RESTRICTED POISSON ALGEBRAS

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Abstract. We re-formulate Bezrukavnikov-Kaledin's definition of a restricted Poisson algebra, provide some natural and interesting examples, and discuss connections with other research topics.

0. Introduction

The Poisson bracket was introduced by Poisson as a tool for classical dynamics in 1809 [Po]. Poisson geometry has become an active research field during the past 50 years. The study of Poisson algebras over \( \mathbb{R} \) or a field of characteristic zero [L-GPV] also has a long history, and is closely related to noncommutative algebra, differential geometry, deformation quantization, number theory, and other areas. The notion of a restricted Poisson algebra was introduced about ten years ago in an important paper of Bezrukavnikov-Kaledin [BK] in the study of deformation quantization in positive characteristic. The project in [BK] is a natural extension of the classical deformation quantization of symplectic (or Poisson) manifolds.

Our first goal is to better understand Bezrukavnikov-Kaledin’s definition via a Lie algebraic approach. We re-interpret their definition in the following way.

Throughout the paper let \( k \) be a base field of characteristic \( p \geq 3 \). All vector spaces and algebras are over \( k \).

Definition 0.1. Let \( (A, \{-, -\}) \) be a Poisson algebra over \( k \).

1. We call \( A \) a weakly restricted Poisson algebra if there is a \( p \)-map operation \( x \mapsto x^{(p)} \) such that \( (A, \{-, -\}, (\cdot)^{(p)}) \) is a restricted Lie algebra.
2. We call \( A \) a restricted Poisson algebra if \( A \) is a weakly restricted Poisson algebra and the \( p \)-map \( (\cdot)^{(p)} \) satisfies

\[
(x^2)^{(p)} = 2x^p x^{(p)}
\]

for all \( x \in A \).

The formulation in (E0.1.1) is slightly simpler than the original definition. We will show that Definition 0.1(2) is equivalent to [BK Definition 1.8] in Lemma 3.7. Generally it is not easy to prove basic properties for restricted Poisson algebras. For example, it is not straightforward to show that the tensor product preserves the restricted Poisson structure. Different formulations are helpful in understanding and proving some elementary properties.

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Since there are several structures on a restricted Poisson algebra, it is delicate to verify all compatibility conditions. There are not many examples given in the literature. Our second goal is to provide several canonical examples from different research subjects. Restricted Poisson algebras can be viewed as a Poisson version of restricted Lie algebras, so the first few examples come from restricted (or modular) Lie theory. Let \( L \) be a restricted Lie algebra over \( k \). Then the trivial extension algebra \( k \oplus L \) (with \( L^2 = 0 \)) is a restricted Poisson algebra. More naturally we have the following.

**Theorem 0.2 (Theorem 6.5)**. Let \( L \) be a restricted Lie algebra over \( k \) and let \( s(L) \) be the \( p \)-truncated symmetric algebra. Then \( s(L) \) admits a natural restricted Poisson structure induced by the restricted Lie structure of \( L \).

To use ideas from Poisson geometry, it is a good idea to extend the restricted Poisson structure to the symmetric algebra of a restricted Lie algebra [Example 6.2]. The following result is slightly more general and useful in other setting.

**Theorem 0.3 (Theorem 6.1)**. Let \( T \) be an index set and \( A = k[x_i | i \in T] \) be a polynomial Poisson algebra. If, for each \( i \in T \), there exists \( \gamma(x_i) \in A \) such that \( \text{ad}_{p-1} x_i = \text{ad}_{\gamma(x_i)} \), then \( A \) admits a restricted Poisson structure \((-)^{(p)} : A \rightarrow A \) such that \( x_i^{(p)} = \gamma(x_i) \) for all \( i \in T \).

The next example comes from deformation theory, which is also considered in [BK]. See \([E7.0.1]\) for the definition of \( M^p_n(f) \).

**Proposition 0.4 (Proposition 7.1)**. Let \( (A, \cdot, \{-,-\}, (-)^{(p)}) \) be a restricted Poisson algebra over \( k \) and let \( (A[[t]], \ast) \) be a deformation quantization of \( A \). If \( M^p_n(f) = 0 \) for \( 1 \leq n \leq p-2 \) and \( f^p \) is central in \( A[[t]] \) for all \( f \in A \), then \( A \) admits a restricted Poisson structure.

A Lie-Rinehart algebra is an algebraic counterpart of a Lie algebroid, and appears naturally in the study of Gerstenhaber algebras, Batalin-Vilkovisky algebras and Maurer-Cartan algebras [Hu1, Hu2]. In this paper, we also study the relationship between restricted Poisson algebras and restricted Lie-Rinehart algebras.

**Theorem 0.5 (Theorem 8.2)**. Let \( (A, \cdot, \{-,-\}, (-)^{(p)}) \) be a restricted Poisson algebra. If the Kähler differential \( \Omega_{A/k} \) is free over \( A \), then \((A, \Omega_{A/k}, (-)^{(p)})\) is a restricted Lie-Rinehart algebra, where the \( p \)-map of \( \Omega_{A/k} \) is determined by

\[
(x du)^{(p)} = x^p du^{(p)} + (x du)^{p-1}(x) du,
\]
for all \( x du \in \Omega_{A/k} \).

The category of restricted Poisson algebras is a symmetric monoidal category. In particular, the tensor product of two restricted Poisson algebras is again a restricted Poisson algebra [Proposition 9.2]. Advances of algebra are tremendously benefited from geometric viewpoint and methods and vice versa. Restricted Poisson algebras are, to some extent, the algebraic counterpart of symplectic differential geometry in positive characteristic. Following this idea, restricted Poisson-Lie groups should correspond to restricted Poisson Hopf algebras which connects both Poisson geometry in positive characteristic and quantum groups at the root of unity. Hence, it is meaningful to introduce the notion of a restricted Poisson Hopf algebra, see Definition 9.3. One natural example of such an algebra is given in Example 9.4.
The paper is organized as follows. Sections 1 and 2 contain basic definitions about restricted Lie algebras and Poisson algebras. In Section 3, we re-introduce the notion of a restricted Poisson algebra. In Sections 4 to 7, we give several natural examples. In Section 8, we prove Theorem 4.3. The notion of a restricted Poisson Hopf algebra is introduced in Section 9. The Appendix contains a combinatorial proof of (7.2.1) which is needed for Example 7.2.

1. Restricted Lie algebras

We give a short review about restricted Lie algebras.

Lie algebras over a field of positive characteristic often admit an additional structure involving a so-called $p$-map. The Lie algebra together with a $p$-map is called a restricted Lie algebra, which was first introduced and systematically studied by Jacobson [J1, J2]. Let $L := (L, [-,-])$ be a Lie algebra over $k$. For convenience, for each $x \in L$, we denote by $\text{ad}_x : L \to L$ the adjoint representation given by $\text{ad}_x(y) = [x,y]$ for all $y \in L$. We recall the definition of a restricted Lie algebra from [J1, Section 1]. As always, we assume that $k$ is of positive characteristic $p \geq 3$.

**Definition 1.1.** [J1] A restricted Lie algebra $(L, (-)[-])$ over $k$ is a Lie algebra $L$ over $k$ together with a $p$-map $(-)[-] : x \mapsto x[-]$ such that the following conditions hold:

1. $\text{ad}_x^p = \text{ad}_x[-]$ for all $x \in L$;
2. $(\lambda x)^[-] = \lambda^p x[-]$ for all $\lambda \in k, x \in L$;
3. $(x + y)[-] = x[-] + y[-] + \Lambda_p(x, y)$, where $\Lambda_p(x, y) = \sum_{i=1}^{p-1} s_i(x,y)$ for all $x, y \in L$ and $s_i(x,y)$ is the coefficient of $t^{i-1}$ in the formal expression $\text{ad}_{x+y}^{p-1}(x)$.

For simplicity of notation, we write all multiple Lie brackets with the notation

$[x_1, [x_2, \cdots, [x_{n-1}, x_n] \cdots]] := [x_1, x_2, \cdots, x_{n-1}, x_n],
$ for $x_1, \cdots, x_n \in L$. Clearly, $\text{ad}^n_x(y) = [x, \cdots, x, y]$ for every $i$. Under this notation, we have

$\sum_{x_k=x \text{ or } y \neq (k|x_k=x)=i-1} [x_1, \cdots, x_{p-2}, y, x],
$ and, hence

$\Lambda_p(x, y) = \sum_{x_k=x \text{ or } y \neq x_{p-1}=y, x=p} \frac{1}{\#(x)}[x_1, \cdots, x_{p-1}, x_p].
$ Note that $\Lambda_p(x, y)$ is denoted by $L(x, y)$ in [BK] and denoted by $\sigma(x, y)$ in [H62].

Another way of understanding $\Lambda_p(x, y)$ is to use the universal enveloping algebra $U(L)$ of the Lie algebra $L$. By [H62], Condition (3) on p. 559),

$\Lambda_p(x, y) = (x + y)^p - x^p - y^p
$ for all $x, y \in L \subset U(L)$, where $(-)^p$ is the multiplicative $p$-th power in $U(L)$.

We give a well-known example which will be used later.
Example 1.2. Let $A$ be an associative algebra over $k$. We denote by $A_L$ the induced Lie algebra with the bracket given by $[x, y] := xy - yx$, for all $x, y \in A$. Then $(A_L, (-)^p)$ is a restricted Lie algebra, where $(-)^p$ is the Frobenius map given by $x \mapsto x^p$.

In [J2, Theorem 11], Jacobson gives a necessary and sufficient condition in which an ordinary Lie algebra over $k$ is restricted.

Lemma 1.3. [J2, Theorem 11] Let $L$ be a Lie algebra with a $k$-basis $\{x_i\}_{i \in I}$ for some index set $I$. Suppose that there exists an element $\gamma(x_i) \in L$ for each $i \in I$ such that $\text{ad}^p_{x_i} = \text{ad}_{\gamma(x_i)}$. Then there exists a unique restricted structure on $L$ such that $x_i^{[p]} = \gamma(x_i)$ for all $i \in I$.

2. Poisson algebras and their enveloping algebras

In this section we recall some definitions. We refer to [L-GPV] for some basics concerning Poisson algebras.

Definition 2.1. [L-GPV, Definition 1.1] Let $A$ be a commutative algebra over $k$. A Poisson structure on $A$ is a Lie bracket $\{ -, - \} : A \otimes A \to A$ such that the following Leibniz rule holds

\[
\{ xy, z \} = x\{ y, z \} + y\{ x, z \}, \quad \forall x, y, z \in A.
\]

The algebra $A$ together with a Poisson structure is called a Poisson algebra.

The Lie bracket $\{ -, - \}$ (which replaces $[-, -]$ in the previous section) is called the Poisson bracket, and the associative multiplication of $A$ is sometimes denoted by $\cdot$. In this paper all Poisson algebras are commutative as an associative algebra.

Recall that the Kähler differentials, denoted by $\Omega_{A/k}$, of a commutative algebra $A$ over $k$ is an $A$-module generated by elements (or symbols) $dx$ for all $x \in A$, and subject to the relations

\[
d(x + y) = dx + dy, \quad d(xy) = xdy + ydx, \quad d\lambda = 0,
\]

where $x, y \in A, \lambda \in k \subseteq A$. When $(A, \{-, -\})$ is a Poisson algebra, the Kähler differentials $\Omega_{A/k}$ admits a Lie algebra structure with Lie bracket given by

\[
[xdu, ydv] = x\{ u, y \}dv + y\{ x, v \}du + xyd\{ u, v \}
\]

for all $xdu, ydv \in \Omega_{A/k}$. Moreover, $A$ is also a Lie module over $\Omega_{A/k}$ with the action given by $(xdu).a = x\{ u, a \}$ for all $xdu \in \Omega_{A/k}, a \in A$. In fact, the pair $(A, \Omega_{A/k})$ is a Lie-Rinehart algebra in the following sense.

Definition 2.2. [Do, Definition 1.5] A Lie-Rinehart algebra over $A$ is a pair $(A, L)$, where $A$ is a commutative associative algebra over $k$, $L$ is a Lie algebra equipped with the structure of an $A$-module together with a map called anchor

\[
\alpha : L \to \text{Der}_k(A)
\]

which is both an $A$-module and a Lie algebra homomorphism such that

\[
[X, aY] = a[X, Y] + \alpha(X)(a)Y
\]
for all \( a \in A \) and \( X, Y \in L \).

Note that, in the situation of Poisson algebra, the anchor map \( \alpha : \Omega_{A/k} \to \text{Der}(A) \) is given by
\[
\alpha(xdu)(z) = x\{u, z\}
\]
for all \( xdu \in \Omega_{A/k} \) and \( z \in A \).

Let \( (A, L) \) be a Lie-Rinehart algebra. In \[Ri\], Rinehart introduced the notion of universal enveloping algebra \( \mathcal{U}(A, L) \) of \( (A, L) \), which is an associative \( k \)-algebra satisfying the appropriate universal property, see \[Hu1\] for more details. We recall the definition next.

Denote by \( A \rtimes L \) the semi-direct product of the Lie algebra \( L \) and the \( L \)-module \( A \). More precisely, \( A \rtimes L \) is the direct sum of \( A \) and \( L \) as a vector space, and the Lie bracket is given by
\[
[(a, X), (b, Y)] = (X(b) - Y(a), [X, Y])
\]
for all \( (a, X), (b, Y) \in A \rtimes L \). Let \( \mathcal{U}(A \rtimes L), \iota \) be the universal enveloping algebra of the Lie algebra \( A \rtimes L \), where \( \iota : A \rtimes L \to \mathcal{U}(A \rtimes L) \) is the canonical embedding. We consider the subalgebra \( \mathcal{U}^+(A \rtimes L) \) (without unit) generated by \( A \rtimes L \). Moreover, \( A \rtimes L \) has the structure of an \( A \)-module via \( a(a', X) = (aa', aX) \) for all \( a, a' \in A \) and \( X \in L \). The (universal) enveloping algebra \( \mathcal{U}(A, L) \) associated to the Lie-Rinehart algebra \( (A, L) \) is defined to be the quotient
\[
\mathcal{U}(A, L) = \frac{\mathcal{U}^+(A \rtimes L)}{(\iota((a,0))\iota((a', X)) - \iota(a(a', X)))}.
\]
Note that \( (1_A, 0) \) becomes the algebra identity of \( \mathcal{U}(A, L) \). There are two canonical maps
\[
\iota_1 : A \to \mathcal{U}(A, L), a \mapsto (a, 0) \quad \text{and} \quad \iota_2 : L \to \mathcal{U}(A, L), X \mapsto (0, X).
\]
Observe that \( \iota_1 \) is an algebra homomorphism and \( \iota_2 \) is a Lie algebra homomorphism. Moreover, we have the following relations
\[
\iota_1(a)\iota_2(X) = \iota_2(aX), \quad \text{and} \quad [\iota_2(X), \iota_1(a)] = \iota_1(X(a))
\]
for all \( a \in A \) and \( X \in L \).

As a consequence of \[Ri\] Theorem 3.1, we have the following.

**Lemma 2.3.** Let \( (A, L) \) be a Lie-Rinehart algebra and \( \mathcal{U}(A, L) \) the enveloping algebra of \( (A, L) \). If \( L \) is a projective \( A \)-module, then the Lie algebra homomorphism \( \iota_2 : L \to \mathcal{U}(A, L) \) is injective.

It is worth spending half page to re-state the above construction for Poisson algebras since it is needed later. Denote by \( A \rtimes \Omega_{A/k} \) the semidirect product of \( A \) and \( \Omega_{A/k} \) with the Lie bracket given by
\[
[(a, xdu), (b, ydv)] = (x\{u, b\} - y\{v, a\}, x\{u, v\}du + y\{x, v\}du + xyd\{u, v\})
\]
for \( (a, xdu), (b, ydv) \in A \rtimes \Omega_{A/k} \). The Poisson enveloping algebra of \( A \), denoted by \( \mathcal{P}(A) \) (which is a new notation), is defined to be the enveloping algebra of the Lie-Rinehart algebra \( (A, \Omega_{A/k}) \), which can be realized as an associated algebra
\[
\mathcal{P}(A) := \mathcal{U}(A, \Omega_{A/k}) = \mathcal{U}^+(A \rtimes \Omega_{A/k})/J,
\]
where $\mathcal{U}(A \times \Omega_{A/k})$ is the universal enveloping algebra of the Lie algebra $A \times \Omega_{A/k}$,
and $J$ is the ideal generated by

(E2.3.1) 
$$(a,0)(b,xdu) - (ab,axdu)$$

for all $a, b \in A, xdu \in \Omega_{A/k}$. Here we have two maps
\[\iota_1: A \to A \times \Omega_{A/k} \to \mathcal{P}(A), \quad \iota_1(a) = (a,0)\]
and
\[\iota_2: \Omega_{A/k} \to A \times \Omega_{A/k} \to \mathcal{P}(A), \quad \iota_2(xdu) = (0,xdu).\]

Then $\iota_1$ and $\iota_2$ are homomorphisms of associative algebras and Lie algebras, respectively. Moreover, we have

(E2.3.2) 
$$\iota_1(\{x,y\}) = [\iota_2(dx), \iota_1(y)],$$

(E2.3.3) 
$$\iota_2(dx) = \iota_1(x)\iota_2(dy) + \iota_1(y)\iota_2(dx)$$

for all $x, y \in A$.

If $\Omega_{A/k}$ is a projective $A$-module, then the canonical map $\iota_2: \Omega_{A/k} \to \mathcal{P}(A)$ is injective [Lemma 2.3]. It follows that $\Omega_{A/k}$ can be seen as a Lie subalgebra of $\mathcal{P}(A)$.

We now recall the definition of a free Poisson algebra, see [Sh, Section 3]. Let $V$ be a $k$-vector space. Let $\text{Lie}(V)$ be the free Lie algebra generated by $V$. The free Poisson algebra generated by $V$, denoted by $FP(V)$, is the symmetric algebra over $\text{Lie}(V)$, namely

(E2.3.4) 
$$FP(V) = k[\text{Lie}(V)].$$

The following universal property is well-known [Sh, Lemma 1, p. 312].

Lemma 2.4. Let $A$ be a Poisson algebra and $V$ be a vector space. Every $k$-linear map $g: V \to A$ extends uniquely to a Poisson algebra morphism $G: FP(V) \to A$ such that $g$ factors through $G$.

In [Sh, Section 3], the notion of a free Poisson algebra is defined by the universal property stated in Lemma 2.4 and then Shestakov proved that the free Poisson algebra can be constructed by using (E2.3.4) [Sh, Lemma 1, p. 312]. In [Sh], Shestakov also considered the super (or $\mathbb{Z}_2$-graded) version of Poisson algebras.

For each associative commutative algebra $A$, let $A^p$ denote the subalgebra generated by $\{f^p \mid f \in A\}$. The free Poisson algebras have the following special property.

Lemma 2.5. Let $A$ be a free Poisson algebra $FP(V)$.

1. $\Omega_{A/k}$ is a free module over $A$. As a consequence, the Lie algebra map $\iota_2: \Omega_{A/k} \to \mathcal{P}(A)$ is injective.
2. The kernel of $d: A \to \Omega_{A/k}$ is $A^p$.

Proof. (1) Since $A$ is a commutative polynomial ring, $\Omega_{A/k}$ is free over $A$. (The proof is omitted). The consequence follows from Lemma 2.3.

(2) Check directly. \[\square\]
Let $V$ be a $k$-vector space. There are two gradings that can naturally be assigned to $FP(V)$. The first one is determined by
\[
\text{deg}_1(x) = 1, \quad \forall \ 0 \neq x \in \text{Lie}(V).
\]
Since $FP(V)$ is the symmetric algebra associated to $\text{Lie}(V)$, the above extends to an $\mathbb{N}$-grading on $FP(V)$. Since the Lie bracket $\{-,-\}$ has degree $-1$, the Poisson bracket on $FP(V)$ has degree $-1$. Note that the multiplication on $FP(V)$ is homogeneous with respect to $\text{deg}_1$.

For the second grading, we assume that
\[
\text{deg}_2(x) = 1, \quad \forall \ 0 \neq x \in V
\]
and make the free Lie algebra $\text{Lie}(V)$ $\mathbb{N}$-graded (namely, $\{-,-\}$ is homogeneous of degree zero). Then we extend the $\mathbb{N}$-grading to $FP(V)$ so that both the Poisson bracket and the multiplication are homogeneous of degree zero.

Let $\{v_i\}_{i \in I}$ be a $k$-basis of $V$ and $\{x_j\}_{j \in J}$ a $k$-basis of $\text{Lie}(V)$. Let $A$ be the free Poisson algebra $FP(V)$ and let $A^c$ be the $A^p$-submodule of $A$ generated by monomials $x_1^{a_1} \cdots x_n^{a_n}$, for $x_1, \cdots, x_n \in \text{Lie}(V)$, which are not in $A^p$.

Recall that
\[
(E2.5.1) \quad \{f_1,f_2,\cdots,f_n\} := \{f_1,\{f_2,\cdots,\{f_{n-1},f_n\}\}\}
\]
for all $f_i \in A$.

Lemma 2.6. Let $A$ be a free Poisson algebra $FP(V)$.

1. Let $f_1, \cdots, f_n$ be polynomials in $v_i$ (not $x_i$). If $p$ does not divide $n-1$, then $\{f_1,f_2,\cdots,f_n\} \in A^c$.
2. Let $f,g$ be polynomials in $v_i$. Then $\Lambda_p(f,g) \in A^c$.
3. The following elements are in $A^c$ for any polynomials in $f,g,h$ in $v_i$:
   (a) $\Lambda_p(f,g), \Lambda_p(f^2, g^2), \Lambda_p(f^2 + g^2, 2fg)$.
   (b) $\Lambda_p(fg,h), \Lambda_p((fg)^2, h^2), \Lambda_p((fg)^2 + h^2, 2fgh)$.
   (c) $\Lambda_p(fg, fh)$.

Proof. (1) By linearity, we may assume that all $f_s$ are monomials in $\{v_i\} \subseteq V$. Then $\text{deg}_1 f_s = \text{deg}_2 f_s$ for $s = 1, \cdots, n$. Let $F := \{f_1,f_2,\cdots,f_n\}$. Then
\[
\text{deg}_1 F = -n + 1 + \text{deg}_2 F.
\]
Since $p$ does not divide $n-1$, $p$ can not divide both $\text{deg}_1 F$ and $\text{deg}_2 F$. This implies that $F \in A^c$.

(2) Note that $\Lambda_p(f,g)$ is a linear combination of terms of the form $E2.5.1$ when $n = p$ and $f_1 = f$ or $g$. By part (1), $\Lambda_p(f,g) \in A^c$.

(3) This is a special case of part (2) for different choices of $f,g$. $\square$

3. Restricted Poisson algebras, Definition

In this section we present a formulation of a restricted Poisson algebra that is equivalent to [HK] Definition 1.8.

Inspired by the notion of a restricted Lie algebra, we first introduce the definition of a weakly restricted Poisson structure over a field $k$ of characteristic $p \geq 3$. 
Definition 3.1. Let \((A, \{\cdot, \cdot\})\) be a Poisson algebra. If \(A\) admits a \(p\)-map \((-)^{(p)}: A \rightarrow A\) such that \((A, \{\cdot, \cdot\}, (-)^{(p)})\) is a restricted Lie algebra, then \(A\) is called a weakly restricted Poisson algebra.

This definition requires no compatibility condition between the \(p\)-map \((-)^{(p)}\) and the multiplication \(\cdot\). We will see that an additional requirement is very natural from a Lie algebraic point of view.

Lemma 3.2. Let \((A, \cdot, \{\cdot, \cdot\})\) be a Poisson algebra and let \(x, y \in A\).

1. If there exists \(\tilde{x}\) and \(\tilde{y}\) in \(A\) such that \(\text{ad}^p_x = \text{ad}\tilde{x}\) and \(\text{ad}^p_y = \text{ad}\tilde{y}\), then
\[
\text{ad}^p_{xy} = \text{ad}_{x\tilde{y} + y\tilde{x} + \Phi_p(x,y)},
\]
where
\[
\Phi_p(x,y) = (x^p + y^p)\Lambda_p(x,y) - \frac{1}{2}(\Lambda_p(x^2,y^2) + \Lambda_p(x^2 + y^2, 2xy)).
\]
In particular, \(\text{ad}^p_{x\tilde{x}} = \text{ad}2x\tilde{x}\).

2. If \((A, \cdot, \{\cdot, \cdot\})\) is a weakly restricted Poisson algebra, then
\[
\text{ad}_{(xy)(\cdot)} = \text{ad}_{x\tilde{y} + y\tilde{x} + \Phi_p(x,y)}.
\]
In particular,
\[
\text{ad}_{(x\tilde{x})(\cdot)} = \text{ad}2x\tilde{x}.
\]

Proof. (1) We first prove the assertion when \(x = y\). By the Leibniz rule, we have \(\text{ad}_{(fg)} = f\text{ad}_g + g\text{ad}_f\) for any \(f, g \in A\). Clearly,
\[
\text{ad}^p_{x\tilde{x}} = (2x\text{ad}_x)^p = (2x)^p(\text{ad}_x)^p = 2x^p\text{ad}_x^p = 2x^p\text{ad}\tilde{x} = \text{ad}2x\tilde{x}.
\]
In the general case, considering the universal enveloping algebra of the Lie algebra \((A, \{\cdot, \cdot\})\) and using \((\text{E1.1.4})\), we get \(\text{ad}_{\Lambda_p(f,g)} = \text{ad}^p_{f\tilde{y} + g\tilde{x}} - \text{ad}^p_{f\tilde{x} - \text{ad}^p_g}\) for any \(f, g \in A\). Therefore,
\[
\text{ad}_{x\tilde{y} + y\tilde{x} + \Phi(x,y)} = \text{ad}_{x\tilde{y} + y\tilde{x} + (x^p + y^p)\Lambda_p(x,y) - \frac{1}{2}(\Lambda_p(x^2,y^2) + \Lambda_p(x^2 + y^2, 2xy))}
+ \frac{1}{2}(\text{ad}^p_{x\tilde{y} + y\tilde{x} + \text{ad}^p_{x\tilde{y} + y\tilde{x} + \Phi(x,y)} - \text{ad}^p_{(x+y)^2})}
+ x^p\text{ad}^p_{x\tilde{y} + y\tilde{x} + \text{ad}^p_{x\tilde{y} + y\tilde{x} + \Phi(x,y)} - (x+y)^p\text{ad}^p_{x\tilde{y} + y\tilde{x} + (x^p + y^p)\text{ad}^p_{x\tilde{y} + y\tilde{x} + \Phi(x,y)} - \text{ad}^p_{(x+y)^2})}
= \text{ad}^p_{x\tilde{y} + y\tilde{x} + \Phi(x,y)},
\]
which completes the proof.

(2) It is an immediate consequence of (1).

Concerning the notation \(\Phi_p\) in \((\text{E3.2.1})\), we also have the following characterization by considering the Poisson enveloping algebra.

Proposition 3.3. Let \(A\) be a Poisson algebra and \(\mathcal{P}(A)\) the Poisson enveloping algebra of \(A\). Then, for all \(x, y \in A\), we have
\[
\text{ad}_{(xy)(\cdot)} = \text{ad}2x\tilde{x}.
\]
Comparing the above two equations, we get

\[ (0, dx^2)^P = (0, x dx)^P = ((2x, 0)(0, dx))^P = (2x, 0)^P(0, dx)^P = 2(x^p, 0)(0, dx)^P \]

and hence

(E3.3.2) \( (\iota_2(dx^2))^P = 2\iota_1(x^p)(\iota_2(dx))^P \)

for any \( x \in A \). It follows that the equation (E3.3.1) holds when \( x = y \).

Considering the Frobenius map of \( \mathcal{P}(A) \), we have

\[
(\iota_2(d(x + y)))^P = (0, d(x + y))^P = ((0, dx) + (0, dy))^P \\
= (0, dx)^P + (0, dy)^P + \Lambda_p((0, dx), (0, dy)) \\
= (\iota_2(dx))^P + (\iota_2(dy))^P + \iota_2(d\Lambda_p(x, y))
\]

since \( \iota_2 \) is a homomorphism of Lie algebras. By the above computation and (E3.3.2), we have

\[
(\iota_2(d(x + y)^2))^P = 2\iota_1((x + y)^p)(\iota_2(d(x + y)))^P \\
= 2\iota_1(x^p + y^p)(\iota_2(dx))^P + (\iota_2(dy))^P + \iota_2(d\Lambda_p(x, y)).
\]

By a direct calculation and (E3.3.2),

\[
(\iota_2(d(x + y)^2))^P = (\iota_2(dx^2 + dy^2 + 2d(xy)))^P \\
= (\iota_2(dx^2 + dy^2))^P + (\iota_2(2d(xy)))^P + \iota_2(d\Lambda_p(x^2 + y^2, 2xy)) \\
= (\iota_2(dx^2))^P + (\iota_2(dy^2))^P + \iota_2(d\Lambda_p(x^2, y^2)) \\
+ 2(\iota_2(d(xy)))^P + \iota_2(d\Lambda_p(x^2 + y^2, 2xy)) \\
= 2\iota_1(x^p)(\iota_2(dx))^P + 2\iota_1(y^p)(\iota_2(dy))^P + \iota_2(d\Lambda_p(x^2, y^2)) \\
+ 2(\iota_2(d(xy)))^P + \iota_2(d\Lambda_p(x^2 + y^2, 2xy))
\]

Comparing the above two equations, we get

\[
(\iota_2(dx^2))^P + \frac{1}{3}(\iota_2(d\Lambda_p(x^2, y^2) + \Lambda_p(x^2 + y^2, 2xy))) \\
= \iota_1(x^p)(\iota_2(dy))^P + \iota_1(y^p)(\iota_2(dx))^P + \iota_1(x^p + y^p)\iota_2(d\Lambda_p(x, y)) \\
= \iota_1(x^p)(\iota_2(dy))^P + \iota_1(y^p)(\iota_2(dx))^P + \iota_2(d((x^p + y^p)\Lambda_p(x, y))).
\]

Therefore,

\[
\nu_2(d\Phi_p(x, y)) = \nu_2(d((x^p + y^p)\Lambda_p(x, y) - \frac{1}{2}(\Lambda_p(x^2, y^2) + \Lambda_p(x^2 + y^2, 2xy)))) \\
= (\iota_2(d(xy)))^P - \iota_1(x^p)(\iota_2(dy))^P - \iota_1(y^p)(\iota_2(dx))^P.
\]

This finishes the proof. \(\square\)

For a weakly restricted Poisson algebra, it is desired to consider some compatibility between the \( p \)-map and the associative multiplication. By removing \( \text{ad} \) from (E3.2.3) (which can be done in some cases), we obtain (E3.4.1) below. Similarly, if we remove \( \text{ad} \) from (E3.2.2), we obtain (E3.5.1) below. Both Lemma 3.2 and Proposition 3.3 suggest the following definition. Following Lemma 3.2, condition (E3.4.1) is forced.
Definition 3.4. Let \((A, \cdot, \{-, -, \}_{\{p\}})\) be a weakly restricted Poisson algebra over \(k\). We call \(A\) a restricted Poisson algebra, if, for every \(x \in A\),

\[(x^2)^{\{p\}} = 2x^{p}x^{\{p\}}.\]  

In this case, the \(p\)-map \((-)^{\{p\}}\) is a restricted Poisson structure on \(A\).

Next we give another description of condition \((E3.4.1)\) which is convenient for some computation.

Proposition 3.5. Let \(A\) be a weakly restricted Poisson algebra.

1. Suppose \((E3.4.1)\) holds. Then \((\lambda 1_A)^{\{p\}} = 0\), for all \(\lambda \in k\).
2. Equation \((E3.4.1)\) holds for all \(x \in A\) if and only if every pair of elements\((x, y)\) in \(A\) satisfies

\[ (xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi_p(x, y). \]

As a consequence, \(A\) is a restricted Poisson algebra if and only if \((E3.5.1)\) holds.
3. Suppose \((E3.5.1)\) holds. Then

\[ (x^n)^{\{p\}} = n x^{(n-1)p} x^{\{p\}} \]

for all \(n\). As a consequence, \((x^p)^{\{p\}} = 0\) for all \(x \in A\).
4. If \((1_A)^{\{p\}} = 0\), then \((E3.5.1)\) holds for pairs \((x, \lambda 1_A)\) and \((\lambda 1_A, x)\) for all \(x \in A\) and all \(\lambda \in k\).

Proof. (1) Clearly, \((1_A)^{\{p\}} = 2 \cdot 1_A^p 1_A^{\{p\}}\) and hence \((1_A)^{\{p\}} = 0\). For every \(\lambda \in k\),

\((\lambda 1_A)^{\{p\}} = \lambda p 1_A^{\{p\}} = 0\).

(2) The “if” part is trivial since \(\Phi_p(x, x) = 0\) for any \(x \in A\). Next, we show the “only if” part. By \((E3.4.1)\) and Definition 1.1(3), we have

\[ ((x + y)^2)^{\{p\}} = 2(x + y)^p (x + y)^{\{p\}} = 2(x^p + y^p) (x^{\{p\}} + y^{\{p\}} + \Lambda_p(x, y)) \]

Since \((A, \{-, -, \}_{\{p\}})\) is a restricted Lie algebra, it follows from Definition 1.1(2,3) that

\[ ((x + y)^2)^{\{p\}} = (x^2 + y^2 + 2xy)^{\{p\}} = (x^2 + y^2)^{\{p\}} + 2p(x^2 y^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy) = (x^2)^{\{p\}} + (y^2)^{\{p\}} + \Lambda_p(x^2, y^2) + 2p(x^2 y^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy) = 2x^{p}x^{\{p\}} + 2y^{p}y^{\{p\}} + \Lambda_p(x^2, y^2) + 2(x^2 y^{\{p\}} + \Lambda_p(x^2 + y^2, 2xy) \]

Comparing the above two equations and using \(2 \neq 0\), we obtain equation \((E3.5.1)\).

(3) This follows by induction.

(4) First of all, \((\lambda 1_A)^{\{p\}} = \lambda p 1_A^{\{p\}} = 0\) for all \(\lambda \in k\). The assertion follows by the fact \(\Phi_p(\lambda 1_A, x) = \Phi_p(x, \lambda 1_A) = 0\).  

\[ \square \]

Remark 3.6. Several remarks are collected below.
(1) As in the paper [BK], we assume that \( p \geq 3 \). So the polynomial \( \Phi_p(x, y) \) in (E3.2.1) is well-defined. When \( p = 3 \), we have
\[
\Phi_3(x, y) = x^2y\{y, y, x\} + xy^2\{x, x, y\} + xy\{x, y\}^2.
\]
For any \( p > 3 \), it is too long to write out all terms like above.

(2) Considering \( \Phi_p(x, y) \) as an element in \( FP(V) \), where \( V = kx \oplus ky \), it is homogeneous of degree \( p + 1 \) with respect to \( \deg_2 \) and homogeneous of degree \( 2p \) with respect to \( \deg_1 \).

(3) In [BK, Definition 1.8], Bezrukavnikov-Kaledin defines a restricted Poisson algebra as a weakly restricted Poisson algebra \((A, \{−, −\}, (−)_{(p)})\) such that the \( p \)-map satisfies
\[
(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + P(x, y)
\]
for all \( x, y \in A \). Here \( P(x, y) \) is a canonical quantized polynomial determined by [BK (1.3)]. We will show that Equation (E3.6.1) is equivalent to (E3.5.1).

(4) The polynomial \( P(x, y) \) is defined implicitly, but it follows from [BK, (1.3)] that \( P(x, x) = 0 \). Therefore a restricted Poisson algebra in the sense of [BK, Definition 1.8] is a restricted Poisson algebra in the sense of Definition 3.4.

(5) There are other interpretations of \( \Phi_p(x, y) \). By using the equation
\[
xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]
\]
we obtain that
\[
(xy)^{\{p\}} = x^p y^{\{p\}} + y^p x^{\{p\}} + \Phi'_p(x, y)
\]
where
\[
\Phi'_p(x, y) = \frac{1}{4}\Lambda_p((x + y)^2, -(x - y)^2) + \frac{1}{2}((x^p + y^p)\Lambda_p(x, y) - (x^p - y^p)\Lambda_p(x, -y))
\]
One can show that \( \Phi_p(x, y) = \Phi'_p(x, y) \) in the free Poisson algebra generated by \( x \) and \( y \).

(6) The following is clear by definition.
(a) \( \Lambda_p(x, y) = \Lambda_p(y, x) \) for all \( x, y \in A \).
(b) If \( \{x, y\} = 0 \), then \( \Lambda_p(x, y) = 0 \).
(c) \( \Phi_p(x, y) = \Phi_p(y, x) \) for all \( x, y \in A \).
(d) If \( \{x, y\} = 0 \), then \( \Phi_p(x, y) = 0 \).

Lemma 3.7. Definitions of restricted Poisson algebras in Definition 3.4 and [BK Definition 1.8] are equivalent.

Proof. Let \( P(x, y) \) be the polynomial defined in [BK (1.3)]. By Proposition 3.5(2), it remains to show that \( P(x, y) = \Phi_p(x, y) \). Let \( Lie(V) \) be the free Lie algebra over a vector space \( V \) and consider the tensor (free) algebra \( T(V) \) as a universal enveloping algebra over \( Lie(V) \). Then we have a Poincaré-Birkhoff-Witt filtration on \( T(V) \). The free quantized algebra \( Q^\bullet(V) \) is the Rees algebra associated to this filtration. By definition, for each \( n \),
\[
F_n := F_n T(V) = k \oplus L^\bullet(V) \oplus (L^\bullet(V))^2 \oplus \cdots \oplus (L^\bullet(V))^n.
\]
Since the multiplication is commutative in a Poisson algebra, we have
\[ (x^p + y^p)^2 + \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y) \]
\[ = (x^p + y^p + \Lambda_p(x, y))^2 \]
\[ = (x + y)^{2p} \]
\[ = (x^2 + y^2 + xy + yx)^p \]
\[ = (x^2 + y^2)^p + (xy + yx)^p + \Lambda_p(x^2 + y^2, xy + yx) \]
\[ = x^{2p} + y^{2p} + \Lambda_p(x^2, y^2) + (xy)^p + (yx)^p + \Lambda_p(xy, yx) \]
\[ + \Lambda_p(x^2 + y^2, xy + yx), \]
and hence
\[ (xy)^p + (yx)^p - x^p y^p - y^p x^p \]
\[ = \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y) - (\Lambda_p(x, y))^2 \]
\[ - \Lambda_p(x^2, y^2) - \Lambda_p(xy, yx) - \Lambda_p(x^2 + y^2, xy + yx). \]
On the other hand,
\[ [x, y]^p = (xy - yx)^p = (xy)^p - (yx)^p + \Lambda_p(xy, -yx). \]
So we have
\[ 2P(x, y) = 2((xy)^p - x^p y^p) \]
\[ = \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y) - \Lambda_p(x^2 + y^2, xy + yx) \]
\[ + (\Lambda_p(x, y))^2 - \Lambda_p(xy, yx) - \Lambda_p(xy, -yx) + [x, y]^p - [x^p, y^p]. \]
In fact, it is easily seen that \( (\Lambda_p(x, y))^2 \in \mathcal{F}_2, [x, y]^p \in \mathcal{F}_p \). On the other hand,
\[ [x^p, y^p] = \text{ad}_x^p(y^p) = -\text{ad}_x^{p-1}(\text{ad}_y^p(x)) \in \mathcal{F}_1, \]
where \( \text{ad}_x(y) = [x, y] \). By the equation (E1.1.3), we have
\[ \Lambda_p(xy, yx) = \sum_{x_k = xy \text{ or } yx} \frac{1}{\#(xy)} \text{ad}_{x_1} \cdots \text{ad}_{x_{p-2}}([yx, xy]). \]
Since \([yx, xy] = [yx, yx + [x, y]] = [yx, [x, y]] \in \mathcal{F}_2 \), we have \( \Lambda_p(xy, yx) \in \mathcal{F}_p \). Similarly, \( \Lambda_p(xy, -yx) \in \mathcal{F}_p \). By definition \([BK (1.3)]\), \( P(x, y) \) is homogeneous of degree \( p + 1 \). Therefore, after removing lower degree components,
\[ 2P(x, y) = \Lambda_p(x, y)(x^p + y^p) + (x^p + y^p)\Lambda_p(x, y) - \Lambda_p(x^2 + y^2, xy + yx). \]
Since the multiplication is commutative in a Poisson algebra, we have
\[ P(x, y) = (x^p + y^p)\Lambda_p(x, y) - \frac{1}{2}(\Lambda_p(x^2, y^2) + \Lambda_p(x^2 + y^2, 2xy)) = \Phi_p(x, y). \]
\[ \square \]
4. Elementary properties and examples

We start with something obvious.

**Definition 4.1.** Let \((A, \cdot, \{-, -\}, (-)^{(p)})\) be a restricted Poisson algebra. A Poisson ideal \(I\) of \(A\) is said to be restricted, if \(x^{(p)} \in I\) for any \(x \in I\).

The proofs of the following three assertions are easy and omitted.

**Lemma 4.2.** Let \(A\) be a restricted Poisson algebra. Suppose that \(I\) is a Poisson ideal of \(A\) that is generated by \(\{x_i \mid i \in S\}\) as an ideal of the commutative ring \(A\). If \(x_i^{(p)} \in I\) for any \(i \in S\), then \(I\) is a restricted Poisson ideal.

**Proposition 4.3.** Let \(A\) be a restricted Poisson algebra and \(I\) a restricted Poisson ideal of \(A\). Then the quotient Poisson algebra \(A/I\) is a restricted Poisson algebra.

As a consequence, we have.

**Corollary 4.4.** Let \(f : A \to A'\) be a homomorphism of restricted Poisson algebras. Then \(\ker f\) is a restricted Poisson ideal of \(A\).

Let \(A^p\) be the subalgebra of \(A\) generated by \(\{f^p \mid f \in A\}\) — the image of the Frobenius map.

**Lemma 4.5.** Let \(A\) be a Poisson algebra and \(f, g, h \in A\). Then the following hold:

1. \(f^p \Phi_p(g, h) - \Phi_p(fg, h) + \Phi_p(f, gh) - h^p \Phi_p(f, g) = 0\).
2. If \(f\) is in the Poisson center of \(A\), then \(f^p \Phi_p(g, h) = \Phi_p(fg, h) = \Phi_p(g, fh)\).
3. \(\Phi_p(f, g + h) - \Phi_p(f, g) - \Phi_p(f, h) = \Lambda_p(fg, fh) - f^p \Lambda_p(g, h)\).

**Proof.** It is clear that (2) is a consequence of (1). It suffices to show assertions (1) and (3) for the free Poisson algebra \(FP(A)\) since there is a surjective Poisson algebra map \(FP(A) \to A\) [Lemma 2.4]. So the hypothesis becomes that \(f, g, h\) are in a \(k\)-space \(V\) sitting inside a free Poisson algebra \(FP(V)\).

When \(A\) is a free Poisson algebra \(FP(V)\), by Lemma 2.6(1), \(\iota_2\) is injective. It follows from Lemma 2.6(2) that

(a) the kernel of the map
\[ A \xrightarrow{\iota_2} \Omega_{A/k} \xrightarrow{\iota_2} \mathcal{P}(A) \]

is \(A^p\).

Let \(\{v_i\}_{i \in S}\) be a basis of the \(V\). Let \(A^c\) be the \(A^p\)-submodule of \(A = FP(V)\) defined before Lemma 2.6. Then

(b) \(A^c \cap A^p = \{0\}\) and \(\Lambda_p(x, y) \in A^c\) for all \(x, y \in k[V]\) by Lemma 2.6(2).

Now we prove (1) and (3) under conditions (a) and (b).

(1) For all \(f, u \in A\), \(d(f^p u) = f^p du\) and \(\iota_2(d(f^p u)) = (f^p, 0)(0, du) \in \mathcal{P}(A)\). By Proposition 3.3 we have
\[ \iota_2(d(f^p \Phi_p(g, h))) = (f^p, 0)(0, d(gh))^p - (fp, 0)(0, dh)^p - (fp, 0)(0, dg)^p, \]

and
for all $f, g, h \in V$. It follows that
\[
\nu_2(d(f \Phi_p(g, h) - \Phi_p(f, g, h) + \Phi_p(f, g) - \Phi_p(f, g)h^p)) = 0.
\]

By condition (a), we get
\[
\nu_2(d(f \Phi_p(g, h) - \Phi_p(f, g, h) + \Phi_p(f, g) - \Phi_p(f, g)h^p)) = 0.
\]

By definition, $X$ is in the $A^p$-submodule generated by $A_p(x, y)$ for all $x, y \in A$, or in $A^c$ as given in condition (b). But $A^p \cap A^c = \{0\}$ by condition (b), we obtain that $X = 0$ and that the desired identity holds.

(3) The proof of part (3) is similar to the proof of (1) and is omitted. \qed

**Proposition 4.6.** Let $A$ be a weakly restricted Poisson algebra.

1. If $(x, y)$ satisfies $\text{[E3.5.1]}$, then so do $(x, \lambda y)$ and $(\lambda x, y)$ for all $\lambda \in k$.
2. Let $f, g, h \in A$. Suppose that $(f, g)$ and $(g, h)$ satisfy $\text{[E3.5.1]}$. Then $(f, g, h)$ satisfies $\text{[E3.5.1]}$ if and only if $(f, gh)$ does.
3. If $(g, f)$ and $(f, h)$ satisfies $\text{[E3.5.1]}$, then so does $(f, g + h)$.

3'. If $(g, f)$ and $(h, f)$ satisfies $\text{[E3.5.1]}$, then so does $(g + h, f)$.
4. Fix an $x \in A$ and let $R_x$ be the set of $y \in A$ such that $(x, y)$ satisfies $\text{[E3.5.1]}$. Then $R_x$ is a $k$-subspace of $A$.

4'. Fix an $x \in A$ and let $L_x$ be the set of $y \in A$ such that $(y, x)$ satisfies $\text{[E3.5.1]}$. Then $L_x$ is a $k$-subspace of $A$.

**Proof.** (1) Assuming $\text{[E3.5.1]}$ for $(x, y)$, we have
\[
(x \lambda y)^{(p)} = \lambda (xy)^{(p)} = \lambda^p(xy)^{(p)}
\]
\[
= \lambda^p(x^p y^{(p)} + y^p x^{(p)} + \Phi_p(x, y))
\]
\[
= x^p(\lambda y)^{(p)} + (\lambda y)^p x^{(p)} + \lambda^p \Phi_p(x, y))
\]
\[
= x^p(\lambda y)^{(p)} + (\lambda y)^p x^{(p)} + \Phi_p(x, \lambda y),
\]
where the last equation is Lemma 4.5 (2). So $(x, \lambda y)$ satisfies $\text{[E3.5.1]}$. Similarly for $(\lambda x, y)$.

(2) By symmetry, we only prove one implication and assume that $(f, g, h)$ satisfies $\text{[E3.5.1]}$. We show next that $(f, g)$ satisfies $\text{[E3.5.1]}$.

\[
(f(gh))^{(p)} = ((f^p h^p)^{(p)} + h^p f^p (g)^{(p)} + \Phi_p(f, g, h))
\]
\[
= (f^p h^p (p) + h^p f^p g^p (p) + \Phi_p(f, g, h) + \Phi_p(f, g, h)) + h^p \Phi_p(f, g, h)
\]
\[
= f^p g^p h^p (p) + f^p h^p g^p (p) + \Phi_p(f, g, h) + h^p \Phi_p(f, g, h)
\]
\[
= f^p (g^p h^p (p) + h^p g^p (p) + \Phi_p(g, h)) + (gh)^p f^p (p) + \Phi_p(f, gh)
\]
\[
= f^p (gh)^p (p) + (gh)^p f^p (p) + \Phi_p(f, gh).
\]
(3) Assume \((f, g)\) and \((f, h)\) satisfies (E3.5.1). Then

\[
(f(h + g))^{[p]} = (fg + fh)^{[p]}
\]

\[
= (fg)^{[p]} + (fh)^{[p]} + \Lambda_p(fg, fh)
\]

\[
= f^p(g^{[p]} + g^p x^{[p]} + \Phi_p(f, g) + x^p h^{[p]} + h^p f^{[p]})
\]

\[
+ \Phi_p(f, h) + \Lambda_p(fg, fh)
\]

\[
= f^p(g^{[p]} + h^{[p]} + \Lambda_p(g, h)) + (g + h)^p f^{[p]} + \Phi_p(f, g + h)
\]

\[
= f^p(g + h)^{[p]} + (g + h)^p f^{[p]} + \Phi_p(f, g + h),
\]

where the second last equality is deduced from Lemma 4.5(3). So \((f, g + h)\) satisfies \((E3.5.1)\).

(3') is equivalent to (3).

(4) Let

\[\mathcal{R}_x = \{ y \in A \mid \text{ (E3.5.1) holds for the pair } (x, y) \}\].

By Proposition 4.6(1), we have

(i) if \(y \in \mathcal{R}_x\), then so is \(\lambda y\) for all \(\lambda \in k\).

By Proposition 4.6(3),

(ii) if \(g, h \in \mathcal{R}_x\), then so is \(g + h\).

By (i) and (ii) above, \(\mathcal{R}_x\) is a \(k\)-subspace of \(A\).

(4') This is true because \(\mathcal{L}_x = \mathcal{R}_x\).

The following result will be used several times.

**Theorem 4.7.** Let \(A\) be a weakly restricted Poisson algebra. Let \(b := \{b_i\}_{i \in \mathbb{G}}\) be a \(k\)-basis of \(A\). If (E3.5.1) holds for every pair \((x, y) \subseteq b\), then \(A\) is a restricted Poisson algebra.

**Proof.** We need to show that (E3.5.1) holds for all \(x, y \in A\). First we fix any \(x \in b\) and let

\[\mathcal{R}_x = \{ y \in A \mid \text{ (E3.5.1) holds for the pair } (x, y) \}\].

By Proposition 4.6(4), \(\mathcal{R}_x\) is a \(k\)-subspace of \(A\). By hypothesis, we see that \(b \subseteq \mathcal{R}_x\). Since \(b\) is a basis of \(A\), \(\mathcal{R}_x = A\).

Next we fix \(y \in A\) and consider

\[\mathcal{L}_y = \{ x \in A \mid \text{ (E3.5.1) holds for the pairs } (x, y) \}\].

Similarly, by Proposition 4.6(4'), \(\mathcal{L}_y\) is a \(k\)-subspace. It contains \(b\) because \(\mathcal{R}_x = A\) for all \(x \in b\) (see the first paragraph). Hence, \(\mathcal{L}_y = A\). This means that (E3.5.1) holds for all pairs \((x, y)\) in \(A\). Therefore \(A\) is a restricted Poisson algebra.

One of the main goals of this paper is to provide some interesting examples of restricted Poisson algebras. In the rest of this section we give some elementary (but nontrivial) examples. We would like to give a gentle warning before the examples. We have checked that all \(p\)-maps given below satisfy (E3.5.1), however our proofs
are tedious computations and therefore omitted. On the other hand, since the \(p\)-maps are explicitly expressed by partial derivatives, one can verify the assertions with enough patience. More sophisticated examples are given in later sections.

**Example 4.8.** Let \(A = \mathbb{k}[x,y]\) be a polynomial algebra in two variables \(x, y\), where the (classical) Poisson bracket is given by

\[
\{f, g\} = f_x g_y - f_y g_x.
\]

for all \(f, g \in A\). For any \(f, g \in A\), \(f_x, f_y\) are the partial derivative of \(f \) with respect to the variables \(x\) and \(y\), respectively. (The bracket defined in (E4.8.1) was the original Poisson bracket studied by many people including Poisson \cite{P} when \(k = \mathbb{R}\).)

(1) Let \(k\) be a base field of characteristic 3. For every \(f \in A\), we define

\[
f^{(3)} = f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy},
\]

where \(f_{xx}, f_{yy}\) and \(f_{xy}\) are the second order partial derivatives of \(f\). Then \((A, \cdot, \{-,-\}, \{-,-\}^{(3)})\) is a restricted Poisson algebra.

(2) Let \(k\) be a base field of characteristic 5. For every \(f \in A\), define

\[
f^{(5)} = f_{x_1 x_2} f_{x_2 x_2} + f_{x_2 x_1} f_{y_1 y_2} + f_{y_1 y_2} f_{x_1 y_2} + f_{y_1 y_2} f_{x_2 y_2} + f_{y_2 y_2} f_{x_1 x_2} + f_{y_2 y_2} f_{x_2 x_2} + f_{y_2 y_2} f_{x_2 x_2} + f_{y_2 y_2} f_{x_1 x_2} + f_{y_2 y_2} f_{x_2 x_2} + f_{y_2 y_2} f_{x_2 x_2}
\]

where \(f_{x_1 x_2} \cdots f_{x_k}\) denotes the \(k\)-th order partial derivative of \(f\) with respect to the variables \(x_1, x_2, \cdots, x_k\). Then \((A, \cdot, \{-,-\}, \{-,-\}^{(5)})\) is a restricted Poisson algebra.

See Example 4.2 for general \(p\). It would be interesting to understand the meaning of (E4.8.2) and (E4.8.3) and to find its connection with other subjects.

The next two are slight generalizations of the previous example.

**Example 4.9.** Suppose \(\text{char} \ k = 3\) and let \(A = \mathbb{k}[x,y]\) be a polynomial Poisson algebra in two variables \(x, y\), where the Poisson bracket is given by

\[
\{f, g\} = \varphi (f_x g_y - f_y g_x),
\]

and \(\varphi = \lambda x + \mu y + \nu\), \(\lambda, \mu, \nu \in \mathbb{k}\). For every \(f \in A\), we define

\[
f^{(3)} = \lambda \varphi f_x f_y^2 + \mu \varphi f_x f_y + \nu \varphi^2 (f_x^2 f_{yy} + f_y^2 f_{xx} + f_x f_y f_{xy}) + \lambda^2 y f_y^3 + \mu^2 x f_x^3.
\]

Then \((A, \cdot, \{-,-\}, \{-,-\}^{(3)})\) is a restricted Poisson algebra.

**Example 4.10.** Suppose \(\text{char} \ k = 3\) and let \(A = \mathbb{k}[x_1, x_2, \cdots, x_n]\) be a Poisson algebra, where the Lie bracket is given by \(\{x_i, x_j\} = 2c_{ij} \in \mathbb{k}\) with \(c_{ij} + c_{ji} = 0\) for \(1 \leq i, j \leq n\). Clearly, \(\{f, g\} = \sum_{1 \leq i, j \leq n} c_{ij} (f_j g_i - f_i g_j)\) for \(f, g \in A\), where \(f_i\) denotes the partial derivative of \(f\) with respect to the variable \(x_i\) for \(i = 1, 2, \cdots, n\). Then \(A\) is a restricted Poisson algebra with the p-map given by

\[
f^{(3)} = \sum_{1 \leq i, j, k, l \leq n} c_{ij} c_{kl} f_i f_k f_{jl}.
\]
for any \( f \in A \), where \( f_{jl} \) is the second partial derivation of \( f \) with respect to the variables \( x_j \) and \( x_l \).

5. Existence and uniqueness of restricted structures

By Lemma 5.2(2), a weakly restricted Poisson structure on a Poisson algebra is very close to a restricted Poisson structure (up to a factor in the Poisson center). In this section, we study the existence and uniqueness of (weakly) restricted Poisson structure. First we consider the trivial extension.

**Lemma 5.1.** Let \( A \) be a Poisson algebra and \( A = k \mathfrak{1}_A \oplus \mathfrak{m} \) as a Lie algebra decomposition.

1. If \( x \mapsto x^{(p)} \) is a restriction \( p \)-map of the Lie algebra \( \mathfrak{m} \), then it can naturally be extended to \( A \) by defining \( 1_A^{(p)} = 0 \). As a consequence, \( A \) is a weakly restricted Poisson algebra.

2. If, further, the \( p \)-map on \( \mathfrak{m} \) satisfies (E3.4.1), then so does the extended \( p \)-map on \( A \). In this case, \( A \) is a restricted Poisson algebra.

**Proof.** (1) This follows from Lemma 1.3. For all \( \lambda \in k \) and \( x \in \mathfrak{m} \), the \( p \)-map is defined by \( (\lambda 1_A + x)^{(p)} = x^{(p)} \).

(2) We check (E3.4.1) for elements in \( A \) as follows:

\[
((\lambda 1_A + x)^2)^{(p)} = (\lambda^2 1_A + 2\lambda x + x^2)^{(p)} = (2\lambda x + x^2)^{(p)} = (2\lambda x)^{(p)} + (x^2)^{(p)} = 2\lambda^p x^{(p)} + 2x^p x^{(p)} = 2(\lambda 1_A + x)^p x^{(p)} = 2(\lambda 1_A + x)^p (\lambda 1_A + x)^{(p)}.
\]

Therefore \( A \) is a restricted Poisson algebra.

The following example is immediate.

**Example 5.2.** (1) Let \( L \) be a restricted Lie algebra and let \( A = k \mathfrak{1}_A \oplus L \) where associate product \( L^2 = 0 \). Then \( A \) is a Poisson algebra in the obvious way. Both sides of (E3.4.1) are zero for elements in \( L \) (since \( L^2 = 0 \)). By Lemma 5.1(2), \( A \) is a restricted Poisson algebra.

(2) Considering a special case when \( L = kx + ky \) is a solvable Lie algebra with \([x, y] = x\). For \( f = \lambda x + \lambda y \in L \), we define the \( p \)-map by \( f^{(p)} = \lambda^{p-1}(\lambda x + \lambda y) \).

It is straightforward to check that \((L, (-)^{(p)})\) is a restricted Lie algebra. Let \( A = k \mathfrak{1}_A \oplus L \). Then, by part (1), \( A \) is a restricted Poisson algebra. As a commutative algebra, \( A = \frac{k[x, y]}{(x^2, xy, y^2)} \) with \( k \)-linear basis \( \{1, x, y\} \). The Poisson bracket is given by \([x, y] = x\).

Let \( L \) be a restricted Lie algebra. It is well known that the \( p \)-map of \( L \) is unique up to a semilinear map from \( L \) to \( Z(L) \), where \( Z(L) \) is the center of \( L \). Recall that a semilinear map \( \gamma : L \to Z(L) \) means that for any \( x, y \in A, \lambda \in k \),

\[
\gamma(x + y) = \gamma(x) + \gamma(y),
\]

\[
\gamma(1_A^{(p)} x) = 0.
\]

\[
\gamma((\lambda 1_A + x)^{(p)}) = \gamma(\lambda 1_A + x)^{(p)} = (\lambda \gamma(1_A) + \gamma(x))^{(p)}.
\]
\[ \gamma(\lambda x) = \lambda^{p}\gamma(x). \]

The following lemma is well-known and easy to prove.

**Lemma 5.3.** Let \((L, (\cdot)^{[p]})\) be a restricted Lie algebra.

1. Let \((-)^{(p)}\) be another restricted Lie structure on \(L\). Then there is a map \(\gamma : L \to Z(L)\) such that \((-)^{(p)} = (-)^{[p]} + \gamma.\)
2. Let \(\gamma\) be a map from \(L\) to \(Z(L)\). Then \((-)^{[p]} + \gamma\) is a restricted Lie structure on \(L\) if and only if \(\gamma\) is a semilinear map from \(L\) to \(Z(L)\).

Let \(A\) be a Poisson algebra over \(k\) and \(Z(A)\) the center of \(A\). Observe that \(Z(A)\) is a left \(A\)-module with the action given by

\[ A \times Z(A) \to Z(A), \quad (a, z) \mapsto a^{p}z. \]

A semilinear map \(\psi : A \to Z(A)\) is called a Frobenius derivation of \(A\) with the value in \(Z(A)\) provided that \(\psi(ab) = a^{p}\psi(b) + b^{p}\psi(a)\) for any \(a, b \in A\). For example, if \(\psi_{0} : A \to A\) is a derivation, then \(\psi : A \to Z(A)\), defined by \(\psi(a) = (\psi_{0}(a))^{p}\) for all \(a \in A\), is a Frobenius derivation of \(A\) with the value in \(Z(A)\).

By Lemma 5.3(1), any two restricted Poisson structures on \(A\) differ by a semilinear map \(\gamma\) which appears in the next proposition.

**Proposition 5.4.** Let \((A, \cdot, \{-, -\}, (-)^{(p)})\) be a restricted Poisson algebra and \(\gamma\) a map from \(A\) to itself. Then the map \((-)^{(p)} + \gamma\) is a restricted Poisson structure if and only if \(\gamma\) is a Frobenius derivation of \(A\) with value in \(Z(A)\).

**Proof.** Let \((-)^{(p)}_{1} : A \to A\) be another \(p\)-map such that \((A, \cdot, \{-, -\}, (-)^{(p)}_{1})\) is also a restricted Poisson algebra. Since \((-)^{(p)}_{1}\) and \((-)^{(p)}\) are restricted structures on Lie algebra \((A, \{-, -\})\), \(\gamma = (-)^{(p)}_{1} - (-)^{(p)}\) is a semilinear map from \(A\) to \(Z(A)\) by Lemma 5.3. Moreover, for any \(x, y \in A\), \((xy)^{(p)}_{1} = x^{p}y^{(p)} + y^{p}x^{(p)} + \Phi_{p}(x, y)\),

\[
 \begin{align*}
 \gamma(xy) &= (xy)^{(p)}_{1} - (xy)^{(p)} \\
 &= x^{p}(y^{(p)}_{1} - y^{(p)}) + y^{p}(x^{(p)}_{1} - x^{(p)}) \\
 &= x^{p}\gamma(y) + y^{p}\gamma(x)
 \end{align*}
\]

It follows that \(\gamma\) is a Frobenius derivation of \(A\) with values in \(Z(A)\).

Conversely, it follows from Lemma 5.3 that the map \((-)^{(p)} + \gamma\) is also a restricted Lie structure on \((A, \{-, -\})\), since \(\gamma\) is a semilinear map from \(A\) to \(Z(A)\) and \((-)^{(p)}\) is a \(p\)-map of Lie algebra \((A, \{-, -\})\). Moreover, for any \(x, y \in A\),

\[
(xy)^{(p)} + \gamma(xy) = x^{p}(y^{(p)} + \gamma(y)) + y^{p}(x^{(p)} + \gamma(x)) + \Phi_{p}(x, y).
\]

It follows that the Poisson algebra \(A\) together with the map \((-)^{(p)} + \gamma\) is a restricted structure. \(\square\)

By Proposition 5.3, the \(p\)-map of a restricted Poisson algebra is unique up to Frobenius derivations.
Remark 5.5. Let \((A, \{ \cdot, \cdot \}, (-)^{(p)})\) be a restricted Poisson algebra and let 
\(\gamma : A \to Z(A)\) be a semilinear map. Suppose that \(\gamma\) is not a Frobenius derivation 
(which is possible for many \(A\)) and defines a new \(p\)-map \((-)^{(p)} = (-)^{(p)} + \gamma\). Then 
by Proposition 5.4, \((A, \{ \cdot, \cdot \}, (-)^{(p)})\) is not a restricted Poisson algebra, but it is 
still a weakly restricted Poisson algebra by Lemma 5.3(2).

6. Restricted Poisson algebras from restricted Lie algebras

We start with a general result.

Theorem 6.1. Let \(A = \mathbb{k}[x_i \mid i \in T]\) be a polynomial Poisson algebra with an index 
set \(T\). If for each \(i \in T\), there exists \(\gamma(x_i) \in A\) such that 
\(\text{ad}^p_{x_i} = \text{ad}_{\gamma(x_i)}\), then \(A\) admits a restricted Poisson structure 
\((-)^{(p)}\) such that \(x_i^{(p)} = \gamma(x_i)\) for all \(i \in T\).

Proof. First we show that \(A\) has a weakly restricted Poisson structure, and then 
verify that the weakly restricted Poisson structure satisfies (E3.5.1).

For the sake of simplicity, we assume that \(T = \{1, 2, \ldots, n\}\). To apply Lemma 1.3, 
we choose a canonical monomial \(k\)-basis of \(A\), which is 
\[\{x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n} \mid i_1, i_2, \ldots, i_n \geq 0\}\].

We define \((x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n})^{(p)}\) inductively on the degree \(i_1 + i_2 + \cdots + i_n\) such that 
\[\text{ad}^p_{(x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n})} = \text{ad}^p_{(x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n})},\]

and therefore get the restricted Lie structure on \((A, \{ \cdot, \cdot \})\) by Lemma 1.3. 
For convenience, we denote \(x^I = x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}\) and \(|I| = i_1 + \cdots + i_n\) for \(I = \{i_1, \ldots, i_n\}\).

If \(|I| = 0\), then \(x^I = 1\), we define \(1^{(p)} = 0\) and if \(|I| = 1\), then \(x^I = x_i\) for some \(1 \leq i \leq n\). We define \(x_i^{(p)} = \gamma(x_i)\) for each \(1 \leq i \leq n\). By hypothesis, 
\[\text{ad}^p_{x_i} = \text{ad}_{\gamma(x_i)}\] for any \(I\) with \(|I| = 0, 1\).

Proceeding by induction and assuming that \((x^I)^{(p)}\) has been defined such that 
\[\text{ad}^p_{x_i} = \text{ad}_{\gamma(x_i)}\] for any \(x^I\) with \(|I| \leq m\). For each monomial \(x^I\) of degree \(m + 1\), we assume that \(k\) is the smallest subscript such that \(i_k \geq 1\) in \(I\), i.e. \(I = (0, \ldots, 0, i_k, \ldots, i_n)\) and define 
\[(E6.1.1) \quad (x^I)^{(p)} = x_k^{(p)}(x_k^{i_k-1}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n})^{(p)} + (x_k^{i_k-1}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n})^p x_k^{(p)} \Phi_p(x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}).\]

By Lemma 5.2(1) for \((x, y)\) = \((x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n})\) and the above definition, we have 
\[\text{ad}^p_{x^I} = \text{ad}_{(x^I)^{(p)}}\] for any \(|I| = m + 1\), which completes the induction. By 
Lemma 1.3 \(A\) has a weakly restricted Poisson structure.

Now let \(b\) be the set of all monomials, which is a \(k\)-basis of \(A\). We prove that 
(E3.5.1) holds for any pair of elements \((x, y)\) in \(b\) by induction on \(\text{deg}x + \text{deg}y\). 
If \(x\) or \(y\) is 1, then (E3.5.1) holds trivially, which also takes care of the case when 
m := \(\text{deg}x + \text{deg}y \leq 1\). Suppose that the assertion holds for \(m\) and now assume 
that \(\text{deg}x + \text{deg}y = m + 1\). Let \(xy = x_k^{i_k}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n}\) where \(i_k > 0\). By (E6.1.1), 
the pair \((x_k, x_k^{i_k-1}x_{k+1}^{i_{k+1}} \cdots x_n^{i_n})\) satisfies (E3.5.1). By symmetry, we may assume 
that \(x = x_kg\). Then the above says that the pair \((x_k, gy)\) satisfies (E3.5.1). By
induction hypothesis, the pairs \((x_k, g)\) and \((g, y)\) satisfy [E3.5.1]. By Proposition 4.6, \((x, y) = (x_k, g, y)\) satisfies [E3.5.1]. By induction, [E3.5.1] holds for any two elements in \(b\). Finally the main statement follows from Theorem 4.7. \(\square\)

As a consequence, we have the following.

**Example 6.2.** Let \(L\) be a restricted Lie algebra. We claim that the polynomial Poisson algebra \(A := \mathbb{k}[L]\) (also denoted by \(S(L)\)) is a restricted Poisson algebra. Let \(\{x_i\}_{i \in I}\) be a basis of \(L\). Then, for each \(i\), there is an \(\gamma(x_i) := x_i^{[p]} \in L\) such that \(\text{ad}_{x_i} = \gamma(x_i)\) when restricted to \(L\). Since \(A\) is a polynomial ring over \(L\), both \(\text{ad}_{x_i}^{[p]}\) and \(\gamma(x_i)\) extends uniquely to derivations of \(A\). Thus \(\text{ad}_{x_i}^{[p]} = \gamma(x_i)\) holds when applying to \(A\). The claim follows from Theorem 6.4 and there is a unique restricted structure \((-)^{(p)}\) on \(A\) such that \(x^{(p)} = x^{[p]}, \forall x \in L\).

Let \(V\) be a vector space. Then the free restricted Lie algebra \(RLie(V)\) can be defined by using the universal property or by taking the restricted Lie subalgebra of the free associative algebra generated by \(V\) with the \(p\)-map being the \(p\)-powering map. Now we can define the free restricted Poisson algebra generated by \(V\).

**Definition 6.3.** Let \(V\) be a \(\mathbb{k}\)-space. The free restricted Poisson algebra generated by \(V\) is defined to be

\[
\text{FRP}(V) = \mathbb{k}[RLie(V)].
\]

The following universal property is standard [Sh, Lemma 1, p. 312].

**Lemma 6.4.** Let \(A\) be a restricted Poisson algebra and \(V\) be a vector space. Every \(\mathbb{k}\)-linear map \(g : V \to A\) extends uniquely to a restricted Poisson algebra morphism \(G : \text{FRP}(V) \to A\) such that \(g\) factors through \(G\).

Continuing Example 6.2 when \(L\) is a restricted Lie algebra over \(\mathbb{k}\) and \(S(L) := \mathbb{k}[L]\) the symmetric algebra on \(L\), then \(S(L)\) admits an induced restricted Poisson structure. One natural setting in positive characteristic is to replace the symmetric algebra \(S(L)\) by the truncated (or small) symmetric algebra \(s(L)\). By definition, when \(L\) has a \(\mathbb{k}\)-basis \(\{x_i\}_{i \in I}\),

\[
(E6.4.1)\quad s(L) = \mathbb{k}[x_i \mid i \in I]/(x_i^{[p]}, \forall i \in I).
\]

It is easily seen that \(s(L)\) admits a Poisson structure with the bracket \(\{f, g\} = \sum \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}\right)\{x_i, x_j\}\) for any \(f, g \in s(L)\). Next we show that \(s(L)\) has a natural restricted Poisson structure.

**Theorem 6.5.** Let \(L\) be a restricted Lie algebra over \(\mathbb{k}\) of characteristic \(p\) and let \(s(L)\) be the Poisson algebra with the bracket induced by \(L\). Then \(s(L)\) admits a natural restricted Poisson structure induced by the \(p\)-map of \(L\).

**Proof.** By Example 6.2 \(S(L)\) has an induced restricted Poisson algebra structure. By [E6.4.1],

\[
s(L) = S(L)/J
\]

where \(J\) is the Poisson ideal generated by \(x_i^{[p]}\) for all \(i \in I\). By Proposition 4.2, \((x_i^{[p]})^{(p)} = 0\). By Lemma 4.2 \(J\) is a restricted Poisson ideal as desired. \(\square\)
7. Restricted Poisson algebras from deformation quantization

We will produce more examples in this section.

Let $A$ be a commutative (associative) algebra. Let $k[[t]]$ be the formal power series ring in one variable $t$. A formal deformation of $A$ means an associative algebra $A[[t]]$ over $k[[t]]$ with multiplication, denoted by $m_t$, satisfying

$$m_t(a \otimes b) = a * b = ab + m_1(a, b)t + \cdots + m_n(a, b)t^n + \cdots,$$

for all $a, b \in A \subset A[[t]]$. We should view $A[[t]]$ as the power series ring in one variable $t$ with coefficients in $A$ where the associative multiplication $m_t$ (or the star product $*$) is induced by a family of $k$-bilinear maps $\{m_i : A \otimes A \rightarrow A\}_{i \geq 0}$ with $m_0(a, b) = ab$.

Define a bilinear map $\{-, -\} : A \otimes A \rightarrow A$ by setting $\{a, b\} = m_1(a, b) - m_1(b, a)$. It is easy to check that $A$ together with the bracket $\{-, -\}$ is a Poisson algebra. Then $(A, \{-, -\})$ is called the classical limit of $(A[[t]], m_t)$, and $(A[[t]], m_t)$ is called a deformation quantization of the Poisson algebra $(A, \{-, -\})$.

For every $f \in A$, we write the $p$-power of $f$ as

$$(E7.0.1) \quad f^{*p} = \sum_{n=0}^{\infty} M_n^p(f)t^n = f^p + M_1^p(f)t + M_2^p(f)t^2 + \cdots \in A[[t]]$$

where $M_i^p(f) \in A$ for all $i = 0, 1, 2, \cdots$.

**Proposition 7.1.** Let $(A, \cdot, \{-, -\})$ be a Poisson algebra over $k$ and let $(A[[t]], *)$ be a deformation quantization of $A$. If $M_n^p(f) = 0$ for $1 \leq n \leq p - 2$ and $f^p$ is central in $A[[t]]$ for all $f \in A$, then $A$ admits a restricted Poisson structure.

**Proof.** Recall that $f^*g = \sum_{n=0}^{\infty} m_n(f, g)t^n \in A[[t]]$ for all $f, g \in A$, where $m_n(f, g) \in A$ for all $n$. By the definition of the deformation quantization,

$$\{f, g\} = \lim_{t \to 0} \frac{f^*g - g^*f}{t} = m_1(f, g) - m_1(g, f).$$

for all $f, g \in A$. Considering the Frobenius map $f \mapsto f^{*p}$ in $A[[t]]$, we get

$$(E7.1.1) \quad [f^{*p}, g]* = \{f, \cdots, f, g\}*_{p \text{ copies}}$$

for all $f, g \in A$.

Since $[f, g]* = \{f, g\}t \pmod{t^2}$ and $[-, -]*$ is $k[[t]]$-bilinear, we have

$$\{f, \cdots, f, g\}*_{p \text{ copies}} \equiv \{f, \cdots, f, g\}t^p \pmod{t^{p+1}}.$$

By assumption, $M_n^p(f) = 0$ for $1 \leq n \leq p - 2$ and $f^p$ is central in $A[[t]]$. Using the fact that $\{E7.1.1\}$ or $\text{ad}_{f^{*p}}(g) = (\text{ad}_f)^p(g)$, it follows that

$$\{M_{p-1}^p(f), g\}t^p = \{f, \cdots, f, g\}t^p \pmod{t^{p+1}}$$
Moreover, for every \( f \in A \) we define \( f^{(p)} = M_{p-1}^{\lambda}(f) \) for any \( f \in A \), and prove that the map \( f \mapsto M_{p-1}^{\lambda}(f) \) gives rise to a restricted Poisson structure on \( A \).

Note that Definition 1.1(1) follows from (E7.1.2). Definition 1.1(2) follows from the fact that \((\lambda f)^{\ast p} = \lambda^{p} f^{\ast p}\). For condition in Definition 1.1(3), we consider the Frobenius map of \( A[[t]] \), and get a restricted Lie structure of \((A[[t]], [-, -]), \ast\). It follows from Example 1.2 that

\[
(f + g)^{\ast p} - f^{\ast p} - g^{\ast p} = \Lambda_{p}^{\ast}(f, g).
\]

Computing the coefficients of \( t^{p-1} \) of the above equation, we get

\[
(E7.1.3) \quad (f + g)^{\langle p \rangle} - f^{\langle p \rangle} - g^{\langle p \rangle} = \Lambda_{p}(f, g)
\]

as desired.

Finally it remains to show (E3.4.1). By assumption, \( M_{n}^{\lambda}(f) = 0 \) for all \( 1 \leq n \leq p - 2 \). We compute the coefficient of \( t^{p-1} \) in the expression of \( f^{\ast,2p} \) as follows:

\[
f^{\ast,2p} = f^{\ast p} \ast f^{\ast p} = (f^{p} + t^{p-1}M_{p-1}^{\lambda}(f) + \cdots) \ast (f^{p} + t^{p-1}M_{p-1}^{\lambda}(f) + \cdots) = f^{2p} + 2f^{p}M_{p-1}^{\lambda}(f)t^{p-1} + \Lambda_{p}^{(f^{2}t^{p-1})}(mod \ t^{p})
\]

Assume that \( f \ast f = f^{2} + tW \), where \( W = m_{1}(f, f) + m_{2}(f, f)t + \cdots \), and it follows that

\[
f^{\ast,2p} = (f^{2})^{\ast p} = (f^{2} + tW)^{\ast p} = (f^{2})^{\ast p} + (tW)^{\ast p} + \Lambda_{p}^{(f^{2}t^{p-1})} \quad (mod \ t^{p})
\]

Therefore, for all \( f \in A \), \( f^{2\langle p \rangle} = 2f^{p}f^{\langle p \rangle} \), which is (E3.4.1).

We now give some explicit examples.

**Example 7.2.** Let \( A = \mathbb{k}[x, y] \) be a Poisson algebra over a field \( \mathbb{k} \) of characteristic \( p \geq 3 \) with the bracket given by \( \{x, y\} = 1 \).

Let \( \mu \) be the multiplication of the commutative algebra \( A[[t]] \). By a direct calculation, the Poisson algebra \( A \) admits a deformation quantization \((A[[t]], \ast)\) with the star product given by

\[
f \ast g = \mu(\exp(t(\partial_{1} \otimes \partial_{2}))(f \otimes g))
\]

for all \( f, g \in A \), where \( \partial_{1} \) and \( \partial_{2} \) are the partial derivatives of \( f \) with respect to the variables \( x \) and \( y \), respectively. To be precise, we have

\[
f \ast g = \sum_{0 \leq n \leq p-1} m_{n}(f, g)t^{n} = \sum_{0 \leq n \leq p-1} \frac{t^{n}}{n!}(\partial_{1}^{n}f)(\partial_{2}^{n}g).
\]

Clearly, \( f^{p} \ast g = f^{p}g = g \ast f^{p} \) for any \( f, g \in A \) and hence \( f^{p} \) is central in \( A[[t]] \). Moreover, for every \( f \in A \), we claim that

\[
(E7.2.1) \quad M_{n}(f) = 0 \quad \text{for} \quad 1 \leq n \leq p - 2.
\]
The proof of the above is given in Appendix. By Proposition 7.1, \( A \) admits a restricted Poisson structure with the p-map \( f^{(p)} = M^p_{p-1}(f) \) for any \( f \in A \). The p-map agrees with \([A[[t]], \ast]\) when \( p = 3 \) and \([A[[t]], \ast]\) when \( p = 5 \).

The next is a generalization of the previous example.

**Example 7.3.** Let \( A = k[x_1, \ldots, x_m] \) be a Poisson algebra with the bracket given by \( \{x_i, x_j\} = c_{ij} \in k \) for \( 1 \leq i < j \leq n \). By direct calculation, a deformation quantization \((A[[t]], \ast)\) of the Poisson algebra \( A \) is given by

\[
f \ast g = \mu \left( \exp \left( \sum_{1 \leq i < j \leq m} c_{ij} \partial_i \otimes \partial_j \right) (f \otimes g) \right)
\]

for all \( f, g \in A \), where \( \partial_i \) is the partial derivative of \( f \) with respect to the variable \( x_i \). This is well-defined by Remark 10.1(2). Clearly, \( f^p \in A \subset A[[t]] \) is central for any \( f \in A \). Being similar to the proof of Example 7.2, we have \( M_n(f) = 0 \) for \( 1 \leq n \leq p - 2 \) and all \( f \in A \). By Proposition 7.1, \( A \) admits a restricted Poisson structure with the p-map \( f^{(p)} = M^p_{p-1}(f) \) for any \( f \in A \). When \( p = 3 \), the p-map is given in Example 7.1.

**Example 7.4.** Let \( B_{2n} = k[x_1, \ldots, x_{2n}] / I \) be the p-truncated polynomial Poisson algebra in \( 2n \) variables over \( k \), where the Poisson bracket is defined by

\[
\{f, g\} = \sum_{i=1}^n (\partial_i(f) \partial_{n+i}(g) - \partial_{n+i}(f) \partial_i(g))
\]

for all \( f, g \in B_{2n} \), and \( I \) is generated by \( x_i^p, i = 1, \ldots, 2n \). In [Sk], Skryabin introduced the notion of the normalized p-map on \((B_{2n}, \{-, -\})\), say, \( 1^{(p)} = 0 \) and \( f^{(p)} \in m^2 \) for all \( f \in m^2 \), where \( m \) is the maximal ideal of \( B_{2n} \) as an associative algebra.

We consider the Poisson algebra \( A = k[x_1, \ldots, x_{2n}] \) in Example 7.3 with the bracket given by \( c_{ij} = \delta_{i+n,j} \) for all \( 1 \leq i < j \leq 2n \). Clearly, \( x_i^p \) is central and \( I \) is a Poisson ideal of \( A \). By Proposition 3.5(3), \( (x_i^p)^{(p)} = 0 \) for all \( i \in I \), and by Lemma 4.2, \( I \) is a restricted Poisson ideal of \( A \). Therefore, it follows from Proposition 4.3 that the Poisson algebra \( B_{2n} \) admits a restricted Poisson structure. Clearly, this p-map is normalized.

8. **Connection with restricted Lie-Rinehart Algebras**

Some definitions concerning Lie-Rinehart algebras were given in Section 2. Let \( A \) be a Poisson algebra and \( \Omega_A/k \) its Kähler differentials. Then the pair \((A, \Omega_A/k)\) is a Lie-Rinehart algebra over \( k \), where the anchor map \( \alpha : \Omega_A/k \to \text{Der}(A) \) is given in [Do]. Dokas introduced the notion of a restricted Lie-Rinehart algebra and study its cohomology theory in [Do]. The goal of this section is to show that the Lie-Rinehart algebra \((A, \Omega_A/k)\) admits a natural restricted structure if the Poisson algebra \( A \) is weakly restricted and \( \Omega_A/k \) is a free module over \( A \).

Let \((L, \{-\})^{[p]} \) and \((L', \{-\})^{[p]} \) be restricted Lie algebras. A map \( f : (L, \{-\})^{[p]} \to (L', \{-\})^{[p]} \) is called a restricted Lie homomorphism, if \( f \) is a Lie algebra homomorphism and satisfies \( f(x^{[p]}) = f(x)^{[p]} \) for all \( x \in L \).
The following definition was introduced by Dokas \cite{Do}.

**Definition 8.1.** \cite{Do} Definition 1.7 A restricted Lie-Rinehart algebra \((A, L, (-)^{[p]}))\) over a commutative \(k\)-algebra \(A\), is a Lie-Rinehart algebra over \(A\) such that

(a) \((L, (-)^{[p]})\) is a restricted Lie algebra over \(k\),
(b) the anchor map is a restricted Lie homomorphism, and
(c) the following relation holds:

\[
(aX)^{[p]} = a^p X^{[p]} + (aX)^{p-1}(a)X
\]

for all \(a \in A\) and \(X \in L\).

We now prove Theorem \[0.5\]

**Theorem 8.2.** Let \((A, \cdot, \{-, -, \}, (-)^{[p]})\) be a weakly restricted Poisson algebra. If the Kähler differential \(\Omega_{A/k}\) is a free, then the Lie-Rinehart algebra \((A, \Omega_{A/k}, (-)^{[p]})\) is restricted, where the \(p\)-map of \(\Omega_{A/k}\) is defined by

\[
(xdu)^{[p]} = x^p du^{[p]} + (xdu)^{p-1}(x)du,
\]

for all \(xdu \in \Omega_{A/k}\).

**Proof.** Since \(\Omega_{A/k}\) is a free \(A\)-module, \(\Omega_{A/k}\) can be embedded into the universal enveloping algebra \(\mathcal{U}(A, \Omega_{A/k})\) \cite[Lemma 2.3]{Do}. By the proof of \cite[Proposition 2.2]{Do}, it suffices to show that

\[
ad_{xdu}^p(ydv) = [x^p du^{[p]} + (xdu)^{p-1}(x)du, ydv]
\]

for all \(xdu, ydv \in \Omega_{A/k}\).

By Hochschild’s relation in \cite[Lemma 1]{Hoc}, we get in \(\mathcal{U}(A, L)\) the relation

\[
(\iota_2(xdu))^{[p]} = \iota_1(x^p)((\iota_2(du))^{[p]} + \iota_2((xdu)^{p-1}(x)du)
\]

for all \(xdu \in \Omega_{A/k}\). Considering the Frobenius map of \(\mathcal{U}(A, L)\), we have

\[
[(\iota_2(du))^{[p]}, \iota_1(y)] = [\iota_2(du), \iota_1(y)] = \iota_1((\text{ad}_u)^p(y)),
\]

and hence \(\iota_2(du)^{p} \iota_1(y) = \iota_1(y) \iota_2(du)^{p} + \iota_1((\text{ad}_u)^p(y))\) for all \(du \in \Omega_{A/k}, y \in A\). Moreover, for \(xdu, ydv \in \Omega_{A/k} \subset \mathcal{U}(A, L)\),

\[
[\iota_1(x^p)((\iota_2(du))^{[p]}), \iota_2(ydv)] = \iota_1(x^p)((\iota_2(du))^{[p]} \iota_1(y) \iota_2(dv) - \iota_1(y) \iota_2(dv) \iota_1(x^p) \iota_2(du))^{[p]}
\]

\[
= \iota_1(x^p)((\iota_1(y) \iota_2(du))^{[p]} + \iota_1((\text{ad}_u)^p(y)) \iota_2(dv)
\]

\[
- \iota_1(y) \iota_1(x^p) \iota_2(dv) + \iota_1(\{v, x^p\}) \iota_2(du))^{[p]}
\]

\[
= \iota_1(x^p)[((\iota_2(du))^{[p]}, \iota_2(dv) + \iota_2(x^p(\text{ad}_u)^p(y)dv)
\]

\[
= \iota_1(x^p) \iota_2(d(\text{ad}_u^p(v))) + \iota_2(x^p(\text{ad}_u)^p(y)dv)
\]

and therefore,

\[
\iota_2(ad_{xdu}^p(ydv)) = [(\iota_2(xdu))^{[p]}, \iota_2(ysv)]
\]

\[
= \iota_1(x^p)((\iota_2(du))^{[p]} + \iota_2((\text{ad}_u)^{p-1}(x)du), \iota_2(ysv)]
\]

\[
= \iota_1(x^p)\iota_2(d(\text{ad}_u^p(v))) + \iota_2(x^p(\text{ad}_u)^p(y)dv)
\]

\[
+ \iota_2((\text{ad}_u)^{p-1}(x)du, ydv]
\]

\[
= \iota_2(x^p yd(\text{ad}_u^p(v))) + \iota_2(x^p(\text{ad}_u)^p(y)dv)
\]
...of two restricted Poisson algebras is again a restricted Poisson algebra.

\begin{equation}
\iota_2([x^p du^{(p)} + \{x|_\alpha\}] du, y dv],
\end{equation}

and hence \(\text{ad}_{\text{ad}}^{(p)} du(y dv) = [x^p du^{(p)} + \{x|_\alpha\}] du, y dv]\) as desired.

For Poisson algebras \(A\) in Examples [4.8, 4.10, 6.2, Theorem 6.5, Examples 7.2, 7.3], it is automatic that \(\Omega_{A/k}\) is free over \(A\).

9. Restricted Poisson Hopf Algebras

We first recall the definition of Poisson Hopf algebras. The notion of a Poisson Hopf algebra was probably first introduced by Drinfel’d [Dr1, Dr2] in 1980s, see also [DHL].

**Definition 9.1.** Let \(A\) be a Poisson algebra. We say that \(A\) is a Poisson Hopf algebra if

1. \(A\) is a Hopf algebra with usual operations \(\Delta, \epsilon, S\).
2. \(\Delta : A \to A \otimes A\) and \(\epsilon : A \to k\) are Poisson algebra morphisms and \(S : A \to A\) is a Poisson algebra anti-automorphism.

To define restricted Poisson Hopf algebras, we need first show that tensor product of two restricted Poisson algebras is again a restricted Poisson algebra.

**Proposition 9.2.** Let \(A\) and \(B\) be two restricted Poisson algebras. Then there is a unique restricted Poisson structure on \(A \otimes B\) such that

\begin{equation}
(a \otimes b)^{(p)} = a^{(p)} \otimes b^{(p)} + a^{p} \otimes b^{(p)}
\end{equation}

for all \(a \in A\) and \(b \in B\).

**Proof.** First of all, it is well-known that \(A \otimes B\) is a Poisson algebra with bracket defined by

\[\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}\]

for all \(a_1, a_2 \in A\) and \(b_1, b_2 \in B\).

Let \(\{a_i\}_{i \in I}\) (respectively, \(\{b_j\}_{j \in J}\)) be a \(k\)-basis of \(A\) (respectively, \(B\)) and assume that \(1_A \in \{a_i\}_{i \in I}\) and \(1_B \in \{b_j\}_{j \in J}\). Then \(\{a_i \otimes b_j\}_{i \in I, j \in J}\) is a \(k\)-basis of \(A \otimes B\).

For any \(a \in A\) and \(b \in B\), \(\text{ad}_{a \otimes b}^{(p)}\) is a derivation. For any \(c \otimes d \in A \otimes B\), we have

\[\text{ad}_{a \otimes b}^{(p)}(c \otimes d) = (1 \otimes d)\text{ad}_{a \otimes b}^{(p)}(c \otimes 1) + (c \otimes 1)\text{ad}_{a \otimes b}^{(p)}(1 \otimes d)\]

\[= (1 \otimes d)(\text{ad}_{a \otimes b}^{(p)}(c \otimes b^{(p)}) + (c \otimes 1)(a^{p} \otimes \text{ad}_{b}^{(p)}(d)))\]

\[= (1 \otimes d)(\text{ad}_{a \otimes b}^{(p)}(c \otimes b^{(p)})) + (c \otimes 1)(a^{p} \otimes \text{ad}_{b}^{(p)}(d))\]

\[= (1 \otimes d)(\text{ad}_{a \otimes b}^{(p)}(c \otimes b^{(p)})) + (c \otimes 1)(\text{ad}_{a}^{(p)} \otimes \text{ad}_{b}^{(p)}(1 \otimes d))\]

\[= (1 \otimes d)(\text{ad}_{a \otimes b}^{(p)}(c \otimes b^{(p)})) + (c \otimes 1)(\text{ad}_{a}^{(p)} \otimes \text{ad}_{b}^{(p)}(1 \otimes d))\]

\[= \text{ad}_{a}^{(p)}(c \otimes b^{(p)}) + a^{p} \otimes b^{(p)}(c \otimes 1)\]

\[= \text{ad}_{a}^{(p)}(c \otimes d) + \text{ad}_{a}^{(p)}(c \otimes d)\]

\[= \text{ad}_{a}^{(p)}(c \otimes d) + \text{ad}_{a}^{(p)}(c \otimes d).\]
In particular,
\[
\text{ad}_{a_i \otimes b_j}^p = \text{ad}_{(a_i, \{p\}) \otimes b_j^p + a_i^p \otimes b_j (p)}
\]
for all \(i\) and \(j\). Since \(\{a_i \otimes b_j\}_{i, j} \subseteq A \otimes B\), by Lemma 6.3, there is a unique weak restricted Poisson structure on \(A \otimes B\) such that
\[
(a_i \otimes b_j)^{(p)} = a_i^{(p)} \otimes b_j^p + a_i^p \otimes b_j (p)
\]
for all \(i, j\), which agrees with (E9.2.1). It remains to show that this weak restricted Poisson structure on \(A \otimes B\) is indeed a restricted Poisson structure and holds.

We first prove (E9.2.1). By (E9.2.2), \((a_i \otimes 1)^{(p)} = a_i^{(p)} \otimes 1\). It follows from Definition 1.1 that
\[
(a \otimes 1)^{(p)} = a^{(p)} \otimes 1
\]
for all \(a \in A\). By symmetry, \((1 \otimes b)^{(p)} = 1 \otimes b^{(p)}\) for all \(b \in B\). Since \(\{a_i \otimes 1, 1 \otimes b_j\} = 0\), (E9.2.2) implies that the pair \((a_i \otimes 1, 1 \otimes b_j)\) satisfies (E3.5.1). By Proposition 4.6(4), \(R_{a_i \otimes 1}\) is a \(k\)-vector space; and by assumption, \(\{b_j\}\) is a \(k\)-basis of \(B\), we have that \(R_{a_i \otimes 1} \supseteq B\). Or, for any \(b \in B\), the pair \((a_i \otimes 1, 1 \otimes b)\) satisfies (E3.5.1). By switching \(a\) and \(b\) and applying the same argument, one sees that any pair \((a \otimes 1, 1 \otimes b)\) satisfies (E3.5.1). This means that
\[
(a \otimes b)^{(p)} = (a \otimes 1)^{(p)}(1 \otimes b)^{p} + (a \otimes 1)^{p}(1 \otimes b)^{(p)} + \Phi_p(a \otimes 1, 1 \otimes b)
\]
\[
= (a \otimes 1)^{(p)}(1 \otimes b)^{p} + (a \otimes 1)^{p}(1 \otimes b)^{(p)}
\]
\[
= a^{(p)} \otimes b^p + a^p \otimes b(p).
\]
So we proved (E9.2.1).

For the rest, we claim that for any pair of elements \((a_i \otimes b_j, a_k \otimes b_l)\), (E3.5.1) holds. By using (E9.2.3), (E3.5.1) holds for all pairs of the form \((a \otimes 1, a' \otimes 1)\). By symmetry, (E3.5.1) holds for all pairs of the form \((1 \otimes b, 1 \otimes b')\). By (E9.2.1), (E3.5.1) holds for pairs of the form \((a \otimes 1, 1 \otimes b)\). Set \(f = a \otimes 1\), \(g = a' \otimes 1\), and \(h = 1 \otimes b\) for any \(a, a' \in A\) and \(b \in B\). Then \((f, g), (g, h)\) and \((f g, h)\) satisfy (E3.5.1). By Proposition 4.6(2), \((f, g h)\) satisfies (E3.5.1). Or equivalently, \((a \otimes 1, a' \otimes h)\) satisfies (E3.5.1). By symmetry, \((1 \otimes b, a \otimes b')\), \((a \otimes b, a' \otimes 1)\) and \((a \otimes b, 1 \otimes b')\) satisfy (E3.5.1). Recycle the letters and let \(f = a \otimes b\), \(g = a' \otimes 1\), and \(h = 1 \otimes b'\). We have that \((f, g), (g, h)\) and \((f g, h)\) all satisfy (E3.5.1). By Proposition 4.6(2), \((f, g h)\) satisfies (E3.5.1). By choosing special \(a, a', b, b'\) we have that \((a_i \otimes b_j, a_k \otimes b_l)\) satisfies (E3.5.1) as desired. This says that every pair of elements from the \(k\)-basis \(\{a_i \otimes b_j\}_{i, j} \subseteq A \otimes B\) satisfies (E3.5.1). By Theorem 4.7, the weak restricted Poisson structure on \(A \otimes B\) is actually a restricted Poisson structure.

The above proof shows that there is a unique restricted Poisson structure on \(A \otimes B\) satisfying (E9.2.2). Since (E9.2.1) is a consequence of (E3.5.1), the assertion follows.

Now it is reasonable to define a restricted Poisson Hopf algebra.

**Definition 9.3.** A restricted Poisson algebra \(H\) is called a **restricted Poisson Hopf algebra** if there are restricted Poisson algebra maps \(\Delta : H \to H \otimes H, \epsilon : H \to k\) and restricted Poisson algebra anti-automorphism \(S : H \to H\) such that \(H\) together with \((\Delta, \epsilon, S)\) becomes a Hopf algebra.
One canonical example is the following.

**Example 9.4.** Let $L$ be a restricted Lie algebra. Then $s(L)$ (given in Theorem 6.5) is a restricted Poisson Hopf algebra with the structure maps determined by

$$
\Delta : x \mapsto x \otimes 1 + 1 \otimes x,
\epsilon : x \mapsto 0,
S : x \mapsto -x
$$
for all $x \in L$. It is straightforward to check that $s(L)$ is a restricted Poisson Hopf algebra. Similarly, $S(L)$ (given in Example 6.2) is a restricted Poisson Hopf algebra with structure maps determined as above.

10. **Appendix: The proof of (E7.2.1).**

Let $A = \mathbb{k}[x, y]$ be the Poisson algebra with the Poisson bracket determined by

$$\{x, y\} = 1.$$

Recall from Example 7.2 that the deformation quantization of $A$ is isomorphic to an associative algebra $(A[[t]], \ast)$ such that the star product of $f, g \in A \subset A[[t]]$ is given by

$$(E10.0.1)\quad f \ast g = \mu(\exp(t(\partial_1 \otimes \partial_y))(f \otimes g)) = \mu\left(\sum_{i=0}^{\infty} \frac{t^i}{i!} \partial_1^i f \otimes \partial_y^i g,\right),$$

where $\mu : A \otimes A \to A$ is the multiplication operation of $A$. Define a sequence of Hasse-Schmidt derivations (or divided power derivations)

$$\partial_1^{(i)} = \frac{1}{i!} \left(\frac{\partial}{\partial x}\right)^i \quad \text{and} \quad \partial_2^{(i)} = \frac{1}{i!} \left(\frac{\partial}{\partial y}\right)^i, \quad \forall \ i \geq 0.$$

Then all of them are $\mathbb{k}$-linear operations from $A$ to $A$. Using these we can re-write part of $E10.0.1$ as

$$(E10.0.2)\quad \exp(t(\partial_1 \otimes \partial_y))(f \otimes g) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \partial_1^{(i)} f \otimes \partial_y^{(i)} g = \sum_{i=0}^{p-1} \frac{t^i}{i!} \partial_1^{(i)} f \partial_2^{(i)} g,$$

which is a sum of finitely many terms. Therefore $E10.0.1$ is well-defined and the summation in $E10.0.1$ is finite.

**Remark 10.1.** Consider a generalization of $E10.0.1$ in $n$ variables. Let $B = \mathbb{k}[x_1, \ldots, x_n]$ and $\partial_i = \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$.

1. For each $c_{ij} \in \mathbb{k}$, $\exp(t c_{ij} \partial_i \otimes \partial_j)(f \otimes g)$ is well-defined for all $f, g \in B$, and it is a sum of finitely many terms as in $E10.0.2$.

2. For a set of $\{c_{ij}\}_{1 \leq i, j \leq n}$,

$$(E10.1.1)\quad \exp(t \sum_{i,j} c_{ij} \partial_i \otimes \partial_j)(f \otimes g) = \prod_{i,j} (\exp(tc_{ij} \partial_i \otimes \partial_j))(f \otimes g),$$

which is well-defined for all $f, g \in B$ and is a sum of finitely many terms in a similar fashion as $E10.0.2$ (but more than $p$ terms in general).
We now go back to the case of two variables. Clearly,

$$\partial \circ \mu = \mu(\partial \otimes \text{id} + \text{id} \otimes \partial)$$

for \(\partial = \frac{\partial}{\partial x} \) or \(\frac{\partial}{\partial y}\), and hence

$$\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right)^n(\mu \otimes \text{id}) = (\mu \otimes \text{id})\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \text{id} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right)^n$$

for all \(n \geq 1\). It follows that

$$\exp(t\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y})(\mu \otimes \text{id}) = (\mu \otimes \text{id})\exp(t\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \text{id} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right)),$$

and therefore,

$$(f \ast g) \ast h = \mu(\exp(t\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}))(f \ast g) \otimes h)
= \mu(\exp(t\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}))(\mu \otimes \text{id}) \exp(t\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \text{id} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right))(f \otimes g \otimes h))
= \mu(\mu \otimes \text{id})(\exp(t\left(\frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} + \text{id} \otimes \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y}\right)))(f \otimes g \otimes h))$$

In general, for \(k \geq 2\) and for \(f_1, \ldots, f_k \in A \subset A[[t]]\),

$$(E10.1.2) \quad f_1 \ast \cdots \ast f_k = \mu^k(\exp(t \sum_{1 \leq i < j \leq k} \partial^j_i(f_1 \otimes \cdots \otimes f_k))).$$

where \(\partial^j_i = \text{id}^\otimes_{i-1} \otimes \frac{\partial}{\partial x} \otimes \text{id}^\otimes_{j-i-1} \otimes \frac{\partial}{\partial y} \otimes \text{id}^\otimes_{k-j}\) is a map from \(A^\otimes_k\) to itself for all \(1 \leq i < j \leq k\), \(\mu^2(a \otimes b) = (ab)\) for all \(a, b \in A\) (extended to a commutative multiplication on \(A[[t]]\)), and \(\mu^k = \mu^2(\mu^{k-1} \otimes \text{id})\), \(k \geq 3\).

Denote by \(M^n_p(f)\) the coefficient of \(t^n\) in \(f \ast \cdots \ast f \in A[[t]]\), see [E7.0.1]. For simplicity, we denote the map

$$\Phi^{i_1, \ldots, i_n}_{j_1, \ldots, j_n} = \mu^p \circ (\partial^j_{i_1} \circ \cdots \circ \partial^j_{i_n}) : A^\otimes_{\sum_{r=1}^p j_r} \otimes_{\sum_{r=1}^p i_r} \cdots \otimes_{\sum_{r=1}^p i_r} f \mapsto A$$

for \(1 \leq i_t < j_t \leq p, t = 1, \ldots, n, n \geq 1\). It follows from the equation \((E10.1.2)\) that

$$(E10.1.3) \quad M^n_p(f) = \frac{1}{n!} \sum_{\sum_{r=1}^p i_r = \sum_{r=1}^p j_r} \Phi^{i_1, \ldots, i_n}_{j_1, \ldots, j_n}(f \otimes \cdots \otimes f)$$

for all \(0 \leq n \leq p - 1\).

Claim 10.2. Retain the above notation. Then \(M^n_p(f) = 0\) for all \(1 \leq n \leq p - 2\) and all \(f \in A\).

This appendix is devoted to the proof of Claim 10.2. We need more notations.

Recall that an oriented graph is a pair \(G = (V(G), E(G))\), where \(V(G)\) is the set of vertices and \(E(G)\) is the set of edges. For \(\alpha \in E(G)\), we denote by \(s(\alpha)\) and \(t(\alpha)\) the source and the target of \(\alpha\), respectively.

Definition 10.3. An oriented graph \(G = (V(G), E(G))\) is called a totally ordered graph (called tograph for short), if the set \(V(G)\) of vertices is totally ordered set with the ordering \(\leq\) and for every \(\alpha \in E(G), s(\alpha) < t(\alpha)\).
Let $G$ be a tograph (possibly with multiple edges). Being similar to usual oriented graphs, for each $v \in V(G)$, we denote the indegree of $v$ by
\[
d^+_G(v) = \#\{\alpha \in E(G) \mid t(\alpha) = v\},
\]
the outdegree of $v$ by
\[
d^-_G(v) = \#\{\alpha \in E(G) \mid s(\alpha) = v\},
\]
the degree of $v$ by
\[
d_G(v) = d^+_G(v) + d^-_G(v).
\]
For $u, v \in V(G)$, we denote by $\nu(u, v)$ the number of the edges with the source $u$ and the target $v$, i.e. $\nu_G(u, v) = \#\{\alpha \in E(G) \mid s(\alpha) = u, t(\alpha) = v\}$.

Let $G$ and $G'$ be tographs. A bijection $f : V(G) \to V(G')$ is called an isomorphism, if $f$ preserves the order of vertices and $\nu_G(u, v) = \nu_{G'}(f(u), f(v))$ for all $u, v \in V(G)$. Two tographs $G$ and $G'$ are said to be isomorphic, denoted by $G \cong G'$, provided that there exists an isomorphism between $G$ and $G'$.

Clearly, the automorphism group of a tograph $G$ is a trivial group since $f$ preserves the order of vertices and $V(G)$ is totally ordered. Suppose that $G_1, \ldots, G_k$ and $G'_1, \ldots, G'_m$ are the connected components of $G$ and $G'$, respectively. Two tographs $G$ and $G'$ are said to be equivalent, and denoted by $G \sim G'$, if $m = k$ and there exists a permutation $\sigma \in S_m$ such that $G_i$ and $G'_{\sigma(i)}$ are isomorphic for each $i = 1, \ldots, m$.

Denote
\[
\Gamma_n = \{(i_1, \ldots, i_n; j_1, \ldots, j_n) \mid 1 \leq i_t < j_t \leq n, t = 1, \ldots, n\}.
\]
For each given $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$, we can assign a tograph, denoted by $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$, where
\begin{itemize}
  \item the set of vertices $V(G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}) = \{1, 2, \ldots, p\}$ with the usual ordering of natural numbers, and
  \item the set of edges $E(G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}) = \{(i_t, j_t) \mid t = 1, \ldots, n\}$.
\end{itemize}
We denote by $G_n$ the set of tographs $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$ for all $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$.

We consider the lexicographical order on the set $\{(i, j) \mid 1 \leq i < j \leq p\}$. To be precise, $(i, j) < (i', j')$ if and only if $i < i'$ or $i = i'$, $j < j'$.

Let $G$ and $G'$ be the tographs associated to elements $(i_1, \ldots, i_n; j_1, \ldots, j_n)$ and $(i'_1, \ldots, i'_n; j'_1, \ldots, j'_n) \in \Gamma_n$, respectively. Clearly, $G = G'$ if and only if there exists a permutation $\sigma \in S_n$ such that $(i'_k, j'_k) = (i_{\sigma(k)}, j_{\sigma(k)})$ for all $k = 1, \ldots, n$. Therefore, for each $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$, there exists a permutation $\sigma \in S_n$ such that $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} = G_{\sigma(j_1), \ldots, \sigma(j_n)}^{i_{\sigma(1)}, \ldots, i_{\sigma(n)}}$ with $(i_{\sigma(1)}, j_{\sigma(1)}) \leq \cdots \leq (i_{\sigma(n)}, j_{\sigma(n)})$.

**Lemma 10.4.** Retain the above notation.

(1) Let $G$ be the tograph associated to $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$. Then
\[
\Phi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f^{\otimes p}) = \frac{\partial^{d_G(1)} f}{\partial x_{d_G(1)}^{d_G(1)} \partial y_{d_G(1)}^{d_G(1)}} \cdots \frac{\partial^{d_G(p)} f}{\partial x_{d_G(p)}^{d_G(p)} \partial y_{d_G(p)}^{d_G(p)}}
\]
where $d_G(i), d^+_G(i)$ and $d^-_G(i)$ are the outdegree, the indegree and the degree of the vertex $i \in V(G)$, respectively.
(2) Let $G$ and $G'$ be the tographs associated to elements $(i_1, \ldots, i_n; j_1, \ldots, j_n)$ and $(i'_1, \ldots, i'_n; j'_1, \ldots, j'_n) \in \Gamma_n$, respectively. Then

$$
\Phi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f \otimes p) = \Phi_{j'_1, \ldots, j'_n}^{i'_1, \ldots, i'_n}(f \otimes p)
$$

for all $f \in A$ if and only if there exists a permutation $\sigma \in \mathfrak{S}_n$ such that

$$(d_G^+(i), d_G^-(i)) = (d_G^+(\sigma(i)), d_G^-(\sigma(i)))$$

for all $i = 1, \ldots, p$.

Proof. (1) By the definition of $\Phi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$, we immediately get the desired equality \[E10.4.1\].

(2) By (1), it is clear. \qed

For convenience, we denote

$$
G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f) = \frac{\partial^{d_G^+(1)} f}{\partial x^{d_G^+(1)} \partial y^{d_G^-(1)}} \cdots \frac{\partial^{d_G^+(p)} f}{\partial x^{d_G^+(p)} \partial y^{d_G^-(p)}},
$$

and hence $\Phi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f \otimes p) = G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f)$.

**Corollary 10.5.** Let $G$ and $G'$ be the tographs in $\mathcal{G}_n$. If $G$ is equivalent to $G'$, then $G(f) = G'(f)$ for any $f \in k[x, y]$.

**Remark 10.6.** The converse of Corollary \[10.5\] does not hold and a counter-example is $G = G_{112}^{(123)}$ and $G' = G_{123}^{(132)}$ when $p = 5$ and $n = 3$.

**Sketch Proof of Claim \[11.2\]** For each $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$, we denote by $N_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$ the number of the tographs which are equivalent to $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$.

We consider the decomposition

$$
G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} = G_{11} \cup \cdots \cup G_{1k_1} \cup \cdots \cup G_{r1} \cup \cdots \cup G_{rk_r},
$$

where $G_{is}$, for $1 \leq s \leq k_i$ and $1 \leq i \leq r$, are connected components of $G$ with $G_{is} \cong G_{jt}$ for all $1 \leq s, t \leq k_i$, and $G_{is} \not\cong G_{jt}$ for $i \neq j$. Denote $|V(G_{is})| = n_i$ for each $i = 1, \ldots, r$. By definition, a tograph $G'$ is equivalent to $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$, if and only if for each $i = 1, \ldots, r$, $G'$ admits $k_i$ connected components being isomorphic to $G_{is}$. Therefore, by combinatorial counting, we have that

$$
N_{j_1, \ldots, j_n}^{i_1, \ldots, i_n} = \frac{p!}{(n_1!)^{k_1} \cdots (n_r!)^{k_r} k_1! \cdots k_r!}
$$

for each $(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n$. Clearly, if $n \leq p - 2$, then the underlying graph of $G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}$ is not connected since $|E(G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n})| = n < p - 1 = |V(G_{j_1, \ldots, j_n}^{i_1, \ldots, i_n})| - 1$. Therefore, $r \geq 2$ and $1 \leq k_i, n_t < p$ for each $t$. Therefore, by \[E10.1.3\],

$$
M_n^p(f) = \frac{1}{n!} \sum_{(i_1, \ldots, i_n; j_1, \ldots, j_n) \in \Gamma_n} \Phi_{j_1, \ldots, j_n}^{i_1, \ldots, i_n}(f \otimes p)
$$

$$
= \frac{1}{n!} \sum_{G \in \mathcal{G}_n} \frac{n!}{\prod_{1 \leq u < v \leq p} \nu(u, v)!} G(f)
$$
\[
\frac{1}{n!} \sum_{[G] \in \mathcal{G}_n/\sim} \frac{n!}{\prod_{1 \leq u < v \leq p} \nu(u,v)! (n_1!)^{k_1} \cdots (n_r!)^{k_r} G(f)} \equiv 0 \pmod{p}
\]

where the sum \( \sum_{[G] \in \mathcal{G}_n/\sim} \) means that one take one element in each equivalence class of \( \mathcal{G}_n \) with respect to the relation \( \sim \). \( \square \)

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