1)\). In particular, by the Sobolev embedding theorem and up to passing to a subsequence, the sequences \((A_\alpha)_\alpha\), \((v_\alpha)_\alpha\) and \((\Phi_\alpha)_\alpha\) converge in \(C^1_{p,\theta}\). We let \(A, v\) and \(\Phi\) be the respective limits of \((A_\alpha)_\alpha\), \((v_\alpha)_\alpha\) and \((\Phi_\alpha)_\alpha\). Then, we can branch again on the asymptotic analysis developed in the
above list of papers. The $C^1$-convergence makes that we can directly apply the following result proved in Li-Zhu [35] when $n = 3$.

**Proposition 10.3** (Li-Zhu [35]). Let $(M, g)$ be a smooth closed 3-manifold and $(p_\alpha)_\alpha$ be a sequence of powers in $(4, 6]$ converging to 6. Let $(h_\alpha)_\alpha$ be a sequence of smooth functions. We assume that $(h_\alpha)_\alpha$ converges $C^1$ to some smooth function $h$. Let $(\varphi_\alpha)_\alpha$ be a sequence of positive solutions of

$$
\Delta_g \varphi_\alpha + h_\alpha \varphi_\alpha = \varphi_\alpha^{p_\alpha - 1}.
$$

Suppose that the operator $\Delta_g + h$ is coercive and that it has positive mass. Then $\|\varphi_\alpha\|_{L^\infty_R} = O(1)$.

At this point we can prove Proposition 10.1 when $\omega \neq 0$ or when $S$ is not constant.

**Proof of Proposition 10.1 when $\omega \neq 0$ or when $S$ is not constant.** Let $(u_\alpha, v_\alpha, A_\alpha)_\alpha$ be a sequence of solutions of (10.1). By Lemma 6.1 it suffices to prove that $(u_\alpha)_\alpha$ is bounded in $L^\infty_R$. We proceed by contradiction and we assume that

$$
\max_M u_\alpha \to +\infty
$$

as $\alpha \to +\infty$. We can rewrite equation (10.5) as

$$
\Delta_g \tilde{u}_\alpha + \frac{2m_0^2}{h^2} \Phi_{\alpha} \tilde{u}_\alpha = \tilde{u}_\alpha^{p_{\alpha} - 1},
$$

where

$$
\tilde{u}_\alpha(x) = \left(\frac{2m_0^2}{h^2}\right)^{\frac{1}{p_{\alpha} - 2}} u_\alpha
$$

for all $\alpha$. As in Hebey and Thizy [26, 27], see also Hebey [25], there exist two positive constants $C, r > 0$, a converging sequence $(x_\alpha)_\alpha$ of points in $M$, and a sequence $(\mu_\alpha)_\alpha$ of positive real numbers converging to 0 such that, up to a subsequence,

$$
\tilde{u}_\alpha(x) \leq C \mu_\alpha^{\frac{p_{\alpha} - 4}{p_{\alpha} - 2}} g(x_\alpha, x)^{-1}
$$

for all $\alpha$ and all $x \in B_{x_\alpha}(r) \setminus \{x_\alpha\}$. Still up to passing to a subsequence we can assume by Proposition 10.2 that there exists $\tilde{u} \in H^1_R$ such that $\tilde{u}_\alpha \rightharpoonup \tilde{u}$ in $H^1_R$. $\tilde{u}_\alpha \to \tilde{u}$ in $L^2_R$ and $\tilde{u}_\alpha \to \tilde{u}$ a.e as $\alpha \to +\infty$. Then, by (10.7),

$$
\Delta_g \tilde{u} + \frac{2m_0^2}{h^2} \Phi \tilde{u} = \tilde{u}^5,
$$

and $\tilde{u} = (2m_0^2/h^2)^{1/4} u$. By (10.8), $\tilde{u} \equiv 0$ in a nonempty part of $M$. Since $\tilde{u} \geq 0$ it follows from the maximum principle that $\tilde{u} \equiv 0$ (the fact that the weak limit has to be zero goes back to Druet [13] and Li and Zhu [35]). Since $\tilde{u} \equiv 0$, and $\tilde{u}_\alpha \to \tilde{u}$ in $L^2_R$, we get that $A \equiv 0$ and $v \equiv 0$ (e.g by the estimates in Lemmas 2.2 and 3.1). Then

$$
\Phi \equiv \frac{h^2}{2m_0^2} |\nabla S|^2 + \omega^2.
$$

When $\omega \neq 0$, or when $S$ is not constant by Lemma 4.2, the operator $\Delta_g + \frac{2m_0^2}{h^2} \Phi$ is coercive. By (0.2) there holds that

$$
\frac{2m_0^2}{h^2} \Phi < \Lambda_g,
$$
where \( \Lambda_g \) is such that \( \Delta_g + \Lambda_g \) has nonnegative mass. By the maximum principle we then get that \( \Delta_g + \frac{2m^2}{h^2} \Phi \) has positive mass. Then we can apply Proposition 10.3 and we get that the sequence \((\tilde{u}_\alpha)_\alpha\) is bounded in \( L^\infty_R \). In other words, (10.6) does not hold true and we get the desired contradiction. This proves Proposition 10.1 when \( \omega \neq 0 \) or when \( S \) is not constant.

It remains to prove Proposition 10.1 when \( \omega = 0 \) and \( S \) is constant. In that case, \( \Phi \equiv 0 \) and thus, \( \Phi_\alpha \to 0 \) in \( C^2_R \) as \( \alpha \to +\infty \). We proceed here using ideas from Hebey and Wei [30]. Since \( u_\alpha > 0 \) in \( M \), we get from the first equation in (10.1) that \( \frac{h^2}{2m^2} \Delta_g + \Phi_\alpha \) is coercive for all \( \alpha \).

**Lemma 10.2.** Let \( G_\alpha : M \times M \setminus D \to \mathbb{R} \) be the Green’s function of \( \frac{h^2}{2m^2} \Delta_g + \Phi_\alpha \), where \( D \) is the diagonal in \( M \times M \). Suppose that \( \omega = 0 \) and that \( \nabla S \equiv 0 \). Then \( \inf_{M \times M \setminus D} G_\alpha \to +\infty \) as \( \alpha \to +\infty \).

**Proof of Lemma 10.2.** Let \( \varepsilon_\alpha = \| \Phi_\alpha \|_{L^\infty_R} \) and \( k_\alpha \in \mathbb{R} \) be such that \( k_\alpha \to +\infty \) and \( \varepsilon_\alpha k_\alpha \to 0 \) as \( \alpha \to +\infty \). Let \( G_\alpha \) be the Green’s function of \( \frac{h^2}{2m^2} \Delta_g + \varepsilon_\alpha \). By the maximum principle, \( G_\alpha \geq \hat{G}_\alpha \) in \( M \times M \setminus D \). We let \( G \geq 0 \) be a Green’s function of \( \frac{h^2}{2m^2} \Delta_g \). For any \( x \in M \), if \( G_x = G(x, \cdot) \), there holds that

\[
\frac{h^2}{2m^2} \Delta_g G_x = \delta_x - \frac{1}{V_g},
\]

where \( V_g \) is the volume of \((M, g)\). Let \( x \in M \) and \( V_\alpha \) solve

\[
\frac{h^2}{2m^2} \Delta_g V_\alpha + \varepsilon_\alpha V_\alpha = \varepsilon_\alpha G_x.
\]

There holds \( \int V_\alpha = \int G_x \) so that, by Poincaré’s inequality and standard estimates on \( G \), \( V_\alpha \) is bounded in \( H^1_R \) uniformly with respect to \( \alpha \). By standard elliptic properties and standard estimates on \( G \), it follows that \( \| V_\alpha \|_{L^\infty_R} \leq C \) for all \( \alpha \) with a bound which is uniform with respect to \( x \). Let \( k_\alpha = \hat{G}_\alpha(x, \cdot) \geq G_x - k_\alpha + V_\alpha \). Then

\[
\frac{h^2}{2m^2} \Delta_g k_\alpha + \varepsilon_\alpha k_\alpha \geq \frac{1}{V_g} - \varepsilon_\alpha k_\alpha
\]

for all \( \alpha \), and by the maximum principle and the above estimates it follows that \( \hat{G}_\alpha(x, \cdot) \geq k_\alpha - C \) for all \( \alpha \) and all \( x \), where \( C \) is independent of \( \alpha \) and \( x \). This proves the lemma.

Now we can prove Proposition 10.1 when \( \omega = 0 \) and \( S \) is constant.

**Proof of Proposition 10.1 when \( \omega = 0 \) and \( S \) is constant.** Let \((u_\alpha, v_\alpha, A_\alpha)_\alpha\) be a sequence of solutions of (10.1). Here again, \( A_\alpha = A(u_\alpha) \) and \( v_\alpha = v(u_\alpha) \). By Lemma 6.1 it suffices to prove that \( (u_\alpha)_\alpha \) is bounded in \( L^\infty_R \). We proceed by contradiction and we assume that

\[
\max_M u_\alpha \to +\infty \quad (10.10)
\]

as \( \alpha \to +\infty \). Then (10.8) still holds true. But also, we refer for instance to the analysis in Hebey and Thizy [26], the following easier estimate (of Gidas-Spruck [21] type) holds true: up to passing to a subsequence,

\[
\mu_\alpha^{-\frac{1}{2}} \tilde{u}_\alpha \exp_{x_\alpha}(\mu_\alpha x) \to \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} \quad (10.11)
\]
in $C_{\text{loc}}^1(\mathbb{R}^3)$ as $\alpha \to +\infty$. Integrating over $B_{x_\alpha}(\mu_\alpha)$, we get that there exists $A > 0$ such that

$$
\int_M u_\alpha^{p_\alpha-1} dv_g \geq A\mu_\alpha^{\frac{p_\alpha-4}{2}}
$$

(10.12)

for all $\alpha$. Branching on the beginning of the proof of Proposition 10.1 when $\omega \neq 0$ or when $S$ is not constant there holds that $\Phi_\alpha \to 0$ in $C_2^R$ as $\alpha \to +\infty$. By (10.5), Lemma 10.2 and (10.12), we then get that

$$
u_\alpha(x) = \int_M G_\alpha(x,y)u_\alpha(y)^{p_\alpha-1}dv_g(y)
\geq AK_\alpha \mu_{\alpha}^{\frac{p_\alpha-4}{2}}
$$

(10.13)

for all $x \in M$, where $K_\alpha \to +\infty$ as $\alpha \to +\infty$. Coming back to (10.8) it follows from (10.13) that

$$K_\alpha \leq C'
$$

(10.14)

for all $\alpha$ and some $C' > 0$ independent of $\alpha$. Obviously (10.14) is in contradiction with the fact that $K_\alpha \to +\infty$ as $\alpha \to +\infty$. In other words (10.10) does not hold true. This proves Proposition 10.1 when $\omega = 0$ and $S$ is constant.

As already mentioned, Theorem 0.2 in the subcritical case easily follows from Lemma 9.1, and Theorem 0.2 in the critical case easily follows from Proposition 10.1.

**Proof of Theorem 0.2.** Let $(u_\alpha, v_\alpha, A_\alpha)_\alpha$ be a sequence of solutions of (0.1) with $u_\alpha \geq 0$ in $M$ for all $\alpha$. By Lemma 4.3, the $u_\alpha$'s, $v_\alpha$'s and $A_\alpha$'s are $C^2$ and, for any $\alpha$, either $u_\alpha \equiv 0$ or $u_\alpha > 0$ in $M$. In particular, up to passing to a subsequence, either we are talking about the trivial solution, or we are talking about a sequence of solutions $(u_\alpha, v_\alpha, A_\alpha)_\alpha$ with $u_\alpha > 0$ in $M$ for all $\alpha$. Theorem 0.2 in the subcritical case is then a direct consequence of Lemma 9.1 while, in the critical case, it is a direct consequence of Proposition 10.1.

We very briefly comment on the sharpness of (0.2). In doing so we enlarge the context from real numbers $\omega \in \mathbb{R}$ to functions $\omega : M \to \mathbb{R}$. An easy remark is that Proposition 10.1 still holds true, with an unchanged proof, if we replace the sequence of positive real numbers $(\omega_\alpha)_\alpha$ converging to $\omega$ by a sequence $(\omega_\alpha)_\alpha$ of smooth functions converging in $C_R^1$ to some smooth function $\omega$. In other words we do have stability if (0.2) holds true, even when we replace the $\omega_\alpha$'s by functions. Another remark is that, in the case of $S^3$, we precisely know what the best $\Lambda_g$ is. There holds that $\Lambda_g = \frac{3}{4}$, a value for which the mass is 0. Let $\Lambda(m_0) = \frac{\sqrt{3\hbar}}{2\sqrt{2m_0}}$. We consider the nonlinear critical equation

$$
\frac{\hbar^2}{2m_0} \Delta_g u + \Lambda(m_0)^2 u = u^5
$$

(10.15)

in $S^3$, with $u > 0$. A final remark, see Hebey and Wei [30], is that the solutions of (10.15) are all known and given by

$$
U_{\varepsilon,x_0} = \frac{3^{1/4}\sqrt{\hbar}}{8^{1/4}\sqrt{m_0}} \left( \frac{\varepsilon}{\varepsilon^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{1/2},
$$
where \( \varepsilon \in (0,1) \), \( r = d_g(x_0, \cdot) \) and \( x_0 \in S^3 \) is arbitrary. These are essentially the solutions of the Yamabe equation on \( S^3 \). We fix \( x_0 \in S^3 \). As one can check,
\[
\lim_{\varepsilon \to 0} \| U_{\varepsilon,x_0} \|_{L^q_R} = +\infty
\]
and the \( U_{\varepsilon,x_0} \)'s blow up as \( \varepsilon \to 0 \). There also holds that \( \| U_{\varepsilon,x_0} \|_{L^q_R} \to 0 \) as \( \varepsilon \to 0 \).

Fix \( S \in C^1_R \) and \( q, m_0, m_1, a > 0 \). By Lemmas 2.2 and 3.1 there holds that \( \| A(U_{\varepsilon,x_0}) \|_{C^1_S} \to 0 \) and \( \| v(U_{\varepsilon,x_0}) \|_{C^1_R} \to 0 \) as \( \varepsilon \to 0 \). Define \( \omega_\varepsilon \in C^2_R \) to be given by
\[
\omega_\varepsilon^2 = \frac{\hbar^2}{2m_0^2} \Lambda_g - \frac{\hbar^2}{2m_0^2} \left( \nabla S - \frac{q}{\hbar} A(U_{\varepsilon,x_0}) \right)^2 - qv(U_{\varepsilon,x_0}).
\]
Such an \( \omega_\varepsilon \) exists if \( S \) is \( C^1 \)-sufficiently close to a constant (in order to have that \( |\nabla S| < \Lambda_g \) in \( M \)) and \( 0 < \varepsilon << 1 \). The triple \( (U_{\varepsilon,x_0}, v(U_{\varepsilon,x_0}), A(U_{\varepsilon,x_0})) \) then solves (0.1) with \( \omega = \omega_\varepsilon \) for all \( \varepsilon \in (0,1) \). Also \( \omega_\varepsilon \to \omega \) in \( C^1_R \), where
\[
\omega^2 = \frac{\hbar^2}{2m_0^2} (\Lambda_g - |\nabla S|),
\]
and we are precisely in the border case of (0.2) since, in that particular situation,
\[
\omega^2 + \frac{\hbar^2}{2m_0^2} |\nabla S|^2 \equiv \frac{\hbar^2}{2m_0^2} \Lambda_g.
\]
In particular, blow-up occurs when (0.2) is not satisfied and (at least) when we are in the more general setting where the \( \omega_\alpha \)'s are allowed to be functions.

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E-mail address: Emmanuel.Hebey@u-cergy.fr