Breakdown of Universality in Quantum Chaotic Transport:  
the Two-Phase Dynamical Fluid Model

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We investigate the transport properties of open quantum chaotic systems in the semiclassical limit. We show how the transmission spectrum, the conductance fluctuations, and their correlations are influenced by the underlying chaotic classical dynamics, and result from the separation of the quantum phase space into a stochastic and a deterministic phase. Consequently, sample-to-sample conductance fluctuations lose their universality, while the persistence of a finite stochastic phase protects the universality of conductance fluctuations under variation of a quantum parameter.

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Universal Conductance Fluctuations (UCF) are arguably one of the most spectacular manifestations of quantum coherence in mesoscopic systems \([\text{\[\text{Pa}\]}\). In metallic samples, the universality of the conductance fluctuations manifests itself in their magnitude, \(\text{rms}(g) = O(e^2/h)\), independently on the sample’s shape and size, its average conductance or the exact configuration of the underlying impurity disorder \([\text{\[\text{Pa}\]}\, \text{\[\text{Pa}\]}\). In ballistic chaotic systems, a similar behavior is observed, which is captured by Random Matrix Theory (RMT) \([\text{\[\text{Pa}\]}\). For an open chaotic cavity connected to two \(N\)-channel leads, and thus having an average classical conductance \(g = N/2\) (we consider spinless fermions and express \(g\) in units of \(e^2/h\)), RMT predicts a universal conductance variance \(\sigma^2(g) = 1/8\) for time-reversal and spin rotational symmetric samples. At the core of the UCF lies the ergodic hypothesis that sample-to-sample fluctuations are equivalent to fluctuations induced by parametric variations (e.g. changing the energy or the magnetic field) within a given sample \([\text{\[\text{Pa}\]}\).

According to the scattering theory, transport properties such as the conductance derive from transmission eigenvalues, i.e. \(g = \sum_{i=1}^{N} T_i\). While coherence effects such as the UCF arise due to nontrivial correlations \((T,T')\), the knowledge of the probability distribution \(P(T)\) of transmission eigenvalues is sufficient to correctly predict, e.g., the average conductance, or the Fano factor \(F \equiv \langle T(1-T)\rangle/\langle T\rangle\) for the shot-noise power \([\text{\[\text{Pa}\]}\). For a ballistic chaotic cavity, RMT predicts \([\text{\[\text{Pa}\]}\)

\[
P_{\text{RMT}}(T) = \frac{1}{\pi} \frac{1}{\sqrt{T(1-T)}}, \tag{1}
\]

and thus \(F = 1/4\). For shot-noise, as well as for UCF, the correct universal behavior is captured by RMT.

The validity of RMT is however generically restricted by the existence of finite time scales. Spectral fluctuations are known to deviate from RMT predictions for energies larger than the inverse period of the shortest periodic orbit for chaotic systems \([\text{\[\text{Pa}\]}\), or than the inverse time of diffusion through the sample in disordered metallic systems \([\text{\[\text{Pa}\]}\). Another time scale which is absent in RMT is the Ehrenfest time \(\tau_E\), i.e. the time it takes for the underlying classical chaotic dynamics (with Lyapunov exponent \(\lambda\)) to stretch an initial narrow wavepacket, of spatial extension given by the Fermi wavelength \(\lambda_F\), to the linear system size \(L\). Defining \(M = L/\lambda_F\), one has \(\tau_E = \lambda^{-1} \ln[M/(2\pi D)^2] \)[\text{\[\text{Pa}\]}, \text{\[\text{Pa}\]}\), with \(\tau_D = M/2N\), the dwell time through the cavity (all times will be measured in units of the time of flight across the cavity). Note, in particular, that the growth of \(\tau_E\) in the semiclassical limit \(M \rightarrow \infty\) is only logarithmic. The emergence of a finite \(\tau_E/\tau_D\) leads to strong deviations from the universal RMT behavior, and in particular to the suppression of shot noise \([\text{\[\text{Pa}\]}, \text{\[\text{Pa}\]}\), or the proximity gap in Andreev billiards \([\text{\[\text{Pa}\]}, \text{\[\text{Pa}\]}\). It has furthermore been predicted that weak localization vanishes at
large $\tau_E/\tau_D$ \cite{17}. Also, in dirty d-wave superconductors, the RMT behavior of the quasiparticle density of states \cite{18} is restored only below an energy scale set by $\tau_E$ \cite{19}.

The suppression of shot-noise for $\tau_E/\tau_D \to \infty$ is due to the disappearance of the stochasticity of quantum mechanical transport, and its replacement by the determinism of classical transport \cite{11,12,13}. Wavepackets traveling on scattering trajectories shorter than $\tau_E$ have no time to diffract, and are thus either fully transmitted or fully reflected. In an open chaotic cavity, different scattering trajectories have in general different dwell times with a distribution $p(t) = \exp(-t/\tau_D)/\tau_D$. For finite $0 < \tau_E/\tau_D \ll \infty$, this suggests that transport is mediated by a two-phase dynamical fluid, consisting of a stochastic phase of relative volume $\alpha \simeq \int_{t_E}^{\infty} p(t) dt = \exp(-\tau_E/\tau_D)$, and a deterministic phase of relative volume $1 - \alpha$. Following this purely classical argument, first expressed in the inset to Fig.1, we expect that the distribution of transmission eigenvalues is given by

$$P_\alpha(T) = \alpha P_{\text{RMT}}(T) + \frac{1-\alpha}{2} [\delta(T)+\delta(1-T)].$$  \hspace{1cm} (2)

This will be confirmed below. We will further show that, quite surprisingly, the two-phase dynamical fluid assumption also correctly describes the behavior of mesoscopic coherent effects, and in particular that it explains the breakdown of universality of the conductance fluctuations when $\tau_E$ becomes comparable to the ergodic time $\tau_0$.

We first summarize our main results. (i) We give full confirmation of Eq. (2) by calculating the integrated distribution of transmission eigenvalues $I(\alpha) \equiv \int_0^T P(T')dT'$. We find that it is very well fitted by (see the inset to Fig.1)

$$I_\alpha(T) = \frac{1-\alpha}{2} (1 + \delta_{1,T}) + \frac{2\alpha}{\pi} \sin^{-1}(\sqrt{T}),$$  \hspace{1cm} (3)

from which we extract $\alpha \simeq \exp(-\tau_E/\tau_D)$. (ii) The conductance fluctuations stay at their universal value, independently on $\tau_E/\tau_D$, under variation of the energy in a given sample. This follows from the survival of a large number of stochastic channels -- even though their relative measure $\alpha \rightarrow 0$ -- which preserves the universality of the conductance fluctuations. (iii) A completely different situation arises when one considers sample-to-sample fluctuations. In this case, one has $\sigma^2(g) \propto (M/M_c)^2$ for $M > M_c$. The scaling parameter $M_c = \tau_D^2 \exp(\lambda)$ is determined by the quantum mechanical resolution of classical phase space structures corresponding to the largest cluster of fully transmitted or reflected neighboring trajectories (see Ref. \cite{12}). (iv) The energy conductance correlator always decays on the universal scale of the Thouless energy, $\xi_E \propto 1/\tau_D$, independently on $\tau_E$. The results (ii) and (iii) show that the ergodic hypothesis breaks down as $\tau_E/\tau_0$ increases. Accordingly, (iv) is somewhat surprising, but will be understood below via a semiclassical argument. All our results and arguments fully confirm the two-phase dynamical fluid model. We note that our conclusion (iii) is in agreement with the very recent finding $\sigma^2(g) \propto M^2$ obtained by Tworzydlo, Tajic and Beenakker \cite{20}. However, (ii) is in complete opposition with their prediction that deviations from $\sigma^2(g) = 1/8$ should occur upon variation of the energy for large $\tau_E/\tau_D$. Points (i) and (iv) are addressed here for the first time.

We consider open systems with fully developed chaotic dynamics, for which $\tau_D \gg 1$. Because $\tau_E$ grows logarithmically with $M$, and since we want to investigate the regime $\tau_E/\tau_D \gtrsim 1$, we model the electron dynamics by the kicked rotator map, which reproduces most of the phenomenology of low-dimensional noninteracting electronic physics \cite{22}. The classical kicked rotator map is given by

\begin{align*}
\bar{x} &= x + p \\
\bar{p} &= p + K \sin(\bar{x}),
\end{align*}

\hspace{1cm} (4)

with $K$ the (dimensionless) kicking strength. It drives the dynamics from fully integrable ($K = 0$) to fully chaotic ($K \gtrsim 7$, with Lyapunov exponent $\lambda \approx \ln(K/2)$). We consider a toroidal classical phase space $x, p \in [0, 2\pi]$, and open the system by defining contacts to ballistic leads via two absorbing phase space strips $[x_L - \delta x, x_L + \delta x]$ and $[x_R - \delta x, x_R + \delta x]$, each of them with a width $2\delta x = \pi/\tau_D$.

Quantizing the map amounts to a discretization of, say, the real space coordinates as $x_m = 2\pi m/M$, $m = 1, \ldots M$. A quantum representation of the map \cite{11} is provided by the unitary $M \times M$ Floquet operator $U$ \cite{21}, which gives the time evolution for one iteration of the map. For our specific choice of the kicked rotator, the Floquet operator has matrix elements

$$U_{m,m'} = M^{-1/2} e^{-(iMK/4\pi)(\cos(2\pi n/M) + \cos(2\pi n'/M))} \times \sum l e^{\pi il(m-m')/M} e^{-(\pi i/2M)l^2}.$$  \hspace{1cm} (5)

The spectrum $\exp(\pm \varepsilon_m)$ of $U$ defines a discrete set of $M$ quasienergies $\varepsilon_m \in [0, 2\pi]$ with an average level spacing $\delta = 2\pi/M$.

Much in the same way as in the Hamiltonian case \cite{22}, a quasienergy-dependent $2N \times 2N$ scattering matrix can be determined from the Floquet operator $U$ as \cite{24}

$$S(\varepsilon) = P[\exp(-i\varepsilon) - U(1 - P^T P)]^{-1} U P^T,$$  \hspace{1cm} (6)

using a $2N \times M$ projection matrix $P$ which describes the coupling to the leads. Its matrix elements are given by

$$P_{n,m} = \begin{cases} 1 & \text{if } n = m \in \{m^{(L)}_i\} \cup \{m^{(R)}_i\} \\ 0 & \text{otherwise} \end{cases}$$  \hspace{1cm} (7)

An ensemble of samples with the same microscopic properties can be defined by varying the position $\{m^{(L,R)}_i\}$, $i = 1, \ldots N$ of the contacts to the left and right leads for fixed $\tau_D = M/2N$ and $K$. 
As usual, the scattering matrix can be written in a four block form in term of $N \times N$ transmission and reflection matrices as

$$S = \begin{pmatrix} r & t \\ r' & t' \end{pmatrix}.$$  \hfill (8)

The spectrum of transmission probabilities is given by the $N$ eigenvalues $T_j$ of $T = \lambda t^t$, from which the dimensionless conductance is obtained, via the Landauer formula $g = \sum_j T_j^2$. Our numerical procedure follows the description given in Ref. [13].

We plot in Fig. 1 various distributions $P(T)$ of transmission eigenvalues. First, it is seen that our model correctly reproduces the RMT distribution of Eq. (1) in the limit $\tau_E/\tau_D \ll 1$. The distribution undergoes strong modifications, however, as $\tau_E/\tau_D$ increases. In particular, more and more weight is accumulated at $T = 0$ and 1. The behavior exhibited by $P(T)$ for finite $\tau_E/\tau_D$ is very similar to that predicted by Eq. (2). To confirm this, we calculate the integrated probability distribution $I(T)$, which presents the advantage of being independent on the size of histogram bins. Results are shown in the inset to Fig. 1. The fitting curves clearly confirm the validity of Eq. (2). The extracted parameter $\alpha$ is found to obey $\alpha \approx \exp(-(1 + \tau_E)/\tau_D)$, for $\tau_E > 0$. We attribute the factor $1 + \tau_E$ in the exponential (and not $\tau_E$) to the discrete nature of the dynamics in our model.

For $\tau_E \to 0$, one is in the UCF regime, where the conductance fluctuates equivalently from sample to sample or as $\varepsilon$ is varied within a given sample. This is no longer the case, however, once $\tau_E$ becomes finite, as is shown in Fig. 2. While $\sigma^2(g) = 1/8$ seems to be preserved when $\varepsilon$ is varied for a given sample, one gets an enormous increase $\sigma^2(g) \propto M^2$ from sample to sample. This behavior derives from the underlying classical dynamics, and can be understood on the basis of a two-phase dynamical fluid, as we now proceed to explain.

While the classical dynamics considered here is fully chaotic, finite-sized phase space structures emerge due to the opening of the cavity. Following Ref. [12], these structures can be visualized by marking which trajectories originating from, say, the left lead, end up being transmitted to the right lead. Such a picture is shown in Fig. 3(a). It is seen that classical trajectories are transmitted by a series of bands exiting the cavity at times $t_j$, and covering a phase space area $A_j \approx \tau_D^2 \exp(-\lambda t_j)$. The shape, position and precise volume of these bands is sample-to-sample dependent. In the semiclassical limit $h_{\text{eff}} = 2\pi/M \to 0$, the effective Planck scale $h_{\text{eff}}$ resolves the $j^{\text{th}}$ band as soon as $h_{\text{eff}} \leq A_j$, or $M \geq 2\pi \tau_D^2 \exp(\lambda t_j)$. Once the largest classical band is resolved, $\sigma^2(g)$ starts to be dominated by the band fluctuations. Each resolved band carrying a growing number $M$ of fully transmitted (or reflected) quantum mechanical modes, one expects a variance $\sigma^2(g) \propto (M/M_c)^2$ for sufficiently large $M > M_c$, with a scaling parameter $M_c = 2\pi \tau_D^2 \exp(\lambda)$ determined by the largest band, exiting the system at the ergodic time $\tau_0 \approx 1$. As shown on Fig. 2, this is precisely what happens [23]. Deviations from the universal behavior emerge for $M \approx M_c$, equivalently when $\tau_E/\tau_0 \approx 1$, i.e. much earlier than the suppression of shot noise (see also Ref. [24]).
We face a completely different situation when varying $\varepsilon$ within a given sample. In a ballistic system like ours, such a change does not modify the classical trajectories, and thus alters only the action phase of each contribution to the semiclassical Green function. Within this picture, the conductance fluctuates only due to long, diffracting orbits with $t > \tau_E$ [26]. These long orbits build up the stochastic phase. Their subset can be viewed as corresponding to an effective stochastic cavity with contacts to leads with $N_{\text{eff}} = \alpha N$ channels. Fixing the microscopic parameters $\tau_D$ and $\lambda$, one has $N_{\text{eff}} \sim M^{-1/18\tau_D} \gg 1$. This means that, despite the prefactor $\alpha$, $N_{\text{eff}}$ is always large enough to guarantee that transport occurs semiclassically, and therefore, one stays always in a regime with universal value $\sigma^2(g) = 1/8 [3]$. A first confirmation of this argument is provided by the numerical data shown in Fig. 2, which indicate a constant behavior of $\sigma^2(g)$, independently on $\tau_E/\tau_D$. To further check this argument, we finally consider the conductance correlator

$$F(\varepsilon) = \sigma^{-2}(g) \langle \delta g(\varepsilon_0) \delta g(\varepsilon_0 + \varepsilon) \rangle.$$  

(9)

As said above, only the phase accumulated after diffractive trajectories (for $t > \tau_E$) contributes to conductance fluctuations [26], and since the subset of diffractive trajectories have an average dwell time given by $\tau_\varepsilon + \tau_D$, they accumulate a relevant relative phase $\propto \varepsilon/\tau_D$. One therefore expects a decay of $F(\varepsilon)$ over the Thouless scale as in the universal regime [1], $\xi \varepsilon \propto 1/\tau_D$, independently on $\tau_E$. This is confirmed by the data shown in Fig. 3(b) and (c).

Our results thus show that the separation of the deterministic and stochastic phases is complete. Beyond a simple explanation of the suppression of shot-noise with $\alpha = \exp(-\tau_E/\tau_D)$ via Eq. [3], the phase separation correctly accounts for the behavior of the conductance variance and correlators in open quantum chaotic systems in the semiclassical limit. Further investigations along the lines initiated here should focus on other effects of mesoscopic coherence, such as the weak-localization corrections, and in particular the magnetoresistance.

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