FIBONACCI IDENTITIES AND FIBONACCI PAIRS

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Abstract. A Fibonacci pair \( F_s(w, x) \) of rank \( s \) is a pair of \( s \times s \) nonsingular matrices such that \( wx = xw \) and that entries of \( aw^n \) and \( axw^m \) are polynomials of Fibonacci or Lucas numbers for some \( a \neq 0 \). We construct identities systematically by the study of \( F_2(w, x) \) and \( F_3(w, x) \).

1. Introduction

A Fibonacci pair \( F_s(w, x) \) of rank \( s \) is a pair of \( s \times s \) nonsingular matrices such that \( wx = xw \) and that entries of \( aw^n \) and \( axw^m \) are polynomials of Fibonacci or Lucas numbers for some \( a \neq 0 \). Existence of \( F_s(w, x) \) such that \( \langle \bar{w}, \bar{x} \rangle \subseteq GL(s, \mathbb{C})/\langle cI_s : c \neq 0 \rangle \) is not cyclic is guaranteed by the following.

Theorem 1.1. Let \( w = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \( x = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \). Then \( wx = xw, \langle \bar{w}, \bar{x} \rangle \) is not cyclic,

\[
wn = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad \text{and} \quad xw^n = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}.
\]

(1.1)

Fibonacci pairs can be constructed easily (see Proposition 2.1 and 5.1). In this article, we give 2 Fibonacci pairs of rank 2 (subsection 1.1) and 2 pairs of rank 3 (Section 5). Note that these pairs are constructed in such a way that the groups \( \langle \bar{w}, \bar{x} \rangle \subseteq GL(s, \mathbb{C})/\langle cI_s : c \neq 0 \rangle \) are not cyclic. For each pair of rank 2, we construct two types of identities, matrix identities (subsection 1.2 and Section 3) and trace identities (subsection 1.3 and Section 4). The two pairs of rank 2 (see (1.4) and (1.5)) are chosen so that many known identities of 3 terms given by Long [12] can be recovered (subsections 3.3 and 4.4). While matrix identities (M1)-(M5) are known, trace identities (T1)-(T7) seem to be less circulated. To illustrate, we give a trace identity as follows ((T6) of Table 5).

\[
L_{2m}^2 + 5F_{2n+1}^2 + 5F_{2m+2n+1}^2 = 5L_{2m}F_{2n+1}F_{2m+2n+1} + 4.
\]

(1.2)

See Section 5 for pairs of rank 3 and their matrix identities. In the case \( z \) is singular, the pair \( S(z, v) \) such that \( vz = vz \) gives interesting identities as well. (1.3) of the following is such an example (see (v) of Section 7).

\[
F_nL_{m+1} + F_{n-1}L_{m-1} + F_{n+1}L_m = L_{n+m+1}.
\]

(1.3)

1.1. Fibonacci pairs of rank \( s \). The first pairs of rank two in this article is \( a = F_r \) and

\[
w = F_r^{-1} \begin{pmatrix} F_{r+1} \\ (-1)^{r+1}F_1 \end{pmatrix}, \quad x = F_r^{-1} \begin{pmatrix} L_r \\ (-1)^{r+1}L_0 \end{pmatrix}.
\]

(1.4)

Note that the trace of \( x \) is 0. Note also that the Fibonacci pair in Theorem 1.1 is a special case of (1.4). The second pair is

\[
a = 2, \quad w = 2^{-1} \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}, \quad x = 2^{-1} \begin{pmatrix} 0 & 10 \\ 2 & 0 \end{pmatrix}.
\]

(1.5)

The detailed construction of the above pairs of rank 2 can be found in Section 3. See Sections 5 and 6 for the construction of \( F_s(w, x) \), where \( s \geq 3 \).
1.2. Matrix identities of Fibonacci pairs of rank \( s \). Let \( F_s(w, x) \) be a Fibonacci pair of rank \( s \) and let \( M, N \in \Delta = \{ w^a, xw^b \} \). The product \( MN \) gives an equation. Take the pair \((M, N) = (w^m, xw^n)\) for instance, the product \( MN \) gives the following equation.

\[
(w^m)(xw^n) = wx^{m+n}. \quad (1.6)
\]

We call equation (1.6) the matrix equation of \((w^m, xw^n)\). Such equation actually gives identities of Fibonacci and Lucas numbers. For instance, if \((w^n, xw^m)\) is given as in Theorem 1.1, then equation \(w^n(xw^m) = xw^{m+n}\) gives the identity

\[
F_{n+1}L_{m+1} + F_nL_m = L_{n+m+1}. \quad (1.7)
\]

To be more precise, (1.1) entry of the following equation after the product \((w^n)(xw^m)\) is simplified gives the identity \(F_{n+1}L_{m+1} + F_nL_m = L_{n+m+1}\).

\[
w^n(xw^m) = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} L_{m+1} & L_m \\ L_m & L_{m-1} \end{pmatrix} = \begin{pmatrix} L_{m+n+1} & L_{m+n} \\ L_{m+n} & L_{m+n-1} \end{pmatrix} = wx^{m+n}. \quad (1.8)
\]

Identity (1.7) is called a matrix identity of \((w^n, xw^m)\). Since \(\Delta\) consists of 2 members, there are 3 matrix equations given as follows.

| Table 1 : Matrix equations of \( F_s(w, x) \) |
|---|---|
| (M, N) | Equations associated with \((M, N)\) |
| \((w^n, w^m)\) | \(w^nw^m = w^{m+n}\) |
| \((w^n, xw^m)\) | \(w^n(xw^m) = xw^{m+n}\) |
| \((xw^n, xw^m)\) | \((xw^n)(xw^m) = x^2w^{m+n}\) |

Matrix identities (such as (1.7)) coming from Table 1 are given in Sections 3 and 5. They are listed as (M1)-(M5) and (N1)-(N5).

1.3. Trace identities of Fibonacci pairs of rank 2. Let \( M \) and \( N \) be given as in subsection 1.2. Set \( A = M/\sqrt{d(M)} \), \( B = N/\sqrt{d(N)} \), where \( d(X) \) is the determinant of \( X \). In the case \( M \) and \( N \) are \(2 \times 2\) matrix, since \( A \) and \( B \) commute with each other, we can show in Section 4 that \( t(AB) + t(A^{-1}B) = t(A)t(B) + t(BAB) + t(A) = t(B)t(AB) \) and

\[
t(A)^2 + t(B)^2 + t(AB)^2 = t(A)t(B)t(AB) + 4, \quad (1.9)
\]

where \( t(X) \) is the trace of \( X \). These 3 identities are called the trace identities of \((M, N)\). See Section 4 for trace identities of \( F_2(w, x) \). They are listed as (T1)-(T11). The following is a combination of a trace identity ((T1) of Table 5) and a matrix identity ((M4) of Section 3).

\[
L_{2n}^2 + L_{2m}^2 + 5F_{2m+2n}^2 = L_{2n}L_{2m}L_{2m+2n}. \quad (1.10)
\]

1.4. Discussion. Our goal is to relate identities to matrices. We are far from being done. For instance, we are unable to construct the identity \( F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3 \) directly by Fibonacci pairs. The closest we can get is \((N5)\) of Table 7

\[
3F_{n-1}^3 - F_n^3 + F_{n+1}^3 + F_{n+1}F_nL_{n-1} = 2L_nF_{2n-1}. \quad (1.11)
\]

An advantage of our method is that one does not have to verify the truth of the identities as they come automatically from the multiplication of matrix equations (see (1.8)). Another advantage is that Fibonacci pairs can be constructed easily (Propositions 2.1 and 5.1).
1.5. **Notations and organisation.** Section 2 explains how $w$ and $x$ are constructed for $F_2(w, x)$. Section 3 gives two Fibonacci pairs of rank 2 and their matrix identities. It is shown that identities (13)-(24) of Long [12] are matrix identities (subsection 3.3). Section 4 studies trace identities of $F_2(w, x)$. It is proved that identities (5)-(9) and (11) of Hoggatt, Jr. and Bergum [7] are trace identities (subsection 4.4). Sections 5 and 6 study $F_s(w, x)$, where $s \geq 3$. $F_n$ is the $n$-th Fibonacci number and $L_m$ is the $m$-th Lucas numbers. Recall that

$$F_0 = 0, F_1 = 1, F_{-r} = (-1)^{r+1} F_r, \quad L_0 = 2, L_1 = 1, L_{-r} = (-1)^r L_r.$$  \hfill (1.12)

The equations in Table 1 are called the *matrix equations* of $F_s(w, x)$ and the identities (such as (1.7)) are called the *matrix identities* of $F_s(w, x)$.

1.6. **Historical background.** Matrices have been used to construct Fibonacci and Lucas identities. See [2]-[5], [9] [13], [14] and [15] for some detailed investigation. A pair of matrices similar to (3.4) has been studied by Prasanta Kumar Ray [13]. To the best of our knowledge, our systematic study of Fibonacci pair $F_2(w, x)$ (such that $(w, x) \in GL(s, \mathbb{C})/\langle cI_s : c \neq 0 \rangle$ is not cyclic) is not in the literature yet.

### 2. Fibonacci Pairs of Rank 2

We give detailed construction of Fibonacci pairs of rank 2 in Proposition 2.1. See Sections 5 and 6 for Fibonacci pairs of ranks 3 or more.

2.1. Let $F_2(w, x)$ be given as in subsection 1.1. Direct calculation shows that the characteristic polynomials of $w$ and $x$ are $X^2 - X - 1$ and $X^2 - 5$ respectively. As a consequence,

$$w^n = F_n w + F_{n-1} I_2.$$  \hfill (2.1)

2.2. The existence of a Fibonacci pair of rank 2 is assured by the following proposition.

**Proposition 2.1.** Let $w$ be a rational matrix whose characteristic polynomial is $X^2 - X - 1$ and let $a, b$ be rational numbers. Then $w^n = F_n w + F_{n-1} I_2$ and $wx = xw$ where $x = aw + b I_2$.

Proposition 2.1 can be proved by induction. Further, the proposition indicates that the construction of a Fibonacci pair $F_2(w, x)$ is easy. For instance, $F_2(w_d, x(a, b))$ is a Fibonacci pair, where $w_d = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $x(a, b) = aw_d + b I_2$. In general, one can always take $w$ to be a matrix similar to the rational canonical form of $X^2 - X - 1$ over $\mathbb{Q}$. To finish the subsection, we give a simple example as follows. Set $w = w_{-2}$. By (2.1), one has

$$w^n = \begin{pmatrix} L_{n+1} & F_n \\ -5F_n & -L_{n-1} \end{pmatrix}, \quad wx = xw$$

where $x = x(1, 2) = w + 2I_2,$  \hfill (2.2)

Hence $F_2(w, x)$ is a Fibonacci pair of rank 2. Note that (2.2) is different from (1.4) and (1.5). Note also that (1.4) and (1.5) are special cases of Proposition 2.1.

2.3. Let $F_2(w, x)$ be given as in (1.4) and (1.5). The set of all matrices commute with $w$ is a vector space of dimension 2 over $\mathbb{C}$. $\{w, x\}$ is a basis of $V$. Let $F_2(w, y)$ be another Fibonacci pair. Then $y = aw + bz$ for some $a, b \in \mathbb{Q}$. Consequently, matrix equations of $F_2(w, y)$ are just combinations of matrix equations of $F_2(w, x)$. In conclusion, $F_2(w, x)$ gives all the matrix equations and identities. Note that $x$ has trace 0 and $x^2$ is a scalar matrix. This simplifies the calculation of the last matrix equation of Table 1. Connections between trace identities (see (1.9)) of $F_2(w, x)$ and $F_2(w, y)$ are quite involved (see subsection 4.4).

Let $F_2(w, x)$ be a Fibonacci pair. See (vi) of Section 7 if $x^2$ is not a scalar matrix.
3. Matrix identities of Fibonacci pairs of rank 2

We shall construct 2 Fibonacci pairs of rank 2 and 5 matrix identities (M1)-(M5). We will see in subsection 3.3 that identities (13)-(24) of Long [12] are consequences of (M1)-(M5).

3.1. Matrix identities (M1)-(M3). Set \( a = F_r \),

\[
 w = F_r^{-1} \begin{pmatrix} F_{r+1} & F_1 \\ (-1)^{r+1}F_1 & (-1)^{r+1}F_{r-1} \end{pmatrix}, \quad x = F_r^{-1} \begin{pmatrix} L_r & L_0 \\ (-1)^{r+1}L_0 & (-1)^{r+1}L_{r-1} \end{pmatrix}. \quad (3.1)
\]

It is clear that \( wx = wx \), trace of \( x \) is 0 and \( x^2 \) is a scalar matrix. By (2.1) and the known identity \( F_{n+r} = F_nF_{r+1} + F_{n-1}F_r \), one has

\[
 w^n = F_r^{-1} \begin{pmatrix} F_{n+r} & F_n \\ (-1)^{r+1}F_n & (-1)^{r+1}F_{n-r} \end{pmatrix}, \quad xw^n = F_r^{-1} \begin{pmatrix} L_{n+r} & L_n \\ (-1)^{r+1}L_n & (-1)^{r+1}L_{n-r} \end{pmatrix}. \quad (3.2)
\]

Hence \( F_2(w,x) \) in (3.1) is a Fibonacci pair. Note that \( w \) and \( w^n \) are closely related to \( F_{n+r} F_{n-r} = (-1)^{n-r} F_2 \). Applying the technique we presented in subsection 1.2 (see (1.7) of subsection 1.2), we obtain 3 matrix identities. They are direct consequences of (3.2) and matrix multiplications (see subsection 1.4 and Table 1). We tabulate our results in the following table.

| \((M,N)\) | Matrix identity associated with \((M,N)\) | Name |
|---|---|---|
| \((w^n,w^m)\) | \(F_{n+r}F_{m+r} + (-1)^{r+1}F_nF_m = F_rF_{n+m+r}\) | (M1) |
| \((w^n,xw^m)\) | \(F_{n+r}L_{m+r} + (-1)^{r+1}F_nL_m = F_rL_{n+m+r}\) | (M2) |
| \((xw^n,xw^m)\) | \(L_{n+r}L_{m+r} + (-1)^{r+1}L_nL_m = 5F_rF_{n+m+r}\) | (M3) |

The table is read as follows. Let \((M,N)\) be given as in the first column. Then one can get an identity given as in the second column by considering the multiplication of \( M \) and \( N \). Take the first row for instance, one has \((w^n)(w^m) = w^{n+m}\) and (1,1) entry of this equation gives \( F_{n+r}F_{m+r} + (-1)^{r+1}F_nF_m = F_rF_{n+m+r}\).

**Discussion.** (i) Recall that \( L_a = (-1)^aL_a \) and \( F_{-a} = (-1)^{a+1}F_a \).

(iii) Since \( wx = wx \) and the characteristic polynomials of \( w \) and \( x \) are \( X^2 - X - 1 \) and \( X^2 - 5 \), the traces of \( w^n \) and \( xw^n \) are \( L_n \) and \( 5F_n \) respectively. Since traces of \( w^n \) and \( xw^n \) can be calculated by (3.2) also, one has

\[
 F_{n+r} + (-1)^{r+1}F_{n-r} = F_rL_n \quad \text{and} \quad L_{n+r} + (-1)^{r+1}L_{n-r} = 5F_rF_n. \quad (3.3)
\]

3.2. Matrix identities (M4) and (M5). To finish our study of Section 3, we consider

\[
 a = 2, \quad w = 2^{-1} \begin{pmatrix} 1 & 5 \\ 1 & 1 \end{pmatrix}, \quad x = 2^{-1} \begin{pmatrix} 0 & 10 \\ 2 & 0 \end{pmatrix}. \quad (3.4)
\]

It is clear that \( xw = wx \) and \( x^2 \) is a scalar matrix. By (2.1), \( w^n \) and \( xw^n \) are given as follows.

\[
 w^n = 2^{-1} \begin{pmatrix} L_n & 5F_n \\ F_n & L_n \end{pmatrix}, \quad xw^n = 2^{-1} \begin{pmatrix} 5F_n & 5L_n \\ L_n & 5F_n \end{pmatrix}. \quad (3.5)
\]

Hence \( F_2(w,x) \) in (3.4) is a Fibonacci pair. Note that the idea of the construction of \( w \) comes from the fact that \( L^2 - 5F^2 = 4(-1)^n \). Since the first matrix equation and the last matrix
equation (see Table 1) give the same matrix identities when $F_2(w, x)$ is given as in (3.4), we list 2 rather than 3 identities. They are direct consequences of (3.5) and matrix multiplications (see subsection 1.4 and Table 1).

Table 3: Matrix identities for $F_2(w, x)$ given as in (3.4)

| $(M, N)$ | Matrix identity associated with $(M, N)$ | Name |
|---------|----------------------------------------|------|
| $(w^n, w^m)$ | $L_nL_m + 5F_nF_m = 2L_{m+n}$ | (M4) |
| $(w^n, xw^m)$ | $L_nF_m + F_nL_m = 2F_{m+n}$ | (M5) |

3.3. Discussion. In [12], Long exhibited 17 identities (labeled as (13)-(24)) by studying $L_n^2 - 5F_n^2 = 4(-1)^n$. The main theme of this subsection is to show that

Proposition 3.1. Identities (13)-(24) of Long [12] are matrix identities or combination of matrix identities (M1)-(M5).

Proof. One sees that 8 of the 17 identities are listed in (M1)-(M5). They are (14 even), (15 even), (16 even), (18), (19), (21), (22), and (23). The remaining 9 identities are combinations of identities (M1)-(M5). Take (24) and (20) of Long [11] for examples. The identities are

$$F_nL_m - L_{n-d}F_{m+d} = (-1)^{n+1}L_dF_{m-n+d}, L_mL_n - 5F_{n-d}F_{m+d} = (-1)^n L_dL_{m-n+d}. \quad (3.6)$$

The first identity is a combination of (M2), and (M5) and the second is a combination of (M1) and (M4). The other 7 identities can be obtained similarly.

4. Trace identities of Fibonacci pairs of rank 2

We will prove three well known trace identities ((4.2), (4.5A) and (4.5B)) and apply such identities to get trace identities of Fibonacci and Lucas numbers.

4.1. Trace function. A matrix $X$ is unimodular if its determinant $d(X)$ equals 1. Let $A$ and $B$ be $2 \times 2$ unimodular matrices. Then

$$t(A)^2 + t(B)^2 + t(AB)^2 = t(A)t(B)t(AB) + t(ABA^{-1}B^{-1}) + 2, \quad (4.1)$$

where $t(X)$ is the trace of $X$. Note that Identity (4.1) holds only if $A$ and $B$ are $2 \times 2$ matrices. In the case $A$ and $B$ commute with each other, one has

Proposition 4.1. Let $A$ and $B$ be $2 \times 2$ unimodular matrices such that $AB = BA$. Then

$$t(A)^2 + t(B)^2 + t(AB)^2 = t(A)t(B)t(AB) + 4, \quad (4.2)$$

Proof. Since $AB = BA$, $A$ and $B$ can be simultaneously upper-triangulated. Since the trace of a matrix is invariant under the change of basis, we may assume that

$$A = \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \quad \text{and that} \quad B = \begin{pmatrix} b & * \\ 0 & b^{-1} \end{pmatrix}. \quad (4.3)$$

Identity (4.2) now follows by straightforward calculation. Note that $t(ABA^{-1}B^{-1}) = 2$. □
4.2. **Trace identities (T1)-(T7).** Let \( F_2(w, x) \) be given as in (3.1) or (3.4) and let \( M, N \in \{ w^n, xw^m \} \). We may apply (4.2) to \( A = M/\sqrt{d(M)} \) and \( B = N/\sqrt{d(N)} \) to get trace identities. As such identities have great resemblance, we present 7 such identities only in the following two tables.

| \( (M, N) \) | Trace identity (4.2) associated with \( (M, N) \) | Name |
|------------|--------------------------------------------------|------|
| \( (w^{2n}, w^{2m}) \) | \( L_{2n}^2 + L_{2m}^2 + L_{2m+2n}^2 = L_{2n}L_{2m}L_{2m+2n} + 4 \) | (T1) |
| \( (wx^{2n}, wx^{2m}) \) | \(-5F_{2n}^2 - 5F_{2m}^2 + L_{2m+2n}^2 = 5F_{2n}F_{2m}L_{2m+2n} + 4 \) | (T2) |
| \( (wx^{2n}, wx^{2m+1}) \) | \(-5F_{2n}^2 + 5F_{2m+1}^2 - L_{2m+2n+1}^2 = -5F_{2n}F_{2m+1}L_{2m+2n+1} + 4 \) | (T3) |
| \( (wx^{2n+1}, wx^{2m+1}) \) | \(5F_{2n+1}^2 + 5F_{2m+1}^2 + L_{2m+2n+2}^2 = 5F_{2n+1}F_{2m+1}L_{2m+2n+2} + 4 \) | (T4) |

Note that (T1) is a special case of the trace identity for \( (w^n, w^m) \) given as follows.

\[( -1)^n F_n^2 + (-1)^m L_m^2 + (-1)^{m+n} L_{m+n}^2 = (-1)^{n+m} L_nL_mL_{m+n} + 4. \] (4.4)

Note also that trace identities for \( (w^n, xw^m) \) and \( (wx^n, xw^m) \) can be obtained also. However, we do not include them here as such identities take more space to present.

| \( (M, N) \) | Trace identity (4.2) associated with \( (M, N) \) | Name |
|------------|--------------------------------------------------|------|
| \( (w^{2n}, w^{2m+1}) \) | \( L_{2n}^2 - 5F_{2m}^2 - 5F_{2m+2n}^2 = -5L_{2n}F_{2m}F_{2m+2n} + 4 \) | (T5) |
| \( (w^{2n+1}, wx^{2m}) \) | \( L_{2n}^2 + 5F_{2m+1}^2 + 5F_{2m+2n+1}^2 = 5L_{2n}F_{2m+1}F_{2m+2n+1} + 4 \) | (T6) |
| \( (w^{2n+1}, wx^{2m+1}) \) | \(-L_{2n+1}^2 - 5F_{2m+1}^2 + 5F_{2m+2n+1}^2 = 5L_{2n+1}F_{2m+1}F_{2m+2n+1} + 4 \) | (T7) |

4.3. **Trace identities (T8)-(T11).** Let \( A \) and \( B \) be two by two unimodular matrices. Then

\[( t(AB) + t(A^{-1}B) = t(A)t(B) \quad (4.5A) \quad \text{and} \quad t(BAB) + t(A) = t(B)t(AB). \quad (4.5B) \]

Note that (4.5A) and (4.5B) can be proved easily if \( AB = BA \) (see the proof of Proposition 4.1). Let \( (M, N) \) be given as in subsection 4.2. We may apply (4.5A) and (4.5B) to \( A = M/\sqrt{d(M)} \) and \( B = N/\sqrt{d(N)} \) to get identities. As such identities have great resemblance, we present 4 such identities only in the following table.

| \( (M, N) \) | Identity (4.5B) for \( (M, N) \) | Name | Identity (4.5A) for \( (M, N) \) | Name |
|------------|---------------------------------|------|---------------------------------|------|
| \( (w^n, w^m) \) | \( L_{2m+n} + (-1)^m L_n = L_mL_{m+n} \) | (T8) | \( L_{m+n} + (-1)^n L_{m-n} = L_mL_n \) | (T9) |
| \( (wx^n, w^m) \) | \( F_{2m+n} + (-1)^m F_n = L_mF_{m+n} \) | (T10) | \( F_{m+n} + (-1)^n F_{m-n} = L_mF_n \) | (T11) |
4.4. **Discussion.** (i) Identities (5)-(9) and (11) of Hoggatt, Jr. and Bergum [7] are trace identities (see Table 6). (ii) Let \( F_2(w, x) \) be given as in (3.1) and let \( y = w + x \). Identity (4.2) for \((M, N) = (w^n, yw^m)\) where \( m \) and \( n \) are even takes the following form

\[
-11L_n^2 + (L_{m+1} + 5F_m)^2 + (L_{m+n+1} + 5F_{m+n})^2 = L_n(L_{m+1} + 5F_m)(L_{m+n+1} + 5F_{m+n}) - 44. \tag{4.6}
\]

Identity (4.6) is a combination of \(11L_n^2 - 11L_{m+1} - 11L_{m+n+1} = -11L_nL_{m+1}L_{m+n+1} + 44 \) (see (4.4)) and a simple but slightly lengthy identity. This is true in general. We leave it to the readers to determine whether one should claim that trace identities of \( F_2(w, y) \) can be obtained from trace identities of \( F_2(w, x) \). We do not work on trace identities associated with \( F_2(w, y) \) \((y = aw + bx)\) as such identities are lengthy (see (4.6) for instance) before simplification.

5. **Fibonacci pairs of rank 3**

We give two Fibonacci pairs \( F_3(w, x) \) and \( F_3(z, v) \) of rank 3. The choice of \( w \) is closely related to the Pascal’s triangle and \( z \) is the rational canonical form of a polynomial.

5.1. **Fibonacci pairs of rank 3.** The existence of \( F_3(w, x) \) is guaranteed by the following proposition which can be proved by induction.

**Proposition 5.1.** Let \( w \) be a \( 3 \times 3 \) rational matrix whose characteristic polynomial is \( f(x) = X^3 - 2X^2 - 2X + 1 \) and let \( a, b, c \) be rational numbers. Then

\[
w^n = F_nF_{n-1}w^2 + F_nF_{n-2}w - F_{n-1}F_{n-2}I_3 \quad \text{and} \quad wx = xw \quad \text{where} \quad x = aw^2 + bw + cI_3. \tag{5.1}
\]

Note that \( f(x) \) is the second auxiliary polynomial of the Fibonacci sequence (see [2], [12]).

**Example 5.2.** Let \( w, x, z, \) and \( v \) be three by three matrices given as follows.

\[
w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \quad v = \begin{pmatrix} -2 & 0 & -1 \\ 1 & -2 & 2 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.2}
\]

Note that \( w \) and \( z \) have the same characteristic polynomial \( X^3 - 2X^2 - 2X + 1 \), the characteristic polynomial of \( x \) is \( (X - 1)(X^2 - 2X - 4) \) and \( z \) is the rational canonical form of \( X^3 - 2X^2 - 2X + 1 \). Note also that \( wz = xw \) and that \( zv = vz \). Applying (5.1), \( w^n \) can be calculated easily.

\[
w^n = \begin{pmatrix} F_n^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n-1}^2 + F_nF_{n+1} & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}, \quad t(w^n) = 2F_{n-1}^2 + F_nF_{n+1} + F_{n+1}^2. \tag{5.3}
\]

Recall that \( t(X) \) is the trace of \( X \). The matrix \( xw^n \) can be calculated easily as well (see (5.4)). As a consequence, \( F_3(w, x) \) is a Fibonacci pair of rank 3.

\[
xw^n = \begin{pmatrix} 2F_{n-1}F_n & F_{n-1}^2 + F_nF_{n+1} & 2F_nF_{n+1} \\ 2F_{n-1}F_n + F_n^2 + F_{n+1}^2 & F_{n-1}^2 + 2F_{n-1}F_n + 3F_nF_{n+1} & F_n^2 + F_{n+1}^2 + F_{n+2}^2 \\ 2F_nF_{n+1} & F_{n+1}^2 - F_nF_{n+1} & 2F_{n+1}F_{n+2} \end{pmatrix}. \tag{5.4}
\]

It is easy to see that trace of \( xw^n \) is \( F_{n-1}^2 + F_{n+1}F_{n+3} + 4F_{2n} \). It is also easy to see that \( F_3(z, v) \) is a Fibonacci pair of rank 3 as well (see (5.6) for \( z^n \)).
5.2. Matrix identities. Equations such as \((w^n)(w^m) = w^{n+m}\), \((w^n)(xw^m) = xw^{n+m}\) and \((xw^n)(xw^m) = x^2w^{n+m}\) give matrix identities. The following table gives 5 of them.

| \((M, N)\) | entry \((r, s)\) | Matrix identity associated with entry \((r, s)\) | Name |
|------------|------------------|---------------------------------------------|------|
| \((w, xw^n)\) | (1, 2) | \(F^2_{n+2} - F^2_n = F_{2n+2}\) | (N1) |
| \((w, xw^n)\) | (2, 1) | \(F^2_{n+2} - F^2_{n-1} = 4F_nF_{n+1}\) | (N2) |
| \((w, xw^n)\) | (2, 2) | \(F^2_{n+1} + F^2_{n+2} = F_nL_{n-1} + F_{n+2}L_n\) | (N3) |
| \((w, xw^n)\) | (3, 2) | \(F^2_{n+3} - F^2_{n+2} = 2F_{2n+2} + 2F_{2n}\) | (N4) |
| \((w^n, xw^n)\) | (1, 1) | \(3F^3_{n-1} - F^3_n + F^3_{n+1} + 2F_{n-1}L_{n-1}L_n = 2L_nF_{2n-1}\) | (N5) |

The first row is read as follows. (1,2) entry of the matrix equation \(w(xw^n) = xw^{n+1}\) gives the identity \(F^2_{n+2} - F^2_n = F_{2n+2}\). The remaining 4 rows can be interpreted similarly. Combination of (N1) and (N4) gives \(F^2_{n+3} - 2F^2_{n+2} - 2F^2_{n+1} + F^2_n = 0\) (Hoggatt, Jr. and Bicknell [6]).

5.3. Trace identities. We shall first prove the following lemma about traces.

**Lemma 5.3.** Suppose that the characteristic polynomial of \(A\) is \(X^3 - 2X^2 - 2X + 1\) and that \(A^n = (a_{ij})\). Then \(a_{11} + a_{22} + a_{33} = L_{2n} + (-1)^n\). Note that \(L_{2n} + (-1)^n = F_{3n}/F_n\).

**Proof.** Eigenvalues of \(A\) are \(-1, \tau = (3+\sqrt{5})/2\) and \(\sigma = (3-\sqrt{5})/2\). Consequently, eigenvalues of \(A^n\) are \((-1)^n\), \(\tau^n\) and \(\sigma^n\). Since \(\tau^n + \sigma^n = L_{2n}\), trace of \(A^n = (a_{ij})\) is \(L_{2n} + (-1)^n\). Hence \(a_{11} + a_{22} + a_{33} = L_{2n} + (-1)^n\). \(\square\)

**Example 5.4.** We give two trace identities, one for \(w^n\) and one for \(z^n\). Characteristic polynomial of \(w\) is \(X^3 - 2X^2 - 2X + 1\). By Lemma 5.3 and (5.3), one has

\[L_{2n} + (-1)^n = 2F^2_{n+1} + F_nF_{n+1} + F^2_{n+1} = 2F^2_{n-1} + F_{n+1}F_{n+2}.\]  

Since the characteristic polynomial of \(z\) is \(X^3 - 2X^2 - 2X + 1\), By (5.1), \(z^n\) is given as follows.

\[z^n = \begin{pmatrix} -F_{n-1}F_{n-2} & -F_{n-1}F_n & -F_{n+1}F_n \\ F_nF_{n-2} & F_{n+1}F_n & F_{n+2}F_n \\ F_{n-1}F_n & F_{n+1}F_n & F_{n+2}F_{n+1} \end{pmatrix}, \quad t(z^n) = 2F^2_{n+1} - F_{n-1}F_{n+2}.\]  

Applying Lemma 5.3 and (5.6), we have the identity \(L_{2n} + (-1)^n = 2F^2_{n+1} - F_{n-1}F_{n+2}\).

**Example 5.5.** Eigenvalues of \(x\) are \(1, 1 + \sqrt{5}\) and \(1 - \sqrt{5}\). Since \(wx = xw\), one can show that eigenvalues of \(xw^n\) are \((-1)^n\), \((1 + \sqrt{5})\tau^n\) and \((1 - \sqrt{5})\sigma^n\). It follows that the trace of \(xw^n\) is \((-1)^n + L_{2n} + 5F_{2n}\). Similar to Example 5.4, expression of \(xw^n\) (see (5.4)) gives

\[F^2_{n-1} + F_{n+1}F_{n+3} = (-1)^n + L_{2n} + F_{2n}.\]  

Note that if \(B = (b_{ij})\) is a conjugate of \(xw^n\), then \(b_{11} + b_{22} + b_{33} = (-1)^n + L_{2n} + 5F_{2n}\).
6. Fibonacci pairs of rank 4 or more

Let $a$ and $b$ be roots of $X^2 - X - 1 = 0$ and $\Phi_r(X) = \prod_{k=0}^{r-1} (X - a^i b^{r-i})$ (see [2], [12]). Let $w$ be the rational canonical form of $\Phi_r(X)$. Similar to Proposition 5.1, one can show that $F_{r+1}(w, x)$ is a Fibonacci pair of rank $r + 1$, where $x = aI_{r+1} + \sum a_i w_i$ for some $a \neq 0$ (see [10]). We do not pursue our study of pairs of rank 4 or more in this article as the identities will be lengthy.

7. Final comment

(i) We have demonstrated how identities can be constructed by studying Fibonacci pairs. Such work can be viewed as an extension of [11] that enables us to generate identities systematically. We hope the readers can construct more identities by applying similar methods.

(ii) It is fairly easy to get new matrix identities other than (M1)-(M5). For instance, if $F_2(w, x)$ is given as in subsection 2.2, where $w = w_2, x = x(1, 2)$, then $(1,1)$ entries of $(w^n)(w^m) = w^{n+m}$ and $(w^n)(xw^m) = xw^{n+m}$ give the following identities.

\[
L_{n+1}L_{m+1} - 5F_n F_m = L_{m+n+1} \quad \text{and} \quad L_{n+1}F_{m+2} - F_n L_{m+1} = F_{m+n+2}.
\]  

(7.1)

Note that identities in (7.1) are not included in (M1)-(M5) of Section 3.

(iii) Let $z^n$ be given as in (5.6). $(1,1)$ and $(2,1)$ entries of $(z^n)(z^n) = z^{2n}$ (after simplification) give the following identities.

\[
L_{n-1}F_{2n-1} = 2F_n^3 - F_{n-1}F_n^2 - 2 \quad \text{and} \quad L_n L_{n-1} = 2F_{n-1}F_{n-2} + F_{n+2}F_n.
\]  

(7.2)

(iv) Let $w$ be given as in Proposition 2.1. Consideration of a pair of $2 \times 2$ matrices $L_2(w, v)$ such that $vw^m = w^{-m}v$ and that entries of $aw^m$ and $aw^n$ are Fibonacci or Lucas numbers ($a \neq 0$) seems less fruitful as $vw = w^{-1}v$ implies that $v$ and $vw^n$ have trace $0$ (4.5A). Can one find pairs of $2 \times 2$ matrices other than $F_2(w, x)$ and $L_2(w, v)$ that give nice identities?

(v) Matrix identities of a pair of matrices $S(z, v)$ such that $z$ is singular and $zv = vz$ are also of interest. For instance, if $z$ and $v$ are given as follows. Then $(1,1)$ entry of $(z^n)(vz^n) = vz^{n+m}$ gives $F_n L_{m+1} + F_{n-1} L_{m-1} + F_{n+1} L_m = L_{n+m+1}$. 

\[
z = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad v = 2I_3 + z, \quad z^n = \begin{pmatrix} F_n & F_{n-1} & F_{n+1} \\ F_{n-2} & F_{n-3} & F_{n-1} \\ F_{n-1} & F_{n-2} & F_n \end{pmatrix}.
\]  

(7.3)

Matrix $z$ is taken from [8]. Note that the characteristic polynomial of $z$ is $X^3 - X^2 - X$. In general, we may take $z$ to be a matrix similar to the rational canonical form of $X^3 - X^2 - X$.

(vi) Let $F_s(w, x)$ be a Fibonacci pair of rank $s$. In the case $x^2$ is not a scalar matrix, one may also consider matrix equation of the form $(x^aw^b)(x^cw^d) = x^{a+c}w^{b+d}$ to get matrix identities. We leave it to the readers if he or she finds it interesting.

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