ON $C^2$ SOLUTION OF THE FREE-TRANSPORT EQUATION IN A DISK

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Abstract. The free transport operator of probability density function $f(t, x, v)$ is one of the most fundamental operator which is widely used in many areas of PDE theory including kinetic theory, in particular. When it comes to general boundary problems in kinetic theory, however, it is well-known that high order regularity is very hard to obtain in general. In this paper, we study the free transport equation in a disk with the specular reflection boundary condition. We obtain initial-boundary compatibility conditions for $C^{1}_{t,x,v}$ and $C^{2}_{t,x,v}$ regularity of the solution. We also provide regularity estimates.

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1. Introduction

The free transport equation (or free transport operator) is one of the most important ones in a wide area of mathematics. When we consider a probability density function $f : \mathbb{R}^{+} \times \Omega \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{+}$, the free transport equation is written by

$$\partial_{t}f + v \cdot \nabla_{x}f = 0.$$ 

Above equation is very simple and has explicit solution $f(t, x, v) = f_{0}(x - vt, v)$ when initial data $f_{0}$ is smooth and spatial domain is $\mathbb{R}^{d}$ or $\mathbb{T}^{d}$. However, if we consider general boundary problems, it becomes
very complicated. One of the most important and ideal boundary conditions in kinetic theory is the specular reflection boundary condition,

\[ f(t, x, v) = f(t, x, R_x v), \quad R_x = I - 2n(x) \otimes n(x), \quad x \in \partial \Omega, \]

where \( n(x) \) is outward unit normal vector on the boundary \( \partial \Omega \) when \( \partial \Omega \) is smooth. \[ \text{(1.1)} \] is motivated by billiard model and we usually analyze the problem through characteristics:

\[ X(s; t, x, v) := \text{position of a particle at time } s \text{ which was at phase space } (t, x, v), \]
\[ V(s; t, x, v) := \text{velocity of a particle at time } s \text{ which was at phase space } (t, x, v), \]

where \( X(s; t, x, v) \) and \( V(s; t, x, v) \) satisfy the following Hamiltonian structure,

\[ \frac{d}{ds} X(s; t, x, v) = V(s; t, x, v), \quad \frac{d}{ds} V(s; t, x, v) = 0, \]

under billiard-like reflection condition on the boundary. Explicit formulation of \( (X(s; t, x, v), V(s; t, x, v)) \) will be given right after Definition \[ \text{1.1} \]. Since \( X(t; t, x, v) = x, \quad V(t; t, x, v) = v \) by definition, we can easily guess the following solution,

\[ f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)), \]

which is same as \( f_0(x - vt, v) \) when \( \Omega = \mathbb{R}^d \). However, unlike to whole space case, regularity of the solution \( f(t, x, v) \) depends on the regularity of trajectory \[ \text{(1.2)} \]. More precisely, then \( X(0; t, x, v) \in \partial \Omega \), differentiability of \[ \text{(1.2)} \] break down in general. This means that for any time \( t > 0 \), there exist corresponding \( (x_*, v_*) \in \Omega \times \mathbb{R}^d \) such that \( f(t, \cdot, \cdot) \) is not differentiable at the point. Or equivalently, for any \( (x, v) \in \Omega \times \mathbb{R}^d \), there exists some corresponding time \( t \) such that \( f(\cdot, x, v) \) is not differentiable at that time.

Now let us consider general kinetic model which has hyperbolic structure, such as hard sphere or general cut-off Boltzmann equations. (Of course, there are lots of other kinetic literature which consider various boundary condition problems.) Although the Boltzmann equation (or other general kinetic equations) is much more complicated than the free transport equation, the recent development of the Boltzmann (or kinetic) boundary problems shows the regularity issue of the problems very well.

In \( \mathbb{T}^3 \) or \( \mathbb{R}^3 \), many results have been known using high order regularity function spaces. We refer to some classical work such as \[ \text{3, 9, 10, 11} \]. (We note that the apriori assumption of \[ \text{3} \] also covers some boundary condition problems, including specular reflection \[ \text{(1.1)} \].) More recently, in the case of non-cutoff Boltzmann equation (which has regularizing effect), it is known that the solution is \( C^\infty \) by \[ \text{16} \].

However, when it comes to general boundary condition problems, a way of getting sufficient high order regularity estimate is not known and low regularity approach has been widely used. By defining mild solution, low regularity \( L^\infty \) solutions have been studied after \[ \text{12} \]. There are lots of references \[ \text{2, 4, 5, 6, 8, 18, 19, 22, 23} \], etc., which deal with low regularity solution of the kinetic equation whose collision operator have regularizing effect such as non-cutoff Boltzmann or Landau equation. In fact, however, there are only a few results known about regularity of the Boltzmann equation with boundary conditions. We refer to \[ \text{14, 15, 17, 20} \].

As briefly explained above, regularity issue of boundary condition problems is very fundamental problem. In fact, even without complicated collision type operators, regularity of the free transport equation with boundary conditions has not been studied thoroughly to the best of author’s knowledge.

1.1. Statements of main theorems. In this paper, we study classical \( C^2_{t,x,v} \) regularity of the free transport equation in a 2D disk,

\[ \partial_t f + v \cdot \nabla_x f = 0, \quad x \in \Omega := \{ x \in \mathbb{R}^2 : |x| < 1 \}, \]

with the specular reflection boundary condition \[ \text{(1.1)} \]. Note that \( n(x) = x \in \partial \Omega \), since we consider unit disk \( \partial \Omega = \{ x \in \mathbb{R}^2 : |x| = 1 \} \). Our aim is to find initial-boundary compatibility conditions of initial data \( f_0 \) for \( C^1 \) and \( C^2 \) regularity of the solution \( f(t, x, v) \). We expect the solution will be mild solution \( f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)) \) surely (See Definition \[ \text{1.1} \] for \( X \) and \( V \).) By performing derivative of \( f \) in terms of \( t, x, v \) directly (up to second order), we will find some conditions of \( f_0 \) which contains first and second derivative in both \( x \) and \( v \). (See \[ \text{18, 19, 10, 11} \].)
In general, for smooth bounded domain $\Omega$, we define
$$\Omega = \{ x \in \mathbb{R}^2 : \xi(x) < 0 \}, \quad \partial \Omega = \{ x \in \mathbb{R}^2 : \xi(x) = 0 \}.$$ 
In the case of unit disk, we may choose
$$\xi(x) = \frac{1}{2} |x|^2 - \frac{1}{2},$$
and hence
$$\nabla \xi(x) = x, \quad \nabla^2 \xi(x) = I.$$ 
Now let us define some notation to precisely describe characteristics $X(s; t, x, v)$ and $V(s; t, x, v)$.

**Definition 1.1.** Considering (1.3), we define basic notations

- $t_b(x, v) := \sup \{ s \geq 0 : x - sv \in \Omega \}$,
- $x_b(x, v) := x - t_b(x, v)v = X(t - t_b(t, x, v); t, x, v)$ Ist bouncing point backward in time,
- $v_b(x, v) := v = \lim_{s \to t_b(t, x, v)} V(t - s; t, x, v)$,
- $t^k(t, x, v) := t^{k-1} - t_b(x^{k-1}, v^{k-1}),$ $k$-th bouncing time backward in time, $t^1(t, x, v) := t - t_b(x, v)$,
- $x^k(x, v) := x^{k-1} - t_b(x^{k-1}, v^{k-1})v^{k-1} = X(t^k; t^{k-1}, x^{k-1}, v^{k-1})$
  $= k$-th bouncing point backward in time, $x_b := x^1$,
- $v^k = R_v v^{k-1} = R_v \lim_{s \to t_b} V(s; t^{k-1}, x^{k-1}, v^{k-1}),$

where $R_v$ is defined in (1.3). We define the specular characteristics as

$$X(s; t, x, v) = \sum_k 1_{s \in (t_{k-1}, t_k)} X(s; t^k, x^k, v^k),$$
$$V(s; t, x, v) = \sum_k 1_{s \in (t_{k-1}, t_k)} V(s; t^k, x^k, v^k).$$

We also use $\gamma_+$ and $\gamma_0$ notation to denote

- $\gamma_+ := \{ (x, v) \in \partial \Omega \times \mathbb{R}^2 : v \cdot n(x) > 0 \}$,
- $\gamma_0 := \{ (x, v) \in \partial \Omega \times \mathbb{R}^2 : v \cdot n(x) = 0 \}$,
- $\gamma_- := \{ (x, v) \in \partial \Omega \times \mathbb{R}^2 : v \cdot n(x) < 0 \}$.

Note that unit disk $\Omega$ is uniformly convex and its linear trajectory (1.5) is well-defined if $x \in \Omega$ (see velocity lemma [12] for example). However, we want to investigate regularity up to boundary $\overline{\Omega}$, so we carefully exclude $\gamma_0$ from $\overline{\Omega} \times \mathbb{R}^2$ since we do not define characteristics starting from (backward in time) $\gamma_0$. Hence, using (1.5) and (1.6), it is natural to write (1.4) as the following mild formulation,

$$f(t, x, v) = f_0(X(0; t, x, v), V(0; t, x, v)), \quad (x, v) \in \mathcal{I} := \{ \overline{\Omega} \times \mathbb{R}^2 \} \setminus \gamma_0.$$

Meanwhile, to study regularity of (1.6), the following quantity is very important,

$$A_{v,y} := \left[ (v \cdot n(y))I + (n(y) \otimes v) \right] \left( I - \frac{v \otimes n(y)}{v \cdot n(y)} \right), \quad (y, v) \in \{ \partial \Omega \times \mathbb{R}^2 \} \setminus \gamma_0.$$ 

Notice that $A_{v,y}$ is a matrix-valued function $A_{v,y} : \{ \mathbb{R}^d \times \partial \Omega \} \setminus \gamma_0 \to \mathbb{R}^d \times \mathbb{R}^d$. ({$d = 2$ in this paper in particular}) In fact, $A_{v,y}$ can be written as

$$A_{v,y} = \nabla_x \left( (v \cdot n(y)n(y)) \right) = (v \cdot n(y)) \nabla_x n(y) + (n(y) \otimes v) \nabla_x n(y),$$

which is identical to (1.7) by (2.2).

Throughout this paper, we denote the $v$-derivative of the $i$-th column of the matrix $A_{v,y}$ by $\nabla_v A^i_{v,y}$, where $A^i$ be the $i$-th column of a matrix $A$ for $1 \leq i \leq d$. For fixed $i$, it means that

$$\nabla_v A^i_{v,x} = (\nabla_v A^i_{v,y})_{y=x},$$

It is important to note that we carefully distinguish between $\nabla_v A^i_{v,x}$ and $\nabla_v A^i_{v,x}(x,v)$. 


Theorem 1.2 \((C^1 \text{ regularity})\). Let \(f_0 \in C^{1}_{x,v}(\overline{\Omega} \times \mathbb{R}^2)\) which satisfies (1.1). If initial data \(f_0\) satisfies
\[
\nabla_x f_0(x,v) + \nabla_v f_0(x,v) \left( \frac{(Q v) \otimes (Q v)}{v \cdot n} \right) R_x = \nabla_x f_0(x,R_xv) + \nabla_v f_0(x,R_xv) \left( \frac{(QR_xv) \otimes (QR_xv)}{R_xv \cdot n} \right), \quad (x,v) \in \gamma_-,
\]
in addition, then \(f(t,x,v)\) defined in (1.6) is a unique \(C^1_{t,x,v}(\mathbb{R}^+ \times I)\) solution of (1.4). We also note that if (1.8) holds, then it also holds for \((x,v) \in \gamma_+\). Here, \(Q\) is counterclockwise rotation by \(\frac{\pi}{2}\) in \(\mathbb{R}^2\).

Theorem 1.3 \((C^2 \text{ regularity})\). Let \(f_0 \in C^{2}_{x,v}(\overline{\Omega} \times \mathbb{R}^2)\) which satisfies (1.1), (1.8), and
\[
\nabla_x f_0(x,R_xv) \parallel (R_xv)^T, \quad \nabla_v f_0(x,R_xv) \parallel (R_xv)^T, \quad \forall (x,v) \in \gamma_-.
\]
(In this case, (1.8) becomes \(\nabla_x f_0(x,v) = \nabla_x f_0(x,R_xv)\). \(A^T\) means transpose of \(A\).) Now, for all \((x,v) \in \gamma_-\), if \(f_0\) satisfies
\[
\begin{align*}
R_x \left[ \nabla_x f_0(x,v) + \nabla_v f_0(x,v) \left( \frac{(Q v) \otimes (Q v)}{v \cdot n} \right) \right] R_x &= \nabla_x f_0(x,R_xv) + \nabla_v f_0(x,R_xv) \left( \frac{(QR_xv) \otimes (QR_xv)}{R_xv \cdot n} \right) + \left( R_x \left[ \nabla_v f_0(x,R_xv) J_1 \right] \nabla_v f_0(x,R_xv) J_2 \right) R_x, \\
R_x \left[ \nabla_x f_0(x,v) + \nabla_v f_0(x,v) \left( \frac{(Q v) \otimes (Q v)}{v \cdot n} + \frac{(Q v) \otimes (Q v)}{v \cdot n} \nabla_x f_0(x,v) \right) \right] R_x &= \nabla_x f_0(x,R_xv) + \nabla_v f_0(x,R_xv) \left( \frac{(QR_xv) \otimes (QR_xv) \otimes (QR_xv)}{R_xv \cdot n} \right) \nabla_x f_0(x,v) + \left( R_x \left[ \nabla_v f_0(x,R_xv) J_1 \right] \nabla_v f_0(x,R_xv) K_1 \right) R_x
\end{align*}
\]
where \(x = (x_1,x_2), v = (v_1,v_2),\) and
\[
J_1 := \frac{1}{v \cdot x} \left[ \begin{array}{c} -4v_2x_1x_2 \\ -2v_2(x_2^2 - x_1^2) \end{array} \right], \quad J_2 := \frac{1}{v \cdot x} \left[ \begin{array}{c} -2v_2(x_2^2 - x_1^2) \\ -4v_2x_1x_2 \end{array} \right],
\]
\[
K_1 := \frac{(v \cdot x)^3}{4v_2^2x_2^3 + 2v_1v_2^2(3x_1x_2^2 - x_1^3) + 2v_2^3(3x_1x_2^2 + x_1^3)} \left[ \begin{array}{c} -4v_1^3x_2^2 + 2v_1x_2^3(3x_1x_2^2 + x_1^3) \\ -2v_1^3x_2^2 - 2v_1x_2^3(3x_1x_2^2 - x_1^3) \end{array} \right],
\]
\[
K_2 := \frac{(v \cdot x)^3}{-4v_1^3x_2^2 - 2v_1x_2^3(3x_1x_2^2 + x_1^3) - 2v_1^3x_2^2 - 2v_1x_2^3(3x_1x_2^2 - x_1^3)} \left[ \begin{array}{c} -4v_1^3x_2^2 + 2v_1x_2^3(3x_1x_2^2 + x_1^3) \\ -2v_1^3x_2^2 - 2v_1x_2^3(3x_1x_2^2 - x_1^3) \end{array} \right],
\]
in addition, then \(f(t,x,v)\) defined in (1.6) is a unique \(C^2_{t,x,v}(\mathbb{R}^+ \times I)\) solution of (1.4).

Remark 1.4. Assume \(f_0\) satisfies (1.1). If \(f_0 \in C^{1}_{x,v}(\overline{\Omega} \times \mathbb{R}^2)\), then
\[
\nabla_x f_0(x,v) = \nabla_x f_0(x,R_xv)R_x, \quad \forall x \in \partial \Omega, \quad \forall v \in \mathbb{R}^2,
\]
also hold. Similarity, if \(f_0 \in C^{2}_{x,v}(\overline{\Omega} \times \mathbb{R}^2)\), then
\[
\nabla_{xv} f_0(x,v) = R_x \nabla_{xv} f_0(x,R_xv)R_x, \quad \forall x \in \partial \Omega, \quad \forall v \in \mathbb{R}^2,
\]
also holds as well as (1.12).

Remark 1.5. By symmetry, Theorem 1.2 and 1.3 also hold for three dimensional sphere if the rotation operator \(Q\) is properly redefined in the plane spanned by \(\{x,v\}\) for \(x \in \mathbb{R},\ v || v \neq 0\).
Theorem 1.6 (Regularity estimates). The $C^1(\mathbb{R}_+ \times I)$ and $C^2(\mathbb{R}_+ \times I)$ solutions of Theorem 1.2 and 1.3 enjoy the following regularity estimates:

$$
\|f\|_{C^1_t, x,v} \lesssim \|f_0\|_{C^1} \frac{|v|}{|v \cdot n(x_b)|^2} (v)^2 (1 + |v| t),
$$

(1.14)

\[\|f\|_{C^2_t, x,v} \lesssim \|f_0\|_{C^2} \frac{|v|^2}{|v \cdot n(x_b)|^4} (v)^2 (1 + |v| t)^2,\]

(1.15)

where $x_b = x_b(x,v)$ and $(v) := 1 + |v|$.

1.2. Brief sketch of proofs and some important remarks. In this paper, our aim is to analyze regularity of mild form (1.6) where characteristic $(X(0; t, x, v), V(0; t, x, v))$ is well-defined (by excluding $\gamma_0$). If backward in time position $X(0; t, x, v) \notin \partial \Omega$, characteristic is also smooth function and we expect that the regularity of (1.6) will be same as initial data $f_0$ by chain rule. When $X(0; t, x, v) \in \partial \Omega$, however, derivative via chain rule does not work anymore because of discontinuous behavior of velocity $V(0; t, x, v)$. Depending on perturbed directions, we obtain different directional derivatives. In fact, we can split directions into two pieces: one gives bouncing and the other does not. See (3.6) and (3.7) for different compatibility condition (1.8). Of course, (1.12) also holds, but (1.12) is gained by taking $v$-derivative of (1.1) directly. We note that both $C^1_x$ and $C^1_v$ conditions yield identical initial compatibility condition (1.8), and the condition for $C^1_t$ is a just a necessary condition for (1.8).

The analysis becomes much more complicated when we study $C^2$ conditions. Nearly all of our analysis consist of precise equalities, instead of estimates. This makes our business much harder. First, let us consider four cases: $\nabla_{xx}, \nabla_{uv}, \nabla_{ux}, \nabla_{vu}$. These yield very complicated initial-boundary compatibility conditions and in particular they contain derivatives of each column of reflection operator $R$ or $\nabla_{x,v}(n(x) \otimes n(x)v)$. It is nearly impossible to give proper geometric interpretation for each term. See (1.2) and (1.3) for example.

Nevertheless, it is quite interesting that the four conditions from $\nabla_{xx}, \nabla_{uv}, \nabla_{ux}, \nabla_{vu}$ can be rearranged with respect to the order of time $t$. By matching all directional derivatives, we obtain (1.21)–(1.24) which contain both second order terms and first order terms. However, the conditions from $\nabla_{xx}, \nabla_{uv}, \nabla_{ux}, \nabla_{vu}$ must satisfy transpose compatibility condition

$$
\nabla^T_{v} = \nabla_{vx} \quad \text{and} \quad \nabla^T_{xx} = \nabla_{xx},
$$

(1.16)

since we hope the solution to be $C^2$. However, it is extremely hard to find any good geometric meaning or properties of some terms like

$$
\nabla_{x}(R^1_{x,x}(x,v)), \quad \nabla_{x}(A^1_{x,x}(x,v)), \quad \text{for} \quad i = 1, 2,
$$

(1.17)

in (1.21)–(1.24). If they do not have any special structures, the way to get compatibility (1.16) is to impose $\nabla_{x,v}f_0(x, Rtv) = 0$ for all $(x,v) \in \gamma_-$. Then $C^1$ compatibility condition (1.8) becomes just trivial. Fortunately, however, the matrices of (1.17) have a rank 1 structure. **More surprisingly**, all the null spaces are spanned by velocity $v$! That is, from Lemma 1.1 and Lemma 1.2

$$
\nabla_{x}(R^1_{x,x}(x,v))v = 0, \quad \nabla_{x}(R^2_{x,x}(x,v))v = 0, \quad \nabla_{x}(-2A^1_{v,x}(x,v))v = 0, \quad \nabla_{x}(-2A^2_{v,x}(x,v))v = 0.
$$

From these interesting results, we can derive necessary conditions (1.9) for transpose compatibility (1.16). By imposing (1.9), we derive $C^2$ conditions as in Theorem 1.3 while keeping $C^1$ condition (1.8) nontrivial. We note that all the conditions that include $\partial_t$ are repetitions of (1.21)–(1.24).

In the last section, we study $C^1$ and $C^2$ regularity estimates of the solution (1.6). Essentially the regularity estimates of the solution come from the regularity estimates of characteristic $(X(0), V(0))$. For $C^1$ of $(X(0), V(0))$, we obtain Lemma 5.1. Note that we can find some cancellation that gives no singular bound for $\nabla_v X(0)$ which was found in [15] for general 3D convex domains. Growth in time need not to be exponential, but it is just linear in time $t$. The second derivative of characteristic is much more complicated and nearly impossible to try to find any cancellation, because of too many terms and combinations that appear. Instead, by studying the most singular terms only, we obtain rough bounds in Lemma 5.8.
2. Preliminaries

Now, let us recall standard matrix notations which will be used in this paper.

**Definition 2.1.** When we perform matrix multiplications throughout this paper, we basically treat a n-dimensional vector \( v \) as a column vector

\[
v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.
\]

For about gradient of a smooth scalar function \( a(x) \), however, we treat n-dimensional vector \( \nabla a \) is a row vector,

\[
\nabla a(x) := (\partial_x a, \partial_{x_2} a, \cdots, \partial_{x_n} a).
\]

For a smooth vector function \( v : \mathbb{R}^n \to \mathbb{R}^m \) with \( v(x) = \left( \begin{array}{c} v_1(x) \\ \vdots \\ v_m(x) \end{array} \right) \), we define \( \nabla_x v(x) \) as \( m \times n \) matrix,

\[
\nabla_x v := \begin{pmatrix}
\partial_1 v_1 & \cdots & \partial_n v_1 \\
\partial_1 v_2 & \cdots & \partial_n v_2 \\
\vdots & \ddots & \vdots \\
\partial_1 v_m & \cdots & \partial_n v_m
\end{pmatrix}_{m \times n} = \begin{pmatrix}
\nabla_x v_1 \\
\vdots \\
\nabla_x v_m
\end{pmatrix}_{m \times n}
\]

We use \( \otimes \) to denote tensor product

\[
a \otimes b := \begin{pmatrix}
a_1 \\
\vdots \\
a_m
\end{pmatrix} \begin{pmatrix}
b_1 & \cdots & b_n
\end{pmatrix}.
\]

**Lemma 2.2.** (1) (Product rule) For scalar function \( a(x) \) and vector function \( v(x) \),

\[
\nabla (a(x)v(x)) = a(x)\nabla v(x) + v \otimes \nabla a(x).
\]

(2) (Chain rule) For vector functions \( v(x) \) and \( w(x) \),

\[
\nabla (v(w(x))) = \nabla v(w(x)) \nabla w(x).
\]

(3) (Product rule) For vector functions,

\[
\nabla (v(x) \cdot w(x)) = v(x) \nabla w(x) + w(x) \nabla v(x).
\]

(4) For matrix \( d \times d \) matrix \( A(x) \) and \( d \times 1 \) vector \( v(x) \),

\[
\nabla_x (A(x)v(x)) = A(x) \nabla v(x) + \begin{pmatrix}
v(x) & \nabla A^1(x) \\
\vdots \\
v(x) & \nabla A^d(x)
\end{pmatrix} = A(x) \nabla v(x) + \sum_{k=1}^{d} \partial_k A(x) E_k,
\]

where \( A^i(x) \) is \( i \)-th row of \( A(x) \) and \( E_k \) is \( d \times d \) matrix whose \( k \)-th column is \( v \) and others are zero. (Here \( \partial_k A(x) \) means elementwise \( x_k \)-derivative of \( A(x) \).) Moreover, if \( A = A(\theta(x)) \) for some smooth \( \theta : \Omega \to \mathbb{R} \),

\[
\nabla_x (A(\theta)v(x)) = A(\theta) \nabla v(x) + \partial_{\theta} A(\theta)v \otimes \nabla_x \theta.
\]
Proof. Only Lemma 2.2 needs some explanation. When \( A = A(\theta(x)) \),

\[
\nabla_x (A(\theta)v(x)) = A(\theta)\nabla v(x) + \sum_{k=1}^{d} \partial_k A(\theta)E_k = A(\theta)\nabla v(x) + \sum_{k=1}^{d} \partial_k A(\theta)\partial_k \theta(x)E_k
\]

\[
= A(\theta)\nabla v(x) + \partial_{\theta} A(\theta) \left( \partial_1 \theta(x)v \cdots \partial_d \theta(x)v \right)
\]

\[
= A(\theta)\nabla v(x) + \partial_{\theta} A(\theta)v \otimes \nabla_x \theta(x).
\]

\[\square\]

Lemma 2.3. We have the following computation where \( x_b = x_b(x,v) \) and \( t_b = t_b(x,v) \).

\[
\nabla_x t_b = \frac{n(x_b)}{v \cdot n(x_b)},
\]

\[
\nabla_v t_b = -t_b \nabla_x t_b = -t_b \frac{n(x_b)}{v \cdot n(x_b)},
\]

\[
\nabla_x x_b = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)},
\]

\[
\nabla_v x_b = -t_b \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right).
\]

Proof. Remind the definition of \( x_b \) and \( t_b \)

\[
x_b = x - t_b v, \quad t_b = \sup \{ s \mid x - sv \in \Omega \}.
\]

Since \( \xi(x) = 0 \) for \( x \in \partial \Omega \), we have \( \xi(x_b) = \xi(x - t_b v) = 0 \). Taking \( x \)-derivative \( \nabla_x \), we get

\[
= (\nabla \xi)(x_b) - [(\nabla \xi)(x_b) \cdot v] \nabla_x t_b = 0,
\]

where the first equality comes from product rule in Lemma 2.2. Thus, we can derive

\[
\nabla_x t_b = \frac{(\nabla \xi)(x_b)}{[(\nabla \xi)(x_b) \cdot v]} = \frac{n(x_b)}{v \cdot n(x_b)}.
\]

Similarly, taking \( v \)-derivative \( \nabla_v \) and product rule in Lemma 2.2 yields

\[
\nabla_v (\xi(x_b)) = (\nabla \xi)(x_b) (-t_b I - v \otimes \nabla_v t_b) = 0,
\]

which implies \( \nabla_v t_b = -t_b \frac{n(x_b)}{v \cdot n(x_b)} \). It follows from the calculation of \( \nabla_x t_b \) and \( \nabla_v t_b \) above that

\[
\nabla_x x_b = \nabla_x (x - t_b v) = I - v \otimes \nabla_x t_b = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)}
\]

\[
\nabla_v x_b = \nabla_v (x - t_b v) = -t_b I - v \otimes \nabla_v t_b = -t_b \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right).
\]

\[\square\]

Lemma 2.4. For \( n(x_b(x,v)) \), we have the following derivative rules,

\[
\nabla_x [n(x_b)] = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)}, \quad \nabla_v [n(x_b)] = -t_b \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right),
\]

where \( x_b = x_b(x,v) \).

Proof. For \( \nabla_x n(x_b) \), we apply the chain rule in Lemma 2.2 to \( (\nabla \xi)(x_b) \) and \( \frac{1}{|\nabla \xi(x_b)|} \) respectively. Because \( \nabla \xi(x) \neq 0 \) at the boundary \( x \in \partial \Omega \) in a circle, it is possible to apply the chain rule to \( \frac{1}{|\nabla \xi(x_b)|} \). Taking \( x \)-derivative \( \nabla_x \), one obtains

\[
\nabla_x [(\nabla \xi)(x_b)] = (\nabla^2 \xi)(x_b) \nabla_x x_b,
\]

\[
\nabla_x \left[ \frac{1}{|\nabla \xi(x_b)|} \right] = -\frac{(\nabla \xi)(x_b)(\nabla^2 \xi)(x_b) \nabla_x x_b}{|\nabla \xi(x_b)|^3}.
\]
Hence,

\[ \nabla_x [n(x_b)] = \nabla_x \left[ \frac{(\nabla \xi)(x_b)}{|(\nabla \xi)(x_b)|} \right] = \frac{1}{|(\nabla \xi)(x_b)|} \nabla_x[(\nabla \xi)(x_b)] + (\nabla \xi)(x_b) \otimes \nabla_x \left[ \frac{1}{|(\nabla \xi)(x_b)|} \right] 
\]

\[ = \frac{1}{|(\nabla \xi)(x_b)|}(\nabla^2 \xi)(x_b) \nabla_x x_b - \nabla \xi(x_b) \otimes \frac{\nabla^2 \xi(x_b)(\nabla \xi(x_b)) \nabla_x x_b}{|(\nabla \xi(x_b))|^3} \]

\[ = \frac{1}{|(\nabla \xi(x_b))|^3} \left( I - n(x_b) \otimes n(x_b) \right) (\nabla^2 \xi)(x_b) \nabla_x x_b. \]

Since \( |\nabla \xi(x_b)| = 1 \) and \( \nabla^2 \xi = I_2 \), we deduce

\[ \nabla_x [n(x_b)] = \left( I - n(x_b) \otimes n(x_b) \right) \left( I - v \otimes n(x_b) \right) \]

\[ = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} - n(x_b) \otimes n(x_b) + n(x_b) \otimes n(x_b) \]

\[ = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)}. \]

The case for \( \nabla_v [n(x_b)] \) is nearly same with extra term \( -t_b \), which comes from Lemma 2.3.

\[ \square \]

3. Initial-boundary compatibility condition for \( C^1_{t,x,v} \)

**Lemma 3.1.** We have the following identities, for \( (x, v) \in \{ \partial \Omega \times \mathbb{R}^d \} \setminus \gamma_0 \),

\[ R_x A_{v,x} = \frac{1}{v \cdot n(x)} Q(v \otimes v)Q^T = \frac{1}{v \cdot n(x)} (Qv) \otimes (Qv), \]

\[ A_{v,x} R_x = \frac{1}{v \cdot n(x)} R_x Q(v \otimes v)Q^T R_x = -\frac{1}{R_x v \cdot n(x)} (QR_x v) \otimes (QR_x v), \]

\[ A_{v,x}^2 = \frac{1}{(v \cdot n(x))^2} (QR_x v) \otimes (QR_x v) (Qv) \otimes (Qv), \]

\[ A_{v,x} v = 0, \]

where \( Q := Q_\frac{\tau}{2} \) is counterclockwise rotation by angle \( \frac{\tau}{2} \).

**Proof.** We compute

\[ R_x A_{v,x} R_x := \left[ (v \cdot n(x))I - (n(x) \otimes v) \right] \left[ I + \frac{v \otimes n(x)}{v \cdot n(x)} \right] \]

\[ = (Qv \otimes Qn(x)) \left[ I + \frac{v \otimes n(x)}{v \cdot n(x)} \right]. \]

Now let us define \( \tau(x) = Q \cdot \frac{\tau}{2} n(x) \) as tangential vector at \( x \in \partial \Omega \). (\( n \) as y-axis and \( \tau \) as x-axis) Then,

\[ R_x A_{v,x} R_x := Qv \otimes \left( -\tau - \frac{v \cdot \tau}{v \cdot n(x)} n(x) \right) \]

\[ = -\frac{1}{v \cdot n(x)} Qv \otimes \left( (v \cdot n(x))\tau + (v \cdot \tau)n(x) \right) \]

\[ = -\frac{1}{v \cdot n(x)} Qv \otimes (R_x Q^T v) \]

\[ = \frac{1}{v \cdot n(x)} Qv \otimes (R_x Qv) \]

\[ = \frac{1}{v \cdot n(x)} Q(v \otimes v)Q^T R_x, \]

and we get (3.1) using \( R_x Q = -R_x Q^T \), because

\[ Q^T R_x Q = I - 2Q^T (n(x) \otimes n(x))Q = I - 2\tau \otimes \tau = -R_x. \]
(3.2) is simply obtained by (3.1). By definition of \( A_{v,x} \) in (1.7), one obtains that
\[
A_{v,x} v = \left[ ((v \cdot n(x)) I + (n(x) \otimes v)) \left( I - \frac{v \otimes n(x)}{v \cdot n(x)} \right) \right] v = ((v \cdot n(x)) I + (n(x) \otimes v)) (v - v) = 0.
\]

Now, throughout this section, we study \( C^1_{v} \) of \( f(t, x, v) \) of (1.6) when
\[
0 = t^1(t, x, v) \text{ or equivalently } t = t_b(x, v).
\]

3.1. \( C^1_v \) condition of \( f \). Since we assume (3.4), \( X(0; t, x, v) = x^1(t, v) = x_b(x, v) \in \partial \Omega \). To derive compatibility condition for \( C^1_v \) of \( f(t, x, v) \), we consider \( \epsilon \)-perturbation and use the following notation for perturbed trajectory:
\[
X^\epsilon(0) := X(0; t, x, v + \epsilon \hat{r}), \quad V^\epsilon(0) := V(0; t, x, v + \epsilon \hat{r}),
\]
where \( \hat{r} \in \mathbb{R}^2 \) is a unit-vector. As \( \epsilon \to 0 \), we simply get
\[
\lim_{\epsilon \to 0} X(0; t, x, v + \epsilon \hat{r}) = x^1(t, v) = x_b(x, v),
\]
from continuity of \( X(0; t, x, v) \) in \( v \). However, \( V(0; t, x, v) \) is not continuous in \( v \) because of (1.1). Explicitly, from Lemma 2.3
\[
\frac{\partial}{\partial \epsilon} t_b(x + \epsilon \hat{r}, v)|_{\epsilon = 0} = \nabla_x t_b(x, v) \cdot \hat{r} = \frac{\hat{r} \cdot n(x_b(x, v))}{v \cdot n(x_b(x, v))}, \quad \text{where } v \cdot n(x_b(x, v)) < 0.
\]
So we define, for fixed \( (x, v), v \neq 0 \),
\[
R_{vel,1} := \{ \hat{r} \in \mathbb{S}^2 : \hat{r} \cdot n(x_b(x, v)) < 0 \}, \\
R_{vel,2} := \{ \hat{r} \in \mathbb{S}^2 : \hat{r} \cdot n(x_b(x, v)) \geq 0 \}.
\]
Then from (3.3), \( \nabla_x t_b(x, v) \cdot \hat{r} > 0 \) when \( \hat{r} \in R_{vel,1} \) and \( \nabla_x t_b(x, v) \cdot \hat{r} \leq 0 \) when \( \hat{r} \in R_{vel,2} \). Therefore, for two unit vectors \( \hat{r}_1 \in R_{vel,1} \) and \( \hat{r}_2 \in R_{vel,2} \), by continuity argument,
\[
\lim_{\epsilon \to 0} V(0; t, x, v + \epsilon \hat{r}_1) = v, \quad \lim_{\epsilon \to 0} V(0; t, x, v + \epsilon \hat{r}_2) = v^1 = R_x v.
\]

We consider directional derivatives with respect to \( \hat{r}_1 \) and \( \hat{r}_2 \). If \( f \) belongs to the \( C^1_v \) class, \( \nabla_v f(t, x, v) \) exists and directional derivatives of \( f \) with respect to \( \hat{r}_1, \hat{r}_2 \) will be \( \nabla_v f(t, x, v)\hat{r}_1, \nabla_v f(t, x, v)\hat{r}_2 \). Using (1.1), we have \( f_0(x^1, v) = f_0(x^1, v^1) \) and hence
\[
\nabla_v f(t, x, v)\hat{r}_1 = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f(t, x, v + \epsilon \hat{r}_1) - f(t, x, v)) \right)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f_0(X(0; t, x, v + \epsilon \hat{r}_1), V(0; t, x, v + \epsilon \hat{r}_1)) - f_0(X(0; t, x, v), V(0; t, x, v))) \right)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f_0(X^\epsilon(0), V^\epsilon(0)) - f_0(X^\epsilon(0), v) + f_0(X^\epsilon(0), v) - f_0(X(0), v)) \right)
\]
\[
= \nabla_x f_0(X(0), v) \cdot \lim_{s \to 0^+} \nabla_x X(s)\hat{r}_1 + \nabla_v f_0(X(0), v) \lim_{s \to 0^+} \nabla_v V(s)\hat{r}_1,
\]
\[
\nabla_v f(t, x, v)\hat{r}_2 = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f(t, x, v + \epsilon \hat{r}_2) - f(t, x, v)) \right)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f_0(X(0; t, x, v + \epsilon \hat{r}_2), V(0; t, x, v + \epsilon \hat{r}_2)) - f_0(X(0; t, x, v), V(0; t, x, v))) \right)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( (f_0(X^\epsilon(0), V^\epsilon(0)) - f_0(X^\epsilon(0), v^1) + f_0(X^\epsilon(0), v^1) - f_0(X(0), v^1)) \right)
\]
\[
= \nabla_x f_0(X(0), v^1) \cdot \lim_{s \to 0^-} \nabla_x X(s)\hat{r}_2 + \nabla_v f_0(X(0), v^1) \lim_{s \to 0^-} \nabla_v V(s)\hat{r}_2,
\]
which implies
\[
\nabla_v f(t, x, v) = \nabla_x f_0(X(0), v) \lim_{s \to 0^+} \nabla_x X(s) + \nabla_v f_0(X(0), v) \lim_{s \to 0^+} \nabla_v V(s),
\]
\[
\nabla_v f(t, x, v) = \nabla_x f_0(X(0), v^1) \lim_{s \to 0^-} \nabla_x X(s) + \nabla_v f_0(X(0), v^1) \lim_{s \to 0^-} \nabla_v V(s).
\]
Since
\[
\lim_{s \to 0^+} \nabla_v X(s) = \lim_{s \to 0^+} \nabla_v (x - v(t - s)) = -t I_{2 \times 2}, \quad \lim_{s \to 0^+} \nabla_v V(s) = \lim_{s \to 0^+} \nabla_v v = I_{2 \times 2},
\]
\[
\nabla_v f(t, x, v) = \nabla_x f_0(X(0), v) \cdot \lim_{s \to 0^+} \nabla_x X(s) + \nabla_v f_0(X(0), v) \cdot \lim_{s \to 0^+} \nabla_v V(s),
\]
\[
\nabla_v f(t, x, v) = \nabla_x f_0(X(0), v^1) \cdot \lim_{s \to 0^-} \nabla_v X(s) + \nabla_v f_0(X(0), v^1) \cdot \lim_{s \to 0^-} \nabla_v V(s).
\]
\[ \nabla_v f(t, x, v) \text{ of } (3.8) \text{ becomes} \]
\[ \nabla_v f(t, x, v) = -t \nabla_x f_0(X(0), v) + \nabla_v f_0(X(0), v). \]  
(3.11)

For (3.9), using the product rule in Lemma 2.2 and (2.23) in Lemma 2.3, we have
\[ \lim_{s \to 0^-} \nabla_v X(s) = \lim_{s \to 0^-} \nabla_v (x^1 - (t^1 + s)v^1) = \lim_{s \to 0^-} \nabla_v x^1 + v^1 \otimes \nabla_v t_b \]
\[ = -t \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - t \frac{v^1 \otimes n(x^1)}{v \cdot n(x^1)} \]
\[ = -t \left( I - \frac{1}{v \cdot n(x^1)} (2(v \cdot n(x^1))n(x^1)) \otimes n(x^1) \right) = -t R_{x^1}, \]
\[ \lim_{s \to 0^-} \nabla_v V(s) = \lim_{s \to 0^-} \nabla_v (R_{x^1}v) \]
\[ = \lim_{s \to 0^-} \left( I - 2(v \cdot n(x^1))\nabla_v n(x^1) - 2n(x^1) \otimes n(x^1) - 2(n(x^1) \otimes v)\nabla_v n(x^1) \right) \]
\[ = R_{x^1} + 2t(2n(x^1)) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) + 2t(n(x^1) \otimes v) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) \]
\[ = R_{x^1} + 2t A_{v,x^1}, \]
where \( A_{v,x^1} \) is defined in (1.7). Hence, using (3.12), \( \nabla_v f(t, x, v) \) in (3.9) becomes
\[ \nabla_v f(t, x, v) = -t \nabla_x f_0(X(0), R_{x^1}v) R_{x^1} \]
\[ + \nabla_v f_0(X(0), R_{x^1}v) R_{x^1} + t \nabla_v f_0(X(0), R_{x^1}v) \left[ 2 \left( (v \cdot n(x^1))I + (n(x^1) \otimes v) \right) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) \right] \]
\[ = -t \nabla_x f_0(x^1, R_{x^1}v) R_{x^1} + \nabla_v f_0(x^1, R_{x^1}v)(R_{x^1} + 2t A_{v,x^1}), \]
(3.13)

where we used \( v \cdot n(x^1) = -v^1 \cdot n(x^1) \). Meanwhile, taking \( \nabla_v \) to specular reflection (1.1) directly, we get
\[ \nabla_v f_0(x, v) = \nabla_v f_0(x, R_x v) R_x, \quad \forall x \in \partial \Omega. \]  
(3.14)

Comparing (3.11), (3.13), and (3.14), we deduce
\[ \nabla_x f_0(x^1, v) = \nabla_x f_0(x^1, R_{x^1}v) R_{x^1} - 2 \nabla_v f_0(x^1, R_{x^1}v) A_{v,x^1}, \quad (x^1, v) \in \gamma_- \]  
(3.15)

3.2. \( C^1_\gamma \) condition of \( f \). Remind we assume (3.4). Similar to previous subsection, we define \( x \)-perturbed trajectory,
\[ X^\epsilon(0) := X(0; t, x + \epsilon \hat{r}, v), \quad V^\epsilon(0) := V(0; t, x + \epsilon \hat{r}, v), \]  
(3.16)
where \( \hat{r} \in \mathbb{R}^2 \) is a unit vector. As \( \epsilon \to 0 \), we simply get
\[ \lim_{\epsilon \to 0} X(0; t, x + \epsilon \hat{r}, v) = x^1(x, v). \]

Similar to previous subsection, using Lemma 2.3
\[ \frac{\partial}{\partial \epsilon} t_b(x, v + \epsilon \hat{r})|_{\epsilon = 0} = \nabla_v t_b(x, v) \cdot \hat{r} = -t_b \frac{\hat{r} \cdot n(x_b(x, v))}{v \cdot n(x_b(x, v))}, \quad \text{where} \quad v \cdot n(x_b(x, v)) < 0. \]  
(3.17)

So we define, for fixed \((x, v), v \neq 0\),
\[ R_{sp,1} := \{ \hat{r} \in S^2 : \hat{r} \cdot n(x_b(x, v)) > 0 \}, \]
\[ R_{sp,2} := \{ \hat{r} \in S^2 : \hat{r} \cdot n(x_b(x, v)) \leq 0 \}. \]  
(3.18)

Then from (3.17), \( \nabla_v t_b(x, v) \cdot \hat{r} > 0 \) when \( \hat{r} \in R_{sp,1} \) and \( \nabla_v t_b(x, v) \cdot \hat{r} \leq 0 \) when \( \hat{r} \in R_{sp,2} \). Therefore, for two unit vectors \( \hat{r}_1 \in R_{sp,1} \) and \( \hat{r}_2 \in R_{sp,2} \), by continuity argument,
\[ \lim_{\epsilon \to 0} V(0; t, x, v + \epsilon \hat{r}_1) = v, \quad \lim_{\epsilon \to 0} V(0; t, x, v + \epsilon \hat{r}_2) = v = R_{x^1} v. \]
Using similar arguments in previous subsection, we obtain
\[
\nabla_x f(t, x, v) \hat{r}_1 = \nabla_x f_0(X(0), v) \lim_{s \to 0^+} \nabla_x X(s) \hat{r}_1 + \nabla_v f_0(X(0), v) \lim_{s \to 0^+} \nabla_v V(s) \hat{r}_1,
\]
\[
\nabla_x f(t, x, v) \hat{r}_2 = \nabla_x f_0(X(0), Rv) \lim_{s \to 0^+} \nabla_x X(s) \hat{r}_2 + \nabla_v f_0(X(0), Rv) \lim_{s \to 0^+} \nabla_v V(s) \hat{r}_2.
\]
(3.19)

Since
\[
\lim_{s \to 0^+} \nabla_x X(s) = I_{2 \times 2}, \quad \lim_{s \to 0^+} \nabla_v V(s) = 0_{2 \times 2},
\]
(3.21)
\[
\nabla_x f(t, x, v) \quad \text{of (3.19) becomes}
\]
\[
\nabla_x f(t, x, v) = \nabla_x f_0(X(0), v).
\]
(3.22)

For \(\nabla_x f(t, x, v)\) of (3.20), we apply the product rule in Lemma 2.2 and 2.3 in Lemma 2.4 to get
\[
\lim_{s \to 0^-} \nabla_x X(s) = \lim_{s \to 0^-} \nabla_x(x^1 - (t^1 + s)v^1) = \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) + \frac{v^1 \otimes n(x^1)}{v \cdot n(x^1)} = R_{x^1},
\]
\[
\lim_{s \to 0^-} \nabla_v V(s) = \lim_{s \to 0^-} \nabla_v(R_{x^1} v)
\]
\[
= \lim_{s \to 0^-} \left( -2(v \cdot n(x^1)) \nabla_x n(x^1) - 2(n(x^1) \otimes v) \nabla_v n(x^1) \right)
\]
\[
= -2(v \cdot n(x^1)) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - 2(n(x^1) \otimes v) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right)
\]
\[
= -2A_{x^1, v^1}.
\]
(3.23)

Hence, using (3.23), \(\nabla_x f(t, x, v)\) in (3.20) becomes
\[
\nabla_x f(t, x, v) = \nabla_x f_0(X(0), R_{x^1} v)R_{x^1} - 2\nabla_v f_0(X(0), R_{x^1} v)A_{x^1, v^1}.
\]
(3.24)

Combining (3.22) and (3.24),
\[
\nabla_x f_0(x^1, v) = \nabla_x f_0(x^1, R_{x^1} v)R_{x^1} - 2\nabla_v f_0(x^1, R_{x^1} v)A_{x^1, v^1}, \quad (x^1, v) \in \gamma_-,
\]
(3.25)

which is identical to (3.15).

3.3. \(C^1_t\) condition of \(f\). To check the \(C^1_t\) condition, we define
\[
X^+(0) := X(0; t + \epsilon, x, v), \quad V^+(0) := V(0; t + \epsilon, x, v).
\]
(3.26)

More specifically,
\[
X^+(0) = x^1 - (t^1 + \epsilon)R_{x^1} v, \quad V^+(0) = R_{x^1} v, \quad \epsilon > 0,
\]
and
\[
X^+(0) = x - (t + \epsilon)v, \quad V^+(0) = v, \quad \epsilon < 0.
\]

Thus, the case \((\epsilon > 0)\) describes the situation after bounce (backward in time) and the case \((\epsilon < 0)\) describes the situation just before bounce (backward in time). Then, for \(\epsilon > 0\),
\[
f_t(t, x, v) = \lim_{\epsilon \to 0^+} \frac{f(t + \epsilon, x, v) - f(t, x, v)}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0^+} \frac{f_0(X^+(0), V^+(0)) - f_0(X(0), V(0))}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0^+} \frac{f_0(X^+(0), R_{x^1} v) - f_0(X(0), R_{x^1} v)}{\epsilon}
\]
\[
= \nabla_x f_0(x^1, R_{x^1} v) \lim_{\epsilon \to 0^+} \frac{X^+(0) - X(0)}{\epsilon}
\]
\[
= -\nabla_x f_0(x^1, R_{x^1} v)R_{x^1} v.
\]
We only consider the situation just before collision and then
\[
\begin{align*}
    f_1(t, x, v) &= \lim_{\epsilon \to 0^-} \frac{f(t + \epsilon, x, v) - f(t, x, v)}{\epsilon} \\
    &= \lim_{\epsilon \to 0^-} \frac{f_0(X^s(0), v) - f_0(X(0), v)}{\epsilon} \\
    &= \nabla_x f_0(x^1, v) \lim_{\epsilon \to 0^-} \frac{X^s(0) - X(0)}{\epsilon} \\
    &= -\nabla_x f_0(x^1, v)v.
\end{align*}
\]

Thus, we derive a $C^1_\tau$ condition
\[
\nabla_x f_0(x^1, v)v = \nabla_x f_0(x^1, R_{x^1}v)R_{x^1}v, \quad (x^1, v) \in \gamma_-.
\] (3.27)

Actually, (3.27) is just particular case of (3.15), because of (3.3).

3.4. Proof of Theorem 1.2

Proof of Theorem 1.2. If $0 \neq t^k$ for any $k \in \mathbb{N}$, then $X(0; t, x, v)$ and $V(0; t, x, v)$ are both smooth function of $(t, x, v)$. By chain rule and $f_0 \in C^1_{t,x,v}$, $f(t, x, v)$ of (1.6) is also $C^1_{t,x,v}$.

Now let us assume $0 = t^k(t, x, v)$ for some $k \in \mathbb{N}$. From discontinuous property of $V(0; t, x, v)$, we consider the following two cases:
\[
\begin{align*}
    \lim_{s \to 0^+} \nabla_v V(s)(\text{or } \nabla_v X(s)) &= \lim_{s \to 0^+} \frac{\partial V(s)(\text{or } \partial X(s))}{\partial(t, k^{-1}, x^{k-1}, v^{k-1})} \frac{\partial(t, k^{-1}, x^{k-1}, v^{k-1})}{\partial v} \\
    \lim_{s \to 0^-} \nabla_v V(s)(\text{or } \nabla_v X(s)) &= \lim_{s \to 0^-} \frac{\partial V(s)(\text{or } \partial X(s))}{\partial(t, k^{-1}, x^{k-1}, v^{k-1})} \frac{\partial(t, k^{-1}, x^{k-1}, v^{k-1})}{\partial v}.
\end{align*}
\]

Since the factor $\frac{\partial(t, k^{-1}, x^{k-1}, v^{k-1})}{\partial v}$ is common factor which is smooth, it suffices to compare above two underbraced terms only. It means that no generality is lost by setting $k = 1$.

Initial-boundary compatibility conditions for $C^1_{t,x,v}$ were obtained in (3.15), (3.25), and (3.27). Since compatibility conditions (3.25) and (3.27) are covered by (3.15), $f(t, x, v) \in C^1_{t,x,v}$ once (3.15) holds. To change (3.15) into more symmetric presentation (1.8), we apply (3.14) and multiply invertible matrix $R_{x^1}$ both sides from the right to obtain
\[
(\nabla_x f_0(x^1, v) + \nabla_v f_0(x^1, v)R_{x^1}A_{v,x^1})R_{x^1} = \nabla_x f_0(x^1, R_{x^1}v) - \nabla_v f_0(x^1, R_{x^1}v)A_{v,x^1}R_{x^1}.
\]

This yields
\[
\left[\nabla_x f_0(x, v) + \nabla_v f_0(x, v)\left(\frac{Qv}{v \cdot n(x)} \otimes \frac{Qv}{v \cdot n(x)}\right)\right]R_{x^1} = \nabla_x f_0(x, R_{x^1}v) + \nabla_v f_0(x, R_{x^1}v)\frac{QR_{x^1}v \otimes QR_{x^1}v}{R_{x^1}v \cdot n(x)}R_{x^1},
\]
by (3.14).

Now we claim that compatibility condition (3.15) also holds for $(x^1, v) \in \gamma_+$. By multiplying $R_{x^1}$ both sides and using $R_{x^1}^2 = I$, $R_{x^1}n(x^1) = -n(x^1)$, and (3.14), we obtain
\[
\nabla_x f_0(x^1, R_{x^1}v) = \nabla_x f_0(x^1, v)R_{x^1} + 2\nabla_v f_0(x^1, v)R_{x^1} \left[\left((v \cdot n(x^1))I + (n(x^1) \otimes v)\right)\left(I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)}\right)\right]R_{x^1}.
\] (3.28)
Since $R_{x^1} = R^T_{x^1}$ (transpose), the underbraced term is written as

\[
R_{x^1} A_{x^1, x^1} R_{x^1} \\
= R_{x^1} \left[ ((v \cdot n(x^1))I + (n(x^1) \otimes v)) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) \right] R_{x^1} \\
= - (R_{x^1} v \cdot n(x^1)) - R_{x^1} v \otimes R_{x^1} n(x^1) + R_{x^1} n(x^1) \otimes R_{x^1} v - \frac{R_{x^1} n(x^1) \otimes R_{x^1} n(x^1)}{v \cdot n(x^1)} |R_{x^1} v|^2 \\
= - \left[ ((R_{x^1} v \cdot n(x^1))I + (n(x^1) \otimes R_{x^1} v)) \left( I - \frac{R_{x^1} v \otimes n(x^1)}{R_{x^1} v \cdot n(x^1)} \right) \right] \\
= - A_{R_{x^1} v, x^1},
\]

and hence (3.28) is identical to (3.15) when $(x^1, v) \in \gamma_+$. $$\square$$

4. Initial-boundary compatibility condition for $C^2_{t,x,v}$

As mentioned in the beginning of the previous section, we treat the problem (1.4) as 2D problem in a unit disk $\{x \in \mathbb{R}^2 : |x| < 1\}$. And, throughout this section, we use the following notation to interchange column and row for notational convenience,

\[
\begin{pmatrix} a \\ b \end{pmatrix} \overset{c \leftrightarrow r}{\equiv} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix} \overset{r \leftrightarrow c}{\equiv} \begin{pmatrix} a \\ b \end{pmatrix}.
\]

Similar to previous section, we assume (3.4), i.e., $0 = t^1(t,x,v)$. We also assume $f_0$ satisfies specular reflection (1.4) and $C^1_{t,x,v}$ compatibility condition (3.15) (or (1.8)) in this section.

4.1. Condition for $\nabla_{xx}$ Similar to previous section, we split perturbed direction into (3.18). We also note that $\nabla_v f(t,x,v)$ can be written as (3.11) or (3.13), which are identical by assuming (3.15). First, using (3.11), $\hat{r}_1$ of (3.18), and notation (5.10)

\[
\nabla_{xx} f(t,x,v) \hat{r}_1 \overset{c \leftrightarrow r}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_v f(t,x+\epsilon \hat{r}_1,v) - \nabla_v f(t,x,v) \right) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_v \left[ f_0(X(0);t,x+\epsilon \hat{r}_1,v),V(0;0,t,x+\epsilon \hat{r}_1,v) \right] - \left( -t \nabla_x f_0(X(0),0) + \nabla_v f_0(X(0),0) \right) \right) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X(0),0) \nabla_v X(0) + \nabla_v f_0(X(0),0) \nabla_v V(0) \right. \\
- \left. \left( -t \nabla_x f_0(X(0),0) + \nabla_v f_0(X(0),0) \right) \right\} \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ -t \nabla_x f_0(X(0),0) - \nabla_v f_0(X(0),0) \right\} + \left[ \nabla_v f_0(X(0),0) - \nabla_v f_0(X(0),0) \right] \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ -t \nabla_x f_0(X(0),0) - \nabla_v f_0(X(0),0) \right\} \left( t \nabla_{xx} f_0(X(0),0) \right) \\
\overset{c \leftrightarrow r}{=} \nabla_{xx} f_0(X(0),0) \lim_{s \to 0^+} \nabla_x X(s)(-t) \hat{r}_1 + \nabla_v f_0(X(0),0) \lim_{s \to 0^+} \nabla_x X(s) \hat{r}_1 \\\n= \left( \nabla_{xx} f_0(x^1,v)(-t) + \nabla_{xx} f_0(x^1,v) \right) \hat{r}_1,
\]
where we have used (3.10), (3.21), \( \nabla_v X^\varepsilon(0) = -tI_2 \), and \( \nabla_v V^\varepsilon(0) = I_2 \). Similarly, using (3.18) and \( \hat{r}_2 \) of (3.18),

\[
\nabla_{xv} f(t, x, v) \overset{\text{res}}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\nabla_{xv} f(t, x + \varepsilon \hat{r}_2, v) - \nabla_{xv} f(t, x, v))
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v X^\varepsilon(0) + \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v V^\varepsilon(0) \right. \\
- \left. (-t \nabla_{xv} f_0(X(0), R_x; v) R_x + \nabla_{xv} f_0(X(0), R_x; v)(R_x + 2tA_{xv; 1})) \right\}
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v X^\varepsilon(0) + t \nabla_{xv} f_0(X(0), R_x; v) R_x \right\}
\]

\[
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v V^\varepsilon(0) - \nabla_{xv} f_0(X(0), R_x; v)(R_x + 2tA_{xv; 1}) \right\}
\]

\[
:= I_{xv; 1} + I_{xv; 2}.
\]

Using (3.12) and (3.23),

\[
I_{xv; 1} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v X^\varepsilon(0) - \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_s \nabla_v X(s) \\
+ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \frac{\partial}{\partial s} \nabla_v X(s) \right\} \left. \right|_{s=0} - \nabla_v X(s) = -tR_x,
\]

\[
= \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v X^\varepsilon(0) - \nabla_v X(0) \right) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) - \nabla_{xv} f_0(X(0), R_x; v) \right)(-tR_x)
\]

\[
\overset{\text{res}}{=} \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v X^\varepsilon(0) - \nabla_v X(0) \right) \right]^T \\
+ (-tR_x) \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{s \to 0} \nabla_v X(s) + \nabla_{xv} f_0(x^1, R_x; v) \lim_{s \to 0} \nabla_v V(s) \right] \hat{r}_2
\]

\[
= \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v X(0; t, x + \varepsilon \hat{r}_2, v) - \nabla_v X(s) \right) \right]^T \\
:= (s)_{xv; 1} \hat{r}_2
\]

and

\[
I_{xv; 2} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_v V^\varepsilon(0) - \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) \nabla_0 \nabla_v V(s) \\
+ \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) (R_x + 2tA_{xv; 1}) \frac{\partial}{\partial s} \nabla_v V(s) \right\}
\]

\[
= \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v V^\varepsilon(0) - \nabla_v V(0) \right) \\
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{xv} f_0(X^\varepsilon(0), V^\varepsilon(0)) - \nabla_{xv} f_0(x^1, R_x; v) \right)(R_x + 2tA_{xv; 1})
\]

\[
\overset{\text{res}}{=} \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v V^\varepsilon(0) - \nabla_v V(0) \right) \right]^T \\
+ (R_x + 2tA_{xv; 1}) \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{s \to 0} \nabla_v X(s) + \nabla_{xv} f_0(x^1, R_x; v) \lim_{s \to 0} \nabla_v V(s) \right] \hat{r}_2
\]

\[
= \left[ \nabla_{xv} f_0(x^1, R_x; v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_v V(0; t, x + \varepsilon \hat{r}_2, v) - \nabla_v V(s) \right) \right]^T \\
:= (s)_{xv; 2} \hat{r}_2
\]

+ (R_x + 2tA_{xv; 1}) \left[ \nabla_{xv} f_0(x^1, R_x; v) \nabla_v f_0(x^1, R_x; v)(-2A_{xv; 1}) \right] \hat{r}_2.
Now we compute two underbraced \((*)_{xv,1}\) and \((*)_{xv,2}\)
\[
(*)_{xv,1}\hat{r}_2 = \left[ \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon} \left( \nabla_v X^0(0) - \nabla_{s_o-0} \nabla_v X(s) \right) \right) \right]^T
\]
\[
= \left[ \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \nabla_x (\partial_{v_1} V(s)) \hat{r}_2, \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \nabla_x (\partial_{v_2} V(s)) \hat{r}_2 \right]^T
\]
\[
(*)_{xv,2}\hat{r}_2 = \left[ \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \nabla_x (\partial_{v_1} V(s)) \hat{r}_2, \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \nabla_x (\partial_{v_2} V(s)) \hat{r}_2 \right]^T
\]
\[
= \left[ \nabla_x f_0(x^1, R_x, v) \nabla_x (-tR_{x}^1(x, v)) \right] \hat{r}_2,
\]
where \(A^i\) means \(i\)th column of matrix \(A\). Similarly,
\[
(*)_{xv,2}\hat{r}_2 = \left[ \nabla_x f_0(x^1, R_x, v) \lim_{\epsilon \to 0} \nabla_x (\partial_{v_1} V(s)) \right] \hat{r}_2
\]
\[
= \left[ \nabla_x f_0(x^1, R_x, v) \nabla_x (-tR_{x}^1(x, v)) \right] \hat{r}_2.
\]

Therefore,
\[
\nabla_v f(t, x, v) = (*)_{xv,1}\hat{r}_1 + (*)_{xv,2}\hat{r}_2
\]
\[
+ (-tR_{x}) \left[ \nabla_{xx} f_0(x^1, R_x, v)R_{x1} + \nabla_{xx} f_0(x^1, R_x, v)(-2A_{xv,1}) \right]
\]
\[
+ (R_{x1} + 2tA_{x,x1}^T) \left[ \nabla_{xx} f_0(x^1, R_x, v)R_{x1} + \nabla_{vv} f_0(x^1, R_x, v)(-2A_{xv,1}) \right].
\]

From (4.1) and (4.2), we get the following compatibility condition
\[
(-t)\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)
\]
\[
= (*)_{xv,1}\hat{r}_1 + (*)_{xv,2}\hat{r}_2
\]
\[
+ (-tR_{x}) \left[ \nabla_{xx} f_0(x^1, R_x, v)R_{x1} + \nabla_{xx} f_0(x^1, R_x, v)(-2A_{xv,1}) \right]
\]
\[
+ (R_{x1} + 2tA_{x,x1}^T) \left[ \nabla_{xx} f_0(x^1, R_x, v)R_{x1} + \nabla_{vv} f_0(x^1, R_x, v)(-2A_{xv,1}) \right].
\]

4.2. Condition for \(\nabla_{vv}\). We split perturbed direction into (3.7). \(\nabla_v f(t, x, v)\) can be written as (3.11) or (3.13). Using (3.11), \(\hat{r}_1\) of (3.7), and notation (3.5),
\[
\nabla_{vv} f(t, x, v)\hat{r}_1
\]
\[
\overset{c^e r}{=} \lim_{\epsilon \to 0} \left\{ -t \left[ \nabla_x f_0(X^0(0), V^0(0)) - \nabla_x f_0(X(0), v) \right] + \left[ \nabla_v f_0(X^0(0), V^0(0)) - \nabla_v f_0(X(0), v) \right] \right\}
\]
\[
= \lim_{\epsilon \to 0} \left\{ -t \left[ \nabla_x f_0(X^0(0), v + \epsilon \hat{r}_1) - \nabla_x f_0(X(0), v) \right] + \left[ \nabla_v f_0(X^0(0), v + \epsilon \hat{r}_1) - \nabla_v f_0(X(0), v) \right] \right\}
\]
\[
= -t \nabla_{xx} f_0(X(0), v) \lim_{s \to 0+} \nabla_v X(s) - t \nabla_{xx} f_0(X(0), v) \lim_{s \to 0+} \nabla_v V(s)
\]
\[
+ \nabla_{xx} f_0(X(0), v) \lim_{s \to 0+} \nabla_v X(s) + \nabla_{vv} f_0(X(0), v) \lim_{s \to 0+} \nabla_v V(s) \hat{r}_1
\]
\[
= \left[ (-t) \nabla_{xx} f_0(x^1, v)(-t) + (-t) \nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)(-t) + \nabla_{vv} f_0(x^1, v)(-t) \right] \hat{r}_1,
\]
where we have used (3.10) and note that we have \(\nabla_{vv} X^0(0) = -I_2\) and \(\nabla_{vv} V^0(0) = I_2\) for \(v + \epsilon \hat{r}\) case also. Similarly, using (3.13), \(\hat{r}_2\) of (3.7), and notation (3.5),
\[
\nabla_{vv} f(t, x, v)\hat{r}_2
\]
\[
\overset{c^e r}{=} \lim_{\epsilon \to 0} \left\{ \nabla_x f_0(X^0(0), V^0(0)) \nabla_v X^0(0) + t \nabla_x f_0(X(0), R_x v) R_{x1} \right\}
\]
\[
+ \lim_{\epsilon \to 0} \left\{ \nabla_v f_0(X^0(0), V^0(0)) \nabla_v V^0(0) - \nabla_v f_0(X(0), R_x v) (R_{x1} + 2tA_{xv,1}) \right\}
\]
\[
:= I_{vv,1} + I_{vv,2},
\]
and each $I_{vv,1,vv,2}$ are estimated by

$$I_{vv,1} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X^\epsilon(0), V^\epsilon(0)) \nabla_v X^\epsilon(0) - \nabla_x f_0(X^\epsilon(0), V^\epsilon(0)) \lim_{s \to 0^-} \nabla_v X(s) \right. + \left. \nabla_x f_0(X^\epsilon(0), V^\epsilon(0)) \lim_{s \to 0^-} \nabla_v X(s) + t \nabla_x f_0(X(0), R_{x_1}v) R_{x_1} \right\}, \quad \lim_{s \to 0^-} \nabla_v X(s) = -t R_{x_1},$$

$$= \left[ \nabla_x f_0(x^1, R_{x_1}v) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_v X^\epsilon(0) - \lim_{s \to 0^-} \nabla_v X(s) \right) \right]^T + (-t R_{x_1}) \left[ \nabla_x f_0(x^1, R_{x_1}v) (-t R_{x_1}) + \nabla_v f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right] \hat{t}_2,$$

$$I_{vv,2} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_v f_0(X^\epsilon(0), V^\epsilon(0)) \nabla_v V^\epsilon(0) - \nabla_v f_0(X^\epsilon(0), V^\epsilon(0)) \lim_{s \to 0^-} \nabla_v V(s) \right. + \left. \nabla_v f_0(X^\epsilon(0), V^\epsilon(0)) (R_{x_1} + 2 t A_{v,x_1}) - \nabla_v f_0(X(0), R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right\}$$

$$= \left[ \nabla_v f_0(x^1, R_{x_1}v) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_v V^\epsilon(0) - \lim_{s \to 0^-} \nabla_v V(s) \right) \right]^T + (R_{x_1} + 2 t A_{v,x_1}^T) \left[ \nabla_x f_0(x^1, R_{x_1}v) (-t R_{x_1}) + \nabla_v f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right] \hat{t}_2.$$

Similar to (142) and (143), using Lemma 2.3,

$$(*)_{vv,1} \hat{t}_2 = \left[ \nabla_x f_0(x^1, R_{x_1}v) \nabla_v (-t R_{x_1}^2(v,x,v)) \right] \hat{t}_2 = t^2 \left[ \nabla_x f_0(x^1, R_{x_1}v) \nabla_x (R_{x_1}^2(v,x,v)) \right] \hat{t}_2, \quad (4.7)$$

$$(*)_{vv,2} \hat{t}_2 = \left[ \nabla_v f_0(x^1, R_{x_1}v) \nabla_v (R_{R_{x_1}}^2(v,x,v) + 2 t A_{v,x_1}^v(v,x,v)) \right] \hat{t}_2 = -t \left[ \nabla_v f_0(x^1, R_{x_1}v) \nabla_x (R_{x_1}^2(v,x,v)) \nabla_v (A_{v,x_1}^v(v,x,v)) \right] \hat{t}_2 + 2 t^2 \left[ \nabla_v f_0(x^1, R_{x_1}v) \nabla_v (A_{v,x_1}^v(v,x,v)) \right] \hat{t}_2. \quad (4.8)$$

Hence, we get

$$\nabla_{vv} f(t, x, v)$$

$$= \left( + (*)_{vv,1} \right) + \left( + (*)_{vv,2} \right) + (-t R_{x_1}) \left[ \nabla_{xx} f_0(x^1, R_{x_1}v) (-t R_{x_1}) + \nabla_{vx} f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right] + (R_{x_1} + 2 t A_{v,x_1}^T) \left[ \nabla_{xx} f_0(x^1, R_{x_1}v) (-t R_{x_1}) + \nabla_{vx} f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right]. \quad (4.9)$$

Then from (4.6) and (4.9) we get the following compatibility condition.

$$( -t ) \nabla_{xx} f_0(x^1, v) (-t) + (-t) \nabla_{vx} f_0(x^1, v) + \nabla_{vv} f_0(x^1, v) (-t) + \nabla_{vv} f_0(x^1, v)$$

$$= \left( + (*)_{vv,1} \right) + \left( + (*)_{vv,2} \right) + (-t R_{x_1}) \left[ \nabla_{xx} f_0(x^1, R_{x_1}v) (-t R_{x_1}) + (-t R_{x_1}) \nabla_{vx} f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right) + (R_{x_1} + 2 t A_{v,x_1}^T) \left[ \nabla_{xx} f_0(x^1, R_{x_1}v) (-t R_{x_1}) + (R_{x_1} + 2 t A_{v,x_1}^T) \nabla_{vx} f_0(x^1, R_{x_1}v) (R_{x_1} + 2 t A_{v,x_1}) \right]. \quad (4.10)$$
4.3. **Condition for** $\nabla_{xx}$. We split perturbed direction into \((3.18)\). $\nabla_x f(t,x,v)$ can be written as \((3.22)\) or \((3.24)\), which are identical by \((3.15)\). Using \((3.22)\), $\hat{r}_1$ of \((3.18)\), and notation \((3.16)\),

\[
\nabla_{xx} f(t,x,v) \hat{r}_1 \overset{c.c.}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f(t,x+\epsilon \hat{r}_1,v) - \nabla_x f(t,x,v) \right)
\]

\[
\begin{align*}
\nabla_{xx} f(t,x,v) \hat{r}_1 &\overset{c.c.}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f(t,x+\epsilon \hat{r}_1,v) - \nabla_x f(t,x,v) \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X(0,t), v; t, x + \epsilon \hat{r}_1, v) - \nabla_x f_0(X(0), v) \right\} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X(0), V(0)) \nabla_x X(0) + \nabla_v f_0(X(0), V(0)) \nabla_v V(0) \right\}
\end{align*}
\]

\[
\hat{r}_1 = \nabla_{xx} f_0(x^1, v) \lim_{s \to 0^+} \nabla_x X(s) \hat{r}_1 = \nabla_{xx} f_0(x^1, v) \hat{r}_1
\]

where we have used \((3.10)\), \((3.21)\), $\nabla_x X(0) = I_2$, and $\nabla_x V(0) = 0$. Similarly, using \((3.24)\), $\hat{r}_2$ of \((3.18)\), and notation \((3.16)\),

\[
\nabla_{xx} f(t,x,v) \hat{r}_2 \overset{c.c.}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f(t,x+\epsilon \hat{r}_2,v) - \nabla_x f(t,x,v) \right)
\]

\[
\begin{align*}
\nabla_{xx} f(t,x,v) \hat{r}_2 &\overset{c.c.}{=} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f(t,x+\epsilon \hat{r}_2,v) - \nabla_x f(t,x,v) \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X(0), V(0)) \nabla_x X(0) + \nabla_v f_0(X(0), V(0)) \nabla_v V(0) \right\}
\end{align*}
\]

\[
\hat{r}_2 = \nabla_{xx} f_0(x^1, v) \lim_{s \to 0^+} \nabla_x X(s) \hat{r}_2
\]

where

\[
\begin{align*}
I_{xx,1} &:= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X(0), V(0)) \nabla_x X(0) - \nabla_x f_0(X(0), V(0)) \lim_{s \to 0^+} \nabla_x X(s) \\
&+ \left( \nabla_x f_0(X(0), V(0)) \nabla_x X(0) - \nabla_x f_0(X(0), V(0)) \nabla_x X(0) \right) \right\} \\
&\overset{r.e.}{=} \left[ \nabla_x f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \left\{ \nabla_x X(0; t, x + \epsilon \hat{r}_2, v) - \lim_{s \to 0^+} \nabla_x X(s) \right\} \right]^{T} \\
&+ R_{x_1} \left[ \nabla_{xx} f_0(x^1, R_{x_1} v) \nabla_x X(0) + \nabla_{xx} f_0(x^1, R_{x_1} v) (-2A_{x_1, x}) \right] \hat{r}_2,
\end{align*}
\]

\[
\begin{align*}
I_{xx,2} &:= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_v f_0(X(0), V(0)) \nabla_x V(0) - \nabla_v f_0(X(0), V(0)) \lim_{s \to 0^+} \nabla_x V(s) \\
&- 2 \nabla_v f_0(X(0), V(0)) A_{v, x_1} + 2 \nabla_v f_0(X(0), R_{x_1} v) A_{v, x_1} \right\} \\
&\overset{r.e.}{=} \left[ \nabla_v f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \left\{ \nabla_x V(0; t, x + \epsilon \hat{r}_2, v) - \lim_{s \to 0^+} \nabla_x V(s) \right\} \right]^{T} \\
&+ (-2A_{x_1, v}) \left\{ \nabla_{xx} f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \nabla_x X(s) + \nabla_{vv} f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \nabla_x V(s) \right\} \hat{r}_2
\end{align*}
\]

\[
\begin{align*}
I_{xx,2} &:= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_v f_0(X(0), V(0)) \nabla_x V(0) - \nabla_v f_0(X(0), V(0)) \lim_{s \to 0^+} \nabla_x V(s) \\
&- 2 \nabla_v f_0(X(0), V(0)) A_{v, x_1} + 2 \nabla_v f_0(X(0), R_{x_1} v) A_{v, x_1} \right\} \\
&\overset{r.e.}{=} \left[ \nabla_v f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \left\{ \nabla_x V(0; t, x + \epsilon \hat{r}_2, v) - \lim_{s \to 0^+} \nabla_x V(s) \right\} \right]^{T} \\
&+ (-2A_{x_1, v}) \left\{ \nabla_{xx} f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \nabla_x X(s) + \nabla_{vv} f_0(x^1, R_{x_1} v) \lim_{s \to 0^+} \nabla_x V(s) \right\} \hat{r}_2
\end{align*}
\]

Similar to \((4.2)\) and \((4.3)\),

\[
I_{xx,1} \hat{r}_2 = \left[ \nabla_x f_0(x^1, R_{x_1} v) \nabla_x X(R_{x_1} x, v) \right] \hat{r}_2
\]

\[
I_{xx,2} \hat{r}_2 = \left[ \nabla_v f_0(x^1, R_{x_1} v) \nabla_v X(R_{x_1} x, v) \right] \hat{r}_2
\]
Then, from (4.11) and (4.14), we get the following compatibility condition

\[
\begin{align*}
\nabla_{xx} f(t, x, v) &= \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) \\
&= \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right)
\end{align*}
\]

Then, from (4.11) and (4.14), we get the following compatibility condition

\[
\nabla_{xx} f_0(x^1, v) = \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right) + \left(\nabla_{xx} f_0(x^1, v) + \nabla_{xx} f_0(x^1, v)\right)
\]

4.4. Condition for \(\nabla_{xx} f\). We split perturbed direction into \((3.7)\). \(\nabla_x f(t, x, v)\) can be written as \((3.22)\) or \((3.24)\). Using \((3.22)\), \(\hat{r}_1\) of \((3.7)\), and notation \((3.5)\),

\[
\begin{align*}
\nabla_{xx} f(t, x, v)\hat{r}_1 &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \nabla_x f(t, x, v + \epsilon \hat{r}_1) - \nabla_x f(t, x, v) \right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \nabla_x f_0(X(0); t, x, v + \epsilon \hat{r}_1), V(0; t, x, v + \epsilon \hat{r}_1) \right) - \nabla_x f_0(X(0), v) \right] \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X^2(0), V^2(0)) \nabla_x X^2(0) + \nabla_v f_0(X^2(0), V^2(0)) \nabla_x V^2(0) - \nabla_x f_0(X(0), v) \right\} \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \nabla_x f_0(X^2(0), v + \epsilon \hat{r}_1) \nabla_x X^2(0) + \nabla_v f_0(X^2(0), v + \epsilon \hat{r}_1) \nabla_x V^2(0) - \nabla_x f_0(X(0), v) \right\}
\end{align*}
\]
where

\[
I_{v,1} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{x} f_{0}(X_{c}(0), V_{c}(0)) \nabla_{x} X_{c}(0) - \nabla_{x} f_{0}(X_{c}(0), V_{c}(0)) \lim_{s \to 0-} \nabla_{x} X(s) \right. \\
+ \left( \nabla_{x} f_{0}(X_{c}(0), V_{c}(0)) R_{x1} - \nabla_{x} f_{0}(X(0), R_{x1}v) R_{x1} \right) \} \\
= \left[ \nabla_{x} f_{0}(x^{1}, R_{x1}v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{x} X_{c}(0) - \lim_{s \to 0-} \nabla_{x} X(s) \right) \right]^{T} \\
+ R_{x1} \left\{ \nabla_{xx} f_{0}(x^{1}, R_{x1}v) \lim_{s \to 0-} \nabla_{v} X(s) + \nabla_{vx} f_{0}(x^{1}, R_{x1}v) \lim_{s \to 0-} \nabla_{v} V(s) \right\} \tilde{r}_{2} \\
= \left[ \nabla_{x} f_{0}(x^{1}, R_{x1}v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{x} X(0; t, x, v + \varepsilon \tilde{r}_{2}) - \lim_{s \to 0-} \nabla_{x} X(s) \right) \right]^{T} \\
+ R_{x1} \left[ \nabla_{xx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2tA_{v,x1}) \right] \tilde{r}_{2},
\]

\[
I_{v,2} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \nabla_{x} f_{0}(X_{c}(0), V_{c}(0)) \nabla_{x} V_{c}(0) - \nabla_{v} f_{0}(X_{c}(0), V_{c}(0)) \lim_{s \to 0-} \nabla_{v} V(s) \right. \\
- 2\nabla_{v} f_{0}(X_{c}(0), V_{c}(0)) A_{v,x1} + 2\nabla_{v} f_{0}(X(0), R_{x1}v) A_{v,x1} \right\} \\
= \left[ \nabla_{v} f_{0}(x^{1}, R_{x1}v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{v} X_{c}(0) - \lim_{s \to 0-} \nabla_{v} X(s) \right) \right]^{T} \\
- \left[ \nabla_{vx} f_{0}(x^{1}, R_{x1}v) \lim_{s \to 0-} \nabla_{v} X(s) + \nabla_{vv} f_{0}(x^{1}, R_{x1}v) \lim_{s \to 0-} \nabla_{v} V(s) \right] \tilde{r}_{2} \\
= \left[ \nabla_{v} f_{0}(x^{1}, R_{x1}v) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \nabla_{v} V(0; t, x, v + \varepsilon \tilde{r}_{2}) - \lim_{s \to 0-} \nabla_{v} X(s) \right) \right]^{T} \\
- (2A_{v,x1}) \left[ \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + \nabla_{vv} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2tA_{v,x1}) \right] \tilde{r}_{2},
\]

Similar to \((4.12)\) and \((4.13)\),

\[
\begin{align*}
(\ast)_{v,1} & \tilde{r}_{2} = \left[ \nabla_{x} f_{0}(x^{1}, R_{x1}v) \nabla_{v}(R_{x1}^{1}(x,v)) \right] \tilde{r}_{2} = -t \left[ \nabla_{x} f_{0}(x^{1}, R_{x1}v) \nabla_{x}(R_{x1}^{1}(x,v)) \right] \tilde{r}_{2}, \\
(\ast)_{v,2} & \tilde{r}_{2} = \left[ \nabla_{v} f_{0}(x^{1}, R_{x1}v) \nabla_{v}(2A_{v,x1}^{1}(x,v)) \right] \tilde{r}_{2} \\
& = 2t \left[ \nabla_{v} f_{0}(x^{1}, R_{x1}v) \nabla_{x}(A_{v,x1}^{1}(x,v)) \right] \tilde{r}_{2}
\end{align*}
\]

Hence,

\[
\nabla_{vx} f(t, x, v) = (\ast)_{v,1} \tilde{r}_{2} + (\ast)_{v,2} \tilde{r}_{2} + R_{x1} \left\{ \nabla_{xx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2A_{v,x1}) \right\} \\
+ (2A_{v,x1}^{1}) \left[ \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + \nabla_{vv} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2A_{v,x1}) \right].
\]

Then from \((4.13)\) and \((4.1)\) we get the following compatibility condition

\[
\begin{align*}
\nabla_{xx} f_{0}(x^{1}, v)(-t) + \nabla_{vx} f_{0}(x^{1}, v) \\
= \left( \ast \right)_{v,1} \tilde{r}_{2} + \left( \ast \right)_{v,2} \tilde{r}_{2} + R_{x1} \nabla_{xx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2A_{v,x1}) \\
+ (2A_{v,x1}^{1}) \nabla_{vx} f_{0}(x^{1}, R_{x1}v)(-tR_{x1}) + (2A_{v,x1}^{1}) \nabla_{vv} f_{0}(x^{1}, R_{x1}v)(R_{x1} + 2A_{v,x1}).
\end{align*}
\]
4.5. Compatibility conditions for transpose: $\nabla^T x = \nabla x$ and $\nabla^T x = \nabla x$. First, we claim that (4.23), (4.24), and (4.26) imply the following four conditions for $(x^1, v) \in \gamma$.

\[
\nabla_{xx} f_0(x^1, v) = R_x \nabla_{xx} f_0(x^1, R_x v) R_x + R_x \nabla_{vv} f_0(x^1, R_x v)(-2A_{v,x^1})
\]

\[
\nabla_{xx} f_0(x^1, v) = R_x \nabla_{xx} f_0(x^1, R_x v) R_x + R_x \nabla_{vv} f_0(x^1, R_x v)(-2A_{v,x^1})
\]

\[
\nabla_{xx} f_0(x^1, v) = R_x \nabla_{xx} f_0(x^1, R_x v) R_x + R_x \nabla_{vv} f_0(x^1, R_x v)(-2A_{v,x^1})
\]

\[
\nabla_{vv} f_0(x^1, v) = R_x \nabla_{vv} f_0(x^1, R_x v) R_x + R_x \nabla_{vv} f_0(x^1, R_x v)(-2A_{v,x^1})
\]

(4.15) is just identical to (4.22). Then applying (4.15) to (4.20) and (4.3), we obtain (4.21) and (4.24), respectively. Finally, applying (4.21) to (4.20), we obtain (4.23) which is true by taking $\nabla_0^2$ to (1.1) directly.

From (4.21)–(4.24), we must check conditions to guarantee necessary conditions, $\nabla^T x = \nabla x$ and $\nabla^T x = \nabla x$.

4.5.1. $\nabla^T x = \nabla x$. From (4.21) and (4.24), we need

\[
\left[ \begin{array}{c} \nabla_{v} f_0(x^1, R_x v) \\nabla_{v} f_0(x^1, R_x v) \end{array} \right] = -2 \left[ \begin{array}{c} \nabla_{v} f_0(x^1, R_x v) \\nabla_{v} f_0(x^1, R_x v) \end{array} \right].
\]

To check (4.25) we explicitly compute $\nabla_x (R_{v,1}^1(x,v))$, $\nabla_x (R_{v,2}^1(x,v))$, $\nabla_x (-2A_{v,1})$, and $\nabla_x (-2A_{v,2})$ in the following Lemma.

Lemma 4.1. Recall reflection operator $R_x$ in (1.1) and $A_{v,x^1}$ in (1.7),

\[
A_{v,x^1} := \left[ \begin{array}{c} (v \cdot n(x^1)) I + (n(x^1) \otimes v) \end{array} \right] \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right).
\]

We write that $A^i$ is the $i$th column of matrix $A$ and $\nabla_v A_{v,y}^i$ be the $v$-derivative of $A_{v,y}^i$ for $1 \leq i \leq 2$ and $(v, y) \in \mathbb{R}^2 \times \partial \Omega$. Then,

\[
\nabla_x (R_{v,1}^1(x,v)) = \left[ \begin{array}{c} -4v_2 n_1 n_2 \\v_2 n_1 n_2 \\v_2 n_1 n_2 \\v \cdot n(x^1) \end{array} \right], \quad \nabla_x (R_{v,2}^1(x,v)) = \left[ \begin{array}{c} -2v_2 (n_2^2 - n_1^2) \\v \cdot n(x^1) \\v \cdot n(x^1) \\v \cdot n(x^1) \end{array} \right],
\]

\[
\nabla_v (-2A_{v,1}^1) = \left[ \begin{array}{c} 2v_1 n_1 \\v_1 n_1 n_2 \\v_1 n_1 n_2 \\v_1 n_1 n_2 \\v \cdot n(x^1) \\v \cdot n(x^1) \\v \cdot n(x^1) \\v \cdot n(x^1) \end{array} \right],
\]

\[
\nabla_v (-2A_{v,2}^1) = \left[ \begin{array}{c} 2v_2 n_1 n_2 \\v_2 n_1 n_2 \\v_2 n_1 n_2 \\v_2 n_1 n_2 \\v \cdot n(x^1) \\v \cdot n(x^1) \\v \cdot n(x^1) \\v \cdot n(x^1) \end{array} \right],
\]

where $v_i$ be the $i$th component of $v$. We denote the $i$th component $n_i(x,v)$ of $n(x^1)$ as $n_i$, that is, $n_i$ depends on $x,v$. Moreover, the following identity holds

\[
\nabla_x (R_{v,1}^1(x,v)) v = 0, \quad \nabla_x (R_{v,2}^1(x,v)) v = 0.
\]
Proof. Remind the definition of the reflection matrix $R_{x^1}$ and $-2A_{v,x^1}$:

$$R_{x^1} = I - 2n(x^1) \otimes n(x^1) = \begin{bmatrix} 1 - 2n_1^2 & -2n_1n_2 \\ -2n_1n_2 & 1 - 2n_2^2 \end{bmatrix},$$

$$-2A_{v,x^1} = -2 \left( (v \cdot n(x^1)I + (n(x^1) \otimes v) \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) \right).$$

To find $\nabla_x(R_{x^1(x,v)}, \nabla_x(R_{x^1(x,v))}).$ we use (2.3) in Lemma 2.4:

$$\nabla_x[n(x^1,v,x)] = I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} = \begin{bmatrix} v_2n_2 & -v_1n_2 \\ -v_1n_2 & v_1n_2 \end{bmatrix},$$

(4.27)

Firstly, we directly calculate $\nabla_x(R_{x^1(x,v)})$ and $\nabla_x(R_{x^1(x,v))}$ using (4.27):

$$\nabla_x(R_{x^1(x,v)}) = \nabla_x \begin{bmatrix} 1 - 2n_2^2 \\ -2n_1n_2 \end{bmatrix} = \begin{bmatrix} -4v_1n_1n_2 \\ -2v_1n_2 \\ -2v_1n_2 \\ 2v_1n_2 \end{bmatrix},$$

$$\nabla_x(R_{x^1(x,v)}) = \nabla_x \begin{bmatrix} -2n_1n_2 \\ 1 - 2n_2^2 \end{bmatrix} = \begin{bmatrix} -2v_2n_2 - n_1^2 \\ -2v_2n_2 - n_1^2 \\ 2v_2n_2 - n_1^2 \\ 2v_2n_2 - n_1^2 \end{bmatrix}. $$

Next, we calculate the $v$-derivative of $-2A_{v,x^1}$:

$$\nabla_v(-2A_{v,x^1}) = \frac{2v_2 n_2 (v \cdot n(x^1)) - 2v_1 n_2 n_1}{(v \cdot n(x^1))^2} - \frac{2v_2 n_1 n_1}{(v \cdot n(x^1))^2}.$$

Similarly, we deduce the $v$-derivative of $[\nabla_x(R_{x^1(x,v)})) and \nabla_x(R_{x^1(x,v))}$, and then (4.20) follows from direct calculation that

$$\nabla_x(R_{x^1(x,v)})v = \begin{bmatrix} \frac{-4v_1n_1n_2}{v \cdot n(x^1)} \\ \frac{-2v_2n_2 - n_1^2}{v \cdot n(x^1)} \end{bmatrix},$$

$$\nabla_x(R_{x^1(x,v)})v = \begin{bmatrix} \frac{4v_1n_1n_2}{v \cdot n(x^1)} \\ \frac{2v_2n_2 - n_1^2}{v \cdot n(x^1)} \end{bmatrix}.$$
Back to the point, we find the condition of $\nabla_v f_0(x^1, Rv)$ satisfying \textbf{4.25}. Since

$$
\begin{bmatrix}
\nabla_v f_0(x^1, Rv)\nabla_x (R^1_{x,v}(x,v)) \\
\nabla_v f_0(x^1, Rv)\nabla_x (R^2_{x,v}(x,v))
\end{bmatrix}^T = \begin{bmatrix}
\nabla_v f_0(x^1, Rv) \frac{\partial}{\partial x_1} (R^1_{x,v}(x,v)) & \nabla_v f_0(x^1, Rv) \frac{\partial}{\partial x_2} (R^2_{x,v}(x,v)) \\
\nabla_v f_0(x^1, Rv) \frac{\partial}{\partial x_1} (R^1_{x,v}(x,v)) & \nabla_v f_0(x^1, Rv) \frac{\partial}{\partial x_2} (R^2_{x,v}(x,v))
\end{bmatrix},
$$

it suffices to find the condition of $\nabla_v f_0(x^1, R_{x,v})$ such that

$$
\begin{align*}
\nabla_v f_0(x^1, R_{x,v}) \left( \frac{\partial}{\partial x_1} (R^1_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^1_{v,x_1}) \right) = 0, & \quad \nabla_v f_0(x^1, R_{x,v}) \left( \frac{\partial}{\partial x_2} (R^1_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^2_{v,x_1}) \right) = 0, \\
\nabla_v f_0(x^1, R_{x,v}) \left( \frac{\partial}{\partial x_1} (R^1_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^1_{v,x_1}) \right) = 0, & \quad \nabla_v f_0(x^1, R_{x,v}) \left( \frac{\partial}{\partial x_2} (R^2_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^2_{v,x_1}) \right) = 0.
\end{align*}
$$

We denote column vectors

$$
K_1 := \frac{\partial}{\partial x_1} (R^1_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^1_{v,x_1}), \quad K_2 := \frac{\partial}{\partial x_2} (R^1_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^2_{v,x_1}),
$$

$$
K_3 := \frac{\partial}{\partial x_1} (R^2_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^1_{v,x_1}), \quad K_4 := \frac{\partial}{\partial x_2} (R^2_{x,v}(x,v)) - \frac{\partial}{\partial v} (-2A^2_{v,x_1}).
$$

To determine whether $\nabla_v f_0(x^1, R_{x,v})$ is a nonzero vector or not for \textbf{4.25}, we need to calculate the following determinant

$$
\det \begin{bmatrix} K_i & K_j \end{bmatrix}, \quad 1 \leq i < j \leq 4.
$$

If every determinant has a value of zero, then $\nabla_v f_0(x^1, R_{x,v})$ satisfying \textbf{4.25} is not the zero vector. We now show that every determinant is 0 and $\nabla_v f_0(x^1, R_{x,v})$ is parallel to a particular direction to satisfy \textbf{4.25}. Using Lemma \textbf{4.1} and $|n(x)| = n_1^2 + n_2^2 = 1$,

(Case 1) $(K_1 \leftrightarrow K_4)$

$$
\begin{align*}
\det \begin{bmatrix} K_1 & K_4 \end{bmatrix} &= \left(\frac{-2}{v \cdot n(x^1)}\right)^2 \det \begin{bmatrix}
2v_2n_1n_2 - \frac{v^2n_1}{v \cdot n(x^1)} & v_1(n_1^2 - n_2^2) - \frac{v^2n_1}{v \cdot n(x^1)} \\
v_2(n_2^2 - n_1^2) - \frac{v^2n_2}{v \cdot n(x^1)} & 2v_1n_1n_2 - \frac{v^2n_2}{v \cdot n(x^1)}
\end{bmatrix} \\
&= \left(\frac{-2}{v \cdot n(x^1)}\right)^2 \left[4v_1v_2n_1n_2^2 + 2v_2^2v_1n_2^2 - \frac{2v_1v_2n_1n_2}{v \cdot n(x^1)} + \frac{v^2v_2^2n_1n_2}{(v \cdot n(x^1))^2} - v_2^2v_1n_2(n_1^2 - n_2^2) - \frac{v^2v_2^2n_1n_2}{v \cdot n(x^1)} + \frac{v^2v_2^2n_1n_2}{(v \cdot n(x^1))^2}\right] \\
&= \left(\frac{-2}{v \cdot n(x^1)}\right)^2 \left(2v_2^2v_1n_1 - \frac{v^2v_2^2n_1n_2}{v \cdot n(x^1)} + \frac{v^2v_2^2n_1n_2}{(v \cdot n(x^1))^2}\right) \\
&= 0,
\end{align*}
$$
\[
\text{(Case 2) } (K_1 \leftrightarrow K_2)
\]
\[
\det \begin{bmatrix} K_1 & K_2 \\ v \cdot n(x^1) & v \cdot n(x^2) \end{bmatrix} = \left(\frac{-2}{v \cdot n(x^1)}\right)^2 \det \begin{bmatrix} 2v_2 n_1 n_2 - \frac{v_3^2 n_1}{v \cdot n(x^1)} & -v_1 n_1 n_2 + v_2 n_2 - \frac{(v_2^2 - v_3^2)n_2^2}{v \cdot n(x^1)} + \frac{2v_1 v_2 n_1 n_2^2}{v \cdot n(x^1)} \\
v_2(n_2^2 - n_1^2) - \frac{v_3^2 n_2}{v \cdot n(x^1)} & v_1 n_2^2 - v_1 n_1 n_2 - \frac{(v_2^2 - v_3^2)n_1 n_2^2}{v \cdot n(x^1)} + \frac{2v_1 v_2 n_1 n_2 n_2^2}{v \cdot n(x^1)} \end{bmatrix} = 0.
\]

\[
\text{(Case 3) } (K_1 \leftrightarrow K_3)
\]
\[
\det \begin{bmatrix} K_1 & K_3 \\ v \cdot n(x^1) & v \cdot n(x^2) \end{bmatrix} = \left(\frac{-2}{v \cdot n(x^1)}\right)^2 \det \begin{bmatrix} 2v_2 n_1 n_2 - \frac{v_3^2 n_1}{v \cdot n(x^1)} & -v_2 n_1^2 - v_1 n_1 n_2 - \frac{(v_2^2 - v_3^2)n_1 n_2^2}{v \cdot n(x^1)} + \frac{2v_1 v_2 n_1 n_2 n_2^2}{v \cdot n(x^1)} \\
v_2(n_2^2 - n_1^2) - \frac{v_3^2 n_2}{v \cdot n(x^1)} & v_1 n_2^2 - v_1 n_1 n_2 - \frac{(v_2^2 - v_3^2)n_1 n_2^2}{v \cdot n(x^1)} + \frac{2v_1 v_2 n_1 n_2 n_2^2}{v \cdot n(x^1)} \end{bmatrix} = 0.
\]

Moreover, from (Case 1) and (Case 2), we deduce
\[
\det \begin{bmatrix} K_2 & K_4 \\ v \cdot n(x^1) & v \cdot n(x^2) \end{bmatrix} = 0.
\]

Likewise, it holds that
\[
\det \begin{bmatrix} K_2 & K_3 \\ v \cdot n(x^1) & v \cdot n(x^2) \end{bmatrix} = 0, \quad \det \begin{bmatrix} K_3 & K_4 \\ v \cdot n(x^1) & v \cdot n(x^2) \end{bmatrix} = 0.
\]

Therefore, it means that we can find a nonzero vector \( \nabla_v f_0(x^1, R_x, v) \) satisfying \( \nabla_v f_0(x^1, R_x, v) \left[ K_1 \right] = 0 \).
\( \nabla_v f_0(x^1, R_x, v) \) is orthogonal to the column vector \( K_1 \). More specifically, \( \nabla_v f_0(x^1, R_x, v)^T \) has the following direction

\[
-2 \frac{v_2 (n_2^2 - n_1^2) + v_1 n_1^2}{v \cdot n(x^1)} = -2 \frac{v_2 n_2 n_1 v_1 n_1^2 (n_2^2 - n_1^2) + 2v_1^2 n_1^2 n_2^2}{(v \cdot n(x^1))^2}
\]

\[
= \frac{2v_1 n_1}{(v \cdot n(x^1))^2} \left[ n_2^2 - n_1^2 \right] \left[ -2n_1 n_2 \right] \left[ v_1 \right] = \frac{2v_1 n_1}{(v \cdot n(x^1))^2} R_x v.
\]

Consequently, for (4.27), we get the following condition

\[
\nabla_v f_0(x, R_x, v) \parallel (R_x v)^T,
\]

for any \( x \in \partial \Omega \).

4.5.2. \( \nabla_{xx}^T = \nabla_{xx} \). From (4.22), we need

\[
\left( \begin{array}{c} \nabla_x f_0(x^1, R_x, v) \nabla_x (R_{x^1}(x,v)) \\ \nabla_x f_0(x^1, R_x, v) \nabla_x (R_{x^2}(x,v)) \end{array} \right) = \left( \begin{array}{c} \nabla_v f_0(x, R_x, v) \nabla_x (-2A_{v,x^1}(x,v)) \\ \nabla_v f_0(x, R_x, v) \nabla_x (-2A_{v,x^2}(x,v)) \end{array} \right)
\]

Thus, it suffices to check that

\[
\nabla_x f_0(x^1, R_x, v) \frac{\partial}{\partial x_2} (R_{x^1}(x,v)) + \nabla_v f_0(x^1, R_x, v) \frac{\partial}{\partial x_1} (-2A_{v,x^1}(x,v))
\]

\[
= \nabla_x f_0(x^1, R_x, v) \frac{\partial}{\partial x_1} (R_{x^2}(x,v)) + \nabla_v f_0(x^1, R_x, v) \frac{\partial}{\partial x_2} (-2A_{v,x^2}(x,v)).
\]

In other words, we have to find the condition of \( \nabla_x f_0(x^1, R_x, v) \) to satisfy

\[
\nabla_x f_0(x^1, R_x, v) \left[ \frac{\partial}{\partial x_2} (R_{x^1}(x,v)) - \frac{\partial}{\partial x_1} (R_{x^2}(x,v)) \right] = \nabla_v f_0(x^1, R_x, v) \left[ \frac{\partial}{\partial x_1} (-2A_{v,x^1}(x,v)) - \frac{\partial}{\partial x_2} (-2A_{v,x^2}(x,v)) \right].
\]

Since we computed \( \nabla_x (R_{x^1}(x,v)), \nabla_x (R_{x^2}(x,v)) \) in Lemma 4.1, we represent \( \nabla_x (-2A_{v,x^1}(x,v)) \) and \( \nabla_x (-2A_{v,x^2}(x,v)) \) by components.

**Lemma 4.2.** Define the matrix \( A_{v,x^1} \) as:

\[
A_{v,x^1} := \left[ (v \cdot n(x^1)) I + (n(x^1) \otimes v) \right] \left[ I - \frac{v \cdot n(x^1)}{v \cdot n(x^1)} \right].
\]

We write that \( A^i \) is the \( i \)th column of matrix \( A \). Then,

\[
\nabla_x (-2A_{v,x^1}(x,v)) = \begin{bmatrix} 4v_1^2 v_2^2 n_1^3 + 2v_1 v_2^2 (3n_1^2 n_2 - n_3^2) + 2v_2^4 (3n_1 n_2^2 + n_1^3) \\ 4v_1^2 v_2^2 n_1^3 + 2v_1 v_2^2 (3n_1^2 n_2 - n_3^2) + 2v_2^4 (3n_1 n_2^2 + n_1^3) \\ (v \cdot n(x^1))^3 \\ (v \cdot n(x^1))^3 \end{bmatrix}
\]

\[
= \begin{bmatrix} -4v_1^2 v_2 n_1^2 - 2v_1 v_2^2 (3n_1^2 n_2 - n_3^2) - 2v_1 v_2^2 (3n_1^2 n_2 + n_1^3) \\ -4v_1^2 v_2 n_1^2 - 2v_1 v_2^2 (3n_1^2 n_2 - n_3^2) - 2v_1 v_2^2 (3n_1^2 n_2 + n_1^3) \\ (v \cdot n(x^1))^3 \\ (v \cdot n(x^1))^3 \end{bmatrix}
\]

where \( v_i \) be the \( i \)th component of \( v \). We denote the \( i \)th component \( n_i(x,v) \) of \( n(x^1) \) as \( n_i \), that is, \( n_i \) depends on \( x, v \). Furthermore, it holds that

\[
\nabla_x (-2A_{v,x^1}(x,v)) v = 0, \quad \nabla_x (-2A_{v,x^2}(x,v)) v = 0.
\]
Proof. We write the matrix \(-2A_{v,x^1}\) by components:

\[
-2A_{v,x^1} = \begin{bmatrix}
-2v_2n_2 - \frac{2v_1v_2n_1n_2}{v \cdot n(x)} + \frac{2v_3^2n_1^2}{v \cdot n(x)} & 2v_1n_2 + \frac{2v_3n_1n_2}{v \cdot n(x)} - \frac{2v_1v_2n_1^2}{v \cdot n(x)} \\
2v_2n_1 - \frac{2v_1v_2n_1n_2}{v \cdot n(x)} + \frac{2v_3^2n_1n_2}{v \cdot n(x)} & -2v_1n_1 - \frac{2v_3^2n_1^2}{v \cdot n(x)} + \frac{2v_1v_2n_1n_2}{v \cdot n(x)} \end{bmatrix}.
\]

For \(\nabla_x(-2A_{v,x^1(x,v)})\), we firstly take a derivative of (1, 1) component of \(-2A_{v,x^1}\) with respect to \(x_1\)

\[
\frac{\partial}{\partial x_1} \left(-2v_2n_2 - \frac{2v_1v_2n_1n_2}{v \cdot n(x)} + \frac{2v_3^2n_1^2}{v \cdot n(x)}\right) = -2v_2\frac{\partial n_2}{\partial x_1} + \frac{\left(-2v_1v_2n_2\frac{\partial n_1}{\partial x_1} - 2v_2v_1n_1\frac{\partial n_2}{\partial x_1}\right)}{v \cdot n(x)} (v \cdot n(x))^2 + 2v_1v_2n_1n_2\left(v_1\frac{\partial n_1}{\partial x_1} + v_2\frac{\partial n_2}{\partial x_1}\right) + \frac{4v_2^3n_1^2}{v \cdot n(x)} + 2v_1v_2(3n_1n_2^2 - n_2^3) + 2v_1^2(3n_1n_2^2 + n_1^3),
\]

where we used (2.3) in Lemma 2.4. Similarly, using (2.3) in Lemma 2.4, we get

\[
\frac{\partial}{\partial x_2} \left(-2v_2n_2 - \frac{2v_1v_2n_1n_2}{v \cdot n(x)} + \frac{2v_3^2n_1^2}{v \cdot n(x)}\right) = -2v_2\frac{\partial n_2}{\partial x_2} + \frac{\left(-2v_1v_2n_2\frac{\partial n_1}{\partial x_2} - 2v_2v_1n_1\frac{\partial n_2}{\partial x_2}\right)}{v \cdot n(x)} (v \cdot n(x))^2 + 2v_1v_2n_1n_2\left(v_1\frac{\partial n_1}{\partial x_2} + v_2\frac{\partial n_2}{\partial x_2}\right) - \frac{4v_2^3n_1^3}{v \cdot n(x)} - 2v_1v_3^2(3n_1n_2^2 - n_2^3) - 2v_1^2v_2^3(3n_1n_2^2 + n_1^3).
\]

Thus, we derived \(\nabla_x(-2A_{v,x^1(x,v)})\). Similar to \(\nabla_x(-2A_{v,x^1(x,v)})\), we can obtain \(\nabla_x(-2A_{v,x^1(x,v)})\), and the details are omitted. By the \(\nabla_x(-2A_{v,x^1(x,v)})\) and \(\nabla_x(-2A_{v,x^1(x,v)})\) formula above, direct calculation gives...
which implies
\[(4.28), \text{ and } (4.31).\]

Now, back to our consideration \((4.29)\). By Lemma \(4.2\), we have
\[
\partial \frac{\partial}{\partial x_2} (-2A_{v,x}^1(x,v)) = \partial \frac{\partial}{\partial x_1} (-2A_{v,x}^1(x,v)),
\]
which implies that
\[
\nabla_x f_0(x^1, R_{x,v}) \left[ \frac{\partial}{\partial x_2} (R_{x}^1(x,v)) - \frac{\partial}{\partial x_1} (R_{x}^2(x,v)) \right] = \frac{2}{n \cdot (n(x))} \nabla_x f_0(x^1, R_{x,v}) \left[ \frac{2v_1n_2 + v_2(n_2^2 - n_1^2)}{v_1(n_2^2 - n_1^2) - 2v_2n_1n_2} \right] = 0.
\]

It means that \(\nabla_x f_0(x, R_{x,v})\) is orthogonal to \(\frac{\partial}{\partial x_2} (R_{x}^1(x,v)) - \frac{\partial}{\partial x_1} (R_{x}^2(x,v))\) and \(\nabla_x f_0(x^1, R_{x,v})^T\) has the following direction
\[
\begin{bmatrix}
-v_1(n_2^2 - n_1^2) + 2v_2n_1n_2 \\
2v_1n_2^2 + v_2(n_2^2 - n_1^2)
\end{bmatrix}
= -\begin{bmatrix}
-n_2^2 - n_1^2 \\
-2n_1n_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= -R_{x,v}^v.
\]

To hold \(\nabla_{xx} f_0(x^1, R_{x,v})^T = \nabla_{xx} f_0(x^1, R_{x,v})\), the following condition
\[
\nabla_x f_0(x, R_{x,v}) \parallel (R_{x,v})^T,
\]
must be satisfied for \(x \in \partial \Omega\).

4.6. Conditions including \(\partial\). In this subsection, we find conditions for \(\partial_t, \partial_s, \nabla_x, \partial_t \nabla_x, \nabla_x \partial_t, \nabla_v \partial_t\). In the last subsection, we show that all these \(\partial\)-including compatibility conditions are covered by \((4.21)-(4.24), (4.28), \text{ and } (4.31)\).

4.6.1. \(\partial_t\). Using the same perturbation \((5.26)\) in \(C^1\) compatibility condition, we derive \(C^2\) compatibility condition. For \(\epsilon > 0\),
\[
\partial_t (f(t + \epsilon, x, v) - f(t, x, v)) = \partial_t (f_0(X^*(0), R_{x,v}) - f_0(X(0), R_{x,v}))
= (\nabla_x f_0(X^*(0), R_{x,v}) - \nabla_x f_0(X(0), R_{x,v}) \nabla_x f_0(X(0), R_{x,v})) (-R_{x,v})
= (-R_{x,v}^v)^T (\nabla_x f_0(X^*(0), R_{x,v}) - \nabla_x f_0(X(0), R_{x,v}))^T,
\]
which implies
\[
f_{tt}(t, x, v) = \lim_{\epsilon \to 0^+} \frac{\partial_t (f(t + \epsilon, x, v) - f(t, x, v))}{\epsilon}
= (-R_{x,v}^v)^T \nabla_{xx} f_0(x^1, R_{x,v}) \lim_{\epsilon \to 0^+} \frac{X^*(0) - X(0)}{\epsilon}
= (-R_{x,v}^v)^T \nabla_{xx} f_0(x^1, R_{x,v})(-R_{x,v}).
\]

On the other hand, for \(\epsilon < 0\), it holds that
\[
\partial_t (f(t + \epsilon, x, v) - f(t, x, v)) = \partial_t (f_0(X^*(0), v) - f_0(X(0), v)) = (\nabla_x f_0(X^*(0), v) - \nabla_x f_0(X(0), v)) (-v)
= (-v)^T (\nabla_x f_0(X^*(0), v) - \nabla_x f_0(X(0), v))^T.
\]
Thus, we have

\[ f_{t\epsilon}(t, x, v) = \lim_{\epsilon \to 0} \frac{\partial_t f(t + \epsilon, x, v) - \partial_t f(t, x, v)}{\epsilon} \]

\[ = (-v)^T \nabla_{xx} f_0(x^1, v) \lim_{\epsilon \to 0} \frac{X'(0) - X(0)}{\epsilon} \]

\[ = (-v)^T \nabla_{xx} f_0(x^1, v)(-v). \]

To sum up, the condition

\[ v^T \nabla_{xx} f_0(x^1, v) = (R_{x^1}v)^T \nabla_{xx} f_0(x^1, R_{x^1}v)(R_{x^1}v), \tag{4.32} \]

must be satisfied to \( f \in C^2_t \).

4.6.2. \( C^2_{t, x} \). We firstly use the perturbation \((3.26)\) for \( \epsilon < 0 \). From \((3.22)\), it holds that

\[ \partial_t [\nabla_x f(t, x, v)] = \lim_{\epsilon \to 0} \frac{\nabla_x f(t + \epsilon, x, v) - \nabla_x f(t, x, v)}{\epsilon} \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x [f_\epsilon(X(0; t + \epsilon, x, v), V(0; t + \epsilon, x, v))] - \nabla_x f_0(X(0), v) \right) \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X'(0), v) - \nabla_x f_0(X(0), v) \right) \]

\[ = -v^T \nabla_{xx} f_0(x^1, v), \tag{4.33} \]

where we used \( \nabla_x X'(0) = I_2 \) and \( \nabla_x V'(0) = 0 \). On the other hand, for \( \epsilon > 0 \),

\[ X'(0) := X(0; t + \epsilon, x, v) = X(0; t, x - \epsilon v, v), \quad V'(0) := V(0; t + \epsilon, x, v) = R_{x^1}v. \]

Similar to previous case \( \nabla_{xx} \), using \((3.23)\) and \((3.24)\),

\[ \partial_t [\nabla_x f(t, x, v)] = \lim_{\epsilon \to 0^+} \frac{\nabla_x f(t + \epsilon, x, v) - \nabla_x f(t, x, v)}{\epsilon} \]

\[ = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \nabla_x [f_\epsilon(X(0; t + \epsilon, x, v), V(0; t + \epsilon, x, v))] - \nabla_x f_0(X(0), R_{x^1}v)A_{v, x^1} \right) \]

\[ = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \nabla_x f_0(X'(0), V'(0))\nabla_x X'(0) + \nabla_v f_0(X'(0), V'(0))\nabla_x V'(0) \right) \]

\[ - \left( \nabla_x f_0(X(0), R_{x^1}v)R_{x^1} - 2\nabla_v f_0(X(0), R_{x^1}v)A_{v, x^1} \right) \]

\[ = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \nabla_x f_0(X'(0), R_{x^1}v)\nabla_x X'(0) - \nabla_x f_0(X(0), R_{x^1}v)R_{x^1} \right) \]

\[ + \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left( \nabla_v f_0(X'(0), R_{x^1}v)\nabla_v V'(0) + 2\nabla_v f_0(X(0), R_{x^1}v)A_{v, x^1} \right) \]

\[ := I_{x, 1} + I_{x, 2}, \]
where

\[
I_{tx,1} := \lim_{\epsilon \to 0^+} \left( \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), R_{x,v})\nabla_x X^*(0) - \nabla_x f_0(X^*(0), R_{x,v}) \lim_{s \to 0^-} \nabla_x X(s) \right) \right)
\]

\[
+ \nabla_x f_0(X^*(0), R_{x,v}) \lim_{s \to 0^-} \nabla_x X(s) - \nabla_x f_0(X(0), R_{x,v})R_{x,v} \right) R_{x,v} \right)
\]

\[
I_{tx,2} := \lim_{\epsilon \to 0^+} \left( \frac{1}{\epsilon} \left( \nabla_v f_0(X^*(0), R_{x,v})\nabla_x V^*(0) - \nabla_v f_0(X^*(0), R_{x,v}) \lim_{s \to 0^-} \nabla_x V(s) \right) \right)
\]

\[
+ \nabla_v f_0(X^*(0), R_{x,v}) \lim_{s \to 0^-} \nabla_x V(s) - 2\nabla_v f_0(X(0), R_{x,v})A_{v,x} \right)
\]

\[
\Rightarrow \quad \left[ \nabla_v f_0(x^1, R_{x,v}) \lim_{\epsilon \to 0^+} \left( \frac{1}{\epsilon} \left( \nabla_x V^*(0) - \nabla_x V(s) \right) \right) \right]^T
\]

\[
+ (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \lim_{\epsilon \to 0^+} \frac{X^*(0) - X(0)}{\epsilon}
\]

\[
\Rightarrow \quad \left[ \nabla_v f_0(x^1, R_{x,v})\nabla_x (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \right] (v) + (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v})(-R_{x,v}).
\]

Thus,

\[
\partial_t [\nabla_x f(t, x, v)] = (v)^T \left[ \nabla_x f_0(x^1, R_{x,v})\nabla_x (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \right] + (-v)^T \left[ \nabla_v f_0(x^1, R_{x,v})\nabla_x (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \right] \nabla_x R_{x,v} + (-v)^T \nabla_{x,v} f_0(x^1, R_{x,v})(-2A_{v,x}).
\]

From (4.33) and (4.34), we have the following condition

\[
(-v)^T \nabla_{x,v} f_0(x^1, v) = (v)^T \left[ \nabla_x f_0(x^1, R_{x,v})\nabla_x (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \right] + (-v)^T \left[ \nabla_v f_0(x^1, R_{x,v})\nabla_x (-2A_{v,x}^T)\nabla_{x,v} f_0(x^1, R_{x,v}) \right] \nabla_x R_{x,v} + (-v)^T \nabla_{x,v} f_0(x^1, R_{x,v})(-2A_{v,x}).
\]

4.6.3. \(C^2_{t,v}\). Similar to \(C^2_{t,x}\), we use (3.11) and the perturbation \(\frac{[x_0]}{\epsilon}\) for \(\epsilon < 0\) to obtain

\[
\partial_t [\nabla_v f(t, x, v)] = \lim_{\epsilon \to 0^-} \frac{\nabla_v f(t + \epsilon, x, v) - \nabla_v f(t, x, v)}{\epsilon}
\]

\[
= \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left( \nabla_v \left[ f_0(X(0; t + \epsilon, x, v), V(0; t + \epsilon, x, v)) \right] - (-t\nabla_x f_0(X(0), v) + \nabla_v f_0(X(0), v)) \right)
\]

\[
= \lim_{\epsilon \to 0^-} \frac{1}{\epsilon} \left( -(-t + \epsilon)\nabla_x f_0(X^*(0), v) + \nabla_v f_0(X^*(0), v) + t\nabla_x f_0(X(0), v) - \nabla_v f_0(X(0), v) \right)
\]

\[
= -\nabla_x f_0(x^1, v) - t(-v)^T \nabla_{x,v} f_0(x^1, v) + (-v)^T \nabla_{x,v} f_0(x^1, v),
\]

where we have used \(\nabla_x X^*(0) = -(t + \epsilon)I_2, \nabla_x V^*(0) = I_2\). For \(\epsilon > 0\), the perturbation \(\frac{[x_0]}{\epsilon}\) becomes

\[
X^*(0) := X(0; t + \epsilon, x, v) = X(0; t, x - \epsilon v, v) = x^1 - (t + \epsilon)R_{x,v}, \quad V^*(0) := V(0; t + \epsilon, x, v) = R_{x,v}.
\]
By product rule, Lemma 2.6.4 and Lemma 2.4, one obtains that

\[
\nabla_v [X'(0)] = \nabla_v [x^1] - (t^1 + \epsilon)R_{x^1} v = -t \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - R_{x^1} v \otimes \nabla_v (R_{x^1} v)
\]
\[
= -t \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - tR_{x^1} v \otimes \frac{n(x^1)}{v \cdot n(x^1)} - \epsilon (R_{x^1} + 2tA_{v,x^1})
\]
\[
= -tR_{x^1} + \epsilon (R_{x^1} + 2tA_{v,x^1})
\]
\[
\nabla_v [V'(0)] = \nabla_v [R_{x^1} v] = R_{x^1} + 2tA_{v,x^1}.
\]

Through the \(v\)-derivative of \(X'(0), V'(0)\) above and (3.13),

\[
\partial_t[\nabla_v f(t, x, v)] = \lim_{\epsilon \to 0^+} \frac{\nabla_v f(t, x, v) - \nabla_v f(t, x, v)}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left[ \nabla_v f_0(X(0; t + \epsilon, x, v), V(0; t + \epsilon, x, v)) \right.
\]
\[
- (-t \nabla f_0(X(0, R_{x^1} v), R_{x^1} v) + \nabla_v f_0(X(0, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1})))
\]
\[
= -\nabla_x f_0(x^1, R_{x^1} v) (R_{x^1} + 2tA_{v,x^1}) - t \left[ \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (\nabla_x f_0(X'(0), R_{x^1} v) - \nabla_x f_0(X'(0), R_{x^1} v)) \right. \]
\[
\left. - \nabla_x f_0(X(0, R_{x^1} v)) \right] (R_{x^1} + 2tA_{v,x^1})
\]
\[
= - (R_{x^1} + 2tA_{v,x^1})^T \nabla_x f_0(x^1, R_{x^1} v)^T - tR_{x^1} \nabla_{xx} f_0(x^1, R_{x^1} v) \lim_{\epsilon \to 0^+} \frac{X'(0) - X(0)}{\epsilon}
\]
\[
= - (R_{x^1} + 2tA_{v,x^1})^T \nabla_{xx} f_0(x^1, R_{x^1} v) \lim_{\epsilon \to 0^+} \frac{X'(0) - X(0)}{\epsilon}
\]
\[
\overset{\text{resp.}}{=} - \nabla_x f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}) - t(-v^T)R_{x^1} \nabla_{xx} f_0(x^1, R_{x^1} v) R_{x^1}
\]
\[
+ (-v^T)R_{x^1} \nabla_v f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}).
\]

Summing (4.36) and (4.37) yields that

\[
- \nabla_x f_0(x^1, v) - t(-v^T)\nabla_{xx} f_0(x^1, v) + (-v^T)\nabla_v f_0(x^1, v)
\]
\[
= -\nabla_x f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}) - t(-v^T)R_{x^1} \nabla_{xx} f_0(x^1, R_{x^1} v) R_{x^1}
\]
\[
+ (-v^T)R_{x^1} \nabla_v f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}).
\] (4.38)

4.6.4. 42. C_{x,t}. Similar to the \(\nabla_v x\) case, using the same perturbation \(\tilde{r}_1\) of (3.18) and (3.22), we have

\[
\nabla_x [\partial_t f(t, x, v)] \tilde{r}_1 = \lim_{\epsilon \to 0} \frac{\partial_t f(t, x + \epsilon \tilde{r}_1, v) - \partial_t f(t, x, v)}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0} \frac{\nabla_x f(t, x + \epsilon \tilde{r}_1, v) - \nabla_x f(t, x, v)}{\epsilon} (-v)
\]
\[
= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \nabla_x f_0(X(0; t, x + \epsilon \tilde{r}_1, v), V(0; t, x + \epsilon \tilde{r}_1, v)) - \nabla_x f_0(X(0, v)) \right] (-v)
\]
\[
= (-v^T)\nabla_{xx} f_0(x^1, v) \tilde{r}_1,
\]
where we have used $\nabla_x X^*(0) = I_2, \nabla_x V^*(0) = 0$. Next, for $\hat{r}_2$ of (3.18), using (3.3) in Lemma 3.2.1, (4.12), and (4.13) gives

$$
\nabla_x [\partial_t f(t, x, v)] \hat{r}_2 = \lim_{\epsilon \to 0} \frac{\partial_t f(t, x + \epsilon \hat{r}_2, v) - \partial_t f(t, x + \epsilon \hat{r}_2, v)}{\epsilon}
$$

$$= \lim_{\epsilon \to 0} \left( \frac{\nabla_x f(t, x + \epsilon \hat{r}_2, v) - \nabla_x f(t, x, v)}{\epsilon} \right) (-v)
$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x [f_0(X(0; t, x + \epsilon \hat{r}_2, v), V(0; t, x + \epsilon \hat{r}_2, v))] - (\nabla_x f_0(X(0), R_{x,v} R_{x,v}) - 2 \nabla_x f_0(X(0), R_{x,v} A_{v,x}))(-v)
$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x X^*(0) - \nabla_x f_0(X(0), R_{x,v} R_{x,v})(-v)
$$

$$+ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x V^*(0) - \nabla_x f_0(X(0), R_{x,v}(-2 A_{v,x}))(v) \right)
$$

$$:= I_{xt,1} + I_{xt,2},
$$

where

$$I_{xt,1} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x X^*(0) - \nabla_x f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_x X(s)
$$

$$+ \nabla_x f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_x X(s) - \nabla_x f_0(X(0), R_{x,v} R_{x,v})(-v)
$$

$$= (-v^T) \begin{bmatrix} \nabla_x f_0(x^1, R_{x,v} v) \nabla_x (R_{x,v}^1(x,v)) \\ \nabla_x f_0(x^1, R_{x,v} v) \nabla_x (R_{x,v}^2(x,v)) \end{bmatrix} \hat{r}_2
$$

$$+ (-v^T) R_{x,v} \nabla_x f_0(x^1, R_{x,v} v) R_{x,v} \hat{r}_2 + (-v^T) R_{x,v} \nabla_v f_0(x^1, R_{x,v} v) (-2 A_{v,x}) \hat{r}_2,
$$

$$I_{xt,2} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x V^*(0) - \nabla_x f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_x V(s)
$$

$$+ \nabla_x f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_x V(s) - \nabla_x f_0(X(0), R_{x,v} v) (-2 A_{v,x})(-v)
$$

$$= (-v^T) \begin{bmatrix} \nabla_v f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^1(x,v)) \\ \nabla_v f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^2(x,v)) \end{bmatrix} \hat{r}_2
$$

$$+ (-v^T) (-2 A_{v,x}^T) \left( \nabla_{x,v} f_0(x^1, R_{x,v} v) R_{x,v} + \nabla_{v,v} f_0(x^1, R_{x,v} v) (-2 A_{v,x}) \right) \hat{r}_2,
$$

$$= (-v^T) \begin{bmatrix} \nabla_{x,v} f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^1(x,v)) \\ \nabla_{v,v} f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^2(x,v)) \end{bmatrix} \hat{r}_2.
$$

To sum up the above, we get the following condition:

$$(-v^T) \nabla_{xv} f_0(x^1, v) = (-v^T) \begin{bmatrix} \nabla_x f_0(x^1, R_{x,v} v) \nabla_x (R_{x,v}^1(x,v)) \\ \nabla_x f_0(x^1, R_{x,v} v) \nabla_x (R_{x,v}^2(x,v)) \end{bmatrix} + (-v^T) \begin{bmatrix} \nabla_v f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^1(x,v)) \\ \nabla_v f_0(x^1, R_{x,v} v) \nabla_x (-2 A_{v,x}^2(x,v)) \end{bmatrix}
$$

$$+ (-v^T) R_{x,v} \nabla_{x,v} f_0(x^1, R_{x,v} v) R_{x,v} + (-v^T) R_{x,v} \nabla_{v,v} f_0(x^1, R_{x,v} v) (-2 A_{v,x}^T).
$$

(4.39)

4.6.5. $C_{v,t}^2$. Using the perturbation $\hat{r}_1$ of (3.18) and (3.22).
\[ \nabla_v (\partial_t f(t, x, v)) \hat{r}_1 = \lim_{\epsilon \to 0} \frac{\partial_t f(t, x, v + \epsilon \hat{r}_1) - \partial_t f(t, x, v)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\nabla_x f(t, x, v + \epsilon \hat{r}_1)(-v + \epsilon \nabla_x f(t, x, v)(-v)) - \nabla_x f(t, x, v)(-v)}{\epsilon} = -\lim_{\epsilon \to 0} \nabla_x f_0(X(0; t, x, v + \epsilon \hat{r}_1), V(0; t, x, v + \epsilon \hat{r}_1)) \hat{r}_1 + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) - \nabla_x f_0(X(0), v)(-v) \right) = -\nabla_x f_0(X(0), v) \hat{r}_1 + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) - \nabla_x f_0(X(0), v)(-v) \right) = -\nabla_x f_0(x^1, v) \hat{r}_1 + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(x^1, v)(-\hat{r}_1) + (-v^T) \nabla_{xx} f_0(x^1, v) \hat{r}_1 \right), \]

where \( X^*(0) := X(0; t, x, v + \epsilon \hat{r}_1) = x - t(v + \epsilon \hat{r}_1), V^*(0) := V(0; t, x, v + \epsilon \hat{r}_1) = v + \epsilon \hat{r}_1 \). Similar to the case \( \nabla_{xx} \), for the perturbation \( \hat{r}_2 \) of (3.18), using (3.23), (3.24) and (3.23) in Lemma [5.1] yields:

\[ \nabla_v (\partial_t f(t, x, v)) \hat{r}_2 = \lim_{\epsilon \to 0} \frac{\partial_t f(t, x, v + \epsilon \hat{r}_2) - \partial_t f(t, x, v)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\nabla_x f(t, x, v + \epsilon \hat{r}_2)(-v + \epsilon \nabla_x f(t, x, v)(-v)) - \nabla_x f(t, x, v)(-v)}{\epsilon} = -\lim_{\epsilon \to 0} \nabla_x f_0(X(0; t, x, v + \epsilon \hat{r}_2), V(0; t, x, v + \epsilon \hat{r}_2)) \hat{r}_2 + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) - \nabla_x f_0(X(0), R_{x^1} v) R_{x^1} (-v) \right) = -\nabla_x f_0(x^1, R_{x^1} v) R_{x^1} + \nabla_v f_0(x^1, R_{x^1} v)(-2A_{v,x^1}) \hat{r}_2 + \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x X^*(0) - \nabla_x f_0(X(0), R_{x^1} v) R_{x^1} (-v) \right) \hat{r}_2 + I_{vt,1} + I_{vt,2}, \]

where

\[ I_{vt,1} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_x f_0(X^*(0), V^*(0)) \nabla_x X^*(0) - \nabla_x f_0(X^*(0), V^*(0)) \right) \lim_{s \to 0} -\nabla_x X(s) + \nabla_x f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_x X(s) - \nabla_x f_0(X(0), R_{x^1} v) R_{x^1} (-v) \right) \hat{r}_2 = (-v^T) \left[ \nabla_x f_0(x^1, R_{x^1} v) R_{x^1} \nabla_v (R_{x^1}^2) \right] \hat{r}_2 + (-v^T) R_{x^1} \left( \nabla_{xx} f_0(x^1, R_{x^1} v)(-tR_{x^1}) + \nabla_{vv} f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}) \right) \hat{r}_2, \]

\[ I_{vt,2} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \nabla_v f_0(X^*(0), V^*(0)) \nabla_v V^*(0) - \nabla_v f_0(X^*(0), V^*(0)) \right) \lim_{s \to 0} -\nabla_v V(s) + \nabla_v f_0(X^*(0), V^*(0)) \lim_{s \to 0} \nabla_v V(s) - \nabla_v f_0(X(0), R_{x^1} v)(-2A_{v,x^1}) (-v) \right) \hat{r}_2 = (-v^T) \left[ \nabla_v f_0(x^1, R_{x^1} v) R_{x^1} \nabla_v (-2A_{v,x^1}^2) \right] \hat{r}_2 + (-v^T) \left( -2A_{v,x^1}^2 \right) \left( \nabla_{xx} f_0(x^1, R_{x^1} v)(-tR_{x^1}) + \nabla_{vv} f_0(x^1, R_{x^1} v)(R_{x^1} + 2tA_{v,x^1}) \right) \hat{r}_2 = (-v^T) \left[ \nabla_v f_0(x^1, R_{x^1} v) R_{x^1} \nabla_v (-2A_{v,x^1}^2) \right] \hat{r}_2.
Thus, we have the following compatibility condition:
\[
\begin{align*}
&\quad -\nabla_x f_0(x^1, v) + tv^T \nabla_{xx} f_0(x^1, v) + (-v^T) \nabla_{vx} f_0(x^1, v) \\
&= \quad -(\nabla_x f_0(x^1, R_{x^1}v)\nabla_x + \nabla_v f_0(x^1, R_{x^1}v)(-2A_{v,x^1})) \\
&\quad + (-v^T) \left[ \nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \right] + (-v^T) \left[ \nabla_v f_0(x^1, R_{x^1}v)\nabla_v (-2A_{v,x^1}(v,x)) \right] \\
&\quad + tv^T R_{x^1} \nabla_{xx} f_0(x^1, R_{x^1}v) R_{x^1} + (-v^T) R_{x^1} \nabla_{vx} f_0(x^1, R_{x^1}v) (R_{x^1} + 2A_{v,x^1}).
\end{align*}
\]  

(4.40)

4.6.6. Derive $C_{tt}, C_{tx}, C_{xt}, C^2_{xx}, C^2_{vx}$ compatibility conditions from (4.21), (4.24), (4.28), and (4.31). So far, we have derived (4.21), (4.24) to satisfy $f \in C_{tt}, C_{tx}, C_{xt}, C^2_{xx}, C^2_{vx}$. In (4.21), (4.24), since $\nabla_{vx} f_0(x^1, v)$ is the same as $\nabla_{xx} f_0(x^1, v)^T$, we need to assume (4.28). Similarly, we obtained (4.28) because $\nabla_{xx} f_0(x^1, v)$ is a symmetric matrix. In this subsection, we will show that the compatibility conditions $C_{tt} (4.32)$, $C_{tx} (4.33)$, $C_{xt} (4.38)$, $C^2_{xx} (4.39)$, and $C^2_{vx} (4.40)$ are induced under (4.21), (4.24), (4.28), and (4.31). Firstly, we consider $C_{tt}$ compatibility condition. Using (3.3) in Lemma 3.1, (4.20), and (4.30), one has

\[
v^T \nabla_{xx} f_0(x^1, v) = v^T \left[ \begin{array}{c} R_{x^1} \nabla_{xx} f_0(x^1, R_{x^1}v) R_{x^1} + R_{x^1} \nabla_{vx} f_0(x^1, R_{x^1}v)(-2A_{v,x^1}) \\
+ (-2A_{v,x^1}) \nabla_{xx} f_0(x^1, R_{x^1}v) R_{x^1} + (-2A_{v,x^1}) \nabla_{vx} f_0(x^1, R_{x^1}v)(-2A_{v,x^1}) \\
+ \left[ \nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \right] - 2 \left[ \nabla_v f_0(x^1, R_{x^1}v)\nabla_v (-2A_{v,x^1}(v,x)) \right] \right] v.
\end{array} \right]
\]

In (4.35), the left-hand side is

\[
(-v^T) \nabla_{xx} f_0(x^1, v) = (-v^T) \left[ \begin{array}{c}
R_{x^1} \nabla_{xx} f_0(x^1, R_{x^1}v) R_{x^1} + R_{x^1} \nabla_{vx} f_0(x^1, R_{x^1}v)(-2A_{v,x^1}) \\
+ (-2A_{v,x^1}) \nabla_{xx} f_0(x^1, R_{x^1}v) R_{x^1} + (-2A_{v,x^1}) \nabla_{vx} f_0(x^1, R_{x^1}v)(-2A_{v,x^1}) \\
+ \left[ \nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \right] - 2 \left[ \nabla_v f_0(x^1, R_{x^1}v)\nabla_v (-2A_{v,x^1}(v,x)) \right] \right] ,
\end{array} \right]
\]

where we have used (4.39). When we assume (4.31), it holds that $\nabla_{xx} f_0(x^1, v)$ is a symmetric matrix. In other words,

\[
\left[ \begin{array}{c}
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \\
\nabla_v f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x))
\end{array} \right] = \left[ \begin{array}{c}
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \\
\nabla_v f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x))
\end{array} \right]^T.
\]

\[
\left[ \begin{array}{c}
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \\
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x))
\end{array} \right] = \left[ \begin{array}{c}
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x)) \\
\nabla_x f_0(x^1, R_{x^1}v)\nabla_v (R_{x^1}^2(v,x))
\end{array} \right]^T.
\]
which implies that
\[ (-v^T) \left( \nabla_x f_0(x, R_{x_1} v) \nabla_x (R^1_{x_1(x,v)}) \right) + (-v^T) \left[ \nabla_v f_0(x, R_{x_1} v) \nabla_x (-2A_{x_1}^{v,x_1(x,v)}) \right] = 0, \]
(4.41)
due to (4.26) and (4.30). Therefore, the left-hand side in (4.35) becomes
\[ (-v^T) \nabla_x f_0(x, R_{x_1} v) + (-v^T) \nabla_x f_0(x, R_{x_1} v) R_{x_1} + (-v^T) \nabla_x f_0(x_1, R_{x_1} v)(-2A_{v,x_1}). \]
(4.42)
Using (4.41), the right-hand side in (4.35) is
\[ (-v^T) \left[ \nabla_x f_0(x_1, R_{x_1} v) \nabla_x (R^1_{x_1(x,v)}) \right] + (-v^T) \left[ \nabla_v f_0(x_1, R_{x_1} v) \nabla_x (-2A_{x_1}^{v,x_1(x,v)}) \right] + (-v^T) R_{x_1} \nabla_x f_0(x_1, R_{x_1} v) R_{x_1} + (-v^T) R_{x_1} \nabla_x f_0(x_1, R_{x_1} v)(-2A_{v,x_1}) \]
(4.43)
From (4.42) and (4.43), we derive (4.35) under the assumption (4.21), (4.24), (4.28), and (4.31). For the left-hand side in (4.38), we use (3.3), the C¹ compatibility condition (3.25), (4.21)–(4.24), and (4.41):
\[ -\nabla_x f_0(x_1, v) + tv^T \nabla_x f_0(x_1, v) + (-v^T) \nabla_x f_0(x_1, v) \]
\[ = -\nabla_x f_0(x_1, v) + tv^T R_{x_1} R_{x_1} - \nabla_x f_0(x_1, v)(-2A_{v,x_1}) \]
\[ + tv^T R_{x_1} \nabla_x f_0(x_1, v) R_{x_1} + tv^T R_{x_1} \nabla_x f_0(x_1, v)(-2A_{v,x_1}) \]
\[ + (-v^T) R_{x_1} \nabla_x f_0(x_1, v) R_{x_1} + (-v^T) R_{x_1} \nabla_x f_0(x_1, v)(-2A_{v,x_1}). \]
Since \( \nabla_x f_0(X_0, v)^T = \nabla_x f_0(R_{x_1} v, v) \) under (4.28), it holds that
\[ \left[ \nabla_v f_0(x_1, v) \nabla_v (-2A_{x_1}^{v,x_1}) \right] = \left[ \nabla_v f_0(x_1, v) \nabla_x (R^1_{x_1}) \right] \cdot \left[ \nabla_v f_0(x_1, v) \nabla_x (-2A_{v,x_1}) \right] \cdot \left[ \nabla_v f_0(x_1, v) \nabla_x (R^2_{x_1}) \right], \]
(4.44)
Since (3.1) in Lemma (3.1) and (4.26), (4.28), and the formula (4.44) above, it follows that
\[ \nabla_v f_0(x_1, v)(-2A_{v,x_1}) = C(R_{x_1} v)^T (-2A_{v,x_1}) = \frac{2C}{v \cdot n(x_1)} v^T (Qv) \otimes (Qv) = 0, \]
(4.45)
where \( C \) is an arbitrary constant. And then, one obtains that
\[ -\nabla_x f_0(x_1, v) + tv^T \nabla_x f_0(x_1, v) + (-v^T) \nabla_x f_0(x_1, v) \]
\[ = -\nabla_x f_0(x_1, v) + tv^T R_{x_1} \nabla_x f_0(x_1, v) R_{x_1} + tv^T R_{x_1} \nabla_x f_0(x_1, v)(-2A_{v,x_1}) \]
(4.46)
By (3.3) and (4.31), the right-hand side in (4.35) is
\[ -\nabla_x f_0(x_1, v) + tv^T R_{x_1} \nabla_x f_0(x_1, v)(R_{x_1} + 2t A_{v,x_1}) \]
\[ + (-v^T) R_{x_1} \nabla_x f_0(x_1, v)(R_{x_1} + 2t A_{v,x_1}) \]
\[ = -\nabla_x f_0(x_1, v) + tv^T R_{x_1} \nabla_x f_0(x_1, v)(-2A_{v,x_1}) \]
(4.47)
where $C$ is an arbitrary constant. Thus, the left-hand side in $4.38$ is the same as the right-hand side in $4.38$ under $4.21, 4.24, 4.28$ and $4.31$. The left-hand side in $4.39$ is as follows:

$$(-v^T)\nabla_{xx}f_0(x^1, v) = (-v^T)R_{xx}\nabla_{xx}f_0(x^1, R_{x^1}v)R_{x^1} + (-v^T)\nabla_{xx}f_0(x^1, R_{x^1}v)(-2A_{v,x^1}),$$

by $4.42$. Using $4.41$, the right-hand side in $4.39$ can be further computed by

$$(-v^T)\left[\nabla_{xx}f_0(x^1, R_{x^1}v)\nabla_x(R_{x^1}v)\right] + (-v^T)\left[\nabla_{xx}f_0(x^1, R_{x^1}v)\nabla_x(-2A_{v,x^1})\right] + (-v^T)R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)R_{x^1} + (-v^T)R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)(-2A_{v,x^1}).$$

Hence, the $4.39$ condition can be deduced by $4.21, 4.24, 4.28$, and $4.31$. Finally, the $4.40$ condition is the last remaining case. The left-hand side in $4.40$ comes from $4.46$:

$$-\nabla_xf_0(x^1, v) + tv^T\nabla_{xx}f_0(x^1, v) + (-v^T)\nabla_{xx}f_0(x^1, v)$$

$$= -\nabla_xf_0(x^1, R_{x^1}v)R_{x^1} + tv^T R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)R_{x^1} + tv^T R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)(-2A_{v,x^1})$$

$$(+ (-v^T)R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)R_{x^1} + (-v^T)R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)(-2A_{v,x^1})).$$

The right-hand side in $4.40$ is simplified as

$$-\nabla_xf_0(x^1, R_{x^1}v)R_{x^1} + tv^T \left[\nabla_{xx}f_0(x^1, R_{x^1}v)\nabla_x(R^2_{x^1}v)\right] + tv^T \left[\nabla_{xx}f_0(x^1, R_{x^1}v)\nabla_x(-2A_{v,x^1})\right]$$

$$+ tv^T R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)R_{x^1} + tv^T R_{x^1}\nabla_{xx}f_0(x^1, R_{x^1}v)(-2A_{v,x^1}).$$

Hence, the $4.40$ condition can be obtained under $4.21, 4.24, 4.28$, and $4.31$.

### 4.7. Proof of Theorem 1.3

**Proof of Theorem 1.3** By the same argument of the proof of Theorem 1.2, it is suffices to set $k = 1$. Through this section, we have shown that $4.21, 4.24, 4.28$, and $4.31$ yields $C^2_{x,v}$ regularity of $f(t, x, v)$ of $4.6$. However, $4.28$ is just obvious consequences from $4.1$ and $4.24$ is identical $4.21$ since we assume $1.9$ which is the same as $4.28$ and $4.31$. So, we omit $4.28$ and $4.24$ in the statement.
Now let us change (4.24) and (4.22) into symmetric forms. First, we multiply (4.24) by \( R_x \) from both left and right. Then applying (4.23) and (3.1), we obtain

\[
R_x \left[ \nabla_x f_0(x^1, v) + \nabla_{yx} f_0(x^1, v) \frac{(Qv) \otimes (Qv)}{v \cdot n(x^1)} \right] R_x = \nabla_x f_0(x^1, R_x v) + \nabla_{yx} f_0(x^1, R_x v) \frac{(QR_x \nabla v \otimes (QR_x \nabla v)}{R_x v \cdot n(x^1)}
\]

Also, plugging the above into (4.22) and using (3.1) again, we obtain

\[
R_x \left[ \nabla_{xx} f_0(x^1, v) + \nabla_{xv} f_0(x^1, v) \frac{(Qv) \otimes (Qv)}{v \cdot n(x^1)} \right] R_x = \nabla_{xx} f_0(x^1, R_x v) + \nabla_{xv} f_0(x^1, R_x v) \frac{(QR_x \nabla v \otimes (QR_x \nabla v)}{R_x v \cdot n(x^1)}
\]

By Lemma 4.1 and Lemma 4.2, \( \nabla_x (R_{x^1(x,v)}) \), \( \nabla_x (R_{x^2(x,v)}) \), \( \nabla_x (A_{v,x^1(x,v)}) \), and \( \nabla_x (A_{v,x^2(x,v)}) \) depend only on \( n(x^1) \) and \( v \). Since \( n(x^1) = x^1 \), we rewrite \( x^1 \) as \( x \) for \( (x,v) \in \gamma_- \) and then we obtain (1.10) and (1.11).

5. Regularity estimate of \( f \)

5.1. First order estimates of characteristics. Using Definition 1.1

\[
V(0;t,x,v) = R_\ell R_{\ell-1} \cdots R_2 R_1 v,
\]

where

\[
R_j = I - 2n(x^j) \otimes n(x^j).
\]

For above \( \ell \),

\[
X(0;t,x,v) = x^\ell - v^\ell t^\ell,
\]

where inductively,

\[
x^k = x^{k-1} - v^{k-1} (t^{k-1} - t^k), \quad 2 \leq k \leq \ell,
\]

and

\[
x^1 = x - v(t - t^1) = x - vt_b.
\]

Or using rotational symmetry, we can also express

\[
x^\ell = Q^\ell n(x^1),
\]

where \( Q_\theta \) is operator (matrix) which means rotation(on the boundary of the disk) by \( \theta \). \( \theta \) is uniquely determined by its first (backward in time) bounce angle \( v \cdot n(x_b) \).

Lemma 5.1. Here, \( \theta \) is the angle at which \( v \) is rotated to \( v^1 \). Moreover, \( \theta > 0 \) is same as the angle of rotation from \( x^k \) to \( x^{k+1} \) for \( k = 1, 2, \cdots, \ell - 1 \). Then, derivatives of \( \theta \) with respect to \( x \) and \( v \) are

\[
\nabla_x \theta = -\frac{2}{\sin \theta} Q - \frac{1}{\sin \theta} n(x^1), \quad \nabla_v \theta = 2 \left( \frac{t_b}{2} - \frac{1}{|v|} \right) Q - \frac{1}{\sin \theta} n(x^1),
\]

provided \( n(x^1) \cdot v \neq 0 \).

Proof. From the definition of \( \theta \),

\[
\cos \left( \frac{\theta}{2} \right) = \sin \left( \frac{\theta}{2} \right) = - \left[ n(x^1) \cdot \frac{v}{|v|} \right].
\]

\[
\]
Thus, taking $\nabla_x$ yields
\[
\frac{1}{2} \cos \frac{\theta}{2} \nabla \frac{\partial}{\partial \theta} = -\frac{v}{|v|} \nabla_x \left( n(x^1) \right) = -\frac{v}{|v|} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) = -\frac{v}{|v|} + \frac{|v|}{v \cdot n(x^1)} n(x^1),
\]
where we used the product rule in Lemma 2.2 and (2.3) in Lemma 2.1. Note that rotating an angle $\phi = \frac{\pi}{2} - \frac{\theta}{2} > 0$ on a normal vector $n(x^1)$ gives the vector $-\frac{n}{|v|}$. In other words, it holds that
\[
-\frac{v}{|v|} = Q_\phi n(x^1), \tag{5.3}
\]
where $Q_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \\ \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \end{bmatrix}$. Thus,
\[
\nabla_x = \frac{2}{\cos \frac{\theta}{2}} \left( Q_\phi - \frac{1}{\sin \frac{\theta}{2}} I \right) n(x^1) = \frac{2}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}} \left[ \begin{array}{c} \sin^2 \frac{\theta}{2} - 1 - \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ \sin^2 \frac{\theta}{2} - 1 - \frac{\theta}{2} \end{array} \right] n(x^1)
\]
\[
= -\frac{2}{\sin \frac{\theta}{2}} \left[ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right] n(x^1) = -\frac{2}{\sin \frac{\theta}{2}} Q_\phi n(x^1).
\]
Similarly, taking the derivative $\nabla_v$ of both sides in (5.2):
\[
\frac{1}{2} \cos \frac{\theta}{2} \nabla_v = -\frac{v}{|v|} \nabla_v \left( n(x^1) \right) - n(x^1) \left( \frac{1}{|v|} - \frac{v \otimes n(x^1)}{|v|^2} \right) = t_b \frac{v}{|v|} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - n(x^1) \left( \frac{1}{|v|} - \frac{v \otimes n(x^1)}{|v|^2} \right)
\]
\[
= t_b \frac{v}{|v|} - t_b \frac{|v|}{v \cdot n(x^1)} n(x^1) - \frac{1}{|v|} n(x^1) + \frac{v \cdot n(x^1)}{|v|^2} v,
\]
where we used the product rule in Lemma 2.2 and (2.3) in Lemma 2.1. From (5.3),
\[
\nabla_v = \frac{2}{\cos \frac{\theta}{2} \sin \frac{\theta}{2}} \left( -t_b \sin \frac{\theta}{2} \left[ Q_\phi - \frac{1}{\sin \frac{\theta}{2}} I \right] n(x^1) + \frac{\sin^2 \frac{\theta}{2}}{|v|} \left[ Q_\phi - \frac{1}{\sin \frac{\theta}{2}} I \right] n(x^1) \right)
\]
\[
= \frac{2}{\sin \frac{\theta}{2}} \left[ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right] n(x^1) - \frac{2}{|v|} \left[ \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right] n(x^1)
\]
\[
= 2 \left( \frac{t_b}{|v|} - \frac{1}{|v|} \right) Q_\phi n(x^1).
\]

**Lemma 5.2.** Let $(t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$. The specular characteristics $X(0; t, x, v)$ and $V(0; t, x, v)$ are defined in Definition 1.7. Whenever $n(x^1) \cdot v \neq 0$, we have derivatives of the characteristics $X(0; t, x, v)$ and $V(0; t, x, v)$:
\[
\nabla_x X(0; t, x, v) = Q_\theta^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) + t_l Q_{\theta^-} (v \otimes \nabla_x \theta) - \frac{1}{|v|} \sin \frac{\theta}{2} Q_\theta (v \cdot n(x^1))
\]
\[
- \frac{|v|(t - t_b - t^l)}{2} Q_{(l-\frac{1}{2})\theta^-} (n(x^1) \otimes \nabla_x \theta),
\]
\[
\nabla_v X(0; t, x, v) = -t_b Q_\theta^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - t_l Q_{\theta^-} + t_l Q_{\theta^-} (v \otimes \nabla_v \theta) + \frac{t_b}{|v|} \sin \frac{\theta}{2} Q_\theta (v \cdot n(x^1))
\]
\[
- \frac{2(t - 1) \sin \frac{\theta}{2} Q_\theta (v \cdot n(x^1)) - |v|(t - t_b - t^l)}{2} Q_{(l-\frac{1}{2})\theta^-} (n(x^1) \otimes \nabla_v \theta), \tag{5.4}
\]
\[
\nabla_x V(0; t, x, v) = -l Q_{\theta^-} (v \otimes \nabla_x \theta),
\]
\[
\nabla_v V(0; t, x, v) = Q_\theta - l Q_{\theta^-} (v \cdot \nabla_v \theta),
\]
where $\theta$ is the angle given in Lemma 5.1. $t_b$ is the backward exit time defined in Definition 1.2. $l$ is the bouncing number, and $Q_\theta$ is a rotation matrix by $\theta$.

**Proof.** Remind
\[
X(0; t, x, v) = x^l - v^l t^l, \quad V(0; t, x, v) = v^l.
\]
Using the rotation matrix \( Q_{\theta} \), \( x^l \) and \( v^l \) can be expressed by
\[
x^l = Q_{\theta}^{l-1} x^1, \quad v^l = Q_{\theta}^{l} v.
\] (5.5)

By the chain rule,
\[
\frac{\partial (X(0; t, x, v), V(0; t, x, v))}{\partial (x, v)} = \frac{\partial (X(0; t, x, v), V(0; t, x, v))}{\partial (t^l, x^l, v^l)} \frac{\partial (t^l, x^l, v^l)}{\partial (x, v)} = \left[ -v^l I - t^l I \right] \left[ \begin{array}{cc} \nabla_x t^l & \nabla_x v^l \\ \nabla_v x^l & \nabla_v v^l \end{array} \right],
\]
where \( I \) is a 2 \( \times \) 2 identity matrix. For the derivative of \( X(0; t, x, v), V(0; t, x, v) \), it is necessary to find the derivative of \( t^l, x^l, \) and \( v^l \). Using the expression (5.5) and (2.1) in Lemma 2.2, we derive
\[
\nabla_x x^l = \nabla_x \left[ Q_{\theta}^{l-1} x^1 \right] = Q_{\theta}^{l-1} \nabla_x x^1 - (l - 1) \left( \begin{array}{c} \sin(l - 1) \theta \\ \cos(l - 1) \theta \\ -\cos(l - 1) \theta \\ \sin(l - 1) \theta \end{array} \right) x^1 \otimes \nabla_x \theta,
\]
\[
\nabla_v x^l = \nabla_v \left[ Q_{\theta}^{l-1} x^1 \right] = Q_{\theta}^{l-1} \nabla_v x^1 - (l - 1) \left( \begin{array}{c} \sin(l - 1) \theta \\ \cos(l - 1) \theta \\ -\cos(l - 1) \theta \\ \sin(l - 1) \theta \end{array} \right) x^1 \otimes \nabla_v \theta,
\]
\[
\nabla_x v^l = \nabla_x \left[ Q_{\theta}^{l} v \right] = -t_b Q_{\theta}^{l-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - (l - 1) \left( \begin{array}{c} \sin(l - 1) \theta \\ \cos(l - 1) \theta \\ -\cos(l - 1) \theta \\ \sin(l - 1) \theta \end{array} \right) x^1 \otimes \nabla_x \theta,
\]
\[
\nabla_v v^l = \nabla_v \left[ Q_{\theta}^{l} v \right] = Q_{\theta}^{l} - l \left( \begin{array}{c} \sin l \theta \\ \cos l \theta \\ -\cos l \theta \\ \sin l \theta \end{array} \right) v \otimes \nabla_v \theta.
\]
For the derivative of \( t^l \), we rewrite it as
\[
t^l = t - (t - t^1) - \sum_{k=1}^{l-1} (t^k - t^{k+1}) = t - t_b - \sum_{k=1}^{l-1} (t^k - t^{k+1}).
\]
Since \( t^k - t^{k+1} = \frac{2 \sin \theta}{|v|} \) for all \( k = 1, 2, \ldots, l - 1 \), it holds that
\[
t^l = t - t_b - \frac{2(l - 1) \sin \theta}{|v|}, \quad l - 1 = \frac{|v|}{2 \sin \theta} (t - t_b - t^l).
\] (5.6)

Taking the derivative of \( t^l \) with respect to \( x, v \)
\[
\nabla_x t^l = -\nabla_x t_b - \frac{(l - 1) \cos \theta}{|v|} \nabla_x \theta = -\frac{n(x^1)}{|v|} - \frac{(l - 1) \cos \theta}{|v|} \nabla_x \theta = \frac{1}{|v|} \frac{n(x^1)}{\sin \theta} - \frac{(l - 1) \cos \theta}{|v|} \nabla_x \theta,
\]
\[
\nabla_v t^l = -\nabla_v t_b + \frac{2(l - 1) \sin \theta}{|v|^3} v - \frac{(l - 1) \cos \theta}{|v|} \nabla_v \theta = t_b \frac{n(x^1)}{|v|^3} + \frac{2(l - 1) \sin \theta}{|v|^3} v - \frac{(l - 1) \cos \theta}{|v|} \nabla_v \theta
\] = \frac{t_b}{|v|^3} \frac{n(x^1)}{\sin \theta} + \frac{2(l - 1) \sin \theta}{|v|^3} v - \frac{(l - 1) \cos \theta}{|v|} \nabla_v \theta.
\] (5.7)

Also note that, from (5.3) and (5.6), we have
\[
-(l - 1) Q_{(l-1)\theta - \frac{\pi}{2}} (x^1 \otimes \nabla \theta) + \frac{(l - 1) \cos \theta}{|v|} Q_{\theta}^{l} (v \otimes \nabla \theta)
\]
\[
= -(l - 1) \left( Q_{(l-1)\theta - \frac{\pi}{2}} + \cos \theta \cos \frac{\pi}{2} Q_{\theta}^{l} \cos \frac{\pi}{2} - \frac{\pi}{2} \right) (n(x^1) \otimes \nabla \theta)
\]
\[
= -(l - 1) Q_{(l-1)\theta - \frac{\pi}{2}} \left[ \begin{array}{cc} \sin \theta \cos \theta & \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta \end{array} \right] (n(x^1) \otimes \nabla \theta)
\]
\[
= \frac{|v|(t - t_b - t^l)}{2} Q_{(l-1)\theta - \pi} (n(x^1) \otimes \nabla \theta).
\] (5.8)
Hence, using (3.3) and $x^1 = n(x^1)$,
\[
\nabla_x X(0; t, x, v) = \nabla_x x^l - t^l \nabla_x v^l - v^l \nabla_x t^l
\]
\[
= Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - (l - 1) \left( \begin{bmatrix} \sin(l - 1)\theta & \cos(l - 1)\theta \\ -\cos(l - 1)\theta & \sin(l - 1)\theta \end{bmatrix} x^1 \right) \otimes \nabla_x \theta 
+ t^l \left( \begin{bmatrix} \sin l\theta & \cos l\theta \\ -\cos l\theta & \sin l\theta \end{bmatrix} v \right) \otimes \nabla_x \theta - \frac{1}{|v|} Q_{\theta}^l \left( v \otimes n(x^1) \right) + \frac{(l - 1) \cos \frac{\theta}{2}}{|v|} Q_{\theta}^l \left( v \otimes \nabla_x \theta \right) 
- (l - 1)Q_{(l-1)\theta-\frac{\pi}{2}}(x^1 \otimes \nabla_x \theta) + \frac{(l - 1) \cos \frac{\theta}{2}}{|v|} Q_{\theta}^l \left( v \otimes \nabla_x \theta \right)
\]
\[
= Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) + t^l Q_{(l-1)\theta-\frac{\pi}{2}}(v \otimes \nabla_x \theta) - \frac{1}{|v|} Q_{\theta}^l \left( v \otimes n(x^1) \right) 
- \frac{|v|(t - t_b - t^l)}{2} Q_{(l-1)\theta-\pi} \left( n(x^1) \otimes \nabla_x \theta \right),
\]

\[
\nabla_v X(0; t, x, v) = \nabla_v x^l - t^l \nabla_v v^l - v^l \nabla_v t^l
\]
\[
= -t_b Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - (l - 1) \left( \begin{bmatrix} \sin(l - 1)\theta & \cos(l - 1)\theta \\ -\cos(l - 1)\theta & \sin(l - 1)\theta \end{bmatrix} x^1 \right) \otimes \nabla_v \theta 
- t^l Q_{\theta} + t^l \left( \begin{bmatrix} \sin l\theta & \cos l\theta \\ -\cos l\theta & \sin l\theta \end{bmatrix} v \right) \otimes \nabla_v \theta 
+ \frac{t_b}{|v|} Q_{\theta} \left( v \otimes n(x^1) \right) - \frac{2(l - 1) \sin \frac{\theta}{2}}{|v|^3} Q_{\theta}^l \left( v \otimes v \right) + \frac{(l - 1) \cos \frac{\theta}{2}}{|v|} Q_{\theta}^l \left( v \otimes \nabla_v \theta \right),
\]
\[
= -t_b Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - t^l Q_{\theta} + t^l Q_{(l-1)\theta-\frac{\pi}{2}}(v \otimes \nabla_v \theta) + \frac{t_b}{|v|} Q_{\theta} \left( v \otimes n(x^1) \right) 
- \frac{2(l - 1) \sin \frac{\theta}{2}}{|v|^3} Q_{\theta}^l \left( v \otimes v \right) - (l - 1)Q_{(l-1)\theta-\frac{\pi}{2}}(x^1 \otimes \nabla_v \theta) + \frac{(l - 1) \cos \frac{\theta}{2}}{|v|} Q_{\theta}^l \left( v \otimes \nabla_v \theta \right)
\]
\[
= -t_b Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x^1)}{v \cdot n(x^1)} \right) - t^l Q_{\theta} + t^l Q_{(l-1)\theta-\frac{\pi}{2}}(v \otimes \nabla_v \theta) + \frac{t_b}{|v|} Q_{\theta} \left( v \otimes n(x^1) \right) 
- \frac{2(l - 1) \sin \frac{\theta}{2}}{|v|^3} Q_{\theta}^l \left( v \otimes v \right) - \frac{|v|(t - t_b - t^l)}{2} Q_{(l-1)\theta-\pi} \left( n(x^1) \otimes \nabla_v \theta \right),
\]

\[
\nabla_x V(0; t, x, v) = \nabla_x v^l = -l \left( \begin{bmatrix} \sin l\theta & \cos l\theta \\ -\cos l\theta & \sin l\theta \end{bmatrix} v \right) \otimes \nabla_x \theta = -l Q_{(l-1)\theta-\pi} \left( v \otimes \nabla_x \theta \right),
\]
\[
\nabla_v V(0; t, x, v) = \nabla_v v^l = Q_{\theta}^l - l \left( \begin{bmatrix} \sin l\theta & \cos l\theta \\ -\cos l\theta & \sin l\theta \end{bmatrix} v \right) \otimes \nabla_v \theta = Q_{\theta}^l - l Q_{(l-1)\theta-\pi} \left( v \otimes \nabla_v \theta \right).
\]

\[\square\]

**Lemma 5.3.** The exit backward time $t_b$ and the $l$-th bouncing backward time $t^l$ are defined in Definition 7.7. Then, it holds that
\[
t_b \leq \frac{2 \sin \frac{\theta}{2}}{|v|}, \quad t^l \leq \frac{2 \sin \frac{\theta}{2}}{|v|}.
\]

**Proof.** Note that
\[
t_b = t - t^l = \frac{|x - x^1|}{|v|}, \quad t^l = \frac{|x^l - X(0; t, x, v)|}{|v^l|}.
\]
Whenever $\theta$ is the angle at which $v$ is rotated to $v^l$, one obtains that
\[
|x - x^1| \leq 2 \sin \frac{\theta}{2}, \quad |x^l - X(0; t, x, v)| \leq 2 \sin \frac{\theta}{2}.
\]
Lemma 5.4. Under the same assumption in Lemma 5.2 we have estimates of derivatives for the characteristics $X(0; t, x, v)$ and $V(0; t, x, v)$

$$|\nabla_x X(0; t, x, v)| \lesssim \frac{|v|}{|v \cdot n(x_b)|} (1 + |v| t),$$

$$|\nabla_v X(0; t, x, v)| \lesssim \frac{1}{|v|} (1 + |v| t),$$

$$|\nabla_x V(0; t, x, v)| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v| t),$$

$$|\nabla_v V(0; t, x, v)| \lesssim \frac{|v|}{|v \cdot n(x_b)|} (1 + |v| t),$$

where $n(x_b)$ is outward unit normal vector at $x_b = x - t_b v \in \partial \Omega$.

Remark 5.5. First order derivatives of characteristics $(X, V)$ for general 3D convex domain was obtained in [15]. Lemma 5.4 is simple version in 2D disk and its singular orders coincide with the results of [15].

Proof. By (5.1) in Lemma 5.2 we have

$$\nabla_x X(0; t, x, v) = Q_{\theta}^{-1} \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) - \frac{|v|}{|v|} t_b - t^l \right) \frac{1}{2} Q_{(l+\frac{1}{2})\theta-x} (n(x_b) \otimes \nabla_x \theta)

+ t^l \frac{1}{2} Q_{\theta-x} (v \otimes \nabla_x \theta) - \frac{1}{|v|} Q_{\theta}(v \otimes n(x_b)).$$

We define a matrix norm by

$$|A| = \max_{i,j} a_{i,j},$$

where $a_{i,j}$ is the $(i, j)$ component of the matrix $A$. Then, we can easily check that

$$|a \otimes b| \leq |a| |b|,$$

for any $a, b \in \mathbb{R}^n$. To find upper bound of $\nabla_x X(0; t, x, v)$, we only need to consider $\nabla_x \theta$ and $t^l \times I$. By (5.1), (5.6) and (5.9),

$$|\nabla_x \theta| = \left| \frac{2}{\sin \frac{\theta}{2}} Q_{\theta-x} n(x_b) \right| \leq \frac{2|v|}{|v \cdot n(x_b)|},$$

$$t^l \times I \leq \frac{2\sin \frac{\theta}{2}}{|v|} \times \left( \frac{|v|}{2\sin \frac{\theta}{2} t + 1} \right) \leq t + \frac{2}{|v|}. \quad (5.10)$$

Using the above inequalities, we derive that

$$|\nabla_x X(0; t, x, v)| \leq \frac{|v|}{|v \cdot n(x_b)|} + \frac{|v| t}{2 |v \cdot n(x_b)|} |\nabla_x \theta| + t^l |v| |\nabla_x \theta| + \frac{1}{|v \cdot n(x_b)|} |v|

\leq 1 + \frac{|v|}{|v \cdot n(x_b)|} + \frac{|v|^2}{|v \cdot n(x_b)|^2} t + \frac{2|v|^2}{|v \cdot n(x_b)|} \left( t + \frac{2}{|v|} \right) + \frac{|v|}{|v \cdot n(x_b)|}

\leq \frac{|v|}{|v \cdot n(x_b)|} (1 + |v| t).$$

□
Remind the derivative $\nabla_v X(0; t, x, v)$ in Lemma 5.2

\[
\nabla_v X(0; t, x, v) = -t_b Q^{-1}_\theta \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) - \frac{|v|(t - t_b - t^l)}{2} Q_{(l-\frac{1}{2})\theta - \pi} (n(x_b) \otimes \nabla_v \theta)
\]

\[
- t^l Q^l_\theta + t^l l Q_{l \theta - \frac{\pi}{2}} (v \otimes \nabla_v \theta) + \frac{t_b}{|v||\sin \frac{\theta}{2}} Q^l_\theta (v \otimes n(x_b)) - \frac{2(l-1)\sin \frac{\theta}{2}}{|v|^3} Q^l_\theta (v \otimes v).
\]

Similarly, to estimate $\nabla_v X(0; t, x, v)$, we need to estimate $\nabla_v \theta$. From (5.11) and (5.10), we directly compute

\[
|\nabla_v \theta| = 2 \left| \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) Q_{-\frac{\theta}{2}} n(x_b) \right| \leq \frac{6}{|v|}.
\]

Thus,

\[
|\nabla_v X(0; t, x, v)| \leq t_b \left( 1 + \frac{|v|}{|v \cdot n(x_b)|} \right) + \frac{|v||t|}{2} |\nabla_v \theta| + t^l + (t^l \times l)|v||\nabla_v \theta| + \frac{t_b}{|v||\sin \frac{\theta}{2}} |v| + \frac{2(l-1)\sin \frac{\theta}{2}}{|v|^3} |v|^2
\]

\[
\leq \frac{2 \sin \frac{\theta}{2}}{|v|} \left( 1 + \frac{|v|}{|v \cdot n(x_b)|} \right) + \frac{|v||t|}{2} \times \frac{6}{|v|} + \frac{2 \sin \frac{\theta}{2}}{|v|} + \frac{6 + \frac{2}{|v|}}{2} + \frac{2 \sin \frac{\theta}{2}}{|v|} |v| + \frac{t - t_b - t^l}{|v|^2} |v|^2
\]

\[
\leq \frac{1}{|v|} (1 + |v||t|),
\]

where we used (5.9) and (5.10). For $\nabla_x V(0; t, x, v)$, using (5.11), (5.12), (5.10), and (5.11) gives

\[
|\nabla_x V(0; t, x, v)| = | -l Q_{l \theta - \frac{\pi}{2}} (v \otimes \nabla_x \theta) | \leq \left( \frac{|v|}{2 \sin \frac{\theta}{2}} |t| + 1 \right) |v||\nabla_x \theta| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v||t|),
\]

\[
|\nabla_v V(0; t, x, v)| = \left| Q^l_\theta - l Q_{l \theta - \frac{\pi}{2}} (v \otimes \nabla_v \theta) \right| \leq 1 + \left( \frac{|v|}{2 \sin \frac{\theta}{2}} |t| + 1 \right) |v||\nabla_v \theta| \lesssim \frac{|v|}{|v \cdot n(x_b)|} (1 + |v||t|).
\]

5.2 Second order estimates of characteristics.

**Lemma 5.6**. $n(x_b)$ is outward unit normal vector at $x_b \in \partial \Omega$. For $(x_b, v) \notin \gamma_0$, it follows that

\[
|\nabla_x [n(x_b)]| \lesssim \frac{|v|}{|v \cdot n(x_b)|}, \quad |\nabla_v [n(x_b)]| \lesssim \frac{1}{|v|}.
\]

**Proof.** We denote the components of $v$ and $n(x_b)$ by $(v_1, v_2)$ and $(n_1, n_2)$. By (2.3) in Lemma 2.4, we have

\[
\nabla_x [n(x_b)] = I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} = \frac{1}{v \cdot n(x_b)} \begin{bmatrix} v_2 n_2 & -v_1 n_2 \\ -v_2 n_1 & v_1 n_1 \end{bmatrix},
\]

\[
\nabla_v [n(x_b)] = -t_b \left( I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) = \frac{-t_b}{v \cdot n(x_b)} \begin{bmatrix} v_2 n_2 & -v_1 n_2 \\ -v_2 n_1 & v_1 n_1 \end{bmatrix},
\]

which by (5.9) is further bounded by

\[
|\nabla_x [n(x_b)]| \lesssim \frac{|v|}{|v \cdot n(x_b)|}, \quad |\nabla_v [n(x_b)]| \lesssim \frac{1}{|v|}.
\]

\[\Box\]

**Lemma 5.7**. The exit backward time $t_b$ and the $l$-th bouncing backward time $t^l$ are defined in Definition [1.7]. Then, we have the following estimates

\[
|\nabla_x t^l| \lesssim \frac{1}{|v||\sin \frac{\theta}{2}|}, \quad |\nabla_v t^l| \lesssim \frac{1}{|v|^2},
\]

\[
|\nabla_x t^l| \lesssim \frac{1}{|v||\sin \frac{\theta}{2}|} (1 + |v||t|), \quad |\nabla_v t^l| \lesssim \frac{1}{|v|^2 |\sin \frac{\theta}{2}|} (1 + |v||t|),
\]

whenever $v \cdot n(x_b) \neq 0$. 

\[\Box\]
Proof. Since \( t^1 = t - t_b \), it follows from Lemma 5.2 that
\[
\nabla_{x} t^1 = -\nabla_{x} t_b = -\frac{n(x_b)}{v \cdot n(x_b)}, \quad \nabla_{v} t^1 = -\nabla_{v} t_b = t_b \frac{n(x_b)}{v \cdot n(x_b)}.
\]
Using the above and (5.9) implies that
\[
|\nabla_x t^1| \lesssim \frac{1}{|v| |\sin \frac{\theta}{2}|}, \quad |\nabla_v t^1| \lesssim \frac{1}{|v|^2}.
\]
By (5.7) in the proof of Lemma 5.2 we have
\[
\nabla_{x} t^1 = \frac{1}{|v| \sin \frac{\theta}{2}} n(x_b) - \frac{(l-1) \cos \frac{\theta}{2}}{|v|} \nabla_{x} \theta,
\]
\[
\nabla_{v} t^1 = -\frac{t_b}{|v| \sin \frac{\theta}{2}} n(x_b) + \frac{2(l-1) \sin \frac{\theta}{2}}{|v|} v - \frac{(l-1) \cos \frac{\theta}{2}}{|v|} \nabla_{v} \theta.
\]
By (5.6) in the proof of Lemma 5.2 the bouncing number \( l \) can be bounded by
\[
l = 1 + \frac{|v|}{2 \sin \frac{\theta}{2}} (t - t_b - t^1) \leq 1 + \frac{|v|}{2 \sin \frac{\theta}{2}} t \lesssim \frac{1}{|v|^2 \sin \frac{\theta}{2}} (1 + |v| t).
\] (5.14)
Then, from (5.9), (5.10), (5.11), and (5.14), one obtains that
\[
|\nabla_x t^1| \lesssim \frac{1}{|v| |\sin \frac{\theta}{2}|} + \frac{1}{|v| \sin^2 \frac{\theta}{2}} (1 + |v| t) \lesssim \frac{1}{|v| \sin^2 \frac{\theta}{2}} (1 + |v| t),
\]
\[
|\nabla_v t^1| \lesssim \frac{1}{|v|^2} + \frac{1}{|v|^2} (1 + |v| t) + \frac{1}{|v|^2} (1 + |v| t) \lesssim \frac{1}{|v|^2} (1 + |v| t).
\]

Lemma 5.8. The characteristics \( X(0; t, x, v) \) and \( V(0; t, x, v) \) are defined in Definition 2.1. Under the same assumption in Lemma 5.2, we have estimates for the second derivatives of characteristics
\[
|\nabla_{xx} X(0; t, x, v)| \lesssim \frac{|v|^4}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2), \quad |\nabla_{vx} X(0; t, x, v)| \lesssim \frac{|v|^2}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2),
\]
\[
|\nabla_{vx} V(0; t, x, v)| \lesssim \frac{|v|^2}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2), \quad |\nabla_{xx} V(0; t, x, v)| \lesssim \frac{1}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2),
\]
\[
|\nabla_{xx} V(0; t, x, v)| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2), \quad |\nabla_{vx} V(0; t, x, v)| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2),
\]
where \(|\nabla_{xx},x,v,v,X(0; t, x, v)|\) or \(|V(0; t, x, v)|\) is given by \( \sup_{i,j} |\nabla_{ij} X(0; t, x, v)| \) or \( |\nabla_{ij} V(0; t, x, v)| \) for \( i, j \in \{x_1, x_2, v_1, v_2\} \).

Proof. We denote the components of \( v \) and \( n(x_b) \) by \( (v_1, v_2) \) and \( (n_1, n_2) \). To estimate \(|\nabla_{xx} X(0; t, x, v)|\), we only consider the component in the matrix \( \nabla_x X(0; t, x, v) \) satisfying inequalities in Lemma 5.4. In fact, from
Lemma 5.2: the (1,1) component \( [\nabla_x X(0; t, x, v)]_{(1,1)} \) of the matrix \( \nabla_x X(0; t, x, v) \) is

\[
[\nabla_x X(0; t, x, v)]_{(1,1)} = \cos((l-1)\theta) \frac{v_2 n_2}{v \cdot n(x_b)} + \sin((l-1)\theta) \frac{v_2 n_1}{v \cdot n(x_b)} \\
+ \frac{|v|(t^1 - t^i)}{\sin \frac{\theta}{2}} \left( -n_1^2 \cos((l-1)\theta) \cos \frac{\theta}{2} - n_1 n_2 \cos((l-\frac{1}{2})\theta) \sin \frac{\theta}{2} + n_1 n_2 \sin((l-\frac{1}{2})\theta) \cos \frac{\theta}{2} \right) \\
+ n_2^2 \sin((l-\frac{1}{2})\theta) \sin \frac{\theta}{2} \\
- \frac{2t_l}{\sin \frac{\theta}{2}} (v_1 n_1 \sin l\theta \cos \frac{\theta}{2} + v_1 n_2 \sin l\theta \sin \frac{\theta}{2} + v_2 n_1 \cos l\theta \cos \frac{\theta}{2} + v_2 n_2 \cos l\theta \sin \frac{\theta}{2}) \\
\leq \frac{|v|}{|v \cdot n(x_b)|} + \frac{1}{|\sin \frac{\theta}{2}|} (1 + |v| t) + \frac{1}{|\sin \frac{\theta}{2}|} \leq \frac{|v|}{|v \cdot n(x_b)|} (1 + |v| t),
\]

where the first inequality comes from (5.9), (5.14), and

\[
t^1 - t^i = \frac{2(l-1)\sin \frac{\theta}{2}}{|v|} \lesssim \frac{1}{|v|} (1 + |v| t).
\]

Similarly, the (1,1) components of matrices \( \nabla_x X(0; t, x, v), \nabla_x V(0; t, x, v), \) and \( \nabla_x V(0; t, x, v) \) satisfy inequalities in Lemma 5.4. Thus, we only consider (1,1) components of derivative matrices for \( X(0; t, x, v) \) and \( V(0; t, x, v) \) to get estimates. When we differentiate \( [\nabla_x X(0; t, x, v)]_{(1,1)} \) with respect to \( x \), the terms containing \( \frac{t_l}{\sin \frac{\theta}{2}} \) are main terms which increases the singularity \( \frac{1}{\sin \frac{\theta}{2}} \) and travel length \( (1 + |v| t) \) order. \( \frac{t_l}{\sin \frac{\theta}{2}} \)

has a singularity order 1 and travel length order 1 because

\[
\left| \frac{t_l}{\sin \frac{\theta}{2}} \right| \lesssim \frac{1}{|\sin \frac{\theta}{2}|} \frac{|\sin \frac{\theta}{2}|}{|v|} \frac{1}{|\sin \frac{\theta}{2}|} (1 + |v| t) = \frac{|v|}{|v \cdot n(x_b)|} (1 + |v| t),
\]

where we have used (5.9) and (5.14). On the other hand, if we take of the term \( \frac{t_l}{\sin \frac{\theta}{2}} \) with respect to \( x \), the singularity and travel length order become 4 and 2 respectively:

\[
\left| \nabla_x \left( \frac{t_l}{\sin \frac{\theta}{2}} \right) \right| = \left| \frac{l}{\sin \frac{\theta}{2}} \nabla_x t^i - \frac{t_l}{\sin ^2 \frac{\theta}{2} t^2} \nabla_x \theta \right| \lesssim \frac{1}{|v| \sin^4 \frac{\theta}{2}} (1 + |v|^2 t^2) + \frac{1}{|v| \sin^3 \frac{\theta}{2}} (1 + |v| t) \\
\lesssim \frac{|v|^3}{|v \cdot n(x_b)|^4} (1 + |v|^2 t^2),
\]

where (5.9), (5.10), (5.13), and (5.14) have been used. Hence, it suffices to estimate the following terms in \( [\nabla_x X(0; t, x, v)]_{(1,1)} \)

\[
- \frac{2t_l}{\sin \frac{\theta}{2}} (v_1 n_1 l\theta \cos \frac{\theta}{2} + v_1 n_2 l\theta \sin \frac{\theta}{2} + v_2 n_1 \cos l\theta \cos \frac{\theta}{2} + v_2 n_2 \cos l\theta \sin \frac{\theta}{2}) := I_1,
\]
to obtain estimate for $|\nabla_{xx}X(0; t, x, v)|$. Taking $x$ derivative to the above terms, one obtains

$$\nabla_x I_1 = \left( \frac{-2t\nabla_{,t}^t}{\sin^2 \frac{\theta}{2}} + \frac{2t\!l \cos \frac{\theta}{2} \nabla_{,\theta} \nabla_{,t}}{2 \sin^2 \frac{\theta}{2}} \right) \left( v_{n_1} \sin \theta \cos \frac{\theta}{2} + v_{n_2} \sin \theta \cos \frac{\theta}{2} + v_{n_3} \cos \theta \sin \frac{\theta}{2} \right)$$

$$- \frac{-2t\!l}{\sin^2 \frac{\theta}{2}} \left( v_{n_1} \sin \theta \cos \frac{\theta}{2} \nabla_{,n_1} + v_{n_2} \cos \theta \sin \frac{\theta}{2} \nabla_{,n_2} + v_{n_3} \cos \theta \sin \frac{\theta}{2} \nabla_{,n_3} \right).$$

Using (5.9), (5.10), (5.12), (5.13), and (5.14), one can further bound the above as

$$|\nabla_x I_1| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^3} (1 + |v|^2 t^2) \times |v| + \frac{1}{|v \cdot n(x_b)|} (1 + |v| t) \times (|v| \|
abla_x n(x_b)| + l|v| |
abla_x \theta|)$$

$$\lesssim \frac{|v|^4}{|v \cdot n(x_b)|} (1 + |v|^2 t^2).$$

Therefore, we get

$$|\nabla_{xx}X(0; t, x, v)| \lesssim \frac{|v|^4}{|v \cdot n(x_b)|} (1 + |v|^2 t^2).$$

For estimate of $|\nabla_{xx}X(0; t, x, v)|$, similar to the case $|\nabla_{xx}X(0; t, x, v)|$, we only consider terms $I_1$. By taking $v$ derivative to $I_1$, we obtain

$$\nabla_v I_1 = \left( \frac{-2t\nabla_{,t}^t}{\sin^2 \frac{\theta}{2}} + \frac{2t\!l \cos \frac{\theta}{2} \nabla_{,\theta} \nabla_{,t}}{2 \sin^2 \frac{\theta}{2}} \right) \left( v_{n_1} \sin \theta \cos \frac{\theta}{2} + v_{n_2} \sin \theta \cos \frac{\theta}{2} + v_{n_3} \cos \theta \sin \frac{\theta}{2} \right)$$

$$- \frac{-2t\!l}{\sin^2 \frac{\theta}{2}} \left( v_{n_1} \sin \theta \cos \frac{\theta}{2} \nabla_{,n_1} + v_{n_2} \cos \theta \sin \frac{\theta}{2} \nabla_{,n_2} + v_{n_3} \cos \theta \sin \frac{\theta}{2} \nabla_{,n_3} \right).$$

Using (5.9), (5.11), (5.12), (5.13), and (5.14) yields that

$$|\nabla_v I_1| \lesssim \left( \frac{1}{|v|^3 \sin^2 \frac{\theta}{2}} + \frac{1}{|v|^3 \sin^2 \frac{\theta}{2}} \right) (1 + |v|^2 t^2) \times |v| + \frac{1}{|v \cdot n(x_b)|} (1 + |v| \|
abla_v n(x_b)| + l|v| |
abla_v \theta|)$$

$$\lesssim \frac{|v|^2}{|v \cdot n(x_b)|} (1 + |v|^2 t^2).$$

Hence, one obtains that

$$|\nabla_{xx}X(0; t, x, v)| \lesssim \frac{|v|^2}{|v \cdot n(x_b)|} (1 + |v|^2 t^2).$$
By Lemma 5.12, we write the $(1, 1)$ component of $\nabla_v X(0; t, x, v)$:

\[
[\nabla_v X(0; t, x, v)]_{(1,1)}
= -t_b \left( \cos(l - 1) \theta \frac{v_2 n_2}{v \cdot n(x_b)} + \sin(l - 1) \theta \frac{v_2 n_1}{v \cdot n(x_b)} \right) - t^l \cos \theta
+ 2lt^l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right)
+ \frac{t_b}{|v| \sin \frac{\theta}{2}} \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} - v_2 n_1 \sin \theta \sin \frac{\theta}{2} - \frac{2(l - 1) \sin \frac{\theta}{2}}{|v|^3} (v_1^2 \cos \theta \cos \frac{\theta}{2} - v_1 v_2 \sin \theta) \right)
- |v|(t^l - t^l) \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( -n_1^2 \cos(l - 1) \theta \cos \frac{\theta}{2} + n_1 n_2 \sin(l - 1) \theta + n_2^2 \sin(l - 1) \theta \sin \frac{\theta}{2} \right).
\]

Similar to $\nabla_x X(0; t, x, v)$, main terms in $\nabla_v X(0; t, x, v)$ are

\[
2lt^l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right) := I_2.
\]

As we take derivative to $\nabla_v X(0; t, x, v)$ with respect to $x$ and $v$, $I_2$ mainly contributes to increase singularity and travel length order. Thus, we only differentiate terms $I_2$ to get estimate for $|\nabla_x X(0; t, x, v)|$ and $|\nabla_v X(0; t, x, v)|$. Firstly, taking $x$ derivative to $I_2$ gives

\[
\nabla_x I_2 = 2l \nabla_x t^l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right)
+ 2lt^l \left( \frac{t_b \cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \nabla_x \theta \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right)
+ 2lt^l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 \sin \theta \cos \frac{\theta}{2} \nabla_x n_1 + v_1 n_1 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_1 n_1 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta \right)
+ v_1 \sin \theta \cos \frac{\theta}{2} \nabla_x n_2 + v_1 n_2 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta + \frac{1}{2} v_1 n_2 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta
+ v_2 \cos \theta \cos \frac{\theta}{2} \nabla_x n_1 - v_2 n_1 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_2 n_1 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta
+ v_2 \cos \theta \cos \frac{\theta}{2} \nabla_x n_2 - v_2 n_2 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta + \frac{1}{2} v_2 n_2 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta \right).
\]

Hence, it follows from 5.39, 5.10, 5.12, 5.13, and 5.14 that

\[
|\nabla_x I_2| \lesssim \frac{1}{|v| \sin \frac{\theta}{2}} \left( 1 + |v|^2 t^2 \right) \times \frac{1}{|v|} \times |v| + \frac{1}{|v|} \left( 1 + |v| t \right) \times \frac{1}{|v| \sin \frac{\theta}{2}} \times |v|
+ \frac{1}{|v|} \left( 1 + |v| t \right) \times \frac{1}{|v| \sin \frac{\theta}{2}} \times |v|
\]

\[
\lesssim \frac{|v|^2}{|v| \cdot n(x_b)} (1 + |v|^2 t^2),
\]

which yields $|\nabla_x X(0; t, x, v)|$ estimate

\[
|\nabla_x X(0; t, x, v)| \lesssim \frac{|v|^2}{|v| \cdot n(x_b)} (1 + |v|^2 t^2).
\]
Similarly, we consider $\nabla_v I_2$:

$$\nabla_v I_2 = 2l \nabla_v I^t \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right)$$

$$+ 2l^t \left( \frac{\nabla_v t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|^2} \frac{v}{|v|^3} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right)$$

$$+ 2l^t \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( n_1 \sin \theta \cos \frac{\theta}{2} \nabla_v n_1 + v_1 n_1 \cos \theta \cos \frac{\theta}{2} \nabla_v n_1 + v_1 n_1 \sin \theta \sin \frac{\theta}{2} \right)$$

By (5.9), (5.11), (5.12), (5.13), and (5.14), the above can be further bounded by

$$|\nabla_v I_2| \lesssim \frac{1}{|v|^2 \sin^2 \frac{\theta}{2}} (1 + |v|^2 t^2) \times |v| \times |v| \times \frac{1}{|v|^2 \sin^2 \frac{\theta}{2}} \times |v| + \frac{1}{|v|^2 \sin^2 \frac{\theta}{2}} (1 + |v|^2 t)$$

$$\lesssim \frac{1}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$

Hence, $|\nabla_v X(0; t, x, v)|$ is bounded by

$$|\nabla_v X(0; t, x, v)| \lesssim \frac{1}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$

To get estimate for $|\nabla_{xx} V(0; t, x, v)|$ and $|\nabla_x V(0; t, x, v)|$, we now consider $|\nabla_x V(0; t, x, v)|_{(1,1)}$:

$$|\nabla_x V(0; t, x, v)|_{(1,1)} = \frac{2l}{\sin \frac{\theta}{2}} \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right),$$

by Lemma 5.2 In $|\nabla_x V(0; t, x, v)|_{(1,1)}$, the main terms are

$$\frac{2l}{\sin \frac{\theta}{2}} v_1 n_1 \sin \theta \cos \frac{\theta}{2} \text{ and } \frac{2l}{\sin \frac{\theta}{2}} v_2 n_1 \cos \theta \cos \frac{\theta}{2},$$

because these terms have the highest singularity order in $|\nabla_x V(0; t, x, v)|_{(1,1)}$. Thus, for $|\nabla_{xx} V(0; t, x, v)|$, we now take $x$ derivative for main terms:

$$\nabla_x \left( \frac{2l}{\sin \frac{\theta}{2}} \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} \right) \right)$$

$$= -\frac{1}{\sin^2 \frac{\theta}{2}} \nabla_{xx} \theta \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} \right)$$

$$+ \frac{2l}{\sin \frac{\theta}{2}} \left( v_1 \sin \theta \cos \frac{\theta}{2} \nabla_x n_1 + v_1 n_1 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_1 n_1 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta \right)$$

$$+ v_2 \cos \theta \cos \frac{\theta}{2} \nabla_x n_1 - v_2 n_1 \sin \theta \cos \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_2 n_1 \cos \theta \sin \frac{\theta}{2} \nabla_x \theta \right)$$

$$:= I_3.$$
By (5.10), (5.12), and (5.14), $I_3$ can be further bounded by

$$|I_3| \lesssim \frac{|v|}{\sin^2 \frac{\theta}{2}} (1 + |v| t) + \frac{|v|}{\sin^2 \frac{\theta}{2}} (1 + |v|^2 t^2) \lesssim \frac{|v|^5}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2),$$

which implies that

$$|\nabla_{xx} V(0; t, x, v)| \lesssim \frac{|v|^5}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$

Similarly, we firstly take $v$ derivative for main terms in $|\nabla_x V(0; t, x, v)|_{(1, 1)}$ and then estimate $v$ derivatives. Then, we deduce

$$|\nabla_{xx} V(0; t, x, v)| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2),$$

where we have used (5.11), (5.12), and (5.14). Lastly, it remains to estimate $|\nabla_{xx} V(0; t, x, v)|$ and $|\nabla_{vv} V(0; t, x, v)|$.

Let us consider the $(1, 1)$ component of $\nabla_v V(0; t, x, v)$:

$$|\nabla_v V(0; t, x, v)|_{(1, 1)} = \cos \theta - 2l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_1 n_2 \sin \theta \sin \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} + v_2 n_2 \cos \theta \sin \frac{\theta}{2} \right),$$

by Lemma 5.2 Similar to previous cases, main terms in $|\nabla_v V(0; t, x, v)|_{(1, 1)}$ are

$$-2l \left( \frac{t_b}{\sin \frac{\theta}{2}} - \frac{1}{|v|} \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} \right) := I_4,$$

by the same reason. Taking $x$ derivative for $I_4$, we get

$$\nabla_x I_4 = -2l \left( \frac{\nabla_x t_b}{\sin \frac{\theta}{2}} - \frac{t_b \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} \nabla_x \theta \right) \left( v_1 n_1 \sin \theta \cos \frac{\theta}{2} + v_2 n_1 \cos \theta \cos \frac{\theta}{2} \right) \nabla_x \theta - \frac{1}{|v|} \left( v_1 \sin \theta \cos \frac{\theta}{2} \nabla_x n_1 + l v_1 n_1 \cos \theta \cos \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_1 n_1 \sin \theta \sin \frac{\theta}{2} \nabla_x \theta + v_2 \cos \theta \cos \frac{\theta}{2} \nabla_x n_1 - l v_2 n_1 \sin \theta \cos \frac{\theta}{2} \nabla_x \theta - \frac{1}{2} v_2 n_1 \cos \theta \sin \frac{\theta}{2} \nabla_x \theta \right).$$

Using (5.9), (5.10), (5.12), (5.13), and (5.14), one obtains that

$$|\nabla_x I_4| \lesssim \frac{1}{|\sin \frac{\theta}{2}|} (1 + |v| t) \times \frac{1}{|v| \sin^2 \frac{\theta}{2}} \times |v| + \frac{1}{|\sin \frac{\theta}{2}|} (1 + |v| t) \times \frac{1}{|v|} \times \frac{|v|}{\sin^2 \frac{\theta}{2}} (1 + |v| t) \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$

Hence, we get estimate for $|\nabla_{xx} V(0; t, x, v)|$

$$|\nabla_{xx} V(0; t, x, v)| \lesssim \frac{|v|^3}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$

Similarly, we take $v$ derivative to main terms $I_4$ and estimate $\nabla_v I_4$ to get $|\nabla_{vv} V(0; t, x, v)|$. From (5.9), (5.11), (5.12), (5.13), and (5.14), we derive

$$|\nabla_{vv} V(0; t, x, v)| \lesssim \frac{|v|}{|v \cdot n(x_b)|^2} (1 + |v|^2 t^2).$$
5.3. Proof of Theorem 1.6

Proof of Theorem 1.6. Step 1 First, we prove $C^1$ estimate. Note that it is easy to derive
\[
\partial_t X(0; t, x, v) = -v^k, \quad \partial_t X(0; t, x, v) = 0, \quad \partial_t V(0; t, x, v) = 0, \quad \partial_t V(0; t, x, v) = 0,
\]
where we assumed $t^{k+1} < 0 < t^k$ for some integer $k$. For $i \in \{t, x, v\}$,
\[
\nabla_i f(t, x, v) = \nabla_x f_0 \nabla_x X(0; t, x, v) + \nabla_v f_0 \nabla_v V(0; t, x, v).
\]
Hence using Lemma 5.3 and (5.15), we obtain
\[
|\partial_t f| \lesssim \|f_0\|_{C^1} |v|, \\
|\nabla_x f| \lesssim \|f_0\|_{C^1} \frac{|v|^2}{|v \cdot n(x_B)|^2} (v)(1 + |v| t), \\
|\nabla_v f| \lesssim \|f_0\|_{C^1} \frac{1}{|v \cdot n(x_B)|} (v)(1 + |v| t),
\]
where $x_B = x_B(x, v)$ and $\langle v \rangle := 1 + |v|$. So we obtain (1.14).

Step 2 Now we compute second order estimate. For $\nabla_{xx} f$, from (5.10), Lemma 5.3 and Lemma 5.5, we obtain
\[
|\nabla_{xx} f| = |\nabla_x (\nabla_x f_0 \nabla_x X(0; t, x, v) + \nabla_v f_0 \nabla_v X(0; t, x, v))|
\lesssim \|f_0\|_{C^2} \left( |\nabla_{xx} X(0)| + |\nabla_{xx} V(0)| \right) + \|f_0\|_{C^2} \left( |\nabla_x X(0)| + |\nabla_x V(0)| \right)
\lesssim \|f_0\|_{C^2} \frac{|v|^4}{|v \cdot n(x_B)|^3} (1 + |v| t)^2,
\]
and
\[
|\nabla_{vx} f| = |\nabla_v (\nabla_x f_0 \nabla_x X(0; t, x, v) + \nabla_v f_0 \nabla_v X(0; t, x, v))|
\lesssim \|f_0\|_{C^2} \left( |\nabla_{vx} X(0)| + |\nabla_{vx} V(0)| \right) + \|f_0\|_{C^2} \left( |\nabla_v X(0)| + |\nabla_v V(0)| \right)
\lesssim \|f_0\|_{C^2} \frac{1}{|v \cdot n(x_B)|} (1 + |v| t)^2,
\]
where $|\nabla_{xx, vx, vV}|$ means sup$_{i,j,k} |\nabla_{ij} X_k(0; t, x, v)|$ for $i, j \in \{x_1, x_2, v_1, v_2\}$ and $k \in \{1, 2\}$. (Also similar for $\nabla_{vV}$.) Combining above three estimates, we obtain (1.15). Second derivative estimates which contains at least one $\partial_t$ also yield same upper bound from (5.15). We omit the details.

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