Bayesian Parameter Identification for Jump Markov Linear Systems

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Abstract

This paper presents a Bayesian method for identification of jump Markov linear systems that is powered by a Markov chain Monte Carlo method called the Gibbs sampler. Unlike maximum likelihood approaches, this method provides the parameter distributions or the variation of likely system responses, which could be useful for analysing the stability margins of control schemes. We also include numerically robust implementation details and examples demonstrating the effectiveness of the proposed algorithm.

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1 Introduction

Switched linear Markovian systems are a system class that naturally handles stochastic regime changes, but still enjoys closed-form solutions from traditional linear estimation. Because of these advantages, the system class has previously gained traction in the areas of telecommunications [42], econometrics [38] [34] [35] [10] [20] [33], [40] [21], fault diagnosis [36], and target tracking [43]. Parameter identification of these systems is a prerequisite to state inference, however literature on parameter identification of certain Switched linear Markovian systems is scarce. In this paper we consider parameter identification for a switched class of systems called jump Markov linear systems (JMLS).

Previously we have presented a paper on parameter estimation for JMLS using a maximum likelihood (ML) approach [6], which used expectation maximisation (EM) to arrive at the most likely model parameters for a given dataset. However, this approach only provides a point estimate with no indication of likely parameter variation, and as it was an EM based algorithm, could only guarantee local convergence. A Bayesian-based approach is capable of offering convergence properties, and additional information about the parameter distributions or variation of the likely responses of the system.

For linear Gaussian (LG) systems, a Bayesian estimate of the deterministic parameters can be calculated by augmenting the state vector with the unknown parameters and using a nonlinear estimator [41] [20]. However, when the complete set of LG parameters requires estimation, it is necessary to use a Markov chain Monte Carlo (MCMC) method, such as Gibbs sampling [48] [13]. This method acknowledges the highly coupled nature between the system states, and the parameter estimates, and samples them jointly.

Gibbs sampling is a powerful tool, useful for sampling from joint distributions e.g. \( p(\theta, \xi) \), where sampling from the conditional distributions \( p(\theta|\xi) \) and \( p(\xi|\theta) \) is much simpler than sampling from the joint distribution directly. For application to switched systems, this requires recursively sampling the JMLS parameters \( \theta \) conditioned on a state trajectory \( \xi \) and vice versa. This approach has previously been used to identify autoregressive (AR) systems [22] [26] [29] [1], change-point models [19] [14] [32], stochastic volatility models [45], and stochastically switched systems operating according to drift processes [34].

Estimating the parameters of switched systems unsurprisingly encounters additional challenges. These include the increased number of parameters to be estimated combined with the exponential complexity of considering each of the \( m \) models being active in each of the \( N \) time steps [5] [2] [11] [31] [4] [30] [12] [7] [37] [38].

An additional challenge is how to quantify the uncertainty about the switching condition of this system. Previous approaches include building the switching condition [16] [15] into a continuous variable, and applying a threshold to determine the active model or using Bernoulli, Beta or Dirichlet distributions as conjugate priors to calculate the posterior distribution. Bernoulli or Beta distributions limit the switched system to only two models [1] [39] [44] [17], but can be nested to remove this restriction [40] [19]. This nesting however, results in a more complex algorithm, and instead [22] [27] [18] [26] [34] [32] [23] uses the far simpler Dirichlet distribution.

A final challenge is presented when considering how models may be reordered. As reordering or ‘relabelling’ of \( m \) models does not affect the likelihood of the parameter estimates if there is a common prior for each model [27] [26] [34] [25] [46], there is an inheritable \( m! \) order symmetry to the parameter estimation problem, where \( ! \) denotes the factorial. [26] discusses the possibility of swapping the identified models randomly, to distribute sampling from each of these \( m! \) regions, but also discusses sorting models upon acceding values of an identified parameter, and how this can explore just one of these symmetric regions if completed correctly.

In this work, we apply the Gibbs sampler to the general JMLS system class. Unlike previous approaches we do not force the system to be univariate [40] [25] [20], support only a small number of models [40] [16] [1], or constrain the system or noise structure [14] [34] [32] [29] [23]. This in part, is made possible by use of the inverse-Wishart conjugate prior such as used within [48] [23], opposed to commonly used inverse-Gamma distribution (e.g. see [1] [18] [40] [26] [22]). Like [22] [27] [18] [26] [34] [32] we use Dirichlet priors on the vectors of the transition matrix.

In order to use Gibbs sampling to target the parameter distribution for this system, we must be able to sample a hybrid state trajectory conditioned on the model parameters. Previous approaches in the literature sample the continuous and discrete latent variables in separate stages [44] [40] [25] [26] [14] [29] [53] [23]. These approaches rely on [24] [27] for sampling the continuous state trajectory, and [22] [55] [18] [27] for the sampling the discrete state sequence. Methods used to sample the discrete sequence include ‘single move’ sampling, where the model used at each time step is sampled
separately by increasing the number of steps within the Gibbs iteration, and ‘multi-move sampling’ where the entire trajectory is sampled as a block [27][40]. Alternative approaches are also possible (see [27]), which include sampling smaller parts of the discrete trajectory [11], or marginalising over the continuous state variables [17] which improves mixing. The separation of discrete and continuous sampling is presumed to have originated from other model classes, which do not allow for such block sampling, or have no benefit e.g. autoregressive (AR) or change-point models. Unlike the aforementioned methods, we use blocked-Gibbs to sample the hybrid trajectory to further improve mixing. This is completed by running a JMLS forward-filter with resampling before backwards construction of the hybrid state trajectory from the resulting hybrid Gaussian mixture.

**The contributions** of this paper are therefore:

1. A method of block sampling the hybrid state trajectory using hybrid Gaussian mixtures.
2. A Gibbs sampler which targets the parameter distribution for JMLS systems, without restriction on state dimension, model or noise structure, or the number of models.
3. A numerically stable implementation of the above.

The proposed methods are self-contained within this paper, which is organised as follows. Section 2 provides a more technical description of the problem. Section 3 provides additional details on how the Gibbs sampler can be applied to this problem. Section 4 details our proposed algorithm. Section 5 provides simulation results, and Section 6 provides some concluding remarks.
2 Problem formulation

Given a finite sequence of exogenous inputs $u_{1:N}$ and a finite sequence of observations $y_{1:N}$, where

\begin{align*}
u_{1:N} &= \{u_1, \ldots, u_N\}, \quad u_k \in \mathbb{R}^{n_u} \quad \forall k = 1, \ldots, N, \\
y_{1:N} &= \{y_1, \ldots, y_N\}, \quad y_k \in \mathbb{R}^{n_y} \quad \forall k = 1, \ldots, N,
\end{align*}

we wish to estimate the distribution $p(\theta|y_{1:N})$, where $\theta$ fully parameterises a JMLS system, written as

\begin{align*}
\begin{bmatrix} y_k \\ x_{k+1} \end{bmatrix} &= \Gamma(z_k) \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} e_k \\ v_k \end{bmatrix}, \quad \Gamma(z_k) = \begin{bmatrix} C(z_k) & D(z_k) \\ A(z_k) & B(z_k) \end{bmatrix}, \\
\begin{bmatrix} e_k \\ v_k \end{bmatrix} &\sim \mathcal{N}(0, \Pi(z_k)), \quad \Pi(z_k) = \begin{bmatrix} R(z_k) & S^T(z_k) \\ S(z_k) & Q(z_k) \end{bmatrix}
\end{align*}

where $x_k \in \mathbb{R}^{n_x}$ is the system state, and the terms $e_k$ and $v_k$ originate from the Gaussian white noise process.

The transition matrix must contain probabilities i.e. $0 \leq T_{i,j} \leq 1$, and satisfy the total law of probability

$$\sum_{i=1}^{m} T_{i,j} = 1 \quad \forall j = 1, \ldots, m.$$ 

Therefore estimation of the distribution,

$$p(\theta|y_{1:N})$$

is required, where $\theta$ comprises of the parameters $\Gamma$ and $\Pi$ for each of the $m$ models and the transition matrix $T$, which governs switching between them i.e.,

$$\theta = \{T, \{\Gamma(i), \Pi(i)\}_{i=1}^{m}\}.$$ 

The following section details how this distribution can be targeted using Gibbs sampling.
3 Markov chain Monte Carlo

The proposed solution is powered by a Markov chain Monte Carlo method called Gibbs sampling [28]. The Gibbs sampler produces samples from a joint distribution by recursively sampling from conditional distributions. The general Gibbs sampling procedure is described in Algorithm 1.

![Algorithm 1 Gibbs sampling](image)

Because of the highly coupled behaviour between the systems states and parameters, the identification problem is made simpler by using a Gibbs sampler targeting the joint posterior $p(\theta, \xi_{1:N+1} | y_{1:N})$, where $\xi_{1:N+1}$ is a hybrid continuous-discrete trajectory, i.e.,

$$
\xi_{1:N+1} = \{\xi_1, \ldots, \xi_{N+1}\},
$$

$$
\xi_k = \{x_k, z_k\}.
$$

The use of the Gibbs sampler for this problem requires efficient sampling from the conditional distributions $p(\theta | \xi_{1:N+1}, y_{1:N})$ and $p(\xi_{1:N+1} | \theta, y_{1:N})$. Efficient sampling from the distribution $p(\xi_{1:N+1} | \theta, y_{1:N})$ is enabled by exploiting the Gaussian mixture (GM) structure to the problem, as explained later.

To enable efficient sampling of the JMLS parameters $\theta$, the parameters were assumed to be distributed according to conjugate priors which allow for $p(\theta | \xi_{1:N+1}, y_{1:N})$ to be calculated using closed-form solutions. These conjugate priors were assumed to be as follows.

The columns of the model transition matrix $T$ were assumed to be distributed according to a Dirichlet distribution

$$
\mathcal{D}(T | \alpha) = \frac{\Gamma(\sum_{i=1}^{m} \alpha_{i,j})}{\prod_{i=1}^{R} \Gamma(\alpha_{i,j})} \prod_{i=1}^{m} (T_{i,j})^{\alpha_{i,j}-1},
$$

where $\alpha$ is a matrix of concentration parameters with elements $\alpha_{i,j} > 0$, and $\Gamma(\cdot)$ is the gamma function.

The covariance matrices were assumed to be distributed according to an inverse-Wishart distribution

$$
\mathcal{W}^{-1}(\Pi(z) | A_z, \nu_z) = \frac{|A_z|^{\nu_z/2}}{2^{(\nu_z+n)/2} \Gamma_n((n+1)/2)} |\Pi(z)|^{-(\nu_z+n+1)/2} \exp \left\{ -\frac{1}{2} \text{tr} \left[ A_z \Pi^{-1}(z) \right] \right\},
$$

where $\Pi(z) \in \mathbb{R}^{n \times n}$ is the positive definite symmetric matrix argument, $A_z \in \mathbb{R}^{n \times n}$ is the positive definite symmetric scale matrix, $\nu_z \in \mathbb{R}$ is the degrees of freedom, which must satisfy $\nu_z > n - 1$, and $\Gamma_n$ is the multivariate gamma function with dimension $n$. A higher degree of freedom indicates greater confidence about the mean value of $\frac{A_z}{\nu_z = n - 1}$ if $\nu_z > n + 1$.

Finally, the deterministic parameters for a model are assumed to be distributed according to a Matrix-Normal distribution with the form

$$
\mathcal{MN}(\Gamma(z) | M_z, \Pi(z), V_z) = \frac{\exp \left\{ -\frac{1}{2} \text{tr} \left[ V_z^{-1}(\Gamma(z) - M_z)^T \Pi^{-1}(z)(\Gamma(z) - M_z) \right] \right\}}{(2\pi)^{n/2}|V_z|^{n/2} |\Pi(z)|^{n/2}}.
$$
with mean $\mathbf{M} \in \mathbb{R}^{n \times p}$, positive definite column covariance matrix $\mathbf{V} \in \mathbb{R}^{p \times p}$, and $\text{tr}(A)$ denoting the trace of matrix $A$. The Matrix Normal is related to the Normal distribution by

$$
\mathcal{MN}(\Gamma|\mathbf{M}, \Pi, \mathbf{V}) = \mathcal{N}(\text{Vec}(\Gamma)|\text{Vec}(\mathbf{M}), \mathbf{V} \otimes \Pi).
$$

(12)
4 The algorithm

The algorithm is composed of recursively sampling latent hybrid trajectories \( \xi_{1:N+1} \) followed by new parameter estimates \( \theta^{f+1} \). We begin with a method for forwards filtering the system, before providing instructions on how this can be used to sample latent state trajectories. The forwards filter of subsection 4.1 and sampling of a latent trajectory from subsection 4.2 is intended to be completed using the \( \theta^f \) parameter set, this is not explicitly written for readability purposes.

All proofs are provided within the appendix.

4.1 Forward-filtering the system

Before filtering each time step, the following transformation made to efficiently handle the cross-covariance term \( S(z_k) \),

\[
\begin{align*}
\mathbf{A}_k(z_k) &= \mathbf{A}(z_k) - \mathbf{J}_k(z_k)\mathbf{C}(z_k), \\
\mathbf{B}_k(z_k) &= [\mathbf{B}(z_k) - \mathbf{J}_k(z_k)\mathbf{D}(z_k) \quad \mathbf{J}_k(z_k)], \\
\mathbf{C}_k(z_k) &= \mathbf{C}(z_k), \\
\mathbf{D}_k(z_k) &= [\mathbf{D}(z_k) - \mathbf{0}_{n_y}], \\
\mathbf{J}_k(z_k) &= H^T(z_k)(\mathbf{R}^{1/2}_k(z_k))^{-T}, \\
\begin{bmatrix}
\mathbf{R}^{1/2}_k(z_k) & H(z_k) \\
0 & \mathbf{Q}_k^{1/2}(z_k)
\end{bmatrix} &= \Pi^{1/2}(z_k), \\
\tilde{u}_k &= \begin{bmatrix} u_k \\ y_k \end{bmatrix},
\end{align*}
\]

where \( A^T \) and \( A^{-1} \) denote matrix transposes and inverses respectively, and \( A^{1/2} \) denotes an upper Cholesky factor of \( A \), i.e. \((A^{1/2})^TA^{1/2} = A\). Notice that the new system uses a different input \( \tilde{u}_k \), and now uses time varying parameters. After applying this transformation we can now calculate the forwards filter distribution using Lemma 4.1.

**Lemma 4.1.** Under the model class 3 with initial prior prediction distribution given by

\[
p(x_1, z_1) = \sum_{i=1}^{M^f_1(z_1)} w^i_{1,0}(z_1) \mathcal{N}\left( x_1 | \mu^i_{1,0}(z_1), \mathbf{P}^i_{1,0}(z_1) \right),
\]

then the subsequent filtering and prediction distributions for \( k = 1, \ldots, N \) are given by

\[
\begin{align*}
p(x_k, z_k | y_{1:k}) &= \sum_{i=1}^{M^f_k(z_k)} w^i_{k,1}(z_k) \mathcal{N}\left( x_k | \mu^i_{k,1}(z_k), \mathbf{P}^i_{k,1}(z_k) \right), \\
p(x_{k+1}, z_{k+1} | y_{1:k}) &= \sum_{j=1}^{M^f_{k+1}(z_{k+1})} w^j_{k+1,1}(z_{k+1}) \mathcal{N}\left( x_{k+1} | \mu^j_{k+1,1}(z_{k+1}), \mathbf{P}^j_{k+1,1}(z_{k+1}) \right),
\end{align*}
\]

respectively, where \( M^f_k(z_k) = M^p_k(z_k) \) and for each \( i = 1, \ldots, M^f_i(z_k) \) and \( z_k = 1, \ldots, m_k \),

\[
\begin{align*}
w^i_{k,1}(z_k) &= \frac{w^i_{k,1}(z_k)}{\sum_{j=1}^{m_k} \sum_{i=1}^{M^f_j(z_k)} w^j_{k,1}(z_k)}, \\
\tilde{w}^i_{k,1}(z_k) &= w^i_{k,1}(z_k) \cdot \mathcal{N}\left( y_k | \eta^i_{k,1}(z_k), \Xi^i_{k,1}(z_k) \right), \\
\mu^i_{k,1}(z_k) &= \mu^i_{k,1}(z_k) + \mathbf{K}^i_{k,1}(z_k)(y_k - \eta^i_{k,1}(z_k)), \\
\eta^i_{k,1}(z_k) &= \mathbf{C}_k(z_k)\mu^i_{k,1}(z_k) + \mathbf{D}_k(z_k)\tilde{u}_k, \\
\mathbf{P}^i_{k,1}(z_k) &= \mathbf{P}^i_{k,1}(z_k) - \mathbf{K}^i_{k,1}(z_k)\mathbf{C}_k(z_k)\mathbf{P}^i_{k,1}(z_k), \\
\mathbf{K}^i_{k,1}(z_k) &= \mathbf{P}^i_{k,1}(z_k)\mathbf{C}_k^T(z_k)\Xi^i_{k,1}(z_k)^{-1}, \\
\Xi^i_{k,1}(z_k) &= \mathbf{C}_k(z_k)\mathbf{P}^i_{k,1}(z_k) + \mathbf{R}_k(z_k),
\end{align*}
\]
then for each $i = 1, \ldots, M^f_i(\tau)$, $\tau = 1, \ldots, m_k$ and $z_{k+1} = 1, \ldots, m_{k+1}$,
\[
\tilde{w}'_{k+1|k}(z_{k+1}) = T_{z_{k+1},r} \cdot w_{k|k}(\tau),
\]
(16a)
\[
\mu'_{k+1|k} = A_k(\tau)\mu_{k|k}(\tau) + B_k(\tau)\tilde{w}_k,
\]
(16b)
\[
P'_{k+1|k} = A_k(\tau)P_{k|k}(\tau)A_k^T(\tau) + Q_k(\tau),
\]
(16c)
where $j$ occupies the next free index. By sampling from the Categorical distribution
\[
\{a, z_{k+1}\} \sim \mathcal{C}(\{\tilde{w}'_{k+1|k}(z_{k+1})\}_{y_k = 1, \ldots, R}),
\]
where $R$ is a user chosen parameter, a larger $R$ increases the the computational cost of the algorithm but also improves the approximated filtered distribution, which otherwise has an exponentially growing number of components. Then the prediction distribution with the reduced number of components is
\[
M^p_{k+1}(z) = \sum_{i=1}^{R} \delta_{z_{k+1|z_i}},
\]
(17)
\[
w'_{k+1|k}(z_{k+1}) \leftarrow \frac{1}{R} \ \ \ \forall i = 1, \ldots, R,
\]
(18)
\[
\mu'_{k+1|k}(z_{k+1}) \leftarrow \mu'_{k+1|k} \ \ \ \forall i = 1, \ldots, R,
\]
(19)
\[
P'_{k+1|k}(z_{k+1}) \leftarrow P'_{k+1|k} \ \ \ \forall i = 1, \ldots, R,
\]
(20)
where $j$ is the next free index.

**Proof.** This lemma is included for completeness and is not considered to be original work. A proof of this lemma is provided in [3]. Notice that when calculating the forward filter distribution we elected to use a resampling procedure, opposed to the merging used in our previous work [3], as merging may generate samples of state trajectories which are infeasible for the system to take. Resampling also has statistical asymptotic convergence properties, while merging does not. For a numerically stable version of Lemma 4.1 consider using log-weights, normalised by a variation of the Log-Sum-Exponent (LSE) trick, and the Q-less QR decomposition $Q(\cdot)$ of
\[
\begin{bmatrix}
R_{11} & R_{12} \\
0 & R_{22}
\end{bmatrix} = Q \left( \begin{bmatrix}
R_k^{1/2}(z_k) & 0 \\
(P_{k|k-1}(z_k))^{1/2}C_k(z_k) & (P_{k|k-1}(z_k))^{1/2}
\end{bmatrix} \right),
\]
(22a)
then it can be shown
\[
(P_{k|k}(z_k))^{1/2} = R_{22},
\]
(22b)
\[
K_{k|k}(z_k) = (R_{11}^{-1}R_{12})^T,
\]
(22c)
\[
(\Xi_{k|k}(z_k))^{1/2} = R_{11}.
\]
(22d)
For the prediction step, consider the decomposition
\[
(P_{k+1|k})^{1/2} = Q \left( \begin{bmatrix}
(P_{k|k}(\ell))^{1/2}A_k^T(\ell) \\
Q_k^{1/2}(\ell)
\end{bmatrix} \right).
\]
(23)
Using these decompositions (22) replaces (15c)-(15g) and (23) replaces (16c). Also note that the filtered likelihood (15b) can be computed directly with $(\Xi_{k|k}(z_k))^{1/2}$ and should not be taken out of square-root form.

**4.2 Sampling the latent variables**

With the forwards filtering stage complete, sampling latent hybrid state trajectories from the distribution $p(\xi_{1:N+1}|\theta^f, y_{1:N})$ can commence. This approach differs significantly from [44, 40, 25, 26, 14, 29, 45, 23], which sample the entire discrete trajectory before sampling the continuous trajectory. By sampling hybrid components, discrete and continuous components are being sampled for each time step before proceeding to sample from another, which improves mixing.
We begin by expressing the target distribution as

\[ p(\xi_1:N+1|\theta^f, y_1:N) = p(\xi_{N+1}|\theta^f, y_1:N) \prod_{k=1}^{N} p(\xi_k|\xi_{k+1:N+1}, \theta^f, y_1:N). \]  

(24)

As the prediction distribution \( p(\xi_{N+1}|\theta^f, y_1:N) \) is readily available from the forward filter outlined in Lemma 4.1, we first sample from it, generating \( x_{N+1}^f \) and \( z_{N+1}^f \). Sampling from this hybrid Gaussian mixture distribution can be completed using Lemma 4.3.

**Lemma 4.2.** We can sample \( x^f \) and \( z^f \) from a hybrid distribution of the form

\[ p(x, z|\cdot) = p(\xi|\cdot) = \sum_{i=1}^{M(z)} w^i(z)N(x|\mu^i(z), P^i(z)) \]

by sampling the hybrid component according to the categorical distribution

\[ \{z^f, i\} \sim C(\{w^i(z)\}_{v_2, v_i}), \]

(25)

then using the newly sampled \( z^f \) and \( i \) we can sample \( x^f \) from the associated Normal distribution,

\[ x^f \sim N(x|\mu^i(z^f), P^i(z^f)). \]

(27)

Using the newly sampled \( x_{N+1}^f \) and \( z_{N+1}^f \) from Lemma 4.2 (24) requires the distribution \( p(\xi_k|\xi_{k+1:N+1}, \theta^f, y_1:N) \) to be constructed for \( k = N, \ldots, 1 \), with sampling of \( x_k^f \) and \( z_k^f \) to occur after the calculation of \( p(\xi_k|\xi_{k+1:N+1}, \theta^f, y_1:N) \). The construction of the distribution \( p(\xi_k|\xi_{k+1:N+1}, \theta^f, y_1:N) \) can be completed using Lemma 4.3 before subsequent use of Lemma 4.2 to sample \( x_k^f \) and \( z_k^f \).

**Lemma 4.3.** Using the component of the sampled trajectory \( x_{k+1}^f \) and \( z_{k+1}^f \) and the forward filtered distribution of the form

\[ p(\xi_k|y_{1:k}) = \sum_{i=1}^{M_f(z_k)} w_{k|k}^i(z_k)N(x_k|\mu_{k|N}^i(z_k), P_{k|N}^i(z_k)), \]

(28)

which is provided by Lemma 4.7 for all \( k \), the target distribution

\[ p(\xi_k|\xi_{k+1:N+1}, \theta^f, y_1:N) = \sum_{i=1}^{M_f(z_k)} w_{k|N}^i(z_k)N(x_k|\mu_{k|N}^i(z_k), P_{k|N}^i(z_k)), \]

(29)

can be constructed where

\[ \tilde{w}_{k|N}^i(z_k) = T_{z_{k+1}, z_k} \cdot w_{k|k}^i(z_k)N(x_{k+1}|h_{k|N}^i(z_k), \Xi_{k|N}^i(z_k)), \]

(30a)

\[ \mu_{k|N}^i(z_k) = \mu_{k|k}^i(z_k) + K_{k|N}^i(z_k)[x_{k+1} - \eta_{k|N}^i(z_k)], \]

(30b)

\[ \eta_{k|N}^i(z_k) = A_k(z_k)\mu_{k|k}^i(z_k) + B_k(z_k)\bar{u}_k, \]

(30c)

\[ P_{k|N}^i(z_k) = (I - K_{k|N}^i(z_k)A_k(z_k))P_{k|k}^i(z_k), \]

(30d)

\[ K_{k|N}^i(z_k) = P_{k|k}^i(z_k)A_k^T(z_k)(\Xi_{k|N}^i(z_k))^{-1}, \]

(30e)

\[ \Xi_{k|N}^i(z_k) = A_k(z_k)P_{k|k}^i(z_k)A_k^T(z_k) + Q_k(z_k), \]

(30f)

and

\[ w_{k|N}^i(z_k) = \frac{\tilde{w}_{k|N}^i(z_k)}{\sum_{z_k=1}^{Z} \sum_{i=1}^{M_f(z_k)} \tilde{w}_{k|N}^i(z_k)}. \]

(31)
For a numerically stable implementation of Lemma 4.3, consider the Q-less QR decomposition

\[
\begin{bmatrix}
R_{11} & R_{12} \\
0 & R_{22}
\end{bmatrix} = Q \begin{bmatrix}
P_{i|k}(z_k) & 0 \\
(P_{i|k}(z_k))^{1/2}A_k^{T}(z_k) & (P_{i|k}(z_k))^{1/2}
\end{bmatrix},
\] (32a)

then it can be shown

\[
(P_{i|N}(z_k))^{1/2} = R_{22},
\] (32b)

\[
K_{i|N}(z_k) = (R_{11}^{-1}R_{12})^{T},
\] (32c)

\[
(\Xi_{i|N}(z_k))^{1/2} = R_{11}.
\] (32d)

Using this decomposition (32) replaces (30d)–(30f), and the likelihood term within (30a) can be directly calculated directly using \((\Xi_{i|N}(z_k))^{1/2}\) and should not be taken out of square-root form.

### 4.3 Conditioned parameter distributions

Conditioned on the sampled hybrid state trajectory \(\xi^t_{1:N+1}\), new parameter estimates \(\theta^{t+1}\) can be sampled from the distribution \(p(\theta^{t+1}|\xi^t_{1:N+1}, y_{1:N})\). Lemma 4.4 provides instructions on how this distribution can be calculated. This distribution has a convenient closed-form because of the assumed conjugate priors.

**Lemma 4.4.** The distribution which parameters are sampled from can be expressed as

\[
p(\{\Gamma(i), \Pi(i)\}_{i=1}^{m}, T|x^t_{1:N+1}, z^t_{1:N+1}, y_{1:N}) \propto \mathcal{D}(T|\alpha) \left( \prod_{i=1}^{m} \mathcal{MN}(\Gamma(i)|\bar{\Lambda}, \bar{\Theta}_i, \bar{V}, \bar{\Psi}_i, \bar{\Sigma}_i) \right),
\] (33)

where the parameters \(\{\nu_i, M_i, V_i, \Lambda_i, \alpha\}\) define the prior for the \(i\)-th model, and the corrected parameters \(\{\bar{\nu}_i, \bar{M}_i, \bar{V}_i, \bar{\Lambda}_i, \bar{\alpha}\}\) for each \(i\) can be calculated using

\[
\bar{\Lambda}_i \triangleq \Lambda_i + \Phi_i - \bar{\Psi}_i \bar{\Sigma}_i^{-1} \bar{\Psi}_i^T,
\] (34a)

\[
\bar{\nu}_i \triangleq \nu_i + N_i,
\] (34b)

\[
\bar{M}_i \triangleq \bar{\Psi}_i \bar{\Sigma}_i^{-1},
\] (34c)

\[
\bar{V}_i \triangleq \bar{\Sigma}_i^{-1},
\] (34d)

\[
\bar{\alpha} = \alpha + u,
\] (34e)

which use the quantities,

\[
\bar{\Sigma}_i \triangleq \Sigma_i + V_i^{-1},
\] (35a)

\[
\bar{\Psi}_i \triangleq \Psi_i + M_i V_i^{-1},
\] (35b)

\[
\bar{\Phi}_i \triangleq \Phi_i + M_i V_i^{-1} M_i^T,
\] (35c)

\[
\Phi_i \triangleq \sum_{k \in G_i} \begin{bmatrix}
y_k \\
x_{k+1}
\end{bmatrix}^T,
\] (35d)

\[
\Psi_i \triangleq \sum_{k \in G_i} \begin{bmatrix}
y_k \\
x_{k+1}
\end{bmatrix}^T,
\] (35e)

\[
\Sigma_i \triangleq \sum_{k \in G_i} \begin{bmatrix}
x_k^T \\
u_k^T
\end{bmatrix},
\] (35f)

\[
N_i = \sum_{k \in G_i} 1,
\] (35g)

\[
u_{j,i} = \sum_{k \in G_{j,i}} 1,
\] (35h)
where \( G_i \) is the set of time steps \( k \) for which \( z_k^i = i \), and \( G_{j;i} \) is the set of time steps \( k \) for which \( z_{k+1}^i = j \) and \( z_k^i = i \). Note that \( u_{j;i} \) denotes the element in the \( j \)-th row and \( i \)-th column of matrix \( u \).

Lemma 4.5 provides a numerically stable implementation of Lemma 4.4.

**Lemma 4.5.** The distribution which parameters are sampled from can be expressed as

\[
p((\mathbf{G}(i), \Pi(i))_{i=1}^{m}, \mathbf{T}, x_{1:N,1+1}, z_{1:N+1,1+1}, y_{1:N}) \propto \mathcal{D}(\mathbf{T} | \alpha) \left( \prod_{i=1}^{m} \mathcal{MN}(\mathbf{G}(i)|\Pi(i), \mathbf{V}_i)\mathcal{W}^{-1}(\Pi(i)|\tilde{\Lambda}_i, \tilde{\nu}_i) \right).
\]

Unlike Lemma 4.4, a slightly different parameter set is used for numerical reasons. The parameters \( \{\nu_i, \mathbf{M}_i, (\mathbf{V}_i^{-1})^{1/2}, \tilde{\Lambda}_i^{1/2}\} \) parametrise the prior the \( i \)-th model, which can be updated to yield \( \{\tilde{\nu}_i, \mathbf{M}_i, (\mathbf{V}_i^{-1})^{1/2}, \tilde{\Lambda}_i^{1/2}\} \) using

\[
\mathbf{M}_i = ((\mathbf{R}_{11}^{-1})^{-1}\mathbf{R}_{12})^T, \quad (\mathbf{V}_i^{-1})^{1/2} = \mathbf{R}_{11}, \quad \tilde{\Lambda}_i^{1/2} = \mathbf{Q}\begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_N^i \\ (\mathbf{V}_i^{-1})^{1/2} \end{bmatrix},
\]

where

\[
\mathbf{R}^i = \begin{bmatrix} \mathbf{R}_{11}^i & \mathbf{R}_{12}^i \\ 0 & \mathbf{R}_{22}^i \end{bmatrix} = \mathbf{Q}\begin{bmatrix} \lambda_1^i \\ \lambda_2^i \\ \vdots \\ \lambda_N^i \\ (\mathbf{V}_i^{-1})^{1/2} \end{bmatrix},
\]

and

\[
\lambda_j^i = [\mathbf{x}_k^T \ y_k^T \ y_k^T \ (\mathbf{x}_{k+1}^T)] \quad \forall k \in G_i,
\]

where \( j \) is the next free index assigning an arbitrary order to elements in the set \( G_i \). Calculation of the parameters \( \tilde{\nu}_i \) and \( \tilde{\alpha} \) are identical to Lemma 4.4.

### 4.4 Sampling new parameters

As the distribution the distribution \( p(\theta|z_{1:N+1}, y_{1:N}) \) can now be calculated from Lemma 4.4 and Lemma 4.5 we now provide instruction on how \( \theta^{t+1} \) may be sampled from such distribution.

We begin by sampling \( \Pi(i) \) from a inverse-Wishart distribution for each model \( i = 1, \ldots, m \), i.e.

\[
\Pi^{t+1}(i) \sim \mathcal{W}^{-1}(\tilde{\Lambda}_i, \tilde{\nu}_i) \quad \forall i = 1, \ldots, m.
\]

Note that it is possible to use \( \tilde{\Lambda}_i^{1/2} \) to sample \( (\Pi^{t+1}(i))^{1/2} \) directly when implementing the recommended numerically stable form. This affords a Bartlett decomposition for efficient calculation when \( \tilde{\nu}_i \) is large or if \( \nu_i \) is not an integer, see [33] for details.

With \( \Pi^{t+1}(i) \) sampled for \( i = 1, \ldots, m \), it is then possible to sample \( \Gamma^{t+1}(i) \) for each model \( i = 1, \ldots, m \) using Lemma 4.6.

**Lemma 4.6.** If we let \( \Gamma^{t+1}(i) \) be determined by

\[
\Gamma^{t+1}(i) = \mathbf{M}_i + ((\Pi^{t+1}(i))^{1/2})^T \mathbf{H}_i \mathbf{V}_i^{1/2}
\]

where each element of \( \mathbf{H}_i \) is distributed i.i.d. according to

\[
h \sim \mathcal{N}(0, 1),
\]

then

\[
\Gamma^{t+1}(i) \sim \mathcal{MN}(\mathbf{M}_i, \Pi^{t+1}(i), \mathbf{V}_i).
\]
Proof. See [33]. Finally sampling the transition matrix $T$ can be completed by sampling from $m$ Dirichlet distributions, all parametrised by $\alpha$. This is completed by sampling each element of $T$ from a Gamma distribution with shape parameter $\alpha_{i,j}$, and scale parameter of 1, i.e.,

$$\tilde{T}_{i,j} \sim \mathcal{G}(\alpha_{i,j}, 1), \quad (46)$$

before normalising over each column,

$$T_{i,j}^{\ell+1} = \frac{\tilde{T}_{i,j}}{\sum_i \tilde{T}_{i,j}}. \quad (47)$$

The procedure for sampling the parameter set is further outlined in Algorithm 2.

Algorithm 2 Sampling the parameters

1: Sample from the inverse-Wishart distribution $\Pi^{\ell+1}(i) \sim \mathcal{W}^{-1}(\Pi(i)|\bar{\Lambda}_i, \bar{\nu}_i) \quad \forall i = 1, \ldots, m.$
2: Using Lemma 4.6 sample from the Matrix-Normal distribution $\Gamma^{\ell+1}(i) \sim \mathcal{MN}(\Gamma(i)|\bar{M}_i, \Pi^{\ell+1}(i), \bar{V}_i) \quad \forall i = 1, \ldots, m.$
3: Sample from the Gamma distribution $\tilde{T}_{i,j} \sim \mathcal{G}(\alpha_{i,j}, 1)$, with shape parameter $\alpha_{i,j}$, and scale parameter of 1 $\forall i = 1, \ldots, m, \forall j = 1, \ldots, m.$
4: Set $T_{i,j}^{\ell+1} = \frac{\tilde{T}_{i,j}}{\sum_i \tilde{T}_{i,j}} \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, m.$

After the new parameters $\theta^{\ell+1}$ have been sampled, the algorithm samples a new latent hybrid state trajectory, by returning to the stage outlined in subsection 4.1.

4.5 Algorithm overview

For clarity, the proposed method is summarised in full by Algorithm 3.

Algorithm 3 Algorithm overview

Require: State prior $p(x_0, z_0)$, prior on model parameters defined by $\{\nu_i, M_i, V_i, \Lambda_i\}_{i=1}^m$, prior on model transitions defined by $\alpha$, initial guess of $\theta^1$, which may be provided using the JMLS EM algorithm [6].

1: for $\ell = 1$ to Max iterations do
2: Forward-filter the system using Lemma 4.1 with (13).
3: Sample a latent trajectory $\xi^\ell \sim p(\xi|\theta^\ell, y_{1:N})$ using the instruction provided in subsection 4.2.
4: Calculate the conditional parameter distribution $p(\theta|\xi^\ell, y_{1:N})$ using Lemma 4.4 and Lemma 4.5.
5: Sample and save new parameter estimate $\theta^{\ell+1} \sim p(\theta|\xi^\ell, y_{1:N})$ using Algorithm 2.
6: end for
7: By the convergence properties of the Gibbs sampler, $\theta^\ell$ samples are now distributed according to the distribution $p(\theta|y_{1:N})$.

12
5 Simulations

In this section we provide two simulations demonstrating the effectiveness of the proposed method.

5.1 Univariate system

In this example we use the proposed method on a univariate JMLS single-input single-output (SISO) system, comprising of two models \( m = 2 \). This ensures that parameters are scalar and distributions can appear on 2D plots. The choice to estimate a univariate system also ensures that certain system matrices (\( A, D, R \)) are free from a similarity transformation \([3, 47, 46, 6]\), as in general there is not a unique solution for these parameters.

The true system used in this example was parameterised by

\[
\begin{align*}
A(1) &= 0.4766, B(1) = -1.207, C(1) = 0.233, \\
D(1) &= -0.8935, Q(1) = 10^{-3}, R(1) = 0.0022, S(1) = 0, \\
A(2) &= -0.1721, B(2) = 1.5330, C(2) = -0.1922, \\
D(2) &= 1.7449, Q(2) = 0.0340, R(2) = 0.0439, S(2) = 0, \\
T &= 
\begin{bmatrix}
0.7 & 0.5 \\
0.3 & 0.5
\end{bmatrix}.
\end{align*}
\]

(48)

Input-output data was generated using these parameters and the input \( u_k \sim \mathcal{N}(0, 1) \) for \( N = 2000 \) time steps before the proposed method was used on the dataset. The proposed method used a resampling step allowing \( R = 5 \) hybrid components per time step, and used uninformative priors on the parameters to ensure the PDFs were highly data driven. The priors chosen were parameterised by

\[
\begin{align*}
M_1 &= M_2 = 0_{2 \times 2}, & & (49a) \\
V_1 &= V_2 = 13 \cdot I_{2 \times 2}, & & (49b) \\
\Lambda_1 &= \Lambda_2 = 10^{-10} \cdot I_{2 \times 2}, & & (49c) \\
\nu_1 = \nu_2 &= 2, & & (49d) \\
\alpha &= I_{2 \times 2}. & & (49e)
\end{align*}
\]

Using these priors, the PDFs produced by the proposed method should have good support of the true parameters, and grow certainty about them with increasing size of dataset. Other alternative algorithms \([40]\) can potentially operate on this system, but due to their use of inverse-Gamma distributions, cannot operate on the example within subsection 5.2. Additionally, as a univariate inverse-Wishart distribution is an inverse-Gamma distribution, these algorithms are equivalent for the univariate case.

The initial parameter set used in the Gibbs sampler is allowed to be chosen arbitrarily. To avoid a lengthy burn-in procedure, the Gibbs algorithm was initialised with values close to those which correspond the maximum likelihood solution. For a real-world problem this could be provided using the EM algorithm \([6]\).

After \( 10^5 \) iterations of the Gibbs algorithm, the parameter samples \( \theta_i \) were used to construct Figure 1 and Figure 2. Figure 1 shows the distribution of diagonal elements of the transition matrix, and represents the probability of models being used for consecutive time steps. As the off-diagonals are constrained by the total law of probability, there is no need to plot them. Whereas Figure 2 shows the distribution of components within \( \{\Gamma(i), \Pi(i)\}_{i=1}^m \) which are free from a similarity transformation.

The proposed method has produced distributions with a large amount of support for the true parameters.

Generation of the both figures for this example, required the models to be sorted. Models were sorted or ‘relabelled’ by comparison of the magnitude of the models Bode response. This is not a core part of the proposed algorithm, as reordering is only required for plotting purposes.
5.2 Multivariate system

In this example, we consider identification of a multivariate three-state SISO JMLS system comprising of three models \( m = 3 \). To the best of the authors knowledge there are no alternative algorithms suitable for this problem, or for generating a ground truth.

As this is a multivariate system, there are infinite state space modes which has equivalent system response. Because of this, plotting the distribution of the model parameters themselves would be somewhat arbitrary. Instead, for the analysis, we provide a variation likely frequency responses of the models. The distribution of the transition matrix however, is free from similarity transformations, but due to the increase in available models can no longer appear on a 2D plot.

The system analysed in this example, described by the discrete transfer functions and transition matrix

\[
H_1(z) = \frac{1.034z^3 - 0.7514z^2 - 0.02663z - 0.1818}{z^3 - 1.186z^2 - 0.4062z + 0.5993},
\]

\[
H_2(z) = \frac{-0.3133z^3 - 0.534z^2 - 0.07519z + 4.543 \times 10^{-5}}{z^3 + 0.7556z^2 + 0.0832z + 0.001395},
\]

\[
H_3(z) = \frac{0.5212z^3 - 0.3133z^2 + 0.03974z + 0.001288}{z^3 - 0.5841z^2 + 0.07541z + 0.001974},
\]

\[
T = \begin{bmatrix}
0.5 & 0.25 & 0.25 \\
0.25 & 0.5 & 0.25 \\
0.25 & 0.25 & 0.5
\end{bmatrix},
\]

was then simulated for \( N = 5000 \) time steps using an input \( u_k \sim \mathcal{N}(0, 1) \). This system was then used to initialise the proposed procedure, and avoid a lengthy burn-in time. For a real-world example, this initial estimate could be obtained using the EM algorithm [6].

The proposed method was then used to identify the system based on the generated input-output data with the filter being allowed to store \( R = 15 \) hybrid Gaussian mixture components. The uninformative priors chosen for identification were...
Figure 2: Estimated model parameter distributions for Example 1. The distribution of parameters free from a similarity transformation are shown in solid blue, whereas the true values are indicated by a dashed red vertical line.
After $10^5$ Gibbs iterations, the samples $\theta^l$ were used to construct Figure[3] and Figure[4]. Figure[3] shows the estimated distribution of model transition probabilities, and Figure[4] shows the variation of expected model responses. As with Example 1, models were sorted using their frequency response before producing these figures. Both of these figures show good support for the model used to generate the data.
Figure 4: Frequency response from the three models from Example 2. The blue line is the true system response, where the red line and shaded red region represents the estimated mean response and 3 standard deviation confidence region respectively.
6 Conclusion

We have developed and demonstrated an effective algorithm for Bayesian parameter identification of JMLS systems. Unlike alternative methods, we have not forced assumptions such as a univariate state or operation according to drift models, and allow all parameters to be estimated without constraints. The proposed method scales easily to an increase in models and state dimension.

The proposed method was deployed for Bayesian estimation of a multivariate JMLS system in subsection 5.2, yielding distributions with good support of the transition matrix and models used to generate the data. To the best of our knowledge, no alternative algorithms are available for this system identification problem.
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A Proof of Lemmata

In this appendix we provide proofs for the Lemma within the paper. These proofs rely on supporting Lemmata within Appendix B.

A.1 Proof of Lemma 4.2

We begin by reintroducing the auxiliary variable $i$ of some hybrid Gaussian mixture, which represents a possible model sequence for this application,

$$p(x, z, i|\cdot) = w^i(z)N(x|\mu^i(z), P^i(z)).$$

(52)

Normally this variable is not of interest and is marginalised to give

$$p(x, z|\cdot) = \sum_{i=1}^{M(z)} p(x, z, i|\cdot) = \sum_{i=1}^{M(z)} w^i(z)N(x|\mu^i(z), P^i(z)).$$

(53)

Using conditional probability, we yield

$$p(x, z, i|\cdot) = p(x|z, i, \cdot)p(z, i|\cdot).$$

(54)

Next we consider the distribution of $p(z, i|\cdot)$

$$p(z, i|\cdot) = \int p(x, z, i|\cdot) \, dx$$

$$= \int w^i(z)N(x|\mu^i(z), P^i(z)) \, dx = w^i(z),$$

(55)

and as such $p(z, i|\cdot)$ is a Categorical distribution, which can be sampled from using

$$\{z^\ell, i^\ell\} \sim C(\{w^i(z)\}_i, \pi).$$

(56)

Next it follows from (52) and (55) that

$$p(x|z, i, \cdot) = \frac{p(x, z, i|\cdot)}{p(z, i|\cdot)} = \frac{w^i(z)N(x|\mu^i(z), P^i(z))}{w^i(z)} = N(x|\mu^i(z), P^i(z)),$$

(57)

and therefore sampling can be completed straight-forwardly using

$$x^\ell \sim N(x|\mu^{i^\ell}(z^\ell), P^{i^\ell}(z^\ell)).$$

(58)
A.2 Proof of Lemma 4.3

For readability, in this proof we utilise the shorthand \( \theta^f = \{ \{ \Gamma^i, \Pi^i \}_i \in \mathbb{N}, T^i \} \). We begin the derivation with

\[
p(x_{1:N+1}, z_{1:N+1} | \{ \{ \Gamma^i, \Pi^i \}_i \in \mathbb{N}, T^i \}, y_{1:N})
= p(x_{1:N+1}, z_{1:N+1} | \theta^f, y_{1:N})
= p(x_{N+1}, z_{N+1} | \theta^f, y_{1:N}) \prod_{k=1}^N p(x_k, z_k | x_{k+1:N+1}, z_{k+1:N+1}, \theta^f, y_{1:N})
= p(x_{N+1}, z_{N+1} | \theta^f, y_{1:N}) \prod_{k=1}^N p(x_k, z_k | x_{k+1}, z_{k+1}, \theta^f, y_{1:N})
= p(x_{N+1}, z_{N+1} | \theta^f, y_{1:N}) \prod_{k=1}^N p(x_k, z_k | x_{k+1}, z_{k+1}, \theta^f, y_{1:k})
= p(x_{N+1}, z_{N+1} | \theta^f, y_{1:N}) \prod_{k=1}^N \frac{p(x_{k+1}, z_{k+1} | x_k, z_k, \theta^f, y_{1:k}) p(x_k, z_k | \theta^f, y_{1:k})}{p(x_{k+1}, z_{k+1} | \theta^f, y_{1:k})}.
\]

(59)

The prediction distribution \( p(x_{N+1}, z_{N+1} | \theta^f, y_{1:N}) \) can be sampled from using Lemma 22. We will now outline sampling the latent variables \( \{ x_k^f, z_k^f \} \) from the distribution

\[
p(x_k, z_k | \theta^f, y_{1:k}) = \sum_{i=1}^{M^f(z_k)} w_i^{f_k} N(x_k | \mu_k^{f_k}(z_k), P_k^{f_k}(z_k)),
\]

(60)

and

\[
p(x_{k+1}, z_{k+1} | x_k, z_k, \theta^f, y_{1:k})
= \mathbb{P}(z_{k+1} | x_{k+1}, x_k, z_k, \theta^f, y_{1:k}) p(x_{k+1} | x_k, z_k, \theta^f, y_{1:k})
= \mathbb{P}(z_{k+1} | z_k, \theta^f) p(x_{k+1} | x_k, z_k, \theta^f, y_{1:k})
= T_{k+1}^{f_k} z_k^f N(x_{k+1} | A_k^f(z_k) x_k + b_k^f(z_k), Q_k^f(z_k)),
\]

(61)

we can rewrite and manipulate the numerator as follows,

\[
p(x_k, z_k | \theta^f, y_{1:k}) p(x_{k+1}, z_{k+1} | x_k, z_k, \theta^f, y_{1:k})
= \sum_{i=1}^{M^f(z_k)} w_i^{f_k} N(x_k | \mu_k^{f_k}(z_k), P_k^{f_k}(z_k)) T_{k+1}^{f_k} z_k^f N(x_{k+1} | A_k^f(z_k) x_k + b_k^f(z_k), Q_k^f(z_k))
= \sum_{i=1}^{M^f(z_k)} w_i^{f_k} z_k^f N(x_k | \mu_k^{f_k}(z_k), P_k^{f_k}(z_k)),
\]

(62)

where \( b_k = B_k \tilde{u}_k^{f_k}(z_k) \), and \( \mu_k^{f_k} \), and \( P_k^{f_k} \) can be computed straight-forwardly as this pattern of Normal distribution terms has a well known solution which is identical the correction step of the weighted Kalman filter used for forward-filtering. Applying this weighted Kalman filter correction yields the following

\[
\tilde{u}_k^{f_k}(z_k) = T_{k+1}^{f_k} z_k^f w_i^{f_k} z_k^f N(x_{k+1} | \eta_k^{f_k}(z_k), \Xi_k^{f_k}(z_k)),
\]

(63a)

\[
\mu_k^{f_k}(z_k) = \mu_k^{f_k}(z_k) + K_k^{f_k}(z_k) \{ x_{k+1} - \eta_k^{f_k}(z_k) \},
\]

(63b)

\[
\eta_k^{f_k}(z_k) = A_k(z_k) \mu_k^{f_k}(z_k) + B_k(z_k) \tilde{u}_k,
\]

(63c)

\[
P_k^{f_k}(z_k) = \left( I - K_k^{f_k}(z_k) A_k(z_k) \right) P_k^{f_k}(z_k),
\]

(63d)

\[
K_k^{f_k}(z_k) = P_k^{f_k}(z_k) A_k^T(z_k) (\Xi_k^{f_k}(z_k))^{-1},
\]

(63e)

\[
\Xi_k^{f_k}(z_k) = A_k(z_k) P_k^{f_k}(z_k) A_k^T(z_k) + Q_k(z_k),
\]

(63f)
Sampling from this hybrid Gaussian mixture can be completed using Lemma 4.2.

Proof of Lemma 4.4

Alternatively, the same numerically stable QR decomposition pattern used within the forward-filter may be used. Notice that the dependence on $\ell$ has been omitted from forward-filter terms within (63) to improve readability. With the numerator of

$$p(x_{k+1}^\ell, z_{k+1}^\ell | x_k, z_k, \theta^\ell, y_{1:k}) p(x_k, z_k | \theta^\ell, y_{1:k})$$

now having a closed form, we can now focus on the denominator

$$p(x_{k+1}^\ell, z_{k+1}^\ell | \theta^\ell, y_{1:k})$$

$$= \sum_{x_k=1}^m \int p(x, z_k | \theta^\ell, y_{1:k}) p(x_{k+1}^\ell, z_{k+1}^\ell | x_k, z_k, \theta^\ell, y_{1:k}) dx_k$$

$$= \sum_{x_k=1}^m \sum_{i=1}^M w^i_{k|N}(z_k) N(x_k | \mu^i_{k|N}(z_k), P^i_{k|N}(z_k))$$

Therefore the distribution $p(x, z_k | x_{k+1:N+1}, z_{k+1:N+1}, \theta^\ell, y_{1:N})$ can be written as

$$p(x, z_k | x_{k+1:N+1}, z_{k+1:N+1}, \theta^\ell, y_{1:N})$$

$$= \frac{p(x_{k+1}^\ell, z_{k+1}^\ell | x_k, z_k, \theta^\ell, y_{1:k}) p(x_k, z_k | \theta^\ell, y_{1:k})}{p(x_{k+1}^\ell, z_{k+1}^\ell | \theta^\ell, y_{1:k})}$$

$$= \frac{\sum_{i=1}^M w^i_{k|N}(z_k) N(x_k | \mu^i_{k|N}(z_k), P^i_{k|N}(z_k))}{\sum_{z_k=1}^m \sum_{i=1}^M w^i_{k|N}(z_k)}$$

where

$$w^i_{k|N}(z_k) = \frac{\tilde{w}^i_{k|N}(z_k)}{\sum_{z_k=1}^m \sum_{i=1}^M \tilde{w}^i_{k|N}(z_k)}.$$ (66)

Sampling from this hybrid Gaussian mixture can be completed using Lemma 4.2.

Proof of Lemma 4.4

For readability, in this proof we utilise the shorthand $\theta = \{\{\Gamma_i, \Pi_i\}_{i=1}^m, T\}$. We begin with

$$p(\theta | x_{1:N+1}, z_{1:N+1}, y_{1:N})$$

$$= \frac{p(x_{1:N+1}^\ell, z_{1:N+1}^\ell, y_{1:N} | \theta) p(\theta)}{p(x_{1:N+1}^\ell, z_{1:N+1}^\ell, y_{1:N})}$$

$$\propto p(x_{1:N+1}^\ell, z_{1:N+1}^\ell, y_{1:N} | \theta) p(\theta)$$

$$= p(\theta) p(x_{1:N}^\ell, z_{1:N}^\ell) \prod_{k=1}^N p(x_{k+1}^\ell, z_{k+1}^\ell | y_k, x_k^\ell, z_k^\ell, y_{1:k-1}, \theta)$$

$$\propto p(\theta) \prod_{k=1}^N p(z_{k+1}^\ell | x_{k+1}^\ell, z_k^\ell, y_{k+1}, \theta) p(x_{k+1}^\ell, y_k | x_k^\ell, z_k^\ell, y_{1:k-1}, \theta) p(x_{k+1}^\ell, y_k | x_k^\ell, z_k^\ell, y_{1:k-1}, \theta),$$ (67)
as $p(x_1^\ell, z_1^\ell)$ is a constant for each iteration. By expanding $\theta$ and exercising conditional independence, we yield

$$p(\theta | x_1^{\ell N+1}, z_1^{\ell N+1}, y_{1:N}) \propto p((\Gamma_i, \Pi_i)_{i=1}^m, T) \prod_{k=1}^N \mathcal{P}(z_k^{\ell+1} | z_k^{\ell}, T) p(x_{k+1}^{\ell}, y_k | x_k^{\ell}, \Gamma_k^{\ell}, \Pi_k^{\ell})$$

$$= (p(T) \prod_{i=1}^m p(\Gamma_i | \Pi_i)) \prod_{k=1}^N \mathcal{P}(z_k^{\ell+1} | z_k^{\ell}, T) p(x_{k+1}^{\ell}, y_k | x_k^{\ell}, \Gamma_k^{\ell}, \Pi_k^{\ell})$$

$$= (p(T) \prod_{i=1}^m \mathcal{P}(z_k^{\ell+1} | z_k^{\ell}, T)(\prod_{i=1}^m p(\Gamma_i | \Pi_i)) \prod_{k=1}^N p(x_{k+1}^{\ell}, y_k | x_k^{\ell}, \Gamma_k^{\ell}, \Pi_k^{\ell})).$$ (68)

Next we consider each of the columns in the $T$ matrix, written as $T_i$, to be conditionally independent, and therefore $p(T) = \prod_{i=1}^m p(T_i)$.

$$p(\theta | x_1^{\ell N+1}, z_1^{\ell N+1}, y_{1:N})$$

$$\propto (\prod_{i=1}^m p(T_i)) \prod_{k=1}^N \mathcal{P}(z_k^{\ell+1} | z_k^{\ell}, T)(\prod_{i=1}^m p(\Gamma_i | \Pi_i)) \prod_{k=1}^N p(x_{k+1}^{\ell}, y_k | x_k^{\ell}, \Gamma_k^{\ell}, \Pi_k^{\ell}))$$

$$= (\prod_{i=1}^m p(T_i) \prod_{k \in G_i} \mathcal{P}(z_k^{\ell+1} | T_i)(\prod_{i=1}^m p(\Gamma_i | \Pi_i)) \prod_{k \in G_i} p(x_{k+1}^{\ell}, y_k | x_k^{\ell}, \Gamma_i, \Pi_i)),$$ (69)

where $G_i$ is the set of time steps in sample $\ell$ which has model $i$ being active, i.e. $i = z_k^\ell, k \in G_i$. Substituting the assumed distributions for these terms yields

$$p(\theta | x_1^{\ell N+1}, z_1^{\ell N+1}, y_{1:N})$$

$$\propto (\prod_{i=1}^m D(T_i | \alpha)) \prod_{k \in G_i} C(z_k^{\ell+1} | T_i)(\prod_{i=1}^m \mathcal{N}(\Gamma_i | M_i, \mathcal{V}_i)) W^{-1}(\Pi_i | \Lambda_i, \nu_i) \prod_{k \in G_i} \mathcal{N}(\left[ \frac{y_k}{x_k^{\ell+1}} \right] | \Gamma_i \left[ \frac{x_k^\ell}{u_k} \right], \Pi_i)),$$ (70)

where $\alpha_i$ denotes the $i$-th column of $\alpha$. We can now use the supporting Lemma [B.1] and Lemma [B.2] for updating the conjugate prior. This yields a solution of the form

$$p((\Gamma_i, \Pi_i)_{i=1}^m, T | x_1^{\ell N+1}, z_1^{\ell N+1}, y_{1:N}) \propto (\prod_{i=1}^m D(T_i | \alpha_i))(\prod_{i=1}^m \mathcal{N}(\Gamma_i | M_i, \mathcal{V}_i)) W^{-1}(\Pi_i | \Lambda_i, \nu_i)).$$ (71)

A sample from this distribution can be taken by sampling from the Dirichlet, Matrix-Normal and Inverse-Wishart distribution for each model, these distributions are parameterised by

$$\tilde{\Lambda}_i \triangleq \Lambda_i + \tilde{\Phi}_i - \tilde{\Psi}_i \Sigma_i^{-1} \tilde{\Phi}_i^T,$$ (72a)

$$\tilde{\nu}_i \triangleq \nu_i + N_i,$$ (72b)

$$\tilde{M}_i \triangleq \tilde{\Psi}_i \Sigma_i^{-1},$$ (72c)

$$\tilde{\mathcal{V}}_i \triangleq \Sigma_i^{-1},$$ (72d)

$$\tilde{\alpha} = \alpha + u,$$ (72e)
which use the quantities

\[ \Sigma_i \triangleq \Sigma_i + V_i^{-1}, \quad (73a) \]
\[ \Psi_i \triangleq \Psi_i + M_i V_i^{-1}, \quad (73b) \]
\[ \Phi_i \triangleq \Phi_i + \Phi_i M_i V_i^{-1} M_i^T, \quad (73c) \]
\[ \Phi_i \triangleq \sum_{k \in G_i} \begin{bmatrix} y_k \\ x_k \end{bmatrix} \begin{bmatrix} y_k \\ x_k \end{bmatrix}^T, \quad (73d) \]
\[ \Psi_i \triangleq \sum_{k \in G_i} \begin{bmatrix} y_k \\ x_k \end{bmatrix} \begin{bmatrix} f_k \\ u_k \end{bmatrix}^T, \quad (73e) \]
\[ \Sigma_i \triangleq \sum_{k \in G_i} \begin{bmatrix} x_k^T \\ u_k^T \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad (73f) \]
\[ N_i = \sum_{k \in G_i} 1, \quad (73g) \]
\[ u_{j,i} = \sum_{k \in G_{j,i}} 1, \quad (73h) \]

where \( G_{j,i} \) is the set of time steps \( k \) for which \( x_{k+1}^j = j \) and \( x_k^i = i \) and \( u_{j,i} \) denotes the element in the \( j \)-th row and \( i \)-th column of \( u \).

\[ \Box \]

**Proof of Lemma 4.5**

In this proof, we relax the notation for readability, as such we have omitted the subscript \( i \) indicating ownership to the \( i \)-th model. All variables in this proof implicitly have this ownership. We begin by defining \( \lambda_k \) as

\[ \lambda_k \triangleq \begin{bmatrix} \zeta_k^T \\ d_k \end{bmatrix}, \quad (74) \]

where

\[ \zeta_k \triangleq \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad d_k \triangleq \begin{bmatrix} y_k \\ x_k \end{bmatrix}. \quad (75) \]

Then by considering the \( Q \)-less QR decomposition on the matrix

\[ \mathcal{R} = Q \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} \begin{bmatrix} \sqrt{V^{-1}} \\ \sqrt{V^{-1} M M^T} \end{bmatrix}, \quad (76) \]

we can expand \( \mathcal{R}^T \mathcal{R} \) as

\[
\mathcal{R}^T \mathcal{R} = \lambda_1^T \lambda_1 + \lambda_2^T \lambda_2 + \cdots + \lambda_N^T \lambda_N
+ \left[ \begin{bmatrix} \zeta_1 \\ d_1 \end{bmatrix} \begin{bmatrix} \zeta_1^T \\ d_1 \end{bmatrix} \right] + \left[ \begin{bmatrix} \zeta_2 \\ d_2 \end{bmatrix} \begin{bmatrix} \zeta_2^T \\ d_2 \end{bmatrix} \right] + \cdots + \left[ \begin{bmatrix} \zeta_N \\ d_N \end{bmatrix} \begin{bmatrix} \zeta_N^T \\ d_N \end{bmatrix} \right]
+ \begin{bmatrix} V^{-1} \\ MV^{-1} \end{bmatrix} \begin{bmatrix} V^{-1} M M^T \\ MV^{-1} M M^T \end{bmatrix}
+ \begin{bmatrix} V^{-1} + \sum_{j=1}^N \zeta_j \zeta_j^T \\ MV^{-1} + \sum_{j=1}^N d_j d_j^T \end{bmatrix} \begin{bmatrix} V^{-1} M M^T + \sum_{j=1}^N \zeta_j d_j^T \\ MV^{-1} M M^T + \sum_{j=1}^N d_j d_j^T \end{bmatrix}
= \Sigma \begin{bmatrix} \Psi \\ \Phi \end{bmatrix}. \quad (77)
\]
Exploiting the upper triangular structure of \( \mathcal{R} \) yields
\[
\mathcal{R}^T \mathcal{R} = \begin{bmatrix} R_{11}^T & 0 \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} = \begin{bmatrix} R_{11}^T R_{11} & R_{11}^T R_{12} \\ R_{12}^T R_{11} & R_{12}^T R_{12} + R_{22}^T R_{22} \end{bmatrix} = \begin{bmatrix} \Sigma & \Psi^T \\ \Psi & \Phi \end{bmatrix}. \tag{78}
\]

Equating parts yields the following three equations,
\[
\begin{align*}
\mathcal{R}_{11}^T \mathcal{R}_{11} &= \bar{\Sigma} \rightarrow \mathcal{R}_{11} = \bar{\Sigma}^{1/2} = (\bar{\mathbf{V}}^{-1})^{1/2}, \\
\mathcal{R}_{11}^T \mathcal{R}_{12} &= \bar{\Psi}^T \rightarrow \mathcal{R}_{12} = \mathcal{R}_{11}^{-T} \bar{\Psi}^T = (\bar{\Sigma}^{1/2})^{-T} \bar{\Psi}^T, \\
\mathcal{R}_{12}^T \mathcal{R}_{12} + \mathcal{R}_{22}^T \mathcal{R}_{22} &= \bar{\Phi} \\
\rightarrow \mathcal{R}_{22}^T \mathcal{R}_{22} &= \Phi - \mathcal{R}_{12}^T \mathcal{R}_{12} \\
&= \Phi - \left( (\bar{\Sigma}^{1/2})^{-T} \bar{\Psi}^T \right)^T \left( (\bar{\Sigma}^{1/2})^{-T} \bar{\Psi}^T \right) \\
&= \Phi - \bar{\Psi} (\bar{\Sigma}^{1/2})^{-1} (\bar{\Sigma}^{1/2})^{-T} \bar{\Psi}^T \\
&= \Phi - \bar{\Psi} ((\bar{\Sigma}^{1/2})^T \bar{\Sigma}^{1/2})^{-1} \bar{\Psi}^T \\
&= \Phi - \bar{\Psi} \bar{\Sigma}^{-1} \bar{\Psi}^T. \tag{79d}
\end{align*}
\]

Now computing \( \mathcal{R}_{11}^{-1} \mathcal{R}_{12} \) yields
\[
\begin{align*}
\mathcal{R}_{11}^{-1} \mathcal{R}_{12} &= \mathcal{R}_{11}^{-1} \mathcal{R}_{11}^{-T} \bar{\Psi}^T \\
&= (\mathcal{R}_{11}^T \mathcal{R}_{11})^{-1} \bar{\Psi}^T \\
&= \bar{\Sigma}^{-1} \bar{\Psi}^T, \tag{80}
\end{align*}
\]

and therefore
\[
(\mathcal{R}_{11}^{-1} \mathcal{R}_{12})^T = \bar{\Psi} \bar{\Sigma}^{-T} = \bar{\Psi} \bar{\Sigma}^{-1} = \bar{\mathbf{M}}, \tag{81}
\]

by the symmetry of \( \bar{\Sigma} \).

Finally by computing an additional QR decomposition with \( \Lambda^{1/2} \), we show
\[
\mathcal{Q} \left( \begin{bmatrix} \mathcal{R}_{22}^2 \\ \Lambda^{1/2} \end{bmatrix} \right) = \left( (\Lambda^{1/2})^T \Lambda^{1/2} + \mathcal{R}_{22}^T \mathcal{R}_{22} \right)^{1/2} \\
= (\Lambda + \Phi - \bar{\Psi} \bar{\Sigma}^{-1} \bar{\Psi}^T)^{1/2} = \Lambda^{1/2}. \tag{82}
\]

B Supporting Lemmata

This appendix contains supporting Lemmata for the paper.

**Lemma B.1.** Suppose that an observation \( y \) was realized from a Categorical distribution, and observed to take the value \( j \). If the Categorical distribution was parameterised by a Dirichlet distribution i.e., \( C(y = j|x)D(x|\alpha) \), then we can apply the following correction to the Dirichlet distribution to yield \( D(x|\bar{\alpha}) \).

\[
D(x|\bar{\alpha}) = C(y = j|x)D(x|\alpha)
\] (83)

where \( \bar{\alpha} = \{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_K\} \), and

\[
\bar{\alpha}_i = \begin{cases} 
\alpha_i + 1 & \text{if } i = j, \\
\alpha_i & \text{otherwise}.
\end{cases}
\] (84)

**Proof.** We begin by expressing the likelihood of the Categorical observation in-terms of the Dirichlet distribution,

\[
C(y = j|x)D(x|\alpha) = x_jD(x|\alpha)
\]

\[
= x_j \Gamma(\sum_{i=1}^{K} \alpha_i) \prod_{i=1}^{K} \Gamma(\alpha_i) \prod_{i=1}^{K} x_i^{\alpha_i - 1}
\]

\[
= \Gamma(\sum_{i=1}^{K} \bar{\alpha}_i) \prod_{i=1}^{K} \Gamma(\bar{\alpha}_i) \prod_{i=1}^{K} x_i^{\bar{\alpha}_i - 1}
\]

where

\[
\bar{\alpha}_i = \begin{cases} 
\alpha_i + 1 & \text{if } i = j, \\
\alpha_i & \text{otherwise}.
\end{cases}
\] (86)

Now by factoring the required normalising constant yields

\[
C(y = j|x)D(x|\alpha) = \frac{\Gamma(\sum_{i=1}^{K} \alpha_i) \prod_{i=1}^{K} \Gamma(\alpha_i) \Gamma(\sum_{i=1}^{K} \bar{\alpha}_i) \prod_{i=1}^{K} \Gamma(\bar{\alpha}_i) \prod_{i=1}^{K} x_i^{\bar{\alpha}_i - 1}}{\Gamma(\sum_{i=1}^{K} \bar{\alpha}_i) \prod_{i=1}^{K} \Gamma(\bar{\alpha}_i)} D(x|\bar{\alpha})
\]

\[
\propto D(x|\bar{\alpha}).
\] (87)

**Lemma B.2.** Given a Matrix-Normal Inverse-Wishart distribution parameterised by \( \{M, V, \Lambda, \nu\} \) describing the distribution of variables \( \Gamma \) and \( \Pi \), then this distribution multiplied by a product of Normal distributions with the form \( \prod_{i=1}^{N} N(d^i|\bar{\xi}^i, \bar{\Sigma}) \) is proportional to another Matrix-Normal distribution i.e.,

\[
M.N(\Gamma|M, \Pi, \bar{V})W^{-1}(\Pi|\bar{\Lambda}, \bar{\nu}) \propto M.N(\Gamma|M, \Pi, V)W^{-1}(\Pi|A, \nu) \prod_{i=1}^{N} N(d^i|\bar{\xi}^i, \Pi),
\] (88)

where we define

\[
\bar{\Lambda} \triangleq A + \Phi - \Phi \Sigma^{-1} \bar{\Psi} \Psi^T, \\
\bar{\nu} \triangleq \nu + N, \\
\bar{M} \triangleq \bar{\Psi} \Sigma^{-1}, \\
\bar{V} \triangleq \Sigma^{-1},
\] (89a-d)
and,

\[ \Sigma \triangleq \Sigma + V^{-1}, \]  
(89e)

\[ \Psi \triangleq \Psi + MV^{-1}, \]  
(89f)

\[ \Phi \triangleq \Phi + MV^{-1}M^T, \]  
(89g)

\[ \Phi \triangleq \sum_{i=1}^{N} d_i d_i^T, \]  
(89h)

\[ \Psi \triangleq \sum_{i=1}^{N} d_i x_i^T, \]  
(89i)

\[ \Sigma \triangleq \sum_{i=1}^{N} x_i x_i^T. \]  
(89j)

**Proof.** We begin with the product of Normal distributions update term,

\[
\prod_{i=1}^{N} \mathcal{N}(d_i | \Gamma x_i, \Pi)
\]

\[
= \prod_{i=1}^{N} \frac{1}{\sqrt{|2\pi \Pi|}} \exp \left\{ -\frac{1}{2} (d_i - \Gamma x_i)^T \Pi^{-1} (d_i - \Gamma x_i) \right\}
\]

\[
= |2\pi \Pi|^{-N/2} \prod_{i=1}^{N} \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} (d_i - \Gamma x_i)(d_i - \Gamma x_i)^T \right] \right\}
\]

\[
= |2\pi \Pi|^{-N/2} \exp \left\{ \sum_{i=1}^{N} \frac{1}{2} \text{tr} \left[ \Pi^{-1} (d_i - \Gamma x_i)(d_i - \Gamma x_i)^T \right] \right\}
\]

\[
= |2\pi \Pi|^{-N/2} \exp \left\{ \frac{1}{2} \text{tr} \left[ \Pi^{-1} (\Phi - \Gamma \Psi^T - \Psi \Gamma^T + \Gamma \Sigma \Gamma^T) \right] \right\}, \quad (90)
\]

where we define \( \Phi \triangleq \sum_{i=1}^{N} d_i d_i^T, \Phi \triangleq \sum_{i=1}^{N} d_i x_i^T, \) and \( \Sigma \triangleq \sum_{i=1}^{N} x_i x_i^T. \)

We now rearrange the Matrix-Normal distribution before combining it with the previous equation,

\[
\mathcal{MN}(\Gamma | M, \Pi, V)
\]

\[
= \exp \left\{ -\frac{1}{2} \text{tr} \left[ V^{-1} (\Gamma - M)^T \Pi^{-1} (\Gamma - M) \right] \right\}
\]

\[
= \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} (\Gamma - M)V^{-1} (\Gamma - M)^T \right] \right\}
\]

\[
= \frac{1}{(2\pi)^{np/2}|V|^{n/2} |\Pi|^{p/2}} \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} \left( \Gamma V^{-1} \Gamma^T - \Gamma V^{-1} M^T - \Gamma V^{-1} \Gamma^T + \Gamma V^{-1} M^T + \Phi - \Gamma \Psi^T - \Psi \Gamma^T + \Gamma \Sigma \Gamma^T \right) \right] \right\}. \quad (91)
\]

By multiplication of (90) and (91)

\[
\mathcal{MN}(\Gamma | M, \Pi, V) \prod_{i=1}^{N} \mathcal{N}(d_i | \Gamma x_i, \Pi)
\]

\[
= \frac{|2\pi \Pi|^{-N/2}}{(2\pi)^{np/2}|V|^{n/2} |\Pi|^{p/2}} \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} \left( \Gamma V^{-1} \Gamma^T - \Gamma V^{-1} M^T - \Gamma V^{-1} \Gamma^T + \Gamma V^{-1} M^T + \Phi - \Gamma \Psi^T - \Psi \Gamma^T + \Gamma \Sigma \Gamma^T \right) \right] \right\}
\]

\[
= \frac{|2\pi \Pi|^{-N/2}}{(2\pi)^{np/2}|V|^{n/2} |\Pi|^{p/2}} \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} \left( \Gamma V^{-1} + \Sigma \right) \Gamma^T - \Gamma (V^{-1} M^T + \Psi^T) - (M V^{-1} + \Psi) \Gamma^T + \Phi + M V^{-1} M^T \right] \right\}
\]

\[
= \frac{|2\pi \Pi|^{-N/2}}{(2\pi)^{np/2}|V|^{n/2} |\Pi|^{p/2}} \exp \left\{ -\frac{1}{2} \text{tr} \left[ \Pi^{-1} \left( \Gamma \Sigma \Gamma^T - \Gamma \Psi^T - \Psi \Gamma^T + \Phi \right) \right] \right\}, \quad (92)
\]
where we define $\Sigma \triangleq \Sigma + V^{-1}, \Phi \triangleq \Phi + MV^{-1}$, and $\Phi \triangleq \Phi + MV^{-1}M^T$. We continue

\[
\mathcal{M}(\Gamma|M, \Pi, V) \prod_{i=1}^{N} \mathcal{N}(d^i | \Gamma x^i, \Pi) = \left[ \frac{|2\pi\Pi|^{-N/2}}{(2\pi)^{np/2}|V|^{n/2}|\Pi|^{p/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}((\Gamma^2 + \Sigma^{-1}T - \Gamma^T \Sigma^{-1} \Sigma^{-T} \Phi^T + \Phi)]) \right) \right],
\]

where we define $\Phi \triangleq \Phi + M V^{-1} M^T$,

\[
\mathcal{M}(\Gamma|M, \Pi, V) \prod_{i=1}^{N} \mathcal{N}(d^i | \Gamma x^i, \Pi) = \left[ \frac{|2\pi\Pi|^{-N/2}}{(2\pi)^{np/2}|V|^{n/2}|\Pi|^{p/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}((\Gamma^2 + \Sigma^{-1}T - \Gamma^T \Sigma^{-1} \Sigma^{-T} \Phi^T + \Phi)]) \right) \right],
\]

(93)

Using the definition of the matrix Normal distribution

\[
\mathcal{M}(\Gamma|M, \Pi, V) \prod_{i=1}^{N} \mathcal{N}(d^i | \Gamma x^i, \Pi) = \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right].
\]

(94)

Therefore by substituting

$MV^{-1} M^T = \Phi \Sigma^{-1} \Sigma^{-T} \Phi^T = \Phi \Sigma^{-1} \Phi^T$ we yield,

\[
\mathcal{M}(\Gamma|M, \Pi, V) \prod_{i=1}^{N} \mathcal{N}(d^i | \Gamma x^i, \Pi) = \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right].
\]

(95)

Next multiplying both sides by the required Inverse-Wishart distribution yields

\[
\mathcal{M}(\Gamma|M, \Pi, V) W^{-1}(\Pi | \Lambda, \nu) \prod_{i=1}^{N} \mathcal{N}(d^i | \Gamma x^i, \Pi)
\]

\[
= \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right] W^{-1}(\Pi | \Lambda, \nu)
\]

\[
= \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right] \frac{|\Lambda|^{\nu/2}}{2^{\nu/2} |\Pi|^{\nu+n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Lambda \Pi^{-1}] \right)
\]

\[
= \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right] \frac{|\Lambda|^{\nu/2}}{2^{\nu/2} |\Pi|^{\nu+n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Lambda \Pi^{-1}] \right)
\]

\[
= \mathcal{M}(\Gamma|M, \Pi, V) \left[ \frac{|2\pi\Pi|^{-N/2}}{|V|^{n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Pi^{-1}(\Phi + MV^{-1} M^T)]) \right) \right] \frac{|\Lambda|^{\nu/2}}{2^{\nu/2} |\Pi|^{\nu+n/2}} \exp\left( -\frac{1}{2} \text{tr}[\Lambda \Pi^{-1}] \right)
\]

(97)
where we define \( \bar{\Lambda} \triangleq \Lambda + \Phi - \Phi \Sigma^{-1} \Psi^T \). Treating the second fraction yields

\[
\frac{2\pi \Pi^{1-N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\nu+n+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)} = \frac{(2\pi)^{-n N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\nu+n+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)} = \frac{(2\pi)^{-n N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\nu+n+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)}.
\]

Then by defining \( \bar{\nu} \triangleq \nu + N \),

\[
\frac{2\pi \Pi^{1-N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\nu+n+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)} = \frac{(2\pi)^{-n N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\bar{\nu}+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)} = \frac{(2\pi)^{-n N/2} |\Lambda|^{\nu/2} |\Pi|^{-(\bar{\nu}+1)/2}}{2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right)}.
\]

Substituting this into (97) yields

\[
\mathcal{M}\mathcal{N}(\Gamma|\textbf{M}, \Pi, \nu) W^{-1}(\Pi|\Lambda, \nu) \prod_{i=1}^{N} \mathcal{N}(d^i|\Gamma x^i, \Pi) = \mathcal{M}\mathcal{N}(\Gamma|\bar{\textbf{M}}, \bar{\Pi}, \nu) \prod_{i=1}^{N} \mathcal{N}(d^i|\bar{\Gamma} x^i, \bar{\Pi}) \exp\left\{ -\frac{1}{2} tr[\bar{\Lambda} \Pi^{-1}] \right\}.
\]

Finally by substituting the definition of the Inverse-Wishart distribution

\[
\mathcal{M}\mathcal{N}(\Gamma|\textbf{M}, \Pi, \nu) W^{-1}(\Pi|\Lambda, \nu) \prod_{i=1}^{N} \mathcal{N}(d^i|\Gamma x^i, \Pi) = (2\pi)^{-n N/2} 2^{\nu n/2} \Gamma_n \left( \frac{\nu}{2} \right) |\Lambda|^{\nu/2} |\Pi|^{-(\bar{\nu}+1)/2} \mathcal{M}\mathcal{N}(\Gamma|\textbf{M}, \Pi, \nu) W^{-1}(\Pi|\bar{\Pi}, \bar{\nu}),
\]

and therefore

\[
\mathcal{M}\mathcal{N}(\Gamma|\textbf{M}, \Pi, \nu) W^{-1}(\Pi|\Lambda, \nu) \prod_{i=1}^{N} \mathcal{N}(d^i|\Gamma x^i, \Pi) \propto \mathcal{M}\mathcal{N}(\Gamma|\textbf{M}, \Pi, \nu) W^{-1}(\Pi|\bar{\Pi}, \bar{\nu}).
\]