Extension of Dirac theory and the classification of elementary particles

Janet Pan$^1$ and Lu Lin$^2$

1. Yale University, P.O. Box 208284, New Haven, CT 06520-8284.

2. Wufeng Institute of Technology, Chia-yi, Taiwan, ROC. E-mail : lulin@mail.wfc.edu.tw

Abstract

The Dirac theory implies the existence of an internal vector space, in addition to spin space. Using Dirac’s coupling of variables in internal space to those in physical space, we construct a new configuration structure for particles in the combined physical plus internal spaces. The importance of this is that the internal degrees of freedom implicit in Dirac’s theory allow a new classification of elementary particles. An important consequence is the prediction of a new type of quark. As expected, our theory groups fermions into doublets, which are then divided into color singlets (leptons) and color triplets (quarks), and which are then further divided into generation singlets and generation triplets. If the Pauli exclusion principle for fermions is also valid within a particle’s internal space, then our theory makes two important predictions. First, we can explain why the widely studied quarks (up, down, charm, strange, top, bottom) cannot be observed as free states in nature. Second, we predict the existence of a new quark which can indeed be observed as a free state in nature, and whose wave function is antisymmetric in internal space. W. M. Fairbank, a Guggenheim Fellow, has published experimental data which supports our second prediction.
Consider the motion of a free electron (or, in general, a fermion) described by a space-time 4-vector $x_\mu$, its momentum $p_\mu$, and Dirac matrices $\alpha_i$, $\beta$, ($i=1$ to $3$, and $\mu=1$ to $4$). The Dirac Hamiltonian [1, 2] of this system is

$$H = \sum \alpha_i \cdot p_i + \beta m, \quad \text{where } \hbar = c = 1.$$  

(1)

The i-th component of the velocity and the i-th component of the coordinate are given as (see Chapter XI of ref. [2])

$$\dot{x}_i = \alpha_i = \dot{\xi}_i + \dot{\eta}_i$$  

(2)

$$x_i = \xi_i + \eta_i$$  

(3)

$$\xi_i = a_i + p_i H^{-1} t$$  

(4)

$$\eta_i = \frac{i}{2} (\alpha_i H^{-1} - p_i H^{-2})$$  

(5)

where $a_i$ is the $\xi_i(t)$ at $t=0$. Since $\alpha_i$ does not commute with the Hamiltonian, each component of the velocity of the electron is not a constant of the motion. Also, since $\alpha_i$ has eigenvalues $\pm 1$, then a measurement of any component of the velocity of the electron will give the speed of light, even when $p_i$ is zero. On the other hand, since the square of the velocity operator is $v_i^2 = \sum \alpha_i^2 = 3c^2$, which is a constant, independent of all $p_i$, then $v_i^2$ commutes with the Hamiltonian and can be simultaneously measured with the Hamiltonian. However, the result, $v_i^2 = 3c^2$, violates relativity, and forces the electron mass $m(v) = m \left(1 - \frac{v_i^2}{c^2}\right)^{-\frac{1}{2}}$ to have an imaginary value, even when the momentum $p_i$ is zero. This does not make sense. This trouble can be understood and reinterpreted as the following. According to Dirac [2], since the $\alpha_i$ and the $\beta$ are independent of, and commute with $x_\mu$ and $p_\mu$, then they must describe some new degree of freedom, belonging to some internal motion of the electron. This implies the existence of an internal space, in which the electron moves, and which is independent of physical space (PS). The Hamiltonian $H$ in Equation. (1) requires that the motion of the electron in physical space is coupled to its motion in internal space. We can assert that the same electron is moving in an enlarged space, which is comprised of physical space plus an internal space. Another way to look at the problem is that the particle energy has two contributions, and these two contributions are coupled by the Dirac Hamiltonian. This is analogous to the case of two classical oscillators, with coordinates $z_1$ and $z_2$, and Hamiltonians $H_1$ and $H_2$. If the oscillators are coupled, then the total Hamiltonian will be $H=H_1+H_2+H_{12}$, and the normal coordinates will be functions of $z_1$ and $z_2$.

Consider an operator $c^2 p_i H^{-1}$, which commutes with the Hamiltonian $H$ and has an eigenvalue $\frac{p_i}{M}$, where $M$ the total mass of the electron. This value is just the value of $v_i = \dot{\xi}_i$ in Equations (2) and (1), and is the usual velocity of the electron. Thus, we can interpret $x_i = \xi_i + \eta_i$ as a generalized coordinate vector, with $\xi_i$ in physical space and $\eta_i$ (which is, in general, complex) in internal space. Then, $\dot{x}_i = c\alpha_i$ is the velocity of the
Denote the vectors in PS, and in IS by nature of a vector in IS. We now proceed to study the representation of \( \alpha \) space (IS). A vector which is comprised of a 4-dimensional physical space (PS) and a 4-dimensional internal space has 4 independent components. Thus, \( \eta_i \) should have a 4th component, \( \eta_4 \), to make a 4-vector \( \eta_{\mu} \). Now we can define our total space as the greater space (GS), which is comprised of a 4-dimensional physical space (PS) and a 4-dimensional internal space (IS). A vector \( x_\mu \) in PS has real \( x_1 \) and imaginary \( x_4 \). However, we do not have any direct information about the nature of a vector in IS. We now proceed to study the nature of a vector in IS.

Denote the vectors in PS, and in IS by \( u_\mu \) and \( \bar{u}_\mu \) respectively, and the time variables corresponding to \( u_4 \) and \( \bar{u}_4 \) by \( t \) and \( \bar{t} \). The Lagrangian of our system can be generalized to contain variables of both PS and IS. For a free particle in classical mechanics, the Lagrangian is a scalar, and is a function of the square of the velocity of the particle. Velocity terms like \( \frac{du_\mu}{dt} \) appear in Equation (2), which implies that, by symmetry, terms like \( \frac{\bar{d}u_\mu}{d\bar{t}} \) should be included in the generalized Lagrangian. This generalized Lagrangian should contain all the velocity terms, \( \frac{du_\mu}{dt} \), \( \frac{du_\mu}{d\bar{t}} \), \( \frac{d\bar{u}_\mu}{dt} \), and \( \frac{d\bar{u}_\mu}{d\bar{t}} \), plus small coupling terms. Since we are interested only in the main qualitative features of the Lagrangian, in order to elucidate the nature of the 4-vector in IS, then we may neglect the small coupling terms and express the Lagrangian as

\[
L = L_1 \left( \frac{du_\mu}{dt} \right) + L_2 \left( \frac{du_\mu}{d\bar{t}} \right) + L_3 \left( \frac{d\bar{u}_\mu}{dt} \right) + L_4 \left( \frac{d\bar{u}_\mu}{d\bar{t}} \right)
\]

(6)

This Lagrangian now allows the system to be treated like a problem with four independent particles. \( L_1 \) describes a physical particle in PS. Each of the other three parts of the Lagrangian has either an unphysical particle or an unphysical space. \( L_2 \) describes a particle in PS, but which is varying with respect to its internal time. The particle described by \( L_2 \) has coordinate variables which should be considered to be independent of the \( u_i \) in \( L_1 \). Since the motion described by \( L_2 \) is unphysical, then the location described by \( L_2 \) should be imaginary. Furthermore, the internal time carried by an unphysical real particle should also be considered as imaginary. For \( L_3 \) and \( L_4 \), we can assume that, when the universe was created, the four vectors were created together by taking all possible real and imaginary combinations of the \( (u_i, t) \) with equal probability. As a result, we have the greater space

\[
GS = P_1 (u_i, t) + P_2 (iu_i, it) + P_3 (\bar{u}_i, i\bar{t}) + P_4 (i\bar{u}_i, i\bar{t})
\]

(7)

where \( u_i, \bar{u}_i, t, \bar{t} \) are real numbers. In 1998, one of the authors suggested this simple idea of a complex space-time, and now we develop it further.

In physical space, if we can describe the system of \( L_1 + L_2 \) of Equation (7), then a well-behaved wave function \( \Phi (x_i, i\bar{x}_i; t, i\bar{t}) \) must exist. For a stationary state, the spatial distribution must be independent of time, so \( \Phi \) can be factored,

\[
\Phi = \Phi (X) \Phi (T), \quad \text{where} \ X = x + i\bar{x} \text{ and } T = t + i\bar{t}
\]

(9)
We can reasonably assume the simple conditions that $\Phi$ is a harmonic function of $(x, i\bar{x})$ and $(t, i\bar{t})$ respectively, and that $\Phi$ has continuous partial derivatives of second order which satisfy the Laplace equations

$$\frac{\partial^2 \Phi(X)}{\partial x^2} + \frac{\partial^2 \Phi(X)}{\partial \bar{x}^2} = 0$$

$$\frac{\partial^2 \Phi(T)}{\partial t^2} + \frac{\partial^2 \Phi(T)}{\partial \bar{t}^2} = 0$$

Since $x$ and $i\bar{x}$ are independent, we can justifiably write the factorization $\Phi(X) = \Phi(x)\Phi(i\bar{x})$, and similarly for $\Phi(T)$. The solutions can be written as

$$\Phi(X) = A \exp(\pm ikx) \exp(\pm ik(i\bar{x}))$$

$$\Phi(T) = B \exp(-i\omega t) \exp(-i\omega(i\bar{t}))$$

In Equation (12) we have chosen the direction of $k$ along the $x$-axes, and the $i\bar{x}$ along the $x$-axes but independent of $x$. In Equation (13), note that we are concerned with a free electron, which is not an anti-particle, and therefore we must take the positive frequency solution for the wave function. This is also true for wave functions in $L_2, L_3, L_4$, because they describe the motion of electrons in different regions of greater space. If we had taken both the positive and the negative frequencies for $\omega$, then the final total wave function would have the form of $a \exp(-i\omega t) + b \exp(i\omega t)$, which is a mixture of a particle and an anti-particle. The latter should be clear because negative frequency solutions correspond to anti-particles. This is easily seen by transforming the Dirac equation in an electromagnetic field by a time-reversal plus a charge conjugation. However, since the charge operator commutes with the Hamiltonian, then the charge eigenvalue can be simultaneously measured with the energy eigenvalue. Thus, the energy eigenfunction must also be a charge eigenfunction. Hence, energy eigenfunctions are not comprised of mixed charge states. That is to say, energy eigenfunctions are not comprised of a mixture of particle and anti-particle states. By setting $\bar{x}=\bar{t}=0$, we get a projected free wave solution in physical space with only real space-time variables, corresponding to a particle with energy and momentum as $E = \hbar \omega$ and $p = \hbar k$.

Consider the four independent particles specified by the four 4-vectors of equation (3). The single particle states are generated by solving, for each particle, a Dirac equation in the corresponding region. In the physical $x_1$-space, there are two single particles, a physical particle and an unphysical particle, while there are two identical particles, both unphysical, in the internal $\bar{x}_1$-space. For a stationary state, a total state function is the product $\psi_1\psi_2\psi_{34}$, where $\psi_1$ and $\psi_2$ are single particle states for particles 1 and 2, and $\psi_{34}$ is a two-particle state for particles 3 and 4. Each $\psi$ has a spatial $x_1$-part and a spin part. The $x_1$-part of each $\psi$ must satisfy equation (10). The solution in equation (12) is the $x_1$-parts of $\psi_1\psi_2$. Since $p_1$ corresponds to physical space, $\psi_1$ must have a good momentum, so there is only one momentum state in $\psi_1$. $\psi_2$ can take two momentum states, those in equation (10). Similarly, for the unphysical regions of $L_3+L_4$, we have $2 \otimes 2$ momentum states. For the spin part of the wave function, since the spin is diagonalized in the rest frame of $L_1$, the particle can only take one spin state in $L_1$. This state cannot be mixed.
Extension of Dirac theory and the classification of elementary particles

with any unphysical spin state. In the other three parts, $L_2$, $L_3$, $L_4$ of unphysical space, each part has two independent spin states. Therefore, for a fermion, with spin states and momentum states taken together, the total number of independent internal states generated from $L_2 + L_3 + L_4$ is

$$\text{Number of internal states} = 2 \bigotimes 2 \bigotimes 2 \bigotimes 2 \bigotimes 2 \bigotimes 2 \quad (14)$$

The first 2 and the last 2 in Equation (15) come from the spin states and the momentum states of $L_2$ while the two brackets come from those states of $L_3$ and $L_4$. The first 2 in Equation (15) means that, all fermions are grouped in doublets, such as $(\nu_e)$, $(\nu_\mu)$, $(\nu_t)$ and $(u \bar{d})$, $(c \bar{s})$, $(t \bar{b})$. The first bracket in Equation (15) signifies that, all the above doublets are divided into color singlets with zero color plus color triplets (each quark has three color states). The second bracket signifies that, the above entities are further divided into a generation singlet which we shall explain later, plus a generation triplet which contain three lepton doublets $(\nu_e)$, $(\nu_\mu)$, $(\nu_t)$ and three quark doublets $(u \bar{d})$, $(c \bar{s})$, $(t \bar{b})$. As far as the last 2 is concerned, we postulate that when our universe was created, the creator might have designed a duality for particles to have a symmetry of electricity and magnetism, such as an electron and a magnetron, a proton and a magnetic proton, a quark and a magnetic quark. However, no experimental evidence exists for magnetic charges, or monopoles. Thus far, experimental evidence is consistent only with asymmetric Maxwell equations: that we have $\nabla \cdot E = 4\pi \rho$, but $\nabla \cdot B = 0$. Could the magnetic matter exist elsewhere in the universe? If it could, then this would vindicate the brilliant theoretical argument of Dirac [4]: that the mere existence of one magnetic monopole in the universe would offer an explanation of the discrete nature of electric charge, and thus solve one of the most profound mysteries of the physical world.

For a one particle state, define the color-spin and the generation-spin as $(s_c, m_c) = (\frac{1}{2}, \pm \frac{1}{2})$ and $(s_g, m_g) = (\frac{1}{2}, \pm \frac{1}{2})$, respectively. Then the two particle color-spin wave functions $\Phi_c(SM)$ can be written as a singlet state $\Phi_c(0, 0)$, and triplet states $\Phi_c(1, 1)$, $\Phi_c(1, 0)$, $\Phi_c(1, -1)$. The singlet is anti-symmetric:

$$\Phi_c(0, 0) = \frac{1}{\sqrt{2}} \left( \phi_1^+ \phi_2^- - \phi_1^- \phi_2^+ \right) \quad (16)$$

The triplet is symmetric:

$$\Phi_c(1, 0) = \frac{1}{\sqrt{2}} \left( \phi_1^+ \phi_2^- + \phi_1^- \phi_2^+ \right) \quad (17)$$

$$\Phi_c(1, 1) = \phi_1^+ \phi_2^+ \quad (18)$$

$$\Phi_c(1, -1) = \phi_1^- \phi_2^- \quad (19)$$

The two particle generation-spin wave functions $\Phi_g(S'M')$ can be written in a similar way.

In the total wave function $\psi_1 \psi_2 \psi_{34}$ of a fermion, $\psi_1$ and $\psi_2$ are single particle wave functions, and $\psi_{34}$ is a two identical particle wave function in internal space, as was
discussed above. We only need to consider $\psi_{34} = \Phi_c(SM)\Phi_g(S'M')$. We assert now that the exclusion principle is also valid in internal space, that the total wave function of a system of identical fermions must be anti-symmetric. All leptons have singlet color wave functions $\Phi_c(0,0)$ with zero color, which are anti-symmetric. The three generations of leptons, $(\nu_e), (\nu_\mu, \mu), (\nu_t, t)$, have symmetric triplet generation wave functions $\Phi_g(1,M)$, $M = 1, 0, -1$, so their total wave function $\Phi_c(00)\Phi_g(1M)$ are anti-symmetric. Therefore, the above three doublets of leptons do not violate the exclusion principle, and are free states in nature. There are also singlet generation leptons $\Phi_g(0,0)$, for which the total wave functions $\Phi_c(00)\Phi_g(00)$ are symmetric. These wave functions violate the exclusion principle, and cannot be found as free states. In this work, we shall not discuss the possibility of having compounds states for this kind of particle.

All quarks have triplet color wave functions, $\Phi_c(1,M)$, $M = 1, 0, -1$, which are symmetric. The three generations of quarks, $(ud), (cs), (tb)$, have symmetric generation wave functions, $\Phi_g(1,M)$, so the total wave functions, $\Phi_c(1M)\Phi_g(1M)$ are symmetric. These wave functions also violate the exclusion principle, and cannot be found as free states. They can only exist in a compound system.

There is one last case, where a quark takes triplet color state $\Phi_c(1,M)$ and a singlet generation state $\Phi_g(00)$. This wave function is anti-symmetric, which does not violate the exclusion principle, and thus this quark can exist as a free state in nature. In 1981, W. M. Fairbank, a Guggenheim Fellow, and his colleagues reported measurements which showed unambiguously the existence of fractional charges of $\pm \frac{1}{3}$. If Fairbank’s measurements are correct, then his quarks could be identified as the quarks which we described as $\Phi_c(1,M)\Phi_g(00)$. However, other groups need to confirm Fairbank’s results, and further work is certainly needed.

References

[1] H. A. Bethe, and R. W. Jackiw, Intermediate quantum mechanics, third ed. 1986, Addison-Wesley Publishing Co. New York.

[2] P. A. M. Dirac, The principles of quantum mechanics, fourth ed. 1958 Oxford Univ. Press, London.

[3] L. Lin, On the possibility of a complex 4-dimensional space-time manifold, Gen. Phys./9804016, (1998).

[4] P. A. M. Dirac, Proc. Roy. Soc. A133, 60 (1931); Phys. Rev. 74, 883 (1948).

[5] G. S. LaRue, W. M. Fairbank, and A. F. Hebard, Phys. Rev. Lett. 38, 1011 (1977).

[6] G. S. LaRue, W. M. Fairbank, and J. D. Phillips, Phys. Rev. Lett. 42, 142, 1019(E) (1979).

[7] G. S. La Rue, J. D. Phillips, and W. M. Fairbank, Phys. Rev. Lett. 46, 967, (1981).