Testing for a pure state with local operations and classical communication

Michael Nathanson
Department of Mathematics and Computer Science, St. Mary’s College, Moraga, CA 94556

We examine the problem of using local operations and classical communication (LOCC) to distinguish a known pure state from an unknown (possibly mixed) state, bounding the error probability from above and below. We study the asymptotic rate of detecting multiple copies of the pure state and show that, if the overlap of the two states is great enough, then they can be distinguished asymptotically as well with LOCC as with global measurements; otherwise, the maximal Schmidt coefficient of the pure state is sufficient to determine the asymptotic error rate.

I. INTRODUCTION

The problem of detecting and distinguishing quantum states is fundamental to quantum information theory. Our ability to distinguish outputs from a quantum algorithm or channel limits the amount of information that we can extract from the process. While Helstrom’s Theorem [1] exactly states the optimal error probability when distinguishing single copies of any two known quantum states, it is only recently that results were definitively established for the asymptotic rate to distinguish many copies of these states [2, 3].

In many quantum protocols, two distant parties are limited to Local Operations and Classical Communication (LOCC) [4], in which case the definitive results for quantum state discrimination cited above no longer apply. In such situations, entanglement is alternately a resource and an impediment to accomplishing information tasks. Specifically in LOCC state discrimination, many nonintuitive results indicate that the presence of entanglement is not always bad, and its absence does not eliminate all difficulties introduced by the limitation to LOCC [5–9].

In this work, we examine the problem of distinguishing a particular pure state from an unknown, possibly mixed state using only LOCC. This problem was addressed by Hayashi, et al., [10] in the case that the pure state was a maximally entangled one. Their analysis exposed two issues: the difficulties that arise from the restriction to LOCC and those that are due to the overlap between the two states. We explore the question of which of these challenges is dominant, especially in the asymptotic paradigm, and how this depends on the entanglement of the pure state.

In Section II we state the problem in detail and present our main results. In Section III we extend the analysis to the case where many copies of our system are present and give asymptotic results. Many of our results also apply in the case that the alternative hypothesis is known; this extension is shown in Section IV. In Sections V and VI we review the symmetries in the problem and construct an LOCC measurement for detecting a particular state. The appendices complete the proofs of the theorems and discuss the impact of the assumptions made in the analysis.

II. TESTING FOR A PURE STATE

Suppose a quantum system $\mathcal{H}$ has been prepared either in the pure state $\rho$ or the (possibly mixed) state $\sigma$, with equal probability. It is well known that the optimal error probability to distinguish $\rho$ and $\sigma$ is given by [1]

$$P_{err}(\rho, \sigma) = \frac{1}{2} \left( 1 - \frac{1}{2} \| \rho - \sigma \|_1 \right) \leq \frac{\theta}{2}$$

(1)

where $\theta = \text{Tr} \rho \sigma$. For any pure $\rho$ and $\theta \in [0, 1]$, there exist $\sigma$ which attain the $\frac{\theta}{2}$ bound; and for any $\sigma$ in this family, the optimal measurement is simply to project onto $\rho$ and its orthogonal complement. This optimal measurement is independent of $\theta$ and of the particular $\sigma$ which saturates the inequality [1].

Suppose now that the same problem is presented to two parties (by convention: Alice and Bob) who share a composite finite-dimensional quantum system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^d \otimes \mathbb{C}^d$. As before, we presume that our system has been prepared as either $\rho$ or $\sigma \in \mathcal{H}_A \otimes \mathcal{H}_B$, but now we wish to determine which one using only Local Operations and Classical...
Communications (LOCC). If $\rho$ and $\sigma$ are both known pure states, then they can be distinguished as well with LOCC as with global operations $[5, 11]$. However, the measurement that achieves this depends critically on both states and cannot be performed if $\sigma$ is unknown.

This leads us to the primary question of the present work, which is how best to distinguish a known pure state from an unknown alternative using LOCC, as discussed in $[10]$. Here, we generalize their results and probe the consequences for understanding entanglement. The new question can be phrased in terms of distinguishing two hypotheses: $H_0$: The system has been prepared in the pure state $\rho$ $H_1$: The system has been prepared in an unknown state $\sigma$ with $\text{Tr}\rho\sigma = \theta$

Any measurement we construct will depend on the bipartite structure of $\rho = |\rho\rangle\langle\rho|$. Without loss of generality, we define the standard bases on $H_A$ and $H_B$ using the Schmidt decomposition of $\rho$:

$$|\rho\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$$

$$\lambda_i \geq \lambda_{i+1} \geq 0, \sum_{i=0}^{d-1} \lambda_i = 1$$

(2)

For simplicity, we will write the maximum Schmidt coefficient as $\lambda := \lambda_0$ and define nonnegative parameters $\alpha$ and $\beta$ implicitly by

$$\lambda_1 = \alpha^2 = \lambda(2\beta - 1)$$

(3)

Thus, $\alpha$ is the geometric mean of the two largest Schmidt coefficients, while $\beta$ is their arithmetic mean.

While our interest lies primarily in LOCC measurements, we will look at the standard nested sets of measurements which are relevant in this context:

$$\text{LOCC}^1 \subset \text{LOCC} \subset \text{SEP} \subset \text{PPT} \subset \text{ALL}$$

(4)

As usual, these refer to those allowing Local Operations and Classical Communication, separable measurements, and measurements with a positive semidefinite partial transpose. $\text{LOCC}^1$ indicates that communication travels in only one predetermined direction.

Letting $X$ equal any of the sets above, we follow the notation in $[12]$ and define

$$P_{X}^{\text{err}}(\rho, \sigma) := \inf_{\{T, I - T\} \in X} \frac{1}{2} \text{Tr}(I - T)\rho + T\sigma$$

(5)

Our ignorance of the identity of $\sigma$ motivates the following definition, which will be the focus of the discussion that follows:

$$P_{X}^{\text{err}}(\rho; \theta) := \inf_{\{T, I - T\} \in X} \sup_{\sigma} \frac{1}{2} \text{Tr}(I - T)\rho + T\sigma$$

(6)

where the supremum is taken over all states $\sigma$ with $\text{Tr}\rho\sigma \leq \theta$. This is a minimax approach--we choose a measurement to minimize the worst possible error probability. From the discussion after (1), we see that $P_{\text{ALL}}^{\text{err}}(\rho; \theta) = \frac{1}{2}\theta$. Our first result gives an upper bound on the minimax error for more restricted classes of measurements by constructing an LOCC measurement that, in analogy with $\{\rho, I - \rho\}$, depends only on $\rho$ and has no Type 1 error:

**Theorem 1 (Existence of an LOCC measurement to detect entangled states)** For any pure state $\rho$, there exists a one-way LOCC measurement $\{\tilde{T}, I - \tilde{T}\}$ with $\text{Tr}\tilde{T}\rho = 1$ such that if the system is actually in the state $\sigma$, the probability of incorrectly detecting $\rho$ is bounded by

$$\text{Tr}\tilde{T}\sigma \leq \frac{\theta + \lambda}{1 + \lambda}$$

(7)

where $\theta = \text{Tr}\rho\sigma$.

In addition, there exists a (bidirectional) LOCC measurement $\{\tilde{T}_2, I - \tilde{T}_2\}$ with $\text{Tr}\tilde{T}_2\rho = 1$ such that if the system is actually in the state $\sigma$, the probability of incorrectly detecting $\rho$ is bounded by

$$\text{Tr}\tilde{T}_2\sigma \leq \frac{\theta + \lambda\beta}{1 + \lambda\beta}$$

(8)
Theorem 2 is proved in Appendix A.

Theorem 2: For any pure state \( \rho \) and PPT measurement \( \{T, I - T\} \), there exists a state \( \sigma \) orthogonal to \( \rho \) such that

\[
\frac{1}{2} \text{Tr}((I - T)\rho + T\sigma) \geq \frac{\lambda \alpha^2}{2 + 7\alpha}
\]

(9)

This means that any fixed measurement we implement will always have a “blind spot” – there will always be a \( \sigma \) which will result in the given error probability. Note that the bound in the theorem is not at all tight as the parameters \( \lambda \) and \( \alpha \) are not sufficient to capture all possible behavior. What will be useful is that our bound is proportional to the Schmidt coefficient \( \lambda \) for fixed \( \alpha \). Note that if \( \rho \) is a product state, then \( \alpha = 0 \) and we recover the fact that zero error is possible. Theorem 2 is proved in Appendix A.

Taken together, Theorems 1 and 2 give us the following.

Theorem 3: For any pure state \( \rho \) with \( \lambda, \alpha, \beta \) as defined in (5):

\[
\frac{1}{2} \theta + (1 - \theta) \left( \frac{\lambda \alpha^2}{2 + 7\alpha} \right) \leq P_{err}^{PPT}(\rho; \theta) \leq P_{err}^{LOCC}(\rho; \theta) \leq \frac{\theta + \lambda}{2(1 + \lambda)}
\]

(10)

A simpler lower bound is also possible which is an increasing function of \( \lambda \):

Lemma 4: For any pure state \( \rho \) and PPT measurement \( \{T, I - T\} \),

\[
\text{Tr}T \geq \frac{1}{\lambda} \text{Tr}T\rho
\]

(11)

This lemma is easily proved: If \( T^{PPT} \geq 0 \) then

\[
\text{Tr}_T = \text{Tr}T^{PPT} \leq \|T^{PPT}\|_1 = \lambda \text{Tr}T
\]

(12)

As a result, if we set \( \sigma = \frac{1}{\sqrt{d^2 - 1}}(I - \rho) \) as the normalized projection onto the orthogonal complement of \( \rho \), then for any PPT measurement \( \{T, I - T\} \),

\[
\text{Tr}(I - T)\rho + T\sigma = 1 - \frac{d^2 \text{Tr}T\rho - \text{Tr}T}{d^2 - 1} \geq 1 - \text{Tr}T\rho \left( \frac{d^2 - \lambda^{-1}}{d^2 - 1} \right) \geq \lambda^{-1} - \frac{1}{d^2 - 1}
\]

This gives us the alternative lower bound:

\[
P_{err}^{PPT}(\rho; 0) \geq P_{err}^{PPT}(\rho; \sigma) \geq \frac{\lambda^{-1} - 1}{2(d^2 - 1)}
\]

(13)

Figure 1 shows possible values for \( P_{err}^{LOCC}(\rho; 0) \) as a function of \( \lambda \). The upper bound in each is the curve \( y = \frac{\lambda}{2(1 + \lambda)} \) from [1]. The lower bounds are given by equations (9) and (13). Note that the range of possible values for \( \lambda \) is governed by the dimension \( d \) and the factor \( \alpha \).
FIG. 1: Possible values for $P_{err}^{LOCC}(\rho; 0)$ for $d = 2$ (left) and $d = 8$ (right). The independent variable is $\lambda \in \left[\frac{1}{d}, \frac{1}{\alpha^2 + 1}\right]$, and $\alpha = 0.6$ is fixed.

In general, the simple bound \[13\] is higher for small values of $\alpha$ and of $d$ while $\[9\]$ is better more often for large dimension. In the case of maximally entangled states, the upper and lower bounds intersect, reproducing the result from $\[10\]$ and showing that for maximally entangled states $\rho$, $P_{PPT}^{err}(\rho; \theta) = P_{LOCC}^{1-err}(\rho; \theta) = d\theta + \frac{1}{2(d + 1)}$ (14)

As we go forward, we will see more examples in which the distinction between PPT and LOCC decreases if $\rho$ is entangled enough. This is not a new observation, but the phenomenon is useful, as the set of PPT measurements is a more tractable superset of LOCC.

III. DETECTING MANY COPIES OF $\rho$: ASYMPTOTICS

Recently Audenaert, et al., and Nussbaum, et al., \[2, 3\] looked at the asymptotic probability of using global operations to distinguish two known mixed states $\rho^{\otimes n}$ and $\sigma^{\otimes n}$ as $n$ goes to infinity. Their work led Matthews and Winter \[12\] to extend this idea to asymptotic discrimination of two states $\rho$ and $\sigma$ restricted to a particular class of operations $X$. They define the Chernoff distance between two states (with respect to a set $X$) as

$$\xi^X(\rho, \sigma) = \lim_{n \to \infty} -\frac{1}{n} \log P^X_{err}(\rho^{\otimes n}, \sigma^{\otimes n})$$ (15)

In addition, they provided an example of mixed states $\rho$ and $\sigma$ for which $\xi^{LOCC}(\rho, \sigma) = -\log P^{LOCC}_{err}(\rho, \sigma)$. In this case, since the asymptotic rate is the same as the single-copy rate; there is nothing gained by entangling the measurement between copies of the system.

Applying these ideas to our current problem, we wish to determine whether our system is in the state $\rho^{\otimes n}$ or $\sigma^{\otimes n}$, where $\sigma_n$ is unknown but $\text{Tr}_{\rho^{\otimes n}}\sigma_n = \theta^n$. This obviously includes the case where $\sigma_n = \sigma^{\otimes n}$ is a product of identical copies but is more general (as explored in Appendix C).

This motivates the definition

$$\xi^X(\rho; \theta) = \lim_{n \to \infty} -\frac{1}{n} \log P^X_{err}(\rho^{\otimes n}; \theta^n)$$ (16)

If $\rho$ is pure, then for any $\sigma$, $\xi^{ALL}(\rho, \sigma) = -\log (\inf_{s \in [0, 1]} \text{Tr} \rho^s \sigma^{1-s}) = -\log \theta$ \[2, 3\]. Now, we see that the measurement $\{\rho^{\otimes n}, I - \rho^{\otimes n}\}$ approaches optimality in the asymptotic limit, no matter what $\sigma$ is. Thus, $\xi^{ALL}(\rho; \theta) = -\log \theta$ for $\theta > 0$. Since the first two Schmidt coefficients of $\rho^{\otimes n}$ are $\lambda^n$ and $\lambda^n \alpha^2$, we can use Theorem 3 to show that

$$\frac{1}{2} \theta^n + (1 - \theta^n) \frac{\lambda^n \alpha^2}{2 + 7\alpha} \leq P^{PPT}_{err}(\rho^n; \theta^n) \leq P^{LOCC^1}_{err}(\rho^n; \theta^n) \leq \frac{\theta^n + \lambda^n}{2(1 + \lambda^n)}$$ (17)
The upper bound on $P_{err}^{LOCC}$ is explored in Figure 2. Taking the log of each term and taking the limit as $n$ goes to infinity, both the upper and lower bounds approach the same values as long as $\alpha \neq 0$:

$$\xi^{LOCC}(\rho, \theta) = \lim_{n \to \infty} -\frac{1}{n} \log P_{err}^{LOCC}(\rho \otimes^n, \theta^n) = -\log(\max(\theta, \lambda))$$ (18)

The same calculation applies equally to $\xi^{PPT}(\rho, \theta)$, which gives the following result:

Theorem 5 For any entangled pure state $\rho$ and any $\theta$:

$$\xi^{LOCC}(\rho, \theta) = \xi^{LOCC}(\rho, \theta) = \xi^{SEP}(\rho, \theta) = \xi^{PPT}(\rho, \theta)$$ (19)

$$= -\log(\max(\theta, \lambda))$$ (20)

That is, this asymptotic problem is equally difficult whether we are restricted to one-way LOCC or simply to PPT measurements. Thus:

- If $\theta \geq \lambda$, then $\xi^{LOCC}(\rho, \theta) = -\log \theta = \xi^{ALL}(\rho, \theta)$ and we can asymptotically detect $\rho$ as well with 1-way LOCC as with global measurements.

- If $\theta < \lambda < 1$, then $\xi^{PPT}(\rho, \theta) > \xi^{ALL}(\rho, \theta)$ and we cannot asymptotically detect $\rho$ as well with PPT measurements as with global ones. In particular, if $\theta < \frac{\lambda}{4}$, we can never detect as well as with PPT or LOCC unless $\rho$ is a product state.

Thus, for any sequence of PPT measurements $\{T_n, I - T_n\}$ on $H \otimes^n$, there exists a sequence of alternative states $\sigma_n$ orthogonal to $\rho \otimes^n$ so that the error probability is at least $O(\lambda^n)$. These states will in general be entangled across the copies of our system; they will not be products like $\rho \otimes^n$. See Appendix C for discussion of this fact.

One way to think about Theorem 5 is that if it is comparatively easy to detect $\rho$ using global measurements, then the challenge presented by LOCC really stands out. If global discrimination is fairly difficult, then the restriction to LOCC doesn’t make as much of an impact, since the problem was already difficult to begin with. What is interesting is that the single parameter $\lambda$ is sufficient to capture all the dependence on $\rho$ and that the rate is always either $\lambda$ or $\theta$, and never in between.

**IV. DISTINGUISHING $\rho$ FROM A KNOWN ALTERNATIVE**

A much more traditional problem is that of distinguishing between two known states $\rho$ and $\sigma$, or between multiple copies of these states: $\rho \otimes^n$ and $\sigma \otimes^n$. As mentioned, Helstrom’s theorem [1] and the quantum Chernoff bound [2,3] give solutions to the single-copy and asymptotic problem when global operations are allowed. When we are restricted to LOCC, the situation is more complex. While we can distinguish two pure states effectively with LOCC [5,11], Matthews and Winter’s example [12] shows two orthogonal mixed states that cannot be distinguished well at all, even asymptotically.

The measurement in Theorem 4 was constructed to deal with the case when $\sigma$ is unknown. However, it can be applied equally well for a known, fixed $\sigma$ to give us the following result:
Corollary 6 (to Theorem 1) For a pure state $\rho$ and any state $\sigma$ with $\text{Tr} \sigma = \theta$, we have

$$P_{\text{err}}^{\text{LOCC}}(\rho, \sigma) \leq \frac{\theta + \lambda}{2(1 + \lambda)} \quad P_{\text{err}}^{\text{LOCC}}(\rho, \sigma) \leq \frac{\theta + \lambda \beta}{2(1 + \lambda \beta)}$$

(21)

Obviously, the theorem and corollary have the same content. The corollary focuses on the problem of distinguishing a fixed pair of states, while the theorem emphasizes that the measurement is independent of $\sigma$.

If we now apply this to the states $\rho \otimes n$ and $\sigma \otimes n$ and take the limit as $n$ goes to infinity, we see that

$$\xi^{\text{LOCC}}(\rho, \sigma) \geq -\log(\max(\lambda, \theta))$$

(22)

which gives us the following result:

Theorem 7 For a pure state $\rho$ with maximal Schmidt coefficient $\lambda$ and any state $\sigma$ with $\text{Tr} \sigma = \theta \geq \lambda$, the asymptotic rate to distinguish $\rho \otimes n$ and $\sigma \otimes n$ with one-way LOCC is the same as with global measurements:

$$\xi^{\text{LOCC}}(\rho, \sigma) = \xi^{\text{ALL}}(\rho, \sigma)$$

(23)

When $\theta \geq \lambda$, the conclusion follows from the fact that

$$-\log \theta = \xi^{\text{ALL}}(\rho, \sigma) \geq \xi^{\text{LOCC}}(\rho, \sigma) \geq -\log \theta$$

(24)

Note that we have not shown the converse–our results don’t preclude the possibility of $\rho$ and $\sigma$ with $\theta < \lambda < 1$ such that $\xi^{\text{LOCC}}(\rho, \sigma) = \xi^{\text{ALL}}(\rho, \sigma)$; and in fact we know that this condition is always true if $\sigma$ is also pure. But if the overlap between $\rho$ and $\sigma$ is big enough, the LOCC restriction is not significant.

V. SYMMETRIZING THE PROBLEM

In this section, we describe the symmetries inherent in our problem so that we may use them to construct the measurement from Theorem 1 and to prove the lower bound in Theorem 2.

Much use was made in [10] of the fact that a standard maximally entangled state is invariant under conjugation with any unitary of the form $(U \otimes U^*)$, where the second term is the entrywise complex conjugate of the first. This is useful because the convex set of LOCC measurements is also invariant under this conjugation, and thus any measurement can be symmetrized by “twirling” (as described in, e.g., [10, 13]).

For a general entangled state $\rho$, $(U \otimes V)\rho(U \otimes V)^* = \rho$ only when $U$ is diagonal in the Schmidt basis of $\rho$ and $V = U$. Though the full twirl is no longer useful, we define the more limited symmetrizing map $\Phi$ by analogy:

$$\Phi(\tau) := \int_{Q_d} (U_x \otimes U_{-x})\tau(U_x \otimes U_{-x})^* dx$$

(25)

where $Q_d$ is the unit hypercube, $x = (x_1, x_2, \ldots, x_d) \in [0, 1]^d$ and $U_x$ is the diagonal matrix with entries $e^{2\pi i x_j}, j = 1 \ldots d$. Note that this map may also be efficiently implemented as a discrete sum:

$$\Phi(\tau) = \frac{1}{p^2} \sum_{k,l=0}^{p-1} (U \otimes U^*)^k(Z \otimes Z^*)^l(U \otimes U^*)^{-k}$$

(26)

where $p \geq d$ is an odd prime number, $\omega$ is a primitive $p$th root of unity, and

$$Z = \sum_{j=0}^{d-1} \omega^j |j\rangle\langle j| \quad U = \sum_{j=0}^{d-1} \omega^{j^2} |j\rangle\langle j|$$

In terms of matrices, $\Phi$ eliminates most of the off-diagonal elements:

$$\Phi(\tau)_{ij,kl} = \begin{cases} \tau_{ij,kl} & \text{if } i = j, k = l \text{ or } i = k, j = l \\ 0 & \text{otherwise} \end{cases}$$
Any hermitian matrix $\Phi(\tau)$ on a bipartite qubit system will be of the form

$$\Phi(\tau) = \begin{pmatrix} a & b_1 & c \\ b_1 & b_2 & d \\ c & d & \tau \end{pmatrix}$$

Note that $\Phi(LOCC) \subset LOCC$; if $\{T, I - T\} \in LOCC$, then Alice and Bob can effectively implement $\{\Phi(T), I - \Phi(T)\}$ using the decomposition \([26]\); they randomly select $(k, l) \in \mathbb{Z}_d^2$ and apply the rotations $U^kZ^l$ and $U^{-k}Z^{-l}$ to their respective systems before implementing $\{T, I - T\}$. $\Phi$ also preserves $LOCC^1$, as Alice can randomly select $(k, l)$ on her own and then send the information to Bob after she’s implemented her measurement.

Because $LOCC$ is closed under $\Phi$ and $\rho$ is invariant under $\Phi = \hat{\Phi}$, the extreme values of $T$ and $\sigma$ will share this symmetry:

$$\min_T \max_\sigma \text{Tr}T\sigma + (I - T)\rho \leq \min_T \max_\sigma \text{Tr}\Phi(T)\sigma + (I - \Phi(T))\rho = \min_T \max_\sigma \text{Tr}\Phi(T)\sigma + (I - T)\Phi(\rho) \leq \min_T \max_\sigma \text{Tr}T\sigma + (I - T)\rho$$

(27)

This implies equality throughout, so the extreme values occur when both $T$ and $\sigma$ share all symmetries with $\rho$ (i.e $T = \Phi(T), \sigma = \Phi(\sigma)$).

Any matrix $T = \Phi(T)$ can be written $T = A + B$, where $A$ operates on the span of the Schmidt basis, and $B$ operates on its orthogonal complement:

$$T = A + B = \sum_{i,j} a_{ij} |i \otimes i\rangle\langle j \otimes j| + \sum_{i \neq j} b_{ij} |i \otimes j\rangle\langle i \otimes j|$$

(28)

where the matrix $A = (a_{ij})$ has $I \geq A \geq 0$ and all $b_{ij} \in [0, 1]$. If in addition $T$ is PPT, then for all $i \neq j$, $|a_{ij}|^2 \leq b_{ij}b_{ji}$. Henceforth, we will write $T = A + B$ to indicate this decomposition.

For illustration: If $d = 2$, we can write

$$T = \begin{pmatrix} a & b_1 & c \\ b_1 & b_2 & d \\ c & d & \tau \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 & c \\ 0 & 0 & d \\ c & d & \tau \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 & 0 \\ b_1 & b_2 & 0 \\ 0 & 0 & \tau \end{pmatrix}$$

This decomposition means that the maximum eigenvector of $(I - \rho)T(I - \rho)$ is either in the Schmidt basis or else a product state of distinct elements in that basis, a fact which we will use in the proof of Theorem \([1]\).

VI. A MEASUREMENT TO DETECT AN ENTANGLED $\rho$

In this section, we construct the LOCC measurement used in Theorem \([1]\).

If $\sigma$ has a large component in the Schmidt basis of $\rho$, a good way to distinguish them is having Alice perform a Von Neumann measurement in basis that is unbiased with respect to the Schmidt basis. With this in mind, we define the measurement $\{Q_0, I - Q_0\}$ as follows: Alice measures in the Fourier basis $\{|\varphi_j\rangle = \frac{1}{\sqrt{d}} \sum_k \omega^{jk}|k\rangle, j = 1 \ldots d\}$, where $\omega$ is a primitive $d$th root of unity. If Alice gets the result $j$, then Bob projects onto the state $|\xi_j\rangle = \sum_k \omega^{-jk}\sqrt{\alpha_k}|k\rangle$, i.e.

$$Q_0 = \sum_j |\varphi_j\rangle\langle \varphi_j| \otimes |\xi_j\rangle\langle \xi_j|$$

This measurement has the nice property that $\text{Tr}Q_0\rho = 1$ and that if $|\sigma\rangle = \sum_i \alpha_i |i \otimes i\rangle$ is a linear combination of vectors in the Schmidt basis, then $\langle \sigma|Q_0|\sigma\rangle = \langle \sigma|\rho|\sigma\rangle = \theta$.

We now symmetrize with the map $\Phi$ to get the measurement $\{Q, I - Q\}$, which retains these nice properties and is also implementable with LOCC:

$$Q := \Phi(Q_0) = d\Phi(|\varphi_0\rangle\langle \varphi_0| \otimes |\xi_0\rangle\langle \xi_0|)$$

$$= \rho + \sum_{i \neq j} \lambda_j |i \otimes j\rangle\langle i \otimes j|$$
Note that \( Q\rho = \rho \), and the other nonzero eigenvalues of \( Q \) are given by the \( \lambda_i \), each with multiplicity \((d - 1)\).

\( Q \) is effective at distinguishing \( \rho \) when \( \sigma \) is written in the Schmidt basis. If \( \sigma \) has no component in the span of the Schmidt basis, then we can easily distinguish it from \( \rho \) with the measurement \( \{ R, I - R \} \) with

\[
R = \sum_k |k\rangle\langle k| \otimes |k\rangle\langle k|
\]

which is simply the projection onto the Schmidt basis of \( \rho \). The decomposition \( [2] \) shows immediately that it is achievable with LOCC. Notice that the projections \( Q \) and \( R \) are orthogonal except on \( \rho \), so that \( QR = RQ = \rho \) and \( 0 \leq Q + R - \rho \leq I \).

Since the set of LOCC measures is convex, the fact that \( Q \) and \( R \) are LOCC implies that for any \( \mu \), the measurement \( T_{\mu} = \mu R + (1 - \mu)Q \) is also LOCC. The eigenvalues of \( T_{\mu} \) are 1 as well as \( \mu \) and \((1 - \mu)\lambda_i \), (each with multiplicity \((d - 1)\)), which implies that

\[
||T_{\mu}(I - \rho)||_{\infty} = \max(\mu, (1 - \mu)\lambda)
\]

We minimize this function at \( \mu = \frac{\lambda}{1 + \lambda} \) in order to define our final measurement:

\[
\hat{T} = \frac{\lambda}{1 + \lambda} R + \frac{1}{1 + \lambda} Q
\]

Since \( \rho \) is an eigenvector of \( \hat{T} \), \( \hat{T} = \rho\hat{T}\rho + (I - \rho)\hat{T}(I - \rho) \) and for any \( \sigma \),

\[
\text{Tr}\hat{T}\sigma = \text{Tr}\rho\sigma + \text{Tr}\sigma(I - \rho)\hat{T}(I - \rho) \leq \theta + ||\sigma(I - \rho)||_1||\hat{T}(I - \rho)||_\infty \leq \theta + (1 - \theta)\frac{\lambda}{1 + \lambda} = \frac{\theta + \lambda}{1 + \lambda}
\]

which was to be shown.

If 2-way LOCC is allowed, we can arbitrarily interchange the roles of Alice and Bob in constructing \( \{ Q_2, I - Q_2 \} \) such that

\[
Q_2 := \frac{1}{2} (Q + SQS)
\]

where \( S \) is the swap operator. \( Q_2 \) still has |\( \rho \rangle \) as an eigenvector with value 1, but now its other eigenvalues are \( \frac{1}{2}(\lambda_i + \lambda_j), i \neq j \). This is bounded above by the arithmetic mean of \( \lambda_0 \) and \( \lambda_1 \), which we have denoted \( \lambda\beta \).

Following through all the previous calculations replacing \( \lambda \) with \( \lambda\beta \) gives us an measurement that can be implemented with 2-way LOCC:

\[
\hat{T}_2 := \frac{\lambda\beta}{1 + \lambda\beta} R + \frac{1}{1 + \lambda\beta} Q_2
\]

such that for any \( \sigma \) with \( \text{Tr}\rho\sigma = \theta \),

\[
\text{Tr}\hat{T}_2\sigma \leq \frac{\theta + \lambda\beta}{1 + \lambda\beta}
\]

Note that in the case \( d = 2 \), \( \lambda\beta = \frac{1}{2} \) and \( \hat{T}_2 = \rho + \frac{1}{3}(I - \rho) \), which is completely symmetric on the orthogonal subspace to \( \rho \). It is not known whether this can be effected in high-dimensions while keeping the error small.

### VII. CONCLUSION

We have examined the problem of detecting a known pure state from an unknown alternative using LOCC measurements. We have constructed a one-way LOCC measurement that depends only on the pure state \( \rho \) and is independent of both the alternative hypothesis \( \sigma \) and the overlap \( \theta = \text{Tr}\rho\sigma \). Surprisingly, this measurement is more effective the more entangled \( \rho \) is and, in fact, is optimal for maximally entangled states. We also constructed two lower bounds showing that any PPT
measurement has states orthogonal to \( \rho \) which are “blind spots” for the measurement. This is another way to articulate the importance of knowing the alternative hypothesis when your measurement set is limited.

Moving into the asymptotic paradigm, we showed that the difficulty of distinguishing a pure \( \rho^\otimes n \) from an unknown alternative with \( \text{Tr}\rho^\otimes n \sigma_n = \theta^n \) is governed solely by the larger of the overlap \( \theta \) and the maximum Schmidt coefficient \( \lambda \). This allows us to conclude that for this asymptotic problem, the restrictions to LOCC and PPT measurements are the equivalent, and that if \( \theta \) is big enough, the results are the same as when global measurements are allowed. Finally, we returned to the more familiar problem of distinguishing two known states and showed that if their overlap is big enough, the restriction to LOCC makes no difference in the asymptotic error rate.

This work continues the exploration of the possibilities and limitations of doing quantum information tasks in an LOCC paradigm and how much is lost by disallowing general global operations. More generally, we hope to continue to improve our understanding of locality and entanglement and the different situations in which entanglement either impedes or enables local tasks.

Acknowledgments

This project began in conversation with Chris King, and I am grateful for his support and suggestions. I have also benefitted from conversation and correspondence with Andreas Winter, Will Matthews, and Keiji Matsumoto, who made me aware of his paper \cite{10}. An early draft of this work was presented at the Joint Meetings of the American and Polish Mathematical Societies in Warsaw, July 2007. I am grateful to Mary Beth Ruskai for the invitation to speak there and to the Saint Mary’s College Faculty Development Fund, which supported my participation in this conference.

Appendix A: Proving the Lower Bound in Theorem 2

Suppose that we fix a PPT measurement \( T \) and we wish to put a lower bound on the error probability in the worst case. That is, we wish to maximize over \( \sigma \) orthogonal to \( \rho \) to find

\[
\max_{\sigma} \text{Tr}(I - T)\rho + T\sigma = 1 - \text{Tr}\rho + ||(I - \rho)T||_{\infty}
\]

In what follows, we will write \( p = \text{Tr}\rho \), and recall that any extreme value is achieved with \( T = \Phi(T) = A + B \) as in (28), which means that \( ||(I - \rho)T||_{\infty} = \max(||A'||_{\infty}, ||B'||_{\infty}) \), where \( A' = (I - \rho)A(I - \rho) \). As mentioned above, we can safely assume that the entries of \( A \) are real, which simplifies the calculation just a bit.

Since \( T \) is PPT, \( \sqrt{b_{ij}b_{ji}} \geq |a_{ij}| \) for all \( i \neq j \). In particular:

\[
||B'||_{\infty} = \max_{i \neq j} b_{ij} \geq \max(b_{01}, b_{10}) \geq |a_{01}| \geq a_{01}
\]

We define \( X = |00\rangle\langle 11| + |11\rangle\langle 00| \) and note that \( \text{Tr}AX = 2a_{01} \leq 2||B'||_{\infty} \). Thus, any lower bound for \( \text{Tr}AX \) gives a lower bound for \( ||B'||_{\infty} \).

We decompose \( A \) in terms of its components parallel and orthogonal to \( \rho \):

\[
A = (\rho + (I - \rho)) A (\rho + (I - \rho))
\]

\[
\text{Tr}AX = \text{Tr}\rho A\rho X + 2\text{Tr}(I - \rho)A\rho X + \text{Tr}(I - \rho)A(I - \rho)X
\]

\[
\text{Tr}\rho A\rho X = 2p\lambda_0\lambda_1 = 2p\lambda\alpha
\]

\[
|\text{Tr}(I - \rho)A\rho X| \leq ||(I - \rho)A^{1/2}||_{\infty}||A^{1/2}\rho||_{1}||\rho X||_{\infty}
\]

\[
= \sqrt{||A'||_{\infty}(\rho|A|\rho)(\lambda_0 + \lambda_1)}
\]

\[
= \sqrt{p\lambda(1 + \alpha^2)||A'||_{\infty}}
\]

\[
|\text{Tr}(I - \rho)A(I - \rho)X| \leq ||(I - \rho)(I - \rho)||_{\infty}||X||_{\infty}
\]

\[
\leq ||A'||_{\infty}
\]

where we use the fact that \( A \geq 0 \) and that \( X \) is rank 2 with one positive and one negative eigenvalue.

Putting it all together gives

\[
\text{Tr}AX \geq 2p\lambda\alpha - 2\sqrt{p\lambda(1 + \alpha^2)||A'||_{\infty} - ||A'||_{\infty}}
\]

If we write \( ||A'||_{\infty} = p\lambda\alpha^2r^2 \) for some positive parameter \( r \), we get

\[
\text{Tr}AX \geq p\lambda\alpha(2 - 2r\sqrt{1 + \alpha^2} - \alpha r^2)
\]
Since \( \|B\|_\infty \geq \frac{1}{2} \Tr AX \), we end up with

\[
\|(I - \rho)(I - \rho)\|_\infty = \max(\|A'\|_\infty, \|B\|_\infty)
\geq \frac{p\lambda\alpha}{2} \min_r \max(2\alpha r^2, 2 - 2r\sqrt{1 + \alpha^2 - \alpha r^2})
= \frac{2p\lambda}{9} \left(1 + 3\alpha + \alpha^2 - \sqrt{(1 + \alpha^2)(1 + \alpha^2 + 6\alpha)}\right)
\geq \frac{2p\lambda}{9} \left(\frac{9\alpha^2}{2 + 7\alpha}\right) = \frac{2p\lambda\alpha^2}{2 + 7\alpha}
\]

Finally, this gets the desired result:

\[
2P_{\text{err}} = \max_\sigma \Tr(I - T)\rho + T\sigma = 1 - p + \|(I - \rho)(I - \rho)\|_\infty \geq 1 - p + \frac{2p\lambda\alpha^2}{2 + 7\alpha} \geq \frac{2\lambda^2}{2 + 7\alpha}
\] (A1)

QED

This inequality is not at all tight in general. The achievement here is showing that we can make the bound proportional to \( \lambda \), since for any \( \lambda < 1 \), \( \lambda^n \rightarrow 0 \). This is what allows us to get the asymptotic results.

Note: In the special case that we initially assume that \( p = \Tr T\rho = 1 \), then \( \rho \) is an eigenvector of \( T \), \( \rho A(I - \rho) = 0 \). This simplifies the calculation considerably, and gives us a bound that is always proportional to \( \lambda \alpha \).

\[
2\alpha_{01} = \Tr AX = \Tr\rho A\rho X + \Tr(I - \rho)A(I - \rho)X
\geq 2\lambda\alpha - \|A'\|_\infty
\]

\[
\|(I - \rho)T\|_\infty = \max(\|A'\|_\infty, \|B\|_\infty)
\geq \max(\|A'\|_\infty, a_{01})
\geq \frac{1}{2} \max(2\|A'\|_\infty, 2\lambda\alpha - \|A'\|_\infty)
\geq \frac{2}{3} \lambda \alpha
\]

Appendix B: Do the a priori probabilities change anything?

Throughout this discussion, we have assumed that the null hypothesis \( \rho \) is true with probability one half. Suppose instead we presume that \( \rho \) occurs with nonzero probability \( \pi_0 \) and \( \sigma \) with nonzero probability \( \pi_1 \).

Since the error with the LOCC measurement in Theorem 1 is strictly one-sided, this doesn’t change the calculation at all. For the lower bound shown, we can adjust (A1) to get

\[
P_{\text{err}} = \max_\sigma \Tr\pi_0(I - T)\rho + \pi_1 T\sigma \geq \pi_0(1 - p) + \pi_1 \frac{2p\lambda\alpha^2}{2 + 7\alpha} \geq \min(\pi_0, \pi_1) \frac{2\lambda^2}{2 + 7\alpha}
\]

So, in the case of distinguishing a single copy, our answer is changed if the probability of \( \rho \) is small enough (which makes sense). However, the a priori probabilities don’t make a difference asymptotically: If \( \rho^\otimes n \) and \( \sigma_n \) appear with nonzero probabilities \( \pi_0 \) and \( \pi_1 \), then for large enough values of \( n \), \( \pi_0 > \pi_1 \frac{2\lambda^2}{2 + 7\alpha} \) and our lower bound is proportional to \( \lambda^n \), as desired.

Appendix C: Entanglement between the copies

Throughout the discussion of asymptotics, we attempted to distinguish \( \rho^\otimes n \) from a general state \( \sigma_n \) with \( \Tr\rho^\otimes n\sigma_n = \theta^n \). This makes sense in some contexts, but often we want to distinguish our \( n \) copies of \( \rho \) from \( n \) copies of some other state \( \sigma \), i.e. we want \( \sigma_n = \sigma^\otimes n \). What would the difference in the results be? The purpose of this appendix is to demonstrate by counterexample the lower bounds do not hold if we insist that \( \sigma_n \) has a product structure between the copies.

Counterexample: Suppose we know that our system is equally likely to be in the state \( \rho^\otimes n \) or an unknown state \( \sigma^\otimes n \) with \( \rho \) pure and \( \Tr\rho\sigma = 0 \).

Let \( T_0 = |0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B \) be the projection onto the maximal Schmidt vector of \( \rho \) and write \( T_1 = I - T_0 \). We apply the measurement \( \{T_0, I - T_0\} \) to each copy of our system. Since the system is in an \( n \)-fold product state, the outputs of these measurements are independent. Thus we have repeated an experiment to distinguish \( \rho \) and \( \sigma \) to generate classical
data, so our probability of identifying $\rho$ and $\sigma$ is governed by the classical Chernoff bound. If $P_{(T_0, T_1)}(\rho^\otimes n, \sigma^\otimes n)$ is the probability of error using the measurement $\{T_0, T_1\}^\otimes n$, then the classical Chernoff bound $C$ is

$$C = \lim_{n \to \infty} -\frac{1}{n} \log P_{(T_0, T_1)}(\rho^\otimes n, \sigma^\otimes n)$$

$$= -\min_{s \in [0,1]} \log \left( P(T_0|\rho)^s P(T_0|\sigma)^{1-s} + P(T_1|\rho)^s P(T_1|\sigma)^{1-s} \right)$$

$$\geq -\min_{s \in [0,1]} \log \left( \lambda^s (1-\lambda)^{1-s} + (1-\lambda)^s \lambda^{1-s} \right)$$

$$= -\log(2\sqrt{\lambda(1-\lambda)})$$

This follows from the fact that $P(T_0|\rho) = \lambda$ and $P(T_0|\sigma) \leq 1-\lambda$ since $\rho + \sigma \leq I$.

This means that for any state $\sigma$ orthogonal to $\rho$ (even if $\sigma$ is unknown),

$$\xi_{LOCC}^1(\rho, \sigma) \geq -\log(2\sqrt{\lambda(1-\lambda)})$$

We can compare this to our minimax definition 0:

$$\xi_{PPT}^1(\rho; 0) = \lim_{n \to \infty} -\frac{1}{n} \log P_{err}^{PPT}(\rho^\otimes n; 0) \leq -\log \lambda$$

Thus, if $\rho$ is close to a product state ($\lambda > \frac{1}{2}$), then

$$\xi_{PPT}^1(\rho; 0) < \xi_{LOCC}^1(\rho, \sigma)$$

for any $\sigma$ that is orthogonal to $\rho$. This means that the orthogonal states guaranteed by Theorem 2 cannot be product states across the copies of the system. Note that argument can be extended to states of the form $\sigma_n = \sigma^1 \otimes \sigma^2 \otimes \cdots \otimes \sigma^n$, not just ones with identical $\sigma$.

On the other hand, if $\rho$ is maximally entangled, then the measurement in [10] is completely symmetric on the orthogonal complement of $\rho$. Thus, the error doesn’t depend on whether $\sigma_n$ is a product or not.

[1] C. W. Helstrom, Quantum Detection and Estimation Theory (Academic, New York, 1976).
[2] K. M. R. Audenaert, J. Calsamiglia, R. Munoz-Tapia, E. Bagan, Ll. Masanes, A. Acín and F. Verstraete, “The Quantum Chernoff Bound,” Phys. Rev. Lett. 98, 160501 (2007).
[3] M. Nussbaum, A. Szkoła, “The Chernoff lower bound for symmetric quantum hypothesis testing,” Ann. Statistics 37 (2), 1040-1057 (2009).
[4] J. Walgate, A. J. Short, L. Hardy and V. Vedral, “Local Distinguishability of Multipartite Orthogonal Quantum States,” Phys. Rev. Lett. 85, 4972 (2000).
[5] J. Walgate, L. Hardy, “Nonlocality, Asymmetry, and Distinguishing Bipartite States,” Phys. Rev. Lett. 89, 147901 (2002).
[6] C. Bennett, D. DiVincenzo, C. Fuchs, T. Mor, E. Rains, P. Shor, J. Smolin, W. Wootters, “Quantum Nonlocality without Entanglement,” Phys. Rev. A 59, 1070 (1999).
[7] M. Nathanson, “Distinguishing bipartite orthogonal quantum states using LOCC: Best and worst cases,” J. Math. Phys. 46, 062103 (2005).
[8] M. Horodecki, A. Sen(De), U. Sen, K. Horodecki, “Local indistinguishability: more nonlocality with less entanglement,” Phys. Rev. Lett. 90, 040402 (2003).
[9] M. Horodecki, K. Matsumoto, Y. Tsuda, “A study of LOCC-detection of a maximally entangled state using hypothesis testing,” J. Phys. A 39, 14427-14446 (2006).
[10] S. Virmani, M. F. Sacchi, M. B. Plenio, D. Markham, “Optimal local discrimination of two multipartite pure states,” Phys. Rev. Lett. 129 (2001).
[11] W. Matthews, A. Winter, “On the Chernoff distance for asymptotic LOCC discrimination of bipartite quantum states,” Commun. Math. Phys. 285, 161174 (2009).
[12] D. P. DiVincenzo, D. W. Leung and B. M. Terhal, “Quantum data hiding,” IEEE Trans. Inf. Theory, 3, pp. 580598 (2002).