Two Problems in Classical Mechanics

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Abstract

A problem about the present structure of dimensional analysis, and another one about the differences between solids and fluids are suggested. Both problems appear to have certain foundational aspects.

1. Why Scaling, and why the given Groups ?

Dimensional Analysis, Bluman & Kumei, is one of those fundamental aspects of Classical Physics which, nevertheless, is often left outside of one’s awareness, since it is taken so much for granted. What it says is that every measurable quantity, say $X$, in Classical Physics has the dimension, denoted by $[X]$, given by a monomial

\[(1.1) \quad [X] = L^\alpha M^\beta T^\gamma \]

where $L$, $M$ and $T$ are, respectively, length, mass and time, and they are the three Fundamental Mechanical Dimensions, while $\alpha$, $\beta$, $\gamma \in \mathbb{R}$ are suitable exponents.

For instance, in the case of velocity $V$, acceleration $A$, or energy $E$, we have, respectively, $[V] = LT^{-1}$, $[A] = LT^{-2}$ and $[E] = L^2MT^{-2}$.

It may, in view of its rather disregarded status, be surprising to see the extent to which the above concept of dimension is not trivial.
A good example in this respect is the computation by Sir Geoffrey Taylor of the amount of energy released by the first ever atomic explosion in New Mexico, USA, in the summer of 1944. All the respective data were, needless to say, classified, and the only information available to Sir Geoffrey was a motion picture of a few seconds, showing the moment of the explosion and the consequent expansion of the spherical fireball, see Bluman & Kumei [pp.9-11].

For a simpler, yet no less surprising example, we recall here a dimensional analysis based proof of the celebrated theorem of Pythagoras. Let be given the triangle $ABC$

\[ \includegraphics{triangle.png} \]

with a right angle at $B$, and let us denote the length of its respective sides by $AB = a$, $BC = b$ and $AC = c$. Further, let us denote by $\psi$ the angle at $A$. Clearly, the area $S(A,B,C)$ of the triangle $ABC$ is perfectly well determined by $c$ and $\psi$, and it is given by a certain two variable function

\[ S(A,B,C) = f(c,\psi) \]

The point in the above is that, in view of obvious geometric reasons, the function $f$ must be quadratic in $c$, namely

\[ f(c,\psi) = c^2 g(\psi) \]

Now, assuming that $AD$ is perpendicular on $AC$, we obtain two right angle triangles $ABD$ and $BDC$ which are similar with the initial triangle $ABC$. And then adding the areas of the two smaller triangles, we obtain

\[ S(A,B,C) = S(A,B,D) + S(B,D,C) \]
thus in terms of the above function $f$, it follows that

$$f(c, \psi) = f(a, \psi) + f(b, \psi)$$

which means that

$$c^2 g(\psi) = a^2 g(\psi) + b^2 g(\psi)$$

in other words

$$c^2 = a^2 + b^2$$

that is, the celebrated theorem of Pythagoras.

One of the most impressive applications of Dimensional Analysis can be found in the setting up of a model for three dimensional turbulence by A Kolmogorov, in 1941.

Let us return now to the general assumption in Classical Physics formulated in (1.1) above. Clearly, that relation is equivalent to saying that

(1.2) \[ [X] \in G^3 \]

where we denoted by $G^3$ the multiplicative group of all monomials $L^a M^\beta T^\gamma$ in (1.1), with the obvious commutative group operation of multiplication

$$ (L^a M^\beta T^\gamma) \cdot (L^{a'} M^{\beta'} T^{\gamma'}) = L^{a+a'} M^{\beta+\beta'} T^{\gamma+\gamma'} $$

In this way, the group $G^3$ is isomorphic with the usual commutative additive group $\mathbb{R}^3$, according to

$$ G^3 \ni L^a M^\beta T^\gamma \leftrightarrow (a, \beta, \gamma) \in \mathbb{R}^3 $$

and the neutral element in $G^3$ is $1 = L^0 M^0 T^0$ which corresponds to the so called dimensionless measurable quantities $X$ of Classical Mechanics, namely, for which we have

$$ [X] = 1 $$
The customary reason which is given for the assumption in (1.1) is based on *scaling*. Namely, it is assumed that the *units* in which one measures quantities in Classical Physics are arbitrary and do not influence the mathematical models which express physical laws. This is why one can simply talk about length, mass and time, and need not specify the respective units in which they are measured.

This, however, clearly conflicts with Quantum Mechanics, where one can no longer consider arbitrarily small quantities.

A second issue related to (1.1) is why precisely those three fundamental mechanical dimensions of length, mass and time? Why not other ones? And if yes, then which other ones?

A third issue one can also raise is the monomial form of the dimensions, as given in (1.1). After all, with the three fundamental mechanical dimensions $L$, $M$ and $T$, one could as well construct other groups.

In this way, we are led to

**Problem 1**

Find answers to the above questions.

2. **What are other differences between Fluids and Solids?**

A long recognized way to classify the various states of matter is to do it according to two criteria, Mandelbrot [p. 123], namely

- flowing versus non-flowing
- fixed versus variable volume

Consequently, we are led to three possible states. Solids have states which are non-flowing and with fixed volume. Liquids have states which are flowing and have fixed volumes. And gases have states which are flowing and with variable volume.
The fourth logical possibility, namely, non-flowing and with variable
volume is considered not to be a possible state of usual matter.

Here, we shall divide the states of matter only in two categories, namely

- solids
- fluids, which consist of liquids or gases

Clearly, therefore, in the above terms, we are led to the following:

**First Difference** between solids and fluids: solids are non-flowing, while fluids are flowing.

In Continuum Mechanics one of the long practiced main differences between the mathematical modelling of solids and fluids is the following. In the respective balance or conservation equations describing them, the unknowns which model the state of the solid are typically *displacements*, while in the case of fluids are *velocities*. Two simple examples illustrate that difference.

The vibrating string, under usual conditions, has the equation

\[ T \frac{\partial^2 U}{\partial x^2}(t, x) = m(x) \frac{\partial^2 U}{\partial t^2}(t, x) + w(t, x), \quad t \geq 0, \quad 0 \leq x \leq L \]

where \( L > 0 \) is the length of the string placed along the \( x \)-axis, \( U(t, x) \) is the lateral *displacement* along the perpendicular \( y \)-axis, \( m(x) \) is the density of the string at the point \( x \), while \( w(t, x) \) is the lateral load at time \( t \) and at the point \( x \).

On the other hand, the shock wave equation is

\[ \partial_t U(t, x) + U(t, x) \partial_x U(t, x) = 0, \quad t \geq 0, \quad x \in \mathbb{R} \]

where \( U(t, x) \) is the *velocity* in the gas at time \( t \) and at the point \( x \).

The usual motivation for this different approach in modelling which prefers displacement in the case of solids and velocity for fluids is that
in fluids displacements can be very large, and then it is more convenient to consider velocities.

In this way we are led to the :

**Second difference** between solids and fluids: solids have equations in displacements, while liquids have equations in velocities.

However, it is obvious that solids and fluids can have significantly different properties and behaviour. And it may appear as quite likely that the respective differences are not taken into account to a sufficient extent when their usual mathematical modelling is performed. Indeed, in Continuum Mechanics typically *three* kind of relations contribute to the making of the

**Usual Mathematical Model** given by

- balance or conservation equations
- stress and strain assumptions
- constitutive relations

However, none of these appear to express clearly and significantly enough the **First Difference** above. Furthermore, one also is lacking a deeper motivation for the **Second Difference**.

In this way, we arrive at the following:

**Problem 2**

Give a simple and precise mathematical formulation of the difference between solids and fluids and add it to the **Usual mathematical Model**.

**Note**

It may turn out that such an augmented mathematical model may be more relevant in the case of fluids. And to the extent that such
would indeed be the case, it may possibly help in a better modelling of turbulence.

References

[1] Bluman G W, Kumei S: Symmetries and Differential Equations. Springer, New York, 1989

[2] Fung Y C: A First Course in Continuum Mechanics. Prentice Hall, Englewood, 1969

[3] Kolmogorov A N, Dokl. Akad. Nauk SSSR, 30, 4, 1941, p. 3201. Translated by Levin V: The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. Proc. R. Soc. London A, 434, 1991, 9-13

[4] Mandelbrot B B: Fractals and Scaling in Finance, Discontinuity, Concentration, Rosk. Springer, New York, 1997