THE PEIRCE AXIOM SCHEME AND SUPREMA

P.L. ROBINSON

Abstract. We present equivalents to the Peirce axiom scheme and highlight its relevance for pairwise suprema.

With the conditional (⊃) as its only logical connective and with modus ponens (MP) as its only inference rule, the Implicational Propositional Calculus (IPC) is commonly founded on the following three axiom schemes:

\( (IPC_0) \quad ((A \supset B) \supset A) \supset A \quad (the \ Peirce \ axiom \ scheme); \)
\( (IPC_1) \quad A \supset (B \supset A) \quad (affirmation \ of \ the \ consequent); \)
\( (IPC_2) \quad [A \supset (B \supset C)] \supset [(A \supset B) \supset (A \supset C)] \quad (self-distributivity). \)

Our aim here is to offer equivalents for the Peirce axiom scheme and highlight a significant role that it plays within the IPC. A convenient DIY introduction to the IPC may be found on pages 24-25 of [4].

Throughout, we shall assume modus ponens and the axiom schemes \( IPC_1 \) and \( IPC_2 \). This comment applies in particular to the statement of theorems: MP, \( IPC_1 \) and \( IPC_2 \) will be implicit assumptions in each of them; whether or not the Peirce scheme \( IPC_0 \) is assumed will be stated explicitly. It is in this sense that our equivalents to the Peirce scheme should be understood: each of them is equivalent to \( IPC_0 \) in the presence of MP, \( IPC_1 \) and \( IPC_2 \).

It will be convenient to begin by recalling some elementary properties of IPC that do not require \( IPC_0 \). Perhaps the most useful of these is the Deduction Theorem (DT): if \( \Gamma, A \vdash B \) (that is, if there is a deduction of \( B \) from \( A \) and the set \( \Gamma \) of well-formed formulas) then \( \Gamma \vdash A \supset B \) (that is, there is a deduction of \( A \supset B \) from \( \Gamma \) alone); as a special case, if \( A \vdash B \) then \( A \supset B \) is a theorem. Because of this, we shall feel free to write \( A \vdash B \) and \( \vdash A \supset B \) interchangeably without mention. A particular consequence of DT is Hypothetical Syllogism (HS): \( A \supset B, B \supset C \vdash A \supset C \) as a derived inference rule; alternatively, if \( A \vdash B \) and \( B \vdash C \) then \( A \vdash C \). Another useful elementary fact is that \( A \supset A \) is a theorem of IPC without \( IPC_0 \); it typically appears as a step on the way to DT (from which metatheorem it otherwise follows at once).

One direction of the following equivalence already appears in [4].

**Theorem 1.** The Peirce axiom scheme \( IPC_0 \) is equivalent to the axiom scheme
\[ (IPC'_0) \quad (A \supset Q) \vdash [(A \supset B) \supset Q] \supset Q. \]

**Proof.** Assume \( IPC_0 \). HS yields
\[ A \supset Q, Q \vdash B \vdash A \supset B \]
whence DT yields
\[ A \supset Q \vdash (Q \supset B) \supset (A \supset B). \]
Consequently, from $A \supset Q$ and $(A \supset B) \supset Q$ as hypotheses, we deduce $(Q \supset B) \supset Q$ by HS. Now, $IPC_0$ gives $\vdash [(Q \supset B) \supset Q] \supset Q$ whence an application of MP produces  
$$A \supset Q, (A \supset B) \supset Q \vdash Q$$
and two applications of DT bring us to $IPC'_0$ as a theorem scheme.

Assume $IPC'_0$ as stated. With $Q$ replaced by $A$ this yields  
$$\vdash (A \supset A) \supset [(A \supset B) \supset A] \supset A.$$  
As $A \supset A$ is a theorem, an application of MP brings us to $IPC_0$ as a theorem scheme.  

The approach to our second equivalent for the Peirce axiom scheme starts from a simple observation within the classical Propositional Calculus. As regards its truth-functional nature, disjunction may be expressed in terms of the conditional: explicitly, all Boolean valuations agree on $A \lor B$ and $(A \supset B) \supset B$. This is one of the less obvious semantic relationships between logical connectives, among the many to which Smullyan draws attention in [5] and elsewhere. On this point, see also D4 in Section 11 of [1].

Accordingly, let us define (as an abbreviation)  
$$A \lor B := (A \supset B) \supset B.$$  
The following is called $\lor$-Introduction in [3].

**Theorem 2.** $A \vdash A \lor B$ and $B \vdash A \lor B$.

**Proof.** For the first deduction, MP gives $A, A \supset B \vdash B$ and then DT gives $A \vdash (A \supset B) \supset B$. The second amounts to an instance of $IPC_1$.

The result of the following theorem is called $\lor$-Elimination in [3].

**Theorem 3.** Assume the Peirce axiom scheme. If $A \vdash Q$ and $B \vdash Q$ then $A \lor B \vdash Q$.

**Proof.** The equivalent to the Peirce scheme presented in Theorem 1 justifies  
$$A \supset Q \vdash [(A \supset B) \supset Q] \supset Q$$
while an application of HS justifies  
$$B \supset Q, (A \supset B) \supset B \vdash (A \supset B) \supset Q.$$  
An application of MP now justifies  
$$A \supset Q, B \supset Q, (A \supset B) \supset B \vdash Q$$  
whence an application of DT justifies  
$$A \supset Q, B \supset Q \vdash [(A \supset B) \supset B] \supset Q$$
or  
$$A \supset Q, B \supset Q \vdash (A \lor B) \supset Q.$$  
Finally, if $A \vdash Q$ and $B \vdash Q$ then $\vdash A \supset Q$ and $\vdash B \supset Q$ so that $\vdash (A \lor B) \supset Q$ and therefore $A \lor B \vdash Q$.  

We may regard this theorem as providing formal justification, on the basis of the Peirce axiom scheme, for the inference rule ‘from $A \vdash Q$ and $B \vdash Q$ infer $A \lor B \vdash Q$’ or  

$$A \supset Q, B \supset Q$$

$$\vdash (A \lor B) \supset Q$$

which we shall refer to as $\lor$E (for $\lor$-Elimination).

We have thus established one direction in the following result, which presents a new equivalent to the Peirce axiom scheme; recall our standing assumption that MP, $IPC_1$ and $IPC_2$ are in force.
Theorem 4. The Peirce axiom scheme $IPC_0$:

$$[(A \supset B) \supset A] \supset A$$

is equivalent to the inference rule $\vee E$:

$$A \supset Q, B \supset Q \vdash (A \vee B) \supset Q.$$

Proof. For the reverse direction, assume the inference rule $\vee E$. Now, hypothesize $(A \supset B) \supset A$. This hypothesis has as a trivial consequence the theorem $A \supset A$. An application of HS to our hypothesis and the instance $B \supset (A \supset B)$ of $IPC_1$ yields the consequence $B \supset A$. Two applications of MP give

$$(A \supset B) \supset A, A \supset B \vdash B$$

whence DT reveals that our hypothesis also has $(A \supset B) \supset B (\equiv A \vee B)$ as a consequence. In summary, $(A \supset B) \supset A$ has as consequences $A \supset A, B \supset A, A \vee B$. Next, our new inference rule $\vee E$ with $Q := A$ provides the further deduction

$$A \supset A, B \supset A \vdash (A \vee B) \supset A.$$ 

Lastly, MP applied to the consequences $A \vee B$ and $(A \vee B) \supset A$ results in

$$(A \supset B) \supset A \vdash A$$

and by DT we arrive at $[(A \supset B) \supset A] \supset A$ as a theorem scheme. \hfill \Box

A reformulation of this new equivalent to the Peirce axiom scheme is of interest.

Let $wf$ denote the set comprising all well-formed formulas. Initially, assume only MP along with $IPC_1$ and $IPC_2$. Declare $A \in wf$ and $B \in wf$ to be (syntactically) equivalent precisely when both $A \vdash B$ and $B \vdash A$ (thus, precisely when both $A \supset B$ and $B \supset A$ are theorems). This evidently defines an equivalence relation $\equiv$ on $wf$; transitivity holds by virtue of HS. Write $L$ for the set of $\equiv$-classes in $wf$ and write $[A]$ for the $\equiv$-class of $A$.

The set $\mathbb{L}$ is partially ordered by declaring that $[A] \leq [B]$ precisely when $A \vdash B$ (equivalently, precisely when $A \supset B$ is a theorem); it is readily checked that this does indeed well-define a partial order. Notice that the poset $(\mathbb{L}, \leq)$ has a top element: namely, $[T]$ where $T$ is any theorem; if $A$ is any well-formed formula then $A \vdash T$ is a triviality. By contrast, $(\mathbb{L}, \leq)$ has no bottom element; semantically, a well-formed formula representing a bottom element must take the value 0 in any Boolean valuation, but each well-formed formula built from $\supset$ alone takes the value 1 when every propositional variable is assigned 1 as its value.

In these terms, we now have the following reformulation of Theorem 4 (with our usual standing assumption).

Theorem 5. The Peirce axiom scheme $IPC_0$ is equivalent to the requirement that each pair $[A], [B]$ in $\mathbb{L}$ has $[A \vee B]$ as its supremum.

Proof. Theorem 2 tells us that $[A]$ and $[B]$ have $[A \vee B]$ as an upper bound. Theorem 3 tells us that if the Peirce axiom scheme holds then $[A \vee B]$ is actually the least upper bound of $[A]$ and $[B]$. The reverse direction of Theorem 4 implies that if $[A \vee B]$ is the least upper bound of $[A]$ and $[B]$ then $IPC_0$ holds. This concludes the proof. \hfill $\Box$

Thus, for the Implicational Propositional Calculus, $(\mathbb{L}, \leq)$ is a (topped, bottomless) join-semilattice in which pairwise suprema are given by

$$\sup([A], [B]) = [(A \supset B) \supset B].$$

For an introductory account of semilattices, we refer to [2].
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE FL 32611 USA

E-mail address: paulr@ufl.edu