The Strategic Perceptron

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Abstract

The classical Perceptron algorithm provides a simple and elegant procedure for learning a linear classifier. In each step, the algorithm observes the samples position and label and may update the current predictor accordingly. In presence of strategic agents, however, the classifier may not be able to observe the true position but a position where the agent pretends to be in order to be classified desirably. Unlike the original setting with perfect knowledge of positions, in this situation the Perceptron algorithm fails to achieve its guarantees, and we illustrate examples with the predictor oscillating between two solutions forever, never reaching a perfect classifier even though one exists. Our main contribution is providing a modified Perceptron-style algorithm which finds a classifier in presence of strategic agents with both ℓ2 and weighted ℓ1 manipulation costs. In our baseline model, knowledge of the manipulation costs is assumed. In our most general model, we relax this assumption and provide an algorithm which learns and refines both the classifier and its cost estimates to achieve good mistake bounds even when manipulation costs are unknown.

1 Introduction

Growing use of machine learning algorithms in decision making has brought the issue of vulnerability to manipulation to the forefront. Automated classification systems make decisions based on perceived attributes of individuals. However, when individuals have information about the classifier, they may try to alter their attributes even artificially to achieve a better outcome. This problem, known as strategic classification, has been formalized in machine learning by Brückner and Scheffer [1], and Hardt et al. [2].

Our goal is to find an online learning algorithm in presence of manipulation. In a strategic classification setting, finding a classifier by an online learning method is challenging particularly for the following two reasons. First, the classifier at each time is the result of manipulated attributes that the algorithm has observed in the past. Second, in each step...
the individuals adapt their behavior to the current classifier which may be different from the behavior of previous individuals.

Finding a linear classifier which is robust to manipulation is even more challenging if the learner does not know the manipulation costs. In this case, on top of estimating the individuals real attributes based on the observed data, the algorithm needs to estimate the costs. Unreasonable estimate of costs may lead to poor performance by the learner as the learner may not be able to distinguish if a classification mistake is due to an improper classifier or improper estimate of costs. This failure to distinguish correctly may lead to deterioration of the classifier and divergence from the optimal solution.

Our Model and Contributions  To isolate the effect of manipulation, we focus on finding a linear classifier when the unmanipulated data are linearly separable; i.e., the feature space is divided into two half spaces with positive data points in one and negative data points in the other, and a nonzero margin between them. In this situation, the well-known Perceptron algorithm learns a linear classifier in a bounded number of mistakes. When individuals can manipulate, however, the situation changes to the following. In each step, the arriving individual wishes to be classified positively. If the individuals feature vector $z$ is not classified as positive with the true attributes, they may choose to suffer a cost and pretend to have a feature vector $x$. We consider two classes of cost functions. The first class is the $\ell_2$ costs setting where the cost is proportional to the Euclidean distance moved. This case represents the setting where individuals when manipulating can take actions that affect multiple attributes. The second class is the weighted $\ell_1$ costs. In this case, there are specific directions in which the individual can move along, and the cost of reaching a destination from an origin is the sum of separate costs paid in each direction. In contrast to the previous case, this case represents the settings where there is a specific action associated with each attribute.

The contributions of the paper are as following:

- We show that the original Perceptron algorithm fails to learn a linear classifier when individuals can manipulate their attributes even when a perfect classifier under manipulation exists. See Example 1.

- We give an online learning algorithm robust to manipulation that finds a linear classifier in a bounded number of mistakes with the knowledge of costs. The number of mistakes is not much larger than the standard Perceptron bound in the non-strategic case, see Theorems 1 and 2.

- We generalize our algorithm to the setting with unknown costs and provide an efficient algorithm with bounded number of mistakes in that setting. See Theorem 3.

Related Work  There is a growing literature on strategic classification. Brückner and Scheffer [1], Hardt et al. [2], Dong et al. [3], and more recently, Chen et al. [4], similar to our paper design efficient learning algorithms that are robust to manipulation. Brückner and Scheffer and later Hardt et al., formalize the strategic classification problem as a Stackelberg competition between a learner and an agent. By using the knowledge about distribution of
agents true features and their cost functions they design near-optimal classifiers. However, the other two papers, similar to us, consider an online learning problem where the learner does not know the distribution of agents true features. In addition to this generalization, Dong et al. [3] consider a model where the learner does not know the agents’ cost functions either. A key difference between [3] and this paper is the assumption on the objective of the agents: we consider agents that wish to be classified as positive, whereas [3] considers agents that wish to increase their dot-product with the hypothesis vector no matter how they are classified. Chen et al. [4] study a model where agents can manipulate in a ball of radius $\delta$ from their real position, where $\delta$ is known. Although there are similarities between the two models, explained in more detail in Section 2, [4] does not consider a fixed utility model. This generality can work against the efficiency of the algorithm, and the performance can be arbitrarily bad depending on the positions of the observed data.

Many other papers, on the other hand, consider objectives other than accuracy in design of their algorithms. Hu et al. [5] focus on a fairness objective and raise the issue that different populations of agents may have different manipulation costs. Braverman and Garg [6], by introducing noise in their classification, design algorithms where agents with different costs are better off not manipulating which tackles the fairness issue. Milly et al. [7] state that the accuracy that strategic classification seeks leads to a raised bar for agents who naturally are qualified and puts a burden on them to prove themselves. Kleinberg and Raghavan [8], Haghtalab et al. [9], Alon et al. [10], Bechavod et al. [11], Shavit et al. [12], and Miller et al. [13] focus on models in which the policy maker is interested in choosing a rule which incentivizes agent(s) to invest their effort into features that truly improve their qualification.

Organization of the Paper. Section 2 introduces the model, overviews the non-strategic setting, and provides examples where the original Perceptron algorithm makes an unbounded number of mistakes. Sections 3 and 4 study the case where the cost of manipulation is known: Section 3 focuses on $\ell_2$ costs and Section 4 on weighted $\ell_1$ costs. Section 5 studies the unknown costs model. Finally, conclusions and some open problems are presented in Section 6.

2 Preliminaries

In the strategic Perceptron problem, a set of linearly separable examples arrive in an online manner, where each example corresponds to a different individual. Each example is a data point in $\mathbb{R}^d$ and shows assessment of the corresponding individual w.r.t. a set of $d$ features, i.e. $i^{th}$ coordinate of a data point denotes assessed value of the $i^{th}$ feature for the corresponding individual.

Individuals have value 1 for being classified as positive, and 0 for being classified as negative. We assume individuals are utility maximizers, and manipulate their corresponding data point if and only if it helps them to get classified positively; where utility is defined as value minus the cost.

Let $z_t$ denote the $t^{th}$ example before manipulation, and $x_t$ denote the observed $t^{th}$ example. We assume there exists a vector $w^*$, such that for each unmanipulated positive example $z_t$ we have $z^T_t w^* \geq 1$, and for each unmanipulated negative example $z_t$ we have $z^T_t w^* \leq -1$;
i.e., a linear separator of margin $\gamma = 1/|w^*|$. In the case of unknown manipulation costs, we assume the value of $\gamma$ is known. Whenever referred to $|w|$, we mean $\ell_2$ norm of $w$.

We consider two settings for manipulation. In the first setting, the cost of manipulation of point $z_t$ to $x_t$ is proportional to the $\ell_2$ distance of the two points. We denote the cost per unit of movement by $c$. We define $\alpha = 1/c$ as the maximum amount data points can afford to move.

We assume $0 \leq \alpha \leq R$ where $R = \max_t |z_t|$. In the second setting, the cost of manipulation is a weighted $\ell_1$ metric, such that the cost of moving from $x_t$ to $z_t$ is equal to $\sum_{i=1}^d c_i |x_{t,i} - z_{t,i}|$. Similarly we define $\alpha_i = 1/c_i$ as the maximum amount data points can afford to move along the $i^{th}$ coordinate vector $e_i$. We consider both scenarios where the cost of manipulation is known or unknown.

2.1 Non-Strategic Setting and the Perceptron Algorithm

As a reminder for the reader we provide the classical Perceptron algorithm here. This algorithm classifies all points with $x_t^T w \geq 0$ as positive, and the rest as negative; updating $w$ when it makes a mistake. The total number of mistakes made by the algorithm is upper bounded by $R^2 |w^*|^2$.

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Algorithm 1: Perceptron Algorithm

    w ← 0;
    for $t = 1, 2, \ldots$ do
        Given example $x_t$, predict $sgn(x_t^T w)$;
        if the prediction was a mistake then
            if $x_t$ was + then $w ← w + x_t$;
            if $x_t$ was − then $w ← w - x_t$;
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2.2 Failure of the Perceptron Algorithm in Strategic Settings

The Perceptron algorithm may make unbounded number of mistakes in the models considered in this paper even when a perfect classifier exists. The following example illustrates this in a setting with $\ell_2$ cost.

**Example 1.** Consider three examples $A = (1, 0)$, $B = (0, -1)$, and $C = (-0.5, -1)$ where $A$ and $B$ have positive labels, and $C$ has a negative label. Suppose that $\alpha = 0.5$. The following scenario of arrival of these examples makes the regular Perceptron algorithm cycle between two classifiers. Suppose $A$ is the first example to arrive, then individuals $B$ and $C$ arrive respectively and repeatedly. After arrival of $A$, $w = (1, 0)$. $B$ does not need to manipulate as it is classified positive with the current classifier. However $C$ manipulates to point $(0, -1)$ and the algorithm mistakenly classifies it as positive. As a consequence, $w$ will be updated to $(1, 0) - (0, -1) = (1, 1)$. With the new classifier, $B$ cannot manipulate to be classified positive. This leads to $w = (1, 1) + (0, -1) = (1, 0)$ and the scenario repeats. However,

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1Chen et al. [4] also consider a model where individuals can move in a ball of fixed radius from their real position. However, they do not focus on a specific utility model.
The failure of the Perceptron algorithm is not restricted to the $\ell_2$ costs model. Example 1 with $\alpha = (0.6,0)$ makes unbounded number of mistakes in the $\ell_1$ costs model as well.

3 Strategic Manipulation with Known $\ell_2$ Costs

In this section, we provide an algorithm for the $\ell_2$ costs setting. At a high level, there are two main ideas to modify and generalize the Perceptron algorithm for this setting. The first modification is raising the bar for a point to be classified as positive. Previously, a nonnegative dot product with the current classifier (a threshold of 0), sufficed for positive classification. However, in the new algorithm, the threshold is a strictly positive value depending on the cost of manipulation. The second modification is using a surrogate for the data points when the classifier updates. Interestingly, we only need to use a surrogate for negative points, and in this case the surrogate is a projection of the point in the opposite direction of manipulation, detected by the algorithm.

Brief description of the algorithm. Algorithm 2 is a modification of the Perceptron algorithm which we call strategic Perceptron. The algorithm starts by predicting all points as positive until it makes a mistake. Note that during this period, individuals do not have incentive to manipulate. From that point on, the algorithm classifies all points with

\[ w = (1,0.5) \] works perfectly for the three points as $B$ can manipulate to be classified positive and $C$ cannot.

In the implementation of the Perceptron algorithm used in this example, the classifier starts with classifying all points as negative until updated for the first time. By changing $A$ to $(-1,0)$ with a negative label, all the steps described in this example work with implementation of Algorithm 1 where the classifier starts with classifying all points as positive until it updates for the first time.
Algorithm 2: Strategic Perceptron for $\ell_2$ costs

\[ w \leftarrow 0; \]

\[ \text{for } t = 1, 2, \cdots \text{ do} \]

\[ \text{Given example } x_t: \]

\[ \begin{align*}
\text{if } |w| \text{ is 0 then} & \quad \text{predict +;}
\text{if the prediction was a mistake then} & \quad w \leftarrow w - x_t;
\text{else} & \quad \text{predict } \text{sgn}(x_t^Tw/|w| - \alpha);
\text{if the prediction was a mistake and } x_t \text{ was + then} & \quad w \leftarrow w + \tilde{x}_t;
\text{if the prediction was a mistake and } x_t \text{ was - then} & \quad w \leftarrow w - \tilde{x}_t;
\end{align*} \]

\[ x_t^Tw/|w| - \alpha \geq 0 \text{ as positive, and the rest as negative. Whenever the algorithm makes a mistake, the predictor } w \text{ is updated with a surrogate value, } \tilde{x}_t, \text{ defined below.} \]

Definition 1 (\( \tilde{x}_t \), surrogate data point in $\ell_2$ setting). We define surrogate data point, $\tilde{x}_t$, as follows.

\[ \tilde{x}_t = \begin{cases} 
 x_t - \alpha \frac{w}{|w|}, & \text{if } x_t \text{ is } - \text{ and } x_t^Tw/|w| = \alpha; \\
 x_t, & \text{if } x_t \text{ is } + \text{ and } x_t^Tw/|w| = \alpha; \\
 x_t, & \text{if } x_t^Tw/|w| > \alpha \text{ or } x_t^Tw/|w| \leq 0.
\end{cases} \]

Observation 1 (manipulation hyperplane). In Algorithm 2, $x_t$ is a manipulated example only if $x_t^Tw/|w| = \alpha$. The reason is as follows. In order to maximize utility, individuals move data points in direction of $w$ and move the point the minimum amount to be classified as positive. Therefore, if with true features they are classified as negative, they only need to move to the line with dot product equal to $\alpha$ and moving to any other location contradicts with utility maximizing.

Observation 2 (forbidden region). For no observed data point, $x_t$, $0 < x_t^Tw/|w| < \alpha$, and therefore $\tilde{x}_t$ does not need to be defined for $0 < x_t^Tw/|w| < \alpha$. The reason is as follows. Suppose there exists a data point $x_t$ such that $0 < x_t^Tw/|w| < \alpha$. This data point either was manipulated or not. If it was manipulated, the manipulation was not rational since it did not help the data point to get classified as positive. If it was not manipulated, this is not rational either. Since the data point has a distance less than $\alpha$ from the classifier and could afford the cost of manipulation to be classified as positive.

We show Algorithm 2 makes at most $(R + \alpha)^2|w^*|^2$ mistakes. First, we need to prove the following lemmas hold.

Lemma 1. For any positive data point $x_t$, $\tilde{x}_t^Tw^* \geq 1$, and for any negative data point $x_t$, $\tilde{x}_t^Tw^* \leq -1$. Also, throughout the execution of Algorithm 2, $w^Tw^* \geq 0$. 

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The algorithm makes a mistake on, inequality is strict, i.e., $x^T w^* > 0$. Second, we show if at the end of step $t - 1$, $w^T w^* \geq 0$, then at step $t$, $\tilde{x}_t^T w^* \geq 1$ for positive points, and $\tilde{\tilde{x}}_t^T w^* \leq -1$ for negative points. Finally, we show if $w^T w^* \geq 0$ at the end of step $t - 1$, and $\tilde{x}_t^T w^* \geq 0$ for positive points, and $\tilde{\tilde{x}}_t^T w^* \leq 0$ for negative points, then $w^T w^* \geq 0$ at the end of step $t$.

The first step is straight-forward. Initially, $w = 0$. While $w = 0$, we have $x^T w = 0$, and arriving examples get classified positively. The first mistake occurs when a negative example $x_t$ arrives, and gets classified as positive. In this case, $w$ gets updated to $w - x_t$. Since $\tilde{x}_t^T w^* \leq -1$, we conclude $(w - x_t) w^* > 0$.

The second step is more involved. By definition of the surrogate values, for any points such that $x_t^T w / |w| \neq \alpha$, we have $\tilde{x}_t = x_t$. By Observation 1, these points are not manipulated, i.e., $x_t = z_t$. This implies $\tilde{x}_t = z_t$ and therefore the claim holds. Thus, we only need to argue for the points on the hyperplane $x_t^T w / |w| = \alpha$. Consider such data points. For the positive data points, we have, $\tilde{x}_t = x_t = z_t + \beta \cdot w / |w|$, where $0 \leq \beta \leq \alpha$. Therefore, $\tilde{x}_t^T w^* = z_t^T w^* + \beta \cdot w^T w^* / |w| \geq z_t^T w^* \geq 1$. The first inequality holds since by assumption of this step, $w^T w^* \geq 0$. On the other hand, for the negative data points we have $\tilde{x}_t = x_t - \alpha \cdot w / |w|$, where $x_t = z_t + \beta \cdot w / |w|$ and $0 \leq \beta \leq \alpha$. This implies $\tilde{x}_t = z_t + (\beta - \alpha) \cdot w / |w|$. By multiplying with $w^*$, we get $\tilde{x}_t^T w^* = z_t^T w^* + (\beta - \alpha) \cdot w^T w^*/|w| \leq z_t^T w^* \leq -1$.

The final step is again straight-forward. Whenever $w$ is updated, for positive points, $w$ gets updated to $w + \tilde{x}_t$, where both $w$ and $\tilde{x}_t$ have nonnegative dot product with $w^*$. For negative points, $w$ gets updated to $w - \tilde{x}_t$, where $w$ has a nonnegative and $\tilde{x}_t$ has a negative dot product with $w^*$.

**Lemma 2.** When Algorithm 2 makes a mistake on a positive example $x_t$, $\tilde{x}_t^T w \leq 0$; and when it makes a mistake on a negative example $x_t$, $\tilde{x}_t^T w \geq 0$.

Proof. The algorithm makes a mistake on a positive example only if $x^T w / |w| < \alpha$. By Observation 2, for no points, $0 < x^T w / |w| < \alpha$. Therefore, for any positive example that the algorithm makes a mistake on, $x^T w \leq 0$. By Definition 1, $\tilde{x}_t = x_t$ for all positive examples. Therefore, $x^T w \leq 0$ implies $\tilde{x}_t^T w \leq 0$.

For negative examples, the algorithm makes a mistake only if $x^T w / |w| \geq \alpha$. If the inequality is strict, i.e., $x^T w / |w| > \alpha$, by Definition 1, $\tilde{x}_t = x_t$, and therefore $\tilde{x}_t^T w \geq 0$. If $x^T w / |w| = \alpha$, again using Definition 1, we have $\tilde{x}_t^T w = 0$.

Next, we show the following theorem holds which gives a bound on the number of mistakes. Proof of the following theorem is along the lines of the proof of the classic Perceptron algorithm.

**Theorem 1.** Algorithm 2 makes at most $(R + \alpha)^2 |w^*|^2$ mistakes, where $R = \max x_t |z_t|$.

Proof. We keep track of two quantities, $w^T w^*$ and $|w|^2$. First, we show that each time we make a mistake, $w^T w^*$ increases by at least 1. If we make a mistake on a positive example then,

$$(w + \tilde{x}_t)^T w^* = w^T w^* + \tilde{x}_t^T w^* \geq w^T w^* + 1;$$
where the last inequality holds by Lemma 1. Similarly, if we make a mistake on a negative example,
\[(w - \tilde{x}_t)^T w^* = w^T w^* - \tilde{x}_t^T w^* \geq w^T w^* + 1.\]
Next, on each mistake we claim that \(|w|^2\) increases by at most \((R + \alpha)^2\). If we make a mistake on a positive example \(x_t\), then we have:
\[(w + \tilde{x}_t)^T(w + \tilde{x}_t) = |w|^2 + 2\tilde{x}_t^T w + |\tilde{x}_t|^2 \leq |w|^2 + |\tilde{x}_t|^2 \leq |w|^2 + (R + \alpha)^2.\]
To understand the middle inequality note that by Lemma 2, when a mistake is made on a positive example \(x_t\), \(\tilde{x}_t^T w \leq 0\). The last inequality comes from \(R = \max_t |z_t|\) implies \(\max_t |\tilde{x}_t| \leq R + \alpha\).
Similarly, if we make a mistake on a negative example \(x_t\), then we have:
\[(w - \tilde{x}_t)^T(w - \tilde{x}_t) = |w|^2 - 2\tilde{x}_t^T w + |\tilde{x}_t|^2 \leq |w|^2 - |\tilde{x}_t|^2 \leq |w|^2 + (R + \alpha)^2.\]
By Lemma 2, when a mistake is made on a negative example \(x_t\), \(\tilde{x}_t^T w \geq 0\), which implies the middle inequality.
Finally, if the algorithm makes \(M\) mistakes, then \(w^T w^* \geq M\) and \(|w|^2 \leq M(R + \alpha)^2\), or equivalently, \(|w| \leq (R + \alpha)\sqrt{M}\). Using the fact that \(w^T w^*/|w^*| \leq |w|\), we have
\[M/|w^*| \leq (R + \alpha)\sqrt{M} \implies \sqrt{M} \leq (R + \alpha)|w^*| \implies M \leq (R + \alpha)^2|w^*|^2.\]

\[\square\]

4 Strategic Manipulation with Known Weighted \(\ell_1\) Costs

In this section, we provide an algorithm for the weighted \(\ell_1\) costs setting. Unlike the \(\ell_2\) case, the modifications to the classical Perceptron algorithm in Algorithm 2 do not suffice; and our algorithm for this setting is more involved. Here is the key difference: In the \(\ell_2\) costs setting, the individuals always manipulate in direction of the current classifier \(w\). However, in the weighted \(\ell_1\) setting this is no longer the case. This brings up two challenges to our approach. First, there may be multiple utility maximizing manipulation directions. Second, the manipulation direction may have a negative dot product with \(w^*\). We overcome these two challenges, and provide an algorithm for this setting.

As a reminder, in the weighted \(\ell_1\) costs setting, there are coordinate unit vectors \(\{e_1, \ldots, e_d\}\) with cost of manipulation \(1/\alpha_i\) along \(e_i\). We need to make one further assumption for this setting. We assume for all \(1 \leq i \leq d\), \(e_i^T w^* \geq 0\). In other words, we assume that each feature is defined so that larger is better.

Brief description of the algorithm. Algorithm 3 starts by predicting all points as positive until it makes a mistake. Note that during this period, individuals do not have incentive to manipulate. From that point on, the algorithm classifies all points \(x_t\) such that \(x_t^T w/|w| - \alpha_i w^T e_i/|w| \geq 0\) as positive, and the rest as negative; where \(e_i\) is the manipulation direction which will be defined later. Similar to Algorithm 2 whenever the algorithm makes a mistakes the predictor \(w\) is updated with a surrogate value, \(\tilde{x}_t\), in Definition 2.
Compared to Algorithm 2 we have two further steps. As discussed above, the first challenge with the weighted \( \ell_1 \) costs is that with an arbitrary \( \mathbf{w} \), there may be multiple utility maximizing manipulation directions, and we may not be able to distinguish along which vector individuals manipulated. Since in the weighted \( \ell_1 \) costs setting, the cost of manipulation can be written as a convex combination of costs in coordinate vectors, there always exists a coordinate vector, \( \mathbf{e}_i \), such that manipulating along that is utility maximizing.

Consider all the coordinate vectors like \( \mathbf{e}_i \) that are utility maximizing, i.e., have the highest \( \alpha_j \cdot \mathbf{w}^T \mathbf{e}_j/|\mathbf{w}| \). To make the manipulation direction unique, we add a tie-breaking step to the algorithm. This step adds a small multiple \( \eta > 0 \), of an arbitrary utility maximization coordinate vector \( \mathbf{e}_i \), to \( \mathbf{w} \) to break the tie. Note that any positive value of \( \eta \) breaks the tie. We set this value in our analysis purposes in Theorem 2 in a way to make sure the number of mistakes our algorithm makes does not increase much.

We need to add another step to address the second challenge: With an arbitrary \( \mathbf{w} \) the direction that the individuals manipulate along may not have a positive dot product with \( \mathbf{w}^* \), i.e., the individuals may choose to move along one of the vectors \( \{-\mathbf{e}_1, \cdots, -\mathbf{e}_d\} \). In order to incentivize individuals to only manipulate along \( \{\mathbf{e}_1, \cdots, \mathbf{e}_d\} \), and not \( \{-\mathbf{e}_1, \cdots, -\mathbf{e}_d\} \), we do the following correction step after each update. If \( \mathbf{e}_j^T \mathbf{w} < 0 \) for any \( \mathbf{e}_j \in \{\mathbf{e}_1, \cdots, \mathbf{e}_d\} \), we set the \( j^{th} \) coordinate of \( \mathbf{w} \) to 0 by adding the smallest multiple of \( \mathbf{e}_j \), denoted by \( \mu_j \), to \( \mathbf{w} \) to make \( \mathbf{e}_j^T \mathbf{w} \) nonnegative. Therefore, \( \mu_j = 0 \) if \( \mathbf{e}_j^T \mathbf{w} \geq 0 \), and \( \mu_j = -\mathbf{e}_j^T \mathbf{w} \), otherwise, implying \( \mu \geq 0 \).

With the unique manipulation direction, similar to the \( \ell_2 \) costs setting, we are now able to choose a surrogate value along the manipulation direction.

**Definition 2** (\( \tilde{x}_t \), surrogate data point in weighted \( \ell_1 \) setting). Let \( \mathbf{e}_i \) be the unique utility maximizing coordinate vector, i.e., \( i = \arg \max \alpha_j \mathbf{w}^T \mathbf{e}_j/|\mathbf{w}| \). We define surrogate data point, \( \tilde{x}_t \), as follows.

\[
\tilde{x}_t = \begin{cases} 
\mathbf{x}_t - \mathbf{e}_i \cdot \alpha_i, & \text{if } \mathbf{x}_t \text{ is } - \text{ and } \frac{\mathbf{x}_t^T \mathbf{w}}{|\mathbf{w}|} = \alpha_i \cdot \frac{\mathbf{w}^T \mathbf{e}_i}{|\mathbf{w}|}; \\
\mathbf{x}_t, & \text{otherwise}. 
\end{cases}
\]

**Lemma 3.** \( \mu_j \leq R + \alpha_j \).

**Proof.** We can show at the end of each round, \( \mathbf{e}_j^T \mathbf{w} \geq 0 \). Initially, \( \mathbf{w} = 0 \), therefore \( \mathbf{e}_j^T \mathbf{w} = 0 \). Suppose at the end of round \( t - 1 \), \( \mathbf{e}_j^T \mathbf{w} \geq 0 \). Assume in round \( t \), \( \mathbf{w} \) gets updated by adding or subtracting \( \tilde{x}_t \) or \( \mathbf{x}_t \). By assumption, the \( j^{th} \) coordinate of \( \mathbf{x}_t \) is in \([-R, R]\), and therefore the \( j^{th} \) coordinate of \( \tilde{x}_t \) is in \([-R - \alpha_j, R + \alpha_j]\). Taken together, \( \mu_j \leq R + \alpha_j \). Note that by adding \( \eta \mathbf{e}_i \) to \( \mathbf{w} \), \( \mathbf{e}_j^T \mathbf{w} \) remains nonnegative.

The following theorem upper bounds the number of mistakes made by Algorithm 3.

**Theorem 2.** Consider a sequence of examples before manipulation \( \mathbf{z}_1, \mathbf{z}_2, \cdots \), which are observed as \( \mathbf{x}_1, \mathbf{x}_2, \cdots \). Consider vector \( \mathbf{w}^* \) such that \( \mathbf{z}_t^T \mathbf{w}^* \geq 1 \) for positive examples, and \( \mathbf{z}_t^T \mathbf{w}^* \leq -1 \) for negative examples. Algorithm 3 makes at most \( (1 + (d + 1)(R + \alpha)^2)|\mathbf{w}^*|^2 \) mistakes, where \( R = \max_i |\mathbf{z}_t_i| \), and \( \alpha = \max\{\alpha_1, \cdots, \alpha_d\} \).

**Proof.** Similar to the proof of Theorem 1 we keep track of two quantities \( \mathbf{w}^T \mathbf{w}^* \) and \( |\mathbf{w}|^2 \).

First, we show each time a mistake is made, \( \mathbf{w}^T \mathbf{w}^* \) increases by at least 1. Then we find an upper bound on the increase of \( |\mathbf{w}|^2 \).
Algorithm 3: Strategic Perceptron for weighted $\ell_1$ costs

\[
\begin{align*}
w &\leftarrow 0; \\
\text{for } t = 1, 2, \cdots \text{ do} & \\
& \text{Given example } x_t: \\
& \text{if } |w| \text{ is } 0 \text{ then} \\
& \quad \text{predict } +; \\
& \quad \text{if the prediction was a mistake then} \\
& \qquad w \leftarrow w - x_t; \\
& \qquad /* \text{Correction Step} */ \\
& \qquad \text{for } j = 1, 2, \cdots, d \text{ do} \\
& \qquad \qquad w \leftarrow w + \mu_j e_j, \text{ where } \mu_j = \max(0, -e_j^T w); \\
& \qquad /* \text{Tie-breaking Step} */ \\
& \qquad i \leftarrow \arg \max_j \alpha_j \cdot \frac{w^T e_j}{|w|}; \\
& \qquad w \leftarrow w + \eta e_i; \\
& \text{else} \\
& \quad \text{predict } \text{sgn}(\frac{w^T x_t}{|w|} - \alpha_i \cdot \frac{w^T e_i}{|w|}); \\
& \quad \text{if the prediction was a mistake and } x_t \text{ was } + \text{ then } w \leftarrow w + \tilde{x}_t; \\
& \quad \text{if the prediction was a mistake and } x_t \text{ was } - \text{ then } w \leftarrow w - \tilde{x}_t; \\
& \quad /* \text{Correction Step} */ \\
& \quad \text{for } j = 1, 2, \cdots, d \text{ do} \\
& \qquad w \leftarrow w + \mu_j e_j \text{ where } \mu_j = \max(0, -e_j^T w); \\
& \qquad /* \text{Tie-breaking Step} */ \\
& \qquad i \leftarrow \arg \max_j \alpha_j \cdot \frac{w^T e_j}{|w|}; \\
& \qquad w \leftarrow w + \eta e_i;
\end{align*}
\]

Starting from the current $w$, the algorithm follows three steps to update: addition/subtraction of $\tilde{x}_t$, the correction step, and the tie-breaking step. As in the algorithm $e_i$ is the manipulation direction.

If the algorithm makes a mistake on a positive example the new value of $w$ is $w + \tilde{x}_t + \eta e_i + \sum_j \mu_j e_j$. Therefore,

\[
\left( w + \tilde{x}_t + \eta e_i + \sum_j \mu_j e_j \right)^T w^* = w^T w^* + \tilde{x}_t^T w^* + \eta e_i^T w^* + \sum_j \mu_j e_j^T w^* \geq w^T w^* + 1;
\]

where the inequality holds because first using the ideas from Lemma 1 $\tilde{x}_t^T w^* \geq 1$ for the positive examples the algorithm makes a mistake on and $\tilde{x}_t^T w^* \leq -1$ for the negative examples the algorithm makes a mistake on, and second, for all $j$, $e_j^T w^* \geq 0$ by assumption, and $\mu_j \geq 0$.

Similarly, If the algorithm makes a mistake on a negative example, we have:

\[
\left( w - \tilde{x}_t + \eta e_i + \sum_j \mu_j e_j \right)^T w^* = w^T w^* - \tilde{x}_t^T w^* + \eta e_i^T w^* + \sum_j \mu_j e_j^T w^* \geq w^T w^* + 1.
\]
Next, on each mistake we claim $|\mathbf{w}|^2$ increases by at most $(d+1)(R+\alpha)^2 + 1$. If the algorithm makes a mistake on a positive example, we have:

$$
\left| \mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i + \sum_j \mu_j \mathbf{e}_j \right|^2 = |\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i|^2 + \left| \sum_j \mu_j \mathbf{e}_j \right|^2 + 2 \left( \sum_j \mu_j \mathbf{e}_j \right)^T (\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i)
$$

$$
= |\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i|^2 + \sum_j |\mu_j \mathbf{e}_j|^2 + 2 \sum_j \mu_j \mathbf{e}_j^T (\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i)
$$

$$
\leq |\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i|^2 + \sum_j |\mu_j \mathbf{e}_j|^2 + 2 \sum_j \eta \mu_j \mathbf{e}_j^T \mathbf{e}_i
$$

$$
= |\mathbf{w} + \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i|^2 + \sum_j |\mu_j|^2 + 2 \eta \mu_i
$$

$$
= |\mathbf{w}|^2 + |\tilde{\mathbf{x}}_t|^2 + |\eta \mathbf{e}_i|^2 + 2 \mathbf{w}^T \tilde{\mathbf{x}}_t + 2 \eta \mathbf{w}^T \mathbf{e}_i + 2 \eta \tilde{\mathbf{x}}_t^T \mathbf{e}_i + \sum_j |\mu_j|^2 + 2 \eta \mu_i
$$

$$
\leq |\mathbf{w}|^2 + (R + \alpha)^2 + \eta^2 + 0 + 2\eta|\mathbf{w}| + 2\eta(R + \alpha) + d(R + \alpha)^2 + 2\eta(R + \alpha)
$$

$$
\leq |\mathbf{w}|^2 + (d+1)(R + \alpha)^2 + \eta^2 + \eta(2|\mathbf{w}| + 4(R + \alpha))
$$

$$
\leq |\mathbf{w}|^2 + (d+1)(R + \alpha)^2 + 1/4 + 1/2
$$

$$
\leq |\mathbf{w}|^2 + (d+1)(R + \alpha)^2 + 1;
$$

where the first equality is the result of expansion. The second uses $\mathbf{e}_j^T \mathbf{e}_k = 0$ for $j \neq k$. The inequality in the third row uses $\mu_j = 0$ when $\mathbf{e}_j^T (\mathbf{w} + \tilde{\mathbf{x}}_t) \geq 0$, and $\mu_j > 0$ when $\mathbf{e}_j^T (\mathbf{w} + \tilde{\mathbf{x}}_t) < 0$, implying $\mu \mathbf{e}_j^T (\mathbf{w} + \tilde{\mathbf{x}}_t) \leq 0$. The fourth row uses $\mathbf{e}_j^T \mathbf{e}_k = 0$ for $k \neq j$ and $\mathbf{e}_j^T \mathbf{e}_j = 1$. The fifth row is the result of expansion. The sixth row substitutes each term with an upper bound using $|\tilde{\mathbf{x}}_t| \leq R + \alpha$ and $\mathbf{w}^T \tilde{\mathbf{x}}_t \leq 0$, similar to the arguments from Lemma 2 and $\mu_j \leq R + \alpha$, by Lemma 3. The eighth row results by setting $\eta = \frac{1}{4|\mathbf{w}|+8(R+\alpha)+2}$. The last row sums up and upper bounds similar terms.

Similarly, if the algorithm makes a mistake on a negative example, we have:

$$
\left| \mathbf{w} - \tilde{\mathbf{x}}_t + \eta \mathbf{e}_i + \sum_j \mu_j \mathbf{e}_j \right|^2 \leq |\mathbf{w}|^2 + (d+1)(R + \alpha)^2 + 1.
$$

Therefore, after each mistake, $|\mathbf{w}|^2$ increases by at most $(d+1)(R + \alpha)^2 + 1$. The rest of the proof is similar to the proof of Theorem 4 concluding that the total number of mistakes is at most $(d+1)(R + \alpha)^2 + 1)|\mathbf{w}^*|^2$. □

## 5 Strategic Manipulation with Unknown Costs

The main result of this section is generalizing our algorithms to the unknown costs setting. The generalization holds for $\ell_2$ costs. However, it does not extend fully to weighted $\ell_1$ costs and only works for a specific case. The algorithm for unknown $\ell_2$ costs is presented in Section 5.1. The case of unknown $\ell_1$ costs is studied in Section 5.2.
5.1 \( \ell_2 \) Costs

In this section, we provide an algorithm that makes at most a bounded number of mistakes when the manipulation cost, \( 1/\alpha \), is unknown. Algorithm 2 is used as a subroutine to evaluate our estimate of \( \alpha \). First, we show Algorithm 2 works efficiently if the estimated value, \( \alpha' \), is in proximity of the real value (when \( \alpha' \) is in the interval of length \( \gamma/2 \) below \( \alpha \)). Using this idea we can run a linear search for \( \alpha \) with step size \( \gamma/2 \). However, we show we can do better than a linear search. The key ingredient that lets us outperform the linear search is the ability to distinguish whether the estimate is below or above the real value. Using this idea we run a binary search to find a proper estimate and come up with an efficient algorithm. Below, we explain these steps more formally.

**Case 1:** \( 0 \leq \alpha - \alpha' \leq \gamma/2 \)

First, we consider the case of \( 0 \leq \alpha - \alpha' \leq \gamma/2 \). Suppose Algorithm 2 takes \( \alpha' \) instead of \( \alpha \) as input. Also, suppose \( \tilde{x}_t \) is defined with respect to \( \alpha' \) instead of \( \alpha \). In Proposition 1 we show if \( 0 \leq \alpha - \alpha' \leq \gamma/2 \), Algorithm 2 with these modifications, makes at most \( 4(R+\alpha'+\gamma/2)^2|w^*|^2 \) mistakes. We need the following two lemmas for proving the proposition. Proofs of Lemma 4 and 5 are along the lines of proofs of Lemma 1 and 2 respectively.

**Lemma 4.** Consider data points \( \tilde{x}_t \) as defined in Definition 1 w.r.t. \( \alpha' \) such that \( 0 \leq \alpha - \alpha' \leq \gamma/2 \). These data points are 1/2-separable; i.e., for positive data points, \( \tilde{x}_t^T w^* \geq 1/2 \); and for negative data points, \( \tilde{x}_t^T w^* \leq -1/2 \). Also, throughout the execution of Algorithm 2 with \( \alpha' \), \( w^T w^* \geq 0 \).

**Proof.** The proof uses the same three steps as Lemma 1. Here, we argue for the second step, i.e., if at the end of step \( t-1 \), \( w^T w^* \geq 0 \), then at step \( t \), \( \tilde{x}_t^T w^* \geq 1/2 \) for positive points, and \( \tilde{x}_t^T w^* \leq -1/2 \) for negative points.

When Algorithm 2 is run with \( \alpha' \), by Definition 1 for any points such that \( x_t^T w/|w| \neq \alpha' \), we have \( \tilde{x}_t = x_t \). By Observation 1 these points are not manipulated, i.e., \( x_t = z_t \). This implies \( \tilde{x}_t = z_t \) which implies the claim for these points. Thus, we only need to argue for the data points such that \( x_t^T w/|w| = \alpha' \). Consider such data points. For the positive data points, we have, \( \tilde{x}_t = x_t = z_t + \beta \cdot w/|w| \), where \( 0 \leq \beta \leq \alpha \). Therefore, \( \tilde{x}_t^T w^* = z_t^T w^* + \beta \cdot w^T w^*/|w| \geq z_t^T w^* \geq 1 \). The first inequality holds because by the assumption of this step, \( w^T w^* \geq 0 \). On the other hand, for the negative data points we have \( \tilde{x}_t = x_t = \alpha' \cdot w/|w| \), where \( x_t = z_t + \beta \cdot w/|w| \) and \( 0 \leq \beta \leq \alpha \). This implies \( \tilde{x}_t = z_t + (\beta - \alpha') \cdot w/|w| \). By multiplying with \( w^* \), we get \( \tilde{x}_t^T w^* = z_t^T w^* + (\beta - \alpha') \cdot w^T w^*/|w| \leq z_t^T w^* + (\alpha - \alpha') \cdot w^T w^*/|w| \). Using \( 0 \leq \alpha - \alpha' \leq \gamma/2 \) and \( \gamma = 1/|w^*| \), we have \( \tilde{x}_t^T w^* \leq z_t^T w^* + w^T w^*/2|w^*||w| \) \leq z_t^T w^* + 1/2 \leq -1/2. \( \square \)

**Lemma 5.** Suppose Algorithm 2 is run with \( \alpha' \) such that \( 0 \leq \alpha - \alpha' \leq \gamma/2 \). When the algorithm makes a mistake on a positive example \( x_t \), \( \tilde{x}_t^T w \leq 0 \); and when it makes a mistake on a negative example \( x_t \), \( \tilde{x}_t^T w \geq 0 \).

**Proof.** First, we consider the positive points. The algorithm makes a mistake on a positive example only if \( x_t^T w/|w| < \alpha' \). Similar to Observation 2 in this case there is a margin
Using Lemma 4 and 5, the rest of the proof is similar to Theorem 1 and is deferred to the Appendix.

Proposition 1. When $0 \leq \alpha - \alpha' \leq \gamma/2$, Algorithm 2 makes at most $4(R + \alpha' + \gamma/2)^2|\mathbf{w}^*|^2$ mistakes.

Proof. Using Lemma 4 and 5, the rest of the proof is similar to Theorem 1 and is deferred to the Appendix.

Case 2: $\alpha < \alpha'$

Suppose $\alpha'$ is larger than $\alpha$. By Observation 2, when Algorithm 2 is run with the real value of $\alpha$, no data point is observed by algorithm in the margin $0 < x_T^T \mathbf{w} / |\mathbf{w}| < \alpha$. However, when the estimate is larger, since we overestimate by how far individuals can manipulate, Observation 2 no longer holds. Therefore, if the algorithm observes a point in the margin $0 < x_T^T \mathbf{w} / |\mathbf{w}| < \alpha'$, we realize that the estimate is large, and we need to refine it. On the other hand, while we have not observed any such points, the algorithm makes at most $(R + \alpha')^2|\mathbf{w}^*|^2$ mistakes. This statement is summarized and proved below.

Proposition 2. Suppose Algorithm 2 is run with $\alpha'$, such that $\alpha' > \alpha$, and is halted if for a data-point $\mathbf{x}_t$, $0 < x_t^T \mathbf{w} / |\mathbf{w}| < \alpha'$. This modified algorithm makes at most $(R + \alpha')^2|\mathbf{w}^*|^2 + 1$ mistakes.

Proof. Similar to the proof of Theorem 1, the maximum number of mistakes Algorithm 2 with estimated manipulation cost $1/\alpha'$ makes on observed data points $\mathbf{x}_t$, where $x_t^T \mathbf{w} / |\mathbf{w}| \leq 0$ or $x_t^T \mathbf{w} / |\mathbf{w}| \geq \alpha'$ is at most $(R + \alpha')^2|\mathbf{w}^*|^2$. If a data point $\mathbf{x}_t$ is observed such that $0 < x_t^T \mathbf{w} / |\mathbf{w}| < \alpha'$, it implies $\alpha' > \alpha$ and the algorithm halts, and at most one more mistake is made on this data point. Therefore, the total number of mistakes is at most $(R + \alpha')^2|\mathbf{w}^*|^2 + 1$.

Case 3: $\alpha' < \alpha - \gamma/2$

We can conclude from Propositions 1, 2 that if the number of mistakes is greater than $\max\{4(R + \alpha' + \gamma/2)^2|\mathbf{w}^*|^2, (R + \alpha')^2|\mathbf{w}^*|^2 + 1\} = 4(R + \alpha' + \gamma/2)^2|\mathbf{w}^*|^2$ then $\alpha' < \alpha - \gamma/2$. Note that the equality holds since the number of mistakes is an integer.

Putting Everything Together

After discussing the three cases, we are now ready to explain Algorithm 4. This algorithm, uses a binary search scheme to find a predictor in a bounded number of mistakes. The algorithm starts with $\alpha' = 0$. For each fixed $\alpha'$ we consider $4(R + \alpha' + \gamma/2)^2|\mathbf{w}^*|^2$ as the maximum number of allowed mistakes. Whenever we exceed this bound using the discussion
Algorithm 4: Strategic Perceptron with unknown manipulation cost

\[ \alpha'' \leftarrow 0, \alpha' \leftarrow 0; \]

\textbf{while} examples are arriving \textbf{do}

\hspace{1em} Run algorithm 2 with estimate \( \alpha' \) on the sequence of arriving examples, halt if

\hspace{2em} \#mistakes > \( 4(R + \alpha' + \gamma/2)^2|w^*|^2 \) or if for an example \( x_t, 0 < \frac{x_t^Tw}{|w|} < \alpha' \);

\hspace{1em} \textbf{if} \ #mistakes > \( 4(R + \alpha' + \gamma/2)^2|w^*|^2 \) \textbf{then}

\hspace{2em} /* guessed value \( \alpha' \) is small. */

\hspace{2em} \( \alpha'' \leftarrow \alpha' \);

\hspace{2em} \( \alpha' \leftarrow \min\{\max\{2\alpha', \gamma/2\}, R\} \);

\hspace{2em} continue;

\hspace{1em} \textbf{else if} for an example \( x_t, 0 < \frac{x_t^Tw}{|w|} < \alpha' \) \textbf{then}

\hspace{2em} /* guessed value \( \alpha' \) is large. */

\hspace{2em} \( \alpha' \leftarrow (\alpha'' + \alpha')/2 \);

\hspace{2em} continue;

in Section 5.1 we learn that \( \alpha' \) is too small. Also whenever we see a data point \( x_t \) such that

\( 0 \leq \frac{x_t^Tw}{|w|} < \alpha' \) as explained above we learn that \( \alpha' \) is too large. Distinguishing between

the cases where \( \alpha' \) is too large or too small allows us to refine the upper bound and lower
bound on \( \alpha' \) until \( 0 \leq \alpha - \alpha' \leq \gamma/2 \). The following theorem shows that the total number of
mistakes is bounded during the whole process.

\textbf{Theorem 3.} Algorithm 4 makes at most \( O(R^2|w^*|^2 \cdot \log(R|w^*|)) \) mistakes.

\textit{Proof.} In Algorithm 4 the candidates for \( \alpha \) are \( \gamma/2 \) apart and the number of them is \( 2R|w^*| \).

Since we are doing a binary search on these candidates, the total number of iterations of
binary search is at most \( \log(2R|w^*|) \). Proposition 1, Proposition 2, and Theorem 1 show that
in each iteration the total number of mistakes is bounded by \( \max\{4(R + \alpha' + \gamma/2)^2|w^*|^2, (R + \alpha')^2|w^*|^2 + 1\} \). Since we are assuming \( \alpha' \leq R \), the total number of mistakes is at most
\( O(R^2|w^*|^2 \cdot \log(R|w^*|)) \) and the proof is complete.

5.2 Weighted \( \ell_1 \) costs

As observed in Section 4, in order for the strategic Perceptron algorithm to work in the
weighted \( \ell_1 \) costs model, it is necessary to identify in what direction the individuals manipu-
late. The tie-breaking step in Algorithm 3 ensured that the manipulation direction is
unique and identifiable. In the unknown costs model, we need to make a guess for the cost in
each direction. Since the guessed values are not accurate, we no longer can use them for
a tie-breaking step and determine the manipulation direction. This restraints us from having
an efficient algorithm for the general case of \( \ell_1 \) costs. However, for a special case where
manipulation is possible in a single direction (finite cost in direction \( e_1 \) and infinite in the
others), the manipulation direction is known and the ideas of Algorithm 4 extend to this case.
6 Conclusions and Open Problems

In this work, we showed that if agents have the ability to manipulate their features within an $\ell_2$ ball or a weighted $\ell_1$ ball in order to be classified as positive, then the classic Perceptron algorithm may fail to achieve a bounded number of mistakes even when a perfect linear classifier exists. We then developed new Perceptron-based algorithms that achieve a finite mistake-bound, not much greater than the classic Perceptron bound in the non-strategic case, in both the $\ell_2$ and weighted $\ell_1$ manipulation setting. In the case that the manipulation costs are unknown to the learner—i.e., the radius of the ball in which agents can modify their features (or the per-coordinate radius in the weighted $\ell_1$ case)—we provide an algorithm for the $\ell_2$ costs setting and a specific case of the weighted $\ell_1$ costs setting.

Our work suggests two main open problems. First, designing an algorithm for the general case of weighted $\ell_1$ costs when the costs of manipulation along each coordinate is unknown. This is challenging because given an observed data point, the learner doesn’t know which direction it may have manipulated from, and this direction will change as the hypothesis classifier changes. Second, for the case of inseparable data points, getting a bound in terms of the hinge-loss of the best separator with respect to the original data points $z_1, z_2, \ldots$. Our ideas in Section 3 can be extended to get a bound in terms of the hinge-loss of the best separator of surrogate data points $\tilde{x}_1, \tilde{x}_2, \ldots$. However, the more interesting question of getting a bound in terms of unmanipulated data points remains open. In the following, we show an example where Algorithm 2 makes an unbounded number of mistakes when data points are not perfectly separable, even though there exists a separator with bounded hinge-loss.

Example 2. Consider data points $z_0 = (-4, -3), z_1 = (-1, -7), z_2 = (3, 2), z_3 = (-1, 7)$, and $z_4 = (3, -2)$ arriving in order; and then the examples $z_1, z_2, z_3, z_4$ repeat forever. Examples $z_0, z_2$ and $z_4$ have positive labels, and $z_1$ and $z_3$ have negative labels. Suppose that $\alpha = 5$. Note that there exists a vertical linear separator that only makes a mistake on $z_0$. However, as shown below, Algorithm 2 will make an unbounded number of mistakes.

Specifically, after arrival of $z_0$, we have $w = (-4, -3)$. Next, $z_1$ arrives, and since $w^T z_1 / |w| = \alpha$, the algorithm makes a mistake on $z_1$ and classifies it as positive. $z_1$ doesn’t even have to manipulate, i.e. $z_1 = x_1$. Surrogate data point $\tilde{x}_1 = (3, -4)$ is created, and is subtracted from $w$ to get $w = (-7, 1)$. Example $z_2$ arrives and since $w^T z_2 < 0$, manipulation does not help, therefore, $x_2 = z_2$. Example $z_2$ gets classified as negative mistakenly. Also, $\tilde{x}_2 = x_2$. Since a mistake is made, $w$ gets updated to $w + \tilde{x}_2 = (-4, 3)$. Next, negative example $z_3$ arrives and since $w^T z_3 / |w| = \alpha$, it gets misclassified as positive, and $x_3 = z_3$. Surrogate data point $\tilde{x}_3 = (3, 4)$ is created and $w$ is updated to $w - \tilde{x}_3 = (-7, -1)$. Next, positive example $z_4$ arrives, and since $w^T z_4 < 0$, it does not manipulate and $z_4 = x_4$. It gets classified as negative mistakenly. Also, $\tilde{x}_4 = x_4$. $w$ is updated to $w + \tilde{x}_4 = (-4, -3)$. If the same four examples arrive over and over again, Algorithm 2 makes an unbounded number of mistakes. However, there exists a linear classifier $w^* = (1, 0)$ which makes only one mistake in this scenario.
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A Proof of Proposition 1

Proof. Similar to the proof of Theorem 1, we keep track of two quantities, \( w^T w^* \), and \(|w|^2\). First, we show each time a mistake is made, \( w^T w^* \) increases by at least \( \frac{1}{2} \). If we make a mistake on a positive example then,

\[(w + \tilde{x}_t)^T w^* = w^T w^* + \tilde{x}_t^T w^* \geq w^T w^* + 1/2.\]

The last inequality holds by Lemma 4. Similarly if we make a mistake on a negative example, then,

\[(w - \tilde{x}_t)^T w^* = w^T w^* - \tilde{x}_t^T w^* \geq w^T w^* + 1/2.\]

Next, on each mistake we claim that \(|w|^2\) increases by at most \((R + \alpha' + \gamma/2)^2\). If we make a mistake on a positive example then we have:

\[(w + \tilde{x}_t)^T(w + \tilde{x}_t) = |w|^2 + 2\tilde{x}_t^T w + |\tilde{x}_t|^2 \leq |w|^2 + |\tilde{x}_t|^2 \leq |w|^2 + (R + \alpha)^2.\]

The middle inequality is the result of applying Lemma 5. The last inequality comes from \( R = \max_t |y_t| \) implying \( \max_t |\tilde{x}_t| \leq R + \alpha \).

Similarly, if we make a mistake on a negative example \( x_t \), then we have,

\[(w - \tilde{x}_t)^T(w - \tilde{x}_t) = |w|^2 - 2\tilde{x}_t^T w + |\tilde{x}_t|^2 \leq |w|^2 + |\tilde{x}_t|^2 \leq |w|^2 + (R + \alpha)^2.\]

The inequalities hold similar to the previous case.

Therefore, if we make \( M \) mistakes, then \( w^T w^* \geq M/2 \) and \(|w|^2 \leq M(R + \alpha)^2\), or equivalently, \(|w| \leq (R + \alpha)\sqrt{M}\). Using the fact that \( w^T w^*/|w^*| \leq |w| \), we have,

\[M/2|w^*| \leq (R + \alpha)\sqrt{M} \implies \sqrt{M} \leq 2(R + \alpha)|w^*| \implies M \leq 4(R + \alpha)^2|w^*|^2 \implies M \leq 4(R + \alpha' + \gamma/2)^2|w^*|^2.\]

And the proof is complete. \( \square \)