A Bicriteria Perspective on $L$-Penalty Approaches – a Corrigendum to Siddiqui and Gabriel’s $L$-Penalty Approach for Solving MPECs

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Abstract
This paper presents a corrigendum to Theorems 2 and 3 in Siddiqui and Gabriel (Netw Spatial Econ 13(2):205–227, 2013). In brief, we revise the claim that their $L$-penalty approach yields a solution satisfying complementarity for any positive value of $L$, in general. This becomes evident when interpreting the $L$-penalty method as a weighted-sum scalarization of a bicriteria optimization problem. We also elaborate further assumptions under which the $L$-penalty approach yields a solution satisfying complementarity.
Keywords  Equilibrium problems · MPEC · $L$-penalty method · Bicriteria optimization

1 Preface

The aim of this paper is twofold. By way of a small counterexample inspired from the application work in Saez-Gallego et al. (2016), we first reveal a flaw in the theoretical development of Siddiqui and Gabriel (2013). Then, we draw from the theory of multi-objective optimization to repair the flaw, explore the limitations of the $L$-penalty method proposed in Siddiqui and Gabriel (2013) and, finally, provide ideas to overcome these limitations. We end this manuscript with a brief literature review highlighting the theoretical and practical importance of MPECs. In this paper we follow the notation of Siddiqui and Gabriel (2013). However, in order to make this paper self-contained, we briefly repeat the relevant formulas from Siddiqui and Gabriel (2013) in the next section indicating by an asterisk those coming from the original paper.

2 Introduction

Following the approach in Siddiqui and Gabriel (2013), we consider a mathematical program with equilibrium constraints given by

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s.t.} & \quad (x, y) \in \Omega \\
& \quad y \in S(x)
\end{align*}
\]

where the continuous variables $x \in \mathbb{R}^{n_x}$ and $y \in \mathbb{R}^{n_y}$ are, respectively, the vector of upper-level and lower-level variables, $f(x, y)$ is the upper-level single-objective function, $\Omega$ is the joint feasible region between these sets of variables and $S(x)$ is the solution set of the lower-level problem that can take the form of an optimization problem, a nonlinear complementarity problem (NCP), or a variational inequality problem (Luo et al. 1996).

The main focus of Siddiqui and Gabriel (2013) is when $S(x)$ is the solution set of an NCP. Then $(1^\ast)$ can be rewritten as

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s.t.} & \quad (x, y) \in \Omega \\
& \quad y \geq 0 \\
& \quad g(x, y) \geq 0 \\
& \quad y^\top g(x, y) = 0
\end{align*}
\]

where $g(x, y) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_y}$ is a vector-valued function. We often make use of the shorthand notation $f = f(x, y)$, $g = g(x, y)$ for convenience.

The set $y^\top g = 0$ is non-convex in $x$, $y$ and can be computationally challenging to find even if $g$ is linear. In this case, the MPEC $(3^\ast)$ can be reformulated using Schur’s decomposition, see $(6^\ast)$, $(8^\ast)$ and $(9^\ast)$ of Siddiqui and Gabriel (2013), where, in brief, new variables $u$ and $v$ are introduced by setting $u = (y + g)/2$ and $v = (y - g)/2$. 

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Then $y = u + v$ and $g = u - v$, thus, $y_ig_i = (u_i + v_i)(u_i - v_i) = u_i^2 - v_i^2 \text{ for every } i = 1, \ldots, n_y$. Since $y \geq 0$ and $g \geq 0$, it follows that $u \geq 0$. Hence, $u_i^2 - v_i^2 = 0$ is equivalent to $u_i - |v_i| = 0$. If $v$ is replaced by two non-negative variables, i.e., $v = v^+ - v^-$ with $v^+, v^- \geq 0$, where (component-wise) at most one is non-zero, the absolute value can be expressed as $|v| = v^+ + v^-$. The corresponding reformulation with SOS 1 variables (special ordered sets of type 1, defined as a set of non-negative variables of which at most one can take a strictly positive value) reads

$$\begin{align*}
\min \quad & f(x, y) \\
\text{s.t.} \quad & (x, y) \in \Omega \\
& y \geq 0 \\
& g(x, y) \geq 0 \\
& u - (v^+ + v^-) = 0 \\
& u = \frac{y + g(x, y)}{2} \\
& v^+ - v^- = \frac{y - g(x, y)}{2}
\end{align*}$$

(9*)

where $v^+, v^-$ are SOS 1 variables.

Formulation (9*) is a viable way to solve the MPEC (1*) as shown in Siddiqui and Gabriel (2013). They also propose an $L$-penalty method of the form

$$\begin{align*}
\min \quad & f(x, y) + \sum_{i=1}^{n_y} L_i (v^+_i + v^-_i) \\
\text{s.t.} \quad & (x, y) \in \Omega \\
& y \geq 0 \\
& g(x, y) \geq 0 \\
& u - (v^+ + v^-) = 0 \\
& u = \frac{y + g(x, y)}{2} \\
& v^+ - v^- = \frac{y - g(x, y)}{2}
\end{align*}$$

(10*)

and $L_i > 0$ for all $i = 1, \ldots, n_y$. Compared to (9*) the SOS 1 property is relaxed and instead a new term weighted by parameters $L_1, \ldots, L_{n_y}$ is added to the objective function. In what follows we use a scalar $L = L_1 = \cdots = L_{n_y}$ whenever a distinction by different parameter values is not necessary.

Since we can replace $v^+_i + v^-_i$ in the objective function by $u_i = (y_i + g_i)/2$, the auxiliary variables $u$, $v^+$, $v^-$ are not required in (10*) and can thus be removed. This yields the simplified but equivalent $L$-penalty formulation (10*).
Theorem 2 of Siddiqui and Gabriel (2013) states that if problem (9*) has a solution and if the KKT-conditions are both necessary and sufficient for (10*), then for any \( L_i > 0 \) and for each \( i \), problem (10*) has a solution where at most one of \((v^+)\_i\) and \((v^-)\_i\) is nonzero (i.e., the SOS 1 property holds).

Translated to (10*\( b \)) this implies that for any \( L_i > 0 \) and for each \( i \), either \( y_i \) or \( g_i \) or both are zero, i.e., complementarity is satisfied. This is because \( v^+_i = \frac{y_i}{2} \) and \( v^-_i = \frac{g_i}{2} \). Unfortunately, Theorem 2 does not hold. The same applies for Theorem 3 of Siddiqui and Gabriel (2013), which states that if the feasible set of (10*) is non-empty and if the maximum of \( \sum_{i=1}^{n_y} v^+_i + v^-_i = \sum_{i=1}^{n_y} (y_i + g_i)/2 \) over this set exists, then there exists an \( L > 0 \) so that for all positive \( \hat{L} \leq L \) a solution to problem (10*) with penalty \( \hat{L} \) also solves (9*), i.e., satisfies complementarity.

The rest of the corrigendum is structured as follows. In Section 3 we present counter-examples to Theorems 2 and 3 of Siddiqui and Gabriel (2013). In Section 4 the failure of the theorems is analyzed from a bicriteria perspective. Section 5 contains a new theorem that guarantees solutions of the proposed \( L \)-penalty method satisfying complementarity under certain assumptions. Section 6 presents a literature review on recent advances in the context of MPECs.

Conclusions are summarized in Section 7.

3 Counter-examples

3.1 Counter-example to Theorem 2

Example 1 (Non-complementarity for certain values of \( L \)) Consider the following instance of Problem (1*) with a parameter \( K > 0 \), inspired by Saez-Gallego et al. (2016):

\[
\begin{align*}
\min \quad & f(x, y) = x_3 + x_4 + x_5 + x_6 \\
\text{s.t.} \quad & x_3 - x_4 = x_1 - 7 \\
& x_5 - x_6 = x_2 - 3 \\
& y_1 = 3 \\
& -4 + y_1 + y_2 - Ky_3 = 0 \\
& g_1(x, y) = 10 - x_1 - x_2 \geq 0 \\
& g_2(x, y) = x_7 - x_2 \geq 0 \\
& g_3(x, y) = Kx_2 \geq 0 \\
& y^\top g = 0 \\
& x, y \geq 0
\end{align*}
\]
The unique optimal solution to (1) is \( x = (7, 3, 0, 0, 0, 0, 3), y = (3, 1, 0) \), which can be seen as follows. Since \( x \) is non-negative, the objective function value is bounded below by zero which is attained if and only if \( x_3 = x_4 = x_5 = x_6 = 0 \), assuming that it is part of a feasible solution. Then, (1b) yields \( x_1 = 7 \) and (1c) yields \( x_2 = 3 \). Furthermore, (1f) yields \( g_1 = 0 \) and (1h) yields \( g_3 = 3 \cdot K > 0 \). Hence, to obtain a solution for which the complementarity conditions hold, i.e. to satisfy (1i), \( y_3 = 0 \) must hold while \( y_1 \geq 0 \) and, in particular, \( y_1 = 3 \) so that constraint (1d) is satisfied. From (1e) we obtain \( y_2 = 1 \). Hence, \( g_2 = 0 \) must hold which implies \( x_7 = x_2 = 3 \). Summarizing, \( x = (7, 3, 0, 0, 0, 0, 3), y = (3, 1, 0) \) is the unique optimal solution for any \( K > 0 \).

We recast problem (1) using the \( L \)-penalty reformulation (10\(^*\)b) with \( L = L_i, i = 1, 2, 3 \), and \( K = 10 \):

\[
\begin{align*}
\min \ x_3 + x_4 + x_5 + x_6 + \frac{L}{2} (y_1 + y_2 + y_3 - x_1 + 8x_2 + x_7 + 10) \\
\text{s.t. } & x_3 - x_4 = x_1 - 7 \quad (2a) \\
& x_5 - x_6 = x_2 - 3 \quad (2b) \\
& y_1 = 3 \quad (2c) \\
& y_1 + y_2 - 10y_3 = 4 \quad (2d) \\
& g_1(x, y) = 10 - x_1 - x_2 \geq 0 \quad (2e) \\
& g_2(x, y) = x_7 - x_2 \geq 0 \quad (2f) \\
& g_3(x, y) = 10x_2 \geq 0 \quad (2g) \\
& x, y \geq 0. \quad (2h)
\end{align*}
\]

For \( L = 1 \) we obtain \( x = (7, 0, 0, 0, 0, 3, 0) \) and \( y = (3, 1, 0) \) as an optimal solution, which yields \( g(x, y) = (3, 0, 0) \). Since \( y_1 \neq 0 \) and \( g_1 \neq 0 \), this solution does not satisfy complementarity, a contradiction to Theorem 2 of Siddiqui and Gabriel (2013).

What goes wrong in the proof of Theorem 2? In the proof from the original paper the KKT conditions for (10\(^*\)) are formulated. It is shown that, if \( v_i^+ \) and \( v_i^- \) are strictly positive, \( L_i = -(\lambda_4) \), where \( \lambda_4 \) denotes the Lagrange multiplier associated with the constraint \( v_i^+ + v_i^- - \frac{\nu_i + \xi_i}{2} = 0 \). The term that corresponds to \( L_i \) in the objective function is cancelled out and it is claimed that an equivalent optimization problem for this recast KKT system exists that contains the constraint \( \frac{\nu_i + \xi_i}{2} = 0 \) instead of \( v_i^+ + v_i^- - \frac{\nu_i + \xi_i}{2} = 0 \). Equivalence, however, is only true as long as the KKT solution remains feasible at optimality, which is not the case when \( v_i^+ > 0, v_i^- > 0 \) and \( \frac{\nu_i + \xi_i}{2} = 0 \). For Example 1 and \( L = 1, y_1 + g_1 = 6 \) and, therefore, the KKT solution of the original problem is not feasible.

### 3.2 Counter-example to Theorem 3

The next counter-example shows that the \( L \)-penalty approach might yield a solution not satisfying complementarity for any \( L > 0 \) in contradiction to Theorem 3.
Example 2 (Non-complementarity for every $L > 0$)

\[ \begin{align*}
\min & \quad -y \\
\text{s.t.} & \quad y \leq 4 \\
& \quad g(x, y) = 10 - 2y \geq 0 \\
& \quad y \geq 0 \\
& \quad y \cdot g = 0
\end{align*} \]  \tag{3}

Since $g \geq 2$, by complementarity the unique optimal solution (with respect to $y$) for (3) is $y = 0$. Consider the objective function of the $L$-penalty method:

\[ f(x, y) + L \cdot f^{\text{pen}}(x, y) = -y + \frac{L}{2} (y + 10 - 2y) = - \left(1 + \frac{L}{2}\right) y + 5L. \]  \tag{4}

For every $L > -2$, the optimal solution of the $L$-penalty approach is $y = 4$, combined with $g \geq 2$ thus, $y \cdot g \neq 0$. It can be easily verified that the example satisfies the assumptions of Theorem 3 from Siddiqui and Gabriel (2013) which, applied to this example, state that $\max\{5 - \frac{y}{2} : y \leq 4, g(x, y) = 10 - 2y \geq 0, y \geq 0\}$ admits a finite optimal solution with a positive objective function value.

The failure of Theorems 2 and 3 of Siddiqui and Gabriel (2013) is explained from a bicriteria perspective in the next section.

4 Bicriteria interpretation of the $L$-penalty formulation

We can interpret the $L$-penalty formulation (10*b)

\[ \begin{align*}
\min & \quad f(x, y) + L \cdot \sum_{i=1}^{n_y} \frac{y_i + g_i(x, y)}{2} \\
\text{s.t.} & \quad (x, y) \in \Gamma
\end{align*} \]

with feasible set $\Gamma = \{(x, y) : (x, y) \in \Omega, y \geq 0, g(x, y) \geq 0\}$ and a scalar parameter $L$ as the weighted-sum formulation of a bicriteria optimization problem of the form

\[ \begin{align*}
\min & \quad f(x, y) \\
\min & \quad f^{\text{pen}}(x, y) \\
\text{s.t.} & \quad (x, y) \in \Gamma,
\end{align*} \]  \tag{5}

where

\[ f^{\text{pen}}(x, y) = \sum_{i=1}^{n_y} \frac{y_i + g_i(x, y)}{2}. \]  \tag{6}

The objective function of (10*b) consists of two terms: the original objective $f(x, y)$ and the $L$-penalty term $f^{\text{pen}}$ which is weighted by $L$. When the two objective functions $f$ and $f^{\text{pen}}$ are conflicting, there does not exist a single solution that optimizes both objectives simultaneously. Instead we deal with a set of solutions that cannot be improved with respect to one criterion without being deteriorated with respect to the other criterion. Such solutions are called efficient, their outcomes nondominated.
In the following, we briefly state common notions from multicriteria optimization in the context of the specific bicriteria problem with objectives $f$ and $f^{pen}$.

**Definition 1 (Efficiency/Nondominance)** Consider the bicriteria optimization problem (5). A solution $(x, y) \in \Gamma$ is called *efficient*, and its outcome $(f(x, y), f^{pen}(x, y))^\top$ is called *nondominated* if there does not exist a solution $(x, y) \in \Gamma$ such that $f(x, y) \leq f(x', y')$ and $f^{pen}(x, y) \leq f^{pen}(x', y')$ with at least one strict inequality.

**Definition 2 (Lexicographic minimum)** Let $(\bar{x}, \bar{y}) \in \Gamma$ be an optimal solution of $\min_{(x, y) \in \Gamma} f(x, y)$. We call an optimal solution of

$$\begin{align*}
\min & \quad f^{pen}(x, y) \\
\text{s.t.} & \quad f(x, y) = f(\bar{x}, \bar{y}) \\
& \quad (x, y) \in \Gamma
\end{align*}$$

*lexicographic minimum* with respect to $(f, f^{pen})$.

The lexicographic minimum with respect to $(f^{pen}, f)$ is determined analogously. Note that every lexicographically optimal solution is efficient. For more details on these basic notions from multicriteria optimization we refer to the textbooks of Chankong and Haimes (1983) and Ehrgott (2005).

If the problem is linear we can easily determine the nondominated set in the following way (see, e.g., Aneja and Nair (1979) for a principle description of the procedure). Therefore, consider again the $L$-penalty formulation (2) from Example 1 for $K = 10$.

We first compute the two *lexicographic minima* with respect to $(f, f^{pen})$ and $(f^{pen}, f)$.

Minimizing only $f$ yields $x_3 = x_4 = x_5 = x_6 = 0$, $x_1 = 7$ from (2b) and $x_2 = 3$ from (2c). Then, $x_7 \geq 3$ from (2g). Moreover, (2f) and (2h) are satisfied. Besides, $y_1 = 3$ from (2d) and $y_2 - 10y_3 = 1$ from (2e). Hence, the set of optimal solutions reads $x = (7, 3, 0, 0, 0, 0, \alpha), y = (3, 1 + 10\beta, \beta)$ with $\alpha \geq 3$ and $\beta \geq 0$. Fixing now this solution and considering the objective $f^{pen}$, thus, minimizing $(11y_3 + x_7 + 31)/2$ with $x_7 \geq 3$ and $y_3 \geq 0$, results in the efficient solution $x = (7, 3, 0, 0, 0, 0, 3), y = (3, 1, 0)$ for which $g = (0, 0, 30)$ and with optimal objective function value $f^{pen} = 17$. Hence, $z^1 = (0, 17)$ is a nondominated point where the first component corresponds to $f$ and the second to $f^{pen}$. Note that it is by coincidence that this solution satisfies complementarity.

In order to determine the second lexicographic minimum we first consider $f^{pen}$. Since $y_1 + y_2 \geq 4$ from (2e) and $y_3, g_1, g_2, g_3 \geq 0$, $f^{pen} \geq 2$ and $f^{pen} = 2$ is obtained for $y_1 = 3$, $y_2 = 1$, $y_3 = 0$ and $g_1 = g_2 = g_3 = 0$. This in turn implies $x_1 = 10$ and $x_2 = x_7 = 0$. The values of $x_3, x_4, x_5, x_6$ are not uniquely determined since these variables do not appear in the $L$-penalty objective. However, when (re-)optimizing the original objective over all solutions with $f^{pen} = 2$, we obtain $x_3 = x_6 = 3, x_4 = x_5 = 0$. Thus, $z^2 = (6, 2)$ with $f = 6$ and $f^{pen} = 2$. 

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is a nondominated point with associated efficient solution \( x = (10, 0, 3, 0, 0, 3, 0) \), \( y = (3, 1, 0) \) and \( g = (0, 0, 0) \).

The two lexicographic minima \( z^1 = (0, 17) \) and \( z^2 = (6, 2) \) give us a range of the outcomes. For any \( L > 0 \) the value of \( f \) will be in \([0, 6]\) and the value of \( f^{pen} \) in \([2, 17]\). However, we do not need to test certain values of \( L \) randomly. According to the procedure proposed in Aneja and Nair (1979) we can use two adjacent nondominated points to derive a value for \( L \) which enables us to search for a new nondominated point between the given two points in a systematic way. Based on \( z^1 = (0, 17) \) and \( z^2 = (6, 2) \) we compute

\[
L = \frac{z^2_1 - z^1_1}{z^2_2 - z^1_2} = \frac{6 - 0}{17 - 2} = 0.4
\]

From the solution of problem (2) with \( L = 0.4 \) we numerically obtain the new nondominated point \( z^3 = (3, 3.5) \). Again, we use pairs of adjacent nondominated points to update \( L \). However, solving (2) with \( L = (z^3_1 - z^1_1)/(z^3_2 - z^1_2) = 2/9 \) and \( L = (z^2_1 - z^1_1)/(z^2_2 - z^1_2) = 2 \), respectively, does not yield a new nondominated point. Hence, the nondominated set consists of the three points \( z^1 = (0, 17) \), \( z^3 = (3, 3.5) \), \( z^2 = (6, 2) \) and all convex combinations between the pairs of adjacent points \((z^1, z^3)\) and \((z^3, z^2)\). A visualization is given in Fig. 1.

An overview of all possible values of \( L \), the resulting objective function values and the solutions \( x, y \) as well as \( g \) is given in Table 1. With increasing values of \( L \) the minimization of \( f^{pen} \) becomes more important. Note that there exist infinitely many optimal solutions for \( L = 2/9 \) and \( L = 2 \). For example, for \( L = 2/9 \), besides the two solutions indicated in Table 1 all solutions \((x, y)\) with \( x = (7, \lambda, 0, 0, 0, 3 - \lambda, \lambda) \), \( 0 < \lambda < 3 \), and \( y = (3, 1, 0) \) are feasible and result in an overall objective function value of \( 34/9 \).

Further note that the optimal solution obtained for \( L \in \left[0, \frac{2}{9}\right) \) equals - by chance - the solution of the original complementarity problem (1). Thus, we state that for

\[ \text{Fig. 1 Nondominated set of Example 1} \]
Table 1 Solutions of the $L$-penalty problem (2) for different choices of $L$ with $K = 10$. At the breakpoints $L = \frac{2}{9}$ and $L = 2$ (infinitely many) alternative solutions exist

| $L$       | $f$ | $f_{pen}$ | $x$       | $y$   | $g$   |
|-----------|-----|-----------|-----------|-------|-------|
| $\left[0, \frac{2}{9}\right]$ | 0   | 17        | $(7,3,0,0,0,0,3)$ | $(3,1,0)$ | $(0,0,30)$ |
| $\left[\frac{2}{9}, 2\right]$ | 3   | $\frac{7}{3}$ | $(7,0,0,0,0,3,0)$ | $(3,1,0)$ | $(3,0,0)$ |
| $[2, \infty)$ | 6   | 2         | $(10,0,3,0,0,3,0)$ | $(3,1,0)$ | $(0,0,0)$ |

this particular problem the $L$-penalty approach and a variation of $L$ would not be required theoretically but complementarity constraint (1i) could simply be ignored. Nevertheless, this example serves to demonstrate the failure of Theorem 2 of Siddiqui and Gabriel (2013) which claims solutions satisfying complementarity for every $L > 0$. In this example, however, for all $L \in (\frac{2}{9}, 2)$ the optimal solution does not respect complementarity since $y_1 > 0$ and $g_1 > 0$.

The two conflicting objectives are one reason for Theorem 2 to fail. We can not expect that complementarity is satisfied for every $L > 0$. A further issue is the fact that the $L$-penalty term as formulated in (10*) does not guarantee complementarity, in general. For sufficiently large values of $L$, solutions with $y$ and $g$ small are obtained, however, there is no guarantee to achieve complementarity as demonstrated by Example 2.

5 Obtaining solutions satisfying complementarity with Siddiqui and Gabriel's $L$-penalty approach

In Example 2 a solution satisfying complementarity exists, however, it is not optimal for (10*) for any $L > 0$. Hence, the question arises under which conditions we can generate solutions satisfying complementarity with Siddiqui and Gabriel’s $L$-penalty approach. We have already stated that $f_{pen}$, which is non-negative by definition, is made smaller (all things being equal) with increasing values of $L$. In case a solution that corresponds to the minimum possible value of $f_{pen}$ satisfies complementarity we can expect it to be generated for a sufficiently large value of $L$. Consider once more Example 1 in which $f_{pen} \geq 2$ and the solution that realizes $f_{pen} = 2$ satisfies complementarity. Indeed, this solution is generated for all $L > 2$. In Example 2, $f_{pen} \geq 3$ but the solution that attains $f_{pen} = 3$ is $y = 4$, $g = 2$, thus, does not satisfy complementarity. This is the reason why no solution for which the complementarity conditions hold is obtained with the $L$-penalty approach of Siddiqui and Gabriel (2013) in this example.

In the following we show that a lexicographic minimal solution with respect to $(f_{pen}, f)$ is generated for sufficiently large values of $L$ under certain conditions. In order to guarantee that parameter $L$ is finite we need the notion of *proper efficiency* or *bounded trade-off* which is defined below for our specific bicriteria application. If the lexicographic minimal solution satisfies complementarity, we can then assure
that the solution of the $L$-penalty approach satisfies complementarity for sufficiently large values of $L$.

**Definition 3 (Proper Efficiency/Nondominance)** A solution $(\bar{x}, \bar{y}) \in \Gamma$ of the bicriteria optimization problem (5) is called *properly efficient* according to Geoffrion (1968) if it is efficient and if there exists a scalar $\bar{L} > 0$ so that for all $(x, y) \in \Gamma$ satisfying $f(x, y) < f(\bar{x}, \bar{y})$ and $f^{\text{pen}}(x, y) > f^{\text{pen}}(\bar{x}, \bar{y})$

$$\frac{f(\bar{x}, \bar{y}) - f(x, y)}{f^{\text{pen}}(x, y) - f^{\text{pen}}(\bar{x}, \bar{y})} \leq \bar{L}$$  \hspace{1cm} (7)

and for all $(x, y) \in \Gamma$ satisfying $f^{\text{pen}}(x, y) < f^{\text{pen}}(\bar{x}, \bar{y})$ and $f(x, y) > f(\bar{x}, \bar{y})$

$$\frac{f^{\text{pen}}(\bar{x}, \bar{y}) - f^{\text{pen}}(x, y)}{f(x, y) - f(\bar{x}, \bar{y})} \leq \bar{L}.$$ \hspace{1cm} (8)

The quotients in (7) and (8) are typically denoted as *trade-offs* between the two objectives (Chankong and Haimes 1983).

Figure 2 illustrates the notion of proper efficiency. In the subfigure on the right, the trade-off is not bounded in $z = (f(\bar{x}, \bar{y}), f^{\text{pen}}(\bar{x}, \bar{y}))$, while it is in the subfigure on the left which depicts a linear problem. Note that every efficient solution is properly efficient in the linear case.

**Theorem 1** Assume that $\Omega$ and $\Gamma = \{(x, y) : (x, y) \in \Omega, y \geq 0, g(x, y) \geq 0\}$ are non-empty and compact. Let $(x_0^0, y_0^0) \in \Gamma$ be lexicographically minimal with respect to $(f^{\text{pen}}, f)$. Moreover, let $(x_0^0, y_0^0)$ be properly efficient with the trade-off bounded by a scalar $\bar{L} > 0$.

Then $(x_0^0, y_0^0)$ is an optimal solution of (10* b) for every $L > \bar{L}$. Moreover, if $(x_0^0, y_0^0)$ satisfies complementarity the optimal solution of (10* b) satisfies complementarity for every $L > \bar{L}$.

![Fig. 2 Illustration of a bounded (left) and an unbounded (right) trade-off in the nondominated point $z = (f(\bar{x}, \bar{y}), f^{\text{pen}}(\bar{x}, \bar{y}))$.](image)
Proof Since \((x^0, y^0) \in \Gamma\) is lexicographically minimal with respect to \((f\text{\,pen}, f)\), it is efficient. Moreover, by definition there is no \((x, y) \in \Gamma\) with \(f\text{\,pen}(x, y) < f\text{\,pen}(\tilde{x}, \tilde{y})\), hence, proper efficiency of this solution means that for all \((x, y) \in \Gamma\) satisfying \(f(x, y) < f(x^0, y^0)\) and \(f\text{\,pen}(x, y) > f\text{\,pen}(x^0, y^0)\)

\[
\frac{f(x^0, y^0) - f(x, y)}{f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0)} \leq L.
\]

We want to show that \((x^0, y^0)\) is an optimal solution of \((10^*b)\) for every \(L > \tilde{L}\). Therefore, we consider the slightly modified objective function

\[
\tilde{f}(x, y) = f(x, y) + L \cdot \left[ f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0) \right]
\]

which only differs from \((10^*b)\) by the constant term \(L \cdot f\text{\,pen}(x^0, y^0)\), hence, yields the same optimal solution set. Now, for every \((x, y) \in \Gamma\) with \(f(x, y) < f(x^0, y^0)\) and \(f\text{\,pen}(x, y) > f\text{\,pen}(x^0, y^0)\) and for every \(L > \tilde{L} > 0\)

\[
\tilde{f}(x, y) = f(x, y) + L \cdot \left[ f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0) \right] \\
> f(x, y) + \tilde{L} \cdot \left[ f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0) \right] \\
\geq f(x, y) + \frac{f(x^0, y^0) - f(x, y)}{f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0)} \cdot \left[ f\text{\,pen}(x, y) - f\text{\,pen}(x^0, y^0) \right] \\
= f\left(x^0, y^0\right) \\
= f\left(x^0, y^0\right) + L \cdot 0 \\
= f\left(x^0, y^0\right) + L \cdot f(x, y) \\
= \tilde{f}\left(x^0, y^0\right).
\]

Since \((x^0, y^0)\) is lexicographically minimal with respect to \((f\text{\,pen}, f)\), there is no \((x, y) \in \Gamma \setminus \{(x^0, y^0)\}\) with \(f\text{\,pen}(x, y) < f\text{\,pen}(x^0, y^0)\) or \(f\text{\,pen}(x, y) = f\text{\,pen}(x^0, y^0)\) and \(f(x, y) < f(x^0, y^0)\). Hence, \((x^0, y^0)\) is minimal for \((10^*b)\) for every \(L > \tilde{L} > 0\). Clearly, if \((x^0, y^0)\) satisfies complementarity, an optimal solution of \((10^*b)\) satisfies complementarity for every \(L > \tilde{L}\). \(\square\)

**Remark 1** An optimal solution of \((10^*b)\) obtained under the assumptions of Theorem 1 is not necessarily optimal for the original complementarity problem \((3^*)\). As can be seen from Example 1 there might be solutions satisfying complementarity with a smaller value of \(f\). According to Table 1 the lexicographic minimum with respect to \((f\text{\,pen}, f)\) is \(x^0 = (10, 0, 3, 0, 0, 3, 0)\), \(y^0 = (3, 1, 0)\) with corresponding \(g = (0, 0, 0)\) and (original) objective function value \(f(x^0, y^0) = 6\). However, the solution \(x = (7, 3, 0, 0, 0, 0, 3)\), \(y = (3, 1, 0)\) with corresponding \(g = (0, 0, 30)\) also satisfies complementarity and yields \(f(x, y) = 0\).

The next example illustrates Theorem 1.
Example 3 (Complementarity for $L > 2$) Consider

$$\begin{align*}
\min & \quad -x_3 - y_2 \\
\text{s.t.} & \quad x_3 \leq 20 \\
& \quad y_2 \leq 10 \\
& \quad g_1(x, y) = 10 - x_1 - x_2 \geq 0 \\
& \quad g_2(x, y) = x_3 - x_2 \geq 0 \\
& \quad y_1, y_2 \geq 0 \\
& \quad g^\top y = 0 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}$$

(9)

and the associated objective of the $L$-penalty formulation

$$f + L \cdot f^{\text{pen}} = -x_3 - y_2 + \frac{L}{2} (y_1 + y_2 + 10 - x_1 - x_2 + x_3 - x_2)$$

$$= \frac{L}{2} y_1 + \left(\frac{L}{2} - 1\right) y_2 - \frac{L}{2} x_1 - L x_2 + \left(\frac{L}{2} - 1\right) x_3 + 5L$$

(10)

Note that the vectors $x = (0, 10, 10), y = (0, 0)$ with corresponding $g_1(x, y) = 0 = g_2(x, y)$ are feasible for (9). It can be easily verified that this solution, which satisfies complementarity, represents the unique lexicographic minimal solution with respect to $(f^{\text{pen}}, f)$. Since the problem is linear, every efficient solution and in particular the lexicographic minimum is properly efficient. Hence, the assumptions of Theorem 1 hold and there must exist a finite $\bar{L} > 0$ such that for every $L > \bar{L}$ an optimal solution of (10$^b$) equals the lexicographic minimum. Moreover, since this solution satisfies complementarity, an optimal solution of (10$^b$) satisfies complementarity.

Indeed, for $L > 2$, $y_1, y_2$ and $x_3$ are chosen as small as possible and $x_1, x_2$ as large as possible (within their bounds), where the higher weight is given to $x_2$. Hence, the lexicographic minimal solution $x = (0, 10, 10), y = (0, 0)$ with $g = (0, 0)$ is optimal for all $L > \bar{L}$ with $\bar{L} = 2$.

For $L \in [0, 2)$ there is an incentive in (10) to choose $y_2$ and $x_3$ as large as possible. Thus, an optimal solution for $L \in [0, 2)$ is $x = (0, 10, 20), y = (0, 10), g = (0, 10)$.

This solution does not satisfy complementarity. For $L = 2$, a solution for which the complementarity conditions hold might be obtained but is not enforced by objective (10).

Finally, note that the assumption that a lexicographic minimal solution with respect to $(f^{\text{pen}}, f)$ satisfies complementarity is sufficient but not necessary for the $L$-penalty method to work, in general. To illustrate this consider the non-linear example presented below motivated from Section 3 of Siddiqui and Gabriel (2013), which shows that the $L$-penalty method might work successfully also for selected values $L > 0$ despite all resulting solutions with $f^{\text{pen}} > 0$ are not lexicographically minimal with respect to $(f^{\text{pen}}, f)$. 

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Example 4 (Complementarity for every $L \geq 0$) Consider

$$
\begin{align*}
\min & \quad (y_1 + y_2 + x - 12) \cdot x \\
\text{s.t.} & \quad g_1(x, y) = x + 2y_1 + y_2 - 12 \geq 0 \\
& \quad g_2(x, y) = x + y_1 + 2y_2 - 12 \geq 0 \\
& \quad y^\top g = 0 \\
& \quad x, y_1, y_2 \geq 0.
\end{align*}
$$

The corresponding $L$-penalty term reads

$$
f^{pen} = 2(y_1 + y_2) + x - 12.
$$

Table 2 summarizes the (numerical) results for different values of $L$. Again, we see that the minimization of the second objective $f^{pen}$ gains importance for increasing values of $L$. Since $g = 0$ is optimal for all evaluated $L \geq 0$, complementarity holds for all these $L$, however, $f^{pen} = 0$ is not satisfied at optimality for those sampled values of $L \leq 10$.

As a practical consequence we propose to first remove the complementarity conditions from the original problem (3*) and solve the resulting problem. In case one obtains a solution satisfying complementarity as in Examples 1 and 4 there is no need to introduce an $L$-penalty formulation or any other approach to reformulate the complementarity condition.

6 The theoretical and practical importance of MPECs

Solution methods for MPECs are an active area of current research. Recently, there have been a number of studies that have advanced the algorithm development and applications of MPECs. Shim et al. (2013) present a branch-and-bound approach for discretely constrained MPECs. Pineda et al. (2018) discuss and investigate the performance of a wide range of general-purpose methods available in the technical literature to solve the MPEC that results from linear bilevel programming problems when their lower level is replaced with its KKT conditions (branch-and-bound, SOS1, regularization, penalization, Big-M reformulation). They also propose a new

| L   | f   | $f^{pen}$ | x   | y   | g         |
|-----|-----|-----------|-----|-----|-----------|
| 0   | -12 | 2         | 6   | (2,2)| (0,0)     |
| 0.0001 | -12 | 2         | 6   | (2,2)| (0,0)     |
| 1   | -11.92 | 1.83      | 6.5 | (1.83,1.83)| (0,0)|
| 10  | -3.67 | 0.33      | 11  | (0.33,0.33)| (0,0)|
| 100 | 0    | 0         | 12  | (0,0)| (0,0)|
method that combines regularization with the Big-M reformulation of the complementarity constraints, which is shown to perform remarkably well in a set of randomly generated problem instances of different size, scaling and sparsity of problem matrices and vectors. Pineda and Morales (to appear) show, by way of a simple counterexample, that the optimal solution found by means of the Big-M reformulation of an MPEC is not necessarily globally optimal even in the case that the Big-M constraints are not tight, which is a widespread belief.

Another recent direction of research deals with DC programming. Most relevant, Jara-Moroni et al. (2018) describe a difference-of-convex approach for solving linear MPECs, which has been applied in topology optimization as well (Kanno 2018). Beyond algorithms, there has been development on the theory of stationarity (Jane and Zhang (2014), Guo et al. (2015)), along with global optimization of such problems (Zhang and Sahinidis 2016). Standard methods, including relaxation methods have also been advanced in recent years (Kanzow and Schwartz 2014a, b).

Applications of MPECs can be found in various fields. In particular, applications in energy have seen a recent uptick, including natural gas networks (Neumann et al. 2015), electric power network oligopoly (Neto et al. (2016), Hesamzadeh et al. (to appear)), market power (Siddiqui and Gabriel (2017), Zerrahn and Huppmann (2017)), binary games (Huppmann and Siddiqui 2018), biofuels (Siddiqui and Christiansen (2016), U-tapao et al. (2016)), energy conservation (Champion and Gabriel 2015) and energy infrastructure planning under uncertainty (Guo and Fan 2017).

7 Conclusions

This corrigendum shows with the help of appropriate counter-examples that Theorems 2 and 3 in Siddiqui and Gabriel (2013) do not hold, in general. A bicriteria analysis helps to understand how the proposed $L$-penalty method works. In particular, the parameter $L$ can be interpreted as the weight of the penalty term with respect to the original objective function. We present a new theorem that guarantees that a solution of the $L$-penalty formulation satisfies complementarity for sufficiently large values of $L$ under the condition that the minimal value of the penalty term is attained by a solution satisfying complementarity and that the trade-off at this solution is bounded. As also shown there are instances in which solutions for which the complementarity conditions hold exist but are not accessible by the $L$-penalty method of Siddiqui and Gabriel (2013) for any $L \geq 0$. In order to reach those solutions, more than one penalty term is required or – in terms of multi-objective optimization – more than two objectives have to be considered. Efficient approaches for solving multi-objective optimization problems with more than two objectives have been proposed in Dächert and Klamroth (2015), Klamroth et al. (2015) and Dächert et al. (2017)). Their application to the problem at hand is left for future research.

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