Abstract

Let \( \mathcal{C} \) be a Krull-Schmidt \((n+2)\)-angulated category and \( \mathcal{A} \) be an \( n \)-extension closed subcategory of \( \mathcal{C} \). Then \( \mathcal{A} \) has the structure of an \( n \)-exangulated category in the sense of Herschend–Liu–Nakaoka. This construction gives \( n \)-exangulated categories which are not \( n \)-exact categories in the sense of Jasso nor \((n+2)\)-angulated categories in the sense of Geiss–Keller–Oppermann in general. As an application, our result can lead to a recent main result of Klapproth.

Keywords: \((n+2)\)-angulated categories; \( n \)-exact categories; \( n \)-extension closed subcategories; \( n \)-exangulated categories

2020 Mathematics Subject Classification: 18G80; 18E10

1 Introduction

In \([GKO]\), Geiss, Keller and Oppermann introduced \((n+2)\)-angulated categories. These are a “higher dimensional” analogue of triangulated categories, in the sense that triangles are replaced by \((n+2)\)-angles, that is, morphism sequences of length \((n+2)\). Thus a 3-angulated category is precisely a triangulated category. An important source of examples of \((n+2)\)-angulated categories are certain cluster tilting subcategories of triangulated categories. Jasso \([Ja]\) introduced \( n \)-exact categories as higher analogs of exact categories. Moreover, he also proved that any \( n \)-cluster-tilting subcategory of an exact category is an \( n \)-exact category.

Let \((\mathcal{C}, \Sigma, \Theta)\) be a Krull-Schmidt \((n+2)\)-angulated category and \( \mathcal{A} \) be an \( n \)-extension closed subcategory of \( \mathcal{C} \). An \( \mathcal{A} \)-conflation is a complex

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1}
\]

with \( A_0, A_1, \ldots, A_{n+1} \in \mathcal{A} \) for which there exists a morphism \( f_{n+1} : A_{n+1} \to \Sigma A_0 \) such that

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0
\]

is an \((n+2)\)-angle in \((\mathcal{C}, \Sigma, \Theta)\). We denote by \( \mathcal{E}_\mathcal{A} \) the class of all \( \mathcal{A} \)-conflations.

Klapproth proved the following result.

---

*This work was supported by the National Natural Science Foundation of China (Grant No. 11901190) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19B239).
Theorem 1.1. [K, Theorem 3.2] Let \((\mathcal{C}, \Sigma, \Theta)\) be a Krull-Schmidt \((n+2)\)-angulated category and \(\mathscr{A}\) be an \(n\)-extension closed subcategory of \(\mathcal{C}\). If \(\mathcal{C}(\Sigma \mathscr{A}, \mathscr{A}) = 0\), then \((\mathscr{A}, \mathcal{E}_\mathscr{A})\) is an \(n\)-exact category.

In particular, when \(n = 1\), it is the main theorem of [D] and it was also rediscovered in [J, Proposition 2.5].

Recently, Herschend, Liu and Nakaoka [HLN] introduced the notion of \(n\)-exangulated categories for any positive integer \(n\). It is not only a higher dimensional analogue of extriangulated categories defined by Nakaoka and Palu [NP], but also gives a common generalization of \(n\)-exact categories in the sense of Jasso [Ja] and \((n+2)\)-angulated categories in the sense of Geiss-Keller-Oppermann [GKO].

Nakaoka and Palu [NP] proved extension closed subcategories of triangulated categories are extriangulated categories. This construction gives extriangulated categories which are not exact and triangulated. Based on this idea, we prove the following conclusion.

Theorem 1.2. (see Theorem 3.4 for details) Let \((\mathcal{C}, \Sigma, \Theta)\) be a Krull-Schmidt \((n+2)\)-angulated category and \(\mathscr{A}\) be an \(n\)-extension closed subcategory of \(\mathcal{C}\). Then \(\mathscr{A}\) has the structure of an \(n\)-exangulated category, induced from that of \(\mathcal{C}\).

From this theorem, we know that any \((n+2)\)-angulated category can be viewed as an \(n\)-exangulated category. This generalizes the Proposition 4.5 in [HLN] and also is a higher counterpart of Nakaoka-Palu’s result.

Herschend, Liu and Nakaoka [HLN, Proposition 4.37] gave a description of when an \(n\)-exangulated category can become \(n\)-exact category. Based on this fact and Theorem 1.1, we give a new proof of Theorem 1.1. Our proof method is to avoid proving that \(\mathcal{E}_\mathscr{A}\) is closed under weak isomorphisms, which is very complicated and difficult.

This article is organized as follows. In Section 2, we review some elementary definitions and facts on \(n\)-exangulated categories. In Section 3, we prove our main result in this article.

2 Preliminaries

In this section, let \(\mathcal{C}\) be an additive category and \(n\) be a positive integer. Suppose that \(\mathcal{C}\) is equipped with an additive bifunctor \(\mathcal{E}: \mathcal{C}^{op} \times \mathcal{C} \to \text{Ab}\), where Ab is the category of abelian groups. Next we briefly recall some definitions and basic properties of \(n\)-exangulated categories from [HLN]. We omit some details here, but the reader can find them in [HLN].

For any pair of objects \(A, C \in \mathcal{C}\), an element \(\delta \in \mathcal{E}(C, A)\) is called an \(\mathcal{E}\)-extension or simply an extension. We also write such \(\delta\) as \(A \delta_C\) when we indicate \(A\) and \(C\). The zero element \(A 0_C = 0 \in \mathcal{E}(C, A)\) is called the split \(\mathcal{E}\)-extension. For any pair of \(\mathcal{E}\)-extensions \(A \delta_C\) and \(A \delta_C'\), let \(\delta \oplus \delta' \in \mathcal{E}(C \oplus C', A \oplus A')\) be the element corresponding to \((\delta, 0, 0, \delta')\) through the natural isomorphism \(\mathcal{E}(C \oplus C', A \oplus A') \simeq \mathcal{E}(C, A) \oplus \mathcal{E}(C, A') \oplus \mathcal{E}(C', A) \oplus \mathcal{E}(C', A')\).

For any \(a \in \mathcal{C}(A, A')\) and \(c \in \mathcal{C}(C', C)\), \(\mathcal{E}(c, A)(\delta) \in \mathcal{E}(C, A')\) and \(\mathcal{E}(c, A)(\delta) \in \mathcal{E}(C', A)\) are simply denoted by \(a \delta\) and \(c \delta\), respectively.
Let \( A\delta_C \) and \( A'\delta'_{C'} \) be any pair of \( \mathcal{E} \)-extensions. A morphism \((a, c): \delta \to \delta'\) of extensions is a pair of morphisms \( a \in \mathcal{E}(A, A') \) and \( c \in \mathcal{E}(C, C') \) in \( \mathcal{E} \), satisfying the equality \( a_c \delta = c^*\delta' \). Then the functoriality of \( \mathcal{E} \) implies \( \mathcal{E}(c, a) = a_c(c^*\delta) = c^*(a_\delta) \).

**Definition 2.1.** [HLN, Definition 2.7] Let \( \mathcal{C}_\mathcal{E} \) be the category of complexes in \( \mathcal{E} \). As its full subcategory, define \( \mathcal{C}^{n+2}_\mathcal{E} \) to be the category of complexes in \( \mathcal{E} \) whose components are zero in the degrees outside of \( \{0, 1, \ldots, n+1\} \). Namely, an object in \( \mathcal{C}^{n+2}_\mathcal{E} \) is a complex \( X^* = \{X_i, d_i^X\} \) of the form

\[
X_0 \xrightarrow{d_0^X} X_1 \xrightarrow{d_1^X} \cdots \xrightarrow{d_{n-1}^X} X_n \xrightarrow{d_n^X} X_{n+1}.
\]

We write a morphism \( f^*: X^* \to Y^* \) simply \( f^* = (f^0, f^1, \ldots, f^{n+1}) \), only indicating the terms of degrees \( 0, 1, \ldots, n+1 \).

**Definition 2.2.** [HLN, Definition 2.23] Let \( s \) be an exact realization of \( \mathcal{E} \).

1. An \( n \)-exangle \( \langle X^*, \delta \rangle \) is called an \( s \)-distinguished \( n \)-exangle if it satisfies \( s(\delta) = [X^*] \). We often simply say distinguished \( n \)-exangle when \( s \) is clear from the context.

2. An object \( X^* \in \mathcal{C}^{n+2}_\mathcal{E} \) is called an \( s \)-conflation or simply a conflation if it realizes some extension \( \delta \in \mathcal{E}(X_{n+1}, X_0) \).

3. A morphism \( f \) in \( \mathcal{E} \) is called an \( s \)-inflation or simply an inflation if it admits some conflation \( X^* \in \mathcal{C}^{n+2}_\mathcal{E} \) satisfying \( d_X^i = f \).

4. A morphism \( g \) in \( \mathcal{E} \) is called an \( s \)-deflation or simply a deflation if it admits some conflation \( X^* \in \mathcal{C}^{n+2}_\mathcal{E} \) satisfying \( d_X^i = g \).

**Definition 2.3.** [HLN, Definition 2.32] An \( n \)-exangulated category is a triplet \( (\mathcal{E}, \mathcal{E}, s) \) of additive category \( \mathcal{E} \), additive bifunctor \( \mathcal{E}^{op} \times \mathcal{E} \to Ab \), and its exact realization \( s \), satisfying the following conditions.

1. (EA1) Let \( A \xrightarrow{f} B \xrightarrow{g} C \) be any sequence of morphisms in \( \mathcal{E} \). If both \( f \) and \( g \) are inflations, then so is \( g \circ f \). Dually, if \( f \) and \( g \) are deflations, then so is \( g \circ f \).

2. (EA2) For \( \rho \in \mathcal{E}(D, A) \) and \( c \in \mathcal{E}(C, D) \), let \( A(X^*, c^*\rho)_C \) and \( A(Y^*, \rho)_D \) be distinguished \( n \)-exangles. Then \( (\id_A, c) \) has a good lift \( f^* \), in the sense that its mapping cone gives a distinguished \( n \)-exangle \( \langle M^f, (d^X_i)_*, \rho \rangle \).

(\text{EA}^{op}) \ Dual of (EA2).

Note that the case \( n = 1 \), a triplet \( (\mathcal{E}, \mathcal{E}, s) \) is a 1-exangulated category if and only if it is an extriangulated category, see [HLN, Proposition 4.3].

**Example 2.4.** From [HLN, Proposition 4.34] and [HLN, Proposition 4.5], we know that \( n \)-exact categories and \( (n+2) \)-angulated categories are \( n \)-exangulated categories. There are some other examples of \( n \)-exangulated categories which are neither \( n \)-exact nor \( (n+2) \)-angulated, see [HLN, LZ, HZZ].

Let \((\mathcal{C}, E, s)\) be an \(n\)-exangulated category and \(F \subseteq E\) be an additive subfunctor (see [HLN, Definition 3.7]). For a realization \(s\) of \(E\), define \(s|_F\) to be the restriction of \(s\) onto \(F\). Namely, it is defined by \(s|_F(\delta) = s(\delta)\) for any \(F\)-extension \(\delta\).

**Lemma 2.5.** [HLN, Proposition 3.16] Let \((\mathcal{C}, E, s)\) be an \(n\)-exangulated category. For any additive subfunctor \(F \subseteq E\), the following statements are equivalent.

1. \((\mathcal{C}, F, s|_F)\) is an \(n\)-exangulated category.
2. \(s|_F\)-inflations are closed under composition.
3. \(s|_F\)-deflations are closed under composition.

### 3 Main result

In this section, when we say that \(\mathcal{A}\) is a subcategory of an additive category \(\mathcal{C}\), we always assume that \(\mathcal{C}\) is full, and closed under isomorphisms, direct sums and direct summands.

We first recall the notion of \(n\)-extension closed from [L1].

**Definition 3.1.** [L1, Definition 3.6] Let \((\mathcal{C}, \Sigma, \Theta)\) be an \((n+2)\)-angulated category. A subcategory \(\mathcal{A}\) of \(\mathcal{C}\) is called \(n\)-extension closed if for each morphism \(f_{n+1}: A_{n+1} \to \Sigma^n A_0\) with \(A_0, A_{n+1} \in \mathcal{A}\), there exists an \((n+2)\)-angle

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0
\]

with terms \(A_1, A_2, \ldots, A_n \in \mathcal{A}\).

The following lemma can be found in [L2, Lemma 2.5].

**Lemma 3.2.** [L2, Lemma 2.5] Let \((\mathcal{C}, \Sigma, \Theta)\) be a Krull-Schmidt \((n+2)\)-angulated category and

\[
A\bullet: A_0 \xrightarrow{(f_0, g_0)} A_1 \oplus B_1 \xrightarrow{(f_1, g_1)} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0
\]

is an \((n+2)\)-angle in \(\mathcal{C}\). If \(g_0 = 0\), then \(A\bullet \simeq A\bullet' \oplus B\bullet\), where

\[
A\bullet': A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0
\]

and

\[
B\bullet: 0 \to B_1 \xrightarrow{1} B_1 \to 0 \to \cdots \to 0 \to 0 \to 0
\]

is two an \((n+2)\)-angle.

Let \((\mathcal{C}, \Sigma, \Theta)\) be an \((n+2)\)-angulated category. Since \(\Sigma: \mathcal{C} \xrightarrow{\sim} \mathcal{C}\) is an automorphism, then \(\Sigma\) gives an additive bifunctor

\[
\mathbb{E}_\Sigma = \mathcal{C}(-, \Sigma-): \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab},
\]

defined by the following.
(i) For any $A, C \in \mathcal{C}$, $E_\Sigma(C, A) = \mathcal{C}(C, \Sigma A)$;

(ii) For any $a \in \mathcal{C}(A, A')$ and $c \in \mathcal{C}(C, C')$, the map $E_\Sigma(c, a) : \mathcal{C}(C, \Sigma A) \to \mathcal{C}(C', \Sigma A)$ sends $\delta \in \mathcal{C}(C, \Sigma A)$ to $c^* a \delta = (\Sigma a) \circ \delta \circ c$.

For each $\delta \in E_\Sigma(C, A)$, we complete it into an $(n + 2)$-angle

$$A \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} C \xrightarrow{\delta} \Sigma A_0$$

Define $s_\Theta(\delta) = [X^*]$ by using $X^* \in \mathcal{C}^{n+2}_{(A, C)}$ given by

$$A \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} C$$

The following result shows that any $(n + 2)$-angulated category can be viewed as an $n$-exangulated category.

**Theorem 3.3.** [HLN, Proposition 4.5] With the above definition, $(\mathcal{C}, E_\Sigma, s_\Theta)$ is an $n$-exangulated category.

Let $(\mathcal{C}, \Sigma, \Theta)$ be an $(n + 2)$-angulated category and $\mathcal{A}$ be an $n$-extension closed subcategory of $\mathcal{C}$. We define $E_\mathcal{A}$ to be the restriction of $E_\Sigma$ onto $\mathcal{A}^{\text{op}} \times \mathcal{A}$, and define $s_\mathcal{A}$ by restricting $s_\Theta$.

The following construction gives $n$-exangulated categories which are not $n$-exact nor $(n + 2)$-angulated in general.

**Theorem 3.4.** Let $(\mathcal{C}, \Sigma, \Theta)$ be a Krull-Schmidt $(n + 2)$-angulated category and $\mathcal{A}$ be an $n$-extension closed subcategory of $\mathcal{C}$. Then $(\mathcal{A}, E_\mathcal{A}, s_\mathcal{A})$ is an $n$-exangulated category.

**Proof.** It is straightforward to verify that $E_\mathcal{A}$ is an additive subfunctor of $E_\Sigma$. By [K, Lemma 3.8], we know that that $s_\mathcal{A}$-inflations are closed under composition. By Lemma 2.5, we have that $(\mathcal{A}, E_\mathcal{A}, s_\mathcal{A})$ is an $n$-exangulated category. \qed

**Proposition 3.5.** [HLN, Proposition 4.37] Let $(\mathcal{C}, E, s)$ be an $n$-exangulated category. Assume that any $s$-inflation is monomorphic, and any $s$-deflation is epimorphic in $\mathcal{C}$. Note that this is equivalent to assuming that any $s$-conflation is $n$-exact sequence. We denote the class of all $s$-conflations by $\mathcal{X}$. If $(\mathcal{C}, E, s)$ satisfies the following conditions (a) and (b) for any pair of morphisms $A \xrightarrow{a} B \xrightarrow{b} C$ in $\mathcal{C}$, then $(\mathcal{C}, \mathcal{X})$ becomes an $n$-exact category in the sense of Jasso [Ja, Definition 4.2].

(a) If $b \circ a$ is an $s$-inflation, then so is $a$.

(b) If $b \circ a$ is an $s$-deflation, then so is $b$.

In Theorem 3.4, we know that $(\mathcal{A}, E_\mathcal{A}, s_\mathcal{A})$ is an $n$-exangulated category. We assume that any $s_\mathcal{A}$-inflation is monomorphic, and any $s_\mathcal{A}$-deflation is epimorphic in $\mathcal{A}$. Note that this is equivalent to assuming that any $s$-conflation is $n$-exact sequence. Thus we denote the class of all $s_\mathcal{A}$-conflations by $\mathcal{E}_\mathcal{A}$.

Now we give a new proof of Klapproth’s result.
Theorem 3.6. [K, Theorem 3.2] Let \((\mathcal{C}, \Sigma, \Theta)\) be a Krull-Schmidt \((n+2)\)-angulated category and \(\mathcal{A}\) be an \(n\)-extension closed subcategory of \(\mathcal{C}\). If \(\mathcal{C}(\mathcal{A}, \mathcal{A}) = 0\), then \((\mathcal{A}, E_{\mathcal{A}}, s_{\mathcal{A}})\) is an \(n\)-exangulated category.

Proof. By Theorem 3.4, we know that \((\mathcal{A}, E_{\mathcal{A}}, s_{\mathcal{A}})\) is an \(n\)-exangulated category.

We first claim that any \(s_{\mathcal{A}}\)-inflation is monomorphic in \(\mathcal{A}\).

Assume that \(f: A \rightarrow B\) is an \(s_{\mathcal{A}}\)-inflation in \(\mathcal{A}\). By definition there exists an \((n+2)\)-angle

\[
A \xrightarrow{f} B \xrightarrow{g_1} C_2 \xrightarrow{g_2} \cdots \xrightarrow{g_n} C_n \xrightarrow{g_{n+1}} C_{n+1} \xrightarrow{g_{n+2}} \Sigma A
\]

with terms \(C_2, C_3, \ldots, C_n \in \mathcal{A}\).

Now we prove that \(f\) is monomorphism in \(\mathcal{A}\). Let \(h: C \rightarrow A\) be a morphism in \(\mathcal{A}\) such that \(fh = 0\). Apply the functor \(\mathcal{C}(C, -)\) to the above \((n+2)\)-angle, we have the following exact sequence:

\[
\mathcal{C}(C, \Sigma^{-1}C_{n+1}) \rightarrow \mathcal{C}(C, A) \xrightarrow{\mathcal{C}(C, f)} \mathcal{C}(C, B)
\]

where the leftmost term vanishes since \(\mathcal{C}(C, \Sigma^{-1}C_{n+1}) \simeq \mathcal{C}(\Sigma C, C_{n+1}) = 0\). This shows that \(\mathcal{C}(C, f)\) is a monomorphism which implies \(h = 0\) since \(fh = 0\). This shows that \(f\) is monomorphism in \(\mathcal{A}\).

In a similar way, one can show that any \(s_{\mathcal{A}}\)-deflation is epimorphic in \(\mathcal{A}\).

For any pair of morphisms \(A \xrightarrow{a} B \xrightarrow{h} C\) in \(\mathcal{A}\), we claim that if \(b \circ a\) is an \(s_{\mathcal{A}}\)-inflation, then so is \(a\). Indeed, since \(h_0 := b \circ a\) is an \(s_{\mathcal{A}}\)-inflation, by definition there exists an \((n+2)\)-angle

\[
A \xrightarrow{a_0} C \xrightarrow{h_1} X_2 \xrightarrow{h_2} \cdots \xrightarrow{h_{n-1}} X_n \xrightarrow{h_n} X_{n+1} \xrightarrow{h_{n+1}} \Sigma A
\]

with terms \(X_2, X_3, \ldots, X_n \in \mathcal{A}\).

For the morphism \(X_{n+1} \xrightarrow{d_{n+1} = \Sigma a_0 h_{n+1}} \Sigma B\) with \(B, C \in \mathcal{A}\), since \(\mathcal{A}\) is \(n\)-extension closed, then there exists an \((n+2)\)-angle

\[
B \xrightarrow{d_0} Y_1 \xrightarrow{d_1} Y_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} Y_n \xrightarrow{d_n} X_{n+1} \xrightarrow{d_{n+1}} \Sigma B
\]

Consider the following commutative diagram of \((n+2)\)-angles

\[
\begin{array}{ccc}
\Sigma^{-1}X_{n+1} & \xrightarrow{h_{-1}} & A \\
\downarrow & & \downarrow a \\
\Sigma^{-1}X_{n+1} & \xrightarrow{d_{-1}} & B \\
\downarrow & & \downarrow d_0 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\downarrow & & \downarrow d_0 \\
\Sigma^{-1}X_{n+1} & \xrightarrow{d_{-1}} & B \\
\downarrow & & \downarrow d_0 \\
\downarrow & & \downarrow \varphi_1 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\downarrow & & \downarrow \varphi_n \\
\Sigma^{-1}X_{n+1} & \xrightarrow{h_{-1}} & A \\
\end{array}
\]

where \(h_{-1} = (-1)^n \Sigma^{-1}h_{n+1}\) and \(d_{-1} = (-1)^n \Sigma^{-1}d_{n+1}\). By [BT, Lemma 4.1], there are \(\varphi_1, \varphi_2, \ldots, \varphi_n\) which give a morphism of \((n+2)\)-angles

\[
\begin{array}{ccc}
\Sigma^{-1}X_{n+1} & \xrightarrow{h_{-1}} & A \\
\downarrow & & \downarrow a \\
\Sigma^{-1}X_{n+1} & \xrightarrow{d_{-1}} & B \\
\downarrow & & \downarrow \varphi_1 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
\downarrow & & \downarrow \varphi_n \\
\Sigma^{-1}X_{n+1} & \xrightarrow{h_{-1}} & A \\
\end{array}
\]
and the sequence

$$A \xrightarrow{(-h_a)} C \oplus B \rightarrow X_2 \oplus Y_1 \rightarrow \cdots \rightarrow X_n \oplus Y_{n-1} \rightarrow Y_n \rightarrow \Sigma A$$

is an $(n+2)$-angle. We observe that $X_2 \oplus Y_1, \cdots, X_n \oplus Y_{n-1}, Y_n \in \mathcal{A}$. This shows that $(-h_a)$ is an $\mathcal{s}_{\mathcal{A}}$-inflation. Since $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) : C \oplus B \rightarrow B \oplus C$ is an isomorphism, it is in particular an an $\mathcal{s}_{\mathcal{A}}$-inflation. Thus $(\begin{smallmatrix} 0 \\ a \end{smallmatrix}) = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) (-h_a)$ is also an $\mathcal{s}_{\mathcal{A}}$-inflation.

Since $(\begin{smallmatrix} 1 \\ -b \\ 0 \end{smallmatrix}) : B \oplus C \rightarrow B \oplus C$ is an isomorphism, it is in particular an an $\mathcal{s}_{\mathcal{A}}$-inflation. Thus $(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}) = (\begin{smallmatrix} 1 \\ -b \\ 0 \end{smallmatrix}) (\begin{smallmatrix} a \\ 0 \end{smallmatrix})$ is also an $\mathcal{s}_{\mathcal{A}}$-inflation. By definition there exists an $(n+2)$-angle

$$A_{\bullet} : A \xrightarrow{(a)} B \oplus C \xrightarrow{(f_1, q_1)} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0$$

with $A_2, A_3, \cdots, A_{n+1} \in \mathcal{A}$. By Lemma 3.2, we obtain $A_{\bullet} \simeq A_{\bullet}' \oplus C_{\bullet}$, where

$$A_{\bullet}' : A \xrightarrow{a} B \xrightarrow{f_{n+1}} A_2 \xrightarrow{f_1} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \Sigma A_0$$

and

$$C_{\bullet} : 0 \rightarrow C \xrightarrow{1} C \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0$$

is two an $(n+2)$-angle. Since $A_2 \in \mathcal{A}$, we also have $A_2 \in \mathcal{A}$. This shows that $a$ is also an $\mathcal{s}_{\mathcal{A}}$-inflation.

Similarly, we also can prove that if $b \circ a$ is an $\mathcal{s}_{\mathcal{A}}$-deflations, then so is $b$.

By Proposition 3.5, we have that $(\mathcal{A}, \mathcal{E}_{\mathcal{A}})$ is an $n$-exact category. \qed

**Remark 3.7.** For the proof of Theorem 3.6, our proof method is to avoid proving that $\mathcal{E}_{\mathcal{A}}$ is closed under weak isomorphisms, which is very complicated and difficult.

**Acknowledgments**

The author would like to thank Yonggang Hu and Tiwei Zhao for the helpful discussions.

**References**

[BT] P. Bergh, M. Thaule. The axioms for $n$-angulated categories. Algebr. Geom. Topol. 13 (2013), no. 4, 2405–2428.

[D] M. J. Dyer. Exact subcategories of triangulated categories. [https://www3.nd.edu/~dyer/papers/extri.pdf](https://www3.nd.edu/~dyer/papers/extri.pdf), preprint, 2005.

[GKO] C. Geiss, B. Keller, S. Oppermann. $n$-angulated categories. J. Reine Angew. Math. 675 (2013), 101–120.

[HLN] M. Herschend, Y. Liu, H. Nakaoka. $n$-exangulated categories (I): Definitions and fundamental properties. J. Algebra 570 (2021), 531–586.
[HZZ] J. Hu, D. Zhang, P. Zhou. Two new classes of $n$-exangulated categories. J. Algebra 568 (2021), 1–21.

[Ja] G. Jasso. $n$-abelian and $n$-exact categories. Math. Z. 283 (2016), no. 3-4, 703–759.

[J] P. Jørgensen. Abelian subcategories of triangulated categories induced by simple minded systems. arXiv: 2010.11799, 2020.

[K] C. Klapproth. $n$-exact categories arising from $(n + 2)$-angulated categories. arXiv: 2108.04596, 2021.

[L1] Z. Lin. $n$-angulated quotient categories induced by mutation pairs. Czechoslovak Math. J. 65(140) (2015), no. 4, 953–968.

[L2] Z. Lin. Idempotent completion of $n$-angulated categories. Appl Categor Struct (2021). https://doi.org/10.1007/s10485-021-09644-y.

[LZ] Y. Liu, P. Zhou. Frobenius $n$-exangulated categories. J. Algebra 559 (2020), 161–183.

[NP] H. Nakaoka, Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Différ. Catég. 60 (2019), no. 2, 117–193.

Panyue Zhou
College of Mathematics, Hunan Institute of Science and Technology, 414006 Yueyang, Hunan, People’s Republic of China.
E-mail: panyuezhou@163.com