Genuinely nonlinear impulsive ultra-parabolic equations and convective heat transfer on a shock wave front

I V Kuznetsov\textsuperscript{1,2} and S A Sazhenkov\textsuperscript{1,2}
\textsuperscript{1} Novosibirsk State University, Pirogova str. 1, Novosibirsk, Russia
\textsuperscript{2} Lavrentyev Institute of Hydrodynamics, ac. Lavrentyev av. 15, Novosibirsk, Russia
E-mail: sazhenkovs@yandex.ru

Abstract. In the present paper, we derive the kinetic equation and impulsive condition and formulate a class of kinetic solutions to impulsive ultra-parabolic equations. Here ultra-parabolic equations are linked with diffusion processes with inertia (convective heat transfer) on a shock wave front.

Key words: ultra-parabolic equation, genuine nonlinearity condition, kinetic solution, convective heat transfer, impulsive condition

1. Introduction
In this paper, we aim to give a description of heat transfer in continuous medium in presence of shock waves, taking into account radiation on a shock wave front [1] in the form of a convective heat transfer. It is known that the maximum entropy production principle leads to the radiative heat transfer on a shock wave front [2]. Moreover, outside of a shock wave front, the heat transfer is conductive. In view of possible application, our research is directed towards study of heat evolution in glaciers. It is important to note that convective heat transfer in a glacier is not a novelty, see [3]. In this paper, we apply the ultra-parabolic equation. Instead of inertia linked with the unidirectional flow [4, 5], we deal with inertia because of the maximum entropy production principle on a shock wave front. Also, it is important to mention impact-induced thermal convection phenomenon [6, Equations (1)–(7)], [7]. Here we assume that a shock wave is propagating in the direction $s$. The dissipation term in the direction $s$ can be ignored. Therefore heat flux $a(x, t, s, u)$ in the $s$ direction depends also on other physical directions $x = (x_1, \ldots, x_d)$, time variable $t$, and the sought specific heat distribution $u$.

We assume that, in a glacier, a shock wave [8] is responsible for convective heat transfer. In this model we include the Bridgman effect in ice [9, 10] which is simulated with the help of an impulsive condition. Some references on impulsive partial differential equations are given in [11]. Considering, in particular, the process of the crater formation at the Yamal Peninsula, we assume that it is linked with ice-melting following degasation on a shock wave [12].

The following report is devoted to kinetic solutions of Problem $\Pi_0$, see sections 2 and 5. We derive the kinetic equation and justify that initial (in $t$) and impulsive conditions hold true. Strong traces of kinetic solutions would be justified in our future research. The questions linked with the uniqueness are still open.
2. Problem II₀ – the basic formulation

Let Ω be a bounded domain of spacial variables \( x \in \mathbb{R}^d \) with a smooth boundary \( \partial \Omega \) (\( \partial \Omega \in C^2 \)). Let \( t \in [0, T] \) and \( s \in [0, S] \) be two independent time-like variables. Here \( T \) and \( S \) are given positive constants. Let \( \tau \in (0, T) \) be a fixed time moment.

Denote \( G_{T,S} := (\Omega \times (0, \tau) \times (0, S)) \cup (\Omega \times (\tau, T) \times (0, S)), \Xi^1 := \Omega \times [0, S], \Xi^2 := \Omega \times [0, T], \Gamma^0 := \Omega \times \{t = 0\} \times [0, S], \Gamma^1 := \Omega \times \{t = \tau\} \times [0, S], \Gamma^2 := \Omega \times [0, \tau] \times \{s = 0\}, \Gamma_S := \Omega \times [0, T] \times \{s = S\}, \Gamma_I := \partial \Omega \times [0, T] \times [0, S]. \)

In this paper we study the following Cauchy-Dirichlet problem.

**Problem II₀.** For arbitrary initial and final data \( u^{(1)}_0 \in C^{0,\alpha}_0(\Xi^1), u^{(2)}_0, u^{(2)}_S \in C^{0,\alpha}_0(\Xi^2) \), and impulsive data \( \beta \in C^{0,\alpha}_0(\Xi^1) \) (\( \alpha \in (0,1) \)), it is necessary to find a function \( u : G_{T,S} \mapsto \mathbb{R} \) satisfying

- quasilinear ultra-parabolic equation
  \[
  \partial_t u + \partial_x a(x, t, s, u) + \text{div}_x \varphi(x, t, s, u) = \Delta_x u + g(x, t, s, u), \quad (x, t, s) \in G_{T,S}, \tag{2.1a}
  \]
- initial condition with respect to time-like variable \( t \)
  \[
  u|_{t=0} = u^{(1)}_0(x, s), \quad (x, s) \in \Xi^1, \tag{2.1b}
  \]
- initial and final conditions with respect to time-like variable \( s \)
  \[
  u|_{s=0} \approx u^{(2)}_0(x, t), \quad u|_{s=S} \approx u^{(2)}_S(x, t), \quad (x, t) \in \Xi^2, \tag{2.1c}
  \]
- homogeneous boundary condition
  \[
  u|_{\Gamma_I} = 0, \tag{2.1d}
  \]
- impulsive condition
  \[
  u(x, \tau + 0, s) = u(x, \tau - 0, s) + \beta(x, s), \quad (x, s) \in \Xi^1. \tag{2.1e}
  \]

By \( C^{0,\alpha}_0(\Xi) \) we standardly denote the space of Hölder continuous with exponent \( \alpha \in (0,1) \) finite functions on a closed set \( \Xi \in \mathbb{R}^N \) supplemented with the norm

\[
||\Phi||_{C^{0,\alpha}_0(\Xi)} = \max_{\zeta \in \Xi} |\Phi(\zeta)| + \sup_{\zeta, \eta \in \Xi, \zeta \neq \eta} \frac{|\Phi(\zeta) - \Phi(\eta)|}{|\zeta - \eta|^\alpha}.
\]

In (2.1c) the relation sign \( \approx \) means that \( u^{(2)}_0 \) and \( u^{(2)}_S \) may be unattained by a solution \( u \) on some parts of the sets \( \Gamma^0_I \) and \( \Gamma^1_I \), respectively. The fact whether \( \approx \) becomes equality (=), or not, is figured out a posteriori, i.e., after a solution of Problem II₀ is constructed somehow.

In the formulation of Problem II₀ the functions \( a = a(x, t, s, \lambda) \) and \( g = g(x, t, s, \lambda) \), and the nonlinear vector-function \( \varphi = (\varphi_1(x, t, s, \lambda), \ldots, \varphi_d(x, t, s, \lambda)) \) are given and satisfy the following conditions.

**Conditions on \( a, \varphi \& g. \)**

(i) There exist constants \( b_1, b_2 > 0 \) such that for all \( (x, t, s) \in G_{T,S} \) and \( \lambda \in \mathbb{R} \) the inequality

\[
(\partial_x a(x, t, s, \lambda) + \text{div}_x \varphi(x, t, s, \lambda) - g(x, t, s, \lambda))\lambda \geq -b_1 \lambda^2 - b_2 \tag{2.2a}
\]

is valid.
(ii) Functions \( a = a(x, t, s, \lambda) \), \( \varphi_i = \varphi_i(x, t, s, \lambda) \), \( g = g(x, t, s, \lambda) \), and their derivatives \( \partial_\lambda a \), \( \partial_{x_j} a \), \( \partial_\lambda \varphi_i \), \( \partial_{x_j} \varphi_i \), and \( \partial_\lambda \varphi_j \) \( (i, j = 1, \ldots, d) \) are Hölder continuous in \( x, t, s \), and \( \lambda \) with continuity exponents \( \gamma, \gamma_2, \gamma \), and \( \gamma (\gamma (\gamma \in (0, 1)) \), respectively, on \( \bar{\Omega} \times [0, \tau] \times [0, S] \times [-M, M] \) and on \( \bar{\Omega} \times [\tau, T] \times [0, S] \times [-M, M] \) for any finite \( M > 0 \).

(iii) Function \( a = a(x, t, s, \lambda) \) satisfies the genuine nonlinearity condition

\[
\text{mes} \{ \lambda \in \mathbb{R} : \xi_1 + \partial_\lambda a(x, t, s, \lambda)\xi_2 = 0 \} = 0 \quad (2.2b)
\]

for every \( (x, t, s, \xi) \in \bar{\Omega} \times [0, T] \times [0, S] \times S^1 \).

By \( S^1 \) we denote the unit circle in \( \mathbb{R}^2 \) centered at the origin, \( S^1 := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1^2 + \xi_2^2 = 1 \} \). In (2.1e) and further in the article we standardly denote \( u(x, \tau + 0, s) := \lim_{t \to \tau^+} u(x, t, s) \), \( u(x, \tau - 0, s) := \lim_{t \to \tau^-} u(x, t, s) \).

**Remark 2.1.** Note that, with \( u|_{s=0}^{(2)} \) and \( u|_{s=\bar{S}}^{(2)} \) on \( \Xi^2 \) on the place of (2.1c), Problem \( \Pi_0 \) becomes ill-posed. Indeed, since for a.e. \( (x, t, s) \in G_{T, S} \) the function \( \lambda \mapsto a(x, t, s, \lambda) \) is nonlinear and, in general, non-monotonous, it may be impossible to equate a solution \( u \) of (2.1a) \( u_0^{(2)} \) and \( u_S^{(2)} \) on the entire sets \( \Gamma_0^2 \) and \( \Gamma_S^2 \). Therefore we set up a more loose non-classical condition (2.1c) following the original idea presented in [13–15], [16, Sec. 2.6–2.8], [17, 18].

In order to deal with the above described sophisticated non-classical features of Problem \( \Pi_0 \), we introduce the notion of a kinetic solution. To this end, we systematically study a strictly parabolic regularized formulation.

## 3. Parabolic regularization. Problem \( \Pi_\varepsilon \)

We construct a kinetic solution of Problem \( \Pi_0 \) as a singular limit of weak solutions \( u_\varepsilon \) of the following strictly parabolic model, as \( \varepsilon \to 0+ \).

**Problem \( \Pi_\varepsilon \).** For arbitrary initial and final data \( u_0^{(1)} \in C_0^{0, \alpha} (\Xi^1) \), \( u_0^{(2)} \), \( u_S^{(2)} \in C_0^{0, \alpha} (\Xi^2) \), and impulsive data \( \beta \in C_0^{0, \alpha} (\Xi^1) (\alpha \in (0, 1) \) is given), it is necessary to find a function \( u_\varepsilon : G_{T, S} \to \mathbb{R} \) satisfying

- **quasilinear parabolic equation**

\[
\partial_t u_\varepsilon + \partial_\lambda a(x, t, s, u_\varepsilon) + \text{div}_x \varphi(x, t, s, u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \partial_\varepsilon^2 u_\varepsilon + g(x, t, s, u_\varepsilon), \quad (x, t, s) \in G_{T, S},
\]

\[
(3.1a)
\]

- **initial condition**

\[
u_\varepsilon|_{t=0} = u_0^{(1)}(x, s), \quad (x, s) \in \Xi^1, \]

\[
(3.1b)
\]

- **boundary conditions**

\[
|_{s=0} = u_0^{(2)}(x, t), \quad u_\varepsilon|_{s=S} = u_S^{(2)}(x, t), \quad (x, t) \in \Xi^2, \]

\[
(3.1c)
\]

\[
|_{t=1} = 0, \quad (3.1d)
\]

- **impulsive condition**

\[
u_\varepsilon(x, \tau + 0, s) = u_\varepsilon(x, \tau - 0, s) + \beta(x, s), \quad (x, s) \in \Xi^1. \]

\[
(3.1e)
\]
Here \( \varepsilon \in (0, 1] \) is an arbitrarily fixed small parameter.

Let us formulate the notion of a weak solution to Problem \( \Pi_\varepsilon \). Let \( \hat{u} \in L^\infty(G_{T,S}) \cap W^1_2(G_{T,S}) \) be an arbitrary extension of \( u_0^{(1)}, u_0^{(2)} \) and \( u_S^{(2)} \) into \( G_{T,S} \) such that \( \hat{u}|_{\Gamma_1} = 0 \).

**Remark 3.1.** Such \( \hat{u} \) may be constructed as the solution of the Dirichlet problem for Laplace’s equation \( \Delta_{x,t,s} \hat{u} = 0 \) in \( G_{T,S} \cup \Gamma_1 \) with boundary data \( \hat{u}|_{\Gamma_1} = 0, \hat{u}|_{\Gamma_0} = u_0^{(1)}, \hat{u}|_{\Gamma'_0} = u_0^{(2)}, \hat{u}|_{\Gamma'_S} = u_S^{(2)} \), for example. This \( \hat{u} \) belongs, in fact, to the space \( C^2(G_{T,S}) \) and is represented in terms of Green’s function [19, Sec. 2.2.4, Th. 12].

**Definition 3.1.** Function \( u_\varepsilon \in L^\infty(G_{T,S}) \cap L^2(0,T; W^2_2(\Xi^1)) \) is called a weak solution of Problem \( \Pi_\varepsilon \), if it satisfies the following demands:

1. The equality \( u_\varepsilon - \hat{u} = 0 \) holds on \( \Gamma_0^{(1)} \cup \Gamma_0^{(2)} \cup \Gamma'_0 \) in the trace sense.
2. One-sided traces \( u_\varepsilon(x, \tau \pm 0, s) \) exist on \( \Gamma' \) and satisfy the impulsive condition (3.1e) for all \((x, s) \in \Xi^1\).
3. The integral equality

\[
\int_{G_{T,S}} \left( -u_\varepsilon \partial_t \phi - a(x,t,s,u_\varepsilon) \partial_s \phi - \varphi(x,t,s,u_\varepsilon) \right) \cdot \nabla_x \phi + \nabla_x \cdot \nabla_x \phi + \varepsilon \partial_s u_\varepsilon \partial_t \phi - g(x,t,s,u_\varepsilon) \right) \, dx dt ds = 0 \tag{3.2}
\]

holds for every \( \phi \in L^\infty(G_{T,S}) \cap \dot{W}^1_2(G_{T,S}) \).

**Proposition 3.1.** Whenever \( u_0^{(1)}, \beta \in C^{0,\alpha}_0(\Xi^1), u_0^{(2)}, u_S^{(2)} \in C^{0,\alpha}_0(\Xi^2) \) (\( \alpha \in (0, 1) \)), and Conditions on \( a, \varphi, g \) hold, for any small fixed \( \varepsilon > 0 \) there is a unique weak generalized solution \( u_\varepsilon = u_\varepsilon(x,t,s) \) of Problem \( \Pi_\varepsilon \) such that \( u_\varepsilon \in H^{\alpha',\gamma'}(\Omega \times [\tau, T] \times [0, S]) \cap H^{\alpha',\gamma'}(\Omega \times [\tau, T] \times [0, S]) \cap W^{2,1}_2(G', \nabla_x u_\varepsilon, \partial_s u_\varepsilon) \in H^{\alpha',\gamma'}(G_{T,S}) \), where \( \alpha', \gamma' \in (0, 1) \) depend on \( \alpha \) and \( \varepsilon \), and \( G' \) is an arbitrary strictly interior subdomain of \( G_{T,S} \).

Moreover, the maximum principle

\[
\|u_\varepsilon\|_{L^\infty(\Xi^1 \times (0, \tau))} \leq \inf_{\xi > b_1} \left\{ e^{\xi \tau} \max \left\{ \|u_0^{(1)}\|_{L^\infty(\Xi^1)}, \|u_0^{(2)}\|_{L^\infty(\Xi^2)}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (0, \tau))}, \right\}, \right. \tag{3.3a}
\]

\[
\|u_\varepsilon\|_{L^\infty(\Xi^1 \times (\tau, T))} \leq \inf_{\xi > b_1} \left\{ e^{\xi \tau} \max \left\{ \|u_\varepsilon(\cdot, \tau - 0, \cdot) + \beta\|_{L^\infty(\Xi^1)}, \|u_0^{(2)}\|_{L^\infty(\Omega \times (\tau, T))}, \right\}, \right. \tag{3.3b}
\]

and the energy estimate

\[
\int_{G_{T,S}} \left( |\nabla_x u_\varepsilon|^2 + \varepsilon |\partial_s u_\varepsilon|^2 \right) \, dx dt ds < C \tag{3.4}
\]

hold. The constant \( C \) does not depend on \( \varepsilon \).
Remark 3.2. Bounds (3.3a) and (3.3b) immediately imply that
\[ \|u_\varepsilon\|_{L^\infty(G,T,S)} \leq M, \]
where the constant \( M \) depends on \( \|u_0^{(1)}\|_{L^\infty(\Omega^1)} \), \( \|u_0^{(2)}\|_{L^\infty(\Omega^2)} \), \( \|u_\varepsilon^{(2)}\|_{L^\infty(\Omega^2)} \), \( \|\nabla\|_{L^\infty(\Omega^1)} \), \( b_1 \) and \( b_2 \), and does not depend on \( \varepsilon \).

In the formulation of Proposition 3.1, by \( H^{\lambda', \lambda'}(G,T,S) \) we standardly denote the spaces of Hölder continuous functions with respect to variables \( (x, s) \) and \( t \) with exponents \( \lambda', \lambda \), and \( \lambda' \), respectively \( \lambda' \in (0,1) \). By \( W^{2,1}_{\lambda}(G') \) we denote the Sobolev space of measurable integrable functions \( v \) on \( G' \) such that \( \partial_t v, \partial_x v, \partial s v, \partial^2_{xx} v, \partial^2_{xx} v, \partial^2_{ss} v \in L^2(G') \), \( i, j = 1, \ldots, d \).

Proof of Proposition 3.1 is based on the well-known theory of quasilinear parabolic equations of the second order [20], [21, Chapter 1, §2, Chapter 5, §6]. On the first step we notice that there is a unique weak generalized solution of Problem \( \Pi_\varepsilon \) for \( t \in (0, \tau) \). On the second step we notice that there exists a weak generalized solution for \( t \in (\tau, T) \) with the Cauchy data \( u_\varepsilon(x, \tau, s) = u_\varepsilon(x, \tau - s, 0) + \beta(x, s) \) on \( \Omega^1 \).

4. Relative compactness of \( \{u_\varepsilon\}_{\varepsilon > 0} \) in \( L^1(G,T,S) \)
The first main result of this paper is the following.

Theorem K. (1) The family of weak generalized solutions \( \{u_\varepsilon\}_{\varepsilon > 0} \) of Problem \( \Pi_\varepsilon \) (in the sense of Definition 3.1) is relatively compact in \( L^1(G,T,S) \), as \( \varepsilon \to 0 \).
(2) Function \( u = \lim_{\varepsilon \to 0} u_\varepsilon \) solves kinetic equation (5.1a) along with nonnegative Radon measures \( m \) and \( n \), and satisfies kinetic initial condition (5.1b) and kinetic impulsive condition (5.1c).

Proof. Relative compactness of the family of weak solutions to Problem \( \Pi \) is proved with the help of the Perthame-Souganidis and Lazar-Mitrovic averaging compactness theorems [22, Theorem 6], [23, Theorem 7]. On the strength of relative compactness, we derive (5.1a) from (3.2) by means of usual routine. Relations (5.1b) and (5.1c) are proved with the help of Aleksic-Mitrovic strong trace existence theorem [24, Theorem 4].

5. Notion of a kinetic solution to Problem \( \Pi_0 \)
Theorem K and the results of [25] motivate us to introduce the notion of kinetic solutions to Problem \( \Pi_0 \). This formulation is the second main result of the paper.

Set
\[ \chi(\lambda; v) = \begin{cases} +1, & \text{for } 0 < \lambda < v, \\ -1, & \text{for } v < \lambda < 0, \\ 0, & \text{elsewhere.} \end{cases} \]

Definition 5.1. Let \( N \in \mathbb{N}, L > 0, \mathcal{O} \) be an open set of \( \mathbb{R}^N \) and a function \( h \in L^\infty(\mathcal{O} \times (-L, L)) \) satisfy the inequality \( 0 \leq h(z, \lambda)\text{sgn}(\lambda) \leq 1 \) for a.e. \( (z, \lambda) \in \mathbb{R}^{N+1} \). We say that \( h \) is a \( \chi \)-function if there exists a function \( v \in L^\infty(\mathcal{O}) \) such that \( h(z, \lambda) = \chi(\lambda; v(z)) \) for a.e. \( z \in \mathcal{O} \).

Lemma 5.1. [26, Lemma 2.1.1]. The following two identities hold true (say for \( \Psi \) measurable on \( \mathcal{O} \times \mathcal{R}_\lambda \) and locally Lipschitz continuous in \( \lambda \), i.e. \( \partial_\lambda \Psi(z, \cdot) \in L^\infty_{\text{loc}}(\mathbb{R}) \)):
(i) \( \int_{\mathcal{R}_\lambda} \partial_\lambda \Psi(z, \lambda) \chi(\lambda; v) d\lambda = \Psi(z, v) - \Psi(z, 0), \) in particular, \( \int_{\mathcal{R}_\lambda} \chi(\lambda; v) d\lambda = v; \)
(ii) \( \int_{\mathcal{R}_\lambda} \partial_\lambda \Psi(z, \lambda) |\chi(\lambda; v) - \chi(\lambda; w)| d\lambda = \text{sgn}(v - w)(\Psi(z, v) - \Psi(z, w)), \) in particular, \( \int_{\mathcal{R}_\lambda} |\chi(\lambda; v) - \chi(\lambda; w)| d\lambda = |v - w|. \)
Additionally, 
(iii) \( |\chi(\lambda; v) - \chi(\lambda; w)| = |\chi(\lambda; v) - \chi(\lambda; w)|^2 \).

The following lemma establishes the link between sequences of \( \chi \)-functions and their limits.

Lemma 5.2. \([27]\).

Let \( O \) be an open set of \( \mathbb{R}^N \) and \( h_n \in L^\infty(O \times (-L, L)) \) be a sequence of \( \chi \)-functions converging weakly to \( h \in L^\infty(O \times (-L, L)) \). Set \( v_n(\cdot) = \int_{-L}^{L} h_n(\cdot, \lambda) \, d\lambda \) and \( v(\cdot) = \int_{-L}^{L} h(\cdot, \lambda) \, d\lambda \).

Then the three assertions are equivalent:
- \( h_n \) converges strongly to \( h \) in \( L^1_{\text{loc}}(O \times (-L, L)) \),
- \( v_n \) converges strongly to \( v \) in \( L^1_{\text{loc}}(O) \),
- \( h \) is a \( \chi \)-function.

Definition 5.2. Function \( u \in L^\infty(G_T,S) \cap L^2((0, T) \times (0, S); \overline{W}^1_2(\Omega)) \) is called a kinetic solution of Problem \( \Pi_0 \) if it satisfies the following kinetic equation and initial, boundary, and impulsive conditions:

- **kinetic equation**

  \[
  \partial_t \chi(\lambda; u(x, t, s)) + \partial_s((\partial_\lambda a(x, t, s, \lambda))\chi(\lambda; u(x, t, s))) + \text{div}_x((\partial_\lambda \varphi(x, t, s, \lambda))\chi(\lambda; u(x, t, s))
  \]

  \[
  \nabla_x \chi(\lambda; u(x, t, s)) + \partial_\lambda a(x, t, s, \lambda) - \partial_s a(x, t, s, \lambda) - \text{div}_x \varphi(x, t, s, \lambda)\chi(\lambda; u(x, t, s))
  \]

  \[
  + \left( \partial_\lambda^2 a(x, t, s, \lambda) + \partial_\lambda \text{div}_x \varphi(x, t, s, \lambda) - \partial_s g(x, t, s, \lambda) \right)\chi(\lambda; u(x, t, s))
  \]

  \[
  - \delta_{\lambda=0} g(x, t, s, \lambda) - \partial_\lambda (m(x, t, s, \lambda) + n(x, t, s, \lambda)) = 0,
  \]

  \[
  (5.1a)
  \]

- **kinetic initial condition**

  \[
  \text{esslim}_{t \to 0^+} \int_{-M}^{M} \int_{\mathbb{R}^1} \left| \chi(\lambda; u(x, t, s)) - \chi(\lambda; u_0^{1}(x, s)) \right| \, dx \, ds \, d\lambda = 0,
  \]

  \[
  (5.1b)
  \]

- **kinetic impulsive condition**

  \[
  \int_{-M}^{M} \chi(\lambda; u(x, \tau + 0, s)) \, d\lambda = \int_{-M}^{M} \chi(\lambda; u(x, \tau - 0, s)) \, d\lambda + \beta(x, s), \quad (x, s) \in \mathbb{R}^{d+1},
  \]

  \[
  (5.1c)
  \]

- **kinetic boundary conditions**

  \[
  \partial_\lambda a(x, t, 0, \lambda)(\chi(\lambda; u_0^{1r}(x, t)) - \chi(\lambda; u_0^{2r}(x, t)))
  \]

  \[
  - \delta_{\lambda=0}(x, t))(a(x, t, 0, u_0^{1r}(x, t)) - a(x, t, 0, u_0^{2r}(x, t))) = \partial_\lambda \mu_0^{(2)}(x, t, \lambda),
  \]

  \[
  (5.1d)
  \]

  \[
  \partial_\lambda a(x, t, S, \lambda)(\chi(\lambda; u_S^{1r}(x, t)) - \chi(\lambda; u_S^{2r}(x, t)))
  \]

  \[
  - \delta_{\lambda=0}(x, t)(a(x, t, S, u_S^{1r}(x, t)) - a(x, t, S, u_S^{2r}(x, t))) = -\partial_\lambda \mu_S^{(2)}(x, t, \lambda),
  \]

  \[
  (5.1e)
  \]
where $m,n \in \mathcal{M}^+ (G_{T,S} \times (-M,M))$, $n = \delta_{(\lambda=-\mu)} |\nabla_x u|^2$; $\mu_0^{(2)}, \mu_S^{(2)} \in \mathcal{M}(\mathbb{R}^2 \times (-M,M))$. Here $\mathcal{M}^+$ denotes the space of finite positive Radon measures and $\mathcal{M}$ denotes the space of finite Radon measures. Functions $u_0^{tr,(2)}$ and $u_S^{tr,(2)}$ are strong traces of a solution $u = u(x,t,s)$ (if any) of the kinetic equation (5.1a) on the planes $\{s = 0\}$ and $\{s = S\}$, respectively:

$$\text{esslim}_{s \to 0+} \int_{\Xi} |u(x,t,s) - u_0^{\text{tr},(2)}(x,t)| dx dt = 0, \quad \text{esslim}_{s \to S-0} \int_{\Xi} |u(x,t,s) - u_S^{\text{tr},(2)}(x,t)| dx dt = 0.$$  (5.1f)

The full proof of existence of a kinetic solution under Conditions on $\alpha$ and $\gamma$ is yet incomplete. In the stationary case ($a = \alpha(\lambda), \varphi = \varphi(\lambda), g \equiv 0$), the existence and uniqueness were proved, see [25]. Existence of strong traces on $\Gamma_0$ and $\Gamma_2$ is not obvious. Even if strong traces do exist, we cannot assume that measures $\mu_0^{(2)}$ and $\mu_S^{(2)}$ are positive measures. Also, the question linked with the uniqueness is open.

6. Conclusion

In this paper we have formulated Theorem K and have given a brief scheme of its proof. As well, we have set up the definition of a kinetic solution to problem $\Pi_0$.

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