On the Free Field Realization of Form Factors

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Abstract

A method to construct free field realizations for the form factors of diagonal factorized scattering theories is described. Form factors are constructed from linear functionals over an associative ‘form factor algebra’, which in particular generate solutions of the deformed Knizhnik-Zamolodchikov equations with parameter $2\pi$. We show that there exists a unique deformation off the (‘Rindler’) value $2\pi$ that preserves the original $S$-matrix and which allows one to realize form factors as vector functionals over an algebra of generalized vertex operators.
1. Introduction

The form factor bootstrap \([1, 2]\) is a universal approach to integrable relativistic QFTs. It takes an implementation of the Wightman axioms in terms of form factors as a starting point. As a consequence of the factorized scattering theory there exists a recursive system of coupled Riemann-Hilbert equations for these form factors, which entail that the Wightman functions built from them have all the desired properties. However, the development of systematic solution techniques for these equations is still in its beginnings. An algebraic solution scheme was proposed in \([3]\): There exists a doublet of form factor algebras \(F_{\pm}(S)\) associated with a given two particle \(S\)-matrix such that suitable (‘\(T\)-invariant’) linear functionals over these algebras define ‘pre-form factors \(f_{\pm}\). The sum \(f = f^+ + f^-\) then automatically satisfies all the form factor axioms, except for the bound state residue axiom. The construction of form factors thus reduces to constructing a realization of \(F_{\pm}(S)\) and to find \(T\)-invariant functionals in this realization. Applied to the case of integrable QFTs with a diagonal factorized scattering theory, the construction also lead to a universal formula for the eigenvalues of the local conserved charges.

The purpose of the present paper is to show how this set-up relates to the vertex operator construction of the eigenvalues \([4]\) and to the proposal \([6, 7]\) to realize form factors as (trace) functionals over an algebra of generalized vertex operators. The functionals in question are ill defined in general, so that some kind of regularization is required. We propose to construct form factors as limits

\[ f_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1) = \lim_{\beta \to 2\pi} f_{a_n\ldots a_1}(\beta) \]

where the functions \(f_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1)\) solve a system of deformed form factor equations. The deformation is uniquely determined by the following two conditions:

i. The ‘deformed form factors’ \(f_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1)\) solve the deformed Knizhnik-Zamolodchikov equations (KZE) with parameter \(i\beta\), i.e.

\[ f_{a_n\ldots a_1}(\theta_n, \ldots, \theta_k + i\beta, \ldots, \theta_1) = (A_k^{(\beta)})_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1) f_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1), \]

\[ (A_k^{(\beta)})_{a_n\ldots a_1}(\theta_n, \ldots, \theta_1) = \prod_{j<k} S_{a_ja_k}(\theta_{jk}) \prod_{j>k} S_{a_ka_j}(\theta_{jk} - i\beta), \]

for the case of a diagonal \(S\)-matrix, considered here.

ii. The factorized scattering theory (i.e. the two particle \(S\)-matrix) remains undeformed. In particular, the deformed form factors satisfy for \(Re \theta_{k+1,k} \neq 0\)

\[ f_{a_n\ldots a_{k+1}a_k\ldots a_1}(\theta_n, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_1) = S_{a_ka_{k+1}}(\theta_{k+1,k}) f_{a_n\ldots a_{k+1}a_k\ldots a_1}(\theta_n, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_1). \]
We shall refer to this deformation as ‘S-matrix preserving deformation’. The original value \( \beta = 2\pi \) has special features in many respects. In particular, it indirectly reflects the thermal properties of the Rindler vacuum. For \( \beta \neq 2\pi \) the functions \( f_{a_n \ldots a_1}^{(\beta)}(\theta_n, \ldots, \theta_1) \) characterize a ‘deformed’ relativistic QFT. We shall make some suggestions on the physical significance of this deformation at the end of the paper. Here we shall be concerned with a purely technical consequence of taking \( \beta \neq 2\pi \): The deformation serves as a regularization, in the sense that for \( \beta/2\pi \) irrational the deformed ‘pre-form factors’ \( f^{(\beta, \pm)}(\theta) \) can be realized as vector functionals over a realization of a deformed form factor doublet \( F^{(\beta)}_{\pm}(S) \). The realization is in terms of generalized vertex operators

\[
f^{(\beta, \pm)}(X^\pm) = \langle O | \rho^{(\beta)}(X^\pm) | 0 \rangle, \quad X^\pm \in F^{(\beta)}_{\pm}(S),
\]

where \( |0\rangle \) is the vacuum of a tensor product of free bosonic Fock spaces and \( |O\rangle \) is a state characterizing the local operator in question. The functionals (1.4) could also be written as trace functionals, in which case the above deformation also serves to regularize the trace. Compared to other regularizations \([7]\) it has the advantage to preserve the conserved charge bootstrap i.e. the relation between the position of the bound state poles in the S-matrix and the spins of the local conserved charges (c.f. section 3).

2. S-matrix preserving deformation of form factors

Suppose a factorized scattering theory to be given with a diagonal S-matrix satisfying the bootstrap equations

\[
S_{ab}(\theta) = S_{ba}(\theta) = S_{ab}(-\theta)^{-1} = S_{ab}^*(-\theta^*) = S_{ba}(\theta)
\]

\[
S_{ab}(i\pi - \theta) = S_{ab}(\theta)
\]

\[
S_{ab}(\theta + i\eta(a)) S_{ab}(\theta + i\eta(b)) S_{ab}(\theta + i\eta(c)) = 1.
\]

(2.1)

Here \( S_{ab}(\theta) \) is the S-matrix element for the elastic scattering of particles of type \( a, b \in \{1, \ldots, r\} \). It is a periodic meromorphic function in the rapidity difference \( \theta = \theta_a - \theta_b \) with period \( 2\pi i \). Bound state poles are situated on the physical sheet \( 0 \leq \mathrm{Im} \theta < \pi \). The first equation expresses hermitian analyticity and (formal) unitarity. The second equation implements crossing invariance in terms of a charge conjugation operation \( a \to \bar{a} \in \{1, \ldots, r\} \). The last equation is the bootstrap equation proper, where the triplet \( (\eta(a), \eta(b), \eta(c)) \) is related to the conventional fusing angles. It is convenient to introduce the scattering phase adopted to the normalization condition at \( \theta = 0 \)

\[
S_{ab}(\theta) = \epsilon_{ab} e^{i\delta_{ab}(\theta)}, \quad S_{ab}(0) = \epsilon_{ab}, \quad \epsilon_{ab} \in \{\pm 1\}, \quad \epsilon_{aa} = -1.
\]

(2.2)
Given a bootstrap $S$-matrix with these properties the form factors associated with $S$ are defined to be solutions of a recursive system of form factor equations\cite{1,2}. These equations admit an algebraic solution scheme in the following sense: There exists a doublet of form factor algebras $F_\pm(S)$ \cite{3} such that ‘$T$-invariant’ linear functionals $f^\pm$ over $F_\pm(S)$ define ‘pre-’form factors. Their sum $f = f^+ + f^-$ then automatically solves all the form factors equations, except the bound state residue axiom. Here we consider a deformation of this construction, motivated by the considerations outlined in the introduction. We first define an algebra $F^{(\beta)}(S)$ and then by specialization $F^{(\beta)}_\pm(S)$. In the case of a diagonal $S$-matrix the former is defined as follows: $F^{(\beta)}(S)$ is an associative algebra with generators $t^\pm_a(\theta)$, $\theta \in \mathbb{C}$, $W_a(\theta)$, $0 \leq Im \theta \leq \beta$, a unit $1$ and the generators $P_\mu$, $\epsilon_{\mu\nu}K$ of the 1+1 dimensional Poincaré algebra. Except for $P_\mu$ all generators transform as scalars under the action of the Poincaré group. The defining relations of $F^{(\beta)}(S)$ then are:

(T) The operators $t_1^+(\theta), \ldots, t_n^+(\theta)$ generate a direct product of abelian algebras, are $2\pi i$-periodic and satisfy $t^+_a(\theta + i\pi) t^+_a(\theta) = 1$. The mixed products $t^+_a(\theta_0) t^-_b(\theta_1)$ satisfy

$$S_{ab}(\theta_0 - 2\pi i + i\beta) t^+_a(\theta_0) t^-_b(\theta_1) = S_{ab}(\theta_1) t^-_b(\theta_1) t^+_a(\theta_0).$$

Further there are linear exchange relations between the $t^\pm$ and the $W$-generators

$$(TW) \quad t^+_a(\theta_0) W_b(\theta_1) = S_{ab}(\theta_0 + 2\pi i - i\beta) W_b(\theta_1) t^+_a(\theta_0),
\quad t^-_a(\theta_0) W_b(\theta_1) = S_{ab}(\theta_0) W_b(\theta_1) t^-_a(\theta_0),$$

valid for generic rapidities. The $W$-generators so far are defined only in the strip $0 \leq Im \theta \leq \beta$. The extension to other strips $\beta k \leq Im \theta \leq \beta(k+1)$, $k \in \mathbb{Z}$ is done by repeated use of the relation

$$(S) \quad W_a(\theta) t^+_a(\theta + i\beta - i\pi) = t^-_a(\theta + i\beta - i\pi) W_a(\theta + i\beta).$$

Finally impose

$$(WW) \quad W_a(\theta_1) W_b(\theta_2) = S_{ab}(\theta_{12}) W_b(\theta_2) W_a(\theta_1), \quad Re \theta_{12} \neq 0.$$

This concludes the definition of the algebra $F^{(\beta)}(S)$. The product of $W$-generators is defined only when all relative rapidities have a non-vanishing real part. For relative rapidities that are purely imaginary, the product of $W$-generators contains simple poles. The algebra $F^{(\beta)}(S)$ in which the $W$-generators in addition satisfy the relation $(R\pm)$ below will be denoted by $F^{(\beta)}_\pm(S)$, respectively. It is convenient to use different symbols $W^+_a(\theta)$ and $W^-_a(\theta)$ for $W$-generators satisfying $(R+)$ and $(R-)$, respectively. The residue conditions then read

$$(R\pm) \quad 2\pi i \text{res}[W^+_a(\theta - i\pi) W^+_b(\theta)] = -\delta_{ab},
\quad 2\pi i \text{res}[W^-_a(\theta + i\pi) W^-_b(\theta)] = -\delta_{ab}.$$

We shall refer to the algebra $F^{(\beta)}_\pm(S)$ as the deformed form factor doublet. Implicit in these definitions, of course, is the presupposition that the above relations define a consistent algebra.
Because of the $\beta$-deformation this does not follow from the results of \cite{3}, but can be proved in a similar way. We proceed with a number of comments on the structure of $F^{(\beta)}_\pm(S)$. First note that by means of (S) and (TW), the residue conditions (R\pm) imply
\begin{align}
2\pi i \text{ res}[W^+_a(\theta + i\beta - i\pi)W^+_b(\theta)] &= \delta_{ab} e_a(\theta + i\beta - 2\pi i), \\
2\pi i \text{ res}[W^-_a(\theta + i\pi - i\beta)W^-_b(\theta)] &= -\delta_{ab} S_{ab}(i\beta) e_a(\theta - \pi i),
\end{align}
where $e_a(\theta) = t^-_a(\theta + i\pi)t^+_a(\theta)$. For $\beta = 2\pi$ this is a central element of $F^{(2\pi)}(S)$; for generic $\beta$ it still commutes with the $W$-generators. For generic $\beta$ one could in principle identify $W^+_a(\theta)$ and $W^-_a(\theta)$ without running into inconsistencies. It is only in the limit $\beta \to 2\pi$ that (R+) and (R-) would then be in conflict with the second and first Eqn. (2.3), respectively. A second remark concerns the inclusion of bound state poles. Recall that multiple products of $W$-generators have been defined only in cases where at most one relative rapidity is purely imaginary. The extension of the product to cases where two or more relative rapidities are purely imaginary is needed for an algebraic implementation of the bound state residue axiom. We shall not attempt to give a complete discussion here. To emphasize the point of the condition $ii.$ in the introduction let us note the residue condition corresponding to simple poles in the $S$-matrix
\begin{equation}
2\pi i \text{ res}[W_a(\theta + i\eta(a))W_b(\theta + i\eta(b))] = \Gamma_{abc} W_c(\theta + i\eta(c) + i\pi),
\end{equation}
where $W$ stands for either $W^+$ or $W^-$ and $\Gamma_{abc}$ is a constant. Similar conditions would be required for higher order poles in the $S$-matrix. Their mutual consistency is non-trivial and will reflect the closure of the $S$-matrix bootstrap. In addition, the product defined in terms of such residue operations will in general not be associative, so that the ‘nesting’ of pairs has to be specified.

Consider now linear functionals over the algebra $F^{(\beta)}(S)$ that are ‘$T$-invariant’ i.e. which satisfy
\begin{align}
f^{(\beta)}(X t^+_a(\theta)) &= f^{(\beta)}(X), \\
f^{(\beta)}(t^-_a(\theta) X) &= \omega(a) f^{(\beta)}(X),
\end{align}
where $X \in F^{(\beta)}(S)$ has rapidities separated from $\theta$ and $\omega(a)$ is a phase. Functionals $f^{(\beta)}$ corresponding to non-trivial local operators in addition satisfy $f^{(\beta)}(1) = 0$. For $\beta = 2\pi$ the sum of two such functionals over $F^{(\beta)}(S)$, respectively, yields solutions of the form factor equations. For generic $\beta$ the most important equations for the deformed form factors are:

(1) Relation between In and Out states: This is the same as in the undeformed case. For later use we remark that the action of the underlying antilinear anti-involution \cite{3} is modified on the $t^+_a(\theta)$ generators. Explicitly
\begin{align}
\sigma(W_a(\theta)) &= W_a(\theta^*), \\
\sigma(t^+_a(\theta)) &= t^+_a(\theta^* + i\beta - 2\pi i).
\end{align}
(2) Undeformed exchange relations (1.3).
(3) $i\beta$-deformed KZE: These are the equations (1.2). If all relative rapidities have a nonvanishing real part, the $k = n$ equation can be rewritten in the form
\begin{equation}
f^{(\beta)}_{a_0 \ldots a_1}(\theta_n + i\beta, \theta_{n-1}, \ldots, \theta_1) = f^{(\beta)}_{a_0 \ldots a_1}(\theta_{n-1}, \ldots, \theta_1, \theta_n).
\end{equation}
Interpreting the rapidities as time variables this equation has the form of a KMS condition for a thermal n-point function with inverse temperature $\beta$. Originally it was this equation, defined for real $\theta_n, \ldots, \theta_1$ (and $\beta = 2\pi$), that was taken as one of the form factor axioms $\text{[3]}$. 

(4) Deformed kinematical residue equations: The construction in $\text{[3]}$ to represent form factors as a sum of two terms $f = f^+ + f^-$ satisfying simple residue equations, remains valid. For $X^\pm = W^\pm_n(\theta_n) \ldots W^\pm_1(\theta_1) \in F^\pm(S)$ set $f^\pm(\theta_n, \ldots, \theta_1) := f^\pm(X^\pm)$. From (R±) one then finds

$$2\pi i \text{res}_{f^\pm} (\theta_n, \ldots, \theta_k + i\pi, \theta_k, \ldots, \theta_1) = -\delta_{\theta_k+1} f^\pm(\theta_n, \ldots, \theta_k+2, \theta_k-1, \ldots, \theta_1),$$

(2.7)
as in the undeformed case. The reversed residue conditions (2.3) are $\beta$-dependent and imply

$$2\pi i \text{res}_{f^\pm} (\theta_n, \ldots, \theta_k + i\beta - i\pi, \theta_k, \ldots, \theta_1) = \delta_{\theta_k+1} \prod_{j > k+1} S_{\theta_k a_j} (\theta_k + i\beta)$$

$$\times \prod_{j < k} S_{\theta_k a_j} (\theta_k + 2\pi i - i\beta) f^\pm(\theta_n, \ldots, \theta_k+2, \theta_k-1, \ldots, \theta_1),$$

(2.8)
together with a similar Eqn. for $f^-$. By considering the sum $f = f^+ + f^-$ one can also work out the deformed counterpart of the kinematical residue axiom. From (2.7), (2.8) it is clear that for generic $\beta$ the kinematical residue axiom splits up into two sets of equations of the form (2.7)(lower case) and (2.8).

(5) Bound state residue axiom: As indicated before we shall not attempt here to give a complete discussion of the bound state residue axiom. The point to emphasize however is that by construction the conserved charge bootstrap principle will be preserved.

All of the above construction has a direct deformed counterpart also in the case of a non-diagonal S-matrix. A special feature of the diagonal case is that the KZE admit the following factorized solutions:

$$f(\theta_n, \ldots, \theta_1) = (-)^{n-1} K(\theta_n, \ldots, \theta_1) \prod_{k > j} \frac{F(\theta_k, \theta_j)}{F(\theta_k + i\pi, \theta_j)} \prod_{j = 1}^{n-1} \frac{\theta_j + i\pi}{\theta_j} ,$$

(2.9)

where $F_{ab}(\theta)$ is the deformed minimal form factor defined below. $K(\theta_n, \ldots, \theta_1)$ are totally symmetric functions of $\theta_n, \ldots, \theta_1$ that are $i\beta$-periodic in each variable and which contain the necessary bound state poles. Such functions can easily (re-)produced as correlators of a collection of free fields. For the part carrying the non-trivial monodromy – here the product of the minimal form factors – this is less obvious. A realization in terms of generalized vertex operators requires that the quantity in question admits a factorization into a product of exponentials. Such factorizations are indeed available for the minimal form factor.

*There are also some non-diagonal theories where this seems to be the case.
3. The minimal form factor as a product of exponentials

The deformed minimal form factor $F^{(\beta)}_{ab}(\theta)$ is uniquely characterized by the following properties:

(i) $F^{(\beta)}_{ab}(\theta)$ is analytic in $0 < Im \theta \leq \beta/2$ and has no zeros in this range. The meromorphic continuation to strips $S_k^{(\beta)}$, $k \neq 0$, $S_k^{(\beta)} = \{ \theta \in \mathbb{C} \mid k\beta/2 < Im \theta \leq (k+1)\beta/2 \}$ is done by means of (ii).

(ii) The following equations hold

$$F^{(\beta)}_{ab}(\theta) = S_{ab}(\theta)F^{(\beta)}_{ab}(-\theta), \quad F^{(\beta)}_{ab}(\theta + i\beta) = F^{(\beta)}_{ba}(-\theta). \quad (3.1)$$

(iii) Normalization: The limit $\lim_{|\theta| \to \infty} F^{(\beta)}_{ab}(\theta)$ exists and $F^{(\beta)}_{ab}(i\beta/2) = 1$.

Following [1] we can write down the unique solution in the following form

$$F^{(\beta)}_{ab}(\theta) = \exp \left\{ \frac{ch^2 (\frac{\theta}{\beta})}{i\pi} \int_0^\infty dt \frac{\frac{1}{t} \ln S_{ab}(\frac{\beta}{\pi} t)}{ch t - ch \frac{2\pi}{\beta} \theta} \right\}, \quad 0 < Im \theta \leq \beta,$$

$$= \exp \left\{ \frac{ch^2 (\frac{\theta}{\beta})}{i\pi} \int_0^\infty dt \frac{\frac{1}{t} \ln S_{ab}(\frac{\beta}{\pi} t)}{ch t - ch \frac{2\pi}{\beta} \theta} \right\} S_{ab}(\frac{2\pi}{\beta} \theta)^{1/2}, \quad \theta \in \mathbb{R}. \quad (3.2)$$

The solution simplifies if one assumes that the scattering phase (2.2) allows for an integral representation of the form

$$\delta_{ab}(\theta) = -\int_0^\infty \frac{dt}{t} h_{ab}(t) \sin \frac{t\theta}{\pi}, \quad 0 < |Im \theta| < \sigma_0 \leq \pi, \quad (3.3)$$

where $\sigma_0$ is the position of the first bound state pole (i.e. $S_{ab}(i\sigma)$ is analytic for $0 < \sigma < \sigma_0$) and $h_{ab}$ is subject to the following conditions

- $h_{ab}$ is real on the real axis and has a meromorphic continuation off the real axis. All poles are simple and are given by $\{ t = \pm i\pi k, \ k \in E \}$ for some subset $E \subset \mathbb{N}$.

- $h_{ab}(t) = h_{ba}(t) = h_{ab}(-t), \quad |h_{ab}(0)| < \infty.$

- $|h_{ab}(t)| \to e^{-|Re t|d_{ab}}, \ |Re t| \to \infty$ for constants $0 < d_{ab} < 1$.

In the last condition we assumed that a possible constant term $k_{ab} \in 2\mathbb{Z}$ in $h_{ab}(t)$ has been split off. Since $\int_0^\infty \frac{dt}{t} \sin \frac{\theta t}{\pi} = \frac{\pi}{2} \text{sign} \ theta$ for $\theta \neq 0$, this amounts to the identification $\epsilon_{ab} = e^{\pm i\pi k_{ab}/2}$.

The integral representation (3.3) is also convenient to derive series expansions of the scattering phase on and off the imaginary $\theta$-axis. Since the constant term of $h_{ab}$ has been split off the
fall-off properties of \( h_{ab} \) are such that series expansions for \( \delta_{ab}(\theta) \) are available just by suitable deformation of the integration contour. This leads to

\[
\delta_{ab}(\theta) = \pm \sum_{n \in \mathcal{P}} \frac{d_{ab}(n)}{n} e^{\pm in\theta}, \quad \pm \text{Re } \theta > 0, \quad 0 \leq \text{Im } \theta < \sigma_0 ,
\]  

(3.4)

where \( d_{ab}(n) = i \text{res}_{t=i\pi n} h_{ab}(t) = -i \frac{h_{ab}(i\pi n)^2}{h_{ab}(i\pi n)} \). On the imaginary axis the expansions (3.4) merge to a Fourier series

\[
\delta_{ab}(i\sigma) = i \sum_{n \in \mathcal{P}} \frac{d_{ab}(n)}{n} \sin n\sigma , \quad 0 < \sigma < 2\pi .
\]  

(3.5)

So far the \( S \)-matrix bootstrap equations (2.1) have not been used. Imposing them on the level of the series expansions (3.4), (3.5) puts constraints on the expansion coefficients \( d_{ab}(n) \). The most prominent (possibly all) solutions of these constraints are those descending from Lie algebraic data. In that case the particles \( a = 1, \ldots, r \) are associated with the Dynkin diagram of a simple Lie algebra \( g \) and the possible fusing angles are selected by the condition

\[
\sum_{l=a, b, c} e^{\pm is\eta(l)} q_{l}^{(s)} = 0 ,
\]  

(3.6)

where \( (q_{1}^{(s)}, \ldots, q_{r}^{(s)})^{T} \) is the normalized real eigenvector of the Cartan matrix with eigenvalue \( 2(1 - \cos \frac{s\pi}{h}) \) (\( h \): Coxeter number, \( s \): exponent). The coefficients \( d_{ab}(n) \) then take the form

\[
d_{ab}(n) = d_{n} q_{a}^{(n)} q_{b}^{(n)} ,
\]  

(3.7)

for real constants \( d_{n} \). The second and third eqn. in (2.1) are satisfied by means of \( q_{\theta}^{(n)} = (-)^{n+1} q_{a}^{(n)} \) and (3.6), respectively. Notice that the coefficients (3.7) vanish unless \( n \) is an exponent of \( g \) modulo \( h \), so that the summations over \( n \in \mathcal{E} \) here correspond to summations over the set of affine exponents. It has been shown in [3, 4] that

\[
I_{a}^{(n)} = e^{n/2} \sqrt{\frac{d_{n}}{n}} q_{a}^{(n)} , \quad n \in \mathcal{E}
\]  

(3.8)

is the exact eigenvalue of the \( n \)-th local conserved charge on a single particle state of type \( a \) at rapidity zero.

Now return to the deformed minimal form factor (3.2). Expressed in terms of the function \( h_{ab}(t) \) it reads

\[
F_{\beta}^{(\beta)}(\theta) = \left( -i \text{sh} \frac{\pi \theta}{\beta} \right)^{k_{ab}/2} \exp \left\{ \int_{0}^{\infty} \frac{dt}{t} h_{ab}(\frac{2\pi t}{\beta}) \frac{\text{sh} t}{\text{sh} t} \sin^{2}(i\pi - \frac{2\pi}{\beta} \theta) \frac{t}{2\pi} \right\} ,
\]  

(3.9)
for $0 < \text{Im} \theta < \beta$. It is sometimes convenient to separate the phase and the modulus of $F^{(\beta)}_{ab}(\theta)$. Writing $\theta = \vartheta + i\sigma$ one has in particular: $\arg F^{(\beta)}_{ab}(\vartheta, 0) = -\frac{\vartheta}{\beta}k_{ab} + \frac{1}{2}\delta_{ab}(\vartheta)$, while for purely imaginary $\theta = i\sigma$ the minimal form factor is real: $\arg F^{(\beta)}_{ab}(0, \sigma) = 0$. For the rest of this paper we will formally set $k_{ab} = 0$ in order to simplify the expressions. The dependence on $k_{ab}$ can easily be restored through eqns (2.2), (3.9).

In the following we will derive two types of series expansions for $\ln F^{(\beta)}_{ab}(\theta)$. The first type is valid for all real positive $\beta$, including $\beta = 2\pi$; the second type is valid only for $\beta/2\pi$ irrational.

On the imaginary $\theta$-axis it is natural to consider Fourier series, while off the imaginary axis expansions in $e^{\pm \theta}$ for $\pm \text{Re} \theta < 0$ will be used. Let us first consider expansions valid for all $\beta > 0$. We claim that

$$\ln F^{(\beta)}_{ab}(i\sigma) = \sum_{n \geq 1} \left[ \cos \frac{2\pi n}{\beta} \sigma + (-)^{n+1} \right] F^{(\beta)}_{ab}(n), \quad 0 < \sigma < \beta,$$

where

$$F^{(\beta)}_{ab}(n) = \int_{0}^{\infty} ds e^{-ns} \delta_{ab}(\frac{\vartheta}{2\pi} s),$$

or in the representation (3.3)

$$F^{(\beta)}_{ab}(n) = -\int_{0}^{\infty} dt \frac{h_{ab}(\frac{2\pi t}{\beta})}{(\pi n)^2 + t^2}.$$

Off the imaginary $\theta$-axis one has

$$\ln F^{(\beta)}_{ab}(\theta) = \sum_{n \geq 1} \left[ e^{\pm n \frac{\vartheta}{\beta}} + (-)^{n+1} \right] F^{(\beta)}_{ab}(n), \quad 0 < \text{Im} \theta < \beta, \quad \pm \text{Re} \theta < 0,$$

with the same coefficients (3.11), (3.12). In order to verify Eqn. (3.10) first note that

$$\sum_{n \geq 1} e^{-nt}[\text{ch} n\theta + (-)^{n+1}] = \frac{\text{th} \frac{\theta}{2} \text{ch} \frac{2\theta}{2}}{\text{th} t - \text{ch} \theta},$$

for $|\text{Re} \theta| < \text{Re} t$. Inserting into Eqn. (3.2) one can exchange the order of summation and integration, which yields (3.10) with coefficients (3.11). In the representation (3.3) a simple integration yields the coefficients in the form (3.12). The derivation of (3.13) runs similarly:

Combining

$$\frac{1 + e^{-\theta}}{(1 + e^{-t})(e^{t} - e^{-\theta})} + \frac{1 + e^{\theta}}{(1 + e^{-t})(e^{t} - e^{\theta})} = \frac{\text{th} \frac{\theta}{2} \text{ch} \frac{2\theta}{2}}{\text{th} t - \text{ch} \theta},$$

and

$$\sum_{n \geq 1} [e^{\pm n \vartheta} + (-)^{n+1}] = \frac{1 + e^{\pm \vartheta}}{(1 + e^{-t})(e^{t} - e^{\pm t})}.$$
one arrives at (3.12).

The expansions (3.10), (3.13) are valid for all $\beta > 0$, in particular for the physical value $\beta = 2\pi$. For practical purposes, however, the expansions (3.10), (3.13) for $\beta = 2\pi$ are not particularly useful. Except for a few simple cases the expansion coefficients $F^{(2\pi)}_{ab}(n)$ cannot be evaluated explicitly (or can only be written in the form of a trigonometric series or trigonometric polynomials with $n$-dependent order). Moreover one would like to relate the coefficients $F^{(2\pi)}_{ab}(n)$ of $\ln F^{(2\pi)}_{ab}(\theta)$ and $d_{ab}(n)$ of $\delta_{ab}(\theta)$. This can be achieved by taking $\beta$ off $2\pi$. The main observation is that for $\beta/2\pi$ irrational the double poles of the integrand in (3.9) are split: By assumption $t \to h_{ab}(2\pi t/\beta)$ has simple poles at \( \{t = \pm i\beta k/2, k \in E\} \) while $t \to 1/\sin t$ has simple poles at $\{t = \pm i\pi n, n \in \mathbb{N}\}$. Thus, for $\beta/2\pi$ irrational both sets of poles do not intersect. This means that all the poles of the integrand are now simple, so that one can derive series expansions of the desired form just by deforming the integration contour. On the imaginary $\theta$-axis one finds

\[
\ln F^{(2\pi)}_{ab}(i\sigma) = \frac{1}{2} \sum_{n \geq 1} \frac{1}{n} h_{ab}(2\pi n) \left( (-)^n - \cos \frac{2\pi n}{\beta} \sigma \right) - \frac{1}{2} \sum_{n \geq 1} \frac{1}{n \sin \frac{1}{2} n} \left( 1 - \cos n\left(\frac{\beta}{2} - \sigma\right) \right), \quad 0 < \sigma < \beta .
\]

(3.14)

Off the imaginary axis the result is

\[
\ln F^{(2\pi)}_{ab}(\theta) = \frac{1}{2} \sum_{n \geq 1} \frac{1}{n} h_{ab}(2\pi n) \left( (-)^n - e^{\pm 2\pi n\theta} \right) - \frac{1}{2} \sum_{n \geq 1} \frac{d_{ab}(n)}{n \sin \frac{1}{2} n} \left( 1 - e^{\pm n(i\beta/2 - \theta)} \right),
\]

for $\pm \Re \theta > 0$ and $0 < \Im \theta < \beta$.

(3.15)

The advantage of the expansions (3.14), (3.15) is that all the expansion coefficients are known explicitly in terms of the function $h_{ab}(t)$. Moreover, the apparently unattractive feature of having a doubling of terms (with expansion coefficients $h_{ab}(i2\pi^2 n/\beta)$ and $d_{ab}(n)$, respectively) corresponds to a decomposition into a part with trivial and with non-trivial monodromy. Before turning to these issues let us discuss the relation to, and consistency with, the series expansions (3.10), (3.13) valid for all $\beta > 0$. To do so, consider the expression (3.12) for the coefficients $F^{(2\pi)}_{ab}(n)$. For $\beta/2\pi$ irrational the integrand in (3.12) has again only simple poles and the integral again can be done by deforming the contour. Using different choices of the contour one obtains the equivalent expressions

\[
F^{(2\pi)}_{ab}(n) = -\frac{1}{2n} h_{ab}(2\pi n) - \frac{\beta}{2\pi^2} \sum_{k \in E} \frac{d_{ab}(k)}{n^2 - (k\beta/2\pi)^2}
\]

\[
= -\frac{1}{n} h_{ab}(2\pi n) - \frac{\beta}{2\pi^2} \sum_{k \in E} \frac{d_{ab}(k)}{n(n - k\beta/2\pi)}
\]
\[-\frac{\beta}{2\pi^2} \sum_{k \in E} \frac{d_{ab}(k)}{n(n+k\beta/2\pi)} , \tag{3.16}\]

for \(0 < \beta < 2\pi\). In particular (3.16) displays the relation between the coefficients \(F_{ab}^{(3)}(n)\) and \(d_{ab}(n)\) searched for. Notice that only in the last of the expressions (3.16) can one take the limit \(\beta \to 2\pi\). Inserting the first expression for \(F_{ab}^{(3)}(n)\) into (3.10) one can justify the exchange in the order of summations. Using then

\[
\sum_{n \geq 1} \left[ \cos \frac{2\pi n \sigma}{\beta} + (-)^{n+1} \right] \frac{1}{n^2 - (k\beta/2\pi)^2} = \frac{2\pi^2}{\beta} \frac{1}{2k \sin \frac{\beta}{2}} \left( 1 - \cos k \left( \frac{\beta}{2} - \sigma \right) \right), \quad 0 < \sigma < \beta < 2\pi ,
\]

for the resummation one recovers (3.14).

Set now

\[
\phi_{ab}^{(\beta)}(\theta) := \sum_{n \geq 1} \frac{d_{ab}(n)}{2n \sin \frac{\beta}{2}} e^{\pm n(i\frac{\beta}{2} - \theta)} , \quad \text{for } \pm \text{Re } \theta > 0 \text{ and } 0 < \text{Im } \theta < \beta , \tag{3.17}
\]

and let \(\nu_{ab}^{(\beta)}(\theta)\) denote the rest of the terms in (3.15), so that \(\ln F_{ab}^{(\beta)}(\theta) = \nu_{ab}^{(\beta)}(\theta) + \phi_{ab}^{(\beta)}(\theta)\). One then verifies that

\[
\nu_{ab}^{(\beta)}(\theta + i\beta) = \nu_{ab}^{(\beta)}(\theta) , \quad \nu_{ab}^{(\beta)}(\theta) = \nu_{ab}^{(\beta)}(-\theta) ,
\]

\[
\phi_{ab}^{(\beta)}(\theta + i\beta) = \phi_{ab}^{(\beta)}(-\theta) , \quad \phi_{ab}^{(\beta)}(\theta) = \phi_{ab}^{(\beta)}(-\theta) + i\delta(\theta) , \quad \text{Re } \theta \neq 0 , \tag{3.18}
\]

using the expansion (3.4) for \(\delta(\theta)\). This means that \(\phi_{ab}^{(\beta)}(\theta)\) carries all the non-trivial monodromy and \(\exp \phi_{ab}^{(\beta)}(\theta)\) is itself a solution to the conditions (i) and (ii) on the \(\beta\)-deformed minimal form factor. The only feature it does not have is a well-defined limit as \(\beta \to 2\pi\). The first term \(\exp \nu_{ab}^{(\beta)}(\theta)\) has trivial monodromy, has again no \(\beta \to 2\pi\) limit, but is designed s.t. the product of both factors does have a limit, which by construction coincides with the original minimal form factor.

Eqns (3.9),(3.14),(3.15) also exhibit the difference to the regularization of the form factors used in [3, 11]. From the viewpoint of the continuum theory, the construction in [3] approximates integrals of the form (3.9) (for \(\beta = 2\pi\)) by a Riemann sum, which is natural in the context of lattice models. The \(S\)-matrix of the asymptotic particles in the scaling limit is only indirectly related to the \(R\)-matrix, so that the analogue of the condition ii. in the introduction does not exist (or, if \(S\) is replaced with \(R\), is empty). In the context of the form factor bootstrap, however, a regularization that violates the condition ii. in the introduction is problematic. A deformation of the \(S\)-matrix will in general also affect the structure of its bound state poles, in which case the consistency (3.6) with the spins of the local conserved charges (i.e. Zamolodchikov’s ‘conserved charge bootstrap’ principle) is spoiled.
4. Free field realization of $F^{(\beta)}(S)$

The feature (3.18) of the series expansion (3.15) suggests to rewrite equation (2.9) in the following form

$$f^{(\beta, \pm)}_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_1) = \tilde{K}^{(\beta, \pm)}_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_1) \prod_{k>j} e^{\phi^{(\beta)}_{a_k a_j}(\theta_{kj})}, \quad (4.1)$$

where

$$\tilde{K}^{(\beta, \pm)}_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_1) = (-)^{n-1} K^{(\beta, \pm)}_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_1) \prod_{k>j} F^{(\beta)}_{a_k a_j}(\mp i\pi) \text{sh} (\theta_{kj} \pm i\pi)^{\frac{1}{\beta}}$$

can be realized as a correlator

$$\tilde{K}^{(\beta, \pm)}_{a_1 \ldots a_n}(\theta_1, \ldots, \theta_1) = \langle O| Q^{\pm}_{a_1}(\theta_1) \ldots Q^{\pm}_{a_n}(\theta_n) | 0 \rangle, \quad (4.2)$$

of $i\beta$-periodic fields $Q^{\pm}_{a_i}(\theta)$ with trivial monodromy. The part carrying the non-trivial monodromy will be realized as

$$\langle \rho I(W_{a_1}(\theta_1)) \ldots \rho I(W_{a_n}(\theta_n)) \rangle = \prod_{k>j} e^{\phi^{(\beta)}_{a_k a_j}(\theta_{kj})}, \quad (4.3)$$

where the operators $\rho I(W_{a_i}(\theta))$ and $\rho I(t^{\pm}_{a_i}(\theta))$ form a realization of $F^{(\beta)}(S)$. In total, the realization of the deformed form factor algebra $F^{(\beta)}_{\pm}(S)$ is a tensor product

$$\rho(W_{a_i}(\theta)) = Q_{a_i}(\theta) \otimes \rho I(W_{a_i}(\theta)), \quad \rho(t^{\pm}_{a_i}(\theta)) = 1 \otimes \rho I(t^{\pm}_{a_i}(\theta)), \quad (4.4)$$

where both factors act on different free field Fock spaces. It remains to construct the realization $\rho I$ of $F^{(\beta)}(S)$. It turns out that the latter can be constructed entirely in terms of the local conserved charges $I^{(n)}$, $n \in E$ together with their images under the antilinear anti-involution $\sigma$ defined in (2.5). The starting point is the relation

$$t^{\pm}_{a_i}(\theta) = \exp \left\{ \pm i \sum_{n \in E} \frac{e^{\mp \theta_n} c_n}{c^{\pm(\theta + \beta)n}} I_{a_i}^{(n)} I^{(n)} \right\}, \quad (4.5)$$

valid for $\pm Re(\theta - \theta_i) > 0$, $\sigma_0 > Im \theta > 0$, acting on a multiparticle state with real rapidities $\theta_1, \ldots, \theta_n$. The operator $t^{-}_{a_i}(\theta)$ can then be computed from the involution (2.5)

$$t^{-}_{a_i}(\theta) = \exp \left\{ \pm i \sum_{n \in E} \frac{c^{\pm(\theta + i\beta)n} I_{a_i}^{(n)} \sigma(I^{(n)})}{c^{\pm(\theta + i\beta)n}} \right\}. \quad (4.6)$$

*For notational simplicity we denote $\rho I(t^{\pm}_{a_i}(\theta))$ by $t^{\pm}_{a_i}(\theta)$ etc. in the following. Since we redefined $S$ by assuming $k_{ab} = 0$ zero modes are absent.
taking into account (3.8) and the correct matching of branches. The relations (T) then fix the
commutator between \( I^{(n)} \) and \( \sigma(I^{(m)}) \). One finds \([\frac{\partial}{\partial x_n}, x_m] = \delta_{mn} \) with the definitions

\[
I^{(n)} = \frac{\partial}{\partial x_n}, \quad \sigma(I^{(n)}) = \mp i|c|^n(1 - e^{\mp i \beta n}) x_n, \tag{4.7}
\]

where the sign options refer to that in (4.5),(4.6). The relations (TW) are equivalent to
\[
[\ln t_a(\theta_0), W_a(\theta_1)] = i\delta_{ab}(\theta_{01} - i\beta) W_a(\theta_1),
\]
\[
[\ln t_a(\theta_0), W_a(\theta_1)] = -i\delta_{ab}(\theta_{10}) W_a(\theta_1),
\]

so that
\[
[\frac{\partial}{\partial x_n}, W_a(\theta)] = I^{(n)} e^{\mp (\theta + i\beta)n} W_a(\theta_1),
\]
\[
[x_n, W_a(\theta)] = \pm i e^{\mp n\theta} \sum_{n \in E} \frac{I^{(n)} e^{\mp i \beta n}}{1 - e^{\mp i \beta n}} \frac{\partial}{\partial x_n} W_a(\theta_1). \tag{4.8}
\]

This implies
\[
W_a(\theta) = \exp \left\{ \sum_{n \in E} I^{(n)} e^{\pm n(\theta + i\beta)} x_n \right\} \exp \left\{ \mp i \sum_{n \in E} I^{(n)} e^{\mp i \beta n} \frac{\partial}{\partial x_n} \right\}. \tag{4.9}
\]

One can now check that the operators (4.5), (4.6) and (4.8) also satisfy the relations (S) and
(WW) so that they yield a realization of \( F^{(\beta)}(S) \). In particular one finds

\[
W_a(\theta_1)W_b(\theta_2) =: W_a(\theta_1)W_b(\theta_2) : e^{\phi_{ab}^{(\beta)}(\theta_{12})}, \quad Re \theta_{12} \neq 0. \tag{4.9}
\]

where : : indicates normal ordering in the Heisenberg algebra generated by \( \frac{\partial}{\partial x_n}, x_n \) and \( \phi_{ab}^{(\beta)}(\theta) \) is defined in (3.17). Thus, if \( \langle , \rangle := \langle , \rangle_\omega \) denotes the canonical sesquilinear form (contravariant
w.r.t. \( \omega(x_n) = \frac{\partial}{\partial x_n} \)) on the Fock space \( \mathbb{C}[x_1, x_2, \ldots] \), equation (4.3) follows, which completes the
construction.

We add two comments: In view of equation (2.6) one might expect the appearence of trace
functionals as in [8, 9]. Because of the Clavelli-Shapiro formula [8, App.C1] however the distinc-
tion between trace- and vector functionals is not an intrinsic one. In upshot, the result
[8, App.C1] can be rephrased as follows: If in a Heisenberg algebra \([d, d^\dagger] = 1\) one uses the
non-standard antilinear anti-involution \( \sigma(d) = \frac{q}{1 - q}d^\dagger \) to define a sesquilinear form contravariant
w.r.t. it (so that e.g. \( \langle d, d \rangle_\sigma = q/(1 - q) \)) the resulting expectation values can be interpreted
as trace functionals w.r.t. \( q^{d^\dagger d} \) and the sesquilinear form contravariant w.r.t. \( \omega(q) = q^\dagger \). More
generally, this phenomenon is related to the GNS construction. A second comment concerns
the possible physical significance of the QFTs defined in terms of the deformed form factors.
Since the scattering theory - and hence the bound state structure - is preserved, the deformation
affects only the kinematical arena. One might therefore expect that the \( \beta \)-deformed QFTs admit
an interpretation as QFTs living on some deformed spacetime.
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