Lagrangian and Eulerian velocity structure functions in hydrodynamic turbulence

K.P. Zybin*, V.A. Sirota

119991 P.N. Lebedev Physical Institute, Russian Academy of Sciences, Moscow, Russia

The Lagrangian and Eulerian transversal velocity structure functions of fully developed fluid turbulence are found basing on the Navier-Stokes equation. The structure functions are shown to obey the scaling relations $S_n^+(\tau) \propto \tau^{\xi_n}$ and $S_n^+(l) \propto l^{\zeta_n}$ inside the inertial range. The scaling exponents $\zeta_n$ and $\xi_n$ are calculated analytically without using dimensional considerations. This result is a significant step toward understanding the nature of hydrodynamic turbulence. The obtained values are in a very good agreement with recent numerical and experimental data.

PACS numbers: 47.10.ad, 47.27.Jv

Understanding of statistical properties of a fully developed turbulence from the Lagrangian and Eulerian points of view has been a challenging theoretical and experimental problem for many years [1–3]. Even a bridge relation between Lagrangian and Eulerian structure function exponents is a theoretical problem. Also, there has been no theory based on the solution of the Navier-Stokes equation up to now. Recent progress in numerical calculations [4]-[6] and experiments [7] encourages to develop a statistical theory. First successful step in this direction was undertaken in [8]-[10]. The aim of this paper is to elaborate the theory of Lagrangian and Eulerian turbulence based on the Navier-Stokes equation. We consider fully developed turbulence in incompressible fluid in non-viscous limit and advance substantially the vortex filament model proposed previously in [8]. On the basis of this model we find scaling exponents of velocity structure functions and derive a bridge relation between the Lagrangian and Eulerian scaling exponents.

The statistics of turbulence deals with different kinds of structure functions. In what follows we are interested in Lagrangian velocity structure functions

$$S_n^+(\tau) = \langle |v(t + \tau) - v(t)|^n \rangle$$

and Eulerian transversal structure functions

$$S_n^+(l) = \langle \left| (v(r + l) - v(r)) \times \frac{1}{l} \right|^n \rangle$$

In the first ones, the velocity difference is taken along the trajectory of one particle and the average is taken over the ensemble of particles. In the second, the velocity difference is taken in different points at the same time, and averaged over all pairs of points. Since the turbulence is assumed to be stationary, both results must not depend on time. We restrict our consideration by the values $\tau, l$ that belong to the inertial range of scales. The usual assumption is that inside the inertial range the structure functions do not depend on geometry of a flow, and have a finite limit as the Reynolds number goes to infinity.

In the previous papers we proposed a theory of vortices formation built on the Navier-Stokes equation [9, 10]. It allowed us to calculate the Lagrangian velocity structure functions of different orders [8, 10]. In this paper we develop and generalize the theory, analyzing a boundary of a vortex filament and introducing a notion of a filament’s age $t_\star$. This allows to calculate the transversal Eulerian structure functions and to find a relation between these two types of structure functions.

The basic idea of our theory is that vortex filaments, i.e., the elongated regions of high vorticity, make the main contribution to structure functions. Thus, not decay of eddies but their stretching is the main process responsible for the formation of structure functions. We will see that studying this stretching allows to obtain ‘intermittent’ scaling exponents practically coinciding with experimental and numerical results.

Below, we recall briefly the basic points of our theory, then we present without derivation the results of the analysis of the basic equation and generalize our previous results for any (not only $\delta$-correlated) random process. Then we apply the results to find the structure functions.

Following [9], we decompose the spatial derivative of velocity into the sum of symmetric and asymmetric parts:

$$\partial_i v_j = \frac{1}{2} \varepsilon_{ijk} \omega_k + b_{ij} \quad b_{ij} = b_{ji}$$

Here $\omega_k$ is the vorticity, and $b_{ij}$ is a traceless (because of incompressibility) symmetric tensor. Differentiating the Euler equation, we then get the equations for all the components of $\omega_i$ and $b_{ij}$. Changing to the quasi-Lagrangian frame moving along the trajectory of a fluid particle, we find the ordinary differential equations de-
scribing the evolution of $\omega_i$ and $b_{ij}$ along the trajectory:

$$\begin{align*}
\dot{b}_{ij} + \frac{1}{2}(\omega_i \omega_j - \omega^2 \delta_{ij}) + b_{ik} b_{kj} + \rho_{ij} & = 0 , \\
\omega_n & = b_{nk}\omega_k
\end{align*}$$

Here $\rho_{ij} = \nabla_i \nabla_j p$, $p$ is the pressure. The time derivative of the second equation gives

$$\dot{\omega}_n = -\rho_{nk}(t)\omega_k$$

along the particle trajectory. This equation is a direct consequence of the Navier-Stokes equation under the limit $\nu \to 0$ without any additional assumptions.

Consider a vortex filament where vorticity is high and the characteristic radius $R$ of the filament is much smaller than the scale of the largest vortices $L$ comparable to the size of the flow:

$$R \sim \langle \nabla \omega/\omega \rangle^{-1} \ll L$$

In general, $\rho_{nk}$ is some complicated integral function of $\omega$. We have shown that inside a vortex filament $\rho_{nk}$ is independent on the local value of $\omega$, and the equation (4) becomes linear.

We now introduce randomness into the equation. Note that in (3) we have $3+6=9$ differential equations for $3+6+6=15$ values $\omega_i$, $b_{ij}$ and $\rho_{ij}$. The condition of incompressibility gives one additional algebraic equation $b_{ii} = 0$. Hence, both in (3) and (4) we have five free functions of time. So, we treat $\rho_{nk}$ as random functions satisfying

$$\langle \rho_{ij}(t) \rho_{nk}(t') \rangle = D_{ijnk}(t-t')$$

Since the flow is assumed to be locally isotropic,

$$D_{ijnk}(\delta t) = (\delta_{in}\delta_{jk} + \delta_{ik}\delta_{jn})D(\delta t) .$$

Normalizing time, we choose $\int D(\tau)d\tau = 1$. Let $D(\delta t)$ be small at time intervals $\delta t$ larger or comparable to the characteristic time $\tau_g$. Neither Gaussian PDF nor $\delta$-correlation is assumed. The only important assumption is that the processes $\rho_{ij}$ are stationary. Then in the limit $t \gg 1$ the statistical moments of vorticity behave as

$$\langle |\omega|^n \rangle \propto e^{\Lambda n t} .$$

The limit of small values $n$ ($\Lambda_n \ll \tau^{-1}_g$) corresponds to a $\delta$-correlated process. In this limit $\Lambda_n$ obeys a linear differential equation of no more than $n$-th order with constant coefficients. We have calculated the values $\Lambda_n$ up to $n \sim 100$. They appeared to satisfy approximately the relation

$$\Lambda_n \simeq \alpha n^{4/3}$$

(For exact values of first score of $\Lambda_n$ see [8].) The non-linear dependence of $\Lambda_n$ (6) is the manifestation of intermittency: $\langle \omega^{2n} \rangle \gg \langle \omega^n \rangle^2$. In the limit of large $n$ ($\Lambda_n \gg \tau^{-1}_g$) we find

$$\Lambda_n \simeq \lambda n .$$

This means that at large values of $n$ intermittency breaks up. This allows to find the structure functions of different orders independently. The detailed discussion of the properties of the equation (4) see in [11].

To illustrate the main ideas, consider a simple model of axially symmetric, non-curved vortex filament with uniform vorticity distribution inside its radius $R$. It contains all the main features of a vortex filament. The difference from the general case is neglecting the curvature and precession of a vortex filament and the gradient of vorticity inside it. We shall see below that the first approximation corresponds to the late stages of a vortex filament’s evolution.

We seek a solution to the Euler equation in the linear form

$$v_\phi = \omega(t) r , \quad v_r = a(t) r , \quad v_z = b(t) z$$

The corresponding pressure is [8]

$$p(r, z, t) = P_1(t)r^2/2 + P_2(t)z^2/2$$

From the Euler equation we then find differential equations for $\omega, a$, and $b$. Combining them, we obtain [8, 10]

$$\ddot{\omega} = -P_2(t) \omega$$

This equation corresponds to (4). In this model the growth of the moments is easy to understand: the amplitude of a solution remains roughly constant for positive $P_2$ and grows exponentially if $P_2$ is negative. Hence, on average the vorticity grows exponentially. This means stretching of the vortex filament. From the Euler equation for (7) it follows that $v_r/r = a = -\dot{\omega}/(2\omega)$, hence the radius of a particle’s orbit decreases as $r(t) \propto \omega^{-1/2}$.

We note that the cross radius $R$ (5) of a vortex filament restricts the region where vorticity is roughly constant. The velocity $\omega R$ on the boundary of the region must be less than the large-scale characteristic velocity pulsations $V_0$: on energetic reasons, the vorticity stops growing in the regions where it becomes comparable. Hence, the equation (4) is linear only inside the radius $R \sim V_0/\omega$; to find the solution outside $R$, feedback effect on $\rho_{ij}$ should be taken into account. As the vorticity increases on average as a function of time, the radius of a filament decreases, $R \sim 1/\omega$. Since the radii $r$ of particles’ orbits change as $1/\sqrt{\omega}$, we have $R/r \propto 1/\sqrt{\omega}$. This means that the approximations of flat vorticity profile and of non-curved axially-symmetric filament improve constantly as the filament compresses. The vorticity continues to grow exponentially (on average) until the cross radius of the filament becomes comparable to the viscous scale. Since we consider a stationary flow, vortex filaments with large radii appear constantly, and at any time there is a set of vortex filaments in different stages of their evolution.

Now we use this model to analyze the transversal structure functions (2). For simplicity, we assume circular
orbits of particles in a filament. Then
\[ \mathbf{v}(t) = \mathbf{r} \times \omega, \quad \delta \mathbf{v} = 1 \times \omega \]
for any pair of points in the vortex filament separated by \( l \). Averaging over all pairs of points in the filament, we have \( \langle \delta \mathbf{v}^n \rangle = l^n \omega^n \).

Now we must take the average over all vortex filaments. For given \( l \), the correlation breaks when \( l \) becomes larger than the radius \( R \) of the filament:
\[ l \geq R \sim V_0/\omega. \tag{8} \]
Hence, calculating \( S^L_n(l) \) we must account only the filaments with \( l \omega < V_0 \). This restriction for vorticity produces the restriction for the filament’s age \( t^\ast \), which results in the set of conditions:
\[ l^n \langle \omega^n \rangle = l^n e^{\Lambda_m t^\ast} \leq V_0^n \]
for all \( m \). Since \( \Lambda_m/m \) grows as a function of \( m \) and \( \Lambda_m/m \to \lambda \) as \( m \to \infty \), we get
\[ t^\ast = \text{inf} \left( \frac{m}{\Lambda_m} \ln \frac{V_0}{t} \right) = \frac{1}{\lambda} \ln \frac{V_0}{t}. \tag{9} \]
It is natural that \( t^\ast \) depends on high-order moments, since they are more sensitive to rare events in intermittent media. Now the structure functions take the form:
\[ S^L_n(l) \sim l^n \langle \omega^n \rangle (t^\ast) \propto l^n e^{\Lambda_m t^\ast} \propto l^{\xi_n}, \tag{10} \]
\[ \xi_n = n - \frac{\Lambda_n}{\lambda} \]
One can see that the cutting parameter \( V_0 \) affects the amplitudes of structure functions but not the scaling law. It means that the boundary of nonlinearity affects on amplitude of structure functions but not their scaling exponents. The vortex filaments with ages less than \( t^\ast \) have size larger then \( l \) and contribute also to the pre-exponents of structure functions.

This particular example reproduces the main aspects of the general case. In the regions where vorticity is high the equation (4) is linear, since \( \rho_k \) does not depend on \( \omega \). The boundary of these "linear" regions is restricted by energetic reasons: in the pairs of points where velocity difference becomes comparable to \( V_0 \) the equation (4) becomes nonlinear, and the exponential growth of vorticity stops. So, the relative velocity inside the vortex filament is restricted by the condition
\[ \Delta \mathbf{v} \leq V_0. \tag{11} \]
The orthogonal component of velocity difference in two points separated by \( l \) inside a filament is \( \Delta \mathbf{v} = 1 \times \omega \).

The condition (11) then restricts the age of the filament by (9), and averaging over all pairs of points gives (10).

One can use analogous consideration to find the Lagrangian structure function exponents. In two near points of a particle’s trajectory separated by time interval \( \tau \) the velocities differ roughly by \( \Delta \mathbf{v} \simeq r_0 \omega^2 \tau \), where \( r_0 \) is the parameter corresponding to momentarily curvature radius of the trajectory. The linear equation (4) and exponential growth of vorticity are only valid for small relative velocities. The condition \( \Delta \mathbf{v} < V_0 \) then gives \( t \leq t^\ast \), where
\[ \tau^m \langle \omega^{2m} \rangle = \tau^m e^{\Lambda_2 \tau^\ast} \leq V_0^m \]
for all \( m \). Making use of \( \lim \frac{\Lambda_m}{m} = \lambda < \infty \), we find
\[ e^{2\Lambda_2 \tau^\ast} = V_0/\tau, \]
so the Lagrangian structure function is
\[ S^L_n(\tau) = \langle \Delta \mathbf{v}^n \rangle \propto \tau^n \langle \omega^{2n}(t^\ast) \rangle \propto \tau^{\xi_n}, \tag{12} \]
\[ \xi_n = n - \frac{\Lambda_{2n}}{2\lambda} \]
In [8] we obtained a similar expression for \( S^L_n(\tau) \). It is identical to (12) if we choose \( \lambda = \Lambda_4/2 \simeq 3.1 \). The derivation in [8] used the assumption that the structure function \( S^L \) was determined by stationary part of the probability density function. In this paper we take the feedback effect into account. This allows to avoid the assumptions of stationarity.

We do not consider longitudinal Eulerian structure functions in this paper. To calculate them, one has to make a complicated analysis of the stochastic equation (3) and to investigate the statistical behavior of \( h_{ij} \), in addition to \( \omega \). From (3) one can see that some linear combinations of \( h_{ij} \) (that do not contribute to the right-hand side of the second equation in (3)) are proportional to \( \omega \). Then the longitudinal scaling exponents may coincide with the transverse ones.

Basing on relations (10) and (12) one can find the bridge relation between Eulerian transverse and Lagrangian scaling exponents:
\[ \Lambda_n (n - \zeta_n) = 2\Lambda_n (n - \xi_n) \tag{13} \]
Let us now compare the predictions of our theory with the results of recent numerical simulations [6] performed for Re \( \simeq 600 \). This is the only DNS that presents both Lagrangian and Eulerian transverse scaling exponents. The scaling exponents are normalized by \( \xi_2 \) and \( \zeta_2 \), respectively (so-called ESS procedure). Our theory has the only adjusting parameter \( \lambda \). We use it to fit the Lagrangian exponent \( \xi_2/\xi_2 \). Then we calculate all other values \( \zeta_n \) and \( \xi_n \) from (10) and (12). The results are presented in Table 1. One can see that the theory is in excellent agreement with the experimental data.

Fig.1 presents the Eulerian scaling exponents (normalized by ESS) as a function of the order \( n \). The theoretical prediction is presented together with experimental data taken from [6] and [12]. We see that, for high \( n \), the longitudinal scaling exponents correspond to the theory better.
than the transversal ones. This is in accord with opinion of the authors of [6], who maintain that the results for high-order transversal exponents are less certain.

Table 1. Scaling exponents normalized by $\xi_2$ in Lagrangian and by $\zeta_3$ in Eulerian case. The results of numerical simulations are cited from [6].

| n   | $\xi_\alpha/\xi_2$ (Lagrange) | $\zeta_n/\zeta_3$ (Euler) |
|-----|--------------------------------|---------------------------|
|     | DNS   | Theory | DNS   | Theory |
| 2   | 0.71 ± 0.01 | 0.72 |
| 4   | 1.26 ± 0.01 | 1.28 |
| 6   | 1.68 ± 0.03 | 1.74 |
| 8   | 1.98 ± 0.10 | 2.13 |
| 10  | 2.25 ± 0.15 | 2.46 |

We stress that to determine $\lambda$, we used the fourth-order Lagrangian scaling exponent. So the theoretical prediction for Eulerian structure functions, which is presented in Fig.1, has no adjusting parameters. We also note that the concavity of the curve corresponds to intermittency of scaling exponents. In our theory it is a direct consequence of intermittency of $\langle \omega^n \rangle$.

![FIG. 1: Eulerian relative scaling exponents $\zeta_\alpha/\zeta_3$. Theory (+ line) and experiment: transversal exponents taken from [6] (V) and longitudinal exponents from [6] (D) and [12] (O).](image)

The expressions (10) and (12) allow not only link the Lagrangian and Eulerian structure functions of the same order, but also to link the scaling exponents of different orders. Namely, using (10) and (12) we express $\lambda$ as a function of $\xi_\alpha/\xi_2$ and of $\zeta_n/\zeta_3$ independently for each $n$. Substituting the scaling exponents obtained in [6], we find the values of $\lambda$ (see Table 2). We see that all the values $\lambda$ obtained from both Lagrangian and Eulerian scaling exponents for all measured $n$ coincide up to the experimental errors.

To summarize, we report in this letter of the first substantial advance in derivation of both Lagrangian and Eulerian scaling exponents basing on the Navier-Stokes equation. We started from the eq. (4), which is a direct consequence of the Navier-Stokes equation in quasi-Lagrangian reference frame. We treat it as a stochastic equation, assuming that $\rho_{ij}$ is a stationary random process. This assumption is natural, since $\rho_{ij}$ is independent of local vorticity. Another important assumption is that validity of the growing solutions of (4) is restricted by energy limitation, which means that velocity differences must be smaller than a large-scale eddy's velocity. Analyzing solutions of (4), we find Lagrangian and Eulerian transversal scaling exponents.

![Table 2. Free parameter $\lambda$ calculated from the relative scaling exponents $\zeta_n/\zeta_3$ (Euler) and $\xi_\alpha/\xi_2$ (Lagrange).](image)

We stress that the theory developed in this paper has only one fitting parameter. No additional propositions were used in calculating both the scaling exponents. Neither dimensional nor phenomenological hypotheses were used. All the obtained relations are consequences of the equation (4) derived directly from the Navier-Stokes equation. Thus, the coincidence between the theoretical predictions, experimental results and numerical calculations is very good.

We are very much obliged to Prof. A.V. Gurevich for his permanent interest to our work and constant support. We are grateful to A.S. Il’in for valuable remarks.

The work was partially supported by the RAS Program "Fundamental Problems of Nonlinear Dynamics".

[1] U. Frisch, 'Turbulence. The legacy of A.N. Kolmogorov', Cambridge Univ. Press, 1995
[2] V.I.Belinicher, V.S.Lvov, A.Pomyalov, I.Procaccia, J.Stat.Phys. 93 (3/4), 797 (1998)
[3] G. Falkovich, K.R. Sreenivasan, Physics Today, 59 (April), 43 (2006)
[4] L.Biferale, E.Bodenschatz, M.Cencini et al Phys.Fluids 20(6):065103 (2008)
[5] A.Arnedo, R.Benzi, J.Berg, et al Phys.Rev.Lett. 100(25): 254504 (2008)
[6] R.Benzi, L.Biferale, R.Fischer et al arXive: 0905.0082v1
[7] F.Toschi and E.Bodenschatz Annu. Rev. Fluid Mech. 41: 375 (2009)
[8] K.P. Zybin, V.A. Sirota, A.S. Il’ in, A.V. Gurevich, Phys. Rev. Lett. 100, 174504 (2008)
[9] K.P. Zybin, V.A. Sirota, A.S. Il’in, A.V. Gurevich, JETP 2007, 105, N 2, p. 455 , arXiv:physics/0612131
[10] K.P. Zybin, V.A. Sirota, A.S. Il’in, A.V. Gurevich, JETP 2008, 134, N 5, p. 1024
[11] K.P. Zybin, V.A. Sirota, A.S. Il’in (in preparing)
[12] Gotoh T., Fukayama D., Nakano T. Phys. Fluids 14, 1065 (2002)