Statistical properties of an ensemble of vortices interacting with a turbulent field

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Abstract

We develop an analytical formalism to determine the statistical properties of a system consisting of an ensemble of vortices with random position in plane interacting with a turbulent field. We calculate the generating functional by path-integral methods. The function space is the statistical ensemble composed of two parts, the first one representing the vortices influenced by the turbulence and the second one the turbulent field scattered by the randomly placed vortices.

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1 Introduction

This work presents a study of the statistical properties of a system of vortices interacting with random waves. This is motivated by the necessity to describe quantitatively the statistics of a turbulent plasma in the regime where structures are generated at random, persist for a certain time and are destroyed by perturbations. This regime is supposed to be reached at stationarity in drift wave turbulence when there is a competition between different space scales. These scales range from the radially extended eddies of the ion temperature gradient (ITG) driven modes to the intermediate $\sqrt{\ln \rho_s}$ interval where the scalar nonlinearity is dominant over the vectorial one and with random fast decays to few $\rho_s$ scales where robust vortex-like structures are generated.

In a physical model the description of these states requires to consider simultaneously the long time response and the much faster events of trapping of the energy in small vortices, condensed from transient Kelvin-Helmholtz instabilities of the locally ordered sheared flow.

The analytical model we propose here is essentially nonperturbative in the sense that the structures are explicitly represented and we take into ac-
count their particular analytical expression (or a reasonable approximation). It is in the spirit of the semiclassical methods, frequently used in field theory. In this sense this treatment presents substantial differences compared with the more usual approaches, aiming in general to obtain a renormalization of the linear response or to calculate the correlations of the fluctuating field using a closure method. When structures are present in the plasma any perturbative approach is inefficient, because it can only go not too far beyond Gaussian statistics, \textit{i.e.} calculate few cumulants beyond the second one. The problem consists in that they take as the base-point for developing perturbative expansions the state of equilibrium of the plasma or a Gaussian ensemble of waves. Or, from this point it is impossible to reach the strongly correlated states of coherent structures. The idea of semiclassical methods is precisely to place the equilibrium state on the structures and then to explore a neighborhood in the space of system’s configurations, to include random waves. This is the approach we will adopt in the present work. To have a tractable problem we assume the simplest case where the vortices are not created or destroyed dynamically. The final outcome from this approach is a list in which a particular dependence of a contribution to the correlation, \textit{e.g.} the exponent $\mu$ in $\langle \phi \phi \rangle_k \sim k^{-\mu}$, is associated to a particular contribution in the physical model: statistics of the gas of vortices, interaction energy, nonlinearity of the physical model, etc.

1.1 The formulation of the problem

In agreement with the experimental observation and numerical simulations it has been found theoretically that the nonlinear differential equations describing strongly nonlinear electrostatic drift waves in two-dimensional plasma can have (1) turbulent solutions, consisting of very irregular fluctuations which can only be described by statistical quantities (irreducible correlations = cumulants); and (2) solutions that are quasi-coherent structures of vortex type, which are remarkably robust and for which we have in certain cases explicit analytical expression. Although there is no conceptual difference between these two aspects, we can say that we have two manifestations of the nonlinearity: one is nonlinear mode coupling and energy transfer between waves; and the second is the generation of organized flow with vortical pattern with strong stability and coherence of form. A first approximation is to break up the field (the electric potential $\phi$) in two distinct elements: vortices and random waves. We will suppose that there is a finite number $N$ of vortices in a two-dimensional plasma and that these vortices have random position. The vortices are individually affected by the turbulent background. The turbulent background, in turn, is affected by the presence
of the structures at random positions. In addition the turbulent background has statistical properties generated by the nonlinear nature of the waves interactions, even at amplitudes below those necessary to condense vortices. Random growth and decay of modes at marginal stability is included as a drive with Gaussian statistics.

Consider the field $\phi^{Vortices}$ $\equiv$ $\phi_V$ of the discrete set of $N$ vortices individually represented by the electric potential $\phi_s^{(a)}(x,y)$

$$\phi_V(x,y) = \sum_{a=1}^{N} \phi_s^{(a)}(x,y)$$ (1)

interacting with a turbulent wave field $\phi^{waves} \equiv \phi$. The total field is

$$\varphi(x,y) = \sum_{a=1}^{N} \phi_s^{(a)}(x,y) + \phi(x,y)$$ (2)

and we want to determine the statistical properties of the field $\varphi(x,y)$. We will construct an action functional and we will calculate the generating functional of the irreducible correlations (cumulants) of the fluctuating field $\varphi$. The action functional is expressed in terms of the field $\varphi$ which we see as composed from vortices and turbulence, Eq.(2). For example, the two-point correlation is composed of four terms

$$\langle \varphi \varphi \rangle = \langle \phi^{Vortices} \phi^{Vortices} \rangle + \langle \phi^{Vortices} \phi^{waves} \rangle + \langle \phi^{waves} \phi^{Vortices} \rangle + \langle \phi^{waves} \phi^{waves} \rangle$$ (3)

Our procedure consists in absorbing the two intermediate terms into the first and the last terms. This is done by calculating the auto-correlations of each component taking into account the presence of the other. In this operation the second and the third terms, although are identical in Eq.(3), are regarded differently: the second term is seen as the contribution of the turbulent background to the auto-correlation of a gas of vortices and the third term is seen as the contribution of a set of vortices to the auto-correlations of a turbulent field.

Therefore we consider that the action functional is composed of two distinct parts: $S_{\varphi V}$ $\equiv$ the action for the system of vortices interacting with the random field; and $S_{\varphi V}$ $\equiv$ the action for the random field interacting with the vortices. Then the statistical ensemble of realizations of the fluctuating field $\varphi(x,y)$ will actually consists of the Cartesian product of two distinct parts. The generating functional will be

$$Z = Z_{\varphi V}Z_{\varphi V}$$ (4)
where each factor is calculated using the action defined above. Although
the generating functional is factorized, which corresponds to splitting the
statistical ensemble into two parts, these two parts are not independent since
each will be calculated such as to include the effect of the field from the other
subsystem.

The interaction with an external current must be included in each of the
action in order to calculate the correlations using functional derivatives. In
the full action the current is introduced by adding a term in the Lagrangian
density, $J\varphi$, where $\varphi$ is the total field, vortices plus random waves. When
we separate the two components we have $\varphi = \phi^{\text{vortices}} + \phi^{\text{waves}}$ and $J\varphi = J\phi^{\text{vortices}} + J\phi^{\text{waves}}$. This expression must be inserted in the action functional
for the full system, before the intermediate terms in Eq.(3) are absorbed into
the first and the last. This means that the same current $J$ will appear in the
final expressions of the two generating functionals Eq.(4)

$$Z[J] = Z_{\varphi}^V [J] Z_{\phi}^V [J]$$

It will be shown below that the value of the field at a particular point can be
obtained by functional derivation to the current at that point $\varphi = \frac{\delta}{\delta J}$. We
obtain the average of the fluctuating field as

$$\langle \phi \rangle = \left. \frac{1}{Z[J=0]} \frac{\delta}{\delta J} Z[J] \right|_{J=0}$$

$$= \left. \frac{1}{Z_{\varphi}^V [J=0]} \frac{\delta}{\delta J} Z_{\varphi}^V [J] \right|_{J=0} + \left. \frac{1}{Z_{\phi}^V [J=0]} \frac{\delta}{\delta J} Z_{\phi}^V [J] \right|_{J=0}$$

The two-point correlation is obtained from the generating functional by applying
two times this functional derivative, with the current in two distinct
points and taking finally the current to zero.

We now explain how this will be done effectively.

1.2 Outline of the present analytical approach

1.2.1 Gas of vortices in the turbulent background

In the case of the vortices interacting with random waves, the basic reasoning
takes into consideration two distinct elements.

The first is concerned with the statistical properties of a collection of
discrete vortices with zero, or very short, range of interaction. At this stage
of the problem the particular shape of the potential distribution in a vortex
is not essential and will be simplified in those situations where only the
positions of the centers are important. The statistical ensemble consists of
the configurations of randomly placed vortices, seen as a dilute gas.
The second aspect is concerned with the interaction between one (generic) vortex and the random waves in the surrounding turbulence and calculates the effect on the form of the potential of the vortex. Instead of the exact solution we will have now a form resulting from the scattering due to the random perturbation produced by the turbulent background.

The statistical properties result from the combination of the two elements, which will now be explained briefly.

The first problem is very similar to the Coulomb gas in two dimensions or to any problem related to dilute gas of interacting particles. The partition function is calculated using the sum of the energy of the individual vortices and the energy of interaction. The first part can be calculated from the exact analytical solution of the differential equation of the model (Eq. (10) below). The second part contains the sum of the energies of pair interaction, a problem that is principally complicated in our case, i.e. in turbulent plasma. Our real vortices are neutral (they are simply a deformation, stable and regular, of the scalar potential of the velocity) and there should be no interaction of Coulomb or Ampere type. However there is an elastic medium between the vortices and any motion of one of them generates sound waves that may couple with the potential of other vortices, influencing their motion. This may suggest that the interaction is of the same nature as the Casimir effect but this is beyond our primary interest here. What we minimally need to include in our partition function is the effect of scattering of the vortices at close encounter, since we do not take into account in the present treatment the variation of the number of vortices due to merging (which would imply a chemical potential). There is an intrinsic spatial scale in all aspects related with drift-wave vortices and this is the sonic Larmor radius, altered by the combined effect of the diamagnetic and a uniform flow. We show below that the pair interaction contains a kernel with fast spatial decay, \( K_0 \left( \frac{r}{\rho_s} \right) \) (the modified Bessel function). There are several reasons to assume that this is a physically correct choice (see [5]).

The second aspect of the problem of vortices acted upon by turbulence is related to the direct modification of the exact vortex by the random waves around it. The case of a single vortex interacting with random waves has been treated in the papers [2], [3]. The starting point is the exact vortex-type solution of the differential equation we investigate. This is of course a non-random object and when it is isolated the statistical ensemble is trivial. However its interaction with random waves of the background turbulence makes it also a fluctuating object. We calculate the statistical properties of the ensemble of states of the fluctuating potential corresponding to the
shapes of the vortex-type solution.

The action functional is extremum at the vortex solution, whose explicit expression we will use. A purely turbulent field also realizes the extremum of the action, but our separation between structures and random waves amounts to an approximative representation of the turbulent field, as a complimentary part to the structures. All configurations consisting of structures with randomly deformed, fluctuating shapes, together with random waves are a very good approximation of reality and must be found in close proximity of the extremum of the action. Then we have to explore the functional space of the system’s configuration in the neighborhood of the structure (the extremum) trying to include as many as possible nearby configurations in the calculation of the generating functional. This will include into the generating functional the turbulent field, besides the structure. In few words, the idea is that the structure and the random waves, although very different in geometry, share a common property: they obtain or are very close to the extremum of the action functional. The technical procedure will consists of expanding the action functional to second order around the structure and to integrate over the space of configurations. This will automatically exclude bad approximations, i.e. the states which are too far from realizing the extremum of the action since for them the Boltzmann weight is exponentially small.

We conclude by noting that this is the standard semiclassical treatment [6]. In this sense it is similar to the treatment of vortex statistics of the Abelian-Higgs model of superfluids [8] and of many other systems.

1.2.2 The turbulent field influenced by random vortices

For the turbulent field interacting with vortices, our approach combines two distinct elements. The first is the inclusion of the vortices as a random perturbation in the generating functional of the turbulent waves. The second is a perturbative treatment of the intrinsic nonlinearity of the turbulent waves, already modified by the inclusion of the effect of the random vortices.

In the first part we follow the similar approaches as for the electron conduction in the presence of random impurities, or as for flexible polymers in porous media. For the case where the vortices have uncorrelated random positions and take at random positive or negative amplitudes of equal magnitude we find that the problem is mapped onto the sine-Gordon model (actually sinh). For our model equation this will amount to a renormalization of the coefficient which plays the role of a physical “mass” of the turbulent field [40], [38], or, in other terms to a shift of the spatial scale from \( \rho_s (1 - v_d/u)^{-1/2} \) to higher values.

The second part consists of a systematic perturbative treatment of the
nonlinearity in order to get, as much as possible, a correct representation of the nonlinear content of the field of the turbulent waves (this means the nonlinear interaction and nonlinear energy transfer between low amplitude random waves \[41, 42, 43\]). The nonlinearity is included in a functional perturbative treatment where the turbulent plasma is driven by random rise and decay of modes at marginal stability. This induces a diffusive behavior of the turbulent field, at lowest order. We develop the treatment to one-loop, which means of order two in the strength of the nonlinearity.

One may inquire if this is not in contradiction with the separation operated at the beginning, where the most characteristic aspect of the nonlinearity, the generation of structure, is treated separately. However, it is clear that there is no danger of overlapping and double counting of the nonlinear effects. Since a statistical perturbative treatment is essentially an expansion in a parameter representing the departure from Gaussian statistics we can only hope to include higher cumulants beyond the second one (which means Gaussianity) but only few of them are accessible to effective calculation. Or, any structure needs a very large number of cumulants since, by definition, is an almost coherent field. It is illusory to try to capture the structure using a perturbative treatment starting with a state of equilibrium (\(i.e.\) no perturbation or, alternatively, a Gaussian collection of linear waves). Since any term of the perturbation can be represented by a Feynman diagram, we face the well known problem that the proliferation of diagrams at high orders leads to an effectively intractable problem. This is actually one reason for the use of the semiclassical methods.

2 The physical model

2.1 The equation

The model of the ion drift instability in magnetically confined plasmas can be formulated using the fluid equations of continuity and momentum conservation for electrons and for ions. It has been shown by a multiple space time scale analysis that the dynamics is dominated by two nonlinearities: the Charney-Hasegawa-Mima type, or vectorial nonlinearity, generated by the ion polarization drift and the Korteweg-De Vries, or scalar nonlinearity, related to the space variation of the density gradient length. The former is of high differential degree and is dominant at small spatial scales, of the order of few sonic Larmor radii \(\rho_s\). The latter is dominant at “mesoscopic” spatial scales, of the order of \(\sqrt{\rho_s L_n}\). The numerical studies \[\ldots\], the fluid-tank experiments \[\ldots\] and the multiple space-time scale analytical analysis
show that the scalar nonlinearity becomes dominant at late regimes in the statistical stationarity of the drift wave turbulence. However the possible manifestation of the two types of nonlinearity rises the problem of structural stability of either regime where only one of these nonlinearity is considered dominant: inclusion of the other strongly changes the behavior of the system. It is then pertinent to consider that the turbulence generated in a realistic regime may include manifestations of both types. The turbulence is dominated by the larger scales sustained by the scalar nonlinearity (described by the Flierl-Petviashvili equation \[^{13}\]) together with robust vortices generated at the scales of few \(\rho_s\), typical the CHM (\(i.e.\) vectorial) nonlinearity \[^{?}\].

When the scalar nonlinearity is prevailing \[^{14}\], \[^{?}\] the equation has the form

\[
\left(1 - \nabla^2\right) \frac{\partial \phi}{\partial t} + v_s \frac{\partial \phi}{\partial y} - v_s T \frac{\partial \phi}{\partial y} = 0
\]

(7)

In a moving frame and restricting to stationarity we obtain

\[
\nabla^2 \phi - \alpha \phi - \beta \phi^2 = 0
\]

(8)

The physical parameters are \[^{9}\], \[^{10}\], \[^{13}\]

\[
\alpha = \frac{1}{\rho_s^2} \left(1 - \frac{v_d}{u}\right), \quad \beta = \frac{c_s^2}{2u^2} \frac{\partial}{\partial x} \left(\frac{1}{L_n}\right)
\]

(9)

where \(\rho_s = c_s/\Omega_i\), \(c_s = (T_e/m_i)^{1/2}\) and the potential is scaled as \(\phi \rightarrow e\phi/T_e\). Here \(L_n\) and \(L_T\) are respectively the gradient lengths of the density and temperature. The velocity is the diamagnetic velocity \(v_d = \rho_s c_s/L_n\). The condition for the validity of this equation are: \((k_x \rho_s)(k \rho_s)^2 \ll \eta_e \rho_s/L_n\), where \(\eta_e = L_n/L_{T_e}\). The coefficients \(\alpha\) and \(\beta\) have the dimension \((\text{length})^{-2}\). This form will be used below.

2.2 The structures

The exact solution of the equation is

\[
\phi_s(y, t; y_0, u) = -3 \left(\frac{u}{v_d} - 1\right) \times \text{sech}^2 \left[\frac{1}{2\rho_s} \left(1 - \frac{v_d}{u}\right)^{1/2} (y - y_0 - ut)\right]
\]

(10)

where the velocity is restricted to the intervals \(u > v_d\) or \(u < 0\). In Ref.\[^{14}\] the radial extension of the solution is estimated as: \((\Delta x)^2 \sim \rho_s L_n\). In our
work we shall assume that \( u \) is close to \( v_d \), \( u \gtrsim v_d \) (i.e. the structures have small amplitudes). The monopolar vortex in this regime is discussed in Ref. [?]. For asymptotic form of the CHM equation see Ref. [5]. We will adopt the one-dimensional section of the solution (i.e. Eq.(10)) when we calculate the eigenmodes of the determinant of the second functional derivative of the action. In conclusion, we consider coherent structures which are monopolar vortices of both signs of vorticity, with equal magnitudes and with random positions in plane.

3 Statistical analysis of the physical system

3.1 General functional framework

The general method for constructing the action functional for a classical stochastic system is described by Martin, Siggia and Rose (MSR) [27], and in path integral formalism, by Jensen [28]. Two reviews by Krommes are very useful references on this point [29], [30]. The functional method has been applied in several concrete problems and references may be consulted for details [31], [32], [4], [33], [2]. We here review few elementary procedures (see [35]).

Consider a differential equation \( F[\phi] = 0 \) whose solution is \( \phi^z \). The unknown function belongs to a space of functions \( \phi(x,y) \). We want to select from this space of functions precisely the one that is the solution of the differential equation and for this we can use the functional Dirac \( \delta \) : \( \delta (\phi - \phi^z) \). This can be represented as a product of usual \( \delta \) functions in every point of space

\[
\delta (\phi - \phi^z) = \prod_{k=1}^{N} \delta [\phi(x_k) - \phi^z(x_k)]
\]

Any operation that will be done on a functional of \( \phi \) can be now particularized to the solution \( \phi^z \) by simply inserting this Dirac functional. For example the calculation of a functional \( \Omega (\phi) \) at the function \( \phi^z \) can be done by a functional integration over the space of all functions, with insertion of this \( \delta \) functional. Using the Fourier representation of the ordinary Dirac functions and going
to the continuous limit we note the appearance of the dual function $\chi$

$$\Omega (\phi^z) = \int \mathcal{D} [\phi] \Omega (\phi) \left( \left. \frac{\delta F}{\delta \phi} \right|_{\phi^z} \right) \delta [F (\phi)]$$

$$= \left( \left. \frac{\delta F}{\delta \phi} \right|_{\phi^z} \right) \int \mathcal{D} [\phi] \mathcal{D} [\chi] \Omega (\phi) \times \exp \left\{ i \int dx \chi (x) F [\phi (x)] \right\}$$

(12)

We will define the partition function as usual, by the functional integral of Boltzmann weights calculated on the base of the MSR action

$$Z [J] = \int \mathcal{D} [\phi] \mathcal{D} [\chi] \exp \left\{ \int dxdy (\chi F [\phi] + J_\phi \phi + J_\chi \chi) \right\}$$

(13)

The functional integration takes into account the fluctuations of the physical field $\phi$ and of its dual $\chi$. The “free-energy” functional is defined by $\exp \{ W [J] \} = Z [J]$ from which the irreducible correlations (cumulants) are calculated by functional derivatives to $J$.

$$\langle \phi (x, y) \rangle = \exp \left\{ -W [J] \right\} \frac{\delta}{\delta J_\phi (x, y)} \exp \{ W [J] \} \bigg|_{J=0}$$

(14)

and similar for higher cumulants. The two-point irreducible correlation for the field is

$$\langle \phi (x, y) \phi (x', y') \rangle$$

$$= \int \mathcal{D} [\phi] \mathcal{D} [\chi] \phi (x, y) \phi (x', y') \times \exp \left\{ \int dxdy (\chi F [\phi] + J_\phi \phi + J_\chi \chi) \right\} \bigg|_{J=0}$$

$$= \frac{1}{Z [J = 0]} \frac{\delta}{\delta J_\phi (x, y)} \frac{\delta}{\delta J_\phi (x', y')} Z [J] \bigg|_{J=0}$$

(15)

This is the general analytical instrument that will be used in the following calculations. The calculations from this work are presented in greater detail in Ref. [?].

4 Vortices with random positions

4.1 The discrete set of $N$ vortices in plane

The action of the discrete set of vortices is determined by the sum of the actions of the individual vortices plus a part that results from the interaction
between them \[8\]. The first part is simply the time integration of the energy, *i.e.* (since time factorizes) the space integration of the expression of the product of the static potentials associated with a single vortex, Eq.(10), and to its dual, which in the end simply means the square of the wave-form of the vortex potential. This quantity is repeated for each of the \(N\) vortices.

The partition function is

\[
Z_V = \frac{1}{N!} \left( \prod_{j=1}^{N} Z_V^{(0)} \right) \sum_{\{\alpha\}} \int \left( \prod_{a=1}^{N} \frac{1}{A} d\mathbf{R}^{(a)} \right)
\]

\[
\times \exp \left[ -\pi \sum_{a=1}^{N} \sum_{b=1}^{N} \int dx dy \int dx' dy' \rho^{(a)}(x, y) \right.
\]

\[
\times K_0 \left( \rho_s^{-1} |\mathbf{R}^{(a)} - \mathbf{R}^{(b)}| \right) \rho^{(b)}_\omega (x', y') \right]
\]

(16)

with the following meaning.

The first factor simply takes into account \(N\) independent vortices with arbitrary positions in plane and expresses the fact that this part of the partition function results from a Cartesian product of the \(N\) statistical ensembles, one for each vortex. The factor \(N!\) takes into account the permutation symmetry. The generating functional for a static vortex with structure given by the interaction with random waves, is calculated in Eq.(65) below. We have \(Z_V^{(0)} = Z_{V, \varphi}[J]\) where we have indicated by the index \(V \varphi\) that the partition function of the vortex includes the interaction vortex-turbulence, and that the expression depends on the external current \(J\), and will contribute to any correlation that we will obtain by functional derivations at \(J\).

The sum is over the set of configurations \(\{\alpha\}\) characterized by random choices of positive and negative vortices.

The integrations over the positions of the centers \(\mathbf{R}^{(a)}\) of the vortices, \(a = 1, N\), express the fact that we allow arbitrary positions in plane, with equal probability. Each integral is normalized with the area of the physically interesting two-dimensional region of the plane, \(A\).

The exponent of the Boltzmann weight contains action resulting from the interaction between vortices. We expect that for a dilute gas of vortices, where the distance between the centers \(\mathbf{R}^{(a)} - \mathbf{R}^{(b)}\) is much larger than the core diameter \(d\), \(\mathbf{R}^{(a)} - \mathbf{R}^{(b)} \gg d\), \(a, b = 1, N\), the interaction is very weak. In order to describe the interaction between vortices we start from the well-known alternative model of the drift waves sustained by the ion polarization drift nonlinearity [13] [25], [26]. In this model it is considered a set of \(N_\omega\) point-like vortices of strength \(\omega_i\) interacting in plane by a short range potential expressed as the function \(K_0\) (modified Bessel function) of the relative
distance between vortices. The potential $\phi^p$ in a point $\mathbf{R}$ is a sum of contributions from all the $N_\omega$ point-vortices $\phi^p(\mathbf{R}) = \sum_{i=1}^{N_\omega} \omega_i K_0 (\rho_s^{-1} |\mathbf{R} - \mathbf{R}_i|)$ and the equations of motion $d\mathbf{R}_i/dt = -\nabla \phi^p \times \hat{e}_z$ (where $\hat{e}_z$ is the versor perpendicular on the plane). The distribution of vorticity of the physical system (in particular the quasi-coherent vortical structures) represents spatial variations of density of these point-like vortices. The interaction between the physical vortices will result from the interaction between the point-like vortices, taking into account the density of these objects. The energy of interaction is

$$H = \sum_{i=1}^{N_\omega} \sum_{j=1}^{N_\omega} \omega_i \omega_j K_0 \left( \rho_s^{-1} |\mathbf{R}_i - \mathbf{R}_j| \right)$$

called the Kirchhoff function. The range of spatial decay of the interaction is the Larmor sonic radius $\rho_s$, which however may be modified to an effective Larmor radius, in the presence of gradients and flow. When we approach the continuum limit $N_\omega \to \infty$ the envelope of the density becomes the physical vorticity $\omega(x, y)$ which, for this stage of the problem is sufficient to be considered as highly concentrated in the cores of the physical vortices and almost vanishing in the rest. Taking the elementary point-vortices of equal strength $\omega_j \equiv \omega_0$ we have that each physical vortex is an integer multiple $N^{(a)}$ of this quantity. Now we will associate with each physical vortex a continuous function, i.e. its vorticity defined on the whole plane, $\rho_\omega^{(a)}(x, y)$, which is, as said, concentrated in $(x, y)$

$$\rho_\omega^{(a)}(x, y) = N^{(a)} \omega_0 \delta \left( \mathbf{R} - \mathbf{R}^{(a)} \right)$$  \hspace{1cm} (17)

Then the energy is

$$H = \sum_{a=1}^{N} \sum_{b=1}^{N} \int dx dy \int dx' dy'$$

$$\times \rho_\omega^{(a)}(x, y) K_0 \left( \frac{|\mathbf{R}^{(a)} - \mathbf{R}^{(b)}|}{\rho_s} \right) \rho_\omega^{(b)}(x', y')$$  \hspace{1cm} (18)

Due to Eq.(17) the interaction energy is only the interaction between the centers $R^{(a)}$ and $R^{(b)}$ of the vortices. The summation proceeds by grouping the point-vortices into physical vortices, then assuming that these (for only this stage of the problem) have $\delta$-function shape and finally formally replacing this $\delta$ with a continuous distribution $\rho_\omega^{(a)}(x, y)$. In this operation a number of infinities arise from the energy of the interaction of the point-vortices which
are grouped into one physical vortex, since the relative distances are zero for them. This singular part can be removed since it does not participate to the functional variations induced by the “external excitation” current \( J \).

For simplification of the computation we now only consider physical vortices of equal amplitude (positive or negative) and then the vorticity distribution \( \rho_\omega^{(a)}(x, y) \) has amplitude \( \omega_v = p\omega_0 \) (\( p \) is an integer), multiplied by the integer \( n^{(a)} \) which can take the values ±1 for positive or negative vorticity. We have \( N^{(a)} = pn^{(a)} \). The sum over the physical vortices’ positions suggests to define a formal unique function of vorticity \( \rho_\omega(x, y) \) for all the \( N \) physical vortices

\[
\rho_\omega(x, y) \equiv \sum_{a=1}^{N} \rho_\omega^{(a)}(x, y) = \sum_{a=1}^{N} \omega_v n^{(a)} \delta(R - R^{(a)})
\]

Further, the energy is normalized with a constant dimensional factor. The interaction part can be rewritten

\[
\exp \left[ -\frac{\pi}{\rho_s^4 \omega_v^2} \sum_{a=1}^{N} \sum_{b=1, a > b}^{N} \int dx dy \int dx' dy' \times \rho_\omega^{(a)}(x, y) K_0 \left( \rho_s^{-1} |R^{(a)} - R^{(b)}| \right) \rho_\omega^{(b)}(x', y') \right] = \exp \left[ -\frac{1}{2\rho_s^4 \omega_v^2} \int dx dy \int dx' dy' \rho_\omega(x, y) G(R - R') \rho_\omega(x', y') \right]
\]

where \( G \) is the kernel of interaction,

\[
G(R - R') = \frac{1}{2\pi} \sum_{a > b} \sum_{a=1}^{N} K_0 \left( \rho_s |R^{(a)} - R^{(b)}| \right) \times \delta \left( R - R^{(a)} \right) \delta \left( R' - R^{(b)} \right)
\]

The differential equation for \( K_0 \) is \((\Delta - 1/\rho_s^2) K_0 (r/\rho_s) = -2\pi \delta(r)\). This helps to replace the interaction part with a Gaussian functional integral, by introducing an auxiliary field \( \psi \)

\[
\exp \left[ -\frac{1}{2\rho_s^4 \omega_v^2} \int dx dy \int dx' dy' \rho_\omega(R) G(R - R') \rho_\omega(R') \right] = \rho_1^{-1} \int \mathcal{D}[\psi] \exp \left\{ -\frac{1}{2} \int dx dy \left[ (\nabla \psi)^2 + \frac{1}{\rho_s^2 \omega_v} \psi^2 \right] \right\} 
\]

\[
\times \exp \left[ i \frac{1}{\rho_s^2 \omega_v} \int dx dy \rho_\omega(R) \psi(R) \right]
\]
with $p_1$ a normalization constant. We make a change of variable in the functional integration $2\pi p\psi \to \chi$ (this also changes the normalization constant $p_1 \to p$) and return to the partition function Eq.(16)

$$Z_V = p^{-1} \int \mathcal{D}[\chi] \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ (\nabla \chi)^2 + \frac{1}{p_s^2} \chi^2 \right] \right\}$$

$$\times \frac{1}{N!} \left( Z_V^{(0)} \right)^N \sum_{\{a\}} \int \left( \prod_{a=1}^N \frac{1}{A} dR^{(a)} \right) \exp \left[ i \sum_a n^{(a)} \chi(R^{(a)}) \right] \tag{22}$$

In the last factor we note that in the sum each term consists of two contributions, corresponding to positive and negative vorticity, $n^{(a)} = \pm 1$, and they are weighted with the same factor, $1/2$

$$\frac{1}{N!} \left( Z_V^{(0)} \right)^N \sum_{\{a\}} \int \left( \prod_{a=1}^N \frac{1}{A} dR^{(a)} \right) \exp \left[ i \sum_a n^{(a)} \chi(R^{(a)}) \right] = \frac{\left( Z_V^{(0)} \right)^N}{N!} \left[ \prod_{a=1}^N \int \frac{1}{A} dR \frac{1}{2} \left( \exp[i \chi(R)] + \exp[-i \chi(R)] \right) \right] \tag{23}$$

In the last line we have removed the upper index $(a)$ since all factors in the product are now identical. For a fixed number $N$ of vortices the partition function $Z_V^{(0)}$ (with nonvanishing contribution to the derivatives to $J$) is decoupled from the other factors and will provide $N$-times the same contribution. The other factors, i.e. the functional integral that contains the interaction between the vortices can only appear in the final answer as a constant, multiplying contributions coming from $Z_V^{(0)}[J]$. When $N$ is arbitrary the partition function must also include a sum over terms each corresponding to a number $N$ of vortices

$$\sum_{N=0}^{\infty} \frac{\left( Z_V^{(0)} \right)^N}{N!} \left[ \int \frac{1}{A} dR \frac{1}{2} \left( \exp[i \chi(R)] + \exp[-i \chi(R)] \right) \right]^N = \exp \left\{ \frac{Z_V^{(0)}}{A} \int dxdy \cos[\chi(x,y)] \right\} \tag{24}$$

Then the partition function becomes

$$Z_V = p^{-1} \int \mathcal{D}[\chi] \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ (\nabla \chi)^2 + \frac{1}{p_s^2} \chi^2 - \frac{8\pi^2}{A} Z_V^{(0)} \cos[\chi(x,y)] \right] \right\} \tag{25}$$
In this expression the quantity $Z_V^{(0)}$ is a functional integral over the space of fields $\phi_s(x, y)$ representing a single vortex. In the absence of the background turbulence the field $\phi_s(x, y)$ is a deterministic quantity [Eq.(10)] and the statistical ensemble is trivially composed of one element. The interaction with the background turbulence induce a fluctuating form and the statistical properties can be obtained from $Z_V^{(0)}$. In other words, $Z_V$ includes two sources of fluctuations: one is the fluctuation of the field of vorticity due to the random positions in plane of the vortices (the gas of vortices) and the other is the fluctuation of the shape of the generic vortex due to interaction with the turbulent background.

If we want to use the expression (25) as a generating functional for correlation we must be able to drop into the functional integral the field representing the vortices, i.e.

$$\phi_V(R) = \sum_a \phi_s^{(a)}(x, y)$$

as in Eq.(15). An external excitation by the current $J$ will produce a change in $Z_V$ from the change of the vorticity of the gas of vortices and from the change of $Z_V^{(0)}$. We have

$$\langle \phi_V(x, y) \rangle = \frac{1}{Z_V[J = 0]} \left[ \left( \frac{\delta Z_V}{\delta J(x, y)} \right)_{\text{vort}} + \frac{\delta Z_V}{\delta Z_V^{(0)}} \frac{\delta Z_V^{(0)}}{\delta J(x, y)} \right]$$

$$\langle \phi_V(x, y) \phi_V(x', y') \rangle = \frac{1}{Z_V[J = 0]} \left[ \left( \frac{\delta^2 Z_V}{\delta J(x, y) \delta J(x', y')} \right)_{\text{vort}} + \frac{\delta^2 Z_V}{\delta (Z_V^{(0)})^2} \frac{\delta Z_V^{(0)}}{\delta J(x, y)} \frac{\delta Z_V^{(0)}}{\delta J(x', y')} + \frac{\delta Z_V}{\delta Z_V^{(0)}} \frac{\delta^2 Z_V^{(0)}}{\delta J(x, y) \delta J(x', y')} \right]$$

The formulas are taken at $J = 0$. The first terms in these equations are related with the fluctuations of the vorticity of the gas of vortices as a continuous version of the discrete set of physical vortices with arbitrary positions in plane. The other terms are related to the fluctuation of the shape of a vortex and in order to calculate these contribution we need the explicit expression of $Z_V$, as a functional of $Z_V^{(0)}$. Further, we will need the detailed expression of $Z_V^{(0)}[J]$ and this will be calculated in the next Section.

In order to obtain the contribution from the fluctuation of the gas of vortices (the first terms in Eqs.(27) and (28)), we introduce a new term in
the action, consisting of the interaction between the vorticity distribution \( \rho_\omega (\mathbf{R}) \) and an external current, \( J_\omega \)

\[
i \frac{1}{\omega_v} \int dx dy [\rho_\omega (x, y) J_\omega (x, y)]
\]

(29)

(the factor \( i \) is introduced for compatibility with Eq.(21)). This current \( J_\omega \) is an external excitation applied on the field of the vorticity and not on the field of potential \( \phi_\nu \) as we would need in the Eq.(5). We may assume that there is a connection between the current \( J_\omega \) and the current \( J \) (which acts on the field \( \phi_\nu \)) but there is no need to specify this relation. Indeed, the Eq.(6) shows that at the end both currents should be taken zero.

The last line of Eq.(21) transforms as follows

\[
\exp \left[ i \frac{1}{\omega_v} \int dx dy \rho_\omega (\mathbf{R}) \psi (\mathbf{R}) \right] \rightarrow \exp \left\{ i \frac{1}{\omega_v} \int dx dy \rho_\omega (\mathbf{R}) [\psi (\mathbf{R}) + J_\omega (\mathbf{R})] \right\}
\]

(30)

All the calculations following Eq.(21) are repeated without modification but in the last term, instead of the function \( \chi (x, y) \) we will have

\[
\cos [\chi (x, y)] \rightarrow \cos [\chi (x, y) + J_\omega (x, y)]
\]

(31)

since this was the term which resulted from the presence of the function \( \rho_\omega (\mathbf{R}) \) in the Eq.(21). Making the change of variable in the functional integration (of Jacobian 1)

\[
\chi \rightarrow \chi - J_\omega
\]

(32)

the integrand in the action is expressed as

\[
-\frac{1}{8\pi^2} \left\{ \nabla (\chi - J_\omega) )^2 + \frac{1}{\rho_s^2} (\chi - J_\omega)^2 - \frac{8\pi^2}{A} Z_V^{(0)} \cos [\chi (x, y)] \right\}
\]

(33)

The two-point correlation of the field of the vorticity fluctuations can be calculated from

\[
\frac{1}{\omega_v^2} \langle \rho_\omega (x, y) \rho_\omega (x', y') \rangle = \frac{1}{Z_V [J_\omega = 0]} \left. \frac{\delta Z_V [J]}{\delta J_\omega (x, y) \delta J_\omega (x', y')} \right|_{J_\omega = 0}
\]

(34)

The squares in the action Eq.(33) are expanded

\[
-\frac{1}{8\pi^2} \left\{ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 - \frac{8\pi^2}{A} Z_V^{(0)} \cos [\chi (x, y)] \right\}
\]

(35)

\[
+ (\nabla J_\omega)^2 + \frac{1}{\rho_s^2} J_\omega^2
\]

\[
-2 (\nabla \chi) (\nabla J_\omega) - \frac{2}{\rho_s^2} \chi J_\omega \right\}
\]
In the part of the action that depends on \( J_\omega \) we make integrations by parts
\[
(\nabla J_\omega)^2 + \frac{1}{\rho_s^2} J_\omega^2 - 2 (\nabla \chi) (\nabla J_\omega) - \frac{2}{\rho_s^2} \chi J_\omega
\]
(36)
\[
\rightarrow -J_\omega (\Delta J_\omega) + \frac{1}{\rho_s^2} J_\omega^2 + 2 J_\omega (\Delta \chi) - \frac{2}{\rho_s^2} \chi J_\omega
\]

The first line at the exponent in Eq.(35) does not contain the current and in the following functional derivations to \( J_\omega \) we will temporary omit it. Consider the application of the first operator of derivation to \( iJ_\omega (x,y) \)
\[
\frac{\delta}{i\delta J_\omega (x,y)} \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ -J_\omega (\Delta J_\omega) + \frac{1}{\rho_s^2} J_\omega^2 + 2 J_\omega (\Delta \chi) - \frac{2}{\rho_s^2} \chi J_\omega \right] \right\}
\]
\[
= \frac{1}{i} \exp \{\ldots\}
\times \rho_s^2 (-1) \frac{1}{8\pi^2} \left( -2\Delta J_\omega + \frac{2}{\rho_s^2} J_\omega + 2\Delta \chi - \frac{2}{\rho_s^2} \chi \right)_{(x,y)}
\]

Every derivation to the current \( J_\omega \) suppresses a space integration and in consequence the result is multiplied with factors \( \rho_s \) which render the space integral dimensionless. The subscript shows that the functions inside the bracket are calculated in the point \((x,y)\).

The second operator of derivation is now applied on Eq.(37)
\[
\frac{\delta}{i\delta J_\omega (x',y')} \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ -J_\omega (\Delta J_\omega) + \frac{1}{\rho_s^2} J_\omega^2 + 2 J_\omega (\Delta \chi) - \frac{2}{\rho_s^2} \chi J_\omega \right] \right\}
\]
\[
\times \frac{1}{i} \rho_s^2 (-1) \frac{1}{8\pi^2} \left( -2\Delta J_\omega + \frac{2}{\rho_s^2} J_\omega + 2\Delta \chi - \frac{2}{\rho_s^2} \chi \right)_{(x',y')}
\]
\[
\times \frac{1}{i} \rho_s^2 (-1) \frac{1}{8\pi^2} \left( -2\Delta J_\omega + \frac{2}{\rho_s^2} J_\omega + 2\Delta \chi - \frac{2}{\rho_s^2} \chi \right)_{(x,y)}
\]
\[
\times \exp \{\ldots\}
\]
\[
+ \frac{1}{i} \rho_s^2 (-1) \frac{1}{8\pi^2} \left[ -2\Delta \delta (x-x', y-y') + \frac{2}{\rho_s^2} \delta (x-x', y-y') \right]
\]
\[
\times \exp \{\ldots\}
\]
At this moment we can take $J \equiv 0$. The exponentials in the Eq(38) are equal to 1. The auto-correlation is

$$\frac{1}{\omega_v^2} \langle \rho_\omega (x, y) \rho_\omega (x', y') \rangle$$

$$= \frac{1}{Z_V [J_\omega = 0]} p^{-1} \int \mathcal{D} [\chi] \exp \left\{ -\frac{1}{8\pi^2} \int dx dy \left[ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 - \frac{8\pi^2}{A} Z_V^{(0)} \cos \left( \chi(x, y) \right) \right] \right\} \times \left\{ \frac{\rho_s^2}{8\pi^2} \left[ -2\Delta \delta (x - x', y - y') + \frac{2}{\rho_s^2} \delta (x - x', y - y') \right] \right. $$

$$- \left. \left( \frac{\rho_s^2}{8\pi^2} \right)^2 \left( 2\Delta \chi - \frac{2}{\rho_s^2} \chi \right)_{(x,y)} \left( 2\Delta \chi - \frac{2}{\rho_s^2} \chi \right)_{(x',y')} \right\}$$

The last two lines in Eq.(39) (the curly bracket) arise from derivation. The normalization gives by definition

$$Z_V [J_\omega = 0] \equiv \langle 1 \rangle$$

$$= p^{-1} \int \mathcal{D} [\chi] \exp \left\{ -\frac{1}{8\pi^2} \int dx dy \left[ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 - \frac{8\pi^2}{A} Z_V^{(0)} \cos \left( \chi(x, y) \right) \right] \right\}$$

The functional integration does not affect the first two terms in the curly bracket in Eq.(39). The functional integral of the last term represents the average of the product of two functions $\chi$.

$$\frac{1}{\omega_v^2} \langle \rho_\omega (x, y) \rho_\omega (x', y') \rangle$$

$$= \frac{\rho_s^2}{4\pi^2} \left[ -\Delta \delta (x - x', y - y') + \frac{1}{\rho_s^2} \delta (x - x', y - y') \right]$$

$$- 4 \left( \frac{\rho_s^2}{8\pi^2} \right)^2 \left( \Delta - \frac{1}{\rho_s^2} \right)_{(x,y)} \left( \Delta - \frac{1}{\rho_s^2} \right)_{(x',y')} \langle \chi (x, y) \chi (x', y') \rangle$$

It is now easier if we use the Fourier representation of the fields, which we denote by the symbol $\sim$. Writing $x \equiv (x, y)$,

$$\frac{1}{\omega_v^2} \langle \rho_\omega (x, y) \rho_\omega (x', y') \rangle = \frac{1}{\omega_v^2} \int dk \exp (ik \cdot x) \int dk' \exp (ik' \cdot x') \langle \rho_\omega (k) \rho_\omega (k') \rangle$$
We have
\[
\frac{1}{\omega_v^2} \int dk \exp (i k \cdot x) \int dk' \exp (i k' \cdot x') \langle \tilde{\rho}_\omega (k) \tilde{\rho}_\omega (k') \rangle \\
= \frac{\rho_s^2}{4\pi^2} \int dk \exp [i k \cdot (x - x')] \left( k^2 + \frac{1}{\rho_s^2} \right) \\
- \left( \frac{\rho_s^2}{4\pi^2} \right)^2 \int dk \exp (i k \cdot x) \int dk' \exp (i k' \cdot x') \\
\times \left( k^2 + \frac{1}{\rho_s^2} \right) \left( k'^2 + \frac{1}{\rho_s^2} \right) \langle \tilde{\chi} (k) \tilde{\chi} (k') \rangle
\]

We can take a fixed reference point
\[
x' = a \\
x = a + x - x'
\]

and write
\[
\frac{1}{\omega_v^2} \int dk \exp [i k \cdot (x - x')] \int dk' \exp [i (k' + k) \cdot a] \langle \tilde{\rho}_\omega (k) \tilde{\rho}_\omega (k') \rangle
\]
\[
= \frac{\rho_s^2}{4\pi^2} \int dk \exp [i k \cdot (x - x')] \left( k^2 + \frac{1}{\rho_s^2} \right) \\
- \left( \frac{\rho_s^2}{4\pi^2} \right)^2 \int dk \exp [i k \cdot (x - x')] \int dk' \exp [i (k' + k) \cdot a] \\
\times \left( k^2 + \frac{1}{\rho_s^2} \right) \left( k'^2 + \frac{1}{\rho_s^2} \right) \langle \tilde{\chi} (k) \tilde{\chi} (k') \rangle
\]

The parameter \( a \) has no particular role: non of our assumption has imposed a nonuniformity of the statistical properties on the plane. Therefore we can integrate Eq.(40) over the position \( a \), \textit{i.e.} on the plane
\[
\frac{1}{A} \int da...
\]

Obviously, this will produce in the left hand side a function \( \delta \)
\[
\delta (k' + k)
\]

after which the integration over the second wavenumber, \( k' \), will impose
\[
k' = -k
\]
For the first term in the right hand side, the integration over \( a \) will have no effect. For the second term the effect is the same as in the left hand side, i.e. we have \( k' = -k \). we will now replace \( x - x' \) by \( x \) and obtain

\[
\begin{align*}
\frac{1}{\omega_v^2} \int dk \exp (i k \cdot x) \langle \tilde{\rho}_\omega (k) \tilde{\rho}_\omega (-k) \rangle \\
= \frac{\rho_s^2}{4\pi^2} \int dk \exp (i k \cdot x) \left( k^2 + \frac{1}{\rho_s^2} \right) \\
- \left( \frac{\rho_s^2}{4\pi^2} \right)^2 \int dk \exp (i k \cdot x) \left( k^2 + \frac{1}{\rho_s^2} \right)^2 \langle \tilde{\chi} (k) \tilde{\chi} (-k) \rangle
\end{align*}
\]

In physical space the correlation also reflects the statistical uniformity,

\[
\frac{1}{\omega_v^2} \int dk \exp (i k \cdot x) \langle \tilde{\rho}_\omega (k) \tilde{\rho}_\omega (-k) \rangle = \frac{1}{\omega_v^2} \langle \rho_\omega (x) \rho_\omega (0) \rangle
\]

The equation is

\[
\begin{align*}
\frac{1}{\omega_v^2} \langle \tilde{\rho}_\omega (k) \tilde{\rho}_\omega (-k) \rangle \\
= \frac{\rho_s^2}{4\pi^2} \left( k^2 + \frac{1}{\rho_s^2} \right) \left[ 1 - \frac{\rho_s^2}{4\pi^2} \left( k^2 + \frac{1}{\rho_s^2} \right) \langle \tilde{\chi} (k) \tilde{\chi} (-k) \rangle \right]
\end{align*}
\]

The second term in the bracket of Eq.(41) contains the two-point correlation of the function \( \chi \) and can be obtained by explicit calculation of the functional integration in Eq.\((??)\). The same analytical problem as for the explicit calculation of \( Z_V [J = 0] \) and \( \langle \chi \chi \rangle_k \equiv \langle \tilde{\chi} (k) \tilde{\chi} (-k) \rangle \) will appear later (for the turbulence scattered by the random vortices) and there we will give some details of calculation. At this moment few explanations are sufficient. For small amplitude of the auxiliary field \( \chi \) the function \( \cos \) is approximated with its first two terms

\[
\cos \chi \approx 1 - \frac{\chi^2}{2}
\]

The constant 1 is only a shift of the action. However it leads to a term that depends on \( Z_V^{(0)} \), which is integrated in the exponential over all volume i.e. the area \( A \) on the plane. In detail, replacing Eq.(42) in Eq.\((??)\)

\[
Z_V [J_\omega = 0] \equiv p^{-1} \int \mathcal{D} [\chi]
\]

\[
\times \exp \left[ -\frac{1}{8\pi^2} \int dxdy \left( -\frac{8\pi^2}{A} Z_V^{(0)} \right) \right] \times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 + \left( \frac{8\pi^2}{A} Z_V^{(0)} \right) \frac{\chi^2}{2} \right] \right\}
\]

\[
21
\]
The first factor can be taken outside the functional integration

\[
\exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left( -\frac{8\pi^2}{A} Z_V^{(0)} \right) \right\}
\]

\[= \exp \left[ Z_V^{(0)} \right] \quad \text{(44)}
\]

Since it is determined by the non-interacting vortices, it must exist even if we would neglect completely the interaction between the vortices taking \( \chi \to 0 \). The rest of the Eq. (43) is the Gaussian functional integral

\[
p^{-1} \int \mathcal{D}[\chi]
\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 + \left( \frac{8\pi^2}{A} Z_V^{(0)} \right) \frac{\chi^2}{2} \right] \right\}
\]

\[= p^{-1} \int \mathcal{D}[\chi]
\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \chi(x,y) \left[ -\Delta + \left( \frac{1}{\rho_s^2} + \frac{4\pi^2}{A} Z_V^{(0)} \right) \right] \chi(x,y) \right\}
\]

\[= q \left[ \det \left( -\Delta + \alpha^2 \right) \right]^{-1/2} \quad \text{(46)}
\]

where

\[\alpha^2 \equiv \frac{1}{\rho_s^2} + \frac{4\pi^2}{A} Z_V^{(0)} \]

The determinant can be calculated explicitly, by the product of the eigenvalues of the operator. This product (besides an infinite factor that will disappear) is convergent. However we keep this formal expression

\[
Z_V = Z_V [J_\omega = 0] = q \exp \left[ Z_V^{(0)} \right] \left[ \det \left( -\Delta + \alpha^2 \right) \right]^{-1/2} \quad \text{(47)}
\]

The second term in the bracket of Eq. (41) contains the two-point correlation of the function \( \chi \) and can be obtained by explicit calculation of the functional integration in Eq. (39). The same analytical problem as for the explicit calculation of \( Z_V [J_\omega = 0] \) (Eq. (??)) and \( \langle \chi \chi \rangle_k \equiv \langle \tilde{\chi}(k) \tilde{\chi}(-k) \rangle \) will appear later (for the turbulence scattered by the random vorticities) and there we will give some details of calculation. At this moment few explanations are sufficient. For small amplitude of the auxiliary field \( \chi \) the function \( \cos \) is approximated with its first two terms

\[
\cos \chi \approx 1 - \frac{\chi^2}{2} \quad \text{(48)}
\]
The constant 1 is only a shift of the action. However it leads to a term that depends on $Z_V^{(0)}$, which is integrated in the exponential over all volume i.e. the area $A$ on the plane. In detail, replacing Eq. (42) in Eq. (??)

$$Z_V[J_\omega = 0] \equiv p^{-1} \int \mathcal{D}[\chi] \exp \left[ -\frac{1}{8\pi^2} \int dxdy \left( \frac{8\pi^2}{A} Z_V^{(0)} \right) \right]$$

$$\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ (\nabla \chi)^2 + \frac{1}{\rho_s^2} \chi^2 + \left( \frac{8\pi^2}{A} Z_V^{(0)} \right) \frac{\chi^2}{2} \right] \right\}$$

The first factor can be taken outside the functional integration and is $\exp \left[ Z_V^{(0)} \right]$. Since it is determined by the non-interacting vortices, it must exist even if we would neglect completely the interaction between the vortices taking $\chi \to 0$. The rest of the Eq.(43) is the Gaussian functional integral. Introducing the notation $\alpha^2 \equiv 1/\rho_s^2 + 4\pi^2 Z_V^{(0)}/A$

$$Z_V[J_\omega = 0] = q \exp \left[ Z_V^{(0)} \right] [\det (-\Delta + \alpha^2)]^{-1/2} \quad (49)$$

and $q$ is a constant. The determinant can be calculated explicitly, by the product of the eigenvalues of the operator. We need this explicit expression because we need the functional dependence of $Z_V = Z_V[J_\omega = 0]$ on $Z_V^{(0)}$ as results from Eq.(28). As will become clear later, the factor with the determinant, which comes from the influence of the fluctuating shape of a vortex on the correlations of the vorticity of a gas of vortices in the plane, is affected by a factor $\rho_s^2/A$, which is small compared with the exponential in Eq.(47). Therefore we calculate in a one dimensional cartesian approximation the eigenvalues, instead of a cylindrical problem. We have to solve

$$\left( -\frac{d^2}{dx^2} + \alpha^2 \right) \eta_n (x,y) = \lambda_n^\eta \eta_n (x,y)$$

on an interval $L$. The eigenvalues are $\lambda_n^\eta = (2\pi n/L)^2 + \alpha^2$, where $n$ is an integer, and we obtain

$$\det (-\Delta + \alpha^2) = \prod_{n=1}^\infty \lambda_n^\eta = \prod_{n=1}^\infty [(2\pi n/L)^2 + \alpha^2]$$

$$= \prod_{n=1}^\infty (2\pi n/L)^2 \prod_{n=1}^\infty \left[ 1 + \frac{\alpha^2 L^2}{n^2} \right]$$

$$= \frac{\sinh (\alpha L/2)}{\alpha L/2} \prod_{n=1}^\infty (2\pi n/L)^2$$

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The infinite product is eliminated since we always use the ratios of $Z_V [J_\omega]$ and $Z_V [J_\omega = 0]$. We also take $L = \sqrt{A}$ and obtain

$$Z_V = q \exp \left[ Z_V^{(0)} \right] \left\{ \frac{\sinh \left[ \left( A\rho_s^{-2} + 4\pi^2 Z_V^{(0)} \right)^{1/2} / 2 \right]}{\left( A\rho_s^{-2} + 4\pi^2 Z_V^{(0)} \right)^{1/2} / 2} \right\}^{-1/2}$$

The quantity $A\rho_s^{-2}$ is very large and an approximation is possible

$$\frac{\sinh \left[ \left( A\rho_s^{-2} + 4\pi^2 Z_V^{(0)} \right)^{1/2} / 2 \right]}{\left( A\rho_s^{-2} + 4\pi^2 Z_V^{(0)} \right)^{1/2} / 2} = \frac{1}{\rho_s^{-1}\sqrt{A} \left( 1 + \frac{1}{2} \frac{4\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right)} \exp \left[ \frac{\rho_s^{-1}\sqrt{A}}{2} \left( 1 + \frac{1}{2} \frac{4\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right) \right]$$

$$\simeq \frac{\exp \left( \rho_s^{-1}\sqrt{A}/2 \right)}{\rho_s^{-1}\sqrt{A}} \exp \left( \frac{\pi^2 Z_V^{(0)}}{\sqrt{A}\rho_s^{-1}} \right) \left( 1 - \frac{2\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right)$$

The first factor is large but constant and can be absorbed into the coefficient $q$. We get from Eq. (51)

$$Z_V = q \exp \left[ Z_V^{(0)} \right] \exp \left( -\frac{\pi^2 Z_V^{(0)}}{2\sqrt{A}\rho_s^{-1}} \right) \left( 1 - \frac{2\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right)^{-1/2}$$

$$= q \exp \left[ Z_V^{(0)} \right] \left( 1 - \frac{\pi^2}{2\sqrt{A}\rho_s^{-1}} \right) \left( 1 + \frac{\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right)$$

We can neglect the second term in the first exponential

$$Z_V = Z_V [J_\omega = 0] = q \left( 1 + \frac{\pi^2 Z_V^{(0)}}{A\rho_s^{-2}} \right) \exp \left[ Z_V^{(0)} \right]$$

The second term in Eq. (41), i.e. the auto-correlation of $\chi$ in $k$-space, may be calculated starting from the real-space correlation

$$\langle \chi (x,y) \chi (x',y') \rangle = \frac{1}{Z_V [J = 0]} p^{-1} \exp \left[ Z_V^{(0)} \right] \int \mathcal{D} [\chi] \chi (x,y) \chi (x',y')$$

$$\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \chi (x,y) (-\Delta + \alpha^2) \chi (x,y) \right\}$$
As usual we return to the form of Eq.(45), and only for this step, we insert an external current $J_e (x, y)$ interacting with $\chi$. The auxiliary functional is denoted $Z_\chi [J_e]$

$$Z_\chi [J_e] = \frac{1}{Z_V [J = 0]} p^{-1} \exp \left[ \frac{Z_V^{(0)}}{Z_V} \right] \int \mathcal{D} [\chi]$$

$$\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ \chi (x, y) \left( -\Delta + \alpha^2 \right) \chi (x, y) + J_e \chi \right] \right\}$$

$$= \frac{1}{Z_V [J = 0]} p^{-1} \exp \left[ Z_V^{(0)} \right] \int \mathcal{D} [\phi]$$

$$\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \phi (x, y) \left( -\Delta + \alpha^2 \right) \phi (x, y)$$

$$\quad -\frac{1}{8\pi^2} \int dxdy \frac{1}{4} J_e (x, y) \left( -\Delta + \alpha^2 \right)^{-1} J_e (x, y) \right\}$$

To obtain the above equation we have made a change of variables $\chi \rightarrow \phi = \chi + \frac{1}{2} (-\Delta + \alpha^2)^{-1} J_e$ of Jacobian 1. The functional integration over $\phi$ can now be carried out and the rest of the expression at the exponent appears in a factor

$$\sim \left[ \det (-\Delta + \alpha^2) \right]^{-1/2} \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \frac{1}{4} J_e (x, y) \left( -\Delta + \alpha^2 \right)^{-1} J_e (x, y) \right\}$$

where the symbol $\sim$ means that there also result constant factors. But these are the same as those contained in the factor $q$ introduced in the Eq.(47). We then have

$$Z_\chi [J_e] = \frac{1}{Z_V [J = 0]} p^{-1} \exp \left[ \frac{Z_V^{(0)}}{Z_V} \right] \int \mathcal{D} [\chi]$$

$$\times \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \left[ \chi (x, y) \left( -\Delta + \alpha^2 \right) \chi (x, y) + J_e \chi \right] \right\}$$

$$= p^{-1} \exp \left\{ -\frac{1}{8\pi^2} \int dxdy \frac{1}{4} J_e (x, y) \left( -\Delta + \alpha^2 \right)^{-1} J_e (x, y) \right\}$$
where we have taken into account Eq.(47). The correlation is

$$\langle \chi(x,y) \chi(x',y') \rangle = \frac{1}{Z_{\chi} [J_e = 0]} \delta Z_{\chi} [J_e] \left[ \frac{\delta}{\delta J_e (x', y')} \right]_{J_e = 0}$$

$$= \frac{1}{Z_{\chi} [J_e = 0]} p^{-1} \frac{\delta}{\delta J_e (x', y')} \left[ \left( -\frac{1}{8\pi^2} 2 \left( -\Delta + \alpha^2 \right)^{-1} J_e (x,y) \right) \exp \{ \ldots \} \right]_{J_e = 0}$$

$$= \frac{1}{Z_{\chi} [J_e = 0]} p^{-1} \left[ -\frac{1}{4\pi^2} \left( -\Delta + \alpha^2 \right)^{-1} \delta (x - x') \exp \{ \ldots \} 
+ \left( -\frac{1}{8\pi^2} 2 \left( -\Delta + \alpha^2 \right)^{-1} J_e (x,y) \right) \left( -\frac{1}{8\pi^2} 2 \left( -\Delta + \alpha^2 \right)^{-1} J_e (x', y') \right) \exp \{ \ldots \} \right]_{J_e = 0}$$

(the factor \( \exp \left[ Z_{(0)}^V \right] \) has not been written since it disappears). Finally we have

$$\langle \chi(x,y) \chi(x',y') \rangle = -\frac{1}{4\pi^2} \left( -\Delta + \alpha^2 \right)^{-1} \delta (x - x')$$

or

$$\langle \chi \chi \rangle_k = \left[ \frac{1}{4\pi^2} \rho_s^2 \left( k^2 + \frac{1}{\rho_s^2} \right) \right]^{-1} \left( \frac{4\pi^2 A Z_{(0)}^V}{\rho_s^2} \right)$$

Introducing this in Eq.(44), the correlation of the field of the scalar potential will have from this a contribution (with normalization)

$$\frac{1}{\phi_0^2} \langle \phi_V \phi_V \rangle_k^{vort} = \frac{1}{k^2 \rho_s^2} \left( 1 + \frac{1}{k^2 \rho_s^2} \right)$$

$$\times \frac{1}{4\pi^2} \left[ 1 - \frac{1 + k^2 \rho_s^2}{1 + k^2 \rho_s^2 + Z_{(0)}^V / 4\pi^2 \rho_s^2 / A} \right]$$

which represents, as said, the first term of the Eq.(28).

The other terms in the correlation \( \langle \phi_V \phi_V \rangle \) (Eq.(28)) are due to the fluctuation of the form of the generic vortex from interaction with the background turbulence. We calculate, using Eq.(57), the derivatives

$$\frac{\delta Z_V}{\delta Z_{(0)}^V} = q \left( 1 + \frac{\pi^2}{A \rho_s^{-2}} + \frac{\pi^2 Z_{(0)}^V}{A \rho_s^{-2}} \right) \exp \left[ Z_{(0)}^V \right]$$

and

$$\frac{\delta^2 Z_V}{\delta [Z_{(0)}^V]^2} = q \left( 1 + \frac{2\pi^2}{A \rho_s^{-2}} + \frac{\pi^2 Z_{(0)}^V}{A \rho_s^{-2}} \right) \exp \left[ Z_{(0)}^V \right]$$

Now we have to calculate the derivatives of \( Z_{(0)}^V \) to \( J \) as shown by the last two terms Eq.(28). This part will be added after \( Z_{(0)}^V \) is calculated, in the next subsection.
4.2 A single vortex interacting with a turbulent environment

The partition function for a single vortex in interaction with turbulence is defined as

\[ Z_V^{(0)} = \mathcal{N}^{-1} \int D[\chi] D[\phi] \exp \{ S_V[\chi, \phi] \} \] (59)

and the equation is the stationary form of the equation used in Ref. [2],

\[ F[\phi] \equiv \nabla_\perp^2 \varphi - \alpha \varphi - \beta \varphi^2 \] (60)

The density of Lagrangean

\[ \mathcal{L}[\chi(x, y), \phi(x, y)] = \chi \left( \nabla_\perp^2 \varphi - \alpha \varphi - \beta \varphi^2 \right) \] (61)

is obtained from the Martin-Siggia-Rose method for classical stochastic systems. Then the action is

\[ S_V[\chi, \phi] = \int dx dy \mathcal{L}[\chi(x, y), \phi(x, y)] \] (62)

As usual we introduce the interaction with the external current \( J = (J_\phi, J_\chi) \). However since there is no need to calculate functional derivatives to \( \chi \), we can only keep \( J \equiv J_\phi \). For uniformity of notation in this paper, we will not use the factor \( i \) in front of the action in contrast with [2]. We have then

\[ Z_V^{(0)}[J] = \mathcal{N}^{-1} \int D[\chi] D[\phi] \exp \left\{ \int dx dy \left[ \chi \left( \nabla_\perp^2 \varphi - \alpha \varphi - \beta \varphi^2 \right) + J_\phi \right] \right\} \] (63)

We need the explicit expression of the functional integral Eq. (63) and this has been obtained in the references [2] and [3]. For convenience we will recall briefly the steps of the calculation restricting to the results we need in the present work.

To calculate the generating functional of the vortex in the background turbulence we proceed in two steps: we solve the Euler-Lagrange equation for the action (62), obtaining the configuration of the system which extremises this action; further, we expand the action to second order in the fluctuations around this extremum (which will include the turbulent field) and integrate. The Euler-Lagrange equations have the solutions

\[ \varphi_{Js}(x, y) \equiv \varphi_s(x, y) \] (64)
\[ \chi_{Js}(x, y) = - \varphi_s(x, y) + \bar{\chi}_J(x, y) \]
The first is the static form of the solution Eq. (10) and does not depend on $J$. The dual function is $-\varphi_s(x, y)$ plus a term resulting from the excitation by $J$ in its equation. This additional term $\tilde{\chi}_J(x, y)$ is calculated by the perturbation of the KdV soliton solution according to the modification of the Inverse Scattering Transform when an inhomogeneous term (i.e. $J$) is included. The action functional is calculated for these two functions

$$S_{Vs}[J] \equiv S_V[\varphi_s(x, y), -\varphi_s(x, y) + \tilde{\chi}_J(x, y)]$$

Then the first part of our calculation is $Z^{(0)}_V[J] \sim N^{-1} \exp\{S_{Vs}[J]\}$. Expanding the action around this extremum

$$S_V[\chi, \varphi; J] = S_V[\varphi_{Js}, \chi_{Js}] + \frac{1}{2} \left( \frac{\delta^2 S_V[J]}{\delta \varphi \delta \chi} \right)_{\varphi_{Js}, \chi_{Js}}$$

we calculate the Gaussian integral and obtain

$$Z^{(0)}_V[J] = N^{-1} \exp\{S_{Vs}[J]\} \left[ \det \left( \frac{\delta^2 S_V[J]}{\delta \varphi \delta \chi} \right)_{\varphi_{Js}, \chi_{Js}} \right]^{-1/2}$$

If we can neglect the advection of vortices by large scale wave-like fluctuations, we can calculate the determinant since the product of the eigenvalues converges without the need for regularization. The result is

$$Z^{(0)}_V[J] = N^{-1} \exp\{S_{Vs}[J]\} \ A \ B \quad (65)$$

Where

$$A = A[J] \equiv \left[ \frac{\beta/2}{\sinh (\beta/2)} \right]^{1/4} \quad (66)$$

$$B = B[J] \equiv \left[ \frac{\sigma/2}{\sin (\sigma/2)} \right]^{1/2} \quad (67)$$

The eigenvalue problem depends functionally on $\tilde{\chi}_J(x, y)$ which implies that $\beta$ and $\sigma$ depend on $J$. Their expressions can be found in [3]. With those detailed formulas we can consider that we have the necessary knowledge to proceed to the calculation of the functional derivatives of $Z^{(0)}_V[J]$ at $J$, using Eq. (65).

$$\frac{1}{Z^{(0)}_V[J = 0]} \frac{\delta Z^{(0)}_V[J]}{\delta J} = \frac{\delta S_{Vs}[J]}{\delta J} + \frac{1}{A} \frac{\delta A}{\delta J} + \frac{1}{B} \frac{\delta B}{\delta J} \quad (68)$$
For the present problem it is sufficient to take the first term as Eq. (10)

\[
\frac{\delta S_{V_s}[J]}{\delta J} \bigg|_{J=0} \simeq \phi_s
\]  

(69)

The next two terms in Eq. (68) represent the averaged, systematic modification of the shape of the field around the vortex due to the mutual interaction. They will be equally neglected, assuming that the main effect is contained in the dispersion of the fluctuations of the shape of the vortex interacting with the random field (which actually is our main concern here). Then

\[
\frac{1}{Z^{(0)}_V[J = 0]} \frac{\delta Z^{(0)}_V[J]}{\delta J} \simeq \phi_s
\]  

(70)

The second derivative to the excitations in two points \(y_1\) and \(y_2\) is

\[
\frac{1}{Z^{(0)}_V[J = 0]} \frac{\delta^2 Z^{(0)}_V[J]}{\delta J(y_2) \delta J(y_1)} = \frac{\delta S_{V_s}[J]}{\delta J(y_2)} \frac{\delta S_{V_s}[J]}{\delta J(y_1)} + \frac{\delta^2 S_{V_s}[J]}{\delta J(y_2) \delta J(y_1)}
\]

\[
+ \frac{1}{A} \frac{\delta A}{\delta J(y_2)} \frac{\delta S_{V_s}[J]}{\delta J(y_1)} + \frac{1}{B} \frac{\delta B}{\delta J(y_2)} \frac{\delta S_{V_s}[J]}{\delta J(y_1)}
\]

\[
+ \frac{1}{A} \frac{\delta A}{\delta J(y_1)} \frac{\delta S_{V_s}[J]}{\delta J(y_2)} + \frac{1}{B} \frac{\delta B}{\delta J(y_1)} \frac{\delta S_{V_s}[J]}{\delta J(y_2)}
\]

\[
+ \frac{1}{A} \frac{\delta^2 A}{\delta J(y_2) \delta J(y_1)} + \frac{1}{B} \frac{\delta^2 B}{\delta J(y_2) \delta J(y_1)}
\]

(71)

Since we have assumed as an acceptable approximation to neglect the averaged change produced by the turbulence on the soliton shape the second
term in the RHS is zero. For the first term we use Eq. (69). We have

\[
\frac{1}{Z_{V}^{(0)}[J = 0]} \frac{\delta^2 Z_{V}^{(0)}[J]}{\delta J(y_2) \delta J(y_1)} = \phi_s(y_2) \phi_s(y_1) + \\
+ \phi_s(y_1) \left[ \frac{\delta}{\delta J(y_2)} \ln A + \frac{\delta}{\delta J(y_2)} \ln B \right] + \phi_s(y_2) \left[ \frac{\delta}{\delta J(y_1)} \ln A + \frac{\delta}{\delta J(y_1)} \ln B \right] \\
\text{\hspace{1cm}} + \frac{\delta \ln A}{\delta J(y_1)} \frac{\delta \ln B}{\delta J(y_2)} + \frac{\delta \ln A}{\delta J(y_2)} \frac{\delta \ln B}{\delta J(y_1)} \\
\text{\hspace{1cm}} + \frac{1}{A} \frac{\delta^2 A}{\delta J(y_2) \delta J(y_1)} + \frac{1}{B} \frac{\delta^2 B}{\delta J(y_2) \delta J(y_1)}
\]

The expressions are complicated (see the Appendix of Ref. [3]) and so numerical calculation of these expression is unavoidable. For small amplitude of the turbulent field the expression can be rewritten

\[
\frac{1}{Z_{V}^{(0)}[J = 0]} \frac{\delta^2 Z_{V}^{(0)}[J]}{\delta J(y_2) \delta J(y_1)} \bigg|_{J = 0} = \phi_s(y_2) \phi_s(y_1) [1 + f(y)]
\]

i.e. in a form that expresses the fact that the two-point correlation is basically the auto-correlation of the potential of the exact soliton modified by a function \( f \) which collects the contributions from the interaction with the random field. In \( k \)-space we have

\[
\frac{1}{Z_{V}^{(0)}[J = 0]} \frac{\delta^2 Z_{V}^{(0)}[J]}{\delta J(y_2) \delta J(y_1)} \bigg|_{J = 0, k} = \phi_s(k) \phi_s(-k) [1 + f(k)]
\]

At the limit where we do not expand the action to include configurations resulting from the interaction vortex-turbulence, we have \( \sigma \to 0 \) and \( \beta \to 0 \) and it results \( A = B = 1 \). In this case \( f \equiv 0 \).

The detailed expressions of these terms are given in the paper [2], [3]. For the purpose of comparisons we will express the spectrum as

\[
\frac{1}{\phi_0^2} \langle \phi_V \phi_V \rangle_k = S(k) [1 + f(k)]
\]

(72)
where $\phi_0$ is amplitude of a vortex, $S(k)$ has been derived by Meiss and Horton [14].

$$S(k) = \left\{12\sqrt{2}\pi^{3/2} kp_s u \frac{\csc h}{v_s} \left[\frac{\pi kp_s}{(1 - v_s/u)^{1/2}}\right]\right\}^2 \quad (73)$$

and $f(k)$ is function that is the correction to the Fourier transform of the squared secant-hyperbolic, produced by the turbulent waves.

Before proceeding further with the calculations based on the Eq.(72) we need to discuss the formal term $f(k)$. Since this term represents the difference from the simple isolated vortex to the vortex perturbed by turbulence, one would like to have a quantitative connection between the amplitude of this term and at least two elements characterizing the background turbulence: (1) the amplitude and (2) the spectrum.

In the way we have conducted the calculations of the generating functional Eqs.(65), (66), (67) the new terms in the expression of the auto-correlation due to the factors $A$ and $B$ are expressed in real space, not in Fourier space. They are obtained from the product of eigenvalues of the operator representing the second order functional derivative of the action, i.e. they are connected with the geometry of the function space around the exact, vortex, nonlinear solution. The determinant of the operator $\delta^2 S_V / (\delta \phi / \delta \chi)$ may be seen as a volume in the function space, centered on the vortex solution. The inverse of any eigenvalue gives an idea of the extension along a particular direction (eigenfunction) in function space. When an eigenvalue is very small, the operator almost vanishes on functions along that direction. At the limit this is a zero mode and corresponds to a translational symmetry of the physical system along that direction. The correlations depend on the sensitivity of this volume (the product of the eigenvalues) on the excitation $J$ applied on the system. The excitation is first manifested in the appearance of $\tilde{\chi}_J(x, y)$. This one consists of a part that will modify the shape of the exact vortex plus the oscillating tail generated when a soliton is perturbed. The latter can be considered as a component of the background turbulence. The “propagation” of the influence from an excitation $J$ can be summarised symbolically in the chain : $J \to \tilde{\chi}_J(x, y) \to$ eigenvalues of the operator $\delta^2 S_V / (\delta \phi / \delta \chi) \to A$ and $B$ (or $\sigma$ and $\beta$). The expressions of $\sigma$ and $\beta$ can be found in [3].
Now we can return to the Eq. (28). Using Eq. (70) the second term is
\[
\frac{1}{Z_V [J = 0]} \delta^2 Z_V \frac{\delta Z_V^{(0)}}{\delta J (x, y)} \frac{\delta Z_V^{(0)}}{\delta J (x', y')}
\]

\[
= \exp \left[ Z_V^{(0)} \right] q \left( 1 + Z_V^{(0)} \rho_s^2 \pi^2 / A \right) \exp \left[ Z_V^{(0)} \right] q \left[ 1 + \left( 2 + Z_V^{(0)} \right) \rho_s^2 \pi^2 / A \right] \delta Z_V (0) V \delta J (x', y')
\]

The third term in Eq. (28) is
\[
\frac{1}{Z_V [j = 0]} \delta Z_V \frac{\delta^2 Z_V^{(0)}}{\delta J (x, y) \delta J (x', y')}
\]

\[
= \exp \left[ Z_V^{(0)} \right] q \left( 1 + Z_V^{(0)} \rho_s^2 \pi^2 / A \right) \left( 1 + \frac{2 \rho_s^2 \pi^2 / A}{1 + Z_V^{(0)} \rho_s^2 \pi^2 / A} \right) \phi_s (x, y) \phi_s (x', y') (1 + f)
\]

In \( \mathbf{k} \) space the two contributions reads
\[
\frac{1}{Z_V [J = 0]} \delta^2 Z_V \frac{\delta Z_V^{(0)}}{\delta J (x, y) \delta J (x', y')} + \frac{1}{Z_V [j = 0]} \delta Z_V \frac{\delta^2 Z_V^{(0)}}{\delta J (x, y) \delta J (x', y')}
\]

\[
= \phi_0^2 S (\mathbf{k}) \left[ 1 + \frac{2 \rho_s^2 \pi^2 / A}{1 + Z_V^{(0)} \rho_s^2 \pi^2 / A} \right] (1 + f) \left( 1 + \frac{\rho_s^2 \pi^2 / A}{1 + Z_V^{(0)} \rho_s^2 \pi^2 / A} \right)
\]

\[
= \phi_0^2 S (\mathbf{k}) \left( 2 + f + \frac{3 + f}{A \rho_s^2 + Z_V^{(0)}} \right)
\]

The results for Eq. (28) can now be collected
\[
\frac{1}{\phi_0^2} \langle \phi_{V'} \phi_{V'} \rangle_{\kappa}^{\text{tor} + \text{cs}}
\]

\[
= \frac{1}{k^2 \rho_s^2} \left( 1 + \frac{1}{k^2 \rho_s^2} \right) \frac{1}{8 \pi^2} \left[ 1 - \frac{\rho_s^2 k^2 + 1}{\rho_s^2 k^2 + 1 + Z_V^{(0)} 4 \pi^2 \rho_s^2 / A} \right] + S (\mathbf{k}) \left( 2 + f + \frac{3 + f}{A \rho_s^2 + Z_V^{(0)}} \right)
\]

\[32\]
We can make few remarks here. If the arbitrary position in plane of the vortices and the interaction between physical vortices were neglected, the only term that would persist is $\exp \left[ Z_V^{(0)} \right]$. The first $k$-dependent factor in Eq. (??) comes from assuming that a statistical ensemble of realizations of the vorticity field is generated from the random positions in plane of the vortices, even reduced at a $\delta$-type shape. In practical terms this may be represented as follows: in a plane, an ensemble of vortices can be placed at arbitrary positions. We construct the statistical ensemble of the realizations of this stochastic system. If we measure in one point the field, it will be zero for most of the realizations and it will be finite when it happens that a vortex is there. This is a random variable. Now, if we measure in two points and collect the results for all realizations, the statistical properties of this quantity (the two-point auto-correlation) has a Fourier transform that is given by the two factors multiplying the square bracket in Eq. (??), divided to $k^4$ (since we have the auto-correlation of the vorticity). When the interaction is considered, the factor in the curly bracket appears.

5  Random field influenced by vortices with random positions

Consider the equation

$$F[\phi] \equiv \nabla_\perp^2 \phi - \alpha \phi - \beta \phi^2 = 0 \quad (76)$$

and extract from the total function the part that is due to the vortices

$$\phi(x, y) = \sum_{a=1}^N \phi_s^{(a)}(x, y) + \phi(x, y) \quad (77)$$

Replacing in the equation we have

$$\nabla_\perp^2 \left[ \sum_{a=1}^N \phi_s^{(a)}(x, y) \right] - \alpha \sum_{a=1}^N \phi_s^{(a)}(x, y) - \beta \left[ \sum_{a=1}^N \phi_s^{(a)}(x, y) \right]^2$$

$$- 2\beta \left[ \sum_{a=1}^N \phi_s^{(a)}(x, y) \right] \phi(x, y)$$

$$+ \nabla_\perp^2 \phi - \alpha \phi - \beta \phi^2$$

$$= 0$$
The first line is zero and we have
\[ \nabla^2 \phi - \left[ \alpha + 2\beta \sum_{a=1}^{N} \phi_s^{(a)}(x, y) \right] \phi - \beta \phi^2 = 0 \] (79)

We write a Lagrangean for the random field according to the MSR procedure
\[ \mathcal{L} [\chi, \phi] = \chi \left\{ \nabla^2 \phi - \left[ \alpha + 2\beta \sum_{a=1}^{N} \phi_s^{(a)}(x, y) \right] \phi - \beta \phi^2 \right\} \] (80)
and the action functional is
\[ S_{\varphi V} [\chi, \phi] = \int dxdy \chi \left\{ \nabla^2 \phi - \left[ \alpha + 2\beta \sum_{a=1}^{N} \phi_s^{(a)}(x, y) \right] \phi - \beta \phi^2 \right\} \] (81)

The generating functional is defined from the functional integral
\[ \mathcal{N}^{-1} \int D [\chi] D [\phi] \exp (-S_{\varphi V} [\chi, \phi]) \] (82)

Now we will modify the action by considering as usual the interaction with external currents,
\[ \Xi [J] \equiv \mathcal{N}^{-1} \int D [\chi] D [\phi] \exp (-S_{\varphi V} [\chi, \phi] + J_\chi \chi + J_\phi \phi) \] (83)

With this functional integral we will have to calculate the free energy functional. The functional \( \Xi [J] \) depends on the function representing the vortices. The vortices are assumed known but their position in plane is random therefore we have to average over them: \(-W [J] = \langle \ln (\Xi [J]) \rangle \).

### 5.1 The average over the positions

To perform the statistical average over the random positions of the vortices. The functional which we have to average is
\[ \Xi = \mathcal{N}^{-1} \int D [\chi] D [\phi] \times \exp \left\{ \int dxdy \left[ (\nabla \chi)(\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 \right. \right. \] 
\[ + \left. \left. \left( 2\beta \sum_{a=1}^{N} \phi_s^{(a)}(x, y) \right) \chi \phi \right] \right\} \] (84)
The part that depends on the positions can be written

\[
\langle \exp \left\{ \int dxdy \left( 2\beta \sum_{a=1}^{N} \phi_s^{(a)} (x, y) \right) \chi \phi \right\} \rangle \quad (85)
\]

\[
= \langle \exp \left\{ \int dxdy 2\beta \phi^s \sum_{a=1}^{N} \delta (r - r_a) \chi \phi \right\} \rangle \quad (75)
\]

\[
= \langle \exp \left[ 2\beta \phi^s \sum_{a=1}^{N} \chi (r_a) \phi (r_a) \right] \rangle
\]

where \( \phi^s \) is now a simple amplitude. Consider the more general situation where we have to average in addition over the amplitudes \( \phi^s \) of the vortices.

If \( \phi^s \) is a stochastic variable we have to perform an average of the type

\[
\langle \exp \left[ 2\beta \phi^s \sum_{a=1}^{N} \chi (r_a) \phi (r_a) \right] \rangle_{\phi^s} \quad (86)
\]

Consider that the (now) random variable \( \phi^s \) has the probability density

\[
g (\phi^s) \quad (87)
\]

Then, restricting for the moment to only the average over \( \phi^s \),

\[
\langle \exp \left[ 2\beta \phi^s \chi (r_a) \phi (r_a) \right] \rangle_{\phi^s} \quad (88)
\]

\[
= \int_{-\infty}^{\infty} d\phi^s g (\phi^s) \exp \left[ 2\beta \phi^s \chi (r_a) \phi (r_a) \right]
\]

\[
= \int_{-\infty}^{\infty} d\phi^s g (\phi^s) \exp (i\lambda \phi^s)
\]

where

\[
i\lambda (r_a) \equiv 2\beta \chi (r_a) \phi (r_a) \quad (89)
\]

Then

\[
\langle \exp \left[ 2\beta \phi^s \chi (r_a) \phi (r_a) \right] \rangle_{\phi^s} = \tilde{g} [\lambda (r_a)] \quad (90)
\]

where \( \tilde{g} \) is the Fourier transform of the probability distribution function \( g \).

We use the notation

\[
\tilde{g} (r_k) \equiv \tilde{g} [\lambda (r_a)] \quad (91)
\]
and we have to calculate
\[
\prod_{a=1}^{N} \langle \tilde{g} (-i2\beta \chi (r_a) \phi (r_a)) \rangle_{r_a} = \prod_{a=1}^{N} \langle \tilde{g} (r_a) \rangle
\] (92)

For this we introduce the function \( h \)
\[
\tilde{g} (r_a) \equiv h (r_a) + 1
\] (93)
and we rewrite the average as
\[
\prod_{a=1}^{N} \langle \tilde{g} (r_a) \rangle
\] (94)
\[
= \prod_{a=1}^{N} \langle h (r_a) + 1 \rangle
\]
\[
= \left\langle \sum_{l=0}^{\infty} \sum_{i_1<i_2<...<i_l} h (r_{i_1}) h (r_{i_2}) ... h (r_{i_l}) \right\rangle
\]
\[
= \exp \left\{ \sum_{l=0}^{\infty} \frac{1}{l!} \int dr_{i_1} dr_{i_2} ... dr_{i_l} h (r_{i_1}) h (r_{i_2}) ... h (r_{i_l}) C^{(l)} (r_{i_1}, r_{i_2}, ..., r_{i_l}) \right\}
\]
where we have introduced the cumulants of the distribution of points \( r_{i_1} \) in the plane.

According to our assumption
\[
C^{(l)} = \begin{cases} 
1/A & l = 1 \\
0 & l > 1 
\end{cases}
\] (95)
where \( A \) is the area in plane. Therefore we have the result of averaging
\[
\prod_{a=1}^{N} \langle \tilde{g} (r_a) \rangle = \exp \left\{ \frac{1}{A} \int dr h (r) \right\} \] (96)
\[
= \exp \left\{ \frac{1}{A} \int dr [\tilde{g} (r) - 1] \right\}
\]
The part of the generating functional which depended on the positions can now be written
\[
\left\langle \exp \left[ 2\beta \phi^s \sum_{a=1}^{N} \chi (r_a) \phi (r_a) \right] \right\rangle_{r_a, \phi^s} = \exp \left\{ \frac{1}{A} \int dr [\tilde{g} (r) - 1] \right\}
\] (97)
and the full partition function is
\[
\langle \Xi \rangle_{\mathbf{r}_a, \phi^s} = N^{-1} \int D[\chi] D[\phi] \times \exp \left\{ \int dxdy \left[ (\nabla \chi) (\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{1}{A} \int d\mathbf{r} [\tilde{g}(\mathbf{r}) - 1] \right] \right\} 
\]

The action at the exponent is
\[
\int dxdy \left\{ (\nabla \chi) (\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{1}{A} [\tilde{g}(x, y) - 1] \right\} 
\]

The function \( \tilde{g}(\mathbf{r}) \) has as argument the expression \(-i2\beta \chi(\mathbf{r}) \phi(\mathbf{r})\).

A reasonable assumption is that the vortices can only be positive or negative and with the same magnitude. Then we have
\[
g(\phi^s) = \frac{1}{2} [\delta (\phi^s - \phi_0) + \delta (\phi^s + \phi_0)] 
\]

and the Fourier transform is
\[
\tilde{g}(q) = \int d\phi^s \exp (iq\phi^s) \frac{1}{2} [\delta (\phi^s - \phi_0) + \delta (\phi^s + \phi_0)] 
\]
\[
= \frac{1}{2} [\exp (iq\phi_0) + \exp (-iq\phi_0)] 
\]
\[
= \cos (q\phi_0) 
\]

This must be calculated for the argument \(-i2\beta \chi(\mathbf{r}) \phi(\mathbf{r})\) and gives
\[
\tilde{g}(r) = \cos [-i2\beta \phi_0 \chi(\mathbf{r}) \phi(\mathbf{r})] = \cosh [2\beta \phi_0 \chi(\mathbf{r}) \phi(\mathbf{r})] 
\]

The functional integral that must be calculated becomes
\[
\langle \Xi \rangle_{\mathbf{r}_b, \phi^s} = N^{-1} \int D[\chi] D[\phi] \times \exp \left\{ \int dxdy \left[ (\nabla \chi)(\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{1}{A} \cosh (2\beta \phi_0 \chi \phi) \right] \right\} 
\]

The system is perturbed with an external current \( J(x, y) \) acting on the field \( \phi(x, y) \).

\[
Z_J \equiv \langle \Xi \rangle_{\mathbf{r}_b, \phi^s}[J] = N^{-1} \int D[\chi] D[\phi] \exp (S_J) 
\]
\[ S_J \equiv \int dxdy \left[ (\nabla \chi)(\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{1}{A} \cosh (2\beta \phi_0 \chi \phi) + J \phi \right] \quad (105) \]

6 Approximation for small amplitude vortices

Consider that the amplitudes of the vortices are not high, \( \phi_0 \). Then

\[ \cosh (2\beta \phi_0 \chi \phi) \simeq 1 + \frac{1}{2} (2\beta \phi_0 \chi \phi)^2 \quad (106) \]

and the action (removing some terms without significance)

\[ S_J \equiv \int dxdy \left[ (\nabla \chi)(\nabla \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{(2\beta \phi_0)^2}{A} \chi^2 \phi^2 + J \phi \right] \quad (107) \]

This action in principle can lead to a perturbative treatment but with two vertices, of order three and of order four, a very difficult and unusual problem. We remark however that

\[ S_J \equiv \int dxdy \left[ -\chi (\nabla^2 \phi) + \alpha \chi \phi + \beta \chi \phi^2 + \frac{(2\beta \phi_0)^2}{A} \chi^2 \phi^2 + J \phi \right] \quad (108) \]

may become quadratic in \( \chi \) and in \( \phi \) (therefore the functional integral is Gaussian) if we succeed to separate the product \( \chi^2 \phi^2 \).

6.0.1 Technical step

Consider the following formula to disentangle the two variables

\[ \exp \left( \frac{1}{2} U^2 \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 - Us \right) \quad (109) \]

then

\[ \exp \left( \frac{1}{2} 2Q \chi^2 \phi^2 \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 - \sqrt{2Q} \chi \phi s \right) \quad (110) \]

and we have the action

\[ Z_J = N^{-1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 \right) \quad (111) \]

\[ \times \int D[\phi] D[\chi] \exp \left\{ \int dxdy \left[ -\chi (\nabla^2 \phi) + \alpha \chi \phi + \beta \chi \phi^2 - \sqrt{2Q} \chi \phi s + J \phi \right] \right\} \]

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or

\[ S_J = \int dx dy [J\phi] + \int dx dy \left[ -\chi (\nabla^2 \phi) + \chi \left( \alpha - s \sqrt{2Q} \right) \phi + \beta \chi \phi^2 \right] \]  \hspace{1cm} (112)

We have used the notation

\[ Q \equiv \frac{(2\beta \phi_0)^2}{A} \]  \hspace{1cm} (113)

The functional integration over \( \chi \) can be done immediately and gives a functional \( \delta \) with the argument the equation in the modified form. If we make the integration over \( \chi \) we obtain

\[ \int D[\chi] \exp \left\{ \int dx dy \left[ -\chi (\nabla^2 \phi) + \chi \left( \alpha - s \sqrt{2Q} \right) \phi + \beta \chi \phi^2 \right] \right\} \]  \hspace{1cm} (114)

\[ = \int D[\chi] \exp \left[ \int dx dy \chi \overline{F}(\phi) \right] \]

\[ = \left[ -i \frac{\delta \overline{F}}{\delta \phi} \right]^{-1} \delta (\phi - \phi^z) \]

where we have introduced the notation

\[ \overline{F}(\phi) \equiv -\nabla^2 \phi + \left( \alpha - s \sqrt{2Q} \right) \phi + \beta \phi^2 \]  \hspace{1cm} (115)

and the function \( \phi^z \) is the solution of the differential equation \( \overline{F}(\phi) = 0 \) i.e.

\[ \overline{F}(\phi^z) = 0 \]  \hspace{1cm} (116)

Inserting this result in the integral over the functions \( \phi \), we have

\[ \mathcal{N}^{-1} \int D[\phi] \exp \left( \int dx dy J\phi \right) \left[ -i \frac{\delta \overline{F}}{\delta \phi} \right]^{-1} \delta (\phi - \phi^z) \]  \hspace{1cm} (117)

\[ = \left[ -i \frac{\delta \overline{F}}{\delta \phi} \right]^{-1} \exp \left( \int dx dy J\phi^z \right) \]

Then

\[ Z_J = \mathcal{N}^{-1} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 \right) \left[ -i \frac{\delta \overline{F}}{\delta \phi} \right]^{-1} \exp \left( \int dx dy J\phi^z \right) \]  \hspace{1cm} (118)
The most important part is to calculate
\[-i \frac{\delta F}{\delta \phi} \bigg|_{\phi^z} = -i \frac{\delta}{\delta \phi} \left[ -\nabla^2 \phi + \left( \alpha - s \sqrt{2Q} \right) \phi + \beta \phi^2 \right] \bigg|_{\phi^z} \tag{119}\]

But this simplifies with the part of the factor
\[\mathcal{N}^{-1} \sim \left| \frac{\delta F}{\delta \phi} \right|_{\phi^z} \tag{120}\]

The fact that the initial factor is written before averaging over the positions and the second factor is obtained after averaging will make a small difference which can only be of higher order. What remains is a factor of normalization that we call \(c^{-1}\).

The functional derivatives are
\[
\frac{\delta Z_j}{\delta J(x,y)} = \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 \right) \bar{\phi}^z(x,y) \exp \left( \int dx dy J \phi \right) \tag{121}
\]
and
\[
\frac{\delta^2 Z_j}{\delta J(x,y) \delta J(x',y')} = \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds \exp \left( -\frac{1}{2} s^2 \right) \times \bar{\phi}^z(x,y) \bar{\phi}^z(x',y') \exp \left( \int dx dy J \phi \right) \tag{122}
\]
and taking
\[J \equiv 0 \tag{123}\]
we can calculate the contribution to the two-point correlation arising from this part of the partition function.

We have to solve
\[-\nabla^2 \phi + \left( \alpha - s \sqrt{2Q} \right) \phi + \beta \phi^2 = 0 \tag{124}\]
for a small amplitude field \(\phi\). This can be done by approximations, starting from the solution of the equation without random centers.

We can see that the immediate effect of the scattering of the turbulent field by the vortices is a renormalization of the coefficient \(\alpha\) of the equation.

The solution that we need for Eq.\((124)\) must reflect the fact that we are examining a turbulent field, in the dynamics of which the randomly placed vortices have been included. We can see that the integration over the auxiliary variable \(s\) must not necessarily be truncated to ensure that \(\alpha_s \equiv \alpha - s \sqrt{2Q}\) remains positive. This is because the field is turbulent and
there is no restriction concerning the existence of the vortex solution which should be imposed to the turbulent component of the field. We also note that the approximation taking only the $s = 0$ contribution in the integral returns us to the problem of the turbulent field without interaction with the random vortices. Although the main contribution (as shown by the Gaussian integration) comes from the free turbulent field itself, the presence of vortices is contained in the rest of the integral, for $|s|$ not close to 0.

These remarks suggest to study the statistical properties of the transformed field, where $\alpha$ is replaced with $\alpha_s$.

7 The background turbulence: perturbative treatment

The vortices represent the strongly nonlinear part of the system and they have been extracted and treated separately as shown in Section IV. However, there is still nonlinear interaction in the remaining random field. This is the nonlinear mode coupling which produces the stationary turbulent states of the system, even in absence of any definite structure formation. This interaction must be taken into account when we analyse the statistical properties of the turbulent field. From general consideration we know what can be expected from this analysis: we will obtain a small departure from pure Gaussian statistics, expressed in nonlinear renormalization of the propagator and of the vertex of the interaction. The presence of random interaction with elements of the system that are beyond our simple model is accounted, as usual, by a noise term acting like a drive for the system of random waves. The noise will be assumed with the simplest (white) statistics

$$\langle \zeta(x, y) \zeta(x', y') \rangle = D \delta(x - x') \delta(y - y')$$ (125)

and should be considered as a random stirring force composed, for example, of random growths and decays of marginally stable modes, thus injecting at random places some energy into the system.

We will provide a standard perturbative treatment for the differential equation

$$-\nabla^2 \phi + (\alpha - s \sqrt{2A}) \phi + \beta \phi^2 = \zeta$$ (126)

with the objective to calculate correlation functions and other statistical properties.

As usual we start by defining the generating functional of the statistical
correlation
\[ \exp (-W[J]) \tag{127} \]
\[ \times \left\langle \int \mathcal{D}[\eta] \mathcal{D}[\phi] \exp \left\{ \int dx dy \eta \right\} \right\rangle \]
\[ \mbox{where we have introduced the notation} \]
\[ \alpha_s \equiv \alpha - s \sqrt{2Q} \tag{128} \]

The second factor can easily be calculated
\[ \left\langle \int \mathcal{D}[\eta] \mathcal{D}[\phi] \exp \left( \int dx dy \eta \zeta \right) \right\rangle \tag{129} \]
\[ = \int \mathcal{D}[\eta] \mathcal{D}[\phi] \exp \left[ \int dx dy (2D\eta^2) \right] \]

In the generating functional there are the two dual functions, \( \eta(x, y) \) and \( \phi(x, y) \), the second being the physical field of the random waves. We have inserted the action related with interaction with “external” currents, \( J_\eta \) and \( J_\phi \) which will permit to express the averages as functional derivatives.

Defining the Lagrangean density (after an integration by parts)
\[ \mathcal{L} = (\nabla \eta)(\nabla \phi) + \eta \alpha_s \phi + \eta \beta \phi^2 + D\eta^2 + J_\eta \eta + J_\phi \phi \tag{130} \]
we will separate into the Gaussian the nonlinear interaction part
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I \tag{131} \]
\[ \mathcal{L}_0 = (\nabla \eta)(\nabla \phi) + \eta \alpha_s \phi + D\eta^2 + J_\eta \eta + J_\phi \phi \tag{132} \]
\[ \mathcal{L}_I = \eta \beta \phi^2 \tag{133} \]
The first part of the Lagrangean is not linear because of the square term, but it is Gaussian therefore it would not pose any problem to the functional integration. We note that we have an order three vertex.

The action functional is correspondingly divided into two parts and can be written, as usual
\[ Z[J] = \int \mathcal{D}[\eta] \mathcal{D}[\phi] \exp \left( \int dx dy \mathcal{L}_0 \right) \exp \left( \int dx dy \mathcal{L}_I \right) \tag{134} \]
\[ = \exp \left( \beta \int dx dy \frac{\delta}{\delta J_\eta} \frac{\delta}{\delta J_\phi} \frac{\delta}{\delta J_\phi} \right) \int \mathcal{D}[\eta] \mathcal{D}[\phi] \exp \left( \int dx dy \mathcal{L}_0 \right) \]
For the linear part we have the Euler-Lagrange equations

\[
\frac{d}{dx} \delta \frac{\delta L_0}{\delta \eta} \frac{\partial \eta}{\partial x} + \frac{d}{dy} \delta \frac{\delta L_0}{\delta \eta} \frac{\partial \eta}{\partial y} - \delta \frac{\delta L_0}{\delta \eta} = 0
\] (135)

\[
\frac{d}{dx} \delta \frac{\delta L_0}{\delta \phi} \frac{\partial \phi}{\partial x} + \frac{d}{dy} \delta \frac{\delta L_0}{\delta \phi} \frac{\partial \phi}{\partial y} - \delta \frac{\delta L_0}{\delta \phi} = 0
\] (136)

The equations are

\[
\Delta \phi - \alpha_s \phi - 2D \eta = J_\eta
\] (137)

and

\[
\Delta \eta - \alpha_s \eta = J_\phi
\] (138)

The latter is an inhomogeneous Helmholtz equation and has the solution

\[
\eta_0 (x, y) = \int dx'dy' G_{\eta\phi} (x, y; x', y') J_\phi (x', y')
\] (139)

in terms of the Green function appropriate for the space domain of our analysis and taking into account the boundary conditions for \( \eta \).

The first Euler-Lagrange equation gives

\[
\Delta \phi - \alpha_s \phi = J_\eta + 2D \eta_0
\] (140)

\[
= J_\eta + 2D \int dx'dy' G_{\eta\phi} (x, y; x', y') J_\phi (x', y')
\]

with the solution

\[
\phi_0 (x, y) = \int dx'dy' G_{\phi\eta} (x, y; x', y') J_\eta (x', y')
\] (141)

\[
+ 2D \int dx'dy'dx''dy'' G_{\phi\eta} (x, y; x'', y'') G_{\eta\phi} (x'', y''; x', y') J_\phi (x', y')
\]

We can introduce the matrix of propagators

\[
G \equiv (G_{ij}) = \begin{pmatrix}
G_{\phi\phi} & G_{\phi\eta} \\
G_{\eta\phi} & G_{\eta\eta}
\end{pmatrix}
\] (142)

where

\[
G_{\phi\phi} (x, y; x', y') = 2D \int dx''dy'' G_{\phi\eta} (x, y; x'', y'') G_{\eta\phi} (x'', y''; x', y')
\] (143)

Introducing the notation

\[
\Psi \equiv \begin{pmatrix}
\phi \\
\eta
\end{pmatrix}
\] (144)
we will write symbolically (summation over repeated indices is implied)
\[ \Psi_i = \int dx' dy' G_{ik} J_k \] (145)

for
\[ \Psi_i = \int dx' dy' G(x, y; x', y') J_k(x', y') \] (146)

We can now calculate the linear part of the action along the system’s configurations given by these solutions. The action will be, of course, a functional of the current.

\[ S_{0J} = \frac{1}{2} \int dxdy \left[ \nabla \eta_0 \cdot \nabla \phi_0 + \alpha_s \eta_0 \phi_0 + 2D \eta_0^2 + J_\eta \eta_0 + J_\phi \phi_0 \right] \] (147)

or
\[ S_{0J} = \frac{1}{2} \int dxdy \left[ \eta_0 (\Delta \phi_0 + \alpha_s \eta_0 \phi_0 + 2D \eta_0^2 + J_\eta \eta_0 + J_\phi \phi_0) \right] \] (148)

We now express the solution by the Green functions
\[ S_{0J} = \frac{1}{2} \int dxdy \int dx' dy' G_{ij} J_i(x, y) J_j(x', y') \] (149)

which can be written using the convention introduced above
\[ S_{0J} = \frac{1}{2} \int dxdy \int dx' dy' G_{ij} J_i(x, y) G_{jk} (x, y; x', y') J_k(x', y') \] (150)

We return to the generating functional
\[ Z[J] = \exp \left( \beta \int dxdy \frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} \frac{\delta}{\delta J_k} \right) \exp (S_{0J}) \] (151)

\[ = \exp \left( \int dxdy C_{ijk} \frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} \frac{\delta}{\delta J_k} \right) \times \exp \left[ \frac{1}{2} \int dxdy \int dx' dy' J_i(x, y) G_{ij} (x, y; x', y') J_j(x', y') \right] \]
where we have introduced, for uniformity of notation
\[ C_{ijk}(x, y) \equiv \beta \delta_{i\eta} \delta_{j\phi} \delta_{k\phi} \tag{152} \]
This is a classical framework for diagrammatic expansion.

In general the functional approach is well suited for statistical problems where the statistical ensemble of the realization of the system’s configurations is produced by the effect of a noise or by a random choice of initial condition, as explained in MSR and Jensen. In the most usual case, where there is a noise acting on the system, using the (known) statistical properties of the noise means implicitly that we make a change of variables, from the field \( \phi \) to the noise \( \zeta \) and this introduces a Jacobian,
\[ J[\phi] = D[\zeta] / D[\phi] \tag{153} \]
The Jacobian cancels all the diagrams that correspond to the vacuum to vacuum transitions, or equal-coordinates correlations. For this reason it is not mentioned. This is explained in Appendix B.

When applying the operator to the second exponential we have to remind that we need at least two open ends with currents \( J_\phi \) since we intend to determine the spectrum. On the other hand, it is clear that any open end with isolated either \( J_\phi \) or \( J_\eta \) would vanish after taking the currents to zero.

The first term is naturally the diffusion driven by the random rise and decay of marginally stable modes, represented here as a noise.

The next term that is useful consists of the product of two vertex-like operators applied on a product of four propagator-like terms
\[ J_{i_1} G_{i_1 i_2} C_{i_2 i_3 i_4} G_{i_3 i_4} G_{i_4 i_5} C_{i_5 i_6 i_7} G_{i_7 i_1} J_{i_1} \tag{154} \]
Obviously, this is the one-loop diagram. We will derivate to the first and last factors (currents) so that, more explicitly, its structure is
\[ G_{\phi\eta} C_{\eta\phi} G_{\phi\phi} G_{\phi\phi} C_{\phi\phi} G_{\phi\phi} \tag{155} \]
We can represent the Green function by its Fourier transform
\[ (\Delta - \alpha_s) G(x, y; x', y') = -\delta (x - x') \delta (y - y') \tag{156} \]
\[ G(x, y; x', y') = \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + \alpha_s} \times \exp [-ik_x (x - x')] \exp [-ik_y (y - y')] \tag{157} \]
We obtain the propagators for the coupling between the field $\phi$ and its dual $\eta$

\[ G_{\phi\eta}(x, y; x', y') = \int dk_x dk_y \frac{1}{k_x^2 + k_y^2 + \alpha_s} \times \exp[-ik_x(x-x')] \exp[-ik_y(y-y')] = G_{\eta\phi}(x, y; x', y') \]  

\[ G_{\phi\phi}(x, y) = 2D \int dx'' dy'' G_{\phi\eta}(x, y; x'', y'') G_{\eta\phi}(x'', y''; x', y') \]  

The intermediate integration over space ($x'', y''$) yields equality of the Fourier variables of the two Green functions. We then have

\[ \tilde{G}_{\phi\phi} = 2D \left( k_x^2 + k_y^2 + \alpha_s \right)^{-2} \]  

\[ \tilde{G}_{\phi\eta} = \tilde{G}_{\eta\phi} = \left( k_x^2 + k_y^2 + \alpha_s \right)^{-1} \]  

The one-loop term is expressed as two integrals over the intermediate momenta at the two vertices where we have a product of three Green functions with the vertex. Conservation of momentum in the loop and overall conservation of the diagram (which means that the $k$ at input line must be the same at the output line) lead to

\[ \langle \phi(x, y) \phi(x', y') \rangle = \int dk_x dk_y \langle \phi \phi \rangle_k \exp[-ik_x(x-x')] \exp[-ik_y(y-y')] \]  

\[ = \frac{2}{(k^2 + \alpha_s)^2} \]  

\[ + \int dk \Gamma \int dp \frac{1}{k^2 + \alpha_s} \frac{2D}{(p^2 + \alpha_s)^2} \frac{2D}{[(k - p)^2 + \alpha_s]^2} \frac{1}{k^2 + \alpha_s} \]  

The coefficient $\Gamma$ represents the multiplicity of this diagram and factors of normalization.

\[ \langle \phi \phi \rangle_k = \Gamma (2D\beta)^2 \left( \frac{1}{k^2 + \alpha_s} \right)^2 \int dp \frac{1}{p^2 + \alpha_s} \frac{1}{[(k - p)^2 + \alpha_s]^2} \]  

\[ = \Gamma (2D\beta)^2 \left( \frac{1}{k^2 + \alpha_s} \right)^2 \int dp \frac{1}{[(k/2 - p + \alpha_s)^2 + \alpha_s]^2} \frac{1}{[(k/2 + p)^2 + \alpha_s]^2} \]  

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We transform the integral
\[ I \equiv \int p dp d\theta \frac{1}{[p^2 - k \cdot p + k^2/4 + \alpha_s]^2} \times \frac{1}{[p^2 + k \cdot p + k^2/4 + \alpha_s]^2} \] (163)

Now we introduce the variables that will replace the constant terms
\[ b \equiv \frac{k^2}{4} + \alpha_s \] (164)
and
\[ k \cdot p = kp \cos \theta \equiv up \] (165)
The integral becomes
\[ I = \int p dp d\theta \frac{1}{[p^2 - up + b]^2} \frac{1}{[p^2 + up + b]^2} \] (166)
We write the denominator in the form
\[ [p^2 - up + b]^2 [p^2 + up + b]^2 = \left[p^4 + 2\gamma p^2 + \delta\right]^2 \] (167)
where
\[ \gamma \equiv 2b - u^2 \] (168)
\[ \delta \equiv b^2 \]
The integral becomes
\[ I = \int d\theta \int_0^\infty \frac{p dp}{(p^4 + 2\gamma p^2 + \delta)^2} \] (169)
we make the substitution
\[ p^2 \rightarrow x \] (170)
\[ I = \int d\theta \frac{1}{2} \int_0^\infty \frac{dx}{(x^2 + 2\gamma x + \delta)^2} \]  

(171)

The integration over the intermediate momentum is

\[ \int_0^\infty \frac{dx}{(x^2 + 2\gamma x + \delta)^2} = \frac{1}{2} \left( \frac{1}{\delta - \gamma^2} \right)^{3/2} \left[ \frac{\pi}{2} - \arctan \left( \frac{\gamma}{\sqrt{\delta - \gamma^2}} \right) \right] + \left( -\frac{\gamma}{2\delta} \right) \frac{1}{\delta - \gamma^2} \]

(172)

Here

\[ \gamma = \frac{b - u^2}{2} \]

(173)

\[ \frac{\gamma}{\sqrt{\delta - \gamma^2}} = \frac{b - u^2/2}{|u| \sqrt{b - u^2/4}} \]

(174)

\[ \frac{\gamma}{\sqrt{\delta - \gamma^2}} = \frac{1}{2k} \frac{k^2 \left( 1 - 2\cos^2 \theta \right) + 4\alpha_s}{|\text{cos } \theta| \sqrt{k^2 \sin^2 \theta + 4\alpha_s}} \]

\[ \delta - \gamma^2 = u^2 \left( b - \frac{u^2}{4} \right) \]

(175)

\[ \frac{\gamma}{\delta} = \frac{4}{k^2} \frac{k^2 \left( 1 - 2\cos^2 \theta \right) + 4\alpha_s}{(k^2 + 4\alpha_s)^2} \]

(176)

Then

\[ \int p dp d\theta \frac{1}{\left[ p^2 - k \cdot p + k^2/4 + \alpha_s \right]^2} \frac{1}{\left[ p^2 + k \cdot p + k^2/4 + \alpha_s \right]^2} \]

(177)

\[ = \int d\theta \left\{ \frac{8}{\left[ k^2 \cos^2 \theta \left( k^2 \sin^2 \theta + \alpha_s \right) \right]^{3/2}} \times \left[ \frac{\pi}{2} - \arctan \left( \frac{k^2 \left( 1 - 2\cos^2 \theta \right) + 4\alpha_s}{2k |\cos \theta| \sqrt{k^2 \sin^2 \theta + 4\alpha_s}} \right) \right] \right\} \]

\[ - \frac{8}{k^2 \cos^2 \theta \left( k^2 \sin^2 \theta + \alpha_s \right)} \frac{k^2 \left( 1 - 2\cos^2 \theta \right) + 4\alpha_s}{(k^2 + 4\alpha_s)^2} \]
We can show that the integral is not singular at the limit
\[ \cos \theta \to 0 \]  
so that the integration over \( \theta \) is safe.

As order of magnitude the integral is dominated by terms like
\[ \sim \Gamma (2D\beta)^2 \frac{1}{k^2 (k^2 + \alpha_s)^{3/2}} \]  
while the diffusive part is
\[ \sim \frac{2D}{(k^2 + \alpha_s)^2} \]  

After this we obtain for the turbulent background field
\[ \langle \phi \phi \rangle_{turbulence}^{k} = a \frac{2D}{(k^2 + \alpha_s)^2} + b (2D\beta)^2 \frac{1}{(k^2 + \alpha_s)^2} \frac{1}{k^2 (k^2 + \alpha_s)^{3/2}} \]

where we have collected the factors in two numbers, \( a \) and \( b \).

8 Summary

It is usually assumed that a turbulent plasma (in particular with embedded structures) should exhibit a spectrum of exponential or algebraic type. There is no universal theoretical basis for such an assumption except for cases where the scaling invariance is justified on physical grounds. We find that it would be more adequate to extract a particular behavior (on different spectral intervals) from expressions like Eq.(75) and Eq.(181), via regression on simple numerical values. This can be done for a particular physical system. Here, however, we will return to the traditional exponents in order to exhibit some associations we consider to be general.

We can now collect the results of the analysis. In the left column we write the approximative behavior of the contributions and in the right column the physical origin, according to this theory.

\[ k^{-2} \]
- gas of vortices  
- (from \( N = 1 \) to closely packed)
- weak interaction of vortices \( K_0 \)

\[ k^{-4} \]
- \( S(k) k^{-2} (\text{weak } \rho_s^2/A) \)  
- background turbulence + vortices \( \frac{2D}{(k^2 + \alpha_s)^2} \)
- perturbed c.s., \( \text{via} \) rare events of large \( N \)

\[ \sim S(k) [1 + f(k)] \]
- geometry of c.s., \( \text{via} \) interaction

\[ b (2D\beta)^2 \frac{1}{(k^2 + \alpha_s)^2} \frac{1}{k^2 (k^2 + \alpha_s)^{3/2}} \]
- one-loop mode coupling
We see two ways in which these results can be useful.

First, the combination of various exponential dependences from the listed contributions will result in an overall dependence with exponents that may be compared with the experiments or numerical simulations [19], [20], [21], [22], [24]. The weight of each contribution is dependent on factors like the shape of the vortices (including the spatial extension compared to the area $A$), amplitude of the vortices, strength of the random drive ($D$). Also in the problem enters as parameter the function of distribution on the plane (assumed here uniform). The energy of interaction between vortices may be reconsidered along the example of vortices in superfluids.

Second, the spectrum can be seen as dominated in different spectral domains by one or another of these contributions. This is compatible with the known fact that the spectrum has regions with different exponents $\mu$.

In both these ways we have to solve an inverse problem, starting from the experimental (or numerical) spectrum and using the above list to map the exponential form to a physical process which may have been at its origin.

Various extensions of this treatment are possible. One can introduce a chemical potential for the description of the statistical equilibrium consisting of generation and suppression of coherent vortices. There are treatments of this type for the conversion of the global rotation of superfluids into localised vortices, an example that may also be useful for the consideration of the zonal flow saturation in tokamak, besides the Kelvin-Helmholtz instability and the collisions.

The treatment by generating functional allows in principle determination of statistical correlations at any high order desired. However, while the functional derivatives can easily produce the $n$-th order cumulant, we must be sure that the generating functional has been calculated with the necessary precision, or, in other words, we must be sure that we have incorporated the physical origin of these correlations. This method should be accompanied by the more standard analysis of closure of the hierarchy of equations for the correlations and these two approaches must be seen as complementary.

As a final remark, the physical model adopted in the present treatment may be extended to cover more complex regimes, with, of course, a certain increase in the analytical work.

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9 Appendix A : physics of the equation

It is of interest to study a realistic two-dimensional model, like Hasegawa-Wakatani or similar. However we need for the beginning a simpler equation and if possible with a vortex solution with known analytical expression.

A possibility is the equation for the ion drift instabilities but here we must specify the scales.

Consider the equations for the ITG model in two-dimensions with adiabatic electrons:

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (v_i n_i) = 0 \tag{182}
\]

\[
\frac{\partial v_i}{\partial t} + (v_i \cdot \nabla) v_i = \frac{e}{m_i} (-\nabla \phi) + \frac{e}{m_i} v_i \times B
\]

We assume the quasineutrality

\[ n_i \approx n_e \tag{183} \]

and the Boltzmann distribution of the electrons along the magnetic field line

\[ n_e = n_0 \exp \left( -\frac{|e| \phi}{T_e} \right) \tag{184} \]

In general the electron temperature can be a function of the radial variable

\[ T_e \equiv T_e(x) \tag{185} \]

The velocity of the ion fluid is perpendicular on the magnetic field and is composed of the diamagnetic, electric and polarization drift terms

\[ v_i = v_{\perp i} \tag{186} \]

\[ = v_{dia,i} + v_E + v_{pol,i} \]

\[ = \frac{T_i}{|e| B n_i} \frac{dn_i}{dr} e_y \]

\[ + \frac{-\nabla \phi \times \hat{n}}{B} \]

\[ - \frac{1}{B \Omega_i} \left( \frac{\partial}{\partial t} + (v_E \cdot \nabla_{\perp}) \right) \nabla_{\perp} \phi \]

The diamagnetic velocity will be neglected. Introducing this velocity into the continuity equation, one obtains an equation for the electrostatic potential \( \phi \).
Before writing this equation we introduce new dimensional units for the variables.

\[ \phi^{\text{phys}} \rightarrow \phi' = \frac{|e| \phi^{\text{phys}}}{T_e} \]  
(187)

\[ (x^{\text{phys}}, y^{\text{phys}}) \rightarrow (x', y') = \left( \frac{x^{\text{phys}}}{\rho_s}, \frac{y^{\text{phys}}}{\rho_s} \right) \]  
(188)

\[ t^{\text{phys}} \rightarrow t' = t^{\text{phys}} \Omega_i \]  
(189)

The new variables \((t, x, y)\) and the function \(\phi\) are non-dimensional. In the following the *primes* are not written.

### 9.1 The equation

With these variables the equation obtained is

\[
\frac{\partial}{\partial t} \left( 1 - \nabla_\perp^2 \right) \phi 
- (-\nabla_\perp \phi \times \hat{n}) \cdot v_d 
\]
\[+ (-\nabla_\perp \phi \times \hat{n}) \cdot v_T \phi \]
\[+ [(-\nabla_\perp \phi \times \hat{n}) \cdot \nabla_\perp] \left( -\nabla_\perp^2 \phi \right) = 0
\]  
(190)

where

\[ v_d \equiv -\nabla_\perp \ln n_0 - \nabla_\perp \ln T_e \]  
(191)

\[ v_T \equiv -\nabla_\perp \ln T_e \]

(This is Eq.(8) from the paper [10]).

### 9.2 No temperature gradient

From the various versions of the nonlinear drift equation, (in particular the Hasegawa-Mima equation) we choose the radially symmetric Flierl-Petviashvili soliton equation:

\[
(1 - \rho_s^2 \nabla_\perp^2) \frac{\partial \phi}{\partial t} + v_d \frac{\partial \phi}{\partial y} - v_T \frac{\partial \phi}{\partial y} = 0
\]  
(192)

where \(\rho_s = c_s/\Omega_i\), \(c_s = (T_e/m_i)^{1/2}\) and the potential is scaled as \(\phi = \frac{L_n}{L_T} \frac{e \phi}{T_e} \). Here \(L_n\) and \(L_T\) are respectively the gradient lengths of the density and temperature. The velocity is the diamagnetic velocity \(v_d = \frac{a_n}{L_n} \). The condition for the validity of this equation are: \((k_x \rho_s) (k \rho_s)^2 \ll \eta_e \frac{\rho_e}{L_n}\), where \(\eta_e = \frac{L_n}{L_T} \).
The exact solution of the equation is
\[ \varphi_s(y, t; y_0, u) = -3 \left( \frac{u}{v_d} - 1 \right) \sec h^2 \left[ \frac{1}{2\rho_s} \left( 1 - \frac{v_d}{u} \right)^{1/2} (y - y_0 - ut) \right] \] (193)
where the velocity is restricted to the intervals \( u > v_d \) or \( u < 0 \). In the Ref. [14] the radial extension of the solution is estimated as: \( (\Delta x)^2 \sim \rho_s L_n \). In our work we shall assume that \( u \) is very close to \( v_d \), \( u \gtrsim v_d \) (i.e. the solitons have small amplitudes).

10 Appendix B : Connection between the MSR formalism and Onsager-Machlup

In our approach the most natural way of proceeding with a stochastic differential equation is to use the MSR type reasoning in the Jensen formulation. The equation is discretized in space and time and selected with \( \delta \) functions in an ensemble of functions (actually in sets of arbitrary numbers at every point of discretization). The result is a functional integral. There is however a particular aspect that needs careful analysis, as mentioned in the previous Subsection. It is the problem of the Jacobian associated with the \( \delta \) functions. This problem is discussed in Ref.[34] and for consistency we include here the essential of their original treatment.

The equation they analyse is in the time domain and is presented in most general form as
\[ \frac{\partial \phi_j (t)}{\partial t} = - (\Gamma_0)_{jk} \frac{\delta H}{\delta \phi_k (t)} + V_j [\phi (t)] + \theta_j \]
where the number of stochastic equations is \( N \), \( H \) is functional of the fields, \( V_j \) is the streaming term which obeys a current-conserving type relation
\[ \delta \frac{\delta V_j [\phi]}{\delta \phi_j} \exp \{-H [\phi]\} = 0 \]
The noise is \( \theta_j \).

The following generating functional can be written
\[ Z_\theta = \int D [\phi_j (t)] \exp \int dt \left[ l_j \phi_j (t) \right] \prod_j \delta \left( \frac{\partial \phi_j (t)}{\partial t} + K_j [\phi (t)] - \theta_j \right) J [\phi] \]
the functions \( l_j (t) \) are currents,
\[ K_j [\phi (t)] \equiv - (\Gamma_0)_{jk} \frac{\delta H}{\delta \phi_k (t)} + V_j [\phi (t)] \]
and \( J[\phi] \) is the Jacobian associated to the Dirac \( \delta \) functions in each point of discretization.

The Jacobian can be written

\[
J = \det \left( \frac{\partial}{\partial t} + \frac{\delta K_{j}[\phi]}{\delta \phi_k} \right) \delta(t - t')
\]

Up to a multiplicative constant

\[
J = \exp \left( \text{Tr} \ln \left( \left( \frac{\partial}{\partial t} + \frac{\delta K}{\delta \phi} \right) \frac{\delta (t - t')}{\delta (t - t')} \right) \right)
\]

or

\[
J = \exp \left( \text{Tr} \ln \left[ 1 + \left( \frac{\partial}{\partial t} \right)^{-1} \delta K(t) \right] \right)
\]

Since the operator \( \left( \frac{\partial}{\partial t} \right)^{-1} \) is retarded, only the lowest order term survives after taking the trace

\[
J = \exp \left[ -\frac{1}{2} \int dt \frac{\delta K_{j}[\phi(t)]}{\delta \phi_j(t)} \right]
\]

The factor \( 1/2 \) comes from value of the \( \Theta \) function at zero.

In the treatment which preserves the dual function \( \hat{\phi} \) associated to \( \phi \) in the functional, there is a part of the action

\[
\hat{\phi} K[\phi]
\]

Then a \( \hat{\phi} \) and a \( \phi \) of the same coupling term from \( \hat{\phi} K[\phi] \) close onto a loop. Since \( G_{\phi \phi} \) is retarded, all these contributions vanish except the one with a single propagator line. This cancels exactly, in all orders, the part coming from the Jacobian.

Then it is used to ignore all such loops and together with the Jacobian.

In conclusion we can compared the two starting points in a functional approach: The one that uses dual functions \( \phi(t) \) and \( \chi(t) \), closer in spirit to MSR; And the approach based on Onsager-Machlup functional, traditionally employed for the determination of the probabilities. Either we keep \( \chi(t) \) and ignore the Jacobian (the first approach) or integrate from the beginning over \( \chi(t) \) and include the Jacobian. The approaches are equivalent but we have followed the first one.
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