The Coulomb Phase in $N = 1$ Gauge Theories
With a LG-Type Superpotential

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Abstract

We consider $N = 1$ supersymmetric gauge theories with a simple classical gauge group, one adjoint $\Phi$, $N_f$ pairs $(Q_i, \bar{Q}_i)$ of (fundamental, anti-fundamental) and a tree-level superpotential with terms of the Landau-Ginzburg form $\bar{Q}_i \Phi^l Q_j$. The quantum moduli space of these models includes a Coulomb branch. We find hyper-elliptic curves that encode the low energy effective gauge coupling for the groups $SO(N_c)$ and $USp(N_c)$ (the corresponding curve for $SU(N_c)$ is already known). As a consistency check, we derive the sub-space of some vacua with massless dyons via confining phase superpotentials. We also discuss the existence and nature of the non-trivial superconformal points appearing when singularities merge in the Coulomb branch.
1. Introduction and Summary

The introduction of supersymmetry into field-theoretical models extends the range of methods that can be used to analyze them. An important feature implied by supersymmetry is that certain quantities are restricted to depend holomorphically on their arguments \([1]\) (for a review, see for example \([2]\)). This opens the way to the use of complex analysis and often leads to exact determination of these quantities. One such quantity is the low energy effective superpotential. Its exact determination leads to the identification of the quantum moduli space of the model. Another such quantity is the low energy effective gauge coupling in the Coulomb branch of a gauge theory. The determination of its dependence on the moduli leads to important qualitative as well as quantitative information about the low energy behavior: the spectrum of charged states, their masses, points of phase transition, non-trivial IR fixed points etc.

In this work, we determine the effective gauge coupling in the Coulomb branch of a large family of \(N = 1\) models. One approach to this problem is to consider directly the Coulomb branch, and combine the restrictions coming from symmetry and from various limits, to arrive at a unique expression. This method led to the determination of the coupling for all the \(N = 2\) models with a simple classical gauge group and fundamental matter \([3]-[12]\). Alternatively, one can approach the Coulomb branch through the points where it meets the other (Higgs/confinement) branches. These are singular points of the Coulomb branch (the effective coupling vanishes there) and, therefore, this approach provides information about the singularity structure of the Coulomb branch which, because of the holomorphicity, determines the gauge coupling to a large extent. The singular points are determined by the low energy effective superpotential, which in turn can be obtained in many cases by symmetry arguments. An efficient method to obtain the superpotential is the “integrating-in” method \([13, 14]\). This method was applied successfully to the \(N = 2\) models described above, to some of their \(N = 1\) generalizations \([15]-[20]\), and also to other models. Here we combine both approaches. Using the first approach, we determine the effective coupling and then use the second approach as a consistency check in some examples.

We consider an \(N = 1\) supersymmetric gauge theory with a simple classical gauge group \(G\), matter content of one adjoint \(\Phi\) and \(N_f\) pairs \((Q_i, \tilde{Q}_i)\) of (fundamental, anti-fundamental) and a tree level superpotential of the Landau-Ginzburg (LG) form

\[
W_{\text{tree}} = \sum_l \text{tr}(h_l Z_l) , \quad (Z_l)_{ij} = \tilde{Q}_i \Phi^l Q_j ,
\]

where \(Z_l\) are gauge invariant operators, and the sources \(h_l\) are complex matrices\([13, 14]\) for \(SU(N_c)\) such terms were considered in \([21]\). Note that the terms with \(l > 1\) are non-renormalizable and, therefore, the model with such terms should be considered as a low energy effective model, valid only for scales below \(\sim h_{0}^{-1}\). Naively, one would expect that

\footnote{For the \(SO(N_c)\) and \(USp(N_c)\) groups, the fundamental and anti-fundamental representations are equivalent, so there are \(2N_f\) fundamentals and \(Z_l, h_l\) are \((2N_f \times 2N_f)\)-dimensional.}

\footnote{When \(h_0\) is diagonal, \(h_1 = 1\) and \(h_l = 0\) for \(l > 1\), the model has \(N = 2\) supersymmetry.}
such terms will be irrelevant in the IR but in fact this is not true. Indeed, consider such a term as a perturbation of a scale invariant model. The vanishing of the beta-function for the gauge coupling implies

\[ 0 = \beta_g \sim b + \sum k_i \gamma_i = k_A(2 + \gamma_\Phi) + 2N_f k_f[(2 + \gamma_Q) - 3] \]

(where \( k_A \) and \( k_f \) are the Dynkin indices of the adjoint and fundamental representations, \( b = 3k_A - \sum k_i = 2k_A - 2N_f k_f \) is minus the one loop coefficient, and \( \gamma_\Phi, \gamma_Q \) are the anomalous dimensions), therefore, at a fixed point

\[ \text{dim}(\tilde{Q}\Phi Q) = (\gamma_Q + 2) + \frac{2}{l}(\gamma_\phi + 2) = 3 - \frac{2 + \gamma_\phi}{4N_f k_f} b' \quad , \quad b' = 2k_A - 2lN_f k_f \quad (1.2) \]

(note that \( b' \) is related to \( b \) by the replacement \( N_f \to lN_f \)). \((2 + \gamma_\phi)\) is the dimension of a gauge invariant operator \( \text{tr}\Phi^2 \) and, therefore, it is positive by unitarity. Consequently, the perturbation \( \tilde{Q}\Phi^2 Q \) is relevant for \( b' > 0 \), marginal for \( b' = 0 \) and irrelevant for \( b' < 0 \). In the following we will analyze only relevant perturbations \( b' > 0 \) (some remarks about the other cases are presented at the end of this section). This implies that \( b > 0 \), which means that the model is asymptotically free and at high enough scales can be treated semi-classically.

Classically, the model with the superpotential \( \mathcal{W} \) has a moduli space of vacua with Coulomb \((Q = \tilde{Q} = 0)\) and Higgs branches. The Coulomb branch is parameterized by (gauge invariant functions of) the vev \( \langle \Phi \rangle \) of \( \Phi \). \( \langle \Phi \rangle \) generically breaks the gauge symmetry to \( U(1)^r \), where \( r \) is the rank of the gauge group. It also contributes to the mass of the quarks \( Q, \tilde{Q} \) through the superpotential \( \mathcal{W} \).

As to the parameterization of the Coulomb branch, the D-term equations imply that \( \Phi \) can be chosen, by a color rotation, to lie in the (complexified) Cartan sub-algebra of the gauge group. Therefore, given a parameterization \( \{\phi_a\} \) of the Cartan sub-algebra, it is convenient to characterize a point in the moduli space by functions of \( \{\phi_a\} \), invariant under the residual gauge freedom (the Weyl group).

The Coulomb branch vacua \((Q = \tilde{Q} = 0)\) survive also in the quantum theory. However, they can be modified by quantum corrections. In the weak-coupling region (where the gauge invariance is broken at high scale) the classical parameterization is valid, therefore, we will adopt for the whole quantum moduli space the same coordinates \( \{\phi_a\} \) as in the classical moduli space, with the understanding that \( a \text{ priori} \) they have the classical meaning (i.e. the classical relation to \( \langle \Phi \rangle \)) only in the semi-classical region.

At a generic point in the Coulomb moduli space, the low energy effective theory is a \( U(1)^r \) gauge theory with no massless charged matter, and the kinetic term in the effective action is of the form

\[ \frac{1}{16\pi} \Im \int d^4 x d^2 \theta \epsilon^{\alpha\beta} W_\alpha W_\beta. \quad (1.3) \]

\(^3\)The following argument is a generalization of an argument given for \( G = SU(N) \) in [21].

\(^4\)See Appendix A.

\(^5\)As discussed in [21] for \( G = SU(N_c) \), the theory has an anomaly-free R-symmetry with \( R_\Phi = 0 \) and \( R_Q = R_{\tilde{Q}} = 1 \). Therefore, any dynamically generated superpotential is quadratic in \( Q, \tilde{Q} \) and cannot lift the Coulomb branch.
The effective gauge coupling matrix \( \tau_{\text{eff}} \) is a function of the moduli, bare parameters and the microscopical coupling constant (represented, in the asymptotically free cases, by the dynamically generated scale \( \Lambda \)). Supersymmetry implies that it is holomorphic (for the Wilsonian effective action), electric-magnetic duality implies that it is not single valued but rather an \( Sp(2r, \mathbb{Z}) \) section and unitarity implies that it is positive-definite (for a review, see for example [2]). All this suggests that \( \tau_{\text{eff}} \) may be identified as the period matrix of a genus \( r \) Riemann surface, parameterized holomorphically by \( \langle \Phi \rangle, h_l \) and \( \Lambda \).

Moreover, as in [4]-[12], we will assume that this surface is a hyper-elliptic curve, i.e., described by

\[
y^2 = K(x; \phi_a, h_l, \Lambda^b) = k \prod_{l=1}^{2r+\delta} (x - x_l) , \quad \delta = 1 \text{ or } 2 \quad \text{(1.4)}
\]

and that \( K \) depends polynomially also on \( \phi_a, h_l \) and the instanton factor \( \Lambda^b \). For the group \( SU(N_c) \), such a description was found in [21]. We will look for the corresponding description for the other classical groups \( SO(N_c) \) and \( USp(N_c) \).

Before proceeding with the derivation of the curves, let us summarize the results:

- **\( SU(N_c) \), \( N_c = r + 1 \): (\( b = 2N_c - N_f \))
  \[
y^2 = P(x)^2 - \Lambda^b H(x) , \quad P(x) = \prod_{a=1}^{r+1} (x - \phi_a) , \quad H(x) = \det h(x) ;
\]

- **\( SO(N_c) \), \( N_c = 2r + \epsilon, \epsilon = 0, 1 \): (\( b = 2(N_c - 2 - N_f) \))
  \[
y^2 = xP(x)^2 + \Lambda^b x^{3-\epsilon} H(x) , \quad P(x) = \prod_{a=1}^{r} (x - \phi^2_a) , \quad H(x) = \det(h(\sqrt{x})) ;
\]

- **\( USp(N_c) \), \( N_c = 2r \): (\( b = N_c + 2 - N_f \))
  \[
y^2 = (xP(x) + \Lambda^b \text{Pf}(h_0))^2 - \Lambda^{2b} H(x) , \quad P(x) = \prod_{a=1}^{r} (x - \phi^2_a) , \quad H(x) = \det(h(\sqrt{x})) .
\]

In all cases, we denote

\[
h(t) = \sum_{l=0}^{t_{\text{max}}} h_l t^l. \quad \text{(1.8)}
\]

For \( SO(N_c) \) and \( USp(N_c) \), because the fundamental and anti-fundamental representations are equivalent, \( h_l \) is \( (2N_f \times 2N_f) \)-dimensional. For \( SU(2) \), being identical to \( USp(2) \), a distinction between fundamentals and anti-fundamentals is also artificial. Nevertheless,
the $SU(2)$ curve in eq. (1.5) is for the special case, where such a distinction is made, \textit{i.e.}, the $2N_f$ quarks are divided into two subsets – fundamentals and “anti-fundamentals” – and $h_l$ (an $(N_f \times N_f)$-dimensional matrix) couples only quarks belonging to different subsets. The $SU(2)$ curve in eq. (1.7) is for the general case.

For the range of parameters in which the model has $N = 2$ supersymmetry (see footnote \[\footnote{4} \]) our results coincide with the previous results \[\footnote{1}, \footnote{2} \]. Note that these curves are indeed of the form (1.4): in all cases, $H(x)$ is a polynomial in $x$ (for the last two cases, this is due to the symmetry properties of $h$) and in the $USp(N_c)$ case the right-hand side is divisible by $x$. The degree of $y^2$ is either $2r + 2$ (in eq. (1.5)) or $2r + 1$ (in eqs. (1.6), (1.7)) and, therefore, the curve has the correct genus.

These curves were originally derived for $b' > 0$, where $b' = 2k_A - 2l_{max}N_fk_f$. However, \textit{a posteriori}, by breaking the gauge symmetry to a subgroup of the same type (as is done in the subsequent sections), one can attempt to derive the curves for $b' \leq 0$. In these cases, there exists, in particular, a term of the form $\Lambda^b G(h)$ ($G$ being a polynomial in the components of $h_l$) which is invariant under all symmetries and, therefore, it is harder to determine its appearance in the curve (in the process of symmetry breaking, it can enter through the matching conditions). Recall, however, that for $l_{max} > 1$, the model is non-renormalizable, valid only at a sufficiently low scale, so $\Lambda$ (and, therefore, also $\Lambda^b G$) is restricted to be sufficiently small, and one may hope that in this range the dependence on $\Lambda^b G$ is weak enough so that it can be ignored. When this is true, the curves have the above form also for $b' \leq 0$ (where $\Lambda^0$, in the $b = 0$ cases, is understood to represent an appropriate modular function of the microscopic gauge coupling).

For $b' < 0$, there is another complication: the degree of $H(x)$ is so high that the curve has genus greater than required (there are more then $2r + 2$ branch points). In the $N = 2$ case this happens when the model is IR free ($b < 0$) and, therefore, valid at scales small compared to $\Lambda$. As was discussed in \[\footnote{4} \], this also implies that the curve is valid only for $x < \Lambda$ ($x < \Lambda^2$ for $SO(N_c)$ and $USp(N_c)$) and in this region there are exactly the right number of branch points. For $l_{max} > 1$, there also exists a restriction to low scales (even when $b \geq 0$), coming from non-renormalizability so, presumably, also in this case the extra branch points are outside the range of validity of the curve.

The outline of the next sections is as follows. In section 2, we derive the curves for $SO(N_c)$ and $USp(N_c)$, using the first approach described above. In section 3, we derive the singularity equations for the curves of $SU(N_c)$ and $SO(N_c)$, approaching from the Higgs-confinement branch, and compare them with the known curves. In section 4, we discuss the existence and nature of points in the Coulomb branch that correspond to non-trivial superconformal theories. Finally, in two appendices, we present some details.

### 2. Determination of the Curves

\[\footnote{4} G(h) \text{ is obtained by expanding } H(x) \text{ in powers of } x \text{ and taking the appropriate coefficient, which exists if the degree of } H(x) \text{ is high enough, and this happens exactly when } b' \leq 0.\]

\[\footnote{4} \text{The case } l_{max} = 1 \text{ is special, but it is essentially the } N = 2 \text{ case.}\]
We assume that $\tau_{\text{eff}}$ is the period matrix of a genus $r$ hyper-elliptic curve of the form \cite{1.4} and that $K$ depends polynomially also on $\phi_a$, $h_l$ and the one instanton factor $\Lambda^b$. We will consider the expansion of $K$ in powers of $\Lambda^b$:

$$K = \sum_{\alpha=0}^{\alpha_{\text{max}}} \Lambda^{b\alpha} K_\alpha(x; \phi_a, h_l) \quad .$$

(2.1)

This obviously depends on the coordinates $(x, y)$ chosen to describe the curve. The coordinates will be chosen to coincide with those of the $N_f = 0$ model.

The curve should be invariant under the (gauge and global) symmetries of the model (since $\tau_{\text{eff}}$ is). This includes all symmetries of the classical model without a superpotential\footnote{Explicit breaking by the superpotential is accounted for by transforming also the parameters $h_l$ and anomalies are compensated by transforming the instanton factor $\Lambda^b.$}. Three $U(1)$ symmetries are particularly useful in the discussion. Adapting the notation

| $Q$ | $\theta$ | $Q$ | $\phi$ |
|-----|----------|-----|--------|
| $Q_\theta$ | 1 | 0 | 0 |
| $Q_Q$ | 0 | 1 | 0 |
| $Q_\Phi$ | 0 | 0 | 1 |

($\theta$ is the fermionic coordinate of superspace), we define

$$R' = Q_\theta + Q_Q$$

$$R = Q_\theta + 2Q_\Phi$$

$$A = Q_Q$$

(2.2)

($R'$ is the symmetry used in footnote\footnote{Explicit breaking by the superpotential is accounted for by transforming also the parameters $h_l$ and anomalies are compensated by transforming the instanton factor $\Lambda^b.$}). Unlike $R'$ which is a symmetry of the full model, both $R$ and $A$ are anomalous and are also broken explicitly by the superpotential. This means that $\Lambda^b$ and $h_l$ have non-trivial $R$ and $A$ charges and, therefore, their appearance in the curve is restricted by these symmetries. Note also that for all the parameters appearing in the curve, $\frac{4}{r} R$ coincides with their mass dimension.

In the following we consider the classical groups case by case.

### 2.1 $USp(N_c), \ N_c = 2r$

#### 2.1.1 Generalities

The fundamental ($N_c$ dimensional) representation of $USp(N_c)$ consists of unitary matrices $U$ that obey

$$U^T J U = J \quad , \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I_r \quad ,$$

(2.3)

where $I_r$ is the $r$-dimensional identity matrix. This means that the fundamental representation is pseudo-real (i.e., $JQ_i$ transforms under the anti-fundamental representation),
so the superpotential (1.1) can take the more general form

\[ W_{\text{tree}} = \sum_l \text{tr}(h_l Z_l), \quad (Z_l)_{ij} = Q_i^T J \Phi^l Q_j, \]  

(2.4)

where \( h_l \) are now \((2N_f \times 2N_f)\)-dimensional. The relation (2.3) also implies that \( J \Phi \) is a symmetric matrix, therefore,

\[ Z_l^T = (-1)^{l+1} Z_l, \quad h_l^T = (-1)^{l+1} h_l. \]  

(2.5)

The vev \( \langle \Phi \rangle \), after gauge rotation, takes the form

\[ \langle \Phi \rangle = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \otimes \text{diag}\{\phi_a\}, \quad \phi_a \in \mathbb{C}. \]  

(2.6)

The residual gauge freedom is generated by permutations and flips of sign of the \( \{\phi_a\} \), so the gauge invariant combinations are \( \{s_{2k}\} \), defined by the generating function

\[ P(x; \phi_a) = \sum_{k=0}^r s_{2k} x^{r-k} = \prod_{a=1}^r (x - \phi_a^2). \]  

(2.7)

The instanton factor is \( \Lambda^b \), where \( b = 2r + 2 - N_f \).

2.1.2 Determining \( K_0 \)

One can give the \( Q \)'s a large mass and integrate them out:

\[ \text{Pf}(h_0) \to \infty, \quad \Lambda_0^{2r+2} = \text{Pf}(h_0) \Lambda^b, \]  

(2.8)

obtaining the \( N_f = 0 \) model (which has \( N = 2 \) supersymmetry), for which the curve is known [12]:

\[ y^2 = x P^2 + O(\Lambda^b). \]  

(2.9)

We choose the coordinates \((x, y)\) of the curve for the \( N_f \neq 0 \) model to be those of the \( N_f = 0 \) model (i.e., in the above limit we should obtain the \( N_f = 0 \) curve with the same \((x, y)\)). This fixes the charges of \( x \) and \( y \) under the symmetries of the model. The charges of the various quantities appearing in the curve under the \( R \) and \( A \) symmetries are [12]:

| \( \theta \) | \( Q \) | \( \phi \) | \( A^6 \) | \( h_l \) | \( x \) | \( y \) |
|---|---|---|---|---|---|---|
| \( R \) | 1 | 0 | 2 | \( 2b \) | 2(1 - \( l \)) | 4 | 2(2r + 1) |
| \( A \) | 0 | 1 | 0 | \( 2N_f \) | -2 | 0 | 0 |

(2.10)

Using the \( A \)-charge we now observe that \( K_0 \) is independent of \( N_f \) and, therefore, given by the \( N_f = 0 \) curve. Indeed, \( K_\alpha \) must have vanishing \( A \)-charge (since \( x \) and \( y \) do). \( K_0 \) is, by definition, independent of \( \Lambda \) and, therefore (by the presumed polynomial dependence), also independent of \( h_l \). In particular, it is invariant under the limit (2.8).

\[ R'(\Phi) = R'(h_l) = R'(\Lambda^b) = 0 \text{ while } R'(W_{\text{eff}}) = 2, \]  

therefore, as stated in the introduction, an effective superpotential cannot be generated in the Coulomb phase, where \( Q = \bar{Q} = 0 \), and the classical moduli space in the Coulomb branch is not lifted quantum mechanically.
2.1.3 \( USp(2\hat{r}) \to USp(2r) \)

In the curve for \( USp(2\hat{r}) \) we set

\[
\hat{\phi}_a = \begin{cases} 
\phi_a & 1 \leq a \leq r \\
\phi'_a + M & r < a \leq \hat{r}
\end{cases}
\]

(2.11)

and take \( M \to \infty \). Classically, for \( \phi_a = 0 \) (and generic \( \phi'_a \)) the gauge symmetry \( USp(2\hat{r}) \) is broken to \( USp(2r) \times U(1)^{\hat{r}-r} \) and \( \phi_a \) parameterize the Coulomb moduli space of a \( USp(2r) \) model. Therefore, we expect to obtain, in the above limit, the curve of the \( USp(2r) \) model. We must identify the details of this model: the fundamentals \( \hat{Q}_i \) of \( USp(2\hat{r}) \) decompose, each, to one fundamental \( Q_i \) of \( USp(2r) \) (consisting of the first \( 2r \) components of \( \hat{Q}_i \)) and \( USp(2r) \)-singlets; the light components of the adjoint \( \hat{\Phi} \) are the components of an adjoint of \( USp(2r) \) (the upper \( 2r \times 2r \) block), so the \( USp(2r) \) model has the same matter content as the \( USp(2\hat{r}) \) one. The tree level superpotential of \( USp(2\hat{r}) \) reduces to

\[
\hat{W}_{\text{tree}} = \sum_l \text{tr} \hat{h}_l \hat{Q}^T \hat{J} \hat{\Phi} \hat{Q}_l \to \sum_l \text{tr} h_l Q^T J \Phi Q,
\]

which implies \( \hat{h}_l = h_l \). Finally, the matching of scales is

\[
\hat{\Lambda} = \Lambda M^{2n'}, \quad n' = \hat{r} - r
\]

(2.12)

(up to a scheme dependent multiplicative constant that can be absorbed in \( \Lambda \)). The curve for the \( USp(2\hat{r}) \) model is

\[
\hat{y}^2 = \hat{x} \hat{P}^2 + \hat{\Lambda} \hat{K}_1 + \hat{\Lambda} \hat{K}_2 + O(\hat{\Lambda}^3)
\]

with

\[
\hat{P} = \prod_l (\hat{x} - \phi_a^2) \prod_{r+1} (\hat{x} - (\phi_a' + M)^2).
\]

To obtain a curve of the form (2.9) in the limit \( M \to \infty \), one must identify \( \hat{x} = x \), \( \hat{y} = y M^{2n'} \), so for \( x \ll M^2 \) the curve is

\[
y^2 = x P^2 + \Lambda \hat{K}_1 / M^{2n'} + \Lambda \hat{K}_2 + O(M^{2n'})
\]

(2.13)

and \( \hat{K}_\alpha \) depends on \( M \) polynomially, through \( \{ \hat{\phi}_a \} = \{ \phi_a \} \cup \{ \phi'_a + M \} \). Since the limit \( M \to \infty \) is finite, we get the following restriction on the general form (2.1):

- \( \hat{K}_\alpha = 0 \) for \( \alpha > 2 \);
- \( \hat{K}_2 \) is independent of \( \hat{\phi}_a \);

(it is independent of \( \phi'_a \) and, since the dependence on \( \hat{\phi}_a \) is only through symmetric functions of them, there cannot be a dependence on \( \phi_a \) either).

\[^{10}\text{This is true at least in the semiclassical region. If the } USp(2r) \text{ model is asymptotically free, then the limit } M \to \infty \text{ corresponds to vanishing coupling at the matching point, so in this case the relation is exact.}\]
These restriction are carried over also to the $USp(2r)$ curve, so we conclude that the curve is of the form
\[ y^2 = xP(x; φ_a)^2 + Λ^bK_1(x; φ_a, h_l) + Λ^{2b}K_2(x; h_l). \] (2.14)

### 2.1.4 The Semi-Classical Quark Singularities

In the semi-classical region (weak coupling), we can substitute $Φ → ⟨Φ⟩$ from (2.6) in the superpotential (2.4) and obtain the effective mass for the fundamentals

\[ W_{\text{tree}} \rightarrow \sum_a Q_T^a \begin{pmatrix} 0 & h^T(-φ_a) \\ -h^T(φ_a) & 0 \end{pmatrix} Q_a \quad , \quad h(t) = \sum_l h_l t^l \] (2.15)

where $Q_i$ is partitioned to 2-component fields $Q_{ia}$ (according to the block structure of $⟨Φ⟩$) and the summation over flavor indices is implicit. Eq. (2.3) implies that

\[ h(-t) = -h(t)^T, \] (2.16)

therefore,

\[ W_{\text{tree}} \rightarrow Q^T m_{\text{eff}} Q \quad , \quad m_{\text{eff}} = -\text{diag} \left\{ \begin{pmatrix} 0 & h(φ_a) \\ h^T(φ_a) & 0 \end{pmatrix} \right\}. \] (2.17)

Whenever $\det(m_{\text{eff}}) = 0$, there is a massless d.o.f. charged under (at least) one of the $U(1)$ factors of the residual gauge freedom. The effective coupling of this $U(1)$ vanishes in the IR, which means that the curve must be singular in such cases and this is characterized by the vanishing of the discriminant $Δ_K$ of the curve. Since (2.17) is a classical expression, we should compare it with the $Λ → 0$ limit of $Δ_K$, but we cannot just substitute $Λ = 0$ since in this case $Δ_K$ vanishes identically. What we need is the leading term, i.e. :

\[ Δ_K = ε^a Δ^0_K + o(ε^a) \quad , \quad ε = Λ^b \] (2.18)

where $Δ^0_K$ is a polynomial in $φ_a$ and $h_l$, independent of $ε$ but non-trivial. The zeros of $Δ^0_K$ are the $Λ → 0$ limits of the singular points in the moduli space and, therefore, $Δ^0_K$ should vanish whenever $\det(m_{\text{eff}}) = 0$. Since

\[ \det(m_{\text{eff}}) = \prod_a H(φ_a^2)^2 \quad , \quad H(x) = \det(h(\sqrt{x})) \]

(note that, because of (2.14), $\det(h(t)) = \det(-h(t))$, hence $H$ is an even polynomial of $\sqrt{x}$ and, therefore, a polynomial in $x$), the requirement that we obtain is

\[ Δ^0_K |H(φ_a^2) \forall a \] (2.19)

(i.e. $H(φ_a^2)$, as a polynomial in $φ_a^2$, is a factor in $Δ^0_K$).

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11Recall that the discriminant $Δ_K$ is a polynomial in the coefficients of $K$ and, therefore, it is a polynomial in $φ_a$, $h_l$ and $ε$. 

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In the calculation of $\Delta K$, it is enough to consider generic values of the parameters (the rest is determined by continuity). We therefore assume $\Delta_{xP} \neq 0$ (which means that all the $\phi_a$’s are different from each other and from 0) and $H(0) \neq 0$ (which means that all the quarks have a non-trivial bare mass). The calculation is done in Appendix B and here we use the results. First we eliminate a few possibilities:

- $K_1|_{x=\phi_a^2} \neq 0$
  
  (a non-trivial polynomial in $\phi_a^2$ and $h_i$.)

  Eqs. (3.9) and (3.10) give

  $$\Delta_K \sim (\Delta_P)^2 \prod_a (x^3 K_1(x))_{x=\phi_a^2} + O(\epsilon)$$

  (as in the Appendix, “$\sim$” means equality up to a numerical constant), which means that

  $$\Delta^0_K \sim (\Delta_P)^2 \prod_a (x^3 K_1(x))_{x=\phi_a^2}. \quad (2.20)$$

  Eq. (2.19) now implies that $x^3 K_1|H$ (as polynomials in $x$), but the $A$-charge of $\Lambda^b H$ is $-2N_f$, which would imply that $\Lambda^b K_1$ is not $A$-invariant. Therefore this possibility is ruled out.

- $K_1 \equiv 0$
  
  Using eqs. (3.9) and (3.10) again, this time with $\epsilon \to \epsilon^2$, and $K_1 \to K_2$, one obtains

  $$\Delta_K \sim (\Delta_P)^2 \prod_a (x^3 K_2(x))_{x=\phi_a^2} + O(\epsilon).$$

  $K_2(x)$ is a non-trivial polynomial (since otherwise the curve would be singular for any $\phi_a$ and $h_i$) and does not depend on $\phi_a$ (as shown in the previous subsection), therefore, for generic $\phi_a$’s, $K_2(\phi_a^2)$ does not vanish and this means that

  $$\Delta^0_K \sim (\Delta_P)^2 \prod_a (x^3 K_2(x))_{x=\phi_a^2}. \quad (2.21)$$

  Eq. (2.19) now implies that $x^3 K_2|H$ (as polynomials in $x$). $A$-charge conservation implies that $x^3 K_2/H$ is independent of $h_i$ and by $R$-charge conservation we obtain $K_2 = H/x$, which is impossible, since $H(0) = \det(h_0) \neq 0$.

  The only other possibility is that $K_1$ is non-trivial but $K_1|_{x=\phi_a^2}$ is. This means that $K_1 = 2PQ$, where $Q(x; \phi_a^2, h_i)$ is non-trivial. Eq. (3.9) and (3.10) gives, again,

  $$\Delta_K \sim (\Delta_P)^2 P(0)^3 \prod_a K(p_a^I),$$

  \footnote{Note that if this is true for one $a$ it is true for all of them, by symmetry.}
but this time $K_1(p_a) = 0$, so we must use eq.  (B.11), which gives

$$K(p'_a) \sim \left( \frac{\bar{H}}{x} \right)_{x=\phi^2_a} + o(\epsilon^2) , \quad \bar{H} = Q^2 - xK_2 . \tag{2.22}$$

Next we show that $\bar{H}(\phi^2_a)$ is a non-trivial polynomial in $\phi^2_a$. By contradiction, assume that $\bar{H}(\phi^2_a) = 0$. Then

$$Q(x; \phi^2_b) |_{x=\phi^2_a} = \phi^2_aK_2(\phi^2_a) ,$$

which implies that $Q(x)$ is independent of $\phi^2_b$ and, moreover,

$$Q = x\bar{Q} , \quad K_2 = x\bar{Q^2} .$$

But this leads to

$$y^2 = x(P + \epsilon\bar{Q})^2 ,$$

which is singular for any $\phi^2_a$. We therefore conclude that $\bar{H}(\phi^2_a)$ is indeed non-trivial, which implies that

$$\Delta^0_K \sim (\Delta_P)^2 \prod_a (x^2\bar{H}(x))_{x=\phi^2_a} . \tag{2.23}$$

Eq. (2.19) now implies, since $H(0) \neq 0$, that $\bar{H}|H$ (as polynomials in $x$) and the conservation of the $A$ and $R$ charges implies that they are proportional. The proportionality constant can be absorbed in $\epsilon$ and $Q$, leading to the curve

$$xy^2 = (xP + \Lambda^bQ)^2 - \Lambda^{2b}H . \tag{2.24}$$

It remains to determine $Q(x)$. The relation $Q^2 = xK_2 + H$ implies that $Q$ is independent of $\phi^2_a$ (since the right hand side is) and $Q(0) = Pf(h_0)$ (since $Q^2 - H$ is divisible by $x$). Then, symmetry ($SU(2N_f)$, $A$ and $R$) restrict $Q$ to the following form

$$Q(x; h_l) = Pf(\sum_i a_i h_2, x^i) \quad (a_0 = 1) ,$$

where $a_i$ are numerical coefficients. This implies that the degree of $Q(x)$ is at most $\frac{1}{2}l_{\text{max}}N_f$ and, since we restrict ourselves to relevant perturbations $l_{\text{max}}N_f < 2r + 2$, deg$Q$ is smaller than deg$(xP)$. This means that, expanding $Q$ in powers of $x$, all the coefficients, except the constant term, can be absorbed in $P$, by redefining the gauge invariant moduli:

$$s_{2k} \rightarrow s_{2k} - \Lambda^b q_{r-k+1} , \quad \text{where } Q(x) = \sum_k q_k(h_2) x^k . \tag{2.25}$$

To summarize, we have determined the curve to be (1.7).

---

13See footnote 12.
2.2 $SO(N_c)$, $N_c = 2r + \epsilon$, $\epsilon = 0, 1$

2.2.1 Generalities

The fundamental ($N_c$ dimensional) representation of $SO(N_c)$ is real, so the superpotential can take the more general form

$$W_{\text{tree}} = \sum_i \text{tr}(h_i Z_i), \quad (Z_i)_{ij} = Q_i^T \Phi Q_j,$$  \hspace{1cm} (2.26)

where $h_i$ are now $(2N_f \times 2N_f)$-dimensional. $\Phi$ is an anti-symmetric matrix, therefore,

$$Z_i^T = (-1)^i Z_i, \quad h_i^T = (-1)^i h_i.$$  \hspace{1cm} (2.27)

The vev $\langle \Phi \rangle$, after gauge rotation, takes the form

$$\langle \Phi \rangle = i \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \otimes \text{diag}\{\phi_a\}, \quad \phi_a \in \mathbb{C}$$  \hspace{1cm} (2.28)

(for odd $N_c$, there is an additional row and column of zeros). The residual gauge freedom is generated by permutations and flips of sign of the $\{\phi_a\}$, so the gauge invariant combinations are $\{s_{2k}\}$, defined, as for $USp(N_c)$, by

$$P(x; \phi_a) = \sum_{k=0}^{r} s_{2k} x^{r-k} = \prod_{a=1}^{r} (x - \phi_a^2).$$  \hspace{1cm} (2.29)

The instanton factor is $\Lambda^b$ with $b = 2(N_c - 2 - N_f)$. The determination of the curve uses arguments identical to the $USp(N_c)$ case, so we will concentrate on the differences in the details.

2.2.2 Determining $K_0$

$K_0$ is determined, by considering the $N_f = 0$ model, to be

$$y^2 = x P^2 + O(\Lambda^b).$$  \hspace{1cm} (2.30)

The charges of the various quantities appearing in the curve under the $R$ and $A$ symmetries are:

|   | $\theta$ | $Q$ | $\phi$ | $\Lambda^b$ | $h_i$ | $x$ | $y$ |
|---|---------|-----|--------|------------|------|----|----|
| $R$ | 1 | 0 | 2 | $2b$ | $2(1 - l)$ | 4 | $2(2r + 1)$ |
| $A$ | 0 | 1 | 0 | $4N_f$ | $-2$ | 0 | 0 |

\[14\] More precisely, for even $N_c$, the flips of signs come in pairs, and this fact is reflected by the existence of another invariant $\text{Pf}(\Phi)$. However, using symmetry arguments, one can show that the curve can depend only on $\text{Pf}(\Phi)^2$, which is $s_{2r}$. 

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2.2.3 \( \text{SO}(\hat{N}_c) \to \text{SO}(N_c), \hat{N}_c - N_c = 2n' \)

In the curve for the \( \text{SO}(\hat{N}_c) \) model we set

\[
\hat{\phi}_a = \begin{cases} 
\phi_a & 1 \leq a \leq r \\
\phi'_a + M & r < a \leq \hat{r}
\end{cases}
\]  

(2.32)

so, in the limit \( M \to \infty \) we should obtain the curve of the \( \text{SO}(N_c) \) model. As in \( \text{USp}(N_c) \), the matter content is the same, \( \hat{h}_l = h_l \) and the matching of scales is

\[
\hat{\Lambda}^b = \Lambda^b M^{4n'} , \quad n' = \hat{r} - r
\]  

(2.33)

(up to a multiplicative constant that can be absorbed in \( \Lambda \)). The curve for the \( \text{SO}(\hat{N}_c) \) model is

\[
\hat{y}^2 = \hat{x} \hat{P}^2 + \hat{\Lambda}^b \hat{K}_1 + \mathcal{O}(\hat{\Lambda}^{2b})
\]

with

\[
\hat{P}(\hat{x}) = \prod_1^r (\hat{x} - \phi_a^2) \prod_{r+1}^{\hat{r}} (\hat{x} - (\phi'_a + M)^2).
\]

To obtain a curve of the form \((2.30)\) in the limit \( M \to \infty \), one must identify \( \hat{x} = x \), \( \hat{y} = y M^{2n'} \), so for \( x \ll M^2 \) the curve is

\[
y^2 = xP^2 + \Lambda^b K_1 + \mathcal{O}(M^{4n'})
\]

and, since the limit \( M \to 0 \) is finite, the curve must be of the form

\[
y^2 = xP(x; \phi_a)^2 + \Lambda^b K_1(x; h_l).
\]  

(2.34)

2.2.4 The Semi-Classical Quark Singularities

In the semi-classical region (weak coupling), we can substitute \( \Phi \to \langle \Phi \rangle \) from \((2.28)\) in the superpotential \((2.26)\) to obtain the effective mass for the fundamentals. We diagonalize \( \langle \Phi \rangle \) by a \( U(N_c) \) rotation (which is a global symmetry)\(^{16}\)

\[
\langle \Phi \rangle \to U^\dagger \langle \Phi \rangle U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \text{diag}\{\phi_a\} , \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \otimes I_r
\]  

(2.35)

and obtain

\[
W_{\text{tree}} \to \sum_a Q_a^T \begin{pmatrix} h^T(-\phi_a) & 0 \\ 0 & h^T(\phi_a) \end{pmatrix} Q_a + Q^{(2r+1)T} h_0^T Q^{(2r+1)} , \quad h(t) = \sum_l h_l t^l
\]  

(2.36)

\(^{15}\)When the \( \text{SO}(N_c) \) model is asymptotically free, this relation is exact; see the footnote in the corresponding section for \( \text{USp}(N_c) \).

\(^{16}\)As in \((2.28)\), we omit the additional trivial row and column for odd \( N_c \).
where the $U$-transformed $Q_i$ is partitioned to 2-component fields $Q_{ia}$ (according to the block structure of $\langle \Phi \rangle$) and the summation over flavor indices is implicit. $Q^{(2r+1)}$ (which appears only for odd $N_c$) is not charged under the $U(1)^r$ residual gauge freedom and, therefore, has no influence on the effective coupling, so we will ignore it in the following. Eq. (2.27) implies that

$$h(-t) = h(t)^T,$$

therefore,

$$W_{\text{tree}} \to Q^T m_{\text{eff}} Q , \quad m_{\text{eff}} = \text{diag}\left( \begin{pmatrix} h(\phi_a) & 0 \\ 0 & h^T(\phi_a) \end{pmatrix} \right).$$

As for $USp(N_c)$, in the calculation of the discriminant $\Delta_K$, we consider the generic case $\Delta_{xP} \neq 0$ (which means that all the $\phi_a$’s are different from each other and from 0) and $H(0) \neq 0$ (which means that all the quarks have a non-trivial bare mass). $K_1(x)$ is a non-trivial polynomial in $x$ which does not depend on $\phi_a$, therefore, $K_1|_{x=\phi_a^2}$ is a non-trivial polynomial in $\phi_a^2$. This is the first case considered for $USp(N_c)$ and it leads to the requirement $x^3 K_1|_H$ where $H(x) = \det(h(\sqrt{x}))$ (which is a polynomial in $x$). Now the $A$ and $R$ symmetries determine $K_1$ up to a constant, which can be absorbed in $\Lambda$. Therefore, the curve is determined to be (1.6).

### 2.3 $SU(N_c)$

The vev $\langle \Phi \rangle$, after a gauge rotation, takes a diagonal form

$$\langle \Phi \rangle = \text{diag}\{\phi_a\} , \quad \phi_a \in \mathbb{C} , \quad \sum_{a=1}^{N_c} \phi_a = 0.$$ (2.39)

The residual gauge freedom is generated by permutations, so the gauge invariant combinations are $\{s_k\}$, defined by the generating function

$$P(x; \phi_a) = \sum_{k=0}^{N_c} s_k x^{N_c-k} = \det(x - \langle \Phi \rangle) = \prod_{a=1}^{N_c} (x - \phi_a).$$ (2.40)

Using similar considerations as for the previous groups, one finds for this model the curve (1.3).

### 3. Vacua With Massless Dyons

The verification that the curve, conjectured to describe the Coulomb branch of the moduli space, indeed gives the correct singularity structure of the moduli space is a restrictive consistency check. The purpose of this section is to consider these singularities. At a singular point a dyon\(^{17}\) (or dyons) becomes massless. Therefore, such a point is a

\(^{17}\)By “dyon” we mean a state charged electrically or magnetically or both.
transition point between the Coulomb branch and the Higgs-confinement branch and, as such, can be approached from the later \[13, 14\].

To isolate these points, we perturb the theory by adding to \(W_{\text{tree}}\) a function of the Coulomb moduli:

\[
W_{\text{tree}} \to \sum_l \text{tr}(h_l Z_l) + g F(\Phi; \xi).
\] (3.1)

(where \(g, \xi\) are parameters and for each \(\xi\), \(F\) is a gauge invariant function of \(\Phi\); we allow a constant term in \(F\)). As a result, most of the Coulomb vacua will be lifted. In fact, only vacua with massless dyons (that undergo condensation) can survive such a perturbation \[17\]. We will look for such vacua and verify that at these points (in the limit \(g \to 0\)) the curve indeed becomes singular. Classically, these vacua are characterized, among the generic Coulomb vacua, by an enhanced residual gauge symmetry. The perturbation \(F\) will be chosen to admit each such vacuum as a solution (of \(dF = 0\) and, therefore, also of \(dW_{\text{tree}} = 0, Q = \bar{Q} = 0\)) for some value of \(\xi\). For all the models considered, a convenient perturbation is \(F = P(x)|_{x = \xi}\), where \(P(x)\) is the generating function for the gauge invariant moduli (a variant of \(\det(t - \langle \Phi\rangle)\)) – the polynomial appearing in the leading term of the curve \[23, 24\].

The quantum vacua of the Higgs-confinement branch are the solutions of the equations of motion \(dW_{\text{eff}} = 0\), obtained from the effective superpotential \(W_{\text{eff}}\), so one way of obtaining them is to find \(W_{\text{eff}}\) and solve the corresponding equations. However, as considered in \[17\], if one recognizes the low energy effective model in the vacuum of interest and knows \(W_L\) (obtained from \(W_{\text{eff}}\) by integrating out of massive fields by their equations of motion but not taking the limit of infinite masses), it is sometimes simpler to obtain the vevs of the moduli directly from \(W_L\).

We will concentrate on the vacua with \(SU(2) \times U(1)^{r-1}\) gauge freedom (vacua with a larger gauge freedom are always on the boundary of the set of the \(SU(2)\) vacua). Generically, all the \(SU(2)\)-charged matter fields in these vacua are massive and can be integrated out, leading to the \(SU(2)\) \(N = 1\) SYM model with two vacua and an effective superpotential

\[
W_d = \pm 2 \Lambda_d^3.
\] (3.2)

Going back up (“integrating in” \[14\]), \(W_L\) is related to \(W_d\) by

\[
W_L = W_d + W_{\text{cl}} + W_\Delta, \quad W_{\text{cl}} = W_{\text{tree}}|_{\varphi_{\text{cl}}},
\] (3.3)

where \(\varphi\) represents, symbolically, the fields integrated in (in our case these are all the matter fields – \(Q\) and \(\bar{Q}\) are integrated in first and then \(\Phi\)), \(\varphi_{\text{cl}}\) is the classical solution \((dW_{\text{tree}}/d\varphi|_{\varphi_{\text{cl}} = 0})\) and \(W_\Delta\) is a possible additional quantum correction. We will assume \(W_\Delta = 0\). In all the cases considered \(W_{\text{cl}}\) vanishes (because of the constant term in \(F\)), therefore, eq. \(3.3\) reduces to \(W_L = W_d\). The inverse Legendre transform equations of the integrating-in procedure lead to the following equations

\[
F|_{\Phi = \langle \Phi \rangle} = \frac{\partial W_d}{\partial g}, \quad g \frac{\partial F}{\partial \xi}|_{\Phi = \langle \Phi \rangle} = \frac{\partial W_d}{\partial \xi}.
\] (3.4)
$W_L$ is proportional to $g$ (this follows from the conservation of the $U(1)$ charge $R'$, defined in (2.2)), therefore, eqs. (3.4) have the form of “singularity equations” for the function (of $\xi$) $F|_{\Phi=\langle \Phi \rangle} - W_L/g$ (the function and its derivative vanishes).

In the following subsections, we provide the details of the above procedure for the groups $SU(N_c)$ and $SO(N_c)$.

### 3.1 $SU(N_c)$, $N_c > 2$

In this model, symmetry enhancement occurs when some of the $\Phi_a$’s coincide. Each such vacuum is a solution of (3.1) with

$$F(\Phi;\xi) = P|_{x=\xi}$$

for some $\xi$, where $P(x)$ is defined in (2.40). When $n \phi_a$’s coincide, the gauge symmetry is enhanced to $SU(n)$. In particular, the $SU(2)$ vacua are

$$\phi_1 = \phi_2 = \xi \neq \phi_a \quad \forall a > 2.$$  

(3.6)

The matching of scales is \[26\]

$$\Lambda^6_d \sim \frac{\text{Pf}(M_Q)M^2_{\Phi}}{\prod_3^{N_c} (\xi - \phi_a)^2} \Lambda^b,$$

(3.7)

where $M_Q$ and $M_{\Phi}$ are the effective masses of the matter fields charged under the residual $SU(2)$ and the denominator represents the masses of the higgsed gluons. $M_Q$ comes from the original superpotential:

$$\sum_l \text{tr}(h_l Z_l) \rightarrow \text{tr}[h(\xi)(\tilde{q}q)] = \frac{1}{2} \text{tr}[M_Q(Q'^T \epsilon Q')],$$

where

$$h(\xi) = \sum_l h_l \xi^l, \quad Q' = (q, \epsilon \tilde{q}^T), \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and $q, \tilde{q}$ consist of the first (color) components of $Q, \tilde{Q}$, respectively. This gives

$$M_Q \sim \begin{pmatrix} 0 & -h(\xi)^T \\ h(\xi) & 0 \end{pmatrix},$$

(3.8)

which leads to

$$\text{Pf}(M_Q) \sim \text{det}(h(x)) \equiv H(\xi).$$

(3.9)

$M_{\Phi}$ comes from the perturbation $F$ and it is given by \[17, 18\] $M_{\Phi} = g \prod_3^{N_c} (\xi - \phi_a)$. Combining all this, we obtain

$$\Lambda^6_d \sim g^2 H(\xi) \Lambda^b,$$

(3.9)

In the determination of the scale matching, we ignore numerical factors. These depend on conventions and can be fixed consistently at the end. In the following $\sim$ will denote equality up to such factors.
and eqs. (3.4) take the form of singularity equations for the functions

\[ P_\pm(x) = P(x) \pm \sqrt{H(x)} \Lambda b \]  

(3.10)

(where we have chosen a normalization for \( \Lambda \)). The solutions are naturally also solutions for the singularity equations for the polynomial \( K = P_+ P_- \) which is indeed the polynomial of the curve (1.3).

### 3.2 \( SO(N_c) \)

There are two types of symmetry enhancement in this model:

- \( n \phi_a^2 \)'s coincide but do not vanish \( \rightarrow SU(n) \);
- \( n \phi_a \)'s vanish \( \rightarrow SO(2n + \epsilon) \) (where \( N_c = 2r + \epsilon \))

and the superpotential (3.1) with

\[ F(\Phi; \xi) = P_{\mid x=\xi} \quad , \quad P(x) = \prod_a (x - \phi_a^2) \]  

(3.11)

admits each of these vacua as solutions, for an appropriate \( \xi \). Each of the above types contains \( SU(2) \) vacua (the second one – \( SO(3) \) – appears only for \( N_c \) odd).

The \( SU(2) \) vacua of the first type are of the form

\[ \phi_1^2 = \phi_2^2 = t^2 \equiv \xi \neq \phi_a^2 \quad \forall a > 2 \quad , \quad \phi_a \neq 0 \quad \forall a. \]  

(3.12)

The matching of scales is

\[ \Lambda_d^6 \sim \text{Pf}(M_Q) g^2 t^{2(2-\epsilon)} \Lambda^b \]  

(3.13)

\( M_Q \) comes from the original superpotential:

\[ \sum_i \text{tr}(h_i Z_i) \rightarrow Q'^T U^T U \left( \begin{array}{cc} h(-t)^T & 0 \\ 0 & h(t)^T \end{array} \right) Q' \quad , \quad U^T U = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \]

where \( U \) is the rotation (2.35) that diagonalizes \( \langle \phi \rangle \) and \( Q' = (q_1, \epsilon q_2) \) consists of the first (color) components of \( U^T Q \), expressed in terms of two \( SU(2) \)-fundamentals \( q_1, q_2 \). This gives

\[ \sum_i \text{tr}(h_i Z_i) \rightarrow \frac{1}{2} \text{tr}[M_Q(q^T \epsilon q)] \quad , \quad M_Q \sim \left( \begin{array}{cc} 0 & -h(t)^T \\ h(t) & 0 \end{array} \right), \]

(a \( 4N_f \times 4N_f \) matrix) which leads to

\[ \text{Pf}(M_Q) = \det(h(t)) \equiv H(\xi). \]  

(3.14)

The scale matching is, therefore,

\[ \Lambda_d^6 \sim g^2 \xi^{2-\epsilon} H(\xi) \Lambda^b \]
and eqs. (3.4) take the form of singularity equations for the functions

\[ P_\pm(x) \sim P(x) \pm \sqrt{x^2 - \epsilon} H(x) \Lambda^b \] (3.15)

(for an appropriate normalization choice). The solutions are naturally also solutions for the singularity equations for the polynomial \( K = xP_+P_- \) which is indeed the polynomial of the curve (1.3).

### 3.3 Vacua with Embedding Index = 2

The \( SO(2r + 1) \rightarrow SO(3) \) vacua are of the form

\[ \phi_1 = 0 \neq \phi_a \quad \forall a > 1. \] (3.16)

The embedding index of this subgroup in \( SO(2r + 1) \) is 2 and the matching of scales is

\[ \Lambda_3^a \sim \frac{\text{Pf}(M_Q)M_\Phi}{\prod \phi_a^4} \Lambda^b, \] (3.17)

where \( M_Q \) and \( M_\Phi \) are the effective masses of the matter fields charged under the residual \( SO(3) \) (2\( N_f + 1 \) vectors) and the denominator represents the masses of the higgsed gluons. In this case, the one instanton contribution is of the same order as the gaugino condensate and, therefore, must be taken into account; it should lead to the singularities at \( x = s_{2r} = 0 \).

Similarly, for \( USp(2r) \), the \( SU(2) \) vacua of the form

\[ \phi_1^2 = \phi_2^2 = t^2 \equiv \xi \neq \phi_a^2 \quad \forall a > 2, \quad \phi_a \neq 0 \quad \forall a \] (3.18)

also have embedding index = 2, and the matching of scales is

\[ \Lambda_6^a \sim \frac{\text{Pf}(M_Q)g^2}{t^4} \Lambda^b. \] (3.19)

Again, the one instanton contribution is important in this branch. In addition, one should consider the \( USp(2) \) vacua at \( \phi_1 = 0 \); this will not be done here.

### 4. Discussion: Non-Trivial Super-Conformal Fixed Points

For every model discussed in this article, its curve is identical in shape (\( x \) and \( \phi_a \) dependence) to an \( N = 2 \) model of the same type (gauge group and matter representations), with a different number of flavors \( N'_f = \text{deg}(H(x)) \), where the quark masses (or the squares of the masses, for \( SO(N_c) \) and \( USp(N_c) \)) are the roots of \( H(x) \) and the scale \( \Lambda \) is rescaled. For example, for \( \det h_{l_{\text{max}}} \neq 0 \), one obtains \( N'_f = l_{\text{max}}N_f \) and \( \Lambda'^{\nu'} = \Lambda^b \det(h_{l_{\text{max}}}) \) (for \( USp(N_c) \), \( \Lambda'^{\nu'} = \Lambda^b \sqrt{\det(h_{l_{\text{max}}})} \)). This means that calculations performed for the


\( N = 2 \) models, using only the curve, apply directly also to the more general models. One implication of this is that the curve passes automatically some other consistency conditions which were not discussed here but were checked for the \( N = 2 \) cases in previous works (e.g. the monodromy structure).

Another important implication is that the Coulomb branch of these models has many points where mutually non-local dyons are massless \(^{27, 28, 29}\) and presumably correspond to non-trivial (interacting) super-conformal field theories (SCFTs). Moreover, the authors of \(^{28, 29}\) also calculated the critical exponents of perturbations around these points. The ratios of these exponents are determined entirely by the curve\(^{20}\), therefore, the same ratios appear in the \( N = 1 \) models discussed here, and these ratios imply that the above distinguished points in the \( N = 1 \) Coulomb phase correspond to many different SCFTs, corresponding to the universality classes listed in \(^{29}\).

For \( l_{\text{max}} > 1 \), these SCFTs seem to be different from the corresponding \( N = 2 \) SCFTs, the difference being in the number of flavors and, therefore, in the global symmetry (this was observed, for the \( SU(N_c) \) case in \(^{21}\)). If this is true, these are new examples of \( N = 1 \) SCFTs. However, this is not the only possibility\(^{21}\). Indeed, it may happen that at a conformal point, the global symmetry and supersymmetry is enhanced. Evidence for supersymmetry enhancement in some cases appear, for example, in \(^{30, 31}\). Therefore, the SCFTs described here may have \( N = 2 \) supersymmetry. In this case, the (ratios of) scaling dimensions suggest that these are actually the SCFTs of the \( N = 2 \) models corresponding to the same curve. Yet, another possibility is that these are new \( N = 2 \) SCFTs. For example, these may be some variants of the “type E” SCFTs – new fixed points with exceptional symmetries – considered in \(^{32}\). In this case, the present model would provide a field-theoretical description of these SCFTs, which is currently unknown. Any one of these alternatives is interesting and invites further study.

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**Note Added**

As this article was being completed, we received the preprint \(^{33}\) which overlaps parts of section 3.

**Appendix A. The One Instanton Factor**

\(^{20}\)To determine the normalization, they used additional \( N = 2 \) information that is not currently available in \( N = 1 \) models.

\(^{21}\)We are grateful to M.R. Plesser for a discussion on this point.
The one-instanton factor in an asymptotically free model is $\Lambda^b$, where $\Lambda$ is the dynamically generated scale and $b$ is proportional to the one-loop coefficient of the gauge coupling beta function. For an $N = 1$ supersymmetric model with matter (hypermultiplets) in (irreducible) representations $R_i$ of the gauge group, $b$ is given by $b = 3k_A - \sum k_i$, where $k_i$ is the Dynkin index of the representation $R_i$ and, in particular, $k_A$ is the index of the adjoint representation. We summarize in this Appendix the Dynkin indices needed for the calculation of the one-instanton factors for the models considered in the text:

| Group     | $k_f$ | $k_A$ | $b = 2k_A - 2N_fk_f$ |
|-----------|-------|-------|----------------------|
| $SU(N_c)$ | $\frac{1}{2}$ | $N_c$ | $2N_c - N_f$         |
| $SO(N_c)$ | 1     | $N_c - 2$ | $2(N_c - 2 - N_f)$ |
| $USp(2r)$ | $\frac{1}{2}$ | $r + 1$ | $2r + 2 - N_f$ |

Appendix B. Calculation of Discriminants

In this Appendix we present the calculation of discriminants used in the text (extending an approach used in [3]). We begin with some generalities (more information can be found in [34]). The discriminant of a polynomial

$$ P(x) = \sum_{i=0}^{r} P_i x^{r-i} = P_0 \prod_{a=1}^{r} (x - p_a) $$

is defined to be

$$ \Delta_P = P_0^{2r-1} \prod_{a<b} (p_a - p_b)^2. $$

(B.1)

It is useful to define also the resultant of two polynomials

$$ \Delta(P, Q) = P_0^s \prod_{a=1}^{r} Q(p_a) = P_0^Q \prod_{a,b} (p_a - q_b) $$

(B.2)

(Where $s$ is the degree of $Q$). Then the discriminant of the product $PQ$ is\footnote{In this Appendix “$\sim$” will mean “equality up to a non-zero numerical constant”, e.g. a sign or a power of $P_0$.}

$$ \Delta_{PQ} \sim \Delta_P \Delta_Q \Delta(P, Q)^2. $$

(B.3)

Also, since $P'(p_a) = P_0 \prod_{b \neq a} (p_a - p_b)$,

$$ \Delta_P \sim \Delta(P, P') \sim \prod_{b=1}^{r-1} P(q_b), $$

(B.4)

where $\{q_b\}$ are the roots of $Q = P'$ (note that the resultant is symmetric in its arguments, up to a sign).
In the following we consider a polynomial

\[ K(x; \epsilon) = \sum_{\alpha} \epsilon^{\alpha} K_\alpha(x), \quad (B.5) \]

where \( \epsilon \) is a small (non-zero) parameter and \( \deg(K) = \deg(K_0) \). We evaluate the discriminant of \( K \) in the leading order in \( \epsilon \). First we summarize the results:

- **\( K_0 = P^2 \):** \( (P(x) = \prod_{a=1}^{r}(x - p_a), \Delta_P \neq 0) \)

  \[ \Delta_K \sim (\Delta_P)^2 \prod_{a=1}^{r} K(p_a')(1 + o(\epsilon^0)) \quad , \]  
  \[ \text{(B.6)} \]

  where

  \[ K(p_a') = \epsilon K_1(p_a) + o(\epsilon) \]  
  \[ \text{(B.7)} \]

  and if \( K_1(p_a) = 0 \) then

  \[ K(p_a') = \epsilon^2 \left[ K_2 - \left( \frac{K_1}{2P} \right)^2 \right]_{x=p_a} + o(\epsilon^2) \]  
  \[ \text{(B.8)} \]

- **\( K_0 = xP^2 \):** \( (P \) as above and also \( P(0) \neq 0) \)

  \[ \Delta_K \sim (\Delta_P)^2 P(0)^3 \prod_{a=1}^{r} K(p_a')(1 + o(\epsilon^0)) \quad , \]  
  \[ \text{(B.9)} \]

  where

  \[ K(p_a') = \epsilon K_1(p_a) + o(\epsilon) \]  
  \[ \text{(B.10)} \]

  and if \( K_1(p_a) = 0 \) then

  \[ K(p_a') = \epsilon^2 \left[ K_2 - \frac{1}{x} \left( \frac{K_1}{2P} \right)^2 \right]_{x=p_a} + o(\epsilon^2) \]  
  \[ \text{(B.11)} \]

**B.1 \( K_0 = P^2 \)**

Using (B.4) we obtain

\[ \Delta_K \sim \prod_{c} K(x_c) \quad , \]  
\[ \text{(B.12)} \]

where \( \{x_c\} \) are the roots of \( K' \). Since \( K' = 2PP' + O(\epsilon) \),

\[ \{x_c\} = \{p_a'\} \cup \{q_b'\} \quad , \quad p_a' = p_a + o(\epsilon^0) \quad , \quad q_b' = q_b + o(\epsilon^0) \quad , \]

where \( \{p_a'\} \) are the roots of \( P \) and \( \{q_b'\} \) are the roots of \( P' \). Using (B.4) again, we obtain

\[ \prod_{b} K(q_b') \sim \left[ \prod_{b} P(q_b) \right]^2 + o(\epsilon^0) \sim (\Delta_P)^2 + o(\epsilon^0) \]  
\[ \text{(B.13)} \]
which leads to (B.6). To estimate $K(p_a')$ we observe that $x = p_a'$ is a solution of

$$0 = K'(x) = (2R(p_a)^2 + o(\epsilon^0))(x - p_a) + \epsilon(K_1'(p_a) + o(\epsilon^0))$$

(where $P(x) = R(x)(x - p_a)$ and $R(p_a) \neq 0$ since $\Delta_P \neq 0$), therefore,

$$(x - p_a) = \mathcal{O}(\epsilon)$$

and this implies (B.7). If $K_1(p_a) = 0$, eq. (B.7) is not useful and we must refine the estimation. Denoting $K_1(x) = S(x)(x - p_a)^m$, $x = p_a'$ is a solution of

$$0 = K'(x) = (2R(p_a)^2 + o(\epsilon^0))(x - p_a) + \epsilon(x - p_a)^{m-1}(mS(p_a) + o(\epsilon^0)) + \epsilon^2(K_2'(p_a) + o(\epsilon^0)),$$

therefore,

$$(x - p_a) = \begin{cases} -\epsilon \frac{S}{2R^2} \bigg|_{x=p_a} + o(\epsilon) & m = 1 \\ -\epsilon^2 \frac{K_1'}{2R^2} \bigg|_{x=p_a} + o(\epsilon^2) & m > 1 \end{cases} \quad (B.14)$$

and this leads to (B.8).

### B.2 $K_0 = xP^2$

#### B.2.1 $K(0) = 0$

In this case, $K = x\tilde{K}$ and $\tilde{K}_0 = P^2$, therefore we can use the results of the previous subsection:

$$\Delta_{\tilde{K}} \sim (\Delta_P)^2 \prod_a \tilde{K}(p_a')(1 + o(\epsilon^0)) \quad , \quad (B.15)$$

where

$$\tilde{K}(p_a') = \epsilon\tilde{K}_1(p_a) + o(\epsilon) \quad (B.16)$$

and for $\tilde{K}_1(p_a) = 0$

$$\tilde{K}(p_a') = \epsilon^2 \left( \tilde{K}_2 - \left( \frac{\tilde{K}_1}{2P} \right)^2 \right) \bigg|_{x=p_a} + o(\epsilon^2) \quad . \quad (B.17)$$

Multiplying (B.16) and (B.17) by $p_a' = p_a + o(\epsilon^0)$ gives (B.10) and (B.11) respectively. Substituting (B.17) in

$$\Delta_K = \Delta_{\tilde{K}}\tilde{K}(0)^2 = \Delta_{\tilde{K}}(P(0)^4 + o(\epsilon^0))$$

and observing that

$$\prod_a K(p_a') = (P(0) + o(\epsilon^0)) \prod_a \tilde{K}(p_a') \quad ,$$

one obtains (B.3).
B.2.2  \( K(0) \neq 0 \)

In this case \( \tilde{K} = xK \) and \( \tilde{P} = xP \) obey \( \tilde{K}_0 = \tilde{P}^2 \) and we can again use the results of the previous subsection:

\[
\Delta \tilde{K} \sim (\Delta \tilde{P})^2 \tilde{K}(p'_0) \prod_a \tilde{K}(p'_a)(1 + o(\epsilon^0)), \quad (B.18)
\]

where

\[
\tilde{K}(p'_a) = \epsilon \tilde{K}_1(p_a) + o(\epsilon) \quad (B.19)
\]

and for \( \tilde{K}_1(p_a) = 0 \),

\[
\tilde{K}(p'_a) = \epsilon^2 \left( \tilde{K}_2 - \left( \frac{\tilde{K}_1}{2P} \right)^2 \right)_{x=p_a} + o(\epsilon^2). \quad (B.20)
\]

\( x = p'_0 = o(\epsilon^0) \) satisfies

\[
0 = (xK)' = (2P(0)^2 + o(\epsilon^0))x + \epsilon(xQ)' \quad , \quad (B.21)
\]

where we use the notation \( K = P^2 + \epsilon Q \). As \( \epsilon \to 0 \), \( (xQ)'_{x=p'_0} \sim \epsilon^\beta \) for some non-negative \( \beta \), so \( (B.21) \) implies \( p'_0 \sim \epsilon^{\beta+1} \). The terms in \( (xQ)'_{x=p'_0} \) are of the form

\[
(xQ)'_{x=p'_0} \sim \epsilon^\alpha (p'_0)^k \sim \epsilon^{\alpha+(\beta+1)k}, \quad k, \alpha, \beta \geq 0
\]

so for the leading term we must have \( \alpha + (\beta + 1)k = \beta \). But this is possible only for \( k = 0 \) (which also implies \( \alpha = \beta \)). Therefore,

\[
(xQ)'_{x=p'_0} = Q(0)(1 + o(\epsilon^0))
\]

and \( (B.21) \) leads to

\[
p'_0 = -\frac{K(0)}{2P(0)^2}(1 + o(\epsilon^0)) \quad. \quad (B.22)
\]

From \( (B.18) \) we obtain

\[
K(0)^2 \Delta K = \Delta \tilde{K} \sim (\Delta \tilde{P})^2 P(0)^2 p'_0 K(0) \prod_a K(p'_a)(1 + o(\epsilon^0)) \quad ,
\]

which leads, after substituting \( (B.22) \), to \( (B.9) \). Dividing \( (B.19) \) and \( (B.20) \) by \( p'_a = p_a + o(\epsilon^0) \) gives \( (B.10) \) and \( (B.11) \) respectively.

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