THE CENTER OF MONOIDAL BICATEGORIES IN 3+1D DIJKGRAAF-WITTEN THEORY

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ABSTRACT. In this work, for a finite group $G$ and a 4-cocycle $\omega \in Z^4(G, k^\times)$, we compute explicitly the center of the monoidal bicategory $2Vec_G^\omega$ of $\omega$-twisted $G$-graded 1-categories of finite dimensional $k$-vector spaces. We show that this center is a braided monoidal bicategory with a trivial Müger center. It turns out that, even in this simple case, we need go beyond semi-strict setting and consider the most general definition of a braided monoidal bicategory. This center gives a precise mathematical description of the topological defects in the associated 3+1D Dijkgraaf-Witten TQFT.

1. INTRODUCTION

The notion of the center of a monoidal 2-category was introduced long time ago \[BN, C, KV\]. A simplest example of a monoidal 2-category is $2Vec$, the 2-category of 1-categories of finite semisimple $\mathcal{V}$-module categories. Here $\mathcal{V}$ is the 1-category of finite dimensional $k$-vector spaces (i.e. $1Vec$). The tensor product in $2Vec$ is the Deligne tensor product. The ground field $k$ is assumed to be $\mathbb{C}$ throughout the paper.

As far as we know, there is, however, no explicit computation of the centers of any non-trivial monoidal 2-categories. In recent years, the demand for such computation from physics becomes paramount. In this work, we consider a very simple case motivated from the physics of 3+1D topological orders. Let $G$ be a finite group and $\omega \in Z^4(G, k^\times)$ a 4-cocycle. Let $2Vec_G^\omega$ be the 2-category (i.e. a strict 2-category) of $G$-graded 1-categories of finite semisimple $\mathcal{V}$-module categories, equipped with a $\omega$-twisted monoidal structure, which makes $2Vec_G^\omega$ a monoidal bicategory (i.e. a weak monoidal 2-category).

We give a definition of the center of monoidal bicategories in Section 2. It is a weak version of Crans’ definition of the center of monoidal 2 categories \[C\]. We use Gurski’s definition of monoidal bicategories and braided monoidal bicategories \[G1, Section 2.4\]. Our first main result is that the center of a monoidal bicategory is a braided monoidal bicategory, see Theorem 2.3. We further compute explicitly the center $Z(2Vec_G^\omega)$ of the monoidal bicategory $2Vec_G^\omega$ in Section 3.

The analogue on the level of 1-categories is known as the twisted Drinfeld double of a finite group $G$. Let $1Vec_G^\chi$ be the $\chi$-twisted monoidal 1-category of $G$-graded finite dimensional $k$-vector spaces for $\chi \in Z^3(G, k^\times)$. Let $Cl$ be the set of conjugacy classes of $G$, and $C_G(h)$ be the centralizer of $h \in G$. There is a the transgression map $\tau_h : C^{k+1}(G, k^\times) \to C^k(C_G(h), k^\times)$. Willerton used it to give a geometric
description of the twisted Drinfeld double, and showed that there is an equivalence of 1-categories:

\[ Z(1\text{Vec}_G) \simeq \bigoplus_{[h] \in Cl} 1\text{Rep}(C_G(h), \tau_h(\chi)), \]

where \(1\text{Rep}(C_G(h), \tau_h(\chi))\) is the 1-category of representations of the central extension of \(C_G(h)\) determined by the 2-cocycle \(\tau_h(\chi)\) \([W]\). Our second result generalizes this from 1-categories to 2-categories.

**Theorem 1.1.** There is an equivalence of 2-categories:

\[ Z(2\text{Vec}_G) \simeq \bigoplus_{[h] \in Cl} 2\text{Rep}(C_G(h), \tau_h(\omega)), \]

where \(2\text{Rep}(C_G(h), \tau_h(\omega))\) is the 2-category of right module categories over the monoidal 1-category \(1\text{Vec}_{\tau_h(\omega)}C_G(h)\).

The underlying category of \(Z(2\text{Vec}_G)\) is a 2-category. Its braided monoidal structure, which makes \(Z(2\text{Vec}_G)\) a braided monoidal bicategory, will be explicitly described in Section 3.2. We expect a similar generalization to \(n\)-categories.

**Conjecture 1.2.** For \(\omega \in Z^{n+2}(G, k^\times)\) and a properly defined notion of an \(n\)-category, we have an equivalence of \(n\)-categories:

\[ Z(n\text{Vec}_G) \simeq \bigoplus_{[h] \in Cl} n\text{Rep}(C_G(h), \tau_h(\omega)). \]

While we are preparing this paper, a beautiful work on the definition of a fusion 2-category by Douglas and Reutter appeared online \([DR]\). They introduced the notion of the 2-categorical idempotent completion, which is used to define that of 2-categorical semisimplicity. Our result further confirms their definition. In particular, \(Z(2\text{Vec}_G)\) is idempotent complete and semisimple. We expect that it is a fusion 2-category.

The unit component \(Z(2\text{Vec}_G)_1\) of \(Z(2\text{Vec}_G)\) will be discussed in Section 3.3. It is a braided monoidal sub-bicategory of \(Z(2\text{Vec}_G)\), and is equivalent to \(2\text{Rep}(G, \tau_1(\omega))\) as braided monoidal bicategories, where \(\tau_1(\omega) \in Z^3(G, k^\times)\) is a coboundary. Therefore, \(Z(2\text{Vec}_G)_1\) is equivalent to the 2-category \(2\text{Rep}(G)\) of module categories over \(1\text{Vec}_G\). Note that \(2\text{Rep}(G)\) is the idempotent completion of the delooping of \(\text{Rep}(G)\) in the sense of Douglas and Reutter.

We next discuss the Müger center of braided monoidal bicategories which is a generalization of Crans’ definition in the semistrict case \([C]\) in Section 3.4. Our third result is that the Müger center of \(Z(2\text{Vec}_G)\) is trivial. Thus \(Z(2\text{Vec}_G)\) should be an example of the yet-to-be-defined modular tensor bicategory.

**Theorem 1.3.** The Müger center of \(Z(2\text{Vec}_G)\) is equivalent to \(2\text{Vec}\) as bicategories.

Our motivations of this work are threefold.

1. It was proposed in \([LKW2]\) that \(Z(2\text{Vec}_G)\) is precisely the bicategory of the topological excitations of a 3+1D topological order. The objects in \(Z(2\text{Vec}_G)\) represent string-like topological excitations, 1-morphisms represent particle-like topological excitations and 2-morphisms represent instantons. We compute \(Z(2\text{Vec}_G)\) explicitly and summarize the result in Theorem 1.1. It is also known that the low energy effective theory
of this 3+1D topological order is the well-known 3+1D Dijkgraaf-Witten TQFT associated to \((G, \omega)\) [DW]. Therefore, Theorem 1.1 also classifies all topological defects in the 3+1D Dijkgraaf-Witten TQFT. In particular, the monoidal 1-category of endomorphisms of the vacuum is equivalent to the category \(\text{Rep}(G)\) of the representations of \(G\).

(2) It is well-known that the topological excitations in a 2+1D topological order form a modular tensor 1-category (MTC). The 3+1D analogue of MTC, i.e. the yet-to-be-defined modular tensor bicategory, should include \(Z(2\text{Vec}_G^\omega)\) as an example. Our second motivation is to find the correct definition of a (braided) fusion bicategory and that of a modular tensor bicategory. It is worthwhile to point out that, even in this simple case, in order to reveal the intertwined relation between the braidings and the 4-cocycle \(\omega\), we need to go beyond the semi-strict setting.

(3) Our third motivation is to find a categorification of conformal blocks by integrating a modular tensor bicategory over 2-dimensional manifolds via factorization homology (see a recent review [AF] and references therein). Douglas and Reutter constructed a state-sum invariant for 4-manifolds associated to any fusion 2-category. We expect that the integration of \(Z(2\text{Vec}_G^\omega)\) is related to their invariant associated to \(2\text{Vec}_G^\omega\).

This work is the first in a series of works on (braided) fusion bicategories. Our long term goal is to develop a mathematical theory of modular tensor bicategories and a physical theory of the condensations of topological excitations in 3+1D topological orders. For example, the forgetful functor \(Z(2\text{Vec}_G^\omega) \to 2\text{Vec}_G^\omega\) is precisely the mathematical description of a physical condensation process.

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2. The Center of Monoidal Bicategories

In this section, we give a definition of the center of monoidal bicategories. It is a weak version of Crans’ definition of the center of monoidal 2 categories in [C]. We use Gurski’s definition of monoidal bicategories and braided monoidal bicategories [G1] Section 2.4]. We prove that the center of a monoidal bicategory is a braided monoidal bicategory in Theorem 2.3.

We briefly recall the notion of a monoidal bicategory which is defined as a tricategory with one object. We refer the reader to [G1] for more detail on tricategories and the coherence. For two bicategories \(B, B'\), let \(\text{Bicat}(B, B')\) denote the tricategory of bicategories, functors, natural transformations and modifications.

Let \(B = (B, \otimes, I, a, r, \pi, \mu, \rho, \lambda)\) be a monoidal bicategory. It consists of the following data:

(1) \(B\) is a bicategory, \(\otimes\) is the monoidal bifunctor in \(\text{Bicat}(B \times B, B)\), and \(I\) is the tensor unit;
(2) \(a\) is the adjoint equivalence in \(\text{Bicat}(B \times B \times B, B)\), consisting of a pair \(a : (\otimes \otimes \cdot) \otimes \cdot \to \cdot \otimes (\cdot \otimes \cdot)\) and its adjoint equivalence \(a^* : \cdot \otimes (\cdot \otimes \cdot) \to (\cdot \otimes \cdot) \otimes \cdot\);
(3) \(l\) and \(r\) are the adjoint equivalences in \(\text{Bicat}(B, B)\), where \(l : I \otimes \cdot \to \cdot\) and \(r : \cdot \otimes I \to \cdot\);
(4) \( \pi \) is the invertible modification for \( a \), and \( \mu, \rho, \lambda \) are the invertible modifications for \( a, l, r \).

It satisfies certain axioms which are omitted here.

We define the center \( \mathcal{Z}(\mathcal{B}) \) in three steps: (1) the bicategory; (2) the monoidal structure; and (3) the braiding.

**Step 1: the bicategory \( \mathcal{Z}(\mathcal{B}) \).**

**Objects.** An object \( \hat{A} = (A, R_{A,-}, R_{(A|-)}) \) consists of an object \( A \) of \( \mathcal{B} \), an adjoint equivalence \( R_{A,-} : A \otimes - \to - \otimes A \) in \( \text{Bicat}(\mathcal{B}, \mathcal{B}) \), and an invertible modification \( R_{(A|-)} \):

\[
\begin{aligned}
(XA)Y & \xrightarrow{\alpha} X(AY) \\
(A)Y & \xrightarrow{\alpha} AX \\
A(AY) & \xrightarrow{\alpha} A(XA) \\
X(YA) & \xrightarrow{\alpha} X(YA)
\end{aligned}
\]

such that the following axiom holds:

\[
(2.1) \quad \begin{aligned}
((XA)Y)Z & \xrightarrow{\alpha} X(AY))Z \xrightarrow{\alpha} (XAY)Z \xrightarrow{\alpha} X((AY)Z) \\
R_{A,X} & \xrightarrow{\alpha} \sim \xrightarrow{\alpha} \sim \xrightarrow{\alpha} \sim \\
(XA)(YZ) & \xrightarrow{\alpha} X(AYZ) \xrightarrow{\alpha} X(YZA) \\
R_{A,X} & \xrightarrow{\alpha} \sim \xrightarrow{\alpha} \sim \xrightarrow{\alpha} \sim \\
(A)(YZ) & \xrightarrow{\alpha} A(XYZ) \xrightarrow{\alpha} (XY)A \xrightarrow{\alpha} X((YZ)A)
\end{aligned}
\]
where the four isomorphisms “∼=” are those defining the naturality of \( a \) in \( B \).

**1-morphisms.** A 1-morphism \((f, R_{f,-}) : (A, R_{A,-}, R_{(A|-,-)}) \to (A', R_{A',-}, R_{(A'|-,|-)})\) consists of a 1-morphism \( f : A \to A' \) in \( B \), and an invertible modification \( R_{f,-} \):

\[
\begin{array}{c}
A'X \xrightarrow{R_{A',X}} XA' \\
\downarrow f \Rightarrow R_{f,X} \downarrow f \\
AX \xrightarrow{R_{A,X}}XA
\end{array}
\]

such that the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{ccc}
(XA')Y & \overset{a}{\longrightarrow} & X(A'Y) \\
\downarrow & & \downarrow & & \downarrow \\
(A'X)Y & \overset{a}{\longrightarrow} & A'(XY) & \overset{R_{A',XY}}{\longrightarrow} & (XY)A' & \overset{a}{\longrightarrow} & X(YA') \\
\downarrow & \underset{\cong}{\scriptsize \Leftrightarrow R_{f,X}} & \downarrow & \cong & \downarrow \underset{\cong}{\scriptsize \Rightarrow R_{f,XY}} & \downarrow \underset{\cong}{\scriptsize \Leftrightarrow R_{f,Y}} \\
(XA)Y & \overset{a}{\longrightarrow} & A(XY) & \overset{R_{A,X}}{\longrightarrow} & (XY)A & \overset{a}{\longrightarrow} & X(YA) \\
\downarrow & \underset{\cong}{\scriptsize \Leftrightarrow R_{f,X}} & \downarrow & \cong & \downarrow \underset{\cong}{\scriptsize \Rightarrow R_{f,XY}} & \downarrow \underset{\cong}{\scriptsize \Leftrightarrow R_{f,Y}} \\
(AX)Y & \overset{a}{\longrightarrow} & A(AY) & \overset{R_{A,Y}}{\longrightarrow} & (AY)A & \overset{a}{\longrightarrow} & X(AY)
\end{array}
\end{array}
\]

where all vertical arrows are 1-morphisms induced by \( f : A \to A' \) in \( B \).

**2-morphisms.** A 2-morphism \( \alpha : (f, R_{f,-}) \Rightarrow (f', R_{f',-}) \) is a 2-morphism \( \alpha : f \Rightarrow f' \) in \( B \) such that \( \alpha \cdot R_{f,-} = R_{f',-} \cdot \alpha \), i.e. the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{ccc}
A'X & \overset{\cong}{\xrightarrow{R_{A',X}}} & XA' \\
\downarrow & \underset{\cong}{\scriptsize \Rightarrow \alpha} & \downarrow f' \\
AX & \overset{\cong}{\xrightarrow{R_{A,X}}} & XA
\end{array}
\end{array}
\]

where the 2-isomorphisms in the front and back are \( R_{f,X} \) and \( R_{f',X} \), respectively.

**Composition of 1-morphisms.** Given two 1-morphisms \((f, R_{f,-}) \) and \((g, R_{g,-})\), the composition

\[
(g, R_{g,-}) \circ (f, R_{f,-}) = (gf, R_{gf,-}),
\]
where $gf$ is the composition in $B$, and $R_{gf,-}$ is given by the following composition of 2-morphisms:

$$
\begin{align*}
A''X & \xrightarrow{R_{A'',X}} XA'' \\
g' & \Rightarrow R_{g,X} \\
A'X & \xrightarrow{R_{A',X}}XA' \\
f & \Rightarrow R_{f,X} \\
AX & \xrightarrow{R_{A,X}}XA.
\end{align*}
$$

The associativity of the compositions of 1-morphisms. There is an associator

$$( (h, R_{h,-}) \circ (g, R_{g,-}) ) \circ (f, R_{f,-}) \Rightarrow (h, R_{h,-}) \circ ((g, R_{g,-}) \circ (f, R_{f,-}))$$

defined by the associator $\alpha_{h,g,f} : (hg)f \Rightarrow h(gf)$ in $B$. It is straightforward to check that $\alpha_{h,g,f}$ gives a well-defined 2-morphism in $Z(B)$, i.e. it satisfies the axiom in (2.3): $\alpha_{h,g,f} \cdot R_{(hg)f,-} = R_{h(gf),-} \cdot \alpha_{h,g,f}$.

Remark 2.1. The associator for the composition of 1-morphisms in the bicategory $B$ is omitted in the diagrams above. By the coherence theorem of bicategories, a pasting diagram of 2-morphisms has a unique value once a choice of the associators has been made.

Remark 2.2. The main difference between our definition and Crans’ definition is that we are working with non-strict bicategories. This non-strictness is absolutely necessary because, as we will show, the braidings intertwine with the associators in a non-trivial way (see Diagram (2.1) and Eq. (3.2)). We do not impose any constraints on the half braidings with the unit object and the identity 1-morphisms.

Step 2: the monoidal structure. We construct a monoidal bicategory $(Z(B), \otimes, I, a, \tilde{a}, \tilde{I}, \tilde{r}, \tilde{\pi}, \tilde{\lambda}, \tilde{\rho})$.

Tensor product of two objects $(A, R_{A,-}, R_{(A)\to,-}) \otimes (B, R_{B,-}, R_{(B)\to,-}) = (AB, R_{AB,-}, R_{(AB)\to,-})$, where $R_{AB,-}$ is an adjoint equivalence given by the composition:

$$(AB) \xrightarrow{a} A(B) \xrightarrow{R_{B,-}} A(-B) \xrightarrow{a^*} (A-)B \xrightarrow{R_{A,-}} (-A)B \xrightarrow{a} -(AB),$$
and \( R_{(AB)[-,-]} \) is an invertible modification:

\[
(2.4)
\]

\[
\begin{array}{c}
\text{Tensor product of an object } \tilde{A} = (A, R_{A,-}, R_{(A)[-,-]}) \text{ and a 1-morphism } (g, R_{g,-}) : (B, R_{B,-}, R_{(B)[-,-]}) \rightarrow (B', R_{B'}-, R_{(B')[-,-]}) \text{ is a 1-morphism } (A\tilde{g}, R_{A\tilde{g},-}) : \tilde{A}B \rightarrow \tilde{A}B', \text{ where } A\tilde{g} : AB \rightarrow AB' \text{ is the 1-morphism in } B, \text{ and } R_{A\tilde{g},-} \text{ is an invertible modification defined by the following diagram:}
\end{array}
\]

\[
\begin{array}{c}
(AB')X \xrightarrow{\alpha} A(B'X) \xrightarrow{\text{R}_{B,-}} A(AB') \xrightarrow{\text{R}_{A,-}} (XA)B' \xrightarrow{\alpha} \text{(AB')Y}
\end{array}
\]

\[
\begin{array}{c}
(AB)X \xrightarrow{\alpha} A(BX) \xrightarrow{\text{R}_{B,-}} A(AB) \xrightarrow{\text{R}_{A,-}} (XA)B \xrightarrow{\alpha} \text{(AB)Y}
\end{array}
\]

where all vertical arrows are 1-morphisms induced by \( \tilde{g} \).

Tensor product of a 1-morphism \((f, R_{f,-}) : \tilde{A} \rightarrow \tilde{A}'\) and an object \( \tilde{B} \) is a 1-morphism \((f\tilde{B}, R_{f\tilde{B},-}) : \tilde{A}\tilde{B} \rightarrow \tilde{A}'\tilde{B}\), where \( f\tilde{B} : AB \rightarrow A'B \) is the 1-morphism in \( B \), and \( R_{f\tilde{B},-} \) is an invertible modification:

\[
\begin{array}{c}
(AB')X \xrightarrow{\alpha} A'X \xrightarrow{\text{R}_{f,-}} A'(AB') \xrightarrow{\alpha} (A'B)X \xrightarrow{\text{R}_{f,-}} X'(AB')
\end{array}
\]

\[
\begin{array}{c}
(AB)X \xrightarrow{\alpha} A'X \xrightarrow{\text{R}_{f,-}} A'(AB) \xrightarrow{\alpha} (A'B)X \xrightarrow{\text{R}_{f,-}} X'(AB)
\end{array}
\]

where all vertical arrows are 1-morphisms induced by \( \tilde{f} \).
The unit \( \tilde{I} = (I, R_{I, -}, R_{(I \rightarrow -)}) \), where \( R_{I, -} \) is an adjoint equivalence \( I \xrightarrow{\ell} - \xrightarrow{\tau} I \), and \( R_{(I \rightarrow -)} \) is an invertible modification:

\[
\begin{align*}
(2.5) \\
\begin{tikzpicture}
\node (A) at (0,0) {\( I(XY) \)};
\node (B) at (2,0) {\( XY \)};
\node (C) at (2,-2) {\( (XY)I \)};
\node (D) at (0,-2) {\( XY \)};
\node (E) at (-1,0) {\( (I)X \)};
\node (F) at (1,0) {\( X(I) \)};
\node (G) at (0,-1) {\( a \)};
\node (H) at (2,-1) {\( r^* \)};
\node (I) at (-1,-2) {\( a \)};
\node (J) at (1,-2) {\( l \)};
\draw[->] (A) to (B);
\draw[->] (B) to (C);
\draw[->] (B) to (D);
\draw[->] (A) to (E);
\draw[->] (A) to (F);
\draw[->] (E) to (I);
\draw[->] (F) to (J);
\draw[->] (I) to (G);
\draw[->] (J) to (H);
\end{tikzpicture}
\end{align*}
\]

An associator \( \tilde{\alpha} : (\tilde{A} \tilde{B})\tilde{C} \rightarrow \tilde{A}(\tilde{B}\tilde{C}) \) is a 1-morphism \( (a, R_{a, -}) \), where \( a : (AB)C \rightarrow A(BC) \) is the associator in \( B \), and \( R_{a, -} \) is an invertible modification:

\[
\begin{align*}
(2.6) \\
\begin{tikzpicture}
\node (A) at (0,0) {\( (AB)(C)X \)};
\node (B) at (2,0) {\( (AB)X \)};
\node (C) at (2,-2) {\( (AB)(X)C \)};
\node (D) at (0,-2) {\( (AB)X \)};
\node (E) at (-1,0) {\( A((BC)X) \)};
\node (F) at (1,0) {\( A(B(C)X) \)};
\node (G) at (0,-1) {\( \alpha \)};
\node (H) at (2,-1) {\( \tau \)};
\node (I) at (-1,-2) {\( \alpha \)};
\node (J) at (1,-2) {\( \equiv \)};
\draw[->] (A) to (B);
\draw[->] (B) to (C);
\draw[->] (B) to (D);
\draw[->] (A) to (E);
\draw[->] (A) to (F);
\draw[->] (E) to (I);
\draw[->] (F) to (J);
\draw[->] (I) to (G);
\draw[->] (J) to (H);
\end{tikzpicture}
\end{align*}
\]

An equivalence \( \tilde{l} : \tilde{I} \tilde{A} \rightarrow \tilde{A} \) is a 1-morphism \( (l, R_{l, -}) \), where \( l : I\tilde{A} \rightarrow \tilde{A} \) is the equivalence in \( \mathcal{B} \), and \( R_{l, -} \) is an invertible modification:

\[
\begin{align*}
(2.7) \\
\begin{tikzpicture}
\node (A) at (0,0) {\( AX \)};
\node (B) at (2,0) {\( XA \)};
\node (C) at (0,-2) {\( (IA)X \)};
\node (D) at (2,-2) {\( X(I)A \)};
\node (E) at (-1,0) {\( I(AX) \)};
\node (F) at (1,0) {\( I(XA) \)};
\node (G) at (0,-1) {\( \tau^* \)};
\node (H) at (2,-1) {\( \mu \)};
\node (I) at (-1,-2) {\( a \)};
\node (J) at (1,-2) {\( \ell \)};
\draw[->] (A) to (B);
\draw[->] (B) to (D);
\draw[->] (B) to (F);
\draw[->] (A) to (C);
\draw[->] (A) to (E);
\draw[->] (E) to (I);
\draw[->] (F) to (J);
\draw[->] (I) to (G);
\draw[->] (J) to (H);
\end{tikzpicture}
\end{align*}
\]
An equivalence \( \tilde{r} : \tilde{A}I \to \tilde{A} \) is a 1-morphism \((r, R_{r,-})\), where \( r : AI \to A \) is the equivalence in \( \mathcal{B} \), and \( R_{r,-} \) is an invertible modification:

\[
\begin{array}{c}
\xymatrix{
AX \ar[r]^{R_{A,X}} & XA \\
(AI)X \ar[r]^{R_{A,I,X}} & (XA)I \ar[r]^{a} & X(AI)
}
\end{array}
\]  

(2.8)

Invertible modifications \( \tilde{\pi}, \tilde{\mu}, \tilde{\lambda}, \tilde{\rho} \) are defined in the same way as in \( \mathcal{B} \). We need to show that they are well-defined 2-morphisms in \( \mathcal{Z}(\mathcal{B}) \), i.e. they satisfy the axiom in (2.3).

We check the case of \( \tilde{\lambda} \) in the following and leave other cases to the reader. The invertible modification \( \tilde{\lambda} : \tilde{l} \Rightarrow \tilde{l} \circ \tilde{a} \) is defined as \( \lambda : l \Rightarrow l \circ a \) in \( \mathcal{B} \). We need to show that the following diagram commutes:

\[
\begin{array}{c}
\xymatrix{
(AB)X \ar[r]^{R_{AB,X}} & X(AB) \\
((IA)B)X \ar[r]^{R_{(IA)B,X}} & X((IA)B)
}
\end{array}
\]  

(2.9)

where the 2-isomorphisms in the front and back are \( R_{l,X} \) and \( R_{loa,X} \), respectively. We decompose the diagram into pieces:

\[
\begin{array}{c}
\xymatrix{
(AB)X \ar[r]^{a} & A(BX) \ar[r]^{R_{B,X}} & A(AXB) \ar[r]^{a^*} & (AX)B \ar[r]^{R_{A,X}} & (XA)B \ar[r]^{a} & X(AB) \\
((IA)B)X \ar[r]^{a} & (IA)(BX) \ar[r]^{R_{B,X}} & (IA)(AXB) \ar[r]^{a^*} & ((IA)X)B \ar[r]^{R_{I,X}} & (X(IA))B \ar[r]^{a} & X((IA)B)
}
\end{array}
\]

The commutativity of each piece follows from the definition of \( R_{l,-} \) in (2.7) and the axioms in \( \mathcal{B} \).

**Step 3: the braiding.**

The braiding of two objects \( \tilde{A} = (A, R_{A,-}, R_{(A|-),?}) \) and \( \tilde{B} = (B, R_{B,-}, R_{(B|-),?}) \) is a 1-morphism \( R_{\tilde{A},\tilde{B}} = (R_{A,B}, R_{R_{A,B},-}) : \tilde{A}B \to \tilde{B}A \) in \( \mathcal{Z}(\mathcal{B}) \), where \( R_{A,B} = R_{A,-}(B) : AB \to BA \) is the adjoint
equivalence in $\mathcal{B}$, and $R_{R_{A,B}^{-1}}$ is an invertible modification:

$$
\begin{aligned}
(BA)X & \xrightarrow{a} B(AX) \xrightarrow{R_{A,X}} B(XA) = (BX)A \xrightarrow{a} (XB)A \xrightarrow{R_{B,X}} X(BA) \\
\Rightarrow R_{(A|B,X)} & \Rightarrow R_{A,-}(R_{B,X}) \Rightarrow R_{A,X,B} \Rightarrow R_{A,B}
\end{aligned}
$$

The braiding of an object $\tilde{A} = (A, R_{A,-}, R_{(A|-,-)})$ and a 1-morphism $(g, R_{g,-})$ is an invertible modification $R_{A,-}(g)$. The braiding of a 1-morphism $(f, R_{f,-})$ and an object $\tilde{B} = (B, R_{B,-}, (B|-,-))$ is an invertible modification $R_{f,-}(B)$.

Two invertible modifications

$$R_{(\tilde{A}|\tilde{B},\tilde{C})} = R_{(A|-,-)}(B, C) = R_{(A|B,C)},$$

and $R_{(\tilde{A},\tilde{B}|\tilde{C})}$ is given by:

$$A(CB) \xrightarrow{a^*} (AC)B \xrightarrow{R_{B,C}} (AC)B \xrightarrow{R_{A,C}} (CA)B$$

So $R_{(\tilde{A},\tilde{B}|\tilde{C})}$ only differs from the identity by the two units $id_{A|B,C} \Rightarrow aa^*$ and $id_{(C|A)B} \Rightarrow a^*a$.

**Theorem 2.3.** The center $Z(\mathcal{B})$ defined above is a braided monoidal bicategory.

**Proof.** See [G1] Section 2.4 for Gurski’s definition of a braided monoidal bicategory. Step 2 makes $Z(\mathcal{B})$ a monoidal bicategory. The adjoint equivalence $R : \otimes \Rightarrow \otimes \circ \tau$ in $\text{Bicat}(Z(\mathcal{B}) \times Z(\mathcal{B}), Z(\mathcal{B}))$, and the invertible modifications $R_{(\tilde{A}|\tilde{B},\tilde{C})}, R_{(\tilde{A},\tilde{B}|\tilde{C})}$ are defined in Step 3.

The four axioms are about 2-isomorphisms in $\text{Hom}((\tilde{A}\tilde{B})\tilde{C})\tilde{D}, \tilde{D}((\tilde{A}\tilde{B})\tilde{C}))$, $\text{Hom}(\tilde{A}(\tilde{B}\tilde{C})\tilde{D}), ((\tilde{B}\tilde{C})\tilde{D})\tilde{A})$, $\text{Hom}((\tilde{A}\tilde{B})(\tilde{C}\tilde{D}), (\tilde{C}\tilde{D})(\tilde{A}\tilde{B}))$, and $\text{Hom}(\tilde{A}\tilde{B})(\tilde{C}\tilde{D}), (\tilde{C}\tilde{D})(\tilde{A}\tilde{B}))$, respectively. The first one follows from the definition of $R_{a,-}$ in the associator $\tilde{a} : (\tilde{A}\tilde{B})\tilde{C} \rightarrow \tilde{A}(\tilde{B}\tilde{C})$ as in (2.7). The second is the same as the axiom in (2.11). The third follows from the definition of $R_{AB,-}$ in the tensor product $\tilde{A}\tilde{B}$ as in (2.4). The last one follows from the definition of $R_{R_{A,B},-}$ in the braiding $\tilde{A}\tilde{B}$ as in (2.11). \[\square\]
A monoidal bicategory is defined as a tricategory with one object. We refer to [GI, Section 4.1] for the definition of tricategories.

Let $\mathcal{V}$ be the 1-category of finite dimensional $k$-vector spaces (i.e., $1Vec$). Let $2Vec$ be the 2-category of 1-categories of finite semisimple $\mathcal{V}$-module categories [Os1]. More precisely, objects in $2Vec$ are of the form $\mathcal{V}^{\oplus n}$, where $\oplus$ is the direct sum; 1-morphisms are the $\mathcal{V}$-module functors; 2-morphisms are $\mathcal{V}$-module natural transformations. The only simple object is $\mathcal{V}$ whose endomorphism 1-category $\text{End}(\mathcal{V}) \cong \mathcal{V}$. The tensor product $\boxtimes$ in $2Vec$ is the Deligne tensor product.

Consider the monoidal bicategory $(2Vec^\omega_G, \boxtimes, I, a, l, r, \pi, \mu, \rho, \lambda)$. The underlying category is a 2-category, i.e., the associators for the compositions of 1-morphisms are all trivial. It is isomorphic to a direct sum of $\mathcal{V}$-copies of $\boxtimes$ tensor product $V_R$ for any $g$ algebras $\mathbb{Z}$.

The invertible modifications $A$ of 1-categories of finite semisimple $\mathcal{V}$-module categories $\omega$ are the 1-category of finite dimensional $k$-vector spaces $1Vec$-module categories $\omega$. More precisely, objects in $2Vec$ are of the form $\mathcal{V}^{\oplus n}$, where $\oplus$ is the direct sum; 1-morphisms are the $\mathcal{V}$-module functors; 2-morphisms are $\mathcal{V}$-module natural transformations. The only simple object is $\mathcal{V}$ whose endomorphism 1-category $\text{End}(\mathcal{V}) \cong \mathcal{V}$. The tensor product $\boxtimes$ in $2Vec$ is the Deligne tensor product.

Consider the monoidal bicategory $(2Vec^\omega_G, \boxtimes, I, a, l, r, \pi, \mu, \lambda)$. The underlying category is a 2-category, i.e., the associators for the compositions of 1-morphisms are all trivial. It is isomorphic to a direct sum of $|G|$ copies of $2Vec$, and the simple objects are $\delta_g$ for $g \in G$. Any object is of the form $A = \bigoplus_{g \in G} A_g$, where $A_g \in 2Vec$ is the $g$-component.

Tensor product of two simple objects is $\delta_g \boxtimes \delta'_g = \delta_{gg'}$. The unit object $I = \delta_1$. The adjoint equivalences $\alpha, \beta, \gamma$ are all identities (i.e., $a, l, r$ and the 2-isomorphisms defining their naturalities are all identities). The invertible modifications $\rho$ and $\lambda$ are determined by $\pi, \mu$ and the axioms. So the monoidal structure is completely determined by $\pi$ and $\mu$. Moreover, $\pi$ is described by a cocycle $\omega \in Z^2(G, k^\times)$:

$$
\begin{align*}
((\delta_x \delta_y) \delta_z) \delta_w & \Rightarrow (\delta_x (\delta_y \delta_z)) \delta_w \\
& \Rightarrow \delta_x ((\delta_y \delta_z) \delta_w) \\
& \Rightarrow \delta_x (\delta_y (\delta_z \delta_w))
\end{align*}
$$

and $\mu$ is described by a 2-cocochain $C^2(G, k^\times)$ which satisfies certain compatibility conditions with $\omega$. We restrict ourselves to the normalized case: (1) $\omega$ is a normalized cocycle, i.e., $\omega(x_1, x_2, x_3, x_4) = 1$ if $x_i = 1$ for some $i$; and (2) the 2-cocochain $\mu$ is trivial, i.e., $\mu(x_1, x_2) = 1$ for all $x_i$. In this case, $\mu, \rho, \lambda$ are all trivial so that the unit is strict. In particular, it means that the invertible modifications defined by (2.5), (2.7), (2.8) are all identities. As a consequence, the diagram (2.9) is automatically commutative.

**Remark 3.1.** It is expected that isomorphism classes of monoidal structures on $2Vec_G$ are classified by $H^4(G, k^\times)$. Any class in $H^4(G, k^\times)$ has a normalized representative. So our restriction to the normalized case is inessential.

### 3.1. The 2-category

We first compute $Z(2Vec^\omega_G)$ as a 2-category. Let $\hat{A} = (A, R_{A,\cdot}, R_{(A|\cdot\cdot\cdot)})$ be an object of $Z(2Vec^\omega_G)$. The half braiding $R_{A,\cdot}$ gives an equivalence of categories $R_{A,g} : A \boxtimes \delta_g \rightarrow \delta_g \boxtimes A$, for any $g \in G$. Moreover, $R_{A,\cdot}(id_{\delta_g}) = id_{R_{A,g}}$ since $2Vec^\omega_G$ is a 2-category. The equation $R_{A,X} \boxtimes Y = R_{A,X} \boxtimes R_{A,Y}$ implies that $R_{A,\cdot}$ is completely determined by the collection $\{R_{A,g}\}$. 

Let $\text{Cl}$ denote the set of conjugacy classes of $G$. We write $h \in c$ and $[h] = c$ if $h \in G$ is in a conjugacy class $c \in \text{Cl}$. Any object of $Z(2\text{Vec}_G^c)$ has a direct sum decomposition $\hat{A} = \bigoplus_{c \in \text{Cl}} \hat{A}_c$ into its $c$-components due to the half braiding. It induces a decomposition $Z(2\text{Vec}_G^c) = \bigoplus_{c \in \text{Cl}} Z(2\text{Vec}_G^c)_c$ of the 2-category.

3.1.1. The component $Z(2\text{Vec}_G^c)_c$. We give an explicit description of one component $Z(2\text{Vec}_G^c)_c$ following Step 1 in Section 2. Let $\{h_1, \ldots, h_s\}$ denote all elements of $G$ in the class $c$.

**Objects.** For an object $\hat{A}_c = (A_c, \tau_{A_c, \ldots, \tau_{A_c, \ldots}})$, its underlying object $A_c = \bigoplus_i A_{h_i}$ in $2\text{Vec}$. The half braiding is a collection of equivalences

$$A_{h_i} \delta_g \delta_{g'} \rightarrow \delta_g \delta_{g'} A_{h_k}$$

for $h_i g = g h_j$. The invertible modification $R_{(A_c|g,g')}: \bigoplus_i R_{(h_i|g,g')}$:

$$R_{h_i,g} \delta_g \rightarrow \delta_g A_{h_k}$$

for $h_i g = gh_j, h_j g' = g'h_k$. Here we omit 1-associators which are all identities. The modifications $R_{(h_i|g,g')}, R_{(h_j|g,g')}, R_{(h_i|g',g'')}, R_{(h_j|g,g'')} $ together with the 4-cocycle $\pi$ should satisfy the axiom in (2.1), for $A = A_{h_i}, X = \delta_g, Y = \delta_{g'}, Z = \delta_{g''}$. All adjoint equivalences $\omega$ are identities so that the four isomorphisms $\gamma \omega$ are identities. This axiom gives an equation of the 2-isomorphisms:

$$R_{(h_i|g,g')} \cdot R_{(h_j|g',g'')} = \tau_{h_i}(\omega(g, g', g'') \cdot R_{(h_i|g,g')}) \cdot R_{(h_j|g,g')},$$

for $h_i g = gh_j, h_j g' = g'h_k, h_k g'' = g''h_i$, and

$$\tau_{h_i}(\omega(g, g', g'')) = \frac{\omega(h_i, g, g', g'') \omega(g, g', h_k, g'')}{\omega(g, g', g'') \omega(g, g', g'', h_i)}.$$

We introduce a handy notation for Equation (3.2): Eq($h_i|g,g',g'')$. It is a consequence of the axiom in (2.1), which can be simplified by omitting 1- and 2-associators as follows:

$$A_{h_i} \delta_g \delta_{g'} \delta_{g''} R_{h_i,g,g''} \rightarrow \delta_g \delta_{g'} \delta_{g''} A_{h_i}.$$
1-morphisms. A 1-morphism is \((f, R_{f,-}) : (A_c, R_{A,-}, R_{(A|-,-)}) \rightarrow (A'_c, R'_{A'_-, -}, R'_{(A'_|-,-)})\) consists of a 1-morphism \(f = \boxplus_i f_i, f_i : A_{h_i} \rightarrow A'_{h_i}\), and an invertible modification \(R_{f,g} = \boxplus_i R_{f_i,g_i}\):

\[
\begin{align*}
A'_h \delta g & \xrightarrow{R'_{h,-}} \delta g A'_{h_j} \\
f_i & \Rightarrow R_{f_i,g_i} \xrightarrow{f_j} \\
A_h \delta g & \xrightarrow{R_{h,-}} \delta g A_{h_j}.
\end{align*}
\]

The invertible modifications \(R_{f_i,g}, R_{f_j,g'}, R_{f_i,g''}\) should satisfy the axiom in (2.2) for \(A = A_{h_i}, A' = A'_{h_i}\), \(X = \delta g, Y = \delta g'. \) The axiom is simplified to the following diagram by omitting identity 1-associators:

\[
\begin{align*}
A'_{h_i} \delta g \delta g' & \xrightarrow{\delta g \delta g' A'_{h_k}} \\
\downarrow f_i & \Rightarrow R_{f_i,g} \xrightarrow{f_k} \\
\downarrow \delta g A'_{h_j} \delta g' & \Rightarrow R_{f_j,g'} \xrightarrow{f_k} \\
A_h \delta g \delta g' & \xrightarrow{\delta g \delta g' A_h} \\
\downarrow \delta g A_{h_j} \delta g' & \Rightarrow R_{h, g} \xrightarrow{\delta g A_{h_j} \delta g'}
\end{align*}
\]

where the 2-isomorphism in the back is \(R_{f_i,g} g'\). We denote this compatibility condition for 1-morphisms as \(\text{Eq1}(h_i|g, g')\).

2-morphisms. A 2-morphism \(\alpha : (f, R_{f,-}) \Rightarrow (f', R_{f',-})\) is a 2-morphism \(\alpha = \boxplus_i \alpha_i, \alpha_i : f_i \Rightarrow f'_i\) which satisfies the axiom in (2.3) for \(A = A_{h_i}, A' = A'_{h_i}, X = \delta g, Y = \delta g'\):

\[
\alpha_j : R_{f_i,g} = R'_{f'_i,g} \cdot \alpha_i.
\]

We denote this compatibility condition for 2-morphisms as \(\text{Eq2}(h_i|g)\).

3.1.2. The restriction to one grading. For an object \(\tilde{A}_c\), its underlying object \(A_c = \boxplus_{h \in c} A_h\) in \(2\text{Vec}\), where \(A_h\) are all equivalent to each other by the requirement of the half braiding. We pick up a grading \(h \in c\), and let \(C_G(h) = \{g \in G|gh = hg\}\) denote the centralizer of \(h\) in \(G\). We focus on the component \(A_h\) and the half braiding with \(\delta_x\) for \(x \in C_G(h)\) in the following.

For \(x \in C_G(h)\), the equivalence \(R_{h,x} : A_h \delta_x \rightarrow \delta_x A_h\) induces an autoequivalence of \(A_h\):

\[
\rho_x : A_h \rightarrow A_h \delta_x \xrightarrow{R_{h,x}} \delta_x A_h \rightarrow A_h,
\]

where the first and last maps are grading shifts in \(2\text{Vec}_G\) which are identities in \(2\text{Vec}\).
For \( x, y \in C_G(h) \), the 2-modification \( R_{(h|x,y)} \) as in (3.1) induces a 2-isomorphism \( m(x, y) : \rho_y \rho_x \Rightarrow \rho_{xy} \) by taking the natural grading shifts to \( A_h \). Thus, the collection \( \{ \rho_x \mid x \in C_G(h) \} \) gives a weak right action of \( C_G(h) \) on the 1-category \( A_h \).

For \( x, y, z \in C_G(h) \), the modifications \( R_{(h|xy,z)}, R_{(h|x,y)}, R_{(h|y,z)}, R_{(h|x,z)} \) satisfy Eq (3.6):

\[
R_{(h|x,yz)} \cdot R_{(h|y,z)} = \frac{\omega(h,x,y,z)\omega(x,y,h,z)}{\omega(x,h,y,z)\omega(x,y,z,h)} R_{(h|xy,z)} \cdot R_{(h|x,y)}.
\]

Note that \( h_i = h_j = h_k = h_l = h \) in this case. Translating to the weak action of \( C_G(h) \) on \( A_h \), the 2-isomorphisms satisfy the following equation:

\[
m(x, yz) \cdot m(y, z) = \frac{\omega(h,x,y,z)\omega(x,y,h,z)}{\omega(x,h,y,z)\omega(x,y,z,h)} m(xy,z) \cdot m(x,y).
\]

The action is associative up to a twisting determined by \( \omega \in Z^4(G, k^\times) \).

Consider the transgression map \( \tau_h : C^{k+1}(G, k^\times) \rightarrow C^k(C_G(h), k^\times) \) defined by:

\[
\tau_h(\omega)(x_1, \ldots, x_k) = \prod_{0 \leq i \leq k} \omega(x_1, \ldots, x_i, h, x_{i+1}, \ldots, x_k)(-1)^i,
\]

for \( x_i \in C_G(h) \). It is straightforward to check that \( \tau_h \) is a chain map. It induces a map between cohomologies which is still denoted by \( \tau_h \). We are mainly interested in the case of \( k = 3 \).

Equation (3.7) can be rewritten as

\[
m(x, yz) \cdot m(y, z) = \tau_h(\omega)m(xy, z) \cdot m(x, y).
\]

It follows that \( A_h \in 2\text{Rep}(C_G(h), \tau_h(\omega)) \), i.e. it is a right module category over the monoidal 1-category \( 1\text{Vec}_{C_G(h)} \). So there is a forgetful map \( Z(2\text{Vec}_G) \rightarrow 2\text{Rep}(C_G(h), \tau_h(\omega)) \) by taking its \( h \)-component.

On the level of morphisms, a 1-morphism \( (f, R_{f, \gamma}) \) restricts to a collection \( \{ R_{f, \rho} : A_h \gamma \rightarrow A'_h \delta \mid x \in C_G(h) \} \) of 2-isomorphisms. This collection defines a 1-morphism in \( 2\text{Rep}(C_G(h), \tau_h(\omega)) \). Similarly, 2-morphisms in \( Z(2\text{Vec}_G) \) restricts to 2-morphisms in \( 2\text{Rep}(C_G(h), \tau_h(\omega)) \). To sum up, we have a forgetful 2-functor

\[
\Phi_h : Z(2\text{Vec}_G) \rightarrow 2\text{Rep}(C_G(h), \tau_h(\omega))
\]

by restricting to the \( h \)-component.

### 3.1.3. The equivalence of the forgetful functor

We show that the forgetful functor \( \Phi_h \) is an equivalence of 2-categories in the following. Fix a set of representatives \( \{ g_i \in G \mid i = 1, \ldots, s \text{ and } g_1 = 1 \} \) for the coset \( C_G(h) \backslash G \). Then \( \{ h_i = g_i^{-1}h g_i \mid i = 1, \ldots, s \} \) are all elements in \( c \), and \( h_1 = h \) is the base point. We construct a 2-functor \( \Psi_h : 2\text{Rep}(C_G(h), \tau_h(\omega)) \rightarrow Z(2\text{Vec}_G) \) in the inverse direction by extending the action of \( C_G(h) \) on \( A_h \) to that of \( G \) on \( A_c \).

**Step 1: Objects.** Let \( M = (M, \rho_x, m(x, y)) \) be an object of \( 2\text{Rep}(C_G(h), \tau_h(\omega)) \), where \( \rho_x \) is the action and \( m(x, y) \) is the 2-modification. We want to extend \( \rho_x, m(x, y) \) from the \( h \)-component to \( h_i \)-component
via the path determined by $g_i$. Define $\Psi_h(M) = (M_c, R_{M,-}, R_{(M|-\gamma)})$ as

$$M_c = \biguplus_i M_{h_i}, \quad M_{h_i} = M,$$

$$R_{M,g} = \biguplus_i R_{h_{i,g}}, \quad R_{h_{i,g}} : M_{h_{i,g}} \xrightarrow{\delta_g} M \xrightarrow{\rho_x} M \xrightarrow{\delta_y} M_{h_{i,j}},$$

where given $i,j$ and $g \in G$, there is a unique $x \in C_G(h)$ such that $g_i g = x g_j$. The 2-modification $R_{(M|g,g')} = \biguplus_i R_{(h_{i,g},g')}$, and $R_{(h_{i,g},g')}$ is defined in the following order:

$$R_{(h|x,y)}, R_{(h|x,g)}, R_{(h|g,g)}, R_{(h|g,g')}, R_{(h_{i,g},g')},$$

where $x, y \in C_G(h)$ and $g, g' \in G$. The initial data is to define $R_{(h|x,y)} = m(x, y)$ and choose any 2-isomorphisms for $R_{(h|x,g)}, R_{(h|g,g)}$ only requiring that $R_{(h|g,1)} = R_{(h|1,g)}$. Eq$(h|x, y, g_i)$ involves four 2-isomorphisms:

$$R_{(h|x,y,g_i)}, R_{(h|y,g_i)}, R_{(h|x,y)}, R_{(h|xy,g_i)}.$$

So $R_{(h|x,g)}$ for $g = y g_i$ is determined by the other three isomorphisms which are already given. Similarly, Eq$(h|x, y, g_i, g')$ uniquely determines $R_{(h|g, g')}$ for $g = x g_i$, and Eq$(h|g_i, g, g')$ uniquely determines $R_{(h_{i,g}, g')}$.

**Lemma 3.2.** The construction $(M_c, R_{M,-}, R_{(M|-\gamma)})$ gives a well-defined object of $Z(\text{2Vec}_G)$.c.

**Proof.** By definition it suffices to show that Eq$(h|g, g', g'')$ in (3.2) holds for all $g, g', g'' \in G$ and all $i = 1,\ldots, s$. The key point is that there is a compatibility condition between Equations

$$\text{Eq}(h|g, g', g''), \text{Eq}(h|g, g', g'' g'''), \text{Eq}(h|g, g' g'', g'''), \text{Eq}(h|g, g', g'''), \text{Eq}(h|g, g', g'''')$$

from the axiom (2.4) for $M_{h_{i,g}} \xrightarrow{\delta_g} M_{h_{i,g}} \xrightarrow{\delta_g} M_{h_{i,j}}$, where $h_i g = gh_j$. We denote this compatibility condition by CC$(h|g, g', g''', g''')$. If any four of the five equations hold then so is the remaining one. We prove that Eq$(h|g, g', g'')$ holds in the following order: (1) $(h|x, y, z), (h|x, y, g_i), (h|x, z, g_i), (h|g, g_i, g'), (h|g, g_i, g')$, and (2) $(h|x, y, g_i), (h|x, g, g'), (h|g, g', g''), (h|g, g', g''')$, where $x, y, z \in C_G(h), g, g', g'' \in G$. The equations in the first group holds from the construction. The condition CC$(h|x, y, z, g_i)$ implies that Eq$(h|x, y, g)$ holds for $g = z g_i$ since the other four equations Eq$(h|x, y, z), Eq(h|x, y, z, g_i), Eq(h|x, y, z, g_i)$, hold. Similarly, the condition CC$(h|x, y, g_i, g')$ implies that Eq$(h|x, g, g')$ holds for $g = y g_i; \text{CC}(h|x, g_i, g', g'')$ implies that Eq$(h|g, g', g'')$ holds for $g = x g_i$; and CC$(h|g_i, g, g', g'')$ implies that Eq$(h|g, g', g'')$ holds. \(\square\)

**Step 2: 1-morphisms.** Let $(f, M_{f,x}) : (M, \rho_x, m(x, y)) \rightarrow (M', \rho'_x, m'(x, y))$ denote a 1-morphism in $2\text{Rep}(C_G(h), \tau_h(\omega))$, where $f : M \rightarrow M'$, and $M_{f,x}$ is the 2-modification for $x \in C_G(h)$. We define $\Psi_h(f, M_{f,x}) = \biguplus_i (f_{i}, R_{f_{i,-}}) : \Psi_h(M) \rightarrow \Psi_h(M')$, where $f_i : M_{h_i} \xrightarrow{\rho_x} M \xrightarrow{\rho'_x} M' \xrightarrow{\rho_x} M_{h_{i'}}$, and $R_{f_{i,g}}$ is the 2-modification for $g \in G$ given below.

The only constraint for a 1-morphism is Eq1$(h|g, g')$ in (3.4) for $A_{h_i} = M_{h_i}, A'_{h_i} = M'_{h_i}$. Eq1$(h|g, g')$ contains five terms $R_{f_{i,g}}, R_{f_{i,g}'}, R_{f_{i,g}''}, R_{(h|g,g')}, R'_{(h|g,g')}$, where $h_i g = gh_j$, and the last two terms are already given. For the first three terms, any two of them determines the remaining one.
We define $R_{f_i,g}$ in the following order: $R_{f_1,x}, R_{f_1,g}, R_{f_1,g}, R_{f_1,g}$ for $x \in C_G(h), g \in G$. Note that $h_1 = h$ is the base point. The initial data is to define $R_{f_1,x} = M_{f,x}$ and $R_{f_1,g} = id$ for all $i = 1, \ldots, s$. Eq(1) $h_1, g, i$ implies that $R_{f_1,x}$ for $g = xg_i$ is uniquely determined by $R_{f_1,x}$ and $R_{f_1,g}$. Eq(1) $h_1, g, i$ implies that $R_{f_1,g}$ is uniquely determined by $R_{f_1,g}$ and $R_{f_1,g}$.

An argument similar to the proof of Lemma 3.2 shows that $\Psi(h(f, R_{f,x})) = \bigoplus (f_i, R_{f_i})$ gives a well-defined $1$-morphism in $Z(2Vec_G^c)$ for each class $c \in Cl$. It suffices to show that Eq(1) $h_1, g, g'$ holds for all $g, g' \in G$. There is a compatibility condition between

$$Eq1(h_1, g, g'), Eq1(h_1, g, g'), Eq1(h_1, g, g'), Eq1(h_1, g, g')$$

from (1) for $M_{h_1, g, g', g''}$. We denote this compatibility condition as $CC1(h_1, g, g', g')$. If any three of the four constraints hold then so is the remaining one. We prove that Eq(1) $h_1, g, g'$ holds in the following order: (1) $h_1, g, g'$, (2) $h_1, g, g'$, (3) $h_1, g, g'$, (4) $h_1, g, g'$, where $x, y \in C_G(h), g, g \in G$.

The constraints in the first group holds from the construction. The condition $CC1(h_1, x, y, g, g)$ implies that Eq(1) $h_1, x, y$ holds for $g = yg_i$. $CC1(h_1, x, g, g')$ implies that Eq(1) $h_1, g, g'$ holds for $g = xg_i$. $CC1(h_1, g, g', g')$ implies that Eq(1) $h_1, g, g'$ holds.

**Step 3: 2-morphisms.** Let $\alpha : (f, M_{f,x}) \Rightarrow (f', M_{f', x})$ be a 2-morphism in $2Rep(C_G(h), \tau_h(\omega))$. We define $\Psi_h(\alpha) = \bigoplus \alpha_i : \Psi_h(f, M_{f,x}) \Rightarrow \Psi_h(f', M_{f', x})$, where $\alpha_i : f_i \Rightarrow f'_i$ is given below. The only constraint for a 2-morphism is Eq(2) $h_1, g, g'$ in (3,5). The term $\alpha_i$ is determined by $\alpha_i$ since $R_{f_1,g}$ and $R_{f'_i,g}$. are isomorphisms.

We define $\alpha_1 = \alpha$ as the 2-morphism in $2Rep(C_G(h), \tau_h(\omega))$, and define $\alpha_i$ from $\alpha_1$ and Eq(2) $h_1, g, i$ for $i = 2, \ldots, s$. A similar argument shows that $\Psi_h(\alpha) = \bigoplus \alpha_i$ gives a well-defined 2-morphism in $Z(2Vec_G^c)$.

We complete the definition of the 2-functor $\Psi_h : 2Rep(C_G(h), \tau_h(\omega)) \Rightarrow Z(2Vec_G^c)$.

To show that $\Phi_h$ and $\Psi_h$ give an equivalence of 2-categories, it is obvious that $\Phi_h \circ \Psi_h$ is the identity 2-functor. It remains to show that $\Psi_h$ is essentially surjective and fully faithful. The proof is similar to the construction of $\Psi_h$ above and we leave it to the reader.

**Theorem 3.3.** There is an equivalence of 2-categories:

$$Z(2Vec_G^c) \simeq \bigoplus_{h \in Cl} 2Rep(C_G(h), \tau_h(\omega)),$$

by choosing one representative $h$ for each class $c \in Cl$. In particular, $Z(2Vec_G^c)$ is semisimple in the sense of Douglas and Reutter [DR].

Any object $\tilde{A}_c$ of $Z(2Vec_G^c)$ is determined by one of its component $A_h$ as an object of $2Rep(C_G(h), \tau_h(\omega))$ from Theorem 3.3. It is known that any indecomposable object of $2Rep(C_G(h), \tau_h(\omega))$ is given by a pair $(H, \psi)$, where $H$ is a subgroup of $C_G(h), \psi \in C^2(H, k^\times)$ such that $d\psi = \tau_h(\omega)^{-1}|_H$. Note that we consider right modules over $1Vec_G^\tau(\omega)$ instead of left modules. More precisely, the object associated to $(H, \psi)$ is $\bigoplus_{s \in H \setminus C_G(h)} \mathcal{V}(s)$, where each component $\mathcal{V}(s) = \mathcal{V}$. The action of $1Vec_G^\tau(\omega)$ is given
by multiplication in \(C_G(h)\) on the right. The stably of \(V(1)\) is equivalent to \(1\text{Vec}_H\), and \(\psi\) determines its 1-associator. Let \(V(H \setminus K) = \bigoplus_{s \in H \setminus K} V(s)\) for \(H < K\).

We express any indecomposable object \(A_c\) as

\[
A(h, H, \psi), \quad \text{where} \ [h] = c, H < C_G(h), \psi \in C^2(H, k^\times), d\psi = \tau_h(\omega)^{-1}|_H.
\]

The presentation is independent of the choice of \(h \in c\): \(A(h, H, \psi) \cong A(g^{-1}hg, g^{-1}Hg, g^*(\psi))\), where \(g^*(\psi) \in C^2(g^{-1}Hg, k^\times)\) is induced by conjugation.

### 3.2. The braided monoidal bicategory

Before we compute the tensor product \(A(h, H, \psi) \boxtimes A(h', H', \psi')\), we first forget about the grading. We have \(A(h, H, \psi) \cong V(H \setminus G)\) as objects in \(2\text{Vec}\). The half braiding induces a weak action of \(1\text{Vec}_G\) on \(V(H \setminus G)\) which is given by multiplication in \(G\) on the right. The tensor product of two weak right \(1\text{Vec}_G\) module categories is given by the Deligne tensor product, and we have

\[
\mathcal{V}(H \setminus G) \boxtimes \mathcal{V}(H' \setminus G) \cong \bigoplus_{t \in H \setminus G / H'} \mathcal{V}(H_t \setminus G),
\]

where the sum is over the double coset \(H \setminus G / H'\), and \(H_t = t^{-1}Ht \cap H'\).

A direct computation from (2.24) shows that \(A(h, H, \psi) \boxtimes A(h', H', \psi')\) contains a component \(A(h_t, H_t, \psi_t)\), where \(t \in H \setminus G / H'\), \(h_t = t^{-1}hth'\), \(H_t = t^{-1}Ht \cap H'\), and

\[
\psi_t = t^*(\psi)|_{H_t} \cdot \psi'|_{H_t} \prod_{0 \leq i \leq j \leq 2} \chi_{ij,t}^{-1} \in C^2(H_t, k^\times).
\]

Here \(\psi_{ij,t}(x_1, x_2) = \omega(\ldots, x_i, t^{-1}ht, \ldots, x_j, h', \ldots)\), for \(0 \leq i \leq j \leq 2\). The underlying 2-category of \(A(h_t, H_t, \psi_t)\) is precisely \(\mathcal{V}(H_t \setminus G)\) in (3.8).

**Lemma 3.4.** Given \(A(h, H, \psi), A(h', H', \psi')\) and \(t \in H \setminus G / H'\), we have \(H_t < C_G(t^{-1}hth')\) and \(d\psi_t = \tau_{t^{-1}hth'}(\omega)^{-1}|_{H_t}\).

**Proof.** We only check the case of \(t = 1\). We have \(H_1 = H \cap H' < C_G(h) \cap C_G(h') < C_G(hh')\). Consider the trivial cochain \(\chi_{kl} = 1 \in C^3(H \cap H', k^\times)\) : \(\chi_{kl}(x_1, x_2, x_3) = d\omega(\ldots, x_k, h, \ldots, x_l, h', \ldots)\), for \(0 \leq k \leq l \leq 3\). A direct computation shows that

\[
\tau_{hh'}(\omega)\tau_h(\omega)^{-1}\tau_{h'}(\omega)^{-1} \prod_{0 \leq i \leq j \leq 2} d\psi_{ij}^{(-1)^{i+j}} = \prod_{0 \leq k \leq l \leq 3} \chi_{kl}^{(-1)^{k+l}} = 1,
\]

when restricting to \(H \cap H'\). So \(d\psi_1 = \tau_{hh'}(\omega)^{-1}|_{H \cap H'}\). \(\square\)

**Proposition 3.5.** The tensor product of two indecomposable objects in \(\mathcal{Z}(2\text{Vec}_G)\) is given by

\[
A(h, H, \psi) \boxtimes A(h', H', \psi') \cong A(h_t, H_t, \psi_t),
\]

where the sum is over \(t \in H \setminus G / H'\).

**Proof.** Lemma 3.4 implies that \(A(h_t, H_t, \psi_t)\) is well-defined. Each component of the right hand side appears in the tensor product at least once. It follows from (3.8) that each of them appears at most once. \(\square\)
The 1-associators in $2\text{Vec}_{G}^{\omega}$ are all identities. In the contrast, a 1-associator $\tilde{a} : (\tilde{A}B)\tilde{C} \to \tilde{A}(\tilde{B}\tilde{C})$ is a 1-morphism $(a, R_{A,-})$, where $a : (AA')A'' \to A(A'A'')$ is the identity in $2\text{Vec}_{G}^{\omega}$, and $R_{A,-}$ is an invertible modification given by Diagram (2.6) which might be nontrivial. The associators $\tilde{l}, \tilde{r}$ are all identities since the 4-cocycle $\omega$ is normalized.

Invertible modifications $\tilde{\pi}, \tilde{\mu}, \tilde{\lambda}, \tilde{\rho}$ are defined in the same way as in $2\text{Vec}_{G}^{\omega}$. In particular, $\tilde{\mu}, \tilde{\lambda}, \tilde{\rho}$ are all identities, and $\tilde{\pi}$ is given by $\omega$.

The braiding of two objects $\tilde{A} = (A, R_{A,-}, R_{(A|-,-)})$ and $\tilde{B} = (B, R_{B,-}, R_{(B|-,-)}')$ is a 1-morphism $R_{\tilde{A},\tilde{B}} = (R_{A,B}, R_{R_{A,-},B,-}) : \tilde{A}\tilde{B} \to \tilde{B}\tilde{A}$ in $Z(2\text{Vec}_{G}^{\omega})$, where $R_{A,B} = R_{A,-}(B) : AB \to BA$ is determined by the half braiding associated to $\tilde{A}$ and the grading of $B$, and $R_{R_{A,-},B,-}$ is an invertible modification given in Diagram (2.10). More precisely, $R_{A,B} = \sqcup R_{h_{i,g}}$:

$$R_{h_{i,g}} : A_{h_{i}}B_{g} \to B_{g}A_{h_{j}}$$

$(x, y) \mapsto (y, \rho_{g}(x))$,

for $x \in A_{h_{i}}, y \in B_{g}$, and $\rho_{g} : A_{h_{i}} \to A_{h_{j}}$ is the action of $G$ on $A$ for $h_{i}g = gh_{j}$.

When $B$ is concentrated in the grading 1, we have

$$R_{A,B} = \Sigma_{A,B} : AB \to BA$$

where $\Sigma_{A,B}$ is the canonical permutation equivalence between the Deligne tensor products which simply permutes the two factors $A$ and $B$ as objects of $2\text{Vec}$.

**Remark 3.6.** The naturality 2-isomorphism $R_{A,f}$ associated to a 1-morphism $f : B \to B'$ is the identity when $B$ and $B'$ are concentrated in the grading 1.

The invertible modifications $R_{(A|B,C)} = R_{(A|-|C)}(B,C) = R_{(A|B,C)}$ is given by the half braiding associated to $\tilde{A}$, and $R_{h_{i},(B|-)}$ is the identity as in Diagram (2.12) since the 1-associators are the identities.

In summary, $Z(2\text{Vec}_{G}^{\omega})$ is a braided monoidal bicategory whose underlying bicategory is given in Theorem 3.3 and the monoidal structure is given by Proposition 3.5 and the braiding structure is given by the half-braidings as explained in Step 3 in Section 2.

**Example 3.7.** Consider $G = \mathbb{Z}_2 = \{1, s\}, \omega = 1$. There are two conjugacy classes: $h = 1, h = s$. We have an equivalence $Z(2\text{Vec}_{\mathbb{Z}_2}^{\omega}) = Z(2\text{Vec}_{\mathbb{Z}_2}^{s\omega}) \cong 2\text{Rep}(\mathbb{Z}_2) \boxtimes 2\text{Rep}(\mathbb{Z}_2)$ of 2-categories from Theorem 4.3. Up to isomorphism $2\text{Rep}(\mathbb{Z}_2)$ has two indecomposable objects: the unit $I$ and the regular representation $T = 1\text{Vec}_{\mathbb{Z}_2}$. A complete set of isomorphism classes of indecomposable objects of $Z(2\text{Vec}_{\mathbb{Z}_2}^{\omega})$ is $\{I, T, I_{s}, T_{s}\}$, where the subscript $s$ denotes the nontrivial grading.

The nontrivial 1-categories of 1-morphisms are

$$\text{End}(I) \simeq \text{Rep}(\mathbb{Z}_2), \text{End}(T) \simeq 1\text{Vec}_{\mathbb{Z}_2}, \text{Hom}(I, T) \simeq 1\text{Vec}, \text{Hom}(T, I) \simeq 1\text{Vec};$$

$$\text{End}(I_{s}) \simeq \text{Rep}(\mathbb{Z}_2), \text{End}(T_{s}) \simeq 1\text{Vec}_{\mathbb{Z}_2}, \text{Hom}(I_{s}, T_{s}) \simeq 1\text{Vec}, \text{Hom}(T_{s}, I_{s}) \simeq 1\text{Vec}.$$
We illustrate these structures in the following quiver:

\[
\begin{array}{ccc}
\text{Rep}(\mathbb{Z}_2) & \circlearrowleft & \text{1Vec} \\
\downarrow & & \downarrow \\
I & \circlearrowleft & T \\
\end{array}
\quad
\begin{array}{ccc}
\text{Rep}(\mathbb{Z}_2) & \circlearrowleft & \text{1Vec} \\
\downarrow & & \downarrow \\
I_s & \circlearrowleft & T_s \\
\end{array}
\]

(3.12)

For the monoidal structure, \(I\) is the unit, and we have

\[
I_s \boxtimes I_s \cong I, \quad I_s \boxtimes T \cong T \boxtimes I_s \cong T_s, \quad T \boxtimes T \cong T_s \boxtimes T_s \cong T \boxplus T, \quad T \boxtimes T_s \cong T_s \boxtimes T \cong T_s \boxplus T_s,
\]

from Proposition\[5.3\] The braiding is given by

\[
R_{X,Y} : \quad XY \rightarrow YX \\
(x, y) \rightarrow (y, \rho_g(x)),
\]

where \(g = 1\) for \(Y = I, T\), \(g = s\) for \(Y = I_s, T_s\), and \(\rho_g : X \rightarrow X\) is the action of \(G\) on \(X\). If \(X = T, T_s\) and \(Y = I_s, T_s\), then \(R_{X,Y} \not= \Sigma_{X,Y}\); otherwise \(R_{X,Y} = \Sigma_{X,Y}\) from \[3.11\].

3.3. **The unit component.** The unit component \(Z(2\text{Vec}_G^\omega)_c\) for \(c = 1\) is a braided monoidal sub-bicategory of \(Z(2\text{Vec}_G^\omega)\). In this case, \(h = 1, C_G(h) = G\) and \(\tau_1(\omega)\) is a coboundary for any \(\omega \in Z^4(G, k^\times)\). If \(\omega\) is normalized, then \(\tau_1(\omega) = 1\). So \(2\text{Rep}(C_G(1), \tau_1(\omega))\) is equivalent to the 2-category \(2\text{Rep}(G)\) of module categories over \(1\text{Vec}_G\). In particular, \(Z(2\text{Vec}_G^\omega)_1 \cong 2\text{Rep}(G)\) as braided monoidal bicategories.

**Corollary 3.8.** There is an inclusion \(2\text{Rep}(G) \hookrightarrow Z(2\text{Vec}_G^\omega)\) of braided monoidal bicategories for any \(\omega \in Z^4(G, k^\times)\).

The 2-category \(2\text{Rep}(G)\) is well studied in \[Oz2\]. More precisely, any indecomposable object of \(2\text{Rep}(G)\) is given by a pair \(A = A(H, \psi)\), where \(H < G\) and \(\psi \in Z^2(H, k^\times)\). The isomorphism class of \(A(H, \psi)\) is determined by the conjugacy class of \(H\) and the cohomological class \([\psi] \in H^2(H, k^\times)\). There are two distinguished objects of \(Z(2\text{Vec}_G^\omega)_1\); one is the unit \(I = A(H, \psi)\) for \(H = G, \psi = 1\); the other one is \(T = A(H, \psi)\) for \(H = 1, \psi = 1\). As objects of \(2\text{Rep}(G)\), \(I = \mathbb{V}\) is the trivial representation, and \(T = 1\text{Vec}_G\) is the regular representation of \(1\text{Vec}_G\). The endomorphism 1-categories are

\[
\text{End}(I) \simeq \text{Rep}(G), \quad \text{End}(T) \simeq 1\text{Vec}_G.
\]

There is a one-to-one correspondence:

\[
\{\text{indecomposable objects of } 2\text{Rep}(G)\} \rightarrow \{\text{fusion categories Morita equivalent to } \text{Rep}(G)\} \\
M \mapsto \text{End}_{2\text{Rep}(G)}(M).
\]

Moreover, bimodules \(\text{Hom}_{2\text{Rep}(G)}(M, N)\) and \(\text{Hom}_{2\text{Rep}(G)}(N, M)\) induces the Morita equivalence between \(\text{End}_{2\text{Rep}(G)}(M)\) and \(\text{End}_{2\text{Rep}(G)}(N)\). Thus, \(2\text{Rep}(G)\) is the idempotent completion of the delooping of
Rep($G$) in the sense of Douglas and Reutter \cite{DR}. We illustrate these structures in the following quiver which is connected.

\[
\begin{array}{c}
\text{End}_{2\text{Rep}(G)}(A(H, \psi)) \\
\text{Rep}(H, \psi^{-1}) \\
A(H, \psi) \\
\text{Rep}(H, \psi) \\
A(G, 1) = I \\
\text{1Vec} \\
\text{Rep}(G) \\
\text{1Vec}_G \end{array}
\]

(3.13)

It follows from Proposition 3.5 that

\[A(H, \psi) \boxtimes A(H', \psi') \cong \boxplus_t A(H_t, \psi_t),\]

where the sum is over \( t \in H \setminus G/H' \), \( H_t = t^{-1}Ht \cap H' \), and \( \psi_t = t^*(\psi)|_{H_t} \cdot \psi'|_{H_t} \) from (3.9) since \( \omega \) is normalized. In particular, \( A(H, \psi) \boxtimes T \cong T \boxtimes A(H, \psi) \cong T^{\boxplus H \setminus G} \). Moreover, the monoidal structure is strictly associative since the invertible modifications in Diagram (2.6) are all identities.

The braiding \( R_{\tilde{A}, \tilde{B}} = (R_{A, B}, R_{R_{A, B}, -}) : \tilde{A}\tilde{B} \to \tilde{B}\tilde{A} \), where \( R_{A, B} = R_{A, -}(B) = \Sigma_{A, B} : AB \to BA \) from (3.11) since \( B \) is concentrated in grading 1, and the invertible modification \( R_{R_{A, B}, -} \) in Diagram (2.10) is the identity.

The invertible modifications \( R_{(\tilde{A}|B), \tilde{C}}, R_{(\tilde{A}, \tilde{B})|\tilde{C}} \) are all identities.

### 3.4. The M"uger center.

We briefly discuss the M"uger center or 2-center of \( 2\text{Rep}(G) \) and \( Z(2\text{Vec}^\omega_G) \). Crans gave a definition of the 2-center of a braided monoidal 2-category in the semistrict case \cite[Section 5.1]{C}. We need a weak version. We propose the following definition without checking the coherence.

**Definition 3.9.** Let \( \mathcal{C} \) be a braided monoidal bicategory. Its M"uger center \( Z_2(\mathcal{C}) \) as a bicategory defined as follows:

1. An object is a pair \((A, v_{A, -})\), where \( A \) is an object of \( \mathcal{C} \), and \( v_{A, -} \) is an invertible modification

\[
\begin{array}{c}
AX \\
R_{A, X} \downarrow \downarrow \psi_{v_{A, X}} \\
\downarrow R_{X, A} \uparrow AX
\end{array}
\]

2. An invertible modification is a pair \((\lambda_{A, B}, \chi_{v_{A, B}})\) such that

\[
\begin{array}{c}
\lambda_{A, B} \\
\chi_{v_{A, B}} \\
B X
\end{array}
\]

\( \lambda_{A, B} \) is an invertible modification such that

\[
\begin{array}{c}
\lambda_{A, B} \\
\lambda_{A, B} \downarrow \downarrow \psi_{\lambda_{A, B}} \\
\downarrow R_{X, A} \uparrow \lambda_{A, B}
\end{array}
\]

\( \chi_{v_{A, B}} \) is an invertible modification such that

\[
\begin{array}{c}
\chi_{v_{A, B}} \\
\chi_{v_{A, B}} \downarrow \downarrow \psi_{\chi_{v_{A, B}}} \\
\downarrow R_{X, A} \uparrow \chi_{v_{A, B}}
\end{array}
\]

3. Composition and identities are defined in a straightforward way.

The M"uger center \( Z_2(\mathcal{C}) \) is a bicategory.
such that the following axiom holds:

\[
\begin{align*}
\text{(3.14)} \quad (AX)Y & \xrightarrow{R_{X,A}} (AX)Y \\
(AX)Y & \xrightarrow{a} (AX)Y \\
X(AY) & \xrightarrow{R_{A,Y}} X(AY) \\
\Rightarrow & \\
\end{align*}
\]

(2) A 1-morphism from \((A, v_{A,-})\) to \((A', v_{A',-})\) is a 1-morphism \(f : A \to A'\) in \(C\) such that the following diagram commutes:

\[
\begin{align*}
\text{(3.15)} \quad A'X & \xrightarrow{\Rightarrow} A'X \\
\xrightarrow{R_{A',X}} & \\
\xrightarrow{\Rightarrow} & \\
\Rightarrow & \\
AX & \xrightarrow{\Rightarrow} AX \\
\xrightarrow{R_{A,X}} & \\
\xrightarrow{\Rightarrow} & \\
\end{align*}
\]

where all vertical arrows are induced by \(f\), and the 2-isomorphism in the back is the identity.

(3) A 2-morphism is defined in the same way as in \(C\).

The monoidal structure, the braiding and the syllepsis structures on \(Z_2(C)\) can be generalized from Crans’ definition in a similar way. We omit the detail here.

**Proposition 3.10.** The Müger center of \(2\text{Rep}(G)\) is equivalent to \(2\text{Rep}(G)\) as bicategories.

**Proof.** Let \((A, v_{A,-})\) be an object of the Müger center of \(2\text{Rep}(G)\). The braiding of \(2\text{Rep}(G)\) is symmetric, i.e. \(R_{X,A} \circ R_{A,X} = id_{AX}\) for any \(X\). We prove that the modification \(v_{A,X} = id_{id_{AX}}\) as follows. Taking \(X = Y = I\) in the axiom (3.14) gives \(v_{A,I}^2 = v_{A,I}\) since \(R_{(A|X,Y)}, R_{(X,Y|A)}\) are identities. It follows that \(v_{A,I}\) is the identity. For any object \(X\) of \(2\text{Rep}(G)\), there exists a nontrivial 1-morphism \(f : I \to X\) since \(2\text{Rep}(G)\) has only one connected component. The naturality of \(v_{A,-}\) associated to \(f\) is described by the
following diagram:

\[
\begin{array}{c}
\xymatrix{
AX & AX \\
| & | \\
\downarrow f & \downarrow f \\
XA & f \\
| & | \\
\downarrow R_{A,f} & \downarrow R_{f,A} \\
AI & AI \\
| & |
\end{array}
\]

Then \( R_{f,A} \) is the identity since \( A \) is concentrated in the grading 1, and \( R_{A,f} \) is the identity from Remark 3.6. It follows that \( v_{A,X} = id_{id_{A,X}} \). Therefore, an object \((A, v_{A,-})\) of the Müger center of \(2Rep(G)\) is completely determined by \( A \) as an object of \(2Rep(G)\). For 1-morphism from \((A, v_{A,-})\) to \((A', v_{A',-})\), any 1-morphism \( f : A \to A' \) in \(2Rep(G)\) satisfies Diagram (3.15) since all 2-isomorphisms are identities there.

Remark 3.11. The 2-category \(2Rep(G)\) has a natural syllepsis structure viewed as the Müger center of \(2Rep(G)\). This syllepsis structure is symmetric in the sense of Crans [C], i.e. \( id_{X,Y} \cdot v_{X,Y} = v_{Y,X} \cdot id_{X,Y} \) as 2-morphisms from \( R_{X,Y} \circ R_{Y,X} \circ R_{X,Y} \) to \( R_{X,Y} \). As a result, \(2Rep(G)\) is an \(E_4\) algebra.

Theorem 3.12. The Müger center of \(Z(2Vec^G_0)\) is equivalent to \(2Vec\) as bicategories.

Proof. Let \((\tilde{A}, v_{\tilde{A},-})\) be an indecomposable object of the Müger center of \(Z(2Vec^G_0)\), where \( v_{\tilde{A},\tilde{X}} : id_{\tilde{A},\tilde{X}} \Rightarrow R_{\tilde{X},\tilde{A}} \circ R_{\tilde{A},\tilde{X}} \) gives an isomorphism between the identity of \( 2Rep\) and the double braiding.

Take \( \tilde{X} = T = A(h, H, \psi) \) for \( h = 1, H = 1, \psi = 1 \). The half braiding \( R_{A,T} = \Sigma_{A,T} \) from (3.11) since \( T \) is concentrated in grading 1. So the other half braiding \( R_{T,\tilde{A}} = \Sigma_{T,\tilde{A}} \). It follows from (3.10) that \( \tilde{A} \) is concentrated in the grading 1 since \( T \) is the regular representation in \(2Rep(G)\). So \( \tilde{A} \) is an object of \(Z(2Vec^G_0)_1 \simeq 2Rep(G)\).

For any \( \tilde{X} \), the half braiding \( R_{\tilde{X},\tilde{A}} = \Sigma_{\tilde{X},\tilde{A}} \) which implies that \( R_{\tilde{X},\tilde{A}} = \Sigma_{\tilde{A},\tilde{X}} \). Then \( \tilde{A} \) has to be the trivial representation in \(2Rep(G)\) by taking \( \tilde{X} = T_h = A(h, H, \psi) \) for \( h \in G, H = 1, \psi = 1 \). We have \( \tilde{A} = A(1, G, \psi) \), where \( \psi \in \mathbb{Z}^2(G, k^\times) \) is determined by \( R_{(\tilde{A},\tilde{X},\tilde{Y})} \). Taking \( \tilde{X} = T_h, \tilde{Y} = T_{h'}, \) the axiom in (3.14) gives

\[
R_{(\tilde{A},\tilde{X},\tilde{Y})} \cdot v_{\tilde{A},\tilde{X},\tilde{Y}} = v_{\tilde{A},\tilde{X},\tilde{Y}} = v_{\tilde{A},\tilde{X},\tilde{Y}} \cdot v_{\tilde{A},\tilde{X},\tilde{Y}},
\]

since \( R_{(\tilde{X},\tilde{Y}|\tilde{A})} \) is the identity. This implies that \( \psi = d\gamma \), where \( \gamma \in \mathbb{Z}^1(G, k^\times) \) is a 1-cochain determined by \( v_{\tilde{A},T_h} \). Therefore, the underlying object \( \tilde{A} \) of \((\tilde{A}, v_{\tilde{A},-})\) is isomorphic to the unit \( I \) in \(Z(2Vec^G_0)\).

We next show that \( I_1 = (I, v_{I,-}) \) and \( I_2 = (I, v'_{I,-}) \) are isomorphic to each other. Define a 1-morphism \((f, R_{f,-}) : I_1 \to I_2\), where \( f = id_I \) and \( R_{f,\tilde{X}} = v_{I,\tilde{X}} \cdot v'_{I,\tilde{X}} \). It is easy to check that \((f, R_{f,-})\) is well-defined and gives an isomorphism. So up to isomorphism there is only one indecomposable object \( I_0 = (I, v_{I,-}), v_{I,\tilde{X}} = id_{id_{\tilde{X}}} \) in the Müger center.
We finally compute \( \text{End}(I_0) \). Let \( f : I \to I \) be a 1-morphism in \( Z(2\text{Vec}_\omega^G) \), i.e. \( f \in \text{End}(I) \cong \text{Rep}(G) \). It follows from (3.15) that \( R_{f, \tilde{X}} \) is the identity for any \( \tilde{X} \) since \( R_{\tilde{X}, f} \) is the identity. So \( f \) has to be the trivial representation in \( \text{Rep}(G) \) by taking \( \tilde{X} = T_\hbar \) as above. We conclude that \( \text{End}(I_0) \cong 1\text{Vec} \). □

Theorem 3.12 is consistent with the expectation that \( Z(2\text{Vec}_\omega^G) \) should be an example of the yet-to-be-defined notion of a unitary modular tensor bicategory. Similar to Definition 3.9, the notion of the relative Müger center of a full subcategory of a braided monoidal bicategory can be defined. A combination of the proofs of Proposition 3.10 and Theorem 3.12 shows that the relative Müger center of \( 2\text{Rep}(G) \) in \( Z(2\text{Vec}_\omega^G) \) is equivalent to \( 2\text{Rep}(G) \). A unitary modular tensor bicategory \( C \), equipped with a braided monoidal fully faithful embedding \( 2\text{Rep}(G) \to C \), is called a modular extension of \( 2\text{Rep}(G) \). Such a modular extension is called minimal if the relative Müger center of \( 2\text{Rep}(G) \) in \( C \) is braided monoidally equivalent to \( 2\text{Rep}(G) \).

By Corollary 3.8, Proposition 3.10 and Theorem 3.12, \( Z(2\text{Vec}_\omega^G) \) is precisely a minimal modular extension of \( 2\text{Rep}(G) \) for \( \omega \in Z^4(G, k^\times) \). Motivated by the classification theory of 2+1D symmetry protected topological orders [LKW1] and its 3+1D analogue [CGLW, LKW2], we propose the following conjecture.

**Conjecture 3.13.** The equivalence classes of minimal modular extensions of \( 2\text{Rep}(G) \) are classified by \( H^4(G, k^\times) \).

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