Asymptotic in a class of network models with an increasing sub-Gamma degree sequence

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Abstract

For the differential privacy under the sub-Gamma noise, we derive the asymptotic properties of a class of network models with binary values with general link function. In this paper, we release the degree sequences of the binary networks under a general noisy mechanism with the discrete Laplace mechanism as a special case. We establish the asymptotic result including both consistency and asymptotically normality of the parameter estimator when the number of parameters goes to infinity in a class of network models. Simulations and a real data example are provided to illustrate asymptotic results.

Key words: Consistency and asymptotically normality; Network data; Differential privacy; sub-Gamma degree sequence

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1 Introduction

In network data analysis, the privacy of network data has received wide attentions as the disclosed data is likely to contain sensitive information about individuals and their social relationships (sexual relationships, email exchanges, money transfers, etc). It is obvious that the analysis of these data is used to address various important problems like disease transmission, fraud detection, accurate marketing, etc. The demand for privacy protection has led to a rapid development
of algorithms for safe release of network data or the aggregation of network data (see Zhou et al. (2008), Yuan et al. (2011), Cutillo et al. (2010), and Lu and Miklau (2014)). At the same time, analysis based on noisy data presents challenges for statistical inference. Moreover, many network data structures are not standardized (see Fienberg (2012), Mosler (2017)). Therefore, parameter estimation and asymptotic property exploration with noisy data are still under study.

The Erdős-Rényi model (see Erdos et al. (1960)) is generally acknowledged as one of the earliest random binary graph models, in which each edge occurs with the same probability independent of any other edge. However, it lacks the ability to capture the extent of degree heterogeneity commonly associated with network data in practice. To capture the heterozygous of nodes’ degree, a class of models have come into use for analyzing binary undirected networks network data. Random binary weighted graph models are often used to study network data and these models capture real networks based on degree sequences [Albert and Barabási (2002)]. The $\beta$-model has undirected binary weighted, which is known for binary arrays whose distribution only depends on the row and column totals (see Britton et al. (2006); Bickel et al. (2011); Zhao et al. (2012); Hillar and Wibisono (2013)). Many scholars have focused on the studies of Maximum Likelihood Estimator (MLE) of the $\beta$-model (see Chatterjee et al. (2011); Blitzstein and Diaconis (2011); Rinaldo et al. (2013)) as the number of parameters goes to infinity. Yan and Xu (2013) further derived its asymptotic normality. A similar model is Fan and Lu (2002) log-linear model, where the edge probability $p_{ij}$ between vertices $i$ and $j$ is $w_i w_j / (\sum_{k=1}^{n} w_k)$ under the normalization constraint $w_i^2 \leq (\sum_{k=1}^{n} w_k) (i = 1, ..., n)$, where $w_i$ is referred to as the weight of vertex $i$. Moreover, Olhede and Wolfe (2012) derived the sampling properties of undirected networks parameterized by independent Bernoulli trials. Karwa and Slavković (2016) paid attention to the discrete Laplace distribution noise addition process through de-noising method and obtained the asymptotic properties of MLE of the $\beta$-model to achieve valid inference. However, asymptotic properties of the parameter estimator in models of this class with a general noisy mechanism are still unknown.

For network data privacy, a common method is to add noise into the degree sequence, such as using Laplace Mechanism to release the sufficiency degree sequence of the graph and to reduce the error between the true sequence and the release sequence [Dwork et al. (2006)]. However, the process of adding noise can easily lead to inconsistent estimation of some parameters. To do this, Karwa and Slavković (2016) paid attention to the discrete Laplace distribution noise addition process through de-noising method and obtained the asymptotic properties of MLE of the $\beta$-model to achieve valid inference. Karwa and Slavković (2016) only considered the case of $\beta$–model with the discrete Laplace distribution noise addition process. In this paper, we consider a general distribution of the noisy variables in a class of binary network models.

In this paper, our main contribution is to provide a method for establishing the asymptotic results in undirected weighted random graph models with a noisy degree sequence, which is different from MLE. Firstly, we show that the estimator of the parameter in undirected weighted random graph models based on the moment equation in which the unobserved original sequence is directly replaced by the noisy degree sequence. Secondly, the probability-mass or density function of the
edge $a_{ij}$ only depends on the sum of $\alpha^*_i$ and $\alpha^*_j$, where $\alpha^*_i$ denotes the strength parameter of vertex $i$. And we consider a general distribution of the noisy variables in a class of binary network models.

For the rest of this article, we state as follows. In Section 2, we give the definitions of edge differential privacy and differential privacy degree sequence. In Section 3, we give a uniform asymptotic result which contains the consistency and asymptotic normality of the moment estimator when the number of parameters goes to infinity. In Section 4, we illustrate several applications of our main results. Proofs are given in the Appendix.

**Notation:** For $0 < q \leq \infty$, we write $|\beta|_q := (\sum_{i=1}^p |\beta_i|^q)^{1/q}$ as the $\ell^q$-norm of a $p$-dimensional vector $\beta$. If $q = \infty$, we have $|\beta|_\infty := \max_{i=1,\ldots,p} |\beta_i|$. For a subset $C \subset \mathbb{R}^n$, let $C^0$ and $\overline{C}$ denote the interior and closure of $C$ in $\mathbb{R}^n$, respectively. Let $\Omega(x, r)$ denote the open ball $\{y : \|x - y\| < r\}$, and $\overline{\Omega(x, r)}$ be its closure.

## 2 Null models for undirected network data

### 2.1 Several Models

Following Yan and Xu (2013), we consider an undirected graph $G_n$ on $n \geq 2$ agents labeled by $1,\ldots,n$. Let $a_{ij} \in \{0,1\}$ be the weight of the undirected edge between $i$ and $j$. That is, if there is a link between $i$ and $j$, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$. Denote $A = (a_{ij})_{n \times n}$ as the symmetric adjacency matrix of $G_n$. We assume that there are no self-loops, i.e., $a_{ii} = 0$. Define $d_i = \sum_{j \neq i} a_{ij}$ as the degree of vertex $i$ and $d = (d_1,\ldots,d_n)^\top$ as the degree sequence of the graph $G$. We suppose that the adjacency matrix $A$ has independent Bernoulli elements such that $P(a_{ij} = 1) = p_{ij}$ and specify corresponding family of probability null models for $A$. Here $\alpha^* = (\alpha^*_1,\alpha^*_2,\ldots,\alpha^*_n)^\top$ is a parameter vector. The parameter $\alpha^*_i$ quantifies the effect of the vertex $i$. To this end, let $\varepsilon(\cdot, \cdot) : \mathbb{R}^2 \mapsto \mathbb{R}$ be a smooth bivariate function satisfying $\varepsilon(x, y) = \varepsilon(y, x)$.

Consider the model $M_\varepsilon$ specified $p_{ij}$ as

$$M_\varepsilon : \log p_{ij} = \alpha^*_i + \alpha^*_j + \varepsilon(\alpha^*_i + \alpha^*_j)$$

so that we obtain a class of log-linear models indexed by $\varepsilon$. In fact, this class encompasses three common choices of link functions:

$$M_{\log} : \log p_{ij} = \alpha^*_i + \alpha^*_j,$$

$$M_{\logit} : \text{logit}(p_{ij}) = \alpha^*_i + \alpha^*_j,$$

where $\text{logit}(x) = \frac{1}{1+e^{-x}},$

$$M_{\text{cloglog}} : \log(-\log(1-p_{ij})) = \alpha^*_i + \alpha^*_j.$$

To see this, set $\varepsilon(\alpha^*_i, \alpha^*_j) = -\log\{1 + \exp(\alpha^*_i + \alpha^*_j)\}$ for the model $M_{\logit}$. As we have seen, the logit-link model $M_{\logit}$ is an undirected version of Holland and Leinhardt (1981) exponential family random graph model without reciprocal parameter. As noted by Yan and Xu (2013), the degree
sequence of $\mathcal{G}$ is sufficient for $\alpha$ in this case, and they derived its asymptotic normality. The log-link model $M_{\log}$ can be considered as an undirected version of expected degree model with $\epsilon(\alpha^*_i, \alpha^*_j) = 0$ constructed by Fan and Lu (2002). Setting $\epsilon(\alpha^*_i, \alpha^*_j) = \log\{1 - \exp(\alpha^*_i + \alpha^*_j)\} - (\alpha^*_i + \alpha^*_j)$, we get the complementary log-log link model $M_{\text{cloglog}}$ (see McCullagh and Nelder (1989)).

2.2 The sub-Exponential/-Gamma noisy sequence

In this section, we will prepare the probability and distribution preliminary for later network analysis. This section can be divided into two parts. In the first part, we are going to recap the definition of a specific type of distributions and state their basic properties.

**Definition 2.1 (Sub-Gaussian distribution).** A random variable $X \in \mathbb{R}$ with mean zero is sub-Gaussian with variance proxy $\sigma^2$ if its MGF satisfies

$$E[\exp(sX)] \leq \exp\left(\frac{\sigma^2 s^2}{2}\right), \quad \forall s \in \mathbb{R}.$$  

In this case we write $X \sim \text{subG} (\sigma^2)$.

Similarly, we define that a random variable is called sub-exponential if its survival function is bounded by that of a particular exponential distribution.

**Definition 2.2 (Sub-exponential distribution).** A random variable $X \in \mathbb{R}$ with mean zero is sub-exponential with parameter $\sigma^2$ (denoted by $X \sim \text{subE}(\lambda)$) if its MGF satisfies

$$Ee^{sX} \leq e^{\frac{s^2 \lambda^2}{2}} \text{ for all } |s| < \frac{1}{\lambda}. \quad (2.5)$$

Obviously, sub-Gaussian random variables are sub-exponential but not vice versa. And there are some equivalent definitions of sub-exponential distributions, see Rigollet and Hütter (2019), which can lead to sub-exponential norm used in concentration inequalities. The equivalent definitions and other details can be seen in the Appendix.

**Definition 2.3 (Sub-exponential norm).** The sub-exponential norm of $X$, denoted $\|X\|_{\psi_1}$, is defined as

$$\|X\|_{\psi_1} = \inf \{t > 0 : E \exp(|X|/t) \leq 2\}. \quad (2.6)$$

The above norm provides us with a useful tool to connect MGF and the defined norm, and hence make it possible to give concentrations for the sub-exponential variables. The following lemma confirms Definition 2.3 would give a concise form of concentrations.

**Lemma 2.1 (Properties of sub-exponential norm).** If $E \exp(|X|/\|X\|_{\psi_1}) \leq 2$, then we have

(a) Tail bounds

$$P\{|X| > t\} \leq 2 \exp(-t/\|X\|_{\psi_1}) \text{ for all } t \geq 0; \quad (2.7)$$
(b) **Moment bounds**

\[ \mathbb{E}|X|^k \leq 2\|X\|_{\psi_1}^k k! \quad \text{for all integer } k \geq 1; \]

(c) *If \( \mathbb{E}X = 0 \), we get the MGF bounds*

\[ \mathbb{E}e^{sX} \leq e^{(2\|X\|_{\psi_1})^2s^2} \quad \text{for all } |s| < 1/(2\|X\|_{\psi_1}) \] (2.8)

which gives \( X \sim \text{subE}(2\|X\|_{\psi_1}) \).

Lemma 2.1(c) implies that the following user-friendly concentration inequality would contain all known constant. One should note that Theorem 2.8.1 of Vershynin (2018) includes an unspecified constant, so it is inefficacious when constructing non-asymptotic confident interval for sub-exponential sample mean.

**Corollary 2.1** (Concentration for sub-exponential sum of r.v.s, Zhang and Chen (2021)). Let \( \{X_i\}_{i=1}^n \) be zero mean independent sub-exponential distribution with \( \|X_i\|_{\psi_1} \leq \infty \). Then for every \( t \geq 0 \),

\[ \mathbb{P}(\sum_{i=1}^n X_i \geq t) \leq 2 \exp\{-\frac{1}{4} \left( \sum_{i=1}^{n} \frac{t^2}{2\|X_i\|_{\psi_1}^2} \right) \wedge \frac{t}{\max_{1 \leq i \leq n} \|X_i\|_{\psi_1}} \}. \]

When network data containing sensitive individuals’ and their relationships’ information can not be directly made public due to privacy concerns, to guarantee the confidence information not to be disclosed, they must be carefully treated before being made public. A simple method to deal with the privacy problem is the anonymization technique by removing identifiers from a network and releasing an anonymized isomorphic network, but it has been demonstrated that it is easy to attack [e.g., Narayanan and Shmatikov (2009), Wondracek et al. (2010), Wang et al. (2019)]. Dwork et al. (2006) developed a rigorous definition of Differential Privacy (DP) to achieve privacy protection. Hay M. and D. (2009) and Karwa and Slavković (2016) used the Laplace mechanism to provide privacy protection in which independent and identically distributed Laplace random variables are added into the input data. This mechanism satisfies “differential privacy” [Dwork et al. (2006)]. Here, we consider a general distribution for the noisy variables with the Laplace distribution as a special case.

**Example 2.1** (Laplace r.v.s). A r.v. \( X \) follows a Laplace distribution (Laplace(\( \mu, b \)), \( \mu \in \mathbb{R}, b > 0 \)) if its probability density function is \( f(x) = \frac{1}{2b}e^{-|x-\mu|/b} \). The Laplace distribution is a distribution of the difference of two independent identical exponential distributed r.vs, thus it is also sub-exponential distributed by using Proposition 6.1(a). The graph of \( f(x) \) are like two exponential distributions which are spliced together back-to-back.

**Example 2.2** (Geometric distributions). The geometric distribution \( X \sim \text{Geo}(q) \) for r.v. \( X \) is given by \( \mathbb{P}(X = k) = (1 - q)q^{k-1}, \quad q \in (0, 1), k = 1, 2, \ldots \). The mean and variance of Geo(\( q \)) are \( \frac{1-q}{q} \) and \( \frac{1-q}{q^2} \) respectively. Apply Lemma 4.3 in Hillar and Wibisono (2013), we have \( (\mathbb{E}|X|^k)^{1/k} < \)
Followed by the triangle inequality applied to the $p$-norm and Jensen’s inequality for $k \geq 1$, we have

$$E[|X - EX|^k]^{1/k} \leq E[|X|^k]^{1/k} + |E[X]| \leq 2E[|X|^k]^{1/k} \leq \frac{4k}{-\log(1-q)}.$$

Then Proposition 6.1(3) shows that the centralized geometric distribution is sub-exponential with $K_3 = \frac{4}{-\log(1-q)}$.

Example 2.3 (Discrete Laplace r.vs). A r.v. $X$ obeys the discrete Laplace distribution with parameter $p \in (0, 1)$, denoted by $DL(p)$, if

$$f_p(k) = P(X = k) = \frac{1 - p}{1 + p} |p|^k, \quad k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

Similar to Laplace distribution, the discrete Laplace r.v. is the difference of two independent identical geometric distributed r.vs (see Proposition 3.1 in Inusah and Kozubowski (2006)). Since geometric distribution is sub-exponential in the previous example, the Proposition 6.1(a) implies that discrete Laplace is also sub-exponential distributed. In differential privacy of network models, the noises are assumed from the discrete Laplace distribution (see Fan et al. (2020) and references therein).

In statistical applications, we sometimes do not expect the bounded assumption in Hoeffding’s inequality, the following Bernstein’s inequality for a sum of independent random variables allows us to estimate the tail probability by a weaker version of exponential condition on the growth of the $k$-moment (like a condition of the exponential MGF) without any assumption of boundedness.

Lemma 2.2 (Bernstein’s inequality). The centred independent random variables $X_1, \ldots, X_n$ satisfy the growth of moments condition

$$E|X_i|^k \leq \frac{1}{2} \nu_i^2 \kappa_i^{-2} k!, \quad (i = 1, 2, \cdots, n), \quad \text{for all } k \geq 2 \quad (2.9)$$

where \( \{\kappa_i\}_{i=1}^n \) and \( \{\nu_i\}_{i=1}^n \) are constants independent of $k$. Denote $\nu_n^2 = \sum_{i=1}^n \nu_i^2$ (the fluctuation of sums) and $\kappa = \max_{1 \leq i \leq n} \kappa_i$. Then we have $Ee^{sX_i} \leq e^{s^2 \nu_n^2/(2 - 2\kappa_i|s|)}$. And for $t > 0$

$$P(|S_n| \geq t) \leq 2 \exp\left(-\frac{t^2}{2\nu_n^2 + 2\kappa t}\right), \quad P(|S_n| \geq \sqrt{2\nu_n^2 t + \kappa t}) \leq 2e^{-t}. \quad (2.10)$$

The proof Bernstein’s inequality for the sum of independent random variables can be founded in p119 of Giné and Nickl (2015).

Like the sub-Gaussian, Boucheron et al. (2013) defines the sub-Gamma r.v. based on the right tail and left tail with variance factor $\nu$ and scale factor $b$. 

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**Definition 2.4** (Sub-Gamma r.v.). A centralized r.v. $X$ is sub-Gamma distributed with variance factor $\nu > 0$ and scale parameter $c > 0$ (denoted by $X \sim \text{sub}\Gamma(\nu, c)$) if

$$\log(\mathbb{E} e^{sX}) \leq \frac{s^2}{2} \frac{\nu}{1 - c|s|}, \quad \forall \ 0 < |s| < c^{-1}. \quad (2.11)$$

The sub-exponential moment condition (2.5) would imply that Bernstein’s moment condition (2.9) is observed as

$$\log(\mathbb{E} e^{sX}) \leq \frac{s^2 \lambda^2}{2} \leq \frac{s^2 \lambda^2}{2(1 - \lambda|s|)}, \quad \forall \ |s| < \lambda. \quad (2.11)$$

**Lemma 2.3** (Concentration for sub-Gamma sum, Section 2.4 of Boucheron et al. (2013)). Let $\{X_i\}_{i=1}^n$ be independent $\{\text{sub}\Gamma(\nu_i, c_i)\}_{i=1}^n$ distributed with zero mean. Define $c = \max_{1 \leq i \leq n} c_i$, then

(a) Closed under addition: $S_n := \sum_{i=1}^n X_i \sim \text{sub}\Gamma(\sum_{i=1}^n \nu_i, c);$

(b) $\mathbb{P}(|S_n| \geq t) \leq 2 \exp\left(-\frac{t^2/2}{\sum_{i=1}^n \nu_i + ct}\right)$ and $\mathbb{P}\{|S_n| > (2t \sum_{i=1}^n \nu_i)^{1/2} + ct\} \leq 2e^{-t}, \quad \forall \ t \geq 0;$

(c) If $X \sim \text{sub}\Gamma(\nu, c)$, the even moments bounds satisfy $\mathbb{E}X^{2k} \leq k!(8\nu)^k + (2k)!(4c)^{2k}, \ k \geq 1.$

The concentration inequalities introduced in above only concerns the linear combinations of independent random variables. For lots of applications in high-dimensional statistics, we have to control the maximum of the $n$ r.v.s when deriving error bounds for the proposed estimator. In our proof of Theorem 3.1, the following maximum inequality is crucial.

**Lemma 2.4** (Concentration for maximum of sub-Gamma random variables). Let $\{X_i\}_{i=1}^n$ be independent $\{\text{sub}\Gamma(\nu_i, c_i)\}_{i=1}^n$ distributed with zero mean. Denote $\max_{i=1,\ldots,n} \nu_i = \nu$ and $\max_{i=1,\ldots,n} c_i = c$, we have

$$\mathbb{E}(\max_{i=1,\ldots,n} |X_i|) \leq \sqrt{2\nu \log(2n)} + c \log(2n). \quad (2.12)$$

### 3 Estimation and its asymptotic properties

In this section, we will derive the asymptotic results for the estimator. This is contrast to Karwa and Slavković (2016), in which the consistency of their differentially private estimator requires the denoised process for $\beta$-model. In Pan and Yan (2020), they used the moment equation to infer the degree parameters with noisy random variables from discrete Laplace distribution in $\beta$-model. In this paper, we first consider a general distribution for the noisy variables with Laplace distribution as a special case. Then we establish an unified asymptotic result including both consistency and asymptotically normality of the parameter estimator with binary weighted network. In undirected binary case, Chatterjee et al. (2011) proved that the maximum likelihood estimator of the $\beta$-model is consistent. Yan and Xu (2013) derived its asymptotic normality. In directed binary case, Yan et al. (2016a) derived the consistency and asymptotic normality of the maximum likelihood
estimator. Luo et al. (2020) further derived a central limit theorem for linear combinations of all the maximum likelihood estimators with an increasing dimension when the edges take finite discrete weight. In this paper, we consider the binary undirected networks with noisy sequence.

Note that $E(a_{ij})$ only depends on the $e^{\alpha_{i}^{*}+\alpha_{j}^{*}+\varepsilon_{ij}(\alpha_{i}^{*}+\alpha_{j}^{*})}$. Let $d = (d_1, \ldots, d_n)^T$ be the degree sequence of graph $G_n$. We assume that random variables $\{e_i\}_{i=1}^n$ are mutually independent and distributed in sub-gamma distributions $\{\text{sub}\Gamma(\nu_i, c_i)\}_{i=1}^n$ with respective parameters $\{(\nu_i, c_i)\}_{i=1}^n$.

Then we observe the noisy sequence $\tilde{d}$ instead of $d$, where

$$\tilde{d}_i = d_i + e_i, \quad i = 1, \ldots, n.$$  \hspace{1cm} (3.1)

We use moment equations to estimate the degree parameter with the noisy sequence $\tilde{d}$ instead of $d$. Define a system of functions:

$$F_i(\alpha) := \tilde{d}_i - E(d_i) = \tilde{d}_i - \sum_{j \neq i} e^{\alpha_i + \alpha_j + \varepsilon_{ij}(\alpha_i + \alpha_j)}, \quad i = 1, \ldots, n,$$

$$F(\alpha) = (F_1(\alpha), \ldots, F_n(\alpha))^T.$$

Now, we define our estimator $\hat{\alpha}$ as the solution to the equation $F(\alpha) = 0$, i.e.,

$$\hat{\alpha} := \{\alpha : F(\alpha) = 0\}$$  \hspace{1cm} (3.2)

It is not hard to see that the estimator is actually induced by the moment equation $\tilde{d} = E(d)$.

These asymptotic results of $\hat{\alpha}$ hold for all $\varepsilon$ satisfying the following condition.

**Assumption 3.1.** For all pairs of node $i$ and node $j$, all choices of $k, l$ and $m$ ($k, l, m = 1, \ldots, n$), the function $\varepsilon_{i,j}$, $\partial \varepsilon_{i,j}/\partial \alpha_k$, $\partial^2 \varepsilon_{i,j}/\partial \alpha_k \partial \alpha_l$, and $\partial^3 \varepsilon_{i,j}/\partial \alpha_k \partial \alpha_l \partial \alpha_m$, are sub-exponential in $\alpha_i + \alpha_j$. That is, there exists a constant $M_0$ such that the absolute values of these functions are bounded by $M_0 \exp(\alpha_i + \alpha_j)$.

Note that the solution to the equation $F(\alpha) = 0$ is precisely the moment estimator. Here, we consider the symmetric parameter space

$$D = \{\alpha \in \mathbb{R}^n : -Q_n \leq \alpha_i + \alpha_j \leq Q_n, \quad Q_n > 0, \quad 1 \leq i < j \leq n\}.$$ 

The uniform consistency of $\hat{\alpha}$ and asymptotic distribution of the parameter estimator are stated as follows, and the proofs are given in Appendix.

**Theorem 3.1 (Consistency).** If

$$e^{Q_n + e^{Q_n}} = \left(\frac{\sqrt{(n-1)\log(n-1)} + \sqrt{2\nu \log(2n)} + c \log(2n)}{n}\right)^{-1/34}$$
and \( \max_{i=1,...,n} v_i = v, \max_{i=1,...,n} c_i = c \), then as \( n \to \infty \), the estimator \( \mathbf{\hat{\alpha}} \) exists and satisfies

\[
\| \mathbf{\hat{\alpha}} - \mathbf{\alpha}^* \|_{\infty} = O_p \left( \frac{1}{n} e^{15Q_n + e^{Q_n}} \left( \sqrt{n \log n + \sqrt{2v \log(2n) + c \log(2n)}} \right) \right) = o_p(1).
\] (3.3)

**Theorem 3.2 (Asymptotic normality).** Using the same notations as Theorem 3.1, if

\[
\frac{1}{n} e^{45Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1) + \sqrt{2v \log(2n) + c \log(2n)}} \right)^2 = o(n^{1/2})
\]

then for any fixed \( k \geq 1 \), as \( n \to \infty \), the vector consisting of the first \( k \) elements of \( (B^{-1})^{1/2}(\mathbf{\hat{\alpha}} - \mathbf{\alpha}^*) \) is asymptotically distributed as \( N(0, \mathbf{I}_k) \), where \( (B^{-1})^{1/2} = \text{diag}(v_{11}^{1/2}, \ldots, v_{mn}^{1/2}) \) with

\[
v_{ii} = \sum_{j \neq i} e^{\alpha_i^* + \alpha_j^* + \varepsilon_{ij}(\alpha_i^*, \alpha_j^*)} \left( 1 + \frac{\partial \varepsilon_{ij}(\alpha_i^*, \alpha_j^*)}{\partial \alpha_i^*} \right).
\]

The proof of the theorem is in Appendix. We use the Newton-Kantovorich Theorem, i.e. Lemma 6.2 in the Appendix, to prove the consistency of the estimator by constructing the Newton’s iterative sequence. This technical step is different from Chatterjee et al. (2011) and yields a simple proof. It requires that the Jacobin matrix \( F'(\mathbf{\alpha}) \) is Lipschitz continuous and restricts the increasing rate of the Lipschitz \( \lambda \) on the dimension \( n \) of \( \mathbf{\alpha} \).

### 4 Numerical studies

In this section, we will evaluate the asymptotic result in Theorems 3.2 through numerical simulations and a real data example. The simulation will be conducted using Hermite distributions which we have introduced and proved its properties under the framework of Section 2.

#### 4.1 Simulation studies

For the simulation study of all models \( M_i \), we set \( \alpha_i^* = i * L/n \) for \( i = 1,...,n \). Other parameter settings in simulation studies are listed as follows. We consider three different values \(-\log(\log(n))^{1/3}, -\log(\log(n))^{1/2}, \) and \(-\log(\log(n))\) for \( L \) in case of model \( M_{\text{log}} \). For model \( M_{\text{logit}} \), we consider three different values \( L = 0, \log(\log(n)) \) and \( \log(n)^{1/2} \). In case of model \( M_{\text{cloglog}} \), we set three different values \( L = \log(\log(\log(n)))^{3/4}, \log(\log(\log(n)))^{5/4} \) and \( \log(\log(\log(n))) \). The noise \( \varepsilon_i (i = 1,...,n) \) is the difference between two independent and identically distributed Hermite distributions (see Definition 6.2). We consider three cases of Hermite distribution where parameters \( a_1 = 0.01, a_2 = (\Lambda - 0.01)/4, m = 2; a_1 = \Lambda - 0.01, a_2 = 0.025, m = 2 \) and \( a_1 = 4 * \Lambda/5, a_2 = \Lambda/5, m = 2 \). Here, we set \( \Lambda = 2 * \exp(-\Lambda_0/2)/(1 - \exp(-\Lambda_0/2))^2 \) and \( \Lambda_0 = 2 \). We consider two values for \( n = 100 \) and \( n = 200 \). Note that by Theorems 3.2, \( \hat{\xi}_{ij} = (\hat{\alpha}_i + \hat{\alpha}_j - (\alpha_i^* + \alpha_j^*))/(1/\hat{v}_{ii} + 1/\hat{v}_{jj})^{1/2} \) is an asymptotically normal distribution, where \( \hat{v}_{ii} \) is the estimator of \( v_{ii} \) by replacing \( \alpha_i \) with \( \hat{\alpha}_i \). The quantile-quantile(QQ) plots of \( \xi_{ij} \) are drawn. We ran 10000 simulations for each scenario.
We simulate with \( n = 100, n = 200 \), three values for \( L \) and three cases of Hermite distribution to find that the QQ-plots for each combination are similar. In order to save the place, we only present the QQ plots of three model in Figure 1, Figure 2 and Figure 3 when \( n = 200, a_1 = 4 \times \Lambda / 5, a_2 = \Lambda / 5, \) and \( m = 2 \) for each case. The horizontal and vertical axes are the theoretical and empirical quantiles respectively, and the straight lines correspond to the reference line \( y = x \). In Figure 1, we first observe that the empirical quantiles agree well with the ones of the standard normality of \( \tilde{\xi}_{ij} \), expect for pair \((n/2, n/2 + 1)\) and \((n-1, n)\) when \( L = -\log(\log(n)) \) in the case of model \( M_{\log} \). For the case of model \( M_{\logit} \), there are no notable derivations from the standard normality for each scenario in Figure 2. We also observe that the empirical quantiles agree well with the ones of the standard normality of \( \tilde{\xi}_{ij} \), expect for pair \((n/2, n/2 + 1)\) and \((n-1, n)\) when \( L = \log(\log(\log(n))) \) in Figure 3.

The coverage probability of the 95\% confidence interval for \( \alpha_i - \alpha_j \), the length of the confidence interval and the frequency that the MLE did not exist are reported in Table 1, Table 2 and Table 3. We can see that the length of estimated confidence interval increases as \( L \) increases for fixed \( n \), and decreases as \( n \) increases for fixed \( L \).

4.2 A data example

We use the KAPFERER TAILOR SHOP network dataset created by Bruce Kapferer which is downloaded from [http://vlado.fmf.uni-lj.si/pub/networks/data/ucinet/ucidata.htm](http://vlado.fmf.uni-lj.si/pub/networks/data/ucinet/ucidata.htm). In each study they obtain measures of social interaction among all actors. In this network data, “1” represents they have a friend relationship between two actors, otherwise, it is denoted as “0”. Because the estimate \( \hat{\alpha} \) does not exist when the degree is zero, we remove the vertex 17 and 22 whose degree is zero before analysis, so the network with the left 37 vertices in each table remains.

When the parameters of three kinds of noise distribution function \( e_i (i = 1, ..., n) \) are set, the parameter estimation is similar in case of model \( M_{\logit}, M_{\log} \) and \( M_{\loglog} \). Thus we here only show the parameter estimation under the case of noise distribution function parameter for \( a_1 = 4 \times \Lambda / 5, a_2 = \Lambda / 5 \) and \( m = 2 \). From Figure 4, we first observe the scatter plots of the noisy degree sequence \( \tilde{d} \) corresponding to the parameter estimation \( \hat{\alpha} \) under model \( M_\varepsilon \) and the value of \( \hat{\alpha} \) increases as the number of \( \tilde{d} \) climbs. The larger the estimated parameters \( \hat{\alpha} \), the actors have more friends.

For these three cases of model \( M_\varepsilon \), the estimated parameters and their standard errors as well as the 95\% confidence intervals and the size of noisy degree sequences are reported in Table 4, Table 5 and Table 6 respectively. The value of estimated parameters reflects the corresponding size of noisy degrees. For example, the large five degrees are 26, 17, 16, 15, 14 for vertices 16, 18, 32, 11, 12 which also have the top five influence parameters at 0.36, −0.05, −0.12, −0.18, −0.25. On the other hand, the five vertices with smallest parameters −2.89, −2.19, −1.79, −1.28, −1.10 have degrees at 1, 2, 3, 5, 6. In Table 5 and Table 6, the larger the parameter \( \hat{\alpha} \), the greater the degree \( \tilde{d} \) of the node. This is the same as the conclusion in Table 4.
5 Summary and discussion

In this paper, we release the degree sequences of the class of binary networks under the sub-Gamma noisy mechanism. We establish the asymptotic result including the consistency and asymptotically normality of the parameter estimator when the number of parameters goes to infinity. By using the Newton-Kantorovich theorem, we try to ignore adding noisy process and obtain the existence and consistency of the parameter estimator satisfying equation $\tilde{d} = E(d)$. Furthermore, we give some simulation results to illustrate that the asymptotic normality behaves well under model $M_{\log}$, $M_{logit}$ and $M_{cloglog}$. However, an edge in networks takes not only binary values but also weighted edges in many scenarios. We will investigate null models for these directed weighted networks in the future. It is worth noting that the conditions imposed on $Q_n, a_1, a_2$, and $m = 2$ may not be the most possible. In particular, the conditions guaranteeing the asymptotic normality are stronger than those guaranteeing the consistency. Simulation studies suggest that the conditions on $Q_n$ might be relaxed. It can be noted that the asymptotic behavior of the parameter estimator depends not only on $Q_n, a_1, a_2$, and $m = 2$, but also on the configuration of all the parameters. We will investigate this in future studies.

In this paper we derived individual parameter asymptotic properties, and we can also study on a linear combination of all the parameter estimation in binary networks with noisy degree sequence in the future work. In our paper, we only consider the model heterogeneity parameter. In network data, the second distinctive feature inherent in most natural networks is the homophily phenomenon. Yan et al. (2019) established the uniform consistency and asymptotic normality of the heterogeneity parameter and homophily parameter estimators. On the other hand, sub-Weibull variables, as an extension of sub-Gamma variables, enable variables have heavier tails, which may also be consider in networks models. Fortunately, there have been some articles investigating concentration of sub-Weibull variables, see Zhang and Wei (2022) for instance. And we further investigate a central limit theorem for a linear combination of all the maximum likelihood estimators of degree parameter when the number of nodes goes to infinity[ Luo et al. (2020)]. We will investigate these aspects in future studies.

6 Appendix

6.1 Two-side discrete compound Poisson and Hermite distributions

The negative binomial random variable belongs to the exponential family when the dispersion parameter is known. But if the parameter in negative binomial random variable $X$ is unknown in real world problems (Zhang and Jia, 2022), it does not belong to the exponential family. Whereas, it is well-known that Poisson and negative binomial distributions belong to the family of discrete infinitely divisible distributions (also named as discrete compound Poisson distributions); see Zhang et al. (2014) and the references therein.
It should be noted that the discrete Laplace random variable is the difference of two i.i.d. geometric distributed random variables (see Proposition 3.1 in Inusah and Kozubowski (2006)). The geometric distribution as a class of infinitely divisible distribution is a special case of discrete compound Poisson distribution. The difference of geometric noise-addition mechanism can be flexibly extended to the difference between two i.i.d. (or independent) discrete compound Poisson random variables (see Definition 4.2 of Zhang and Li (2016)). In fact, the difference between two independent discrete compound Poisson random variables follows the infinitely divisible distributions with integer support; see Chapter IV of Steutel and Van Harn (2003).

**Definition 6.1.** We say that $Y$ is discrete compound Poisson (DCP) distributed if the characteristic function of $Y$ is

$$
\varphi_Y(t) = \mathbb{E}e^{itY} = \exp\left\{\sum_{k=1}^{\infty} \alpha_k \lambda(e^{itk} - 1)\right\} \quad (t \in \mathbb{R}),
$$

(6.1)

where $(\alpha_1 \lambda, \alpha_2 \lambda, \ldots)$ are infinite-dimensional parameters satisfying $\sum_{i=1}^{\infty} \alpha_k = 1$, $\alpha_k \geq 0$, $\lambda > 0$. We denote it as $Y \sim \text{DCP}(\alpha_1 \lambda, \alpha_2 \lambda, \ldots)$.

Based on the (6.1), the discrete compound Poisson random variable $X_i$ has the weighted Poisson decomposition

$$X_i := \sum_{k=1}^{\infty} k N_k, \quad \text{where } N_k^{\text{ind.}} \sim \text{Poisson}(\alpha_k \lambda),$$

so we have $\mathbb{E}X = \lambda \sum_{k=1}^{\infty} k \alpha_k$.

Prékopa (1952) first considered the difference between two composed Poisson distributions (we name it two-side composed Poisson distribution in the following context), of which the characteristic function is

$$
\varphi(t) = \exp\left\{\sum_{k \in \mathbb{Z}\setminus\{0\}} C_k (e^{i\lambda_k t} - 1)\right\},
$$

(6.2)

where $C_k, \lambda_k \geq 0$ and $\sum_{k \in \mathbb{Z}\setminus\{0\}} C_k < \infty$. We notice (6.2) degenerates to (6.1) when $\lambda_k = k (k \geq 1)$ and $C_k = 0 (k \leq 0)$.

Notice the characteristic function of Levy process $Z(t)$ can be represented by

$$
\varphi(\theta) = \mathbb{E}[e^{i\theta Z(t)}] = \exp\left\{ait\theta - \frac{1}{2} \sigma^2 t^2 \theta^2 + t \int_{\mathbb{R}\setminus0} (e^{i\theta x} - 1 - i\theta x \mathbb{1}_{|x|<1}) w(dx)\right\}
$$

using Levy-Khinchine formula, where $a \in \mathbb{R}$, $\sigma > 0$ and $\mathbb{1}_{|x|<1}$ as the indicator function. $w$ is usually referred as the Levy measure, which is a non-negative measure that satisfies $\int_{\mathbb{R}\setminus0} \min\{x^2, 1\} w(dx) < \infty$.

Construct the following Levy measure

$$w(dx) = \alpha_k \lambda d\delta_k,$$

where $\sum_{k \in \mathbb{Z}\setminus\{0\}} \alpha_k = 1$ and $\delta_k$ is the
Dirac measure. According to the definition of \(w(dx)\), we have

\[
a \alpha \theta - \frac{1}{2} \sigma^2 \theta^2 = t \int_{\mathbb{R} \setminus 0} (i \theta x \mathbb{1}_{|x|<1}) w(dx) = 0.
\]

Therefore, the characteristic function of two-side CPD denoted by \(Z\) is

\[
E[e^{itZ}] = \exp \left\{ \lambda \sum_{k=1}^{\infty} \alpha_k (e^{ikt} - 1) + \mu \sum_{k=1}^{\infty} \beta_k (e^{-ikt} - 1) \right\},
\]

which can also be used as the definition of two-side CPD. We say a random variable has two-side CPD if the characteristic function of it satisfies the (6.3).

Notice the discrete composed Poisson distribution can be generated by Poisson process. Similarly, the two-side CPD can be generated by the difference between two Poisson processes with different parameters. We introduce the two-side Poisson distribution later.

When the characteristic function of random variable \(Z\) satisfies (6.3) with \(\alpha_1 = \beta_1 = 1\) and \(\alpha_k = \beta_k = 0 (k > 1)\), we say \(Z\) satisfies tow-side Poisson distribution, of which the characteristic function is \(\exp \{ \lambda (e^{it} - 1) + \mu (e^{-it} - 1) \}\) and the p.d.f. is in the form

\[
P(Z = k) = e^{-\lambda + \mu} \left( \frac{\lambda}{\mu} \right)^{k/2} I_{|k|} (2 \sqrt{\lambda \mu}), \quad k \in \mathbb{Z},
\]

where \(I_n(x)\) is the modified Bessel function of the first kind. \(I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+n)!} (\frac{x}{2})^{2k+n}\) satisfies:

(i) the expansion \(\exp \{ \frac{1}{2} x(z + z^{-1}) \} = \sum_{n \in \mathbb{Z}} I_n(x) z^n\); (ii) \(I_n(x) = I_{-n}(x)\).

Two special cases occur when we set \(\mu = 0\) and \(\lambda = 0\) in a two-side Poisson distribution \(Z\) respectively. When \(\mu = 0\), \(Z\) degenerates to Poisson distribution, consistent with the fact that

\[
P(Z = k) = \lim_{\lambda \to 0} \left( \frac{\lambda}{\mu} \right)^{k/2} \sum_{i=0}^{\infty} e^{\lambda + \mu |i|} (i + |k|)! = \lim_{\lambda \to 0} \left( \frac{\lambda}{\mu} \right)^{k/2} \frac{e^{\lambda |k|}}{e^{|k|} |k|!} = \frac{\lambda^k e^{-\lambda}}{k!} (k \geq 0).
\]

When \(\mu = 0\), we derive the p.d.f. of \(Z\) that

\[
P(Z = k) = \lim_{\lambda \to 0} \left( \frac{\lambda}{\mu} \right)^{k/2} \sum_{i=0}^{\infty} \frac{(\sqrt{\lambda \mu})^{2i+|k|}}{e^{\lambda + \mu |i|} (i + |k|)!} = \lim_{\lambda \to 0} \frac{\lambda^{k/2} \mu^{-|k|/2} e^{\lambda |k|/2}}{e^{|k|} |k|!} = \frac{\mu^{-k} e^{-\mu}}{(-k)!} (k \leq 0),
\]

which shows that \(Z\) is in a negative Poisson distribution.

If \(\alpha_k = 0\) for \(k \geq 3\) in Definition 6.1, there is a special kind of DCP which is often used in social analysis and biology and is called Hermite distribution.
Definition 6.2. We say that $Y$ is in a Hermite distribution with parameters $(\Lambda, a_1, a_2)^\top$ if

$$Y := N_1 + 2N_2,$$

where $N_k \overset{ind.}{\sim} \text{Poisson}(\Lambda a_k)$

with $a_1, a_2 > 0$ and $a_1 + a_2 = 1$. We denote it by $Y \sim \text{Herm}(\Lambda, a_1, a_2)$.

It can be seen that Hermite distribution is actually DCP with only first two active elements. In the next result, we show the sub-Gamma concentration for the sum of independent Hermite random variables. Hence the two-side Hermite distribution also enjoys sub-Gamma concentration.

Theorem 6.1. Given $Y_i \sim \text{DCP} \left(\alpha(i)(\lambda(i), \ldots, \alpha(i)(\lambda(i))\right)$ independently for $i = 1, 2, \ldots, n$, and $\sigma_i^2 := \text{var} Y_i = \lambda(i) \sum_{k=1}^r k^2 \alpha_k(i)$, for non-random weights $\{w_i\}_{i=1}^n$ with $w = \max_{1 \leq i \leq n} |w_i| > 0$, we have

$$\text{P} \left\{ \left| \sum_{i=1}^n w_i(Y_i - \mu_i) \right| \geq w \left[ (2x \sum_{i=1}^n \sigma_i^2)^{1/2} + rx/3 \right] \right\} \leq 2e^{-x}, \quad \forall x > 0. \quad (6.4)$$

Specially, if $Y_i \overset{iid.}{\sim} \text{Herm}(\Lambda, a_1, a_2)$, we have

$$\text{P} \left\{ \left| \sum_{i=1}^n w_i(Y_i - \mu_i) \right| \geq \sqrt{\frac{2x\Lambda(a_1 + 4a_2)}{n} + \frac{2x}{3n}} \right\} \leq 2e^{-x}, \quad \forall x > 0.$$

Proof. To apply the above lemma, we need to evaluate the log-moment-generating function of centered DCP random variables. Let $\mu_i := EY_i = \lambda(i) \sum_{k=1}^r k \alpha_k(i)$, we have

$$\log \text{E}e^{sw_i(Y_i - \mu_i)} = -sw_i\mu_i + \log \text{E}e^{sw_iY_i} = -sw_i\mu_i + \log e^{\lambda(i) \sum_{k=1}^r \alpha_k(i)(e^{kw_i} - 1)}$$

$$= \lambda(i) \sum_{k=1}^r \alpha_k(i)(e^{kw_i} - ksw_i - 1). \quad (6.5)$$

Hence, one can derive that

$$\log \text{E}e^{sw_i(Y_i - \mu_i)} \leq \sum_{k=1}^r \alpha_k(i)(\lambda(i) k^2 w_i^2 s^2) = \frac{ws^2 \lambda(i) \sum_{k=1}^r k^2 \alpha_k(i)}{2(1 - ksw_i)} \leq \frac{\sigma_i^2 w^2 s^2}{2(1 - rw|s|/3)} \leq \frac{3}{rw}, \quad |s| \leq \frac{3}{rw}. \quad (6.6)$$

where $\sigma_i^2 = \lambda(i) \sum_{k=1}^r k^2 \alpha_k(i)$. Thus (6.6) implies that $w_i(Y_i - \mu_i) \sim \text{sub}\Gamma(w^2\sigma_i^2, rw/3)$.

From Proposition 2.3(a), we obtain $S_n^w \sim \text{sub}\Gamma(w^2 \sum_{k=1}^n \sigma_i^2, rw/3)$. Then applying Proposition 2.3(b), we get (6.4).

6.2 Proofs in Section 2

First, we would like to introduce some equivalent definitions of sub-exponential variables. The detailed discussions and proofs can be seen in Rigollet and Hütter (2019).
Proposition 6.1 (Characterizations of sub-exponential distributions). Let $X$ be a r.v. in $\mathbb{R}$ with $EX = 0$. Then the following properties are equivalent (the parameters $K_i > 0$ are equal up to a constant factor)

(1) The tails of $X$ satisfy $P\{|X| \geq t\} \leq 2e^{-t/K_1}$ for all $t \geq 0$;

(2) The MGF of $X$ satisfies $Ee^{tX} \leq e^{K_2^2t^2}$ for all $|t| \leq \frac{1}{K_2}$;

(3) The moments of $X$ satisfy $(E|X|^k)^{1/k} \leq K_3k$ for integer $k \geq 1$;

(4) The MGF of $|X|$ satisfies $Ee^{l|X|} \leq e^{K_4l}$ for all $0 \leq l \leq \frac{1}{K_4}$;

(5) The MGF of $|X|$ is bounded at some point: $Ee^{|X|/K_5} \leq 2$.

Lemma 6.1 (Concentration for weighted sub-exponential sum). Let $\{X_i\}_{i=1}^n$ be independent and distributed with zero mean. Define $\lambda = \max_{1 \leq i \leq n} \lambda_i > 0$ and the non-random vector $w := (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ with $w := \max_{1 \leq i \leq n} |w_i| > 0$, we have

(a) Closed under addition: $\sum_{i=1}^n w_iX_i \sim \text{subE}(\|w\|_2\lambda)$;

(b) $P(\sum_{i=1}^n w_iX_i \geq t) \leq 2e^{-\frac{t^2}{\|w\|_2^2\lambda^2} - \frac{t}{\|w\|_2\lambda}} = \begin{cases} 2e^{-\frac{t^2}{\|w\|_2^2\lambda^2}}, & 0 \leq t \leq \frac{\|w\|_2^2\lambda}{w} \\ 2e^{-\frac{t}{2\|w\|_2\lambda}}, & t > \frac{\|w\|_2^2\lambda}{w} \end{cases}$.

Proof. See Rigollet and Hütter (2019). \hfill \Box

6.2.1 The proof of Lemma 2.1 and Corollary 2.1

(a). To verified (2.7), using exponential Markov’s inequality, we have

$$P(|X| \geq t) = P\left(e^{t|X|/\|X\|_{\psi_1}} \geq e^{t/\|X\|_{\psi_1}}\right) \leq e^{-t/\|X\|_{\psi_1}}Ee^{t|X|/\|X\|_{\psi_1}} \leq 2e^{-t/\|X\|_{\psi_1}}$$

where the last inequality stems from the definition of sub-exponential norm.

(b). From (2.7), we get

$$E|X|^k = \int_0^\infty P(|X| \geq t)kt^{k-1}dt \leq 2k \int_0^\infty e^{-t/\|X\|_{\psi_1}}t^{k-1}dt,$$

[let $s = t/\|X\|_{\psi_1}$] = $2k \int_0^\infty e^{-s(s\|X\|_{\psi_1})}s^{k-1}ds = 2\|X\|_{\psi_1}^k k! \Gamma(k - 1) = 2\|X\|_{\psi_1}^k k!$. 

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Applying Taylor’s expansion to MGF, we have

\[ E \exp(sX) = E \left( 1 + sX + \sum_{k=2}^{\infty} \frac{(sX)_k}{k!} \right) = 1 + \sum_{k=2}^{\infty} \frac{s^kE X^k}{k!} \]

[Applying (b)] \( \leq 1 + 2 \sum_{k=2}^{\infty} (s \|X\|_{\psi_1})^k = 1 + \frac{2(s \|X\|_{\psi_1})^2}{1 - s \|X\|_{\psi_1}}, \quad (|s \|X\|_{\psi_1}| < 1) \)

\[ \leq 1 + 4(s \|X\|_{\psi_1})^2 \leq e^{(2\|X\|_{\psi_1})^2 s^2}, \quad \text{if } |s| < 1/(2 \|X\|_{\psi_1}). \]

Therefore, \( X \sim \text{subE}(2 \|X\|_{\psi_1}) \). And to prove Corollary 2.1, we only need to note that if \( E \exp(|X|/\|X\|_{\psi_1}) \leq 2 \), then \( X \sim \text{subE}(2 \|X\|_{\psi_1}) \) by using Lemma 2.1(c). The result follows Lemma 6.1 (b).

### 6.2.2 Proof of Lemma 2.4

The Jensen’s inequality and \( e^{|x|} \leq e^x + e^{-x} \) imply that

\[ \exp(sE \max_{i=1,\ldots,n} |X_i|) \leq E \exp(\max_{i=1,\ldots,n} s|X_i|) = E \max_{i=1,\ldots,n} e^{s|X_i|} \leq E \max_{i=1,\ldots,n} (e^{-sX_i} + e^{sX_i}) \leq \sum_{i=1}^{n} E(e^{-sX_i} + e^{sX_i}) \leq 2n \cdot \exp\left( \frac{s^2}{2 \frac{1 - c|s|}{1}} \right) \]

where the last inequality is deduced by \( X_i \sim \Gamma(\nu_i, c_i) \) and \( -X_i \sim \Gamma(\nu_i, c_i) \).

By (6.7) and taking logarithm, we have

\[ E \max_{i=1,\ldots,n} |X_i| \leq \inf_{|s| < c^{-1}} \frac{\log(2n) + \frac{s^2}{2 \frac{1 - c|s|}{1}}}{s} \leq \inf_{|s| < c^{-1}} \left\{ \frac{\log(2n)}{s} (1 - c|s|) + \frac{s}{2 \frac{1 - c|s|}{1}} + c \log(2n) \right\} = \sqrt{2} \log(2n) + c \log(2n). \]

This completes the proof of Lemma 2.4.

### 6.3 Proofs in Section 3

For a subset of \( C \subset \mathbb{R}^n \), denote \( C^0 \) and \( \overline{C} \) as the interior and closure of \( C \) respectively. For a vector \( x = (x_1, \ldots, x_n) \top \in \mathbb{R}^n \), denote \( \|x\|_{\infty} = \max_{1 \leq i \leq n} |x_i| \) as the \( \ell_{\infty} \)-norm of \( x \). For a \( n \times n \) matrix \( J = (J_{ij}) \), let \( \|J\|_{\infty} \) be the matrix norm induced by the \( \ell_{\infty} \)-norm on vectors in \( \mathbb{R}^n \), i.e.,

\[ \|J\|_{\infty} = \max_{x \neq 0} \frac{\|Jx\|_{\infty}}{\|x\|_{\infty}} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |J_{ij}|. \]

Let \( D \) be an open convex subset of \( \mathbb{R}^n \). We say a \( n \times n \) function matrix \( G(x) \) whose elements \( G_{ij}(x) \) are functions on vectors \( x \), is Lipschitz continuous on \( D \) if there exists a real number of \( \lambda \)
such that for any \( v \in \mathbb{R}^n \) and any \( x, y \in D \),

\[
\|G(x)(v) - G(y)(v)\|_\infty \leq \lambda \|x - y\|_\infty \|v\|_\infty, \tag{6.9}
\]

where \( \lambda \) may depend on \( n \) but is independent of \( x \) and \( y \). For every fixed \( n \), \( \lambda \) is a constant. Given \( m, M > 0 \), we say an \( n \times n \) matrix \( V = (v_{ij}) \) belongs to the matrix class \( \mathcal{L}_n(m, M) \) if \( V \) is a diagonally balanced matrix with positive elements bounded by \( m \) and \( M \),

\[
v_{ii} = \sum_{j \neq i} v_{ij}, \quad i = 1, \ldots, n, \quad m \leq v_{ij} \leq M, \quad i, j = 1, \ldots, n, \quad i \neq j. \tag{6.10}
\]

We use \( V \) to denote the Jacobian matrix induced by the moment equations and show that it belongs to the matrix class \( \mathcal{L}_n(m, M) \). We require the inverse of \( V \), which doesn’t have a closed form. Yan and Xu (2013) proposed approximating the inverse \( V^{-1} \) of \( V \) by a matrix \( S = (s_{ij}) \), where

\[
s_{ij} = \frac{\delta_{ij}}{v_{ii}}, \tag{6.11}
\]

in which \( \delta_{ij} = 1 \) when \( i = j \) and \( \delta_{ij} = 0 \) when \( i \neq j \).

### 6.3.1 Preliminaries

Before the beginning of the proof, we introduce the preliminary results that will be used in the proofs. For a subset \( C \subseteq \mathbb{R}^n \), let \( C^0 \) and \( \overline{C} \) denote the interior and closure of \( C \) in \( \mathbb{R}^n \), respectively. Denote \( \Omega(x, r) \) as the open ball \( \{y : \|x - y\| < r\} \) and \( \overline{\Omega(x, r)} \) as its closure. We use Newton’s iterative sequence to prove the existence and consistency of the moment estimates relying on results of Gragg and Tapia (1974).

**Lemma 6.2.** Let \( F(x) = (F_1(x), \ldots, F_n(x))^\top \) be a function vector on \( x \in \mathbb{R}^n \). Assume that the Jacobian matrix \( F'(x) \) is Lipschitz continuous on an open convex set \( D \) with Lipschitz constant \( \lambda \). Given \( x_0 \in D \), assume that \( [F'(x_0)]^{-1} \) exists, we have

\[
\| [F'(x_0)]^{-1} \|_\infty \leq \mathcal{K}, \quad \| [F'(x_0)]^{-1} F(x_0) \|_\infty \leq \delta, \quad h = 2\mathcal{K}\lambda \delta \leq 1,
\]

\[
\Omega(x_0, t^*) \subset D^0, \quad t^* := \frac{2}{h} (1 - \sqrt{1 - h}) \delta = \frac{2}{1 + \sqrt{1 - h}} \delta \leq 2\delta,
\]

where \( \mathcal{K} \) and \( \delta \) are positive constants that may depend on \( x_0 \) and the dimension \( n \) of \( x_0 \). Then the Newton iterates \( x_{k+1} = x_k - [F'(x_k)]^{-1} F(x_k) \) exist and \( x_k \in \Omega(x_0, t^*) \subset D^0 \) for all \( k \geq 0 \); \( \hat{x} = \lim x_k \) exists, \( \hat{x} \in \Omega(x_0, t^*) \subset D \) and \( F(\hat{x}) = 0 \). Thus if \( t^* \to 0 \), then \( \|\hat{x} - x_0\| = o(1) \).

**Lemma 6.3.** For a matrix \( A = (a_{ij}) \), define \( \|A\| := \max_{i,j} |a_{ij}| \). If \( V \in \mathcal{L}_n(m, M) \) at (6.10) and \( n \) is large enough, we get

\[
\|V^{-1} - S\| \leq \frac{c_1 M^2}{m^3(n - 1)^2},
\]
where $S$ is defined at (6.11) and $c_1$ is a constant that does not depend on $M$, $m$, and $n$.

Lemma 6.4. If $V \in \mathcal{L}_n(m, M)$, for $n$ which is large enough,

$$
\|V^{-1}\|_\infty \leq \|V^{-1} - S\|_\infty + \|S\|_\infty \leq \frac{c_1 n M^2}{m^3 (n - 1)^2} + \frac{1}{m} \left( \frac{1}{n(n - 1)} + \frac{1}{n - 1} \right) \leq \frac{c_2 M^2}{nm},
$$

where $c_2$ is a constant that does not depend on $M$, $m$, and $n$.

6.3.2 Proof of the Theorem 3.1

Then the Jacobian matrix $F' (\alpha)$ of $F(\alpha)$ can be calculated as follows. For $i, j = 1, \ldots, n$,

$$
\frac{\partial F_i}{\partial \alpha_i} = -\sum_{j \neq i} e^{\alpha_i + \alpha_j + \varepsilon_{i,j}(\alpha_i + \alpha_j)} \left( 1 + \frac{\partial \varepsilon_{i,j}(\alpha_i + \alpha_j)}{\partial \alpha_i} \right), \quad i = 1, \ldots, n,
$$

$$
\frac{\partial F_i}{\partial \alpha_j} = -e^{\alpha_i + \alpha_j + \varepsilon_{i,j}(\alpha_i + \alpha_j)} \left( 1 + \frac{\partial \varepsilon_{i,j}(\alpha_i + \alpha_j)}{\partial \alpha_j} \right), \quad j = 1, \ldots, n, \quad j \neq i.
$$

Following Assumption 3.1, $-C_0 e^{Q_n} \leq \varepsilon_{i,j} \leq C_0 e^{Q_n}$ and $-C_1 e^{Q_n} \leq \partial \varepsilon_{i,j}/\partial \alpha_j \leq C_1 e^{Q_n}$, where $C_0$ and $C_0$ are positive constants, we have

$$
-e^{Q_n + C_0 e^{Q_n}} (1 + C_1 e^{Q_n}) \leq \frac{\partial F_i}{\partial \alpha_j} \leq -e^{-Q_n - C_0 e^{Q_n}} (1 - C_1 e^{Q_n}).
$$

So for any $i \neq j$, we have the following inequality:

$$
m \leq \frac{\partial F_i}{\partial \alpha_j} \leq M.
$$

It is not difficult to verify that $-F'_{i,j}(\alpha^*) \in \mathcal{L}_n(m, M)$ where $m = e^{-Q_n - C_0 e^{Q_n}} (1 - C_1 e^{Q_n})$, $M = e^{Q_n + C_0 e^{Q_n}} (1 + C_1 e^{Q_n})$.

The following lemma assures that the condition holds with a large probability.

Lemma 6.5. With probability approaching one, the following holds:

$$
\max_{i=1, \ldots, n} |\bar{d}_i - E(d_i)| = O\left( \sqrt{(n - 1) \log(n - 1)} + \sqrt{2v \log(2n) + c \log(2n)} \right).
$$

Proof. Note that $\{e_{i}\}_{i=1}^{n}$ are mutually independent and distributed in sub-gamma distributions $\Gamma(v_i, c_i)(i = 1, \ldots, n)$ with respective parameters $(v_i, c_i)(i = 1, \ldots, n)$. Let $\max_{i=1, \ldots, n} v_i = v$ and $\max_{i=1, \ldots, n} c_i = c$. By Lemma 2.4, for each $i = 1, \ldots, n$, we have

$$
E(\max_{i=1, \ldots, n} |e_i|) \leq \sqrt{2v \log(2n) + c \log(2n)} \quad (6.12)
$$

Following Yan et al. (2016b) in (C5) that they show the following inequality holds with probability
approaching one:

\[
\max_{i=1,\ldots,n} |d_i - E(d_i)| \leq O_p(\sqrt{(n-1) \log(n-1)}),
\]

(6.13)

we have

\[
\max_{i=1,\ldots,n} |\tilde{d}_i - E(d_i)| \leq \max_i |d_i - E(d_i)| + \max_i |e_i|
\]

\[
= O_p\left(\sqrt{(n-1) \log(n-1) + 2v \log(2n) + c \log(2n)}\right),
\]

(6.14)

and it is what we need to prove.

Now, we present the proof of Theorem 3.1.

\textbf{Proof.} Let

\[
g_{ij}(\alpha) = \left(\frac{\partial^2 F_i}{\partial \alpha_i \partial \alpha_j}, \ldots, \frac{\partial^2 F_i}{\partial \alpha_n \partial \alpha_j}\right)^\top.
\]

It is easy to verify that

\[
\frac{\partial^2 F_i}{\partial \alpha^2} = \sum_{j \neq i}^n e^{\alpha_i + \alpha_j + \varepsilon_{i,j}(\alpha_i, \alpha_j)} \left[\left(1 + \frac{\partial \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_i}\right)^2 + \frac{\partial^2 \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_i^2}\right],
\]

\[i = 1, \ldots, n,
\]

and

\[
\frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_i}
\]

\[
eq e^{\alpha_i + \alpha_j + \varepsilon_{i,j}(\alpha_i, \alpha_j)} \left[\left(1 + \frac{\partial \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_j}\right)\left(1 + \frac{\partial \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_i}\right) + \frac{\partial^2 \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_j \partial \alpha_i}\right],
\]

\[j = 1, \ldots, n, \quad j \neq i.
\]

Following Assumption 3.1 that

\[-C_2 e^{Q_n} \leq \frac{\partial^2 \varepsilon_{i,j}(\alpha_i, \alpha_j)}{\partial \alpha_j \partial \alpha_i} \leq C_2 e^{Q_n}\]

where \(C_2\) is a positive constant, we have

\[
\left|\frac{\partial^2 F_i}{\partial \alpha_j \partial \alpha_i}\right| \leq e^{Q_n + Q_n} \left(1 + 2C_1 e^{Q_n} + C_1^2 e^{2Q_n} + C_2 e^{Q_n}\right).
\]

(6.15)

Let \(M_1 = e^{Q_n + Q_n} \left(1 + 2C_1 e^{Q_n} + C_1^2 e^{2Q_n} + C_2 e^{Q_n}\right)\). This leads to \(\|g_{ii}(\alpha)\|_1 \leq 2(n - 1)M_1\), where \(\|x\|_1 = \sum_i |x_i|\) for a general vector \(x\). On the other hand, we note that when \(i \neq j\) and \(k \neq i, j\), there exists

\[
\frac{\partial^2 F_i}{\partial \alpha_k \partial \alpha_j} = 0
\]
which leads to that $\|g_{ij}(\alpha)\|_1 \leq 2M_1$ when $i \neq j$. Consequently, for any vector $v$,

$$\max_i \sum_j \left[ \frac{\partial F_i}{\partial \alpha_j}(x) - \frac{\partial F_i}{\partial \alpha_j}(y) \right] v_j \leq \|v\|_\infty \max_i \sum_j \left| \frac{\partial F_i}{\partial \alpha_j}(x) - \frac{\partial F_i}{\partial \alpha_j}(y) \right|$$

$$= \|v\|_\infty \max_i \sum_j \left| \int_0^1 g_i(tx + (1-t)y) \, dt \right|$$

$$= \|v\|_\infty \|x - y\|_\infty \max_i \sum_j \max_{\alpha \in D} \|g_{ij}(\alpha)\|_1$$

$$\leq 4M_1(n-1)\|v\|_\infty \|x - y\|_\infty$$

It shows that $F'(x)$ is Lipschitz continuous with the lipschitz coefficient $\lambda = 4M_1(n-1)$. For any $\alpha \in D$, we can define the Newton’s iterative sequence with the starting point $\alpha^{(0)} := \alpha$, i.e.,

$$\alpha^{(k+1)} = \alpha^{(k)} - [F' (\alpha^{(k)})]^{-1} F(\alpha^{(k)}), \quad k = 0, 1, ...$$

which shows that $F'(\alpha) \in L_\infty(m, M)$ with

$$m = e^{-Q_n - C_0 e^{-Q_n}} (1 - C_1 e^{-Q_n}), \quad M = e^{Q_n + C_0 e^{Q_n}} (1 + C_1 e^{Q_n}).$$

By lemma 6.3 and $M^2/m^3 = o(n)$, we have

$$\|F'(\alpha)\|_\infty \leq \|F'(\alpha)\|_\infty \|F(\alpha)\|_\infty \leq \left[ \frac{c_1 M^2}{m^3 (n-1)^2} + \frac{1}{m(n-1)} \right] \|F(\alpha)\|_\infty \leq O_p \left( \frac{c_2 M^2}{nm^3} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2v \log(2n) + c \log(2n)} \right) \right)$$

$$\leq O_p \left( \frac{1}{n} e^{14Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2v \log(2n) + c \log(2n)} \right) \right).$$

Hence

$$\kappa = O \left( \frac{1}{n} e^{4Q_n + e^{Q_n}} \right), \quad \delta = O \left( \frac{1}{n} e^{14Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2v \log(2n) + c \log(2n)} \right) \right),$$

and thus

$$h = 2\kappa \delta = O \left( \frac{1}{n} e^{5Q_n + e^{Q_n}} \right) \times (n-1)O(e^{4Q_n + e^{Q_n}})$$

$$\times O \left( \frac{1}{n} e^{14Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2v \log(2n) + c \log(2n)} \right) \right).$$

If $e^{Q_n + e^{Q_n}} = \left( \frac{(\sqrt{(n-1) \log(n-1)} + \sqrt{2v \log(2n) + c \log(2n)})}{n} \right)^{-1/23}$ and $n \to \infty$, we have $h = o(1)$. This
verifies the conditions in Lemma 6.2. Therefore, \(\lim_{k \to \infty} \alpha^{(k)}\) exists, and it is exactly \(\hat{\alpha}\). By Lemma 6.2, it satisfies

\[
\|\hat{\alpha} - \alpha^*\|_\infty \leq 2\delta = O_p\left(\frac{1}{n}e^{16Q_n + e^{Q_n}} \left(\sqrt{(n - 1)\log(n - 1)} + \sqrt{2v\log(2n)} + c\log(2n)\right)\right).
\]

This is the consistency we need to prove.

6.3.3 Proof of the Theorem 2

To prove Theorem 3.2, we should introduce the following lemma and proposition.

Lemma 6.6. If \(V \in \mathcal{L}_n(m, M)\), \(W = V^{-1} - S\) and \(U = \text{cov}\left[W\{d - E(d)\}\right]\), then

\[
\|U\| \leq \|V^{-1} - S\| + \frac{2M}{m^2(n - 1)^2}.
\]

Proof. The proof of lemma 6.6 is similar to that of Proposition 1 in Yan et al. (2015), so we omit it.

Proposition 6.2. Assume that

\(\text{(C1) } V := \text{var}(\tilde{d}) \in \mathcal{L}_n(m, M);\)

\(\text{(C2) } (\tilde{d}_i - E(d_i))/v_{ii}^{1/2}\) are asymptotically standard normal as \(n \to \infty.\)

If \(M/m^2 = o(n)\), then for any fixed \(k\), the first \(k\) elements of \(S(\tilde{d} - E(d))\) are asymptotically normal distribution with mean zero and the covariance is given by the upper \(k \times k\) submatrix of the diagonal matrix \(B = \text{diag}(1/v_{11}, \ldots, 1/v_{nn})\), where \(S\) is the approximate inverse of \(V\) defined at (6.11).

In Proposition 1 in Yan et al. (2016b), they show that if \(M/m^2 = o(n)\), then the vector \((d_1 - E(d_1), \ldots, d_r - E(d_r))^T\) is asymptotically normally distributed with mean zero and covariance matrix \(\text{diag}(v_{11}, \ldots, v_{rr})\) for a fixed \(r \geq 1.\) Note that random variables \(\{e_i\}_{i=1}^n\) are mutually independent and distributed by sub-gamma distributions with respective parameters \((v_i, c_i)(i = 1, \ldots, n)\). Recall that \(\max_{i=1,\ldots,n} v_i = v\) and \(\max_{i=1,\ldots,n} c_i = c\) for any \(\tau > 0\), by Chebyshev’s inequality, we have

\[
P\left(\frac{|e_i|}{v_{ii}} > \tau\right) = P(|e_i| > \tau v_{ii}) \leq \frac{\text{var}(e_i)}{\tau^2(v_{ii})^2}.
\]

According to Boucheron et al. (2013), we have \(E X^2 \leq 8(v + c^2)\). Then \(\text{var}(e_i)/\tau^2(v_{ii})^2 \leq \frac{(8(v + c^2))}{\tau^2(v_{ii})^2}\) holds. If \(M/m^2 = o(n)\), we get

\[
(v_{ii})^{1/2}[S(\tilde{d} - E(d))]_i = \frac{d_i - E(d_i)}{(v_{ii})^{1/2}} + \frac{e_i}{v_{ii}} = \frac{d_i - E(d_i)}{v_{ii}^{1/2}} + o_p(1)
\]

Therefore, for any fixed \(k\), \((\tilde{d}_i - E(d_i))/(v_{ii})^{1/2}, i = 1, \ldots, k\), are asymptotically independent and standard normal distributions.
**Proof of Theorem 3.2.** Let \( \tilde{\gamma}_{ij} = \hat{\alpha}_i + \hat{\alpha}_j - \alpha_i^* - \alpha_j^* \) and assume

\[
\max_{i \neq j} |\tilde{\gamma}_{ij}| = O\left( \frac{1}{n} e^{14Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + 2\sqrt{2} \log(2n) + c \log(2n) \right) \right). \tag{6.16}
\]

For \( i = 1, \ldots, n \), by Taylor’s expansion, we have

\[
\tilde{d}_i - E(d_i) = \sum_{j \neq i} (e^{\hat{\alpha}_i + \hat{\alpha}_j + \epsilon_{i,j}(\hat{\alpha}_i, \hat{\alpha}_j)} - e^{\alpha_i^* + \alpha_j^* + \epsilon_{i,j}(\alpha_i^*, \alpha_j^*)})
\]

\[
= \sum_{j \neq i} \left[ (e^{\alpha_i^* + \alpha_j^* + \epsilon_{i,j}(\alpha_i^*, \alpha_j^*)})^\prime (e^{\hat{\alpha}_i + \hat{\alpha}_j + \epsilon_{i,j}(\hat{\alpha}_i, \hat{\alpha}_j)} - e^{\alpha_i^* + \alpha_j^* + \epsilon_{i,j}(\alpha_i^*, \alpha_j^*)}) \right] + h_i,
\]

where \( h_i = \frac{1}{2} \sum_{j \neq i} (e^{\alpha_i + \alpha_j + \epsilon_{i,j}(\alpha_i, \alpha_j)})^\prime |_{\alpha_i + \alpha_j = \theta_{ij}} [(\hat{\alpha}_i + \hat{\alpha}_j) - (\alpha_i^* + \alpha_j^*)]^2 \) and \( \theta_{ij} = t_{ij}(\alpha_i^* + \alpha_j^*) + (1 - t_{ij})(\hat{\alpha}_i + \hat{\alpha}_j) \), \( 0 < t_{ij} < 1 \).

Writing the above expressions in matrices, we have

\[
\tilde{d} - Ed = V(\hat{\alpha} - \alpha) + h.
\]

Equivalently,

\[
\hat{\alpha} - \alpha = V^{-1}(\tilde{d} - Ed) + V^{-1}h
\]

\[
= S(\tilde{d} - Ed) + W(\tilde{d} - Ed) + V^{-1}h,
\]

where \( h = (h_1, \ldots, h_n)^T \). Now that \( \mu''(\theta_{ij}) = O(e^{4Q_n}) \), then we get

\[
|h_i| \leq \frac{1}{2} (n - 1) e^{4Q_n} \tilde{\gamma}_{ij}^2,
\]

Therefore,

\[
|(V^{-1}h)_i| = |(Sh)_i| + |(Wh)_i|
\]

\[
\leq \max_i \left| h_i \right| + \|W\| \sum_i |h_i|
\]

\[
\leq O\left( \frac{3e^{4Q_n} \tilde{\gamma}_{ij}^2}{2n} + \frac{c_1 M^2}{m^3(n-1)^2} \times \frac{1}{2} n(n-1) e^{4Q_n} \tilde{\gamma}_{ij}^2 \right)
\]

\[
\leq O\left( \frac{m^2 + C_2 M^2}{2m^3} e^{4Q_n} \tilde{\gamma}_{ij}^2 \right)
\]

\[
= O\left( \frac{1}{n} e^{4Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2} \log(2n) + c \log(2n) \right)^2 \right).
\]

If \( \frac{1}{n} e^{4Q_n + e^{Q_n}} \left( \sqrt{(n-1) \log(n-1)} + \sqrt{2} \log(2n) + c \log(2n) \right)^2 = o(n^{1/2}) \), then, \( (V^{-1}h)_i = o(n^{-1/2}) \),
by lemma 6.6, we have

\[
\text{var}[W\{d_i - E(d_i)\} + W\{e_i\}] \\
= U_{ii} + 2 \text{cov}([W\{d_i - E(d_i)\}_i, W\{e_i\}_i] + \text{var}(\Sigma_j W_{ij} e_j) \\
\leq O\left(\frac{e^{6Q_n}}{(n-1)^2}\right) + 2\Sigma_j w_{ij}^2 \text{cov}(d_j - E(d_j), e_j) + 8(v + c^2)n\|W\|^2 \\
\leq O\left(\frac{e^{6Q_n}}{(n-1)^2} + \frac{e^{6Q_n}8(v + c^2)}{(n-1)^3}\right).
\]

If \(8e^{6Q_n}(v + c^2) = o(n^{1/2})\), by Chebyshev’s inequality, we obtain that

\[
P\left(\frac{[W\{\tilde{d} - E(d)\}]_i}{n^{-1/2}} > \epsilon\right) \leq \frac{n \text{var}[W\{\tilde{d} - E(d)\}]_i}{\epsilon^2} = o(n^{-1/2})
\]

For arbitrarily given \(\epsilon > 0\), it shows that

\[
[W\{\tilde{d} - E(d)\}]_i = o(n^{-1/2}) \quad (6.17)
\]

By the first part of this theorem, (6.16) holds with probability approaching 1. Consequently, by (6.17), we have

\[
(\hat{\alpha} - \alpha^*)_i = [S(\tilde{d} - E(d))]_i + o_p(n^{-1/2}).
\]

Therefore, Theorem 3.2 follows after Proposition 6.2.

\[
\square
\]

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Figure 1: In case of model $M_{log}$, the QQ plots of $\xi_{ij}$ with red color for $\hat{\xi}_{ij}$ ($n = 100$ and $\epsilon = 2$).
Figure 2: In case of model $M_{logit}$, the QQ plots of $\xi_{ij}$ with red color for $\hat{\xi}_{ij}$ ($n = 100$ and $\epsilon = 2$).
Figure 3: In case of model $M_{\text{loglog}}$, the QQ plots of $\xi_{ij}$ with red color for $\hat{\xi}_{ij}$ ($n = 100$ and $\epsilon = 2$).
Table 1: In case of $M_{\log}$, estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair $(i, j)$ as well as the length of confidence intervals , and the probabilities that the parameter estimator does not exist, multiplied by 100.

| $n$   | $(i, j)$, $a_1 = 0.01$, $a_2 = (\Lambda - 0.01)/4$, $m = 2$ | $-\log(\log(t))^{1/3}$ | $-\log(\log(t))^{1/2}$ | $-\log(\log(t))$ |
|-------|-------------------------------------------------|---------------------|-------------------|---------------------|
| 100   | (1,2)                                           | 98.64/0.41/0        | 98.69/0.43/0.22   | 96.42/0.41/3.34     |
|       | (50,51)                                         | 96.10/0.55/0        | 96.13/0.59/0.22   | 92.59/0.60/3.34     |
|       | (99,100)                                        | 94.14/0.73/0        | 93.67/0.80/0.22   | 91.04/0.90/3.34     |
| 200   | (1,2)                                           | 98.98/0.30/0        | 99.00/0.30/0      | 96.53/0.28/0.14     |
|       | (100,101)                                       | 96.94/0.40/0        | 96.12/0.42/0      | 91.86/0.43/0.14     |
|       | (199,200)                                       | 95.58/0.54/0        | 94.58/0.58/0      | 88.96/0.65/0.14     |
|       | $a_1 = \Lambda - 0.01$, $a_2 = 0.025$, $m = 2$ |                     |                   |                     |
| 100   | (1,2)                                           | 98.68/0.41/0        | 98.72/0.44/0.08   | 96.52/0.42/2.42     |
|       | (50,51)                                         | 95.82/0.55/0        | 96.32/0.59/0.08   | 92.60/0.61/2.42     |
|       | (99,100)                                        | 94.10/0.73/0        | 93.55/0.80/0.08   | 90.65/0.90/2.42     |
| 200   | (1,2)                                           | 99.08/0.30/0        | 98.26/0.30/0      | 96.74/0.28/0.06     |
|       | (100,101)                                       | 97.42/0.40/0        | 96.30/0.42/0      | 91.61/0.43/0.06     |
|       | (199,200)                                       | 95.62/0.54/0        | 95.40/0.57/0      | 88.69/0.65/0.06     |
|       | $a_1 = 4 \cdot \Lambda/5$, $a_2 = \Lambda/5$, $m = 2$ |                     |                   |                     |
| 100   | (1,2)                                           | 98.60/0.42/0.4     | 98.74/0.44/0.1    | 96.53/0.42/3.16     |
|       | (50,51)                                         | 96.26/0.55/0.04    | 96.06/0.59/0.1    | 96.27/0.61/3.16     |
|       | (99,100)                                        | 93.70/0.73/0.04    | 94.65/0.80/0.1    | 90.27/0.90/3.16     |
| 200   | (1,2)                                           | 98.82/0.30/0       | 98.86/0.30/0      | 96.43/0.28/0.05     |
|       | (100,101)                                       | 97.16/0.40/0       | 96.50/0.42/0      | 92.53/0.43/0.05     |
|       | (199,200)                                       | 95.28/0.54/0       | 94.48/0.58/0      | 88.36/0.65/0.05     |
Table 2: In case of $M_{logit}$, estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair $(i,j)$ as well as the length of confidence intervals, and the probabilities that the parameter estimator does not exist, multiplied by 100.

| n   | (i, j)         | 0    | log(log(n)) | log(n)$^{1/2}$ |
|-----|----------------|------|-------------|----------------|
|     | $a_1 = 0.01$, $a_2 = (\Lambda - 0.01)/4, m = 2$ |      |             |                |
| 100 | (1,2)          | 93.02/0.60/0 | 92.51/0.63/1.9 | 92.07/0.68/44.24 |
|     | (50,51)        | 93.16/0.60/0 | 91.76/0.76/1.9 | 91.60/0.94/44.24 |
|     | (99,100)       | 92.98/0.60/0 | 90.74/1.03/1.9 | 96.10/1.63/44.24 |
| 200 | (1,2)          | 93.94/0.40/0 | 93.48/0.48/0.024 | 93.60/0.48/14.00 |
|     | (100,101)      | 93.56/0.40/0 | 93.02/0.55/0.04  | 92.49/0.68/14.00 |
|     | (199,200)      | 94.16/0.40/0 | 92.86/0.75/0.04  | 94.21/1.12/14.00 |
|     | $a_1 = \Lambda - 0.01$, $a_2 = 0.025, m = 2$ |      |             |                |
| 100 | (1,2)          | 93.10/0.58/0 | 92.45/0.63/1.5 | 92.50/0.68/45.1  |
|     | (50,51)        | 93.00/0.58/0 | 91.53/0.76/1.5 | 91.51/0.94/45.1  |
|     | (99,100)       | 93.22/0.58/0 | 91.15/1.02/1.5 | 95.70/1.60/45.1  |
| 200 | (1,2)          | 94.04/0.40/0 | 93.86/0.45/0.03 | 93.76/0.48/11.98 |
|     | (100,101)      | 94.25/0.40/0 | 93.80/0.55/0.03 | 92.69/0.68/11.98 |
|     | (199,200)      | 94.15/0.40/0 | 92.51/0.76/0.03 | 95.02/1.12/11.98 |
|     | $a_1 = 4 \times \Lambda/5$, $a_2 = \Lambda/5, m = 2$ |      |             |                |
| 100 | (1,2)          | 92.48/0.58/0 | 92.27/0.63/1.62 | 92.46/0.68/44.58 |
|     | (50,51)        | 92.64/0.58/0 | 90.93/0.76/1.62 | 90.47/0.94/44.58 |
|     | (99,100)       | 92.88/0.58/0 | 90.28/1.03/1.62 | 95.23/1.59/44.58 |
| 200 | (1,2)          | 93.81/0.40/0 | 93.50/0.45/0.024 | 93.80/0.48/12.67 |
|     | (100,101)      | 94.00/0.40/0 | 93.25/0.55/0.04  | 92.61/0.68/12.67 |
|     | (199,200)      | 94.27/0.40/0 | 92.51/0.75/0.04  | 95.00/1.12/12.67 |
Table 3: In case of $M_{\text{cloglog}}$, estimated coverage probabilities of $\alpha_i - \alpha_j$ for pair $(i, j)$ as well as the length of confidence intervals , and the probabilities that the parameter estimator does not exist, multiplied by 100.

| $n$  | $(i, j)$     | $\log(\log(\log(t)))^{3/2}$ | $\log(\log(\log(t)))^{5/4}$ | $\log(\log(\log(t)))$ |
|------|--------------|-------------------------------|-------------------------------|------------------------|
| 100  | (1,2)        | 86.86/0.47/0                  | 89.21/0.48/0                  | 91.24/0.50/0           |
|      | (50,51)      | 88.16/0.45/0                  | 91.52/0.47/0                  | 93.50/0.50/0           |
|      | (99,100)     | 89.46/0.45/0                  | 92.70/0.47/0                  | 96.06/0.52/0           |
| 200  | (1,2)        | 91.58/0.34/0                  | 92.42/0.35/0                  | 97.08/0.36/0           |
|      | (100,101)    | 93.63/0.34/0                  | 95.38/0.36/0                  | 98.54/0.37/0           |
|      | (199,200)    | 95.66/0.34/0                  | 96.78/0.38/0                  | 99.18/0.41/0           |
|      | $a_1 = 0.01$, $a_2 = (\Lambda - 0.01)/4, m = 2$ | | | |
| 100  | (1,2)        | 86.64/0.47/0                  | 89.33/0.45/0                  | 91.34/0.45/0           |
|      | (50,51)      | 88.06/0.48/0                  | 91.46/0.47/0                  | 93.52/0.47/0           |
|      | (99,100)     | 89.54/0.50/0                  | 92.84/0.50/0                  | 95.90/0.52/0           |
| 200  | (1,2)        | 91.48/0.34/0                  | 92.22/0.35/0                  | 97.16/0.36/0           |
|      | (100,101)    | 93.88/0.34/0                  | 95.66/0.36/0                  | 98.66/0.37/0           |
|      | (199,200)    | 95.40/0.34/0                  | 96.78/0.38/0                  | 99.24/0.41/0           |
|      | $a_1 = \Lambda - 0.01, a_2 = 0.025, m = 2$ | | | |
| 100  | (1,2)        | 86.02/0.47/0                  | 88.72/0.48/0                  | 91.02/0.50/0           |
|      | (50,51)      | 88.32/0.45/0                  | 91.18/0.47/0                  | 93.56/0.50/0           |
|      | (99,100)     | 89.72/0.45/0                  | 92.22/0.47/0                  | 95.72/0.52/0           |
| 200  | (1,2)        | 91.42/0.34/0                  | 92.06/0.35/0                  | 96.70/0.36/0           |
|      | (100,101)    | 93.82/0.34/0                  | 95.54/0.36/0                  | 98.66/0.37/0           |
|      | (199,200)    | 95.18/0.34/0                  | 96.94/0.38/0                  | 99.12/0.41/0           |
|      | $a_1 = 4 \times \Lambda/5, a_2 = \Lambda/5, m = 2$ | | | |
Table 4: The Bruce Kapferer network dataset: In case of $M_{\log}$, the parameter estimator $\tilde{\alpha}_{\log}$ in model $M_{\log}$, $\tilde{\alpha}_{\log}$, 95% confidence intervals (in square brackets) and their standard errors (in parentheses).

| Vertex | degree | $\tilde{\alpha}_{\log}$ | Vertex | degree | $\tilde{\alpha}_{\log}$ |
|--------|--------|----------------|--------|--------|----------------|
| 1      | -2.50 [-3.63, -0.76] (0.73) | 20 | -0.59 [-1.24, 0.06] (0.33) | 10 |
| 2      | -1.28 [-2.19, -0.37] (0.47) | 21 | -1.28 [-2.19, -0.37] (0.47) | 5 |
| 3      | -0.49 [-1.11, 0.13] (0.32) | 22 | -0.49 [-1.11, 0.13] (0.32) | 11 |
| 4      | -0.49 [-1.11, 0.13] (0.32) | 23 | -2.50 [-3.63, -0.76] (0.73) | 2 |
| 5      | -1.79 [-2.96, -0.62] (0.60) | 24 | -2.50 [-3.63, -0.76] (0.73) | 2 |
| 6      | -1.10 [-1.93, -0.27] (0.42) | 25 | -1.79 [-2.96, -0.62] (0.60) | 3 |
| 7      | -1.10 [-1.93, -0.27] (0.42) | 26 | -0.69 [-1.38, 0.01] (0.35) | 9 |
| 8      | -1.79 [-2.96, -0.62] (0.60) | 27 | -0.69 [-1.38, 0.01] (0.35) | 9 |
| 9      | -0.69 [-1.38, 0.01] (0.35) | 28 | -0.69 [-1.38, 0.01] (0.35) | 9 |
| 10     | -2.89 [-4.91, -0.87] (1.03) | 29 | -1.10 [-1.93, -0.27] (0.42) | 6 |
| 11     | -0.18 [-0.72, 0.35] (0.27) | 30 | -0.12 [-0.64, 0.40] (0.26) | 16 |
| 12     | -0.25 [-0.80, 0.30] (0.28) | 31 | -0.59 [-1.24, 0.06] (0.33) | 16 |
| 13     | -0.59 [-1.24, 0.06] (0.33) | 32 | -0.12 [-0.64, 0.40] (0.26) | 16 |
| 14     | -0.81 [-1.53, -0.09] (0.37) | 33 | -1.10 [-1.93, -0.27] (0.42) | 6 |
| 15     | -1.28 [-2.19, -0.37] (0.47) | 34 | -0.59 [-1.24, 0.06] (0.33) | 10 |
| 16     | 0.37 [-0.05, 0.78] (0.21) | 35 | -1.10 [-1.93, -0.27] (0.42) | 6 |
| 17     | -0.94 [-1.72, -0.17] (0.39) | 36 | -0.69 [-1.38, 0.01] (0.35) | 9 |
| 18     | -0.06 [-0.56, 0.45] (0.26) | 37 | -1.28 [-2.19, -0.37] (0.47) | 5 |
| 19     | -2.89 [-4.91, -0.87] (1.03) | 38 | -1.10 [-1.93, -0.27] (0.42) | 6 |

Table 5: The Bruce Kapferer network dataset: In case of $M_{\logit}$, the parameter estimator $\tilde{\alpha}_{\logit}$ in model $M_{\logit}$, $\tilde{\alpha}_{\logit}$, 95% confidence intervals (in square brackets) and their standard errors (in parentheses).

| Vertex | degree | $\tilde{\alpha}_{\logit}$ | Vertex | degree | $\tilde{\alpha}_{\logit}$ |
|--------|--------|----------------|--------|--------|----------------|
| 1      | -2.52 [-4.02, -1.03] (0.76) | 20 | -0.32 [-1.12, 0.49] (0.41) | 10 |
| 2      | -1.38 [-2.39, -0.36] (0.52) | 21 | -1.38 [-2.39, -0.36] (0.52) | 5 |
| 3      | -0.14 [-0.93, 0.64] (0.40) | 22 | -0.14 [-0.93, 0.64] (0.40) | 11 |
| 4      | -0.14 [-0.93, 0.64] (0.40) | 23 | -2.52 [-4.02, -1.03] (0.76) | 2 |
| 5      | -2.04 [-3.29, -0.78] (0.64) | 24 | -2.52 [-4.02, -1.03] (0.76) | 2 |
| 6      | -1.12 [-2.07, -0.17] (0.48) | 25 | -2.04 [-3.29, -0.78] (0.64) | 3 |
| 7      | -1.12 [-2.07, -0.17] (0.48) | 26 | -0.50 [-1.32, 0.33] (0.42) | 9 |
| 8      | -2.04 [-3.29, -0.78] (0.64) | 27 | -0.50 [-1.32, 0.33] (0.42) | 9 |
| 9      | -0.50 [-1.32, 0.33] (0.42) | 28 | -0.32 [-1.12, 0.49] (0.41) | 10 |
| 10     | -3.30 [-5.35, -1.26] (1.04) | 29 | -1.12 [-2.07, -0.17] (0.48) | 6 |
| 11     | 0.47 [-0.26, 1.21] (0.38) | 30 | -0.62 [-0.11, 1.35] (0.37) | 16 |
| 12     | 0.33 [-0.42, 1.07] (0.38) | 31 | -0.32 [-1.12, 0.49] (0.41) | 10 |
| 13     | -0.32 [-1.12, 0.49] (0.41) | 32 | 0.62 [-0.11, 1.35] (0.37) | 16 |
| 14     | -0.69 [-1.55, 0.17] (0.44) | 33 | -1.12 [-2.07, -0.17] (0.48) | 6 |
| 15     | -1.38 [-2.39, -0.36] (0.52) | 34 | -0.32 [-1.12, 0.49] (0.41) | 10 |
| 16     | 2.10 [1.29, 2.91] (0.41) | 35 | -1.12 [-2.07, -0.17] (0.48) | 6 |
| 17     | -0.89 [-1.79, 0.00] (0.46) | 36 | -0.50 [-1.32, 0.33] (0.42) | 9 |
| 18     | 0.76 [0.03, 1.49] (0.37) | 37 | -1.38 [-2.39, -0.36] (0.52) | 5 |
| 19     | -3.30 [-5.35, -1.26] (1.04) | 38 | -1.12 [-2.07, -0.17] (0.48) | 6 |
Table 6: The Bruce Kapferer network dataset: In case of $M_{cloglog}$, the parameter estimator $\tilde{\alpha}_{cloglog}$ in model $M_{cloglog}$, $\tilde{\alpha}_{cloglog}$,95% confidence intervals (in square brackets) and their standard errors (in parentheses).

| Vertex | degree | $\tilde{\alpha}_{cloglog}$ | Vertex | degree | $\tilde{\alpha}_{cloglog}$ |
|--------|--------|-----------------------------|--------|--------|-----------------------------|
|        |        | In case of $M_{cloglog}$ and $\epsilon = 2$ |        |        |                             |
| 1      | -1.19  | [-2.25, -0.14] (0.54)       | 2      | 20     | -0.17 [-0.85, 0.52] (0.35)  | 10     |
| 2      | -0.70  | [-1.50, 0.09] (0.40)        | 5      | 21     | -0.70 [-1.50, 0.09] (0.40)  | 5      |
| 3      | -0.07  | [-0.75, 0.60] (0.34)        | 11     | 22     | -0.07 [-0.75, 0.60] (0.34)  | 11     |
| 4      | -0.07  | [-0.75, 0.60] (0.34)        | 11     | 23     | -1.19 [-2.25, -0.14] (0.54) | 2      |
| 5      | -0.10  | [-1.93, -0.07] (0.47)       | 3      | 24     | -1.19 [-2.25, -0.14] (0.54) | 2      |
| 6      | -0.58  | [-1.34, 0.16] (0.39)        | 6      | 25     | -0.10 [-1.93, -0.07] (0.47) | 3      |
| 7      | -0.58  | [-1.34, 0.16] (0.39)        | 6      | 26     | -0.26 [-0.96, 0.43] (0.35)  | 9      |
| 8      | -0.10  | [-1.93, -0.07] (0.47)       | 3      | 27     | -0.26 [-0.96, 0.43] (0.35)  | 9      |
| 9      | -0.26  | [-0.96, 0.43] (0.35)        | 9      | 28     | -0.17 [-0.85, 0.52] (0.35)  | 10     |
| 10     | -1.48  | [-2.81, -0.15] (0.68)       | 1      | 29     | -0.58 [-1.34, 0.16] (0.39)  | 6      |
| 11     | 0.26   | [-0.40, 0.93] (0.34)        | 15     | 30     | 0.34 [-0.33, 1.01] (0.34)   | 16     |
| 12     | 0.18   | [-0.48, 0.85] (0.34)        | 14     | 31     | -0.17 [-0.85, 0.52] (0.35)  | 10     |
| 13     | -0.17  | [-0.85, 0.52] (0.35)        | 10     | 32     | 0.34 [-0.33, 1.01] (0.34)   | 16     |
| 14     | -0.36  | [-1.07, 0.35] (0.36)        | 8      | 33     | -0.58 [-1.34, 0.16] (0.39)  | 6      |
| 15     | -0.70  | [-1.50, 0.09] (0.40)        | 5      | 34     | -0.17 [-0.85, 0.52] (0.35)  | 10     |
| 16     | 1.05   | [0.31, 1.79] (0.38)         | 26     | 35     | -0.58 [-1.34, 0.16] (0.39)  | 6      |
| 17     | -0.47  | [-1.20, 0.26] (0.37)        | 7      | 36     | -0.26 [-0.96, 0.43] (0.35)  | 9      |
| 18     | 0.42   | [-0.25, 1.09] (0.34)        | 17     | 37     | -0.70 [-1.50, 0.09] (0.40)  | 5      |
| 19     | -1.48  | [-2.81, -0.15] (0.68)       | 1      |        |                             |        |
Figure 4: The scatter plots \( n = 39, \epsilon = 2 \). The \( \tilde{d} \) denotes the noisy degree sequences and \( \tilde{\alpha} \) denotes the corresponding the parameter estimation.