ENDOMORPHISMS OF EXCEPTIONAL $\mathcal{D}$-ELLIPTIC SHEAVES

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Abstract. We relate the endomorphism rings of certain $\mathcal{D}$-elliptic sheaves of finite characteristic to hereditary orders in central division algebras over function fields.

1. Introduction

The endomorphism rings of abelian varieties have long been a subject of intensive investigation in number theory. One of the earliest results in this area was the determination by Deuring of the endomorphism rings of elliptic curves over finite fields. His results were later generalized to higher dimensional abelian varieties by Honda, Tate and Waterhouse [16]. These results have important applications, e.g., they play a key role in calculations of local zeta functions of Shimura varieties.

In [3], Drinfeld introduced a certain function field analogue of abelian varieties; these objects are now called Drinfeld modules. Denote by $\mathbb{F}_q$ the finite field with $q$ elements. Let $X$ be a smooth, projective, geometrically connected curve defined over $\mathbb{F}_q$. Let $F = \mathbb{F}_q(X)$ be the function field of $X$. Fix a place $\infty$ of $F$ (in Drinfeld’s theory this plays the role of an archimedean place). Let $o \neq \infty$ be another place of $F$. Denote by $\mathbb{F}_o$ the residue field at $o$. In [6], Drinfeld proved the analogue of Honda-Tate theorem for Drinfeld modules defined over extensions of $\mathbb{F}_o$. In [9], Gekeler extended Drinfeld’s results, in particular, he proved that for a rank-$d$ supersingular Drinfeld module $\phi$ over $\mathbb{F}_o$ the endomorphism ring $\text{End}(\phi)$ is a maximal order in the central division algebra over $F$ of dimension $d^2$, which is ramified exactly at $o$ and $\infty$ with invariants $-1/d$ and $1/d$, respectively. Moreover, there is a bijection between the isomorphism classes of rank-$d$ supersingular Drinfeld modules over $\mathbb{F}_o$ and the left ideal classes of $\text{End}(\phi)$ (see [9, Thm. 4.3]).

In [13], Laumon, Rapoport and Stuhler introduced the notion of $\mathcal{D}$-elliptic sheaves, which is a generalization of the notion of Drinfeld modules. (One can think of these objects as function field analogues of abelian varieties equipped with an action of a maximal order in a simple algebra over $\mathbb{Q}$.) In Laumon-Rapoport-Stuhler theory one needs to fix a central simple algebra $D$ over $F$ of dimension $d^2$ which is split at $\infty$, and a maximal $\mathcal{O}_X$-order $\mathcal{D}$ in $D$ (see [22] for definitions). In [13] Ch. 9, the authors develop the analogue of Honda-Tate theory for $\mathcal{D}$-elliptic sheaves over $\mathbb{F}_o$ with zero $o$ and pole $\infty$, assuming $D$ is split at $o$. The assumption that $D$ is split at $o$ is not superficial. When $D$ is ramified at $o$, to obtain a reasonable theory of $\mathcal{D}$-elliptic sheaves with zero $o$ and pole $\infty$, one has to assume at least that $D \otimes_F F_o$ is the division algebra with invariant $1/d$ over $F_o$, where $F_o$ is the completion of $F$ at $o$. Such $\mathcal{D}$-elliptic sheaves play a crucial role in the function

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field analogue of Čerednik-Drinfeld uniformization theory developed by Hausberger [11].

Assume $D_o := D \otimes_F F_o$ is the $d^2$-dimensional central division algebra with invariant $1/d$ over $F_o$. In this paper we define a subclass of $D$-elliptic sheaves over $F_o$, which we call exceptional, and which are distinguished by a particularly simple relationship between the actions of $D_o$ and the Frobenius at $o$; see Definition 5.1. In general, exceptional $D$-elliptic sheaves do not correspond to points on the moduli schemes constructed in [13] or [11], so they are not very natural from moduli-theoretic point of view. Nevertheless, we show that the theory of endomorphism rings of these objects is similar to the theory of endomorphism rings of supersingular Drinfeld modules. The main result is the following (see Theorems 5.3 and 5.4):

**Theorem 1.1.** Let $E$ be an exceptional $D$-elliptic sheaf over $F_o$ of type $f$. Then $\text{End}(E)$ is a hereditary $\mathcal{O}_X$-order in the central division algebra $\bar{D}$ over $F$ with invariants

$$\text{inv}_x(\bar{D}) = \begin{cases} 
1/d, & x = \infty; \\
0, & x = o; \\
\text{inv}_x(D), & x \neq o, \infty.
\end{cases}$$

This order is maximal at every place $x \neq o$, and at $o$ it is isomorphic to a hereditary order of type $f$. There is a bijection between the set of isomorphism classes of exceptional $D$-elliptic sheaves over $F_o$ of type $f$ and the isomorphism classes of locally free rank-1 right $\text{End}(E)$-modules.

The type of an exceptional $D$-elliptic sheaf is determined by the action of the Frobenius at $o$, and the type of a hereditary order determines the order up to an isomorphism (see §2). In §6 we use Theorem 1.1 to prove a mass-formula for exceptional $D$-elliptic sheaves, and discuss a geometric application of this formula. In §7 we explain how the argument in the proof of Theorem 1.1 can be used to prove a theorem about endomorphism rings of supersingular $D$-elliptic sheaves over $F_o$, which implies Gekeler’s result mentioned earlier as a special case (in §7 we assume that $D$ is split at $o$).

**Notation.** Unless specified otherwise, the following notation is fixed throughout the article.

- $k$ is a fixed algebraic closure of $\mathbb{F}_q$ and $Fr_q : k \rightarrow k$ is the automorphism $x \mapsto x^q$.
- $|X|$ denotes the set of closed points on $X$ (equiv. the set of places of $F$).
- For $x \in |X|$, $\mathcal{O}_x$ is the completion of $\mathcal{O}_{X,x}$, and $F_x$ (resp. $\mathbb{F}_x$) is the fraction field (resp. the residue field) of $\mathcal{O}_x$. The degree of $x$ is $\text{deg}(x) := [\mathbb{F}_x : \mathbb{F}_q]$, and $q_x := q^{\text{deg}(x)} = \#\mathbb{F}_x$. We fix a uniformizer $\pi_x$ of $\mathcal{O}_x$.
- $\mathcal{A}_F := \prod_{x \in |X|} F_x$ denotes the adele ring of $F$, and for a set of places $S \subset |X|$, $\mathcal{A}_F^S := \prod_{x \in |X| - S} F_x$ denotes the adele ring outside of $S$.
- The zeta-function of $X$ is

$$\zeta_X(s) = \prod_{x \in |X|} (1 - q_x^{-s})^{-1}, \quad s \in \mathbb{C}.$$ 

- For any ring $R$ we denote by $R^\times$ its subgroup of units.
- $\mathbb{M}_d$ denotes the ring of $d \times d$ matrices.
2. Orders

For the convenience of the reader, we recall some basic definitions and facts concerning orders over Dedekind domains. A standard reference for these topics is [14].

Let \( R \) be a Dedekind domain with quotient field \( K \) and let \( A \) be a central simple \( K \)-algebra. For any finite dimensional \( K \)-vector space \( V \), a full \( R \)-lattice in \( V \) is a finitely generated \( R \)-submodule \( M \) in \( V \) such that \( K \otimes_R M \cong V \). An \( R \)-order in the \( K \)-algebra \( A \) is a subring \( \Lambda \) of \( A \), having the same unity element as \( A \), and such that \( \Lambda \) is a full \( R \)-lattice in \( A \). A maximal \( R \)-order in \( A \) is an \( R \)-order which is not contained in any other \( R \)-order in \( A \). A hereditary \( R \)-order in \( A \) is an \( R \)-order \( \Lambda \) which is a hereditary ring, i.e., every left (equiv. right) ideal of \( \Lambda \) is a projective \( \Lambda \)-module. Maximal orders are hereditary. Being maximal or hereditary are local properties for orders: an \( R \)-order \( \Lambda \) in \( A \) is maximal (resp. hereditary) if and only if \( \Lambda_p := \Lambda \otimes_R R_p \) is a maximal (resp. hereditary) \( R_p \)-order in \( A_p := A \otimes_K K_p \) for all prime ideals \( p \subset R \), where \( R_p \) and \( K_p \) are the \( p \)-adic completions of \( R \) and \( K \).

Let \( I \) be a full \( R \)-lattice in \( A \). Define the left order of \( I \)

\[ O_I(I) = \{ a \in A \mid aI \subseteq I \} \]

It is easy to see that \( O_I(I) \) is an \( R \)-order in \( A \). One similarly defines the right order \( O_r(I) \) of \( I \).

Assume \( R \) is a complete discrete valuation ring with a uniformizer \( \pi \) and fraction field \( K \). Let \( f = (f_0, \ldots, f_{d-1}) \) be a \( d \)-tuple of non-negative integers such that \( \sum_{i=0}^{d-1} f_i = d \). Denote by \( \mathbb{M}_d(f, R) \) the subgroup of \( \mathbb{M}_d(R) \) consisting of matrices of the form \((m_{ij})\), where \( m_{ij} \) ranges over all \( f_i \times f_j \) matrices with entries in \( R \) if \( i \geq j \), and over all \( f_i \times f_j \) matrices with entries in \( \pi R \) if \( i < j \) (a block of size 0 is assumed to be empty), e.g., if \( f = (d, 0, \ldots, 0) \) then \( \mathbb{M}_d(f, R) = \mathbb{M}_d(R) \).

**Theorem 2.1.** Let \( \Lambda \) be an \( R \)-order in \( \mathbb{M}_d(K) \). \( \Lambda \) is maximal if and only if there is an invertible element \( u \in \mathbb{M}_d(K) \) such that \( u\Lambda u^{-1} = \mathbb{M}_d(R) \); \( \Lambda \) is hereditary if and only if \( u\Lambda u^{-1} = \mathbb{M}_d(f, R) \) for some \( f \) (uniquely determined up to permutation of its entries).

**Proof.** See Theorems 17.3 and 39.14 in [14]. \( \square \)

When \( \Lambda \) is a hereditary order as in Theorem 2.1, we shall call \( f \) the type of \( \Lambda \). (This is slightly different from the terminology used in [14] p. 360.)

Let \( A \) be a central simple algebra over \( F \). An \( \mathcal{O}_X \)-order in \( A \) is a coherent locally free sheaf \( \mathcal{A} \) of \( \mathcal{O}_X \)-algebras with generic fibre \( A \). The \( \mathcal{O}_X \)-order \( \mathcal{A} \) is maximal (resp. hereditary) if for every open affine \( U = \text{Spec}(R) \subset X \) the set of sections \( \mathcal{A}(U) := \Gamma(U, \mathcal{A}) \) is a maximal (resp. hereditary) \( R \)-order in \( A \). For \( x \in |X| \) we denote \( \mathcal{A}_x := A \otimes_F F_x \) and \( \mathcal{A}_x := A \otimes_{\mathcal{O}_X} \mathcal{O}_x \), so \( \mathcal{A}_x \) is isomorphic to a subring of \( \mathcal{A}_x \). \( \mathcal{A} \) is maximal (resp. hereditary) if and only if \( \mathcal{A}_x \) is a maximal (resp. hereditary) \( \mathcal{O}_x \)-order in \( \mathcal{A}_x \).

Let \( \mathcal{A} \) be a hereditary \( \mathcal{O}_X \)-order. Let \( \mathcal{I} \) be a coherent sheaf on \( X \) which is a locally free rank-1 right \( \mathcal{A} \)-module. (The action of \( \mathcal{A} \) on \( \mathcal{I} \) extends the action of \( \mathcal{O}_X \).) The generic fibre \( \mathcal{I} \otimes_{\mathcal{O}_X} F \) is isomorphic to \( A \) as an \( F \)-vector space. Define a sheaf \( O_t(\mathcal{I}) \) on \( X \) as follows. For an open affine \( U \subset X \) let

\[ O_t(\mathcal{I})(U) = O_t(\mathcal{I}(U)) \]
It is easy to see that $O_{\ell}(I)$ is an $O_N$-order in $A$, which is locally isomorphic to $A$.

3. **Dieudonné modules**

Let $R$ be a complete discrete valuation ring of positive characteristic $p > 0$ and residue field $\mathbb{F}_q$. Fix a uniformizer $\pi$ of $R$ and identify $R = \mathbb{F}_q[[\pi]]$. Let $K$ be the fraction field of $R$. Let $\mathcal{R} = R \widehat{\otimes}_{\mathbb{F}_q} k \cong k[[\pi]]$ be the completion of the maximal unramified extension of $R$, and $\mathcal{K} = K \widehat{\otimes}_{\mathbb{F}_q} k \cong k[[\pi]]$ be the field of fractions of $\mathcal{R}$. We will denote the canonical lifting of $\mathbb{F}_q[\pi] \to \text{Aut}(\mathcal{K})$ by the same symbol, so

$$\text{Fr}_q \left( \sum_{i=0}^{\infty} a_i \pi^i \right) = \sum_{i=0}^{\infty} a_i^q \pi^i, \quad n \in \mathbb{Z}.$$ 

The following definition and Theorem 3.2 below are due to Drinfeld [8]; see also [12, §2.4].

**Definition 3.1.** A **Dieudonné $R$-module over $k$** is a free $\mathcal{R}$-module of finite rank $M$ endowed with an injective $\mathbb{F}_q$-linear map $\varphi : M \to M$ such that the cokernel of $\varphi$ is finite dimensional as a $k$-vector space. The **rank** of $(M, \varphi)$ is the rank of $M$ as a $\mathcal{R}$-module. A **Dieudonné $K$-module over $k$** is a finite dimensional $K$-vector space $N$ endowed with a bijective $\mathbb{F}_q$-linear map $\varphi : N \to N$. The **rank** of $(N, \varphi)$ is the dimension of $N$ as a $K$-vector space. A **morphism** of Dieudonné $R$-modules (resp. $K$-modules) over $k$ is a linear map between the underlying $\mathcal{R}$-modules (resp. $K$-vector spaces) which commutes with the $\mathbb{F}_q$-linear maps $\varphi$. If $(M, \varphi)$ is a Dieudonné $R$-module over $k$, then $(K \otimes_R M, K \otimes_R \varphi)$ is a Dieudonné $K$-module over $k$.

Let $K\{\tau\}$ be the non-commutative polynomial ring with commutation rule $\tau \cdot a = \text{Fr}_q(a)\tau$, $a \in K$. For each pair of integers $(r, s)$ with $r \geq 1$ and $(r, s) = 1$, let

$$N_{r,s} = K\{\tau\}/K\{\tau\}(\tau^r - \pi^s).$$ 

Then $(N_{r,s}, \varphi_{r,s})$, where $\varphi_{r,s}$ is the left multiplication by $\tau$, is a Dieudonné $K$-module over $k$ of rank $r$.

**Theorem 3.2.** The category of Dieudonné $K$-modules over $k$ is $K$-linear and semi-simple. Its simple objects are $(N_{r,s}, \varphi_{r,s})$, $r, s \in \mathbb{Z}$, $r \geq 1$, $(r, s) = 1$. The $K$-algebra of endomorphisms $D_{r,s} = \text{End}(N_{r,s}, \varphi_{r,s})$ of such an object is the central division algebra over $K$ with invariant $-s/r$.

This theorem implies that given a Dieudonné $K$-module $(N, \varphi)$, its endomorphism algebra $\text{End}(N, \varphi)$ is a finite dimensional semi-simple $K$-algebra such that the center of each simple component is $K$. It is clear that for a Dieudonné $R$-module $(M, \varphi)$, the endomorphism ring $\text{End}(M, \varphi)$ is an $R$-order in $\text{End}(N, \varphi)$, where $(N, \varphi) = K \otimes (M, \varphi)$.

**Proposition 3.3.** Let $(M, \varphi)$ be a Dieudonné $R$-module over $k$ of rank $n$. Suppose $\varphi(M) = M$. Then

$$(M, \varphi) \cong (R^n \widehat{\otimes}_{\mathbb{F}_q} k, \text{Id} \widehat{\otimes}_{\mathbb{F}_q} \text{Fr}_q).$$

**Proof.** This is proven in [8] Prop. 2.5] using $\pi$-divisible groups. An alternative argument is as follows. Let $(N, \varphi)$ be the associated Dieudonné $K$-module. By [12] Prop. 2.4.6], the assumption of the proposition is equivalent to

$$(N, \varphi) \cong (N_{1,0}, \varphi_{1,0})^n.$$
Hence
\[(N, \varphi) \cong (N^\varphi \hat{\otimes}_{\mathbb{F}_q} k, \text{Id} \hat{\otimes}_{\mathbb{F}_q} \text{Fr}_q),\]
where \(N^\varphi = \{ a \in N \mid \varphi(a) = a \}\) is an \(n\)-dimensional \(K\)-vector space. Since \(N = M \otimes K, M^\varphi := M \cap N^\varphi\) is a full \(R\)-lattice in \(N^\varphi\) and \(M = M^\varphi \hat{\otimes}_{\mathbb{F}_q} k\).

Let \(D\) be the \(d^2\)-dimensional central division algebra over \(K\) with invariant \(1/d\). Let \(D\) be the maximal \(R\)-order in \(D\). Denote by \(R_d = \mathbb{F}_q[[\pi]]\) the ring of integers of the degree \(d\) unramified extension of \(K\). We can identify \(D\) with the \(R\)-algebra \(R_d[\Pi]\) of non-commutative formal power series in the indeterminate \(\Pi\) satisfying the relations
\[
\Pi a = \text{Fr}_q(a)\Pi \quad \text{for any } a \in R_d,
\]
\[
\Pi^d = \pi.
\]

**Definition 3.4.** A Dieudonné \(R\)-module \((M, \varphi)\) over \(k\) is connected if for all large enough positive integers \(m, \varphi^m(M) \subset \pi M\). This is equivalent to saying that \((N_{1,0}, \varphi_{1,0})\) does not appear in the decomposition of the associated Dieudonné \(K\)-module into simple factors. A Dieudonné \(D\)-module over \(k\) is a rank-\(d^2\) connected Dieudonné \(R\)-module \((M, \varphi)\) over \(k\) equipped with a right \(D\)-action which commutes with \(\varphi\) and extends the natural action of \(R\). A morphism of Dieudonné \(D\)-modules is a morphism of the underlying Dieudonné \(R\)-modules which commutes with the action of \(D\). For each Dieudonné \(D\)-module \((M, \varphi)\) over \(k\) there is an associated Dieudonné \(D\)-module \((N, \varphi) = K \otimes (M, \varphi)\).

A Dieudonné \(D\)-module \((M, \varphi)\) over \(k\) is naturally a right \(D\hat{\otimes}_{\mathbb{F}_q} k\)-module. Fix an embedding \(\mathbb{F}_q \hookrightarrow k\). Since \(\mathbb{F}_q\) is also embedded in \(D\), we obtain a grading
\[M = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} M_i,\]
where \(M_i = \{ m \in M \mid m(\lambda \hat{\otimes} 1) = m(1 \hat{\otimes} \lambda')\), \(\lambda \in \mathbb{F}_q\}\}. Each \(M_i\) is a free finite rank \(\mathcal{R}\)-module. The action of \(\Pi \hat{\otimes} 1\) on \(M\) induces injective linear maps \(\Pi : M_i \rightarrow M_{i+1},\ i \in \mathbb{Z}/d\mathbb{Z}\). The composition
\[M_i \xrightarrow{\Pi} M_{i+1} \xrightarrow{\Pi} \cdots \xrightarrow{\Pi} M_{i+d-1} \xrightarrow{\Pi} M_{i+d} = M_i\]
is \(M_i(\pi \hat{\otimes} 1) = \pi M_i\). In particular, all \(M_i\) have the same rank over \(\mathcal{R}\), which must be \(d\), since the rank of \(M\) is \(d^2\). Since \(\dim_k(\text{coker}(\pi)) = d^2\), we have \(\dim_k(\text{coker}(\Pi)) = d\).

Similarly, \(\varphi\) induces injective \(\text{Fr}_q\)-linear maps
\[M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \cdots \xrightarrow{\varphi_{i+d-1}} M_{i+d} = M_i.\]
Let \(f_i := \dim_k(\text{coker}(\varphi_i))\), so \(\sum_{i=0}^{d-1} f_i = \dim_k(\text{coker}(\varphi))\). We call the ordered \(d\)-tuple \(f = (f_0, \ldots, f_{d-1})\) the type of \(M\).

**Definition 3.5.** (cf. [10], [11]) A Dieudonné \(D\)-module \((M, \varphi)\) over \(k\) is exceptional if \(\text{Im}(\varphi) = \text{Im}(\Pi)\). \((M, \varphi)\) is special if \(f_i = 1\) for all \(i\). \((M, \varphi)\) is superspecial if it is special and exceptional.

Note that \((M, \varphi)\) being exceptional is equivalent to \(\text{Im}(\varphi_i) = \text{Im}(\Pi_i)\) for all \(i \in \mathbb{Z}/d\mathbb{Z}\), i.e., every index of \(M\) is critical in the terminology of [10] Def. II.1.3]. In particular, if \((M, \varphi)\) is exceptional of type \(f\) then \(\sum_{i=0}^{d-1} f_i = d\).
Proposition 3.6. Let \((M, \varphi)\) be an exceptional Dieudonné \(D\)-module over \(k\) of type \(f\). Then \(\text{End}_D(M, \varphi) \cong M_d(f, R)\). In particular, \(\text{End}_D(M, \varphi)\) is a hereditary \(R\)-order in \(\text{End}_D(N, \varphi) \cong M_d(K)\).

Proof. Using the injections \(\Pi_i\), we can identify all \(M_i \otimes K\) with the same \(d\)-dimensional \(K\)-vector space \(V\). Then \(\Pi_i\) induces a bijective linear map \(V \to V\), and \(\varphi_i\) induces a bijective \(\text{Fr}_q\)-linear map. Consider \(\Pi_i^{-1} \circ \varphi_i : V \to V\) as a bijective \(\text{Fr}_q\)-linear map. Since \((M, \varphi)\) is exceptional, \(\text{Im}(\Pi_i) = \text{Im}(\varphi_i)\) for all \(i \in \mathbb{Z}/d\mathbb{Z}\).

Hence \(\Pi_i^{-1} \circ \varphi_i\) is bijective on \(M_i\). By Proposition 3.3, there are \(d\) full \(R\)-lattices \(\Lambda_i\) in \(K^d\), \(i \in \mathbb{Z}/d\mathbb{Z}\), such that \(M_i = \Lambda_i \hat{\otimes} k\). Since the action of \(D\) commute with the action of \(\varphi\), we have \(\varphi_{i+1} \circ \Pi_i = \Pi_{i+1} \circ \varphi_i\). This implies that the identifications \(M_i = \Lambda_i \hat{\otimes} k\) can be made compatibly so that

\[
\Lambda_0 \xrightarrow{\pi_0} \Lambda_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{d-2}} \Lambda_{d-1} \xrightarrow{\pi_{d-1}} \Lambda_0,
\]

where \(\pi_i\)'s are injections, \(\Pi_i = \pi_i \hat{\otimes} k\), \(\varphi_i = \pi_i \hat{\otimes} \text{Fr}_q\). The inclusions \(\pi_i\) satisfy

\[
\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{i+d-1} = \pi,
\]

so each \(\text{coker}(\pi_i)\) has no nilpotents and \(\dim_k(\text{coker}(\pi_i)) = f_i\).

Now it is easy to see that giving an endomorphism of \((M, \varphi)\) commuting with the action of \(D\) is equivalent to giving an endomorphism \(g\) of \(K^d\) which preserves the flag of lattices (3.1). Such endomorphisms form an \(R\)-algebra isomorphic to \(M_d(k, R)\). That this is a hereditary order in \(M_d(K)\) follows from Theorem 2.1. □

Every Dieudonné \(D\)-module over \(k\) satisfies the properties in [10] p. 20, hence corresponds to a formal \(D\)-module of height \(d^2\). It is instructive to give explicit examples of such formal modules. What follows below is motivated by [10] I.4.2.

The underlying formal group is isomorphic to \(\mathcal{G}_{a,k}^d\), where \(\mathcal{G}_{a,k} = \text{Spf}(k[[t]])\) is the formal additive group. Denote by \(\tau\) the Frobenius isogeny of \(\mathcal{G}_{a,k}\) corresponding to \(t \mapsto t^a\). To give a formal \(D\)-module essentially amounts to giving an embedding

\[
\Phi : F_q^d[[\Pi]] = D \hookrightarrow \text{End}(\mathcal{G}_{a,k}^d) \cong M_d(k\{\{\tau\}\}),
\]

where \(k\{\{\tau\}\}\) is the non-commutative ring of formal power series in \(\tau\) satisfying \(\tau a = a \tau\) for all \(a \in k\).

Now let \((M, \varphi)\) be an exceptional Dieudonné \(D\)-module of type \(f\). Being exceptional, i.e., \(\text{Im}(\Pi) = \text{Im}(\varphi)\), translates into

\[
\Phi(\Pi) = \tau \cdot \text{Id}.
\]

The type translates into

\[
\Phi(\lambda) = \text{diag}(\chi_{ij}(\lambda))_{0 \leq j \leq d-1, 1 \leq i \leq f_j}, \lambda \in F_q^d,
\]

where \(\chi_{ij}(\lambda) = \lambda^{q^j}\) if \(f_j \neq 0\) and is omitted from \(\text{diag}(\cdot)\) otherwise. For example, if \(d = 3\) and \(f = (2, 0, 1)\) then

\[
\Phi(\Pi) = \begin{pmatrix}
\tau & 0 & 0 \\
0 & \tau & 0 \\
0 & 0 & \tau
\end{pmatrix}
\quad \text{and} \quad
\Phi(\lambda) = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^{q^2}
\end{pmatrix}.
\]

The endomorphism ring \(\text{End}_D(M, \varphi)\) is isomorphic to the opposite algebra of the centralizer of \(\Phi(D)\) in \(M_d(k\{\{\tau\}\})\). One can check as in [10] I.4.2 that this centralizer is isomorphic to \(M_d(f, R)^{\text{opp}}\).
4. \(\mathcal{D}\)-ELLiptic sheaves

In this section we recall the definition of \(\mathcal{D}\)-elliptic sheaves of finite characteristic and their basic properties as given in [13, Ch. 9].

Fix a closed point \(\infty \in |X|\). Let \(D\) be a central simple algebra over \(F\) of dimension \(d^2\). Assume \(D\) is split at \(\infty\), i.e., \(D \otimes_F F_\infty \cong M_d(F_\infty)\). Fix a maximal \(\mathcal{O}_X\)-order \(\mathcal{D}\) in \(D\). Denote by Ram \(\subset |X|\) the set of places where \(D\) is ramified; hence for all \(x \not\in\) Ram the couple \((D_x, D_\infty)\) is isomorphic to \((M_d(F_x), M_d(O_x))\). Fix another closed point \(o \in |X| - \infty\), and an embedding \(\mathbb{F}_o \hookrightarrow k\). Let \(z\) be the morphism determined by these choices

\[
\begin{align*}
z : \text{Spec}(k) & \rightarrow \text{Spec}(\mathbb{F}_o) 
\hookrightarrow X.
\end{align*}
\]

**Definition 4.1.** A \(\mathcal{D}\)-elliptic sheaf of characteristic \(o\) over \(k\) is a sequence \(E = (E_i, j_i, t_i)_{i \in \mathbb{Z}}\), where \(E_i\) is a locally-free \(\mathcal{O}_X \otimes_k k\)-module of rank \(d^2\), equipped with a right action of \(\mathcal{D}\) which extends the \(\mathcal{O}_X\)-action, and

\[
\begin{align*}
\jmath_i : E_i & \hookrightarrow E_{i+1} \\
t_i : E_i := (\text{Id}_X \otimes \text{Fr}_q)^* E_i & \hookrightarrow E_{i+1}
\end{align*}
\]

are injective \(\mathcal{D}\)-linear homomorphisms. Moreover, for each \(i \in \mathbb{Z}\) the following conditions hold:

1. The diagram

\[
\begin{array}{ccc}
E_i & \xrightarrow{\jmath_i} & E_{i+1} \\
\downarrow t_{i-1} & & \downarrow t_i \\
\tau E_{i-1} & \xrightarrow{\tau_j=\tau_{j-1}} & \tau E_i
\end{array}
\]

commutes;

2. \(E_{i+d-\text{deg}(\infty)} = E_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)\), and the inclusion

\[
E_i \xrightarrow{\jmath_i} E_{i+1} \xrightarrow{\jmath_{i+1}} \cdots \rightarrow E_{i+d-\text{deg}(\infty)} = E_i \otimes_{\mathcal{O}_X} \mathcal{O}_X(\infty)
\]

is induced by \(\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\infty)\);

3. \(\dim_k H^0(X \otimes k, \ker j_i) = d\);

4. \(E_i/t_{i-1}(\tau E_{i-1}) = z_i \mathcal{H}_i\), where \(\mathcal{H}_i\) is a \(d\)-dimensional \(k\)-vector space.

When \(o \not\in\) Ram, the previous definition is exactly the one found in [13, p. 260]. When \(o \in\) Ram this definition is not restrictive enough, but for our purposes it is adequate to take it as a starting point.

**Definition 4.2.** Let DES be the category whose objects are the \(\mathcal{D}\)-elliptic sheaves of characteristic \(o\) over \(k\), and a morphism between two objects in this category

\[
\psi = (\psi_i)_{i \in \mathbb{Z}} : E' = (E'_i, j'_i, t'_i)_{i \in \mathbb{Z}} \rightarrow E'' = (E''_i, j''_i, t''_i)_{i \in \mathbb{Z}}
\]

is a sequence of sheaf morphisms \(\psi_i : E'_i \rightarrow E''_i\) which are compatible with the action of \(\mathcal{D}\) and commute with the morphisms \(j_i\) and \(t_i\):

\[
\psi_{i+1} \circ j'_i = j''_i \circ \psi_i \quad \text{and} \quad \psi_i \circ t'_{i-1} = t''_i \circ \psi_{i-1}.
\]

Denote by \(\text{Hom}(E', E'')\) the set of all morphisms \(E' \rightarrow E''\), and let \(\text{End}(E) = \text{Hom}(E, E)\).
Definition 4.3 ([3]). A \( \varphi \)-space over \( k \) is a finite dimensional \( F \otimes_{\mathbb{Q}_l} k \)-vector space \( N \) equipped with a bijective \( F \otimes_{\mathbb{Q}_l} Fr_q \)-linear map \( \varphi : N \to N \). A morphism \( \alpha \) between two \( \varphi \)-spaces \( (N', \varphi') \) and \( (N'', \varphi'') \) is a \( F \otimes_{\mathbb{Q}_l} k \)-linear map \( N' \to N'' \) such that \( \varphi'' \circ \alpha = \alpha \circ \varphi' \).

Let \( \mathbb{E} \in \text{DES} \). Denote \( N = H^0(\text{Spec}(F \otimes_{\mathbb{Q}_l} k), \mathcal{E}_0) \). This is a free \( D \otimes_{\mathbb{Z}_l} k \)-module of rank \( 1 \). The \( t \)'s induce a bijective \( F \otimes_{\mathbb{Q}_l} Fr_q \)-linear map \( \varphi : N \to N \), compatible with the action of \( D \) on the right, so to \( \mathbb{E} \) one can attach a \( \varphi \)-space over \( k \) equipped with an action of \( D \), which commutes with \( \varphi \). This action induces an \( F \)-algebra homomorphism

\[
\iota : D^{\text{opp}} \to \text{End}(N, \varphi).
\]

We denote by \( \text{End}_D(N, \varphi) \) the \( F \)-algebra of endomorphisms of \( (N, \varphi) \) which commute with the action of \( D \). The triple \( (N, \varphi, \iota) \) is called the generic fibre of \( \mathbb{E} \) ([13 Def. 9.2]). It is independent of the choice of \( E \) since the sheaves \( \mathcal{E}_i \) are isomorphic over \( (X - \infty) \otimes k \) via \( j \)'s.

For \( x \in |X| \), denote \( M_x := H^0(\text{Spec}(O_x \otimes \mathbb{Q}_l), \mathcal{E}_0) \). This is a free \( O_x \otimes_{\mathbb{Q}_l} k \)-module of rank \( d^2 \) with a right action of \( D_x \). Let \( N_x = F_x \otimes_{O_x} M_x \). The \( t \)'s induce a bijective \( F_x \otimes_{\mathbb{Q}_l} Fr_q \)-linear map \( \varphi_x : N_x \to N_x \), compatible with the action of \( D_x \). The pair \( (N_x, \varphi_x) \) is the Dieudonné module of \( \mathbb{E} \) at \( x \). The \( F_x \)-algebra of endomorphisms of \( (N_x, \varphi_x) \) commute with the action of \( D_x \) will be denoted by \( \text{End}_{D_x}(N_x, \varphi_x) \). Note that \( (N_x, \varphi_x) = (F_x \otimes_{\mathbb{Q}_l} N, F_x \otimes_{\mathbb{Q}_l} \varphi) \).

As easily follows from definitions, the lattices \( M_x \) have the following properties (see [13 Lem. 9.3]):

(M1) If \( x = \infty \), then

\[
M_\infty \subset \varphi_\infty(M_\infty)
\]

\[
dim_k(\varphi_\infty(M_\infty)/M_\infty) = d
\]

\[
\varphi_\infty^{d \deg(\infty)}(M_\infty) = \pi^{-1}_\infty M_\infty.
\]

(M2) If \( x = o \), then

\[
\pi_o M_o \subset \varphi_o(M_o) \subset M_o
\]

and the \( \mathbb{F}_q \otimes_{\mathbb{Q}_l} k \)-module \( M_o/\varphi_o(M_o) \) is of length \( d \) and is supported on the connected component of \( \text{Spec}(\mathbb{F}_q \otimes_{\mathbb{Q}_l} k) \) which is the image of \( z \).

(M3) If \( x \neq o, \infty \), then

\[
\varphi_x(M_x) = M_x.
\]

(M4) Some basis of \( N \) generates \( M_x \) in \( N_x \) for all but finitely many \( x \in |X| \).

Definition 4.4. Let \( \text{DMod} \) be the category whose objects are the pairs

\[(N, \varphi, \iota), (M_x)_{x \in |X|} \]

where \( (N, \varphi) \) is a \( \varphi \)-space of rank \( d^2 \) over \( F \otimes k \), \( \iota : D^{\text{opp}} \to \text{End}(N, \varphi) \) is an \( F \)-algebra homomorphism, and \( (M_x)_{x \in |X|} \) is a collection of \( D_x \)-lattices in \( (N_x, \varphi_x) = (F_x \otimes_{\mathbb{Q}_l} N, F_x \otimes_{\mathbb{Q}_l} \varphi) \) which satisfy (M1)-(M4). A morphism \( \alpha \) between two such objects

\[
\alpha : ((N', \varphi', \iota'), (M'_x)_{x \in |X|}) \to ((N'', \varphi'', \iota''), (M''_x)_{x \in |X|})
\]

is a morphism of the \( \varphi \)-spaces \( \alpha : (N', \varphi') \to (N'', \varphi'') \) such that

\[
\iota'' \circ \alpha = \alpha \circ \iota'
\]

and

\[
\alpha \otimes F_x(M'_x) \subset M''_x
\]

for all \( x \in |X| \).
Proposition 4.5. The functor $\text{DES} \to \text{DMod}$ which associates to a $D$-elliptic sheaf of characteristic $0$ over $k$ its generic fibre along with the lattices $M_x$ in its Dieudonné modules is an equivalence of categories.

Proof. From the description of a locally free sheaf on a curve through lattices in its generic fibre, it follows that the functor in question is fully faithful. Now let $((N, \varphi, \iota), (M_x)_{x \in [X]}) \in \text{DMod}$ and $i \in \mathbb{Z}$. Define a sheaf $\mathcal{E}_i$ on $X \otimes_{\mathbb{F}_q} k$ as follows. Let $U \subset X$ be an open affine. If $\infty \notin |U|$, then let

$$\mathcal{E}_i(U \otimes_{\mathbb{F}_q} k) := \bigcap_{x \in |U|} (N \cap M_x),$$

where the inner intersections are taken in $N_x$, and the outer in $N$. If $\infty \in |U|$, let

$$\mathcal{E}_i(U \otimes_{\mathbb{F}_q} k) := \left( \bigcap_{x \in |U| - \infty} (N \cap M_x) \right) \bigcap (\varphi^i_\infty(M_\infty) \cap N).$$

Thanks to (M4), $\mathcal{E}_i$ is a locally-free $O_{X \otimes_{\mathbb{F}_q} k}$-module of rank $d^2$. The inclusions $\varphi_\infty^i(M_\infty) \subset \varphi_\infty^{i+1}(M_\infty)$ induce inclusions $j_i : \mathcal{E}_i \hookrightarrow \mathcal{E}_{i+1}$. The action of $\varphi$ on $N$ induces homomorphisms $t_i : \mathcal{E}_i \to \mathcal{E}_{i+1}$. The action of $D$ on $(N, \varphi)$ and $D_x$ on $M_x$, defines an action of $D$ on $\mathcal{E}_i$ compatible with $j_i$ and $t_i$. Finally, the conditions (M1)-(M3) ensure that $(\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}} \in \text{DES}$. Hence our functor is essentially surjective. $\square$

Let $x \in |X|$ and $r := [\mathbb{F}_x : \mathbb{F}_q]$. Since $N_x$ is a free $F_x \otimes_{\mathbb{F}_q} k$-module, by fixing an embedding $\mathbb{F}_x \to k$, we obtain two actions of $\mathbb{F}_x$ on $N_x$. These actions induce a grading

$$N_x = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} N_{x,i},$$

where $N_{x,i} = \{ a \in N_x \mid (\lambda^i \otimes 1) a = (1 \otimes \lambda) a, \lambda \in \mathbb{F}_x \}$. Now $\varphi_x$ maps $N_{x,i}$ bijectively into $N_{x,i+1}$, and $N_{x,0}$ is an $F_x \otimes_{\mathbb{F}_q} k$-vector space. Hence $(N_{x,0}, \varphi_x^o)$ is a Dieudonné $F_x$-module over $k$ in the sense of [B]. We can recover $(N_x, \varphi_x)$ uniquely from $(N_{x,0}, \varphi_x^o)$ since as $F_x \otimes_{\mathbb{F}_q} k$-vector space

$$N_x \cong \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} N_{x,0}$$

with the action of $\varphi_x$ given by

$$(a_0, a_1, \ldots, a_{r-1}) \mapsto (\varphi_x^o(a_{r-1}), a_0, \ldots, a_{r-2}).$$

Finally, since the action of $D_x$ commutes with $\varphi_x$, $(N_{x,0}, \varphi_x^o)$ is a Dieudonné $D_x$-module over $k$ and $\text{End}_{D_x}(N_{x,0}, \varphi_x) = \text{End}_{D_x}(N_{x,0}, \varphi_x^o)$. Similar argument applies also to the lattices $M_x (x \neq \infty)$ and produces Dieudonné $D_x$-modules over $k$.

5. Endomorphism rings

Let $D$ be as in [4]. Assume $D$ is a division algebra such that $D_o$ is the $d^2$-dimensional central division algebra over $F_o$ with invariant $1/d$. In this case $D_o$ is
the unique maximal order of $D_o$ which we identify with $\mathbb{F}_{o}^{(d)}[\Pi_o]$. Here $\mathbb{F}_{o}^{(d)}$ is the degree $d$ extension of $\mathbb{F}_{o}$ and
\[
\Pi_o a = \mathbb{F}_{q}^{\deg(o)}(a)\Pi_o,
\]
\[
\Pi_o^d = \pi_o.
\]

**Definition 5.1.** Let $E \in \text{DES}$. We say that $E$ is exceptional if
\[
\varphi_o^{\deg(o)}(M_o) = M_o \cdot \Pi_o.
\]
Clearly $E$ is exceptional if and only if the Dieudonné $D_o$-module $(M_o, \varphi_o^{\deg(o)})$ associated to $(M_o, \varphi_o)$ is exceptional in the sense of Definition 5.5. The type of an exceptional $E$ is the type of $(M_o, \varphi_o^{\deg(o)})$. Similarly, we say that $E$ is special (resp. superspecial) if $(M_o, \varphi_o^{\deg(o)})$ is special (resp. superspecial).

**Remark 5.2.** Exceptional $D$-elliptic sheaves do not correspond to points on the moduli schemes constructed in [11], unless they are superspecial.

Let $\bar{D}$ be the central division algebra over $F$ with invariants
\[
\text{(5.1)} \quad \text{inv}_x(\bar{D}) = \begin{cases} 
1/d, & x = \infty; \\
0, & x = o; \\
\text{inv}_x(D), & x \neq o, \infty.
\end{cases}
\]

**Theorem 5.3.** If $E$ is exceptional of type $\mathbf{f}$, then $\text{End}(E)$ has a natural structure of a hereditary $\mathcal{O}_X$-order in $\bar{D}$. This order is maximal at every $x \in |X| - o$, and at $o$ it is isomorphic to $\mathcal{M}_d(\mathbf{f}, \mathcal{O}_o)$.

**Proof.** Let $((N, \varphi, t), (M_x)_{x \in |X|}) \in \text{DMod}$ be the object attached to $E$ by Proposition 4.5. Giving an endomorphism of $E$ is equivalent to giving
\[
\psi \in \text{End}(N, \varphi, t) = \text{End}_D(N, \varphi)
\]
such that $\psi \otimes_F F_x \in \text{End}(N_x, \varphi_x)$ preserves the lattice $M_x$ for all $x \in |X|$. Let $(\bar{F}, \bar{\Pi})$ be the $\varphi$-pair of $(N, \varphi)$; see [13] App. A for the definition. Since $E$ is exceptional
\[
\varphi_o^{d \deg(o)}(M_\infty) = \pi_{\infty}^{-1} M_\infty,
\]
\[
\varphi_o^{\deg(o)}(M_o) = \pi_o M_o,
\]
\[
\varphi_x(M_x) = M_x, \quad \text{if } x \neq o, \infty.
\]
Let $h$ be the class number of $X$. The divisor $h(\deg(o) - \deg(o)\infty)$ is principal, so from the previous equalities $\varphi^{dh} \in F$. By construction of $(\bar{F}, \bar{\Pi})$, this implies $\bar{F} = F$ and $\bar{\Pi} \in F$ has valuations
\[
\text{(5.2)} \quad \text{ord}_x(\bar{\Pi}) = \begin{cases} 
1/d \deg(o), & x = o; \\
-1/d \deg(\infty), & x = \infty; \\
0, & x \neq o, \infty.
\end{cases}
\]
Since $(N, \varphi)$ is isotypical [13 Lem. 9.6], [5.2] and [13] Thm. A.6] imply that $A = \text{End}(N, \varphi)$ is the central simple algebra over $F$ of dimension $d^4$ with invariants
\[
\text{inv}_o(A) = -1/d, \text{inv}_\infty(A) = 1/d, \text{inv}_x(A) = 0, \quad x \neq o, \infty. \quad \text{End}_D(N, \varphi) \text{ is exactly the centralizer of } \iota(D^{\text{pp}}) \text{ in } \text{End}(N, \varphi). \quad \text{By the double centralizer theorem [14 Cor. 7.14]}
\]
\[
\text{End}_D(N, \varphi) \otimes_F D^{\text{pp}} \cong A.
\]
This implies that $\text{End}_D(N, \varphi)$ is the central simple algebra over $F$ of dimension $d^2$ with invariants

$$\text{inv}_x(\text{End}_D(N, \varphi)) = \text{inv}_x(A) - \text{inv}_x(D^\text{opp}) = \text{inv}_x(A) + \text{inv}_x(D) \mod \mathbb{Z}.$$}

Comparing the invariants, we see that $\text{End}_D(N, \varphi) \cong \bar{D}$.

If $x \neq o$. The same isomorphism for $x = o$ follows from Proposition 3.6. Hence (5.3) becomes an isomorphism after tensoring with $\mathcal{O}$, and therefore is an isomorphism itself. To finish the proof we need to show that $\text{End}_{\mathcal{D}_x}(M_x, \varphi_x)$ is maximal for every $x \neq o$, and is hereditary for $x = o$.

If $x = o$, then by Proposition 3.6,

$$\text{End}_{\mathcal{D}_x}(M_x, \varphi_x) \cong \text{End}_{\mathcal{D}_x}(M_{x,0}, \varphi_{x,0}^{\text{deg}(x)}) \cong \mathcal{M}_{d}(f, \mathcal{O}_x).$$

If $x \neq o, \infty$, then $\varphi_x(M_x) = M_x$. By Proposition 3.3,

$$(M_{x,0}, \varphi_{x,0}^{\text{deg}(x)}) \cong (A_x \otimes F_{\mathbb{F}_q}, \text{Id} \otimes F_{\mathbb{F}_q}^{\text{deg}(x)}),$$

where $A_x$ is a free $\mathcal{O}_x$-module of rank $d^2$. The action of $\mathcal{D}_x$ commutes with $\varphi_x$, so $\mathcal{D}_x$ is in the right order of the full $\mathcal{O}_x$-lattice $A_x$ in $D_x$. Since $\mathcal{D}_x$ is maximal, the left order $O_1(A_x)$ of $A_x$ is also maximal in $D_x \cong \bar{D}_x$; see [14 (17.6)]. On the other hand, $O_1(A_x) \subseteq \text{End}_{\mathcal{D}_x}(M_{x,0}, \varphi_{x,0}^{\text{deg}(x)})$, which forces $\text{End}_{\mathcal{D}_x}(M_x, \varphi_x)$ to be maximal.

Finally, let $x = \infty$. By [13 Lem. 9.8],

$$(N_x, \varphi_x) \cong (N_{d,-1}, \varphi_{d,-1})^d,$$

with the action of $\mathcal{D}_x$ being the natural right action of $\mathcal{M}_{d}(O_{\infty})$. By definition, this action preserves $M_x$, so Morita equivalence [14 Ch. 4] reduces the problem to showing the following:

In the notation of [13] if $M$ is a free $\mathcal{R}$-module in $N_{d,-1}$ such that $M \otimes_R K = N_{d,-1}$ and $M \subseteq \varphi_{d,-1}(M)$, then $\text{End}(M, \varphi_{d,-1})$ is a maximal order in the central division algebra over $K$ with invariant $1/d$. Let $M$ and $M'$ be any two such lattices. By [13 Prop. B.10], $M' = \varphi_{d,-1}^n(M)$ for some $n \in \mathbb{Z}$, so $\text{End}(M, \varphi_{d,-1}) = \text{End}(M', \varphi_{d,-1})$. Therefore, we can assume that $M$ is the free left $\mathcal{R}$-module generated by $1, \tau, \ldots, \tau^{d-1}$ in $K[\tau]/K[\tau](\tau^d - \pi^{-1})$ with $\varphi_{d,-1} = \tau$. Note that

$$R_d\{\tau^{-1}\}/R_d\{\tau^{-1}\}(\tau^{-d} - \pi) \subseteq \text{End}(M, \tau)^\text{opp}$$

But the left hand-side is a maximal order, so $\text{End}(M, \tau)$ is also a maximal order. $\square$

**Theorem 5.4.** Assume $E \in \text{DES}$ is exceptional. The map

$$E' \rightarrow I = \text{Hom}(E, E')$$
establishes a bijection between the set of isomorphism classes of exceptional $D$-elliptic sheaves $E'$ of the same type as $E$ and the isomorphism classes of locally free rank-1 right $\text{End}(E)$-modules. Under this bijection,

$$\text{End}(E') \cong O_{\ell}(I).$$

**Proof.** Denote $\mathcal{A} := \text{End}(E)$. Let $E'$ be an exceptional $D$-elliptic sheaf of the same type as $E$. Let $((N, \varphi, \iota), (M_x)_{x \in |X|})$ and $((N', \varphi', \iota'), (M'_x)_{x \in |X|})$ be the objects in $\text{DMod}_E$ attached to $E$ and $E'$, respectively, under the equivalence of Proposition $\ref{1.3}$.

From the proof of Theorem $\ref{6.3}$ we know that the $\varphi$-pairs $(\bar{F}, \bar{\Pi})$ associated to the generic fibres of $E$ and $E'$ are the same. By $\ref{13}$ (9.12), this implies that the generic fibres $(N, \varphi, \iota)$ and $(N', \varphi', \iota')$ are isomorphic. (In the terminology of $\ref{13}$ this is equivalent to saying that $E$ and $E'$ are isogenous.) Hence the Dieudonné modules $(N_x, \varphi_x)$ and $(N'_x, \varphi'_x)$ of $E$ and $E'$ are also isomorphic for all $x \in |X|$. Consider the $D_x$-lattices $M_x \subset N_x$ and $M'_x \subset N'_x$. We claim that there is an isomorphism $\alpha_x : (N_x, \varphi_x) \cong (N'_x, \varphi'_x)$ which commutes with $D_x$ and $\alpha(M_x) = M'_x$. When $x \neq 0, \infty$, this follows from Proposition $3.3$ and the fact that any two maximal orders in $D_x$ are conjugate. When $x = 0$, the claim follows from the proof of Proposition $10.5$ using the assumption that $E$ and $E'$ have the same type. Finally, when $x = \infty$ this follows from $\ref{13}$ Prop. B.10]. Moreover, thanks to (M4), if we fix an isomorphism $\alpha : (N, \varphi, \iota) \cong (N', \varphi', \iota')$, then for almost all $x$ we can take $\alpha_x = \alpha \otimes F_x$. The argument which shows that $\ref{6.3}$ is an isomorphism also implies that for $x \in |X|$ $\text{Hom}(E, E') \otimes_{\mathcal{O}_x} \mathcal{O}_x \cong \text{Hom}_{\mathcal{D}_x}((M_x, \varphi_x), (M'_x, \varphi'_x))$.

Now from what was said above we conclude that $\text{Hom}(E, E')$ is a locally free rank-1 right $\mathcal{A}$-module.

Conversely, let $I$ be a locally free rank-1 right $\mathcal{A}$-module. Define $(N', \varphi', \iota') = (N, \varphi, \iota)$ and $M'_x = I_x \otimes_{\mathcal{A}} M_x$, $x \in |X|$. It is easy to check that the pair $((N', \varphi', \iota'), (M'_x)_{x \in |X|})$ belongs to $\text{DMod}$, hence defines a $D$-elliptic sheaf of characteristic $0$ over $k$. We denote this $D$-elliptic sheaf by $I \otimes_{\mathcal{A}} E$. Since $(M_0, \varphi_0) \cong (M'_0, \varphi'_0)$, $I \otimes_{\mathcal{A}} E$ is exceptional of the same type as $E$.

There are natural morphisms $I \to \text{Hom}(E, I \otimes_{\mathcal{A}} E)$ and $\text{Hom}(E, E') \otimes_{\mathcal{A}} E \to E'$, which are locally isomorphisms, so the two constructions are inverses of each other, and the bijection of the theorem follows.

Finally, it is clear that $O_{\ell}(I) \subset \text{End}(I \otimes_{\mathcal{A}} E)$, and since both sides are locally isomorphic hereditary orders, an equality must hold. $\square$

6. Mass-formula

Denote by $\mathfrak{x}_f$ the set of isomorphism classes of exceptional $D$-elliptic sheaves of characteristic $0$ over $k$ of type $f$. Using Proposition $\ref{1.3}$ one can easily show that $\mathfrak{x}_f \neq \emptyset$; cf. the proof of $\ref{13}$ Thm. 9.13. Let $E \in \mathfrak{x}_f$. Denote $\mathcal{A} := \text{End}(E)$. Let $\mathcal{A} := \mathcal{D} \otimes_{\mathcal{A}} \mathcal{A}$ and

$$A(k_f) := \prod_{x \in |X|} A_x \hookrightarrow \mathcal{D}(k_f).$$
The ring $\tilde{D}(F)$ embeds diagonally into $\tilde{D}(\mathbb{A}_F)$. One consequence of Theorem 5.3 is that there is a bijection between $\mathfrak{X}_F$ and the double coset space

$$\tilde{D}(F)^\times \setminus \tilde{D}(\mathbb{A}_F)^\times / A(\mathbb{A}_F)^\times.$$  

The order of this double coset space is infinite, which accounts for the fact that there is a natural free action of $\mathbb{Z}$ on the category of $\mathcal{D}$-elliptic sheaves. In order to obtain “finite” spaces while trying to classify $\mathcal{D}$-elliptic sheaves, e.g. moduli schemes of finite type over $F$, one has to mod out by this action as in [13].

Let $(\mathcal{E}_i, j_i, t_i)_{i \in \mathbb{Z}}$ be a $\mathcal{D}$-elliptic sheaf. The group $\mathbb{Z}$ acts by “shifting the indices”:

$$[n](\mathcal{E}_i, j_i, t_i) = (\mathcal{E}_i', j_i', t_i')_{i \in \mathbb{Z}}$$

with $\mathcal{E}_i' = \mathcal{E}_{i+n}$, $j_i' = j_{i+n}$, $t_i' = t_{i+n}$. It is clear that $\mathbb{Z}$ preserves the set $\mathfrak{X}_F$.

Since $\tilde{D}_\infty$ is a division algebra, the composition of the reduced norm with the valuation at $\infty$ gives an isomorphism $D_\infty^\times / A_\infty^\times \cong \mathbb{Z}$.

**Lemma 6.1.** The action of $n \in \mathbb{Z}$ on the double coset space (6.1) corresponding to the action of $n$ on $\mathfrak{X}_F$ is the translation by $n$ on $\tilde{D}_\infty^\times / A_\infty^\times \cong \mathbb{Z}$.

**Proof.** Let $((N, \varphi, \iota), (M_x)_{x \in |X|})$ and $((N', \varphi', \iota), (M'_x)_{x \in |X|}) \in \text{DMod}$ be the objects attached to $\mathcal{E}$ and $\mathcal{E}' = [n] \cdot \mathcal{E}$, respectively, under the equivalence of Proposition 4.5. Since the restriction of $\mathcal{E}_i$ to $(X - \infty) \otimes k$ does not depend on $i$, $(N', \varphi') = (N, \varphi)$, $(M'_x, \varphi'_x) = (M_x, \varphi_x)$ if $x \neq \infty$, and $(M'_\infty, \varphi'_\infty) = (\varphi_\infty^0, M_\infty, \varphi_\infty)$. Let $\Pi_\infty$ be a generator of the maximal ideal of $A_\infty$. We conclude that the action of $1 \in \mathbb{Z}$ on $\mathfrak{X}_F$ corresponds to multiplication by $\Pi_\infty$ on the double coset space (6.1). Since $\Pi_\infty$ maps to 1 under the homomorphism $\text{ord}_\infty \circ \text{Nr} : D_\infty^\times \to \mathbb{Z}$, the lemma follows. □

Form (6.1) and Lemma 6.1, we conclude that there is a bijection between $\mathfrak{X}_F / \mathbb{Z}$ and the double coset space

$$\tilde{D}(F)^\times \setminus \tilde{D}(\mathbb{A}_F)^\times / A(\mathbb{A}_F)^\times,$$

where $\tilde{D}(\mathbb{A}_F) = \tilde{D} \otimes_F \mathbb{A}_F^\times$ and $A(\mathbb{A}_F)^\times := \prod_{x \in |X| - \infty} A_x$. By the strong approximation theorem for $\tilde{D}^\times$, this double coset space has finite cardinality. Unfortunately, in general, an explicit expression for class numbers of hereditary orders over Dedekind domains is not known (e.g. the order of the double coset space above). But one can at least give an estimate on this number using an analogue of Eichler’s mass-formula.

Denote by $\mathcal{A}^\times$ the sheaf of invertible elements in $\mathcal{A}$. Let

$$\text{Aut}(\mathcal{E}) := \Gamma(X, \mathcal{A}^\times).$$

Let $R = \Gamma(X - \infty, \mathcal{O}_X)$ and $\Lambda = \Gamma(X - \infty, \mathcal{A})$. Then $\Lambda$ is a hereditary $R$-order in $\tilde{D}$. Clearly $\Lambda^\times = \Gamma(X - \infty, \mathcal{A}^\times)$.

**Lemma 6.2.** The natural restriction map $\text{Aut}(\mathcal{E}) \to \Lambda^\times$ is an isomorphism, and $\Lambda^\times \cong \mathbb{F}_q^\times$ for some $s$ dividing $d$.

**Proof.** First we show that $\Lambda^\times \cong \mathbb{F}_q^\times$ for some $s$ dividing $d$. Let $Z$ be the center of $\tilde{D}$ as an algebraic group. Then $G = \tilde{D}^\times / Z^\times$ is a projective algebraic variety over $F$. Since $\tilde{D}_\infty$ is a division algebra, $G(F_\infty)$ is compact in $\infty$-adic topology, and contains $\Lambda^\times / R^\times$ as a discrete subgroup. Hence $\Lambda^\times / R^\times$ is finite, and as $R^\times \cong \mathbb{F}_q^\times$ is finite, $\Lambda^\times$ is finite. Let $\lambda \in \Lambda^\times$. Since $\lambda^n = 1$ for some $n$, $\lambda$ is algebraic over $\mathbb{F}_q$. Conversely, it is clear that if $\lambda \in \Lambda$ is algebraic over $\mathbb{F}_q$ and $\lambda \neq 0$, then $\lambda \in \Lambda^\times$. 

Let \( \Lambda^{alg} \) be the subset of \( \Lambda \) consisting of elements which are algebraic over \( F_q \). It is easy to show that \( \Lambda^{alg} \) is a field extension of \( F_q \); see [2, p. 383]. Let \( \Lambda^{alg} \cong F_q^s \). Then \( F_q \cdot F \) is a field extension of \( F \) of degree \( s \) contained in \( \bar{D} \). This implies that \( s \) divides \( d \) (see [12, Prop. A.1.4]), so \( \Lambda^{alg} - 0 \cong F_q^s \).

Let \( \bar{D}_\infty \) be a division algebra, \( A_\infty \) is its unique maximal order which is characterized as being the integral closure of \( O_\infty \) in \( \bar{D}_\infty \). As \( \lambda \) is obviously integral, \( \lambda \in A_\infty \), so \( \lambda \) extends to a global section of \( A_\infty \). This implies that \( \text{Aut}(E) \to \Lambda^{alg} \) is surjective. It is clear that a section of \( A_\infty \) generically generates a finite extension of \( F_q \). Hence if such a section is 1 on \( X - \infty \), then it is identically 1, so \( \text{Aut}(E) \to \Lambda^{alg} \) is also injective. \( \square \)

Let \( E_1, \ldots, E_h \) be representatives of \( \mathcal{X}_f / \mathbb{Z} \), and let \( w_i = \# \text{Aut}(E_i), 1 \leq i \leq h \). Consider the sum

\[
\text{Mass}(f) := (q - 1) \sum_{i=1}^{h} \frac{1}{w_i}.
\]

The double coset space \( (6.2) \) is in bijection with isomorphism classes of locally free rank-1 right \( \Lambda \)-modules. Let \( I_1, \ldots, I_h \) represent the isomorphism classes of such modules. Let

\[
\Lambda_i = \Gamma(X - \infty, \text{End}(E_i)), \quad 1 \leq i \leq h.
\]

From Theorem 5.4 one deduces that \( O_\ell(I_i) = \Lambda_i \). Hence using Lemma 6.2

\[
\text{Mass}(f) = \sum_{i=1}^{h} (\Lambda_i^\times : R^\times)^{-1}.
\]

According to [2], it is possible to give a formula for this last sum in terms of the invariants of \( F, D \) and \( f = (f_0, \ldots, f_{d-1}) \). For \( x \neq o \), \( D_x \cong M_{\kappa_x}(\Delta_x) \), where \( \Delta_x \) is a central division algebra over \( F_x \) of index \( e_x \). We always have \( \kappa_x e_x = d \), and \( e_x = 1 \) if \( x \not\in \text{Ram} \). Let

\[
T^o = \prod_{x \in \text{Ram}, x \neq o} \prod_{1 \leq j \leq d-1} (q_j^x - 1)
\]

and

\[
T_o = \frac{\prod_{1 \leq j \leq d} (q_j^o - 1)}{\prod_{0 \leq i \leq d-1} \prod_{1 \leq j \leq d} (q_j^o - 1)}.
\]

If we denote by \( h(A) \) the class number of \( A \), then [2, (1)] specializes to

\[
(6.3) \quad \text{Mass}(f) = h(A) \cdot T^o \cdot T_o \cdot \prod_{i=1}^{d-1} \zeta_X(-i).
\]

From this we get our desired explicit estimate of the order of \( (6.2) \):

\[
\text{Mass}(f) \leq \#(\mathcal{X}_f / \mathbb{Z}) \leq \frac{q^d - 1}{q - 1} \cdot \text{Mass}(f).
\]

We end this section with a geometric application of previous results. Let \( d = 2 \), and fix a closed finite subscheme \( n \neq \emptyset \) of \( X - \infty - o \). Denote by \( E_\ell(D, n) \) the modular curve of \( D \)-elliptic sheaves which are special at \( o \) in the sense of [11], equipped with level-\( n \) structures, modulo the action of \( \mathbb{Z} \).
Remark 6.3. The definition of $\mathcal{D}$-elliptic sheaves in \[11\] includes a “normalization” condition which requires the Euler-Poincaré characteristic of $\mathcal{E}_0$ to be in the interval $[0, d)$. The resulting category is equivalent to the quotient of the category of $\mathcal{D}$-elliptic sheaves by the action of $\mathbb{Z}$ as is done in \[13\].

According to Theorems 6.4 and 8.1 in \[11\], $\mathcal{E}\ell\mathcal{D}_n$ is a fine moduli scheme which is projective of relative dimension 1 over $X' = (X - n - \infty - \text{Ram}) \cup \{o\}$. It is smooth over $X'$ except at $o$. The fibre of $\mathcal{E}\ell\mathcal{D}_n$ over $o$ is a reduced singular curve whose only singular points are ordinary double points, and whose normalization is a disjoint union of finitely many rational curves.

Proposition 6.4. The singular points of $\mathcal{E}\ell\mathcal{D}_n \times X'$, Spec($\mathcal{O}_o$) are represented by the isomorphisms classes of pairs $(\mathcal{E}, \theta_n)$, where $\mathcal{E}$ is a superspecial $\mathcal{D}$-elliptic sheaf of characteristic $o$ over $k$, and $\theta_n$ is a level-$n$ structure on $\mathcal{E}$.

Proof. Denote $Y := \mathcal{E}\ell\mathcal{D}_n \times X'$, Spec($\mathcal{O}_o$). As follows from \[10\] Prop. 4.1.2, \[11\] Prop. 2.16 and \[7\] Prop. 2.1, a $\mathcal{D}$-elliptic sheaf of characteristic $o$ over $k$ is special in the sense of Definition 5.1 if and only if it is special in sense of \[11\] Def. 3.5. Hence $Y$ classifies the pairs $(\mathcal{E}, \theta_n)$, where $\mathcal{E}$ is a special $\mathcal{D}$-elliptic sheaf over $k$ in the sense of Definition 5.1 and $\theta_n$ is a level-$n$ structure on $\mathcal{E}$.

Let $T := \hat{T}^2 \otimes F_k$, where $\hat{T}^2$ is the formal scheme over Spf($\mathcal{O}_o$) corresponding to Drinfeld’s upper-half plane. By a theorem of Drinfeld \[4\], $T$ parametrizes special Dieudonné $\mathcal{D}$-modules over $k$ (“special” in the sense of Definition 5.5) equipped with some extra data. Proposition II.2.7.1 and the main result of Chapter III in \[10\] imply that the singular points of $T$ correspond exactly to superspecial Dieudonné $\mathcal{D}$-modules.

Finally, Hausberger’s uniformization theorem \[11\] Thm. 8.1 relates $T$ and $Y$ as functors. This theorem, combined with the previous two paragraphs, implies that a closed point on $Y$ corresponding to $(\mathcal{E}, \theta_n)$ is singular if and only if $\mathcal{E}$ is superspecial. \[\square\]

Let $\mathcal{D}_n = \mathcal{D} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / N)$, where $N$ be the ideal sheaf of $n$. $F_q^\times$ embeds diagonally into $\mathcal{D}_n^\times$. Denote $d(n) = \#(\mathcal{D}_n^\times / F_q^\times)$.

Corollary 6.5. The number of singular points on $\mathcal{E}\ell\mathcal{D}_n \times X'$, Spec($\mathcal{O}_o$) is equal to

$$d(n) \cdot b(A) \cdot \zeta_X(-1) \cdot (q_o + 1) \cdot \prod_{x \in \text{Ram} - o} (q_c - 1).$$

Proof. A superspecial $\mathcal{D}$-elliptic sheaf over $k$ is the same thing as an exceptional $\mathcal{D}$-elliptic sheaf of type $f = (1, 1)$ (we assume $d = 2$). Fix such a $\mathcal{D}$-elliptic sheaf $\mathcal{E}$ and let $w = \#\text{Aut}(\mathcal{E})$. The number of all level-$n$ structures on $\mathcal{E}$, up to an isomorphism, is equal to $d(n) \frac{\varphi^w}{w}$. This combined with Proposition 6.3 implies that the number of singular points on $\mathcal{E}\ell\mathcal{D}_n \times X'$, Spec($\mathcal{O}_o$) is equal to $d(n) \cdot \text{Mass}(1, 1)$. Now the corollary follows from \[10\]. \[\square\]

7. Supersingular $\mathcal{D}$-Elliptic Sheaves

We keep the notation and assumptions of \[11\]. In this section we assume $o \notin \text{Ram}$, i.e., $D_o \cong M_d(F_o)$.

Definition 7.1. $\mathcal{E} \in \text{DES}$ is supersingular if for all large enough integers $n$

$$\varphi^n_o(M_o) \subset \pi_o M_o.$$
Let $\bar{D}$ be the central division algebra over $F$ with invariants

$$\text{inv}_x(\bar{D}) = \begin{cases} 1/d, & x = \infty; \\ -1/d, & x = o; \\ \text{inv}_x(D), & x \neq o, \infty. \end{cases}$$

**Theorem 7.2.** If $E$ is a supersingular $D$-elliptic sheaf of characteristic $o$ over $k$, then $\text{End}(E)$ is a maximal $\mathcal{O}_X$-order in $\bar{D}$. There is a bijection between the set of isomorphism classes of supersingular $D$-elliptic sheaves over $k$ and the isomorphism classes of locally free rank-1 right $\text{End}(E)$-modules.

**Proof.** The proof is mostly the same as the proof of Theorems 5.3 and 5.4. Two places where the argument needs to be slightly modified are the following:

First, one shows that $\text{End}(E)$ is generically isomorphic to $\bar{D}$ in this case by using the argument in the proof of Proposition 9.9 and Corollary 9.10 in [13].

Second, $(N_o, \varphi^\text{deg(o)}_o) \cong (N_d, \varphi^d_1)^d$, by [5, Lem. 9.8]. Now to show that $\text{End}(E)$ is maximal at $o$, one can proceed as in the $x = \infty$ case of Theorem 5.3.

Suppose $D = \mathbb{M}_d(F)$ and $\mathcal{D} = \mathbb{M}_d(\mathcal{O}_X)$. Let $A = \Gamma(X - \infty, \mathcal{O}_X)$. Morita equivalence establishes an equivalence between the category of $\mathcal{D}$-elliptic sheaves of characteristic $o$ over $k$, and the category of rank-$d$ elliptic sheaves over $k$ with pole at $\infty$; cf. [1, 3.1.4]. On the other hand, by a theorem of Drinfeld this latter category modulo the action of $\mathbb{Z}$ is equivalent to the category of rank-$d$ Drinfeld $A$-modules over $k$; see [3] and [1, 5.2]. Hence the category of $\mathcal{D}$-elliptic sheaves over $k$ modulo the action of $\mathbb{Z}$ is equivalent to the category of rank-$d$ Drinfeld $A$-modules over $k$. Under this equivalence, supersingular $\mathcal{D}$-elliptic sheaves correspond to supersingular Drinfeld modules (see [9] for the definition of supersingular Drinfeld modules). Indeed, the supersingular $\mathcal{D}$-elliptic sheaves and Drinfeld modules are uniquely characterized by the fact that their endomorphism algebra is $\bar{D}$. One concludes that in this case Theorem 7.2 specializes to [9, Thm. 4.3].

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