ASYMPTOTICS OF SINGULARLY PERTURBED DAMPED WAVE EQUATIONS WITH SUPER-CUBIC EXPONENT

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Abstract. This work is devoted to studying the relations between the asymptotic behavior for a class of hyperbolic equations with super-cubic nonlinearity and a class of heat equations, where the problem is considered in a smooth bounded three dimensional domain. Based on the extension of the Strichartz estimates to the case of bounded domain, we show the regularity of the pullback, uniform, and cocycle attractors for the non-autonomous dynamical system given by hyperbolic equation. Then we prove that all types of non-autonomous attractors converge, upper semicontinuously, to the natural extension global attractor of the limit parabolic equations.

1. Introduction. A popular approach to study the dissipative dynamics is related to the concept of the attractors. By definition, attractor is a compact invariant set attracting the images of all bounded sets of the phase space as time tends to infinity. The damped wave equation is a typical dissipative system which is of a great permanent interest for both theoretical and applied points of view. They arise as an evolutionary mathematical model in various systems for the relevant physical application, including electrodynamics, models in relativistic quantum mechanics nonlinear elasticity, see more in [15].

The long time behavior of the weakly damped wave equations has been widely studied for both autonomous and non-autonomous cases, ([2, 8, 11, 16, 18]). To prove the existence of attractors corresponding to the semigroup or process generated by weakly damped wave equations, one has to guarantee the global well-posedness for this problem which depends strongly on the growth rate of the nonlinearity. It is well known that the global unique solvability in the energy space has been established for the sub-cubic or cubic growth rate of nonlinearity in the bounded smooth domain in $\mathbb{R}^3$ ([1, 30]), and they also studied the dynamical behavior of weakly damped wave equations. For a long time, the cubic growth rate of nonlinearity had been considered as a critical one for weakly damped wave equations in the bounded smooth domain in $\mathbb{R}^3$ since weakly damped wave equations endowed with super-cubic nonlinearity encounters lack of uniqueness as the 3D Navier-Stokes equation does. Researchers did a lot of efforts to overcome this difficulty, J. M. Ball studied the regularity of solutions for the generalized semiflow in [4]. Zelik proved

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the existence of a global attractor in the trajectory dynamical system acting on the space of all properly defined weak energy solutions in [31]. Carvalho in [9] constructed the global weak solutions of this problem as the limits as the solutions of wave equations involving the strong damping term. However due to the establishment of Strichartz estimates and generalization of Morawetz-Pohozhaev identity to the case of bounded domain in [5]. In the sense of Shatah-Struwe solution, the global well-posedness of quintic wave equation in smooth bounded domains in $\mathbb{R}^3$ has been obtained in [7, 20] and they also established the global attractor for weakly damped wave equations in the case of quintic or sub-quintic growth rates of nonlinearity in [20].

One of the most important aspects to be taken into account is to study the behavior of attractors under perturbations([29, 17, 13, 26, 25]). The perturbation are expressed through some continuity properties of the dynamical systems, which together with suitable natural conditions will result in a continuity property of the attractors.

In this paper, we consider the following perturbation problem:

$$
\begin{align*}
\begin{cases}
  u_t = \Delta u + f_0(u) & \text{in } \Omega, \ t > \tau, \\
  u|_{t=\tau} = u_0, \ u_t|_{t=\tau} = u_1, & \text{in } \Omega, \\
  u|_{\partial\Omega} = 0, & \text{in } \Omega, \\
  u|_{t=\tau} = u_0, \ u_t|_{t=\tau} = u_1, & \text{in } \Omega \\
  \end{cases}
\end{align*}
$$

(1.1)

and for $\varepsilon > 0$,

$$
\begin{align*}
\begin{cases}
  \varepsilon u_{\varepsilon t} + u_\varepsilon^t = \Delta u_\varepsilon + f_\varepsilon(t, u_\varepsilon) & \text{in } \Omega, \ t > \tau, \\
  u_\varepsilon|_{t=\tau} = u_0, \ u_{\varepsilon t}|_{t=\tau} = u_1, & \text{in } \Omega, \\
  u_{\varepsilon}|_{\partial\Omega} = 0, & \text{in } \Omega, \\
  \end{cases}
\end{align*}
$$

(1.2)

where $\Delta$ denotes the standard Laplacian, $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, $\varepsilon$ is a positive parameter which will tend to zero and the initial data $(u_0, u_1)$ is taken from the standard energy space $H_0^1(\Omega) \times L^2(\Omega)$. We assume that for $\varepsilon > 0$ the nonlinearity $f_\varepsilon(t, s) : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable with respect to the first variable $t$ and twice differentiable in the second variable $s$, $f_0(s) : \mathbb{R} \to \mathbb{R} \in C^2(\mathbb{R}, \mathbb{R})$, and satisfies the following super-cubic growth condition for every constant $R > 0$:

$$
\begin{align*}
  f_\varepsilon(t, s) & \sim s^{5-\kappa}, \text{ for every } \varepsilon > 0 \text{ and } |s| \leq R, \\
  f_0(s) & \sim s^{5-\kappa}, \text{ for every } |s| \leq R,
\end{align*}
$$

(1.3)

where $0 < \kappa \leq 1$. The nonlinearity $f_0$ is also assumed to satisfy the condition:

$$
\limsup_{|s| \to \infty} \frac{f_0(s)}{s} < \mu_1, \quad \forall \ s \in \mathbb{R},
$$

(1.4)

where $\mu_1 > 0$ denotes the first eigenvalue of the operator $-\Delta$ on $\Omega$ with homogeneous Dirichlet boundary conditions and without loss of generality we may assume that $f_0(0) = 0$. Then we can deduce that there exist constants $0 \leq c_0 < \mu_1$ and $c_0' \in \mathbb{R}$ such that ([23])

$$
\begin{align*}
sf_0(s) & \leq c_0'|s| + c_0|s|^2, \text{ for all } s \in \mathbb{R},
\end{align*}
$$

(1.5)

Equations (1.2) can model heat conduction corresponding to the Maxwell-Cattaneo law, and (1.1) corresponding to the classical heat conduction law([10]). Physical experiments demonstrate that the parameter changes as the material changes, for example the parameter $\varepsilon$ may be large for sand or very small for metals. The weakly damped wave equations (1.2) converge to parabolic equation (1.1) as $\varepsilon$ goes to zero, study the continuity of asymptotic behaviors of this kind problems.
would answer a natural question that whether the dynamics of problems described by the Maxwell-Cattaneo law is close to the dynamics described by the Fourier law when the parameter \( \varepsilon \) is small ([14]). Considering the family of autonomous weakly damped wave equations with sub-cubic or cubic nonlinearity, continuity of global attractors was proved in [17] and so was the exponential attractors in [13], they later improved their results in [26] by proving the exponential attractors is Hölder continuous with respect to \( \varepsilon \). For the non-autonomous case with sub-cubic or cubic nonlinearity, continuity of pullback, uniform, and cocycle attractors was considered in [14].

As to the singularly perturbed damped wave equations with super-cubic growth nonlinearity, global attractors \( A_{\varepsilon_{tr}} \) corresponding to trajectory dynamical system acting on the space of all properly defined weak energy solutions was considered in [31]. In this paper, we extend the result for super-cubic nonlinearity and consider the continuity of pullback, uniform, and cocycle attractors corresponding to the processes generated by the Shatah-Struwe solutions of (1.2).

Note that the limit equation (1.1) is a parabolic equation, therefore the phase spaces for the perturbed and unperturbed equations are not the same. Because the parabolic equation has a regularization effect, we can consider the weak solution of (1.1) and its derivative w.r.t. \( t \) as a vector solution. Then we construct a compact set through the vector solution which the global attractor corresponding to (1.1) can be embedded in naturally. Furthermore, under the above hypotheses, we can yield that the non-autonomous attractors corresponding to the process \( S_\varepsilon(t, \tau) \) generated by the Shatah-Struwe solution of (1.2) can be bounded in more regular phase spaces. Finally, with assumptions of the nonlinearity \( f_\varepsilon \) with respect to \( \varepsilon \), we can prove the upper semicontinuity of pullback, uniform, and cocycle attractors for the non-autonomous dynamical system given by hyperbolic equation with fast growing dissipative nonlinearities on a bounded domain in \( \mathbb{R}^3 \).

The paper is organized as follows. The preliminary things, including the key notations and some technical lemmas are recalled in Section 2. The key properties and uniform asymptotic smoothness estimates with respect to \( \varepsilon \) of the Shatah-Struwe solution of the hyperbolic equation (1.2) are stated in Section 3. The following Section 4 is devoted to prove our main result, the upper semicontinuous convergence.

2. Preliminaries. In this section, we briefly recall some basic concepts related with non-autonomous dynamical systems and state the Strichartz estimates for weakly damped wave equationss (1.2) from [20]. Given a metric space \( (X, d_X) \), denote by \( B_{X,r}(x) \) the ball of radius \( r \) centered at \( x \),

\[
B_{X,r}(x) = \{ y \in X : d_X(x, y) < r \}.
\]

And the Hausdorff semi-distance between two subsets \( A \) and \( B \) of \( X \) defined as

\[
dist_X(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b).
\]

**Definition 2.1.** ([19]) Let \( (X, d_X) \) be a complete metric space. A process \( S(\cdot, \cdot) \) on \( X \) is a two-parameter family of maps \( S(t, \tau) : X \to X, \tau \in \mathbb{R}, t \geq \tau, \) such that

1. \( S(t, t) = Id, \)
2. \( S(t, \tau) \circ S(\tau, s) = S(t, s) \) for every \( t \geq \tau \geq s, \)
3. \( S(t, \tau)x \) is continuous in \( t, \tau \) and \( x \) for every \( t \geq \tau \) and \( x \in X. \)

If the property of a evolution process \( \{S(t, \tau)\}_{t \geq \tau} \) is independent of \( t - \tau \), i.e., \( S(t, \tau) = S(t + s, \tau + s) \) for every \( s \in \mathbb{R} \), then a semigroup \( \{T(t)\}_{t \geq 0} \) generated by
a autonomous dynamical system can be defined as:
\[ T(t) \triangleq S(t+\tau,\tau) = S(t,0) \quad \text{for every } t \geq 0. \]

Generally speaking, there are three natural ways to characterize the long-time behavior for a non-autonomous dynamical system, the first one is to consider the limit of \{S(t,\tau)\}_{t \geq \tau} for a fixed \( t \to -\infty \), i.e. the pullback attractor, the second one is to examine the limit of \{S(t,\tau)\}_{t \geq \tau} as \( t \to \infty \), i.e. the forward attractor, while the third one is to abstract a family of symbol functions from the structure of the non-autonomous dynamical system and study the cocycle attractor and the uniform attractor. We state a formal definition of the pullback attractor in the following:

**Definition 2.2.** \([19]\) We call a family of compact sets \( \mathcal{A}(t) \) in \( X \) the pullback attractor for the process \( \{S(t,\tau)\}_{t \geq \tau} \) if

1. \( \mathcal{A}(t) \) is invariant, i.e. \( S(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t) \) for all \( t \geq \tau \),
2. \( \mathcal{A}(t) \) is pullback attracting, for every bounded set \( B \subset X \) and \( t \in \mathbb{R} \)
   \[ \text{dist}_X (S(t,\tau)B, \mathcal{A}(t)) \to 0 \quad \text{as } \tau \to \infty, \]
3. \( \mathcal{A}(t) \) is minimal in the sense that if \( \mathcal{A}'(t) \) is any other family of compact sets that satisfies (1) and (2) then \( \mathcal{A}(t) \subset \mathcal{A}'(t) \) for all \( t \in \mathbb{R} \).

The definition of the uniform attractor involved with cocycle which is defined below:

**Definition 2.3.** \([11]\) Let \( (X,d) \) and \( (\Sigma,\rho) \) be two metric spaces. A non-autonomous dynamical system denoted by \( (\phi,\theta)_{(X,\Sigma)} \) consists of two ingredients:

1. A driving semigroup \( \{\theta_t: t \geq 0\} \) of continuous maps of \( \Sigma \) into itself,
2. A cocycle on \( \theta \) is a map \( \varphi: \mathbb{R}^+ \times \Sigma \to \mathcal{C}(X) \) such that the following properties hold:
   - (i) \( \varphi(0,p) = Id \) for all \( p \in \Sigma \),
   - (ii) the map \((t,p) \mapsto \varphi(t,p)x\) from \( \mathbb{R}^+ \times \Sigma \) to \( X \) is continuous for all \( x \in X \),
   - (iii) cocycle property, i.e. \( \varphi(t+\tau,p) = \varphi(t,\theta_\tau p)\varphi(\tau,p) \) for all \( t,\tau \geq 0 \) and \( p \in \Sigma \).

**Definition 2.4.** A set \( \mathcal{A} \subset X \) is the uniform attractor for the non-autonomous dynamical system \( (\varphi,\theta)_{(X,\Sigma)} \) if it is the minimal compact set such that
\[
\lim_{t \to \infty} \sup_{p \in \Gamma} \text{dist}_X (\varphi(t,p)B, \mathcal{A}) = 0,
\]
for every bounded subset \( B \) of \( X \) and bounded set \( \Gamma \) of \( \Sigma \).

The next definition is the so-called cocycle attractor.

**Definition 2.5.** \([14]\) A non-autonomous set is a family \( \{D(p)\}_{p \in \Sigma} \) of subsets of \( X \) indexed in \( \Sigma \). We say that \( \{D(p)\}_{p \in \Sigma} \) is an open (closed, compact, nonempty) non-autonomous set if each fiber \( D(p) \) is an open (closed, compact, nonempty) subset of \( X \).

**Definition 2.6.** \([14]\) A compact non-autonomous set \( \{A(p)\}_{p \in \Sigma} \) is called a cocycle attractor for the non-autonomous dynamical system \( (\varphi,\theta)_{X,\Sigma} \) if

1. \( \{A(p)\}_{p \in \Sigma} \) is invariant under \( (\varphi,\theta)_{(X,\Sigma)} \), that is \( \varphi(t,p)A(p) = A(\theta_t p) \), for all \( p \in \Sigma \) and \( t \geq 0 \),
(2) \( \{A(p)\}_{p \in \Sigma} \) pullback attracts all bounded subsets \( B \subset X \), that is,

\[
\lim_{t \to \infty} \text{dist}_X(\varphi(t, \theta_{-1} p) B, A(p)) = 0.
\]

For a family of processes \( \{S_\lambda(\cdot, \cdot)\}_{\lambda \in \Lambda} \) where \( \lambda \) is a parameter in a metric space \( \Lambda \). Once the condition that the processes \( S_\lambda(\cdot, \cdot) \) converge (in some appropriate sense) to \( S_{\lambda_0}(\cdot, \cdot) \) as \( \lambda \to \lambda_0 \) is assumed, we can study the continuity of the corresponding non-autonomous attractors.

**Definition 2.7.** ([19]) Let \( X \) and \( \Lambda \) be metric spaces and \( \{\mathcal{A}_\lambda\}_{\lambda \in \Lambda} \) a family of subsets of \( X \). We say that the family \( \mathcal{A}_\lambda \) is upper semicontinuous as \( \lambda \to \lambda_0 \) if

\[
\lim_{\lambda \to \lambda_0} \text{dist}_X(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = 0.
\]

We say that \( \mathcal{A}_\lambda \) is lower semicontinuous as \( \lambda \to \lambda_0 \) if

\[
\lim_{\lambda \to \lambda_0} \text{dist}_X(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) = 0.
\]

\( \mathcal{A}_\lambda \) is continuous as \( \lambda \to \lambda_0 \) if it is both upper and lower semicontinuous as \( \lambda \to \lambda_0 \).

The following result shows that (semi)continuity with respect to \( \lambda \in \Lambda \) at \( \lambda_0 \) is completely characterized by the behavior of sequences \( \{\mathcal{A}_\lambda\}_{\lambda \in \Lambda} \) where \( \lambda_n \to \lambda_0 \).

**Lemma 2.8** ([19]). Let \( X \) and \( \Lambda \) be metric spaces and \( \{\mathcal{A}_\lambda\}_{\lambda \in \Lambda} \) be a family of compact subsets of \( X \). Then

1. \( \mathcal{A}_\lambda \) is upper semicontinuous at \( \lambda_0 \) if and only if, whenever \( \lambda_n \to \lambda_0 \) as \( n \to \infty \), any sequence \( \{x_n\}_{n \in \mathbb{N}} \in \{\mathcal{A}_{\lambda_n}\}_{n \in \mathbb{N}} \) has a convergence sequence whose limit belongs to \( \mathcal{A}_{\lambda_0} \).
2. \( \mathcal{A}_\lambda \) is lower semicontinuous at \( \lambda_0 \) if and only if, whenever \( x_0 \in \mathcal{A}_{\lambda_0} \) and \( \lambda_n \to \lambda_0 \), there is a sequence \( x_n \in \mathcal{A}_{\lambda_n} \) such that \( x_n \to x_0 \) as \( n \to \infty \).

The operator \( A \triangleq -\Delta \) is positive, self-adjoint in \( L^2(\Omega) \), and has compact inverse. Consequently, there exists an orthonormal basis of \( L^2(\Omega) \) of eigenfunctions \( \{\psi_j\} \in H^1_0(\Omega), j \in \mathbb{N} \) of \( A \) with eigenvalues

\[
0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots, \quad \text{and} \quad \mu_j \to \infty \text{ as } j \to \infty.
\]

Following the theory that constructs the fractional power spaces associated with \( A \) in [19], we denote for every \( \alpha > 0 \)

\[
X^{2\alpha} = (D(A^\alpha), \langle \cdot, \cdot \rangle_{X^\alpha}) = \left\{ \psi = \sum_{j \in \mathbb{N}} c_j \psi_j, \ c_j \in \mathbb{R}, \ \left| \sum_{j \in \mathbb{N}} \mu_j^{2\alpha} c_j^2 \right| < \infty \right\},
\]

where the inner product in \( X^\alpha \) is given by \( \langle u, v \rangle_{X^\alpha} = \langle A^\alpha u, A^\alpha v \rangle_{X^\alpha} \) for every \( u, v \in D(A^\alpha) \) and

\[
A^\alpha \psi = A^\alpha \sum_{j \in \mathbb{N}} c_j \psi_j = \sum_{j \in \mathbb{N}} \mu_j^\alpha c_j \psi_j.
\]

With this notation, we see that \( X^1 = H^1_0(\Omega) \), \( X^0 = L^2(\Omega) \) and \( X^2 = H^2(\Omega) \cap H^1_0(\Omega) \).
3. **A prior estimates.** In this section, we will derive the estimates needed to prove the upper semicontinuity of the uniform attractor. Note that during the calculation, the constant $e$ may vary from line to line and we denote $E$ as $X^1 \times X^0$ and $E^\delta$ as $X^{1+\delta} \times X^\delta$.

The following conditions are needed for the upper semicontinuous convergence of attractors:

$$\limsup_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} |f_\varepsilon(t, s) - f_0(s)| = 0, \quad \text{for every } s \in \mathbb{R}, \quad (3.1)$$

and

$$\sup_{\varepsilon \in [0, 1]} \sup_{t \in \mathbb{R}} \sup_{s \in \mathbb{R}} \left( |f_\varepsilon(t, s) - f_0(s)| + \left| \frac{\partial f_\varepsilon}{\partial s}(t, s) - f'_0(s) \right| \right) < \infty. \quad (3.2)$$

We also assume that $t \mapsto \frac{\partial f_\varepsilon}{\partial t}(t, s)$ is uniformly continuous w.r.t. $s \in \mathbb{R}$ from $\mathbb{R}$ to $\mathbb{R}$, $s \mapsto \frac{\partial^2 f_\varepsilon}{\partial t^2}(t, s)$ is uniformly continuous w.r.t. $t \in \mathbb{R}$ and

$$\left| \frac{\partial f_\varepsilon}{\partial t}(t, s) \right| \leq c(1 + |s|^{\frac{3}{2}}), \quad \text{for every } s \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (3.3)$$

where the constant $c$ does not depend on $\varepsilon$.

The symbol space for the problem (1.2) can be constructed in a general way, for $\varepsilon \in (0, 1]$ define:

$$\Sigma_\varepsilon \triangleq \{ \theta_\varepsilon f_\varepsilon : t \in \mathbb{R} \},$$

where the closure is in sense of the metric of the uniform convergence on compact sets ([2]) and $\theta_\varepsilon$ is the translation operator, that is $\theta_\varepsilon f(\cdot) = f(t + \cdot)$.

We can verify that for $\varepsilon \in (0, 1]$, $p_\varepsilon \in \Sigma_\varepsilon$ satisfies all the conditions assumed for $f_\varepsilon$ and also for every $R > 0$ the following property is valid ([14]):

$$\lim_{\varepsilon \to 0} \sup_{|s| \leq R} \sup_{t \in \mathbb{R}} \left( |p_\varepsilon(t, s) - f_0(s)| + \left| \frac{\partial p_\varepsilon}{\partial s}(t, s) - f'_0(s) \right| \right) = 0.$$

### 3.1. Local existence of the Shatah Struwe solution

In this section, we recall some properties Shatah-Struwe solutions for weakly damped wave equations with super-cubic nonlinearity. We begin by giving the definition of Shatah-Struwe solution.

**Definition 3.1.** A function $u(t)$ is a weak solution of (1.2) if $\xi_u \triangleq (u, u_t) \in L^\infty(\tau, T; E)$ and satisfies the following equation in the distribution sense:

$$\frac{d^2}{dt^2} \langle u(t), v \rangle + \frac{d}{dt} \langle u(t), v \rangle = \langle \nabla u(t), \nabla v \rangle + \langle f(u), v \rangle \quad \forall \ v \in H^1_0(\Omega), \ t \in (\tau, T),$$

where $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\Omega)$.

**Definition 3.2.** A weak solution $u(t)$, $t \in [\tau, T]$ is said to be a Shatah-Struwe solution of the problem (1.2) if $u \in L^4(\tau, T; L^{12}(\Omega))$ for all $T > \tau$.

**Theorem 3.3.** We assume that the function $f_\varepsilon$ satisfies the growth restriction (1.3). Then for every initial $\xi^\varepsilon_t \triangleq (u_0, u_1) \in E$ and some $T > \tau$, there exists a unique Shatah-Struwe solution $\xi^\varepsilon_u(t) = (u^\varepsilon(t), u^\varepsilon_t(t))$ for problem (1.2) satisfies the following estimates:

$$\|\xi^\varepsilon_u(t)\|_E + \|u^\varepsilon\|_{L^4(\max\{\tau, t-\delta\}, t; L^{12}(\Omega))} \leq Q_1(\|\xi^\varepsilon_0\|_E), \quad \tau < t < T, \quad (3.4)$$

where $Q_1$ is a monotone function which independent of $\tau, t, \omega$ and $u^\varepsilon$.

The proof of this theorem follows exactly from [20].
3.2. Global dissipation. First we prove the dynamical system (1.2) is dissipative in \(E\) in the sense that there exist the absorbing sets independent of \(\varepsilon\).

Notice that the assumption (1.5) implies that there exist constants \(0 < \nu < 1\) and \(c_1 \in \mathbb{R}\) such that
\[
(f_0(u), u) \leq (1 - \nu)\|u\|_{H^1_0(\Omega)}^2 + c_1, \quad \text{for all } u \in H^1_0(\Omega).
\] (3.5)

**Theorem 3.4 (Dissipation in the energy space).** Let assumptions (1.3), (1.4) and (3.1) hold, there exists a positive constant \(R_0\) and for every \(r_0 \geq 0\), a positive constant \(t_0 = t_0(r_0)\) (independent of \(\varepsilon\)) such that, for every \(T > t_0\) and any Shatah-Struwe solution \(u^\varepsilon(t)\) of (1.2) with \((\varepsilon\|u_1\|_{L^2(\Omega)}^2 + \|u_0\|_{H^1(\Omega)}^2)^{1/2} \leq r_0\), we have the following estimates:
\[
(\varepsilon\|u^\varepsilon(T)\|_{L^2(\Omega)}^2 + \|u^\varepsilon(T)\|_{H^1(\Omega)}^2)^{1/2} \leq R_0, \quad \text{for every } 0 < \varepsilon \leq 1, \tau \in \mathbb{R}.
\]

**Proof.** Let \((u^\varepsilon(t), u^\varepsilon_\tau(t))\) be the corresponding Shatah-Struwe solution for (1.2) with initial data \((u^\varepsilon(\tau), u^\varepsilon_\tau(\tau)) = (u_0, u_1)\) for \(t \geq \tau\).

Multiplying (1.2) by \(u^\varepsilon + \beta u^\varepsilon\) with small positive number \(\beta\) which will be choose appropriately later, and applying (3.5), then we get:
\[
\begin{align*}
\frac{d}{dt}\left(\frac{1}{2}\varepsilon\|u^\varepsilon_t(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + \varepsilon\beta\langle u^\varepsilon_t(t), u^\varepsilon(t) \rangle\right) & - \langle F_0(u^\varepsilon(t)), 1 \rangle + \frac{1}{2}\beta\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 \\
+ \beta\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + (1 - \varepsilon\beta)\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 & = \langle f_\varepsilon(t), u^\varepsilon(t) \rangle - f_0(u^\varepsilon(t)), u^\varepsilon(t) \rangle \\
& \leq C_{\varepsilon_0} + c_0\|u^\varepsilon_t(t)\|_{L^2(\Omega)}^2 + \beta\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + \beta\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + \beta c_1,
\end{align*}
\]
where \(F_0(s) = \int_0^s f_0(r)dr\) and the constants \(\varepsilon_0 > 0\) and \(\varepsilon_0' > 0\) small enough.

Choose \(0 < \beta < \frac{1}{2(\varepsilon + \nu)\varepsilon}\) one can deduce that
\[
\begin{align*}
\frac{d}{dt}E(u^\varepsilon_t(t), u^\varepsilon(t)) & + \beta\nu(\varepsilon\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2) \\
& \leq C,
\end{align*}
\]
where \(C\) is a constant independent of \(\varepsilon\) and
\[
E(u^\varepsilon_t(t), u^\varepsilon(t)) \equiv \varepsilon\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2 - 2\langle F_0(u^\varepsilon(t)), 1 \rangle \\
& + 2\beta\varepsilon\langle u^\varepsilon(t), u^\varepsilon(t) \rangle + \beta\|u^\varepsilon(t)\|_{L^2(\Omega)}^2.
\]

We estimate \(E\) from below using (3.5):
\[
\begin{align*}
E(u^\varepsilon_t(t), u^\varepsilon(t)) & \geq \varepsilon\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 - (1 - \nu)\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 - c_1 + 2\beta\varepsilon\langle u^\varepsilon(t), u^\varepsilon(t) \rangle + \beta\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 \\
& = \varepsilon\|u^\varepsilon_t(t)\|_{L^2(\Omega)}^2 + \nu\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 - c_1 + 2\beta\varepsilon\langle u^\varepsilon(t), u^\varepsilon(t) \rangle + \beta\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 \\
& \geq \varepsilon\|u^\varepsilon_t(t)\|_{L^2(\Omega)}^2 + \nu\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 - c_1 - \beta\varepsilon^2\|u^\varepsilon(t)\|_{L^2(\Omega)}^2,
\end{align*}
\]
then we let \(\beta \leq \frac{1}{2}\varepsilon\Rightarrow E(u^\varepsilon_t(t), u^\varepsilon(t)) \geq \frac{1}{2}\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \nu\|u^\varepsilon(t)\|_{H^1(\Omega)}^2 - c_1.
\]

We also estimate the energy \(E(u^\varepsilon_t(t), u^\varepsilon(t))\) from above by using growth condition (1.3).
\[
0 \leq E(u^\varepsilon_t(t), u^\varepsilon(t)) \leq c(\varepsilon\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + \|u^\varepsilon\|_{X^\varepsilon}^2 + 1) \\
\leq c(\varepsilon\|u^\varepsilon(t)\|^6_{L^2(\Omega)} + \|u^\varepsilon(t)\|^6_{H^1(\Omega)} + 1),
\]
\[
(E(u^\varepsilon_t(t), u^\varepsilon(t)))^{\frac{1}{6}} \leq c(\varepsilon\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2 + 1).
\]
Let $V_\varepsilon(t) = E(u_\varepsilon^r(t), u^r(t)) + c_1$, 

$$
\frac{d}{dt}V_\varepsilon(t) + k_1 (V_\varepsilon(t))^{\frac{2}{p-2}} \leq c,
$$

(3.6)

where the constants $k_1$, $c$ independent of $\varepsilon$.

This theorem is proved if we show that there exists a positive constant $R_0$ and for every $t_0 \geq 0$, a positive constant $c(t_0)$ (independent of $\varepsilon$) such that for all $0 < \varepsilon \leq 1$:

$$
V_\varepsilon(t) \leq R_0, \quad \text{for every } t - \tau \geq t(r_0),
$$
as long as $V_\varepsilon(\tau) = V_\varepsilon(u_0, u_1) \leq r_0$.

If $0 \leq V_\varepsilon(t) \leq (\varepsilon_1^\alpha)^{\frac{6}{4-\alpha}}$ where $c$ is the same as in the inequality (3.6), then

$$
V_\varepsilon(t) \leq \left( \frac{c}{k_1} \right)^{\frac{6}{4-\alpha}} \quad \text{for every } t \geq \tau.
$$

Otherwise, for $(\varepsilon_1^\alpha)^{\frac{6}{4-\alpha}} \leq V_\varepsilon(t) \leq r_0$ there will be an obvious contradiction argument shows that for every $\eta > 0$,

$$
V_\varepsilon(t) \leq \left( \frac{c}{k_1} + \eta \right)^{\frac{6}{4-\alpha}} \quad \text{for all } t - \tau \geq t'(r_0, \eta),
$$

where $t'(r_0, \eta) = \frac{1}{\eta k_1} \left( r_0 - \left( \frac{c}{k_1} + \eta \right)^{\frac{6}{4-\alpha}} \right)$. Hence this theorem is proved. \(\square\)

**Theorem 3.5 (Uniform bound for the Strichartz norm).** Let assumptions (1.3), (1.4) and (3.1) hold, Strichartz estimates for the Shatah-Struwe solution (1.2) is independent of $\varepsilon$, that is there exists a constant $R_1$ independent of $\varepsilon$ such that for every $t \geq \tau$

$$
\|u^r(t)\|_{L^3(t, t+1; L^{12})(\Omega)} \leq R_1.
$$

**Proof.** The existence of Shatah-Struwe solution for (1.2) was constructed by Galerkin method in [20]. Let $P_N : L^2(\Omega) \rightarrow L^2(\Omega)$ be the orthoprojector to the linear subspace spanned by the first $N$ eigenfunctions $\{\psi_i\}_{1 \leq i \leq N}$. Then the Galerkin approximation to the problem (1.2) is the following with $u_N^\varepsilon(t) \in P_N L^2(\Omega)$

$$
\begin{cases}
\varepsilon v_{N,t,tt} + v_{N,t} = \Delta u_N^\varepsilon + P_N f_\varepsilon(t, u^\varepsilon_N) & \text{in } \Omega, \quad t > \tau, \\
u_N^\varepsilon|_{t=\tau} = P_N u_0, \quad u_{N,t}|_{t=\tau} = P_N u_1, & \text{in } \Omega, \\
u_N^\varepsilon|_{\partial\Omega} = 0, \quad t \geq \tau.
\end{cases}
$$

(3.7)

The key ingredient of this theorem is to guarantee that the estimate for the extra regularity is independent of $\varepsilon$. Follow the discussions in [20] we only have to illustrate that the estimate for (3.7) is independent of $\varepsilon$.

Split the solution $u_N^\varepsilon(t) = v_N^\varepsilon(t) + w_N^\varepsilon(t)$ where $v_N^\varepsilon(t)$ solves the linear problem

$$
\begin{cases}
\varepsilon v_{N,t,tt} + v_{N,t} = \Delta v_N^\varepsilon & \text{in } \Omega, \quad t > \tau, \\
v_N^\varepsilon|_{t=\tau} = P_N u_0, \quad v_{N,t}|_{t=\tau} = P_N u_1, & \text{in } \Omega, \\
v_N^\varepsilon|_{\partial\Omega} = 0, \quad t \geq \tau,
\end{cases}
$$

(3.8)

and $w_N^\varepsilon(t)$ is a reminder which satisfies

$$
\begin{cases}
\varepsilon w_{N,t,tt} + w_{N,t} = \Delta w_N^\varepsilon + P_N f_\varepsilon(t, u^\varepsilon_N) & \text{in } \Omega, \quad t > \tau, \\
w_N^\varepsilon|_{t=\tau} = 0, \quad w_{N,t}|_{t=\tau} = 0, & \text{in } \Omega, \\
w_N^\varepsilon|_{\partial\Omega} = 0, \quad t \geq \tau.
\end{cases}
$$

(3.9)
Due to the Corollary 2.4 in [20] and Theorem 3.4, for every \( \eta > 0 \) there exists a \( T = T(\eta) > 0 \) such that

\[
\varepsilon \| v^\varepsilon_{\eta,t}(t) \|_{L^2(\Omega)} + \| v^\varepsilon_{\eta}(t) \|_{X^1} \leq C, \quad \| v^\varepsilon_{\eta}(t) \|_{L^p(\tau,t;L^{12}(\Omega))} \leq \eta,
\]

where \( C \) is independent of \( N \) and \( \varepsilon \). Then apply the growth condition (1.3) and interpolation inequality, follow the same procedure in [20] we end up with

\[
Y^\varepsilon_N(t) + \| w^\varepsilon_N(t) \|_{L^p(\tau,t;L^{12}(\Omega))} \leq C(t^\tau + \eta^4) + CY^\varepsilon_N(t)^5, \quad t - \tau \leq T(\eta), \quad Y^\varepsilon_N(\tau) = 0,
\]

where \( (\eta > 0) \) is independent of \( t, \tau, \varepsilon, N \) and

\[
Y^\varepsilon_N(t) \triangleq \varepsilon \| w^\varepsilon_N(t) \|_{L^2(\Omega)} + \| w^\varepsilon_N(t) \|_{X^1} + \| w^\varepsilon_N(t) \|_{L^p(\tau,t;L^{10}(\Omega))}.
\]

Thus for every \( \eta \in \mathbb{R} \) by choosing \( \eta \) and \( T(\eta) \) small enough to satisfy \( C(2C(t^\tau + \eta^4)) \leq C(t^\tau + \eta^4) \quad \text{for} \quad t - \tau \leq T(\eta) \), we get the uniform estimate

\[
\varepsilon \| w^\varepsilon_N(t) \|_{L^2(\Omega)} + \| w^\varepsilon_N(t) \|_{X^1} + \| w^\varepsilon_N(t) \|_{L^p(\tau,t;L^{10}(\Omega))} \leq C_1,
\]

where \( C_1 \) is independent of \( t, \tau, \varepsilon, N \). Then together with (3.10) we can conclude that there exists a constant \( R_1 \) independent of \( t, \tau, \varepsilon, N \) for every \( \tau \in \mathbb{R} \) and \( t - \tau \leq T(\eta) \) the following inequality hold:

\[
\varepsilon \| u^\varepsilon_N(t) \|_{L^2(\Omega)} + \| u^\varepsilon_N(t) \|_{X^1} + \| u^\varepsilon_N(t) \|_{L^p(\tau,t;L^{12}(\Omega))} \leq R_1.
\]

Passing the limit \( N \rightarrow \infty \) in a standard way, the assertion follows.

Therefore, the solutions of (1.2) is uniformly decaying in the energy and Strichartz norms. Thus we can define the processes generated of problem (1.2) as

\[
S^\varepsilon_z(t,\tau)(u_0, u_1) \triangleq (u^\varepsilon(t), u^\varepsilon(\tau)),
\]

where \( (u^\varepsilon(t), u^\varepsilon(\tau)) \) is the Shatah Struwe solution in the Theorem 3.3.

3.3. Regularity estimates. In this section, we obtain the uniform estimates (w.r.t. \( \varepsilon \)) on the uniform attractors in fractional spaces in needed as an intermediate step for further regularity results and estimates. Following [31], for some sufficiently large real number \( L > 0 \) we split the process \( \{S^\varepsilon_z(t,\tau)\}_{t \geq \tau} \) into the sum of two components:

\[
\begin{align*}
\begin{cases}
\varepsilon v^\varepsilon_t + v^\varepsilon + Lv^\varepsilon(t) = \Delta v^\varepsilon + f_0(\varepsilon) & \text{in } \Omega, \quad t > \tau, \\
v^\varepsilon|_{t=\tau} = u_0, \quad v^\varepsilon_t|_{t=\tau} = u_1, & \text{in } \Omega, \\
v^\varepsilon|_{\partial\Omega} = 0, & \text{in } \Omega,
\end{cases}
\quad \text{(3.11)}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\varepsilon w^\varepsilon_t + w^\varepsilon = Lw^\varepsilon(t) + \Delta w^\varepsilon + f_\varepsilon(t, u^\varepsilon) - f_0(\varepsilon) & \text{in } \Omega, \quad t > \tau, \\
w^\varepsilon|_{t=\tau} = 0, \quad w^\varepsilon_t|_{t=\tau} = 0, & \text{in } \Omega, \\
w^\varepsilon|_{\partial\Omega} = 0, & \text{in } \Omega,
\end{cases}
\quad \text{(3.12)}
\end{align*}
\]

We study the problem (3.11) first, the component that is exponentially decaying in the energy and Strichartz norms.

**Proposition 3.6.** Let assumptions (1.3), (1.4) and (3.1) hold, \( B_{R_0} \) be the set in Theorem 3.4 and \( (u^\varepsilon(t), u^\varepsilon(\tau)) \) \( \in B_{R_0} \) for all \( t \geq \tau \). Then there exists a positive number \( L \) such that the Shatah-Struwe solution \( v^\varepsilon \) of (3.11) satisfies

\[
\varepsilon \| v^\varepsilon(t) \|_{L^2(\Omega)} + \| v^\varepsilon(t) \|_{X^1}^2 + \| v^\varepsilon(t) \|_{L^p(t,t+1;L^{12}(\Omega))} \leq \rho e^{-\alpha(t-\tau)} \quad \text{for every } t \geq \tau,
\]

for some positive constants \( \rho, \alpha > 0 \) independent of \( t, \tau, \varepsilon \).
Proof. Note that the dissipation assumption (1.4) and $f_0(0)$ imply that for every $v \in \mathbb{R}$
\[ f_0(v)v \leq k|v|^2 \]  
(3.14)
\[ F_0(v) = \int_0^v f_0(s)ds \leq \frac{k}{2}|v|^2, \]  
(3.15)
for some positive constant $k$.

Multiplying equation (3.11) by $v_\varepsilon(t) + \delta v^\varepsilon(t)$, integrating over $\Omega$ and arguing as in Theorem 3.4, we obtain that for $L > K$,
\[ \frac{d}{dt} E_1(v_\varepsilon(t), v^\varepsilon(t)) + \alpha E_1(v_\varepsilon(t), v^\varepsilon(t)) \leq 0, \]
where $\alpha$ is a positive constant and
\[ E_1(v_\varepsilon(t), v^\varepsilon(t)) = \varepsilon\|v_\varepsilon(t)\|_{L^2(\Omega)}^2 + \|v^\varepsilon(t)\|_{X^1}^2 + \left(\frac{1}{4} + L\right)\|v^\varepsilon(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\varepsilon\beta\langle v^\varepsilon(t), v_\varepsilon(t) \rangle \]
\[ - 2 \int_\Omega F_0(v^\varepsilon)dx. \]

By the Gronwall lemma, we have
\[ E_1(v_\varepsilon(t), v^\varepsilon(t)) \leq E_1(u_0, u_1)e^{-\alpha(t-\tau)}. \]

Moreover, take $0 < \beta < \min\{\frac{1}{4}, \frac{\alpha}{4}\}$ we have
\[ \frac{1}{4} \left(\|v^\varepsilon(t)\|_{X^1}^2 + \varepsilon\|v_\varepsilon(t)\|_{L^2(\Omega)}^2\right) \leq \frac{1}{2}\|v^\varepsilon(t)\|_{X^1}^2 + 2\beta\varepsilon\langle v^\varepsilon(t), v_\varepsilon(t) \rangle + \frac{1}{2}\varepsilon\|v_\varepsilon(t)\|_{L^2(\Omega)}^2 \]
\[ \leq \frac{3}{4}\left(\|v^\varepsilon(t)\|_{X^1}^2 + \varepsilon\|v_\varepsilon(t)\|_{L^2(\Omega)}^2\right), \]
and
\[ \left| \int_\Omega F_0(v^\varepsilon)dx \right| \leq c\left(\|v^\varepsilon(t)\|_{L^2(\Omega)}^{6-\varepsilon} + 1\right) \leq c\|v^\varepsilon(t)\|_{X^1}^2, \]
where $c = c(B_{R_0}, \Omega)$ independent of $\varepsilon$.

With the estimates above we can conclude that $E_1(v_\varepsilon(t), v^\varepsilon(t))$ and $\varepsilon\|v_\varepsilon(t)\|_{L^2(\Omega)}^2 + \|v^\varepsilon(t)\|_{X^1}$ are equivalent norms. Furthermore, the dissipative estimate for $\|v^\varepsilon(t)\|_{L^1(t, t+1; L^2(\Omega))}$ is obtained in Corollary 4.2 in [20]. Thus the assertion follows.

We now study the equation (3.12), the component that can be obtained in the more regular phase space.

**Proposition 3.7.** Let assumptions (1.3), (1.4), (3.1) and (3.2) hold. Then for every $(u_0, u_1) \in B_{R_0}$ where $B_{R_0}$ is in Theorem 3.4, there exist positive constants $c$ and $\delta$ independent of $\varepsilon$ such that for every $\tau \in \mathbb{R}$ the following estimate is valid:
\[ \varepsilon\|u_\varepsilon(t)\|_{X^1}^2 + \|w^\varepsilon(t)\|_{X^{1+1}}^2 + \|w^\varepsilon(t)\|_{L^1(t, t+1; W^{1, 12}(\Omega))} \leq ce^{-\delta(t-\tau)}, \]  
for every $t \geq \tau$.

**Proof.** Let $u^\varepsilon(t)$ be the Shatah-Struwe solutions of (1.2), then according to the previous conclusion we have
\[ \varepsilon\|u_\varepsilon(t)\|_{L^2(\Omega)}^2 + \|w^\varepsilon(t)\|_{X^{1+1}}^2 + \|w^\varepsilon(t)\|_{L^1(t, t+1; L^2(\Omega))} \leq R, \]
for every $t \geq \tau$, where $R$ is independent of $\varepsilon, t, \tau$.

In order to get the estimates for smooth component, we use the fact that
\[ |\frac{\partial}{\partial u} f_\varepsilon(t, u)| \leq c(1 + |u|^{4-\kappa}) \]
where 0 < \kappa < 1.

By the interpolation inequality
\[ \|u\|_{L^\theta_p(s,t;\mathbf{L}^{2,p}(\Omega))} \leq c\|u\|_{L^\theta_p(s,t;\mathbf{L}^{2,p}(\Omega))} \cdot \|u\|_{L^\infty(s,t;X^1)}^{1-\theta}, \] for all \( t > s. \)

Let \( \theta = \frac{4}{5}, \) then we obtain
\[
\|f_\varepsilon(t, u^\varepsilon)\|_{L^4(t, t+1; L^2(\Omega))} \leq c \left( 1 + \int_t^{t+1} \|u^\varepsilon(t)\|^5_{L^1(\Omega)} \, ds \right)
\leq c \left( 1 + \|u^\varepsilon(t)\|_{L^4(t, t+1; L^2(\Omega))} \cdot \|u^\varepsilon(t)\|_{L^\infty(t, t+1; X^1)} \right)
\leq c \left( 1 + \|u^\varepsilon(t)\|_{L^4(t, t+1; L^2(\Omega))} \right),
\]
where the constant \( c \) is independent of \( \varepsilon \) and may vary from line to line.

On the other hand, we get
\[
\|f_\varepsilon(t, u^\varepsilon)\|_{L^p(\Omega)} = \| \frac{\partial f_\varepsilon(t, u^\varepsilon(t))}{\partial u} \|_{L^p(\Omega)}
\leq c \left( 1 + \|u^\varepsilon(t)\|^{3-\alpha}_{L^p(\Omega)} \right) \|\nabla u^\varepsilon(t)\|_{L^p(\Omega)}
\leq c \left( 1 + \|u^\varepsilon(t)\|^{3-\alpha}_{L^p(\Omega)} \right) \|\nabla u^\varepsilon(t)\|_{L^2(\Omega)},
\]
where \( \frac{1}{p} = \frac{1}{2} + \frac{3-\alpha}{12} \) and the constant \( c \) independent of \( \varepsilon \) and may vary from line to line.

Thus for every \( t \geq \tau \) we have \( \|f_\varepsilon(t, u^\varepsilon)\|_{L^4(t, t+1; W^{1,p}(\Omega))} \leq K_0, \) where \( K_0 \) is a constant independent of \( \varepsilon, t, \tau. \)

By using the embedding \( W^{1,p} \hookrightarrow X^\delta \) with \( \frac{1}{2} = \frac{1}{\kappa} - \frac{1-\delta}{3} \) and \( \delta = \frac{10}{3\kappa}, \) we can arrive at the estimate
\[
\|f_\varepsilon(t, u^\varepsilon)\|_{L^4(t, t+1; X^\delta)} \leq K_1, \quad \text{for every } t \geq \tau, \tag{3.16}
\]
where \( K_1 \) is a constant independent of \( \varepsilon, t, \tau. \) By the similar procedure, we can conclude that
\[
\|f_0(u^\varepsilon)\|_{L^4(t, t+1; X^\delta)} \leq K_2, \quad \text{for every } t \geq \tau, \tag{3.17}
\]
where \( K_2 \) is a constant independent of \( \varepsilon, t, \tau. \)

Multiplying (3.12) by \( A^\delta(w^\varepsilon + \beta w^\varepsilon) \) where \( \beta > 0 \) will be chosen appropriately later, we have
\[
\frac{d}{dt} \left( \frac{1}{2} \varepsilon \|w^\varepsilon(t)\|_{X^\delta}^2 + \frac{1}{2} \|w^\varepsilon(t)\|_{X^{\delta+1}}^2 + \varepsilon \beta \|w^\varepsilon(t)\|_{X^\delta}^2 \right)
+ (1 - \varepsilon \beta) \|w^\varepsilon(t)\|_{X^\delta}^2 + \beta \|w^\varepsilon(t)\|_{X^{\delta+1}}^2
= \langle Lw^\varepsilon(t), A^\delta(w^\varepsilon + \beta w^\varepsilon) \rangle + \langle f_\varepsilon(t, u^\varepsilon), A^\delta(w^\varepsilon + \beta w^\varepsilon) \rangle
- \langle f_0(v^\varepsilon(t)), A^\delta(w^\varepsilon + \beta w^\varepsilon) \rangle
\triangleq I_1 + I_2 + I_3.
\]

By using Poincare inequality and Cauchy inequality, we will estimate the terms one by one. Since \( 2\delta < 1, \) we have
\[
I_1 = \langle Lw^\varepsilon(t), A^\delta(w^\varepsilon(t) + \beta w^\varepsilon(t)) \rangle
= \langle LA^\delta v^\varepsilon, w^\varepsilon(t) + \beta w^\varepsilon(t) \rangle
\leq C \|A^\delta v^\varepsilon(t)\|_{L^2(\Omega)} \|w^\varepsilon(t) + \beta w^\varepsilon(t)\|_{L^2(\Omega)}
\leq C,
\]
where the constant $C > 0$ varies from line to line and is independent of $\tau, t, \varepsilon$.

\[ I_2 = \langle f_\varepsilon(t, u^\varepsilon), A^\delta (w^\varepsilon + \beta w^\varepsilon) \rangle \]
\[ \leq \| f_\varepsilon(t, u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^s} + C_\delta \| f_\varepsilon(t, u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^{s+1}}^{1} \]
\[ \leq C_{\beta_{1}, \beta_{2}} \| f_\varepsilon(t, u^\varepsilon) \|_{X^s}^{2} + \beta_1 \| w^\varepsilon(t) \|_{X^s}^{2} + C_\delta \| f_\varepsilon(t, u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^{s+1}}^{1}, \]

where the constants $\beta_{1}, \beta_{2}, C_\delta, C_{\beta_{1}, \beta_{2}} > 0$ independent of $\tau, t, \varepsilon$.

\[ I_3 = \langle f_0(u^\varepsilon(t)), A^\delta (w^\varepsilon + \beta w^\varepsilon) \rangle \]
\[ \leq \| f_0(u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^s} + C_\delta \| f_0(u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^{s+1}}^{1} \]
\[ \leq C_{\beta_{3}, \beta_{4}} \| f_0(u^\varepsilon) \|_{X^s}^{2} + \beta_3 \| w^\varepsilon(t) \|_{X^s}^{2} + C_\delta \| f_0(u^\varepsilon) \|_{X^s} \| w^\varepsilon(t) \|_{X^{s+1}}^{1}, \]

where the constants $\beta_{1}, \beta_{2}, C_\delta, C_{\beta_{1}, \beta_{2}} > 0$ independent of $\tau, t, \varepsilon$.

Therefore, by taking $\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}$ small enough to satisfy $\beta_1 + \beta_2 < 1 - \varepsilon \beta$ and $\beta_3 + \beta_4 < \frac{1}{\varepsilon}$, we have

\[ \frac{d}{dt} E_2(w^\varepsilon(t), w_i^\varepsilon(t)) + \alpha E_2(w^\varepsilon(t), w_i^\varepsilon(t)) \leq K_3, \]

where $\alpha > 0$, $K_3 > 0$ independent of $\tau, t, \varepsilon$ and

\[ E_2(w^\varepsilon(t), w_i^\varepsilon(t)) \]
\[ = \frac{1}{2} \| w_i^\varepsilon(t) \|_{X^s}^{2} + \frac{1}{2} \| w^\varepsilon(t) \|_{X^{s+1}}^{2} + \frac{1}{2} \beta \| w^\varepsilon(t) \|_{X^s}^{2} + \varepsilon \beta \| w_i^\varepsilon(t), A^\delta w^\varepsilon(t) \|. \]

Apply the Gronwall inequality, then we deduce that

\[ E_2(w^\varepsilon(t), w_i^\varepsilon(t)) \leq K_3 \left( 1 - e^{-\alpha(t-\tau)} \right) \leq K_3 \quad \text{for every} \ t \geq \tau. \]

For the extra regularity, let $z = A^\frac{1}{2} u^\varepsilon$, then we have:

\[ \varepsilon z_{tt} + z_t - \Delta z = L(-\Delta)^\frac{1}{2} v^\varepsilon + (-\Delta)^{\frac{1}{2}} f_\varepsilon(t, u^\varepsilon) - (-\Delta)^{\frac{1}{2}} f_0(u^\varepsilon) = F. \]

According to the Strichartz estimate in [5] we only have to verify that $F \in L^1(t, t + 1; L^\frac{8}{\delta}_2(\Omega))$. Since $A^\frac{1}{2} v(t)$ is bounded in $X^{1-\delta}$, we have

\[ \| A^\frac{1}{2} v(t) \|_{L^\frac{8}{\delta}_2(\Omega)} \leq \| A^\frac{1}{2} v(t) \|_{L^{\frac{6}{3\delta}}(\Omega)} \leq \| A^\frac{1}{2} v(t) \|_{X^{1-\delta}}, \]

where embedding index $q$ can be defined as $\frac{1}{q} = \frac{1}{2} - \frac{1-\delta}{3} = \frac{1+2\delta}{6}$. And also notice that $A^\frac{1}{2} f_\varepsilon(t, u^\varepsilon)$ is bounded in $L^1(t, t + 1; L^2(\Omega))$, we have

\[ \| A^\frac{1}{2} f_\varepsilon(t, u^\varepsilon) \|_{L^\frac{8}{\delta}_2(\Omega)} \leq \| A^\frac{1}{2} f_\varepsilon(t, u^\varepsilon) \|_{L^2(\Omega)}. \]

The estimate for $A^\frac{1}{2} f_0(u^\varepsilon)$ is the same as $A^\frac{1}{2} f_\varepsilon(t, u^\varepsilon)$. Therefore the extra regularity follows.

Moreover, as discussed before, $E_2(w^\varepsilon(t), w_i^\varepsilon(t))$ and $\varepsilon \| w_i^\varepsilon(t) \|_{X^s}^{2} + \| w^\varepsilon(t) \|_{X^{s+1}}^{2} + \| w^\varepsilon(t) \|_{X^s}^{2}$ are equivalent norms once $\beta < \frac{1}{2\varepsilon}$. Thus, the assertion follows.

**Proposition 3.8 (Regularity of the absorbing set).** Let assumptions (1.3), (1.4), (3.1) and (3.2) hold. For any $(u_0, u_1) \in B_{R_0}$ where $B_{R_0}$ be as in Theorem 3.4, there exists a positive constant $C$ which is independent $t, \tau, \varepsilon$, such that for every $\tau \in \mathbb{R}$ the following estimate is valid:

\[ \varepsilon \| u_i^\varepsilon(t) \|_{X^1}^2 + \| u_i^\varepsilon(t) \|_{X^1}^2 + \| u^\varepsilon(t) \|_{H^2(\Omega)}^2 + \| u^\varepsilon(t) \|_{L^4(t, t + 1; W^{1, 12})} \leq C, \quad \text{for every} \ t \geq \tau. \]
Proof. Following the bootstraping method from Babin and Vishik in [3], take \((u_0, u_1) \in E^\delta\) satisfies
\[
\varepsilon \|u_1\|_{X^\delta} + \|u_0\|_{X^{1+\delta}} + \|u_0\|_{L^4(t, t+1; W^{4,12})} \leq R
\]
where \(R\) is the lower bound for \(w^\varepsilon(t)\) obtained in Proposition 3.7.

Apply the operator \(A^\delta\) for the equation (3.11) we get
\[
\|u(t)\|_{X^\delta} + \|\nabla u(t)\|_{X^\delta} \leq Ce^{-\mu(t-\tau)}, \quad \text{for every } t \geq \tau,
\]
for some constants \(C > 0\) and \(\mu > 0\) independent of \(t, \tau, \varepsilon\).

Together with Proposition 3.7, the dynamical processes \(\{S_\varepsilon(t, \tau)\}_{t \geq \tau}\) generated by (1.2) is well-defined and dissipative in the higher energy \(E^\delta\) as well.

In particular, we have \(\nabla u \in X^\delta\) then by the Sobolev’s embedding we can obtain \(\|\nabla u\|_{X^\delta} \leq \|\nabla u\|_{X^{\delta}} \leq \frac{1}{2} \geq \frac{1}{2} - \frac{\delta}{4}\). Arguing through the same way in Proposition 3.7, we improve the estimate (3.16) as
\[
\|f_\varepsilon(t, u^\varepsilon)\|_{L^1(t, t+1; X^{\delta+1})} \leq K_4,
\]
where the constant \(K_4\) is independent of \(t, \tau, \varepsilon\) and \(\delta_1 = \delta + \frac{n}{4} > \delta\).

Finally, iterating the procedure in Proposition 3.7 we can conclude that \(\{S_\varepsilon(t, \tau)\}_{t \geq \tau}\) is well-defined and dissipative in the higher space \(E^{\delta_n}\) with \(\delta_n = \frac{1}{4}n\) in \(n\) steps and \(\delta_n\) can reach the value 1 in finitely steps. Thus the Proposition is proved.

**Remark 3.9.** It is easy to check the conclusions above hold if we replace \(f_\varepsilon\) by \(p_\varepsilon \in \Sigma_\varepsilon\). Then due to the infinite dynamical system theory, we can deduce from Theorem 3.4, Proposition 3.6 and Proposition 3.7 that there exists uniform attractor \(\mathcal{A}_\varepsilon \in \mathcal{E}\) for problem (1.2) also the cocycle attractors \(\{A_\varepsilon(p)\}_{p_\varepsilon \in \Sigma_\varepsilon}\) and pullback attractors \(\{A_\varepsilon(t)\}_{t \in \mathbb{R}}\).

**Theorem 3.10 (Regularity of the uniform attractor).** Let assumptions (1.3), (1.4), (3.1) and (3.2) hold. Then the dynamical processes \(\{S_\varepsilon(t, \tau)\}_{t \geq \tau}\) generated by (1.2) admit a uniform attractor \(\mathcal{A}_\varepsilon\) in \(X^2 \times X^1\) and there exists a positive constant \(K_5\) independent of \(t, \tau, \varepsilon\) such that for every \((\varphi, \psi) \in \mathcal{A}_\varepsilon\),
\[
\|\varphi\|_{X^1}^2 + \|\psi\|_{X^2}^2 \leq K_5, \quad \text{for every } t \geq \tau.
\]

**Proof.** In Proposition 3.8, we showed that the dynamic process \(\{S_\varepsilon(t, \tau)\}_{t \geq \tau}\) generated by (1.2) is dissipative in \(X^2 \times X^1\) and that its asymptotic compactness can be deduced in \(X^2 \times X^1\) by the decomposition method. According to the theory of infinite dynamical system ([27, 19]), the dynamical processes \(\{S_\varepsilon(t, \tau)\}_{t \geq \tau}\) admits a compact uniform attractor \(\mathcal{A}_\varepsilon\) in \(X^2 \times X^1\).

**Remark 3.11.** We can show that the cocycle attactors \(\{A_\varepsilon(p)\}_{p_\varepsilon \in \Sigma_\varepsilon}\) and pullback attractors \(\{A_\varepsilon(t)\}_{t \in \mathbb{R}}\) can be embedded in more regular space \(E^1 = X^2 \times X^1\). The proof is the same as we did to the uniform attractor \(\mathcal{A}_\varepsilon\) in Theorem 3.10.

To conclude this section, we end up with a uniform estimate on \(\varepsilon \|u_{tt}(t)\|_{L^2(\Omega)}^2\) where \((u^\varepsilon(t), u_1^\varepsilon)\) is a complete trajectory in \(\mathcal{A}_\varepsilon\).

**Proposition 3.12.** Let assumptions (1.3), (1.4) and (3.1)-(3.3) hold. Then for a complete trajectory \((u^\varepsilon(t), u_1^\varepsilon)\) in \(\mathcal{A}_\varepsilon\), for every \(\tau \in \mathbb{R}\)
\[
\varepsilon \|u_{tt}(t)\|_{L^2(\Omega)}^2 \leq K_6, \quad \text{for every } t \geq \tau,
\]
where \(K_6\) is independent of \(t, \tau, \varepsilon\).
Proof. From Proposition 3.7, the dynamical processes
\[ S_{\varepsilon}(t, \tau)(u_0, u_1) = (u_{\varepsilon}(t), u_{\varepsilon}^*(t)) \in \mathcal{C}([\tau, \infty); X^{2} \times X^{1}) \]
for every \((u_0, u_1) \in \mathcal{A}_{\varepsilon}.

We may consider the following hyperbolic equation with \(w(t) = u_{\varepsilon}^*(t)\)
\[
\begin{cases}
\varepsilon w_{tt} + w_t - \Delta w = \partial f_{\varepsilon}(t, u^\varepsilon) + \partial f_{\varepsilon}(t, u^\varepsilon) & \text{in } \Omega, \ t > \tau, \\
|w|_{\tau=\tau} = 0, \ w|_{\tau=\tau} = \frac{1}{2} (f_{\varepsilon}(\tau, u_{\varepsilon}^*(u_0)) - u_1 + \Delta u_0), & \text{in } \Omega, \\
w|_{\partial \Omega} = 0, \ t \geq \tau.
\end{cases}
\]
(3.18)

Taking into account that \((u_{\varepsilon}^*(t), u_{\varepsilon}^*(t)) \in \mathcal{A}_{\varepsilon},\) using the growth condition (1.3) and
the embedding \(H^2(\Omega) \hookrightarrow C(\bar{\Omega})\) we can deduce that:
\[
\frac{\partial f_{\varepsilon}}{\partial t}(t, u^\varepsilon), \ \frac{\partial f_{\varepsilon}}{\partial u^\varepsilon}(t, u^\varepsilon) \in C^0([\tau, \infty); L^2(\Omega)).
\]
Then there is a unique Shatah-Struwe solution \(w(t)\) of (3.18) and \((w(t), w_{t}(t)) \in C^0([\tau, \infty); X^{1} \times L^2(\Omega)).\)

Multiplying (3.18) by \(w_{t}(t) + \beta w(t)\) where \(\beta > 0\) will be chosen appropriately
smaller, we can easily get
\[
\frac{d}{dt} E_3(w^\varepsilon(t), w_{\varepsilon}^*(t)) + \alpha' E_3(w^\varepsilon(t), w_{\varepsilon}^*(t)) \leq K_{6},
\]
(3.19)
where \(\alpha' > 0, \ K_{6} \) independent of \(t, \tau, \varepsilon,\) and
\[
E_3(w^\varepsilon(t), w_{\varepsilon}^*(t)) = \frac{1}{2} \varepsilon ||w_{t}(t)||_{L^2(\Omega)}^2 + \frac{1}{2} \parallel w(t) \parallel_{X^{\frac{1}{2}}}^2 + \varepsilon \beta (w(t), w_{t}(t)) + \parallel w(t) \parallel_{L^2(\Omega)}^2.
\]
By Gronwall inequality, (3.19) gives us
\[
E_3(w^\varepsilon(t), w_{\varepsilon}^*(t)) \leq K_{6} e^{-\alpha'(t-\tau)} \leq K_{6}, \ \text{for every } t \geq \tau.
\]
Therefore this Proposition is proved. \(\square\)

Remark 3.13. It is easy to check the Proposition 3.12 hold if we replace \(f_{\varepsilon}\)
by \(p_{\varepsilon} \in \Sigma_{\varepsilon}.

4. Limiting asymptotic behavior. In this section, we proved our main theorem
in this paper, that is the upper semicontinuity of the pullback, cocycle and uniform
attractors generated by the Shatah-Struwe solutions of (1.2). For the parabolic
equation (1.1), the global existence of weak solution is established in [12]. Therefore
we can define the semigroup \(T(t) : H^1_0(\Omega) \to H^1_0(\Omega)\) generated by problem (1.1),
\(T(t)u_0 = u(t)\) where \(u(t)\) is the weak solution of (1.1). Under the above hypotheses,
there is a global attractor \(A_0\) in \(H^1_0(\Omega)\) for \(\{T(t)\}_{t \geq 0}\) (see in [27]). Due to the
regularization effect of the parabolic equation, we can deduce that \(A_0 \subset H^{2}(\Omega) \cap \)
\(H^1_0(\Omega).\) Therefore, define
\[
\mathcal{A}_0 = \{(u(t), u_t(t)) : u(t) \in A_0, u_t(t) = \Delta u + f_0(u)\}.
\]
The set \(\mathcal{A}_0\) is a natural embedding of the attractor \(A_0 \subset H^1_0(\Omega)\) into \(\mathcal{E}.

Theorem 4.1 (Upper semicontinuity of the uniform attractor). Let ass-
sumptions (1.3), (1.4) and (3.1)-(3.3) hold, we have that the uniform attractors \(\mathcal{A}_{\varepsilon}\)
converge upper semicontinuously to the attractor \(\mathcal{A}_0\) when \(\varepsilon \to 0,\) that is
\[
\lim_{\varepsilon \to 0} \text{dist}(\mathcal{A}_{\varepsilon}, \mathcal{A}_0) = 0.
\]
Proof. Let \((u_{\epsilon}^n(t), u_{\epsilon}^{n,\epsilon}(t)) \in \mathcal{M}_\epsilon\) be Shatah-Struwe solutions of (1.2) with \(\epsilon = \epsilon_n\) and \(\epsilon \to 0\). Then by Theorem 3.10,

\[
\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} u_{\epsilon}^{n}(t) \text{ is a procompact set in } X^1,
\]

and the family of mappings \(u_{\epsilon}^{n}(t) \in C^0(\mathbb{R}; X^1), n \geq 0\) is equicontinuous from \(\mathbb{R}\) into \(X^1\). Let \(J_m, m \geq 0\) be a sequence of compact intervals of \(\mathbb{R}\) such that

\[
J_m \subset J_{m+1}, \quad m \geq 0 \quad \text{and} \quad \bigcup_{m \in \mathbb{Z}} J_m = \mathbb{R}.
\]

According to the Arzelà-Ascoli theorem that there exists a subsequence \(u_{\epsilon,m+1}^n(t)\) of \(u_{\epsilon}^{n}(t)\) such that \(u_{\epsilon,m+1}^n(t)\) converges to \(\bar{u}(t)\) in \(C^0(J_m, X^1)\) and \(u_{\epsilon}^{n}(t)\) is also a subsequence of \(u_{\epsilon,m}^n\), \(m \geq 0\). Then by taking a diagonal subsequences in the usual way, there exists a subsequence of positive numbers \(\epsilon_j\) of \(\epsilon_n\) and corresponding subsequence \(u_{\epsilon,j}^n(t)\) of \(u_{\epsilon}^{n}(t)\), which are Shatah-Struwe solutions of (1.2) with \(\epsilon = \epsilon_j\), such that

\[
u_{\epsilon,j}^n(t) \to \bar{u}(t) \quad \text{in } C^0(J; X^1) \quad \text{for every compact interval } J \subset \mathbb{R}, \quad (4.1)
\]

and \(\bar{u}(t) \in C^0(J; X^1)\).

On the other hand, \(u_{\epsilon,j}^n(t)\) satisfies:

\[
u_{\epsilon,j}^n(t) = -\epsilon_j \nu_{\epsilon,j}^{n,tt}(t) + \Delta \nu_{\epsilon,j}^n(t) + p_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n).
\]

We will pass to the limit with \(n \to \infty\) in the right hand side of (4.2) one by one.

For any \(\varphi \in C^\infty_{\mathbb{C}}\), first by the Proposition 3.12 we have

\[
\lim_{\epsilon_j \to 0} \sup_{t \in \mathbb{R}} |\langle \epsilon_j \nu_{\epsilon,j}^{n,tt}(t), \varphi \rangle| \\
\leq \lim_{\epsilon_j \to 0} \sup_{t \in \mathbb{R}} \epsilon_j \| \nu_{\epsilon,j}^{n,tt}(t) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
= 0.
\]

Then from the (4.1) we have,
\[
\langle -\Delta \nu_{\epsilon,j}^n(t), \varphi \rangle - \langle -\Delta \bar{u}(t), \varphi \rangle \\
= \langle \nabla \nu_{\epsilon,j}^n(t) - \nabla \bar{u}(t), \varphi \rangle \\
\leq \| \nu_{\epsilon,j}^n(t) - \bar{u}(t) \|_{X^1} \| \varphi \|_{X^1} 
\to 0 \quad \text{as } j \to \infty.
\]

Finally, due to the property (1.3),(3.1) and a uniform pointwise bound on \(u_{\epsilon,j}^n(t)\) obtained from Theorem 3.10, we have,

\[
\sup_{t \in J} \langle p_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_0(\bar{u}), \varphi \rangle \\
\leq \sup_{t \in J} \| p_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_0(\bar{u}) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
\leq \sup_{t \in J} \| p_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
+ \sup_{t \in J} \| f_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_0(\bar{u}) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
+ \sup_{t \in J} \| f_0(\bar{u}_{\epsilon,j}^n) - f_0(\bar{u}) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
\leq \sup_{t \in J} \| p_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
+ \sup_{t \in J} \| f_{\epsilon,j}(t, \bar{u}_{\epsilon,j}^n) - f_0(\bar{u}) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)} \\
+ \sup_{t \in J} \| f_0(\bar{u}_{\epsilon,j}^n) - f_0(\bar{u}) \|_{L^2(\Omega)} \| \varphi \|_{L^2(\Omega)}
\]
we can conclude that the sense of distributions for every bounded interval I in C

Due to Theorem 3.10 and the property (4.1) we obtain

Thus we have proved the convergence of the Shatah-Struwe solution.

where u^{\varepsilon}_{j,n}(t) we proved that

We now turn to verify that \bar{u}_t = -\Delta \bar{u} + f_0(\bar{u}).

Also \tilde{z}(t) = u^{\varepsilon}_{j,n}(t) - \bar{u}(t) satisfies

\begin{align*}
&\varepsilon_n z_{tt} + z_t - \Delta z = p_n(t, u^{\varepsilon}_{j,n}) - f_0(\bar{u}) - \varepsilon \bar{u}_{tt} := G_{\varepsilon_{j,n}}(t) \quad \text{in } \Omega, \ t > \tau, \\
&\tilde{z}|_{t=\tau} = 0, \ \tilde{z}|_{t=\tau} = 0, \quad \text{in } \Omega, \\
&\tilde{z}|_{\partial \Omega} = 0, \ t \geq \tau.
\end{align*}

We can obtain the estimate in the following from the Strichartz type estimates for wave equations in bounded domains (see [5]):

\[ ||z(t)||_{L^{4}(\tau, t; L^{12}(\Omega))} \leq C ||G_{\varepsilon_{j,n}}(t)||_{L^{1}(\tau, t; L^{2}(\Omega))}, \ \text{for every } t \geq \tau, \quad (4.3) \]

where C is independent of \varepsilon. By the similar argument as before we can deduce that

\[ ||z(t)||_{L^{4}(\tau, t; L^{12}(\Omega))} \to 0 \quad \text{as } n_j \to \infty. \]

Thus we have proved the convergence of the Shatah-Struwe solution.

We now turn to verify that \bar{u}(t) is a complete bounded trajectory of the parabolic equation (1.1). Due to Theorem 3.10 and the property (4.1) we obtain

\[ \sup_{t \in \mathbb{R}} ||\bar{u}(t)||_{X^1} \leq R, \]

where the constant R > 0 is a positive independent of t, \tau, \varepsilon, that is, \bar{u}(t) \in C^0_0(\mathbb{R}; X^1). Moreover, with the estimate for ||f_0(\bar{u})|| deduced in Proposition 3.8, we can conclude that

\[ \bar{u}_t(t) \in C^0_0(\mathbb{R}; X^{-1}). \]

Furthermore, the convergence of \tilde{u}^{\varepsilon}_{j,n}(t) to \bar{u}_t(t) in C^0(J; X^{-1}) for every compact interval J \subset \mathbb{R} implies the convergence can take place in C^0(J; L^2(\Omega)) since the bound for ||\tilde{u}^{\varepsilon}_{j,n}(t)||_{X^1} is independent of t, \tau, \varepsilon proved the last section. And also we can deduce that \bar{u}_t(t) \in C^0(\mathbb{R}; L^2(\Omega)).

Due to the regularization effect of the operator \(-\Delta\) and the equality

\[ -\Delta \bar{u} = -\bar{u}_t - f_0(\bar{u}), \]

we obtain \bar{u}(t) \in L^{\infty}(\mathbb{R}; H^2(\Omega)). In conclusion, for the solution \bar{u}(t) of (1.1) with u(\tau) = \bar{u}(\tau) we proved that

\[ \bar{u}(t) \in L^{\infty}(\mathbb{R}; H^2(\Omega)), \ \bar{u}_t(t) \in L^{\infty}(\mathbb{R}; L^2(\Omega)). \]
Therefore, we have \((\bar{u}(t), \bar{u}_t(t)) \in \mathcal{A}_0\).

According to the relationship between the uniform attractor, the cocycle attractor and the pullback attractor showed in [6], the following result follows.

**Theorem 4.2 (Upper semicontinuity of the pullback and cocycle attractors).** The families of cocycle and pullback attractors \(\{A_\varepsilon(p)\}_{p \in \Sigma_\varepsilon}\) and \(\{A_\varepsilon(t)\}_{t \in \mathbb{R}}\) for the process defined by equation (1.2) are upper semicontinuous at \(\varepsilon = 0\), and we have

\[
\lim_{\varepsilon \to 0} \sup_{p \in \Sigma_\varepsilon} \text{dist}(A_\varepsilon(p), \mathcal{A}_0) = 0,
\]

\[
\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} \text{dist}(A_\varepsilon(t), \mathcal{A}_0) = 0.
\]

5. **Conclusion.** To summarize, we established the upper semicontinuity of pullback, cocycle and uniform attractors admitted by the non-autonomous dynamical system generated of weakly damped wave equation to the natural extension of the global attractor corresponding to the parabolic equation. Since the well-posedness of weakly damped wave equations (1.2) with super-cubic nonlinearity is failed in natural energy space, the dynamical system is generated by the Shatah-Struwe solution of (1.2). Therefore, we consider not only the convergence of the solutions under the energy norm, but also the convergence under the Strichartz norm. This result shows that the Shatah Struwe solution shares the similar dynamic behavior as the weak solution in sub-cubic or cubic case which complements the property of the Shatah Struwe solution in the infinite dynamical theory.

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**REFERENCES**

[1] J. Arrieta, A. N. Carvalho and J. K. Hale, A damped hyperbolic equation with critical exponent, *Communications in Partial Differential Equations*, 17 (1992), 841–866.

[2] A. V. Babin, M. I. Vishik, Regular attractors of semi-groups and evolution equations, *J. Math. Pures Appl.*, 62 (1983), 441–491.

[3] A. V. Babin, M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and Its Applications, 25. North-Holland Publishing Co., Amsterdam, 1992.

[4] J. M. Ball, On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations, *Journal of Differential Equations*, 27 (1978), 224–265.

[5] M. D. Blair, H. F. Smith, C. D. Sogge, Strichartz estimates for the wave equation on manifolds with boundary, *Annales de l’Institut Henri Poincare (C) Non Lineaire*, 26 (2009), 1817–1829.

[6] M. C. Bortolan, A. N. Carvalho, J. A. Langa, Structure of attractors for skew product semiflows, *Journal of Differential Equations*, 257 (2014), 490–522.

[7] N. Burq, G. Lebeau, F. Planchon, Global existence for energy critical waves in 3-D domains, *Journal of the American Mathematical Society*, 21 (2008), 831–845.

[8] A. N. Carvalho, J. A. Langa, J. C. Robinson, *Attractors for Infinite-Dimensional Non-Autonomous Dynamical Systems*, Springer, 2013.

[9] A. N. Carvalho, J. W. Cholewa, T. Dlotko, Damped wave equations with fast growing dissipative nonlinearities, *Discrete and Continuous Dynamical Systems-A*, 24 (2009), 1147–1165.

[10] C. I. Christov, P. M. Jordan, Heat conduction paradox involving second-sound propagation in moving media, *Physical Review Letters*, 94 (2005), 154301.

[11] V. V. Chepyzhov, M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Soc., 2002.

[12] L. C. Evans, *Partial Differential Equations*, Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.
[13] P. Fabrie, C. Galusinski, A. Miranville, S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, *Discrete and Continuous Dynamical Systems*, 10 (2004), 211–238.
[14] M. M. Freitas, P. Kalita, J. A. Langa, Continuity of non-autonomous attractors for hyperbolic perturbation of parabolic equations, *Journal of Differential Equations*, 264 (2018), 1886–1945.
[15] T. Gallay, G. Raugel, Scaling variables and asymptotic expansions in damped wave equations, *J. Differential Equations*, 150 (1998), 42–97.
[16] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, Providence, RI, 1988.
[17] J. K. Hale, G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *Journal of Differential Equations*, 73 (1988), 197–214.
[18] A. Haraux, Two remarks on hyperbolic dissipative problems, *Nonlinear Partial Differential Equations and their Applications. College de France seminar*, 7 (1983–1984).
[19] L. T. Hoang, E. J. Olson, J. C. Robinson, Continuity of pullback and uniform attractors, *Journal of Differential Equations*, 264 (2018), 4067–4093.
[20] V. Kalantarov, A. Savostianov, S. Zelik, Attractors for damped quintic wave equations in bounded domains, *Annales Henri Poincaré*, 17 (2016), 2555–2584.
[21] S. Konabe, T. Nikuni, Coarse-grained finite-temperature theory for the bose condensate in optical lattices, *Journal of Low Temperature Physics*, 150 (2008), 12–46.
[22] A. C. Lazer, P. J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Rev.*, 32 (1990), 537–578.
[23] D. Li, Q. Chang, C. Sun, Pullback attractors for a critical degenerate wave equation with time-dependent damping, To appear.
[24] Y. Lv, W. Wang, Limiting dynamics for stochastic wave equations, *Journal of Differential Equations*, 244 (2008), 1–23.
[25] C. Matheus, M. C. Bortolan, A. N. Carvalho, J. A. Langa, *Attractors Under Autonomous and Non-autonomous Perturbations*, American Mathematical Society, 2020.
[26] A. Miranville, V. Pata, S. Zelik, Exponential attractors for singularly perturbed damped wave equations: A simple construction, *Asymptotic Analysis*, 53 (2007), 1–12.
[27] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, 2001.
[28] G. Somieski, Shimmy analysis of a simple aircraft nose landing gear model using different mathematical methods, *Aerospace Science and Technology*, 1 (1997), 545–555.
[29] Y. Wang, C. Zhong, Upper semicontinuity of global attractors for damped wave equations, *Asymptotic Analysis*, 91 (2015), 1–10.
[30] S. Zelik, Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent, *Communications on Pure and Applied Analysis*, 3 (2004), 921–934.
[31] S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, *Discrete and Continuous Dynamical Systems*, 11 (2004), 351–392.

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