Sequential mechanism design

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based on joint works with

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Executive Summary

**Mechanism design**: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like.

Important question: how to avoid *manipulations*?

This can be done, but is *costly*.

Our objective: *minimize* these costs.

We study the problem in *sequential* setting for
- public project problem,
- single unit auctions.
Recap: Direct Mechanisms (1)

Given:

- set of decisions $D$,
- for each player $i$ a set of types $\Theta_i$,
- initial utility function $v_i : D \times \Theta_i \rightarrow \mathbb{R}$. 
Recap: Direct Mechanisms (2)

We consider the following sequence of events:

- Each player $i$ has an initial utility $v_i(d, \theta_i)$, and a type (e.g., valuation of an item) $\theta_i$,
- Each player $i$ announces to the central authority a type (e.g., a bid) $\theta'_i$,
- The central authority computes decision and taxes
  \[
  d := f(\theta'_1, \ldots, \theta'_n) \text{ and } (t_1, \ldots, t_n) := t(\theta'_1, \ldots, \theta'_n),
  \]
  and communicates to each player $i$ the pair $(d, t_i)$.
- Player's final utility: $u_i((f, t)(\theta), \theta_i) := v_i(f(\theta), \theta_i) + t_i(\theta)$.
- Social welfare: $\sum_{i=1}^{n} u_i((f, t)(\theta), \theta_i)$.
Recap: Direct Mechanisms (3)

A direct mechanism \((f, t)\) is

- **feasible** if always \(\sum_{i=1}^{n} t_i(\theta) \leq 0.\)
  
  *(External funding not needed.)*

- **incentive compatible** if no player is better off when submitting a false type \((\theta'_i \neq \theta_i).\)

  *(Manipulations do not pay off or truth-telling is a dominant strategy.)*
Public Project Problem

Each person is asked to report his or her willingness to pay for the project, and the project is undertaken if and only if the aggregate reported willingness to pay exceeds the cost of the project.

(15 October 2007, The Royal Swedish Academy of Sciences, Press Release, Scientific Background)
Public Project Problem Formally

\[ D = \{0, 1\}, \]

for each player \( i \)

\[ \Theta_i = [0, c], \text{ where } c > 0, \]

\[ v_i(d, \theta_i) := d(\theta_i - \frac{c}{n}), \]

\[ f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases} \]
Incentive Compatibility

Theorem (Clarke ’71):

\[ t_i(\theta'_i, \theta_{-i}) := \begin{cases} 
\min(0, \frac{n-1}{n}c - \sum_{k \neq i} \theta_k) & \text{if } \sum_{k \neq i} \theta_k + \theta'_i < c \\
\min(0, \sum_{k \neq i} \theta_k - \frac{n-1}{n}c) & \text{otherwise}
\end{cases} \]

yields an incentive compatible mechanism.

Example

c = 300.

| player | type | submitted type | tax | \( u_i \) |
|--------|------|----------------|-----|----------|
| A      | 110  | 110            | −10 | 0        |
| B      | 80   | 80             | 0   | −20      |
| C      | 110  | 110            | −10 | 0        |
Theorem [Apt, Conitzer, Guo, Markakis, WINE’08]
Consider the public project problem. No direct mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Clarke’s tax.
However . . .

Clarke’s tax is not optimal in the public project problem when the payments per player can differ.

Note: Pivotal mechanism then ceases to be anonymous.
arg\sup \theta := \mu_i(\theta_i = \max_{j \in \{1, \ldots, n\}} \theta_j).

\begin{itemize}
  \item $D = \{1, \ldots, n\}$,
  \item for each player $i$
    \begin{itemize}
      \item $\Theta_i = \mathbb{R}_+$,
      \item $v_i(d, \theta_i) := \begin{cases} 
        \theta_i & \text{if } d = i \\
        0 & \text{otherwise}
      \end{cases}$
      \item $f(\theta) := \arg\sup \theta$.
    \end{itemize}
\end{itemize}
Vickrey Auction as a Direct Mechanism

\( \theta^* \): the reordering of \( \theta \) in descending order.

\[
t_i^V(\theta) := \begin{cases} 
-\theta^*_2 & \text{if } i = \text{argsmax } \theta \\
0 & \text{otherwise}
\end{cases}
\]

Example:

| player | bid | tax to authority | \( u_i \) |
|--------|-----|-----------------|--------|
| A      | 18  | 0               | 0      |
| B      | 24  | -21             | 3      |
| C      | 21  | 0               | 0      |

**Theorem**: Vickrey auction is incentive compatible.
Bailey-Cavallo Mechanism

\[ t_i(\theta) := t_i^V(\theta) + \frac{(\theta-i)^*}{n} \]

**Example:**

| player | bid | tax to authority | \( u_i \) | why? |
|--------|-----|------------------|-----------|------|
| A      | 18  | 0                | 7         | (= 1/3 of 21) |
| B      | 24  | -2               | 9         | (= 24 - 2 - 7 - 6) |
| C      | 21  | 0                | 6         | (= 1/3 of 18) |

**Theorem:** Bailey-Cavallo mechanism is feasible and incentive compatible.

**Warning:** Bailey-Cavallo mechanism does not satisfy the participation constraint.
Theorem [Apt, Conitzer, Guo, Markakis, WINE’08]

Consider the sealed bid auction. No tax-based mechanism exists that is

- feasible,
- incentive compatible,
- ‘better’ than Bailey-Cavallo mechanism.
Groves Auctions

A sealed bid auction with redistribution:

\[ t_i(\theta) := t_i^V(\theta) + r_i(\theta_{-i}). \]

**Theorem** [Groves ’73] Each Groves auction is incentive compatible.
Players move sequentially.

Player $i$ submits his/her type after he has seen the types of players $1, \ldots, i - 1$.

The decisions and taxes are computed using a given direct based mechanism.
Assume a sequential mechanism $Seq$.

- A strategy of player $i$ in $Seq$:

$$s_i : \Theta_1 \times \ldots \times \Theta_i \rightarrow \Theta_i.$$ 

- Strategy $s_i(\cdot)$ of player $i$ is optimal in $Seq$ if for all $\theta \in \Theta$ and $\theta_i' \in \Theta_i$

$$u_i((f, t)(s_i(\theta_1, \ldots, \theta_i), \theta_i), \theta_i) \geq u_i((f, t)(\theta_i', \theta_i), \theta_i).$$
Intuitions

- Strategy of player $j$ is memoryless if it does not depend on the types of players $1, \ldots, j - 1$.

- Then $s_i(\cdot)$ is optimal iff for all $\theta \in \Theta$ it yields a best response to all joint strategies of players $j \neq i$ assuming players $i + 1, \ldots, n$ use memoryless strategies (or move jointly with player $i$).

- In particular, an optimal strategy is a best response to truth-telling by players $j \neq i$. 
Optimality Result (3)

Theorem [Apt, Estévez-Fernández, SAGT’09]

Consider public project problem and Clarke’s tax.

Strategy

\[ s_i(\theta_1, \ldots, \theta_i) := \begin{cases} 
\theta_i & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i < n, \\
0 (!) & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i = n, \\
c (!) & \text{if } \sum_{j=1}^{i} \theta_j \geq c
\end{cases} \]

is optimal for player \( i \) in the sequential pivotal mechanism.

Under certain natural circumstances \( s_i \) simultaneously maximizes the final utility of the other players.
Example 1

\( c = 300 \).

**Pivotal mechanism:**

| player | type | submitted type | tax | \( u_i \) |
|--------|------|----------------|-----|-----------|
| A      | 110  | 110            | −10 | 0         |
| B      | 80   | 80             | 0   | −20       |
| C      | 110  | 110            | −10 | 0         |

**Now:**

| player | type | submitted type | tax | \( u_i \) |
|--------|------|----------------|-----|-----------|
| A      | 110  | 110            | 0   | 10        |
| B      | 80   | 80             | 0   | −20       |
| C      | 110  | 300            | −10 | 0         |
$c = 300$. 

Pivotal mechanism:

| player | type | submitted type | tax  | $u_i$ |
|--------|------|---------------|------|-------|
| A      | 110  | 110           | 0    | 0     |
| B      | 80   | 80            | $-10$| $-10$ |
| C      | 100  | 100           | 0    | 0     |

Now:

| player | type | submitted type | tax  | $u_i$ |
|--------|------|---------------|------|-------|
| A      | 110  | 110           | 0    | 0     |
| B      | 80   | 80            | 0    | 0     |
| C      | 100  | 0             | 0    | 0     |
Optimality Result (4)

Theorem [Apt, Estévez-Fernández, SAGT’09]

Consider public project problem and Clarke’s tax.

Strategy

\[ s_i(\theta_1, \ldots, \theta_i) := \begin{cases} 
\theta_i & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i < n, \\
0 (!) & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i = n, \\
0 (!!) & \text{if } \sum_{j=1}^{i} \theta_j = c, \theta_i > \frac{c}{n} \text{ and } i = n, \\
c (!) & \text{otherwise}
\]  

is optimal for player \( i \) in the sequential pivotal mechanism.

When all players follow \( s_i(\cdot) \), maximal social welfare is generated in the universe of optimal strategies.
Example 3

\[ c = 300. \]

Before:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{player} & \text{type} & \text{submitted type} & \text{tax} & u_i \\
\hline
A & 110 & 110 & 0 & 10 \\
B & 80 & 80 & 0 & -20 \\
C & 110 & 300 & -10 & 0 \\
\hline
\end{array}
\]

Now:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{player} & \text{type} & \text{submitted type} & \text{tax} & u_i \\
\hline
A & 110 & 110 & 0 & 0 \\
B & 80 & 80 & 0 & 0 \\
C & 110 & 0 & 0 & 0 \\
\hline
\end{array}
\]
Proof Idea (1)

Lemma 1 Let $s'_i(\cdot)$ be an optimal strategy for player $i$.

- Suppose $\sum_{j=1}^{i} \theta_j < c$ and $i < n$. Then $s'_i(\theta_1, \ldots, \theta_i) = \theta_i$.

- Suppose $\sum_{j=1}^{i} \theta_j < c$ and $i = n$. Then $\sum_{j=1}^{n-1} \theta_j + s'_i(\theta_1, \ldots, \theta_n) < c$.

- Suppose $\sum_{j=1}^{i} \theta_j = c$ and $i < n$. Then $s'_i(\theta_1, \ldots, \theta_i) \geq \theta_i$.

- Suppose $\sum_{j=1}^{i} \theta_j > c$. Then $\sum_{j=1}^{i-1} \theta_j + s'_i(\theta_1, \ldots, \theta_i) \geq c$.

Proof In each case by case by case analysis.
Proof Idea (2)

Lemma 2 \( s_i(.) \) maximizes social welfare in the universe of optimal strategies, assuming that players who follow \( i \) are truthful.

Proof By Lemma 1 and case analysis:

Case 1 \( \sum_{j=1}^{i} \theta_j < c \) and \( i < n \).

Case 2 \( \sum_{j=1}^{i} \theta_j < c \) and \( i = n \)

Case 3 \( \sum_{j=1}^{i} \theta_j = c, \ \theta_i > \frac{c}{n} \) and \( i = n \).

Case 4 \( \sum_{j=1}^{i} \theta_j = c, \ \theta_i \leq \frac{c}{n} \) and \( i = n \).

Case 5 \( (\sum_{j=1}^{i} \theta_j = c \text{ and } i < n) \text{ or } \sum_{j=1}^{i} \theta_j > c. \)
Suppose players submit their strategies simultaneously, for each vector of initial types their final utilities are determined using the pivotal mechanism.

Game-theoretic interpretation: sequential pre-Bayesian games.

**Theorem** Vectors of strategies from Theorems 1 and 2 form a Nash equilibrium in the universe of optimal strategies.

The result does not hold if deviations to non-optimal strategies are allowed.
Optimal Strategies in Seq. Groves Auctions

\[ \bar{\theta}_i := \max_{j \in \{1, \ldots, i-1\}} \theta_j. \]

Lemma

\( s_i(\cdot) \) is an optimal strategy for player \( i \) iff the following holds:

- Suppose \( \theta_i > \bar{\theta}_i \) and \( i < n \). Then \( s_i(\theta_1, \ldots, \theta_i) = \theta_i. \)
- Suppose \( \theta_i > \bar{\theta}_i \) and \( i = n \). Then \( s_i(\theta_1, \ldots, \theta_i) > \bar{\theta}_i. \)
- Suppose \( \theta_i \leq \bar{\theta}_i \) and \( i < n \). Then \( s_i(\theta_1, \ldots, \theta_i) \leq \bar{\theta}_i. \)
- Suppose \( \theta_i < \bar{\theta}_i \) and \( i = n \). Then \( s_i(\theta_1, \ldots, \theta_i) \leq \bar{\theta}_i. \)
Optimality Result (5)

Theorem ([Apt, Markakis, AAMAS ’09]).

- **Strategy**

  \[ s_i(\theta_1, \ldots, \theta_i) := \begin{cases} \theta_i & \text{if } \theta_i > \max_{j \in \{1, \ldots, i-1\}} \theta_j, \\ 0(!) & \text{otherwise} \end{cases} \]

  is optimal for player \( i \) in the sequential Vickrey auction.

- **When all** players follow \( s_i(\cdot) \), **maximal** social welfare is generated in the universe of optimal strategies.
### Example

**Before:**

| player | bid | tax to authority | $u_i$ |
|--------|-----|-----------------|------|
| A      | 18  | 0               | 0    |
| B      | 24  | $-21$           | 3    |
| C      | 21  | 0               | 0    |

**Now:**

| player | bid | tax to authority | $u_i$ |
|--------|-----|-----------------|------|
| A      | 18  | 0               | 0    |
| B      | 24  | $-18$           | 6    |
| C      | 0   | 0               | 0    |

Social welfare: 3 vs 6.
Optimality Result (6)

Theorem ([Apt, Markakis, WINE’09]).

\[ s_i(\theta_1, \ldots, \theta_i) := \begin{cases} 
\theta_i & \text{if } \theta_i > \max_{j \in \{1, \ldots, i-1\}} \theta_j \\
(\theta_1, \ldots, \theta_{i-1})^*_{1(!)} & \text{if } \theta_i \leq \max_{j \in \{1, \ldots, i-1\}} \theta_j \text{ and } i \leq n - 1 \\
(\theta_1, \ldots, \theta_{i-1})^*_{2(!)} & \text{otherwise} 
\end{cases} \]

is optimal for player \( i \) in the sequential Bailey-Cavallo mechanism.

When all players follow \( s_i(\cdot) \), maximal social welfare is generated in the universe of optimal strategies.
# Example: Bailey-Cavallo mechanism

## Before:

| player | type | tax to authority | $u_i$ | why? |
|--------|------|------------------|-------|------|
| A      | 18   | 0                | 7     | (= 21/3) |
| B      | 24   | -2               | 9     | (= 24 - 21 + 18/3) |
| C      | 21   | 0                | 6     | (= 18/3) |

## Now:

| player | type | tax to authority | $u_i$ | why? |
|--------|------|------------------|-------|------|
| A      | 18   | 0                | 6     | (= 18/3) |
| B      | 24   | 0                | 12    | (= 24 - 18 + 18/3) |
| C      | 18   | 0                | 6     | (= 18/3) |

Social welfare: 22 vs 24.
Safety-level Equilibrium

- Introduced in [Ashlagi, Monderer, Tennenholtz ’06] for pre-Bayesian games.
- Given $\theta \leq i \in \Theta \leq i$ and $s(\cdot)$

$$\min_{\theta > i \in \Theta > i} u_i((f, t)([s(\cdot), \theta]), \theta_i)$$

is the guaranteed final utility for player $i$.
- $s(\cdot) \succeq_i s'(\cdot)$ iff for all $\theta \leq i \in \Theta \leq i$

$$\min_{\theta > i \in \Theta > i} u_i((f, t)([s(\cdot), \theta]), \theta_i) \geq \min_{\theta > i \in \Theta > i} u_i((f, t)([s'(\cdot), \theta]), \theta_i).$$
- $s(\cdot)$ is safety-level equilibrium if for all $i$ and $s'_i(\cdot)$

$$(s_i(\cdot), s_{-i}(\cdot)) \succeq_i (s'_i(\cdot), s_{-i}(\cdot)).$$
Implementation in Safety-level Equilibrium

Theorem ([Apt, Markakis, WINE’09]).

- Introduced vector $s(\cdot)$ of strategies in sequential Vickrey auction forms a safety-level equilibrium.
- Introduced vector $s(\cdot)$ of strategies in sequential Bailey-Cavallo mechanism forms a safety-level equilibrium.
Conclusions

- Social welfare can be increased if the players move sequentially.

- Dalai Lama:

  The intelligent way to be selfish is to work for the welfare of others.

*Microeconomics: Behavior, Institutions, and Evolution*, S. Bowles ’04.
THANK YOU