PERTURBATIONS OF GRAPHS FOR NEWTON MAPS

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Abstract. We consider graphs consisting of finitely many internal rays for degener-
ating Newton maps and state a convergence result. As an application, we prove that
a hyperbolic component in the moduli space of quartic Newton maps is bounded if
and only if every element has degree 2 on the immediate basin of each root. This pro-
vides the first complete description of bounded hyperbolic components in a complex
2-dimensional moduli space.

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1. Introduction

For \( d \geq 2 \), denote by \( \text{Rat}_d \) the space of rational maps of degree \( d \) in one complex
variable. Via parametrizing coefficients, the space \( \text{Rat}_d \) is an open dense subset of the
\( 2d+1 \)-dimensional complex projective space \( \mathbb{P}^{2d+1} \). The boundary \( \partial \text{Rat}_d := \mathbb{P}^{2d+1} \setminus \text{Rat}_d \)
consists of so-called degenerate rational maps. A sequence in \( \text{Rat}_d \) is degenerate if its
limit is a degenerate rational map. It is of interest to understand the interplay of
dynamics for a degenerate sequence and its limit. The goal of this paper is to explore
this interplay in a significant slice of \( \text{Rat}_d \), namely Newton family. We show that under
natural assumptions, the dynamics preserves stably when Newton maps approach to
\( \partial \text{Rat}_d \). Once this result is at our disposal, we can describe completely the boundedness
of hyperbolic components in the moduli space of quartic Newton maps.

1.1. Statements of main results. For a degree \( d \geq 2 \) complex polynomial \( P(z) \) with
simple roots, its Newton map is defined by

\[
\hat{f}_P(z) = z - \frac{P(z)}{P'(z)}.
\]

Denote by \( \text{NM}_d \) the space of degree \( d \) Newton maps. It follows that \( \text{NM}_d \) is a \( d \)-
dimensional subspace in \( \text{Rat}_d \) and hence in \( \mathbb{P}^{2d+1} \). Let \( \overline{\text{NM}_d} \) be the closure of \( \text{NM}_d \) in
\( \mathbb{P}^{2d+1} \). For \( f \in \overline{\text{NM}_d} \), denote \( \hat{f} \) the reduction of \( f \), see Section 2.1. We are interested
in the case that \( \hat{f} \) has degree at least 2, see Lemma 5.6. Then \( \hat{f} \) is a Newton map for
a polynomial with possible multiple roots. For more details, we refer [22].

Now consider the basin of roots of \( \hat{f} \). Let \( \mathcal{U} \) be a set consisting of finitely many
components of such basins. The boundary of each \( U \in \mathcal{U} \) is locally connected [6, 30].
Provided that \( \hat{f} \) is forward invariant and postcritically finite on \( \bigcup_{U \in \mathcal{U}} U \), each \( U \in \mathcal{U} \)
carries landed internal rays \( I_{(u,\omega)}(t) \) of \( \hat{f} \) for \( t \in \mathbb{R}/\mathbb{Z} \), where \( u \in U \) is the center of \( U \).

\begin{center}
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\end{center}
Let $\Gamma$ be a connected graph consisting of finitely many (pre)periodic internal rays in $\bigcup_{U \in \mathcal{U}} U$, that is
$$\Gamma := \bigcup_{(U,t) \in \mathcal{V} \times \mathcal{T}} I_{(U,u)}(t)$$
for some finite subsets $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{T} \subseteq \mathbb{Q}$. The canonical paradigms of such graphs are the Newton graphs (see Section 4.1) formulated recently by Drach e.t [5] and the alternative graphs for cubic Newton maps (see Section 4.2) based on Roesch’s work in [26].

Since $f \in \text{NM}_d$, let $\{f_n\}_{n \geq 1} \subset \text{NM}_d$ be a sequence such that $f_n$ converges to $f$. If the convergence is under the dynamically weak Carathéodory topology, see Definition 3.1, a Böttcher coordinate of $\hat{f}$ on $U \in \mathcal{U}$ naturally deduces a Böttcher coordinate of $f_n$ on the deformation $(U_n, u_n)$ of $(U, u)$, see Section 3.1. Then we can define the corresponding internal rays in $U_n$, which either land on $\partial U_n$ or terminate at $f_n$-iterated preimages of critical points in $U_n$, see Section 3.2. It follows that we obtain the perturbation
$$\Gamma_n := \bigcup_{(U,t) \in \mathcal{V} \times \mathcal{T}} I_{(U_n,u_n)}(t)$$
of $\Gamma$. For examples satisfying the above conditions, see Lemma 3.3.

Under natural assumptions, we prove the graphs $\Gamma_n$ converge to $\Gamma$ in the Hausdorff metric topology:

**Theorem 1.1.** Let $f, f_n, \Gamma$ and $\Gamma_n$ be as above. Suppose that
1. $\deg(f|_{\tilde{U}}) = \deg(f_n|_{\tilde{U}_n})$ if $\tilde{U} \in \mathcal{V}$ is the immediate basin of a root of $\hat{f}$, and
2. for each $U \in \mathcal{V}$ and $t \in \mathcal{T}$, the orbit of the landing point of $I_{(U,u)}(t)$ is eventually repelling periodic and avoids the critical points of $\hat{f}$.

Then for all large $n$, the graph $\Gamma_n$ is homeomorphic to $\Gamma$, and $\Gamma_n$ converges to $\Gamma$ as $n \to \infty$.

The technique of perturbations of internal rays already appear in complex dynamics for the non-degenerate maps, see e.g. [11, 12, 25]. Theorem 1.1 generalizes it to the degenerate case within the Newton family. The key point of the proof, differing from the non-degenerate case, is an elaborate argument to the internal rays landing at holes of $f$.

In principle, our above theorem provides a combinational method to study degenerate sequences of Newton maps in the parameter space and hence that in moduli space. In certain sense it asserts that, under the assumptions, part of the dynamics of the degenerate map $\hat{f}$ embeds into the dynamics of non-degenerate maps $f_n$’s. Thus it allows us to control the dynamics of $f_n$ by that of $\hat{f}$.

Now we apply Theorem 1.1 to study the boundedness of hyperbolic components in the moduli space of quartic Newton maps. Since the point $\infty$ is the unique repelling fixed point for Newton maps, the moduli space of degree $d$ Newton maps is defined by
$$\text{nm}_d := \text{NM}_d/\text{Aut}(\mathbb{C}),$$
modulo the action by conjugation of the group of affine maps. We mention here that the space $\text{nm}_d$ has complex dimension $d - 2$. Recall that a rational map is **hyperbolic** if each critical point converges under iteration to a (super)attracting cycle, equivalently, it is uniformly expanding in a neighborhood of its Julia set, see [17] Section 3.4]. The
space of hyperbolic Newton maps descends an open subset in \( \mathbb{nm}_d \), and each component of this subset is a **hyperbolic component** in \( \mathbb{nm}_d \). Endowing \( \mathbb{nm}_d \) the quotient topology, we say a hyperbolic component in \( \mathbb{nm}_d \) is **bounded** if it has compact closure in \( \mathbb{nm}_d \), and **unbounded** otherwise.

A hyperbolic component \( \mathcal{H} \subset \mathbb{nm}_d \) is of **immediate escaping type** if each element in \( \mathcal{H} \) is the conjugacy class of a Newton map having degree at least 3 in the immediate basin of some root.

**Theorem 1.2.** Let \( \mathcal{H} \subset \mathbb{nm}_4 \) be a hyperbolic component. Then \( \mathcal{H} \) is unbounded if and only if \( \mathcal{H} \) is of immediate escaping type.

**Figure 1.** The \( c \)-plane for the family of Newton maps \( f_{P_c} \) for the polynomials \( P_c(z) = z^4/12 - cz^3/6 + (4c - 3)z/12 + (3 - 4c)/12 \), see [23, Figure 1]. The critical points of \( f_{P_c} \) are the four roots of \( P_c(z) \), 0 and \( c \). The map \( f_{P_c} \) has a superattracting 2-cycle \( 0 \rightarrow 1 \rightarrow 0 \). The letters indicate the types of hyperbolic components, see Section 5.1. Our result asserts that the hyperbolic components indicated by A, B, C or FE1 are bounded in \( \mathbb{nm}_4 \).

For the boundedness of hyperbolic components, motivated by a result of Kleinian groups [29, Theorem 1.2], McMullen [10] conjectures that every hyperbolic component with Sierpiński Julia set is bounded in the moduli space of degree \( d \) rational maps. In his celebrated work [19, Remark 7.2], Milnor proposed the study of this topic in quadratic case. If the moduli space has complex dimension at least 2, there are only few already known results: for a hyperbolic component in the moduli space of bicritical rational maps, if each element possesses two distinct (super)attracting cycles of period at least 2, then it is bounded, see [7, Theorem 1] and [24, Theorem 1.1]; for quartic Newton maps, the second author and Pilgrim proved that a hyperbolic component in \( \mathbb{nm}_4 \) is bounded if each element has two distinct (super)attracting cycles of period at least 2 [23, Main Theorem].

All the previous known bounded hyperbolic components are of so-called type D, that is each element has maximal number of (super)attracting cycles. We point out
here that the type D components are semi-algebraic, but the components of other types are possible transcendental objects, see [18, Theorem 1 and Conjecture 2]. Our boundedness result gives the first non semi-algebraic bounded hyperbolic components in a complex 2-dimensional moduli space. Moreover, it strengthens the result [23, Theorem 1.3].

1.2. Strategy of the proof of Theorem 1.2. One direction of Theorem 1.2 is the result [23, Theorem 1.4]: if $\mathcal{H}$ is of immediate escaping type, then $\mathcal{H}$ is unbounded. Now we give an overview of the proof of the reverse implication. Differing from the analytic argument in [7] and the arithmetic argument in [23, 24], our argument relies on the combinatorial properties of Newton maps and applies Theorem 1.1. The proof goes by contradiction as follows. Suppose $\mathcal{H}$ is unbounded and not of immediate escaping type. Then we obtain a unbounded sequence $[f_n] \in \mathcal{H}$. Passing to a subsequence, we can assume that $[f_n]$ has a lift $f_n \in \text{NM}_4$ such that $f_n$ converges to $f \in \partial\text{NM}_4$ with reduction $\hat{f}$ having degree 2 or 3 and no roots of $f_n$ collide as $n \to \infty$, see Lemma 5.6. It follows that at least one non-fixed critical point $c_n$ of $f_n$ diverging to $\infty$. We derive a contradiction case by case.

**Case 1:** $\deg \hat{f} = 2$. In this case, we consider rational internal rays in the immediate basins of the roots of $\hat{f}$ and the corresponding perturbations for $f_n$. Theorem 1.1 implies that $\deg f_n = 2$ and hence leads to a contradiction.

**Case 2:** $\deg \hat{f} = 3$ and $\mathcal{H}$ is of type A, B, C or D. It turns out that the Newton graphs of $\hat{f}$ are disjoint with the unique non-fixed critical point $c$ of $\hat{f}$. Applying Theorem 1.1 to the Newton graphs of $\hat{f}$, we bound the immediate basins of the (super)attracting cycles of periods at least 2 for $f_n$. We obtain a contradiction by arguing the location of forward orbit of the critical point $c_n$.

**Case 3:** $\deg \hat{f} = 3$ and $\mathcal{H}$ is of type FE1 or FE2. In this case, the critical point $c$ could be an iterated preimage of $\infty$. Then we can not apply Theorem 1.1 directly to the Newton graphs as in the previous case. Alternatively, using Rosech’s results in [26] on cut angles, we construct a natural Jordan curve $\mathcal{C}$ consisting of (pre)periodic internal rays of $\hat{f}$ such that the orbit of $\mathcal{C}$ is away from the critical point $c$. Then Theorem 1.1 works for the curve $\mathcal{C}$. Thus, we can continue to analyze the location of the related critical points and the corresponding Fatou components of $f_n$, and obtain a contradiction.

We remark that our proof of Theorem 1.2 highly relies on the behavior of the critical point $c_n$ for $f_n$, see Lemma 5.3. We do not expect an analogy of such behavior holding for Newton maps of higher degrees. But it would be interesting to apply Theorem 1.1 to investigate the boundedness of hyperbolic components in $\text{nm}_d$ for $d \geq 5$.

Structure of the paper. This paper is organized as follows. In Section 2 we introduce the relevant preliminaries about degenerate rational maps and Newton maps. Section 3 contains the proof of Theorem 1.1. In Section 4, we investigate some dynamical graphs for Newton maps, and in Section 5 we prove Theorem 1.2.

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2. Preliminaries

In this section, we give background materials. In Section 2.1, we provide basic definitions and properties of degenerate rational maps. Section 2.2 concerns the (degenerate) Newton maps.

2.1. Degenerate rational maps. As mentioned in the introduction, the space \( \text{Rat}_d \) is naturally identified to an open and dense subset of \( \mathbb{P}^{2d+1} \). We say each element \( f \in \mathbb{P}^{2d+1} \setminus \text{Rat}_d \) is a degenerate rational map of degree \( d \). For such \( f \), there exist two degree \( d \) homogeneous polynomials \( F(X,Y) \) and \( G(X,Y) \) in \( \mathbb{C}[X,Y] \) such that \( f = [F : G] \) in homogeneous coordinates and \( H_f := \gcd(F,G) \) is a polynomial in \( \mathbb{C}[X,Y] \) of degree at least 1. We can rewrite

\[ f = H_f \hat{f}, \]

where \( \hat{f} \) is a rational map of degree less than \( d \). We say each zero of \( H_f \) is a hole of \( f \) and denote by \( \text{Hole}(f) \) the set of holes of \( f \). Moreover, we call \( \hat{f} \) the reduction of \( f \).

For convenience, if \( f \) is a rational map of degree \( d \), we define \( H_f = 1 \) and then \( \hat{f} = f \).

Let \( \{f_n\}_{n \geq 1} \) be a sequence of degree \( d \geq 1 \) rational maps. If \( f_n \) converges to \( f = H_f \hat{f} \in \mathbb{P}^{2d+1} \). Then \( f_n \) converges locally uniformly to \( \hat{f} \) outside \( \text{Hole}(f) \).

Conversely, combining [2, Lemma 2.8] and [22, Lemma 2.2.3], we have

**Lemma 2.2.** Let \( \{f_n\}_{n \geq 1} \) be a sequence of degree \( d \geq 1 \) rational maps and let \( S \subset \mathbb{P}^1 \) be a finite set. Suppose \( f_n \) converges locally uniformly to a map \( \hat{f} \) on \( \mathbb{P}^1 \setminus S \). Then \( \hat{f} \) is rational map of degree at most \( d \). Moreover, there exists a homogeneous polynomial \( H_f \) of degree \( d - \deg \hat{f} \) such that \( f_n \) converges to \( f := H_f \hat{f} \) and \( \text{Hole}(f) \subset S \).

Suppose that each \( f_n \) possesses a cycle of fixed period. If the limit of these cycles is away from the holes of \( f \), Lemma 2.1 immediately implies that this limit is also a cycle for \( \hat{f} \). We state as follows and omit the proof.

**Lemma 2.3.** Let \( \{f_n\}_{n \geq 1} \) be a sequence of degree \( d \geq 2 \) rational maps. Suppose that \( f_n \) converges to \( f = H_f \hat{f} \in \mathbb{P}^{2d+1} \) with \( \deg \hat{f} \geq 1 \). Assume \( \mathcal{O}_n \) is a cycle of \( f_n \) of period \( m \geq 1 \) and suppose that \( \mathcal{O}_n \) converges to \( \mathcal{O} \) in \( \mathbb{P}^1 \). If \( \mathcal{O} \cap \text{Hole}(f) = \emptyset \), then \( \mathcal{O} \) is a cycle of \( \hat{f} \) of period \( q \) with \( q \mid m \). Furthermore, (1) if \( \mathcal{O}_n \) is attracting, then \( \mathcal{O} \) is non-repelling; (2) if \( q < m \), then \( \mathcal{O} \) is parabolic.

If the limit intersects the holes of \( f \), we have the following basins shrinking result.
Lemma 2.4 ([23] Proposition 2.8]). Let \( \{f_n\}_{n \geq 1} \) be a sequence of degree \( d \geq 2 \) rational maps. Assume that \( f_n \) converges to \( f = H_f \hat{f} \in \mathbb{P}^{2d+1} \). Assume \( \deg \hat{f} \geq 2 \) and \( \infty \in \text{Hole}(f) \) is a fixed point of \( \hat{f} \). Let \( \{z_n^{(0)}, \ldots, z_n^{(m-1)}\} \) be a (super)attracting cycle of \( f_n \) of period \( m \geq 2 \), and let \( U_n^{(k)} \) be the Fatou component containing \( z_n^{(k)} \). Suppose \( z_n^{(k)} \to z^{(k)} \) for \( k = 0, \ldots, m-1 \) with \( z^{(0)} = \infty \) and \( z^{(i)} \neq \infty \) for some \( 1 \leq i \leq m-1 \). Then

1. \( U_n^{(0)} \) converges to \( \infty \) in the sense that, for any \( \epsilon > 0 \), the component \( U_n^{(0)} \) is contained in the disk \( \{z : \rho(z, \infty) < \epsilon\} \) for all large \( n \), where \( \rho \) is the sphere metric; and
2. there exists a neighborhood \( V \) of \( \infty \) such that \( U_n^{(i)} \cap V = \emptyset \) for all large \( n \).

Now we state a straightforward result about the perturbations of periodic points.

Lemma 2.5. Let \( f = H_f \hat{f} \in \mathbb{P}^{2d+1} \) with \( \deg \hat{f} \geq 1 \). Then the following holds.

1. For \( z_0 \in \hat{\mathbb{C}} \) and \( j \geq 1 \), denote by \( z_i := \hat{f}(z_0) \) for \( 0 \leq i \leq j \). Suppose \( z_i \) avoids the critical point of \( \hat{f} \) for all \( 0 \leq i \leq j-1 \). Let \( z_j(g) \) be a holomorphic map defined in a neighborhood of \( f \in \mathbb{P}^{2d+1} \) with \( z_j(f) = z_j \). Then for each \( 0 \leq i \leq j-1 \), there exists a holomorphic map \( z_i(g) \) defined in a neighborhood of \( f \) such that \( z_i(f) = z_i \) and \( \hat{g}^{j-i}(z_i(g)) = z_j(g) \). Moreover, if \( z_i \) avoids the holes of \( f \) for all \( 0 \leq i \leq j-1 \), then \( z_i(g) \) is the unique point near \( z_i \) such that \( \hat{g}^{j-i}(z_i(g)) = z_j(g) \), which implies \( z_i(g) = \hat{g}^{j-i}(z_i(0)) \) for all \( 0 \leq i \leq j-1 \).
2. Let \( \mathcal{O} = \{\xi_0, \ldots, \xi_{k-1}\} \) be an attracting (resp. repelling) cycle of \( \hat{f} \). If \( \mathcal{O} \cap \text{Hole}(f) = \emptyset \), then for each \( g \) close to \( f \), there exists a unique attracting (resp. repelling) cycle \( \mathcal{O}(g) := \{\xi_0(g), \ldots, \xi_{k-1}(g)\} \) of \( g \) such that each \( \xi_i(g) \) is a holomorphic map near \( f \) with \( \xi_i(f) = \xi_i \).

Proof. By pre and post composition of Möbius transformations, we can assume \( z_0, \ldots, z_j \in \mathbb{C} \). For \( g = H_g \hat{g} \in \mathbb{P}^{2d+1} \) close to \( f \), we have \( \deg \hat{g} \geq 1 \). Then for \( 0 \leq i \leq j-1 \), the iteration \( \hat{g}^{j-i} \) is well-defined, see [3, Lemma 2.2]. Consider the holomorphic function

\[ F_i(g, z) := g^{j-i} - z_j \]

on \( \Lambda_f \times D(z_j) \), where \( \Lambda_f \subseteq \mathbb{P}^{2d+1} \) is a neighborhood of \( f \) and \( D(z_j) \subseteq \mathbb{C} \) is a neighborhood of \( z_j \). By the assumptions, we have that \( F_i(f, z_i) = 0 \) and

\[ \frac{\partial F_i}{\partial z}|_{(f, z_i)} = (\hat{g}^{j-i})'(z_i) \neq 0. \]

Then the Implicit Function Theorem implies there exists a holomorphic function \( z_i(g) \) near \( f \) such that \( \hat{g}^{j-i}(z_i(g)) = z_j(g) \). If \( \{z_0, \ldots, z_{j-1}\} \cap \text{Hole}(f) = \emptyset \), the function \( \hat{g}^{j-i} \) is holomorphic in \( z \) in a fixed neighborhood of \( z_i \) for each \( g \) close to \( f \). It follows from Hurwitz’s Theorem (see [3]) that \( g^{j-i}(z) - z_j(g) \) has a unique root near \( z_i \) for \( g \) close to \( f \). Thus statement (1) follows.

For statement (2), note that the cycle \( \mathcal{O} \cap \text{Hole}(f) = \emptyset \). Applying the Implicit Function Theorem on \( G(g, z) := g^k(z) - z \), we obtain the expected cycle \( \mathcal{O}(g) \) of \( g \) for \( g \) close to \( f \). \( \square \)

For \( f = H_f \hat{f} \in \mathbb{P}^{2d+1} \), assume \( \hat{f} \) has an attracting cycle \( \mathcal{O} \) and denote by \( \Omega \) the immediate basin of \( \mathcal{O} \). If \( \Omega \cap \text{Hole}(f) = \emptyset \), Lemma 2.5 implies that for \( g \) close to \( f \), the map \( \hat{g} \) has an attracting cycle \( \mathcal{O}(g) \). Denote by \( \Omega(g) \) the immediate basin of \( \mathcal{O}(g) \). Then we have
Lemma 2.6. Let \( E \subset \Omega \) be any compact set. Then \( E \subseteq \Omega(g) \) for any \( g \) sufficiently close to \( f \).

This above result is well-known in the case that \( f \) is a rational map of degree \( d \), see [4, Lemma 6.3]. Our assumption \( \Omega \cap \text{Hole}(f) = \emptyset \) guarantees that the argument in the non-degenerate case also works in our case. Here we omit the proof.

2.2. Newton maps. For a degree \( d \geq 2 \) complex polynomial \( P(z) \) with simple roots, its Newton map

\[
f_P(z) := z - \frac{P(z)}{P'(z)}
\]

is a degree \( d \) rational map having \( d \) superattracting fixed points at the roots of \( P \). The only other fixed point is at \( \infty \). The Holomorphic Index Formula (see [20, Theorem 12.4]) asserts that the point \( \infty \) is the unique repelling fixed point of \( f_P \). The critical points of \( f_P \) are the roots of \( P \) and the zeros of \( P'' \). Moreover, the poles of \( f_P \) are the zeros of \( P' \).

Recall that \( \text{NM}_d \) is the space of degree \( d \) Newton maps and \( \overline{\text{NM}_d} \) is the closure of \( \text{NM}_d \) in \( \mathbb{P}^{2d+1} \). Then for each \( f = H_f \hat{f} \in \overline{\text{NM}_d} \), there exists a degree at most \( d \) polynomial \( Q \) with possible multiple roots such that \( \hat{f} \) is the Newton map of \( Q \). Each root \( r \) of \( Q \) is a (super)attracting fixed point of \( \hat{f} \) with multiplier \( 1 - 1/n_r \), where \( n_r \) is the multiplicity of \( r \) as a zero of \( Q \). Moreover, again \( \hat{f} \) has only one more fixed point at \( \infty \), which is repelling. It follows that each hole of \( f \) is either a multiple root of \( Q \) or \( \infty \). Furthermore, \( \infty \in \text{Hole}(f) \) if and only if \( \deg Q < d \). For more details about degenerate Newton maps, we refer [22].

Conversely, the following result, which was originally due to Head [13], gives a criterion to determine whether a rational map is a reduction of a (degenerate) Newton map. The criterion concerns only the fixed points and the corresponding multipliers.

Proposition 2.7. A rational map \( \hat{g} \) of degree \( d \geq 2 \) is a reduction of a (degenerate) Newton map of degree at least \( d \) if and only if \( \hat{g} \) has \( d + 1 \) distinct fixed points \( r_1, \ldots, r_d, \infty \) such that each \( r_i \) has multiplier of the form \( 1 - 1/n_i \) with \( n_i \in \mathbb{N} \).

For \( f = H_f \hat{f} \in \overline{\text{NM}_d} \) with \( \deg \hat{f} \geq 2 \), the Fatou components of \( \hat{f} \) have well-studied topological structure. According to Shishikura [28], all Fatou components of \( \hat{f} \) are simply connected, and hence the Julia set of \( \hat{f} \) is connected. Moreover, the boundary of each component of the basins of roots is locally connected, see [6] and [30].

3. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. We define Böttcher coordinates on the deformations in Section 3.1 and prove the convergence of Böttcher coordinates (Proposition 3.5). To do that, we introduce the dynamically weak Carathéodory topology (Definition 3.1). In Section 3.2, we use the Böttcher coordinates on the deformations to define the corresponding internal rays, and then show a convergence result on these rays (Proposition 3.6). Finally, we prove Theorem 1.1 in Section 3.3.
3.1. Perturbation of Böttcher coordinates. Let \( f = H_f \hat{f} \in \mathbb{NM}_d \) with \( \deg \hat{f} \geq 2 \) and denote by \( \Omega_f \) the union of basins of the roots of \( \hat{f} \). Let \( \mathcal{U} \) be a finite subset of components of \( \Omega_f \) such that if \( U \in \mathcal{U} \), then \( \hat{f}(U) \in \mathcal{U} \). Recall that \( \hat{f} \) is postcritically finite on \( \bigcup_{U \in \mathcal{U}} U \) if the critical points in any \( U \in \mathcal{U} \) have finite orbits. For such \( \hat{f} \) and \( \mathcal{U} \), one can choose a system of Böttcher coordinates \( \{ \phi_U : U \to \mathbb{D} \}_{U \in \mathcal{U}} \) satisfying
\[
\phi_{\hat{f}(U)} \circ \hat{f} \circ \phi_U^{-1}(z) = z^{d_U}, z \in \mathbb{D}, \text{ where } d_U := \deg(\hat{f}|_U).
\]

Let \( \{ f_n \}_{n \geq 1} \) be a sequence in \( \mathbb{NM}_d \) such that \( f_n \) converges to \( f \). Since \( \hat{f} \) is postcritically finite on \( \bigcup_{U \in \mathcal{U}} U \), the Fatou components in \( \hat{f}^{-1}(U) \) are disjoint with the holes of \( f \).

Indeed, a possible hole of \( f \) is either \( \infty \) or an attracting fixed point, see Section 2.2. Hence, by Lemma 2.6, for \( U \in \mathcal{U} \), its center \( u := \phi_U^{-1}(0) \) belongs to a component \( U_n \) of \( \Omega_{f_n} \) for all large \( n \). We call such \( U_n \) the deformation of \( U \) at \( f_n \). In this subsection, under natural assumptions, we define a Böttcher coordinate \( \phi_{U_n} \) on the deformations \( U_n \) of \( U \) and show a convergence result of \( \phi_{U_n} \).

We first recall the definition of weak Carathéodory topologies on set of pointed sets and set of holomorphic functions, respectively. Let \( \mathcal{V} \) be a set of open simply-connected pointed sets \( (V, v) \) in \( \mathbb{C} \). The weak Carathéodory topology on \( \mathcal{V} \) is defined by the following convergence: \( (V_n, v_n) \) converges to \( (V, v) \) if and only if (i) \( v_n \) converges to \( v \); and (ii) for any compact \( K \subset V \), we have \( K \subset V_n \) for all large \( n \). Denote by \( \mathcal{G} \) the set of holomorphic functions defined on \( (V, v) \in \mathcal{V} \). Then the weak Carathéodory topology on \( \mathcal{G} \) is defined as follows. Let \( g : (V, v) \to \mathbb{C} \) and \( g_n : (V_n, v_n) \to \mathbb{C} \) be functions in \( \mathcal{G} \). We say \( g_n \) converges to \( g \) if (i) \( (V_n, v_n) \) converges to \( (V, v) \) in \( \mathcal{V} \); and (ii) \( g_n \) converges to \( g \) uniformly on compact subsets of \( V \) as \( n \to \infty \).

Back to Newton maps, let \( \{ f_n \}_{n \geq 1} \) and \( f \) be as above. For \( U \in \mathcal{U} \), consider its center \( u \in U \). We use the following definition.

**Definition 3.1.** We say \( f_n \) converges to \( f \) on \( \mathcal{U} \) under the dynamically weak Carathéodory topology if for each \( U \in \mathcal{U} \), there exists a point \( u_n \) in the deformation \( U_n \) of \( U \) such that

1. the restrictions \( f_n : (U_n, u_n) \to \mathbb{C} \) converges to \( \hat{f} : (U, u) \to \mathbb{C} \) under weak Carathéodory topology;
2. the point \( u_n \) is (pre)periodic under \( f_n \) with the same (pre)period as that of \( u \) under \( \hat{f} \); and
3. the local degrees \( \deg_u \hat{f} = \deg_{u_n} f_n \).

We call such \( u_n \) (if exists) a center of \( U_n \), and call \( (U_n, u_n) \) the deformation of \( (U, u) \) at \( f_n \). To abuse notations, we denote the set of pointed sets \( (U, u) \) with \( U \in \mathcal{U} \) also by \( \mathcal{U} \). Set
\[
\mathcal{U}_n := \{(U_n, u_n) : (U_n, u_n) \text{ is the deformation of } (U, u) \in \mathcal{U}\}.
\]

It may happen that the set \( \mathcal{U}_n \) contains several distinct centers:

**Remark 3.2.** If a critical point \( c \) of \( \hat{f} \) is contained in the boundaries of distinct \( (U, u) \) and \( (U', u') \) in \( \mathcal{U} \), it is possible that \( U_n \) coincides with \( U'_n \) and it contains the critical point of \( f_n \) perturbed from \( c \) (see Figure 2). In this case, both \( u_n \) and \( u'_n \) are centers of \( U_n = U'_n \), and hence \( f_n \) is not postcritically finite on the union of \( U_n \) with \( (U_n, u_n) \in \mathcal{U}_n \).
The following result states a natural sufficient condition for the convergence of $f_n$ under the dynamically weak Carathéodory topology, which we use repeatedly in Section 5.

**Lemma 3.3.** Let $f_n$, $f$ and $U$ be as above. Assume that $\hat{f}$ has degree 2 on every immediate basin of roots in $U$ and degree 1 on all other elements in $U$. Then $f_n$ converges to $f$ on $U$ under the dynamically weak Carathéodory topology.

**Proof.** Since every $U \in U$ avoids the poles of $f$, by Lemmas 2.5 and 2.6, we have the following: for every $(U, u) \in U$,

1. there exists a unique (pre)periodic point $u_n$ of $f_n$ near $u$ with the same (pre)period as that of $u$ such that $u_n \to u$. In particular, if $U$ is the immediate basin of a root of $\hat{f}$, then $u_n$ is the root of $f_n$ contained in $U_n$ (the deformation of $U$ at $f_n$);
2. any compact subset of $U$ is contained in $U_n$ for all large $n$;
3. given $k \geq 1$, the $k$-th derivative $f_n^{(k)}(u_n)$ converges to $f^{(k)}(u)$ as $n \to \infty$.

The statements (1) and (2) imply that $f_n : (U_n, u_n) \to \mathbb{C}$ converges to $\hat{f} : (U, u) \to \mathbb{C}$ under weak Carathéodory topology. If $\hat{f}(U) = U$, then $U$ is an immediate basin of some root of $\hat{f}$. By statement (3), it follows that $f_n^{(k)}(u_n) = 0$ but $f_n^{(k)}(u_n) \neq 0$. Hence $\deg_{u_n} f_n = 2 = \deg_{u_n} \hat{f}$. If $\hat{f}(U) \neq U$, then $U$ is not an immediate basin of any root of
be the maximal radius such that $f'_n(u_n) \neq 0$. Hence $\deg_{u_n} f_n = 1 = \deg u \hat{f}$. This completes the proof.

From now on, we assume $f_n$ converges to $f$ on $\mathcal{U}$ under the dynamically weak Carathéodory topology. Since $\hat{f}$ is postcritically finite on $\bigcup_{U \in \mathcal{U}} U$, by Lemma 2.1 we have the following straight forward result and omit the proof.

**Lemma 3.4.** If $(U_n, u_n)$ is the deformation of $(U, u) \in \mathcal{U}$ at $f_n$, then $(f_n(U_n), f_n(u_n))$ is the deformation of $(\hat{f}(U), \hat{f}(u))$.

The above lemma suggests that for each $(U_n, u_n) \in \mathcal{U}_n$, we have a Böttcher coordinate $\phi_{(U_n, u_n)}$ near $u_n$ such that

$$\phi_{(U_n, u_n)}(z)^{du} = \phi_{(f_n(U_n), f_n(u_n))} \circ f_n(z)$$

(3.1)

for $z$ near $u_n$, and that

$$\phi'_{(U_n, u_n)}(u_n) \to \phi'_{(U, u)}(u) \text{ as } n \to \infty.$$  

(3.2)

The map $\phi_{(U_n, u_n)}$ extends conformally until meeting an iterated preimage of critical points of $f_n$. Then there exists a maximum $r_n \leq 1$ such that $\psi_{(U_n, u_n)} := \phi_{(U_n, u_n)}^{-1} : \mathbb{D}_{r_n} \to U_n$ is conformal.

Denote by $\psi_{(U, u)}^r$ the inverse of $\phi_{(U, u)}$. The following result asserts that $\psi_{(U_n, u_n)}$ converges to $\psi_{(U, u)}$ locally uniformly on $\mathbb{D}$.

**Proposition 3.5.** For $(U, u) \in \mathcal{U}$, let $(U_n, u_n) \in \mathcal{U}_n$ be the deformation of $(U, u)$. Then $\psi_{(U_n, u_n)}$ converges to $\psi_{(U, u)}$ locally uniformly on $\mathbb{D}$.

**Proof.** It is sufficient to show that for any given $0 < r < 1$, the maps $\psi_{(U_n, u_n)}$ uniformly converge to $\psi_{(U, u)}$ on $\mathbb{D}_r := \{ z : |z| \leq r \}$. We first assume $\hat{f}(U, u) = (U, u)$, equivalently $\hat{f}(u) = u$. Then $f_n(U_n, u_n) = (U_n, u_n)$ for all large $n$. Given any $r \in (0, 1)$, let $r_1 \in (r, 1)$ and denote by $U(r_1) := \psi_{(U, u)}(\mathbb{D}_{r_1})$. By Lemma 2.6 we have $U(r_1) \subseteq U_n$ for all large $n$. Since $u$ is the unique critical point of $\hat{f}$ in $U(r_1)$, the Böttcher coordinate $\phi_{(U_n, u_n)}$ extends to $U(r_1)$.

Note that $\{ \phi_{(U_n, u_n)} \}_{n \geq 1}$ is a normal family on $U(r_1)$. Let $\phi_{(U_n, u_n)}$ be any converging subsequence and denote the limit by $\phi$. By Equation (3.1) and Lemma 2.1 it follows that on $U(r_1)$ we have

$$z^{du} \circ \phi = \phi \circ \hat{f}.$$ 

Hence, $\phi$ is a Böttcher coordinate of $\hat{f}$ on $U$. According to the convergence (3.2), we obtain $\phi = \phi_{(U, u)}$. By the arbitrariness of $\phi_{(U_n, u_n)}$, the sequence $\phi_{(U_n, u_n)}$ converges uniformly to $\phi_{(U, u)}$ in $U(r_1)$. As a consequence, the image $\phi_{(U_n, u_n)}(U(r_1))$ contains $\mathbb{D}_r$ for all large $n$, and $\psi_{(U_n, u_n)}$ converges uniformly to $\psi_{(U, u)}$ on $\mathbb{D}_r$.

In the general case, by inductively using the argument above, we can prove the conclusion.

3.2. Perturbation of internal rays. In previous subsection, we perturb a Böttcher coordinate in $(U_n, u_n) \in \mathcal{U}$ to obtain a Böttcher coordinate $\phi_{(U_n, u_n)}$ in $(U_n, u_n) \in \mathcal{U}_n$. In this subsection, we use the inverse map $\psi_{(U_n, u_n)}$ to define the internal rays in $(U_n, u_n)$ and prove a convergence result on internal rays.

Now we define internal rays of $f_n$ in $(U_n, u_n)$ as follows. For each $\theta \in \mathbb{R} / \mathbb{Z}$, let $r_\theta$ be the maximal radius such that $\psi_{(U_n, u_n)}$ extends along $(0, r_\theta)e^{2\pi i \theta}$. If $r_\theta < 1$, then arc
\[ \psi_{(U_n,u_n)}((0, r_\theta) e^{2\pi i \theta}) \text{ terminates at an iterated preimage of critical points of } f_n, \text{ and if } r_\theta = 1, \text{ the arc } \psi_{(U_n,u_n)}((0,1) e^{2\pi i \theta}) \text{ accumulates, factually lands on } \partial U_n. \] In the latter case, we call
\[ I_{(U_n,u_n)}(\theta) := \psi_{(U_n,u_n)}([0,1] e^{2\pi i \theta}) \]
the landed internal ray in \((U_n,u_n)\) of angle \(\theta\). Note that \(f_n\) sends a landed internal ray of \((U_n,u_n)\) to a landed internal ray of \((f(U_n), f(u_n))\). Also, since \(U_n\) may contain more than one centers, it may posses several groups of landed interval rays. In this case, each such ray starts from a center of \(U_n\) and rays from distinct groups are disjoint (see Figure 2).

The following result asserts that the internal rays of eventually periodic angles converge.

**Proposition 3.6.** For \((U,u) \in \mathcal{U}\), assume that the internal ray \(I_{(U,u)}(\theta)\) of angle \(\theta\) lands at an eventually repelling periodic point. For all large \(n\), suppose that \(I_{(U_n,u_n)}(\theta)\) is a landed internal ray in \((U_n,u_n) \in \mathcal{U}_n\). Then \(I_{(U_n,u_n)}(\theta) \to I_{(U,u)}(\theta)\) as \(n \to \infty\).

**Proof.** To ease notation, we write \(I(\theta), f_n(\theta), \psi\) and \(I_{(U,u)}(\theta), I_{(U_n,u_n)}(\theta), \psi_{(U,u)}\) and \(\psi_{(U_n,u_n)}\), respectively. Set \(\delta := \text{deg}(\hat{f}_U)\) and let \(z_0\) be the landing point of \(I(\theta)\). It is sufficient to show that, given any \(\eta > 0\), for all large \(n\), we have \(d_H(I(\theta), I_n(\theta)) < \eta\), where \(d_H\) is the Hausdorff metric.

First assume that \(I(\theta)\) is periodic of period \(p \geq 1\). Then \(u\) is a (super)attracting fixed point of \(\hat{f}\). Define
\[ D_\epsilon := \{ z \in \hat{C} : \rho(z, z_0) < \epsilon \}, \]
where \(\rho\) is the spherical metric. Shrinking \(\epsilon\) if necessary, we may assume \(\hat{f}|_{D_\epsilon}\) is injective and \(\overline{D_\epsilon} \subseteq \hat{f}(D_\epsilon)\). We claim that for any sufficiently large \(n\) and any component \(D'_n\) of \(f_n^{-p}(D_\epsilon)\), either \(\overline{D'_n} \subseteq D_\epsilon\) or \(\overline{D'_n} \subseteq \hat{C} \setminus D_\epsilon\). Indeed, if \(p > 1\), the landing point \(z_0\) of \(I(\theta)\) is not a hole of \(f^p\), see Section 2.1 and [3] Lemma 2.2. It follows from Lemma 2.1 that \(f_n^p\) converges uniformly to \(f^p\) near \(z_0\), and hence \(f_n^p|_{D_\epsilon}\) is injective and \(\overline{D'_n} \subseteq f_n^p(D_\epsilon)\) for all large \(n\). Then in this case the claim follows. Now we consider the case that \(p = 1\). Then \(z_0 = \infty\). If \(z_0 = \infty\) is not a hole of \(f\), the claim follows by previous argument. If \(z_0 = \infty\) is a hole of \(f\), then \(f_n\) fails to converge uniformly to \(\hat{f}\) near \(\infty\). In this case, we prove the claim by contradiction. Suppose that the claim fails. Then there exists a subsequence, denoted also by \(\{f_n\}\), such that for each \(f_n\), there exists a component \(D'_n\) of \(f_n^{-1}(D_\epsilon)\) with \(\overline{D'_n} \cap \partial D_\epsilon \neq \emptyset\). Choose a point \(w_n \in \overline{D'_n} \cap \partial D_\epsilon\). Passing to subsequence if necessary, we may assume \(w_n \to w\). Then \(w \in \partial D_\epsilon\). By Lemma 2.1 the sequence \(f_n\) converges uniformly to \(\hat{f}\) on \(\partial D_\epsilon\). It follows that as \(n \to \infty\),
\[ f_n(w_n) \to \hat{f}(w). \]
Note that \(\overline{D_\epsilon} \subseteq f(D_\epsilon)\). Then
\[ \hat{f}(\partial D_\epsilon) \cap \overline{D_\epsilon} = \emptyset. \]
We have that \(f(w) \not\in \overline{D_\epsilon}\). However,
\[ f_n(w_n) \in f_n(D'_n) = \overline{D_\epsilon}, \]
which implies \(f(w) \in \overline{D_\epsilon}\). It is a contradiction. Therefore, the claim holds.

Since \(I(\theta)\) lands at \(z_0\), there exists \(0 < r < 1\) such that
\[ \psi((r,1) e^{2\pi i \theta}) \subseteq U \cap D_\epsilon. \]
Pick $0 < s < 1$ such that $s^\delta > r$. Then the segment $\psi([s^\delta, s[e^{2\pi i\theta}]) \subseteq I(\theta)$ belongs to $U \cap D$. It follows from Proposition 3.5 that for all large $n$,

$$d_H(\psi_n([0, s]e^{2\pi i\theta}), \psi([0, s]e^{2\pi i\theta})) < \epsilon. \tag{3.3}$$

Define

$$\gamma_{n,0} : [0, 1] \to \psi_n([s^\delta, s]e^{2\pi i\theta})$$

to be an arc such that $\gamma_{n,0}(0) = \psi_n(s^\delta e^{2\pi i\theta})$ and $\gamma_{n,0}(1) = \psi_n(se^{2\pi i\theta})$. Then

$$\gamma_{n,0}(0, 1] \subseteq D \cap U.$$

Note that $f_n^p(\gamma_{n,0}(1)) = \gamma_{n,0}(0)$. Lift $\gamma_{n,0}$ to an arc $\gamma_{n,1}$ based at $\gamma_{n,0}(1)$. Inductively, we obtain a sequence of arcs $\gamma_{n,k}$ such that $\gamma_{n,k+1}$ is a lift by $f_n$ of $\gamma_{n,k}$ based at the endpoint of $\gamma_{n,k}$ which is not in $\gamma_{n,k-1}$.

Now we claim that for sufficiently large $n$, the arc $\gamma_{n,k} \subseteq D$. We prove the claim by induction on $k$. The claim holds for $k = 0$ by the definition of $\gamma_{n,0}$. Suppose that for $k \geq 0$, the arc $\gamma_{n,k} \subseteq D$. Since $\gamma_{n,k+1}$ is a preimage of $\gamma_{n,k}$ under $f_n$, there exists a component $D'$ of $f_n^{-1}(D)$ containing $\gamma_{n,k+1}$. Since the intersection point of $\gamma_{n,k+1} \subseteq D'$ and $\gamma_{n,k} \subseteq D$ belongs to $D$, it follows that $D' \cap D \neq \emptyset$. By the previous claim, we have $D' \subseteq D$, and hence $\gamma_{n,k+1} \subseteq D$, which completes the induction.

Note that for all large $n$,

$$I_n(\theta) = \psi_n([0, s]e^{2\pi i\theta}) \cup \{z_n\},$$

where $z_n$ is the landing point of $I_n(\theta)$. According to estimate (3.3) and the fact that $\gamma_{n,k} \subseteq D$, we have

$$d_H(I(\theta), I_n(\theta)) < \epsilon.$$

By choosing $\epsilon < \eta$, we prove the proposition under the periodicity assumption.

In the strictly preperiodic case, we set $(V, v) := \hat{f}(U, u)$ and $I_V(\theta') = \hat{f}(I(\theta))$. Let $(V_n, v_n)$ be the deformation of $(V, v)$ with $f_n(U_n, u_n) = (V_n, v_n)$. Inductively, it is sufficient to prove $d_H(I(\theta), I_n(\theta)) < \epsilon$ under the assumption that $\lim_{n \to \infty} d_H(I_V(\theta'), I_{V_n}(\theta')) = 0$.

Define $D_\epsilon$ as above. By Proposition 3.5 there exists $0 < s < 1$ such that for all large $n$,

$$d_H(\psi_n([0, s]e^{2\pi i\theta}), \psi([0, s]e^{2\pi i\theta})) < \epsilon \text{ and } \psi_n(se^{2\pi i\theta}) \in D_\epsilon.$$

Denote by $L_n' := \psi_{(V_n, v_n)}([s^\delta, 1]e^{2\pi i\theta})$ and $L' := \psi_{(V, v)}([s^\delta, 1]e^{2\pi i\theta})$, respectively. Since $I_{V_n}(\theta') \to I_V(\theta')$, we have $L_n'$ and $L'$ are contained in $\hat{f}(D_\epsilon)$ for large $n$. Since $I_n(\theta)$ is a landed internal ray for all large $n$, there is a lift $L_n$ of $L_n'$ based at the point $\psi_n(se^{2\pi i\theta})$. Denote by $L$ the lift of $L'$ based at the point $\psi(se^{2\pi i\theta})$. Note that in this case we have $z_0 \notin \text{Hole}(\hat{f})$. Then $f_n$ converges uniformly to $\hat{f}$ on $D$. Thus for sufficiently large $n$,

$$f(D_\epsilon) \subseteq f_n(D_{2\epsilon}).$$

Hence we have $L_n \subseteq D_{2\epsilon}$ and $L \subseteq D_{2\epsilon}$. Note $I(\theta) = \psi([0, s]e^{2\pi i\theta}) \cup L$ and $I_n(\theta) = \psi_n([0, s]e^{2\pi i\theta}) \cup L_n$. It follow that

$$d_H(I(\theta), I_n(\theta)) < 2\epsilon.$$

Choose $\epsilon < \eta/2$. This completes the proof. \qed
3.3. **Proof of Theorem 1.1.** Now we begin to prove the Theorem 1.1. To apply Proposition 3.6, we first show the rays \( I_{(U,u_n)}(t) \) land on \( \partial U_n \).

**Lemma 3.7.** Under the assumptions in Theorem 1.1, let \( U \in \mathcal{V} \) and \( t \in T \). Then all large \( n \), the rays \( I_{(U,u_n)}(t) \) are landed internal rays.

**Proof.** We first consider the case that \( U \) is the immediate basin of a root of \( \hat{f} \). By the Definition 3.1, the assumption \( \deg \hat{f}|_U = \deg f_n|_{U_n} \) implies that the center \( u_n \) is the unique critical point of \( f_n \) in \( U_n \). Then the conclusion follows immediately.

Now we consider the case that \( U \) is not the immediate basin of a root of \( \hat{f} \). We set \( (V,v) := \hat{f}(U,u) \) and \( I_{(V,v)}(t') := \hat{f}(I_{(U,u)}(t)) \). By an induction argument, it suffices to show \( I_{(U,u_n)}(t) \) lands on \( \partial U_n \) under the assumption that \( I_{(V,v_n)}(t') \) lands on \( \partial V_n \) for all large \( n \). Fix notations as in the proof of Proposition 3.6. Since the orbit of the Julia points in \( \Gamma \) is homeomorphic to \( \Gamma \) for large \( n \), it follows from Lemma 2.5 and Propsosition 3.6 that \( \psi_{(V,v_n)}(s \epsilon^{2\pi i t'}) \in \partial V_n \).

It follows that \( I_{(U,u_n)}(t) \) land on \( \partial U_n \). \( \square \)

Now we prove Theorem 1.1

**Proof of Theorem 1.1.** By Proposition 3.6 and Lemma 3.7, we have that for each \( ((U,u),t) \in \mathcal{V} \times T \), the internal rays \( I_{(U,u_n)}(t) \) converges to \( I_{(U,u)}(t) \) as \( n \to \infty \). It follows immediately that \( \Gamma_n \) converges to \( \Gamma \) as \( n \to \infty \). It remains to check that \( \Gamma_n \) is homeomorphic to \( \Gamma \) for large \( n \). It is sufficient to show that for any \( ((U,u),t) \) and \( ((U',u'),t') \) in \( \mathcal{V} \times T \), the rays \( I_{(U,u)}(\theta) \) and \( I_{(U',u')}(\theta') \) land at a common point if and only if \( I_{(U,u_n)}(t) \) and \( I_{(U',u'_n)}(t') \) land at a common point for all large \( n \). It immediately follows from Lemma 2.5 and Proposition 3.6 since the orbits of the Julia points in \( \Gamma \) are away from the critical points of \( \hat{f} \). \( \square \)

4. **Invariant graphs for Newton maps**

In this section, we introduce suitable dynamical graphs of Newton maps for later use to prove Theorem 1.2. In Section 4.1, we recall the Newton graphs given by Drach et al. In Section 4.2, we first state Roesch’s result on cut angles and then construct invariant graphs differing from the Newton graphs for cubic Newton maps. In Section 4.3, we generalize Roesch’s cut angles result to quartic Newton maps.

4.1. **Newton graphs.** Let \( f \in \text{NM}_d \) with \( d \geq 2 \). Recall that \( \Omega_f \) is the union of basins of its roots. Assume that \( f \) is postcritically finite on \( \Omega_f \). The dynamics of \( f \) can be characterized by an invariant graph what is so-called Newton graph. Such graph was first constructed in [5] and then applied to study the dynamics of corresponding maps, see [6, 9, 10, 14, 15, 30]. In this subsection, we state briefly the construction of Newton graphs and list some properties.

Let \( r \) be a root of \( f \) and denote by \( \Omega_f(r) \) its immediate attracting basin. The fixed internal rays in \( \Omega_f(r) \) land at fixed points in \( \partial \Omega_f(r) \). Since the only Julia fixed point of \( f \) is at \( \infty \), all fixed internal rays in \( \Omega_f \) have a common landing point at \( \infty \). We denote...
by $\Delta_0$ the union of all fixed internal rays in $\Omega_f$ together with $\infty$. Then $f(\Delta_0) = \Delta_0$. For any $m \geq 0$, denote by $\Delta_m$ the connected component of $f^{-m}(\Delta_0)$ that contains $\infty$. Following [5], we call $\Delta_m$ the Newton graph of $f$ at level $m$. The vertex set of $\Delta_m$ consists of iterated preimages of fixed points of $f$ contained in $\Delta_n$.

A crucial property for Newton graphs is the following.

**Lemma 4.1** ([5 Theorem 3.4]). There exists $M \geq 0$ such that the Newton graph $\Delta_M$ contains all poles of $f$. Hence $\Delta_{m+1} = f^{-1}(\Delta_m)$ and $\Delta_m \subseteq \Delta_{m+1}$ for any $m \geq M$.

The Newton graphs induce naturally a puzzle structure for $f$ on $\hat{\mathbb{C}}$. Let $\Delta_f$ denote the Newton graph of $f$ with the least level such that $\Delta_f$ contains all poles and all critical points that map to fixed points under iteration. Set $X_0$ the complement of the union of the disks $\{z \in U : \phi_U(z) < 1/2\}$ for all connected components $U$ of $\Omega_f$ with $U \cap \Delta_f \neq \emptyset$, where $\phi_U$ is the Böttcher coordinate on $U$. Define $G_0 := (\Delta_f \cap X_0) \cup \partial X_0$. Then $G_0$ is a finite graph consisting of segments of internal rays and equipotential lines in $\Omega_f$. For each $m \geq 0$, we define $X_m := f^{-m}(X_0)$ and $G_m := f^{-m}(G_0)$. Then each $X_m$ is connected and the interior $\text{int}(X_m)$ contains the Julia set $J_f$ of $f$. For each $m \geq 0$, the closures of the components of $X_m \setminus G_m$ are called puzzle pieces of level $m$. It follows that the puzzle pieces of different levels have a nested structure. For each $z \in J_f$, denote by $E_m(z)$ the union of puzzle pieces of level $m$ which contains $z$. Then $z \in \text{int}(E_m(z))$. Moreover, $E_m(z)$ are puzzle pieces for all $m$ if and only if $z$ is not an iterated preimage of $\infty$.

**Proposition 4.2** ([6 Corollary 1.2] and [30 Theorem 1.1]). If $z$ is on the boundary of a component of $\Omega_f$, then

$$\bigcap_{m \geq 0} E_m(z) = \{z\}.$$  

In particular, the boundary of any component of basins of the roots is locally connected.

### 4.2. An alternative graph for cubic Newton maps.

In this subsection, we focus on the case that $f \in \text{NM}_3$. Except some special cases, we construct an invariant graph away from the unique non-fixed critical point. Our graph is based on Roesch’s work in [26 Section 3] and differs from the Newton graphs introduced above.

Let $r_1, r_2$ and $r_3$ be the roots of $f$ and let $\Omega_1$, $\Omega_2$ and $\Omega_3$ be the corresponding immediate basins, respectively. Note that $f$ has another critical point denoted by $c$. In this subsection, we always assume the $c \not\in \Omega_1 \cup \Omega_2 \cup \Omega_3$ and $c$ is not a pole, that is $f(c) \neq \infty$.

Under the assumptions, we have that $f$ has two distinct poles, denoted by $\xi_1$ and $\xi_2$. An orientation argument implies that $\partial \Omega_1$, $\partial \Omega_2$ and $\partial \Omega_3$ can not intersect at a common pole. By counting the preimages of $\Omega_i$’s, we have that there is a unique pole where exact two of $\partial \Omega_i$’s intersect. Up to reindexing, we can assume $\xi_1 \in \partial \Omega_1 \cap \partial \Omega_2$. It follows that $\xi_2 \notin \partial \Omega_1 \cup \partial \Omega_2$.

For $i = 1, 2$ and $3$, denote by $I_i(\theta)$ the internal ray in $\Omega_i$ of angle $\theta \in \mathbb{R}/\mathbb{Z}$. Following Roesch [26], we say an angle $\theta$ is a cut angle in $\Omega_i$ if there exists $\theta' \in \mathbb{R}/\mathbb{Z}$ such that $I_1(\theta)$ and $I_2(\theta')$ land at a common point. It turns out that $\theta$ is a cut angle in $\Omega_1$ if and only if $1 - \theta$ is a cut angle in $\Omega_2$. For the basin $\Omega_3$, the only cut angle is 0. Let $\Theta$ be the set of cut angles in $\Omega_1$. It follows immediately that $0, 1/2 \in \Theta$. Label $\Omega_1$ such that
\(\Omega_3\) and \(I_{\Omega_1}(\theta)\) are in the same complementary component of the curve
\[
\gamma(0, 1/2) := I_1(0) \cup I_1(1/2) \cup I_2(0) \cup I_2(1/2)
\]
for any \(\theta \in (0, 1/2)\), and define
\[
\alpha := \inf \{\theta : \theta \in \Theta\},
\]
where \(\inf\) is obtained under the order by identifying \(\mathbb{R}/\mathbb{Z}\) with \((0, 1]\). In fact, the locally connectivity of \(\partial \Omega_1\) and \(\partial \Omega_2\) implies that \(\alpha \in \Theta\).

Now we summarize the properties of the cut angles for later use. We use the following notations. Let \(\Omega_1^{(1)}\) be the preimage of \(\Omega_i\) disjoint from \(\Omega_i\). Then \(c \notin \Omega_1^{(1)}\). For \(j \geq 1\), if \(\Omega_1^{(j)}\) is a domain such that \(f^j : \Omega_1^{(j)} \to \Omega_i\) is a homeomorphism, then an internal ray \(I_i(\theta)\) in \(\Omega_i\) deduces an internal ray \(I_1^{(j)}(\theta)\) in \(\Omega_1^{(j)}\) satisfying \(I_1^{(j)}(\theta) = f^{-j}(I_i(\theta))\).

**Lemma 4.3 ([26, Section 3]).** Fix the notations as above. The following statements hold.

1. If the orbit of a rational angle \(\theta\) is contained in \([\alpha, 1]\), then \(\theta \in \Theta\).
2. The angle \(0 < \alpha < 1/2\). Furthermore, the periodic angles \(1 - 1/(2^n - 1)\) belong to \(\Theta\) for all large \(n\).
3. Assume \(0 < \theta < 1/2\) with \(2\theta \in \Theta\). Then \(\theta + 1/2 \in \Theta\). Furthermore, if \(\theta \in \Theta\), then \(I_1^{(1)}(2\theta)\) and \(I_2^{(1)}(1 - 2\theta)\) land at a common point; if \(\theta \notin \Theta\), then \(I_1(\theta)\) and \(I_2(1 - 2\theta)\) land at a common point, as well as \(I_2(1 - \theta)\) and \(I_1^{(1)}(2\theta)\). The two landing points are distinct.
4. The curve \(\gamma(0, 1/2)\) defined in \((4.1)\) separates \(\Omega_3\) and \(\Omega_3^{(1)}\).
5. Let \(0 < \theta < 1/2\) with \(2\theta \in \Theta\). If \(\theta \notin \Theta\), then the curve
\[
I_1(1/2) \cup I_1(\theta) \cup I_2^{(1)}(1 - 2\theta) \cup I_2^{(1)}(0) \cup I_1^{(1)}(0) \cup I_1^{(1)}(2\theta) \cup I_2(1 - \theta) \cup I_2(1/2)
\]
separates \(c\) and \(\infty\).

Figure [3] provides an example to illuminate the curves in the above lemma.

Let \(\gamma(0, 1/2)\) be as in \((4.1)\). Then the complement of \(\gamma(0, 1/2)\) in \(\hat{\mathcal{C}}\) contains two components. Denote by \(D\) the one that is disjoint with \(\Omega_3\). It follows from Lemma 4.3 (4) that \(\Omega_3^{(1)} \subset D\).

By Lemma 4.3 (2), we can choose a rational angle \(\theta \in (0, 1/2)\) satisfying

(i) \(\theta \notin \Theta\), but \(2\theta \in \Theta\),

(ii) there exists \(k \geq 1\) such that \(\eta := 2^k\theta \in (1/2, 1]\), and

(iii) the orbit of the landing point of \(I_1(\theta)\) avoids \(c\) and \(\infty\).

Define
\[
\mathcal{L} := I_3(0) \cup I_3(1/2) \cup I_1(0) \cup I_1(\theta) \cup I_2(0) \cup I_2(1 - \theta) \cup I_2^{(1)}(1 - 2\theta) \cup I_2^{(1)}(0) \cup I_1^{(1)}(0) \cup I_1^{(1)}(2\theta).
\]

Then Lemma 4.3 (3) implies that \(\mathcal{L}\) is a connected graph. Moreover, \(\hat{\mathcal{C}} \setminus \mathcal{L}\) has three components. We label \(W\) the one disjoint with \(\Omega_3\). In the remaining two components, we label \(W_-\) the one intersecting with \(\Omega_1\) and label \(W_+\) the one intersecting with \(\Omega_2\) (see Figure [4]). By Lemma 4.3 (5), it immediately follows that \(D \cup \Omega_3^{(1)} \subseteq W\) and \(c \in W \setminus \overline{D}\). In particular, \(\xi_1 \in W\). Moreover, we have \(I_3(3/4) \subseteq W_-\) and \(I_3(1/4) \subseteq W_+\).

Now consider the components of \(f^{-1}(\Omega_1^{(1)})\) and \(f^{-1}(\Omega_2^{(1)})\). Note that \(f^{-1}(\Omega_2^{(1)})\) has a component whose boundary contains the landing point of \(I_1((1 + \theta)/2)\). Since \(I_1((1 + \theta)/2) \subset \Omega_2\) and \(\Omega_2 \not\subset W_+\), we have \(\Omega_2^{(1)} \not\subset W_+\).
Figure 3. The dynamical plane of the Newton map for the polynomial $z^3/3 - z^2/2 + 1$. The curve $\gamma(0, 1/2)$ consists of the internal rays $I_1(0), I_1(1/2), I_2(0)$ and $I_2(1/2)$. The angle $\theta \notin \Theta$ but $2\theta \in \Theta$. A curve in Lemma 4.3 (5) consists of indicated internal rays except the ones in $\gamma(0, 1/2)$. In this section, we continue to use this example in the subsequent figures.

Figure 4. The curve $\mathcal{L}$ consists of the indicated internal rays except $I_1(1/2)$ and $I_2(1/2)$. The boundary of $D$ consists of $I_1(0), I_1(1/2), I_2(1/2)$ and $I_2(0)$.

$\theta/2) \subset D$, this component is also contained in $D$. Hence it does not contain $c$ since $c \in W \setminus \overline{D}$. Note that the landing points of $I_3(1/4)$ and $I_3(3/4)$ are contained in the boundaries of the two remaining components of $f^{-1}(\Omega_2^{(1)})$, respectively. We denote
by $\Omega_2^{(2)}$ the component whose boundary contains the landing point of $I_3(3/4)$. Then $I_2^{(2)}(0)$ and $I_3(3/4)$ land at a common point. Moreover, $\Omega_2^{(2)} \subseteq W_-$ since $I_3(3/4) \subseteq W_-$. It follows that $c \not\in \Omega_2^{(2)}$. By Lemma 4.3 (3), we have $I_1(\theta)$ and $I_2^{(1)}(1 - 2\theta)$ land at a common point. It follows that $I_1(\theta/2)$ and $I_2^{(2)}(1 - 2\theta)$ land at a common point since $I_1(\theta/2) \subseteq W_-$. Similarly, denote by $\Omega_1^{(2)}$ the component of $f^{-1}(\Omega_1^{(1)})$ contained in $W_+$. Then $c \not\in \Omega_1^{(2)}$. Moreover, $I_1^{(2)}(0)$ and $I_3(1/4)$ land at a common point, as well as $I_1^{(2)}(2\theta)$ and $I_2(1 - \theta/2)$. Define the Jordan curve

$$C := I_3(1/4) \cup I_3(3/4) \cup I_2^{(2)}(0) \cup I_2^{(2)}(1 - 2\theta) \cup I_1(\theta/2) \cup I_1(\eta) \cup I_2(1 - \eta) \cup I_2(1 - \theta/2) \cup I_1^{(2)}(2\theta) \cup I_1^{(2)}(0).$$

![Figure 5. The curve C consists of the indicated internal rays. For this \( \theta \), we have \( \eta = 2\theta \).](image)

We show that the critical point $c$ is not in the iterations of $C$ and separated by $C$ from $\infty$. More precisely, we have the following.

**Lemma 4.4.** Let $C$ be as above. Then the following statements hold.

1. The orbit of any Julia point in $C$ is disjoint with the critical points of $f$.
2. Denote by $V$ the bounded component of $\hat{C} \setminus C$. Then
   $$\overline{\Omega}_1^{(1)} \cup \overline{\Omega}_2^{(1)} \cup \overline{\Omega}_3^{(1)} \cup \{\xi_1, \xi_2, c\} \subseteq V$$

**Proof.** The Julia points in $C$ are the landing points of $I_3(1/4), I_3(3/4), I_1(\theta/2), I_1(\eta)$ and $I_2(1 - \theta/2)$. By the choice of $\theta$, the orbits of the landing points of $I_1(\theta/2), I_1(\eta)$ and $I_2(1 - \theta/2)$ are away from $c$. Since

$$c \in W \setminus \{\infty, \xi_2\} \subseteq \hat{C} \setminus \overline{\Omega}_3,$$

it follows that $c \not\in \partial\Omega_3$, and hence the orbits of the landing points of $I_3(1/4)$ and $I_3(3/4)$ are disjoint with $c$. Then statement (1) holds.
The statement (2) follows immediately from the construction of \( \mathcal{C} \) and Lemma 4.3 (4),(5).

Since \( \theta \) is rational, there is a positive integer \( k > 1 \) such that the graph
\[
G := \bigcup_{j=0}^{k} f^j(\mathcal{C})
\]
is invariant. Lemma 4.4 immediately implies that \( c \notin G \). Moreover, obviously our graph \( G \) is distinct from the Newton graphs of \( f \).

4.3. **Cut angles for quartic Newton maps.** In this subsection, we generalize part of results in [26] Section 3 from cubic case to quartic case. Throughout this subsection, we assume that \( f \in \text{NM}_4 \) has degree 2 in the immediate basin of each root.

Let \( r_1, r_2, r_3 \) and \( r_4 \) be the roots of \( f \) and denote by \( \Omega_1, \Omega_2, \Omega_3 \) and \( \Omega_4 \) the corresponding immediate basins. Then there exist \( 1 \leq i < j \leq 4 \) such that \( \partial \Omega_i \cap \partial \Omega_j \) contains a pole. Hence the internal rays \( I_i(1/2) \) and \( I_j(1/2) \) land at a common point. We say that \( f \) is of separable type if there exist \( 1 \leq i < j \leq 4 \) such that \( I_i(1/2) \) and \( I_j(1/2) \) land at a common pole and each component of \( \widehat{\mathbb{C}} \setminus \gamma(0,1/2) \) contains a pole of \( f \), where
\[
\gamma(0,1/2) := I_i(0) \cup I_i(1/2) \cup I_j(0) \cup I_j(1/2).
\]

If \( f \) is not of separable type, we can choose \( 1 \leq i < j \leq 4 \) such that \( I_i(1/2) \) and \( I_j(1/2) \) land at a common pole, but a component \( \hat{D} \) of \( \widehat{\mathbb{C}} \setminus \gamma(0,1/2) \) does not contain a pole of \( f \). Relabeling the roots of \( f \), we set \( i = 1 \) and \( j = 2 \). Furthermore, we can set \( I_1(\theta) \in \hat{D} \) if and only if \( \theta \in (1/2,1) \). Hence \( I_2(\theta') \in \hat{D} \) if and only if \( \theta' \in (0,1/2) \).

We now consider the cut angles in \( \Omega_1 \). An angle \( \theta \in \mathbb{R}/\mathbb{Z} \) is a cut angle in \( \Omega_1 \) if there exists \( \theta' \in \mathbb{R}/\mathbb{Z} \) such that \( I_1(\theta) \) and \( I_2(\theta') \) land at a common point. If \( \theta \) is a cut angle in \( \Omega_1 \), then the corresponding \( \theta' = 1 - \theta \). Denote by \( \Theta \) the set of all cut angles in \( \Omega_1 \) and set \( \alpha := \inf\{\theta : \theta \in \Theta\} \), where \( \inf \) is obtained under the order by identifying \( \mathbb{R}/\mathbb{Z} \) with \((0,1]\). Since \( \widehat{\mathbb{C}} \setminus \mathcal{D} \) contains \( \Omega_3 \cup \Omega_4 \), it follows that \( \alpha > 0 \). By the locally connectivity of \( \partial \Omega_1 \) and \( \partial \Omega_2 \), we have \( \alpha \in \Theta \) and \( \Theta \) is a closed set in \( \mathbb{R}/\mathbb{Z} \).

Now we state some properties of the cut angles. Since we are interesting in hyperbolic maps, see Section 5 we further assume that \( f \) is hyperbolic in the following result.

**Proposition 4.5.** Let \( f \) be hyperbolic and not of separable type. With the above notations, the following statements hold.

1. For any \( \theta \in \Theta \), \((\theta+1)/2 \in \Theta \).
2. Let \( \theta \) be a periodic angle. If the orbit of \( \theta \) belongs to \((\alpha,1)\), then \( \theta \in \Theta \).
3. The angles \( \alpha \in (0,1/2) \) and there exist periodic angles in \((\alpha,1/2) \cap \Theta \).

**Proof.** For statement (1), since \((\theta+1)/2 > 1/2 \), the internal rays \( I_1((\theta+1)/2) \subseteq \hat{D} \). Suppose \((\theta+1)/2 \notin \Theta \). Since \( f(I_1((\theta+1)/2)) = I_1(\theta) \) and \( \theta \in \Theta \), there exists a component \( \Omega_2^{(1)} \) of \( f^{-1}(\Omega_2) \) disjoint with \( \Omega_2 \) such that \( \Omega_2^{(1)} \) contains the landing point of \( I_1((\theta+1)/2) \). Note that \( f \) is hyperbolic and hence the landing point of \( I_2(1/2) \) is not a critical point. It follows that \( \Omega_2^{(1)} \subseteq \hat{D} \). Hence \( D \) contains a pole of \( f \). It contradicts to the choice of \( D \).
To prove statement (2), let \( p \) be the period of the angle \( \theta \). Under the assumptions of \( f \), the unique fixed angle is 0. It follows that \( p > 1 \). Define

\[
\gamma(0, \alpha) := I_1(0) \cup I_1(\alpha) \cup I_2(0) \cup I_2(1 - \alpha).
\]

Since \( \alpha \leq 1/2 \), there exists a component of \( \hat{\mathbb{C}} \setminus \gamma(0, \alpha) \) containing \( D \). Denote this component by \( W \). It follows that the only possible pole of \( f \) contained in \( W \) is the common landing point of \( I_1(1/2) \) and \( I_2(1/2) \). Hence the only component of \( f^{-1}(\Omega_1) \) (resp. \( f^{-1}(\Omega_2) \)) intersecting with \( W \) is \( \Omega_1 \) (resp. \( \Omega_2 \)) itself.

For each \( 0 \leq i \leq p \), denote by \( z_i \) the landing point of \( I_1(2^i \theta) \) and by \( w_i \) the landing point of \( I_2(2^i(1 - \theta)) = I_2(1 - 2^i \theta) \). Since \( \theta \) is \( p \)-periodic, the points \( z_0, \ldots, z_{p-1} \) (resp. \( w_0, \ldots, w_{p-1} \)) are pairwise disjoint and \( z_0 = z_p \) (resp. \( w_0 = w_p \)). Moreover, the assumption of \( \theta \) implies that \( z_0, \ldots, z_{p-1}, w_0, \ldots, w_{p-1} \in W \). Suppose \( \theta \notin \Theta \). Then \( z_0 \neq w_0 \). As \( \Theta \) is closed, we can choose an arc \( \ell_0 \) in \( W \setminus \{ I_1(t) \cup I_2(1 - t) : t \in \Theta \} \) joining the points \( z_0 = z_p \) and \( w_0 = w_p \) such that \( \ell_0 \) is disjoint with \( \Omega_1 \cup \Omega_2 \). Let \( \ell_1 \) be the lift of \( \ell_0 \) based at \( z_{p-1} \). By the choice of \( \ell_0 \), we have

\[
\ell_1 \subset W \setminus \{ I_1(t) \cup I_2(1 - t) : t \in \Theta \}
\]

and

\[
\ell_1 \cap (\Omega_1 \cup \Omega_2) = \emptyset.
\]

Note that the endpoint of \( \ell_1 \) is on the boundary of a preimage of \( \Omega_2 \). By the previous paragraph, this preimage is \( \Omega_2 \) itself. Note also that \( w_{p-1} \) is the unique preimage of \( w_p \) on \( \partial \Omega_2 \) such that \( w_{p-1} \) and \( z_{p-1} \) are in the same component of \( W \setminus (I_1(1/2) \cup I_2(1/2)) \). Hence the endpoint of \( \ell_1 \) is \( w_{p-1} \).

Inductively, for each \( m \geq 1 \), we get an arc \( \ell_{mp} \subset W \) joining \( z_0 \) and \( w_0 \) which is a lift of \( \ell_0 \) by \( f^{mp} \). Choose \( \ell_0 \) such that it does not intersect the closure of the forward orbits of the critical points of \( f \). Since \( f \) is hyperbolic, it is uniformly expanding near its Julia set. It follows that the length of \( \ell_{mp} \) converges to 0 as \( m \to \infty \). Then \( z_0 = w_0 \), a contradiction. Hence \( \theta \in \Theta \) and statement (2) follows.

Now we prove statement (3). Note that \( \alpha \in (0, 1/2] \). Suppose, on the contrary, that \( \alpha = 1/2 \). According to statement (1), the angles \( 1 - 1/2^n \in \Theta \) for all \( n \geq 1 \). Choose an angle \( \eta \in \Theta \) close to 1 and define

\[
\gamma(0, \eta) := I_1(0) \cup I_1(\eta) \cup I_2(1 - \eta) \cup I_2(0).
\]

Let \( D_\eta \) be component of \( \hat{\mathbb{C}} \setminus \gamma(0, \eta) \) contained in \( D \). We can choose \( \eta \) sufficiently close to 1 such that \( D_\eta \) contains no critical values of \( f \). Since \( \alpha = 1/2 \), then \( I_1(\eta/2) \) and \( I_2(1 - \eta/2) \) land at distinct points. Denote by \( \Omega_1^{(1)} \) the component of \( f^{-1}(\Omega_1) \) such that \( I_2(1 - \eta/2) \) and \( I_1^{(1)}(\eta) \) land at a common point and denote by \( \Omega_2^{(1)} \) the component of \( f^{-1}(\Omega_2) \) such that \( I_1(\eta/2) \) and \( I_2^{(1)}(1 - \eta) \) land at a common point. Since \( f \) is hyperbolic, its Julia set contains no critical points. It follows that there exists a component \( D_\eta' \) of \( f^{-1}(D_\eta) \) whose boundary contains the arc

\[
I_1(1/2) \cup I_2(1/2) \cup I_1(\eta/2) \cup I_2(1 - \eta/2) \cup I_2^{(1)}(1 - \eta) \cup I_1^{(1)}(\eta).
\]

Note that the two arcs \( I_1(\eta/2) \cup I_2^{(1)}(1 - \eta) \) and \( I_2(1 - \eta/2) \cup I_1^{(1)}(\eta) \) are disjoint and mapped to the same arc \( I_1(\eta) \cup I_2(1 - \eta) \) under \( f \). Then the proper map \( f : D_\eta' \to D_\eta \) has degree at least 2. It implies that \( D_\eta' \) contains at least one critical point. Hence \( D_\eta \) contains a critical value. It contradicts to the choice of \( D_\eta \).
For the second part of statement (3), let \( \theta_n := 1 - 1/(2^n - 1) \). Then \( \theta_n \) is periodic with period \( n \). If \( 0 \leq i < n - 1 \), we have
\[
2^i \theta_n = 1 - 2^i/(2^n - 1) \in (1/2, 1).
\]
For \( i = n - 1 \), we have
\[
2^{n-1} \theta_n = \frac{1}{2} \left(1 - \frac{1}{2^{n-1}}\right) \in (0, 1/2).
\]
Since \( \alpha < 1/2 \), it follows that \( 2^{n-1} \theta_n \in (\alpha, 1) \) for sufficiently large \( n \). Then \( \theta_n \in \Theta \) by statement (2), and hence \( 2^{n-1} \theta_n \) is as required. \( \square \)

5. The boundedness of hyperbolic components

In this section, we aim to prove Theorem 1.2. In Section 5.1, we classify the hyperbolic components into several types and state known boundedness results. Section 5.2 contains two key lemmas for the proof of Theorem 1.2: one concerns the orbit of a critical point and the limit of an attracting cycle; the other one concerns the combinatorial property of the limit function. Then we prove Theorem 1.2 in Section 5.3.

5.1. Classification of hyperbolic components and known results. Let \( f \in \text{NM}_4 \) be the Newton map of the quartic polynomial \( P \). Then the finite fixed points of \( f \) are the zeros of \( P \), and the critical points of \( f \) are the zeros of \( P \) and zeros of \( P'' \). Hence zeros of \( P \) are the superattracting fixed points of \( f \). We call any other (super)attracting cycles of \( f \) is a free (super)attracting cycle. Then any free (super)attracting cycle has period at least 2. Moreover, we say a critical point \( c \) of \( f \) is additional if \( P''(c) = 0 \). Hence \( f \) has two additional critical points, counted with multiplicity. According to the orbits of the additional critical points, the hyperbolic components in the moduli space \( \text{nm}_4 := \text{NM}_4/\text{Aut}(\mathbb{C}) \) belong to the following seven types, see [23]. Same classification is also for hyperbolic components in \( \text{NM}_4 \).

**Type A. Adjacent critical points.** The two additional critical points belong to the same component of the immediate basin of a free (super)attracting cycle.

**Type B. Bitransitive.** Each of the two additional critical points belongs to the immediate basin of a free (super)attracting period cycle, with two distinct components.

**Type C. Capture.** Only one additional critical point belongs to the immediate basin of a free (super)attracting cycle, but the orbit of the other additional critical point eventually lies in this immediate basin.

**Type D. Disjoint (super)attracting orbits.** The two additional critical points belong to the immediate basins of two distinct free (super)attracting cycles.

**Type IE. Immediate Escape.** Some additional critical point in the immediate basin of a root.

**Type FE1. One Future Escape.** Only one additional critical point in the basin (but not immediate basin) of a root, while the other additional critical point is in the immediate basin of a free (super)attracting cycle.

**Type FE2. Two Future Escape.** The two additional critical points belong to the basins (but not immediate basins) of one or two roots.

The above classification is an analogy of that for quadratic rational maps [19] and for cubic polynomials [21].
Recall that a hyperbolic component in \( nm_4 \) is bounded if it has a compact closure in \( nm_4 \). Since the type D hyperbolic components have semi-algebraic boundaries, an arithmetic argument shows that such components are bounded:

**Proposition 5.1** ([23, Main Theorem]). The hyperbolic components of type D in \( nm_4 \) are bounded.

In contrast, all hyperbolic components of type IE are unbounded.

**Proposition 5.2** ([23, Theorem 1.4]). Let \( \mathcal{H} \subset nm_4 \) be a hyperbolic component. If \( \mathcal{H} \) is of type IE, then \( \mathcal{H} \) is unbounded in \( nm_4 \).

In the remainder of this section, we give more bounded hyperbolic components in \( nm_4 \). In fact, we show the condition in Proposition 5.2 is also necessary.

### 5.2. Key lemmas

To prove Theorem 1.2, we need two key lemmas.

Let \( \{ f_n \} \subset NM_4 \) be a sequence converging to \( f = H_f \hat{f} \in \overline{NM}_4 \) such that \( \text{Hole}(f) = \{ \infty \} \) and \( \deg \hat{f} = 3 \). Then \( f_n \) has a unique additional critical point \( c_n \) converging to \( \infty \) as \( n \to \infty \). We suppose that all \( f_n \)s are in a same hyperbolic component in \( NM_4 \) and assume that \( f_n \) has an attracting cycle \( \mathcal{O}_n = \{ w_n^{(0)}, \ldots, w_n^{(m-1)} \} \) of period \( m \geq 2 \). Our first lemma states the orbit of \( c_n \) and the limit of \( \mathcal{O}_n \).

**Lemma 5.3.** Let \( f_n, f, c_n \) and \( \mathcal{O}_n \) be as above. Then the following statements holds:

1. Given any \( k \geq 0 \) and small \( \epsilon > 0 \), the points \( c_n, f_n(c_n), \ldots, f_n^k(c_n) \) are in the \( \epsilon \)-neighborhood of \( \infty \) for all large \( n \).
2. Suppose \( \mathcal{O}_n \) converges to \( \mathcal{O} \) as \( n \to \infty \). Then \( \mathcal{O} \neq \{ \infty \} \).
3. If \( \infty \in \mathcal{O} \), then \( c_n \) is not in the immediate basin of \( \mathcal{O}_n \).

**Proof.** Denote by \( r_{1,n}, r_{2,n}, r_{3,n} \) and \( r_{4,n} \) the roots of \( f_n \). Since \( \text{Hole}(f) = \{ \infty \} \) and \( \deg \hat{f} = 3 \), we may assume \( r_{4,n} \to \infty \), as \( n \to \infty \), and for \( 1 \leq i \leq 3 \), the point \( r_{i,n} \) is outside the \( \epsilon \)-neighborhood of \( \infty \) for all large \( n \). Define \( M_n(z) := r_{4,n} z \) and let \( g_n := M_n^{-1} \circ f_n \circ M_n \). Then \( g_n \in NM_4 \) with roots at \( r_{1,n}/r_{4,n}, r_{2,n}/r_{4,n}, r_{3,n}/r_{4,n} \) and 1. Let \( g = H_f \hat{g} \) be the degenerate Newton map of the polynomial \( z^3(z-1) \). Then \( g_n \) converges locally uniformly to \( \hat{g} \) away from \( \text{Hole}(g) = \{ 0 \} \). Note that \( \hat{g} \) has a critical point at \( \hat{c} = 1/2 \) and \( \hat{c} \) is attracted to the attracting fixed point 0. Given any \( k \geq 0 \), the point \( \hat{c} \) is not in \( \text{Hole}(g^k) = \cup_{i=0}^{k-1} \hat{g}^{-i}(0) \). It follows that there exists \( \epsilon_0 = \epsilon_0(k) > 0 \) such that \( |\hat{g}^j(\hat{c})| > \epsilon_0 \) for all \( 0 \leq j \leq k \). By Lemma [2.1](#), we have \( |g_n^k(c_n)| > \epsilon_0 \) for all large \( n \). Note that for the maps \( f_n \), we have \( f_n^k(c_n) = M_n(g_n^k(c_n)) \). It follows that \( |f_n^k(c_n)| > r_{4,n} \epsilon_0 \). Thus, statement (1) follows.

For statement (2), suppose to the contrary that \( \mathcal{O} = \{ \infty \} \). Then all \( w_i^{(0)} \)'s converge to \( \infty \). In the following argument, we may pass to subsequences if necessary to obtain limits. Relabeling the indices, we may assume \( w_i^{(0)}/w_i^{(0)} \) dose not converge to 0 for all \( 0 \leq i \leq m - 1 \). Write \( L_n(z) = w_n^{(0)} z \). Then

\[
\mathcal{O}' := \{ 1, \frac{w_n^{(1)}}{w_n^{(0)}}, \ldots, \frac{w_n^{(n-1)}}{w_n^{(0)}} \}
\]

is an attracting cycle of \( h_n := L_n^{-1} \circ f_n \circ L_n \in NM_4 \). Denote by \( \mathcal{O}' \) the limit of \( \mathcal{O}' \). Then \( 0 \notin \mathcal{O}' \). Assume that \( h_n \to h = H_f \hat{h} \in NM_4 \). Note that \( \text{Hole}(h) \subset \{ 0, \infty \} \) and \( 1 \leq \deg \hat{h} \leq 2 \).
If \( \deg \hat{h} = 1 \), then at least three roots of \( h_n \) collide to 0 as \( n \to \infty \) and the remaining root either collides to 0 or diverges to \( \infty \). For otherwise, \( \hat{h} \) would have degree 2. It follows that \( \hat{h}(z) = 3z/4 \) or \( h(z) = 2z/3 \). Thus, \( \hat{h} \) has an attracting fixed point at 0 and a repelling fixed point at \( \infty \). Moreover, \( \text{Hole}(h^j) = \text{Hole}(h) \subset \{ 0, \infty \} \) for all \( j \geq 1 \). It follows that \( \mathcal{O}' \cap \text{Hole}(h) = \emptyset \). Then by Lemma 2.3 the set \( \mathcal{O}' \) is a non-repelling cycle of \( \hat{h} \). Note 1 \( \in \mathcal{O}' \) is not a fixed point of \( \hat{h} \). It is a contradiction since all the periodic points of \( \hat{h} \) are fixed points.

If \( \deg \hat{h} = 2 \), then \( \text{Hole}(h) = \{ 0 \} \). Moreover, \( \hat{h} \) has an attracting fixed point at 0, a superattracting fixed point at the limit cycle of \( \hat{h} \) and a repelling fixed point at \( \infty \). Since \( 0 \notin \mathcal{O}' \), then \( \mathcal{O}' \cap \text{Hole}(h) = \emptyset \). By Lemma 2.3 the set \( \mathcal{O}' \) is a non-repelling cycle of \( \hat{h} \) of period at least 2. It follows that \( \hat{h} \) has at most 3 non-repelling cycles: two (super)attracting fixed points 0 and \( \infty \), and one non-repelling cycle \( \mathcal{O}' \). It contradicts to the Fatou-Shishikura inequality (see [27]) which asserts that \( \hat{f} \) has at most 2 non-repelling cycles. Therefore, we have \( \mathcal{O} \neq \{ \infty \} \) and the conclusion follows.

Now we prove statement (3). For \( 0 \leq j \leq m - 1 \), denote by \( U(w_n^{(j)}) \) the Fatou component of \( f_n \) containing \( w_n^{(j)} \). Suppose to the contrary that \( c_n \in \bigcup_{j=0}^{m-1} U(w_n^{(j)}) \).

Then \( c_n \in U(w_n^{(j_0)}) \) for some \( 0 \leq j_0 \leq m - 1 \). By relabeling the index, we can assume that \( j_0 = 0 \). If \( w_n^{(0)} \to \infty \), by statement (2), there exists \( 1 \leq j \leq m - 1 \) such that \( w_n^{(j)} \not\to \infty \). It follows from Lemma 2.4 that the basin \( U(w_n^{(j)}) \) stays outside a neighborhood of \( \infty \) for all large \( n \). Since \( f_n^j(c_n) \in U(w_n^{(j)}) \), statement (1) implies that \( c_n \not\to \infty \). If \( w_n^{(0)} \not\to \infty \), some \( w_n^{(\ell)} \) with \( 1 \leq \ell \leq m - 1 \) must converge to \( \infty \) since \( \infty \in \mathcal{O} \). Again by Lemma 2.4 the basin \( U(w_n^{(\ell)}) \) stay outside a neighborhood of \( \infty \) for all large \( n \). Hence \( c_n \not\to \infty \). It contradicts to the assumption that \( c_n \to \infty \). Hence \( c_n \notin \bigcup_{j=0}^{m-1} U(w_n^{(j)}) \).

\[ \Box \]

**Corollary 5.4.** Let \( f_n, f, c_n, \mathcal{O}_n \) and \( \mathcal{O} \) be as in Lemma 5.3 and let \( \hat{\mathcal{H}} \subset \text{NM}_4 \) be the hyperbolic component containing \( f_n \)'s. Assume \( \hat{\mathcal{H}} \) is of type A, B, C or D. If \( c_n \) is in the basin of \( \mathcal{O}_n \), then \( c_n \notin \mathcal{O} \).

**Proof.** If \( \hat{\mathcal{H}} \) is of type A, B or D, then \( c_n \) is in the immediate basin of \( \mathcal{O}_n \). By Lemma 5.3 (3), it follows that \( \mathcal{O} \subset \mathcal{C} \). If \( \mathcal{H} \) is of type C, suppose \( \infty \in \mathcal{O} \). By Lemma 5.3 (2), there exist periodic points \( w_n^{(i)} \) and \( w_n^{(j)} \) in \( \mathcal{O}_n \) such that \( w_n^{(i)} \to \infty \) but \( w_n^{(j)} \not\to \infty \). It follows from Lemma 2.4 that the basin \( U(w_n^{(j)}) \) stays outside a neighborhood of \( \infty \) for all large \( n \). Moreover, by Lemma 5.3 (3), the critical point \( c_n \) is not in the immediate basin of \( \mathcal{O}_n \). Then there exists \( k \), independent of \( n \), such that \( f_n^k(c_n) \in U(w_n^{(j)}) \). It contradicts to Lemma 5.3 (1). Hence \( \infty \notin \mathcal{O} \).

\[ \Box \]

Recall from Section 4.3 that a quartic Newton map \( f \in \text{NM}_4 \) is of separable type if \( f \) has two distinct immediate basins \( \Omega_i \) and \( \Omega_j \) of roots such that the corresponding internal rays \( I_i(1/2) \in \Omega_i \) and \( I_j(1/2) \in \Omega_j \) land at a pole and the curve \( I_i(0) \cup I_i(1/2) \cup I_j(1/2) \cup I_j(0) \) separates the remaining poles of \( f \). We say a hyperbolic component \( \mathcal{H} \) of \( \text{nm}_4 \) is of **separable type** if each element in \( \mathcal{H} \) is of separable type, equivalently, there is an element of separable type in \( \mathcal{H} \). Otherwise, we say \( \mathcal{H} \) is of **inseparable type**.

Our next key lemma asserts that a non type IE hyperbolic component is of inseparable type under extra assumption on its lift.
Lemma 5.5. Let \( \mathcal{H} \subset \mathbb{N}M_4 \) be a non type IE hyperbolic component and let \( \widetilde{\mathcal{H}} \subset \mathbb{N}M_4 \) be a lift of \( \mathcal{H} \). Suppose there exists a sequence \( \{ f_n \} \subset \widetilde{\mathcal{H}} \) such that \( f_n \) converges to \( f = H_I \tilde{f} \in \mathbb{N}M_4 \) with \( \text{Hole}(f) = \{ \infty \} \) and \( \deg(f) = 3 \). Then \( \mathcal{H} \) is of inseparable type. Moreover, all poles of \( \tilde{f} \) are simple.

Proof. By the assumptions, \( \tilde{f} \) has three roots, denoted by \( r_1, r_2 \) and \( r_3 \) respectively. Let \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) be the corresponding immediate basins. Moreover, since \( \mathcal{H} \) is of non type IE, the map \( \tilde{f} \) has a unique critical point \( c \) with \( c \not\in \bigcup_{i=1}^3 \Omega_i \). By Lemma 3.3, the sequence \( f_n \), converges \( f \) on \( \{ \Omega_1, \Omega_2, \Omega_3 \} \) under the dynamically weak Carathéodory topology. Relabeling \( r_i,s \), we may assume that there exists a pole of \( \tilde{f} \) in the intersection \( \partial \Omega_1 \cap \partial \Omega_2 \). For \( 1 \leq i \leq 3 \), denote by \( (\Omega_i, r_{i,n}) \) the deformation of \( (\Omega_i, r_i) \) at \( f_n \). Then each \( r_{i,n} \) is a root of \( f_n \) and \( \partial \Omega_{1,n} \cap \partial \Omega_{2,n} \) contains a pole of \( f_n \). Let \( r_{4,n} \) be the remaining root of \( f_n \). Then \( r_{4,n} \rightarrow \infty \), as \( n \rightarrow \infty \). Denote by \( \Omega_{4,n} \) its immediate basin.

On the contrary, we assume \( \mathcal{H} \) is of separable type. Consider the internal rays in \( \Omega_{1,n} \) and \( \Omega_{2,n} \). Set

\[
\gamma_n(0, 1/2) := I_{1,n}(0) \cup I_{1,n}(1/2) \cup I_{2,n}(0) \cup I_{2,n}(1/2).
\]

Then each component of \( \widetilde{\mathcal{C}} \setminus \gamma_n(0, 1/2) \) contains a pole of \( f_n \), and hence contains \( \Omega_{3,n} \) or \( \Omega_{4,n} \). We denote by \( D_n \) the one containing \( \Omega_{4,n} \), and assume that \( I_{1,n}(\theta) \subseteq D_n \) if and only if \( \theta \in (1/2, 1) \).

Since \( \Omega_{4,n} \subset D_n \), there exists a minimal \( k \geq 2 \) such that the landing point \( z_n \) of \( I_{2,n}(1/2^k) \) is not in \( \partial \Omega_{1,n} \). Let \( \Omega_{1,n}^{(1)} \) be the component of \( f_n^{-1}(\Omega_{1,n}) \) such that \( z_n \in \partial \Omega_{1,n}^{(1)} \).

Then \( \Omega_{1,n}^{(1)} \neq \Omega_{1,n} \) and \( \Omega_{1,n}^{(1)} \subset D_n \). Note that \( \Omega_{1,n}^{(1)} \) contains no critical points. For otherwise, \( \partial \Omega_{1,n}^{(1)} \) and hence \( D_n \) would contain two poles of \( f_n \), which is impossible.

Then \( \partial \Omega_{1,n}^{(1)} \) contains a unique pole of \( f_n \), which coincides with the one on \( \partial \Omega_{4,n} \). Set \( I_{1,n}^{(1)}(t) \) the internal ray in \( \Omega_{1,n}^{(1)} \) landing at \( z_n \). By Proposition 3.6, the landing point \( z_n \) of \( I_{2,n}(1/2^k) \) converges to the landing point \( z \) of \( I_2(1/2^k) \). Note that the pole of \( f_n \) in \( \partial \Omega_{1,n} \cap \partial \Omega_{2,n} \) (resp. \( \partial \Omega_{3,n} \)) converges to the pole of \( \tilde{f} \) in \( \partial \Omega_1 \cap \partial \Omega_2 \) (resp. \( \partial \Omega_3 \)).

Thus, the pole of \( f_n \) in \( \partial \Omega_{4,n} \cap \partial \Omega_{1,n}^{(1)} \) converges to \( \infty \), as \( n \rightarrow \infty \). For otherwise, these poles converge to poles of \( \tilde{f} \), contradicting to \( \deg(\tilde{f}) = 3 \). Similarly, the center of \( \Omega_{1,n}^{(1)} \) converges to \( \infty \). Then, passing to subsequences if necessary, we have that the arcs \( I_{1,n}^{(1)}(t) \) converge to a continuum \( \ell \) containing \( \infty \) and \( z \).

Recall that \( \psi_{1,n}^{(1)} : \mathbb{D} \rightarrow \Omega_{1,n}^{(1)} \) and \( \psi_{1,n} : \mathbb{D} \rightarrow \Omega_{1,n} \) are the inverses of the Böttcher coordinates on \( \Omega_{1,n}^{(1)} \) and \( \Omega_{1,n} \), respectively. Let \( q \) be any point in \( \ell \setminus \{ \infty \} \). There exists \( q_n \in I_{1,n}^{(1)}(t) \) with \( q_n \rightarrow q \). We write \( q_n = \psi_{1,n}^{(1)}(s_ne^{2\pi it}) \). Since \( q \neq \infty \), we have \( f_n(q_n) \rightarrow \tilde{f}(q) \). Note that

\[
f_n(q_n) = f_n \circ \psi_{1,n}^{(1)}(s_ne^{2\pi it}) = \psi_{1,n}(s_ne^{2\pi it}) \in I_{1,n}(t).
\]

Since \( I_{1,n}(t) \rightarrow I_1(t) \), the point \( \tilde{f}(q) \) belongs to \( I_1(t) \). We claim in fact that \( \tilde{f}(q) \in \partial \Omega_1 \). Otherwise, \( q \) belongs to either \( \Omega_1 \) or the other component \( \Omega_{1,n}^{(1)} \) of \( f^{-1}(\Omega_1) \). Note that \( \Omega_{1,n}^{(1)} \cap D_n = \emptyset \) for large \( n \). By Lemma 2.6, we have \( q_n \not\in \Omega_{1,n} \). It is a contradiction. By this claim, any point in \( \ell \setminus \{ \infty \} \) maps under \( \tilde{f} \) to the landing point of \( I_1(t) \). It is impossible. Thus, \( \mathcal{H} \) is of inseparable type.
Now we show all poles of $\hat{f}$ are simple. Let $\Theta$ be the set of angles $\theta$ such that $I_{1,n}(\theta)$ and $I_{2,n}(1-\theta)$ land at a common point. Since $\mathcal{H}$ is of inseparable type, by Proposition 4.3 (3), there exists a periodic angle $\theta \in \Theta \cap (0,1/2)$. According Proposition 3.6, the internal rays $I_1(\theta)$ and $I_2(1-\theta)$ land at a common point. This implies $c$ cannot be a pole of $\hat{f}$: since otherwise $c$ is a common point of $\partial\Omega_i$, $i = 1, 2, 3$, impossible.

5.3. Proof of Theorem 1.2. To prove Theorem 1.2, we first state the following lift result.

Lemma 5.6. For $d \geq 3$, let $[g_n] \in \text{nm}_d$ be a sequence such that $[g_n] \to \infty$. Then there exists a sequence $f_n \in \text{NM}_d$ such that $[f_n] = [g_n]$ and $f_n$ converges to $f = H_f \hat{f} \in \partial\text{NM}_d$ with $\text{Hole}(f) = \{\infty\}$ and $\deg \hat{f} \geq 2$. Moreover, if all $[g_n]$s are contained in a same hyperbolic component in $\text{nm}_d$, then $f_n$s are contained in a same hyperbolic component in $\text{NM}_d$.

Proof. Since $[g_n] \to \infty$, there exists a subsequence $g_{n_i}$ such that $g_{n_i}$ converges to an element in $\partial\text{NM}_d$. We first normalize the roots of $g_{n_i}$ by affine maps to obtain a sequence $\tilde{g}_{n_i} \in \text{NM}_d$ such that 0 and 1 are two roots of $\tilde{g}_{n_i}$. Note $[\tilde{g}_{n_i}] = [g_{n_i}]$. It follows that $[\tilde{g}_{n_i}] \to \infty$ and hence $\{\tilde{g}_{n_i}\}$ contains a subsequence converging to an element in $\partial\text{NM}_d$. We also denote this subsequence by $\{\tilde{g}_{n_i}\}$. We can further assume all roots of $\tilde{g}_{n_i}$ converge in $\hat{\mathbb{C}}$. Denote by $r_{1,n_i}, \ldots, r_{d,n_i}$ the roots of $\tilde{g}_{n_i}$. Choose $1 \leq m_0 < m_1 \leq d$ such that $|r_{m_0,n_i} - r_{m_1,n_i}| = O(|r_{\ell,n_i} - r_{k,n_i}|)$ for all $1 \leq \ell < k \leq d$ with $r_{\ell,n_i} \not\to \infty$ and $r_{k,n_i} \not\to \infty$, as $n_i \to \infty$. Define $M_{n_i}(z) := \frac{z - r_{m_1,n_i}}{r_{m_0,n_i} - r_{m_1,n_i}}$ and set $f_{n_i} := M_{n_i} \circ \tilde{g}_{n_i} \circ M_{n_i}^{-1}$. Then $f_{n_i}$ has roots at 0, 1 and no roots colliding in $\mathbb{C}$. Then the sequence $f_{n_i}$ is the desired sequence.

Now we claim that each hyperbolic component in $\text{nm}_d$ has a unique lift in $\text{NM}_d$. Indeed, if $[g] \in \text{nm}_d$ is contained in a hyperbolic component, let $f \in \text{NM}_d$ be such that $[f] = [g]$ and $f \neq g$. There exists an affine map $M(z)$ such that $g = M \circ f \circ M^{-1}$. Let $\gamma : [0,1] \to \text{Aut}(\mathbb{C})$ be a curve such that $\gamma(0) = id$ and $\gamma(1) = M$. We obtain a curve $g_t := \gamma(t) \circ g \circ (\gamma(t))^{-1} \in \text{NM}_d$
of hyperbolic Newton maps with $g_0 = g$ and $g_1 = f$. Then the claim holds. Thus if $[g_n]$s are contained in a hyperbolic component $\mathcal{H} \subset \text{nm}_d$, the above claim implies immediately that $f_{n_i}$s are contained in the unique lift $\hat{\mathcal{H}} \subset \text{NM}_d$ of $\mathcal{H}$.

Proof of Theorem 1.2. By Proposition 5.2, it suffices to show that if $\mathcal{H} \subset \text{nm}_4$ is not of type IE, then $\mathcal{H}$ is bounded in $\text{nm}_4$. The proof goes by contradiction.

Suppose $\mathcal{H}$ is unbounded. Let $\{(f_n)\}_{n \geq 0}$ be a degenerated sequence in $\mathcal{H}$. Passing to a subsequence, by Lemma 5.6, we can assume that all $f_n$ belong to a hyperbolic component in $\text{NM}_4$, and $f_n$ converges to $f = H_f \hat{f} \in \text{NM}_4$ with $\text{Hole}(f) = \{\infty\}$ and $\deg \hat{f} = 2$ or 3. We deduce the contradiction case by case.

Case 1: $\deg \hat{f} = 2$.

Let $(\Omega_1, r_1)$ and $(\Omega_2, r_2)$ be the immediate basins of roots of $\hat{f}$. By Lemma 3.3, we have that $f_n$ converges to $f$ on $\{(\Omega_1, \Omega_2)\}$ under the dynamically weak Carathéodory
topology. Denote by $(\Omega_{1,n}, r_{1,n})$ and $(\Omega_{2,n}, r_{1,n})$ the deformations of $(\Omega_1, r_1)$ and $(\Omega_2, r_2)$ at $f_n$, respectively. In this case, the Julia set of $\hat{f}$ is
\[ J_f = \partial \Omega_1 = \partial \Omega_2, \]
which is a Jordan curve and contains no critical points. Given any rational angle $\theta$, the internal rays $I_1(\theta)$ and $I_2(1-\theta)$ land at a common point. By Theorem 1.1 for all large $n$, the internal rays $I_{1,n}(\theta)$ and $I_{1,n}(1-\theta)$ land at a common point. Since all $f_n$ belong to the same hyperbolic component, we get that the internal rays $I_{1,0}(t)$ and $I_{2,0}(1-t)$ of $f_0$ land together for all $t \in \mathbb{Q}$. Then the boundaries $\partial \Omega_{1,0}$ and $\partial \Omega_{1,0}$ coincide. It follows that $f_0$ is conjugate to $z \mapsto z^2$, which is a contradiction.

Case 2: deg $\hat{f} = 3$.

In this case, $\hat{f} \in \text{NM}_3$. Moreover, the unique additional critical point $\hat{c}$ of $\hat{f}$ is not in the immediate basins of the roots of $\hat{f}$. For otherwise, $f_n$ would possess an additional critical point in the immediate basin of some root, which is a contradiction.

Let $c_n$ be the additional critical point of $f_n$ such that $c_n$ converges to $\infty$. Now we proceed our argument according to the type of $\mathcal{H}$.

Case 2.(i): $\mathcal{H}$ is of type A,B,C or D.

Let $\mathcal{O}_n$ be the free (super)attracting cycle of $f_n$ such that $c_n$ is in the basin of $\mathcal{O}_n$. Denote $\mathcal{O}$ the limit of $\mathcal{O}_n$. By Corollary 2.3, we have that $\mathcal{O} \subseteq \mathbb{C}$. Then by Lemma 2.3, the set $\mathcal{O}$ is a non-repelling cycle of $\hat{f}$ of period at least 2. It follows that the critical point $\hat{c}$ is not an iterated preimage of $\infty$ under $\hat{f}$. Moreover, $\hat{f}$ is postcritically finite on $\Omega_f$.

Consider the Newton graph $\Delta_m(\hat{f})$ of $\hat{f}$ at level $m$. By Proposition 4.2, for a sufficiently large $m$, there exists a Jordan curve $\gamma \subseteq \Delta_m(\hat{f})$ such that the orbit $\mathcal{O}$ is contained in the bounded component of $\hat{C} \setminus \gamma$. Let $\mathcal{U}$ be the collection of components of $\Omega_f$ intersecting $\Delta_m(\hat{f})$. Then $\hat{f}(U) \in \mathcal{U}$ for $U \in \mathcal{U}$. By Lemma 3.3, the sequence $f_n$ converges to $f$ on $U$ under the dynamically weak Carathéodory topology.

Set $\delta := d_H(\infty, \gamma)$. By Theorem 1.1, the curve $\gamma$ is perturbed to a Jordan curve $\gamma_n \subseteq \Delta_m(f_n)$ such that $\mathcal{O}_n$ is contained in the bounded component of $\hat{C} \setminus \gamma_n$ and $d_H(\gamma_n, \gamma) < \delta/3$ for all large $n$. Since the immediate basin of $\mathcal{O}_n$ is disjoint with $\Delta_m(f_n)$ for all $n$, it is contained in the bounded component of $\hat{C} \setminus \gamma_n$.

If $\mathcal{H}$ is of type A, B or D, then the critical point $c_n$ is in the immediate basin of $\mathcal{O}_n$. The above argument immediately implies that the distance between $c_n$ and $\infty$ is at least $\delta/3$, a contradiction to $c_n \to \infty$.

If $\mathcal{H}$ is of type C, since critical point $c_n$ converges to $\infty$, the above argument implies that $c_n$ is not in the immediate basins of $\mathcal{O}_n$. In this case, there exists $k > 0$ such that $f_k^n(c_n)$ belongs to the immediate basin of $\mathcal{O}_n$ for all $n$, which stays outside the $\delta/3$ neighborhood of $\infty$. It contradicts to Lemma 5.3 (1).

Case 2.(ii): $\mathcal{H}$ is of type FE1 or FE2.

In priori, differing from Case 2.(i), the additional critical point of $\hat{f}$ may be an iterated preimage of $\infty$. So the assumptions of Theorem 1.1 may fail for the Newton graphs of $\hat{f}$. Alternatively, we apply Theorem 1.1 to the Jordan curve $\mathcal{C}$ constructed in Section 4.2 in the following argument.

By Lemma 5.5, the additional critical point $\hat{c}$ of $\hat{f}$ is not a pole. We can thus use the results in Section 4.2. Inheriting the notations in Section 4.2, by Lemma 4.4, we obtain
a Jordan curve $C$ consisting of some internal rays in $\Omega_1, \Omega_2, \Omega_3, \Omega^{(2)}_1$ and $\Omega^{(2)}_2$ such that
the orbits of the landing points of these rays are disjoint with the critical points of $\hat{f}$
and the bounded component of $\hat{C} \setminus C$ contains $\Omega^{(1)}_1, \Omega^{(1)}_2, \Omega^{(1)}_3, c$ and the poles of $\hat{f}$.

Set
\[ \mathcal{U} := \{\Omega_1, \Omega_2, \Omega_3, \Omega^{(1)}_1, \Omega^{(1)}_2, \Omega^{(2)}_1, \Omega^{(2)}_2\}. \]

Then $\hat{f}(U) \in \mathcal{U}$ for $U \in \mathcal{U}$. Moreover, by Lemma 3.3 we have $f_n$ converges to $f$
on $\mathcal{U}$ under the dynamically weak Carathéodory topology. By applying Theorem 1.1
to $C$, for all large $n$, we obtain a Jordan curve $C_n$ consisting of internal rays of $f_n$
in $\Omega_{1,n} \cup \Omega_{2,n} \cup \Omega_{3,n} \cup \Omega^{(2)}_{1,n} \cup \Omega^{(2)}_{2,n}$ with the same angles as those of $\hat{f}$ in $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega^{(2)}_1 \cup \Omega^{(2)}_2$.

Then the bounded component of $\hat{C} \setminus C_n$ contains $\Omega^{(1)}_{1,n}, \Omega^{(1)}_{2,n}, \Omega^{(1)}_{3,n}$, two poles of $f_n$ and
the closures of the two preimages of $\Omega_{4,n}$ disjoint with $\Omega_{4,n}$. Moreover, the unbounded
component of $\hat{C} \setminus C_n$ contains $\Omega_{4,n}$.

For the additional critical point $c_n$ of $f_n$ with $c_n \to \infty$, we claim that there exists a minimal integer $k \geq 1$ such that
\[ f^n_i(c_n) \in \Omega_{1,n} \cup \Omega_{2,n} \cup \Omega_{3,n} \cup \Omega_{4,n}. \]

To prove this claim, it suffices to consider the case that $f_n$ has a free (super)attracting
cycle $\mathcal{O}_n$. Suppose $\mathcal{O}_n$ converges to $\mathcal{O}$. If $\infty \in \mathcal{O}$, the claim follows from Lemma 5.3
(3). If $\mathcal{O} \subseteq \mathbb{C}$, by Lemma 2.3 the set $\mathcal{O}$ is the non-repelling cycle of $\hat{f}$ of period at
least 2. It follows that $\hat{f}^j(c) \neq \infty$ for all $j \geq 0$. Moreover, $\hat{f}$ is postcritically finite in
the basins of the roots. With the same argument in Case 2.(i), we obtain that the immediate basin of $\mathcal{O}_n$ is disjoint with a fixed neighborhood of $\infty$. Hence the claim follows since $c_n \to \infty$.

By Lemma 5.3 (1), for each $0 \leq i \leq k - 1$ and all large $n$, the Fatou component
$U(f^n_i(c_n))$ containing $f^n_i(c_n)$ is not contained in the bounded domain of $\hat{C} \setminus C_n$. Furthermore, none of these Fatou components intersects $C_n$. Indeed, if $U(f^n_i(c_n))$ intersects
$C_n$ for some $0 \leq i \leq k - 1$, then $U(f^n_i(c_n))$ coincides with either $\Omega^{(2)}_{1,n}$ or $\Omega^{(2)}_{2,n}$. It then
follows that $U(f^{i+1}_n(c_n))$ coincides with either $\Omega^{(1)}_{1,n}$ or $\Omega^{(1)}_{2,n}$. Note $\Omega^{(1)}_{1,n}$ and $\Omega^{(1)}_{2,n}$ are both
in the bounded component of $\hat{C} \setminus C_n$. It contradicts to Lemma 5.3 (1). Therefore, for
$0 \leq i \leq k - 1$, the component $U(f^n_i(c_n))$ is contained in the unbounded component of
$\hat{C} \setminus C_n$.

By previous argument, the closure of any non-fixed preimage of $\Omega_{1,n}, \Omega_{2,n}, \Omega_{3,n}$ or
$\Omega_{4,n}$ either belongs to the bounded component of $\hat{C} \setminus C_n$, or intersects with $\partial \Omega_{4,n}$ at a
pole. Then
\[ \partial U(f^{k-1}_n(c_n)) \cap \partial \Omega_{4,n} \neq \emptyset. \]

Note that $\Omega_{4,n}$ is the unique component of $f^{-1}_n(\Omega_{4,n})$ contained in the unbounded
component of $\hat{C} \setminus C_n$. Since each $U(f^n_i(c_n))$ is in the unbounded component of $\hat{C} \setminus C_n$, then for all $0 \leq i \leq k - 1$,
\[ \partial U(f^n_i(c_n)) \cap \partial \Omega_{4,n} \neq \emptyset. \]

Moreover, we claim in fact that $k \geq 2$. Indeed, if $k = 1$, then the Fatou component
$U(c_n)$ contains two poles of $f_n$. Note that bounded component of $\hat{C} \setminus C_n$ contains two
poles of $f_n$ and it complement contains the other pole. We then get a contradiction
since $U(c_n)$ is contained in the unbounded component of $\hat{C} \setminus C_n$. 
Note that all $f_n$ are in the same hyperbolic component, then all quantities defined for $f_n$ and properties satisfied by $f_n$ for $n$ large also hold for $f_0$. We deduce the contradiction by $f_0$. Suppose $\partial U(f_0(c_0))$ intersects $\partial \Omega_{4,0}$ at the landing point of $I_{4,0}(\theta)$. Since $U(c_0)$ contains a critical point and is contained in the unbounded component of $\hat{\Omega} \setminus \mathcal{C}_0$, the intersection $\partial U(c_0) \cap \partial \Omega_{4,0}$ contains the landing points of $I_{4,0}(\theta/2)$ and $I_{4,0}(1+\theta)/2)$. We consider an arc $\gamma_1 \subset \overline{U(c_0)}$ joining these two landing points and avoiding the orbits of critical points of $f_0$. Let $\gamma_2$ be the lift of $\gamma_1$ based at the landing point of $I_{4,0}(\theta/2^2)$. Since $\gamma_1$ does not intersect with $\mathcal{C}_0$, the endpoint of $\gamma_2$ belongs to $\partial \Omega_{4,0}$. Note also that the preimages of $\gamma_1(1)$ on $\partial \Omega_{4,0}$ are the landing points of the internal rays in $\Omega_{4,0}$ of angles $(1+\theta)/4$ or $(3+\theta)/4$. Since $(1+\theta)/4 \in (\theta/2, (1+\theta)/2)$, it follows that the endpoint of $\gamma_2$ is the landing point of $I_{4,0}((3+\theta)/4)$. 

Inductively, for every $m \geq 1$, define $\gamma_{m+1}$ to be the lift of $\gamma_m$ based at the landing point of $I_{4,0}(\theta/2^{m+1})$. Then the endpoint of $\gamma_{m+1}$ is the landing point of $I_{4,0}(1-(1-\theta)/2^{m+1})$. Note that for large $m$, each $\gamma_m$ is an arc joining two points of $\partial \Omega_{4,0}$ in different components of $\partial \Omega_{4,0} \setminus (I_{4,0}(0) \cup I_{4,0}(1/2))$ near $\infty$. Moreover, the intersection of $\gamma_m$ and $\overline{\Omega}_{1,2} \cup \overline{\Omega}_{2,0} \cup \overline{\Omega}_{3,0} \cup \overline{\Omega}_{4,0}$ is the endpoints of $\gamma_m$. It follows that the length of $\gamma_m$ have a positive infinitum as $m \to \infty$. However, since $f_0$ is uniformly expanding near the Julia set, the length of $\gamma_m$ decrease to 0 as $m \to \infty$. It is a contradiction. \[\square\]

References

[1] X. Buff and L. Tan, Dynamical convergence and polynomial vector fields, J. Differential Geom., 77 (2007), pp. 1–41.
[2] G. Cui and L. Tan, Hyperbolic-parabolic deformations of rational maps, Sci. China Math., 61 (2018), pp. 2157–2220.
[3] L. DeMarco, Iteration at the boundary of the space of rational maps, Duke Math. J., 130 (2005), pp. 169–197.
[4] A. Douady, Does a Julia set depend continuously on the polynomial?, in Complex dynamical systems (Cincinnati, OH, 1994), vol. 49 of Proc. Sympos. Appl. Math., Amer. Math. Soc., Providence, RI, 1994, pp. 91–138.
[5] K. Drach, R. J. Mikukich, Yauhen, and D. Schleicher, A combinatorial classification of postcritically fixed newton maps, Ergodic Theory Dynam. Systems, 39 (2019), pp. 2983–3014.
[6] K. Drach and D. Schleicher, Rigidity of Newton dynamics, arXiv e-prints, (2018), p. arXiv:1812.11919.
[7] A. L. Epstein, Bounded hyperbolic components of quadratic rational maps, Ergodic Theory Dynam. Systems, 20 (2000), pp. 727–748.
[8] T. W. Gamelin, Complex analysis, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2001.
[9] Y. Gao, Density of hyperbolicity of real Newton maps, arXiv e-prints, (2019), p. arXiv:1906.03556.
[10] Y. Gao, On the core entropy of Newton maps, arXiv e-prints, (2019), p. arXiv:1906.01523.
[11] Y. Gao and T. Giulio, The core entropy for polynomials of higher degree, arXiv e-prints, (2017), p. arXiv:1703.08703.
[12] L. Goldberg and J. Milnor, Fixed point portraits of polynomial maps, part ii: Fixed point portraits, Experiment. Math., 26 (1993), pp. 51–98.
[13] J. Head, The combinatorics of Newtons method for cubic polynomials, Thesis, Cornell University, (1987).
[14] R. Lodge, Y. Mikukich, and D. Schleicher, A classification of postcritically finite Newton maps, arXiv e-prints, (2015), p. arXiv:1510.02771.
[15] ———, Combinatorial properties of Newton maps, arXiv e-prints, (2015), p. arXiv:1510.02761.
C. McMullen, *Automorphisms of rational maps*, in Holomorphic functions and moduli, Vol. I (Berkeley, CA, 1986), vol. 10 of Math. Sci. Res. Inst. Publ., Springer, New York, 1988, pp. 31–60.

C. T. McMullen, *Complex dynamics and renormalization*, vol. 135 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1994.

J. Milnor, *Hyperbolic component boundaries*, http://www.math.stonybrook.edu/jack/HCBkoreaPrint.pdf.

———, *Geometry and dynamics of quadratic rational maps*, Experiment. Math., 2 (1993), pp. 37–83. With an appendix by the author and Lei Tan.

———, *Dynamics in one complex variable*, vol. 160 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, third ed., 2006.

H. Nie, *Iteration at the Boundary of Newton Maps*, ProQuest LLC, Ann Arbor, MI, 2018. Thesis (Ph.D.)–Indiana University.

H. Nie and K. M. Pilgrim, *Boundedness of Hyperbolic Components of Newton Maps*, accepted to Israel Journal of Mathematics.

———, *Bounded hyperbolic components of bicritical rational maps*, arXiv e-prints, (2019), p. arXiv:1903.08873.

P. Roesch, *Holomorphic motions and puzzles (following m. shishikura)*, The Mandelbrot Set, Theme and Variations, 274 (2000), pp. 117–132.

———, *On local connectivity for the Julia set of rational maps: Newton’s famous example*, Ann. of Math. (2), 168 (2008), pp. 127–174.

M. Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. École Norm. Sup. (4), 20 (1987), pp. 1–29.

———, *The connectivity of the Julia set and fixed points*, in Complex dynamics, A K Peters, Wellesley, MA, 2009, pp. 257–276.

W. P. Thurston, *Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds*, Ann. of Math. (2), 124 (1986), pp. 203–246.

X. Wang, Y. Yin, and J. Zeng, *Dynamics of Newton maps*, arXiv e-prints, (2018), p. arXiv:1805.11478.