THE EVOLUTION OF VISCOUS INCLINED DISKS IN AXISYMMETRIC AND TRIAXIAL GALAXIES

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ABSTRACT

We have used a set of equations developed by Pringle to follow the evolution of a viscous twisted disk in a galaxy-like potential that is stationary or tumbling relative to inertial space. In an axisymmetric potential, the disk settles to the equatorial plane at a rate largely determined by the coefficient $v_x$, associated with shear perpendicular to the local disk plane. If the disk is initially close to the galaxy equator, then the rate at which the inclination decays is well described by the analytic formula of Steiman-Cameron & Durisen; in a highly inclined disk, “breaking waves” of curvature steepen as they propagate through the disk, rendering the numerical treatment untrustworthy. In a triaxial potential that is stationary in inertial space, settling is faster than in an oblate or prolate galaxy, since the disk twists simultaneously about two perpendicular axes. If the figure of the potential tumbles about one of its principal axes, the viscous disk can settle into a warped state in which gas at each radius follows a stable tilted orbit, which precesses in such a way as to remain stationary relative to the underlying galaxy.

Subject heading: accretion, accretion disks — galaxies: kinematics and dynamics — galaxies: structure

1. INTRODUCTION

Warped and twisted disks are common in the galactic context. In most spiral galaxies, the neutral hydrogen disk is warped out of the plane of the central galaxy, often by tens of degrees. Models for galactic warps have been primarily gravitational in nature, ignoring gas processes within the disk (see, for a review, Binney 1992). Early-type galaxies usually contain little gas, but when gas is present, it is often found in fairly ordered disks or rings, orbiting at a large angle to the apparent symmetry planes of the underlying galaxy; the polar rings cataloged by Whitmore et al. (1990) are striking examples. On a smaller scale, the centers of galaxies often contain warped or tilted gas disks: within the central kiloparsecs of our own Milky Way, H I and molecular gas follow orbits tipped with respect to the outer disk (e.g., Burton 1992); H II regions in the bulge of M31 occupy a tilted disk (Ciardullo et al. 1988); and a number of barred galaxies, including NGC 2217 (Bettoni, Fasano, & Galletta 1990), have warped and twisted central disks of gas and dust.

These gas disks generally appear thin and have a regular velocity structure, which implies that they have settled dissipatively at least partway toward an equilibrium state. Their evolution presents a problem in three-dimensional hydrodynamics, so that all theoretical investigations necessarily involve many approximations. Christodoulou & Tohline (1991) adopted a hydrodynamic approach to gas disks in an oblate potential; even with a fairly coarse computational grid, their simulations required the use of a supercomputer. Particle-based techniques, such as the smoothed-particle scheme developed by Lucy (1977), allow high resolution where material is present, while avoiding the waste of machine time in following empty regions of the computational volume. Such schemes have been used by Tubbs (1980), Hernquist & Katz (1989), Varnas (1990), and Katz & Rix (1992) to follow the time development of gas disks in oblate and prolate potentials, while Habe & Ikeuchi (1985, 1988) used them to study the settling of disks in prolate and triaxial potentials. In these schemes, the time step is limited by the need to follow the orbital motion of individual fluid particles, while the inflow and settling perpendicular to the local disk plane are much slower. Steiman-Cameron & Durisen (1988) developed a system of analytic equations that describes the time development of a twisted viscous gas disk, assuming the disk to be in centrifugal balance and thus removing the need to integrate the fast orbital motion, which they used to investigate the timescale on which an initially tilted gas disk would settle to the equatorial plane of an oblate galaxy. Such an orbit-averaging method has the limitation that it can follow only motions that are slow compared to the orbital velocities, and could not treat the collapse and rapid infall of a gas disk.

Investigations of the time development of twisted galactic disks are also hampered by our ignorance of the nature of the dissipation. Molecular viscosity is negligible, and the complex dissipative processes in the interstellar medium cannot be simply characterized. In modeling the disk dynamics, one generally develops a simple description of viscosity as a process that dissipates energy and causes diffusion of momentum, and one hopes that the results are at least qualitatively independent of the details of that process. In the context of planetary rings, we know that this hope is false—some forms of dissipation make the rings spread, and others cause them to become narrower (see, e.g., § 5.3.2 of Goldreich & Tremaine 1982; Wisdom & Tremaine 1988). Comparing the predictions of the various numerical schemes provides a test of whether the effects found in simulations are dependent on the form of viscosity used; behavior found in common may be generic and indicative of what is to be expected in real galaxies. Christodoulou et al. (1992)
make such a comparison of hydrodynamic and particle-based simulations, and note various points of agreement.

Pringle (1992) has recently developed an orbit-averaging scheme in which gas inflow and twisting are controlled by two viscous coefficients, the first controls forces depending on shear in the plane of the disk, while the second controls forces perpendicular to the local disk. Pringle’s formulation is somewhat more simple than that of Steinman-Cameron & Durisen (1988), and it has the additional advantage that it explicitly conserves local angular momentum. Here we apply Pringle’s equations in discretized form to investigate the time evolution of warped and tilted gas disks in axisymmetric and triaxial galactic potentials. We calculate rates of settling into equilibrium states, examine the inflow associated with settling, search for long-lived warped states, and compare our results with those of hydrodynamic and particle-based simulations. The equations are briefly described in § 2 below. Section 3 then considers the settling of a gas disk initially tilted away from the symmetry plane of an axisymmetric potential. In § 4, we discuss triaxial potentials held stationary in inertial space. In § 5, we discuss tumbling triaxial potentials. Section 6 summarizes our results and conclusions.

2. EQUATIONS OF MOTION

Following Pringle (1992), we approximate a vertically thin disk of centrifugally supported material by a set of discrete circular rings, radius $R$, with angular momentum pointing along the unit normal $l(R, t)$ of the ring. When $l$ varies with radius, the rings are inclined and azimuthally twisted relative to each other; we shall refer to this system as a “twisted disk.” We assume that the orbital angular velocity $\Omega(R)$ around the orbit is set by the gravitational potential of an underlying mass distribution and that the disk contributes negligibly to the gravitational forces. The annulus of material between radii $R - \Delta R/2$ and $R + \Delta R/2$, therefore, has angular momentum $d\Delta l \equiv 2\pi R \Sigma \Delta R R^2 \Omega(R)$, where $\Sigma(R, t)$ is the surface mass density of the disk. The evolution of the angular momentum density $L \equiv \Sigma R^2 \Omega$, under the influence of viscosity and of an external torque $N(R, L, t)$, is then given by Pringle’s equation (2.6):

$$\frac{\partial}{\partial t} L(R, t) = \frac{1}{R} \frac{\partial}{\partial R} \left\{ \frac{\{l \Sigma \Omega \} \{R^2 \Omega \}}{\Sigma (\partial l / \partial R)(R^2 \Omega)} \right\}$$

$$+ \frac{1}{2} \frac{\partial}{\partial R} \left\{ \frac{l v_1 R}{l^2} \right\}$$

$$+ \frac{1}{2} \frac{\partial}{\partial R} \left\{ \frac{l v_1 R^2}{l^2} (\partial l / \partial R) \frac{l}{\partial l} \right\}$$

$$+ \frac{1}{2} \frac{\partial}{\partial R} \left\{ \frac{l v_1 R}{l^2} \right\}$$

$$+ N(R, L, t),$$

where $\Omega \equiv d\Omega/dR$. The two coefficients, $v_1$ and $v_2$, correspond to viscosity acting on the azimuthal and vertical shear, respectively; thus, $v_1$ is the shear viscosity normally associated with planar accretion disks, while $v_2$ controls the damping of out-of-plane motions associated with a twist in the disk. Since $\Omega(R)$ is given, the surface density $\Sigma(R, t)$ is then specified by the magnitude of $L$.

This first-order equation clearly conserves angular momentum when external torques are absent. The derivation assumes that the angular momentum of each ring lies entirely along the normal to the orbital plane, so equation (1) describes only the slow precession of the disk and does not follow fast nutational motions. The rate of change of angular momentum due to the external torque $N$ should be small enough not to violate this assumption. When $\partial l / \partial R = 0$, the equation reduces to the standard form for a flat disk (e.g., Pringle 1981). The first two terms are diffusive, while the third term is advective, with the advective velocity $V_{adv}$ given by

$$V_{adv} = \frac{1}{2} \frac{v_2 R^2 \Omega}{(\partial l / \partial R)(R^2 \Omega)} - \frac{v_1 \Omega}{\Omega}.$$  

(2)

If the disk is flat or if $v_2 = 0$, then $V_{adv}$ is positive (angular momentum is advected outward); it becomes negative when the curvature $d\partial / dR$ is strong.

To discretize equation (1), we follow Pringle (1992) in using a first-order explicit scheme that treats the Cartesian $(x, y, z)$ components of the angular momentum separately, spacing our grid points evenly in radius. The standard forward time centered space differencing is used for the two diffusive terms, with zero-torque boundary conditions at the inner and outer edges of the radial grid, as outlined by Pringle, in such a way that angular momentum is conserved exactly under the action of the first two terms of equation (1). The third, advective, term is treated by upstream differencing (e.g., Press et al. 1992), in such a way that angular momentum is lost or gained if $V_{adv}$ changes sign in the middle of the grid; this can be a problem in the settling disks that we study here. The zero-torque boundary conditions require $d\partial / dR = 0$ at the inner and outer edges of the disk, so the advective velocity is always positive there; angular momentum (and therefore mass) flows outward off the grid at a rate proportional to the local density. Therefore, we choose the outer boundary to lie well beyond most of the mass of the disk. The time step $\Delta t$ must be chosen short enough to satisfy the Courant condition on the diffusive terms:

$$\Delta t < (\Delta R)^2 \max \left\{ \frac{v_1 R^2 (-\Omega)}{(\partial l / \partial R)(R^2 \Omega)}, \frac{1}{2} v_2 \right\}^{-1}.$$  

(3)

Thus, the integration scheme is mathematically simple, but computationally it is relatively inefficient. It is not at all clear what the appropriate viscosity for a galactic gas disk should be. An estimate may be made by considering a disk made up of gas clouds, which collide with mean free path $\lambda$; the coefficients of kinematic viscosity are then given by $v \sim \lambda v_r$, where $v_r$ is the one-dimensional cloud velocity dispersion (e.g., Shu 1992). The disk is thin, so that if the cloud velocity dispersion is close to isotropic, random speeds are well below the local circular velocity: $v_r \ll R \Omega(R)$. The maximum distance they can move between collisions is then given by the epicycle amplitude, $\lambda < v_r / \kappa$, where $\kappa$ is the epicyclic frequency (Binney & Tremaine 1987). When the rotation curve is flat, with constant linear speed $V$, $\kappa = 2^{1/2} V / R$. This sets an upper limit to the velocity $v \leq v_{max} \sim v_{rms} R / V^{1/2}$, where $v_{rms}$ is now the three-dimensional cloud velocity dispersion; with Steinman-Cameron & Durisen (1988), we adopt

$$v_{max} = 0.2 v_{rms} R / V.$$  

(4)

We choose units in such a way as to measure lengths in kiloparsecs and velocities in units of 100 km s$^{-1}$; therefore, our time unit is about $10^7$ yr. Typical velocity dispersions in the cold H I layer of our Galaxy and others are about 5–10
km s\(^{-1}\) (Burton 1992). Taking \(V_{\text{ms}} = 10\ \text{km s}^{-1}, R = 5\ \text{kpc}\), and a rotation speed of 250 km s\(^{-1}\), the viscosities \(v_1\) and \(v_2\) should not exceed about \(4 \times 10^{-5}\) in our units, or 0.4 kpc km s\(^{-1}\).

3. OBLATE OR PROLATE GALAXY POTENTIAL

In an axisymmetric galaxy potential, the precessional torque may be written as where the unit vector \(z\) points along the polar axis of the galaxy. The scale-free models of Richstone (1980; see also Toomre 1982) describe axisymmetric potentials in which the linear speed \(V\) of orbital motion is independent of radius: \(\Omega(R) = V/R\). Richstone's model, which Toomre labels by \((n = 0, m = 1)\), has a potential \(\Phi\) that constant on concentric similar spheroids: at (spherical) radius \(R\) and height \(z\) above the equatorial plane,

\[
\Phi(R, z) = \left(\frac{V^2}{2}\right) \ln \left[ \frac{R^2}{R_0^2} \left( 1 + \eta P_2 \left( \frac{z}{R} \right) \right) \right] \\
\approx \left(\frac{V^2}{2}\right) \ln \left[ \frac{R}{R_0} + \frac{\eta}{2} + \frac{P_2}{2} \left( \frac{z}{R} \right) \right],
\]

where \(P_2\) is the Legendre polynomial and the parameter \(\eta\) controls the asphericity. Positive \(\eta\) corresponds to an oblate mass distribution and \(\eta < 0\) to a prolate system, and the approximate equality holds when \(\eta\) is small. The scaling radius \(R_0\) does not appear in the expressions for forces and torques. Averaging over a uniform circular ring at radius \(R\), with inclination \(i = \arccos (i \cdot z)\) to the equatorial plane, yields a torque per unit mass

\[
N = -\frac{3V}{4R} \eta (z \cdot \mathbf{t}) (z \times \mathbf{L}).
\]

If viscosity is absent, then this will cause the orbits of material at radius \(R\) to precess about the \(z\)-axis at the rate

\[
\Omega_p(R, t) = -\frac{3V}{4R} \eta (z \cdot \mathbf{t})
\]

(cf. Sparke 1986). If \(i \propto R\), then the entire warped disk precesses rigidly without change of shape.

The smoothed particle hydrodynamic simulations of Katz, as reported in § 2.2 and Figure 1 of Christodoulou et al. (1992), followed a near-polar ring initially inclined by 80° to the equatorial plane of both an oblate and a prolate potential. In the oblate potential, the ring locked into a steadily precessing state, warping down toward the equatorial plane with inclination approximately following the relation \(\cos i \propto R\). This warped state persisted for at least 30 orbital periods. In the prolate potential, the ring behaved very differently: it initially developed a warp up toward the pole, after which the material collapsed within five rotation periods into the center. The addition of self-gravity to the ring material did not change these results.

We can compare these results with the predictions of equation (1). In an axisymmetric galaxy, the precession-inducing torque has no component along the axis \(z\). Under the interchange \((N, x, y) \rightarrow (\theta, N, -x, -y)\), the \(z\)-component of equation (1) for \(\partial L/\partial t\) is unchanged, while the \(x\) and \(y\) components reverse their signs. Thus, the inclination \(i\) must evolve in a prolate potential exactly as a disk with twist angles mirror reflected through the \(z\)-axis would evolve in the corresponding oblate potential with equal and opposite flattening. In particular, if the disk evolves to a steadily precessing state in an oblate potential, it should reach a corresponding steady state in a prolate potential, with the same inclination to the \(z\)-axis, but with reversed sense of precession and azimuthal twist. The effective viscous interaction in the computations described by Christodoulou et al. (1992) is a complex mixture of “pressure” forces and cooling. The fact that those calculations show a difference between the ring evolution in oblate and prolate galaxy potentials, while equation (1) insists that the behavior must be the same but for a reversal of helicity, implies that this aspect of the disk evolution may be sensitive to the particular form of viscosity chosen.

The inclination, the surface density, the speed of radial inflow \(V_r\) calculated according to equation (2.3) of Pringle (1992), and the advective speed \(V_{adv}\) are shown in Figure 1a for an annulus of material settling from an initial inclination of 30° in a potential with \(\eta = 0.2\). The density \(\Sigma(R)\) was initially given by a Gaussian, truncated at the inner and outer edges, which implies that the radial speed \(V_r\) is not smooth at the boundaries. We also tried runs where \(\Sigma(R)\) is modified so that \(V_r\) approaches zero smoothly at the edges, which requires the derivative of \(\Sigma(R)R^2 \partial V_r/\partial R\) to go to zero there. Our results were almost identical, and since the condition required for smooth \(V_r\) does not seem physically natural, we have used the truncated Gaussian form for \(\Sigma(R)\).

As seen in Figure 1a, the innermost rings settle fastest toward the equatorial plane. The nonmonotonic behavior in the inner edge is a consequence of the stress-free boundary condition: the ring at the edge feels a torque only from the penultimate ring and therefore takes longer to settle than rings that experience torques from material on both sides. Inflow is slow: both the radial and advective speeds are less than 1% of the circular speed in the disk. For \(r > 0\), the minima of \(V_r\) reflect those of \(V_{adv}\), which occur where the disk is actively settling and \(d\ell/\partial R\) is appreciable; inflow is most rapid in these regions. The advective speed is positive in the inner, settled part of the disk but becomes negative farther out, so we lose about 2% of the mass and 1% of the \(z\)-angular momentum during the run. As expected, a run identical except for the sign of \(\eta\) gives identical results apart from a reversal of all the azimuthal angles.

To test the influence of our boundary conditions, we computed the settling of an annulus narrow enough so that the density at the edge was extremely low, only \(2 \times 10^{-21}\) of its value in the middle of the annulus. Figure 1b shows the inclination and surface density for a computation identical to that of Figure 1a, except that the surface density is given initially by a truncated Gaussian that is 3 times narrower. Gas flows inward more quickly in response to the steep density gradient, so the decrease in inclination is somewhat slower than that in Figure 1a. However, the very similar nature of the settling shows that our results are not strongly influenced by effects at the boundaries.

Figure 2 shows the deviation of the azimuth of the line of nodes in the computation of Figure 1a, from that expected for a freely precessing disk with an inclination of 30°. At any given time, the orbits of material near the inner edge of the disk have regressed somewhat less than they would have in the absence of viscosity because matter has flowed in from larger radii where regression is slower. In the outermost region, where material has moved outward, more regression has taken place than if viscosity had been absent. But the deviations from free precession are small, consistent with
Fig. 1.—Settling of an initially coplanar disk inclined at 30° to the equator of an oblate galaxy potential corresponding to \( V = 250 \text{ km s}^{-1} \), \( \eta = 0.2 \); the viscous coefficients \( \nu_1 = \nu_2 = 10^{-3} \), and the computational grid contain 100 rings equally spaced in radius. Time \( t = 600 \) at the end of the run corresponds to \( 6 \) Gyr, and the time step was 0.01, corresponding to \( 10^5 \) yr. (a) Top left panel shows the inclination, which at each radius decreases with time; top right panel shows the density; lower left panel shows \( \nu_1 = \nu_2 = 10^{-3} \); shows \( V_{\text{adv}} \). In the top right panel, the inclination is constant, the surface density is a Gaussian in radius, and both \( V_R \) and \( V_{\text{adv}} \) are given by straight line segments. (b) Inclination and surface density for an otherwise identical run starting with a narrower annulus of gas.
At steps 1–5 of the computation shown in Fig. 1a, the fractional difference $\Delta \omega$ between the azimuth $\omega$ of the line of nodes of the disk at radius $R$ and that expected in free precession at an inclination of $30^\circ$: $\Delta \omega(t) \equiv (\omega - \Omega_p t)/\omega$. We have set $\Delta \omega(0) = 0$.

$\Omega_p = -0.75 \eta \Omega(r) \cos 30^\circ$

Fig. 2.—At steps 1–5 of the computation shown in Fig. 1a, the fractional difference $\Delta \omega$ between the azimuth $\omega$ of the line of nodes of the disk at radius $R$, and that expected in free precession at an inclination of $30^\circ$: $\Delta \omega(t) \equiv (\omega - \Omega_p t)/\omega$. We have set $\Delta \omega(0) = 0$.

The conclusions reached § IIIb of Steiman-Cameron & Durisen (1988): as long as the viscosity is well below a critical level, here corresponding to $\nu_{\text{crit}} \approx \eta VR = 1.5 - 5$ in our units, viscous torques have little effect on the rate of precession. However, viscosity strong enough to prevent differential precession will cause rapid inflow on the timescale characteristic of that precession.

Figure 3 shows the effect of reducing the in-plane viscosity $\nu_1$ by a factor of 10: settling proceeds much as before, but inflow is much reduced. The sharp dip in both radial and advective speeds reflects a breaking wave instability, which we discuss further below. Here it occurs in a region of the disk that contains very little mass and does not affect settling over the rest of the disk. When the “twisting” viscosity $\nu_2$ is reduced, Figure 4 shows that the inclination decays more slowly, but inflow remains much as in Figure 1a. At this low inclination, the inflow and settling are not strongly coupled.

Habe & Ikeuchi (1985) used a smoothed-particle hydrodynamic code to investigate the settling of an initially inclined annulus of gas in a prolate potential. They found that an annulus initially tilted by $10^\circ$ settled to a vertically
thin ring in the equatorial plane within 10 rotation periods with little infall, but a disk initially inclined at 80° collapsed to one-third of its initial radial extent as it settled. Varnas (1990), also using a smoothed-particle code, found that infall was more rapid in disks with a higher initial tilt, as did Steiman-Cameron & Durisen (1988), using their orbit-averaging equations. Figure 5a, for an initial inclination of 60°, indeed shows stronger infall, with mass building up at the inner edge of the grid. The curves of radial and advective velocity are very similar to each other; radial motion is largely controlled by the local disk curvature $dl/dR$, and strong infall is confined to regions of the disk in which the inclination changes rapidly.

However, we cannot give a reliable estimate of the infall rate. Because $V_{\text{adv}}$ changes abruptly from positive to negative in the middle of the grid, 26% of the initial mass and 5% of the initial z-angular momentum in our calculation is lost. A finer computational grid causes the jump in $V_{\text{adv}}$ to steepen, as shown in Figure 5b, worsens the angular momentum loss, and hastens the infall. We can see from equation (2) how the problem develops: in regions of strong curvature, angular momentum is advected inward at a faster rate, leading to a still greater steepening of the curvature, and a behavior similar to the breaking of water waves (e.g., Whitham 1974). The disk curvature grows to the limit set by the finite resolution of the radial grid, so that with a finer mesh the waves steepen still further and angular momentum conservation worsens. Reducing $\nu_2$ relative to $\nu_1$ does not help; only the $\nu_2$ viscosity can smooth out the twisting caused by differential precession, so that when it is small the disk builds up a large curvature and the same breaking waves are seen. It appears that this set of equations cannot be used to treat the settling of disks through a large angle to the plane about which they precess. We found good results (mass conservation holding to a few percent and improving with a finer grid, and flow quantities converging to limiting values with a finer grid) for initially coplanar disks only while the inclination changes remain below about 40°.

We can compare the settling of our model disks with a simple analytic formula derived by Steiman-Cameron & Durisen (1990) in the limit of weak viscosity. They showed that an initially tilted disk should settle toward the equatorial plane of an oblate galaxy potential approximately according to their

$$i \propto e^{-\left(\frac{t}{\tau_e}\right)^3}, \quad \tau_e^{-3} = \frac{(l/6\Omega_p/dR)^2}{(\epsilon/6\Omega_p/dR)^2}. \quad (8)$$

We recover their formula if we set $\nu = \nu_2$, $\nu_1 = 0$ in equation (1) and assume that the curvature in the ring is small, so that the diffusive term containing $\nu_2$ is much larger than the advective term, which is quadratic in $|dl/dR|$. Furthermore, we must assume that, subsequently, the azimuth...
of the line of nodes of the ring changes much more rapidly with radius than any other quantities, and that the inclination is small in such a way that can be evaluated in the equatorial plane. In that case, \[ \frac{\partial}{\partial t} L(R, t) \approx \frac{v_2}{R} \frac{\partial^2 L}{\partial R^2} + \Omega_{\phi}(R, i = 0) \times L(R, t) . \] (9)

In a freely precessing disk that is initially coplanar, the radial derivative in equation (9) eventually grows as \( (d\Omega_{\phi}/dR)^2 t^2 L \); writing the angular momentum as the product of the freely precessing motion and a slow decay in amplitude of the tilt shows that when the inclination remains small, the off-axis angular momentum \( L \sin i \) decays exponentially according to equation (8). Figure 6 compares the prediction of this equation to the computed decay in inclination for the disk of and for a Figure 1 similar run with both coefficients of viscosity reduced by a factor of 10. The analytic prediction approximates the disk inclination well for low viscosity, but the fit becomes worse at higher viscosity, when shows that appreciable inflow takes place.

Finally, Figure 7 shows the development of a disk from an initial state with a straight line of nodes and inclination given by \( \cos i \propto R \); in the absence of viscosity, this disk would precess rigidly without change of shape. Katz & Rix (1992), using smoothed-particle hydrodynamics to investigate highly inclined gas disks in an oblate galaxy potential, found that, unless the random speeds of the particles were subjected to a strong "cooling," particle collisions were frequent and most of the material sank toward the center. When cooling was included, the gas settled into an annulus warped approximately according to \( \cos i \propto R \), with only a small dispersion in velocity. Since particles in the cooled disk follow near-circular paths, the orbit-averaged equations discussed here ought to give a good representation of this behavior. But instead we find that viscous diffusion is destabilizing; as material diffuses inward and outward, the inclination of the inner disk decreases and that of the outer disk rises, and eventually sufficient differential precession occurs that the disk starts to settle and inflow becomes rapid. As the viscosity is reduced, so the rate of both inflow and settling diminishes. According to the scheme considered here, an initially tilted disk can never settle into a state in which \( \cos i \propto R \); this aspect of the disk behavior may depend on the character of the dissipative scheme used in the computation.

4. TRIAXIAL GALAXY POTENTIAL

To model a triaxial galaxy potential, we include a second torque term of the same form as in equation (6):
Choosing $x$ to lie along the longest axis of the potential and $z$ to lie along the shortest, we have $\eta_x < 0$ and $\eta_z > 0$. In the absence of viscosity, each ring now precesses about either the $x$ or the $z$ axis, depending on its initial orientation. Because the $x$ and $z$ components of $\mathbf{L}$ have the same $X \cdot \mathbf{L}$ dependence on the respective components of $\mathbf{L}$, in free precession

$$\frac{\partial (\mathbf{L} \cdot \mathbf{L})}{\partial t} = 0 ;$$

(11)

the conserved quantity $\mathbf{L} \cdot \mathbf{L}$ can be used to derive the precession trajectories of $\mathbf{L}$, shown in Figure 8 for a model with $\eta_x = -\eta_z$. The extremal values of $\mathbf{L} \cdot \mathbf{L}$ are $(\pm 3V \eta_x/4R)$, when the angular momentum is aligned or antialigned with the $z$-axis, and $(\pm 3V \eta_z/4R)$, when it lies along the $x$-direction. Closed loops of constant $\Omega_\phi \cdot \mathbf{L}$ corresponding to precession about the $z$-axis circle the extremal point at $\mathbf{L} = (0, 0, \pm 1)$; loops around $\mathbf{L} = (\pm 1, 0, 0)$ represent precession about the $x$-axis. The points $(0, \pm 1, 0)$ are saddle points, where no precession occurs—there is no starting inclination for which the ring precesses about the intermediate $y$-axis, since particle orbits circling the inter-

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**Fig. 8**

Disk inclination as a function of radius (solid lines), compared with the prediction of eq. (8), given by the dashed lines. (a) From the computation of Fig. 1a, at $t = 0, 150, 300$. (b) At the same times for a computation with the same parameters, but with viscosity 10 times smaller.
As in Fig. 1a, but with a disk that initially is warped following \( \cos i = \propto R \), in such a way that it would precess freely at the rate \( \Omega_0 = -0.035 \), or \( 3.5 \text{ km s}^{-1} \text{ kpc}^{-1} \) retrograde to the \( z \)-component of the disk angular momentum, in the absence of viscosity. This configuration is destabilized by viscosity. Time \( t = 600 \) at the end of the run, and the viscosity was 0.01. Different line types correspond to equally spaced time intervals: at \( t = 0 \), the inclination is constant, the surface density is a Gaussian in radius, and both \( V_R \) and \( V_{\text{adv}} \) are given by straight line segments.

The initial inclination is 30°, and the azimuth \( \omega \) of the line of nodes of the disk, at which it cuts the \( x-y \) plane, is 45°: \( \Omega_\phi \cdot I \) is equal to 0.625 of its extremal value, and the disk precesses about the \( z \)-axis. The wavy pattern in the inclination reflects the fact that the free precession trajectories of Figure 8 do not maintain a fixed inclination as the azimuth changes; since the precession rate drops with radius, the azimuthal dependence appears as a radial variation. Observations of tilted gas disks, which appear to have been twisted by precession (S0 galaxy NGC 4753 [Steiman-Cameron, Kormendy, & Durisen 1992], NGC 660 [van Driel et al. 1995], and NGC 3718 [Schwarz 1985; see also Cox & Sparke 1996]) would allow one to limit the triaxiality of the potential, since the disk inclination should then depend systematically on azimuth.
Steiman-Cameron et al. (1992) were able to model the warped dust lane in NGC 4753 as a twisted disk in which the inner gas had precessed by almost two complete revolutions relative to the outer material in the oblate potential of the galaxy, with no appreciable settling or noticeable dependence on azimuth. This suggests that the mass distribution in that galaxy is almost axisymmetric.

The rings making up the tilted disk of Figure 9 settle to the x-y plane at a rate somewhat faster than in the oblate potential because the disk is now being twisted about both x and z axes. The mean inclination is fairly well fit by the Steiman-Cameron & Durisen (1990) formula (8), with mean squared precession rate \( (d\Omega_x/dR)^2 \) set equal to 2 times the square of the mean z-precession derivative \( (V_R/4R^2)g_z \) (3), (1) the factor of 2 accounts for twisting due to the x-torque. Inflow speeds are larger than in Figure 1a; again, most of the inflow takes place during the time that the region of the disk that contains most of the mass is actively settling.

Figure 10 shows the development of a disk starting at \( i = 80^\circ, \omega = 45^\circ \), which precesses about the x-axis and eventually settles into the y-z plane. This initial state is somewhat further from the plane into which the disk settles, with \( \Omega_p \cdot \hat{I} \) equal to only 0.455 of the extremal value reached by a disk in the y-z plane. The inflow is correspondingly faster, a hint of the “breaking wave” instability is seen in the early stages of the calculation, and the conservation of mass is somewhat worse than for Figure 9. Equation (8) again gives a good estimate of the rate of settling.

5. TUMBLING TRIAXIAL GALAXY POTENTIAL

If the galaxy potential of § 4 is taken to tumble about the axis z at some rate \( \Omega_z \), then the torque will contain a component normal to the instantaneous direction of the long axis of the potential. We now have

\[
N = \Omega_p \times L = -\frac{3V}{4R} \left[ \eta_z (\hat{x} \cdot \hat{L}) + \eta_z (\hat{z} \times \hat{L}) \right],
\]

with \( \eta_z < 0, \eta_z > 0 \). In free precession with no viscous forces,

\[
\frac{d}{dt} (\Omega_p \cdot \hat{L}) = -\frac{3V}{2R} \eta_z (\hat{x} \cdot \hat{L}) \frac{\partial \hat{x}}{\partial t} + 2\Omega_z \frac{\partial \hat{L}}{\partial t}.
\]
its extremal values correspond to orientations $I$, at which the ring precesses along with the figure of the potential. Habe & Ikeuchi (1988) used their smoothed-particle hydrodynamic code to follow the settling of a tilted gas disk in prolate and triaxial potentials tumbling about the short axis, and we can compare our results with theirs.

The positions of the extrema of $(\Omega_x - 2\Omega_z) \cdot I$ depend on the tumbling rate $\Omega$, which without loss of generality may be taken as positive. When tumbling is slow, $0 < \Omega < 4R/3V_\eta$, the poles $I = (0, 0, \pm 1)$ are both stable fixed points of the precessional motion, while the saddle points, which were at $I = (0, \pm 1, 0) = 0$ in the static potential, now migrate toward the retrograde pole and are at $I_x = 0, I_z = -4R\Omega/3V_\eta$. The stable points at $I = (\pm 1, 0, 0)$ move to

$$I_x = 0, \quad I_z = \cos i = -4R\Omega/\left[3V(\eta_x - \eta_z)\right]; \quad (14)$$

these orientations correspond to the stable anomalous retrograde orbits found by Heisler, Merritt, & Schwarzschild (1982) and by Tohline & Durisen (1982). This is region B of the parameter space defined by Habe & Ikeuchi (1988). When the figure of the galaxy tumbles somewhat faster, so that $4R/3V_\eta < \Omega < 4R/\left[3V(\eta_x - \eta_z)\right]$, the direct pole $I = (0, 0, 1)$ is still stable, but the two unstable fixed points have merged with the retrograde pole, which is now itself unstable. The two stable points given by equation (14) persist until $\Omega$ increases to the point that they merge with the retrograde pole $I = (0, 0, -1)$. At faster tumbling rates, both poles are stable and there are no other fixed points of precession. This regime corresponds to Habe & Ikeuchi's region C.

Figure 11 shows the inclination of a disk with the same initial parameters as Figure 9, tilted by 30° away from the $x$-$y$ plane of a potential with $\eta_x = -0.2, \eta_z = 0.2$, but now tumbling at a rate $\Omega = 0.07$ about the $z$-axis. All the rings making up the disk precess about that axis, and the disk settles into the $x$-$y$ plane in a very similar manner to that in a stationary potential. The behavior is very similar to that in model TA of Habe & Ikeuchi (1988).

Since the inclination $i$ of the stably precessing tilted orbits at a given tumbling rate depends on radius according to equation (14), an inclined disk of material that precesses rigidly along with the tumbling potential must have azimuth $\omega = 90°$ and warp according to $\cos i \propto R$. Van Albada, Kotanyi, & Schwarzschild (1982) have suggested that the warped and twisted minor-axis dust lanes observed in a number of elliptical galaxies, including NGC 5128 (Cen A), can be explained as material that has settled onto this sequence of tilted orbits. Figure 12 shows the development of an initially coplanar disk with $i = 130°$ and azimuth $\omega = -90°$, in the potential of Figure 11, but with the disk
rotation now retrograde to the figure tumbling. The rings making up this disk indeed settle to the warped shape that precesses along with the potential. The large downward spike in $V_r$ marks the point at which the line of nodes of the disk twists by a complete revolution as it settled. The rate of settling into this stable warped state is approximately the same as for the disk settling to the equatorial plane. Our results are very similar to those found by Habe & Ikeuchi (1988) for their model TF.

We did not run a simulation comparable to model TB of Habe & Ikeuchi (1988), in which the potential tumbles so fast that the gas disk lies outside the corotation radius, so that the rotation period of orbits in the disk exceeds that of the tumbling figure. Analysis of single-particle orbits beyond corotation (e.g., Mulder 1983) reveals a family of stable prograde orbits tilted away from the $x$-$y$ plane, analogous to the slowly precessing retrograde family of van Albada et al. (1982). But the total angular momentum of a ring of gas following one of these prograde orbits would be dominated by the precessional contribution—in conflict with the approximation used in deriving equation (1)—that the angular momentum of the disk material is nearly aligned with the local normal to the surface. Thus, our computational scheme is unsuitable for checking Habe & Ikeuchi’s (1988) finding that a tilted prograde disk initially stationary in inertial space did not settle onto Mulder’s family of stable orbits.

6. SUMMARY

We have used a set of equations developed by Pringle (1992) to follow the evolution of a viscous twisted disk in a galaxy-like potential that is stationary or tumbling relative to inertial space. These equations represent the result of averaging viscous torques and the gravitational force from the underlying galaxy over the orbit of each gas element in the disk. In an axisymmetric potential, we find that the disk settles to the equatorial plane at a rate determined largely by the coefficient $v_2$ associated with shear perpendicular to the local disk plane. If the disk is initially close to the galaxy equator, then the settling depends on viscosity and the rate of differential precession in a way that is well described by an analytic formula given by Steiman-Cameron & Durisen (1990). In a highly inclined disk, however, breaking waves of curvature steepen as they propagate through the disk, up to the maximum gradient allowed by the radial grid. Angular momentum and mass are lost from the calculation where the advective speed $V_{adv}$ changes sign, and the computed curvature does not approach a smooth limit as the grid is made finer.

The predictions of Pringle’s (1992) orbit-averaged equa-
Inclination to the $z$-axis, surface density, inflow speed $V_R$, and azimuth of the line of nodes, relative to the longest axis of the tumbling potential, for a disk initially inclined at $50^\circ$ to the $x$-$y$ plane and rotating retrograde with respect to the tumbling of the galaxy. Time $t = 600$ at the end of the run, and the time step was 0.01. Different line types correspond to five equally spaced time intervals: at $t = 0$, the inclination is constant, the surface density is a Gaussian in radius, $V_R$ is given by straight line segments, and the relative azimuth is $\varphi = 90^\circ$.

The different behavior of viscous disks in oblate as opposed to prolate potentials found by Christodoulou et al. (1992) and Katz & Rix (1992) may well depend on aspects of their dissipative scheme that are not reflected in the orbit-averaged equations. This is surprising, since the particles in those calculations follow nearly circular orbits, in such a way that one would expect orbit-averaging to give a good representation of the dynamics. In particular, we find that viscosity cannot lock a tilted ring into a steadily precessing warped state with inclination $i$ given by $\cos i \propto R$. This configuration is neutrally stable for a freely precessing disk, but according to the orbit-averaged scheme, it is destabilized by viscosity.

In a triaxial potential, which is stationary in inertial space, stable orbits exist about both the longest and shortest axes, but not about the intermediate axis; an orbit that is not in one of the symmetry planes must precess about either the long or the short axis. We find that, as expected, an inclined disk settles into a plane perpendicular to that about which the individual gas orbits precess. The inclination at each radius of a precessing disk must vary with azimuth as the ring precesses. This could be useful as a diagnostic for triaxiality in the potentials of galaxies such as NGC 4753 (Steiman-Cameron et al. 1992), in which a tilted gas disk has been strongly twisted by precession. We also find that the disk settles somewhat faster than in an axisymmetric galaxy, since it twists simultaneously about two perpendicular axes.

If the figure of the potential tumbles about one of its principal axes, then there is a stable tilted orbit at each radius (up to some maximum), precessing so as to remain stationary relative to the underlying potential. In a system tumbling about the shortest axis, these orbits have angular momentum that is retrograde with respect to the tumbling of the figure. We find that an initially prograde viscous disk in such a potential settles into the plane normal to the short axis, while an initially retrograde disk settles toward a warped shape composed of orbits that at each radius precess so as to remain stationary relative to the underlying galaxy. The time required to settle into such a warped state...
is approximately the same as for settling to a principal plane of the potential.

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