Confidence intervals for the normal mean utilizing prior information

David Farchione and Paul Kabaila*

*Department of Mathematics and Statistics, La Trobe University, Victoria 3086, Australia

Abstract

Consider $X_1, X_2, \ldots, X_n$ that are independent and identically $N(\mu, \sigma^2)$ distributed. Suppose that we have uncertain prior information that $\mu = 0$. We answer the question: to what extent can a frequentist $1 - \alpha$ confidence interval for $\mu$ utilize this prior information?

Keywords: Frequentist confidence intervals; prior information; normal mean

* Corresponding author. Address: Department of Mathematics and Statistics, La Trobe University, Victoria 3086, Australia; Tel.: +61-3-9479-2594; fax: +61-3-9479-2466.
E-mail address: P.Kabaila@latrobe.edu.au.
1. Introduction

Suppose that $X_1, \ldots, X_n$ are independent and identically $N(\mu, \sigma^2)$ distributed. The parameter of interest is $\mu$. Also suppose that previous experience with similar data sets and/or scientific background and expert opinion suggest that $\mu = a$, where $a$ is a specified number. Without loss of generality we assume that $a = 0$. Our aim is to answer the following question. To what extent can a frequentist $1 - \alpha$ confidence interval (i.e. a confidence interval whose coverage probability has infimum $1 - \alpha$) utilize this prior information?

For the sake of simplicity, we first deal with the case that $\sigma^2$ is known. We find a confidence interval for $\mu$ by first finding a confidence interval for $\theta = \sqrt{n}\mu/\sigma$. Let $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $X = \bar{X}/(\sigma/\sqrt{n})$, so that $X \sim N(\theta, 1)$. Suppose that $I = [\ell(X), u(X)]$ is a $1 - \alpha$ confidence interval for $\theta$ i.e. $P(\theta \in I) \geq 1 - \alpha$ for all $\theta$. The confidence interval for $\mu$ that corresponds to this confidence interval for $\theta$ is $J = [(\sigma/\sqrt{n})\ell(X), (\sigma/\sqrt{n})u(X)]$. Pratt (1961, 1963) considers $X \sim N(\theta, 1)$ and presents confidence intervals for $\theta$ that (a) have a pre-specified minimum coverage $1 - \alpha$ and (b) are shorter than the usual confidence interval when $\theta = 0$. The $1 - \alpha$ confidence interval for $\mu$ that has the smallest possible expected length when $\mu = 0$ is derived by Pratt (1961) and is

$$\left[\min\left(0, \bar{X} - z_{a}\frac{\sigma}{\sqrt{n}}\right), \max\left(0, \bar{X} + z_{a}\frac{\sigma}{\sqrt{n}}\right)\right] \tag{1}$$

where $z_{a}$ is defined by $P(Z > z_{a}) = a$ for $Z \sim N(0, 1)$.

This confidence interval for $\mu$ has the following analogue for the case that $\sigma^2$ is unknown

$$\left[\min\left(0, \bar{X} - t_{a,n-1}\frac{S}{\sqrt{n}}\right), \max\left(0, \bar{X} + t_{a,n-1}\frac{S}{\sqrt{n}}\right)\right] \tag{2}$$

where $t_{a,m}$ is defined by $P(T > t_{a,m}) = a$ for $T \sim t_{a,m}$ and $S^2 = (n - 1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. This analogue is given by Brown et al (1995) and has been described e.g. by Bofinger (1985).

These confidence intervals have two major problems. The first problem is that the expected lengths of these confidence intervals diverge to $\infty$ as $|\mu| \to \infty$. This unpleasant
feature means that if the prior information happens to be badly incorrect (i.e. $\mu$ happens to be very far from 0) then these confidence intervals perform extremely poorly. The second problem is that neither of these confidence intervals approaches the corresponding standard confidence interval when the data strongly indicates that the prior information about $\mu$ is incorrect. Surely, if the data strongly indicate that this prior information is incorrect then we should be using something very close to the standard $1 - \alpha$ confidence interval for $\mu$. For example, when $\sigma^2$ is known and $|X| > 10$ then we should be using the standard confidence interval $[\bar X - z_{\alpha/2}(\sigma/\sqrt{n}), \bar X + z_{\alpha/2}(\sigma/\sqrt{n})]$ for $\mu$.

In this paper we describe confidence intervals for $\mu$ that do not suffer from these problems. Similarly to Hodges and Lehmann (1952) and Bickel (1983, 1984), our aim is to utilize the uncertain prior information in the frequentist inference of interest, whilst providing a safeguard in case this prior information happens to be incorrect. Our $1 - \alpha$ confidence intervals have the following desirable properties. They have expected lengths that (a) are relatively small when the prior information that $\mu = 0$ is correct and (b) have a maximum value that is not too large. They also coincide with the corresponding standard $1 - \alpha$ confidence interval when the data happens to strongly contradict the prior information about $\mu$. In Sections 2, 3 and 4 we deal with the case that $\sigma^2$ is known, by applying the methodology of Pratt (1961) with a novel weight function. In Sections 5 and 6 we use the same novel weight function, combined with invariance and a new computationally-based approach, to deal with the case that $\sigma^2$ is unknown.

Finally, consider point and interval estimators utilizing uncertain prior information in linear regression. Bickel (1984) presents a comprehensive analysis of point estimators in a very general setting. He also analyzes the coverage properties of some fixed-width confidence intervals, assuming the covariance matrix of the error vector is known. Tuck (2006) develops a new variable-width confidence interval analogous to $\square$. The methodology described in Sections 5 and 6 of the present paper, leading to variable-width confidence intervals, can be extended to the linear regression context (Kabaila and Giri (2007)).

2. The known variance case

Assume that $\sigma^2$ is known. In the introduction we defined the random variable $X$,
which has an $N(\theta, 1)$ distribution, and the $1 - \alpha$ confidence intervals $I$ and $J$ for $\theta$ and $\mu$ respectively. Observe that $P_\mu(\mu \in I) = P_\theta(\theta \in I)$, so that the confidence intervals $I$ and $J$ have the same minimum coverage probabilities. Furthermore, $E_\mu($length of $J) = (\sigma/\sqrt{n})E_\theta($length of $I)$ when $\theta = \sqrt{n}\mu/\sigma$. In other words, the expected length of $J$ is proportional to the expected length of $I$. We therefore focus on the case that $X$ has an $N(\theta, 1)$ distribution and we have uncertain prior information that $\theta = 0$.

Let $C(X)$ be a $1 - \alpha$ confidence region for $\theta$. Define $A(\theta)$ by $\theta \in C(X)$ if and only if $x \in A(\theta)$. Here $A(\theta)$ is the acceptance region for the null hypothesis that $\theta$ is the true parameter value. Let $L(C(X))$ denote the sum of the lengths of the intervals making up $C(X)$. Also let our aim be to minimise the average expected length

$$\int E_\theta(L(C(X)))d\nu(\theta).$$

(3)

for a specified weight function $\nu$. We use $\phi$ to denote the $N(0, 1)$ density function. As proved by Pratt (1961), the solution to this problem is to choose the acceptance region $A(\theta)$ to consist of those values of $x$ such that

$$\frac{\int_{-\infty}^{\infty} \phi(x - \theta)d\nu(\theta)}{\phi(x - \theta)} < c_\alpha(\theta)$$

where $c_\alpha(\theta)$ is chosen such that $P_\theta(X \in A(\theta)) = 1 - \alpha$. For some weight functions $\nu$ the average expected length is infinite, even for the confidence interval corresponding to the acceptance regions obtained in this way. However, the criterion

$$\int (E_\theta(L(C(X))) - 2z_{\alpha/2})d\nu(\theta)$$

takes the (finite) value 0 when $C(X)$ is the standard $1 - \alpha$ confidence interval for $\theta$. It is straightforward to show that the minimization of this criterion leads to the same formula for $A(\theta)$ as the (formal) minimization of (3).

As pointed out by Pratt (1961), the standard $1 - \alpha$ confidence interval for $\theta$,

$$[X - z_{\alpha/2}, X + z_{\alpha/2}],$$

(4)

is the $1 - \alpha$ confidence interval that minimizes the average expected length when $\nu(x) = x$ for all $x$. Also, as pointed out by Pratt (1961), the $1 - \alpha$ confidence interval (1) for $\theta$ is
the 1 − α confidence interval that minimizes the average expected length when ν = H where H is the unit step function defined by H(x) = 0 for x < 0 and H(x) = 1 for x ≥ 0.

Now consider a weight function that is a mixture of the weight functions ν(x) = x and ν = H. It is plausible that if we minimise the average expected length using this weight function then we will obtain a confidence interval that (a) has relatively small expected length when θ = 0 and (b) overcomes the weaknesses of Pratt’s interval (1). So, we consider the 1 − α confidence interval that minimises the average expected length when the weight function ν is

\[ \nu(x) = wx + H(x) \quad \text{for all } x. \] (5)

Here, w is a fixed nonnegative number. We call this the ‘mixed interval.’

In this case, the acceptance region A(θ) corresponding to the confidence region C(X) minimizing the average expected length (3) consists of the values of x such that

\[ \frac{w + \phi(x)}{\phi(x - \theta)} - c_\alpha(\theta) < 0, \]

where \( c_\alpha(\theta) \) is chosen such that \( P_\theta(X \in A(\theta)) = 1 - \alpha \). Define

\[ g(x, c, \theta) = \frac{w + \phi(x)}{\phi(x - \theta)} - c. \]

Also define \( B(c, \theta) = \{ x : g(x, c, \theta) < 0 \} \). Obviously, \( c_\alpha(\theta) \) is the value of c such that \( P_\theta(X \in B(c, \theta)) = 1 - \alpha \). To analyse the properties of the acceptance region A(θ) we will need the following theorem.

**Theorem 2.1.** For every fixed \( w > 0, \theta \in \mathbb{R} \) and \( c > 0 \), the following is true. The set \( B(c, \theta) \) is either (a) the empty set or (b) an interval with finite endpoints.

**Proof.** Fix \( w > 0, \theta \in \mathbb{R} \) and \( c > 0 \). Observe that \( g(x, c, \theta) \to \infty \) as \( |x| \to \infty \). The result will be proved by showing that \( \partial g(x, c, \theta)/\partial x \) is an increasing function of \( x \in \mathbb{R} \).

Now \( g(x, c, \theta) = \exp(\frac{1}{2} \theta^2) g^*(x, \theta) - c \), where \( g^*(x, \theta) = w^* \exp(\frac{1}{2} x^2 - \theta x) + \exp(-\theta x) \) and \( w^* = \sqrt{2\pi} w \). Note that

\[ \frac{\partial g^*(x, \theta)}{\partial x} = \exp(-\frac{1}{2} \theta^2) w^*(x - \theta) \exp(\frac{1}{2} (x - \theta)^2) - \theta \exp(-\theta x). \]
This is an increasing function of $x$. Hence $\partial g(x, c, \theta)/\partial x$ is an increasing function of $x$.

□

This theorem leads to the very important property of $A(\theta)$ described in the following corollary, whose proof is omitted for the sake of brevity.

**Corollary 2.1.** For every fixed $w > 0$ and $\theta \in \mathbb{R}$, the following is true. The $1 - \alpha$ acceptance region $A(\theta)$ is an interval with finite endpoints.

The computation of the acceptance region $A(\theta)$ for given $w > 0$ and $\theta \in \mathbb{R}$ is facilitated by the following result.

**Theorem 2.2.** For every $w > 0$ and $\theta \in \mathbb{R}$,

$$w \sqrt{2\pi} \exp\left(\frac{1}{2}z_{\alpha/2}^2\right) \leq c_\alpha(\theta) \leq (w \sqrt{2\pi} + 1) \exp\left(\frac{1}{2}z_{\alpha/2}^2\right).$$

**Proof.** The result follows from the fact that for every $w > 0$ and $\theta \in \mathbb{R}$ the following is true. For every $x \in \mathbb{R}$,

$$\frac{w}{\phi(x - \theta)} \leq \frac{w + \phi(x)}{\phi(x - \theta)} \leq \frac{w + (1/\sqrt{2\pi})}{\phi(x - \theta)}.$$

□

The following theorem describes an important property of the confidence set $C(x)$. The proof of this theorem is omitted, for the sake of brevity.

**Theorem 2.3.** For every $w > 0$ the following is true. The $1 - \alpha$ confidence set $C(x)$ is an interval for all sufficiently large $|x|$, with endpoints approaching those of the standard $1 - \alpha$ confidence interval $[x - z_{\alpha/2}, x + z_{\alpha/2}]$ as $|x| \to \infty$.

3. Numerical comparison of the intervals for the known variance case

We continue with our consideration of the case that $\sigma^2$ is known. As described in the introduction, we reduce this case to the problem of finding a $1 - \alpha$ confidence interval for $\theta$ based on $X \sim N(\theta, 1)$. We denote the standard $1 - \alpha$ confidence interval (4) by $C_S(X)$. Also, we denote Pratt’s interval (11) by $C_P(X)$.

Consider the case that the weight function $\nu$ is given by (5). This weight function is a mixture of the weight functions $\nu(x) = x$ and $\nu = H$ that lead to $C_S$ and $C_P$.
respectively. For $1 - \alpha = 0.95$ and each $w \in \{0.01, 0.1, 1\}$, Corollary 2.1 and Theorem 2.2 were used to compute the acceptance regions $A(\theta)$, corresponding to the 0.95 confidence sets minimizing the average expected length \([32]\), for a fine grid of values of $\theta$. For each of these values of $w$, the confidence region corresponding to $A(\theta)$ was found to always be an interval. We denote the 0.95 confidence interval minimizing the average expected length when $\nu$ is given by \([33]\) (which we have called the mixed interval) by $C_w^{(M)}(X)$. All the computations for the present paper were performed with programs written in MATLAB, using the Optimization and Statistics toolboxes.

We use $C_S$ as the standard against with other $1 - \alpha$ confidence intervals for $\theta$ are judged. The efficiency of $C_S$ relative to $C$, a $1 - \alpha$ confidence interval for $\theta$, for a given value of $\theta$ is defined to be

$$e(\theta, C_S, C) = \left(\frac{E_\theta(L(C(X)))}{E_\theta(L(C_S(X)))}\right)^2.$$ 

Let $X = (X_1, X_2, \ldots, X_n)$. Note that a $1 - \alpha$ confidence interval $C(X)$ for $\theta$ corresponds to a $1 - \alpha$ confidence interval $D(X)$ for $\mu$ that is obtained by multiplying the endpoints of $C(X)$ by $\sigma/\sqrt{n}$. Let $D_S(X)$ denote the standard $1 - \alpha$ confidence interval for $\mu$. We define the efficiency of $D_S$ relative to $D$ as \(\left(E(L(D(X)))/E(L(D_S(X)))\right)^2\) and note that this is the same function of $\theta$ as $e(\theta, C_S, C)$.

Figure 1 shows plots of the efficiency of $C_S$ relative to $C_w^{(M)}$ for $w = 1$, $w = 0.1$, $w = 0.01$ and $w = 0$, when $1 - \alpha = 0.95$. Note that Pratt’s interval $C_P$ is equal to the mixed interval $C_w^{(M)}$ for $w = 0$. Also, the standard interval $C_S$ may be viewed as the mixed interval $C_w^{(M)}$ in the limiting case $w \to \infty$. Even for $w = 1$, which is not a particularly large value of $w$, $C_w^{(M)}$ is fairly close to $C_S$.

The minimum over all $1 - \alpha$ confidence intervals $C$ of $e(0, C_S, C)$ is 0.7223 and this minimum is achieved by Pratt’s interval $C_P$. However, as noted in the introduction, this interval suffers the severe problems that (a) $e(\theta, C_S, C_P) \to \infty$ as $|\theta| \to \infty$ and (b) it does not revert to the standard interval when $|X| \to \infty$. The mixed interval $C_w^{(M)}$ with $w = 0.1$ is far preferable. For this interval, $e(0, C_S, C_w^{(M)}) = 0.8016$, which is not that much larger than 0.7223. Also, for this interval, $e(\theta, C_S, C_w^{(M)})$ never exceeds 1.2095. Finally,
in accordance with Theorem 2.3, this interval approaches the standard interval $C_S$ as $|X| \to \infty$. Of course, the value of $w$ can be chosen so as to reflect the strength of the prior information that $\theta = 0$.

Figure 1: Plots of the efficiency $e(\theta, C_S, C_M^{w})$ of the standard interval $C_S$ relative to the mixed interval $C_M^{w}$ for $w = 1$, $w = 0.1$, $w = 0.01$ and $w = 0$ when $1 - \alpha = 0.95$.

4. Invariance properties of the confidence interval in the known variance case

In this section we first describe the invariance properties that we expect the confidence interval $J$ (defined in the introduction) to possess. Traditional invariance arguments (see e.g. Casella and Berger (2002, section 6.4) do not include considerations of the available prior information. The novelty in the present section is that the invariance arguments need to take proper account of the uncertain prior information that $\mu = 0$.

We first describe an invariance property that $J$ already possesses. The model that $X_1, \ldots, X_n$ are independent and identically $N(\mu, \sigma^2)$ distributed may be re-expressed as follows. Define $Y_i = aX_i$ for $i = 1, \ldots, n$ where $a > 0$. Thus $Y_1, \ldots, Y_n$ are independent and identically $N(\tilde{\mu}, \tilde{\sigma}^2)$ distributed where $\tilde{\mu} = a\mu$ and $\tilde{\sigma} = a\sigma$. Define $Y = \bar{Y}/(\tilde{\sigma}/\sqrt{n})$. 
The prior information $\mu = 0$ may be re-expressed as $\bar{\mu} = 0$. The re-expressed model and prior information have the same form as the original model and prior information. Thus the confidence interval $\tilde{J} = \left[ (\bar{\sigma}/\sqrt{n})\ell(Y), (\bar{\sigma}/\sqrt{n})u(Y) \right]$ for $\bar{\mu}$ must lead to a confidence interval for $\mu$ that is identical to $J$. It is easily seen that this requirement is satisfied.

Next, we describe an invariance property that $J$ should possess and conclude from this that the equality $\ell(x) = -u(-x)$ should hold for all $x \in \mathbb{R}$. The model that $X_1, \ldots, X_n$ are independent and identically $N(\mu, \sigma^2)$ distributed may be re-expressed as follows. Define $Y_i = -X_i$ for $i = 1, \ldots, n$. Thus $Y_1, \ldots, Y_n$ are independent and identically $N(\bar{\mu}, \sigma^2)$ distributed where $\bar{\mu} = -\mu$. Define $Y = \bar{Y}/(\sigma/\sqrt{n})$. The prior information $\mu = 0$ may be re-expressed as $\bar{\mu} = 0$. The re-expressed model and prior information have the same form as the original model and prior information. Thus the confidence interval $\tilde{J} = \left[ (\sigma/\sqrt{n})\ell(Y), (\sigma/\sqrt{n})u(Y) \right]$ for $\bar{\mu}$ must lead to a confidence interval for $\mu$ that is identical to $J$. It is easily seen that this requirement is satisfied if and only if $\ell(x) = -u(-x)$ for all $x \in \mathbb{R}$. Note that both Pratt’s interval (1) and the mixed interval defined in Section 2 satisfy this requirement.

5. Invariance arguments in the unknown variance case

We now consider the case that $\sigma^2$ is unknown. This is the case that is usually encountered in practice. The standard $1 - \alpha$ confidence interval for $\mu$ is

$$\left[ \bar{X} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}} \right]. \quad (6)$$

A natural analogue of the confidence interval $J$ for $\mu$ is the confidence interval

$$K = \left[ \frac{S}{\sqrt{n}} a \left( \bar{X}/S/\sqrt{n} \right), \frac{S}{\sqrt{n}} b \left( \bar{X}/S/\sqrt{n} \right) \right]$$

for $\mu$. Note that both (6) and (2) have this form. Suppose that our uncertain prior information is that $\mu = 0$. Using the same model transformations as in Section 4, invariance arguments show that the equality $a(x) = -b(-x)$ must hold for all $x \in \mathbb{R}$. In other words,

$$K = \left[ -\frac{S}{\sqrt{n}} b \left( -\bar{X}/S/\sqrt{n} \right), \frac{S}{\sqrt{n}} b \left( \bar{X}/S/\sqrt{n} \right) \right]. \quad (7)$$
The constraint that the upper endpoint of this confidence interval is never less than the lower endpoint implies that \( b(x) \geq -b(-x) \) for all \( x \in \mathbb{R} \). It also seems reasonable to require that \( b \) is a strictly increasing function.

6. Computation of the interval for the unknown variance case

In this section we provide computationally convenient expressions that are used to calculate the mixed interval for the unknown variance case. We illustrate the performance of this interval and compare its performance with the corresponding mixed interval when \( \sigma^2 \) is known for the case that \( n = 24 \) and \( 1 - \alpha = 0.95 \).

Suppose that \( \sigma^2 \) is unknown. As in Section 3, let \( \mathbf{X} = (X_1, X_2, \ldots, X_n) \). Also let \( G(\mathbf{X}) \) be a confidence interval for \( \mu \) that is of the form (7). Our aim is to minimize the average expected length of \( G(\mathbf{X}) \) for a given weight function \( \nu \), such that the coverage probability of \( G(\mathbf{X}) \) is at least \( 1 - \alpha \) for all \( \mu \in \mathbb{R} \). Let \( \theta = \sqrt{n\mu}/\sigma \). As we show shortly, the coverage probability \( P(\mu \in G(\mathbf{X})) \) is a function of \( \theta \). The expected length of \( G(\mathbf{X}) \) is a function of \((\mu, \sigma^2)\). However, we will introduce a scaled expected length of \( G(\mathbf{X}) \) which is a function of \( \theta \). By using this scaled expected length, instead of the expected length, we will be able to achieve our aim by considering only quantities that are functions of \( \theta \) (cf. Kabaila (1998, 2005)).

The coverage probability of \( G(\mathbf{X}) \) is a function of \( \theta \) and is given by

\[
P(\mu \in G(\mathbf{X})) = P \left( -Rb \left( \frac{-X}{R} \right) \leq \theta \leq Rb \left( \frac{X}{R} \right) \right)
\]  

(8)

where \( X = \sqrt{n\bar{X}}/\sigma \sim N(\theta, 1) \) and \( R = S/\sigma \). Note that \( X \) and \( R \) are independent random variables. We assume that \( b \) is a strictly increasing function. This implies that \( b^{-1} \) exists. A computationally convenient expression for the right hand side of (8) is

\[
\int_0^\infty \left( \Phi \left( -rb^{-1} \left( \frac{-\theta}{r} \right) \right) - \Phi \left( rb^{-1} \left( \frac{\theta}{r} \right) \right) \right) f_R(r)dr
\]  

(9)

where \( \Phi \) is the N(0,1) cumulative distribution function and \( f_R \) denotes the density of \( R \).

We introduce the scaled expected length of \( G(\mathbf{X}) \) which is a function of \( \theta \) and is defined to be

\[
\frac{\sqrt{n}}{\sigma} E(L(G(\mathbf{X}))) = E \left( R \left( b \left( \frac{X}{R} \right) + b \left( \frac{-X}{R} \right) \right) \right).
\]

10
A computationally convenient expression for this scaled expected length is

\[
\int_0^\infty \int_{-\infty}^\infty \left( b\left( \frac{x}{r} \right) + b\left( \frac{-x}{r} \right) \right) \phi(x - \theta) \, dx \, r f_R(r) \, dr.
\] (10)

We use the weight function (5). As with the \( \sigma^2 \) known case, for \( w > 0 \), the average scaled expected length criterion is infinite even for the standard confidence interval (6). Therefore, similarly to Section 2, we replace this criterion by the following criterion

\[
\int \left( \frac{\sqrt{n}}{\sigma} E \left( L(G(X)) \right) - 2t_{\alpha/2,n-1}E(R) \right) d\nu(\theta)
\] (11)

which takes the (finite) value 0 when \( G(X) \) is the standard confidence interval (6). Substituting the expression (10) for the scaled expected length into (11) we obtain

\[
\int_0^\infty \int_{-\infty}^\infty \left( b\left( \frac{x}{r} \right) + b\left( \frac{-x}{r} \right) - 2t_{\alpha/2,n-1} \right) (w + \phi(x)) \, dx \, r f_R(r) \, dr.
\] (12)

Remember that we require the confidence interval to coincide with the standard \( 1 - \alpha \) confidence interval when the data happens to strongly contradict the prior information about \( \mu \). The statistic \( \sqrt{nX}/S \) provides an indication of how far \( \sqrt{n}\mu/\sigma \) is from 0. We therefore satisfy this requirement by setting \( b(y) = y + t_{\alpha/2,n-1} \) for all \( |y| \geq q \) where \( q \) is a specified positive number (which is chosen to be sufficiently large). Thus \( b(x/r) + b(-x/r) - 2t_{\alpha/2,n-1} = 0 \) for all \( |x|/r \geq q \). Changing the variable of integration from \( x \) to \( y = x/r \), (12) can now be expressed in the computationally convenient form

\[
\int_0^\infty \int_{-q}^q \left( b(y) + b(-y) - 2t_{\alpha/2,n-1} \right) (w + \phi(ry)) \, dy \, r^2 f_R(r) \, dr.
\] (13)

For computational ease, we restrict the function \( b(y) \) to be a cubic spline in the interval \([-q,q]\). This spline is required to take the value \(-q + t_{\alpha/2,n-1} \) at \( y = -q \) and \( q + t_{\alpha/2,n-1} \) at \( y = q \) and has knots that are equally spaced between \(-q \) and \( q \). In addition, the derivative of this spline is constrained to be 1 at both \( y = -q \) and \( y = q \).

We minimize (13) with respect to the function \( b \), subject to the constraints on \( b \) described at the end of Section 5 and the constraint that (9) is at least \( 1 - \alpha \) for all \( \theta \in \mathbb{R} \). We denote the minimizing function \( b \) by \( b^w_M \). We call the confidence interval for \( \mu \) corresponding to \( b^w_M \) the mixed interval and denote it by \( G^w_M \). We denote the standard
confidence interval (6) by \( G_S \). Similarly to Section 3, we use \( G_S \) as the standard against with other \( 1 - \alpha \) confidence intervals for \( \mu \) are judged. The efficiency of \( G_S \) relative to \( G \), a \( 1 - \alpha \) confidence interval for \( \mu \), is defined to be \( \left( \frac{E(L(G(X)))}{E(L(G_S(X)))} \right)^2 \) which is a function of \( \theta \).

To illustrate the performance of the mixed interval and to compare its performance with the corresponding mixed interval when \( \sigma^2 \) is known, we consider the case that \( n = 24 \) and \( 1 - \alpha = 0.95 \). For the computation of \( G_M^w \), we chose \( q = 8 \) with the knots of the cubic spline at \(-8, -7, \ldots, 7, 8\). We also chose \( w = 0.1 \). The efficiency of \( G_S \) relative to \( G_M^w \) is shown in the right panel of Figure 2. When the prior information is correct i.e. \( \mu = 0 \) the efficiency of \( G_S \) relative to \( G_M^w \) is 0.8013. Also, the efficiency of \( G_S \) relative to \( G_M^w \) never exceeds 1.1930. Furthermore, \( G_M^w \) reverts to the standard \( 1 - \alpha \) confidence interval when the prior information happens to be badly incorrect. This is reflected in the fact that the efficiency of \( G_S \) relative to \( G_M^w \) approaches 1 as \( |\theta| \to \infty \). It is notable that the coverage probability of the confidence interval \( G_M^w \) was found to be 0.95 to an extremely good approximation throughout the parameter space. Now \( n = 24 \) is quite large and so \( S \) will be probabilistically close to \( \sigma \). We therefore expect that the efficiency of \( G_S \) relative to \( G_M^w \) to be similar to the efficiency of \( D_S \) relative to \( D_M^w \) when \( w = 0.1 \). This expectation is confirmed by the left panel of Figure 2.

![Figure 2: These plots concern the case that \( n = 24 \), \( 1 - \alpha = 0.95 \) and \( w = 0.1 \). The left panel is a plot of the efficiency of \( D_S \) relative to \( D_M^w \) as a function of \( \theta \). The right panel is a plot of the efficiency of \( G_S \) relative to \( G_M^w \) as a function of \( \theta \).](#)
References

Bickel, P.J., (1983). Minimax estimation of the mean of a normal distribution subject to doing well at a point, in: M.H. Rizvi, J.S. Rustagi and D. Siegmund, eds. Recent Advances in Statistics, Academic Press, New York, pp. 511–528.

Bickel, P.J., (1984). Parametric robustness: small biases can be worthwhile. Annals of Statistics 12, 864–879.

Bofinger, E., 1985. Expanded confidence intervals. Communications in Statistics: Theory and Methods 14, 1849–1864.

Brown, L.D., Casella, G., Hwang, J.T.G., (1995). Optimal confidence sets, bioequivalence and the Limacon of Pascal. Journal of the American Statistical Association 90, 880–889.

Casella, G., Berger, R. L., (2002). Statistical Inference, 2nd ed. Duxbury, Pacific Grove, California.

Hodges, J.L., Lehmann, E.L., (1952). The use of previous experience in reaching statistical decisions. Annals of Mathematical Statistics 23, 396–407.

Kabaila, P., (1998). Valid confidence intervals in regression after variable selection. Econometric Theory 14, 463–482.

Kabaila, P., (2005). Assessment of a preliminary F-test solution to the Behrens-Fisher problem. Communications in Statistics: Theory and Methods 34, 291–302.

Kabaila, P., Giri, K., (2007). Large sample confidence intervals in regression utilizing prior information. La Trobe University, Department of Mathematics and Statistics, Technical Report No. 2007–1, Jan 2007.

Pratt, J.W., (1961). Length of confidence intervals. Journal of the American Statistical Association 56, 549–657.

Pratt, J.W., (1963). Shorter confidence intervals for the mean of a normal distribution with known variance. Annals of Mathematical Statistics 34, 574–586.

Tuck, J., (2006). Confidence intervals incorporating prior information. PhD thesis, August 2006, Department of Mathematics and Statistics, La Trobe University.