Response in kinetic Ising model to oscillating magnetic fields

Kwan-tai Leung and Zoltán Nédé

Institute of Physics, Academia Sinica, Taipei, Taiwan 11529, R.O.C.

(Last revised February 14, 2022)

Ising models obeying Glauber dynamics in an oscillating magnetic field are analyzed. In the context of stochastic resonance, the response in the magnetization is calculated by means of both a mean-field theory with linear-response approximation, and the time-dependent Ginzburg-Landau equation. Analytic results for the temperature and frequency dependent response, including the resonance temperature, compare favorably with simulation data.

PACS numbers: 64.60.Ht, 05.40.+j, 05.50.+q

I. INTRODUCTION

Ising models with Glauber dynamics in an oscillating magnetic field were recently considered with Monte Carlo (MC) simulations in [1] and [2]. The phenomenon of stochastic resonance [3] was revealed, exhibiting a characteristic peak in the correlation function \( C(T) \) between the external oscillating magnetic field and the magnetization \( M(t) \) versus the temperature \( T \) of the system. The resonance temperature \( T_r \) (the temperature at which \( C(T) \) has a maximum) was systematically computed as a function of the driving period, lattice size and driving amplitude, both for two-dimensional (2D) [1] and three-dimensional (3D) [2] systems. The one-dimensional (1D) case was analysed by Brey and Prados [4] in a linear-field approximation.

The present work is a natural continuation of those studies, considering analytically the 2D and 3D cases. We will present two approaches. The mean-field theory with linear response approximation will be discussed first. Then in 2D where the mean-field theory is not as good as in other dimensions, a more refined time-dependent Ginzburg-Landau (TDGL) approach will be presented, with significant improvements.

Recently, kinetic Ising systems in oscillating external fields have also been examined both experimentally and theoretically in [5]. The focus was on properties below the zero-field critical point, such as the frequency dependence of the probability distributions for the hysteresis-loop area and the residence time. The latter quantity for small systems in moderately weak fields suggests further evidences of stochastic resonance. Very recently, finite-size effects versus driving frequency are analyzed as a dynamical critical phenomena [6]. In contrast to these works, ours is focused on the temperature dependence above the zero-field critical point.

II. MEAN-FIELD THEORY AND LINEAR-RESPONSE APPROXIMATION

Our starting point is the master equation for the kinetic Ising model obeying Glauber dynamics [7]:

\[
P(\sigma; t + 1) - P(\sigma; t) = \sum_{\sigma'} [w(\sigma' \to \sigma)P(\sigma'; t) - w(\sigma \to \sigma')P(\sigma; t)],
\]

where \( P(\sigma; t) \) is the joint probability of finding the spin configuration \( \sigma \) at time \( t \), and \( w's \) are the transition rates between two configurations which differ by one spin flip. For the heat-bath algorithm, the rate function is chosen as

\[
w(\sigma \to \sigma') = \frac{1}{1 + e^{-\beta[E(\sigma') - E(\sigma)]}},
\]

with \( \beta = 1/T \) (hereafter the Boltzmann constant \( k \equiv 1 \)), and \( E(\sigma) = E_\sigma \) is the energy of \( \sigma \) in a magnetic field \( h \):

\[
E_\sigma = -J \sum_{\text{nn}} S_i S_j - h(t) \sum_i S_i,
\]

where \( h(t) = A \sin(\omega t) \) and \( \sum_{\text{nn}} \) denotes a summation over nearest neighbors in a square or cubic lattice.

Let us denote the configuration \( \sigma \) by the values of the spins \( S_1, S_2, \ldots, S_V \), with system volume given by \( V = N^d \). \( d \) is the spatial dimension of the system and \( N \) is its linear size. Since \( S_1 = \pm 1 \), it is easy to rewrite (1) as

\[
\frac{d}{dt} P(S_1, S_2, \ldots, S_V; t) = - \sum_{j=1}^V w_j(S_j)P(S_1, S_2, \ldots, S_V; t)
\]

\[
+ \sum_{j=1}^V w_j(-S_j)P(S_1, S_2, \ldots, S_V; t)
\]

with

\[
w_j(S_j) = \frac{1}{2} [1 - S_j \tanh(E_j/T)],
\]

\[
E_j = J \sum_{k=1}^z S_k + h,
\]
with the phase shift and amplitude given by the basic equation for the Glauber dynamics:

$$\frac{d}{dt} < S_i >= - < S_i > + < \tanh(E_i/T) > .$$  \hspace{1cm} (5)$$

Invoking the mean-field approximation, we replace $E_i$ by $J_z < S > + h$ to get:

$$\frac{d}{dt} < S >= - < S > + \tanh[(h + T^\text{MF}_c < S >/T)],$$  \hspace{1cm} (6)$$

where $T^\text{MF}_c = J_z$ is the mean-field critical temperature. In the absence of $h$, the magnetization is given by the stationary solution of the well-known equation:

$$< S >_0 = \tanh[T^\text{MF}_c < S >_0 /T].$$  \hspace{1cm} (7)$$

For small $h(t)$, we may use the linear-response theory in (6) by first writing $< S > (t) = < S >_0 + \Delta S (t)$ and considering the $h/T \ll 1$ and $\Delta S/T \ll 1$ limits. Performing the Taylor expansion and keeping only the first-order terms, equation (6) becomes:

$$\frac{d}{dt} \Delta S = - \frac{\Delta S}{\tau^\text{MF}} + \frac{A}{T}(1- < S >^2_0) \sin(\omega t),$$  \hspace{1cm} (8)$$

where

$$\tau^\text{MF} = \frac{1}{1 - T^\text{MF}_c(1- < S >^2_0)}$$  \hspace{1cm} (9)$$

is the relaxation time. The solution can be found easily:

$$\Delta S(t) = \Delta S_0 \sin(\omega t - \theta^\text{MF}),$$  \hspace{1cm} (10)$$

with the phase shift and amplitude given by

$$\theta^\text{MF} = \arctan(\omega \tau^\text{MF}),$$  \hspace{1cm} (11)$$

$$\Delta S_0 = \frac{A}{T}(1- < S >^2_0) \frac{1}{\sqrt{1+\omega^2}}.$$  \hspace{1cm} (12)$$

The correlation function between the total magnetization $M = V < S >$ and the external field $h(t)$ can be computed:

$$C = \frac{M(t)h(t)}{\langle V/W/2\pi \rangle \int_0^{2\pi/\omega} \Delta S(t)h(t)dt}$$

$$= \frac{VA^2}{2T}(1- < S >^2_0) \frac{\tau^\text{MF}}{1 + \omega^2 \tau^2_{\text{MF}}}. $$  \hspace{1cm} (13)$$

Here the overline denotes a temporal average over a period $P = 2\pi/\omega$. In the $T > T^\text{MF}_c$ domain, $< S >_0 = 0$, thus $C$ becomes:

$$C_{T>T^\text{MF}_c} = \frac{VA^2}{2} \frac{T - T^\text{MF}_c}{(T - T^\text{MF}_c)^2 + \omega^2 T^2}.$$  \hspace{1cm} (14)$$

III. TIME-DEPENDENT GINZBURG-LANDAU APPROACH

Before comparing (13) to simulations, we present an alternative, continuum approach to compute $C$. For an Ising system with non-conservative order parameter (model $A$), the time-dependent Ginzburg-Landau (TDGL) equation for the local magnetization density $\phi(\vec{r}, t)$ takes the following form:

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\delta H}{\delta \phi} + \zeta,$$  \hspace{1cm} (15)$$

$$H = \int d\vec{r} \left\{ \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} u \phi^2 + \frac{g}{4!} \phi^4 \right\}, $$  \hspace{1cm} (16)$$

where $H$ is the coarse grained Hamiltonian. For our present purpose, the white noise $\zeta(\vec{r}, t)$ which accounts for the effect of thermal fluctuations is irrelevant. Conventionally, parameters $\Gamma$, $u$ and $g$ in (16) are understood to be obtained by coarse graining the microscopic dynamics (1). For critical properties, the important temperature dependence in these parameters lies in $u \propto T - T^\text{GL}_c$, giving rise to the spontaneous symmetry breaking below the critical temperature $T^\text{GL}_c$. In order to compare with simulations, more precise dependences on $T$ are required. To this end, we outline here a refined mean-field approach in the continuum limit. The same approach has been successfully applied to the two-species driven diffusive systems (1). This approximation is expected to be good outside the critical region. However, this turns out to be not a serious handicap because the presence of an oscillating field prevents the system from building up critical correlations.

In a mean-field approximation, the joint probabilities in (10) are factorized into singlet probabilities $p(\vec{r}, t)$ for finding the spin up at site $\vec{r}$ at time $t$. This effectively produces the power series expansion of $H$ in $\phi$. $1-p$ gives the probability of finding the spin down. The continuum limit involves an expansion in the derivatives, such as:

$$p(x \pm 1, y; t) \rightarrow p(x, y; t) \pm \frac{\partial p(x, y; t)}{\partial x} + \frac{1}{2} \frac{\partial^2 p(x, y; t)}{\partial x^2} + \cdots$$

By identifying $p$ as $(\phi+1)/2$, we obtain from (1) a kinetic equation for $\phi$ after some algebra. For $h = 0$, we find precisely the deterministic part of (15) with:

$$\Gamma = \frac{1}{8} (-2 W_4 + 2 W_{-4} - W_8 + W_{-8}),$$  \hspace{1cm} (17)$$

$$u = \frac{1}{8 \pi} (6 W_0 + 12 W_4 - 4 W_{-4} + 5 W_8 - 3 W_{-8}),$$  \hspace{1cm} (18)$$

$$g = \frac{3}{2 \Gamma} (-6 W_0 - 4 W_4 + 4 W_{-4} + 5 W_8 + W_{-8}),$$  \hspace{1cm} (19)$$

where $W_n \equiv 1/(1 + e^{n \beta J})$ contains the desired explicit $T$ dependence. When a small uniform field $h$ is applied, to $O(h)$ (neglecting a $\phi^5$ term) we have finally the deterministic kinetic equation:

$$\frac{\partial \phi}{\partial t} = -\Gamma \frac{\phi}{2} + \frac{g}{4!} \phi^4 - \frac{1}{2} u \phi^2,$$
\[ \frac{\partial \phi}{\partial t} = -\Gamma \left\{ -\nabla^2 \phi + u \phi + \frac{g}{6} \phi^3 - \mu h \right\}, \]  

where \( \mu = \beta (3W_0^2 + 4W_1 W_{-1} + W_8 W_{-8})/2\Gamma \). It is useful to note that \( \Gamma, g \) and \( \mu \) in (20) are positive definite for all \( T \), whereas \( u \) has one zero at \( T_{c,\text{GL}} \approx 3.0901J \approx 1.3618T_c \), where \( T_c = -2/\ln(\sqrt{2} - 1)J \approx 2.2692J \) is exact. This is an improvement over \( T_{c,\text{MF}}^{\text{MF}} = 4J \) from the last section. Moreover, we reproduce the first few terms of the high-temperature series expansions of thermodynamic quantities such as the susceptibility and the relaxation time. In the \( \beta \to 0 \) limit, we recover the mean-field results of the last section: \( u \approx 1/\beta J - 4, \Gamma \approx \beta J, g \approx 48(\beta J)^2 \), and \( \mu \approx 1/J \).

For small \( h \) and \( T > T_{c,\text{GL}}^{\text{GL}} \), the nonlinear term \( g\phi^3 \) in (21) is negligible. The total magnetization \( M(t) = \int d\vec{r} \phi(\vec{r},t) = \phi(\vec{q} = 0,t) \) in response to an external field can then be computed easily, where \( \phi \) denotes the spatial Fourier transform of \( \phi \). It satisfies \( \partial M/\partial t = -\Gamma u M + \Gamma \mu \phi(\vec{q} = 0,t) \). We readily find

\[ M(t) = \frac{V\mu AT}{(\Gamma u)^2 + \omega^2} \sin(\omega t - \theta_{\text{GL}}), \]  

where the phase shift is \( \theta_{\text{GL}} = \arctan(\omega/\Gamma u) \). The correlation function with \( h \) is then given by

\[ C_{T>T_{c,\text{GL}}} = \frac{VA^2 T^2 \mu u}{2[(\Gamma u)^2 + \omega^2]} \]  

Note that this coincides with the mean-field result (14) in the high-temperature limit.

For \( T < T_{c,\text{GL}}^{\text{GL}} \), the term proportional to \( g \) is needed to break the symmetry, leading to the spontaneous magnetization \( m = \sqrt{-6u/g} \). Linearizing about \( m \), we find precisely the same form of \( C \) as above and below the respective \( T_{c,\text{GL}} \), as shown in Fig. 1. This double-peak structure in \( C \) is consistent with simulations for larger lattice sizes (up to \( N = 200 \) for 2D and \( N = 40 \) for 3D) and with smaller steps in \( T \) than reported in [1] and [2]. The reason for missing the peak below \( T_{c,\text{GL}} \) in our earlier simulations is due to the use of small lattice sizes. Note that the peak below \( T_{c,\text{GL}} \) is much smaller than the one above and its position is less sensitive to the driving period. The reason for the overestimated theoretical values of the peaks below \( T_{c,\text{GL}} \) may be accounted for by the frustration of the system to order in the presence of \( h(t) \); such frustration stemming from metastability has not been taken into account explicitly in our theories below \( T_{c,\text{GL}} \) when we seek the symmetry-breaking solutions.

We believe that this also explains the discrepancy at \( T_{c,\text{GL}} \), where simulations show a small but finite \( C(T) \). Finite-size effects are not of great concern here because, as mentioned above, the correlation length even at \( T_{c,\text{GL}} \) is truncated by \( h \). In simulations, we have checked the convergence in \( C(T) \) for \( N \geq 50 \) in 2D.

Focusing on \( T > T_{c,\text{GL}} \) from now on, the TDGL predictions for \( C(T) \) are more accurate than those of the mean-field theory in general. They both converge to the simulations in the tails at \( T \gg T_{c} \) (see Fig. 1). In 3D the mean-field theory is already acceptable.

Turning our attention to the amplitude dependence, reploting the simulation data from [1] and [2] suggests that the height of the peak \( C(T_r) \propto A^2 \) in agreement with [4] and [2]. For not too large frequencies and small \( A \), the theoretical proportionality constant agrees well with simulations. For example, the slope of \( C(T_r)/(VT_c) \) versus \( A^2/T_c^2 \) for \( P = 50 \) in 2D gives 0.92 from simulations [1], 0.96 from TDGL and 0.99 from mean-field approach. In 3D the same slope is 0.88 from simulations [2], and 1.29 from mean-field approach (In 3D the comparison is worse because \( T_r \) is much closer now to \( T_c \).) This proportionality is a manifestation of the linear response of the system to \( h \), which breaks down at large enough amplitudes. Our new simulations show that this happens for \( A/T_c > 0.15 \) in 2D for \( P = 40 \).

A quantity of significant interest is the resonance temperature \( T_r(P) \). It can be determined analytically from (13)

\[ T_r^{\text{MF}} = T_c^{\text{MF}} \left( 1 + \sqrt{1 - \frac{1}{\omega^2 + 1}} \right), \]  

and numerically from (21) for \( T_r^{\text{GL}} \). These together with simulation results are presented in Fig. 2. The agreements are reasonable. As expected, the mean-field approximation is quite good in 3D but in 2D the TDGL approximation is better.

The results in Fig. 2 confirm the earlier observation in [1] and [2] that for \( P \to \infty \) we get \( T_r \to T_c \). This result is also consistent with the one obtained by Brey and Prados [1] in 1D where the above limit becomes \( T_r \to T_c \). In the opposite limit \( P \leq 1 \) (in unit of Monte Carlo steps \( P \geq 1 \) both the theory in 1D [1] and our approximations in 2D and 3D suggest \( T_r \to \) const. Unfortunately, in [1] and [2] the wrong conclusion \( T_r \to \) const. was drawn in this limit. Similarly, the position of the peak below \( T_c \) also converges to \( T_c \) in the \( P \to \infty \) limit.

In passing, we also derive [1] the relationship between the correlation function and the hysteresis-loop area \( A \):

\[ A = 2\pi C \left| \tan \theta \right| \]  

where \( \theta \) is the phase shift between \( h \) and \( M \). This then relates our results of \( C \) to that of \( A \) as observed in [1].
V. CONCLUSIONS

Using mean-field with linear-response and TDGL approximations, the characteristics of the resonance peaks observed in kinetic Ising models in oscillating magnetic fields are reproduced. New simulations improve earlier results by confirming the analytically predicted double peaks. Focusing mostly on the behavior above $T_c$ (where our approaches work better), we determine the dependence of the resonance temperature as a function of driving frequency and amplitude. We confirm the already predicted result in $[1,2]$ that $T_r \to T_c$ for the limit of practically interesting driving frequencies ($P \to \infty$), and corrected the wrong extrapolation in the opposite limit $P \to 1$. We introduce a refined TDGL approach which improves significantly the mean-field results in 2D, but in 3D the mean-field approximation is already acceptable. We have thus demonstrated that the stochastic resonance in kinetic Ising models above $T_c$ can be understood by means of rather simple theoretical approaches for small driving amplitudes.

VI. ACKNOWLEDGMENTS

We are grateful to NSC of ROC for their support through the grant NSC87-2112-M-001-006.

* on leave from: Babes-Bolyai University, Dept. of Theoretical Physics, str. Kogalniceanu 1, RO-3400, Cluj-Napoca, Romania.

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FIG. 1. $C(T)/(VT_c)$ versus temperature for $P = 40$ and $A = 0.05T_c$ for 2D in (a) and 3D in (b). Dots are MC simulation results in 2D ($N = 200$), triangles are MC simulations in 3D ($N = 40$), continuous line is from TDGL approximation and the dashed line is the mean-field result. The higher peak for TDGL than mean-field theory below $T_c$ is due to our dropping the $\phi^5$ term in (20).
FIG. 2. Resonance temperature above $T_c$ versus driving period $P$ for $A = 0.05$, on absolute scale $T_r/J$ in (a) and on relative scale $T_r/T_c$ in (b). The long-dashed and short-dashed lines in (a) are the mean-field results for 3D and 2D respectively, in the rest the symbols mean the same as in Fig. 1.