Work done by friction during a complete inelastic collision

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Abstract. The work done by the Coulomb frictional force has been addressed by several authors, but still persist some open issues. Friction increases the microscopic energy of a system, and the work done by it needs to satisfy the First Law of Thermodynamics. We present an approach that allows a smooth transition between the works calculated within the domains of newtonian mechanics and thermodynamics. This is done for a symmetric complete inelastic collision where it is possible to calculate the work done by friction on each colliding body. Within certain approximation this collision allows to track analytically some of its physical quantities such as momentum and kinetic energy.

1. A COMPLETE INELASTIC COLLISION

Two blocks of the same size, of lengths $L$ and masses $M_1$ and $M_2$, can slide on a frictionless surface that is divided into two floors. Between both blocks there is a Coulomb friction force $f_k = \pm \mu_k N$, where $\mu_k$ is the coefficient of kinetic friction and $N$ is the magnitude of the normal force between both blocks. At $t = 0$ block-1 front end reaches the step, as shown in Fig. 1, starting to slide on and to drag block-2. For reasons of symmetry we study a collision where at

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{The collision starts at $t = 0$. The initial velocity $v_0$ needs to be determined.}
\end{figure}

the end of it both blocks get stuck together exactly one on top of the other.
1.1. Boundary conditions
We describe this collision in the laboratory reference frame, placing its origin at the step. The collision starts at \( t = 0 \), when each block center of mass are \( x_1(0) = -L/2 \) and \( x_2(0) = L/2 \), and their respective velocities are \( v_1(0) = v_0 \) and \( v_2(0) = 0 \), where \( v_0 \) is to be determined. The upper block leaves the upper floor at \( t = t' \). While by definition \( x_1(t') = L/2 \), \( x_2(t') \) is to be determined. The velocities \( v_1(t') \) and \( v_2(t') \) are unknowns. Both blocks get stuck together at \( t = t'' \). By conservation of momentum \( v_1(t'') = v_2(t'') \), but \( x_1(t'') = x_2(t'') \) are unknowns.

1.2. Equations of motion
During \( 0 \leq t \leq t' \) the magnitude of \( f_k \) increases steadily from zero to its maximum value, and the blocks motion is coupled. During \( t' \leq t \leq t'' \), \( f_k \) is constant and the blocks motion becomes uncoupled.

1.2.1. Equations of motion for \( 0 \leq t \leq t' \) Let us denote by \( \beta, \gamma \) and \( \delta \) the lengths of the upper block that are respectively on top of the upper floor, on top of the gap between the step and the lower block, and on top of the lower block:

\[
\beta(t) = \frac{L}{2} - x_1(t), \\
\gamma(t) = x_2(t) - \frac{L}{2}, \\
\delta(t) = L + x_1(t) - x_2(t).
\]

The length \( \gamma(t) \) is supported proportionally by the upper floor and the lower block, so the length of the upper block that is supported by the lower one is \( \sigma(t) = \delta(t) + p(t)\delta(t) \) where \( p(t) = (2x_2(t) - L)/(3L - 2x_2(t)) \). The magnitude of the normal force between both blocks is

\[
N(t) = 2\frac{L + x_1(t) - x_2(t)}{3L - 2x_2(t)}M_1g,
\]

and the dynamic equations for \( x_1 \) and \( x_2 \) satisfy the following coupled set of second order nonlinear differential equations:

\[
M_1\frac{d^2x_1}{dt^2} = -2\mu_k\frac{L + x_1(t) - x_2(t)}{3L - 2x_2(t)}M_1g, \\
M_2\frac{d^2x_2}{dt^2} = +2\mu_k\frac{L + x_1(t) - x_2(t)}{3L - 2x_2(t)}M_1g.
\]

Making the approximation that \( \gamma(t) \) is supported evenly by the upper floor and the lower block, we arrive to a coupled set of linear differential equations with analytic solutions. Within this approximation \( \sigma(t) = \delta(t) + (\gamma(t)/2) \). Therefore, the magnitude of \( N \) will be approximated by

\[
N(t) = \frac{3L + 4x_1(t) - 2x_2(t)}{4L}M_1g,
\]

and

\[
M_1\frac{d^2x_1}{dt^2} = -\mu_k\frac{3L + 4x_1(t) - 2x_2(t)}{4L}M_1g, \\
M_2\frac{d^2x_2}{dt^2} = +\mu_k\frac{3L + 4x_1(t) - 2x_2(t)}{4L}M_1g.
\]
1.2.2. Equations of motion for $t' \leq t \leq t''$ Within this interval the upper block is supported totally by the lower one, so $f_k = \pm \mu_k M_1 g$, and

$$M_1 \frac{d^2 x_1}{dt^2} = -\mu_k M_1 g$$  \hspace{1cm} (10)$$

$$M_2 \frac{d^2 x_2}{dt^2} = +\mu_k M_1 g.$$  \hspace{1cm} (11)

1.3. Solution for two identical blocks

For two identical blocks, $t' = \tau \theta$, where $\tau = \sqrt{2L/3\mu_k g}$, and $\theta$ satisfies the transcendental equation

$$16\theta^2 + 16\theta \sin \theta - 16 \cos^2 \theta + 24 \cos \theta - 35 = 0.$$  \hspace{1cm} (12)

The initial velocity is

$$v_0 = \frac{3}{\theta + 2 \sin \theta} \left( \frac{L}{\tau} \right),$$  \hspace{1cm} (13)

and the collision time is

$$t'' = t' + \frac{3}{4} \left( \frac{4 \cos \theta - 1}{\theta + 2 \sin \theta} \right) \tau.$$  \hspace{1cm} (14)

Each block center of mass position satisfies

$$x_1(t) = \frac{L}{2} + \frac{1}{3} t_0 \left( t + 2 \tau \sin \left( \frac{t}{\tau} \right) \right)$$  \hspace{1cm} (15)$$

$$x_2(t) = \frac{L}{2} + \frac{2}{3} t_0 \left( t - \tau \sin \left( \frac{t}{\tau} \right) \right)$$  \hspace{1cm} (16)

for $0 \leq t \leq t'$, and

$$x_1(t) = x_1(t') + v_1(t') \left( t - t' \right) - \frac{1}{2} \mu_k g \left( t - t' \right)^2$$  \hspace{1cm} (17)$$

$$x_2(t) = x_2(t') + v_2(t') \left( t - t' \right) + \frac{1}{2} \mu_k g \left( t - t' \right)^2$$  \hspace{1cm} (18)

for $t' \leq t \leq t''$.

2. WORK DONE BY FRICTION

Within the realm of newtonian mechanics the work done by $f_k$ on each block is the integral of the product of $f_k$ times each block center of mass displacement. This is known as pseudo-work [1] and corresponds to each block center of mass kinetic energy change. When dealing with friction, the work needs to be generalized to satisfy the First Law of Thermodynamics. [2, 3, 4]

The system formed by both blocks is isolated i.e. there is no external work done on it nor heat transfer with the adiabatic floor or the air. Its total energy can be written as the sum of each block center of mass kinetic energy, which we call macroscopic energy, plus the kinetic and potential energies of the atoms vibrating around their equilibrium positions within each block, which we refer to as microscopic energy. During the collision the system macroscopic energy decreases by $\Delta E_{\text{mac}} = -(1/2)K_0$. This drop is compensated exactly by an increase in the system microscopic energy:

$$\Delta E_{\text{mic}} = (1/2)K_0.$$  \hspace{1cm} (19)
2.1. Thermodynamic work done by $f_k$ on the upper block in the laboratory frame

To start with, let us consider just the upper block as the system. This system is open. In general, the energy of any open system can be affected both by external work and by heat transfer. The external work can affects both the kinetic energy of the block center of mass and the microscopic energy within it.

Both blocks have the same heat capacity and before the collision they have the same temperature. As the collision takes place any heat flow from the upper-block-near-front toward the lower block is compensated, by symmetry, by an inverse heat flow from the lower-block-rear-end toward the upper block. Therefore the net heat transfer through each block boundary is zero.

The only force that does external work on the upper block is $f_k$ and we find it convenient to split this work into pseudo-work plus microscopic work.

Between $0 \leq t \leq t'$ occurs displacement I, and between $t' \leq t \leq t''$ occurs displacement II. From Eq. (15) $f_k$ on the upper block during its displacement I is

$$f_{k \text{ in I}}^1(t) = -\mu_k N(t),$$

where

$$N(t) = Mg \left( \frac{3}{\theta + 2\sin \theta} \right) \sin \left( \sqrt{\frac{3\mu_k g}{2L}} t \right),$$

is the normal force magnitude. In displacement II $f_k$ is

$$f_{k \text{ in II}}^1(x) = -\mu_k Mg.$$ (22)

The pseudo-works during displacements I and II are

$$W_{\text{in I}}^1 = -\frac{1}{9} K_0 (1 - \cos \theta + \sin^2 \theta),$$

$$W_{\text{in II}}^1 = -\frac{4}{5} K_0 \left( \cos^2 \theta + \cos \theta - \frac{5}{16} \right).$$ (24)

Therefore the total pseudo-work on the upper block is

$$W_{\text{pseudo-work on block 1}} = -\frac{3}{4} K_0.$$ (25)

By symmetry the increase (19) must be distributed evenly between both blocks. Therefore $\Delta E_{\text{mic, 1}} = (1/4) K_0$. Since there is no heat contribution, this microscopic energy increase corresponds just to microscopic work done by friction on the upper block, i.e.,

$$W_{\text{microscopic on block 1}} = \frac{1}{4} K_0.$$ (26)

The thermodynamic work on the upper block is the sum of (25) and (26):

$$W_{\text{thermodynamic on block 1}} = -\frac{1}{2} K_0.$$ (27)

This is the total energy change of the upper block.
2.2. Thermodynamic work done by $f_k$ on the lower block in the laboratory frame

The pseudo-works on the lower block are

$$W_{\text{on } 1}^{\text{on } 2} = \frac{4}{9} K_0 \left( 2 - 2 \cos \theta - \sin^2 \theta \right),$$

and

$$W_{\text{on } 2}^{\text{on } 2} = \frac{4}{9} K_0 \left( \cos^2 \theta + \cos \theta - \frac{5}{18} \right)
- \frac{2}{27} K_0 (\theta - \sin \theta)(\theta + 2 \sin \theta).$$

Adding Eqs. (28) and (29) we get

$$W_{\text{pseudo-work on block 2}}^{\text{whole displacement}} = \frac{1}{4} K_0.$$  \hspace{1cm} (30)

The microscopic work on the lower block is

$$W_{\text{microscopic on block 2}}^{\text{whole displacement}} = \frac{1}{4} K_0.$$  \hspace{1cm} (31)

Therefore, the thermodynamic work on the lower block is

$$W_{\text{thermodynamic on block 2}}^{\text{whole displacement}} = \frac{1}{2} K_0.$$  \hspace{1cm} (32)

Adding expressions (27) and (32), we see that the total work done by friction on the whole system is zero.

3. CONCLUDING REMARKS

We have studied a particular complete inelastic collision where, by symmetry, the net heat transfer from one block to the other is zero. This allows to calculate the microscopic work and, therefore, the thermodynamic work on each block separately. For calculating it, we generalized the mechanical work adding a new term to become the thermodynamical work. Note that in this problem it would have been not possible to follow the approach of Sherwood and Bernard of introducing an effective displacement.

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