Berry-Esseen’s central limit theorem for non-causal linear processes in Hilbert space

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Abstract

Let $H$ be a real separable Hilbert space and $(a_k)_{k \in \mathbb{Z}}$ a sequence of bounded linear operators from $H$ to $H$. We consider the linear process $X$ defined for any $k$ in $\mathbb{Z}$ by

$$X_k = \sum_{j \in \mathbb{Z}} a_j (\varepsilon_{k-j})$$

where $(\varepsilon_k)_{k \in \mathbb{Z}}$ is a sequence of i.i.d. centered $H$-valued random variables. We investigate the rate of convergence in the CLT for $X$ and in particular we obtain the usual Berry-Esseen’s bound provided that $\sum_{j \in \mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)} < +\infty$ and $\varepsilon_0$ belongs to $L^\infty_H$.

Short title: Berry-Esseen’s CLT for Hilbertian linear processes.

Key words: Central limit theorem, Berry-Esseen bound, linear process, Hilbert space.

1 Introduction and notations

Let $(H, \| \cdot \|_H)$ be a separable real Hilbert space and $(\mathcal{L}, \| \cdot \|_{\mathcal{L}(H)})$ be the class of bounded linear operators from $H$ to $H$ with its usual uniform norm. Consider a sequence $(\varepsilon_k)_{k \in \mathbb{Z}}$ of i.i.d. centered random variables, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with values in $H$. If $(a_k)_{k \in \mathbb{Z}}$ is a sequence in $\mathcal{L}$, we define the (non-causal) linear process $X = (X_k)_{k \in \mathbb{Z}}$ in $H$ by

$$X_k = \sum_{j \in \mathbb{Z}} a_j (\varepsilon_{k-j}), \quad k \in \mathbb{Z}. \quad (1)$$

If $\sum_{j \in \mathbb{Z}} \|a_j\|_{\mathcal{L}(H)} < \infty$ and $E\|\varepsilon_0\|_H < +\infty$ then the series in (1) converges almost surely and in $L^1_H(\Omega, \mathcal{A}, \mathbb{P})$ (see Bosq [2]). The condition $\sum_{j \in \mathbb{Z}} \|a_j\|_{\mathcal{L}(H)} < \infty$ is know to be sharp.
for the \( \sqrt{n} \)-normalized partial sums of \( X \) to satisfies a CLT provided that \( (\varepsilon_k)_{k \in \mathbb{Z}} \) are i.i.d. centered having finite second moments (see Merlevede et al. [6]). In this work, we investigate the rate of convergence in the CLT for \( X \) under the condition

\[
\sum_{j \in \mathbb{Z}} |j|^\tau \|a_j\|_{L(H)} < \infty
\]

with \( \tau = 1 \) when \( (\varepsilon_k)_{k \in \mathbb{Z}} \) are assumed to be i.i.d. centered and such that \( \varepsilon_0 \) belongs to \( L^\infty_H \) and \( \tau = 1/2 \) when \( (\varepsilon_k)_{k \in \mathbb{Z}} \) are i.i.d. centered and such that \( \varepsilon_0 \) belongs to some Orlicz space \( L_{H,\psi} \) (see section 2). This problem was previously studied (with \( \tau = 1 \) in Condition (2)) by Bosq [3] for (causal) Hilbert linear processes but a mistake in his proof was pointed out by V. Paulauskas [7]. However, in the particular case of Hilbertian autoregressive processes of order 1, Bosq [4] obtained the usual Berry-Esseen inequality provided that \( (\varepsilon_k)_{k \in \mathbb{Z}} \) are i.i.d. centered with \( \varepsilon_0 \) in \( L^\infty_H \).

2 Main result

In the sequel, \( C_{\varepsilon_0} \) is the autocovariance operator of \( \varepsilon_0 \), \( A := \sum_{j \in \mathbb{Z}} a_j \) and \( A^* \) is the adjoint of \( A \). For any sequence \( Z = (Z_k)_{k \in \mathbb{Z}} \) of random variables with values in \( H \) we denote

\[
\Delta_n(Z) = \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \left\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Z_k \right\|_H \leq t \right) - \mathbb{P} \left( \|N\|_H \leq t \right) \right|
\]

where \( N \sim \mathcal{N}(0, AC_{\varepsilon_0}A^*) \).

For any \( j \in \mathbb{Z} \), denote \( c_{j,n} = \sum_{i=1}^{n} b_{i-j} \) where \( b_i = a_i \) for any \( i \neq 0 \) and \( b_0 = a_0 - A \).

**Lemma 1** For any positive integer \( n \),

\[
\sum_{k=1}^{n} X_k = A \left( \sum_{k=1}^{n} \varepsilon_k \right) + Q_n + R_n
\]

where \( Q_n = \sum_{k=1}^{n} \sum_{|j| > n} a_{k-j} (\varepsilon_j) \) and \( R_n = \sum_{|j| \leq n} c_{j,n}(\varepsilon_j) \).

Recall that a Young function \( \psi \) is a real convex nondecreasing function defined on \( \mathbb{R}^+ \) which satisfies \( \lim_{t \to +\infty} \psi(t) = +\infty \) and \( \psi(0) = 0 \). We define the Orlicz space \( L_{H,\varphi} \) as the space of \( H \)-valued random variables \( Z \) defined on the probability space \( (\Omega, F, \mathbb{P}) \) such
that $E[\psi(\|Z\|_H/c)] < +\infty$ for some $c > 0$. The Orlicz space $L_{H,\psi}$ equipped with the so-called Luxemburg norm $\|\cdot\|_\psi$ defined for any $H$-valued random variable $Z$ by

$$\|Z\|_\psi = \inf\{ c > 0 ; E[\psi(\|Z\|_H/c)] \leq 1 \}$$

is a Banach space. In the sequel, $c(N)$ denotes a bound of the density of $N(0, AC_{\bar{c}_0} A^*)$ (see Davydov et al. [5]). Our main result is the following.

**Theorem 1** Let $(\varepsilon_k)_{k \in \mathbb{Z}}$ be a sequence of i.i.d. centered $H$-valued random variables and let $X$ be the Hilbertian linear process defined by \(1\).

1) If $\varepsilon_0$ belongs to $L_H^\infty$ and $\sum_{j \in \mathbb{Z}} |j| \|a_j\|_{L(H)} < \infty$ then

$$\Delta_n(X) \leq \frac{c_1}{\sqrt{n}}$$

where $c_1 = c_2 + 14c(N)\|\varepsilon_0\|_\infty \sum_{j \in \mathbb{Z}} |j| \|a_j\|_{L(H)}$ and $c_2$ is a positive constant which depend only on the distribution of $\varepsilon_0$.

2) If $\psi$ is a Young function then

$$\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \varphi\left(\frac{c(N)\|Q_n + R_n\|_\psi}{\sqrt{n}}\right)$$

where $\varphi(x) = xh^{-1}(1/x)$ and $h(x) = x\psi(x)$ for any real $x > 0$.

The inequality \(1\) ensures a rate of convergence to zero for $\Delta_n(X)$ as $n$ goes to infinity provided that $\Delta_n(A(\varepsilon))$ goes to zero as $n$ goes to infinity and a bound for $\|Q_n + R_n\|_\psi$ exists. As just an illustration, we have the following corollary.

**Corollary 1** Assume that $(\varepsilon_k)_{k \in \mathbb{Z}}$ are i.i.d. centered $H$-valued random variables and that the condition \(2\) holds with $\tau = 1/2$.

1) If $\varepsilon_0$ belongs to $L_{H,\psi_1}$ then $\Delta_n(X) = O\left(\frac{\log n}{n}\right)$ where $\psi_1(x) = \exp(x) - 1$.

2) If $\varepsilon_0$ belongs to $L_H^r$ for $r \geq 3$ then $\Delta_n(X) = O\left(n^{-\frac{r-2}{3(r+1)}}\right)$. 

3
3 Proofs

Proof of Lemma 1. For any positive integer \( n \), we have

\[
R_n = \sum_{j=-n}^{n} c_{j,n}(\varepsilon_j) = \sum_{k=1}^{n} \sum_{j=-n}^{n} b_{k-j}(\varepsilon_j)
\]

\[
= \sum_{k=1}^{n} \sum_{j \in [-n,n] \setminus \{k\}} a_{k-j}(\varepsilon_j) + (a_0 - A) \left( \sum_{k=1}^{n} \varepsilon_k \right)
\]

\[
= \sum_{k=1}^{n} \sum_{j=-n}^{n} a_{k-j}(\varepsilon_j) - A \left( \sum_{k=1}^{n} \varepsilon_k \right)
\]

\[
= -Q_n + \sum_{k=1}^{n} X_k - A \left( \sum_{k=1}^{n} \varepsilon_k \right).
\]

The proof of Lemma 1 is complete.

Proof of Theorem 1. Let \( \lambda > 0 \) and \( t > 0 \) be fixed and denote \( U = A \left( \sum_{k=1}^{n} \varepsilon_k / \sqrt{n} \right) \) and \( V = (Q_n + R_n) / \sqrt{n} \). So \( U + V = \sum_{k=1}^{n} X_k / \sqrt{n} \) and

\[
\mathbb{P}(\|U + V\|_H \leq t) \leq \mathbb{P}(\|U\|_H \leq t + \lambda) + \mathbb{P}(\|V\|_H \geq \lambda)
\]

(5)

For \( \lambda_0 = 2\|V\|_\infty \), we obtain

\[
\mathbb{P}(\|U + V\|_H \leq t) - \mathbb{P}(\|N\|_H \leq t) \leq \mathbb{P}(\|U\|_H \leq t + \lambda_0) - \mathbb{P}(\|N\|_H \leq t).
\]

If \( c(N) \) denotes a bound for the density of \( \|N\|_H \) (see Davydov et al. [5]) then

\[
\Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \frac{2c(N)\|Q_n + R_n\|_\infty}{\sqrt{n}}.
\]

Noting that

\[
Q_n = \sum_{j \geq n+2} a_j \left( \sum_{k=1-j}^{n-1} \varepsilon_k \right) + \sum_{j < 0} a_j \left( \sum_{k=1}^{n-j} \varepsilon_k \right)
\]

(6)

and

\[
R_n = R'_n + R''_n
\]

(7)

where

\[
R'_n = \sum_{j=-n}^{n} a_j \left( \sum_{k=1}^{n-j} \varepsilon_k \right) - \sum_{j < -n} a_j \left( \sum_{k=1}^{n} \varepsilon_k \right) - \sum_{j > 0} a_j \left( \sum_{k=1}^{n-j+1} \varepsilon_k \right)
\]
and
\[ R''_n = \sum_{j=1}^{n} a_j \left( \sum_{k=-j+1}^{0} \varepsilon_k \right) + \sum_{j=n+1}^{2n} a_j \left( \sum_{k=-n}^{n-j} \varepsilon_k \right), \]
we derive that \( \|Q_n + R_n\|_{\infty} \leq 7\|\varepsilon_0\|_{\infty} \sum_{j \in \mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)} \) and consequently
\[ \Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \frac{14c(N)\|\varepsilon_0\|_{\infty} \sum_{j \in \mathbb{Z}} |j| \|a_j\|_{\mathcal{L}(H)}}{\sqrt{n}}. \]

Combining the last inequality with the Berry-Esseen inequality for i.i.d. centered \( H \)-valued random variables (see Yurinski [11] or Bosq [2], Theorem 2.9) we obtain (3).

In the other part, if \( \psi \) is a Young function we have \( \mathbb{P}(\|V\|_{H} \geq \lambda) \leq \frac{1}{\psi(\lambda/\|V\|_{\psi})} \) and keeping in mind inequality (5), we derive
\[ \Delta_n(X) \leq \Delta_n(A(\varepsilon)) + c(N)\lambda + \frac{1}{\psi(\lambda/\|V\|_{\psi})}. \]
Noting that \( c(N)\lambda = \frac{1}{\psi(\lambda/\|V\|_{\psi})} \) if and only if \( \lambda = \frac{\varphi(c(N)\|V\|_{\psi})}{c(N)} \) where \( \varphi \) is defined by \( \varphi(x) = xh^{-1}(1/x) \) and \( h \) by \( h(x) = x\psi(x) \), we conclude
\[ \Delta_n(X) \leq \Delta_n(A(\varepsilon)) + \varphi \left( \frac{c(N)\|Q_n + R_n\|_{\psi}}{\sqrt{n}} \right). \]

The proof of Theorem 1 is complete.

Proof of Corollary 1. Assume that \( \|\varepsilon_0\|_{\psi_1} < \infty \) where \( \psi_1 \) is the Young function defined by \( \psi_1(x) = \exp(x) - 1 \). There exists \( a > 0 \) such that \( E(\exp(a\|\varepsilon_0\|_{H})) \leq 2 \). So, there exist (see Arak and Zaizsev [1]) constants \( B \) and \( L \) such that
\[ E\|\varepsilon_0\|_{H}^m \leq \frac{m!}{2} B^2 L^{m-2}, \quad m = 2, 3, 4, ... \]
Applying Pinelis-Sakhanenko inequality (see Pinelis and Sakhanenko [?] or Bosq [2]), we obtain
\[ \mathbb{P} \left( \left\| \sum_{k=p}^{q} \varepsilon_k \right\|_{H} \geq x \right) \leq \exp \left( -\frac{x^2}{2(q-p+1)B^2 + 2xL} \right), \quad x > 0 \]
and using Lemma 2.2.10 in Van Der Vaart and Wellner [?], there exists a universal constant \( K \) such that
\[ \left\| \sum_{k=p}^{q} \varepsilon_k \right\|_{\psi_1} \leq K \left( L + B\sqrt{q-p+1} \right) \]
Combining (6), (7) and (8), we derive $\|Q_n + R_n\|_{\psi_1} \leq C \sum_{j \in \mathbb{Z}} \sqrt{|j|} \|a_j\|_{\mathcal{L}(H)}$ where the constant $C$ does not depend on $n$. Keeping in mind the Berry-Esseen’s central limit theorem for i.i.d. centered $H$-valued random variables (see Yurinski [11] or Bosq [2], Theorem 2.9), we apply Theorem 1 with the Young function $\psi_1$. Since the function $\varphi$ defined by $\varphi(x) = xh^{-1}(1/x)$ with $h(x) = x\psi_1(x)$ satisfies
$$\lim_{x \to 0} \frac{\varphi(x)}{x \log(1 + \frac{1}{x})} = 0,$$
we derive $\Delta_n(X) = O\left(\frac{\log n}{\sqrt{n}}\right)$.

Now, assume that $\|\varepsilon_0\|_{r} < \infty$ for some $r \geq 3$. Applying Pinelis inequality (see Pinelis [8]), there exists a universal constant $K$ such that
$$\left\| \sum_{k=p}^{q} \varepsilon_k \right\|_{r} \leq K \left( r \left( \sum_{k=p}^{q} E\|\varepsilon_k\|_{H}^{r} \right)^{1/r} + \sqrt{r} \left( \sum_{k=p}^{q} E\|\varepsilon_k\|_{H}^{2} \right)^{1/2} \right)$$
and consequently
$$\left\| \sum_{k=p}^{q} \varepsilon_k \right\|_{r} \leq 2Kr\|\varepsilon_0\|_{r} \sqrt{q - p + 1}. \tag{9}$$

Combining (6), (7) and (9), we derive $\|Q_n + R_n\|_{r} \leq C \sum_{j \in \mathbb{Z}} \sqrt{|j|} \|a_j\|_{\mathcal{L}(H)}$ where the constant $C$ does not depend on $n$. Again, applying Berry-Esseen’s CLT (see Yurinski [11] or Bosq [2], Theorem 2.9) and Theorem 1 with the Young function $\psi(x) = x^r$ and the function $\varphi$ given by $\varphi(x) = x^{r/(r+1)}$, we obtain $\Delta_n(X) = O\left( n^{-\frac{r}{2(r+1)}} \right)$. The proof of Corollary 1 is complete.

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