DUAL CANONICAL BASES FOR THE QUANTUM SPECIAL LINEAR GROUP AND INVARIANT SUBALGEBRAS

HECHUN ZHANG AND R. B. ZHANG

ABSTRACT. A string basis is constructed for each subalgebra of invariants of the function algebra on the quantum special linear group. By analyzing the string basis for a particular subalgebra of invariants, we obtain a “canonical basis” for every finite dimensional irreducible $U_q(\mathfrak{sl}_n)$-module. It is also shown that the algebra of functions on any quantum homogeneous space is generated by quantum minors.

1. Introduction

The theory of crystal and canonical bases was initiated and developed by Kashiwara [9, 10] and Lusztig [14, 17]. Many remarkable features are known for canonical bases, among which the positivity property seems to have the deepest implications. Variations of canonical bases were also introduced and investigated in the literature. In [4], a dual basis of the canonical basis of the modified quantum enveloping algebra of type $A$ was investigated under the name global IC basis. In [12], the dual canonical basis of $U_q(\mathfrak{n}_+)$ was constructed by using the so-called quantum shuffles, and was shown to be related to the representation theories of Hecke algebras and quantum affine algebras [13] (see also [19]). In [1] Berenstein and Zelevinsky conjectured a multiplicative property, which states that two dual canonical basis elements $b_1, b_2$ of $U_q(\mathfrak{n}_+)$ $q$-commute if and only if $b_1b_2 = q^mb$ for some $b$ in the dual canonical basis and some integer $m$. The conjecture was studied by using Hall algebra techniques in reference [21]. It was observed that a large portion of the dual canonical basis enjoyed the multiplicative property, see also [3]. However, counter-examples are given to Berenstein and Zelevinsky’s conjecture in [11] by finding some so-called imaginary vectors. Reference [11] also introduced the notion of string bases, which was designed to realize the so-called good bases. See [18] for the existence of good bases.

In [22] the dual basis of the canonical basis of the modified quantum enveloping algebra was constructed and the case of type $A$ was extensively investigated. It was shown that the dual canonical basis is invariant under the multiplication of certain $q$ central elements. More importantly, it was shown in [7] that the product of any two dual canonical basis elements is a $\mathbb{Z}_+[q, q^{-1}]$ combination of elements of the dual canonical basis, establishing multiplicative positivity of the dual canonical basis (see Theorem 2.6).

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In this note we use the dual canonical basis constructed in [22] for the algebra of functions $O_q(SL(n))$ on the quantum special linear group to study its subalgebras of invariants. Recall that $O_q(SL(n))$ admits two natural left actions of the quantized universal enveloping algebra $U_q(\mathfrak{sl}(n))$ respectively corresponding to the left and right translations familiar in the classical context of Lie groups. Given any subalgebra $C$ of $U_q(\mathfrak{sl}(n))$ which is at the same time a two-sided co-ideal, one can show [6] that the subspace of $O_q(SL(n))$ invariant under left translation with respect to $C$ forms a subalgebra. Such invariant subalgebras of $O_q(SL(n))$ include the algebras of functions on quantum homogeneous spaces [20, 6] as special cases.

A string basis (in the sense of [1]) is constructed for each subalgebra of invariants in this paper. In the case when $C$ is $U_q(n_-)$, we obtain from the string basis a “canonical basis” for every irreducible finite dimensional $U_q(\mathfrak{sl}(n))$-module (see Section 4 for details). When the subalgebra of invariants defines a quantum homogeneous space, we show that it is generated by quantum minors of some particular form.

The arrangement of the paper is as follows. In Section 2, we recall the construction of the dual canonical basis of $O_q(SL(n))$ in [22]. In Section 3, we introduce a new type of bases for $O_q(SL(n))$, which are suitable for studying the actions of the Kashiwara operators. Section 4 contains the main results of the paper. We first show that any subalgebra of invariants is spanned by a subset of the dual canonical basis of $O_q(SL(n))$ [This result is stated in a more precise manner in Theorem 4.4]. We then derive from this fact the results discussed in the last paragraph.

We mention that our approach to the canonical bases of finite dimensional irreducible $U_q(\mathfrak{sl}(n))$-modules is very different from the usual approach taken in canonical basis theory. Here we first realize every finite dimensional irreducible $U_q(\mathfrak{sl}(n))$-module by using the quantum Borel-Weil theorem, then construct a canonical basis for it by exploiting Theorem 4.4.

2. The Construction of the Basis of $O_q(M(n))$

Throughout the paper, the base field is $\mathbb{Q}(q)$, where $q$ is an indeterminate over the rational numbers. The coordinate algebra $O_q(M(n))$ of the quantum matrix is an associative algebra, generated by elements $x_{ij}, i, j = 1, 2, \cdots, n$, subject to the following defining relations:

\begin{equation}
\begin{aligned}
x_{ij}x_{ik} & = q^2x_{ik}x_{ij} \text{ if } j < k, \\
x_{ij}x_{kj} & = q^2x_{kj}x_{ij} \text{ if } i < k, \\
x_{ij}x_{st} & = x_{st}x_{ij} \text{ if } i > s, j < t, \\
x_{ij}x_{st} & = x_{st}x_{ij} + (q^2 - q^{-2})x_{it}x_{sj} \text{ if } i < s, j < t.
\end{aligned}
\end{equation}

For any matrix $A = (a_{ij})_{i,j=1}^n \in M_n(\mathbb{Z}_+) \ (\mathbb{Z}_+ = \{0, 1, \cdots\})$ we define a monomial $x^A$ by

\begin{equation}
x^A = \Pi_{i,j=1}^nx_{ij}^{a_{ij}},
\end{equation}
where the factors are arranged in the lexicographic order on $I(n) = \{(i, j) \mid i, j = 1, \ldots, n\}$. More explicitly,

$$x^A = x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}} x_1^{a_{21}} \cdots x_n^{a_{2n}} x_n^{a_{nn}}.$$  

Since the algebra $O_q(M(n))$ is an iterated Ore extension (see, e.g., pp11-14 in [3] for a discussion on iterated Ore extensions), the set \{ $x^A \mid A \in M_n(\mathbb{Z}_+)$ \} is a basis of the algebra $O_q(M(n))$.

Let us recall the construction of the dual canonical basis in [22]. From the defining relations (2.1) of the algebra $O_q(M(n))$, it is easy to show the following lemma.

**Lemma 2.1.** (1) The mapping

$$- : x_{ij} \mapsto x_{ij}, \quad q \mapsto q^{-1}$$

extends to an algebra anti-automorphism of $O_q(M(n))$ regarded as an algebra over $\mathbb{Q}$.

(2) The mapping

$$\sigma : x_{ij} \mapsto x_{ji}$$

extends to an algebra automorphism of $O_q(M(n))$ regarded as an algebra over $K = \mathbb{Q}(q)$.

**Remark 2.2.** For the relation between the bar action in the present form and the bar action in the papers by Kashiwara, see [22].

Given $A = (a_{ij})_{n \times n} \in M_n(\mathbb{Z}_+)$, we define the row sum $ro(A)$ and the column sum $co(A)$ of the matrix, respectively, by

$$ro(A) = (\sum_j a_{1j}, \ldots, \sum_j a_{nj}) = (r_1, r_2, \ldots, r_n),$$

$$co(A) = (\sum_j a_{j1}, \ldots, \sum_j a_{jn}) = (c_1, c_2, \ldots, c_n).$$

Let $M = (m_{ij}) \in M_n(\mathbb{Z}_+)$. If $m_{ij}m_{st} \geq 1$ for two pairs of indices $i, j$ and $s, t$ satisfying $i < s, j < t$, we define a new matrix $M' = (m'_{uv}) \in M_n(\mathbb{Z}_+)$ with

$$m'_{ij} = m_{ij} - 1, \quad m'_{st} = m_{st} - 1,$$

$$m'_{it} = m_{it} + 1, \quad m'_{st} = m_{st} + 1,$$

$$m'_{uv} = m_{uv}, \quad \text{for all other entries.}$$

We say that the matrix $M'$ is obtained from the matrix $M$ by a $2 \times 2$ sub-matrix transformation. Using this we may define a partial order on the set $M_n(\mathbb{Z}_+)$ such that $M \prec N$ if $M$ can be obtained from $N$ by a sequence of $2 \times 2$ sub-matrix transformations. Note that we have $ro(M) = ro(N)$, $co(M) = co(N)$.

From the defining relations (2.1) of the algebra $O_q(M(n))$, we have

$$\overline{x^A} = E(A)x^A + \sum_B c_{BA}x^B,$$
where
\[ E(A) = q^{-2(\sum_i \sum_{j>k} a_{ij}a_{ik} + \sum_i \sum_{j>k} a_{ji}a_{ki})} \]
and \( c_{BA} \in \mathbb{Z}[q, q^{-1}] \), which is nonzero only if \( B < A \).

Let \( D(A) = q^{-\sum_i \sum_{j>k} a_{ij}a_{ik} - \sum_i \sum_{j>k} a_{ji}a_{ki}} \) and let \( x(A) = D(A)x^A \). Set
\[ \mathcal{L}^* = \bigoplus_{A \in M_n(\mathbb{Z})} \mathbb{Z}[q]x(A). \]

In [22], a basis of the algebra \( \mathcal{O}_q(M(n)) \) is constructed which has the following property:

**Theorem 2.3.** There is a unique basis \( B^* = \{ b(A) | A \in M_n(\mathbb{Z}) \} \) of \( \mathcal{L}^* \) such that:
1. \( \overline{b(A)} = b(A) \) for all \( A \).
2. \( b(A) = x(A) + \sum_{B \prec A} c_{BA} x(B) \) where \( c_{BA} \in q\mathbb{Z}[q] \).

**Definition 2.4.** The basis \( B^* \) of the algebra \( \mathcal{O}_q(M(n)) \) is called the dual canonical basis of \( \mathcal{O}_q(M(n)) \).

The quantum determinant \( \det_q \) is defined in the usual way by:
\[ \det_q = \sum_{\sigma \in S_n} (-q^2)^{l(\sigma)} x_{1\sigma(1)}x_{2\sigma(2)} \cdots x_{n\sigma(n)}. \]

It is known that \( \det_q \) is a central and group-like element of the algebra \( \mathcal{O}_q(M(n)) \).

Let \( m \leq n \) be a positive integer. Let \( I = \{i_1, i_2, \cdots, i_m\} \) and \( J = \{j_1, j_2, \cdots, j_m\} \) be any two subsets of \( \{1, 2, \cdots, n\} \), each having cardinality \( m \). By examining the defining relations (2.1) one can see that the subalgebra of \( \mathcal{O}_q(M(n)) \) generated by the elements \( x_{ij} \) with \( i \in I, j \in J \), is isomorphic to \( \mathcal{O}_q(M(m)) \), so we can talk about its determinant. Such a determinant is called a quantum minor, and will be denoted by \( \det_q(I, J) \).

In [22], it was also proved that

**Proposition 2.5.** All of the quantum minors are dual canonical basis elements. Moreover, the basis \( B^* \) is invariant under the multiplication of the quantum determinant. Furthermore, \( \sigma(B^*) = B^* \).

The algebra of functions \( \mathcal{O}_q(SL(n)) \) on the quantum special linear group will be identified with \( \mathcal{O}_q(M(n))/\langle \det_q - 1 \rangle \). For convenience of reference, we denote by
\[ T : \mathcal{O}_q(M(n)) \to \mathcal{O}_q(M(n))/\langle \det_q - 1 \rangle \]
the canonical map. Then the image \( \bar{B}^* = T(B^*) \) of the set \( B^* \) forms a canonical basis of \( \mathcal{O}_q(SL(n)) \), which is called the dual canonical basis of \( \mathcal{O}_q(SL(n)) \).

In [7], it was proved that the dual canonical basis \( \bar{B}^* \) enjoys a remarkable property, the positivity of the multiplication.

**Theorem 2.6.** For any \( b_1, b_2 \in \bar{B}^* \), we have
\[ b_1b_2 = \sum_{b \in \bar{B}^*, n \in \mathbb{Z}} c_{b_1,b_2,b,n} q^nb, \]
where \( c_{b_1,b_2,b,n} \in \mathbb{Z}_+ \) are zero except for finitely many \( b,n \).
The following formulas are needed for the computation involving the Kashiwara operators. Define that \( [m] = \frac{q^{-4m} - 1}{q-1} \).

**Lemma 2.7.** For any \( i < k, j < l \),

\[
x_{kl} x^s_{ij} = x^s_{ij} x_{kl} + (q^{2-4s} - q^{-2}) x_{il} x^{s-1}_{kj},
\]

\[
x_{kl} x^s_{ij} = x^s_{ij} x_{kl} + (q^{2-4s} - q^{-2}) x_{il} x^{s-1}_{kj}.
\]

Furthermore,

\[
(2.5) \quad (x_{ij} x_{kl} - q x_{il} x_{kj})^s = \sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^m x_{kj}^m x_{kl}^{s-m}.
\]

**Proof:** We prove the third equation only, using induction on \( s \). The case with \( s = 1 \) is clearly true. Assume that the equation holds for \( s \). Then

\[
(x_{ij} x_{kl} - q x_{il} x_{kj})^{s+1} = (x_{ij} x_{kl} - q x_{il} x_{kj}) \sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^m x_{kj}^m x_{kl}^{s-m}.
\]

The right hand side of the equation can be express as

\[
\sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^m x_{kj}^m x_{kl}^{s-m} + \sum_{m=0}^{s} (-q^2)^{m+1} \binom{s}{m} \frac{q^{4m+1(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^{m+1} x_{kj}^m x_{kl}^{s-m} + \sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m+1(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^m x_{kj}^{m+1} x_{kl}^{s-m} + \sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m+2(m-s)}}{q^{-4}} x_{ij}^{s-m} x_{il}^m x_{kj}^m x_{kl}^{s-m}.
\]

The coefficient of \( x_{ij}^{s+1-m} x_{il}^m x_{kj}^m x_{kl}^{s+1-m} \) is

\[
(-q^2)^m \binom{s}{m} \frac{q^{4m(m-s-1)}}{q^{-4}} + (-q^2)^{m+1} \binom{s}{m} \frac{q^{4m+1(m-s)}}{q^{-4}} + (-q^2)^m \binom{s}{m} \frac{q^{4m+1(m-s-1)}}{q^{-4}} + (-q^2)^m \binom{s}{m} \frac{q^{4m+2(m-s-1)}}{q^{-4}}
\]

which can be simplified into \( \binom{s+1}{m} q^{4m(m-s-1)} \).

**Remark 2.8.** The third equation in the above lemma can be re-written as

\[
\left[ b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^s = \sum_{m=0}^{s} (-q^2)^m \binom{s}{m} \frac{q^{4m}}{q^{-4}} x \begin{pmatrix} s-m & m \\ m & s-m \end{pmatrix}.
\]
3. Kashiwara operators

Let $A_{n-1} = (a_{ij})_{i,j=1}^{n-1}$ be the Cartan matrix of type $A$. The quantized enveloping algebra $U_q(A_{n-1})$ is the unital associative algebra generated by $E_i, F_i, K_i, K_i^{-1}$ $(1 \leq i \leq n-1)$ with relations

\begin{align*}
K_i K_j &= K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i \\
E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}} \\
K_i E_j &= q^{2a_{ij}} E_j K_i, \quad K_i F_j = q^{-2a_{ij}} F_j K_i \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left(1 - \frac{a_{ij}}{s}\right) \frac{q^2}{q} E_i^{1-a_{ij}-s} E_j^{s} E_i &= 0 \quad (i \neq j) \\
\sum_{s=0}^{1-a_{ij}} (-1)^s \left(1 - \frac{a_{ij}}{s}\right) \frac{q^2}{q} F_i^{1-a_{ij}-s} F_j^{s} F_i &= 0 \quad (i \neq j).
\end{align*}

As is well known, $U_q(A_{n-1})$ is a Hopf algebra. We take the following co-multiplication $\Delta$, the co-unit $\epsilon$, and the antipode $S$ respectively defined by

\begin{align*}
\Delta(K_i) &= K_i \otimes K_i, \\
\Delta(E_i) &= E_i \otimes 1 + K_i^2 \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-2} + 1 \otimes F_i, \\
\epsilon(K_i) &= 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0, \\
S(K_i) &= K_i^{-1}, \quad S(E_i) = -K_i^{-2} E_i, \quad S(F_i) = -F_i K_i^2
\end{align*}

(3.1)

There are two natural left actions of $U_q(A_{n-1})$ on the quantized function algebra $O_q(SL(n))$, which correspond to left and right translations in the classical setting. These actions are respectively defined, for all $x \in U_q(A_{n-1}), f \in O_q(SL(n))$, by

\begin{align*}
R_x(f) &= \sum_{(f)} f_{(1)} < f_{(2)}, x >, \\
L_x(f) &= \sum_{(f)} < f_{(1)}, S(x) > f_{(2)},
\end{align*}

where $S$ is the antipode of $U_q(A_{n-1})$. Here we have used Sweedler’s notation $\Delta(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$. Note that $L_x$ and $R_x$ act on $O_q(SL(n))$ as generalized derivations. More precisely, for $f, g \in O_q(SL(n))$,

\begin{align*}
L_x(fg) &= \sum_{(x)} L_{x_{(1)}}(f) \cdot L_{x_{(2)}}(g), \\
R_x(fg) &= \sum_{(x)} R_{x_{(1)}}(f) \cdot R_{x_{(2)}}(g).
\end{align*}

Furthermore, the two left actions commute.

A quantum analogue of the Peter-Weyl theorem states the following:
Theorem 3.1. As a left $L_{U_q(A_{n-1})} \otimes R_{U_q(A_{n-1})}$-module,

$$O_q(SL(n)) \cong \bigoplus_{\lambda \in P^+} L^* (\lambda) \otimes L(\lambda).$$

Any $u \otimes v \in L^* (\lambda) \otimes L(\lambda)$ can be viewed as a linear functional on $U_q(A_{n-1})$ as follows:

$$(u \otimes v)(x) = u(xv), \quad \forall x \in U_q(A_{n-1}).$$

The theorem follows from the complete reducibility of finite dimensional $U_q(A_{n-1})$-modules and the fact that all finite dimensional $U_q(A_{n-1})$-modules are direct sums of submodules of tensor powers of the natural module. In fact the theorem holds for all quantized enveloping algebras of finite dimensional simple Lie algebras. See [20] for a proof in the case of type $A$, and also see [10] for the general case.

The following observation will be useful for the remainder of the paper. There exists an algebra anti-involutions of $\omega : U_q(A_{n-1}) \rightarrow U_q(A_{n-1})$ given by

$$\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(K_i) = K_i, \quad \omega(q) = q.$$

Then $\theta = \omega \circ S^{-1}$ is an algebra involution of $U_q(A_{n-1})$. Using $\theta$ we can construct another left action of $U_q(A_{n-1})$ on $O_q(SL(n))$:

$$U_q(A_{n-1}) \otimes O_q(SL(n)) \rightarrow O_q(SL(n)), \quad x \otimes f \mapsto L_{\theta(x)}(f).$$

This left action is much simpler than $L$ itself. Let us define Kashiwara operators for this left action. To this end, we need to have a suitable basis.

Proposition 3.2. There exists an basis of the algebra $O_q(M(n))$ consisting of the elements of the form

$$q^m x \begin{pmatrix} 0 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} & 0 & \cdots & 0 \end{pmatrix} \times \Pi M_{ij} \Pi_{i \geq j} q^{a_{ij} \cdot j} x_{ij}^a,$$

where $a_{ij} \in \mathbb{Z}^+$ for all $i, j$, and the $2 \times 2$ quantum minors $M_{ij}$ are of the form $\det_q(\{1,2\}, \{i,j\})$. The product $\Pi M_{ij}$ of quantum minors is arranged according to the lexicographic order, namely, $M_{ij} \geq M_{st}$ if $j > t$ or $j = t$ and $i \geq s$. The integer $m$ is uniquely determined by the entries $a_{ij}$ and the $2 \times 2$ minors. The transition matrix between this new basis and the PBW basis consisting of the modified monomials is of the form:

$$\begin{pmatrix} 1 & \cdots & q \mathbb{Z}[q] \\ 0 & 1 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Proof: We will build the basis step by step. First note that the elements $x_{ij}$ for $i \geq 3$ are annihilated by the left actions of both $E_1$ and $F_1$, and so they will contribute nothing under the Kashiwara operators which will be defined later. Hence, we may put aside the elements $x_{ij}$ for $i \geq 3$ and only consider the elements $x_{i1}, x_{i2}$ for $l = 1, 2, \cdots, n$.

For a PBW basis element $x(A)$,
• if \( a_{1,n-1}a_{2,n} = 0 \), we just keep it unchanged.
• If \( a_{1,n-1} \geq a_{2,n} \geq 1 \), we can rewrite the modified monomial \( x(A) \) as
  \[
  x(A) = q^{2a_{n}(a_{1,n}+a_{2,n}-1)}D(A)x_{11}^{a_{11}} \cdots x_{1,n-2}^{a_{1,n-2}}x_{1,n-1}^{a_{1,n-1}}x_{1,n}^{a_{1,n}}
  \times x_{21}^{a_{21}} \cdots x_{2,n-2}^{a_{2,n-2}}x_{2,n-1}^{a_{2,n-1}}M_{n-1,n}^{a_{2,n-1}} + \sum_{B<A} c_{BA}x(B).
  \]

An elementary computation shows that \( c_{BA} \in q\mathbb{Z}[q] \) with the help of equation (2.5).

• If \( 1 \leq a_{1,n-1} < a_{2,n} \), we can rewrite the modified monomial \( x(A) \) as
  \[
  x(A) = q^{a_{1,n-1}(a_{1,n}+a_{2,n}-1)}D(A)x_{11}^{a_{11}} \cdots x_{1,n-2}^{a_{1,n-2}}x_{1,n-1}^{a_{1,n-1}}x_{1,n}^{a_{1,n}}
  \times x_{21}^{a_{21}} \cdots x_{2,n-2}^{a_{2,n-2}}x_{2,n-1}^{a_{2,n-1}}M_{n-1,n}^{a_{1,n-1}} + \sum_{B<A} c_{BA}x(B).
  \]

By using equation (2.5), a simple computation shows that \( c_{BA} \in q\mathbb{Z}[q] \).

This means that we construct a new basis by replacing the sub-word \( x_{1,n-1}x_{2,n} \) by the \( 2 \times 2 \) minor \( M_{n-1,n} \). Clearly, the transition matrix between the new basis and the PBW basis of modified monomials is of the form

\[
\begin{pmatrix}
1 & \cdots & q\mathbb{Z}[q] \\
0 & 1 & \cdots \\
\cdots & \cdots & \cdots \\
0 & \cdots & 1
\end{pmatrix}.
\]

Repeat the above procedure according to the lexicographic order, replacing the sub-word \( x_{1,j}x_{2,k} \) by the \( 2 \times 2 \) minor \( M_{jk} \), we get the desired basis. Note that in each step the transition between the old basis and the new one is of the form

\[
\begin{pmatrix}
1 & \cdots & q\mathbb{Z}[q] \\
0 & 1 & \cdots \\
\cdots & \cdots & \cdots \\
0 & \cdots & 1
\end{pmatrix},
\]

the transition matrix between the PBW basis of modified monomials and our final basis is composition of the matrices of the above form. This completes the proof. \( \square \)

Denote by \( x(A)_i \) the resulting basis element obtained from \( x(A) \). By doing the same thing to the \( i \)th and \( (i+1) \)th rows, we get a basis \( \{ x(A)_i | A \in M_n(\mathbb{Z}_+) \} \). The order of \( M_n(\mathbb{Z}_+) \) gives an order on the basis

\[
B_i := \{ x(A)_i | A \in M_n(\mathbb{Z}_+) \}.
\]

From the transition matrix between \( B_i \) and the PBW basis of modified monomials, we get

\[
\overline{x(A)}_i = x(A)_i + \sum_{B<A} e_{BA}x(B)_i
\]

with \( e_{BA} \in \mathbb{Z}[q, q^{-1}] \).
Remark 3.3. Since the transition matrices of all these various bases are all of the form

\[
\begin{pmatrix}
1 & \cdots & q \mathbb{Z}[q] \\
0 & 1 & \cdots \\
\cdots & \cdots & \cdots \\
0 & \cdots & 1
\end{pmatrix},
\]

we have \( \mathcal{L}^* = \oplus \mathbb{Z}[q] x(A_i) = \oplus \mathbb{Z}[q] x(A_i), \) for all \( i. \) The canonical basis \( B^* \) can be constructed starting from any basis \( \{ x(A_i) | A \in M_n(\mathbb{Z}_+) \} \) for any \( i. \) More explicitly, for any matrix \( A, \) the element \( b(A) \) can be written as

\[
b(A) = x(A)_i + \sum_{B < A} c_{B,A,i} x(B)_i
\]

with \( c_{B,A,i} \in \mathbb{Q}[q]. \) This again justifies the word canonical.

Let us consider Kashiwara operators. Note that when examining the actions of \( \tilde{E}_1 \) and \( \tilde{F}_1 \) on \( x(A)_1, \) we can ignore the \( 2 \times 2 \) minors and those \( x_{ij} \) for \( i \geq 3 \) in \( x(A)_1. \) Now the Kashiwara operators \( \tilde{E}_1 \) and \( \tilde{F}_1 \) for the left action are defined as follows:

\[
\tilde{E}_1 \left( x \begin{pmatrix} 0 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} & 0 & \cdots & 0 \end{pmatrix} \right) = \sum_k q^{\sum_{t=1}^{k-1} 2a_{2t} - x} \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2k-1} & a_{2k} - 1 & a_{2r} & 0 & \cdots & 0 \end{pmatrix}
\]

where the summation is over all the \( k \) such that \( a_{2k} \geq 1. \) In particular, if \( a_{2r} \geq 1, \) the right hand side of the above equation will contain the term

\[
q^{\sum_{t=1}^{r-1} 2a_{2t} - x} \begin{pmatrix} 0 & \cdots & 0 & a_{1r} + 1 & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} - 1 & 0 & \cdots & 0 \end{pmatrix}.
\]

Similarly, we also have

\[
\tilde{F}_1 \left( x \begin{pmatrix} 0 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} & 0 & \cdots & 0 \end{pmatrix} \right) = \sum_k q^{\sum_{t=k}^{r} 2a_{2t} - x} \begin{pmatrix} 0 & \cdots & 0 & a_{1r} & a_{1,r+1} & \cdots & a_{1k} - 1 & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,r-1} & a_{2r} & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}
\]

where the summation is over the \( k \) such that \( a_{1k} \geq 1. \)

Proposition 3.4. For any \( b(A) \in B^* \) and Kashiwara operators \( \tilde{E}_i \) and \( \tilde{F}_i, \) \( \tilde{E}_i(b(A)) = 0 \) if and only if \( \tilde{E}_i(x(A)_i) = 0, \) and \( \tilde{F}_i(b(A)) = 0 \) if and only if \( \tilde{F}_i(x(A)_i) = 0. \)

Proof: The element \( b(A) \) can be written as

\[
b(A) = x(A)_i + \sum_{B < A} c_{B,A,i} x(B)_i.
\]

(3.3)
We only prove the statement for $\tilde{E}_1$, the other cases are pretty much the same. The equation $\tilde{E}_1 x(A)_1 = 0$ implies that $x(A)_1$ is of the form:
\[
q^s x \left( \begin{array}{cccccc}
0 & \cdots & 0 & a_{11} & \cdots & a_{1n} \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{array} \right) \times \text{some } 2 \times 2 \text{ minors } \times \Pi_{i \geq 3,j} x_{ij}^{a_{ij}}.
\]

Note that if we set $i = 1$ in equation \(3.3\), then each $x(B)_1$ on the right hand side is associated with a $2 \times n$ matrix with the second row being zero. This can be seen by considering $\tilde{\pi}(A)_1$. Using the commutation relations among the $x_{ij}$ we see that $\tilde{\pi}(A)_1$ can be expressed as a combination of such $x(B)_1$'s. Hence by the construction of the dual canonical basis, all of the terms in $b(A)$ are of the form $x(B)_1$ associated with a $2 \times n$ matrix with the second row being zero. Such terms are all annihilated by $\tilde{E}_1$.

Conversely, if $\tilde{E}_1 x(A)_1 \neq 0$, then $x(A)_1$ is of the form
\[
q^s x \left( \begin{array}{cccccc}
0 & \cdots & 0 & a_{11} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2r} & 0 & \cdots & 0
\end{array} \right) \times \text{some } 2 \times 2 \text{ minors } \times \Pi_{i \geq 3,j} x_{ij}^{a_{ij}},
\]
with some $a_{2j}$ nonzero. Without losing generality, we may assume that $a_{21} \neq 0$. Then the leading term of $\tilde{E}_1 x(A)_1$ is of the form
\[
q^s x \left( \begin{array}{cccc}
1 & \cdots & 0 & a_{11} & \cdots & a_{1n} \\
a_{21} - 1 & \cdots & a_{2r} & 0 & \cdots & 0
\end{array} \right) \times \text{some } 2 \times 2 \text{ minors } \times \Pi_{i \geq 3,j} x_{ij}^{a_{ij}}.
\]

In order to obtain a term of this form by applying $\tilde{E}_1$ to some $x(B)_1$, then $x(B)_1$ must be of the form
\[
q^s x \left( \begin{array}{cccc}
1 & \cdots & \cdots & a_{11} & \cdots & a_{1n} \\
a_{21} - 1 & \cdots & a_{2k} + 1 & \cdots & \cdots & \cdots
\end{array} \right) \times \text{some } 2 \times 2 \text{ minors } \times \Pi_{i \geq 3,j} x_{ij}^{a_{ij}},
\]
where $s \in \mathbb{Z}$. As we have already discussed, such terms can never appear in $b(A)$. Therefore, the leading term of $\tilde{E}_1 x(A)_1$ can not be cancelled by terms coming from $\tilde{E}_i x(B)_i$ for $B < A$. Hence $\tilde{E}_i(b(A)) \neq 0$.

The Kashiwara operators are compatible with the action \(3.2\) of Chevalley generators on the quantized function algebra in the following sense:

**Lemma 3.5.** Let $x \in \mathbb{C}_q(SL(n))$. Then $L_{\theta(E_i)}(x) = 0$ if and only if $\tilde{E}_i(x) = 0$. Similarly, $L_{\theta(F_i)}(x) = 0$ if and only if $\tilde{F}_i(x) = 0$.

**Proof:** We shall only prove the case when $i = j = 1$. The other cases are exactly the same. For convenience we simply write $L_{\theta(u)}(x)$ as $u(x)$. For a basis element
\[
x(A)_1 = q^s x \left( \begin{array}{cccc}
0 & \cdots & 0 & a_{11} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2r-1} & a_{2r} & \cdots & 0
\end{array} \right) \times \text{some } 2 \times 2 \text{ minors } \times \Pi_{i \geq 3,j} x_{ij}^{a_{ij}},
\]
both $E_1(x(A)_1) = 0$ and $\tilde{E}_1(x(A)_1) = 0$ equivalent to $a_{2i} = 0$ for all $i$.

Write $x = \sum_A c_A x(A)_1$. Suppose $E_1(x) = 0$. Choose a term $x(A)_1$ with nonzero coefficient $c_A$ and is maximal with respect to this property. If $E_1(x(A)_1) \neq 0$, the same argument as in the above Proposition shows that
the leading term in $E_1(x(A)_1)$ can not be cancelled by the rest of the terms in the expression of $E_1(x)$ which is a contradiction! Hence, all terms in the expression of $x$ are killed by $E_1$ which is equivalent to say that all terms are killed by $E_1.$

The statement for $F_1$ and $F_1$ can be proved similarly. □

From the definition of the Kashiwara operators, we can easily deduce that

**Proposition 3.6.** For any $b^* \in B^*$, if $\tilde{E}_i(b^*) \neq 0$, then there exists an element $b^*_1 \in B^*$ such that $\tilde{E}_i(b^*) = b^*_1 \mod qL^*$. If $\tilde{F}_i(b^*) \neq 0$, then there exists an element $b^*_2 \in B^*$ such that $\tilde{F}_i(b^*) = b^*_2 \mod qL^*$.

**Proof:** If $\tilde{E}_i(b^*) \neq 0$, then the coefficient of the leading term in $\tilde{E}_i(b^*) \neq 0$ is 1, which means there exist a $x(B)_1$ such that

$$\tilde{E}_i(b^*) = x(B)_1 \mod qL^*.$$  

Hence, $\tilde{E}_i(b^*) = b(B) \mod qL^*$.  

Similarly, we can prove the statement for $\tilde{F}_1$. □

**Remark 3.7.** Applying the automorphism $\sigma$, we can define the Kashiwara operators for the right action and the same statements for the left action hold for the right action.

4. INVARIANT SUBALGEBRAS AND STRING BASES

To simplify notations, we set $G = SL(n)$ and $\mathfrak{g} = \mathfrak{sl}_n$ in this section. Any subset $S$ of the generators $\{E_i, F_i, K^\pm_i \mid i = 1, 2, \ldots, n - 1\}$ generate a subalgebra $U_S$ of $U_q(\mathfrak{g})$. Note that $\theta(S)$ also generates a subalgebra $U_{\theta(S)} = \theta(U_S)$ of $U_q(\mathfrak{g})$.

**Definition 4.1.** $O_q(G)^S := \{f \in O_q(SL(n)) \mid L_x(f) = \epsilon(x)f, \forall x \in U_S\}$.

This is the subalgebra of $O_q(G)$ consisting of the elements which are invariant under the left action $L$ of $U_S$. As the left translation $L$ and right translation $R$ commute, $O_q(G)^S$ forms a left $U_q(A_{n-1})$-module under $R$. The following lemma easily follows from the definition of $O_q(G)^S$ and Lemma 3.5.

**Lemma 4.2.** If $E_i, F_j \in \theta(S)$, then

$$\tilde{E}_i(f) = 0, \quad \tilde{F}_j(f) = 0, \quad \forall f \in O_q(G)^S,$$

where $\tilde{E}_i$ and $\tilde{F}_j$ are the Kashiwara operators associated with $E_i$ and $F_j$.

In an analogous way, we shall also define the subset of dual canonical basis elements which are invariant with respect to $U_S$:

**Definition 4.3.** $(\check{B}^*)^S := \{b \in \check{B}^* \mid L_x(b) = \epsilon(x)b, \forall x \in U_S\}$.

The following theorem is the main result of this paper.

**Theorem 4.4.** The subset $(\check{B}^*)^S$ of $B^*$ forms a basis of the subalgebra of invariants $O_q(G)^S$.  

Theorem 4.6. The subspace \( U_\lambda \) of \( U \) which have weight \( \lambda \) is the subspace consisting of elements with highest weight \( \lambda \) in \( U \). 

Example: \( \mathbb{Z}_+[q, q^{-1}] \) denote the \( \mathbb{Z}_+[q, q^{-1}] \) span of \( (\bar{B}^*)^S \). Then for all \( f, g \in \langle (\bar{B}^*)^S \rangle_+ \), 

\[
fg \in \langle (\bar{B}^*)^S \rangle_+, \quad \text{and} \quad R_{E_i}(f) \in \langle (\bar{B}^*)^S \rangle_+, \forall i.
\]

The first property follows from Theorem 2.6. The second property can be seen by embedding \( B^* \) into a dual canonical basis of the strictly lower triangular subalgebra \( U_q(A_{2n-1}) \) (i.e., the subalgebra generated by the Chevalley generators \( F_i \)'s) of \( U_q(A_{2n-1}) \). The embedding was established in [7]. It is known that the dual canonical basis of \( U_q(A_{2n-1}) \) satisfies the second relation of (1.1).

Recall that in [1] Berenstein and Zelevinsky introduced the notion of a string basis, which is a canonical basis with the \( \mathbb{Z}_+[q, q^{-1}] \) span satisfying properties analogous to (1.1).

Lemma 4.5. \( (\bar{B}^*)^S \) forms a string basis for \( \mathcal{O}_q(G)^S \).

Let us now consider in some detail examples of the subset \( S \) and the associated invariant subalgebra \( \mathcal{O}_q(G)^S \).

4.1. Example: \( S = \{ F_i \mid i = 1, 2, \ldots, n-1 \} \). In this case we denote the subalgebra \( U_S \) by \( U_q(n) \), and \( \mathcal{O}_q(G)^S \) by \( \mathcal{O}_q(G)^{U_q(n)} \). Denote by \( \mathcal{O}_q(G)^{U_q(n)} \) the subspace consisting of elements with \( L_{U_q(A_{n-1})} \)-weight \(-\lambda \) (the negative of \( \lambda \)). The result below easily follows from the quantum Peter-Weyl theorem.

Theorem 4.6. The subspace \( \mathcal{O}_q(G)^{U_q(n)}_\lambda \), \( \lambda \in P_+ \), forms an irreducible left \( U_q(A_{n-1}) \)-submodule with highest weight \( \lambda \) under the action \( R \). Furthermore, we have the \( U_q(A_{n-1}) \)-module isomorphism 

\[
\mathcal{O}_q(G)^{U_q(n)} = \bigoplus_{\lambda \in P_+} \mathcal{O}_q(G)^{U_q(n)}_\lambda.
\]

The first statement of the theorem is a variation of the quantum analogue of the celebrated Borel-Weil theorem.

Let \( (\bar{B}^*)^S_{\nu} \) denote the subset of \( (\bar{B}^*)^S_{\nu} \) consisting of elements which have weight \(-\lambda \) under the \( L \)-action of \( U_q(A_{n-1}) \). Set 

\[
\langle (\bar{B}^*)^S_{\nu} \rangle_+ = \mathbb{Z}_+[q, q^{-1}](\bar{B}^*)^S_{\nu}, \quad \nu \in P_+.
\]

Since every element of \( \bar{B}^* \) has a definite weight, we have the following result.
Theorem 4.7. For every $\lambda \in P_+$, $(\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$ forms a basis of $O_q(G)_{U_q(U^q_{\lambda}(n-1))}$. Furthermore, for all $f \in (\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$ and $g \in (\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$, we have $fg$ belongs to $(\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$.

By Theorem 4.6, every finite dimensional irreducible $U_q(A_{n-1})$-module can be realized in terms of $O_q(G)_{U_q(U^q_{\lambda}(n-1))}$ with the appropriate highest weight $\lambda \in P_+$. Thus Theorem 4.7 gives rise to a basis for each finite dimensional irreducible $U_q(A_{n-1})$-module.

Let $\Lambda_1, \ldots, \Lambda_{n-1}$ be the fundamental dominant weights. The natural representation is the simple modules $L(\Lambda_i)$ which is dual to $L(\Lambda_{n-1})$ and both are of dimension $n$.

The quantum minor

$$\Delta_s := det_q(\{n-s+1, \cdots, n\}, \{1, 2, \cdots, s\})$$

is of weight $\Lambda^*_s$ (the highest weight of the dual module of $L(\Lambda_s)$) with respect to the $L$-action of $U_q(A_{n-1})$, and weight $\Lambda_s$ under the $R$-action of $U_q(A_{n-1})$. It is easy to see that $\Delta_s$ is annihilated by all $E_i$ and $R_E$ for $i = 1, 2, \cdots, n-1$, thus is the highest weight vector of the submodule $L(\Lambda_s)^* \otimes L(\Lambda_s)$ in $O_q(SL_n)$.

Moreover, for arbitrary dominant weight

$$\lambda = \ell_1\Lambda_1 + \ell_2\Lambda + \cdots + \ell_{n-1}\Lambda_{n-1},$$

the submodule $L(\lambda)^* \otimes L(\lambda)$ in $O_q(SL_n)$ is generated by a monomial

$$\Delta_{\ell_1}^\ell_1 \Delta_{\ell_2}^\ell_2 \cdots \Delta_{\ell_{n-1}}^\ell_{n-1}$$

which is again a highest weight vector. All such monomials are dual canonical basis elements by [22, Theorem 5.2].

Remark 4.8. The basis $\langle (\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))} \rangle_+$ seems to deserve the name of canonical basis as its elements are bar-invariant $\mathbb{Z}[q]$ combinations of PBW basis elements.

4.2. Example: $S = \{E_i, F_i \mid i \in \Theta\} \cup \{K_i^{\pm 1} \mid j = 1, 2, \cdots, n-1\}$. Here $\Theta$ is any set of $\{1, 2, \ldots, n-1\}$. In this case, we shall denote $U_S$ by $U_q(\mathfrak{t})$, which is isomorphic to $U_q(\mathfrak{gl}_{k_1}) \otimes U_q(\mathfrak{gl}_{k_2}) \otimes \cdots \otimes U_q(\mathfrak{gl}_{k_n})$ for some $t \leq n$, where $k_i \geq 1$ and $\sum_{i=1}^t k_i = n-1$. To be more specific, let $N_s = \sum_{r=1}^s k_r$, then we may assume that $E_i, F_i$ for $N_{s-1} \leq i < N_s$ belong to the $U_q(\mathfrak{gl}_{k_s})$ subalgebra of $U_q(\mathfrak{t})$. We shall denote $O_q(G)^S$ by $O_q(K \backslash G)$, which is the algebra of functions on some quantum homogeneous space $[3]$ determined by $U_q(\mathfrak{t})$.

To describe $O_q(K \backslash G)$, we first consider the pre-image of $(\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$ under the map $T$ (defined by (2.4)). Needless to say, $T^{-1}(\mathcal{B}_S^*)_{U_q(U^q_{\lambda}(n-1))}$ is a subset of $\mathcal{B}_S$. Each element of it spans a 1-dimensional $U_q(\mathfrak{t})$-module with respect to the restriction of the left action $L$. 


Under $L_{U_q(\mathfrak{t})}$, the $\mathbb{Q}(q)$ span of $x_{ij}$, $i, i = 1, 2, \ldots, n$, decomposes into a direct sum of submodules
\[
\sum_{ij=1}^{n} \mathbb{Q}(q)x_{ij} = \bigoplus_{s=1}^{t} V^n_s; \quad V^n_s = \sum_{i=N_s-1+1}^{N_s} \sum_{j=1}^{n} \mathbb{Q}(q)x_{ij}.
\]
Each $V^n_s$ is $n$ copies of the natural module over the $U_q(\mathfrak{gl}_{k_s})$ subalgebra in $U_q(\mathfrak{t})$, and with all the other $U_q(\mathfrak{gl}_{k_s})$-module, the subalgebra of $O_q(M(n))$ generated by $V^n_s$ is isomorphic to the direct sum of the $q$-symmetrized tensor powers of $n$ copies of the natural module. The only possible 1-dimensional $U_q(\mathfrak{gl}_{k_s})$-submodules must come from the determinant module of $U_q(\mathfrak{gl}_{k_s})$. Therefore, inside the subalgebra of $O_q(M(n))$ generated by $V^n_s$, every 1-dimensional $U_q(\mathfrak{gl}_{k_s})$-submodule is a $\mathbb{Q}(q)$-linear combination of products of quantum determinants of $k_s \times k_s$ sub-matrices of
\[
\begin{bmatrix}
x_{N_s-1+1,1} & x_{N_s-1+1,2} & \cdots & x_{N_s-1+1,n} \\
x_{N_s-1+2,1} & x_{N_s-1+2,2} & \cdots & x_{N_s-1+2,n} \\
\cdots & \cdots & \cdots & \cdots \\
x_{N_s,1} & x_{N_s,2} & \cdots & x_{N_s+1,n}
\end{bmatrix}.
\]

Such quantum determinants are quantum minors of the form $\det_q(I_s, J(s))$, where $I_s = \{N_s-1+1, N_s-1+2, \ldots, N_s\}$, and $J(s)$ is any subset of $\{1, 2, \ldots, n\}$ of cardinality $k_s$.

It therefore follows that every element of $T^{-1}((\mathcal{B}^*)_{U_q(A_{n-1})})$ is a $\mathbb{Q}[q, q^{-1}]$ combination of products of quantum minors $\det_q(I_s, J(s))$, $s = 1, 2, \ldots, t$, and the quantum determinant $\det_q$. Thus all elements of $(\mathcal{B}^*)_{U_q(A_{n-1})}$ can be expressed in terms of the images of the quantum minors under $T$.

The above discussions have established the following result.

**Lemma 4.9.** The subalgebra $O_q(K \setminus G)$ of invariants is generated by quantum minors of the form $T(\det_q(I_s, J(s)))$, $s = 1, 2, \ldots, t$, where $I_s = \{N_s-1 + 1, N_s-1+2, \ldots, N_s\}$, and $J(s)$ is any subset of $\{1, 2, \ldots, n\}$ of cardinality $k_s$.

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China

E-mail address: hzhang@math.tsinghua.edu.cn

School of Mathematics and Statistics, University of Sydney, Australia

E-mail address: rzhang@maths.usyd.edu.au