On fixed points of locally and pointwise contractive set-valued maps with an application to the existence of Nash equilibrium in games

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Abstract. We establish the existence of fixed points for set-valued maps defined on metric spaces and satisfying a pointwise or a local version of Banach’s contraction property. As an application, we demonstrate the existence of Nash equilibrium in a general class of strategic games played on metric spaces of strategies.

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1. Introduction

The study of fixed points in metric spaces constitutes an important part of analysis and has many uses within and outside of mathematics. The modern theory of games originates in von Neumann and Morgenstern [30]. Strategic games have several applications of game theory to biology, control theory, engineering, economics, and related fields. The existence of Nash equilibrium for finite games, viewed as fixed points of certain set-valued maps, was first demonstrated by Nash [24]. Related proofs often use some version of the fixed-point theorem in Kakutani [16]. For games with infinite-dimensional strategy spaces, existence results often employ a related generalization by Glicksberg. In addition to requiring that the set-valued map is convex-valued and has a closed graph, these results rely on the assumption that the domain of the set-valued map is convex.

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We extend recent results on the existence of fixed points for maps satisfying pointwise and local contraction properties to set-valued maps defined on metric spaces. As an illustration, we use these results to demonstrate the existence of pure-strategy Nash equilibrium for a certain class of strategic games, which we term contractive games. Our treatment allows for the domain of the set-valued map to be rather general (possibly infinitely dimensional) metric spaces. Our most general results require that the set-valued map is compact-valued, that it satisfies a pointwise version of Banach’s contraction mapping property, and that its domain is a compact, rectifiable path-connected metric space.

Local and pointwise contraction properties are, of course, weaker than assuming that the set-valued map is a contraction globally. As a result, completeness of the underlying metric space will not be sufficient for the existence of a fixed point, thus, that of a Nash equilibrium. While we use existence of Nash equilibrium in strategic games as an illustration of our main results, these techniques can be used in other applications, including the existence of value functions in set-valued dynamic programming.

2. Review of main concepts

A classic result by Banach [3] established the existence of a unique fixed point in complete metric spaces for maps satisfying a contraction property. Since the beginning of the twentieth century, there has been a well-developed theory of set-valued maps, that is maps $F : X \to 2^X$, see, for example, Kuratowski [17]. Selected recent references, to name a few, include Gorniewicz [12], Aubin and Frankowska [2], Arutyunov and Obukhovskii [1], and Malkoun and Olver [18]. Fixed points for set-valued maps are defined as points $x^*$, such that $x^* \in F(x^*)$. A celebrated result due to Kakutani [16] established the existence of fixed points for set-valued maps satisfying certain continuity and convexity properties. Kakutani’s result relies on the domain being convex (a finite-dimensional simplex). The notion of a contraction can be applied to set-valued maps, for example using a Hausdorff distance. Using this notion, generalizations of Banach’s existence result for set-valued maps can be obtained as in Nadler [23]. Unlike in the single-valued case, uniqueness is not guaranteed.

There have been attempts to generalize Banach’s result for single-valued maps satisfying weaker, pointwise or local notions of the standard contraction property; see Ciesielski and Jasinski [8] for a recent comprehensive review. These results typically require additional conditions to the completeness of the underlying metric space, such as compactness and some notion of connectedness. These restrictions are topological in nature and do not require convexity. Here, we investigate under what conditions pointwise or local contraction properties are sufficient for the existence of fixed points for set-valued maps. The common thread in the argument used involves two steps. First, using a Hausdorff distance provides us with a complete metric space. We can
then define set-valued maps and their pointwise or local contraction properties in that space. The second step involves imposing additional topological properties on the space and defining a new metric under which the underlying metric space remains complete, and the local or pointwise set-valued contraction becomes a global set-valued contraction. Related properties have been previously established for maps, and this paper demonstrates that, subject to appropriate modifications, they can be extended to set-valued maps.

Throughout the paper, we let \((X, d)\), \(X \neq \emptyset\), be a metric space. Additional restrictions on \(X\) will be imposed as needed in what follows. Banach [3] demonstrated that if \((X, d)\) is complete, and \(f : X \to X\) satisfies the \textit{contraction property} that there is \(\beta \in [0, 1)\), such that for all \(x, y \in X\),

\[
d(f(x), f(y)) \leq \beta d(x, y),
\]

then there exists a unique \(x^* \in X\), such that \(x^* = f(x^*)\). The map \(f\) is sometimes referred to as a \textit{contraction of modulus} \(\beta\). A number of generalizations of Banach’s contraction property have been investigated. In what follows, we will concern ourselves with the implications of extending the local and pointwise versions studied in Ciesielski and Jasinński [8] to the study of set-valued maps. We will make use of the following notions.

**Definition 1.** \(X\) is \textit{convex} if, for all \(x, y \in X\), there exists \(z \in X\), such that

\[
d(x, y) = d(x, z) + d(z, y).
\]

In the case where \(X\) is complete and convex, whenever \(x, y \in X\), then \(X\) also contains a subset whose boundary points are \(x\) and \(y\), and which is isometric to an interval of length \(d(x, y)\). The following notion of convexity is commonly employed in metric spaces.

**Definition 2.** \(X\) is \textit{d-convex} if, for all \(x, y \in X\), there exists a homeomorphism (path) \(p : [0, 1] \to X\) with \(p(0) = x\), \(p(1) = y\) and such that for any \(0 \leq t_1 \leq t_2 \leq t_3 \leq 1\), we have

\[
d(p(t_1), p(t_3)) = d(p(t_1), p(t_2)) + d(p(t_2), p(t_3)).
\]

Thus, if \(X\) is \(d\)-convex, then for any \(x, y \in X\), there is an interval \([x, y] \subseteq X\) that is the image of a homeomorphism \(p : [0, 1] \to X\).

**Definition 3.** Fix \(r > 0\). \(X\) is \textit{r-chainable} if, given any \(x, y \in X\), there are finitely many points \(z_0, \ldots, z_n \in X\), such that \(z_0 = x\), \(z_n = y\), and \(d(z_{i-1}, z_i) < r\), for all \(i = 1, \ldots, n\).

Finally, we will employ the following.

**Definition 4.** Let \(\Pi\) be the set of all possible partitions \(0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\) of \([0, 1]\), \(n < \infty\). \(X\) is \textit{(rectifiably) path-connected} if, for all \(x, y \in X\), there exists a continuous function (path) \(p : [0, 1] \to X\), such that \(p(0) = x\), \(p(1) = y\), and whose length

\[
l(p) := \sup_{\pi \in \Pi} \sum_{i=1}^n d(p(t_{i-1}), p(t_i)) < \infty.
\]

In other words, \(X\) is rectifiably path-connected if every two points of \(X\) can be connected by a path \(p\) in \(X\) of finite length, \(l(p)\). Later, we will use
\( l(p^{[a,b]}) \) to indicate the restriction of a path \( p : [0,1] \to X \) on \( [a,b] \), where \( 0 \leq a < b \leq 1 \).

While Banach’s Theorem makes no use of connectedness properties, the situation is more delicate once only local or pointwise contraction properties are assumed. Indeed, absent the rectifiable path connectedness assumption, Ciesielski and Jasinskic [6] used a Cantor set-like structure to construct a compact \( X \subset \mathbb{R} \) containing no isolated points, and a smooth bijection \( f : X \to X \), such that \( f \) has zero slope everywhere, but it admits no fixed (or periodic) points.

In what follows, we will restrict attention to set-valued maps. Let \( (X,d) \) be a metric space with \( X \not= \emptyset \). A set-valued map is a map \( F : X \to 2^X \setminus \emptyset \). A fixed point of \( F \) is an \( x^* \in X \), such that \( x^* \in F(x^*) \). We will let \( \text{Fix}(F) \) stand for the set of fixed points of \( F \) in \( X \). We begin with some basic concepts related to set-valued maps.

Let \( C(X) := \{ Y \subseteq X : Y \not= \emptyset, Y \text{ closed and bounded} \} \), and let \( K(X) := \{ Y \subseteq X : Y \not= \emptyset, Y \text{ compact} \} \). Let \( d \) be a metric on \( X \). For any \( X_1, X_2 \in C(X) \), the Hausdorff distance is defined as follows. Let \( d(y,X_1) := \inf_{x \in X_1} d(x,y) \) (1)

Then

\[ H(X_1, X_2) := \max \left\{ \sup_{x \in X_1} d(x, X_2), \sup_{y \in X_2} d(y, X_1) \right\}. \] (2)

Equivalently, letting \( B_r(X_1) := \bigcup_{x \in X_1} \{ z \in X : d(x,z) \leq r \} \),

we have

\[ H(X_1, X_2) := \inf \{ r : X_1 \subset B_r(X_2) \text{ and } X_2 \subset B_r(X_1) \} \]. (4)

Using the Hausdorff distance, Nadler [23] generalized the notions of Lipschitz and contraction maps to set-valued maps.

**Definition 5.** Let \( (X,d_X) \) and \( (Y,d_Y) \) be metric spaces. A set-valued map \( F : X \to Y \) is a set-valued Lipschitz map if, for all \( x, y \in X \), \( H(F(x), F(y)) \leq \alpha d(x,y) \), where \( \alpha \in [0, \infty) \) is the Lipschitz constant. \( F \) is a set-valued contraction of modulus \( \beta \) if it is a set-valued Lipschitz map with Lipschitz constant \( \beta \in [0, 1) \).

Nadler [23] demonstrated the existence of fixed points for set-valued maps defined on complete metric spaces and satisfying the above contraction property. Several generalizations have been studied since; see, for example, Reich [25,26]. We now turn to the main focus of this paper, which concerns extending results on the existence of fixed points of pointwise and local contractions to set-valued maps. The following defines iterates of set-valued maps. Let \( X, Y, Z \) be metric spaces.
Definition 6. Let \( F_1 : X \to Y \) and \( F_2 : Y \to Z \). The composition of \( F_1 \) and \( F_2 \) is defined by: 
\((F_2 \circ F_1)(x) := \bigcup \{ F_2(y) : y \in F_1(x) \}\), for all \( x \in X \). Proceeding recursively, we obtain: 
\( F^{(n)}(x) := \underbrace{F \circ F \ldots \circ F}_{n\text{-times}} \). We remark that if \( F_1 \) is a contraction of modulus \( \beta_1 \) and \( F_2 \) is a contraction of modulus \( \beta_2 \), then \((F_2 \circ F_1)\) is a contraction map of modulus \( \beta_1 \cdot \beta_2 \).

The following definitions extend the related notions in Ciesielski and Jasinski [8] to set-valued maps.

Definition 7. A set-valued map \( F : X \to X \) is a set-valued pointwise contraction if, for all \( x \in X \), there exists \( \beta_x \in [0,1) \) and an open neighborhood \( N(x) \subseteq X \), such that for all \( y \in N(x) \), 
\[ H(F(x), F(y)) \leq \beta_x d(x, y). \]

The set-valued map \( F \) is a set-valued uniform pointwise contraction if it is a set-valued pointwise contraction and \( \beta \) does not depend on \( x \).

A stronger property is defined in the following.

Definition 8. A set-valued map \( F : X \to X \) is a \((\beta_x, r_x)\)-local contraction if, for all \( x \in X \), there exists \( r_x > 0 \) and \( \beta_x \in [0,1) \), such that \( F \) is a set-valued contraction with modulus \( \beta_x \) in \( B_{r_x}(x) \). The set-valued map \( F \) is a \((\beta, r)\)-uniform local contraction if \( \beta \) and \( r \) do not depend on \( x \).

A pointwise/local contraction is also a uniform pointwise/uniform local contraction if the underlying space is compact. A (uniform) local contraction is a (uniform) pointwise contraction, but the reverse implication does not necessarily hold.

3. Fixed points of locally contractive set-valued maps

As before, we let \((X, d)\) be a metric space with \( X \neq \emptyset \). We have the following.

Proposition 9. Let \( X \) be convex and complete and let \( F : X \to K(X) \) be a set-valued \((\beta, r)\)-uniform local contraction on \( X \). Then, \( \text{Fix}(F) \neq \emptyset \).

Proof. It is sufficient to show that \( F \) is a set-valued contraction of modulus \( \beta \) on \( X \). The conclusion then follows from Nadler [23]. To this end, let \( x, y \in X \). By convexity of \( X \), given any \( r > 0 \), there are points \( z_0 = x, z_1, \ldots, z_{n-1}, z_n = y \), such that 
\[ d(x, y) = \sum d(z_{i-1}, z_i) \text{ and } d(z_{i-1}, z_i) < r. \]
Then
\[ H(F(x), F(y)) \leq \sum_{i=1}^{n} H(F(z_{i-1}), F(z_i)) \]
\[ \leq \beta \sum_{i=1}^{n} d(z_{i-1}, z_i) = \beta d(x, y). \]

Thus, \( F \) is a set-valued contraction. \( \square \)

Next, we assume that \( X \) is compact and \( r \)-chainable and present an existence proof for the case where \( F \) is a (not necessarily uniform) locally contractive set-valued map. We will use the following.
Lemma 10. Let $X$ be $r$-chainable. For each $x, y \in X$, define the function \( d_c : X \times X \rightarrow \mathbb{R}_+ \) by

\[
d_c(x, y) := \inf \left\{ \sum_{i=1}^{n} d(z_{i-1}, z_i) : z_0 = x, z_1, \ldots, z_n = y \text{ is an } r\text{-chain from } x \text{ to } y \right\}. \tag{6}
\]

Then, (a) \( d_c : X \times X \rightarrow \mathbb{R}_+ \) is a metric on \( X \) and (b) \( d_c \) is topologically equivalent to \( d \). Thus, the space \( (X, d_c) \) is complete whenever \( (X, d) \) is complete.

Proof. (a) Clearly, \( d_c(x, y) \geq 0 \), \( d_c(x, y) = 0 \) if and only if \( x = y \), and \( d_c(x, y) = d_c(y, x) \) by construction. For the triangle inequality, fix \( x, y, z \in X \) and \( \epsilon > 0 \). Let \( w_0 = x, w_1, \ldots, w_{n-1}, w_n = y \) be an \( r \)-chain from \( x \) to \( y \) and \( v_0 = y, v_1, \ldots, v_{n-1}, v_n = z \) be an \( r \)-chain from \( y \) to \( z \), such that

\[
d_c(x, y) \geq \sum_{i=1}^{n} d(w_{i-1}, w_i) - \epsilon \tag{7}
\]

\[
d_c(y, z) \geq \sum_{i=1}^{n} d(v_{i-1}, v_i) - \epsilon. \tag{8}
\]

Then, \( w_0, w_1, \ldots, w_{n-1}, w_n, v_0, v_1, \ldots, v_{n-1}, v_n \) is an \( r \)-chain from \( x \) to \( z \) and

\[
d_c(x, y) + d_c(y, z) \geq \sum_{i=1}^{n} d(w_{i-1}, w_i) - \epsilon + \sum_{i=1}^{n} d(v_{i-1}, v_i) - \epsilon \geq d_c(x, z) - 2\epsilon. \tag{9}
\]

Since \( \epsilon > 0 \) was arbitrary, the triangle inequality follows.

(b) Clearly, \( d(x, y) \leq d_c(x, y) \), for all \( x, y \in X \). In addition, when \( d(x, y) < r \), we have

\[
d(x, y) \leq d_c(x, y)
\]

\[
= \inf \left\{ \sum_{i=1}^{n} d(z_{i-1}, z_i) : z_0 = x, \ldots, z_n = y \text{ is an } r\text{-chain from } x \text{ to } y \right\}
\]

\[
= \inf \{ d(z_0, z_1) : z_0 = x, z_1 = y \}
\]

\[
= d(x, y). \tag{10}
\]

Hence, the two metrics are topologically equivalent and \( (X, d_c) \) is a complete metric space whenever \( (X, d) \) is complete. \hfill \Box

Let \( X \) be compact and \( F : X \rightarrow K(X) \) be a set-valued \((\beta_x, r_x)\)-local contraction on \( X \). The set \( \{ B_{r_x}(x) : x \in X \} \) forms an open covering of \( X \). Since \( X \) is compact, it has a finite subcover: \( \mathcal{B} = \{ B_{r_x}(x) : x \in X \} \). Let \( \beta = \max \{ \beta_x : B_{r_x}^c(x) \in \mathcal{B} \} < 1 \) and \( 0 < r = \min \{ r_x : x \in X \} \). Then, \( F \) is a \((\beta, r)\)-uniform local contraction. Given such an \( F \), we have the following.

Proposition 11. Let \( X \) be compact and let \( F : X \rightarrow K(X) \) be a set-valued \((\beta_x, r_x)\)-local contraction on \( X \). Then, (a) if \( X \) is \( r \)-chainable, then \( \text{Fix}(F) \neq \emptyset \); (b) if \( X \) is the finite union of connected components that are pairwise disjoint and \( F(x) \) is connected, for all \( x \), then \( \text{Fix}(F^{(n)}) \neq \emptyset \).
Proof. (a) By Lemma 10, it suffices to show that the local contraction property implies that $F : X \to K(X)$ is a contraction with respect to metric $d_c$ and the corresponding Hausdorff distance, $H_c$. Let $x, y \in X$ and $z_0 = x, z_1, \ldots, z_{n-1}, z_n = y$ be an $r$-chain from $x$ to $y$. Since $d(z_{i-1}, z_i) < r$, for all $i$, we have that $H(F(z_{i-1}), F(z_i)) \leq \beta d(z_{i-1}, z_i) < r$, where $r$ and $\beta$ are as discussed above. Thus

$$H_c(F(x), F(y)) \leq \sum_{i=1}^{n} H_c(F(z_{i-1}), F(z_i))$$

$$= \sum_{i=1}^{n} H(F(z_{i-1}), F(z_i))$$

$$\leq \beta \sum_{i=1}^{n} d(z_{i-1}, z_i)$$

$$= \beta d_c(x, y). \quad (11)$$

Since $z_0 = x, z_1, \ldots, z_{n-1}, z_n = y$ was an arbitrary $r$-chain, we have that $H_c(F(x), F(y)) \leq \beta d_c(x, y)$, i.e., $F$ is a set-valued contraction with respect to $d_c$ and $H_c$.

(b) Now, suppose that $X = X_1 \cup \cdots \cup X_n$, where $X_i$ is connected, for all $i$, and $X_i \cap X_j = \emptyset$, $i \neq j$. Since $F$ is a contraction and $F(x)$ is connected, we have that $F^{(n)}(X_j)$ is connected, for any $j$. Since $X$ consists of finitely many disconnected components, for all $j$, there exist $k$, $l$, and $m$, such that $F^{(k)}(X_j) \cap X_m \neq \emptyset$ and $F^{(k+l)}(X_j) \cap X_m \neq \emptyset$, where $X_m$ is the same connected component of $X$. Thus, $F^{(l)}(X_m) \subset X_m$. Since $X_m$ is connected, it is $r$-chainable. Applying part (a) of the Proposition to $F^{(l)}$ on $X_m$, we conclude that there exists an $x^* \in X_m \subset X$, such that $x^* \in F^{(l)}(x^*)$. $\square$

4. Fixed points of pointwise contractive set-valued maps

In what follows we consider two cases. First, we will assume that $(X, d)$ is $d$-convex. Then, we will impose the condition that $X$ is rectifiably path-connected. For the first case, we have the following.

Proposition 12. Let $X$ be complete and $d$-convex and let $F : X \to K(X)$ be a uniform pointwise set-valued contraction on $X$. Then, $\text{Fix}(F) \neq \emptyset$.

Proof. Since $F$ is a uniform pointwise contraction, for all $x \in X$, there exists $\beta \in [0, 1)$ and an open neighborhood $N_c(x) \subseteq X$, such that if $y \in N_c(x)$, then $H(F(x), F(y)) \leq \beta d(x, y)$. It is sufficient to show that for all $x \in X$, there exists $\beta \in [0, 1)$, such that $F$ is a set-valued contraction with modulus $\beta$ on $X$.

Let $x, y \in X$. Since $X$ is $d$-convex, there exists a continuous $p : [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$. Since $F$ is a uniform pointwise contraction on $X$, for each $t \in [0, 1]$, there is $\epsilon_t > 0$, such that if $d(p(t), x) < \epsilon_t$, then $H(F \circ p(t), F(x)) \leq \beta d(p(t), x)$. Since $p$ is continuous, for all $t \in (0, 1)$, there exists $\delta_t > 0$ with $B_{\delta_t}(t) \subseteq (0, 1)$ and such that $t' \in B_{\delta_t}(t)$ implies $d(p(t), p(t')) < \epsilon_t$. Similarly, for the endpoints $t = 0, 1$, we can choose $\delta_0, \delta_1 > 0$, such that
B_{\delta_0}(0) \subseteq [0, 1), B_{\delta_1}(1) \subseteq (0, 1], and t' \in B_{\delta_i}(i) implies d(p(i), p(t')) < \epsilon_i, i = 0, 1. The collection \{B_{\delta_i}(t) \subset (0, 1) : t \in [0, 1]\} forms an open cover of [0, 1]. Since [0, 1] is connected, there is a sequence of sets \{B_{\delta_i}(t_i)\} such that [0, 1] \subset \bigcup_{i=1}^{n} B_{\delta_i}(t_i), 0 \in B_{\delta_0}(t_0), 1 \in B_{\delta_1}(t_1), and B_{\delta_i}(t_i) \cap B_{\delta_j}(t_j) \neq \emptyset, if |i - j| \leq 1.

Since B_{\delta_i}(t_i) is a symmetric interval centered at t_i, we can choose c_i \in B_{\delta_{i-1}}(t_{i-1}) \cap B_{\delta_i}(t_i), so that t_{i-1} < c_i < t_i, i = 1, \ldots, n. Choosing \delta_{t_i} > 0, such that d(p(t_i), p(t'_i)) < \epsilon_{t_i} if t'_i \in B_{\delta_i}(t_i), we have that d(p(t_{i-1}), p(c_i)) < \epsilon_{t_{i-1}} and d(p(c_i), p(t_i)) < \epsilon_{t_i}. Since F is a uniform pointwise contraction, we have that for all i

\[ H(F \circ p(t_{i-1}), F \circ p(t_i)) \leq H(F \circ p(t_{i-1}), F \circ p(c_i)) + H(F \circ p(c_i), F \circ p(t_i)) \leq \beta d(p(t_{i-1}), p(c_i)) + \beta d(p(c_i), p(t_i)) = \beta d(p(t_{i-1}), p(t_i)). \]

Thus

\[ H(F(x), F(y)) = H(F \circ p(t_0), F \circ p(t_n)) \leq \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)) \leq \beta \sum_{i=1}^{n} d(p(t_{i-1}), p(t_i)) \leq \beta d(p(0), p(1)) = \beta d(x, y). \]

Hence, F is a set-valued contraction on X.

Next, we assume that X is rectifiably path-connected. Recall that, given 0 < a < b < 1, a rectifiable path is a continuous map from a closed real interval [a, b] into X, whose length is given by

\[ l(p^{[a,b]}) = \sup_{\pi \in \Pi} \left\{ \sum_{i=1}^{n} d(p(t_{i-1}), p(t_i)) : \pi = \{t_0 = a, \ldots, t_n = b\} \right\} < \infty. \]

If p is a rectifiable path, then for all c \in [a, b], we have that \(l(p^{[a,b]}) = l(p^{[a,c]}) + l(p^{[c,b]}).\) First, using the Hausdorff distance, we extend the notion of length to set-valued paths to demonstrate the following.

**Lemma 13.** Let F : X \rightarrow K(X) be a set-valued uniform pointwise contraction with modulus \(\beta \in [0, 1]\) and let p : [0, 1] \rightarrow X be a path with \(l(p) < \infty.\) Then, \(l(F \circ p) \leq \beta l(p) < \infty, \) where

\[ l(F \circ p) := \sup_{\pi \in \Pi} \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)). \]

**Proof.** Let F : X \rightarrow K(X) be a set-valued uniform pointwise contraction and consider a rectifiable path p : [0, 1] \rightarrow X. We first let 0 < a < b < 1 and show that \(H(F \circ p(a), F \circ p(b)) \leq \beta l(p^{[a,b]})\). Rectifiability and the contraction
property imply a partition \( \pi = \{ t_i : t_0 = a, \ldots, t_{i-1}, t_i, \ldots, t_n = b \} \) and \( s_i \in (t_{i-1}, t_i), i = 1, \ldots, n \), such that
\[
H(F \circ p(t_{i-1}), F \circ p(t_i)) \leq H(F \circ p(t_{i-1}), F \circ p(s_i)) + H(F \circ p(s_i), F \circ p(t_i)) \leq \beta [d(p(t_{i-1}), p(s_i)) + d(p(s_i), p(t_i))].
\] (15)
For all \( 0 < s_1 < s_2 < 1, d(s_1, s_2) \leq l(p^{[s_1, s_2]}) \). Thus, (15) implies that for all \( i = 1, \ldots, n \)
\[
H(F \circ p(t_{i-1}), F \circ p(t_i)) \leq \beta [l(p^{[t_{i-1}, s_i]}) + l(p^{[s_i, t_i]})] = \beta l(p^{[t_{i-1}, t_i]}). \] (16)
Hence
\[
H(F \circ p(a), F \circ p(b)) = H(F \circ p(t_0), F \circ p(t_n)) \\
\leq \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)) \\
\leq \beta \sum_{i=1}^{n} l(p^{[t_{i-1}, t_i]}) = \beta l(p). \] (17)
Thus, \( H(F \circ p(a), F \circ p(b)) \leq \beta l(p^{[a, b]}) \), for all \( 0 < a < b < 1 \). Next, consider \( \pi = \{ t_i : t_0 = 0, \ldots, t_{i-1}, t_i, \ldots, t_n = 1 \} \). Expression (16) gives
\[
\sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)) \leq \beta \sum_{i=1}^{n} l(p^{[t_{i-1}, t_i]}) = \beta l(p). \] (18)
Hence, \( \beta l(p) \) is an upper bound of the set
\[
\left\{ \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)) : \pi = \{ t_0 = 0, \ldots, t_n = 1 \} \right\}. \] (19)
By the least upper bound property of \( \mathbb{R} \), the sup of this set, \( l(F \circ p) \), exists and satisfies \( l(F \circ p) \leq \beta l(p) \). \( \square \)

We now have the following.

**Proposition 14.** Let \( X \) be complete and rectifiably path-connected and let \( F : X \to K(X) \) be a set-valued uniform pointwise contraction with modulus \( \beta \in [0, 1) \). Then, \( \text{Fix}(F) \neq \emptyset \).

**Proof.** Let \( x, y \in X \) and, as before and let \( p : [0, 1] \to X \) be such that \( p(0) = x \) and \( p(1) = y \). Define
\[
d_r(x, y) := \inf_{p} \{ l(p) : p \text{ is a rectifiable path from } x \text{ to } y \}
\]
\[
= \inf_{p} \sup_{\pi \in \Pi} \sum_{i=1}^{n} d(p(t_{i-1}), p(t_i)) < \infty. \] (20)
The function \( d_r : X \times X \to \mathbb{R}^+ \) is a metric on \( X \). Let \( H_r \) be the Hausdorff distance associated with \( d_r \). It remains to show that: (a) \( F : X \to K(X) \) is a contraction with respect to the metrics \( d_r \) and \( H_r \), and (b) \( (X, d_r) \) is complete, if \( (X, d) \) is complete. As before, the existence of a fixed point of \( F \) then follows from Nadler [23].
For (a), let \(x, y \in X\), and \(\epsilon > 0\) be given. Since \(X\) is rectifiably path-connected, there exists a rectifiable path \(p\) from \(x\) to \(y\), such that \(l(p) \leq d_r(x, y) + \epsilon\). Then, (20) and Lemma 13 imply

\[
H_r(F(x), F(y)) \leq \sup_{\pi \in \Pi} \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_i)) = l(F \circ p) \leq \beta l(p) \leq \beta(d_r(x, y) + \epsilon). \tag{21}
\]

Since \(\epsilon\) was arbitrary, \(H_r(F(x), F(y)) \leq \beta d_r(x, y)\), for all \(x, y \in X\). Thus, \(F\) is a contraction on \(X\).

For (b), let \(\{x_n\}\) be a Cauchy sequence in \((X, d_r)\). Since \(d(x, y) \leq d_r(x, y)\), for all \(x, y \in X\), \(\{x_n\}\) is also a Cauchy sequence in \((X, d)\). Since \((X, d)\) is complete, there exists \(\bar{x} \in X\), such that \(\lim_{n \to \infty} d(x_n, \bar{x}) = 0\). It is sufficient to show that \(\lim_{n \to \infty} d_r(x_n, \bar{x}) = 0\). Since \(\{x_n\}\) is Cauchy in \((X, d_r)\), there exists a sequence \(\{\epsilon_i\}\), where \(\epsilon_i > 0\), for all \(i\), and \(\sum_{i=1}^{\infty} \epsilon_i < \infty\), and an increasing sequence \(\{N_i\}\), such that if \(m, n \geq N_i\) then \(d_r(x_n, x_m) < \epsilon_i\). Moving to a subsequence if necessary, \(d_r(x_{n_k}, x_{n_k+1}) < \epsilon_{n_k}\), for \(n_k > N_{n_k}\). Thus, for \(n_k > N_{n_k}\), we have

\[
d_r(x_{n_k}, \bar{x}) \leq \sum_{i=k}^{\infty} d_r(x_{n_i}, x_{n_i+1}) \leq \sum_{i=k}^{\infty} \epsilon_i. \tag{22}
\]

Hence, \(\lim_{n_k \to \infty} d_r(x_{n_k}, \bar{x}) = 0\) and, since \(\{x_{n_k}\}\) is a subsequence of a Cauchy sequence, \(\lim_{n \to \infty} d_r(x_n, \bar{x}) = 0\), as desired. The result follows, since \(F\) is a contraction defined on the complete metric space, \((X, d_r)\). \(\square\)

We next turn our attention to the case where \(F\) is a set-valued pointwise contraction. For \(x, y \in X\), we will again let \(d_r : X \times X \to \mathbb{R}_+\) be defined by

\[
d_r(x, y) := \inf\{l(p) : p \text{ is a rectifiable path from } x \text{ to } y\}. \tag{23}
\]

As discussed earlier, \(d_r\) is a metric on \(X\) and \((X, d_r)\) is complete, since \((X, d)\) is complete. Note that since \(d_r\) is not necessarily topologically equivalent to \(d\), the space \((X, d_r)\) might not be compact. In what follows, we let \(K(X) := \{Y \subseteq X : Y \neq \emptyset, Y \text{ compact}\}\) with respect to the metric \(d_r\), and we again let \(H_r\) denote the Hausdorff distance associated with \(d_r\). We wish to establish the following.

**Proposition 15.** Let \(X\) be compact and rectifiably path-connected and let \(F : X \to K(X)\) be a set-valued pointwise contraction. Then, \(\text{Fix}(F) \neq \emptyset\).

The proof follows the steps in Ciesielski and Jasinski [7]. Let \(X\) be as stated above. For any \(Y \subseteq X\), \(F : Y \to K(Y)\), and \(x \in Y\), define \(s(F(x)) := \limsup_{y \to x} \frac{H(F(x), F(y))}{d(x, y)}\), if \(x\) is a limit point of \(Y\), and \(s(F(x)) = 0\), otherwise. Clearly, \(F : X \to K(X)\) is a set-valued pointwise contraction if and only if \(s(F) < 1\), for all \(x \in X\). We have the following analog of Lemma 13.

**Lemma 16.** Let \(Y\) be the range of a rectifiable path \(p\) in \(X\) and let \(F : X \to K(X)\) be such that \(s(F(x)) < \beta \in [0,1]\), for all \(x \in Y\). Then, \(l(F \circ p) \leq \beta l(p)\).
\(\beta l(p) < \infty\), where

\[
l(F \circ p) := \sup_{p} \sum_{i=1}^{n} H(F \circ p(t_{i-1}), F \circ p(t_{i})).
\]

**Proof.** As before, we let \(l(p^{[a,b]})\) denote the restriction of a path \(p : [0,1] \to X\) on \([a,b]\), where \(0 \leq a < b \leq 1\). We first show that for any \(\epsilon > 0\), we have that 

\[
H(F \circ p(a), F \circ p(b)) \leq (\beta + \epsilon)l(p^{[a,b]}).
\]

Since \(s(F(p(v))) \leq \beta < 1\), for every \(v \in [a,b]\), there exists \(\delta_{v} > 0\) and \(B_{\delta_{v}}(v)\), such that for all \(w \in B_{\delta_{v}}(v) \cap [a,b]\), we have

\[
H(F \circ p(v), F \circ p(w)) \leq (\beta + \epsilon)d(p(v), p(w)). \tag{24}
\]

Since \([a,b]\) is compact, it has a minimal finite subcover \(\mathcal{B} = \{B(v_{i}) : v_{i} \in [a,b]\}\). Let \((v_{1}, v_{3}, \ldots, v_{2n-1})\) list the \(v_{i} \in [a,b]\) in increasing order. Since \(\mathcal{B}\) is minimal, for all \(1 \leq i \leq n\), there exists \(v_{2i} \in B(v_{2i-1}) \cap B(v_{2i+1}) \cap (v_{2i-1}, v_{2i+1})\), with \(v_{0} = a \in B(v_{1})\) and \(v_{2n} = b \in B(v_{2n-1})\). We thus obtain a partition \(a = v_{0} < v_{1} < v_{2} < \cdots < v_{2n-1} \leq v_{2n} = b\) with \(v_{2i}, v_{2i+2} \in B(v_{2i+1})\), for all \(0 \leq i \leq n\). Inequality (24) then implies

\[
H(F \circ p(a), F \circ p(b)) \leq \sum_{i=0}^{2n} H(F \circ p(v_{i}), F \circ p(v_{i+1}))
\]

\[
\leq \sum_{i=0}^{2n} (\beta + \epsilon)d(p(v_{i}), p(v_{i+1})) \leq (\beta + \epsilon)l(p^{[a,b]}). \tag{25}
\]

Next, let \(\pi = \{t_{i} : t_{0} = 0, \ldots, t_{i-1}, t_{i}, \ldots, t_{n} = 1\}\) and \(\epsilon > 0\), such that

\[
l(F \circ p) \leq \sum_{i=1}^{n} H(F \circ p(t_{i}), F \circ p(t_{i+1})) + \epsilon. \tag{26}
\]

Inequality (25) implies

\[
l(F \circ p) \leq \sum_{i=1}^{n} H(F \circ p(t_{i}), F \circ p(t_{i+1})) + \epsilon
\]

\[
\leq \sum_{i=1}^{n} (\beta + \epsilon)l(p^{[t_{i-1}, t_{i}]}) + \epsilon = (\beta + \epsilon)l(p) + \epsilon. \tag{27}
\]

Since \(\epsilon > 0\) was arbitrary, the conclusion that \(l(F \circ p) \leq \beta l(p)\) follows. \(\square\)

We will also make use of the following.

**Lemma 17.** Let \(X\) be compact and let \(x, y \in X\). Suppose there is a rectifiable path \(p : [0,1] \to X\). Then, (a) \(p\) admits a reparametrization, i.e., a path \(\overline{p} : [0,1] \to X\) with the same length and range, such that, for all \(t \in [0,1]\), \(l(\overline{p}^{[0,t]}) = tl(p)\); (b) there is a geodesic, i.e., a rectifiable path of minimum length, \(L < \infty\), from \(x\) to \(y\).

**Proof.** (a) Consider an arbitrary rectifiable path \(p : [0,1] \to X\). Rescale \(t\) by letting \(\overline{t} := \frac{l(p^{[0,t]})}{l(p)}\) and define the mapping

\[
\overline{p} := \left\{ \left( \frac{l(p^{[0,t]})}{l(p)}, p(t) \right) : t \in [0,1] \right\}. \tag{28}
\]
We then have
\[ l(\overline{p}^{[0, t]}) = l(p^{[0, t]}) = tl(p) = tl(\overline{p}). \] (29)

Thus, by rescaling the length of the segments of \( p \), if needed, we can always create a path, \( \overline{p} \), with the same range and total length, such that \( l(\overline{p}^{[0, t]}) = tl(\overline{p}) \), \( t \in [0, 1] \).

(b) Let \( x, y \in X \) and let \( L = \inf \{ l(p) : p \text{ is a rectifiable path from } x \text{ to } y \} \). Choose a sequence of rectifiable paths \( \{p_n\} \) from \( x \) to \( y \), such that for all \( n \), \( l(p_n) = tl(p_n^{[0, t]}) \), for \( t \in [0, 1] \). Since \( X \) is compact, applying Fact 1 to \( \{p_n\} \), there is a subsequence \( \{p_{n_k}\} \) converging uniformly to a rectifiable path \( p : [0, 1] \to X \) with \( l(p) = L \). \( \square \)

Next, we have the following.

**Definition 18.** A set-valued map \( F : X \to K(X) \) is **shrinking** if for all \( x, y \in X \), \( H(F(x), F(y)) < d(x, y) \).

The following lemma establishes a connection between set-valued pointwise contractions and set-valued maps satisfying the above shrinking property.

**Lemma 19.** Let \( X \) be compact and let \( F : (X, d) \to (K(X), H) \) be a set-valued pointwise contraction. Then, \( F : (X, d_r) \to (K(X), H_r) \) is shrinking.

**Proof.** Fix \( x, y \in X \). We need to show that \( H_r(F(x), F(y)) < d_r(x, y) \). By Lemma 17(b), any two points in a compact metric space that can be joined by a rectifiable curve can be joined by a length minimizer. Thus, there is a path \( p : [0, 1] \to X \), such that \( p(0) = x, p(1) = y \), and \( l(p) = d_r(x, y) = \inf \{ l(p) : p \text{ is a rectifiable path from } x \text{ to } y \} \). Since \( H_r(F(x), F(y)) \leq l(F \circ p) \), it is sufficient to show that \( l(F \circ p) < l(p) \).

Let \( Y = p([0, 1]) \subseteq X \). Given \( n > 0 \) and \( \delta > 0 \), define the set
\[
M(n) := \{ x \in Y : \frac{H(F(x), F(x'))}{d(x, x')} \leq \frac{n - 1}{n}, \text{ for all } x' \in B_{1/n}(x) \subseteq Y \}. \] (30)
The set \( M(n) \) is closed in \( X \). Since \( F \) is a pointwise set-valued contraction, \( Y = \bigcup_n M(n) \). Thus, \( Y \) is an at most countable union of closed sets. By the weak form of the Baire Category Theorem (see Royden and Fitzpatrick 2017), there exists \( n \), such that \( int(M(n)) \neq \emptyset \) in \( Y \). Since \( p \) is continuous, there is an interval \( [a, b] \subseteq p^{-1}(int(M(n))) \subseteq [0, 1] \). For any \( x \in Y = p([a, b]) \), we have \( s(F(x)) \leq \frac{n - 1}{n} \). Hence, by Lemma 16, \( l(F \circ p^{[a, b]}) \leq \frac{n - 1}{n} l(p^{[a, b]}) \).

Moreover, \( s(F(x)) < 1 \), for all \( x \in X \), so Lemma 16 also implies that \( l(F \circ p^{[v_1, v_2]}) \leq l(p^{[v_1, v_2]}) \), for any \( 0 \leq v_1 \leq v_2 \leq 1 \). Thus
\[
l(p) = l(p^{[0, a]}) + l(p^{[a, b]}) + l(p^{[b, 1]}) > l(p^{[0, a]}) + \frac{n - 1}{n} l(p^{[a, b]}) + l(p^{[b, 1]}) \geq l(F \circ p^{[0, a]}) + l(F \circ p^{[a, b]}) + l(F \circ p^{[b, 1]}) = l(F \circ p). \] (31)
We conclude that
\[ H_r(F(x), F(y)) \leq l(F \circ p) < l(p) = d_r(x, y). \]
Therefore, \( F \) is shrinking.

To complete the proof, we will make use of the following [22]. Let \( X \) be a compact metric space, and for all \( n \), let \( p_n : [0, 1] \to X \) be a rectifiable path, such that, for all \( t \in [0, 1] \), \( l([p_n^0, t]) = tl(p_n) \). If \( L := \liminf l(p_n) < \infty \), then there is a subsequence \( \{p_{n_k}\} \) converging uniformly to a rectifiable path \( p : [0, 1] \to X \) with \( l(p) \leq L \).

**Proof.** Returning to the proof of the Proposition, let
\[ L = \inf\{H_r(x, F(x)) : x \in X\}. \tag{32} \]
It remains to show that there exists an \( x^* \in X \), such that \( H_r(x^*, F(x^*)) = L = 0 \), i.e., \( x^* \in F(x^*) \). By Lemma 19, \( H_r(F(x), F(y)) < d_r(x, y) \), for all \( x, y \in X \). Fix a sequence \( \{x_n\} \), such that, for all \( n \), \( x_{n+1} \in F(x_n) \) is chosen, such that \( d_r(x_n, x_{n+1}) \leq H_r(x_n, F(x_n)) \). Then
\[ \liminf_{n \to \infty} d_r(x_n, x_{n+1}) \leq \liminf_{n \to \infty} H_r(x_n, F(x_n)) = L. \]
By rectifiability and Lemma 17(b), for every \( n \), there exists a path \( p_n : [0, 1] \to X \), \( x_n \mapsto x_{n+1} \in F(x_n) \), of (minimizing) length \( d_r(x_n, x_{n+1}) = \inf\{l(p_n) : p_n \) is a rectifiable path from \( x_n \) to \( x_{n+1}\} \). Thus
\[ \liminf_{n \to \infty} l(p_n) = \liminf_{n \to \infty} d_r(x_n, x_{n+1}) \leq L. \]
Using Lemma 17(a) to reparametrize, if necessary, we have that \( l(p_n) = tl(p_n^0, t) \), for all \( n \) and \( t \in [0, 1] \). Myers [22] implies that \( \{p_n\} \) admits a subsequence \( \{p_{n_k}\} \) that converges uniformly to a rectifiable path \( p : [0, 1] \to X \) with \( l(p) \leq L \). Let \( x^* = \lim_{k \to \infty} p_n(k)(0) = \lim_{k \to \infty} x_{n_k} \). Then, \( p \) is a path from \( p(0) = x^* \) to \( p(1) = \lim_{k \to \infty} p_n(k)(1) = \lim_{k \to \infty} x_{n_{k+1}} \in \lim_{k \to \infty} F(x_{n_k}) = F(x^*) \), where the last implication follows from the fact that, since it is compact-valued and Lipschitz, \( F \) has a closed graph. Thus, \( H_r(x^*, F(x^*)) \leq l(p) \leq L \). Finally, note that we must have \( L = 0 \), since, otherwise, \( H_r(F(x^*), F(F(x^*))) < H_r(x^*, F(x^*)) \), contradicting that \( L = \inf\{H_r(x, F(x)) : x \in X\} \). Hence, \( H_r(x^*, F(x^*)) = 0 \), and \( x^* \in F(x^*) \) as desired. We conclude that \( Fix(F) \neq \emptyset \).

5. An application: pure-strategy Nash equilibria in metric spaces

A game in strategic form, \( \Gamma \), is defined by \( \Gamma = \langle I, X_i, u_i \rangle \), where \( I = \{1, \ldots, n\} \) is the set of players, \( X_i \) is player \( i \)'s (pure) strategy set, and \( u_i : X_1 \times \cdots \times X_n \to \mathbb{R} \) is the payoff of player \( i \), \( i = 1, \ldots, n \). Each player’s payoff depends on his strategy, as well as on that of the other players. We assume that the players’ strategy sets are metric spaces, \( (X_i, d_i) \), and use \( x = (x_1, \ldots, x_n) := (x_i, x_{-i}) \) to denote the strategy profile, where \( x_i \in X_i \), and \( x_{-i} \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n \), for all \( i \). Thus, the payoff of each player \( i \) can be written as \( (x_i, x_{-i}) \mapsto u_i(x_i, x_{-i}) \). Let \( X = X_1 \times \cdots \times X_n \),
and let $d$ be the product metric associated with the metrics $d_i \ , \ i = 1, \ldots , n$, and similarly for $d_r$. For all $i$, the best response of player $i$, $BR_i : X \rightarrow X_i$, is defined by the set-valued map

$$BR_i(x) := \{ x_i \in X_i : u_i(x_i, x_{-i}) \geq u_i(x_i', x_{-i}) , \text{ for all } x_i' \in X_i \}. \ (33)$$

The best response for the game $\Gamma$, $BR : X \rightarrow X$, is defined by the set-valued map $BR(x) := BR_1(x) \times \cdots \times BR_n(x)$. We have the following. A (pure strategy) Nash equilibrium for $\Gamma$ is a strategy profile $x^* \in X$, such that $x^* \in BR(x^*)$.

The finite Cartesian product of compact sets is compact, and similarly for rectifiably path-connected sets. The following follows directly from the existence results established in the previous sections.

**Corollary 20.** Suppose that either one of the following conditions hold: (a) $(X, d)$ is compact and $r$-chainable, and $BR(x)$ is a local set-valued contraction on $X$; (b) $(X, d)$ is complete and rectifiably path-connected, and $BR(x)$ is a uniform pointwise set-valued contraction on $X$; (c) $(X, d_r)$ is compact and rectifiably path-connected, and $BR(x)$ is a pointwise set-valued contraction on $X$. Then, $\text{Fix}(BR(x)) \neq \emptyset$.

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