Cylindrical Shape Decomposition Algorithm for 3D Segmentation

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Abstract

Shape decomposition is a fundamental problem in geometry processing where an arbitrary object is regarded as an arrangement of simple primitives or semantic components. The application of 3D shape decomposition in the context of image segmentation, however, is not well-studied. In this paper, we develop a shape decomposition algorithm called cylindrical shape decomposition (CSD) to be applied for the segmentation of tubular structures in large-scale 3D images. CSD starts by partitioning the curve skeleton of a tubular object into maximal-length sub-skeletons, minimizing an orientation objective. Each sub-skeleton corresponds to a semantic component. To determine boundaries between the semantic components, CSD searches for critical points where the object cross-section substantially changes. CSD then cuts the object at critical points and assigns the same label to those object parts which are along the same sub-skeleton, defining a semantic tubular component. CSD further rectify/reconstructs these semantic components using generalized cylinders. We demonstrate the application of CSD in the segmentation of large-scale 3D electron microscopy image datasets of myelinated axons, the decomposition of vascular networks, and synthetic objects. We also compare CSD to other state-of-the-art decomposition techniques in these applications. These experiments indicate that CSD outperforms other decomposition techniques and achieves a promising performance.

Keywords: Shape decomposition, Cylinder decomposition, Generalized cylinder, Tubular object, Image segmentation, Electron microscopy

1. Introduction

Shape decomposition is a fundamental problem in geometry processing in which an arbitrary object is regarded as an arrangement of simple primitives \cite{1,2} or semantic components \cite{3,4}. The application of shape decomposition mainly concerns disciplines such as object recognition and retrieval \cite{5,6}, shape reconstruction \cite{7}, shape clustering \cite{8}, or modeling \cite{9}. The application of shape decomposition in the context of image segmentation, however, is not well-studied. In this paper, we develop a decomposition algorithm called cylindrical shape decomposition (CSD) to apply for the segmentation of tubular structures in large-scale 3D image datasets.

Recent advances in biomedical imaging have made possible acquiring three-dimensional (3D) images of unprecedented size, from gigabytes to hundreds of terabytes \textsuperscript{10,12}. Typically, image segmentation of such datasets has been based on bottom-up strategies, where the image is first over-segmented into supervoxels, and subsequently, supervoxels are merged \textsuperscript{13-15}. However, the process is error-prone because both the over-segmentation and subsequent merge are usually based on local optimization process as opposed to meeting a global objective. In addition, the computational complexity of these methods depends on the number of supervoxels, which prohibitively increases time and memory consumption when encountering a large number (tens of thousands) of objects in large-scale datasets. Our idea is instead to approach the segmentation problem based on a top-down strategy, where an under-segmentation is followed by the subsequent split using \textit{a-priori} knowledge of the shapes to be segmented. This strategy enables dividing a large-scale image dataset into sub-domains, geometry/topology of which can be analyzed on a global objective. Also, these sub-domains can be processed independently and in parallel, addressing computation time and memory limitations. However, to manage the split operation, we need a new 3D shape decomposition algorithm. In particular, tubular structures are ubiquitous in biomedical image datasets: blood vessels in computed tomography angiography \textsuperscript{10,16} or neuronal structures in electron microscopy (EM) images \textsuperscript{11,17}. We apply top-down design to segment tubular structures in large-scale 3D image datasets.
datasets. For that, we first elevate tubular structures in the 3D image dataset. Tubular structures can be elevated on the foreground by the curvilinear-enhancement filters, e.g. the steerable filters [18, 19], orientation score-based diffusion filters [20, 21], or machine learning techniques using deep-convolutional neural networks [22]. We assume that a thresholding-based preliminary segmentation of a 3D image with enhanced tubular structures includes under-segmented tubular objects. Therefore, we apply CSD to semantically decompose and correct the under-segmented tubular objects, i.e., subsequent split.

Figure 1a shows an under-segmented tubular object, i.e., an object that is a union of several semantic tubular components. We propose to decompose the object into its semantic components using its curve skeleton and cross-sectional analysis. For that, we partition the curve skeleton of the object into maximal-length sub-skeletons, minimizing an orientation objective. Each sub-skeleton corresponds with a semantic tubular component. To determine the boundaries between the semantic components, we analyze the object cross-sections along its sub-skeletons, searching for critical points. A critical point is such a point that the object cross-section changes substantially. We restrict the cross-sectional analysis to intervals, called decomposition intervals, in the proximity of junction-points where skeleton branches connect. We cut the object at critical points, and assign the same label to those parts that are along the same sub-skeleton, defining a semantic tubular component. We further rectify/reconstruct these semantic components using generalized cylinders.

In this paper, our contributions are 1) an algorithm that partitions the skeleton graph of an object into distinct paths with maximum length and straightness, 2) introducing decomposition intervals to restrict cross-sectional analysis, 3) using Hausdorff distance to directly calculate cross-sectional changes compared to an average curve, and 4) applying generalized cylinders to reconstruct the decomposed semantic tubular components. We demonstrate the application of CSD in the segmentation of large-scale 3D-EM datasets of myelinated axons, the decomposition of vascular networks, and synthetic objects. We also compare CSD to other state-of-the-art decomposition techniques, approximate convex decomposition [1] and skeleton to surface mapping [23], in these applications. These experiments indicate that CSD outperforms other decomposition techniques and achieves a promising performance. The first example is shown in Fig. 1a that compares our CSD method to approximate convex decomposition.

2. Related works

Shape decomposition has been extensively studied in the computer vision. In the following, we review most relevant techniques. We distinguish between two types of shape decomposition techniques: 1) representing an object in terms of geometrically simple primitives, such as ellipsoids, generalized cylinders, or components with homogeneous characteristics [1, 2, 7, 24–28], and 2) decomposing an object into its semantic-components using object skeleton or Reeb graph [3, 4, 23].

Simple primitives are homogeneous components with a compact representation and efficient computation. Examples of primitives with a simple parametric representation include ellipsoids [24] and straight cylinders [25]. Higher-level primitives can be generated by trading-off the representation simplicity. We can obtain smaller number of decomposition components by applying higher-level primitives, e.g., tubular primitives [26], convex components [1, 27], generalized cylinders [2], and generalized sweep components [7, 28]. For example, tubular primitives in Plumber [26] are constructed applying a seeded region growing with heuristically set of sphere positions and radii [26]. Plumber mainly extracts ideal tubular components of an object and those object parts which are not classified as tubes are defined as body parts of the object [26]. Convexity-based methods are an interesting class of primitive-based decomposition techniques. These techniques were developed based on the human tendency of dividing objects into parts around concave regions [29]. The approximate convex components in [1] were merged on a volumetric consistency measure to form semantic parts. The merging objective, however, does
not address local volumetric differences within a semantic component and over-segments objects. Generalized cylinder decomposition is an alternative to the convex decomposition, quantifying cylindricity. Zhou et al. \cite{zhou2016quantifying} introduced a quantitative measure for the cylindricity following minimum description length principle \cite{zhang2007geodesic} as a measure of the skeleton straightness and the variation among the profiles. The global objective for merging local generalized cylinders is to minimize the cylindricity. The approach yields excellent decomposition in synthetic objects. However, applying the method on large-scale objects from biomedical images is computationally expensive for two reasons: 1) a large number of generalized cylinders should be fit into the object iteratively, and 2) the large number of generated generalized cylinders should be merged to satisfy the cylindricity objective.

Another set of primitives-based decomposition techniques is based on cross-sectional sweeping. These algorithms, computationally less demanding compared to the convexity and generalized cylinder methods, sweep an object to generate homogeneous sweeping components. Goyal et al. \cite{goyal2007cross} study generates local prominent cross-sections from initial seed points. This method is semi-automated and requires user interactions for varying the density of the cross-sections in different regions of an object, and also for avoiding the creation of prominent cross-sections in regions with no sweep evidence. As well, at intersections this method generates partial cross-sections, increasing the number of components. Li et al. \cite{li2009sweeping} study is based on sweeping the object along its curve skeleton in search for critical points. In this method, the geometric function over which cross-sectional changes are measured is the perimeter of cross-sections, which is overly sensitive to the surface noise. As well, in the absence of an objective to merge local object parts, this method can over-segment the object to all surface protrusions.

To decompose an object into its semantic-components, the curve skeleton or Reeb graph of an object can be utilized. Both concepts are object descriptors and they are able to guide the decomposition. The curve skeleton is a 1D representation of a 3D object \cite{hachisuka2000curve}, encoding its topology and geometry. The Reeb graph tracks topology changes in level sets of a scalar function \cite{sun2001reeb}. Reniers et al. \cite{reniers2007multi} developed a multi-scale skeleton extraction technique based on a global importance measure. The authors provided an object decomposition by defining skeleton-to-boundary mapping based on the shortest geodesics. The distance-based mapping, however, does not provide accurate boundaries between object parts. Au et al. \cite{au2017object} extract the curve skeleton by applying an implicit Laplacian smoothing with global positional constraints preserving the mesh connectivity and key features. The authors provide an object decomposition with an approximate measure of thickness about extracted curve skeletons. The method automatically assigns a cut to every skeleton branch, or semi-automatically enables the user to specify the desired number of segments. The automatic one-to-one assignment yields a sensitive decomposition to the skeletonization process which is prone to having spurious branches. Berretti et al. \cite{berretti2011sweeping} achieves the decomposition of a 3D mesh based on Reeb graph construction and refinement. The Reeb graph captures the surface topology and protrusions, and the refinement step uses curvature information and adjacency to the graph critical points to localize part boundaries. The boundaries, however, are not minimized over an internal energy to generate smooth cuts.

3. Preliminary definitions

In this section, we define the core concepts used in the paper as there are no generally accepted definitions for most of them.

**Object.** An object $\Omega \subset \mathbb{R}^3$ is a nonempty bounded open set. We assume that its boundary $\partial \Omega$ is homeomorphic to a 2-sphere. For a discrete object, which is a result of a foreground segmentation, we define a 3D binary image as $I : X \subset \mathbb{Z}^3 \rightarrow \{0, 1\}$, and a segmented object $\Omega := \{x \in X : I(x) = 1\}$, where $X$ is the image domain. Throughout the paper, $\Omega$, $\partial \Omega$, and $x$ are in $\mathbb{R}^3$ unless defined otherwise.

**Curve skeleton.** Given $\Omega$ and $\partial \Omega$, the curve skeleton $\Upsilon \subset \Omega$ is defined as a locus of centers of maximal inscribed balls \cite{hachisuka2000curve}. A ball $B(x, r)$ centered at $x \in \Omega$ with radius $r$ is maximally inscribed if its surface touches $\partial \Omega$ in at least two distinct points. Formally, $B$ is a maximal inscribed ball in $\Omega$ if $\forall B', B \subseteq B' \subseteq \Omega \Rightarrow B' = B$.

**Curve skeleton point type.** We distinguish three types of points on the curve skeleton of an object: 1) regular-points that have exactly two neighbor points on skeleton, 2) end-points that have exactly one neighbor point on skeleton, and 3) junction-points that have three or more neighbor points on skeleton \cite{hachisuka2000curve}. We denote the collection of junction-points as $J$ where $j \in J$ and the collection of end-points as $O$ where $o \in O$.

**Skeleton branch.** Removing junction points $J$ from the curve skeleton $\Upsilon$ results in disconnected simple curves, known as skeleton branches. The collection of skeleton branches is denoted as $\Gamma$ and a skeleton branch...
is $γ ∈ Γ$. For $γ(t) : [0, 1] → \mathbb{R}^3$, its arc-length is written as $l = \int_0^1 |γ′(t)| \, dt$ with the convention $γ′(t) := \frac{d}{dt} γ(t)$.

**Skeleton graph.** The topology of curve-skeleton $Γ$ can be represented as a connected acyclic undirected graph (i.e., a tree) $G_Γ = (V, E, L)$. There is a one-to-one map between skeleton branches in $Γ$ and edges in $E$ and a one-to-one map between the union of end and junction points $(O ∪ J)$ and vertices in $V$. This means that for each branch $γ ∈ Γ$ we associate exactly one edge in $e$ in $G_Γ$. $L ⊂ \mathbb{R}^+$ is the set of edge lengths. The length of an edge is the arc-length of its associated skeleton branch.

**Walk, path.** A walk is a finite or infinite sequence of edges which joins a sequence of vertices. A finite walk is a sequence of vertices $v_0, v_1, \ldots, v_n$ such that $v_i = v_{i−1}v_i$ for $i = 1, \ldots, n$. The vertex sequence of the walk is $(v_0, v_1, \ldots, v_n)$. A path is a walk in which all vertices are distinct.

**Sub-skeleton** is a path in the curve skeleton domain. If $W = \{e_1, e_2, \ldots, e_n\}$ is a path in the skeleton graph, and $\{γ_1, γ_2, \ldots, γ_n\}$ are corresponding skeleton branches, then $ψ = \bigcup_i γ_i ⊆ Γ$ is a sub-skeleton.

**Critical point.** A point on a sub-skeleton at which the cross-sectional contour of the object changes substantially. We provide a formal definition in section 6.2.

**Cut.** A closed simple curve $C ⊂ ∂Ω$ is a cutting-curve if $∂Ω \setminus C$ is not connected. Cut means removal of a cutting-curve from the surface.

4. **Outline of the CSD algorithm**

The outline of the CSD algorithm is as follows:

1. Define the curve skeleton of a given object (section 5.1).
2. Partition the curve skeleton of the object into sub-skeletons (section 5.2).
3. Define decomposition intervals to restrict the object sweep (section 6.1).
4. Find critical points by sweeping the object (section 6.2).
5. Cut the object into parts and reconstruct the object between parts that have the same label using generalized cylinders (section 7).

5. **Skeleton partitioning**

We use the curve skeleton of an object to drive the decomposition. For that, we partition the skeleton graph into several distinct paths union of which covers the skeleton graph. Partitioning the skeleton graph, by extension, partitions the curve skeleton into sub-skeletons. And each sub-skeleton corresponds to exactly one semantic object component.

5.1. **Curve skeleton**

To determine the curve skeleton of an object $Ω$ with sub-voxel precision, we apply a method from [33, 34]. The algorithm initiates by determining a point $x^∗ ∈ Ω$ with the largest distance from the object surface $∂Ω$ inside the object domain. This point is used to determine a skeleton branch $γ(t) : [0, 1] → \mathbb{R}^3$, starting at $x_s$, the furthest geodesic point from $x^∗$ in $Ω$, and ending at $x^∗$. A cost function $F$ is defined to enforce the path to run at the middle of $Ω$, where $F$ should increase if the path moves away from the center. To determine $F$, we find the the distance field $D(x)$ from $∂Ω$, and assign $F = 1 − \left(\frac{D(x)}{D(x^∗)}\right)^2$. The distance field $D(x)$ is determined by solving an Eikonal equation on the object domain $Ω$ using the fast marching method [35]. Starting at $x_s$, the skeleton branch $γ$ is traced by a back-tracking procedure on $F$ to reach $x^∗$, written as

$$γ = \arg \min_P \int_{x_s}^{x^∗} F(P(t)) \, dt,$$

where $t$ traces the path $P$. For the back-tracking procedure, we use Euler scheme, which solves the ordinary differential equation with a sub-voxel accuracy. This process is repeated to determine further branches which form the curve skeleton of the object. But rather than using the single point $x^∗$ as the start point for the fast marching method, all points in the previously calculated branches are used as start points. We propagate a new wave from the start points with the speed $F$ to update $x_s$. The point $x_s$ is now the furthest point from the current state of the curve skeleton and the start point of the new branch. Applying a back-tracking algorithm from the updated $x_s$ defines the new skeleton branch. Figure 2 shows skeletons of a synthetic object and a vascular network.

5.2. **Skeleton graph decomposition**

Detecting the branches of the skeleton of an object is not sufficient for the decomposition of the skeleton of that object as several branches are often required to represent one semantic component of the object. An example is shown on Fig. 2(a) where, e.g., $γ_1, γ_2, γ_3$ represent the same semantic component of the object.

We decompose the skeleton graph of the object into several distinct paths union of which covers the set of
graph edges. Formally, we decompose the skeleton graph $G_T(V, E, L)$ into $m$ paths $W_i, i = 1, \ldots, m$ so that $\cup_i W_i = E$ and $W_i \cap W_j = \emptyset \forall i, k = 1, \ldots, m, i \neq k$. To determine the paths we require four conditions: 1) the path contains the longest edge not associated to any sub-skeleton, 2) the path has the maximum number of edges, 3) the associated angle between two successive edges is bigger than $\theta_e$, and 4) the path minimizes the orientation cost function $\Theta$ written as

$$W = \{e_1, \ldots, e_n\}$$
$$\Theta(W) = \sum_{s=1}^{n-1} \pi - \angle(e_s, e_{s+1}),$$

where $\angle()$ is the angle between the two line segments connecting endpoints of the skeleton branches associated with edge $e_s$ and $e_{s+1}$. The angle between line segments lies in range $[0, \pi]$. We used Algorithm 1 to determine the skeleton graph decomposition of the synthetic object $n=7$ into three paths $m=3$. Each path is equivalent to a sub-skeleton.

Figure 2: The curve skeleton of an object is the union of all skeleton branches. (a) The skeleton of the synthetic object, size: 800 x 400 x 70 voxels, 7 branches. The blue filled-circles show junction-points, and the red filled-circles show end-points. The skeleton graph of this object is $G_T(V, E, L)$, where $V = \{v_1, \ldots, v_T\}$, $V = \{v_0, \ldots, v_T\}$, and $L = \{l_1, \ldots, l_T\}$. (b) The skeleton of a vascular network, size: 256 x 256 x 256 voxels, 20 branches. Skeleton branches are color coded.

6. Cylindrical decomposition

In this section, we propose a method to decompose an object into parts and junction-surfaces by cutting the object at critical points. To determine critical points, we sweep the object along sub-skeletons in decomposition intervals to find locations where the object geometry changes substantially (see Fig. 6.4).

6.1. Decomposition interval

We restrict the sweep of the object along each sub-skeleton to decomposition intervals in the proximity of a junction point $j$ on sub-skeleton $\psi$ as illustrated in Fig. 6.4a. It is convenient to work with parametrized sub-skeleton $\psi(t) : [0, 1] \rightarrow \mathbb{R}^3$. We define two decomposition intervals $[t_j^r, t_j^l]$ and $[t_j^c, t_j^r]$ for each junction point as in Fig. 6.4a. To determine the boundaries of a decomposition interval, we define an upper threshold $r_s$ and a lower threshold $r_c$. We specify $r_s$ and $r_c$ based on the radius of the maximal inscribed ball at $j$ and two factors $\alpha_j \geq 1$ and $\alpha_c \geq 0$ where $\alpha_j \geq \alpha_c$, as $r_s = \alpha_j x r$ and $r_c = \alpha_c x r$.

To determine the thresholds, we use the signed arc-length from $j$. Define $t_j$ so that $j = \psi(t_j)$. Then $t_j^r (t_j^c)$ is such a point on the sub-skeleton that signed arc-length from $t_j$ to $t_j^r(t_j^c)$ equals $r_s (r_c)$.

Input: $G_T = (V, E, L); \theta_e$.
Output: Collection of distinct paths $\Lambda$.

\begin{algorithm}
\begin{algorithmic}
\State $\Lambda \leftarrow \emptyset$
\While{$E \neq \emptyset$}
\State $W \leftarrow \emptyset; e^* \leftarrow \text{longest } e \in E$
\State $V^* \leftarrow \{\text{incident vertices to } e^*\}$
\State $W \leftarrow W \cup \{e^*\}$
\ForAll{$v \in V^*$}
\State $e^{ref} \leftarrow e^*$
\State $\varphi_{next} \leftarrow v$
\While{$\deg(\varphi_{next}) > 1$ and $e^{ref} \neq \emptyset$}
\State $CE \leftarrow \{\text{edges connected to } \varphi_{next} \} \setminus e^{ref}$
\State $e_{next} \leftarrow \emptyset$
\ForAll{$e^{ref} \in CE$}
\State $\varphi_{next} \leftarrow v^*$, where $e^{ref} = (v_2, \varphi_{next})$
\State $W \leftarrow W \cup \{e^{ref}\}$
\EndFor
\State $\Lambda \leftarrow \Lambda \cup \{W\}$ and $E \leftarrow E \setminus W$
\EndWhile
\EndForAll
\EndWhile
\EndWhile
\EndAlgorithm

Algorithm 1: Decomposing the set of edges of $G_T$ into distinct paths. A vertex and an edge are called incident if the vertex is on one of the two vertices the edge connects.

Figure 2: The curve skeleton of an object is the union of all skeleton branches. (a) The skeleton of the synthetic object, size: 800 x 400 x 70 voxels, 7 branches. The blue filled-circles show junction-points, and the red filled-circles show end-points. The skeleton graph of this object is $G_T(V, E, L)$, where $V = \{v_1, \ldots, v_T\}$, $V = \{v_0, \ldots, v_T\}$, and $L = \{l_1, \ldots, l_T\}$. (b) The skeleton of a vascular network, size: 256 x 256 x 256 voxels, 20 branches. Skeleton branches are color coded.
point on the sub-skeleton that signed arc-length from \( t_j \) to \( t_j' \) (\( t_j'' \)) equals \( r_j \). We have \( t_j'' < t_j' < t_j < t_j' \).

The upper and lower thresholds may imply arc-distances outside parametrization limits of \( \psi \). If the arch-length from \( \psi(0) \) to \( \psi(t_j) \) is smaller than \( r_j \) (\( r_j \)) we assign \( t_j'' = 0 \) (\( t_j' = 0 \)). And if the arch-length from \( \psi(t_j) \) to \( \psi(1) \) is smaller than \( r_j \) (\( r_j \)) we assign \( t_j' = 1 \) (\( t_j'' = 1 \)).

Also, when a junction-point is at the either ends of a sub-skeleton, e.g., in a T-shape object, we define only one decomposition interval. Therefore, if \( \psi(t_j) = 0 \) (\( \psi(t_j) = 1 \)) the only interval that we define is \( [t_j'', t_j'] \) (\( [t_j', t_j'' \) ). Figure 3B shows decomposition intervals in the proximity of \( j_1 \) and \( j_2 \) on sub-skeletons \( \psi_1, \psi_2 \), and \( \psi_3 \).

6.2. Critical point

A critical point on a sub-skeleton is such a point that the cross-sectional contour of the object at this point changes substantially (Fig. 3). We use Hausdorff metric to compare geometrical changes between cross-sectional contours in a decomposition interval. The Hausdorff distance between two curves \( C_1 \) and \( C_2 \) is calculated as

\[
\mathcal{H}(C_1, C_2) = \max \{ \sup_{p \in C_1} \inf_{q \in C_2} d(p, q), \sup_{q \in C_2} \inf_{p \in C_1} d(p, q) \},
\]

where \( d(.) \) is the Euclidean distance between two points. We sweep \( \partial \Omega \) by a cross-sectional plane \( \mathcal{P} \subset \mathbb{R}^3 \) to extract the cross-sectional contours. A cross-sectional plane \( \mathcal{P}(t) \) is a plane orthogonal to \( \psi \) at every point \( t \) along \( \psi \). The plane normal is equal to the tangent vector to \( \psi \) at point \( \psi(t) \). We sweep \( \partial \Omega \) by \( \mathcal{P} \) along \( \psi \) in \( [t_j'', t_j'] \) interval starting at \( t_j'' \) toward \( t_j' \), and in \( [t_j', t_j''] \) interval starting at \( t_j' \) toward \( t_j'' \) as illustrated in Fig. 4B

Let \( \mathcal{P}(t) \) intersects \( \partial \Omega \) at an inquiry point \( t \). Since we assumed that \( \partial \Omega \) is homeomorphic to a 2-sphere, the cross-sectional contour \( C(t) \) : \([0, 1] \rightarrow \mathbb{R}^2 \) is a simple closed curve, where \( C(0) = C(1) \). Translating \( \mathcal{P} \) along \( \psi(t) \) with \( t \) moving in decomposition intervals, we compare the Hausdorff distance between the cross-sectional contour at \( t \) denoted as \( C \), with the average of visited cross-sectional contours \( \mu \).

To find the average curve \( \mu \) between two nearly similar curves \( C_1 \) and \( C_2 \), we first need a one-to-one mapping between \( C_1 \) and \( C_2 \). We use OrthoMap \( OM \) method from [60]. Consider that \( C_1 \) is parameterized by \( s \). To each point \( C_1(s) \) of \( C_1 \), the \( OM(C_1, C_2) \) associates the closest point \( C_2(s') \) on \( C_2 \) that lies on the line passing through \( C_1(s) \) and having for direction the normal \( \nu(C_1(s)) \) at \( C_1(s) \). Having this mapping, then each point \( C_2(s') \) of \( C_2 \) may be expressed as the normal offset \( C_1(s) + d(s')\nu(C_1(s)) \) of \( C_1(s) \). We say that \( C_1(s) \) is the closest normal projection of \( C_2(s') \) onto \( C_1 \) and can express \( C_2 \) as a deformation of \( C_1 \) completely defined by the normal displacement field \( d(s') \). The average curve obtained by OrthoMap correspondence is asymmetric: \( OM(C_1, C_2) \) is not necessarily equal to \( OM(C_2, C_1) \). Therefore, we consistently take \( C_1 \) as an already visited curve, \( C_2 \) as the new cross-sectional curve, and define the average curve over \( OM(C_1, C_2) \).
We normalize the Hausdorff distance $\mathcal{H}(C_r, \mu)$ to the range $[0, 1]$ and denote it as $H_s(t)$. For that we first find a point interior to $C_i$ denoted as $\kappa$. We define $\kappa \in \mathbb{R}^2$ to be the intersection of $\mathcal{P}$ and $\psi$ at point $t$. Defining $d_C(\kappa) = \sup_{q \in C_i} d(\kappa, C_i)$, we write $H_s(t)$ as

$$H_s(t) = \frac{\mathcal{H}(C_r, \mu)}{\mathcal{H}(C_r, \mu) + d_C(\kappa)}.$$ (4)

We define a similarity threshold between cross-sectional contours as $\theta_H$. While sweeping $\partial \Omega$ along $\psi$ from $t^c_1$ to $t^c_2$, if $H_s(t) < \theta_H$, the inquiry continues to the next point. However, if $H_s(t) \geq \theta_H$ the inquiry stops at $t$ and the point is called a critical point, denoted as $t_{c_1}$ ($t_{c_2}$) as shown in Fig. 5. In $[t^c_1, t^c_2]$ $([t^c_2, t^c_1])$, if at no inquiry point $H_s(t)$ exceeds $\theta_H$, we define the $t_{c_1}$ ($t_{c_2}$) as the point with minimum arc-distance $r (-r)$ to $\psi(t)$ at which $H_s$ is maximum.

![Figure 5: Sweeping the object surface along the sub-skeleton $\phi_1$ at junction-point $j_1$ between $[t^c_1, t^c_2]$ and $[t^c_2, t^c_1]$. At any decomposition interval, if $H_s < \theta_H$ the inquiry continues to the next point. If $H_s(t) \geq \theta_H$ the inquiry stops at $t$ and the point is called a critical point. The critical point in the first interval is denoted as $t_{c_1}$, and in the second interval is denoted as $t_{c_2}$.](image)

7. Object reconstruction

We cut the object at all critical points and decompose $\partial \Omega$ into $n$ parts, $n$ is the number of skeleton branches, and $\delta$ junction-surfaces, $\delta$ is the number of junction-points. We distinguish between an object part and a junction-surface such that the interior of a junction-surface includes a junction-point. The final decomposition step is to discard junction-surfaces and assign the same label to those object parts that are along the same sub-skeleton to obtain $m$ semantic tubular components, $m$ is the number of sub-sketches. As we discard the junction-surfaces, we reconstruct the semantic tubular components using generalized cylinders. A generalized cylinder $\Phi(u, \zeta) : [0, 1]^2 \rightarrow \mathbb{R}^3$ represents an elongated surface on an arbitrary axis and smoothly varying cross-sections [37]. In Cartesian coordinates $x_1, x_2, x_3$, the axis is parametrized by $u$ as $\zeta(u) = (x_1(u), x_2(u), x_3(u))$ and cross-section boundary is represented as $C_u(\zeta) = (x_1(u, \zeta), x_2(u, \zeta))$. To construct $\Phi$, we apply a translational sweep along $\zeta(u)$ using closed simple curves $C_u(\zeta)$ written as

$$\Phi(u, \zeta) := \{\zeta(u) \in \mathbb{R}^3, C_u(\zeta) \in \mathbb{R}^2 : u, \zeta \in [0, 1]\}.$$ (5)

To obtain a parametric representation of generalized cylinders, it is convenient to employ a local coordinate system defined with origin at each point of $\zeta(u)$. A convenient choice is the Frenet-Serret frame which is suitable for describing the kinematic properties of a particle moving along a continuous, differentiable curve in $\mathbb{R}^3$. The Frenet-Serret frame is an orthonormal basis composed of three unit vectors $e_T, e_N, e_B$, where $e_T$ is the unit tangent vector, and $e_N$ and $e_B$ are the unit normal and unit binormal vectors, respectively. By defining the cross-section in the Frenet-Serret frame, we form a parametric representation of generalized cylinders [38] as follows:

$$\Phi(u, \zeta) = \zeta(u) + x_1(u, \zeta)e_N(u) + x_2(u, \zeta)e_B(u)$$ (6)

To define $C_u(\zeta)$, we use homotopy between two curves $C_{c_1}(\zeta)$ and $C_{c_2}(\zeta)$, where the curves are obtained by cross-sectioning the object surface at critical points $t_{c_1}$ and $t_{c_2}$, respectively (see Fig. 6). Let the simple closed curves $C_{c_1}(\zeta)$ and $C_{c_2}(\zeta)$ in $\mathbb{R}^2$ be homotopic with a continuous map $h : [0, 1]^2 \rightarrow \mathbb{R}^2$. So, we write:

$$h(0, \zeta) = C_{c_1}(\zeta), \quad h(1, \zeta) = C_{c_2}(\zeta), \quad \forall \zeta \in [0, 1],$$ (7)

$$h(u, 0) = h(u, 1), \quad \forall u \in [0, 1],$$ (8)

where $h$ is called a homotopy from $C_{c_1}(\zeta)$ to $C_{c_2}(\zeta)$. We denote a cross-section at a point along $\zeta(u)$ as $C_u := h(u, \cdot)$. Note that, $\mathbb{R}^2$ is simply connected space. We use a linear homotopy between $C_{c_1}(\zeta)$ to $C_{c_2}(\zeta)$ defined as:

$$h(u, \zeta) = (1 - u) C_{c_1}(\zeta) + u C_{c_2}(\zeta)$$ (9)

where the computation on the right side is in $\mathbb{R}^2$. Equation 9 essentially indicates that we are moving from $C_{c_1}(\zeta)$ to $C_{c_2}(\zeta)$ along a straight lines. To define the curve $\zeta(u)$, we use an interpolation between $\psi(t_{c_1})$ and $\psi(t_{c_2})$. Figure 6b shows $\Phi$ on different choices of $\zeta$.

8. Experimental results

In this section, we evaluate the effect of different parameters on the decomposition, present CSD applica-
tion in the decomposition of tubular objects in biomedical images, and show its performance for more general objects.

8.1. Parameter setting

In our experiments, we fix the value of $\alpha_e = 1$ (defined at section 6.1) which means that $t^e_0$ ($t^e_*$) is a point such that its distance to a junction-point is equal to the radius of the maximal inscribed ball at that junction-point. We examine the effect of $\alpha_s$ (defined at section 6.1) which determines $t^s_0$ ($t^s_*$) in decomposition intervals and set $\theta_H = 0.7$ (defined at section 6.2). We use a linear interpolation to define $\xi$ (defined at section 7), and set $\theta_s = 0^\circ$ (defined at section 5.2). Figure 7 shows the decomposition of the synthetic tubular object at $\alpha_s = 10, 20, 30$. For smaller values of $\alpha_s$, sweeping the object starts from points closer to junction-points. The decomposition/reconstruction at $\alpha_s = 10$ is faithful to the original object because the critical points are detected close to the junction-points. However, when $\alpha_s$ is equal to 30, CSD does not tolerate gradient changes of the tube diameter. This suggests that $\alpha_s = 30$ is a big value for starting the cross-sectional sweep. Setting $\alpha_s$ to small values provides more accurate decomposition results, but it also increases the risk that the cross-sectional sweep starts at a point very close to the junction, resulting in inaccurate boundaries between semantic tubular parts. This is important when we apply CSD on tubular objects extracted from biomedical images, because 1) a preliminary foreground segmentation contains surface noise, and 2) the curve skeleton may not exactly lie in the center of the object. In both cases, the radius of the maximum inscribed ball at a junction-point can be measured smaller than its true value. For $\alpha_s$ and $\alpha_e$, we suggest values in range [5, 20] and [0.5, 2], respectively.

We also examine the effect of $\theta_H$, which determines the similarity threshold between a cross-sectional contour and $\mu$. Figure 8 shows the decomposition of the synthetic object at $\theta_H = 0.6, 0.7, 0.8$. To better demonstrate the effect of $\theta_H$, we set $\alpha_s = 30$ which we earlier showed that the decomposition/reconstruction at that $\alpha_s = 30$ is not faithful to the original object. We set $\alpha_e = 1$ and used a linear interpolation to define $\xi$. We set $\theta_s = 0^\circ$. At $\theta_H = 0.6$, CSD is sensitive to the gradient increase of the tube diameter, hence the critical points are detected distant from junction-points. However, at $\theta_H = 0.7$ and $\theta_H = 0.8$ the tolerance of CSD to cross-sectional changes increases, thus the decomposition becomes more accurate despite distant starting points for cross-sectional sweeping. Note that, increasing $\theta_H$ elevates the tolerance of CSD to the cross-sectional changes very quickly, e.g., at $\theta_H = 0.9$ the algorithm tolerates 9 times difference between a cross-section and $\mu$, and at $\theta_H = 0.95$, it tolerates 19 times difference. We suggest $\theta_H$ to be in range [0.7, 0.85].

To partition the skeleton graph, we assume that at a
vertex the angle between two successive edges is bigger than \( \theta_c \). By setting \( \theta_c \) to bigger values we emphasize the straightness of a path, but then a path may not be maximal-length. Figure 9 shows how \( \theta_c \) affects the number of object partitions. At \( \theta_c = 0^\circ \), we obtain maximal-length paths and minimum number of object partitions (Fig. 2b). By increasing \( \theta_c \), less edges are allowed to append while defining the paths which increases the number of object partitions. Figure 9c shows that decomposition for \( \theta_c = 135^\circ \) yields 4 semantic components. At \( \theta_c = 180^\circ \), every edge in the skeleton graph is a path, hence producing the maximum number of semantic components which is equal to the number of skeleton branches.

![Tubular object](image)

(a) Tubular object

![Path with \( \theta_c = 0^\circ \), \( m = 3 \)](image)

(b) \( \theta_c = 0^\circ \), \( m = 3 \)

![Path with \( \theta_c = 135^\circ \), \( m = 4 \)](image)

(c) \( \theta_c = 135^\circ \), \( m = 4 \)

![Path with \( \theta_c = 180^\circ \), \( m = 5 \)](image)

(d) \( \theta_c = 180^\circ \), \( m = 5 \)

Figure 9: The angle between two successive edges in a path should be bigger than \( \theta_c \). (a) A synthetic tubular object, size: 300 \( \times \) 300 \( \times \) 300 voxels. (b) Setting \( \theta_c = 0^\circ \) produces maximal-length paths, minimum number of object partitions \( m = 3 \). (c) At \( \theta_c = 135^\circ \), the number of semantic components increases to \( m = 4 \). (d) At \( \theta_c = 180^\circ \), every edge in the skeleton graph is a path, hence producing the maximum number of semantic components which is equal to the number of skeleton branches, \( m = 5 \).

8.2. Axon segmentation in EM volumes

The main purpose of developing CSD is to segment tens of thousands of myelinated axons in large-scale white matter 3D-EM datasets of size 4000 \( \times \) 2000 \( \times \) 1300 voxels. We generate a probability map of myelinated axons using deep convolutional neural networks (for details refer to [39]). We threshold the probability map and using connected component analysis obtain a preliminary foreground segmentation of myelinated axons. Figure 10 (first column) shows examples of myelinated axons after connected component analysis with an under-segmentation error(s): an axon intersects other axons or merges with the extra-axonal space. We apply CSD to evaluate every preliminary segment for the under-segmentation error. If CSD recognizes the error in a preliminary segmented component, it decomposes the segmented component into its semantic components. Figure 10 shows the performance of CSD in decomposition of axons and also compares our method to the approximate convex decomposition [11] and skeleton to surface mapping [23] approaches. The approximate convex decomposition algorithm functions on point cloud representation of objects. To apply it on large-scale objects, we first down-sample the point cloud representation of objects to 50 000 points, enabling the decomposition to be performed in a reasonable time (less than 10 minutes per object). Figure 10 (second column) shows that approximate convex method over-segments the objects. We perform skeleton to surface mapping decomposition based on Voronoi partitioning of the surface, using Euclidean distance to skeleton branches (Fig. 10 (third column)). Since skeleton captures the geometry and topology of an object, skeleton to surface mapping decomposes the object close-to-semantic, but it does not recognize junction-surfaces for the further object reconstruction. Figure 10 (last column) shows our decomposition of myelinated axons. CSD provides minimum number of segments and a semantic segmentation of myelinated axons. The objects are reconstructed at junction-surfaces using generalized cylinders \( (\alpha_s = 10, \alpha_c = 1.5, \theta_B = 0.85, \text{spline interpolation to define } \xi, \text{ and } \theta_c = 90^\circ) \). Figure 11 shows the complete segmentation of myelinated axons in a large-scale 3D-EM dataset, where CSD scans, decomposes, and reconstructs about 30 000 myelinated axons.

8.3. Decomposition of vascular networks

We examine our method, approximate shape decomposition [11], and skeleton to surface mapping on a vascular network (acquired from [https://github.com/cbm755/fast_marching_kroon](https://github.com/cbm755/fast_marching_kroon)) as shown in Fig. 12a. Figure 2b shows 20 skeleton branches extracted from the vascular network. Figure 12b shows that approximate convex decomposition over-segments the vascular network. Figure 12c shows that surface to
Figure 10: First column: examples of foreground segmentation of myelinated axons with under-segmentation. Second column: decomposition using approximate convex method. The point cloud representation of objects are first down-sampled to 50,000 points to enable the decomposition task in a reasonable time. The method over-segments the objects. Third column: skeleton to surface mapping based on Voronoi partitioning of the surface using skeleton branches. Last column: CSD decomposition provides minimum number of segments and a semantic segmentation of myelinated axons. The objects are reconstructed at junction-surfaces using generalized cylinders ($\alpha_r = 10$, $\alpha_e = 1.5$, $\theta_R = 0.85$, spline interpolation to define $\zeta$, $\theta_t = 90^\circ$). Objects inside boxes are magnified.
Figure 11: (a) A large-scale 3D-EM dataset of the white matter. The size of the dataset is 4055 × 2002 × 1292 voxels in x, y and z directions, respectively. (b) A 3D rendering of myelinated axons (at one-third of the original resolution). CSD evaluates a preliminary segment for under-segmentation error(s), and if required decomposes and reconstructs under-segmented myelinated axon. (c) A 3D rendering of myelinated axons sampled at different locations illustrating the diversity of thickness and orientation in segmented axons.

skeleton mapping decomposes the object into 20 segments based on Euclidean distance to skeleton benches, but the boundaries between object parts are not accurate. For example, Fig. 12c (magnified box) shows that where the thin vessel (green partition) bends on the thick vessel (red partition), skeleton to surface mapping erroneously assigns a section of the thick vessel to the thin vessel; the section which has smaller distance to the skeleton of the thin vessel. CSD decomposes the object into 8 semantic components and reconstructs the object at the junction-surfaces ($\alpha_s = 4, \alpha_e = 1, \theta_H = 0.85$, spline interpolation to define $\zeta, \theta_c = 90^\circ$).

8.4. Decomposition of synthetic objects

To demonstrate the general applicability of our algorithm, we examine CSD on synthetic objects with voxel-based representation from McGill 3D Shape Benchmark [6]. These objects have a wider sense of tubularity, as for example tables or birds are composed of flat object parts. We decompose the objects by increasing the length of decomposition intervals, bigger values of $\alpha_s$, smaller values of $\alpha_e$, increasing $\theta_H$ to tolerate bigger cross-sectional changes, and dropping the object reconstruction step. Figure 13 shows a gallery of decomposition on a mixture of objects with articulating parts, such as humans, spiders, pliers and objects with moderate articulation, such as tables, and dinosaurs, and airplanes. A grey-colored partition in decomposed objects of Fig. 13 depicts the junction-surfaces. Table 1 shows how many skeleton branches and object partitions are extracted from different objects.

8.5. Computation time

The time complexity of the sub-voxel precise skeletonization is $O(n N_\Omega \log N_\Omega)$, where $n$ is the number of skeleton branches and $N_\Omega$ is the number of voxels in a discrete $\Omega$. The $N_\Omega \log N_\Omega$ factor is from the fast marching algorithm [35]. The time complexity to determine a critical point is $O(N_p)$ where $N_p$ is the number of points along the sub-skeleton in the decomposition interval that the critical point belongs. Defining the critical points is independent of $N_\Omega$. The complexity of the method is measured through the number of basic arithmetic operations performed; other factors that may also influence the execution time, such as the number of memory accesses or memory consumption, have not been considered.

On a $2 \times$ Intel Xeon E5 2630 CPU 2.4 GHz machine with 512 GB RAM using Python 2.7, skeletonization of a myelinated axon (Fig. 10, first row) consumes 117 s and defining its critical points 353 s. Down-sampling skeleton partitions with a factor have minimum effect on the results but reduces the search for critical points significantly, e.g., down-sampling with a factor 1/5 reduces the time to 75 secs. On the same machine, skeletonization of synthetic objects (Fig. 13) consumes 4-6 secs and defining the critical points 8-15 secs.
Figure 12: (a) A vascular network as a complex tubular structure with 20 object parts (the skeleton of the vascular network is presented in Fig. (b)). (b) Approximate convex decomposition over-segments the vascular network. (c) Surface to skeleton mapping decomposes the object into 20 object parts. The boundaries between object parts are not accurate. (d) CSD decomposes the object into 8 semantic components and reconstructs the object at junction-surfaces. Objects inside boxes are magnified.

9. Conclusion

In this paper, we propose the application of 3D shape decomposition in image segmentation. We apply CSD to rectify under-segmented tubular objects in the foreground segmentation of large-scale image datasets. We drive the decomposition on the skeleton graph of objects to determine their semantic tubular components. For that, we partition the skeleton graph into maximal-length paths, minimizing an orientation objective. Each path corresponds with a sub-skeleton, and each sub-skeleton corresponds with a semantic component. To determine fine boundaries between semantic components, we propose to sweep the object cross-sections along sub-skeletons and determine the critical points. We use Hausdorff distance to directly compare cross-sectional curves with the average of visited curves. The search for critical points is restricted to decomposition intervals defined in the proximity of junction-points. We obtain the semantic decomposition of the object by cutting it into parts and subsequently, give the same label to parts along the same sub-skeleton. The final step in CSD is to reconstruct the object using generalized cylinders. We demonstrate the application of CSD for the segmentation of large-scale 3D-EM datasets of myelinated axons, the decomposition of vascular networks, and synthetic objects. Comparing CSD to other state-of-the-art decomposition techniques in these applications shows that CSD outperforms these techniques.

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Figure 13: A gallery of CSD decomposition of synthetic objects. Recognized semantic components are given the same color, e.g., the table legs and the wings of birds are one semantic component. The grey-colored partition(s) of the objects shows junction-surfaces.

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Table 1: The number of object partitions $n$, semantic components $m$, junction-surfaces $\delta$ from Fig. 13, and the time to determine the curve skeleton (Skel time) and critical points (CP time). The order of the objects in the table is based on Fig. 1 from left to right, starting from the top-left object.

| Model       | $n$ | $m$ | $\delta$ | Skel time (s) | CP time (s) |
|-------------|-----|-----|----------|---------------|-------------|
| Human       | 6   | 3   | 2        | 4             | 12          |
| Four limb   | 5   | 3   | 1        | 4             | 11          |
| Bird        | 6   | 3   | 1        | 3             | 10          |
| Four limb   | 6   | 3   | 1        | 4             | 13          |
| Pliers      | 4   | 2   | 1        | 4             | 8           |
| Table       | 7   | 2   | 1        | 1             | 10          |
| Airplane    | 3   | 2   | 1        | 4             | 9           |
| Dinosaur    | 7   | 3   | 2        | 5             | 10          |
| Spider      | 9   | 5   | 1        | 6             | 14          |
| Pliers      | 4   | 2   | 1        | 4             | 8           |
| Octopus     | 8   | 4   | 1        | 6             | 14          |
| Octopus     | 9   | 4   | 1        | 6             | 15          |
| Human       | 6   | 3   | 2        | 4             | 12          |
| Four limb   | 7   | 3   | 1        | 4             | 11          |
| Table       | 6   | 2   | 1        | 4             | 10          |
| Airplane    | 9   | 3   | 2        | 4             | 9           |

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