LOCAL $C^0$-ESTIMATE AND EXISTENCE THEOREMS FOR SOME PRESCRIBED CURVATURE PROBLEMS ON COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS

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Abstract. In this article we study a class of prescribed curvature problems on complete noncompact Riemannian manifolds. To be precise, we derive local $C^0$-estimate under an asymptotic condition which is in effect optimal, and prove the existence of complete conformal metrics with prescribed curvature functions. A key ingredient of our strategy is Aviles-McOwen’s result or its fully nonlinear version on the existence of complete conformal metrics with prescribed curvature functions on manifolds with boundary.

1. Introduction

An important problem in conformal geometry is to determine which function could arise as a curvature function of a metric that is complete and conformally equivalent to a given metric. A special case, known as Yamabe problem, conjectures that any closed Riemannian manifold of dimension $\geq 3$ can be conformally deformed to achieve constant scalar curvature. The Yamabe problem on closed manifolds is valid according to the work of Trudinger [29], Aubin [1] and Schoen [27], while for general complete noncompact Riemannian manifolds the corresponding Yamabe problem is not always true as shown by Jin [17] who constructed some counterexamples. This reveals that there are great differences between the conformal change of Riemannian metrics on closed manifolds and those on complete noncompact manifolds. Therefore, it is reasonable to investigate which complete noncompact Riemannian manifold could admit a complete conformal metric with the scalar curvature being a constant or a more general function. The problem in this and related topic has attracted enormous interest, see for instance [2, 3, 8, 18, 24] and references therein.

This paper is devoted to study a fully nonlinear version and then extend it to higher order curvatures.

Without specific clarification, we assume throughout this article that $(M, g)$ is a complete noncompact Riemannian manifold of dimension $n \geq 3$ with the Levi-Civita connection $\nabla$. For a Riemannian metric $g$, we denote the sectional, Ricci and scalar curvature tensors by $\text{Sec}_g$, $\text{Ric}_g$ and $R_g$, respectively.

Partial results are stated as follows.

Research supported in part by NSFC grant 11801587.
Let $\sigma_k$ be the $k$-th elementary symmetric function, $\Gamma_k$ be the $k$-th Gårding’s cone
\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \forall 1 \leq j \leq k \}. \]

**Theorem 1.1.** On a complete noncompact Riemannian manifold $(M, g)$ satisfying
\[ \sigma_k(\lambda(-g^{-1}\text{Ric}_g)) \geq \delta > 0, \quad \lambda(-g^{-1}\text{Ric}_g) \in \Gamma_k \]
for some $2 \leq k \leq n$ and for some constant $\delta > 0$, there exists at least one smooth complete conformal metric $\tilde{g} = e^{2u}g$ with $\lambda(-\tilde{g}^{-1}\text{Ric}_{\tilde{g}}) \in \Gamma_k$ and
\[ \sigma_k(\lambda(-\tilde{g}^{-1}\text{Ric}_{\tilde{g}})) = \psi \text{ in } M, \]
provided that $\psi$ is a positive smooth function having a uniform bound $0 < \psi < \Lambda$, where $\Lambda$ is a positive constant.

By studying $\sigma_k$ curvature equation for conformal deformation of Einstein tensor, $G_g = \text{Ric}_g - \frac{R}{2}g$, we also deduce

**Theorem 1.2.** Let $(M, g)$ be a complete noncompact Riemannian manifold with $\text{Sec}_g \leq -\delta < 0$ for some constant $\delta > 0$. Then for each $2 \leq k \leq n-1$ and for any smooth positive function $\psi$ with $0 < \psi < \Lambda$ for some constant $\Lambda > 0$, $M$ admits a smooth complete conformal metric $\tilde{g} = e^{2u}g$ with
\[ \sigma_k(\lambda(\tilde{g}^{-1}G_{\tilde{g}})) = \psi, \quad \lambda(\tilde{g}^{-1}G_{\tilde{g}}) \in \Gamma_k \text{ in } M. \]

In fact we are primarily concerned with a more general problem of finding a complete conformal metric with a prescribed curvature function

\[ f(\lambda(\tilde{g}^{-1}A^\tau_{\alpha}g)) = \psi \text{ in } M, \quad 0 < \psi \in C^\infty(M), \]

\[ \tilde{g} = e^{2u}g \text{ is complete and admissible (i.e. } \lambda(\tilde{g}^{-1}A^\tau_{\alpha}g) \in \Gamma), \]

which is generated by a smooth symmetric function $f$ of eigenvalues of $A^\tau_{\alpha}g$ with respect to $\tilde{g}$, where
\[ A^\tau_{\alpha}g = \frac{\alpha}{n-2}(\text{Ric}_{\tilde{g}} - \frac{\tau R_{\tilde{g}}}{2(n-1)}g), \quad \alpha = \pm 1, \quad \tau \in \mathbb{R}. \]

The equation is a fully nonlinear equation of similar type to Hessian equations going back to the work of Caffarelli-Nirenberg-Spruck [5] which studied Dirichlet problem on smooth bounded domains of Euclidean spaces. In addition, $f$ is defined in an open symmetric and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at origin, $\partial \Gamma \neq \emptyset$, $\Gamma_n \subseteq \Gamma \subset \Gamma_1$.

Moreover, the following three conditions are required:

\[ f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n, \]

\[ f \text{ is concave in } \Gamma, \]

\[ f(t\lambda) = tf(\lambda) \text{ for any } \lambda \in \Gamma, \quad t > 0. \]
Without loss of generality, we assume throughout the article that
\[ f|_{\partial \Gamma} \equiv 0, \ f|_{\Gamma} > 0 \text{ and } f(\vec{1}) = 1, \]
where \( \vec{1} = (1, \cdots, 1) \in \mathbb{R}^n \), whenever \( f \) is homogeneous of degree one. This implies
\[ (1.5) \quad \sum_{i=1}^{n} \lambda_i \geq nf(\lambda), \quad \lambda \in \Gamma. \]

Typical examples satisfying (1.2), (1.3), (1.4) are given by
\[ f = \left( \frac{C_l^n \sigma_k / (C_k^n \sigma_l)}{n-n_1} \right)^{1/(k-l)}, \]
where \( 0 \leq l < k \leq n \), \( \Gamma = \Gamma_k \), where \( \sigma_0 = 1 \) and \( C_k^n = n!/(n-k)!k! \).

The \( k \)-Yamabe problem for conformal deformation of Schouten tensor \( A_g = \frac{1}{n-2} (Ric_g - \frac{R_n}{2(n-1)^2} g) \) was proposed by Viaclovsky [32] and since then it has drawn much attention as the interest from geometry, see for instance [6, 11, 14, 16, 20, 25].

Before stating our more general results, we shall recall some result on partial uniform ellipticity.

**Definition 1.3** ([34]). For the cone \( \Gamma \), we denote
\[ \kappa_\Gamma := \max \{ k : (-\alpha_1, \cdots, -\alpha_k, \alpha_{k+1}, \cdots, \alpha_n) \in \Gamma, \text{ where } \alpha_j > 0, \forall 1 \leq j \leq n \}. \]

Obviously, \( \kappa_\Gamma \) is an integer with \( 0 \leq \kappa_\Gamma \leq n-1 \). It was shown in [34] that the constant \( \kappa_\Gamma \) measures the partial uniform ellipticity of \( f \):

**Lemma 1.4** ([34]). Let \( f \) and \( \Gamma \) be as above. Suppose \( f \) satisfies (1.2), (1.3) and (1.6)
\[ \text{For each } \sigma < \sup_{\Gamma} f \text{ and } \lambda \in \Gamma, \lim_{t \to +\infty} f(t\lambda) > \sigma. \]

Then there is a universally positive constant \( \vartheta_\Gamma \) depending only on \( \Gamma \), such that for each \( \lambda \in \Gamma \) with order \( \lambda_1 \leq \cdots \leq \lambda_n \),
\[ (1.7) \quad f_i(\lambda) \geq \vartheta_\Gamma \sum_{j=1}^{n} f_j(\lambda) \text{ for } 1 \leq i \leq \kappa_\Gamma + 1. \]

Moreover, the statement (1.7) is sharp and cannot be further improved.

Throughout this present paper, \( \vartheta_\Gamma \) and \( \kappa_\Gamma \) always stand for the constants in Lemma 1.4 and Definition 1.3 respectively.

We now impose an appropriate and delicate restriction to the parameters in \( A_g^{\tau,\alpha} \):
\[ (1.8) \quad \tau < 1, \quad \text{if } \alpha = -1; \]
\[ \tau > 1 + (n-2)(1 - \kappa_\Gamma \vartheta_\Gamma), \quad \text{if } \alpha = 1, \]
which further yields
\[ (1.9) \quad \alpha(n\tau + 2 - 2n) > 0. \]

(It would be worthwhile to note that if \( \alpha = 1 \) equation (1.1) fails to be elliptic for general \( \tau < n-1, \tau \neq 1 \); surprisingly assumption (1.8) allows \( 1 + (n-2)(1 - \kappa_\Gamma \vartheta_\Gamma) < \tau < n-1 \) in the case \( \Gamma \neq \Gamma_n \).) Together with Lemma 1.4 we can propose an effective
way to understand the structure of linearized operator for equation (1.1) and then establish the local estimates for gradient and Hessian of solutions (Theorem 3.1). With such local estimates at hand, the remaining goal is to derive local zero order estimate for solutions to approximate Dirichlet problems.

In an attempt to derive local bound of solutions from below, we assume there exists an admissible complete (noncompact) conformal metric \( g \in [g] \) satisfying

\[
\lambda(g^{-1}A_{g}^{\tau,\alpha}) \in \Gamma \text{ in } M,
\]

where \([g] = \{e^{2v}g : v \in C^2(M)\}\) is the \(C^2\)-smooth complete conformal class of \(g\), and that such \(g\) further satisfies a key asymptotic property at infinity: There exists a compact subset \(K_0\) of \(M\) and a constant \(\Lambda_0 > 0\) such that

\[
\frac{f(\lambda(g^{-1}A_{g}^{\tau,\alpha}))}{\psi} \geq \Lambda_0 \text{ holds uniformly in } M \setminus K_0.
\]

This asymptotic property near infinity can be viewed as a subsolution, which is only used to derive local bound for approximate solutions from below (Theorem 4.3). The assumption (1.11) is in effect sufficient and necessary for the existence result:

- The metric we expect to obtain in Theorem 1.5 tautologically obeys (1.11).
- There is a counterexample on nonexistence of such a complete conformal metric, in which (1.11) is not satisfied. For some nonexistence results on scalar and negative Ricci curvatures we are referred for example to [18, 24, 28].

For the rest issue of deriving local bound of solutions from above, it requires to construct supersolutions. In this paper we apply a theorem due to Aviles-McOwen [4], which extensively extends a famous result of Loewner-Nirenberg [23], or its fully nonlinear version as shown in Theorem 3.9 below (partially proved in [34]) to achieve the goal and thus provide a straightforward and simple approach to establish local bound of approximate solutions from above (Theorem 4.2). It would be worthwhile to note that our strategy does work without any further geometric restrictions to underlying manifolds as well as to prescribed curvature function, to be compared with those in [2, 18] for scalar curvature equation and in [10] for a generalization to \(\sigma_k\) curvature equations for some \((0,2)\)-type tensors. This is new even for Euclidean spaces, and it also works for prescribed scalar curvature equation (see Theorem 4.4).

As a result, we prove the main result. (Throughout the paper, we always assume that \(f\) satisfies (1.2), (1.3), (1.4), and \(\psi\) is the prescribed curvature function without specific clarification).

**Theorem 1.5.** For the \(\tau\) satisfying (1.8), we assume there is a complete conformal metric to satisfy (1.10) and (1.11). Then for such \(\tau\), there exists at least one smooth function \(u_\infty \in C^\infty(M)\) such that \(g_\infty = e^{2u_\infty}g\) is a smooth admissible complete conformal metric on \(M\) satisfying (1.1).
This theorem asserts the existence of a smooth admissible complete conformal metric solving the prescribed curvature equation (1.1), provided that there is a $C^2$-smooth admissible complete conformal metric $g$ satisfying (1.11). That is, all of the geometric and analytic obstructions to the solvability of (1.1) are in fact embodied in the assumption of the asymptotic condition (1.11).

Immediately we have Theorem 1.1, Corollaries 1.6 and 1.7. Together with (2.1), we further derive Theorem 1.2 and Corollaries 1.8, 1.9.

Corollary 1.6. Let $0 < \psi \in C^\infty(M)$, and we assume in addition that $\text{Ric}_g \leq -\delta \psi$ for some constant $\delta > 0$. Then $M$ admits a smooth complete conformal metric $\tilde{g} = e^{2u}g$ satisfying $f(\lambda(-\tilde{g}^{-1}\text{Ric}_\tilde{g})) = \psi$, $\lambda(-\tilde{g}^{-1}\text{Ric}_\tilde{g}) \in \Gamma$.

Corollary 1.7. Assume $\psi$ is a smooth positive function. Suppose in addition that $\text{Ric}_g \leq 0$, $R_g \leq -\delta \psi$ for some constant $\delta > 0$. Then for such $\psi$ and for $\alpha = -1$, $\tau < 0$, there exists a smooth complete conformal metric satisfying (1.1).

Corollary 1.8. Let $\alpha = 1$, $\tau > n - 1$. Suppose $\text{Sec}_g \leq 0$ and $R_g \leq -\delta \psi$ for a smooth positive function $\psi$ for some constant $\delta > 0$. Then for such $\psi$, there is a smooth complete conformal metric $\tilde{g} = e^{2u}g$ solving (1.1).

Corollary 1.9. Let $0 < \psi \in C^\infty(M)$. Suppose $\Gamma \neq \Gamma_n$ and $\text{Sec}_g \leq -\delta \psi$ for some constant $\delta > 0$. Then for such $\psi$, there is a smooth complete conformal metric $\tilde{g} = e^{2u}g$ with $\lambda(\tilde{g}^{-1}G_\tilde{g}) \in \Gamma$ solving $f(\lambda(\tilde{g}^{-1}G_\tilde{g})) = \psi$ in $M$.

Remark 1.10. Notice the rescaling invariance: $A^{\tau,\alpha}_g = A^{\tau,\alpha}_t$ for all constants $t > 0$. The constant $\Lambda_0$ in (1.11) can be chosen as $\Lambda_0 = 1$.

Remark 1.11. In Theorem 1.5, $\tau$ can be replaced by a smooth function obeying (1.8).

The article is organized as follows. In Section 2, we collect some useful notation and formulas, and briefly discuss some results on the partial uniform ellipticity. In Section 3, we obtain local estimates for gradient and Hessian by confirming the fully uniform ellipticity of a more general class of fully nonlinear elliptic equations which includes equation (1.1) as a special case. Using the local estimate, we obtain existence results of solutions for such equations when the background space is a compact manifold with boundary. In Section 4, we derive local $C^0$-estimate for approximate solutions and complete the proof of our main existence theorem by using an approximate process. In Section 5, by confirming the asymptotic property near infinity (1.11), we obtain on Euclidean spaces the existence of smooth complete metrics of prescribed curvature functions with a restriction to decay ratio. In Section 6, we briefly study the conformal deformation of $-A_g$ and then apply it to Einstein tensor and certain modified Schouten tensors by constructing certain cones of type 2.

2. Preliminaries

2.1. Notation. Let $e_1, \ldots, e_n$ be a local frame on $M$. We denote $g_{ij} = g(e_i, e_j), (g^{ij}) = (g_{ij})^{-1}$. Under Levi-Civita connection $\nabla$ of $(M, g)$, $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$, and $\Gamma^k_{ij}$ denote
the Christoffel symbols and $\Gamma^k_{ij} = \Gamma^k_{ji}$. For simplicity we write $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i(\nabla_j) - \Gamma^k_{ij} \nabla_k$. On $(M, g)$ one also defines the curvature tensor by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z.$$ 

Under a locally unit orthogonal frame $e_1, \ldots, e_n$, i.e. $g_{ij} = \delta_{ij}$,

$$\text{Ric}_{\tilde{g}}(e_i, e_i) = \sum_{j=1}^{n} g(R(e_i, e_j)e_i, e_j), \quad R_g = \sum_{i=1}^{n} \text{Ric}_{g}(e_i, e_i),$$

thus one can check

$$(2.1) \quad \text{Ric}_{\tilde{g}}(e_i, e_i) = R_g - \frac{1}{2} \sum_{k,l\neq i} g(R(e_k, e_l)e_k, e_l), \quad \forall 1 \leq i \leq n.$$ 

**2.2. Some formulas for conformal change.** Under the conformal change $\tilde{g} = e^{2u}g$,

$$Ric_{\tilde{g}} = Ric_g - \Delta u g - (n-2)\nabla^2 u - (n-2)|\nabla u|^2 g + (n-2)du \otimes du,$$

$$e^{2u}R_{\tilde{g}} = R_g - 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2,$$

where and hereafter $\Delta u$, $\nabla^2 u$ and $\nabla u$ are respectively the Laplacian, Hessian and gradient of $u$ with respect to $g$, $|\nabla u|^2 = g^{ij}\nabla_i u \nabla_j u$. Thus

$$(2.2) \quad A_{\tilde{g}}^{\tau,\alpha} = A_g^{\tau,\alpha} + \frac{\alpha(\tau - 1)}{n-2} \Delta u g - \frac{\alpha(\tau - 2)}{2} |\nabla u|^2 g + \alpha du \otimes du.$$ 

**2.3. Existence of complete conformal metric with scalar curvature $-1$.** A key tool is a result due to Aviles-McOwen [4] who extended a theorem of Loewner-Nirenberg [23] from smooth bounded domains $\Omega \subset \mathbb{R}^n$ to general compact Riemannian manifolds with boundary. Namely,

**Theorem 2.1.** Let $(M, g)$ be a compact Riemannian manifold with smooth boundary, then $(M, g)$ has a complete conformal metric with scalar curvature $-1$.

**2.4. Partial uniform ellipticity revisited.** For $\sigma < \sup_{\Gamma} f$, we denote

$$(2.3) \quad \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}.$$ 

So $\Gamma \setminus \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) \leq \sigma \}$.

The following lemma is a refinement of Lemma 3.4 in [33] (also Lemma 2.1 of [34]).

**Lemma 2.2.** Let $\sup_{\partial \Gamma} f < \tau_0 < \sup_{\Gamma} f$ be fixed. For the $f$ satisfying (1.3), and

$$(2.4) \quad f_i(\lambda) \geq 0, \quad \forall 1 \leq i \leq n, \quad \forall \lambda \in \Gamma,$$

the following two statements are equivalent each other:

$$(2.5) \quad \lim_{t \to +\infty} f(t\lambda) > \tau_0, \quad \forall \lambda \in \Gamma.$$ 

$$(2.6) \quad \sum_{i=1}^{n} f_i(\lambda)\mu_i > 0, \quad \forall \lambda \in \Gamma \setminus \Gamma^{\tau_0}, \quad \forall \mu \in \Gamma.$$
Proof. (2.5) $\Rightarrow$ (2.6): The proof is different from that given in [33]. For given $\lambda \in \Gamma \setminus \Gamma_{\tau_0}$ and $\mu \in \Gamma$, by the concavity we have

$$t \sum_{i=1}^{n} f_i(\lambda) \mu_i \geq \sum_{i=1}^{n} f_i(\lambda) \lambda_i + f(t\mu) - f(\lambda).$$

Let $\mu = \lambda$ and $t \gg 1$, we have $\sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0$. Thus (2.6) holds.

(2.5) $\Leftarrow$ (2.6): Let $f(c(\tau_0)\vec{1}) = \tau_0$. For any $\lambda \in \Gamma$, if $t$ is sufficiently large then $t\lambda - c(\tau_0)\vec{1} \in \Gamma$. So $f(t\lambda) - \tau_0 \geq \sum_{i=1}^{n} f_i(t\lambda)(t\lambda_i - c(\tau_0)) > 0$ which yields (2.5). □

Using Lemma 2.2 as in [34], we have the following results which slightly refine Lemma 1.4 above and Proposition 8.1 of [34], respectively.

**Proposition 2.3.** In addition to (1.3), (2.4), we assume (2.5) holds for some $\sup_{\partial \Gamma} f < \tau_0 < \sup_{\Gamma} f$. Then for any $\lambda \in \Gamma \setminus \Gamma_{\tau_0}$ with order $\lambda_1 \leq \cdots \leq \lambda_n$ and for $1 \leq i \leq \kappa_{\Gamma} + 1$,

$$f_i(\lambda) \geq \vartheta_{\Gamma} \sum_{j=1}^{n} f_j(\lambda),$$

holds uniformly for a universally positive constant $\vartheta_{\Gamma}$ depending only on $\Gamma$.

**Proof.** The proof is essentially the same as that of [34], we present the detail for the choice of the constant $\vartheta_{\Gamma}$ in Lemma 1.4 and Proposition 2.3 as well.

Let $\lambda \in \Gamma \setminus \Gamma_{\tau_0}$ and we assume $\lambda_1 \leq \cdots \leq \lambda_n$. If $\Gamma = \Gamma_n$, equivalently to $\kappa_{\Gamma} = 0$, then the statement is true since

$$f_1(\lambda) \geq \frac{1}{n} \sum_{i=1}^{n} f_i(\lambda).$$

For $\Gamma \neq \Gamma_n$, $\kappa_{\Gamma} \geq 1$, let $\alpha_1, \cdots, \alpha_n$ be $n$ strictly positive constants such that

$$(-\alpha_1, \cdots, -\alpha_{\kappa_{\Gamma}}, \alpha_{\kappa_{\Gamma}+1}, \cdots, \alpha_n) \in \Gamma.$$

From (2.6),

$$- \sum_{i=1}^{\kappa_{\Gamma}} \alpha_i f_i(\lambda) + \sum_{i=\kappa_{\Gamma}+1}^{n} \alpha_i f_i(\lambda) > 0,$$

which yields $f_{\kappa_{\Gamma}+1}(\lambda) > \frac{\alpha_1}{\sum_{i=\kappa_{\Gamma}+1}^{n} \alpha_i} f_1(\lambda)$. By (2.7) and iteration, one derives

$$f_{\kappa_{\Gamma}+1}(\lambda) \geq \frac{\alpha_1}{(\sum_{i=\kappa_{\Gamma}+1}^{n} \alpha_i - \sum_{i=2}^{\kappa_{\Gamma}} \alpha_i)} f_1(\lambda).$$

From the discussion above, we see $\vartheta_{\Gamma}$ can be achieved as $\vartheta_{\Gamma} = \frac{\alpha_1}{n(\sum_{i=\kappa_{\Gamma}+1}^{n} \alpha_i - \sum_{i=2}^{\kappa_{\Gamma}} \alpha_i)}$. □

**Remark 2.4.** Obviously, $0 \leq \kappa_{\Gamma} \leq n - 1$ and $0 < \vartheta_{\Gamma} \leq \frac{1}{n}$; furthermore, $\vartheta_{\Gamma} = \frac{1}{n}$ and $\kappa_{\Gamma} = n - 1$ cannot occur simultaneously.
Proposition 2.5. For the $f$ satisfying (1.3), (2.4) and (2.5) for some $\sup_{\partial \Gamma} f < \tau_0 < \sup_{\Gamma} f$, then for each $\lambda \in \Gamma \setminus \Gamma^{\tau_0}$ we have

\[(2.8) \quad f_i(\lambda) \geq \vartheta \Gamma \sum_{j=1}^{n} f_j(\lambda) \text{ if } \lambda_i \leq 0.\]

Moreover, (2.5) can be removed for $n = 2$.

Proof. Fix $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma \setminus \Gamma^{\tau_0}$. Without loss of generality, we assume $\lambda_n \geq \cdots \geq \lambda_1$. By the definition of $\kappa \Gamma$, $\lambda_{\kappa \Gamma + 1} > 0$. Therefore, $i \leq \kappa \Gamma$ if $\lambda_i \leq 0$, which deduces (2.8). □

Proposition 2.5 is very useful. Together with $\sum_{i=1}^{n} f_i(\lambda) \geq \kappa(\tau_0) > 0, \lambda \in \Gamma \setminus \Gamma^{\tau_0}$ (since $\sum_{i=1}^{n} f_i(\lambda) \lambda_i > 0$), this proposition confirms a key assumption which is used to derive (local and global) a priori estimates for solutions to various partial differential equations from differential geometry. For more details we refer to [7, 13, 22, 26, 30, 31].

2.5. Proof of (1.9). We verify (1.9) that plays an important role in this paper.

Case 1: $\alpha = -1$. As we assume in (1.8), $\tau < 1$. Then $n\tau + 2 - 2n < 2 - n < 0$.

Case 2: $\alpha = 1$. We have $\tau > 1 + (n - 2)(1 - \kappa \Gamma \vartheta \Gamma)$ according to (1.8). Since $0 < \vartheta \Gamma \leq \frac{1}{n}$ and $0 \leq \kappa \Gamma \leq n - 1$, we see $\tau > \frac{2n - 2}{n}$. Thus $n\tau + 2 - 2n > 0$.

3. Local estimates and existence of solutions for some fully nonlinear equations on compact manifolds with boundary

Throughout this section, we assume that $(M, g)$ is a $n$-dimensional ($n \geq 3$) compact Riemannian manifold with smooth boundary $\partial M$, $\bar{M} = M \cup \partial M$, $A$ is a smooth symmetric $(0, 2)$-type tensor on $\bar{M}$. Let

\[(3.1) \quad V[u] = A + \Delta u g - g(x)\nabla^2 u + a(x)|\nabla u|^2 g + b(x)du \otimes du + c(x)L(du),\]

where $g(x), a(x), b(x)$ and $c(x)$ are smooth functions, and $L(du)$ is a smooth symmetric $(0, 2)$-type tensor depending linearly on $du$. In addition, we replace (1.4) by

\[(3.2) \quad f(t\lambda) = t^{\gamma} f(\lambda), \quad f(\lambda) > 0, \quad f|_{\partial \Gamma} = 0,\]

for some $0 < \gamma \leq 1$, for any $\lambda \in \Gamma$, $t > 0$.

We consider the equation

\[(3.3) \quad F(V[u]) := f(\lambda(g^{-1}V[u])) = \psi e^{2\gamma u} \text{ in } M, \quad 0 < \psi \in C^\infty(\bar{M}).\]

In order to study (3.3) in the framework of elliptic equations, we are going to look for solutions in class of admissible functions satisfying

\[(3.4) \quad \lambda(g^{-1}V[u]) \in \Gamma \text{ in } \bar{M}, \quad u \in C^2(\bar{M}).\]
3.1. Local and boundary estimates for equation (3.3). We have local and boundary estimates for admissible solutions for (3.3) under the assumption that
\begin{equation}
0 < \varrho(x) < \frac{1}{1 - \kappa_\Gamma \vartheta_\Gamma}, \text{ or } \varrho(x) < 0 \text{ in } \bar{M}.
\end{equation}
Together with $0 < \vartheta_\Gamma \leq \frac{1}{n}$, $0 \leq \kappa_\Gamma \leq n - 1$ this condition implies
\begin{equation}
\varrho(x) < n,
\end{equation}
which will play an important role in the construction of locally upper barriers as we will show below.

**Theorem 3.1.** Suppose (1.2), (1.3), (3.2) and (3.5) hold. Let $B_r \subset M$ be a geodesic ball of radius of $r > 0$. Let $u \in C^4(B_r)$ be an admissible solution of (3.3) in $B_r$, then
\begin{equation}
\sup_{B_{r/2}} (|\nabla u|^2 + |\nabla^2 u|) \leq C/r^2,
\end{equation}
where $C$ is a uniformly positive constant depending on $|u|_{C^0(B_r)}$ and other known data in $B_r$.

3.1.1. Fully uniform ellipticity. The linearized operator $\mathcal{L}$ of (3.3) at $u$ is given by
\begin{equation}
\mathcal{L}v = F^{ij}g_{ij}\Delta v - \varrho(x)F^{ij}\nabla_{ij} v + 2a(x)g(\nabla u, \nabla v)F^{ij}g_{ij}
+ b(x)F^{ij}(\nabla_i u \nabla_j v + \nabla_i v \nabla_j u) + c(x)F^{ij}(L(du))_{ij}
= (F^{pq}g_{pq}g^{ij} - \varrho(x)F^{ij})\nabla_{ij} v + \text{lower order terms},
\end{equation}
where $F^{ij} = \frac{\partial g}{\partial V}(V)$, $V = V[u]$.

The eigenvalues of matrix $\{F^{pq}g_{pq}g^{ij} - \varrho(x)F^{ij}\}$ with respect to $\{g^{ij}\}$ are precisely
\begin{equation}
\sum_{j=1}^n f_j - \varrho(x)f_i, \cdots, \sum_{j=1}^n f_j - \varrho(x)f_n.
\end{equation}

Next, we prove that (3.3) is fully uniform ellipticity, provided (3.5) holds. That is

**Proposition 3.2.** There is $\theta > 0$ so that
\begin{equation}
\sum_{j=1}^n f_j - \varrho(x)f_i \geq \theta \sum_{j=1}^n f_j, \quad \forall i.
\end{equation}

**Proof.** If $\varrho < 0$, (3.8) holds for $\theta = 1$. In what follows we assume $0 < \varrho(x) < \frac{1}{1 - \kappa_\Gamma \vartheta_\Gamma}$.

As a corollary of Lemma 1.4, $\sum_{j=1}^n f_j - f_i \geq \kappa_\Gamma \vartheta_\Gamma \sum_{j=1}^n f_j$, i.e.,
\begin{equation}
\sum_{j=1}^n f_j \geq (1 - \varrho(x)(1 - \kappa_\Gamma \vartheta_\Gamma)) \sum_{j=1}^n f_j + \varrho(x)f_i
\end{equation}
for all $1 \leq i \leq n$. This proves (3.8).
With Proposition 3.2 above at hand, Theorem 3.1 follows as a corollary of Theorems 5.1 and 5.3 of [34]; furthermore, by Theorem 5.4 there, we obtain the following boundary estimate.

**Theorem 3.3.** Suppose \((1.2), (1.3), (3.2), (3.5)\) hold. Let \(u \in C^3(M) \cap C^2(\overline{M})\) be an admissible solution to equation \((3.3)\) with \(u = \varphi\) on \(\partial M\), \(\varphi \in C^3(\overline{M})\). Then

\[
\sup_{\partial M} |\nabla^2 u| \leq C
\]

holds for a uniformly positive constant \(C\) depending on 
\((\inf_{\overline{M}} (1 - \varrho(x)(1 - \kappa T \vartheta_T))^{-1}, |u|_{C^1(\overline{M})}, |\varphi|_{C^3(\overline{M})}, \text{and other known data})\).

### 3.2. Existence results.

In this subsection, we assume there exists a \(C^2\)-admissible function \(w\) satisfying \((3.4)\); furthermore we assume there is some positive constant \(\beta\) such that

\[
(3.9) \quad \frac{1}{\beta} + a(x) > 0,
\]

\[
(3.10) \quad b(x) - \frac{\varrho(x)}{\beta} \geq 0, \text{ or } a(x) + b(x) + \frac{1 - \varrho(x)}{\beta} > 0
\]

which are used to construct locally lower/upper barriers. In particular, such two conditions hold for \(0 < \beta \ll 1\), given \(\varrho(x) < 0\) or \(0 < \varrho(x) < 1\). We prove

**Theorem 3.4.** Suppose \((1.2), (1.3), (3.2), (3.5), (3.9), (3.10)\) hold. For any \(\varphi \in C^\infty(\partial M)\), there is a unique smooth admissible function \(u\) to solve \((3.3)\) with

\[
(3.11) \quad u|_{\partial M} = \varphi,
\]

provided that there exists a \(C^2\)-admissible function \(w\) satisfying \((3.4)\).

### 3.2.1. \(C^0\)-estimate.

**Lemma 3.5.** Let \(w, v \in C^2(\overline{M})\) be admissible functions with

\[
f(\lambda(V[w])) \geq \psi e^{2\gamma w}, \quad f(\lambda(V[v])) \leq \psi e^{2\gamma v} \text{ in } M,
\]

\[
(w - v)|_{\partial M} \leq 0,
\]

then \(w - v \leq 0\) in \(M\).

**Proof.** Suppose that there is an interior point \(x_0 \in M\) such that \(0 < (w - v)(x_0) = \sup_M (w - v)\), therefore at \(x_0\),

\[
\nabla^2 v \geq \nabla^2 w, \quad \nabla v = \nabla w.
\]

Combining with Proposition 3.2,

\[
f(\lambda(V[v])) \geq f(\lambda(V[w])),
\]

which further yields \(v(x_0) \geq w(x_0)\). This contradicts to \(w(x_0) > v(x_0)\). \(\square\)
Lemma 3.6. Let \( u \in C^2(M) \) be an admissible solution to Dirichlet problem (3.3) and (3.11), then
\[
\begin{align*}
\inf_M (u - w) &\geq \min \left\{ \inf_{\partial M} (\varphi - w), \frac{1}{2\gamma} \inf_M \frac{f(\lambda (g^{-1}V[w]))}{\psi} \right\}, \\
\sup_M (u - w) &\leq \max \left\{ \sup_{\partial M} (\varphi - w), \frac{1}{2\gamma} \sup_M \frac{f(\lambda (g^{-1}V[w]))}{\psi} \right\}.
\end{align*}
\]
(3.12)

Proof. The proof is based on maximum principle. We omit the detail here. \( \square \)

3.2.2. Local barriers and boundary estimate for gradient. Let \( \rho(x) \) be the distance function from \( x \) to boundary \( \partial M \) with respect to \( g \),
\[ \Omega_\delta = \{ x \in M : \rho(x) < \delta \} \]
Notice \( \rho \) is smooth in \( \Omega_\delta \) when \( 0 < \delta \ll 1 \), and \( |\nabla \rho| = 1 \) on \( \partial M \).

Locally lower barrier. We use \( \rho \) to construct local barriers. Let
\[ w = \beta \log \frac{\delta^2}{\delta^2 + \rho}. \]
Then \( e^{2\gamma w} = (\frac{\beta^2}{\delta^2 + \rho})^{2\beta \gamma} \). First, a straightforward computation gives

Lemma 3.7.
\[
V[w] = A + \frac{\beta(1 + a\beta)}{(\delta^2 + \rho)^2} |\nabla \rho|^2 g + \frac{\beta(b\beta - g)}{\delta(\delta^2 + \rho)^2} d\rho \otimes d\rho
\]
\[ + cL(\frac{-\beta}{\delta^2 + \rho} d\rho) - \frac{\beta}{\delta^2 + \rho} (\Delta \rho g - \rho \nabla^2 \rho), \]
\[ V[w + \varphi] = V[w] + c(L(d(w + \varphi)) - L(dw)) + (\Delta \varphi g - \rho \nabla^2 \varphi)
\]
\[ + a \left( |\nabla \varphi|^2 - \frac{2\beta}{\delta^2 + \rho} g(\nabla \varphi, \nabla \rho) \right) g + bd\varphi \otimes d\varphi
\]
\[ - \frac{\beta b}{\delta^2 + \rho} (d\varphi \otimes d\rho + d\rho \otimes d\varphi)
\]
Using this lemma, if (3.9) and (3.10) hold then there is a positive constant \( c_0 \) such that for small \( \delta \),
\[ V[w] \geq \frac{c_0}{(\delta^2 + \rho)^2} |\nabla \rho|^2 g, \quad V[w + \varphi] \geq \frac{c_0}{2(\delta^2 + \rho)^2} |\nabla \rho|^2 g \text{ on } \Omega_\delta, \]
and so
\[ f(\lambda (g^{-1}V[w + \varphi])) \geq \psi e^{2\gamma (w + \varphi)} \text{ in } \Omega_\delta. \]
One can check \( (w + \varphi - u)_{|\partial M} = 0 \), and
\[ (w + \varphi - u)_{|\rho=\delta} = \beta \log \frac{\delta}{1 + \delta} + (\varphi - u)_{|\rho=\delta} < 0 \text{ if } \delta \ll 1. \]
Here we use Lemma 3.6. Now we have a local lower barrier \( w + \varphi \) near boundary.
Locally upper barrier. Let \( v = \beta' \log(1 + \frac{\rho}{\delta}) + \varphi \).

\[
\text{tr}(g^{-1}V[v]) = \frac{\beta' |\nabla \rho|^2}{(\delta^2 + \rho)^2} \left\{ -n + \varrho + (na + b)\beta' \right\} + \frac{\beta'(n - \varrho)}{\delta^2 + \rho} \Delta \rho
\]

\[
+ 2(na + b)\beta' \delta \rho \nabla \varphi + |\nabla \varphi|^2 + \text{tr}(g^{-1}A)
\]

By (3.6), \(-n + \varrho + (na + b)\beta' \leq -\frac{\varrho}{\delta^2} < 0\) if \( 0 < \beta' \ll 1 \). Note \( |\nabla \rho| = 1 \) on \( \partial\Omega \).

So we can choose \( \delta \) small sufficiently so that \( \text{tr}(g^{-1}V[v]) \leq 0 \) in \( \Omega_\delta \). It is easy to see \( (v - u)|_{\partial \Omega} = 0 \). On \( \{ \rho = \delta \} \), by Lemma 3.6 again, \( v - u = \beta' \log(1 + \frac{\rho}{\delta}) + \varphi - u > 0 \) for \( 0 < \delta \ll 1 \).

Therefore, near the boundary we now obtain a locally upper barrier

\[
v = \beta' \log(1 + \frac{\rho}{\delta}) + \varphi.
\]

As a consequence, we obtain the gradient estimate at boundary \( \sup_{\partial M} |\nabla u| \leq C \).

By standard continuity method, we can prove Theorem 3.4. The conformal metrics with prescribed boundary metric is also obtained.

**Theorem 3.8.** Suppose (1.2), (1.3), (1.4), (1.8) hold, and \( 0 < \psi \in C^\infty(\bar{M}) \). If, in addition, there is \( u \in C^2(\bar{M}) \) so that \( g = e^{2u}g \) obeys

\[
(3.13) \quad \lambda(g^{-1}A^{\alpha,\beta}) \in \Gamma \text{ in } \bar{M}.
\]

For a Riemannian metric \( h \) on \( \partial M \) which is conformal to \( g|_{\partial M} \), there is a smooth conformal metric \( \tilde{g} \) with prescribed boundary condition \( \tilde{g}|_{\partial M} = h \) to satisfy (1.1).

**Proof.** It requires to verify (3.9)-(3.10) given assumption (1.8) holds. In our case \( \varrho = \frac{n-2}{\tau - 1}, a = \frac{(n-2)(\tau - 2)}{2(\tau - 1)}, b = \frac{n-2}{\tau - 1}, c = 0 \). First, if \( \tau < 1 \) and \( \alpha = -1 \) then \( \rho < 0 \) and so (3.9) and (3.10) hold for \( 0 < \beta \ll 1 \); while for \( \tau > 1 + (n-2)(1 - \kappa_\tau \bar{\vartheta}) \) and \( \alpha = 1 \), we can verify (3.9) and (3.10) hold for \( \beta = 1 \). Here we use (1.9). \( \square \)

### 3.3. Complete conformal metrics on manifolds on boundary.

The following theorem provides a straightforward approach to local bound of solutions from above.

**Theorem 3.9.** Suppose (1.2), (1.3), (1.4), (1.8), (3.13) hold and that \( 0 < \psi \in C^\infty(\bar{M}) \). Then there exists at least one smooth complete conformal metric \( g_\infty = e^{2u_\infty}g \) satisfying (1.1).

**Proof.** From Theorem 3.8 there exists a \( u_k \in C^\infty(\bar{M}) \) satisfying

\[
(3.14) \quad f(\lambda(g^{-1}A^{\alpha,\beta}_{2k})) = \psi e^{2u_k}, \quad \tilde{g}_k = e^{2u_k}g \text{ in } M, \quad u_k = \log k \text{ on } \partial M.
\]

It follows from (1.5) that

\[
n\psi e^{2u_k} \leq \text{tr}(g^{-1}A^{\alpha,\beta}_{2k}) = \frac{\lambda(g^{-1}A^{\alpha,\beta}_{2k})(2(n-1)\Delta u_k + (n-1)(n-2)|\nabla u_k|^2 - R_g)}{(2n - 1)(n-2)}.
\]
Theorem 2.1 yields that there is a smooth function \( \tilde{u} \in C^\infty(M) \) so that
\[
2(n - 1)\Delta \tilde{u} + (n - 1)(n - 2)|\nabla \tilde{u}|^2 - R_g = e^{2\tilde{u}} \text{ in } M, \quad \lim_{x \to \partial M} \tilde{u}(x) = +\infty.
\]
The maximum principle yields
\[
(3.16) \quad u_k \leq u_{k+1} \leq \tilde{u} + \frac{1}{2} \log \frac{\alpha(n\tau + 2 - 2n)}{2(n-1)(n-2)\inf_M \psi} \text{ in } M \text{ for } k \geq 1.
\]
As a result, for any compact subset \( K \subset \subset M \),
\[
|u_k|_{C^0(K)} \leq C_1(K) \text{ for all } k, \quad \text{for } C_1(K) \text{ being independent of } k.
\]
Given any compact subset \( K \subset \subset M \), we choose a compact subset \( K_1 \subset \subset M \) such that \( K \subset \subset K_1 \). By Theorem 3.1 and Evans-Krylov theorem,
\[
|u_k|_{C^{2,\alpha}(K)} \leq C_2(K, K_1) \text{ for all } k, \quad \text{where } C_2(K, K_1) \text{ depends not on } k.
\]
Combining with Schauder theory, let \( k \to +\infty \), there is a smoothly admissible function \( u_\infty \) in \( M \) to solve
\[
f(\lambda(g^{-1}A_{g_{\hat{\rho}}}^{\tau, \alpha} \hat{\rho})) = \psi e^{2u_\infty}, \quad g_\infty = e^{2u_\infty} g \text{ in } M, \quad \lim_{x \to \partial M} u_\infty(x) = +\infty.
\]
Indeed, \( u_\infty(x) = \lim_{k \to +\infty} u_k(x) \) for all \( x \in M \). By (3.16), one has in \( M \)
\[
(3.17) \quad u_\infty \leq \tilde{u} + \frac{1}{2} \log \frac{\alpha(n\tau + 2 - 2n)}{2(n-1)(n-2)\inf_M \psi}.
\]
Next we prove \( g_\infty \) is complete. Let \( h_k = h_k(\rho) = \log \frac{k^2}{k\rho + \delta^2} \). The straightforward computation gives \( h''_k = h^2_k = \frac{k^2}{(k\rho + \delta^2)^2} \) and
\[
\frac{\alpha(n-1)}{n-2} \Delta h_k g - \alpha \nabla^2 h_k - \frac{\alpha(n-2)}{2} |\nabla h_k|^2 g + \alpha dh_k \otimes dh_k
\]
\[
= \frac{\alpha(n\tau + 2 - 2n)}{(n-2)} h''_k |\nabla \rho|^2 g + \alpha h'_k \left( \frac{\tau - 1}{n-2} \Delta \rho g - \nabla^2 \rho \right).
\]
Let \( \hat{g}_k = e^{2h_k} g \) near boundary. Then, near the boundary \( \Omega_\delta, 0 < \delta \ll 1, A_{g_{\hat{\rho}}}^{\tau, \alpha} \geq c_0 h''_k g \) for some constant \( c_0 > 0 \) (here we use (1.9) and \( |\nabla \rho| = 1 \) on \( \partial M \)), and
\[
f(\lambda(g^{-1}A_{g_{\hat{\rho}}}^{\tau, \alpha} \hat{\rho})) \geq \frac{c_0 k^2}{(k\rho + \delta^2)^2} \geq \frac{\psi \delta^4 k^2}{(k\rho + \delta^2)^2} = \psi e^{2h_k}.
\]
On \( \{ \rho = \delta \} \), \( h_k(\delta) = \log \frac{k\delta}{k\rho + \delta^2} \to -\infty \) as \( \delta \to 0 \). Again, maximum principle yields \( u_k \geq \log \frac{k^2}{k\rho + \delta^2} \) near boundary. Thus \( u_\infty + \log \rho \geq -C_0 \) for some \( C_0 \) near the boundary. The metric \( g_\infty = e^{2u_\infty} g \) is complete. \( \square \)
4. LOCAL ZERO ORDER ESTIMATE AND PROOF OF MAIN RESULTS

Let \( \{M_k\}_{k=1}^{\infty} \) be an exhaustion domains of \( M \) with
\[
M = \bigcup_{k=1}^{\infty} M_k, \quad \bar{M}_k = M_k \cup \partial M_k, \quad \bar{M}_k \subset \subset M_{k+1},
\]
\( \bar{M}_k \) is a compact \( n \)-manifold with smooth boundary.

Without loss of generality, we can assume that the admissible complete conformal metric \( g = e^{2u}g \) satisfying assumption \((1.11)\) is smooth, i.e. \( u \in C^\infty(M) \).

Let’s consider a sequence of approximate Dirichlet problems
\[
(4.1) \quad f(\lambda(g^{-1}A_{\bar{g}}^{\tau,\alpha})) = \psi e^{2u} \text{ in } M_k, \quad \bar{g} = e^{2u}, \quad u = \underline{u} \text{ on } \partial M_k.
\]
where \( A_{\bar{g}}^{\tau,\alpha} \) obeys the formula \((2.2)\) under the conformal change. According to Theorem 3.8, for each \( k \), there is a unique smooth admissible conformal metric \( g_k = e^{2u_k}g \) to satisfy \((4.1)\) on \( M_k \).

**Remark 4.1.** If \( u \) is only \( C^2 \), then on each \( \partial M_k \), the boundary value condition \( (u - \underline{u})|_{\partial M_k} = 0 \) should be replaced by \( (u - u_k)|_{\partial M_k} = 0 \), where \( u_k \) is a smooth function on \( \bar{M}_k \) and satisfies
\[
|u - u_k|_{C^2(\bar{M}_k)} \leq \frac{1}{1 + k},
\]
\[
f(\lambda(g^{-1}A_{g_k}^{\tau,\alpha})) \geq \frac{k}{1 + k} f(\lambda(g^{-1}A_{\bar{g}}^{\tau,\alpha})), \quad g_k = e^{2u_k}g, \quad \lambda(g^{-1}A_{g_k}^{\tau,\alpha}) \in \Gamma \text{ in } \bar{M}_k.
\]

4.1. **Local zero order estimate.** To complete the proof of main results, it suffices to prove local zero order estimate for approximate Dirichlet problems \((4.1)\).

**Theorem 4.2.** Let \( K \subset \subset M \) be a compact subset of \( M \), and assume \( K \subset \subset M_m \) for some \( m \in \mathbb{N} \). Then there is a uniformly positive constant \( C \) depending on \( K \) and \( M_m \) such that for each \( u_k (k \geq m) \) solving \((4.1)\) on \( M_k \), we have
\[
\sup_K u_k \leq C.
\]

**Proof.** Here we present two proofs.

**First proof.** According to Theorem 3.9, there is a \( w_m \in C^\infty(M_m) \) satisfying
\[
f(\lambda(g^{-1}A_{\bar{g}}^{\tau,\alpha})) = \psi e^{2w_m}, \quad \bar{g}_m = e^{2w_m}g, \quad \lambda(g^{-1}A_{\bar{g}_m}^{\tau,\alpha}) \in \Gamma \text{ in } M_m, \quad \lim_{x \to \partial M_m} w_m(x) = +\infty.
\]
Applying the maximum principle, for any \( k \geq m \), the (admissible) solution \( u_k \) to \((4.1)\) shall satisfy
\[
u_k \leq w_m \text{ in } M_m,
\]
which then completes the first proof of Theorem 4.2.

**Second proof.** A key ingredient is \((1.5)\). As a result, for all \( k \), in \( M_k \)
\[
\text{tr}(g^{-1}A_{g_k}^{\tau,\alpha}) = \frac{\alpha(n\tau + 2 - 2n)}{2(n-1)(n-2)} \left\{ 2(n-1)\Delta u_k + (n-1)(n-2)|\nabla u_k|^2 - R_g \right\} \geq n\psi e^{2u_k}.
\]
Another key tool is Theorem 2.1 which yields that there is a smooth function $\tilde{u}_m \in C^\infty(M_m)$ so that

$$2(n-1)\Delta \tilde{u}_m + (n-1)(n-2)|\nabla \tilde{u}_m|^2 - R_g = e^{2\tilde{u}_m} \text{ in } M_m, \quad \lim_{x \to \partial M_m} \tilde{u}_m(x) = +\infty.$$  

We observe that for all $k \geq m$ the solution $\tilde{u}_m$ may provide a supersolution of Dirichlet problem (4.1) on $M_k$. The maximum principle gives for each $k \geq m,$

$$u_k \leq \tilde{u}_m + \frac{1}{2} \log \left( \frac{\alpha(n \tau + 2 - 2n)}{2n(n-1)(n-2) \inf_{M_m} \psi} \right) \text{ in } M_m.$$  

The second proof is complete.

From assumption (1.11) there is a uniformly positive constant $\Lambda_1$ such that

$$f(\lambda(g^{-1} A_2^\tau \alpha)) \geq \Lambda_1 \psi e^{2u} \text{ in } M \text{ for some } \Lambda_1 > 0, \quad g = e^{2u}.$$

By comparison principle, we can derive $\inf_{M_k}(u_k - u) \geq \min \{0, \frac{1}{2} \log \Lambda_1 \}.$ That is

**Theorem 4.3.** For the solution $u_k$ to (4.1), we have

$$u_k \geq u + \frac{1}{2} \min \{0, \log \Lambda_1 \} \text{ in } M_k.$$  

4.2. **Completion of proof of Theorem 1.5.** Let $K \subset \subset K_1 \subset \subset M$ be two distinct compact subsets of $M$, let $u_k$ be the admissible solution to Dirichlet problem (4.1) on $M_k$. For $K_1$ there is $k_0$ such that $K_1 \subset \subset M_{k_0}$. By Theorems 4.2 and 4.3 there is a uniformly positive constant $C_0$ depending not on $k$ such that for any $k \geq k_0,$

$$\sup_{K_1} |u_k| \leq C_0,$$

then according to Theorem 3.1

$$|u_k|_{C^2(K)} \leq C_1, \quad \forall k \geq k_0$$

holds for a uniformly positive constant $C_1$ depending not on $k$. The Evans-Krylov theorem [9, 19] and classical Schauder theory give

$$|u_k|_{C^{l,\alpha}(K)} \leq C_l = C_l(K, K_1), \quad \forall k \geq k_0, \ l \geq 2, \text{ for some } 0 < \alpha < 1.$$  

By diagonal process, we obtain a desired solution $u_\infty \in C^\infty(M)$ to equation (1.1).

From Theorem 4.3

$$u_\infty \geq u + \frac{1}{2} \min \{0, \log \Lambda_1 \} \text{ in } M.$$  

Combining with the completeness of $g = e^{2u}g$, we know $g_\infty = e^{2u_\infty}g$ is complete. This completes the proof of Theorem 1.5.

The method also works for prescribed scalar curvature equation.
Theorem 4.4. Assume \( R_g < 0 \). Let \( \psi \) be a smooth positive function which is bounded from above in terms of \( -R_g \), i.e. there is a constant \( \delta > 0 \) such that \( 0 < \psi \leq -\delta R_g \). Then there exists a smooth complete conformal metric \( \tilde{g} \) with \( R_{\tilde{g}} = -\psi \).

5. Complete conformal metrics on Euclidean spaces

There are two fundamental models of complete manifolds: standard hyperbolic space \((\mathbb{H}^n, g_{-1})\) and flat Euclidean space \((\mathbb{R}^n, g_0)\).

In this section we deal with the latter one, and the purpose is to prove the existence of complete metric that is conformal to the standard Euclidean metric. The main ingredient is to construct certain admissible complete smooth conformal metrics on \( \mathbb{R}^n \) which then confirms assumption \([1.11]\) of Theorem 1.5.

Let \( (x_1, \ldots, x_n) \) be the standard coordinate systems, and \( r^2 = |x|^2 = \sum_{i=1}^n x_i^2 \). Let \( h \) be a smooth radial symmetric function:

\[
h = \beta \log(1 + r^2) \quad \text{for} \quad \beta > 0.
\]

Since \( g_0 \) is flat, \( A_{\tau,\alpha}^{g_0} \equiv 0 \). Let \( g = e^{2h} g_0 \), then

\[
A_{\tau,\alpha}^g = \frac{\alpha(\tau - 1)}{n - 2} \Delta h g - \frac{\alpha(\tau - 2)}{2} |Dh|^2 g + \alpha dh \otimes dh.
\]

Some simple computations (with respect to the flat metric \( g_0 \)) give

\[
|Dr^2| = 4r^2, \quad D^2r^2 = 2g_0,
\]

\[
dh \otimes dh = \frac{\beta^2}{(1 + r^2)^2} dr^2 \otimes dr^2, \quad |Dh|^2 = \frac{4\beta^2 r^2}{(1 + r^2)^2},
\]

\[
D^2h = \frac{2\beta}{1 + r^2} g_0 - \frac{\beta}{(1 + r^2)^2} dr^2 \otimes dr^2,
\]

\[
\Delta h = \frac{2n\beta}{1 + r^2} - \frac{4\beta r^2}{(1 + r^2)^2} = \frac{2\beta(n - 2)r^2 + 2\beta n}{(1 + r^2)^2}.
\]

Therefore, we obtain

\[
\frac{\tau - 1}{n - 2} \Delta h g_0 - D^2h + \frac{\tau - 2}{2} |Dh|^2 g_0 + dh \otimes dh
\]

\[
= \frac{2\beta}{(1 + r^2)^2} \left( (\tau - 2)(1 + \beta)r^2 + \frac{n(\tau - 2) + 2}{n - 2} \right) g_0 + \frac{\beta(1 + \beta)}{(1 + r^2)^2} dr^2 \otimes dr^2,
\]

with the eigenvalues (with respect to \( g_0 \)):

\[
\lambda_1 = \cdots = \lambda_{n-1} = \frac{2\beta}{(1 + r^2)^2} \left( (\tau - 2)(1 + \beta)r^2 + \frac{n(\tau - 2) + 2}{n - 2} \right),
\]

\[
\lambda_n = \frac{2\beta}{(1 + r^2)^2} \left( \tau(1 + \beta)r^2 + \frac{n(\tau - 2) + 2}{n - 2} \right).
\]
5.1. **Modified Schouten tensors.** With replacing (1.8) by a stronger condition

\[(5.2)\]

\[\tau < 0, \quad \text{if } \alpha = -1;\]
\[\tau > \max\{2, 1 + (n - 2)(1 - \kappa \Gamma \vartheta \Gamma)\}, \quad \text{if } \alpha = 1,\]

we prove

**Theorem 5.1.** Suppose (1.2), (1.3), (1.4), (5.2) hold. Let \(\delta > 0\) be fixed. For any smooth function \(\psi\) with

\[(5.3)\]

\[0 < \psi(x) \leq \Lambda_2 |x|^{-2-\delta} \quad \text{in } \mathbb{R}^n \quad \text{whenever } |x| \gg 1.\]

There is a smooth admissible complete metric \(\tilde{g}\) that is conformal to \(g_0\) such that

\[f(\lambda(\tilde{g}^{-1} A_{\tilde{g}}^{\tau, \alpha})) = \psi, \quad \lambda(\tilde{g}^{-1} A_{\tilde{g}}^{\tau, \alpha}) \in \Gamma\text{ in } \mathbb{R}^n.\]

**Proof.** Given (5.2) we can check that \(\lambda(g_0^{-1} A_{g_0}^{\tau, \alpha}) \geq c_0 r^{-2} g_0\) for some \(c_0 > 0\). The proof is complete by using Theorem 1.5. \(\square\)

5.2. **Ricci tensor.** In this case, \(\tau = 0, \alpha = -1\). We divide this case into two subcases.

**Theorem 5.2.** Let \(f\) satisfy (1.2), (1.3), (1.4), and we assume that for some \(\delta > 0\), \(\psi\) satisfies (5.3). Then there exists a smooth admissible complete conformal metric \(\tilde{g} = e^{2u} g_0\) satisfying with

\[f(\lambda(-\tilde{g}^{-1} \text{Ric}_{\tilde{g}})) = \psi, \quad \lambda(-\tilde{g}^{-1} \text{Ric}_{\tilde{g}}) \in \Gamma\text{ in } \mathbb{R}^n,\]

provided that \(\Gamma \neq \Gamma_n\).

**Proof.** The eigenvalues of \(-\text{Ric}_g\) with respect to \(g_0\) are as follows:

\[\lambda_1 = \cdots = \lambda_{n-1} = \frac{2\beta}{(1 + r^2)^2} \left(2(1 + \beta)r^2 + \frac{2n - 2}{n - 2}\right) \geq \frac{4\beta(1 + \beta)r^2}{(1 + r^2)^2},\]
\[\lambda_n = \frac{4(n - 1)\beta}{(1 + r^2)^2(n - 2)}.\]

Since \(\Gamma \neq \Gamma_n, (1, \cdots, 1, 0) \in \Gamma\) and then \(c_0 := f(1, \cdots, 1, 0) > 0\) is well defined. So

\[f(\lambda(-g_0^{-1} \text{Ric}_{\tilde{g}})) \geq \frac{4\beta(1 + \beta)c_0^2r^2}{(1 + r^2)^2},\]

that is

\[f(\lambda(-\tilde{g}^{-1} \text{Ric}_{\tilde{g}})) \geq \frac{4\beta(1 + \beta)c_0^2r^2}{(1 + r^2)^2 + 2\beta}.\]

This completes the proof by setting \(\beta = \frac{\delta}{4}\). \(\square\)

However, it is much more complicated for the case \(\Gamma = \Gamma_n\). While for special cases \(f = (\sigma_{n,k})^{1/(n-k)}, 0 \leq k < n\), where \(\sigma_{n,k} = C_n^k \sigma_n / \sigma_k\), we obtain the following theorem.
Theorem 5.3. There exists a smooth complete metric $\tilde{g}$, which is conformal to $g_0$, with negative Ricci curvature and
$$\sigma_{n,k}(\tilde{g}^{-1}\text{Ric}_\tilde{g}) = \psi \text{ in } \mathbb{R}^n, \quad 0 \leq k < n,$$
provided $\psi \in C^\infty(\mathbb{R}^n)$, $0 < \psi(x) \leq \Lambda_2|x|^{-2(\nu_1-k)}$ for $|x| \gg 1$, for some $\delta > 0$.

When $f$ is the $(k,l)$-quotient functions, $f = (C_k^l \sigma_k / C_l^k \sigma_l)^{1/(k-l)}$, the problem for negative Ricci tensor on Euclidean spaces was discussed in [28] by constructing radial symmetric subsolutions and supersolutions.

6. Conformal deformation of the $-A_g$

The conformally prescribed curvature problem for the $-A_g$ is in general rather hard to handle, since the counterexample of interior estimates of solutions to some prescribed curvature equations for conformal deformation of $-A_g$ [25].

Recall the definition of type 2 cone given by Caffarelli-Nirenberg-Spruck [5].

Definition 6.1 ([5]). $\Gamma$ is said to be of type 1 if the positive $\lambda_i$ axes belong to $\partial \Gamma$; otherwise it is called of type 2.

Assuming that $\Gamma$ is of type 2, we study the conformal deformation of $-A_g$,

$$f(\lambda(-\tilde{g}^{-1}A_{\tilde{g}})) = \psi, \quad \lambda(\tilde{g}^{-1}A_{\tilde{g}}) \in \Gamma, \quad \tilde{g} = e^{2u}g \text{ in } M.$$  

Under the conformal change $\tilde{g} = e^{2u}g$, by (2.2), the equation (6.1) is reduced to

$$f \left( \lambda \left( g^{-1}(\nabla^2 u + \frac{1}{2}|
abla u|^2 g - du \otimes du - A_g) \right) \right) = \psi e^{2u}.$$  

From Definitions 6.1 and 1.3 $\kappa_\Gamma = n - 1$ is equivalent to $\Gamma$ is of type 2. According to Lemma 1.4, $f$ is fully uniform ellipticity, provided (1.2), (1.3), (1.6) hold, that is

$$f_i(\lambda) \geq \partial_\Gamma \sum_{j=1}^{n} f_j(\lambda), \quad \forall 1 \leq i \leq n,$$

which then allows us to derive the local estimates for gradient and Hessian of solutions.

For the case when $(M,g)$ is a compact Riemannian manifold with boundary, as in Section 3 the locally lower and upper barriers are respectively given by $h^+ = \log \frac{k\delta^2}{\kappa_\Gamma}$ and $h^- = \beta' \log(1 + \frac{2\beta'}{\delta})$ ($0 < \beta' \ll 1$). And then we can prove

Theorem 6.2. Let $(M,g)$ be a compact Riemannian manifold with smooth boundary $\partial M$ and $\lambda(-g^{-1}A_g) \in \Gamma$ in $M$. Suppose (1.2), (1.3), (1.4) hold and that $\psi \in C^\infty(M)$, $\psi > 0$ in $M$. If, in addition, the corresponding cone $\Gamma$ is of type 2, there is on $M$ a smooth complete conformal metric $g_\infty = e^{2u_\infty}g$ satisfying (6.1).

Theorem 6.3. Suppose (1.2), (1.3), (1.4) hold, and the corresponding cone $\Gamma$ is of type 2. Assume that $(M,g)$ is a complete noncompact Riemannian manifold with

$$f(\lambda(-g^{-1}A_g)) \geq \delta \psi, \quad \lambda(-g^{-1}A_g) \in \Gamma \text{ holds for some } \delta > 0, \text{ in } M.$$
Then there exists at least one smooth function \( u_\infty \in C^\infty(M) \) such that \( g_\infty = e^{2u_\infty} g \) is a smooth admissible complete metric satisfying (6.1).

The results are applicable to conformal deformation of Einstein tensor and of modified Schouten tensors \( A^\tau_{\alpha} \) for \( \tau > n - 1, \alpha = 1 \), and \( \tau < 1, \alpha = -1 \).

6.0.1. Application to Ricci tensor and certain modified Schouten tensors. Let \( \varrho \) be a fixed constant with \( \varrho < 1 \) and \( \varrho \neq 0 \). Set \( \mu_i = \frac{1}{n-\varrho}(\sum_{j=1}^{n} \lambda_j - \varrho \lambda_i) \). Let \( \tilde{f}(\lambda_1, \cdots, \lambda_n) = f(\mu_1, \cdots, \mu_n) \). If \( f \) is a homogeneous function of degree one then so is \( \tilde{f} \) in the corresponding cone \( \tilde{\Gamma} = \{ \lambda : \mu = (\mu_1, \cdots, \mu_n) \in \Gamma \} \). We can check that \( \kappa_{\tilde{\Gamma}} = n - 1 \) for any \( \Gamma \). Again, a straightforward computation gives

\[
\text{tr}(-g^{-1}A_g)g - g(-A_g) = \frac{\varrho}{n-2} \text{Ric}_g - \frac{(n-2+\varrho)R_g}{2(n-1)(n-2)} g.
\]

As consequences, we derive many results. If \( \varrho < 0 \) then we get the results on \( A^\tau_{\alpha} \) for \( \tau < 1, \alpha = -1 \); in particular for the special case of \( \varrho = -(n-2) \), we obtain the main results proved in [12, 15] on a compact Riemannian manifold with boundary. When \( 0 < \varrho < 1 \) we can obtain the main results in [21] for modified Schouten tensors \( A^\tau_{\alpha} \) with \( \alpha = 1, \tau > n - 1 \) on compact manifolds with boundary.

6.0.2. Application to the Einstein tensor. Let \( \mu_i = \frac{1}{n-1}(\sum_{j \neq i} \lambda_j) \). Given a homogeneous function of degree one, we let \( \tilde{f}(\lambda_1, \cdots, \lambda_n) = f(\mu_1, \cdots, \mu_n) \). Then \( \tilde{f} \) is a homogeneous function of degree one with \( \tilde{\Gamma} = \{ \lambda : \mu = (\mu_1, \cdots, \mu_n) \in \Gamma \} \). We can check that when \( \Gamma \neq \Gamma_n, \kappa_{\tilde{\Gamma}} = n - 1 \), and so \( f \) is of fully uniform ellipticity according to Lemma [14]. By a simple computation, one derives

\[
\text{tr}(-g^{-1}A_g)g - g(-A_g) = \frac{1}{n-2}(\text{Ric}_g - \frac{R_g}{2} g) = \frac{1}{n-2} G_g.
\]

As an application, we obtain Theorem 1.1 of [34] on compact manifolds with boundary.

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