FOUR DIMENSIONAL ELASTICITY AND
GENERAL RELATIVITY

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Abstract

It has been shown that the extension of the elasticity theory in
more than three dimensions allows a description of space-time as a
properly stressed medium, even recovering the Minkowski metric in
the case of uniaxial stress. The fundamental equation for the metric
in the theory is shown to be the equilibrium equation for the medium.
Examples of spherical and cylindrical symmetries in four dimensions
are considered, evidencing convergencies and divergencies with the
classical general relativity theory. Finally the possible meaning of the
dynamics of the four dimensional elastic medium is discussed.

1 Introduction

The tensorial theory of elasticity in three dimensions has some apparent sim-
ilarity with the classical general relativity theory. The question I have ad-
dressed in this paper is whether or not this formal analogy may correspond
to something more profound than the mere use of symmetric tensors in both
cases. In fact many authors have tried and introduced elasticity into general
relativity, casting the general equations into relativistic form. This was usually made for "practical" purposes, in order to describe the
dynamic behaviour of astrophysical bodies in relativistic conditions, the interaction of gravitational waves with bar antennas, the propagation of shock waves in viscoelastic media and the like.

Here the approach is different, because the space-time itself will be looked at as an elastic medium. Some hint in that direction can be found in the literature, for instance in [7] where Gerlach and Scott introduce a sort of "elasticity of the metric", though in connection with the presence of matter.

The guiding idea of this paper is as follows: suppose space-time is a four dimensional elastic medium. This, when unstrained, is perfectly homogeneous and isotropic. The fundamental symmetry around any point inside it is GL(4,R). Apply now some stresses to the medium: the symmetry will be broken and reduced. In particular if the applied stress is one-dimensional the consequence will be that one particular direction is specialized: could this be time? Otherwise stated: is it possible that a uniaxial stress reduces the GL(4,R) symmetry to SO(3,1)?

This approach, as we shall see, is indeed viable at a first level but leads (I would say 'of course') to final equilibrium equations which are different from those of general relativity. First of all the linear theory of elasticity in any number of dimensions leads inevitably to linear equations. The description of space-time that comes out as a result is "static" i.e. perfectly deterministic. Any dynamics of a four-dimensional medium needs a fifth evolution parameter and the evolution itself would bring about modifications both in the past and the future of a given event.

In what follows I shall review the theory of elasticity and show how it can be brought to describe a reasonable space-time.

2 Instant review of the elasticity theory

2.1 Strain

Suppose you have an n-dimensional elastic medium. In the absence of any strain the geometry inside it is Euclidean (or at least, assume it is). The squared distance between two nearby points is

$$dl_o^2 = \epsilon_{\alpha\beta}dx^\alpha dx^\beta \quad (1)$$

where $\epsilon_{\mu\nu}$ is the metric tensor of the unstrained medium.
Introduce now an infinitesimal strain. Any point will in general be displaced by a small vector \( \vec{w} \), varying from place to place. As a consequence the new squared distance between two points will be written:

\[
dl^2 = (\epsilon_{\alpha\beta} + d\epsilon_{\alpha\beta}) (dx^\alpha + dw^\alpha)(dx^\beta + dw^\beta)
\] (2)

The \( x \)'s are still referred to the unperturbed background and the \( \vec{w} \)'s as well as \( \epsilon \)'s are functions of the \( x \)'s.

In explicit form one has:

\[
\left\{ \begin{array}{l}
\delta\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta,\mu}w^\mu \\
\delta w^\alpha = w^\alpha_{,\mu}dx^\mu
\end{array} \right.
\] (3)

Commas denote partial derivatives.

Now eq. (2) may be written:

\[
dl^2 = dx^\alpha dx^\beta (\epsilon_{\alpha\beta} + \epsilon_{\alpha\mu}w^\mu_{,\beta} + \epsilon_{\beta\mu}w^\mu_{,\alpha} + \epsilon_{\mu\nu}w^\mu_{,\alpha}w^\nu_{,\beta} + \epsilon_{\alpha\beta,\mu}w^\mu + \epsilon_{\alpha\mu,\nu}w^\nu w^\mu_{,\beta} + \epsilon_{\mu\beta,\nu}w^\nu w^\mu_{,\alpha} + \epsilon_{\mu\nu,\lambda}w^\lambda w^\mu_{,\alpha}w^\nu_{,\beta})
\] (4)

Usually part of the content of the brackets in (4) is identified with the strain tensor \( u_{\mu\nu} \), which is manifestly symmetric.

Now (4) becomes:

\[
dl^2 = (\epsilon_{\alpha\beta} + u_{\alpha\beta})dx^\alpha dx^\beta
\] (5)

The almost obvious identification:

\[
g_{\mu\nu} \equiv \epsilon_{\mu\nu} + u_{\mu\nu}
\] (6)

leads to

\[
dl^2 = g_{\mu\nu}dx^\mu dx^\nu
\] (7)

The symmetric tensor \( g_{\mu\nu} \) is now the metric tensor of the strained medium.

All this, as said, has been written using the unperturbed euclidean coordinates. It is more natural to have recourse to internal or intrinsic coordinates (those attached to the medium); these (let us call them \( \xi \)'s) will in general be functions of the \( x \)'s. The geometric nature of the objects used in the theory is such that it is possible to recast everything in terms of the \( \xi \)'s by standard coordinate transformations for tensors. In practice we can simply read formulas from (6) to (7) as if the \( x \)'s were \( \xi \)'s and nothing changes, consequently we shall continue to use \( x \)'s in the new meaning[8]. By the way in the base situation (unstrained medium) \( x \)'s and \( \xi \)'s coincide.
2.2 Stress

In the classical theory of elasticity a stress tensor is introduced whose element $\sigma^{\alpha\beta}$ has the meaning of the $\alpha$-component of the force per unit surface acting on a surface element orthogonal to the $\beta$-direction. Assuming the range of the stresses to be zero (propagation only by surface interaction), $\sigma^{\mu\nu}$ is symmetric. The next step is to link stresses and strains. This can be done via the so called Hooke’s law, which is indeed a linearization stating the proportionality between stresses and strains. The theory may be found in any text book on elasticity such as for instance [9]. The Hooke’s law in $n$ dimensions is expressed by the equivalent formulae:

$$
\begin{align*}
\sigma_{\alpha\beta} &= (K - \frac{2\mu}{n}) \epsilon_{\alpha\beta} \epsilon^{\nu\lambda} u_{\lambda\nu} + 2\mu u_{\alpha\beta} \\
u_{\alpha\beta} &= \left(\frac{1}{n^2 K} - \frac{1}{2\mu}\right) \epsilon_{\alpha\beta} \epsilon^{\nu\lambda} \sigma_{\nu\lambda} + \frac{1}{2\mu} \sigma_{\alpha\beta}
\end{align*}
$$

(8)

$K$ is the uniform compression modulus; $\mu$ is the shear modulus. Reasonable restrictions to the values of $K$ and $\mu$ are:

$$K > 0, \mu > 0$$

(9)

In general in a linearized theory of elasticity in any dimensions two independent parameters are enough to describe the properties of the medium. The parameters in use may be variously combined to produce others such as the first Lamé coefficient $\lambda$ (the second is $\mu$), the Young modulus $E$, the Poisson coefficient $\sigma$.

3 Equilibrium conditions

In our homogeneous stressed medium equilibrium is attained when the following equation holds:

$$\sigma_{\alpha\beta,\beta} + f_{\alpha} = 0$$

(10)

Now $f_{\alpha}$ represents the $\alpha$-component of any force per unit volume; considering the linearity of the Hooke’s law indices are raised and lowered using $\epsilon_{\mu\nu}$’s.

Combining (8), (11) and (8) one obtains:

$$
\begin{align*}
\left[\left(K - \frac{2\mu}{n}\right) \left(\epsilon^{\nu\lambda} g_{\lambda\nu} - n\right) - 2\mu\right] \epsilon_{\alpha\beta,\beta} + \left(K - \frac{2\mu}{n}\right) \epsilon_{\alpha\beta} \left(\epsilon^{\mu\nu} g_{\mu\nu}\right)_{\beta} + 2\mu g_{\alpha\beta,\beta} &= -f_{\alpha}
\end{align*}
$$

(11)
Using Cartesian coordinates (and Euclidean background geometry) (11) simplifies to
\[
(K - \frac{2\mu}{n}) g_{\alpha\alpha,\mu} + 2\mu g_{\mu\nu,\nu} = -f_\mu
\] (12)

Formulae (11) or (12) are \(n\) equations in \(n(n + 1)/2\) unknowns, consequently the problem is underdetermined. Suitable boundary conditions are needed.

4 Uniaxial stress.

Let us now suppose that in our homogeneous \(n\)-dimensional medium a uniform stress is applied along an arbitrary direction: let us call the corresponding axis the \(\tau\) axis. The stress tensor (as referred to a Cartesian coordinates system) is, in our conditions:

\[
\begin{align*}
\sigma_{oo} &= p \\
\sigma_{\alpha\beta} &= 0 \quad \alpha \neq \beta \\
\sigma_{ii} &= \Sigma
\end{align*}
\] (13)

The index number 0 corresponds to \(\tau\), latin indices run from 1 to \(n - 1\). \(\Sigma\) and \(p\) are constants; \(p > 0\) means traction, \(p < 0\) means compression.

Looking at eq. (12) we see that any \(g_{\mu\nu} = \text{constant}\) is a solution of the equilibrium equation. To actually solve the problem we have to directly deduce the \(g_{\mu\nu}\)'s from (6) and the Hooke's law (8):

\[
\begin{align*}
\begin{cases}
  u_{oo} = \frac{1}{n} \left[ \left( \frac{1}{nK} + \frac{n-1}{2\mu} \right) p + (n - 1) \left( \frac{1}{nK} - \frac{1}{2\mu} \right) \Sigma \right] \\
  u_{\alpha\beta} = 0 \\
  u_{ii} = \frac{1}{n} \left[ \left( \frac{1}{nK} - \frac{1}{2\mu} \right) p + \left( \frac{n-1}{nK} + \frac{1}{2\mu} \right) \Sigma \right]
\end{cases}
\end{align*}
\] (14)

Now applying (8) we see that it is \(g_{\mu\nu} = \eta_{\mu\nu}\), where \(\eta_{\mu\nu}\) is the Minkowski metric tensor, whenever

\[
\begin{align*}
\begin{cases}
p = \frac{n-1}{n} (2\mu - nK) \\
\Sigma = -\frac{2\mu + nK(n-1)}{n}
\end{cases}
\] (15)

In four dimensions it is:

\[
\begin{align*}
\begin{cases}
p = \frac{3}{2} (\mu - 2K) \\
\Sigma = -\frac{1}{2} (\mu + 6K)
\end{cases}
\] (16)
Considering conditions (9) it comes out that $\Sigma < 0$ in any case, which means transverse compression. This is consistent with what we know from three-dimensional elasticity if $p > 0$, i.e. if there is traction along the $\tau$ axis. The parameter $p$ is actually greater than zero when

$$\mu > 2K$$

(17)

Minkowski space-time looks like a four-dimensional medium with suitable elastic properties stretched along the time axis. Before the application of the stress no difference exists among the various coordinates, so there is no "time"; once the stress is there one of the coordinates, measured along any axis within the light cone about $\tau$, becomes no longer interchangeable with the others: this, from the intrinsic viewpoint, is time.

5 Spatially flat expanding universe.

Another interesting case is that of an open expanding universe. The corresponding conformally flat metric, in Cartesian coordinates, may be written as:

$$ds^2 = \alpha^2(\tau) \left( d\tau^2 - dx^2 - dy^2 - dz^2 \right)$$

(18)

Introducing the metric (18) into (12) it is easily verified that a nontrivial solution is found for $\alpha(\tau)$ if:

$$f_i = 0, \quad f_o = F = \text{constant}$$

The solution is:

$$\alpha^2(\tau) = \frac{F}{2K - 3\mu} \tau + \text{constant}$$

(19)

and is consistent with the existence of a uniform volume field orthogonal to any space section of the four-dimensional elastic medium.

Rescaling time according to the equation:

$$\alpha(\tau) d\tau = dt$$

the line element assumes its synchronous form:
\[ ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \] (20)

with

\[ a^2(t) = \left| \frac{F}{3\mu - 2K} \right|^{2/3} t^{2/3} \] (21)

As it can be seen the time dependence of the space scale factor is the same as that for a matter dominated Friedmann universe [10].

The solution we have found corresponds to a spherically symmetric situation in four dimensions. There is one center of symmetry (the big bang) and any radial axis may be used as a time axis. Instead a more general positive or negative space curvature Robertson Walker metric does not comply with this symmetry and is no solution to eq. (11).

6 Rotation symmetry about an axis.

This is the typical situation which in general relativity leads, in the static case, to the Schwarzschild solution. A general form for a metric with this symmetry is:

\[
g = \begin{pmatrix}
    f(r, \tau) & 0 & 0 & 0 \\
    0 & -h(r, \tau) & 0 & 0 \\
    0 & 0 & -r^2 & 0 \\
    0 & 0 & 0 & -r^2 \sin^2 \vartheta
\end{pmatrix} \quad (22)
\]

Cylindrical coordinates \( \tau, r, \vartheta, \varphi \) have been used. The corresponding expression for the \( \epsilon \)'s is:

\[
\epsilon = \begin{pmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & r^2 & 0 \\
    0 & 0 & 0 & r^2 \sin^2 \vartheta
\end{pmatrix} \quad (23)
\]

Inserting (22) and (23) into (11) leads to a couple of independent equations:

\[
\begin{align*}
(K - 2\mu) \left( \frac{f}{r} - \frac{\dot{h}}{r^2} \right) + 2\mu \frac{\ddot{f}}{r} &= -f_r \\
(K - 2\mu) \left( \frac{\dot{f}}{r} - \frac{\dot{h}}{r^2} \right) - 2\mu \frac{\ddot{f}}{r} &= -f_r
\end{align*} \quad (24)
\]
Dots stand for partial derivatives with respect to $\tau$ and primes for partial derivatives with respect to $r$.

The static case (independence of $f$ and $h$ from $\tau$) would require $f_o$ to be zero, whereas non trivial solutions exist only if $f_r \neq 0$. To actually solve the problem one needs to impose the distribution of strains or stresses on a suitable surface, remembering also that it should be $f, h > 0$.

At the same results one can arrive starting from the solution of the uniaxial stress case and letting $p$ and $\Sigma$ depend on $\tau$ and $r$.

7 Discussion.

It has been shown that the solutions for the equilibrium conditions inside a four-dimensional elastic medium stressed in any way may provide reasonable forms for the metric of space-time in various symmetry conditions. There are however some problems: one is that of signature.

Treating the case of uniaxial stress we saw that it is possible to recover the Minkowski metric. In four dimensions (14) and (16) lead to the strain tensor components:

\[
\begin{align*}
  u_{oo} &= 0 \\
  u_{ii} &= -2
\end{align*}
\]  

(25)

However we know that the strain tensor is defined starting from a strain vector field according to (4). In the case of uniaxial symmetry the explicit form of the strain tensor in the background euclidean coordinates is:

\[
  u_{\mu \nu} = w_{,\mu,\nu} + w_{,\nu,\mu} + w_{,\alpha,\mu} w_{,\alpha,\nu}
\]  

(26)

Introducing (25) into (26) and solving for the $w$’s one obtains:

\[
\begin{align*}
  w^o &= \text{constant} - 2\tau \\
  w^r &= (-1 \mp i) r
\end{align*}
\]  

(27)

While the strain tensor is real the strain vector field is complex: this is the price to be payed for the Minkowski signature.

Another important point to remind is that the theory, from Hooke’s law on, is linear. This implies that only weak field regions may be described this way. It is possible to attain better approximations for stronger fields having recourse to non linear elasticity. The starting point is the development of
the Helmholtz free energy $F$ of the medium in powers of the strains, where Hooke’s law comes from. The next approximation after the linear one is:

$$F = F_0 + \frac{\lambda}{2} (u^\alpha_\alpha)^2 + \mu u_{\alpha\beta} u^{\alpha\beta} + \frac{\nu}{3} (u^\alpha_\alpha)^3 + \pi u^\alpha_\alpha u^\beta_\beta u^{\beta\alpha} + \rho u^\alpha_\alpha u^\beta_\beta u^{\alpha\beta} + ... \ (28)$$

Three new parameters ($\nu, \pi, \rho$) have been introduced to characterize the behaviour of the medium. Now it is no longer allowed to raise and lower indices using simply the $\epsilon$’s; the $g_{\mu\nu}$’s must be used after developing them up to first order in the $u$’s. Everything is much more complicated but it may be managed.

Finally we may remark that our treatment of an equilibrium condition corresponds to a perfectly static situation, i.e. to an entirely deterministic universe where the histories coincide with the flux lines of the strain vector field. However any elastic medium has not only statics but also dynamics: it may vibrate and has characteristic internal frequencies. If we consider a four dimensional elastic medium, vibrations have meaning of course only with respect to some appropriate evolution parameter, let us call it $T$: something like the good old newtonian time. Four-dimensional observers have of course no means to measure $T$: their clocks actually measure what we called $\tau$ (or t), though now $\tau$, as well as the other space coordinates, parametrically depend on $T$. An influence of the vibrations in four plus one dimensions may however be seen from inside the four-dimensional world.

Suppose for instance that in the medium there are a couple of points hold fixed, whereas the rest undergoes elastic vibrations. Any strain flux line (or history) going from one point to the other is continuously modified by the vibrations in $T$. An internal observer, not being able to perceive $T$, will notice that there are many nearby histories and what in four plus one dimensions is a $T$ evolution for him may well be transformed into different probabilities to be attached to the various histories. If the four-dimensional observer wants to forecast future he will be led to average over histories. A remarkable feature is the fact that for a given vibrating point both future and past (in $\tau$ or $t$) vary in $T$.

I think that this viewpoint may provide a new approach to quantum mechanics and in any case is worth further investigation.

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