Local well-posedness for Euler-Poisson fluids with non-zero heat conduction

Jiang Xu*

Department of Mathematics,
Nanjing University of Aeronautics and Astronautics,
Nanjing 211106, P.R.China

Abstract

We consider the multidimensional Euler-Poisson equations with non-zero heat conduction, which consist of a coupled hyperbolic-parabolic-elliptic system of balance laws. We make a deep analysis on the coupling effects and establish a local well-posedness of classical solutions to the Cauchy problem pertaining to data in the critical Besov space. Proof mainly relies on a standard iteration argument. To achieve it, a new Moser-type inequality is developed by the Bony’ decomposition.

Keywords: local well-posedness, Euler-Poisson equations, classical solutions, Besov spaces

AMS subject classification: 35M10; 35Q35; 76X05

1 Introduction and main results

The ongoing miniaturization of semiconductor devices, some high field phenomena such as hot electron effects, impact ionization and heat generation appear inside the devices. The traditional drift-diffusion model employed for numerical simulation does not provide an adequate description of these effects. Consequently, the hydrodynamical model for semiconductors was introduced, which can be derived from the Boltzmann equation by moment method based on the shifted Maxwellian ansatz for the equilibrium phase space distribution. Precisely, the hydrodynamical model takes the form of the following compressible Euler-Poisson equations (see, e.g., [26]):

\[
\begin{align*}
&\partial_t n + \text{div}(nu) = 0, \\
&\partial_t (nu) + \text{div}
(nu \otimes u) + \nabla P = n\nabla \Phi - \frac{nu}{\tau_p}, \\
&\partial_t W + \text{div}(Wu + Pu) - \text{div}(\kappa \nabla T) = nu \cdot \nabla \Phi - \frac{W - W}{\tau_w}, \\
&\lambda^2 \Delta \Phi = n - \bar{n},
\end{align*}
\]

(1.1)

*E-mail: jiangxu_79@yahoo.com.cn, jiangxu_79@muaa.edu.cn
for \((t, x) \in [0, +\infty) \times \mathbb{R}^N (N = 2, 3)\). Here, \(n(t, x) > 0\) denotes the electron density, \(u(t, x) \in \mathbb{R}^N\) electron velocity and \(W(t, x)\) energy density. \(\Phi = \Phi(t, x)\) represents the electrostatic potential generated by the Coulomb force from electrons and background ions. \(P = nT\) is the pressure of electron fluid where \(T(t, x)\) is the temperature of electrons. The energy density \(W\) satisfies \(W = \frac{n|u|^2}{2} + \frac{\gamma}{\gamma - 1} (\gamma > 1)\) and \(\bar{W} = \frac{nT_L}{\gamma - 1}\) is the ambient device energy, where \(T_L > 0\) is a given ambient device temperature. \(\bar{n} > 0\) is the doping profile which stands for the density of fixed, positively charged background ions. The scaled coefficient \(\tau_p, \tau_w\) and \(\lambda\) are the momentum relaxation-time, energy relaxation-time and the Debye length, respectively. The coefficient \(\kappa\) is the heat conductivity, which generally depends on the electron density and temperature. For the sake of simplicity, we assume it to be one constant.

The full hydrodynamical model (1.1) for the balance laws of the density, velocity, temperature and the electric potential consists of a quasi-linear hyperbolic-parabolic-elliptic system, which contains the damping relaxation, heat conduction and electric dissipation. The interaction of these special effects makes it complicated to understand the qualitative behavior of solutions, however, many efforts were made by various authors, see [1, 2, 3, 7, 11, 14, 15, 16, 17, 18, 19, 22, 23, 25, 27, 28, 29, 33, 35] and the references therein, for issues of well-posedness of steady-state solutions or classical, large time behavior and singular limit problems.

In this paper, we are concerned with the well-posedness of classical solutions starting with smooth initial data under the coupled effects. In the one space dimension, Chen, Jerome, and Zhang [7] first considered the initial boundary problem of (1.1) and established the local existence of smooth solutions. Furthermore, they showed that smoothness in local solutions can be extended globally in time for smooth initial data near a constant state. This result indicated that the relaxation effect could prevent the development of shock waves for the case of smooth initial data with small oscillation. Hsiao and Wang [19] considered the corresponding non-constant steady state solutions to (1.1) and it was shown that the solutions were exponentially locally asymptotically stable. For the Cauchy problem of (1.1) with large smooth initial data, it was proved in [28] the solution generally develops a singularity, shock waves, and hence no global classical solution exists, which is due to the strong hyperbolicity, even the damping relaxation and the heat conduction (parabolicity) can’t prevent the formation of singularity. The effect of the Poisson coupling (ellipticity) is smoothing and it decisively affects the stationary states of the Euler-Poisson equations (1.1), see [11, 13].

Physically, it is more important and more interesting to study (1.1) in several space dimensions where were expected to get some similar results as the one-dimension case. Hsiao, Jiang and Zhang [17] first studied the Cauchy-Neumann problem of (1.1). Using the classical energy approach, they established the global exponential stability of small smooth solutions near the constant equilibrium. Subsequently, Li [22] extended their results to the non-constant equilibrium.

Recently, we started a program to investigate the hydrodynamic model for semiconductor from the point of view of Fourier analysis, which is more careful and refined manner. For instance, the method enables us to understand the Poisson coupling effect well, which plays a key role in the low frequency of density. Such a fact explains the global exponential stability of small smooth solutions in essential. Up to now, we have achieved some results in this direction, see [12, 30, 31, 32, 34]. In these works, we focused on the case of \(\kappa = 0\) mainly and the Euler-Poisson equations (1.1) can be reduced to the pure hyperbolic form of the balance law with a non-local source term by virtue of the Green’s formulation. However, the case of \(\kappa \neq 0\) is not
the trivial one based on the following considerations:

(1) The Euler-Poisson equations (1.1) has not scaling invariance, pertinent to the compressible Navier-Stokes equations in [9, 10]. Thus, we are going to choose the non-homogeneous Besov spaces $B^{N/p}_{p,1}(1 \leq p < \infty)$ as the basic functional setting (in $x$), which are the critical spaces embedding in the space of Lipschitz functions;

(2) For the critical case of regularity index, the classical existence theory for generally hyperbolic systems established by Kato and Majda [21, 24] fails. Ifrimie [20] first gave the contribution for general hyperbolic systems, however, due to the complicated coupling, the result of Ifrimie can not be applied directly;

(3) The inequality in Proposition 2.6 giving parabolic regularity in the framework of Besov spaces depends on given time $T$ (except for the case of $\alpha = 1$ and $\alpha_1 = \infty$), which may preclude from proving global existence results (even if small data).

Therefore, as the first step, we establish a local existence result of Cauchy problem pertaining to data in the critical Besov spaces for the Euler-Poisson equations (1.1) with $\kappa \neq 0$. For this purpose, the initial conditions for $n, u$ and $\mathcal{T}$, and a boundary condition for $\Phi$ are equipped:

\[(n, u, T)(x, 0) = (n_0, u_0, T_0)(x), \quad x \in \mathbb{R}^N, \quad (1.2)\]

\[\lim_{|x| \to +\infty} \Phi(t, x) = 0, \quad a. e. \ t > 0, \quad (1.3)\]

where the homogeneous boundary condition for $\Phi$ means that the semiconductor device is in equilibrium at infinity.

The local existence of a solution stems from the standard iterative method. In comparison with that in [10], there are some differences in the proof of local existence. First, the estimate of density $n$ does not follow from the estimates for the transport equation in Besov spaces directly, since the velocity $u$ in the momentum equations has not higher regularity. Actually, to obtain the desired frequency-localization estimate of $(n, u)$, we introduce a function change to reduce (1.1) to a part symmetric hyperbolic form, and take full advantage of linear hyperbolic theory in the framework of Besov spaces and hyperbolic energy method for dyadic blocks. Consequently, this restricts us to the space case of $p = 2$ in (1). Second, we show that the approximate solution sequence is a Cauchy sequence in some norm to prove the convergence rather than using compactness arguments. In the meantime, in order to overcome the difficulty arising from the heat conduction term, we develop a more general version of the classical Moser-type inequality in Proposition 2.3, the reader is refer to the Appendix. According to the new Moser-type inequality, the heat conduction term can be estimated ultimately, for details, see (3.34)-(3.35). Finally, by a careful analysis on the coupling effects, the uniqueness of classical solutions is shown in the appropriately larger spaces.

Through this paper, the regularity index $\sigma = 1 + N/2$. Our main result is stated as follows.

**Theorem 1.1.** Let $\bar{n}, \mathcal{T}_L > 0$ be the constant reference density and temperature. Suppose that $n_0 - \bar{n}, u_0, \nabla \Phi(\cdot, 0) \in B^\sigma_{2,1}(\mathbb{R}^N)$ with $n_0 > 0$ and $\mathcal{T}_0 - \mathcal{T}_L \in B^{\sigma+1}_{2,1}(\mathbb{R}^N)$. Then there exists a time $T_1 > 0$ such that
(i) Existence: the system $[1.1]-[1.3]$ has a solution $(n, u, T, \nabla \Phi)$ belongs to \( (n, u, T, \nabla \Phi) \in C^1([0, T_1] \times \mathbb{R}^N) \) with \( n > 0 \) for all \( t \in [0, T_1] \).

Furthermore, the solution \((n, u, T, \nabla \Phi)\) satisfies
\[
(n - \bar{n}, u, \nabla \Phi) \in \bar{C}_T(B_{2,1}^s(\mathbb{R}^N)) \times \left( \bar{C}_T(B_{2,1}^s(\mathbb{R}^N)) \right)^N \times \left( \bar{C}_T(B_{2,1}^s(\mathbb{R}^N)) \right)^N
\]
and
\[
T - T_L \in \bar{C}_T(B_{2,1}^{s+1}(\mathbb{R}^N)).
\]

(ii) Uniqueness: the solution \((n, u, T, \nabla \Phi)\) is unique in the spaces
\[
\bar{L}_T^\infty(B_{2,1}^s) \times \left( \bar{L}_T^\infty(B_{2,1}^{s-1}) \right)^N \times \bar{L}_T^1(B_{2,1}^s) \times \left( \bar{L}_T^\infty(B_{2,1}^{s-1}) \right)^N,
\]
In addition, there exists a constant \( C_0 > 0 \) depending only on \( \bar{n}, T_L, N, \gamma \) such that
\[
\| (n - \bar{n}, u, \nabla \Phi) \|_{\bar{L}_T^\infty(B_{2,1}^s)} + \| T - T_L \|_{\bar{L}_T^\infty(B_{2,1}^{s+1})} \leq C_0 M,
\]
where \( M := \| (n_0 - \bar{n}, u_0, \nabla \Phi(\cdot, 0)) \|_{B_{2,1}^s} + \| \bar{T}_0 - T_L \|_{B_{2,1}^{s+1}} \) and \( \nabla \Phi(\cdot, 0) := \nabla \Delta^{-1}(n_0 - \bar{n}) \).

Remark 1.1. The symbol \( \nabla \Delta^{-1} \) means
\[
\nabla \Delta^{-1} f = \int_{\mathbb{R}^N} \nabla_x G(x - y) f(y) dy,
\]
where \( G(x, y) \) is a solution to \( \Delta_x G(x, y) = \delta(x - y) \) with \( x, y \in \mathbb{R}^N \).

Remark 1.2. Unlike \cite{10}, no smallness condition on the initial density is required, hence the local well-posedness in critical spaces holds for any initial density bounded away from zero. Let us mention that the heat conductivity \( \kappa \) in the proof is assumed to be large appropriately, see the following inequalities (3.21) and (3.36).

Remark 1.3. The local existence results in the framework of Sobolev spaces \( H^s \) with higher regularity \( s > 1 + N/2 \) were obtained by the standard contraction mapping principle, see, e.g., [7]. Theorem 1.1 deals with the limit case of regularity \( (\sigma = 1 + N/2) \), which is a natural generalization of their results. To the best of our knowledge, the global well-posedness and large-time behavior of (1.1) in critical spaces still remain unsolved, since the inequality of parabolic regularity in Proposition 2.6 depends on the time \( T \), which are under current consideration.

The rest of this paper unfolds as follows. In Section 2, we briefly review the Littlewood-Paley decomposition theory and the characterization of Besov spaces and Chemin-Lerner’s spaces. Section 3 is dedicated to the proof of the local well-posedness of classical solutions in critical spaces. Finally, the paper ends with an appendix, where we prove a Moser-type inequality with aid of the Bony’s composition.

Notations. Throughout this paper, \( C > 0 \) is a harmless constant. Denote by \( C([0, T], X) \) (resp., \( C^1([0, T], X) \)) the space of continuous (resp., continuously differentiable) functions on \([0, T]\) with values in a Banach space \( X \). For simplicity, the notation \( \|(a, b, c, d)\|_X \) means \( \|a\|_X + \|b\|_X + \|c\|_X + \|d\|_X \), where \( a, b, c, d \in X \). We shall omit the space dependence, since all functional spaces are considered in \( \mathbb{R}^N \). Moreover, the integral \( \int_{\mathbb{R}^N} f dx \) is labeled as \( \int f \) without any ambiguity.
2 Tools

The proofs of most of the results presented in this paper require a dyadic decomposition of Fourier variable. Let us recall briefly the Littlewood-Paley decomposition theory and the characterization of Besov spaces and Chemin-Lerner’s spaces, see for instance [5, 8] for more details.

Let \((\varphi, \chi)\) be a couple of smooth functions valued in \([0, 1]\) such that \(\varphi\) is supported in the shell \(C(0, \frac{2}{3}, \frac{8}{3}) = \{\xi \in \mathbb{R}^N | \frac{2}{3} \leq |\xi| \leq \frac{8}{3}\}\), \(\chi\) is supported in the ball \(B(0, \frac{4}{3}) = \{\xi \in \mathbb{R}^N | ||\xi|| \leq \frac{4}{3}\}\) and

\[
\chi(\xi) + \sum_{q=0}^{\infty} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^N.
\]

Let \(S'\) be the dual space of the Schwartz class \(S\). For \(f \in S'\), the nonhomogeneous dyadic blocks are defined as follows:

\[
\Delta_{-1} f := \chi(D)f = \tilde{\omega} * f \quad \text{with} \quad \tilde{\omega} = \mathcal{F}^{-1}\chi;
\]

\[
\Delta_q f := \varphi(2^{-q}D)f = 2^q d \int \omega(2^q y) f(x-y)dy \quad \text{with} \quad \omega = \mathcal{F}^{-1}\varphi, \quad \text{if} \quad q \geq 0,
\]

where \(*\) the convolution operator and \(\mathcal{F}^{-1}\) the inverse Fourier transform. The nonhomogeneous Littlewood-Paley decomposition is

\[
f = \sum_{q \geq -1} \Delta_q f \quad \forall f \in S'.
\]

Define the low frequency cut-off by

\[
S_q f := \sum_{p \leq q-1} \Delta_p f.
\]

Of course, \(S_0 f = \Delta_{-1} f\). The above Littlewood-Paley decomposition is almost orthogonal in \(L^2\).

**Proposition 2.1.** For any \(f \in S'(\mathbb{R}^N)\) and \(g \in S'(\mathbb{R}^d)\), the following properties hold:

\[
\Delta_p \Delta_q f \equiv 0 \quad \text{if} \quad |p-q| \geq 2,
\]

\[
\Delta_q (S_{p-1} f \Delta_p g) \equiv 0 \quad \text{if} \quad |p-q| \geq 5.
\]

Having defined the linear operators \(\Delta_q(q \geq -1)\), we give the definition of Besov spaces and Bony’s decomposition.

**Definition 2.1.** Let \(1 \leq p \leq \infty\) and \(s \in \mathbb{R}\). For \(1 \leq r < \infty\), Besov spaces \(B^s_{p,r} \subset S'\) are defined by

\[
f \in B^s_{p,r} \iff \|f\|_{B^s_{p,r}} =: \left( \sum_{q \geq -1} (2^{qs}\|\Delta_q f\|_{L^p})^r \right)^{\frac{1}{r}} < \infty
\]

and \(B^s_{p,\infty} \subset S'\) are defined by

\[
f \in B^s_{p,\infty} \iff \|f\|_{B^s_{p,\infty}} =: \sup_{q \geq -1} 2^{qs}\|\Delta_q f\|_{L^p} < \infty.
\]
Definition 2.2. Let \( f, g \) be two temperate distributions. The product \( f \cdot g \) has the Bony’s decomposition:
\[
f \cdot g = T_{fg} + T_{gf} + R(f, g),
\]
where \( T_{fg} \) is paraproduct of \( g \) by \( f \),
\[
T_{fg} = \sum_{p \leq q - 2} \Delta_p f \Delta_q g = \sum_q S_{q-1} f \Delta_q v
\]
and the remainder \( R(f, g) \) is denoted by
\[
R(f, g) = \sum_q \Delta_q f \tilde{\Delta}_q g \quad \text{with} \quad \tilde{\Delta}_q := \Delta_{q-1} + \Delta_q + \Delta_{q+1}.
\]

As regards the remainder of para-product, we have the following results.

Proposition 2.2. Let \((s_1, s_2) \in \mathbb{R}^2\) and \(1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty\). Assume that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1, \quad \text{and} \quad s_1 + s_2 > 0.
\]
Then the remainder \( R \) maps \( B_{s_1^{p_1}, r_1} \times B_{s_2^{p_2}, r_2} \) in \( B_{s_1 + s_2 + d(1/1, 1/1)} \) and there exists a constant \( C \) such that
\[
\|R(f, g)\|_{B_{s_1 + s_2 + d(1/1, 1/1)}} \leq C\|f\|_{B_{s_1^{p_1}, r_1}} \|g\|_{B_{s_2^{p_2}, r_2}}.
\]

Some conclusions will be used in subsequent analysis. The first one is the classical Bernstein’s inequality.

Lemma 2.1. Let \( k \in \mathbb{N} \) and \( 0 < R_1 < R_2 \). There exists a constant \( C \), depending only on \( R_1, R_2 \) and \( d \), such that for all \( 1 \leq a \leq b \leq \infty \) and \( f \in L^a \),
\[
\sup \text{Supp } F f \subset B(0, R_1 \lambda) \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+d(1/a - 1/b)} \|f\|_{L^a};
\]
\[
\sup \text{Supp } F f \subset C(0, R_1 \lambda, R_2 \lambda) \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^k \|f\|_{L^a}.
\]
Here \( F f \) represents the Fourier transform on \( f \).

As a direct corollary of the above inequality, we have

Remark 2.1. For all multi-index \( \alpha \), it holds that
\[
\|\partial^\alpha f\|_{B^s_{p,r}} \leq C \|f\|_{B^s_{p,r+|\alpha|}}.
\]

The second one is the embedding properties in Besov spaces.
Lemma 2.2. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then

$$B^s_{p,r} \hookrightarrow B^{	ilde{s}}_{p,r} \quad \text{whenever } \tilde{s} < s \text{ or } \tilde{s} = s \text{ and } r \leq \tilde{r};$$

$$B^s_{p,r} \hookrightarrow B^{s-N(\frac{1}{p} - \frac{1}{\tilde{p}})}_{p,r} \quad \text{whenever } \tilde{p} > p;$$

$$B^d_{p,1}(1 \leq p < \infty) \hookrightarrow C_0, \quad B^0_{\infty,1} \hookrightarrow C \cap L^\infty,$$

where $C_0$ is the space of continuous bounded functions which decay at infinity.

The third one is the traditional Moser-type inequality.

Proposition 2.3. Let $s > 0$ and $1 \leq p, r \leq \infty$. Then $B^s_{p,r} \cap L^\infty$ is an algebra. Furthermore, it holds that

$$\|fg\|_{B^s_{p,r}} \leq C(\|f\|_{L^\infty} \|g\|_{B^s_{p,r}} + \|g\|_{L^\infty} \|f\|_{B^s_{p,r}}).$$

On the other hand, we present the definition of Chemin-Lerner’s spaces first introduced by J.-Y. Chemin and N. Lerner [6], which is the refinement of the spaces $L^q_T(B^s_{p,r})$.

Definition 2.3. For $T > 0, s \in \mathbb{R}, 1 \leq r, \rho \leq \infty$, set (with the usual convention if $r = \infty$)

$$\|f\|_{L^q_T(B^s_{p,r})} := \left(\sum_{q \geq -1} (2^{qs}\|\Delta_q f\|_{L^q_T(L^p)})^r\right)^{\frac{1}{r}}.$$

Then we define the space $\tilde{L}^q_T(B^s_{p,r})$ as the completion of $S$ over $(0, T) \times \mathbb{R}^d$ by the above norm.

Furthermore, we define

$$\tilde{C}_T(B^s_{p,r}) := \tilde{L}^\infty_T(B^s_{p,r}) \cap C([0, T], B^s_{p,r}),$$

where the index $T$ will be omitted when $T = +\infty$. Let us emphasize that

Remark 2.2. According to Minkowski’s inequality, it holds that

$$\|f\|_{L^q_T(B^s_{p,r})} \leq \|f\|_{L^q_T(B^s_{p,r})} \quad \text{if } r \geq \rho; \quad \|f\|_{L^q_T(B^s_{p,r})} \geq \|f\|_{L^q_T(B^s_{p,r})} \quad \text{if } r \leq \rho.$$

Then, we state the property of continuity for product in Chemin-Lerner’s spaces $\tilde{L}^q_T(B^s_{p,r})$.

Proposition 2.4. The following estimate holds:

$$\|fg\|_{\tilde{L}^q_T(B^s_{p,r})} \leq C(\|f\|_{L^p_T(L^\infty)} \|g\|_{\tilde{L}^q_T(B^s_{p,r})} + \|g\|_{L^p_T(L^\infty)} \|f\|_{\tilde{L}^q_T(B^s_{p,r})})$$

whenever $s > 0, 1 \leq p \leq \infty, 1 \leq \rho, \rho_1, \rho_2, \rho_3, \rho_4 \leq \infty$ and

$$\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4}.$$

As a direct corollary, one has

$$\|fg\|_{\tilde{L}^q_T(B^s_{p,r})} \leq C\|f\|_{\tilde{L}^p_T(B^s_{p,r})} \|g\|_{\tilde{L}^q_T(B^s_{p,r})}$$

whenever $s \geq N/p, \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. 

7
In addition, the estimates of commutators in $\widetilde{L}^p_T(B^{s}_{p,r})$ spaces are also frequently used in the subsequent analysis. The indices $s, p$ behave just as in the stationary case \cite{8, 12} whereas the time exponent $\rho$ behaves according to Hölder inequality.

**Lemma 2.3.** Let $1 \leq p < \infty$ and $1 \leq \rho \leq \infty$, then the following inequalities are true:

$$2^{qs}\|[f, \Delta_q]A g\|_{L^p_T(L^p)} \leq \left\{ \begin{array}{ll}
C_c q \|f\|_{\tilde{L}^p_T(B^{s}_{p,1})} \|g\|_{\tilde{L}^p_T(B^{s}_{p,1})}, & s = 1 + N/p,
C_c q \|f\|_{\tilde{L}^p_T(B^{s}_{p,1})} \|g\|_{\tilde{L}^p_T(B^{s+1}_{p,1})}, & s = N/p,
C_c q \|f\|_{\tilde{L}^p_T(B^{s}_{p,1})} \|g\|_{\tilde{L}^p_T(B^{s+1}_{p,1})}, & s = N/p,
\end{array} \right.$$  

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$, the operator $A = \text{div}$ or $\nabla$, $C$ is a generic constant, and $c_q$ denotes a sequence such that $\|\|c_q\||_{11} \leq 1, \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$.

In the symmetrization, we shall face with some composition functions. To estimate them, the following continuity result for compositions is necessary.

**Proposition 2.5.** Let $s > 0, 1 \leq p, r, \rho \leq \infty$, $F \in W^{[\alpha]+1, \infty}_{\text{loc}}([1, \infty])$ with $F(0) = 0$, $T \in (0, \infty)$ and $v \in \tilde{L}^p_T(B^{s}_{p,r}) \cap \tilde{L}^\infty_T(L^\infty)$. Then

$$\|F(v)\|_{\tilde{L}^p_T(B^{s}_{p,r})} \leq C(1 + \|v\|_{\tilde{L}^\infty_T(L^\infty)})^{[\alpha]+1} \|v\|_{\tilde{L}^p_T(B^{s}_{p,r})}.$$  

Finally, we give the estimate of heat equation to end up this section.

**Proposition 2.6.** Let $s \in \mathbb{R}$ and $1 \leq \alpha, p, r \leq \infty$. Let $T > 0, u_0 \in B^{s}_{p,r}$ and $f \in \tilde{L}^p_T(B^{s-\frac{2}{\alpha}}_{p,r})$. Then the problem of heat equation

$$\partial_t u - \mu \Delta u = f, \quad u|_{t=0} = u_0$$

has a unique solution $u \in \tilde{L}^p_T(B^{s+\frac{2}{\alpha}}_{p,r}) \cap \tilde{L}^\infty_T(B^{s}_{p,r})$ and there exists a constant $C$ depending only on $N$ and such that for all $1 \leq \alpha_1 \leq \alpha_2$, we have

$$\mu^{\frac{1}{\alpha_1}}\|u\|_{\tilde{L}^p_T(B^{s+\frac{2}{\alpha}}_{p,r})} \leq C \left\{ \left(1 + T^{\frac{1}{\alpha_1}}\right)\|u_0\|_{B^{s}_{p,r}} + (1 + T^{1+\frac{1}{\alpha_1} - \frac{1}{\alpha}})\mu^{\frac{1}{\alpha_1}-1}\|f\|_{\tilde{L}^p_T(B^{s-\frac{2}{\alpha}}_{p,r})} \right\}.$$  

In addition, if $r$ is finite then $u$ belongs to $C([0, T]; B^{s}_{p,r})$.

### 3 Well-posedness for $\kappa \neq 0$

In this section, using the frequency-localization methods, we give the proof of main result.

**The proof of Theorem 1.1** The coefficients $\tau_p, T_w, \lambda$ are assumed to be one. For classical solutions, (1.1) can be changed into the following system in $(n, u, T, \Phi)$:

$$\begin{aligned}
\partial_t n + \text{div}(nu) &= 0, \\
\nu \partial_t u + n(u \cdot \nabla)u + \nabla(nT) = n
\nabla \Phi - nu, \\
n\partial_t T + nu \cdot \nabla T + (\gamma - 1)n T \text{div} u = (\gamma - 1)\kappa \Delta T + \frac{\gamma - 1}{2} n|u|^2 - n(T - T_L), \\
\Delta \Phi = n - \bar{n},
\end{aligned}$$

(3.1)
In order to obtain the effective frequency-localization estimate on \((n, u)\), we introduce a function change

\[
\begin{pmatrix}
\rho \\
u \\
\theta \\
E
\end{pmatrix} = \begin{pmatrix}
\ln n - \ln \bar{n} \\
u \\
T - T_L \\
\nabla \Phi
\end{pmatrix}.
\]

Then the new variable \((\rho, u, \theta, E)\) satisfies

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho + \text{div} u &= 0, \\
\partial_t u + T_L \nabla \rho + (u \cdot \nabla) u + \nabla \theta + \theta \nabla \rho &= E - u, \\
\partial_t \theta - \frac{(\gamma - 1) \kappa}{n} \Delta \theta + u \cdot \nabla \theta &= h_1(\rho) \Delta \theta - (\gamma - 1)(T_L + \theta) \text{div} u + \frac{\gamma - 1}{2} |u|^2 - \theta, \\
\partial_t E &= -\nabla \Delta^{-1} \text{div}(h_2(\rho) u + \bar{n} u),
\end{align*}
\]

(3.2)

where \(h_1(\rho), h_2(\rho)\) defined by

\[
h_1(\rho) = \frac{(\gamma - 1) \kappa}{n} \left(1 - \exp(-\rho)\right) \quad \text{and} \quad h_2(\rho) = \bar{n}(\exp(\rho) - 1)
\]

are two smooth functions on the interval \((-\infty, \infty)\). The non-local term \(\nabla \Delta^{-1} \nabla \cdot f\) is the product of Riesz transforms on \(f\). Here and below, we set \(\bar{\kappa} = \frac{(\gamma - 1) \kappa}{n}\) for simplicity.

The initial data \((1.2)\) become

\[
(\rho, u, \theta, E)(x, 0) = (\ln n_0 - \ln \bar{n}, u_0, \theta_0 - T_L, \nabla \Delta^{-1}(n_0 - \bar{n})), \quad x \in \mathbb{R}^N.
\]

(3.3)

**Remark 3.1.** The variable transform is from the open set \(\{(n, u, T, E) \in (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N\}\) to the whole space \(\{ (\rho, u, \theta, E) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N\}\). It is easy to show that for classical solutions \((n, u, T, E)\) away from vacuum, \((1.1) - (1.2)\) is equivalent to \((3.2) - (3.3)\).

The proof of the local well-posedness stems from a standard iterative process. First of all, we consider the linear coupled system of hyperbolic-parabolic form

\[
\begin{align*}
\partial_t \eta + v \cdot \nabla \eta + \text{div} \eta &= 0, \\
\partial_t \xi + T_L \nabla \xi + (v \cdot \nabla) \xi &= f, \\
\partial_t \gamma - \kappa \Delta \gamma &= g, \\
\partial_t E &= -\nabla \Delta^{-1} \text{div} h,
\end{align*}
\]

(3.4)

subject to the initial data

\[
(\eta, \xi, \gamma, E)|_{t=0} = (\eta_0, \xi_0, \gamma_0, E_0),
\]

(3.5)

where \(v, f, h : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N\) and \(g : \mathbb{R}^+ \times \mathbb{R}^N \rightarrow \mathbb{R}\).

For the system \((3.4) - (3.5)\), we have the following conclusion.

**Proposition 3.1.** Let \(p \in [1, +\infty]\), \(r \in [1, +\infty]\), \(s_1 > 0\) and \(s_2 \in \mathbb{R}\). Suppose that \((\eta_0, \xi_0, \gamma_0, E_0) \in B^{s_1}_{2, r}, \eta_0 \in B^{s_2}_{p, r}, \eta_0 \in C([0, T], B^{s_1}_{2, r}), \xi \in L^1(0, T; B^{s_2}_{p, r})\) and

\[
\nabla v \in \left\{ \begin{array}{ll}
C([0, T], B^{s_1 - 1}_{2, r}) & \text{if } s_1 > 1 + N/2, \text{or } s_1 = 1 + N/2 \text{ and } r = 1; \\
C([0, T], B^{s_1 + \varepsilon}_{2, r}) & \text{for some } \varepsilon > 0 \text{ if } s_1 = 1 + N/2 \text{ and } r > 1; \\
C([0, T], B^{s_1}_{2, 0} \cap L^\infty) & \text{if } 0 < s_1 < 1 + N/2;
\end{array} \right.
\]

9
for any given \( T > 0 \). Then the system \((3.4)-(3.5)\) has a unique solution \((\bar{\rho}, \bar{u}, \bar{\theta}, \bar{E})\) satisfying

\[
(\bar{\rho}, \bar{u}, \bar{E}) \in \tilde{C}_T(B_{2,r}^{\infty}) \quad \text{and} \quad \bar{\theta} \in \tilde{C}_T(B_{p,r}^{\infty}).
\]

Proof. Note that the \( L^2 \)-boundedness of Riesz transform, Proposition 3.1 is the direct consequence of Proposition 2.6 and Theorem 4.15 in the recent book [5]. \( \Box \)

In what follows, the proof of Theorem 1.1 is divided into several steps, since it is a bit longer.

**Step 1: approximate solutions**

We use a standard iterative process to build a solution. Starting from \((\rho^0, u^0, \theta^0, E^0) := (0, 0, 0, 0)\). Then we define by induction a solution sequence \(\{(\rho^m, u^m, \theta^m, E^m)\}_{m \in \mathbb{N}}\) by solving the following linear equations

\[
\begin{aligned}
\partial_t \rho^{m+1} + u^m \cdot \nabla \rho^{m+1} + \text{div} u^{m+1} &= 0, \\
\partial_t u^{m+1} + T_L \Delta \theta^{m+1} + (u^m \cdot \nabla) u^{m+1} &= -\nabla \theta^m - \theta^m \nabla \rho^m + E^m - u^m, \\
\partial_t \theta^{m+1} - \delta \Delta \theta^{m+1} &= -u^m \cdot \nabla \theta^m + h(\rho^m) \Delta \theta^m - (\gamma - 1)(T_L + \theta^m) \text{div} u^m + \frac{\gamma - 1}{2} |u^m|^2 - \theta^m, \\
\partial_t E^{m+1} &= -\nabla \Delta^{-1} \text{div} \{ h(\rho^m) u^m + \bar{u} u^m \},
\end{aligned}
\]

with the initial data

\[
(\rho^{m+1}, u^{m+1}, \theta^{m+1}, E^{m+1})(x, 0) = (S_{m+1} \rho_0, S_{m+1} u_0, S_{m+1} \theta_0, S_{m+1} E_0), \quad x \in \mathbb{R}^N.
\]

Since all the data belong to \( B_{2,r}^\infty \), Proposition 3.1 enable us to show by induction that the above Cauchy problem has a global solution which belongs to \( \tilde{C}(B_{2,r}^\infty) \).

**Step 2: uniform bounds**

Set

\[
E_T^\sigma := \tilde{C}_T(B_{2,1}^\sigma) \times \left( \tilde{C}_T(B_{2,1}^{\sigma + 1}) \times \left( \tilde{C}_T(B_{2,1}^{\sigma}) \right) \right)^N
\]

for \( T > 0 \). We hope to find a time \( T \) such that the approximate solution \(\{(\rho^m, u^m, \theta^m, E^m)\}_{m \in \mathbb{N}}\) is uniformly bounded in \( E_T^\sigma \).

First, by applying the operator \( \Delta_q(q \geq -1) \) to the first two equations of (3.6), we infer that for \((\Delta_q \rho^{m+1}, \Delta_q u^{m+1})\)

\[
\begin{aligned}
\partial_t \Delta_q \rho^{m+1} + (u^m \cdot \nabla) \Delta_q \rho^{m+1} + \Delta_q \text{div} u^{m+1} &= [u^m, \Delta_q] \cdot \nabla \rho^{m+1}, \\
\partial_t \Delta_q u^{m+1} + T_L \Delta_q \nabla \rho^{m+1} + (u^m \cdot \nabla) \Delta_q u^{m+1} &= -\Delta_q \nabla \theta^m + [u^m, \Delta_q] \cdot \nabla u^{m+1} - \rho^m \Delta_q \theta^m + \nabla |\rho^m, \Delta_q| \theta^m + \Delta_q E^m - \Delta_q u^m,
\end{aligned}
\]

where the commutator \([\cdot, \cdot]\) is defined by \([f, g] = fg - gf\).

Then multiplying the first equation of Eqs. (3.6) by \(T_L \Delta_q \rho^{m+1}\), the second one by \(\Delta_q u^{m+1}\), and adding the resulting equations together, after integrating it over \(\mathbb{R}^N\), we have

\[
\frac{1}{2} \frac{d}{dt} \left( T_L \|\Delta_q \rho^{m+1}\|_{L^2}^2 + \|\Delta_q u^{m+1}\|_{L^2}^2 \right)
\]
\[
\begin{align*}
= \frac{1}{2} \int \text{div} u^m (T_L |\Delta_\rho \rho^{m+1}|^2 + |\Delta_\rho u^{m+1}|^2) + \int T_L [u^m, \Delta_\rho] \cdot \nabla \rho^{m+1} \Delta_\rho \rho^{m+1} & \\- \int \Delta_\rho \nabla \rho^m \cdot \Delta_\rho u^{m+1} + \int [u^m, \Delta_\rho] \cdot \nabla u^{m+1} \Delta_\rho u^{m+1} - \int \nabla \rho^m \cdot \Delta_\rho \rho^{m+1} \Delta_\rho \theta^m & \\+ \int [\nabla \rho^m, \Delta_\rho] \theta^m \cdot \Delta_\rho u^{m+1} + \int \Delta_\rho F^m \cdot \Delta_\rho u^{m+1} - \int \Delta_\rho u^m \cdot \Delta_\rho u^{m+1} & \\
\leq \frac{1}{2} \|\nabla u^m\|_{L^\infty} (\|\Delta_\rho \rho^{m+1}\|_{L^2}^2 + \|\Delta_\rho u^{m+1}\|_{L^2}^2 + T_L \|\nabla \rho^m, \Delta_\rho \| L^2 \|\Delta_\rho \rho^{m+1}\|_{L^2} & \\+ \|\Delta_\rho \nabla \rho^m\|_{L^2} + \|\nabla u^m, \Delta_\rho\|_{L^2} + \|\|\Delta_\rho \rho^m\|_{L^\infty} \|\Delta_\rho \theta^m\|_{L^2} & \\+ \|\Delta_\rho \theta^m\|_{L^2} + \|\|\Delta_\rho \rho^m\|_{L^\infty} \|\Delta_\rho \theta^m\|_{L^2} & \\+ \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} + \|\|\nabla u^m, \Delta_\rho\|_{L^2} & \\+ \|\|\Delta_\rho \theta^m\|_{L^2} & \end{align*}
\]

where we have used Cauchy-Schwarz’s inequality.

Dividing (3.9) by \( (T_L \|\Delta_\rho \rho^{m+1}\|_{L^2}^2 + \|\Delta_\rho u^{m+1}\|_{L^2}^2 + \varepsilon) \) \((\varepsilon > 0 \text{ is a small quantity})\), we get

\[
\frac{d}{dt} (T_L \|\Delta_\rho \rho^{m+1}\|_{L^2}^2 + \|\Delta_\rho u^{m+1}\|_{L^2}^2 + \varepsilon) \frac{1}{2} \leq C \|\nabla u^m\|_{L^\infty} \|\Delta_\rho \rho^{m+1}\|_{L^2} + \|\Delta_\rho u^{m+1}\|_{L^2} + \|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \rho^{m+1}\|_{L^2} + \|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \rho^{m+1}\|_{L^2} + \|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \theta^m\|_{L^2} + \|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \theta^m\|_{L^2} + \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} + \|\|\nabla u^m, \Delta_\rho\|_{L^2} + \|\|\Delta_\rho \theta^m\|_{L^2} & (3.10)
\]

where \( C > 0 \) here and below denotes a uniform constant independent of \( m \). Integrating (3.10) with respect to the variable \( t \in [0, T] \), then taking \( \varepsilon \to 0 \), we arrive at

\[
\begin{align*}
\|\Delta_\rho \rho^{m+1}(t)\|_{L^2} + \|\Delta_\rho u^{m+1}(t)\|_{L^2} & \leq C \|\Delta_\rho \rho^{m+1}\|_{L^2} + \|\Delta_\rho u^{m+1}\|_{L^2} + C \int_0^t \|\nabla u^m(\tau)\|_{L^\infty} \|\Delta_\rho \rho^{m+1}(\tau)\|_{L^2} + \|\Delta_\rho u^{m+1}(\tau)\|_{L^2} d\tau & \\+ C \int_0^t \|\Delta_\rho \nabla \rho^m\|_{L^2} + \|\|\nabla \rho^m\|_{L\infty} \|\Delta_\rho \theta^m\|_{L^2} + \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \theta^m\|_{L^2} & \\+ \|\|\Delta_\rho \rho^m, \Delta_\rho\|_{L^2} + \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \theta^m\|_{L^2} & \\+ \|\|\Delta_\rho \theta^m\|_{L^2} + \|\|\Delta_\rho \rho^m, \Delta_\rho\|_{L^2} \|\Delta_\rho \theta^m\|_{L^2} & \\+ \|\|\Delta_\rho \theta^m\|_{L^2} & \end{align*}
\]

Multiply the factor \( 2^\sigma (\sigma = 1 + N/2) \) on both sides of (3.11) to obtain

\[
\begin{align*}
2^\sigma \|\Delta_\rho \rho^{m+1}(t)\|_{L^2} + 2^\sigma \|\Delta_\rho u^{m+1}(t)\|_{L^2} & \leq C 2^\sigma \|\Delta_\rho \rho^{m+1}\|_{L^2} + \|\Delta_\rho u^{m+1}\|_{L^2} + C \int_0^t \|u^m(\tau)\|_{B_{2,1}^\sigma} 2^\sigma \|\Delta_\rho \rho^{m+1}(\tau)\|_{L^2} & \\+ \|\Delta_\rho u^{m+1}(\tau)\|_{L^2} d\tau + C \int_0^t c_0(\tau) \|u^m(\tau)\|_{B_{2,1}^{\sigma+1}} \|\rho^{m+1}(\tau)\|_{B_{2,1}^{\sigma+1}} + \|u^{m+1}(\tau)\|_{B_{2,1}^{\sigma+1}} d\tau & \\+ C \int_0^t c_0(\tau) \|\rho^m\|_{B_{2,1}^{\sigma+1}} \|\theta^m\|_{B_{2,1}^{\sigma+1}} d\tau + C \int_0^t 2^\sigma \|\Delta_\rho \nabla \rho^m\|_{L^2} & \\+ \|\Delta_\rho \theta^m\|_{L^2} + \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} + \|\|\nabla \rho^m, \Delta_\rho\|_{L^2} & \end{align*}
\]

(3.12)
where we used Remark 2.1 and Lemmas 2.2, 2.3 and \( \{c_q\} \) denotes some sequence which satisfies \( \|c_q\|_{t_1} \leq 1 \) although each \( \{c_q\} \) is possibly different in (3.12).

Summing up (3.12) on \( q \geq -1 \) implies
\[
\| (\rho^{m+1}, u^{m+1}) \|_{\bar{L}_T^\infty(B_{2,1}^s)} \\
\leq C(\| (\rho_0^{m+1}, u_0^{m+1}) \|_{B_{2,1}^s} + C \int_0^T \| u^m(t) \|_{B_{2,1}^s}(\rho^{m+1}, u^{m+1})(t)\|_{\bar{L}_T^\infty(B_{2,1}^s)} dt \\
+ C \int_0^T (1 + \| \rho^m \|_{B_{2,1}^s}) \| \theta^m \|_{B_{2,1}^{s+1}} + \| (u^m, E^m) \|_{B_{2,1}^s}) dt.
\]
(3.13)

Then it follows from Gronwall's inequality that
\[
\| (\rho^{m+1}, u^{m+1}) \|_{\bar{L}_T^\infty(B_{2,1}^s)} \\
\leq Ce^{CZ^m(T)} \left\{ \| (\rho_0, u_0) \|_{B_{2,1}^s} \\
+ \int_0^T e^{-CZ^m(t)} \left( (1 + \| \rho^m(t) \|_{B_{2,1}^s}) \| \theta^m(t) \|_{B_{2,1}^{s+1}} + \| (u^m, E^m)(t) \|_{B_{2,1}^s}) dt \right\},
\]
(3.14)
with \( Z^m(T) := \int_0^T \| u^m(t) \|_{B_{2,1}^s} dt. \)

On the other hand, by the last equation of (3.6), with the aid of Proposition 2.5 we can obtain
\[
\| E^{m+1} \|_{\bar{L}_T^\infty(B_{2,1}^s)} \\
\leq C \left( \| E_0^{m+1} \|_{B_{2,1}^s} + \int_0^T \| \nabla \Delta^{-1} \text{div}(h_2(\rho^m)u^m + \bar{n}u^m) \|_{B_{2,1}^s} dt \right) \\
\leq C \left( \| E_0 \|_{B_{2,1}^s} + \int_0^T (1 + \| \rho^m \|_{B_{2,1}^s}) \| u^m \|_{B_{2,1}^s} dt \right),
\]
(3.15)
where we have used the \( L^2 \)-boundedness of nonlocal (but zero order) operator \( \nabla \Delta^{-1} \text{div} \).

Taking \( \alpha_1 = \alpha = \infty, \ s = \sigma + 1, \ p = 2 \) and \( \tau = 1 \) in Proposition 2.6 and applying the resulting inequality to the third equation of (3.6), we have
\[
\| \theta^{m+1} \|_{\bar{L}_T^\infty(B_{2,1}^{s+1})} \leq C \left( \| \theta_0^{m+1} \|_{B_{2,1}^{s+1}} + (1 + T)^{\kappa-1} \| F_1^m \|_{\bar{L}_T^\infty(B_{2,1}^{s-1})} \right),
\]
(3.16)
where \( F_1^m := -u^m \cdot \nabla \theta^m + h_1(\rho^m)\Delta \theta^m - (\gamma - 1)(\mathcal{T}_L + \theta^m) \text{div} u^m + \frac{\gamma - 1}{2} |u^m|^2 - \theta^m. \) From Propositions 2.4, 2.5 and Lemma 2.2 we are led to
\[
\| \theta^{m+1} \|_{\bar{L}_T^\infty(B_{2,1}^{s+1})} \\
\leq C \left\{ \| \theta_0 \|_{B_{2,1}^{s+1}} + (1 + T)^{\kappa-1} \left( (1 + \| \theta^m \|_{\bar{L}_T^\infty(B_{2,1}^s)}) \| \theta^m \|_{\bar{L}_T^\infty(B_{2,1}^{s+1})} + \right. \right. \\
\left. \left. + (1 + \| u^m \|_{\bar{L}_T^\infty(B_{2,1}^s)}) \| u^m \|_{\bar{L}_T^\infty(B_{2,1}^s)} \right\}.
\]
(3.17)
Note that although the above constant \( C \) may depend on \( N \), it is nothing to do with \( m \), so we obtain the following uniform estimates.
**Lemma 3.1.** There exists a time $T_1 > 0$ (independent of $m$) such that
\[
\| (\rho^m, u^m, E^m) \|_{L_T^\infty(B_2^\sigma)} + \| \theta^m \|_{L_T^\infty(B_2^{\sigma+1})} \leq C_1 A,
\]  
for all $m \in \mathbb{N} \cup \{0\}$, provided that $\tilde{\kappa} > 0$ is sufficiently large, where the constant $C_1 > 0$ independent of $m$ and $A := \| (\rho_0, u_0, E_0) \|_{B_2^\sigma} + \| \theta_0 \|_{B_2^{\sigma+1}}$.

**Proof.** Indeed, the claim follows from the standard induction. First, we see that (3.18) holds for $m = 0$. Suppose that (3.18) holds for any $m > 0$, we expect to prove it is also true for $m + 1$.

Together with the assumption, by (3.14)-(3.15) and (3.17), we get
\[
\| (\rho^{m+1}, u^{m+1}) \|_{L_T^\infty(B_2^\sigma)} \leq C \left[ A + T \max \{ 2C_1 A, 2(C_1 A)^2 \} \right] e^{CC_1 A T},
\]  
(3.19)

\[
\| E^{m+1} \|_{L_T^\infty(B_2^\sigma)} \leq C \left[ A + T \max \{ 2C_1 A, 2(C_1 A)^2 \} \right],
\]  
(3.20)

\[
\| \theta^{m+1} \|_{L_T^\infty(B_2^{\sigma+1})} \leq C \left[ A + (1 + T)\tilde{\kappa}^{-1} \max \{ 4C_1 A, 4(C_1 A)^2 \} \right]
\leq C \left[ A + T \max \{ 4C_1 A, 4(C_1 A)^2 \} \right],
\]  
(3.21)

where we suffice to take $\tilde{\kappa}$ satisfying $\tilde{\kappa} \geq \frac{(1+T)}{\kappa}$ (T to be determined) in the last step of the inequality (3.21). Combining with (3.19)-(3.21), we have
\[
\| (\rho^{m+1}, u^{m+1}, E^{m+1}) \|_{L_T^\infty(B_2^\sigma)} + \| \theta^{m+1} \|_{L_T^\infty(B_2^{\sigma+1})}
\leq 3C \left[ A + T \max \{ 4C_1 A, 4(C_1 A)^2 \} \right] e^{CC_1 A T}.
\]  
(3.22)

Furthermore, if we choose $T_1$ satisfying
\[
0 < T_1 \leq \min \left\{ \frac{\ln(C_1 - 6C)}{CC_1 A}, T_0 \right\} \big( C_1 > 1 + 6C \big),
\]
where $T_0$ is the root of algebra equation
\[
e^{CC_1 A T} = \frac{1}{6C \max \{ 4, 4C_1 A \} T_0},
\]
then $\| (\rho^{m+1}, u^{m+1}, E^{m+1}) \|_{L_T^\infty(B_2^\sigma)} + \| \theta^{m+1} \|_{L_T^\infty(B_2^{\sigma+1})} \leq C_1 A$ is followed, which concludes the proof of the assertion. \(\square\)

That is, we find a time $T_1 > 0$ (independent of $m$) such that the sequence $\{ (\rho^m, u^m, \theta^m, E^m) \}_{m \in \mathbb{N}}$ is uniformly bounded in $E_T^{\sigma}$.

**Step 3: convergence**
Next, it will be shown that $\{ (\rho^m, u^m, \theta^m, E^m) \}_{m \in \mathbb{N}}$ is a Cauchy sequence in $E_T^{\sigma}$.  

13
Define
\[ \delta \rho^{m+1} = \rho^{m+p+1} - \rho^{m+1}, \quad \delta u^{m+1} = u^{m+p+1} - u^{m+1}, \]
\[ \delta \theta^{m+1} = \theta^{m+p+1} - \theta^{m+1}, \quad \delta E^{m+1} = E^{m+p+1} - E^{m+1}, \]
for any \((m, p) \in \mathbb{N}^2\).

Take the difference between the equation \(3.6\) for the \((m+p+1)\)-th step and the \((m+1)\)-th step to give
\[
\begin{align*}
\partial_t \delta \rho^{m+1} + u^{m+p} \cdot \nabla \delta \rho^{m+1} + \delta u^{m} \cdot \rho^{m+1} + \text{div} \delta u^{m+1} &= 0, \\
\partial_t \delta u^{m+1} + T_L \nabla \delta \rho^{m+1} + (u^{m+p} \cdot \nabla) \delta u^{m+1} + (\delta u^{m} \cdot \nabla) u^{m+1} &= -\nabla \delta \theta^m - \theta^m \delta \rho^m - \nabla \rho^{m+p} \theta^m + \delta E^m - \delta u^m, \\
\partial_t \delta \theta^{m+1} - \tilde{h} \Delta \delta \rho^{m+1} &= -\delta u^{m} \cdot \nabla \rho^{m+p} - u^{m} \nabla \delta \theta^m + [h_1(\rho^{m+p}) - h_1(\rho^m)] \Delta \delta \rho^{m+1} \\
&\quad + h_2(\rho^m) \Delta \delta \theta^m - (\gamma - 1)(T_L + \theta^m) \text{div} \delta u^{m} \\
&\quad - \theta^m \text{div} u^{m+p} + \tilde{h}_2(u^{m+p} + u^m) \delta u^m - \delta \theta^m, \\
\partial_t \delta E^{m+1} &= -\nabla \Delta^{-1} \text{div} \{[h_2(\rho^{m+p}) - h_2(\rho^m)] u^{m+p} + h_2(\rho^m) \delta u^m + \tilde{h} \delta u^m\},
\end{align*}
\]
subject to the initial data
\[ (\delta \rho^{m+1}, \delta u^{m+1}, \delta \theta^{m+1}, \delta E^{m+1})(x, 0) = [S_{m+p+1} - S_{m+1}](\rho_0, u_0, \theta_0, E_0), \quad x \in \mathbb{R}^N. \] (3.24)

Applying the operator \(\Delta_q (q \geq -1)\) to the first two equations of \((3.23)\) gives
\[
\begin{align*}
\partial_t \Delta_q \delta \rho^{m+1} + (u^{m+p} \cdot \nabla) \Delta_q \delta \rho^{m+1} + \Delta_q \text{div} \delta u^{m+1} &= [u^{m+p}, \Delta_q] \cdot \nabla \rho^{m+1} - \Delta_q (\delta u^m \cdot \nabla \rho^{m+1}), \\
\partial_t \Delta_q \delta u^{m+1} + T_L \Delta_q \nabla \delta \rho^{m+1} + (u^{m+p} \cdot \nabla) \Delta_q \delta u^{m+1} &= [u^{m+p}, \Delta_q] \cdot \nabla \delta u^{m+1} - \Delta_q (\delta u^m \cdot \nabla u^{m+1}) - \Delta_q \nabla \delta \theta^m, \\
\partial_t \Delta_q \delta \theta^{m+1} - \tilde{h} \Delta \delta \rho^{m+1} &= -\delta u^{m} \cdot \nabla \rho^{m+p} - u^{m} \nabla \delta \theta^m + [h_1(\rho^{m+p}) - h_1(\rho^m)] \Delta \delta \rho^{m+1} \\
&\quad + h_2(\rho^m) \Delta \delta \theta^m - (\gamma - 1)(T_L + \theta^m) \text{div} \delta u^{m} \\
&\quad - \theta^m \text{div} u^{m+p} + \tilde{h}_2(u^{m+p} + u^m) \delta u^m - \delta \theta^m, \\
\partial_t \Delta_q \delta E^{m+1} &= -\nabla \Delta^{-1} \text{div} \{[h_2(\rho^{m+p}) - h_2(\rho^m)] u^{m+p} + h_2(\rho^m) \delta u^m + \tilde{h} \delta u^m\},
\end{align*}
\]
By multiplying the first equation of Eqs. \((3.25)\) by \(T_L \Delta_q \delta \rho^{m+1}\), the second one by \(\Delta_q \delta u^{m+1}\), and adding the resulting equations together, after integrating it over \(\mathbb{R}^N\), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} &\left( T_L \| \Delta_q \delta \rho^{m+1} \|_{L^2}^2 + \| \Delta_q \delta u^{m+1} \|_{L^2}^2 \right) \\
&\leq \| \nabla u^{m+p} \|_{L^\infty} \left( T_L \| \Delta_q \delta \rho^{m+1} \|_{L^2}^2 + \| \Delta_q \delta u^{m+1} \|_{L^2}^2 \right) \\
&\quad + \left( T_L \| [u^{m+p}, \Delta_q] \cdot \nabla \rho^{m+1} \|_{L^2} + \| \Delta_q (\delta u^m \cdot \nabla \rho^{m+1}) \|_{L^2} \right) \| \Delta_q \delta \rho^{m+1} \|_{L^2} \\
&\quad + \left( \| [u^{m+p}, \Delta_q] \cdot \nabla \delta u^{m+1} \|_{L^2} + \| \Delta_q (\delta u^m \cdot \nabla u^{m+1}) \|_{L^2} + \| \Delta_q \nabla \delta \theta^m \|_{L^2} \right) \\
&\quad + \| \Delta_q (\delta \theta^m \nabla \rho^{m+p}) \|_{L^2} + \| \delta \rho^m \|_{L^\infty} \left( \| \Delta_q \nabla \theta^m \|_{L^2} + 2^q \| \Delta_q \theta^m \|_{L^2} \right) \\
&\quad + \| \nabla \delta \rho^m, \Delta_q \theta^m \|_{L^2} + \| \Delta_q \delta E^m \|_{L^2} + \| \Delta_q \delta u^m \|_{L^2} \right) \| \Delta_q \delta u^{m+1} \|_{L^2},
\end{align*}
\]
where we have bounded the integration
\[ - \int \Delta_q \theta^m \nabla \delta \rho^m \cdot \Delta_q \delta u^{m+1} \]
\[
\begin{align*}
&= \int \delta \rho^m \Delta q \nabla \theta^m \cdot \Delta q \delta u^{m+1} + \delta \rho^m \Delta q \theta^m \cdot \Delta q \text{div} \delta u^{m+1} \\
&\leq \|\delta \rho^m\|_{L^\infty} \left( \|\Delta q \nabla \theta^m\|_{L^2} + 2^q \|\Delta q \theta^m\|_{L^2} \right) \|\Delta q \delta u^{m+1}\|_{L^2}.
\end{align*}
\]

Similar to the estimate of (3.10), we can obtain
\[
\begin{align*}
\frac{d}{dt} \left( T_L \|\Delta q \delta \rho^{m+1}\|_{L^2}^2 + \|\Delta q \delta u^{m+1}\|_{L^2}^2 + \varepsilon \right) &\leq \left\| \nabla u^{m+p} \right\|_{L^\infty} \left( T_L \|\Delta q \delta \rho^{m+1}\|_{L^2} + \|\Delta q \delta u^{m+1}\|_{L^2} \right) + \left( T_L \|[u^{m+p}, \Delta q] \cdot \nabla \delta \rho^{m+1}\|_{L^2} \right. \\
&\quad + \|\Delta q (\delta u^m \cdot \nabla \rho^{m+1})\|_{L^2} + \left\{ \|[u^{m+p}, \Delta q] \cdot \nabla \delta u^{m+1}\|_{L^2} + \|\Delta q (\delta u^m \cdot \nabla u^{m+1})\|_{L^2} \right. \\
&\quad + \|\Delta q \nabla \theta^m\|_{L^2} + \|\Delta q (\delta \theta^m \nabla \rho^{m+1})\|_{L^2} + \|\delta \rho^m\|_{L^\infty} \left( \|\Delta q \nabla \theta^m\|_{L^2} + 2^q \|\Delta q \theta^m\|_{L^2} \right) \\
&\quad \left. + \left\| \nabla \delta \rho^m, \Delta q \right\|_{L^2} \right\} + \|\Delta q \delta \text{E}^m\|_{L^2} + \|\Delta q \delta u^m\|_{L^2} \right) \right.,
\end{align*}
\]

where \( \varepsilon > 0 \) is a small quantity.

Integrating (3.27) with respect to the variable \( t \in [0, T_1] \), then taking \( \varepsilon \to 0 \), we arrive at
\[
\begin{align*}
&\|\Delta q \delta \rho^{m+1}(t)\|_{L^2} + \|\Delta q \delta u^{m+1}(t)\|_{L^2} \\
&\leq C \left( \|\Delta q \delta \rho^0\|_{L^2} + \|\Delta q \delta u^0\|_{L^2} \right) + C \int_0^t \left\| \nabla u^{m+p} \right\|_{L^\infty} \left( \|\Delta q \delta \rho^{m+1}\|_{L^2} \right. \\
&\quad + \|\Delta q \delta u^{m+1}\|_{L^2} \right) \left. + C \int_0^t \left( \|[u^{m+p}, \Delta q] \cdot \nabla \delta \rho^{m+1}\|_{L^2} + \|\Delta q (\delta u^m \cdot \nabla \rho^{m+1})\|_{L^2} \right. \\
&\quad + \|\Delta q (\delta \theta^m \nabla \rho^{m+1})\|_{L^2} + \|\delta \rho^m\|_{L^\infty} \left( \|\Delta q \nabla \theta^m\|_{L^2} + 2^q \|\Delta q \theta^m\|_{L^2} \right) \\
&\quad \left. + \left\| \nabla \delta \rho^m, \Delta q \right\|_{L^2} \right\} + \|\Delta q \delta \text{E}^m\|_{L^2} + \|\Delta q \delta u^m\|_{L^2} \right). \tag{3.28}
\end{align*}
\]

By multiplying the factor \( 2^{q(\sigma-1)} \) on both sides of the resulting inequality (3.28), we obtain
\[
\begin{align*}
&2^{q(\sigma-1)} \left( \|\Delta q \delta \rho^{m+1}(t)\|_{L^2} + \|\Delta q \delta u^{m+1}(t)\|_{L^2} \right) \\
&\leq C 2^{q(\sigma-1)} \left( \|\Delta q \delta \rho^0\|_{L^2} + \|\Delta q \delta u^0\|_{L^2} \right) \\
&\quad + C \int_0^t \left\| \nabla u^{m+p} \right\|_{L^\infty} 2^{q(\sigma-1)} \left( \|\Delta q \delta \rho^{m+1}\|_{L^2} + \|\Delta q \delta u^{m+1}\|_{L^2} \right) \\
&\quad + C \int_0^t \left( c_q \|[u^{m+p}, B_{2,1}^q] \|_{B_{2,1}^q} \|\delta \rho^{m+1}\|_{B_{2,1}^q} + c_q \|\delta u^m\|_{B_{2,1}^q} \right. \\
&\quad + c_q \|\delta \theta^m\|_{B_{2,1}^q} + c_q \|\delta \theta^m\|_{B_{2,1}^q} \|\rho^{m+1}\|_{B_{2,1}^q} + c_q \|\delta \theta^m\|_{B_{2,1}^q} \|\theta^m\|_{B_{2,1}^q} \right) \\
&\quad + c_q \|\nabla \theta^m\|_{B_{2,1}^q} + c_q \|\delta \theta^m\|_{B_{2,1}^q} \|\rho^{m+1}\|_{B_{2,1}^q} + c_q \|\delta \theta^m\|_{B_{2,1}^q} \|\theta^m\|_{B_{2,1}^q} \right) \left. + 2^{q(\sigma-1)} \|\Delta q \delta \text{E}^m\|_{L^2} + 2^{q(\sigma-1)} \|\Delta q \delta u^m\|_{L^2} \right), \tag{3.29}
\end{align*}
\]

where \( \{c_q\} \) denotes some sequence which satisfies \( \|\{c_q\}\|_{l^1} \leq 1 \).
Summing up \((3.29)\) on \(q \geq -1\), it is not difficult to get
\[
\| (\delta \rho^{m+1}, \delta u_{m+1}) \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} \\
\leq C \| (\rho_0, u_0) \|_{B_{2,1}^\sigma} \\
+ C \int_0^{T_1} \| u^{m+p} \|_{B_{2,1}^\sigma} \| (\delta \rho^{m+1}, \delta u^{m+1}) \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} dt \\
+ C \int_0^{T_1} \| \delta u^m \|_{B_{2,1}^\sigma} (1 + \| \rho^{m+1} \|_{B_{2,1}^\sigma} + \| u^{m+1} \|_{B_{2,1}^\sigma}) dt \\
+ C \int_0^{T_1} \| \delta \theta^m \|_{B_{2,1}^\sigma} (1 + \| \rho^{m+p} \|_{B_{2,1}^\sigma}) dt \\
+ C \int_0^{T_1} \left( \| (\delta \rho^m, \delta u^m, \delta E^m) \|_{B_{2,1}^\sigma} + \| \delta \theta^m \|_{B_{2,1}^\sigma} \right) \\
\times \left( 1 + \| (\rho^{m+1}, \rho^{m+p}, u^{m+1}, \theta^m) \|_{B_{2,1}^\sigma} \right) dt,
\]
\[(3.30)\]

where we have used Lemma \([2.1]\) and Remark \([2.1]\).

With the aid of Gronwall’s inequality, we immediately deduce that
\[
\| (\delta \rho^{m+1}, \delta u^{m+1}) \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} \\
\leq C e^{C Z^{m+p}(T_1)} \left\{ 2^{-m} \| (\rho_0, u_0) \|_{B_{2,1}^\sigma} \\
+ \int_0^{T_1} e^{-C Z^{m+p}(t)} \left( \| (\delta \rho^m, \delta u^m, \delta E^m) \|_{B_{2,1}^\sigma} + \| \delta \theta^m \|_{B_{2,1}^\sigma} \right) \\
\times \left( 1 + \| (\rho^{m+1}, \rho^{m+p}, u^{m+1}, \theta^m) \|_{B_{2,1}^\sigma} \right) dt \right\} \\
\leq C e^{C T_1} \left\{ 2^{-m} + T_1 \left( \| (\delta \rho^m, \delta u^m, \delta E^m) \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} + \| \delta \theta^m \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} \right) \right\},
\]
\[(3.31)\]

where we have noticed Remark \([2.2]\) and the fact that the sequence \(\{(\rho^m, u^m, \theta^m, E^m)\}_{m \in \mathbb{N}}\) is uniformly bounded in \(E_{T_1}^\sigma\).

From the last equation of \((3.23)\), we get directly
\[
\| \delta E^{m+1} \|_{L_{T_1}^\infty (B_{2,1}^\sigma)} \\
\leq \| \delta E^0_{\sigma} \|_{B_{2,1}^\sigma} + \int_0^{T_1} \left( \| \delta \rho^m \|_{B_{2,1}^\sigma} \| u^{m+p} \|_{B_{2,1}^\sigma} + \| \delta u^m \|_{B_{2,1}^\sigma} (1 + \| \rho^m \|_{B_{2,1}^\sigma}) \right) dt \\
\leq C Z^{-m} + C T_1 \| (\delta \rho^m, \delta u^m) \|_{L_{T_1}^\infty (B_{2,1}^\sigma)}.
\]
\[(3.32)\]

Using Proposition \([2.6]\) (taking \(\alpha_1 = \alpha = \infty, s = \sigma, p = 2\) and \(r = 1\)), we have
\[
\| \delta \theta^{m+1} \|_{L_{T_1}^\infty (B_{2,1}^\sigma)}
\]
The second term in (3.34) can be further estimated as

\[ F_2^m := -\delta u^m \cdot \nabla \theta^{m+p} - u^m \nabla \delta \theta^m + [h_1(\rho^{m+p}) - h_1(\rho^m)] \Delta \theta^{m+p} \\
+ h_1(\rho^m) \Delta \delta \theta^m - (\gamma - 1)(T_L + \theta^m) \text{div} \delta u - \delta \theta^m \text{div} u^{m+p} \\
+ \frac{\gamma - 1}{2} (u^{m+p} + u^m) \delta u - \delta \theta^m. \]

In bounding \( F_2^m \), each product term can be estimated effectively with the help of the standard Moser-type inequality (Proposition 2.3), except for the term \( h_1(\rho^m) \Delta \delta \theta^m \). Here, we develop a Moser-type inequality of general form to estimate \( h_1(\rho^m) \Delta \delta \theta^m \), which will be shown in the Appendix, see Proposition 4.1. According to it, we can reach

\[ \|h_1(\rho^m) \Delta \delta \theta^m\|_{B_{2,1}^{\sigma - 2}} \leq C(\|h_1(\rho^m)\|_{L^\infty} \|\Delta \delta \theta^m\|_{B_{2,1}^{\sigma -2}} + \|\Delta \delta \theta^m\|_{L^2} \|h_1(\rho^m)\|_{B_{\infty,1}^{\sigma - 2}}). \tag{3.34} \]

The second term in (3.34) can be further estimated as

\[
\[\]
\[\]
\[
\|\Delta \delta \theta^m\|_{L^2} \|h_1(\rho^m)\|_{B_{\infty,1}^{\sigma - 2}} \\
\leq C \|\Delta \delta \theta^m\|_{B_{2,1}^{\sigma - 2}} \|h_1(\rho^m)\|_{B_{N,1}^{\sigma - 1}} (\sigma = 1 + N/2, \; N \geq 2) \\
\leq C \|\delta \theta^m\|_{B_{2,1}^{\sigma - 2}} \|h_1(\rho^m)\|_{B_{2,1}^{\sigma - 2}} \\
\leq C \|\delta \theta^m\|_{B_{2,1}^{\sigma - 2}} \|\rho^m\|_{B_{2,1}^{\sigma - 2}} \tag{3.35}
\]

where we used the embedding properties \( B_{2,1}^{\sigma - 2} \hookrightarrow L^2 \) and \( B_{N,1}^{\sigma - 1} \hookrightarrow B_{\infty,1}^{\sigma - 2} \). To ensure \( B_{2,1}^{\sigma - 2} \hookrightarrow B_{2,1}^{\sigma - 2} \), \( N - 1 < 1 + N/2 \) i.e. \( N \leq 4 \) is required in the last second step of (3.35).

Combining (3.33) and (3.35) and recalling on the choice of \( \tilde{\kappa}(\tilde{\kappa} > \frac{1 + T_1}{T_1}) \), we conclude that

\[ \|\delta \theta^{m+1}\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \leq C \left\{ 2^{-m} + T_1 \left( \|\delta \rho^m, \delta u^m\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} + \|\delta \theta^m\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \right) \right\} \tag{3.36} \]

Therefore, together with (3.31)-(3.32) and (3.36), we end up with

\[ \|\delta \rho^{m+1}, \delta u^{m+1}, \delta E^{m+1}\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} + \|\delta \theta^{m+1}\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \leq C_{T_1} \left\{ 2^{-m} + T_1 \left( \|\delta \rho^m, \delta u^m, \delta E^m\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} + \|\delta \theta^m\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \right) \right\} \tag{3.37} \]

where \( C_{T_1} := Ce^{CT_1} \). Arguing by induction, one can easily deduce that

\[ \|\delta \rho^{m+1}, \delta u^{m+1}, \delta E^{m+1}\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} + \|\delta \theta^{m+1}\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \leq \frac{(T_1C_{T_1})^{m+1}}{(m + 1)!} \left( \|\delta \rho^p, \delta u^p, \delta E^p\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} + \|\delta \theta^p\|_{L^\infty_{T_1}(B_{2,1}^{\sigma - 2})} \right) + C_{T_1} \sum_{k=0}^{m} 2^{-(m-k)} \frac{(T_1C_{T_1})^k}{k!}. \tag{3.38} \]
As \( \|\delta \rho^p, \delta u^p, \delta E^p\|_{L^\infty_T(B_{2,1}^{2,-1})} + \|\delta \theta^p\|_{L^\infty_T(B_{2,1}^{2,-1})} \) can be bounded independent of \( p \), we further take \( T_1 \) so small that

\[
\frac{(T_1 C_T)^{m+1}}{(m+1)!} \leq C 2^{-m} \quad \text{and} \quad \frac{C T_1 (T_1 C_T)^k}{k!} \leq 4^{-k}.
\]

Thus we conclude that there exists some constant \( C_2 > 0 \) (independent of \( m \)) such that

\[
\|(\delta \rho^{m+1}, \delta u^{m+1}, \delta E^{m+1})\|_{L^\infty_T(B_{2,1}^{2,-1})} + \|\delta \theta^{m+1}\|_{L^\infty_T(B_{2,1}^{2,-1})} \leq C 2^{-m},
\]

which implies \{\( (\rho^m, u^m, \theta^m, E^m) \}_{m \in \mathbb{N}} \) is a Cauchy sequence in \( E_{T_1}^{\sigma,-1} \). Therefore, there exists some function \( (\rho, u, \theta, E) \) in \( E_{T_1}^{\sigma,-1} \) such that

\[
\{(\rho^m, u^m, \theta^m, E^m)\} \to (\rho, u, \theta, E) \quad \text{strongly in} \quad E_{T_1}^{\sigma,-1}.
\]

**Step 4: the solution \((\rho, u, \theta, E)\)**

In this step we show that \((\rho, u, \theta, E) \in E_{T_1}^{\sigma,-1} \) is a solution of the system (3.2)-(3.3). Fatou’s property ensures that \((\rho, u, \theta, E) \in L^\infty_T(B_{2,1}^{2}) \times (L^\infty_T(B_{2,1}^{2,1}))^N \times L^\infty_T(B_{2,1}^{2,1}) \times (L^\infty_T(B_{2,1}^{2,1}))^N \), since \{\( (\rho^m, u^m, \theta^m, E^m) \)\}_{m \in \mathbb{N}} is also uniformly bounded in the spaces \( L^\infty_T(B_{2,1}^{2}) \times (L^\infty_T(B_{2,1}^{2,1}))^N \times L^\infty_T(B_{2,1}^{2,1}) \times (L^\infty_T(B_{2,1}^{2,1}))^N \).

On the other hand, \{\( (\rho^m, u^m, E^m) \)\}_{m \in \mathbb{N}} converges to \((\rho, u, E) \) in \( C([0,T_1]; B_{2,1}^{\sigma,-1}) \) and \( \{\theta^m\}_{m \in \mathbb{N}} \) converges to \( \theta \) in \( C([0,T_1]; B_{2,1}^{\sigma}) \). These properties of strong convergence enable us to pass to the limits in the systems (3.2)-(3.3) and conclude that \((\rho, u, \theta, E) \) to the system (3.2)-(3.3). Now, what remains is to check \((\rho, u, \theta, E) \) also belongs to \( C([0,T_1]; B_{2,1}^{\sigma,-1}) \times (C([0,T_1]; B_{2,1}^{\sigma}))^N \times C([0,T_1]; B_{2,1}^{\sigma}) \times (C([0,T_1]; B_{2,1}^{\sigma}))^N \). Indeed, for instance, we easily achieve that the map \( t \mapsto \|\Delta_q \rho(t)\|_{L^2} \) is continuous on \([0,T_1] \), since \( \rho \in C([0,T_1]; B_{2,1}^{\sigma,-1}) \). Then we have \( \Delta_q \rho(t) \in C([0,T_1]; B_{2,1}^{\sigma}) \) for all \( q \geq -1 \). Note that \( \rho \in L^\infty_T(B_{2,1}^{2}) \), the series \( \sum_{q \geq -1} 2^{q\sigma} \|\Delta_q \rho(t)\|_{L^2} \) converges uniformly on \([0,T_1] \), which yields \( \rho \in C([0,T_1]; B_{2,1}^{\sigma}) \). The same arguments are valid for the other variables \((u, \theta, E) \). Hence, we finish the existence part of solutions.

**Step 5: uniqueness**

Let \( \tilde{\rho} = \rho_1 - \rho_2 \), \( \tilde{u} = u_1 - u_2 \), \( \tilde{\theta} = \theta_1 - \theta_2 \), \( \tilde{E} = E_1 - E_2 \) where \((\rho_1, u_1, \theta_1, E_1)^T \) and \((\rho_2, u_2, \theta_2, E_2)^T \) are two solutions to the system (3.2)-(3.3) subject to the same initial data, respectively. Then the error solution \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{E})^T \) satisfies

\[
\begin{align*}
\partial_t \tilde{\rho} + \text{div} \tilde{u} &= -u_1 \cdot \nabla \tilde{\rho} - \tilde{u} \cdot \nabla \rho_2, \\
\partial_t \tilde{u} + T_L \nabla \tilde{\rho} &= -\nabla \tilde{\theta} - u_1 \cdot \nabla \tilde{u} - \tilde{u} \nabla u_2 - \theta_1 \nabla \tilde{\rho} - \tilde{\theta} \nabla \rho_2 + \tilde{E} - \tilde{u}, \\
\partial_t \tilde{\theta} - \tilde{n} \Delta \tilde{\theta} &= -u_1 \cdot \nabla \tilde{\theta} - \tilde{u} \nabla \theta_2 + [h_1(\rho_1) - h_1(\rho_2)] \Delta \theta_1 + h_1(\rho_2) \Delta \tilde{\theta} - (\gamma - 1)(T_L + \theta_1) \text{div} \tilde{u}_1 - (\gamma - 1) \tilde{\theta} \text{div} u_2 + \frac{\gamma - 1}{2} \tilde{u}(u_1 + u_2) - \tilde{\theta}, \\
\partial_t \tilde{E} &= -\nabla \Delta^{-1} \nabla \cdot [(h_2(\rho_1) - h_2(\rho_2)) u_1 + (h_2(\rho_2) + \tilde{n}) \tilde{u}].
\end{align*}
\]

As previously, following from the proof of Cauchy sequence, we obtain the inequalities:

\[
\|(\tilde{\rho}, \tilde{u})\|_{L^\infty_T(B_{2,1}^{2,-1})} \leq C \int_0^{T_1} \left( \|(\tilde{\rho}, \tilde{u}, \tilde{E})\|_{B_{2,1}^{2,-1}} \right) \left( 1 + \rho_1, \rho_2, u_1, u_2, \theta_1 \right)_{B_{2,1}^{2,-1}} \right) \right) \right) dt 
\]
and
\[ \| \tilde{E} \|_{L^\infty_t(B_{2,1}^{s-1})} \leq C \int_0^{T_1} \| (\tilde{\rho}, \tilde{u}) \|_{B_{2,1}^{\sigma(-1)}} \left( 1 + \| (\rho_2, u_1) \|_{B_{2,1}^s} \right) dt. \] (3.42)

According to Proposition 2.5 (taking \( \alpha_1 = \alpha = 1, s = \sigma - 2, p = 2 \) and \( r = 1 \)), we have
\[ \| \tilde{\theta} \|_{L^1_t(B_{2,1}^s)} \leq C T_1 \left\{ \left( 1 + \| (\rho_2, u_1, u_2) \|_{L^\infty_t(B_{2,1}^{s-1})} \right) \| \tilde{\theta} \|_{L^1_t(B_{2,1}^s)} \right. \]
\[ + \left. \int_0^{T_1} \| (\tilde{\rho}, \tilde{u}) \|_{B_{2,1}^{\sigma(-1)}} (1 + \| (u_1, u_2) \|_{B_{2,1}^s} + \| (\theta_1, \theta_2) \|_{B_{2,1}^s}) dt \} \right\}. \] (3.43)

We further choose \( T_1 \) satisfying
\[ T_1 \leq \frac{1}{2C(1 + \| (\rho_2, u_1, u_2) \|_{L^\infty_t(B_{2,1}^{s-1})})}, \]
and obtain
\[ \| \tilde{\theta} \|_{L^1_t(B_{2,1}^s)} \leq C \int_0^{T_1} \| (\tilde{\rho}, \tilde{u}) \|_{B_{2,1}^{\sigma(-1)}} (1 + \| (u_1, u_2) \|_{B_{2,1}^s} + \| (\theta_1, \theta_2) \|_{B_{2,1}^s}) dt. \] (3.44)

Note that \( L^1_t(B_{2,1}^s) \equiv L^1_t(B_{2,1}^s) \), we insert (3.44) into (3.44) to get after combining (3.42)
\[ \| (\tilde{\rho}, \tilde{u}, \tilde{E}) \|_{L^\infty_t(B_{2,1}^{s-1})} \]
\[ \leq C \int_0^{T_1} \| (\tilde{\rho}, \tilde{u}, \tilde{E}) \|_{L^\infty_t(B_{2,1}^{s-1})} \left( 1 + \| (\rho_1, \rho_2, u_1, u_2) \|_{B_{2,1}^s} + \| (\theta_1, \theta_2) \|_{B_{2,1}^s} \right) dt. \] (3.45)

Gronwall’s inequality gives \( (\tilde{\rho}, \tilde{u}, \tilde{E}) \equiv 0 \) immediately. Substituting it into (3.44), \( \tilde{\theta} = 0 \) is also followed.

Finally, by Remark 3.1 we can arrive at Theorem 1.1 satisfying the inequality (1.4). Hence the proof of Theorem 1.1 is complete.

4 Appendix

In the last section, we give the crucial Moser-type inequality in the non-homogeneous Besov spaces and Chemin-Lerner’s spaces. For the homogeneous version, which has been remarked by Zhou in [36].

Proposition 4.1. Let \( s > 0 \) and \( 1 \leq p, r, p_1, p_2, p_3, p_4 \leq \infty \). Assume that \( f \in L^{p_1} \cap B_{p_4,r}^s \) and \( g \in L^{p_3} \cap B_{p_2,r}^s \) with
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]

Then it holds that
\[ \| fg \|_{B_{p,r}^s} \leq C(\| f \|_{L^{p_1}} \| g \|_{B_{p_2,r}^s} + \| g \|_{L^{p_3}} \| f \|_{B_{p_4,r}^s}). \] (4.1)

In particular, whenever \( s \geq N/p \), there holds
\[ \| fg \|_{B_{p,r}^s} \leq C \| f \|_{B_{p,r}^s} \| g \|_{B_{p,r}^s}. \] (4.2)
Proof. From Bony's decomposition, we have

\[ fg = Tfg + Tgf + R(f,g). \]

It follows from Proposition 2.1 that

\[
2^{q_s} \| \Delta_q Tfg \|_{L^p} \leq 2^{q_s} \sum_{|q' - q| \leq 4} \| \Delta_q (S_{q'} f \Delta_{q'} g) \|_{L^p} \leq 2 \sum_{|q' - q| \leq 4} 2^{(q-q')^s} \| S_{q'} f \|_{L^{p_1}} 2^{q' s} \| \Delta_{q'} g \|_{L^{p_2}} \left( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \right) \leq C \sum_{|q' - q| \leq 4} 2^{q' s} \| f \|_{L^{p_1}} 2^{q' s} \| \Delta_{q'} g \|_{L^{p_2}} \]

(4.3)

where \( c_{q1} := \sum_{|q' - q| \leq 4} \frac{2^{q' s} \| \Delta_{q'} g \|_{L^{p_2}}}{\| f \|_{B^{s}_{p_2,r}}} \) satisfies \( \| c_{q1} \|_{L^p} \leq 1 \). Similarly,

\[
2^{q_s} \| \Delta_q Tgf \|_{L^p} \leq 2^{q_s} \sum_{|q' - q| \leq 4} \| \Delta_q (S_{q'} g \Delta_{q'} f) \|_{L^p} \leq C \sum_{|q' - q| \leq 4} 2^{q' s} \| f \|_{L^{p_1}} 2^{q' s} \| \Delta_{q'} g \|_{L^{p_2}} \]

(4.4)

where \( c_{q2} := \sum_{|q' - q| \leq 4} \frac{2^{q' s} \| \Delta_{q'} f \|_{L^{p_2}}}{\| g \|_{B^{s}_{p_2,r}}} \) satisfies \( \| c_{q2} \|_{L^p} \leq 1 \).

On the other hand, from Proposition 2.2, we arrive at

\[
2^{q_s} \| \Delta_q R(f, g) \|_{L^p} \leq C \| R(f, g) \|_{B^s_{p,q}} \leq C \| f \|_{B^s_{p_1,\infty}} \| g \|_{B^s_{p_2,\infty}} \]

(4.5)

where \( c_{q3} := \frac{2^{q_s} \| \Delta_q R(f, g) \|_{L^p}}{\| R(f, g) \|_{B^s_{p,q}}} \) satisfies \( \| c_{q3} \|_{L^p} \leq 1 \). In the last step, we used the embedding property \( B^0_{p_1,1} \hookrightarrow L^{p_1} \hookrightarrow B^0_{p_1,\infty} \).

Hence, (4.1) follows from (4.3)-(4.5). Moreover, the embedding properties in Lemma 2.2 give (4.2) directly.

It is not difficult to generalize Proposition 4.1 to the framework of Chemin-Lerner’s spaces \( \tilde{L}^s_T(B^s_{p,r}) \). The indices \( s, p, r \) behave just as the stationary case whereas the time exponent \( \rho \) behaves according to Hölder’s inequality, which is given by a proposition for clarity.

**Proposition 4.2.** The following estimate holds:

\[
\| fg \|_{\tilde{L}^p_T(B^s_{p,r})} \leq C (\| f \|_{L^{p_1}_T(B^s_{p_1,r})} \| g \|_{L^{p_3}_T(B^s_{p_3,r})} + \| g \|_{L^{p_1}_T(B^s_{p_1,r})} \| f \|_{L^{p_3}_T(B^s_{p_3,r})})
\]

whenever \( s > 0, 1 \leq p, r \leq \infty, 1 \leq p_1, p_2, p_3, p_4 \leq \infty, 1 \leq \rho, p_1, p_2, p_3, p_4 \leq \infty \) with

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},
\]

20
and

\[ \frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4}. \]

As a direct corollary, one has

\[ \|fg\|_{L^p_T(B^r_{p,r})} \leq C\|f\|_{L^p_{T1}(B^r_{p,r})}\|g\|_{L^p_{T2}(B^r_{p,r})} \]

whenever \( s \geq N/p. \)

**Acknowledgement**

The research of Jiang Xu is partially supported by the NSFC (11001127), China Postdoctoral Science Foundation (20110490134) and NUAA Research Funding (NS2010204).

**References**

[1] G. Alì, Global existence of smooth solutions of the \( N \)-dimensional Euler-Possion model, *SIAM J. Math. Anal.*, 35 (2003) 389-422.

[2] G. Alì, D. Bini and R. Nionero, Global existence and relaxation limit for smooth solutions to the Euler-Possion model for semiconductors, *SIAM J. Math. Anal.*, 32 (2000) 572-587.

[3] G. Alì, L. Chen, A. Jüngel, and Y. J. Peng, The zero-electron-mass limit in the hydrodynamic model for plasmas, *Nonlinear Anal. TMA*, 72 (2010) 4415-4427.

[4] P. Amster, M. P. Beccar Varela, A. Jüngel and M. C. Mariani, Subsonic solutions to a one-dimensional non-isentropic model for semiconductors, *Journal of Mathematical Analysis and Applications*, 258 (2001) 52-62.

[5] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Berlin, Heidelberg: Springer-Verlag, 2011.

[6] J.-Y. Chemin, Théorèmes d’unicité pour le système de Navier-Stokes tridimensionnel, *Journal d’Analyse Mathématique*, 77 (1999) 25-50.

[7] G. Q. Chen, J. W. Jerome and B. Zhang, Existence and the singular relaxation limit for the inviscid hydrodynamic energy model. *Modeling and Computation for Application in Mathematics, science, and Engineering* (Evanston, IL, 1996); 189–215, Numer. Math. Sci. Comput., Oxford Univ.Press: New York, 1998.

[8] R. Danchin, *Fourier Analysis Methods for PDE’s*, (Lecture Notes), 2005.

[9] R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Inventiones Mathematicae* 141 (2000) 579-614.

[10] R. Danchin, Local theory in critical spaces for flows of compressible viscous and heat-conductive gases, *Comm. P. D. E.* 26 (2001) 1183-1233.
[11] P. Degond and P. A. Markowich, A steady-state potential flow model for semiconductors, *Ann. Mat. Pura Appl.* IV (1993) 87-98.

[12] D. Y. Fang, J. Xu and T. Zhang, Global exponential stability of classical solutions to the hydrodynamic model for semiconductors, *Math. Models Methods Appl. Sci.* 17 (2007), 1507-1530.

[13] I. Gamba, Stationary transonic solutions of a one-dimensional hydrodynamic model for semiconductor, *Comm. Partial Diff. Equns* 17 (1992), 553-577.

[14] Y. Guo, Smooth irritational flows in the large to the Euler-Poisson system in $\mathbb{R}^{3+1}$, *Commun. Math. Phys.* 195 (1998) 249-265.

[15] Y. Guo and W. Strauss, Stability of semiconductor states with insulating and contact boundary conditions, *Arch. Rational Mech. Anal.*, 179 (2005), 1-30.

[16] I. Gasser and R. Natalini, The energy transport and the drift diffusion equations as relaxation limits of the hydrodynamic model for semiconductors, *Quart. Appl. Math.*, 57 (1999), 269-282.

[17] L. Hsiao, S. Jiang and P. Zhang, Global existence and exponential stability of smooth solutions to a full hydrodynamic model to semiconductors, *Monatshefte für Mathematik*, 136 (2002) 269-285.

[18] L. Hsiao, P. A. Markowich and S. Wang, The asymptotic behavior of globally smooth solutions of the multidimensional isentropic hydrodynamic model for semiconductors, *J. Differential Equations*, 192 (2003) 111-133.

[19] L. Hsiao and S. Wang, Asymptotic behavior of global smooth solutions to the full 1D hydrodynamic model for semiconductors, *Math. Model Methods Appl. Sci.*, 12 (2002) 777-796.

[20] D. Iftimie, The resolution of the Navier-Stokes equations in anisotropic spaces, *Revista Matemática Iberoamericana* 15 (1999), 1-36.

[21] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rational Mech. Anal.*, 58 (1975) 181-205.

[22] Y. P. Li, Global existence and asymptotic behavior for a multidimensional nonisentropic hydrodynamic semiconductor model with the heat source, *J. Differential Equations*, 225 (2006) 134-167.

[23] T. Luo, R. Natalini and Z. P. Xin, Large time behavior of the solutions to a hydrodynamic model for semiconductors, *SIAM J. Appl. Math.*, 59 (1998) 810-830.

[24] A. Majda, *Compressible Fluid Flow and Conservation laws in Several Space Variables* (Springer-Verlag: Berlin/New York, 1984).

[25] P. Marcati and R. Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equations, *Arch. Ration. Mech. Anal.*, 129 (1995) 129-145.
[26] P. A. Markowich, C. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Vienna, Springer-Verlag, 1990.

[27] S. Wang, Quasineutral limit of Euler-Poisson system with and without viscosity. *Comm. Partial Differential Equations* **29**, (2004) 419-456.

[28] D. H. Wang and G. Q. Chen, Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation, *J. Differential Equations*, **144** (1998) 44-65.

[29] J. Xu, Energy-transport limit of the hydrodynamic model for semiconductors, *Math. Models Methods Appl. Sci.*, **20** (2010) 937-954.

[30] J. Xu, Relaxation-time limit in the isothermal hydrodynamic model for semiconductors, *SIAM J. Math. Anal.*, **40** (2009) 1979-1991.

[31] J. Xu, Well-posedness and stability of classical solutions to the multidimensional full hydrodynamic model for semiconductors, *Comm. Pure Appl. Anal.*, **8** (2009) 1073-1092.

[32] J. Xu and W.-A. Yong, Relaxation-time limits of non-isentropic hydrodynamic models for semiconductors, *J. Differential Equations*, **247** (2009) 1777-1795.

[33] J. Xu and W.-A. Yong, Zero-relaxation limit of non-isentropic hydrodynamic models for semiconductors, *Discrete Contin. Dyn. Syst.*, **25** (2009) 1319-1332.

[34] J. Xu and T. Zhang, Zero-electron-mass limit of Euler-Poisson equations, submitted (2010).

[35] W.-A. Yong, Diffusive relaxation limit of multidimensional isentropic hydrodynamical models for semiconductors, *SIAM J. Appl. Math.*, **64** (2004) 1737-1748.

[36] Y. Zhou, Local well-posedness for the incompressible Euler equations in the critical Besov spaces, *Ann. Inst. Fourier*, **54** (2004) 773-786.