Analytically solvable potentials for $\gamma$-unstable nuclei.

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Abstract. An analytical solution of the collective Bohr equation with a Coulomb-like and a Kratzer-like $\gamma$-unstable potential in quadrupole deformation space is presented. Eigenvalues and eigenfunctions are given in closed form and transition rates are calculated for the two cases. The corresponding $SO(2,1) \times SO(5)$ algebraic structure is discussed.

PACS numbers: 21.60.Fw, 21.10.Re

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1. Introduction

Large interest has been recently raised by the analytic solution of the $\gamma$-unstable collective Bohr Hamiltonian in the case of a square well potential in the $\beta$ variable [1]. This situation describes the shape phase transition between spherical and $\gamma$-unstable nuclei and it has been associated with the E(5) group. Similarly an approximate separation of variables has been proposed [2] as a solution for the transition between spherical and axially deformed shapes which has been referred to as X(5). Experimental confirmation of the actual occurrence of such situations was found for instance in the examination of the level scheme of $^{134}$Ba for E(5) and $^{152}$Sm for X(5) [3, 4].

The occurrence of dynamic symmetries is associated with systems where the Hamiltonian is solely written in terms of Casimir operators of a chain of subalgebras. The wider algebra of this chain is the one that settles the nature of the problem and its general form bears all the information on the system in a compact way. The non-unique way in which the algebra is split in chains of subalgebras exhibits some peculiar feature of the system leading to exact analytical results, that may help in elucidating experimental findings and data trends. Concerning ourselves to nuclei, an example of occurrence of dynamical symmetries is provided by the Interacting Boson Model. With particular choices of the general SU(6) Hamiltonian one obtains limiting situations that are analytically solvable (O(6), U(5) and SU(3) cases). Moving around in the nuclear chart or along a chain of isotopes, nuclei can undergo phase transitions from one dynamical symmetry to another. The interest of E(5) and X(5) lies in the fact that they are related to new analytically solvable situations corresponding precisely to the critical phase transition points.

In the specific case of the transition from spherical to $\gamma$-unstable nuclei, the E(5) description assumes, for the transition potential in the $\beta$ variable, an infinite square-well potential [1], a case that has been generalized by Caprio with the introduction of a finite square well [5]. Other choices are possible, like the Davidson potential studied by Elliott [6] and later by Rowe [7], that also generates an analytic vibration-rotation spectrum. The work of Rowe is very much akin to the one discussed here, displaying the same $SO(2,1) \times SO(5)$ algebraic structure.

The purpose of this investigation is to show that, within the condition of $\gamma$-instability, there are other classes of analytically solvable potentials.

The Bohr Hamiltonian [5] will be our starting point:

$$H = -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{1}{\beta^2 \sin 3\gamma} \frac{\partial}{\partial \gamma} \sin 3\gamma \frac{\partial}{\partial \gamma} \right] f(\beta) + V(\beta, \gamma).$$

Following the standard procedure, when the potential depends only upon $\beta$, i.e. $V(\beta, \gamma) = U(\beta)$, one can separate variables as in Wileyts and Jean [9], obtaining a system of two differential equations, one containing the $\gamma$ variable and the three Euler angles, the other containing only the $\beta$ variable. The spectrum is determined by the solution of the latter, namely

$$\left\{ -\frac{\hbar^2}{2B} \left[ \frac{1}{\beta^4} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \frac{\Lambda^2}{\beta^2} \right] + U(\beta) \right\} f(\beta) = Ef(\beta)$$

(2)
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where $\hat{\Lambda}^2$ is the Casimir operator of SO(5) ($\tau$ being the associated quantum number) and we rewrite here the differential equation in the $\beta$ variable in the second order standard form as:

$$\chi''(\beta) + \left\{ \epsilon - u(\beta) - \frac{(\tau + 3/2)^2}{\beta^2} + \frac{1}{4\beta^2} \right\} \chi(\beta) = 0$$  (3)

where $\chi(\beta) = \beta^2 f(\beta)$, while $\epsilon = \frac{2B}{\hbar}E$ and $u = \frac{2B}{\hbar}U$ are the reduced energies and potentials. In this note we will point out that this equation displays analytic solutions with the choice of the Coulomb and the Kratzer potentials in the quadrupole deformation parameter $\beta$. The former has a very simple form diverging in zero, while the latter has been widely used in the early stages of quantum theory to describe interactions within ions in configuration space and has a minimum for a finite value of $\beta$. In both cases we have in fact the possibility to regain, with some simple mathematical steps, the well-known Whittaker’s standard form for eq. (3), and hence to obtain analytic solutions.

2. Coulomb-like potential

Inserting the potential $u_C(\beta) = -\frac{A}{\beta}$, $A > 0$ (4) in equation (3) and with the substitutions $\varepsilon = -\epsilon$, $x = 2\sqrt{\varepsilon}\beta$, $k = A/(2\sqrt{\varepsilon})$ and $\mu = \tau + 3/2$ the previous equation takes the Whittaker’s standard form [10]:

$$\chi''(x) + \left\{ -\frac{1}{4} + \frac{k}{x} + \frac{(1/4 - \mu^2)}{x^2} \right\} \chi(x) = 0$$  (5)

The solution for negative energies, that is regular in the origin, may be found (as in [10, 11]) to be the Whittaker’s function $M_{k,\mu}(x)$:

$$\chi_{k,\mu}(x) = N_{\tau,\xi} x^{(2\mu+1)} e^{-x/2} 1_F(\mu + 1/2 - k, 2\mu + 1; x)$$  (6)

The normalization $N_{\tau,\xi}$ of these states can be obtained by quadrature § from:

$$\int_0^\infty \chi(\beta)^2 d\beta = \int_0^\infty \beta^4 f(\beta)^2 d\beta = 1$$  (7)

where the volume element in the $\beta$ variable is $\beta^4 d\beta$ [8]. The hypergeometric function for $x \to \infty$ is in general proportional to $e^x$ and hence diverges. However, it is not divergent when it becomes an associated Laguerre polynomial, that is to say when the first parameter $\mu + 1/2 - k = \tau + 2 - A/(2\sqrt{\varepsilon})$ is a negative integer, $-\xi$. This leads to a condition that fixes the spectrum as

$$\varepsilon_{\tau,\xi} = \frac{A^2/4}{(\tau + \xi + 2)^2}$$  (8)

Now $\tau + \xi$ works as a single quantum number $n$ for the energies, but the shape of the wavefunctions depend on $\tau$ and $\xi$ separately. This spectrum is depicted in figure 1 where the energy of the first two states have been fixed respectively to 0 and 1 and this is sufficient to settle the energy scale $\varepsilon'$. The $(4^+, 2^+)$ doublet with $n = 2$ has an

§ The general integral of two hypergeometric from 0 to $\infty$ is not known, but one can find the normalization constants in an analytical way since the hypergeometrics reduce to Laguerre polynomials.
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Figure 1. Spectrum of the Coulomb-like potential. The energy scale ($\epsilon'$) is chosen by fixing the energy of the first two states respectively to 0 and 1. The transition rate for $2_{1+}^+ \rightarrow 0_{0,0}^+$ has been fixed to 100. Some selected quadrupole transitions are shown in the figure for simplicity.

For the sake of completeness we give the wavefunctions with the reduction of the hypergeometric to associated Laguerre polynomials:

$$\chi(x)_{\tau,\xi} = N_{\tau,\xi} x^{2\mu+1} e^{-x/2} \frac{\xi!}{(2\mu+1)!} L^{(2\mu)}_{\xi}(x)$$ (9)

where the denominator is a Pochhammer symbol.

3. Kratzer-like potential

We move now to the study of the Kratzer potential (see figure 2)

$$u_K(\beta) = -2D \left( \frac{\beta_0}{\beta} - \frac{1}{2} \frac{\beta_0^2}{\beta^2} \right)$$ (10)

where $D$ represents the depth of the minimum, located in $\beta_0$. We may also write it as $u_K(\beta) = u_C(\beta) + B/\beta^2$ for later purpose. Inserting the cited potential in (8), with the substitutions $\varepsilon = -\epsilon$, $x = 2\sqrt{\beta_0} \beta$, $k = D\beta_0/(\sqrt{\varepsilon})$ and $\mu^2 = (\tau + 3/2)^2 + D\beta_0^2$, the equation takes again the Whittaker's standard form. The regular solutions are again
Whittaker’s functions with the new substitutions and the same arguments apply for the properties of convergence. Now $\mu + 1/2 - k = \sqrt{(\tau + 3/2)^2 + D\beta_0^2 + 1/2 - D\beta_0/\sqrt{\varepsilon}}$ must be a negative integer, $-\xi$. The spectrum worked out from the last requirement reads:

$$
\varepsilon_{\tau,\xi} = \frac{D^2\beta_0^2}{(\lambda + \xi)^2} = \frac{D^2\beta_0^2}{\left(\sqrt{(\tau + 3/2)^2 + D\beta_0^2 + 1/2} + \xi\right)^2} = \frac{A^2/4}{(\sqrt{(\tau + 3/2)^2 + B + 1/2} + \xi)^2} \tag{11}
$$

with $\xi = 0, 1, 2, \ldots$. The proper set of quantum numbers that characterizes the eigenvalues is $\tau, \xi, L, M$. Notice that each $\tau, \xi$ state may be degenerate with respect to the angular momentum, according to the Wilets and Jean rules [8]. The $\tau$ quantum number is contained in $\lambda = \sqrt{(\tau + 3/2)^2 + D\beta_0^2 + 1/2}$ and the $\xi$ quantum number has the same meaning as in the paper of Iachello (shifted by one unity) being connected with the zeros of the Whittaker’s function that are determined by the zeros of the hypergeometric functions and give rise to the different bands. If one displays the spectrum given in formula (11) imposing that the ground state ($\tau = 0, \xi = 0$) is at zero energy and that the ($\tau = 1, \xi = 0$) state has $\varepsilon = 1$, it is evident that one has to play with the position of the minimum $\beta_0$ and the depth $D$ of the potential. In fig. 4 we study the dependence upon the depth of the potential well. It is seen from the lowest few states of the first two bands that there are clearly two limiting cases: when the depth goes to zero the spectrum becomes equivalent to the already discussed $1/\beta$ case, that we wish to call the Coulomb-like limit, while when $D$ tends to infinity all the $\xi > 0$ bands escape to an infinite energy and the $\xi = 0$ band has a spectrum that follows a simple $\beta$-rigid, $\gamma$-soft rule: $\varepsilon_\tau = \tau(\tau + 3)/4$. In fig. 4 we study instead the evolution of the spectra with $\beta_0$ for a fixed value of $D$. Again the same two limits are seen when the deformation parameter is very small or very large. There is a qualitative equivalence between the case with small $\beta_0$, the case with small $D$ and the Coulomb-like case. Moreover when $\beta_0$ or $D$ tend to infinity the situation is equivalent. Apart from the limiting cases the most interesting situations are realized for spectra.
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that have a reasonable $\beta_0$ (that is usually smaller than 0.5) and a $\mathcal{D}$ freely adjustable that can widen the relative distances in the $\xi = 0$ band as well as move the lowest state of the second $\xi = 1$ band.

4. Transitions

Electromagnetic transition rates (see [1, 5, 12]) are defined as reduced matrix elements of the transition operator $T(E\lambda)$

\[
B(E\lambda; \xi_i, \tau_i, L_i \rightarrow \xi_f, \tau_f, L_f) = \frac{|\langle \xi_i, \tau_i, L_i | T(E\lambda) | \xi_f, \tau_f, L_f \rangle|^2}{(2L_i + 1)}
\]

and the quadrupole transition operator, to the first order, reads:

\[
T(E2, \mu) \propto \beta \left[ D^{(2)}_{\mu,0} \cos \gamma + (D^{(2)}_{\mu,+2} + D^{(2)}_{\mu,-2}) \frac{\sin \gamma}{\sqrt{2}} \right]
\]

The matrix elements have been calculated for a few selected transitions and are displayed directly on the figs. [6, 8] and [4]. The $\beta$ part of the integration factorizes out and the $\{\gamma, \theta_i\}$ part has been calculated in the standard way. Taking into account all the transitions between a given $\tau$ state and one of his neighbouring (for instance $\tau - 1$) states, it suffices to calculate explicitly one transition, the others being obtained
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Figure 4. Evolution of spectra with fixed $D = 10$ and increasing $\beta_0$. The first two bands ($\xi = 0, 1$) are displayed with their lowest states ($\tau = 0, 1, 2, ..$). The various substates are not displayed for the sake of simplicity. The selected transition rates are the same than in the preceding figure.

by fixed geometrical factors coming from the $\gamma$-angular part \[13\]. It is interesting to note that the transition from the first $2^+$ state of the second band to the ground state of the system is very small if compared to the reference transition, evidencing almost no overlap between the two wavefunctions in the $\beta$ variable.

For simplicity in the last two figures we have given only the following transition rates: $6^+_2 \rightarrow 2^+_2$, $4^+_2 \rightarrow 2^+_1$, and $2^+_1 \rightarrow 0^+_1$, relatively to the transition $2^+_1 \rightarrow 0^+_0$ that has been given the reference value of 100. It is interesting to note that in both cases (fixing one parameter and changing the other) the two evidenced limits have also common values for B(E2)’s and the two Coulomb-like limits for the Kratzer potential show the same values of the exact Coulomb-like case in fig. 11.

5. Algebraic structure

We discuss now the group theoretical interpretation of the hamiltonians that we have solved directly in the previous sections. We will consider the quantization of the collective model, restricted to quadrupole deformations only ($\lambda = 2$), by defining pairs of canonically conjugate operators on the Hilbert space $\hat{\alpha}_\mu = \alpha_\mu$ and $\hat{\pi}^\mu = -i \frac{\partial}{\partial \alpha^\mu}$ (having dropped any $\hbar$ and using a covariant/contravariant notation). These operators obey the Heisenberg-Weyl commutation relations $[\hat{\alpha}_\mu, \hat{\pi}^\nu] = i \delta_{\mu,\nu} \hat{1}$. If $\hat{\alpha}$ is defined as the vector whose five components are the operators defined above, the scalar product

$$\langle \alpha | \beta \rangle = \frac{1}{\sqrt{\text{Vol}(\mathcal{H})}} \int d\alpha \langle \alpha | \beta \rangle$$

where $\text{Vol}(\mathcal{H})$ is the volume of the Hilbert space.
may be written as
\[ \hat{\alpha} \cdot \hat{\alpha} = \sum_{\mu} \alpha^\mu_\alpha \alpha^\mu_{-\mu} = \sum_{\mu} |\alpha^\mu|^2 \] (14)
where the last equivalence is a consequence of the property:\n\[ \alpha^\mu = \alpha^\mu_\alpha = (-1)^{\mu} \alpha^\mu_{-\mu}.\]
Notice that\n\[ \beta^2 = \hat{\alpha} \cdot \hat{\alpha}. \] We need also to consider the parameters\n\[ a^\mu_\mu \] and\n\[ p^\mu_\mu \] which play the role of the\n\[ \alpha^\mu_\mu \] and\n\[ \pi^\mu_\mu \] in the intrinsic frame of reference, and have the useful\nproperty that\n\[ a^\mu = a^{\mu}_{-\mu}. \] Since the transformations between these two sets of variables\nare unitary, the commutation relations are preserved as well as the scalar products\n(for example\n\[ \sum_{\mu} \pi^\mu_{\pi \mu} = \sum_{\mu} p^\mu_{p \mu}. \]
Taking the reduced quantities, equation (2) may be recast in the form
\[ \left[ \pi^2 + u(\beta) - \epsilon \right] f(\beta) = 0 \] (15)
where (see [3])
\[ \pi^2 = \hat{\pi} \cdot \hat{\pi} = \hat{p} \cdot \hat{p} = -\frac{1}{\beta^2} \frac{\partial}{\partial \beta} \beta^4 \frac{\partial}{\partial \beta} + \hat{\Lambda}^2. \] (16)
For both the Coulomb-like and Kratzer-like potentials the hamiltonian\n\[ H \] is\n\[ SO(5) \] invariant. In addition it displays a spectrum generating algebra since one may\nconstruct three operators [7] that are infinitesimal generators for the corresponding\ngroup:
\[ \hat{Z}_1 = 4\beta \left( \hat{p}^2 + \frac{B}{\beta^2} \right) \hat{Z}_2 = \beta \hat{Z}_3 = 2(\hat{\alpha} \cdot \hat{p} - i) \] (17)
which have the commutation relations of the four non-compact isomorphic Lie algebras\nsu(1,1) ~ so(2,1) ~ sl(2,R) ~ sp(2,R) (using Wybourne’s notation for symplectic group\ndimensions [15]):
\[ [\hat{Z}_1, \hat{Z}_2] = -4i \hat{Z}_3 \quad [\hat{Z}_3, \hat{Z}_2] = -2i \hat{Z}_1 \quad [\hat{Z}_3, \hat{Z}_1] = 2i \hat{Z}_2. \] (18)
With the potential\n\[ u(\beta) = u_K(\beta) \] (that contains also the Coulomb-like case, when\n\[ B = 0 \] ) the operator\n\[ \beta H \] is in fact expressible as a linear combination of the elements of the algebra of\nsu(1,1), namely in the form
\[ \beta H = \hat{Z}_1/4 - A. \] (19)
Now one can define new operators\n\[ \hat{X}_i \] with\n\[ i = 1, 2, 3 \] by means of a linear transformation:
\[ \hat{X}_1 = \frac{1}{4}(\hat{Z}_1 - \hat{Z}_2) \quad \hat{X}_2 = \frac{1}{2} \hat{Z}_3 \quad \hat{X}_3 = \frac{1}{4}(\hat{Z}_1 + \hat{Z}_2) \] (20)
which satisfy the following commutation relations:
\[ [\hat{X}_1, \hat{X}_2] = -i \hat{X}_3 \quad [\hat{X}_2, \hat{X}_3] = i \hat{X}_1 \quad [\hat{X}_3, \hat{X}_1] = i \hat{X}_2. \] (21)
The eigenvalue equation for the Bohr hamiltonian is now, having multiplied by\n\[ \beta \] on the left,
\[ \beta(H - \epsilon) \Psi = 0. \] (22)
Using equation (16) and the definitions of the\n\[ \hat{X}_i \] and\n\[ \hat{Z}_i \] operators, we rewrite the last equation as
\[ \left[ (1 + 4\epsilon)\hat{X}_1 - 2A + (1 - 4\epsilon)\hat{X}_3 \right] \Psi = 0 \] (23)
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and following the procedure in [16] we can perform a (1,3) hyperbolic rotation of an angle $\theta$ to diagonalize the eigenvalue equation and choosing $\tgh(\theta) = -(1+4\epsilon)/(1-4\epsilon)$ (valid for $\epsilon < 0$) we obtain

$$\hat{X}_3 \tilde{\Psi} = \frac{A}{\sqrt{-4\epsilon}} \tilde{\Psi},$$

(24)

where $\tilde{\Psi}$ is the rotated wavefunction. The Casimir operator of the so(2,1) algebra is evaluated to be:

$$\hat{C}_2 = \hat{\Lambda}^2 + \hat{X}_- - \hat{X}_+ + B + 2$$

(25)

with eigenvalue $\tau(\tau + 3) + B + 2$. The two last equations must be compared with the two following eigenvalue equations (for unitary representations $D^+$ [17]):

$$\hat{X}_3 | \phi, \xi \rangle = (\xi - \phi) | \phi, \xi \rangle$$

$$\hat{C}_2 | \phi, \xi \rangle = \phi(\phi + 1) | \phi, \xi \rangle.$$ 

(26)

This comparison yields a spectrum of the form:

$$\epsilon_{\tau,\xi} = -\frac{A^2/4}{(\sqrt{(\tau + 3/2)^2 + B + 1/2 + \xi})^2}$$

(27)

that coincides with the one found from the direct solution of the differential equation with a Kratzer-like potential. It also contains as a special case ($B = 0$) the Coulomb-like case.

The algebra associated with the SO(5) group is the so-called degeneracy algebra (according to the definitions in [15]) and the group SO(2,1) is associated with the spectrum generating algebra. The relevant chain of subalgebras that gives the labels of the set of orthonormal states $\{ | \xi \tau \alpha \xi L M \rangle \}$ is explicitly given as [7, 14]:

$$SU(1,1) \times SO(5) \supset U(1) \times SO(3) \supset SO(2)$$

(28)

where $\lambda$ is an SU(1,1) lowest weight and $\alpha$ indexes the SO(3) multiplicity. These basis diagonalize the Hamiltonian (19) given above. We can thus state that the problem studied so far displays a SO(2,1)$\times$SO(5) dynamical algebra.

6. Final remarks

In this paper we have solved the Bohr Hamiltonian for two specific $\gamma$–unstable potentials that yield analytical solutions. We have given the corresponding spectra and wavefunctions in closed form and the most important transition probabilities. Rather interesting looks the case of the Kratzer-like potential that may be given in terms of two unrelated parameters $A$ and $B$, as $-A/\beta + B/\beta^2$. Changing the value of $B$ from zero to a finite value, one can describe situations ranging from spherical to quadrupole deformed shape. The critical point is actually the potential with $B \rightarrow 0$, that is the Coulomb-like limit discussed above. The same dynamical symmetry discussed briefly in the former section, is thus effective for a class of different potentials, that, depending on some parameter, may describe very different situations. We would like to remark that these potentials parallel the case of the family of potentials of the form:

$$u_H(\beta) = A\beta^2$$

(29)

and

$$u_D(\beta) = u_H(\beta) + \frac{B}{\beta^2}$$

(30)
where $H$ stays for harmonic and $D$ for Davidson, that also lead to solvable Bohr’s hamiltonians and are furthermore both characterized by a $\text{SO}(2,1) \times \text{SO}(5)$ dynamical group \[7\]. Other interesting possibilities, such as linear combinations of powers (to be solved with Frobenius method), arise in the same spirit of this paper.

The $\beta$–part of the spectrum of the potentials that have been discussed here was combined with the condition of $\gamma$–instability. We will show in a forthcoming paper that approximate solutions can also be obtained for the same functional dependence on $\beta$, but with a $\gamma$–dependence favouring axial symmetry. This case will be therefore the homologous of the situation associated with the occurrence of $X(5)$ in the approximate solution of reference \[2\].

Acknowledgments

We wish to acknowledge valuable correspondence and discussions with J.M.Arias, F.Iachello and especially with D.J.Rowe.

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