Response: Commentary: A Remark on the Fractional Integral Operators and the Image Formulas of Generalized Lommel-Wright Function

Ritu Agarwal1*, Sonal Jain1, Ravi P. Agarwal2 and Dumitru Baleanu3,4

1 Department of Mathematics, Malaviya National Institute of Technology, Jaipur, India, 2 Department of Mathematics, Texas A&M University, Kingsville, TX, United States, 3 Department of Mathematics, Çankaya University, Ankara, Turkey, 4 Institute of Space Sciences, Magurele-Bucharest, Romania

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A Commentary on

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In view of the commentary by Kiryakova [1] regarding our paper Agarwal et al. [2] published in “Frontiers in Physics” (2018), we reply to the commentary by proving the worth of our results and providing justification to all the points raised within commentary. Besides, our original contributions are better pointed out.

1. INTRODUCTION

We recall that numerous real world phenomena are closely related to the fractional order extensions of the Bessel function \( J_\nu(z) \) which in turn are generalized by the Lommel-Wright function \( s \) and their special cases \( J_\nu^s(z), J_\nu^s,\kappa(z) \), where \( \kappa > 0 \) is a fractional parameter.

Substantial problems of mechanics, physics, astronomy, and various other engineering fields leads us to the generalized Bessel, Lommel, Struve, and Lommel-Wright functions.

In our paper Agarwal et al. [2], the image formulas for the generalized Lommel-Wright function involving the Saigo-Maeda fractional integral operators, in term of the Fox-Wright function have been established. Following that certain theorems, with the outcomes achieved for the aforementioned functions in relation with the integral transforms like Beta transform, pathway transform, Laplace transform and Whittaker transform, have been proved. Interesting consequences of our results would involve the Saigo fractional integral operators \( I_{\gamma,\tau,\eta}^s \) and \( I_{x,\infty}^{s_1,s_2} \) which can be deduced from the theorems in Agarwal et al. [2] by appropriately applying the relationships between these operators. Our paper contains results that are mathematically correct and novel in nature.

2. REPLY TO COMMENTARY

(a) As mentioned by Virginia Kiryakova in her commentary in the first paragraph [[1], p.3, Equation (8)] that Appell function \( F_3 \) can be expressed as Meijer G-function, it is to bring to the notice that almost all the special functions are expressible in the form of (G-function is a
provided the existence conditions are satisfied.

Lot of work has been done on the various types of integrals and fractional operators involving H-function (see, e.g., [3, 5–8]). The limitations of H-function are due to the large number of parameters involved, its applications to the real world problems can be found only through its special cases. The large number of parameters interact each other and are not preferable from the experimental viewpoint. In fact, numerical simulation is also difficult for the said generalized functions, namely, H-function and G-function.

(b) Paragraph 2 [[1], p. 3] highlights that the Marichev-Saigo-Maeda (MSM) fractional operator is actually composition of three Erdélyi-Kober (EK) operators. If it is so, we can raise the question on the novelty of the MSM operator. In our opinion, the composition of the operators is entirely different concept. These two operators are not linearly connected. In mathematics research, we know that even a different representation of the result/function is important. For example, a simple Beta function has so many representations and each one is important at its place. The fractional integral operators of Erdélyi-Kober are special cases of the operators of the fractional integral introduced by Marichev-Saigo-Maeda as can be observed from the following definitions:

For $\nu, \nu', \tau, \tau', \eta \in \mathbb{C}$ and $x > 0$, the generalized operators of Marichev-Saigo-Maeda are defined using the Appell's function in the kernel as follows:

\[
\left( I_{0,0}^{\nu,\nu',\tau,\tau',\eta} h \right)(\omega) = \frac{\omega^{-\nu}}{\Gamma(\eta)} \int_0^{\omega} (\omega - t)^{-1} t^{-\nu'} F_3 \\
\left( \nu, \nu', \tau, \tau', \eta; \frac{1}{t} - 1, \frac{1}{t} - 1 \right) h(t) dt, \quad (\Re(\eta) > 0),
\]

and

\[
\left( K_{0,0}^{\nu,\nu',\tau,\tau',\eta} h \right)(\omega) = \frac{\omega^{-\nu}}{\Gamma(\eta)} \int_{\omega}^{\infty} (t - \omega)^{\eta-1} t^{-\nu'} F_3 \\
\left( \nu, \nu', \tau, \tau', \eta; \frac{1}{t} - 1, \frac{1}{t} - 1 \right) h(t) dt, \quad (\Re(\eta) > 0).
\]

The fractional integral operators of Erdélyi-Kober type are described as mentioned below [9]:

\[
\left( I_{0,0}^{\nu,\nu',\tau,\tau',\eta} h \right)(\omega) = \frac{\omega^{-\nu-\eta}}{\Gamma(\gamma)} \int_0^{\omega} (\omega - t)^{-1} t^{-\eta} h(t) dt, \quad (\Re(\gamma) > 0)
\]

and

\[
\left( K_{0,0}^{\nu,\nu',\tau,\tau',\eta} h \right)(\omega) = \frac{\omega^{\eta}}{\Gamma(\gamma)} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\eta} h(t) dt, \quad (\Re(\gamma) > 0).
\]

where $h(.)$ is so restricted that both the above integrals in (6) and (7) exist.

(c) Again, as per the last paragraph [1, p.3] in the commentary, the result in Agarwal et al. [2, Theorem 3.2] is obvious. This implies that majority of the research work done so far by many researchers worldwide is obvious because all the new results are derived from/based upon the earlier obtained results. For application purpose, a researcher (from a different domain like physics, engineering etc.) search for direct results, doesn’t want to derive all the mathematics behind those results. Here is the Theorem 3.2 and its proof as in Agarwal et al. [2]:

**Theorem 3.2.** Let $\gamma, \gamma', \tau, \eta, \rho, \varphi \in \mathbb{C}, \varphi > 0, m \in \mathbb{N}, \Re(\rho) > 0, \Re(s) > 0, \sigma > 1$ and $y > 0$ be the parameters satisfying

\[
\Re(\varphi) > 0, \quad \Re(\sigma) > -1, \quad \Re(s) > 0, \\
\Re(\rho + \sigma) > \max(0, \Re(\gamma + \gamma' + \tau - \rho), \Re(\gamma' - \tau'))
\]

then the $P_s$-transform formula is given by:

\[
P_{s} \left[ \frac{\omega^{\nu'-\nu} \omega^{\nu'-\nu}}{\Gamma(\eta)} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\nu'} F_3 \\
\left( A + \rho, A + \sigma + \rho - \gamma' - \tau, A + \sigma + \rho - \gamma', A + \nu - \gamma' - \tau, A + \nu - \gamma', A + \nu - \gamma' - \tau, 1/0 Requests convertible to 2024-02-22
\right) \right] = \frac{\omega^{-\nu-\sigma+2\varphi}}{2\omega + 2\varphi}
\]

\[
\left[ \frac{\omega^{\nu'-\nu} \omega^{\nu'-\nu}}{\Gamma(\eta)} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\nu'} F_3 \\
\left( A + \rho, A + \sigma + \rho - \gamma' - \tau, A + \sigma + \rho - \gamma', A + \nu - \gamma' - \tau, A + \nu - \gamma', A + \nu - \gamma' - \tau, 1/0 Requests convertible to 2024-02-22
\right) \right] = \frac{\omega^{-\nu-\sigma+2\varphi}}{2\omega + 2\varphi}
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\left[ \frac{\omega^{\nu'-\nu} \omega^{\nu'-\nu}}{\Gamma(\eta)} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\nu'} F_3 \\
\left( A + \rho, A + \sigma + \rho - \gamma' - \tau, A + \sigma + \rho - \gamma', A + \nu - \gamma' - \tau, A + \nu - \gamma', A + \nu - \gamma' - \tau, 1/0 Requests convertible to 2024-02-22
\right) \right] = \frac{\omega^{-\nu-\sigma+2\varphi}}{2\omega + 2\varphi}
\]

such that $A = \rho + \sigma + 2\varphi$ and $\Lambda(\sigma; s) = \left( \frac{\sigma - 1}{\ln(1 + (\sigma - 1)n)} \right)

Proof: Let us denote the left-hand side of the formula (9) as $\mathcal{E}$. Applying [[2], Equation 1.30] to [[2], Equation 3.2] we get,

\[
\mathcal{E} = \int_0^{\infty} [1 + (\tau - 1)s]^{-1} 2^{1-\nu'} \omega^{\nu'-\nu} \omega^{\nu'-\nu} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\nu'} F_3 \\
\left( A + \rho, A + \sigma + \rho - \gamma' - \tau, A + \sigma + \rho - \gamma', A + \nu - \gamma' - \tau, A + \nu - \gamma', A + \nu - \gamma' - \tau, 1/0 Requests convertible to 2024-02-22
\right) = \frac{\omega^{-\nu-\sigma+2\varphi}}{2\omega + 2\varphi}
\]

Here, applying the [[2], Equation 2.4] to the integral, we get

\[
\mathcal{E} = \int_0^{\infty} [1 + (\tau - 1)s]^{-1} 2^{1-\nu'} \omega^{\nu'-\nu} \omega^{\nu'-\nu} \int_{\omega}^{\infty} (t - \omega)^{-1} t^{-\nu'} F_3 \\
\left( A + \rho, A + \sigma + \rho - \gamma' - \tau, A + \sigma + \rho - \gamma', A + \nu - \gamma' - \tau, A + \nu - \gamma', A + \nu - \gamma' - \tau, 1/0 Requests convertible to 2024-02-22
\right) = \frac{\omega^{-\nu-\sigma+2\varphi}}{2\omega + 2\varphi}
\]
particular investigation done in Kiryakova [1] on our correctly
within the comment. Therefore, we disagree with the aim of a
functions. . . " However, only our paper has been pointed out
devoted to evaluation of the images of classes of special
works
paragraph of Kiryakova [1], section 1], "The commented
mentioned by the author of the commentary in the first
(calculus operators (see for example some of new related works on
calculus in the classical way, which has comparatively lesser
We proved that our results reported in Agarwal et al. [2] are
3. CONCLUSION
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\[ Z = \frac{\Gamma(A\tau' - \gamma' - \tau + 2p)}{\Gamma(\sigma + \rho + 1 + \varphi p)(\Gamma(\sigma + 1 + p)^{m}} \times \int_{0}^{\infty} \frac{(y)^{2p}}{[1 + (\sigma - 1)s]^{-\frac{\pi}{2}}} z^{\sigma + 2 \varphi + 2p + 1} dz. \]

Using the result [[2], Equation 1.31] and swapping the order of the integration and the summation, we arrive at,E-K integral [13], the domain gets narrowed and hence it is not always suitable for describing the dynamics of the real
world phenomenon. Besides, these indices interact each other, reducing the practical usage of the above said functions. In our opinions, the construction suggested by Kiryakova [13] will face problems while it will be applied to the real world problems mainly because:
- contains many indices and they interact each other and from experimental view point the researchers will not prefer to work
with many indices;
- the laws of Nature are simple and do not require non-physical operators to describe them. So far, we were unable to find any numerical implementation suggested by the generalization in Kiryakova [13, 14].

Thus, proof of the Theorem 3.2 clearly illustrates that its not as straightforward as claimed in Agarwal et al. [2]. It can be seen clearly that our work is new and the results are distinctive.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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