Families of congruences for fractional partition functions modulo powers of primes

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Abstract

Recently, Chan and Wang studied the fractional partition function and found several infinite classes of congruences satisfied by the corresponding coefficients. In this paper, we find new families of congruences modulo powers of primes using the Rogers-Ramanujan continued fraction and some dissection formulae of certain $q$-products. We also find analogous congruences in the coefficients of the fractional powers of the generating function for the 2-color partition function.

Keywords: Partition, $n$-color partition, $n$-colored partition, Fractional partition function, Congruence, Rogers-Ramanujan continued fraction

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1 Introduction

For complex numbers $a$ and $q$ with $|q| < 1$, define the standard $q$-product by

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - a q^j).$$

In the sequel, for brevity, we set $E_n := (q^n; q^n)_\infty$ for integers $n \geq 1$.

A partition $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer $n$ is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The partition function $p(n)$ is defined as the number of partitions of $n$. It is well known that the generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{E_1},$$

where by convention $p(0) = 1$. The arithmetic properties of the partition function $p(n)$ have been studied extensively after Ramanujan [27–29] found his famous congruences modulo 5, 7, and 11, namely, for all $n \geq 0$,

$$p(5n + 4) \equiv 0 \pmod{5},$$

$$p(7n + 5) \equiv 0 \pmod{7},$$

$$p(11n + 6) \equiv 0 \pmod{11}.$$
\[ p(7n + 5) \equiv 0 \pmod{7}, \]
\[ p(11n + 6) \equiv 0 \pmod{11}. \]

For any non-zero rational number \( \alpha \), define the fractional partition function \( p_{\alpha}(n) \) by
\[ \sum_{n=0}^{\infty} p_{\alpha}(n) q^n := E_1^{\alpha}. \]

Clearly, \( p_{-1}(n) = p(n) \). For some work on congruences for \( p_{\alpha}(n) \) for non-zero integers \( \alpha \neq 1 \), see \([2,3,6,8,18,19,24–26]\). Note that \( p_{-\alpha}(n) \) for a positive integer \( \alpha \) counts the \( \alpha \)-colored partitions of \( n \) in which each part can have at most \( \alpha \) colors.

For the rest of the paper, we write \( \alpha = a/b \), where \( a, b \in \mathbb{Z} \), \( b \geq 1 \), and \( \gcd(a, b) = 1 \).

Recently, Chan and Wang \([12]\) studied the function \( p_{\alpha}(n) \) for non-integral rational \( \alpha \) and found infinite classes of congruences for the function. Firstly, they established the following theorem in order to show that it is meaningful to explore congruences for \( p_{\alpha}(n) \) modulo powers of any prime \( \ell \) not dividing \( \text{denom}(\alpha) \).

**Theorem 1** (Chan and Wang \([12, \text{Theorem 1.1}]\)) For any prime \( \ell \), we have
\[ \text{denom}(p_{\alpha}(n)) = b^n \prod_{\ell \mid b} \ell^{\text{ord}_\ell(n)}. \]

Secondly, using known series expansion of \( E_1^{d} \) for \( d \in \{1, 3, 4, 6, 8, 10, 14, 26\} \), Chan and Wang proved the following theorem which gives Ramanujan-type congruences for \( p_{\alpha}(n) \) for non-zero rational numbers \( \alpha \).

**Theorem 2** (Chan and Wang \([12, \text{Theorem 1.2}]\)) Suppose that \( a, b, d \in \mathbb{Z}, b \geq 1, \) and \( \gcd(a, b) = 1 \). Let \( \ell \) be a prime divisor of \( a + db \) and \( 0 \leq r < \ell \). Suppose that \( d, \ell, \) and \( r \) satisfy any of the following conditions:

(i) \( d = 1 \) and \( \left( \frac{24r + 1}{\ell} \right) = -1 \);
(ii) \( d = 3 \) and \( \left( \frac{8r + 1}{\ell} \right) = -1 \) or \( 8r + 1 \equiv 0 \pmod{\ell} \);
(iii) \( d \in \{4, 8, 14\}, \ell \equiv 5 \pmod{6}, \) and \( 24r + d \equiv 0 \pmod{\ell} \);
(iv) \( d \in \{6, 10\}, \ell \geq 5, \ell \equiv 3 \pmod{4}, \) and \( 24r + d \equiv 0 \pmod{\ell} \);
(v) \( d = 26, \ell \equiv 11 \pmod{12}, \) and \( 24r + d \equiv 0 \pmod{\ell} \).

Then, for all \( n \geq 0 \), we have
\[ p_{-\alpha}(\ell n + r) \equiv 0 \pmod{\ell}. \] (1.1)

Chan and Wang \([12]\) also found three additional congruences modulo 25 and 49, and conjectured \([12, \text{Conjectures 3.1 and 3.2}]\) seventeen more congruences. Chan and Wang proved one of those seventeen congruences using modular forms and speculated that additional congruences might be proved in a similar way.

Recently, Xia and Zhu \([31]\) proved many of the congruences conjectured by Chan and Wang \([12]\) and also discovered new congruences for \( p_{\alpha}(n) \). They used Ramanujan’s modular equations of fifth, seventh, and thirteenth orders. In this paper, we first present another method to prove the conjectural congruences modulo powers of 5 by Chan and Wang.
Furthermore, the method gives new congruences. We use a dissection formula for $E_1$ and some identities involving the Rogers-Ramanujan continued fraction $R(q)$, defined by

$$R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \cdots}}} = q^{1/5} \frac{(q; q^5)_\infty}{(q^2; q^5)_\infty} \frac{(q^3; q^5)_\infty}{(q^4; q^5)_\infty}, \quad |q| < 1. \quad (1.2)$$

Note that for a power series $P(q)$ in $q$, an $n$-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{n-1} q^j P_j(q^n),$$

where $P_j(q)$ are power series in $q^n$.

Chan and Wang [12] listed their congruences for $p_\alpha(n)$ modulo 5 when $2 \leq \alpha \leq 5$. In the following theorem, we provide five new congruences for $p_\alpha(n)$ that arise due to our method involving the Rogers-Ramanujan continued fraction when the denominator of $\alpha$ is 6.

**Theorem 3** For all $n \geq 0$, we have

$$p_{-1/6} (5^2 n + r) \equiv 0 \pmod{5^2}, \quad \text{where } r \in \{9, 14, 19, 24\},$$

$$p_{1/6} (5^3 n + r) \equiv 0 \pmod{5^3}, \quad \text{where } r \in \{96, 121\},$$

$$p_{-5/6} (5^3 n + r) \equiv 0 \pmod{5^2}, \quad \text{where } r \in \{45, 70, 95, 120\},$$

$$p_{5/6} (5^2 n + r) \equiv 0 \pmod{5^3}, \quad \text{where } r \in \{15, 20\},$$

$$p_{5/6} (5^3 n + r) \equiv 0 \pmod{5^4}, \quad \text{where } r \in \{65, 70\}. \quad (1.3)$$

Before stating the next theorem, we recall some recent work on $p_\alpha(n)$. Due to Euler and Jacobi, $qE_2$ and $qE_3^3$ are known to be lacunary. Using these facts, Bevilacqua, Chandran, and Choi [7] generalized (1.1) for higher powers of $\ell$ for $d=1$ and 3. They also arrived at many congruences of the form

$$p_\alpha \left( \ell^k n + r \right) \equiv 0 \pmod{\ell^k} \quad \text{for all } n \geq 0.$$  

We call such congruences $\ell^k$-balanced, or more generally balanced. For even integers $d > 0$, Serre [30] proved that $(q^{1/24}E_1)^d$ is lacunary if and only if $d \in \{2, 4, 6, 8, 10, 14, 26\}$.Using Serre’s results, Choi [16] proved congruences for higher powers of $\ell$ of (1.1) for $d \in \{2, 4, 6, 8, 10, 14, 26\}$. It is to be noted that very recently, Iskander, Jain, and Talvola [23] found the exact formulæ for $p_\alpha(n)$ using the circle method.

In this paper, the 3-dissection of $E_3^3$, the $\ell$-dissection of $E_1$ for any prime $\ell \geq 5$, and the 5-, 7-, and 11-dissections of $E_1^d$ for $d = 2, 3, 4, 6, 8, 14$ are employed to find a new infinite class of congruences for arbitrary powers of any prime $\ell \geq 5$ and several new infinite classes of congruences for arbitrary powers of 3, 5, 7, and 11. We list the congruences that we find using the dissection formulæ in the following theorem and Appendix A.

**Theorem 4** Let $b, k,$ and $s$ be non-zero positive integers such that $s \leq \lfloor k/2 \rfloor$.  


(i) For primes \( \ell \geq 5 \), let \( \gcd(\ell, b) = 1 \). Then, for all \( n \geq 0 \), we have
\[
p - \left( \frac{\ell^k - b}{b} \right) \equiv 0 \pmod{\ell^{k-s+1}}, \quad \text{where } r \in \{1, 2, \ldots, \ell\} - \{\lfloor \ell/24 \rfloor \}.
\] (1.4)

(ii) For \( \ell \in \{3, 5, 7\} \), let \( \gcd(\ell, b) = 1 \). Then, for all \( n \geq 0 \), we have
\[
p - \left( \frac{\ell^k - 3b}{b} \right) \equiv 0 \pmod{\ell^{k-s+1}}, \quad \text{where } r \in \{1, 2, \ldots, \ell\}.
\] (1.5)

(iii) Let \( \gcd(3, b) = 1 \). Then, for all \( n \geq 0 \), we have
\[
p - \left( \frac{3^k - 6b}{b} \right) \equiv 0 \pmod{3^k}, \quad \text{where } r \in \{1, 2\}.
\] (1.6)

Example 1 We show that for some choices of \( \ell, b, \) and \( k \), the modulus of (1.4) given in Theorem 4 is sharp. Choosing \( \ell = 5, \) \( b = 1567, \) and \( k = 5 \) in (1.4), we conclude that for all \( n \geq 0, \)
\[
p = \frac{5^2n + 5r + 1}{5^4}, \quad \text{when } s = 1,
\] (1.7)
\[
p = \frac{5^4n + 5^3r + 26}{5^7}, \quad \text{when } s = 2,
\] (1.8)
where \( r \in \{1, 2, 3, 4\} \). Using Wolfram’s Mathematica, we find that
\[
p = \frac{5^4 \cdot 257158702906265183}{14805189781918095169}.
\]
Therefore, the power of 5 in the modulus of (1.7) is sharp for some choices of \( \ell, b, \) and \( k \).
The same can be shown for (1.8).

Remark 1 Note that the arithmetic progressions in each of (1.4)–(1.6) for different values of \( s \) do not overlap. We explain this phenomenon in Remark 6.

Remark 2 In (1.4) and (1.5), as \( s \) decreases, the moduli of the congruences become larger, and the arithmetic progressions become denser. But in (1.6), as \( s \) decreases, the modulus of the congruence remains the same and the arithmetic progression becomes denser. Remarks 1 and 2 apply analogously to Theorem 6 as well.

When \( k \) is odd in Theorem 4, a particular identity, namely (3.24) in the proof of (1.6), gives the following additional congruence.

Corollary 1 Let \( k \geq 1 \) and \( b \) be two integers such that \( k \) is odd and \( \gcd(3, b) = 1 \). Then, for all \( n \geq 0 \), we have
\[
p - \left( \frac{3^k - 6b}{b} \right) \equiv 0 \pmod{3^k}.
\] (1.9)
Remark 3 If we choose \( b = 1 \) in (1.4)–(1.6) and (1.9), then we find infinitely many congruences satisfied by \( p_{-\alpha}(n) \), where \( \alpha \in \{\ell^k - 1, 3^k - 3, 5^k - 3, 7^k - 3\} \) for any prime \( \ell \geq 5 \), integers \( k \geq 1 \), and \( \alpha = 3^k - 6 \) for odd integers \( k \).

Motivated by \( p_\alpha(n) \), in this paper, we also consider the fractional powers of the generating function of the 2-color partition function. For any non-zero rational number \( \alpha \) and integer \( \beta > 1 \), we define the fractional 2-color partition function \( p_{[1, \beta; \alpha]}(n) \) by

\[
\sum_{n=0}^{\infty} p_{[1, \beta; \alpha]}(n) q^n := (E_1 E_\beta)^\alpha.
\]

From Theorem 1, the denominators of both \( p_{[1, \beta; \alpha]}(n) \) and \( \alpha \) have the same prime divisors. Therefore, for non-integral rational numbers \( \alpha \), we study congruences for \( p_{[1, \beta; \alpha]}(n) \) modulo powers of primes \( \ell \) not dividing \( \text{denom}(\alpha) \).

Clearly, \( p_{[1, \beta; -1]}(n) \) is the number of 2-color partitions\(^1\) of \( n \) where one of the colors appears only in parts that are multiples of \( \beta \). Recently, \( p_{[1, \beta; -1]}(n) \) has been studied prominently using Ramanujan’s cubic continued fractions, Rogers-Ramanujan continued fractions, modular forms, and the huffing operator analysis [22]. (See [1, 9–11, 13–15, 20, 21, 32]). In particular, H. C. Chan [9] proved Ramanujan-type congruences such as

\[
p_{[1, 2; -1]}(3n + 2) \equiv 0 \pmod{3}, \text{ for all } n \geq 0, \tag{1.10}
\]

and Ahmed, Baruah, and Dastidar [1] and Chern [14] established that when \( \beta \in \{2, 3, 4, 5, 7, 8, 10, 15, 17, 20\} \) and \( \beta + r = 24 \), then, for all \( n \geq 0 \),

\[
p_{[1, \beta; -1]}(25n + r) \equiv 0 \pmod{5}. \tag{1.10}
\]

We find the following theorem analogous to Theorem 2 using some known formulae involving \( E_1, E_2, \) and \( E_4 \).

**Theorem 5** Suppose that \( a, b, d, \beta \in \mathbb{Z}, b \geq 1, \) and \( \gcd(a, b) = 1 \). Let \( \ell \) be an odd prime divisor of \( a + db \) and \( 0 \leq \gamma < \ell \). Suppose that \( \ell, d, r, \) and \( \gamma \) satisfy any of the following conditions:

(i) \( \beta = 2, d = 2, \ell \equiv 3 \pmod{4}, \) and \( 4\gamma + 1 \equiv 0 \pmod{\ell} \);
(ii) \( \beta = 2, d = 3, \ell \equiv 5 \text{ or } 7 \pmod{8}, \) and \( 8\gamma + 3 \equiv 0 \pmod{\ell} \);
(iii) \( \beta = 3, d = 3, \ell \equiv 2 \pmod{3}, \) and \( 2\gamma + 1 \equiv 0 \pmod{\ell} \);
(iv) \( \beta = 4, d = 2, \ell \equiv 3 \pmod{4}, \) and \( 12\gamma + 5 \equiv 0 \pmod{\ell} \);
(v) \( \beta = 4, d = 3, \ell \equiv 3 \pmod{4}, \) and \( 8\gamma + 5 \equiv 0 \pmod{\ell} \);
(vi) \( \beta = 4, a = 5^k - 3b \) for integers \( k \geq 1, \ell = 5, \) and \( \gamma \in \{2, 3\} \).

Then, for all \( n \geq 0 \), we have

\[
p_{[1, \beta; -\alpha]}(\ell n + \gamma) \equiv 0 \pmod{\ell}. \tag{1.11}
\]

\(^1\)It is to be noted that the 2-color partition function \( p_{[1, \beta; -1]}(n) \) and the 2-colored partition function \( p_{-2}(n) \) defined above are different.
**Example 2** We provide an example in which Theorem 5 gives a sharp bound in $\ell$. We choose $\beta = d = 2$, $\ell = 11$, $a = 1$, $b = 5$, and $\gamma = 8$ so that Condition (i) in Theorem 5 is satisfied. Then, for all $n \geq 0$, we have

$$p_{[1, 2, -1/5]}(11n + 8) \equiv 0 \pmod{11}.$$  \hspace{1cm} (1.12)

Using Wolfram’s Mathematica, we find that

$$p_{[1, 2, -1/5]}(8) = \frac{11 \cdot 39767}{1953125},$$

which assures that the power of the modulus of (1.12) is sharp with the bound given by Theorem 5.

**Remark 4** H. C. Chan’s congruence (1.10) follows immediately from (1.11) when we find the following theorem analogous to Theorem 4.

**Theorem 6** Let $k$ and $s$ be non-zero positive integers such that $s \leq \lfloor k/2 \rfloor$, and $\ell \in \{3, 5, 7\}$.

(i) Let $\gcd(\ell, b) = 1$. Then, for all $n \geq 0$, we have

$$p_{[1, 2, -1/(\ell^k - b)]}(\ell^{2s} \cdot n + \ell^{2s-1} \cdot r + \frac{\ell^{2s} - 1}{8}) \equiv 0 \pmod{\ell^{k-2s+1}}, \text{ where } r \in \{1, 2, \ldots, \ell\}. \hspace{1cm} (1.13)$$

(ii) Let $\gcd(5, b) = 1$. Then, for all $n \geq 0$, we have

$$p_{[1, 3, -(5^k - b)]}(5^{2s} \cdot n + 5^{2s-1} \cdot r + \frac{2 \cdot 5^{2s-1} - 1}{3}) \equiv 0 \pmod{5^{k-s+1}}, \text{ where } r \in \{0, 2, 3, 4\}. \hspace{1cm} (1.14)$$

(iii) Let $\gcd(5, b) = 1$. Then, for all $n \geq 0$, we have

$$p_{[1, 3, -(5^k - 3b)]}(5^{2s} \cdot n + 5^{2s-1} \cdot r + \frac{5^{2s-1} - 1}{2}) \equiv 0 \pmod{5^k}, \text{ where } r \in \{0, 1, 3, 4\}. \hspace{1cm} (1.15)$$

**Example 3** We show that for some choices of $\ell$, $b$, and $k$, the modulus given in (1.13) is also sharp. We choose $\ell = 3$, $b = 17$, and $k = 3$ in (1.13) so that for all $n \geq 0$, we have

$$p_{[1, 2, -10/17]}(3^2n + 3r + 1) \equiv 0 \pmod{3^2}, \text{ where } r \in \{1, 2\}. \hspace{1cm} (1.16)$$
Using Wolfram’s Mathematica, we have

\[ p_{[1,2; -10/17]}(4) = \frac{3^2 \cdot 34465}{83521} \]

Therefore, the power of 3 in the modulus of \((1.16)\) is sharp.

Similar to Corollary 1, when \(k \) is odd in Theorem 6, extension of the proof of \((1.15)\) gives the following \(5^k\)-balanced congruence.

**Corollary 2** Let \(k \geq 1\) and \(b\) be two integers such that \(k \) is odd and \(\gcd(5^k, b) = 1\). Then, for all \(n \geq 0\), we have

\[ p_{[1,3; (5^k - 3b)/b]} \left(5^k \cdot n + \frac{5^k - 1}{2}\right) \equiv 0 \pmod{5^k} \quad (1.17) \]

**Remark 5** As in Remark 3, setting \(b = 1\) in \((1.13)-(1.17)\), we can deduce corresponding congruences for the \(2\alpha\)-color partition function \(p_{[1,\beta; -\alpha]}(n)\), where \(\alpha \in \{3^k - 1, 5^k - 1, 5^k - 2, 5^k - 3, 7^k - 1\}\).

In Appendix B, we present additional congruences for \(p_{[1,\beta; \alpha]}(n)\) along with the key identities required to prove them. Those congruences can be obtained using similar arguments used to prove \((1.13)-(1.17)\).

We organize the paper in the following way. In Sect. 2, we present some preliminary results useful for the proofs of our congruences. In Sect. 3, we prove Theorems 3–6. In the appendices, we present additional congruences for the fractional partition functions \(p_{\alpha}(n)\) and \(p_{[1,\beta; \alpha]}(n)\) that can be proved using the methods of this paper.

### 2 Preliminaries

For a power series \(\sum_{n=0}^{\infty} P(n)q^n\) and integers \(0 \leq r < s\), we define the extraction operator \(U_{sn+r}\) as

\[ U_{sn+r} \left( \sum_{n=0}^{\infty} P(n)q^n \right) = \sum_{n=0}^{\infty} P(sn + r)q^n. \]

It is evident that

\[ U_{sn+r} \left( \sum_{n=0}^{\infty} P(n)q^n \right) \left( \sum_{n=0}^{\infty} P(n)q^m \right) = \left( \sum_{n=0}^{\infty} P(sn + r)q^n \right) \left( \sum_{n=0}^{\infty} P(n)q^m \right), \]

which we use frequently in Sect. 3. While proving Theorems 3–6, we use the following lemma in order to take advantage of \(U_{sn+r}\).

**Lemma 1** ([12, p. 64, Lemma 2.1]) Let \(z = x/y\), where \(x, y \in \mathbb{Z}, y \geq 1\), and \(\gcd(x, y) = 1\). Let \(\ell\) be a prime not dividing \(y\). Then, for any positive integers \(j\) and \(n\), we have

\[ E_{\ell}^j \equiv E_{\ell n}^{\ell - 1} \pmod{\ell^j}. \]

The 3-dissections of \(E_1^3\) and \(E_1E_2\), which are useful in the proofs of Theorem 4, \((1.13)\), and \((4.16)\), are given in the next lemma.
Lemma 2 ( [4, p. 345, Entry 1(iv)] and [20, p. 132]) We have

\[ E_1^3 = E_3 \left( \frac{E_6 E_9^3}{E_3 E_{18}^3} - 3q + 4q^3 \frac{E_3^2 E_6}{E_5 E_9} \right), \quad (2.1) \]

\[ E_1 E_2 = E_3 E_{18} \left( \frac{E_6 E_3}{E_3 E_{18}^3} - q - 2q^3 \frac{E_3 E_9}{E_5 E_9} \right), \quad (2.2) \]

The \( n \)-dissection of \( E_1 \) for \( n \equiv \pm 1 \pmod{6} \) stated in the following lemma is exploited to prove (1.4).

Lemma 3 ( [4, p. 274, Theorem 12.1]) Let \( n \geq 5 \) be an integer such that \( n \equiv \pm 1 \pmod{6} \). Let \( n = 6m + (−1)^i \) for some integer \( i \geq 1 \), then, we have

\[ E_1 = E_{25n} \left( (-1)^m q^{(n^2-1)/24} \sum_{j=1}^{(n-1)/2} (-1)^{j+m} q^{3j-3m-(-1)^j/2} \left( \frac{q^{2m}; q^{m^2}}{(q^m; q^{m^2}) \infty} \right) \left( \frac{q^{n^2-2m}; q^{n^2}}{(q^{n^2-nm}; q^{n^2}) \infty} \right) \right) \]

Setting \( n = 5 \) in Lemma 3 and then using (1.2), we have the next corollary.

Corollary 3 Let \( R(q) := q^{1/5}/R(q) \). Then, for all \( n \geq 1 \), we have

\[ E_n = E_{25n} \left( \frac{R(q^{5n}) - q^n - \frac{q^{2n}}{R(q^{5n})}}{R(q^{5n})} \right). \quad (2.3) \]

We require the following identities for the Rogers-Ramanujan continued fraction while proving Theorems 3 and 6.

Lemma 4 ( [5, Chapter 7, Theorem 7.4.4] and [1, pp. 193–194]) We have

\[ R_5(q) - \frac{q^2}{R_5(q)} = 11q + \frac{E_6^6}{E_5^5}, \quad (2.4) \]

\[ R_5(q)R(q^3) - \frac{R_5^2(q^3)}{R(q)} + q^2 \frac{R(q)}{R_5^2(q^3)} - \frac{q^2}{R_5(q)R(q^3)} = 3q. \quad (2.5) \]

For the proof of Theorem 3, we make use of the following definition.

Definition 1 For \( k \geq 1 \), let

\[ x_k := R_5^k(q) + (-1)^k \frac{q^{2k}}{R_5^k(q)}. \quad (2.6) \]

A recurrence relation involving \( x_k \) is helpful in expressing \( x_k \) in terms of \( E_1 \) and \( E_5 \) for higher values of \( k \), which are required in the proof of Theorem 3. Simplifying the expression \( x_1 x_{k-1} + q^2 x_{k-2} \) and using (2.4) and (2.6), we deduce the recurrence relation of \( x_k \) in the next lemma.

Lemma 5 For \( k \geq 3 \), we have

\[ x_k = x_1 x_{k-1} + q^2 x_{k-2}, \]

with

\[ x_1 = 11q + \frac{E_6}{E_5} \quad \text{and} \quad x_2 = 123q^2 + 22q \frac{E_6}{E_5} + \frac{E_6^6}{E_5^5}. \quad (2.7) \]
We end this section referring to some well-known series representations, which are used in proving Theorem 5.

**Lemma 6** ([17, pp. 15–17]) We have

\[ E_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{((6n+1)^2-1)/24}, \]  

\[ E_3^1 = \sum_{n=-\infty}^{\infty} (4n + 1)q^{((4n+1)^2-1)/8}, \]  

\[ E_2^1 = \sum_{n=-\infty}^{\infty} q^{((4n+1)^2-1)/8}, \]  

\[ E_2^2 E_4^1 = \sum_{n=-\infty}^{\infty} (3n + 1)q^{((3n+1)^2-1)/3}. \]

### 3 Proofs of the theorems and corollaries

**Proof of Theorem 3** The proofs of all the congruences are similar in nature. Therefore, we prove only (1.3) in detail. By Lemma 1, we have

\[ \sum_{n=0}^{\infty} p^{-5/6}(5)q^n = E_1^{5/6} E_1^{20} E_5^{5/6} E_1^{20} E_5 (\mod 5^2). \]  

Employing the 5-dissection of \( E_1 \) given by (2.3) of Corollary 3 in (3.1) and applying \( U_{5n} \), we obtain

\[ \sum_{n=0}^{\infty} p^{-5/6}(5)q^n = \frac{E_1^{5/6} E_1^{20} E_5}{E_1^5} \left( R^{20}(q) + \frac{q^8}{R^{20}(q)} \right) + 456q \left( R^{15}(q) - \frac{q^6}{R^{15}(q)} \right) 
+ 52972q^2 \left( R^{10}(q) - \frac{q^4}{R^{10}(q)} \right) 
+ 224808q^3 \left( R^5(q) - \frac{q^2}{R^5(q)} \right) + 1813055q^4 \right) \mod 5^2. \]

Now, on account of Definition 1, (3.2) reduces to

\[ \sum_{n=0}^{\infty} p^{-5/6}q^n = \frac{E_1^{5/6} E_1^{20} E_5}{E_1^5} (x_3 + 6qx_3 + 22q^2x_2 + 8q^3x_1 + 20q^4) \mod 5^2. \]

We use Lemma 5 to find that

\[ x_3 = 1364q^3 + 366q^2 E_6^1 E_5^1 E_9^1 \]  

\[ x_4 = 15127q^4 + 5412q^3 E_6^2 E_5^2 + 730q^2 E_6^2 E_5^2 E_1^4 + 44q E_1^4 E_5^2 E_1^4 + E_1^4 E_5^2. \]

Using the expressions of \( x_1, x_2, x_3, \) and \( x_4 \) given by (2.7), (3.4), and (3.5) in (3.3), and then using Lemma 1, we obtain

\[ \sum_{n=0}^{\infty} p^{-5/6}(5)q^n = \frac{E_1^{19}}{E_5^1} \left( E_1^{19} + 50q E_1^{13} E_5^2 + 950q^2 E_7 E_5^2 + 8100q^3 E_1 E_5^{14} + 26125q^4 E_2^2 E_1^4 \right) \]
\[ \equiv \frac{E_2^{24}}{E_5^{5/6}} \equiv \frac{E_2^{24}}{E_5^{5/6}} \pmod{5^2}. \] (3.6)

Again, applying the 5-dissection of \( E_1 \) due to (2.3) of Corollary 3 in (3.6) and applying \( U_{5n+4} \), we find that

\[
\sum_{n=0}^{\infty} p_{-5/6(25n+20)} q^n \equiv \frac{E_2^{24}}{E_5^{5/6}} \left( 5x_4 + 5x_3 q + 10x_2 q^2 + 15x_1 q^3 \right) \pmod{5^2},
\]

which with the help of (2.7), (3.4), and (3.5) results in

\[
\sum_{n=0}^{\infty} p_{-5/6(25n+20)} q^n \equiv \frac{1}{E_5^{5/6}} \left( 5E_1^{20} + 225qE_1^{14}E_5^6 + 3825q^2E_1^8E_5^{12} + 29125q^3E_2^2E_5^{18} + 83850q^4E_5^{24} \right) \pmod{5^2}
\equiv 5 \frac{E_1^{20}}{E_5^{5/6}} \pmod{5^2}. \] (3.7)

Due to the presence of a factor 5 in the right side of (3.7), it is enough to study \( E_1^{20}/E_5^{5/6} \) under modulo 5 only. Therefore, by Lemma 1, (3.7) becomes

\[
\sum_{n=0}^{\infty} p_{-5/6(25n+20)} q^n \equiv 5 \frac{E_5^{5/6}}{E_5^{5/6}} \pmod{5^2}. \] (3.8)

In (3.8), we see that there is no term of the form \( q^{5r+r} \) for \( r \in \{1, 2, 3, 4\} \). So, applying \( U_{5n+r} \) for \( r \in \{1, 2, 3, 4\} \) on (3.8), we arrive at (1.3). \( \square \)

**Proof of Theorem 4** Proofs of all the congruences in Theorem 4 are similar. Therefore, we prove (1.4) and (1.6) only. Then, using the proof of (1.6), we prove Corollary 1.

First, we prove (1.4). For any prime \( \ell \geq 5 \), the operator \( U_{\ell n+(\ell^2-24\ell\lfloor \ell/24 \rfloor)-1/24} \) is well defined as \( \lfloor \ell/24 \rfloor \) is the least integer such that \( \ell^2 - 24\ell \lfloor \ell/24 \rfloor - 1 \) is a divisor of \( \ell^2 - 1 \). In the \( \ell \)-dissection of \( E_1 \) given by Lemma 3, we see that \( (j-m)(3j-3m+1)/2 \neq \ell^2 - 1 \) for any \( j \) such that \( 1 \leq j \leq (\ell - 1)/2 \) and \( \ell = 6m \pm 1 \). For if \( (j-m)(3j-3m+1)/2 = \ell^2 - 1 \) for \( j \), then \( 3j = \ell \), which is a contradiction. Therefore, the powers of \( q \) are never of the form \( \ell n + (\ell^2 - 1)/24 \) in the following sum

\[
\sum_{j=1}^{(\ell-1)/2} (-1)^{j+m} q^{(j-m)(3j-3m+1)/2} (q^{2\ell}; q^{2\ell})_\infty (q^{2\ell^2-2\ell}; q^{2\ell^2})_\infty. \] (3.9)

In particular, powers of \( q \) of the form \( \ell n + (\ell^2 - 1)/24 - \ell \lfloor \ell/24 \rfloor = \ell(n - \lfloor \ell/24 \rfloor) + (\ell^2 - 1)/24 \) are not present in (3.9). Therefore, employing the \( \ell \)-dissection of \( E_1 \) given by Lemma 3, we have

\[
U_{\ell n+(\ell^2-24\ell\lfloor \ell/24 \rfloor)-1/24} (E_1) = (-1)^m q^{\ell/24} E_\ell. \] (3.10)

Now, by Lemma 1, we have

\[
\sum_{n=0}^{\infty} p_{-(\ell k-b)/b(n)} q^n = E_1^{-(\ell k-b)/b} = E_1^{E_1^{\ell k-1}/b} \equiv \frac{E_1}{E_\ell^{\ell k-1}/b} \pmod{\ell^{k-2k+1}}, \] (3.11)
where $s \geq 1$ is an integer. First, we take $s = 1$ in (3.11). Employing $U_{\ell n + (\ell^2 - 24\ell \ell/|\ell/24| - 1)/24}$ on (3.11) and then using (3.10), we obtain
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell n + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) q^n \equiv (-1)^m q^{\ell/24} \frac{E_\ell}{E_1^{\ell-1/b}} \pmod{\ell^{k-1}},
\]
which by Lemma 1 gives
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell n + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) q^n \equiv (-1)^m q^{\ell/24} \frac{E_\ell}{E_1^{\ell-2/b}} \pmod{\ell^{k-1}}.
\]
(3.12)

Applying $U_{\ell n + \ell}$ for $0 \leq r < \ell$, $r \neq |\ell/24|$ in (3.12) yields
\[
p_{-(\ell - b)/b} \left( \ell^2 n + \ell r + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) \equiv 0 \pmod{\ell^{k-1}},
\]
(3.13)
where $0 \leq r < \ell$, $r \neq |\ell/24|$. Thus, (1.4) is true for $s = 1$.

Now, taking $s = 2$ in (3.11), we have
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} (n) q^n \equiv \frac{E_1}{E_1^{\ell-1/b}} \pmod{\ell^{k-3}}.
\]
(3.14)

So, like (3.12), the following identity arises from (3.14):
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell n + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) q^n \equiv (-1)^m q^{\ell/24} \frac{E_\ell}{E_1^{\ell-2/b}} \pmod{\ell^{k-3}}.
\]
(3.15)

Applying the operator $U_{\ell n + |\ell/24|}$ on (3.15) and then using Lemma 1, we find that
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell^2 n + \ell |\ell/24| + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) q^n \equiv (-1)^m \frac{E_1}{E_1^{\ell-2/b}} \equiv (-1)^m \frac{E_1}{E_1^{\ell-3/b}} \pmod{\ell^{k-3}}.
\]
(3.16)

Again, employing $U_{\ell n + (\ell^2 - 24\ell \ell/|\ell/24| - 1)/24}$ on (3.16) and then using (3.10), we deduce that
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell^3 n + \ell |\ell/24| + \ell^2 \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} + \frac{\ell^2 - 24\ell \ell/|\ell/24| - 1}{24} \right) q^n \equiv q^{\ell/24} \frac{E_\ell}{E_1^{\ell-3/b}} \pmod{\ell^{k-3}},
\]
which by Lemma 1 and simplification, reduces to
\[
\sum_{n=0}^{\infty} p_{-(\ell - b)/b} \left( \ell^3 n + \frac{\ell^4 - 24\ell^3 \ell/|\ell/24| - 1}{24} \right) q^n \equiv q^{\ell/24} \frac{E_\ell}{E_1^{\ell-4/b}} \pmod{\ell^{k-3}}.
\]
It is evident that (3.17), on using $U_{ln+r}$ for $0 \leq r < \ell, r \neq \lfloor \ell/24 \rfloor$, gives

$$p_{-(\ell^k-b)/b}\left(\ell^4 n + \lfloor \ell/24 \rfloor r + \frac{\ell^4 - 24\ell^3 \lfloor \ell/24 \rfloor - 1}{24}\right) \equiv 0 \pmod{\ell^{k-3}},$$

where $0 \leq r < \ell, r \neq \lfloor \ell/24 \rfloor$. Thus, (1.4) is true for $s = 2$.

The other cases of $s \geq 3$ can similarly be accomplished by an iterative method described below.

| Cases of $s$ | Beginning of iterations | End of iterations |
|--------------|-------------------------|-------------------|
| $s = 3$      | (3.11) when $s = 3$, then proceed with steps similar to (3.14)–(3.17) | (3.18) when $s = 3$ |
| $s = 4$      | (3.11) when $s = 4$, then proceed with steps similar to (3.14)–(3.17) | (3.18) when $s = 4$ |
| ...          | ...                     | ...               |
| $s = \lfloor k/2 \rfloor$ | (3.11) when $s = \lfloor k/2 \rfloor$, then proceed with steps similar to (3.14)–(3.17) | (3.18) when $s = \lfloor k/2 \rfloor$ |

In general, by the iterative method we obtain

$$\sum_{n=0}^{\infty} P_{-(\ell^k-b)/b}\left(\ell^{2s-1} n + \frac{\ell - 24[\ell/24]\ell^{2s-1} - 1}{24}\right) q^n \equiv (-1)^{ms} q^{\lfloor \ell/24 \rfloor} \frac{E_{\ell}}{E_{\ell}^{k-2s/b}} \left(\mod \ell^{k-2s+1}\right),$$

which gives

$$p_{-(\ell^k-b)/b}\left(\ell^{2s} \cdot n + \ell^{2s-1} \cdot r + \frac{(\ell - 24[\ell/24])\ell^{2s-1} - 1}{24}\right) \equiv 0 \pmod{\ell^{k-2s+1}},$$

where $0 \leq r < \ell, r \neq \lfloor \ell/24 \rfloor$. Note that the maximum value of $s$ is $\lfloor k/2 \rfloor$, because if $s > \lfloor k/2 \rfloor$, then in the denominator of the right hand side of (3.18), we have a negative power of $\ell$. Therefore, (1.4) is true for $s \leq \lfloor k/2 \rfloor$. Thus, we have completed the proof of (1.4).

To prove (1.6), we require the following lemma which will be proved inductively with respect to $s$.

**Lemma 7** For integers $1 \leq s \leq \lfloor k/2 \rfloor$, we have

$$\sum_{n=0}^{\infty} P_{-(3^s-6b)/b}\left(3^{2s-1} n + \frac{3^{2s} - 1}{4}\right) q^n \equiv 3^{2s} \frac{E_3^6}{E_3^{3s-2s/b}} \left(\mod 3^k\right).$$

**Proof of Lemma 7** Squaring both sides of (2.1) given in Lemma 2 and then using the extraction operator $U_{3n+2}$, we have

$$U_{3n+2}\left(E_1^6\right) = 3^2 E_3^6.$$

(3.20)
By Lemma 1, we have
\[
\sum_{n=0}^{\infty} p_{-(3^k-6b)/b}(n) q^n = E_1^{-\left(3^k-6b\right)/b} \equiv \frac{E_1^6}{E_3^{3^k/b}} \left(\text{mod } 3^k\right).
\]
(3.21)

Applying $U_{3n+2}$ to (3.21) and simplifying using (3.20) and Lemma 1, we find that
\[
\sum_{n=0}^{\infty} p_{-(3^k-6b)/b}(3n+2) q^n \equiv 3^2 \frac{E_3^6}{E_1^{3^k-1/b}} \equiv 3^2 \frac{E_3^6}{E_3^{3^{k-2}/b}} \left(\text{mod } 3^k\right).
\]
Thus, (3.19) is true for $s = 1$.

Now, let us assume that (3.19) is true for some integer $s - 1$, where $s \geq 2$. Then, we have
\[
\sum_{n=0}^{\infty} p_{-(3^k-6b)/b}\left(3^{2(s-1)-1}n + \frac{3^{2(s-1)} - 1}{4}\right) q^n \equiv 3^{2(s-1)} \frac{E_3^6}{E_3^{3^{k-2(s-1)-1}/b}} \left(\text{mod } 3^k\right).
\]
(3.22)

Using $U_{3n}$ and Lemma 1 on (3.22), we obtain
\[
\sum_{n=0}^{\infty} p_{-(3^k-6b)/b}\left(3^{2(s-1)-1}n + \frac{3^{2(s-1)} - 1}{4}\right) q^n \equiv 3^{2(s-1)} \frac{E_3^6}{E_1^{3^{k-2(s-1)-1}/b}} \equiv 3^{2(s-1)} \frac{E_3^6}{E_3^{3^{k-2(s-1)-1}/b}} \left(\text{mod } 3^k\right).
\]
(3.23)

Applying $U_{3n+2}$ to (3.23) and simplifying using (3.20) and Lemma 1, we find that
\[
\sum_{n=0}^{\infty} p_{-(3^k-6b)/b}\left(3^{2s-1}n + 2 \cdot 3^{2(s-1)} + \frac{3^{2(s-1)} - 1}{4}\right) q^n \equiv 3^{2s} \frac{E_3^6}{E_1^{3^{k-2(s-1)+1}/b}} \equiv 3^{2s} \frac{E_3^6}{E_3^{3^{k-2s}/b}} \left(\text{mod } 3^k\right).
\]
(3.24)

Thus, (3.19) is true for $s$ as well. Again, the maximum value of $s$ is $\lfloor k/2 \rfloor$, because if $s > \lfloor k/2 \rfloor$ in (3.24), then in the denominator of the right hand side of (3.24), we have a negative power of 3. Thus, by induction, (3.19) is true which completes the proof of Lemma 7.

We are now ready to prove (1.6). Employing $U_{3n+r}$ for $r \in \{1, 2\}$ on (3.19) given in Lemma 7, we have
\[
p_{-(3^k-6b)/b}\left(3^{2s} \cdot n + 3^{2s-1} \cdot r + \frac{3^{2s} - 1}{4}\right) \equiv 0 \left(\text{mod } 3^k\right),
\]
(3.25)
where $r \in \{1, 2\}$ which is (1.6). Thus, we have finished the proof of Theorem 4. $\Box$
Proof of Corollary 1 We extend the proof of (1.6) with $k$ depending on the parity of $k$.

**Case 1:** $k$ is even. Since $k - 2\lfloor k/2 \rfloor$ vanishes when $k$ is even, therefore, setting $s = \lfloor k/2 \rfloor$ in (3.24), we have

$$
\sum_{n=0}^{\infty} p_{(3^k-6b)/b} \left( 3^{k-1} \cdot n + \frac{3^k - 1}{4} \right) q^n \equiv 3^k \frac{E_3^6}{E_3^{1/b}} \pmod{3^k},
$$

which gives

$$
p_{(3^k-6b)/b} \left( 3^{k-1} \cdot n + \frac{3^k - 1}{4} \right) \equiv 0 \pmod{3^k},
$$

which is a stronger property than (3.25) when $s = \lfloor k/2 \rfloor$.

**Case 2:** $k$ is odd. Since $k - 2\lfloor k/2 \rfloor = 1$, putting $s = \lfloor k/2 \rfloor$ in (3.24), we have

$$
\sum_{n=0}^{\infty} p_{(3^k-6b)/b} \left( 3^{2\lfloor k/2 \rfloor-1} \cdot n + \frac{3^{2\lfloor k/2 \rfloor} - 1}{4} \right) q^n \equiv 3^{2\lfloor k/2 \rfloor} \frac{E_3^6}{E_3^{1/b}} \pmod{3^k}.
$$

Applying $U_{3n}$ on (3.26) and then by Lemma 1, we obtain

$$
\sum_{n=0}^{\infty} p_{(3^k-6b)/b} \left( 3^{2\lfloor k/2 \rfloor} \cdot n + \frac{3^{2\lfloor k/2 \rfloor} - 1}{4} \right) q^n \equiv 3^{2\lfloor k/2 \rfloor} \frac{E_3^6}{E_3^{1/b}} \equiv 3^{2\lfloor k/2 \rfloor} \frac{E_3^6}{E_1^{1/b}} \pmod{3^k}.
$$

Finally, employing $U_{3n+2}$ and (3.20) on (3.27), we arrive at

$$
\sum_{n=0}^{\infty} p_{(3^k-6b)/b} \left( 3^{2\lfloor k/2 \rfloor+1} \cdot n + \frac{3^{2\lfloor k/2 \rfloor+1+1} - 1}{4} \right) q^n \equiv 3^{2\lfloor k/2 \rfloor+2} \frac{E_3^6}{E_1^{1/b}} \pmod{3^k},
$$

which evidently gives

$$
p_{(3^k-6b)/b} \left( 3^k \cdot n + \frac{3^{k+1} - 1}{4} \right) \equiv 0 \pmod{3^k},
$$

which is (1.9).

**Remark 6** It is crucial to note that while obtaining (3.13) from (3.12), we employ $U_{\ell n+r}$, $0 \leq r < \ell$, $r \neq \lfloor \ell/24 \rfloor$, which gives the case $s = 1$ of (1.4). But, to obtain the case $s = 2$ of (1.4), we apply $U_{\ell n+\lfloor \ell/24 \rfloor}$ on (3.15). Since $\ell n + r \neq \ell n + \lfloor \ell/24 \rfloor$, we obtain different arithmetic progressions of (1.4) when $s = 1$ and $s = 2$. In general, as a consequence of applying the operators $U_{\ell n+r}$ for $0 \leq r < \ell$, $r \neq \lfloor \ell/24 \rfloor$ and $U_{\ell n+\lfloor \ell/24 \rfloor}$, the arithmetic progressions of (1.4) do not overlap for different values of $s$. This phenomenon happens in the proof of the other congruences of Theorems 4 and 6 as well.

**Proof of Theorem 5** Since $\ell \mid (a + db)$, we assume that $a + db = \ell m$ for some integer $m$. Furthermore, since $\gcd(a, b) = 1$, it follows that $\gcd(b, \ell) = 1$. By Lemma 1, we have

$$
\sum_{n=0}^{\infty} p_{[1,\beta, -a/b]}(n) q^n = (E_1 E_\beta)^{-a/b} = \frac{(E_1 E_\beta)^d}{(E_1 E_\beta)^{\ell m/b}} \equiv \frac{(E_1 E_\beta)^d}{(E_1 E_\beta)^{\ell m/b}} \pmod{\ell}.
$$
We now prove (1.11) satisfying the conditions (i) and (vi). By (2.9) and (2.10) of Lemma 6, we have

\[(E_1 E_2)^2 = E_1^3 \cdot E_2^2 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (4n + 1)q^{(4n+1)^2+(4m+1)^2-2/8}. \quad (3.29)\]

Note that \(N = ((4n + 1)^2 + (4m + 1)^2 - 2)/8\) is equivalent to \(8N + 2 = (4n + 1)^2 + (4m + 1)^2\). Since \(\ell \equiv 3 \pmod{4}\), \(\left(\frac{-1}{\ell}\right) = -1\). Therefore, \(8N + 2 \equiv 0 \pmod{\ell}\), equivalently, \(4N + 1 \equiv 0 \pmod{\ell}\) if and only if \(4n + 1 \equiv 0 \pmod{\ell}\) and \(4m + 1 \equiv 0 \pmod{\ell}\). Using (3.29) in (3.28) and then applying \(U_{8n+4}\) on both sides, we arrive at (1.11) when Condition (i) is satisfied.

Letting \(\beta = 4\), \(a = 5^k - 3b\) for integers \(k \geq 1\), and \(\ell = 5\) in (3.28), we have

\[\sum_{n=0}^{\infty} P_{4, -(5^k - 3b)/b}(n)q^n = \frac{(E_1 E_4)^3}{(E_1 E_4)^{5^k/b}} = \frac{(E_1 E_4)^3}{(E_5 E_{20})^{5^k-1/b}} \pmod{5}. \quad (3.30)\]

From (2.3) of Corollary 3, it is evident that

\[E_1^3 = E_2^5 \left(R^3 \left(q^5\right) - 3qR^2 \left(q^3\right) + 5q^2 - \frac{3q^5}{R^2 \left(q^3\right)} - \frac{q^6}{R^3 \left(q^5\right)}\right). \quad (3.31)\]

Employing the 5-dissections of \(E_1^3\) and \(E_4^3\) given by (3.31) in (3.30), and then applying \(U_{5n+2}\) and \(U_{5n+3}\), we obtain

\[\sum_{n=0}^{\infty} P_{4, -(5^k - 3b)/b}(5n + 2)q^n \equiv 5 \cdot \frac{(E_5 E_{20})^3}{(E_1 E_4)^{5^k-1/b}} \left(-3q \left(R^2(q^4) + \frac{q^2}{R^2(q)}\right) + q^2 \left(R^3(q) - \frac{q^3}{R^3(q^4)}\right)\right) \pmod{5}, \quad (3.32)\]

and

\[\sum_{n=0}^{\infty} P_{4, -(5^k - 3b)/b}(5n + 3)q^n \equiv 5 \cdot \frac{(E_5 E_{20})^3}{(E_1 E_4)^{5^k-1/b}} \left( \left(R^3(q^4) - \frac{q^3}{R^3(q)}\right) - 3q^2 \left(R^2(q) - \frac{q^2}{R^2(q)}\right)\right) \pmod{5}, \quad (3.33)\]

respectively, which readily imply (1.11) when Condition (vi) is satisfied.

The proof of (1.11) with conditions (ii)–(v) of Theorem 5 can similarly be achieved. Therefore, we put the key steps required to prove (1.11) for those conditions in Appendix C.

\[\square\]

**Proof of Theorem 6** The methods of proving the congruences in Theorem 6 are similar to those used in Theorem 4. In the following, we only present the proof of (1.14).

Multiplying the 5-dissections of \(E_1^2\) and \(E_3^2\) given by (2.3) and then employing \(U_{5n+3}\), we have

\[U_{5n+3} \left(E_1^2 E_3^2\right) = E_2^2 E_1^5 \left(q - 2 \left(R^2(q)R(q^3) - \frac{R^2(q^3)}{R(q)} + q^2 \frac{R(q)}{R^2(q^3)} - \frac{q^2}{R^2(q)R(q^3)}\right)\right), \quad (3.34)\]

which, by (2.5), reduces to

\[U_{5n+3} \left(E_1^2 E_3^2\right) = -5qE_2^2 E_1^5. \quad (3.35)\]
By Lemma 1, we have

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(n)q^n = (E_1E_3)^{5k/b} = \frac{(E_1E_3)^2}{(E_5E_15)^{5k-1/b}} \pmod{5^{k-\varepsilon+1}},
\]

(3.33)

where \( s \geq 1 \) is an integer. Taking \( s = 1 \), employing \( U_{5n+3} \) on (3.33), and then using (3.32), we obtain

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5n+3)q^n = -5q\frac{(E_5E_15)^2}{(E_5E_15)^{5k-1/b}} \pmod{5^k},
\]

which by Lemma 1 gives

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5n+3)q^n = -5q\frac{(E_5E_15)^2}{(E_5E_15)^{5k-2/b}} \pmod{5^k}.
\]

(3.34)

From (3.34), applying \( U_{5n+r} \) for \( r \in \{0, 2, 3, 4\} \), we have

\[
p_{\{1,3,\ldots,5k-2b/b\}}(5^2n+5r+3) \equiv 0 \pmod{5^k},
\]

where \( r \in \{0, 2, 3, 4\} \). Thus, (1.14) is true for \( s = 1 \).

Now, taking \( s = 2 \) in (3.33), we have

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(n)q^n = \frac{(E_1E_3)^2}{(E_5E_15)^{5k-1/b}} \pmod{5^{k-1}}.
\]

(3.35)

From (3.35), like (3.34), it is easy to obtain

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5n+3)q^n = -5q\frac{(E_5E_15)^2}{(E_5E_15)^{5k-2/b}} \pmod{5^{k-1}}.
\]

(3.36)

Applying \( U_{5n+1} \) on (3.36) and then using Lemma 1, we find that

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5^2n+8)q^n = -5\frac{(E_1E_3)^2}{(E_5E_3)^{5k-2/b}} \equiv -5\frac{(E_1E_3)^2}{(E_5E_15)^{5k-3/b}} \pmod{5^{k-1}}.
\]

(3.37)

Again, employing \( U_{5n+3} \) on (3.37) and then using (3.32), we deduce that

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5^3n+83)q^n \equiv 5^2q\frac{(E_5E_15)^2}{(E_1E_3)^{5k-3/b}} \pmod{5^{k-1}},
\]

which by Lemma 1 gives

\[
\sum_{n=0}^{\infty} p_{\{1,3,\ldots,5k-2b/b\}}(5^3n+83)q^n \equiv 5^2q\frac{(E_5E_15)^2}{(E_5E_15)^{5k-4/b}} \pmod{5^{k-1}}.
\]

(3.38)
It is evident from (3.38) that
\[ P_{[1,3;-(5^k-2b)/b]} \left( 5^4 n + 5^3 r + 83 \right) \equiv 0 \pmod{5^{k-1}}, \]
where \( r \in \{0, 2, 3, 4\}. \) Thus, (1.14) is true for \( s = 2. \)

The other cases of \( s \geq 3 \) can similarly be accomplished by the iterative method described in the proof of Theorem 4. By the iterative method, we obtain
\[ \sum_{n=0}^{\infty} P_{[1,3;-(5^k-2b)/b]} \left( 5^{2s-1} n + \frac{2 \cdot 5^{2s-1} - 1}{3} \right) q^n \equiv (-5)^s q \frac{(E_5 E_{15})^2}{(E_5 E_{15})^{2s-2}/b} \pmod{5^{k-s+1}}. \] (3.39)

Note that the maximum value of \( s \) is \( \lfloor k/2 \rfloor \), because if \( s > \lfloor k/2 \rfloor \), we have a negative power of 5 in the denominator of the right hand side of (3.39). On using \( U_{5n+r} \) for \( r \in \{0, 2, 3, 4\} \) in (3.39), it follows evidently that
\[ P_{[1,3;-(5^k-2b)/b]} \left( 5^{2s} n + 5^{2s-1} r + \frac{2 \cdot 5^{2s-1} - 1}{3} \right) \equiv 0 \pmod{5^{k-s+1}}, \]
where \( r \in \{0, 2, 3, 4\} \), which is (1.14). \( \square \)

Corollary 2 can be proved after proving (1.15) of Theorem 6 in the same manner as we prove Corollary 1 from the proof of (1.6) of Theorem 4.

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4 Appendix
4.1 Appendix A
The following congruences can be proved by employing the methods given in the proofs of Theorem 4 and Corollary 1. Let \( k \) and \( s \) be non-zero positive integers such that \( s \leq \lfloor k/2 \rfloor \) and \( b \)'s in (4.1)–(4.10) are co-prime to the moduli. Then, for all \( n \geq 0 \), we have
\[ P_{-(5^k-2b)/b} \left( 5^{2s} n + 5^{2s-1} r + \frac{5^{2s} - 1}{12} \right) \equiv 0 \pmod{5^{k-2s+1}}, \quad \text{where} \ r \in \{1, 2, 3, 4\} \quad (4.1) \]
\[ P_{-(7^k-2b)/b} \left( 7^{2s} n + 7^{2s-1} r + \frac{7^{2s} - 1}{12} \right) \equiv 0 \pmod{7^{k-2s+1}}, \quad \text{where} \ r \in \{1, 2, \ldots, 6\} \quad (4.2) \]
\[ P_{-(11^k-2b)/b} \left( 11^{2s} n + 11^{2s-1} r + \frac{11^{2s} - 1}{12} \right) \equiv 0 \pmod{11^{k-2s+1}}, \quad \text{where} \ r \in \{1, 2, \ldots, 10\} \quad (4.3) \]
\[ P_{-(5^k-4b)/b} \left( 5^{2s} n + 5^{2s-1} r + \frac{5^{2s} - 1}{6} \right) \]
\begin{equation}
\equiv 0 \pmod{5^k}, \quad \text{where } r \in \{0, 2, 3, 4\}, \tag{4.5}
\end{equation}

\begin{equation}
\equiv 0 \pmod{5^k}, \quad \text{where } r \in \{0, 1, 3, 4\}, \tag{4.6}
\end{equation}

\begin{equation}
\equiv 0 \pmod{7^k}, \quad \text{where } r \in \{0, 2, 3, 4, 5, 6\}, \tag{4.7}
\end{equation}

and when \( k \) is odd,

\begin{equation}
\equiv 0 \pmod{5^k}, \tag{4.8}
\end{equation}

\begin{equation}
\equiv 0 \pmod{5^k}, \tag{4.9}
\end{equation}

\begin{equation}
\equiv 0 \pmod{7^k}. \tag{4.10}
\end{equation}

The identities, lemmas, and corollaries used to prove (4.1)–(4.10) are given in the following chart.

| Congruence | Relevant identity | Used Lemma/Corollary |
|------------|------------------|----------------------|
| \((1.5), \ell = 3\) | \(U_{3n+1} (E_3^2) = -3E_3^3\) | Lemma 2, (2.1) |
| \((1.5), \ell = 5\) | \(U_{5n+3} (E_5^3) = 5E_5^3\) | \((3.31)\) |
| \((1.5), \ell = 7\) | \(U_{7n+6} (E_7^3) = -7E_7^3\) | Lemma 3, \(n = 7\) |
| \((4.1)\) | \(U_{5n+2} (E_5^3) = -E_5^2\) | Corollary 3, \((2.3)\) |
| \((4.2)\) | \(U_{7n+4} (E_7^3) = E_7^3\) | Lemma 3, \(n = 7\) |
| \((4.3)\) | \(U_{11n+10} (E_{11}^2) = E_{11}^2\) | Lemma 3, \(n = 11\) |
| \((4.4)\) | \(U_{5n+4} (E_5^3) = -5E_5^3\) | Corollary 3, \((2.3)\) |
| \((4.5)\) | \(U_{5n+3} (E_5^3) = -5^3 qE_5^8\) | Corollary 3, \((2.3)\) |
| \((4.6)\) | \(U_{5n+4} (E_5^{14}) = -5^6 q^2 E_5^{14}\) | Corollary 3, \((2.3)\) |
| \((4.7)\) | \(U_{7n+5} (E_7^5) = 7^2 qE_7^5\) | Lemma 3, \(n = 7\) |

4.2 Appendix B

The following congruences can also be proved by using the methods explained in the proof of Theorems 4 and 6. Let \( k \) and \( s \) be positive integers such that \( s \leq \lfloor k/2 \rfloor \) and \( b \)'s in (4.11)–(4.19) are co-prime to the moduli. Then, for all \( n \geq 0 \), we have

\begin{equation}
P_{[13;\ldots,(5^s-b)/b]} \left( 5^{2s} \cdot n + 5^{2s-1} \cdot r + \frac{5^{2s} - 1}{6} \right)
\equiv 0 \pmod{5^{k-2s+1}}, \quad \text{where } r \in \{1, 2, 3, 4\}, \tag{4.11}
\end{equation}
The identities, lemmas, and corollaries used to prove (4.11)–(4.19) are given in the following chart.

| Congruence | Relevant identity | Used Lemma/Corollary |
|------------|------------------|---------------------|
| (1.13), $\ell = 3$ | $U_{3n+1} (E_1 E_2) = -E_3 E_6$ | (2.2) |
| (1.13), $\ell = 5$ | $U_{5n+3} (E_1 E_2) = E_5 E_{10}$ | (2.3) |
| (1.13), $\ell = 7$ | $U_{7n+6} (E_1 E_2) = E_7 E_{14}$ | Lemma 3, $n = 7$ |
| (1.15) | $U_{5n+2} (E_1^2 E_3^2) = 5^2 q^2 E_2^3 E_1^3$ | (3.31) |
| (4.11) | $U_{5n+4} (E_1 E_3) = E_5 E_{15}$ | Corollary 3, (2.3) |
| (4.12) | $U_{11n+9} (E_1 E_3) = q E_{11} E_{33}$ | Lemma 3, $n = 11$ |
| (4.13) | $U_{7n+3} (E_1 E_4) = q E_7 E_{28}$ | Lemma 3, $n = 7$ |
| (4.14) | $U_{11n+3} (E_1 E_4) = q^2 E_{11} E_{44}$ | Lemma 3, $n = 11$ |
| (4.15) | $U_{3n+2} (E_1^2 E_2^2) = -3 E_2^2 E_1^2$ | (2.2) |
| (4.16) | $U_{3n+2} (E_1^2 E_2^2) = -3^2 q E_3^3 E_6$ | Lemma 2, (2.2) |
| (4.17) | $U_{5n+4} (E_1^2 E_2^2) = 5^2 q E_2^3 E_1^3$ | Corollary 3, (2.3) |
4.3 Appendix C

The following steps are required to establish the conditions (ii)–(v) of Theorem 5.

**Condition (ii).** Use (2.9) of Lemma 6 to obtain the expression

\[ N = \frac{(4n + 1)^2 + 2(4m + 1)^2 - 3}{8}, \]

equivalent to \( 8N + 3 = (4n + 1)^2 + 2(4m + 1)^2 \). If \( \ell \equiv 5 \text{ or } 7 \pmod{8} \), then

\[ \left( \frac{-2}{\ell} \right) = -1. \]

It follows that \( 8N + 3 \equiv 0 \pmod{\ell} \) if and only if \( 4n + 1 \equiv 0 \pmod{\ell} \) and \( 4m + 1 \equiv 0 \pmod{\ell} \).

**Condition (iii).** Use (2.9) of Lemma 6 to obtain the expression

\[ N = \frac{(4n + 1)^2 + 3(4m + 1)^2 - 4}{8}, \]

equivalent to \( 8N + 4 = (4n + 1)^2 + 3(4m + 1)^2 \). If \( \ell \equiv 2 \pmod{3} \), then

\[ \left( \frac{-3}{\ell} \right) = -1. \]

It follows that \( 8N + 4 \equiv 0 \pmod{\ell} \), equivalently, \( 2N + 1 \equiv 0 \pmod{\ell} \) if and only if \( 4n + 1 \equiv 0 \pmod{\ell} \) and \( 4m + 1 \equiv 0 \pmod{\ell} \).

**Condition (iv).** Use (2.8) and (2.11) of Lemma 6 to obtain the expression

\[ N = \frac{(6n + 1)^2 + 4(3m + 1)^2 - 5}{12}, \]

equivalent to \( 12N + 5 = (6n + 1)^2 + 4(3m + 1)^2 \). If \( \ell \equiv 3 \pmod{4} \), then

\[ \left( \frac{-1}{\ell} \right) = -1. \]

It follows that \( 12N + 5 \equiv 0 \pmod{\ell} \) if and only if \( 6n + 1 \equiv 0 \pmod{\ell} \) and \( 3m + 1 \equiv 0 \pmod{\ell} \).

**Condition (v).** Use (2.9) of Lemma 6 to obtain the expression

\[ N = \frac{(4n + 1)^2 + 4(4m + 1)^2 - 5}{8}, \]

equivalent to \( 8N + 5 = (4n + 1)^2 + 4(4m + 1)^2 \). If \( \ell \equiv 3 \pmod{4} \), then

\[ \left( \frac{-1}{\ell} \right) = -1. \]

It follows that \( 8N + 5 \equiv 0 \pmod{\ell} \) if and only if \( 4n + 1 \equiv 0 \pmod{\ell} \) and \( 4m + 1 \equiv 0 \pmod{\ell} \).

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