Bohm-Bell type experiments: Classical probability approach to (no-)signaling and applications to quantum physics and psychology

Andrei Khrennikov\textsuperscript{1,2}, Alexander Alodjants\textsuperscript{1}

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\textsuperscript{1} National Research University for Information Technology, Mechanics and Optics (ITMO), department, St. Petersburg, 197101, Russia
\textsuperscript{2} Linnaeus University, International Center for Mathematical Modeling in Physics and Cognitive Sciences, Växjö, SE-351 95, Sweden

Abstract

We consider the problem of representation of quantum states and observables in the framework of classical probability theory (Kolmogorov’s measure-theoretic axiomatics, 1933). Our aim is to show that, in spite of the common opinion, correlations of observables $A_1, A_2$ and $B_1, B_2$ involved in the experiments of the Bohm-Bell type can be expressed as correlations of classical random variables $a_1, a_2$ and $b_1, b_2$. The crucial point is that correlations $\langle A_i, B_j \rangle$ should be treated as conditional on the selection of the pairs $(i, j)$. The setting selection procedure is based on two random generators $R_A$ and $R_B$. They are also considered as observables, supplementary to the “basic observables” $A_1, A_2$ and $B_1, B_2$. These observables are absent in the standard description, e.g., in the scheme for derivation of the CHSH-inequality. We represent them by classical random variables $r_a$ and $r_b$. Following the recent works of Dzhafarov and collaborators, we apply our conditional correlation approach to characterize (no-)signaling in the classical probabilistic framework. Consideration the Bohm-Bell experimental scheme in the presence of signaling is important for applications outside quantum mechanics, e.g., in psychology and social science.
1 Introduction

This paper is related to the old foundational problem of quantum mechanics: whether it is possible to represent quantum states by classical probability (CP) distributions and quantum observables by random variables. (In fact, we analyze the general measurement scheme involving compatible and incompatible observables which need not be described by the quantum formalism. But our starting point is construction of the CP-representation for quantum mechanics.)

1.1 Towards CP-representation

The first step to study the problem of CP-embedding of quantum mechanics was done by Wigner [64] who tried to construct the joint probability distribution (jpd) of the position and momentum observables. However, Wigner’s function can take negative values. We also mention the Husimi-Kano Q-function [2,16] and Glauber-Sudarshan function [4,5]. These functions are widely applied to quantum mechanics and field theory (e.g., [6]). However, they cannot be described by CP-theory. The first CP-representation of quantum mechanics based on of symplectic tomogram was constructed in [7] (see also [8,9]).

Another construction of the CP-representation of quantum mechanics is based on so-called prequantum classical statistical field theory [10]-[13]. In this theory quantum states (density operators) are represented by covariation operators of random fields valued in complex Hilbert space $H$ (the state space of the quantum formalism); quantum observables (Hermitian operators) are represented by quadratic forms of such fields. The classical→quantum map has the form:

$$B \rightarrow \rho = B/\text{Tr}B, \ f_A \rightarrow A, \ \text{for the quadratic form } f_A(\phi) = \langle \phi | A | \phi \rangle. \quad (1)$$

This representation suffers of violation of the spectral postulate: the range of values of a quadratic form differs from the spectrum of the corresponding operator. To solve this problem, prequantum classical statistical field theory was completed by the corresponding measurement theory based on the detectors of the threshold type [14,15]. However, the majority of physicists would not be convinced that quantum effects can be reduced to behavior...
of classical random field combined with threshold detection (although experimenters typically recognize the role of detection thresholds in quantum measurements).

It should be honestly said the tomographic and random field approaches were practically ignored by the quantum foundational community. Nowadays it is commonly believed that CP-theory, see Kolmogorov [16], cannot serve to represent quantum observables. The roots of this belief lie in the previous unsuccessful attempts to construct the CP-representation for quantum mechanics, starting with Wigner’s attempt.

1.2 Bell’s type no-go statements

However, the main argument against the possibility to proceed with the CP-representation is based on no-go theorems. The first no-go theorem was proven by von Neumann [17] (German edition -1933): the theorem on nonexistence of dispersion free states. This theorem was strongly criticized by Bell [18] who pointed to non-physicality of von Neumann’s rule for correspondence between classical and quantum probabilistic structures (probabilities→states, random variables→Hermitian operators), cf. sections 2.4, 3.4. Bell’s own no-go theorem [19, 18] has much better reputation than von Neumann’s theorem and it has the very big impact to quantum foundations, quantum information, and quantum technology (at the same time it generated a plenty of critical papers, see, e.g., [20]-[24] for some recent publications). Bell proposed the CP-description of the Bohm-Bell type experiments. This approach is known as the hidden variables description. Since it is very difficult to test experimentally the original Bell inequality (see [25, 26] for a discussion), Clauser, Horne, Shimony, & Holt (CHSH) [27] modified Bell’s approach on the basis of the CHSH-inequality. (In spite of a rather common opinion, this modification does not equivalent to the original Bell approach.) We denote the CP-model proposed by them by the symbol $\mathcal{M}_{BCHSH}$ (see section 2.3).

Bell emphasized the role of nonlocality [19, 18]. However, Fine [28, 29] showed that the CHSH-inequality is satisfied if and only if the assumption on the existence of the jpd for the four observables $A_1, A_2, B_1, B_2$ involved in the experiment, see section 2.1. The latter is equivalent to using CP-theory. Therefore a violation of the CHSH-inequality inequality by quantum (theoretical and experimental) probabilities implies inapplicability of CP-theory. Erroneously inapplicability of one concrete CP-model, namely, $\mathcal{M}_{BCHSH}$, to describe the Bohm-Bell type experiments was commonly treated as inapplicability of CP in general.

Nevertheless, as was shown by Khrennikov and coauthors [30, 31] and re-
ently by Dzhafarov and coauthors \cite{32}-\cite{38}, the Bohm-Bell type experiments can be modeled with the aid of the CP-representation of quantum observables. However, such CP-models are not so straightforward as $\mathcal{M}_{\text{BCHSH}}$. Denote the models developed in \cite{31} and in \cite{32}-\cite{38} by the symbols $\mathcal{M}_{\text{KH}}$ and $\mathcal{M}_{\text{DZ}}$, respectively.

### 1.3 Conditional probability approach

The basic distinguishing feature of $\mathcal{M}_{\text{KH}}$ is taking into account the conditional nature of quantum probabilities. Generally, we follow Ballentine \cite{39, 40}, especially his paper \cite{41}. In the previous works \cite{30}, \cite{31} and the present paper conditioning is modeled with the aid of the random generators selecting the experimental settings. They are represented as random variables (RVs) $r_a, r_b$ which are supplementary to the “basic” RVs $a_1, a_2, b_1, b_2$ (see sections \ref{2.3}, \ref{3.2}). These RVs are absent in $\mathcal{M}_{\text{BCHSH}}$. At the same time the random generators play the crucial role in the real experimental design of such experiments. We remark that Bohr emphasized that in modeling quantum phenomena all components of the experimental arrangement should be taken into account \cite{42, 43}. Thus ignoring the random generators makes a model without them (as, e.g., $\mathcal{M}_{\text{BCHSH}}$) inadequate to the real physical situation.

However, model $\mathcal{M}_{\text{BCHSH}}$ need not be rigidly coupled to conditioning on selection of experimental settings. A variety of conditioning can lead to violation of the Bell type inequalities. In particular, in the random field theory \cite{14, 15} endowed with the threshold detection scheme this is conditioning on joint detection or more generally on a time widow for coupling two clicks of spatially separated detectors, see \cite{44} for the general discussion.

We remark that CP-conditioning is one of the forms of the mathematical representation of context-dependence, dependence of outputs of observables on components of the experimental context (see again Bohr \cite{42}). Thus $\mathcal{M}_{\text{KH}}$ can be treated as a contextual model of the Bohm-Bell type experiments. However, one has to be very careful with the use of the notion “contextuality”. Here it matches the Copenhagen interpretation of quantum mechanics in its original Bohr’s understanding \cite{43}, cf. with the notion of “quantum contextuality” based on the Bell type tests. In contrast to the latter, Bohr’s type contextuality is not mystical - it is straightforwardly coupled to the experimental arrangement. We can also move another way around: to start with a general contextual probabilistic model and then to describe the class of such models which can be represented in complex Hilbert space $H$, see \cite{45}-\cite{47} (“constructive wave function approach”).
1.4 CP-representations in the presence of signaling

Model $\mathcal{M}_{DZ}$ does not contain explicit counterparts of the random generators for setting’s selection. It is based on contextual coupling of random variables corresponding to the choice of experimental settings. In spite of different mathematical structures, both models, $\mathcal{M}_{KH}$ and $\mathcal{M}_{DZ}$, represent the procedure of experimental settings’ selection: $\mathcal{M}_{KH}$ with the aid of the random generators, $\mathcal{M}_{DZ}$ with the aid of contextual indexing of RVs representing observables.

Model $\mathcal{M}_{DZ}$ was applied to study contextuality in the CP-framework with the especial emphasis of the possibility to proceed in the presence of signaling $\text{[32]-[38]}$. (Contextuality studied by Dzhafarov and the coauthors is the natural extension of the notion of quantum contextuality based on the Bell type tests.) We remark that signaling is absent in quantum mechanics. Therefore contextuality theory developed in $\text{[32]-[38]}$ and known as contextuality by default (CbD) is more general than the standard theory of quantum contextuality. In particular, the standard Bell type inequalities are modified by including the signaling contribution. They are known as the Bell-Dzhafarov-Kujala (BDK) inequalities. This generality provides the possibility to apply CbD outside physics, especially in psychology $\text{[48]-[51]}$, where the condition of no-signaling is generally violated $\text{[33, 36]}$.

Papers $\text{[30], [31]}$ were aimed to show the existence of the CP-representation for the Bohm-Bell experiment with genuine quantum systems. In these papers model $\mathcal{M}_{KH}$ was presented in the very concrete framework coupled to classical versus quantum discussion on the CHSH-inequality. This rigid coupling with quantum mechanics led to ignoring the possibility to use model $\mathcal{M}_{KH}$ even in the presence of signaling. Consistent CP-treatment of (no-)signaling in model $\mathcal{M}_{DZ}$ motivated the authors of this paper to analyze (no-)signaling issue on the basis of $\mathcal{M}_{KH}$. And we found very clear CP-interpretation of no-signaling: independence of RVs $a_1, a_2, r_a$ representing Alice’s observables and random generator from RV $r_b$ representing the random generator for selecting Bob’s observables. Thus no-signaling has clear probabilistic meaning.

In contrast to papers $\text{[30], [31]}$, in this paper we proceed in very general abstract framework which can be used both in physics and outside it, e.g., in psychology. (See $\text{[48]-[51]}$ for consideration of the Bell type inequalities in psychology.)

Finally, we point to recently published paper of Margareta Manjko and Vladimir Manjko $\text{[52]}$ presenting a very general scheme of the CP-representation of quantum states and observables.
2 Bohm-Bell type experiment: traditional description

2.1 Description of (four) observables

In the observational framework for the Bohm-Bell type experiments, there are considered four observables $A_1$, $A_2$, $B_1$, $B_2$ taking values $\pm 1$. It is assumed that the pairs of observables $(A_i, B_j), i, j = 1, 2$, can be measured jointly, i.e., $A$-observables are compatible with $B$-observables. However, the observables in pairs $A_1$, $A_2$ and $B_1$, $B_2$ are incompatible, i.e., they cannot be jointly measured. Thus probability distributions $p_{A_iB_j}$ are well defined theoretically by quantum mechanics and they can be verified experimentally; probability distributions $p_{A_1A_2}$ and $p_{B_1B_2}$ are not defined by quantum mechanics and, hence, the question of their experimental verification does not arise.

We stress that, although our starting point is quantum mechanics and the Bohm-Bell experiment for measurement of spin of electrons or polarization of photons, we need not to restrict our scheme to quantum observables. It is applicable to any measurement design involving compatible and incompatible observables, see, e.g., [48]-[51] for such experimental design in psychology. Here compatibility (incompatibility) is understood as the possibility (impossibility) of joint measurement and determination of jpd.

2.2 Terminology: observational, empirical, epistemic, and ontic

This section can be useful for experts in quantum foundations. But, in principle, one can skip it and jump directly to section 2.3.

To be completely careful, in physics we should distinguish empirical probabilities obtained in experiments and theoretical probabilities given by the quantum formalism. The former are given by frequencies of outputs of observations. The von Mises frequency probability theory [53, 54] is the best formalism to handle them [55]. The quantum theoretical probabilities are given by the Born rule. However, since the applicability of the quantum formalism was confirmed by numerous experiments, in physics we can identify quantum empirical and theoretical probabilities. (In principle, we should consider two sets of probabilities, $p_{A_iB_j}^{exp}$ and $p_{A_iB_j}^{QM}$.) However, we want to present a very general framework covering even experiments outside physics, e.g., in psychology [48]-[51]. Generally we interpret observables and corresponding probabilities empirically.

The Bell type inequalities cannot be derived in the observational frame-
work (neither empirical nor theoretical). To derive them, one has to operate in the CP-framework. Here CP-theory is understood as the measure-theoretical approach to probability proposed by Kolmogorov in 1933 [16]. We remark that the Bell type inequalities cannot be derived [55] by using the von Mises frequency probability theory [53, 54]. (In principle, this theory also can be considered as a CP-theory.) We also remark that von Neumann treated quantum probabilities in the von Mises framework [17]. So, the Bell argument is about comparison of the Hilbert space and measure-theoretic representations of probabilities.

We also make a remark on the used terminology. In philosophy and quantum foundations, it is common to consider the epistemic and ontic descriptions of natural (or mental) phenomena, Atmanspacher and Primas [56]. At the epistemic level we represent our knowledge about phenomena. The ontic level is related to “reality as it is when nobody observes it.”

In this paper we speak about the “observational framework”. This is more or less the same as the epistemic framework. But epistemic is typically related to a theoretical model. So, quantum mechanics is an epistemic model. Our “observational framework” covers not only theoretical models, but even “rough experimental data”.

Although the term “ontic” is well established in philosophy as well as in quantum foundations (and was widely used by one of the coauthors of this paper, e.g., [13]), it seems that often the use of the notion of “reality as it is” is really misleading. We can speak only about models and the ontic level of description is still our own (typically mathematical) description.

We prefer to speak about “counterfactual components of a model”. The ontic level of description is characterized by the presence of counterfactuals.

2.3 Classical probability model (BCHSH) for the Bohm-Bell experiment: four random variables

Let \((\Lambda, \mathcal{F}, P)\) be some probability space [16]. Here \(\Lambda\) is the set of hidden variables (or in mathematics “elementary events”), \(\mathcal{F}\) is a \(\sigma\)-algebra of events, \(P\) is a probability measure on \(\mathcal{F}\).

The notion of a \(\sigma\)-algebra can be disturbing for physicists. We remark that if \(\Lambda\) is finite, then \(\mathcal{F}\) is the collection of all its subsets. In CP-modeling the CHSH framework it can be assumed that \(\Lambda\) is finite.

Consider two pairs of random variables \(a_1, a_2 : \Lambda \to \{\pm 1\}\) and \(b_1, b_2 : \Lambda \to \{\pm 1\}\). These random variables are associated with observables \(A_1, A_2, B_1, B_2\). This is the Bell type CP-model for the observational framework presented in section 2.1. Denote this CP-model by \(\mathcal{M}_{BCHSH}\).
We remark that the jpd of four random variables $a_1, a_2, b_1, b_2$ is well defined:

$$P_{a_1a_2b_1b_2}(\alpha_1, \alpha_2, \beta_1, \beta_2) = P(\lambda : a_1(\lambda) = \alpha_1, a_2(\lambda) = \alpha_2, b_1(\lambda) = \beta_1, b_2(\lambda) = \beta_2),$$

where $\alpha_i, \beta_j = \pm 1$.

In model $\mathcal{M}_{BCHSH}$, one can form the CHSH linear combination of the correlations of the pairs of random variables $a_i, b_j$

$$B = \langle a_1b_1 \rangle - \langle a_1b_2 \rangle + \langle a_2b_1 \rangle + \langle a_2b_2 \rangle$$

and prove the CHSH-inequality:

$$|B| \leq 2.$$  

Here

$$\langle a_ib_j \rangle \equiv E(a_ib_j) = \int_{\Lambda} a_i(\lambda)b_j(\lambda)dP(\lambda) = \sum_{\alpha,\beta} \alpha\beta P_{a_ib_j}(\alpha, \beta).$$

We remark that probabilities for the joint measurements of $a$ and $b$ observables can be represented as the marginal probabilities for the quadruple jpd, e.g., $P_{a_ib_1}(\alpha, \beta) = \sum_{x,y} P_{a_1a_2b_1b_2}(\alpha, x, \beta, y)$. This representation plays the crucial role in the derivation of CHSH-inequality \[3\]. Moreover, by Fine’s theorem[?] the existence of the jpd is equivalent to satisfying the CHSH-inequality.

In principle, we can select $\Lambda$ as the set of vectors $\lambda = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ with coordinates $\pm 1$. Here probability $P$ is given by jpd; events are all possible subsets of this $\Lambda$.

We remark that model $\mathcal{M}_{BCHSH}$ contains counterfactual components, e.g., the joint presence of the values of the incompatible observables, say $a_1(\lambda), a_2(\lambda)$. Consequently pair-wise jpdfs $P_{a_1a_2}$ and $P_{b_1b_2}$ as well quadrupole jpdf $P_{a_1a_2b_1b_2}$ are also counterfactual. By using the ontic-epistemic terminology we can say that $\mathcal{M}_{BCHSH}$ is an ontic model for the epistemic model - quantum mechanics.

Now consider the observational probabilities $p_{A,B}$. The BCHSH-coupling between the observational and CP descriptions is straightforward, it will be presented in the next section.

2.4 BCHSH-rule for correspondence between observational and classical probabilities

The observational framework (section 2.1) is coupled with CP-model $\mathcal{M}_{BCHSH}$ by the following correspondence rule:
The observational probabilities $p_{A_i B_j}$ are identified with the CP-probabilities $P_{a_i b_j}$.

This coupling leads to contradiction, because the CHSH linear combination composed of observational correlations (either experimental or quantum theoretical):

$$B_{\text{observational}} = \langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle$$ \hspace{1cm} (5)

can violate CHSH-inequality (3); generally

$$|B_{\text{observational}}| > 2.$$ \hspace{1cm} (6)

One can conclude that CP-model $M_{\text{BCHSH}}$ is not adequate neither to the quantum (epistemic) model nor to the experimental situation.

This mismatching related to concrete CP-model $M_{\text{BCHSH}}$ and the BCHSH correspondence rule is commonly interpreted too generally: as the impossibility of the CP-description of quantum phenomena, impossibility to represent quantum states by probability measures and quantum observables (generally incompatible) by classical random variables.

We remark that typically physicists speak about realism and locality. In the CP-framework, realism is encoded in the functional representation of observables by random variables, locality (noncontextuality) is encoded in single indexing of random variables, say $a_i$ is indexed by solely its experimental setting $i$ (see [38]). For PBS, this is the concrete orientation angle $\theta_i$. Thus the statement about mismatching of model $M_{\text{BCHSH}}$ and quantum mechanics is typically stated as mismatching between local realism and quantum mechanics.

2.5 Missed component of experimental arrangement

In the CHSH observational framework, the correlations composing quantity $B_{\text{observational}}$ cannot be measured jointly. The concrete experiment can be performed only for one fixed pair of indexes $(i, j)$, experimental settings (orientations of PBSs). Generally these settings are selected randomly, by using two random generators $R_A$ and $R_B$ taking values 1, 2. What are the theoretical counterparts of these random generators in $M_{\text{BCHSH}}$? They are absent. So, CP-model $M_{\text{BCHSH}}$ is inadequate the observational framework.

One sort of randomness, namely, generated by $R_A, R_B$ is missed. We shall present another CP-model corresponding the real experimental situation: the observational BCHSH-framework (section 2.1) with supplementary observables $R_A, R_B$. 9
By proceeding in this way we follow the Copenhagen interpretation of quantum mechanics. Bohr always emphasized: *all components of the experimental arrangement (context) have to be taken into account* [42]. Experimenters strictly follow the Copenhagen interpretation. Random generators play the fundamental role in the experiments of Bohm-Bell type. However, these generators are absent in CP-model $\mathcal{M}_{BCHSH}$.

3 Bohm-Bell type experiments: taking into account random generators

At the observational level, we plan to complete the standard description of the Bohm-Bell type experiments (section 2.1) by taking into account the aforementioned “missed components of the experimental arrangement”. Then we shall construct a CP-model which will be adequate to the completed observational framework. It will take into account “missed component of randomness”. Denote such a CP-model under construction by $\mathcal{M}_{KH}$.

3.1 Description of (six) observables

Following Bohr, we treat random generators $R_A$ and $R_B$ as a part of experimental arrangement. Instead of the observational framework with four observables (section 2.1) $A_1, A_2, B_1, B_2$, we consider the framework with six observables $A_1, A_2, B_1, B_2, R_A, R_B$. The latter two observables are compatible, i.e., they can be jointly measurable; moreover, they are compatible with each of four “basic observables” $A_1, A_2, B_1, B_2$ (see [57] for the mathematical representation of these six observables within the quantum operator formalism). In principle, in the real experimental situation one can assume that observables $R_A$ and $R_B$ are independent. For for the moment, we proceed without this assumption.

To improve the visibility of the role of random generators, in physics we can consider the experimental design of the pioneer experiment performed by Aspect, see [58]. In the modern experimental design, there are two beam splitters, one on the $A$-side and another on the $B$-side, and two devices for random selection of orientations on the corresponding sides. Aspect considered four beam splitters and two switchers preceding corresponding pairs of beam splitters. The $A$-switcher selects randomly one of the beam splitters on the $A$-side; the $B$-switcher selects randomly one of the beam splitters on the $B$-side (switchers open optical channels to corresponding beam splitters). For this design, it is natural to introduce the additional value of observables, we set $A_i = 0$ ($B_j = 0$) if its input channel is closed by the random switcher.
We consider the ideal experiment with 100 % of efficiency of the whole experimental scheme, i.e., including detector, beam splitters, an optical fibers.

3.2 Complete CP-model: six random variables

Let again $(\Lambda, \mathcal{F}, P)$ be some probability space. We want to introduce random variables $a_1, a_2, b_1, b_2$ associated with observables $A_1, A_2, B_1, B_2$, but not so straightforwardly as in $\mathcal{M}_{\text{BCHSH}}$. Additionally, we consider two random variables $r_A, r_B : \Lambda \to \{1, 2\}$ associated with the random generators. Besides of values $\pm 1$, random variables $a_1, a_2, b_1, b_2$ can take the value zero.

The zero-value is determined by governing selections of measurement settings, i.e., $A_1, A_2, B_1, B_2$, by random generators $R_A$ and $R_B$. In our CP-model, it has the form:

- $a_i = 0$ (with probability one), if the $i$-setting was not selected, i.e., $r_A \neq i$;
- $b_j = 0$ (with probability one), if the $j$-setting was not selected, i.e., $r_B \neq j$.

We remark that in our model the zero-value has nothing to do with detection’s inefficiency (as is often considered in modeling the Bohm-Bell experiment). We model the experimental situation with detectors having 100% efficiency.

3.3 Constraints on joint probabilities implied by matching condition

In terms of probability the condition of $a - r_a$ matching can be written as follows:

$$P(a_i = 0|r_a = j) = 1, i \neq j.$$  \hfill (7)

It implies that

$$P(a_i = \alpha|r_a = j) = 0, \alpha = \pm 1, i \neq j.$$  \hfill (8)

Thus RV $a_i$ cannot take values $\pm 1$ if $r_a \neq i$. This is the CP-presentation of the impossibility to measure observable $A_i$ if random generator $R_A \neq i$. Equality (8) implies

$$P(a_i = \alpha, r_a = j) = 0, \alpha = \pm 1, i \neq j.$$  \hfill (9)

In the same way, the condition of $b - r_a$ matching can be written as follows:

$$P(b_i = 0|r_b = j) = 1, i \neq j.$$  \hfill (10)
This condition implies

\[ P(b_i = \beta, r_b = j) = 0, \beta = \pm 1, i \neq j. \]  

(11)

From equalities (7), (10), we obtain

\[ P(a_i = 0, r_a = j) = P(r_a = j), \ P(b_i = 0, r_b = j) = P(r_b = j), i \neq j. \]  

(12)

In turn, these equalities imply

\[ P(a_i = 0, r_a = i) = P(r_a = i), \ P(b_i = 0, r_b = i) = P(r_b = i). \]  

(13)

The jpd of six random variables \( a_1, a_2, b_1, b_2, r_A, r_B \) is well defined:

\[ P_{a_1a_2b_1b_2r_ar_b}(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \]

\[ = P(\lambda : a_1(\lambda) = \alpha_1, a_2(\lambda) = \alpha_2, b_1(\lambda) = \beta_1, b_2(\lambda) = \beta_2, r_A(\lambda) = \gamma_1, r_B(\lambda) = \gamma_2), \]

where \( \alpha_i, \beta_j = 0, \pm 1, \gamma_k = 1, 2. \)

The matching condition implies that, e.g., \( P_{a_1a_2b_1b_2r_ar_b}(\alpha_1, \pm 1, \beta_1, \beta_2, 1, \gamma_2) = 0. \) Thus only 16 components of the jpd are different from zero:

\[ P_{a_1a_2b_1b_2}(\alpha, 0, \beta, 0, 1, 1), P_{a_1a_2b_1b_2}(\alpha, 0, 0, \beta, 1, 2), \]

\[ P_{a_1a_2b_1b_2}(0, \alpha, \beta, 0, 2, 1), P_{a_1a_2b_1b_2}(0, \alpha, 0, \beta, 2, 2), \]

where \( \alpha, \beta = \pm 1. \)

### 3.4 Correspondence between observational and classical conditional probabilities

Now consider the observational probabilities \( p_{A_iB_j}. \) These are probabilities for the fixed pair of experimental settings \((i, j). \) Their counterparts in CP-model \( \mathcal{M}_{KH} \) are obtained by conditioning on the fixed values of random variables \( r_A \) and \( r_B. \) The rule of correspondence between observational and CP-probabilities is based on the following identification:

\[ p_{A_iB_j}(\alpha, \beta) = P(a_i = \alpha, b_j = \beta|r_A = i, r_B = j), \]  

(14)

where \( \alpha, \beta = \pm 1. \) Thus

\[ p_{A_iB_j}(\alpha, \beta) = \frac{P(\lambda \in \Lambda : a_i(\lambda) = \alpha, b_j(\lambda) = \beta, r_A(\lambda) = i, r_B(\lambda) = j)}{P(\lambda \in \Lambda : r_A(\lambda) = i, r_B(\lambda) = j)}. \]  

(15)

This correspondence rule for the “basic observables” is completed by the similar rule for random generators \( R_A \) and \( R_B : \)

\[ p_{R_AR_B}(i, j) = P(\lambda \in \Lambda : r_a(\lambda) = i, r_b(\lambda) = j). \]  

(16)
3.5 Violation of the CHSH-inequality by conditional correlations

Conditioning on the selection of experimental settings plays the crucial role. The CP-correlations are based on the conditional probabilities

\[ \langle a_i b_j \rangle \equiv E(a_i b_j | r_A = i, r_B = j) = \sum_{\alpha, \beta = \pm 1} \alpha \beta \rho(a_i = \alpha, b_j = \beta | r_A = i, r_B = j). \]

We can form the CHSH linear combination of conditional correlations of RVs:

\[ \tilde{B} = \langle a_1 b_1 \rangle - \langle a_1 b_2 \rangle + \langle a_2 b_2 \rangle + \langle a_2 b_1 \rangle. \]  

It is possible to find such classical probability spaces that \(|\tilde{B}| > 2\).

Since each conditional probability is also a probability measure and since RVs \(a_i, b_j\) take values in \([-1, +1]\), the conditional expectations \(E(a_i b_j | r_A = i, r_B = j)\) are bounded by 1, so

\[ |\tilde{B}| \leq 4. \]

Thus the common claim on mismatching of the CP-description with quantum mechanics and experimental data was not justified.

In principle, one can consider linear combination \(B\) composed of correlations \(\langle a_1 b_1 \rangle\) which are not conditioned on selection of experimental settings. Such \(B\) satisfies the CHSH-inequality. But such correlations cannot be identified with experimental ones.

3.6 Construction of jpd from observational probabilities

Correspondence rules (14), (16) imply

\[ P(a_1 a_2 b_1 b_2 r_A r_B (\alpha_1, \alpha_2, \beta_1, \beta_2, i, j) = p_{A_i B_j} (\alpha, \beta) p_{R_A R_B} (i, j), \alpha, \beta = \pm 1, \]  

From this equality, we can determine all nonzero components the jpd:

\[ p(\alpha, 0, \beta, 0, 1, 1) = p_{A_1 B_1} (\alpha, \beta) p_{R_A R_B} (1, 1), p(\alpha, 0, 0, \beta, 1, 2) = p_{A_1 B_2} (\alpha, \beta) p_{R_A R_B} (1, 2), \]

\[ p(0, \alpha, \beta, 0, 2, 1) = p_{A_2 B_1} (\alpha, \beta) p_{R_A R_B} (2, 1), p(0, \alpha, 0, \beta, 2, 2) = p_{A_2 B_2} (\alpha, \beta) p_{R_A R_B} (2, 2) \]

In model \(M_{KH}\), the jpd is completely determined by observational (epistemic) probabilities. In contrast to \(M_{CHSH}\), there are no counterfactual probabilities.
4 (No-)signaling

4.1 No-signaling in quantum physics

In the observational framework for the Bohm-Bell type experiment, the condition of no-signaling is formulated in the probabilistic terms. There is no-signaling, from the B-side to the A-side, if the A-marginals of jpdfs $p_{A,B,j}$

$$M_{ij}(\alpha) = \sum_{\beta = \pm 1} p_{A,B,j}(\alpha, \beta), \ i = 1, 2,$$

(20)
do not depend on the index $j$.

This notion of signaling need not be rigidly coupled to quantum observables. It can be applied to any measurement design in that $A_i$ is compatible with both $B_j$, $j = 1, 2$, but $B_1$ and $B_2$ are incompatible, i.e., we are not able to perform their joint measurement. No-signaling from the A-side to the B-side is defined in the same way.

In physics, signaling is often understood as real signaling from the B-side to the A-side and even, what is worse, from the B system to the A-system. By constructing the CP-model, we can clarify the meaning of (no-)signaling at the level of RVs and then observations.

4.2 No-signaling as condition of independence of random variables

Now we proceed with CP-model $M_{KH}$. Let us fix $r_a = i$. For any value $r_b = j$, consider conditional $a_i$-marginal

$$m_{ij}(\alpha) = \sum_{\beta} P(a_i = \alpha, b_j = \beta | r_a = i, r_b = j), i = 1, 2.$$

(21)

By correspondence rules (14), (16)

$$M_{ij}(\alpha) = m_{ij}(\alpha).$$

(22)
The marginal $m_{ij}(\alpha)$ does not depend on the $j$-settings governed by $r_b$ under the following assumption:

\textbf{I}_{a_i}: The pair of RVs $a_i, r_a$ does not depend on RV $r_b$.

Under this assumption

$$m_{ij}(\alpha) = P(a_i = \alpha | r_a = i).$$

(23)
This is the conditional-probability version of no-signaling for \( a_i \). To prove equality (23), we first remark
\[
m_{ij}(\alpha) = P(a_i = \alpha | r_a = i, r_b = j)
\]
(since the conditional probability is a probability measure). Hence,
\[
m_{ij}(\alpha) = \frac{P(a_i = \alpha, r_a = i, r_b = j)}{P(r_a = i, r_b = j)} = \frac{P(a_i = \alpha, r_a = i)P(r_b = j)}{P(r_a = i)P(r_b = j)}
\]
and this proves (23).

Now, let us assume that RVs \( r_a \) and \( r_b \) are independent. (From the experimental viewpoint, this is the very natural assumption.) Suppose that, for \( \alpha = \pm 1 \), the marginal \( m_{ij}(\alpha) \) does not depend on \( j \). Generally this marginal can be represented in the form:
\[
m_{ij}(\alpha) = \frac{P(a_i = \alpha, r_a = i, r_b = j)}{P(r_a = i, r_b = j)} = \frac{P(a_i = \alpha, r_a = i)P(r_b = j)}{P(r_a = i)}.
\]
The right-hand side does not depend on \( j \) only if \( P(a_i = \alpha, r_a = i | r_b = j) = P(a_i = \alpha, r_a = i) \) (see appendix). This is the condition of independence of the pair of RVs \( a_i, r_a \) from RV \( r_b \).

In the same way, consider the assumption
\[
I_{b_j} \text{ The pair of random variables } b_j, r_b \text{ does not depend on } r_a.
\]
Under this assumption
\[
m_{ij}(\beta) = \sum_{\alpha} P(a_i = \alpha, b_j = \beta | r_a = i, r_b = j) = P(b_j = \beta | r_b = j).
\]
This is the conditional version of no-signaling for random variable \( b_j \).

The CP-presentation of no-signaling in terms of conditional probabilities, see \( I_a, I_{b_j} \), explains the meaning of signaling. For example, \( b \rightarrow a \) signaling means either interdependence of random generators \( r_a \) and \( r_b \), or dependence of \( a \)-RVs on random generator \( r_b \).

Under the assumption of independence of RVs \( r_a \) and \( r_b \) representing the random generators, \( b \rightarrow a \) signaling has the meaning of dependence of \( a \)-variables on random generator \( r_b \), i.e., the latter governs not only \( b \)-variables, but even the \( a \)-variables.

### 4.3 Interpretation of no-signaling: from random variables to observables

By using (26) we can lift the CP-interpretation of no-signaling to the level of observables. Let us consider the case of independent random generators \( R_A \)
and $R_B$ represented by independent RVs $r_a$ and $r_b$. The absence of $B \rightarrow A$ signaling for observables, i.e., independence $M_{ij}(\alpha)$ from index $j$, is equivalent to the absence of $b \rightarrow a$ signaling RVs. Hence, at the observational level $B \rightarrow A$ no-signaling has the meaning of independence of $A$-observables from selection of experimental settings governed by random generator $R_B$.

We stress that $\mathcal{M}_{KH}$ can serve as a CP-model for quantum probabilities, i.e., probabilities described by the quantum formalism with the aid of the Born rule. Thus the absence of signaling in the quantum description of the Bohm-Bell experiment has very natural CP-explanation: selection of $A$-settings depends only on the random generator $R_A$ and selection of $B$-settings depends only on the random generator $R_B$.

### 4.4 (No-) signaling in experiments in quantum physics and psychology

In quantum physics the problem of the presence of signaling patterns in statistical data collected in the Bohm-Bell type experiments was highlighted in the work [59] (it seems, it was the first paper on this problem). Since the quantum formalism predicts the absence of signaling, such signaling patterns were considered as a consequence of the improper experimental performance. After the pioneer paper [59], experimenters started to pay attention to signaling. Tremendous efforts of experimenters to eliminate technicalities which may lead to signaling were culminated in the breakdown experiments of Vienna’s group [60] and NIST’s group [61]. (Unfortunately, the first experiment claiming to be loophole free [62] suffers of strong signaling, see [63].)

As was found by Dzhafarov and the coauthors, see, e.g., [33, 36], the psychological experiments of the Bohm-Bell type generated statistical data with statistically non-negligible signaling patterns. (These are experiments to test quantum contextuality in the psychological analogs of the Bell-Bohm type experiments [48]-[51]. So, the issue of nonlocality is not involved.) In psychology we do not have theoretical justification of the absence of signaling. Therefore it is not clear whether the mental signaling is a consequence of improper experimental design and performance or this is the fundamental feature of experiments with humans.

### 5 Concluding remarks

We presented the brief review on CP-representations of quantum probability. Then the paper was concentrated on one special representation based on the conditional probability interpretation of quantum probabilities [31]. The
formalism of the latter article was described in the very general framework covering the experimental schemes of the Bohm-Bell type. Such experimental schemes need not be coupled to quantum physics. In particular, they can be realized for experiments with humans. As was found by Dzhafarov and the coauthors, the latter experiments are characterized by the presence of statistically significant signaling patterns. In this paper, we analyzed the CP-meaning of signaling in the conditional probabilistic model. We found that signaling can be described as simply dependence of random variables.

We highlight the basic impacts of the CP-representation of the experimental schemes of the Bohm-Bell type:

1. It demystifies quantum probability theory - representation of probabilities by complex amplitudes and observables by Hermitian operators:

2. It justifies the use of CP-based mathematical statistics for analysis of data from quantum experiments.

3. It clarifies the meaning of (no-)signaling as independence-dependence of classical random variables.

Finally, we emphasize once again the foundational impact of Ballentine’s works [39]-[41] on the conditional probabilistic interpretation of quantum probabilities. These works stimulated development of contextual probability theory [47]. As was found in [31], the quantum contextual probabilities generated in experiments of the Bohm-Bell type can be even represented as classical probabilities (see also [32]-[38]).

Appendix

Consider two RVs $X$ and $Y$. Here $X$ is an arbitrary discrete RV, $X = x_1, ..., x_m$, and $Y$ is a dichotomous RV, $Y = 1, 2$. Suppose that, for each $x_i$
Conditional probability $P(X = x | Y = j)$ does not depend on $j$. We want to show that this implies that, in fact,

$$P(X = x | Y = j) = P(X = x),$$

(27)
i.e., that RVs $X$ and $Y$ are independent.

Set $A_x = \{ \lambda \in \Lambda : X(\lambda) = x \}$ and $B_j = \{ \lambda \in \Lambda : Y(\lambda) = j \}$. We have

$$P(A_x | B_1) = P(A_x | B_2),$$
i.e.

$$P(A_x \cap B_1) = \frac{P(B_1)}{P(B_2)} P(A_x \cap B_2),$$
or

$$P(A_x \cap B_1) = \frac{P(B_1)}{P(B_2)} [P(A_x) - P(A_x \cap B_1)],$$
i.e.

$$P(A_x \cap B_1) \left[ 1 + \frac{P(B_1)}{P(B_2)} \right] = \frac{P(B_1)}{P(B_2)} P(A_x).$$

Thus we obtained

$$P(A_x \cap B_1) = P(B_1) P(A_x)$$

This also implies that $P(A_x \cap B_2) = P(B_2) P(A_x)$. Hence, equality (27) holds and RVs $X$ and $Y$ are independent.

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