A NOTE ON HOMOLOGICAL DIMENSION OF A FAMILY
OF COHERENT SHEAVES

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We prove a theorem on how a conclusion on homological dimension
of the family of coherent sheaves on a scheme can be done from homo-
logical dimension of the restriction of this family to the reduction
of the base.

Bibliography: 5 items.
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module.

To the blessed memory of my Mum

The problem we solve in the present note is

how to conclude about homological dimension of the family of sheaves on the
family of schemes if it is known about homological dimension of the reduction?

As usually if T is a scheme with structure sheaf \(\mathcal{O}_T\) then its reduction is a
scheme consisting of the same topological space but with structure sheaf equal
to \(\mathcal{O}_{T_{\text{red}}} := \mathcal{O}_T/\text{Nil}(\mathcal{O}_T)\) where \(\text{Nil}(\mathcal{O}_T)\) is nilradical of \(\mathcal{O}_T\). This is called
for brevity a reduced scheme structure. The \(\mathcal{O}_T\)-module epimorphism onto the
quotient module sheaf \(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_{T_{\text{red}}}\) induces a canonical closed immersion \(T_{\text{red}} \hookrightarrow T\) of schemes. From now T is a base scheme of a flat morphism of finite type
\(f : X \rightarrow T\) of Noetherian schemes. We introduce notations \(X_{\text{red}} := X \times_T T_{\text{red}}, f_{\text{red}} : X_{\text{red}} \rightarrow T_{\text{red}},\) i.e. \(X_{\text{red}}\) is a restriction of the family \(X\) to the reduction \(T_{\text{red}}\) as to a closed subscheme in \(T\). Actually the scheme \(X_{\text{red}}\) can be nonreduced but \(f_{\text{red}}\) is flat morphism as a morphism obtained by a base change of a flat
morphism. Now let \(E\) be a coherent \(\mathcal{O}_X\)-module and let \(E\) be flat as \(\mathcal{O}_T\)-
module. This is a standard situation in various problems of algebraic geometry
when families of sheaves are considered, especially in moduli problems. Denote
\(E_{\text{red}} = E \otimes_{\mathcal{O}_T} \mathcal{O}_{T_{\text{red}}}\).

Let \(A\) be a commutative ring, \(M\) \(A\)-module. Also introduce parallel algebraic
notations: a reduction \(A_{\text{red}} := A/\text{Nil}A\) of the ring \(A\) is its quotient ring over its
nilradical. If a commutative ring \(B\) is an \(A\)-algebra then we denote \(B \otimes_A A_{\text{red}} =: B_{\text{red}}\). Hence \(B_{\text{red}}\) is not obliged to be reduced but it is \(A_{\text{red}}\)-flat whenever \(B\) is
\(A\)-flat, by well-known change-of-ring theorem. Also denote \(M \otimes_A A_{\text{red}} =: M_{\text{red}}\).
We say that $M$ has homological dimension not greater than $n$ (notation: $\text{hd}_A M \leq n$) if one of following equivalent conditions holds \cite{3} Ch. 7, theorem 1.1:

1) for all $A$-modules $N$ $\text{Ext}^{n+1}_A(M, N) = 0$,
2) in any exact $A$-sequence $0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to M \to 0$ if $E_j$ are projective for $0 \leq j \leq n - 1$ then $E_n$ is also projective,
3) there is a projective $A$-resolution of length $n$, i.e. there exists an exact sequence

$$0 \to E_n \to \cdots \to E_1 \to E_0 \to M \to 0$$

with $E_j$ projective for $0 \leq j \leq n$.

Since if $A$ is local ring then any projective $A$-module is free \cite{1} theorem 19.2 and comment thereafter), Then when working with coherent sheaves on schemes we speak of locally free resolutions instead of projective ones.

As usually the symbol $\text{hd}_X E$ means homological dimension of the coherent sheaf $E$ as $\mathcal{O}_X$-module.

We prove the following well-expected result.

**Theorem 1. Algebraic version.** Let $f^\#: A \to B$ be local homomorphism of local Noetherian rings and $B$ is flat as $A$-algebra. Let $M$ is $B$-module of finite type which is flat over $A$. Then following assertions are equivalent:

1) $\text{hd}_B M \leq n$,
2) $\text{hd}_{B_{\text{red}}} M_{\text{red}} \leq n$.

**Theorem 2. Sheaf version.** Let $f: X \to T$ be a flat morphism of Noetherian schemes, $E$ coherent $\mathcal{O}_X$-module which is flat over $T$. Then following assertions are equivalent:

1) $\text{hd}_X E \leq n$,
2) $\text{hd}_{X_{\text{red}}} E_{\text{red}} \leq n$.

**Remark 1.** The case $n = 1$ for a trivial family of schemes over a field was considered in the author’s paper \cite{5}.

**Remark 2.** Since both theorems are just versions of the same result their proofs are transferred literally to each other and we prove an algebraic version.

**Proof of Theorem 1.** For the implication 1)$\Rightarrow$2) we do not need locality and consider $B$-exact sequence

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$$

with $E_j$ free for $j \geq 0$. Cutting it into triples we have

$$0 \to E_n \to E_{n-1} \to M^{(n-1)} \to 0,$$

$$\cdots \cdots \cdots \cdots \cdots \cdots ,$$

$$0 \to M^{(j+1)} \to E_j \to M^{(j)} \to 0,$$

$$\cdots \cdots \cdots \cdots \cdots \cdots ,$$

$$0 \to M^{(1)} \to E_0 \to M \to 0.$$
Since $M$ and $E_j$, $j \geq 0$, are flat $A$-modules \cite{4} Ch. 3, sect. 7, transitivity (1)] we have that $M^{(j)}$ are $A$-flat for $j \geq 1$. Hence tensoring by $\otimes_B B_{\text{red}} = \otimes_A A_{\text{red}}$ ("cancelation formula") we have $\text{Tor}_i^A(M^{(j)}, A_{\text{red}}) = \text{Tor}_i^A(M, A_{\text{red}}) = 0$ and come to exact sequence

$$0 \to E_{\text{red}} \to E_{(n-1)\text{red}} \to \cdots \to E_{1\text{red}} \to E_{0\text{red}} \to M_{\text{red}} \to 0$$

with $E_{j\text{red}}$ free as $B_{\text{red}}$-modules for $0 \leq j \leq n$.

For the opposite implication we organize induction on $n$. Assume that the theorem is true for coherent $B$-sheaves of homological dimension not greater than $n - 1$. Let we are given an $B$-module epimorphism $E_0 \to M$ where $E_0$ is free and let $M' := \ker(E_0 \to M)$. Let $\text{hd}_B M_{\text{red}} \leq n$. We are to conclude that $\text{hd}_B M \leq n$. By the exact $B$-triple

$$0 \to M' \to E_0 \to M \to 0$$

and tensoring by $\otimes_B B_{\text{red}} = \otimes_A A_{\text{red}}$, since $M$ is $A$-flat then $\text{Tor}_i^A(M, A_{\text{red}}) = 0$ for $i > 0$ and hence we come to the exact $A_{\text{red}}$-triple

$$0 \to M'_{\text{red}} \to E_{\text{red}} \to M_{\text{red}} \to 0$$

where $\text{hd}_{A_{\text{red}}} M'_{\text{red}} \leq n - 1$. By flatness of $E_0$ as $B$-module and of the homomorphism $f^*$ the term $E_0$ is $A$-flat \cite{4} Ch. 3, sect. 7, transitivity (1)] and hence $M'$ is also flat as $A$-module. By the inductive assumption $\text{hd}_B M' \leq n - 1$ and hence $\text{hd}_B M \leq n$.

For the base of induction set $n = 0$. This means that $M$ is flat as $A$-module and $M_{\text{red}}$ is free as $A_{\text{red}}$-module.

Apply the following result from A. Grothendieck's SGA:

**Proposition 1.** \cite{2} Ch. IV, Corollaire 5.9] Let $A \to B \to C$ be local homomorphisms of local Noetherian rings, $M$ be $C$-module of finite type. Assume that $B$ is flat over $A$ and $k$ is a residue field of $A$. Then following assertions are equivalent:

(i) $M$ is $B$-flat;

(ii) $M$ is $A$-flat and $M \otimes_A k$ is $B \otimes_A k$-flat.

For our purposes set $B \to C$ to be an identity isomorphism, $M$ is flat over $A$ and $M_{\text{red}}$ is free (i.e. flat) as $A_{\text{red}}$-module. Then $M \otimes_A k = M_{\text{red}} \otimes_{A_{\text{red}}} k$, $B \otimes_A k = B_{\text{red}} \otimes_{A_{\text{red}}} k$, and $M_{\text{red}} \otimes_{A_{\text{red}}} k$ is flat over $B_{\text{red}} \otimes_{A_{\text{red}}} k$ because $M_{\text{red}}$ is free over $B_{\text{red}}$. From this we conclude that $M$ is free as $B$-module. This completes the proof of the theorem 1.

**References**

[1] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Math. 150. Springer – Verlag, New York, 1995.
[2] A. Grothendieck, Seminaire de Geometrie Algebrique du Bois Marie, 1960 – 61. I: Revetements etales et groupe fondamental. Lecture Notes in Math., Springer – Verlag, Berlin – Heidelberg – New York, 1971.

[3] S. Maclane, Homology. Die Grundlehren der Mathematischen Wissenschaften. Band 114. Springer – Verlag, Berlin – Göttingen – Heidelberg, 1963.

[4] H. Matsumura, Commutative Ring Theory, Cambridge Univ. Press, Cambridge, 1986.

[5] N. V. Timofeeva, On a morphism of compactifications of moduli scheme of vector bundles // arXiv:1308.0111v4.