ON THE BRAUER-MANIN OBSTRUCTION APPLIED TO RAMIFIED COVERS.

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Abstract

The Brauer-Manin obstruction is used to explain the failure of the local-global principle for algebraic varieties. In 1999 Skorobogatov gave the first example of a variety that does not satisfy the local-global principle which is not explained by the Brauer-Manin obstruction. He did so by applying the Brauer-Manin obstruction to étale covers of the variety, and thus defining a finer obstruction. In 2008 Poonen gave the first example of failure of the local-global principle which cannot be explained for by Skorobogatov’s étale-Brauer obstruction. However, Poonen’s construction was not accompanied by a definition of a new finer obstruction. In this paper I shall present a possible definition for such an obstruction by allowing to apply the Brauer-Manin obstruction to some ramified covers as well, and show that this new obstruction can explain Poonen counterexample in the case of a totally imaginary number field.

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Call a $X$ variety nice if it is smooth, projective, and geometrically integral. Now given a nice variety $X$ over a global field $k$, a prominent problem is to understand the set $X(k)$, e.g. to decide whether $X(k) = \emptyset$. As a first approximation one can consider the set $X(\mathbb{A}_k) \supset X(k)$, where $\mathbb{A}_k$ is the adeles ring of $k$. It is a classical theorem of Minkowski and Hasse that if $X$ is a quadratic form then $X(\mathbb{A}_k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$. When a variety $X$ satisfy this property we say that it satisfy the Hasse principle. In the 1940’s Lind and Reichardt ([Lin40], [Rei42]) gave examples of genus 1 curves that does not satisfy the Hasse principle. More counterexamples to the Hasse principle where given throughout the years until in 1971 Manin ([Man70]) describe a general obstruction to the Hasse principle, that explained all the examples that was known to that date. The obstruction (known as the Brauer-Manin obstruction) is defined by considering a set $X(\mathbb{A}_k)^{Br}$, which satisfy, $X(k) \subset X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)$. If $X$ is a counterexample to the Hasse principle we say that it is accounted for or explained by the Brauer-Manin obstruction if $\emptyset = X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k) \neq \emptyset$.

In 1999 Skorobogatov ([Sko99]) defined a refinement of the Brauer-Manin obstruction (also known as the étale-Brauer-Manin obstruction) and used it to show an example of a variety $X$ such that $X(\mathbb{A}_k)^{Br} \neq \emptyset$ but $X(k) = \emptyset$. Namely he described a set $X(k) \subset X(\mathbb{A}_k)^{\text{Et,Br}} \subset X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)$ and found a variety $X$ such that $X(\mathbb{A}_k)^{\text{Et,Br}} = \emptyset$ but $X(\mathbb{A}_k)^{Br} \neq \emptyset$.

In his paper from 2008 [Poo08] Poonen constructed the first and currently only known example of a variety $X$ such that $X(\mathbb{A}_k)^{\text{Et,Br}} \neq \emptyset$ but $X(k) = \emptyset$. However, Poonen’s method of showing that $X(k) = \emptyset$ relies on the details of his specific construction and is not explained by a new finer obstruction. Therefore, one wonders could Poonen’s counterexample can be accounted for by an additional refinement of the $X(\mathbb{A}_k)^{\text{Et,Br}}$. Namely can we give a general definition of a set $X(k) \subset X(\mathbb{A}_k)^{\text{new}} \subset X(\mathbb{A}_k)^{\text{Et,Br}}$, such that Poonen’s variety $X$, satisfy $X(\mathbb{A}_k)^{\text{new}} \emptyset$, in this paper I shall give a suggestion for such a refinement.
The results are presented in this paper only for global fields without real embeddings, i.e. for function fields and totally imaginary number fields, but I believe that this restriction is not essential.

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1. Ramified Covers and Brauer-Manin Obstruction

In Skorobogatov presented the étale-Brauer-Manin obstruction. In this section we shall give a slight generalization of it, to be applied in our case. Let $X$ be a projective variety over a global field $k$ and $\pi : Y \to X$ a connected étale-Galois cover. The idea of the étale-Brauer-Manin obstruction is that if there is a rational point on $X$ then there must be some twisting of $\pi : Y \to X$ that has a rational point.

**Definition 1.1.** Let $X$ be a projective variety and $D \subset X$ an effective divisor. We say that $\pi : Y \to X$, is an almost Galois cover over $X$ ramified at $D$ of degree $d$, if:

(i) $Y$ is connected.

(ii) $\pi$ is a surjective finite morphism of generic degree $d$.

(iii) $\pi : Y \to X$ has a group of automorphisms $G$, such that $|G| = d$.

(iv) The ramification locus of $\pi : Y \to X$ is a subset of $D$.

Now let $D$ be a divisor such that $D(k) = \emptyset$, and $\pi : Y \to X$ be an almost Galois cover over $X$ ramified at $D$, note that like in the case of a usual Galois cover, since there is a bijection between Galois covers of $X \setminus D$ and almost Galois covers of $X$ ramified at $D$ one can twist $\pi : Y \to X$ by an element of $H^1(k, G)$ and get a new such cover. Now assume that $X(k) \neq \emptyset$ then if $x \in X(k)$, since $x \notin D$, $\pi^{-1}(x)$ is an principle homogenous space of $G$, so one can twist $\pi : Y \to X$ such
that over $x$ lies a rational points. Therefore if one show that for every twist $Y_\sigma$ we have $Y_\sigma(k) = \emptyset$, then one knows that $X(k) = \emptyset$.

Now given a projective variety and $D$ a divisor in $X$ such that $D(k) = \emptyset$ one can defined the set:

$$X(k) \subset X(\mathbb{A})^{\text{ét,Br} \sim D} = \bigcap_{Y} \bigcup_{\sigma} \pi_\sigma(Y_\sigma(\mathbb{A})^{\Br}).$$

where the union is taken over all the possible twists of $Y$ and the intersection is taken over all the almost Galois covers over $X$ ramified at $D$. When $X(\mathbb{A})^{\text{ét,Br} \sim D}$ we shall say that the absence of rational points is explained by the $(\text{ét,Br} \sim D)$-obstruction.

In this paper we shall show that (under some conditions) for the variety $X$ that Poonen defines in [Poo08], one can choose a divisor $D \subset X$ such that $D(k) = \emptyset$ and $X(\mathbb{A})^{\text{ét,-Br} \sim D} = \emptyset$ and thus explain the absence of rational points on $X$.

2. Conic bundles

In this section we shall present some construction of conic bundles on a nice variety $B$, and some of its property. This construction appeared in [Poo08] §4 and Poonen used it in order to build his counterexample. We base our notation here on his, and add some notations of our own.

In this section, $k$ is any field of characteristic not 2. Let $B$ be a nice $k$-variety. Let $\mathcal{L}$ be a line sheaf on $B$. Let $\mathcal{E}$ be the rank-3 vector sheaf $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{L}$ on $B$. Let $a \in k^\times$ and let $s \in \Gamma(B, L^{\otimes 2})$ be a nonzero global section. The zero locus of

$$1 \oplus (-a) \oplus (-s) \in \Gamma(B, \mathcal{O} \oplus \mathcal{O} \oplus L^{\otimes 2}) \subset \Gamma(B, \text{Sym}^2 \mathcal{E})$$

in $\mathbb{P}\mathcal{E}$ is a projective geometrically integral scheme $X = X(B, \mathcal{L}, a, s)$ with a morphism $\alpha : X \to B$. We shall call $(\mathcal{L}, s, a) \in \text{Div}B \times \Gamma(B, L^{\otimes 2}) \times k^\times$ a Conic Bundle Datum on $B$, and we call $X$ the that total space of $(\mathcal{L}, s, a)$ and denote $X = \text{Tot}_B(\mathcal{L}, s, a)$. If $U$ is a dense open subscheme of $B$ with a trivialization $\mathcal{L}|_U \cong \mathcal{O}_U$ and we identify $s|_U$ with an element of $\Gamma(U, \mathcal{O}_U)$, then the affine scheme defined by $y^2 - az^2 = s|_U$ in $A^2_U$ is a dense open subscheme of $X$. Therefore we call $X$ the conic bundle given by $y^2 - az^2 = s$. In the special case where $B = \mathbb{P}^1$, $\mathcal{L} = \mathcal{O}(2)$, and the homogeneous form $s \in \Gamma(\mathbb{P}^1, \mathcal{O}(4))$
is separable, $X$ is called the Châtelet surface given by $y^2 - az^2 = s(x)$, where $s(x) \in k[x]$ denotes a dehomogenization of $s$. Returning to the general case, we let $Z$ be the subscheme $s = 0$ of $B$. We call $Z$ the degeneracy locus of the conic bundle $(\mathcal{L}, s, a)$. Each fiber of $\alpha$ above a point of $B - Z$ is a smooth plane conic, and each fiber above a geometric point of $Z$ is a union of two projective lines crossing transversely at a point. A local calculation shows that if $Z$ is smooth over $k$, then $X$ is smooth over $k$.

**Lemma 2.1.** The generic fiber $\bar{X}_\eta$ of $\bar{X} \to \bar{B}$ is isomorphic to $\mathbb{P}^1(\bar{k})$.

**Proof.** It is a smooth plane conic, and it has a rational point since $a$ is a square in $\bar{k} \subset \kappa(\bar{B})$. □

**Lemma 2.2.** Let $B$ be a nice $k$-variety, and $(\mathcal{L}, s, a)$ be a conic bundle datum on $B$, denote the corresponding bundle $\pi : X \to B$ and the generic point of $B$ by $\eta$. Now let $Z$ be the degeneracy locus. Assume that $Z$ is the union of the irreducible components $Z = \bigcup_{1 \leq i \leq r} Z_i$, then there is a natural exact sequence of Galois modules.

$$(2.1) \quad 0 \to \bigoplus ZZ_i \xrightarrow{\rho_1} \text{Pic } B \oplus \bigoplus ZZ_i^+ \oplus \bigoplus ZZ_i^- \xrightarrow{\rho_2} \text{Pic } X \xrightarrow{\rho_3} \text{Pic } X_\eta \to 0$$

where $\rho_4$ is a natural section of $\rho_3$.

**Proof.** Denote by $F_i^\pm$ the divisors that lay over $Z_i$, and defined by the additional condition that $y = \pm \sqrt{az}$. Now, define $\rho_1$ by $\rho_1(Z_i) = (-Z_i, Z_i^+, Z_i^-))$. Define $\rho_2(M, 0, 0) = \pi^* M, \rho_2(0, Z_i^+, 0) = F_i^+, \rho_2(0, Z_i^-, 0) = F_i^-$. Let $\rho_3$ be restriction. Each $\rho_i$ is $\Gamma_k$-equivariant. Given a prime divisor $D$ on $\bar{X}_\eta$, we take $\rho_4(D)$ to be its Zariski closure in $X$. $\rho_4(D)$ restricts to give $D$ on $\bar{X}_\eta$, thus $\rho_4$ is indeed a section of $\rho_3$ and thus $\rho_3$ is indeed surjective. The kernel of $\rho_3$ is generated by the classes of vertical prime divisors of $X$; in fact, there is exactly one above each prime divisor of $B$ except that above each $Z_i \in \text{Div } B$ we have $F_i^+, F_i^- \in \text{Div } X$. This proves exactness at Pic $X$. Now, a rational function on $X$ with vertical divisor must be the pullback of a rational function on $B$. The
previous two sentences prove exactness at $\text{Pic} \; B \oplus \bigoplus \mathbb{Z} \mathbb{Z}_i^+ \oplus \bigoplus \mathbb{Z} \mathbb{Z}_i^-$. Injectivity of $\rho_1$ is trivial.

3. Poonen’s Counter Example

Poonen’s construction can be done over any global field $k$ of characteristic different from 2. We shall follow his construction in this section. Let $a \in k^\times$, and let $\tilde{P}_\infty(x), \tilde{P}_0(x) \in k[x]$ be relatively prime separable degree-4 polynomials such that the (nice) Châtelet surface $\mathcal{V}_\infty$ given by $y^2 - az^2 = \tilde{P}_\infty(x)$ over $k$ satisfies $\mathcal{V}_\infty(\mathbb{A}_k) \neq \emptyset$ but $\mathcal{V}_\infty(k) = \emptyset$. (Such Châtelet surfaces exist over any global field $k$ of characteristic different from 2: see [Poo08, Proposition 5.1 and 11]. If $k = \mathbb{Q}$, one may use the original example from [Isk71], with $a = -1$ and $\tilde{P}_\infty(x) := (x^2 - 2)(3 - x^2)$.) Let $P_\infty(w, x)$ and $P_0(w, x)$ be the homogenizations of $\tilde{P}_\infty$ and $\tilde{P}_0$. Define $L = O(1, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and define

$$s_1 := u^2 P_\infty(w, x) + v^2 P_0(w, x) \in \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, L \otimes 2),$$

where the two copies of $\mathbb{P}^1$ have homogeneous coordinates $(u : v)$ and $(w : x)$, respectively. Let $Z_1 \in \mathbb{P}^1 \times \mathbb{P}^1$ be the zero locus of $s_1$. Let $F \in \mathbb{P}^1$ be the (finite) branch locus of the first projection $Z_1 \rightarrow \mathbb{P}^1$, i.e.

$$F := \{(u : v) \in \mathbb{P}^1 | u^2 P_\infty(w, x) + v^2 P_0(w, x) \text{ Has a multiple root}\}$$

Let $\alpha_1 : \mathcal{V} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle given by $y^2 - az^2 = s_1$, i.e, the conic bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the datum $(O(1, 2), a, s_1)$. Composing $\alpha_1$ with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ yields a morphism $\beta_1 : \mathcal{V} \rightarrow \mathbb{P}^1$ whose fiber above $\infty := (1 : 0)$ is the Châtelet surface $\mathcal{V}_\infty$ defined earlier. Now Let $C$ be a nice curve over $k$ such that $C(k)$ is finite and nonempty. Choose a dominant morphism $\gamma : C \rightarrow \mathbb{P}^1$, étale above $F$, such that $\gamma(C(k)) = \{\infty\}$. Define the fiber product $X := \mathcal{V} \times_{\mathbb{P}^1} C$ and morphisms $\alpha$ and $\beta$ as in the diagram

$$
\begin{array}{c}
\begin{array}{c}
X \\
\downarrow \alpha \\
C \times \mathbb{P}^1 \\
\downarrow 1^{st} \\
C
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow \gamma \\
\rightarrow \alpha_1 \\
\rightarrow (\gamma, 1) \\
\rightarrow 1^{st}
\end{array}
\end{array}
\begin{array}{c}
\rightarrow \mathbb{P}^1 \\
\rightarrow \mathbb{P}^1 \times \mathbb{P}^1
\end{array}
\end{array}$$
Each map labeled $1^{st}$ is the first projection.

By composing vertically, we get

\[(3.2) \quad \xymatrix{ X \ar[r]_{\beta} \ar[d]_{\beta_1} & Y \\ C \ar[r]_{\gamma} & \mathbb{P}^1 } \]

$X$ is the variety the satisfy $X(\mathbb{A}_k)^{\acute{E}t, Br} \neq \emptyset$ but $X(k) = \emptyset$. Note that $X$ can also be considered as the variety corresponding to the datum $(O(1, 2), a, s_1)$ pulled back via $(\gamma, 1)$ to $C \times \mathbb{P}^1$.

4. Our construction

In this section we present the construction we use to explain the absence of rational points on $X$ by taking the variant of étale-Brauer-Manin obstruction defined in §2. All the notations will agree with those of the previous section.

First we shall show that almost Galois coverings behave the pull-back in good way namely:

**Lemma 4.1.** Let $X$ be a projective variety, $D \subset X$ a divisor and $\pi : Y \rightarrow X$ an almost Galois cover ramified at $D$. Further assume that $D(k) = \emptyset$ and $\rho : Z \rightarrow X$, is a map. then $\pi^* : Y \times_X Z \rightarrow Z$ is a almost Galois cover ramified at $\rho^{-1}(D)$ and $\rho^{-1}(D)(k) = \emptyset$.

**Proof.** clear. \[\square\]

Now let $F' := \gamma^{-1}(F) \subset C$, and denote $C' := C \setminus F'$, note that $C'$ is a non-projective curve. Now let $D := \beta^{-1}(F')$, Note that $\infty \notin F$, So $C(k) \cap F' = \emptyset$. Thus $D$ has no connected components stable under $\Gamma_k$.

Therefore it is clear that $D(k) = \emptyset$. We shall use the $(\acute{E}t, Br \sim D)$-obstruction defined in section §[1] to show that $X(k) = \emptyset$.

Let $E' \subset (\mathbb{P}^1 \setminus F) \times (\mathbb{P}^1)^4$ be the curve defined by
\[u^2P_{\infty}(w_i, x_i) + v^2P_0(w_i, x_i) = 0, \quad 1 \leq i \leq 4, (w_i : x_i) \neq (w_j : x_j), i \neq j, 1 \leq i, j \leq 4\]

where $(u : v)$ are the projective coordinates of $\mathbb{P}^1 \setminus F$ and $(w_i : x_i), 1 \leq i \leq 4$ are the projective coordinates of the 4 copies of $\mathbb{P}^1$. One can check that $E'$ is a smooth connected curve and the first projection $E' \overset{1^{st}}{\longrightarrow} \mathbb{P}^1 \setminus F$, gives $E'$ structure of étale Galois covering of $\mathbb{P}^1 \setminus F$ under
the action of $S_4$ that acts on the fibres by permuting the coordinates of $(w_i : x_i), 1 \leq i \leq 4$. Since every family of curves has a unique projective smooth member, one can construct an almost Galois cover of $\mathbb{P}^1$ ramified at $F$ of degree 24 $E \to \mathbb{P}^1$ such that when restricted to $\mathbb{P}^1 \setminus F$ is exactly $E' \to \mathbb{P}^1 \setminus F$. Now the $k$-twists of $E \to \mathbb{P}^1$ are classified by $H^1(k, S_4) = \text{Hom}(\Gamma_k, S_4)/ \sim$, where $\sim$ is the conjugate relation.

More concretely for every $\phi : \Gamma_k \to S_4$ define $E_{\phi}$ to be the $k$-form of $E$ with the Galois action on $E$ that when restricted to $E'$ is the usual one twisted by $\Gamma_k$ permuting $(w_i : x_i), 1 \leq i \leq 4$ via $\phi$.

Now for every $\phi : \Gamma_k \to S_4$ define $X_{\phi} : \gamma \times_{\mathbb{P}^1} E_{\phi}$, and $X_{\phi} := X \times_C C_{\phi}$.

By Lemma 4.1, $X_{\phi}$ is a complete family of twists of an almost Galois cover of $X$ of degree 24 ramified $D, (\text{recall } D(k) = \emptyset)$ therefore to explain the fact that $X(\mathbb{Q}) = \emptyset$ it is enough to show, $X_{\phi}(\mathbb{A})^{Br} = \emptyset$ for every $\phi \in H^1(\Gamma_k, S_4)$.

5. Reduction to $X_{\phi_{\infty}}$

**Lemma 5.1.** Assume that $C(k) = C(\mathbb{A})^{Br}$ then for every $\phi \in H^1(k, S_4)$, $C_{\phi}(k) = C_{\phi}(\mathbb{A})^{Br}$.

**Proof.** Note that we have a non-constant map $\pi_{\phi} : C_{\phi} \to C$. Now the proof will relay on Stoll’s results in [Sto07]. In [Sto07] Stoll, defines for a variety $X$ the set $X(\mathbb{A})^{f-ab}$, and proves that if $X$ is a curve $X(\mathbb{A})^{f-ab} = X(\mathbb{A})^{Br}$ (Corollary 7.3 [Sto07]). Now by Proposition 8.5 [Sto07] and the existence of the map $\pi_{\phi} : C_{\phi} \to C$, we have that $C(\mathbb{A})^{f-ab} = C(\mathbb{A})^{Br} = C(k)$ implies $C_{\phi}(\mathbb{A})^{Br} = C_{\phi}(\mathbb{A})^{f-ab} = C_{\phi}(k)$.

$\square$

Denote now by $\phi_{\infty} \in H^1(\gamma, S_4)$ the map $\gamma_k \to S_4$ defined by the Galois action on the 4 roots of $P_{\infty}$.

**Lemma 5.2.** Let $\phi \in H^1(\Gamma_k, S_4)$ be such that $\phi \neq \phi_{\infty}$ then $C_{\phi}(k) = \emptyset$

**Proof.** Recall that $C_{\phi} := C \times_{\mathbb{P}^1} E_{\phi}$. Denote $\pi_{\phi} : E_{\phi} \to \mathbb{P}^1$. Since $\phi \neq \phi_{\infty}$ we get that $E_{\phi}(k) \cap \pi_{\phi}^{-1}(\infty) = \emptyset$. Now Since $\gamma(C(k)) = \infty$ we get $C_{\phi}(k) = \emptyset$.

$\square$
Now denote by $\rho_\phi : X_\phi \to C_\phi$ the map defined earlier, we have for every $\phi \in H^1(k,S_4)$,
\[
\rho_\phi(X_\phi(\mathbb{A})^{Br}) \subset C_\phi(\mathbb{A})^{Br} = C_\phi(k).
\]
So we get that for $\phi \neq \phi_\infty$, $X_\phi(\mathbb{A})^{Br} = \emptyset$.

6. THE PROOF THAT $X_{\phi_\infty}(\mathbb{A})^{Br} = \emptyset$.

In this section we shall prove that if $k$ does’nt have real places (i.e. $k$ is a function field or a totaly imaginary number field) then $X_{\phi_\infty}(\mathbb{A})^{Br} = X_{\phi_\infty}(\mathbb{A})^{Br} = \emptyset$.

Let $p \in C_{\phi_\infty}(k)$, the fiber $\rho_{\phi_\infty}^{-1}(p)$ is isomorphic to the Châtelet surface $\mathcal{V}_\infty$. We shall denote by $\rho_\phi : \mathcal{V}_\infty \to X_{\phi_\infty}$, the corresponding natural isomorphism onto the fiber $\rho_{\phi_\infty}^{-1}(p)$. Recall that $\mathcal{V}_\infty$ satisfy $\mathcal{V}_\infty(\mathbb{A})^{Br} = \emptyset$.

**Lemma 6.1.** Let $k$ be global field with no real embeddings. Let $x \in X_{\phi_\infty}(\mathbb{A})^{Br}$, there exists $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A})^{Br})$.

**Proof.** From functoriality and Lemma 5.1 we get
\[
\rho_{\phi_\infty}(x) \in \rho_{\phi_\infty}(X_{\phi_\infty}(\mathbb{A})^{Br}) \subset C_{\phi_\infty}(\mathbb{A})^{Br} = C_{\phi_\infty}(k)
\]
we denote $p = \rho_{\phi_\infty}(x) \in C_{\phi_\infty}(k)$, now it is clear that in all but maybe the infinite places $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A})^{Br})$. So it remain to deal with the infinite places which by assumption are all complex. But since both $X_{\phi_\infty}$ and $\mathcal{V}_\infty$ are geometrically integral, taking connected components reduce $X(\mathbb{C})$ and $\mathcal{V}_\infty(\mathbb{C})$ to a single point. $\square$

**Lemma 6.2.** Let $p \in C_{\phi_\infty}(k)$, then the map
\[
\rho_p^* : Br(X_{\phi_\infty}) \to Br(\mathcal{V}_\infty)
\]
is surjective.

We will prove Lemma 6.2 in section § 7

**Lemma 6.3.** Let $k$ be global field with no real embeddings , $X_{\phi_\infty}(\mathbb{A})^{Br} = \emptyset$.

**Proof.** Assume that $X_{\phi_\infty}(\mathbb{A})^{Br} \neq \emptyset$. Let $x \in X_{\phi_\infty}(\mathbb{A})^{Br}$. By Lemma 6.1 there exists $p \in C_{\phi_\infty}(k)$ such that $x \in \rho_p(\mathcal{V}_\infty(\mathbb{A})^{Br})$, let $y \in \mathcal{V}_\infty(\mathbb{A})^{Br}$ be such that $\rho_p(y) = x$. We shall show that $y \in \mathcal{V}_\infty(\mathbb{A})^{Br}$. Indeed
let $b \in Br(V_\infty)$ by Lemma 6.2 there exists $\tilde{b} \in Br(X'_{\phi\infty})$ such that $\rho^*_p(\tilde{b}) = b$, now $(y, b) = (y, \rho^*_p(\tilde{b})) = (\rho_p(y), \tilde{b}) = (x, \tilde{b}) = 0$. But by assumption $x \in X_{\phi\infty}(\mathbb{A})^{Br}$, so we have $(y, b) = (x, \tilde{b}) = 0$. Thus we have $y \in V_\infty(\mathbb{A})^{Br} = \emptyset$ which is a contradiction. 

7. The surjectivity of $\rho^*_p$

In this section we shall prove the statement of Lemma 6.2.

Lemma 7.1. Let $p \in C_{\phi\infty}(k)$ and $\rho_p : V_\infty \rightarrow X_{\phi\infty}$ the corresponding map as above. Then the map of Galois modules

$$\rho^*_p : \text{Pic}(X_{\phi\infty}) \rightarrow \text{Pic}(V_\infty)$$

has a section.

Proof. Consider the map $\phi_p : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times C_{\phi\infty}$ defined by $a \mapsto (a, p)$. It is clear that the map $\rho_p : V_\infty \rightarrow X_{\phi\infty}$ comes from pulling back the conic bundle datum defining $X_{\phi\infty}$ over $\mathbb{P}^1 \times C_{\phi\infty}$ by this map. Denote $B = \mathbb{P}^1 \times C_{\phi\infty}$ and consider the following commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \rightarrow & \bigoplus \mathbb{Z}Z_i & \rightarrow & \text{Pic} B \oplus \bigoplus \mathbb{Z}Z_i^+ \oplus \bigoplus \mathbb{Z}Z_i^- & \rightarrow & \text{Pic} X_{\phi\infty} \rightarrow & Z \rightarrow & 0 \\
& & s_1 \downarrow \downarrow & & \downarrow s_2 \bigg\downarrow & & \downarrow \rho^*_p \bigg\downarrow \downarrow & & \\
0 & \rightarrow & \bigoplus \mathbb{Z}W_i & \rightarrow & \text{Pic} \mathbb{P}^1 \oplus \bigoplus \mathbb{Z}W_i^+ \oplus \bigoplus \mathbb{Z}W_i^- & \rightarrow & \text{Pic} V_\infty \rightarrow & Z \rightarrow & 0 
\end{array}
$$

where $Z$ is the degeneracy locus of $X_{\phi\infty}$ over $B$ and $W$ is the degeneracy locus of $V_\infty$ over $\mathbb{P}^1$. The existence of a section for $\rho^*_p$ follows by diagram chasing by the existence of the compatible sections $s_1$ and $s_2$. We shall first explain the existence of $s_1$. Every $W_i$ $(1 \leq i \leq 4)$ is a point that corresponds to a different of root of the polynomial $P_\infty(x, w)$.

and it is clear that one can choose $\hat{Z}_i^\pm$ to be the Zariski closer of the zero set of $y \pm \sqrt{a}z, w, x - x_iw$. Now $p = (c, ((x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0)), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0)) \in C(k) \times_{\mathbb{P}^1(k)} E_{\phi\infty}(k)$, since $\gamma(C(k)) = \{\infty\}$, the four points $\{(x_1^0 : w_1^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0), (x_2^0 : w_3^0)\}$ are exactly the four different roots of $P_\infty(x, w)$ and thus the existence of $s_1$ is clear. The existence of $s_2$ is clear from the same reason together with the obvious fact that the map $\phi_p : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times C_{\phi\infty}$ has a section. \qed
Lemma 7.2 (Lemma 6.2). Let $p \in C_{\phi_{\infty}}(k)$, then the map
\[
\rho_p^*: Br(X_{\phi_{\infty}}) \rightarrow Br(V_{\infty})
\]
is surjective.

Proof. Denote by $s_p : \text{Pic}(V_{\infty}) \rightarrow \text{Pic}(X_{\phi_{\infty}})$ the section of $\rho_p^* : \text{Pic}(X_{\phi_{\infty}}) \rightarrow \text{Pic}(V_{\infty})$. It is clear that $s_p$ induces a section of the map $\rho_p^{**} : H^1(k, \text{Pic}(X_{\phi_{\infty}})) \rightarrow H^1(k, \text{Pic}(V_{\infty}))$. Therefore the map $\rho_p^* : Br_1(X_{\phi_{\infty}}) \rightarrow Br_1(V_{\infty})$ is surjective. But since $Br_1(V_{\infty}) = Br(V_{\infty})$, we have that $\rho_p^* : Br(X_{\phi_{\infty}}) \rightarrow Br(V_{\infty})$ is surjective. 

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