A Comment on Duality Transformations and (Discrete) Gauge Symmetries in Four-Dimensional Strings

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Abstract

We discuss the relationship between target space modular invariance and discrete gauge symmetries in four-dimensional orbifold-like strings. First we derive the modular transformation properties of various string vertex operators of the massless string fields. Then we find that for supersymmetric compactifications the action of the duality elements, leaving invariant the multicritical points, corresponds to a combination of finite Kähler and gauge transformations. However, those finite gauge transformations are not elements of a remnant discrete gauge symmetry. We suggest that, at least in the case of Gepner models corresponding to tensor products of identical minimal models, the duality element leaving invariant the multicritical point corresponds to the $\mathbb{Z}_{k+2}$ symmetry of any of the minimal $N = 2$ models appearing in the tensor product.
In four-dimensional strings, moduli fields in general have non-vanishing charges under local gauge symmetries. This implies that parts of the local gauge symmetries are spontaneously broken at generic points in moduli space. However usually, a discrete gauge symmetry \cite{1} survives this spontaneous symmetry breakdown. On the other hand, target space duality transformations \cite{2}, in particular target space modular transformations \cite{3}, act non-trivially on many light four-dimensional string fields. Moreover, particular elements of the duality group act with simple, constant phases on the fields. Therefore duality symmetries act similarly to discrete symmetries. (We will show that they act like $R$-symmetries.) The similarities between discrete gauge symmetries and some duality symmetries naively suggest considering these duality symmetries as just another example of discrete gauge symmetries. However, this naive suggestion has to be qualified. For example, it is well known that the massless spectrum of orbifold models typically has duality anomalies \cite{4},\cite{5},\cite{5},\cite{6},\cite{7} whereas the enhanced gauge (discrete or continuous) symmetries are anomaly-free (at least for (2,2) models). Thus things are not so simple and, although indeed there is a connection between duality and discrete gauge transformations, the identification is not straightforward.

The intention of this letter is to clarify the relation between target space modular transformations, broken gauge symmetries and discrete gauge groups in four-dimensional string models. As a specific example we consider the $\mathbb{Z}_3$ orbifold. \footnote{For previous discussions on the relation between duality symmetries and broken gauge symmetries in the $\mathbb{Z}_3$ orbifold see \cite{9},\cite{10}.} However the discussion can be easily extended to other models.

Let us determine the transformation properties of the vertex operators of various fields under target space modular transformations $T \rightarrow \frac{aT - ib}{icT + d}$ for the $\mathbb{Z}_3$ orbifold. As explained in \cite{10}, these transformation rules can be derived from the action of the modular transformations on the momentum and winding numbers and from the subsequent action on the Narain lattice vectors. Specifically, consider the left (right) moving complex coordinate $X^{+i}_{L(R)}$ ($i = 1, 2, 3$) which is associated with the $i^{th}$ 2-dimensional torus of the $\mathbb{Z}_3$ orbifold. For a general $PSL(2,\mathbb{Z})$ transformation one finds

$$X^{+i}_{L}(\bar{z}) \rightarrow \lambda_i \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{-1/2} X^{+i}_{L}(\bar{z}), \hspace{1cm} X^{+i}_{R}(z) \rightarrow \lambda_i \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/2} X^{+i}_{R}(z),$$

(1)
where $\lambda_i$ is a $T_i$-independent phase that depends on the parameters $c_i, d_i$: e.g. $\lambda = \rho = e^{2\pi i/3}$ for $c = d = 1$ (ST transformation).

Next consider the corresponding right-moving world sheet fermions $\psi^{+i}_R$ with conformal dimension $h_R = 1/2$. Their transformation behaviour can be deduced from the requirement [9] that the right-moving world sheet supersymmetry commutes with the target space modular transformations. This requirement follows from the fact that the action of the right-moving supercurrent connects equivalent (picture-changed) physical string states. The right-moving world sheet supercurrent has the form

$$S_R(z) = \sum_{i=1}^{3} (\psi^{+i}_R \partial X^{-i}_R + \psi^{-i}_R \partial X^{+i}_R)(z). \quad (2)$$

Demanding $S_R$ to be invariant under modular transformations, one obtains

$$\psi^{+i}_R(z) \rightarrow \lambda_i \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/2} \psi^{+i}_R(z). \quad (3)$$

For many purposes it is very convenient to bosonize the fermions $\psi^{+i}_R$: $\psi^{+i}_R(z) = e^{iH^i_R(z)}$. Then modular transformations act on the two-dimensional bosons $H^i_R$ as

$$H^i_R(z) \rightarrow H^i_R(z) - i \log \lambda_i \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/2}. \quad (4)$$

Next consider the bosonic twist field vertex operators $\sigma^{i}_{\alpha_i}(\tau, z)$ of conformal dimension $h_L = h_R = 1/9$. Each twist field vertex is associated with one of the three fixed points $\alpha_i$ ($\alpha_i = 1, \ldots, 3$) of each complex $i$. The twists fields $\sigma_{\alpha_i}$ of different fixed points transform into linear combinations under target space modular transformations [11]. In addition there is a common field-dependent phase factor, which was determined in ref.[12]. In total one gets

$$\sigma^{i}_{\alpha_i}(\tau, z) \rightarrow \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/6} A_{\alpha_i, \beta_i} \sigma^{i}_{\beta_i}(\tau, z), \quad (5)$$

Specifically for the ST transformation the matrix $A$ has the form

$$A = \frac{i}{\sqrt{3}} \begin{pmatrix} \rho & \bar{\rho} & \bar{\rho} \\ \rho & \rho & 1 \\ \rho & 1 & \rho \end{pmatrix}. \quad (6)$$
Now we are ready to examine the modular transformation properties of some specific massless string states. First the vertex operator associated with the modulus $T_i$ has the form (we will only show the internal parts of the vertex operators)

$$\phi_T^i(\bar{z}, z) \sim \partial X_L^{-i}(\bar{z}) \partial X_R^{+i}(z).$$

(7)

Thus $\phi_T^i$ transforms under $PSL(2, \mathbb{Z})$ as

$$\phi_T^i(\bar{z}, z) \rightarrow \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/2} \phi_T^i(\bar{z}, z).$$

(8)

Next consider the scalar fields $\phi_{1,2}^i$ whose mass depends on the background parameters $T_i$. They become massless at the critical point $T_c = -ie^{2\pi i/3}$. The corresponding vertex operators look like

$$\phi_{1,2}^i(\bar{z}, z) \sim V_{1,2}(\bar{z}, z, T_i) \partial X_R^{+i}(z);$$

(9)

$V_{1,2}(\bar{z}, z, T_i)$ is a conformal field that depends on the background $T_i$ (see [9],[10] for details). Thus they transform as

$$\phi_{1,2}^i(\bar{z}, z) \rightarrow \lambda_i \left[ \frac{-ic_i T_i + d_i}{ic_i T_i + d_i} \right]^{1/2} \phi_{1,2}^i(\bar{z}, z).$$

(10)

In the untwisted sector of the $\mathbb{Z}_3$ orbifold, each complex plane is associated with three fields, which transform under the $27$ representation of $E_6$. Their vertex operators are obtained by action with the left-moving supercurrent on the vertex operators of the moduli $T_i$:

$$\phi_{27U}^{ij}(\bar{z}, z) \sim \psi_L^{+j}(\bar{z}) \partial X_R^{+i}(z).$$

(11)

(We left out some part of the vertex operator associated with the gauge group $E_6$.) Thus we obtain that the fields $\phi_{27U}^{ij}$ transform in the same way as the fields $\phi_{1,2}^i$ (see eq.(10)).

Next let us consider the fields in the twisted sector of the $\mathbb{Z}_3$ orbifold. First we have 27 fields, associated with the 27 fixed points of the $\mathbb{Z}_3$ orbifold, which transform like the $27$ representation of $E_6$. Their vertex operators are built by the product of
the three bosonic twist fields $\sigma^i_{\alpha_i}$. In addition the twist also acts on the left and right moving world sheet fermions. In total one obtains

$$\phi_{27T}^{\alpha_1\alpha_2\alpha_3}(z, \bar{z}) \sim \prod_{i=1}^{3} e^{iH^i_L(z)/3} \sigma^i_{\alpha_i}(\bar{z}, z) e^{iH^i_R(z)/3}. \tag{12}$$

Thus these fields transform as

$$\phi_{27T}^{\alpha_1\alpha_2\alpha_3}(z, \bar{z}) \rightarrow \prod_{i=1}^{3} \lambda^{2/3} \left[ \frac{-ic_iT_i + d_i}{ic_iT_i + d_i} \right]^{1/3} A_{\alpha_1\beta_1} A_{\alpha_2\beta_2} A_{\alpha_3\beta_3} \phi_{27T}^{\beta_1\beta_2\beta_3}(z, \bar{z}). \tag{13}$$

Finally there are 81 $E_6$ singlets in the twisted sector. They contain left moving twisted oscillators. (Some linear combinations of them, the twisted moduli, are obtained by acting with $S_L$ on $\phi_{27T}$.) Their vertex operators look like

$$\phi_{1T}^{i\alpha_1\alpha_2\alpha_3}(z, \bar{z}) \sim e^{-i2H^i_L(z)/3} e^{iH^i_L(z)/3} e^{iH^i_R(z)/3} T^i_{\alpha_i}(\bar{z}, z) \sigma^j_{\alpha_j}(\bar{z}, z) \sigma^k_{\alpha_k}(\bar{z}, z) \prod_{l=1}^{3} e^{iH^i_R(z)/3}. \tag{14}$$

Here $i \neq j \neq k$ and $T^i$ is an excited twist field obtained by the operator product of $\partial X^i_L$ and $\sigma^i$.

Then $\phi_{1T}^{i\alpha_1\alpha_2\alpha_3}$ transforms as

$$\phi_{1T}^{i\alpha_1\alpha_2\alpha_3}(z, \bar{z})_{1T} \rightarrow \lambda^{5/3} \left[ \frac{-ic_iT_i + d_i}{ic_iT_i + d_i} \right]^{5/6} \lambda^{2/3} \left[ \frac{-ic_jT_j + d_j}{ic_jT_j + d_j} \right]^{1/3} \lambda^{2/3} \left[ \frac{-ic_kT_k + d_k}{ic_kT_k + d_k} \right]^{1/3} A_{\alpha_1\beta_1} A_{\alpha_2\beta_2} A_{\alpha_3\beta_3} \phi_{1T}^{i\beta_1\beta_2\beta_3}(z, \bar{z}), \tag{15}$$

So far we considered the transformation rules of the scalar components of the chiral superfields. To obtain the action of the modular group on the corresponding space-time fermions, one has to examine the vertex operator of the space-time supercharge. Its internal part has the form

$$Q(z) \sim \prod_{i=1}^{3} e^{-iH^i_R(z)/2}. \tag{16}$$
Therefore \( Q \) transforms as

\[
Q(z) \to \prod_{i=1}^{3} \chi_i^{1/2} \left[ \frac{-ic_iT_i + d_i}{ic_iT_i + d_i} \right]^{-1/4} Q(z).
\]

(17)

Thus we recognize that the fermions get an additional phase under modular transformations. Therefore target space modular transformations act like an \( R \) symmetry.

From the field theory point of view, we will identify this additional phase as a Kähler transformation.

To summarize this analysis, the modular transformation behaviour of the massless string fields has the following form (up to the field-independent phase):

\[
\phi_s \to \prod_{i=1}^{3} \left[ \frac{-ic_iT_i + d_i}{ic_iT_i + d_i} \right]^{-n_s_i/2} \phi_s.
\]

(18)

\( n_s_i \) is called the modular weight vector of \( \phi_s \). Specifically, comparing with our previous results, we have

\[
\begin{align*}
\phi^i_T : & \quad \vec{n} = -2\vec{e}_i \\
\phi^i_{1,2}, \phi^{ij}_{27U} : & \quad \vec{n} = -\vec{e}_i \\
\phi_{27T} : & \quad \vec{n} = (-2/3, -2/3, -2/3) \\
\phi^i_{1T} : & \quad \vec{n} = (-2/3, -2/3, -2/3) - \vec{e}_i \\
Q : & \quad \vec{n} = (1/2, 1/2, 1/2).
\end{align*}
\]

(19)

(The \( \vec{e}_i \) are the 3-dimensional unit-vectors.) Notice that the above numbers for \( \vec{n} \) correspond to the modular weights of massless states discussed in ref. [7].

Now let us consider the particular element, \( \gamma = ST \), of the target space modular group acting as \( T_i \to \frac{1}{T_{i-1}} \), i.e. \( a_i = 0, b_i = -1, c_i = d_i = 1 \). This transformation leaves the critical point \( T_c = -ie^{2\pi i/3} \) invariant. Moreover, one can find a basis in which the \( ST \) transformation acts diagonally on particular linear combinations of the twist fields of the form \( \sum_{\alpha} c_{\alpha} \sigma_{\alpha}^i \). Specifically, the \( ST \) charges of these particular linear combinations of twist fields are given by the eigenvalues of the matrix in eq.(6):

\[ A' = \text{diag} (\vec{\rho}, \vec{\rho}, \rho) \]. Thus, at the critical point, all fields transform under the \( ST \) modular transformation as

\[
\phi_s \to e^{2\pi i Q_{ST}^i} \phi_s.
\]

(20)
We call $Q_{iST}$ the duality charge of each field. With $\frac{\sqrt{3}T_{c}+1}{iT_{c}+1} = e^{A\pi i/3}$ one obtains the following duality charges:

\begin{align}
\phi_{T}^i : & \quad Q_{iST} = 2/3 \\
\phi_{1,2}^i, \phi_{27U}^i : & \quad Q_{iST} = 2/3 \\
\phi_{27}^{1,2} : & \quad Q_{iST} = 0 \\
\phi_{27}^3 : & \quad Q_{iST} = 2/3 \\
\phi_{1T}^{i,1,2} : & \quad Q_{iST} = 0 \\
\phi_{1T}^3 : & \quad Q_{iST} = 2/3
\end{align}

On the other hand, all fields are characterized by certain $U(1)$ gauge charges. First each complex plane is associated with an enhanced $U(1)_1^i \times U(1)_2^i$ gauge group which is left unbroken at the critical point $T_i = T_c$. As discussed in refs. [13], [14], at the critical point it is possible to ‘rebosonize’ those parts of the vertex operators that involve the torus coordinates $X_{L,R}^{\pm i}$. In this new, so-called covariant lattice basis the left-moving (and also the right-moving) part of the vertex operators of all fields we have considered so far can be written as

\[ V_L(\tau) \sim \prod_{i=1}^{3} \exp\left( i(Q_1^i Y_1^i(\tau) + Q_2^i Y_2^i(\tau)) \right), \]

where $Y_{1,2}^i$ are the covariant lattice coordinates. The conformal dimensions are just given by $h_L = \frac{1}{2} \sum_{i=1}^{3} \left( (Q_1^i)^2 + (Q_2^i)^2 \right)$. For example, the left-moving torus coordinates can be written as $i\partial X_{L}^{\pm i}(\tau) = 1/\sqrt{3} \sum_{\alpha} \exp(\pm i\alpha \cdot \bar{Y}(\tau))$, where $\pm \alpha$ are the six root vectors of $SU(3)$ with $\alpha^2 = 2$. The gauge bosons of the enhanced $U(1)^6$ gauge group, which are massless for $T = T_c$, correspond just to $\partial Y_{1,2}^i(\tau)$. In the twisted sectors, the fields with definite $U(1)$ charges just correspond to those linear combinations of twist fields, on which $ST$ acts in a diagonal way.

Let us consider the particular $U(1)^i$ subgroup of $U(1)_1^i \times U(1)_2^i$, with charge $Q_{U(1)}^i$, defined by the following linear combination

\[ Q_{U(1)}^i = \frac{\sqrt{2}}{3} Q_1^i + \frac{2}{\sqrt{6}} Q_2^i. \]
In fact, these charges $Q^i_1$ and $Q^i_2$ are just given by the charges that appear in the vertex operators eq.(22) in the covariant lattice basis. The various fields have the following $U(1)$ gauge charges $[13]$:

\[
\begin{align*}
\phi^i_T, \phi^i_{1,2} & : \quad Q^i_1 = \sqrt{2}, \quad Q^i_2 = 0, \quad Q^i_{U(1)} = 2/3 \\
Q^i_1 &= -\frac{1}{\sqrt{2}}, \quad Q^i_2 = \pm \frac{3}{\sqrt{6}}, \quad Q^i_{U(1)} = 2/3 \\
\phi^{ij}_{27U} & : \quad Q^i_1 = 0, \quad Q^i_2 = 0, \quad Q^i_{U(1)} = 0 \\
\phi^1_{27T} & : \quad Q^i_1 = \frac{1}{3\sqrt{2}}, \quad Q^i_2 = -\frac{1}{\sqrt{6}}, \quad Q^i_{U(1)} = -2/9 \\
\phi^2_{27T} & : \quad Q^i_1 = -\frac{1}{3\sqrt{2}}, \quad Q^i_2 = 0, \quad Q^i_{U(1)} = -2/9 \\
\phi^3_{27T} & : \quad Q^i_1 = \frac{1}{3\sqrt{2}}, \quad Q^i_2 = \frac{1}{\sqrt{6}}, \quad Q^i_{U(1)} = 4/9 \\
\phi^1_{1T} & : \quad Q^i_1 = -\frac{2}{3\sqrt{2}}, \quad Q^i_2 = \frac{2}{\sqrt{6}}, \quad Q^i_{U(1)} = 4/9 \\
\phi^2_{1T} & : \quad Q^i_1 = \frac{2\sqrt{2}}{3}, \quad Q^i_2 = 0, \quad Q^i_{U(1)} = 4/9 \\
\phi^3_{1T} & : \quad Q^i_1 = -\frac{2}{3\sqrt{2}}, \quad Q^i_2 = -\frac{2}{\sqrt{6}}, \quad Q^i_{U(1)} = 1/9 \\
\end{align*}
\]

Second each complex plane is associated with a $U(1)$ holonomy charge $Q_{U(1)}$. This charge is a linear combination of the superconformal $U(1)$ inside $E_6$ and of the two Cartan subalgebra $U(1)'s$ of $SU(3)$: $Q_{superconf} = \sum_{i=1}^3 Q^i_{U(1)}, Q^{SU(3)}_1 = \frac{1}{\sqrt{2}}(Q^1_{U(1)} - Q^2_{U(1)}), Q^{SU(3)}_2 = \frac{1}{\sqrt{6}}(Q^1_{U(1)} + Q^2_{U(1)} - 2Q^3_{U(1)}); Q^i_{U(1)}$ is determined by the left-moving world sheet fermions $\psi^i_L$. Specifically, the left-moving fermionic part of the vertex operators has the form $e^{i\frac{1}{4}Q^i_{U(1)}H^i_L}$. Thus we obtain:

\[
\begin{align*}
T_i & : \quad Q^i_{U(1)} = 0 \\
\phi^i, \phi^{ij}_{1,2} & : \quad Q^i_{U(1)} = 0 \\
\phi^{ij}_{27U} & : \quad Q^i_{U(1)} = 2/3 \\
\phi^{27T} & : \quad Q^i_{U(1)} = 2/9 \\
\phi^{1T} & : \quad Q^i_{U(1)} = -4/9. \\
\end{align*}
\]
Now comparing eqs. (21), (24) and (25), we recognized that the charges obey the following relation

\[ Q_{ST}^i = Q_{U(1)}^i + Q_{Hol}^i. \]  

(26)

For the fermions, an additional phase is involved. Thus an ST duality transformation acts like a linear combination of an enhanced U(1) gauge transformation and a U(1) gauge holonomy transformation. Moreover, the ST duality transformation acts like an R-symmetry. Equation (26) becomes clear when the left-moving supercurrent \( S_L \) is investigated. It has charges \( Q_{U(1)} = 2/3, Q_{Hol} = -2/3 \). Specifically, the vertex operator of \( S_L \) in the covariant lattice basis has the form

\[ S_L \sim \sum_{i=1}^{3} \left[ \exp \left( i\sqrt{2}Y_1^i \right) + \exp \left( i(-\frac{1}{\sqrt{2}}Y_1^i + \frac{3}{\sqrt{6}}Y_2^i) \right) + \exp \left( i(-\frac{1}{\sqrt{2}}Y_1^i - \frac{3}{\sqrt{6}}Y_2^i) \right) \right] \exp(-iH_L^i). \]

Then, the requirement that \( Q_{ST} = 0 \) for \( S_L \) implies eq.(26).

It is also interesting to consider the overall target space modular transformations, i.e. simultaneous transformations on all \( T_i \) with \( a_1 = a_2 = a_3 \) etc. With eq.(26) it is easy to see that an overall ST transformation acts exactly like a linear combination of all \( U(1)^9 \) gauge transformations.

Note that the \( \mathbb{Z}_3 \) orbifold can be equivalently constructed by tensoring together nine \( c = 1, N = 2 \) superconformal models. Using this method [15], one always constructs the theory at the multicritical point \( T_i = T_c \) with enhanced \( U(1)^9 \) gauge symmetry. In this case, the massless bosonic states of the model may be labelled by giving the nine \( q \)-charges of the nine chiral states with \( (l, q, s) = (1, 1, 0) \) appearing in the tensor product. For example (see e.g. ref.[16]), the untwisted \( 27 \)'s correspond to charges \( (q_1, q_2, \ldots, q_9) = (1, 1, 1; 0, 0, 0; 1, 1, 0; 0, 0, 0), (0, 0, 0; 1, 1, 1; 0, 0, 0), (0, 0, 0; 0, 0, 0; 1, 1, 1), \ldots \). Twisted \( 27 \)'s correspond to states labelled \( (1,0,0; 1,0,0; 1,0,0), \) where the underlining indicates permutations. We have grouped the charges in sets of three to explicitly show the correspondence of each set of three factors with one complex dimension. The symmetry assignments of each massless state with respect to the \( \mathbb{Z}_{k+2} \) symmetries of the model are obtained by computing the scalar product of vectors of the form \( \gamma = (\gamma_1, \ldots, \gamma_9) \) \( (\gamma_i \text{ integers}) \) with the \( (q_1, \ldots, q_9) \) vectors of each state using a diagonal metric with \( g_{ij} = -\delta_{ij}/(k+2) \), \( i, j = 1, \ldots, 9 \). In particular, we find that the action in the first complex plane of the ST generator on the massless bosonic fields of the \( \mathbb{Z}_3 \) orbifold (eq.(21)) is identical to the symmetry generated by \( \gamma = (1, 0, 0; 0, 0, 0; 0, 0, 0), \) as the reader may easily check.
The above fact suggests that, at least for Gepner models \[15\] of the type \((k = n)^m\) (\(m\) identical copies of a \(k = n\) minimal model), the duality generator that leaves invariant the multicritical point corresponds to the \(Z_{k+2}\) symmetry generated by symmetries of the type \(\gamma = (1, 0, \ldots, 0)\).

Now let us discuss the target space modular transformations and their relation to the \(U(1)\) gauge transformations within the 4-dimensional effective field theory of the orbifold compactified heterotic string. The kinetic energies of the moduli \(T_i\) and the chiral “matter” fields \(A_s\) are determined by the Kähler potential of the following (tree level) form:

\[
K = -3 \sum_{i=1}^3 \log(T_i + \overline{T}_i) + 3 \prod_{i=1}^3 (T_i + \overline{T}_i)^{n_i^s} |A_s|^2. \tag{27}
\]

Invariance of the matter kinetic energies under target space modular transformations requires that the chiral superfields as well as their bosonic components (we denote them by the same symbol \(A_s\)) transform like (up to constant matrices and phases):

\[
A_s \rightarrow \prod_{i=1}^3 (ic_i T_i + d_i)^{n_i^s} A_s. \tag{28}
\]

Therefore we call the numbers \(n_i^s\) the modular weights of the matter fields. The modular weights \(n_i^s\), i.e. the kinetic energies of the matter fields, were previously computed \[17\], \[7\], \[18\] by comparing string calculations with the effective Lagrangian, and for the matter fields of the \(Z_3\) orbifold the result is given in eq.(19).

The transformation behaviour of the fermionic components of the chiral fields in the supergravity Lagrangian follows from the action of the target space modular group on the Kähler potential. Specifically \(K\) transforms with a Kähler transformation like

\[
K \rightarrow K + \Lambda + \overline{\Lambda}, \quad \Lambda = \sum_{i=1}^3 (ic_i T_i + d_i). \tag{29}
\]

Then it follows \[19\] that the fermions \(\psi_{A_s}\) transform with an additional Kähler phase as

\[
\psi_{A_s} \rightarrow e^{\frac{i}{4}(\Lambda - \overline{\Lambda})}(ic_i T_i + d_i)^{n_i^s} \psi_{A_s} = \left[\frac{-i c_i \overline{T}_i + d_i}{i c_i T_i + d_i}\right]^{-1/4} (ic_i T_i + d_i)^{n_i^s} \psi_{A_s}. \tag{30}
\]
Likewise, the gauginos $\psi_\lambda$ transform as

$$\psi_\lambda \to e^{\frac{1}{4}(\Lambda-\bar{\Lambda})} \psi_\lambda = \left[ \frac{-ic_i \bar{T}_i + d_i}{ic_i T_i + d_i} \right]^{1/4} \psi_\lambda.$$  \hspace{1cm} (31)

It is now easy to show that the transformation rules of the fields $A_s$ in the supergravity Lagrangian agree with the rules we have obtained in the string theory by examining the corresponding vertex operators. In the string basis, the fields $\phi_s$ are represented by vertex operators, which create normalized states with canonical kinetic energies. Therefore to relate the string fields with the supergravity fields, one has to perform the following non-holomorphic field redefinition (except for the moduli fields $T_i$, see [10]):

$$\phi_s = \prod_{i=1}^{3} (T_i + \bar{T}_i)^{n_i/2} A_s.$$  \hspace{1cm} (32)

Then, using eq. (28), we immediately obtain the correct field-dependent phase eq. (18) of the string vertex operators. Moreover, the string theory also provides the information about the field-independent phases and matrices, which cannot be obtained by considering the effective supergravity Lagrangian. Analogously, the Kähler phase $e^{\frac{1}{4}(\Lambda-\bar{\Lambda})}$ just corresponds to the non-trivial modular transformation behaviour, eq. (17), of the space-time supercharge in the string basis.

Let us now show that also within the effective supergravity Lagrangian, the $ST$-modular transformations act like a linear combination of field-independent $U(1)$ gauge transformations plus, for the fermions, a constant Kähler phase on the chiral fields. To achieve this, one has to perform a field redefinition to a new supergravity field basis, which allows to couple the charged chiral fields to the $U(1)$ vector gauge fields. Specifically, for the moduli fields $T_i$ one has to perform the following holomorphic field redefinition [10]:

$$\tilde{T}_i = \frac{T_c - T_i}{T_c + T_i}, \quad T_c = -ie^{2\pi i/3}.$$  \hspace{1cm} (33)

Then $\tilde{T}_i$ transforms under the $ST$ transformation $T_i \to \frac{1}{T_i - T}$ as

$$\tilde{T}_i \to e^{4\pi i/3} \tilde{T}_i.$$  \hspace{1cm} (34)

Thus $\tilde{T}_i$ transforms under $ST$ exactly like the vertex operator $\phi_{T_i}$ at the critical point $T_c$, i.e. $ST$ acts on $\tilde{T}_i$ like a $U(1)$ gauge transformation. In fact in the literature about
modular functions (see e.g. [20]) the variable $\tilde{T}^3_i$ is nothing else than the uniformizing variable around the elliptic fixed point $T_c = -ie^{2i\pi/3}$ of the element $\gamma = ST$. In general, the uniformizing variables are conveniently used to expand meromorphic functions $F$ around the fixed points of modular transformations. A function $F$ is single valued if it can be expressed in terms of integer powers of $\tilde{T}^3_i$. This is the so-called uniformization.

Similarly, the matter fields have to be redefined as follows

$$\tilde{A}_s = \left[ \frac{\sqrt{T_c + T_c}}{T_c + T} \right]^{-n_s} A_s.$$ (35)

Then the matter fields transform under $ST$ with a constant phase like displayed in eqs.(20) and (21). For the fermions similar field redefinitions can be performed and one obtains that the fermions transform with an additional constant phase, which shows that $ST$ acts like an $R$-symmetry in the supergravity Lagrangian.

The Kähler potential in the new field basis has the following form:

$$\tilde{K} = -\sum_{i=1}^3 \log(1 - |\tilde{T}_i|^2) + \prod_{i=1}^3 (1 - |\tilde{T}_i|^2)^{n_s} |\tilde{A}_s|^2.$$ (36)

Note that in this field basis the Kähler gauge function is a purely imaginary number, $\tilde{\Lambda} = -4i\pi/3$. Therefore $ST$ does not act on $\tilde{K}$.

It is obvious that the Lagrangian can now be made locally gauge invariant under the $U(1)^9$ gauge symmetry by the gauge covariant replacement $\tilde{A}_s \rightarrow \tilde{A}_s \exp(\sum_{a=1}^9 Q^a_s V_a)$ (here $\tilde{A}_s$ also includes $\tilde{T}_i$), where $V_a$ are the $U(1)^9$ vector fields and the $Q^a_s$ are the corresponding charges. Since the moduli fields $\tilde{T}_i$ are charged under $U(1)_1^i \times U(1)_2^i$ (see eq.(24)), non-vanishing vacuum expectation values of these fields spontaneously break these $U(1)$ symmetries, and the corresponding gauge bosons become massive. (See refs.[9],[10] for details). (Non-vanishing vacuum expectation values of twisted moduli also break the two further $U(1)$’s, which are linear combinations of $U(1)^9_{Hol}$. Moreover, in general there are also moduli that are charged under $E_6$.) However an inspection of the $U(1)$ charges of the various fields shows that $U(1)_1^i \times U(1)_2^i$ is not broken completely, but the Lagrangian is still invariant under a discrete gauge symmetry $\mathbb{Z}_3 \times \mathbb{Z}_3$. Consider for example the group $U(1)^i$ defined by eq.(23), with charges as displayed in eq.(24). Since all untwisted fields, including the
symmetry-breaking field $\tilde{T}_i$, have charge 2/3, whereas all twisted fields have charges of units 1/9, a discrete $\mathbb{Z}_3$ symmetry remains unbroken. Untwisted fields are neutral under this discrete gauge symmetry, whereas twisted fields have $\mathbb{Z}_3$ charges 1/3, 2/3. This result has to be compared with the discrete $\mathbb{Z}_3$ group generated by the modular element $ST$. Looking at the $ST$ charges of all fields, eq.(21), we see that the $ST$ discrete group cannot be identified with the discrete gauge group discussed above, since the symmetry-breaking field $\tilde{T}_i$ is not inert under $ST$.

Finally let us mention the interesting possibility [21] that the discrete gauge groups are anomalous. In this case the corresponding anomaly of the underlying $U(1)$ must be cancelled by the Green-Schwarz mechanism, i.e. by a non-trivial gauge transformation of the universal axion field. Similarly target space duality transformations, which involve an additional Kähler phase for the fermions, may be anomalous [4],[5],[6],[7]. Then the axion transforms non-trivially under modular transformations. However note that there is no direct relationship between anomalous discrete gauge symmetries [21] and anomalous target space modular transformations [4],[5],[6],[7]. In fact, for the $\mathbb{Z}_3$ orbifold, looking at the massless spectrum target space modular transformations are anomalous, whereas the enhanced $U(1)$ symmetries and thus the discrete gauge symmetries are anomaly-free.

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