CURVE COUNTING AND DT/PT CORRESPONDENCE FOR CALABI-YAU 4-FOLDS

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Abstract. Recently, Cao-Maulik-Toda defined stable pair invariants of a compact Calabi-Yau 4-fold X. Their invariants are conjecturally related to the Gopakumar-Vafa type invariants of X defined using Gromov-Witten theory by Klemm-Pandharipande. In this paper, we consider curve counting invariants of X using Hilbert schemes of curves and conjecture a DT/PT correspondence which relates these to stable pair invariants of X.

After providing evidence in the compact case, we define analogous invariants for toric Calabi-Yau 4-folds using a localization formula. We formulate a vertex formalism for both theories and conjecture a relation between the (fully equivariant) DT/PT vertex, which we check in several cases. This relation implies a DT/PT correspondence for toric Calabi-Yau 4-folds with primary insertions.

0. Introduction

0.1. GW/GV invariants. Gromov-Witten invariants of a complex smooth projective variety X are rational numbers (virtually) counting stable maps from curves to X. Let X be a Calabi-Yau 4-fold\(^1\) then the virtual dimension formula shows that these invariants vanish unless \(g = 0\) or \(1\). Since Gromov-Witten invariants involve multiple covers, their enumerative meaning is a priori unclear. In [16], Klemm and Pandharipande defined Gopakumar-Vafa type invariants of X, in terms of Gromov-Witten invariants of X, and conjectured their integrality.

More precisely, let \(\overline{M}_{g,m}(X,\beta)\) be the moduli space of stable maps \(f : C \to X\), where C is a (connected) nodal curve of arithmetic genus \(g\) with \(m\) marked points satisfying \(\int_C [\beta] = \beta \in H_2(X)\). This space has a virtual class with virtual dimension \(1 - g + m\). Consider the following Gromov-Witten invariants (with primary insertions)

\[
GW_{0,\beta}(X)(\gamma_1, \ldots, \gamma_m) := \int_{[\overline{M}_{0,m}(X,\beta)]^{\text{vir}}} \prod_{i=1}^m \text{ev}_i^*(\gamma_i) \in \mathbb{Q},
\]

\[
GW_{1,\beta}(X) := \int_{[\overline{M}_{1,0}(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q},
\]

where \(\gamma_1, \ldots, \gamma_m \in H^*(X,\mathbb{Z})\) and \(\text{ev}_i : \overline{M}_{0,m}(X,\beta) \to X\) is evaluation map at the \(i\)th marked point. Klemm-Pandharipande defined genus 0 and 1 Gopakumar-Vafa type invariants

\[
n_0,\beta(X)(\gamma_1, \ldots, \gamma_m), \quad n_{1,\beta}(X)
\]

by the following two “multiple cover formulae”

\[
\sum_{\beta > 0} GW_{0,\beta}(X)(\gamma_1, \ldots, \gamma_m) q^\beta = \sum_{\beta > 0} n_{0,\beta}(X)(\gamma_1, \ldots, \gamma_m) \sum_{d=1}^\infty d^{m-3} q^{d\beta}
\]

\[
\sum_{\beta > 0} GW_{1,\beta}(X) q^\beta = \sum_{\beta > 0} n_{1,\beta}(X) \sum_{d=1}^\infty \frac{\sigma_1(d)}{d} q^{d\beta}
\]

\[
+ \frac{1}{24} \sum_{\beta > 0} n_{0,\beta}(X)(c_2(X)) \log(1 - q^\beta) - \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1+\beta_2}).
\]

Here the sums are over all non-zero effective curve classes in \(H_2(X)\) and \(\sigma_1(d) = \sum_{i \mid d} i\). Moreover, \(m_{\beta_1, \beta_2} \in \mathbb{Z}\) are so-called meeting invariants which are inductively determined by the genus 0 Gromov-Witten invariants of X. Klemm-Pandharipande’s integrality conjecture states that \(n_{0,\beta}(X)(\gamma_1, \ldots, \gamma_m)\) and \(n_{1,\beta}(X)\) are integers, which they verify in examples using virtual localization or mirror symmetry [16].

\(^1\)In this paper, a Calabi-Yau 4-fold is a complex smooth projective 4-fold \(X\) satisfying \(K_X \cong \mathcal{O}_X\).
0.2. **PT/GV correspondence.** Donaldson-Thomas theory of Calabi-Yau 4-folds was introduced by Cao-Leung [11] and Borisov-Joyce [4]. In [14], Cao-Maulik-Toda defined stable pair invariants of Calabi-Yau 4-folds, using the methods for constructing virtual classes of [11, 4], and proposed an interpretation of (0.1) in terms of these invariants.

More precisely, denote by \( P_n(X, \beta) \) the moduli space of stable pairs \( \{ s : O_X \to F \} \) in the sense of Pandharipande-Thomas [21], where \( F \) is a pure dimension 1 sheaf on \( X \), \( s \) is a section with 0-dimensional cokernel, and \( \text{ch}(F) = (0, 0, 0, \beta, n) \). This moduli space carries a virtual class \( [P_n(X, \beta)]^\text{vir} \in H_{2n}(P_n(X, \beta), \mathbb{Z}) \), which depends on a choice of orientation of a certain (real) line bundle on \( P \). On each connected component of \( P_n(X, \beta) \), there are two choices of orientation, which affects the corresponding contribution to the virtual class by a sign. Define

\[
\tau : H^*(X, \mathbb{Z}) \to H^{*+2}(P_n(X, \beta), \mathbb{Z}), \quad \tau(\gamma) = \pi_P(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})),
\]

where \( \pi_X \) and \( \pi_P \) are projections from \( X \times P_n(X, \beta) \) to its factors and \( \mathbb{F}^* = \{ \pi_X^* O_X \to \mathbb{F} \} \) is the universal stable pair on \( X \times P_n(X, \beta) \). Since \( \mathbb{F} \) is of pure dimension 1 on each fibre over \( P_n(X, \beta), \text{ch}_3(\mathbb{F}) \) is Poincaré dual to the class of the scheme-theoretic support \( C \) of \( \mathbb{F} \), which is a Cohen-Macaulay curve on each fibre over \( P_n(X, \beta) \).

Stable pair invariants (with primary insertions) of \( X \) are defined by

\[
P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) := \int_{[P_n(X, \beta)]^\text{vir}} \prod_{i=1}^{\beta} \tau(\gamma_i) \in \mathbb{Z}, \quad P_{0, \beta}(X) := \int_{[P_0(X, \beta)]^\text{vir}} 1 \in \mathbb{Z},
\]

for all \( \gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z}) \). In the following conjectures, when the curve class \( \beta \) is zero, we set \( P_{0,0}(X) := 1, P_{n,0}(X)(\gamma_1, \ldots, \gamma_m) = n_{0,0}(X)(\gamma_1, \ldots, \gamma_m) := 0 \) when \( n > 0 \).

**Conjecture 0.1.** (Part of Conjecture 0.1 and Section 0.7 of [14]) Let \( X \) be a Calabi-Yau 4-fold, \( \beta \in H_2(X) \), \( \gamma_1 = \cdots = \gamma_n = \gamma \in H^4(X, \mathbb{Z}) \), and \( n \in \mathbb{Z}_{>0} \). Then there exist choices of orientations such that

\[
P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_n) = \beta_{\beta_0 + \beta_1 + \cdots + \beta_n = \beta} \prod_{i=1}^{n_{0,0}} P_{0, \beta}(X) \cdot \prod_{i=1}^{n} n_{0, \beta_i}(X)(\gamma).
\]

**Conjecture 0.2.** (Conjecture 0.2 of [14]) Let \( X \) be a Calabi-Yau 4-fold and \( \beta \in H_2(X) \). Then there exist choices of orientation such that

\[
\sum_{\beta \geq 0} P_{0, \beta}(X) q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1, \beta}(X)},
\]

where \( M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n} \) denotes the MacMahon function.

0.3. **DT/PT correspondence.** In this paper, we study curve counting invariants of \( X \) using Hilbert schemes and conjecture an explicit DT/PT correspondence. As a consequence, we get an interpretation of the Gopakumar-Vafa type invariants \( n_{0, \beta}(X)(\gamma_1, \ldots, \gamma_m) \) and \( n_{1, \beta}(X) \) in terms of virtual counts of ideal sheaves of curves (closed subschemes of dimension \( \leq 1 \)) on \( X \).

Let \( I_n(X, \beta) \) denote the Hilbert scheme of closed subschemes \( C \subseteq X \) of dimension \( \leq 1 \) such that \( \text{ch}(O_C) = (0, 0, 0, \beta, n) \). Similar to [14], there exists a virtual class

\[
[I_n(X, \beta)]^\text{vir} \in H_{2n}(I_n(X, \beta), \mathbb{Z}),
\]

in the sense of Borisov-Joyce [4], which depends on a choice of orientation of a certain (real) line bundle on \( I_n(X, \beta) \). As in the stable pairs case, we consider primary insertion

\[
\tau : H^*(X, \mathbb{Z}) \to H^{*+2}(I_n(X, \beta), \mathbb{Z}), \quad \tau(\gamma) = \pi_I(\pi_X^* \gamma \cup \text{ch}_3(O_Z)),
\]

where \( \pi_X, \pi_I \) are projections from \( X \times I_n(X, \beta) \) to its factors and \( Z \subseteq X \times I_n(X, \beta) \) is the universal subscheme. Note that \( \text{ch}_3(O_Z) \) is Poincaré dual to the class of \( Z \). We define curve counting invariants (with primary insertions) of \( X \) by

\[
I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) := \int_{[I_n(X, \beta)]^\text{vir}} \prod_{i=1}^{\beta} \tau(\gamma_i) \in \mathbb{Z}, \quad I_{0, \beta}(X) := \int_{[I_0(X, \beta)]^\text{vir}} 1 \in \mathbb{Z}.
\]
On Calabi-Yau 3-folds, curve counting and stable pair invariants are related by the DT/PT correspondence formulated in \cite{21} and proved in \cite{24,25} using wall-crossing and Hall algebra techniques. We conjecture an analogue on Calabi-Yau 4-folds.

**Conjecture 0.3.** (Conjecture \cite{13}) Let $X$ be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X)$, $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$, and $n \in \mathbb{Z}$. Then there exist choices of orientation such that

1. $I_{0, \beta}(X) = P_{0, \beta}(X)$,
2. $I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m)$.

In the Calabi-Yau 3-folds case, one requires corrections from “floating points” in DT/PT correspondence. This is not needed in the case of Calabi-Yau 4-folds because the (“complex”) virtual dimension of $I_n(X, \beta)$ and $P_n(X, \beta)$ is $n$ and the primary insertion does not see floating points. A heuristic derivation for “ideal Calabi-Yau 4-folds”, on which all curves deform in expected dimensions, is presented in Section 1.3.

The bulk of evidence in this paper comes from toric Calabi-Yau 4-folds discussed in the next section. For compact Calabi-Yau 4-folds we can check a few cases, where the moduli spaces $P_n(X, \beta)$ and $I_n(X, \beta)$ coincide (see also \cite{13} \cite{11}):

- Suppose $\beta \in H_2(X)$ has the property that any element $C$ of the Chow group Chow$_\beta(X)$ satisfies $\chi(O_C) = 1$, i.e. is rational. Then Conjecture 0.3 holds for $n = 0, 1$. Examples include (1) irreducible curve classes on complete intersection Calabi-Yau 4-folds in products of projective spaces, (2) degree two classes on sextic 4-folds.
- Suppose $X \to \mathbb{P}^3$ is a Weierstrass elliptically fibred Calabi-Yau 4-fold and $\beta = r[F]$ is a multiple fibre class for some $r > 0$. Then Conjecture 0.3 holds for $n = 0$.
- Suppose $X = Y \times E$, where $Y$ is a smooth projective Calabi-Yau 3-fold, $E$ is an elliptic curve, and $\beta = r[E]$, where $[E]$ denotes the class of a fibre $\{pt\} \times E$ and $r > 0$. Then Conjecture 0.3 holds for $n = 0$.

### 0.4. Equivariant DT/PT correspondence

Let $X$ be a toric Calabi-Yau 4-fold\footnote{By this we mean a smooth quasi-projective toric 4-fold $X$ satisfying $K_X \cong \mathcal{O}_X$ and $H^{>0}(\mathcal{O}_X) = 0$.} Denote by $(\mathbb{C}^*)^4$ the dense open torus of $X$. Let $T \subseteq (\mathbb{C}^*)^4$ be the 3-dimensional subtorus which preserves the Calabi-Yau volume form. Since $X$ is not compact, the Hilbert schemes $I_n(X, \beta)$ are in general non-compact as well, because “floating points can move off to infinity”. Nevertheless, the fixed locus

$$I_n(X, \beta)^T = I_n(X, \beta)^{(\mathbb{C}^*)^4}$$

consist of finitely many isolated reduced points (Lemma 21 and 22).

For $\gamma_1, \ldots, \gamma_m \in H_T^*(X, \mathbb{Q})$, we define $T$-equivariant curve counting invariants of $X$

$$I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Q}\langle\lambda_1, \lambda_2, \lambda_3, \lambda_4\rangle / \langle\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\rangle \cong \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)$$

by using a localization formula (see Definition 2.7). Here $\lambda_1, \ldots, \lambda_4$ are the equivariant parameters of $(\mathbb{C}^*)^4$, which satisfy the relation $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ on the Calabi-Yau torus $T$.

This definition involves a sum over the elements of the fixed locus and depends on a choice of sign at each torus fixed point. Therefore, we have $2^\#I_n(X, \beta)^T$ choices of signs in total. When $\beta = 0$, $I_n(X, 0)$ is the Hilbert scheme of $n$ points on $X$, which was studied by Nekrasov (with $K$-theoretic insertions \cite{19} and the authors (with tautological insertions \cite{9}).

Contrary to $I_n(X, \beta)$, the moduli space $P_n(X, \beta)$ may not have 0-dimensional fixed locus. Nevertheless, there are many interesting cases for which $P_n(X, \beta)^{(\mathbb{C}^*)^4}$ is 0-dimensional, for instance when $X$ is a local toric curve or surface. Then (Proposition 2.9)

$$P_n(X, \beta)^T = P_n(X, \beta)^{(\mathbb{C}^*)^4}$$

consists of finitely many isolated reduced points. For $\gamma_1, \ldots, \gamma_m \in H_T^2(X)$, we may define $T$-equivariant stable pair invariants of $X$ by a localization formula as well (Definition 2.9)

$$P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3).$$

We form generating series

$$I_\beta(X; T)(\gamma_1, \ldots, \gamma_m) := \sum_{n \in \mathbb{Z}} I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) \ q^n \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)[[q]],$$

$$P_\beta(X; T)(\gamma_1, \ldots, \gamma_m) := \sum_{n \in \mathbb{Z}} P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) \ q^n \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)[[q]].$$

We denote the generating series without insertions by $I_\beta(X; T)$, $P_\beta(X; T)$ and set $I_{0, 0}(X; T) = 1$. 

Remark 0.6. By a conjecture of Nekrasov (Conj. 2.15), there exist choices of signs such that

\[ V_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q), V_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)(q), \]

where \( \lambda, \mu, \nu, \rho \) are plane partitions (i.e. 3-dimensional piles of boxes) and we only define the PT vertex when at most two of \( \lambda, \mu, \nu, \rho \) are non-empty. Roughly speaking, these are the DT/PT partition functions on \( X = \mathbb{C}^4 \) for which the underlying Cohen-Macaulay support curve is fixed and determined by plane partitions \( \lambda, \mu, \nu, \rho \) along the coordinate axes of \( \mathbb{C}^4 \).

The following is our DT/PT correspondence at the level of the 4-fold vertex:

**Conjecture 0.4.** (Conjecture 2.13) For any plane partitions \( \lambda, \mu, \nu, \rho \), at most two of which are non-empty, there are choices of signs such that

\[ V_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q) = V_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) V_{\emptyset,\emptyset,\emptyset,\emptyset}^{\text{DT}}(q). \]

Moreover, for each \( n \), the choice of signs for which the coefficients of \( q^n \) of LHS and RHS agree is unique up to an overall sign.

The series \( V_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q), V_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) \) are Laurent series. They have the same leading term \( q^\ell \) for some \( \ell \in \mathbb{Z} \) depending on \( \lambda, \mu, \nu, \rho \). We define \( V_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q) = q^{-\ell} V_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q) \) and \( V_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) = q^{-\ell} V_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) \), which have leading term \( q^0 \) (though in general with coefficient \( \neq 1 \)). Using an implementation into Maple we gathered the following evidence:

**Proposition 0.5.** (Proposition 2.14) There are choices of signs such that

\[ \tilde{V}_{\lambda,\mu,\nu,\rho}^{\text{DT}}(q) = \tilde{V}_{\lambda,\mu,\nu,\rho}^{\text{PT}}(q) \tilde{V}_{\emptyset,\emptyset,\emptyset,\emptyset}^{\text{DT}}(q) \mod q^N \]

in the following cases:

- for any \( |\lambda| + |\mu| + |\nu| + |\rho| \leq 1 \) and \( N = 4 \),
- for any \( |\lambda| + |\mu| + |\nu| + |\rho| \leq 2 \) and \( N = 4 \),
- for any \( |\lambda| + |\mu| + |\nu| + |\rho| \leq 3 \) and \( N = 3 \),
- for any \( |\lambda| + |\mu| + |\nu| + |\rho| \leq 4 \) and \( N = 3 \).

In each of these cases, the choice of signs for which the coefficients of \( q^n \) of LHS and RHS agree is unique up to overall sign.

The last case involves a genuine asymptotic plane partition (i.e. 3D pile of boxes of height 2).

**Remark 0.6.** By a conjecture of Nekrasov (Conj. 2.15), there exist choices of signs such that

\[ V^{\text{DT}}_{\emptyset,\emptyset,\emptyset,\emptyset}(q) = \exp \left( q \frac{1 + \lambda_1 + \lambda_2}{\lambda_1 \lambda_2} \right). \]

The above equality has a \( K \)-theoretic enhancement found by Nekrasov [19]. It is therefore natural to ask for a \( K \)-theoretic enhancement of Conjecture 0.4 as well.

Conjecture 0.4 implies the following DT/PT correspondence for toric Calabi-Yau 4-folds.

**Theorem 0.7.** (Theorem 2.18) Assume Conjecture 0.4 holds. Let \( X \) be a toric Calabi-Yau 4-fold, \( \beta \in H_2(X) \) such that \( P_n(X, \beta)(\mathbb{C}^\times)^s \) is 0-dimensional for all \( n \in \mathbb{Z} \), and let \( \gamma_1, \ldots, \gamma_m \in H_1^{\vir}(X) \). Then there exist choices of signs such that

\[ \frac{I_{\beta}(X; T)(\gamma_1, \ldots, \gamma_m)}{I_0(X; T)} = P_{\beta}(X; T)(\gamma_1, \ldots, \gamma_m). \]

In particular, without insertions we have

\[ \frac{I_{\beta}(X; T)}{I_0(X; T)} = P_{\beta}(X; T). \]

Combining Proposition 0.5 and Theorem 0.7, we obtain verifications of the DT/PT correspondence for several local toric geometries (see Section 2.8 for details).

Although the Hilbert schemes \( I_n(X, \beta) \) are rarely compact, there are many cases where the moduli space \( P_n(X, \beta) \) is compact. Then the virtual class \( [P_n(X, \beta)]^{\vir} \) of the previous section is defined. If \( P_n(X, \beta)(\mathbb{C}^\times)^s \) is also 0-dimensional, the invariants of this section and the invariants of the previous section are expected to be related by a 4-fold virtual localization formula

\[ P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z} \]
for any $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$ admitting $T$-equivariant lifts and satisfying $\sum \deg \tau(\gamma_i) = 2n$, and for appropriately chosen signs on RHS. Similarly, for any $n < 0$ we have $[P_n(X, \beta)]^{\text{vir}} = 0$ (by (0.2)) and therefore we expect that there exist choices of signs such that
\begin{equation}
(0.5) \quad P_{n, \beta}(X; T) = 0, \quad \text{for all } n < 0.
\end{equation}

For the following toric Calabi-Yau 4-folds, $P_n(X, \beta)$ is compact and $P_n(X, \beta)^{(C^*)^4}$ is isolated for all $\beta \in H_2(X)$ and $n \in \mathbb{Z}$:

- $X = \text{Tot}_{P_1}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$, i.e. the total space of $\mathcal{O}_{P_2}(-1) \oplus \mathcal{O}_{P_2}(-2)$

- $X = \text{Tot}_{P_1 \times P_1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$.

The following two theorems provide further motivation for Conjecture 0.3.

**Theorem 0.8.** (Theorem 2.19) Let $X$ be a toric Calabi-Yau 4-fold and $\beta \in H_2(X)$. Suppose $P_n(X, \beta)$ is proper and $P_n(X, \beta)^{(C^*)^4}$ is 0-dimensional for all $n \in \mathbb{Z}$. Assume the following:

1. the DT/PT vertex correspondence (Conjecture 0.4) holds,
2. holds for $\beta$ and $n = 0$, and (0.3) holds for $\beta$ and all $n < 0$,
3. the signs of (1) and (2) can be chosen compatibly.

Then $I_{n, \beta}(X; T) = P_{n, \beta}(X) \in \mathbb{Z}$.

**Theorem 0.9.** (Theorem 2.20) Let $X$ be a toric Calabi-Yau 4-fold satisfying $\int_X c^*_3(X) = 0$ and let $\beta \in H_2(X)$. Let $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$ admitting $T$-equivariant lifts and satisfying $\sum_i \deg \tau(\gamma_i) = 2n > 0$. Suppose $P_n(X, \beta)$ is proper, and $P_n(X, \beta)^{(C^*)^4}$ is 0-dimensional for all $\chi \in \mathbb{Z}$. Assume the following:

1. the DT/PT vertex correspondence (Conjecture 0.4) holds,
2. Nekrasov’s conjecture (Remark 0.6) holds,
3. holds for $\beta$, $\gamma_1, \ldots, \gamma_m$, $n$,
4. the signs of (1), (2), and (3) can be chosen compatibly.

Then $I_{n, \beta}(X; T) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}$.

**Remark 0.10.** The spaces $P_n(X, \beta)$ are compact and the fixed loci $P_n(X, \beta)^{(C^*)^4}$ are 0-dimensional for all $n \in \mathbb{Z}$ when $X = \text{Tot}_{P_2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$ or $\text{Tot}_{P_1 \times P_1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$. In the second case, the equivariant integral $\int_X c^*_3(X)$ is zero. In Section 2.8 we discuss these two local surfaces and the local curve $\text{Tot}_{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$ in more detail. In the range where we checked Conjecture 0.4 (Proposition 0.5) and Nekrasov’s formula (Remark 0.6), we also verified the compatibility of signs. This leads to several values of $\beta, n$ for which Theorem 0.7 follows unconditionally. In an appendix, we also prove the virtual localization formula in some cases.

### 0.5. DT/PT generating series of a local curve

Let $X = \text{Tot}_{P_2}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$. Then $H_2(X) \cong \mathbb{Z}$ is freely generated by the class of the zero section. For any $d \geq 1$ and $n \in \mathbb{Z}$, we consider (equivariant) stable pair invariant without insertions
\[ P_{n, d}(X; T) \in Z(\lambda_1, \lambda_2, \lambda_3). \]

Motivated by [13], we conjecture the following expression for the generating series:

**Conjecture 0.11.** (Conjecture 2.21) For $X = \text{Tot}_{P_2}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$, there exist choices of signs such that the following equation holds
\[ \sum_{n,d \geq 0} P_{n, d}(X; T) q^n y^d = \exp \left( \frac{q y}{\lambda_2} \right), \]
where $\lambda_2$ is the equivariant parameter for the $C^*$-action on the first fibre $P_2$, and $P_{0,0}(X; T) := 1$.

By using the vertex formalism and Maple calculations, we obtain the following verifications:

**Proposition 0.12.** (Proposition 2.23) Conjecture 0.11 holds in the following cases:

- for any $n \leq d$,
- $d = 1$ and modulo $q^6$,
- $d = 2$ and modulo $q^6$,
- $d = 3$ and modulo $q^6$,
- $d = 4$ and modulo $q^6$.

As an application, we obtain a generating series for the curve counting invariants of $X = \text{Tot}_{P_2}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$. Assume Nekrasov’s formula (0.3), Conjectures 0.3 0.11 and assume the signs can be chosen compatibly. Then
\[ \sum_{n,d \geq 0} I_{n, d}(X; T) q^n y^d = \exp \left( \frac{q}{\lambda_2} \left( y + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_3} (\lambda_1 + \lambda_3) (\lambda_2 + \lambda_3) + \frac{\lambda_2 (\lambda_1 - \lambda_2) (\lambda_1 + \lambda_2 + \lambda_3)}{\lambda_1 (\lambda_1 + \lambda_3) (\lambda_2 + \lambda_3)} \right) \right). \]
0.6. Notation and conventions. In this paper, all varieties and schemes are defined over \( \mathbb{C} \). For \( \mathcal{F}, \mathcal{G} \in \mathbb{D}^b(\text{Coh}(\mathcal{X})) \), we denote by \( \text{ext}^i(\mathcal{F}, \mathcal{G}) \) the dimension of \( \text{Ext}^i_{\mathcal{X}}(\mathcal{F}, \mathcal{G}) \). A class \( \beta \in H_2(X, \mathbb{Z}) \) is called irreducible (resp. primitive) if it is not the sum of two non-zero effective classes (resp. if it is not a positive integer multiple of an effective class).

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1. DT/PT for compact Calabi-Yau 4-folds

1.1. DT invariants of Calabi-Yau 4-folds. Let us first introduce the set-up of Donaldson-Thomas invariants of smooth projective Calabi-Yau 4-folds \( X \). We fix an ample divisor \( \omega \) on \( X \) and take a cohomology class \( v \in H^*(X, \mathbb{Q}) \).

The coarse moduli space \( M_\omega(v) \) of \( \omega \)-Gieseker semistable sheaves \( E \) on \( X \) with \( \text{ch}(E) = v \) exists as a projective scheme. We always assume that \( M_\omega(v) \) is a fine moduli space, i.e. any point \( [E] \in M_\omega(v) \) is stable and there is a universal family

\[
E \in \text{Coh}(X \times M_\omega(v))
\]

flat over \( M_\omega(v) \). For instance, the moduli space of 1-dimensional stable sheaves \( E \) on \( X \) with \( [E] = \beta \), \( \chi(E) = 1 \) and Hilbert schemes of closed subschemes satisfy this assumption [3, 9, 13].

In [4, 11], under certain hypotheses, the authors construct a virtual class

\[
[M_\omega(v)]^\text{vir} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}),
\]

where \( \chi(-,-) \) denotes the Euler pairing. Notice that this class is not necessarily algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point \( [E] \in M_\omega(v) \), a half-dimensional real subspace

\[
\text{Ext}^2_{\mathcal{X}}(E, E) \subseteq \text{Ext}^2(E, E)
\]

of the usual obstruction space \( \text{Ext}^2(E, E) \), on which the quadratic form \( Q \) defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of the form

\[
\kappa_\pm = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \to \text{Ext}^1_{\mathcal{X}}(E, E),
\]

where \( \kappa \) is the Kuranishi map for \( M_\omega(v) \) at \( [E] \) and \( \pi_+ \) denotes projection on the first factor of the decomposition \( \text{Ext}^2(E, E) = \text{Ext}^2_{\mathcal{X}}(E, E) \oplus \sqrt{-1} \cdot \text{Ext}^1_{\mathcal{X}}(E, E) \).

In [11], local models are glued in three special cases:

1. when \( M_\omega(v) \) consists of locally free sheaves only,
2. when \( M_\omega(v) \) is smooth,
3. when \( M_\omega(v) \) is a shifted cotangent bundle of a derived smooth scheme.

In each case, the corresponding virtual classes are constructed using either gauge theory or algebro-geometric perfect obstruction theory.

The general gluing construction is due to Borisov-Joyce [1], based on Pantev-Töen-Vaquié-Vezzosi’s theory of shifted symplectic geometry [23] and Joyce’s theory of derived \( C^\infty \)-geometry.

The corresponding virtual class is constructed using Joyce’s D-manifold theory (a machinery similar to Fukaya-Oh-Ohta-Ono’s theory of Kuranishi space structures used for defining Lagrangian Floer theory).

The examples of this section only involve the virtual class constructions in situations (2) and (3) mentioned above. We briefly review them:

- When \( M_\omega(v) \) is smooth, the obstruction sheaf \( \text{Ob} \to M_\omega(v) \) is a vector bundle endowed with a quadratic form \( Q \) via Serre duality. Then the virtual class is given by

\[
[M_\omega(v)]^\text{vir} = \text{PD}(e(\text{Ob}, Q)).
\]

Here \( e(\text{Ob}, Q) \) is the half-Euler class of \( (\text{Ob}, Q) \) (i.e. the Euler class of its real form \( \text{Ob}_+ \)), and \( \text{PD}(-) \) denotes Poincaré dual. Note that the half-Euler class satisfies

\[
e(\text{Ob}, Q)^2 = (-1)^{\text{rk}(\text{Ob})} e(\text{Ob}), \quad \text{if rk}(\text{Ob}) \text{ is even},
\]

\[
e(\text{Ob}, Q) = 0, \quad \text{if rk}(\text{Ob}) \text{ is odd}.
\]
• Suppose $M_\omega(v)$ is a the shifted cotangent bundle of a derived smooth scheme. Roughly speaking, this means that at any closed point $[F] \in M_\omega(v)$, we have a Kuranishi map of the form

$$\kappa: \text{Ext}^1(F, F) \to \text{Ext}^2(F, F) = V_F \oplus V_F^\ast,$$

where $\kappa$ factors through a maximal isotropic subspace $V_F$ of $(\text{Ext}^2(F, F), Q)$. Then the virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_F\}_{F \in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_F\}_{F \in M_\omega(v)}$ over $M_\omega(v)$.

**On orientations.** To construct the above virtual class $[\text{vir}]$ with coefficients in $\mathbb{Z}$ (instead of $\mathbb{Z}_2$), we need an orientability result for $M_\omega(v)$, which can be stated as follows. Let

$$\mathcal{L} := \det(\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \in \text{Pic}(M_\omega(v)), \quad \pi_M: X \times M_\omega(v) \to M_\omega(v)$$

be the determinant line bundle of $M_\omega(v)$, which is equipped with a symmetric pairing $Q$ induced by Serre duality. An orientation of $(\mathcal{L}, Q)$ is a reduction of its structure group from $\text{SO}(1, \mathbb{C})$ to $\text{SO}(1, \mathbb{C}) = \{1\}$. In other words, we require a choice of square root of the isomorphism

$$Q: \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_{M_\omega(v)}.$$

Given such a choice, one can construct a virtual class $[\text{vir}]$. Existence of orientations was first proved when the Calabi-Yau 4-fold $X$ satisfies $\text{Hol}(X) = \text{SU}(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$ [12], and was recently generalized to arbitrary Calabi-Yau 4-folds [8]. Notice that, if orientations exist, the collection of orientations forms a torsor for $H^0(M_\omega(v), \mathbb{Z}_2)$.

### 1.2. DT/PT correspondence

Let $X$ be a smooth projective Calabi-Yau 4-fold and $\beta \in H_2(X)$. Let $I_n(X, \beta)$ be the Hilbert scheme of closed subschemes $Z \subseteq X$ of dimension $\leq 1$ with $\text{ch}(\mathcal{O}_Z) = (0, 0, 0, \beta, n)$. These Hilbert schemes are isomorphic to moduli spaces of rank 1 torsion free sheaves on $X$ with trivial determinant. On the latter, one can construct a virtual class

$$[I_n(X, \beta)]_{\text{vir}} \in H_2\!(I_n(X, \beta)),$$

in the sense of Borisov-Joyce [4] (as in [14] Theorem 1.4)). This virtual class depends on the choice of orientation discussed above.

We consider primary insertions

$$(1.2) \quad \tau: H^s(X, \mathbb{Z}) \to H^{s-2}(I_n(X, \beta), \mathbb{Z}), \quad \tau(\gamma) = \pi_{I_n}(\pi_X^* \gamma \cup \text{ch}_3(\mathcal{O}_Z)),$$

where $\pi_X, \pi_I$ are projections from $X \times I_n(X, \beta)$ to the corresponding factors, $Z \subseteq X \times I_n(X, \beta)$ is the universal subscheme, and $\text{ch}_3(\mathcal{O}_Z)$ is the Poincaré dual of the fundamental cycle of $Z$.

For any $\gamma_1, \ldots, \gamma_m \in H^n(X, \mathbb{Z})$, we define curve counting invariants (with primary insertions) of $X$ as follows

$$I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) := \int_{[I_n(X, \beta)]_{\text{vir}}} \prod_{i=1}^m \tau(\gamma_i) \in \mathbb{Z}, \quad I_{0, \beta}(X) := \int_{[I_0(X, \beta)]_{\text{vir}}} 1 \in \mathbb{Z}.$$

We propose the following DT/PT correspondence on compact Calabi-Yau 4-folds:

**Conjecture 1.1.** Let $X$ be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X)$, $\gamma_1, \ldots, \gamma_m \in H^n(X, \mathbb{Z})$, and $n \in \mathbb{Z}$. Then there exist choices of orientation such that

$$(1) \quad I_{0, \beta}(X) = P_{0, \beta}(X), \quad (2) \quad I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m).$$

### 1.3. Heuristic argument

In this subsection, we give a heuristic argument to explain part of Conjecture 1.1. The argument is similar to the one used in [14].

Let $X$ be an “ideal” smooth projective Calabi-Yau 4-fold, in the sense that curves of $X$ deform in families of expected dimensions, and have expected generic properties as follows:

1. Connected reduced curves with arithmetic genus 0 (rational curves). Any rational curve in $X$ is a chain of smooth $\mathbb{P}^1$’s with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$ and moves in a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}$.

2. Connected reduced curves with arithmetic genus 1. Any such curve $E$ in $X$ is smooth (i.e. an elliptic curve) and super-rigid, i.e. its normal bundle is $L_1 \oplus L_2 \oplus L_3$ for general degree zero line bundle $L_1$ on $E$ satisfying $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_E$. Also, any two elliptic curves on $X$ are disjoint and disjoint from all families of rational curves on $X$.

3. There are no connected, reduced curves in $X$ of arithmetic genus $g \geq 2$. 
Let compact examples.

1.4. We claim \( I_0(X, \beta) \equiv P_0(X, \beta) \). For \([I_Z] \in I_0(X, \beta)\), the torsion subsheaf \( T_0 \subseteq \mathcal{O}_Z \) gives an exact sequence

\[
0 \to T_0 \to \mathcal{O}_Z \to \mathcal{O}_C \to 0,
\]

where \( T_0 \) is 0-dimensional and \( C \) is Cohen-Macaulay. From our “ideal” assumptions, a 1-dimensional Cohen-Macaulay scheme \( C_0 \) supported in one of our families of rational curves (resp. elliptic curves) has \( \chi(O_{C_0}) \geq 1 \) (resp. \( \chi(O_{C_0}) \geq 0 \)). Since \( \chi(O_Z) = 0 \), \( C \) can only be supported on some elliptic curves in \( X \) and \( T_0 = 0 \). Thus \([I_Z] = [I_C] \in P_0(X, \beta)\) defines a stable pair.

Conversely, for \([I^* = \{\mathcal{O}_X \to F\}] \in P_0(X, \beta)\), we have a short exact sequence

\[
0 \to \mathcal{O}_C \to F \to Q \to 0,
\]

where \( C \) is Cohen-Macaulay and \( Q \) is 0-dimensional. By a similar reasoning, we know \( Q = 0 \), \( C \) is supported on some elliptic curves in \( X \), and \([I_C] \in I_0(X, \beta)\). The virtual classes get identified under this isomorphism and hence \( I_{0, \beta}(X) = P_{0, \beta}(X) \).

- \( I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \). Here we can only justify the case when \( m = n \) and \( \gamma_1 = \cdots = \gamma_m = \gamma \in H^4(X) \). For \( n \geq 1 \), we want to compute

\[
\int_{[I_n(X, \beta)]^{\text{vir}}} \tau(\gamma)^n, \quad \gamma \in H^4(X, Z),
\]

when \( X \) is an ideal smooth projective Calabi-Yau 4-fold. Let \( \{Z_i\}_{i=1}^n \) be (mutually distinct) 4-cycles which represent the class \( \gamma \). For dimension reasons, we may assume that for any \( i \neq j \) the rational curves meeting with \( Z_i \) are disjoint from those with \( Z_j \). The insertions cut down the moduli space to finitely many elements whose support intersect all \( \{Z_i\}_{i=1}^n \). We denote the moduli space of such “incident” 1-dimensional subschemes by

\[
Q_n(X, \beta; \{Z_i\}_{i=1}^n) \subseteq I_n(X, \beta).
\]

Then we claim that

\[
(1.3) \quad Q_n(X, \beta; \{Z_i\}_{i=1}^n) = \prod_{\beta_0 + \beta_1 + \cdots + \beta_n = \beta} I_0(X, \beta_0) \times R_1(X, \beta_1; Z_1) \times \cdots \times R_1(X, \beta_n; Z_n),
\]

where \( R_1(X, \beta_i; Z_i) \) is the moduli space of 1-dimensional subschemes supported on rational curves (in class \( \beta_i \)) which meet with \( Z_i \).

In fact, for \( Z \in Q_n(X, \beta; \{Z_i\}_{i=1}^n) \), we have a torsion filtration of \( \mathcal{O}_Z \):

\[
0 \to T_0 \to \mathcal{O}_Z \to \mathcal{O}_C \to 0,
\]

where \( T_0 \) is 0-dimensional and \( C \) is Cohen-Macaulay. Note that \( \mathcal{O}_C \) decomposes into a direct sum \( \bigoplus_{i=0}^n \mathcal{O}_{C_i} \), where \( C_i \) is supported on elliptic curves and each \( C_i \) for \( 1 \leq i \leq n \) is supported on \( \text{disjoint} \) rational curves which meet with \( Z_i \). As explained before, a Cohen-Macaulay scheme \( C' \) supported on a family of rational curves (resp. elliptic curves) satisfies \( \chi(O_{C'}) \geq 1 \) (resp. \( \chi(O_{C'}) \geq 0 \)), so \( \chi(O_{C_i}) \geq 0 \), \( \chi(O_{C_i}) \geq 1 \) for all \( 1 \leq i \leq n \). Since \( \chi(O_Z) = 0 \), we have \( \chi(O_{C_i}) = 0 \), \( \chi(O_{C_i}) = 1 \) for all \( i = 1, \ldots, n \), and \( T_0 = 0 \). Therefore \( \text{(1.3)} \) holds.

Moreover, each \( R_1(X, \beta_i; Z_i) \) consists of finitely many rational curves which meet with \( Z_i \), whose number is exactly \( n_{0, \beta_i}(\gamma) \). By counting the number of points in \( I_0(X, \beta_0) \) and \( R_1(X, \beta_1; Z_1) \), for all \( 1 \leq i \leq n \), we obtain

\[
I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) := \int_{[I_n(X, \beta)]^{\text{vir}}} \tau(\gamma)^n = \int_{[Q_n(X, \beta; \{Z_i\}_{i=1}^n)]^{\text{vir}}} 1
\]

Combining with \( I_{n, \beta}(X) = P_{n, \beta}(X) \) explained above, we see that the above equality coincides with the formula in Conjecture \( [11] \), so \( I_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \) for \( n > 0 \).

1.4. Compact examples. In this section, we verify Conjecture \( [11] \) in some simple examples.

**Lemma 1.2.** Let \( X \) be a smooth projective variety and \( \beta \in H_2(X) \) an irreducible class. If any element \( C \) of the Chow variety \( \text{Chow}_\beta(X) \) is a smooth rational curve, then \( I_0(X, \beta) \) is empty.
Proof. Given \([I_Z] \in I_0(X, \beta)\) with torsion filtration \(T_0 \subseteq O_Z\) such that \(O_C := O_Z/T_0\), we have
\[
0 = \chi(O_Z) = \chi(O_C) + \chi(T_0) \geq \chi(O_C).
\]
This contradicts \(C \cong \mathbb{P}^1\).

The above lemma also holds when \(I_0(X, \beta)\) is replaced by \(P_0(X, \beta)\) (see [14, Lem. 2.7]). So for any irreducible curve class \(\beta\) for which all \(C \in \text{Chow}_\beta(X)\) are smooth rational, the equality \(I_{0, \beta}(X) = P_{0, \beta}(X)\) obviously holds. This is the case in the following examples:

**Proposition 1.3.** Conjecture [14] (1) holds for any irreducible class \(\beta \in H_2(X)\) when:

1. \(X\) is one of the quintic fibrations in [16].
2. \(X\) a complete intersection in a product of projective spaces.
3. \(X\) one of the complete intersection in Grassmannian varieties in [15].

**Remark 1.4.** \(I_{0, \beta}(X) = P_{0, \beta}(X)\) also holds for degree two class on sextic 4-folds (by [7], [14, Prop. 2.2]).

Next, we discuss some non-primitive examples on elliptic fibrations. For \(Y = \mathbb{P}^3\), we take general elements
\[
u \in H^0(Y, \mathcal{O}_Y(-4K_Y)), \quad \nu' \in H^0(Y, \mathcal{O}_Y(-6K_Y)).
\]
Let \(X\) be a smooth projective Calabi-Yau 4-fold with an elliptic fibration\(\pi: X \to Y\) given by the Weierstrass equation
\[
zy^2 = x^3 + uxz^2 + vz^3
\]
in the \(\mathbb{P}^2\)-bundle
\[
P(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y) \to Y,
\]
where \([x : y : z]\) are homogeneous coordinates in the fibres of the above projective bundle. A general fiber of \(\pi\) is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve. Moreover, \(\pi\) admits a section \(\iota: Y \hookrightarrow X\) whose image corresponds to the fibre point \([0 : 1 : 0]\). We denote the class of the fibre by \([\iota] \in H_2(X)\).

**Lemma 1.5.** For any \(r \geq 1\), there exists an isomorphism
\[
I_0(X, r[\iota]) \cong \text{Hilb}^r(\mathbb{P}^3),
\]
under which the virtual class is given by
\[
[I_0(X, r[\iota])]^{\text{vir}} = (-1)^r \cdot [\text{Hilb}^r(\mathbb{P}^3)]^{\text{vir}},
\]
for a certain choice of orientation on the LHS.

**Proof.** The proof is similar to the stable pair case [14, Lemma 2.5]. We show that the natural morphism
\[
\pi^*: \text{Hilb}^r(\mathbb{P}^3) \to I_0(X, r[\iota])
\]
is an isomorphism. Let \([I_Z] \in I_0(X, r[\iota])\) be an ideal sheaf and \(T_0 \subseteq O_Z\) the torsion subsheaf. Denote \(O_C := O_Z/T_0\), then
\[
0 = \chi(O_Z) = \chi(O_C) + \chi(T_0) \geq \chi(O_C).
\]
Fix a polarization on \(X\). By the Harder-Narasimhan filtration, we have
\[
0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = O_C,
\]
where the quotients \(E_i = F_i/F_{i-1}\) are stable with decreasing reduced Hilbert polynomials
\[
p(E_1) \geq p(E_2) \geq \cdots \geq p(E_n).
\]
Note \(\text{ch}(E_i) = (0, 0, 0, r_i[\iota], \chi(E_i))\) for some \(r_i \geq 1\), so the above inequalities are equivalent to
\[
\frac{\chi(E_1)}{r_1} \geq \frac{\chi(E_2)}{r_2} \geq \cdots \geq \frac{\chi(E_n)}{r_n}.
\]
By [13, Lem. 2.2], stability of \(E_i\) implies that it is scheme theoretically supported on some fibre \(\pi^{-1}(p_i)\) of \(\pi\), i.e. \(E_i = (\tau_{p_i})_*(E'_i)\) for some \(\tau_{p_i}: X_{p_i} \hookrightarrow X\) and stable sheaf \(E'_i \in \text{Coh}(X_{p_i})\).

Since \(s: O_X \to O_C\) is surjective, so is the composition \(O_X \to O_C \to E_n\). By adjunction, we obtain an isomorphism
\[
\text{Hom}_X(O_X, E_n) \cong \text{Hom}_{X_{p_n}}(O_{X_{p_n}}, E'_n) \neq 0,
\]
which implies that \( p(E_a') \geq 0 \), hence \( \chi(E_a) \geq 0 \). Therefore

\[
0 \geq \chi(\mathcal{O}_C) = \sum_{i=1}^{n} \chi(E_i) \geq 0,
\]

which implies \( T_0 = 0 \) and \( I_Z = I_C \) is a stable pair by (1.5). The remaining argument is the same as the stable pair case (ref. [14] Lemma 2.5). \( \square \)

In particular, we conclude:

**Proposition 1.6.** Conjecture [L] (1) holds in the following cases:

1. \( X \) the Weierstrass elliptic fibration [1.4] and \( \beta = r[f] \) for any \( r > 1 \).
2. \( X = Y \times E \) is the product of a smooth projective Calabi-Yau 3-fold \( Y \) and an elliptic curve \( E \), and \( \beta = r[E] \), for any \( r > 1 \), where \( [E] \) denotes the class of \([\text{pt}] \times E \).

**Proof.** (1) By Lemma [1.5] and [14] Lem. 2.5, \( P_{0,\beta}(X) = I_{0,\beta}(X) \). (2) A similar result as Lemma [1.5] holds in this case. By comparing with [14] Lem. 2.14, we are done. \( \square \)

Next, we check \( I_{1,\beta}(X) = P_{1,\beta}(X) \) when any \( C \in \text{Chow}_\beta(X) \) is a rational curve. Then any \( [I_Z] \in I_1(X, \beta) \) is a stable pair and \( I_1(X, \beta) \cong P_1(X, \beta) \). Moreover, the virtual classes are identified under this isomorphism.

**Proposition 1.7.** Conjecture [L] (2) holds for \( n = 1 \) and \( \beta \in H_2(X) \) in the following cases:

1. \( X \) a complete intersection in a product of projective spaces and \( \beta \) irreducible,
2. \( X = Y \times E \) and \( \beta \in H_2(Y) \subseteq H_2(X) \) irreducible, where \( Y \) is a smooth complete intersection Calabi-Yau 3-fold in a product of projective spaces and \( E \) an elliptic curve,
3. \( X \) a generic sextic 4-fold and \( \beta = 2[l] \), where \( [l] \in H_2(X) \) is the class of a line.

**Proof.** For cases (1) and (3), we have an inclusion \( \iota : X \hookrightarrow \prod_{i=1}^r \mathbb{P}^{n_i} \) and the Lefschetz hyperplane theorem implies

\[
\iota_* : H_2(X) \cong \bigoplus_{i=1}^r \mathbb{Z} \cdot [l_i],
\]

where \([l_i]\) is the class of a line on \( \mathbb{P}^{n_i} \). In case (2) the same holds with \( X \) replaced by \( Y \). The only irreducible curve classes on \( X \) (resp. \( Y \)) are \([l_1], \ldots, [l_r]\) and any \( C \in \text{Chow}_\beta(X) \) (resp. \( \text{Chow}_\beta(Y) \)) is a smooth rational curve, hence \( \chi(\mathcal{O}_C) = 1 \). In case (3), any \( C \in \text{Chow}_\beta(X) \) is a smooth conic or union of two distinct lines [7] Prop. 1.4 hence also \( \chi(\mathcal{O}_C) = 1 \).

For \( Z \in I_1(X, \beta) \), we have a torsion filtration

\[
0 \to T_0 \to O_Z \to O_C \to 0,
\]

where \( T_0 \) is 0-dimensional and \( C \) is Cohen-Macaulay. In the above cases, any \( C \in \text{Chow}_\beta(X) \) satisfies \( \chi(\mathcal{O}_C) = 1 \). So \( T_0 = 0 \) and \( Z = C \) is Cohen-Macaulay, so \( I_Z = I_C \) is a stable pair.

Conversely, for \([I^*] = [O_X \to F] \in P_1(X, \beta)\), we have an exact sequence

\[
0 \to O_C \to F \to Q \to 0,
\]

where \( C = \text{supp}(F) \) and \( Q \) is 0-dimensional. Then \( \chi(\mathcal{O}_C) = 1 \) implies \( Q = 0 \), so we have \([I^*] = [I_C] \in I_1(X, \beta) \).

Working in families, we obtain an isomorphism

\[
I_1(X, \beta) \cong P_1(X, \beta),
\]

under which the virtual classes are identified. Capping with insertions gives the result. \( \square \)

2. DT/PT on toric Calabi-Yau 4-folds

In this section, we study equivariant analogues of the DT/PT correspondence for toric Calabi-Yau 4-folds. After studying fixed loci, we define equivariant curve counting invariants and stable pair invariants, which are rational functions of the equivariant parameters. We formulate a vertex formalism for both invariants and conjecture a relation between the (fully equivariant) DT/PT vertex, which we check in several cases. This implies a DT/PT correspondence for toric Calabi-Yau 4-folds with primary insertions. For local surfaces, we consider cases where the correspondence reduces to an equality of numbers providing good motivation for Conjecture [L].
2.1. Fixed loci of Hilbert schemes. Let $X$ be a smooth quasi-projective toric Calabi-Yau 4-fold. By this we mean a smooth quasi-projective toric 4-fold $X$ satisfying $K_X \cong \mathcal{O}_X$ and $H^{-0}(\mathcal{O}_X) = 0$. We also assume the fan contains cones of dimension 4. Such cones correspond to $(\mathbb{C}^*)^4$-invariant affine open subsets (equivariantly) isomorphic to $\mathbb{C}^4$. Important examples are the following:

- $X = \text{Tot}_{P^1 \times P^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$,
- $X = \text{Tot}_{P^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$,
- $X = \text{Tot}_{P^1}(\mathcal{O}(a, b, c))$, with $a + b + c = -2$.

Let $\Delta(X)$ be the polytope corresponding to $X$ and denote the collection of its vertices and edges by $V(X)$, $E(X)$ respectively. The elements $\alpha \in V(X)$ correspond to the $(\mathbb{C}^*)^4$-fixed points $p_\alpha \in X$. Each such fixed point lies in a maximal $(\mathbb{C}^*)^4$-invariant affine open subset $\mathbb{C}^4 \cong U_\alpha \subseteq X$. The elements $\alpha \beta \in E(X)$ (connecting vertices $\alpha, \beta$) correspond to the $(\mathbb{C}^*)^4$-invariant lines $\mathbb{P}^1 \cong C_{\alpha \beta} \subseteq X$. Such a line has normal bundle

$$N_{C_{\alpha \beta}/X} \cong \mathcal{O}_\alpha(m_{\alpha \beta}) \oplus \mathcal{O}_{\beta}(m'_{\alpha \beta}) \oplus \mathcal{O}_{\alpha}(m''_{\alpha \beta}),$$

(2.1)

where the first isomorphism follows from the splitting principle and the second equality follows from the fact that $X$ is Calabi-Yau.

The action of the dense open torus $(\mathbb{C}^*)^4$, and its Calabi-Yau subtorus $T \subseteq (\mathbb{C}^*)^4$, lift to the Hilbert scheme $I_n(X, \beta)$. Although $I_n(X, \beta)$ is in general not proper, its $(\mathbb{C}^*)^4$-fixed and $T$-fixed loci are proper. In fact, these loci coincide and are 0-dimensional and reduced.

**Lemma 2.1.** At the level of closed points, we have

$$I_n(X, \beta)^T = I_n(X, \beta)((\mathbb{C}^*)^4),$$

which consists of finitely many points.

**Proof.** Similarly as in [3 Lemma 3.1], we cover $X$ by maximal $(\mathbb{C}^*)^4$-invariant affine open subsets $\{U_\alpha \cong \mathbb{C}^4\}_{\alpha \in V(X)}$ with centres at $(\mathbb{C}^*)^4$-fixed points $p_\alpha$. There exist coordinates $x_1, x_2, x_3, x_4$ on $U_\alpha \cong \mathbb{C}^4$, such that the action of $t \in (\mathbb{C}^*)^4$ on $U_\alpha$ is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4 \text{ and } t \in (\mathbb{C}^*)^4.$$

Then the Calabi-Yau torus is given by

$$T = \{t \in (\mathbb{C}^*)^4 \mid t_1 t_2 t_3 t_4 = 1\}$$

and we see that $U_\alpha$ is also $T$-invariant. Therefore it suffices to prove the lemma for $X = U_\alpha = \mathbb{C}^4$ with the standard torus action.

The $(\mathbb{C}^*)^4$-invariant ideals in $\mathbb{C}[x_1, x_2, x_3, x_4]$ are precisely the monomial ideals. Clearly

$$I_n(X, \beta)^T \supseteq I_n(X, \beta)((\mathbb{C}^*)^4).$$

By considering the weight of $x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}$ under the action of $t \in \mathbb{C}^4$, it is easy to see that any $T$-invariant ideal $I \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$ is of form

$$I = \langle x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4} f_1(x_1 x_2 x_3 x_4), \ldots, x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4} f_l(x_1 x_2 x_3 x_4) \rangle,$$

where $\{f_i(y)\}$ are polynomials of one variable with constant coefficient 1 and $n_{ij} \in \mathbb{Z}_{\geq 0}$.

Suppose $I$ is $T$-invariant and corresponds to a one-dimensional subscheme $Z$. If $x_1 x_2 x_3 x_4 \neq 0$ on $Z$, then $f_i(x_1 x_2 x_3 x_4) = 0$ on $Z$ for all $i = 1, \ldots, l$. However such a system of equations cannot cut out a subscheme of dimension $\leq 1$. So $x_1 x_2 x_3 x_4 = 0$ on $Z$ and $f_i(x_1 x_2 x_3 x_4) \neq 0$ on $Z$ for all $i = 1, \ldots, l$. Hence the open subset

$$U = \{f_1(x_1 x_2 x_3 x_4) = 0\} \cap \ldots \cap \{f_l(x_1 x_2 x_3 x_4) = 0\}$$

contains $Z$. The polynomials $f_i(x_1 x_2 x_3)$ become invertible elements after restriction to $U$, hence

$$I|_U = \langle x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}, \ldots, x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4} \rangle.$$

We conclude that

$$I = \langle x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}, \ldots, x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4} \rangle,$$

which shows $I_n(X, \beta)^T \subseteq I_n(X, \beta)((\mathbb{C}^*)^4)$. $\square$

Similarly to [3 I, Lem. 6] and [3 Lem. 3.4], we have the following.


Lemma 2.2. For any $Z \in I_n(X, \beta)^T$, we have an isomorphism of $T$-representations
\[ \text{Ext}^0(I_Z, \mathcal{O}_Z) \cong \text{Ext}^1(I_Z, I_Z). \]
Moreover, $\text{Ext}^0(I_Z, \mathcal{O}_Z)^T = 0$. In particular, the scheme $I_n(X, \beta)^T = I_n(X, \beta)(\mathbb{C}^*)^4$ consists of finitely many reduced points.

We characterize the elements $I_n(X, \beta)^T$ by collections of so-called solid partitions.

Definition 2.3. A solid partition $\pi$ is a sequence $\pi = \{\pi_{ijk} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\}_{i,j,k \geq 1}$ satisfying:
\[ \pi_{ijk} \geq \pi_{i+1,j,k}, \quad \pi_{ijk} \geq \pi_{i,j+1,k}, \quad \pi_{ijk} \geq \pi_{i,j,k+1} \quad \forall \ i, j, k \geq 1. \]
This extends the notions of a plane partition $\lambda = \{\lambda_{ij}\}_{i,j \geq 1}$ (which we visualize as a pile of boxes in $\mathbb{R}^3$ where $\lambda_{ij}$ is the height along the $z$-axis) and (ordinary) partitions $\lambda = \{\lambda_i\}_{i \geq 1}$ (which we visualize as a pile of squares in $\mathbb{R}^2$ where $\lambda_i$ is the height along the $y$-axis). Given a solid partition $\pi = \{\pi_{ijk}\}_{i,j,k \geq 1}$, there exist unique plane partitions $\lambda, \mu, \nu, \rho$ such that
\[ \pi_{ijk} = \lambda_{jk}, \quad \forall \ i \geq 0, \ j, k \geq 1, \]
\[ \pi_{ijk} = \mu_{ik}, \quad \forall \ j \geq 0, \ i, k \geq 1, \]
\[ \pi_{ijk} = \nu_{ij}, \quad \forall \ k \geq 0, \ i, j \geq 1, \]
\[ \pi_{ijk} = \infty \iff k = \rho_{ij}, \quad \forall \ i, j, k \geq 1. \]
We refer to $\lambda, \mu, \nu, \rho$ as the asymptotic plane partitions associated to $\pi$ in directions $1, 2, 3, 4$ respectively. We call $\pi$ point-like, when $\lambda = \mu = \nu = \rho = \emptyset$. Then the size of $\pi$ is defined by
\[ |\pi| := \sum_{1 \leq i, j, k \leq N} \pi_{ijk}. \]
We call $\pi$ curve-like when $\lambda, \mu, \nu, \rho$ have finite size $|\lambda|, |\mu|, |\nu|, |\rho|$. When $\pi$ is curve-like, we define its renormalized volume (similar to [18]) as follows. For any $N \gg 0$
\[ |\pi| := \sum_{1 \leq i, j, k \leq N} \pi_{ijk} - (|\lambda| + |\mu| + |\nu|) \cdot N, \]
which is independent of $N \gg 0$.

Let $[Z] \in I_n(X, \beta)^T$. Suppose $\mathbb{C}^4 \cong U \subseteq X$ is a maximal $(\mathbb{C}^*)^4$-invariant affine open subset. Then there are coordinates $(x_1, x_2, x_3, x_4)$ such that
\[ t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4 \text{ and } t \in (\mathbb{C}^*)^4. \]
The restriction $Z|_U$ is cut out by a $(\mathbb{C}^*)^4$-invariant ideal $I_Z|_U \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$. Curve-like solid partitions $\pi$ are in bijective correspondence to $(\mathbb{C}^*)^4$-invariant ideal $I_{Z_{\pi}} \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$ cutting out subschemes $Z_{\pi} \subseteq \mathbb{C}^4$ of dimension $\leq 1$ via the following formula
\[ I_{Z_{\pi}} = \left\langle x_1^{t_1-1}x_2^{t_2-1}x_3^{t_3-1}x_4^{t_4-1} \pi_{ijk} : i, j, k \geq 1 \right\rangle. \]
Therefore, $[Z] \in I_n(X, \beta)^T$ determines a collection of curve-like solid partitions $\{\pi^{(\alpha)}\}_{\alpha = 1}^{e(X)}$, where $e(X)$ is the topological Euler characteristic of $X$, which equals the number of $(\mathbb{C}^*)^4$-fixed points of $X$. Let $\alpha, \beta \in E(X)$ and consider the corresponding $(\mathbb{C}^*)^4$-invariant line $C_{\alpha \beta} \cong \mathbb{P}^1$. Suppose this line given by $\{x_2 = x_3 = x_4 = 0\}$ in both charts $U_{\alpha}, U_{\beta}$. Let $\lambda^{(\alpha)}, \lambda^{(\beta)}$ be the asymptotic plane partitions of $\pi^{(\alpha)}, \pi^{(\beta)}$ along the $x_1$-axes in both charts. Then
\[ (2.3) \quad \lambda^{(\alpha)}_{ij} = \lambda^{(\beta)}_{ij} =: \lambda^{(\alpha \beta)}_{ij} \quad \forall \ i, j \geq 1. \]
When a collection of curve-like solid partitions $\{\pi^{(\alpha)}\}_{\alpha = 1}^{e(X)}$ satisfies (2.3), for all $\alpha, \beta = 1, \cdots, e(X)$, we say that it satisfies the gluing condition. Therefore, we obtain a bijective correspondence
\[ \left\{ \pi = \{\pi^{(\alpha)}\}_{\alpha = 1}^{e(X)} : \pi^{(\alpha)} \text{ curve-like and satisfying (2.3)} \right\} \]
\[ \cong \bigcup_{\beta \in H_2(X), n \in \mathbb{Z}} I_n(X, \beta)^T. \]
Suppose $\beta \neq 0$ is effective. Then for any $Z \in I_n(X, \beta)^T$, there exists a maximal Cohen-Macaulay subscheme $C \subseteq Z$ such that the cokernel in the short exact sequence
\[ 0 \to I_Z \to I_C \to I_C/I_Z \to 0 \]

is 0-dimensional. As usual, the restriction $C|_{U_\alpha}$ corresponds to a curve-like solid partition $\pi$ with asymptotics $\lambda, \mu, \nu, \rho$. Since $C|_{U_\alpha}$ has no embedded points, the solid partition $\pi$ is entirely determined by the asymptotics $\lambda, \mu, \nu, \rho$ as follows

\[
\pi_{ijk} = \begin{cases} 
\infty & \text{if } 1 \leq k \leq \rho_j \\
\max\{\lambda_{jk}, \mu_{jk}, \nu_{ij}\} & \text{otherwise.} 
\end{cases}
\]

(2.4)

For any plane partition of finite size, and $m, m', m'' \in \mathbb{Z}$, we define

\[
f_{m, m', m''}(\lambda) := \sum_{i, j \geq 1} \sum_{k=1} \left(1 - m(i - 1) - m'(j - 1) - m''(k - 1)\right).
\]

For any $\alpha \beta \in E(X)$ and plane partition $\lambda$ satisfying $|\lambda| < \infty$, we set

\[
f(\alpha, \beta) := f_{m, m', m''}(\lambda),
\]

where $m, m', m''$ were defined in (2.1).

**Lemma 2.4.** Let $X$ be a smooth toric Calabi-Yau 4-fold and $Z \subseteq X$ a $(\mathbb{C}^*)^4$-invariant closed subscheme of dimension $\leq 1$. Then

\[
\chi(\mathcal{O}_Z) = \sum_{\alpha \in V(X)} |\pi^{(\alpha)}| + \sum_{\alpha \beta \in E(X)} f(\alpha, \beta).
\]

**Proof.** Denote by $\{\pi^{(\alpha)}\}_{\alpha \in V(X)}$ the collection of solid partitions corresponding to $Z$ and denote the corresponding asymptotic plane partitions by $\{\lambda^{(\alpha \beta)}\}_{\alpha \beta \in E(X)}$. For each $\alpha \in V(X)$, the renormalized volume of $\pi^{(\alpha)}$ is given by

\[
|\pi^{(\alpha)}| = \sum_{(i, j, k, l) \in \pi} \left(1 - \#\{\text{legs containing } (i, j, k, l)\}\right),
\]

where $(i, j, k, l) \in \pi^{(\alpha)}$ means $l = \pi_{ijk} \neq 0$.

Suppose $C \subseteq Z$ is the maximal Cohen-Macaulay subcurve. Each plane partition $\lambda^{(\alpha \beta)}$ determines an associated Cohen-Macaulay curve on $X$, such that its solid partitions in each chart are entirely determined by the single asymptotic plane partition $\lambda^{(\alpha \beta)}$. We denote this Cohen-Macaulay curve by $\lambda^{(\alpha \beta)}C_{\alpha \beta} \subseteq X$. Its underlying reduced curve by $C_{\alpha \beta} \equiv \mathbb{P}^1$. Then

\[
C \subseteq \bigcup_{\alpha \beta \in E(X)} \lambda^{(\alpha \beta)}C_{\alpha \beta}
\]

and by inclusion-exclusion, we have

\[
\chi(\mathcal{O}_C) = \sum_{\alpha \beta \in E(X)} \chi(\mathcal{O}_{\lambda^{(\alpha \beta)}C_{\alpha \beta}}) - \sum_{\alpha \in V(X)} \sum_{(i, j, k, l) \in \pi^{(\alpha)}} \#\{\text{legs containing } (i, j, k, l)\}.
\]

Furthermore, we have

\[
\chi(\mathcal{O}_{\lambda^{(\alpha \beta)}C_{\alpha \beta}}) = \sum_{(i, j, k) \in \lambda^{(\alpha \beta)}} \chi(\mathcal{O}_{\mathbb{P}^4}(-im_{\alpha \beta}) \oplus \mathcal{O}_{\mathbb{P}^4}(-jm_{\alpha \beta}) \oplus \mathcal{O}_{\mathbb{P}^4}(-km_{\alpha \beta}))
\]

\[
= \sum_{(i, j, k) \in \lambda^{(\alpha \beta)}} \left(1 - im_{\alpha \beta} - jm_{\alpha \beta} - km_{\alpha \beta}\right).
\]

The lemma follows from the fact that the kernel of $\mathcal{O}_Z \to \mathcal{O}_C$ equals $I_C/I_Z$ and

\[
\chi(I_C/I_Z) = \sum_{\alpha \in V(X)} \chi(I_{C|_{U_\alpha}}/I_{Z|_{U_\alpha}}) = \sum_{\alpha \in V(X)} \sum_{(i, j, k, l) \in \pi^{(\alpha)}} 1.
\]

\[
\square
\]

2.2. Fixed loci of stable pairs moduli spaces. Let $X$ be a toric Calabi-Yau 4-fold. Then the action of $(\mathbb{C}^*)^4$ lifts to the stable pairs space $P_{\alpha}(X, \beta)$. We will describe the fixed loci $P_{\alpha}(X, \beta)^{(\mathbb{C}^*)^4}$ and $P_{\alpha}(X, \beta)^T$ closely following Pandharipande-Thomas [22].

Given a stable pair $(F, s)$ on $X$, the scheme-theoretic support $C_F := \text{supp}(F)$ of $F$ is a Cohen-Macaulay curve [21 Lem. 1.6]. Stable pairs with Cohen-Macaulay support curve $C$ can be described as follows [21 Prop. 1.8]:

Let $m \subseteq \mathcal{O}_C$ be the ideal of a finite union of closed points on $C$. A stable pair $(F, s)$ such that $C_F = C$ and $\text{supp}(Q)_{\text{red}} \subseteq \text{supp}(\mathcal{O}_C/m)$ is equivalent to a subsheaf of $\lim_{\rightarrow} \mathcal{H}om(m^r, \mathcal{O}_C)/\mathcal{O}_C$. 

This uses the natural inclusions
\[ \text{Hom}(m^r, \mathcal{O}_C) \hookrightarrow \text{Hom}(m^{r+1}, \mathcal{O}_C) \]
\[ \mathcal{O}_C \hookrightarrow \text{Hom}(m^r, \mathcal{O}_C) \]
induced by \( m^{r+1} \subseteq m^r \subseteq \mathcal{O}_C \).

Suppose \( [(F, \sigma)] \in P_n(X, \beta)(\mathbb{C}^*)^4 \), then \( C_F \) is \((\mathbb{C}^*)^4\)-fixed and therefore determines to curve-like solid partitions \( \{ \pi^{(a)} \}_{a \in V(X)} \) as in the previous section. Consider a maximal \((\mathbb{C}^*)^4\)-invariant affine open subset \( \mathbb{C}^4 \cong U_\alpha \subseteq X \). Denote the asymptotic plane partitions of \( \pi := \pi^{(a)} \) in directions 1, 2, 3, 4 by \( \lambda, \mu, \nu, \rho \). Just like in (2.2), these correspond to \((\mathbb{C}^*)^4\)-invariant ideals
\[ I_{Z_\lambda} \subseteq \mathbb{C}[x_2, x_3, x_4], \]
\[ I_{Z_\mu} \subseteq \mathbb{C}[x_1, x_3, x_4], \]
\[ I_{Z_\nu} \subseteq \mathbb{C}[x_1, x_2, x_4], \]
\[ I_{Z_\rho} \subseteq \mathbb{C}[x_1, x_2, x_3]. \]

Define the following \( \mathbb{C}[x_1, x_2, x_3, x_4] \)-modules
\[ M_1 := \mathbb{C}[x_1, x_1^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_2, x_3, x_4]/I_{Z_\lambda}, \]
\[ M_2 := \mathbb{C}[x_2, x_2^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_3, x_4]/I_{Z_\mu}, \]
\[ M_3 := \mathbb{C}[x_3, x_3^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_4]/I_{Z_\nu}, \]
\[ M_4 := \mathbb{C}[x_4, x_4^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_3]/I_{Z_\rho}. \]

Then (2.2) Sect. 2.4]
\[ \lim_{\rightarrow} \text{Hom}(m^r, \mathcal{O}_{C(U_\alpha)}) \cong \bigoplus_{i=1}^4 M_i := M, \]
where \( m = (x_1, x_2, x_3, x_4) \subseteq \mathbb{C}[x_1, x_2, x_3, x_4] \). Each module \( M_i \) comes from a ring and therefore has a unit 1, which is homogeneous of degree \((0, 0, 0, 0)\) with respect to the character group \( X((\mathbb{C}^*)^4) = \mathbb{Z}^4 \). We form the quotient
\[ (2.5) \quad \frac{M}{\langle (1, 1, 1, 1) \rangle}. \]

Then \((\mathbb{C}^*)^4\)-equivariant stable pairs on \( U_\alpha \cong \mathbb{C}^4 \) correspond to \((\mathbb{C}^*)^4\)-invariant \( \mathbb{C}[x_1, x_2, x_3, x_4] \)-submodules of (2.3).

**Combinatorial description** \( M/\langle (1, 1, 1, 1) \rangle \) Denote the character group of \((\mathbb{C}^*)^4 \) by \( X((\mathbb{C}^*)^4) = \mathbb{Z}^4 \). For each module \( M_i \), the weights \( w \in \mathbb{Z}^4 \) of its non-zero eigenspaces determine an infinite “leg” \( \text{Leg}_i \subseteq \mathbb{Z}^4 \) along the \( x_i \)-axis. For each weight \( w \in \mathbb{Z}^4 \), introduce four independent vectors \( 1_w, 2_w, 3_w, 4_w \). Then the \( \mathbb{C}[x_1, x_2, x_3, x_4] \)-module structure on \( M/\langle (1, 1, 1, 1) \rangle \) is determined by
\[ x_j \cdot i_w = i_{w+e_j}, \]
where \( i, j = 1, \ldots, 4 \) and \( e_1, \ldots, e_4 \) are the standard basis vectors of \( \mathbb{Z}^4 \). Similar to the 3-fold case Sect. 2.5], we define regions
\[ I^+ \cup II \cup III \cup IV \cup I^- = \bigcup_{i=1}^4 \text{Leg}_i \subseteq \mathbb{Z}^4, \]
as follows:

- **I^+** consists of all weights \( w \in \mathbb{Z}^4 \) with all coordinates non-negative and which lie in precisely one leg. If \( w \in I^+ \), then the corresponding weight space of \( M/\langle (1, 1, 1, 1) \rangle \) is 0-dimensional.
- **I^-** consists of all weights \( w \in \mathbb{Z}^4 \) with at least one negative coordinate. If \( w \in I^- \) is supported in \( \text{Leg}_i \), then the corresponding weight space of \( M/\langle (1, 1, 1, 1) \rangle \) is 1-dimensional
\[ C \cong \mathbb{C} \cdot i_w \subseteq M/\langle (1, 1, 1, 1) \rangle. \]
- **II** consists of all weights \( w \in \mathbb{Z}^4 \), which lie in precisely two legs. If \( w \in II \) is supported in \( \text{Leg}_i \) and \( \text{Leg}_j \), then the corresponding weight space of \( M/\langle (1, 1, 1, 1) \rangle \) is 1-dimensional
\[ C \cong \mathbb{C} \cdot i_w \oplus \mathbb{C} \cdot j_w \subseteq M/\langle (1, 1, 1, 1) \rangle. \]
- **III** consists of all weights \( w \in \mathbb{Z}^4 \), which lie in precisely three legs. If \( w \in III \) is supported in \( \text{Leg}_i, \text{Leg}_j, \) and \( \text{Leg}_k \), then the corresponding weight space of \( M/\langle (1, 1, 1, 1) \rangle \) is 2-dimensional
\[ C^2 \cong \mathbb{C} \cdot i_w \oplus \mathbb{C} \cdot j_w \oplus \mathbb{C} \cdot k_w \subseteq M/\langle (1, 1, 1, 1) \rangle. \]
• IV consists of all weights \( w \in \mathbb{Z}^4 \), which lie in all four legs. If \( w \in \text{IV} \), then the corresponding weight space of \( M/((1,1,1,1)) \) is 3-dimensional.

\[
\mathbb{C}^3 \cong \mathbb{C} \cdot 1_w \oplus \mathbb{C} \cdot 2_w \oplus \mathbb{C} \cdot 3_w \oplus \mathbb{C} \cdot 4_w / \mathbb{C} \cdot (1_w + 2_w + 3_w + 4_w) \subseteq M/((1,1,1,1)).
\]

**Box configurations.** A box configuration is a finite collection of weights \( B \subseteq \Pi \cup \Pi \cup \Pi \cup \Pi \) satisfying the following property:

if \( w = (w_1, w_2, w_3, w_4) \in \Pi \cup \Pi \cup \Pi \cup \Pi \) and one of \((w_1 - 1, w_2, w_3, w_4), (w_1, w_2 - 1, w_3, w_4), (w_1, w_2, w_3 - 1, w_4), \) or \((w_1, w_2, w_3, w_4 - 1)\) lies in \( B \) then \( w \in B \).

Such a box configuration determines a \((\mathbb{C}^*)^4\)-invariant submodule of \( M/((1,1,1,1)) \) and therefore a \((\mathbb{C}^*)^4\)-invariant stable pair on \( U_\alpha \cong \mathbb{C}^4 \) with cokernel of length

\[
\#(B \cap \Pi) + 2 \cdot \#(B \cap \Pi) + 3 \cdot \#(B \cap \Pi) + 4 \cdot \#(B \cap \Pi).
\]

The box configurations defined in this section by no means describe all \((\mathbb{C}^*)^4\)-invariant submodules of \( M/((1,1,1,1)) \). From now on, we restrict attention to the case where the fixed loci \( P_n(X, \beta)(\mathbb{C}^*)^4 \) are 0-dimensional for all \( n \). Then the restriction of any \([F, s] \in P_n(X, \beta)(\mathbb{C}^*)^4\) to each \( U_\alpha \) has a Cohen-Macaulay support curve with at most two asymptotic plane partitions (as we prove below) and is described by a box configuration as above.

**Proposition 2.5.** Suppose \( P_n(X, \beta)(\mathbb{C}^*)^4 \) is 0-dimensional for all \( n \in \mathbb{Z} \). Then for any \((F, s) \in P_n(X, \beta)(\mathbb{C}^*)^4\) and any \( \alpha \in V(X) \), the Cohen-Macaulay curve \( C_F|_{U_\alpha} \) has at most two asymptotic plane partitions.

**Proof.** Suppose there is an \( \alpha \in V(X) \), such that the module \( M/((1,1,1,1)) \) associated to \( C_F|_{U_\alpha} \), for some \( \alpha \in V(X) \), has three or four non-empty asymptotic plane partitions.

**Case 1:** \( C_F|_{U_\alpha} \) has four non-empty asymptotic plane partitions. Construct the following \((\mathbb{C}^*)^4\)-invariant submodule of \( M/((1,1,1,1)) \). At \( w = (0,0,0,0) \), choose any 1-dimensional subspace

\[
V_0 \subseteq \mathbb{C} \cdot 1_w \oplus \mathbb{C} \cdot 2_w \oplus \mathbb{C} \cdot 3_w \oplus \mathbb{C} \cdot 4_w / \mathbb{C} \cdot (1_w + 2_w + 3_w + 4_w)
\]

and for any \( w \in (\Pi \cup \Pi \cup \Pi \cup \Pi \) \) \( \setminus \{0\} \), we define

\[
V_w := \begin{cases} 
(M/((1,1,1,1)))_w & \text{if } w_1, w_2, w_3, w_4 \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

This determines a \((\mathbb{C}^*)^4\)-invariant submodule of \( M/((1,1,1,1)) \). By varying \( V_0 \) over all possible 1-dimensional subspaces, we obtain a connected component of \( \bigcup_\alpha P_n(X, \beta)(\mathbb{C}^*)^4 \) with \( \mathbb{P}^2 \) as its underlying reduced variety.

**Case 2:** \( C_F|_{U_\alpha} \) has three non-empty asymptotic plane partitions, say in directions 1,2,3. The same strategy as above works, this time choosing a 1-dimensional subspace

\[
V_0 \subseteq \mathbb{C} \cdot 1_w \oplus \mathbb{C} \cdot 2_w \oplus \mathbb{C} \cdot 3_w / \mathbb{C} \cdot (1_w + 2_w + 3_w)
\]

and setting \( V_w = (M/((1,1,1,1)))_w \) for all \( w \in (\Pi \cup \Pi \cup \Pi \cup \Pi \) \( \setminus \{0\} \) with \( w_1, w_2, w_3, w_4 \geq 0 \), and \( V_0 = 0 \) otherwise. This gives a connected component with underlying reduced variety \( \mathbb{P}^1 \).

**Proposition 2.6.** Suppose \( P_n(X, \beta)(\mathbb{C}^*)^4 \) is 0-dimensional for all \( n \in \mathbb{Z} \). Then, at the level of schemes, \( P_n(X, \beta)(\mathbb{C}^*)^4 \) consists of finitely many reduced points for all \( n \in \mathbb{Z} \).

**Proof.** Suppose \([I^* = \mathcal{O}_X \to F] \in P_n(X, \beta)(\mathbb{C}^*)^4 \). As before, we denote the open cover by maximal \((\mathbb{C}^*)^4\)-invariant affine open subsets by \( \{U_\alpha\}_{\alpha \in V(X)} \). Let \( I^*_\alpha := I^*|_{U_\alpha}, F_\alpha = F|_{U_\alpha}, \) and \( Q_\alpha = Q|_{U_\alpha} \) (where \( Q \) is the cokernel of the stable pair). The proof consists of three parts, which closely follows [22] Sect. 3.1, 3.2, but now one dimension higher.

**Step 1:** The restriction map

\[
\text{Hom}_X(I^*, F) \to \bigoplus_{\alpha \in V(X)} \text{Hom}_{U_\alpha}(I^*_\alpha, F_\alpha)
\]

is an isomorphism after taking \((\mathbb{C}^*)^4\)-fixed parts. This follows from sequence [22] (3.2) and the fact that \( \text{Hom}_{U_\alpha}(I_{C_\alpha}, F_\alpha)(\mathbb{C}^*)^4 = 0 \). This vanishing is obvious because there are no \((\mathbb{C}^*)^4\)-weights \( w \) such that the corresponding weight spaces \( (I_{C_\alpha})_w \) and \( (F_\alpha)_w \) are both non-zero.

\[^3\text{In J. Bryan’s words: gravity pulls in the (1,1,1,1)-direction.}\]
The restriction map \(26\) is also an isomorphism after taking \(T\)-fixed parts. This requires the less obvious fact that \(\text{Hom}_{U_n}(I_{C_n}, F_n)^T = 0\). This vanishing follows from the analog of \([22\) Lem. 2\)], the proof of which immediately adapts to our setting (and is very similar to the manipulation with Haiman arrows detailed in \([9\) Lem. 3.4\]).

**Step 2:** There are isomorphisms

\[
\text{Hom}_{U_n}(I^*_\alpha, F_n)^T \cong \text{Ext}^1_{U_n}(Q_\alpha, F_n)^{T} \cong \text{Hom}_{U_n}(Q_\alpha, M/F_n)^{T}
\]

by \([22\) (3.2) & Lem. 1\]. Since we have at most two legs coming together (Corollary \(23\)), there are no \((\mathbb{C}^*)^4\)-weights \(w\) such that the weight spaces \((Q_\alpha)_w\) and \((M/F_n)_w\) are both non-zero. Therefore

\[
\text{Hom}_{U_n}(Q_\alpha, M/F_n)^{\mathbb{C}^*^4} = 0.
\]

Note that we do not require the analogue of \([22\) Prop. 4\], which deals with the more complicated case of labelled boxes. Hence \(\text{Hom}_X(I^*, F)^{\mathbb{C}^*^4} = 0\) and \(P_n(X, \beta)^{\mathbb{C}^*^4}\) consists of finitely many reduced points.

**Step 3:** For the Calabi-Yau torus \(T \subseteq (\mathbb{C}^*)^4\), we also have

\[
\text{Hom}_{U_n}(I^*_\alpha, F_n)^T \cong \text{Ext}^1_{U_n}(Q_\alpha, F_n)^{T} \cong \text{Hom}_{U_n}(Q_\alpha, M/F_n)^{T}
\]

by \([22\) Lem. 3\]. Decompose \(\text{Hom}_{U_n}(Q_\alpha, M/F_n)^T\) with respect to the 1-dimensional torus \((\mathbb{C}^*)^4/T \cong \mathbb{C}^*\). Suppose there exists a non-zero homogeneous morphism \(\phi \in \text{Hom}_{U_n}(Q_\alpha, M/F_n)^T\) of weight \(n \in \mathbb{Z}\). Since we have at most two legs coming together (Corollary \(23\)), we must have \(n < 0\). Since \(\phi\) is assumed non-zero, it sends some monomial \(x_1^{w_1}x_2^{w_2}x_3^{w_3}x_4^{w_4}\) of a box of \(Q_\alpha\) to \(\lambda x_1^{w_1+n}x_2^{w_2+n}x_3^{w_3+n}x_4^{w_4+n} \in M/F_n\) for \(w\) in some leg, say Leg\(_1\), and some \(\lambda \neq 0\). Consider such a \(x_1^{w_1}x_2^{w_2}x_3^{w_3}x_4^{w_4}\) with minimal \(w_1\) and maximal \(w_2\). Case 1: \(Q_\alpha\) supports a box at \((w_1 + n, w_2 + n + 1, w_3 + n, w_4 + n)\). This is not possible by minimality of \(w_1\) (recall \(n < 0\)).

**Case 2:** \(Q_\alpha\) does not support a box at \((w_1 + 1, w_2 + n + 1, w_3 + n, w_4)\). Then

\[
\lambda x_1^{w_1+n}x_2^{w_2+n+1}x_3^{w_3+n}x_4^{w_4+n} = x_2 \cdot \phi(x_1^{w_1}x_2^{w_2}x_3^{w_3}x_4^{w_4}) = \phi(x_1^{w_1}x_2^{w_2+1}x_3^{w_3}x_4^{w_4}) = \phi(0) = 0,
\]

where \(x_1^{w_1}x_2^{w_2+1}x_3^{w_3}x_4^{w_4} = 0 \in Q_\alpha\) by maximality of \(w_2\). This is a contradiction because \((w_1, w_2 + 1, w_3, w_4)\) does not support a box of \(Q_\alpha\), so \(0 \neq x_1^{w_1+n}x_2^{w_2+1+n}x_3^{w_3+n}x_4^{w_4+n} \in M/F_n\). \(\square\)

### 2.3. Equivariant invariants.

Following \([11\) 9 14\], we study curve counting and stable pair invariants of toric Calabi-Yau 4-folds \(X\). These are defined by \(T\)-localization, because \(X\) is not proper and therefore \(I_n(X, \beta)\) is in general not proper either.

Let \(\bullet\) denote \(\text{Spec} \mathbb{C}\) with trivial \((\mathbb{C}^*)^4\)-action. We denote by \(\mathbb{C} \otimes t_i\) the 1-dimensional \((\mathbb{C}^*)^4\)-representation with weight \(t_i\) and we write \(\lambda_i \in H^*(\mathbb{C}^*^4, \bullet)\) for its \((\mathbb{C}^*)^4\)-equivariant first Chern class. Then

\[
H^*_T(\bullet) = \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \lambda_4],
\]

Recall that \(T \subseteq (\mathbb{C}^*)^4\) denotes the Calabi-Yau torus, which is defined as the subtorus preserving the Calabi-Yau volume form. Note that the Serre duality pairing \(\text{Ext}^i(E, E) \cong \text{Ext}^{4-i}(E, E)^*\) is a \(T\)-equivariant isomorphism for any \(T\)-equivariant coherent sheaf on \(X\).

For each of the finitely many fixed points \(Z \in I_n(X, \beta)^T = I_n(X, \beta)^{(\mathbb{C}^*)^4}\) (Lemma \(22\)), one can form a complex vector bundle

\[
ET \times_T \text{Ext}^i(I_Z, I_Z)
\]

whose Euler class is the \(T\)-equivariant Euler class \(e_T(\text{Ext}^i(I_Z, I_Z))\).

When \(i = 2\), the Serre duality pairing on \(\text{Ext}^2(I_Z, I_Z)\) induces a non-degenerate quadratic form \(Q\) on \(ET \times_T \text{Ext}^2(I_Z, I_Z)\), because \(T\) preserves the Calabi-Yau volume form. We define

\[
e_T(\text{Ext}^2(I_Z, I_Z), Q) \in \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3]
\]
as the half Euler class\(^3\) of \((ET \times_T \text{Ext}^2(I_Z, I_Z), Q)\), which satisfies
\[
(2.8) \quad (-1)^{\frac{i}{2}} \text{ext}^2(I_Z, I_Z) e_T \left( \text{Ext}^2(I_Z, I_Z) \right) = \left( e_T \left( \text{Ext}^2(I_Z, I_Z), Q \right) \right)^2.
\]

The half Euler class \(\text{(2.7)}\) depends on a choice of orientation on a positive real form, or—equivalently—a choice of square root of \(\text{(2.8)}\).

Similar to [9] Sect. 3], we define equivariant curve counting invariants as follows.

**Definition 2.7.** Let \(X\) be a toric Calabi-Yau 4-fold and \(\beta \in H_2(X)\). For \(\gamma_1, \ldots, \gamma_m \in H_2^+(X, \mathbb{Q})\), we define
\[
I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m)_{o(\mathcal{L})} := \sum_{Z \in I_{n, \beta}(X, \beta)^T} (-1)^{o(\mathcal{L})}|Z| \sqrt{(-1)^{\frac{i}{2}} \text{ext}^2(I_Z, I_Z) e_T (\text{Ext}^2(I_Z, I_Z))} \prod_{i=1}^m \tau(\gamma_i)|Z|
\]
\[
eq \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \cong \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),
\]
where \(o(\mathcal{L})|Z\) denotes a choice of sign for each \(Z \in I_{n, \beta}^T\) and \(\tau\) is the insertion \(\text{(1.2)}\). We often drop \(o(\mathcal{L})\) from the notation. When there is no insertion, we write \(I_{n, \beta}(X; T)\) for the invariant. We also form the generating series
\[
I_{\beta}(X; T)(\gamma_1, \ldots, \gamma_m) := \sum_{n \in \mathbb{Z}} I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) q^n \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)(q).
\]

**Remark 2.8.** The number of choices of orientation in the previous definition is \(2^{\# I_{n, \beta}(X)^T}\).

There is a parallel story for stable pairs. For \([I^*] = \{O_X \to F\} \in P_n(X, \beta)^{(C^*)^4}\), we can define the equivariant Euler classes \(e_T(\text{Ext}^4([I^*], [I^*]))\) and a square root
\[
\sqrt{(-1)^{\frac{i}{2}} \text{ext}^2([I^*], [I^*]) e_T (\text{Ext}^2([I^*], [I^*]))} \in \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3].
\]

We define equivariant stable pair invariants when \(P_n(X, \beta)^{(C^*)^4}\) is 0-dimensional for all \(n \in \mathbb{Z}\). Then \(P_n(X, \beta)^T = P_n(X, \beta)^{(C^*)^4}\) consists of finitely many reduced points (Proposition 2.6). Recall that this happens in several interesting examples, e.g. when \(X\) is a local toric curve or surface.

**Definition 2.9.** Let \(X\) be a toric Calabi-Yau 4-fold and let \(\beta \in H_2(X)\). Suppose \(P_n(X, \beta)^{(C^*)^4}\) is 0-dimensional for all \(n \in \mathbb{Z}\). For \(\gamma_1, \ldots, \gamma_m \in H_2^+(X)\), we define
\[
P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m)_{o(\mathcal{L})} := \sum_{[I^*] \in P_n(X, \beta)^T} (-1)^{o(\mathcal{L})}|[I^*]| \sqrt{(-1)^{\frac{i}{2}} \text{ext}^2([I^*], [I^*]) e_T (\text{Ext}^2([I^*], [I^*]))} \prod_{i=1}^m \tau(\gamma_i)|[I^*]|
\]
\[
eq \frac{\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \cong \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),
\]
where \(o(\mathcal{L})|[I^*]\) denotes a choice of sign for each \([I^*] \in P_n(X, \beta)^T\) and \(\tau\) is the insertion \(\text{(1.2)}\). We often drop \(o(\mathcal{L})\) from the notation. When there is no insertion, we write \(P_{n, \beta}(X; T)\) for the invariant. We also form generating series
\[
P_{\beta}(X; T)(\gamma_1, \ldots, \gamma_m) := \sum_{n \in \mathbb{Z}} P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) q^n \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)(q).
\]

**Remark 2.10.** Unlike the compact case, where \(n\) is the virtual dimension, it is not true in general that \(I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m)\) and \(P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m)\) are zero when \(n < 0\). E.g. for \(X = \text{Tot}_{\mathbb{P}^2 \times \mathbb{P}^1}(O(-1, -1) \oplus O(-1, -1))\), \(\beta = (3, 2), n = -1\), we have \(P_{n, \beta}(X; T) = 0\) only for \(2^6\) out of \(2^{12}\) possible choices of signs.

2.4. Vertex formalism. We now develop a vertex formalism, which reduces the calculation of the invariants of the previous section to \(\mathbb{C}^4\). In the case of equivariant curve counting invariants, we closely follow the 3-fold case developed by Maulik-Nekrasov-Okounkov-Pandharipande [13]. For \(\beta = 0\), this was carried out by the authors in [9]. In the case of equivariant stable pair invariants, we follow the 3-folds case developed by Pandharipande-Thomas [22].

Let \(X\) be a toric Calabi-Yau 4-fold and consider the cover \(\{U_\alpha\}_{\alpha \in V(X)}\) by maximal \((C^*)^4\)-invariant affine open subsets. Let \(E = I_Z\), with \(Z \in I_n(X, \beta)^{(C^*)^4}\) or \(E = [I^*] \in P_n(X, \beta)^{(C^*)^4}\). For each \(\alpha \in V(X)\), the scheme \(Z|_{U_\alpha}\) corresponds to a solid partition and the cokernel of

\[^3\text{i.e. the Euler class of its positive real form, which is a half rank real subbundle on which } Q \text{ is real and positive definite.}\]
$I^i|_{U_a} = \{ \mathcal{O}_{U_a} \to F^i|_{U_a} \}$ corresponds to a box configuration as described in Sections 2.4 and 2.2. When $E = I^i$, we assume that, for all $\alpha \in V(X)$, the Cohen-Macaulay support curve $C_{\alpha}|_{U_a}$ is described by a solid partition with at most two non-empty asymptotic plane partitions. We want to calculate

$$-\text{RHom}(E, E)_0 \in K_T(\bullet).$$

In fact, we calculate the class of this complex in $K_{(\mathbb{C}^*)^4}(\bullet)$. We will use the exact triangle

$$E \to \mathcal{O}_X \to E',$$

where $E' = O_Z$ when $E = I_Z$, and $E' = F$ when $E = I^i$. In either case, $E'$ is 1-dimensional. Let $\Gamma(-)$ denote the global section functor. Let $U_{\alpha\beta} = U_{\alpha} \cap U_{\beta}, U_{\alpha\beta\gamma} = U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$, etc., and let $E_{\alpha} := E|_{U_{\alpha}}, E_{\alpha\beta} := E|_{U_{\alpha\beta}}$ etc. The local-to-global spectral sequence and calculation of sheaf cohomology with respect to the Čech cover $\{U_{\alpha}\}_{\alpha \in V(X)}$ gives

$$-\text{RHom}(E, E)_0 = \sum_{\alpha \in V(X), i} (-1)^i \left( \Gamma(U_{\alpha}, \mathcal{O}_{U_{\alpha}}) \to \Gamma(U_{\alpha}, \mathcal{O}_{U_{\alpha}}) \right),$$

where $E|_{U_{\alpha\beta\gamma\cdots}} = \mathcal{O}_{U_{\alpha\beta\gamma\cdots}}$,

which can be seen from (2.6) and the fact that $E'$ is 1-dimensional. Hence we are reduced to determining

$$-\text{RHom}_{U_{\alpha}}(E_{\alpha}, E_{\alpha})_0 = \sum_{i} (-1)^i \left( \Gamma(U_{\alpha}, \mathcal{O}_{U_{\alpha}}) \to \Gamma(U_{\alpha}, \mathcal{O}_{U_{\alpha}}) \right),$$

$$-\text{RHom}_{U_{\alpha\beta}}(E_{\alpha\beta}, E_{\alpha\beta})_0 = \sum_{i} (-1)^i \left( \Gamma(U_{\alpha\beta}, \mathcal{O}_{U_{\alpha\beta}}) \to \Gamma(U_{\alpha\beta}, \mathcal{O}_{U_{\alpha\beta}}) \right).$$

**Vertex term.** On $U_a \cong \mathbb{C}^4$, we use coordinates $x_1, x_2, x_3, x_4$ such that the $(\mathbb{C}^*)^4$-action is

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4 \text{ and } t \in (\mathbb{C}^*)^4.$$

Let $R := \Gamma(\mathcal{O}_{U_a}) \cong \mathbb{C}[x_1, x_2, x_3, x_4].$ Consider the class $[E_{\alpha}]$ in the equivariant $K$-group $K_{(\mathbb{C}^*)^4}(U_a)$. There exists a ring isomorphism

$$K_{(\mathbb{C}^*)^4}(U_{\alpha}) \cong \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm, t_4^\pm],$$

where $[R]$ corresponds to 1. The Laurent polynomial $\mathcal{P}(E_{\alpha})$ corresponding to $[E_{\alpha}]$ via this isomorphism is called the Poincaré polynomial of $E_{\alpha}$. For any $w = (w_1, w_2, w_3, w_4) \in X((\mathbb{C}^*)^4) = \mathbb{Z}^4$, we use multi-index notation

$$t^w := t_1^{w_1} t_2^{w_2} t_3^{w_3} t_4^{w_4}.$$

Then $[R \otimes t^w] \in K_{(\mathbb{C}^*)^4}(U_{\alpha})$ corresponds to $t^w \in \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm, t_4^\pm]$.

Define an involution $(\cdot)^\ast$ on $K_{(\mathbb{C}^*)^4}(U_{\alpha})$ by $Z^\ast := (Z^\ast)^\ast$.

The trace map

$$\text{tr} : K_{(\mathbb{C}^*)^4}(U_{\alpha}) \to \mathbb{Z}((t_1, t_2, t_3, t_4))$$

is the map induced from sending a $(\mathbb{C}^*)^4$-equivariant $R$-module to its underlying $(\mathbb{C}^*)^4$-representation. As in [18], we take a $(\mathbb{C}^*)^4$-equivariant free resolution

$$0 \to F_0 \to \cdots \to F_1 \to E_{\alpha} \to 0,$$

where

$$F_i = \bigoplus_j R \otimes t^{d_{ij}},$$

for certain $d_{ij} \in \mathbb{Z}^4$. Then

$$\mathcal{P}(E_{\alpha}) = \sum_{i,j} (-1)^j d_{ij}, \quad \text{(2.10)}$$

Denote the $(\mathbb{C}^*)^4$-character of $E_{\alpha}|_{U_a}$ by $Z_{\alpha} := \text{tr} E_{\alpha}|_{U_a}$.
When $E' = O_Z$, the scheme $Z|_{U_\alpha}$ corresponds to a solid partition $\pi^{(\alpha)}$ as described in Section 2.1. Then

$$Z_\alpha = \sum_{i,j,k \geq 1} \sum_{l=1}^{\delta(j)} t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1}. \quad (2.11)$$

When $E' = F$, we use the short exact sequence

$$0 \to O_C \to F \to Q \to 0,$$

where $C$ is the Cohen-Macaulay support curve and $Q$ is the cokernel. Then

$$Z_\alpha = \text{tr}_{O_C|_{U_\alpha}} + \text{tr}_{Q|_{U_\alpha}}, \quad (2.12)$$

where $O_C|_{U_\alpha}$ is described by a solid partition $\pi^{(\alpha)}$ and $Q|_{U_\alpha}$ is described by a box configuration $B^{(\alpha)}$ as in Section 2.2. Then $\text{tr}_{O_C|_{U_\alpha}}$ is given by the RHS of (2.11). Moreover, $\text{tr}_{Q|_{U_\alpha}}$ is the sum of $t^w$ over all $w \in B^{(\alpha)}$.

In both cases, $Z_\alpha$ can be expressed in terms of the Poincaré polynomial of $E_\alpha$ as follows

$$Z_\alpha = \text{tr}_{O_{U_\alpha}} - E_\alpha = \frac{1 - P(E_\alpha)}{(1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)}. \quad (2.13)$$

We obtain

$$\text{RHom}(E_\alpha, E_\alpha) = \sum_{i,j,k,l} (-1)^{i+k} \text{Hom}(R \otimes t^{d_{ij}}, R \otimes t^{d_{kl}})$$

$$= \sum_{i,j,k,l} (-1)^{i+k} R \otimes t^{d_{kl} - d_{ij}}$$

$$= P(E_\alpha)P(E_\alpha),$$

where we used (2.10) for the third equality. Eliminating $P(E_\alpha)$ by using (2.13), we conclude

$$\text{tr}_{-\text{RHom}(E_\alpha, E_\alpha)_0} = Z_\alpha + \frac{Z_\alpha}{t_1 t_2 t_3 t_4} - \frac{Z_\alpha Z_\alpha (1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4)}{t_1 t_2 t_3 t_4}. \quad (2.14)$$

**Edge term.** Let $\alpha \beta \in E(X)$. On $U_{\alpha \beta} \cong \mathbb{C}^* \times \mathbb{C}^3$, we use coordinates $x_1, x_2, x_3, x_4$ such that the $(\mathbb{C}^*)^4$-action is given by

$$t \cdot x_i = t_i x_i, \quad \text{for all } i = 1, 2, 3, 4 \text{ and } t \in (\mathbb{C}^*)^4.$$ 

Let $R := \Gamma(O_{U_{\alpha \beta}}) \cong \mathbb{C}[x_1, x_1^{-1}] \otimes \mathbb{C}[x_2, x_3, x_4]$. As in [18], we define

$$\delta(t_1) := \sum_{k \in \mathbb{Z}} t_1^k.$$

Similar to the vertex calculation, we set

$$Z_{\alpha \beta} := \text{tr}_{E'|_{U_{\alpha \beta}}}.$$

In both cases $E = I_Z$ and $E = I^*$, we have an underlying Cohen-Macaulay support curve $C$ and $C|_{U_\alpha}$, $C|_{U_\beta}$ are described by solid partitions $\pi^{(\alpha)}$, $\pi^{(\beta)}$ respectively. Suppose in both charts $U_\alpha \cong \mathbb{C}^4$ and $U_\beta \cong \mathbb{C}^4$, the line $C_{\alpha \beta} \cong \mathbb{P}^1$ is given by $\{x_2 = x_3 = x_4 = 0\}$. Then $\pi^{(\alpha)}$, $\pi^{(\beta)}$ give rise to the same asymptotic plane partition $\lambda^{(\alpha \beta)}$ along the $x_1$-direction, by gluing condition (2.3), and

$$Z_{\alpha \beta} = \sum_{j,k \geq 1} \sum_{l=1}^{\delta(j)} t_2^{-1} t_3^{-1} t_4^{-1}. \quad (2.15)$$

A computation similar to the vertex case yields

$$- \text{tr}_{-\text{RHom}(E_\alpha, E_\alpha)_0} = \delta(t_1) \left(-Z_{\alpha \beta} + \frac{Z_{\alpha \beta}}{t_2 t_3 t_4} - \frac{Z_{\alpha \beta} Z_{\alpha \beta} (1 - t_2)(1 - t_3)(1 - t_4)}{t_2 t_3 t_4}\right). \quad (2.16)$$

**Redistribution.** Expressions (2.14) and (2.16) are formal Laurent series in $t_1, t_2, t_3, t_4$. However, MNOP [18] give a method to redefine these terms in such a way that we get Laurent polynomials in $t_1, t_2, t_3, t_4$. This process is known as redistribution. Define

$$F_{\alpha \beta} := -Z_{\alpha \beta} + \frac{Z_{\alpha \beta}}{t_2 t_3 t_4} - \frac{Z_{\alpha \beta} Z_{\alpha \beta} (1 - t_2)(1 - t_3)(1 - t_4)}{t_2 t_3 t_4}. \quad (2.17)$$
For any \( \alpha \in V(X) \), denote by \( C_{\alpha \beta_1}, C_{\alpha \beta_2}, C_{\alpha \beta_3}, C_{\alpha \beta_4} \) the four \((\mathbb{C}^*)^2\)-invariant lines passing through the \((\mathbb{C}^*)^4\)-fixed point \( p_\alpha \). Then we define

\[
V_\alpha := \text{tr}-\text{RHom}(E_0, E_\alpha) + \sum_{i=1}^4 \frac{F_{\alpha \beta}(t_i, t_{i'}, t_{i''})}{1-t_i},
\]

where \( \{t_i, t_{i'}, t_{i''}\} = \{t_1, t_2, t_3, t_4\} \). For any \( \alpha \in E(X) \), use coordinates as in the treatment of the edge term above. Then we define

\[
E_{\alpha \beta} := t_1^{-1} \frac{F_{\alpha \beta}(t_2, t_3, t_4)}{1-t_1^{-1}} - \frac{F_{\alpha \beta}(t_2 t_1^{-m_{\alpha \beta}}, t_3 t_4^{-m_{\alpha \beta}}, t_4 t_1^{-m_{\alpha \beta}})}{1-t_1^{-1}},
\]

where \( (t_1, t_2, t_3, t_4) \mapsto (t_1^{-1}, t_2 t_1^{-m_{\alpha \beta}}, t_3 t_4^{-m_{\alpha \beta}}, t_4 t_1^{-m_{\alpha \beta}}) \) corresponds to the coordinate transformation \( U_\alpha \to U_\beta \) and \( m_{\alpha \beta}, m'_{\alpha \beta}, m''_{\alpha \beta} \) are the weights of the normal bundle of \( C_{\alpha \beta} \) defined in \((2.18)\). The result of this redistribution is

\[
V_\alpha \in \mathbb{Z}[t_{1}^{\pm}, t_{2}^{\pm}, t_{3}^{\pm}, t_{4}^{\pm}], \quad E_{\alpha \beta} \in \mathbb{Z}[t_{1}^{\pm}, t_{3}^{\pm}, t_{4}^{\pm}],
\]

which is proved precisely as in \([18]\) Lem. 9. When \( E = I_Z \), we write \( V_\alpha^{\text{DT}} \) for \( V_\alpha \) and \( E_{\alpha \beta}^{\text{DT}} \) for \( E_{\alpha \beta} \). When \( E = \text{I}^* \), we write \( V_\alpha^{\text{PT}} \) for \( V_\alpha \) and \( E_{\alpha \beta}^{\text{PT}} \) for \( E_{\alpha \beta} \). Note that \( E_{\alpha \beta}^{\text{DT}} = E_{\alpha \beta}^{\text{PT}} \). We summarize:

**Proposition 2.11.** Let \( X \) be a toric Calabi-Yau 4-fold and \( \beta \in H_2(X) \). Let \( E = I_Z \), with \( Z \in I_\alpha(X, \beta)^{(\mathbb{C}^*)^4} \) or \( E = [\text{I}^*] \in P_\alpha(X, \beta)^{(\mathbb{C}^*)^4} \). In the second case, we assume the Cohen-Macaulay support curve \( C \) has the following property: for all \( \alpha \in V(X) \), the solid partition associated to \( C|_{U_\alpha} \) has at most two non-empty asymptotic plane partitions. Then

\[
\text{tr}-\text{RHom}(E, E_\alpha) = \sum_{\alpha \in V(X)} V_\alpha + \sum_{\alpha \in E(X)} E_{\alpha \beta},
\]

where \( V_\alpha \) and \( E_{\alpha \beta} \) are Laurent polynomials for all \( \alpha \in V(X) \) and \( \alpha \beta \in E(X) \).

2.5. **Equivariant DT/PT correspondence.** Consider plane partitions \( \lambda, \mu, \nu, \rho \) of finite size at most two of which are non-empty. This data determines a \((\mathbb{C}^*)^4\)-fixed Cohen-Macaulay curve \( C \subseteq \mathbb{C}^4 \) with solid partition defined by \((2.11)\). We denote this solid partition by \( \pi_{\text{CM}}(\lambda, \mu, \nu, \rho) \). We are interested in:

- All \((\mathbb{C}^*)^4\)-invariant closed subschemes \( Z \subseteq \mathbb{C}^4 \) with underlying maximal Cohen-Macaulay subcurve \( C \). These correspond to solid partitions \( \pi \) with asymptotic plane partitions \( \lambda, \mu, \nu, \rho \) in directions 1, 2, 3, 4 (Section 2.1). We denote the collection of such solid partitions by \( \Pi^\text{DT}(\lambda, \mu, \nu, \rho) \). To any solid partition \( \pi \in \Pi^\text{DT}(\lambda, \mu, \nu, \rho) \), we associate a character \( Z_\pi \) defined by RHS of \((2.11)\). This in turn determines a Laurent polynomial

\[
V_\pi^{\text{DT}} \in \mathbb{Z}[t_{1}^{\pm}, t_{2}^{\pm}, t_{3}^{\pm}, t_{4}^{\pm}]
\]

by RHS of \((2.18)\) (via \((2.17), (2.14)\)).

- All \((\mathbb{C}^*)^4\)-invariant stable pairs \((F, s)\) on \( \mathbb{C}^4 \) with underlying Cohen-Macaulay support curve \( C \). These correspond to box configurations as described in Section 2.2. Denote the collection of such box configurations by \( \Pi^\text{PT}(\lambda, \mu, \nu, \rho) \). To any box configuration \( B \in \Pi^\text{PT}(\lambda, \mu, \nu, \rho) \), we associate a character \( Z_B \) defined by the RHS of \((2.12)\), where the Cohen-Macaulay part is given by \((2.11)\), for solid partition \( \pi_{\text{CM}}(\lambda, \mu, \nu, \rho) \), and the cokernel part is the sum of \( t^w \) over all \( w \in B \). This determines a Laurent polynomial

\[
V_\pi^{\text{PT}} \in \mathbb{Z}[t_{1}^{\pm}, t_{2}^{\pm}, t_{3}^{\pm}, t_{4}^{\pm}]
\]

by RHS of \((2.18)\) (via \((2.17), (2.14)\)).

From the definitions, one readily calculates (in both the DT/PT case)

\[
\nabla_\pi = V_\pi \cdot t_1 t_2 t_3 t_4.
\]

Therefore, after restricting to \( \mathbb{Z}[t_1, t_2, t_3, t_4]/(t_1 t_2 t_3 t_4) \), his expression has a square root. This allows us to make the following definition:

**Definition 2.12.** Consider plane partitions \( \lambda, \mu, \nu, \rho \) of finite size at most two of which are non-empty. Define the equivariant 4-fold DT vertex by

\[
V_{\lambda \mu \nu \rho}^{\text{DT}}(q, \omega) := \sum_{\pi \in \Pi^\text{DT}(\lambda, \mu, \nu, \rho)} (-1)^{o(\pi)} \cdot \sqrt{-e_T(\omega) q^{\frac{1}{2}}},
\]

where \( o(\pi) \) gives a choice of sign for each \( \pi \). \( e_T(\cdot) \) is the equivariant Euler class, and \( \omega \) denotes renormalized volume. Moreover, \((-1)^{o(\pi)}\) denotes the unique sign for which \((-1)^{o(\pi)} e_T(-V_\pi^{\text{DT}})\) is a square in \( \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3) \).
Define the equivariant 4-fold PT vertex by
\[
\mathcal{V}^\text{PT}_{\lambda\mu\nu}(q)_{\partial(L)} := \sum_{B \in \Pi^\text{PT}_{\lambda'\mu'\nu'}} (-1)^{o(B)} n \sqrt{(-1)^{o_B} e_T(-\mathcal{V}^\text{PT}_B)} q^{B} [\pi_{\text{CM}}(\lambda, \mu, \nu, \rho)] \in Q(\lambda_1, \lambda_2, \lambda_3)((q)),
\]
where \(o(L)|_B = 0\) gives a choice of sign for each \(B\), \(|B|\) denotes the total number of boxes in the box configuration, and \(|\pi_{\text{CM}}(\lambda, \mu, \nu, \rho)|\) denotes renormalized volume. Moreover, \((-1)^{o_B}\) denotes the unique sign for which \((-1)^{o_B} e_T(-\mathcal{V}^\text{PT}_B)\) is a square in \(Q(\lambda_1, \lambda_2, \lambda_3)\). We often omit \(o(L)\) from the notation. We also define “normalizations”
\[
\tilde{\mathcal{V}}^\text{DT}_{\lambda\mu\nu}(q) := q^{-|\pi_{\text{CM}}(\lambda, \mu, \nu, \rho)|} \mathcal{V}^\text{DT}_{\lambda\mu\nu}(q) \in Q(\lambda_1, \lambda_2, \lambda_3)[q],
\]
\[
\tilde{\mathcal{V}}^\text{PT}_{\lambda\mu\nu}(q) := q^{-|\pi_{\text{CM}}(\lambda, \mu, \nu, \rho)|} \mathcal{V}^\text{PT}_{\lambda\mu\nu}(q) \in Q(\lambda_1, \lambda_2, \lambda_3)[q].
\]

Analogously, we associate to the plane partition \(\lambda\) the character \(Z_\lambda\) defined by the RHS of Corollary 2.15. We define the edge term
\[
E^\text{DT}_{\lambda, o(L)} = E^\text{PT}_{\lambda, o(L)} \in Q(\lambda_1, \lambda_2, \lambda_3)
\]
by taking the square root of the (signed) Euler class of minus the RHS of Corollary 2.14 (via Corollary 2.17). As before, this definition depends on a sign \((-1)^{o(L)}\lambda\), where \(o(L)|_\lambda = 0, 1\). We usually omit this dependence from the notation.

The vertex formalism reduces the calculation of equivariant curve counting and stable pair invariants of any toric Calabi-Yau 4-fold to a combinatorial expression involving \(\mathcal{V}_{\lambda\mu\nu}\) and \(E_\lambda\). Rather than writing this down in general, we discuss an illustrative example. Suppose \(X\) is the total space of \(\mathcal{O}_\mathbb{P}^2(-1) \oplus \mathcal{O}_\mathbb{P}^2(-2)\). Let \(\beta = [P^1]\). Then Lemma 2.4 and the vertex formalism (Section 2.3) imply
\[
I_\beta(X; T) = \sum_{|\lambda| + |\mu| + |\nu| = d} q^{f_{1-1-2}(\lambda) + f_{1-1-2}(\mu) + f_{1-1-2}(\nu)} E^\text{DT}_{\lambda, \mu, \nu} \mathcal{V}^\text{DT}_{\lambda\mu\nu} \mathcal{V}^\text{DT}_{\mu\nu} \mathcal{V}^\text{DT}_{\nu}|(\lambda, \mu, \nu), (\lambda, \mu, \nu), (\lambda, \mu, \nu)
\]
where the sum is over all point-like plane partitions \(\lambda, \mu, \nu\) satisfying \(|\lambda| + |\mu| + |\nu| = d\). Here the choice of signs for \(I_\beta(X; T)\) is determined by the choice of signs in each vertex and edge term. The same expression holds for \(I_\beta(X; T)\) replacing DT by PT.

We conjecture the following DT/PT vertex correspondence:

**Conjecture 2.13.** For any plane partitions \(\lambda, \mu, \nu, \rho\), at most two of which are non-empty, there are choices of signs such that
\[
\mathcal{V}^\text{DT}_{\lambda\mu\nu}(q) = \mathcal{V}^\text{PT}_{\lambda\mu\nu}(q) \mathcal{V}^\text{DT}_{\mu\nu}(q).
\]
Moreover, for each \(n\), the choice of signs for which the coefficients of \(q^n\) of LHS and RHS agree is unique up to an overall sign.

By using an implementation into Maple, we verified the following cases:

**Proposition 2.14.** There are choices of signs such that
\[
\mathcal{V}^\text{DT}_{\lambda\mu\nu}(q) = \mathcal{V}^\text{PT}_{\lambda\mu\nu}(q) \mathcal{V}^\text{DT}_{\mu\nu}(q) \mod q^N
\]
in the following cases:

- for any \(|\lambda| + |\mu| + |\nu| + |\rho| \leq 1\) and \(N = 4\),
- for any \(|\lambda| + |\mu| + |\nu| + |\rho| \leq 2\) and \(N = 4\),
- for any \(|\lambda| + |\mu| + |\nu| + |\rho| \leq 3\) and \(N = 3\),
- for any \(|\lambda| + |\mu| + |\nu| + |\rho| \leq 4\) and \(N = 3\).

In each of these cases, the choice of signs for which the coefficients of \(q^n\) of LHS and RHS agree is unique up to an overall sign.

When \(\beta = 0\), the space \(I_n(X, \beta)\) reduces to the Hilbert scheme \(\text{Hilb}^n(X)\) of \(n\) points previously studied in[0][19]. When \(X = \mathbb{P}^4\), we have \(I_0(X; T) = \mathcal{V}^\text{DT}_{\mu\nu}(q)\). A closed expression for this generating series was conjectured by Nekrasov [19]:

**Conjecture 2.15.** (Nekrasov [19]) There exist unique choices of signs such that the following formula holds
\[
\mathcal{V}^\text{DT}_{\mu\nu}(q) = \exp \left( q \frac{\prod_{i=1}^3 (\lambda_i + \lambda_2)(\lambda_i + \lambda_3)}{\prod_{i=1}^3 (\lambda_1 + \lambda_2 + \lambda_3)} \right).
\]
Remark 2.16. The existence part of the conjecture was verified modulo $q^7$ in [19] and later modulo $q^{17}$ in [20]. The uniqueness was conjectured in [9, Appendix B] and checked modulo $q^2$. We also verified that in all cases of Proposition 2.14 one of the two choices of signs is compatible with the choice of signs in Nekrasov’s conjecture. Note that

\[
\frac{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}{\lambda_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3)} = - \int_{\mathbb{C}^4} c_T^3 (\mathbb{C}^4),
\]

where \( c_T^3 \) is equivariant push-forward to a point and \( c_T^3 (\mathbb{C}^4) \) denotes the equivariant third Chern class of the tangent bundle \( T_{\mathbb{C}^4} \).

Let \( X \) be any toric Calabi-Yau 4-fold. The vertex formalism (for \( \beta = 0 \)), Atiyah-Bott localization on \( X \), and Nekrasov’s conjecture at once imply the following result about \( \text{Hilb}^3 (X) \):

**Proposition 2.17.** Assuming Conjecture 2.10 holds, then for any toric Calabi-Yau 4-fold \( X \), there exist choices of signs such that

\[
I_0(X; T) = \exp \left( - q \int_X c_T^3 (X) \right).
\]

2.6. Primary insertions. Let \( X \) be a toric Calabi-Yau 4-fold and \( \gamma \in H^*_T (X, \mathbb{Q}) \). Consider \( [E] \in M^{(\mathbb{C}^*)^4} \), where either \((Z, M) = ([I_Z], I_n (X, \beta))\) or \((|E|, M) = ([I^* = O_X \to F]), I_n (X, \beta))\). Denote the underlying Cohom-Macaulay curve by \( C \). We are interested in the restriction

\[
\tau (\gamma) \big|_{[E]}.
\]

Going around the following diagram both ways

\[
\begin{array}{ccc}
{[E]} \times X' & \xrightarrow{\pi_X} & X \\
\downarrow f_X & & \downarrow \pi_M \\
& & M
\end{array}
\]

we find

\[
(2.20) \quad \tau (\gamma) \big|_{[E]} = \int_X \gamma \cdot \text{ch}_3 (O_C),
\]

where \( f_X \) denotes equivariant push-forward to a point \( \bullet \). Crucially, we used

\[
(2.21) \quad \text{ch}_3^T (O_Z) = \text{ch}_3^T (O_C) = \text{ch}_3^T (F),
\]

so \( \tau (\gamma) \big|_{[E]} \) is given by the same expression in the DT and PT case. This is no longer the case for higher descendent insertions, which are not considered in this paper. Applying Atiyah-Bott localization to \( 2.21 \) gives

\[
(2.22) \quad \tau (\gamma) \big|_{[E]} = \sum_{\alpha \in V(X)} \text{ch}_3^T (Z_\alpha) \cdot (1 - t_1 (\alpha))(1 - t_2 (\alpha))(1 - t_3 (\alpha))(1 - t_4 (\alpha)) \cdot \frac{\gamma |_{p_\alpha}}{\epsilon (F_X |_{p_\alpha})},
\]

where \( t_i (\alpha) \) are the characters of the \((\mathbb{C}^*)^4\)-action on chart \( U_\alpha \) and \( Z_\alpha \) is given by \( \pi^{(\alpha)} \) the solid partition corresponding to \( C \mid_{U_\alpha} \). Insertions for stable pair invariants were studied on 3-folds in [22, Sect. 6].

Similar to the previous paragraph, the vertex formalism can also be used to calculate arbitrary equivariant curve counting and stable pairs invariants with primary insertions on toric Calabi-Yau 4-folds. Again we illustrate this for \( X \) the total space of \( O_{\mathbb{P}^2} (-1) \oplus O_{\mathbb{P}^2} (-2) \) and \( \beta = d [\mathbb{P}^1] \).

For any point-like plane partitions \( \lambda, \mu, \nu \), denote by \( C_{\lambda, \mu, \nu} \) the \((\mathbb{C}^*)^4\)-invariant Cohen-Macaulay curve supported on the \((\mathbb{C}^*)^4\)-invariant lines \([\mathbb{P}^1] \cup [\mathbb{P}^1] \cup [\mathbb{P}^1] \subseteq \mathbb{P}^2 \subseteq X \) with “cross-sections” \( \lambda, \mu, \nu \) along these lines. We denote its character in chart \( U_\alpha \) by \( (C_{\lambda, \mu, \nu} |_{U_\alpha}) \). Then Lemma 2.24. The vertex formalism (Section 2.4), and (2.22) imply

\[
I_\beta (X; T) (\gamma_1, \ldots, \gamma_m) = \sum_{\lambda, \mu, \nu} q^{f_{\lambda, \mu, \nu}} E_{\lambda, \mu, \nu} \text{ch}_3^T \left( (C_{\lambda, \mu, \nu}) |_{p_\alpha} \right) \frac{\gamma |_{p_\alpha}}{\epsilon (F_X |_{p_\alpha})},
\]

for all \( \gamma_1, \ldots, \gamma_m \in H^*_T (X, \mathbb{Q}) \). The choice of signs for \( I_\beta (X; T) (\gamma_1, \ldots, \gamma_m) \) are determined by RHS. The same expression holds for \( P_\beta (X; T) (\gamma_1, \ldots, \gamma_m) \) when replacing DT by PT.
2.7. Consequences. The DT/PT vertex correspondence (Conjecture \ref{conj:2.13}) implies the following equivariant DT/PT correspondence on toric Calabi-Yau 4-folds.

**Theorem 2.18.** Assume Conjecture \ref{conj:2.13} holds. Let $X$ be a toric Calabi-Yau 4-fold, $\beta \in H_2(X)$ such that $P_n(X, \beta)^{(C)^t}$ is 0-dimensional for all $n \in \mathbb{Z}$, and let $\gamma_1, \ldots, \gamma_m \in H_2(X)$. Then there exist choices of signs such that

$$I_\beta(X; T)(\gamma_1, \ldots, \gamma_m) = \frac{P_\beta(X; T)(\gamma_1, \ldots, \gamma_m)}{I_0(X; T)}.$$

In particular, without insertions we have

$$I_\beta(X; T) = \frac{P_\beta(X; T)}{I_0(X; T)}.$$

**Proof.** In order to keep the notation sufficiently simple, we consider the case $X$ is the total space of $\mathcal{O}_\mathbb{P}^1(1) \oplus \mathcal{O}_\mathbb{P}^1(-2)$ and $\beta = d[\mathbb{P}^1]$. The general case follows similarly, after setting up the right notation. As in the previous paragraph, for any point-like plane partitions $\lambda, \mu, \nu$, let $C_{\lambda\mu\nu}$ be the $(C^*)^4$-invariant Cohen-Macaulay curve supported on the $(C^*)^4$-invariant lines $\mathbb{P}^1 \cup \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq X$ and with “cross-sections” $\lambda, \mu, \nu$ along these lines. We denote its character in chart $U_0$ by $(C_{\lambda\mu\nu})_*$. Then Conjecture \ref{conj:2.13} implies that there exist choices of signs such that

$$I_\beta(X; T)(\gamma_1, \ldots, \gamma_m) = \sum_{k, \mu, \nu, \lambda} \tau^{f_1 - 1 - 2(\lambda) + f_1 - 1 - 2(\mu) + f_1 - 1 - 2(\nu)} \mathbf{E}_\lambda^{DT} |_{(\lambda, \lambda, 0, \lambda, 0)} \mathbf{V}_\mu^{DT} |_{(\lambda, 0, 0, \lambda, 0)} \mathbf{E}_\nu^{DT} |_{(\lambda, 0, 0, \lambda, 0)} \mathbf{V}_\lambda^{DT} |_{(\lambda, 0, 0, \lambda, 0)},$$

where $\tau$ is the DT/PT vertex correspondence (Conjecture \ref{conj:2.13}) holds, and all

$$\lambda < \nu < \mu < \lambda.$$

Besides Conjecture \ref{conj:2.13} this uses the fact that $\mathbf{E}_\lambda^{DT} = \mathbf{E}_\lambda^{PT}$ (the DT/PT edges coincide) and $\tau(\gamma)([x]_2) = \tau(\gamma)([x]_2)$ for any $[x]_2 \in I_0(X, \beta)^{(C)^t}$ and $[\gamma]_2 \in P_n(X, \beta)^{(C)^t}$ with the same underlying Cohen-Macaulay curve \ref{conj:2.13}. \hfill \square

The next two theorems can be seen as motivation for Conjecture \ref{conj:1.1}.

**Theorem 2.19.** Let $X$ be a toric Calabi-Yau 4-fold and $\beta \in H_2(X)$. Suppose $P_n(X, \beta)$ is proper and $P_n(X, \beta)^{(C)^t}$ is 0-dimensional for all $n \in \mathbb{Z}$. Assume the following:

1. the DT/PT vertex correspondence (Conjecture \ref{conj:2.13}) holds,
2. \ref{conj:2.13} holds for $\beta$ and $n = 0$, and \ref{conj:2.13} holds for $\beta$ and all $n \leq 0$,
3. the signs of (1) and (2) can be chosen compatibly.

Then $I_{0, \beta}(X; T) = P_{0, \beta}(X) \in \mathbb{Z}$.

**Proof.** Theorem \ref{conj:2.13} implies

$$I_{0, \beta}(X; T) = \sum_{k=0}^{\infty} I_{k, 0}(X; T) \cdot P_{-k, \beta}(X; T),$$

which is a finite sum because $P_{-k}(X, \beta)^{(C)^t}$ is $\emptyset$ for $k \gg 0$. For all $k > 0$, the virtual class $[P_{-k}(X, \beta)]^{vir}$ has negative virtual dimension. Therefore \ref{conj:2.13} implies

$$0 = [P_{-k}(X, \beta)]^{vir} = P_{-k, \beta}(X; T)$$

for all $k > 0$. We conclude $I_{0, \beta}(X; T) = P_{0, \beta}(X; T) = P_{0, \beta}(X) \in \mathbb{Z}$, where the second equality follows from the localization formula \ref{eqn:1.4} \hfill \square
Theorem 2.20. Let $X$ be a toric Calabi-Yau 4-fold satisfying $\int_X c_3^T(X) = 0$ and let $\beta \in H_2(X)$. Let $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$ admitting $T$-equivariant lifts and satisfying $\sum_i \deg \tau(\gamma_i) = 2n > 0$. Suppose $P_\chi(X, \beta)$ is proper, and $P_\chi(X, \beta)^{\mathbb{C}^*}^{\dag}$ is 0-dimensional for all $\chi \in \mathbb{Z}$. Assume the following:

1. the DT/PT vertex correspondence (Conjecture 2.13) holds,
2. Nekrasov’s conjecture (Conjecture 2.17) holds,
3. (0.4) holds for $\beta, \gamma_1, \ldots, \gamma_m, n$,
4. the signs of (1), (2), and (3) can be chosen compatibly.

Then $I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}$.

Proof. Choose insertions $\gamma_1, \ldots, \gamma_m \in H^*(X, \mathbb{Z})$ (admitting $T$-equivariant lifts) with the property that $\sum_i \deg \tau(\gamma_i) = 2n > 0$. Theorem 2.18 implies

\begin{equation}
I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) = \sum_{k=0}^{\infty} I_{k,0}(X; T) \cdot P_{n-k, \beta}(X; T)(\gamma_1, \ldots, \gamma_m),
\end{equation}

which is a finite sum because $P_{-k}(X, \beta)^{\mathbb{C}^*}^{\dag} = \emptyset$ for $k \gg 0$. Nekrasov’s conjecture and $\int_X c_3^T(X) = 0$ imply

$I_0(X; T) = \exp \left( -q \int_X c_3^T(X) \right) = 1$.

Hence $I_{k,0}(X; T) = 0$ for all $k > 0$ and (2.23) reduces to

$I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}$,

where the second equality follows from (0.4). \hfill \Box

2.8. Examples. We now discuss some cases where the results of Theorem 2.18, 2.19 and 2.20 hold unconditionally. In each case the verification is for curve classes $\beta$ of total degree $\leq 4$ and holomorphic Euler characteristic $n$ with $n \leq N$ for some $N$. The cases $n = 0, 1$ are particularly interesting cases in view of the PT/GV correspondence of [14] discussed in Section 0.2.

Let $X = \text{Tot}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2))$. Then (the proof of) Theorem 2.18 implies that there exist choices of signs such that

\begin{equation}
I_\beta(X; T)(\gamma_1, \ldots, \gamma_m) = I_0(X; T) \cdot P_\beta(X; T)(\gamma_1, \ldots, \gamma_m) \mod q^N
\end{equation}

in the following cases:

- $\beta = 1$ and $N = 5$,
- $\beta = 2$ and $N = 6$,
- $\beta = 3$ and $N = 6$,
- $\beta = 4$ and $N = 7$.

Let $X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$. Then (the proof of) Theorem 2.18 implies that there exist choices of signs such that (2.24) holds in the following cases:

- $\beta = 1$ and $N = 5$,
- $\beta = 2$ and $N = 5$,
- $\beta = 3$ and $N = 3$,
- $\beta = 4$ and $N = 1$.

Note that $P_n(X, \beta)$ is proper for all $n \in \mathbb{Z}$, because $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ has no sections. Assuming the virtual localization formulae (0.4) (for $\beta = 3, 4$ and $n = 0$), (0.5) (for $\beta = 3, 4$ and $n < 0$), and compatibility of signs, we also conclude that $I_{\beta,0}(X; T) = P_{\beta,0}(X) \in \mathbb{Z}$ for $\beta = 3, 4$. In App. A, we prove the virtual localization formula when all stable pairs are scheme theoretically supported in the zero section and the moduli space is smooth, which holds for $\beta = 3, 4$, $n = 0$.

Let $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$. Then (the proof of) Theorem 2.18 implies that there exist choices of signs such that (2.24) holds in the following cases:

- $\beta = (1, 0)$ and $N = 5$,
- $\beta = (2, 0)$ and $N = 6$,
- $\beta = (1, 1)$ and $N = 5$,
- $\beta = (3, 0)$ and $N = 6$,
- $\beta = (2, 1)$ and $N = 4$,
- $\beta = (4, 0)$ and $N = 7$,
- $\beta = (3, 1)$ and $N = 4$,
- $\beta = (2, 2)$ and $N = 3$. 
Again, $P_n(X, \beta)$ is proper for all $n \in \mathbb{Z}$, because $O(-1, -1) \oplus O(-1, -1)$ has no sections. Also note that $\int_X c_i^\ast (X) = 0$ (by direct calculation). Assuming the virtual localization formulae (0.4) (for $\beta = (2, 2)$ and $n = 0$), (0.5) (for $\beta = (2, 2)$ and $n < 0$), and compatibility of signs, we also conclude that $I_{(2, 2), 0}(X; T) = P_{(2, 2), 0}(X) \in \mathbb{Z}$. For $\beta = (2, 2)$ and $n = 0$, all stable pairs are scheme theoretically supported in the zero section and the moduli space is smooth, so the virtual localization formula holds by App. A.

Let $\beta, n > 0$ be in the previous list. Let $\gamma_1, \ldots, \gamma_m \in H^\ast(X, \mathbb{Z})$ admitting $T$-equivariant lifts and satisfying $\sum_i \deg \tau(\gamma_i) = 2n > 0$. In this range, Nekrasov’s conjecture (Conjecture 2.15) holds. Assuming (0.4) (for $\beta, \gamma_1, \ldots, \gamma_m, n$) and compatibility of signs, we conclude $I_{n, \beta}(X; T)(\gamma_1, \ldots, \gamma_m) = P_{n, \beta}(X)(\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}$. Moreover, in the following cases all stable pairs are scheme theoretically supported in the zero section, so the virtual localization formula is established in Appendix A:

- $\beta = (1, 0)$ and any $n$
- $\beta = (2, 0)$ and $n \leq 2$
- $\beta = (1, 1)$ and any $n$
- $\beta = (3, 0)$ and $n \leq 3$
- $\beta = (2, 1)$ and $n \leq 2$
- $\beta = (4, 0)$ and $n \leq 4$
- $\beta = (3, 1)$ and $n \leq 2$
- $\beta = (2, 2)$ and $n \leq 2$

2.9. DT/PT generating series of a local curve. Consider the toric 4-fold $X = \text{Tot}_{\mathbb{P}^1}(O(l_1) \oplus O(l_2) \oplus O(l_3))$.

Suppose $l_1 + l_2 + l_3 = -2$, then $X$ is Calabi-Yau. There are two maximal $(\mathbb{C}^\ast)^4$-invariant affine open subsets and the transition map is given by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1^{-1}, x_2x_1^{-l_1}, x_3x_1^{-l_2}, x_4x_1^{-l_3}),$$

where

$$t \cdot (x_1, x_2, x_3, x_4) = (t_1x_1, t_2x_2, t_3x_3, t_4x_4) \quad \text{for all } t \in (\mathbb{C}^\ast)^4.$$

Using the identification $H_2(X) \cong \mathbb{Z}$ (where the class of the zero section $[\mathbb{P}^1]$ corresponds to 1), we have (equivariant) stable pair invariants

$$P_{n, d}(X; T) \in \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \cong \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3),$$

for all $n, d \in \mathbb{Z}$. For $l_1 = 0$ and $l_2 = l_3 = -1$, and motivated by [14], we propose the following:

**Conjecture 2.21.** For $X = \text{Tot}_{\mathbb{P}^1}(O \oplus (-1) \oplus (-1))$, there exist choices of signs such that the following equation holds

$$\sum_{n, d \geq 0} P_{n, d}(X; T) q^n y^d = \exp \left( \frac{ny}{\lambda_2} \right),$$

where $\lambda_2$ is the equivariant parameter for the $\mathbb{C}^\ast$-action on the first fibre $O_{\mathbb{P}^1}$, and $P_{0, 0}(X; T) := 1$.

**Remark 2.22.** The above conjecture is equivalent to

$$P_{n, n}(X; T) = \frac{1}{n! \lambda_2^n}, \quad P_{n, d}(X; T) = 0, \quad \text{if } n \neq d.$$

which could be viewed as an equivariant analogue of Conjecture 0.1.

Using the vertex formalism and a Maple implementation, we obtain the following:

**Proposition 2.23.** Conjecture 2.21 (formulated as in Rem. 2.22) is true in the following cases:

- for any $n \leq d$,
- $d = 1$ and modulo $q_0$,
- $d = 2$ and modulo $q_0$,
- $d = 3$ and modulo $q_0$,
- $d = 4$ and modulo $q_7$.  

Proof. By Lemma 2.7, \( P_n(X,d) \subset \mathbb{C} \) is empty unless \( n \geq d \). For \( n = d \), the moduli space \( P_n(X,d) \subset \mathbb{C} \) only contains one element, namely the Cohen-Macaulay curve obtained by the \( n \) times thickening of the zero section into the \( O \)-direction. Equation (2.1) gives

\[
F = \frac{t_i}{1-t_i} \left[ \sum_{i=0}^{n-1} t_i^2 + \frac{1}{t_i t_2 t_3 t_4} \sum_{i=0}^{n-1} t_i^2 \sum_{j=0}^{n-1} t_j^2 \right] - \frac{1}{1-t_i} \left[ \sum_{i=0}^{n-1} t_i^2 + \frac{1}{t_i t_2 t_3 t_4} \sum_{i=0}^{n-1} t_i^2 \sum_{j=0}^{n-1} t_j^2 \right] = \sum_{i=0}^{n-1} t_i^2 + \sum_{i=0}^{n-1} t_i^2 - (1-t_i) (1-t_j) (1-t_k) \left( \sum_{i=0}^{n-1} t_i^2 + \sum_{i=0}^{n-1} t_i^2 \right) = \frac{1-t_i^2-1-t_j^2}{1-t_i-1-t_j},
\]

where we used the relation \( t_1 t_2 t_3 t_4 = 1 \) in the second and third equality. Taking the square root of the (signed) Euler class of minus this expression gives \( \pm 1/(nt_i) \), as desired.

Other cases were checked using the vertex formalism and a Maple implementation. □

As an application, we calculate all curve counting invariants of \( \text{Tot} P_1(O \oplus O(-1) \oplus O(-1)) \).

The vertex formalism (Section 2.4) at once implies:

**Corollary 2.24.** Let \( X = \text{Tot} P_1(O \oplus O(-1) \oplus O(-1)) \). Assume the following:

1. the DT/PT vertex correspondence holds (Conjecture 2.5),
2. Nekrasov’s conjecture holds (Conjecture 2.6),
3. the local curve conjecture holds (Conjecture 2.7),
4. the choices of signs of (1)-(3) are compatible.

Then there exist choices of signs such that

\[
\sum_{n,d \geq 0} I_{n,d}(X;T) q^n t^d = \exp \left( \frac{q}{\lambda_2} \left( y + \frac{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)}{\lambda_1 \lambda_3 (\lambda_1 + \lambda_2 + \lambda_3)} + \frac{\lambda_3 (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + \lambda_3)}{\lambda_1 (\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \right) \right).
\]

**Appendix A. Virtual localization and relative Hilbert schemes**

Consider the local surfaces \( X = \text{Tot} P_2(O(-1) \oplus (2)) \), \( \text{Tot} P_1 \times P_1(O(-1, -1) \oplus O(-1, -1)) \) and denote the zero section by \( S \subseteq X \). For curve classes of small degree, it is shown in [10] Prop. 2.1] that all stable pairs are scheme theoretically supported on \( S \subseteq X \) and \( P_n(S, \beta) \cong P_n(S, \beta) \) is smooth, in which case the stable pair invariants can be computed using relative Hilbert schemes.

**Theorem A.1.** Let \( X = \text{Tot} P_2(O(-1) \oplus (2)) \) or \( \text{Tot} P_1 \times P_1(O(-1, -1) \oplus O(-1, -1)) \) and denote the zero section by \( S \subseteq X \). Suppose \( \beta \in H_2(X) \) and \( n \in \mathbb{Z} \) are chosen such that \( P_n(S, \beta) \cong P_n(S, \beta) \) is smooth. Then the 4-fold virtual localization formula (3.3) holds for some choice of orientation.

Proof. Write \( L_1 \) for the first fibre direction of \( X \) and \( L_2 \) for the second fibre direction. Then \( L_1 \times L_2 \cong K_S \). Let \( P_S \subset P_n(S, \beta) \) and denote the universal stable pair on \( P_S \times S \) by \( \pi_S : \pi_S \to \{O_{P_S} \to S \to F\} \). There exists a choice of orientation such that (e.g. [14] Prop. 3.7, [10])

\[
[P_n(S, \beta)]^\chi = [P_n(S, \beta)]^\chi \cdot e \left( - \text{RHom}_{(F, F \boxtimes L_1)} \right),
\]

where the virtual class of \( P_n(S, \beta) \) on RHS is defined using surface stable pair deformation-obstruction theory as in [14] Appendix] and \( \pi_{P_S} : P_S \times S \to P_S \) denotes the projection. The complex \( - \text{RHom}_{(F, F \boxtimes L_1)} \) is concentrated in degree one. Then

\[
\int_{[P_n(S, \beta)]^\chi} \tau(\gamma) = \int_{[P_n(S, \beta)]^\chi} \tau(\gamma) \cdot e \left( - \text{RHom}_{(F, F \boxtimes L_1)} \right)
\]

\[
= \int_{[P_n(S, \beta)]^\chi} \tau(\gamma) \cdot e \left( - \text{RHom}_{(F, F \boxtimes L_1)} \right) \cdot e(\text{Ext}^1_{P_S}(\pi_S^* F, F))
\]

\[
= \sum_{[\pi_S : O_{P_S} \to F] \in P_n(S, \beta) \subset \mathbb{C}} \tau(\gamma) \cdot e \left( - \text{RHom}_{(F, F \boxtimes L_1)} \right) \cdot e(\text{RHom}_{(F, F \boxtimes L_1)}),
\]
where we use smoothness of the moduli space \([3\) Prop. 5.6\] in the second equality and Atiyah-Bott localization \([2\] for the action of \((\mathbb{C}^*)^4\) on \(P_S\) in the third equality.

Let \(P_X := P_n(X, \beta)\), then \(P_X \cong P_S\) by assumption, and therefore
\[
\text{Ext}^2_S(I^*_S, F) \cong \text{Ext}^2_X(I^*_X, I^*_X)_0,
\]
where \([I^*_S = \{\mathcal{O}_S \to F\}] \in P_n(S, \beta), \ I^*_X = \{\mathcal{O}_X \to \iota_* \mathcal{O}_S \to \iota_* F\},\) and \(\iota : S \hookrightarrow X\) denotes inclusion of the zero section. As shown in \([13\) Prop. 3.7\] and \([10\], at the level of obstruction spaces, we have an inclusion of a maximal isotropic subspace
\[
\text{Ext}^2_S(I^*_S, F) \oplus \text{Ext}^1_S(F, F \otimes L_1) \hookrightarrow \text{Ext}^2_X(I^*_X, I^*_X)_0
\]
with respect to the quadratic form given by Serre duality. Hence
\[
\sum_{[I^*_S = \{\mathcal{O}_S \to F\}] \in P_n(S, \beta)} \tau(\gamma)[I^*_S] \cdot e(-\text{RHom}_S(F, F \otimes L_1)) \cdot e(-\text{RHom}_S(I^*_S, F)) = \sum_{[I^*_S] \in P_n(X, \beta)} \tau(\gamma)[I^*_S] \cdot \sqrt{(-1)^n} \cdot e(\text{RHom}_X(I^*_S, I^*_S)_0). \tag{\*} \]

**Remark A.2.** Although Theorem A.1 establishes the 4-fold localization formula \((0.4)\) when \(P_n(X, \beta) \cong P_n(S, \beta)\) is smooth, it is not obvious that the choice of signs on RHS of \((0.4)\) is compatible with the choice of signs in Conjecture 2.13 and Theorem 2.13.

**References**

[1] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa. The topological vertex, Comm. Math. Phys. 254 (2005), 425-478.

[2] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1982) 523-615.

[3] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. 128 (1997), 45-88.

[4] D. Borisov and D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves, Adv. in Math. 314, (2017), 601-648.

[5] T. Bridgeland, Hall algebras and curve-counting invariants, J. Amer. Math. Soc. 24 (2011), 969-998.

[6] Y. Cao, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, arXiv:1801.06310.

[7] Y. Cao, Counting conics on sextic 4-folds, to appear in Math. Res. Lett. arXiv:1805.04696.

[8] Y. Cao, J. Gross and D. Joyce, Orientability of moduli spaces of Spin(7)-instantons and coherent sheaves on Calabi-Yau 4-folds, arXiv:1811.09658.

[9] Y. Cao and M. Kool, Zero-dimensional Donaldson-Thomas invariants of Calabi-Yau 4-folds, Adv. in Math. 338 (2018), 601-648.

[10] Y. Cao, M. Kool and S. Monavari, Stable pair invariants on toric CY 4-folds I: local surfaces, preprint, March 2019.

[11] Y. Cao and N. C. Leung, Donaldson-Thomas theory for Calabi-Yau 4-folds, arXiv:1407.7659.

[12] Y. Cao and N. C. Leung, Orientability for gauge theories on Calabi-Yau manifolds, Adv. in Math. 314, (2017), 48-70.

[13] Y. Cao, D. Maulik and Y. Toda, Genus zero Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, Adv. in Math. 338, (2018), 41-92.

[14] Y. Cao, D. Maulik and Y. Toda, Stable pairs and Gopakumar-Vafa type invariants for Calabi-Yau 4-folds, arXiv:1902.00003.

[15] A. Gerhardus, H. Jockers, Quantum periods of Calabi-Yau fourfolds, Nucl. Phys. B 913 (2016) 425-474.

[16] A. Klemm and R. Pandharipande, Enumerative geometry of Calabi-Yau 4-folds, Comm. Math. Phys. 281, 621-653 (2008).

[17] M. Kool and R. P. Thomas, Reduced classes and curve counting on surfaces I: theory, Algebr. Geom. 1 (2014) no. 3 334-383.

[18] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, Gromov-Witten and Donaldson-Thomas theory I, Compositio Math. 142 (2006) 1263-1285.

[19] N. Nekrasov, Magnificent Four, arXiv:1711.08128 [hep-th].

[20] N. Nekrasov and N. Piazzalunga Magnificent Four with Colors, arXiv:1808.05296 [hep-th].

[21] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), 407-447.

[22] R. Pandharipande and R. P. Thomas, The 3-fold vertex via stable pairs, Geom. Topol. 13 (2009) 1835-1876.

[23] T. Pantev, B. Toën, M. Vaquié and G. Vezzosi, Shifted symplectic structures, Publ. Math. I.H.E.S. 117 (2013), 271-328.

[24] Y. Toda, Curve counting theories via stable objects I: DT/PT correspondence, J. Amer. Math. Soc. 23 (2010), 1119-1157.