OPERATIONS ON SPECTRAL PARTITION LIE ALGEBRAS AND TAQ COHOMOLOGY

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ABSTRACT. We determine all natural operations and their relations on the homotopy groups of spectral partition Lie algebras, as well as mod p Topological André-Quillen cohomology operations at any prime. We construct unary operations and a shifted restricted Lie algebra structure on the homotopy groups of spectral partition Lie algebras. Then we prove a composition law for the unary operations, as well as a compatibility condition between unary operations and the shifted Lie bracket with restriction. Comparing with Brantner-Mathew’s result on the ranks of the homotopy groups of free spectral partition Lie algebras, we deduce that these generate all natural operations and obtain the target category. Since \( \mathbb{F}_p \)-linear TAQ cohomology operations coincide with operations on the homotopy groups of spectral partition Lie algebras, we deduce the structure of \( \mathbb{F}_p \)-linear TAQ cohomology, thereby clarifying unpublished results by Kriz and Basterra-Mandell. As a corollary, we determine all natural operations on mod p \( \mathbb{S} \)-linear TAQ cohomology, along with relations among the operations.

1. INTRODUCTION

In this paper, we give explicit constructions of all natural operations on the homotopy groups of spectral partition Lie algebras over \( H\mathbb{F}_p \), as well as mod p Topological André-Quillen cohomology operations. We then determine all relations among the operations.

The project started as an attempt to refine a result of Brantner and Mathew on the ranks of the homotopy groups of the free spectral partition Lie algebras in [BM19]. Roughly speaking, spectral partition Lie algebras over \( H\mathbb{F}_p \) are divided power algebras Koszul dual to non-unital \( \mathbb{E}_\infty \)-\( H\mathbb{F}_p \)-algebras, implementing the Koszul duality between the non-unital \( \mathbb{E}_\infty \)-operad and the spectral Lie operad. Since spectral partition Lie algebras are algebras over a certain monad \( \pi_\mathbb{F}_p, \mathbb{E}_\infty \), the homotopy groups of free spectral partition Lie algebras \( \operatorname{Lie}_{\mathbb{F}_p, \mathbb{E}_\infty}^\pi (\Sigma^i H\mathbb{F}_p \oplus \cdots \oplus \Sigma^k H\mathbb{F}_p) \) parametrize all natural \( k \)-ary operations on the homotopy groups of spectral partition Lie algebras as \( (i_1, \ldots, i_k) \) varies. In [BM19, Theorem 1.20], Brantner and Mathew obtained bases for homotopy groups of free spectral partition Lie algebras on single generators via an isotropy spectral sequence studied by Arone, Dwyer, and Lesh ([ADL13]). Then they propagated the result to multiple generators by means of an EHP sequence and a decomposition of the partition complex developed by Arone and Brantner in [AB21]. Nonetheless, this method did not provide explicit descriptions of the nature of the operations, nor were the relations among the operations clarified.

On the other hand, \( \mathbb{F}_p \)-linear TAQ cohomology \( \operatorname{TAQ}^*(R, H\mathbb{F}_p; H\mathbb{F}_p) \) of \( \mathbb{E}_\infty \)-\( H\mathbb{F}_p \)-algebras \( R \) has representing objects trivial square-zero extensions. Hence reduced \( \mathbb{F}_p \)-linear TAQ cohomology groups of trivial algebras \( H\mathbb{F}_p \oplus \Sigma^i H\mathbb{F}_p \oplus \cdots \oplus \Sigma^k H\mathbb{F}_p \) parametrize all natural \( k \)-ary operations. By [BM19, Proposition 5.35], there is an isomorphism

\[
\pi_*(\operatorname{Lie}_{\mathbb{F}_p, \mathbb{E}_\infty}^\pi (\Sigma^i H\mathbb{F}_p \oplus \cdots \oplus \Sigma^k H\mathbb{F}_p)) \oplus \mathbb{F}_p \cong \operatorname{TAQ}^{-*}(H\mathbb{F}_p \oplus \Sigma^{-i_1} H\mathbb{F}_p \oplus \cdots \oplus \Sigma^{-i_k} H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p).
\]

Hence natural operations on the homotopy groups of spectral partition Lie algebras agree with cohomology operations on the (reduced) \( \mathbb{F}_p \)-linear TAQ cohomology of \( \mathbb{E}_\infty \)-\( H\mathbb{F}_p \)-algebras. In unpublished work, Kriz computed the \( \mathbb{F}_2 \)-linear TAQ cohomology on a connective generator in [Kri93,
Theorem 1.10]. Arone and Mahowald computed the dimensions of unary operations on \( \mathbb{F}_p \)-linear TAQ cohomology of connective objects in terms of free modules over subalgebras of the mod \( p \) Steenrod algebra in [AM99, Theorem 3.16 and 3.17]. Around the same time, Basterra and Mandell announced a computation of unary operations and their relations as the Koszul dual to Dyer-Lashof operations on \( \mathbb{F}_p \)-linear TAQ cohomology of connective objects for \( p > 2 \) and observed a shifted restricted Lie algebra structure, but a proof never appeared, cf. [Law20, Example 1.8.8].

In this paper, we use a dual bar spectral sequence and the machinery of classical Koszul duality developed by Priddy ([Pri70]) and Fresse ([Fre04]) to identify the power ring of additive operations on the homotopy groups of spectral partition Lie algebras and \( \mathbb{F}_p \)-linear TAQ cohomology. Roughly speaking, this power ring is given by a collection of unstable Ext groups over the Dyer-Lashof algebra, with composition product given by a sheared Yoneda product. The verification of the law of composition relies crucially on a result of Brantner in [Br17], which demonstrates the compatibility of the algebraic Koszul duality on the \( E^2 \)-page of the (dual) bar spectral sequence with the monadic Koszul duality that the \( E^\infty \)-page assembles to. As the degree of a homotopy class gets arbitrarily large, the algebra of additive unary operations on that class approaches the Koszul dual algebra of the Dyer-Lashof algebra.

Then we construct a shifted Lie bracket on the homotopy groups of spectral partition Lie algebras, and use a homotopy fixed points spectral sequence to detect a restriction map on the shifted Lie algebra. The restriction on an odd degree homotopy class when \( p = 2 \) coincides with the bottom unary operation up to a unit. Furthermore, brackets of unary operations that are not iterations of the restriction always vanish. Comparing with Brantner and Mathew’s computation ([BM19, Theorem 1.20]), we deduce that the dual bar spectral sequence collapses in the universal cases. Hence we conclude that these are all the natural operations and obtain the optimal target category.

As an immediate corollary, we determine the structure of operations on the mod \( p \) TAQ cohomology \( \mathrm{TAQ}^n(\_; \mathbb{S}; \mathbb{H} \mathbb{F}_p) \) of \( \mathbb{E}_\infty \)-\( \mathbb{S} \)-algebras. The \( \mathbb{S} \)-linear TAQ cohomology operations consist of \( \mathbb{F}_p \)-linear TAQ cohomology operations, as well as mod \( p \) Steenrod operations. The relations among the unary operations are given by the Adem relations and the Nishida relations on mod \( p \) cohomology ([Nis68]).

1.1. Statement of results. In section 2, we summarize recent results in [BM19] on spectral partition Lie algebras over \( \mathbb{H} \mathbb{F}_p \). These are algebras over a certain monad \( \mathrm{Lie}^\pi_{\mathbb{F}_p \mathbb{E}_\infty} \) on the category of \( \mathbb{H} \mathbb{F}_p \)-modules. Then we describe the relation between spectral partition Lie algebras and the mod \( p \) TAQ spectrum \( \mathrm{TAQ}(R, \mathbb{H} \mathbb{F}_p; \mathbb{H} \mathbb{F}_p) \) of \( \mathbb{E}_\infty \)-\( \mathbb{H} \mathbb{F}_p \)-algebras \( R \) constructed by Basterra ([Bas99]). The \( n \)th mod \( p \) TAQ cohomology group of \( R \) is given by

\[
\mathrm{TAQ}^n(R, \mathbb{H} \mathbb{F}_p; \mathbb{H} \mathbb{F}_p) = [\Sigma^{-n}\mathrm{TAQ}(R, \mathbb{H} \mathbb{F}_p; \mathbb{H} \mathbb{F}_p), \mathbb{H} \mathbb{F}_p]_{\text{Mod}_{\mathbb{H} \mathbb{F}_p}},
\]

with \( \mathrm{TAQ}^n(R, \mathbb{H} \mathbb{F}_p; \mathbb{H} \mathbb{F}_p) \) the reduced mod \( p \) TAQ cohomology. For any \( m \) and tuple \( (i_1, \ldots, i_k) \) of integers, there is an isomorphism

\[
\pi_m(\mathrm{Lie}^\pi_{\mathbb{F}_p \mathbb{E}_\infty}(\Sigma^{i_1} \mathbb{H} \mathbb{F}_p \oplus \cdots \oplus \Sigma^{i_k} \mathbb{H} \mathbb{F}_p)) \cong \mathrm{TAQ}^{-m}(\mathbb{H} \mathbb{F}_p \oplus \Sigma^{-i_1} \mathbb{H} \mathbb{F}_p \oplus \cdots \oplus \Sigma^{-i_k} \mathbb{H} \mathbb{F}_p; \mathbb{H} \mathbb{F}_p)
\]

by [BM19, Proposition 5.35]. The left hand side is the group of natural transformations

\[
\prod_{i=1}^{k} \pi_i(-) \to \pi_m(-)
\]
of functors from the category of spectral partition Lie algebras to Sets, whereas the right hand side is the group of cohomology operations

\[ \prod_{i=1}^{k} \text{TAQ}^{-i}(-, H \mathbb{F}_p; H \mathbb{F}_p) \to \text{TAQ}^{-m}(-, H \mathbb{F}_p; H \mathbb{F}_p). \]

Hence the structure of natural operations on the homotopy groups of spectral partition Lie algebras over \( H \mathbb{F}_p \) coincide with that on the reduced version of TAQ cohomology \( \text{TAQ}_Q(A) \) for \( A \) a non-unital \( \mathbb{E}_\infty - H \mathbb{F}_p \)-algebra.

Denote by \( \mathbb{E}^\text{nu} \) the non-unital \( \mathbb{E}_\infty \)-operad. In section 3, we examine the bar spectral sequence

\[ \tilde{E}_s^2 = \pi_s(\text{Bar}_*(\text{id}, \text{Poly}_R, \pi_*(A))) \Rightarrow \pi_{s+t}(|\text{Bar}_*(\text{id}, \mathbb{E}^\text{nu} \otimes H \mathbb{F}_p, A)|) = \text{TAQ}_{s+t}(A) \]

and the dual bar spectral sequence

\[ E_s^2 = \pi_s(\text{Bar}_*(\text{id}, \text{Poly}_R, \pi_*(A))^\vee) \Rightarrow \pi_{s+t}(|\text{Bar}_*(\text{id}, \mathbb{E}^\text{nu} \otimes H \mathbb{F}_p, A)|) = \text{TAQ}^{-s-t}(A) \]

for bounded above \( \mathbb{E}^\text{nu} - H \mathbb{F}_p \)-algebras \( A \) of finite type. Here \( \text{Poly}_R \) is the monad associated to the free functor that first takes the graded polynomial algebra on the free unstable module over the Dyer-Lashof algebra, then identifies \( x^{\otimes 2} \) with the bottom operation \( Q^{|x|/2} \) if \( p = 2 \) and \( x^{\otimes p} \) with the bottom operation \( Q^{|x|/2} \) if \( p > 2 \) and \( |x| \) is even, and finally imposes the Cartan formula.

When \( A \) is a direct sum of shifts of \( H \mathbb{F}_p \) considered as a trivial \( \mathbb{E}^\text{nu} - H \mathbb{F}_p \)-algebra, the \( E^2 \)-page of the dual bar spectral sequence parametrizes natural operations and their relations on the Andrée-Quillen cohomology

\[ \text{AQ}^*_{\text{Poly}_R}(M) := \pi_*(\text{Bar}_*(\text{id}, \text{Poly}_R, M)^\vee) \]

of algebras \( M \) over the monad \( \text{Poly}_R \), whereas the \( E^\infty \)-page is isomorphic to the homotopy groups of the free spectral partition Lie algebra on \( A^\vee \).

In section 4, we compute the (dual) bar spectral sequence when \( A \) is a direct sum of shifts of \( H \mathbb{F}_p \). The main idea is the following: first we find a suitable factorization of the indecomposables functor \( Q^\text{Poly}_R \), which allows us to replace the \( E^1 \)-page with a much smaller bi-simplicial object; then we compute the \( E^2 \)-page as the homology of the total complex using the machinery of classical Koszul duality developed by Priddy ([Pri70]) and Fresse ([Fre04]). This strategy is a modification of the method in [BHK19, section 4.4] and was employed in [Zh21, section 3.3] to compute the mod 2 André-Quillen homology of spectral Lie algebras. The additive unary operations on \( \text{AQ}^*_{\text{Poly}_R}(M) \) are parametrized by a ringoid \( (\mathcal{R}')^! \) that encodes certain unstable Ext groups over the Dyer-Lashof algebra \( \mathcal{R} \). The composition product is given by juxtaposition, which corresponds to the Yoneda product on (unstable) Ext groups.

**Definition 1.1.** (Definition 4.1 and 4.10) The ringoid \( (\mathcal{R}')^! \) is defined as follows:

1. For \( p = 2 \), let \( \mathcal{F} \) be the ringoid with objects \( \mathbb{Z}_{\leq 0} \times \mathbb{Z} \) whose morphisms are freely generated over \( \mathbb{F}_2 \) under juxtaposition by the following elements: for all \( i, j \) satisfying \( i > -j \) and \( s \leq 0 \), there is an element \( (Q')^s \in \mathcal{F}((s, j), (s - 1, j - i)) \) of weight 2.

The ringoid \( (\mathcal{R}')^! \) is the quotient of \( \mathcal{F} \) by the ideal generated by the Adem relations

\[ (Q'^a)^*(Q'^b)^* = \sum_{a+b-c>2c, c>-j} \binom{b-c-1}{a-2c-1}(Q'^{a+b-c})^*(Q'^c)^* \]

for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq 2b \), \( b > -j \) and \( a > b - j \) in \( \mathcal{F}((s, j), (s - 2, j - a - b)) \).
(2) For \( p > 2 \), let \( \mathcal{F} \) be the ringoid with objects \( \mathbb{Z}_{\geq 0} \times \mathbb{Z} \) and morphisms freely generated over \( \mathbb{F}_p \) under juxtaposition on the following elements: for \( 2i > -j \) and any \( s \leq 0 \) there are elements \((Q^i)^* \in \mathcal{F}((s, j), (s-1, j-(2(p-1)i))\) and \((\beta Q^i)^* \in \mathcal{F}((s, j), (s-1, j-(2(p-1)i)+1))\).

The ringoid \((\mathcal{R}')^1\) is the quotient of \( \mathcal{F} \) by the ideal generated by the Adem relations

\[
(Q^i)^*(Q^j)^* = \sum_{a+b-c=pc, 2c>-k} (-1)^{a-c} \left( \frac{(p-1)(b-c)-1}{a-pc} \right) (Q^{a+b-c})^*(Q^c)^*,
\]

in \( \mathcal{F}((s, j), (s-2, j-(2(p-1)a-2(p-1)b))) \) for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq pb, 2b > -j, 2a > 2(p-1)b-j \).

\[
(\beta Q^i)^*(Q^j)^* = \sum_{a+b-c=pc, 2c>-k} (-1)^{a-c} \left( \frac{(p-1)(b-c)-1}{a-pc} \right) (Q^{a+b-c})^*(\beta Q^c)^*
- \sum_{a+b-c=pc, 2c>-k} (-1)^{a-c} \left( \frac{(p-1)(b-c)-1}{a-pc} \right) (\beta Q^{a+b-c})^*(Q^c)^* = 0
\]

in \( \mathcal{F}((s, j), (s-2, j-(2(p-1)a-2(p-1)b+1))) \) for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq pb, 2b > -j, 2a > 2(p-1)b-j \), and

\[
(\beta^\varepsilon Q^i)^*(\beta Q^j)^* = \sum_{a+b-c=pc, 2c>-k} (-1)^{a-c} \left( \frac{(p-1)(b-c)-1}{a-pc} \right) (\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^*
\]

in \( \mathcal{F}((s, j), (s-2, j-(2(p-1)a-2(p-1)b+\varepsilon+1))) \) for \( \varepsilon \in \{0, 1\} \) and \( a, b \in \mathbb{Z} \) satisfying \( a < pb, 2b > -j, 2a > 2(p-1)b-j \).

**Proposition 1.2.** (Proposition 4.9 and 4.15) The André-Quillen cohomology \( AQ^*_{\text{Poly}}(M) \) of algebras \( M \) over the monad \( \text{Poly}_R \) admits a shifted Lie bracket that vanishes on additive unary operations. When \( p = 2 \), additive unary operations on \( AQ^*_{\text{Poly}}(M) \) are parametrized by the ringoid \((\mathcal{R}')^1\). There is also a nonadditive unary operation acting as the restriction on the shifted Lie bracket. When \( p > 2 \), (additive) unary operations on \( AQ^*_{\text{Poly}}(M) \) of algebras \( M \) are parametrized by the ringoid \((\mathcal{R}')^1\) together with self-brackets on even degree classes.

We deduce that the spectral sequences collapse on the \( E^2 \)-page with no extension problems by comparing with the bases in [BM19, Theorem 1.20].

Section 5 is devoted to determining the composition law and all relations among the operations on the homotopy groups of free spectral partition Lie algebras and reduced mod \( p \) TAQ cohomology. While the dimensions of operations on the homotopy groups of free spectral partition Lie algebras agree with the dimensions of operations on the André-Quillen cohomology \( AQ^*_{\text{Poly}}(-) \), the composition product of the former differs a priori from the composition product of the latter. Inspired by the work of Brantner regarding the structure of operations on the Lubin-Tate theory of spectral Lie algebras in [Br17, section 4], we encode additive unary operations on the homotopy groups of free spectral partition Lie algebras with a power ring. Roughly speaking, this power ring arises from equipping the ringoid \((\mathcal{R}')^1\) in Definition 1.1 with a twisted composition product.

**Definition 1.3.** (Definition 5.4) We define the power ring \( \mathcal{P} \) to be the collection of \( \mathbb{F}_p \)-modules \( \{\mathcal{P}_k[w]\}_{(j,k,w) \in \mathbb{Z}^2 \times \mathbb{Z}_{\geq 0}} \), where \( \mathcal{P}_k[w] := (\mathcal{R}')^1((0,k),(-w,j+w)) \) for \( w > 0 \), \( \mathcal{P}_k[0] := \mathbb{F}_p \{t_i\} \) for all \( i \), and \( \mathcal{P}_i[0] = \emptyset \) for all \( i \neq j \). The associative and unital composition product \( \mathcal{P}_j[v] \times \mathcal{P}_k[w] \rightarrow \mathcal{P}_{j+k}[v+w] \) is given by the sheared Yoneda product.
The homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_p$-modules. Equivalently, for any $\mathbb{E}_\infty^\text{un}-H\mathbb{F}_p^\ast$-algebra $A$, there is a shifted (restricted) Lie bracket

$$[-, -] : \pi_n(A) \otimes \pi_n(A) \to \pi_{n+1}(A).$$

Now we can state the main result of this paper, which records all natural operations and their relations on the homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_p$ and the reduced TAQ cohomology of $\mathbb{E}_\infty^\text{un}-H\mathbb{F}_p^\ast$-algebras.

**Theorem 1.4.** (Theorem 5.5 and 5.13)

1. The homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_2$, or the reduced TAQ cohomology of an $\mathbb{E}_\infty^\text{un}-H\mathbb{F}_2$-algebra, form a module over the power ring $\mathcal{P}$ of additive unary operations.

2. The weight 2 additive operations are given by the collection $R^i \in \mathcal{P}_j^{i-1}[1]$ for all $i, j \in \mathbb{Z}$ satisfying $i > -j + 1$, subject to the Adem relations

$$R^a R^b = \sum_{a+b-c \geq 2c, c > -j+1} \binom{b-c-1}{a-2c} R^{a+b-c} R^c$$

in $\mathcal{P}_j^{i-a-b}[2]$ for all $a, b \in \mathbb{Z}$ satisfying $-j < a < 2b$ and $b > -j + 1$.

3. There is a nonadditive unary operation $R^{-|x|+1}(x)$ for any homotopy class $x$ that serves as the restriction $x^{[2]}$ on $x$. The restriction on a sum of classes $x$ and $y$ in different degrees is given by

$$(x + y)^{[2]} = R^{-|x|+1}(x) + R^{-|y|+1}(y) + [x, y].$$

The bracket is compatible with the unary operations in the sense that $[y, \alpha(x)] = 0$ for any homotopy class $x, y$ and unary operation $\alpha$ of positive weight that is not an iteration of the restriction.
The operations $R^i$ and the shifted restricted Lie bracket generate all natural operations under the above relations. A basis for unary operations on a degree $j$ class is given by the collection of all monomials $R^i R^j \cdots R^k$ such that $i_j > -j$ and $i_m \geq 2i_{m+1}$ for $1 \leq m < l$.

**Theorem 1.5.** (Theorem 5.6 and 5.13)

1. The homotopy groups of a spectral partition Lie algebra over $H\mathbb{F}_p$, or the reduced TAQ cohomology of any $E_{\infty-}H\mathbb{F}_p$-algebra, form a module over the power ring $P$.

2. The weight $p$ unary operations are given by the collection $\beta^e R^i \in P_j^{(2(p-1)i-\varepsilon)}[1]$ for $\varepsilon = 0, 1$ and any $2i > -j$, subject to the Adem relations

$$
\beta^e R^a \beta^b = \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c+1} \left( \frac{(p-1)(b-c) - 1}{a - pc - 1} \right) \beta^e R^{a+b-c} \beta^c
$$

in $P_j^{(2(p-1)(a+b)-2)}[2]$ for all $a, b, c \in \mathbb{Z}$ satisfying $a \leq pb$, $2b > -j$, $2a > 2(p-1)b - j$,

$$
R^a \beta^b = \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c} \left( \frac{(p-1)(b-c)}{a - pc} \right) \beta^e p^{a+b-c} R^c
$$

- \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a - pc - 1} \right) R^{a+b-c} \beta^c
$$

in $P_j^{(2(p-1)(a+b)-2)}[2]$ for all $a, b, c \in \mathbb{Z}$ satisfying $a \leq pb$, $2b > -j$, $2a > 2(p-1)b + 1 - j$,

$$
\beta^e R^a \beta^b = \sum_{a+b-c \geq pc, 2c > -j} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a - pc} \right) \beta^e R^{a+b-c} R^c
$$

in $P_j^{(2(p-1)(a+b)-2)}[2]$ for all $a, b, c \in \mathbb{Z}$ satisfying $a < pb$, $2b > -j$, $2a > 2(p-1)b - j$, and $\varepsilon \in \{0, 1\}$.

3. For all odd $j$ and $x$ a homotopy class in degree $j$, the restriction $x^{[p]}$ is the bottom operation $R^{(-j+1)/2}(x)$ up to a unit $\lambda_j$. The restriction map on a sum of classes $x$ and $y$ in odd degrees $j \neq k$ is given by

$$(x+y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} s_i f(x, y),$$

where $s_i$ is the coefficient of $t^{i-1}$ in the formal expression $\ad((x+y)^{p-1})$. Furthermore, $[y, x^\varepsilon(x)] = 0$ for any homotopy class $x, y$ and $\alpha$ a unary operation of positive weight, unless $x$ is in odd degree and $\alpha$ an iteration of the restriction.

4. The operations $\beta^e R^i$ and the shifted restricted Lie bracket generate all natural operations under the above relations. A basis for unary operations on a degree $j$ class with $j$ odd is given by all monomials $\beta^e R^i \beta^e R^j \cdots \beta^e R^k$ such that $2i_j > -j$ and $i_m \geq p_i_{m+1} + \varepsilon_{m+1}$ for $1 \leq m < l$. If $j$ is even, a basis is given by $\beta^e R^i \beta^e R^j \cdots \beta^e R^k B^e$ such that $2i_j > -(1+\varepsilon)j - \varepsilon$ and $i_m \geq p_i_{m+1} + \varepsilon_{m+1}$ for $1 \leq m < l$.

The key step in the proof of Theorem 5.5 and 5.6 is checking that the composition product in $P$ is the law of composition for additive operations on $\pi_*(\Lie_{\mathbb{F}_p}(A))$. The proof relies on a general result of Brantner ([Br17, Theorem 3.5.1 and 4.3.2]), which explains the compatibility of the structure of additive operations on the $E^2$-page of the (dual) bar spectral sequence with that on the homotopy groups of the monadic bar construction the $E^\infty$-page assemble to.

When the degree of a class gets arbitrarily large, we observe in Corollary 5.7 that the additive operations on that class form an algebra Koszul dual to the Dyer-Lashof algebra, which is isomorphic
to the extended Steenrod algebra but with \( S_0 \equiv 0 \). Indeed, the Dyer-Lashof algebra is isomorphic to the extended Steenrod algebra for restricted Lie algebras, cf. \([KL83]\). For filtration reasons, the restriction on the shifted Lie bracket is not visible on the \( E^2 \)-page of the dual bar spectral sequence, i.e. the André-Quillen cohomology associated to the monad \( \text{Poly}_R \), when \( p > 2 \). Instead, we use a homotopy fixed points spectral sequence to detect the restriction map in Lemma 5.12. Thus we recover and clarify the unpublished computations of Kriz and Basterra-Mandell on connective objects via a different method.

**Remark 1.6.** The shifted Lie brackets on the mod \( p \) homology and Lubin-Tate theory of spectral Lie algebras always vanish on unary operations of positive weights, as was shown by Antolín-Camarena ([AC20]), Brantner ([Br17]), and Kjaer ([Kja18]). The slight difference from our situation comes from the fact that all unary operations on spectral Lie algebras are additive, whereas the restriction map for the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras is a nonadditive unary operation. See Remark 5.14.

**Remark 1.7.** To identify the unit \( \lambda_j \) by which the bottom operation and the restriction on an odd class \( x \) in degree \( j \) differ when \( p > 2 \), we expect that a chain-level understanding of the operations involved is necessary. We hope to construct these operations on the explicit chain model of spectral partition Lie algebra obtained by Brantner, Campos, and Nuiten in [BCN21, Definition 4.43].

**Remark 1.8.** In [Kja18], Kjaer computed the dimensions of the odd primary homology groups of free spectral Lie algebras and conjectured that the relations among the unary operations are given by the mixed Adem relations in [CLM76, II.3], cf. [Kja18, Conjecture 5.3]. One approach to proving these relations is to imitate Behrens’ computation of the quadratic relations in [Beh12, Section 1.4-1.5] when \( p = 2 \). The difficulty lies in working out a formula for the induced map of the transfer \( \Sigma_2 \to \Sigma_p \wr \Sigma_p \) on mod \( p \) homology groups of the extended power constructions. This formula was obtained by Kuhn in [Kuh85] when \( p = 2 \), but remains elusive when \( p > 2 \). The method in this paper suggests an alternative approach to resolving the question. Namely, one could construct a sifted-colimit-preserving monad \( C \) that parametrizes divided power \( \mathbb{E}_\infty^\nu-HF_p \)-algebras following the axiomatic argument in [BM19, Section 4], determine the structure of natural operations on the homotopy groups of algebras over \( C \), and feed it into the dual bar spectral sequence converging to \( \pi_*(-\text{Bar}_\nu(\text{id},C,A)) \) with \( A \) any trivial algebra over \( C \) of finite type.

Therefore the target category for the homotopy groups of a spectral partition Lie algebra or the reduce mod \( p \) TAQ cohomology is the category of \( P \)-sLie\(^p\)-algebras (Definition 5.15), i.e., modules \( L \) over the power ring \( P \) together with a shifted restricted Lie bracket satisfying the conditions in part (3) of Theorem 1.4 and 1.5. Denote by \( \text{Free}^{\text{sLie}\,^p} \) the free \( P \)-sLie\(^p\)-algebra functor.

As an immediate application, we obtain a computation of natural operations and relations on the mod \( p \) TAQ cohomology \( \text{TAQ}^\nu(R,S;HF_p) \) of \( \mathbb{E}_\infty-S \)-algebras \( R \), which is based on conversations with Tyler Lawson.

Since the functor \( \text{TAQ}^\nu(-,S;HF_p) \) has representing object the trivial square-zero extension \( S \oplus \Sigma^iHF_p \) for all \( i \), operations and relations are again parametrized by the mod \( p \) TAQ cohomology on the trivial square-zero extensions \( S \oplus \Sigma^iHF_p \). Using the base change formula

\[
\text{TAQ}(-,S;HF_p) \otimes HF_p \simeq \text{TAQ}(- \otimes HF_p,HF_p;HF_p),
\]

we deduce immediately from Theorem 1.4 and 1.5 the structure of natural operations on the mod \( p \) TAQ cohomology \( \mathbb{E}_\infty-S \)-algebras.
Theorem 1.9. (Corollary 6.1, Proposition 6.2) For any tuple \((i_1, \ldots, i_k)\) of integers, the \(k\)-ary cohomology operations
\[
\prod_{i=1}^{k} \operatorname{TAQ}^i(-, S; H\mathbb{F}_p) \rightarrow \operatorname{TAQ}^m(-, S; H\mathbb{F}_p).
\]
are parametrized by the homological degree \(-m\) part of \(\text{Free}^{\Delta \operatorname{Lie}\mathcal{P}}(\Sigma^{-i_1} A \oplus \cdots \oplus \Sigma^{-i_k} A)\), where \(A\) is the Steenrod algebra graded homologically. All operations vanish on the unit except for scalar multiplication. The Steenrod operations commute with the bracket via the Cartan formula and the \(\mathbb{F}_p\)-linear \(\operatorname{TAQ}\) cohomology operations via the Nishida relations on cohomology of the second extended power:

1. For \(p = 2\) we have
\[
\operatorname{Sq}^a[x, y] = \sum_i [\operatorname{Sq}^i(x), \operatorname{Sq}^{a-i}(y)],
\]
\[
\operatorname{Sq}^a R^{-|x|+1}(x) = \sum \left(\frac{|x|-c}{a-2c}\right) R^{a+|x|+1-c} \operatorname{Sq}^c(x) + \sum_{l<k,l+k=a} [\operatorname{Sq}^l(x), \operatorname{Sq}^k(x)],
\]
\[
\operatorname{Sq}^a R^b(x) = \sum \left(\frac{b-1-c}{a-2c}\right) R^{a+b-c} \operatorname{Sq}^c(x), \ b > -|x| + 1.
\]

2. For \(p > 2\) we have
\[
\operatorname{P}^a[x, y] = \sum_i [\operatorname{P}^i(x), \operatorname{P}^{a-i}(y)], \ \beta \operatorname{P}^a[x, y] = \sum_i ([\beta \operatorname{P}^i(x), \operatorname{P}^{a-i}(y)] + [\operatorname{P}^i(x), \beta \operatorname{P}^{a-i}(y)]).
\]

For any class \(x\) and all \(2j > -|x| + 1\), the Nishida relations are
\[
\operatorname{P}^a \beta \operatorname{R}^i = (-1)^{n-i} \sum_i \left(\frac{(j-i)(p-1)}{n-pi}\right) \beta \operatorname{R}^{a+j-i} \operatorname{P}^i + (-1)^{n-i} \sum_i \left(\frac{(j-i)(p-1)-1}{n-pi-1}\right) \operatorname{R}^{a+j-i} \beta \operatorname{P}^i,
\]
\[
\operatorname{P}^a \operatorname{R}^i = (-1)^{n-i} \sum_i \left(\frac{(j-i)(p-1)-1}{n-pi}\right) \operatorname{R}^{a+j-i} \operatorname{P}^i,
\]
as well as
\[
\operatorname{P}^a \operatorname{R}^i(x) = (-1)^{n-i} \sum_i \left(\frac{(j-i)(p-1)-1}{n-pi}\right) \operatorname{R}^{a+j-i} \operatorname{P}^i(x)
\]
\[
+ \frac{1}{\lambda_{|x|}} \sum_{\sigma(1) \leq \cdots \leq \sigma(p) \leq \cdots \leq \sigma(n)} [[\cdots [[\operatorname{P}^{\sigma(1)}(x), \operatorname{P}^{\sigma(2)}(x)], \operatorname{P}^{\sigma(3)}(x)], \ldots], \operatorname{P}^{\sigma(p)}(x)]
\]
when the degree of \(x\) is odd and \(2j = -|x| + 1\), where the bracket term sums over all nondecreasing sequences \(l = (0 \leq i_1 \leq i_2 \leq \cdots \leq i_p)\) with \(i_1 + i_2 + \cdots + i_p = n\), and \(\lambda_{|x|}\) is a fixed unit given in Theorem 1.5. (3).

Remark 1.10. In parallel to the theory of spectral partition Lie algebras, Brantner and Mathew developed a derived version called partition Lie algebras, which are algebras over a certain monad \(\text{Lie}_{\mathbb{F}_p}^\Delta\) on the derived category of chain complexes over \(\mathbb{F}_p\). They showed that cocommutative, finite type partition Lie algebras over \(\mathbb{F}_p\) serve as the Koszul dual to complete local Noetherian simplicial commutative \(\mathbb{F}_p\)-algebras ([BM19, Theorem 1.11 and 1.13]). They then computed the ranks of homotopy groups of free partition Lie algebras in [BM19, Theorem 1.16]. While partition Lie algebras are more delicate than their spectral counterparts studied in this paper, we are hopeful that a similar strategy can be employed to obtain relations among the operations on the homotopy groups of partition Lie algebras. This will require as input the structure of natural operations on the homotopy groups of simplicial and cosimplicial commutative \(\mathbb{F}_p\)-algebras. Note that for \(p = 2\), Goerss
identified all operations and their relations on the homotopy groups of coconnective partition Lie algebras in [Goe90, Theorem H] via a Quillen spectral sequence and a Hilton-Milnor type argument, with input the structure of unary operations on the homotopy groups simplicial commutative \( \mathbb{F}_2 \)-algebras by the works of Cartan ([Car58]), Bousfield ([Bou68]), Dwyer ([Dwy80]). For \( p > 2 \), a basis of unary operations was computed by Nakaoka in the simplicial case ([Nak57, Nak58]), and the quadratic relations among the operations were determined by Bousfield in [Bou68, Theorem 8.9]. In the cosimplicial case, Priddy determined the unary operations and their relations in [Pri73] at all primes.

1.2. Conventions. The grading convention is homological unless for cohomology groups or otherwise stated. We assume that every object is graded and weighted whenever it makes sense. For instance, \( \text{Mod}_{\mathbb{F}_p} \) stands for the ordinary category of weighted graded \( \mathbb{F}_p \)-modules, and \( \text{Mod}_{\mathbb{F}^p \mathbb{Z}} \) the \((\infty,\infty)\)-category of weighted graded modules over \( H\mathbb{F}_p \). A weighted graded \( \mathbb{F}_2 \)-module (resp \( H\mathbb{F}^p \)-module) \( M \) is an \( \mathbb{N} \)-indexed collection of \( \mathbb{Z} \)-graded \( \mathbb{F}_p \)-modules \( \{ M(w) \}_{w \in \mathbb{N}} \). The weight grading of an element \( x \in M(w) \) is \( w \). Morphisms are weight preserving morphisms of graded \( \mathbb{F}_p \)-modules (\( H\mathbb{F}^p \)-modules). The Day convolution \( \otimes \) makes \( \text{Mod}_{\mathbb{F}_p} \) (\( \text{Mod}_{\mathbb{F}^p \mathbb{Z}} \)) a symmetric monoidal category. The Koszul sign rule \( x \otimes y = (-1)^{|x||y|} y \otimes x \) for the symmetric monoidal product \( \otimes \) depends only on the internal grading and not the weight grading.

Similarly, a shifted Lie algebra \( L \) over \( \mathbb{F}_p \) is a weighted graded \( \mathbb{F}_p \)-module equipped with a shifted Lie bracket \( [-,-] : L_n \otimes L_m \to L_{n+m-1} \) that adds weights, as well as satisfying graded commutativity \( [x,y] = (-1)^{|x||y|} [y,x] \) and the graded Jacobi identity
\[
(-1)^{|x||z|} [x,[y,z]] + (-1)^{|y||z|} [y,[z,x]] + (-1)^{|z||x|} [z,[x,y]] = 0.
\]
If \( p = 2 \), then we further require that \( [x,x] = 0 \) for all \( x \). If \( p = 3 \), then we further require that \( [x,[x,x]] = 0 \) for all \( x \).

We use \( \pi_n(-) \) to denote the functor taking the \( n \)th homotopy group of a spectrum, \( H\mathbb{F}_p \)-module, or simplicial \( \mathbb{F}_p \)-module, as well as the functor taking the \( n \)th homology group of a chain complex over \( \mathbb{F}_p \).

We use \( \mathbb{E}_\infty \) and \( \mathbb{E}^\text{nu}_\infty \) to denote respectively the unital and non-unital commutative operad in the category of spectra, i.e. the category of \( S \)-algebras, and \( \mathbb{P} = \mathbb{E}^\text{nu}_\infty \otimes H\mathbb{F}_p \) the nonunital commutative operad in \( \text{Mod}_{H\mathbb{F}_p} \). Often we abuse notations and denote by \( O \) the monad associated to the free \( O \)-algebra functor when \( O = \mathbb{E}_\infty, \mathbb{E}^\text{nu}_\infty, \mathbb{P} \).

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2. Preliminaries

2.1. Spectral Lie operad. We begin with a brief review of the spectral Lie operad \( \partial_* (\text{Id}) \). Ching ([Chi05]) and Salvatore showed that the Goowillie derivatives \( \{ \partial_n (\text{Id}) \} \) of the identity functor \( \text{Id} : \text{Top}_* \to \text{Top}_* \) form an operad \( \partial_* (\text{Id}) \) in Spectra. This operad is Koszul dual to the nonunital commutative operad \( \mathbb{E}^\text{nu}_\infty \) via the operadic bar construction
\[
\partial_* (\text{Id}) \simeq \mathbb{D} \text{Bar}(1, \mathbb{E}^\text{nu}_\infty, 1).
\]
For a description of the operadic bar construction, see [Chi05] for a topological model using trees and [Br17, Appendix D] for an \( \infty \)-categorical construction along with a comparison with the topological model.
The \( n \)-th derivative \( \partial_n(\text{Id}) \) admits an explicit description due to Arone and Mahowald ([AM99]), following the work of Johnson ([Joh95]). Let \( \mathcal{P}_n \) be the poset of partitions of the set \( \underline{n} = \{1, 2, \ldots, n\} \) ordered by refinements, equipped with a \( \Sigma_n \)-action induced from that on \( n \). Denote by \( \hat{0} \) the discrete partition and \( \hat{1} \) the partition \( \{n\} \). Set \( \Pi_n = \mathcal{P}_n - \{\hat{0}, \hat{1}\} \). Regarding a poset \( \mathcal{P} \) as a category, we obtain via the nerve construction a simplicial set \( N_\bullet(\mathcal{P}) \). The partition complex \( \Sigma|\Pi_n| \circ \), the reduced-unreduced suspension of the realization of \( \Pi_n \), is modeled by the simplicial set 

\[
N_\bullet(\mathcal{P}_n)/(N_\bullet(\mathcal{P}_n - \hat{0}) \cup N_\bullet(\mathcal{P}_n - \hat{1}))
\]

for \( n \geq 2 \) and the simplicial 0-circle \( S^0 \) for \( n = 1 \). Then there is an equivalence 

\[
\partial_n(\text{Id}) \simeq \mathbb{D}(\Sigma|\Pi_n| \circ)
\]

of spectra with \( \Sigma_n \)-action, where \( \mathbb{D} \) denotes the Spanier-Whitehead dual of a spectrum.

2.2. **Spectral partition Lie algebras.** Motivated by the theory of classical operadic Koszul duality ([GK94]), the natural next step is to formulate a Koszul duality theorem between suitable categories of algebras over the Koszul dual pair \( \mathbb{P} \)-modules of finite type, i.e. \( \mathbb{H}_p \)-modules with degree-wise finite homotopy groups. Denote by \( \text{Mod}_{\mathbb{H}_p}^{\text{ft}} \subseteq \text{Mod}_{\mathbb{H}_p} \) the subcategory spanned by coconnective objects. Let \( \mathbb{P} \) be the nonunital commutative operad in \( \text{Mod}_{\mathbb{H}_p} \).

There is an adjunction 

\[
\text{Alg}_{\mathbb{P}}(\text{Mod}_{\mathbb{H}_p}^{\text{ft}}) \xrightarrow{\cot} \text{Mod}_{\mathbb{H}_p}^{\text{ft}},
\]

where the functor \( \text{sqz} \) sends an object \( M \) to the \( \mathbb{P} \)-algebra \( M \) as a trivial square-zero extension. The restriction of this adjunction to the subcategory \( \text{Mod}_{\mathbb{H}_p}^{\text{ft}, \leq 0} \) defines a sifted-colimit-preserving monad \( (M \mapsto \cot(\text{sqz}(M)^\vee)^\vee) \) on \( \text{Mod}_{\mathbb{H}_p}^{\text{ft}, \leq 0} \).

**Definition 2.1.** [BM19, Definition 5.32] The spectral partition Lie monad \( \text{Lie}_{\mathbb{P}^{\text{ft}}, \mathbb{E}_n}^{\mathbb{H}_p} \) is the unique sifted-colimit-preserving monad 

\[
\text{Lie}_{\mathbb{P}^{\text{ft}}, \mathbb{E}_n}^{\mathbb{H}_p} : \text{Mod}_{\mathbb{H}_p}^{\text{ft}} \to \text{Mod}_{\mathbb{H}_p}^{\text{ft}}
\]

extending the monad \( (M \mapsto \cot(\text{sqz}(M)^\vee)^\vee) \) on \( \text{Mod}_{\mathbb{H}_p}^{\text{ft}, \leq 0} \).

Algebras over the monad \( \text{Lie}_{\mathbb{P}^{\text{ft}}, \mathbb{E}_n}^{\mathbb{H}_p} \) are called spectral partition Lie algebras. The free spectral partition Lie algebras on bounded above objects admit an explicit description.

**Proposition 2.2.** [BM19, Proposition 5.35] For \( V \) a bounded above \( \mathbb{H}_p \)-module,

\[
\text{Lie}_{\mathbb{P}^{\text{ft}}, \mathbb{E}_n}^{\mathbb{H}_p}(V) \simeq \mathbb{D}|\text{Bar}_\bullet(\text{id}, \mathbb{P}, V^\vee)| \simeq \bigoplus_{n \geq 1} \left( (\partial_n(\text{id}) \otimes \mathbb{H}_p) \otimes (V)^{\otimes n} \right)_{h\Sigma_n}.
\]

The above formula makes it clear that spectral partition Lie algebras are *not* algebras over the spectral Lie operad, as the structural map of an algebra \( L \) over the spectral Lie operad in \( \text{Mod}_{\mathbb{H}_p} \) is given by

\[
\text{Free}_{\mathbb{P}}(\partial_n(\text{id}) \otimes \mathbb{H}_p)(L) \simeq \bigoplus_{n \geq 1} \left( (\partial_n(\text{id}) \otimes \mathbb{H}_p) \otimes (L)^{\otimes n} \right)_{h\Sigma_n} \to L.
\]
Heuristically, spectral partition Lie algebras are the dual of divided power coalgebras over the co-operad $\text{Bar}(1,\mathbb{P},1)$, and hence candidates for the Koszul dual of $\mathbb{F}_p\otimes\mathbb{F}_p$-algebras. To formulate the precise Koszul duality statement, we need to introduce one more technical condition.

**Definition 2.3.** An $\mathbb{E}_\infty\mathbb{F}_p$-algebra $A$ is complete local Noetherian if

1. $\pi_0(A)$ is a complete local Noetherian ring;
2. $A$ is connective and $\pi_n(A)$ is a finitely-generated module over $\pi_0(A)$ for all $n \geq 0$.

Now we can state a restricted version of of the main results by Brantner and Mathew.

**Theorem 2.4.** [BM19, Theorem 1.19] There is an equivalence of $\infty$-categories between complete local Noetherian $\mathbb{E}_\infty\mathbb{F}_p$-algebras and the $\infty$-category of coconnective spectral partition Lie algebras of finite type.

### 2.3. Relation to TAQ cohomology

Spectral partition Lie algebras are closely related to the $\mathbb{F}_p$-linear TAQ spectrum. Inspired by the unpublished work of Kriz ([Kri93]), Basterra constructed in [Bas99] the topological André-Quillen homology object $\text{TAQ}(R,A;B)$ for a fixed map of $\mathbb{E}_\infty$-algebras $A \to B$ and any object $R$ in the category of $\mathbb{E}_\infty$-algebras between $A$ and $B$. For any object $R$ in the category of $\mathbb{E}_\infty\mathbb{S}$-algebras with a map to $\mathbb{H}\mathbb{F}_p$, we obtain the TAQ spectrum

$$\text{TAQ}(R,\mathbb{S};\mathbb{H}\mathbb{F}_p) \simeq |\text{Bar}_{\bullet}(\mathbb{H}\mathbb{F}_p \otimes (-),\mathbb{E}_\infty,R)|,$$

cf. [Bas99, section 5] and [Law20, Proposition 1.8.9]. There is a base change formula to $\mathbb{F}_p$-linear TAQ spectrum

$$\text{TAQ}(R,\mathbb{S};\mathbb{H}\mathbb{F}_p) \otimes_{\mathbb{S}} \mathbb{H}\mathbb{F}_p \simeq \text{TAQ}(R \otimes_{\mathbb{S}} \mathbb{H}\mathbb{F}_p,\mathbb{H}\mathbb{F}_p;\mathbb{H}\mathbb{F}_p)$$

for $R$ any $\mathbb{E}_\infty\mathbb{S}$-algebra. The $n$th $\mathbb{F}_p$-linear TAQ cohomology is defined to be

$$\text{TAQ}^n(R,\mathbb{H}\mathbb{F}_p,\mathbb{H}\mathbb{F}_p) = [\Sigma^{-n}\text{TAQ}(R,\mathbb{H}\mathbb{F}_p;\mathbb{H}\mathbb{F}_p),\mathbb{H}\mathbb{F}_p]_{\text{Mod}\mathbb{H}\mathbb{F}_p}$$

for $R$ any $\mathbb{E}_\infty\mathbb{F}_p$-algebra.

In this paper we work with the nonunital $\mathbb{F}_p$-linear version

$$\overline{\text{TAQ}}(A) \simeq |\text{Bar}_{\bullet}(\text{id},\mathbb{P},A)|,$$

where $\mathbb{P}$ is the nonunital $\mathbb{E}_\infty$-operad in $\text{Mod}\mathbb{H}\mathbb{F}_p$ and $A$ a $\mathbb{P}$-algebra. We call this the reduced mod $p$ TAQ spectrum of $A$, since

$$\overline{\text{TAQ}}(A) \oplus \mathbb{H}\mathbb{F}_p \simeq \text{TAQ}(\mathbb{H}\mathbb{F}_p \oplus A,\mathbb{H}\mathbb{F}_p;\mathbb{H}\mathbb{F}_p).$$

Thus the reduced mod $p$ TAQ cohomology group $\overline{\text{TAQ}}^n(A) := [\Sigma^{-n}\text{TAQ}(A),\mathbb{H}\mathbb{F}_p]_{\text{Mod}\mathbb{H}\mathbb{F}_p}$ differ from the $\mathbb{F}_p$-linear TAQ cohomology group $\text{TAQ}^n(A \oplus \mathbb{H}\mathbb{F}_p,\mathbb{H}\mathbb{F}_p;\mathbb{H}\mathbb{F}_p)$ only when $n = 0$ by a copy of $\mathbb{F}_p$.

By Proposition 2.2, when $A$ is a bounded above $\mathbb{H}\mathbb{F}_p$-module of finite type considered as a trivial $\mathbb{P}$-algebra, there is an equivalence

$$\overline{\text{TAQ}}^n(A^\vee) \cong \pi_{-n}((\mathbb{P}|\text{Bar}_{\bullet}(\text{id},\mathbb{P},A^\vee))) \cong \pi_{-n}(\text{Lie}_{\mathbb{F}_p,\mathbb{E}_\infty}(A)).$$

Going forward we will often omit $\mathbb{F}_p$-linear and mod $p$ when there is no ambiguity regarding which version of TAQ cohomology is concerned.
2.4. Operations. The goal of this paper is to understand natural operations and their relations on the homotopy groups of spectral partition Lie algebras and mod $p$ TAQ cohomology. First we record a few general remarks about operations on algebras over a monad, adapted from Lawson’s excellent survey ([Law20, section 1.4]) on the theory of operations for algebras over operads.

Given a monad $T$ on $\text{Mod}_{HF_p}$, we define an operation on $T$-algebras to be a natural transformation $\pi_m(-) \to \pi_n(-)$ of functors $\text{hAlg}_T(\text{Mod}_{HF_p}) \to \text{Sets}$ for some $m,n$. Here $\text{hAlg}_T(\text{Mod}_{HF_p})$ is the homotopy category of $T$-algebras over $\text{Mod}_{HF_p}$. Let $\text{Op}(m;n)$ be the set of operations for fixed $m,n$. It follows from the universal property of free algebras that for any $T$-algebra $A$,

$$\pi_m(A) \cong \text{Map}_{\text{Alg}_T(\text{Mod}_{HF_p})}(\text{Free}^T(\Sigma^m HF_p), A).$$

Hence $\text{Free}^T(\Sigma^m HF_p)$ is the representing object for the functor $\pi_m(-)$ on $\text{hAlg}_T(\text{Mod}_{HF_p})$.

By the Yoneda Lemma, the set of operations $\text{Op}(m;n)$, or equivalently natural transformations $\pi_m(-) \to \pi_n(-)$ in $\text{hAlg}_T(\text{Mod}_{HF_p})$, is isomorphic to $\pi_m(\text{Free}^T(\Sigma^m HF_p))$. Explicitly, given an operation $\alpha \in \pi_n(\text{Free}^T(\Sigma^m HF_p))$ and a class $x \in \pi_m(A)$ with $A$ a $T$-algebra, we obtain a class $\alpha(x)$ in $\pi_n(A)$ via the pullback

$$\pi_m(A) \cong \text{Map}_{\text{Alg}_T(\text{Mod}_{HF_p})}(\text{Free}^T(\Sigma^m HF_p), A) \xrightarrow{\alpha^*} \text{Map}_{\text{Alg}_T(\text{Mod}_{HF_p})}(\text{Free}^T(\Sigma^m HF_p), A) \cong \pi_n(A).$$

Therefore, to understand the unary operations on $T$-algebras and their relations, we need to first compute $\pi_n(\text{Free}^T(\Sigma^m HF_p))$ as an algebra for all $m$. Then we need to understand the composition product on unary operations

$$\pi_n(\text{Free}^T(\Sigma^m HF_p)) \times \pi_m(\text{Free}^T(\Sigma^i HF_p)) \to \pi_n(\text{Free}^T(\Sigma^i HF_p))$$

for all $l,m,n$, which corresponds to composing two natural transformations $\pi_l(-) \to \pi_m(-)$ and $\pi_m(-) \to \pi_n(-)$ of functors on $\text{hAlg}_T(\text{Mod}_{HF_p})$. In general, natural $k$-ary operations $\prod_{i=1}^k \pi_i(-) \to \pi_n(-)$ are parametrized by the homotopy groups

$$\pi_n(\text{Free}^T(\Sigma^i HF_p \oplus \cdots \oplus \Sigma^k HF_p))$$

for all $k$-tuples $(i_1, \ldots, i_k)$.

Here we specialize to $T = \text{Lie}^{\pi}_{HF_p, \mathbb{E}_n}$. The decomposition of the free algebra over $\text{Lie}^{\pi}_{HF_p, \mathbb{E}_n}$ into homogeneous pieces in Proposition 2.2 allows us to impose a weight grading on the operations on the homotopy groups of $\text{Lie}^{\pi}_{HF_p, \mathbb{E}_n}$-algebras in the usual sense.

On the other hand, the mod $p$ TAQ cohomology functor $\text{TAQ}(-, HF_p; HF_p)$ on $\text{Mod}_{HF_p}$ has as representing objects the trivial square-zero extensions. ([Law20, section 1.8]) Therefore, for any $m$ and tuple $(i_1, \ldots, i_k)$, the group of cohomology operations

$$\prod_{i=1}^k \text{TAQ}^i(-, HF_p; HF_p) \to \text{TAQ}^m(-, HF_p; HF_p)$$

is the given by $\text{TAQ}^m(HF_p \oplus \Sigma^i HF_p \oplus \cdots \Sigma^k HF_p, HF_p; HF_p)$.

Note that all operations vanish on the unit except for scalar multiplication. Since

$$\text{TAQ}^i(A^\vee) \cong \pi_{-n}(\text{Bar}_*(\text{id}_A; A^\vee))$$

for all $n$ when $A$ is a bounded above $HF_p$-module of finite type considered as a trivial $\mathbb{P}$-algebra, natural operations and their relations on the reduced mod $p$ TAQ cohomology, or equivalently, the mod $p$ TAQ cohomology away from the unit, agree with those on the homotopy group of spectral partition Lie algebras up to a change of grading conventions.
Brantner and Mathew obtained bases of the homotopy groups of free spectral partition Lie algebras on $\Sigma^k H\mathbb{F}_p$ via an isotropy spectral sequence as in [ADL13, Example 1.3]. Then they propagated the result to any direct sum of shifts of $H\mathbb{F}_p$ using a Hilton-Milnor-type decomposition of the partition complex and an EHP sequence developed in [AB21].

**Definition 2.5.** We say a word $w$ in letters $\{x_1, \ldots, x_k\}$ is a Lyndon word if it is smaller than any of its cyclic rotations in the lexicographic order with $x_1 < \cdots < x_k$. Write $B(n_1, \ldots, n_k)$ for the set of Lyndon words in which the letter $x_i$ appears precisely $n_i$ times.

Note that the collection of all Lyndon words in letters $x_1, \ldots, x_k$ produces a basis for the free *totally-isotropic* Lie algebra over $\mathbb{F}_p$ on $k$ generators, where totally-isotropic means there are no non-vanishing self-brackets.

**Theorem 2.6.** [BM19, Theorem 1.20] The $\mathbb{F}_p$-vector space $\pi_s(\text{Lie}_{\mathbb{F}_p,\Sigma}^n(\Sigma^i H\mathbb{F}_p \oplus \cdots \oplus \Sigma^k H\mathbb{F}_p))$ has a basis indexed by sequences $(i_1, \ldots, i_k, e, w)$. Here $w \in B(n_1, \ldots, n_m)$ is a Lyndon word. We have $e \in \{0, 1\}$, where $t = 1$ if $p$ is odd and $\deg(w) := \sum (l_i - 1)n_i + 1$ is even. Otherwise, $t = 0$. The integers $i_1, \ldots, i_k$ satisfy:

1. Each $i_j$ is congruent to 0 or 1 modulo $2(p - 1)$.
2. For all $1 \leq j < k$, we have $i_j < pi_{j+1}$.
3. We have $i_k \leq (p - 1)(1 + e)\deg(w) - t$. The homological degree of $(i_1, \ldots, i_k, e, w)$ is $((1 + e)\deg(w) - e) + i_1 + \cdots + i_k - k$.

Nonetheless, their method does not yield explicit descriptions of the nature of the operations, nor are the composition products or the relations among the operations clarified. In the rest of the paper, we use a dual bar spectral sequence to compute the relations among the unary operations.

By comparing with the bases in the above theorem, we show that the spectral sequences of interest degenerate. Then we define a composition product among unary operations and construct a shifted Lie bracket with restrictions on the homotopy group of spectral partition Lie algebras. Finally we deduce that these are all the natural operations and construct the target category.

### 3. A BAR SPECTRAL SEQUENCE

This section serves as a preliminary examination of the bar spectral sequence for a $\mathbb{P}$-algebra $A$, where $\mathbb{P}$ is the $\mathbb{E}^\infty_\omega$-operad in $\text{Mod}_{H\mathbb{F}_p}$, obtained by the skeletal filtration of the geometric realization of the bar construction

$$E^2_{s,t} = \pi_s(\pi_t(\text{Bar}_\bullet(\text{id}, \mathbb{P}, A))) \Rightarrow \pi_{s+t}(\pi_t(\text{Bar}_\bullet(\text{id}, \mathbb{P}, A))) \equiv \text{TAQ}_{s+t}(A).$$

Similarly, there is a dual bar spectral sequence

$$E^2_{s,t} = \pi_s(\pi_t(\text{Bar}_\circ(\text{id}, \mathbb{P}, A)^\vee)) \Rightarrow \pi_{s+t}(\pi_t(\text{Bar}_\circ(\text{id}, \mathbb{P}, A)))$$

When $A = \Sigma^j H\mathbb{F}_p$ is a trivial $\mathbb{P}$-algebra, the $E^\infty$-page records unary operations on a degree $j$ class in the homotopy group of any spectral partition Lie algebra and those on a degree $-j$ cohomology class in the reduced mod $p$ TAQ cohomology.

#### 3.1. The Dyer-Lashof algebra

Dyer-Lashof operations are natural unary operations on the mod $p$ homology of infinite loop spaces and $\mathbb{E}_\infty$-algebras in Spectra. These operations and their relations were computed by Araki-Kudo ([KA56]), Dyer-Lashof ([DL62]), Cohen-Lada-May ([CLM76]), and Bruner-May- McClure-Steinberger ([BMMS86]). Denote by $\mathcal{R}$ the non-unital mod $p$ Dyer-Lashof algebra.
Proposition 3.1. [CLM76, I.1], [BMMS86, III.1] At \( p = 2 \), the Dyer-Lashof algebra \( \mathcal{R} \) is generated by operations \( Q^i \) in degree \( i \) and weight 2 subject to the Adem relations

\[
Q^r Q^s = \sum_{r+s-i \leq 2i} (\begin{pmatrix} i-s-1 \\ 2i-r \end{pmatrix}) Q^{r+s-i} Q^i
\]

for \( r > 2s \).

For \( p \) an odd prime, the mod \( p \) Dyer-Lashof algebra is generated by operations \( \beta^s Q^i \) in degree \( 2(p-1)i - \varepsilon \) and weight \( p \) for \( \varepsilon \in \{0,1\} \) and all \( i \), subject to the Adem relations

\[
Q^r Q^s = \sum_{r+s-i \leq pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i,
\]

\[
\beta Q^r Q^s = \sum_{r+s-i \leq pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} \beta Q^{r+s-i} Q^i,
\]

for \( r > ps \) and

\[
Q^r \beta Q^s = \sum_{r+s-i < pi} (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i - \sum_{r+s-i \leq pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i,
\]

\[
\beta Q^r \beta Q^s = -\sum_{r+s-i < pi} (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} \beta Q^{r+s-i} \beta Q^i
\]

for \( r \geq ps \).

We say that a module \( M \) over the Dyer-Lashof algebra \( \mathcal{R} \) is unstable (or allowable) if the following conditions hold:

1. When \( p = 2 \), for any nonempty sequence of operation \( Q^j = Q^{i_1} \cdots Q^{i_k} \) and \( x \in M \) of degree \( j \), if \( i_1 - i_{l+1} - \cdots - i_k < j \) for some \( 1 \leq l \leq k \) then \( Q^j(x) = 0 \).

2. When \( p > 2 \), for any \( x \in M \) of degree \( j \) and any nonempty sequence of operation \( \alpha = \beta^{\varepsilon_1} Q^{i_1} \cdots \beta^{\varepsilon_k} Q^{i_k} \), if \( 2i_m - \varepsilon_m < j + 2(p-1)i_{m+1} + \cdots + 2(p-1)i_k - \varepsilon_1 - \cdots - \varepsilon_k + x \) for some \( 1 \leq m \leq k \) then \( \alpha(x) = 0 \).

Denote by \( \text{Mod}_{\mathcal{R}} \) the category of unstable \( \mathcal{R} \)-modules.

Definition 3.2. A Poly\( \mathcal{R} \)-algebra \( M \) is a (graded weighted) polynomial algebra over \( \mathbb{F}_p \) with an unstable \( \mathcal{R} \)-module structure that is compatible with the commutative product \( \otimes \) in the sense that

1. The Cartan formula is satisfied: \( Q^j(x \otimes y) = \sum_j Q^j(x) \otimes Q^{j-j}(y) \);

2. We further require that \( Q^{i/2}(x) = x^\otimes p \) for all even degree \( x \in M \) when \( p > 2 \), and \( Q^{i/2}(x) = x^\otimes 2 \) when \( p = 2 \).

Let Poly\( \mathcal{R} \) be the category of Poly\( \mathcal{R} \)-algebras. A classical result by May and McClure tells us that this is the target category for the mod \( p \) homology of non-unital \( E_\infty-H\mathbb{F}_p \)-algebras.

Theorem 3.3. [May72, BMMS86] For any \( \mathcal{P} \)-algebra \( X \), there is an isomorphism

\[
\pi_*(\text{Free}_{\mathcal{P}}(X)) \cong \text{Free}_{\text{Mod}_{\mathcal{R}}}^{\text{Poly}_{\mathcal{R}}}(\pi_*(X))
\]

of free Poly\( \mathcal{R} \)-algebras.
The free $\text{Poly}_R$-algebra on an $\mathbb{F}_p$-module $M$ can be obtained by taking the polynomial algebra on $\text{Free}_{\text{Mod}_{\mathbb{F}_p}}^R(M)$, identifying $Q^{[k]/2}(x) = x^{2^k}$ for all even $x \in M$ when $p > 2$ and $Q^{[k]}(x) = x^{2^k}$ when $p = 2$, and finally imposing the Cartan formula. Denote again by $\text{Poly}_R$ the monad coming from the free-forgetful adjunction

$$
\begin{array}{c}
\text{Mod}_{\mathbb{F}_p} \\
\text{Poly}_R
\end{array} 
\xrightarrow{\text{Free}_{\text{Mod}_{\mathbb{F}_p}}^R}
\xleftarrow{\text{U}_{\text{Mod}_{\mathbb{F}_p}}}
\text{Poly}_R.
$$

### 3.2. A bar spectral sequence

By repeatedly applying the equivalence in Theorem 3.3, the bar spectral sequence for a $\mathbb{P}$-algebra $A$ with $M = \pi_*(A)$ can be rewritten as

$$
\bar{E}^2_{s,t} = \pi_*(\text{Bar}^{s}(\text{id}, \text{Poly}_R, M))_t \Rightarrow \pi_{s+t}(|\text{Bar}^{s}(\text{id}, \mathbb{P}, A)|) = \text{TAQ}_{s+t}(A).
$$

Similarly, the dual bar spectral sequence takes the form

$$
E^2_{s,t} = \pi_*(\text{Bar}^{s}(\text{id}, \text{Poly}_R, M)^{\vee})_t \Rightarrow \pi_{s+t}(\mathbb{D}[\text{Bar}^{s}(\text{id}, \mathbb{P}, A)]).
$$

The $E^\infty$-page is the reduced mod $p$ TAQ cohomology $\text{TAQ}^\infty(A)$, or the homotopy group of the spectral partition Lie algebra $\mathbb{D}[\text{Bar}^{s}(\text{id}, \mathbb{P}, A)]$ when $A$ is a trivial bounded below $\mathbb{P}$-algebra of finite type. Since we will only be concerned with bounded above objects of finite type over $H\mathbb{F}_p$, we can switch freely between the two version by taking linear dual when computing the second page. It is easier to work with the bar construction, so we will focus on the bar spectral sequence.

Observe that the $E^2$-page of the bar spectral sequence

$$
\bar{E}^2_{s,t} = \pi_*(\text{Bar}^{s}(\text{id}, \text{Poly}_R, M))_t = \pi_*(\text{L}Q^P_{\text{Mod}_{\mathbb{F}_p}}(M))_t
$$

is the André-Quillen homology of $M$ with respect to the monad $\text{Poly}_R$. Our plan is to find a suitable factorization of the indecomposable functor $Q^P_{\text{Mod}_{\mathbb{F}_p}}$ to separate the unary and binary structures of the monad $\text{Poly}_R$. This will allow us to replace the bar construction computing its total left derived functor by a smaller double complex that is amenable to Koszul-duality-type computations. We will follow the same strategy in [Zh21, section 3.3], which is inspired by [BHK19, section 4.4]. The subtlety lies in the bottom non-vanishing Dyer-Lashof operations, which appear in both the unary and binary structures of $\text{Poly}_R$.

### 3.3. The derived indecomposable functor

Before diving into the computation, we briefly recall without proof the homotopy theory of monads on the category of weighted graded $\mathbb{F}_p$-modules and especially the two-sided bar construction for simplicial objects, following closely Sections 3.1.4.2 and 4.3 in [BHK19]. For the general theory, see for instance Sections 3.1 and 3.2 of [JN14].

Let $T$ be an augmented monad on the category $\text{Mod}_{\mathbb{F}_p}$ of weighted graded $\mathbb{F}_p$-modules. Denote by $\text{Alg}_T(\text{Mod}_{\mathbb{F}_p})$ the category of $T$-algebras. The forgetful functor $U : \text{Alg}_T(\text{Mod}_{\mathbb{F}_p}) \to \text{Mod}_{\mathbb{F}_p}$ admits a left adjoint, the free functor $\text{Free}_T : \text{Mod}_{\mathbb{F}_p} \to \text{Alg}_T(\text{Mod}_{\mathbb{F}_p})$.

Levelwise application of this pair of adjunction gives rise to a adjunction between the corresponding categories of simplicial objects, denoted again by $\text{Free}_T \dashv U$, as well as a monad $T$ on the category of simplicial weighted graded $\mathbb{F}_p$-modules with the standard cofibrantly generated model structure. Then this adjunction induces a right transferred model structure on the category of simplicial $T$-algebras, with weak equivalences and fibrations defined on the underlying simplicial weighted graded $\mathbb{F}_p$-modules.
Denote by \( \text{triv} : \text{Mod}_{F_p} = \text{Alg}_{\text{id}}(\text{Mod}_{F_p}) \to \text{Alg}_T(\text{Mod}_{F_p}) \) the inclusion of trivial \( T \)-algebras, which is induced by the augmentation. It has a left adjoint \( Q^{T} : \text{Alg}_{T}(\text{Mod}_{F_p}) \to \text{Mod}_{F_p} \), the indecomposable functor with respect to the \( T \)-algebra structure. Applying this pair of adjunction levelwise to the corresponding categories of simplicial objects, we obtain a Quillen adjunction denoted again as \( Q^{T} \circ \text{triv} \). The total left derived functor \( \mathbb{L}Q^{T} \) of \( Q^{T} \) can be computed by the following standard recipe.

**Construction 3.4.** Given a right functor \( R : \text{Mod}_{F_p} \to \mathcal{D} \) over \( T \), and a simplicial object \( A_* \) in \( \text{Alg}_{T}(\text{Mod}_{F_p}) \), one can apply the two-sided bar construction \( \text{Bar}_*(R,T,-) \) levelwise to \( A_* \). The diagonal of the resulting bisimplicial complex is a simplicial object in \( \mathcal{D} \), denoted by \( \text{Bar}_*(R,T,A_*) \).

In particular, if we regard a \( T \)-algebra \( A \) as a simplicial \( T \)-algebra concentrated in simplicial degree 0, denoted also as \( A \) by abuse of notation, then \( \text{Bar}_*(R,T,A) \) agrees with the usual two-sided bar construction.

The free resolution \( \text{Bar}_*(\text{Free}^T,T,A) \) is a cofibrant replacement of \( A \) in the category of simplicial \( T \)-algebras. The left derived functor of a functor \( F \) can be computed by applying \( F \) levelwise to a cofibrant replacement. Hence

\[
\mathbb{L}Q^{T}(A) \simeq Q^{T}\text{Bar}_*(\text{Free}^T,T,A) = \text{Bar}_*(\text{id},T,A).
\]

### 3.4. A smaller complex for the \( E^1 \)-page.

We disentangle the unary and binary structure by defining a structure that heuristically discard the bottom non-vanishing Dyer-Lashof operations on any element in a module over the Dyer-Lashof algebra when \( p = 2 \) and on any even class when \( p > 2 \).

Let \( \text{Mod}_{R'} \) be category of modules over the Dyer-Lashof algebra \( R' \) with unstability conditions \( Q^i(x) = 0 \) for \( i \leq |x| \) when \( p = 2 \) and \( 2i \leq |x| \) when \( p > 2 \). Then the indecomposable functor \( Q^{\text{Poly}_{R'}}_{\text{Mod}_{F_p}} \) factors as \( Q^{\text{Mod}_{G'}}_{\text{Mod}_{F_p}} \circ Q^{\text{Poly}_{R'}}_{\text{Mod}_{G'}} \) sitting in the composite adjunction with the inclusion/trivial functor.

\[
\begin{array}{ccc}
\text{Mod}_{F_p} & \xleftarrow{Q^{\text{Mod}_{G'}}_{\text{Mod}_{F_p}}} & \text{Mod}_{R'} \xleftarrow{Q^{\text{Poly}_{R'}}_{\text{Mod}_{G'}}} & \text{Poly}_{R'}
\end{array}
\]

In particular, there is an isomorphism

\[
U^{\text{Poly}_{R'}}_{\text{Mod}_{F_p}} \circ Q^{\text{Poly}_{R'}}_{\text{Mod}_{G'}} \cong Q^{\text{Poly}_{R'}}_{\text{Mod}_{F_p}} \circ U^{\text{Poly}_{R'}}_{\text{Mod}_{F_p}}
\]

Denote by \( \text{Poly}_{F_p} \) the monad corresponding to the free graded polynomial algebra functor on \( \text{Mod}_{F_p} \). We want to use the factorization \( Q^{\text{Poly}_{R'}}_{\text{Mod}_{F_p}} \cong Q^{\text{Mod}_{G'}}_{\text{Mod}_{F_p}} \circ Q^{\text{Poly}_{R'}}_{\text{Mod}_{G'}} \) to obtain a double complex that is more computable than the bar complex \( \text{Bar}_*(\text{id},\text{Poly}_{R},M) \).

**Lemma 3.5.** There is a weak equivalence of simplicial \( R' \)-modules

\[
\mathbb{L}Q^{\text{Poly}_{R'}}_{\text{Mod}_{G'}}(M) \simeq \text{Bar}_*(\text{id},\text{Poly}_{F_p},M)
\]

for any \( F_p \)-module \( M \) considered as a trivial \( \text{Poly}_{R} \)-algebra.

**Proof.** There is a map of augmented monads \( \text{Poly}_{R} \to \text{Poly}_{F_p} \to \text{id} \), the first of which kills all Dyer-Lashof operations that are not the bottom operations \( Q^{12}(x) = x^{2p} \) when \( p = 2 \) or the bottom operations \( Q^{2i}(x) = x^{2i} \) on even classes when \( p > 2 \). When \( M \) is an \( F_p \)-module considered as a trivial \( \text{Poly}_{R} \)-algebra, the map \( \text{Poly}_{R}(M) \to \text{Poly}_{F_p}(M) \to M \) is a map of \( \text{Poly}_{R} \)-algebras if we regard \( \text{Poly}_{F_p}(M) \) as a \( \text{Poly}_{R} \)-algebra where all Dyer-Lashof operations vanish except for the bottom
operations when $p = 2$ and the bottom operations on even classes when $p > 2$. Therefore we obtain a map of free bar resolutions

$$
\Psi : \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M) \to \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_{\mathbb{F}_p}, M),
$$

which is a weak equivalence of simplicial $\operatorname{Poly}_R$-algebras.

Next we want to show that applying $\Psi_{\operatorname{Mod}_{\mathbb{F}_p}}$ preserves this weak equivalence, i.e.

$$
\mathbb{L}Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R}(M) \simeq Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M) \to Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_{\mathbb{F}_p}, M)
$$

is a weak equivalence of simplicial $\mathbb{R}'$-modules. This is equivalent to showing that the underlying map of simplicial $\mathbb{F}_p$-modules $U_{\operatorname{Mod}_{\mathbb{F}_p}}(\Psi)$ is a weak equivalence.

Using the isomorphism

$$
U_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \circ Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \cong Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}} \circ U_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}},
$$

we can rewrite $U_{\operatorname{Mod}_{\mathbb{F}_p}}(\Psi)$ as

$$
U_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \circ Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M) \simeq Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}} \circ U_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M)
$$

$$
\simeq Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}} \circ U_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_{\mathbb{F}_p}, M) \simeq \operatorname{Bar}_\bullet(\operatorname{id}, \operatorname{Poly}_{\mathbb{F}_p}, M).
$$

This is indeed a weak equivalence since we are applying $Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_{\mathbb{F}_p}}$ to a free simplicial $\operatorname{Poly}_{\mathbb{F}_p}$-algebra on both sides. Therefore we obtain a weak equivalence of simplicial $\mathbb{R}'$-modules

$$
\mathbb{L}Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R}(M) \cong \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M) \simeq \operatorname{Bar}_\bullet(\operatorname{id}, \operatorname{Poly}_{\mathbb{F}_p}, M)
$$

as desired.

Let $\mathcal{A}_{\mathbb{R}'}$ be the additive monad associated to the free $\mathbb{R}'$-module functor. Therefore the André-Quillen homology of an algebra $M$ over the monad $\operatorname{Poly}_R$, i.e. the $E^2$-page of the bar spectral sequence

$$
E_{s,t}^2 \cong \pi_s(\mathbb{L}Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R}(M)) \cong \pi_s(Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M)) \cong \pi_s(Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R} \operatorname{Bar}_\bullet(\operatorname{Free}_{\operatorname{Mod}_{\mathbb{F}_p}}, \operatorname{Poly}_R, M)) \cong \pi_s(Q_{\operatorname{Mod}_{\mathbb{F}_p}}^{\operatorname{Poly}_R}(\operatorname{Bar}_\bullet(\operatorname{id}, \operatorname{Poly}_{\mathbb{F}_p}, M)))
$$

can be computed as the homotopy group of the double complex $\operatorname{Bar}_\bullet(\operatorname{id}, \mathcal{A}_{\mathbb{R}'}, \operatorname{Bar}_\bullet(\operatorname{id}, \operatorname{Poly}_{\mathbb{F}_p}, M))$.

4. Computing the dual bar spectral sequence

In this section, we compute the $E^2$-page of the (dual) bar spectral sequences in the universal case, i.e., when $A = \Sigma^0\mathbb{H}^F_p \oplus \cdots \oplus \Sigma^0\mathbb{H}^F_p$. Then a comparison with Theorem 2.6 allows us to deduce the degeneration of the spectral sequence in these cases. There is a non trivial distinction between the cases $p = 2$ and $p > 2$ regarding the restriction on shifted Lie algebra structures and the notations are rather different, so we record them separately.
4.1. The $E^2$-page at $p = 2$. We will utilize Priddy’s machinery on algebraic Koszul duality in [Pri70, Theorem 2.5]. Since the Dyer-Lashof algebra $\mathcal{R}$ is a Koszul algebra, the Ext group

$$\text{Ext}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \pi_*((\text{Bar}_{\bullet}(\mathbb{F}_2, \mathcal{R}, \mathbb{F}_2))^{\vee})$$

is the Koszul dual algebra of $\mathcal{R}$. The Koszul generators are given by the collection

$$(Q^i)^* := [(Q^i)^{\vee}]_{1} \in \text{Ext}_{\mathcal{R}}^{-1,s}^{1,s}(\mathbb{F}_2, \mathbb{F}_2)$$

with homological bidegree $(-1, -i)$ and weight 2, with composition given by juxtaposition, which corresponds to the Yoneda product on Ext groups, cf. [Pri70, p.42] and [McC01, Theorem 9.8]. The quadratic relations among the generators are the Koszul dual of the Adem relations (Proposition 3.1), i.e.

$$(1) (Q^a)^*(Q^b)^* = \sum_{a+b-c=2\epsilon} \left( \frac{b-c-1}{a-2c-1} \right) (Q^{a+b-c})^*(Q^c)^*$$

for $a \leq 2b$.

We are interested in the unstable Ext group

$$\text{UnExt}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \Sigma^j\mathbb{F}_2) = \pi_*((\text{Bar}_{\bullet}(\text{id}, A_{\mathcal{R}'}, \Sigma^{-j}\mathbb{F}_2))^{\vee}),$$

which is a variant of the Ext group

$$\text{Ext}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \pi_*((\text{Bar}_{\bullet}(\mathbb{F}_2, \mathcal{R}, \mathbb{F}_2))^{\vee})$$

obtained by regarding $\Sigma^{-j}\mathbb{F}_2$ as an unstable trivial module over $\mathcal{R}$ and imposing the unstablitly conditions $[Q^i]\alpha = 0$ for $j \leq |\alpha|$ in the bar complex, cf. [BC70, §3]. To incorporate the Koszul dual of the unstability conditions as well as the simplicial grading, we introduce the following ringoid.

**Definition 4.1.** Let $\mathcal{F}$ be the ringoid with objects the $\mathbb{Z}_{\leq 0} \times \mathbb{Z}$ and morphisms freely generated over $\mathbb{F}_2$ under juxtaposition by the following elements: for any $s \leq 0$ and all $i, j$ satisfying $i > j$, there is an element $(Q^i)^* \in \mathcal{F}((s, j), (s-1, j-i))$ of weight 2. Let $(\mathcal{R}')^!$ be the quotient of $\mathcal{F}$ by the ideal generated by the relations

$$(2) (Q^a)^*(Q^b)^* = \sum_{a+b-c=2\epsilon, c > -j} \left( \frac{b-c-1}{a-2c-1} \right) (Q^{a+b-c})^*(Q^c)^*$$

for all $a, b$ satisfying $a \leq 2b$, $b > -j$ and $a > b - j$ in $\mathcal{F}((s, j), (s-2, j-a-b))$.

The first grading corresponds to the homological degree in Ext, or equivalently the filtration degree in the dual bar spectral sequence. The second grading is the topological degree.

**Remark 4.2.** There is an evident isomorphism $(\mathcal{R}')^!(\mathcal{R}')^!((s, i), (s', j)) \cong (\mathcal{R}')^!((s-r, i), (s'-r, j))$ for any $i, j, s, s'$ and $r$ such that $s-r < 0$. For any $t > 0$, there is an injection

$$\text{sus}^!(\mathcal{R}')^!(s, i), (s', j)) \hookrightarrow (\mathcal{R}')^!(s, i+t), (s', j+t)),$$

since more operations are defined on classes with higher homological degree.

**Remark 4.3.** Note the relations are always well defined on both sides: if $b \geq -j+1$, then $a \geq b - j + 1 \geq -2j + 2$, so $\frac{a+b}{2} \geq -j + 1$ and the right hand side is never empty.

The unstable Ext group $\text{UnExt}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \Sigma^j\mathbb{F}_2)$ is thus the underlying bigraded $\mathbb{F}_2$-module of the free $(\mathcal{R}')^!$-module $(\mathcal{R}')^!(0, j, -)$. We grade the Ext groups homologically.

On the other hand, the Tor group $\text{Tor}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is a coalgebra $(\mathcal{R}')^!$ generated by classes in

$$\text{Tor}_{\mathcal{R}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{[Q]^1, i \in \mathbb{N}\}.$$
The unstable Tor group

\[ \text{UnTor}^{*,*}(\mathbb{F}_2, \Sigma/\mathbb{F}_2) = \pi_*(\text{Bar}_\bullet(id, \mathcal{A}_{\mathcal{R}'}, \Sigma/\mathbb{F}_2)) \]

is thus a coalgebra generated under juxtaposition by elements in

\[ \text{UnTor}^{1,*}(\mathbb{F}_2, \Sigma/\mathbb{F}_2) = \{(Q)[i], i > j\}, \]

which is a comodule over the co-ringoid \((\mathcal{R}')^\vee\).

Back to computing the \(E_2\)-page. Denote by \(\text{coFree}^{(\mathcal{R})^\vee}\) the functor that takes the cofree \((\mathcal{R})^\vee\)-comodule. Similarly, \(\text{Free}^{(\mathcal{R})^\vee}\) is the functor that takes the free \((\mathcal{R})^\vee\)-module with a simplicial grading, i.e. the class \((Q^i)^*(Q^j)^* \cdots (Q^k)^*(x) \in \text{Free}^{(\mathcal{R})^\vee}(M)\) has simplicial degree \(-k\) for \(x \in M\).

Since \(\mathcal{A}_{\mathcal{R}'}\) is an additive monad, there is no nontrivial simplicial operations on the total left derived functor. Furthermore, the homological (vertical) and simplicial (horizontal) differentials do not mix, i.e. the targets of the vertical differential never involve elements from the inner bar complex and vice versa. Therefore we can deduce the following:

**Lemma 4.4.** Suppose that \(V_\bullet\) is a trivial simplicial \(\mathcal{R}'\)-module. Then

\[ \pi_*(\text{Bar}_\bullet(id, \mathcal{A}_{\mathcal{R}'}, V_\bullet)) = \text{coFree}^{(\mathcal{R})^\vee}(\pi_*(V_\bullet)). \]

In our case, we are interested in the trivial simplicial \(\mathcal{R}'\)-module \(V_\bullet = \text{Bar}_\bullet(id, \text{Poly}_{\mathbb{F}_2}, M)\) where \(M\) is a direct sum of shifts of \(\mathbb{F}_2\) as trivial \(\text{Poly}_{\mathbb{F}_2}\)-modules.

**Definition 4.5.** (cf. [Jac41], [Fre00] for the unshifted version.) A shifted restricted Lie algebra over \(\mathbb{F}_2\), denoted as a \(\text{sLie}_{\mathbb{F}_2}^p\)-algebra, is a graded \(\mathbb{F}_2\)-module \(L = L_\bullet\) with a shifted Lie bracket \(L_m \otimes L_n \to L_{m+n-1}\) and a restriction map \(x \mapsto x^{[2]}\) with \(x^{[2]} \in L_{2|x|-1}\), satisfying the following identities:

1. \(\text{ad}(x^{[2]}) = \text{ad}(x)^2\) for all \(x \in L\);
2. For all \(x, y \in L\), \((x+y)^{[p]} = x^{[p]} + y^{[p]} + [x,y]\).

Here \(\text{ad}(x)\) stands for the adjoint representation, i.e., the self-map \(y \mapsto [y,x]\) on \(L\).

Let \(\text{Free}^\text{sLie}_{\mathbb{F}_2}^p\) be the associated free functor. Given an \(\mathbb{F}_2\)-module \(M\) with basis \(\{x_1, \ldots, x_k\}\), a basis for \(\text{Free}^\text{sLie}_{\mathbb{F}_2}^p(M)\) is given by

\[ \{u, u^{[2]}, (u^{[2]})^{[2]}, \ldots\}, \]

where \(u\) ranges over Lyndon words in letters \(x_1, \ldots, x_k\). (See, for instance, [BKS05, section 2].)

Note that the monad \(\text{Poly}_{\mathbb{F}_2}\) is the monad associated with the free functor of the commutative operad \(\text{Comm} \in \text{Mod}_{\mathbb{F}_2}\). Denote by \(\text{sLie}\) the shifted Lie operad in the category chain complexes over \(\mathbb{F}_2\), i.e. \(\text{sLie}(n)\) sits in homological dimension \(1 - n\) with trivial differentials. The comparison morphism \(\bigoplus_n (\text{sLie}(n)^\vee \otimes M)^{\Sigma_n} \to \text{Bar}_\bullet(id, \text{Comm}, M)\) of the shifted Harrison complex and the bar construction (cf. [Fre04, 5.2.3, 6.6]) is an isomorphism on homotopy when \(M\) is an \(\mathbb{F}_2\)-module considered as a trivial algebra over \(\text{Comm}\): any cycle on the left-hand side is a sum of all way to put \(n-1\) nested parenthesis on a fixed sequence \(x_1, \ldots, x_n\) of \(n\) classes in \(M\), such that each nesting represents taking the polynomial product with one more class on a different simplicial level and no nesting is trivial, and this cycle has preimage the bracket \([\cdots [[x_1, x_2], x_3], \ldots, x_n]\). Taking linear dual, self-brackets become restrictions by [Fre00, Theorem 0.1], so we have

\[ \pi_*(\text{Bar}_\bullet(id, \text{Comm}, M)) = \pi_*(\text{Bar}_\bullet(id, \text{Poly}_{\mathbb{F}_2}, M)) \cong \text{coFree}^{\text{co-sLie}_{\mathbb{F}_2}^p}(M), \]

\[ \pi_*(\text{Bar}_\bullet(id, \text{Comm}, M)^\vee) = \bigoplus_n (\text{sLie}(n) \otimes (M^\vee)^{\Sigma_n})^\Sigma_n \cong \text{Free}^\text{sLie}_{\mathbb{F}_2}^p(M^\vee) \]
for any trivial algebra $M$ over Comm. On the other hand, a shifted Lie coalgebra over $\mathbb{F}_2$ has a shifted Lie cobracket $L_m \otimes L_n \rightarrow L_{m+n+1}$ satisfying the co-Jacobi identity. The shift reflects the simplicial degree in the (co)bar resolution. Similarly a co-sLies$_2$-algebra stands for shifted coLie-algebra over $\mathbb{F}_2$.

4.2. The $E^\infty$-page for $p = 2$. In the case where $A = \Sigma H\mathbb{F}_p$, the $E^\infty$-page records all unary operations on a degree $-j$ class in the homotopy groups of spectral partition Lie algebras. Note that

$$\pi_*(\mathbb{D}\text{Bar}_*(\text{id}, \text{Poly}_{\mathbb{F}_2}, \pi_*(\Sigma^j H\mathbb{F}_p))) = \text{Free}^{\text{sLies}_2}_*(\Sigma^{-j}\mathbb{F}_2)$$

has exactly one class $x^{[2]}$ in $\pi_0$ of weight $2^s$ for all $s \geq 0$. Therefore the dual bar spectral sequence simplifies to

$$E^2_{s,t} = \text{Free}^{(\mathcal{R}^j)}(\mathbb{F}_2\{x^{[2]}_i, s \geq 0\}) \Rightarrow \pi_{s+t}(\mathbb{D}\text{Bar}_*(\text{id}, \mathbb{F}_2, \pi_*(\Sigma^j H\mathbb{F}_p)))$$

Note that the $E^2$-page is concentrated in weight $2^k$ for $k \in \mathbb{N}$, and the weight $2^k$ part is concentrated on a single line $s = -k$. Hence the spectral sequences collapses on the second page and there are no extension problems. Therefore we have found all the unary operations on a degree $j$ class for any $j$.

In particular, we see that the restriction $x^{[2]}$ is represented by the cycle $(Q^{|s|})^*|x$ on the $E^1$-page $\text{Bar}_*(\text{id}, \text{Poly}_{\mathcal{R}^j}, \mathbb{F}_2)^\vee$. On the other hand, the relations in (1) never involve the bottom operation: on a class $x$ of degree $j$, if $c = -j$ then the coefficient $\left(\begin{smallmatrix}b+j-1 \\ a+j-1 \end{smallmatrix}\right)$ of $(Q^{a+b-c})^*(Q^c)^*$ is nonzero only if $a + 2j - 1 \leq b + j - 1$, which is impossible since $a > b - j$.

Hence the ringoid $(\mathcal{R}^j)^\vee$ is the ringoid of additive operations. Now we define a new ringoid that takes into account the restriction being an unary operation.

Definition 4.6. Let $\mathcal{F}$ be the ringoid with objects $\mathbb{Z}_{\leq 0} \times \mathbb{Z}$ whose morphisms are freely generated over $\mathbb{F}_2$ under juxtaposition by the following elements: for all $i, j$ satisfying $i \geq -j$ and $s \leq 0$, there is an element $(Q^j)^* \in \mathcal{F}((s, j), (s-1, j-i))$ of weight 2.

Let $\mathcal{R}^j$ be the quotient of $\mathcal{F}$ by the ideal generated by the relations $(Q^{a-j})^*(Q^s)^* = 0$ for all $j, s$ and $(Q^a)^* \in \mathcal{F}((s, j), (s-1, j-a))$ with $a > -j$, and the Adem relations

$$\sum_{a+b-c+2c < -j} \binom{b-c-1}{a-2c-1} (Q^{a+b-c})^*(Q^c)^* = 0$$

for all $a, b$ satisfying $a \leq 2b, b > -j$ and $a > b - j$ in $\mathcal{F}((s, j), (s-2, j-a-b))$.

The $E^2$-page of the dual bar spectral sequence on one generator has the following structure.

Definition 4.7. An sLies$_{\mathcal{R}^j}$-algebra is an $\mathbb{F}_2$-module $M$ with an $\mathcal{R}^j$-module structure and a shifted Lie bracket

$$\left[\ , \right] : M_{s,t} \otimes M_{s', t'} \rightarrow M_{s+s'-1, t+t'}$$

with restriction $(-)^{[2]}$ satisfying the following conditions:

1. The bottom operation $(Q^{-|s|})^*(x) = x^{[2]}$ is the restriction for any $x$;
2. $[x, \alpha(y)] = 0$ for any $x, y$ and non-empty sequence $\alpha$ of $(Q^i)^*$ unless $\alpha$ is an iteration of the restriction map.

Let $\text{Free}^{\text{sLies}_{\mathcal{R}^j}}$ denote the functor that first takes the free sLies$_{\mathcal{R}^j}$-algebra on a bigraded $\mathbb{F}_2$-module $M$, then takes the free $\mathcal{R}^j$-module on the underlying graded $\mathbb{F}_2$-module of $\text{Free}^{\text{sLies}_{\mathcal{R}^j}}(M)$, and finally identifies the restriction with the bottom operation $x^{[2]} = (Q^{-|s|})^*(x)$ for all $x \in \text{Free}^{\text{sLies}_{\mathcal{R}^j}}(M)$. Therefore we can deduce that:
Proposition 4.8. For $A = \Sigma^j H\mathbb{F}_2$, the bar spectral sequence has $E^\infty$-page

$$E^\infty_{s,t} \simeq 
E^2_{s,t} \simeq \coFree^{(R^\vee)} \coFree^{s\Lie_2^F}(\Sigma^j \mathbb{F}_2).$$

The dual bar spectral sequence for $A$ has $E^\infty$-page

$$E^\infty_{s,t} \simeq E^2_{s,t} \simeq \Free^{s\Lie_K^F}(\Sigma^{-j} \mathbb{F}_2).$$

A basis is given by the monomials $(Q^i)^*(Q^j)^* \cdots (Q^k)^*(x)$, where $i_s \geq |x| = -j$ and $i_j > 2i_{j+1}$ for all $1 \leq l < s$.

Now we can compute the $E^\infty$-page of the (dual) bar spectral sequences in the universal case, and deduce the set of $k$-ary natural operations of all $k$. A priori, knowing the composition product and relations among operations on the André-Quillen cohomology $\AQ_{\text{Poly}}^\pi(-)$ does not imply knowledge of the relations on the homotopy groups of spectral partition Lie algebra and mod 2 TAQ cohomology. The composition product on the later differs from that on the former, as we will see in Theorem 5.5.

Proposition 4.9. Let $A = \Sigma^j H\mathbb{F}_2 \oplus \Sigma^j H\mathbb{F}_2 \oplus \cdots \oplus \Sigma^j H\mathbb{F}_2$ be a trivial $\mathbb{P}$-algebra. Then the dual bar spectral sequence for $\pi_\ast(\Lie_{\mathbb{F}_2,\mathbb{E}_m}^\pi(A^\vee)) \cong \TAQ(A)$ has $E^\infty$-page

$$E^\infty_{s,t} \simeq E^2_{s,t} \simeq \Free^{s\Lie_K^F}(\Sigma^{-j} \mathbb{F}_2 \oplus \cdots \oplus \Sigma^{-j} \mathbb{F}_2).$$

Proof. The dual bar spectral sequence simplifies to

$$E^2_{s,t} = \Free^{s\Lie_K^F}(\Sigma^{-j} \mathbb{F}_2 \oplus \cdots \oplus \Sigma^{-j} \mathbb{F}_2) \Rightarrow \pi_{s+t}(\mathbb{D}|\Bar_\ast(\id, \mathbb{P}, A)|).$$

A priori we can’t deduce that the spectral sequence collapses using a sparsity argument when $k > 1$, since the $E^2$-page is concentrated on multiple lines at $p^m$ when $m > 1$. However, the $E^\infty$-page of the dual bar spectral sequence is the homotopy group of

$$\Lie_{\mathbb{F}_p,\mathbb{E}_m}^\pi(A^\vee) \cong \mathbb{D}|\Bar_\ast(\id, \mathbb{P}, A)|,$$

the free spectral partition Lie algebra on $A^\vee$. Comparing the basis of the $E^2$-page with Theorem 2.6, which is a basis of $\Lie_{\mathbb{F}_2,\mathbb{E}_m}^\pi(A^\vee)$, we deduce that $E^2 \cong E^\infty$, so the spectral sequence collapses on the second page and there are no extension problems. \hfill \Box

4.3. The dual bar spectral sequence for odd primes. In the section, we apply the same analysis to the odd primary case.

Again utilizing Priddy’s machinery on algebraic Koszul duality in [Pri70, Theorem 2.5], we deduce that $\Ext_R^I(\mathbb{F}_p, \mathbb{F}_p) = \pi_\ast((\Bar_\ast(\mathbb{F}_p, R^\vee, \mathbb{F}_p))^\vee)$ is isomorphic as an algebra to the Koszul dual $(R^\vee)^!$ of $R^\vee$. The Koszul generators are given by the collection

$$(\beta^e Q^i)^* := [(\beta^e Q^i)^\vee] 1 \in \Ext_{R^!}^I(\mathbb{F}_p, \mathbb{F}_p), \epsilon \in \{0, 1\}, i \in \mathbb{N}$$

where $(\beta^e Q^i)^*$ has homological bidegree $(-1, -2(p - 1)i + \epsilon)$ and weight $p$. Composition is given by juxtaposition corresponding to the Yoneda product. The quadratic relations are the Koszul dual of the Adem relations (Proposition 3.1), i.e.

$$(Q^a)^*(Q^b)^* + \sum_{a+b-c=p} (-1)^{a-c} \left( \frac{(p-1)(b-c)-1}{a-pc-1} \right) (Q^{a+b-c})^* (Q^c)^* = 0$$
for \( a \leq pb \),

\[
(\beta Q^a)^*(Q^b)^* = \sum_{a+b-c \geq pc} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (Q^{a+b-c})^*(\beta Q^c)^* \\
+ \sum_{a+b-c > pc} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (\beta Q^{a+b-c})^*(Q^c)^* = 0
\]

for \( a \leq pb \),

\[
(\beta^\varepsilon Q^a)^*(\beta Q^b)^* = \sum_{a+b-c \geq pc} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^* = 0
\]

for \( \varepsilon \in \{0, 1\} \) and \( a < pb \), cf. [KL83].

Analogous to the case \( p = 2 \), we dualize the unstability condition and record the simplicial grading using a ringoid.

**Definition 4.10.** Let \( \mathcal{F} \) be the ringoid with objects \( \mathbb{Z}_{\leq 0} \times \mathbb{Z} \) and morphisms freely generated over \( \mathbb{F}_p \) under juxtaposition by the following elements: for \( 2i - j \) and \( s \leq 0 \) there are elements \((Q^i)^* \in \mathcal{F}(s, j), (s-1, j-2(p-1)i)\) and \((\beta Q^i)^* \in \mathcal{F}(s, j), (s-1, j-2(p-1)i+1)\). We suppress the first grading for ease of notation when there is no ambiguity.

Let \((\mathcal{R}')^1\) be the quotient of \( \mathcal{F} \) by the ideal generated by the following quadratic relations for all \( s \leq 0 \):

\[
(Q^i)^*(Q^j)^* = \sum_{a+b-c \geq pc} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (Q^{a+b-c})^*(Q^c)^*,
\]

in \( \mathcal{F}(j, j-2(p-1)a-2(p-1)b) \) for all \( a, b \in \mathbb{Z} \) and satisfying \( a \leq pb \), \( 2b > -j \), \( 2a > 2(p-1)b - j \),

\[
(\beta Q^i)^*(Q^j)^* = \sum_{a+b-c \geq pc, 2c > -k} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (Q^{a+b-c})^*(\beta Q^c)^* \\
- \sum_{a+b-c > pc, 2c > -k} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc-1} \right) (\beta Q^{a+b-c})^*(Q^c)^* = 0
\]

in \( \mathcal{F}(j, j-2(p-1)a-2(p-1)b+1) \) for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq pb \), \( 2b > -j \), \( 2a > 2(p-1)b - j \),

and

\[
(\beta^\varepsilon Q^i)^*(\beta Q^j)^* = \sum_{a+b-c \geq pc, 2c > -k} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc} \right) (\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^* \\
- \sum_{a+b-c > pc, 2c > -k} (-1)^{a-c} \left( \frac{(p-1)(b-c) - 1}{a-pc} \right) (\beta^\varepsilon Q^{a+b-c})^*(\beta Q^c)^* = 0
\]

in \( \mathcal{F}(j, j-2(p-1)a-2(p-1)b+\varepsilon + 1) \) for \( \varepsilon \in \{0, 1\} \) and \( a, b \in \mathbb{Z} \) satisfying \( a < pb \), \( 2b > -j \), \( 2a > 2(p-1)b - j \).

A basis for \((\mathcal{R}')^1((s, j), (s-k, -))\) is given by sequences \((\beta \varepsilon_1 Q^i_1)^* (\beta \varepsilon_2 Q^i_2)^* \cdots (\beta \varepsilon_k Q^i_k)^* \) where \( 2i_l > -j \) and \( i_l > pi_{l+1} - \varepsilon \) for \( 1 \leq l < k \).

**Remark 4.11.** Similar to the case \( p = 2 \), there is an isomorphism

\[
(\mathcal{R}')^1((s, i), (s', j)) \cong (\mathcal{R}')^1((s-r, i), (s'-r, j))
\]

for any \( i, j, s, s' \) and \( r \) such that \( s-r < 0 \). For any \( t > 0 \), there is an injection

\[
\text{susp}^t : (\mathcal{R}')^1((s, i), (s', j)) \hookrightarrow (\mathcal{R}')^1((s, i+t), (s', j+t))
\]

since more operations are defined on classes with higher homological degree.
Lemma 4.12. Suppose that \( V_* \) is a trivial simplicial \( R' \)-module. Then

\[
\pi_*(\text{Bar}_*(\text{id}, A_{R'}, V_*)) = \text{coFree}((R')^\vee)(\pi_*(V_*)).
\]

Now we run the spectral sequence for a trivial \( \mathbb{P} \)-algebra \( A \). First we look at the dual bar spectral sequences for \( A = \Sigma^{/H\mathbb{F}_p} \), which parametrizes unary operations on a degree \( -j \) class the homotopy groups of spectral partition Lie algebra.

Proposition 4.13. Let \( A = \Sigma^{/H\mathbb{F}_p} \). If \( j \) is odd, then \( \pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{Z}}^\pi(\Sigma^{/H\mathbb{F}_p})) \cong \text{Free}((R')^\vee)(\Sigma^{-j}\mathbb{F}_p) \). If \( j \) is even, then \( \pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{Z}}^\pi(\Sigma^{/H\mathbb{F}_p})) \cong \text{Free}((R')^\vee)(\Sigma^{-j}\mathbb{F}_p \oplus \Sigma^{-2j-1}\mathbb{F}_p) \).

Proof. If \( A = \Sigma^{/H\mathbb{F}_p} \) with \( j \) odd, then \( \text{Bar}_*(\text{id}, \text{Poly}_{\mathbb{F}_p}, \Sigma^{/H\mathbb{F}_p}) \) is the constant simplicial object on \( \Sigma^{/H\mathbb{F}_p} \) due to graded commutativity. The dual bar spectral sequence simplifies to

\[
E^3_{s,t} = \text{Free}((R')^\vee)(\Sigma^{-j}\mathbb{F}_p) \Rightarrow \pi_{s-t}(\text{Bar}_*(\text{id}, \mathbb{P}, A)) \cong \pi_{s+t}(\text{Lie}_{\mathbb{F}_p, \mathbb{Z}}^\pi(\Sigma^{/H\mathbb{F}_p})).
\]

Note that the \( E^2 \)-page is concentrated in weight \( p^k \) for \( k \in \mathbb{N} \), and the weight \( p^k \) part is concentrated on a single line \( s = -k \). Hence the spectral sequence collapses on the second page and there are no extension problems.

If \( A = \Sigma^{/H\mathbb{F}_p} \) with \( j \) even, then

\[
\pi_*(\text{Bar}_*(\text{id}, \text{Poly}_{\mathbb{F}_p}, \Sigma^{/H\mathbb{F}_p})) \cong \mathbb{F}_p \{ x, [x,x] \}
\]

with a weight 1 class \( x \in \pi_0 \) and a weight 2 class \( [x,x] \in \pi_1 \). On the \( E^2 \)-page of the dual bar spectral sequence

\[
E^2_{s,t} = \text{Free}((R')^\vee)\left( \mathbb{F}_p \{ x, [x,x] \} \right) \Rightarrow \pi_{s+t}(\text{Bar}_*(\text{id}, \mathbb{P}, A))
\]
is concentrated in weights $p^k$ and $2p^k$, and at each $k$ concentrated on the line $s = -k$ at weight $p^k$ and the line $s = -k - 1$ at weight $2p^k$. Again there are no further differentials or extension problems. \hfill \Box

\textbf{Remark 4.14.} (1) As in the case $p = 2$, knowing the relations among unary operations on the André-Quillen cohomology $Aq^\text{Poly}_n(\ldots)$ does not immediately yield knowledge of the relations on the homotopy groups of spectral partition Lie algebra and mod $p$ TAQ cohomology. The composition product on the later differs from that on the former, as we will see in Theorem 5.6.

(2) While we expect to see a shifted restricted Lie algebra, the restriction on an odd class is not detected on the $E^2$-page for filtration reasons, as we shall see in Lemma 5.12 in the next section.

In general, we have the odd primary counterpart of Proposition 4.9.

\textbf{Proposition 4.15.} Suppose that $A = \Sigma^l H\mathbb{F}_p \oplus \Sigma^l H\mathbb{F}_p \oplus \cdots \oplus \Sigma^l H\mathbb{F}_p$ is a trivial $\mathbb{P}$-algebra. Then the $E^\infty$-page of the dual bar spectral sequence for $\pi_n(Lie_{\mathbb{F}_p, \mathbb{F}_p}(A)) \cong TAQ^*\pi_n(A)$ has $E^\infty$-page

$$E^\infty_{s,t} \cong E^2_{s,t} \cong \text{Free}^{(R^*)}(\text{Free}^{s\text{Lie}_p}(\Sigma^l H\mathbb{F}_p \oplus \cdots \oplus \Sigma^l H\mathbb{F}_p)),$$

Proof. The comparison morphism $\bigoplus_n (s\text{Lie}(n)^* \otimes M)^{\otimes n} \rightarrow \text{Bar}_\bullet(\text{id}, \text{Comm}, M)$ of the shifted Harrison complex and the bar construction (cf. [Fre04, 5.2.3, 6.2]) is surjective on homotopy when $M$ is an $\mathbb{F}_p$-module considered as a trivial algebra: any cycle on the left-hand side is an alternating sum of all way to put $n - 1$ nested parenthesis on a fixed sequence $x_1, \ldots, x_n$ of $n$ classes in $M$, such that each nesting represents taking the polynomial product with one more class on a different simplicial level and no nesting is trivial, and this cycle has preimage the bracket $[\cdots[x_1, x_2], x_3, \ldots, x_n]$. The graded commutativity and the vanishing of self-brackets on odd degree classes correspond to the graded commutativity of the polynomial product. Since the restriction on odd classes are zero by Proposition 4.13, taking linear dual yields

$$\pi_*(\text{Bar}_\bullet(\text{id}, \text{Comm}, M)^*) \cong \text{Free}^{s\text{Lie}_p}(M^*),$$

for any trivial algebra $M$ over Comm and the dual bar spectral sequence simplifies to

$$E^2_{s,t} = \text{Free}^{(R^*)}(\text{Free}^{s\text{Lie}_p}(\Sigma^l H\mathbb{F}_p \oplus \cdots \oplus \Sigma^l H\mathbb{F}_p)) \Rightarrow \pi_{s+t}(\text{Bar}_\bullet(\text{id}, \mathbb{P}, A^*)).$$

As in the case $p = 2$, a priori we can’t deduce that the spectral sequence collapses using a sparsity argument when $k > 1$, since the $E^2$-page is concentrated on multiple lines at $p^m$ when $m > 1$. Nonetheless, comparing the basis of the $E^2$-page with Theorem 2.6, we deduce that $E^2 \cong E^\infty$, so the spectral sequence collapses on the second pages and there are no extension problems. \hfill \Box

5. OPERATIONS AND THEIR RELATIONS

In this section, we construct all natural operations on the homotopy groups of spectral partition Lie algebras and mod $p$ TAQ (co)homology. It follows from a general result of Brantner ([Br17, Theorem 3.5.1 and 4.3.2]) that composition product of additive operations on the homotopy groups of spectral partition Lie algebras agrees, up to a shearing, with the Yoneda product on the $E^2$-page of the dual bar spectral sequence. This allows us to deduce all relations among the unary operations in Theorem 5.5 and 5.6. Then we construct a shifted Lie algebra structure and prove the existence of a restriction map in Lemma 5.12 when $p > 2$. Finally we deduce all relations among unary operations and the bracket in Theorem 5.13.
5.1. Unary operations. In the dual bar spectral sequence
\[ E^2_{s,t} \cong \text{Free}^{\text{Lie}_p} R(\Sigma/j\Sigma F)_s \to \pi_{s+t}(\text{Bar}_*(\text{id} F, \Sigma^{-1} H F)) \cong \pi_{s+t}(\text{Lie}_{p,\Sigma}^\Sigma (\Sigma/j\Sigma F)), \quad p = 2 \]
\[ E^2_{s,t} \cong \text{Free}^{(R')}^n \text{Free}^{\text{Lie}_p} R(\Sigma/j\Sigma F)_s \to \pi_{s+t}(\text{Bar}_*(\text{id} F, \Sigma^{-1} H F)) \cong \pi_{s+t}(\text{Lie}_{p,\Sigma}^\Sigma (\Sigma/j\Sigma F)), \quad p > 2, \]
the \( E^2 \)-page is generated by a single class \( x \in E^0_{i,j} \) under unary operations
\[ (Q^i)^* : E^2_{s,i} \to E^2_{s-1,i}, \quad i > -t \]
for \( p = 2 \), and
\[ (\beta^c Q^i)^* : E^2_{s,i} \to E^2_{s-1,2(p-1)i+1}, \quad 2i > -t \]
where \( c \in \{0,1\} \), as well as a shifted self-bracket if \( j \) is even for \( p > 2 \). The additive unary operations, excluding the self-bracket, and their relations are encoded by the ringoid \((R')^i\) in Definition 4.1 and 4.10.

The \( E^\infty \)-page is the homotopy groups of the free spectral partition Lie algebra on \( \Sigma/JF \). Hence, it parametrizes unary operations on a degree \( j \) homotopy class of a spectral partition Lie algebra and a degree \(-j\) cohomology class in \( \text{mod} \) \( p \) TAQ cohomology. Now we give a concrete description of unary operations on the homotopy groups of any spectral partition Lie algebra \( A \).

**Construction 5.1.** Suppose that \( \xi : \Sigma/\Sigma F \to A \) represents a homotopy class \( x \in \pi_j(A) \).

1. Suppose that \( p = 2 \). For any sequence
\[ \alpha = (Q^i)^*(Q^j)^* \cdots (Q^k)^* \in (R^i)((0,j), (-k,j-m)) \]
with \( i_1 + \cdots + i_k = m \), there is a unique class \( R^{i_1,i_2,\ldots,i_k}(x) \in \pi_{j-m-k}(A) \) given by
\[ \Sigma^{j-m-k} F_2 \cong \text{Free}^{R^i}(\Sigma F_2) \hookrightarrow \text{Free}^{\text{Lie}_p} R(\Sigma F)_2 \]
\[ \cong \pi_*(\text{Lie}_{p,\Sigma}^\Sigma (\Sigma/j\Sigma F)) \xrightarrow{\xi} \pi_*(\text{Lie}_{p,\Sigma}^\Sigma (A)) \to \pi_*(A). \]

2. Suppose that \( p > 2 \). For any sequence
\[ \alpha = (\beta^c Q^i)^*(\beta^c Q^j)^* \cdots (\beta^c Q^k)^* \in (R^i)((0,j), (-k,j-m)) \]
with \( m = 2(p-1)i_1 + \cdots + 2(p-1)i_k - \epsilon_1 - \cdots - \epsilon_k \), there is a unique class \( R^{i_1,i_2,\ldots,i_k}(x) \in \pi_{j-m-k}(A) \) given by
\[ \Sigma^{j-m-k} F_p \cong \text{Free}^{(R')}^n \text{Free}^{(R')}^n \text{Free}^{\text{Lie}_p} (\Sigma F_2) \]
\[ \cong \pi_*(\text{Lie}_{p,\Sigma}^\Sigma (\Sigma/j\Sigma F)) \xrightarrow{\xi} \pi_*(\text{Lie}_{p,\Sigma}^\Sigma (A)) \to \pi_*(A). \]

If \( j \) is even, then there is a unique class \( B(x) \in \pi_{j+2}(A) \) given by
\[ \Sigma^{2j-1} \xrightarrow{\text{Free}^{\text{Lie}_p} (\Sigma F_2)} \text{Free}^{(R')^i} \text{Free}^{\text{Lie}_p} (\Sigma F_2) \cong \pi_*(\text{Lie}_{p,\Sigma}^\Sigma (\Sigma/j\Sigma F)) \to \pi_*(A) \]

Translating to cohomological grading, for any \( \Sigma/\Sigma F \)-algebra \( A \), there are unary operations
\[ R^{i_1,i_2,\ldots,i_k} : \text{TAQ}^i(A) \to \text{TAQ}^{j+i_1+i_2+\cdots+i_k}(A), \quad i > j \]
for \( p = 2 \) and
\[ R^{i_1,i_2,\ldots,i_k} : \text{TAQ}^i(A) \to \text{TAQ}^{j+2(p-1)(i_1+i_2+\cdots+i_k)+\epsilon_1+\cdots+\epsilon_k}(A), \quad 2i > j \]
where \( \epsilon \in \{0,1\} \) as well as a self-bracket
\[ B : \text{TAQ}^i(A) \to \text{TAQ}^{2j+1}(A) \]
when \( j \) is even for \( p > 2 \).
Remark 5.2. These operations are stable in the sense that any cohomology operation \( \alpha : \overline{\text{TAQ}}^m(A) \to \overline{\text{TAQ}}^{m+1}(A) \) agrees with \( \alpha : \overline{\text{TAQ}}^{m-1}(A) \to \overline{\text{TAQ}}^{m}(A) \) under cohomological desuspension, or equivalently \( \alpha : \pi_m(L) \to \pi_{m-1}(L) \) agrees with \( \alpha : \pi_{m+1}(L) \to \pi_{m}(L) \) for any spectral partition Lie algebra \( L \). This is straightforward to check on the \( E^2 \)-page of the dual bar spectral sequence in the universal cases using the fact that Dyer-Lashof operations satisfy this notion of stability.

A convenient way to encode the structure of additive operations is via a power ring, as was introduced in [Br17] to encode additive unary operations on the Lubin-Tate theory of spectral Lie algebras. Note that our convention differs in that we switch to a logarithmic grading convention for the weight grading.

Definition 5.3. [Br17, Definition 3.17] A power ring is a collection \( P = \{ P^i_j(w) \}_{(j,k,w) \in \mathbb{Z}_2 \times \mathbb{Z}_{>0}} \) of abelian groups with elements \( t_i \in P^i_i[0] \) for all \( i \), equipped with associative and unital composition maps \( P^i_j[w] \times P^j_k[w] \to P^i_k[v+w] \).

A module over the power ring \( P \) is a (weighted) \( \mathbb{F}_p \)-module \( M \) with structure maps \( P^i_j[w] \otimes M_j[w] \to M_i(p^w) \) that are compatible with the composition maps in \( P \).

Definition 5.4. The collection \( \{ P^i_j[w] := (\mathcal{R}^i)'((0,k),(-w,j+w)), w > 0 \} \), along with \( P^i_i[0] := \mathbb{F}_p \{ t_i \} \) for all \( i \) and \( P^i_i[0] = \emptyset \) for \( i \neq j \), defines a power ring \( P \), with composition product given by the sheared Yoneda product

\[
P^i_j[v] \times P^j_k[w] \xrightarrow{\cong} (\mathcal{R}^i)'((0,j),(-v,i+v)) \times (\mathcal{R}^j)'((0,k),(-w,j+w))
\]

for \( v,w > 0 \), as well as isomorphisms \( P^i_j[w] \times P^j_i[0] \xrightarrow{\cong} P^i_j[w] \) and \( P^i_i[0] \times P^j_j[w] \xrightarrow{\cong} P^j_i[w] \) exhibiting \( t_i \) as a two-sided unit.

The first map is an injection on the left factor because operations are stable under suspension and here \( w \geq 0 \), cf. Remark 4.2 and 4.11. The last map is the composition in the ringoid (\( \mathcal{R}^i \)', i.e., juxtaposition corresponding to the Yoneda product on Ext groups.

Explicitly, when \( p = 2 \), the \( \mathbb{F}_2 \)-module \( P^i_j[w] \) consists of operations \( R^{i_1 \cdots i_w} \) such that \( j - i_1 - \cdots - i_w = k \) and \( i_l - 1 > i_{l+1} + \cdots + i_w - j - (w - l) \) for all \( 1 \leq l \leq w \), subject to the relations in (\( \mathcal{R}^i \)'((0,j),(-w,k+w))). The composition product sends \( R^{i_1 \cdots i_w} \circ R^{i_{w+1} \cdots i_k} \) to the juxtaposition \( R^{i_1 \cdots i_{w+1} \cdots i_k} \). The weight 2 additive operations are given by the collection \( R^i \in P^i_j[j+1][1] \) for any \( i > j + 1 \).

Theorem 5.5. The homotopy groups of a spectral partition Lie algebra over \( \text{H} \mathbb{F}_2 \), or the reduced \( \text{TAQ} \) cohomology of an \( \mathbb{E}_\infty \)-\( \text{H} \mathbb{F}_2 \)-algebra form a left module over the power ring \( P \) of additive unary
operations. The relations among the weight 2 additive operations are given by the Adem relations
\[ R^a R^b = \sum_{a+b-c \geq 2c, c > -j+1} \binom{b-c-1}{a-2c} R^{a+b-c} R^c \]
in \( \mathcal{P}_j^{j-a-b}[2] \) for all \( a, b \in \mathbb{Z} \) satisfying \( b-j < a < 2b \) and \( b > -j+1 \).

A basis for unary operations on a degree \( j \) class is given by all monomials \( R^i \) such that \( i_j > -j \) and \( i_m \geq 2i_{m+1} \) for \( 1 \leq m < l \).

When \( p > 2 \), the weight \( p \) unary operations are given by the collection of elements \( \beta^\epsilon R^i := R^{(i,\epsilon)} \in \mathcal{P}_j^{j-2(p-1)i-\epsilon} \) for \( \epsilon = 0, 1 \) and any \( 2i > -j \).

**Theorem 5.6.** The homotopy groups of a spectral partition Lie algebra \( A \) over \( \mathbb{H}_p \), or the reduced TAQ cohomology of any \( \mathbb{E}_n \)-\( \mathbb{H}_p \)-algebra, form a module over the power ring \( \mathcal{P} \) of unary operations. The relations among the weight \( p \) operations are given by the Adem relations
\[ \beta R^a \beta R^b = \sum_{a+b-c \geq 2c, c > -j} (-1)^{a-c+1} \binom{(p-1)(b-c)-1}{a-pc-1} \beta R^{a+b-c} R^c \]
in \( \mathcal{F}_p (j, j-2(p-1)a-2(p-1)b-2) \) for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq pb \), \( 2b > -j \), and \( 2a > 2(p-1)b-j \).

\[ R^a \beta R^b = \sum_{a+b-c \geq 2c, c > -j} (-1)^{a-c} \binom{(p-1)(b-c)}{a-pc} \beta R^{a+b-c} R^c - \sum_{a+b-c \geq 2c, c > -j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc-1} R^{a+b-c} R^c \]
in \( \mathcal{F}_p (j, j-2(p-1)a-2(p-1)b-1) \) for all \( a, b \in \mathbb{Z} \) satisfying \( a \leq pb \), \( 2b > -j \), and \( 2a > 2(p-1)b-1-j \).

\[ \beta^\epsilon R^a R^b = \sum_{a+b-c \geq 2c, c > -j} (-1)^{a-c} \binom{(p-1)(b-c)-1}{a-pc} \beta^\epsilon R^{a+b-c} R^c \]
in \( \mathcal{P}_j^{j-2(p-1)a-2(p-1)b-\epsilon}[2] \) for all \( a, b \in \mathbb{Z} \) satisfying \( a < pb \), \( 2b > -j \), \( 2a > 2(p-1)b-j \), and \( \epsilon \in \{0, 1\} \).

A basis for unary operations on a degree \( j \) class with \( j \) odd is given by the collection of all monomials \( \beta^\epsilon R^i \beta^\epsilon R^i \cdots \beta^\epsilon R^i \) such that \( 2i_j > -j \) and \( i_m \geq p i_{m+1} + \epsilon_{m+1} \) for \( 1 \leq m < l \). If \( j \) is even, a basis is given by the collection \( \beta^\epsilon R^i \beta^\epsilon R^i \cdots \beta^\epsilon R^i \beta^k \) such that \( 2i_j > -(1+ \epsilon)j-\epsilon \) and \( i_m \geq p i_{m+1} + \epsilon_{m+1} \) for \( 1 \leq m < l \).

**Proof of Theorem 5.5 and 5.6.** By construction, the set \( \mathcal{P}_j [w] \) is isomorphic to the image of
\[ (\mathcal{R}^j)^{(0, j), (w, w+i)} \subset E_{\epsilon, \pi}^2 \]
in \( \pi_i (\mathcal{L}(\Sigma \mathcal{H} \mathcal{F}_p)) \) via the collapse of the dual bar spectral sequence
\[ E_{\epsilon, \pi}^2 \cong \text{Free}^{\mathcal{R}^j} (\Sigma \mathcal{H} \mathcal{F}_p) \Rightarrow \pi_\epsilon (\text{Lie}_p^\mathcal{R} (\Sigma \mathcal{H} \mathcal{F}_p)) \]
and for \( p > 2 \)
\[ E_{\epsilon, \pi}^2 \cong \text{Free}^{(\mathcal{R}^j)} \text{Free}^{\mathcal{R}^j} (\Sigma \mathcal{H} \mathcal{F}_p) \Rightarrow \pi_\epsilon (\text{Lie}_p^\mathcal{R} (\Sigma \mathcal{H} \mathcal{F}_p)). \]

We need to verify that compositions of unary operations on the homotopy groups of spectral partition Lie algebras is reflected by the composition product of the power ring \( \mathcal{P} \).
For ease of notations, we will use $\mathcal{L}$ to denote the monad $\text{Lie}_F^{\mathbb{P}_p, \mathbb{P}_m}$ throughout this proof. The unary operations on the homotopy groups of algebras over $\mathcal{L}$, other than the self-brackets on even classes when $p > 2$, are concentrated in weights $p^n$ for $n \geq 1$ by Proposition 4.8 and 4.13. When $A$ is bounded above, they live in the homotopy groups of the summands

$$\mathcal{L}[n](A) := (\partial_{p^n}(\text{Id} \otimes H F_p^\delta) \otimes (A)^{\otimes p^n}) \overset{i_{p^n}}{\longrightarrow} \mathcal{L}(A)$$

by Proposition 2.2. The composition $\beta \circ \alpha$ of two unary operations

$$\alpha \in \mathcal{P}_k[w] \subseteq \pi_j(\mathcal{L}[w](\Sigma^k H F_p^\delta)), \quad \beta \in \mathcal{P}_v[v] \subseteq \pi_i(\mathcal{L}[v](\Sigma^l H F_p^\delta)),$$

considered as maps $\Sigma^k H F_p^\delta \rightarrow \mathcal{L}[v](\Sigma^l H F_p^\delta)$ and $\beta : \Sigma^l H F_p^\delta \rightarrow \mathcal{L}[v](\Sigma^k H F_p^\delta)$, is given by

$$\Sigma^l H F_p^\delta \overset{\beta}{\longrightarrow} \mathcal{L}[v](\Sigma^l H F_p^\delta) \overset{\mathcal{L}[v](\alpha)}{\longrightarrow} \mathcal{L}[v]\circ\mathcal{L}[w](\Sigma^k H F_p^\delta) \rightarrow \mathcal{L}[v + w](\Sigma^k H F_p^\delta).$$

The last map is induced by the weight $p^{v+w}$ summand

$$(\partial_{p^n}(\text{Id} \otimes H F_p^\delta) \otimes ((\partial_{p^{n'}}(\text{Id} \otimes H F_p^\delta) \otimes (\Sigma^k H F_p^\delta)^{\otimes p^n}))^{\otimes p^n} \rightarrow (\partial_{p^{v+w}}(\text{Id} \otimes H F_p^\delta) \otimes (\Sigma^k H F_p^\delta)^{\otimes p^{v+w}})$$

of the structure map of the monad $\mathcal{L} \circ \mathcal{L} \rightarrow \mathcal{L}$ on any bounded above object $A$.

Let $a \in \mathcal{R}_k((0, k), (-w, j + w))$, $b \in \mathcal{R}_v((0, j), (-v, i + v))$ be the unique preimages under the isomorphisms in Proposition 4.8 and 4.13 of $\alpha$ and $\beta$ on the $E^2$-pages of the dual bar spectral sequences converging respectively to $\pi_*(\mathcal{L}(\Sigma^k H F_p^\delta))$ and $\pi_*(\mathcal{L}(\Sigma^l H F_p^\delta))$.

Since $\pi_* (\mathcal{L}(\Sigma^k H F_p^\delta))$ is bounded above and of finite type, we can run the dual bar spectral sequence for the $H F_p^\delta$-module $A = \mathcal{L}(\Sigma^k H F_p^\delta)$ converging to $\pi_*(\mathcal{L}(\Sigma^l H F_p^\delta))$. The spectral sequence collapses on the $E^2$-page

$$E^2 \cong \text{Free}^{\mathsf{Lie}_{R^l}}(\pi_*(\mathcal{L}(\Sigma^k H F_p^\delta))) \cong \text{Free}^{\mathsf{Lie}_{R^l}}(\mathcal{L}(\Sigma^k H F_p^\delta)), \quad p = 2,$$

$$E^2 \cong \text{Free}^{(R')_l} \text{Free}^{\mathsf{Lie}_{R^l}}(\pi_*(\mathcal{L}(\Sigma^k H F_p^\delta))) \cong \text{Free}^{(R')_l} \text{Free}^{\mathsf{Lie}_{R^l}}(\mathcal{L}(\Sigma^k H F_p^\delta)), \quad p > 2$$

by comparing with Theorem 2.6 in the limiting case.

The map $\mathcal{L} \circ \mathcal{L}(\Sigma^k H F_p^\delta) \rightarrow \mathcal{L}(\Sigma^l H F_p^\delta)$ coming from the monad composition induces a map of the $E^2$-pages of the dual bar spectral sequences

$$\text{Free}^{\mathsf{Lie}_{R^l}} \circ \text{Free}^{\mathsf{Lie}_{R^l}}(\Sigma^k H F_p^\delta) \rightarrow \text{Free}^{\mathsf{Lie}_{R^l}}(\Sigma^k H F_p^\delta), \quad p = 2,$$

$$\text{Free}^{(R')_l} \text{Free}^{\mathsf{Lie}_{R^l}}(\Sigma^k H F_p^\delta) \rightarrow \text{Free}^{(R')_l} \text{Free}^{\mathsf{Lie}_{R^l}}(\Sigma^k H F_p^\delta), \quad p > 2.$$

We need to understand the restriction of the above maps to the additive part, i.e., the horizontal maps of the diagram

$$\text{Free}^{(R')_l} \circ \text{Free}^{(R')_l}(\Sigma^k H F_p^\delta) \longrightarrow \text{Free}^{(R')_l}(\Sigma^k H F_p^\delta) \quad \text{Free}^{(R')_l} \circ \text{Free}^{(R')_l}(\Sigma^k H F_p^\delta) \longrightarrow \text{Free}^{(R')_l}(\Sigma^k H F_p^\delta).$$

Recall that $\text{Free}^{(R')_l}(\Sigma^k H F_p^\delta)$ is by construction isomorphic to the unstable Ext group

$$\text{UnExt}_{R^l}^{\mathbb{P}_m}(F_p^\delta, \Sigma^k H F_p^\delta) \cong \pi_*(\text{Bar}_\bullet(\text{Id}, \Lambda_{R^l}, \Sigma^{-k} F_p^\delta)^\vee).$$

We will make use of a general result of Brantner that follows from [Br17, Theorem 3.5.1 and 4.3.2]: Suppose that $T$ is an additive monad on $\text{Mod}_{F_p^\delta}$ associated to the free (unstable) module
functor over an algebra \( R \). Then composition map of the monad \( \text{Bar}_a(id,T,-) \) is compatible with the Yoneda product on the (unstable) Ext groups over \( R \) up to a shearing.

Here \( \mathcal{A}_R \) is an additive monad associated with the free functor that takes the unstable module over the Koszul algebra \( R \). It follows that the top map is a sheared Yoneda product on (unstable) Ext groups. More precisely, given \( \rho \in \mathcal{R}'((0,j),(-v,i+v)) \cong \text{UnExt}_{R'}^{-v,i+v}(\mathbb{F}_p,\Sigma^j\mathbb{F}_p) \) and \( \mu \in \mathcal{R}'((0,k),(-w,j+w)) \cong \text{UnExt}_{R'}^{-w,j+w}(\mathbb{F}_p,\Sigma^k\mathbb{F}_p) \), the top map produces an element

\[
b \circ \rho \mu \in (\mathcal{R}')^1((0,k),(-v-w,i+v+w)) \cong \text{UnExt}_{R'}^{-v-w,i+v+w}(\mathbb{F}_p,\Sigma^k\mathbb{F}_p)
\]

via the composite

\[
\begin{align*}
(\mathcal{R}')^1((0,j),(-v,i+v)) \times (\mathcal{R}')^1((0,k),(-w,j+w)) &\xrightarrow{\text{susp}^n \times \text{id}} (\mathcal{R}')^1((0,j+w),(-v,i+v+w)) \times (\mathcal{R}')^1((0,k),(-w,j+w)) \\
(\mathcal{R}')^1((0,j+w),(-v,i+v+w)) \times (\mathcal{R}')^1((0,k),(-w,j+w)) &\xrightarrow{\cong} (\mathcal{R}')^1((0,k),(-v-w,i+v+w)).
\end{align*}
\]

The first map is an injection on the left factor because operations are stable under suspension and here \( w \geq 0 \), cf. Remark 4.2 and 4.11. The last map is the composition in the ringoid \( (\mathcal{R}')^1 \), i.e. juxtaposition corresponding to the Yoneda product on unstable Ext groups. This is exactly the composition product in \( \mathcal{P} \).

Therefore the map (4) lifts to a map along the \( E^2 \)-pages of the respective dual bar spectral sequences, given explicitly by \( b|_{x_j} \mapsto b|_{(a|x_k)} \) and \( b|_{(a|x_k)} \mapsto b|_{a|x_k} \). Here we use \( \circ \) to denote juxtaposition in \( (\mathcal{R}')^1 \) and \( x_k \) the generator for \( \Sigma^k\mathbb{F}_p \). Passing to the \( E^\infty \)-pages, we deduce that there is a commutative diagram

\[
\begin{CD}
\mathcal{P}_j^1[v] \times \mathcal{P}_j^1[w] @>\ast>> \mathcal{P}_j^1[v+w] \\
@VV\cong V \quad @VV\cong V \\
(\mathcal{R}')^1((0,j),(-v,i+v)) \times (\mathcal{R}')^1((0,k),(-w,j+w)) @>\cup>> (\mathcal{R}')^1((0,k),(-v-w,i+v+w)) \\
@VV\cong V \quad @VV\cong V \\
(\mathcal{R}')^1((0,k),(-w,j+w)) \times (\mathcal{R}')^1((0,k),(-w,j+w)) @>\circ>> (\mathcal{R}')^1((0,k),(-v-w,i+v+w)) \\
@VV\pi_j(\mathcal{L}(\Sigma^j\mathbb{H}\mathbb{F}_p)) \times \pi_j(\mathcal{L}(\Sigma^k\mathbb{H}\mathbb{F}_p)) V \quad @VV\pi_{j+i}(\mathcal{L}(\Sigma^k\mathbb{H}\mathbb{F}_p)) V
\end{CD}
\]

as desired. The first two horizontal maps, i.e. the composition product in the power ring \( \mathcal{P} \), are given by the sheared Yoneda product, and the bottom horizontal map is given explicitly by the map (4).
In particular, the composition product \( \mathcal{P}_{j-a}^{j-a} \times \mathcal{P}_{j-a}^{j-a} \rightarrow \mathcal{P}_{j-a}^{j-a} \) sends \((R^a, R^b)\) to \((R^{a+b})^j \times (R^{a+b})^j \) for \( p > 2 \), and \((\beta^a R^{b-e} \beta^b R^{d-e})\) to \(R^{(b-e) - (d-e)} = (\beta^a \beta^b) Q^{d-e} \) for \( p > 2 \). When \( p = 2 \), a basis for additive unary operations on a degree \( j \) class is given by all monomials \((Q^i)^*(Q^j)^* \cdots (Q^k)^* \in (R^j)^l((0, j), (s, m))\) such that \( i_s \geq -j + 1 \) and \( i_l > 2i_{l+1} \) for all \( 1 \leq l < s \), cf. Proposition 4.8. Any such monomial is the image of the (well-defined) iterated composition \( R^{i+1} R^{j+1} \cdots R^{k+1} \) in \( P^{j-m-s} \). Hence every additive unary operation \( R^{(i_1, \ldots, i_s)} \) can be written as a linear combination of compositions \( R^{i_1} \cdots R^{i_s} \) of operations in \( P^s \). The case \( p > 2 \) is analogous.

\[ \square \]

**Corollary 5.7.** When \( j \) gets arbitrarily large, we deduce that the algebra of unary operations on a degree \( j \) homotopy class of a spectral partition Lie algebra, or equivalently a degree \( j \) class in mod \( p \) TAQ cohomology, is the Koszul dual algebra of the mod \( p \) Dyer-Lashof algebra.

This is because when the degree \( j \) of a class \( x \) gets arbitrarily large, all unary operations and relations are defined on \( x \), while the shifted Lie brackets and restrictions vanish for degree reasons. For \( p > 2 \) one needs be careful about the precise duality. The Dyer-Lashof operation \( Q^j \) is sent to \( \beta^j \) in cohomological degree \( 2(p-1)j + 1 \) and \( \beta Q^j \) to \( R^j \) in cohomological degree \( 2(p-1)j \) since the Bockstein homomorphism increases cohomological degree by one.

### 5.2. Shifted restricted Lie algebra structure

Next we examine the shifted restricted Lie algebra structure on the homotopy groups of spectral partition Lie algebras and the reduced mod \( p \) TAQ cohomology. Some of the methods in this section are inspired by the thesis works of Antolín-Camarena and Brantner regarding the shifted Lie algebra structure on the mod 2 homology and the Lubin Tate theory of spectral Lie algebras in [AC20, Br17].

We showed in Proposition 4.9 and 4.15 that in the case of a free \( \mathbb{F}_p \)-\( \mathcal{E}_n \)-algebra on a bounded \( \mathbb{F}_p \)-module \( A \) as a trivial \( \mathcal{P} \)-algebra, there is a shifted (restricted) Lie bracket on the André-Quillen cohomology of the trivial Poly\( \mathcal{P} \)-algebra \( \pi_*(A) \), which is the \( E^2 \)-page of the dual bar spectral sequence that converges to \( \text{TAQ}^{-*} (A) \cong \pi_*(\text{Lie}_{\mathbb{F}_p, \mathcal{E}_n}(A^\vee)) \). Now we show that there is a shifted restricted Lie algebra structure on the homotopy groups of any spectral partition Lie algebra, or the TAQ cohomology of any \( \mathcal{E}_n \)-\( \mathbb{F}_p \)-algebra \( A \). The shifted Lie algebra structure exists at the level of \( H \mathbb{F}_p \)-modules and agrees with the shifted Lie algebra structure on the \( E^2 \)-page.

Recall that the second Goodwillie derivative of the identity functor \( \partial_2(\text{Id}) \simeq S^{-1} \) is a naive \( \Sigma_2 \)-spectrum with trivial \( \Sigma_2 \)-action. By Proposition 2.2, the weight 2 part of the free spectral partition Lie algebra on a bounded \( H \mathbb{F}_p \)-module \( A \) is

\[
\big( (\partial_2(\text{Id}) \otimes H \mathbb{F}_p) \otimes A \big)^{h \Sigma_2} \simeq (\partial_2(\text{Id}) \otimes H \mathbb{F}_p) \otimes A \overset{\ell_2}{\rightarrow} \text{Lie}_{\mathbb{F}_p, \mathcal{E}_n}(A) \simeq \bigoplus_n (\partial_n(\text{Id}) \otimes H \mathbb{F}_p) \otimes A \overset{h \Sigma_n}{\rightarrow} (A)^{\otimes n}.
\]

Here \( \ell_2 \) denotes the inclusion of the weight \( n \) homogeneous piece.

The binary operation \([-,-]\) representing the shifted Lie bracket is encoded by the weight two part of the structure map

\[
\xi_2 : S^{-1} \otimes (A^{\otimes 2})^{h \Sigma_2} \simeq (\partial_2(\text{Id}) \otimes H \mathbb{F}_p) \otimes A^{\otimes 2} \overset{\ell_2}{\rightarrow} \text{Lie}_{\mathbb{F}_p, \mathcal{E}_n}(A) \rightarrow A,
\]

explicitly given as follows.
Construction 5.8. For \( x : \Sigma^j HF_p \to A, y : \Sigma^k HF_p \to A \) representing two homotopy classes of a spectral partition Lie algebra \( A \), we have a map of Lie\(_{\mathbb{F}_p, \mathbb{E}_n}\)-algebras

\[
\theta : \text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_p \oplus \Sigma^k HF_p) \to \text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (A) \to A,
\]

where the second map is the Lie\(_{\mathbb{F}_p, \mathbb{E}_n}\)-algebra structural map of \( A \). Write \( X = \Sigma^j HF_p \) and \( Y = \Sigma^k HF_p \). There is a binary operation \([-,-]\) on \( \pi_*(A) \) represented by the map

\[
\theta \circ t_2 : \partial_2(\text{Id}) \otimes (X \otimes Y)^{\oplus 2} \subset \text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_p \oplus \Sigma^k HF_p) \to A,
\]

which sends the pair of homotopy classes \( x \) and \( y \) to a class \([x,y] \in \pi_*(A)\).

At the level of the dual bar spectral sequence, the binary operation \([x,y] \in \pi_*(A)\) is represented uniquely up to a nonzero scalar by the weight 2 part of the composite

\[
\Sigma^{-1}(\Sigma^j F_2 \oplus \Sigma^k F_2) \to \text{Free}_{\mathbb{F}_p}^k (\Sigma^j F_2 \oplus \Sigma^k F_2) \leftarrow \text{Free}_{\mathbb{F}_p}^k (\Sigma^j HF_p \oplus \Sigma^k HF_p)
\]

\[
\cong \pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_2 \oplus \Sigma^k HF_2)) \overset{\theta}{\to} \pi_*(A),
\]

\[
\Sigma^{-1}(\Sigma^j F_2 \oplus \Sigma^k F_2) \to \text{Free}_{\mathbb{F}_p}^k (\Sigma^j F_2 \oplus \Sigma^k F_2) \leftarrow \text{Free}_{\mathbb{F}_p}^k (\Sigma^j HF_p \oplus \Sigma^k HF_p)
\]

\[
\cong \pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_2 \oplus \Sigma^k HF_2)) \overset{\theta}{\to} \pi_*(A),
\]

for \( p > 2 \).

More precisely, since the dual bar spectral sequence converging to \( \pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_2 \oplus \Sigma^k HF_2)) \) collapses on the \( E^2 \)-page with no extension problems, there are unique preimages of \( x,y \) on the \( E^2 \)-page, which we again call \( x,y \) by abuse of notations. The binary operation \([-,-]\) is then given at all primes \( p \) by

\[
\Sigma^{-1}(\Sigma^j F_p \oplus \Sigma^k F_p) \to \mathbb{F}_p \{[x,y]\} \leftarrow \text{wt}_2 \left( \text{Free}_{\mathbb{F}_p}^k (\Sigma^j F_2 \oplus \Sigma^k F_2) \right)
\]

\[
\cong \pi_*(S^{-1} \otimes (\Sigma^j HF_p \oplus \Sigma^k HF_p)^{\oplus 2})
\]

\[
\overset{(t_2)}{\to} \text{wt}_2[\pi_*(\text{Lie}_{\mathbb{F}_p, \mathbb{E}_n}^k (\Sigma^j HF_2 \oplus \Sigma^k HF_2))] \overset{\theta}{\to} \pi_*(A)
\]

up to a nonzero scalar \( c \). We fix a choice of the generator for the shifted Lie bracket on the \( E^2 \)-page of the dual bar spectral sequence so that \( c = 1 \), and by abuse of notation we use \([-,-]\) to denote both the binary operation on \( \pi_*(A) \) and the shifted Lie bracket on the \( E^2 \)-page.

Translating to cohomological grading, we constructed a binary operation

\[
[-,-] : \text{TAQ}^{\mathfrak{m}}(A) \otimes \text{TAQ}^{\mathfrak{n}}(A) \to \text{TAQ}^{\mathfrak{m}+\mathfrak{n}+1}(A)
\]

for any \( \mathbb{E}_{\mathfrak{nu}} \)-\( HF_p \)-algebra \( A \).

First we show that this binary operation is indeed a shifted Lie bracket in a general sense.

**Proposition 5.9.** The binary operation \([-,-]\) in Construction 5.8 satisfies graded commutativity \([x,y] = (-1)^{|x||y|}[y,x]\) and the graded Jacobi identity

\[
(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|y||z|}[z,[x,y]] = 0.
\]
We shall see in the next proposition that \([x,x] = 0\) for all \(x\) at \(p = 2\) and \([x,[x,x]] = 0\) for all \(x\) at \(p = 3\). Hence \([-,-]\) equips the homotopy groups of any spectral partition Lie algebra with a shifted Lie algebra structure as is the convention of this paper.

**Proof.** Graded commutativity \([x,y] = (-1)^{|x||y|}[y,x]\) follows by construction, with the sign coming from the induced action of the transposition \((12) \in \Sigma_2\). To check the graded Jacobi identity of the shifted bracket, we use an argument adapted from [AC20]. Let \(A = \Sigma^j H^p \oplus \Sigma^k H^p \oplus \Sigma^l H^p\). The iteration \([[-,-],-]\) of the binary operation \([-,-]\) is given by the weight 3 summand

\[
\begin{align*}
(\partial_2(\text{Id}) \otimes H^p) &\otimes \left( (\partial_1(\text{Id}) \otimes H^p, \otimes A) \otimes ((\partial_2(\text{Id}) \otimes H^p, \otimes A^{\otimes 2}) \right) \\
&\overset{5}{\longrightarrow} \text{wt}_3 \left[ (\partial_2(\text{Id}) \otimes H^p) \otimes \left( (\partial_1(\text{Id}) \otimes H^p, \otimes A) \otimes ((\partial_2(\text{Id}) \otimes H^p, \otimes A^{\otimes 2}) \right) \right] \\
&\rightarrow (\partial_3(\text{Id}) \otimes H^p) \otimes A^{\otimes 3}
\end{align*}
\]

of the monad composition \(\text{Lie}^p_{\Sigma,\Sigma} \circ \text{Lie}^p_{\Sigma,\Sigma} \rightarrow \text{Lie}^p_{\Sigma,\Sigma}\) applied to \(A\). Since \(\partial_2(\text{Id}) \simeq S^{-1}\) and \(\partial_1(\text{Id}) \simeq S\) both have trivial actions by the symmetric groups, we deduce that the source of the above structure map is equivalent to

\[
(\partial_2(\text{Id}) \otimes (\partial_1(\text{Id}) \otimes \partial_2(\text{Id}))) \otimes H^p \otimes (A^{\otimes 3})^{\Sigma_2}.
\]

Denote by \(\nu\) the structure map \(\partial_2(\text{Id}) \otimes (\partial_1(\text{Id}) \otimes \partial_2(\text{Id})) \rightarrow \partial_3(\text{Id})\). The graded Jacobi identity

\[
(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||z|}[y,[z,x]] + (-1)^{|z||x|}[z,[x,y]] = 0
\]

is then equivalent to showing that \(\nu + (\sigma)_* \nu + (\sigma)_*^2 \nu\) is null-homotopic, where \(\sigma\) is the induced action of the cyclic permutation \((123)\). It was proved in [AC20, Proposition 5.2] that \(\nu + (\sigma)_* \nu + (\sigma)_*^2 \nu\) is null-homotopic. \(\square\)

Next we investigate the interaction between the shifted Lie bracket \([-,-]\) and the unary operation in Construction 5.1.

**Proposition 5.10.** Given any \(x,y \in \pi_* (A)\) and unary operation \(\alpha\) of positive weight, we have \([x,\alpha(y)] = 0\) unless one of the following condition is satisfied:

1. \(p = 2\) and \(\alpha\) is an iteration of bottom unary operations on \(y\). The bottom unary operation on a class \(y\) is given by \(R^{-[y]+1}\) in Construction 5.1, which equips the shifted Lie bracket with a restriction map \(x \mapsto x^{[2]}\). In particular \([x,x] = 0\) for all \(x\).

2. \(p > 2\) and \(y\) is in odd degree \(\alpha\) is an iteration of bottom unary operations on \(y\). The bottom unary operation on an odd class \(y\) is given by \(R^{(-[y]+1)/2}\) in Construction 5.1. In particular \([x,[x,x]] = 0\) for all \(x\) when \(p = 3\).

**Proof.** We use an argument adapted from [Br17, Proposition 4.3.15]. Suppose that \(\alpha\) is a nonempty sequence of operations with weight \(\nu\) divisible by \(p\), and \(x: \Sigma^j H^p \rightarrow A, y: \Sigma^k H^p \rightarrow A\) representing two homotopy classes of a spectral partition Lie algebra \(A\). The operation \([x,\alpha(y)]\) is encoded
by the weight $w+1$ part of the structure map $\text{Lie}_p^\Sigma \circ \text{Lie}_p^\Sigma \to \text{Lie}_p^\Sigma$, i.e.,

$$\Sigma^{j+k+|\alpha|-1} H F_p \to d_2 \otimes \left( \left( d_1 \otimes \Sigma^j H F_p \right) \otimes \left( d_w \otimes \left( \Sigma^k H F_p \right)^{\otimes w} \right) \right) \quad \text{(6)}$$

$$\xrightarrow{\text{w}_{\Sigma^{j+k+|\alpha|-1}}} \text{wt}_{w+1} \left( \left( d_1 \otimes \Sigma^j H F_p \right) \otimes \left( d_w \otimes \left( \Sigma^k H F_p \right)^{\otimes w} \right) \right)^{\otimes 2}$$

$$\to d_{w+1} \otimes \left( \Sigma^j H F_p \otimes \Sigma^{kw} H F_p \right) \to \text{Lie}_p^\Sigma \left( \Sigma^j H F_p \otimes \Sigma^{kw} H F_p \right) \to \text{Lie}_p^\Sigma(A) \to A,$$

where we write $d_n$ for $\partial_n(\text{Id}) \otimes H F_p$ for ease of notation. Note that the action obtained by restriction to $\Sigma_1 \times \Sigma_w \subset \Sigma_{w+1}$ on $\partial_{w+1}(\text{Id})$ is freely induced from the action of the trivial subgroup on $S^{-w}$. For any finite group $G$, we have $(\text{Ind}_G^F(X) \otimes Y)^{hG} \simeq X \otimes Y$ for all $G$-spectra $Y$ by the Wirthmüller isomorphism. Hence

$$\left( \partial_{w+1}(\text{Id}) \otimes H F_p \right)^{h(\Sigma_1 \times \Sigma_w)} \otimes \Sigma^{j+kw} H F_p \simeq \Sigma^{j+(k-1)w} H F_p.$$

In particular, the $F_p$-module of weight $w+1$ operations on spectral partition Lie algebras coming from the bracket of one weight one operation and one weight $w$ operation is one-dimensional. Note that $[[\cdots [[x,y],y] \cdots],y]$, where we take the bracket with $y$ exactly $w$ times, is a class of weight $w+1$ obtained by taking the bracket of $y$ with a length $w$ bracket $[[\cdots [[x,y],y] \cdots],y]$ where we take the bracket with $y$ exactly $w-1$ times in

$$\text{Free}^{\text{Lie}_p^\Sigma} \circ \text{Free}^{\text{Lie}_p^\Sigma} \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \subset \text{Free}^{\text{Lie}_p^\Sigma} \circ \text{Free}^{\text{Lie}_p^\Sigma} \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \cong \pi_* \left( \text{Lie}_p^\Sigma \circ \text{Lie}_p^\Sigma \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \right) \xrightarrow{\theta} \pi_* (A)$$

when $p = 2$ and

$$\text{Free}^{\text{Lie}_p^\Sigma} \circ \text{Free}^{\text{Lie}_p^\Sigma} \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \subset \text{Free}^{\text{Lie}_p^\Sigma} \circ \text{Free}^{\text{Lie}_p^\Sigma} \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \cong \pi_* \left( \text{Lie}_p^\Sigma \circ \text{Lie}_p^\Sigma \left( \Sigma^j H F_p \otimes \Sigma^k H F_p \right) \right) \xrightarrow{\theta} \pi_* (A)$$

for $p > 2$. In the free case this class is nonzero, so we obtain a generator $\gamma$ of the $F_p$-module of weight $w+1$ operations coming from the bracket of one weight one operation and one weight $w$ operation.

If $\alpha$ represents an iteration of the self-bracket, then $[x, \alpha(y)] = 0$ by the Jacobi identity when $p \neq 3$. When $p = 3$ and $|x| = k$ is even, a degree count shows that $[x, [x,x]] = 0$ since the weight 3 part of the $E^2$-page of the dual bar spectral sequence

$$E^2 \cong \text{Free}^{\Sigma^k H F_p} \circ \text{Free}^{\Sigma^k H F_p} \circ \text{Free}^{\Sigma^k H F_p} \Rightarrow \pi_* \left( \text{Lie}_p^\Sigma \circ \text{Lie}_p^\Sigma \left( \Sigma^j H F_p \right) \right)$$

has nothing in total degree $3k - 2$.

Suppose that $\alpha$ is not an iteration of the self-bracket. If $\alpha$ is not an iteration of the bottom operation $R^{-|x|+1}$ on $x$ when $p = 2$, or an iteration of the bottom operation $R^{-|x|+1}/2$ on odd $x$ when $p > 2$. Then a comparison of topological degrees shows that $[x, \alpha(y)]$ has strictly smaller topological degree than the generator $\gamma = [[\cdots [[x,y],y] \cdots]],y$ of weight $w+1$ operations coming from the bracket of one weight one operation and one weight $w$ operation. Therefore it has to be zero.

If $p = 2$ and $\alpha$ is the bottom operation $R^{-|y|+1}$ on $y$, then we know that $[x, (Q^{-|y|})^\alpha(y)] = [[x,y],y]$ on the $E^m \cong E^2$-page of the dual bar spectral sequence converging to $\pi_* \left( \text{Lie}_p^\Sigma \circ \text{Lie}_p^\Sigma \left( \Sigma^j H F_p \right) \right)$.
Lemma 5.12. Here \( \text{ad} \) (when \( A \) stands for the self-map \( \Sigma \rightarrow \Sigma \)) of the map of \( E^2 \)-pages of the dual bar spectral sequences for the monad composition map

\[
\text{Lie}_{\mathbb{F}_p, \Sigma n} \circ \text{Lie}_{\mathbb{F}_p, \Sigma n} \circ \text{Lie}_{\mathbb{F}_p, \Sigma n} (\Sigma/\mathbb{F}_p \oplus \Sigma/\mathbb{F}_p) \rightarrow \text{Lie}_{\mathbb{F}_p, \Sigma n} (\Sigma/\mathbb{F}_p \oplus \Sigma/\mathbb{F}_p),
\]

of the map of \( E^2 \)-pages of the dual bar spectral sequences for the monad composition map

\[
\text{Lie}_{\mathbb{F}_p, \Sigma n} \circ \text{Lie}_{\mathbb{F}_p, \Sigma n} \circ \text{Lie}_{\mathbb{F}_p, \Sigma n} (\Sigma/\mathbb{F}_p \oplus \Sigma/\mathbb{F}_p) \rightarrow \text{Lie}_{\mathbb{F}_p, \Sigma n} (\Sigma/\mathbb{F}_p \oplus \Sigma/\mathbb{F}_p).
\]

Passing to the \( E^\infty \)-pages of the dual bar spectral sequences of the composition map, we deduce that \([x, R^{-|x|+1}(y)] = \gamma = [[x], y], \) i.e., the bottom operation serves as the restriction for the bracket. By induction we conclude that if \( \alpha \) is the \( n \)-th iteration of the restriction, then \([x, \alpha(y)] = \gamma \) in weight \( 2n + 1 \). Note that there is only one unary operation of degree \(-|x| + 1\) on any class \( x \) up to a scalar, and the restriction is such an unary operation. Hence we deduce that \([x, x] = 0\) since taking self-bracket is an additive operation while the restriction is not.

When \( p > 2 \), we expect to see a shifted restricted Lie algebra structure on the \( E^\infty \)-page analogous to the case \( p = 2 \) with the restriction given by the bottom operation, as was noted in [BCN21, Remark 4.49] and by Basterra and Mandell, cf. [Law20, Example 1.8.8].

Definition 5.11. (cf. [Jac41], [Fre00]) A shifted restricted Lie algebra over \( \mathbb{F}_p \), denoted as a \( \text{sLie}_{\mathbb{F}_p} \)-algebra, is a graded \( \mathbb{F}_p \)-module \( L = L_0 \) with a shifted Lie bracket \( L_m \otimes L_n \rightarrow L_{m+n-1} \) and a restriction map \( x \mapsto x^{[p]} \) with \( x^{[p]} \in L_{[p]|x|-p+1} \) whenever \( |x| \) is odd, satisfying the following identities:

1. \( (cx)^{[p]} = c^{p} x^{[p]} \) for all odd degree \( x \in L \) and \( c \in \mathbb{F}_p \);
2. \( \text{ad}(x^{[p]}) = \text{ad}(x) \) for all odd degree \( x \in L \);
3. \( \text{For all odd degree } x, y \in L, (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i}{i} (x, y), \) where \( s_i \) is the coefficient of \( t_i \) in the formal expression \( \text{ad}(tx+y)^{p-1}(x) \).

Here \( \text{ad}(x) \) stands for the self-map \( y \mapsto [y, x] \) on \( L \).

Lemma 5.12. If \( j \) is odd, then \( \text{T AQ}^\infty (\Sigma^{-j} H \mathbb{F}_p) \cong \pi_\ast (\text{Lie}_{\mathbb{F}_p, \Sigma n} (\Sigma/\mathbb{F}_p)) \) admits a restriction \( x \mapsto x^{[p]} \) that coincides with the bottom operation \( R^{(-j+1)/2} \) on the generator \( x \) in degree \( j \) up to a unit \( \lambda_j \) that depends only on \( j \). If \( j \) is even, such a map does not exist. In general the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras admits a restriction that on any odd degree class \( x \) is represented by the bottom operation \( R^{(-|x|+1)/2} \) up to a unit \( \lambda_{|x|} \).

Proof. It suffices to check the cases of single generators and two generators. Since the category of \( H \mathbb{F}_p \)-modules is equivalent to the derived category of chain complexes over \( \mathbb{F}_p \) ([Lur17, 7.1.1.16]), we can think of \( \pi_{n+r} (\mathbb{D}[\text{Bar}_\ast (\text{id}, \mathbb{F}_p, A)]) \) as the homology of the chain complex

\[
C = C_n (\mathbb{D}[\text{Bar}_\ast (\text{id}, \mathbb{F}_p, A)]) \cong \bigoplus_n (\text{slie}(n) \otimes (A^\vee)^{\otimes n}) h\Sigma_n
\]

when \( A \) is bounded by [BCN21, Remark 4.49], and its homotopy group is a shifted restricted Lie algebra. Here \( \text{sLie} \) is the shifted Lie operad in the dg-category, with \( \text{sLie}(r) \) concentrated in dimension \( 1 - r \), weight \( r \). It remains to show that the restriction is nonzero and identify the restriction on the generator \( x \) of \( \Sigma/\mathbb{F}_p \).

There is a homotopy fixed points spectral sequence

\[
E^2_{s,t} = H^s (\Sigma_p, \pi_r (\text{slie}(p) \otimes (\Sigma/\mathbb{F}_p)^{\otimes p})) \Rightarrow \pi_{s-t} \left( (\text{slie}(p) \otimes (\Sigma/\mathbb{F}_p)^{\otimes p}) h\Sigma_p \right),
\]

where \( \Sigma/\mathbb{F}_p \) is considered as a one-dimensional chain complex over \( \mathbb{F}_p \) concentrated in homological degree \( j \) with no differentials. Since \( \pi_r (\text{slie}(p) \otimes (\Sigma/\mathbb{F}_p)^{\otimes p}) = 0 \) unless \( t = p j + 1 - p \), the \( E^2 \)-page
of the homotopy fixed points spectral is concentrated on a single line \( t = pj + 1 - p \). Hence the sequence collapses and there are no extension problems.

Taking \( s = 0 \), we deduce that

\[
\pi_{pn+1-p}(\text{sLie}(p) \otimes (\Sigma^j \mathbb{F}_p) \otimes \Sigma^p) \cong H^0(\Sigma^p; \text{sLie}(p) \otimes (\Sigma^j \mathbb{F}_p) \otimes \Sigma^p),
\]

where the right hand side has coefficients in \( \text{Mod}_{\mathbb{F}_p} \). By [Fre00, Theorem 1.2.5], if \( j \) is odd then

\[
H^0(\Sigma^p; \text{sLie}(p) \otimes (\Sigma^j \mathbb{F}_p) \otimes \Sigma^p) \cong (\text{sLie}(p) \otimes (\Sigma^j \mathbb{F}_p) \otimes \Sigma^p \Sigma^p \cong (\Sigma^{-1} \text{Lie}(p) \otimes (\Sigma^{j+1} \mathbb{F}_p) \otimes \Sigma^p)
\]

contains an element that serves as the restriction \( x^{[p]} \) in the free shifted restricted Lie algebra on \( \Sigma^j \mathbb{F}_p \). Since \( x^{[p]} \) is in topological degree \( pj + 1 - p \) and the only element in the weight \( p \) part of \( \pi_{pj+1-p}(\text{sLie}_{\mathbb{F}_p, \Sigma^j}(\Sigma^j \mathbb{F}_p)) \) comes from the bottom operation \( \beta Q^{(j+1)/2} \) up to a unit \( \lambda \) on the \( E^2 \)-page of the dual bar spectral sequence, we conclude that the restriction on \( x \) is given by the image of \( \lambda R^{(j+1)/2} \) in \( \pi_{pj+1-p}(\text{sLie}_{\mathbb{F}_p, \Sigma^j}(\Sigma^j \mathbb{F}_p)) \). This unit \( \lambda \) is fixed for any class of a given degree \( j \) by functoriality of the restriction map and the bottom Dyer-Lashof operation on an odd class \( x \) as the \( p \)-fold Massey product on \( x \).

The class \( x \) does not admit a restriction when \( j \) is even because \( \pi_{pj+1-p}(\text{sLie}_{\mathbb{F}_p, \Sigma^j}(\Sigma^j \mathbb{F}_p)) = 0 \), which is as expected for a shifted restricted Lie algebra.

Next we take \( A = \Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p \), with \( j \) odd. There is a homotopy fixed points spectral sequence

\[
E^2_{s,t} = \bigoplus_n H^s(\Sigma_n; \pi_t(\text{sLie}(n) \otimes (\Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p) \otimes n)) \Rightarrow \bigoplus_n \pi_{t-s}(\text{sLie}(n) \otimes (\Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p) \otimes n)^{H\Sigma_n}.
\]

On the line \( s = 0 \), we have

\[
\bigoplus_n H^0(\Sigma_n; \text{sLie}(n) \otimes (\Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p) \otimes n)) \cong \bigoplus_n (\text{sLie}(n) \otimes (\Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p) \otimes n)^{H\Sigma_n},
\]

which is the free shifted restricted Lie algebra on the \( \mathbb{F}_p \)-module \( \Sigma^j \mathbb{F}_p \otimes \Sigma^k \mathbb{F}_p \) by [Fre00, Theorem 1.2.5]. A prior the bracket on the \( E^2 \)-page of the homotopy fixed points spectral sequence survives to a bracket on the \( E^\infty \)-page that agrees with the shifted Lie bracket \([-,-]\) in Construction 5.8 up to a nonzero scalar \( c \). Hence we choose a generator for \( \text{sLie}(2) \) so that \( c = 1 \), and by abuse of notation we also denote the bracket on the \( E^2 \)-page by \([-,-]\). The \((p+1)\)-th summand of the spectral sequence collapses on the \( E^2 \)-page with no extension problems, since the group cohomology of \( \Sigma_n \) with coefficients in \( \text{Mod}_{\mathbb{F}_p} \) is concentrated in degree 0 when \( n \) is coprime to \( p \). From the computation on one generator, we deduce that the restriction \( x^{[p]} \) on the generator \( x \) of \( \Sigma^j \mathbb{F}_p \) on the \( E^2 \)-page of the homotopy fixed points spectral sequence survives to the element \( \lambda_j R^{(j+1)/2} \) with \( \lambda_j \) the unit given in the first part of the proof. Furthermore, the identity \( [y, x^{[p]}] = [[\cdots [[y, x], x], \cdots], x] \) on the \( E^2 \)-page survives to an identity in \( \pi_{pj+k-p}(\text{sLie}_{\mathbb{F}_p, \Sigma^j}(\Sigma^j \mathbb{F}_p \otimes \Sigma^k H \mathbb{F}_p)) \).

Similarly, suppose that \( j \) and \( k \) are both odd and let \( x, y \) represent the generator of \( \Sigma^j \mathbb{F}_p \) and \( \Sigma^k \mathbb{F}_p \) on the \( E^2 \)-page above. Then the identity \( (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i (x,y) \) on the line \( s = 0 \) of the \( E^2 \)-page survives to on identity \( (x+y)^{[p]} = \lambda_j R^{(j+1)/2} (x) + \lambda_k R^{(k+1)/2} (y) + \sum_{i=1}^{p-1} s_i (x,y) \) on the \( E^\infty \)-page via the collapse of the homotopy fixed points spectral sequence, where \( s_i \) is the coefficient of \( t^{i-1} \) in the formal expression \( \text{ad}(tx+y)^{p-1}(x) \). In particular, if \( j = k \) then

\[
\lambda_j R^{(j+1)/2} (x+y) = \lambda_j R^{(j+1)/2} (x) + \lambda_j R^{(j+1)/2} (y) + \sum_{i=1}^{p-1} s_i (x,y),
\]
so the bottom operation on an odd class is not additive in general even though it lifts to an additive operation on the $E^2$-page of the dual bar spectral sequence.

Finally, we want to show that the collection $x^{[p]} := \lambda_j R^{|-|x|+1|/2}(x)$ for $|x| = j$ odd and $\lambda_j$ a unit depending only on $j$ defines a restriction map $(-)^{[p]}$ for the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras by extending to linear sums of classes $x,y$ with $|x| = j \neq |y| = k$ odd via $(x+y)^{[p]} := \lambda_j R^{|-|x|+1|/2}(x) + \lambda_k R^{|-|y|+1|/2}(y) + \sum_{i=1}^{p-1} \delta_i(x,y)$. The $p$th iteration of $[-,x]$ on a class $y$ is encoded by a summand in the weight $p+1$ part of the iterated monad composition

$$(\text{Lie}_{\mathbb{F}_p, E_n}^p)^{\otimes p} \to (\text{Lie}_{\mathbb{F}_p, E_n}^p)^{\otimes p-1} \to \cdots \to \text{Lie}_{\mathbb{F}_p, E_n}^p$$

applied to $H\Sigma^j \mathbb{F}_p \oplus H\Sigma^k \mathbb{F}_p$. Explicitly, this summand is the $(p-1)$-th iteration of $\partial_2(\text{Id}) \otimes (\partial_1(\text{Id}) \otimes (-))$ on $\partial_2(\text{Id})$. Note that the last step $\text{Lie}_{\mathbb{F}_p, E_n}^p \circ \text{Lie}_{\mathbb{F}_p, E_n}^p \to \text{Lie}_{\mathbb{F}_p, E_n}^p$ of the above chain of compositions is

$$(\partial_2(\text{Id}) \otimes H\mathbb{F}_p) \otimes \left((\partial_1(\text{Id}) \otimes H\mathbb{F}_p) \otimes \big((\partial_1(\text{Id}) \otimes H\mathbb{F}_p) \otimes (\Sigma^j H\mathbb{F}_p)^{\otimes p}\big)\right)$$

$$\to \left(\partial_{p+1}(\text{Id}) \otimes H\mathbb{F}_p\right)^{\otimes (\Sigma^j H\mathbb{F}_p \otimes \Sigma^k H\mathbb{F}_p)}$$

as in (6), which we showed to be one-dimensional in Proposition 5.10 with $[\cdots [y, x], x, \cdots]$, $x$ where $x$ appears $p$ times a generator. The monad composition induces a map of homotopy fixed points spectral sequences, both of which collapse on the $E^2$-page at weight $p+1$ with no extension problems. Hence we get a map that is the weight $p+1$ part of

$$\bigoplus_n \left(\text{sLie}(n) \otimes (\bigoplus_m \text{sLie}(m) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes m})^\otimes n\right)^{\Sigma_n} \to \bigoplus_n \left(\text{sLie}(n) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes n} \right)^{\Sigma_n}$$

along the line $s = 0$ on the $E^2$-pages. This map has as summand

$$\text{sLie}(2) \otimes \left((\text{sLie}(1) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes 1})^{\Sigma_1} \oplus (\text{sLie}(p) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes p})^{\Sigma_p}\right)^{\otimes 2}$$

$$\to \left(\text{sLie}(p+1) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes p+1}\right)^{\Sigma_{p+1},}$$

with further summand

$$\phi : \text{sLie}(2) \otimes \left(\Sigma^j \mathbb{F}_p \otimes (\text{sLie}(p) \otimes (\Sigma^k \mathbb{F}_p)^{\otimes p})^{\Sigma_p}\right) \to (\text{sLie}(p+1) \otimes (\Sigma^j \mathbb{F}_p \oplus \Sigma^k \mathbb{F}_p)^{\otimes p+1})^{\Sigma_{p+1}}.$$ 

The map $\phi$ agrees with the construction 5.8 on the $E^\infty$-page, i.e., it is the evaluation of the free shifted restricted Lie bracket. The image of $[y, R^{|-|x|+1|/2}(x)]$ under $\phi$ is $[\cdots [y, x], x, \cdots] x$ up to a unit $\lambda_{[x]}$. Hence on the $E^\infty$-page we have $[y, \lambda_{[x]} R^{|-|x|+1|/2}(x)] = [\cdots [y, x], x, \cdots] x$ as desired. \Box

To sum up, we have the following theorem.

**Theorem 5.13.** The binary operation $[-, -]$ constructed above equips the homotopy groups of any spectral partition Lie algebra $A$ with a shifted restricted Lie algebra bracket

$$[-, -] : \pi_j(A) \otimes \pi_k(A) \to \pi_{j+k-1}(A)$$

over $\mathbb{F}_p$. If $p = 2$, for all $j$ and $x \in \pi_j(A)$ the restriction $x^{[2]}$ is represented by the bottom operation $R^{-j+1}(x)$. The restriction map on a sum of classes $x, y$ in degrees $j \neq k$ is given by

$$(x+y)^{[2]} = \lambda_j R^{-j+1}(x) + \lambda_k R^{-k+1}(y) + [x, y].$$
If \( p > 2 \), for all odd \( j \) and \( x \in \pi_j(A) \) the restriction \( x^{[p]} \) is the bottom operation \( R^{(-j+1)/2}(x) \) up to a unit \( \lambda_j \). The restriction map on a sum of classes \( x, y \) in degrees \( j \neq k \) are given by

\[
(x + y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} s_i(x, y),
\]

where \( s_i \) is the coefficient of \( t^{i-1} \) in the formal expression \( \text{ad}(tx + y)^{p-1}(x) \).

The bracket is compatible with the unary operations in Theorem 5.5 and 5.6 in the sense that \([x, \alpha(y)] = 0 \) for \( x, y \in \pi_n(A) \) and any unary operations \( \alpha \) of positive weight that is not an iteration of the restriction map. Equivalently, for any \( \mathbb{E}_n^n \)-\( \mathbb{F}_p \)-algebra \( A \), there is a shifted Lie bracket with restriction

\[
[-, -] : \text{TAQ}^j(A) \otimes \text{TAQ}^k(A) \rightarrow \text{TAQ}^{j+k+1}(A)
\]

satisfying the above conditions.

**Remark 5.14.** The interaction between the unary operations and the shifted Lie bracket on the homotopy groups of spectral partition Lie algebras differ from that on the homology of spectral Lie algebras. It was shown in [AC20, Br17, Kja18] that on the mod \( p \) homology groups and the Lubin-Tate theory of spectral Lie algebras, the bracket \([x, \alpha(y)]\) always vanishes if \( \alpha \) is a unary operation of positive weight. Note that unary operations on spectral Lie algebras are all additive. In comparison, on the homotopy groups of spectral partition Lie algebras the bracket \([x, \alpha(y)]\) does not necessarily vanish when \( \alpha \) is an iteration of the restriction map, i.e., a non-additive unary operation. For instance, when \( p = 2 \), the bottom non-vanishing operation \( \tilde{Q}^{[1]} \) on a mod \( 2 \) homology class \( x \) of a free spectral Lie algebra is identified with the nonzero self-bracket \([x, x]\) by [AC20, Lemma 6.4]. Hence \([y, \tilde{Q}^{[1]}(x)] = 0\) by the Jacobi identity for all \( x, y \). In comparison, the bottom operation \( R^{[-1]+1} \) on a class in the homotopy group of a free spectral partition Lie algebra over \( \mathbb{F}_2 \) represents the restriction on \( x \), so \([y, R^{[-1]+1}(x)] = [y, x, x]\) is nonzero. Whereas self-brackets always vanish in shifted restricted Lie algebras over \( \mathbb{F}_2 \), cf. [Fre00, Remark 1.2.9].

5.3. **Generation.** Finally we put all the structures together to obtain the optimal target category for the homotopy group of spectral partition Lie algebras, or equivalently the reduced mod \( p \) TAQ cohomology.

**Definition 5.15.** A \( \mathcal{P} \)-sLie\( p \)-algebra \( L \) is a module over the power ring \( \mathcal{P} \), together with a shifted Lie bracket and a restriction \((-)^{[p]}\), that satisfies the following conditions:

1. If \( p = 2 \), for all \( j \) and \( x \in \pi_j(A) \) the restriction \( x^{[2]} \) is given by the bottom operation \( R^{(-j+1)}(x) \). The restriction map on a sum of classes \( x, y \) is given by

\[
(x + y)^{[2]} = R^{(-j+1)}(x) + R^{(-k+1)}(y) + [x, y],
\]

If \( p > 2 \), for all odd \( j \) and \( x \in \pi_j(A) \) the restriction \( x^{[p]} \) is up to a unit \( \lambda_j \) the bottom operation \( R^{(-j+1)/2}(x) \). The restriction map on the sum of classes \( x, y \) in degrees \( j \neq k \) are given by

\[
(x + y)^{[p]} = \lambda_j R^{(-j+1)/2}(x) + \lambda_k R^{(-k+1)/2}(y) + \sum_{i=1}^{p-1} s_i(x, y),
\]

where \( s_i \) is the coefficient of \( t^{i-1} \) in the formal expression \( \text{ad}(tx + y)^{p-1}(x) \);

2. The bracket \([y, \alpha(x)]\) vanishes for any \( x, y \in L \) and \( \alpha \) a unary operation of positive weight, unless \( \alpha \) is an iteration of the restriction map.

Denote by \( s\text{Lie}_p^p \) the category of \( \mathcal{P} \)-sLie\( p \)-algebras.
Hence the homotopy group every spectral partition Lie algebra, or the reduced TAQ cohomology of any $\mathbb{E}_\infty$-$HF_p$-algebra, has the structure of an $s\text{Lie}^p$-algebra. The free $\mathcal{P}$-$s\text{Lie}^p$-algebra functor $\text{Free}^{s\text{Lie}^p}$ on a $F_p$-module $M$ can be computed as follows: first we take the free shifted restricted Lie algebra over $F_p$, then take the free $\mathcal{P}$-module on $\text{Free}^{s\text{Lie}^p}(M)$. If $p = 2$ then we define the bottom operation $R^{-|x|+1}(x)$ to be the restriction $x[^2]$; if $p > 2$, we identify the restriction $x \mapsto x[|p|$ with the bottom operation $R^{(-|x|+1)/2}(x)$ up to a unit $\lambda_{|x|}$ for any odd degree $x$. Finally we extend the shifted Lie bracket and the restriction map to the quotient of $\text{Free}^{\mathcal{P}} \text{Free}^{s\text{Lie}^p}(M)$ by the above identification, subject to the conditions in Definition 5.16.

Note that when $p > 2$, given an $F_p$-module $M$ with basis $\{x_1, \ldots, x_k\}$, a basis for $\text{Free}^{s\text{Lie}^p}(M)$ is given by

$$\{v\} \cup \{u, u[^p], (u[^p])[^p], \ldots\},$$

where $u$ ranges over shifted brackets represented by Lyndon words in letters $x_1, \ldots, x_k$ with odd degree, and $v$ those with even degree. (See, for instance, [BKS05, section 2].) Proposition 4.9 and 4.15 immediately imply the following.

**Corollary 5.16.** The canonical map of $\mathcal{P}$-$s\text{Lie}^p$-algebras

$$\alpha : \text{Free}^{s\text{Lie}^p} \pi_\ast(A) \to \pi_\ast(\text{Lie}^p_{F_p, \mathbb{E}_\infty}(A))$$

is an isomorphism when $A$ is any direct sum of shifts of $HF_p$'s.

Hence we have identified the target category for the homotopy groups of spectral partition Lie algebras and the reduced TAQ cohomology of any $\mathbb{E}_\infty$-$HF_p$-algebra.

6. OPERATIONS ON MOD $p$ TAQ COHOMOLOGY

In the last section, we record a computation of all natural operations on the mod $p$ TAQ cohomology $\text{TAQ}^\ast(-, \mathbb{S}; HF_p)$ of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras, as well as determining their relations. The results in this section are largely inspired by conversations with Tyler Lawson.

Recall from [Law20, 1.8] that the mod $p$ TAQ homology of an $\mathbb{E}_\infty$-$\mathbb{S}$-algebra $R$ can be computed by

$$\text{TAQ}_\ast(R, \mathbb{S}; HF_p) \simeq \pi_\ast(\bar{\text{Bar}}_\ast(HF_p \otimes \text{id}, \mathbb{E}_\infty, R))).$$

For any $\mathbb{E}_\infty$-$\mathbb{S}$-algebra $A$, the reduced mod $p$ TAQ cohomology of an $\mathbb{E}_\infty$-$\mathbb{S}$-algebra $\mathbb{S} \oplus A$ is the same as the mod $p$ TAQ cohomology groups

$$\text{TAQ}^\ast(A, \mathbb{S}; HF_p) := [\Sigma^{-n}\bar{\text{Bar}}_\ast(\text{id}, \mathbb{E}_\infty, A), HF_p]_{Sp}. $$

When $A$ is of finite type, $\text{TAQ}^\ast(A, \mathbb{S}; HF_p) \simeq \pi_{-\ast}(\mathbb{D}[\bar{\text{Bar}}_\ast(HF_p \otimes \text{id}, \mathbb{E}_\infty, A)])$. Since all operations vanish on the unit of the mod $p$ TAQ cohomology except for multiplication by units, we will again compute operations on $\text{TAQ}^\ast(-, \mathbb{S}; HF_p)$ by throwing away the base point and computing the dual bar spectral sequence on representing objects.

**Corollary 6.1.** Unary operations on a degree $j$ cohomology class in the reduced mod $p$ $\mathbb{S}$-linear TAQ cohomology $\text{TAQ}^\ast(A, \mathbb{S}; HF_p)$ of $\mathbb{E}_\infty$-$\mathbb{S}$-algebras $A$ are parametrized by the free $\mathcal{P}$-$s\text{Lie}^p$-algebra $\text{Free}^{s\text{Lie}^p}(\Sigma^{-j}A)$, where $A$ is the mod $p$ Steenrod algebra with homological grading.

In general, for any tuple $(i_1, \ldots, i_k)$, the $k$-ary cohomology operations

$$\prod_{i=1}^{k} \text{TAQ}^\ast(\Sigma^{-i_j}A) \to \text{TAQ}^\ast(\Sigma^{-i_k}A)$$

for $i_1 < \ldots < i_k$.
away from the unit are parametrized by the homological degree $-m$ part of 
\[ \text{Free}^{\text{Lie}_p}(\Sigma^{-i} \mathcal{A} \oplus \cdots \oplus \Sigma^{-k} \mathcal{A}) . \]

**Proof.** The representing objects for the mod $p$ TAQ cohomology functor $\text{TAQ}^\ast (-, \mathbb{F}_p; H\mathbb{F}_p)$ are the trivial square-zero extensions $\mathbb{F}_p \oplus \Sigma^i H\mathbb{F}_p \oplus \cdots \oplus \Sigma^m H\mathbb{F}_p$. To compute the unary operations, we plug in the trivial algebras $\mathbb{F}_p \oplus \Sigma H\mathbb{F}_p$. There is a base change formula
\[ \text{TAQ}(R, \mathbb{F}_p; H\mathbb{F}_p) \simeq \text{TAQ}(R \otimes H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p) , \]
so unary operations on a degree $j$ cohomology class are parametrized by the reduced mod $p$ TAQ cohomology $\text{TAQ}(\Sigma H\mathbb{F}_p \otimes H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p)$.

It follows from the limiting case of Proposition 4.9 and 4.15 that the dual bar spectral sequence takes the form
\[ E_{2}^{s,t} = \pi_s(\text{Bar}_*(\text{id}, \text{Poly}_R, \pi_*((\Sigma H\mathbb{F}_p \otimes H\mathbb{F}_p))^\vee)) \Rightarrow \text{TAQ}^{-s-t}(\Sigma H\mathbb{F}_p \otimes H\mathbb{F}_p, H\mathbb{F}_p; H\mathbb{F}_p) \]
and collapses on the $E^{2}$-page. Hence we deduce that
\[ E_{s,t}^{\infty} \cong E_{s,t}^{2} \cong \text{Free}^{\text{Lie}_p}(\Sigma^{-j} \mathcal{A}) \cong \text{Free}^{\text{Lie}_p}(\Sigma^{-j} \mathcal{A}), \quad p = 2 , \]
\[ E_{s,t}^{\infty} \cong E_{s,t}^{2} \cong \text{Free}^{(R')} \text{Free}^{\text{Lie}_p}(\Sigma^{-j} \mathcal{A}) \cong \text{Free}^{\text{Lie}_p}(\Sigma^{-j} \mathcal{A}), \quad p > 2 . \]
The computation for $k > 1$ is similar. \hfill \square

The Steenrod operations commute with the bracket via the usual Cartan formula
\[ \begin{align*}
Sq^a[x, y] &= \sum_i [Sq^i(x), Sq^{-i}(y)], \text{ for } p = 2 , \\
P^a[x, y] &= \sum_i [P^i(x), P^{-i}(y)], \quad \beta P^a[x, y] = \sum_i ([\beta P^i(x), P^{-i}(y)] + [P^i(x), \beta P^{-i}(y)]) 
\end{align*} \]
for $p > 2$.

Finally we deduce the relations between the Steenrod operations and the unary $\mathbb{F}_p$-linear TAQ cohomology operations.

**Proposition 6.2.** The Steenrod operations commute with unary $\mathbb{F}_p$-linear TAQ cohomology operations $R^i$ via the Nishida relations on mod $p$ cohomology, i.e.,
\[ \begin{align*}
Sq^a R^{a+1}[x] &= \sum_{i} \binom{j-c}{a-2c} R^{a+j+1-c} Sq^i(x) + \sum_{l<k, l+k=a} [Sq^i(x), Sq^k(x)] , \\
Sq^a R^b(x) &= \sum_{i} \binom{b-1-c}{a-2c} R^{a+b-c} Sq^i(x), \quad b > -|x| + 1 
\end{align*} \]
for $p = 2$. For $p > 2$ we have
\[ \begin{align*}
P^a \beta R^i(x) &= (-1)^{n-i} \sum_{i} \binom{j-i(p-1)}{n-p i} \beta R^{a+j-i} P^i(x) \\
&\quad + (-1)^{n-i} \sum_{i} \binom{j-i(p-1)-1}{n-p i-1} R^{n+j-i} \beta P^i(x) , \\
P^a R^i(x) &= (-1)^{n-i} \sum_{i} \binom{j-i(p-1)}{n-p i} R^{n+j-i} P^i(x) 
\end{align*} \]
for all $2j > -|x| + 1$, as well as
\[
P^i R^j(x) = (-1)^{n-i} \sum_{i} \binom{(j-i)(p-1) - 1}{n-pi} R^{n+j-i} P^i(x)
+ \frac{1}{4^{k+1}} \sum_{I, \sigma \in \Sigma_p, \sigma(1) = 1} \left[ \cdots \left[ P^i_{\sigma(1)}(x), P^i_{\sigma(2)}(x), \ldots, P^i_{\sigma(p)}(x) \right] \right]
\]
when the degree of $x$ is odd and $2j = -|x| + 1$, where the bracket term sums over all nondecreasing sequences $I = (0 \leq i_1 \leq i_2 \leq \ldots \leq i_p)$ with $i_1 + i_2 + \cdots + i_p = n$ for $p > 2$.

Recall that $\lambda_{|x|}$ is the unit by which bottom operation on an odd degree class $x$ differs from the restriction $x[p]$ on $x$, cf. Lemma 5.12.

**Remark 6.3.** Note that the commuting relations between the Steenrod operations and the TAQ cohomology operations $R^i$ coincide with the Adem relations for Steenrod algebras, thereby reinforcing the heuristics that the operations $R^i$ are extended Steenrod operations.

**Proof of Proposition 6.2.** Since the operations $R^i = (Q^{|i-1|})^*$ come from the linear dual of the Dyer-Lashof operations $Q^{|i-1|}$, the Steenrod operations commute with $R^i$ via the Nishida relations on cohomology. When $p = 2$ the relations are worked out explicitly, for example, by Miller in [Mil16]. The Nishida relations for applying a Steenrod operation to the bottom operation on $x$ involves an extra bracket term because the bottom operation is the restriction on $x$.

For $p > 2$, the Nishida relations on cohomology can be read off from Theorem 3 and its corollary in Nishida’s original paper [Nis68]:
\[
P^n \beta R^i = (-1)^{n-i} \sum_{i} \binom{(j-i)(p-1) - 1}{n-pi} \beta R^{n+j-i} P^i + (-1)^{n-i} \sum_{i} \binom{(j-i)(p-1) - 1}{n-pi - 1} R^{n+j-i} \beta P^i,
\]
\[
P^n R^i = (-1)^{n-i} \sum_{i} \binom{(j-i)(p-1) - 1}{n-pi - 1} R^{n+j-i} P^i.
\]

Analogous to the case $p = 2$, when $x$ is a class in odd degree, the Nishida relations for the steenrod action on the bottom class $P^0 R^{(-|x|+1)/2}(x)$ involve extra bracket terms since $\lambda_{|x|} R^{(-|x|+1)/2}(x)$ is the restriction on $x$.

In order to determine the extra bracket terms, we need an explicit expression for the restriction map on an odd class. In the setting of unshifted graded $\mathbb{F}_p$-modules, this is worked out by Fresse in [Fre00, Remark 1.2.8]. Note that there is an embedding of the Lie operad into the associative operad Assoc. Furthermore, there is an identity
\[
\sum_{\sigma \in \Sigma_p} X_{\sigma(1)} \cdots X_{\sigma(p)} = \sum_{\sigma \in \Sigma_p, \sigma(1) = 1} \langle \cdots \langle X_{\sigma(1)}, X_{\sigma(2)} \rangle, X_{\sigma(3)} \cdots \rangle, X_{\sigma(p)} \rangle
\]
in the associative operad, where $\langle x, y \rangle = xy - yx$ is the commutator. For $x \in V$ in even degree, the $p$th power on $x$ is given by
\[
\sum_{\sigma \in \Sigma_p} X_{\sigma(1)} \cdots X_{\sigma(p)} \otimes x^{\otimes p} \in (\text{Assoc}(p) \otimes V^{\otimes p})_{\Sigma_p} \cong (\text{Assoc}(p) \otimes V^{\otimes p})_{\Sigma_p}.
\]

Using the identity (7), we can pull back the $p$th power on $x$ along the embedding
\[
(Lie(p) \otimes V^{\otimes p})_{\Sigma_p} \hookrightarrow (\text{Assoc}(p) \otimes V^{\otimes p})_{\Sigma_p}.
\]
The resulting element is the restriction on $x$ in the free restricted Lie algebra on $V$, i.e.,
\[
x^{[p]} = \left( \sum_{\sigma \in \Sigma_p, \sigma(1) = 1} \left[ \cdots \left[ X_{\sigma(1)}, X_{\sigma(2)} \right], \cdots \right], X_{\sigma(p)} \right) \otimes x^{\otimes p} \in (\text{Lie}(p) \otimes V^{\otimes p})_{\Sigma_p}.
\]
Since we are working with shifted graded \( \mathbb{F}_p \)-modules, the commutator in the shifted graded associative algebra is \( [x,y] = xy - (-1)^{|x||y|-|x||y|}yx \). If \( x, y \) are both in odd degrees, then \( [x,y] = xy - yx \). Hence the identity (7) pulls back to the restriction map (8) on an odd class \( x \) in the free shifted graded restricted Lie algebra over \( \mathbb{F}_p \). Now we apply the Steenrod operation \( \text{P}^\mu \) to the \( \mu \)th power on \( x \) and use the Cartan formula. Note that the Steenrod operations \( \text{P}^\mu \) raises degree by an even number, so none of the signs are altered. Pulling back to the free shifted restricted Lie algebra, we deduce that the bracket terms in the Nishida relation for \( \text{P}^\mu R^{(1)}(-|x|+1)/2(x) \) consists of terms

\[
\sum_{\sigma \in \Sigma_\mu, |\sigma(1)| = 1} \left[ [\cdots [\text{P}^{i_{\sigma(1)}}(x), \text{P}^{i_{\sigma(2)}}(x)], \text{P}^{i_{\sigma(3)}}(x)] \cdots , \text{P}^{i_{\sigma(\mu)}}(x) \right]
\]

for all nondecreasing sequences \( 0 \leq i_1 \leq i_2 \leq \ldots \leq i_\mu \) with \( i_1 + i_2 + \cdots + i_\mu = n \). \( \square \)

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