Finite Dimensional Approximations in Geometry

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Abstract

In low dimensional topology, we have some invariants defined by using solutions of some nonlinear elliptic operators. The invariants could be understood as Euler class or degree in the ordinary cohomology, in infinite dimensional setting. Instead of looking at the solutions, if we can regard some kind of homotopy class of the operator itself as an invariant, then the refined version of the invariant is understood as Euler class or degree in cohomotopy theory. This idea can be carried out for the Seiberg-Witten equation on 4-dimensional manifolds and we have some applications to 4-dimensional topology.

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1. Introduction

The purpose of this paper is to review the recent developments in a formal framework to extract topological information from nonlinear elliptic operators.

We also explain some applications of the idea to 4-dimensional topology by using the Seiberg-Witten theory.

A prototype is the notion of index for linear elliptic operators. In this introduction we explain this linear case. Later we mainly explain the Seiberg-Witten case.

Let $D : \Gamma(E^0) \to \Gamma(E^1)$ be an linear elliptic operator on a close manifold $X$. The index $\text{ind} \, D$ is defined to be

$$\text{ind} \, D = \dim \text{Ker} \, D - \dim \text{Coker} \, D.$$

We can extend this definition as follows. Take any decomposition $D = L \oplus L' : V^0 \oplus W^0 \to V^1 \oplus W^1$. such that $L : V^0 \to V^1$ is a linear map between two finite
dimensional vector spaces and that $L' : W^0 \to W^1$ is an isomorphism between infinite dimensional vector spaces. Then we have

$$\text{ind } D = \dim V^0 - \dim V^1.$$  

It is easy to check that the right-hand-side is independent of the choice of the decomposition. For example we have decomposition satisfying $V^0 = \text{Ker } D$, $V^1 \cong \text{Coker } D$, $L = 0$, which gives the former definition of the index.

An important property of $\text{ind } D$ is its invariance under continuous variation of $D$. This property is closely related to the above well-definedness.

Another way to understand this property is to consider the whole space of Fredholm maps. Then the given map $D$ sits in the space and the $\text{ind } D$ is nothing but the label of the connected component containing $D$.

In other words, there are presumably three possible attitudes:

1. The essential data is “supported” on $\text{Ker } D$ and $\text{Coker } D$.
2. It is convenient to look at “some” finite dimensional approximation. $L : V^0 \to V^1$.
3. The essential data is the whole map $D : \Gamma(E^0) \to \Gamma(E^1)$.

When one considers a family of elliptic operators and tries to define the index of the family, it is not enough to look at their kernels and cokernels.

It is tempting to regard the finite dimensional approximation as a topological version of the notion of “low energy effective theory” in physics. In this story, the whole map $D$ would be regarded as a given original theory.

In this paper we explain a nonlinear version of the notion of index which is formulated by using finite dimensional approximations.

2. Non-linear cases

While every elliptic operator on a closed manifold has its index as topological invariant, it is quite rare that a nonlinear elliptic operator gives some topological invariant.

We have three examples of this type of invariants: the Donaldson invariant, the Gromov-Ruan-Witten invariant and the Seiberg-Witten invariant. Moreover, the Casson invariant is regarded a variant of the Donaldson invariant. Some other finite type invariants for 3-manifolds are also supposed to be related to these kinds of invariant [37].

Even for these cases, however, it is not obvious how to proceed to obtain nonlinear version of index in full generality.

Let us first give several examples of finite dimensional approximations.

1. C. Conley and E. Zehnder solved the Arnold conjecture for torus by reducing a certain variational problem to a finite dimensional Morse theory [10].
2. Casson’s definition of the Casson invariant. Taubes gave an interpretation of the Casson invariant via gauge theory [41]. In other words, Casson’s construction gave a finite dimensional approximation of the gauge theoretical setting
by Taubes. (The statement of the Atiyah-Floer conjecture could be regarded as a partial finite dimensional approximation along fibers.)

3. Seiberg-Witten equation. The moduli space of Seiberg-Witten equation is known to be compact for closed 4-manifolds. This enables us to globalize the Kuranishi construction to obtain finite dimensional approximations [18], [3].

4. Seiberg-Witten-Floer theory. C. Manolescu and P.B. Kronheimer defined Floer homotopy type for Seiberg-Witten theory, which is formulated as spectrum [32], [27].

5. Kontsevich explained an idea to define invariants of 3-manifolds by using configuration spaces. This idea was realized by Fukaya [13], Bott-Cattaneo [5], [6], and Kuperberg-Thurston [28]. Formally the configuration spaces appear as finite approximations of certain path spaces.

3. Kuranishi construction

While the index is regarded as the infinitesimal information of a nonlinear elliptic operator, its local information is given by the Kuranishi map, which has been used to describe local structure in various moduli problems [29].

A few years ago the Arnold conjecture was solved in a fairly general setting and the Gromov-Ruan-Witten invariant was defined for general symplectic manifolds. These works were done by several groups independently [14], [31], [34], [38]. A key of their arguments was to construct virtual moduli cycle over $\mathbb{Q}$. In their case, the point is to glue local structure to obtain some global data to define invariants. Since their invariants are defined by evaluating cohomology classes, it was enough to have the virtual moduli cycle.

4. Global approximation

The notion of Fukaya-Ono’s Kuranishi structure or Ruan’s virtual neighborhood is defined as equivalence class of collections of maps, which define the moduli space. The collection of maps is necessary because the moduli space as topological space is not enough to recover the nature of the singularity on it.

The data depend on the choice of various choice of auxiliary data. When we change the data, the change of the moduli space is supposed to be given by a cobordism, even with the extra structure we have to look at.

Suppose we would like to regard this structure itself as an invariant. Then we have to identify the place where the invariant lives. Since cobordism classes are identified by Pontrjagin-Thom construction, what we need would be a certain stable version of Pontrjagin-Thom construction.

In the case of symplectic geometry or Donaldson’s theory, this construction has not been done. A main problem seems to describe a finite dimensional approximation of the ambient space where the compactification of the moduli space lies. (The same problem occurs for Kotschick-Morgan conjecture.) Since the compactification is fairly complicated, it is not straightforward to identify the finite dimensional approximation.
However in the Seiberg-Witten case, the moduli spaces are known to be compact for closed 4-manifolds and it is not necessary to take any further compactifications.

Let us briefly recall the Seiberg-Witten equation for a closed Spin$^c$ manifold $X$. For simplicity we assume $b_1(X) = 0$. Let $W = W^0 \oplus W^1$ be the spinor bundle and $\mathcal{A}$ be the space of connections on $\det W^0 \cong \det W^1$. Then the Seiberg-Witten equation is given by a map

$$\Gamma(W^0) \times \mathcal{A} \to \Gamma(W^1) \times \Gamma(\Lambda^+)$$

where $\Gamma(\Lambda^+)$ is the self-dual 2-forms for a fixed Riemannian metric. This is an $U(1)$-equivariant map. The inverse image of 0 divided by $S^1$ is the moduli space, which is known to be compact.

A finite approximation of the above map is defined by global version of the Kuranishi construction. The approximation is a proper $U(1)$-equivariant map

$$\mathbb{C}^{c_0} \oplus \mathbb{R}^{d_0} \to \mathbb{C}^{c_1} \oplus \mathbb{R}^{d_1}$$

for some natural numbers $c_0, c_1, d_0$ and $d_1$. The differences $c_0 - c_1$ and $d_0 - d_1$ depends only on the topology of $X$ and its spin$^c$-structure.

The invariant we have is the stable homotopy class of the above $U(1)$-equivariant proper map, or equivalently, the $U(1)$-equivariant map from the sphere $S(\mathbb{C}^{c_0} \oplus \mathbb{R}^{d_0})$ to the sphere $S(\mathbb{C}^{c_1} \oplus \mathbb{R}^{d_1})$.

S. Bauer and the author pointed out that the invariant constructed above is a refinement of the usual Seiberg-Witten invariant [3].

5. 4-dimensional topology and Seiberg-Witten theory

We explain some applications of the finite dimensional approximation to 4-dimensional topology.

(1) Bauer’s connected sum formula [2]

Suppose $X$ is the connected sum of $X_0$ and $X_1$. If the neck of the connected sum is long enough, it is known that the moduli space of the solution of the Seiberg-Witten equation (or anti-self-dual equation) for $X$ is identified with the product of the moduli spaces for $X_0$ and $X_1$. When $X_1 = \mathbb{CP}^2$, then this gives the blowing-up formula. When $b^+(X_0), b^-(X_1) \geq 1$, this gives vanishing of the Seiberg-Witten (or the Donaldson) invariant of $X = X_0 \# X_1$. Bauer essentially showed that the product formula holds true for the virtual neighborhood of the moduli spaces, if we use Ruan’s terminology. In the language of stable maps between spheres, “product” becomes “join”. In particular Bauer’s formula gives the blowing-up formula for the refined invariant. When $b^+(X_0), b^-(X_1) \geq 1$, the join is torsion. It is, however, not necessary zero. In this way Bauer gave many new examples of 4-manifolds which are homeomorphic but not diffeomorphic to each other.

Ishida-Lebrun [24] [25] obtained some applications of the connected sum formula to Riemannian geometry.
(2) Intersection form of spin 4-manifolds

When 4-manifold is spin, we have certain extra symmetry, and the place where the invariant lives is a set of $Pin(2)$-equivariant stable maps [13].

When $X$ is a closed spin 4-manifold with $b_1(X) = 0$, the Seiberg-Witten map for the spin structure is a $Pin(2)$-equivariant map formally given by

$$H^\infty \oplus \tilde{\mathbb{R}}^\infty \rightarrow H^\infty \oplus \tilde{\mathbb{R}}^\infty,$$

where $\tilde{\mathbb{R}}$ is the non-trivial 1-dimensional real representation space of $Pin(2)$. and $H$ is the 4-dimensional real irreducible representation space of $Pin(2)$. Let $Z/4$ be the subgroup of $Pin(2)$ generated by an element in $Pin(2) \setminus U(1)$. The differences of the power $\infty$'s are given by the index of some elliptic operators.

A finite dimensional approximation is given by a $Pin(2)$-equivariant proper map

$$H^{c_0} \oplus \tilde{\mathbb{R}}^{d_0} \rightarrow H^{c_1} \oplus \tilde{\mathbb{R}}^{d_1},$$

for some $c_0, c_1, d_0, d_1$ satisfying

$$c_0 - c_1 = -\frac{\text{sign}(X)}{16}, \quad d_0 - d_1 = b^+(X).$$

This existence implies some inequality between the signature and the second Betti number.

To obtain the inequality explicitly we can use the following results.

**Theorem** Suppose $k > 0$ and $k \equiv a \mod 4$ for $a = 0, 1, 2, 3$. Then there does not exist a $G$-equivariant continuous map from $S(H^{k+x} \oplus \tilde{\mathbb{R}}^y)$ to $S(H^{k+x} \oplus \tilde{\mathbb{R}}^{2k+a'-1+y})$. for the following $G$ and $a'$.

1. (B. Schmidt [39] see also [40], [11], [33]) $G = Z/4$ and $a' = a$ for $a = 1, 2, 3$.
2. (F. - Y. Kametani [21]) $G = Pin(2)$ and $a' = 3$ for $a = 0$.

From the above non-existence results, we have the following inequality, which is a partial result towards the 11/8-conjecture $b^+ \geq 3|\text{sign}(X)/16|$. 

**Theorem** Let $X$ be a closed spin 4-manifold with $\text{sign}(X) = -16k < 0$. If $k \equiv a \mod 4$ for $a = 0, 1, 2, 3$, then we have $b^+ \geq 2k+b$, where $a' = a$ if $a = 1, 2, 3$ and $a' = 3$ if $a = 0$.

Equivariant version and $V$-manifold version can be formulated similarly [7], [12], [16], [11]. There are some applications of these extended versions:

1. C. Bohr [1] and R. Lee - T.-J. Li [30] investigated the intersection forms of closed even 4-manifolds which are not spin.
2. Y. Fukumoto, M. Ue and the author [16], [15], [17], [42], and N. Saveliev [36] investigated homology cobordisms groups of homology 3-spheres.

When $b_1 > 0$, we can construct another closed spin 4-manifold with $b_1 = 0$ without changing the intersection form. It implies that we can assume $b_1 = 0$ to obtain restriction on the intersection form. However when the intersection form on $H^1(X)$ is non-trivial, we may have a stronger restriction. Y. Kametani, H. Matsue, N. Minami and the author found that such a phenomenon actually occurs if there are $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^1(X, \mathbb{Z})$ such that $\langle \prod \alpha_i, [X] \rangle$ is odd [22].
6. Seiberg-Witten-Floer homotopy type

Recently C. Manolescu and P. B. Kronheimer extends the above formulation for closed 4-manifolds to the relative version \[32\], \[27\]. Let us explain their theory briefly.

We mentioned that Conley-Zehnder used a finite dimensional approximation of a Morse function on an infinite dimensional space to approach the Arnold conjecture for torus. Following this line, Conley extended the notion of Morse index and defined the Conley index for compact isolated set \[9\]. The Conley index is not a number, but a homotopy type of spaces. Floer extracted some information from the Conley index just by looking at some finite dimensional skeleton of the Conley index under some assumption. Floer’s formulation has the advantage that the Floer homology is defined even when the Conley index is not rigorously defined.

On the other hand R. L. Cohen, J. D. S. Jones and G. B. Segal tried to define certain stable homotopy type directly which should be an extended version of the Conley index \[8\]. They called it the Floer homotopy type. At that time the Floer homology was defined only for the Donaldson theory and the Gromov-Ruan-Witten theory. In these theories the moduli spaces are non-compact in general. This cause a serious difficulty to carry out their program.

In the Seiberg-Witten theory, we have a strong compactness for the moduli spaces. Manolescu and Kronheimer succeeded to construct the Floer homotopy type as spectra for the Seiberg-Witten theory by using this compactness.

They also defined relative invariant for 4-manifolds with boundary is also defined and it extends the invariant in \[9\].

7. Concluding remarks

The idea of finite dimensional approximation is closely related to the notion of “low energy effective theory” in physics. Actually the approximation should be regarded just as a part of the vast notion which we can deal with rigorously or mathematically.

Since Witten’s realization of Donaldson theory as a TQFT, the formal relation between mathematically regorous definition of invariants and their formal path integral expressions has suggested many things. For instance, the well-definedness of the Donaldson invariant is based on the fact that the formal dimension of the moduli space increases when the instanton number goes up. This fact seems equivalent to the other fact that the pure Yang-Mills theory is asymptotically free (for N=2 SUSY theory) and its renormalized theory does exists.

In the case of the finite dimensional approximations of Seiberg-Witten theory, the suspension maps give relations between many choices of approximations. If we use some generalized cohomology theories to detect our invariants, the suspension maps induces the Thom isomorphisms, or integrations along fibers. If we compare this setting with physics, the family of integrations look quite similar to the renormalization group. It seems the Thom classes which play the role of vacua. In this sense, one could say that the family of finite approximations is a topological version
of the renormalization group. This topological setting is very limited. It, however, has one advantage: Usually the path integral expression is supposed to take values in real or complex numbers. On the other hand our invariants could take values in torsions.

Let us conclude this survey by giving three open problems.

1. What is the correct formulation of the geography of spin 4-manifolds with $b_1 > 0$? (If the intersection on $H^1$ is complicated enough, then $\text{sign}(X)$ would have stronger restriction.)

2. When an oriented closed 3-manifold is a link of isolated algebraic singular point, construct a canonical Galois group action on some completion of the Floer homotopy type of Kronheimer-Manolescu. (This problem was suggested by a hand-written manuscript by D. Johnson in which Casson-type invariants were defined.)

3. The Seiberg-Witten map is quadratic. Extract non-topological information from this structure. (Is it possible to approach the 11/8-conjecture from this point of view?)

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