A performance study of some approximation algorithms for minimum dominating set in a graph

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Abstract

We implement and test the performances of several approximation algorithms for computing the minimum dominating set of a graph. These algorithms are the standard greedy algorithm, the recent LP rounding algorithms and a hybrid algorithm that we design by combining the greedy and LP rounding algorithms. All algorithms perform better than anticipated in their theoretical analysis, and have small performance ratios, measured as the size of output divided by the LP objective lower-bound. However, each may have advantages over the others. For instance, LP rounding algorithm normally outperforms the other algorithms on sparse real-world graphs. On a graph with 400,000+ vertices, LP rounding took less than 15 seconds of CPU time to generate a solution with performance ratio 1.011, while the greedy and hybrid algorithms generated solutions of performance ratio 1.12 in similar time. For synthetic graphs, the hybrid algorithm normally outperforms the others, whereas for hypercubes and k-Queens graphs, greedy outperforms the rest. Another advantage of the hybrid algorithm is to solve very large problems where LP solvers crash, as we observed on a real-world graph with 7.7 million+ vertices.

1 Introduction and Summary

Domination theory has its roots in the k-Queens problem in 18th century. Later in 1957, Berge [4] formally introduced the domination number of a graph. The problem of computing the domination number of a graph has extensive applications including the design of telecommunication networks, facility location, and social networks. We refer the reader to the book by Haynes, Hedetniemi, and Slater [22] as a general reference in domination theory.

We assume that the reader is familiar with general concepts of graph theory as in [12], the theory of algorithms as in [11], and linear and integer programming concepts as in [14], respectively. Throughout this paper $G = (V, E)$ denotes an undirected graph on vertex set $V$ and edge set $E$ with $n = |V|$ and $m = |E|$. Two vertices $x, y \in V$ where $x \neq y$ are adjacent (or they are neighbors) if $x, y \in E$. For any $x \in V$, degree of $x$, denoted by $deg(x)$ is the number of vertices adjacent to $x$ in $G$. For any $x \in V$, let $N(x)$ denote the set of all vertices in $G$ that are adjacent to $x$. Let $N[x]$ denote $N(x) \cup \{x\}$. Arboricity of $G$, denoted by $a(G)$ is the minimum number of spanning acyclic subgraphs of $G$ that $E$ can be partitioned into. By a theorem of Nash Williams, $a(G) = \max_S \left[ \frac{m_S}{n_S-1} \right]$, where $n_S$ and $m_S$ are the number of vertices and edges, respectively, of
the induced subgraph on the vertex set $S$. Consequently $m \leq a(G)(n - 1)$, and thus $a(G)$ measures how dense $G$ is. It is known that $a(G)$ can be computed in polynomial time.

Let $D \subseteq V$. $D$ is a dominating set if for every $x \in V \setminus D$ there exists $y \in D$ such that $(x, y) \in E$. The domination number of $G$, denoted by $\gamma(G)$, is the cardinality of a minimum (smallest) dominating set of $G$. Computing $\gamma(G)$ is known to be an NP-Hard problem even for unit disc graphs and grids.

1.1 Greedy approximation algorithm

A simple greedy algorithm attributed to Chvatal [9] and Lovas [25] (for approximating the set cover problem) is known to approximate $\gamma(G)$ within a multiplicative factor of $H(\Delta(G))$ from its optimal value, where $\Delta(G)$ is maximum degree of $G$ and $H(k) = \sum_{i=1}^{k} (1/i)$ is the $k$-th harmonic number. The algorithm initially labels all vertices uncovered. At iteration one, the algorithm selects a vertex $v_1$ of maximum degree in $G$, places $v_1$ in a set $D$, and labels all vertices adjacent to it as covered. In general, at iteration $i \geq 2$, the algorithm selects a vertex $v_i \in V \setminus \{v_1, v_2, ..., v_{i-1}\}$ with the largest number of uncovered vertices adjacent to it, adds $v_i$ to $D$, and labels all of its uncovered adjacent vertices as covered. The algorithm stops when $D$ becomes a dominating set. It is easy to implement the algorithm in $O(n + m)$ time. It is known that approximating $\gamma(G)$ within a factor $(1 - \varepsilon)ln(\Delta)$ from the optimal is NP-hard [17]. Hence, no algorithm for approximation $\gamma(G)$ can improve the asymptotic worst case performance ratio achieved by the greedy algorithm. Different variations of the greedy algorithm to approximate $\gamma(G)$ are developed and some are tested in practice; See work of Chalupa [9], Campan et. al. [8], Eubank et. al. [18], Parekh [20], Sanchis [27], and Siebertz [28].

Below are two examples of worst-case graphs (one sparse and one dense) for greedy algorithm which are derived from an instance of set cover problem provided in [6]. For both instances, the solutions provided by the greedy algorithm are actually $O(ln(\Delta))$ times the optimal.

Example 1.1.

Let $p \geq 2$ be an integer and for $i = 1, 2, ..., p$, let $S_i$ be a star on $2^i$ vertices. Consider a graph $G$ on $n = 2^{p+1}$ vertices whose vertices are the disjoint union of the vertices of the $S_i$’s ($i = 1, 2, ..., p$) plus two additional vertices $t_1$ and $t_2$. Now, place edges from $t_1$ and $t_2$ to the first half of the vertices in each $S_i$ (including the root), and the second half of the vertices in each $S_i$, respectively. Note that the root of each $S_i$ has degree $2^i$ and the degree of both $t_1$ and $t_2$ is $2^p - 1$. Initially, greedy chooses the root of $S_p$ which can cover $2^p + 1$ vertices (including itself). Generally, at iteration $i \geq 2$, there is a tie between the root of $S_{p+1-i}$ and $t_2$ since each can cover $2^{p-2}$ uncovered vertices. If tie breaking does not result in selecting $t_2$, there will be a tie in every iteration until the algorithm returns the set of $S_i$’s ($i = 1, 2, ..., p$). This dominating set has cardinality $p = \log(\Delta) - 1$, but $\gamma(G) = 2$, since $\{t_1, t_2\}$ is a minimum dominating set. Note that $G$ is a planar graph.

\footnote{Note that $ln(k + 1) \leq H(k) \leq ln(x) + 1$.}
Example 1.2.

Let $p \geq 2$ be an integer, and let $G$ be a graph with vertices $V_1 \cup V_2$, where $V_1 = \{s_1, s_2, ..., s_p, t_1, t_2\}$ and $V_2 = \{v_1, v_2, ..., v_{2p+1-2}\}$. Now make $V_1$ a clique and $V_2$ an independent set of vertices, respectively. Next, consider a linear ordering $L$ on $V_2$: for $i = 1, 2, ..., p$, the set of neighbors of $s_i$ in $V_2$, denoted by $W_i$, has cardinality $2^i$ and is disjoint from $W_k$, for any $k \leq i$. Finally, for $i = 1, 2, ..., p$ place edges between $t_1$ and the first half of the vertices in each $W_i$, and place edges between $t_2$ and the second half of the vertices in each $W_i$. Now note that the greedy algorithm will be forced to pick the vertices $s_p, s_{p-1}, ..., s_1$, in that order but the minimum dominating set in $G$ is $\{t_1, t_2\}$ and $\Delta = 2^p + p + 1$.

1.2 Linear programming rounding approximation algorithms

One can formulate the computation of $\gamma(G)$ as an integer programming problem stated below. However, since integer programming problems are known to be NP-hard [23], the direct applications of the integer programming method would not be computationally fruitful.

| IP1: |
|------|
| Minimize $I = \sum_{v \in V} x_v$ |
| Subject to $\sum_{u \in N[v]} x_u \geq 1$, $\forall v \in V$ |
| $x_v \in \{0, 1\}$, $\forall v \in V$ |

Now observe that by relaxing the integer program IP1 one obtains the following linear program.

| LP1: |
|------|
| Minimize $L = \sum_{v \in V} x_v$ |
| Subject to $\sum_{u \in N[v]} x_u \geq 1$, $\forall v \in V$ |
| $0 \leq x_v \leq 1$, $\forall v \in V$ |

Note that $L^* \leq \gamma(G) = I^*$, where $L^*$ and $I^*$ are the values of $L$ and $I$ at optimality. Since the class of linear programming problems are solvable in polynomial time [24], LP1 can be solved in polynomial time. Very recently, Bansal and Umboh [3] and Dvok [16] have shown that an appropriate rounding of fractional solutions of LP1 gives integer solutions to IP1 whose values are at most $3 \cdot a(G) \cdot L^*$ and $(2 \cdot a(G) + 1) \cdot L^*$, respectively, in polynomial time. Hence, for sparse graphs (graphs with bounded arboricity), one can get a better approximation ratio than $O(ln(\Delta))$ which is achieved by the greedy algorithm. To our knowledge, and in contrast to the greedy algorithm, the performances of the LP rounding approaches have not been tested in practice.
1.3 Other approximation algorithms

There are other approximation algorithms for very specific classes of graphs including planar graphs which have better than constant performance ratio in the worst case but are more complex than algorithms described here. See [28] for a brief reference to some related papers.

1.4 Our work

Greedy is simple and fast, since it can be implemented in linear time. Its performance ratio in the worst case scenario is logarithmic. Linear programming works in polynomial time but is more time consuming than greedy. For sparse graphs, recent linear programming rounding methods in [3, 16] have a constant performance ratio, but there have not been any experimental study of their performances.

In this paper, we implement three types of algorithms and compare and contrast their performances in practice. These algorithms are the greedy algorithm, the LP rounding algorithms, and a hybrid algorithm that combines the greedy and LP approach. The hybrid algorithm first solves the problem using the greedy algorithm and finds a dominating set $D, |D| = d$. It then takes a portion of vertices in $D$, forces their weights to be 1 in linear program LP1, solves the resulting (partial) linear program, and then properly rounds the solution to the partial LP. Finally, it returns the rounded solution plus the portion of the greedy solution that was forced to LP1.

1.5 Environment, implementation and datasets

We used a laptop with modest computational power - 8th generation Intel i5 (1.6GHz) and 8GB RAM - to perform the experiments. We implemented the $O(n + m)$ time version of the greedy algorithm in C++. We used IBM Decision Optimization CPLEX Modeling (DOCPLEX) for Python to solve the LP relaxation of the problem. Python and DOCPLEX were used to implement the LP rounding and hybrid algorithms.

The graph generator at [1] was used to create the planar graphs, trees, k-planar graphs (graphs embedded in the plane with at most $k$ crossings per edge), and k-trees (graphs with tree width $k$ with largest number of edges) up to 20,000 vertices. The k-Queens graphs, hypercubes (up to 12 dimensions) and graph implementations of the cases described in 1.1 and 1.2 were created ourselves. We also used publicly available Google+ and Pokec social-network graphs, as well as real-world DIMACS Graphs with up to more than 7,700,000 vertices.

https://snap.stanford.edu/data/com-Youtube.html [8]
https://github.com/joklawitter/GraphGenerators [1]
http://davidchalupa.github.io/research/data/social.html [9]
https://www.cc.gatech.edu/dimacs10/downloads.shtml [2]

1.6 Our results

Through experimentation, all algorithms perform better than anticipated in their theoretical analysis, particularly with respect to the performance ratios (measured with respect to the LP objective lower-bound). However, each may have advantages over the others for specific data sets. For instance, LP rounding normally outperforms the other algorithms on real-world graphs. On a graph with 400,000+ vertices, LP rounding took less than 15 seconds of CPU time to generate a solution with performance ratio 1.011, while the greedy and hybrid algorithms...
generated solutions of performance ratio 1.12 in similar time. For synthetic graphs (generated k-trees, k-planar) the hybrid algorithm normally outperforms the others, whereas for hypercubes and k-Queens graphs, the greedy outperforms the rest. Particularly, on the 12-dimensional hypercube, greedy finds a solution with performance ratio 1.7 in 0.01 seconds. On the other hand, the LP rounding and hybrid algorithms produce solutions with performance ratio 13 and 3.3 using 7.5 and 0.08 seconds of CPU time, respectively. It is notable that greedy gives optimal results in some cases where the domination number is known. Specifically, the greedy algorithm produces an optimal solution on hypercubes with dimensions $d = 2^k - 1$ where $k=1, 2, 3, \text{ and } 4$. The hybrid algorithm can solve very large problems when the size of LP1 becomes formidable in practice. For instance, the hybrid algorithm solved a real-world graph with 7.7 million+ vertices in 106 seconds of CPU time with a performance ratio of 2.0075. The LP solver crashed on this problem.

This paper is organized as follows. In section two, we formally describe LP rounding and hybrid algorithms. When the size of problem is so large that LP1 cannot be solved in practice, then $L^*$ cannot be computed, and hence the performance ratio of the hybrid algorithm cannot be determined. We resolved this problem by decomposing LP1 in to two smaller linear programs so each of them has an objective value not exceeding $L^*$ and used the maximum objective value of the two smaller LP’s, instead of $L^*$, to measure the performance ratio of the hybrid algorithm. Section 3, 4, and 5 contains results for Planar, k-Planar, and k-Tree graphs, hypercubes and k-Queen graphs, and real-world graphs respectively.

2 Linear Programming and hybrid approach

The following algorithm is due to Bansal and Umboh [3].

Algorithm $A_1$ ([3])
Solve LP1, and let $H$ be the set of all vertices that have weight at least 1/(3\(a(G)\)), where $a(G)$ is the arboricity of graph $G$. Let $U$ be the set of all vertices not adjacent to any vertex in $H$ and returns $H \cup U$.

Dvok[15, 16] studied $d$-domination problem, that is, when a vertex dominates all vertices at distance at most $d$ from it and its combinatorial dual, or a 2$d$-independent set [1]. In [16] he employed the LP rounding approach of Bansal and Umboh, as a part of his frame work and consequently, for $d = 1$, he improved the approximation ratio of Algorithm $A_1$ by showing that the algorithm $A_2$ given below provides a $2a(G) + 1$ approximation.

Algorithm $A_2$ ([16])
Solve LP1, and let $H$ be the set of all vertices that have weight at least 1/(2\(a(G) + 1\)), where $a(G)$ is the arboricity of graph $G$. Let $U$ be the set of all vertices that are not adjacent to any vertex of $H$ and return $H \cup U$.

Remark 2.1. Graph $G$ in example 1.1 is planar, so $a(G) \leq 3$. Thus, algorithms $A_1$ and $A_2$ have a worst-case performance ratio of nine and seven respectively, whereas greedy exhibits a worst-case $O(\log(n))$ performance ratio. Throughout our experiments, rounding algorithms returned an optimal solution of size two for both examples, whereas greedy returned a set of size three for Example 1.1. Furthermore, in Example 1.2, it can be verified that $a(G) \geq (p + 2)/2$ for graph $G$ and hence in theory the worse case performance ratios of the rounding algorithms are not constant either. Interestingly enough, in our experiments, $L^*$ was always two for graphs of
Example 1.2, and LP rounding algorithms also always found a solution of size two which is the optimal value. Thus the performance ratio was always one and much smaller than the predicted worst case.

Next, we provide a description of the decomposition approach for approximating LP1 and our hybrid algorithm. Recall that a separation in $G = (V, E)$ is a partition $A \cup B \cup C$ of $V$ so that no vertex of $A$ is adjacent to any vertex of $C$. In this case $B$ is called a vertex separator in $G$. Let $X = \{x_v | v \in V\}$ be a feasible solution to LP1, and let $V' \subseteq V$. Then $X(V')$ denotes $\sum_{v \in V'} x_v$.

**Lemma 2.1.** Let $A \cup B \cup C$ be a separation in $G = (V, E)$ and consider the following linear programs:

**LP2:**

\[
\begin{align*}
\text{Minimize} & \quad M = \sum_{v \in A \cup B} x_v \\
\text{Subject to} & \quad \sum_{u \in N[v]} x_u \geq 1, \forall v \in A \\
& \quad 0 \leq x_v \leq 1, \forall v \in A \cup B
\end{align*}
\]

**LP3:**

\[
\begin{align*}
\text{Minimize} & \quad N = \sum_{v \in C \cup B} x_v \\
\text{Subject to} & \quad \sum_{u \in N[v]} x_u \geq 1, \forall v \in C \\
& \quad 0 \leq x_v \leq 1, \forall v \in B \cup C
\end{align*}
\]

Then $\max\{M^*, N^*\} \leq L^*$.

**Proof.** Let $X = \{x_v | v \in V\}$ be an optimal solution to LP1. Note that the restrictions of $X$ to $A \cup B$ and $C \cup B$ give feasible solutions for LP3 and LP2 of values $X(B \cup C)$ and $X(B \cup A)$, and hence the claim for the lower bound on $L^*$ follows.

Note that in LP2, LP3 the constraints are not written for all variables, and rounding method in [3] may not directly be applied.

**Theorem 2.1.** Let $G = (V, E)$, let $A \subset V$, let $B = E(A)$ and let $C = V - (A \cup B)$. Let $X$ be an optimal solution for LP3, and let $X(C)$ denote the sum of the weights assigned to all vertices in $C$. Then there is a dominating set in $G$ of size at most $|A| + 3a(G)X(C) \leq |A| + 3a(G)N^*$.

**Proof.** Let $H$ be the set of all vertices $v$ in $C$ with $x(v) \geq \frac{1}{3a}$, and let $U = C - (H \cup E(H))$. Now apply the method in [3] to $C$ to obtain a rounded solution, or a dominating set $D$, of at most $|U| + |H| \leq 3a(G)X(C)$ vertices in $C$. Finally, note that $A \cup D$ is a dominating set in $G$ with cardinality at most $|A| + 3a(G)X(C) \leq |A| + 3a(G)N^*$.
Algorithm $H$ (Hybrid Algorithm)
Apply the greedy algorithm to $G$ to obtain a dominating set $D = \{x_1, x_2, ..., x_d\}$, and let $S = \{x_1, x_2, ..., x_{\alpha.d}\}$ be the first $\alpha.d$ vertices in $D$. Now solve the following linear program on the induced subgraph of $G$ with the vertex set $V - \{S\}$.

\[
\begin{align*}
\text{Minimize } J &= \sum_{v \in V - \{S\}} x_v \\
\text{Subject to } &\sum_{u \in N[v]} x_u \geq 1, \forall v \in V - \{S \cup N[S]\} \\
&0 \leq x_v \leq 1, \forall v \in V - S
\end{align*}
\]

Next, let $A = S, B = E(S)$ and $C = V - (A \cup B)$, and apply the rounding scheme in algorithms $A_1$ or $A_2$ to $C$, and let $H$ and $U$ be corresponding sets, and output the set $S \cup H \cup U$.

Remark 2.2. Note that by Theorem 2.1 Algorithm $H$ can be implemented in polynomial time. Furthermore, $|S \cup H \cup U| \leq \alpha.d + 3a(G)N^* \leq \alpha.(ln(\Delta) + 1) + 3a(G).\gamma(G)$, and thus Algorithm $H$ has a bounded performance ratio.

3 Performance on Planar Graphs, k-Planar Graphs, and k-Trees

In this section, we compare the performance ratios of Greedy, $A_1$, $A_2$, $A_1$ Hybrid, and $A_2$ Hybrid on planar graphs, k-planar graphs k-trees. In Tables 2 and 3, we present the performance of the algorithms on k-trees where $k = \lfloor V^{0.25}\rfloor$ and k-planar graphs where $k = \lfloor ln(|V|)\rfloor$, respectively. These graphs are dense. We also present the algorithms’ performance on sparse k-trees and sparse k-planar graphs in tables 4 and 5. The planar graphs k-trees, and k-planar graphs were all made using † described in section 1.5.

In most cases, the $A_1$ and $A_2$ variants of the hybrid algorithm outperformed the others, producing the lowest performance ratio to the LP lower bound $L^*$. Greedy performs close to hybrid and outperforms it for the larger dense k-trees and a few of the k-planar graphs. The LP-rounding algorithms performed the worst across the board. All algorithms were able to compute dominating sets in less than 2 seconds across the different types of graphs and their range of sizes.

The arboricity of each of the planar graphs is at most 3. For k-trees, we use $\lceil k - (k/2)(k-1) \rceil$ for arboricity. For k-planar graphs, we use the upper bound of $\lceil 8\sqrt{k} \rceil$ on arboricity.

### Table 1: Results for Planar Graphs

| $n, m$ | $L^*$ | Greedy/$L^*$ | $A_1/L^*$ | $A_1$ Hybrid/$L^*$ | $A_2/L^*$ | $A_2$ Hybrid/$L^*$ |
|-------|-------|-------------|-----------|-------------------|-----------|-------------------|
| 2000, 5980 | 316.93 | 1.12 | 1.40 | 1.11 | 1.39 | 1.11 |
| 4000, 11972 | 620.72 | 1.16 | 1.35 | 1.14 | 1.34 | 1.14 |
| 6000, 17978 | 942.59 | 1.13 | 1.29 | 1.13 | 1.29 | 1.13 |
| 8000, 23974 | 1239.16 | 1.14 | 1.41 | 1.13 | 1.40 | 1.13 |
| 10000, 29972 | 1579.06 | 1.13 | 1.27 | 1.13 | 1.27 | 1.13 |
| 12000, 35973 | 1874.66 | 1.13 | 1.36 | 1.12 | 1.35 | 1.12 |
| 14000, 41974 | 2185.35 | 1.14 | 1.33 | 1.14 | 1.32 | 1.14 |
| 16000, 47975 | 2514.62 | 1.14 | 1.33 | 1.13 | 1.33 | 1.13 |
| 18000, 53971 | 2811.98 | 1.15 | 1.35 | 1.14 | 1.35 | 1.14 |
| 20000, 59971 | 3127.20 | 1.14 | 1.32 | 1.13 | 1.31 | 1.13 |
Table 2: Results for $k$-Trees where $k = \sqrt[0.25]{|V|}$

| $n, m$   | $L^*$ | Greedy $/L^*$ | $A_1/L^*$ | $A_1$ Hybrid $/L^*$ | $A_2/L^*$ | $A_2$ Hybrid $/L^*$ |
|----------|-------|---------------|-----------|---------------------|-----------|---------------------|
| 2000, 13972 | 15.00 | 1.07 | 1.20 | 1.00 | 1.20 | 1.00 |
| 4000, 31964 | 10.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 6000, 53955 | 11.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8000, 71955 | 13.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 10000, 99945 | 11.19 | 1.07 | 2.23 | 1.07 | 2.23 | 1.07 |
| 12000, 119945 | 12.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 14000, 139945 | 18.50 | 1.08 | 1.89 | 1.14 | 1.89 | 1.14 |
| 16000, 175934 | 11.25 | 1.16 | 1.60 | 1.33 | 1.60 | 1.33 |
| 18000, 197934 | 11.00 | 1.18 | 2.00 | 1.18 | 2.00 | 1.18 |
| 20000, 219934 | 10.50 | 1.14 | 1.43 | 1.43 | 1.43 | 1.43 |

Table 3: Results for $k$-Planar Graphs where $k = \ln (|V|)$

| $n, m$   | $L^*$ | Greedy $/L^*$ | $A_1/L^*$ | $A_1$ Hybrid $/L^*$ | $A_2/L^*$ | $A_2$ Hybrid $/L^*$ |
|----------|-------|---------------|-----------|---------------------|-----------|---------------------|
| 2000, 12986 | 151.97 | 1.26 | 2.16 | 1.24 | 2.11 | 1.24 |
| 4000, 27254 | 289.69 | 1.27 | 2.65 | 1.29 | 2.64 | 1.29 |
| 6000, 40885 | 431.77 | 1.26 | 2.50 | 1.26 | 2.50 | 1.26 |
| 8000, 54568 | 568.01 | 1.24 | 2.57 | 1.25 | 2.57 | 1.25 |
| 10000, 71414 | 684.20 | 1.27 | 2.57 | 1.28 | 2.56 | 1.28 |
| 12000, 85580 | 821.65 | 1.26 | 2.62 | 1.27 | 2.62 | 1.27 |
| 14000, 100241 | 957.77 | 1.25 | 2.47 | 1.26 | 2.46 | 1.26 |
| 16000, 114270 | 1098.18 | 1.27 | 2.21 | 1.27 | 2.21 | 1.27 |
| 18000, 128725 | 1238.09 | 1.27 | 2.23 | 1.27 | 2.22 | 1.27 |
| 20000, 142891 | 1368.44 | 1.26 | 2.24 | 1.25 | 2.23 | 1.25 |

Table 4: Results for $k$-Trees where $k = 5$

| $n, m$   | $L^*$ | Greedy $/L^*$ | $A_1/L^*$ | $A_1$ Hybrid $/L^*$ | $A_2/L^*$ | $A_2$ Hybrid $/L^*$ |
|----------|-------|---------------|-----------|---------------------|-----------|---------------------|
| 2000, 9985 | 39.00 | 1.05 | 1.08 | 1.05 | 1.08 | 1.05 |
| 4000, 19985 | 70.50 | 1.04 | 1.06 | 1.04 | 1.06 | 1.04 |
| 6000, 29985 | 90.83 | 1.03 | 1.17 | 1.03 | 1.17 | 1.03 |
| 8000, 39985 | 132.25 | 1.03 | 1.07 | 1.03 | 1.07 | 1.03 |
| 10000, 49985 | 158.00 | 1.03 | 1.03 | 1.03 | 1.03 | 1.03 |
| 12000, 59985 | 209.67 | 1.02 | 1.08 | 1.02 | 1.08 | 1.02 |
| 14000, 69985 | 225.58 | 1.04 | 1.09 | 1.04 | 1.09 | 1.04 |
| 16000, 79985 | 270.25 | 1.02 | 1.09 | 1.02 | 1.09 | 1.02 |
| 18000, 89985 | 291.83 | 1.02 | 1.06 | 1.02 | 1.06 | 1.02 |
| 20000, 99985 | 339.58 | 1.04 | 1.08 | 1.04 | 1.08 | 1.04 |

Table 5: Results for $k$-Planar Graphs where $k = 5$

| $n, m$   | $L^*$ | Greedy $/L^*$ | $A_1/L^*$ | $A_1$ Hybrid $/L^*$ | $A_2/L^*$ | $A_2$ Hybrid $/L^*$ |
|----------|-------|---------------|-----------|---------------------|-----------|---------------------|
| 2000, 11465 | 171.42 | 1.19 | 1.65 | 1.20 | 1.65 | 1.20 |
| 4000, 23033 | 336.57 | 1.21 | 1.63 | 1.22 | 1.63 | 1.22 |
| 6000, 34577 | 510.02 | 1.24 | 2.20 | 1.25 | 2.19 | 1.25 |
| 8000, 46130 | 680.88 | 1.25 | 1.91 | 1.25 | 1.91 | 1.25 |
| 10000, 57786 | 840.92 | 1.23 | 2.12 | 1.24 | 2.10 | 1.24 |
| 12000, 69220 | 1019.54 | 1.23 | 2.02 | 1.22 | 2.02 | 1.22 |
| 14000, 80680 | 1181.05 | 1.22 | 1.90 | 1.22 | 1.90 | 1.22 |
| 16000, 92300 | 1355.13 | 1.23 | 2.03 | 1.23 | 2.03 | 1.23 |
| 18000, 103862 | 1516.14 | 1.24 | 1.99 | 1.24 | 1.99 | 1.24 |
| 20000, 115354 | 1689.35 | 1.22 | 2.08 | 1.21 | 2.08 | 1.21 |

4 Performance on Hypercubes and $k$-Queen Graphs

In this section, we present the performance of Greedy, $A_1$, $A_2$, $A_1$ Hybrid, and $A_2$ Hybrid on hypercubes from 5-12 dimensions and $k$-Queens graphs.
Table 6 compares the performance ratios of the algorithms on hypercubes. We use the arboricity for hypercubes \( a = \lceil d/2 + 1 \rceil \) for LP rounding and hybrid [21]. For k-Queens graphs, arboricity is unknown, so we use the upper bound \( 3(k - 1) \), where \( k \) is the length of the chessboard.

For both hypercubes and k-Queens graphs, Greedy performs the best, followed by \( A_1 \) Hybrid and \( A_2 \) Hybrid. \( A_1 \) and \( A_2 \) LP rounding perform the worst by far. This is not surprising as LP Rounding approaches are known to in general perform worse on dense graphs than sparse graphs. Solutions were computed in under 8 seconds for all graphs and algorithms.

| \( n, m \) | \( L^* \) | \( \text{Greedy/}L^* \) | \( A_1/L^* \) | \( A_1 \text{ Hybrid/}L^* \) | \( A_2/L^* \) | \( A_2 \text{ Hybrid/}L^* \) |
|---|---|---|---|---|---|---|
| 5, 80 | 5.33 | 1.50 | 3.00 | 1.50 | 3.00 | 1.50 |
| 6, 192 | 9.14 | 1.75 | 7.00 | 1.75 | 7.00 | 1.75 |
| 7, 448 | 16.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 8, 1024 | 28.44 | 1.13 | 9.00 | 1.13 | 9.00 | 1.13 |
| 9, 2304 | 51.20 | 1.25 | 7.07 | 2.99 | 7.07 | 2.99 |
| 10, 5120 | 93.09 | 1.38 | 11.00 | 2.70 | 11.00 | 2.70 |
| 11, 11264 | 170.67 | 1.50 | 6.59 | 2.85 | 6.59 | 2.85 |
| 12, 24576 | 315.08 | 1.63 | 13.00 | 3.14 | 13.00 | 3.14 |

Table 7: Results for k-Queens Graphs

| \( n, m \) | \( L^* \) | \( \text{Greedy/}L^* \) | \( A_1/L^* \) | \( A_1 \text{ Hybrid/}L^* \) | \( A_2/L^* \) | \( A_2 \text{ Hybrid/}L^* \) |
|---|---|---|---|---|---|---|
| 225, 5180 | 4.89 | 2.05 | 38.45 | 6.75 | 36.40 | 6.75 |
| 256, 6320 | 5.19 | 1.93 | 46.98 | 7.70 | 43.90 | 7.12 |
| 289, 7616 | 5.50 | 1.82 | 45.84 | 8.91 | 44.03 | 8.91 |
| 324, 9078 | 5.80 | 1.90 | 50.34 | 9.83 | 48.27 | 9.83 |
| 361, 10716 | 6.10 | 1.97 | 52.42 | 9.67 | 50.78 | 9.67 |
| 400, 12540 | 6.41 | 2.03 | 56.81 | 10.14 | 53.06 | 9.68 |
| 441, 14560 | 6.71 | 1.94 | 59.89 | 11.32 | 56.91 | 11.17 |
| 484, 16786 | 7.02 | 2.00 | 63.86 | 9.55 | 59.29 | 9.12 |
| 529, 19228 | 7.32 | 1.91 | 65.83 | 10.38 | 62.83 | 10.11 |
| 576, 21896 | 7.62 | 1.97 | 70.82 | 11.93 | 64.00 | 11.67 |
| 625, 24800 | 7.93 | 2.02 | 74.15 | 10.47 | 69.61 | 10.34 |
| 676, 27950 | 8.23 | 1.94 | 76.27 | 11.78 | 68.50 | 11.30 |
| 729, 31356 | 8.54 | 1.87 | 80.80 | 11.83 | 74.48 | 11.13 |
| 784, 35028 | 8.84 | 1.92 | 80.07 | 14.82 | 74.64 | 14.25 |
| 841, 38976 | 9.15 | 1.97 | 85.81 | 12.02 | 78.81 | 11.70 |
| 900, 43210 | 9.45 | 2.01 | 87.18 | 12.91 | 81.26 | 12.38 |

5 Performance on Real-World Graphs

In this section, we present the performance of LP rounding, greedy, and hybrid on the real-world social network graphs from Google+ [9], Pokec [9], and DIMACS [2]. Each of these graphs are sparse, but their arboricity is unknown. Since arboricity is unknown, we experiment with the threshold applied during LP rounding and hybrid, starting with \( 1/3a' \), where \( a' = \lceil |E|/(|V| - 1) \rceil \) is a lower bound on arboricity. We call LP Rounding with this threshold Algorithm \( A_1' \). Similarly, Algorithm \( A_2' \) has threshold \( 1/2a' + 1 \). Through experimentation, the best threshold which we found was \( 2/a' \); the resulting Algorithm is called \( A_3 \).

In Table 8, we compare the solution size of \( A_1' \), \( A_2' \), and \( A_3 \), along with their hybrid analogs and greedy, to the LP lower bound \( L^* \) on the Google+ graphs. Table 9 compares the same algorithms on the Pokec graphs. In Table 10, we compare the performance ratio to the LP lower bound for these algorithms on 3 social network graphs from DIMACS. In Tables 8, 9 and 10, LP Rounding performs better than the greedy and hybrid approaches, with greedy being the worst out of the algorithms tested. Out of the LP rounding approaches, \( A_3 \) performs the best.
Table 8: Results for Google+ Graphs

| n, m       | L*  | Greedy | A₁' | A₂' | A₃ | A₁' Hybrid | A₂' Hybrid | A₃ Hybrid |
|------------|-----|--------|-----|-----|----|------------|------------|----------|
| 500, 1000  | 42  | 42     | 42  | 42  | 42 | 42         | 42         | 42       |
| 2000, 5343 | 170 | 176    | 170 | 170 | 170| 176        | 176        | 176      |
| 10000, 33954| 860 | 864    | 864 | 864 | 893| 893        | 893        | 893      |
| 20000, 81352| 1715| 1730   | 1716| 1800| 1800| 1800       | 1800       | 1800     |
| 50000, 231583| 4565| 4651   | 4607| 4585| 4790| 4790       | 4790       | 4790     |

Table 9: Results for Pokec Graphs

| n, m       | L*  | Greedy | A₁' | A₂' | A₃ | A₁' Hybrid | A₂' Hybrid | A₃ Hybrid |
|------------|-----|--------|-----|-----|----|------------|------------|----------|
| 500, 993   | 16  | 16     | 16  | 16  | 16 | 16         | 16         | 16       |
| 2000, 5893 | 75  | 75     | 75  | 75  | 75 | 75         | 75         | 75       |
| 10000, 44745| 413 | 413    | 413 | 413 | 413| 413        | 413        | 413      |
| 20000, 102826| 921| 921    | 921 | 921 | 923| 923        | 923        | 923      |
| 50000, 281726| 2706| 2773   | 2712| 2712| 2757| 2757       | 2757       | 2743     |

Compared to the best results from [9], which used a randomized local search algorithm that is run for up to one hour, LP Rounding approaches generally produced a smaller or as good solution using significantly less run-time at less than 0.5 seconds for each graph.

Table 10: Results for DIMACS Graphs

| Graph               | n, m      | L*    | Greedy/L* | A₁′/L* | A₁′ Hybrid/L* | A₂′/L* | A₂′ Hybrid/L* | A₃/L* | A₃ Hybrid/L* |
|---------------------|-----------|-------|-----------|--------|---------------|--------|---------------|-------|--------------|
| coAuthorsDBLP       | 299067, 977676| 43969.00 | 1.02     | 1.02   | 1.02          | 1.02   | 1.02          | 1.02  | 1.02         |
| coPapersCiteseer    | 434102, 16036720| 26040.92 | 1.12     | 1.01   | 1.12          | 1.01   | 1.12          | 1.01  | 1.12         |
| citationCiteseer    | 268495, 1156647| 43318.85 | 1.04     | 1.03   | 1.04          | 1.03   | 1.04          | 1.04  | 1.04         |

Table 11 shows an example of a 7 million+ vertices graph where A₁ and A₂ cannot be run as a result of the large size. For hybrid approaches, using the first d/2 vertices from the greedy solution, where d is the size of the greedy solution, resulted in the use of too much memory. We instead used the first 3d/4 vertices from the greedy solution. Both A₁ Hybrid and A₂ Hybrid performed better than greedy. Greedy took 14 seconds to produce a solution while hybrid took 107 seconds. max{M*, N*} is provided as a lower bound on L*, and therefore, γ(G).

Table 11: Results for the DIMACS Great Britain Street Network

| n, m     | M*      | N*      | max{M*, N*} | Greedy | A₁ Hybrid | A₂ Hybrid |
|-----------|---------|---------|-------------|--------|-----------|-----------|
| 7733822, 8156517 | 1314133 | 1357189 | 1357189     | 2732935| 2724608   | 2724608   |
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