Probability Theories with Dynamic Causal Structure: A New Framework for Quantum Gravity

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Abstract

Quantum theory is a probabilistic theory with fixed causal structure. General relativity is a deterministic theory but where the causal structure is dynamic. It is reasonable to expect that quantum gravity will be a probabilistic theory with dynamic causal structure. The purpose of this paper is to present a framework for such a probability calculus. We define an operational notion of space-time, this being composed of elementary regions. Central to this formalism is an object we call the causaloid. This object captures information about causal structure implicit in the data by quantifying the way in which the number of measurements required to establish a state for a composite region is reduced when there is a causal connection between the component regions. This formalism puts all elementary regions on an equal footing. It does not require that we impose fixed causal structure. In particular, it is not necessary to assume the existence of a background time. The causaloid formalism does for probability theory something analogous to what Riemannian calculus does for geometry. Remarkably, given the causaloid, we can calculate all relevant probabilities and so the causaloid is sufficient to specify the predictive aspect of a physical theory. We show how certain causaloids can be represented by suggestive diagrams and we show how to represent both classical probability theory and quantum theory by a causaloid. We do not give a causaloid formulation for general relativity though we speculate that this is possible. The causaloid formalism is likely to be very powerful since the basic equations remain unchanged when we go between different theories - the differences between these theories being contained in the specification of the causaloid alone. The work presented here suggests a research program aimed at finding a theory of quantum gravity. The idea is to use the causaloid formalism along with principles taken from the two theories to marry the dynamic causal structure of general relativity with the probabilistic structure of quantum theory.
1 Introduction

The two great pillars of twentieth century physics are general relativity (GR) and quantum theory (QT) and both have enjoyed considerable empirical success. It so happens that the domain where general relativity has been well verified corresponds to situations where quantum effects (such as superposition) are negligible. And similarly, the domain where quantum theory has been verified corresponds to situations where general relativistic effects (such as matter dependent curvature of space time) are negligible. Sufficiently sophisticated experiments would be able to probe domains where both quantum and general relativistic effects are significant. However, each theory is formulated in a way that requires that the particular effects of the other can be ignored and so, in such domains, we would not be able to make predictions. What is required is a new theory, a theory of quantum gravity (QG), which reduces to GR or to QT in the situation where quantum effects or where general relativistic effects, respectively, are small. The problem is that we want to proceed from two less fundamental theories (GR and QT) to a more fundamental theory (QG). How can we do this? One approach is to try to formulate one theory entirely in the terms of the other. For example, we might try to “quantize general relativity”. This is likely to work when one theory is clearly less fundamental than the other. However, GR and QT each bring fundamental notions to the table that cannot easily be accommodated in terms of the structures available in the other theory. Instead a more even handed approach seems favourable. This is problematic. It seems unlikely that we can combine two mathematical formulations of two different theories in an even handed way without stepping outside those mathematical formulations. Hence we adopt the following strategy. We will pick out essential conceptual properties of each theory and try to find a mathematical framework which can accommodate them. A historical example of this approach is provided by Einstein himself in his invention of special relativity which resulted from an attempt to combine Newtonian physics with electromagnetism. From Newtonian physics he took the Galilean principle of invariance for inertial frames and from electromagnetism he took the fact that the speed of light is independent of the source. These facts were set apart from their mathematical formulation in their original theories. Thus, Einstein stated Galileo’s principle in words rather than giving it the usual mathematical expression as the Galilean transformations. It was only having done this that he was able to avoid the mess associated with earlier attempts to reconcile Newtonian physics with electromagnetism in terms of the properties of an ether. Indeed, these earlier attempts were an attempt to formulate electromagnetism in within the Newtonian framework.

With the implementation of this approach in mind, we note the following.

1. General relativity is a deterministic theory with dynamic causal structure.
2. Quantum theory is a probabilistic theory with fixed causal structure.

Once the probabilistic cat is out of the bag it is unlikely that we will go back to a fundamentally deterministic theory. Likewise, once we have dynamic causal
structure it is unlikely that a more fundamental theory will have an underlying fixed causal structure. Hence, we require a mathematical framework for physical theories with the following properties:

1. It is probabilistic.
2. It admits dynamic causal structure.

In this paper we will find such a framework. We will show how QT can be formulated in this framework. We also expect to be able to formulate GR in the framework though we do not give an explicit construction. But, of course, the real point of this exercise is that we should be able to formulate a theory of QG in this framework. And, further, this framework should make this job easier. We will suggest possible approaches to finding a theory of QG within this framework.

2 Overview

In GR we introduce coordinates $x^\mu$. We can consider intervals $\delta x^\mu$ in these coordinates. We do not say up front which of these intervals (or which linear combinations of these intervals) is time-like. It is only after solving for the metric that we can do this. The causal structure is dynamic. In quantum theory, on the other hand, we must specify the causal structure in advance. One way to see this is to consider different ways in which we might put two operators, $\hat{A}$ and $\hat{B}$, together. If the two regions corresponding to these operators are space-like separated then we use the tensor product $\hat{A} \otimes \hat{B}$. If the two regions are immediately sequential (time-like) then we write $\hat{B} \hat{A}$. In order to know what type of product to take we need to know the causal structure in advance. We seek a new type of product which unifies these two products (along with any other products in QT) and puts them on an equal footing.

The approach taken in this paper is operational. We define an operational notion of space-time consisting of elementary regions $R_x$. An arbitrary region $R_1$ may consist of many elementary regions. In region $R_1$ we may perform some action which we denote by $F_{R_1}$ (for example we may set a Stern-Gerlach apparatus to measure spin along a certain direction) and observe something $X_{R_1}$ (the outcome of the spin measurement for example). Consider two disjoint regions $R_1$ and $R_2$. Our basic objective is to find a formalism which allows us to calculate the probability for something in one region, $R_1$, conditioned on what happened in another region $R_2$ if this probability is well defined (we will explain what we mean by a “well defined probability” in Sec. 3). Namely we want to be able to calculate all probabilities of the form

$$\text{prob}(X_{R_1} | F_{R_1}, X_{R_2}, F_{R_2})$$

when well defined. We would like the formalism that does this to put every elementary region $R_x$ on an equal footing.
To this end we introduce vectors $r_{(X_{R_1}, F_{R_1})}(R_1)$ for $R_1$ (these are analogous to operators in QT). Such vectors are defined for any region including the elementary regions $R_x$. Given any composite region such as $R_1 \cup R_2$ we can find the corresponding $r$ vector using the causaloid product

$$r_{(X_{R_1 \cup R_2}, F_{R_1 \cup R_2})}(R_1 \cup R_2) = r_{(X_{R_1}, F_{R_1})}(R_1) \otimes^A r_{(X_{R_2}, F_{R_2})}(R_2) \quad (2)$$

This means that $r$ vectors for any region can be built out of $r$ vectors for the elementary regions, $R_x$, comprising this region. The causaloid product is given by the causaloid which we will describe briefly in a moment. The $r$ vectors for the elementary regions themselves are also given by the causaloid. If this formalism is applied to QT then the causaloid product unifies the different products in QT mentioned above.

We find that the probability

$$\text{prob}(X_{R_1}|X_{R_2}, F_{R_1}, F_{R_2})$$

is well defined if and only if

$$v \equiv r_{(X_{R_1}, F_{R_1})} \otimes^A r_{(X_{R_2}, F_{R_2})}$$

is parallel to

$$u \equiv \sum_{Y_{R_1}} r_{(Y_{R_1}, F_{R_1})} \otimes^A r_{(X_{R_2}, F_{R_2})}$$

(where the sum is over all possible observations, $Y_{R_1}$, in $R_1$ consistent with action $F_{R_1}$) and this probability is given by

$$\text{prob}(X_{R_1}|X_{R_2}, F_{R_1}, F_{R_2}) = \frac{|v|}{|u|} \quad (3)$$

where $|a|$ denotes the length of the vector $a$.

The causaloid is theory specific and is given by providing a means to calculate certain matrices (called lambda matrices). The lambda matrices quantify the way in which the number of measurements to determine the state is reduced due to correlations implied by the theory. We have lambda matrices for each elementary region (called local lambda matrices) and we have lambda matrices for every subset of elementary regions. Hence, at this general level all elementary regions are put on an equal footing. In any specific theory we will expect that some lambda matrices will follow from others. In QT, for example, it turns out that we only need local lambda matrices and lambda matrices for pairs of adjacent regions. From these we can calculate all other lambda matrices. In a particular theory we will expect to break the symmetry between elementary regions by virtue of some particular choice of lambda matrices. For example, in QT the lambda matrix associated with a pair of adjacent elementary region is different to the lambda matrix associated with a pair of non-adjacent elementary regions. However, the fact that we start with a formalism that does not impose any particular such structure from the very beginning puts us in a strong position to make progress in finding a theory of QG.
The causaloid framework does not have, as a fundamental notion, the idea of a state evolving in time. However, standard physical theories such as QT do. Thus, to help us put the QT in the causaloid framework, we will show how to recover a notion of a state evolving in time in the causaloid framework. Having done this we are able to put classical probability theory and quantum theory into the causaloid framework. Having pulled QT into the causaloid framework we can leave behind the problematic notion of a state at time $t$.

Any attempt to find a theory of QG in this program is likely to start by putting GR into the framework. We discuss how this might be done before considering issues that arise in QG.

The important new technical results in this paper are contained in Sec. 12 to Sec. 29. These sections are fairly self-contained and the impatient reader can jump straight to those sections on a first reading of this paper (though perhaps skimming the earlier sections).

3 Data

We are looking for a framework for physical theories. But what is a physical theory and what does it do for us? There are many possible answers to these questions. But we take the following to be true:

Assertion: A physical theory, whatever else it does, must correlate recorded data.

A physical theory may do much more than this. For example it may provide a picture of reality. It may satisfy our need for explanation by being based on some simple principles. But for a physical theory to have any empirical content, it must at least correlate recorded data. This sounds like a rather weak assertion. But, as we will see, it provides us with a strong starting point for the construction of the framework we seek.

The assertion above leaves unspecified what the word "correlate" means. This could be deterministic correlation, probabilistic correlation, or conceivably something else. Since we will be interested in probabilistic theories, we will take this to mean probabilistic correlation.

What is data? We can compile the following list of properties that data has.

1. Data is a record of (i) actions and (ii) observations. For example it might record statements like (i) I lifted the rock and let go (an action), and (ii) it fell and hit my toe (an observation).

2. Data is recorded by physical means. For example it may be written on bits of paper, stored in a computer’s memory, or stored in the brain of the experimentalist.

3. Data is robust (it is unlikely to randomly change).

4. Data can be copied so that new physical records exist.
5. Data can be translated (e.g. English to French or binary to base 10).

6. Data can be moved around (e.g. in wires or on bits of paper).

7. Data can be processed (e.g. to check that it is correlated according to some physical theory).

The physicality of data may concern us a little - especially in those situations where we expect the physical systems which store, transport, and process the data to interfere with the physical experiment we are performing. To deal with this concern we make the following assumption

**The indifference to data principle:** It is always possible to find physical devices capable of storing, transporting, and processing data such that (to within some arbitrarily small error) the probabilities obtained in an experiment do not depend on the detailed configuration of these devices where this detailed configuration corresponds to the particular data and programming (for the program which will process the data) whilst it is being stored, transported, and processed. Such physical devices will be called *low key.*

Without such a principle the probabilities might depend on whether the experiment is conducted by an Englishman or a Frenchman (since the same data in English or French will have a different detailed physical configuration). This principle does not imply that the presence of the physical device which stores and processes the data has no effect on the experiment. But rather that any such effect does not depend on the detail of the data being stored. For example, a computer being used to record and process data from a nearby gravitationally sensitive experiment has mass and therefore will effect the experiment in question. However, this effect will not depend on the detailed configuration of the computer which corresponds to the data (or at least that effect will be arbitrarily small). Consequently the principle does not forbid the physical data devices from being part of the experiment. For example, we could throw a computer from the leaning tower of Pisa to gain information about how the computer falls. It might collect data through a camera about the time it passes successive levels of the building. In this case the data device is actually part of the experiment and the principle still applies. This means that we do not need to put the observer (for observer read “physical data devices”) outside the system under investigation. The observer can be a part of the system they are investigating so long as they can store and process data in a low key manner. In fact, one might even argue that we must always regard the observer as part of the system we are observing. How could the data end up being collected otherwise? Of course, there are certain situations where we have reasons to regard the observer as being outside the system under consideration - namely those situations where the probabilities measured do not depend on the bulk properties of the observer. However, the important point is that we can do physics when this is not the case so long as we can have low key data devices. It is easy to imagine data processing devices which are not low key. For example, we could use a computer which...
stores information in the configuration of a number of large rocks rather than a
standard electronic computer to store and process data about a nearby
gravitationally sensitive experiment. Then the probabilities would depend on
the detail of the data.

The fact that we start with considerations about data where data is a collec-
tions of actions and observations puts us in an operational or instrumental mode
of thinking. Operationalism played a big role in the discovery of both relativity
theory and QT. There are different ways of thinking about operationalism. We
can either take it to be fundamental and assert that physical theories are about
the behaviour of instruments and nothing more. Or we can take it to be a
methodology aimed at finding a theory in which the fundamental entities are
beyond the operational realm. In the latter case operationalism helps us put
in place a scaffolding from which we can attempt to construct the fundamental
theory. Once the scaffolding has served its purpose it can be removed leaving
the fundamental theory partially or fully constructed. The physicist operates
best as a philosophical opportunist (and indeed as a mathematical opportunist).
For this reason we will not commit to either point of view for the time being
noting only that the methodology of operationalism serves our purposes. Inde-
deep, operationalism is an important weapon in our armory when we are faced
with trying to reconcile apparently irreconcilable theories. A likely reason for
any such apparent irreconcilability is that we are making some unwarranted
assumptions beyond the operational realm. It was through careful operational
reasoning that Einstein was able to see that absolute simultaneity is unneces-
sary (since it has no operational counterpart). The operational methodology is
a way of not making wrong statements. If we are lucky we can use it to make
progress.

4 Remarks on quantum theory

When all is said and done, quantum theory provides a way of calculating prob-
abilities. It is a probability calculus. Hence, its natural predecessor is not
Newtonian mechanics or any other branch of classical physics, but rather what
might be called classical probability theory (CProbT). Thus, in the same way
that CProbT can be applied to various physical situations from classical physics
(such as systems of interacting spins, particles in a gas, electromagnetic fields...) to
calculate probabilities, quantum theory can be applied to various different
physical situations (interacting quantum spins, a quantum particle in a poten-
tial well, quantum fields, ....) to calculate probabilities. Quantum theory is, like
classical probability theory, a meta theory with many realizations for different
physical situations.

One particular realization of QT is what might be called quantum mechanics
(QM). This is the non-relativistic theory for multi-particle systems in which we
introduce a wavefunction $\psi(x_1, x_2, \cdots, t)$ and the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_i \frac{\hbar^2}{2m_i} \nabla_i^2 \psi + V(x_1, x_2, \cdots, t)\psi$$

(4)

to evolve the wavefunction. QM is an example of QT.

Quantum field theory (QFT) is another example of QT. The basic framework of quantum theory (which we will present in detail below) consisting of an evolving state $\hat{\rho}$ that acts on a Hilbert space $\mathcal{H}$ is capable of expressing both non-relativistic quantum mechanics and relativistic QFT (see pg. 49 of [1]). This is clearest in the formulation of QFT in which we write down a superwavefunctional $\Psi(\phi(x))$ which we can regard as a linear superposition of basis states where these basis states correspond to definite configurations of the field $\phi(x)$.

This addresses a common misconception. We should regard QFT as a special case of QT rather than something like the converse. Thus we should not think of QT as a limiting case of QFT - though we might attempt to derive QM as the limit of QFT. It is not the case that QT, thus understood, is necessarily non-relativistic. The only point that should be added to these remarks is that QFT requires an infinite dimensional Hilbert space whereas we can do a lot in non-relativistic scenarios with finite dimensional Hilbert spaces. However, this is a technical rather than conceptual point and, in any case, there are good reasons to believe that a theory of quantum gravity will have something like a finite dimensional Hilbert space. In our discussion of QT we will stick with finite dimensional Hilbert spaces both for technical simplicity and because we are dealing with finite data sets. Issues related to infinities and continuities will be discussed in Sec. 33.

In going from QT to QFT we have to add much additional structure to QT. However, the deep conceptual novelties of QT are evident without going to QFT. For this reason it seems reasonable to look at ways to combine QT (rather than QFT) with GR. This way we can hope to import these conceptual novelties into a theory of QG without getting distracted by the additional structure of QFT. Ultimately we would require that a theory of QG incorporate QFT (at some appropriate level of approximation). However, we take the attitude that this is not likely to be important in the early days of the construction of QG and it may even be possible to fully construct QG before taking on this consideration.

5 Basic framework for operational theories

We want to give a simple operational formulation of CProbT and QT and for this purpose we present a framework which works for probabilistic theories that admit a universal background time.

The basic scenario we consider is that shown in Fig. 1. This consists of a sequence of operations on the system. We can represent these operations by boxes. Each box has a knob on it which can be used to vary the operation implemented. At each operation we have the possibility of extracting some
outcome \( l \) (this is data). Each operation can be regarded alternatively as (a) a preparation (since it outputs a system), (b) a transformation (since it transforms the state of a system), and (c) a measurement (since it inputs a system and outputs an outcome \( l \)). The same can be said of any sequence of such operations. We can, strictly, only regard an operation as a preparation if it outputs the system in a definite state. We define the state in the following way.

**The state** associated with a preparation is that thing represented by any mathematical object that can be used to calculate the probability for every outcome of every measurement that may be performed on the system.

Given this definition we can define the state to be represented by a list of all probabilities,

\[
P = \begin{pmatrix}
\vdots \\
p_\alpha \\
\vdots 
\end{pmatrix}
\]  

(5)

where \( \alpha \) labels every outcome of every possible measurement. We note that we can write

\[
p_\alpha = R_\alpha \cdot P
\]  

(6)

where \( R_\alpha \) has a 1 in position \( \alpha \) and 0’s everywhere else. The object \( P \) contains a lot of information. In general we would expect a physical theory to afford some simplification so that some entries in \( P \) can be calculated from other entries. In fact we can insist that this be done by a linear formula so we have a state given
by

\[ \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_K \end{pmatrix} \]  

(7)
such that

\[ p_\alpha = \mathbf{r}_\alpha \cdot \mathbf{p} \]  

(8)
The \( p_k \)'s are the probabilities associated with a set of fiducial measurement outcomes. We take \( K \) to be the minimum number of entries in \( \mathbf{p} \) that makes it possible to write the state by a linear formula like this. That this will always be possible is clear since we have (6) as a last resort.

A special measurement is the identity measurement \( \mathbf{r}_I \) corresponding to the measurement whose result is positive if any result is seen. In the case that the state is normalized we have \( \mathbf{r}_I \cdot \mathbf{p} = 1 \). However, for technical reasons, we will not normalize the state after each step.

It follows from the fact that these probabilities are given by a linear formula that the transformation of the state is given by a linear formula. Thus, if we obtain result \( l \) the new state is

\[ \mathbf{p} \rightarrow Z_l \mathbf{p} \]  

(9)
where \( Z_l \) is a \( K \times K \) real matrix. This is clear since each component of the new state must be given by a linear combination of the components in the initial state. The probability of outcome \( l \) is

\[ \text{prob}_l = \frac{\mathbf{r}_l \cdot Z_l \mathbf{p}}{\mathbf{r}_I \cdot \mathbf{p}} \]  

(10)

The state can be normalized by dividing by \( \mathbf{r}_I \cdot \mathbf{p} \). However, this introduces unnecessary non-linearities in the evolution of the state. It is more convenient to allow the state to be unnormalized and use the above formula for calculating probabilities.

We may have more than one system. We need a way of setting up the framework for such composite systems. Both CProbT and QT turn out to be simple in this respect. The state of a composite system is given by specifying joint probabilities \( p_{k_1, k_2} \) with \( k_1 = 1 \) to \( K_1 \) and \( k_2 = 1 \) to \( K_2 \).

6 Brief summary of classical probability theory

Consider a classical system which can be in one of \( N \) distinguishable configurations (for example a bit has \( N = 2 \)). We can write the state of this system by specifying a probability, \( p_n \), for each configuration, \( n \).

\[ \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{pmatrix} \]  

(11)
Note that $K = N$. We define the \textit{identity measurement vector}, $r_I$, for CProbT by

$$ r_I = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (12) $$

Now we can state postulates of CProbT in compact form.

1. The state of a system is given by $p \in S_N$ where $S_N$ is defined by (i) $p_n \geq 0$ and (ii) $r_I \cdot p \leq 1$.

2. The state $p$ for a composite system 12 made from systems 1 and 2 has components $p_{n_1 n_2}$ and belongs to $S_{N_1 N_2}$.

3. Any operation which transforms the state of a system and has classical outcomes labeled by $l$ is associated with a set of $N \times N$ matrices $Z_l$ which (i) map $S_N$ into $S_N$, and (ii) have the properties that $r_I \cdot Z_l p \leq r_I \cdot p$ and $r_I \cdot (\sum_l Z_l) p = r_I \cdot p$ for all states. The probability of outcome $l$ is

$$ \text{prob}_l = \frac{r_I \cdot Z_l p}{r_I \cdot p} \quad (13) $$

and the state after outcome $l$ is observed is

$$ p \rightarrow Z_l p \quad (14) $$

A few notes of clarification are useful here. We deliberately do not impose the normalization condition $r_I \cdot p = 1$ (though we impose the condition $r_I \cdot p \leq 1$ to keep these vectors bounded). First it is not necessary to normalize since the denominator on the RHS of (13) ensures that the sum of probabilities over all outcomes adds up to 1. Second, it is useful to allow the freedom not to normalize since then we can regard $Z_l p$ as a new state even though this new state is not normalized. We can, if we wish, normalize a state by dividing it by $r_I \cdot p$.

7 \hspace{1em} \textbf{Brief summary of quantum theory}

The postulates of quantum theory (stripped of additional structure pertaining to particular applications) can be written in the following compact form.

1. The state of a system is given by a positive operator $\hat{\rho}$ acting on a complex Hilbert space $\mathcal{H}$ with $\text{trace}(\hat{\rho}) \leq 1$.

2. A composite system 12 made from systems 1 and 2 has Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. 
3. Any operation which transforms the state of a system and has classical outcomes labelled by \( l \) is associated with a set of trace non-increasing completely positive linear maps (also known as superoperators), \( \mathcal{S}(\cdot) \), where \( \sum_l \mathcal{S}_l \) is trace preserving. The probability of outcome \( l \) is

\[
\text{prob}_l = \frac{\text{trace}(\mathcal{S}_l(\hat{\rho}))}{\text{trace}(\hat{\rho})}
\]  

and the state after outcome \( l \) is observed is

\[
\hat{\rho} \to \mathcal{S}_l(\hat{\rho})
\]

A completely positive linear map \( \mathcal{S} \) is one which, when extended to a composite system 12 as \( \mathcal{S} \otimes I \) (where \( I \) is the identity map on system 2), leaves the state for the total system \( \hat{\rho}_{12} \) positive regardless of the initial state of the total system and the dimension of system 2. This property is required for the internal consistency of the theory. There are two familiar special cases of superoperators. First there are unitary maps,

\[
\mathcal{S}(\hat{\rho}) = \hat{U} \hat{\rho} \hat{U}^\dagger
\]

where \( \hat{U} \) is a unitary operator (satisfying \( \hat{U} \hat{U}^\dagger = I \)). We can understand this as an example of postulate 2 above where \( l \) only takes one value (so we always see the same result). Second there are projection operators,

\[
\mathcal{S}(\rho) = \hat{P} \rho \hat{P}
\]

where \( \hat{P} \) is a projection operator (satisfying \( \hat{P} \hat{P} = \hat{P} \)). The first example is trace preserving and the second example is trace decreasing. There are many other types of superoperator which, in one way or another, extrapolate between these two extremes.

States can normalized at any time by applying the following formula

\[
\hat{\rho} \to \frac{1}{\text{trace}(\hat{\rho})} \hat{\rho}
\]

In calculating the probabilities for outcomes of measurements we are not necessarily interested in the evolution of the state of the system during measurement. To this end we can associate a set, \( \{ \hat{A}_l \} \), of positive operators with

\[
\text{prob}_l = \text{trace}(\hat{A}_l \hat{\rho}) = \text{trace}(\mathcal{S}_l(\hat{\rho}))
\]

if the state is normalized. There is a many to one linear map between \( \mathcal{S}_l \) and \( \hat{A}_l \) (which can be written down explicitly). Usually positive operators \( \hat{A} \) (or something like them) are introduced in the postulates. However, this is not necessary.
8 Quantum theory formulated in similar way to classical probability theory

We can reformulate QT in a way which resembles CProbT. We define

\[ \hat{A} = \begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \vdots \\ \hat{A}_{N^2} \end{pmatrix} \]  

where \( \hat{A}_k \) for \( k = 1 \) to \( N^2 \) are a fixed fiducial set of linearly independent positive operators which span the space of positive operators acting on an \( N \) dimensional Hilbert space \( \mathcal{H}_N \). We define the identity measurement vector, \( \mathbf{r}_I \), for QT by

\[ \mathbf{r}_I \cdot \hat{A} = I \]  

where \( I \) is the \( N^2 \times N^2 \) identity matrix. Since the operators \( \hat{A}_k \) form a complete linearly independent set, \( \mathbf{r}_I \) is unique. Now we can restate the postulates of QT.

1. The state of a system is given by \( \mathbf{p} \in S^Q_N \) where \( S^Q_N \) is the set of \( \mathbf{p} \) for which (i) we can write \( \mathbf{p} = \text{trace}(\hat{A} \hat{\rho}) \) where \( \hat{\rho} \) is a positive operator acting on \( \mathcal{H}_N \) and (ii) \( \mathbf{r}_I \cdot \mathbf{p} \leq 1 \).

2. The states for a composite system 12 made from systems 1 and 2 belong to \( S^Q_{N_1 N_2} \) derived from \( \hat{A}_1 \otimes \hat{A}_2 \) whose elements act on \( \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2} \).

3. Any operation which transforms of the state of a system and has classical outcomes labelled by \( l \) is associated with a set \( \mathbf{Z}_l \) of \( N^2 \times N^2 \) matrices which (i) are such that \( \mathbf{Z}_l \otimes I \) maps \( S^Q_{N_1 N_2} \) into \( S^Q_{N M} \) for any ancillary system of any \( M \), and (ii) have the property that \( \mathbf{r}_I \cdot \mathbf{Z}_l \mathbf{p} \leq \mathbf{r}_I \cdot \mathbf{p} \) and also \( \mathbf{r}_I \cdot (\sum_l \mathbf{Z}_l)\mathbf{p} = \mathbf{r}_I \cdot \mathbf{p} \) for all states. The probability of outcome \( l \) is

\[ \text{prob}_l = \frac{\mathbf{r}_I \cdot \mathbf{Z}_l \mathbf{p}}{\mathbf{r}_I \cdot \mathbf{p}} \]  

and the state after outcome \( l \) is observed is

\[ \mathbf{p} \rightarrow \mathbf{Z}_l \mathbf{p} \]  

Note that for QT we have \( K = N^2 \). It is easy to show that

\[ Z = \text{trace}(\hat{A} \hat{S}(\hat{A}^T))[\text{trace}(\hat{A} \hat{A}^T)]^{-1} \]  

where the superscript \( T \) denotes transpose.
9 Reasonable postulates for quantum theory

The postulates of QT, unlike those of CProbT, are rather abstract. It is not clear where they come from. However, QT can be obtained by a set of what might be regarded as reasonable postulates [2]. Before stating these we need a few definitions.

The maximum number of reliably distinguishable states, \( N \), is defined to be equal to the maximum number of members of any set of states which have the property that there exists some measurement device which can be used to distinguish the states in a single shot measurement (so that the sets of outcomes possible for each state in this set are disjoint).

This quantity captures the information carrying capacity of the system (in QT it is simply the dimension of the Hilbert space). More exactly, we might say that the information carrying capacity of the system is \( \log_2 N \) bits. We will say that

A system is constrained to have information carrying capacity \( \log_2 M \) if the states are such that, for a measurement set to distinguish \( N \) distinguishable states, we only ever obtain outcomes associated with some given subset of \( M \) of these states.

We want to define a useful notion

Equivalence classes. Two operations belong to the same equivalence class if replacing one by the other gives rise to the same probabilities

An example might be measuring the polarization of a photon with a polarizing beamsplitter or a calcite crystal each orientated at an angle \( \theta \). If one device is replaced by the other (with an appropriate identification of outcomes between the two devices) then the probabilities remain the same.

The spin degree of freedom of an electron and the polarization degree of freedom of a photon are, in each case, described by a Hilbert space of dimension 2. There is a sense in which these two systems have the same properties. We define this idea as follows

Two systems have the same properties if there is a mapping between equivalence classes of operations such that under this mapping we get the same probabilities for outcomes for each type of system.

This mapping might take us from an experiment involving an electron’s spin degree of freedom to another experiment involving a photon’s polarization degree of freedom. For example, the electron could be prepared with spin up along the \( z \) direction, sent through a magnetic field which (acting as a transformation) rotates the spin through 20° in the \( zx \) plane. Then the electron impinges on a
Stern-Gerlach device orientated in the $x$ direction. Under a mapping between equivalence classes, this might correspond to an experiment in which a photon is prepared with vertical polarization. The photon passes through a crystal which rotates its polarization through $10^\circ$ and then onto a polarizing beamsplitter orientated at angle $45^\circ$. The probabilities seen would be the same in each case (note that the angles must be halved for the photon since orthogonal states are vertical and horizontal rather than up and down).

We define

**Pure states** are states which cannot be simulated by mixtures of other distinct states

Also we define

**A reversible transformation** is a transformation, $T$, for which there exists another transformation $T^{-1}$ which is such that if $T$ is applied, then $T^{-1}$, the over all transformation leaves the the incoming state unchanged (it is the identity transformation) for all incoming states

and

**A continuous transformation** is one which can be enacted by sequential application of infinitely many infinitesimal transformations where an infinitesimal transformation is one which has the property that it only has an infinitesimal effect on the probability associated with any given outcome for any measurement that may be performed on the state

Given these definitions, we are in a position to state the postulates

**Postulate 0 Probabilities.** Relative frequencies (measured by taking the proportion of times a particular outcome is observed) tend to the same value (which we call the probability) for any case where a given measurement is performed on a ensemble of $n$ systems prepared by some given preparation in the limit as $n$ becomes infinite.

**Postulate 1 Information.** There exist systems for which $N = 1, 2, \cdots$, and, furthermore, systems having, or constrained to have the same information carrying capacity have the same properties.

**Postulate 2 Composite systems.** A composite system 12 consisting of subsystems 1 and 2 satisfies $N_{12} = N_1 N_2$ and $K_{12} = K_1 K_2$.

**Postulate 3 Continuity.** There exists a continuous reversible transformation on a system between any two pure states of that system for systems of any dimension $N$.

**Postulate 4 Simplicity.** For each given $N$, $K$ takes the minimum value consistent with the other axioms.
The crucial postulate here is Postulate 3. If the single word “continuous” is dropped from these axioms, then we get classical probability theory rather than quantum theory.

These axioms give rise to the full structure of quantum theory with operators on finite dimensional complex Hilbert space as described above. The construction theorem is simple but rather lengthy and the reader is referred to [2] for details. However, the main ideas are as follows. If follows from Postulate 1 that $K = K(N)$ and $K(N + 1) > K(N)$ (this second point is not obvious). It then follows from Postulate 2, after a little number theory, that $K = N^r$ where $r = 1, 2, 3, \ldots$. By the simplicity postulate we should take the smallest value of $r$ that works. First we try $r = 1$ but this fails because of the continuity postulate. Then we try $r = 2$ and this works. Thus, we have $K = N^2$. (As an aside, if we dropped the word “continuity” we get $r = 1$ and hence $K = N$. This leads very quickly to classical probability theory.) Next we take the simplest nontrivial case $N = 2$, and $K = 4$. If we just consider normalized states then rather than 4 probabilities we have three. We apply the group of continuous reversible transformations (implied by the continuity postulate) to show that the set of states must live inside a ball (with pure states on the surface). This is the Bloch ball of quantum theory for a two dimensional Hilbert space. Thus, we get the correct space of states for two dimensional Hilbert space. We now apply the information postulate to the general $N$ case to impose that the state restricted to any two dimensional Hilbert space behaves as a state for Hilbert space of dimension 2. By this method we can construct the space of states for general $N$. Various considerations give us the correct space of measurements and transformations and the tensor product rule and, thereby, we reconstruct quantum theory for finite $N$.

10 Remarks on general relativity

General relativity was a result of yet another attempt to make two theories consistent, namely special relativity and Newton’s theory of gravitation. Einstein gives various reasons that Galileo’s principle of invariance is not sufficiently general since it applies only to inertial frames. He replaces it with the following

The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (generally covariant) [3].

He then employs the equivalence principle to argue that the metric $g_{\mu\nu}$ represents the gravitational field. He sets up the mathematics of tensors as objects which have the property that a physical law expressible by setting all the components of a tensor equal to zero is generally covariant. Out of these ideas he is able, by an ingenious chain of reasoning, to obtain field equations for GR which can be expressed as

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$ (26)
$G_{\mu\nu}$ is Einstein tensor and is a measure of the local curvature concocted out of derivatives of the metric. $T_{\mu\nu}$ is the stress-energy tensor and is determined by the local properties of matter.

We can distinguish two roles for the metric.

1. It determines the set of local frames in which gravitational fields vanish and inertial physics applies locally. This property is given by the local value of the metric without taking into account its variation with $x^\mu$.

2. It tells us how to compare local fields at two different points by providing a way to parallel transport such fields. This it does via the connection $\Gamma^\alpha_{\mu\nu}$ and the machinery of covariant differentiation. This property does depend on the variation of the metric with $x^\mu$.

The metric also determines causal structure. It tells us whether two events are space-like or time-like and thereby determines whether we can send a signal from one to the other. However, the metric is a dynamical feature of the theory. It is determined by solving the Einstein field equations. Hence the causal structure is dynamical. We can try to express this in more operational terms. Given local events with local labels $x^\mu$ (which may be abstract or read off some real physical reference frame) there is no way in general to say, in advance (that is without solving the equations), whether it is possible to send a signal from one to another. In non-gravitational physics we have fixed causal structure. For example, in SR the metric is fixed and so we know the causal structure in advance.

GR is a deterministic theory. In classical physics we can always introduce probabilities simply by applying CProbT. However, CProbT assumes a fixed causal structure just as QT does. In particular it assumes a fixed background time. Hence, we would not expect to be able to apply CProbT to GR in a straightforward manner. Indeed, we could consider the program of unifying CProbT and GR to find what we might call probabilistic general relativity (ProbGR). This program might be expected share many of the same difficulties we face in the program to find a theory of QG. We have taken as our goal to find a framework for probabilistic theories which admit dynamic causal structure. This framework should include ProbGR as well as QG. Hence, we will need to introduce further principles to get QG rather than ProbGR. For such principles we can look to the differences between CProbT and QT. These differences are especially clear in the formulation of QT in Sec. 5 which looks similar to CProbT and in the postulates introduced in Sec. 9 (see [4] for a discussion of the differences between CProbT and QT).

11 Remarks on the problem of finding a theory of quantum gravity

The most obvious issue that arises when attempting to combine QT with GR is that QT has a state on a space-like surface that evolves with respect to
an external time whereas in GR time is part of a four dimensional manifold whose behavior is dynamically determined by Einstein’s field equations. We can formulate GR in terms of a state evolving in time - namely a canonical formulation \([5, 6]\). Such formulations are rather messy (having to deal with the fact that time is not a non-dynamical external variable) and break the elegance of Einstein’s manifestly covariant formulation. Given that Einstein’s chain of reasoning depended crucially on treating all four space-time coordinates on an equal footing it is likely to be at least difficult to construct QG if we make the move of going from a four dimensional manifold \(\mathcal{M}\) to an artificial splitting into a three dimensional spatial manifold \(\Sigma\) and a one dimensional time manifold \(R\). But there is a further reason coming from quantum theory that suggests it may be impossible. If the causal structure is dynamically determined then what constitutes a space-like surface must also be dynamically determined. However, in quantum theory we expect any dynamics to be subject to quantum uncertainty. Hence, we would expect the property of whether a surface is space-like or not to be subject to uncertainty. It is not just that we must treat space and time on an equal footing but also that there may not even be a matter-of-fact as to what is space and what is time even after we have solved the equations (see \([7, 8]\)). To this end we will give a framework (which admits a formulation of quantum theory) which does not take as fundamental the notion of an evolving state. The framework will, though, allow us to construct states evolving through a sequence of surfaces. However, these surfaces need not be space-like (indeed, there may not even be a useful notion of space-like).

Another way of thinking about these issues is by considering how we might probe causal structure. The most obvious way to do this is to use light signals since they probe the light cone structure which underpins causal structure in GR. In GR we are typically interested in cases where the presence of light represents only a small perturbation and so we freely employ counterfactual reasoning in which we imagine sending or not sending light signals without having to consider the effect this has on the solution to Einstein’s field equations. On the other hand, in QT, the presence or not of even a single photon can have a dramatic effect on what is seen. The most clear example of this is provided in \([9]\) in which an interference effect involving one photon depends on whether the path of another photon is blocked or not. In QT we have a fixed background causal structure (which might be implemented for example by bolting apparatuses down to a rigid structure) and so there is no need to employ such reasoning about the counterfactual transmission of photons for the purposes of understanding causal structure. However, in QG, we will not assume that there is a fixed background causal structure. We cannot assume that two regions of space-time have a certain causal relationship in the absence of any photon being transmitted between them just because we know that their causal relationship would be fixed if a photon were to be so transmitted. This line of thinking lends separate support to the possibility mentioned above that there may not even be a matter-of-fact as to what is space and what is time even after the equations have been solved.

A more mathematical handle on this issue can be gained by considering the
various ways in quantum theory we can put together pairs of operators $\hat{A}$ and $\hat{B}$. We can form the product $\hat{A}\hat{B}$. We can take the tensor product $\hat{A} \otimes \hat{B}$. A third slightly more subtle example is $\hat{A}\hat{B}$ where $\hat{?}$ stands for an unknown operator. The first and third of these two examples correspond to a time-like situation whereas the second case corresponds to a space-like situation (or at least an equal time situation). Each of these cases is treated on a different footing in QT. In GR we initially treat space and time on an equal footing. Thus, we introduce four space-time coordinates $x^\mu$ (with $\mu = 0, 1, 2, 3$) giving rise to the intervals $\delta x^\mu$. We do this without saying which of these intervals (or which linear combinations of them) are time-like. We then solve Einstein’s field equations and obtain a metric $g_{\mu\nu}$. From the metric (which has a Lorentzian signature) we can identify which linear combinations pertain to time-like intervals. But, at least in principle, we do this after we have solved the field equations. Thus, by analogy, if we are to treat space and time on an equal footing in QG as we do in GR then we would also want to put those objects in the theory of QG which correspond to the three types of product in QT mentioned above on an equal footing. This should already be an issue in special relativistic quantum theory though since the causal structure is fixed in advance it is not essential we attend to it. But in QG it is likely to be quite essential. In fact, in QG the issue is likely to be even more serious than it is in GR since it may be, as mentioned above, that even after the equations have been solved, we are unable to identify what intervals are space-like and what are time-like. In order to address this issue we will define a new type of product which unifies all these types of products in quantum theory (and their counterparts in more general probabilistic frameworks) putting them on an equal footing.

We should ultimately be interested in experiments to test a theory of QG. Before we get to real experiments it is interesting to consider gedankenexperiments which illuminate the conceptual structure of a theory. As we noted in the introduction, a theory of QG would be necessary in a situation in which we could not neglect the particular effects of both GR and QT. The type of gedankenexperiment in which this is going to be the case will be one where we simultaneously have dynamic causal structure and quantum superposition. Such a situation occurs when we look for quantum interference of a massive object which goes into a superposition of being in two places at once. Gedankenexperiments of this type have been discussed by Penrose [10] and there has been some effort to design a realizable version of this type of experiment [11].

### 12 Setting the scene

We repeat our assertion from Sec. 8: A physical theory, whatever else it does, must correlate recorded data. Data is a record of actions and observations taken during an experiment. We will assume that this data is recorded onto cards. Each card will record a small amount of *proximate* data. We will illustrate what we mean by this soon with examples. Thus the cards represent something analogous to Einsteinian events. One piece of data recorded on any given card
Figure 2: A source of electrons is followed by four Stern-Gerlach apparatuses shown schematically here. Data is recorded onto a card at each apparatus.

will be something we will regard as representing or being analogous to space-time location. Of course, it is not necessary to record the data onto cards. It could be recorded in a computer’s memory, or by any other means. But this story with the cards will help us in setting up the mathematical framework we seek.

We will consider examples where the data recorded on each card is of the form \((x, a, s)\). The first piece of data, \(x\), is an observation and represents location. For example, it could be the space-time position read off some real physical reference frame such as a GPS system \([12]\) or a reference mollusc \([13]\). Or it might be some other data that we are simply going to regard as representing location. The second piece of data, \(a\), represents some actions. For example it might correspond to the configuration of some knobs we have freedom in setting. The third piece of data, \(s\), represents some further local observations. Here are two examples of this type

1. Imagine we have a sequence of four Stern-Gerlach apparatuses (shown schematically in Fig. 2) labelled \(x = 1, 2, 3, 4\) preceded by a source of electrons labelled \(x = 0\). We can set each Stern-Gerlach apparatus to measure spin along the direction \(a\). Then we record the outcome \(s = \pm 1/2\). Thus we get a card from each Stern-Gerlach apparatus with \((x, a, s)\) written on it. We would also want to extract a card (or maybe a set of cards) from the apparatus which prepares the electrons. For example we could write \((0, a, s)\) where \(a = \text{“source appropriately constructed”}\), and \(s = \text{“observations consistent with source’s proper functioning seen”}\). From each run of the experiment we would collect a stack of five cards. We could vary the settings \(a\) on the Stern-Gerlach apparatuses. To extract probabilities we would want to run the experiment many times.

2. Imagine we have a number of probes labelled \(n = 1, 2, \ldots, N\) drifting in space (see Fig. 3). Each probe has a clock which ticks out times \(t_n = 1, 2, \ldots, T\). Each probe has knobs on it which correspond to the
Figure 3: We have a number of probes (two shown here) drifting in space. Data is collected onto a card at each probe at each tick of the probe’s clock.

settings of some measurement apparatuses on the probe. We let the different configurations of these knobs be labelled by $a$. Further, each probe has some meters which record the outcomes of those experiments. We let $s$ label these outcomes. On each card we can record the data $(x = (n, t_n, t^m_n), a, s)$ where $t^m_n$ is the retarded time of probe $m$ as seen at probe $n$. We may want to put more information into $x$, such as the observed relative orientations. And we may want to reduce the amount of information in $x$, for example we could only record the retarded times of three specific probes (providing something like a GPS system). We take one card from each probe for each tick of the probe clock. At the end of each run of the experiment we will have a stack of $NT$ cards. The clocks can be reset and the experiment repeated many times to obtain probabilities.

In these cases the number of cards in the stack is the same from one run to the next but this need not be the case.

We introduce the important notion of a *procedure*, $F$, which tells the exper-
mentalist what actions to implement - that is how to set the knobs. In our examples above (where we record data \((x, a, s)\) onto the cards) we allow the choice of \(a\) to depend on \(x\) by some function \(a = F(x)\). After a procedure has been implemented we will end up with a stack, \(X\), of cards. This stack contains all the data for the experiment.

We may have more complicated ways of recording data onto cards. A useful example is where we record \((x, q, a, s)\) onto the cards where \(q\) represents some observations we do not want to regard as part of \(x\) which we make immediately before implementing the action \(a\). In this case we can have \(a = F(x, q)\).

We may wish to condition \(F\) at \(x\) on some data collected “previously” at \(x'\). For example we might want to have \(a = F(x, s')\) where \(s'\) is data recorded at \(x'\). However, since we do not want to assume we have fixed causal structure, it is a matter of the physical dynamics as to whether data recorded at \(x'\) will be available at \(x\). Therefore it makes no sense to allow this functional dependence. Rather, any such dependence must be implemented physically. For example, \(q\) could be equal to the retarded value of \(s'\) seen at \(x\). Then we can write \(a = F(x, q)\).

It is possible that some cards in the stack will be repeated. This could happen for example if the clocks in the second example above were faulty and sometimes ticked out the same value twice. To get round this we replace repeated cards by a new card having the same data plus the multiplicity appended to \(s\).

### 13 Thinking inside the box

After each run of the experiment we will have a stack of cards which we denote by \(X\). We can bundle these together and attach a tag to this bundle specifying \(F\). This can be repeated many times for each possible \(F\). We imagine that all these tagged bundles are sent to a man in a sealed room who analyzes them (see Fig. 4). The man cannot look outside the sealed room for extra clues. Hence, all concepts must be defined in terms of the cards themselves. The point of this story is that it enforces a particular kind of honesty. The man is not able to introduce what Einstein called “factitious causes” such as an unobservable global inertial reference frame since he is forced to work with the cards.

The order of the cards in any particular stack does not, in itself, constitute recorded data and is of no significance. Likewise the order the bundled stacks arrive in the sealed room is also of no significance. Thus, in his analysis of the cards, the man in the sealed room should not use these orderings. We could imagine that each stack is shuffled before being bundled and the bundles are then shuffled before being sent to the sealed room. Of course, it is significant which bundle a particular card belongs to and so we should not allow cards from different bundles to get mixed up.

To aid our analysis we begin with a few simple definitions.

**The stack, \(X\),** is the set of cards collected in one run of an experiment.
The full pack, \( V \) is the set of all logically possible cards when all possible procedures are taken into account. It is possible that some cards never actually occur in any stack because of the nature of the physical theory but we include them in \( V \) anyway.

The procedure will be specified by that set of cards \( F \) which is the subset of all cards in \( V \) which are consistent with the procedure \( F \). For example, if the data on each card is of the form \((x, a, s)\) then the set \( F \) is all cards of the form \((x, F(x), s)\). We deliberately use the same symbol, \( F \), to denote the abstract procedure \( F \), the function \( F(x) \), and the set \( F \) since it will be clear from context which meaning is intended.

We have

\[
X \subseteq F \subseteq V
\]

We note that these definitions are in terms of the cards as required.

As described we imagine repeating the experiment many times. In Sec. we will suggest an approach that does not involve repeating the experiment as suggested here.
14 Regions

We continue to provide definitions in terms of the cards. We define the notion of a region. The region $R_O$ is specified by the set of cards from $V$ having $x \in O$. We define $R_x$ to be an elementary region consisting only of the cards having $x$ on them. Regions can be regarded as the seat of actions. In a region we have an independent choice of which action to implement. This captures the notion of space-time regions as places where we have local choices.

When we have a particular run of the experiment we end up with a stack $X$ of data. We can allocate this data to the appropriate regions. Then we have a picture of what happened as laid out in a kind of space-time. For a region $R_1$ (which we take to be shorthand for $R_{O_1}$) we define the stack in $R_1$ to be

$$X_{R_1} = X \cap R_1$$

(28)

These are the cards from the stack, $X$, that belong to region $R_1$.

We define the procedure in $R_1$ to be

$$F_{R_1} = F \cap R_1$$

(29)

these are the cards from the set $F$ which belong to $R_1$. Clearly

$$X_{R_1} \subseteq F_{R_1} \subseteq R_1$$

(30)

Given $(X_{R_1}, F_{R_1})$ we know “what was done” ($F_{R_1}$) and “what was seen” ($X_{R_1}$) in region $R_1$.

15 Statement of objective

We seek to find a formalism to calculate conditional probabilities of the form

$$\text{prob}(X_{R_1} | X_{R_2}, F_{R_1}, F_{R_2})$$

(31)

when these probabilities are well defined without imposing any particular causal structure in advance. Of course, any particular theory we might cast in terms of this formalism is likely to have some sort of causal structure built in and this will be evident in the particular form the mathematical objects in the formalism end up taking.

If the above probability is given a frequency interpretation it is equal to

$$\frac{N(X_{R_1}, X_{R_2}, F_{R_1}, F_{R_2})}{N(X_{R_2}, F_{R_1}, F_{R_2})}$$

(32)

in the limit as the denominator becomes large. Here $N(\cdot)$ is the number of stacks satisfying the given condition.

This probability may not be well defined if there is insufficient conditioning. To see this consider the example with four Stern-Gerlach apparatuses illustrated in Fig. 2. Let $R_1$ consist of all the cards associated with the fourth Stern-Gerlach
apparatus (having $x = 4$) and let $R_2$ consist of all the cards associated with the second apparatus (having $x = 2$). Then we cannot expect this probability to be well defined since we do not know what angle the third Stern-Gerlach apparatus has been set at. In such cases we do not require the formalism to predict any value for the probability.

To aid our considerations we will restrict our attention to a region $R$ for which all probabilities

$$\text{prob}(X_{R}|F_{R}, C)$$

are well defined where $C$ is some condition on the cards outside $R$. We will call such a region, $R$, a *predictively well defined region* with respect to the conditioning $C$. This region can be very big (consisting of a substantial fraction of the cards from $V$). We will assume that we are only interested in probabilities inside this region and, since it is always implicit, we will drop the $C$ writing

$$\text{prob}(X_{R}|F_{R})$$

An example of a predictively well defined region might be provided by the data coming out of a quantum optical laboratory. We would have to set up the laboratory, the optical table, the lasers, the various optical elements and the computers to record data. Having set this up we would want to keep the doors of the laboratory shut, the shutters on the windows down, and condition on many other actions and observations to isolate the experiment from stray light. All this data could go into $C$. In setting up the formalism we will assume that $C$ is always satisfied and that it is the purpose of a physical theory to correlate what goes on inside $R$. Having set up the formalism we will discuss ways to avoid having to have a predictively well defined region and having to conditionize on $C$ (see Sec. 22 and Sec. 30).

16 States

Now consider region $R_1$ inside the predictively well defined region $R$ (see Fig. 5(a)). Then (34) can be written

$$p = \text{prob}(X_{R_1} \cup X_{R-R_1}|F_{R_1} \cup F_{R-R_1})$$

We will regard $(X_{R-R_1}, F_{R-R_1})$ which happens in $R-R_1$ as a *generalized preparation* for region $R_1$. Associated with this generalized preparation for $R_1$ is a state which we will define shortly. We will regard $(X_{R_1}, F_{R_1})$ which happens in $R_1$ as a *measurement*. We will label measurements in $R_1$ with $\alpha_1$. Then we can write

$$p_{\alpha_1} = \text{prob}(X_{R_1}^{\alpha_1} \cup X_{R-R_1}|F_{R_1}^{\alpha_1} \cup F_{R-R_1})$$

We define

*The state* for $R_1$ associated with a generalized preparation in $R-R_1$ is defined to be that thing represented by any mathematical object which can be used to predict $p_{\alpha_1}$ for all $\alpha_1$. 25
Note that quite deliberately we define this for the joint probabilities in (36) rather than the conditional probabilities \( \text{prob}(X_{R^1} | X_{R-R^1}, F_{R^1} \cup F_{R-R^1}) \) even though the latter may seem more natural. The reason for this is that introducing conditional probabilities requires normalization by Bayes formula which introduces nonlinearities. It turns out that these nonlinearities would represent an insurmountable problem in the case where we have dynamic causal structure and so it is better to work with the joint probabilities. As we will see, we can use Bayes formula in the final step when calculating conditional probabilities so there is no problem.

Given the above definition we could write the state as

\[
\mathbf{P}(R) = \begin{pmatrix}
\vdots \\
p_{\alpha_1} \\
\vdots
\end{pmatrix}
\]  

(37)

We can then write

\[
p_{\alpha_1} = R_{\alpha_1}(R) \cdot \mathbf{P}(R)
\]

(38)

where the vector \( R_{\alpha_1}(R) \) has a 1 in position \( \alpha_1 \) and 0’s in all other positions.

Now, we would expect a physical theory to have some structure such that it is not necessary to list all probabilities as in \( \mathbf{P}(R) \) but rather only some subset of them. Thus, we pick out a set of fiducial measurements \( (X_{R^1}, F_{R^1}^1) \), where \( k_1 \in \Omega_1 \), such that we can write a general probability by means of a linear
where
\[ p(\mathbf{R}_1) = \begin{pmatrix} \vdots \\ p_{k_1} \\ \vdots \end{pmatrix} \quad \text{where} \quad k_1 \in \Omega_1 \] (40)

now represents the state with
\[ p_{k_1} = \text{prob}(X^{R_1}_{k_1} \cup X_{R-R_1} \cup F^{k_1}_{R_1} \cup F_{R-R_1}) \] (41)

and where \( K_1 = |\Omega_1| \) is taken to be the minimum number of probabilities in such a list. It is clear we can always do this since, as a last resort, we have (38). Omega sets such as \( \Omega_1 \) will play a big role in this paper. They are not, in general, unique but we can always pick one and stick with it.

The vector \( \mathbf{r}_{\alpha_1}(R_1) \) is associated with the measurement \( (X_{R_1}^{a_1}, F_{R_1}^{a_1}) \) in \( R_1 \). The fiducial measurements are represented by
\[ \mathbf{r}_{k_1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \text{for all} \quad k_1 \in \Omega_1 \] (42)

where the 1 is in the \( k_1 \) position since this is the only way to ensure that \( \mathbf{r}_{k_1} \cdot \mathbf{p} = p_{k_1} \) as required.

We define \( \Lambda^{k_1}_{a_1} \) by
\[ \mathbf{r}_{a_1} = \sum_{k_1 \in \Omega_1} \Lambda^{k_1}_{a_1} \mathbf{r}_{k_1} \] (43)

We will define further lambda matrices in the next section. Like omega sets, they will play a central role in this work. They give a quantitative handle on the amount of compression the physical theory provides. Given (42) it is clear that here the lambda matrices are just the components of the vector \( \mathbf{r}_{a_1} \). That is
\[ \mathbf{r}_{a_1}|_{k_1} = r_{a_1}^{k_1} = \Lambda^{k_1}_{a_1} \] (44)

We will sometimes drop the \( \alpha \)'s writing \( \mathbf{r} \) with components \( r_{k_1} \). The \( \alpha \)'s are then understood to be implicit. Note it also follows from the definition of the \( \Lambda \) matrix that
\[ \Lambda^{k_1}_{k'_1} = \delta^{k_1}_{k'_1} \quad \text{for} \quad k'_1, k_1 \in \Omega_1 \] (45)

where \( \delta^{k_1}_{k'_1} \) equals 1 if the subscript is equal to the superscript and 0 otherwise.
17 Composite Regions

Now consider two disjoint regions $R_1$ and $R_2$ in $R$ (see Fig. 5(b)). We have

$$\text{prob}(X_{R_1} \cup X_{R_2} \cup X_{R-R_1-R_2} | F_{R_1} \cup F_{R_2} \cup F_{R-R_1-R_2}) = r(R_1) \cdot p(R_1)$$

$$= \sum_{k_1} r_{k_1}(R_1) p_{k_1}(R_1)$$

$$= \sum_{k_1} r_{k_1}(R_1) r(R_2) \cdot p_{k_1}(R_2)$$

$$= \sum_{k_1, k_2} r_{k_1} r_{k_2} p_{k_1, k_2}$$

(46)

where $p_{k_1}(R_2)$ is the state in $R_2$ given the generalized preparation $(X_{R_1}^{k_1} \cup X_{R-R_1-R_2}^{k_1}, F_{R_1}^{k_1} \cup F_{R-R_1-R_2}^{k_1})$ in region $R - R_2$ and where

$$p_{k_1, k_2} = \text{prob}(X_{R_1}^{k_1} \cup X_{R_2}^{k_2} \cup X_{R-R_1-R_2}^{k_1} | F_{R_1}^{k_1} \cup F_{R_2}^{k_2} \cup F_{R-R_1-R_2}^{k_1})$$

(47)

This means that we can write the probability for any measurement for the composite region $R_1 \cup R_2$ in terms of a linear sum of joint probabilities $p_{k_1, k_2}$ with $k_1, k_2 \in \Omega_1 \times \Omega_2$. It may even be the case that we do not need all of these probabilities. There may be some further compression (though still maintaining that we have a linear sum). Hence we have the result that a fiducial set of measurements for the composite region $R_1 \cup R_2$ is given by

$$(X_{R_1}^{k_1} \cup X_{R_2}^{k_2}, F_{R_1}^{k_1} \cup F_{R_2}^{k_2})$$

for $k_1, k_2 \in \Omega_{12} \subseteq \Omega_1 \times \Omega_2$

(48)

This result is the central to the whole approach in this paper.

If the behaviour in the two regions is not causally connected then we expect that $\Omega_{12} = \Omega_1 \times \Omega_2$. On the other hand, if there is a strong causal connection then we can have $|\Omega_{12}| = |\Omega_1| = |\Omega_2|$. The relationships between these $\Omega$ sets gives us a combinatoric handle on the causal structure. We seek, however, a more quantitative handle.

To this end we define $\Lambda_{l_1 l_2}^{k_1 k_2}$ by

$$r_{l_1 l_2} = \sum_{k_1, k_2 \in \Omega_{12}} \Lambda_{l_1 l_2}^{k_1 k_2} r_{k_1, k_2} \quad \text{for} \quad l_1, l_2 \in \Omega_1 \times \Omega_2$$

(49)

We will adopt the convention of labelling the elements of the post compression omega set with $k$’s and the elements of the pre-compression product set with $l$’s as in this equation. As before, the fiducial measurements are represented by

$$r_{k_1, k_2} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
1 \\
\vdots \\
0
\end{pmatrix} \quad \text{for all} \quad k_1, k_2 \in \Omega_{12}$$

(50)

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where the 1 is in the \( k_1k_2 \) position. It follows by taking the \( k_1k_2 \) component of (49) that

\[
r_{l_1l_2|k_1k_2} = r_{k_1k_2} = \Lambda_{k_1k_2}^{l_1l_2}
\]  
(51)

This is similar to (44) above.

We can use lambda matrices to calculate an arbitrary \( r \) for the composite system. To see this we start by putting (44) in (46) and reinserting \( \alpha \)’s

\[
r_{l_1l_2} = \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} \Lambda_{l_1l_2}^{l_1l_2} p_{l_1l_2}
\]  
(52)

\[
r_{l_1l_2} = \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} \Lambda_{l_1l_2}^{l_1l_2} \cdot \sum_{k_1k_2 \in \Omega_{12}} \Lambda_{k_1k_2}^{l_1l_2} r_{k_1k_2} \cdot p
\]  
(53)

\[
r_{l_1l_2} = \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} \Lambda_{l_1l_2}^{l_1l_2} \sum_{k_1k_2 \in \Omega_{12}} \Lambda_{k_1k_2}^{l_1l_2} \cdot \Lambda_{k_1k_2}^{k_1k_2} r_{k_1k_2} \cdot p
\]  
(54)

Since this must be true for all \( p \) we have,

\[
r_{\alpha_1\alpha_2} = \sum_{k_1k_2 \in \Omega_{12}} \left( \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} \Lambda_{l_1l_2}^{l_1l_2} \Lambda_{l_1l_2}^{l_1l_2} \Lambda_{k_1k_2}^{l_1l_2} \right) \Lambda_{k_1k_2}^{l_1l_2}
\]  
(55)

and hence, in view of (50) above, the components of \( r_{\alpha_1\alpha_2} \) are

\[
r_{\alpha_1\alpha_2|k_1k_2} = \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} \Lambda_{l_1l_2}^{l_1l_2} \Lambda_{l_1l_2}^{l_1l_2} \Lambda_{k_1k_2}^{l_1l_2}
\]  
(56)

This is consistent with (51) above given (45).

We can generalize this for \( N \) regions. We define lambda matrices for multi-region composites by

\[
r_{l_1\ldots l_N} = \sum_{k_1\ldots k_N \in \Omega_{12}} \Lambda_{k_1\ldots k_N}^{l_1\ldots l_N} r_{k_1\ldots k_2} \text{ for } l_1\ldots l_2 \in \Omega_1 \times \cdots \times \Omega_N
\]  
(57)

And then it is easy to show that

\[
r_{\alpha_1\ldots \alpha_N|k_1\ldots k_N} = \sum_{l_1\ldots l_N \in \Omega_1 \times \cdots \times \Omega_N} \Lambda_{\alpha_1\ldots \alpha_N}^{l_1\ldots l_N} \Lambda_{\alpha_1\ldots \alpha_N}^{l_1\ldots l_N} \Lambda_{k_1\ldots k_2}^{l_1\ldots l_N}
\]  
(58)

Hence, given the lambda matrices we have a way of calculating the components of \( r \) vectors for one region (44), two regions (56), and multi-regions (58).

18 The Causaloid

In the previous section we discussed composite regions made from regions \( R_1, R_2, \ldots \). The smallest component regions are the elementary regions \( R_x \). Any region, \( R_O \), is composed of elementary regions. A general measurement in this
region is labelled with $\alpha_{O}$. But since a general measurement decomposes into local measurements at each component elementary region we can write
\[
\alpha_{O} = \alpha_{x}\alpha_{x'} \cdots \alpha_{x''} \quad \text{where} \quad O = \{x, x', \cdots x''\}
\] (59)

For each of these elementary regions we will have a local lambda matrix $\Lambda^{k_{x}}_{\alpha_{x}}(x, \Omega_{x})$ with $k_{x} \in \Omega_{x}$. We include the argument $x$ for clarity and the argument $\Omega_{x}$ since the choice of omega set is not, in general, unique. For the region $R_{O}$ we have lambda matrix
\[
\Lambda^{k_{O}}_{\alpha_{O}}(O, \Omega_{O})
\] (60)
where
\[
l_{O} \equiv l_{x}l_{x'} \cdots l_{x''} \in \Omega_{x} \times \Omega_{x'} \times \cdots \times \Omega_{x''} \quad \text{where} \quad O = \{x, x', \cdots x''\}
\] (61)
and
\[
k_{O} \equiv k_{x}k_{x'} \cdots k_{x''} \in \Omega_{O} \quad \text{where} \quad O = \{x, x', \cdots x''\}
\] (62)

We will sometimes have reason to consider a lambda matrix as in (60) but where $O = \{x\}$. That is $\Lambda^{k_{x}}_{\alpha_{x}}$. By convention $l$’s are in the product set and $k$’s are in the new omega set. But in this case there is only one omega set, namely $\Omega_{x}$ to take a product over. Thus, we have
\[
\Lambda^{k_{x}}_{l_{x}} = \delta^{k_{x}}_{l_{x}} \quad \text{with} \quad l_{x}, k_{x} \in \Omega_{x}
\] (63)

It is worth noting that if we know $\Lambda^{k_{O}}_{\alpha_{O}}(O, \Omega_{O})$ for one omega set then, since the lambda matrix contains all relevant linear dependencies, we can calculate (i) all other omega sets and (ii) the lambda matrix for any other omega set for the given region (we will spell out the method for doing this in Sec. 24). Hence, it is enough to know the lambda matrix for one omega set for each region. The lambda matrices can be used to calculate an arbitrary measurement vector using the results of the previous sections applied to elementary regions. From (44) and (58)
\[
\begin{align*}
\mathbf{r}_{\alpha_{x}}|_{k_{x}} &= \Lambda^{k_{x}}_{\alpha_{x}} \\
\mathbf{r}_{\alpha_{O}}|_{k_{O}} &= \sum_{l_{O}} \Lambda^{l_{x}}_{\alpha_{x}} \Lambda^{l_{x'}}_{\alpha_{x'}} \cdots \Lambda^{l_{x''}}_{\alpha_{x''}} \Lambda^{k_{O}}_{l_{O}}
\end{align*}
\] (64)

We now come to the central mathematical object in the approach to be taken in this paper.

**The causaloid** for a predictively well defined region $R$ made up of elementary regions $R_{x}$ is defined to be that thing represented by any mathematical object which can be used to obtain $\mathbf{r}_{\alpha_{O}}(R_{O})$ for all measurements $\alpha_{O}$ in region $R_{O}$ for all $R_{O} \subseteq R$.

In view of the above results, one mathematical object which specifies the causaloid is
\[
\begin{bmatrix}
\Lambda^{k_{x}}_{\alpha_{x}}(x, \Omega_{x}) & \text{for one } \Omega_{x} \text{ for each } R_{x} \\
\Lambda^{k_{O}}_{\alpha_{O}}(O, \Omega_{O}) & \text{for one } \Omega_{O} \text{ for each non-elementary } R_{O} \subseteq R
\end{bmatrix}
\] (65)
This lists all \( \Lambda \) matrices. However, we might expect any given physical theory to have some structure such that some \( \Lambda \) matrices can be calculated from others. If this is the case then we might expect that we can take some subset of the \( \Lambda \) matrices, labelled by \( j \) and write

\[
\Lambda = [\Lambda(j) : j = 1 \text{ to } J] \text{RULES}
\]

where RULES are rules for deducing a general \( \Lambda \) from the given \( \Lambda \)'s. We will see that we can achieve quite considerable compression of this nature in the cases of CProbT and QT.

19 The causaloid product

As we noted in Sec. 11, there are three basic ways of putting two operators together in quantum theory: \( AB, A \otimes B \), and \( A \otimes B \). We noted there that it would be desirable to treat these on an equal footing. To this end we now define the causaloid product for our framework. At this stage we are working in a general framework. However, we will see later that this unifies all these types of product for quantum theory. Let \( r_{\alpha_1} \) be a measurement vector in \( \mathcal{R}_1 \) (corresponding to \( O_1 \)) and let \( r_{\alpha_2} \) be a measurement vector in \( \mathcal{R}_2 \) (corresponding to \( O_2 \)) and let the regions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be non-overlapping. We define the causaloid product \( \otimes^\Lambda \) by

\[
r_{\alpha_1 \alpha_2} = r_{\alpha_1} \otimes^\Lambda r_{\alpha_2}
\]

Strictly we should write

\[
(r_{\alpha_1 \alpha_2}, \mathcal{O}_1 \cup \mathcal{O}_2) = (r_{\alpha_1}, \mathcal{O}_1) \otimes^\Lambda (r_{\alpha_2}, \mathcal{O}_2)
\]

since the causaloid product needs to know which region it is addressing but for brevity we will stick with (67) the regions being implicit in the labels \( \alpha_1 \) and \( \alpha_2 \).

The components of the LHS are obtained from the components of the vectors on the RHS by applying (44) and (56)

\[
r_{\alpha_1 \alpha_2}|k_{1}k_{2} = \sum_{l_1l_2 \in \Omega_1 \times \Omega_2} (r_{\alpha_1}|l_1)(r_{\alpha_2}|l_2) \Lambda_{l_1l_2}^{k_{1}k_{2}}
\]

Now, the lambda matrix \( \Lambda_{l_1l_2}^{k_{1}k_{2}} \) is given by the causaloid. To see this we can reinsert the \( \mathcal{O}'s \) writing it as \( \Lambda_{l_1l_2}^{k_{1}k_{2}} \) which, in view of (61, 62), is the same as \( \Lambda_{l_1l_2}^{k_{1}k_{2}} \). Now, in fact the causaloid gives this lambda matrix for the \( l \)'s in the product set over all elementary product sets \( \Omega_x \)'s (as in (61)) whereas we only require \( l \)'s in the subset \( \Omega_{\mathcal{O}_1 \cup \mathcal{O}_2} \). We use only those components of \( \Lambda_{l_1l_2}^{k_{1}k_{2}} \) we need to take the causaloid product.

We wish to discuss two special cases.

1. Omega sets multiply. \( \Omega_{12} = \Omega_1 \times \Omega_2 \) so that \(|\Omega_{12}| = |\Omega_1||\Omega_2|\).
2. Omega sets do not multiply. \( \Omega_{12} \subset \Omega_1 \times \Omega_2 \) so that \(|\Omega_{12}| < |\Omega_1||\Omega_2|\).
First note that it follows immediately from the definition of the lambda matrix in (49) that

\[ \Lambda_{l_1 l_2}^{k_1 k_2} = \delta_{l_1 l_2}^{k_1 k_2} \quad \text{for } l_1 l_2, k_1 k_2 \in \Omega_{12} \]  

(70)

where \( \delta_{l_1 l_2}^{k_1 k_2} \) equals 1 if the subscripts and superscripts are equal and 0 otherwise. Hence, we can write (69) as

\[ r_{\alpha_1 \alpha_2}^{k_1 k_2} = (r_{\alpha_1}^{k_1}) (r_{\alpha_2}^{k_2}) + \sum_{l_1 l_2 \in \Omega_1 \times \Omega_2 - \Omega_{12}} (r_{\alpha_1}^{l_1}) (r_{\alpha_2}^{l_2}) \Lambda_{l_1 l_2}^{k_1 k_2} \]  

(71)

It follows that

\[ \text{if } \Omega_{12} = \Omega_1 \times \Omega_2 \quad \text{then} \quad r_{\alpha_1 \alpha_2} = r_{\alpha_1} \otimes r_{\alpha_2} \]  

(72)

where \( \otimes \) denotes the ordinary tensor product. Hence we see that the ordinary tensor product is a special case of the causaloid product when the omega sets multiply. We will see that, in quantum theory, typically omega sets will multiply. This corresponds to the products \( A \otimes B \) and \( \hat{A} \otimes \hat{B} \) from quantum theory which have the property that the total number of real parameters after taking the product is equal to the product of the number from each operator (so we have \( |\Omega_{12}| = |\Omega_1||\Omega_2| \)). Omega sets do not multiply when we have strong causal dependence so that what happens in one region depends, at least partially, on what is done in the other region in a way that cannot be altered by what is done in the rest of \( R \). In quantum theory we see this when we take the product \( \hat{A} \hat{B} \). Then the total number of real parameters in the product is equal to the number in \( A \) and \( B \) separately (this is basically \( |\Omega_{12}| = |\Omega_1| = |\Omega_2| \)). Typically strong causal dependence indicates that two regions are sufficiently “close” that what is done outside of these regions cannot interfere with the causal dependence. We will say that the two regions are causally adjacent in these cases.

### 20 Using the causaloid to make predictions

The causaloid is so named because it is an object which gives us a quantitative handle on the causal structure as was seen in the previous section. What is surprising is that, given the causaloid, we can calculate any probability pertaining to the predictively well defined region \( R \) so long as that probability is well defined. To see this note that if we have disjoint regions \( R_1 \) and \( R_2 \) we can write

\[ \text{prob}(X_{R_1}, X_{R_2}, F_{R_1}, F_{R_2}) = \frac{r_{(X_{R_1}, F_{R_1})} \otimes \Lambda r_{(X_{R_2}, F_{R_2})} \cdot p(R_1 \cup R_2)}{\sum_{Y_{R_1}} r_{(Y_{R_1}, F_{R_1})} \otimes \Lambda r_{(X_{R_2}, F_{R_2})} \cdot p(R_1 \cup R_2)} \]  

(73)

(this is basically Bayes formula). The sum in the denominator is over all possible stacks \( Y_{R_1} \) in \( R_1 \) consistent with \( F_{R_1} \), that is all \( Y_{R_1} \subseteq F_{R_1} \). All probabilities pertaining to region \( R \) are of this form. Now, if this probability is well defined then it does not depend on what is outside \( R_1 \cup R_2 \). Hence, it does not depend
on the state $\mathbf{p}(R_1 \cup R_2)$. The space of possible states spans the full dimensionality of the vector space by definition since we have a minimal fiducial set of measurements specifying the components of the state. Hence

The probability

$$\text{prob}(X_{R_1}|X_{R_2}, F_{R_1}, F_{R_2})$$

is well defined if and only if

$$\mathbf{v} \equiv r(X_{R_1}, F_{R_1}) \otimes^A r(X_{R_2}, F_{R_2})$$

is parallel to

$$\mathbf{u} \equiv \sum_{Y_{R_1} \subseteq F_{R_1}} \mathbf{r}(Y_{R_1}, F_{R_1}) \otimes^A \mathbf{r}(X_{R_2}, F_{R_2})$$

and this probability is given by

$$\text{prob}(X_{R_1}|X_{R_2}, F_{R_1}, F_{R_2}) = \frac{|\mathbf{v}|}{|\mathbf{u}|} \quad (74)$$

where $|\mathbf{a}|$ denotes the length of the vector $\mathbf{a}$.

In fact, since the two vectors are parallel, we can simply take the ratio of any given component of the two vectors as long as the denominator is nonzero. We can write $\mathbf{v} = p \mathbf{u}$ where $p$ is the above probability.

One concern might be that it will be a rare state of affairs that these vectors are parallel and so the formalism is rarely useful. Since we have only set ourselves the task of calculating probabilities when they are well defined we are not compelled to address this. However, it turns out that the situation is not so bad. In fact we can always make $R_2$ big enough that we have well defined probabilities. To see this consider the extreme case where $R_2 = R - R_1$. Then we have $\mathbf{p}(R_1 \cup R_2) = \mathbf{p}(R)$. Now, we only have one preparation for the predictively well defined region $R$, namely the condition $C$ being true on the cards outside $R$. Since we are always taking this to be true we can only have one state. This means that it must be specified by a single component, that is $\mathbf{p}(R) = (p_1)$ where $(p_1)$ is a single component vector. The number $p_1$ cancels out in (73) and so the probability is well defined.

If the two vectors are not exactly parallel the probability will not be well defined but it may be be approximately well defined. Indeed, we can use (64) to place bounds on the probability. Define $\mathbf{v}^\parallel$ and $\mathbf{v}^\perp$ as the components of $\mathbf{v}$ parallel and perpendicular to $\mathbf{u}$ respectively. Then it is easy to show that

$$\frac{|\mathbf{v}^\parallel|}{|\mathbf{u}|} - \frac{|\mathbf{v}^\perp|}{|\mathbf{v}| \cos \phi} \leq \text{prob}(X_{R_1}|X_{R_2}, F_{R_1}, F_{R_2}) \leq \frac{|\mathbf{v}^\parallel|}{|\mathbf{u}|} + \frac{|\mathbf{v}^\perp|}{|\mathbf{v}| \cos \phi} \quad (75)$$

where $\phi$ is the angle between $\mathbf{v}$ and $\mathbf{v}^\perp$ (we get these bounds using $|\mathbf{v} \cdot \mathbf{p}| \leq |\mathbf{u} \cdot \mathbf{p}|$ and considering $\mathbf{p}$ parallel to $\mathbf{v}^\perp$).
21 Physical theories and the causaloid formalism

In the causaloid formalism

1. We have the causaloid, $\Lambda$, which is theory specific. The causaloid depends on the boundary conditions $C$ outside $R$. These might only be relevant when we are “close” to the boundary (QT appears to be of this nature). In this case, modulo what happens at the boundary, we can say that the causaloid fully characterizes the physical theory (at least its predictive component).

2. We have some basic equations which are theory non-specific. These are (64) for calculating a general $r$ from the causaloid, (69) for forming the causaloid product, and Bayes formula in the form given in (73) above (or we can use (74) given that appropriate conditions are satisfied).

Given the theory specific part and the theory independent equations we can make predictions. This framework is very general and we would expect any physical theory to fit into it (perhaps with some minor modifications concerning the way the data is collected onto cards). Hence we see that we have a potentially very powerful formalism in which the theory specifics are all put into one object. This is likely to be useful if we are hoping to combine different physical theories.

22 The open causaloid

In typical situations we will have some elementary regions which can be regarded as being at the boundary. Typically we might expect to have to use special mathematical techniques to deal with these. However, if the region $R$ is sufficiently big then we are most likely to be interested in probabilities which do not pertain to the these boundary regions. For this reason we define the open causaloid.

**The open causaloid** for a predictively well defined region $R$ made up of elementary regions $R_x$ with boundary elementary regions $R_{x_b}$ with $x_b \in O_b$ is defined to be that thing represented by any mathematical object which can be used to obtain $r_{\alpha_O}(R_O)$ for all measurements $\alpha_O$ in region $R_O$ for all $R_O \subseteq R - R_{O_b}$.

We can use the open causaloid to calculate all well defined probabilities excluding those which pertain to the boundary. If we make the region $R$ sufficiently big then we can be sure that any regions like $R_1$ and $R_2$ of interest (and for which we want to calculate conditional probabilities) do not overlap with the boundary regions. In this case the open causaloid is as useful as the causaloid itself. Indeed, given that we have already restricted our attention to $R$ in defining the causaloid, it does not matter much if we restrict slightly further to $R - R_{O_b}$. In view of the remarks in the last section, the open causaloid is likely
to be characteristic of the physical theory without being especially influenced by boundary conditions outside \( R \). We can, further, envisage letting the boundary tend to infinity so that the open causaloid and the causaloid become equivalent.

### 23 Some results concerning lambda matrices

The causaloid is either specified by giving all lambda matrices or just giving a few lambda matrices and then using some RULES to calculate all others. If we want to use such RULES then it will be useful to have some results relating lambda matrices.

First we note that when omega sets multiply so do lambda matrices.

\[
\Lambda^{k_1 k_2}_{l_1 l_2} = \Lambda^{k_1}_{l_1} \Lambda^{k_2}_{l_2} \quad \text{if} \quad \Omega_{12} = \Omega_1 \times \Omega_2
\]

(76)

where the \( l_1 \) might belong to any subset of the full set of allowed measurements (that are labelled by \( \alpha_1 \)), and likewise for \( l_2 \). This follows from (72) using (44) and restricting to the given sets for the \( l \)’s.

Next we give the following result.

\[
\Lambda^{k_1 k_2 k_3}_{l_1 l_2 l_3} = \sum_{k_2' \in \Omega_{2y}} \Lambda^{k_1}_{l_1} \Lambda^{k_2}_{l_2} \Lambda^{k_3}_{l_3} \quad \text{if} \quad \Omega_{123} = \Omega_{12} \times \Omega_{23} \quad \text{and} \quad \Omega_{23} = \Omega_{2y} \times \Omega_{2y}
\]

(77)

where the notation \( \Omega_{23} \) means that we form the set of all \( k_3 \) for which there exists \( k_2 k_3 \in \Omega_{23} \). This generalizes as

\[
\Lambda^{k_1 k_2 k_3 k_4}_{l_1 l_2 l_3 l_4} = \sum_{k_2' \in \Omega_{2y}, k_3' \in \Omega_{3y}} \Lambda^{k_1}_{l_1} \Lambda^{k_2}_{l_2} \Lambda^{k_3}_{l_3} \Lambda^{k_4}_{l_4} \quad \text{if} \quad \Omega_{1234} = \Omega_{12} \times \Omega_{23} \times \Omega_{34}
\]

(78)

and so on. We will now prove (77) (we obtain (78) using the same proof technique). We have

\[
r_{l_1 l_2 l_3} \cdot p = r_{l_2 l_3} \cdot p_{l_1} = \sum_{k_2' k_3 \in \Omega_{23}} \Lambda^{k_2}_{l_2 l_3} r_{k_2' k_3} \cdot p_{l_1} = \sum_{k_2' k_3 \in \Omega_{23}} \Lambda^{k_2}_{l_2 l_3} r_{l_1 k_2 k_3} \cdot p = \sum_{k_1 k_2 k_3 \in \Omega_{123}} \sum_{k_2' k_3 \in \Omega_{23}} \Lambda^{k_1}_{l_1 l_2} \Lambda^{k_2'}_{l_2 l_3} \Lambda^{k_3}_{l_3} r_{k_1 k_2 k_3} \cdot p
\]

(79)

where \( p_{l_1} \) is the state for region \( R_2 \cup R_3 \) given that the preparation was \((X_{R_1}^l \cup X_{R-R_1-R_2-R_3}, F_{R_1}^l \cup F_{R-R_1-R_2-R_3}) \) in \( R - R_2 - R_3 \). Also note that we have used the same method in the last line that was used in the first three lines but for \( p_{k_3} \). Since this is true for any \( p \) we have

\[
r_{l_1 l_2 l_3} = \sum_{k_1 k_2 k_3 \in \Omega_{123}} \sum_{k_2' k_3 \in \Omega_{23}} \Lambda^{k_1}_{l_1 l_2} \Lambda^{k_2'}_{l_2 l_3} r_{k_1 k_2 k_3}
\]

(80)
We also have, by definition,

\[
\mathbf{r}_{l_1l_2l_3} = \sum_{k_1k_2k_3 \in \Omega_{l_1l_2l_3}} \Lambda^{k_1k_2k_3}_{l_1l_2l_3} \mathbf{r}_{k_1k_2k_3}
\]

Comparing the last two equations gives us (77).

We have found some relations between lambda matrices when these lambda matrices have certain properties (such as having omega sets which satisfy the given properties). This proves that we do not have complete freedom to choose lambda matrices independently of one another. It should be possible to characterize all possible relationships between lambda matrices so we know how much freedom we have in specifying the causaloid. These constraints are likely to give us deep insight into the possible nature of physical theories.

### 24 Transforming lambda matrices

We write \( \Lambda^{k_0}_{\mathcal{O}}(\mathcal{O}, \Omega_\mathcal{O}) \). We noted in Sec. 18 that since the lambda matrix contains all relevant linear dependencies we can calculate (i) all other omega sets and (ii) the lambda matrix for any other omega set. For example, we might want to check that \( \Omega'_\mathcal{O} \) is an omega set and then calculate \( \Lambda^{k_0'}_{\mathcal{O}}(\mathcal{O}, \Omega'_\mathcal{O}) \). To do this is easy. First we form the square matrix

\[
\Lambda^{k_0}_{\mathcal{O}}(\mathcal{O}, \Omega_\mathcal{O}) \quad k'_\mathcal{O} \in \Omega'_\mathcal{O}
\]

If this has an inverse then \( \Omega'_\mathcal{O} \) is an omega set. Then it is easy to show that

\[
\Lambda^{k_0}_{\mathcal{O}}(\mathcal{O}, \Omega'_\mathcal{O}) = \sum_{k_0} [\Lambda^{k_0}_{\mathcal{O}}(\mathcal{O}, \Omega_\mathcal{O})]^{-1} \Lambda^{k_0}_{\mathcal{O}}(\mathcal{O}, \Omega_\mathcal{O})
\]

by considering the equations by which the lambda matrices are defined. Similar remarks apply to local lambda matrices \( \Lambda^{k_x}_{\mathcal{O}}(x, \Omega_x) \).

Note that if \( \Omega'_\mathcal{O} = \Omega_\mathcal{O} \) then the matrix in (85) is equal to the identity. Indeed, this is how we can deduce the omega set from the lambda matrix.

### 25 Introducing time

#### 25.1 Foliations

It is common in physics to think of a state evolving in “time”. We will show how we can recover this notion in the causaloid formalism. This will be a useful exercise if we wish to make contact with those physical theories, such as QT, that take the notion of an evolving state as fundamental. We will find, however, that this formalism admits a much more general framework for evolving states. In particular there is no requirement that the time slices are space-like.

If we wish to think of the state evolving in the region \( \mathcal{O} \) then we must introduce introduce a time label \( t = 0, 1, \ldots T \) and a set of nested subsets of \( \mathcal{O} \)

\[
\mathcal{O} \equiv R(0) \supset R(1) \supset \cdots \supset R(t) \supset R(t+1) \supset \cdots \supset R(T) \equiv \emptyset
\]

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We will call this a *foliation*. It is a feature of the present approach that we are free to allocate the nested set in this foliation any way we wish - even ways that would not correspond to our usual notion of time. However, we expect that certain foliations will be well behaved - namely those that correspond to a good choice of $t$. We define an *elementary time-slice*

$$R_t = R(t+1) - R(t) \tag{88}$$

The elementary time-slice $R_t$ consists of all the cards in $R$ between times $t$ and $t+1$. Note the notational difference between $R(t)$ and $R_t$. We use an argument to denote what happens after time $t$ and a subscript to denote what happens between times $t$ and $t+1$.

### 25.2 States evolving in time

We can write

$$p(t) \equiv p(R(t)) \tag{89}$$

for “the state at time $t$”. Given this state we can calculate probabilities for what happens at times after $t$. If we know the causaloid then can find omega sets $\Omega(t) \equiv \Omega[R(t)]$. The components of the state are $p_k(t)$ with $k(t) \in \Omega(t)$. The notation $k(t)$ is a little unnatural. $k_t$ would be more natural, but we reserve this for elementary time-slices. We understand the $t$ argument on $k(t)$ to tell us that these $k$’s are in the omega set $\Omega(t)$.

The state will evolve from time $t$ to time $t+1$. The transformation will depend on what was done and what was seen in the elementary time slice $R_t$. We denote this by $(X_t, F_t)$ and have the associated vector $r_{(X_t, F_t)}(R_t)$. Since $R(t+1) = R_t \cup R(t)$ we can calculate the components of $p(t+1)$ from $p(t)$.

$$p_{k(t+1)} = r_{k(t+1)}(t+1) \otimes^A r_{(X_t, F_t)}(R_t) \cdot p(t) \tag{90}$$

where $k(t+1) \in \Omega(t+1)$. Hence we can write,

$$p(t+1) = Z_{(X_t, F_t)}(t+1, t)p(t) \tag{91}$$

where the elements of the $|\Omega(t+1)| \times |\Omega(t)|$ real matrix $Z$ are given by

$$Z_{(X_t, F_t)}(t+1, t)_{k(t+1)k(t)} = [r_{k(t+1)}(t) \otimes^A r_{(X_t, F_t)}(R_t)]_{k(t)} \tag{92}$$

We can calculate these from the causaloid. We can label each $(X_t, F_t)$ in the elementary time-slice $R_t$ with $\alpha_t$. Thus will write the transformation matrix for time $t$ to $t+1$ as $Z_{\alpha_t}$ or as $Z_t$ if we are suppressing the $\alpha$’s.

Since there is only one preparation for $R$ (namely that implicit in the condition $C$) we know that the state at time $t = 0$ is

$$p(0) \equiv p(R) = (p_1) \tag{93}$$

where $(p_1)$ is a single component vector. Hence, we can calculate a general state by

$$p(t) = Z(t, 0)p(0) \tag{94}$$

$$37$$
with
\[ Z(t, 0) = Z_t Z_{t-1} \cdots Z_0 \]  
(95)

We see that the causaloid provides us with a notion of a state evolving and tells us how to evolve it. The only quantity left undetermined by the causaloid is the component \( p_1 \). But this will cancel when we use Bayes formula to calculate conditional probabilities in \( R \) and so need not be determined.

### 25.3 Obtaining lambda matrices from transformation matrices

We can write \((X_R, F_R)\) in \( R \) as the union of \((X_t, F_t)\) in \( R_t \) over \( t = 0 \) to \( T - 1 \) or denote it with the collection of \( \alpha \) labels \( \alpha_{T-1} \cdots \alpha_0 \). Then

\[
\text{prob}(X_R, F_R) = \text{prob}(\alpha_{T-1} \cdots \alpha_0) = r_{\alpha_{T-1}} Z_{\alpha_{T-2}} \cdots Z_{\alpha_0} \cdot p(0) \]  
(96)

For notational simplicity we perform the following replacements

\[ r_{\alpha_{T-1}} \rightarrow Z_{\alpha_{T-1}} \text{ and } Z_{\alpha_0} \cdot p(0) \rightarrow Z_{\alpha_0} \]  
(97)

Hence we have (suppressing most \( \alpha \)'s)

\[
\text{prob}(X_t^\alpha \cup X_{R-R_t}, F_t^\alpha \cup F_{R-R_t}) \equiv p_{\alpha_t} = Z_{T-1} Z_{T-2} \cdots Z_{\alpha_t} \cdots Z_0 \]  
(98)

Corresponding to the elementary region \( R_t \) is a state \( p_t \) with a generalized preparation \((X_{R-R_t}, F_{R-R_t})\) in the region \( R - R_t \). Note that the generalized preparation contains a part to the past of \( t \) and a part to the future of \( t + 1 \) (at this level the framework is similar to the time-symmetric formulation of Aharonov, Bergmann and Lebowitz [14]). The state \( p_t \) has components

\[ p_{k_t} = Z_{T-1} \cdots Z_{k_t} \cdots Z_0 \]  
(99)

We can write an arbitrary \( Z \) at time \( t \) in terms of the linearly independent set \( Z_{k_t} \) with \( k_t \in \Omega_t \). That is

\[ Z_{\alpha_t} = \sum_{k_t \in \Omega_t} \rho_{k_t}^{\alpha_t} Z_{k_t} \]  
(100)

Putting (99) and (100) into (98) gives

\[ p_{\alpha_t} = r_{\alpha_t} \cdot p_t \]  
(101)

as required. This justifies the use of \( r_{k_t} \) in (100) and it justifies labelling the linearly independent set of \( Z_{k_t} \)'s with \( k_t \in \Omega_t \). This means that we can write

\[ Z_{\alpha_t} = \sum_{k_t \in \Omega_t} \Lambda_{\alpha_t}^{k_t} Z_{k_t} \]  
(102)

using (14).
Now consider adjacent elementary time-slices $R_t$ and $R_{t+1}$. We have

$$p_{l_{t+1}l_t} = Z_{l_{t+1}} Z_{l_t} \cdots Z_{l_{t+1}} Z_{l_t} \cdots Z_0 \quad \text{for} \quad l_{t+1}l_t \in \Omega_{t+1} \times \Omega_t$$  \hspace{1cm} (103)

There will be a state $p_{l+1,t}$ for the composite region made from $R_t$ and $R_{t+1}$ associated with a generalized preparation in the region $R_t - R_{t+1} - R_t$. This state has components

$$p_{k_{t+1}k_t} = Z_{l_{t+1}} Z_{l_t} \cdots Z_{k_{t+1}} Z_{k_t} \cdots Z_0 \quad \text{with} \quad k_{t+1}k_t \in \Omega_{t+1,t}$$  \hspace{1cm} (104)

We can write

$$Z_{l_{t+1}} Z_{l_t} = \sum_{k_{t+1}k_t \in \Omega_{t+1,t}} r_{k_{t+1}k_t}^{l_{t+1}l_t} Z_{k_{t+1}} Z_{k_t}$$  \hspace{1cm} (105)

Putting (104) and (105) into (103) gives

$$p_{l_{t+1}l_t} = r_{l_{t+1}l_t} \cdot p_{l_{t+1},t}$$  \hspace{1cm} (106)

Hence we can write

$$Z_{l_{t+1}} Z_{l_t} = \sum_{k_{t+1}k_t \in \Omega_{t+1,t}} \Lambda_{l_{t+1}l_t}^{k_{t+1}k_t} Z_{k_{t+1}} Z_{k_t}$$  \hspace{1cm} (107)

using (61).

This method will also work for more than two sequential elementary time-slices. In general

$$Z_{l_{t+\tau}} \cdots Z_{l_{t}+\tau} Z_{l_t} = \sum_{k_{t+\tau}k_t \in \Omega_{t+1,t}} \Lambda_{l_{t+\tau}l_{t+\tau}}^{k_{t+\tau}k_t} Z_{k_{t+\tau}} \cdots Z_{k_t} Z_{k_t}$$  \hspace{1cm} (108)

for $\tau$ sequential regions $R_t$ to $R_{t+\tau}$.

If a theory provides transformation matrices $Z$’s we can use (102) and (105) to obtain lambda matrices $\Lambda_{l_{t+\tau}l_{t+\tau}}^{k_{t+\tau}k_t}$ and $\Lambda_{l_{t+\tau}l_{t+\tau}}^{k_{t+\tau}k_t}$. However, this is not sufficient to fully specify the causaloid (or even the open causaloid) since (i) it does not tell us how to calculate lambda matrices for a non-sequential set of elementary time-slices and (ii) elementary time-slices may contain many elementary regions $R_x$ and we do not know how to obtain lambda matrices between these. Up till now everything we have done has been quite general. In particular, all this works for any choice of nested regions $R(t)$ or, equivalently, for any choice of disjoint elementary time-slices $R_t$. To deal with (ii) we need to add spatial structure which we will deal with later. In the mean time we would like to make progress on (i) and to that end we will introduce some extra assumptions which are true in QT and CProbT for the natural choice of time slicing.

### 25.4 Some assumptions

We make two assumptions which happen to be true in CProbT and QT.

**Assumption 1:** We assume $|\Omega(t)| = K = \text{constant}$ for all $t$ except the end points $t = 0$ and $t = T$ where we must have $|\Omega(0)| = |\Omega(T)| = 1$. 

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**Assumption 2:** We assume that \(|\Omega_t| = |\Omega(t)||\Omega(t+1)|\) so that we have the maximum possible number of linearly independent matrices \(Z_{k_t}\).

The first assumption follows from symmetry under time translations (except at the end points). Both assumptions taken together imply that \(|\Omega_t| = K^2\) for \(t = 1\) to \(T - 1\) and \(|\Omega_{0,T-1}| = K\) for the first and last time-slice. Consequently the non-end point transformation matrices are \(K \times K\) (that is square matrices) and the matrices at the end points are \(1 \times K\) for \(R_{T-1}\) and \(K \times 1\) for \(R_0\) (that is they are row and column vectors respectively).

Now consider two non-sequential time-slices \(R_t\) and \(R_{t'}\) with \(t' > t + 1\). We have

\[
p_{l_t^t l_t} = Z_{T-1} \cdots Z_{l_t^t} Z(t' - 1, t + 1) Z_{l_t^t} \cdots Z_0\quad \text{for}\quad l_t^t, l_t \in \Omega_t \times \Omega_t
\]

where \(Z(t' - 1, t + 1)\) is the transformation from \(t + 1\) to \(t' - 1\). We can define the linear operator \([Z_{l_t^t}, Z_{l_t}]\) by

\[
[Z_{l_t^t}, Z_{l_t}] Z(t' - 1, t + 1) = Z_{l_t^t} Z(t' - 1, t + 1) Z_{l_t}
\]

It can be shown that it follows from assumptions 1 and 2 that

\[
[Z_{l_t^t}, Z_{l_t}]\quad \text{for}\quad l_t^t, l_t \in \Omega_t \times \Omega_t\quad \text{are linearly independent}
\]

Hence,

\[
\Omega_{t,t'} = \Omega_{t'} \times \Omega_t
\]

where \(\Omega_{t,t'}\) is the omega set for region \(R_{t'} \cup R_t\). When omega sets multiply then so do lambda matrices (see (110)). Hence,

\[
\Lambda_{l_t^t, l_t}^{k_t^t} = \Lambda_{l_t^t}^{k_t^t} \Lambda_{l_t}^{k_t}
\]

where \(t' > t + 1\). This result generalizes in two respects. First it works if we replace the elementary time slices by clumps of any number of sequential time slices. For example

\[
\Lambda_{l_t^t, l_t}^{k_t^t k_t k_t -1} = \Lambda_{l_t^t}^{k_t^t} \Lambda_{l_t l_t}^{k_t k_t -1}
\]

where \(t' > t + 1\). Second, it works for more than two non-sequential clumps. For example,

\[
\Lambda_{l_t^t l_t l_t l_t}^{k_t^t k_t k_t k_t -1} = \Lambda_{l_t^t l_t l_t l_t}^{k_t^t k_t -1} \Lambda_{l_t l_t l_t l_t}^{k_t k_t -1}
\]

where \(t'' - 1 > t' > t + 1\) (so we have non-sequential clumps). This second generalization requires proving a generalization of (111).

We can now summarize as follows. If we have a physical theory for which there exists a choice of nested subsets \(R(t)\) such that the above two assumptions are satisfied and if this theory provides us with transformation matrices \(Z\) then we can use the above results to calculate lambda matrices for all unions of elementary time-slices. Given an arbitrary such union of elementary time-slices we proceed as follows. First we use equation (108) to obtain, from the \(Z\)'s, a lambda matrix for each clump of sequential time-slices. We then multiply these lambda matrices to get the desired lambda matrix. We can also obtain local lambda matrices, \(\Lambda_\alpha^k\), for each elementary time-slice using (102).
25.5 Calculating lambda matrices from more basic lambda matrices

In the previous subsection we need to use $Z$’s to calculate lambda matrices. We would like to leave the $Z$’s behind since, from our point of view, they belong to a less fundamental way of thinking about this structure. In this subsection we will show how we can get lambda matrices for arbitrary unions of elementary time-slices (excluding the first and the last elementary time-slice) starting only with local lambda matrices, $\Lambda^k_{t,i}$, for each $R_t$ and lambda matrices, $\Lambda^{k_{t+1},k_t}_{t+1,i}$, for pairs of sequential $R_t$. But to do this we need to add one more assumption.

**Assumption 3:** We assume that one at least one of the allowed transformation matrices, $Z_t$, for each elementary time-slice (except for the first and the last), $R_t$, is invertible so $Z_t^{-1}$ exists.

Note, we do not require that $Z_t^{-1}$ is in the set of allowed transformation matrices. In $R_t$ we have the fiducial matrices $Z_{k_t}$ with $k_t \in \Omega_t$. We let

$$\Omega_t = (1, 2, 3, \ldots, L) \quad \text{for} \quad 0 < t < T - 1$$

where we have $L = K^2$. Employing the above assumption, we can, without loss of generality, choose the first fiducial matrix, $Z_1$, to be invertible for each elementary time-slice except the first and last. Now consider a clump of sequential elementary time-slices from $R_t$ to $R_{t'}-1$ (with $t > 0$ and $t' < T$) where we implement $Z_1$ for each elementary time-slice except the first in the clump where we implement $Z_{k_t}$. The corresponding matrices

$$Z_1Z_1\ldots Z_1Z_{k_t} \quad \text{with} \quad k_t \in \Omega_t$$

form a linearly independent set in terms of which we can expand general transformations $Z(t', t)$ from $t > 0$ to $t' < T$. Hence, we can say that a fiducial set is given by

$$\Omega_{t,t+1,\ldots,t'-1} = (111\ldots1, 211\ldots1, \ldots, L11\ldots1)$$

It is very simple to verify that these omega sets satisfy the conditions given for 77 and its generalizations (such as 78) to hold. Hence, using this method we can calculate the lambda matrix for an arbitrary sequential clump using just the lambda matrices for pairwise sequential regions (so long as we exclude the first and the last elementary time-slices). We can then apply the methods of the last section to get the lambda matrix for a completely arbitrary set of elementary time-slices (though still excluding the first and last elementary time-slice).

25.6 A basic causaloid diagram

Assume that the elementary time-slices are, in fact, elementary regions. Then we have developed sufficient machinery to calculate the open causaloid (where the first and last elementary time-slices are regarded as being in the boundary region). We can summarize this with a diagram, which we will call a causaloid diagram. This diagram is shown in Fig. 6(a) consisting of nodes, links, and a sisterline whose significance is the following.
Figure 6: A causaloid diagram for (a) a single system (b) two interacting systems, and (c,d) a number of systems interacting.
1. **Nodes** correspond to elementary regions $R_t$ and are dressed with $\Lambda_{\alpha t}^{k_t}$. From this lambda matrix we can deduce the omega set $\Omega_t = (1, 2, \cdots, L)$ associated with the node.

2. **Links** join sequential regions and are dressed with $\Lambda_{t,t+1}^{k_t k_{t+1}}$. From this lambda matrix we can deduce the omega set $\Omega_{t,t+1} = (11, 21, \cdots, L1)$ associated with the link.

3. The **sisterline** (the thin line running along the side) denotes the set of omega sets $\Omega_{t,t+1,\cdots,t+\tau} = (111\cdots1, 211\cdots1, \cdots, L11\cdots1)$. This line is to the right as we go up. This corresponds to the direction implicit in this set of omega sets.

We exclude the first and last elementary time-slices from this diagram. Using the causaloid diagram we can determine the open causaloid since we can calculate any lambda matrix (except those pertaining to the first and last elementary time-slices). A lambda matrix for arbitrary $O$ can be calculated using the clumping method obtained above. We first identify all clumps of sequential $t$’s in $O$. Then we use (77) and its generalizations (such as (78)) to calculate lambda matrices for each sequential clump. Then we multiply these lambda matrices to get the lambda matrix for $O$. For clumps consisting of a single member we use (63) before multiplying the lambda matrices.

If we are starting with a theory which is expressed in terms of transformation matrices (we regard such a theory as less fundamental) then we can calculate the lambda matrices for nodes and links using (102) and (107). We will show how to do this for CProbT and QT in a later section. Once we have these lambda matrices we can disregard the transformation matrix formalism and work with the causaloid formalism instead (as long as Assumptions 1, 2, and 3 are true).

### Adding spatial structure

To add spatial structure we will use the notion of interacting **systems**. We will label systems with $i = 1, 2, \ldots$. We can regard the situation depicted in Fig. 4(a) as corresponding to a single system. Now consider the causaloid diagram shown in Fig. 4(b). This depicts what we can regard as two systems interacting. We label these systems $i$ and $j$. These labels become attached to the corresponding sister lines. We have two types of node. Nodes at crossing points (there is only one such node in Fig. 4(b)) and nodes at non-crossing points. We can think of crossing points as having two systems present which may (depending on what local procedure is carried out) be interacting. More complicated situations involving several interacting systems are shown in the causaloid diagrams in Fig. 4(c,d). For simplicity we will restrict the maximum number of systems in any given elementary region to two so we never have more than two systems crossing through a node. The methods we will present can quite easily be generalized to situations in which we relax this constraint.
The nodes are labelled by \( x \) which we think of as a space-time label. For each system we have a sequence of \( x \)'s (those picked out by the corresponding sister line). Our intention is to find a way to go from theories (like CProT and QT) which have transformation matrices to the causaloid formalism. Thus, for each system we have a sequence of matrices \( Z^i_x \) and local omega sets \( \Omega^i_x \). We now introduce the following assumption

**Assumption 4:** The matrices \( Z^i_{k_l^l} \equiv Z^{i,k_l^l} \) with \( k_l^l \in \Omega^i_x \times \Omega^j_x \) form a complete fiducial set at any crossing node, \( x \). Here \( \otimes \) is defined in the usual way (so that \((A \otimes B)(C \otimes D) = AC \otimes BD\)).

It follows from Assumption 4 that we can write a general transformation matrix at a crossing node as

\[
Z_{\alpha_x} = \sum_{k_l^l \in \Omega^i_x \times \Omega^j_x} \Lambda^{k_l^l}_{\alpha_x} Z^{i,k_l^l} \otimes Z^{j,k_l^l} \quad (119)
\]

Thus, at the crossing node \( x \) we have

\[
\Omega_x = \Omega^i_x \times \Omega^j_x \quad (120)
\]

This equation implies that \( |\Omega_x| = |\Omega^i_x| |\Omega^j_x| \). We can interpret this to mean that when two systems are put together the number of properties is simply the product of the number of properties of each system - we do not lose or gain properties. Equation (119) tells about how systems interact. If we can write

\[
\Lambda^{k_l^l}_{\alpha_x} = \Lambda^{k_l^l}_{\alpha^i_x} \Lambda^{k_l^l}_{\alpha^j_x} \quad (121)
\]

then we can write all transformation matrices at \( x \) as \( Z^{i,j}_{\alpha_x} \otimes Z^{j,i}_{\alpha_x} \) and consequently the two systems actually cannot interact. It would further follow that we can always write

\[
(X_{R_x}, F_{R_x}) = (X_{R^i_x} \cup X_{R^j_x}, F_{R^i_x} \cup F_{R^j_x}) \quad (122)
\]

at \( R_x \) and, hence, that we can regard anything we might do in \( R_x \) as being composed of separate actions on the two systems. In this case we might as well regard \( x, i \) and \( x, j \) as corresponding to separate elementary regions - there is no reason to have a crossing node. Interaction at \( R_x \) requires that \( \Lambda^{k_l^l}_{\alpha_x} \) does not factorize (i.e. this is required for something interesting to be going on at the crossing node \( x \)). In particular, it means that there are some \((X_{R_x}, F_{R_x})\) at \( R_x \) which cannot be regarded as being composed of separate actions on the two systems. It is interesting that we have a form of interaction between two systems here even though the omega sets multiply simply because the local lambda matrix does not factorize.

If local omega sets multiply for two systems at a crossing node then it is reasonable that omega sets for larger regions for two systems will multiply. Thus, we assume

**Assumption 5:** Omega sets \( \Omega^i_{\mathcal{O}_1} \) and \( \Omega^j_{\mathcal{O}_2} \) multiply where \( i \neq j \) and \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) may overlap.
With this final assumption in place we will be able to calculate the open causaloid. First let us summarize.

1. Each **non-crossing node** is dressed with $\Lambda_{x_i}^{k_i}$. The $i$ label is that of the sisterline passing by this node. From the lambda matrix we can deduce the omega set $\Omega_{x_i} = (1, 2, \cdots, L_i)$ associated with the node.

2. Each **crossing node** is dressed with $\Lambda_{x_i}^{k_i,k_j}$. The $i$ and $j$ labels are those of the sister lines crossing by this node. From the lambda matrix we can deduce the omega set $\Omega_{x_i} \times \Omega_{x_j}$ associated with this node.

3. Each **link** is dressed with $\Lambda_{x_i}^{k_i,k_i'}$. The $i$ is that of the sister line running along side this link. The $x$ and $x'$ are those of the nodes at either end of the link. From this lambda matrix we can deduce the omega set $\Omega_{x,x'}$ associated with this link.

4. Each **sister line** is associated with a system and has a label $i$. Associated with the sister line is a set of omega sets

$$\Omega_{x,x',\cdots,x''}^i = (11 \cdots 1, 211 \cdots 1, \cdots, L_11 \cdots 1)$$

for system $i$ here $x, x', \ldots, x''$ are sequential nodes along the line which have the line running to the right as we go along the sequence.

Basically we have a lambda matrix for each node and each link along with some rules about how omega sets for composite regions are formed. From these we can calculate the lambda matrix for an arbitrary composite region $R_O$ as follows.

1. Identify all clumps of nodes in $O$ which are sequential along a given sister line $i$. For each of these clumps apply the procedure outlined in the previous section - namely applying (78) and its generalizations (such as (79)) to obtain lambda matrices for system $i$ for these clumps. For clumps consisting of a single node we have $\Lambda_{x_i}^{k_i} = \delta_{x_i}^{k_i}$ as in (88).

2. Repeat this for each sister line.

3. Now multiply all the lambda matrices calculated in steps 1 and 2 to get the lambda matrix for $R_O$.

The local lambda matrices for the elementary regions are already given and hence we can calculate all lambda matrices. This means that the causaloid diagram dressed with lambda matrices in the manner described is a way of representing the open causaloid. To determine the open causaloid we need only specify a small subset of all the lambda matrices. We then have the above RULES for calculating other lambda matrices. We call these RULES the **clumping method**.

In physical theories it is typical that we have symmetry such that the lambda matrix associated with equivalent objects (non-crossing nodes, crossing nodes, or links) in the causaloid diagram could be identical. In this case we can represent the open causaloid by **just three lambda matrices** and the appropriate causaloid diagram.
27 Time, space, and systems

In the previous section we employed a picture of systems inhabiting space evolving in time and interacting when they meet at the same space-time location. This picture underpins much theoretical thought in physics. However, from the point of view of the causaloid formalism this picture need not be regarded as fundamental. Rather, it provides an organizing principle which is useful to calculate the causaloid. If we start with a suitable causaloid then we may be able to work backwards and derive this picture. If we regard the causaloid as fundamental then we should contend with the possibility that this picture is derived. Further, it may turn out that this picture is of limited or no use in calculating some causaloids (and maybe the causaloid for QG is such an example). We should therefore be wary of attempting to derive physical theories from this picture.

Already in the above examples the causaloid can have properties which weaken the notion of system. In particular, we note that it is possible to represent the same causaloid by different causaloid diagrams. We give a possible example in Fig. 7.

28 The causaloid for classical probability theory

We will consider a number of interacting classical bits. An example of a bit would be a ball which can be in box 1 or box 2.

First we consider a single bit. This has a state given by

\[ p(t) = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \]  

(123)
(see Sec. 6). We represent this system by a sequence of nodes labelled $t$ as in the causaloid diagram shown in Fig. 6(a). We can evolve the state by acting on it with a sequence of transformations $Z_{\alpha_t}$. These transformation matrices must satisfy the properties outlined in Sec. 6. To see what this entails first we can write

$$Z_{\alpha_t} = \begin{pmatrix} z_{11}^{\alpha_t} & z_{12}^{\alpha_t} \\ z_{21}^{\alpha_t} & z_{22}^{\alpha_t} \end{pmatrix}$$  (124)

We can interpret $z_{mn}^{\alpha_t}$ as the probability that the ball jumps to box $m$ and outcome $\alpha_t$ happens given that the ball is in box $n$. Hence,

$$0 \leq z_{mn}^{\alpha_t} \quad \text{and} \quad \sum_m z_{mn}^{\alpha_t} \leq 1 \quad \text{(125)}$$

For each value of the label $\alpha_t$ we will have a different realization of $Z$ consistent with these constraints. This space of allowed $Z$’s is continuous and hence the set of labels $\alpha_t$ will be infinite. However, $\alpha_t$ is supposed to label actual data that may be recorded on a card and so must actually belong to a finite set. Thus, in practice we will only include a finite set of possible $Z$’s in the set we can actually implement. Typically recording $\alpha_t$ will involve reading numbers off scales to some accuracy and the finite resolution in doing this will lead to a finite set.

The following allowed $Z$’s can serve as fiducial matrices

$$Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{(126)}$$

Note that $Z_1$ is invertible satisfying Assumption 3. In words, $Z_1$ is the identity transformation and leaves the system unchanged. $Z_2$ is the transformation associated with a measurement to see if the ball is in box 1 which leaves the ball in box 1 afterwards. This means that in a backwards time direction it also measures to see if the ball is in box 1. $Z_3$ is the transformation associated with a measurement to see if the ball is in box 1 and which flips it into box 2 if it is. In the backwards time direction it looks to see if the ball is in box 2 and flips it into box 1. $Z_4$ is similar to $Z_3$ with the box labels interchanged.

Something interesting has happened here. We have a classical bit and so the number of reliably distinguishable states is $N = 2$. However, since we are in a setting where the generalized preparation of region $R_t$ is both to the past and the future, we have $K_t \equiv |\Omega_t| = N^2$. In the quantum case we will have $K_t \equiv |\Omega_t| = N^4$. The point is that in this generalized preparation setting it is the transformation matrices $Z$ that map linearly to the $r$ vectors.

We can write

$$Z_{\alpha_t} = \sum_{k_t=1}^{4} \Lambda_{\alpha_t}^k Z_{k_t}$$  (127)

This equation can be solved to give the components of the local lambda matrix $\Lambda_{\alpha_t}^k$, in terms of the matrix elements of $Z_{\alpha_t}$.

$$\Lambda_{\alpha_t}^1 = z_{22}^{\alpha_t}, \quad \Lambda_{\alpha_t}^2 = z_{11}^{\alpha_t} - z_{22}^{\alpha_t}, \quad \Lambda_{\alpha_t}^3 = z_{21}^{\alpha_t}, \quad \Lambda_{\alpha_t}^4 = z_{12}^{\alpha_t} \quad \text{(128)}$$
The real importance of these expressions is given by the fact that we know the space of allowed \( Z_{\alpha t} \) (given in (125)). Given this we can now calculate the space of allowed \( \Lambda_{k t} \). The purpose of the \( \alpha_t \) label is then simply to label each element in this space.

Now consider two sequential times \( t \) and \( t + 1 \). An omega set is given by

\[
\Omega_{t+1,t} = (11, 12, 13, 14)
\]

(129)

We have

\[
Z_{l^t} Z_{l} = \sum_{k^t k \in \Omega_{t+1,t}} \Lambda_{l^t l}^{k^t} Z_{k^t} Z_k
\]

(130)

We can solve these equations explicitly to obtain \( \Lambda_{l^t l}^{k^t} \). Omitting the trivial elements (\( \Lambda_{l^t l}^{k^t} = \delta_{l^t l}^{k^t} \) for \( l^t, k^t \in (11, 12, 13, 14) \)) we have

\[
\begin{array}{cccccccccccc}
\Lambda & l^t l & 21 & 22 & 23 & 24 & 31 & 32 & 33 & 34 & 41 & 42 & 43 & 44 \\
k^t k & & & & & & & & & & & & \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
13 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
\]

(131)

We have now calculated the lambda matrices for nodes and links and hence we have enough to specify the open causaloid.

Now consider having a number of bits interacting according to one of the causaloid diagrams in Fig. 6(b-d). We switch to labelling nodes by \( x \) and label different systems by different \( i \). We have already calculated lambda matrices for non-crossing nodes and links. To calculate the lambda matrix for a crossing node we have to solve

\[
Z_{x \alpha x} = \sum_{k^i \in \Omega_{x} \times \Omega_{x}} \Lambda_{x \alpha x}^{k^i} Z_{k^i} \otimes Z_{k^i}
\]

(132)

The constraints on the space of \( Z_{x \alpha x} \) (namely that elements are positive and the sum of each column is no greater than 1) induces a constraint on the space of \( \Lambda_{x \alpha x}^{k^i} \) and the point of \( \alpha_x \) is simply to label elements in this space.

We have explicitly calculated the lambda matrix for a non-crossing node and for a link and shown how to calculate it for a crossing-node. Given these three lambda matrices and the causaloid diagram we have fully specified the open causaloid and therefore provided a complete predictive framework for CProbT.

29 The causaloid for quantum theory

We could proceed in an exactly analogous way in QT as we did in CProbT. Thus, we could take a number of interacting qubits. Of course, we don’t have to restrict ourselves to qubits. We could have systems whose Hilbert space
dimension is different from 2 and we could have systems of various Hilbert space dimension.

Thus, first let us consider a single system with associated Hilbert space of dimension $N$. We have a causoidal diagram as shown in Fig. 6(a). If we follow exactly the technique above then we would use $Z$ matrices for quantum theory. We saw in Sec. 8 how QT can be formulated with $Z$ matrices. However, we can instead proceed with superoperators, $\$, which are more familiar. In fact there is an invertible linear map between superoperators and $Z$ matrices (25) so we can switch between the two objects at any time. First we choose a fiducial set of linearly independent superoperators for each $t$

$$\$ \, \text{for} \quad k_t \in \Omega_t = (1, 2, \ldots, N^4) \quad (133)$$

Then, since there is a linear map between $Z$’s and $\$’s, we can write

$$\sum_{k_t \in \Omega_t} \Lambda^{k_t}_{\alpha_t} \$ k_t \quad (134)$$

instead (102). Similarly we can write

$$\$_{l_{t+1}} \$ _l_t = \sum_{k_{t+1}k_t \in \Omega_{t+1,t}} \Lambda^{k_{t+1}k_t}_{l_{t+1}l_t} \$ _{k_{t+1}} \$ _{k_t} \quad (135)$$

instead of (107) where

$$\Omega_{t+1,t} = (11, 12, 13, \ldots, 1N^4) \quad (136)$$

We can solve (154) to find the space of $\Lambda^{k_t}_{\alpha_t}$ from the known space of the superoperators. Then the $\alpha_t$ label is used to label each point in this space (or at least a large set of points consistent with the resolution of the experiment). We can also solve (155) to get the lambda matrix for pairs of sequential time-slices (which we are taking to be elementary regions). This matrix will be $|\Omega_t \times \Omega_t|$ by $|\Omega_{t+1,t}|$. That is it will be $N^8 \times N^4$. In the case of a qubit this is $256 \times 16$. This is a rather big object (though not too big). This size can be thought of as the price we pay for working in a framework (the causoidal formalism) capable of expressing any physical theory (at least any physical theory which correlates data as described in Sec. 15). We now know the lambda matrices for nodes and links and so have specified the open causoid for this quantum system of dimension $N$.

We can now consider many quantum systems interacting as shown in Fig. 6(b-d). The $i$th such system has Hilbert space dimension $N_i$. We switch to labelling nodes with $x$’s. The above calculations provide us with the lambda matrices for non-crossing nodes and for links. We only need to find the lambda matrix for crossing nodes. This is given by solving

$$\sum_{k_{l}k_{l}^{'}} \Lambda^{k_{l}k_{l}'}_{\alpha_{x}x} \$ _{k_{l}} \otimes \$ _{k_{l}'} \quad (137)$$

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which is obtained from (119). On solving this we obtain the space of $\Lambda^{k^i,k^j}$ from the known space of $'$s for the two systems. We label points in this space with $\alpha$ (up to the resolution of the experiment). Given this lambda matrix we now have the open causaloid and so can leave the usual quantum formalism behind.

30 The causaloid without boundary conditions

The causaloid is defined for a predictively well defined region $R$ with a condition $C$ on the cards outside $R$. The open causaloid is defined to exclude cards in the boundary region $R_O$. The idea is that condition $C$ is only relevant to this boundary region - conditional probabilities the remainder of $R$ are unaffected by $C$. If we are restricting our attention to $R - R_O$ then we might ask why we had condition $C$ in the first place. If we are not interested in the cards that go into verifying $C$ why even collect these cards? Put another way, can we simply identify $R - R_O$ with the full pack $V$? If we retrace our steps we can see that the reason we introduced $C$ was so we could have conditional probabilities of the form $\text{Prob}(X_R,F_R|C)$. Completely unconditional probabilities make no sense. However, although we use probabilities which are only conditioned on $C$ as intermediate steps in our construction of the open causaloid, when we use the open causaloid to calculate probabilities we are calculating the probability of something in $R - R_O$ conditionalized on something else that happened in $R - R_O$. The conditioning on $C$ is also implicit, but if we accept that conditional probabilities in $R - R_O$ are independent of $C$, then this conditioning on $C$ is actually redundant. This motivates us to now define the causaloid in the following way.

**The causaloid** for the full pack $V$ made up of elementary regions $R_x$ is, if it exists, defined to be that thing represented by any mathematical object which can be used to obtain $r_{\alpha}(R_O)$ for all measurements $\alpha$ in region $R_O$ for all $R_O \subseteq V$ where these $r$ vectors can be used to calculate conditional probabilities using (73).

When we say that we use (73) to calculate conditional probabilities we mean that we look to the case where the probabilities are independent of the state $p(R_1 \cup R_2)$ which basically means that we use (74) or (75).

If we look back at the two cases we have explicitly worked out, CProbT and QT, then we see that we can actually regard the open causaloid as the causaloid for the full pack $V$. By defining this object we have effectively removed the boundary condition and so have a much more useful object. The causaloid for the full pack $V$ fully characterizes the physical theory without the qualifications given in Sec. 24. However, we cannot actually be sure that there will exist such a causaloid. It is possible that conditions outside the region of the experiment will always influence conditional probabilities inside $V$ and, if these are not taken into account, we cannot have well defined probabilities in $V$ and so cannot have a causaloid for $V$. One way to avoid this would be to take the causaloid to correspond to the whole universe so there can be no possibility of influences.
31 Dynamic causal structure

The causaloid is a fixed object. Yet at the same time we have not assumed any fixed causal structure in deriving the causaloid formalism. That is to say we have not specified any particular causal ordering between the elementary regions. In this sense we must have allowed the possibility of dynamic causal structure. It interesting to see a little more explicitly how this can work in the causaloid formalism.

The best way to see this is as follows. Given a causaloid for $R$ (we could consider it for $R - R_{Ob}$, or $V$ instead) we can imagine that we have collected data $(X_{R-R}, F_{R-R})$ in region $R - \bar{R}$. We can regard this new data as conditioning, like $C$, for a new causaloid in $\bar{R}$. Let us call this new causaloid $\bar{\Lambda}$. We could alternatively imagine collecting different data, $(X'_{R-R}, F'_{R-R})$, in the same region $R - \bar{R}$ and obtain the causaloid $\bar{\Lambda}'$ for the same region $\bar{R}$. This is shown in Fig. 8. Both $\bar{\Lambda}$ and $\bar{\Lambda}'$ can be calculated from the original causaloid. Now, it is possible that the causal structure evident in $\bar{\Lambda}$ is quite different to that evident in $\bar{\Lambda}'$. For example, it could be that some subset of nodes in $\bar{R}$ has the causaloid diagram diagram shown in Fig. 9(a) if the causaloid is $\bar{\Lambda}$ whereas the same nodes have the causaloid diagram shown in Fig. 9(b) if the causaloid is $\bar{\Lambda}'$ (note though that a general causaloid cannot be represented by causaloid

Figure 8: Dynamic causal structure can manifest itself when the effective causaloid in $\bar{R}$ depends on the data in $R - \bar{R}$. If we do this we get into various issues such as what it means to have repeatability and what it means to take data for the whole universe. We will discuss these issues in Sec. 36.
Figure 9: Depending on data collected in some other region, the causal structure among these four nodes may be as in (a) or (b).

diagrams of this type).

Consideration of the causaloid diagrams for CProb T and QT leads us to the conclusion that these theories do not have dynamic causal structure of this type. We can see that this type of reconditionalization will not change the pattern of links. This supports our starting assertion that CProbT and QT have fixed causal structure.

Dynamic causal structure is likely to be quite generic for causaloids. However, it is unlikely to be as clear cut as the hypothetical example we just discussed. In general we cannot expect the sort of clear cut causal structure we see evident in the causaloid diagrams of Fig. 9. In general, the causal relationship between nodes may be more complicated than can be represented by pairwise links. Thus, when we speak of “causal structure” we do not necessarily intend to imply that we have well defined causal structure of the type that allows us to determine whether two nodes are separated by a time-like or a space-like interval.

The causaloid can be thought of as containing all potential causal structures. The particular causal structure that gets manifested depends on what data is collected and even after data has been collected it may not be useful or even possible to say retrospectively that two regions were time-like or space-like separated.
### 32 Problems with putting general relativity in the causaloid framework

General relativity is, ultimately an empirical theory concerning data and so it should be possible to put it in the framework here or one very similar to it. However, as things stand, there are a number of obstacles.

General relativity is based on a space-time continuum whereas the structure we have described, being based on actual data, is discrete. There are various approaches we could take to dealing with this. First, we could attempt to find a continuous version of the causaloid formalism. This would involve taking limits to pass from a discrete to a continuous structure. Alternatively, we could use a discrete version of GR such as the Regge calculus [15] (for other discrete approaches see [16], [17]). Since our ultimate objective is to formulate a theory of QG which is likely to be discrete and then show that GR emerges as a limiting case we might well be satisfied with a discrete version of GR. We will discuss issues pertaining to the continuum in the next section.

General relativity is deterministic. If general relativity had an external time we could use CProbT and form probabilistic mixtures of pure states. CProbT is well defined and so this would be a fairly straightforward matter. Deterministic GR would then simply correspond to the evolution of pure states in this framework. However, since we do not have external time this option is not open to us. Rather we have to exploit the causaloid structure. Despite the fact that a theory of quantum gravity would require us to address the issue of probabilities in the context of GR, there appears to be very little work in this area. One issue is that in GR we tend to find a solution for a whole space-time whereas probabilities are often understood in the context of a repeatable situation. Even if we could create many copies of the universe, we have only have access to data from one copy. This issue, at least, can be addressed (see Sec. 35).

We have introduced the notion of agency. We allow the possibility of alternative actions - such as the setting of knobs. Agency is actually a very common notion in physics. In Newtonian physics we speak of forces being applied. In quantum theory we think of performing different measurements. In both classical and quantum physics, we can remove the need for agency by, for example, including a full description of the agents in the Hamiltonian. Even so, we might ask whether we can really understand physical theories without some notion of agency. An equation in physics tells us what would happen were various counterfactual possibilities realized. In quantum theory the notion of agency is especially entrenched. The quantum state is, one might argue, best understood as a list of probabilities pertaining to incompatible measurements - that is pertaining to different possible actions - and is therefore difficult to interpret without agency. The notion of agency is not naturally incorporated into Einstein’s field equations. However, we could attempt to incorporate it by interpreting the eventual effects of tiny differences in $T_{\mu\nu}$ well below the experimental resolution we are working to as corresponding to different choices of an agent. For example, whether a knob is set in one position or another depends
on tiny differences in the brain of the experimentalist. A special case is where there is only ever one choice of action. And consequently no-agency is a special case of agency. This means, at least formally, there is no problem of putting GR into an agency based framework. However, if do not have active agency we lose something potentially quite important. The notion of causal structure is best understood from an agency point of view. Thus, we can ask whether measurement outcomes in one location depend on what action is implemented in another location. We cannot employ this understanding of causality if we do not have an active notion of agency.

In GR coordinates, \( x^\mu \), are usually understood to be abstract. It follows from the principle of general covariance that, having found a solution \((g_{\mu\nu}, T_{\mu\nu})\) on some manifold \(M\), the correct description of nature is given by forming the equivalence class over all diffeomorphisms of that manifold (smooth mappings of the points in \(M\) onto itself). The abstract coordinates, \( x^\mu \), themselves have no physical meaning. However, the \( x \) which appears on the cards in the causaloid formalism is an actual observation and must be read off an actual physical system. In one sense this is better. The cards have actual data on them and, in this sense, must correspond to observables. In some sense we are (from the point of view of GR) already working with diffeomorphism invariant objects. However, this leaves the problem of how to put GR in the causaloid formalism. One approach to this problem is to attempt to introduce actual physical coordinates into GR. Einstein, in a semi-popular account of GR [13], spoke of a “reference-mollusc” as a way of giving physical meaning to the abstract coordinates. Thus he imagined many small clocks attached to a non-rigid reference body such that infinitesimally displaced clocks have readings that are infinitesimally close (see also [18]). We could consider many different molluscs. Einstein then states “The general principle of relativity requires that all these molluscs can be used as reference-bodies with equal right and equal success in the formulation of the general laws.” Another physical reference frame is provided by the fact that the universe is, at a fine grained level, highly non-isotropic and inhomogeneous. Therefore the view from each point is different. This provides a way of physically labelling each space-time point (or at least each small region). A modern version of a physical reference frame is the GPS system. Any space-time point could be labelled with the retarded times received from four appropriately positioned clocks. Rovelli has considered how one might go about measuring the metric using such a GPS system [12]. General relativity has the property, in common with other pre-quantum physical theories, that it is fairly harmless to consider counterfactuals. Thus, even if there is no reference mollusc, we can consider counterfactually, what would have happened if there had been one. Rather than having \( T_{\mu\nu} \) we have \( T_{\mu\nu} + T_{\mu\nu}^{\text{mollusc}} \). If \( T_{\mu\nu}^{\text{mollusc}} \) is sufficiently small compared to \( T_{\mu\nu} \) then the two solutions of Einstein’s field equations will differ very little. Hence, we can draw empirical conclusions about the solution with no mollusc by looking at the solution with a mollusc. In quantum theory such counterfactuals are notoriously more tricky. Two solutions differing only in a single photon can be quite radically different. A successful attempt to find a theory of QG should embrace rather than shy away from this issue.
Newton invented differential calculus for the purpose of understanding the motion of particles. In doing so he considered a ratio \( \frac{\delta x}{\delta t} \) and took the limit as \( \delta t \to 0 \). There are two ways we might understand this limit. We might imagine that time is ontologically continuous and so it makes sense to consider smaller and smaller time intervals. Alternatively, we might take an operational approach. Thus, we might imagine that there is no limit to how accurately we can measure \( \delta x \) and \( \delta t \). If we take an operational approach then we know that there is certainly a practical limit to how accurately we can measure these intervals and, furthermore, there may be limits in principle.

In practice rather than directly measuring \( \delta x \) and \( \delta t \) to some incredible precision we often perform a measurement over a much longer time and extrapolate back to get a value of \( \frac{\delta x}{\delta t} \). For example, we may deduce the transverse velocity of a particle emerging from a hole in a card from the position, \( y \), it hits a screen placed at some distance. We will have some model of the motion of the particles which will allow this deduction. Tiny differences in this transverse velocity will correspond to big differences in \( y \). And this is really the key point. The reason we want to imagine that we have a well defined ratio \( \frac{\delta x}{\delta t} \) is that, even though it is essentially impossible to measure it as defined, our models predict that tiny differences in the ratio correspond to big differences at later times. There is a danger that the quantity \( \frac{\delta x}{\delta t} \) only derives it meaning from the model and this process of extrapolation and that it does not actually have either the ontological or operational meaning we allot to it.

Even though very small differences in \( \frac{\delta x}{\delta t} \) may not be measurable, the much bigger differences in \( y \) at the screen are. Ideally we would like a calculus which is not based on quantities that are not directly measurable (and may have no meaning) but is still predictively useful. How are we to account for the measurable differences in \( y \) other than in terms of small differences in some quantity \( \frac{\delta x}{\delta t} \) we cannot measure? In fact, in developing the causaloid formalism, we have already given an answer to this. We developed the notion of fiducial measurements. Thus, though we might be able to measure a large set of quantities, it is only necessary to measure a small fiducial set to deduce all the others. Thus, rather than relating \( y \) to a quantity that cannot actually be measured, we can simply relate it to other quantities like \( y \) which can actually be measured. The causaloid formalism gives a consistent way of doing this (though for probabilities). For example, we might imagine that we can apply various fields to particles emerging from the hole, place the screen at various distances, and so on. We would like to know the position the particle is detected on the screen for all these various things we might do. These various positions will be related and the causaloid formalism provides the appropriate mathematical machinery for relating them.

One reason we seek to define quantities like \( \frac{\delta x}{\delta t} \) is that we want to know the state at a given time \( t \). The usual notion of “state at time \( t \)” is that it pertains to some ontological state of affairs at time \( t \) which can be measured, in principle, at time \( t \) or at least within some short time \( t \) to \( t + \delta t \). But it is not
necessary that nature admits such notions at a fundamental level. And, even if nature does not admit these notions, we will still be able to do empirical science using the causaloid formalism. In the causaloid formalism the basic object, \( \Lambda \), is built out of lambda matrices. These matrices pertain to operationally defined elementary regions at the level of experimental apparatuses. We can use the causaloid to give meaning to the state at time \( t \) by choosing a foliation as in (87). The state at time \( t \) is \( p(t) \equiv p(R(t)) \) where \( R(t) \) contains everything of interest that comes after time \( t \). The fiducial measurements \( \Omega(t) \) are in \( R(t) \). A stronger constraint would be to demand that the fiducial measurements actually fall in \( R_t \equiv R(t+1) - R(t) \). That is we might demand that it be possible to establish the state at time \( t \) by measurements carried out during the small time interval \( t \) to \( t+1 \). This requirement, or something like it, would seem to be a feature of all current physical theories. However, there is no reason to demand it in principle. If we drop this constraint then move beyond the type of situation envisaged by Newton where the state at time \( t \) is specified by quantities measurable in principle during the time interval \( t \) to \( t + \delta t \). Thus, we see that the causaloid formalism provides us with a new calculus capable of dealing with situations where Newton’s differential calculus would be inappropriate.

The advantage of differential calculus and the implied ontology is that, where it works, it affords a simple picture of reality which allows significant symmetries to be applied. We can hope that increased familiarity with the causaloid approach may achieve something similar.

In classical physics, including GR, the distinction between ontological notions of space and time and the operational support for them is not an important one. There are no fundamental limitations coming from classical theories on how small apparatuses which might measure \( \delta x \) and \( \delta t \) can be. In QT the situation is a little more subtle. QT as applied to say systems of atoms does predict a scale which suggests a limit on how small apparatuses might be. However, we can imagine that there are other fields which can probe nature on a smaller scale and there is no limit coming from QT as to how small this scale might be. In these cases we can always imagine in principle probing on much smaller scales than the characteristic scales of the physics being considered. It is only when we get to QG that we hit in principle limits to our ability to probe nature directly at smaller and smaller scales. Any instrument used to probe at these small scales will necessarily have mass and energy. As \( \delta x \) and \( \delta t \) become smaller the associated energy and momentum will lead to black hole formation. This happens at the Planck scale. Thus in QG we expect for the first time a clear break down in our ability to give operational support for ontological notions of continuous time and space (see also [19]). We should be wary of introducing ontological notions which are not backed up, at least in principle, by operational procedures since we risk introducing factitious elements into our theory. Hence, in QG we expect that we will have to use a different calculus. The causaloid calculus provides a way forward here.

The formulation of ProbGR is likely to be useful in formulating QG. However, it is worth noting that this likely operational breakdown of the continuum distinguishes QG from ProbGR and constitutes an extra problem that we must
deal with.

It is often stated that experiments to test a theory of QG will involve probing nature at the Planck scale. It is no coincidence that apparatuses we might construct to do this would have to be very big. As illustrated above, postulated variation at a small scale shows up at a large scale and we might even doubt that there is any ontological meaning to talking about what is happening on this small scale. The fiducial measurements in the causaloid formulation for such an experiment will, we expect, be at a much larger scale than the Planck scale.

34 Ideas on how to formulate GR and QG in the causaloid framework

In the causaloid framework the lambda matrices that make up the causaloid tell us everything. In a formulation of GR they must therefore replace both $g_{\mu\nu}$ and $T_{\mu\nu}$. In this respect it is interesting to note that the two roles of $g_{\mu\nu}$ pointed out in Sec. 10 have analogues in the causaloid formalism. Thus, the local lambda matrices tell us about local physics as does the value of $g_{\mu\nu}$ at a point, and the composite system lambda matrices tell us about how elementary regions become correlated as does the connection $\Gamma^\alpha_{\mu\nu}$ (which depends on the local variation of $g_{\mu\nu}$). There are two approaches we could take to formulating GR in the causaloid framework. First we could attempt to put a discrete version of GR (such as the Regge calculus) in the framework. Secondly, we could attempt to rederive GR from scratch in this new framework perhaps by imitating appropriate aspects of Einstein’s original derivation of GR. The second approach is likely to be more fundamental though the first approach may help us gain important insights.

Data is collected on cards and is typically of the form $(x,a,s)$. We can think of $x$ as playing two roles. First, it provides a label that differentiates the elementary regions and second, it provides a local orientation in four-dimensional space-time. The action $a$ will typically have some direction associated with it. Thus, we might measure the spin along a certain axis. If we want think relativistically we should use a four vector $a_{\mu}$ to describe this measurement (this being measured relative to the local orientation provided by $x$). Similarly the outcome $s$ will also have a direction and should be denoted with a four vector $s_{\nu}$. The measurement $(X_{R_x}, F_{R_x})$ would therefore be associated with two four-vectors. It is reasonable, therefore, to suppose that a fiducial set is given just by taking such measurements associated with the sixteen components (as happens in tensorial analysis). Thus, we can select a subset of all measurements where $a$ and $s$ are orientated along the $\mu$ and $\nu$ directions respectively. This will lead to fiducial measurements $k_{\mu\nu}$. For each $\mu\nu$ there will be many such $k$’s corresponding to each possible value of $a$ and $s$. The situation may be more complicated since $a$ and $s$ may have more $\mu$-type labels. Motivated by the discussion in Sec. 26 we could attempt to formalize this by saying that in each elementary region $R_{x}$ there exist fields $k_{\mu}^{i}(x)$ where

$$\Omega(x) = \Omega^{r}(x) \times \Omega^{l}(x) \times \cdots \times \Omega^{n}(x)$$  \hspace{1cm} (138)
with
\[ \Omega^i(x) = (\text{All } k^i_{\mu\nu}(x) \text{ for given } x, i) \]  \hspace{1cm} (139)

Using suggestive notation we could have fields \( g_{\mu\nu} \) and \( T_{\mu\nu} \). We would then have local lambda matrices
\[ \Lambda_{g_{\mu\nu}}(x)T_{\sigma\tau}(x) \]  \hspace{1cm} (140)

and composite system matrices
\[ \Lambda_{h_{\mu\nu}}(x)h_{\sigma\tau}(x') \quad \text{and} \quad \Lambda_{S_{\mu\nu}}(x)S_{\sigma\tau}(x') \]  \hspace{1cm} (141)

where we use \( h \) and \( S \) to label the precompression elements in the product omega sets. We can form similar objects for composite systems composed of more than two elementary regions. Since we do not have fixed causal structure we do not expect the methods of Sec. 26 to work. If these initial steps are correct then the problem of formulating both GR and QG in this framework is to find the corresponding causaloids.

In general relativity, the principle of general covariance requires that the form of the laws, expressed as equations between tensorial objects, is invariant under general coordinate transformations. We could attempt to do something similar in the causaloid formalism. Thus, let us imagine that the causaloid is determined by solving some equations. Then there are two levels at which we could demand something like general covariance. First, at a general level. We can write down the causaloid with respect to an arbitrary set of omega sets for each region \( R_0 \). We could require that the equations which determine the causaloid take the same form for any choice of omega sets. Second, and less generally, we could just require that these equations are invariant under general transformations of the coordinates \( x^\mu \to x'^\mu \) which induces a transformation \( k^i_{\mu\nu}(x) \to k'^i_{\mu\nu}(x) \) of the local fiducial measurements, which, in turn, induces a transformation \( \Omega^i(x) \to \Omega'^i(x) \) in the local omega sets.

At least in the case of GR we should seek a way to implement the principle of equivalence. As a first stab at this we might require that there always exists a coordinate transformation inducing a transformation to local omega sets \( \Omega^i_{\text{GR}}(x) \) such that the local lambda matrices predict special relativistic physics in \( R_x \). However, since we are in a probabilistic context we have to admit the possibility that we have uncertainty as to what this local frame is. A sufficiently precise measurement in \( R_x \) should be able to establish what the set of local inertial frames is for each field \( i \). We can let this measurement be associated with a set of measurement vectors \( r^i_{g_{\mu\nu}} \) where \( g_{\mu\nu} \) labels the local inertial frames which leave \( g_{\mu\nu} \) in Minkowski form. The equivalence principle requires that the fields are correlated so that, when there is certainty as to what the local inertial frame is for each field, there is agreement. Thus,

\[ \text{If } r^i_{g_{\mu\nu}}(x) \cdot p(x) = 1 \text{ and } r^j_{h_{\mu\nu}}(x) \cdot p(x) = 1 \text{ then } h_{\mu\nu} = g_{\mu\nu} \]  \hspace{1cm} (142)

This should work in GR. However, in QG we expect to get pure states which are superpositions of other pure states. Thus, if we have \( p_1(x) \) and \( p_2(x) \)
each of which agrees with special relativistic physics but with different $g_{\mu\nu}$, then we expect other pure states corresponding to some sort of superposition of these for which there is no transformation to special relativistic physics. The implementation of the equivalence principle in QG would therefore seem to require a preferred basis with respect to which it holds.

If we are successful in formulating GR (actually ProbGR) in the causaloid framework we can then attempt to formulate QG. It is quite likely that some of the differences between ProbGR and QG mirror the differences between CProbT and QT and this might give us a strong handle on how to obtain QG from ProbGR. There are two key differences between CProbT and QT.

First, in CProbT we have $K = N$ whereas in QT we have $K = N^2$ (in the sense discussed in Sec. 9). Thus, if we are to build up a complete set of fiducial measurements in QG it is likely that we will want to add extra measurements to those GR. In fact the situation is a little more complicated for the causaloid than it was in Sec. 9. Since the preparation for the elementary region $R_x$ is both to the future and the past (and the sides) of $R_x$ it is the transformation matrices which map linearly to the $r$ vectors and hence, as discussed in Sec. 28, we have $|\Omega_x| = N^2$ in CProbT and $|\Omega_x| = N^4$ in QT. To get from CProbT to QT we need to add two new fiducial $r$’s for each pair of fiducial measurements in CProbT.

The second key difference is that QT has a continuous set of reversible transformations whereas CProbT has only a discrete set. This has the effect of filling out the space of pure states in QT. In QT we have unitary transformations. This is unlikely to survive in QG since we do not have a fixed background to evolve the state with respect to. Thus, information about the state is likely to leak out into the degrees of freedom which represent our frame of reference. For a transformation to be reversible on the other hand we require that no information about the state leaks out. However, we may have something which very well approximates reversible unitary transformation for sufficiently small regions of space time.

In the above discussion we used $g_{\mu\nu}$ and $T_{\mu\nu}$. However, operational quantities would be better than these. Operational quantities would involve the meeting of test particles, the behaviour of rays of light, and so on [20, 21, 22].

35 The universal causaloid

We have defined a few notions of causaloid: (i) the causaloid for a predictively well defined region $R$; (ii) the open causaloid; and (iii) the causaloid for the whole pack $V$ (this may not actually exist). We now wish to introduce a further notion - the universal causaloid. The motivation for this is to remove the need for repeatability. We repeat an experiment many times and bundle the cards from each run together forming a stack. The fact that we are able to bundle the cards separately indicates that, actually, there is some additional marker which could constitute recorded data that distinguishes the cards from one run to the next. For example, in the case of the probes floating in space illustrated in Fig.
we reset the clocks and take note of the fact that we have done this. All this is rather artificial. Why should we bundle our data into stacks coming from different repetitions of an experiment? It would, one might imagine, be more natural simply to collect one big stack of data which might or might not be regarded as coming from repetitions of an experiment. The problem with this is that it is unclear how we might interpret probabilities. The following approach seems reasonable. Associated with any proposition $A$ concerning the data that might be collected is some vector $r_A$. In testing the data to see whether $A$ is true we will be testing its truth among a complete set of mutually exclusive propositions $A, A', \ldots, A''$. We define $r_A^I = r_A + r_A' + \cdots + r_A''$. We will say that $A$ is true if
\[ r_A \approx r_A^I \] (143)
we use the symbol $\approx$ because we can never expect experimental data to give absolute support for a proposition. We can decide in advance just how exactly equal we require these two vectors to be.

To illustrate how this can apply to the case of probabilities consider the vectors
\[ r_n \equiv r_{(X_{1n}, F_{1n})} \otimes^A r_{(X_{2n}, F_{2n})} \] (144)
pertaining to in the disjoint regions $R_n \equiv R_{1n} \cup R_{2n}$ for $n = 1$ to $N$ where $N$ is big. Further, define
\[ r_n^I \equiv \sum_{Y_{1n} \subseteq F_1} r_{(Y_{1n}, F_{1n})} \otimes^A r_{(X_{2n}, F_{2n})} \] (145)
We define $r_n = r_n^I - r_n$. Now assume that
\[ r_n = p r_n^I \] (146)
for all $n$. We see that $r_n$ is like $v$ and $r_n^I$ is like $u$ of Sec. 20. In our previous language we would say that this means that the probability of $X_{1n}$ in $R_{1n}$ given see $X_{2n}$ in $R_{2n}$ and we perform procedure $F_{1n} \cup F_{2n}$ in $R_n$ is $p$. But we can turn this into a statement in the form of (143). Thus, consider
\[ r_A \equiv \sum_{(p-\Delta p)N < |S| < (p+\Delta p)N} \left( \bigotimes_{n \in S}^A r_n \right) \otimes^A \left( \bigotimes_{n \in \bar{S}}^A r_n \right) \] (147)
This is the vector corresponding to the property that $pN$ out of the $N$ regions $R_n$ have outcome $X_{R_n}$ to within $\pm \Delta pN$. We also have
\[ r_n^I \equiv \bigotimes_n^A r_n \] (148)
Using (146) we obtain
\[ r_A = \left( \sum_{(p-\Delta p)N < |S| < (p+\Delta p)N} [p^n(1-p)^{N-n}] \right) r_A^I \] (149)
From the theory of binomial distributions we have

\[
\sum_{(p-\Delta p)N<|S|<(p+\Delta p)N} [p^n(1-p)^{N-n}] \approx 1 - O(1/[\Delta p\sqrt{N}]) \quad (150)
\]

Hence, if \( N \) is sufficiently large, condition (143) is satisfied and we can say that the proposition is true.

This means that we do not need to make repeatability intrinsic to the definition of the causaloid. Rather, we can simply define a universal causaloid so that we can look for properties that are true (to within some small error). We define

**The universal causaloid** for a region made up of elementary regions \( R \) is, if it exists, defined to be that thing represented by any mathematical object which can be used to calculate vectors \( r_A \) for any proposition \( A \) concerning the data collected in these elementary regions such that if the proposition is true (to within some small error) we have \( r_A \approx r_A' \) where \( r_A' = r_A + r_{A'} + \cdots + r_{A''} \) and \( A, A', \ldots, A'' \) is a complete set of mutually exclusive propositions.

We see that using this object we can recover the notion that we have probabilities by using the argument above in reverse (though see [23] for a cautionary tale on this subject). However, the universal causaloid is potentially a richer object. We can formulate many questions about the data as "is proposition \( A \) true?" pertaining to situations where we have not repeated the same experiment many times. The universal causaloid should enable us to answer all such questions.

One problem with the universal causaloid is that we cannot directly measure it (unlike the earlier causaloids we defined) since we do not have repeatability. It is repeatability that allows us to obtain probabilities for different procedures and hence calculate the lambda matrices. However, we can suppose that there are certain symmetries and deduce the causaloid that way. The causaloid for CProbT and QT will simply be that found by allowing a causaloid diagram such as that in Fig. 6(c) to extend indefinitely to include possible repetitions of the experiment. We may imagine that the causaloid extends indefinitely into the future and arbitrarily far into the past. Indeed, we might think of the universal causaloid as corresponding to the entire history of the universe (this would be essential if we want to consider cosmology). However, we have the problem that we cannot expect to collect cards from such an arbitrarily large region and send them to a sealed room. The universal causaloid, as a mathematical object, transcends the limited domain in which the causaloid was first conceived. This can be regarded as removing some of the operational scaffolding we had originally erected to help find a more general probability calculus. Having removed this scaffolding we are still able to use the universal causaloid to make predictions.
36 The principle of counterfactual indifference

As we develop the causaloid formalism we should look for simplifying assumptions. One possible such assumption is the following.

**The principle of counterfactual indifference** states that the probability of $E$ does not depend on what action would have been implemented had $E'$ happened instead if we condition on cases where $E'$ did not happen (as long as the device implementing this action is low key).

For example, imagine Alice tosses a dice then a coin. Then the probability that a coin comes up heads cannot depend on the fact that had a six come up she intended to bend the coin in a particular way if we only consider those cases where a six did not come up. Indeed, in a different procedure she might have intended not bend the coin had the six come up. If the principle of counterfactual indifference were false in this case, then somebody could deduce Alice’s intention from data in which a six never comes up and where she never implements her intention. But this would contradict the principle of indifference to data since such intentions are part of the programming and correspond to the detail of the way in which information is stored (in this case Alice’s intentions as manifested in her brain). And indeed a little thought shows that the principle of counterfactual indifference is a consequence of the principle of indifference to data (given that low key physical systems are used to process data). The principle of counterfactual indifference implies

$$r(x_1,f_1) = r(x_1,f'_1) \quad \text{where} \quad X_1 \subseteq F_1,F'_1$$

since both procedures $F_1$ and $F'_1$ amount to doing the same thing if we see $X_1$. We do, indeed, have this property in CProbT and QT.

37 Comparison with other approaches to QG

The approach outlined in this paper aims at finding a framework for a theory of QG. As such, it is quite possible that other approaches to QG will fit within this framework. However, there are two key aspects of the causaloid framework which should be compared with other approaches.

1. We deal with data that may be collected in actual experiments. The approach here is “top down” rather than “bottom up”. All other approaches to QG start with some ideas about the structure of space and time at very small scales (usually the Planck scale) and then attempt to build up.

2. The causaloid formalism is more general than quantum theory. We can attempt to treat QT and GR in an even handed way rather than requiring GR to fit fully in the framework of QT. Most other approaches to QG take the basic form of QT unchanged.
The two main approaches to QG are string theory \[24\] and quantum loop gravity \[25, 26, 27\] though there are other approaches.

String theory assumes a fixed non-dynamical background space-time and attempts to obtain a perturbative version of quantum gravity (along with the other fundamental forces). It is difficult to see how, when we go beyond the pertubative domain, this approach could have truly dynamic causal structure as it must. The basic picture appears to be that of unitary evolution in standard quantum theory. Thus, string theory fits in the usual quantum framework.

Quantum loop gravity is canonical approach. A canonical formulation of GR is quantized. Thus, it is fundamentally a 3 + 1 approach and this appears to be the conceptual origin of some of the mathematical problems faced by the program. In treating space and time on a different footing we break the elegance of Einstein’s fundamentally covariant approach (this seems to be ok for canonical QG if we adopt the Newton-Cartan approach where \(c \rightarrow \infty\) and we have natural 3 + 1 splitting \[28\]). Rather than forcing GR into a canonical framework it seems more natural to require that QT be put in a manifestly covariant framework. This appears to be problematic since the notion of a state across space evolving in time is basic to the usual formulation of QT. However, the causaloid framework allows us to go to a more fundamental formalism in which we do not have a state evolving in time.

A new approach emerging from the theory of quantum loop gravity is the spin-foam approach (see \[29\] for a short review). In this approach spin-foams represent four dimensional histories in space time. The evolution between two times is represented by an amplitude weighted sum over such spin-foams. In this approach we see graphs dressed with matrices. However, the causaloid diagrams are quite different since they are fixed for a given theory. The notion of a history is clearly better than that of a state at a given time so far as providing a manifestly covariant formulation is concerned. However, a history is a rather big thing. It involves all possible events between the two times of interest. The causaloid formalism deals with matrices between elementary regions. In the case that there exist RULES we may only need to specify local lambda matrices and lambda matrices for pairs of regions (as in QT). This is closer to Einstein’s original approach than providing an amplitude for an entire history is.

Another approach is the causal set approach of Sorkin \[30\]. In this approach the fundamental notion is of points with causal relations between them. Causal sets are taken to obey certain axioms so that they form partially ordered sets. These sets are represented by a graph with nodes joined by links. The partially ordered sets of the causal set approach are actually very different to the causaloid for various reasons. First, these partially ordered sets are supposed to provide a picture of space-time at the Planck scale. Second, the links are not dressed with matrices and so causal relationship between the points is not as rich as that between elementary regions in the causaloid formalism. And third, a given causal set is meant to provide one possible history (rather like a spin-foam) whereas the causaloid is a fixed object (though one which contains a way of calculating probabilities for all histories).
A more recent approach is due to Lloyd [31]. He suggests that quantum gravity should be modelled by a quantum computation. In so doing he is able to implement dynamic causal structure. However, since the whole process is embedded in standard unitary quantum theory, there is still a background time.

Another approach is non-commutative geometry pioneered by Connes [32]. The basic idea is that operators like $\hat{x}$, $\hat{y}$, and $\hat{z}$, at a point do not commute. This appears to fit within the framework of QT as we have defined it (since we said nothing about commutation relations). In standard QT the operators $A \otimes I$ and $I \otimes B$ will commute by virtue of the way the tensor product is defined. In the causaloid formalism the analogous objects will not, in general, commute if $\otimes$ is replaced by $\otimes^A$. In this sense there may be a connection between the causaloid formalism and non-commutative geometry.

One problem which is common to most approaches which start with a Planck scale picture is that it is difficult to account for the four dimensional appearance of our world at a macroscopic level (Smolin calls this the “inverse problem” [33]). Since the approach in this paper starts at the macroscopic level, it may allow us to circumvent this problem in the same way Einstein does in GR. Thus, we would not attempt to prove that space-time is four dimensional at the macroscopic level but put this in by hand. This is not an option in Planck scale approaches to QG because the constraint that a four dimensional world emerges at the macroscopic scale has no obvious expression at the Plank scale.

The best approach, however, may be to combine an approach which posits some properties at a Planck scale with the causaloid approach. By working in both directions we might hope to constrain the theory in enough different ways that it becomes unique.

38 Conclusions

We have developed a framework for probabilistic theories which allow dynamic causal structure. Central to this is an object called the causaloid. This object is theory specific and we have calculated the causaloid for classical probability theory and quantum theory. We have not calculated it for GR though we presented some preliminary ideas. The results in this paper suggest the following program for finding a theory of QG.

1. Formulate probabilistic general relativity (ProbGR) in the causaloid framework. This will involve finding RULES to calculate the causaloid from a basic set of lambda matrices.

2. Construct $\mathbf{r}_{\alpha_x}$ for quantum gravity from the $\mathbf{r}_{\alpha_x}$ of ProbGR in such a way that we go from $|\Omega_x| = N^2$ to $|\Omega_x| = N^4$.

3. Find RULES for QG from the RULES for ProbGR.

A particular issue we will have to pay attention to is that in ProbGR we have continuous space-time whereas we do not expect to be able to give operational
support to this notion in QG. This may force us to be more radical in the
construction of QG than is suggested by the above three steps.

The causaloid formalism contains some elements in common with the Aharonov,
Bergmann, and Lebowitz (ABL) time-symmetric formulation of QT [14]. In-
deed, it may even be regarded as a radical generalization of the ABL formulation.
The ABL approach has led to a number of fascinating results where naive
reasoning leads to counterintuitive though correct results. For example, if a par-
ticle is tunnelling through a potential barrier then, when it is in the “forbidden”
region, one might naively reason that its kinetic energy (total energy minus po-
tential energy) is negative (even though it should always be positive). It turns
out [34] that a certain type of measurement of the kinetic energy (called a weak
measurement) will actually give negative readings if the state of the particle
is preselected in its half evolved state and postselected in the forbidden region
(using the ABL approach). The causaloid formalism might be expected to put
such counterintuitive properties in an even more general setting and this may
contribute to our understanding of them.

The approach taken here attempts to combine the early operational philo-
osophy of Einstein as applied to GR with the operationalism of Bohr as applied to
QT (see [35] for a discussion of how Einstein and Bohr might have engaged in a
more constructive debate). We do this primarily for methodological reasons to
obtain a mathematical framework which might be suitable for a theory of QG
without committing ourselves to operationalism as a philosophy of physics. In
fact it is interesting just how close this early philosophy of Einstein is to the
later philosophy of Bohr. Einstein said

The law of causality has not the significance of a statement as to the
world of experience, except when observable facts ultimately appear
as causes and effects [3].

and

All our space-time verifications invariably amount to a determina-
tion of space-time coincidences. (…) Moreover, the results of our
measurings are nothing but verifications of such meetings of the
material points of our measuring instruments with other material
points, coincidences between the hands of a clock and points on the
clock dial, and observed point-events happening at the same place
and the same time [3].

Bohr said

However far the phenomena transcend the scope of classical physical
explanation, the account of all evidence must be expressed in clas-
sical terms. (…) The argument is simply that by the word experiment
we refer to a situation where we can tell others what we have done
and what we have learned and that, therefore, the account of the ex-
perimental arrangements and of the results of the observations must
be expressed in unambiguous language with suitable application of
the terminology of classical physics [36].
While Einstein might have felt uncomfortable with the lack of ontological clarity of Bohr’s interpretation, there is a striking similarity between these sentiments. This underlines the power of operationalism as a methodology.

The formalism here was developed specifically in the hope of finding a theory of QG. However it may find application in other areas, even outside physics. Indeed it may be useful in any situation where there is reason to believe that a straightforward analysis in terms of a state evolving through time is inadequate. An example might be where we are trying to model the behaviour of a system that is better able to predict the future than we are. The behaviour of such a system would appear to depend on the future in ways we could not account for in a purely forward in time way and the causaloid formalism might be useful here. One possible example of a system of this nature would be the financial markets.

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