Correlators in tensor models from character calculus

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Abstract

We explain how the calculations of arXiv:1704.08648, which provided the first evidence for non-trivial structures of Gaussian correlators in tensor models, are efficiently performed with the help of the (Hurwitz) character calculus. This emphasizes a close similarity between technical methods in matrix and tensor models and supports a hope to understand the emerging structures in very similar terms. We claim that the $2^m$-fold Gaussian correlators of rank $r$ tensors are given by $r$-linear combinations of dimensions with the Young diagrams of size $m$. The coefficients are made from the characters of the symmetric group $S_m$ and their exact form depends on the choice of the correlator and on the symmetries of the model. As the simplest application of this new knowledge, we provide simple expressions for correlators in the Aristotelian tensor model as tri-linear combinations of dimensions.

1 Introduction

Emerging interest \cite{1}-\cite{20} to tensor models \cite{21} allows one to begin their systematic study. In the framework of non-linear algebra \cite{22}, one does not expect any essential difference between the tensor and matrix calculi, and the only difference is that the latter is well developed, while the former one, not. Within the systematic approach, the development should proceed in steps, and the first step is evaluating the Gaussian correlators \cite{23-25} targeted at finding the underlying structures and their adequate analytic description. In \cite{20}, we demonstrated that the structures are indeed present and, non-surprisingly, similar to those in matrix models. To reveal them in full generality and beauty, one, however, needs to evaluate a lot of quantities, and thus needs an efficient technique for this. The goal of this letter is to claim that the most effective calculus of this kind based on the character expansions \cite{26} and Hurwitz theory \textit{a la} \cite{27} is directly extended from matrix models case \cite{28-30} to the tensor case. A very similar observation is also made in a very recent paper \cite{31}. In the present letter, we show how the complicated expressions from \cite{20} are drastically simplified by use of the character/Hurwitz calculus.

As explained in big detail in \cite{20} and \cite{30}, the simplest for the Gaussian calculus is the rectangular complex matrix model (RCM) \cite{32-34}, its tensor liftings are now called \textit{rainbow models} \cite{8}. The field in this model is the $N_1 \times N_2$ matrix $M$, and the correlators are labeled by Young diagrams $\Lambda = \{m_1 \geq m_2 \geq \ldots \geq m_l > 0\}$:

\begin{equation}
O_\Lambda = \left\langle \prod_{p=1}^{l_\Lambda} (\text{Tr} M \bar{M})^{m_p} \right\rangle
\end{equation}

where $\bar{M} = M^\dagger$ is Hermitian conjugate, i.e. an $N_2 \times N_1$ matrix, while the averages are defined as the $2N_1N_2$-fold integrals

\begin{equation}
\left\langle \ldots \right\rangle = \int e^{-\text{Tr} M \bar{M}} d^2 M, \quad \text{with} \quad d^2 M = \prod_{a=1}^{N_1} \prod_{\alpha=1}^{N_2} d^2 M_{a\alpha}
\end{equation}

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In [20,29] and [30] we described general formulas for the correlators [1]. They possess a simple interpretation in terms of the Hurwitz calculus and were actually present also in [28]. Moreover, their tensorial generalization was also just considered in [31], unfortunately, without comparison to the results of [20]. Schematically, expressions of this kind for various models look like (see the main text for the notation)

Hermitian matrix model:
\[
O^{(2m)}_\sigma = \sum_{\gamma} N^{\#(\sigma \gamma)}_{S_m}
\]

Rectangular matrix model:
\[
O^{(m)}_\sigma = \sum_{\gamma} N^{\#(\sigma \gamma)}_{S_m} N_{1,2}^{\#(\sigma \gamma)}
\]

Rainbow tensor model:
\[
O^{(m)}_{\sigma_1 \otimes \ldots \otimes \sigma_r} = \sum_{\gamma} \prod_{s=1}^r N^{\#(\sigma_s \gamma)}_{S_m}
\]

The Hurwitz/character calculus allows one to re-express the sums over permutations in these formulas through multi-linear combinations of dimensions, like it was done in [30] in the matrix models case. As we demonstrate in this letter, the same is possible in the tensor case, and the formulas are equally simple and powerful.

2 Description in terms of permutations: RCM

Within the Hurwitz approach, one interprets the Young diagram \( R = \{ r_p \} \); \( r_1 \geq r_2 \geq \ldots \geq r_k \geq 0 \), \( r_i \in \mathbb{Z} \) as that labeling a representation of the symmetric groups \( S_m \) with \( m = |R| = \sum_{p=1}^k r_p \) (and also a conjugacy class in the representation) rather than that of the linear groups (the essence of this calculus is exactly the explicit description of the Schur-Weyl duality between these two interpretations). An efficient parametrization of the correlators is possible in terms of permutations \( \sigma \in S_m \),

\[
O_\sigma = \left< \prod_{p=1}^{l_A} (\text{Tr} M M)_{m_p} \right> = \left< \prod_{i=1}^{m} M_{a_i a_i} \bar{M}^{a_i a_i} \right> \tag{3}
\]

for example,

\[
O_{[1]} = \left< \text{Tr} M \bar{M} \right> = \left< M_{a a} \bar{M}^{a a} \right>, \quad \sigma = \text{id} = (1) = [1] \in S_1
\]

\[
O_{[1]} = \left< (\text{Tr} M \bar{M})^2 \right> = \left< M_{a_1 a_1} M_{a_2 a_2} \bar{M}^{a_1 a_1} \bar{M}^{a_2 a_2} \right>, \quad \sigma = \text{id}(1)(2) = [11] \in S_2
\]

\[
O_{[2]} = \left< \text{Tr} (M \bar{M})^2 \right> = \left< M_{a_1 a_1} M_{a_2 a_2} \bar{M}^{a_1 a_1} \bar{M}^{a_2 a_2} \right>, \quad \sigma = (12) = [2] \in S_2
\]

\[
\ldots
\]

More generally, \( O_{[m]} = \left< (\text{Tr} M \bar{M})^m \right> \) is associated with \( \sigma = \text{id} = [1^m] \in S_m \), while \( O_{[m]} = \left< \text{Tr} (M \bar{M})^m \right> \), with the longest cycle \( \sigma = (12 \ldots m) \in S_m \). In general, the number \( l_A \) of traces is the number of cycles in the associated \( \sigma \in S_{|A|} \) (the number of lines in the Young diagram \( A \) describing the conjugacy class of \( \sigma \)).

It remains to apply the Wick theorem

\[
\left< \prod_{i=1}^{m} M_{a_i a_i} \bar{M}^{b_i b_i} \right> = \sum_{\gamma} \prod_{i=1}^{m} \delta_{a_i}^{b_i} \delta_{\gamma a_i}^{\gamma b_i} \tag{5}
\]

in order to obtain

\[
O^{RCM}_\sigma = \frac{1}{|\sigma|!} \sum_{\gamma} N^{\#(\gamma)}_{S_m} N^{\#(\gamma \circ \sigma)}_{S_m} \tag{6}
\]

where \( \#(\gamma) \) is the number of cycles in the permutation \( \gamma \). Now one can use the standard identity [35]

\[
p_\gamma = \sum_{R \vdash |\gamma|} \psi_R(\gamma) \chi_R(p) \tag{7}
\]
Here the sum goes over all Young diagrams with $|\gamma|$ boxes, $\chi_R^p$ is the Schur function (the character of the linear group $GL(N)$), $p_{\gamma} = \prod_{i=1}^{l_{\gamma}} p_{\gamma_i}$ and $\psi_R(\gamma)$ is the character of the symmetric group $S_{|R|}$. Choosing all $p_k = N$, we immediately come to the formula

$$N^\#_\gamma = N^{l_{\gamma}} = \sum_{R \vdash |\gamma|} D_R(N) \psi_R(\gamma)$$

(8)

where $D_R(N)$ are dimensions of representation $R$ of the Lie algebra $GL(N)$, i.e. the value of character at unity.

Further, one can use that, for the symmetric group, $\gamma$ and $\gamma^{-1}$ belong to the same conjugacy class and apply the standard character orthogonality relation (valid for any finite group) [36]

$$\sum_{\gamma} \psi_R(\gamma) \psi_Q(\gamma \circ \sigma) = \sum_{\gamma} \psi_R(\gamma^{-1}) \psi_Q(\gamma \circ \sigma) = |R|! \frac{\psi_R(id)}{|R|!} = \frac{\psi_R([1^{|R|}])}{|R|!}$$

(9)

in order to convert the sum (6) into

$$O_{\sigma}^{RCM} = \sum_{R \vdash m} \frac{D_R(N_1) D_R(N_2)}{d_R} \psi_R(\sigma)$$

(10)

of [29] and [30].

3 Description in terms of permutations: Hermitian matrix model

The only difference in this case is that one does not distinguish between $\hat{M}$ and $M$, which makes the Wick theorem (5) a little more involved: the sum now goes only over permutations $\gamma$ which has $m$ cycles of length 2, i.e. $\gamma \in [2^m] \subset S_{2m}$. This leads to a minor complication in the counterpart of (10) [28,30]:

$$O_{\sigma}^{HM} = \sum_{R \vdash m} \varphi_R([2^m]) : D_R(N) \cdot \psi_R(\sigma)$$

(11)

where $\varphi_R(\mu)$ are again the symmetric group characters, but with a slightly different normalization, see [27].

The Hermitian model enumerates equilateral triangulations which can be considered as ramified coverings of Riemann sphere. Thus, it is directly related to the ordinary Hurwitz numbers, Belyi functions, dessins d’enfants and Galois groups, see [37]- [41] for just some papers on various aspects of these relations, and especially [28] for their connection to formulas like (6).

4 Description in terms of permutations: rainbow tensor models

Of course, the calculation in sec.2 remains exactly the same when $M$ is substituted by an arbitrary $N_1 \times \ldots \times N_r$ tensor of rank $r$. The relevant operators are now labeled by sets of $r-1$ permutations $\bar{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_r \in S_m\}$, acting on the corresponding indices in $M$. The answer for the correlator is just the obvious generalization of (6):

$$O_{\bar{\sigma}}^{\text{rainbow}} = \sum_{\gamma \in S_m} \prod_{s=1}^{r} N_{\gamma^s}^\#(\gamma \circ \sigma_s)$$

(12)

and the counterpart of (10) is

$$O_{\bar{\sigma}}^{\text{rainbow}} = \sum_{R \vdash m} \prod_{s=1}^{r} D_{R_s}(N_s) \psi_R(\bar{\sigma})$$

(13)

where

$$\psi_R(\bar{\sigma}) = \sum_{\gamma \in S_m} \prod_{s=1}^{r} \psi_R(\gamma \circ \sigma_s)$$

(14)

Unfortunately, there are no known poly-linear counterparts of the orthogonality relation as in the matrix model case of $r = 2$ in order to simplify this formula, however, it is easy to evaluate in every particular example. Of
course, one of the $r$ permutations $\sigma_s$, say, $\sigma_1$ can be absorbed into $\gamma$, and the correlator depends only on the $r - 1$ “ratios” $\sigma_1^{-1} \circ \sigma_s$. Eq. (13) seems to be in accord with the claim of [31].

In the case of non-rainbow models with lower symmetry, these formulas get a little more complicated, in exactly the same way as described in sec.3: some $N$-dependent dimensions $D_{R_s}(N_s)$ are substituted by symmetric characters, trading the disappearing symmetry parameters $N_s$ for the symmetry-breaking insertions made from $\varphi_R$.

5 Examples

**Hermitian matrix model:**

\[
\left\langle (\text{Tr } X^2) \right\rangle = \mathcal{O}_{[1]} = \sum_{\gamma \in [2] \in S_2} N^\#(\gamma [11]) = N^\#([2]) = N
\]

\[
\left\langle \text{Tr } (X^2) \right\rangle = \mathcal{O}_{[2]} = \sum_{\gamma \in [2] \in S_2} N^\#(\gamma [2]) = N^\#([11]) = N^2
\]

**Complex matrix model:**

\[
\left\langle (\text{Tr } XX^2) \right\rangle = \mathcal{O}_{id \otimes [1]} = \sum_{\gamma \in S_2} N^1_1 N^\#(\gamma [11]) = N^1_1 N^\#([11]) + N^1_2 N^\#([2]) = N_1^2 N_2 + N_1 N_2
\]

\[
\left\langle \text{Tr } (XX)^2 \right\rangle = \mathcal{O}_{id \otimes [2]} = \sum_{\gamma \in S_2} N^1_1 N^\#(\gamma [2]) = N^1_1 N^\#([11]) + N^1_2 N^\#([2]) = N_1^2 N_2 + N_1 N_2 = N_1 N_2 (N_1 + N_2)
\]

**Aristotelian tensor model** [20]: The simplest operators in this rainbow model of the rank-3 tensor $M_{abc}$ with the action $M_{abc} \tilde{M}_{abc}$ are

\[
K_m = \tilde{M}_{a_1b_1c_1} M_{a_2b_2c_2} M_{a_3b_3c_3} \ldots M_{a_mb_mc_m} \tilde{M}_{a_mb_mc_m},
\]

associated with the permutations

\[
\sigma_{K_m} = \text{id} \otimes (12\ldots m) \otimes (12\ldots m) \in S_{m}^{\otimes 3},
\]

\[
K_3 = \begin{array}{c}
\end{array}
\]

\[
K_{2,2} = \begin{array}{c}
\end{array}
\]

and $K_{m,n}$, associated with

\[
\sigma_{K_{m,n}} = \text{id} \otimes (12\ldots m + n - 1) \otimes (12\ldots m)(m + 1\ldots m + n - 1) \in S_{m+n-1}^{\otimes 3}
\]

For the average of the simplest $K_2$ we have:

\[
\left\langle K_2 \right\rangle = \mathcal{O}_{id \otimes [2] \otimes [2]} = \sum_{\gamma \in S_2} N^1_1 N^\#(\gamma [2]) N^\#([2]) = N_1^2 N_2^2 N_3 + N_1 N_2^3 N_3 = N_1 N_2 N_3 \left( N_1 + N_2 N_3 \right)
\]
For the cases of $\mathcal{K}_3$ and $\mathcal{K}_{2,2}$ we need multiplication table in $S_3$:

| $\gamma$ | $\#(\text{cycles})$ | $\gamma \circ (123)$ | $\#(\text{cycles})$ | $\gamma \circ (12)$ | $\#(\text{cycles})$ |
|----------|---------------------|----------------------|----------------------|---------------------|----------------------|
| $id$     | 3                   | (123)                | 1                    | (12)                | 2                    |
| (12)     | 2                   | (13)                 | 2                    | $id$                | 3                    |
| (13)     | 2                   | (23)                 | 2                    | (132)               | 1                    |
| (23)     | 2                   | (12)                 | 2                    | (123)               | 1                    |
| (123)    | 1                   | (132)                | 1                    | (23)                | 2                    |
| (132)    | 1                   | $id$                 | 3                    | (13)                | 2                    |

Then

$$\left\langle \mathcal{K}_3 \right\rangle = O_{id \otimes (123) \otimes (123)} = \sum_{\gamma \in S_3} N_1^{\#(\gamma)} N_2^{\#(\gamma \circ (123))} N_3^{\#(\gamma \circ (123))} =$$

$$= N_1^3 N_2 N_3 + 3 N_1^2 N_2^2 N_3^2 + N_1 N_2 N_3 + N_1 N_2^3 N_3^3 = N_1 N_2 N_3 \left( N_2^2 N_3^2 + 3 N_1 N_2 N_3 + N_1^2 + 1 \right) \quad (24)$$

$$\left\langle \mathcal{K}_{2,2} \right\rangle = O_{id \otimes (123) \otimes (12)} = \sum_{\gamma \in S_3} N_1^{\#(\gamma)} N_2^{\#(\gamma \circ (123))} N_3^{\#(\gamma \circ (12))} =$$

$$= N_1^3 N_2 N_3^3 + N_1^2 N_2^2 N_3^3 + 2 N_1^2 N_2^2 N_3^3 + N_1 N_2 N_3^2 + N_1 N_2^3 N_3^2 \quad (25)$$

All these expressions, indeed, coincide with the answers from [20].

6 On the structure of tensor model correlators

Eq. (13) implies that the correlator in tensor model is actually a multi-linear combination of dimensions $D_{R_n}(N_1)$ with all $R_n$ of the same size $m$. By itself, it is a non-trivial prediction of the Hurwitz formalism, and it is indeed true, at least for all the examples explicitly evaluated in [20] for the Aristotelian model, the simplest non-trivial one within the rainbow class. For example,

$$\left\langle \mathcal{K}_1 \right\rangle = D_{[1]}(N_1) D_{[1]}(N_2 N_3) = D_{[1]} \otimes D_{[1]} \otimes D_{[1]} = N_1 N_2 N_3 \quad (26)$$

$$\left\langle \mathcal{K}_2 \right\rangle = 2 D_{[2]}(N_1) D_{[2]}(N_2 N_3) - 2 D_{[1]}(N_1) D_{[1]}(N_2 N_3) =$$

$$= 2 \left( D_{[2]} \otimes D_{[2]} \otimes D_{[2]} + D_{[2]} \otimes D_{[11]} \otimes D_{[11]} - D_{[11]} \otimes D_{[2]} \otimes D_{[1]} - D_{[1]} \otimes D_{[1]} \otimes D_{[2]} \right) =$$

$$= \frac{N_1 N_2 N_3}{4} \left( (N_1 + 1)(N_2 + 1)(N_3 + 1) + (N_1 + 1)(N_2 - 1)(N_3 - 1) - (N_1 - 1)(N_2 + 1)(N_3 - 1) - (N_1 - 1)(N_2 - 1)(N_3 + 1) \right) = N_1 N_2 N_3 \left( N_1 + N_2 N_3 \right) \quad (27)$$

The first formulas here, bilinear in dimensions are pure matrix model ones, the new and general, structurally relevant for arbitrary operators (in particular, more complicated than $\mathcal{K}_m$) are tri-linear expressions in boxes. The tensor products mean that the tree dimensions are evaluated at three different values of $N$. Note the asymmetry between $N_1$ and $N_2$, $N_3$ coming from the asymmetry of the operator $\mathcal{K}_m$. Further,

$$\left\langle \mathcal{K}_3 \right\rangle = 6 D_{[3]}(N_1) D_{[3]}(N_2 N_3) - 3 D_{[21]}(N_1) D_{[21]}(N_2 N_3) + 6 D_{[11]}(N_1) D_{[11]}(N_2 N_3) =$$

$$= 6 D_{[3]} \otimes D_{[3]} \otimes D_{[3]} + D_{[21]} \otimes D_{[21]} + D_{[11]} \otimes D_{[11]} \right) -$$

$$- 3 D_{[21]} \otimes D_{[21]} + D_{[21]} \otimes D_{[3]} + D_{[21]} \otimes D_{[21]} + D_{[21]} \otimes D_{[11]} + D_{[11]} \otimes D_{[21]} +$$

$$+ 6 D_{[11]} \otimes D_{[21]} + D_{[21]} \otimes D_{[11]} + D_{[21]} \otimes D_{[21]} + D_{[11]} \otimes D_{[3]} \right) \quad (28)$$

$$\left\langle \mathcal{K}_{2,2} \right\rangle = \left( 6 D_{[3]} \otimes D_{[3]} - 3 D_{[21]} \otimes D_{[21]} + 6 D_{[11]} \otimes D_{[11]} \right) \otimes D_{[3]} -$$

$$- \left( 6 D_{[3]} \otimes D_{[11]} - 3 D_{[21]} \otimes D_{[21]} + 6 D_{[11]} \otimes D_{[3]} \right) \otimes D_{[11]} \right) \quad (29)$$
Note that there is no matrix model expression (bilinear in dimensions) for $K_{2,2}$, but the tensor model tri-linear expression exists.

Such formulas allow one to introduce new time variables substituting each dimension in the tensor product by a character: $D_R(N_s) \rightarrow \chi_R(p^{(s)})$, the original formulas then arise from these extended quantities at the "classical topological locus" 

$$p^{(s)} = N_s.$$

7 Conclusion

In this paper we explained a simple method [31] to calculate Gaussian averages in tensor models, which was basically used in evaluation of correlators in [20]. It directly extends the Hurwitz (character) approach to matrix models [27–30] and knot calculus [43].

Technically, it requires knowledge of just two things:
(a) parametrization of operators by permutations and
(b) multiplication table for permutations, actually available in MAPLE and Mathematica.

This method proved to be very efficient in matrix model theory [27–30] and it will be equally powerful in application to tensor models. As emphasized in [20], having explicit formulas for correlators opens a possibility to study recursion relations and genus-like expansions [23,44], Ward identities (Virasoro-like constraints) [45], the AMM/EO topological recursions [46] and many other structures which are already showing up in the tensor models, very much in the same way as they did in the matrix model case [47].

One more aspect of the story is its close relation with the arborescent calculus [48–50] in knot theory. Examples in the present text were rather simple, the evaluation of more sophisticated correlators in tensor models will include the Racah matrices, not just representation decompositions.

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