DECOMPOSITION OF GENERALIZED PETERSEN GRAPHS INTO CLAWS, CYCLES AND PATHS

M. SUBBULAKSHMI\textsuperscript{1}, I. VALLIAMMAL\textsuperscript{2,†,∗}

\textsuperscript{1}Department of Mathematics, G.V.N. College, Kovilpatti, Thoothukudi, Tamil Nadu-628502, India
\textsuperscript{2}Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-12, Tamil Nadu, India

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. Let $G = (V, E)$ be a finite graph with $n$ vertices. Let $n$ and $k$ be positive integers where $n \geq 3$ and $1 \leq k < \frac{n}{2}$. The Generalized Petersen Graph $GP(n,k)$ is a graph with vertex set $\{u_0, u_1, u_2, \ldots, u_{n-1}, v_0, v_1, v_2, \ldots, v_{n-1}\}$ and edge-set consisting of all edges of the form $u_iu_{i+1}, u_iv_i$ and $v_iv_{i+k}$ where $0 \leq i \leq n - 1$, the subscripts being reduced modulo $n$. Obviously $GP(n,k)$ is always a cubic graph and $GP(5,2)$ is the well-known Petersen graph.

In this paper, we show that $GP(n,1), n \geq 3$ can be decomposed into $n$ copies of $S_3$ if $n$ is even and $P_4$ and $(n-1)$ copies of $S_3$ if $n$ is odd. Also, we show that $GP(n,2), n \geq 5$ can be decomposed into $\frac{n}{2}$ copies of $S_3$, 2 copies of $C_2$ and $\frac{n}{2}$ copies of $P_5$ if $n$ is even and $C_n$, $P_4$, $\left\lceil \frac{n}{2} \right\rceil$ copies of $S_3$ and $(\left\lfloor \frac{n}{2} \right\rfloor - 1)$ copies of $P_2$ if $n$ is odd.

$GP(n,2), n \geq 5$ and $n = 3d, d = 2, 3, \ldots$ can be decomposed into $2d$ copies of $S_3$ and $d$ copies of $P_4$. $GP(n,2), n \geq 5$ and $n = 4d, d = 2, 3, \ldots$ can be decomposed into $3d$ copies of $S_3$ and $d$ copies of $P_4$. $GP(n,3), n \geq 8$ can be decomposed into $n$ copies of $S_3$ if $n$ is even and $P_6, P_2$ and $(n-2)$ copies of $S_3$ if $n$ is odd.

Keywords: Generalized Petersen Graph; Decomposition; claw.

2010 AMS Subject Classification: 05C70, 05C51, 05C38.

\textsuperscript{∗}Corresponding author
E-mail address: valli.vasanthi@gmail.email
\textsuperscript{†}Reg. No: 18222052092006
Received December 16, 2020
1. **Introduction**

The origin of graph decomposition is from the combinatorial problems most of which emerged in the 19th century. The first one was the prize question for the year in the *Lady's and Gentlemen's Diary* of 1844 stated by W.S.B.Woolhouse: Determine the number of combinations that can be made out of $n$ symbols, $p$ symbols in each such that no combination of $q$ symbols which may appear in any one of them may appear in any other. When every $q$ symbols appear in exactly one of the $p$ subsets, such a configuration is known as a *Steiner system*. When $q = 2$ and $p = 3$, the configuration is known as *Steiner Triple system*. In 1847, T.P.Kirkman settled the existence question for the *Steiner Triple system*. It is equivalent to the decomposition of $K_n$ into triangles.

The other best problems are *Kirkman’s problem* of 15 strolling school girls, *Dudeney problem* of 9 handcuffed prisoners, *Euler’s problem* of 36 army officers, *Kirkman’s problem of knights*, *Lucas’ dance around problem* and *the four color problem*. However, the earliest works in the direction are not explicitly related to *graph decompositions*. The paper dealing directly with *graph decomposition* appeared after the turn of the 19th century. Since then, the interest in graph decomposition has been on increase and a real up surge is witnessed after 1950. Now a days, *graph decomposition* rank among the most prominent areas of graph theory and combinatorics. Many types of decomposition have been well studied in the literature. There are lot of applications of *decomposition of graph* which include group testings, DNA library screening, scheduling problems, sharing scheme and synchronous optical networks.

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [8] and to study about the decomposition of graphs into paths, stars and cycles is referred to [1], [2], [3] and [4].

As usual $C_n$ denotes the cycle of length $n$, $P_{n+1}$ denotes the path of length $n$ and $S_3$ denotes the claw.
2. Basic Definitions and Examples of Generalized Petersen Graphs

In this section, we see some basic definitions of graph decomposition, Generalized Petersen Graph and examples of Generalized Petersen Graph.

Let \( L = \{H_1, H_2, \ldots, H_r\} \) be a family of subgraphs of \( G \). An \( L \)-decomposition of \( G \) is an edge-disjoint decomposition of \( G \) into positive integer \( \alpha_i \) copies of \( H_i \), where \( i \in \{1, 2, \ldots, r\} \). Furthermore, if each \( H_i \) (\( i \in \{1, 2, \ldots, r\} \)) is isomorphic to a graph \( H \), we say that \( G \) has an \( H \)-decomposition. It is easily seen that \( \sum_{i=1}^{r} \alpha_i \cdot e(H_i) = e(G) \) is one of the obvious necessary conditions for the existence of a \( \{H_1, H_2, \ldots, H_r\} \)-decomposition of \( G \). For convenience, we call the equation, \( \sum_{i=1}^{r} \alpha_i \cdot e(H_i) = e(G) \), a necessary sum condition.

Let \( n \) and \( k \) be positive integers, \( n \geq 3 \) and \( 1 \leq k < \frac{n}{2} \). The Generalized Petersen Graph \( GP(n,k) \) is a graph with vertex set \( \{u_0, u_1, u_2, \ldots, u_{n-1}, v_0, v_1, v_2, \ldots, v_{n-1}\} \) and edge-set consisting of all edges of the form \( u_iu_{i+1}, u_iv_i \) and \( v_iv_{i+k} \) where \( 0 \leq i \leq n-1 \), the subscripts being reduced modulo \( n \).

Obviously \( GP(n,k) \) is always a cubic graph and \( GP(5,2) \) is the well-known Petersen graph. Thus \( GP(5,2) \) is the Petersen graph and is represented in Figure 1(d).

Also \( GP(3,1), \ GP(4,1), \ GP(5,1) \) and \( G(10,4) \) are represented in Figure 1(a), Figure 1(b), Figure 1(c) and Figure 1(e).

3. Decomposition of Generalized Petersen Graph \( GP(n,1) \) into Claws and Path

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph \( GP(n,1) \) into claws and paths.

**Theorem 3.1.** \( GP(n,1), n \geq 3 \) can be decomposed into \( n \) copies of \( S_3 \) if \( n \) is even and \( P_4 \) and \( (n-1) \) copies of \( S_3 \) if \( n \) is odd.
Proof. Let $V(GP(n, 1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and $E(GP(n, 1)) = \{e_0, e_1, ..., e_{n-1}, e'_0, e'_1, ..., e'_{n-1}, e''_0, e''_1, ..., e''_{n-1}\}$ where $e_i$ and $e'_i$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e''_i$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Case 1. If $n$ is even. That is $n = 2d, d = 2, 3, ...$

Let $E_i = \{e_i, e''_i, e'_{i+1}\}$ where $i = 0, 2, 4, ..., 2d - 2$ and $E_j = \{e'_j, e''_j, e'_{j+1}\}$ where $j = 1, 3, 5, ..., 2d - 1$. 

FIGURE 1. Examples of Generalized Peterson Graphs
Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms $d$ disjoint claws. Therefore $GP(n, 1)$ can be decomposed into $d + d = 2d = n$ disjoint claws. Hence $GP(n, 1)$ can be decomposed into $n$ distinct claws.

**Case 2.** If $n$ is odd. That is $n = 2d - 1, d = 2, 3, 4, \ldots$

Let $E_i = \{e_i, e''_{i+1}, e_{i+1}\}$ where $i = 0, 2, 4, \ldots, 2d - 4$ and $E_j = \{e'_j, e''_{j+1}, e'_{j+1}\}$ where $j = 1, 3, 5, \ldots, 2d - 3$ and $E_k = \{e_{2d-2}, e''_0, e'_0\}$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms $d$ disjoint claws and the edge induced subgraph $< E_k >$ forms a path of length 4 (i.e, $P_4$). Therefore $GP(n, 1)$ can be decomposed into $d + d = 2d = n - 1$ disjoint claws and $P_4$. Hence $GP(n, 1)$ can be decomposed into $n - 1$ disjoint claws and $P_4$. □

**Illustration:** The above theorem can be explained through the following Figure 2.

![Figure 2](image)

The above two figures represents decomposition of $GP(8, 1)$ into 8 claws and decomposition of $GP(9, 1)$ into 8 claws and a path $P_4$ respectively.

All edges of the claws and path differentiated in the above Figure 2.
4. Decomposition of Generalized Petersen Graph $GP(n, 2)$ into Claws and Paths

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph $GP(n, 2)$ into claws, cycles and paths when $n$ is a multiple of 2, 3 and 4 respectively.

**Theorem 4.1.** $GP(n, 2), n \geq 5$ can be decomposed into $\frac{n}{2}$ copies of $S_3$, 2 copies of $C_2$ and $\frac{n}{2}$ copies of $P_2$ if $n$ is even and $C_n$, $P_4$, $\left\lfloor \frac{n}{4} \right\rfloor$ copies of $S_3$ and $\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right)$ copies of $P_2$ if $n$ is odd.

**Proof.** Let $V(GP(n, 1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and $E(GP(n, 2)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+2} / 0 \leq i \leq n-1\}$ with indices taken modulo $n$.

Now, $E(GP(n, 2)) = \{e_0, e_1', ..., e_{n-1}, e_0', e_1', ..., e_{n-1}'\}$ where $e_i$ and $e_i' \leq i \leq n-1$ are edges in outer and inner cycles and $e_i$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

**Case 1.** If $n$ is even. That is $n = 2d$, $d = 3,4, ...$

Let $E_i = \{e_i, e_{i+1}', e_{i+1}\}$ where $i = 0, 2, 4, ..., 2d - 2$, $E_j = \{e_0', e_2', e_4', ..., e_{2d-2}'\}$, $E_k = \{e_1', e_3', e_5', ..., e_{2d-1}'\}$ and $E_l = \{e_0'', e_2'', e_4'', ..., e_{2d-2}'\}$.

Then, the edge induced subgraph $< E_i >$ forms $d = \frac{n}{2}$ disjoint claws, each edge induced subgraph $< E_j >$ and $< E_k >$ forms two disjoint cycles of length $d = \frac{n}{2}$ and the edge induced subgraph $< E_l >$ forms $d = \frac{n}{2}$ disjoint path of length one (i.e, $P_2$). Hence $GP(n, 2)$ can be decomposed into $\frac{n}{2}S_3, 2C_2$ and $\frac{n}{2}P_2$.

**Case 2.** If $n$ is odd. That is $n = 2d - 1$, $d = 3,4, ...$

Let $E_i = \{e_i, e_{i+1}', e_{i+1}\}$ where $i = 0, 2, 4, ..., 2d - 4$, $E_j = \{e_0', e_2', e_4', ..., e_{2d-2}', e_1', e_3', e_5', ..., e_{2d-3}'\}$, $E_k = \{e_{2d-2}'', e_{2d-2}', e_0''\}$ and $E_l = \{e_2'', e_4'', e_6'', ..., e_{2d-4}''\}$.

Then, the edge induced subgraph $< E_i >$ forms $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint claws, the edge induced subgraph $< E_j >$ forms a cycle of length $n$, the edge induced subgraph $< E_k >$ forms a path of length three (i.e, $P_4$) and the edge induced subgraph $< E_l >$ forms $\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right)$ disjoint path of length one (i.e, $P_2$). Hence $GP(n, 2)$ can be decomposed into $\left\lfloor \frac{n}{2} \right\rfloor$ claws, $C_n, P_4$ and $\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right)P_2$. □

**Theorem 4.2.** $GP(n, 2), n \geq 5$ and $n = 3d, d = 2, 3, ...$ can be decomposed into $2d$ copies of $S_3$ and $d$ copies of $P_4$. 

Proof. Let $V(GP(n, 1)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and $E(GP(n, 2)) = \{u_iu_{i+1}, u_iv_i, v_{i+2}v_i / 0 \leq i \leq n-1\}$ with indices taken modulo $n$. Now, $E(GP(n, 2)) = \{e_0, e_1, \ldots, e_{n-1}, e_0', e_1', \ldots, e_{n-1}'\}$ where $e_i$ and $e'_i$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e''_i$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Given $n \geq 5$ and $n = 3d$, $d = 2, 3, \ldots$

Let $E_i = \{e_i, e''_{i+1}, e_{i+1}\}$ where $i = 0, 3, 6, \ldots, 3d - 3$, $E_j = \{e_j', e_{j+2}', e_{j+2}\}$ where $j = 1, 4, 7, \ldots, 3d - 2$, and $E_k = \{e_k, e_k', e_k'\}$ where $k = 2, 5, 8, \ldots, 3d - 1$.

Then each edge induced subgraph $< E_i >$ and $< E_j >$ forms $d$ disjoint claws and the edge induced subgraph $< E_k >$ forms $d$ disjoint paths of length 3 (i.e, $P_3$). Hence $GP(n, 2), (n \geq 5, n = 3d)$ can be decomposed into $2d$ claws and $dP_4$. $\square$

**Theorem 4.3.** $GP(n, 2), n \geq 5$ and $n = 4d, d = 2, 3, \ldots$ can be decomposed into $3d$ copies of $S_3$ and $d$ copies of $P_4$.

Proof. Let $V(GP(n, 1)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and $E(GP(n, 2)) = \{u_iu_{i+1}, u_iv_i, v_{i+2}v_i / 0 \leq i \leq n-1\}$ with indices taken modulo $n$. Now, $E(GP(n, 2)) = \{e_0, e_1, \ldots, e_{n-1}, e_0', e_1', \ldots, e_{n-1}'\}$ where $e_i$ and $e'_i$, $0 \leq i \leq n-1$ are edges in outer and inner cycles and $e''_i$, $0 \leq i \leq n-1$ are intermediate edges in the two cycles.

Given $n \geq 5$ and $n = 4d$, $d = 2, 3, \ldots$

Let $E_i = \{e_i, e''_{i+1}, e_{i+1}\}$ where $i = 0, 4, 8, \ldots, 4d - 4$, $E_j = \{e_j', e_{j+2}', e_{j+2}\}$ where $j = 1, 4, 7, \ldots, 3d - 2$, and $E_k = \{e_{k-1}, e_{k-1}', e_k\}$ where $k = 3, 7, \ldots, 4d - 1$.

Then, the edge induced subgraph $< E_i >$ forms $d$ disjoint claws, the edge induced subgraph $< E_j >$ forms $2d$ disjoint claws and the edge induced subgraph $< E_k >$ forms $d$ disjoint path of length three (i.e, $P_3$). Hence $GP(n, 2), (n \geq 5, n = 4d)$ can be decomposed into $3d$ claws and $dP_4$. $\square$
5. Decomposition of Generalized Petersen Graph $GP(n,3)$ into Claws and Paths

In this section, we characterize the theorem of decomposition of Generalized Petersen Graph $GP(n,3)$ into claws and paths.

**Theorem 5.1.** $GP(n,3), n \geq 8$ can be decomposed into $n$ copies of $S_3$ if $n$ is even and $P_6$, $P_2$ and $(n - 2)$ copies of $S_3$ if $n$ is odd.

**Proof.** Let $V(GP(n,1)) = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and $E(GP(n,2)) = \{u_iu_{i+1}, u_iv_i, v_{i+3} \mid 0 \leq i \leq n - 1\}$ with indices taken modulo $n$.

Now, $E(GP(n,2)) = \left\{e_0, e_1, ..., e_{n-1}, e'_0, e'_1, ..., e'_{n-1}, e''_0, e''_1, ..., e''_{n-1}\right\}$ where $e_i$ and $e'_i$, $0 \leq i \leq n - 1$ are edges in outer and inner cycles and $e''_i$, $0 \leq i \leq n - 1$ are intermediate edges in the two cycles.

**Case 1.** If $n$ is even and $n \geq 8$.

Let $E_i = \left\{e_i, e''_{i+1}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., n - 2$,

and $E_j = \left\{e'_j, e''_{j+3}, e'_{j+3}\right\}$ where $j = 1, 3, 5, ..., n - 1$.

Then each edge induced subgraph $<E_i>$ and $<E_j>$ forms $\frac{n}{2}$ disjoint claws. Hence $GP(n,3), n \geq 8$ can be decomposed into $nS_3$ if $n$ is even.

**Case 2.** If $n$ is odd and $n \geq 8$.

Let $E_i = \left\{e_i, e''_{i+1}, e_{i+1}\right\}$ where $i = 0, 2, 4, ..., n - 3$, $E_j = \left\{e'_j, e''_{j+3}, e'_{j+3}\right\} \cup \left\{e''_{n-4}, e''_{n-1}, e''_{0}, e'_{0}\right\}$ where $j = 1, 3, 5, ..., n - 6$, $E_k = \left\{e''_{n-4}, e''_{n-1}, e''_{0}, e'_{0}\right\}$ and $E_l = \left\{e''_{n-2}\right\}$.

Then, the edge induced subgraph $<E_i>$ forms $\left\lfloor \frac{n}{2} \right\rfloor$ disjoint claws, the edge induced subgraph $<E_j>$ forms $\left\lfloor \frac{n}{2} \right\rfloor - 1$ disjoint claws, the edge induced subgraph $<E_k>$ forms a path of length five (i.e, $P_6$) and the edge induced subgraph $<E_l>$ forms a path of length one (i.e, $P_2$). Hence $GP(n,3), n \geq 8$ can be decomposed into and $(n - 2)$ disjoint claws, $P_6$ and $P_2$ if $n$ is odd. \(\square\)

**Illustration:** The above theorem can be explained through the following Figure 3.

All edges of the claws and path differentiated in Figure 3.

**Remark 5.2.** This construction does not work with $GP(7,3)$.

$GP(7,3)$ can be decomposed into $5S_3, C_5$ and $P_2$. 
The above two figures represents decomposition of $GP(8, 3)$ into 8 claws and decomposition of $GP(9, 1)$ into 7 claws, a path $P_6$ and a path $P_2$ respectively.

**Proof.** Let $E_i = \{e_i, e''_{i+1}, e_{i+1}\}$ where $i = 0, 2, 4$, $E_j = \{e'_j, e''_{j+3}, e''_{j+3}\} \cup \{e'_{n-1}, e''_2, e_2\}$ where $j = 1$, $E_k = \{e'_{n-4}, e''_{n-1}, e_{n-1}, e''_0, e_0\}$ and $E_l = \{e'_{n-2}\}$.

Then the edge induced subgraph $<E_i>$ forms $\left\lfloor \frac{n}{2} \right\rfloor = 3$ disjoint claws, the edge induced subgraph $<E_j>$ forms $\left\lfloor \frac{n}{2} \right\rfloor - 1 = 2$ disjoint claws, the edge induced subgraph $<E_k>$ forms a cycle of length five (i.e, $C_5$) and the edge induced subgraph $<E_l>$ forms a path of length one (i.e, $P_2$). Hence $GP(7, 3)$ can be decomposed into $5S_3, C_5$ and $P_2$.

**Illustration:** The above remark-5.2 can be explained through the following Figure 4.
The above figure represents decomposition of $GP(7, 3)$ into 5 claws, a cycle $C_5$ and a path $P_2$ respectively.

All edges of the claws, cycle and path differentiated in the above Figure 4.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

**REFERENCES**

[1] T.W. Shyu, Decompositions of Complete Graphs into Paths and Cycles, Ars Comb. 97 (2010), 257-270.
[2] T.W. Shyu, Decompositions of Complete Graphs into Paths and Stars, Discrete Math. 310 (2010), 2164-2169.
[3] S. Arumugam, I. Sahul Hamid and V.M. Abraham, Decomposition of graphs into Paths and Cycles, J. Discrete Math. 2013 (2013), 721051.
[4] T.W. Shyu, Decompositions of Complete Graphs into Cycles and Stars, Graphs Comb. 29 (2013), 301-313.
[5] R. Nedela, M. Škoviera, Which generalized petersen graphs are cayley graphs? J. Graph Theory. 19 (1995), 1–11.
[6] M. Sujatha, R. Sasikala, R. Mathivanan, A study on Decomposition of graphs, Emerging Trends in Pure and Applied Mathematics (ETPAM-2018), March 2018.

[7] M.E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory, 6 (1969), 152-164.

[8] J. A. Bondy, U.S.R. Murty, Graph theory with applications, The Macmillan Press Ltd, New York, 1976.

[9] D. West, Introduction to Graph Theory, third edn. Prentice Hall, Saddle River, 2007.

[10] M. Subbulakshmi, I. Valliammal, Decomposition of Generalized Petersen Graphs into Paths and Cycles, SSRG Int. J. Math. Trends Technol. Special Issue NCPAM (2019), 62-67.