ON A CALDERÓN-ZYGMUND COMMUTATOR-TYPE ESTIMATE

MIRKO TARULLI AND J. MICHAEL WILSON

Abstract
In this paper we extend a Calderón-Zygmund commutator-type estimate. This estimate enables us to prove an embedding result concerning weighted function spaces.

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1. Introduction and statement of results

The theory of commutators has been extensively studied. They were introduced by R. Coifman, R. Rochberg and G. Weiss in the paper [8], in which it is proved that, given a singular integral $T$ with standard kernel (we refer to [10] and [25] for more details), the operator $[b,T] = bT - Tb$ is bounded in $L^p$, $1 < p < \infty$ if $b$ is a $BMO$ function; the converse implication is due to S. Janson, who stated it in [17]. We refer also to the remarkable paper of C. Pérez [18], in which he underlined that $[b,T]$ is more singular than the operator $T$, proving that this commutator does not satisfy the corresponding $(1,1)$ estimate, but a weaker one, and to the paper of A. Uchiyama [30] in which is proved that the commutator of $T$ and $b$ is a compact operator provided that $b \in VMO$. We also have a more general form of commutator operators; we refer to the papers of A. P. Calderón [4] and [5], in which the author proved that these kinds of operators are bounded on $L^2$ and, under certain condition, they are of Calderón-Zygmund-type (see also his survey paper [6] and to the paper of R. Coifman and Y. Meyer [9]).

In this paper we generalize a Calderón-Zygmund commutator-type estimate introduced by T. Tao in the paper [28] and successively presented in a more extended form by T. Tao and I. Rodnianski in the paper [21] which reads

\begin{equation}
\| [P_k, f] g \|_{L^r} \leq C 2^{-k} \| \nabla f \|_{L^s} \| g \|_{L^p},
\end{equation}

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for any Schwartz functions \( f, g \) whenever \( 1 \leq r, p, q \leq \infty \), \( 1/r = 1/q + 1/p \) and with \( [P_k, f] g = P_k(f g) - f P_k(g) \). Here \( P_k \) is the \( k \)th Littlewood-Paley projection (see Section 2 for the definition).

We prove that this estimate can be improved by considering not only derivatives of first order—the operator \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_n}) \) appearing on the right of (1.1)—but also introducing a homogeneous differential operator of fractional order \( s \) belonging to a bounded interval of \( \mathbb{R} \). This is connected to the fact that the structure of the commutator operator is very rich, and that we can use the good properties of the kernels of such operators coupled with some cancelation involving terms in its integral representation. On the other hand, we can also exploit the relation between differential operators of negative fractional order and \( L^p \) \((1 < p < \infty)\) bounded singular integral operators with homogeneous kernels.

Our main result in this direction is:

**Theorem 1.1.** Given a function \( P \in L^1(\mathbb{R}^n) \) which satisfies, for \( \varepsilon > 0, 0 < s < 1, \) and \( j \in \mathbb{N} \setminus \{0\} \),

\[
\int_{2^{j-1} \leq |x| \leq 2^j} |P(x)| \, dx \lesssim 2^{-j(\varepsilon + s)},
\]

\[
\int_{|x| \leq 1} |P(x)| \, dx \lesssim 1,
\]

if we set \( P_k = 2^{nk} P(2^k x) \), for all \( k \in \mathbb{Z} \), we have, for any \( f, g \in \mathcal{S}(\mathbb{R}^n) \),

\[
\left\| \int P_k(y)(f(x) - f(y))g(x-y) \, dy \right\|_{L^r} \lesssim 2^{-ks} \| |D|^s f\|_{L^p} \| g\|_{L^q},
\]

provided that \( 1 \leq p, q, r \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \).

The theorem implies:

**Corollary 1.1.** The estimate

\[
\| [P_k, f] g \|_{L^r} \leq C 2^{-sk} \| |D|^s f\|_{L^p} \| g\|_{L^q},
\]

holds for all \( k \in \mathbb{Z} \), all functions \( f, g \in \mathcal{S}(\mathbb{R}^n) \) and all \( 1 \leq p, q, r \leq \infty \) such that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \) and \( 0 < s < 1 \).

We apply these commutator estimates to problems of well-posedness and local and global existence for a class of evolution equations. In particular, we are interested in certain function spaces that come up in obtaining the relevant a priori estimates for these equations. These equations can be written in the general form,

\[
\begin{cases}
  iu_t(t,x) + P(D)u(t,x) = F, & t \geq 0, \ x \in \mathbb{R}^n \\
  u(0,x) = u_0(x),
\end{cases}
\]

where the symbol \( P(\xi) \) (it could be either a scalar or a real matrix) has some regularity properties. Some of the most important a priori estimates
on solutions of problems of this type have the form

\[(1.6) \quad \|D_x^\alpha v\|_{L^2(B_w^\gamma)} \lesssim \|f\|_{\mathcal{H}} + \|D_x^\beta F\|_{L^2(B_w^\gamma)},\]

with \(\alpha, \beta \in \mathbb{R}\) satisfying some relations, \(B_w\) is a suitable weighted Banach space (and \(B_w^\gamma\) is its dual space), and \(\mathcal{H}\) is a Hilbert space. In fact, for the case of the Schrödinger equation (i.e., with \(P(D) = -\Delta\) in (1.5)), C. Kenig, G. Ponce and L. Vega established the so-called smoothing estimates in (15) (see also the papers of A. Ruiz and L. Vega [23], B. Perthame and L. Vega [19], and V. Georgiev and M. Tarulli [12]). These were later extended to more general second order Schrödinger equations in [16]. The smoothing estimate reads:

\[(1.7) \quad \sup_m 2^{-m/2} \left\| D_x^{1/2} u \right\|_{L^2(B^2(|x| \sim 2^m))} \leq C \|u_0\|_{L^2} + \sum_m 2^m \left\| D_x^{-1/2} F \right\|_{L^2(B^2(|x| \sim 2^m))},\]

and corresponds to (1.6) if one chooses \(\alpha = 1/2, \beta = -1/2\) and \(B_w = L^1(\mathbb{Z}, 2^{k\delta} dk, L^2(|x| \sim 2^m))\). Chihara proved in [7] that similar estimates can be obtained for the solutions of a larger class of partial differential equations of the form (1.6). More precisely, if we take \(\alpha = d/2, \beta = -d/2\) and \(B_w = L^2(\mathbb{R}^n, \langle x \rangle^{d/2} dx) := L^2_\delta\) the estimates are

\[(1.8) \quad \left\| D_x^{d/2} u \right\|_{L^2_\delta L^2_\delta} \lesssim \|u_0\|_{L^2} + \left\| D_x^{-d/2} F \right\|_{L^2_\delta L^2_\delta},\]

with \(\langle x \rangle = \sqrt{1 + |x|^2}, d > 1, \delta > 0\), where the assumptions on \(P(D)\) reflect the dispersive nature of the equation (1.5); or in other words, its symbol \(P(\xi)\) behaves like \(|\xi|^m, m > 1\). We point out that this estimate is also valid if one uses some rotation invariant norms, just seen for the case of the Schrödinger equation, and which give scale-invariant estimates. There are analogous estimates, transposed to the framework of the wave equation, which correspond to the case \(P(D) = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}\), \(u_0 = (f, g)\), in (1.5), as showed by N. Burq in [3] and M. Tarulli in [27]. These estimates have the form

\[(1.9) \quad \left\| D_x^\gamma (u, D_x^{-1} \partial_t u) \right\|_{L^2_\delta(\mathbb{R}, L^2_{\delta, \Delta} \times L^2_{\delta, \Delta})} \lesssim \|f, g\|_{\dot{H}_\gamma \times \dot{H}_{\gamma - 1}} + \left\| D_x^\gamma F \right\|_{L^2_\delta L^2_\delta},\]

which are valid as well if one uses norms that give scale-invariant estimates, instead of weighted Sobolev norms.

On the other hand, many authors (see, e.g., [21] or [13]) underline the necessity of using Besov (weighted) spaces instead of Sobolev (weighted) ones, because they allow one to better control some nonlinear terms arising in the evolution problem (1.5). (An example is the case of the magnetic potential perturbation \(F = -A(x) \cdot \nabla u\), with some decay property assumed.
on the function $A(x)$; for more detail we refer again to [11]). The Besov norms give not only natural scale-invariant estimates, important for many problems in partial differential equations, but also enable us to exploit the Littlewood-Paley theory on which they are built. There is a vast literature on this subject. We have found many results on equivalences of Besov norms and Sobolev norms (we refer to [2]), but very few regarding the relation between weighted Besov norms and weighted Sobolev norms. A first attempt to do this was due to V. Georgiev, A. Stefanov and M. Tarulli [11]; we generalize those results here. Motivated by this, we shall show that weighted $L^2$-based Sobolev spaces, defined as the closure of Schwartz functions $\phi$ with respect to the norms

\begin{equation}
\sum_m 2^{m/2} \left\| D_x^{\gamma-d/2} \phi(x) \right\|_{L^2(|x| \sim 2^m)}
\end{equation}

are embedded in suitable weighted Besov spaces $Y^{\gamma,d}$, defined as the closure of functions $\phi(x) \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with respect to the norm

\begin{equation}
\|\phi\|_{Y^{\gamma,d}} := \left( \sum_k 2^{\gamma k} \|\phi\|_{Y_{d,k}}^2 \right)^{1/2},
\end{equation}

In (1.11) the spaces $Y_{d,k}$ are defined by the norm:

\begin{equation}
\|\phi\|_{Y_{d,k}} = 2^{-dk/2} \sum_m 2^{m/2} \|\varphi(2^{-m} \cdot \phi_m)\|_{L^2_x},
\end{equation}

where $\phi_m = P_k \phi$ (see again section 2). Similarly, we have the Banach space $Y'_{d,\gamma}$, with norm

\begin{equation}
\|\phi\|_{Y'_{d,\gamma}} := \left( \sum_k 2^{\gamma k} \|\phi\|_{Y'_{d,k}}^2 \right)^{1/2},
\end{equation}

where the spaces $Y'_{d,k}$, equipped with the norm

\begin{equation}
\|\phi\|_{Y'_{d,k}} = 2^{dk/2} \sup_m 2^{-m/2} \|\varphi(2^{-m} \cdot \phi_m)\|_{L^2_x},
\end{equation}

are duals to the spaces $Y_{d,k}$.

From the fact that $\|F\|_{L^2(|x| \sim 2^m)} \sim \|\varphi(2^{-m} \cdot F)\|_{L^2}$ we can replace at will, $\|F\|_{L^2(|x| \sim 2^m)}$, with the comparable expression, $\|\varphi(2^{-m} \cdot F)\|_{L^2}$. We will often do this in the sequel, in order to make use of localizations in both the space and frequency variables.

We shall show:

\begin{enumerate}
\item As noted in [11], the expressions $\phi \to \|\phi\|_{Y_{d,k}}$ are not faithfull norms, in the sense that may be zero, even for some $\phi \neq 0$. On the other hand, they satisfy all the other norm requirements and $\phi \to (\sum_k 2^{\gamma k} \|\phi_k\|_{Y_{d,k}}^2)^{1/2}$ is a norm!
\end{enumerate}
Theorem 1.2. Assume that \( n \geq 2 \). There is a constant \( C = C(n) \), so that for every function \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( \gamma, d \geq 0 \), we have

\[
\| \phi \|_{Y^{\gamma,d}} \leq C \sum_m 2^{m/2} \| \varphi(2^{-m} \cdot) D_x^{\gamma-d/2} \phi(t,x) \|_{L^2_x}.
\]

We conclude this section proving the following corollary.

Corollary 1.2. Assume that \( n \geq 2 \). There is a constant \( C = C(n) \), so that for every function \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( \gamma, d \geq 0 \), we have

\[
\sup_m 2^{-m/2} \| \varphi(2^{-m} \cdot) D_x^{\gamma+d/2} \phi(t,x) \|_{L^2_x} \leq C_n \| \phi \|_{Y^{\gamma,d}}.
\]

Proof. This corollary is an easy consequence of the previous one. We can give its proof in one line. Consider the mapping \( \phi \rightarrow D_x^{\gamma} \phi \), and take the estimate dual to (1.15). That proves (1.16). \( \square \)

Remark 1. We see that we can also assume that the functions \( \phi \) are dependent on the time variable, that is \( \phi = \phi(t,x) \). We just replace in (1.10), (1.12) and (1.14) the \( L^2_x \) norm by the mixed one \( L^2_t L^2_x \), this doesn’t cause confusion, because the whole proof only depends on what happens in the space variable (see, for more details, the proofs of Lemma 4.1 and Theorem (1.2). The estimates (1.15) and (1.16) remain valid and defined in this way, the spaces \( Y^{\gamma,d} \) (now based on \( L^2_t L^2_x \)) are related to the others listed above and connected with the evolution problem (1.5). If we choose \( \gamma = 0 \), we are in the case of (1.8). Furthermore, the smoothing spaces in (1.7) are a particular case of the previous one; that is. \( \gamma = 0 \) and \( d = 1 \) while the others in (1.8) are derived if we pick \( \gamma = 0 \) and \( d = \tilde{d} \). Finally, for the wave equation, we may consider \( \gamma = \tilde{\gamma} - 1/2 \) and \( d = 1 \). The authors say that it is possible to use these results in many other situations.

Remark 2. Theorem (1.2) and Corollary (1.2) generalize completely the estimates proved in Lemma 1 in the paper [11], which correspond to (1.15) and (1.16) in the case \( \gamma = 0 \) and \( d = 1 \).

1.1. Outline of paper. Section 2 is devoted to presenting the ancillary theory used in the whole paper. In Section 3 we prove Theorem 1.1 using two important introductory lemmas. In Section 4 we give the proof of the embedding Theorem 1.2. Finally, in the Appendix we present an alternative proof of Lemma 3.2.

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2. Notations and preliminaries

In this section we introduce some important tools and useful notation
1. Given any two positive real number \(a, b\), we write \(a \lesssim b\) to indicate \(a \leq Cb\), with \(C > 0\). In the same way we say that \(a \sim b\), if there exist positive constants \(c_1, c_2\), so that \(c_1a \leq b \leq c_2a\).

2. The Fourier transform and its inverse are defined by
   \[
   \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} \, dx
   \]
   \[
   f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x \cdot \xi} \, d\xi.
   \]

3. Consider a positive, decreasing, function \(\psi : \mathbb{R}^+ \to \mathbb{R}^+\), which is smooth away from zero, supported in \(\{\xi : 0 \leq \xi \leq 2\}\) and equals 1 for all \(0 \leq \xi \leq 1\). Set \(\varphi(\xi) \equiv \psi(\xi) - \psi(2\xi)\). The function \(\varphi\) is smooth and satisfies:
   \[
   \text{supp} \varphi(\xi) \subseteq \{1/2 \leq |\xi| \leq 2\}, \quad \sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad \forall \xi \neq 0
   \]
   We use \(\varphi(\xi)\) to create a frequency space localization. Similarly, we can introduce the notion of physical space localization by means of the function \(\varphi(x)\) which satisfies \((2.1)\). In higher dimensions, we slightly abuse notation and denote a function with similar properties by the same symbol; namely, \(\varphi(\xi) = \varphi(|\xi|)\), \(\psi(x) = \psi(|x|)\) etc. Note that for \(n > 1\), \(\psi : \mathbb{R}^n \to \mathbb{R}\) is a smooth function even at zero. The \(k^{th}\) Littlewood-Paley projection operator \(P_k, P_{\leq k}\) is defined by
   \[
   \hat{P}_k f(\xi) = \varphi(2^{-k}\xi) \hat{f}(\xi),
   \]
   and
   \[
   \hat{P}_{\leq k} f(\xi) = \psi(2^{-k}\xi) \hat{f}(\xi),
   \]
   we also write \(P_{< k}\) for \(P_{\leq k-1}\). Let us observe that the kernel of \(P_k\) is integrable, smooth and real-valued for every \(k\) and that every \(P_k\) is bounded on every \(L^p\) space, \(1 \leq p \leq \infty\), and that it commutes with constant coefficient differential operators. We may also consider \(P_{> k} := \sum_{l > k} P_l\), which essentially restricts the Fourier transform to frequencies \(\gtrsim 2^k\). In the paper we shall use, without any discussion, the notation \(f_k, f_{\leq k}, f_{< k}\), to indicate respectively \(P_k f, P_{\leq k} f, P_{< k} f\), and, for any fixed \(N \geq 1\), we usually write \(f_{k-N} \leq \leq (K+N)\) for \(P_{\leq k+N} f - P_{< k-N} f\). An obvious modification give also the operators \(P_{\leq k+N} f - P_{< k-N} f\) and consequently we have \(f_{k-N} \leq \leq (K+N)\). We notice also that one can obtain the representations
   \[
   P_k f(x) = 2^{nk} \int \hat{\varphi}(2^k y) f(x - y) dy
   \]
   and
   \[
   P_{\leq k} f(x) = 2^{nk} \int \hat{\psi}(2^k y) f(x - y) dy,
   \]
for any $f \in \mathcal{S}(\mathbb{R}^n)$. Another helpful observation is that, for the differential operator $D_x^s$, defined via the multiplier $|\xi|^s$, we have the identity
\[ D_x^s P_k u = 2^{ks} \tilde{P}_k u, \]
where $\tilde{\varphi}(\xi) = \varphi(\xi)|\xi|^s$. Notice that the operator $\tilde{P}_k$ has kernel satisfying the same estimate of $P_k$. In view of this we say that the two operators are equivalent and we make no difference in the use of them.

5. The Littlewood-Paley theory just presented enables us to introduce the Bernstein inequality for $\mathbb{R}^n$. Given a function $f \in \mathcal{S}(\mathbb{R}^n)$, for any dyadic number $K$ the estimate
\[ \| P_{\leq K} f \|_{L^q} \lesssim K^n \left( \frac{1}{p} \right) \| P_{\leq K} f \|_{L^p} \]
is valid provided that $1 \leq p \leq q \leq \infty$. This inequality remains true if we replace the operator $P_{\leq K}$ by the other one $P_K$ introduced above.

6. We say a function $f$ belongs to $L^r(X)$ if the norm
\[ \| f \|_{L^r(X)} = \left( \int_X |f(x)|^r \, dx \right)^{1/r} \]
is finite; we will usually omit the domain of integration if $X = \mathbb{R}^n$. In a similar way we say a function $u$ belongs to $L^q_t L^r_x$ if the mixed space-time norm,
\[ \| u \|_{L^q_t L^r_x} = \left( \int_\mathbb{R} \left( \int_{\mathbb{R}^n} |u(t,x)|^r \, dx \right)^{q/r} \, dt \right)^{1/q}, \]
is finite.

7. The Hardy-Littlewood maximal function is defined by
\[ M(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, dy, \]
where $B(x,r)$ is the ball of radius $r$ centered at $x$ and $\mu$ is the classical Lebesgue measure on $\mathbb{R}^n$. We remember also that the operator $M$ is bounded on $L^p$ for $1 < p \leq \infty$ (we refer to [10] or [25] for more details).

3. Proof of the Theorem

First of all we need the following lemma

**Lemma 3.1.** Given any function $f \in \mathcal{S}(\mathbb{R}^n)$ and real number $0 < s < 1$, there exists a positive constant $C_{s,n}$ so that, for all $x$ and $y$ in $\mathbb{R}^n$,
\[ |f(x) - f(y)| \leq C_{s,n} |x - y|^s (M(|D|^s f)(x) + M(|D|^s f)(y)), \]
where $M(\cdot)$ is the Hardy-Littlewood maximal operator.
Proof. If \( f \in \mathcal{S}(\mathbb{R}^n) \), we can easily calculate its inverse Fourier transform, obtaining
\[
(3.2) \quad f(x) = \int \hat{f}(\xi) e^{2\pi i (x, \xi)} d\xi = \int |\xi|^{-s} |\xi|^s \hat{f}(\xi) e^{2\pi i (x, \xi)} d\xi.
\]
If \( 0 < \alpha < n \),
\[
|\xi|^{-\alpha} = c_{\alpha,n} |\xi|^{a-n}(\xi).
\]
Therefore, we can rewrite the right-hand side of (3.2) as
\[
(3.3) \quad \int |\xi|^{-\alpha} \left( |\xi|^s \hat{f}(\xi) \right) e^{2\pi i (x, \xi)} d\xi = c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} dt.
\]
Combining (3.2) and (3.4), we have the representation
\[
(3.4) \quad f(x) = c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} dy,
\]
so, we can write:
\[
(3.5) \quad f(x) - f(y) = c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} dt - c_{s,n} \int \frac{|D|^s f(t)}{|y-t|^{n-s}} dt =
\]
\[
= c_{s,n} \int |D|^s f(t) \left( \frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}} \right) dt
\]
\[
= I + II,
\]
where
\[
(3.6) \quad I = c_{s,n} \int_{|x-t| \leq 3|x-y|} |D|^s f(t) \left( \frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}} \right) dt,
\]
and
\[
(3.7) \quad II = c_{s,n} \int_{|x-t| \geq 3|x-y|} |D|^s f(t) \left( \frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}} \right) dt.
\]
We treat \( I \) first. We notice that it is easy to obtain the following set inclusions:
\[
\{ y : |x-t| \leq 3|x-y| \} \subseteq \{ y : |t-y| \leq 4|x-y| \}.
\]
We can write:
\[
|I| \leq c_{s,n} \left( \int_{|x-t| \leq 3|x-y|} \frac{|D|^s f(t)|}{|x-t|^{n-s}} + \int_{|t-y| \leq 4|x-y|} \frac{|D|^s f(t)|}{|y-t|^{n-s}} dt \right).
\]
Given any \( R \in \mathbb{R}_+ \) and \( 0 < s < 1 \), we introduce the following function:
\[
(3.8) \quad \Phi_1^R(z) = \frac{1}{|z|^{n-s}} \chi_{\{|z| \leq 4R\}};\]
\footnote{The kernel in the integral (3.3) is in generally referred to as a Riesz potential.}
where $\chi_{\{|z| \leq 4R\}}$ is the characteristic function of the set $\{|z| \leq 4R\}$. We need to use the pointwise estimate given in the following proposition (see section 2.2, Chapter III, [25], or section 4, Chapter II, [10]).

**Proposition 3.1.** if $\phi$ is a positive, radial decreasing function (on $(0, \infty)$) and integrable, then we have

$$\sup_{R > 0} |\Phi^R * g(z)| \leq \|\Phi^R\|_{L^1} M(g)(z).$$

In our case, from this lemma we get

$$\int_{|z - t| \leq R} |g(t)| \leq \|\Phi^R\|_{L^1} M(g)(z),$$

for all $g \in S(\mathbb{R}^n)$. Then, coupling the estimates (3.9), (3.10) with taking $|x - y| \sim R$, we obtain the inequalities

$$|I| \leq C_{s,n} (\|\Phi^R\|_{L^1} M(|D|^s f)(x) + \|\Phi^R\|_{L^1} M(|D|^s f)(y)) \leq C_{s,n} R^s (M(|D|^s f)(x) + M(|D|^s f)(y)).$$

where the third inequality in the above estimate (3.10) is achieved using the fact that $R^s = c_n \|\Phi^R\|_{L^1}$.

Now we look at $II$. We have from (3.7) the following estimate

$$II \leq c_{s,n} \int_{|x - t| \geq 3|x - y|} \left| |D|^s f(t) \right| \frac{1}{|x - t|^{n-s}} - \frac{1}{|y - t|^{n-s}} dt.$$

We notice now that if $|x - y| \geq 3|x - t|$, then the triangle inequality easily yields

$$2|x - y| \leq |x - t|, \quad |y - t| \leq 2|x - t|, \quad 2|x - t| \leq 3|y - t|,$$

which amount to saying that $|x - t| \sim |y - t|$. This enables us to obtain the following pointwise estimates:

$$\left| \frac{1}{|x - t|^{n-s}} - \frac{1}{|y - t|^{n-s}} \right| \lesssim \frac{|x - t|^{n-s} - |y - t|^{2n-2s}}{|x - t|^{n-s}} \lesssim \frac{|y - t|^{n-s-1}}{|x - t|^{2n-2s}}.$$

Replacing the estimate (3.12) in the inequality (3.11), we arrive at

$$|II| \leq c_{s,n} |x - y| \int_{|x - t| \geq 3|x - y|} \left| |D|^s f(t) \right| \frac{|x - t|^{n-s}}{|x - t|^{n-s+1}} dt.$$
Let us introduce, for $R \in \mathbb{R}_+$ and $0 < s < 1$, the following function
\begin{equation}
\Phi_R^2(z) = \frac{1}{|z|^{n-s+1}} \chi_{\{|z| \geq 4R\}},
\end{equation}
where $\chi_{\{|z| \geq 4R\}}$ is the characteristic function of the set $\{|z| \leq 4R\}$, by the same argument used to prove the inequality (3.9), we see that the right hand side of the (3.13) satisfies the following
\begin{equation}
\int_{|z-t| \geq R} \frac{|D^s f(t)|}{|z-t|^{n-s}} dt \leq \|\Phi_R^2\|_{L^1} M(|D^s f)(x)\).
\end{equation}
We thus have the following chain of inequalities
\begin{equation}
|II| \leq C_{s,n} R \left(\|\Phi_R^2\|_{L^1} M(|D^s f)(x) + \|\Phi_R^2\|_{L^1} M(|D^s f)(y)\) \leq
\end{equation}
\begin{equation}
C_{s,n} R^s (M(|D^s f)(x) + M(|D^s f)(y)).
\end{equation}
with $R^{s-1} = c_n \|\Phi_R^2\|_{L^1}$ and $|x-y| \sim R$. Finally, coupling the estimates (3.10) and (3.16), we obtain the bound (3.1), as desired. □

Remark 3. We notice that this lemma can be read as “local Hölder continuity.” Up to this point, we have considered functions that are small at infinity. It is quite natural to ask what happens for more general ones. Given a function $f$ such that $|D^s f| \in L^\infty$, we obtain from (3.1) the following:

\begin{equation}
\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^s} < \infty.
\end{equation}
This means that the function is Hölder continuous of order $0 < s < 1$. Furthermore, the Hölder continuous norm is related to another one, involving the frequency decomposition (2.1). Set

\[ \|f\|_{Lip(s)} = \sup_k 2^{sk} \|P_k f\|_{L^\infty}. \]
We know that this norm and the norm (3.17) are equivalent, provided that $0 < s < 1$. Thus, from these considerations, we get that $\|f\|_{Lip(s)}$ is bounded whenever $|D^s f| \in L^\infty$. On the other hand if we choose a weak assumption $|D^s f| \in L^p$, for any $p < \infty$ we get another kind of continuity, more precisely

\begin{equation}
\sup_{x \neq y} \left\| \frac{|f(x) - f(y)|}{|x-y|^s} \right\|_{L^p} < \infty.
\end{equation}
This underlines the fact that the pointwise estimate (3.1) is more general and could be applied to other cases.

Lemma 3.1 allows us to prove:
Lemma 3.2. Let be two function \( f, g \in \mathcal{S}(\mathbb{R}^n) \) and a function \( h \in L^1(\mathbb{R}^n) \) with supp\( h \subseteq B(0, R) \). Define

\[
\mathcal{H}(x) = \int h(y)(f(x) - f(x - y))g(x - y)\,dy,
\]

then there exists a positive constant \( C_{s,n} \), so that the following estimate

\[
\|\mathcal{H}\|_{L^r} = C_{s,n}R^s\|h\|_{L^1}\|D^s f\|_{L^p}\|g\|_{L^q},
\]

is satisfied, provided that \( 1 \leq p, q, r \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \) and \( 0 < s < 1 \).

Proof. We split the proof in two cases, \( 1 < p \leq \infty \), and \( p = 1 \).

Case 1 \( < p \leq \infty \). From the definition of the function \( h \), we can rewrite the function \( \mathcal{H}(x) \) as

\[
\mathcal{H}(x) = \int_{|y| \leq R} h(y)(f(x) - f(x - y))g(x - y)\,dy,
\]

and thus, using the the Hölder inequality, one has the bound

\[
\|\mathcal{H}\|_{L^r} \leq C\|h\|_{L^1}\sup_{|y| \leq R} \left( \int |(f(x) - f(x - y))g(x - y)|^r \right)^{1/r} \leq C\|h\|_{L^1}\sup_{|y| \leq R} \left( \int |f(x) - f(x - y)|^p \right)^{1/q} \|g\|_{L^q}.
\]

Now, using the pointwise estimate (3.1) in the (3.22), we obtain

\[
\|\mathcal{H}\|_{L^r} \leq C\|h\|_{L^1}R^s \sup_{|y| \leq R} \|M(|D^s f| \cdot + M(|D^s f| \cdot - y))\|_{L^p}\|g\|_{L^q}.
\]

(3.23)

Finally, by the well-known estimate for the Hardy-Littlewood maximal function \( \|M(f)\|_{L^p} \lesssim \|f\|_{L^p} \), with \( 1 < p \leq \infty \), we obtain (3.20) as desired.

Case \( p = 1 \). Consider, for fixed \( R \in \mathbb{R}_+ \), the function \( \Phi \in L^1(\mathbb{R}^n) \) given by

\[
\Phi = \Phi_1^R + \Phi_2^R,
\]

where \( \Phi_1^R \) and \( \Phi_2^R \) are as in (3.8) and (3.14). From (3.5), (3.10) and (3.16) we have

\[
|f(x) - f(x - y)| \leq |I + II| \lesssim (\Phi * |D^s f(x)| + \Phi * |D^s f(x - y)|),
\]

(3.25)

with \( I \) and \( II \) as in the (3.6) and (3.7). From the pointwise estimate (3.25) and following now the same line of the proof of the previous case we arrive
at
\[ \|H\|_{L^r} \leq C\|h\|_{L^1} \sup_{|y| \leq R} \|\Phi \ast |D|^s f(\cdot)| + \Phi \ast |D|^s f(\cdot - y)\|_{L^p} \|g\|_{L^q}. \]
(3.26)
Finally an application of the Young inequality gives the desired estimate (3.20).

3.1. Proof of the Theorem 1.1 Let us introduce the smooth, positive functions
\[ \chi_j(x) = \varphi(2^{-j}x), \ j \geq 1, \ \chi_0(x) = \sum_{j \leq 0} \varphi(2^{-j}x), \]
with \( \varphi \) defined as in (2.1), and set
\[ P = \sum_{j \geq 0} P\chi_j = \sum_{j \geq 0} Q_j, \]
(3.27)
where \( Q_j = P\chi_j \). Consider now, for any \( k \in \mathbb{Z} \), the rescaled functions
\[ (Q_j)_{2^{-k}} = 2^{-jk}Q_j(2^kx) \] so that \( \text{supp} \ (Q_j)_{2^{-k}} \subseteq \{|x| \leq 2^{-k}2^j\} \), and they also satisfy:
\[ \int |(Q_j)_{2^{-k}}| \ dx \lesssim 2^{-j(s+\epsilon)}. \]
(3.28)
To prove (1.2) start with the fact that
\[ \left\| \int P_k(y)(f(x) - f(y))g(x - y) \, dy \right\|_{L^r} \]
is less than or equal to
\[ \sum_{j \geq 0} \left\| \int (Q_j)_{2^{-k}}(f(x) - f(y))g(x - y) \, dy \right\|_{L^r}. \]
(3.29)
Because of Lemma 3.2, each term of the series (3.29) is less than or equal to a constant times
\[ 2^{(j-k)s}\|(Q_j)_{2^{-k}}\|_{L^1}\|D|^s f\|_{L^p}\|g\|_{L^q} \lesssim 2^{(j-k)s}2^{-j(s+\epsilon)}\|D|^s f\|_{L^p}\|g\|_{L^q} \sim 2^{-ks}2^{-j\epsilon}\|D|^s f\|_{L^p}\|g\|_{L^q}, \]
and therefore (3.29) is no bigger than a constant times \( 2^{-ks}\|D|^s f\|_{L^p}\|g\|_{L^q} \). This complete the proof of the theorem.

4. Proof of the embedding Theorem 1.2

First we introduce the following lemma, in which we improve a result proved in \([11]\). Namely, we have:
Lemma 4.1. If \( f \) is a Schwartz function satisfying the property \( \text{supp} \hat{f}(\xi) \subseteq \{|\xi| \lesssim 2^k\} \), then

\[
\sum_m 2^{am} \|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \leq C\|D\|^{-a} \|f\|_{L^2}
\]

holds for all integers \( k \) and all \( a \in (-n/2, 1) \).

Proof. We shall distinguish two cases. For \( m \leq -k \) we use the estimate (1.4)—with \( r = 2, p = \infty, q = 2, 0 < s < 1 \)—to obtain

\[
\|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \lesssim 2^{-ks} \|[D]^s \varphi(2^{-m} \cdot)\|_{L^2} \|f\|_{L^\infty}.
\]

From Young’s inequality we get

\[
\|D\|^s \varphi(2^{-m} \cdot)\|_{L^2} \leq C 2^{m(n/2 - s)};
\]

and, from Bernstein’s inequality, we derive

\[
\|f\|_{L^\infty} \lesssim 2^{nk/2} \|f\|_{L^2}.
\]

Indeed, from these two estimates we obtain

\[
2^{am} \|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \lesssim 2^{m(a+n/2-s)2k(n/2-s)} \|f\|_{L^2},
\]

Now, given \( a \in (-n/2, 1) \), we can find an \( s \in (0, 1) \) so that \( a + n/2 - s > 0 \). With this \( s \) chosen, we get the estimates:

\[
\sum_{m \leq -k} 2^{am} \|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \lesssim 2^{k(n/2-s)} \sum_{m \leq -k} 2^{m(a+n/2-s)} \|f\|_{L^2} \lesssim 2^{-ak} \|f\|_{L^2}.
\]

For \( m > -k \) we follow a similar argument. We replace (4.3) by the following commutator estimate

\[
\|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \lesssim 2^{-ks} \|[D]^s \varphi(2^{-m} \cdot)\|_{L^\infty} \|f\|_{L^2},
\]

obtained from (1.4)—with \( r = 2, p = 2, q = \infty \). In this way, since Young’s inequality gives

\[
\|[D]^s \varphi(2^{-m} \cdot)\|_{L^\infty} \lesssim 2^{-ms},
\]

we arrive at

\[
\sum_{m > -k} 2^{am} \|[P_k, \varphi(2^{-m} \cdot)]f\|_{L^2} \lesssim 2^{-ks} \sum_{m > -k} 2^{m(a-s)} \|f\|_{L^2} \lesssim 2^{-ak} \|f\|_{L^2}
\]

where we this time we choose \( s \in (0, 1) \) so that \( a - s < 0 \). Finally, this estimate and (4.4) lead to (4.1). \( \square \)
4.1. **Proof of Theorem 1.2.** Taking into account the definition of \( Y_{\gamma,d} \) and the mapping \( \phi \to D_{x}^{\gamma+d/2} \phi = g \), we need to show that, for every Schwartz function \( g \),

\[
\sum_{k} \left( \sum_{m} 2^{m/2} \| \varphi(2^{-m} \cdot) \tilde{P}_{k} g \|_{L_{x}^{2}} \right)^{2} \lesssim \left( \sum_{m} 2^{m/2} \| \varphi(2^{-m} \cdot) g \|_{L_{x}^{2}} \right)^{2},
\]

(4.7)

where \( \tilde{P}_{k} \) is the operator defined as in (2.4). We distinguish two cases. For \( m < -k \) we get

\[
\sum_{k} \left( \sum_{m < -k} 2^{m/2} \| \varphi(2^{-m} \cdot) \tilde{P}_{k} g \|_{L_{x}^{2}} \right)^{2} \lesssim \sum_{k} 2^{-k} \| \tilde{P}_{k} g \|_{L_{x}^{2}}^{2} \lesssim \| D_{x}^{-1/2} g \|_{L_{x}^{2}}
\]

(4.8)

In the remaining case, \( m > -k \), the decomposition

\[
\varphi(2^{-m} x) \tilde{P}_{k} g = \tilde{P}_{k}(\varphi(2^{-m} \cdot) g) - [\tilde{P}_{k}, \varphi(2^{-m} \cdot)] g.
\]

(4.9)

is valid. For the first term on the right-hand side of (4.9), we get the estimate

\[
\left( \sum_{k} \left( \sum_{m > -k} 2^{m/2} \| \varphi(2^{-m} \cdot) \tilde{P}_{k} g \|_{L_{x}^{2}} \right)^{2} \right)^{1/2} \lesssim \sum_{m} 2^{m/2} \| \varphi(2^{-m} \cdot) g \|_{L_{x}^{2}} \sim \sum_{m} 2^{m/2} \| \varphi(2^{-m} \cdot) g \|_{L_{x}^{2}}.
\]

(4.10)

It remains to estimate the commutator in (4.9). In order to do that we use the following relation

\[
[\tilde{P}_{k}, \varphi(2^{-m} \cdot)] g = \sum_{l \leq k-N} \tilde{P}_{k}(\varphi(2^{-m} \cdot) P_{l} g) + [\tilde{P}_{k}, \varphi(2^{-m} \cdot)] g_{k-N < < k+N} + \sum_{l \geq k+N} \tilde{P}_{k}(\varphi(2^{-m} \cdot) P_{l} g),
\]

(4.11)

with \( N \geq 3 \), and in general we refer to the first term on the right-hand side of the above identity as "high-low interactions", while the second and the third are the "low-high" interactions and "high-high interaction". The second term arising above is easy to handle. By the commutator estimate (4.11) (with \( a = 1/2 \), we obtain

\[
\sum_{m > -k} 2^{m/2} \| [\tilde{P}_{k}, \varphi(2^{-m} \cdot)] g_{k-N < < k+N} \|_{L_{x}^{2}} \lesssim 2^{-k/2} \| g_{k-N < < k+N} \|_{L_{x}^{2}}.
\]

(4.12)
Squaring this estimate and taking the sum over $k$ we find

$$
(4.13) \quad \sum_k \left( \sum_{m > -k} 2^{m/2} \| \varphi(2^{-m} \cdot) g_{k-N < \cdot < k+N} \|_{L^2} \right)^2 \lesssim \| D_x^{-1/2} g \|_{L^2_x}^2,
$$

Now we need to estimate the first and the third terms on the right-hand side of (4.11). In order to do that by the Plancherel identity and the properties of the operator $P_l$, we obtain the inequalities

$$
\| \tilde{P}_k(\varphi(2^{-m} \cdot) P_l g) \|_{L^2_x} \lesssim \| \tilde{P}_k(\varphi(2^{-m} \cdot) \tilde{P}_l * P_l g) \|_{L^2_x} \lesssim \left\| \int K_{k,m,l}(\xi,\eta) \tilde{P}_l g(\eta) d\eta \right\|_{L^2_x},
$$

with $\tilde{P}_l = P_{l-1} + P_l + P_{l+1}$ and kernel $K_{k,m,l}(\xi,\eta)$, which satisfies

$$
|K_{k,m,l}(\xi,\eta)| \leq C 2^{nm} \frac{\varphi_k(\xi) \varphi_l(\eta)}{2^{M(\max(k,l)+m)}},
$$

for some $M > 0$ large enough. From these two estimates we get (for more detail on the argument used here we refer to the paper [12], Lemmas 5.1 and 8.1):

$$
(4.14) \quad \| \tilde{P}_k(\varphi(2^{-m} \cdot) P_l g) \|_{L^2_x} \lesssim 2^{-(M(\max(k,l)+m))} \| P_l g \|_{L^2_x}.
$$

for somewhat smaller $M > 0$. We start with the term with indices $k,l$ in the set $\{l \geq k + N\}$. Let us note that $l \geq k + N \geq -m + N$; that is, $l + m \geq N$. By the triangle inequality and the estimate (4.14), we get, for the corresponding terms in (4.11), the following:

$$
(4.15) \quad \sum_k \left( \sum_{m > -k} 2^{m/2} \| \sum_{l \geq k+N} \tilde{P}_k(\varphi(2^{-m} \cdot) P_l g) \|_{L^2_x} \right)^2 \lesssim \sum_k \left( \sum_{m > -k} 2^{m/2} \sum_{l \geq k+N} 2^{-(l+m)M} \| P_l g \|_{L^2_x} \right)^2.
$$

Now, taking into account that the chains of inequalities

$$
\frac{m}{2} - (l + m)M \leq \frac{m}{2} - l - m - (l + m)(M - 1) \leq -\frac{m}{2} - l - (l + m)(M - 1) \leq -\frac{m}{2} - \frac{l}{2} - (l + m)(M - 1) \leq -\left(\frac{m}{2} + \frac{l}{2}\right) - \frac{l}{2} - (l + m)(M - 1) \leq -\frac{l}{2} - (l - k)(M - 1),
$$

we find

$$
\sum_{m > -k} 2^{m/2} \sum_{l \geq k+N} 2^{-(l+m)M} \| P_l g \|_{L^2_x} \lesssim \sum_{m > -k} 2^{m/2} \sum_{l \geq k+N} 2^{-(l+k)(M - 1)} \| P_l g \|_{L^2_x}.
$$

and

$$
\sum_k \left( \sum_{m > -k} 2^{m/2} \| \sum_{l \geq k+N} \tilde{P}_k(\varphi(2^{-m} \cdot) P_l g) \|_{L^2_x} \right)^2 \lesssim \sum_k \left( \sum_{m > -k} 2^{m/2} \sum_{l \geq k+N} 2^{-(l+k)(M - 1)} \| P_l g \|_{L^2_x} \right)^2.
$$

This completes the proof of (4.1) for $S$ and $T$.
are valid, we see that (4.15) can be controlled by the following

\begin{equation}
(4.16) \quad \sum_{l_1,l_2} 2^{-l_1/2} \| P_{l_1} g \|_{L^2_x}^2 2^{l_2/2} \| P_{l_2} g \|_{L^2_x}^2 \sum_{k \leq \min(l_1,l_2) - N} 2^{-2(M-1)(l_1-l_2)} \leq \frac{1}{\| D_x^{-1/2} g \|_{L^2_x}^2}.
\end{equation}

We claim that if \( k \leq \min(l_1, l_2) - N \), then \( l_1 + l_2 - 2k \geq |l_1 - l_2| \). By symmetry, it suffices assume \( l_2 \geq l_1 \), say \( l_2 = l_1 + r \) with \( r \geq 0 \) and \( k = l_1 - N - d \) with \( d \geq 0 \). From this we get

\[ l_1 + l_2 - 2k = 2l_1 + r - 2l_1 + 2N + 2d = r + 2N + 2d \geq r = |l_1 - l_2|, \]

and by this it is easy to bound (4.16) by

\begin{equation}
(4.17) \quad \sum_{l_1,l_2} 2^{-l_1/2} \| P_{l_1} g \|_{L^2_x}^2 2^{-l_2/2} \| P_{l_2} g \|_{L^2_x}^2 2^{-(M-1)r} \leq \frac{1}{\| D_x^{-1/2} g \|_{L^2_x}^2}.
\end{equation}

So, from the estimates (4.15), (4.16) and (4.17) one get the desired

\begin{equation}
(4.18) \quad \sum_k \left( \sum_{m < -k} 2^{m/2} \left( \sum_{l \geq k + N} \tilde{P}_k(\varphi(2^{-m} \cdot) P_l g) \right) \right)^2 \leq \frac{1}{\| D_x^{-1/2} g \|_{L^2_x}^2}.
\end{equation}

It remains to evaluate the first term on the right hand side of the (4.11). We follow the same line of the proof of the previous term, but we consider \( l \leq k - N \). It is enough here to notice again that \( k + m > 0 \) and use the triangle inequality and the estimate (4.13). First we notice that in this regime we have

\[ \frac{m}{2} - (k + m)M \leq \frac{m}{2} - \frac{k}{2} - \frac{m}{2} - (k + m) \left( M - \frac{1}{2} \right) \leq -\frac{k}{2} - (k + m) \left( M - \frac{1}{2} \right) \leq -\frac{k}{2}, \]

and, choosing now \( l_2 = l_1 + r \), \( r \geq 0 \) as in the previous case but with \( k = l_1 + r + N + d \), \( d \geq 0 \),

\[ l_1 + l_2 - 2k = 2l_1 + r - 2l_1 - 2r - 2N - 2d \leq r = |l_1 - l_2| \]
This gives

\[(4.19)\]
\[
\sum_k \left( \sum_{m \geq -k}^\infty 2^{m/2} \| \sum_{l \leq k-N} \tilde{P}_k(\varphi(2^{-m}\cdot)P_l g) \|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \sum_k \left( \sum_{m \geq -k} \sum_{l \leq k-N} 2^{-(k+m)M} \| P_l g \|_{L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \sum_k \left( \sum_{l \leq k-N} 2^{-(k-l_2)} \| P_l g \|_{L_x^2}^2 \right)^{\frac{1}{2}} = \sum_{l_1, l_2} 2^{-l_1/2} \| P_{l_1} g \|_{L_x^2}^2 \lesssim \sum_{l_1, l_2} 2^{-l_1/2} \| P_{l_1} g \|_{L_x^2}^2 \lesssim \sum_{l_1, l_2} 2^{-l_1} \| P_{l_1} g \|_{L_x^2}^2 \lesssim \| D_x^{-1/2} g \|_{L_x^2}^2.
\]

However, by Sobolev embedding and Hölder’s inequality we obtain

\[(4.20)\]
\[
\| D_x^{-1/2} g \|_{L_x^2} \lesssim \| g \|_{L_x^{2n}} = \left( \sum_m \int_{2^{m-1} \leq |x| \leq 2^m} |g|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}} \lesssim \left( \sum_m (2^{mn})^{\frac{n}{n+1}} \int_{2^{m-1} \leq |x| \leq 2^m} |g|^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}} \lesssim \left( \sum_m 2^{m/n+1} \| \varphi(2^{-m}\cdot)g \|_{L_x^2}^{\frac{n+1}{2n}} \right)^{\frac{n+1}{2n}} \lesssim \sum_m 2^{m/2} \| \varphi(2^{-m}\cdot)g \|_{L_x^2},
\]

where the last estimate comes from the fact that $n \geq 2$ and the property $L^q L^p \subset L^p L^q$ if $q \leq p$.

Thus, the estimates estimate (4.8), (4.13), (4.18), (4.15), (4.19) with (4.20) and (4.10) give the proof of the Theorem 1.2.

\[\Box\]

\textbf{Remark 4.} Note that from the argument in the proof of the theorem above one can easily get the following estimate:

\[(4.21)\]
\[
\| P(D) f_k \|_{Y^{\gamma,d}} \lesssim \| f_k \|_{Y^{\gamma,d}} = 2^{\gamma k} \| f \|_{Y_{d,k}}, \quad \forall k \in \mathbb{Z},
\]
valid for any pseudodifferential operator with symbol $P(\xi) \in C^\infty_0(\mathbb{R}^n)$.

\textbf{Remark 5.} Some generalizations of the weighted Sobolev norms used in the previous Theorem can be seen in Theorem 1.6 and Theorem 1.7 in [12].
5. Appendix

In this section we shall show that the method used to prove the Lemma 3.2, case $p = 1$, can be applied in order to achieve the general result for $p \in (1, \infty]$. However the use of the Lemma 3.1 (we referred at this as "local Hölder continuity") is of interest in its own right, and such kind has not appeared in the literature. So, here we have the following

**Lemma 5.1** (Version II of Lemma 3.2). Let be $f, g$ two functions in $S(R^n)$ and $h$ a function in $L^1(R^n)$ with $\text{supp } h \subseteq B(0, R)$. Define

\[
H(x) = \int h(y)(f(x) - f(x - y))g(x - y) \, dy,
\]

then there exists a positive constant $C_{s,n}$, so that the following estimate

\[
\|H\|_{L^r} = C_{s,n}R^s\|h\|_{L^1}\|D^s f\|_{L^p}\|g\|_{L^q},
\]

is satisfied, provided that $1 \leq p, q, r \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $0 < s < 1$.

**Proof.** We consider now the general case $1 \leq p \leq \infty$.

Following the line of the proof of the case $p = 1$, we use again the function $\Phi \in L^1(R^n)$, with $R \in R_+$ fixed, given by

\[
\Phi = \Phi_1^R + \Phi_2^R,
\]

where $\Phi_1^R$ and $\Phi_2^R$ are defined in (3.8) and (3.14). We repeat the proof of the Lemma 3.1, using (3.5), (3.10) (3.16) (notice that we didn’t use the Proposition 3.1 involving the Hardy-Littlewood maximal operator) and we arrive at the identity (3.25). We get easily, by this, the estimate

\[
\|H\|_{L^r} \leq C\|h\|_{L^1}\sup_{|y| \leq R} \|\Phi \ast |D|^s f(\cdot)\| + \Phi \ast \|D|^s f(\cdot - y)\|_{L^p}\|g\|_{L^q}.
\]

(5.3)

Now, by the Young inequality again, we bound the sum on the left hand side of the above estimate (5.3) in the following way

\[
\sup_{|y| \leq R} \|\Phi \ast |D|^s f(\cdot)\| + \Phi \ast \|D|^s f(\cdot - y)\|_{L^p} \lesssim R^s\|D|^s f\|_{L^p}.
\]

(5.4)

This estimate and the (5.3) give the estimate (5.2). The alternative proof is now complete.

□

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Mirko Tarulli, Dipartimento di Matematica Università di Pisa, Largo B. Pontecorvo 5, 56100 Pisa, Italy
E-mail address: tarulli@mail.dm.unipi.it

J. Michael Wilson, Mathematics Department University of Vermont 16 Colchester Ave. Burlington, Vermont 05405, USA
E-mail address: wilson@cem.ums.edu