Dunkl Operators and Related Special Functions

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1 Introduction

This is a preliminary version of a chapter for the volume titled “Multivariable Special Functions,” edited by T. Koornwinder, in the Askey-Bateman project.

Functions like the exponential, Chebyshev polynomials, and monomial symmetric polynomials are pre-eminent among all special functions. They have simple definitions and can be expressed using easily specified integers like \(n!\). Families of functions like Gegenbauer, Jacobi and Jack symmetric polynomials and Bessel functions are labeled by parameters. These could be unspecified transcendental numbers or drawn from large sets of real numbers, for example the complement of \(\{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}\). One aim of this chapter is to provide a harmonic analysis setting in which parameters play a natural role. The basic objects are finite reflection (Coxeter) groups and algebras of operators on polynomials which generalize the algebra of partial differential operators. These algebras have as many parameters as the number of conjugacy classes of reflections in the associated groups. The first-order operators have acquired the name “Dunkl operators.”

Coxeter groups are related to root systems. This chapter begins with a presentation of these systems, the definition of a Coxeter group and the classification of the indecomposable systems. Then the theory of the operators is developed in detail with an emphasis on inner products and their relation to Macdonald-Opdam integrals, a generalization of the Fourier transform, the Laplacian operator and harmonic polynomials, and applications to particular groups to study Gegenbauer, Jacobi and nonsymmetric Jack polynomials.

For \(x, y \in \mathbb{R}^d\) the inner product is \(\langle x, y \rangle = \sum_{j=1}^{d} x_j y_j\), and the norm is \(|x| = \langle x, x \rangle^{1/2}\). A matrix \(w = (w_{ij})_{i,j=1}^{d}\) is called orthogonal if \(ww^T = I_d\). The group of orthogonal \(d \times d\) matrices is denoted by \(O(d)\). The standard unit basis vectors of \(\mathbb{R}^d\) are denoted by \(\varepsilon_i, 1 \leq i \leq d\). The nonnegative integers \(\{0, 1, 2, \ldots\}\) are denoted by \(\mathbb{N}_0\), and the cardinality of a set \(S\) is denoted by \(\#S\).
Definition 1.1 For \( a \in \mathbb{R}^d \setminus \{0\} \), the reflection along \( a \), denoted by \( s_a \), is defined by

\[
s_a x := x - 2 \frac{\langle x, a \rangle}{|a|^2} a.
\]

Writing \( s_a = I_d - 2 (a^T a)^{-1} a a^T \) shows that \( s_a = s_a^T \) and \( s_a^2 = I_d \), that is, \( s_a \in O(d) \). The matrix entries of \( s_a \) are \((s_a)_{ij} = \delta_{ij} - 2 a_i a_j / |a|^2\). The hyperplane \( a^\perp = \{ x : \langle x, a \rangle = 0 \} \) is the invariant set for \( s_a \). Also \( s_a a = -a \), and any nonzero multiple of \( a \) determines the same reflection. The reflection \( s_a \) has one eigenvector for the eigenvalue \(-1\), and \( d-1 \) independent eigenvectors for the eigenvalue \( +1 \) and \( \det s_a = -1 \).

For \( a, b \in \mathbb{R}^d \setminus \{0\} \), let \( \cos \angle (a, b) := \langle a, b \rangle / (|a| \cdot |b|) \); if \( a, b \) are linearly independent in \( \mathbb{R}^d \) then \( s_a s_b \) is a plane rotation in \( \text{span}_\mathbb{R} \{a, b\} \) through an angle \( 2 \angle (a, b) \). Consequently for given \( m = 1, 2, 3, \ldots \), \((s_a s_b)^m = I_d \) if and only if \( \cos \angle (a, b) = \cos \left( \frac{\pi j}{m} \right) \) for some integer \( j \). Two reflections \( s_a \) and \( s_b \) commute if and only if \( \langle a, b \rangle = 0 \), since \((s_a s_b)^{-1} = s_b s_a \), in general. The conjugate of a reflection is again a reflection: suppose \( w \in O(d) \) and \( a \in \mathbb{R}^d \setminus \{0\} \) then \( ws_aw^{-1} = s_wa \).

2 Root Systems

Definition 2.1 A root system is a finite set \( R \subset \mathbb{R}^d \setminus \{0\} \) such that \( a, b \in R \) implies \( s_b a \in R \). If, additionally, \( \mathbb{R}a \cap R = \{ \pm a \} \) for each \( a \in R \) then \( R \) is said to be reduced. The rank of \( R \) is defined to be \( \dim(\text{span}_\mathbb{R}(R)) \).

Note that \( a \in R \) implies \( -a = s_a a \in R \), for any root system. By choosing some \( a_0 \in \mathbb{R}^d \) such that \( \langle a, a_0 \rangle \neq 0 \) for all \( a \in R \) one defines the set of positive roots to be \( R_+ = \{ a \in R : \langle a, a_0 \rangle > 0 \} \). Also set \( R_- = -R_+ \) so that \( R = R_+ \cup R_- \), the disjoint union of the sets of positive and negative roots.

Definition 2.2 The Coxeter, or finite reflection, group \( W(R) \) is defined to be the subgroup of \( O(d) \) generated by \( \{ s_a : a \in R_+ \} \).

The group \( W(R) \) is finite because each \( w \in W(R) \) fixes the orthogonal complement of \( \text{span}_\mathbb{R}(R) \) pointwise, and for each \( a \in R \) the orbit \( W(R)a \subset R \), hence is finite.

There is a distinguished set \( S = \{ r_1, \ldots, r_n \} \), \( n = \dim(\text{span}_\mathbb{R}(R)) \), of positive roots, called the simple roots, such that \( S \) is a basis for \( \text{span}_\mathbb{R}(R) \) and \( a \in R_+ \) implies \( a = \sum_{i=1}^n c_i r_i \) with \( c_i \geq 0 \) (see [12, Theorem 1.3]). The corresponding reflections \( s_i = s_{r_i} \) \((i = 1, \ldots, n) \) are called simple reflections.

Definition 2.3 The length of \( w \in W(R) \) is \( \ell(w) := \#(wR_+ \cap R_-) \).
Equivalently (see [12 Corollary 1.7]), $\ell(w)$ equals the number of factors in the shortest product $w = s_{i_1}s_{i_2}\ldots s_{i_m}$ for expressing $w$ in terms of simple reflections. In particular, $w$ has length one if and only if it is a simple reflection.

For purpose of studying the group $W(R)$ one can replace $R$ by a reduced root system (example: $\{\sqrt{2}a/|a| : a \in R\}$ is a commonly used normalization).

**Definition 2.4** The discriminant, or alternating polynomial, of the reduced root system $R$ is

$$a_R(x) := \prod_{a \in R_+} (x, a).$$

If $R$ is reduced and $b \in R_+$ then $\langle sbx, b \rangle = -\langle x, b \rangle$ and $\# \{a \in R_+ \setminus \{b\} : sba \in -R_+\}$ is even. It follows that $a_R(sbx) = -a_R(x)$, and furthermore that $a_R(wx) = \det(w)a_R(x)$, $w \in W(R)$.

The set of reflections in $W(R)$ is exactly $\{s_a : a \in R_+\}$. This is a consequence of a divisibility property: if $b \in \mathbb{R}^d \setminus \{0\}$ and $p$ is a polynomial such that $p(sb,x) = -p(x)$ for all $x \in \mathbb{R}^d$, then $p(x)$ is divisible by $\langle x, b \rangle$ (without loss of generality assume that $b = e_1$; let $p(x) = \sum_{j=0}^n x_j^2 p_j(x_2, \ldots, x_d)$, then $p(sb,x) = \sum_{j=0}^n -x_1^2 p_j(x_2, \ldots, x_d)$ and $p(sb,x) = -p(x)$ implies $p_j = 0$ unless $j$ is odd). If $w = sb \in W(R)$ is a reflection then $\det w = -1$ and so $a_R(wx) = -a_R(x)$ and hence $a_R(x)$ is divisible by $\langle x, b \rangle$. Linear factors are irreducible and the unique factorization theorem shows that some multiple of $b$ is an element of $R_+$.

**Definition 2.5** The root system $R$ is called crystallographic if $2\langle a, b \rangle/|b|^2 \in \mathbb{Z}$ for all $a, b \in R$.

In the crystallographic case $R$ is in the $\mathbb{Z}$-lattice generated by the simple roots and $W(R)$ acts on this lattice (see [11 §10.1]).

If $R$ can be expressed as a disjoint union of non-empty sets $R_1 \cup R_2$ with $\langle a, b \rangle = 0$ for every $a \in R_1, b \in R_2$ then each $R_i$ ($i = 1, 2$) is itself a root system and $W(R) = W(R_1) \times W(R_2)$, a direct product. Furthermore, $W(R_1)$ and $W(R_2)$ act on the orthogonal subspaces $\text{span}_\mathbb{R}(R_1)$ and $\text{span}_\mathbb{R}(R_2)$, respectively. In this case, the root system $R$ and the reflection group $W(R)$ are called decomposable. Otherwise the system and group are indecomposable (also called irreducible). There is a complete classification of indecomposable finite reflection groups.

Assume that the rank of $R$ is $d$, that is, $\text{span}_\mathbb{R}(R) = \mathbb{R}^d$. The set of hyperplanes $H = \{a^\perp : a \in R_+\}$ divides $\mathbb{R}^d$ into connected (open) components, called chambers. The order of the group equals the number of chambers (see [13 Theorems 1.4, 1.8]). Recall that $R_+ = \{a \in R : \langle a, a_0 \rangle > 0\}$ for some $a_0 \in \mathbb{R}^d$. The connected component of $\mathbb{R}^d \setminus H$ which contains $a_0$ is called the fundamental chamber. The simple roots correspond to the bounding hyperplanes of this chamber and they form a basis of $\mathbb{R}^d$ (note that the definitions of chamber and fundamental chamber extend to the situation where $\text{span}_\mathbb{R}(R)$ is embedded in a higher-dimensional space). The simple reflections $s_i$, $i = 1, \ldots, d$, correspond to the simple roots. Let $m_{ij}$ be the order of $s_is_j$ (clearly $m_{ii} = 1$ and $m_{ij} = 2$ if and only if $s_is_j = s_js_i$ for $i \neq j$). The group $W(R)$ is isomorphic to the abstract group generated by $\{s_i : 1 \leq i \leq d\}$ subject to the relations $(s_is_j)^{m_{ij}} = 1$ (see [12 Theorem 1.9]).
The Coxeter diagram is a graphical way of displaying the relations: it is a graph with \( d \) nodes corresponding to the simple reflections, the nodes \( i \) and \( j \) are joined with an edge when \( m_{ij} > 2 \); the edge is labeled by \( m_{ij} \) when \( m_{ij} > 3 \). The root system is indecomposable if and only if the Coxeter diagram is connected. There follow brief descriptions of the indecomposable root systems and the corresponding groups. The rank is indicated by the subscript. The systems are crystallographic except for \( H_3, H_4 \) and \( I_2(m), m \notin \{2, 3, 4, 6\} \).

2.1 Type \( A_{d-1} \)

The root system is \( R = \{\varepsilon_i - \varepsilon_j : i \neq j\} \subset \mathbb{R}^d \). The span is \( (\sum_{i=1}^{d} \varepsilon_i)^\perp \), thus the rank is \( d - 1 \). The reflection \( s_{ij} = s_{\varepsilon_i - \varepsilon_j} \) interchanges the components \( x_i \) and \( x_j \) of each \( x \in \mathbb{R}^d \) and is called a transposition, often denoted by \( (ij) \). Thus \( W(R) \) is the symmetric (or permutation) group \( S_d \) of \( d \) objects. Choose \( a_0 = \sum_{i=1}^{d} (d + 1 - i)\varepsilon_i \) then \( R_+ = \{\varepsilon_i - \varepsilon_j : i < j\} \) and the simple roots are \( \{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq d - 1\} \). The corresponding reflections are the adjacent transpositions \( s_i = (i, i+1) \). The structure constants satisfy \( m_{i,i+1} = 3 \) and \( m_{ij} < 3 \) otherwise. The Coxeter diagram is

\[ \circ - \circ - \circ - \cdots - \circ \]

The alternating polynomial is

\[ a_R(x) = \prod_{a \in R_+} \langle x, a \rangle = \prod_{1 \leq i < j \leq d} (x_i - x_j). \]

The fundamental chamber is \( \{x : x_1 > x_2 > \cdots > x_d\} \).

2.2 Type \( B_d \)

The root system is \( R = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq d\} \cup \{\pm\varepsilon_i : 1 \leq i \leq d\} \). For \( d = 1 \), \( R = \{\pm\varepsilon_1\} \) which is essentially the same as \( A_1 \). The group \( W(B_d) \) is the full symmetry group of the hyperoctahedron \( \{\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_d\} \subset \mathbb{R}^d \) (also of the hypercube) and is thus called the hyperoctahedral group. Its elements are the \( d \times d \) generalized permutation matrices with entries \( \pm 1 \) (that is, each row and each column has exactly one nonzero element \( \pm 1 \)). With the same \( a_0 \) as used for \( A_{d-1} \), the positive root system is \( R_+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : i < j\} \cup \{\varepsilon_i : 1 \leq i \leq d\} \) and the simple roots are \( \{\varepsilon_i - \varepsilon_{i+1} : 1 < d\} \cup \{\varepsilon_d\} \). The order of \( s_{\varepsilon_{d-1} - \varepsilon_d} s_{\varepsilon_d} \) is 4. The Coxeter diagram is \( \circ - \circ - \circ - \cdots - \circ - \circ \)

The alternating polynomial is

\[ a_R(x) = \prod_{1 \leq i < j \leq d} (x_i^2 - x_j^2). \]

The fundamental chamber is \( \{x : x_1 > x_2 > \cdots > x_d > 0\} \).
2.3 Types $C_d$ and $BC_d$

The root system $C_d$ is $\{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq d\} \cup \{2\varepsilon_i : 1 \leq i \leq d\}$, and the system $BC_d$ is $\{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq d\} \cup \{2\varepsilon_i, \varepsilon_i : 1 \leq i \leq d\}$. The latter system is not reduced. Both of these systems generate the same group as $W(B_d)$. The system $C_d$ has to be distinguished from $B_d$ because of the crystallographic condition in Definition 2.5. For $d = 1, 2$ there is no essential distinction.

2.4 Type $D_d$

For $d \geq 4$ the root system is $R = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq d\}$, a subset of $B_d$. The group $W(D_d)$ is the subgroup of $W(B_d)$ fixing the polynomial $\prod_{j=1}^{d} x_j$. The simple roots are $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i < d\} \cup \{\varepsilon_{d-1} + \varepsilon_d\}$ and the Coxeter diagram is

\[ \circ - \circ - \circ - \cdots - \circ \circ \]

The alternating polynomial is

\[ a_R(x) = \prod_{1 \leq i < j \leq d} (x_i^2 - x_j^2) \]

and the fundamental chamber is $\{x : x_1 > x_2 > \cdots > |x_d|\}$.

2.5 Type $F_4$

The root system is $R_1 \cup R_2$ where $R_1 = \{\pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 4\}$ and $R_2 = \{\pm \varepsilon_i : 1 \leq i \leq 4\} \cup \{\frac{1}{4}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}$. Each set contains 24 roots. The group $W(F_4)$ contains $W(B_4)$ as a subgroup of index 3. The simple roots are $\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{4}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)$ and the Coxeter diagram is

\[ \circ - \circ - \circ - \circ \]

With the orthogonal coordinates $y_1 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$, $y_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)$, $y_3 = \frac{1}{\sqrt{2}}(-x_3 + x_4)$, $y_4 = \frac{1}{\sqrt{2}}(x_3 + x_4)$, the alternating polynomial is

\[ a_R(x) = 2^{-6} \prod_{1 \leq i < j \leq 4} (x_i^2 - x_j^2) (y_i^2 - y_j^2) \].

2.6 Type $G_2$

This system is a subset $R_1 \cup R_2$ of $\left\{ \sum_{i=1}^{3} x_i \varepsilon_i : \sum_{i=1}^{3} x_i = 0 \right\} \subset \mathbb{R}^3$, where $R_1 = \{\pm (\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq 3\}$ and $R_2 = \{\pm (2\varepsilon_i - \varepsilon_j - \varepsilon_k) : \{i, j, k\} = \{1, 2, 3\}\}$. The simple roots are $\varepsilon_1 - \varepsilon_2$, $-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and the Coxeter diagram is

\[ \circ 6 \circ \]
2.7 Types $E_6, E_7, E_8$

The root system $E_8$ equals $R_1 \cup R_2$ where

$$R_1 = \{ \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 8 \},$$

$$R_2 = \left\{ \frac{1}{2} \sum_{i=1}^{8} (-1)^{ni} \varepsilon_i : n_i \in \{0, 1\}, \sum_{i=1}^{8} n_i = 0 \mod 2 \right\},$$

with $\#R_1 = 112$ and $\#R_2 = 128$. The simple roots are $r_1 = \frac{1}{2} \left( \varepsilon_1 - \sum_{i=2}^{7} \varepsilon_i + \varepsilon_8 \right)$, $r_2 = \varepsilon_1 + \varepsilon_2$, $r_i = \varepsilon_{i-1} - \varepsilon_{i-2}$ for $3 \leq i \leq 8$. The systems $E_6$ and $E_7$ consist of the elements of $R$ which lie in $\text{span}\{r_i : 1 \leq i \leq 6\}$ and $\text{span}\{r_i : 1 \leq i \leq 7\}$ respectively. (Because these systems are crystallographic, the spans are over $\mathbb{Z}$.) For more details on these systems see [12] pp. 42-43.

2.8 Type $I_2(m)$

These are the dihedral systems corresponding to symmetry groups of regular $m$-gons in $\mathbb{R}^2$ for $m \geq 3$. Using a complex coordinate system $z = x_1 + i x_2$ and $\bar{z} = x_1 - i x_2$, the map $z \mapsto ze^{i\theta}$ is a rotation through the angle $\theta$, and the reflection along $(\sin \phi, -\cos \phi)$ is $z \mapsto ze^{2i\phi}$. The reflection along $v_{(j)} = \left( \sin \frac{\pi j}{m}, -\cos \frac{\pi j}{m} \right)$ corresponds to $s_j : z \mapsto ze^{2\pi ij/m}$, for $1 \leq j \leq 2m$; note that $v_{(m+j)} = -v_{(j)}$. For $a_0 = (\cos \frac{\pi}{2m}, \sin \frac{\pi}{2m})$, the positive roots are $\{ v_{(j)} : 1 \leq j \leq m \}$ and the simple roots are $v_{(1)}$, $v_{(m)}$. Then $s_1 s_m$ maps $z$ to $ze^{2\pi jm/m}$, and has period $m$. The Coxeter diagram is $\circ - \circ$. Since $s_j s_n s_j = s_{2j-n}$ for any $n$, $j$, there are two conjugacy classes of reflections $\{ s_{2i} \}$, $\{ s_{2i+1} \}$ when $m$ is even, and one class when $m$ is odd. There are 3 special cases: the groups $W(I_2(3)), W(I_2(4)), W(I_2(6))$ are isomorphic to $W(A_2), W(B_2), W(G_2)$, respectively. The alternating polynomial is a multiple of $(z^m - \bar{z}^m)/i$.

2.9 Type $H_3$

Let $\tau = (1 + \sqrt{5})/2$ (so $\tau^2 = \tau + 1$). Take the positive root system to be $R_+ = \{ (2,0,0), (0,2,0), (0,0,2), (\tau, \pm \tau^-1, \pm \tau), (\pm 1, \tau, \pm \tau^-1), (\tau^-1, \pm 1, \tau), (-\tau^-1, 1, \tau), (\tau^-1, \tau, -\tau) \}$, thus $\#R_+ = 15$. The root system $R = R_+ \cup (-R_+)$ as a configuration in $\mathbb{R}^3$ is called the icosidodecahedron. The group $W(H_3)$ is the symmetry group of the icosahedron $Q_{12} = \{(0, \pm \tau, \pm 1), (\pm 1, \pm \tau, 0), (\pm \tau, 0, \pm \tau^-1), (\pm \tau^-1, \pm \tau, 0), (\pm 1, \pm 1, \mp 1)\}$ (12 vertices, 20 triangular faces) and of the dodecahedron $Q_{20} = \{(0, \pm \tau^-1, \pm \tau^-1), (\tau, 0, \pm \tau), (\pm \tau, 0, \pm \tau^1), (\pm \tau^-1, \pm \tau, 0), (\pm 1, \pm 1, \pm 1)\}$ (20 vertices, 12 pentagonal faces). The simple roots are $(\tau, -\tau^-1, -1), (-1, \tau, -\tau^-1), (\tau^-1, -1, \tau)$ and the Coxeter diagram is $\circ - \circ - \circ$.

2.10 Type $H_4$

This root system has 60 positive roots, and the Coxeter diagram $\circ - \circ - \circ - \circ$ . The group $W(H_4)$ is the symmetry group of the 600-cell and is sometimes called the hecatonicosahedral group.
2.11 Miscellaneous Results

For any root system, the subgroup generated by a subset of simple reflections (that is, the result of deleting one or more nodes from the Coxeter diagram) is called a parabolic subgroup of $W(R)$. More generally, any conjugate of such a subgroup is also called parabolic. Therefore, for any $a \in \mathbb{R}^d$ the stabilizer $W_a = \{ w \in W(R) : wa = a \}$ is parabolic. The number of conjugacy classes of reflections equals the number of connected components of the Coxeter diagram after all edges with an even label have been removed.

3 Invariant Polynomials

For $w \in O(d)$ and $p \in \Pi^d$, the space of polynomials on $\mathbb{R}^d$, let $(wp)(x) = p(w^{-1}x)$ (thus $((w_1w_2)p)(x) = (w_1(w_2p))(x)$, $w_1, w_2 \in O(d)$). For a finite subgroup $G$ of $O(d)$ let $\Pi^G$ denote the space of $G$-invariant polynomials $\{ p \in \Pi^d : wp = p \text{ for all } w \in G \}$.

When $G$ is a finite reflection group $W(R)$, $\Pi^G$ has an elegant structure; there is a set of algebraically independent homogeneous generators, whose degrees are fundamental constants associated with $R$.

**Theorem 3.1** (See [12, Theorems 3.5, 3.9,]) Suppose $R$ is a root system in $\mathbb{R}^d$ then there exist $d$ algebraically independent $W(R)$-invariant homogeneous polynomials $\{ q_j : 1 \leq j \leq d \}$ of degrees $n_j$, such that $\Pi^{W(R)}$ is the ring of polynomials generated by $\{ q_j \}$. Furthermore, $\#W(R) = n_1n_2\cdots n_d$ and the number of reflections in $W(R)$ is $\sum_{j=1}^{d}(n_j - 1)$.

For an indecomposable root system $R$ of rank $d$ the numbers $\{n_j : 1 \leq j \leq d\}$ are called the fundamental degrees of $W(R)$. The coefficient of $t^k$ in the product $\prod_{j=1}^{d}(1 + (n_j - 1)t)$ is the number of elements of $W(R)$ whose fixed point set is of codimension $k$ (see [12, Remark 3.9]). Here are the structural constants (see [12, Table 3.1]):

| Type | $\#R$ | $\#W(R)$ | $n_1, \ldots, n_d$ |
|------|-------|-----------|-------------------|
| $A_d$ | $\frac{d(d+1)}{2}$ | $(d+1)!$ | 2, 3, $\ldots$, $d+1$ |
| $B_d$ | $d^2$ | $2^d d!$ | 2, 4, 6, $\ldots$, $2d$ |
| $D_d$ | $(d-1)$ | $2^{d-1}d!$ | 2, 4, 6, $\ldots$, $2(d-1), d$ |
| $G_2$ | 6 | 12 | 2, 6 |
| $H_3$ | 15 | 120 | 2, 6, 10 |
| $F_4$ | 24 | 1152 | 2, 6, 8, 12 |
| $H_4$ | 60 | 14400 | 2, 12, 20, 30 |
| $E_6$ | 36 | $2^i \times 3^4 \times 5$ | 2, 5, 6, 8, 9, 12 |
| $E_7$ | 63 | $2^{10} \times 3^4 \times 5 \times 7$ | 2, 6, 8, 10, 12, 14, 18 |
| $E_8$ | 120 | $2^{14} \times 3^4 \times 5^2 \times 7$ | 2, 8, 12, 14, 18, 20, 24, 30 |
| $I_2(m)$ | $m$ | $2m$ | 2, $m$ |
4 Dunkl operators

Throughout this section $R$ denotes a reduced root system contained in $\mathbb{R}^d$ and $G = W(R)$, a subgroup of $O(d)$. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ let $|\alpha| := \sum_{i=1}^d \alpha_i$ and $x^\alpha := \prod_{i=1}^d x_i^{\alpha_i}$, a monomial of degree $|\alpha|$. Let $\Pi^d_\alpha := \text{span} \{ x^\alpha : \alpha \in \mathbb{N}^d, |\alpha| = n \}$, the space of homogeneous polynomials of degree $n$, $n \in \mathbb{N}_0$, where $F$ is some extension field of $\mathbb{R}$ containing the parameter values.

Definition 4.1 A multiplicity function on $R$ is a $G$-invariant function $\kappa$ with values in $\mathbb{R}$ or a transcendental extension of $\mathbb{Q}$, that is, $a \in R, w \in G$ implies $\kappa(wa) = \kappa(a)$. Note $\kappa(-a) = \kappa(a)$ since $s_\alpha a = -a$.

Suppose $a \in \mathbb{R}^d \setminus \{0\}$ and $w \in O(d)$ then $ws_a w^{-1} = s_{wa}$. Thus, $\kappa$ can be considered as a function on the reflections $\{ s_a : a \in R_+ \}$ which is constant on conjugacy classes. In the sequel $\kappa$ denotes a multiplicity function of $G$.

For any reflection $s_a$ and polynomial $p \in \Pi^d$ the polynomial $p(x) - s_a p(x)$ vanishes on $a\perp$ hence is divisible by $\langle x, a \rangle$. The gradient is denoted by $\nabla$.

Definition 4.2 For $p \in \Pi^d$ and $x \in \mathbb{R}^d$ let

$$\nabla_\kappa p(x) := \nabla p(x) + \sum_{v \in R_+} \kappa(v) \frac{p(x) - s_v p(x)}{\langle x, v \rangle} v,$$

and for $a \in \mathbb{R}^d$ let $D_a p(x) := \langle \nabla_\kappa p(x), a \rangle$. $D_a$ is a Dunkl operator. For $1 \leq i \leq d$ denote $D_{\alpha_i}$ by $D_i$.

Thus each $D_a$ is an operator on polynomials, and maps $\Pi^d_n$ into $\Pi^d_{n-1}$, $n \in \mathbb{N}_0$ (that is, $D_a$ is homogeneous of degree $-1$). The important properties of $\{ D_a : a \in \mathbb{R}^d \}$ are $G$-covariance and commutativity.

Proposition 4.3 Let $w \in G$ and $a \in \mathbb{R}^d$, then (as operators on $\Pi^d$) $w^{-1} D_a w = D_{wa}$.

Proof. Let $p \in \Pi^d$ then

$$wD_a w^{-1} p(x) = w \langle \nabla w^{-1} p(x), a \rangle + \frac{1}{2} w \sum_{v \in R} \kappa(v) \frac{p(wx) - p(w_\alpha x)}{\langle x, v \rangle} \langle v, a \rangle$$

$$= \langle \nabla p(x), wa \rangle + \frac{1}{2} w \sum_{v \in R} \kappa(v) \frac{p(x) - p(w_\alpha x)}{\langle w^{-1} x, v \rangle} \langle v, a \rangle$$

$$= \langle \nabla p(x), wa \rangle + \frac{1}{2} \sum_{u \in R} \kappa(u) \langle w^{-1} u \rangle \frac{p(x) - p(s_\alpha x)}{\langle x, u \rangle} \langle u, wa \rangle$$

$$= D_{wa} p(x).$$
In the reflection part of $D_a$ the sum over $R_+$ can be replaced by $\frac{1}{2}$ of the sum over $R$. Then the summation variable $v \in R$ is replaced by $v = w^{-1}u$. ■

For $t \in \mathbb{R}^d$ let $m_t$ denote the multiplier operator on $\Pi^d$ given by

$$(m_tp)(x) := \langle t, x \rangle p(x).$$

The commutator of two operators $A, B$ on $\Pi^d$ is $[A, B] := AB - BA$. The identities $[[A, B], C] = [[A, C], B] - [[B, C], A]$ and $[A^2, B] = A[A, B] + [A, B]A$ are used below.

**Proposition 4.4** For $a, t \in \mathbb{R}^d$

$$[D_a, m_t] = \langle a, t \rangle + 2 \sum_{v \in R_+} \kappa(v) \frac{\langle a, v \rangle \langle t, v \rangle |v|^2}{|v|^2} s_v.$$ 

**Proof.** Let $p \in \Pi^d$. Then $\langle \nabla (\langle t, x \rangle p(x)), a \rangle - \langle t, x \rangle \langle \nabla p(x), a \rangle = \langle t, a \rangle p(x)$. Next for any $v \in R_+$ we have

$$\frac{\langle t, x \rangle p(x) - \langle t, s_v x \rangle p(s_v x)}{\langle x, v \rangle} - \frac{\langle t, x \rangle p(x) - p(s_v x)}{\langle x, v \rangle} = \frac{p(s_v x)}{\langle x, v \rangle} ((\langle t, x \rangle - \langle s_v t, x \rangle) = 2\frac{p(s_v x) \langle x, v \rangle}{\langle x, v \rangle |v|^2} \langle t, v \rangle.$$ 

This proves the formula. ■

**Lemma 4.5** For $b \in \mathbb{R}^d$ and $v \in R$,

$$[s_v, D_b] = \frac{\langle b, v \rangle}{|v|^2} s_v D_v.$$ 

**Proof.** Indeed by Proposition 4.3 $[s_v, D_a] = s_v D_a - D_a s_v = s_v (D_a - D_{sv a}) = s_v \left(2\frac{\langle a, v \rangle}{|v|^2} D_v\right).$ ■

**Theorem 4.6** If $a, b \in \mathbb{R}^d$ then $[D_a, D_b] = 0$.

**Proof.** Let $t \in \mathbb{R}^d$, we will show $[[D_a, D_b], m_t] = 0$. Indeed

$$[[D_a, D_b], m_t] = [[D_a, m_t], D_b] - [[D_b, m_t], D_a],$$

and

$$[[D_a, m_t], D_b] = [\langle a, t \rangle, D_b] + 2 \sum_{v \in R_+} \kappa(v) \frac{\langle a, v \rangle \langle t, v \rangle}{|v|^2} [s_v, D_b]$$

$$= 4 \sum_{v \in R_+} \kappa(v) \frac{\langle a, v \rangle \langle t, v \rangle \langle b, v \rangle |v|^2}{|v|^2} s_v D_v,$$
which is symmetric in \( a, b \). The algebra generated by \( \{ m_t : t \in \mathbb{R}^d \} \) is \( \Pi^d \). For any \( p \in \Pi^d \) we have \([D_a, D_b]p = p[D_a, D_b] 1 = 0\). ■

This method of proof is due to Etingof [8, Theorem 2.15] (the result was first established by Dunkl [1]). One important consequence is that the operators \( D_a \) generate a commutative algebra.

**Definition 4.7** Let \( A_k \) denote the algebra of operators on \( \Pi^d \) generated by \( \{ D_i : 1 \leq i \leq d \} \). Let \( \rho \) denote the homomorphism \( \Pi^d \to A_k \) given by \( pp(x_1, \ldots, x_d) = p(D_1, \ldots, D_d), p \in \Pi^d \).

The map \( \rho \) is an isomorphism.

**Proposition 4.8** If \( w \in G \) and \( p \in \Pi^d \) then \( \rho(wp) = wp(p)w^{-1} \).

**Proof.** It suffices to show this for first-degree polynomials. For \( t \in \mathbb{R}^d \) let \( p_t(x) = \langle t, x \rangle \), then \( \rho p_t = \sum_{i=1}^d t_i D_i = D_t \). Also \( wp_t(x) = \langle t, w^{-1}x \rangle = \langle wt, x \rangle = p_{wt}(x) \). Then \( \rho(wp_t) = \rho(p_{wt}) = D_{wt} = wD_tw^{-1} = wp(p)w^{-1} \) by Proposition 4.3 ■

There is a Laplacian-type operator in the algebra \( A_k \). This operator is an important part of the analysis of the \( L^2 \)-theory associated to the \( G \)-invariant weight functions.

**Definition 4.9** For \( p \in \Pi^d \) the Dunkl Laplacian \( \Delta_k \) is given by

\[
\Delta_k p(x) := \Delta p(x) + 2 \sum_{v \in R_+} \kappa(v) \left\{ \frac{\langle \nabla p(x), v \rangle}{\langle x, v \rangle} - \frac{|v|^2 p(x) - p(s_v x)}{2 \langle x, v \rangle^2} \right\}.
\]

**Theorem 4.10** \( \Delta_k = \sum_{i=1}^d D_i^2 \), and \( [\Delta_k, m_t] = 2D_t \) for each \( t \in \mathbb{R}^d \).

**Proof.** We use the same method as in the proof of Theorem 4.6 that is, we show \( [\Delta_k, m_t] = \sum_{i=1}^d [D_i^2, m_t] \) for \( t \in \mathbb{R}^d \). Clearly \( \Delta_k 1 = 0 = \sum_{i=1}^d D_i^2 1 \). We have

\[
\sum_{i=1}^d [D_i^2, m_t] = \sum_{i=1}^d \{ D_i [D_i, m_t] + [D_i, m_t] D_i \}
\]

\[
= \sum_{i=1}^d \left\{ 2t_i D_i + 2 \sum_{v \in R_+} \kappa(v) \frac{\langle t, v \rangle v_i}{|v|^2} (D_is_v + s_v D_i) \right\}
\]

\[
= 2D_t + 2 \sum_{v \in R_+} \kappa(v) \frac{\langle t, v \rangle}{|v|^2} (D_v s_v + s_v D_v) = 2D_t,
\]

because \( D_v s_v + s_v D_v = D_v s_v + D_{s_v v} s_v \) and \( D_{s_v v} = -D_v \). Next \( [\Delta, m_t] = 2 \sum_{i=1}^d t_i \frac{\partial}{\partial x_i} \). For \( v \in R_+ \) let \( T_v \) denote the operator defined in \( \{ \} \) in the formula for \( \Delta_k \), then

\[
[T_v, m_t] p(x) = \frac{\langle t, v \rangle}{\langle x, v \rangle} p(x) - \frac{|v|^2}{2} \frac{\langle x, t \rangle - \langle s_v x, t \rangle}{\langle x, v \rangle^2} p(s_v x)
\]

\[
= \frac{\langle t, v \rangle}{\langle x, v \rangle} (p(x) - p(s_v x)).
\]

10
since \( \langle x, t \rangle - \langle s_vx, t \rangle = \frac{2}{|v|^2} \langle x, v \rangle \langle t, v \rangle \), for any \( p \in \Pi^d \). Thus \([\Delta_\kappa, m_t] = 2D_t p\).

**Corollary 4.11** \( \Delta_\kappa \in \mathcal{A}_\kappa \) and \([\Delta_\kappa, w] = 0 \) for \( w \in G \).

**Proof.** For any reflection group \( p_2(x) = |x|^2 \) is \( G \)-invariant. The Theorem shows that \( \Delta_\kappa = \rho(p_2) \in \mathcal{A}_\kappa \) and thus \( w\Delta_\kappa w^{-1} = \Delta_\kappa \).

There is a natural bilinear \( G \)-invariant form on \( \Pi^d \) associated with Dunkl operators. We will show that the form is symmetric, and positive-definite when \( \kappa \geq 0 \), that is, \( \kappa(v) \geq 0 \) for all \( v \in R \). The proof involves a number of ingredients.

**Lemma 4.12** For each \( n \in \mathbb{N}_0 \) the space \( \Pi^d_n \) is a direct sum of eigenvectors of \( \sum_{i=1}^d x_i D_i \).

**Proof.** If \( p \in \Pi^d \) then

\[
\sum_{i=1}^d x_i D_i p(x) = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} p(x) + \sum_{v \in R_+} \kappa(v)(p(x) - s_v p(x)).
\]

The space \( \Pi^d_n \) is a \( G \)-module under the action \( w \mapsto (p \mapsto wp) \) for \( w \in G, p \in \Pi^d_n \). So \( \Pi^d_n = \sum_{j=1}^m \oplus M_j \) where each \( M_j \) is an irreducible \( G \)-submodule. There are constants \( c_j(\kappa) \) such that

\[
\sum_{v \in R_+} \kappa(v)(1-s_v)p = c_j(\kappa)p \quad \text{for each } p \in M_j, 1 \leq j \leq m,
\]

because \( \sum_{v \in R_+} \kappa(v)(1-s_v) \) is in the center of the group algebra of \( G \), (it is a sum over conjugacy classes and \( c_j(\kappa) = \sum_{v \in R_+} \kappa(v)c_{j,v} \) where \( c_{j,v} \in \mathbb{Q} \) and the map \( v \mapsto c_{j,v} \) is constant on \( G \)-orbits in \( R \)). Thus

\[
\sum_{i=1}^d x_i D_i p = (n + c_j(\kappa))p \quad \text{for } p \in M_j.
\]

**Definition 4.13** For \( p, q \in \Pi^d \) let \( \langle p, q \rangle_\kappa := \rho(p)q(x)|_{x=0} \), (the polynomial \( \rho(p)q(x) \) is evaluated at \( x = 0 \)).

**Theorem 4.14** The pairing \( \langle \cdot , \cdot \rangle_\kappa \) has the following properties:

1. if \( p \in \Pi^d_n, q \in \Pi^d_m \) and \( m \neq n \) then \( \langle p, q \rangle_\kappa = 0 \);
2. if \( w \in G \) and \( p, q, r \in \Pi^d \) then \( \langle wp, wq \rangle_\kappa = \langle p, q \rangle_\kappa \) and \( \langle rp, q \rangle_\kappa = \langle p, \rho(r)q \rangle_\kappa \);
3. the form is bilinear and symmetric.

**Proof.** If \( p \in \Pi^d_n, q \in \Pi^d_m \) and \( m \geq n \) then \( \rho(p)q \in \Pi^d_{m-n} \) and vanishes at \( x = 0 \) if \( m > n \); if \( m < n \) then \( \rho(p)q = 0 \); a nonzero value at \( x = 0 \) is possible only if \( m = n \) and then \( \rho(p)q \) is a constant. For part (2) we may assume \( p, q \in \Pi^d_n \) for some \( n \) and \( \langle p, q \rangle_\kappa = \rho(p)q \) (a constant). By Proposition \( 4.3 \) \( \rho(wp) = wp(p)w^{-1} \) thus \( \langle wp, wq \rangle_\kappa = w\rho(p)w^{-1}wq = \rho(wp)q = \langle p, q \rangle_\kappa \) (because \( w1 = 1 \)). That \( \langle rp, q \rangle_\kappa = \langle p, \rho(r)q \rangle_\kappa \) follows easily from the definition. Use induction on part (3): for constants \( p_1, p_2 \) the form equals \( \langle p_1, p_2 \rangle_\kappa = p_1p_2 \); assume the form is symmetric on \( \sum_{m=0}^n \Pi^d_m \) for some \( n \) and let \( p, q \in \Pi^d_{n+1} \). Using Lemma \( 4.12 \) suppose \( p \) and \( q \) are eigenfunctions of \( \sum_{v \in R_+} \kappa(v)(1-s_v) \) with eigenvalues \( c_1(\kappa) \) and \( c_2(\kappa) \) respectively then

\[
c_1(\kappa)\langle p, q \rangle_\kappa = \sum_{v \in R_+} \kappa(v)\langle (1-s_v)p, q \rangle_\kappa = \sum_{v \in R_+} \kappa(v)\langle p, (1-s_v)q \rangle_\kappa
\]

\[
= c_2(\kappa)\langle p, q \rangle_\kappa.
\]
Finally the parameter space (the dimension equals the number of $G$-orbits in $R$). Thus $\langle p, q \rangle_\kappa = 0$ (a symmetric relation). Now assume $c_1(\kappa) = c_2(\kappa)$, then

\[
(n + c_1(\kappa)) \langle p, q \rangle_\kappa = \left( \sum_{i=1}^{d} x_i \langle D_i p, q \rangle_\kappa \right) = \sum_{i=1}^{d} \langle D_i p, D_i q \rangle_\kappa \\
= \sum_{i=1}^{d} \langle D_i q, D_i p \rangle_\kappa = \left( \sum_{i=1}^{d} x_i \langle D_i q, p \rangle_\kappa \right) \\
= (n + c_1(\kappa)) \langle q, p \rangle_\kappa ,
\]

where the inductive hypothesis was used to imply $\langle D_i p, D_i q \rangle_\kappa = \langle D_i q, D_i p \rangle_\kappa$. For fixed $p, q$ $(\langle p, q \rangle_\kappa - \langle q, p \rangle_\kappa)$ is a polynomial in the values of $\kappa$ and $n + c_1(\kappa) = 0$ defines a hyperplane in the parameter space (the dimension equals the number of $G$-orbits in $R$). Thus $\langle p, q \rangle_\kappa = \langle q, p \rangle_\kappa$ for all $\kappa$. 

**Corollary 4.15** Suppose $\langle \cdot, \cdot \rangle_1$ is a bilinear symmetric form on $\Pi^d$ such that $\langle 1, 1 \rangle_1 = 1$ and $\langle x_i p, q \rangle_1 = \langle p, D_i q \rangle_1$ for all $p, q \in \Pi^d$ and $1 \leq i \leq d$, then $\langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_\kappa$.

**Proof.** Let $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$ for some $n \geq 0$. Let $q \in \sum_{m=0}^{n-1} \Pi_m^d$. By hypothesis $\langle x^\alpha, q \rangle_1 = \langle 1, \rho (x^\alpha) q \rangle_1 = 0$. If $q \in \Pi_n^d$ then $\langle x^\alpha, q \rangle_1 = \langle 1, \rho (x^\alpha) q \rangle_1 = \langle 1, \rho (x^\alpha) q \rangle_\kappa = \langle x^\alpha, q \rangle_\kappa$. By linearity and symmetry the proof is complete. 

**4.1 The Gaussian form**

The operator $e^{\Delta_{\kappa}/2}$ maps $\Pi_n^d$ into $\sum_{m=0}^{n} \Pi_m^d$ and its inverse is $e^{-\Delta_{\kappa}/2}$.

**Definition 4.16** The Gaussian form on $\Pi^d$ is given by

\[
\langle p, q \rangle_g := \left( e^{\Delta_{\kappa}/2} p, e^{\Delta_{\kappa}/2} q \right)_\kappa .
\]

**Proposition 4.17** The Gaussian form is symmetric, bilinear and if $p, q \in \Pi^d$ then $\langle wp, wq \rangle_g = \langle p, q \rangle_g$ for all $w \in G$, $\langle D_i p, q \rangle_g = \langle p, (x_i - D_i) q \rangle_g$ and $\langle x_i p, q \rangle_g = \langle p, x_i q \rangle_g$ for $1 \leq i \leq d$.

**Proof.** The first claim follows from the property $w \Delta_{\kappa} = \Delta_{\kappa} w$. By Theorem 4.10 $[\Delta_{\kappa}, m_{x_i}] = 2D_i$ ( $m_{x_i}$ denotes multiplication by $x_i$). Using the general formula $[A^n, B] = A^n B + [A, B] A^{n-1}$ repeatedly shows $[\Delta_{\kappa}, m_{x_i}] = 2n D_i \Delta_{\kappa}^{n-1}$. This implies $e^{\Delta_{\kappa}/2} x_i p (x) - x_i e^{\Delta_{\kappa}/2} p (x) = D_i e^{\Delta_{\kappa}/2} p (x)$ for any $p \in \Pi^d$. Thus

\[
\langle D_i p, q \rangle_g = \left( D_i e^{\Delta_{\kappa}/2} p, e^{\Delta_{\kappa}/2} q \right)_\kappa = \left( e^{\Delta_{\kappa}/2} p, x_i e^{\Delta_{\kappa}/2} q \right)_\kappa \\
= \left( e^{\Delta_{\kappa}/2} p, e^{\Delta_{\kappa}/2} (x_i - D_i) q \right)_\kappa = \langle p, (x_i - D_i) q \rangle_g .
\]

Finally $\langle p, x_i q \rangle_g = \langle D_i p, q \rangle_g + \langle p, D_i q \rangle_g$, which is symmetric in $p, q$. 

12
Thus the Gaussian form satisfies $\langle p, q \rangle_g = \langle 1, pq \rangle_g$. This suggests that there may be an integral formula; this is indeed the situation when $\kappa \geq 0$. The properties in the Proposition imply a uniqueness result by “reading the proof backwards”; set $\langle p, q \rangle_1 = \langle e^{-\Delta/2} p, e^{-\Delta/2} q \rangle_g$ and use Corollary 4.15.

**Definition 4.18** For $\kappa \geq 0$ the fundamental $G$-invariant weight function is

$$w_\kappa(x) := \prod_{v \in R^+} |\langle x, v \rangle|^{2\kappa(v)} , x \in \mathbb{R}^d.$$  

The $G$-invariance is a consequence of the definition of multiplicity functions.

**Definition 4.19** For $p, q \in \Pi^d$ and $\kappa \geq 0$ let

$$\langle p, q \rangle_2 := \frac{c_\kappa}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p(x) q(x) w_\kappa(x) e^{-|x|^2/2} dx,$$

where $c_\kappa$ is the normalizing constant resulting in $\langle 1, 1 \rangle_2 = 1$.

The constant $c_\kappa$ is related to the Macdonald-Mehta integral. This will be discussed below. It is not involved in the following.

**Theorem 4.20** If $p, q \in \Pi^d$, $1 \leq i \leq d$ then $\langle D_i p, q \rangle_2 = \langle p, (x_i - D_i) q \rangle_2$. The forms $\langle \cdot, \cdot \rangle_2$ and $\langle \cdot, \cdot \rangle_g$ are equal when $\kappa \geq 0$.

**Proof.** Let $p, q \in \Pi^d, 1 \leq i \leq d$. Note

$$\frac{\partial}{\partial x_i} \left( w_\kappa(x) e^{-|x|^2/2} \right) = \left( -x_i + 2 \sum_{v \in R^+} \frac{\kappa(v)}{\langle x, v \rangle} v_i \right) w_\kappa(x) e^{-|x|^2/2},$$

(in the special case $0 < \kappa(v) < \frac{1}{2}$ the formula is valid provided $\langle x, v \rangle \neq 0$.) It suffices to show the following integral vanishes:

$$\int_{\mathbb{R}^d} (qD_i p + pD_i q - x_i pq) w_\kappa e^{-|x|^2/2} dx$$

$$= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (pq) w_\kappa e^{-|x|^2/2} dx - \int_{\mathbb{R}^d} x_i pq w_\kappa e^{-|x|^2/2} dx$$

$$+ \sum_{v \in R^+} \kappa(v) v_i \int_{\mathbb{R}^d} \frac{2p(x) q(x) - p(s_v x) q(x) - p(x) q(s_v x)}{\langle x, v \rangle} w_\kappa e^{-|x|^2/2} dx$$

$$= - \sum_{v \in R^+} \kappa(v) v_i \int_{\mathbb{R}^d} \frac{p(s_v x) q(x) + p(x) q(s_v x)}{\langle x, v \rangle} w_\kappa e^{-|x|^2/2} dx = 0,$$
because in each term the integrand is odd under the action of a reflection $s_v$. Integration by parts and exponential decay shows $$\int_{\mathbb{R}^d} \frac{\partial}{\partial x^i} (pq) \, w_\kappa e^{-|x|^2/2} \, dx = -\int_{\mathbb{R}^d} pq \frac{\partial}{\partial x^i} (w_\kappa e^{-|x|^2/2}) \, dx.$$ The terms in the sum over $R_+$ have the singularity $|\langle x, v \rangle|^2 \kappa(v)^{-1}$ which is integrable for $\kappa(v) > 0$ (the terms with $\kappa(v) = 0$ do not appear).

As a consequence of the Theorem, and of the invertibility of $e^{\Delta_\kappa/2}$ on polynomials it follows that the forms $\langle \cdot, \cdot \rangle_\kappa$ and $\langle \cdot, \cdot \rangle_g$ are positive-definite when $\kappa \geq 0$.

There is an elegant formula for $c_\kappa$ when there is only one $G$-orbit in $R$. Recall the discriminant $a_R(x) = \prod_{v \in R_+} \langle x, v \rangle$.

**Theorem 4.21** Suppose $G$ has just one conjugacy class of reflections, $|v|^2 = 2$ for each $v \in R$ and $\kappa_0 > 0$, then

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} |a_R(x)|^{2\kappa_0} e^{-|x|^2/2} \, dx = \prod_{i=1}^d \frac{\Gamma(1 + n_i \kappa_0)}{\Gamma(1 + \kappa_0)},$$

where $\{n_i : 1 \leq i \leq d\}$ is the set of fundamental degrees of $G$ (see Theorem 3.1).

(If the rank of $G$ is less than $d$ then some of the degrees equal 1.) The integral for the symmetric group can be deduced from Selberg’s integral formula. Macdonald conjectured a formula for $c_\kappa$ for any reflection group. Opdam \[18\] proved the formula for all cases, except for a constant (independent of $\kappa$) multiple for $H_3$ and $H_4$; the constant was verified by Garvan with a computer-assisted proof. Etingof \[7\] extended Opdam’s method to a general proof for the one-class type.

**Corollary 4.22** With the notation as in the Theorem and $\kappa(v) = \kappa_0$ for all $v \in R$

$$\langle a_R, a_R \rangle_\kappa = \langle a_R, a_R \rangle_g = \#G \prod_{i=1}^d \prod_{j=1}^{n_i} (j + n_i \kappa_0).$$

**Proof.** The polynomial $\Delta_\kappa a_R = 0$ because it is alternating and of degree $\#R_+ - 2$, and $a_R$ is the minimal degree nonzero alternating polynomial. Thus $e^{\Delta_\kappa/2} a_R = a_R$. From the definition of $c_\kappa$ it follows that

$$\langle a_R, a_R \rangle_g = \frac{c_{\kappa_0}}{c_{\kappa_0 + 1}} = (1 + \kappa_0)^{-d} \prod_{i=1}^d \frac{\Gamma(1 + n_i + n_i \kappa_0)}{\Gamma(1 + n_i \kappa_0)} = (1 + \kappa_0)^{-d} \prod_{i=1}^d \prod_{j=1}^{n_i} (j + n_i \kappa_0) = \prod_{i=1}^d \prod_{j=1}^{n_i} (j + n_i \kappa_0),$$

and $\prod_{i=1}^d n_i = \#G$. \[ \square \]
The formula is valid for all $\kappa_0 \in \mathbb{C}$ because it is a polynomial identity.

The indecomposable reflection groups with two classes of reflections consist of $I_2(2m)$ with $m \geq 2$, $B_d$ and $F_4$. To get a convenient expression for the dihedral group $I_2(2m)$ let $z = x_1 + ix_2$ and denote the values of $\kappa$ by $\kappa_0$ and $\kappa_1$; write the weight function as $w_\kappa(x) = |z^m - \overline{z}^m|^{2\kappa_0} |z^m + \overline{z}^m|^{2\kappa_1}$. Then

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} w_\kappa(x) e^{-|x|^2/2} dx = 2^{m(\kappa_0 + \kappa_1)} \frac{\Gamma(1 + 2\kappa_0) \Gamma(1 + 2\kappa_1) \Gamma(1 + m(\kappa_0 + \kappa_1))}{\Gamma(1 + \kappa_0) \Gamma(1 + \kappa_1) \Gamma(1 + \kappa_0 + \kappa_1)}.$$

For the hyperoctahedral group $B_d$ (see Section 2.2 for the root system) the integral is

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} \prod_{i=1}^d |x_i|^{2\kappa_1} \prod_{1 \leq i < j \leq d} \left| x_i^2 - x_j^2 \right|^{2\kappa_0} e^{-|x|^2/2} dx$$

$$= 2^d((d-1)\kappa_0 + \kappa_1) \prod_{i=1}^d \frac{\Gamma(1 + i\kappa_0) \Gamma((i - 1)\kappa_0 + \kappa_1 + 1/2)}{\Gamma(1 + \kappa_0) \Gamma(1/2)}.$$

In the formula $\Gamma(1/2)$ is used in place of $\sqrt{\pi}$ for the sake of appearance. The formula can be derived from Selberg’s integral.

Using the notation of Section 2.5 for the group $F_4$ (a special case of Opdam’s result)

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \prod_{1 \leq i < j \leq 4} \left| x_i^2 - x_j^2 \right|^{2\kappa_1} \prod_{1 \leq i < j \leq 4} \left| y_i^2 - y_j^2 \right|^{2\kappa_2} e^{-|x|^2/2} dx$$

$$= 2^{12(\kappa_1 + \kappa_2)} \frac{\Gamma(2\kappa_1 + \kappa_2 + 1/2) \Gamma(\kappa_1 + 2\kappa_2 + 1/2) \Gamma(3\kappa_1 + 3\kappa_2 + 1/2)}{\Gamma(1/2)^3} \times \frac{\Gamma(4\kappa_1 + 4\kappa_2 + 1) \Gamma(2\kappa_1 + 1) \Gamma(3\kappa_1 + 1)}{\Gamma(\kappa_1 + 2 + 1)^2 \Gamma(\kappa_1 + 1)^2}.$$

The formula agrees with the simpler single-class result when $\kappa_1 = \kappa_2$, and with the $D_4$ value when $\kappa_2 = 0$ (by use of the Gamma function duplication formula).

5 Harmonic polynomials

The Gaussian form is an important part of our analysis. Accordingly the polynomials in the kernel of $\Delta_\kappa$ have properties relevant to the two forms.

**Definition 5.1** Let $H_\kappa := \{ p \in \Pi^d : \Delta_\kappa p = 0 \}$ and $H_{\kappa,n} := H_\kappa \cap \Pi_{\kappa}^d$ for $n = 0, 1, 2, \ldots$. These are the spaces of harmonic and harmonic homogeneous polynomials, respectively. Let $\gamma_\kappa := \sum_{v \in R_\kappa} \kappa(v)$ ($w_\kappa$ is positively homogeneous of degree $2\gamma_\kappa$).
For convenience we let $|x|^2$ denote both the polynomial in $\Pi_d^2$ and the corresponding multiplier operator. We will show that $\Pi_d^n = \sum_{j=0}^{|n/2|} \oplus |x|^{2j} H_{\kappa,n-2j}$ for each $n \geq 2$ provided that $\gamma_\kappa + \frac{d}{2} \notin -\mathbb{N}_0$. Trivially $\Pi_d^n = H_{\kappa,n}$ for $n = 0, 1$. Note that the proof can not use any nonsingularity property of the form $\langle \cdot, \cdot \rangle_\kappa$.

Lemma 5.2 For $m = 1, 2, 3, \ldots$

$$\left[ \Delta_\kappa |x|^{2m} \right] = 2m |x|^{2(m-1)} \left( 2m - 2 + 2\gamma_\kappa + 2 \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} \right).$$

Corollary 5.3 If $m, n, k = 1, 2, 3, \ldots$ and $p \in H_{\kappa,n}$ then

1) $\Delta_\kappa \left( |x|^{2m} p(x) \right) = 2m (2m - 2 + d + 2\gamma_\kappa + 2n) |x|^{2m-2} p(x)$,

2) $\Delta_k^k \left( |x|^{2m} p(x) \right) = 4k (-m)_k (1 - m - d/2 - \gamma_\kappa - n)_k |x|^{2m-2k} p(x)$.

Part 2) implies that $\Delta_k^k \left( |x|^{2m} p(x) \right) = 0$ if $k > m$. This leads to an orthogonality relation.

Proposition 5.4 Suppose $m, k \leq \frac{n}{2}, n \geq 2, p \in H_{\kappa,n-2k}$, and $q \in H_{\kappa,n-2m}$; if $m \neq k$ then $\langle |x|^{2k} p, |x|^{2m} q \rangle_\kappa = 0$.

Proof. By the symmetry of the form we may assume $k > m$; then $\langle |x|^{2k} p, |x|^{2m} q \rangle_\kappa = \langle p, \Delta_k^k |x|^{2m} q \rangle_\kappa = 0$. ■

Definition 5.5 Suppose $\gamma_\kappa + \frac{d}{2} \neq 0, -1, -2, \ldots$ and $n = 2, 3, 4, \ldots$ then let

$$\pi_{\kappa,n} \colon= \sum_{j=0}^{|n/2|} \frac{1}{4j! (-\gamma_\kappa - n + 2 + d/2)_j} |x|^{2j} \Delta_j^\kappa;$$

for $n = 0, 1$ let $\pi_{\kappa,n} \colon= I$.

The following is a version of Dixon’s summation theorem (see [17, 16.4.4]).

Lemma 5.6 Suppose $k \in \mathbb{N}_0$ and $a, b$ satisfy $a + 1, a - b + 1 \notin -\mathbb{N}_0$ then

$$3F_2 \left( \begin{array}{c} -k, a, b \\ k + a + 1, a - b + 1 \end{array} ; 1 \right) = \frac{(a + 1)_k (\frac{a}{2} - b + 1)_k}{(\frac{a}{2} + 1)_k (a - b + 1)_k}$$
Proposition 5.7 If $\gamma_k + \frac{d}{2} \not\in -N_0$ and $p \in \Pi_n^d$, $n = 2, 3, 4, \ldots$ then $\pi_{\kappa,n} p \in H_{\kappa,n}$; if $p \in H_{\kappa,n}$ then $p = \pi_{\kappa,n} p$, that is, $\pi_{\kappa,n}$ is a projection. Furthermore

$$ p = \sum_{j=0}^{[n/2]} \frac{1}{4^{j!} (\gamma_k + d/2 + n - 2j)_j} |x|^{2j} \pi_{\kappa,n-2j} (\Delta_k^j p). $$

Proof. The first part is a consequence of Lemma 5.2. The proof of the expansion formula depends on Lemma 5.6. Set $a = 1 - n - \gamma_k - \frac{d}{2}$ and $b = \frac{a}{2} + 1$ then the coefficient of $(4^k k!)^{-1} |x|^{2k} \Delta_k^k$ in the right side is

$$ \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(1-a-2j)_j (a+1+2j)_{k-j}} = \sum_{j=0}^{k} \frac{(-1)^j (a+1)_{2j}}{j! (a+j)_{j+1} (a+1+k)_{k-j}} $$

which vanishes for $k \geq 1$ because of the term $(\frac{a}{2} - b + 1)_k = (0)_k$ in the summation formula. In the transformation $(a+1)_{2j+1} = \frac{(a+2j+1)}{a+2j} = (a/2)_{j+1}$ was used in the last step. The sum equals 1 when $k = 0$. The identity holds for generic $\gamma_k$, and the hypothesis $\gamma_k + \frac{d}{2} \not\in -N_0$ insures that the terms in $\pi_{\kappa,n-2j}$ are well-defined.

This establishes the validity of $\Pi_n^d = \bigoplus_{j=0}^{[n/2]} \oplus |x|^{2j} H_{\kappa,n-2j}$, provided $\gamma_k + \frac{d}{2} \not\in -N_0$. (Argue by induction that $H_{\kappa,n} \cap |x|^2 \Pi_{n-2} = \{0\}$.) To transfer these results to the Gaussian form let $p \in H_{\kappa,n}, s \in \mathbb{R}$ and evaluate

$$ e^{s \Delta_k} (|x|^{2m} p(x)) = \sum_{j=0}^{m} \frac{1}{j!} s^j \Delta_k^j (|x|^{2m} p(x)) $$

$$ \sum_{j=0}^{m} \frac{1}{j!} (4s)^j (-m)_j (1 - m - d/2 - \gamma_k - n)_j |x|^{2m-2j} p(x) $$

$$ = (4s)^m m! L_m^{(\alpha)} \left( \frac{-|x|^2}{4s} \right) p(x), $$

where $\alpha = \gamma_k + \frac{d}{2} + n - 1$ and $L_m^{(\alpha)}$ denotes the Laguerre polynomial of degree $m$ and index $\alpha$; it is defined provided $\alpha + 1 \not\in -N_0$ and part of an orthogonal family of polynomials if $\alpha > -1$. Here we use $s = -\frac{1}{2}$.

We list the conditions on $\gamma_k$ specialized to some reflection groups:
• $I_2(2m): m (\kappa_0 + \kappa_1) + 1 \notin \mathbb{N}_0$;
• $A_{d-1} \subset \mathbb{R}^d$: $d ( (d - 1) \kappa + 1) \notin \mathbb{N}_0$;
• $B_d$: $d ( (d - 1) \kappa_0 + \kappa_1 + \frac{1}{2}) \notin \mathbb{N}_0$.

When $\kappa \geq 0$ the Gaussian form can be related to an integral over the unit sphere, and there is an analog of the spherical harmonics. Let $\omega$ denote the normalized rotation-invariant measure on the surface of the unit sphere $S_{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \}$. Suppose $f$ is continuous and integrable over $\mathbb{R}^d$ then

$$\int_{\mathbb{R}^d} f(x) \, dx = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty r^{d-1} \, dr \int_{S_{d-1}} f(ru) \, d\omega(u).$$

The constant multiplier is evaluated by setting $f(x) = e^{-|x|^2/2}$. Now suppose $f$ is positively homogeneous of degree $\beta$ (that is, $f(tx) = t^\beta f(x)$ for $t > 0$) and $\beta + d > 1$ then

$$(2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-|x|^2/2} \, dx = 2^{\beta/2} \frac{\Gamma\left(\frac{\beta+d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \int_{S_{d-1}} f(u) \, d\omega(u).$$

To normalize the measure $w_\kappa d\omega$ set $f = w_\kappa$ (with $\beta = 2\gamma_\kappa$) and let

$$c_{\kappa,S}^{-1} := \int_{S_{d-1}} w_\kappa \, d\omega = 2^{-\gamma_\kappa} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\gamma_\kappa + \frac{d}{2}\right)} c_{\kappa}^{-1}.$$ 

Observe that the condition $\gamma_\kappa + \frac{d}{2} \notin \mathbb{N}_0$ appears again.

**Proposition 5.8** Suppose $p \in \mathcal{H}_\kappa,n$ and $q \in \mathcal{H}_\kappa,m$,
1) if $m \neq n$ then $c_{\kappa,S} \int_{S_{d-1}} pq w_\kappa d\omega = 0$;
2) if $m = n$ then

$$c_{\kappa,S} \int_{S_{d-1}} pq w_\kappa d\omega = \frac{c_n (2\pi)^{-d/2}}{2^n (\gamma_\kappa + \frac{d}{2})_n} \int_{\mathbb{R}^d} p(x) q(x) w_\kappa(x) e^{-|x|^2/2} \, dx$$

$$= \frac{1}{2^n (\gamma_\kappa + \frac{d}{2})_n} \langle p, q \rangle_\kappa.$$

That is, the spaces $\{ \mathcal{H}_\kappa,n : n \in \mathbb{N}_0 \}$ are pairwise orthogonal in $L^2(S_{d-1}, w_\kappa d\omega)$. By Proposition 5.7 each polynomial agrees with a harmonic one on $S_{d-1}$ and so $L^2(S_{d-1}, w_\kappa d\omega) = \sum_{n=0}^\infty \mathcal{H}_\kappa,n$ by the density of polynomials. In the next section we consider the reproducing and Poisson kernels.

By a version of Hamburger’s theorem $\Pi^d$ is dense in $L^2\left(\mathbb{R}^d, w_\kappa(x) e^{-|x|^2/2} dx\right)$. There exist orthogonal bases consisting of products of harmonic polynomials and Laguerre polynomials with argument $|x|^2/2$. In general there is no explicit orthogonal basis for $\mathcal{H}_\kappa,n$. 

18
Definition 5.9 For $n \in \mathbb{N}_0$ let $X_n := \text{span} \left\{ p(x) L^{(\alpha_n)}_{m}\left(\frac{|x|^2}{2}\right) : p \in \mathcal{H}_{\kappa,n}, m \in \mathbb{N}_0 \right\}$, where $\alpha_n = \gamma_n + \frac{d}{2} + n - 1$.

Suppose $k, l, m, n \in \mathbb{N}_0$ and $p \in \mathcal{H}_{\kappa,n}, q \in \mathcal{H}_{\kappa,l}$ then \( \left\langle x^{2m} p(x), x^{2k} q(x) \right\rangle_\kappa = 0 \) unless $n = l$ and $m = k$; if $2m + n \neq 2k + l$ this follows from part (1) of Theorem 4.14 Proposition 5.4 applies if $2m + n = 2k + l$ and $m \neq k$. By equation 5.1 with $s = -\frac{1}{2}$ \( \left\langle L^{(\alpha_n)}_m \left(\frac{|x|^2}{2}\right) p(x), L^{(\alpha_l)}_k \left(\frac{|x|^2}{2}\right) q(x) \right\rangle_g = 0 \) unless $n = l$ and $m = k$. Thus

\[
L^2 \left( \mathbb{R}^d, w_\kappa(x) e^{-|x|^2/2} dx \right) = \sum_{n=0}^{\infty} \oplus X_n.
\]

Suppose $p, q \in \mathcal{H}_{\kappa,n}$ and $k, m \in \mathbb{N}_0$ then

\[
\left\langle L^{(\alpha_n)}_m \left(\frac{|x|^2}{2}\right) p(x), L^{(\alpha_l)}_k \left(\frac{|x|^2}{2}\right) q(x) \right\rangle_g = \delta_{mk} \frac{(\alpha_n + 1)_m}{m!} \langle p, q \rangle_g,
\]

and \( \langle p, q \rangle_g = 2^n (\gamma_n + \frac{d}{2})_n c_{\kappa,S} \int_{\mathbb{R}^{d-1}} pq w_\kappa d\omega \). These formulae show how an orthogonal basis for $\mathcal{H}_{\kappa,n}$ can be used to produce such a basis for $L^2 \left( \mathbb{R}^d, w_\kappa(x) e^{-|x|^2/2} dx \right)$.

6 The intertwining operator and the Dunkl kernel

Several important objects can be defined when the form $\left\langle \cdot, \cdot \right\rangle_\kappa$ is nondegenerate for specific numerical values of $\kappa$. We refer to “generic” $\kappa$ when $\kappa$ has some transcendental value (formal parameter), and to “specific” $\kappa$ when $\kappa$ takes on real values. The form $\left\langle \cdot, \cdot \right\rangle_\kappa$ is defined for all $\kappa$, and so is the following operator.

Definition 6.1 The operator $V^0_\kappa$ on $\Pi^d$ is given by

\[
V^0_\kappa p(y) := \left\langle e^{(y,x)}, p(x) \right\rangle_\kappa, p \in \Pi^d,
\]

\[
V^0_\kappa p(y) = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| = n} \frac{1}{\alpha!} y^\alpha D^\alpha p, p \in \Pi^d_n.
\]

The second equation is the explicit effect of the formal operator defined in the first equation. Note $e^{(y,x)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{|\alpha| = n} \binom{n}{\alpha} x^\alpha y^\alpha$ (where $\alpha! = \prod_{i=1}^{d} \alpha_i!$ and $\binom{n}{\alpha} = \frac{n!}{\alpha!}$) and $D^\alpha p = \left\langle x^\alpha, p(x) \right\rangle$ for $p \in \Pi^d_n, |\alpha| = n \in \mathbb{N}_0$. Also $V^0_\kappa 1 = 1$.

Proposition 6.2 If $1 \leq i \leq d$ and $p \in \Pi^d$ then $\frac{\partial}{\partial x_i} V^0_\kappa p(x) = V^0_\kappa D_i p(x)$. If $w \in G$ then $V^0_\kappa wp = wV^0_\kappa p$. 

19
Definition 6.6 \( \kappa \) that the singular set is \( \{ -1 \} \).

For a specific \( \kappa \) of degree \( n \), of \( \Pi^d \) homogeneous subspaces. We have

\[ \langle G, x, y \rangle = \int_{x, y} \left( \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_i} \kappa \right) (1 - s_v). \]

If \( p \in \text{Rad}(\kappa) \) then \( \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} p(x) \in \text{Rad}(\kappa) \) by part (1), and hence \( \text{Rad}(\kappa) \) is the sum of its homogeneous subspaces.

Part (1) implies that \( \text{Rad}(\kappa) \) is a rational Cherednik algebra (an abstract algebra isomorphic to the algebra of operators on \( \Pi^d \) generated by the multipliers \( x_i \), the operators \( D_i \) and the group \( G \)). We can now set up the key decomposition of the parameter space. Multiplicity functions can be identified with points in \( \mathbb{R}^c \) where \( c \) is the number of \( G \)-orbits in \( R \).

Definition 6.5 Let \( \Lambda^0 := \{ \kappa : \text{Rad}(\kappa) \neq \{0\} \} \), the singular set, and let \( \Lambda^{\text{reg}} := \{ \kappa : \text{Rad}(\kappa) = \{0\} \} \), the regular set.

As a result of the papers of Opdam [18] and Dunkl, de Jeu and Opdam [5] there is a concise description of the singular set for indecomposable reflection groups: The value of the integral \( \int_{\mathbb{R}^d} w_\kappa(x) e^{-|x|^2/2} dx \) is a meromorphic function of \( \kappa \); the integral is defined for \( \kappa \geq 0 \) but the value extends analytically to \( \mathbb{C} \). The poles coincide with the singular set. For the one-class type the singular set is \( \{ -j/n_m : j \in \mathbb{N}_0, 1 \leq m \leq d, j/n_m \notin \mathbb{Z} \} \), where the rank of \( G \) is \( d \) and the fundamental degrees are \( n_1, \ldots, n_d \). The realization of the Gaussian form as an integral shows that \( \kappa \geq 0 \) implies \( \kappa \in \Lambda^{\text{reg}} \). In the next paragraphs we use superscripts \((x), (y)\) to indicate the variable on which an operator acts.

Definition 6.6 For \( \kappa \in \Lambda^{\text{reg}} \) let \( V_\kappa := (V_\kappa^0)^{-1} \), the intertwining operator, and let \( K_{\kappa,n}(x,y) := \frac{1}{n!} V_\kappa^0(x,y) \). The polynomial \( K_{\kappa,n}(x,y) \) is homogeneous of degree \( n \) in both \( x \) and \( y \).
Theorem 6.7 \( K_{\kappa,n} \) and \( V_\kappa \) have the following properties:

1) if \( p \in \Pi^d \) and \( 1 \leq i \leq d \) then \( D_i (V_\kappa p)(x) = V_\kappa \left( \frac{\partial}{\partial x_i} p(x) \right) \); if \( w \in G \) then \( wV_\kappa = V_\kappa w \);

2) \( V_\kappa \) maps \( \Pi_n^d \) one-to-one onto \( \Pi_n^d \) for each \( n \);

3) \( D_i^{(y)} K_{\kappa,n} (x,y) = x_i K_{\kappa,n-1} (x,y) \);

4) \( \langle K_{\kappa,n} (x,\cdot) , p \rangle_\kappa = p(x) \) for \( p \in \Pi_n^d \);

5) \( K_{\kappa,n} (x,y) = K_{\kappa,n} (y,x) \) for all \( x,y \in \mathbb{R}^d \), and \( K_{\kappa,n} (wx,wy) = K_{\kappa,n} (x,y) \) for each \( w \in G \).

Proof. Parts (1) and (3) are straightforward. Part (2) holds because \( V_\kappa^0 \) maps \( \Pi_n^d \) into \( \Pi_n^d \) and its inverse exists. For part (4) let \( \partial_\alpha^\kappa = \prod_{i=1}^d \left( \frac{\partial}{\partial y_i} \right)^{\alpha_i} ; \) if \( p \in \Pi_n^d \) then \( p(x) = \sum_{\alpha \in \mathbb{N}_0^d,|\alpha|=n} \frac{1}{\alpha!} x^\alpha p \left( \partial_\alpha^\kappa y^\alpha \right) \). Apply \( V_\kappa^0 \) to both sides (and the right side is independent of \( y \)) thus

\[
\begin{align*}
p(x) &= V_\kappa^0 p(x) = \sum_{\alpha \in \mathbb{N}_0^d,|\alpha|=n} \frac{1}{\alpha!} x^\alpha V_\kappa^0 (\partial_\alpha^\kappa y^\alpha) \\
&= \sum_{\alpha \in \mathbb{N}_0^d,|\alpha|=n} \frac{1}{\alpha!} x^\alpha \left( D^{(y)} \right) V_\kappa^0 (y^\alpha) = \langle p, K_{\kappa,n} (x,\cdot) \rangle_\kappa ;
\end{align*}
\]

and the form \( \langle \cdot , \cdot \rangle_\kappa \) is symmetric. For any \( p,q \in \Pi_n^d \), by part (4),

\[
\begin{align*}
\langle p,q \rangle_\kappa &= \left\langle K_{\kappa,n} \left( \cdot , D^{(y)} \right) p(y), q \right\rangle_\kappa = K_{\kappa,n} \left( D^{(x)}, D^{(y)} \right) p(y) q(x) \\
&= \langle q,p \rangle_\kappa .
\end{align*}
\]

This implies \( K_{\kappa,n} (x,y) = K_{\kappa,n} (y,x) \), because \( \kappa \in \Lambda_{reg} \). Let \( f_x(y) = K_{\kappa,n} (x,y) \), \( w \in G \) and \( p \in \Pi_n^d \), then \( p(x) = \langle f_x, p \rangle_\kappa \) and

\[
\begin{align*}
wp(x) &= p(w^{-1}x) = \langle f_{w^{-1}x}, p \rangle_\kappa \\
&= \langle f_x, wp \rangle_\kappa = \langle w^{-1}f_x, p \rangle_\kappa ;
\end{align*}
\]

thus \( K_{\kappa,n} (x,wy) = w^{-1}f_x = f_{w^{-1}x} = K_{\kappa,n} (w^{-1}x,y) \). \( \blacksquare \)

Denote the formal sum \( \sum_{n=0}^{\infty} K_{\kappa,n} (x,y) \) by \( K_\kappa (x,y) \). The question now arises: does the series converge in a useful way? There are some strong results for \( \kappa \geq 0 \). For \( x,y \in \mathbb{R}^d \) let \( \rho (x,y) = \max_{w \in G} |\langle x,wy \rangle| \).

Theorem 6.8 Suppose \( \kappa \geq 0 \) and \( x,y \in \mathbb{R}^d \), then \( |K_{\kappa,n} (x,y)| \leq \frac{1}{n!} \rho (x,y)^n \) for all \( n \in N_0 \), the series for \( K_\kappa \) converges uniformly and absolutely on compact subsets of \( \mathbb{R}^d \times \mathbb{R}^d \), and \( |K_\kappa (x,y)| \leq e^{\rho (x,y)} \).

Theorem 6.9 Suppose \( \kappa \geq 0 \) and \( x \in \mathbb{R}^d \) then there exists a Baire probability measure \( \mu_x \) with \( spt (\mu_x) \subset co \{ wx : w \in G \} \) (the convex hull) such that \( V_\kappa p(x) = \int_{\mathbb{R}^d} pd\mu_x \) for each \( p \in \Pi^d \).
Corollary 6.10 If $x, y \in \mathbb{R}^d$ then $K_\kappa(x, y) > 0$ and $|K_\kappa(x, iy)| \leq 1$.

Proof. By Fubini’s theorem summation and integration can be interchanged in $K_\kappa(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} V^{(y)}_\kappa(x, y)^n = \int_{\mathbb{R}^d} e^{i(x,z)} d\mu_{\kappa}(z)$. By homogeneity $K_\kappa(x, iy) = \sum_{n=0}^{\infty} i^n K_{\kappa,n}(x, y) = \int_{\mathbb{R}^d} e^{i(x,z)} d\mu_{\kappa}(z)$. \hfill $\blacksquare$

Theorem 6.8 was shown in Dunkl’s paper [2] constructing $K_\kappa$, which is now called the Dunkl kernel. Later Rösler [20] proved Theorem 6.9. The inequality $|K_\kappa(x, iy)| \leq 1$ will be used in the later section on the Dunkl transform. There is a mean-value type result for $K_\kappa$.

Proposition 6.11 Suppose $p \in \Pi^d$ then

$$\frac{c_\kappa}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} V_{\kappa}(p) w_\kappa(x) e^{-|x|^2/2} dx = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} p(x) e^{-|x|^2/2} dx.$$

Proof. By Theorem 4.20 the left side equals $\langle e^{\Delta/2} V_\kappa(p), e^{\Delta/2} 1 \rangle_\kappa = \langle V_\kappa e^{\Delta/2} p, 1 \rangle_\kappa = \langle e^{\Delta/2} p, 1 \rangle_0$ which equals the right side (the subscript 0 indicates $\kappa = 0$). \hfill $\blacksquare$

Corollary 6.12 If $f \in C(\{x \in \mathbb{R}^d : |x| \leq 1\})$ and $\kappa > 0$ then

$$c_{\kappa,S} \int_{S_{d-1}} (V_{\kappa} f) w_{\kappa} d\omega = 2 \frac{\Gamma(\gamma_{\kappa} + d/2)}{\Gamma(\gamma_{\kappa}) \Gamma(d/2)} \int_{|x| \leq 1} f(x) \left(1 - |x|^2\right)^{\gamma_{\kappa}-1} dx.$$

This is proven by applying the Proposition to a homogeneous polynomial and then evaluating the two integrals in spherical polar coordinates (see Proposition 5.8 only the even degree case need be computed). The formula extends to continuous functions by Theorem 6.9. This result is due to Xu [22].

Opdam [18] defined a Bessel function in the multiplicity function context. His approach was through $G$-invariant differential operators commuting with $\Delta + 2 \sum_{v \in \Pi_+} \kappa(v) \langle \bar{v}, \bar{x} \rangle$ (the differential part of $\Delta_\kappa$). The result is that $J_G(x, y) = \frac{1}{\#G} \sum_{w \in G} K_\kappa(wx, y)$ is real-entire in $x, y$ for each $\kappa \in \Lambda^{reg}$, and $J_G(x, y)$ is meromorphic in $\kappa$ with poles on $\Lambda^0$. In the paper [18] Rem. 6.12] Opdam observed that $J_G$ can be interpreted as a spherical function on a Euclidean symmetric space, when $G$ is a Weyl group and $\kappa$ takes values in certain discrete sets.

The properties of $K_{\kappa,n}$ described in Theorem 6.7 extend to $K_\kappa$:

1) $D_i^{(y)} K_{\kappa}(x, y) = x_i K_{\kappa}(x, y)$ for $1 \leq i \leq d$,
2) $\langle K_{\kappa}(x, \cdot ), p \rangle_\kappa = p(x)$ for $p \in \Pi^d$;
3) $K_{\kappa}(x, y) = K_{\kappa}(y, x)$ for all $x, y \in \mathbb{R}^d$, and $K_{\kappa}(wx, wy) = K_{\kappa}(x, y)$ for each $w \in G$.

Property (3) shows that $J_G(wx, y) = J_G(x, wy) = J_G(x, y)$ for all $w \in G$. 

22
6.1 Example: $Z_2$

The objects described above can be stated explicitly for the smallest reflection group. We use $\kappa$ for the value of the multiplicity function and suppress the subscript “1” (for example, in $x_1$). Throughout $n \in \mathbb{N}_0$.

1) $D\kappa (x) = \partial_x \kappa (x) + \kappa (x) \frac{p(x) - p(-x)}{2}$,
2) $\langle x^{2n}, x^{2n} \rangle = 2^n n! (k + \frac{1}{2})^{n-1} (\kappa + \frac{1}{2})^{n+1}$,
3) $V_\kappa x^{2n} = \frac{(k + \frac{1}{2})^{n-1}}{(\frac{1}{2})^{n+1}} x^{2n}$, $V_\kappa x^{2n+1} = \frac{(k + \frac{1}{2})^{n+1}}{(\frac{1}{2})^{n+1}} x^{2n+1}$,
4) $\Lambda^0 = \left\{ -\frac{1}{2}, -\frac{3}{2}, \ldots \right\}$; if $\kappa = -\frac{1}{2} - n$ then $\text{Rad} (\kappa) = \text{span} \{ x^m : m \geq 2n + 1 \}$;
5) $\langle p, q \rangle = 2^{-\kappa - 1/2} \Gamma (k + \frac{1}{2})^{-1} \int_0^\infty p(x) q(x) |x|^{2\kappa} e^{-x^2/2} dx$, valid for $\kappa > -\frac{1}{2}$;
6) $\text{e}^{-D^2/2} x^{2n} = (-1)^m n! 2^n \frac{\Gamma (k^{(n-1/2)})}{\Gamma (k^{(n^{(n+1/2)})})} \left( \frac{x^2}{2} \right)$, $\text{e}^{-D^2/2} x^{2n+1} = (-1)^n n! 2^n x \frac{\Gamma (k^{(n+1/2)})}{\Gamma (k^{(n+1/2)})} \left( \frac{x^2}{2} \right)$;
7) $V_\kappa p(x) = \frac{\Gamma (\frac{1}{2}) \Gamma (k)}{\Gamma (k + \frac{1}{2})} \int_0^1 p(x^t) (1 + t)^{k} (1 - t)^{(k-1)} dt$, for $\kappa > 0$;
8) $K_\kappa (x, y) = \sum_{n=0}^\infty \frac{1}{n! (k + \frac{1}{2})^{n}} \left( \frac{2y}{x} \right)^{2n} + \frac{xy}{1 + 2k} \sum_{n=0}^\infty \frac{1}{(k + \frac{1}{2})^{(n)}} \frac{(2y^n)}{n^n}.$

Part (7) can be shown by substituting $p(x) = x^n$ in the integral and using part (3). In part (8) note that the modified Bessel function $I_{\kappa-1/2} (x) = \frac{(x/2)^{\kappa-1/2}}{\Gamma (k + \frac{1}{2})} \sum_{n=0}^\infty \frac{1}{n! (k + \frac{1}{2})^{n}} \left( \frac{2x^n}{y^n} \right).$ This partly explains the use of “Bessel” in naming $J_\kappa (x, y)$.

6.2 Asymptotic properties of the Dunkl kernel

De Jeu and Rößler [14] proved the following results concerning the limiting behavior of $K_\kappa (x, y)$ as $x \to \infty$, for $\kappa \geq 0$. The variable $x$ is restricted to the inside of a chamber. The fundamental chamber corresponds to the positive root system $R_+$, also let $\delta > 0$, then define

$C := \left\{ x \in \mathbb{R}^d : \langle x, v \rangle > 0 \forall v \in R_+ \right\},$

$C_\delta := \left\{ x \in \mathbb{R}^d : \langle x, v \rangle > \delta |x| \forall v \in R_+ \right\}.$

The walls of $C$ are the hyperplanes $v^\perp$ for the simple roots $v$.

**Theorem 6.13** For each $w \in G$ there is a constant $A_w$, such that for all $y \in C$

$$\lim_{x \in \mathbb{C}_\delta, |x| \to \infty} \sqrt{w_\kappa (x) w_\kappa (y)} e^{-i\langle x, wy \rangle} K_\kappa (ix, wy) = A_w.$$  

Recall $\gamma_\kappa = \sum_{v \in R_+} \kappa (v)$. For $z \in C$ with $\text{Re} z \geq 0$ let $z^{\gamma_\kappa}$ denote the principal branch ($1^{\gamma_\kappa} = 1$). (See Definition [14] for $\gamma_\kappa$.)

**Theorem 6.14** The constant $A_1 = (i^{\gamma_\kappa} \kappa)^{-1}$ and for $x, y \in C$

$$\lim_{\text{Re} z \geq 0, z \to \infty} z^{\gamma_\kappa} e^{-z \langle x, y \rangle} K_\kappa (z x, y) = \frac{1}{c_\kappa \sqrt{w_\kappa (x) w_\kappa (y)}}.$$
This limit is used in the context of a heat kernel.

6.3 The heat kernel

For functions defined on $\mathbb{R}^d \times (0, \infty)$ the generalized heat equation is
\[
\Delta_\kappa u (x, t) - \frac{\partial}{\partial t} u (x, t) = 0.
\]

The associated boundary-value problem is to find the solution $u$ such that $u (x, 0) = f (x)$ where $f$ is a given bounded continuous function on $\mathbb{R}^d$.

Definition 6.15 For $x, y \in \mathbb{R}^d$ and $t > 0$ the generalized heat kernel $\Gamma_\kappa$ is given by
\[
\Gamma_\kappa (t, x, y) := \frac{c_\kappa}{(2t)^{n+d/2} (2\pi)^{d/2}} \exp \left( -\frac{|x|^2 + |y|^2}{4t} \right) K_\kappa \left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right).
\]

Definition 6.16 For a bounded continuous function $f$ on $\mathbb{R}^d$ and $t > 0$ let
\[
H (t) f (x) := \int_{\mathbb{R}^d} f (y) \Gamma_\kappa (t, x, y) w_\kappa (y) \, dy.
\]

Theorem 6.17 Suppose $f \in \mathcal{S} (\mathbb{R}^d)$ (the Schwartz space) then $H (t) f \in \mathcal{S} (\mathbb{R}^d)$ for all $t > 0$, $H (s) H (t) f = H (s + t) f$ for all $s, t > 0$, and $\lim_{t \to 0^+} \sup_x |H (t) f (x) - f (x)| = 0$. Furthermore the function $u (x, t) = H (t) f (x)$ for $t > 0$, $= f (x)$ for $t = 0$, solves the boundary-value problem.

These results are due to Rösler [19]. Theorem 6.14 implies
\[
\lim_{t \to 0^+} \sqrt{w_\kappa (x) w_\kappa (y)} \Gamma_\kappa (t, x, y) \Gamma_0 (t, x, y) = 1
\]
for all $x, y \in C$.

There is an associated càdlàg Markov process $X = (X_t)_{t \geq 0}$ with infinitesimal generator $\frac{1}{2} \Delta_\kappa$. The semigroup densities are
\[
p_t^\kappa (x, y) := \Gamma_\kappa \left( \frac{t}{2}, x, y \right) w_\kappa (y).
\]

For further details see Rösler and Voit [21], Gallardo and Yor [9], and the monograph [10].
7 The Dunkl transform

The Dunkl kernel is used to define a generalization of the Fourier transform. The Fourier integral kernel $e^{-i(x,y)}$ is replaced by $K_{\kappa}(x,-iy)w_{\kappa}(x)$. Throughout this section $\kappa \geq 0$. Recall $|K_{\kappa}(x,-iy)| \leq 1$ for all $x, y \in \mathbb{R}^d$.

**Definition 7.1** For $f \in L^1(\mathbb{R}^d, w_{\kappa}(x)\,dx)$ let

$$\mathcal{F}f(y) := \frac{c_{\kappa}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) K_{\kappa}(x,-iy) w_{\kappa}(x) \,dx.$$ 

By finding a set of eigenfunctions of $\mathcal{F}$ which is dense in $L^2(\mathbb{R}^d, w_{\kappa}(x)\,dx)$ (by Hamburger’s theorem) we show that $\mathcal{F}$ is an $L^2$-isometry, has period 4, and $\mathcal{F}x_j = iD_j\mathcal{F}$ for $1 \leq j \leq d$. Convergence arguments, mostly depending on the dominated convergence theorem, are omitted, and appropriate smoothness restrictions on functions are implicitly assumed.

**Theorem 7.2** Let $f(x) = p(x) L_m^{(\alpha)}(|x|^2) e^{-|x|^2/2}$ where $m, n \in \mathbb{N}_0$, $\alpha = n + \frac{d}{2} + \kappa - 1$ and $p \in \mathcal{H}_{\kappa,n}$, then $\mathcal{F}f(y) = (-i)^{n+2m} f(y), y \in \mathbb{R}^d$.

**Proof.** Suppose $q$ is an arbitrary polynomial of degree $n$, for some $n$, and $N \geq n$ then the formula $\langle q, \sum_{j=0}^{N} K_{\kappa,j}(\cdot, u) \rangle_n = q(u)$ is valid for all $u \in \mathbb{C}^d$, since it is a polynomial relation. By Theorem 5.20 and letting $N \to \infty$

$$\frac{c_{\kappa}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\Delta_{\kappa}/2} q(x) e^{-\Delta_{\kappa}^{(x)}/2} K_{\kappa}(x,u) e^{-|x|^2/2} w_{\kappa}(x) \,dx = q(u),$$

and $e^{-\Delta_{\kappa}^{(x)}/2} K_{\kappa}(x,u) = \exp\left(-\frac{1}{2} \sum_{j=1}^{d} u_j^2\right) K_{\kappa}(x,u)$; thus

$$\frac{c_{\kappa}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\Delta_{\kappa}/2} q(x) K_{\kappa}(x,u) e^{-|x|^2/2} w_{\kappa}(x) \,dx = \exp\left(\frac{1}{2} \sum_{j=1}^{d} u_j^2\right) q(u).$$

Set $q(x) = e^{\Delta_{\kappa}/4}\left(|x|^{2m} p(x)\right) = m! L_m^{(\alpha)}\left(-\sum_{j=1}^{d} x_j^2\right) p(x)$ (by equation 5.1), and set $u = -iy$ in the previous equation. In the left side we have $e^{-\Delta_{\kappa}/4}\left(|x|^{2m} p(x)\right) e^{-|x|^2/2} = (-1)^m m! f(x)$; and the right side equals $m!e^{-|y|^2/2} L_m^{(\alpha)}\left(|y|^2\right)p(-iy) = (-i)^n m! f(y)$. \[\square\]

**Corollary 7.3** If $f \in L^2(\mathbb{R}^d, w_{\kappa}(x)\,dx)$ then $\int_{\mathbb{R}^d} |\mathcal{F}f(y)|^2 w_{\kappa}(y) \,dy = \int_{\mathbb{R}^d} |f(y)|^2 w_{\kappa}(x) \,dx$, and $\mathcal{F}^2 f(x) = f(-x)$ for almost all $x \in \mathbb{R}^d$.\[25\]
Suppose \( f(x), |x| f(x) \in L^1(\mathbb{R}^d, w_\kappa(x) \, dx) \) and \( 1 \leq j \leq d \) then
\[
D_j \mathcal{F} f(y) = -i \mathcal{F} (x_j f(x))(y), y \in \mathbb{R}^d,
\]

since \( K_\kappa(x,-iy) = K_\kappa(-ix,y) \). The \( G \)-invariance property of \( K_\kappa \) implies that \( w \mathcal{F} = \mathcal{F} w \) for all \( w \in G \).

The transform and the \( L^2 \)-isometry result first appeared in [3]. The uniform boundedness of \( |K_\kappa(x,-iy)| \) was shown by de Jeu [13]. For the special case \( d = 1, G = \mathbb{Z}_2 \) and even functions \( f \) on \( \mathbb{R} \) the transform \( \mathcal{F} \) essentially coincides with the classical Hankel transform.

## 8 The Poisson kernel

In this section \( \kappa \geq 0 \). There is a natural boundary value problem for harmonic functions. Given \( f \in C(S_{d-1}) \) find the function \( P[f] \) which is smooth on \( \{ x : |x| < 1 \} \), \( \Delta_\kappa P[f] = 0 \) and \( \lim_{r \to 1^-} P[f](rx) = f(x) \) for \( x \in S_{d-1} \). We outline the argument for polynomial functions on \( S_{d-1} \).

Let \( n \in \mathbb{N}_0 \) and \( x, y \in \mathbb{R}^d \). By the definition of \( \pi_{\kappa,n} \) and Theorem 6.73
\[
\pi_{\kappa,n}^{(x)} K_{\kappa,n}(x,y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-\gamma_\kappa - n + 2 - d/2)^j} |x|^{2j} |y|^{2j} K_{\kappa,n-2j}(x,y),
\]

Applying \( \pi_{\kappa,n}^{(x)} \) to the reproducing equation \( \langle p, K_{\kappa,n}(x, \cdot) \rangle_\kappa = p(x) \) for \( p \in \Pi_n^{(d)} \) we obtain
\[
\pi_{\kappa,n} P(x) = \langle p, \pi_{\kappa,n}^{(x)} K_{\kappa,n}(x, \cdot) \rangle_\kappa = \frac{c_\kappa}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\Delta_\kappa/2} p(y) \pi_{\kappa,n}^{(x)} K_{\kappa,n}(x,y) e^{-|x|^2/2} w_\kappa(y) \, dy
\]
\[
= 2^n \left( \gamma_\kappa + \frac{d}{2} \right)_n c_{\kappa,S} \int_{S_{d-1}} p(y) \pi_{\kappa,n}^{(x)} K_{\kappa,n}(x,y) w_\kappa(y) \, d\omega(y),
\]

by Proposition 5.8 and because \( e^{-\Delta_\kappa/2} p(y) = p(y) + p'(y) \) where \( p' \) is of degree \( \leq n - 2 \) and is thus orthogonal to \( \pi_{\kappa,n}^{(x)} K_{\kappa,n}(x,y) \).

**Definition 8.1** For \( n \in \mathbb{N}_0 \) and \( x, y \in \mathbb{R}^d \) let
\[
P_{\kappa,n}(x,y) := 2^n \left( \gamma_\kappa + \frac{d}{2} \right)_n \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{4^j j! (-\gamma_\kappa - n + 2 - d/2)^j} |x|^{2j} |y|^{2j} K_{\kappa,n-2j}(x,y).
\]

By the decomposition 5.7 any polynomial \( p \) satisfies \( p(x) = \sum_{n,k \geq 0} |x|^{2k} p_{n,k}(x) \) where each \( p_{n,k} \in \mathcal{H}_{\kappa,n} \), and so \( p \) agrees with the harmonic polynomial \( q(x) = \sum_{n,k \geq 0} p_{n,k}(x) \) on \( S_{d-1} \).

Thus
\[
q(x) = c_{\kappa,S} \int_{S_{d-1}} p(y) \sum_{j=0}^{N} P_{\kappa,j}(x,y) w_\kappa(y) \, d\omega(y),
\]

where \( N \) is sufficiently large. The series converges for \( |x| < 1, |y| = 1 \).
Theorem 8.2 For $|x| < 1, |y| = 1$

$$\sum_{j=0}^{\infty} P_{\kappa,j} (x, y) = V_{\kappa}^{(y)} \left( \frac{1 - |x|^2}{1 - 2 \langle x, y \rangle + |x|^2} \right).$$

The result follows from expanding the right hand side as a series in $V_{\kappa}^{(y)} (\langle x, y \rangle^j)$. The left side of the equation is thus the Poisson kernel for harmonic functions in the unit ball. For fixed $x$ with $|x| < 1$ the denominator in the $V_{\kappa}^{(y)}$-term does not vanish for $|y| \leq 1$. There is a formula for $P_{\kappa,n} (x, y)$ restricted to $|x| = 1$:

$$V_{\kappa}^{(x)} \frac{n + \alpha C_{n}^{\alpha}}{\alpha} (\langle x, y / |y| \rangle) |y|^n$$

$$= 2^n \left( \gamma_n + \frac{d}{2} \right) \sum_{j=0}^{[n/2]} 4^j j! (-\gamma_n - n + 2 - d/2)_j |y|^{2j} K_{\kappa,n-2j} (x, y),$$

where $C_{n}^{\alpha}$ is the Gegenbauer polynomial of degree $n$ and index $\alpha = \gamma_n + \frac{d}{2} - 1$. (For the exceptional case $\alpha = 0$, replace the left side by $V_{\kappa}^{(x)} 2T_n (\langle x, y / |y| \rangle) |y|^n$ for $n \geq 1$; $T_n$ is the Chebyshev polynomial of the first kind). The formula is suggested by a generating function for these polynomials. Maslouhi and Youssfi [16] studied the properties of the Poisson kernel in connection with $L^p$-type convergence and with a generalized translation.

9 Harmonic polynomials for $\mathbb{R}^2$

This section exhibits the classical Gegenbauer and Jacobi polynomials as spherical harmonics for the groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ with root systems $\{ \pm \varepsilon_2 \}$ and $\{ \pm \varepsilon_1, \pm \varepsilon_2 \}$ respectively. We mention here that the structure associated with the group $\mathbb{Z}_2^d$, $R = \{ \pm \varepsilon_i : 1 \leq i \leq d \}$ and $\kappa (\varepsilon_i) = \kappa_i$ can be analyzed in a similar way and leads to multi-variable Jacobi polynomials orthogonal on a simplex; see Chapter 2.

Polar coordinates will be used: $x_1 = r \cos \theta, x_2 = r \sin \theta, r \geq 0, -\pi \leq \theta \leq \pi$.

9.1 One parameter

Let $R = \{ \pm \varepsilon_2 \}$ and $\kappa > -\frac{1}{2}$, then

1) $w_\kappa (x) = |x_2|^{2\kappa} = r^{2\kappa} |\sin \theta|^{2\kappa}$,

2) $D_1 p (x) = \frac{\partial}{\partial x_1} p (x), D_2 p (x) = \frac{\partial}{\partial x_2} p (x) + \kappa \frac{p(x)-p(x_1,-x_2)}{x_2}$,

3) $\Delta_\kappa p (x) = \Delta p (x) + \frac{\kappa}{x_2} \left( 2 \frac{\partial}{\partial x_2} p (x) - \frac{p(x)-p(x_1,-x_2)}{x_2} \right).$
For \( n \geq 1 \) the space \( \mathcal{H}_{\kappa,n} \) contains an orthogonal basis consisting of two polynomials, \( p_{n,0} \) being even and \( p_{n,1} \) being odd in \( x_2 \):

\[
\begin{align*}
 p_{n,0}(x) &= r^n C_{n}^\kappa(\cos \theta), \\
 p_{n,1}(x) &= r^n \sin \theta C_{n-1}^{\kappa+1}(\cos \theta).
\end{align*}
\]

Let \( a_\kappa = \frac{\Gamma(\kappa+1)}{2\sqrt{\pi}\Gamma(\kappa+1/2)} \), the normalizing constant such that \( a_\kappa \int_{-\pi}^{\pi} |\sin \theta|^{2\kappa} \, d\theta = 1 \); then

\[
\begin{align*}
 a_\kappa \int_{-\pi}^{\pi} p_{n,0}(\cos \theta, \sin \theta)^2 |\sin \theta|^{2\kappa} \, d\theta &= \frac{\kappa(2\kappa)_n}{(n+\kappa)_n n!}, \\
 a_\kappa \int_{-\pi}^{\pi} p_{n,1}(\cos \theta, \sin \theta)^2 |\sin \theta|^{2\kappa} \, d\theta &= \frac{(\kappa+\frac{1}{2})(2\kappa+2)n-1}{(n+\kappa)(n-1)!}.
\end{align*}
\]

If the Gegenbauer polynomials are replaced by \( P_n^\kappa = \frac{n!}{(2\kappa)_n} C_n^\kappa \) (similarly for \( C_n^{\kappa+1} \), these are normalized by \( P_n^\kappa(1) = 1 \) then at \( \kappa = 0 \) one obtains \( p_{n,0} = r^n \cos n\theta \) and \( p_{n,1} = r^n \sin n\theta \).

### 9.2 Two parameters

Let \( R = \{ \pm \varepsilon_1, \pm \varepsilon_2 \} \) and \( \kappa_1, \kappa_2 > -\frac{1}{2} \), then

1) \( w_\kappa(x) = |x_1|^{2\kappa_1} |x_2|^{2\kappa_2} = r^{2\kappa_1+2\kappa_2} |\cos \theta|^{2\kappa_1} |\sin \theta|^{2\kappa_2} \),

2) \( D_1 p(x) = \frac{\partial}{\partial x_1} p(x) + \kappa_1 \frac{p(x)-p(-x_1,x_2)}{x_2} \), \( D_2 p(x) = \frac{\partial}{\partial x_2} p(x) + \kappa_2 \frac{p(x)-p(x_1,-x_2)}{x_2} \),

3) \( \Delta_\kappa p(x) = \Delta p(x) + \frac{\kappa_1}{x_1} \left( \frac{2}{x_1} \frac{\partial}{\partial x_1} p(x) - \frac{p(x)-p(-x_1,x_2)}{x_2} \right) + \frac{\kappa_2}{x_2} \left( \frac{2}{x_2} \frac{\partial}{\partial x_2} p(x) - \frac{p(x)-p(x_1,-x_2)}{x_2} \right) \).

There are 4 families of harmonic polynomials. The first subscript indicates the degree and the second subscript is used to indicate the parity type, for example 01 denotes “even in \( x_1 \), odd in \( x_2 \).”

\[
\begin{align*}
 p_{2n,00}(x) &= r^{2n} P_n^{(\kappa_2-1/2,\kappa_1-1/2)}(\cos 2\theta), \\
 p_{2n,11}(x) &= r^{2n} \sin 2\theta P_{n-1}^{(\kappa_2+1/2,\kappa_1+1/2)}(\cos 2\theta), \\
 p_{2n+1,10}(x) &= r^{2n+1} \cos \theta P_{n-1}^{(\kappa_2-1/2,\kappa_1+1/2)}(\cos 2\theta), \\
 p_{2n+1,01}(x) &= r^{2n+1} \sin \theta P_{n-1}^{(\kappa_2+1/2,\kappa_1-1/2)}(\cos 2\theta).
\end{align*}
\]

The norms with respect to \( L^2 \left([\pi, \pi], |\cos \theta|^{2\kappa_1} |\sin \theta|^{2\kappa_2} \, d\theta \right) \) can be computed from

\[
\int_{-\pi}^{\pi} \left\{ P_n^{(\alpha,\beta)}(\cos 2\theta) \right\}^2 |\cos \theta|^{2\alpha+1} |\sin \theta|^{2\beta+1} \, d\theta = \frac{4\Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} (\alpha+1)_n (\beta+1)_n (\alpha+\beta+n+1), n \in \mathbb{N}_0.
\]

28
9.3 Dihedral groups

The harmonic polynomials for the general dihedral groups can be expressed using the two previous types and complex coordinates \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \). Interpret \( p(z^m) \) as \( p(\text{Re} z^m, \text{Im} z^m) \). For the one-parameter case \( I_2(m) \) with \( m \) odd the harmonic polynomials are spanned by

1) \( p_{n,0}(z^m), p_{n,1}(z^m) \), of degree \( nm \).
2) \( \text{Re} p_{nm+j}(z), \text{Im} p_{nm+j}(z) \) where \( p_{nm+j}(z) := z^j \left( \frac{n+2\kappa}{2\kappa} p_{n,0}(z^m) + ip_{n,1}(z^m) \right) \), for \( 1 \leq j < m \).

For the two-parameter case \( I_2(2m) \) with \( w_\kappa(z) = |z^m + \bar{z}^m|^{2\kappa 1} |z^m - \bar{z}^m|^{2\kappa 2} \) let

\[
q_{2n}(z) := p_{2n,00}(z) + \frac{1}{2} p_{2n,11}(z), \\
q_{2n+1}(z) := \left( n + \kappa_2 + \frac{1}{2} \right) p_{2n+1,10}(z) + i \left( n + \kappa_1 + \frac{1}{2} \right) p_{2n+1,01}(z).
\]

The harmonic polynomials are spanned by \( z^j q_n(z^m) \) and \( \bar{z}^j q_n(z^m) \) for \( n \in \mathbb{N}_0, 0 \leq j \leq m \) (note \( q_0(z) = 1 \)). These results are from [1].

10 Nonsymmetric Jack polynomials

For the symmetric group \( S_d \) acting naturally on \( \mathbb{R}^d \) there is an elegant orthogonal basis for \( \Pi^d \) with respect to the form \( \langle \cdot, \cdot \rangle_\kappa \). The basis consists of nonsymmetric Jack polynomials, so named because their symmetrization (summing over an \( S_d \)-orbit) yields the Jack polynomials, with parameter \( 1/\kappa \). The construction of the basis depends on commuting self-adjoint “Cherednik-Dunkl” operators and an ordering of the monomial basis with respect to which the operators are represented by triangular matrices.

Recall the notation from Section 2.1. \((i, j)\) denotes the transposition \( s_{e_i-e_j} \) and \( s_i = (i, i+1) \) for \( 1 \leq i < d \). Interpret \( S_d \) as the set of bijections on \( \{1, 2, \ldots, d\} \) then the action on \( \mathbb{R}^d \) is given by \( (w^{-1}x)_i = x_{w(i)} \) and the action on monomials is \( w(x^\alpha) = x^\omega w^{-1} \) where \( \alpha w^{-1}_i = \alpha_{w^{-1}(i)} \) for \( w \in S_d, 1 \leq i \leq d, \alpha \in \mathbb{N}_0^d \).

**Definition 10.1** The set of partitions (of length \( \leq d \)) is

\[ \mathbb{N}_0^{d,+} := \left\{ \lambda \in \mathbb{N}_0^d : \lambda_i \geq \lambda_{i+1}, 1 \leq i < d \right\}, \]

and for \( \alpha \in \mathbb{N}_0^d \) let \( \alpha^+ \) denote the unique partition such that \( \alpha^+ = \alpha w \) for some \( w \in S_d \).

**Definition 10.2** For \( \alpha \in \mathbb{N}_0^d \) and \( 1 \leq i \leq d \) let

\[ w_\alpha(i) := \# \{ j : \alpha_j > \alpha_i \} + \# \{ j : 1 \leq j \leq i, \alpha_j = \alpha_i \} \]

be the rank function.
Let Proposition 10.4 The operators \( U_\alpha \) and \( U_{\alpha'} \) be one-to-one on \( \{1, 2, \ldots, d\} \), hence \( U_\alpha \in S_d \). Also \( \alpha \) is a partition if and only if \( w_\alpha (i) = j \) for all \( i \). In general \( \alpha w_\alpha^{-1} = \alpha' \) because \( (\alpha w_\alpha^{-1})_i = \alpha w_\alpha^{-1}(i) \) for \( 1 \leq i \leq d \).

There is one conjugacy class of reflections and we use \( \kappa \) for the value of the multiplicity function. For \( p \in \Pi^d \) and \( 1 \leq i \leq d \)

\[
D_i p (x) = \frac{\partial}{\partial x_i} p (x) + \kappa \sum_{j \neq i} p (x) - p ((i, j) x) / (x_i - x_j).
\]

The commutation relations from Proposition 10.5 (using \( x_i \) to denote the multiplication operator) become

\[
[D_i, x_i] = 1 + \kappa \sum_{j \neq i} (i, j), \quad (10.1)
\]

\[
[D_j, x_i] = -\kappa (i, j), \quad j \neq i.
\]

The order on compositions is derived from the dominance order on partitions.

**Definition 10.3** For \( \alpha, \beta \in \mathbb{N}_0^d \) the partial order \( \alpha \triangleright \beta \) (\( \alpha \) dominates \( \beta \)) means that \( \alpha \neq \beta \) and \( \sum_{i=1}^j \alpha_i \geq \sum_{i=1}^j \beta_i \) for \( 1 \leq j \leq d \); and \( \alpha \triangleright \beta \) means that \( |\alpha| = |\beta| \) and either \( \alpha^+ \triangleright \beta^+ \) or \( \alpha^+ = \beta^+ \) and \( \alpha \triangleright \beta \).

For example \((5, 1, 4) \triangleright (1, 5, 4) \triangleright (4, 3, 3)\), while \((1, 5, 4) \) and \((6, 2, 2)\) are not comparable in \( \triangleright \). The following hold for \( \alpha \in \mathbb{N}_0^d \): (1) \( \alpha^+ \triangleright \alpha \), (2) if \( \alpha_i > \alpha_j \) and \( i < j \) then \( \alpha \triangleright \alpha (i, j) \), (3) if \( 1 \leq m < \alpha_i - \alpha_j \) then \( \alpha^+ \triangleright (\alpha - m (\varepsilon_i - \varepsilon_j))^+ \).

The Cherednik-Dunkl operators are extensions of the Jucys-Murphy elements; for \( 1 \leq i \leq d \) let

\[
U_i := D_i x_i - \kappa \sum_{1 \leq j < i} (i, j).
\]

**Proposition 10.4** The operators \( U_i \) satisfy:
1) \( \{U_i, U_j\} = 0 \) for \( 1 \leq i, j \leq d \),
2) \( \langle U_i p, q \rangle \kappa = \langle p, U_i q \rangle \kappa \) for \( p, q \in \Pi^d \),
3) \( s_j U_i s_j = U_i \) for \( j \neq i - 1, i \) and \( s_i U_i s_i = U_{i+1} + \kappa s_i \) for \( 1 \leq i \leq d \).

**Proposition 10.5** Let \( \alpha \in \mathbb{N}_0^d \) and \( 1 \leq i \leq d \), then

\[
U_i x^\alpha = ((d - w_\alpha (i)) \kappa + \alpha_i + 1) x^\alpha + q_{\alpha, i} (x),
\]

where \( q_{\alpha, i} (x) \) is a sum of terms \( \pm \kappa x^{\alpha(i,j)} \) with \( \alpha \triangleright \alpha (i, j) \) and \( j \neq i \).
This shows that the matrix representing $U_i$ on the monomial basis of $\Pi^n_d$ for any $n \in \mathbb{N}_0$ is triangular (recall any partial order can be embedded in a total order) and the eigenvalues of $U_i$ are \{$(d - w_\alpha(i)) \kappa + \alpha_i + 1 : \alpha \in \mathbb{N}^d_0, |\alpha| = n$\}.

For $\alpha \in \mathbb{N}^d_0$ and $1 \leq i \leq d$ let

$$
\xi_i(\alpha) := (d - w_\alpha(i)) \kappa + \alpha_i + 1.
$$

To assert that a commuting collection of triangular matrices have a basis of joint eigenvectors a separation property suffices: if $\alpha, \beta \in \mathbb{N}^d_0$ and $\alpha \neq \beta$ then $\xi_i(\alpha) \neq \xi_i(\beta)$ for some $i$. This condition is satisfied if $\kappa$ is generic or $\kappa > 0$ (in this case the eigenvalues $\{\xi_i(\alpha) : 1 \leq i \leq d\}$ are pairwise distinct, and the largest value is $(d - 1) \kappa + \alpha_i^d$; if $\xi_i(\alpha) = \xi_i(\beta)$ for all $i$ then $\alpha_j = \beta_j$ for $j = w_\alpha^{-1}(1) = w_\beta^{-1}(1)$, and so on).

**Theorem 10.6** Suppose $\kappa$ is generic or $\kappa > 0$, then for each $\alpha \in \mathbb{N}^N_0$, there is a unique simultaneous eigenfunction $\zeta_\alpha$ such that $U_i \zeta_\alpha = \xi_i(\alpha) \zeta_\alpha$ for $1 \leq i \leq d$ and

$$
\zeta_\alpha = x^\alpha + \sum_{\alpha > \beta} A_{\beta \alpha} x^\beta,
$$

with coefficients $A_{\beta \alpha} \in \mathbb{Q}(\kappa)$.

These eigenfunctions are called nonsymmetric Jack polynomials, and form a basis of $\Pi_n^d$ by the triangularity property. If $\alpha \neq \beta$ then $\xi_i(\alpha) \neq \xi_i(\beta)$ for some $i$ and $\xi_i(\alpha) \langle \zeta_\alpha, \zeta_\beta \rangle_\kappa = \langle U_i \zeta_\alpha, \zeta_\beta \rangle_\kappa = \langle \zeta_\alpha, U_i \zeta_\beta \rangle_\kappa = \xi_i(\beta) \langle \zeta_\alpha, \zeta_\beta \rangle_\kappa$ and thus $\langle \zeta_\alpha, \zeta_\beta \rangle_\kappa = 0$. The formula for $\langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa$ is more complicated.

Suppose $\alpha \in \mathbb{N}^d_0$ and $\alpha_i < \alpha_{i+1}$ then $\alpha s_i \triangleright \alpha$ and $w_\alpha s_i = w_\alpha s_i$; let $p = s_i \zeta_\alpha - c \zeta_\alpha$ where $c \in \mathbb{Q}(\kappa)$ and is to be determined. By Proposition 10.4 $U_j p = \xi_j(\alpha) p$ for $j \neq i, i + 1$ and the leading term (with respect to $\triangleright$) in $p$ is $x^{\alpha s_i}$. Solve the equation $U_i p = \xi_{i+1}(\alpha) p$ for $c$ by using $U_i s_i = s_i U_{i+1} + \kappa$ to obtain $c = \frac{\kappa}{\xi_i(\alpha) - \xi_{i+1}(\alpha)}$. This implies that $U_{i+1} p = \xi_{i+1}(\alpha) p, p = \zeta_\alpha s_i$, and

$$
\begin{align*}
\xi_i(\alpha) s_i \zeta_\alpha &= \zeta_\alpha + \zeta_\alpha s_i, \\
\langle s_i \zeta_\alpha, s_i \zeta_\alpha \rangle_\kappa &= (1 - c^2) \zeta_\alpha - c \zeta_\alpha s_i, \\
\langle \zeta_\alpha s_i, \zeta_\alpha s_i \rangle_\kappa &= (1 - c^2) \langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa.
\end{align*}
$$

The last equation follows from $\langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa = \langle s_i \zeta_\alpha, s_i \zeta_\alpha \rangle_\kappa = c^2 \langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa + \langle \zeta_\alpha s_i, \zeta_\alpha s_i \rangle_\kappa$. The other ingredient is a raising operator. From the commutations (10.1) we obtain:

$$
\begin{align*}
U_i x_d &= x_d (U_i - \kappa (i, d)), & 1 \leq i < d, \\
U_d x_d &= x_d (1 + D_d x_d).
\end{align*}
$$

Let $\theta_d := s_1 s_2 \ldots s_{d-1}$ thus $\theta_d(d) = 1$ and $\theta_d(i) = i + 1$ for $1 \leq i < d$ (a cyclic shift). Then

$$
\begin{align*}
U_i x_d &= x_d (\theta_d^{-1} U_{i+1} \theta_d), & 1 \leq i < d, \\
U_d x_d &= x_d (1 + \theta_d^{-1} U_i \theta_d).
\end{align*}
$$

31
If $p$ satisfies $\mathcal{U}_i p = \lambda_i p$ for $1 \leq i \leq d$ then $\mathcal{U}_i (x_d \theta_d^{-1} f) = \lambda_{i+1} (x_d \theta_d^{-1} f)$ for $1 \leq i < d$ and $\mathcal{U}_d (x_d \theta_d^{-1} f) = (\lambda_1 + 1) (x_d \theta_d f)$. For $\alpha \in \mathbb{N}_0^d$ let $\phi (\alpha) := (\alpha_2, \alpha_3, \ldots, \alpha_d, \alpha_1 + 1)$, then $x_d \theta_d^{-1} x^\alpha = x^{\phi (\alpha)}$.

**Proposition 10.7** If $\alpha \in \mathbb{N}_0^d$ then $\zeta_{\phi (\alpha)} = x_d \theta_d^{-1} \zeta_\alpha$ and
\[
\langle \zeta_{\phi (\alpha)}, \zeta_{\phi (\alpha)} \rangle_\kappa = ((d - w_\alpha (1)) \kappa + \alpha_1 + 1) \langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa.
\]

**Proof.** The first part is shown by identifying the eigenvalues $\xi_i (\phi (\alpha))$. Also $\langle \zeta_{\phi (\alpha)}, \zeta_{\phi (\alpha)} \rangle_\kappa = \langle \theta_d^{-1} \zeta_\alpha, \mathcal{D}_d x_d \theta_d^{-1} \zeta_\alpha \rangle_\kappa = \langle \theta_d^{-1} \zeta_\alpha, \theta_d^{-1} \mathcal{D}_1 x_1 \zeta_\alpha \rangle_\kappa = \xi_1 (\alpha) \langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa$. $\blacksquare$

The norm formula involves a hook-length product. For $\alpha \in \mathbb{N}_0^d$ let $\ell (\alpha) = \max \{ j : \alpha_j > 0 \}$, the length of $\alpha$. For a point $(i, j)$ in the Ferrers diagram $\{(k, l) \in \mathbb{N}_0^d : 1 \leq k \leq \ell (\alpha), 1 \leq l \leq \alpha_k\}$ and $t \in \mathbb{Q} (\kappa)$ let
\[
h(\alpha, t; i, j) := \alpha_i - j + t + \kappa \# \{ l : l > i, j \leq \alpha_l \leq \alpha_i \}
+ \kappa \# \{ l : l < i, j \leq \alpha_l + 1 \leq \alpha_i \},
\]
and let
\[
h(\alpha, t) := \prod_{i=1}^{\ell (\alpha)} \prod_{j=1}^{\alpha_i} h(\alpha, t; i, j).
\]
For $\lambda \in \mathbb{N}_0^{d+}$ let $(t)_\lambda := \prod_{i=1}^{d} (t - (i - 1) \kappa)_\lambda$, the generalized Pochhammer symbol.

**Theorem 10.8** For $\alpha \in \mathbb{N}_0^d$,
\[
\langle \zeta_\alpha, \zeta_\alpha \rangle_\kappa = (d \kappa + 1)_{\alpha_1} \frac{h(\alpha, 1)}{h(\alpha, \kappa + 1)},
\]
\[
\zeta_\alpha (1, \ldots, 1) = \frac{(d \kappa + 1)_{\alpha_1}}{h(\alpha, \kappa + 1)}.
\]

These formulae are proved by induction starting with $\alpha = 0$ and using the steps $\alpha \to \phi (\alpha), \alpha \to \alpha s_i$ for $\alpha_i < \alpha_{i+1}$ (it suffices to use $\lambda \to \phi (\lambda)$ with $\lambda \in \mathbb{N}_0^{d+}$ in the computation). The number of such steps in any sequence linking 0 to $\alpha$ equals
\[
|\alpha| + \frac{1}{2} \sum_{1 \leq i < j \leq d} \{|\alpha_i - \alpha_j| + |\alpha_i - \alpha_j + 1| - 1\}.
\]

There are two explicit results for $\mathcal{D}_i \zeta_\alpha$. Recall $\theta_m = s_1 s_2 \ldots s_{m-1}$ for $m \leq d$.

**Proposition 10.9** Suppose $\alpha \in \mathbb{N}_0^d$ and $\ell (\alpha) = m$; let $\bar{\alpha} = (\alpha_m - 1, \alpha_1, \ldots, \alpha_{m-1}, 0, \ldots)$ and $\beta_m = \alpha_m - w_\alpha (m) \kappa$ then
\[
\mathcal{D}_i \zeta_\alpha = 0, \ m < i \leq d,
\]
\[
\mathcal{D}_m \zeta_\alpha = \frac{(m \kappa + \beta_m)((d + 1) \kappa + \beta_m)}{(m + 1) \kappa + \beta_m} \theta_{m-1} \zeta_{\bar{\alpha}}.
\]

32
This result is from [4] Proposition 3.17.

Symmetric polynomials (that is, $S^d$-invariant) have bases labeled by partitions $\lambda \in \mathbb{N}_0^d$. We describe bases whose elements are mutually orthogonal in $\langle \cdot, \cdot \rangle_\kappa$. For a given $\lambda \in \mathbb{N}_0^d$, one can consider $\sum_{w \in S_d} w \zeta_\lambda$ or a sum $\sum_{\alpha^+ = \lambda} b_\alpha \zeta_\alpha$ with suitable coefficients $\{b_\alpha\}$. Let $\lambda^R = (\lambda_d, \lambda_{d-1}, \ldots, \lambda_1)$, thus $\lambda^R$ is the unique $>_\lambda$-minimum in $\{\alpha : \alpha^+ = \lambda\}$. No term $x^\alpha$ with $\alpha^+ = \lambda$ except $\alpha = \lambda^R$ appears in $\zeta_\lambda$. Also let $n_\lambda = \# \{w \in S_d : \lambda w = \lambda\}$.

**Definition 10.10** For $\lambda \in \mathbb{N}_0^{d+}$ let $j_\lambda := h(\lambda, 1) \sum_{\alpha^+ = \lambda} \frac{\zeta_\alpha}{h(\alpha, 1)}$.

**Theorem 10.11** For $\lambda \in \mathbb{N}_0^{d+}$, $j_\lambda$ has the following properties:
1) $j_\lambda$ is symmetric, and the coefficient of $x^\lambda$ in $j_\lambda$ is 1,
2) $j_\lambda = \frac{h(\lambda, \kappa + 1)}{n_\lambda h(\lambda^R, \kappa + 1)} \sum_{w \in S_d} w \zeta_\lambda = \frac{1}{n_\lambda} \sum_{w \in S_d} w \zeta_{\lambda^R}$,
3) $\langle j_\lambda, j_\lambda \rangle_\kappa = (dk + 1) \frac{d! h(\lambda, 1)}{n_\lambda h(\lambda^R, \kappa + 1)}$ and $j_\lambda(1, 1, \ldots, 1) = \frac{d! (dk + 1)_\lambda}{n_\lambda h(\lambda^R, \kappa + 1)}$.

Knop and Sahi [15] proved the norm formulas and also showed that the coefficient of each monomial in $h(\alpha, \kappa + 1) \zeta_\alpha$ is a polynomial in $\kappa$ with nonnegative coefficients.

For $\lambda \in \mathbb{N}_0^{d+}$, $j_\lambda$ is a scalar multiple of the Jack polynomial $J_\lambda(x; \frac{1}{\kappa})$. For more details on the nonsymmetric Jack polynomials and their applications to Calogero-Moser-Sutherland systems and the groups of type $B$ see the monograph [6] Chapters 8,9.

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