Explicit gauge invariant quantization of the Schwinger model on
a circle in the functional Schrödinger representation

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Abstract

We solve the Schwinger model on a circle by first finding the explicit groundstate
functional(s). Having done this, we give the structure of the Hilbert space and derive
bosonization formulae in this formalism.

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1 Introduction and Discussion

Massless QED in 1+1 dimensions was first studied by Schwinger \cite{1}. Since then it has been considered by numerous authors. In particular it has been studied on the circle by \cite{2, 3, 4}.

The spectrum of the theory is equivalent to that of a massive free boson. From a fermionic point of view this result is non-trivial. In this paper we are treating the Schwinger model in a functional representation. This has some nice properties. We treat gauge-invariance in Dirac’s way i.e. we let states be annihilated by the constraint operator and we only allow operators commuting with the constraint operator. In a functional representation this can be done explicitly. Furthermore, the functional representation space is large enough to contain the inequivalent Fock spaces that always exist in quantum field theory. In the case of the Schwinger model this is desirable since the solution of the model usually involves a Bogoliubov transformation that takes you out of a chosen Fockspace into another. The functional representation allows you to treat all these Fock spaces in a general setting.

One of the main motivations for this paper is the study of the groundstate functional for the Schwinger model, expressed in terms of fermionic variables. The idea is that the groundstate in more complicated theories, like the massive Schwinger model or higher dimensional QED, may be of the same or a similar form as that of the Schwinger model. Even if this does not turn out to be the case, one might use it as a variational ansatz for such theories.

So what does the groundstate look like? The groundstate cannot be gaussian in a fermionic representation since the spectrum is not that of a free fermion. It turns out that the groundstate is a two parameter functional where one of the parameters is the covariance associated with the free (gaussian) groundstate and the other parameter is induced by the interaction. We would like to stress that the groundstate is annihilated by Gauss’ law. This holds in particular for the free groundstate. There is therefore no need of a modified Gauss’ law as suggested in \cite{5}.

The non-gaussian character of the groundstate has some unpleasant properties. Trying to evaluate expectation values directly, using our general expression for the inner product, one runs into very hard calculations. However for numerical calculations our inner product is well suited. To calculate expectation values in a simple way we can use bosonization.

The sequence of deriving various statements of the model is different in this work than in previous works. We start out by finding the groundstate. This fixes the irreducible representation within the functional representation that we are using. Having done this, we derive the anomalous chiral current algebra and find the creation operators and the structure of the Hilbert space. We also give a proof of bosonization by showing that the action of the fermionic operators and the corresponding bosonic ones have the same action on the explicit groundstate. Finally we calculate the different 2-point correlators of the theory.

This is a technical paper. We have tried to keep it readable by putting all proofs and lengthy calculations in appendices.
2 Hamiltonian

The Hamiltonian of massless QED on a circle of circumference \( L \) is,

\[
\hat{H} = \int_0^L dx \left( \frac{1}{2} \hat{\mathcal{E}}^2(x) - \hat{\psi}^\dagger(x) \gamma (i\partial_x - e\hat{A}(x)) \hat{\psi}(x) \right),
\]

where \( \gamma = \gamma^0 \gamma^1 \). In an explicit representation take \( \gamma^0 = \sigma_1, \gamma^1 = -i \sigma_2 \) and hence \( \gamma = \sigma_3 \).

One also has the first class constraint operator, Gauss’ law:

\[
\hat{G}(x) = \partial_x \hat{\mathcal{E}}(x) - e \frac{1}{2} [\hat{\psi}^\dagger(x), \hat{\psi}(x)] = \partial_x \hat{\mathcal{E}}(x) - e j_0(x) \approx 0.
\]

Furthermore, boundary conditions needs to be specified. We choose,

\[
\hat{A}(x + L) = \hat{A}(x),
\]

\[
\hat{\mathcal{E}}(x + L) = \hat{\mathcal{E}}(x),
\]

\[
\hat{\psi}(x + L) = e^{-2\pi i \frac{x}{L}} \hat{\psi}(x).
\]

Since \( \hat{\mathcal{E}} \) is periodic, (2) implies that the total electric charge \( \hat{Q}_0 = e \int_0^L dx j_0(x) \) vanishes on physical states. Define the transverse fields,

\[
\hat{A}_T = \frac{1}{L} \int_0^L dx \hat{A}(x),
\]

\[
\hat{\mathcal{E}}_T = \frac{1}{L} \int_0^L dx \hat{\mathcal{E}}(x),
\]

\[
\hat{\psi}_T(x) = \exp \left( ie \int_0^x dx' \hat{A}(x') - i e x \hat{A}_T + i 2\pi \alpha x / L \right) \hat{\psi}(x).
\]

With this definition, \( \hat{\psi}_T \) is periodic, \( \hat{\psi}_T(x + L) = \hat{\psi}_T(x) \). Under a gauge transformation \( \Lambda(x) = e^{i \lambda(x)} \), the fields transform as

\[
\hat{\mathcal{E}}'(x) = \hat{\mathcal{E}}(x),
\]

\[
\hat{A}'(x) = \hat{A}(x) - \frac{1}{e} \partial_x \lambda(x),
\]

\[
\hat{\psi}'(x) = \Lambda(x) \hat{\psi}(x).
\]

Under a small gauge transformation, \( \lambda(x) \) periodic and \( \lambda(0) = 0 \), we find that \( \hat{A}_T \) and \( \hat{\psi}_T \) are invariant. Under a global transformation \( \lambda(x) = \lambda(0), \hat{A}_T \) is still invariant while \( \hat{\psi}'_T = e^{i \lambda(0)} \hat{\psi}_T \). Finally, for a large transformation (picking a representative for each class), \( \lambda(x) = 2\pi n x / L \), we have

\[
\hat{A}'_T = \hat{A}_T - \frac{2\pi n}{e L},
\]

\[
\hat{\psi}'_T(x) = e^{i 2\pi n x / L} \hat{\psi}_T(x).
\]

The fields satisfy the following nonvanishing commutators and anticommutators:

\[
[\hat{A}(x), \hat{\mathcal{E}}(y)] = i \delta(x - y),
\]

\[
\{ \hat{\psi}_\alpha(x), \hat{\psi}_\beta^\dagger(y) \} = \delta_{\alpha \beta} \delta(x - y).
\]
Let us also introduce the operator \( \hat{J} \), which commutes with \( \hat{G} \),
\[
\hat{J}_{\alpha\beta}(x, y) = \frac{i}{2}[\hat{\psi}^\dagger_{T,\alpha}(x), \hat{\psi}_{T,\beta}(y)].
\] (15)

Using (14) we obtain,
\[
[\hat{A}_T, \hat{E}_T] = \frac{i}{L},
\] (16)
\[
[\hat{J}_{\alpha\beta}(x, y), \hat{J}_{\alpha'\beta'}(x', y')] = \delta_{\beta\beta'}\delta(y - x')\delta_{\alpha\alpha'}\delta(x - y') - \delta_{\alpha\alpha'}\delta(x - y')\delta_{\beta\beta'}(x', y'),
\] (17)
\[
[\hat{J}(x, y), \hat{E}_L(z)] = e \left( \frac{y - x}{L} - \int_x^y dx' \delta(x' - z) \right) \hat{J}(x, y),
\] (18)
\[
[\hat{J}(x, y), \hat{E}_T] = 0.
\] (19)

Note that (17) only holds on all states as long as \( x \neq y \) or \( x' \neq y' \). The constraint operator now reads,
\[
\hat{G}(x) = \partial_x \hat{E}_L(x) - e \hat{j}_0(x) = \partial_x \hat{E}_L(x) - e\frac{i}{2}[\hat{\psi}^\dagger_T(x), \hat{\psi}_T(x)] = \partial_x \hat{E}_L(x) - e\hat{J}_{\alpha\alpha}(x, x),
\] (20)

where \( \hat{E}_L(x) = \hat{E}(x) - \hat{E}_T \). Hence on the set of physical states, defined such that \( \hat{G}(x) = 0 \) on that set, we may express \( \hat{E}_L(x) \) in terms of \( \hat{j}_0(x) = \hat{J}_{\alpha\alpha}(x, x) \). Doing this one may write the Hamiltonian on the set of physical states as,
\[
\hat{H} = \frac{L}{2} \hat{E}_T^2 - i \int_0^L dx \lim_{y \to x}(\partial_y + i\hat{b})\gamma_{\alpha\beta}\hat{J}_{\alpha\beta}(x, y) + \int_0^L dx \int_0^L dy \hat{V}(x - y)\hat{j}_0(x)\hat{j}_0(y),
\] (21)

where \( \hat{b} = \hat{A}_T - \frac{2\pi\alpha}{cL} \) and
\[
\hat{V}(x) = \frac{e^2L}{4\pi^2} \sum_{n>0} \frac{1}{n^2} \cos \frac{2\pi nx}{L}.
\] (22)

### 3 Functional representation

To implement the canonical (anti)commutation relations (14) we will use the functional Schrödinger representation. For fermions we will use the reducible representation first found in [4].

Let the bosonic operators \( \hat{A}(x), \hat{E}(x) \) and the fermionic operators \( \hat{\psi}_\alpha(x), \hat{\psi}^\dagger_{\alpha}(x) \) act on wavefunctionals \( \Psi(A, \eta^*, \eta) \) of a real bosonic field \( A(x) \), a complex Grassmann field \( \eta(x) \) and its complex conjugate \( \eta^*(x) \) according to

\[
\hat{A}(x) \leftrightarrow A(x),
\] (23)
\[
\hat{E}(x) \leftrightarrow \frac{i}{\sqrt{2}} \frac{\delta}{\delta A(x)},
\] (24)
\[
\hat{\psi}_\alpha(x) \leftrightarrow \frac{1}{\sqrt{2}}(\eta_\alpha(x) + \frac{\delta}{\delta \eta^*_\alpha(x)}),
\] (25)
\[
\hat{\psi}^\dagger_{\alpha}(x) \leftrightarrow \frac{1}{\sqrt{2}}(\eta^*_\alpha(x) + \frac{\delta}{\delta \eta^*_\alpha(x)}).
\] (26)
A general wave functional may be viewed as an overlap with a product of a bosonic field state and a Grassman field state,

$$|A\eta^*\eta\rangle = |A\rangle \otimes |\eta^*\eta\rangle,$$

$$\langle A\eta^*\eta|A'\eta'^*\eta'\rangle = \delta(A - A') \exp \left[ \int dx \left( \eta^*_\alpha(x)\eta'_{\alpha}(x) - \eta'^*_\alpha(x)\eta_{\alpha}(x) \right) \right]. \quad (27)$$

Here, and in the following, left out integration limits means that the integral should be taken from 0 to \(L\). Using the field states, the partition of unity and the inner product are given by functional integration \([6, 7]\),

$$\hat{1} = \int DA |A\rangle \langle A| \otimes \int D\eta D\eta' |\eta\rangle \langle \eta| \langle \eta^*\eta'\rangle \langle \eta'\eta\rangle \Psi_1(A, \eta^*, \eta) \Psi_2(A, \eta'^*, \eta'). \quad (28)$$

where we have set \(D^2\eta = D\eta^* D\eta\).

## 4 Gauge-invariant states

Gauge-invariant states are annihilated by Gauss’ law,

$$\hat{G}(x)\Psi(A, \eta^*, \eta) = 0. \quad (30)$$

These states are invariant under the transformations that are generated by \(\hat{G}\), i.e. the small and the global gauge transformations. Gauge-invariant wavefunctionals have been found in \([8]\). A general gauge-invariant functional is parameterized by a family of distributions denoted \(f\) and has the following form (spinor indices are summed over):

$$\Psi_f(A, \eta, \eta^*) = \left[ f^{(0)}(A_T) + \sum_{a=1}^{\infty} \frac{1}{a!} \int dx dy f^{(a)}(A_T; x_1, y_1, \ldots, x_a, y_a) \eta^*_T(x_1) \eta_T(y_1) \cdots \eta^*_T(x_a) \eta_T(y_a) \right] \times \exp \left[ \int dx \eta^*_T(x) M \eta_T(x) \right], \quad (31)$$

where the real variable \(A_T\) and the Grassman field \(\eta_{T,\alpha}(x)\) are defined in analogy with \(\hat{A}_T\) and \(\hat{\psi}_T(x)\), i.e.

$$A_T = \frac{1}{L} \int_0^L dx A(x) \quad (32)$$

$$\eta_{T,\alpha}(x) = \exp \left[ i e \int_0^x dx' A(x') - i e x A_T + i 2\pi\alpha x/L \right] \eta_{\alpha}(x). \quad (33)$$

A few comments are in order. The constant matrix \(M\) in \([8]\) must satisfy \(\text{tr} M = 0\) and \(M^2 = 1\). These conditions do not uniquely determine \(M\). However, due to the
reducibility of the representation different choices of \( M \) give functionals all representing the same physical state. Furthermore, the matrix \( M \) determines the matrix \( \tilde{\gamma} \) through \( \tilde{\gamma} = \frac{1}{2}(1 - M)\gamma(1 + M) \). A convenient choice of \( M \) giving a gaussian free groundstate is \( M = i\gamma^1 \). In the following that choice is understood. In the gauge-invariant state (31), the parameterizing distributions multiply "base states",

\[
\eta^*_T(x_1)\tilde{\gamma}\eta^*_T(y_1)\cdots\eta^*_T(x_a)\tilde{\gamma}\eta^*_T(y_a) \exp \left[ \int dx \eta^*_T(x)M\eta_T(x) \right].
\]

(34)

The base states are invariant under the exchange of pairs \((x_i, y_i) \leftrightarrow (x_j, y_j)\) but change sign under exchange of \(x_i \leftrightarrow x_j\) or \(y_i \leftrightarrow y_j\) separately. We will demand that the distributions \( f(a) \) satisfy

\[
f(a)(A_T; x_1, y_1, \ldots, x_j, y_j, \ldots) = f(a)(A_T; x_j, y_j, \ldots, x_i, y_i, \ldots).
\]

(35)

It is also worth mentioning that gauge-invariant states characterized by translation-invariant distributions (in the spatial coordinates) have zero total momentum.

**Action of physical operators on gauge-invariant states**

Physical (gauge-invariant) operators may be constructed from the operator \( \hat{J}_{\alpha\beta}(x,y) \). Let the operator \( A \cdot \hat{J}(x,y) = A_{\alpha\beta}\hat{J}_{\alpha\beta}(x,y) \) act on a general gauge-invariant state parameterized by the family \( f \). The state produced will be a gauge-invariant state parameterized by a new family, say \( f_A \), i.e.

\[
A \cdot \hat{J}(x,y)\Psi_f = \Psi_{f_A}
\]

(36)

The resulting families \( f_A \) may be expressed in terms of the original family \( f \). In [8] the families \( f_A \) were found for \( A \) equal to \( 1, \gamma^0, \gamma^1, \gamma \) respectively. For families having the property (35) the result is

\[
f_1(a)(A_T; x_1, y_1, \ldots, x_a, y_a) = \\
\sum_{b=1}^{a} \left\{ \delta(x - x_b)f(a)(A_T; x_1, y_1, \ldots, y_b, \ldots, x_a, y_a) \\
-\delta(y - y_b)f(a)(A_T; x_1, y_1, \ldots, x_b, x, \ldots, x_a, y_a) \right\},
\]

(37)

\[
f_{\gamma^0}(A_T; x_1, y_1, \ldots, x_a, y_a) = \\
-\frac{i}{a}f(a+1)(A_T; x_1, y_1, \ldots, x_a, y_a, y, x) \\
+i\sum_{b=1}^{a} \left\{ f(a+1)(A_T; x_1, y_1, \ldots, x_b, x, y, y_b, \ldots, x_a, y_a) \\
+ f(a-1)(A_T; x_1, y_1, \ldots, f_b, y_b, \ldots, x_a, y_a)\delta(x - x_b)\delta(y - y_b) \right\},
\]

(38)
\[ f^{(a)}_{\gamma} (A_T; x_1, y_1, \ldots, x_a, y_a) = \\
- i \delta(x - y) f^{(a)} (A_T; x_1, y_1, \ldots, x_a, y_a) \\
+ i \sum_{b=1}^{a} \left\{ \delta(x - x_b) f^{(a)} (A_T; x_1, y_1, \ldots, y_b, \ldots, x_a, y_a) \\
\right. \\
\left. + \delta(y - y_b) f^{(a)} (A_T; x_1, y_1, \ldots, x_b, x, \ldots, x_a, y_a) \right\}, \tag{39} \]

\[ f^{(a)}_{\gamma} (A_T; x_1, y_1, \ldots, x_a, y_a) = \\
f^{(a+1)} (A_T; x_1, y_1, \ldots, x_a, y_a, y, x) \\
- \sum_{b=1}^{a} f^{(a+1)} (A_T; x_1, y_1, \ldots, x_b, x, y, y_b, \ldots, x_a, y_a) \\
+ \sum_{b=1}^{a} f^{(a-1)} (A_T; x_1, y_1, \ldots, x_b, y_b, \ldots, x_a, y_a) \delta(x - x_b) \delta(y - y_b). \tag{40} \]

These expressions are also valid for \( a = 0 \) if sums ranging from one to zero are set to zero.

**Inner product and gauge-invariant states**

Gauge-invariant state functionals are completely specified by a family of distributions. The inner product \( (29) \), when involving gauge-invariant states only, must therefore be a mapping from a pair of families to the complex numbers. We will see how that comes about.

First, considering \( (29) \), we observe that \( DA = \text{const } dA_T DA_L, D^2 \eta' D^2 \eta = D^2 \eta' T D^2 \eta_T \) and \( \langle \eta'^* \eta^* \eta'^* \eta' \rangle = \langle \eta^*_T \eta_T \eta'^*_T \eta'^*_T \rangle \). Then, after these changes of variables, all dependence of \( A_L \) has disappeared from the integrand and the integral over \( A_L \) gives just another divergent constant. The inner product \( (29) \) thus becomes

\[ \langle \Psi_g | \Psi_f \rangle = N \int_{-\infty}^{\infty} dA_T \int D^2 \eta'_T D^2 \eta_T \langle \eta'^*_T \eta_T | \eta'^*_T \eta'^*_T \rangle \Psi^*_g (A_T, \eta'^*_T, \eta_T) \Psi_f (A_T, \eta'^*_T, \eta'_T). \tag{41} \]

On inserting the expression \( (31) \) for the two general gauge-invariant functionals, it is possible to do the fermionic integrals. After dropping the factor \( N \) and a factor \( \text{det}(2I) \) which emerges in the calculation \( (I_{\alpha\beta}(x,y) = \delta_{\alpha\beta} \delta(x - y)) \) we arrive at the final form of the inner product on the space of gauge-invariant functionals:

\[ \langle \Psi_g | \Psi_f \rangle = \int_{-\infty}^{\infty} dA_T \left\{ \tilde{g}^{(0)} (A_T)^* f^{(0)} (A_T) \right. \right.
\left. + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^3 x \, d^3 y \, \epsilon_{i_1 \ldots i_a} g^{(a)} (A_T; x_1, y_{i_1}, \ldots, x_a, y_{i_a})^* f^{(a)} (A_T; x_1, y_1, \ldots, x_a, y_a) \right\}. \tag{42} \]

We will return to this expression below. Another way of deriving \( (42) \) is by making an appropriate ansatz and then using the hermiticity properties of the various gauge invariant operators to constrain the ansatz as was done in \( [\text{8}] \).
5 Groundstates

Having set up the formalism we are now ready to find eigenstates of \( \hat{H} \). Using (43) and (44) one finds the action of \( \hat{H} \) on a general gauge-invariant state \( \Psi_f(A_T, \eta_T, \eta_T^+) \). We have

\[
\hat{H} \Psi_f(A_T, \eta_T, \eta_T^+) = \Psi_f'(A_T, \eta_T, \eta_T^+),
\]

where the family \( f' \) is given, in terms of the family \( f \), by (remember that \( b = A_T - \frac{2\pi a}{eL} \))

\[
f'^{(a)}(A_T; x_1, y_1, \ldots, x_a, y_a) = -\frac{1}{2L} \partial^2_{A_T} f^{(a)}(A_T; x_1, y_1, \ldots, x_a, y_a)
\]

\[
-\imath \int dx \lim_{y \rightarrow x} (\partial_y + ie b) f^{(a+1)}(A_T; x_1, y_1, \ldots, x_a, y_a, y, x)
\]

\[
+\imath \int dx \lim_{y \rightarrow x} (\partial_y + ie b) \sum_{b=1}^a f^{(a+1)}(A_T; x_1, y_1, \ldots, x_b, y_b, \ldots, x_a, y_a)
\]

\[
-\imath \sum_{b=1}^a f^{(a-1)}(A_T; x_1, y_1, \ldots, x_b, y_b, \ldots, x_a, y_a) (\partial_y + ie b) \delta (x_b - y_b)
\]

\[
+ V(x_1, y_1, \ldots, x_a, y_a) f^{(a)}(A_T; x_1, y_1, \ldots, x_a, y_a).
\]

(44)

This expression holds for \( a = 0 \) if sums ranging from one to zero are set to zero. The potential \( V \) is defined as

\[
V(x_1, y_1, \ldots, x_a, y_a) = \sum_{i,j=1}^a V(x_i - y_j) - \sum_{j>i=1}^a [V(x_i - x_j) + V(y_i - y_j)],
\]

(45)

where

\[
V(x) = \frac{e^2 L}{2\pi^2} \sum_{n>0} \frac{1}{n^2} \left( 1 - \cos \frac{2\pi nx}{L} \right) = 2[\hat{V}(0) - \hat{V}(x)].
\]

(46)

(50)

There is a simple physical interpretation of \( V \). It gives rise to an attractive interaction between states of different charge, \((xy)\), and a repulsive interaction between states of equal charge, \((xx)\) or \((yy)\).

Demanding that \( \Psi_f(A_T, \eta_T, \eta_T^+) \) is an eigenstate of \( \hat{H} \) with eigenvalue \( E \), i.e.

\[
f'^{(a)} = Ef^{(a)}, \ a = 0, 1, 2, \ldots,
\]

(47)

leads by (44) to a complicated set of hierarchy equations coupling different levels (we call \( a \) the level).

It turns out that there are eigenstates that factorize in an electromagnetic (EM) part and a fermionic (F) part (this is not true if we add a mass term to the Hamiltonian), i.e.

\[
\Psi(A_T \eta_T^+ \eta_T) = \Psi(A_T) \Psi(\eta_T^+ \eta_T),
\]

\[
|\Psi\rangle = |\Psi\rangle_{EM} \otimes |\Psi\rangle_F.
\]

(48)

(49)

Accordingly their families also factorize,

\[
f^{(a)}(A_T, x_1, y_1, \ldots, x_a, y_a) = \Psi(A_T)f^{(a)}(x_1, y_1, \ldots, x_a, y_a), \ a = 0, 1, 2, \ldots
\]

(50)
For such states the strategy is: First, split the hamiltonian in an electromagnetic and a fermionic part defined through $\hat{H} = \frac{1}{2}E_T^2 + \hat{H}_F$ and find $A_T$-independent eigenstates of $\hat{H}_F$ with eigenvalue $E(A_T)$.

This corresponds to solving (14) ignoring the kinetic term for $A_T$ in the expression for $f^{(a)}$ given in (14). Second, having found such eigenstates of $\hat{H}_F$, (17) and (14) collapse into

$$\frac{1}{2L} \partial_{A_T}^2 \Psi(A_T) + E(A_T) \Psi(A_T) = E \Psi(A_T),$$

which is then solved.

We will pursue the strategy outlined above. In Appendix A we show that the states $\Psi_N(\eta_T^* \eta_T)$, parameterized by the $A_T$-independent family $f_N$ are eigenstates of $\hat{H}_F$ with eigenvalues $E_N(A_T)$. The family is given by $f^{(0)}_N = 1$ and for $a \neq 0$,

$$f^{(a)}_N(x_1, y_1, \ldots, x_a, y_a) = \Omega_N(x_1 - y_1) \cdots \Omega_N(x_a - y_a) \Phi^{(a)}(x_1, y_1, \ldots, x_a, y_a),$$

where

$$\Omega_N(x) = \frac{1}{L} \sum_n \text{sgn}(n + N)e^{ip_n x},$$

$$\Phi^{(a)}(x_1, y_1, \ldots, x_a, y_a) = \exp \left( \sum_{i,j=1}^a \varphi(x_i - y_j) - \sum_{j>i=1}^a \varphi(x_i - x_j) + \varphi(y_i - y_j) \right)$$

and

$$\varphi(x) = - \sum_{n>0} \frac{1}{n \eta_n} \left( \sqrt{p_n^2 + M^2} - p_n \right) (1 - \cos p_n x).$$

Here we have introduced the notation $p_n = \frac{2\pi n}{L}$ for discrete momenta and the standard notation $M = e/\sqrt{\pi}$. Note that when $e = 0$, $f^{(a)}_N$ is just a product of $\Omega_N$’s and the corresponding functional is gaussian. The eigenvalue $E_N(A_T)$ is, after regularization,

$$E_N(A_T) = -\frac{\pi^2}{6L} + \frac{2\pi}{L} \left( \frac{eA_T L}{2\pi} - N - \frac{1}{2} - \alpha \right)^2 + \sum_{n>0} \left( \sqrt{p_n^2 + M^2} - p_n \right).$$

From the form of the eigenenergy we see that the remaining quantum mechanical system (51) is a harmonic oscillator with well-known solutions. Denote its lowest energy state by $\Psi_N(A_T)$ and its eigenvalue by $E_N$. Thus,

$$\Psi_N(A_T) = \exp \left( -\frac{2\pi}{ML} \left( \frac{eA_T L}{2\pi} - N - \frac{1}{2} - \alpha \right)^2 \right)$$

$$E_N = -\frac{\pi}{6L} + \frac{M}{2} + \sum_{n>0} \left( \sqrt{p_n^2 + M^2} - p_n \right).$$

Hence the states $\Psi_N(A_T \eta_T^* \eta_T) = \Psi_N(A_T) \Psi_N(\eta_T^* \eta_T)$, parameterized by the family $\Psi_N(A_T)f_N$, are eigenstates of $\hat{H}$, all with the same energy $E = E_N$. These states are the groundstates of $\hat{H}$. The corresponding kets will be denoted $|\Psi_N\rangle$. 

8
The groundstates $|\Psi_N\rangle$ have an important property. Under a large gauge transformation $\lambda(x) = 2\pi x/L$, we have that
\begin{align}
A_T' &= A_T - \frac{2\pi}{eL}, \\
\eta_T'(x) &= e^{i2\pi x/L}\eta_T(x),
\end{align}
which implies that $|\Psi_N\rangle$ is mapped into $|\Psi_{N+1}\rangle$. Hence a state transforming only by a phase under large gauge transformations is the $\theta$-vacuum $|\theta\rangle$ defined as,
\begin{equation}
|\theta\rangle = \sum_N e^{-iN\theta} |\Psi_N\rangle.
\end{equation}

### 6 Currents, charges and creation operators

In the last section we found the explicit groundstate(s) of the model. The entire Hilbert space may now be constructed by the action of physical operators on the groundstate(s).

The different groundstates $|\Psi_N\rangle$ all define different representation spaces of the algebra of the physical operators. These representation spaces need not be orthogonal. However, using a set of well-known creation operators one may construct orthogonal Fock spaces from the different groundstates. Then, an additional operator connects the different Fock spaces.

In this section we will define some operators and investigate their properties within the functional representation. These operators will be needed when calculating expectation values in the next section.

We start by defining the currents and their fourier transforms. Let
\begin{align}
\hat{J}_0(x) &= 1 \cdot \hat{J}(x, x), \\
\hat{J}_5(x) &= \gamma \cdot \hat{J}(x, x), \\
\hat{J}_\pm(x) &= \frac{1}{2} (\hat{J}_0(x) \pm \hat{J}_5(x)) = \frac{1}{L} \sum_n \hat{J}_\pm(n) e^{ip_n x}.
\end{align}

Hermiticity demands that $\hat{J}_\pm^\dagger(n) = \hat{J}_\pm(-n)$. When acting on the representation space(s) defined by the groundstate(s) $|\Psi_N\rangle$, the chiral currents have a well-known anomalous commutator, a Schwinger term,
\begin{equation}
[\hat{J}_\pm(x), \hat{J}_\pm(y)] = \pm \frac{1}{2\pi i} \delta'(x - y).
\end{equation}

In Appendix B we verify this algebra in the functional representation by calculating its action on the explicit groundstate $\Psi_N(A_T^\ast \eta_T^\ast \eta_T)$. In momentum space the algebra reads
\begin{equation}
[\hat{J}_\pm(n), \hat{J}_\pm^\dagger(m)] = \pm n\delta_{n,m}.
\end{equation}

The chiral charge is defined as,
\begin{equation}
\hat{Q}_5 = \int dx \hat{J}_5(x).
\end{equation}
Let us act with \( \hat{Q}_5 \) on a groundstate. By (40) we obtain
\[
\hat{Q}_5 \Psi_N(A_T \eta_T^* \eta_T) = L \Omega_N(0) \Psi_N(A_T \eta_T^* \eta_T).
\] (68)
The expression for \( \Omega_N(0) \) contains a sum which is not absolutely convergent. To make the sum well-defined we use for consistency the exponential regularization used in appendix A. We find
\[
\hat{Q}_5 \Psi_N(A_T \eta_T^* \eta_T) = 2 \left( e^{A_T L \frac{2\pi}{2\pi}} - N - \frac{1}{2} - \alpha \right) \Psi_N(A_T \eta_T^* \eta_T).
\] (69)

From the above expression we deduce the following commutators, valid when acting on the representation spaces defined by the groundstates:
\[
[\hat{Q}_5, A \cdot J(x,y)] = [\gamma, A] \cdot J(x,y),
\] (70)
\[
[\hat{Q}_5, \hat{E}_T] = i \frac{e}{\pi}.
\] (71)

In particular note the anomalous commutator (71). On the representation spaces defined by the groundstates \( |\Psi_N\rangle \) we may thus write
\[
\hat{Q}_5 = 2 \left( e^{A_T L \frac{2\pi}{2\pi}} - \hat{N} - \frac{1}{2} - \alpha \right),
\] (72)

where the operator \( \hat{N} \) is defined through its action on the groundstate(s),
\[
\hat{N} |\Psi_N\rangle_{EM} = |\Psi_N\rangle_{EM}, \quad \hat{N} |\Psi_N\rangle_F = N |\Psi_N\rangle_F,
\] (73)

and through its commutator with physical operators,
\[
[\hat{N}, A \cdot J(x,y)] = [-\frac{i}{2} \hat{Q}_5, A \cdot J(x,y)] = \frac{i}{2} [A, \gamma] \cdot J(x,y),
\] (74)
\[
[\hat{N}, \hat{A}_T] = [\hat{N}, \hat{E}_T] = 0.
\] (75)

Note that the regularization has made \( \hat{Q}_5 \) invariant under large gauge-transformations. Furthermore, the groundstate \( |\Psi_N\rangle \) is not an eigenstate of \( \hat{Q}_5 \).

One can find creation operators \( \hat{a}_\pm^\dagger (n) \), which are related to the currents through a Bogoliubov transform. We have
\[
\begin{bmatrix}
\hat{a}_+^\dagger (n) \\
\hat{a}_-^\dagger (n)
\end{bmatrix} = \frac{i}{2} \begin{bmatrix}
\kappa_n + \kappa_n^{-1} & \kappa_n - \kappa_n^{-1} \\
\kappa_n - \kappa_n^{-1} & \kappa_n + \kappa_n^{-1}
\end{bmatrix} \begin{bmatrix}
\hat{j}_+^\dagger (n) \\
\hat{j}_-^\dagger (n)
\end{bmatrix}, \quad n \neq 0,
\] (76)

where \( \kappa_n = \left[ 1 + \frac{M^2}{p_n^2} \right]^{\frac{1}{2}} \) and as before \( \hat{a}_\pm^\dagger (n) = \hat{a}_\pm (-n) \). When acting on the representation spaces defined by the groundstates, the algebra (68) leads to
\[
[\hat{a}_\pm (n), \hat{a}_\pm^\dagger (m)] = n \delta_{n,m},
\] (77)
\[
[\hat{H}, \hat{a}_\pm^\dagger (n)] = \sqrt{p_n^2 + M^2} \hat{a}_\pm (n) , \quad n > 0.
\] (78)
Also, one may check that the operators $\hat{a}_\pm(n), n > 0$ annihilate $|\Psi_N\rangle$ and that $\hat{a}^\dagger_+(n), n > 0$ ($\hat{a}^\dagger_-(n), n > 0$) create states with positive (negative) momenta.

Apart from the creation operators above there is the creation operator related to the electromagnetic sector creating states with zero momentum,

$$\hat{a}^\dagger = \sqrt{\frac{L}{2M}} \left(M\hat{A}_T - i\hat{E}_T - \frac{2\pi M}{eL}(N + \frac{1}{2} + \alpha)\right) = \sqrt{\frac{L}{2M}} \left(\frac{\pi M}{eL}\hat{Q}_5 - i\hat{E}_T\right). \quad (79)$$

Furthermore, $\hat{a}$ annihilates $|\Psi_N\rangle$ and

$$[a, a^\dagger] = 1, \quad (80)$$
$$[\hat{H}, a^\dagger] = Ma^\dagger. \quad (81)$$

The operator $a^\dagger$ together with the operators $\hat{a}^\dagger_+(n)$ are the creation operators building up the infinite set of orthogonal Fock spaces from the different groundstates. Since all the creation operators commute with $\hat{N}$ a Fock space consisting of states transforming with a phase under large gauge-transformations may be built upon the theta vacuum.

The spectrum is that of a free massive boson. In the following section we will calculate expectation values and further examine the concept of bosonization in the context of the functional representation.

### 7 Expectation values and bosonization

Let us first discuss some general properties of overlaps. States with different eigenvalues of $\hat{N}$ are orthogonal since $\hat{N}$ is hermitian. By calculating the $N$-charge of various operators one may then figure out between what states these operators have non-vanishing expectation values. From (74) we have e.g.

$$[\hat{N}, \frac{1}{2}(\gamma^0 \pm \gamma^1) \cdot \hat{J}(x,y)] = 0, \quad (82)$$
$$[\hat{N}, \frac{1}{2}(\gamma^0 \pm \gamma^1) \cdot \hat{J}(x,y)] = \pm \frac{1}{2}(\gamma^0 \pm \gamma^1) \cdot \hat{J}(x,y). \quad (83)$$

Consider thus e.g.,

$$\frac{\langle \Psi_N | \frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x,y) | \Psi_{N+1}\rangle}{\| |\Psi_N\rangle \| \| |\Psi_{N+1}\rangle \|} = \frac{\langle \Psi_N | \frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x,y) | \Psi_{N+1}\rangle}{\langle \Psi_N | \Psi_N \rangle}. \quad (84)$$

Since $|\Psi_N\rangle = |\Psi_N\rangle_{EM} \otimes |\Psi_N\rangle_F$ is a product state we can evaluate the fermionic and electromagnetic part separately. Trying to use the inner product (12) to evaluate (84) directly, one runs into very hard calculations. This is due to the non-gaussian character of $|\Psi_N\rangle_F$. The fermionic vacuum amplitude $F\langle \Psi_N | \Psi_N \rangle_F$ (which happens to be a very complicated object) doesn’t factorize from the numerator of (84) in a straightforward manner. However, to show that (12) works in principle, we have evaluated (84) to first
order in $\varphi(x)$. One finds for the fermionic part:

$$
F(P_N | \frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x,y) | P_{N+1}) = 
= \frac{i}{L} \sum_{a=0}^{\infty} \left[ \left( \frac{\lambda - 1}{a} \right) + \sum_{n>0} \left( \frac{\lambda - 1}{a} \right) C_n \left( 1 - \cos p_n(x-y) \right) + \left( \frac{\lambda - 3}{a} \right) 4C_n(n-1) \right]
$$

$$
= \frac{i}{L} 2^\lambda \left( 1 + \sum_{n>0} nC_n - C_n \cos p_n(x-y) \right),
$$

(85)

where $\lambda = L\delta(0) = \sum n$ is assumed to be regularized in a suitable manner and where $C_n = (1 - \sqrt{1 + M^2/p_n^2})/n$. Hence the fermionic part of (84) to first order in $C_n$ becomes,

$$
\frac{F(P_N | \frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x,y) | P_{N+1})}{F(P_N | P_N)} = \frac{i}{L} \left( 1 - \sum_{n>0} C_n \cos p_n(x-y) \right).
$$

(87)

In order to evaluate (84) exactly, and to verify (87), we will use bosonization. In appendix C we show the following formulae by calculating their action on the explicit groundstate in the functional representation:

$$
\frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x,y) = \frac{i}{L} 2^{\frac{\lambda}{2}} \left( x-y \right) e^{\frac{2\pi i}{L} (x-y)} e^{\hat{\phi}_+(x) - \hat{\phi}_+(y) - \hat{\phi}_-(x) + \hat{\phi}_-(y)},
$$

(88)

$$
\frac{1}{2}(\gamma^0 + \gamma^1) \cdot \hat{J}(x,y) = -\frac{i}{L} 2^{\frac{\lambda}{2}} \left( x-y \right) e^{\hat{\phi}_+(x) - \hat{\phi}_+(y) - \hat{\phi}_-(x) + \hat{\phi}_-(y) \hat{S}^\dagger},
$$

(89)

$$
(\gamma \pm 1) \cdot \hat{J}(x,y) = \Omega_0(y - x) e^{-\frac{2\pi i}{L} (x-y)} e^{\hat{\phi}_+(x) - \hat{\phi}_+(y) - \hat{\phi}_-(x) + \hat{\phi}_-(y)}.
$$

(90)

Here we have defined $\hat{\phi}_\pm(x) = \sum_{n>0} \frac{\gamma}{\alpha} \hat{\gamma}_\pm(\pm n)e^{\pm ip_n x}$. The shift operator $\hat{S}$ shifts the fermionic part of $|P_N\rangle$ into the fermionic part of $|P_{N-1}\rangle$,

$$
\hat{S}|P_N\rangle = |P_{N-1}\rangle,
$$

(91)

$$
\hat{S}^\dagger|P_{N-1}\rangle = |P_N\rangle,
$$

(92)

and leaves the electromagnetic part invariant, i.e. the state $\langle \hat{S}|P_N\rangle(A_T, \eta_T, \eta_T^\dagger)$ is parameterized by the family $\Psi_N(A_T)_{f_{N-1}}$. Having the bosonization formulae it is a simple task to calculate vacuum expectation values. One expresses the currents in terms of annihilators and creators and normal orders the exponentials using $e^{A}e^{B} = e^{A+B}e^{[A,B]}$.

In view of what we will eventually evaluate, namely $\theta$ expectation values, define the operator $\hat{J}_I$ invariant under large gauge transformations,

$$
\hat{J}_I(x,y) = e^{-ie\hat{A}_T(x-y)} \hat{J}(x,y).
$$

(93)

We obtain,

$$
\langle \Psi_N | \frac{1}{2}(\gamma^0 \pm \gamma^1) \cdot \hat{J}_I(x,y) | \Psi_N \rangle = \pm \frac{i}{L} \delta_{\gamma,0} e^{-\frac{2\pi i}{L} (x-y)} e^{-\frac{n_\gamma}{2} (x-y)^2} g_1(x-y),
$$

(94)

$$
\langle \Psi_N | \frac{1}{2}(\gamma \pm 1) \cdot \hat{J}_I(x,y) | \Psi_N \rangle = \delta_{\gamma,0} \Omega_0(y-x) e^{-\frac{2\pi i}{L} (x-y)} e^{-\frac{n_\gamma}{2} (x-y)^2} g_2(x-y),
$$

(95)
where

\[
g_1(x) = \exp \left( \sum_{n>0} \frac{1}{n} \left( 1 - \frac{1}{2} (\kappa_n^2 + \kappa_n^{-2}) + \frac{1}{2} (\kappa_n^2 - \kappa_n^{-2}) \cos p_n x \right) \right),
\]

\[
g_2(x) = \exp \left( \sum_{n>0} \frac{1}{n} \left( 1 - \frac{1}{2} (\kappa_n^2 + \kappa_n^{-2}) \right) \left( 1 - \cos p_n x \right) \right).
\]

We also recall that \( \kappa_n = \left[ 1 + \frac{M^2}{p_n^2} \right]^{\frac{1}{4}} \). By expanding \( g_1(x) \) to first order in \( C_n = \frac{1}{n} (1 - \kappa_n^2) \) it is easy to see that the fermionic part of (94) and (87) agrees. Furthermore when \( x = y \) (94) agrees with the result found in [3] up to an irrelevant phase which can be absorbed in the definition of \( |\Psi_N\rangle \). Finally defining \( \theta \) expectation values by

\[
\langle A \rangle_\theta = \frac{\langle \theta | A | \theta \rangle}{\langle \theta | \theta \rangle}
\]

one obtains

\[
\langle \frac{1}{2} (\gamma^0 \pm \gamma^1) \cdot \hat{J}_l (x, y) \rangle_\theta = \mp \frac{i}{L} e^{\pm i\theta} e^{-\frac{2\pi i}{L} (x-y) - \frac{\pi M}{L} (x-y)^2} g_1(x - y),
\]

\[
\langle 1 \cdot \hat{J}_l(x, y) \rangle_\theta = 0,
\]

\[
\langle \gamma \cdot \hat{J}_l (x, y) \rangle_\theta = \Omega_0(y - x) e^{-\frac{2\pi i}{L} (x-y) - \frac{\pi M}{L} (x-y)^2} g_2(x - y).
\]

To conclude this paper we use (98) to evaluate the chiral condensate,

\[
\langle \hat{\psi}(x) \hat{\psi}(x) \rangle_\theta = -2 \sin \theta e^{-\frac{\pi}{M}} g_1(0) \rightarrow -\frac{M}{2\pi} e^\gamma \sin \theta, \quad L \rightarrow \infty.
\]
A Eigenstates

We want to find eigenstates of $\hat{H}_F$, the fermionic part of the Hamiltonian. Let these states be parameterized by the $A_T$-independent family $f_N$ where $N$ is an integer. Denoting the eigenenergy by $E_N(A_T)$ we should solve

$$\hat{H}_F \Psi_{f_N}(\eta_T, \eta_T^*) = E_N(A_T) \Psi_{f_N}(\eta_T, \eta_T^*),$$

(102)

or in terms of the family $f_N$,

$$f_N^{(a)} = E_N(A_T) f_N^{(a)}$$

(103)

where $f_N^{(a)}$ is calculated using (44) ignoring the kinetic term for $A_T$. Make the ansatz

$$f_N^{(0)} = 1$$

and for $a \neq 0$,

$$f_N^{(a)}(x_1, y_1, \ldots, x_a, y_a) = \Phi^{(a)}(x_1, y_1, \ldots, x_a, y_a) \Omega_N(x_1 - y_1) \cdots \Omega_N(x_a - y_a),$$

(104)

where

$$\Omega_N(x) = -\frac{1}{L} \sum_n \text{sgn}(n + N) e^{ip_n x},$$

(105)

$$\Phi^{(a)}(x_1, y_1, \ldots, x_a, y_a) = \exp \left[ \sum_{i,j=1}^a \varphi(x_i - y_j) - \sum_{j>i}^a (\varphi(x_i - x_j) + \varphi(y_i - y_j)) \right]$$

(106)

and

$$\varphi(x) = \sum_{n>0} C_n (1 - \cos p_n x).$$

(107)

Furthermore we have defined,

$$\text{sgn}(n) = \begin{cases} 1, & n \geq 0 \\ -1, & n < 0 \end{cases}.$$ 

(108)

For some purposes it is convenient to rewrite $\Omega_N$ in the following manner,

$$\Omega_N(x) = \delta(x) + \omega_N(x),$$

(109)

where

$$\omega_N(x) = -\frac{2}{L} \sum_{n \geq -N} e^{ip_n x} = -\frac{2}{L} e^{-ip_N x} \sum_{n \geq 0} e^{ip_n x} = e^{-ip_N x} \omega_0(x).$$

(110)

Some important properties of $\Omega_N$ are established by Fourier transforming the product $g(x)\Omega_N(x)$ for any (periodic) function $g(x)$:

$$\lim_{x \to 0} (g(x)\Omega_N(x)) = \frac{1}{i\pi} g'(0) + g(0)\Omega_N(0),$$

(111)

$$\lim_{x \to 0} (g(x)\Omega_N'(x)) = \frac{i}{2\pi} g''(0) + g(0)\Omega_N'(0).$$

(112)
The singular quantities $\Omega_N(0)$ and $\Omega_N'(0)$ have to be regularized in a suitable manner for them to make sense. When analyzing (14) we will also use the following property of $\Phi$,

$$
\Phi^{(a+1)}(x_1, y_1, \ldots, x_a, y, x) =
\Phi^{(a+1)}(x_1, y_1, \ldots, x_b, x, y, y_b, \ldots, x_a, y_a)
= \Phi^{(a)}(x_1, y_1, \ldots, x_a, y_a)
\times \exp [\varphi(y - x) + \sum_{i=1}^{a} \varphi(x - x_i) - \varphi(y - x_i) + \varphi(y - y_i) - \varphi(x - y_i)].
$$

(113)

The eigenvalue $E_N(A_T)$ is easy to find. By (103) we have that $E_N(A_T) = f_N^{(0)}$. Remembering to ignore the kinetic term, (14) leads to

$$
E_N(A_T) = -i \int dy \lim_{y \rightarrow x} (\partial_y + ieb) f_N^{(1)}(y, x) = -\frac{L}{2\pi} \varphi''(0) + L(eb\Omega_N(0) - i\Omega_N'(0)),
$$

(114)

having used (111) and (112). We will regularize the singular part of $E_N$ by exponential regularization. Hence write,

$$
L(eb\Omega_N(0) - i\Omega_N'(0)) = -\frac{2\pi}{L} \sum_n (n + \frac{ebL}{2\pi}) \text{sgn}(n + N) e^{-|n + \frac{ebL}{2\pi}|}
= -\frac{2\pi}{L} \left[ \frac{2}{e^2} + \frac{1}{12} - \left( \frac{ebL}{2\pi} - N - \frac{1}{2} \right)^2 \right] + O(\epsilon).
$$

(115)

Subtracting the pole in $\epsilon$ and letting $\epsilon \rightarrow 0$ the expression for the regularized energy is,

$$
E_N(A_T) = -\frac{L}{2\pi} \varphi''(0) - \frac{\pi}{6L} + \frac{2\pi}{L} \left( \frac{eA_T L}{2\pi} - N - \frac{1}{2} - \alpha \right)^2.
$$

(116)

Now let’s find the eigenstates. Using (14) and (114) the equation (103) with $a = 1$ reduces to

$$
-i(\partial_x + ieb)\delta(x_1 - y_1) + W(x_1 - y_1)\Omega_N(x_1 - y_1)
+ i \int dx \Omega_N(x_1 - x)(\partial_x + ieb + \varphi'(x - y_1) - \varphi'(x - x_1))\Omega_N(x - x_1) = 0,
$$

(117)

where $W(x) = V(x) + D(x)$ and

$$
D(x_1 - y_1) = -\frac{1}{2\pi} \int dx (\varphi'(x - y_1) - \varphi'(x - x_1))^2 = -\sum_{n>0} C_n^2 n p_n (1 - \cos p_n(x_1 - y_1)).
$$

Similarly one finds for $a = 2$, using (117),

$$
0 = i \int dx \left[ \Omega_N(x_1 - y_1)[\delta(x_2 - y_2)\delta(x_2 - x) - \Omega_N(x_2 - x)\Omega_N(x - y_2)]
\times (\varphi'(x - x_1) - \varphi'(x - y_1)) + (1 \leftrightarrow 2) \right]
+ \Omega_N(x_1 - y_1)\Omega_N(x_2 - y_2)[W(x_1 - y_2) + W(x_2 - y_1) - W(x_1 - x_2) - W(y_1 - y_2)].
$$

(118)

(15)
If (117) and (118) are satisfied then (103) is satisfied for all \(a\). Using (109) one may obtain the identity,

\[
i \int dx \varphi'(x - x_1) \Omega_N(x_2 - x) \Omega_N(x - y_2) = i \delta(x_2 - y_2) \varphi'(x_2 - x_1) + \Omega_N(x_2 - y_2)(G(x_2 - x_1) - G(y_2 - x_1)),
\]

where,

\[
G(x) = \sum_{n>0} C_n p_n (1 - \cos p_n x).
\]

Hence (118) becomes,

\[
0 = \Omega_N(x_1 - y_1) \Omega_N(x_2 - y_2) \left( W(x_1 - y_2) + 2G(x_1 - y_2) + W(x_2 - y_1) + 2G(x_2 - y_1) - W(x_1 - x_2) - 2G(x_1 - x_2) - W(y_1 - y_2) - 2G(y_1 - y_2) \right) \quad (121)
\]

i.e. \(W(x) + 2G(x) = 0\) which leads to a quadratic equation for \(C_n\). Keeping only the root giving a convergent series for \(\varphi\) we end up with,

\[
C_n = \frac{1}{n} \left( 1 - \sqrt{1 + \frac{e^2 L^2}{4n^2 \pi^3}} \right) = -\frac{1}{n p_n} (\sqrt{p_n^2 + e^2/\pi} - p_n).
\]

By (103) one has,

\[
- i(\partial_x + i e b) \delta(x_1 - y_1) + i \int dx \Omega_N(x_1 - x) (\partial_x + i e b) \Omega_N(x - x_1) = 0,
\]

thus reducing (117) to \(W(x_1 - y_1) + 2G(x_1 - y_1) = 0\) having used (119).

### B Anomalous commutators

We will verify the chiral current algebra on the groundstate. First regularize the current operators by point splitting

\[
\hat{j}_\pm(x) = A_\pm \cdot \hat{J}(x, x) \rightarrow \hat{j}_\pm(x, \epsilon) = A_\pm \cdot \hat{J}(x, x + \epsilon)
\]

where \(A_\pm = \frac{1}{2} (1 \pm \gamma)\). From (17) it then follows that

\[
[j_\pm(x, \epsilon), j_\pm(y, \epsilon)] = \delta(x - y + \epsilon) A_\pm \cdot \hat{J}(x, y + \epsilon) - \delta(x - y - \epsilon) A_\pm \cdot \hat{J}(y, x + \epsilon)
\]

Now act with the commutator on the groundstate \(\Psi_N(A_T, \eta_T^*, \eta_T)\) parameterized by the family \(\Psi_N(A_T) f_N\). Let the resulting state be parameterized by a family \(\Psi_N(A_T) f_\pm\), i.e.

\[
\lim_{\epsilon \to 0} [j_\pm(x, \epsilon), j_\pm(y, \epsilon)] \Psi_N(A_T, \eta_T^*, \eta_T) = \Psi_N(A_T) \Psi_{f_\pm} (\eta_T^*, \eta_T)
\]

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By (37), (10) and (125) we obtain
\[
\begin{align*}
    f^{(0)}_{\pm} &= \pm \frac{1}{2} \lim_{\epsilon \to 0} \left\{ \delta(x - y + \epsilon) f^{(1)}_{N}(y + \epsilon, x) - \delta(x - y - \epsilon) f^{(1)}_{N}(x + \epsilon, y) \right\} \\
    &= \pm \frac{1}{2} \lim_{\epsilon \to 0} \left[ \{ \delta(x - y + \epsilon) - \delta(x - y - \epsilon) \} \Omega(2\epsilon) e^{\varphi(2\epsilon)} \right] \\
    &= \pm \frac{1}{2\pi i} \frac{\partial}{\partial (2\epsilon)} \left[ \{ \delta(x + y + \epsilon) - \delta(x + y - \epsilon) \} e^{\varphi(2\epsilon)} \right]_{\epsilon=0} \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(0)}_{N}
\end{align*}
\]

(127)

Here we have used the property (111) of \( \Omega_{N} \) and that \( \varphi'(0) = 0 \). For the higher levels we simply have
\[
\begin{align*}
    f^{(a)}_{\pm}(x_1, y_1, \ldots, x_a, y_a) &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a)}_{N}(x_1, y_1, \ldots, x_a, y_a) \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a+1)}_{N}(x_1, y_1, \ldots, x_a, y_a, x, y + \epsilon) \\
    &\quad - \delta(x - y - \epsilon) f^{(a+1)}_{N}(x_1, y_1, \ldots, x_a, y_a, x + \epsilon, y) \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a)}_{N}(x_1, y_1, \ldots, x_a, y_a)
\end{align*}
\]

(128)

All other terms in (37) and (10) are regular as \( \epsilon \to 0 \) and automatically cancel. A similar calculation to the one above yields
\[
\begin{align*}
    f^{(a)}_{\pm}(x_1, y_1, \ldots, x_a, y_a) &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a)}_{N}(x_1, y_1, \ldots, x_a, y_a) \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a+1)}_{N}(x_1, y_1, \ldots, x_a, y_a, x, y + \epsilon) \\
    &\quad - \delta(x - y - \epsilon) f^{(a+1)}_{N}(x_1, y_1, \ldots, x_a, y_a, x + \epsilon, y) \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) f^{(a)}_{N}(x_1, y_1, \ldots, x_a, y_a)
\end{align*}
\]

(129)

and therefore we have
\[
\begin{align*}
    [\hat{J}_{\pm}(x), \hat{J}_{\pm}(y)] \Psi_{N}(A_T, \eta^*_T, \eta_T) &= \pm \frac{1}{2\pi i} \delta'(x - y) \Psi_{N}(A_T, \eta^*_T, \eta_T) \\
    &= \pm \frac{1}{2\pi i} \delta'(x - y) \Psi_{N}(A_T, \eta^*_T, \eta_T)
\end{align*}
\]

(130)

Since the commutator is a \( c \)-number, this result holds not only for the groundstate but for the entire representation space defined by the groundstate(s).

C Boseonization

To prove the bosonization formulas, we simply show that the fermionic and bosonic operators have the same action on the groundstate \( |\Psi_{N}\rangle \). Acting with the different fermionic operators on \( |\Psi_{N}\rangle \) produce states parameterized by the family \( \Psi_{N}(A_T) \). The action of \( \frac{i}{2}(\gamma^0 \pm \gamma^1) \cdot \hat{J}(x, y) \) gives
\[
\begin{align*}
    f^{(0)} &= -\frac{i}{2} (\Omega_{N}(y - x) \pm \delta(y - x)) e^{\varphi(y-x)}, \\
    f^{(a)}(x_1, y_1, \ldots, x_a, y_a) &= \Phi^{(a+1)}(x_1, y_1, \ldots, x_a, y_a, y, x) \\
    &\times \left\{ -\frac{i}{2} (\Omega_{N}(y - x) \pm \delta(y - x)) \prod_{i=1}^{a} \Omega_{N}(x_i - y_i) \\
    &+ \frac{i}{2} \sum_{i=1}^{a} (\Omega_{N}(x_i - x) \pm \delta(x_i - x))(\Omega_{N}(y - y_i) \pm \delta(y - y_i)) \prod_{j \neq i} \Omega_{N}(x_j - y_j) \right\}
\end{align*}
\]

(131)

(132)
and for \( \frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x, y) \) we obtain
\[
f^{(0)} = \frac{1}{2} \Omega_N(y - x) e^{\varphi(y - x)},
\]
where
\[
f^{(a)}(x_1, y_1, \ldots, x_a, y_a) = \Phi^{(a+1)}(x_1, y_1, \ldots, x_a, y_a, x)
\]
and
\[
x \prod_{i=1}^{a} \Omega_N(x_i - y_i)
\]
we obtain
\[
-\frac{1}{2} \sum_{i=1}^{a} (\Omega_n(x_i - x) \mp \delta(x_i - x))(\Omega_n(y - y_i) \pm \delta(y - y_i)) \prod_{j \neq i} \Omega_N(x_j - y_j).
\]

In the following, let \( z \) denote the quantity \( z = e^{i \frac{2\pi}{L} \hat{z}} \). We have,
\[
\Omega_N(x) + \delta(x) = z^{-N_x} \sum_{n>0} z^{-nx} = z^{-N_x} \frac{2z^{-x}}{L 1 - z^{-x}},
\]
\[
\Omega_N(x) - \delta(x) = \omega_N(x) = -z^{-N_x} \sum_{n\geq0} z^{nx} = -z^{-N_x} \frac{1}{L 1 - z^x}.
\]
To avoid the pole in \( z = 1 \) one may regularize by e.g.
\[
\sum_{n\geq0} z^{nx} e^{-en}.
\]

We will assume that such a regularization has been done in otherwise ill defined expressions. Also define,
\[
\xi(x) = \sum_{n\geq0} \frac{1}{n} z^{nx} = -\log (1 - z^x).
\]
The identity \( e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \) valid when \([A, B]\) is a c-number will be used repeatedly.

We will prove now prove that
\[
\frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x, y) = \frac{i}{L} \hat{S} e^{\frac{2\pi i}{L}(x-y)} e^{\delta_+^+(x) - \delta_-^+(y) - \delta_-(x) + \delta_+^-(y)},
\]
which may be rewritten to yield
\[
\frac{1}{2}(\gamma^0 - \gamma^1) \cdot \hat{J}(x, y) = \frac{i}{L} \hat{S} z^{N(x-y)} e^{\xi(0)} e^{A},
\]
where
\[
A = \frac{1}{4} \int dx' \left( [\xi(x' - x) - \xi(y - x') - \xi(x - x') + \xi(x' - y)] \hat{J}_0(x') + [\xi(x' - x) + \xi(y - x') - \xi(x - x') - \xi(x' - y)] \hat{J}_5(x') \right).
\]
The action of \( A \) produces a state which can be written in the form \( B |\Psi_N\rangle \) having defined \( B \) as,
\[
B = \int dx' \left( \xi(x' - x) - \xi(y - x') + \varphi(x - x') - \varphi(y - x') \right) \hat{J}_0(x').
\]
Thus we have,
\[ e^{A}|\Psi_N\rangle = e^{A}e^{-A+B}|\Psi_N\rangle = e^{\frac{1}{2}[A,B]}e^{B}|\Psi_N\rangle, \]  
(142)
where the commutator is \( \frac{1}{2}[A,B] = \xi(y-x) - \xi(0) + \varphi(y-x) \). Hence the action of the right hand side of (139) on \( |\Psi_N\rangle \) by (37) gives a state parameterized by a family \( \Psi(\hat{A}_T)f'' \), where

\[ f''(0) = \frac{i}{L} z^{N(x-y)} e^{\xi(y-x)+\varphi(y-z)} \]  
(143)

\[ f''(x_1, y_1, \ldots, x_a, y_a) = \frac{i}{L} z^{N(x-y)} e^{\xi(y-x)+\varphi(y-z)} \Phi(a)(x_1, y_1, \ldots, x_a, y_a) \]
\[ \times \prod_{i=1}^{a} \Omega_{N-1}(x_i - y_i) \exp[\xi(x_i - x) - \xi(y-x_i) - \xi(y_i - x) + \xi(y-y_i) + \varphi(x - x_i) - \varphi(y - x_i) + \varphi(y - y_i) - \varphi(x - y_i)] \]  
(144)

Clearly, by (136) and \( e^{\xi(x)} = \frac{1}{1-ze^{x}} \), (143) and (134) coincide. For \( a \neq 0 \) consider,

\[ \Omega_{N-1}(x_i - y_i) \exp[\xi(x_i - x) - \xi(y-x_i) - \xi(y_i - x) + \xi(y-y_i)] \]
\[ = \delta(x_i - y_i) + \omega_N(x_i - y_i) z^{x_i-y_i} \frac{(1 - z^{x_i-x})(1 - z^{y-y_i})}{(1 - z^{x_i-x})(1 - z^{y-y_i})} \]
\[ = \delta(x_i - y_i) + \omega_N(x_i - y_i) \frac{(1 - z^{x_i-x})(1 - z^{y-y_i}) - (1 - z^{x_i-y_i})(1 - z^{y-x})}{(1 - z^{x_i-x})(1 - z^{y-y_i})} \]
\[ = \Omega_N(x_i - y_i) + \frac{L}{2} z^{-N(x_i-y_i)}(1 - z^{y-x})\omega_0(x_i - x)\omega_0(y - y_i). \]  
(145)

Now since the base states (34) change sign under the transformation \( x_i \leftrightarrow x_j \) or \( y_i \leftrightarrow y_j \) all terms in \( f''(a) \) invariant under this transformation will vanish when contracted with a base state. Dropping such symmetric terms and using (113) one sees that (144) and (132) coincide. For the other bosonic operators similar calculations lead to the same action as the fermionic operators on the groundstate. To prove equivalence for all states in the same representation space as the groundstate one also has to check the commutator algebra of the bosonic operators with the fermionic ones. We will leave this calculation out as it is well known.

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