GENERALIZED FOSTER’S IDENTITIES

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Abstract. Foster’s network theorems and their extensions to higher orders involve resistance values and conductances. We establish identities concerning voltage values and conductances. Our identities are analogous to the extended Foster’s identities.

1. Introduction

Foster’s first identity, which is also known as Foster’s network theorem, was initially proved by R. M. Foster [Fo1] in 1949 by using Kirchhoff’s rule. In 1990, D. Copersmith, P. Doyle, P. Raghavan, and M. Snir [CDRS] gave another proof of this theorem by using random walks on graphs. Independently, in 1991, P. Tetali [TP1] gave a third proof of this theorem again by using random walks on graphs. (See also [BB, Theorem 25 and exercise 23 in Chapter IX]). Later in 1994, Tetali [TP2] gave a fourth probabilistic proof by using an elementary identity for ergodic Markov chains.

In the late 1980’s, T. Chinburg and R. Rumely introduced a canonical measure $\mu_{can}$ on a metrized graph. As a consequence of the fact that $\mu_{can}$ has total mass 1, Chinburg and Rumely [CR, remark on pg. 26] obtained an identity. In 2003, M. Baker and X. Faber [BF, Corollary 6] showed that this identity is equivalent to the Foster’s first identity. This became another interesting proof of the Foster’s first identity.

In another direction, L. Weinberg [W] in 1958, D. J. Klein and M. Randić [D-M, Corollary C1] in 1993, and R. Bapat [RB1, Lemma 2] in 2004 proved identities that are equivalent to the Foster’s first identity. The properties of discrete Laplacian and its pseudo inverse were used in the articles [D-M] and [RB1].

In 1961, Foster [Fo2] proved another identity which we call Foster’s second identity. In 2002, J. L. Palacios [P] gave a probabilistic proof of the Foster’s second identity, generalizing the arguments used in [TP2]. He also showed how to extend the Foster’s identities to graph paths with more edges and he proved a third identity. The Foster’s first and second identities, and the third identity due to Palacios are about the sums of the resistance values over the graph paths consisting of one, two and three edges, respectively.

In 2007, we [C1, Section 5.5] proved four identities that involve voltage values rather than resistance values. Our method was to use the properties of discrete laplacian and its pseudo inverse. As a corollary to these identities, we obtained Foster’s 1st, 2nd, 3rd, and 4th identities.

In 2008, E. Bendito, A. Carmona, A. M. Encinas and J. M. Gesto [BCEG, Proposition 2.3] extended the Foster’s identities to higher orders. Their formula contains all previously known Foster’s identities as specific cases.

Key words and phrases. Electrical networks, generalized Foster’s identities, Foster’s Network Theorem, the voltage function, the resistance function, discrete Laplacian, pseudo inverse, metrized graphs.

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In this paper, we extend the identities concerning voltage values [C1, Section 5.5] to higher orders. This can be found in Theorem 4.2, which is our main result. In this way, we generalize the extended Foster’s identities. We provide two proofs for Theorem 4.2. Both of the proofs rely on the properties of the discrete Laplacian and its pseudo inverse. In the first proof, we use a remarkable relation between the resistance values and the pseudo inverse \( L^+ \) of the discrete Laplacian [RB1] [RB2] (see also Lemma 3.4) to express the voltage values by using \( L^+ \) (see Lemma 3.5). Similarly, in the second proof, we use an equally interesting relation between the resistance values and some “equilibrium measures” \( \nu^i \)'s [BCEG, Proposition 2.1] to express the voltage values by using \( \nu^i \)'s (see Proposition 4.4). More information about the equilibrium measure can be found in the article [BCEG] and the related references therein.

Note that there is a 1−1 correspondence between the metrized graphs and the equivalence class of finite connected weighted graphs in which the weight of an edge is the reciprocal of its length [BR, Lemma 2.2]. We can also view such a graph as a resistive electrical network in which the resistance of each edge is the same as its length. We will work with metrized graphs in this paper. However, the results and their proofs are valid for the corresponding weighted graphs and the resistive electrical networks.

2. THE VOLTAGE AND RESISTANCE FUNCTIONS

A metrized graph \( \Gamma \) is a finite connected graph equipped with a distinguished parametrization of each of its edges. One can find other definitions of metrized graphs in the articles [CR], [BR], [Zh, Appendix] and [BF].

A metrized graph can have multiple edges and self-loops. For any given \( p \in \Gamma \), the number of directions emanating from \( p \) will be called the valence of \( p \), and will be denoted by \( \nu(p) \). By definition, there can be only finitely many \( p \in \Gamma \) with \( \nu(p) \neq 2 \).

For a metrized graph \( \Gamma \), we will denote its set of vertices by \( V(\Gamma) \). We require that \( V(\Gamma) \) be finite and non-empty and that \( p \in V(\Gamma) \) for each \( p \in \Gamma \) if \( \nu(p) \neq 2 \). For a given metrized graph \( \Gamma \), it is possible to enlarge the vertex set \( V(\Gamma) \) by considering more additional points of valence 2 as vertices.

For a given metrized graph \( \Gamma \), the set of edges of \( \Gamma \) is the set of closed line segments with end points in \( V(\Gamma) \). We will denote the set of edges of \( \Gamma \) by \( E(\Gamma) \). We will denote \( \#(V(\Gamma)) \) and \( \#(E(\Gamma)) \) by \( n \) and \( e \), respectively if there is no danger of confusion. We denote the length of an edge \( e_i \in E(\Gamma) \) by \( L_i \).

In the article [CR], the voltage function \( j_z(x, y) \) is defined and studied as a function of \( x, y, z \in \Gamma \). For fixed \( z \) and \( y \) it has the following physical interpretation: when \( \Gamma \) is viewed as a resistive electric circuit with terminals at \( z \) and \( y \), with the resistance in each edge given by its length, then \( j_z(x, y) \) is the voltage difference between \( x \) and \( z \), when unit current enters at \( y \) and exits at \( z \) (with reference voltage 0 at \( z \)).

For any \( x, y, z \in \Gamma \), the voltage function \( j_x(y, z) \) on \( \Gamma \) is a symmetric function in \( y \) and \( z \), i.e., \( j_x(y, z) = j_x(z, y) \). Moreover, for any \( x, y, z \in \Gamma \) it satisfies \( j_x(x, z) = 0 \) and \( j_x(y, y) = r(x, y) \), where \( r(x, y) \) is the resistance function on \( \Gamma \) (for more information see the articles [CR] and [BR, sec 1.5 and sec 6], and [Zh, Appendix]).

For any \( x \in \Gamma \), by circuit theory we can transform \( \Gamma \) to an \( Y \)-shaped graph with the same resistances between \( x, p, \) and \( q \) as in \( \Gamma \) (see [C2, Section 2] for more details). This is shown in Figure 1 with the corresponding voltage values on each segment. Therefore,
Then it follows from (1) that

\[ 2j_p(x, q) = r(p, x) + r(p, q) - r(q, x), \quad \text{for any } p, q, x \in \Gamma. \]

3. THE DISCRETE LAPLACIAN L AND THE PSEUDO INVERSE L^+

To have a well-defined discrete Laplacian matrix L for a metrized graph \( \Gamma \), we first choose a vertex set \( V(\Gamma) \) for \( \Gamma \) in such a way that there are no self-loops, and no multiple edges connecting any two vertices. This can be done for any graph by enlarging the vertex set by inserting more valence two vertices whenever needed. We will call such a vertex set \( V(\Gamma) \) optimal. If distinct vertices \( p \) and \( q \) are the end points of an edge, we call them adjacent vertices.

Note that a weighted graph corresponding to a metrized graph with optimal vertex set does not have self loops and multiple edges. Such graphs are usually called simple graphs in the literature.

Let \( \Gamma \) be a graph with \( e \) edges and with an optimal vertex set \( V(\Gamma) \) containing \( n \) vertices. Fix an ordering of the vertices in \( V(\Gamma) \). Let \( \{L_1, L_2, \cdots, L_e\} \) be a labeling of the edge lengths. The \( n \times n \) matrix \( A = (a_{pq}) \) given by

\[
a_{pq} = \begin{cases} 
0 & \text{if } p = q, \text{ or } p \text{ and } q \text{ are not adjacent}, \\
\frac{1}{L_k} & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are connected by an edge of length } L_k.
\end{cases}
\]

is called the adjacency matrix of \( \Gamma \). Let \( D = \text{diag}(d_{pp}) \) be the \( n \times n \) diagonal matrix given by \( d_{pp} = \sum_{s \in V(\Gamma)} a_{ps} \). Then \( L := D - A \) is defined to be the discrete Laplacian matrix of \( \Gamma \). That is, \( L = (l_{pq}) \) where

\[
l_{pq} = \begin{cases} 
0 & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are not adjacent}, \\
-\frac{1}{L_k} & \text{if } p \neq q, \text{ and } p \text{ and } q \text{ are connected by an edge of length } L_k, \\
-\sum_{s \in V(\Gamma) - \{p\}} l_{ps} & \text{if } p = q
\end{cases}
\]

The discrete Laplacian matrix is also known as the generalized (or the weighted) Laplacian matrix in the literature.

Throughout this paper, all matrices will have entries in \( \mathbb{R} \). Given a matrix \( M \), let \( M^T \), \( \text{tr}(M) \), be the transpose and trace of \( M \), respectively. Let \( I_n \) be the \( n \times n \) identity matrix, and let \( O \) be the zero matrix (with the appropriate size if it is not specified).
A matrix $M$ is called doubly centered, if both row and column sums are 0. That is, $M$ is doubly centered iff $MY = O$ and $Y^TM = O$, where $Y = [1, 1, \cdots, 1]^T$.

**Remark 3.1.** For any graph $\Gamma$, the discrete Laplacian matrix $L$ is symmetric and doubly centered. That is, $\sum_{p \in V(\Gamma)} l_{pq} = 0$, for each $q \in V(\Gamma)$, and $l_{pq} = l_{qp}$ for each $p, q \in V(\Gamma)$.

In our case, $\Gamma$ is connected by definition. Thus, $L$ is of rank $n - 1$. Although $L$ is not invertible, it has generalized inverses. In particular, it has the pseudo inverse $L^+$, also known as the Moore-Penrose generalized inverse, which is uniquely determined by the following properties:

$i)$ $LL^+L = L$,  
$ii)$ $L^+LL^+ = L^+$,  
$iii)$ $(LL^+)^T = LL^+$,  
$iv)$ $(L^+L)^T = L^+L$.

The following properties hold for both $L$ and $L^+$:

$i)$ $L$ and $L^+$ are symmetric,  
$ii)$ $L$ and $L^+$ are doubly centered,  
$iii)$ $L$ and $L^+$ are EP matrices,  
$iv)$ $L$ and $L^+$ are positive semi-definite.

For more information about $L$ and $L^+$, see the article [C3] and references therein.

I. Gutman and W. Xiao [I-W, Lemma 3] obtained the following result when $L$ arises from a graph where each edge length is 1.

**Lemma 3.2.** [D-M] Equation 2.9] Let $J$ be an $n \times n$ matrix having each entries 1, and let $L$ be the discrete Laplacian of a graph (not necessarily with equal edge lengths). Then $LL^+ = L^+L = I - \frac{1}{n}J$.

As an immediate consequence of Lemma 3.2, we have

**Corollary 3.3.** Let $\Gamma$ be a graph and let $L$ be the corresponding discrete Laplacian matrix of size $n \times n$. Then for any $p, q \in V(\Gamma)$, $\sum_{s \in V(\Gamma)} l_{ps}^+l_{sq}^+ = \begin{cases} -\frac{1}{n} & \text{if } p \neq q; \\ \frac{n-1}{n} & \text{if } p = q \end{cases}$.

The resistance distance has been studied in chemical literature and in computer science. See the articles [D-M] and [RB1]. The following fact plays an important role for the rest of the paper:

**Lemma 3.4.** [RB2] [RB1] Let $\Gamma$ be a graph with discrete Laplacian $L$ and the resistance function $r(x, y)$, and let $H$ be a generalized inverse of $L$, i.e., $LHL = L$. Then $r(p, q) = H_{pp} - H_{pq} + H_{qp}$, for any $p, q \in V(\Gamma)$. In particular, for the pseudo inverse $L^+$ we have $r(p, q) = l_{pp}^+ - 2l_{pq}^+ + l_{qq}^+$, for any $p, q \in V(\Gamma)$.

Note that Lemma 3.4 for the pseudo inverse $L^+$ follows from [D-M, Theorem A]. Similarly, Babić et al [BKLNT] has Lemma 3.4 for $L^+$ and the graphs with all edges of length 1.

Now, we can easily derive the following analogous result about the voltage function:

**Lemma 3.5.** Let $\Gamma$ be a graph with the discrete Laplacian $L$ and the voltage function $j_x(y, z)$. Then for any $p, q, s \in V(\Gamma)$, $j_p(q, s) = l_{pp}^+ - l_{pq}^+ - l_{ps}^+ + l_{qs}^+$.

**Proof.** By equation (2), $2j_p(q, s) = r(q, p) + r(s, p) - r(q, s)$. Then the result follows from Lemma 3.4. \qed
Let $\Gamma$ be a metrized graph with an optimal vertex set $V(\Gamma)$ containing $n$ vertices as before. Recall that if we consider $\Gamma$ as a resistive electric circuit, the resistance of an edge is given by its length. If $L_i$ is the length of an edge $e_i$ with end points $p$ and $q$, then we have $C_{pq} = C_{qp} = \frac{1}{L_i}$ as the conductance of the edge $e_i$. We write $q \sim p$ when the vertex $q$ is adjacent to the vertex $p$, i.e., when $p$ and $q$ are the end points of an edge. We set $C_{pp} = 0$ and $C_{pq} = 0$ if $p$ and $q$ are not adjacent, and define $C_p := \sum_{q \in V(\Gamma), q \sim p} C_{pq}$. It follows from the definitions that $C_p = l_{pp}$ and $C_{pq} = -l_{pq}$ if $p \neq q$, where $L = (l_{pq})$ is the discrete Laplacian of $\Gamma$.

For any $s \in V(\Gamma)$, we [C1 Section 5.5] proved the following identities by using Lemma 3.5 and the properties of $L$ and $L^+$:

$$
\sum_{p,q \in V(\Gamma), p \sim q, p < q} j_p(q, s)C_{pq} = \frac{v - 1}{2},
$$

$$
\sum_{p, q, w \in V(\Gamma), w \sim p \sim q, w < q} j_w(q, s) \frac{C_{wp}C_{pq}}{C_p} = \frac{v - 2}{2},
$$

$$
\sum_{p, q, t, w \in V(\Gamma), w \sim p \sim q \sim t, w < t} j_w(t, s) \frac{C_{wp}C_{pq}C_{qt}}{C_pC_q} = \frac{v - 3}{2} + \frac{1}{2} \sum_{p, q \in V(\Gamma), p \sim q} \frac{C_{pq}^2}{C_pC_q},
$$

$$
\sum_{p, q, t, u, w \in V(\Gamma), u \sim p \sim w \sim q \sim t, u < t} j_u(t, s) \frac{C_{wp}C_{pq}C_{wq}C_{qt}}{C_pC_wC_q} = \frac{v - 4}{2} + \frac{1}{2} \sum_{p, q \in V(\Gamma), p \sim q} \frac{C_{pq}^2}{C_pC_q} + \frac{1}{2} \sum_{p, q, w \in V(\Gamma), p \sim q \sim w \sim p} \frac{C_{pw}C_{wq}C_{qp}}{C_pC_wC_q}.
$$

Note that the formulas above are for graph paths containing at most 4 edges. A certain pattern involving conductances can be seen in these formulas. In the rest of the paper, we will extend formulas above, and obtain a new proof of the extended Foster’s identities as a corollary.

Let $P = (p_{it})$ be the $n \times n$ matrix given by $p_{it} = \frac{C_{it}}{C_i}$, and let $P^k = (p_{it}^{(k)})$ be the $k$-th power of $P$. It follows from the definitions that $p_{it}^{(k+1)} = \sum_{m_1, \ldots, m_k=1}^n \frac{C_{i m_1} C_{m_1 m_2} \cdots C_{m_k t}}{C_i C_{m_1} \cdots C_{m_k}}$ whenever $k \geq 1$. Moreover, we have $\sum_{t=1}^n p_{it}^{(k)} = 1$ and that $\sum_{i=1}^n C_t p_{it}^{(k)} = C_t$ for any $k \geq 1$. As it is also given in the article [BCEG], $P$ is known as transition probability matrix of the reversible Markov chain associated to $\Gamma$ when $\Gamma$ is considered as an electrical network, and $P^k$ is known as the $k$-step transition probability matrix. The value $p_{it}^{(k)}$ is the probability that the Markov chain attains vertex $t$ at $k$-th step after starting from vertex $i$.

**Proposition 4.1.** For any $k \geq 1$, we have $\sum_{i, t=1}^n C_{i t} l_{it}^{+} p_{it}^{(k+1)} = 1 - \text{tr}(P^k) + \sum_{i, t=1}^n C_{i t} l_{it}^{+} p_{it}^{(k)}$. 
Proof. We first note that

\begin{equation}
\sum_{i=1}^{n} l_{it}^{+} C_{iq} = t_{gt}^{+} l_{qt} - \sum_{i=1}^{n} l_{it}^{+} l_{iq} = C_{q} t_{qt}^{+} - \begin{cases} -\frac{1}{n} & \text{if } t \neq q; \\ \frac{1}{n} & \text{if } t = q. \end{cases}
\end{equation}

The last equality follows from Corollary 3.3. For any \( k \geq 1 \), we have

\begin{align*}
\sum_{i, t=1}^{n} C_{it}^{+} p_{it}^{(k+1)} &= \sum_{t, m_1, \ldots, m_k=1}^{n} \frac{C_{m_1 m_2} \cdots C_{m_k}}{C_{m_1} \cdots C_{m_k}} \sum_{i=1}^{n} l_{it}^{+} C_{im_1}, \\
&= \sum_{i, t=1}^{n} C_{it}^{+} p_{it}^{(k)} + \frac{1}{n} \sum_{t, m_1, \ldots, m_k=1}^{n} \sum_{t \neq m_1}^{n} \frac{C_{m_1 m_2} \cdots C_{m_k t}}{C_{m_1} \cdots C_{m_k}} - \frac{n-1}{n} \sum_{m_1=1}^{n} p_{m_1 t}^{(k)} - \text{tr}(P^k) \\
&= \sum_{i, t=1}^{n} C_{it}^{+} p_{it}^{(k)} + \frac{1}{n} \sum_{t, m_1=1}^{n} p_{m_1 t}^{(k)} - \text{tr}(P^k).
\end{align*}

Then the result follows from the fact that \( \sum_{m_1=1}^{n} p_{m_1 t}^{(k)} = \sum_{m_1=1}^{n} 1 = n \) for any \( k \geq 1 \). \( \square \)

We have

\begin{equation}
\sum_{i, t=1}^{n} C_{it}^{+} p_{it} = \sum_{i, t=1}^{n} l_{it}^{+} C_{it} = -\sum_{i, t=1}^{n} l_{it}^{+} l_{it} + \sum_{i}^{n} C_{it}^{+} = -n + 1 + \sum_{i}^{n} C_{it}^{+},
\end{equation}

where the last equality follows from Corollary 3.3. Thus, the successive application of Proposition 4.1 gives the following equality for any \( k \geq 1 \):

\begin{equation}
\sum_{i, t=1}^{n} C_{it}^{+} p_{it}^{(k)} = k - n - \sum_{i=1}^{k-1} \text{tr}(P^i) + \sum_{i=1}^{n} C_{it}^{+}.
\end{equation}

Our main result is the following theorem which gives an identity analogues to the extended Foster’s identities. Unlike extended Foster’s identities which involve the resistance values, our formula involves the voltage values.

**Theorem 4.2.** Let \( s \in V(\Gamma) \). For any \( k \geq 1 \), we have \( \sum_{i, t=1}^{n} C_{i j_i (s, t)} p_{it}^{(k)} = n - k + \sum_{i=1}^{k-1} \text{tr}(P^i) \).

Proof. Let \( s \) be a given vertex in \( V(\Gamma) \). For each \( k \geq 1 \), we have

\begin{align*}
\sum_{i, t=1}^{n} C_{i j_i (s, t)} p_{it}^{(k)} &= \sum_{i, t=1}^{n} C_{i} (l_{it}^{+} - l_{it}^{-} - l_{it}^{+} + l_{it}^{-}) p_{it}^{(k)}, \quad \text{by Lemma 3.5} \\
&= \sum_{i=1}^{n} C_{i} (l_{it}^{+} - l_{it}^{-} - l_{it}^{+} + l_{it}^{-}) p_{it}^{(k)}, \quad \text{since } \sum_{i=1}^{n} p_{i1}^{(k)} = 1. \\
&= \sum_{i=1}^{n} C_{i} l_{it}^{+} - \sum_{i, t=1}^{n} C_{i} l_{it}^{+} p_{it}^{(k)}, \quad \text{since } \sum_{i=1}^{n} C_{i} p_{it}^{(k)} = C_{t}.
\end{align*}

Then the result follows from equation (4). \( \square \)
Next, we will state the extended Foster’s identities as a corollary of Theorem 4.2.

**Corollary 4.3.** For any \( k \geq 1 \), we have \[
\frac{1}{2} \sum_{i,t=1}^{n} C_i r(i,t) p_{it}^{(k)} = n - k + \sum_{i=1}^{k-1} \text{tr}(P_i).
\]

**Proof.** By equation (1), we have \( r(i,t) = j_i(t,s) + j_t(i,s) \) for any \( s \in \Gamma \). Thus, the result follows from Theorem 4.2. \( \square \)

Note that the formula given in Corollary 4.3 was originally proved by E. Bendito, A. Carmona, A. M. Encinas and J. M. Gesto [BCEG, Proposition 2.3]. Next, we will use their methods to provide a second proof for Theorem 4.2. First, we recall some related definitions and results.

Let \( e \) be the \( n \times 1 \) vector whose entries equal to 1, and let \( e_i \) denote the \( n \times 1 \) \( i \)-th unit vector for each \( i = 1, \ldots, n \). That is, \( e_i^k = 1 \) if \( k = i \) and \( e_i^k = 0 \) if \( k \neq i \). Suppose \( \nu^i \) is the unique solution of the linear system \( Lu = e - ne^i \) satisfying \( \nu^i = 0 \). The solution \( \nu^i \) is called the equilibrium measure of the set \( V(\Gamma) \backslash \{i\} \).

Since \( r(i,t) = \frac{1}{n}(\nu^i s + \nu^s i + \nu^t i - \nu^s t - \nu^t s) \) [BCEG, Proposition 2.1], we obtain the following proposition by using equation (2):

**Proposition 4.4.** For any \( i, j, t \) in \( V(\Gamma) \), we have
\[
2 j_i(s,t) = \frac{1}{n} \left( \nu^i s + \nu^s i + \nu^t i - \nu^s t - \nu^t s \right).
\]

On the other hand, we have [BCEG] Proof of Proposition 2.3] that
\[
\frac{1}{n} \sum_{i,t=1}^{n} C_i \nu^i p_{it}^{(k)} = n - k + \sum_{i=1}^{k-1} \text{tr}(P_i).
\] (5)

The second proof of Theorem 4.2 will be as follows. For any \( k \geq 1 \), we have
\[
\sum_{i,t=1}^{n} C_i j_i(s,t) p_{it}^{(k)} = \frac{1}{2n} \sum_{i,t=1}^{n} C_i \left( \nu^i s + \nu^s i + \nu^t i - \nu^s t - \nu^t s \right) p_{it}^{(k)}, \quad \text{by Proposition 4.3}
\]
\[
= \frac{1}{n} \sum_{i,t=1}^{n} C_i \nu^i p_{it}^{(k)}, \quad \text{since } C_i p_{it}^{(k)} = C_t p_{ti}^{(k)} \text{ and } \sum_{t=1}^{n} p_{it}^{(k)} = 1.
\]
\[
= n - k + \sum_{i=1}^{k-1} \text{tr}(P_i), \quad \text{by equation (5)}.
\]

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