Model Selection for Generic Reinforcement Learning

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Abstract

We address the problem of model selection for the finite horizon episodic Reinforcement Learning (RL) problem where the transition kernel $P^*$ belongs to a family of models $\mathcal{P}$ with finite metric entropy. In the model selection framework, instead of $P^*$, we are given $M$ nested families of transition kernels $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_M$. We propose and analyze a novel algorithm, namely Adaptive Reinforcement Learning (General) (ARL-GEN) that adapts to the smallest such family where the true transition kernel $P^*$ lies. ARL-GEN uses the Upper Confidence Reinforcement Learning (UCRL) algorithm with value targeted regression as a blackbox and puts a model selection module at the beginning of each epoch. Under a mild separability assumption on the model classes, we show that ARL-GEN obtains a regret of $O(d^*_E H^2 + \sqrt{T d^*_EM^*H^2})$, with high probability, where $H$ is the horizon length, $T$ is the total number of steps, $d^*_E$ is the Eluder dimension and $M^*$ is the metric entropy corresponding to $P^*$. Note that this regret scaling matches that of an oracle that knows $P^*$ in advance. We show that the cost of model selection for ARL-GEN is an additive term in the regret having a weak dependence on $T$. Subsequently, we remove the separability assumption and consider the setup of linear mixture MDPs, where the transition kernel $P^*$ has a linear function approximation. With this low rank structure, we propose novel adaptive algorithms for model selection, and obtain (order-wise) regret identical to that of an oracle with knowledge of the true model class.

Keywords: Model selection, Reinforcement Learning, Function approximation

Full paper: The full paper is available at https://tinyurl.com/3tfrzcju

1. INTRODUCTION

A Markov decision process (MDP) (Puterman, 2014) is a classical framework to model a reinforcement learning (RL) environment, where an agent interacts with the environment by taking successive decisions and observe rewards. One of the objectives in RL is to maximize the total reward accumulated over multiple rounds, or equivalently minimize the regret in comparison with an optimal policy (Jaksch et al., 2010). Regret minimization is useful in several sequential decision-making problems such as portfolio allocation and sequential investment, dynamic resource allocation in communication systems, recommendation systems, etc. In these settings, there is no separate budget to purely explore the unknown environment; rather, exploration and exploitation need to be carefully balanced.

In many applications (e.g., AlphaGo, robotics), the space of states and actions can be very large or even infinite, which makes RL challenging, particularly in generalizing learnt knowledge across unseen states and actions. In recent years, we have witnessed an explosion in the RL literature to
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tackle this challenge, both in theory (see, e.g., Osband and Van Roy (2014); Chowdhury and Gopalan (2019); Ayoub et al. (2020); Wang et al. (2020); Kakade et al. (2020)), and in practice (see, e.g., Mnih et al. (2013); Williams et al. (2017)). The most related work to ours is by Ayoub et al. (2020), which proposes an algorithm, namely UCRL-VTR, for model-based RL without any structural assumptions, and it is based on the upper confidence RL and value-targeted regression principles. The regret of UCRL-VTR depends on the eluder dimension (Russo and Van Roy, 2013) and the metric entropy of the corresponding family of distributions \( \mathcal{P} \) in which the unknown transition model \( P^* \) lies. In most practical cases, however, the class \( \mathcal{P} \) given to (or estimated by) the RL agent is quite pessimistic; meaning that \( P^* \) actually lies in a small subset of \( \mathcal{P} \) (e.g., in the game of Go, the learning is possible without the need for visiting all the states (Silver et al., 2017)). This issue becomes more interpretable in the setup where \( P^* \) assumes a low-rank structure via a linear model (see, e.g., Jin et al. (2019); Yang and Wang (2019); Jia et al. (2020); Zhou et al. (2020)). RL problems with linear function approximation are often parameterized by a \( \theta^* \in \mathbb{R}^d \). In this setting, the regret usually depends on (a) the norm \( \|\theta^*\| \) and (b) the dimension \( d \). In particular, Jia et al. (2020) show that the UCRL-VTR algorithm, when instantiated in the linear setting, achieves a regret of \( O(bd\sqrt{HT^2}) \), where \( b \) is an upper bound over \( \|\theta^*\| \), \( H \) is the horizon length and \( T \) is the total number of steps. In this setting, the choice of \( \|\theta^*\| \) and dimension or sparsity of \( \theta^* \) is crucial. If these quantities are under-specified, the regret bounds may fail to hold, and the learning algorithms may incur linear regret. Furthermore, if these quantities are over-specified (which is the case in most RL applications, e.g., the linear quadratic regulators (Abbasi-Yadkori and Szepesvári, 2011)), the regret bounds are unnecessarily large. Hence, one needs algorithms that exploit the structure of the problem and adapt to the problem complexity with high confidence.

The problem of model selection can be formally stated as follows – we are given a family of \( M \) nested hypothesis classes \( \mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_M \), where each class posits a plausible model class for the underlying RL problem. The true model \( P^* \) lies in a model class \( \mathcal{P}_{m^*} \), which is assumed to be contained in the family of nested classes. Model selection guarantees refer to algorithms whose regret scales in the complexity of the smallest model class containing the true model \( P^* \), even though the algorithm is not aware of that a priori. Model selection is well studied in the contextual bandit setting. In this setting, minimax optimal regret guarantees can be obtained by exploiting the structure of the problem along with an eigenvalue assumption (Chatterji et al., 2019; Foster et al., 2019; Ghosh et al., 2021a). In this work, we address the problem of model selection in RL environments. We consider both the setups where the underlying transition distribution has no structural assumption as well as when it admits a low rank linear function approximation. In the RL framework, the question of model selection has received little attention. In a series of works, Pacchiano et al. (2020a,b) consider the corralling framework of Agarwal et al. (2017) for contextual bandits and reinforcement learning. While the corralling framework is versatile, the price for this is that the cost of model selection is multiplicative rather than additive. In particular, for the special case of linear bandits and linear reinforcement learning, the regret scales as \( \sqrt{T} \) in time with an additional multiplicative factor of \( \sqrt{M} \), while the regret scaling with time is strictly larger than \( \sqrt{T} \) in the general contextual bandit. These papers treat all the hypothesis classes as bandit arms, and hence work in a (restricted) partial information setting, and as a consequence explore a lot, yielding worse regret. On the other hand, we consider all \( M \) classes at once (full information setting) and do inference, and hence explore less and obtain lower regret.

Very recently, Lee et al. (2020) study the problem of model selection in RL with function approximation. Similar to the active-arm elimination technique employed in standard multi-armed
bandit (MAB) problems (Even-Dar et al., 2006), the authors eliminate the model classes that are dubbed misspecified, and obtain a regret of $O(T^{2/3})$. On the other hand, our framework is quite different in the sense that we consider model selection for RL with general transition structure. Moreover, our regret scales as $O(\sqrt{T})$. Note that the model selection guarantees we obtain in the linear MDPs are partly influenced by Ghosh et al. (2021a), where model selection for linear contextual bandits are discussed. However, there are a couple of subtle differences: (a) for linear contextual framework, one can perform pure exploration, and Ghosh et al. (2021a) crucially leverages that and (b) the contexts in linear contextual framework is assumed to be i.i.d, whereas for linear MDPs, the contexts are implicit and depend on states, actions and transition probabilities.

Outline and Contributions: The first part of the paper deals with the setup where we consider any general model class that are totally bounded, i.e., for arbitrary precision, the metric entropy is bounded. Notice that this encompasses a significantly larger class of problems compared to the problems with function approximation. Assuming a nested family of transition kernels, we propose an adaptive algorithm, namely Adaptive Reinforcement Learning-General (ARL-GEN). Assuming the transition families are well-separated, ARL-GEN constructs a test statistic and thresholds it to identify the correct family. We show that this simple scheme achieves the regret $\hat{O}(d^*\theta^2 + \sqrt{d^*\theta^2H^2T})$, where $d^*$ is the eluder dimension and $\theta^*$ is the metric entropy corresponding to the transition family $P^*_m$ in which the true model $P^*$ lies. The regret bound shows that ARL-GEN adapts to the true problem complexity, and the cost of model selection is only $O(\log T)$, which is minimal compared to the total regret. In the second part of the paper, we focus on the linear function approximation framework, where the transition kernel is parameterized by a vector $\theta^* \in \mathbb{R}^d$. With norm ($\|\theta^*\|$) and sparsity ($\|\theta^*\|_0$) as problem complexity parameters, we propose two algorithms, namely Adaptive Reinforcement Learning-Linear (norm) and Adaptive Reinforcement Learning-Linear (dim), respectively, that adapt to these complexities – meaning that the regret depends on the actual problem complexities $\|\theta^*\|$ and $\|\theta^*\|_0$. Here also, the costs of model selection are shown to be minor lower order terms.

1.1. Related Work

Model Selection in Online Learning: Model selection for bandits are only recently being studied (Ghosh et al., 2017; Chatterji et al., 2019). These works aim to identify whether a given problem instance comes from contextual or standard setting. For linear contextual bandits, with the dimension of the underlying parameter as a complexity measure, Foster et al. (2019); Ghosh et al. (2021a) propose efficient algorithms that adapts to the true dimension of the problem. While Foster et al. (2019) obtains a regret of $O(T^{2/3})$, Ghosh et al. (2021a) obtains a $O(\sqrt{T})$ regret (however, the regret of Ghosh et al. (2021a) depends on several problem dependent quantities and hence not instance uniform). Later on, these guarantees are extended to the generic contextual bandit problems without linear structure (Ghosh et al., 2021c; Krishnamurthy and Athey, 2021), where $O(\sqrt{T})$ regret guarantees are obtained. The algorithm Corral was proposed in Agarwal et al. (2017), where the optimal algorithm for each model class is casted as an expert, and the forecaster obtains low regret with respect to the best expert (best model class). The generality of this framework has rendered it fruitful in a variety of different settings; see, for example Agarwal et al. (2017); Arora et al. (2021).

RL with Function Approximation: Regret minimization in RL under function approximation is first considered in Osband and Van Roy (2014). It makes explicit model-based assumptions and
the regret bound depends on the eluder dimensions of the models. In contrast, Yang and Wang (2019) considers a low-rank linear transition model and propose a model-based algorithm with regret $O(\sqrt{dH^3T})$. Another line of work parameterizes the Q-functions directly, using state-action feature maps, and develop model-free algorithms with regret $O(\min\{d^{\frac{1}{2}}H, dH\sqrt{T}\})$ by bypassing the need for fully learning the transition model (Jin et al., 2019; Wang et al., 2019; Zanette et al., 2020). A recent line of work (Wang et al., 2020; Yang et al., 2020) generalize these approaches by designing algorithms that work with general and neural function approximations, respectively.

1.2. Preliminaries

**Notation:** For any $n \in \mathbb{N}$, $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. $\gamma_{\min}(A)$ denotes the minimum eigenvalue of the matrix $A$. $\mathbb{B}_d^+ \equiv \mathbb{R}^d_+$ denotes the unit ball in $\mathbb{R}^d$ and $\mathbb{S}_d^+$ denotes the set of all $d \times d$ positive definite matrices. For functions $f, g : \mathcal{X} \rightarrow \mathbb{R}$, $(f - g)(x) := f(x) - g(x)$ and $(f - g)^2(x) := (f(x) - g(x))^2$ for any $x \in \mathcal{X}$. For any $P : \mathcal{Z} \rightarrow \Delta(\mathcal{X})$, we denote $(Pf)(z) := \int_{\mathcal{X}} f(x)P(x|z)dx$ for any $z \in \mathcal{Z}$, where $\Delta(\mathcal{X})$ denotes the set of signed distributions over $\mathcal{X}$.

**Regret Minimization in Episodic MDPs:** An episodic MDP is denoted by $\mathcal{M}(S, A, H, P^*, r)$, where $S$ is the state space, $A$ is the action space (both possibly infinite), $H$ is the length of each episode, $P^* : S \times A \rightarrow \Delta(S)$ is an (unknown) transition kernel (a function mapping state-action pairs to signed distribution over the state space) and $r : S \times A \rightarrow [0, 1]$ is a (known) reward function. In episodic MDPs, a (deterministic) policy $\pi$ is given by a collection of $H$ functions $(\pi_1, \ldots, \pi_H)$, where each $\pi_h : S \rightarrow A$ maps a state $s$ to an action $a$. In each episode, an initial state $s_1$ is first picked by the environment (assumed to be fixed and history independent). Then, at each step $h \in [H]$, the agent observes the state $s_h$, picks an action $a_h$ according to $\pi_h$, receives a reward $r(s_h, a_h)$, and then transitions to the next state $s_{h+1}$, which is drawn from the conditional distribution $P^*(s_{h+1}|s_h, a_h)$. The episode ends when the terminal state $s_{H+1}$ is reached. For each state-action pair $(s, a) \in S \times A$ and step $h \in [H]$, we define action values $Q^\pi_h(s, a)$ and and state values $V^\pi_h(s)$ corresponding to a policy $\pi$ as

$$Q^\pi_h(s, a) = r(s, a) + \mathbb{E}\left[ \sum_{h' = h+1}^H r(s_{h'}, \pi_{h'}(s_{h'})) | s_h = s, a_h = a \right], \quad V^\pi_h(s) = Q^\pi_h(s, \pi_h(s)),$$

where the expectation is with respect to the randomness of the transition distribution $P^*$. It is not hard to see that $Q^\pi_h$ and $V^\pi_h$ satisfy the Bellman equations:

$$Q^\pi_h(s, a) = r(s, a) + (P^* V^{\pi}_{h+1})(s, a), \quad \forall h \in [H], \quad \text{with } V^{\pi}_{H+1}(s) = 0 \text{ for all } s \in S.$$

A policy $\pi^*$ is said to be optimal if it maximizes the value for all states $s$ and step $h$ simultaneously, and the corresponding optimal value function is denoted by $V^*_h(s) = \sup_{\pi \in \Pi} V^\pi_h(s)$ for all $h \in [H]$, where the supremum is over all (non-stationary) policies. The agent interacts with the environment for $K$ episodes to learn the unknown transition kernel $P^*$ and thus, in turn, the optimal policy $\pi^*$. At each episode $k \geq 1$, the agent chooses a policy $\pi^k := (\pi^k_1, \ldots, \pi^k_H)$ and a trajectory $(s_{h}^k, a_{h}^k, r(s_{h}^k, a_{h}^k), s_{h+1}^k)_{h \in [H]}$ is generated. The performance of the learning agent is measured by the cumulative (pseudo) regret accumulated over $K$ episodes, defined as

$$R(T) := \sum_{k=1}^K \left[ V^{\pi^*_k}_1(s_1^k) - V^{\pi^k_1}_1(s_1^1) \right],$$

where $T = KH$ is total steps in $K$ episodes.
2. GENERAL MDPs

In this section, we consider general MDPs without any structural assumption on the unknown transition kernel $P^*$. In the standard setting (Ayoub et al., 2020), it is assumed that $P^*$ belongs to a known family of transition models $\mathcal{P}$. Here, in contrast to the standard setting, we do not have the knowledge of $\mathcal{P}$. Instead, we are given $M$ nested families of transition kernels $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_M$. The smallest such family where the true transition kernel $P^*$ lies is denoted by $\mathcal{P}_{m^*}$, where $m^* \in [M]$. However, we do not know the index $m^*$, and our goal is to propose adaptive algorithms such that the regret depends on the complexity of the family $\mathcal{P}_{m^*}$. In order to achieve this, we need a separability condition on the nested model classes.

Assumption 1 (Separability) There exists a constant $\Delta > 0$ such that for any bounded function $V : \mathcal{S} \rightarrow [0, H]$, transition kernel $P \in \mathcal{P}_{m^* - 1}$, and state-action pair $(s, a) \in \mathcal{S} \times \mathcal{A}$, we have

$$((PV)(s, a) - (P^*V)(s, a))^2 \geq \Delta$$

This assumption ensures that given a state-action pair, expected values under the true model is well-separated from expected values under any model from non-realizable classes. We need this to hold uniformly over all states and actions since we do not assume any additional structure over state-action spaces. Note that separability is standard and assumptions of similar nature appear in a wide range of model selection problems, specially in the setting of contextual bandits (Ghosh et al., 2021c; Krishnamurthy and Athey, 2021). Separability condition is also quite standard in statistics, specifically in the area of clustering and latent variable modelling (Balakrishnan et al., 2017; Yi et al., 2016; Ghosh et al., 2019).

Note that the assumption breaks down for any constant function $V$. However, we will be invoking this assumption with the value functions computed by the learning algorithm (see (1)). For reward functions that vary sufficiently with states and actions, and transition kernels that admit densities, the chance of getting hit by constant value functions is admissibly low. In case the rewards are constant, every policy would anyway incur zero regret rendering the learning problem trivial.

The value functions appear in the separability assumption in the first place since we are interested in minimizing the regret. Instead, if one cares only about learning the true model, then separability of transition kernels under some suitable notion of distance (e.g., the KL-divergence) might suffice. Note that in Ghosh et al. (2021c); Krishnamurthy and Athey (2021), the regret is defined in terms of the regression function and hence the separability is assumed on the regression function itself.

Model selection without separability is kept as an interesting future work.

2.1. Algorithm: Adaptive Reinforcement Learning - General (ARL-GEN)

In this section, we provide a novel model selection algorithm ARL-GEN (Algorithm 1) that use successive refinements over epochs. We use UCRL-VTR algorithm of Ayoub et al. (2020) as our base algorithm, and add a model selection module at the beginning of each epoch. In other words, over multiple epochs, we successively refine our estimates of the proper model class where the true transition kernel $P^*$ lies.

The Base Algorithm: UCRL-VTR, in its general form, takes a family of transition models $\mathcal{P}$ and a confidence level $\delta \in (0, 1]$ as its input. At each episode $k$, it maintains a (high-probability) confidence set $B_{k-1} \subset \mathcal{P}$ for the unknown model $P^*$ and use it for optimistic planning. First, it finds the transition kernel $P_k = \arg\max_{P \in B_{k-1}} V^*_P(s^k_1)$, where $V^*_P$ denote the optimal value function...
of an MDP with transition kernel \( P \) at step \( h \). \textsc{ucrl-vtr} then computes, at each step \( h \), the optimal value function \( V^k_h := V^*_{P,h} \) under the kernel \( P_h \) using dynamic programming. Specifically, starting with \( V^k_{H+1}(s,a) = 0 \) for all pairs \((s,a)\), it defines for all steps \( h = H \) down to 1,

\[
Q^k_h(s,a) = r(s,a) + (P_hV^k_{h+1})(s,a), \quad V^k_h(s) = \max_{a \in A} Q^k_h(s,a).
\]

(1)

Then, at each step \( h \), \textsc{ucrl-vtr} takes the action that maximizes the \( Q \)-function estimate, i.e. it chooses \( a^k_h = \arg\max_{a \in A} Q^k_h(s^k_h,a) \). Now, the confidence set is updated using all the data gathered in the episode. First, \textsc{ucrl-vtr} computes an estimate of \( P^* \) by employing a non-linear value-targeted regression model with data \( \{(s^j_h,a^j_h,V^j_{h+1}(s^j_{h+1})\}_{j \in [k], h \in [H]} \). Note that \( \mathbb{E}[V^k_{h+1}(s^k_{h+1})|\mathcal{G}^k_{h-1}] = (P^*V^k_{h+1})(s^k_h,a^k_h) \), where \( \mathcal{G}^k_{h-1} \) denotes the total number of episodes completed before epoch \( s^k_{h+1} \) is observed. This naturally leads to the estimate \( \hat{P}_k = \arg\min_{P \in \mathcal{P}} \mathcal{L}_k(P) \), where

\[
\mathcal{L}_k(P) := \sum_{j=1}^k \sum_{h=1}^H \left(V^j_{h+1}(s^j_{h+1}) - (PV^j_{h+1})(s^j_h,a^j_h)\right)^2.
\]

(2)

The confidence set \( \mathcal{B}_k \) is then updated by enumerating the set of all transition kernels \( P \in \mathcal{P} \) satisfying \( \sum_{j=1}^k \sum_{h=1}^H \left((PV^j_{h+1})(s^j_h,a^j_h) - (PV^j_{h+1})(s^j_h,a^j_h)\right)^2 \leq \beta_k(\delta) \) with the confidence width being defined as \( \beta_k(\delta) := 8H^2\log\left(\frac{2N(\mathcal{P}, \mathcal{B}_k, \delta)}{H^2}\right) + 4H^2 + \left(2 + \sqrt{2 log\left(\frac{4kh(\delta+1)}{\delta}\right)}\right) \), where \( N(\mathcal{P}, \cdot, \cdot) \) denotes the covering number of the family \( \mathcal{P} \). Then, one can show that \( P^* \) lies in the confidence set \( \mathcal{B}_k \) in all episodes \( k \) with probability at least \( 1 - \delta \). Here, we consider a slight different expression of \( \beta_k(\delta) \) as compared to Ayoub et al. (2020), but the proof essentially follows the same technique. Please refer to Appendix A for further details, and for all the proofs of this section.

**Our Approach:** We consider doubling epochs - at each epoch \( i \geq 1 \), \textsc{ucrl-vtr} is run for \( k_i = 2^i \) episodes. At the beginning of \( i \)-th epoch, using all the data of previous epochs, we add a model selection module as follows. First, we compute, for each family \( \mathcal{P}_m \), the transition kernel \( \tilde{P}^{(i)}_{m,j} \), that minimizes the empirical loss \( \mathcal{L}_{\tau_{i-1}}(P) \) over all \( P \in \mathcal{P}_m \) (see (2)), where \( \tau_{i-1} := \sum_{j=1}^{k_i-1} k_j \) denotes the total number of episodes completed before epoch \( i \). Next, we compute the average empirical loss \( T^{(i)}_{m} := \frac{1}{\tau_{i-1}} \mathcal{L}_{\tau_{i-1}}(\tilde{P}^{(i)}_{m,j}) \) for the model \( \tilde{P}^{(i)}_{m,j} \). Finally, we compare \( T^{(i)}_{m} \) to a pre-calculated threshold \( \gamma_i \), and pick the transition family for which \( T^{(i)}_{m} \) falls below such threshold (with smallest \( m \), see Algorithm 1). After selecting the family, we run \textsc{ucrl-vtr} for this family with confidence level \( \delta_i = \frac{\delta}{2^i} \), where \( \delta \in (0,1] \) is a parameter of the algorithm.

### 2.2. Analysis of \textsc{arl-gen}

First, we present our main result which states that the model selection procedure of \textsc{arl-gen} (Algorithm 1) succeeds with high probability after a certain number of epochs. To this end, we denote by \( M_m := \log(\mathcal{N}(\mathcal{P}_m, 1/T, \| \cdot \|_{\infty,1})) \) the metric entropy (with scale \( 1/T \)) of the family \( \mathcal{P}_m \).

**Lemma 1 (Model selection of \textsc{arl-gen})** Fix a \( \delta \in (0,1] \) and suppose Assumption 1 holds. Suppose the thresholds are set as \( \gamma_i = T^{(M)}_{m} + \sqrt{\frac{\log M_m}{2T/\delta}} \), where \( T^{(M)}_{m} \) is the test statistic for the \( M \)-th model class, and \( i \) is the epoch number. Then, with probability at least \( 1 - 3M\delta \), \textsc{arl-gen} identifies the
Algorithm 1 Adaptive Reinforcement Learning - General - ARL-GEN

1: **Input**: Confidence parameter $\delta$, function classes $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots \subset \mathcal{P}_M$, thresholds $\{\gamma_i\}_{i \geq 1}$
2: for epochs $i = 1, 2, \ldots$ do
3: Set $\tau_{i-1} = \sum_{j=1}^{i-1} k_j$
4: for function classes $m = 1, 2, \ldots, M$ do
5: Compute $\hat{P}^{(i)}_m = \arg\min_{P \in \mathcal{P}_m} \frac{\sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} (V^k_{h+1}(s^k_{h+1}) - (PV^k_{h+1})(s^k_{h}, a^k_{h}))^2}{(\tau_{i-1} - 1)}$
6: Compute $T^{(i)}_m = \frac{\sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} (V^k_{h+1}(s^k_{h+1}) - (\hat{P}^{(i)}_m V^k_{h+1})(s^k_{h}, a^k_{h}))^2}{(\tau_{i-1} - 1)}$
7: end for
8: Set $m^{(i)} = \min \{m \in [M] : T^{(i)}_m \leq \gamma_i\}$, $k_i = 2^i$ and $\delta_i = \delta/2^i$
9: Run UCRL-VTR for the family $\mathcal{P}_{m^{(i)}}$ for $k_i$ episodes with confidence level $\delta_i$
10: end for

correct model class $\mathcal{P}_{m^*}$ from epoch $i \geq i^*$, where epoch length of $i^*$ satisfies

$$2^{i^*} \geq C \max \left\{ \frac{H^3 \log K}{\Delta^2}, \frac{1}{\delta^2}, H \left( M + \log \frac{1}{\delta^2} \right) \right\}$$

for a sufficiently large universal constant $C > 1$.

**Remark 1 (Dependence on the biggest class)** Note that we choose a threshold that depends on the epoch number and the test statistic of the biggest class. Here we crucially exploit the fact that the biggest class always contains the true model class and use this to design the threshold.

**Proof idea.** In order to do model selection, we first obtain upper bounds on the test statistics $T^{(i)}_m$ for model classes that includes $\mathcal{P}^*$. We accomplish this by carefully defining a martingale difference sequence that depends on the value function estimates $V^k_{h+1}$, and invoking Azuma-Hoeffding inequality. We then obtain a lower bound on $T^{(i)}_m$ for model classes not containing $\mathcal{P}^*$ by leveraging Assumption 1 (separability). Combining the above two bounds yields the desired result.

**Regret Bound:** In order to present our regret bound, we define, for each model model class $\mathcal{P}_m$, a collection of functions $\mathcal{F}_m := \{f : S \times A \times \mathcal{V}_m \rightarrow \mathbb{R}\}$ such that any $f \in \mathcal{F}_m$ satisfies $f(s, a, V) = (PV)(s, a)$ for some $P \in \mathcal{P}_m$, where $\mathcal{V}_m := \{V^*_h : P \in \mathcal{P}_m, h \in [H]\}$ denotes the set of optimal value functions under the transition family $\mathcal{P}_m$. By one-to-one correspondence, we have $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_M$, and the complexities of these function classes determine the learning complexity of the RL problem under consideration. We characterize the complexity of each function class $\mathcal{F}_m$ by its eluder dimension, which is defined as follows.

**Definition 1 (Eluder dimension)** The $\varepsilon$-eluder dimension $\dim_\varepsilon(\mathcal{F}_m, \varepsilon)$ of the function class $\mathcal{F}_m$ is the length of the longest sequence $\{(s_i, a_i, V_i)\}_{i=1}^n \subseteq S \times A \times \mathcal{V}_m$ of state-action-optimal value function tuples under the transition family $\mathcal{P}_m$ such that for some $\varepsilon' \geq \varepsilon$ and for each $i \in \{2, \ldots, n\}$, we have $\sup_{f_1, f_2 \in \mathcal{F}_m} (f_1 - f_2)(s_i, a_i, V_i) > \varepsilon'$, given that $\sqrt{\sum_{i=1}^{n-1} (f_1 - f_2)^2(s_i, a_i, V_i) \leq \varepsilon'}$.

The notion of eluder dimension was first introduced by Russo and Van Roy (2013) to characterize the complexity of function classes, and since then it has been widely used (Osband and Van Roy, 2014; Ayoub et al., 2020; Wang et al., 2020). We denote by $d^r_{\varepsilon} = \dim_\varepsilon(\mathcal{F}_{m^*}, 1/T)$ the $(1/T)$-eluder dimension of the function class $\mathcal{F}_{m^*}$ corresponding to the (realizable) family $\mathcal{P}_{m^*}$. Further-
more, we denote by $M^* := M_{m^*}$ the metric entropy of the family $\mathcal{P}_{m^*}$. Then, armed with Lemma 1, we obtain the following regret bound.\(^2\)

**Theorem 1 (Cumulative regret of ARL-GEN)** Suppose the conditions of Lemma 1 hold. Then, for any $\delta \in (0, 1]$, running ARL-GEN for $K$ episodes yields a regret bound

$$R(T) \leq O\left(\max\left\{\frac{H^3 \log K}{\Delta^2}, \log \frac{1}{\delta}\right\} + O\left(H^2 d_{\mathcal{E}} \log K + H \sqrt{T d_{\mathcal{E}} \left(M^* \log \frac{1}{\delta}\right) \log K \log \frac{T}{\delta}}\right)\right)$$

with probability at least $1 - 3M\delta - 2\delta$. \(^3\)

The first term in the regret bound captures the cost of model selection – the cost suffered before accumulating enough samples to infer the correct model class (with high probability). It has weak (logarithmic) dependence on the number of episodes $K$ and hence considered as a minor term, in the setting where $K$ is large. Hence, model selection is essentially free up to log factors. Let us now have a close look at this term. It depends on the metric entropy of the biggest model class $\mathcal{P}_M$. This stems from the fact that the thresholds $\{\gamma_i\}_{i \geq 1}$ depends on the test statistic of $\mathcal{P}_M$ (see Remark 1). We believe that, without additional assumptions, one can’t get rid of this (minor) dependence on the complexity of the biggest class.

The second term is the major one ($\sqrt{T}$ dependence on total number of steps), which essentially is the cost of learning the true kernel $P^*$. Since in this phase, we basically run UCRL-VTR for the correct model class, our regret guarantee matches to that of an oracle with the apriori knowledge of the correct class. Note that if we simply run a non model-adaptive algorithm (e.g. UCRL-VTR) for this problem, the regret would be $\tilde{O}(H \sqrt{T d_{\mathcal{E}, M} \log M})$, where $d_{\mathcal{E}, M}$ denotes the eluder dimension of the largest model class $\mathcal{P}_M$. In contrast, by successively testing and thresholding, our algorithm adapts to the complexity of the smallest function class containing the true model class.

**Remark 2 (Dependence on $\Delta$)** Dependence on the separation $\Delta$ is reflected in the minor term of the regret bound. If the separation is small, it is difficult for ARL-GEN to separate out the model classes. Hence, it requires additional exploration, and as a result the regret increases. It is worth noting that ARL-GEN does not require any knowledge of $\Delta$. Rather, it adapts to the separation present in the problem. Another interesting fact of Theorem 1 is that it does not require any minimum separation across model classes. This is in sharp contrast with existing results in statistics (see, e.g. Balakrishnan et al. (2017); Yi et al. (2016)). Even if $\Delta$ is quite small, Theorem 1 gives a model selection guarantee. Now, the cost of separation appears anyways in the minor term, and hence in the long run, it does not effect the overall performance of the algorithm.

### 3. Linear Kernel MDPs

In this section, we consider a special class of MDPs called *linear kernel MDPs* (Jia et al., 2020). Roughly speaking, it means that the transition kernel $P^*$ can be represented as a linear function of a given feature map $\phi : S \times A \times S \rightarrow \mathbb{R}^d$. Formally, we have the following definition.

2. For ease of representation, we assume the transition kernel $P^*$ to be fixed for all steps $h$. Our results extends naturally to the setting, where there are $H$ different kernels. This would only add a multiplicative $\sqrt{H}$ factor in the regret bound (Jin et al., 2019). Moreover, our results can be extended to the setting where the rewards are also unknown.

3. One can choose $\delta = 1/poly(M)$ to obtain a high-probability bound which only adds an extra log $M$ factor.
The MDP is parameterized by the unknown parameter $\theta^*$, and a natural measure of complexity of the problem is the dimension or sparsity (number of non-zero coordinates) of $\theta^*$. We propose an adaptive algorithm ARL-LIN(dim) that tailors to the sparsity $\|\theta^*\|_0$ of $\theta^*$ (Algorithm 2).

The Base Algorithm: We take the algorithm of Jia et al. (2020) as our base algorithm, which is an adaptation of UCRL-VTR for linear kernel MDPs (henceforth, denoted as UCRL-VTR-LIN). UCRL-VTR-LIN takes the upper bound $b$ of $\|\theta^*\|$ and a confidence level $\delta \in (0, 1]$ as its input. Now, let us have a look at how UCRL-VTR-LIN constructs the confidence ellipsoid at the $k$-th episode. Observe that at step $h$ of episode $k$, we have $\langle P^* \phi_{V_{k+1}^h}(s_h^k, a_h^k), \theta^* \rangle$, where $V_{k+1}^h$ is an estimate of the value function constructed using all the data received before episode $k$. An estimate $\hat{\theta}_k$ of $\theta^*$ is then computed by solving the following optimization problem

$$
\min_{\theta \in \mathbb{R}^d} \sum_{j=1}^k \sum_{h=1}^H \left( V_{k+1}^j (s_h^j, a_h^j) - \langle \phi_{V_{k+1}^j} (s_h^j, a_h^j), \theta \rangle \right)^2 + \|\theta\|^2. \tag{3}
$$

The confidence ellipsoid is then constructed as $B_k = \left\{ \theta \in \mathbb{R}^d : \left\| \Sigma_k^{1/2} (\hat{\theta} - \theta) \right\| \leq \beta_k(\delta) \right\}$, where $\Sigma_k = I + \sum_{j=1}^k \sum_{h=1}^H \phi_{V_{k+1}^j} (s_h^j, a_h^j) \phi_{V_{k+1}^j} (s_h^j, a_h^j)^T$ and $\beta_k(\delta) = O \left( (b^2 + H^2 \log(kH)^2 \log^2(k^2H/\delta)) \right)$. Hence, the $Q$-function estimates (see (1)) of UCRL-VTR-LIN at the $k$-th episode take the form

$$
Q_{k}^h(s, a) = r(s, a) + \left\langle \phi_{V_{k+1}^h}(s, a), \hat{\theta}_{k-1} \right\rangle + \sqrt{\beta_{k-1}(\delta)} \left\| \Sigma_{k-1}^{-1/2} \phi_{V_{k+1}^h}(s, a) \right\|.
$$

Using these, UCRL-VTR-LIN defines the value estimates as $V_{h}^k(s) = \min\{ \max_{a \in A} Q_{h}^k(s, a), H \}$ to keep those bounded. Then, Jia et al. (2020) show that $\theta^*$ lies in the confidence ellipsoid $B_k$ in all episodes $k$ with probability at least $1 - \delta$.

Our approach: The proposed algorithm works over multiple epochs, and we use diminishing thresholds to estimate the support of $\theta^*$. The algorithm is parameterized by the initial phase length $k_0$, and the confidence level $\delta \in (0, 1]$. ARL-LIN(dim) proceeds in epochs numbered $0, 1, \ldots$, increasing with time. Each epoch $i$ is divided into two phases - (i) a regret minimization phase lasting $36^i k_0$ episodes, (ii) followed by a support estimation phase lasting $6^i \sqrt{k_0}$ episodes. Thus,
each epoch $i$ lasts for a total of $36^i k_0 + 6^i \lceil \sqrt{k_0} \rceil$ episodes. At the beginning of epoch $i \geq 0$, $D^{(i)} \subseteq [d]$ denotes the set of ‘active coordinates’ – the estimate of non-zero coordinates of $\theta^*$.

In the regret minimization phase of epoch $i$, a fresh instance of UCRL-VTR-LIN is spawned, with the dimensions restricted only to the set $D^{(i)}$ and confidence level $\delta_i := \frac{\delta}{2^i}$. On the other hand, in the support estimation phase, we continue running the UCRL-VTR-LIN algorithm in full $d$ dimension, from the point where we left off in epoch $i - 1$. Concretely, one should think of the support estimation phases over epochs as a single run of UCRL-VTR-LIN in the full $d$ dimension with confidence level $\delta$ and norm upper bound $b$. At the end of each epoch $i \geq 0$, let $\tau_i := \sum_{j=0}^{i} 6^j \lceil \sqrt{k_0} \rceil$ denote the total number of episodes run in the support estimation phases. Then, ARL-LIN(dim) forms an estimate of $\theta^*$ as $\hat{\theta}^{(i+1)} := \hat{\theta}_{\tau_i}$, where, for any $k \geq 1$, $\hat{\theta}_k$ is as defined in (3). The active coordinate set $D^{(i+1)}$ for the next epoch is then the coordinates of $\hat{\theta}^{(i+1)}$ with magnitude exceeding $(0.5)^{i+1}$. By this careful choice of exploration periods and thresholds, we show that the estimated support of $\theta^*$ is equal to the true support, for all but finitely many epochs. Thus, after a finite number of epochs, the true support of $\theta^*$ locks-in, and thereafter the agent incurs the regret that an oracle knowing the true support would incur. Hence, the extra regret we incur with respect to an the oracle (which knows the support of $\theta^*$) is small.

Now, we informally state the regret bound of ARL-LIN(dim). The formal statement is deferred to Appendix B.1, Theorem 2. We let $d^* = ||\theta^*||_0$ to denote the sparsity of $\theta^*$.

Proposition 1 (Informal regret bound of ARL-LIN(dim)) Under a technical assumption on the feature mapping $\phi$, if ARL-LIN(dim) is run for $K$ episodes with suitably chosen initial phase length $k_0$, its regret satisfies (with high probability) the following:

$$R(T) = \mathcal{O}\left( \frac{H k_0}{\gamma^{0.18}} + \left(b d^* \sqrt{H^3 T} + b d H^2 K^{1/4}\right) \text{polylog } T \right),$$

where $\gamma = \min \{|\theta^*(j)|: \theta^*(j) \neq 0\}$ with $\theta^*(j)$ denoting the $j$-th coordinate of $\theta^*$.

Note that the cost of model selection is $\mathcal{O}\left( \frac{H k_0}{\gamma^{0.18}} + b d H^2 K^{1/4}\right)$, which has a weaker dependence on the number of episodes $K$ than the leading term, and hence has a minor influence on the performance of the algorithm in the long run.

Remark 3 (Dependence on $\gamma$) The regret bound depends on $\gamma$ – the minimum absolute non-zero entry of $\theta^*$, and hence is instance-dependent. At this point, it is worth mentioning that we haven’t optimized over the choice of epoch lengths and dependence on $\gamma$. In particular, choosing the support estimation period as $2^i \lceil \sqrt{k_0} \rceil$, the regret minimization period as $4^i k_0$ and the threshold as $(0.9)^i$, one can ensure the support locks-in after only 2 epochs. However, the dependence on $\gamma$ in this case becomes worse. Therefore, the support estimation period, regret minimization period and threshold selection can be kept as tuning parameters.

Norm as complexity measure: In Appendix B.2, we take the norm, $||\theta^*||$, of the unknown parameter $\theta^*$ as a measure of complexity of linear kernel MDPs. We develop an algorithm ARL-LIN(norm) which adapts to $||\theta^*||$ (see Algorithm 3). This is in sharp contrast to non-adaptive algorithms (e.g. UCRL-VTR-LIN of Jia et al. (2020)) that requires an upper bound on the norm of $\theta^*$. In this setting also, we show that model selection is (order-wise) free. Please refer to Theorem 3 for the exact regret bound of ARL-LIN(norm).
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Appendix

Appendix A. DETAILS FOR SECTION 2

A.1. Confidence Sets in UCRL-VTR

We first describe how the confidence sets are constructed in UCRL-VTR. Note that the procedure is similar to that done in Ayoub et al. (2020), but with a slight difference. Specifically, we define the confidence width as a function of complexity of the transition family \( \mathcal{P} \) on which \( P^* \) lies; rather than the complexity of a value-dependent function class induced by \( \mathcal{P} \) (as done in Ayoub et al. (2020)). We emphasize that this small change makes the model selection procedure easier to understand without any effect on the regret.

Let us define, for any two transition kernels \( P, P' \in \mathcal{P} \) and any episode \( k \geq 1 \), the following

\[
L_k(P) = \sum_{j=1}^{k} \sum_{h=1}^{H} \left( V^j_{h+1}(s^j_{h+1}) - (PV^j_{h+1})(s^j_{h+1}, a^j_h) \right)^2 ,
\]

\[
L_k(P, P') = \sum_{j=1}^{k} \sum_{h=1}^{H} \left( (PV^j_{h+1})(s^j_{h+1}, a^j_h) - (P'V^j_{h+1})(s^j_{h+1}, a^j_h) \right)^2 .
\]

Then, the confidence set at the end of episode \( k \) is constructed as

\[
B_k = \left\{ P \in \mathcal{P} \mid L_k(P, \hat{P}_k) \leq \beta_k(\delta) \right\} ,
\]

where \( \hat{P}_k = \arg\min_{P \in \mathcal{P}} L_k(P) \) denotes an estimate of \( P^* \) after \( k \) episodes. The confidence width \( \beta_k(\delta) \equiv \beta_k(\mathcal{P}, \delta) \) is set as

\[
\beta_k(\delta) := \begin{cases} 
8H^2 \log \left( \frac{2|\mathcal{P}|}{\delta} \right) & \text{if } \mathcal{P} \text{ is finite}, \\
8H^2 \log \left( \frac{2N(h, k)}{\delta} \right) + 4H^2 \left( 2 + 2 \log \left( \frac{4kH(kH+1)}{\delta} \right) \right) & \text{if } \mathcal{P} \text{ is infinite}.
\end{cases}
\]

Lemma 2 (Concentration of \( P^* \)) For any \( \delta \in (0, 1] \), with probability at least \( 1 - \delta \), uniformly over all episodes \( k \geq 1 \), we have \( P^* \in B_k \).

Proof First, we define, for any fixed \( P \in \mathcal{P} \) and \( (h, k) \in [H] \times [K] \), the quantity

\[
Z^k_h := 2 \left( (P^*V^k_{h+1})(s^k_h, a^k_h) - (PV^k_{h+1})(s^k_h, a^k_h) \right) \left( V^k_{h+1}(s^k_{h+1}) - (P^*V^k_{h+1})(s^k_h, a^k_h) \right) .
\]

Then, we have

\[
L_k(\hat{P}_k) = L_k(P^*) + L_k(P^*, \hat{P}_k) + \sum_{j=1}^{k} \sum_{h=1}^{H} Z^j_h \hat{v}_h .
\]

Using the notation \( y^k_h = V^k_{h+1}(s^k_{h+1}) \), we can rewrite \( Z^k_h \) as

\[
Z^k_h := 2 \left( (P^*V^k_{h+1})(s^k_h, a^k_h) - (PV^k_{h+1})(s^k_h, a^k_h) \right) \left( y^k_h - \mathbb{E}[y^k_h|G^k_{h-1}] \right) ,
\]

where \( G^k_{h-1} \) denotes the \( \sigma \)-field summarising all the information available just before \( s^k_{h+1} \) is observed. Note that \( Z^k_h \) is \( G^k_h \)-measurable. Moreover, since \( V^k_{h+1} \in [0, H] \), \( Z^k_h \) is \( 2H\left| (P^*V^k_{h+1})(s^k_h, a^k_h) - (PV^k_{h+1})(s^k_h, a^k_h) \right| \)-sub-Gaussian conditioned on \( G^k_{h-1} \). Therefore, for any \( \lambda < 0 \), with probability at
least 1 − δ, we have
\[ \forall k \geq 1, \sum_{j=1}^{k} \sum_{h=1}^{H} Z_{h}^{j,P} \geq \frac{1}{\lambda} \log(1/\delta) + \frac{\lambda}{2} \cdot 4H^2 \sum_{j=1}^{k} \sum_{h=1}^{H} \left( (P^* V_{h}^{j}) (s_h^j, a_h^j) - (PV_{h+1}^{j}) (s_h^j, a_h^j) \right)^2 . \]

Setting \( \lambda = -1/(4H^2) \), we obtain for any fixed \( P \in \mathcal{P} \), the following:
\[ \forall k \geq 1, \sum_{j=1}^{k} \sum_{h=1}^{H} Z_{h}^{j,P} \geq -4H^2 \log \left( \frac{1}{\delta} \right) - \frac{1}{2} \mathcal{L}_k(P^*, P) . \]

with probability at least 1 − δ. We consider both the cases – when \( \mathcal{P} \) is finite and when \( \mathcal{P} \) is infinite.

**Case 1 – finite \( \mathcal{P} \):** We take a union bound over all \( P \in \mathcal{P} \) in (5) to obtain that
\[ \forall k \geq 1, \forall P \in \mathcal{P}, \sum_{j=1}^{k} \sum_{h=1}^{H} Z_{h}^{j,P} \geq -4H^2 \log \left( \frac{|\mathcal{P}|}{\delta} \right) - \frac{1}{2} \mathcal{L}_k(P^*, P) . \]

with probability at least 1 − δ. By construction, \( \hat{\mathcal{P}}_k \in \mathcal{P} \) and \( \mathcal{L}_k(\hat{\mathcal{P}}_k) \leq \mathcal{L}_k(P^*) \). Therefore, from (4), we have
\[ \forall k \geq 1, \mathcal{L}_k(P^*, \hat{\mathcal{P}}_k) \leq 8H^2 \log \left( \frac{|\mathcal{P}|}{\delta} \right) \]

with probability at least 1 − δ, which proves the result for finite \( \mathcal{P} \).

**Case 2 – infinite \( \mathcal{P} \):** Fix some \( \alpha > 0 \). Let \( \mathcal{P}^{\alpha} \) denotes an \((\alpha, \|\cdot\|_\infty, 1)\) cover of \( \mathcal{P} \), i.e., for any \( P \in \mathcal{P} \), there exists an \( P^{\alpha} \) in \( \mathcal{P}^{\alpha} \) such that \( \|P^{\alpha} - P\|_\infty := \sup_{s,a} \int_S |P^{\alpha}(s' | s, a) - P(s' | s, a)| ds' \leq \alpha \).

Now, we take a union bound over all \( P^{\alpha} \in \mathcal{P}^{\alpha} \) in (5) to obtain that
\[ \forall k \geq 1, \forall P^{\alpha} \in \mathcal{P}^{\alpha}, \sum_{j=1}^{k} \sum_{h=1}^{H} Z_{h}^{j,P^{\alpha}} \geq -4H^2 \log \left( \frac{|P^{\alpha}|}{\delta} \right) - \frac{1}{2} \mathcal{L}_k(P^*, P^{\alpha}) . \]

with probability at least 1 − δ, and thus, in turn,
\[ \forall k \geq 1, \forall P \in \mathcal{P}, \sum_{j=1}^{k} \sum_{h=1}^{H} Z_{h}^{j,P} \geq -4H^2 \log \left( \frac{|P^{\alpha}|}{\delta} \right) - \frac{1}{2} \mathcal{L}_k(P^*, P) + \zeta_k^{\alpha}(P) . \]

with probability at least 1 − δ, where \( \zeta_k^{\alpha}(P) \) denotes the discretization error:
\[ \zeta_k^{\alpha}(P) = \sum_{j=1}^{k} \sum_{h=1}^{H} \left( Z_{h}^{j,P} - Z_{h}^{j,P^{\alpha}} \right) + \frac{1}{2} \mathcal{L}_k(P^*, P) - \frac{1}{2} \mathcal{L}_k(P^*, P^{\alpha}) \]
\[ = \sum_{j=1}^{k} \sum_{h=1}^{H} \left( 2g_h^j \left( (P^{\alpha} V_{h+1}^{j})(s_h^j, a_h^j) - (PV_{h+1}^{j})(s_h^j, a_h^j) \right) + \frac{1}{2} (PV_{h+1}^{j})^2 (s_h^j, a_h^j) - \frac{1}{2} (P^{\alpha} V_{h+1}^{j})^2 (s_h^j, a_h^j) \right) . \]

Since \( \|P - P^{\alpha}\|_\infty, 1 \leq \alpha \) and \( \|V_{h+1}^k\|_\infty \leq H \), we have
\[ \left( (P^{\alpha} V_{h+1}^{k})(s, a) - (PV_{h+1}^{k})(s, a) \right) \leq aH , \]
bounded for all episodes \( k \) where the last step holds for any \( \alpha \) with probability at least 1. Therefore, we can upper bound the discretization error as

\[
|\zeta_k^\alpha(P)| \leq 2\alpha H \sum_{j=1}^k \sum_{h=1}^H |y_h^j| + \sum_{j=1}^k \sum_{h=1}^H \left(\alpha H^2 + \frac{\alpha^2 H^2}{2}\right)
\]

\[
\leq 2\alpha H \sum_{j=1}^k \sum_{h=1}^H |y_h^j| + \mathbb{E}[y_h^j|G_{h-1}]| + \sum_{j=1}^k \sum_{h=1}^H \left(3\alpha H^2 + \frac{\alpha^2 H^2}{2}\right).
\]

Since \( y_h^k - \mathbb{E}[y_h^k|G_{h-1}] \) is \( H \)-sub-Gaussian conditioned on \( G_{h-1} \), we have

\[
\forall k \geq 1, \forall h \in [H], \quad |y_h^k - \mathbb{E}[y_h^k|G_{h-1}]| \leq H \sqrt{2 \log \frac{2kH(k+1)}{\delta}}
\]

with probability at least 1 - \( \delta \). Therefore, with probability at least 1 - \( \delta \), the discretization error is bounded for all episodes \( k \geq 1 \) as

\[
|\zeta_k^\alpha(P)| \leq kH \left(2\alpha H^2 \sqrt{2 \log \frac{2kH(k+1)}{\delta}} + 3\alpha H^2 + \frac{\alpha^2 H^2}{2}\right)
\]

\[
\leq \alpha kH \left(2H^2 \sqrt{2 \log \frac{2kH(k+1)}{\delta}} + 4H^2\right),
\]

where the last step holds for any \( \alpha \leq 1 \). Therefore, from (7), we have

\[
\forall k \geq 1, \forall P \in \mathcal{P}, \sum_{j=1}^k \sum_{h=1}^H Z_h^j P \geq -4H^2 \log \left(\frac{|P^\alpha|}{\delta}\right) - \frac{1}{2} \mathcal{L}_k(P^*, P)
\]

\[
- \alpha kH \left(2H^2 \sqrt{2 \log \frac{2kH(k+1)}{\delta}} + 4H^2\right), \quad (8)
\]

with probability at least 1 - 2\( \delta \). Now, setting \( \alpha = \frac{1}{k\mathcal{P}} \), we obtain, from (4), that

\[
\forall k \geq 1, \mathcal{L}_k(P^*, \hat{P}_k) \leq 8H^2 \log \left(\frac{2\mathcal{N}(\mathcal{P}, \frac{1}{k\mathcal{P}}, ||\cdot||_{\infty,1})}{\delta}\right) + 4H^2 \left(2 + \sqrt{2 \log \frac{4kH(k+1)}{\delta}}\right)
\]

with probability at least 1 - \( \delta \), which proves the result for infinite \( \mathcal{P} \).

\[\Box\]

**A.2. Model Selection in ARL-GEN**

First, we find concentration bounds on the test statistics \( T_m^{(i)} \) for all epochs \( i \geq 1 \) and class indexes \( m \in [M] \), which are crucial to prove the model selection guarantee (Lemma 1) of ARL-GEN.
1. Realizable model classes: Fix a class index \( m \geq m^* \). In this case, the true model \( P^* \in \mathcal{P}_m \). Therefore, we can upper bound the empirical at epoch \( i \) as

\[
T_m^{(i)} \leq \frac{1}{\tau_{i-1} H} \sum_{k=1}^{H} \sum_{h=1}^{\tau_{i-1}} \left( V_{h+1}^k(s_{h+1}^k) - (P_m^* V_{h+1}^k(s_h^k, a_h^k)) \right)^2 = \frac{1}{\tau_{i-1} H} \sum_{k=1}^{H} \sum_{h=1}^{\tau_{i-1}} \left( y_h^k - \mathbb{E}[y_h^k|G_{h-1}^k] \right)^2.
\]

Now, we define the random variable \( m_h^k := (y_h^k - \mathbb{E}[y_h^k|G_{h-1}^k])^2 \). We use the notation \( \mathbb{E}[m_h^k|G_{h-1}^k] = \text{var}[y_h^k|G_{h-1}^k] = \sigma^2 \). Moreover, note that \( (m_h^k - \mathbb{E}[m_h^k|G_{h-1}^k])_{k, h} \) is a martingale difference sequence adapted to the filtration \( G_{h}^k \), with absolute values \( |m_h^k - \mathbb{E}[m_h^k|G_{h-1}^k]| \leq H^2 \) for all \( k, h \). Therefore, by the Azuma-Hoeffding inequality, with probability at least \( 1 - \delta/2^i \),

\[
\sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} m_h^k \leq \sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} \mathbb{E}[m_h^k|G_{h-1}^k] + H^2 \sqrt{2\tau_{i-1} H \log(2^i/\delta)}.
\]

Now, using a union bound, along with the definition of \( T_m^{(i)} \), with probability at least \( 1 - \delta \), for any class index \( m \geq m^* \), we have

\[
\forall i \geq 1, \quad T_m^{(i)} \leq \sigma^2 + H^{3/2} \sqrt{\frac{2 \log(2^i/\delta)}{\tau_{i-1}}}
\]

(9)

Let us now look at the definition of \( T_m^{(i)} \). We have

\[
T_m^{(i)} = \frac{1}{\tau_{i-1} H} \sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} \left( V_{h+1}^k(s_{h+1}^k) - (P_m^* V_{h+1}^k(s_h^k, a_h^k)) \right)^2.
\]

We write

\[
\frac{1}{\tau_{i-1} H} \sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} \left( V_{h+1}^k(s_{h+1}^k) - (P_m^* V_{h+1}^k(s_h^k, a_h^k)) \right)^2 - \frac{1}{\tau_{i-1} H} \sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} \left( V_{h+1}^k(s_{h+1}^k) - (\hat{P}_m V_{h+1}^k(s_h^k, a_h^k)) \right)^2
\]

\[
= \frac{1}{\tau_{i-1} H} \sum_{k=1}^{\tau_{i-1}} \sum_{h=1}^{H} \left[ (\hat{P}_m V_{h+1}^k(s_h^k, a_h^k) - (P_m^* V_{h+1}^k(s_h^k, a_h^k))(2V_{h+1}^k(s_{h+1}^k) - (P_m^* V_{h+1}^k(s_h^k, a_h^k)) - (\hat{P}_m V_{h+1}^k(s_h^k, a_h^k)) \right]
\]

Taking expectations (w.r.t the true model \( P^* \)), we obtain

\[
\mathbb{E}T_m^{(i)} \geq \sigma^2,
\]

and hence, using similar martingale difference construction, we obtain

\[
T_m^{(i)} \geq \sigma^2 - H^{3/2} \sqrt{\frac{2 \log(2^i/\delta)}{\tau_{i-1}}}
\]

with probability at least \( 1 - \delta \).
2. Non-realizable model classes: Fix a class index $m < m^*$. In this case, the true model $P^* \not\in \mathcal{P}_m$. We can decompose the empirical risk at epoch $i$ as $T_m^{(i)} = T_{m,1}^{(i)} + T_{m,2}^{(i)} + T_{m,3}^{(i)}$, where

$$T_{m,1}^{(i)} = \frac{1}{\tau_i - 1} \sum_{k=1}^{\tau_i - 1} \sum_{h=1}^{H} \left( V_{h+1}(s_{h+1}^k) - (P^* V_{h+1})(s_{h}^k, a_{h}^k) \right)^2,$$

$$T_{m,2}^{(i)} = \frac{1}{\tau_i - 1} \sum_{k=1}^{\tau_i - 1} \sum_{h=1}^{H} \left( (P^* V_{h+1})(s_{h}^k, a_{h}^k) - (\hat{P}_m^{(i)} V_{h+1})(s_{h}^k, a_{h}^k) \right)^2,$$

$$T_{m,3}^{(i)} = \frac{1}{\tau_i - 1} \sum_{k=1}^{\tau_i - 1} \sum_{h=1}^{H} 2 \left( (P^* V_{h+1})(s_{h}^k, a_{h}^k) - (\hat{P}_m^{(i)} V_{h+1})(s_{h}^k, a_{h}^k) \right) \left( V_{h+1}(s_{h+1}^k) - (P^* V_{h+1})(s_{h}^k, a_{h}^k) \right).$$

First, using a similar argument as in (9), with probability at least $1 - \delta$, we obtain

$$\forall i \geq 1, \quad T_{m,1}^{(i)} \geq \sigma^2 - H^{3/2} \sqrt{\frac{2 \log(2^i/\delta)}{\tau_i - 1}}.$$

Next, by Assumption 1, we have

$$\forall i \geq 1, \quad T_{m,2}^{(i)} \geq \Delta.$$ 

Now, we turn to bound the term $T_{m,3}^{(i)}$. We consider both the cases – when $\mathcal{P}$ is finite and when $\mathcal{P}$ is infinite.

Case 1 – finite model classes: Note that $\hat{P}_m^{(i)} \in \mathcal{P}_m$. Then, from (6), we have

$$\forall i \geq 1, \quad T_{m,3}^{(i)} \geq - \frac{4H}{\tau_i - 1} \log \left( \frac{\left| \mathcal{P}_m \right|}{\delta} \right) - \frac{1}{2} T_{m,2}^{(i)}$$

with probability at least $1 - \delta$. Now, combining all the three terms together and using a union bound, we obtain the following for any class index $m \leq m^* - 1$:

$$\forall i \geq 1, \quad T_m^{(i)} \geq \sigma^2 + \frac{1}{2} \Delta - H^{3/2} \sqrt{\frac{2 \log(2^i/\delta)}{\tau_i - 1}} - \frac{4H}{\tau_i - 1} \log \left( \frac{\left| \mathcal{P}_m \right|}{\delta} \right).$$

with probability at least $1 - 2\delta$.

Case 2 – infinite model classes: We follow a similar procedure, using (8), to obtain the following for any class index $m \leq m^* - 1$:

$$\forall i \geq 1, \quad T_m^{(i)} \geq \sigma^2 + \frac{1}{2} \Delta - H^{3/2} \sqrt{\frac{2 \log(2^i/\delta)}{\tau_i - 1}} - \frac{4H}{\tau_i - 1} \log \left( \frac{\mathcal{N}(\mathcal{P}_m, \alpha, \|\cdot\|_{\infty,1})}{\delta} \right)$$

$$- \alpha \left( 2H^2 \sqrt{2 \log \left( \frac{2 \tau_i - 1 H (\tau_i - 1 H + 1)}{\delta} \right) + 4H^2} \right)$$

with probability at least $1 - 3\delta$.

A.2.1. Proof of Lemma 1

We are now ready to prove Lemma 1, which presents the model selection Guarantee of ARL-GEN for infinite model classes $\{\mathcal{P}_m\}_{m \in [M]}$. Here, at the same time, we prove a similar (and simpler) result for finite model classes also.
First, note that we consider doubling epochs $k_i = 2^i$, which implies $\tau_{i+1} = \sum_{j=1}^{i+1} k_j = \sum_{j=1}^{i} 2^j = 2^{i+1} - 1$. With this, the number of epochs is given by $N = \lceil \log_2(K+1) - 1 \rceil = O(\log K)$.

Let us now consider finite model classes $\{P_m\}_{m \in [M]}$.

**Case 1 – finite model classes:** First, we combine (9) and (10), and take a union bound over all $m \in [M]$ to obtain

$$\forall m \geq m^*, \forall i \geq 1, \sigma^2 - H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}} \leq T_m^{(i)} \leq \sigma^2 + H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}}$$

$$\forall m \leq m^* - 1, \forall i \geq 1, \quad T_m^{(i)} \geq \sigma^2 + \frac{1}{2} \Delta - H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}} - \frac{4H}{\tau_{i-1}} \log \left( \frac{|P_m|}{\delta} \right)$$

with probability at least $1 - 2M \delta$, where we have used that $\log(2^i/\delta) \leq N \log(2/\delta)$ for all $i$ and $|P_m| \leq |P_M|$ for all $m$. Now, suppose for some epoch $i^*$, satisfies

$$2^{i^*} \geq C \max \left\{ \frac{2H^3 \log K}{\Delta^2} \log(2/\delta), 4H \log \left( \frac{|P_M|}{\delta} \right) \right\}.$$

where $C$ is a sufficiently large universal constant. Then, we have

$$\forall m \geq m^*, \forall i \geq i^*, \quad \sigma^2 - \frac{c_0}{2\sigma^2} \leq T_m^{(i)} \leq \sigma^2 + \frac{c_0}{2\sigma^2}$$

$$\forall m \leq m^* - 1, \forall i \geq i^*, \quad T_m^{(i)} \geq \sigma^2 + \frac{1}{2} \Delta - \frac{c_1}{2}\sigma^2$$

with probability at least $1 - 3M \delta$. Note that with the chosen threshold $\gamma_i$, for all $i \geq i^*$

$$\sigma^2 - \frac{c_0}{2\sigma^2} + \frac{\sqrt{1}}{2\sigma^2} \leq \gamma_i \leq \sigma^2 + \frac{c_0}{2\sigma^2} + \frac{\sqrt{1}}{2\sigma^2}$$

$$\leq \sigma^2 + \frac{1}{2} \Delta - \frac{c_1}{2}\sigma^2,$$

where the last inequality comes from the choice of $2^{i^*}$. With this, we have

$$\forall m \geq m^*, \forall i \geq i^*, \quad T_m^{(i)} \leq \gamma_i$$

$$\forall m \leq m^* - 1, \forall i \geq i^*, \quad T_m^{(i)} \geq \gamma_i$$

with probability at least $1 - 3M \delta$. The above equation implies that $m^{(i)} = m^*$ for all epochs $i \geq i^*$.

Now, we focus on infinite model classes $\{P_m\}_{m \in [M]}$, for which Lemma 1 is stated.

**Case 2 – infinite model classes:** First, we combine (9) and (11), and take a union bound over all $m \in [M]$ to obtain

$$\forall m \geq m^*, \forall i \geq 1, \sigma^2 - H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}} \leq T_m^{(i)} \leq \sigma^2 + H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}}$$

$$\forall m \leq m^* - 1, \forall i \geq 1, \quad T_m^{(i)} \geq \sigma^2 + \frac{1}{2} \Delta - H^{3/2} \sqrt{\frac{2N \log(2/\delta)}{\tau_{i-1}}} - \frac{4H}{\tau_{i-1}} \log \left( \frac{N(P_M, \alpha, \| \cdot \|_{\infty, 1})}{\delta} \right)$$

$$- \alpha \left( 2H^2 \sqrt{2 \log \left( \frac{2KH(KH + 1)}{\delta} \right)} + 4H^2 \right)$$

$\alpha$
with probability at least $1 - 3M\delta$. Suppose for some epoch $i^*$, we have,

$$2^{i^*} \geq C \max \left\{ \frac{H^3 \log K}{\Delta^2} \log(2/\delta), 4H \log \left( \frac{N(P_M, \alpha, \|\cdot\|_{\infty, 1})}{\delta} \right) \right\}$$

where $C > 1$ is a sufficiently large universal constant. Then, with the choice of threshold $\gamma_i$, and doing the same calculation as above, we obtain

$$\forall m \geq m^*, \forall i \geq i^*, \quad T_m^{(i)} \leq \gamma_i \quad \text{and} \quad \forall m \leq m^* - 1, \forall i \geq i^*, \quad T_m^{(i)} \geq \gamma_i$$

with probability at least $1 - 3M\delta$. The above equation implies that $m^{(i)} = m^*$ for all epochs $i \geq i^*$, proving the result.

### A.3. Regret Bound of ARL–GEN (Proof of Theorem 1)

Lemma 1 implies that as soon as we reach $i^*$, ARL–GEN identifies the model class with high probability, i.e., for each $i \geq i^*$, we have $m^{(i)} = m^*$. However, before that, we do not have any guarantee on the regret performance of ARL–GEN. Since at every episode the regret can be at most $H$, the cumulative regret up until the $i^*$ epoch is upper bounded by $\tau_{i^* - 1}H$, which is at most

$$O \left( \max \left\{ H^4 \log(K) \log(1/\delta), H^2 \log \left( \frac{N(P_M, \alpha, \|\cdot\|_{\infty, 1})}{\delta} \right) \right\} \right),$$

if the model classes are infinite, and

$$O \left( \max \left\{ H^4 \log(K) \log(1/\delta), H^2 \log \left( \frac{P_M}{\delta} \right) \right\} \right),$$

if the model classes are finite. Note that this is the cost we pay for model selection.

Now, let us bound the regret of ARL–GEN from epoch $i^*$ onward. Let $R_{\text{UCRL-VTR}}^{(i)}(k_i, \delta_i, P_m^{(i)})$ denote the cumulative regret of UCRL–VTR, when it is run for $k_i$ episodes with confidence level $\delta_i$ for the family $P_m^{(i)}$. Now, using the result of Ayoub et al. (2020), we have

$$R_{\text{UCRL-VTR}}^{(i)}(k_i, \delta_i, P_m^{(i)}) \leq 1 + H^2 \dim_{\mathcal{E}} \left( \mathcal{F}_{m^{(i)}}, \frac{1}{k_iH} \right) + 4 \sqrt{\beta_{k_i}(P_m^{(i)}, \delta_i) \dim_{\mathcal{E}} \left( \mathcal{F}_{m^{(i)}}, \frac{1}{k_iH} \right)} k_iH$$

$$+ H \sqrt{2k_iH \log(1/\delta_i)}$$

with probability at least $1 - 2\delta_i$. With this and Lemma 1, the regret of ARL–GEN after $K$ episodes (i.e., after $T = KH$ timesteps) is given by

$$R(T) \leq \tau_{i^* - 1}H + \sum_{i = i^*}^{N} R_{\text{UCRL-VTR}}^{(i)}(k_i, \delta_i, P_m^{(i)})$$

$$\leq \tau_{i^* - 1}H + N + \sum_{i = i^*}^{N} H^2 \dim_{\mathcal{E}} \left( \mathcal{F}_{m^*}, \frac{1}{k_iH} \right) + 4 \sum_{i = i^*}^{N} \sqrt{\beta_{k_i}(P^{m^*}, \delta_i) \dim_{\mathcal{E}} \left( \mathcal{F}_{m^*}, \frac{1}{k_iH} \right)} k_iH$$

$$+ \sum_{i = i^*}^{N} H \sqrt{2k_iH \log(1/\delta_i)}.$$
the last term in the above expression. Substituting $\delta_i = \delta/2^i$, we have
\[
\sum_{i=i^*}^{N} H \sqrt{2k_i H \log(1/\delta_i)} = \sum_{i=i^*}^{N} H \sqrt{2k_i H \log(2/\delta)}
\]
\[
\leq H \sqrt{2HN \log(2/\delta)} \sum_{i=1}^{N} \sqrt{k_i}
\]
\[
= \mathcal{O} \left( H \sqrt{KH N \log(1/\delta)} \right) = \mathcal{O} \left( H \sqrt{T \log K \log(1/\delta)} \right),
\]
where we have used that the total number of epochs $N = \mathcal{O}(\log K)$, and that
\[
\sum_{i=1}^{N} \sqrt{k_i} = \sqrt{k_N} \left( 1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \ldots + \frac{N}{N} \right) \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{k_N} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{K}.
\]
Next, we can upper bound the third to last term in the regret expression as
\[
\sum_{i=i^*}^{N} H^2 \dim_{\mathcal{F}} \left( \mathcal{F}_{m^*}, \frac{1}{k_i H} \right) \leq H^2 N \log \left( \mathcal{F}_{m^*}, \frac{1}{KH} \right) = \mathcal{O} \left( H^2 d^*_E \log K \right).
\]
Now, notice that, by substituting $\delta_i = \delta/2^i$, we can upper bound $\beta_{k_i}(\mathcal{P}_{m^*}, \delta_i)$ as follows:
\[
\beta_{k_i}(\mathcal{P}_{m^*}, \delta_i) = \mathcal{O} \left( H^2 \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{k_i H}, \| \cdot \|_{\infty,1})}{\delta} \right) + H^2 \left( 1 + \frac{k_i H (k_i H + 1)}{\delta} \right) \right)
\]
\[
\leq \mathcal{O} \left( H^2 N \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{KH}, \| \cdot \|_{\infty,1})}{\delta} \right) + H^2 \left( 1 + \sqrt{\log(1/\delta)} \right) \right)
\]
\[
\leq \mathcal{O} \left( H^2 \log K \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{KH}, \| \cdot \|_{\infty,1})}{\delta} \right) + H^2 \sqrt{\log K \log(KH/\delta)} \right)
\]
for infinite model classes, and
\[
\beta_{k_i}(\mathcal{P}_{m^*}, \delta_i) = \mathcal{O} \left( H^2 \log K \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{KH}, \| \cdot \|_{\infty,1})}{\delta} \right) \right)
\]
for finite model classes. With this, the second to last term in the regret expression can be upper bounded as
\[
\sum_{i=i^*}^{N} 4 \sqrt{\beta_{k_i}(\mathcal{P}_{m^*}, \delta_i) \dim_{\mathcal{F}} \left( \mathcal{F}_{d_i}, \frac{1}{k_i H} \right)} k_i H
\]
\[
\leq \mathcal{O} \left( H \sqrt{\log K \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{KH}, \| \cdot \|_{\infty,1})}{\delta} \right) + \sqrt{\log K \log(KH/\delta)} \sqrt{\dim_{\mathcal{F}} \left( \mathcal{F}_{m^*}, \frac{1}{KH} \right)} \sum_{i=i^*}^{N} \sqrt{k_i H} \right)
\]
\[
\leq \mathcal{O} \left( H \sqrt{\log K \log \left( \frac{\mathcal{N}(\mathcal{P}_{m^*}, \frac{1}{KH}, \| \cdot \|_{\infty,1})}{\delta} \right) \log \left( \frac{KH}{\delta} \right)} \sqrt{KH \dim_{\mathcal{F}} \left( \mathcal{F}_{m^*}, \frac{1}{KH} \right)} \right)
\]
\[
= \mathcal{O} \left( H \sqrt{Td^*_E (M^* + \log(1/\delta)) \log K \log(T/\delta)} \right)
\]
for infinite model classes. Similarly, for finite model classes, we can upper bound this term by
\[
\mathcal{O} \left( H \sqrt{Td^*_E \log \left( \frac{|\mathcal{P}_{m^*}|}{\delta} \right) \log K \right).
\]
Hence, for infinite model classes, the final regret bound can be written as
\[
R(T) = \mathcal{O} \left( \max \left\{ H^4 \log(K) \log(1/\delta), H^2 \log \left( \frac{2 \mathbb{N}(\mathcal{P}_M, 1/T, \|\cdot\|_{\infty,1})}{\delta} \right) \right\} \right) \\
+ \mathcal{O} \left( H^2 d^*_C \log K + H \sqrt{T d^*_C (\mathbb{M}^* + \log(1/\delta)) \log K \log(T/\delta)} \right).
\]
The above regret bound holds with probability greater than
\[
1 - 3M\delta - \sum_{i=1}^{N} \frac{2\delta}{2^i - 1} \geq 1 - 3M\delta - \sum_{i \geq 1} \frac{2\delta}{2^i - 1} = 1 - 3M\delta - 2\delta,
\]
which completes the proof of Theorem 1.

Similarly, for finite model classes, the final regret bound can be written as
\[
R(T) = \mathcal{O} \left( \max \left\{ H^4 \log(K) \log(1/\delta), H^2 \log \left( \frac{\mathbb{P}_M}{\delta} \right) \right\} \right) \\
+ \mathcal{O} \left( H^2 d^*_C \log K + H \sqrt{T d^*_C \log \left( \frac{\mathbb{P}_{m^*}}{\delta} \right) \log K} \right),
\]
which holds with probability greater than \(1 - 2M\delta - 2\delta\).

**Appendix B. DETAILS FOR SECTION 3**

In this setting, we do not need any separability condition like Assumption 1. Instead, we make the following assumptions on the feature map \(\phi\). To this end, for any function \(V : S \to \mathbb{R}\) and state-action pair \((s, a)\), we first define the function \(\phi_V(s, a) := \int_S \phi(s, a, s') V(s') ds'\). We then introduce a function class \(\mathcal{V}\), which is the collection of all possible maps
\[
s \mapsto \min_{a \in A} \left\{ H, \max_{a \in A} \left( r(s, a) + \langle \psi(s, a), \theta \rangle + \eta \|\psi(s, a)\|_W \right) \right\},
\]
where \(\psi : S \times A \to \mathbb{R}^d, \theta \in \mathbb{R}^d, W \in \mathbb{S}^+_{d} \) and \(\eta > 0\) parameterize these maps. Note that the value functions computed by our algorithms will belong to this class.

**Assumption 2** For any bounded function \(V : S \to [0, H]\) and any state-action pair \((s, a) \in S \times A\), we have \(\|\phi_V(s, a)\| \leq 1\). Furthermore, there exists a \(\Sigma \in \mathbb{S}^+_{d}\) and a \(\rho_{\min} > 0\) such that for all \(V \in \mathcal{V}, h \in [H], k \in [K]\), the following holds almost surely:
\[
\mathbb{E} \left[ \phi_V(s^k_h, a^k_h) \phi_V(s^k_h, a^k_h)^\top | \mathcal{G}_{k-1} \right] = \Sigma \succeq \rho_{\min} I,
\]
where \((s^k_h, a^k_h)\) is the state-action pair visited in step \(h\) of episode \(k\) and \(\mathcal{G}_{k-1}\) denotes the \(\sigma\)-field summarizing the information available before the start of episode \(k\).

We emphasize that similar assumptions have featured in model selection for stochastic contextual bandits (Chatterji et al., 2019; Foster et al., 2019; Ghosh et al., 2021a). The assumption ensures that the confidence ball of the estimate of \(\theta^*\) shrinks at a certain rate. We emphasize here that for model selection problems the assumption becomes crucial since we care about the rate of estimate shrinkage as well as the regret performance. Note that classical algorithms, like OFUL for bandits (Abbasi-Yadkori et al., 2011), UCRL-VTR-LIN for RL (Ayoub et al., 2020) only care about regret minimization and hence conditions similar to the above assumption are not needed. The above observation and similar eigenvalue assumption first featured in Chatterji et al. (2019), for model
selection between linear and standard bandits. Later Foster et al. (2019); Ghosh et al. (2021a) used this for model selection in (linear) contextual bandits. Recently, Ghosh et al. (2021b) use this condition for careful model estimation and clustering for contextual bandit problems. In fact the necessity of an assumption like this is posed as an open problem in Foster et al. (2020).

First, using Assumption 2, we obtain the following result, which is crucial to understand the rate at which the confidence ellipsoid of $\theta^*$ shrinks.

**Lemma 3** Fix a $\delta \in (0, 1]$, and suppose that Assumption 2 holds. Then, with probability at least $1 - \delta$, uniformly over all $k \in \{\tau_{\min}(\delta), \ldots, K\}$, we have $\gamma_{\min}(\Sigma_k) \geq 1 + \frac{\rho_{\min} k H}{2}$, where $\tau_{\min}(\delta) := \left(\frac{16}{\rho_{\min}} + \frac{8}{\rho_{\min}}\right) \log \left(\frac{2dKH}{\delta}\right)$.

**Proof** We follow a similar proof technique as used in Chatterji et al. (2019) in the setting of contextual linear bandits using Assumption 2 and the matrix Freedman inequality. First, note that by Assumption 2, we have $\left\|\phi_{V_{h+1}}((s_{h,k}^j, a_{h,k}^j))\right\| \leq 1$, and $\mathbb{E}[\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top | G'_{k-1}] = \Sigma \geq \rho_{\min} I$ for all $h \in [H]$ and $k \in [K]$. Now, fix an $h \in [H]$, and define the following matrix martingale

$$Z_{h,k} = \sum_{j=1}^k \phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top - k\Sigma,$$

with $Z_{h,0} = 0$. Next, consider the martingale difference sequence

$$Y_{h,k} = Z_{h,k} - Z_{h,k-1} = \phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top - \Sigma.$$

Since $\left\|\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)\right\| \leq 1$, we have

$$\left\|\Sigma\right\|_{op} = \left\|\mathbb{E}[\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top | G'_{k-1}]\right\|_{op} \leq 1,$$

and as a result

$$\left\|Y_{h,k}\right\|_{op} = \left\|\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top - \Sigma\right\|_{op} \leq 2.$$

Furthermore, a simple calculation yields

$$\left\|\mathbb{E}[Y_{h,k}Y_{h,k}^\top | G'_{k-1}]\right\|_{op} = \left\|\mathbb{E}[Y_{h,k}^\top Y_{h,k} | G'_{k-1}]\right\|_{op}$$

$$= \left\|\mathbb{E}\left[\left\|\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)^2\phi_{V_{h+1}}(s_{h,k}^j, a_{h,k}^j)(s_{h,k}^j, a_{h,k}^j)^\top | G'_{k-1}\right\|^2 - \Sigma^2\right]\right\| \leq 2.$$

Now, applying matrix Freedman inequality (Theorem 13 of Chatterji et al. (2019)) with $R = 2$, $\omega^2 = 2k$, $u = \rho_{\min} k/2$, we obtain

$$\mathbb{P}\left[\left\|Z_{h,k}\right\|_{op} \geq \frac{\rho_{\min} k}{2}\right] \leq \frac{\delta}{KH},$$

for any $k \geq \left(\frac{16}{\rho_{\min}} + \frac{8}{\rho_{\min}}\right) \log(2dKH/\delta)$. Note that the above concentration bound holds for any $h \in [H]$. Now, we define $Z_k = \sum_{h=1}^H Z_{h,k}$. Then, applying a union bound, we obtain

$$\left\|Z_k\right\|_{op} \leq \sum_{h=1}^H \left\|Z_{h,k}\right\|_{op} \leq \frac{\rho_{\min} kH}{2}.$$
for a given \( k \in \{ \tau_{\min}(\delta), \ldots, K \} \), with probability at least \( 1 - \delta/K \). By Assumption 2, we have \( \gamma_{\min}(kH\Sigma) \geq \rho_{\min}kH \). Hence, using Weyl's inequality, we obtain

\[
\gamma_{\min}\left( \sum_{j=1}^{k} \sum_{h=1}^{H} \phi_{V_{k+1}}^j (s_h^j, a_h^j) \phi_{V_{k+1}\top} (s_h^j, a_h^j) \right) \geq \rho_{\min}Hk/2
\]

for a given \( k \in \{ \tau_{\min}(\delta), \ldots, K \} \), with probability at least \( 1 - \delta/K \). Now, the result follows by taking a union bound, and by noting that \( \Sigma_k = I + \sum_{j=1}^{k} \sum_{h=1}^{H} \phi_{V_{k+1}}^j (s_h^j, a_h^j) \phi_{V_{k+1}\top} (s_h^j, a_h^j) \).

\[\sqrt{\Sigma} \]

**Lemma 5** Suppose Assumption 2 holds, and for any \( \delta \in (0, 1] \), the initial phase length \( k_0 \) satisfies

\[
\sqrt{k_0} = \tau_{\min}(\delta) + O\left( \frac{b^2Hd\log(2KH/\delta)}{\rho_{\min}(0.5)^2} \right),
\]

where \( \tau_{\min}(\delta) \) is as defined in Lemma 3. Then, for all epochs \( i \geq 10 \), we have \( \mathbb{P}\left[ ||\hat{\theta}^{(i)} - \theta^*||_\infty \geq (0.5)^i \right] \leq \frac{\delta}{2^i} \).

**Proof** Note that, for each epoch \( i \geq 1 \), \( \hat{\theta}^{(i)} \) is computed by considering the samples of UCRL-VTR-LIN over \( \tau_{i-1} \) episodes, where \( \tau_{i-1} = \sum_{j=0}^{i-1} 6^j \lceil \sqrt{k_0} \rceil \). If \( \tau_{\min}(\delta) \leq \tau_{i-1} \leq K \) and \( \tau_{i-1} = \Omega\left( \frac{b^2Hd\log(2KH/\delta)}{\rho_{\min}(0.5)^2} \right) \),

**B.1. Analysis of ARL-LIN (dim)**

First, we prove the following concentration result on the estimates of \( \theta^* \) in the sup-norm, which is important in designing the model selection procedure of ARL-LIN (dim).

**Lemma 4** Suppose Assumption 2 holds. Also, suppose that \( \hat{\theta}_\tau \) is the estimate of \( \theta^* \) after running UCRL-VTR-LIN in full \( d \)-dimension for \( \tau \) episodes with norm upper bound \( b \) and confidence level \( \delta \), where \( \tau \in \{ \tau_{\min}(\delta), \ldots, K \} \) and \( \tau_{\min}(\delta) = \left( \frac{16}{\rho_{\min}} + \frac{8}{\rho_{\min}^2} \right) \log(2dKH/\delta) \). Furthermore, for any \( \varepsilon \in (0, 1) \), let \( \tau = \Omega\left( \frac{b^2Hd\log^2(K^2H/\delta)}{\rho_{\min}^2\varepsilon^2} \right) \). Then, we have

\[
\mathbb{P}\left[ ||\hat{\theta}_\tau - \theta^*||_\infty \geq \varepsilon \right] \leq 2\delta.
\]

**Proof** By Lemma 3 and the properties of UCRL-VTR-LIN, we obtain for all \( \tau \in \{ \tau_{\min}(\delta), \ldots, K \} \),

\[
||\hat{\theta}_\tau - \theta^*||_\infty \leq ||\hat{\theta}_\tau - \theta^*|| \leq \frac{\sqrt{\beta_\tau(\delta)}}{1 + \rho_{\min}H\tau/2}
\]

with probability at least \( 1 - 2\delta \). Let us now look at the confidence radius

\[
\beta_\tau(\delta) = O\left( b^2 + H^2d\log(\tau H) \right) \log^2(\tau^2H/\delta) \leq C b^2H^2d\log(KH) \log^2(K^2H/\delta)
\]

for some positive constant \( C \). With this, the above equation can be written as

\[
||\hat{\theta}_\tau - \theta^*||_\infty \leq \sqrt{C} \frac{bHd\log(KH) \log^2(K^2H/\delta)}{\rho_{\min}H\tau/2}.
\]

Now setting \( \tau \geq 2C \frac{b^2Hd\log^2(K^2H/\delta) \log(KH)}{\rho_{\min}^2\varepsilon^2} \), and using the fact that \( \tau < K \), we get the result.
then by Lemma 4, we have
\[ P \left[ \| \hat{\theta}^{(i)} - \theta^* \|_{\infty} \geq (0.5)^i \right] \leq \frac{\delta}{2^{i-1}}. \]
Consider the epoch \( i = 1 \). In this case, \( \hat{\theta}^{(1)} \) is computed by samples of UCRL-VTR-LIN over \( \tau_0 = \lceil \sqrt{k_0} \rceil \) episodes. Then, the choice \( \sqrt{k_0} = \tau_{\text{min}}(\delta) + \mathcal{O} \left( \frac{b^2 H d \log^2(K^2 H/\delta) \log(K H)}{\rho_{\text{min}}(0.5)^2} \right) \) (we have added \( \tau_{\text{min}}(\delta) \) to make the calculations easier), ensures that
\[ P \left[ \| \hat{\theta}^1 - \theta^* \|_{\infty} \geq 0.5 \right] \leq \delta. \]
Note that, we require \( \tau_{i-1} \geq i 4^i \lceil \sqrt{k_0} \rceil \). The proof is concluded if we can show that this holds for epochs \( i \geq 10 \). To this end, we note that
\[ \tau_{i-1} = \sum_{j=0}^{i-1} 6^j \lceil \sqrt{k_0} \rceil = i 4^i \lceil \sqrt{k_0} \rceil \geq i 4^i \lceil k_0 \rceil, \]
where the last inequality holds for all \( i \geq 10 \).

Armed with the above results, we have the following regret bound for ARL-LIN(dim).

**Theorem 2 (Cumulative regret of ARL-LIN(dim))** Suppose ARL-LIN(dim) is run with parameter \( k_0 \) chosen as in Lemma 5 for \( K \) episodes. Then, with probability at least \( 1 - 3\delta \), its regret
\[ R(T) = \mathcal{O} \left( \frac{H k_0}{\gamma 5.18} + \left( bd \sqrt{H^3 T} + b d H^2 K^{1/4} \right) \text{polylog}(T/\delta) \right), \]
where \( \gamma = \min \{ |\theta^*(j)| : \theta^*(j) \neq 0 \} \) with \( \theta^*(j) \) denoting the \( j \)-th coordinate of \( \theta^* \).

**Proof** We first calculate the probability of the event \( \mathcal{E} = \bigcap_{i \geq 10} \left\{ \| \hat{\theta}^{(i)} - \theta^* \|_{\infty} \leq (0.5)^i \right\} \), which follows from Lemma 4 by a straightforward union bound. Specifically, we have
\[ P[\mathcal{E}] = P \left[ \bigcap_{i \geq 10} \left\{ \| \hat{\theta}^{(i)} - \theta^* \|_{\infty} \leq (0.5)^i \right\} \right] = 1 - P \left[ \bigcup_{i \geq 10} \left\{ \| \hat{\theta}^{(i)} - \theta^* \|_{\infty} \geq (0.5)^i \right\} \right] \geq 1 - \sum_{i \geq 10} P \left[ \| \hat{\theta}^{(i)} - \theta^* \|_{\infty} \geq (0.5)^i \right] \geq 1 - \sum_{i \geq 10} \frac{\delta}{2^{i-1}} \geq 1 - \sum_{i \geq 10} \frac{\delta}{2^{i-1}} = 1 - \delta. \]
Now, consider the phase \( i(\gamma) := \max \left\{ 10, \log_2 \left( \frac{1}{\gamma} \right) \right\} \). Note that when event \( \mathcal{E} \) holds, then for all epochs \( i \geq i(\gamma) \), \( D^{(i)} \) is the correct set of \( d^* \) non-zero coordinates of \( \theta^* \). Thus, with probability at least \( 1 - \delta \), the cumulative regret of ARL-LIN(dim) after \( K \) episodes is given by
\[ R(T) \leq H \sum_{j=0}^{i(\gamma)-1} 36^j k_0 + \sum_{j=i(\gamma)}^{N} R_{d}^{\text{UCRL-VTR-LIN}}(36^j k_0, \delta_j, b) + R_{d}^{\text{UCRL-VTR-LIN}} \left( \sum_{j=0}^{N} 6^j \lceil \sqrt{k_0} \rceil, \delta, b \right), \]
where \( N \) denotes the total number of epochs. Note that \( N = \mathcal{O} \left( \log_{36} \left( \frac{K}{k_0} \right) \right) \), and hence \( \sum_{j=0}^{N} 6^j \lceil \sqrt{k_0} \rceil = \mathcal{O} \left( \sqrt{K} \right) \). Then, the third term in the above regret expression can be upper bounded by \( R_{d}^{\text{UCRL-VTR-LIN}}(\sqrt{K}, \delta, b) \). Here, the subscript \( d \) denotes that UCRL-VTR-LIN in full
\( d \)-coordinates during the support estimation phases of all epochs \( j \geq 0 \). Thus, using the result of \text{Jia et al. (2020)}, the regret suffered by \text{ARL-LIN (dim)} during all the support estimation phases can be upper bounded as

\[
R_d^{\text{UCRL-VTR-LIN}} \left( \sum_{j=0}^{N} 6^j \left\lceil \sqrt{k_0} \right\rceil, \delta, b \right) \leq R_d^{\text{UCRL-VTR-LIN}} (\sqrt{K}, \delta, b) = O \left( bdH^2K^{1/4} \log(\sqrt{KH}) \log(KH/\delta) \right)
\]

with probability at least \( 1 - \delta \).

Now, we turn to upper bound the second term of the regret expression. Here, the subscript \( d^* \) denotes that \text{UCRL-VTR-LIN} is run in only \( d^* \)-coordinates (with high probability) during the regret minimization phases of all epochs \( j \geq i(\gamma) \). Now, using the result of \text{Jia et al. (2020)}, for all epochs \( j \geq i(\gamma) \), we have

\[
R_{ds}^{\text{UCRL-VTR-LIN}} (36^j k_0, \delta_j, b) = O \left( bd^* H^2 \sqrt{36^j k_0} \log(36^j k_0 H) \log(36^{2j} k_0^2 H/\delta_j) \right)
\]

with probability at least \( 1 - \delta_j \). Substituting \( \delta_j = \delta/2^j \), the regret suffered by \text{ARL-LIN (dim)} during all the regret minimization phases can be upper bounded as

\[
\sum_{j=i(\gamma)}^{N} R_d^{\text{UCRL-VTR-LIN}} (36^j k_0, \delta_j, b) = \sum_{j=i(\gamma)}^{N} O \left( bd^* H^2 \sqrt{36^j k_0} \log(j) \log(k_0 H) \log(k_0^2 H/\delta) \right)
\]

\[
\leq O \left( bd^* H^2 \log(N) \log(k_0 H) \log(k_0^2 H/\delta) \right) \sum_{j=i(\gamma)}^{N} 6^j \left\lceil \sqrt{k_0} \right\rceil
\]

\[
= O \left( bd^* H^2 \sqrt{K} \log(K/k_0) \log(k_0 H) \log(k_0^2 H/\delta) \right)
\]

with probability at least \( 1 - \sum_{j\geq i(\gamma)} \delta/2^j \geq 1 - \delta \). Here, we have used that the total number of epochs \( N = O \left( \left\lceil \log_{36} \left( \frac{K}{k_0} \right) \right\rceil \right) = O(\log(K/k_0)) \).

Putting everything together, the regret of \text{ARL-LIN (dim)} is upper bounded as

\[
R(T) \leq Hk_0 36^{i(\gamma)} + O \left( bd^* \sqrt{H^3 T} \log(K/k_0) \log(k_0 H) \log(k_0^2 H/\delta) \right)
\]

\[
+ O \left( bdH^2 K^{1/4} \log(\sqrt{KH}) \log(KH/\delta) \right)
\]

\[
= O \left( \frac{H}{35.18} k_0 + \left( bd^* \sqrt{H^3 T} + bdH^2 K^{1/4} \right) \log(K/k_0) \polylog(T/\delta) \right)
\]

with probability at least \( 1 - 3\delta \). Here, in the last step, we have used that \( 36 \leq 2^{5.18} \), which completes the proof.

\[ \Box \]

\textbf{B.2. Norm As Complexity Measure}

In this section, we define \( \| \theta^* \| \) as a measure of complexity of the problem. In linear kernel MDPs, if \( \theta^* \) is close to 0 (with small \( \| \theta^* \| \)), the set of states \( s' \) for the next step will have a small cardinality. Similarly, when \( \theta^* \) is away from 0 (with large \( \| \theta^* \| \)), the above-mentioned cardinality will be quite large. Hence, we see that \( \| \theta^* \| \) serves as a natural measure of complexity.
Algorithm 3 Adaptive Reinforcement Learning - Linear (norm) – ARL-LIN (norm)

1: Input: An upper bound $b$ of $\|\theta^*\|$, initial epoch length $k_1$, confidence level $\delta \in (0, 1]$
2: Initialize estimate of $\|\theta^*\|$ as $b^{(1)} = b$, set $\delta_1 = \delta$
3: for epochs $i = 1, 2 \ldots$ do
4: Play UCRL-VTR-LIN with norm estimate $b^{(i)}$ for $k_i$ episodes with confidence level $\delta_i$
5: Refine estimate of $\|\theta^*\|$ as $b^{(i+1)} = \max_{\theta \in B_{k_i}} \|\theta\|$
6: Set $k_{i+1} = 2k_i$, $\delta_{i+1} = \frac{\delta_i}{2}$
7: end for

B.2.1. ALGORITHM: ADAPTIVE REINFORCEMENT LEARNING - LINEAR (NORM)

We propose and analyze an algorithm (Algorithm 3), that adapts to the problem complexity $\|\theta^*\|$, and as a result, the regret obtained will depend on $\|\theta^*\|$. In prior work (Jia et al., 2020), usually it is assumed that $\theta^*$ lies in a norm ball with known radius, i.e., $\|\theta^*\| \leq b$. This is a non-adaptive algorithm and the algorithm uses $b$ as a proxy for the problem complexity, which can be a huge over-estimate. In sharp contrast, we start with this over estimate of $\|\theta^*\|$, and successively refine this estimate over multiple epochs. We show that this refinement strategy yields a consistent sequence of estimates of $\|\theta^*\|$, and as a consequence, our regret bound depends on $\|\theta^*\|$, but not on its upper bound $b$.

Our Approach: Similar to ARL-GEN, we consider doubling epochs - at each epoch $i \geq 1$, UCRL-VTR-LIN is run for $k_i = 2^{i-1}k_1$ episodes with confidence level $\delta_i = \frac{\delta}{2^{i-1}}$ and norm estimate $b^{(i)}$, where the initial epoch length $k_1$ and confidence level $\delta$ are parameters of the algorithm. We begin with $b$ as an initial (over) estimate of $\|\theta^*\|$, and at the end of the $i$-th epoch, based on the confidence set built, we choose the new estimate as $b^{(i+1)} = \max_{\theta \in B_{k_i}} \|\theta\|$. We argue that this sequence of estimates is indeed consistent, and as a result, the regret depends on $\|\theta^*\|$.

B.2.2. ANALYSIS OF ARL-LIN (NORM)

First, we present the main result of this section. We show that the norm estimates $b^{(i)}$ computed by ARL-LIN (norm) (Algorithm 3) indeed converges to the true norm $\|\theta^*\|$ at an exponential rate with high probability.

Lemma 6 (Convergence of norm estimates) Suppose Assumption 2 holds. Also, suppose that, for any $\delta \in (0, 1]$, the length $k_1$ of the initial epoch satisfies

$$k_1 \geq \max \left\{ \tau_{\min}(\delta) \log_2 (1 + K/k_1), C \left( b \max\{p, q\} \right)^2 d \right\},$$

where $p = O\left( \frac{1}{\sqrt{\tau_{\min} H}} \right)$, $q = O\left( \sqrt{\frac{H \log(k_1 H) \log^2(k_1^2 H/d)}{\tau_{\min} p_{\min}}} \right)$, $\tau_{\min}(\delta)$ is as defined in Lemma 3, and $C > 1$ is some sufficiently large universal constant. Then, with probability exceeding $1 - 4\delta$, the sequence $\{b^{(i)}\}_{i=1}^\infty$ converges to $\|\theta^*\|$ at a rate $O\left( \frac{1^{3/2}}{2^i} \right)$, and $b^{(i)} \leq c_1 \|\theta^*\| + c_2$, where $c_1, c_2 > 0$ are universal constants.
Proof We consider doubling epochs, with epoch lengths \(k_i = 2^{i-1}k_1\) for \(i \in \{1, \ldots, N\}\), where \(k_1\) is the initial epoch length and \(N\) is the number of epochs. From the doubling principle, we obtain

\[
\sum_{i=1}^{N} 2^{i-1}k_1 = K \implies N = \log_2 \left( 1 + \frac{K}{k_1} \right) = \mathcal{O} \left( \log(K/k_1) \right).
\]

Now, let us consider the \(i\)-th epoch, and let \(\hat{\theta}_{k_i}\) be the least square estimate of \(\theta^*\) at the end of epoch \(i\), which is the estimate computed by UCRL-VTR-LIN after \(k_i\) episodes. The confidence interval at the end of epoch \(i\), i.e., after UCRL-VTR-LIN is run with a norm estimate \(b(i)\) for \(k_i\) episodes with confidence level \(\delta_i\), is given by

\[
B_{k_i} = \left\{ \theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_{k_i}\|_{\Sigma_{k_i}} \leq \sqrt{\beta_{k_i}(\delta_i)} \right\}.
\]

Here \(\beta_{k_i}(\delta_i)\) denotes the radius and \(\Sigma_{k_i}\) denotes the shape of the ellipsoid. Using Lemma 3, one can rewrite \(B_{k_i}\) as

\[
B_{k_i} = \left\{ \theta \in \mathbb{R}^d : \|\theta - \hat{\theta}_{k_i}\| \leq \frac{\sqrt{\beta_{k_i}(\delta_i)}}{\sqrt{1 + \rho_{\min}Hk_i/2}} \right\},
\]

with probability at least \(1 - \delta_i = 1 - \delta/2^{i-1}\). Here, from Lemma 3, we use the fact that \(\gamma_{\min}(\Sigma_{k_i}) \geq 1 + \rho_{\min}Hk_i/2\), provided \(k_i \geq \tau_{\min}(\delta/2^{i-1})\). To ensure this condition, we take (the sufficient condition) \(k_1 \geq \tau_{\min}(\delta)N\). Hence, with \(k_1\) satisfying \(k_i \geq \tau_{\min}(\delta)\log_2 \left( 1 + \frac{K}{k_1} \right)\), we ensure that \(\gamma_{\min}(\Sigma_{k_i}) \geq 1 + \rho_{\min}Hk_i/2\). Also, we know that \(\theta^* \in B_{k_i}\) with probability at least \(1 - \delta_i\). Hence, we obtain

\[
\|\hat{\theta}_{k_i}\| \leq \|\theta^*\| + \frac{\sqrt{\beta_{k_i}(\delta_i)}}{\sqrt{1 + \rho_{\min}Hk_i/2}}
\]

with probability at least \(1 - 2\delta_i\). Now, recall that at the end of the \(i\)-th epoch, \(\text{ARL-LIN(norm)}\) set the estimate of \(\|\theta^*\|\) to

\[
b^{(i+1)} = \max_{\theta \in B_{k_i}} \|\theta\|.
\]

From the definition of \(B_{k_i}\), we obtain

\[
b^{(i+1)} = \|\hat{\theta}_{k_i}\| + \frac{\sqrt{\beta_{k_i}(\delta_i)}}{\sqrt{1 + \rho_{\min}Hk_i/2}} \leq \|\theta^*\| + 2 \frac{\sqrt{\beta_{k_i}(\delta_i)}}{\sqrt{1 + \rho_{\min}Hk_i/2}}
\]

with probability exceeding \(1 - 2\delta_i\). Let us now look at the confidence radius

\[
\beta_{k_i}(\delta_i) = O \left( (b(i))^2 + H^2d\log(k_iH)\log^2(k_i^2H/\delta_i) \right).
\]

We now substitute \(k_i = 2^{i-1}k_1\) and \(\delta_i = \frac{\delta}{2^i}\) to obtain

\[
\sqrt{1 + \rho_{\min}Hk_i/2} \geq 2^{1/2} \sqrt{\rho_{\min}Hk_1}, \quad \text{and}
\]

\[
\frac{\sqrt{\beta_{k_i}(\delta_i)}}{\sqrt{1 + \rho_{\min}Hk_i/2}} \leq \frac{C_1}{2} \frac{b(i)}{2^{1/2} \sqrt{\rho_{\min}Hk_1}} + \frac{C_2}{2} \frac{\delta^{3/2}}{2^{1/2} \sqrt{\rho_{\min}Hk_1}} \left( H \sqrt{d\log(k_iH)\log^2(k_i^2H/\delta_i)} \right)
\]

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for some universal constants $C_1, C_2$. Using this, we obtain

$$b^{(i+1)} \leq \|\theta^*\| + C_1 \frac{b^{(i)}}{2^{-i/2} \sqrt{\rho_{\text{min}} H k_1}} + C_2 \frac{i^{3/2}}{2^{-i/2} \sqrt{\rho_{\text{min}} H k_1}} \left( H \sqrt{d \log(k_1 H) \log^2(k_2 H/\delta_1)} \right)$$

$$= \|\theta^*\| + b^{(i)} \frac{p}{2^{-i/2}} \sqrt{\frac{1}{k_1}} + \frac{i^{3/2} q}{2^{-i/2}} \sqrt{\frac{d}{k_1}},$$

with probability at least $1 - 2\delta_i$, where

$$p = \frac{C_1}{\sqrt{\rho_{\text{min}} H}} \text{ and } q = \frac{C_2}{\sqrt{\rho_{\text{min}} H}} \sqrt{H \log(k_1 H) \log^2(k_2 H/\delta_1)}.$$

Hence, we obtain, with probability at least $1 - 2\delta_i$,

$$b^{(i+1)} - b^{(i)} \leq \|\theta^*\| - \left( 1 - \frac{p}{2^{-i/2}} \sqrt{\frac{1}{k_1}} \right) b^{(i)} + \frac{i^{3/2} q}{2^{-i/2}} \sqrt{\frac{d}{k_1}}.$$

By construction, $b^{(i)} \geq \|\theta^*\|$ (since $\theta^* \in B_{k_i}$). Hence, provided $k_1 > \frac{4p^2}{2}$, we have

$$b^{(i+1)} - b^{(i)} \leq \frac{p}{2^{-i/2}} \sqrt{\frac{1}{k_1} \|\theta^*\|} + \frac{i^{3/2} q}{2^{-i/2}} \sqrt{\frac{d}{k_1}},$$

with probability at least $1 - 2\delta_i$. From the above expression, we have

$$\sup_i b^{(i)} < \infty$$

with probability greater than or equal to

$$1 - \sum_i 2\delta_i = 1 - \sum_i 2\delta/2^{-i-1} = 1 - 4\delta.$$

From the expression of $b^{(i+1)}$ and using the above fact in conjunction yield

$$\lim_{i \to \infty} b^{(i)} \leq \|\theta^*\|.$$

However, by construction $b^{(i)} \geq \|\theta^*\|$. Using this, along with the above equation, we obtain

$$\lim_{i \to \infty} b^{(i)} = \|\theta^*\|,$$

with probability exceeding $1 - 4\delta$. So, the sequence $\{b^{(1)}, b^{(2)}, \ldots\}$ converges to $\|\theta^*\|$ with probability at least $1 - 4\delta$, and hence our successive refinement algorithm is consistent.

**Rate of Convergence:** Since

$$b^{(i+1)} - b^{(i)} = \tilde{O} \left( \frac{i^{3/2}}{2^i} \right),$$

with high probability, the rate of convergence of the sequence $\{b^{(i)}\}_{i=1}^{\infty}$ is exponential in the number of epochs.

**Uniform upper bound on $b^{(i)}$:** We now compute a uniform upper bound on $b^{(i)}$ for all $i$. Consider the sequences $\left\{ \frac{i^{3/2}}{2^{-i/2}} \right\}_{i=1}^{\infty}$ and $\left\{ \frac{1}{2^{-i/2}} \right\}_{i=1}^{\infty}$, and let $t_j$ and $u_j$ denote the $j$-th term of the sequences respectively. It is easy to check that $\sup_j t_i < \infty$ and $\sup_j u_i < \infty$, and that the sequences $\{t_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$ are convergent. With this new notation, we have

$$b^{(2)} \leq \|\theta^*\| + u_1 \frac{pb^{(1)}}{\sqrt{k_1}} + t_1 q \frac{\sqrt{d}}{\sqrt{k_1}}.$$
with probability at least $1 - 2\delta$. Similarly, for $b^{(3)}$, we have
\[
    b^{(3)} \leq \|\theta^*\| + u_2 \frac{p b^{(2)}}{\sqrt{k_1}} + t_2 \frac{q \sqrt{d}}{\sqrt{k_1}} \\
    \leq \left( 1 + u_2 \frac{p}{\sqrt{k_1}} \right) \|\theta^*\| + \left( u_1 u_2 \frac{p}{\sqrt{k_1}} b^{(1)} \right) + \left( t_1 u_2 \frac{p}{\sqrt{k_1}} + t_2 \frac{q \sqrt{d}}{\sqrt{k_1}} \right)
\]
with probability at least $1 - 2\delta - \delta = 1 - 3\delta$. Similarly, we write expressions for $b^{(4)}, b^{(5)}, \ldots$. Now, provided $k_1 \geq C \left( \max\{p, q\} b^{(1)} \right)^2 / d$, where $C$ is a sufficiently large constant, the expression for $b^{(i)}$ can be upper-bounded as
\[
    b^{(i)} \leq c_1 \|\theta^*\| + c_2
\]
for all $i$, where $c_1, c_2 > 0$ are some universal constants, which are obtained from summing an infinite geometric series with decaying step size. The above expression holds with probability at least $1 - \sum_i 2\delta_i = 1 - 4\delta$, which completes the proof.

Armed with the above result, we finally focus on the regret bound for $\text{ARL-LIN(norm)}$.

**Theorem 3 (Cumulative regret of $\text{ARL-LIN(norm)}$)** Fix any $\delta \in (0, 1]$, and suppose that the hypothesis of Lemma 6 holds. Then, with probability exceeding $1 - 6\delta$, $\text{ARL-LIN(norm)}$ enjoys the regret bound
\[
    R(T) = \tilde{O} \left( \|\theta^*\| d \sqrt{H^3 T \log(k_1 H) \log(k_1^2 H / \delta)} \right),
\]
where $T = KH$ denotes the total number of rounds, and $\tilde{O}$ hides a $\text{polylog}(K/k_1)$ factor.

**Proof** The cumulative regret is given by
\[
    R(T) \leq \sum_{i=1}^{N} R^{\text{UCRL-VTR-LIN}}(k_i, \delta_i, b^{(i)}),
\]
where $N$ denotes the total number of epochs $R^{\text{UCRL-VTR-LIN}}(k_i, \delta_i, b^{(i)})$ denotes the cumulative regret of UCRL-VTR-LIN, when it is run with confidence level $\delta_i$ and norm upper bound $b^{(i)}$ for $k_i$ episodes. Using the result of Jia et al. (2020), we have
\[
    R^{\text{UCRL-VTR-LIN}}(k_i, \delta_i, b^{(i)}) = O \left( b_i d H^2 / \sqrt{k_i} \log(k_i H) \log(k_i^2 H / \delta_i) \right)
\]
with probability at least $1 - \delta_i$. Now, using Lemma 6, we obtain
\[
    R(T) \leq (c_1 \|\theta^*\| + c_2) \sum_{i=1}^{N} \mathcal{O} \left( d H^2 \sqrt{k_i} \log(k_i H) \log(k_i^2 H / \delta_i) \right)
\]
with probability at least $1 - 4\delta - \sum_i \delta_i$. Substituting $k_i = 2^{i-1} k_1$ and $\delta_i = \delta / 2^{i-1}$, we obtain
\[
    R(T) \leq (c_1 \|\theta^*\| + c_2) \sum_{i=1}^{N} \mathcal{O} \left( d H^2 \sqrt{k_i} \text{poly}(i) \log(k_1 H) \log(k_1^2 H / \delta) \right)
\]
with probability at least $1 - 4\delta - 2\delta = 1 - 6\delta$. Using the above expression, we obtain

$$R(T) \leq (c_1\|\theta^*\| + c_2) \mathcal{O} \left( dH^2 \log(k_1 H) \log(k_1^2 H/\delta) \right) \sum_{i=1}^{N} \text{poly}(i) \sqrt{k_i}$$

$$\leq (c_1\|\theta^*\| + c_2) \mathcal{O} \left( dH^2 \log(k_1 H) \log(k_1^2 H/\delta) \right) \text{poly}(N) \sum_{i=1}^{N} \sqrt{k_i}$$

$$\leq (c_1\|\theta^*\| + c_2) \mathcal{O} \left( dH^2 \log(k_1 H) \log(k_1^2 H/\delta) \right) \text{polylog}(K/k_1) \sum_{i=1}^{N} \sqrt{k_i}$$

$$\leq (c_1\|\theta^*\| + c_2) \mathcal{O} \left( dH^2 \log(k_1 H) \log(k_1^2 H/\delta) \right) \text{polylog}(K/k_1) \sqrt{K}$$

$$= \mathcal{O} \left( \|\theta^*\| d\sqrt{H^3 T \log(k_1 H) \log(k_1^2 H/\delta) \text{polylog}(K/k_1)} \right),$$

where we have used that $N = \mathcal{O} (\log(K/k_1))$, $\sum_{i=1}^{N} \sqrt{k_i} = \mathcal{O}(\sqrt{K})$, and $T = KH$. The above regret bound holds with probability greater than or equal to $1 - 6\delta$, which completes the proof. ■