THE COMPLEXITY OF FINDING FAIR INDEPENDENT SETS IN CYCLES

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Abstract. Let $G$ be a cycle graph and let $V_1, \ldots, V_m$ be a partition of its vertex set into $m$ sets. An independent set $S$ of $G$ is said to fairly represent the partition if $|S \cap V_i| \geq \frac{1}{2} |V_i| - 1$ for all $i \in [m]$. It is known that for every cycle and every partition of its vertex set, there exists an independent set that fairly represents the partition (Aharoni et al. 2017). We prove that the problem of finding such an independent set is PPA-complete. As an application, we show that the problem of finding a monochromatic edge in a Schrijver graph, given a succinct representation of a coloring that uses fewer colors than its chromatic number, is PPA-complete as well. The work is motivated by the computational aspects of the ‘cycle plus triangles’ problem and of its extensions.

Keywords. Fair independent sets in cycles, the complexity class PPA, Schrijver graphs

Subject classification. 68Q17, 68R10

1. Introduction

In 1986, Du, Hsu, and Hwang conjectured that if a graph on $3n$ vertices is the disjoint union of a Hamilton cycle of length $3n$ and $n$ pairwise vertex-disjoint triangles then its independence number is $n$. The conjecture has become known as the ‘cycle plus triangles’ problem and has been strengthened by Erdős (1990), who conjectured that such a graph is 3-colorable. Fleischner & Stiebitz (1992) confirmed these conjectures in a strong form and proved, using an algebraic approach of Alon & Tarsi (1992), that such a graph is
in fact 3-choosable. Their proof can also be viewed as an application of Alon’s Combinatorial Nullstellensatz technique (1999). Slightly later, an alternative elementary proof of the 3-coloring result was given by Sachs (1993). However, none of these proofs supplies an efficient algorithm that given a graph on $3n$ vertices whose set of edges is the disjoint union of a Hamilton cycle and $n$ pairwise vertex-disjoint triangles finds a 3-coloring of the graph or an independent set of size $n$. Questions on the computational aspects of the problem were posed in several works over the years (see, e.g., Aharoni et al. (2017); Alon (2002); Bérczi & Kobayashi (2017); Fleischner & Stiebitz (1997)).

A natural extension of the problem of Du et al. (1993) is the following. Let $G$ be a cycle and let $V_1, \ldots, V_m$ be a partition of its vertex set into $m$ sets. We are interested in an independent set of $G$ that (almost) fairly represents the given partition, that is, an independent set $S$ of $G$ satisfying $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m] = \{1, \ldots, m\}$. The existence of such an independent set was proved in a work of Aharoni, Alon, Berger, Chudnovsky, Kotlar, Loebl, and Ziv (2017). For the special case where all the sets $V_i$ are of size 3, the proof technique of Aharoni et al. (2017) allowed them to show that there are two disjoint independent sets that fairly represent the partition, providing a new proof of a stronger form of the original conjecture of Du et al. (1993). The results of Aharoni et al. (2017) were then extended in a work of Alishahi & Meunier (2017). A special case of one of their results is the following.

**Theorem 1.1 (Alishahi & Meunier 2017).** Let $G$ be a cycle on $n$ vertices and let $V_1, \ldots, V_m$ be a partition of its vertex set into $m$ sets. Suppose that $n$ and $m$ have the same parity. Then, there exist two disjoint independent sets $S_1$ and $S_2$ of $G$ covering all vertices but one from each $V_i$ such that for each $j \in \{1, 2\}$, it holds that $|S_j \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$.

As shown by Black et al. (2020), analogs of Theorem 1.1 for paths and for partitions into sets of odd sizes can also be proved using the approach of Aharoni et al. (2017).

It is interesting to mention that although the statements of
Theorem 1.1 and of its aforementioned variants are purely combinatorial; all of their known proofs are based on tools from topology. The use of topological methods in combinatorics was initiated by Lovász (1978) who applied the Borsuk–Ulam theorem (Borsuk 1933) from algebraic topology to prove a conjecture of Kneser (1955) on the chromatic number of Kneser graphs. For integers $n \geq 2k$, the Kneser graph $K(n, k)$ is the graph whose vertices are all the $k$-subsets of $[n]$ where two sets are adjacent if they are disjoint. It was proved in (Lovász 1978) that the chromatic number of $K(n, k)$ is $n - 2k + 2$, a result that was strengthened and generalized by several researchers (see, e.g., Matoušek (2007, Chapter 3)). One such strengthening was obtained by Schrijver (1978), who studied the subgraph of $K(n, k)$ induced by the collection of all $k$-subsets of $[n]$ with no two consecutive elements modulo $n$. This graph is denoted by $S(n, k)$ and is commonly referred to as the Schrijver graph. It was proved in (Schrijver 1978), again by a topological argument, that the chromatic number of $S(n, k)$ is equal to that of $K(n, k)$. As for Theorem 1.1, the proof of Alishahi & Meunier (2017) employs the Octahedral Tucker lemma that was applied by Matoušek (2004) in an alternative proof of Kneser’s conjecture and can be viewed as a combinatorial formulation of the Borsuk–Ulam theorem (see also Ziegler (2002)). The approach of Aharoni et al. (2017) and of Black et al. (2020), however, is based on a direct application of the chromatic number of the Schrijver graph. As before, these proofs are not constructive, in the sense that they do not suggest efficient algorithms for the corresponding search problems. Understanding the computational complexity of these problems is the main motivation for the current work.

In 1994, Papadimitriou has initiated the study of the complexity of total search problems in view of the mathematical argument that lies at the existence proof of their solutions. Let TFNP be the complexity class, defined by Megiddo & Papadimitriou (1991), of the total search problems in NP, that is, the class of search problems in which a solution is guaranteed to exist and can be verified in polynomial running time. Papadimitriou (1994) has introduced several subclasses of TFNP, each of which consists of the total search problems that can be reduced to a problem that rep-
resents some mathematical argument. For example, the class PPA (polynomial parity argument) corresponds to the simple fact that every graph with maximum degree 2 that has a vertex of degree 1 must have another degree 1 vertex. Hence, PPA is defined as the class of all problems in TFNP that can be efficiently reduced to the LEAF problem, in which given a succinct representation of a graph with maximum degree 2 and given a vertex of degree 1 in the graph, the goal is to find another such vertex. The class PPAD (polynomial parity argument in directed graphs) is defined similarly with respect to directed graphs. Another complexity class defined in (Papadimitriou 1994) is PPP (polynomial pigeonhole principle) whose underlying mathematical argument is the pigeonhole principle. Additional examples of complexity classes defined in this way are PLS (polynomial local search, Johnson et al. 1988), CLS (continuous local search, Daskalakis & Papadimitriou 2011), and EOPL (end of potential line, Fearnley et al. 2020).

The complexity class PPAD is known to perfectly capture the complexity of many important search problems. Notable examples of PPAD-complete problems are those associated with Sperner’s lemma (Chen & Deng 2009; Papadimitriou 1994), the Nash equilibrium theorem (Chen et al. 2009; Daskalakis et al. 2009), the envy-free cake-cutting theorem (Deng et al. 2012), and the hairy ball theorem (Goldberg & Hollender 2021). For PPA, the undirected analog of PPAD, until recently no ‘natural’ complete problems were known, where by ‘natural’ we mean that their definitions do not involve circuits and Turing machines. In the last few years, the situation was changed following a breakthrough result of Filos-Ratsikas & Goldberg (2018, 2019), who proved that the consensus-halving problem with an inverse-polynomial precision parameter is PPA-complete (see also Filos-Ratsikas et al. (2020)) and used it to derive the PPA-completeness of the classical splitting necklace with two thieves and discrete sandwich problems. This was obtained building on the PPA-hardness, proved by Aisenberg et al. (2020), of the search problem associated with Tucker’s lemma. The variant of the problem that corresponds to the octahedral Tucker’s lemma was suggested for study by Pálvölgyi (2009) and proved to be PPA-complete by Deng, Feng, and Kulkarni (2017). The PPA-
completeness of the consensus-halving problem was improved to a constant precision parameter in a recent work of Deligkas et al. (2022a). Additional examples of PPA-complete problems can be found, for instance, in the works of Belovs et al. (2017), Schnider (2021), and Deligkas et al. (2022b).

1.1. Our contribution. The present work initiates the study of the complexity of finding independent sets that fairly represent a given partition of the vertex set of a cycle. It is motivated by the computational aspects of combinatorial existence statements, such as the ‘cycle plus triangles’ conjecture of Du et al. (1993) proved by Fleischner & Stiebitz (1992) and its extensions by Aharoni et al. (2017), Alishahi & Meunier (2017), and Black et al. (2020). As mentioned before, the challenge of understanding the complexity of the corresponding search problems was explicitly raised by several authors, including Fleischner & Stiebitz (1997), Alon (2002), and Aharoni et al. (2017). In this work, we demonstrate that this research avenue may illuminate interesting connections between this family of problems and the complexity class PPA. As an application, we determine the complexity of finding a monochromatic edge in Schrijver graphs colored by fewer colors than the chromatic number.

We start by introducing the fair independent set in cycle problem, which we denote by FAIR-IS-CYCLE and define as follows.

**Definition 1.2** (Fair Independent Set in Cycle Problem). In the FAIR-IS-CYCLE problem, the input consists of a cycle $G$ and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets. The goal is to find an independent set $S$ of $G$ satisfying $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$.

The existence of a solution to every input of FAIR-IS-CYCLE is guaranteed by a result of Aharoni et al. (2017, Theorem 1.8). Since such a solution can be verified in polynomial running time, the total search problem FAIR-IS-CYCLE lies in the complexity class TFNP. We prove that the class PPA captures the complexity of the problem.
Theorem 1.3. The Fair-IS-Cycle problem is PPA-complete.

In view of the ‘cycle plus triangles’ problem, it would be interesting to understand the complexity of the Fair-IS-Cycle problem restricted to partitions into sets of size 3. While Theorem 1.3 immediately implies that this restricted problem lies in PPA, the question of determining its precise complexity remains open.

We proceed by considering the search problem associated with Theorem 1.1. In the fair splitting of cycle problem, denoted Fair-Split-Cycle, we are given a cycle and a partition of its vertex set and the goal is to find two disjoint independent sets that fairly represent the partition and cover all vertices but one from every part of the partition. We define below an approximate version of this problem, in which the fairness requirement is replaced with the relaxed notion of $\varepsilon$-fairness, where the independent sets should include at least $\frac{1}{2} - \varepsilon$ fraction of the vertices of every part.

Definition 1.4 (Approximate Fair Splitting of Cycle Problem).

In the $\varepsilon$-Fair-Split-Cycle problem with parameter $\varepsilon \geq 0$, the input consists of a cycle $G$ on $n$ vertices and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets, such that $n$ and $m$ have the same parity. The goal is to find two disjoint independent sets $S_1$ and $S_2$ of $G$ covering all vertices but one from each $V_i$ such that for each $j \in \{1, 2\}$, it holds that $|S_j \cap V_i| \geq (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1$ for all $i \in [m]$. For $\varepsilon = 0$, the problem is denoted by Fair-Split-Cycle.

The existence of a solution to every input of $\varepsilon$-Fair-Split-Cycle, already for $\varepsilon = 0$, is guaranteed by Theorem 1.1 proved by Alishahi & Meunier (2017). For $\varepsilon = 0$, Fair-Split-Cycle is at least as hard as Fair-IS-Cycle (see Lemma 2.12). Yet, it turns out that Fair-Split-Cycle lies in PPA and is thus also PPA-complete.

Theorem 1.5. The Fair-Split-Cycle problem is PPA-complete.

In fact, using the recent work (Deligkas et al. 2022a), we also obtain the following PPA-completeness result for the approximate version of the problem (see Remark 2.11).

Theorem 1.6. There exists an absolute constant $\varepsilon > 0$ for which the $\varepsilon$-Fair-Split-Cycle problem is PPA-complete.
We finally consider the complexity of the SCHRIJVER problem. In this problem, we are given a succinct representation of a coloring of the Schrijver graph $S(n, k)$ with $n - 2k + 1$ colors, which is one less than its chromatic number (Schrijver 1978), and the goal is to find a monochromatic edge (see Definition 3.1). The study of the SCHRIJVER problem is motivated by a question raised by Deng et al. (2017) regarding the complexity of the analog KNESER problem for Kneser graphs. Note that the latter is not harder than the SCHRIJVER problem, because $S(n, k)$ is a subgraph of $K(n, k)$ with the same chromatic number. As an application of our Theorem 1.3, we prove the following.

**Theorem 1.7.** The SCHRIJVER problem is PPA-complete.

It would be interesting to determine the computational complexity of the KNESER problem and to decide whether it is PPA-complete, as suggested by Deng et al. (2017). It would also be interesting to prove unconditional lower bounds on the query complexity of algorithms for the KNESER and SCHRIJVER problems in the black-box input model, where the input is given as an oracle access. We note that the study of the KNESER problem is motivated by its connections to a resource allocation problem called Agreeable Set, that was introduced by Manurangsi & Suksompong (2019) and further studied in Goldberg et al. (2020); Haviv (2022a). From an algorithmic point of view, it was shown in the recent works (Haviv 2022a,b) that there exist randomized algorithms for the KNESER and SCHRIJVER problems with running time $n^{O(1)} \cdot k^{O(k)}$ on graphs $K(n, k)$ and $S(n, k)$ respectively; hence, these problems are fixed-parameter tractable with respect to the parameter $k$. It would be nice to further explore algorithms for these problems as well as for the other problems studied in the current work.

**1.2. Overview of proofs.** To obtain our results, we present a chain of reductions, as described in Figure 1.1. Our starting point is the consensus-halving problem with precision parameter $\varepsilon$, in which given a collection of $m$ probability measures on the interval $[0, 1]$ the goal is to partition the interval into two pieces using relatively few cuts, so that each of the measures has the same
mass on the two pieces up to an error of $\varepsilon$ (see Definition 2.1). It is known that every instance of this problem has a solution with at most $m$ cuts even for $\varepsilon = 0$ (Alon & West 1986; Goldberg & West 1985; Simmons & Su 2003) and that the problem of finding such a solution is PPA-hard for some constant $\varepsilon > 0$ (Deligkas et al. 2022a).

In Section 2, we reduce the consensus-halving problem to an intermediate variant of the $\varepsilon$-FAIR-SPLIT-CYCLE problem, which we call $\varepsilon$-FAIR-SPLIT-PATH′ (see Definition 2.4). Then, we use this reduction to obtain our hardness results for the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems. The reduction borrows a discretization argument that was used in (Filos-Ratsikas & Goldberg 2018) to reduce the consensus-halving problem to the splitting necklace problem with two thieves. This argument enables us to transform a consensus-halving instance into a path and a partition of its vertex set, for which the goal is to partition the path using relatively few cuts into two parts, each of which contains roughly half of the vertices of every set in the partition. In order to relate this problem to independent sets that fairly represent the partition, we need an additional simple trick. Between every two consecutive vertices of the path, we add a new vertex and put all the new vertices in a new set added to the partition of the vertex set. We then argue, roughly speaking, that two disjoint independent sets in the obtained path,
which fairly represent the partition and cover almost all of the vertices, can be used to obtain a solution to the original instance. The high-level idea is that those few vertices that are uncovered by the two independent sets can be viewed as cuts, and every path between two such vertices alternates between the two given independent sets. By construction, it means that only one of the two independent sets contains in such a path original vertices (that is, vertices that were not added in the last phase of the reduction); hence, every such path can be naturally assigned to one of the two pieces required by the consensus-halving problem. Combining our reduction with the known hardness results of consensus-halving, we derive the PPA-hardness of FAIR-IS-CYCLE and of ε-FAIR-SPLIT-CYCLE for a constant ε > 0, as needed for Theorem 1.3, Theorem 1.5, and Theorem 1.6.

In Section 3, we introduce and study the SCHRIJVER problem. We reduce the FAIR-IS-CYCLE problem to the SCHRIJVER problem, implying that the latter is PPA-hard. The reduction follows an argument of Aharoni et al. (2017) who used the chromatic number of the Schrijver graph (Schrijver 1978) to prove the existence of the independent set required in FAIR-IS-CYCLE. Finally, employing arguments of Meunier (2011) and Alishahi & Meunier (2017), we reduce the SCHRIJVER and FAIR-SPLIT-CYCLE problems to the search problem associated with the Octahedral Tucker lemma (see Definition 3.3). Since it is known, already from (Papadimitriou 1994), that this problem lies in PPA, we get that FAIR-IS-CYCLE, FAIR-SPLIT-CYCLE, and SCHRIJVER are all members of PPA, completing the proofs of Theorem 1.3, Theorem 1.5, Theorem 1.6, and Theorem 1.7.

We remark that one could consider appropriate analogs of the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems for paths rather than for cycles and obtain similar results. We have chosen to focus here on the cycle setting, motivated by the computational aspects of the ‘cycle plus triangles’ problem (Du et al. 1993; Erdős 1990; Fleischner & Stiebitz 1992).
2. Fair independent sets in cycles

In this section, we prove our hardness results for the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems. We first recall the definition of the consensus-halving problem and state its hardness result from (Deligkas et al. 2022a). Then, we present an efficient reduction from this problem to an intermediate problem, which is used to obtain the hardness results of Theorem 1.3, Theorem 1.5, and Theorem 1.6.

2.1. Consensus-halving. Consider the following variant of the consensus-halving problem, denoted CON-HALVING.

**Definition 2.1 (Consensus-Halving Problem).** For a precision parameter \(\varepsilon = \varepsilon(m)\), the input of the \(\varepsilon\)-CON-HALVING\((m, \ell)\) problem consists of \(m\) probability measures \(\mu_1, \ldots, \mu_m\) on the interval \(I = [0, 1]\), given by their density functions. The goal is to partition the interval \(I\) using at most \(\ell\) cuts into two (not necessarily connected) pieces \(I^+\) and \(I^-\), so that for every \(i \in [m]\) it holds that \(|\mu_i(I^+) - \mu_i(I^-)| \leq \varepsilon\).

For \(\ell \geq m\), every input of \(\varepsilon\)-CON-HALVING\((m, \ell)\) has a solution even for \(\varepsilon = 0\) (Simmons & Su 2003). We will rely on the hardness result of CON-HALVING stated below. Here, a function on an interval is said to be *piecewise constant* if its domain can be partitioned into a finite set of intervals such that the function is constant on each of them. We refer to the intervals of the partition on which the function is nonzero as the *blocks* of the function. Note that a piecewise constant function can be explicitly represented by the endpoints and the values of its blocks.

**Theorem 2.2** (Deligkas et al. 2022a, Theorem 1.3). There exists a constant \(\varepsilon > 0\) such that for every constant \(c \geq 0\), the \(\varepsilon\)-CON-HALVING\((m, m + c)\) problem, restricted to inputs with piecewise constant density functions with at most 3 blocks, is PPA-hard.

**Remark 2.3.** We note that, as explained in (Filos-Ratsikas et al. 2020), the constant \(c\) given in Theorem 2.2 can be replaced by \(m^{1-\alpha}\) for any constant \(\alpha > 0\). This stronger hardness, however, is not
required to obtain our results. We also note that our results do not rely on the fact that the hardness given in Theorem 2.2 holds for instances with density functions with at most 3 blocks, as proved in (Deligkas et al. 2022a), rather than polynomially many blocks.

2.2. The main reduction. To obtain our hardness results for the FAIR-IS-CYCLE and FAIR-SPLIT-CYCLE problems, we consider the following intermediate problem.

**Definition 2.4.** In the $\varepsilon$-FAIR-SPLIT-PATH’ problem with parameter $\varepsilon \geq 0$, the input consists of a path $G$ and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets such that $|V_i|$ is odd for all $i \in [m]$. The goal is to find two disjoint independent sets $S_1$ and $S_2$ of $G$ covering all but at most $m$ of the vertices of $G$ such that

$$|S_1 \cap V_i| \in \left[\left(\frac{1}{2} - \varepsilon\right) \cdot |V_i| - 1, \left(\frac{1}{2} + \varepsilon\right) \cdot |V_i|\right]$$

for all $i \in [m]$. For the case of $\varepsilon = 0$, the problem is denoted by FAIR-SPLIT-PATH’.

Note that the $\varepsilon$-FAIR-SPLIT-PATH’ problem ($\varepsilon \geq 0$) differs from the $\varepsilon$-FAIR-SPLIT-CYCLE problem (see Definition 1.4) in the following respects: (a) The input graph is a path rather than a cycle, (b) an $\varepsilon$-fairness property is required only for the independent set $S_1$ rather than for both $S_1$ and $S_2$, (c) there is no requirement regarding the sets $V_i$ to which the vertices that are uncovered by $S_1$ and $S_2$ belong, and (d) the sets $V_i$ are required to be of odd sizes. Yet, every instance of the $\varepsilon$-FAIR-SPLIT-PATH’ problem has a solution already for $\varepsilon = 0$, as follows from Theorem 1.1 applied to the cycle obtained by connecting the endpoints of the given path by an edge.

We turn to prove the following.

**Theorem 2.5.** Let $p$ be a polynomial and suppose that $\varepsilon = \varepsilon(m)$ is bounded from below by some inverse-polynomial in $m$. Then, for any constant $\alpha \in [0, 1)$, the $\varepsilon$-CON-HALVING$(m, m + 1)$ problem, restricted to inputs with piecewise constant density functions with at most $p(m)$ blocks, is polynomial-time reducible to the $\frac{\alpha \varepsilon}{2}$-FAIR-SPLIT-PATH’ problem.
Proof. Consider an instance of $\varepsilon$-CON-HALVING$(m, m+1)$ consisting of $m$ probability measures $\mu_1, \ldots, \mu_m$ on the interval $I = [0, 1]$, given by their piecewise constant density functions $g_1, \ldots, g_m$, each of which has at most $p(m)$ blocks. Fix any constant $\alpha \in [0, 1)$. The reduction constructs an instance of $\frac{\alpha\varepsilon}{2}$-FAIR-SPLIT-PATH', namely, a path $G$ and a partition $V_1, \ldots, V_{m+1}$ of its vertex set into $m + 1$ sets of odd sizes.

We start with a high-level description of the reduction. First, borrowing a discretization argument of (Filos-Ratsikas & Goldberg 2018), the reduction associates with every density function $g_i$ a collection $V_i$ of vertices located in the (at most $p(m)$) intervals on which $g_i$ is nonzero. To do so, we partition every block of $g_i$ into subintervals such that the measure of $\mu_i$ on each of them is $\delta$, where $\delta > 0$ is some small parameter (assuming, for now, that the measure of $\mu_i$ on every block is an integer multiple of $\delta$). At the middle of every such subinterval we locate a vertex and put it in $V_i$. Then, we construct a path $G$ that alternates between the vertices of $V_1 \cup \cdots \cup V_m$ ordered according to their locations in $I$ and additional vertices which we put in another set $V_{m+1}$. We also take care of the requirement that each $|V_i|$ is odd.

The intuitive idea behind this reduction is the following. Suppose that we are given a solution to the constructed instance, i.e., two disjoint independent sets $S_1$ and $S_2$ of the path $G$ covering all but $m + 1$ of the vertices such that $S_1$ contains roughly half of the vertices of $V_i$ for each $i \in [m + 1]$. Observe that by removing from $G$ the $m + 1$ vertices that do not belong to $S_1 \cup S_2$, we essentially get a partition of the vertices of $S_1 \cup S_2$ into $m + 2$ paths. Since $S_1$ and $S_2$ are independent sets in $G$, it follows that each such path alternates between $S_1$ and $S_2$. However, recalling that $G$ alternates between $V_1 \cup \cdots \cup V_m$ and $V_{m+1}$, it follows that ignoring the vertices of $V_{m+1}$, each such path contains either only vertices of $S_1$ or only vertices of $S_2$. Now, one can view the $m + 1$ locations of the vertices that do not belong to $S_1 \cup S_2$ as cuts in the interval $I$ which partition it into $m + 2$ subintervals, each of which includes vertices from either $S_1$ or $S_2$ (again, ignoring the vertices of $V_{m+1}$). Let $I^+$ and $I^-$ be the pieces of $I$ obtained from the subintervals that correspond to $S_1$ and $S_2$, respectively. Since the number of
vertices from $V_i$ in every path is approximately proportional to the measure of $\mu_i$ in the corresponding subinterval, it can be shown that the probability measure of $\mu_i$ on $I^+$ is approximately $\frac{1}{2}$. This yields that the probability measure $\mu_i$ is approximately equal on the pieces $I^+$ and $I^-$, as needed for the CON-HALVING($m, m + 1$) problem.

We turn to the formal description of the reduction. For an illustration, see Figure 2.1. Define $\delta = \frac{(1-\alpha)\varepsilon}{2(p(m) + m + 3)}$. The reduction acts as follows.

1. For every $i \in [m]$, do the following:
   - We are given a partition of the interval $I$ into intervals such that on at most $p(m)$ of them the function $g_i$ is equal to a nonzero value and is zero everywhere else. For every such interval, let $\gamma$ denote the volume of $g_i$ on it, and divide it into $\lceil \gamma/\delta \rceil$ subintervals of volume $\delta$ each, possibly besides the last one whose volume might be smaller. We refer to a subinterval of volume smaller than $\delta$ as an imperfect subinterval. The number of imperfect subintervals associated with $g_i$ is clearly at most $p(m)$. At the middle point of every subinterval of $g_i$, locate a vertex and put it in the set $V_i$.
   - If the number of vertices in $V_i$ is even, then add another vertex to $V_i$ and locate it arbitrarily in $I$.
   - Note that, by $\mu_i(I) = 1$, we have
     \begin{equation}
     |V_i| \cdot \delta \in [1, 1 + (p(m) + 1) \cdot \delta].
     \end{equation}

2. Consider the path on the vertices of $V_1 \cup \cdots \cup V_m$ ordered according to their locations in the interval $I$, breaking ties arbitrarily.

3. Add a new vertex before every vertex in this path, locate it at the middle of the subinterval between its two adjacent vertices (where the first new vertex is located at 0), and put these new vertices in the set $V_{m+1}$. If the number of vertices in $V_{m+1}$ is even then add one more vertex to the end of the
path, locate it at 1, and put it in $V_{m+1}$ as well. Denote by $G$ the obtained path, and note that $G$ alternates between $V_1 \cup \cdots \cup V_m$ and $V_{m+1}$.

4. The output of the reduction is the path $G$ and the partition $V_1, \ldots, V_{m+1}$ of its vertex set $V$ into $m+1$ sets. By construction, $|V_i|$ is odd for every $i \in [m+1]$.

It is easy to verify that the reduction can be implemented in polynomial running time. Indeed, every density function $g_i$ is piecewise constant with at most $p(m)$ blocks; hence, for every $i \in [m]$ the number of vertices that the reduction defines for $V_i$ is at most $1/\delta + p(m) + 1$, and the latter is polynomial in the input size because of the definition of $\delta$ and the fact that $\varepsilon$ is at least inverse-polynomial in $m$. The additional set $V_{m+1}$ doubles the number of vertices, possibly with one extra vertex, preserving the construction polynomial in the input size.

Figure 2.1: An illustration of the reduction for $m = 2$ (Theorem 2.5). Given the density functions $g_1$ and $g_2$, the reduction produces a path $G$ and a partition of its vertex set into three sets of odd sizes: $V_1$ (gray), $V_2$ (black), and $V_3$ (white). The path $G$ alternates between $V_1 \cup V_2$ and $V_3$.

We turn to prove the correctness of the reduction, that is, that a solution to the constructed instance of $\frac{\alpha \varepsilon}{2}$-FAIR-SPLIT-PATH’ can be used to efficiently compute a solution to the original instance of $\varepsilon$-CON-HALVING$(m, m+1)$. Suppose we are given a
solution to $\frac{\alpha - \varepsilon}{2} \cdot \text{FAIR-SPLIT-PATH}'$ for the path $G$ and the partition $V_1, \ldots, V_{m+1}$ of its vertex set $V$. Such a solution consists of two disjoint independent sets $S_1$ and $S_2$ of $G$ covering all but at most $m+1$ of the vertices of $G$ such that

$$|S_1 \cap V_i| \in \left[ \left( \frac{1}{2} - \frac{\alpha - \varepsilon}{2} \right) \cdot |V_i| - 1, \left( \frac{1}{2} + \frac{\alpha - \varepsilon}{2} \right) \cdot |V_i| \right]$$

for all $i \in [m+1]$. Let $S_3 = V \setminus (S_1 \cup S_2)$. It can be assumed that $|S_3| = m+1$ (otherwise, remove some arbitrary vertices from $S_2$). Denote the vertices of $S_3$ by $u_1, \ldots, u_{m+1}$ ordered according to their order in $G$. Let $P_1, \ldots, P_{m+2}$ be the $m+2$ paths obtained from $G$ by removing the vertices of $S_3$ (where some of the paths might be empty). Since $S_1$ and $S_2$ are independent sets, every path $P_j$ alternates between $S_1$ and $S_2$. By our construction, this implies that in every path $P_j$ either the vertices of $S_1$ are from $V \setminus V_{m+1}$ and those of $S_2$ are from $V_{m+1}$, or the vertices of $S_2$ are from $V \setminus V_{m+1}$ and those of $S_1$ are from $V_{m+1}$. We define $b_j = 1$ in the former case and $b_j = 2$ in the latter. Thus, for every $i \in [m]$, the number of vertices of $V_i$ that appear in the paths $P_j$ with $b_j = 1$ is precisely $|S_1 \cap V_i|$.

Now, let $\beta_1, \ldots, \beta_{m+1} \in I$ be the locations of the vertices $u_1, \ldots, u_{m+1}$ in the interval $I$ as defined by the reduction. We interpret these locations as $m+1$ cuts of the interval $I$. Set $\beta_0 = 0$ and $\beta_{m+2} = 1$, and for every $j \in [m+2]$, let $I_j$ denote the interval $[\beta_{j-1}, \beta_j]$. Consider the partition of $I$ into two pieces $I^+$ and $I^-$, where $I^+$ includes all the parts $I_j$ with $b_j = 1$ and $I^-$ includes all the parts $I_j$ with $b_j = 2$. We claim that this partition, which is obtained using $m+1$ cuts in $I$, forms a valid solution to the original instance of $\varepsilon$-CON-HALVING$(m, m+1)$. To this end, we show that for every $i \in [m]$ it holds that $|\mu_i(I^+) - \frac{1}{2}| \leq \frac{\varepsilon}{2}$, which is equivalent to $|\mu_i(I^+) - \mu_i(I^-)| \leq \varepsilon$.

Fix some $i \in [m]$. We turn to estimate the quantity $\mu_i(I^+)$, i.e., the total measure of $\mu_i$ on the intervals $I_j$ with $b_j = 1$. By our construction, every vertex of $V_i$ corresponds to a subinterval whose measure by $\mu_i$ is $\delta$ (except for at most $p(m)+1$ of them). Since the intervals of $I^+$ correspond to the paths $P_j$ whose vertices in $V \setminus V_{m+1}$ are precisely the vertices of $S_1 \setminus V_{m+1}$, one would expect $\mu_i(I^+)$ to measure the number of vertices in $S_1 \cap V_i$, with a contribution of $\delta$
per every such vertex. This suggests an estimation of $|S_1 \cap V_i| \cdot \delta$ for $\mu_i(I^+)$. This estimation, however, is not accurate for the following reasons:

- The set $V_i$ might include vertices that correspond to imperfect subintervals whose measure by $\mu_i$ is smaller than $\delta$. Since there are at most $p(m)$ such vertices in $V_i$, they can cause an error of at most $p(m) \cdot \delta$ in the above estimation.

- To make sure that $|V_i|$ is odd, the reduction might add one extra vertex to $V_i$. This might cause an error of at most $\delta$ in the above estimation.

- The precise locations $\beta_j$ of the cuts of $I$ might fall inside subintervals that correspond to vertices of $V_i$. Since the subintervals that correspond to vertices of $V_i$ are disjoint, every such cut can cause an error of at most $\delta$ in the above estimation, and since there are $m + 1$ cuts the error here is bounded by $(m + 1) \cdot \delta$.

We conclude that $\mu_i(I^+)$ differs from the aforementioned estimation $|S_1 \cap V_i| \cdot \delta$ by not more than $(p(m) + m + 2) \cdot \delta$. Combining (2.6) and (2.7), it can be verified that

\[
\left| |S_1 \cap V_i| \cdot \delta - \frac{1}{2} \right| \leq \frac{\alpha \cdot \epsilon}{2} + (p(m) + 1) \cdot \delta,
\]

hence

\[
\left| \mu_i(I^+) - \frac{1}{2} \right| \leq \left| \mu_i(I^+) - |S_1 \cap V_i| \cdot \delta \right| + \left| |S_1 \cap V_i| \cdot \delta - \frac{1}{2} \right| \\
\leq (p(m) + m + 2) \cdot \delta + \frac{\alpha \cdot \epsilon}{2} + (p(m) + 1) \cdot \delta \\
= \frac{\alpha \cdot \epsilon}{2} + (2p(m) + m + 3) \cdot \delta = \frac{\epsilon}{2},
\]

where the last equality holds by the definition of $\delta$. This completes the proof. \(\square\)

We establish the following result.
Theorem 2.8. There exists an absolute constant $\varepsilon > 0$ for which the $\varepsilon$-Fair-Split-Path' problem is PPA-hard.

Proof. By Theorem 2.2, the $\varepsilon$-Con-Halving$(m, m + 1)$ problem is PPA-hard for input density functions that are piecewise constant with at most 3 blocks, where $\varepsilon > 0$ is some constant. By Theorem 2.5, for any $\alpha \in [0, 1)$, this problem is polynomial-time reducible to the $\frac{\alpha \cdot \varepsilon}{2}$-Fair-Split-Path' problem, implying the assertion of the theorem. □

2.3. Hardness of Fair-IS-Cycle and Fair-Split-Cycle. With Theorem 2.8 at hand, we are ready to derive the hardness of the Fair-IS-Cycle and Fair-Split-Cycle problems (see Definition 1.2 and Definition 1.4).

Theorem 2.9. The Fair-IS-Cycle problem is PPA-hard.

Proof. By Theorem 2.8, the $\varepsilon$-Fair-Split-Path' problem is PPA-hard for some $\varepsilon > 0$. It thus follows that Fair-Split-Path', with $\varepsilon = 0$, is PPA-hard as well. Hence, to prove the theorem, it suffices to show that Fair-Split-Path' is polynomial-time reducible to Fair-IS-Cycle.

Consider an instance of Fair-Split-Path', that is, a path $G$ on $n$ vertices and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets such that $|V_i|$ is odd for all $i \in [m]$. The reduction simply returns the cycle $G'$, obtained from the path $G$ by connecting its endpoints by an edge, and the same partition $V_1, \ldots, V_m$ of its vertex set. For correctness, suppose that we are given a solution to this instance of Fair-IS-Cycle, i.e., an independent set $S_1$ of $G'$ satisfying $|S_1 \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$. Since each $|V_i|$ is odd, it can be assumed that $|S_1 \cap V_i| = \frac{1}{2} \cdot (|V_i| - 1)$ for all $i \in [m]$ (by removing some vertices from $S_1$ if needed), implying that

$$|S_1| = \sum_{i=1}^{m} |S_1 \cap V_i| = \frac{1}{2} \cdot \sum_{i=1}^{m} (|V_i| - 1) = \frac{n - m}{2}.$$ 

For every vertex of $S_1$ consider the vertex that follows it in the cycle $G'$ (say, oriented clockwise), and let $S_2$ be the set of vertices that follow those of $S_1$. Since $S_1$ is an independent set in $G'$, we...
get that \( S_2 \) is another independent set in \( G' \) which is disjoint from \( S_1 \) and has the same size. We obtain that

\[
|S_1 \cup S_2| = |S_1| + |S_2| = 2 \cdot \frac{n-m}{2} = n - m;
\]

hence, \( S_1 \) and \( S_2 \) are two disjoint independent sets of \( G' \) covering all but \( m \) of its vertices. In particular, \( S_1 \) and \( S_2 \) are independent sets in the path \( G \), and as such, they form a valid solution to the FAIR-SPLIT-PATH' instance. This solution can clearly be constructed in polynomial running time given \( S_1 \), completing the proof. □

**Theorem 2.10.** There exists a constant \( \varepsilon > 0 \) for which the \( \varepsilon \)-FAIR-SPLIT-CYCLE problem is PPA-hard.

**Proof.** By Theorem 2.8, the \( \varepsilon \)-FAIR-SPLIT-PATH' problem is PPA-hard for some constant \( \varepsilon > 0 \). It thus suffices to show that for every \( \varepsilon \geq 0 \), the \( \varepsilon \)-FAIR-SPLIT-PATH' problem is polynomial-time reducible to the \( \varepsilon \)-FAIR-SPLIT-CYCLE problem.

Consider again the reduction that given a path \( G \) and a partition \( V_1, \ldots, V_m \) of its vertex set into sets of odd sizes returns the cycle \( G' \), obtained from the path \( G \) by connecting its endpoints by an edge, and the same partition \( V_1, \ldots, V_m \). Since the sets of the partition have odd sizes, it follows that the number of vertices and the number of sets in the partition have the same parity; hence, the reduction provides an appropriate instance of the \( \varepsilon \)-FAIR-SPLIT-CYCLE problem.

For correctness, consider a solution to the constructed instance, i.e., two disjoint independent sets \( S_1 \) and \( S_2 \) of \( G' \) covering all vertices but one from each part \( V_i \) such that for each \( j \in \{1, 2\} \), it holds that \( |S_j \cap V_i| \geq (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1 \) for all \( i \in [m] \). We claim that \( S_1 \) and \( S_2 \) form a valid solution to the original \( \varepsilon \)-FAIR-SPLIT-PATH' instance. Indeed, an independent set in \( G' \) is also an independent set in \( G \). In addition, the set \( S_1 \) satisfies

\[
|S_1 \cap V_i| \in \left[ (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1, (\frac{1}{2} + \varepsilon) \cdot |V_i| \right]
\]

for all \( i \in [m] \), where the upper bound holds because

\[
|S_1 \cap V_i| = |V_i| - |S_2 \cap V_i| - 1 \leq |V_i| - \left( (\frac{1}{2} - \varepsilon) \cdot |V_i| - 1 \right) - 1 = (\frac{1}{2} + \varepsilon) \cdot |V_i|.
\]
This completes the proof. \qed

**Remark 2.11.** The PPA-hardness of $\varepsilon$-Con-Halving$(m, m + 1)$, given by Theorem 2.2, was proved in (Deligkas et al. 2022a) for any constant $\varepsilon < 0.2$. It thus follows from the proofs of Theorem 2.8 and Theorem 2.10 that $\varepsilon$-Fair-Split-Cycle is PPA-hard for any constant $\varepsilon < 0.1$.

Theorem 2.10 implies that the Fair-Split-Cycle problem, with $\varepsilon = 0$, is PPA-hard. The following simple lemma shows that this hardness result can also be derived from the hardness of the Fair-IS-Cycle problem, given in Theorem 2.9.

**Lemma 2.12.** The Fair-IS-Cycle problem is polynomial-time reducible to the Fair-Split-Cycle problem.

**Proof.** Consider an instance of Fair-IS-Cycle, that is, a cycle $G$ on $n$ vertices and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets. If $n$ and $m$ have the same parity then the reduction returns the input as is. Otherwise, there exists some $j \in [m]$ for which the size of $V_j$ is even. In this case, the reduction adds to the cycle $G$ a new vertex located between two arbitrary consecutive vertices and puts it in $V_j$. Now, the number of vertices and the number of sets in the partition have the same parity, so the reduction can output the obtained cycle and partition.

We turn to prove the correctness of the reduction. If the given instance of Fair-IS-Cycle satisfies that $n$ and $m$ have the same parity, then its solution as an instance of Fair-Split-Cycle includes two disjoint independent sets that fairly represent the partition, and each of them also forms a solution as an instance of Fair-IS-Cycle. So suppose that $n$ and $m$ have a different parity, and let $j \in [m]$ denote the index for which the reduction adds a vertex to $V_j$. Let $u$ denote the added vertex, and define $V'_i = V_i$ for $i \in [m] \setminus \{j\}$ and $V'_j = V_j \cup \{u\}$. Now, a solution to the constructed instance of Fair-Split-Cycle includes two disjoint independent sets that fairly represent the partition. Clearly, at least one of the sets does not include both neighbors of $u$. Letting $S'$ denote such a set, it follows that the set $S = S' \setminus \{u\}$ is independent in
the original given cycle. For every $i \in [m] \setminus \{j\}$, it holds that $|S \cap V_i| = |S' \cap V_i'| \geq \frac{1}{2} \cdot |V_i'| - 1 = \frac{1}{2} \cdot |V_i| - 1$. It further holds that $|S \cap V_j| \geq |S' \cap V_j'| - 1 \geq \frac{1}{2} \cdot |V_j'| - 2$.

Since $|V_j|$ is even, it follows that $|V_j'|$ is odd; hence,

$$|S \cap V_j| \geq \frac{1}{2} \cdot |V_j'| - \frac{3}{2} = \frac{1}{2} \cdot |V_j| - 1.$$

This implies that $S$ is a solution to the original instance of the FAIR-IS-CYCLE problem, and we are done. \qed

3. The Schrijver problem

In this section, we introduce and study the SCHRIJVER problem, a natural analog of the KNESER problem defined by Deng et al. (2017).

We start with some definitions. A set $A \subseteq [n]$ is said to be stable if it does not contain two consecutive elements modulo $n$ (that is, if $i \in A$ then $i + 1 \notin A$, and if $n \in A$ then $1 \notin A$). In other words, a stable subset of $[n]$ is an independent set in the cycle on $n$ vertices with the numbering from 1 to $n$ along the cycle. For integers $n \geq 2k$, let $\binom{[n]}{k}_{\text{stab}}$ denote the collection of all stable $k$-subsets of $[n]$. Recall that the Schrijver graph $S(n, k)$ is the graph on the vertex set $\binom{[n]}{k}_{\text{stab}}$, where two sets are adjacent if they are disjoint. We define the search problem SCHRIJVER as follows.

**Definition 3.1 (Schrijver Graph Problem).** In the SCHRIJVER problem, the input consists of a Boolean circuit that represents a coloring

$$c : \binom{[n]}{k}_{\text{stab}} \to [n - 2k + 1]$$

of the Schrijver graph $S(n, k)$ using $n - 2k + 1$ colors, where $n$ and $k$ are integers satisfying $n \geq 2k$. The goal is to find a monochromatic edge, i.e., two disjoint sets $S_1, S_2 \in \binom{[n]}{k}_{\text{stab}}$ such that $c(S_1) = c(S_2)$.

As mentioned earlier, it was proved by Schrijver (1978) that the chromatic number of $S(n, k)$ is precisely $n - 2k + 2$. Therefore, every input to the SCHRIJVER problem has a solution.
3.1. From Fair-IS-Cycle to Schrijver. The following theorem is used to obtain the hardness result for the SCHRIJVER problem. The proof applies an argument of Aharoni et al. (2017) (see also Black et al. (2020)).

**Theorem 3.2.** The Fair-IS-Cycle problem is polynomial-time reducible to the Schrijver problem.

**Proof.** Consider an instance of the Fair-IS-Cycle problem, namely, a cycle $G$ and a partition $V_1, \ldots, V_m$ of its vertex set into $m$ sets. For every $i \in [m]$, let $V'_i$ be the set obtained from $V_i$ by removing one arbitrary vertex if $|V_i|$ is even, and let $V'_i = V_i$ otherwise. Since the size of every set $V'_i$ is odd, we can write $|V'_i| = 2r_i + 1$ for an integer $r_i \geq 0$. Let $G'$ be the cycle obtained from $G$ by removing the vertices that do not belong to the sets $V'_i$ and connecting the remaining vertices according to their order in $G$. Letting $n$ denote the number of vertices in $G'$, it can be assumed that its vertex set is $[n]$ with the numbering from 1 to $n$ along the cycle. Let $k = \sum_{i=1}^m r_i$, and notice that $n = 2k + m$. Define a coloring $c$ of the Schrijver graph $S(n, k)$ as follows. The color $c(S)$ of a vertex $S \in \binom{[n]}{k}_{\text{stab}}$ is defined as the smallest integer $i \in [m]$ for which $|S \cap V'_i| > r_i$ in case that such an $i$ exists, and $m+1$ otherwise. This gives us a coloring of $S(n, k)$ with $n - 2k + 1$ colors, and thus an instance of the SCHRIJVER problem. It can be seen that a Boolean circuit that computes the coloring $c$ can be constructed in polynomial running time.

To prove the correctness of the reduction, consider a solution to the constructed SCHRIJVER instance, i.e., two disjoint sets $S_1, S_2 \in \binom{[n]}{k}_{\text{stab}}$ with $c(S_1) = c(S_2)$. It is impossible that for some $i \in [m]$ it holds that $|S_1 \cap V'_i| > r_i$ and $|S_2 \cap V'_i| > r_i$, because $S_1$ and $S_2$ are disjoint and $|V'_i| = 2r_i + 1$. It follows that $c(S_1) = c(S_2) = m+1$, meaning that $|S_1 \cap V'_i| \leq r_i$ and $|S_2 \cap V'_i| \leq r_i$ for all $i \in [m]$. Since $|S_1| = |S_2| = k$, it follows that $|S_1 \cap V'_i| = r_i$ and $|S_2 \cap V'_i| = r_i$ for all $i \in [m]$; hence, $S_1$ and $S_2$ are two disjoint independent sets of $G'$ covering all vertices but one from each $V'_i$ and for each $j \in \{1, 2\}$, we have $|S_j \cap V'_i| = \frac{1}{2} \cdot (|V'_i| - 1) \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$. Since $S_1$ and $S_2$ are also independent sets of the original cycle $G$, each
of them forms a valid solution to the FAIR-IS-CYCLE instance, completing the proof. □

3.2. Membership in PPA. We now show that the SCHRIJVER and FAIR-SPLIT-CYCLE problems lie in PPA by reductions to the search problem associated with the Octahedral Tucker lemma. The reductions follow the proofs of the corresponding mathematical statements by Meunier (2011) and by Alishahi & Meunier (2017), and we describe them here essentially for completeness.

We start with some notation, following (De Loera et al. 2019, Section 2). The partial order ≤ on the set \{+,-,0\} is defined by 0 ≤ + and by 0 ≤ −, where + and − are incomparable. The definition is extended to vectors, so that for two vectors \(x, y\) in \{+,-,0\}^n, we have \(x \preceq y\) if for all \(i \in [n]\) it holds that \(x_i \preceq y_i\) (equivalently, \(x_i = y_i\) whenever \(x_i \neq 0\)). The Octahedral Tucker lemma, given implicitly in (Matoušek 2004) and explicitly in (Ziegler 2002), says that for any function

\[
\lambda : \{+,-,0\}^n \setminus \{0\} \rightarrow \{±1, \ldots, ±(n−1)\}
\]

satisfying \(\lambda(−x) = −\lambda(x)\) for all \(x\), there exist vectors \(x, y\) such that \(x \preceq y\) and \(\lambda(x) = −\lambda(y)\). Note that this corresponds to the general Tucker’s lemma applied to (the boundary of) the barycentric subdivision of the \(n\)-cube whose vertex set can be identified with \{+,-,0\}^n (see Ziegler (2002)). The lemma guarantees the existence of a solution to every input of the following search problem, denoted OCTAHEDRAL-TUCKER.

Definition 3.3 (Octahedral Tucker Problem). In the problem OCTAHEDRAL-TUCKER, the input consists of a Boolean circuit that represents a function

\[
\lambda : \{+,-,0\}^n \setminus \{0\} \rightarrow \{±1, ±2, \ldots, ±(n−1)\}
\]

satisfying \(\lambda(−x) = −\lambda(x)\) for all \(x\). The goal is to find vectors \(x, y\) such that \(x \preceq y\) and \(\lambda(x) = −\lambda(y)\).

The OCTAHEDRAL-TUCKER problem is known to be PPA-complete (Deng et al. 2017), where its membership in PPA essentially follows already from (Papadimitriou 1994) (see also Deng et al. (2017, Appendix A) and Aisenberg et al. (2020, Section 3)).
**Proposition 3.4.** The Octahedral-Tucker problem lies in PPA.

For a vector $x \in \{+,-,0\}^n$, we let $x^+ = \{i \in [n] \mid x_i = +\}$ and $x^- = \{i \in [n] \mid x_i = -\}$. We further let $\text{alt}(x)$ denote the maximum length of an alternating subsequence of $x$, that is, the largest integer $\ell$ for which there exist indices $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, such that $x_{i_j} \in \{+,-\}$ for all $j \in [\ell]$ and $x_{i_j} \neq x_{i_{j+1}}$ for all $j \in [\ell - 1]$. We clearly have $\text{alt}(x) = \text{alt}(-x)$ for every $x \in \{+,-,0\}^n$.

For a given vector $x \in \{+,-,0\}^n$, we let $A(x)$ denote the vector in $\{+,-,0\}^n$ defined as follows. Let $I = \{i \in [n] \mid x_i = 0\}$. We first define $A(x)_i = 0$ for every $i \in I$. Next, consider the restriction $x_T$ of $x$ to the entries whose indices are in $\overline{I} = [n] \setminus I$, and notice that $x_T$ can be viewed as a sequence of maximal blocks of $+$’s and of $-$’s. The restriction $A(x)_T$ of $A(x)$ to the entries of $\overline{I}$ is defined as the vector obtained from $x_T$ by replacing all the symbols to zeros but the first symbol of each block. For example, for the vector $x = (+,+,0,+,+,−,0,−,+,0)$ we have $x_T = (+,+,+,−,−,−,+)\); hence, $A(x) = (+,0,0,0,−,0,0,0,+,0)$.

Observe that for every vector $x \in \{+,-,0\}^n$, the vector $A(x)$ satisfies $A(x) \preceq x$ as well as $\text{alt}(x) = \text{alt}(A(x)) = |A(x)^+| + |A(x)^-|$. Further, since $A(x)^+$ and $A(x)^-$ alternate, each of them includes no consecutive numbers. For an integer $r$ satisfying $r \leq \text{alt}(x)$, we let $A_r(x)$ denote the vector obtained from $A(x)$ by changing its last $\text{alt}(x) - r$ nonzero values to zeros. It clearly holds that $A_r(x) \preceq x$ and that $\text{alt}(A_r(x)) = |A_r(x)^+| + |A_r(x)^-| = r$. Observe that the quantity $\text{alt}(x)$ and the vectors $A_r(x)$ for $r \leq \text{alt}(x)$ can be computed in polynomial running time given $x$.

We reduce the Schrijver problem to Octahedral-Tucker, applying an argument of Meunier (2011).

**Theorem 3.5.** The Schrijver problem is polynomial-time reducible to the Octahedral-Tucker problem.

**Proof.** Consider an instance of the Schrijver problem, that...
is, a Boolean circuit that represents a coloring
\[ c : \binom{[n]}{k} \rightarrow [n - 2k + 1] \]
of the Schrijver graph \( S(n, k) \) using \( n - 2k + 1 \) colors. Based on this coloring, we construct an instance of the Octahedral-Tucker problem given by the function
\[ \lambda : \{+,-,0\}^n \setminus \{0\} \rightarrow \{\pm1, \pm2, \ldots, \pm(n-1)\} \]
defined as follows. For a given vector \( x \in \{+,-,0\}^n \setminus \{0\} \), we consider the following two cases:

1. \( \text{alt}(x) \leq 2k - 1 \).
   In this case, we define \( \lambda(x) = +\text{alt}(x) \) if the first nonzero value of \( x \) is +, and \( \lambda(x) = -\text{alt}(x) \) otherwise.

2. \( \text{alt}(x) \geq 2k \).
   In this case, let \( z = A_{2k}(x) \) and recall that
   \[ \text{alt}(z) = |z^+| + |z^-| = 2k. \]
   Observe that \( z^+ \) and \( z^- \) are two vertices of the Schrijver graph \( S(n, k) \). If \( c(z^+) < c(z^-) \) then we define \( \lambda(x) = +(c(z^+) + 2k - 1) \), and if \( c(z^-) < c(z^+) \) then we define \( \lambda(x) = -(c(z^-) + 2k - 1) \). Otherwise, we define \( \lambda(x) \) to be either \( +(n-1) \) or \( -(n-1) \), according to whether the first nonzero value of \( x \) is + or -, respectively.

Since the given coloring \( c \) uses the elements of \([n - 2k + 1]\) as colors, it follows that the function \( \lambda \) returns values from \( \{\pm1, \ldots, \pm(n-1)\} \). We further claim that \( \lambda(-x) = -\lambda(x) \) for all \( x \in \{+,-,0\}^n \setminus \{0\} \). Indeed, this follows from the definition of \( \lambda \) combined with the simple fact that for every \( x \) and \( r \leq \text{alt}(x) \), we have \( \text{alt}(x) = \text{alt}(-x) \) and \( A_r(x) = -A_r(-x) \). It is easy to verify that a Boolean circuit that computes the function \( \lambda \) can be constructed in polynomial running time.

We turn to prove the correctness of the reduction. Suppose we are given a solution to the constructed Octahedral-Tucker
instance, i.e., two vectors \( x, y \in \{+, -, 0\}^n \setminus \{0\} \) with \( x \preceq y \) and \( \lambda(x) = -\lambda(y) \). First observe that for a vector \( w \) with \( \text{alt}(w) \geq 2k \), such that the vector \( z = A_{2k}(w) \) satisfies \( c(z^+) = c(z^-) \), the sets \( z^+ \) and \( z^- \) are two adjacent vertices in the Schrijver graph \( S(n, k) \) and thus they form a monochromatic edge in this graph. Hence, if at least one of the vectors \( x \) and \( y \) satisfies these conditions, which correspond to the very last sub-case of Case 2 in the definition of \( \lambda \), then we are done. Otherwise, by the definition of \( \lambda \), either both \( \text{alt}(x) \) and \( \text{alt}(y) \) are at most \( 2k - 1 \) (Case 1) or they are both not (Case 2). However, it is easy to verify that if \( x \preceq y \) and \( \text{alt}(x) = \text{alt}(y) \), then the first nonzero values of \( x \) and \( y \) are equal; hence, both \( \text{alt}(x) \) and \( \text{alt}(y) \) must be at least \( 2k \) (Case 2). Assume without loss of generality that \( \lambda(x) \) is positive. By \( \lambda(x) = -\lambda(y) \) it follows that \( c(A_{2k}(x)^+) = c(A_{2k}(y)^-) \). Using again the fact that \( x \preceq y \), it follows that \( A_{2k}(x)^+ \) and \( A_{2k}(y)^- \) are adjacent vertices in the Schrijver graph \( S(n, k) \), providing the required monochromatic edge. □

We finally reduce the \textsc{Fair-Split-Cycle} problem (see Definition 1.4) to \textsc{Octahedral-Tucker}, applying an argument of Alishahi & Meunier (2017).

**Theorem 3.6.** The \textsc{Fair-Split-Cycle} problem is polynomial-time reducible to the \textsc{Octahedral-Tucker} problem.

**Proof.** Consider an instance of the \textsc{Fair-Split-Cycle} problem, that is, a cycle \( G \) on the vertex set \([n]\) and a partition \( V_1, \ldots, V_m \) of \([n]\) into \( m \) sets, such that \( n \) and \( m \) have the same parity. It can be assumed that \( n > m \). We construct an instance of the \textsc{Octahedral-Tucker} problem given by the function

\[
\lambda: \{+, -, 0\}^n \setminus \{0\} \to \{\pm 1, \pm 2, \ldots, \pm (n-1)\}
\]

defined as follows. For a given vector \( x \in \{+, -, 0\}^n \setminus \{0\} \), let \( J(x) \) denote the collection of indices \( i \in [m] \) satisfying

\[
|x^+ \cap V_i| = |x^- \cap V_i| = \frac{|V_i|}{2} \quad \text{or} \quad \max(|x^+ \cap V_i|, |x^- \cap V_i|) > \frac{|V_i|}{2},
\]

and consider the following two cases.
1. \( J(x) = \emptyset \).
   In this case, we define \( \lambda(x) = +\text{alt}(x) \) if the first nonzero value of \( x \) is +, and \( \lambda(x) = -\text{alt}(x) \) otherwise. Note that by \( J(x) = \emptyset \) it follows that \( |x^+ \cup x^-| \leq n - m \).

2. \( J(x) \neq \emptyset \).
   In this case, let \( i \) be the largest element of \( J(x) \). If \( |x^+ \cap V_i| = |x^- \cap V_i| = \lfloor \frac{|V_i|}{2} \rfloor \), then we define \( \lambda(x) = + (i + n - m - 1) \) in the case where the smallest element of \( (x^+ \cup x^-) \cap V_i \) is in \( x^+ \) and \( \lambda(x) = -(i + n - m - 1) \) otherwise. If, however, it holds that \( \max(|x^+ \cap V_i|, |x^- \cap V_i|) > \lfloor \frac{|V_i|}{2} \rfloor \) then we define \( \lambda(x) = +(i + n - m - 1) \) in the case where \( |x^+ \cap V_i| > \lfloor \frac{|V_i|}{2} \rfloor \) and \( \lambda(x) = -(i + n - m - 1) \) otherwise.

By combining the definition of the function \( \lambda \) with the fact that \( n > m \), it follows that \( \lambda \) returns values from \( \{\pm 1, \ldots, \pm (n - 1)\} \).

We further claim that \( \lambda(-x) = -\lambda(x) \) for all \( x \in \{+, -, 0\}^n \setminus \{0\} \). This indeed follows from the definition of \( \lambda \) combined with the fact that for every \( x \), we have \( \text{alt}(x) = \text{alt}(-x) \) and \( J(x) = J(-x) \). It is easy to verify that a Boolean circuit that computes the function \( \lambda \) can be constructed in polynomial running time.

We turn to prove the correctness of the reduction. Suppose we are given a solution to the constructed Octahedral-Tucker instance, i.e., two vectors \( x, y \in \{+, -, 0\}^n \setminus \{0\} \) with \( x \preceq y \) and \( \lambda(x) = -\lambda(y) \). By the definition of \( \lambda \), it is impossible that \( J(x) = J(y) = \emptyset \), because if \( x \preceq y \) and \( \text{alt}(x) = \text{alt}(y) \) then the first nonzero values of \( x \) and \( y \) are equal. It is also impossible that \( J(x) \) and \( J(y) \) are both nonempty, because \( |\lambda(x)| = |\lambda(y)| \) would imply that the largest element of \( J(x) \) is equal to that of \( J(y) \); hence, by \( x \preceq y \), \( \lambda(x) \) and \( \lambda(y) \) have the same sign. By \( x \preceq y \), we are left with the case where \( J(x) = \emptyset \) and \( J(y) \neq \emptyset \). It follows that for some \( i \in [m] \), we have

\[
\text{alt}(x) = |\lambda(x)| = |\lambda(y)| = i + n - m - 1 \geq n - m.
\]

Let \( S_1 = x^+ \) and \( S_2 = x^- \). By \( J(x) = \emptyset \), \( |S_1 \cap V_i| + |S_2 \cap V_i| \leq |V_i| - 1 \) for all \( i \in [m] \), and using \( \text{alt}(x) \geq n - m \) we get that \( |S_1 \cup S_2| = n - m \) and thus \( |S_1 \cap V_i| + |S_2 \cap V_i| = |V_i| - 1 \) for all \( i \in [m] \). This means that \( S_1 \) and \( S_2 \) cover all the vertices of \( G \) but one from each \( V_i \), so
by \( J(x) = \emptyset \), each of them includes at least \( \frac{1}{2} \cdot |V_i| - 1 \) elements of \( V_i \). Moreover, the sets \( S_1 \) and \( S_2 \) alternate, so since \( n - m \) is even, we get that they both form independent sets in the cycle \( G \). Hence, \( S_1 \) and \( S_2 \) form a valid solution to the given instance of FAIR-SPLIT-CYCLE, and this solution can be constructed in polynomial running time given \( x \) and \( y \). \( \square \)

3.3. Putting it all together. We finally show that the presented reductions complete the proofs of our results (see Figure 1.1). Indeed, the FAIR-IS-CYCLE problem is PPA-hard by Theorem 2.9 and is polynomial-time reducible to the SCHRIJVER problem by Theorem 3.2. By Theorem 3.5, the latter is efficiently reducible to the OCTAHEDRAL-TUCKER problem, which by Proposition 3.4 lies in PPA. It thus follows that the FAIR-IS-CYCLE and SCHRIJVER problems are PPA-complete, as required for Theorem 1.3 and Theorem 1.7. In addition, by Theorem 2.10, there exists a constant \( \varepsilon > 0 \) for which the \( \varepsilon \)-FAIR-SPLIT-CYCLE problem is PPA-hard. The \( \varepsilon \)-FAIR-SPLIT-CYCLE problem lies in PPA, even for \( \varepsilon = 0 \), as follows by combining Theorem 3.6 with Proposition 3.4. This confirms Theorem 1.5 and Theorem 1.6.

Acknowledgements

We are grateful to Aris Filos-Ratsikas and Alexander Golovnev for helpful discussions and to the anonymous referees for their useful suggestions and comments. The research is supported in part by the Israel Science Foundation (grant No. 1218/20).

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Manuscript received 9 February 2021

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