REGULARITY AND IRREGULARITY OF SUPERPROCESSES
WITH \((1 + \beta)\)-STABLE BRANCHING MECHANISM

LEONID MYTNIK AND VITALI WACHTEL

Abstract. We would like to give an overview of results on regularity, or better
to say "irregularity", properties of densities at fixed times of super-Brownian
motion with \((1 + \beta)\)-stable branching for \(\beta < 1\). First, the following

\[ \text{dichotomy for the density is shown: it is continuous in the dimension } d = 1 \text{ and locally unbounded in all higher dimensions where it exists.} \]

Then in \(d = 1\) we determine pointwise and local Hölder exponents of the density, and calculate
the multifractal spectrum corresponding to pointwise Hölder exponents.

1. Introduction, main results and discussion

1.1. Model and motivation. This paper is devoted to regularity and fractal
properties of superprocesses with \((1 + \beta)\)-branching. Regularity properties of functions
is the most classical question in analysis. Typically one is interested in such prop-
erties as continuity/discontinuity and differentiability. Starting from Weierstrass,
who constructed an example of a continuous but nowhere differentiable function,
people got more and more interested in such 'strange' properties of functions. Tra-
jectories of stochastic processes give a rich source of such functions. The most
classical example is the Brownian motion: almost every path of the Brownian mo-
tion is continuous but nowhere differentiable.

In order to measure the regularity of a function \(f\) at point \(x_0\), we need to intro-
duce so-called Hölder classes \(C^\eta(x_0)\). One says that \(f \in C^\eta(x_0)\), \(\eta > 0\) if the exists
a polynomial \(P\) of degree \([\eta]\) such that
\[
|f(x) - P(x - x_0)| = O(|x - x_0|^\eta).
\]

For \(\eta \in (0, 1)\) the above definition coincides with the definition of Hölder continuity
with index \(\eta\) at \(x_0\). With the definition of \(f \in C^\eta(x_0)\) at hand, let us define the
pointwise Hölder exponent of \(f\) at \(x_0\):
\[
H_f(x_0) := \sup\{\eta > 0 : f \in C^\eta(x_0)\},
\]
and we set it to 0 if \(f \notin C^\eta(x_0)\) for all \(\eta > 0\). To simplify the exposition we will
sometimes call \(H_f(x_0)\) the Hölder exponent of \(f\) at \(x_0\).

It is well known that the Weierstrass functions have the same Hölder exponent
at all points. The same is true for the Brownian motion: the pointwise Hölder
exponent at all times is equal to \(1/2\) with probability one. However, there exist
functions with Hölder exponent changing from point to point. In such a case one
speaks of a multifractal function. For some classical examples of deterministic
multifractal functions we refer to Jaffard [15]. In studying multifractal functions

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people are interested in the 'size' of the set of points with given Hölder exponent. To measure these sizes for different Hölder exponents of a function $f$ one introduces the following function

$$D(\eta) = \dim \{ x_0 : H_f(x_0) = \eta \}, \quad (2)$$

where $\dim(A)$ denotes the Hausdorff dimension of the set $A$. The mapping $\eta \mapsto D(\eta)$ reveals the so-called multifractal spectrum related to pointwise Hölder exponents of $f$. A standard example of a multifractal random function is given by a Levy process with infinite Levy measure. Its multifractal spectrum was determined by Jaffard in [16]. In general, the multifractal analysis of random functions and measures has attracted attention for many years and has been studied for example in Dembo et al. [4], Durand [5], Hu and Taylor [12], Klenke and Mörters [17].

All these examples are related to Levy processes, which have independent increments. Whenever the independence structure is lost, the analysis becomes much more complicated. However, there are still some examples of stochastic processes without independent increments, for which the rigorous analysis of multifractal spectrum is possible. The multifractal spectrum of measures defined on branching random trees has been studied by Mörters and Shieh [21], Berestycki, Berestycki and Schweinsberg [2], and more recently by Balança [1]. The analysis of multifractal spectrum has been also done in some variations for measure-valued branching processes, see, e.g, Le Gall and Perkins [20], Perkins and Taylor [24], and recently by Mytnik and Wachtel [23]. One of the aims of this review is to describe results on multifractal spectrum and other regularity properties of measure-valued branching processes, and show the methods of proofs. We shall do it in the particular case of $(1 + \beta)$-stable super-Brownian motion, whose densities in low dimensions turn out to have a very non-trivial regularity structure.

Before we start with the precise definition of these processes, we need to introduce the following notation. $\mathcal{M}$ is the space of all Radon measures on $\mathbb{R}^d$ and $\mathcal{M}_f$ is the space of finite measures on $\mathbb{R}^d$ with weak topology ($\Rightarrow$ denotes weak convergence). In general if $F$ is a set of functions, write $F_+$ or $F^+$ for non-negative functions in $F$. For any metric space $E$, let $C_E$ (respectively, $D_E$) denote the space of continuous (respectively, càdlàg) $E$-valued paths with compact-open (respectively, Skorokhod) topology. The integral of a function $\phi$ with respect to a measure $\mu$ is written as $\langle \mu, \phi \rangle$ or $\langle \phi, \mu \rangle$ or $\mu(\phi)$. We use $c$ (or $C$) to denote a positive, finite constant whose value may vary from place to place. A constant of the form $c(a, b, \ldots)$ means that this constant depends on parameters $a, b, \ldots$. Moreover, $c(\#)$ will denote a constant appearing in formula line (or array) ($\#$).

Let $(\Omega, \mathcal{F}, \mathcal{F}, \mathbf{P})$ be the probability space with filtration, which is sufficiently large to contain all the processes defined below. Let $C(E)$ denote the space of continuous functions on $E$ and let $C_0(E)$ be the space of bounded functions in $C(E)$. Let $C_0^n = C_0^n(\mathbb{R}^d)$ denote the subspace of functions in $C_0 = C_0(\mathbb{R}^d)$ whose partial derivatives of order $n$ or less are also in $C_0$. A càdlàg adapted measure-valued process $X$ is called a super-Brownian motion with $(1 + \beta)$-stable branching if $X$
satisfies the following martingale problem. For every $\varphi \in C^2_b$ and every $f \in C^2(\mathbb{R})$,
\[
\begin{align*}
    f((X_t, \varphi)) - f((X_0, \varphi)) - \frac{1}{2} \int_0^t f'((X_s, \varphi))(X_s, \Delta \varphi) ds \\
    - \int_0^t \left( \int_{\mathbb{R}^d \times (0, \infty)} (f((X_s, \varphi) + r \varphi(x)) - f((X_s, \varphi)) - f'((X_s, \varphi)) r \varphi(x)) n(dr)X_s(dx) \right) ds
\end{align*}
\]  
(3)

is an $\mathcal{F}_t$-martingale, where
\[
    n(dr) = \frac{\beta(\beta + 1)}{\Gamma(1 - \beta)} r^{-2-\beta} dr.
\]  
(4)

There is also an analytic description of this process: For every positive $\varphi \in C^2_b$ one has
\[
    \mathbb{E} e^{-\langle X_t, \varphi \rangle} = e^{-\langle X_0, u \rangle},
\]
where $u$ is the solution to the equation
\[
\begin{align*}
    \frac{d}{dt} u &= \Delta u - u^{1+\beta},
\end{align*}
\]  
(5)

with the initial condition $\varphi$.

If $\beta = 1$, $X$ has continuous sample $\mathcal{M}_t$-valued paths, while for $0 < \beta < 1$, $X$ is a.s. discontinuous and has jumps all of the form $\Delta X_t = \delta_{\zeta(t)} m(t)$ and the set of jump times is dense in $[0, \zeta)$, where $\zeta = \inf \{ t : \langle X_t, 1 \rangle = 0 \}$ is the lifetime of $X$ (see, for example, Section 6.2.2 of [3]). For $t > 0$ fixed, $X_t$ is absolutely continuous a.s. if and only if $d < 2/\beta$ (see [7] and Theorem 8.3.1 of [3]). If $\beta = 1$, and $d = 1$, then much more can be said — $X_t$ is absolutely continuous for all $t > 0$ a.s. and has a density $X(t, x)$ which is jointly continuous on $(0, \infty) \times \mathbb{R}$ (see [18], [25]). In view of the jumps of $X$ (described above) if $0 < \beta < 1$, we see that $X_t$ cannot have a density for a dense set of times a.s. and the regularity properties of the densities are very intriguing. In this work we consider the “stable branching” case of $0 < \beta < 1$ and consider the question:

What are the regularity properties of the density of $X$ at fixed times $t$?

The analytic methods used in [22] to prove the existence of a density at a fixed time do not shed any light on its regularity properties. However recently there have been developed techniques that allowed to treat these questions. To the best of our knowledge, the regularity properties of the densities for super-Brownian motion with $\beta$-stable branching were first studied in Mytnik and Perkins [22]. It was shown there, that there is a continuous version of the density if and only if $d = 1$. Moreover, when $d > 1$ the density is very badly behaved. Note that in the case of $\beta = 1$, the density of super-Brownian motion $X_t(dx)$ exists only in dimension $d = 1$ and the density has a version which is Hölder continuous with any exponent smaller than $1/2$ (see Konno and Shiga [18]).

Now consider the case $\beta < 1$. In a series of papers of Fleischmann, Mytnik, and Wachtel [8], [9] and Mytnik and Wachtel [23] the properties of the density were studied for a superprocess with $\beta$-stable branching with an $\alpha$-stable motion, the so-called $(\alpha, d, \beta)$-superprocess. The case of $\alpha = 2$, clearly corresponds to the super-Brownian motion. In [8], the results of Mytnik and Perkins [22] were extended to the case of $\alpha$-stable motion. In particular, it was shown that there is a dichotomy for the density function of the measure (in what follows, we just say the “density of the measure”): There is a continuous version of the density of $X_t(dx)$ if $d = 1$ and $\alpha > 1 + \beta$, but otherwise the density is unbounded on open
sets of positive $X_t(dx)$-measure. Moreover, in the case of continuity ($d = 1$ and $\alpha > 1 + \beta$), Hölder regularity properties of the density had been studied in [8], [9], [23]. It turned out that on any set of positive $X_t(dx)$ measure, there are points with different pointwise Hölder exponents. In [23] the Hausdorff dimensions of sets containing the points with certain Hölder exponents were computed: this reveals the multifractal spectrum related to pointwise Hölder exponents.

The main purpose of this paper is to give concise exposition of the results on the regularity properties of densities of superprocesses with stable branching. On top of it we will also prove some new results that give a more complete picture of regularity properties.

1.2. Results on regularity properties of the densities of super-Brownian motion with stable branching. As we have mentioned above we are interested in the regularity properties of the $(\alpha, d, \beta)$-superprocess with $\beta \in (0, 1)$. In this paper we will consider the particular case of

$$\alpha = 2,$$

that is, the case of super-Brownian motion. We do it in order to simplify the exposition, however the proofs go through also in the case of $\alpha$-stable motion process. The enthusiastic reader who is interested in this general case is invited to go through the series of papers [8], [9], [23].

So, from now on we assume (6) and

$$\beta < 1.$$

The first result deals with the dichotomy of the density of super-Brownian motion, see [22] and [8]. Recall that, by [7], the density for fixed times $t > 0$, exists if and only if $d < 2/\beta$.

**Theorem 1 (Dichotomy for densities).** Let $d < 2/\beta$. Fix $t > 0$ and $X_0 = \mu \in \mathcal{M}_t$.

(a): If $d = 1$ then with probability one, there is a continuous version $\tilde{X}_t$ of the density function of the measure $X_t(dx)$.

(b): If $d > 1$, then with probability one, for all open $U \subseteq \mathbb{R}^d$,

$$\|X_t\|_U = \infty \text{ whenever } X_t(U) > 0.$$

For the later results, we assume that $d = 1$, $\beta < 1$, that is, there is a continuous version of the density at fixed time $t$. This density, with a slight abuse of notation, will be also denoted by $X_t(x)$, $x \in \mathbb{R}$.

In the next theorem the first regularity properties of the density $X_t(\cdot)$, in dimension $d = 1$, are revealed (see [8]).

**Theorem 2 (Local Hölder continuity).** Let $d = 1$. Fix $t > 0$ and $X_0 = \mu \in \mathcal{M}_t$.

(a): For each $\eta < \eta_c := \frac{2}{1+\beta} - 1$, this version $X_t(\cdot)$ is locally Hölder continuous of index $\eta$:

$$\sup_{x_1, x_2 \in K, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} < \infty, \text{ compact } K \subset \mathbb{R}.$$
(b): For every \( \eta \geq \eta_c \) with probability one, for any open \( U \subseteq \mathbb{R} \),
\[
\sup_{x_1, x_2 \in U, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} = \infty \quad \text{whenever} \quad X_t(U) > 0.
\]

One of the consequences of the above theorem is that the so-called optimal index for local Hölder continuity of \( X_t \) equals to \( \eta_c \) (see e.g. Section 2.2 in \[27\] for the discussion about local Holder index of continuity).

The main part of this paper is devoted to studying pointwise regularity properties of \( X_t \) which differ drastically from their local regularity properties. In particular, we are interested in pointwise Hölder exponent at fixed points and in multifractal spectrum.

Let us fix \( t > 0 \) and return to the continuous density \( X_t \) of the \((2,1,\beta)\)-superprocess. In what follows, \( H_X(x) \) will denote the pointwise Hölder exponent of \( X_t \) at \( x \in \mathbb{R} \).

**Theorem 3 (Pointwise Hölder exponent at fixed points).** Let \( d = 1 \). Fix \( t > 0, x \in \mathbb{R} \) and \( X_0 = \mu \in M_t \). Define \( \bar{\eta}_c := \frac{3}{1+\beta} - 1 \). If \( \bar{\eta}_c \neq 1 \) then, for every fixed \( x \),
\[
H_X(x) = \bar{\eta}_c, \quad \mathbf{P} \text{- a.s. on } \{X_t(x) > 0\}.
\]

**Remark 4.** The above result was proved in \[9\] for the case of \( \beta > 1/2 \). This is the case for which \( H_X(x) < 1 \). In the case of \( \beta \leq 1/2 \), it was shown in \[9\], that for any fixed point \( x \in \mathbb{R} \)
\[
H_X(x) \geq 1 \quad \mathbf{P} \text{- a.s. on } \{X_t(x) > 0\}.
\]

Thus, Theorem 3 strengthens the result from \[9\] by determining the pointwise Hölder exponent at fixed points for any \( \beta \in (0,1) \setminus \{\frac{1}{2}\} \).

The above results immediately imply that almost every realization of \( X_t \) has points with different pointwise exponents of continuity. For example, it follows from Theorem 2 that one can find (random) points \( x \) where \( H_X(x) = \eta_c \). Moreover, it follows from Theorem 3 that there are also points \( x \) where \( H_X(x) = \bar{\eta}_c > \eta_c \). This indicates that we are dealing with random multifractal function \( x \mapsto X_t(x) \). To study its multifractal spectrum, for any open \( U \subseteq \mathbb{R} \) and any \( \eta \in (\eta_c, \bar{\eta}_c] \) define a random set
\[
\mathcal{E}_{U,X,\eta} := \{x \in U : H_X(x) = \eta\}
\]
and let \( D_U(\eta) \) denote its Hausdorff dimension (similarly to \[2\]).

The function \( \eta \mapsto D_U(\eta) \) reveals the multifractal spectrum related to pointwise Hölder exponents of \( X_t(\cdot) \). This spectrum is determined in the next theorem (see \[23\] ) which also claims its independence on \( U \).

**Theorem 5 (Multifractal spectrum).** Fix \( t > 0 \), and \( X_0 = \mu \in M_t \). Let \( d = 1 \). Then, for any \( \eta \in [\eta_c, \bar{\eta}_c] \setminus \{1\} \) and any open set \( U \), with probability one,
\[
D_U(\eta) = (\beta + 1)(\eta - \eta_c)
\]
whenever \( X_t(U) > 0 \).

**Remark 6.** It should be emphasized that the result in Theorem 5 is not uniform in \( \eta \). More precisely, an event of zero probability, on which (7) can fail, is not necessarily the same for different values of the exponent \( \eta \). The question, whether there exists a zero set \( M \) such that (7) holds for all \( \eta \) and all \( \omega \in M^c \), remains open. Note that the uniformity of multifractal spectrum in \( \eta \) has been obtained by
Remark 7. The proof of the above theorem fails in the case $\eta = 1$, and it is a bit disappointing. Formally, it happens for some technical reasons, but one has also to note, that this point is critical: it is the borderline between differentiable and non-differentiable functions. However we still believe that the function $D_U(\cdot)$ can be continuously extended to $\eta = 1$, i.e., $D_U(1) = (\beta + 1)(1 - \eta_c)$ almost surely on $\{X_t(U) > 0\}$.

Remark 8. The condition $\beta < 1$ excludes the case of the quadratic super-Brownian motion, i.e., $\beta = 1$. But it is a known “folklore” result that the super-Brownian motion $X_t(\cdot)$ is almost surely monofractal on any open set of strictly positive density. That is, $P$-a.s., for any $x$ with $X_t(x) > 0$ we have $H_X(x) = 1/2$. For the fact that $H_X(x) \geq 1/2$, for any $x$, see Konno and Shiga [18] and Walsh [28]. To get that $H_X(x) \leq 1/2$ on the event $\{X_t(x) > 0\}$ one can show that

$$\limsup_{\delta \to 0} \frac{|X_t(x + \delta) - X_t(x)|}{\delta^{\eta}} = \infty \quad \text{for all } x \text{ such that } X_t(x) > 0, \ P \text{- a.s.},$$

for every $\eta > 1/2$. This result follows from the fact that for $\beta = 1$ the noise driving the corresponding stochastic equation for $X_t$ is Gaussian (see (0.4) in [18]) in contrast to the case of $\beta < 1$ considered here, where we have driving discontinuous noise with Lévy type intensity of jumps.

**Organization of the article.** Beyond the description of the regularity properties of the densities of $(2, d, \beta)$-superprocesses, which is given above, one of the main goals of the article is to provide the approach for proving these properties. We will also show how to use this approach to verify some of the results mentioned above. In particular we will give main elements of the proofs of Theorems 1, 3, 5. As for the missing details and the proof of Theorem 2 we refer the reader to corresponding papers.

Now we will say a few of words about the organization of the material in the following sections. In Section 2 we give the representation of $(2, d, \beta)$-superprocess as a solution to certain martingale problem and describe the approach for studying the regularity of the superprocess. Section 3 collects certain properties of $(2, d, \beta)$-superprocesses which later are used for the proofs. Section 4 is very important: it gives precise estimates on the sizes of jumps of $(2, d, \beta)$-superprocesses, which in turn are crucial for deriving the regularity properties. Sections 5, 6, 7 are devoted to the partial proofs of Theorems 1, 2, 5. Since many of the proofs are technical, at the beginning of several sections and subsections we give heuristic explanations of our results, which, as we hope, will provide the reader with some intuition about the results and their proofs.

2. Stochastic representation for $X$ and description of the approach for determining regularity

Let us start with formal definition of $(2, d, \beta)$ superprocess. A càdlàg adapted measure-valued process $X$ is called an $(2, d, \beta)$ superprocess, or super-Brownian motion with $(1 + \beta)$-stable branching, if $X$ satisfies the following martingale problem.
For every $\varphi \in C^2_b(\mathbb{R}^d)$ and every $f \in C^2(\mathbb{R})$,

$$f((X_t, \varphi)) - f((X_0, \varphi)) - \frac{1}{2} \int_0^t f'(\langle X_s, \varphi \rangle)(X_s, \Delta \varphi)\,ds$$

$$- \int_0^t \left( \int_{D(0, \infty)} \left( f((X_s, \varphi) + r\varphi(x)) - f((X_s, \varphi)) - f'(\langle X_s, \varphi \rangle)r\varphi(x) \right) n(dr)X_s(dx) \right)\,ds$$

is an $\mathcal{F}_t$-martingale. The jump measure $n(dr)$ is defined in (4).

The following lemma contains a semimartingale decomposition of $X$ which includes stochastic integrals with respect to discontinuous martingale measures.

**Lemma 9.** Fix $X_0 = \mu \in \mathcal{M}_f$.

(a) **(Discontinuities):** Define the random measure

$$N := \sum_{s \in J} \delta(s, \Delta X_s),$$

where $J$ denotes the set of all jump times of $X$. Then there exists a random counting measure $N(d(s,x,r))$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ such that

$$\int_{\mathbb{R}_+} \int_{\mathcal{M}_f} G(s,\mu)N(ds,d\mu) = \int_0^\infty \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} G(s,r\delta_x)N(ds,dx,dr),$$

for any bounded continuous $G$ on $\mathbb{R}_+ \times \mathcal{M}_f$. That is, all discontinuities of the process $X$ are jumps upwards of the form $r\delta_x$.

(b) **(Jump intensities):** The compensator $\tilde{N}$ of $N$ is given by

$$\tilde{N}(d(s,x,r)) = dsX_s(dx)\,n(dr),$$

that is, $\tilde{N} := N - \tilde{N}$ is a martingale measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$.

(c) **(Martingale decomposition):** For all $\varphi \in C^2_b$ and $t \geq 0$,

$$\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t ds \langle X_s, \Delta \varphi \rangle + M_t(\varphi)$$

with discontinuous martingale

$$t \mapsto M_t(\varphi) := \int_0^t \int_{\mathcal{M}_f} \langle \mu, \varphi \rangle(N - \tilde{N})(d(s,d\mu))$$

$$= \int_{[0,t] \times \mathbb{R}^d \times \mathbb{R}_+} r\varphi(x)\tilde{N}(d(s,x,r)).$$

The martingale decomposition of $X$ in the above lemma is basically proven in Dawson [3, Section 6.1]. However for the sake of completeness we will reprove it here. Some ideas are taken also from [19].

**Proof.** Since $X$ satisfies the martingale problem (8) one can easily get (by formally taking $f(x) = x$) that

$$\langle X_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t ds \langle X_s, \Delta \varphi \rangle + \tilde{M}_t(\varphi)$$

where $\tilde{M}_t(\varphi)$ is a local martingale. Moreover, by taking again $f((X_t, \varphi))$ for $f \in C^2_b(\mathbb{R})$, applying the Itô formula and comparing the terms with (8) one can easily
see that for each $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$, $\tilde{M}_t(\varphi)$ is a purely discontinuous martingale, with the compensator measure $\tilde{N}$ given by

$$\int_0^t \int_{\mathbb{R}^d} f(s,v)\tilde{N}(ds,dv) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s,r\varphi(x)) X_s(dx)n(dr)ds, \quad t \geq 0$$

(14)

for any bounded continuous $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. This means that if $s$ is a jump time for $\langle X, \varphi \rangle$, then

$$\Delta \langle X_s, \varphi \rangle = \Delta \tilde{M}_s(\varphi) = r\varphi(x)$$

(15)

for some $r$ and $x \in \mathbb{R}^d$ (“distributed according to $g X_s(dx)n(dr)$). Since this holds for any test function $\varphi$, by putting all together we infer that if $s > 0$ is the jump time for the measure-valued process $X$, then $\Delta X_s = r\delta_x$ for some $r$ and $x \in \mathbb{R}^d$.

Let $\mathcal{N}$ be as in (16). Then by above description of of jumps of $X$, it is clear that there exists a random counting measure $N$ such that

$$N := \sum_{(s,x,r) : s \in J, \Delta X_s = r\delta_x} \delta(s,x,r),$$

(16)

and (a) follows.

In order to obtain (11), we first get $\hat{N}$ — the compensator of $\mathcal{N}$. It is defined as follows. For any nonnegative predictable function $F$ on $\mathbb{R}^d \times \Omega \times \mathcal{M}_t$, $\hat{N}$ satisfies the following quality

$$E_\mu \left[ \int_{\mathbb{R}^d} \int_{\mathcal{M}_t} F(s,\omega,\mu)\mathcal{N}(ds,d\mu) \right] = E_\mu \left[ \int_{\mathbb{R}^d} \int_{\mathcal{M}_t} F(s,\omega,\mu)\hat{N}(ds,d\mu) \right].$$

(17)

We will show that, in fact, $\hat{N}$ is defined by the equality

$$\int_{\mathbb{R}^d} \int_{\mathcal{M}_t} G(s,\mu)\hat{N}(ds,d\mu) = \int_0^\infty ds \int_{\mathbb{R}^d} n(dr) \int_{\mathbb{R}^d} X_s(dx) G(s,r\delta_x),$$

(18)

which holds for any bounded continuous function $G$ on $\mathbb{R}^d \times \mathcal{M}_t$. To show (18), for any bounded nonnegative continuous $\varphi$ define random measure

$$\mathcal{N}_\varphi := \sum_{s \in J} \delta(s,\Delta(X_s,\varphi)), $$

(15)

By (15) we have that the compensator measure of $\mathcal{N}_\varphi$ is $\hat{N}_\varphi$. Clearly for any bounded continuous $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, and $\varphi \in \mathcal{C}_b^2$

$$\int_0^t \int_{\mathcal{M}_t} f(s,\varphi)\mathcal{N}(ds,d\mu) = \int_0^t \int_{\mathbb{R}^d} f(s,v)\mathcal{N}_\varphi(ds,dv), \quad \forall t \geq 0,$$

This immediately implies, that the corresponding compensator measures also satisfy

$$\int_0^t \int_{\mathcal{M}_t} f(s,\varphi)\hat{N}(ds,d\mu)$$

$$= \int_0^t \int_{\mathbb{R}^d} f(s,v)\hat{N}_\varphi(ds,dv)$$

$$= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(s,r\varphi(x)) X_s(dx)n(dr)ds,$$

(19)

where the second equality follows by (14). Since collection of functions in the form $f(s,\varphi)$, is dense in the space of bounded continuous functions on $\mathbb{R}^d \times \mathcal{M}_t$,
(18) follows. Now (11) follows by (a) and the definition of $N$. This finishes the proof of (b).

To show (13) it is enough to derive only the first inequality since the second one is immediate by the definition of $N$. Let us first identify the class of functions for which the stochastic integral with respect to $(\mathcal{N} - \tilde{\mathcal{N}})(ds, d\mu)$ is well defined. Let $F$ be a measurable function on $\mathbb{R}^+ \times \mathcal{M}_t$ such that, for every $t \geq 0$,

$$
\mathbb{E}_\mu \left[ \left( \sum_{s \in J \cap [0,t]} F(s, \Delta X_s)^2 \right)^{1/2} \right] < \infty.
$$

(20)

Following (14) (Section II.1d), we can then define the stochastic integral of $F$ with respect to the compensated measure $\mathcal{N} - \tilde{\mathcal{N}}$,

$$
\int_0^t F(s, \mu) (\mathcal{N} - \tilde{\mathcal{N}})(ds, d\mu),
$$

as the unique purely discontinuous martingale (vanishing at time 0) whose jumps are indistinguishable of the process $1_F(s) F(s, \Delta X_s)$.

We shall be interested in the special case where $F(s, \mu) = F_\phi(s, \mu) \equiv \int \phi(s, x) \mu(dx)$ for some measurable function $\phi$ on $\mathbb{R}^+ \times \mathbb{R}^d$ (some convention is needed when $\int |\phi(s, x)| \mu(dx) = \infty$, but this will be irrelevant in what follows). If $\phi$ is bounded, then it is easy to see that condition (20) holds. Indeed, we can bound separately

$$
\mathbb{E}_\mu \left[ \left( \sum_{s \leq t} \langle \Delta X_s, 1 \rangle^2 1_{\{\langle \Delta X_s, 1 \rangle \leq 1\}} \right)^{1/2} \right] \leq \mathbb{E}_\mu \left[ \sum_{s \leq t} \langle \Delta X_s, 1 \rangle^2 \right] \leq \left( \int_{[0,1]} r^2 n(dr) \mathbb{E}_\mu \left[ \int_0^t \langle X_s, 1 \rangle ds \right] \right)^{1/2} < \infty,
$$

and, using the simple inequality $a_1^2 + \cdots + a_n^2 \leq (a_1 + \cdots + a_n)^2$ for any nonnegative reals $a_1, \ldots, a_n$,

$$
\mathbb{E}_\mu \left[ \left( \sum_{s \leq t} \langle \Delta X_s, 1 \rangle^2 1_{\{\langle \Delta X_s, 1 \rangle > 1\}} \right)^{1/2} \right] \leq \mathbb{E}_\mu \left[ \sum_{s \leq t} \langle \Delta X_s, 1 \rangle \right] \leq \int_{1, \infty} r n(dr) \mathbb{E}_\mu \left[ \int_0^t \langle X_s, 1 \rangle ds \right] < \infty.
$$

In both cases, we have used (17) and the fact that $\mathbb{E}_\mu[\langle X_t, 1 \rangle] \leq \langle \mu, 1 \rangle$.

To simplify notation, we write

$$
M_t(\phi) = \int_0^t \int_{\mathbb{R}^d} \phi(s, x) M(ds, dx) \equiv \int_0^t F_\phi(s, \mu) (\mathcal{N} - \tilde{\mathcal{N}})(ds, d\mu),
$$

whenever (20) holds for $F = F_\phi$. This is consistent with the notation of the introduction. Indeed, if $\phi(s, x) = \varphi(x)$ where $\varphi \in C^2_0(\mathbb{R})$, then by the very definition, $M_t(\phi)$ is a purely discontinuous martingale with the same jumps as the process $\langle X_t, \varphi \rangle$. Since the same holds for the process

$$
\tilde{M}_t(\varphi) = \langle X_t, \varphi \rangle - \langle X_0, \varphi \rangle - \frac{1}{2} \int_0^t \langle X_s, \Delta \varphi \rangle ds
$$

(see Théorème 7 in [6]) we get that $M_t(\phi) = \tilde{M}_t(\varphi)$. □ □
Let \( \{p_t(x), t \geq 0, x \in \mathbb{R}^d\} \) denote the continuous transition kernel related to the Laplacian \( \Delta \) in \( \mathbb{R}^d \), and \( (S_t, t \geq 0) \) the related semigroup, that is,

\[
S_t f(x) = \int_{\mathbb{R}^d} p_t(x - y) f(y) dy \quad \text{for any bounded function } f
\]

and

\[
S_t \nu(x) = \int_{\mathbb{R}^d} p_t(x - y) \nu(dy) \quad \text{for any finite measure } \nu.
\]

Fix \( X_0 = \mu \in \mathcal{M}_f \{0\} \). Recall that if \( d = 1 \) then \( X_t(dx) \) is a.s. absolutely continuous for every fixed \( t > 0 \) (see [7]). In what follows till the end of the section we will consider the case of \( d = 1 \). From the Green function representation related to (12) (see, e.g., [8, (1.9)]) we obtain the following representation of a version of the density function of \( X_t(dx) \) in \( d = 1 \) (see, e.g., [8, (1.12)]):

\[
X_t(x) = \mu * p_t(x) + \int_{(0,t] \times \mathbb{R}} M(d(u,y)) p_{t-u}(y-x)
\]

\[
= \mu * p_t(x) + Z_t(x), \quad x \in \mathbb{R},
\]

where

\[
Z_s(x) = \int_{(0,s] \times \mathbb{R}} M(d(u,y)) p_{t-u}(y-x), \quad 0 \leq s \leq t.
\]

Note that although \( \{Z_s\}_{s \leq t} \) depends on \( t \), it does not appear in the notation since \( t \) is fixed throughout the paper. \( M(d(s,y)) \) in (21) is the martingale measure related to (13). Note that by Lemma 1.7 of [8] the class of “legitimate” integrands with respect to the martingale measure \( M(d(s,y)) \) includes the set of functions \( \psi \) such that for some \( p \in (1 + \beta, 2) \),

\[
\int_0^T ds \int_{\mathbb{R}} dx S_s \mu(x) |\psi(s,x)|^p < \infty, \quad \forall T > 0.
\]

We let \( \mathcal{L}_p^\text{loc} \) denote the space of equivalence classes of measurable functions satisfying (23). For \( \beta < 1 \), it is easy to check that, for any \( t > 0, z \in \mathbb{R} \),

\[
(s,x) \mapsto p_{t-s}(z-x) 1_{s \leq t}
\]

is in \( \mathcal{L}_p^\text{loc} \) for any \( p \in (1 + \beta, 2) \), and hence the stochastic integral in the representation (21) is well defined.

\( \mu * p_t(x) \) is obviously twice differentiable. Thus, the regularity properties of \( X_t(\cdot) \) including its multifractal structure coincide with that of \( Z \). Recalling the definitions of \( Z \) and \( M(ds,dy) \), we see that there is a “competition” between branching and motion: jumps of the martingale measure \( M \) try to destroy smoothness of \( X_t(\cdot) \) and \( p \) tries to make \( X_t(\cdot) \) smoother. Thus, it is natural to expect, that \( \{x : H_Z(x) = \eta\} \) can be described by jumps of a certain order depending on \( \eta \).

Next we connect the martingale measure \( M \) with spectrally positive \( 1 + \beta \)-stable processes.

Let \( L = \{L_t : t \geq 0\} \) denote a spectrally positive stable process of index \( 1 + \beta \). Per definition, \( L \) is an \( \mathbb{R} \)-valued time-homogeneous process with independent increments and with Laplace transform given by

\[
\mathbb{E} e^{-\lambda L_t} = e^{\lambda^{1+\beta}}, \quad \lambda, t \geq 0.
\]
Note that $L$ is the unique (in law) solution to the following martingale problem:

$$t \mapsto e^{-\lambda L_t} - \int_0^t ds \ e^{-\lambda L_s} \chi_{1+\beta} \text{ is a martingale for any } \lambda > 0. \quad (25)$$

**Lemma 10.** Let $d = 1$. Suppose $p \in (1 + \beta, 2)$ and let $\psi \in \mathcal{L}_{loc}^p(\mu)$ with $\psi \geq 0$. Then there exists a spectrally positive $(1 + \beta)$-stable process $\{L_t : t \geq 0\}$ such that

$$U_t(\psi) := \int_{(0, t] \times \mathbb{R}} M(d(s, y)) \psi(s, y) = L_{T(t)}, \quad t \geq 0,$$

where $T(t) := \int_0^t ds \int_{\mathbb{R}} X_s(dy) \ (\psi(s, y))^{1+\beta}$.

**Proof.** Let us write Itô’s formula for $e^{-U_t(\psi)}$:

$$e^{-U_t(\psi)} - 1 = \text{local martingale}$$

$$+ \int_0^t ds \ e^{-U_{s}(\psi)} \int_{\mathbb{R}} X_s(dy) \int_0^\infty n(dr) \left( e^{-r \psi(s, y)} - 1 + r \psi(s, y) \right).$$

Define $\tau(t) := T^{-1}(t)$ and put $t^* := \inf\{ t : \tau(t) = \infty \}$. Then it is easy to get for every $v > 0$,

$$e^{-v \psi_U(t)}(\psi) = 1 + \int_0^t ds \ e^{-v \psi_U(s)} \frac{X_{\tau(s)}(v^{1+\beta} \psi^{1+\beta}(s, \cdot))}{X_{\tau(s)}(\psi^{1+\beta}(s, \cdot))} + \text{loc. mart.}$$

$$= 1 + \int_0^t ds \ e^{-v \psi_U(s)} v^{1+\beta} + \text{loc. mart.}, \quad t \leq t^*.$$

Since the local martingale is bounded, it is in fact a martingale. Let $\tilde{L}$ denote a spectrally positive process of index $1 + \beta$, independent of $X$. Define

$$L_t := \begin{cases} U_{\tau(t)}(\psi), & t \leq t^*, \\ U_{\tau(t^*)}(\psi) + \tilde{L}_{t-t^*}, & t > t^* \text{ (if } t^* < \infty). \end{cases}$$

Then we can easily get that $L$ satisfies the martingale problem (25) with $\kappa$ replaced by $1 + \beta$. Now by time change back we obtain

$$U_t(\psi) = \tilde{L}_{T(t)} = L_{T(t)},$$

finishing the proof. \hfill \Box \hfill \Box

Having this result we may represent the increment $Z_t(x_1) - Z_t(x_2)$ as a difference of two stable processes. More precisely, for every fixed pair $(x_1, x_2)$ there exist spectrally positive stable processes $L^+$ and $L^-$ such that

$$Z_t(x_1) - Z_t(x_2) = L^+_{T_{+}(x_1, x_2)} - L^-_{T_{-}(x_1, x_2)} \quad (26)$$

where

$$T_{\pm}(x_1, x_2) = \int_0^1 du \int_{\mathbb{R}} X_u(dy) \ (p_{t-u}(x_1 - y) - p_{t-u}(x_2 - y))^\pm)^{1+\beta}. \quad (27)$$

It is clear from Lemma 10 that every jump $r \delta_{s,y}$ of the martingale measure $M$ produces a jump of one of those stable processes:

- If $p_{t-s}(x_1 - y) > p_{t-s}(x_2 - y)$ then $L^+$ has a jump of size $r(p_{t-u}(x_1 - y) - p_{t-u}(x_2 - y))$;
- If $p_{t-s}(x_1 - y) < p_{t-s}(x_2 - y)$ then $L^-$ has a jump of size $r(p_{t-u}(x_2 - y) - p_{t-u}(x_1 - y))$. 

Therefore, representation \( [26] \) gives a handy tool for the study of the influence of jumps of \( M \) on the behavior of the increment \( Z_t(x_1) - Z_t(x_2) \). Moreover, it becomes clear that one needs to know good estimates for the difference of kernels \( p_{t-u}(x_1 - y) - p_{t-u}(x_2 - y) \) and for the tails of spectrally positive stable processes.

For Hölder exponents \( \eta > 1 \) we cannot use \( [26] \), since for exponents greater than 1 one has to subtract a polynomial correction. Instead of \( Z(x_1) - Z(x_2) \) we shall consider

\[
Z_s(x_1, x_2) := Z_s(x_1) - Z_s(x_2) - (x_1 - x_2) \int_0^s \int_\mathbb{R} M(du, uy) \frac{\partial}{\partial y} p_{t-u}(x_2 - y)
\]

\[
= \int_0^s \int_\mathbb{R} M(du, uy) Q_{t-u}(x_1 - y, x_2 - y), \quad 0 \leq s \leq t,
\]

where

\[
q_s(x, y) := p_s(x) - p_s(y) - (x - y) \frac{\partial}{\partial y} p_s(y).
\]

Here we may again apply Lemma \( [10] \) to obtain a representation for \( Z(x_1, x_2) \) in terms of difference of spectrally positive stable processes, similarly to \( [26] \) which gives the representation for \( Z_s(x_1) - Z_s(x_2) \). The only difference to \( [26] \) is that \( p_{t-u}(x_1 - y) - p_{t-u}(x_2 - y) \) in \( [27] \) is replaced by \( q_{t-u}(x_1 - y, x_2 - y) \).

3. SOME SIMPLE PROPERTIES OF \((2,d,\beta)\)-SUPERPROCESSES

In this section we collect some estimates on \((2,d,\beta)\)-superprocesses which are needed for the implementation of the program described in Section 1.

We start with a lemma where we give some left continuity properties of \((2,d,\beta)\)-superprocess at fixed times, in dimensions \( d < 2/\beta \).

**Lemma 11.** Let \( d < 2/\beta \), and \( B \) be an arbitrary open ball in \( \mathbb{R}^d \). Then, for a fixed \( t > 0 \),

\[
\lim_{s \to t} X_s(B) = X_t(B), \quad \mathbb{P} \text{-a.s.}
\]

**Proof.** Since \( t \) is fixed, \( X \) is continuous at \( t \) with probability 1. Therefore,

\[
X_t(B) \leq \liminf_{s \to t} X_s(B) \leq \limsup_{s \to t} X_s(B) \leq \limsup_{s \to t} X_s(D) \leq X_t(D)
\]

with \( D \) denoting the closure of \( B \). But since \( X_t(dx) \) is absolutely continuous with respect to Lebesgue measure, we have \( X_t(B) = X_t(D) \). Thus the proof is finished. \( \square \)

In the next lemma we give a simple test for explosion of an integral involving \( \{X_s(B)\}_{s \leq t} \) whereas \( B \) in an open ball in \( \mathbb{R}^d \).

**Lemma 12.** Let \( d < 2/\beta \), and \( B \) be an arbitrary open ball in \( \mathbb{R}^d \). Let \( f : (0, t) \to (0, \infty) \) be measurable and assume that

\[
\int_{t-\delta}^t ds \ f(t-s) = \infty \text{ for all sufficiently small } \delta \in (0, t).
\]

Then for these \( \delta \)

\[
\int_{t-\delta}^t ds \ X_s(B) f(t-s) = \infty \quad \mathbb{P} \text{-a.s. on the event } \{X_t(B) > 0\}.
\]
Proof. Fix $\delta$ as in the lemma. Fix also $\omega$ such that $X_t(B) > 0$ and $X_s(B) \to X_t(B)$ as $s \uparrow t$. For this $\omega$, there is an $\varepsilon \in (0, \delta)$ such that $X_s(B) > \varepsilon$ for all $s \in (t - \varepsilon, t)$. Hence
\[
\int_{t-\delta}^{t} ds \ X_s(B) f(t-s) \geq \varepsilon \int_{t-\varepsilon}^{t} ds f(t-s) = \infty,
\]
and we are done. \hfill $\square$ \hfill $\square$

Now we will study the properties of $(2,1,\beta)$-superprocess in dimension $d = 1$. We start with moment estimates on the spatial increments of $Z_t$ defined in (22). Until the end of this section we consider the case of $d = 1$.

Lemma 13. Let $d = 1$. For each $q \in (1, 1 + \beta)$ and $\delta < \min \{1, (2 - \beta)/(1 + \beta)\}$,
\[
\mathbb{E}[Z_t(x_1) - Z_t(x_2)]^q \leq C |x_1 - x_2|^{\delta q}, \quad x_1, x_2 \in \mathbb{R}.
\]

Proof. Applying (3.1) from (22) with
\[
\phi(s,y) = p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y),
\]
we get
\[
\begin{align*}
\mathbb{E}[Z_t(x_1) - Z_t(x_2)]^q & \leq C \left[ \left( \int_0^t ds \int_\mathbb{R} S_s \mu(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^q \right)^{q/\theta} \\
& \quad + \int_0^t ds \int_\mathbb{R} S_s \mu(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^q \right]. \\
& \quad \text{For every } \varepsilon \in (1, 3), \\
& \quad \int_0^t ds \int_\mathbb{R} S_s \mu(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\varepsilon \\
& \quad = \int_\mathbb{R} \mu(dz) \int_0^t ds \int_\mathbb{R} dy \ p_s(y-z) |p_{t-s}(x_1 - z) - p_{t-s}(x_2 - z)|^\varepsilon \\
& \quad = \int_\mathbb{R} \mu(dz) \int_0^t ds \int_\mathbb{R} dy \ p_s(y) |p_{t-s}(x_1 - z - y) - p_{t-s}(x_2 - z - y)|^\varepsilon.
\end{align*}
\]
Using Lemma 43 we get for every positive $\delta < \min \{1, (3-\varepsilon)/\varepsilon \}$,
\[
\begin{align*}
\int_0^t ds \int_\mathbb{R} S_s \mu(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\varepsilon & \leq C |x_1 - x_2|^{\delta \varepsilon} \int_\mathbb{R} \mu(dz) \left( p_t((x_1 - z)/2) + p_t((x_2 - z)/2) \right) \leq C |x_1 - x_2|^{\delta \varepsilon},
\end{align*}
\]
since $\mu, t$ are fixed. Applying this bound to both summands at the right hand side of (30) finishes the proof of the lemma. \hfill $\square$ \hfill $\square$

Bounds on moments of spatial increments of $Z_t$, from the previous lemma, clearly give the same bounds on spatial increments of $X_t$ itself. However, on top of this, they immediately give the bounds on the moments of the supremum of $X_t(\cdot)$ on compact spatial sets: this is done in the next lemma.
Lemma 14. Let $d = 1$. If $K \subset \mathbb{R}$ is a compact and $q \in (1, 1 + \beta)$ then
\[
E \left( \sup_{x \in K} X_t(x) \right)^q < \infty.
\]

Proof. By Jensen’s inequality, we may additionally assume that $q > 1$. It follows from [21] that
\[
\left( \sup_{x \in K} X_t(x) \right)^q \leq 4 \left( \left( \sup_{x \in K} \mu * p_t(x) \right)^q + \sup_{x \in K} |Z_t(x)|^q \right).
\]

Clearly, the first term at the right hand side is finite. Furthermore, according to Corollary 1.2 of Walsh [28], Lemma 13 implies that
\[
E \sup_{x \in K} |Z_t(x)|^q < \infty.
\]

This completes the proof. □ □

From the above lemma one can immediately see that for any fixed $t$, $X_t(\cdot)$ is bounded on compacts. However, this is clearly not the case, if one start considering $X_s(x)$ as a function of $(s, x)$ with $s \leq t$. The reason is obvious: as we have discussed in the introduction, the measure-valued process $X_s(dx)$ has jumps in the form of atomic measures, and if $\langle X_t, 1 \rangle > 0$, the set of jump times is dense in $[0, t]$. However, it turns out that if one “smooths” a bit $X_s$ by taking its convolution with the heat kernel $p_{c(t-s)}(x-\cdot)$, then the resulting function of $(s, x)$ is a.s. bounded on compacts for $c$ large enough. This not obvious result is given in the next lemma.

Lemma 15. Let $d = 1$. Fix a non-empty compact $K \subset \mathbb{R}$. Then
\[
V(K) := \sup_{0 \leq s \leq t, x \in K} S_{4(t-s)} X_s(x) < \infty \quad P - a.s.
\]

Proof. Assume that the statement of the lemma does not hold, i.e. there exists an event $A$ of positive probability such that $\sup_{0 \leq s \leq t, x \in K} S_{4(t-s)} X_s(x) = \infty$ for every $\omega \in A$. Let $n \geq 1$. Put
\[
\tau_n := \begin{cases} 
\inf \{ s < t : \text{there exists } x \in K \text{ such that } S_{4(t-s)} X_s(x) > n \}, & \omega \in A, \\
\infty, & \omega \in A^c.
\end{cases}
\]

If $\omega \in A$, choose $x_n = x_n(\omega) \in K$ such that $S_{4(t-\tau_n)} X_{\tau_n}(x_n) > n$, whereas if $\omega \in A^c$, take any $x_n = x_n(\omega) \in K$. Using the strong Markov property gives
\[
E S_{3(t-\tau_n)} X_t(x_n) = E E \left[ S_{3(t-\tau_n)} X_t(x_n) \big| \mathcal{F}_{\tau_n} \right] 
= E S_{3(t-\tau_n)} S_{(t-\tau_n)} X_{\tau_n}(x_n) = E S_{4(t-\tau_n)} X_{\tau_n}(x_n).
\]

From the definition of $(\tau_n, x_n)$ we get
\[
E S_{4(t-\tau_n)} X_{\tau_n}(x_n) \geq n P(A) \to \infty \quad \text{as } n \uparrow \infty.
\]

In order to get a contradiction, we want to prove boundedness in $n$ of the expectation in (31). Choosing a compact $K_1 \supset K$ satisfying $\text{dist}(K, (K_1)^c) \geq 1$, we
have
\[
\mathbb{E}S_{3(t-s)}X_t(x_n) = \mathbb{E}\int_{K_1} dy \ X_t(y) p_{3(t-s)}(x_n - y) + \mathbb{E}\int_{(K_1)^c} dy \ X_t(y) p_{3(t-s)}(x_n - y) \\
\leq \mathbb{E} \sup_{y \in K_1} X_t(y) + \mathbb{E}X_t(\mathbb{R}) \sup_{y \in (K_1)^c, x \in K, 0 \leq s \leq t} p_{3s}(x - y).
\]

By our choice of \( K_1 \) we obtain the bound
\[
\mathbb{E}S_{3(t-s)}X_t(x_n) \leq \mathbb{E} \sup_{y \in K_1} X_t(y) + C = C,
\]
the last step again by Lemma 14. Altogether, (31) is bounded in \( n \), and the proof is finished. \( \square \)

An easy application of the previous lemma is the following result.

**Lemma 16.** Let \( d = 1 \). Fix any non-empty bounded \( K \subset \mathbb{R} \). Then
\[
W_K := \sup_{(c,s,x) : c \geq 1, 0 \vee (t-c^{-1/2}) \leq s < t, x \in K} \frac{X_s(B_{c(t-s)^{1/2}}(x))}{c(t-s)^{1/2}} < \infty \quad \mathbb{P} \text{-a.s.}
\]

**Proof.** Every ball of radius \( c(t-s)^{1/2} \) can be covered with at most \( [c] + 1 \) balls of radius \( (t-s)^{1/2} \). Therefore,
\[
\sup_{(c,s,x) : c \geq 1, 0 \vee (t-c^{-1/2}) \leq s < t, x \in K} \frac{X_s(B_{c(t-s)^{1/2}}(x))}{c(t-s)^{1/2}} \\
\leq 2 \sup_{(s,x) : 0 < s \leq t, x \in K_1} \frac{X_s(B_{(t-s)^{1/2}}(x))}{(t-s)^{1/2}},
\]
where \( K_1 := \{ x : \text{dist}(x, \overline{K}) \leq 1 \} \) with \( \overline{K} \) denoting the closure of \( K \). (The restriction \( s \geq t - c^{-1/2} \) is imposed to have all centers \( x \) of the balls \( B_{(t-s)^{1/2}}(x) \) in \( K_1 \).)

We further note that
\[
S_{t-s}X_s(x) = \int_\mathbb{R} dy \ p_{t-s}(x - y)X_s(y) \geq \int_{B_{(t-s)^{1/2}}(x)} dy \ p_{t-s}(x - y)X_s(y).
\]
Using the monotonicity and the scaling property of \( p \), we get the bound
\[
S_{t-s}X_s(x) \geq (t-s)^{-1/2}p_1(1)X_s(B_{(t-s)^{1/2}}(x)).
\]
Consequently,
\[
\sup_{(s,x) : 0 < s \leq t, x \in K_1} \frac{X_s(B_{(t-s)^{1/2}}(x))}{(t-s)^{1/2}} \leq \frac{1}{p_1(1)} \sup_{(s,x) : 0 < s \leq t, x \in K_1} S_{t-s}X_s(x).
\]
It was proved in Lemma 14 that the random variable at the right hand side is finite. Thus, the lemma is proved. \( \square \)

The boundedness of the smoothed density will play a crucial role in the analysis of the time changes \( T_{\pm}(x_1, x_2) \) described in the previous section, see \([26]-[29]\) and discussion there. The next lemma provides necessary tools to obtain pointwise upper bounds for \( T_{\pm}(x_1, x_2) \): by taking \( \theta = 1 + \beta \) in \([33]\) and \([34]\) below we get estimates for \( T_{\pm}(x_1, x_2) \).
Lemma 17. Let $d = 1$. Fix $\theta \in [1, 3)$, $\delta \in [0, 1]$ with $\delta < (3 - \theta)/\theta$, and a non-empty compact $K \subset \mathbb{R}$. Then

$$
\int_0^t ds \int \mathbb{R} X_s(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta \leq CV|x_1 - x_2|^{\delta \theta}, \quad x_1, x_2 \in K, \quad \mathbb{P}\text{-a.s.,}
$$

(33)

with $V = V(K)$ from Lemma 15. Moreover, for every $\theta \in [1, 2)$ and $\delta \in (0, (3 - 2\theta)/\theta)$,

$$
\int_0^t ds \int \mathbb{R} X_s(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y) - (x_1 - x_2) \frac{\partial}{\partial x_2} p_{t-s}(x_2 - y)|^\theta
\leq CV|x_1 - x_2|^{1+\delta}, \quad x_1, x_2 \in K, \quad \mathbb{P}\text{-a.s.}
$$

(34)

Proof. Using (130) gives

$$
\int_0^t ds \int \mathbb{R} X_s(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\leq C|x_1 - x_2|^{\delta \theta} \times 
\int_0^t ds (t-s)^{-(\delta \theta + \theta - 1)/2} \int \mathbb{R} X_s(dy) \left(p_{t-s}((x_1 - y)/2) + p_{t-s}((x_2 - y)/2)\right),
$$

uniformly in $x_1, x_2 \in \mathbb{R}$. Recalling the scaling property of the kernel $p$, we get

$$
\int_0^t ds \int \mathbb{R} X_s(dy) |p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\leq C|x_1 - x_2|^{\delta \theta} \int_0^t ds (t-s)^{-(\delta \theta + \theta - 1)/2} \left(S_{4(t-s)} X_s(x_1) + S_{4(t-s)} X_s(x_2)\right).
$$

We complete the proof of (33) by applying Lemma 15. To derive (34) it suffices to replace (130) by (133) in the computations we used to prove (33).

□ □ □

In Lemmas 15 and 16 we have obtained uniform on compact sets upper bounds for the “smoothed” densities. Now we turn to the analysis of this smoothed density near a fixed spatial point. Without loss of generality we choose fixed point $x = 0$ in the next lemma.

Lemma 18. Let $d = 1$. For all $c, \theta > 0$,

$$
\mathbb{P}\left(X_t(0) > \theta, \liminf_{s \uparrow t} S_{t-s} X_s\left(c \left(t-s\right)^{1/2}\right) \leq \theta\right) = 0.
$$

Proof. For brevity, set

$$
A := \left\{\liminf_{s \uparrow t} S_{t-s} X_s\left(c \left(t-s\right)^{1/2}\right) \leq \theta\right\}
$$

and for $n > 1/t$ define the stopping times

$$
\tau_n := \left\{\inf\left\{s \in (t - 1/n, t) : S_{t-s} X_s\left(c \left(t-s\right)^{1/2}\right) \leq \theta + 1/n\right\}, \quad \omega \in A, \right. \right.
$$
$$
\left. \left. t, \quad \omega \in A^c. \right\}
$$

Define also

$$
x_n := c \left(t - \tau_n\right)^{1/2}.
$$

Then, using the strong Markov property, we get

$$
\mathbb{E}[X_t(x_n) \mid \mathcal{F}_{\tau_n}] = S_{t-\tau_n} X_{\tau_n}(x_n) = X_{t}(0) 1_{A^c} + S_{t-\tau_n} X_{\tau_n}(x_n) 1_A.
$$

(35)
We next note that \( x_n \to 0 \) almost surely as \( n \uparrow \infty \). This implies, in view of the continuity of \( X_t \) at zero, that \( X_t(x_n) \to X_t(0) \) almost surely. Recalling that
\[
\mathbb{E} \sup_{|x| \leq 1} X_t(x) < \infty,
\]
in view of Corollary 2.8 of [8], we conclude that
\[
X_t(x_n) \xrightarrow{n \uparrow \infty} X_t(0) \quad \text{in } \mathcal{L}_1.
\]
This, in its turn, implies that
\[
\mathbb{E} \left[ X_t(x_n) \mid \mathcal{F}_{\tau_n} \right] - \mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{\tau_n} \right] \to 0 \quad \text{in } \mathcal{L}_1.
\] (36)
Furthermore, it follows from the well known Levy theorem on convergence of conditional expectations that
\[
\mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{\tau_n} \right] \xrightarrow{n \uparrow \infty} \mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{\infty} \right] \quad \text{in } \mathcal{L}_1,
\]
where \( \mathcal{F}_\infty := \sigma \left( \bigcup_{n>1/t} \mathcal{F}_{\tau_n} \right) \).
Noting that \( \tau_n \uparrow t \), we conclude that
\[
\mathcal{F}_{t-} \subseteq \mathcal{F}_\infty \subseteq \mathcal{F}_t.
\]
Since \( X_t(0) \) is continuous at fixed \( t \) a.s., we have \( X_t(0) = \mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{t-} \right] \) almost surely. Consequently, \( \mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{\infty} \right] = X_t(0) \) almost surely, and we get, as a result,
\[
\mathbb{E} \left[ X_t(0) \mid \mathcal{F}_{\tau_n} \right] \xrightarrow{n \uparrow \infty} X_t(0) \quad \text{in } \mathcal{L}_1.
\] (37)
Combining (36) and (37), we have
\[
\mathbb{E} \left[ X_t(x_n) \mid \mathcal{F}_{\tau_n} \right] \xrightarrow{n \uparrow \infty} X_t(0) \quad \text{in } \mathcal{L}_1.
\]
From this convergence and from (35) we finally get
\[
\mathbb{E} \left[ 1_A \mid X_t(0) - S_{t-\tau_n} X_{\tau_n}(x_n) \right] \xrightarrow{n \uparrow \infty} 0.
\]
Since \( S_{t-\tau_n} X_{\tau_n}(x_n) \leq \theta + 1/n \) on \( A \), for all \( n > 1/t \), the latter convergence implies that \( X_t(0) \leq \theta \) almost surely on the event \( A \). Thus, the proof is finished. □ □

4. ANALYSIS OF JUMPS OF SUPERPROCESSES.

This section is devoted to the analysis of jumps of \((2,d,\beta)\)-superprocesses. The results of this section will be crucial for proofs of main theorems, since regularity properties of \((2,d,\beta)\)-superprocesses depend heavily on presence and intensity of big jumps at certain locations.

Let \( \Delta X_s := X_s - X_{s-} \) denote the jumps of the measure-valued process \( X \). Also let \( |\Delta X_s| = \langle \Delta X_s, 1 \rangle \) be the size of the jump, and with some abuse of notation \( |\Delta X_s(x)| \) denotes the size of jump at a point \( (s,x) \).

The results of the section are a bit technical, however let us explain briefly the main bounds we are going to obtain. Recall that \( t > 0 \) is fixed. First we would like to verify that the largest jump at the proximity of time \( t \) is of the order
\[
|\Delta X_s| \sim (t - s)^{1/(1+\beta)},
\] (38)
for \( s < t \). The exact lower and upper bounds are given in Lemmas 19 and 20. Note that the jump of order \((t - s)^{1/(1+\beta)}\) happens at the some “random” spatial point. Whenever one asks about the size of the maximal jump at the proximity of a given
Lemma 19. Let $d < 2/\beta$ and $B$ be an open ball in $\mathbb{R}^d$. For each $\varepsilon \in (0, t)$,
\[
P\left( \Delta X_s(B) > (t-s)^{1+\beta} \log \frac{1}{t-s} \right. \text{ for some } s \in (t-\varepsilon, t) \big| X_t(B) > 0 \big) = 1 \]

Proof. It suffices to show that
\[
P\left( \Delta X_s(B) \leq (t-s)^{1+\beta} \log \frac{1}{t-s} \right. \text{ for all } s \in (t-\varepsilon, t), X_t(B) > 0 \big) = 0. \tag{40} \]

For $u \in (0, \varepsilon]$ define
\[
\Pi_u := N\left( (s, x, r) : s \in (t-\varepsilon, t-\varepsilon+u), x \in B, r > (t-s)^{1+\beta} \log \frac{1}{t-s}, (s, x, r) \right),
\]
with the random measure $N$ introduced in Lemma 9. Then
\[
\left\{ \Delta X_s(B) \leq (t-s)^{1+\beta} \log \frac{1}{t-s} \right\} = \{ \Pi_u = 0 \}. \tag{41} \]

From a classical time change result for counting processes (see e.g. Theorem 10.33 in Jacod [13]), we conclude that there exists a standard Poisson process $A = \{ A(v) : v \geq 0 \}$ such that
\[
\Pi_u = A \left( \int_{t-\varepsilon}^{t-\varepsilon+u} ds \, X_s(B) \int_{(t-s)^{1+\beta} \log \frac{1}{t-s}}^{\infty} n(dr) \right) = A \left( \frac{c_\beta}{1+\beta} \int_{t-\varepsilon}^{t-\varepsilon+u} ds \, X_s(B) \frac{1}{(t-s) \log \frac{1}{t-s}} \right),
\]
where $c_\beta := \frac{\beta(\beta+1)}{(1-\beta)}$. (By this definition, $n(dr) = c_\beta r^{-2-\beta} dr$.) Then
\[
P(\Pi_u = 0, X_t(B) > 0) \leq P \left( \int_{t-\varepsilon}^{t} ds \, X_s(B) \frac{1}{(t-s) \log \frac{1}{t-s}} < \infty, X_t(B) > 0 \right).
\]

It is easy to check that
\[
\int_{t-\delta}^{t} ds \frac{1}{(t-s) \log \frac{1}{t-s}} = \infty \text{ for all } \delta \in (0, \varepsilon).
\]

Therefore, by Lemma 12,
\[
\int_{t-\varepsilon}^{t} ds \, X_s(B) \frac{1}{(t-s) \log \frac{1}{t-s}} = \infty \text{ on } \{ X_t(B) > 0 \}.
\]

As a result we have $P(\Pi_u = 0, X_t(B) > 0) = 0$. Combining this with (41) we get (40). □
The next result complements the previous lemma: it gives with probability close to one an upper bound for the sizes of jumps.

**Lemma 20.** Let $d = 1$. Let $\varepsilon > 0$ and $\gamma \in (0, (1 + \beta)^{-1})$. There exists a constant $c(\varepsilon, \gamma) \in c(\varepsilon, \gamma)$ such that

$$
P\left(|\Delta X_s| > c(\varepsilon, \gamma) (t-s)^{(1+\beta)^{-1} - \gamma} \text{ for some } s < t\right) \leq \varepsilon. \tag{42}
$$

**Proof.** Recall the random measure $N$ from Lemma 9(a). For any $c > 0$, set

$$
Y_n := N\left(\left[\left(1 - 2^{-n}\right)t, (1 - 2^{-n-1})t\right] \times \mathbb{R}^d \times (c2^{-\lambda t^\lambda}, \infty)\right), \quad n \geq 1, \tag{43}
$$

where $\lambda := (1 + \beta)^{-1} - \gamma$. It is easy to see that

$$
P\left(|\Delta X_s| > c(t-s)^\lambda \text{ for some } s < t\right) \leq P\left(\sum_{n=0}^{\infty} Y_n \geq 1\right) \leq \sum_{n=0}^{\infty} EY_n, \tag{45}
$$

where in the last step we have used the classical Markov inequality. From the formula for the compensator $\hat{N}$ of $N$ in Lemma 9(b),

$$
EY_n = c(\varepsilon, \gamma) \int_{(1-2^{-n})t}^{(1-2^{-n-1})t} ds \int_{\mathbb{R}^d} \int_0^{\infty} dr \ r^{-2-\beta}, \quad n \geq 1. \tag{46}
$$

Now

$$
EY_s = X_0(\mathbb{R}^d) =: c(\varepsilon, \gamma). \tag{47}
$$

Consequently,

$$
EY_n \leq \frac{c(\varepsilon, \gamma)}{1 + \beta} c(\varepsilon, \gamma) c^{-1-\beta} 2^{-(n+1)(1+\beta)} \lambda^{-1+\beta}. \tag{48}
$$

Analogous calculations show that (48) remains valid also in the case $n = 0$. Therefore,

$$
\sum_{n=0}^{\infty} EY_n \leq \frac{c(\varepsilon, \gamma)}{1 + \beta} c(\varepsilon, \gamma) c^{-1-\beta} \lambda^{-1+\beta} \sum_{n=0}^{\infty} 2^{-(n+1)(1+\beta)} = \frac{c(\varepsilon, \gamma)}{1 + \beta} c(\varepsilon, \gamma) c^{-1-\beta} \lambda^{-1+\beta} \frac{2^{-\gamma(1+\beta)}}{1 - 2^{-\gamma(1+\beta)}}. \tag{49}
$$

Choosing $c = c(\varepsilon, \gamma)$ such that the expression in (49) equals $\varepsilon$, and combining with (45), the proof is complete. \hfill \square

Put

$$
f_{s,x} := \log\left((t-s)^{-1}\right) 1_{\{x \neq 0\}} \log\{|x|^{-1}\}. \tag{50}
$$

In the following lemma and corollary we obtain suitable upper bounds for maximal jumps which occur close to 0.

**Lemma 21.** Let $d = 1$. Fix $X_0 = \mu \in M_0 \setminus \{0\}$. Let $\varepsilon > 0$ and $q > 0$. Then there exists a constant $c(\varepsilon, q) \in c(\varepsilon, q)$ such that

$$
P\left(|\Delta X_s(x)| > c(\varepsilon, q) (t-s)|x|^{1+\ell} f_{s,x}^{\alpha} \text{ for some } s < t, \quad x \in B_{1/e}(0)\right) \leq \varepsilon, \tag{51}
$$

where

$$
\ell := \frac{1}{1+\beta} + q. \tag{52}
$$
Proof. For any $c > 0$ (later to be specialized to some $c \in [0, 1]$ set
\[ Y := N \big((s, x, r) : (s, x) \in (0, t) \times B_{1/e}(0), r \geq c ((t-s)|x|)^{1/(1+\beta)}(f_{s,x})^\ell \big), \]
Clearly,
\[ P \Big( \Delta X_s(x) > c ((t-s)|x|)^{1/(1+\beta)}(f_{s,x})^\ell \text{ for some } s < t \text{ and } x \in B_{1/e}(0) \Big) = P(Y \geq 1) \leq EY, \]
where in the last step we have used the classical Markov inequality. From (11),
\[ EY = c_\beta E \int_0^t ds \int_R X_s(dx) 1_{B_{1/e}(0)}(x) \int_c ((t-s)|x|)^{1/(1+\beta)}(f_{s,x})^\ell dr r^{-2-\beta} \]
\[ = c_\beta \frac{c^{-1-\beta}}{1+\beta} \int_0^t ds (t-s)^{-1} \log^{-1-q(1+\beta)}((t-s)-1) \]
\[ \times \int_R \mathbf{E} X_s(dx) 1_{B_{1/e}(0)}(x) |x|^{-1} \log^{-1-q(1+\beta)}(|x|-1). \]
Now, writing $C$ for a generic constant (which may change from place to place),
\[ \int_R \mathbf{E} X_s(dx) 1_{B_{1/e}(0)}(x) |x|^{-1} \log^{-1-q(1+\beta)}(|x|-1) \]
\[ \leq \int_R \mu(dy) \int dx p_s(x-y) 1_{B_{1/e}(0)}(x) |x|^{-1} \log^{-1-q(1+\beta)}(|x|-1) \]
\[ \leq c \mu(\mathbb{R}) s^{-1/2} \int_R 1_{B_{1/e}(0)}(x) |x|^{-1} \log^{-1-q(1+\beta)}(|x|-1) \]
\[ =: c^{55} s^{-1/2}, \]
where $c^{55} = c^{55}(q)$ (recall that $t$ is fixed). Consequently,
\[ EY \leq c_\beta c^{55} c^{-1-\beta} \int_0^t ds s^{-1/2} (t-s)^{-1} \log^{-1-q(1+\beta)}((t-s)-1) \]
\[ =: c^{56} c^{-1-\beta} \]
with $c^{56} = c^{56}(q)$. Choose now $c$ such that the latter expression equals $\varepsilon$ and write $c^{56}$ instead of $c$. Recalling (53), the proof is complete. \qed

Since $\sup_{0<y<1} y^\gamma \log^\ell \frac{1}{y} < \infty$ for every $\gamma > 0$, the following corollary is immediate from Lemma 22.

Corollary 22. Fix $X_0 = \mu \in \mathcal{M}_1 \backslash \{0\}$. Let $d = 1$. Let $\varepsilon > 0$ and $\gamma \in (0, (1+\beta)^{-1})$. There exists a constant $c^{57} = c^{57}(\varepsilon, \gamma)$ such that
\[ P \Big( \Delta X_s(x) > c^{57} ((t-s)|x|)^{1/(1+\beta)} \text{ for some } s < t \text{ and } x \in B_{2}(0) \Big) \leq \varepsilon. \]

Introduce the event
\[ D_\theta := \bigg\{ X_t(0) > \theta, \sup_{0<s\leq t} X_s(\mathbb{R}) \leq \theta^{-1}, W_{B_t}(0) \leq \theta^{-1} \bigg\}. \]
In the next lemma we obtain a lower bound on a jump which occur close to time $t$ and to the spatial point $z = 0$.\]
Lemma 23. Let $d = 1$. For each $\theta > 0$ there exists a constant $c_{58} = c_{58}(\theta) \geq 1$ such that

$$
P \left( \Delta X_s(y) > Q \left( y (t-s)^{1/(1+\beta)} \log^{1/(1+\beta)} \left( \frac{(t-s)^{-1}}{D} \right) \right) \right.
$$

for some $s \in (t-\varepsilon, t)$ and $\frac{c_{58}}{2} (t-s)^{1/2} \leq y \leq \frac{3c_{58}}{2} (t-s)^{1/2} |D_\theta| = 1$

for all $\varepsilon \in (0, t \wedge 1/8)$, $Q > 0$.

Proof. Analogously to the proof of Lemma 19, to show that the number of jumps is greater than zero almost surely on some event, it is enough to show the divergence of a certain integral on that event or even on a bigger one. Specifically here, it suffices to verify that there exists $c = c_{58}$ such that

$$
I_{\varepsilon,c} := \int_{t-\varepsilon}^{t} \frac{ds}{(t-s) \log((t-s)^{-1})} \int_{D^{1/2}} \frac{d\gamma}{y} \frac{y^{-1} X_s(y)}{p_{t-s}(c (t-s)^{1/2} - y)} = \infty
$$

almost surely on the event $D_\theta$.

The mapping $\varepsilon \mapsto I_{\varepsilon,c}$ is nonincreasing. Therefore, we shall additionally assume, without loss of generality, that $\varepsilon \leq c^{-1/2}$ and this in turn implies that $c(t-s)^{1/2} \leq 1$ for all $s \in (t-\varepsilon, t)$. So, in what follows, in the proof of the lemma we will assume without loss of generality that given $c$, we choose $\varepsilon$ so that

$$
c(t-s)^{1/2} \leq 1, \quad \forall s \in (t-\varepsilon, t).
$$

Since $y \leq \frac{3c}{2} (t-s)^{1/2}$ and $p_s(x) \leq p_s(0)$ for all $x \in \mathbb{R}$, we have

$$
I_{\varepsilon,c} \geq \frac{2}{3c} \int_{t-\varepsilon}^{t} \frac{ds}{(t-s)^{3/2} \log((t-s)^{-1})}
$$

$$
\times \int_{D^{1/2}} \frac{d\gamma}{y} \frac{p_{t-s}(c (t-s)^{1/2} - y)}{p_{t-s}(0)} X_s(y).
$$

Then, using the scaling property of the kernel $p$, we obtain

$$
I_{\varepsilon,c} \geq \frac{2}{3c p_1(0)} \int_{t-\varepsilon}^{t} \frac{ds}{(t-s) \log((t-s)^{-1})} \left( S_{t-s} X_s \left( c (t-s)^{1/2} \right) \right.
$$

$$
- \int_{|y-c (t-s)^{1/2}| > \frac{3c}{2} (t-s)^{1/2}} dy \ p_{t-s}(c (t-s)^{1/2} - y) X_s(y))
$$

(59)

Since we are in dimension one, if

$$
y \in \tilde{D}_{s,j} := \left\{ z : c \left( \frac{1}{2} + j \right) (t-s)^{1/2} < |z - c (t-s)^{1/2}| \right\}
$$

$$
< c \left( \frac{1}{2} + j \right) (t-s)^{1/2} \right\},
$$

then

$$
p_{t-s}(c (t-s)^{1/2} - y) \leq p_{t-s}(c (j + 1/2) (t-s)^{1/2})
$$

$$
= (t-s)^{-1/2} p_1(c (j + 1/2)) = (2\pi)^{-1/2} (t-s)^{-1/2} e^{-c^2(j + 1/2)^2/2}.
$$
From this bound we conclude that
\[
\int_{|y-c(t-s)/2|^2 > \varphi(t-s)^2} dy \ p_{t-s} (c(t-s)^{1/2} - y) 1_{B_2(0)}(y) X_s(y) \\
\leq (2\pi)^{-1/2} (t-s)^{-1/2} \sum_{j=0}^{\infty} e^{-c^2(1/2+j)^2/2} \int_{D_{n,j}} dy 1_{B_2(0)}(y) X_s(y).
\]

Now recall again that the spatial dimension equals to one and hence for any \( j \geq 0 \) the set \( D_{n,j} \) in (61) is the union of two balls of radius \( c(t-s)^{1/2} \). If furthermore \( D_{n,j} \cap B_2(0) \neq \emptyset \), then, in view of the assumption \( c(t-s)^{1/2} \leq 1 \), the centers of those balls lie in \( B_2(0) \). Therefore, we can apply Lemma 16 to bound the integral
\[
\int_{D_{n,j}} dy 1_{B_2(0)}(y) X_s(y) \leq 2c(t-s)^{1/2} W_{B_2(0)} \text{ and obtain}
\]
\[
\int_{|y-c(t-s)/2|^2 > \varphi(t-s)^2} dy \ p_{t-s} (c(t-s)^{1/2} - y) 1_{B_2(0)}(y) X_s(y) \\
\leq \frac{2c}{(2\pi)^{1/2}} W_{B_2(0)} \sum_{j=0}^{\infty} e^{-c^2(1/2+j)^2/2} \leq CW_{B_2(0)}c^{-2}. \tag{61}
\]

Furthermore, if \( |y| \geq 2 \) and \( (t-s) \leq c^{-2} \), then
\[
p_{t-s} (c(t-s)^{1/2} - y) \leq p_{t-s}(1) = (t-s)^{-1/2}p_1((t-s)^{-1/2}) = (2\pi)^{-1/2}(t-s)^{-1/2}e^{-1/2(t-s)}.
\]

This implies that
\[
\int_{\mathbb{R}\setminus B_2(0)} dy \ p_{t-s} (c(t-s)^{1/2} - y) X_s(y) \leq (2\pi)^{-1/2}(t-s)^{-1/2}e^{-1/2(t-s)} X_s(\mathbb{R}) \\
\leq Cc^{-2}X_s(\mathbb{R}).
\]

Combining this bound with (61), we obtain
\[
\int_{|y-c(t-s)/2|^2 > \varphi(t-s)^2} dy \ p_{t-s} (c(t-s)^{1/2} - y) X_s(y) \\
\leq Cc^{-2} \left( W_{B_2(0)} + \sup_{0<s\leq t} X_s(\mathbb{R}) \right).
\]

Thus, we can choose \( c \) so large that the right hand side in the previous inequality does not exceed \( \theta/2 \). Since, in view of Lemma 18
\[
\liminf_{s \uparrow t} S_{t-s} X_s(c(t-s)^{1/2}) > \theta,
\]
we finally get
\[
\liminf_{s \uparrow t} \left( S_{t-s} X_s(c(t-s)^{1/2}) \\
- \int_{|y-c(t-s)/2|^2 > \varphi(t-s)^2} dy \ p_{t-s} (c(t-s)^{1/2} - y) X_s(y) \right) \geq \theta/2.
\]

From this bound and (59), the desired property of \( I_{c,c} \) follows.
Fix any \( \theta > 0 \), and to simplify notation write \( c := c_{[\theta]} \). For all \( n \) sufficiently large, say \( n \geq N_0 \), define

\[
A_n := \left\{ \Delta X_s \left( \left( \frac{C}{2} 2^{-n}, \frac{3C}{2} 2^{-n} \right) \right) \geq 2^{-(\bar{\eta}+1)n} n^{1/(1+\beta)} \right\}
\]

for some \( s \in (t - 2^{-2n}, t - 2^{-2(n+1)}) \). (62)

Based on Lemma 23 we will show in the following lemma that, if \( X_t(0) > 0 \) then there exist infinitely many jumps \( \Delta X_s(x) \) which are greater than \( ((t-s)|x|)^{1/(1+\beta)} \) with \( x \sim (t-s)^{1/2} \). To be more precise, we show that \( A_n \) occur infinitely often.

**Lemma 24.** We have

\[
P(A_n \text{ infinitely often } \mid D_\theta) = 1.
\]

**Proof.** If \( y \in \left( \frac{C}{2} (t-s)^{1/2}, \frac{3C}{2} (t-s)^{1/2} \right) \) and \( s \in (t - 2^{-2n}, t - 2^{-2(n+1)}) \), then

\[
((t-s)y \log((t-s)^{-1}))^{1/(1+\beta)} \geq \left( 2^{-2(\bar{\eta}+1)} \frac{C}{2} 2^{-n} 2n \log 2 \right)^{1/(1+\beta)} = c_{[63]} 2^{-(\bar{\eta}+1)n} n^{1/(1+\beta)}.
\]

This implies that

\[
A_n \supseteq \left\{ \Delta X_s(y) \geq c_{[63]} ((t-s)y \log((t-s)^{-1}))^{1/(1+\beta)} \right\}
\]

for some \( s \in (t - 2^{-2n}, t - 2^{-2(n+1)}) \) and \( y \in \left( \frac{C}{2} (t-s)^{1/2}, \frac{3C}{2} (t-s)^{1/2} \right) \). (64)

Consequently, from (64) we get

\[
\bigcup_{n=N}^{\infty} A_n \supseteq \left\{ \Delta X_s(y) \geq c_{[63]} ((t-s)y \log((t-s)^{-1}))^{1/(1+\beta)} \right\}
\]

for some \( s \in (t - 2^{-2N}, t) \) and \( y \in \left( \frac{C}{2} (t-s)^{1/2}, \frac{3C}{2} (t-s)^{1/2} \right) \) for all \( N > N_0 \). Applying Lemma 23 and using the monotonicity of the union in \( N \), we get

\[
P \left( \bigcup_{n=N}^{\infty} A_n \mid D_\theta \right) = 1 \quad \text{for all } N \geq N_0.
\]

This completes the proof. \( \square \)

5. **Dichotomy for densities**

5.1. **Proof of Theorem (1a).** The non-random part \( \mu * p_t(x) \) is continuous. The continuity of \( Z_t(\cdot) \) follows from the classical Kolmogorov criteria. Indeed, it suffices to show that there exist \( \theta, q \) and \( \delta \) as in Lemma 14 such that \( \delta q > 1 \). But this is immediate from the observation

\[
\sup_{\delta < \min \{1, (3-\theta)/\beta \}, \beta \in (1+\beta, 2), q \in (1,1+\beta)} \delta q = \min \{1 + \beta, 2 - \beta \} > 1.
\]
Remark 25. Combining Lemma 19 with Corollary 1.2 in Walsh [28], we infer that $Z_t(\cdot)$ is Hölder continuous of all orders smaller than $\delta - 1/q$. Noting that
\[
\sup_{\delta<\min\{1,(3-\theta)/\theta\},\theta\in(1+\beta,2),\eta\in(1,1+\beta)} (\delta - 1/q) = \min\left\{ 1, \frac{2-\beta}{1+\beta} \right\} - \frac{1}{1+\beta},
\]
we see that $Z$ is Hölder continuous of all orders smaller than $\min\{\beta, 1-\beta\}/(1+\beta)$. In other words, we proved Theorem 2 for $\beta \geq 1/2$. ⊤

5.2. Proof of Theorem 1(b). Throughout this subsection we assume that $d > 1$. Recall that $t > 0$ and $X_0 = \mu \in \mathcal{M}_1 \setminus \{0\}$ are fixed. We want to verify that for each version of the density function $X_t$ the property
\[
||X_t||_B = \infty \quad \text{P-a.s. on the event } \{X_t(B) > 0\} \tag{65}
\]
holds whenever $B$ is a fixed open ball in $\mathbb{R}^d$. Having this relation for every open ball we may prove Theorem 1(b) by the following simple argument: Let fix $\omega$ outside a null set so that (65) is valid for any ball with rational center and rational radius. If $U$ is an open set with $X_t(U) > 0$ then there exists a ball $B$ with rational center and rational radius such that $B \subset U$ and $X_t(B) > 0$. Consequently, $||X_t||_U(\omega) = ||X_t||_B(\omega) = \infty$.

To get (65) we first show that on the event $\{X_t(B) > 0\}$ there are always sufficiently “big” jumps of $X$ on $B$ that occur close to time $t$. This is done in Lemma 19 above. Then with the help of properties of the log-Laplace equation derived in Lemma 20 we are able to show that the “big” jumps are large enough to ensure the unboundedness of the density at time $t$. Loosely speaking the density is getting unbounded in the proximity of big jumps. As we have seen in the previous section, the largest jump at time $s < t$ is of order $(t-s)^{1/(1+\beta)}$. Suppose this jump occurs at spatial point $x$. Since a jump occurring at time $s$ is smeared out by the kernel $p_{t-s}$, we have the following estimate for the value of the density at time $t$ and spatial point $x$:
\[
X_t(x) \approx (t-s)^{1/(1+\beta)} p_{t-s}(0) \approx (t-s)^{1/(1+\beta)-d/2}. \tag{66}
\]

From (66) it is clear that the density should explode in any dimension $d > 1$. In the rest of the section we justify this heuristic.

Set $\varepsilon_n := 2^{-n}$, $n \geq 1$. Then we choose open balls $B_n \uparrow B$ such that
\[
\overline{B_n} \subset B_{n+1} \subset B \quad \text{and} \quad \sup_{y \in B_n, x \in B_n, 0 < s \leq \varepsilon_n} p_s(x-y) \rightarrow 0. \tag{67}
\]

Fix $n \geq 1$ such that $\varepsilon_n < t$. Set, for brevity,
\[
\tau_n := \inf \left\{ s \in (t-\varepsilon_n,t) : \Delta X_s(B_n) > (t-s)^{\frac{1}{\gamma}} \log \frac{\varepsilon_n}{t-s} \right\}.
\]

It follows from Lemma 19 that
\[
P(\tau_n = \infty) \leq P(X(B_n) = 0), \quad n \geq 1. \tag{68}
\]

In order to obtain a lower bound for $||X_t||_B$ we use the following inequality
\[
||X_t||_B \geq \int_B dy \ X_t(y)p_u(y-x), \quad x \in B, \ u > 0. \tag{69}
\]
On the event \( \{ \tau_n < t \} \), denote by \( \zeta_n \) the spatial location in \( B_n \) of the jump at time \( \tau_n \), and by \( r_n \) the size of the jump, meaning that \( \Delta X_{\tau_n} = r_n \delta_{\zeta_n} \). Then specializing (69),

\[
\|X_t\|_B \geq \int_B dy \ X_t(y) p_{t-\tau_n}(y - \zeta_n) \quad \text{on the event} \quad \{ \tau_n < t \}.
\]

(70)

From the strong Markov property at time \( \tau_n \), together with the branching property of superprocesses, we know that conditionally on \( \{ \tau_n < t \} \), the process \( \{ X_{\tau_n+u} : u \geq 0 \} \) is bounded below in distribution by \( \{ \tilde{X}^n_u : u \geq 0 \} \), where \( \tilde{X}^n \) is a super-Brownian motion with initial value \( r_n \delta_{\zeta_n} \). Hence, from (70) we get

\[
\mathbb{E} \exp\{ - \|X_t\|_B \} \leq \mathbb{E} 1_{\{\tau_n < t\}} \exp\left\{ - \int_B dy \ X_t(y) p_{t-\tau_n}(y - \zeta_n) \right\} + \mathbb{P}(\tau_n = \infty)
\]

\[
\leq \mathbb{E} 1_{\{\tau_n < t\}} \mathbb{E}_{r_n,\delta_{\zeta_n}} \exp\left\{ - \int_B dy \ X_{t-\tau_n}(y) p_{t-\tau_n}(y - \zeta_n) \right\} + \mathbb{P}(\tau_n = \infty).
\]

Note that on the event \( \{ \tau_n < t \} \), we have

\[
r_n \geq (t - \tau_n) \log \frac{1}{t - \tau_n} =: h_\beta(t - \tau_n).
\]

(72)

We now claim that

\[
\lim_{n \uparrow \infty} \sup_{0 < s < \varepsilon_n, \ x \in B_n, \ r \geq h_\beta(s)} \mathbb{E}_{r,\delta_x} \exp\left\{ - \int_B dy \ X_s(y)p_s(y - x) \right\} = 0.
\]

(73)

To verify (73), let \( s \in (0, \varepsilon_n), \ x \in B_n \) and \( r \geq h_\beta(s) \). Then, using the Laplace transition functional of the superprocess we get

\[
\mathbb{E}_{r,\delta_x} \exp\left\{ - \int_B dy \ X_s(y)p_s(y - x) \right\} = \exp\left\{ - r v^n_{s,x}(s, x) \right\}
\]

\[
\leq \exp\left\{ - h_\beta(s) v^n_{s,x}(s, x) \right\},
\]

(74)

where the non-negative function \( v^n_{s,x} = \{ v^n_{s,x}(s', x') : s' > 0, \ x' \in \mathbb{R}^d \} \) solves the log-Laplace integral equation

\[
v^n_{s,x}(s', x') = \int_{\mathbb{R}^d} dy \ p_{s'}(y - x') 1_B(y) p_s(y - x)
\]

\[
- \int_0^{s'} ds' \int_{\mathbb{R}^d} dy \ p_{s'-s'}(y - x')(v^n_{s,x}(s', y))^{1+\beta}
\]

(75)

related to (5).

**Lemma 26.** If \( d > 1 \) then

\[
\lim_{n \uparrow \infty} \left( \inf_{0 < s < \varepsilon_n, \ x \in B_n} h_\beta(s) v^n_{s,x}(s, x) \right) = +\infty.
\]

(76)

**Proof.** We start with a determination of the asymptotics of the first term at the right hand side of the log-Laplace equation (73) at \( (s', x') = (s, x) \). Note that

\[
\int_{\mathbb{R}^d} dy \ p_s(y - x) 1_B(y) p_s(y - x)
\]

(77)

\[
= \int_{\mathbb{R}^d} dy \ p_s(y - x) p_s(y - x) - \int_{B^c} dy \ p_s(y - x) p_s(y - x).
\]
In the latter formula line, the first term equals \( p_{2s}(0) = Cs^{-d/2} \), whereas the second one is bounded from above by

\[
\sup_{0 < s < \varepsilon_n, x \in B_n, y \in B^c} p_s(y - x) \longrightarrow 0, \quad \text{as } n \uparrow \infty \tag{78}
\]

where the last convergence follows by assumption (67) on \( B_n \). Hence from (75) and (77) we obtain

\[
\int_{\mathbb{R}^d} dy \, p_s(y - x) 1_B(y) p_s(y - x) = Cs^{-d/2} + o(1) \quad \text{as } n \uparrow \infty, \quad \tag{79}
\]

uniformly in \( s \in (0, \varepsilon_n) \) and \( x \in B_n \).

To simplify notation, we write \( v^n := v^n_{s,x} \). Next, since \( v^n \) is non-negative we drop the non-linear term in from (75) to get the upper bound

\[
v^n(s', x') \leq \int_{\mathbb{R}^d} dy \, p_{s'}(y - x') p_s(y - x) = p_{s+s}(x - x'). \tag{80}
\]

Then we have

\[
\begin{align*}
\int_0^s dr' \int_{\mathbb{R}^d} dy \, p_{s-r'}(y - x) (v^n(r', y))^{1+\beta} \\
\leq \int_0^s dr' \int_{\mathbb{R}^d} dy \, p_{s-r'}(y - x) (p_{r+s}(x - y))^{1+\beta} \\
\leq (p_s(0))^{\beta} \int_0^s dr' \int_{\mathbb{R}^d} dy \, p_{s-r'}(y - x) p_{r+s}(x - y) \\
= (p_s(0))^{\beta} \int_0^s dr' \, p_{2s}(0) = Cs^{1-d(1+\beta)/2}.
\end{align*}
\]

Summarizing, by (75), (79) and (80),

\[
v^n(s, x) \geq Cs^{-d/2} + o(1) - Cs^{1-d(1+\beta)/2} \tag{81}
\]

uniformly in \( s \in (0, \varepsilon_n) \) and \( x \in B_n \). According to the general assumption \( d < 2/\beta \), we conclude that the right hand side of (81) behaves like \( Cs^{-d/2} \) as \( s \downarrow 0 \), uniformly in \( s \in (0, \varepsilon_n) \). Now recalling definition (72) as well as our assumption that \( d > 1 \) we immediately get

\[
\lim_{n \uparrow \infty} \inf_{0 < s < \varepsilon_n} h_{\beta}(s) s^{-d/2} = +\infty.
\]

By (81), this implies (73), and the proof of the lemma is finished. \( \square \) \( \square \)

We are now in position to complete the proof of Theorem 1(b). The claim (73) readily follows from estimates (74) and (79). Moreover, according to (73), by passing to the limit \( n \uparrow \infty \) in the right hand side of (71), and then using (68), we arrive at

\[
\mathbb{E} \exp \left\{ - \|X_t\|_B \right\} \leq \sup_{n \uparrow \infty} \mathbb{P} \left( \tau_n = \infty \right) \leq \sup_{n \uparrow \infty} \mathbb{P} \left( X_t(B_n) = 0 \right).
\]

Since the event \( \{X_t(B) = 0\} \) is the non-increasing limit as \( n \uparrow \infty \) of the events \( \{X_t(B_n) = 0\} \) we get

\[
\mathbb{E} \exp \left\{ - \|X_t\|_B \right\} \leq \mathbb{P} \left( X_t(B) = 0 \right).
\]

Since obviously \( \|X_t\|_B = 0 \) if and only if \( X_t(B) = 0 \), we see that (65) follows from this last bound. The proof of Theorem 1(b) is finished for \( U = B \).
6. Pointwise Hölder exponent at a given point: proof of Theorem 3

Let us first give a heuristic explanation for the value of $\bar{\eta}_k$. According to Lemmas 21 and 23 the maximal jump at time $s$ and spatial point $x$ near point $z = 0$ is of order $((t-s)|x|)^{1/(1+\beta)}$. Due to the scaling properties of the heat kernel, the jump that has a decisive effect on the pointwise Hölder exponent at $z = 0$ should occur at distance

$$|x| \approx (t-s)^{1/2}. \quad (82)$$

Then the size of this jump $r$ is of order

$$((t-s)|x|)^{1/(1+\beta)} \approx |x|^{3/(1+\beta)}. \quad (84)$$

Therefore the convolution of the jump $r\delta_x$ with $p_{t-s}(x-\cdot) - p_{t-s}(0 - \cdot)$ is of order

$$|x|^{3/(1+\beta)}(p_{t-s}(0) - p_{t-s}(|x|)) \approx |x|^{3/(1+\beta) - 1}. \quad (83)$$

In the last step we used (82). This leads then to the result that difference of values of the density at points $x$ and $0$ is of the same order. Then the pointwise Hölder exponent at $0$ should be

$$\frac{3}{1+\beta} - 1 = \bar{\eta}_k.$$  

This heuristic works for $\bar{\eta}_k < 1$. In the case $\bar{\eta}_k > 1$ the density becomes differentiable. For this reason one has to convolute $r\delta_x$ with $q_{t-s}(x - \cdot, 0 - \cdot)$, where $q$ is defined in (29). This convolution is also of the order

$$|x|^{3/(1+\beta)}q_{t-s}(0, x) | \approx |x|^{3/(1+\beta) - 1}. \quad (83)$$

Again, we arrive at the same value of the pointwise Hölder exponent $\bar{\eta}_k$.

Since the case $\bar{\eta}_k < 1$ has been studied in [9], we shall concentrate here on the case $\bar{\eta}_k > 1$. In other words, we shall assume that $\beta < 1/2$. Under this assumption, the function $\frac{p_t}{p_{t-s}}p_{t-s}(x_2 - y)$ is integrable with respect to $M(du, dy)$. Consequently, we may consider $Z_k(x_1, x_2)$ defined in (28) with $q$ defined in (29).

6.1. Proof of the lower bound for $H_X(0)$. To get a lower bound for $H_X(0)$ it suffices to show that, for every positive $\gamma$,

$$\sup_{0 < x < 1} \frac{|Z_k(x, 0)|}{|x|^{\bar{\eta}_k - \gamma}} < \infty$$

with probability one.

Let $\Delta Z_k(x_1, x_2)$ denote the jump of $Z(x_1, x_2)$:

$$\Delta Z_k(x_1, x_2) := Z_k(x_1, x_2) - Z_k(-x_1, x_2).$$

Denote

$$A^*_1 := \{ |\Delta X_s| \leq c_{12} (t-s)^{(1+\beta)^{-1} - \gamma} \text{ for all } s < t \} \cap \{ V(B_1(0)) \leq c_\varepsilon \} \cap \{ \Delta X_s(x) \leq c_{14} (t-s)|x|^{1+\beta - \gamma} \text{ for all } s < t \text{ and } x \in B_2(0) \}$$

with $V$ defined in Lemma 15. According to Lemma 15 there exists $c_\varepsilon$ such that $P(V(B_1(0)) > c_\varepsilon) < \varepsilon$. Combining this with Lemma 20 and Corollary 22 we conclude that

$$P(A^*_1) > 1 - 3\varepsilon.$$  \quad (84)$$

In the next lemma we derive an upper bound for jumps of $Z_k(0, x)$. Afterwords, in Lemma 28 we derive an upper bound for the values of $Z_k(0, x)$, which confirms the upper bound part of (83).
Lemma 27. On the event \( A_1^c \) we have, for all \( s \leq t \) and all \( x \in \mathbb{R} \),
\[
|\Delta Z_s(x,0)| \leq C|x|^{\eta - 3\gamma}.
\]

Proof. Let \((y,s,r)\) be the point of an arbitrary jump of the measure \( \mathcal{N} \) with \( s \leq t \). Then for the corresponding jump of \( Z(x,0) \) we have the following bound
\[
|\Delta Z_s(x,0)| \leq r|q_{t-s}(x-y, -y)|. \tag{85}
\]

On \( A_1^c \) we have
\[
r \leq C(t-s)^{\frac{1}{1+\gamma}}|y|^{\frac{1}{1+\gamma}}.
\]

Applying (133) with \( \delta = \eta_c - 1 - 3\gamma \) to \(|q_{t-s}(x-y, -y)|\), we obtain
\[
|\Delta Z_s(x,0)| \leq C(t-s)^{\frac{1}{1+\gamma}}|y|^{\frac{1}{1+\gamma}}|x|^{\eta - 3\gamma} \frac{|x|}{(t-s)^{\frac{3}{2}}}(p_{t-s}(-y/2) + p_{t-s}((x-y)/2))
\]
\[
= C|x|^{\eta - 3\gamma} \frac{|y|}{(t-s)^{1/2}} \left( p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) + p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \right). \tag{86}
\]

Assume first that \(|x| \leq (t-s)^{1/2}\). If \(|y| > 2(t-s)^{1/2}\) then \(|x-y| > |y|/2\) and, consequently,
\[
\left( \frac{|y|}{(t-s)^{1/2}} \right)^{\frac{1}{1+\gamma}} \left( p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) + p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \right) \leq 2 \left( \frac{|y|}{(t-s)^{1/2}} \right)^{\frac{1}{1+\gamma}} p_1 \left( \frac{|y|}{4(t-s)^{1/2}} \right).
\]

Noting that the function on the right hand side is bounded, we conclude from (86) that
\[
|\Delta Z_s(x,0)| \leq C|x|^{\eta - 3\gamma} \tag{87}
\]
for \(|x| \leq (t-s)^{1/2}\) and \(|y| > 2(t-s)^{1/2}\). Moreover, if \(|y| \leq 2(t-s)^{1/2}\) then
\[
\left( \frac{|y|}{(t-s)^{1/2}} \right)^{\frac{1}{1+\gamma}} \left( p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) + p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \right) \leq 2^{1+\beta}.
\]

Consequently, (87) is valid also for \(|y| \leq 2(t-s)^{1/2}\) and \(|x| \leq (t-s)^{1/2}\).

Assume now that \(|x| > (t-s)^{1/2}\) and \(|y| > 2(t-s)^{1/2}\). In this case we bound \(q_{t-s}\) in a completely different way:
\[
|q_{t-s}(x-y, -y)| \leq |p_{t-s}(x-y) - p_{t-s}(-y)| + |x| \left| \frac{\partial}{\partial y} p_{t-s}(-y) \right|.
\]

Applying now (130) with \( \delta = \eta_c - 2\gamma \) and (131), we see that \(|q_{t-s}(x-y, -y)|\) is bounded by
\[
C|x|^{\eta - 2\gamma} (t-s)^{-\frac{1}{1+\gamma}} \left( p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) + p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \right)
\]
\[
+ C \frac{|x|}{t-s} \eta \frac{-y}{2(t-s)^{1/2}}.
\]
Consequently,
\[
|\Delta Z_s(x,0)| \leq C|y|^{\frac{1}{1+\beta}-\gamma}|x|^\eta-2\gamma \left( p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) + p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \right) \\
+ C|x||y|^{\frac{1}{1+\beta}-\gamma}(t-s)^{1-\gamma}p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right). \tag{88}
\]

Since \( u^{\frac{1}{1+\beta}-\gamma}p_1(u) \) is bounded, the term in the second line in (88) does not exceed
\[
C|x|(t-s)^{\frac{3}{1+\beta}-\frac{3\gamma}{2}}.
\]

As a result, for \( |x| > (t-s)^{1/2} \) we get
\[
|y|^{\frac{1}{1+\beta}-\gamma}(t-s)^{1-\gamma}p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) \leq C|x|^{\frac{1}{1+\beta}-\frac{1}{2}} = C|x|^{\frac{1}{1+\beta}-\frac{1}{2}}. \tag{89}
\]

By the same argument,
\[
|y|^{\frac{1}{1+\beta}-\gamma}|x|^\eta-2\gamma p_1 \left( \frac{-y}{2(t-s)^{1/2}} \right) \leq C|x|^{\frac{1}{1+\beta}-\gamma}. \tag{90}
\]

We next note that if \(|y| > 2|x|\) then \(|y-x| > |y|/2\) and, consequently,
\[
p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \leq p_1 \left( \frac{x-y}{4(t-s)^{1/2}} \right).
\]

Therefore,
\[
|y|^{\frac{1}{1+\beta}-\gamma}|x|^\eta-2\gamma p_1 \left( \frac{x-y}{2(t-s)^{1/2}} \right) \leq C|x|^{\frac{1}{1+\beta}-\gamma}. \tag{91}
\]

for \(|y| > 2|x|\). But if \(|y| \leq 2|x|\) then (91) is obvious. Combining (88) – (91) we conclude that (87) holds for \(|x| > (t-s)^{1/2}\). This completes the proof. □ □

**Lemma 28.** For every fixed \( x \) with \(|x| < 1\) we have
\[
P \left( |Z_t(x,0)| > r|x|^{\frac{\eta-2\gamma}{\gamma}}, A_1 \right) \leq \left( \frac{C}{r|x|^\gamma} \right)^{r|x|^{1-\gamma}/C}, \quad r > 0.
\]

**Proof.** According to Lemma 10 there exist spectrally positive \((1+\beta)\)-stable processes \( L^+(x) \) and \( L^-(x) \) such that
\[
Z_t(x,0) = L^+_{T^+_t(x)} - L^-_{T^-_t(x)}, \tag{92}
\]
where
\[
T^\pm_t(x) = \int_0^t du \int_\mathbb{R} X_u(dy) \left( q_{t-u}(x-y,-y) \right)^{1+\beta}.
\]

By (31) with \( \theta = 1+\beta \) and \( \delta = (1-2\beta-\gamma)/(1+\beta) \), there exists \( C = C(\varepsilon,\gamma) \) such that
\[
T^\pm_t(x) \leq C|x|^{2-\beta-\varepsilon_1} \quad \text{on} \ A_1^\varepsilon. \tag{93}
\]
Applying Lemma 27 and (93), we obtain
\[
\mathbf{P}\left( L^R_{\mathcal{T}_\pm}(x) \geq r|x|^{-2\gamma}, A_1^\varepsilon \right) \\
\leq \mathbf{P}\left( L^R_{\mathcal{T}_\pm}(x) \geq r|x|^{-2\gamma}, A_1^\varepsilon, \sup_{x \in \mathcal{T}_\pm(x)} \Delta L^R_s \leq C|x|^{-3\gamma} \right) \\
\leq \mathbf{P}\left( \sup_{x \in \mathcal{T}_\pm(x)} L^R_s(x) \geq r|x|^{-2\gamma}, A_1^\varepsilon, \sup_{0 \leq s \leq t} \Delta L^R_s \leq C|x|^{-3\gamma} \right).
\]

Applying now Lemma 46, we get
\[
\mathbf{P}\left( L^R_{\mathcal{T}_\pm}(x) \geq r|x|^{-2\gamma}, A_1^\varepsilon \right) \leq \left( \frac{C|x|^{\beta-\gamma}}{r|x|^{-2\gamma}} \right)^{r|x|^{-\gamma/C}} = \left( \frac{C}{r} \right)^{r|x|^{-\gamma/C}}.
\]

Thus, the proof is finished. \( \square \) \( \square \)

Taking into account (83), we see that the lower bound for \( H_x(0) \) will be proven if we show that, for every \( \varepsilon > 0 \),
\[
\sup_{0 < x < 1} \frac{|Z_t(x, 0)|}{x^{\bar{\eta}_e-\gamma}} < \infty \quad \text{on} \quad A_1^\varepsilon.
\] (94)

Fix some \( q \in (0, 1) \) and note that
\[
\left\{ \sup_{0 < x < 1} \frac{|Z_t(x, 0)|}{x^{\bar{\eta}_e-\gamma}} > k \right\} \subseteq \bigcup_{n=1}^\infty \left\{ \sup_{x \in I_n} |Z_t(x, 0)| > \frac{k}{2^q} n^{-\bar{\eta}_s-\gamma} \right\},
\]
where \( I_n := \{ x : (n+1)^{-q} \leq x < n^{-q} \} \). Moreover, by the triangle inequality,
\[
|Z_t(x, 0)| \leq |Z_t(n^{-q}, 0)| + |Z_t(x) - Z_t(n^{-q})| + (n^{-q} - x)|W|, \quad x \in I_n,
\]
where
\[
W := \int_0^t \int_\mathbb{R} M \left( d(u, y) \frac{\partial}{\partial y} p_{t-u}(-y) \right)
\]
Consequently, for all \( n \geq 1 \),
\[
\left\{ \sup_{x \in I_n} |Z_t(x, 0)| > \frac{k}{2^q} n^{-\bar{\eta}_s-\gamma} \right\} \subseteq \left\{ \sup_{x \in I_n} |Z_t(x) - Z_t(n^{-q})| > \frac{k}{3 \cdot 2^q} n^{-q} \right\}
\]
\[
\cup \left\{ |Z_t(n^{-q}, 0)| > \frac{k}{3 \cdot 2^q} n^{-\bar{\eta}_s-\gamma} \right\} \cup \left\{ |W| > \frac{k n^{-\bar{\eta}_s-\gamma}}{3 \cdot 2^q (n^{-q} - (n+1)^{-q})} \right\},
\]
Note that, for all \( R > 0 \),
\[
\left\{ \sup_{0 < x < 1} \frac{|Z_t(x) - Z_t(y)|}{x - y} \leq R \right\} \subseteq \left\{ |Z_t(x) - Z_t(n^{-q})| \leq R q^{\bar{\eta}_s-\gamma} / (q+1) n^{-q}, \quad x \in I_n \right\}.
\]
This implies that

\[
\left\{ \sup_{0 < x < 1} \left| Z_t(x, 0) \right| > k \right\} \subseteq \left\{ \sup_{0 < x < y < 1} \frac{\left| Z_t(x) - Z_t(y) \right|}{|x - y|^{q(\bar{\eta}_c - \gamma)/(q+1)}} > c(q)k \right\}
\]

\[
\cup \bigcup_{n=1}^{\infty} \left\{ \left| Z_t(n^{-q}, 0) \right| > \frac{k}{3 \cdot 2^q} n^{-q(\bar{\eta}_c - \gamma)} \right\} \cup \{ |W| > c(q)k \}.
\]

where \( c(q) \) is some positive constant.

If we choose \( q \) so small that \( (\bar{\eta}_c - \gamma)q/(q + 1) < \eta_c \), then

\[
\lim_{k \to \infty} \mathbb{P} \left( \sup_{0 < x < y < 1} \frac{\left| Z_t(x) - Z_t(y) \right|}{|x - y|^{q(\bar{\eta}_c - \gamma)/(q+1)}} > c(q)k \right) = 0,
\]

since, by Theorem [2] \( Z \) is locally Hölder continuous of every index smaller than \( \eta_c \).

Furthermore,

\[
\lim_{k \to \infty} \mathbb{P}(|W| > c(q)k) = 0.
\]

Finally, applying Lemma [28] conclude that

\[
\lim_{k \to \infty} \mathbb{P} \left( \bigcup_{n=1}^{\infty} \left\{ \left| Z_t(n^{-q}, 0) \right| > \frac{k}{3 \cdot 2^q} n^{-q(\bar{\eta}_c - \gamma)} \right\} \right) = 0.
\]

Thus, (94) is shown.

6.2. Proof of the optimality of \( \bar{\eta}_c \). Now it is time to explain the detailed strategy of the optimality proof. Define

\[
A_2^\varepsilon := \left\{ \Delta X_s(y) \leq c(\ell) \left( (t - s)|y| \right)^{1/(1+\beta)} (f_{s,x})^\ell \right\} \cap \left\{ \Delta X_s(y) \leq c(\ell) (t - s)^{1/(1+\beta)} - \gamma \right\} \cap \{ V(B_2(0)) \leq c_\varepsilon \},
\]

where \( f_{s,x} \) and \( \ell \) are defined in [50] and [52], respectively. Further, according to Lemma [13] \( \mathbb{P}(V(B_2(0)) > c_\varepsilon) \to 0 \) for any \( c_\varepsilon \to \infty \) as \( \varepsilon \to 0 \). Note that \( D_\theta \uparrow \{ X_t(0) > 0 \} \) as \( \theta \downarrow 0 \). Moreover, by Lemmata [20] and [15] \( A_2^\varepsilon \uparrow \Omega \) as \( \varepsilon \downarrow 0 \). Hence, for the proof of Theorem [3 b) it is sufficient to show that

\[
\mathbb{P} \left( \sup_{x \in B_1(0), x \neq 0} \frac{|X_t(x) - X_t(0)|}{|x|^\bar{\eta}_c} = \infty \right| D_\theta \cap A_2^\varepsilon = 1.
\]

Since \( \mu * p_t(x) \) is Lipschitz continuous at 0, the latter will follow from the equality

\[
\mathbb{P} \left( Z_t(c 2^{-n-2}, 0) \geq 2^{-\bar{\eta}_c n} n^{1/(1+\beta)} - \varepsilon \right| D_\theta \cap A_2^\varepsilon = 1,
\]

where we choose \( c = c(55) \).

To verify (96), we will again use the LM "the" instead of "our" method of representing \( Z \) as a time-changed stable process. To be more precise, applying (92) with \( x = c 2^{-n-2} \) (for \( n \) sufficiently large) and using \( n \)-dependent notation as \( L_n^+, T_n, \pm \) (and \( \varphi_n, \pm \)), we have

\[
Z_t(c 2^{-n-2}, 0) = L_n^+(T_n, +) - L_n^-(T_n, -).
\]

It follows from this representation that (96) is a consequence in the following statement.
Proposition 29. For almost every $\omega \in D_B \cap A_2^2$ there exists a subsequence $n_j$ such that

$$L_{n_j}^+ (T_{n_j},+) \geq 2^{1-\eta_n} n_j^{1/(1+\beta) - \varepsilon} \quad \text{and} \quad L_{n_j}^- (T_{n_j},-) \leq 2^{-\eta_n} n_j^{1/(1+\beta) - \varepsilon}.$$ 

Let us define the following events

$$B_n^+ := \{ L_n^+ (T_n,+) \geq 2^{1-\eta_n} n^{1/(1+\beta) - \varepsilon} \}, \quad B_n^- := \{ L_n^- (T_n,-) \leq 2^{-\eta_n} n^{1/(1+\beta) - \varepsilon} \}.$$

and

$$B_n := B_n^+ \cap B_n^-.$$

Then, obviously, Proposition 29 will follow once we verify

$$\lim_{N \uparrow \infty} P \left( \bigcup_{n=N}^{\infty} (B_n \cap A_n) \bigg| D_B \cap A_2^2 \right) = 1,$$

where $A_n$ were defined in (62). Taking into account Lemma 24 we conclude that to get (97) we have to show

$$\lim_{N \uparrow \infty} P \left( \bigcup_{n=N}^{\infty} (B_n^c \cap A_n) \bigg| D_B \cap A_2^2 \right) = 0.$$

Let us explain briefly the meaning of (97). By Lemma 24 we know that there exists a sequence of big jumps

$$\Delta X_s \left( \frac{c}{2} 2^{-n_j}, \frac{3c}{2} 2^{-n_j} \right) \geq 2^{-(7\beta + 1)n_j} n_j^{1/(1+\beta)}$$

for some $s \in (t - 2^{-2n_j}, t - 2^{-2(n_j+1)})$. (97) implies that these jumps guarantee big values of $L_n^+ (T_n,+) - L_n^- (T_n,-)$ for some subsequence of $\{n_j\}$. And this is the main consequence of Proposition 29.

Now we will present two lemmas, from which (98) will follow immediately. To this end, split

$$B_n^c \cap A_n = (B_n^{+,c} \cap A_n) \cup (B_n^{-,c} \cap A_n).$$

Lemma 30. We have

$$\lim_{N \uparrow \infty} \sum_{n=N}^{\infty} P (B_n^{+,c} \cap A_n \cap A_2^2) = 0.$$
Lemma 31. We have

\[
\lim_{N \uparrow \infty} \sum_{n=N}^{\infty} P(B_n^{c} \cap A_n \cap A_n^{2} \cap D_n) = 0.
\]

Now we are ready to finish

Proof of Proposition 29. Combining Lemmata 30 and 31, we conclude that there exists a subsequence \( \{n_j\} \) with properties described in Proposition 29. \( \square \)

The remaining part of the paper will be devoted to the proof of Lemma 31 and we prepare now for it.

One can easily see that \( B_n^{c} \) is a subset of a union of two events (with the obvious correspondence):

\[
B_n^{c} \subseteq U_1^1 \cup U_2^2 := \{ \Delta L_n^- > 2^{-\tilde{\eta}_n} n^{1/(1+\beta)-2\varepsilon}\} \cup \{ \Delta L_n^- \leq 2^{-\tilde{\eta}_n} n^{1/(1+\beta)-2\varepsilon}, L_n^{-}(T_{n,-}) > 2^{-\tilde{\eta}_n} n^{1/(1+\beta)-\varepsilon}\},
\]

where

\[
\Delta L_n^- := \sup_{0<s \leq T_{n,-}} \Delta L_n^- (s).
\]

The occurrence of the event \( U_1^1 \) means that \( L_n^- \) has big jumps. If \( U_2^2 \) occurs, it means that \( L_n^- \) gets large without big jumps. It is well-known that stable processes without big jumps can not achieve large values. Thus, the statement of the next lemma is not surprising.

Lemma 32. We have

\[
\lim_{N \uparrow \infty} \sum_{n=N}^{\infty} P(U_2^2 \cap A_n^1) = 0.
\]

We omit the proof of this lemma as well, since its crucial part related to bounding of \( P(U_2^2 \cap A_n^1) \) is a repetition of the proof of Lemma 5.6 in [8] (again with obvious simplifications).

Lemma 33. There exist constants \( \rho \) and \( \xi \) such that, for all sufficiently large values of \( n \),

\[
A_n^2 \cap A_n \cap U_n^1 \subseteq A_n^c \cap E_n(\rho, \xi),
\]

where

\[
E_n(\rho, \xi) := \left\{ \text{There exist at least two jumps of } M \text{ of the form } r\delta_{(s,y)} \text{ such that } \begin{align*}
 r &\geq \left( (t-s) \max \{ (t-s)^{1/2}, |y| \} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\varepsilon} ((t-s)^{-1}), \\
 |y| &\leq (t-s)^{1/2} \log^{\xi} ((t-s)^{-1}), \ s \in [t-2^{-2n} n^\rho, t-2^{-2n} n^{-\rho}] \right\}.
\]

Proof. By the definition of \( A_n \), there exists a jump of \( M \) of the form \( r\delta_{(s,y)} \) with \( r, s \) as in \( E_n(\rho, \xi) \), and \( y > c2^{-n-1} \). Furthermore, noting that \( \varphi_{n-} (y) = 0 \) for \( y \geq c2^{-n-3} \), we see that the jumps \( r\delta_{(s,y)} \) of \( M \) contribute to \( L_n^{-}(T_{n,-}) \) if and only if \( y < c2^{-n-3} \). Thus, to prove the lemma it is sufficient to show that \( U_n^1 \) yields
the existence of at least one further jump of \( M \) on the half-line \( \{ y < c 2^{-n-3} \} \) with properties mentioned in the statement. Denote

\[
D := \left\{ (r, s, y) : r \geq \left( (t - s) \max \{ (t - s)^{1/2}, |y| \} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\epsilon} ((t - s)^{-1}),
\right.
\]

\[
y \in \left[ -(t - s)^{1/2} \log^\xi ((t - s)^{-1}), c 2^{-n-3} \right],
\]

\[
s \in \left[ t - 2^{-2n} n^\rho, t - 2^{-2n} n^{-\rho} \right].
\]

Then we need to show that \( U_n^1 \) implies the existence of a jump \( r\delta_{(s,y)} \) of \( M \) with

\( (r, s, y) \in D \).

Note that

\[
D = D_1 \cap D_2 \cap D_3
\]

\[
:= \left\{ (r, s, y) : r \geq 0, \ s \in [0, t], \ y \in \left( -(t - s)^{1/2} \log^\xi ((t - s)^{-1}), c 2^{-n-3} \right) \right\}
\]

\[
\cap \left\{ (r, s, y) : r \geq 0, \ y \in (-\infty, c 2^{-n-3}), \ s \in \left[ t - 2^{-2n} n^\rho, t - 2^{-2n} n^{-\rho} \right] \right\}
\]

\[
\cap \left\{ (r, s, y) : y \in (-\infty, c 2^{-n-3}), \ s \in [0, t],
\right.
\]

\[
r \geq \left( (t - s) \max \{ (t - s)^{1/2}, |y| \} \right)^{1/(1+\beta)} \log^{1/(1+\beta)-2\epsilon} ((t - s)^{-1}) \right\}.
\]

Therefore,

\[
D^c \cap \{ y < c 2^{-n-3} \} = \left( D_1^c \cap \{ y < c 2^{-n-3} \} \right) \cup \left( D_1^c \cap D_2^c \right) \cup \left( D_1 \cap D_2 \cap D_3^c \right),
\]

where the complements are defined with respect to the set

\[
\{(r, s, y) : r \geq 0, \ s \in [0, t], \ y \in \mathbb{R} \}.
\]

We first show that any jumps of \( M \) in \( D_1^c \cap \{ y < c 2^{-n-3} \} \) cannot be the course of a jump of \( L_n^\delta \) such that \( U_n^1 \) holds. Indeed, using the last inequality in Lemma 105 with \( \delta = \bar{\gamma}_e \), we get for \( y < c 2^{-n-3} \) the inequality

\[
(\rho t - s (c 2^{-n-2}, 0)) \leq C 2^{-\bar{\gamma}_e n} (t - s)^{-\bar{\gamma}_e/2} \rho t - s (y/2)
\]

\[
\leq C 2^{-\bar{\gamma}_e n} (t - s)^{-1+\bar{\gamma}_e/2} \exp \left\{ - \frac{y^2}{8(t - s)} \right\}
\]

\[
\leq C 2^{-\bar{\gamma}_e n} (t - s)^{1-\bar{\gamma}_e/2} |y|^{-3}, \quad (101)
\]

in the second step we used the scaling property of the kernel \( p \), and in the last step we have used the trivial bound \( e^{-x} \leq x^{-3/2} \).

Further, by (106), on the set \( A_2^\delta \) we have

\[
\Delta X_s(y) \leq C \left( |y| (t - s)^{1/(1+\beta)} (f_{s,y})^\xi \right)^k, \quad |y| \leq 1/e, \quad (102)
\]

and

\[
\Delta X_s(y) \leq C (t - s)^{1/(1+\beta) - \gamma}, \quad |y| > 1/e, \quad (103)
\]

and recall that \( f_{s,x} = \log((t - s)^{-1}) 1_{\{x \neq 0\}} \log(|x|^{-1}) \). Combining (101) and (102), we conclude that the corresponding jump of \( L_n^\delta \), henceforth denoted by \( \Delta L_n^\delta \gamma_{(s,y)} \), is bounded by

\[
C 2^{-\bar{\gamma}_e n} (t - s)^{1-\bar{\gamma}_e/2+q} \log^{1+q} ((t - s)^{-1}) |y|^{-3+q} \log^{1+q} |y|^{-1}.
\]
From this bound and (102) we obtain

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{-\bar{n}} n \log \frac{t \log \frac{t}{s}}{r\delta} \left( |y|^{-1} \right). \]

Choosing \( \xi \geq \frac{2+2q(1+\beta)}{3(1+\beta)} \), we see that

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{-\bar{n}} n, \quad |y| < 1/e. \] (104)

Moreover, if \( y < -1/e \), then it follows from (101) and (103) that the jump \( \Delta L_n^-[r\delta(s,y)] \) is bounded by

\[ C 2^{-\bar{n}} n (t-s)^{1-n/2+\frac{1}{t\log\frac{t}{s}}} |y|^{-3} \leq C 2^{-\bar{n}} n. \] (105)

Combining (104) and (105), we see that all the jumps of \( M \) in \( D_1^t \cap \{ y < c 2^{-n/3} \} \) do not produce jumps of \( L_n \) such that \( U_1^t \) holds.

We next assume that \( M \) has a jump \( r\delta(s,y) \) in \( D_1 \cap D_2^s \). If, additionally, \( s \leq t - 2^{-2n} n^\rho \), then, using the last inequality in Lemma 14 with \( \delta = 1 \), we get

\[ (q_{t-s}(c2^{-n-2},0))^- \leq C 2^{-2n} (t-s)^{-3/2}. \]

From this bound and (102) we obtain

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{2n} (t-s)^{-3/2+\frac{1}{t\log\frac{t}{s}}} \log \left( |y|^{1/2} \log \frac{|y|^{1/2} \log \frac{|y|^{1/2}}{r\delta}}{|(t-s)^{-1}|} \right) \leq C 2^{2n} (t-s)^{-3/2+\frac{1}{t\log\frac{t}{s}}} \log \left( |y|^{1/2} \log \frac{|y|^{1/2} \log \frac{|y|^{1/2}}{r\delta}}{|(t-s)^{-1}|} \right). \]

Using the assumption \( t-s \geq 2^{-2n} n^\rho \), we arrive at the inequality

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{-\bar{n}} n^{2+2\frac{1}{t\log\frac{t}{s}}+2q(\bar{n}-2)}. \]

From this we see that if we choose \( \rho \geq \frac{2(2q+(2+\xi)(1+\beta))}{2-\bar{n}} \) then the jumps of \( L_n^-[r\delta(s,y)] \) are bounded by \( C 2^{-\bar{n}} n \), and hence \( U_1^t \) does not occur.

Using (130) with \( \delta = 1 \) and (131), one can easily derive

\[ (q_{t-s}(c2^{-n-2},0))^- \leq C 2^{-n} (t-s)^{-1}. \]

Then for \( y \in \left( -(t-s)^{1/2} \log \xi \left( (t-s)^{-1} \right), \ c2^{-n-3} \right) \) and \( t-s \leq 2^{-2n} n^{-\rho} \) we have the inequality

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{-n} (t-s)^{-1} \left( |y| \left( (t-s) \right) \right) \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right) \]

\[ \leq C 2^{-n} (t-s)^{-1} \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right) \]

\[ = C 2^{-n} (t-s)^{(\bar{n}-1)/2} \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right) \]

\[ \leq C 2^{-\bar{n}} n^{\rho(\bar{n}-1)/2+2n+2q+2q}. \]

Choosing \( \rho \geq \frac{2(3+2q(1+\beta))}{1+\beta} (\bar{n}-1) \), we conclude that \( \Delta L_n^-[r\delta(s,y)] \leq C 2^{-\bar{n}} n \), and again \( U_1^t \) does not occur.

Finally, it remains to consider the jumps of \( M \) in \( D_1 \cap D_2 \cap D_2^s \). If the value of the jump does not exceed \( (t-s)^{1/2+\log \frac{1}{r\delta}} \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right) \), then it follows from (133) with \( \delta = \bar{n} - 1 \) that

\[ \Delta L_n^-[r\delta(s,y)] \leq C 2^{-\bar{n}} n (t-s)^{-1} \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right) \]

\[ \leq C 2^{-\bar{n}} n \log \frac{2n+2q}{r\delta} \left( (t-s)^{-1} \right). \]
Then, on $D_2$, that is, for $t - s > 2^{-2n}n^{-\rho}$,
$$
\Delta L_n^t \left[ r \bar{\delta}(s,y) \right] \leq C 2^{-\bar{\delta}_n}n^{\frac{1}{1+\rho} - 2\varepsilon}.
$$
(106)

Furthermore, if $y < -(t-s)^{1/2}$ and the value of the jump is less than $\left( \frac{|y|(t-s)}{1+\log(t-s)^{-1}} \right)^{-2\varepsilon}$, then, we get
$$
\Delta L_n^t \left[ r \bar{\delta}(s,y) \right] \leq C 2^{-\bar{\delta}_n}(t-s)^{-1} \left( \frac{\log(1+\log(t-s)^{-1})}{(t-s)} \right)^{-2\varepsilon} |y|^{-3+\frac{1}{1+\rho}}.
$$

Then, on $D_2$, that is, for $t - s > 2^{-2n}n^{-\rho}$,
$$
\Delta L_n^t \left[ r \bar{\delta}(s,y) \right] \leq C 2^{-\bar{\delta}_n}n^{\frac{1}{1+\rho} - 2\varepsilon}.
$$
(107)

By (106) and (107), we see that the jumps of $M$ in $D_1 \cap D_2 \cap D_3$ do not produce jumps such that $U_1^t$ holds. Combining all the above we conclude that to have $\Delta L_n^t \left[ r \bar{\delta}(s,y) \right] > C 2^{-\bar{\delta}_n}n^{\frac{1}{1+\rho} - 2\varepsilon}$ it is necessary to have a jump in $D_1 \cap D_2 \cap D_3$. Thus, the proof is finished.

Now we are ready to finish

**Proof of Lemma 31** In view of the Lemmas 32 and 33 it suffices to show that
$$
\lim_{N \to \infty} \sum_{n=N}^{\infty} P \left( E_n(\rho, \xi) \cap A_2^c \cap D_\theta \right) = 0.
$$
(108)

The intensity of the jumps in $D$ [the set defined in (100) and satisfying conditions in $E_n(\rho, \xi)$] is given by
$$
\int_{t-2^{-2n}n^{-\rho}}^{t} ds \int_{|y| \leq \left( t-s \right)^{1/2}} X_s(dy) \log^2(1+\log(1+\log(t-s)^{-1})) \frac{(t-s)^{-1}}{(t-s) \max \{(t-s)^{1/2}, |y|\}}.
$$
(109)

Since in (108) we are interested in a limit as $N \to \infty$, we may assume that $n$ is such that $(t-s)^{1/2} \log^2((t-s)^{-1}) \leq 1$ for $s \geq t - 2^{-2n}n^\rho$. We next note that
$$
\int_{|y| \leq (t-s)^{1/2}} X_s(dy) \max \{(t-s)^{1/2}, |y|\}
$$
$$
= (t-s)^{-1/2} X_s\left( -(t-s)^{1/2}, (t-s)^{1/2} \right) \leq \theta^{-1}
$$
on $D_\theta$. Further, for every $j \geq 1$ satisfying $j \leq \log^2((t-s)^{-1})$,
$$
\int_{j(t-s)^{1/2} \leq |y| \leq \left( j+1 \right)(t-s)^{1/2}} X_s(dy) \max \{(t-s)^{1/2}, |y|\}
$$
$$
\leq j^{-1}(t-s)^{-1/2} X_s\left( \left\{ y : j(t-s)^{1/2} \leq |y| \leq \left( j+1 \right)(t-s)^{1/2} \right\} \right).
$$

Since the set $\left\{ y : j(t-s)^{1/2} \leq |y| \leq \left( j+1 \right)(t-s)^{1/2} \right\}$ is the union of two balls with radius $\frac{1}{2} (t-s)^{-1/2}$ and centers in $B_2(0)$, we can apply Lemma 16 with $c = 1$ to get
$$
\int_{j(t-s)^{1/2} \leq |y| \leq \left( j+1 \right)(t-s)^{1/2}} X_s(dy) \max \{(t-s)^{1/2}, |y|\} \leq 2 \theta^{-1} j^{-1}.
$$
on $D_\theta$. As a result, on the event $D_\theta$ we get the inequality
\[
\int_{|y| \leq (t-s)^{1/2} \log((t-s)^{-1})} X_t(dy) \leq \frac{1}{\max\{(t-s)^{1/2}, |y|\}} \leq C\theta^{-1} \log\left(\left|\log((t-s)^{-1})\right|\right).
\]
Substituting this into (109), we conclude that the intensity of the jumps is bounded by
\[
C\theta^{-1} \int_{t-2^{-2n}\rho}^{t-2^{-2n}\rho} ds \frac{\log^{2\epsilon(1+\beta)-1}((t-s)^{-1} \log\log((t-s)^{-1}))}{(t-s)}.
\]
Simple calculations show that the latter expression is less than
\[
C\theta^{-1} n^{2\epsilon(1+\beta)-1} \log^{1+2\epsilon(1+\beta)} n.
\]
Consequently, since $E_n(\rho, \xi)$ holds when there are two jumps in $D$, we have
\[
\mathbf{P}\left( E_n(\rho, \xi) \cap A_{\epsilon}^2 \cap D_\theta \right) \leq C\theta^{-2} n^{4\epsilon(1+\beta)-2} \log^{2+4\epsilon(1+\beta)} n.
\]
Because $\epsilon < 1/8 \leq 1/4(1+\beta)$, the sequence $\mathbf{P}(E_n(\rho, \xi) \cap A_{\epsilon}^2 \cap D_\theta)$ is summable, and the proof of the lemma is complete. □

7. ELEMENTS OF THE PROOF OF THEOREM 5

The spectrum of singularities of $X_t$ coincides with that of $Z$. Consequently, to prove Theorem 5, we have to determine Hausdorff dimensions of the sets
\[
\mathcal{E}_Z, \eta := \{x \in (0,1) : H_Z(x) = \eta\},
\]
\[
\tilde{\mathcal{E}}_Z, \eta := \{x \in (0,1) : H_Z(x) \leq \eta\}
\]
and this is done in the next two subsections.

As usual, we give some heuristic arguments, which should explain the result. Using heuristic arguments that led to (83), one can easily show that a jump of order $(t-s)^{\nu}$ occurring between times $t-s$ and $t$ implies that (if there are no other "big" jumps nearby) the pointwise Hölder exponent should be $2\nu - 1$ in the ball of radius $(t-s)^{1/2}$ centered at the spatial position of this jump. In other words, in order to have $H_X(x) = \eta$ at a point $x$, a jump of order $(t-s)^{(\eta+1)/2}$ should appear, whose distance to $x$ is less or equal to $(t-s)^{1/2}$.

From the formula for the compensator we infer that the number of such jumps, $N_\eta$, is of order
\[
N_\eta \approx \int_{t-s}^{t} \, du \int_{(t-s)^{(\eta+1)/2}} r^{-2-\beta} \, dr \approx (t-s)^{-(\eta+1)(1+\beta)/2} \int_{t-s}^{t} \, du X_u((0,1)).
\]
It turns out that the random measure $X_u$ can be replaced by the Lebesgue measure multiplied by a random factor. (The proof of this fact is one of the main technical difficulties.) As a result, the number of jumps $N_\eta$, leading to pointwise Hölder exponent $\eta$ at certain points, is of order
\[
(t-s)^{-(\eta+1)(1+\beta)/2+1}.
\]
Since every such jump effects the regularity in the ball of radius \( r \approx (t - s)^{1/2} \), the corresponding Hausdorff dimension \( \alpha \) of the set of points \( x \), with \( H_X(x) = \eta \), can be obtained from the relation
\[
N_r r^\alpha \approx ((t - s)^{1/2})^\alpha (t - s)^{-(\gamma(1 + \beta)/2 + 1)} \approx 1.
\]
This gives
\[
\alpha = (\eta + 1)(1 + \beta) - 2 = (\eta - \eta_c)(1 + \beta),
\]
which coincides with (7).

Proposition 34. For every \( \eta \in [\eta_c, \overline{\eta}_c] \),
\[
\dim(\mathcal{E}_{Z, \eta}) \leq \dim(\tilde{\mathcal{E}}_{Z, \eta}) \leq (1 + \beta)(\eta - \eta_c), \quad \mathbf{P} \text{- a.s.}
\]

We need to introduce an additional notation. In what follows, for any \( \eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\} \), we fix an arbitrary small \( \gamma = \gamma(\eta) \in (0, \frac{10^{-7} \eta_c}{\alpha}) \) such that
\[
\gamma < \begin{cases} 
10^{-2} \min\{1 - \eta, \eta\}, & \text{if } \eta < 1, \\
10^{-2} \min\{\eta - 1, 2 - \eta\}, & \text{if } \eta > 1,
\end{cases}
\]
and define
\[
S_\eta := \left\{ x \in (0, 1) : \text{there exists a sequence } (s_n, y_n) \to (t, x) \right. \\
\left. \qquad \text{with } \Delta X_{s_n}((y_n)) \geq (t - s_n)^{\frac{10^{-7} \eta_c}{\alpha} - \gamma}|x - y_n|^{\eta - \eta_c} \right\}.
\]

To prove the above proposition we have to verify the following two lemmas.

Lemma 35. For every \( \eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\} \) we have
\[
\mathbf{P} \left( H_Z(x) \geq \eta - 4\gamma \text{ for all } x \in (0, 1) \setminus S_\eta \right) = 1.
\]

Lemma 36. For every \( \eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\} \) we have
\[
\dim(S_\eta) \leq (1 + \beta)(\eta - \eta_c), \quad \mathbf{P} \text{- a.s.}
\]

The aim of the Lemma 35 is to show that for any \( \varepsilon > 0 \) sufficiently small, outside the set \( S_{\eta + 4\gamma - \varepsilon} \) the H"older exponent is larger than \( \eta + \varepsilon \), and hence \( \tilde{\mathcal{E}}_{Z, \eta} \subset S_{\eta + 4\gamma - \varepsilon} \). Therefore Lemma 35 gives immediately the upper bound on dimension of \( \tilde{\mathcal{E}}_{Z, \eta} \). We can formalize it and immediately get

Proof of Proposition 34. It follows easily from Lemma 35 that \( \tilde{\mathcal{E}}_{Z, \eta} \subset S_{\eta + 4\gamma + \varepsilon} \) for every \( \eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\} \) and every \( \varepsilon > 0 \) sufficiently small. Therefore,
\[
\dim(\tilde{\mathcal{E}}_{Z, \eta}) \leq \lim_{\varepsilon \to 0} \dim(S_{\eta + 4\gamma + \varepsilon}).
\]

Using Lemma 36 we then get
\[
\dim(\tilde{\mathcal{E}}_{Z, \eta}) \leq (1 + \beta)(\eta + 4\gamma - \eta_c), \quad \mathbf{P} \text{- a.s.}
\]

Since \( \gamma \) can be chosen arbitrary small, the result for \( \eta \neq 1 \) follows immediately. The inequality for \( \eta = 1 \) follows from the monotonicity in \( \eta \) of the sets \( \tilde{\mathcal{E}}_{Z, \eta} \).

Let \( \varepsilon \in (0, \eta_c/2) \) be arbitrarily small. We introduce a new “good” event \( A_{\varepsilon}^2 \) which will be frequently used throughout the proofs. On this event, with high probability,
V = V(B_2(0)) by Lemma [15] is bounded by a constant, and there is a bound on the sizes of jumps. By Lemma [20], there exists a constant C^{110} = C^{110}(\varepsilon, \gamma) such that

$$P(|\Delta X_s| > C^{110}(t - s)^{(1 + \beta)^{-1} - \gamma} \text{ for some } s < t) \leq \varepsilon/3. \quad (110)$$

Then we fix another constant C^{111} = C^{111}(\varepsilon, \gamma) such that

$$P(V \leq C^{111}) \geq 1 - \varepsilon/3. \quad (111)$$

Recall that, by Theorem 1.2 in [3], x \mapsto X_t(x) is P-a.s. Hölder continuous with any exponent less than \eta. Hence we can define a constant C^{112} = C^{112}(\varepsilon) such that

$$P \left( \sup_{x_1, x_2 \in (0,1), x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^{\eta - \varepsilon}} \leq C^{112} \right) \geq 1 - \varepsilon/3. \quad (112)$$

Now we are ready to give the proof of Lemma 36.

**Proof of Lemma 36.** To every jump (s, y, r) of the measure \mathcal{N} (in what follows in the paper we will usually call them simply “jumps”) with

$$(s, y, r) \in D_{j,n} := [t - 2^{-j}, t - 2^{-j-1}) \times (0,1) \times [2^{-n-1}, 2^{-n})$$

we assign the ball

$$B^{(s,y,r)} := B \left( y, \left( \frac{2^{-n}}{(2^{-j-1})^{1/(\eta - \gamma)}} \right) \right). \quad (114)$$

We used here the obvious notation B(y, \delta) for the ball in \mathbb{R} with the center at y and radius \delta. Define \eta_0(j) := j(\frac{1}{1 + \beta} - \frac{\gamma}{\beta}). It follows from (110) and (113) that, on \mathcal{A}^2, there are no jumps bigger than 2^{-n_0(j)} in the time interval [t - 2^{-j}, t - 2^{-j-1}).

It is easy to see that every point from \mathcal{S}_0 is contained in infinitely many balls B^{(s,y,r)}. Therefore, for every j \geq 1, the set

$$\bigcup_{j \geq J, n \geq 1} \bigcup_{(s,y,r) \in D_{j,n}} B^{(s,y,r)}$$

covers \mathcal{S}_0. From (110) and (113) we conclude that, on \mathcal{A}^2, there are no jumps bigger than C^{110}2^{-j(1/(1 + \beta) - \gamma)} in the time interval s \in [t - 2^{-j}, t - 2^{-j-1}) for any j \geq 1. Define \eta_0(j) := j(\frac{1}{1 + \beta} - \frac{\gamma}{\beta}). Clearly, there exists \mathcal{J}_0 such that for all j \geq \mathcal{J}_0 there are no jumps bigger than 2^{-n_0(j)} in the time interval [t - 2^{-j}, t - 2^{-j-1}). Hence, for every j \geq \mathcal{J}_0, the set

$$\mathcal{S}_0(j) := \bigcup_{j \geq \mathcal{J}_0, n \geq n_0(j)} \bigcup_{(s,y,r) \in D_{j,n}} B^{(s,y,r)}$$

covers \mathcal{S}_0 for every \omega \in \mathcal{A}^2.
It follows from the formula for the compensator that, on the event 
\{\sup_{s \leq t} X_s((0,1)) \leq N\}, the intensity of jumps with \((s, y, r) \in D_{j,n}\) is bounded by
\[ N2^{-j-1} \int_{2^{-j-1}}^{2^{-j}} c_\beta r^{-2-\beta} dr = \frac{Nc_\beta(2^{1+\beta} - 1)}{2(1+\beta)} 2n(1+\beta)^{-j} =: \lambda_{j,n}. \]
Therefore, the intensity of jumps with \((s, y, r) \in \cup_{n=n_0(j)}^{n_1(j)} D_{j,n} := \tilde{D}_j\), where \(n_1(j) = j [\frac{1}{1+\beta} + \frac{\gamma}{2}]\), is bounded by
\[ \sum_{n=n_0(j)}^{n_1(j)} \lambda_{j,n} \leq \frac{Nc_\beta 2\beta}{(\beta + 1)} 2^{j(1+\beta)\gamma/4} =: \Lambda_j. \]
The number of such jumps does not exceed \(2\Lambda_j\), with the probability \(1 - e^{-(1-2\log 2)\lambda_j}\).
This is immediate from the exponential Chebyshev inequality applied to Poisson distributed random variables. Analogously, the number of jumps with \((s, y, r) \in D_{j,n}\) does not exceed \(2\lambda_{j,n}\) with the probability at least \(1 - e^{-(1-2\log 2)\lambda_{j,n}}\). Since
\[ \sum_j \left( e^{-(1-2\log 2)\Lambda_j} + \sum_{n=n_1(j)}^{\infty} e^{-(1-2\log 2)\lambda_{j,n}} \right) < \infty, \]
we conclude, applying the Borel-Cantelli lemma, that, for almost every \(\omega\) from the set \(A_2^c \cap \{\sup_{s \leq t} X_s((0,1)) \leq N\}\), there exists \(J(\omega)\) such that for all \(j \geq J(\omega)\) and \(n \geq n_1(j)\), the numbers of jumps in \(\tilde{D}_j\) and in \(D_{j,n}\) are bounded by \(2\Lambda_j\) and \(2\lambda_{j,n}\) respectively.

The radius of every ball corresponding to the jump in \(\tilde{D}_j\) is bounded by \(r_j := C2^{-\frac{\beta}{2n_0(j)}}\). Thus, one can easily see that
\[ \sum_{j=1}^{\infty} \left( 2\Lambda_j r_j^\theta + \sum_{n=n_1(j)}^{\infty} 2\lambda_{j,n} \left( \frac{2^{-n}}{(2^{-j-1})^{\left(\frac{1}{1+\beta}\right)\gamma}} \right)^{\theta/(\eta-\eta_c)} \right) < \infty \]
for every \(\theta > (1+\beta)(\eta-\eta_c)\). This yields the desired bound for the Hausdorff dimension for almost every \(\omega \in A_2^c \cap \{\sup_{s \leq t} X_s((0,1)) \leq N\}\). Letting \(N \to \infty\) and \(\varepsilon \to 0\) completes the proof. \(\square\)

The proof of Lemma 38 is omitted since it goes similarly to the proof of Theorem 39(a) and all the modifications come from the necessity of dealing with numerous random points. For details of the proof of Lemma 38, we refer the interested reader to the proof of Lemma 3.2 in [23].

7.2. Lower bound for the Hausdorff dimension. The aim of this section is to describe the main steps in the proof of the following proposition. The full proof of it is given in the Section 4 of [23].

Proposition 37. For every \(\eta \in (\eta_c, \eta_c) \setminus \{1\}\),
\[ \dim(\mathcal{E}_{Z,\eta}) \geq (1+\beta)(\eta - \eta_c), \quad \mathbb{P} - \text{a.s. on } \{X_t((0,1)) > 0\}. \]

Remark 38. Clearly the above proposition together with Proposition 34 finishes the proof of Theorem 5.
The proof of the lower bound is much more involved than the proof of the upper one. Let us give short description of the strategy. First we state two lemmas that give some uniform estimates on "masses" of \(X_s\) of dyadic intervals at times \(s\) close to \(t\). These lemmas imply that \(X_s(dx)\) for times \(s\) close to \(t\) is very close to Lebesgue measure with the density bounded from above and away of zero. This is very helpful for constructing a set \(J_{\eta,1}\) with \(\dim(J_{\eta,1}) \geq (\beta + 1)(\eta - \eta_c)\), on which we show existence of "big" jumps of \(X\) that occur close to time \(t\). These jumps are "encoded" in the jumps of the auxiliary processes \(L_{n,t,r}^{+}\) and they, in fact, "may" destroy the Hölder continuity of \(X_t(\cdot)\) on \(J_{\eta,1}\) for any index greater or equal to \(\eta\) (see Proposition 43 and the proof of Proposition 37).

In the next two lemmas we give some bounds for \(X_s(I_n^{(n)}(k))\), where

\[
I_n^{(n)} := [k2^{-n}, (k + 1)2^{-n}).
\]

In what follows, fix some \(m > 3/2\), \((115)\) and let \(\theta \in (0,1)\) be arbitrarily small. Define

\[
\begin{align*}
O_n := \{\omega : & \text{ there exists } k \in [0, 2^n - 1] \text{ such that } \\
& \sup_{s \in (t - 2^{-2n+4m/3}, t)} X_s(I_n^{(n)}) \geq 2^{-n4m/3} \} 
\end{align*}
\]

and

\[
\begin{align*}
B_n = B_n(\theta) := \{\omega : & \text{ there exists } k \in [0, 2^n - 1] \text{ such that } \\
& I_n^{(n)} \cap \{x : X_t(x) \geq \theta\} \neq \emptyset \\
& \text{and } \inf_{s \in (t - 2^{-2n-2m}, t)} X_s(I_n^{(n)}) \leq 2^{-n2m} \}.
\end{align*}
\]

**Lemma 39.** There exists a constant \(C\) such that

\[
P(O_n) \leq Cn^{-2m/3}, \quad n \geq 1.
\]

Recall the definition of \(A_3^\varepsilon\) given in (113).

**Lemma 40.** There exists a constant \(C = C(m)\) such that, for every \(\theta \in (0,1),
\[
P(B_n(\theta) \cap A_3^\varepsilon) \leq C\theta^{-1}n^{-\alpha m/3}, \quad n \geq \tilde{n}(\theta),
\]

for some \(\tilde{n}(\theta)\) sufficiently large.

The proofs of the above lemmas are technical and hence we omit them. Let us just mention that the proof of Lemma 43 is an almost word-by-word repetition of the proof of Lemma 5.5 in [8], and for the proof of Lemma 40 we refer the reader to the proof of Lemma 6.7 in [23].

**7.2.1. Analysis of the set of jumps which destroy the Hölder continuity.** In this subsection we construct a set \(J_{\eta,1}\) such that its Hausdorff dimension is bounded from below by \((\beta + 1)(\eta - \eta_c)\) and in the vicinity of each \(x \in J_{\eta,1}\) there are jumps of \(X\) which destroy the Hölder continuity at \(x\) for any index greater than \(\eta\).

We first introduce \(J_{\eta,1}\) and prove the lower bound for its dimension. Set

\[
q := \frac{5m}{(\beta + 1)(\eta - \eta_c)}
\]
and define
\[ A_k^{(n)} := \left\{ \Delta X_s(I_{k-2^n}^{(n)} - 2^{-(n+1)n}) \geq 2^{-(n+1)n} \right\} \]
for some \( s \in \left[ t - 2^{-2n-2}n^{-2m}, t - 2^{-(n+1)(n+1)-2m} \right) \).

\[ J_{k,r}^{(n)} := \left[ \frac{k}{2^n} - (n^q2^{-n})^r, \frac{k+1}{2^n} + (n^q2^{-n})^r \right]. \]

Let us introduce the following notation. For a Borel set \( B \) and an event \( E \) define a random set
\[ B1_E(\omega) := \begin{cases} B, & \omega \in E, \\ \emptyset, & \omega \notin E. \end{cases} \]

Now we are ready to define random sets
\[ J_{n,r} := \limsup_{n \to \infty} 2^n \sum_{k=2^n+2}^{2^n-1} J_{k,r}^{(n)} I_{A_k^{(n)}}, \quad r > 0. \]

As we have mentioned already we are interested in getting the lower bound on Hausdorff dimension of \( J_{n,1} \). The standard procedure for this is as follows. First show that a bit "inflated" set \( J_{n,r} \), for certain \( r \in (0,1) \), contains open intervals. This would imply a lower bound on the Hausdorff dimension of \( J_{n,1} \) (see Lemma 41 and Theorem 2 from [16] where a similar strategy was implemented). Thus to get a sharper bound on Hausdorff dimension of \( J_{n,1} \) one should try to take \( r \) as large as possible. In the next lemma we show that, in fact, one can choose \( r = (\beta + 1)(\eta - \eta_c) \).

**Lemma 41.** On the event \( A_3^c \),
\[ \{ x \in (0,1) : X_t(x) \geq \theta \} \subseteq J_{n,(\beta+1)(\eta - \eta_c)} \), \( P \)-a.s.
for every \( \theta \in (0,1) \).

**Proof.** Fix an arbitrary \( \theta \in (0,1) \). We estimate the probability of the event \( E_n \cap A_3^c \), where
\[ E_n := \left\{ \omega : \{ x \in (0,1) : X_t(x) \geq \theta \} \subseteq \bigcup_{k=2^n+2}^{2^n-1} J_{k,(\beta+1)(\eta - \eta_c)} I_{A_k^{(n)}} \right\}. \]

To prove the lemma it is enough to show that the sequence \( P(E_n \cap A_3^c) \) is summable. It follows from Lemma 40 that, for all \( n \geq \tilde{n}(\theta) \),
\[ P(E_n \cap A_3^c) \leq P(E_n \cap B_n \cap A_3^c) + P(E_n \cap B_n^c \cap A_3^c) \leq C\theta^{-1}n^{-2m/3} + P(E_n \cap B_n^c \cap A_3^c). \] (116)

For any \( k = 0, \ldots, 2^n - 1 \), the compensator measure \( \tilde{\mathcal{N}}(dr,dy,ds) \) of the random measure \( \mathcal{N}(dr,dy,ds) \) (the jump measure for \( X \) — see Lemma 40), on \( J_1^{(n)} \times I_k^{(n)} \times J_2^{(n)} \) is given by the formula
\[ 1\{(r,y,s) \in J_1^{(n)} \times I_k^{(n)} \times J_2^{(n)}\} n(dr)X_s(dy)ds. \] (117)

If \( k \in K_{\theta} := \{ l : I_l^{(n)} \cap \{ x \in (0,1) : X_t(x) \geq \theta \} \neq \emptyset \}, \)
then, by the definition of $B_n$, we have
\[ X_n(I^n_k) \geq 2^{-n} n^{-2m}, \quad \text{for } s \in J^{(n)}_2, \text{ on the event } A^c_3 \cap B^c_n. \] (118)

Define the measure $\tilde{\Gamma}(dr, dy, ds)$ on $\mathbb{R} \times (0, 1) \times \mathbb{R}_+$, as follows,
\[ \tilde{\Gamma}(dr, dy, ds) := n(dr) n^{-2m} dy ds. \] (119)

Then, by (117) and (118), on $A^c_3 \cap B^c_n$, as follows,
\[ \tilde{\Gamma}(dr, dy, ds) \leq n(dr) n^{-2m} dy ds. \]

By standard arguments it is easy to construct the Poisson point process $\Gamma(dr, dx, ds)$ on $\mathbb{R} \times (0, 1) \times \mathbb{R}_+$ with intensity measure $\tilde{\Gamma}$ given by (119) on the whole space $\mathbb{R} \times (0, 1) \times \mathbb{R}_+$ such that on $A^c_3 \cap B^c_n$,
\[ \Gamma(dr, I^n_k, J^{(n)}_2) \leq N(dr, I^n_k, J^{(n)}_2) \]
for $r \in J^{(n)}_1$ and $k \in K_\theta$.

Now, define
\[ \xi^{(n)}_k = 1 \{ r(J^{(n)}_1 \times I^n_k \times J^{(n)}_2) \geq 1 \}, \quad k \geq 2n^q + 2. \]

Clearly, on $A^c_3 \cap B^c_n$ and for $k$ such that $k - 2n^q - 2 \in K_\theta$,
\[ \xi^{(n)}_k \leq A^{(n)}_k. \]

Moreover, by construction $\{\xi^{(n)}_k\}^{2n+2n^q+1}_{k=2n^q+2}$ is a collection of independent identically distributed Bernoulli random variables with success probabilities
\[ p^{(n)} := \tilde{\Gamma} \left( J^{(n)}_1 \times I^n_k \times J^{(n)}_2 \right) = C 2^{(q-m)(1+\beta)n-n^q-4m}. \]

From the above coupling with the Poisson point process $\Gamma$, it is easy to see that
\[ \mathbf{P}(E^c_n \cap B^c_n \cap A^c_3) \leq \mathbf{P}(\tilde{E}^c_n), \] (120)

where
\[ \tilde{E}_n := \left\{ (0, 1) \subseteq \bigcup_{k=2n^q+2}^{2^n+2n^q+1} J^{(n)}_k \{ \xi^{(n)}_k = 1 \} \right\}. \]

Let $L(n)$ denote the length of the longest run of zeros in the sequence $\{\xi^{(n)}_k\}^{2n+2n^q+1}_{k=2n^q+2}$. Clearly,
\[ \mathbf{P}(\tilde{E}^c_n) \leq \mathbf{P}(L(n) \geq 2^{-n-(\beta+1)(q-m)n^5m}) \]
and it is also obvious that
\[ \mathbf{P}(L(n) \geq j) \leq 2^n p^{(n)}(1 - p^{(n)})^j, \quad \forall j \geq 1. \]

Use this with the fact that, by (115), $m > 1$, to get that
\[ \mathbf{P}(\tilde{E}^c_n) \leq \exp \left\{ -\frac{1}{2} n^m \right\} \] (121)
for all $n$ sufficiently large. Combining (116), (120) and (121), we conclude that the sequence $\mathbb{P}(E_n^c \cap A_n^c)$ is summable. Applying Borel-Cantelli, we complete the proof of the lemma. \qed 

Define 

$$h_\eta(x) := x^{(\beta+1)(\eta-\eta_0)} \log^2 \frac{1}{x}$$

and 

$$\mathcal{H}_\eta(A) := \liminf_{\epsilon \to 0} \left\{ \sum_{j=1}^{\infty} h_\eta(|I_j|), A \in \bigcup_{j=1}^{\infty} I_j \text{ and } |I_j| \leq \epsilon \right\}.$$ 

Combining Lemma 41 and Theorem 2 from [16], one can easily get 

**Corollary 42.** On the event $A_3^c \cap \{X_t(0,1)) > 0\}$, 

$$\mathcal{H}_\eta(J_{\eta,1}) > 0, \ \mathbb{P} \text{-a.s.}$$

and, consequently, on $A_3^c \cap \{X_t(0,1)) > 0\}$,

$$\dim(J_{\eta,1}) \geq (\beta+1)(\eta-\eta_c), \ \mathbb{P} \text{-a.s.}$$

**Proof.** Fix any $\theta \in (0,1)$. If $\omega \in A_3^c$ is such that $B_\theta := \{x \in (0,1) : X_t(x) \geq \theta\}$ is not empty, then by the local Hölder continuity of $X_t(\cdot)$ there exists an open interval $(x_1(\omega), x_2(\omega)) \subset B_{\theta/2}$. Moreover, in view of Lemma 41

$$(x_1(\omega), x_2(\omega)) \subset J_{\eta,(\beta+1)(\eta-\eta_0)}(\omega), \ \mathbb{P} \text{-a.s.}$$

on the event $A_3^c \cap \{B_\theta \text{ is not empty}\}$. Thus, we may apply Theorem 2 from [16] to the set $(x_1(\omega), x_2(\omega))$, which gives 

$$\mathcal{H}_\eta((x_1(\omega), x_2(\omega)) \cap J_{\eta,1}) > 0, \ \mathbb{P} \text{-a.s.}$$

on the event $A_3^c \cap \{B_\theta \text{ is not empty}\}$. Thus,

$$\dim((x_1(\omega), x_2(\omega)) \cap J_{\eta,1}) \geq (\beta+1)(\eta-\eta_c), \ \mathbb{P} \text{-a.s.}$$

on the event $A_3^c \cap \{B_\theta \text{ is not empty}\}$. Due to the monotonicity of $\mathcal{H}_\eta(\cdot)$ and $\dim(\cdot)$, we conclude that $\mathcal{H}_\eta(J_{\eta,1}) > 0$ and $\dim(J_{\eta,1}) \geq (\beta+1)(\eta-\eta_c), \ \mathbb{P} \text{-a.s.}$ on the event $A_3^c \cap \{B_\theta \text{ is not empty}\}$. Noting that $1_{\{B_\theta \text{ is not empty}\}} \uparrow 1_{\{X_t(0,1)) > 0\}}$ as $\theta \downarrow 0$, $\mathbb{P}$-a.s., we complete the proof. \qed

Now we turn to the second part of the present subsection. By construction of $J_{\eta,1}$ we know that to the left of every point $x \in J_{\eta,1}$ there exist big jumps of $X$ at time $s$ “close” to $t$: such jumps are defined by the events $A_k^{(n)}$. We would like to show that these jumps will result in destroying the Hölder continuity of any index greater than $\eta$ at the point $x$. To this end, we will introduce auxiliary processes $L_{n,y,x}^\pm$ that are indexed by points $(y,x)$ on a grid finer than $\{k2^{-n}, k = 0, 1, \ldots\}$. That is, take some integer $Q > 1$ (note, that eventually $Q$ will be chosen large enough, depending on $\eta$). Define 

$$Z^\eta_s(x_1, x_2) := \int_0^s \int_{\mathbb{R}} M(d(u,y)) p^{\eta}_{t-u}(x_1 - y, x_2 - y), \quad s \in [0,t],$$

where 

$$p^{\eta}_{t}(x,y) := \begin{cases} 
    p_s(x) - p_s(y), & \text{if } \eta \leq 1, \\
    p_s(x) - p_s(y) - (x-y) \frac{\partial p_s(y)}{\partial y}, & \text{if } \eta \in (1,\bar{\eta}_c).
\end{cases}$$
According to (26) and (28), for every \( x, y \in 2^{-Q_n} \mathbb{Z} \), there exist spectrally positive \((1 + \beta)-\)stable processes \( L_{n,y,x}^+ \) such that
\[
Z^\eta_s(y,x) = L_{n,y,x}^+(T_{n,y,x}^+(s)) - L_{n,y,x}^-(T_{n,y,x}^-(s)) =: L_{n,y,x}^+ - L_{n,y,x}^-, \tag{122}
\]
where
\[
T_{n,y,x}^\pm(s) = \int_0^s \int_{\mathbb{R}} X_u(dz) \left( (P_{t-u}(y-z,x-z))^{\pm1+\beta} \right), \quad s \leq t.
\]
In what follows let \([z]\) denote the integer part of \( z \) for \( z \in \mathbb{R} \). The crucial ingredient for the proof of the lower bound is the following proposition.

**Proposition 43.** Fix arbitrary \( \eta \in (\eta_c, \overline{\eta}_c) \setminus \{1\} \) and \( Q > 1 \). For \( \mathbb{P} \)-a.s. \( \omega \) on \( A_{\eta} \), there exists a set \( G_{\eta} \subseteq [0,1] \) with
\[
\dim(G_{\eta}) < (\beta + 1)(\eta - \eta_c) \tag{123}
\]
such that the following holds. For \( \mathbb{P} \)-a.s. \( \omega \) on \( A_{\eta} \), for every \( x \in J_{\eta,1} \setminus G_{\eta} \), there exist a (random) sequences \( n_j = n_j(x), \ j \geq 1 \), and \( (x_{n_j}, y_{n_j}) = (x_{n_j}(x), y_{n_j}(x)), \ j \geq 1 \) with
\[
x_{n_j} = 2^{-Q_n} [2^{Q_n} x], \quad j \geq 1, \quad |y_{n_j} - x_{n_j}| \leq C n_j^{\theta} 2^{-n_j}, \quad j \geq 1,
\]
such that
\[
L_{n_j,y_{n_j},x_{n_j}}^+ \geq n_j^{m_2} 2^{-n_j}, \quad L_{n_j,y_{n_j},x_{n_j}}^- \leq 2^{-(\eta n_j - 1)},
\]
for all \( n_j \) sufficiently large.

Note that in the above proposition we do not give precise definition of \( y_{n_j} \), but it is chosen in a way that it is "close" to the spatial position of a "big" jump that is supposed to destroy the Hölder continuity at \( x \). As for the point \( x_{n_j} \), it is chosen to be "close" to \( x \) itself.

Similarly to Proposition 29, Proposition 43 deals with possible compensation effects. In contrast to the case of fixed points considered in Proposition 29 we cannot show that the compensation does not happen. But we derive an upper bound for the Hausdorff dimension of the set \( G_{\eta} \), on which such a compensation may occur. The dimension of \( G_{\eta} \) turns to be strictly smaller than \((\beta + 1)(\eta - \eta_c)\), see (123).

The proof of Proposition 43 is rather technical and uses the same ideas as the proof of Proposition 29. The major additional difficulty comes from the need to consider random sets. To overcome it we use Borel-Cantelli arguments. We refer the reader to Section 4.3 in [23] where the proofs of results leading to Proposition 43 are given. Now we will explain how Proposition 43 implies the proof of Proposition 37 for the case of \( \eta < 1 \) (the proof for the case of \( \eta > 1 \) goes along the similar lines, see Section 4.4 in [23]).

**Proof of Proposition 37 for \( \eta < 1 \).** Fix arbitrary \( \eta \in (\eta_c, \min(\overline{\eta}_c, 1)) \). Also fix
\[
Q = \left[ \frac{\eta}{\eta_c} + 2 \right],
\]
where as usual \([z]\) denotes the integer part of \(z\). Let \(G_{n}\) be as in Proposition 43. If \(x\) is an arbitrary point in \(J_{n,1} \setminus G_{n}\), then let \(\{n_{j}(x)\}_{j \geq 1}\) and \(\{(x_{n_{j}}(x), y_{n_{j}}(x))\}_{j \geq 1}\) be the sequences constructed in Proposition 43. Then Proposition 43 implies that,

\[
\liminf_{j \to \infty} 2^{(\eta + \delta)n_{j}} \left| Z_{n_{j}}(x, x_{n_{j}}(x)) \right| = \infty, \quad \forall x \in J_{n,1} \setminus G_{n}, \; \mathbb{P} - \text{a.s. on } A_{n}^{3}.
\]

(124)

for any \(\delta > 0\). Recall that, \(X_{i}(\cdot)\) and \(Z_{i}(\cdot)\) are H"{o}lder continuous with any exponent less than \(\eta_{k}\) at every point of \((0, 1)\). Therefore, recalling that \(Q > 4\frac{A}{n}\), we have

\[
\limsup_{j \to \infty} 2^{(\eta + \delta)n_{j}} \left| Z_{n_{j}}^{n}(x, x_{n_{j}}(x)) \right| = \lim_{j \to \infty} C(\omega) 2^{-\frac{1}{2}Q n_{j}} 2^{(\eta + \delta)n_{j}} = 0, \quad \mathbb{P} - \text{a.s. on } A_{n}^{3}.
\]

(125)

Therefore, for any \(x\) in \(J_{n,1} \setminus G_{n}\), we have

\[
|Z_{n_{j}}^{n}(y_{n_{j}}(x), x)| \geq |Z_{n_{j}}^{n}(y_{n_{j}}(x), x_{n_{j}}(x))| - |Z_{n_{j}}^{n}(x_{n_{j}}(x), x)|, \quad j \geq 1.
\]

(126)

Therefore, combining (124) and (125), (126) we conclude that

\[
H_{Z}(x) \leq \eta, \quad \forall x \in J_{n,1} \setminus G_{n}, \; \mathbb{P} - \text{a.s. on } A_{n}^{3}.
\]

(127)

We know, by Lemma 35, that

\[
H_{Z}(x) \geq \eta - 2\gamma - 2\rho \quad \forall x \in (0, 1) \setminus S_{n-2\rho}, \; \mathbb{P} - \text{a.s.},
\]

This and (127) imply that on \(A_{n}^{3}, \mathbb{P} - \text{a.s.}, \)

\[
\eta - 2\gamma - 2\rho \leq H_{Z}(x) \leq \eta \quad \forall x \in (J_{n,1} \setminus S_{n-2\rho}) \setminus G_{n}.
\]

(128)

It follows easily from Lemma 35, Corollary 42, and Lemma 17 that on \(A_{n}^{3}\)

\[
\dim \left( (J_{n,1} \setminus S_{n-2\rho}) \setminus G_{n} \right) \geq (\beta + 1)(\eta - \eta_{k}), \; \mathbb{P} - \text{a.s.}
\]

Thus, by (128),

\[
\dim \{ x : H_{Z}(x) \leq \eta \} \geq (\beta + 1)(\eta - \eta_{k}), \quad \mathbb{P} - \text{a.s.}\]

It is clear that

\[
\{ x : H_{Z}(x) = \eta \} \cup \bigcup_{n=n_{0}}^{\infty} \{ x : H_{Z}(x) \in (\eta - n^{-1}, \eta - (n + 1)^{-1}] \}
\]

\[
= \{ x : \eta - n_{0}^{-1} \leq H_{Z}(x) \leq \eta \}.
\]

Consequently,

\[
H_{\eta}(\{ x : \eta - n_{0}^{-1} \leq H_{Z}(x) \leq \eta \})
\]

\[
= H_{\eta}(\{ x : H_{Z}(x) = \eta \})
\]

\[
+ \sum_{n=n_{0}}^{\infty} H_{\eta}(\{ x : H_{Z}(x) \in (\eta - n^{-1}, \eta - (n + 1)^{-1}] \}).
\]

Since the dimensions of \(S_{n-2\rho}\) and \(G_{n}\) are smaller than \(\eta\), the \(H_{\eta}\)-measure of these sets equals zero. Applying Corollary 42, we then conclude that on \(A_{n}^{3}\)

\[
H_{\eta}(J_{n,1} \setminus S_{n-2\rho}) \setminus G_{n}) > 0, \; \mathbb{P} - \text{a.s.}
\]

And in view of (128), \(H_{\eta}(\{ x : \eta - n_{0}^{-1} \leq H_{Z}(x) \leq \eta \}) > 0.\) Furthermore, it follows from Proposition 44 that dimension of the set \(\{ x : H_{Z}(x) \in (\eta - n^{-1}, \eta - (n + 1)^{-1}] \}\)
is bounded from above by \((\beta + 1)(\eta - (n+1)^{-1} - \eta_c)\). Hence, the definition of \(\mathcal{H}_n\) immediately yields
\[
\mathcal{H}_n\{(x : H_x(x) \in (\eta - n^{-1}, \eta - (n+1)^{-1})\} = 0, \text{ on } A_3^\epsilon, \text{ P-a.s.}
\]
for all \(n \geq n_0\). As a result we have
\[
\mathcal{H}_n\{(x : H_x(x) = \eta)\} > 0 \quad \text{P-a.s. on } A_3^\epsilon.
\]
(129)
Since \(\epsilon > 0\) was arbitrary, this implies that (129) is satisfied on the whole probability space \(P\)-a.s. From this we get that
\[
\dim\{x : H_x(x) = \eta\} \geq (\beta + 1)(\eta - \eta_c), \quad P\text{-a.s.}
\]
\[\square\]

**Appendix A. Estimates for the transition kernel of the one-dimensional Brownian motion**

We start with the following estimates for \(p_t\) which are taken from Rosen [26].

**Lemma 44.** Let \(d = 1\). For each \(\delta \in (0, 1]\) there exists a constant \(C\) such that
\[
|p_t(x) - p_t(y)| \leq C\frac{|x - y|^{\delta}}{t^{d/2}}\left(p_t(x/2) + p_t(y/2)\right),
\]
(130)
\[
\left|\frac{\partial p_t(x)}{\partial x}\right| \leq Ct^{-1/2}p_t(x/2),
\]
(131)
\[
\left|\frac{\partial p_t(x)}{\partial x} - \frac{\partial p_t(y)}{\partial y}\right| \leq C\frac{|x - y|^{\delta}}{t^{1+\delta/2}}\left(p_t(x/2) + p_t(y/2)\right),
\]
(132)
\[
\left|p_t(x) - p_t(y) - (x - y)\frac{\partial p_t(y)}{\partial y}\right| \leq C\frac{|x - y|^{1+\delta}}{t^{1+\delta/2}}\left(p_t(x/2) + p_t(y/2)\right)
\]
(133)
for all \(t > 0\) and \(x, y \in \mathbb{R}\).

The next lemma is a simple corollary of the previous one.

**Lemma 45.** Let \(d = 1\). If \(\theta \in (1, 3)\) and \(\delta \in (0, 1]\) satisfy \(\delta < (3 - \theta)/\theta\), then
\[
\int_0^t ds \int_\mathbb{R} dy \quad p_s(y)|p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\]
\[
\leq C(1 + t)|x_1 - x_2|^{\delta\theta}(p_{t/2} + p_{t/2}), \quad t > 0, \quad x_1, x_2 \in \mathbb{R}.
\]

**Proof.** By Lemma 44 for every \(\delta \in [0, 1]\),
\[
|p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\]
\[
\leq C\frac{|x_1 - x_2|^{\delta\theta}}{(t - s)^{\delta\theta/2}}\left(p_{t-s}(x_1/2 + p_{t-s}(x_2/2))\right)^\theta,
\]
t \(s \geq 0, x_1, x_2, y \in \mathbb{R}\). Noting that \(p_{t-s}(\cdot) \leq C(t - s)^{-1/2}\), we obtain
\[
|p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\]
\[
\leq C\frac{|x_1 - x_2|^{\delta\theta}}{(t - s)^{\delta\theta/2}}\left(p_{t-s}(x_1/2 + p_{t-s}(x_2/2))\right),
\]
(134)
t > s \geq 0, x_1, x_2, y \in \mathbb{R}$. Therefore,
\[
\int_0^t ds \int_\mathbb{R} dyp_s(y)|p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta \leq C|x_1 - x_2|^{\delta \theta}
\]
\[
\times \int_0^t ds \ (t - s)^{-((\delta + \theta - 1)/2)} \int_\mathbb{R} dy \ p_s(y)\left(p_{t-s}(x_1 - y/2) + p_{t-s}(x_2 - y/2)\right).
\]
By scaling of the kernel \( p \),
\[
\int_\mathbb{R} dyp_s(y)p_{t-s}((x - y)/2) = \frac{1}{2}\int_\mathbb{R} dy \ p_{s/4}(y/2)p_{t-s}((x_2 - y)/2)
= \frac{1}{2} p_{s/4+t-s}(x/2) = \frac{1}{2} (s/4 + t - s)^{-1/2} p_1((s/4 + t - s)^{-1/2}x/2)
\leq t^{-1/2} p_1(t^{-1/2}x/2) = p_t(x/2),
\]
since \( t/4 \leq s + t/4 - s \leq t \).

As a result we have the inequality
\[
\int_0^t ds \int_\mathbb{R} dyp_s(y)|p_{t-s}(x_1 - y) - p_{t-s}(x_2 - y)|^\theta
\leq C|x_1 - x_2|^{\delta \theta} (p_t(x_1/2) + p_t(x_2/2)) \int_0^t ds \ s^{-(\delta + \theta - 1)/2}.
\]
Noting that the latter integral is bounded by \( C(1 + t) \), since \( (\delta + \theta - 1) < 2 \), we get the desired inequality.

**Appendix B. Probability inequalities for a spectrally positive stable process**

Let \( L \) be a spectrally positive stable process of index \( \kappa \) with Laplace transform given by (24). Let \( \Delta L_s := L_s - L_{s-} > 0 \) denote the jumps of \( L \).

**Lemma 46.** We have
\[
P\left( \sup_{0 \leq u \leq t} L_u 1\{\sup_{0 \leq v \leq u} \Delta L_v \leq y \} \geq x \right) \leq \left( \frac{Ct}{xy^{\kappa-1}} \right)^{x/y}, \quad t > 0, \ x, y > 0.
\]

**Proof.** Since for \( r > 0 \) fixed, \( \{L_{rt}: t \geq 0\} \) is equal to \( r^{1/\kappa} L \) in law, for the proof we may assume that \( t = 1 \). Let \( \{\xi_i: i \geq 1\} \) denote a family of independent copies of \( L_1 \). Set
\[
W_{ns} := \sum_{1 \leq k \leq ns} \xi_k, \quad L^{(n)}_s := n^{-1/\kappa} W_{ns}, \quad 0 \leq s \leq 1, \ n \geq 1.
\]

Denote by \( D_{[0,1]} \) the Skorokhod space of càdlàg functions \( f: [0,1] \to \mathbb{R} \). For fixed \( y > 0 \), let \( H: D_{[0,1]} \to \mathbb{R} \) be defined by
\[
H(f) := \sup_{0 \leq u \leq 1} f(u) 1\{\sup_{0 \leq v \leq u} \Delta f(v) \leq y \}, \quad f \in D_{[0,1]}.
\]
It is easy to verify that \( H \) is continuous on the set \( D_{[0,1]} \setminus J_y \), where \( J_y := \{f \in D_{[0,1]} : \Delta f(v) = y \text{ for some } v \in [0,1]\} \). Since \( P(L \in J_y) = 0 \), from the invariance principle (see, e.g., Gikhman and Skorokhod [H], Theorem 9.6.2) for \( L^{(n)} \) we conclude that
\[
P(H(L) \geq x) = \lim_{n \to \infty} P(H(L^{(n)}) \geq x), \quad x > 0.
\]
Consequently, the lemma will be proved if we show that
\[
\mathbb{P}\left( \sup_{0 \leq n \leq 1} W_{nn} \{ \max_{1 \leq k \leq n} \xi_k \leq y n^{1/\kappa} \} \geq x n^{1/\kappa} \right)
\leq \left( \frac{C}{xy^{k-1}} \right)^{x/y}, \quad x, y > 0, \quad n \geq 1.
\] (135)

To this end, for fixed \( y', h \geq 0 \), we consider the sequence
\[ \Lambda_0 := 1, \quad \Lambda_n := e^{hW_n} \{ \max_{1 \leq k \leq n} \xi_k \leq y' \}, \quad n \geq 1. \]

It is easy to see that
\[
\mathbb{E}\{ \Lambda_{n+1} | \Lambda_n = e^{hu} \} = e^{hu} \mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \quad \text{for all} \quad u \in \mathbb{R}
\]
and that
\[ \mathbb{E}\{ \Lambda_{n+1} | \Lambda_n = 0 \} = 0. \]

In other words,
\[
\mathbb{E}\{ \Lambda_{n+1} | \Lambda_n \} = \Lambda_n \mathbb{E}\{ e^{hL_1}; L_1 \leq y' \}. \quad (136)
\]

This means that \( \{ \Lambda_n : n \geq 1 \} \) is a supermartingale (submartingale) if \( h \) satisfies \( \mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \leq 1 \) (respectively \( \mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \geq 1 \)). If \( \Lambda_n \) is a submartingale, then by Doob’s inequality,
\[
\mathbb{P}\left( \max_{1 \leq k \leq n} \Lambda_k \geq e^{hx'} \right) \leq e^{-hx'} \mathbb{E}\{ \Lambda_n \}, \quad x' > 0.
\]

But if \( \Lambda_n \) is a supermartingale, then
\[
\mathbb{P}\left( \max_{1 \leq k \leq n} \Lambda_k \geq e^{hx'} \right) \leq e^{-hx'} \mathbb{E}\{ \Lambda_0 \} = e^{-hx'}, \quad x' > 0.
\]

From these inequalities and (136) we get
\[
\mathbb{P}\left( \max_{1 \leq k \leq n} \Lambda_k \geq e^{hx'} \right) \leq e^{-hx'} \max\left\{ 1, \left( \mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \right)^n \right\}. \quad (137)
\]

It was proved by Fuk and Nagaev [10] (see the first formula in the proof of Theorem 4 there) that
\[
\mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \leq 1 + h\mathbb{E}\{ L_1; L_1 \leq y' \} + \frac{e^{hy'} - 1 - hy'}{(y')^2} V(y'), \quad h, y' > 0,
\]
where \( V(y') := \int_{y'}^{\infty} \mathbb{P}(L_1 \in du) u^2 > 0. \) Noting that the assumption \( \mathbb{E}L_1 = 0 \) yields that \( \mathbb{E}\{ L_1; L_1 \leq y' \} \leq 0 \), we obtain
\[
\mathbb{E}\{ e^{hL_1}; L_1 \leq y' \} \leq 1 + \frac{e^{hy'} - 1 - hy'}{(y')^2} V(y'), \quad h, y' > 0. \quad (138)
\]

Now note that
\[
\left\{ \max_{1 \leq k \leq n} W_k 1\{ \max_{1 \leq i \leq k} \xi_i \leq y' \} \geq x' \right\} = \left\{ \max_{1 \leq k \leq n} e^{hW_k} 1\{ \max_{1 \leq i \leq k} \xi_i \leq y' \} \geq e^{hx'} \right\}
\leq \left\{ \max_{1 \leq k \leq n} \Lambda_k \geq e^{hx'} \right\}. \quad (139)
\]

Thus, combining (139), (138), and (137), we get
\[
\mathbb{P}\left( \max_{1 \leq k \leq n} W_k 1\{ \max_{1 \leq i \leq k} \xi_i \leq y' \} \geq x' \right) \leq \mathbb{P}\left( \max_{1 \leq k \leq n} \Lambda_k \geq e^{hx'} \right)
\leq \exp\left\{ -hx' + \frac{e^{hy'} - 1 - hy'}{(y')^2} n V(y') \right\}.
\]
Choosing $h := (y')^{-1} \log \left( 1 + x'y'/n V(y') \right)$, we arrive, after some elementary calculations, at the bound
\[
P \left( \max_{1 \leq k \leq n} W_k \mathbf{1} \left\{ \max_{1 \leq i \leq k} \xi_i \leq y' \right\} \geq x' \right) \leq \left( \frac{e n V(y')}{x'y'} \right)^{x'/y'}, \quad x', y' > 0.
\]
Since $P(L_1 > u) \sim C u^{-\kappa}$ as $u \uparrow \infty$, we have $V(y') \leq C (y')^{2-\kappa}$ for all $y' > 0$. Therefore,
\[
P \left( \max_{1 \leq k \leq n} W_k \mathbf{1} \left\{ \max_{1 \leq i \leq k} \xi_i \leq y' \right\} \geq x' \right) \leq \left( \frac{C n}{x'(y')^{\kappa-1}} \right)^{x'/y'}, \quad x', y' > 0. \tag{140}
\]
Choosing finally $x' = x n^{1/\kappa}$, $y' = y n^{1/\kappa}$, we get \cite{FleischmannMytnikWachtel2010} from (140). Thus, the proof of the lemma is complete. \qed \qed

Lemma 47. There is a constant $c_\kappa$ such that
\[
P \left( \inf_{u \leq t} L_u < -x \right) \leq \exp \left\{ - c_\kappa \frac{x^{\kappa/(\kappa-1)}}{\kappa/(\kappa-1)} \right\}, \quad x, t > 0.
\]

Proof. It is easy to see that for all $h > 0$,
\[
P \left( \inf_{u \leq t} L_u < -x \right) = P \left( \sup_{s \leq t} e^{-h L_u} > e^{hx} \right).
\]
Applying Doob's inequality to the submartingale $t \mapsto e^{-h L_t}$, we obtain
\[
P \left( \inf_{u \leq t} L_u < -x \right) \leq e^{-hx} E e^{-h L_t}.
\]
Taking into account definition (24), we have
\[
P \left( \inf_{u \leq t} L_u < -x \right) \leq \exp \left\{ -hx + th^\kappa \right\}.
\]
Minimizing the function $h \mapsto -hx + th^\kappa$, we get the inequality in the lemma with $c_\kappa = (\kappa - 1)/(\kappa)^{\kappa/(\kappa-1)}$. \qed \qed

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Faculty of Industrial Engineering and Management, Technion Israel Institute of Technology, Haifa 32000, Israel

E-mail address: leonid@ie.technion.ac.il

Institut für Mathematik, Universität Augsburg, 86135 Augsburg, Germany

E-mail address: vitali.vachtel@mathematik.uni-augsburg.de