The Sparse Awakens: Streaming Algorithms for Matching Size Estimation in Sparse Graphs

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Abstract

Estimating the size of the maximum matching is a canonical problem in graph algorithms, and one that has attracted extensive study over a range of different computational models. We present improved streaming algorithms for approximating the size of maximum matching with sparse (bounded arboricity) graphs.

• (Insert-Only Streams) We present a one-pass algorithm that takes \( O(c \log^2 n) \) space and approximates the size of the maximum matching in graphs with arboricity \( c \) within a factor of \( O(c) \). This improves significantly upon the state-of-the-art \( \tilde{O}(cn^{2/3}) \)-space streaming algorithms.

• (Dynamic Streams) Given a dynamic graph stream (i.e., inserts and deletes) of edges of an underlying \( c \)-bounded arboricity graph, we present an one-pass algorithm that uses space \( \tilde{O}(c^{2/3}n^{2/3}) \) and returns an \( O(c) \)-estimator for the size of the maximum matching. This algorithm improves the state-of-the-art \( \tilde{O}(cn^{4/5}) \)-space algorithms, where the \( \tilde{O}(\cdot) \) notation hides logarithmic in \( n \) dependencies.

In contrast to the previous works, our results take more advantage of the streaming access to the input and characterize the matching size based on the ordering of the edges in the stream in addition to the degree distributions and structural properties of the sparse graphs.

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1 Introduction

Problems related to (maximum) matchings in graph have a long history in Combinatorics and Computer Science. They arise in many contexts, from choosing which advertisements to display to online users, to characterizing properties of chemical compounds. Stable matchings have a suite of applications, from assigning students to universities, to arranging organ donations. These have been addressed in a variety of different computation models, from the traditional RAM model, to more recent sublinear (property testing) and external memory (MapReduce) models. Matching has also been studied for a number of classes of input graph: including general graphs, bipartite graphs, weighted graphs, and those with some sparsity structure.

We focus on the streaming case, where each edge is seen once only, and we are restricted to space sublinear in the size of the graph (i.e., no. of its vertices). In this case, the objective is to find (approximately) the size of the matching. Even here, results for general graphs are either weak or make assumptions about the input or the stream order. In this work, we seek to improve the guarantees by restricting to graphs that have some measure of sparsity – bounded arboricity, or bounded degree. This aligns with reality, where most massive graphs have asymptotically fewer than $O(n^2)$ edges.

Recently, Kapralov, Khanna, and Sudan [19] developed a streaming algorithm which computes an estimate of matching size for general graphs within a factor of $O(polylog(n))$ in the random-order streaming model using $O(polylog(n))$ space. In the random-order model, the input stream is assumed to be chosen uniformly at random from the set of all possible permutations of the edges. Esfandiari et al. [10] were the first to study streaming algorithms for estimating the size of matching in bounded arboricity graphs in the adversarial-order streaming model, where the algorithm is required to provide a good approximation for any ordering of edges. Graph arboricity is a measure to quantify the density of a given graph. A graph $G(V, E)$ has arboricity $c$ if the set $E$ of its edges can be partitioned into at most $c$ forests. Since a forest on $n$ nodes has at most $n − 1$ edges, a graph with arboricity $c$ can have at most $c(n − 1)$ edges. Indeed, by a result of Nash-Williams [28, 29] this holds for any subgraph of a $c$-bounded arboricity graph $G$. Formally, the Nash-Williams Theorem [28, 29] states that $c = \max_{U \subseteq V} \{|E(U)|/(|U| − 1)\}$, where $|U|$ and $|E(U)|$ are the number of nodes and edges in the subgraph with nodes $U$, respectively. Several important families of graphs have constant arboricity. Examples include planar graphs (that have arboricity $c = 3$), bounded genus graphs, bounded treewidth graphs, and more generally, graphs that exclude a fixed minor.

The important observation in [10] is that the size of matching in bounded arboricity graphs can be approximately characterized by the number of high degree vertices (vertices with degree above a fixed threshold) and the number of so called shallow edges (edges with both low degree endpoints). This characterization allows for estimation of the matching size in sublinear space by taking samples from the vertices and edges of the graph. The work of [10] implements the characterization in $O(cn^{2/3})$ space giving a $O(c)$ approximation of the matching size. Subsequent works [2, 26] consider alternative characterizations and improve upon the approximation factor however they do not result in major space improvements.

1.1 Our Contributions

We present major improvements in the space usage of streaming algorithms for sparse graphs ($c$-bounded Arboricity Graphs). Our main result is a polylog space algorithm that beats the $n^\varepsilon$ space bound of prior algorithms. More precisely, we show:

**Theorem 1.** Let $G(V, E)$ be a graph with arboricity bounded by $c$. Let $S$ be an (adversarial order) insertion-only stream of the edges of the underlying graph $G$. Let $M^*$ be the size of the maximum matching of $G$ (or $S$ interchangeably). Then, there is a randomized 1-pass streaming algorithm that outputs a $(22.5c + 6)(1 + \varepsilon)$-approximation to $M^*$ with probability at least $1 − \delta$ and takes $O(\frac{c \varepsilon}{2\varepsilon} \log(\frac{1}{\varepsilon}) \log cn \log n)$ space.

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1It can be shown that for an $H$-minor-free graph, the arboricity number is $O(h \sqrt{n})$ where $h$ is the number of vertices of $H$. [22]
For the case of dynamic streams (i.e., streams of inserts and deletes of edges), we design a different algorithm using $\tilde{O}(c^{10/3}n^{2/3})$ space which improves the $\tilde{O}(cn^{1/5})$-space dynamic (insertion/deletion) streaming algorithms of [2][3]. The following theorem states this result (proved in Section 3.3).

**Theorem 2.** Let $G(V, E)$ be a graph with the arboricity bounded by $c$. Let $M^*$ be the size of the maximum matching of $G$. Let $S$ be a dynamic stream of edge insertions and deletions of the underlying graph $G$ of length at most $O(cn)$. Let $\beta = \mu((2\mu)/\left(\mu - 2c + 1\right)) + 1$ where $\mu > 2c$. Then, there exists a streaming algorithm that takes $O(\frac{\beta^{4/3}(nc/\mu)^{2/3}}{\varepsilon^{\gamma/3}})$ space in expectation and outputs a $(1 - \varepsilon)\beta$ approximation of $M^*$ with probability at least 0.86.

Our algorithms for bounded arboricity graphs are based on two novel streaming-friendly characterizations of the maximum matching size. The first characterization is a modification of the characterization in [2] which approximates the size of the maximum matching by $h_\mu + s_\mu$ where $h_\mu$ is defined as the number of high degree vertices (vertices with degree more than a threshold $\mu$) and $s_\mu$ is the number of shallow edges (edges with low degree endpoints). While $h_\mu$ can be easily approximated by sampling the vertices and checking if they are high degree or not, approximating $s_\mu$ in sublinear space is a challenge because in one pass we cannot determine if a sampled edge is shallow or not. The work of [9] resolves this issue by sampling the edges at a high rate and manages to implement their characterization in $\tilde{O}(cn^{2/3})$ space for adversarial insert-only streams.

To bring the space usage down to $\tilde{O}(c^{2.5}n^{1/2})$ (for insert-only streams), we modify the formulation of the above characterization. We still need to approximate $h_\mu$ but instead of $s_\mu$ we approximate $n_L$, the number of non-isolated vertices in the induced subgraph $G_L$ defined over the low degree vertices. Note that $s_\mu$ is the number of shallow edges in $G_L$. This subtle change of definition turns out to be immensely helpful. Similar to $h_\mu$ we only need to sample the nodes and check if their degrees are below a certain threshold or not, however we carry the additional constraint that we have to avoid counting the nodes in $G_L$ that are isolated (have only high degree nodes as neighbors). To satisfy this additional constraint, our algorithm stores the neighbors of the sampled vertices along with a counter for each that maintains their degree in the rest of the stream. Although we only obtain a lower bound on the degree of the neighbors, as it turns out the lower bound information on the degree is still useful because we can ensure the number of false positives that contribute to our estimate is within a certain limit. As result, we can approximate $h_\mu + n_L$ using $\tilde{O}(cn^{1/2})$ space which gives a $(2c + 1)(2c + 2)$ approximation of the maximum matching size after choosing appropriate values for $\mu$ and other parameters. This characterization is of particular importances, as it can be adapted to work under edge deletions as well as long as the number of deletions is bounded by $O(cn)$. Details of the characterization and the associated algorithms are given in Lemma 4 and Section 3.2.

To obtain a $\text{polylog}(n)$ space algorithm (and prove the claim of Theorem 1), we give a totally new characterization. This characterization, unlike the previous ones that only depend on the parameters of the graph, also takes the ordering of the edges in the stream into account. Roughly speaking, we characterize the size of a maximum matching by the number of edges in the stream that have few neighbor edges in the rest of the stream. To understand the connection with maximum matching, consider the following simple special case. Suppose the input graph $G$ is a forest composed of $k$ disjoint stars. Observe that the maximum matching on this graph is just to pick one edge from each star. We relate this to a combinatorial characterization that arises from the sequence of edges in the stream: no matter how we order the edges of $G$ in the stream, from each star there is exactly one edge that has no neighboring edges in the remainder of the stream (in other words, the last edge of the star in the stream). Our characterization generalizes this idea to graphs with arboricity bounded by $c$ by counting the $\alpha$-good edges, i.e. edges that have at most $\alpha = 6c$ neighbors in the remainder of the stream. We prove this characterization gives an $O(\varepsilon)$ approximation of the maximum matching size. More important, a nice feature of this characterization is that it can be implemented in $\text{polylog}(n)$ space if one allows a $1 + \varepsilon$ approximation. The implementation adapts an idea
Table 1: Known results for estimating the size of a maximum matching in data streams. (*) In some entries, an $(1 + \varepsilon)$ multiplicative factor has been suppressed for concision. (**) The $\tilde{\mathcal{O}}(.)$ notation hides logarithmic dependencies.

from the well-known $L_0$ sampling algorithm. It runs $O(\log n)$ parallel threads each sampling the stream at a different rate. At the end, a thread “wins” that has sampled roughly $\Theta(\log n)$ elements from the $\alpha$-good edges (samples the edges with a rate of $\frac{\log n}{k}$ where $k$ is the number of $\alpha$-good edges). The threads that under-sample will end up with few edges or nothing while the ones that have oversampled will keep too many $\alpha$-good edges and will be terminated as result.

Table 1 summarizes the known and new results for estimating the size of a maximum matching.

### 1.2 Further Related Streaming Work

In the classical offline model, where we assume we have enough space to store all vertices and edges of a graph $G = (V, E)$, the problem of computing the maximum matching of $G$ has been extensively studied. The best result in this model is the 30-years-old algorithm due to Micali and Vazirani [27] with running time $O(m\sqrt{n})$, where $n = |V|$ and $m = |E|$. A matching of size within $(1 - \varepsilon)$ factor of a maximum cardinality matching can be found in $O(m/\varepsilon)$ time [17] [27]. Very recently, Duan and Pettie [5] develop a $(1 - \varepsilon)$-approximate maximum weighted matching algorithm in time $O(m/\varepsilon)$.

The question of approximating the maximum cardinality matching has been extensively studied in the streaming model. An $O(n)$-space greedy algorithm trivially obtains a maximal matching, which is a 2-approximation for the maximum cardinality matching [11]. A natural question is whether one can beat the approximation factor of the greedy algorithm with $O(n \text{polylog}(n))$ space. Recently, it was shown that obtaining an approximation factor better than $\frac{\varepsilon}{\varepsilon - 1} \simeq 1.58$ in one pass requires $n^{1 + \Omega(1/ \log \log n)}$ space [12] [13], even in bipartite graphs and in the vertex-arrival model, where the vertices arrive in the stream together with their incident edges. This setting has also been studied in the context of online algorithms, where each arriving vertex has to be either matched or discarded irrevocably upon arrival. Seminal work due to Karp, Vazirani and Vazirani [20] gives an online algorithm with $\frac{\varepsilon}{\varepsilon - 1}$ approximation factor in this model.

Closing the gap between the upper bound of 2 and the lower bound of $\frac{\varepsilon}{\varepsilon - 1}$ remains one of the most appealing open problems in the graph streaming area (see [31]). The factor of 2 can be improved on if one either considers the random-order model or allows for two passes [21]. By allowing even more passes, the approximation factor can be improved to multiplicative $(1 - \varepsilon)$-approximation via finding and applying augmenting paths with successive passes [24] [25] [6] [7] [1].

Another line of research [11] [24] [32] [8] has explored the question of approximating the maximum-weight matching in one pass and $O(n \text{polylog}(n))$ space. Currently, the best known approximation factor equals $4 + \varepsilon$ (for any positive constant $\varepsilon$) [4].
2 Preliminaries and Notations

Let $G(V, E)$ be an undirected unweighted graph with $n = |V|$ vertices and $m = |E|$ edges. For a vertex $v \in V$, let $\deg_G(v)$ denote the degree of vertex $v$ in $G$. A matching $M$ of $G$ is a set of pairwise non-adjacent edges, i.e., no two edges share a common edge. Edges in $M$ are called matched edges; the other edges are called unmatched. A maximum matching of graph $G(V, E)$ is a matching of maximum size. Throughout the paper, when we fix a maximum matching of $G(V, E)$, we denote it by $M^*$. A matching $M$ of $G$ is maximal if it is not a proper subset of any other matching in graph $G$. Abusing the notation, we sometimes use $M^*$ and $M$ for the size the maximum and maximal matching, respectively. It is well-known (see for example [23]) that the size of a maximal matching is at least half of the size of a maximum matching, i.e., $M \geq M^*/2$. Thus, we say a maximal matching is a 2-approximation of a maximum matching of $G$. It is known [23] that the simple greedy algorithm, where we include the newly arrived edge if none of its endpoint are already matched, returns a maximal matching.

3 Algorithms for Bounded Arboricity Graphs

Throughout this section, $h_{\mu}$ denotes the number of vertices in graph $G = (V, E)$ that have degree above $\mu$. Let $G_L = (V', E')$ be an induced subgraph of $G$ where $V' = \{v | \deg_G(v) \leq \mu\}$ and $(u, v) \in E'$ iff $u$ and $v$ are both in $V'$. Note that $G_L$ might have isolated vertices. In the following we let $M_{\mu}$ denote the size of maximum matching in $G_L$.

3.1 Characterization lemmas

Lemma 3 ([9]). For a $c$-bounded arboricity graph $G(V, E)$ and $\mu > 2c$, we have $h_{\mu} \leq \frac{2\mu}{\mu - 2c + 1}M^*$.

Lemma 4. For a $c$-bounded arboricity graph $G(V, E)$ and $\mu > 2c$, we have

$$M^* \leq h_{\mu} + M_{\mu} \leq \left(\frac{2\mu}{\mu - 2c + 1} + 1\right)M^* .$$

Proof. The lower bound is easy to see: every edge of a maximum matching either has an endpoint with degree more than $\mu$ or both of its endpoints are vertices with degree at most $\mu$. The number of matched edges of the first type is bounded by $h_{\mu}$ whereas the number of matched edges of the second type are bounded by $M_{\mu}$.

To prove the upper bound, we use the fact $M_{\mu} \leq M^*$ and Lemma 3.

Definition 5. Let $S = (e_1, \ldots, e_m)$ be a sequence of edges. We say the edge $e_i = (u, v)$ is $\alpha$-good with respect to $S$ if $\max\{d_i(u), d_i(v)\} \leq \alpha$ where $d_i(x)$ is defined as the number of the neighbors of $x$ that appear after the $i$-th location in the stream.

Lemma 6. Let $\mu > 2c$ be a (large enough) parameter. Let $E_\alpha$ be the set of $\alpha$-good edges in an edge stream for a graph with arboricity at most $c$. We have:

$$\left(\frac{1}{2} - \frac{c}{\mu + 1}\right)M^* \leq |E_\alpha| \leq \left(\frac{5}{4}\alpha + 2\right)M^* ,$$

where $\alpha = \max\{\mu - 1, \frac{4\alpha(\mu + 1)}{\mu + 1 - 2c}\}$. In particular for $\mu = 6c - 1$, we have

$$M^* \leq 3|E_6c| \leq (22.5c + 6)M^* .$$

Proof. First we prove the lower bound on $|E_\alpha|$. In particular we show a relation involving the number of edges where both endpoints have low degree, $s_\mu = |\{e = (u, v) | e \in E, \deg(u) \leq \mu, \deg(v) \leq \mu\}|$:
\[
\left(\frac{1}{2} - \frac{c}{\mu + 1}\right) h_\mu + s_\mu \leq |E_\alpha|.
\]

The claim in the lemma follows from the relatively loose bound that \(M^* \leq h_\mu + s_\mu\). Let \(H\) be the set of vertices in the graph with degree above \(\mu\) and let \(L = V \setminus H\). Recall that \(h_\mu = |H|\). Let \(H_0\) be the vertices in \(H\) that have no neighbor in \(L\), and let \(H_1 = H \setminus H_0\). First we notice that \(|H_1| \geq\left(1 - \frac{2c}{\mu}\right)|H|\). To see this, let \(E'\) be the edges with at least one endpoint in \(H_0\). By definition, every node in \(H_0\) has degree at least \(\mu + 1\), so we have \(|E'| \geq \frac{\mu + 1}{2}|H_0|\). At the same time, the total number of edges in the stream induced by the nodes \(H\) is at most \(c(|H| - 1)\), using the arboricity assumption. Therefore,

\[
c(|H| - 1) \geq |E'| \geq \frac{\mu + 1}{2}|H_0|
\]

It follows that \(|H_0| \leq \frac{2c}{\mu + 1}(|H| - 1)\) which further implies that

\[
|H_1| \geq \left(1 - \frac{2c}{\mu + 1}\right)|H| = \left(1 - \frac{2c}{\mu + 1}\right)h_\mu.
\] (1)

Now let \(d_H(v)\) be the degree of \(v\) in the induced subgraph \(H\). We have \(\sum_{v \in H_1} d_H(v) \leq 2c|H|\), again using the arboricity bound and the fact that summing over degrees counts each edge at most twice. Therefore, taking the average over nodes in \(H_1\),

\[
d_H(v) \leq \frac{2c}{1 - \frac{2c}{\mu + 1}}
\]

for \(v \in H_1\). Consequently, at least half of the vertices in \(H_1\) have their \(d_H\) bounded by \(\frac{4c(\mu+1)}{\mu+1-2c}\) (via the Markov inequality). Let \(H'_1\) be those vertices. For each \(v \in H'_1\) we find an \(\alpha\)-good edge. Let \(e^* = (v, u)\) be the last edge in the stream where \(u \in L\). Then, there cannot be too many edges that neighbor \((v, u)\) and come after it in the stream: the total number of edges that share an endpoint with \(e^*\) in the rest of the stream is bounded by \(\max\{\mu - 1, \frac{4c(\mu+1)}{\mu+1-2c}\}\). Consequently, for \(\alpha = \max\{\mu - 1, \frac{4c(\mu+1)}{\mu+1-2c}\}\), we have \(|E_\alpha| \geq \left(\frac{1}{2} - \frac{c}{\mu + 1}\right)h_\mu\), based on the set of \(|H_1|/2\) edges in \(H'_1\) and using (1). For \(\alpha \geq \mu\), \(E_\alpha\) also contains the disjoint set of edges from \(L \times L\), which are all guaranteed to be \(\alpha\)-good since both their endpoints have degree bounded by \(\mu\). Therefore \(|E_\alpha| \geq s_\mu + \left(\frac{1}{2} - \frac{c}{\mu + 1}\right)h_\mu\) as claimed.

To prove the upper bound on \(|E_\alpha|\), we notice that the subgraph containing only the edges in \(E_\alpha\) has degree at most \(\alpha + 1\). Such a graph has a matching size of at least \(\frac{4|E_\alpha|}{5(\alpha+1)+3}\) \([14]\). It follows that \(|E_\alpha| \leq \frac{5\alpha+8}{\alpha+1} M^*\). This finishes the proof of the lemma.

Last, we note that in the special case of trees (or more generally, graph streams which represent forests), a tighter approximation bound follows (for which our algorithms specified below will also apply).

**Lemma 7.** For trees we have \(M^* \leq |E_1| \leq 2M^*\).

**Proof.** Let \(T = (V, E)\) be a tree with maximum matching size \(M^*\). The upper bound follows by considering \(E_1\): the subgraph of \(T\) formed by \(E_1\) has degree at most 2, and since we are considering trees, the set \(E_1\) can have no cycles and so consists of paths. Hence, \(|E_1| \leq 2M^*\).

For the lower bound, we use induction on the number of nodes. Suppose the claim \(|E_1| \geq M^*\) is true for all trees on \(n\) nodes. We want to show that the claim remains true for trees on \(n + 1\) nodes. The base case \(n = 2\) is trivially true. Given a tree \(T = (V, E)\) with \(n + 1\) nodes, there is always a leaf \(w \in V\), that is connected to a node \(u\) where \(u\) has at most one non-leaf neighbor. If we remove the edge \((u, w)\) from the tree, we get a tree \(T'\) with \(n\) nodes and by our induction hypothesis, \(|E_1(T')| \geq M^*(T')\) no matter how the stream is ordered. Fix some ordering of the stream for \(T'\). We claim after inserting the edge \((u, w)\) in the stream (anywhere) we will have \(|E_1(T)| \geq M^*(T)|\). Why? We have two cases to consider.

1. \(w\) has no sibling in \(T\). In this case \(E_1(T) = E_1(T') \cup \{(u, w)\}\). This is because \(u\) must have been a
Lemma 9. Let \( (u, w) \) does not cause any other edge to lose the 1-goodness property. If follows that the size of \( E_1 \) increases by 1 while \( M^* \) increases by at most 1.

(2) \( w \) has a sibling. In this case for sure \( M^* \) does not increase. Although there may be a concern that the size of \( E_1 \) could drop, below we show that adding a leaf to the stream of the edges of a tree does not cause the size of the set \( E_1 \) to drop. This is enough to show that in this case as well \( |E_1(T)| \geq M^*(T) \).

To see why adding a leaf to the stream of edges cannot reduce \( |E_1| \), assume we insert an edge \( e = (u, w) \) in the stream where \( w \) is a newly added leaf. If \( u \) has no 1-good edges incident on it, then \( E_1 \) remains as it was. If \( u \) has one 1-good edge on it and adding \( e = (u, w) \) causes it to be kicked out of \( E_1 \), it means \( e \) is admitted as a new member of \( E_1 \). So the loss is accounted for. If \( u \) has two 1-good edges on it, say \( e_1 \) and \( e_2 \), we show adding \( e \) cannot cause them to be ejected from \( E_1 \). To see this, assume to the contrary that it could, and suppose (without loss of generality) the edges come in the following order in the stream: \( \ldots, e_2, \ldots, e_1, \ldots, e, \ldots \). The edge \( e_2 \) clearly cannot be part of \( E_1 \). But \( e_1 \) must have a neighboring edge \( e_3 \) incident on \( u \), that follow it. But that means \( e_2 \) was already out before adding \( e \) to the stream. A contradiction. Finally we note that \( u \) cannot have more than two 1-good edges on it. This finishes the proof.

3.2 \( \tilde{O}(\sqrt{n}) \) space algorithm for insert-only streams

In this section, first we present Algorithm 1 that estimates \( M_\mu + h_\mu \) and prove the following theorem.

**Theorem 8.** Let \( G(V, E) \) be a graph with the arboricity bounded by \( c \). Let \( S \) be an (adversarial order) insertion-only stream of the edges of the underlying graph \( G \). Let \( \beta = \mu((2\mu)/(\mu - 2c + 1) + 1) \) where \( \mu > 2c \). Then, there exists an insertion-only streaming algorithm (Algorithm 1) that takes \( \tilde{O}(\sqrt{\sqrt{cn}/\epsilon} \log n) \) space in expectation and outputs a \((1 + \epsilon)\beta\) approximation of \( M^* \) with probability at least 0.86, where \( M^* \) is a maximum matching of \( G \).

**Algorithm 1: Estimate-\( M_\mu + h_\mu \)**

**Initialization:** Each node is sampled to set \( S \) with probability \( p \) (determined below).

**Stream Processing:**

\[ \text{for all the edges } e = (u, v) \text{ in the stream do} \]

\[ \text{if } u \in S \text{ or } v \in S \text{ then} \]

\[ \text{store } e \text{ in } H \text{ if } u \in S \text{ then} \]

\[ \text{increment } d(u) \text{ else increment } l(u) \text{ if numv } \in S \text{ then} \]

\[ \text{increment } d(v) \text{ else increment } l(v) \]

**Post Processing:**

\[ \text{Let } S_1 = \{ v \in S \mid d(v) \leq \mu, \exists w \in \Gamma(v) : d(w) + l(w) \leq \mu \} \]

\[ \text{Let } S_2 \text{ be the set of vertices } \{ v \mid v \in S, d(v) > \mu \} \]

\[ \text{return } s = (|S_1| + |S_2|)/p \]

For each \( w \) in \( \Gamma(S) \) (the set of neighbors of nodes in \( S \)), the algorithm maintains \( l(w) \), the number of occurrences of \( w \) observed since (the first) \( v \in S \) such that \( w \in \Gamma(\{v\}) \) was added. Note that in this algorithm, \( l(w) \) is a lower bound on the degree of \( w \). For the output, \( S_1 \) is the subset of nodes in \( S \) whose degree is bounded by \( \mu \) and additionally there is at least one neighbor of \( v, w \), whose observed degree \((d(w) \text{ or } l(w))\) is at most \( \mu \). Meanwhile, \( S_2 \) is the “high degree” nodes in \( S \).

**Lemma 9.** Let \( \epsilon \in (0, 1) \) and \( \beta = \mu((2\mu)/(\mu - 2c + 1) + 1) \). Algorithm 1 outputs \( s \) where \((1 - \epsilon)M^* \leq s \leq (1 + \epsilon)\beta M^* \) with probability at least \( 1 - \epsilon \).
Proof. First we prove the following bounds on $E(s)$. $M_\mu + h_\mu \leq E(s) \leq \mu(M_\mu + h_\mu)$. Let $L$ be the set of vertices in $G$ that have degree at most $\mu$ and let $G_L$ be the induced graph on $L$. Let $H = V \setminus L$. Note that $G_L$ might have isolated vertices. Let $N$ be the non-isolated vertices in $G_L$. It is clear that if the algorithm samples $v \in N$, $v$ will be in $S_1$. Likewise, if it samples a vertex $w \in H$, $w$ will be in $S_2$. Given the fact that $|H| = h_\mu$ and $|N| \geq M_\mu$, this proves the lower bound on $E(s)$.

The expectation may be above $M_\mu$, as the algorithm may pick an isolated vertex in $G_L$ (a vertex that is only connected to the high-degree vertices) and include it in $S_1$ because one of its high-degree neighbours $w$ was identified as low degree, i.e., $w \in \Gamma(S)$ and $|\Gamma(w)| \leq \mu$ but $w \in H$. Let $u \in H$ and let $U = \{a_1, \ldots, a_\mu\}$ be the last $\mu$ neighbours of $u$ according to the ordering of the edges in the stream. The algorithm can only identify $u$ as low degree when it picks a sample from $U$ and no samples from $\Gamma(u) \setminus U$. This restricts the number of unwanted isolated vertices to at most $\mu h_\mu$. Together with the fact that $|N| \leq \mu M_\mu$, it establishes the upper bound on $E(s)$.

Now using a Chernoff bound, $Pr[|s - E(s)| \leq \lambda E(s)] \leq e^{-\lambda^2(M_\mu + h_\mu)\mu/4} \leq e^{-\lambda^2M_\mu\mu/4}$. Therefore with probability at least $1 - e^{-\lambda^2\mu^2\mu/4}$,

$$
(M_\mu + h_\mu) - \lambda(M_\mu + h_\mu) \leq s \leq \mu(1 + \lambda)(M_\mu + h_\mu)
$$

(2)

Setting $\lambda = \varepsilon/\beta$ and putting this and Lemma 4 together, we derive the statement of the lemma.

Algorithm 2: Estimate-$M^*$

**Initialization:** Let $\varepsilon \in (0, 1)$ and $t = \lceil \beta \sqrt{\text{nc}/\varepsilon} \rceil$ where $\beta$ is as defined in Lemma 9.

**Stream Processing:** Do the following tasks in parallel:

1. Greedily keep a maximal matching of size at most $r \leq t$ (and terminate this task if this size bound is exceeded).

2. Run the Estimate-$M_\mu + h_\mu$ procedure (Algorithm 1) with parameter $p \geq 8/\sqrt{\varepsilon t}$ where $\lambda = \varepsilon/\beta$.

**Post processing:** If $r < t$ then output $2r$ as the estimate for $M^*$, otherwise output the result of the Estimate-$M_\mu + h_\mu$ procedure.

Proof of Theorem 8: Suppose $M^* < t$. Clearly the size of the maximal matching $r$ obtained by the first task will be less than $t$. In this case, $M^* \leq M^* \leq 2M^*$. Now suppose $M^* \geq t$. By Lemma 4 we will have $M_\mu + h_\mu \geq t$ and hence by Lemma 9 with probability at least $1 - e^{-2} \geq 0.86$, the output of the algorithm will be within the promised bounds. The expected space of the algorithm is $O((t + pnc)\log n)$. Setting $t = \beta \sqrt{\text{nc}/\varepsilon}$ to balance the space costs, the space complexity of the algorithm will be $O(\frac{2\sqrt{\text{nc}}}{\varepsilon} \log n)$ as claimed.

3.3 $O(n^{2/3})$ space algorithm for insertion/deletion streams

Algorithms 1 and 2 form the basis of our solution in the more general case where the stream contains deletions of edges as well. In the case of Algorithm 1 the algorithm has to maintain the induced subgraph on $S$ and the edges of the cut $(S, \Gamma(S))$. However if we allow arbitrary number of insertions and deletions, the size of the cut $(S, \Gamma(S))$ can grow as large as $O(n)$ even when $|S| = 1$. This is because each node at some intermediate point could become high degree and then lose its neighbours because of the subsequent deletion of edges. Therefore here in order to limit the space usage of the algorithm, we make the assumptions that number of deletions is bounded by $O(cn)$. Since the processed graph has arboricity at most $c$ this forces the number of insertions to be $O(cn)$ as well. Under this assumption, if we pick a random vertex, still, in expectation the number of neighbours is bounded by $O(c)$.
Another complication arises from the fact that, with edge deletions, a vertex added to $\Gamma(S)$ might become isolated at some point. In this case, we discard it from $\Gamma(S)$. Additionally for each vertex in $S \cup \Gamma(S)$, the counters $d(v)$ (or $l(v)$ depending on if it belongs to $S$ or $\Gamma(S)$) can be maintained as before. The space complexity of the algorithm remains $O(pnc \log n)$ in expectation as long as the arboricity factor remains within $O(\epsilon)$ in the intermediate graphs. In the case of Algorithm 2, we need to keep a maximal matching of size $O(t)$. This can be done in $O(t^2)$ space using a randomized algorithm [3]. Setting $t$ at $\left(\frac{8\beta nc}{\epsilon^2}\right)^{1/3}$ to rebalance the space costs, we obtain the result of Theorem 2.

3.4 The $O(\log^2 n)$ space algorithm for insert-only streams

In this section we present our polylog space algorithm by presenting an algorithm for estimating $|E_\alpha|$ within $(1 + \varepsilon)$ factor. Our algorithm is similar in spirit to the known $L_0$ sampling strategy. It runs $O(\log n)$ parallel threads each sampling the stream at a different rate. At the end, a thread “wins” that has sampled roughly $\Theta(\log n)$ elements from $|E_\alpha|$ (samples the edges with a rate of $\frac{\log n}{|E_\alpha|}$). The threads that under-sample will end up with few edges or nothing while the ones that have oversampled will keep too many elements of $E_\alpha$ and will be aborted as result.

First we give a simple procedure (Algorithm 3) that is the building block of the algorithm.

**Algorithm 3:** The $\alpha$-good test

```
Initialization: given the edge $e = (u, v)$ in the stream, let $r(u) = 0$ and $r(v) = 0$.
forall the subsequent edges $e' = (t, w)$ do
  if $t = u$ or $w = u$ then increment $r(u)$ if $t = v$ or $w = v$ then increment $r(v)$ if
  max{$r(u), r(v)$} $> \alpha$ then terminate and report NOT $\alpha$-good
```

Now we present the main algorithm (Algorithm 4) followed by its analysis.

**Algorithm 4:** An algorithm for approximating $|E_\alpha|$

```
Initialization: $\forall i.X_i = \emptyset$ \> $X_i$ represents the current set of sampled $\alpha$-good edges.
Stream Processing:
forall the levels $i \in \{0, 1, \ldots, \lceil \log_{1+\varepsilon} m \rceil\}$ in parallel do
  forall the edges $e$ do
    Feed $e$ to the active $\alpha$-good tests and update $X_i$
    With probability $p_i = \frac{1}{(1+\varepsilon)^k}$ add $e$ to $X_i$ and start a $\alpha$-good test for $e$.
    Let $|X_i|$ be the number of active $\alpha$-good tests within this level.
    if $|X_i| > \tau = \frac{64\beta^2 \log n}{\varepsilon^2}$ then terminate level $i$
Post processing:
if $|X_0| \leq \tau$ then
  return $|X_0|$
else \> $|X_0| > \tau$
  let $j$ be the smallest integer such that $X_j \leq \frac{8 \log n (1+\varepsilon)}{\varepsilon^2}$ and the $j$-th level was not terminated;
  if there is no such $j$ then return FAIL else return $\frac{X_j}{p_i}$
```

**Lemma 10.** With high probability, Algorithm 4 outputs a $1 \pm O(\varepsilon)$ approximation of $|E_\alpha|$.

**Proof.** It is clear that if $|X_0| \leq \tau$ then $X_0 = E_\alpha$ and the algorithm makes no error. In case $|X_0| > \tau$, we claim that $|E_\alpha| > \frac{\varepsilon}{2\alpha^2}\tau$. To prove this let $t$ be the time step where $|X_0|$ exceeds $\tau$ and let $G_t = (V, E(t))$...
be the graph where \( E(t) = \{e_1, \ldots, e_t\} \). Clearly \( M^*(G) \geq M^*(G_t) \) because the size of matching only increases. Abusing the notation, let \( E_\alpha(G_t) \) denote the set of \( \alpha \)-good edges at time \( t \). By Lemma 6 we have

\[
\tau < |E_\alpha(G_t)| \leq \left( \frac{5}{4} \alpha + 2 \right) M^*(G_t) \leq 4\alpha M^*(G) \leq \frac{2\mu}{\mu - 2c} 4\alpha |E_\alpha| \leq \frac{2\alpha^2}{c} |E_\alpha|.
\]

This proves the claim.

Therefore in the following we assume that \( |E_\alpha| > \frac{c}{\alpha^2} \tau \). Let \( \tau' = \frac{8 \log n}{\varepsilon^2} \) and let \( i^* \) be the integer such that \( (1 + \varepsilon)^{i^* - 1} \tau' \leq |E_\alpha| \leq (1 + \varepsilon)^i \tau' \).

Assuming the \( i^* \)-th level does not terminate before the end, we have \( \frac{\tau'}{1+\varepsilon} \leq E[|X_{i^*}|] \leq \tau' \). By a Chernoff bound, for each \( i \) we have (again assuming we do not terminate the corresponding level)

\[
\Pr[|X_i| - E(|X_i|) \geq \varepsilon E(|X_i|)] \leq \exp \left( -\frac{\varepsilon^2 |E_\alpha|}{4} \right).
\]

Therefore, \( \Pr[|X_{i^*}| - E(|X_{i^*}|) \geq \varepsilon E(|X_{i^*}|)] \leq \exp \left( -\frac{\varepsilon^2 |E_\alpha|}{2(1+\varepsilon)^{\tau'}} \right) \leq \exp \left( \frac{2\log n}{1 + \varepsilon} \right) \leq O(n^{-1}).
\]

As a result, with high probability \( |X_{i^*}| \leq \frac{8 \log n (1+\varepsilon)}{\varepsilon^2} \). Moreover for all \( i < i^* - 1 \), the corresponding levels either terminate prematurely or in the end we will have \( |X_i| > \frac{8 \log n (1+\varepsilon)}{\varepsilon^2} \) with high probability. Consequently \( j \in \{i^*, i^* - 1\} \). It remains to prove that runs corresponding to \( i^* \) and \( i^* - 1 \) will survive until the end with high probability. We prove this for \( i^* \). The case of \( i^* - 1 \) is similar.

Consider a fixed time \( t \) in the stream and let \( X_{i^*}^{(t)} \) be the set of sampled \( \alpha \)-good edges at time \( t \) corresponding to the \( i^* \)-th level. Note that \( X_{i^*}^{(t)} \) contains the a subset of \( \alpha \)-good edges with respect to the stream \( S_t = (e_1, \ldots, e_t) \). From the definition of \( i^* \) and our earlier observations we have

\[
E[|X_{i^*}^{(t)}|] = \frac{|E_\alpha(G_t)|}{(1+\varepsilon)^{i^*}} \leq \frac{2\alpha^2 |E_\alpha|}{c(1+\varepsilon)^{i^*}} \leq \frac{2\alpha^2 \tau'}{c}.
\]

By the Chernoff inequality for \( \delta \geq 1 \),

\[
\Pr \left[ |X_{i^*}^{(t)}| \geq (1+\delta)E(|X_{i^*}^{(t)}|) = \tau \right] \leq \exp \left( -\frac{\delta^2}{3} E(|X_{i^*}^{(t)}|) \right).
\]

From \( \delta = \frac{\tau}{E(|X_{i^*}^{(t)}|)} - 1 = \frac{\tau (1+\varepsilon)^{i^*}}{|E_\alpha(G_t)|} - 1 \), we get

\[
\Pr \left[ |X_{i^*}^{(t)}| \geq \tau \right] \leq \exp \left( -\frac{\tau}{3} + \frac{|E_\alpha(G_t)|}{(1+\varepsilon)^{i^*}} \right) \leq \exp \left( -\frac{\tau}{3} + \frac{2\alpha^2 \tau'}{c} \right).
\]

For \( \tau \geq \frac{8 c^2 \tau'^*}{\varepsilon^2} \), the term inside the exponent is smaller than \( -2 \log n \). It also satisfies \( \delta \geq 1 \). After applying the union bound, for all \( t \) the size of \( X_{i^*}^{(t)} \) is bounded by \( \tau = \frac{64 \alpha^2 \log n}{\varepsilon^2 c^2} \) with high probability. This finished the proof of the lemma.

Next, putting everything together, we prove Theorem 1.

**Proof of Theorem 1:** The theorem follows from Lemmas 6 and 10 and taking \( \alpha = \mu + 1 = 6c \). Observe that the space cost of Algorithm 4 can be bounded: we have \( \log_{1+\varepsilon} m \) levels where each level runs at most \( \tau \) concurrent \( \alpha \)-good tests otherwise it will be terminated. Each \( \alpha \)-good test keeps an edge and two counters and as result it occupies \( O(1) \) space. Consequently the space usage of the algorithm is bounded by \( O(\tau \log_{1+\varepsilon} m) \). The space bound in the theorem follows from the facts that \( \tau = O(\frac{c}{\varepsilon^2} \log n) \) for \( \alpha = 6c \) and \( m \leq cn \).

**References**

[1] K. J. Ahn, S. Guha, and A. McGregor. Analyzing graph structure via linear measurements. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 459–467, 2012.
[2] M. Bury and C. Schwiegelshohn. Sublinear estimation of weighted matchings in dynamic data streams. In *Proceedings of the 23rd Annual European Symposium on Algorithms (ESA)*, pages 263–274, 2015.

[3] R. Chitnis, G. Cormode, H. Esfandiari, M.T. Hajiaghayi, A. McGregor, M. Monemizadeh, and S. Vorotnikova. Kernelization via sampling with applications to finding matchings and related problems in dynamic graph streams. In *Proceedings of the 27th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1326–1344, 2016.

[4] M. Crouch and D. S. Stubbs. Improved streaming algorithms for weighted matching, via unweighted matching. In *Proceedings of the 17th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM)*, pages 96–104, 2014.

[5] R. Duan and S. Pettie. Linear-time approximation for maximum weight matchings. *Journal of the ACM*, 61(1):1–23, 2014.

[6] S. Eggert, L. Kliemann, P. Munstermann, and A. Srivastav. Bipartite graph matchings in the semi-streaming model. *Algorithmica*, 63(1-2):490–508, 2012.

[7] Sebastian Eggert, Lasse Kliemann, Peter Munstermann, and Anand Srivastav. Bipartite matching in the semi-streaming model. *Algorithmica*, 63(1-2):490–508, 2012.

[8] Leah Epstein, Asaf Levin, Julián Mestre, and Danny Segev. Improved approximation guarantees for weighted matching in the semi-streaming model. *SIAM J. Discrete Math.*, 25(3):1251–1265, 2011.

[9] H. Esfandiari, M.T. Hajiaghayi, V. Liaghat, M. Monemizadeh, and K. Onak. Streaming algorithms for estimating the matching size in planar graphs and beyond. In *Proceedings of the 26th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2015.

[10] Hossein Esfandiari, Mohammad Taghi Hajiaghayi, Vahid Liaghat, Morteza Monemizadeh, and Krzysztof Onak. Streaming algorithms for estimating the matching size in planar graphs and beyond. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2015, San Diego, CA, USA, January 4-6, 2015*, pages 1217–1233, 2015.

[11] J. Feigenbaum, S. Kannan, A. McGregor, S. Suri, and J. Zhang. On graph problems in a semi-streaming model. *Theoretical Computer Science*, 348(2):207–216, 2005.

[12] Ashish Goel, Michael Kapralov, and Sanjeev Khanna. On the communication and streaming complexity of maximum bipartite matching. In *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 468–485, 2012.

[13] Oded Goldreich and Dana Ron. Property testing in bounded degree graphs. *Algorithmica*, 32(2):302–343, 2002.

[14] Yijie Han. Matching for graphs of bounded degree. In *Frontiers in Algorithmics, Second Annual International Workshop, FAW 2008, Changsha, China, June 19-21, 2008, Proceedings*, pages 171–173, 2008.

[15] Yijie Han. Matching for graphs of bounded degree. In *Frontiers in Algorithmics*, pages 171–173. Springer, 2008.

[16] A. Hassidim, J. A. Kelner, H. N. Nguyen, and K. Onak. Local graph partitions for approximation and testing. In *Proceedings of the 50th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 22–31, 2009.
[17] John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.*, 2(4):225–231, 1973.

[18] M. Kapralov. Better bounds for matchings in the streaming model. *Proceedings of the 23rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1679–1697, 2013.

[19] M. Kapralov, S. Khanna, and M. Sudan. Approximating matching size from random streams. *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 734–751, 2014.

[20] R. M. Karp, U. V. Vazirani, and V. V. Vazirani. An optimal algorithm for on-line bipartite matching. *Proceedings of the 22nd Annual ACM Symposium on Theory of Computing (STOC)*, pages 352–358, 1990.

[21] C. Konrad, F. Magniez, and C. Mathieu. Maximum matching in semi-streaming with few passes. In *Proceedings of the 11th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM)*, pages 231–242, 2012.

[22] Alexandr V. Kostochka. Lower bound of the hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.

[23] L. Lovasz and M.D. Plummer. Matching theory. In *North-Holland, Amsterdam-New York*, 1986.

[24] A. McGregor. Finding graph matchings in data streams. In *Proceedings of the 8th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM)*, pages 170–181, 2005.

[25] A. McGregor. Graph mining on streams. In *Encyclopedia of Database Systems*, pages 1271–1275. Springer, 2009.

[26] A. McGregor and S. Vorotnikova. Planar matching in streams revisited. In *Proceedings of the 19th International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX)*, 2016.

[27] S. Micali and V. V. Vazirani. An $o(\sqrt{|V||e|})$ algorithm for finding maximum matching in general graphs. *Proceedings of the 21st IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 17–27, 1980.

[28] C. St. J. A. Nash-Williams. Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, 36(1):445–450, 1961.

[29] C. St. J. A. Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 39(1):12, 1964.

[30] H. N. Nguyen and K. Onak. Constant-time approximation algorithms via local improvements. In *Proceedings of the 49th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 327–336, 2008.

[31] List of open problems in sublinear algorithms: Problem 60. [http://sublinear.info/60](http://sublinear.info/60)

[32] Mariano Zelke. Weighted matching in the semi-streaming model. *Algorithmica*, 62(1-2):1–20, 2012.