PROPERTIES OF OPTIMAL PATHS IN FIRST PASSAGE PERCOLATION

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Abstract. In this paper, we study some properties of optimal paths in the first passage percolation on \( \mathbb{Z}^d \) and show the following: (1) the number of optimal paths has an exponentially growth if the distribution has an atom; (2) the means of intersection and union of optimal paths are linear in the distance. For the proofs, we use the resampling argument introduced in [J. van den Berg and H. Kesten. Inequalities for the time constant in first-passage percolation. Ann. Appl. Probab. 56-80, 1993] with suitable adaptations.

1. Introduction

We consider the first passage percolation (FPP) on the lattice \( \mathbb{Z}^d \) with \( d \geq 2 \). The model is defined as follows. The vertices are the elements of \( \mathbb{Z}^d \). Let us denote by \( E^d \) the set of edges:

\[ E^d = \{ \{v, w\} | v, w \in \mathbb{Z}^d, |v - w|_1 = 1 \}, \]

where we set \( |v - w|_1 = \sum_{i=1}^{d} |v_i - w_i| \) for \( v = (v_1, \cdots, v_d) \), \( w = (w_1, \cdots, w_d) \). Note that we consider non–oriented edges in this paper, i.e. \( \langle v, w \rangle = \langle w, v \rangle \) and we sometimes regard \( \langle v, w \rangle \) as a subset of \( \mathbb{Z}^d \) with some abuse of notation.

We assign a non-negative random variable \( \tau_e \) to each edge \( e \in E^d \), called the passage time of \( e \). The collection \( \tau = \{ \tau_e \}_{e \in E^d} \) is assumed to be independent and identically distributed with common distribution function \( F \). A path \( \Gamma \) is a finite sequence of vertices \( (x_1, \cdots, x_l) \subset \mathbb{Z}^d \) such that for any \( i \in \{1, \cdots, l - 1\} \), \( \{x_i, x_{i+1}\} \in E^d \). It is convenient to regard a path as a subset of edges in the following way:

\[ \Gamma = (x_i)_{i=1}^l = (\{x_i, x_{i+1}\})_{i=1}^{l-1}. \]

Given a path \( \Gamma \), we define the passage time of \( \Gamma \) as

\[ t(\Gamma) = \sum_{e \in \Gamma} \tau_e. \]

Given two vertices \( v, w \in \mathbb{Z}^d \), we define the first passage time from \( v \) to \( w \) as

\[ t(v, w) = \inf_{\Gamma : v \rightarrow w} t(\Gamma), \]

where the infimum is over all finite paths \( \Gamma \) starting at \( v \) and ending at \( w \). A path from \( v \) to \( w \) is said to be optimal if it attains the first passage time \( t(v, w) \), i.e. \( t(\Gamma) = t(v, w) \). Denote by \( \mathcal{O}(v, w) \) the set of all self–avoiding optimal paths from \( v \) to \( w \). (Since if the distribution \( F \) has an atom at 0, the number of optimal paths can be infinity, we only consider self–avoiding optimal paths in this paper.) Hereafter, we simply call them optimal paths.

Definition 1 ([2]). Distribution \( F \) is said to be useful if \( \mathbb{E}[\tau_e] < \infty \) and

\[ F(F^-) < \begin{cases} p_c(d) & \text{if } F^- = 0 \\ \tilde{p}_c(d) & \text{otherwise}. \end{cases} \]

where \( p_c(d) \) and \( \tilde{p}_c(d) \) stand for the critical probability for \( d \)-dimensional percolation and oriented percolation model, respectively and \( F^- \) is the infimum of the support of \( F \).
It is well–known that if $F$ is useful, for any $v, w \in \mathbb{Z}^d$, there exists an optimal path from $v$ to $w$, i.e. $\#O(v, w) \geq 1$, with probability one (see, e.g. [10]. One can see it from Lemma 2 therein). It is worth noting that this problem is open for general distribution (see Question 21 in [1]). If $F$ is continuous, i.e. $P(\tau_e = a) = 0$ for any $a \in \mathbb{R}$, then the optimal path is uniquely determined almost surely. Indeed, it is easy to see that for two different paths $\Gamma_1, \Gamma_2$, $P(t(\Gamma_1) = t(\Gamma_2)) = 0$.

Since the cardinality of finite paths is countable, it follows that $P(\forall v, w \in \mathbb{Z}^d, \#O(v, w) = 1) = 1$.

Contrarily, if $F$ has an atom, there can be multiple optimal paths. We study the cardinality, the intersection and the union of optimal paths.

Let $(e_i)_{i=1}^d$ be the canonical basis. We only consider the optimal paths from 0 to $N e_1$, though all of the results also hold for any direction. For the sake of simplicity, we write $O_N = O(0, N e_1)$.

1.1. The number of optimal paths.

**Theorem 1.** Suppose that $d \geq 2$ and $F$ is useful and there exists $\alpha \in [0, \infty)$ such that $P(\tau_e = \alpha) > 0$. Then, there exists $c > 0$ such that
\[
\liminf_{N \to \infty} N^{-1} \log \#O_N > c \quad a.s.
\]

**Remark 1.** The similar statement of Theorem 1 was proved for the directed last passage percolation with finite support distributions in [5] (see section 1.3.1 for the details).

We can also obtain the corresponding upper bound of (1.2) but with a different constant.

**Theorem 2.** If $E[\tau_e^2] < \infty$ and $F$ is useful, then there exists a non–random constant $C > 0$ such that
\[
\limsup_{N \to \infty} N^{-1} \log \#[O_N] < C \quad a.s.
\]

It is certainly desirable to have the existence of $\lim_{N \to \infty} N^{-1} \log \#[O_N]$, but it seems to be much harder problem.

1.2. Intersection and union of optimal paths. Theorem 1 tells us that there are exponentially many open paths. Our next results partially reveal how the optimal paths are distributed in the space.

**Theorem 3.** Suppose that $F$ is useful. Then there exists $c > 0$ such that for any $N \in \mathbb{N}$,
\[
E \left[ \# \left( \bigcap_{\Gamma \in O_N} \Gamma \right) \right] \geq c N,
\]
where we regard a path $\Gamma$ as a set of edges as in (1.1).

**Corollary 1.** Suppose that $F$ is useful and $E[\tau_e^2] < \infty$. Then there exists $c > 0$ such that for any $N \in \mathbb{N}$,
\[
P \left( \# \left( \bigcap_{\Gamma \in O_N} \Gamma \right) \geq c N \right) \geq c.
\]

This shows that there are lots of pivotal edges on optimal paths with positive probability. We believe that the event of the left hand side holds with high probability.
Theorem 4. Suppose that $F$ is useful and there exists $\ell > 2(d - 1)$ such that $\mathbb{E}[\tau_1^F] < \infty$. Then there exists $C > 0$ such that for any $N \in \mathbb{N}$,
\[\mathbb{E} \left[ z \left( \bigcup_{\Gamma \in \mathcal{G}} \Gamma \right) \right] \leq CN.\]

Remark 2. Similar statement to Theorem 4 was proved in [12] (see section 1.3.2 for the details).

Remark 3. The moment condition of Theorem 4 is used in Lemma 9 and Lemma 11.

1.3. Historical background and related works. First passage percolation is a model of the spread of a fluid through some random medium, which was introduced by Hammersley and Welsh in [3]. Since it is easy to check that $t(\cdot, \cdot)$ is pseudometric and moreover, if $F(0) = 0$, exactly a metric almost surely, we can naturally regard the model as a random metric space. Hence mathematical objects of interest are the asymptotic behavior of the first passage time $t(v, w)$ (metric) as $|v - w|_1 \to \infty$ and its optimal paths (geodesics). Over 50 years, there has been significant progress for these problems but there still remains many interesting problems (see [1] for more on the background and open problems).

1.3.1. Number of optimal paths. There has been revived interest on the number of optimal (or maximizing) paths in directed last passage percolation and oriented percolation [5, 6, 7, 9]. (Here assigning $\tau_\epsilon = 0$ if it is open and $\tau_\epsilon = \infty$ otherwise, an open path for oriented percolation can be seen to be an optimal path.) Especially, in [5], it was proved that the number of maximizing paths of directed last passage percolation with finite support distributions has an exponential growth. The proof was based on the multi-valued map principle (MVMP). In this paper, we prove a similar result by another method, van den Berg-Kesten’s resampling argument. Remark that our techniques do not work in critical and supercritical regime, i.e. $\mathbb{P}(\tau_e = 0) \geq p_c(d)$ due to the lack of Lemma 2. In supercritical regime, the number of optimal paths should be infinity with high probability. Even in this case, MVMP seems to be applicable, though not straight forwardly.

1.3.2. Intersection and union of optimal paths. In directed polymer models, the overlap of independent polymers naturally arise in the analysis of the free energy and it has received much interest in this field (see e.g. [3]). In the model which allows us to use Malliavin calculus, especially integration by parts, much is known for the overlaps (see [4] and reference therein). However, to my knowledge, little is known for general setting. For the union of optimal paths, it was showed in [11] that the same estimate of Theorem 4 for optimal paths from the origin to the boundary of the bounded area if $F$ is subcritical Bernoulli distribution, that is the case $\mathbb{P}(\tau_e \in \{0, 1\}) = 1$ and $\mathbb{P}(\tau_e = 0) < p_c(d)$. The proof in [12] is based on the Russo formula, which seems to be specific to the Bernoulli case.

Note that our results strongly suggest that all optimal paths from 0 to $Ne_1$ are contained by some thin sausage (see Figure 1.2). In other words, in practice, these optimal paths should be represented by one optimal path.

1.4. Notation and terminology. This subsection collects some notations and terminologies for the proof.

- We use $c > 0$ for a small constant and $C > 0$ for a large constant. They may change from line to line.
- Given a path $\gamma = (x_i)_{i=1}^n$, we define a new path as $\gamma[x_m, n] = (x_i)_{i=m}^n$.
- Given two paths $\gamma_1 = (x_i)_{i=1}^m$ and $\gamma_2 = (y_i)_{i=1}^n$ with $x_l = y_1$, we denote the connected path by $\gamma_1 \oplus \gamma_2$, i.e. $\gamma_1 \oplus \gamma_2 = (x_1, \ldots, x_l, y_1, \ldots, y_n)$.
- Given $x, y \in \mathbb{R}^d$, we define $d_\infty(x, y) = \max\{|x_i - y_i| : i = 1, \ldots, d\}$. It is useful to extend the definition as
  \[d_\infty(A, B) = \inf\{d_\infty(x, y) : x \in A, y \in B\}\ \text{for } A, B \subset \mathbb{R}^d.\]
- When $A = \{x\}$, we write $d_\infty(x, B)$.
- For $x \in \mathbb{R}^d$ and $r > 0$, denote the closed ball whose center is $x$ and radius is $\delta$ by $B(x, r)$. 
2. Proof for Theorem 1

Given \( e = (x, y) \), we sometimes write \( \tau_e = \tau(x, y) \) in this section.

**Definition 2.** Given a path \( \gamma = (x_0, \cdots, x_l) \), let us define the reflection of \( x_i \) across \( \gamma \) as \( x_i^* = x_{i-1} + (x_{i+1} - x_i) \) with the convention \( x_0^* = x_0 \) and \( x_l^* = x_1 \). \( x \in \mathbb{Z}^d \) is said to be \( \mathbb{G} \)-turn for \( \gamma \) if there exists \( i \in \{1, \cdots, l-1\} \) such that \( x = x_i, \tau(x_{i-1}, x_i) + \tau(x_i, x_{i+1}) = \tau(x_i^*, x_{i+1}) + \tau(x_{i-1}, x_i^*) \), \( x_i - x_{i-1} \) is perpendicular to \( x_{i+1} - x_i \) and \( x_i^* \notin \gamma \).

**Definition 3.** We set the attached passage time as \( t^+(\gamma) = t(\gamma) + \beta \mathbb{E}[\{x_i \mid x_i \text{ is } \mathbb{G} \text{-turn for } \gamma\}] \) with a constant \( \beta > 0 \) to be chosen in Lemma 5 and denote by \( t^+(0, N e_1) \) the first passage time from 0 to \( N e_1 \) corresponding to \( t^+(\cdot) \). We call it the attached first passage time.

**Lemma 1.**

\[
\mathbb{E} \left[ \min \{ \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \mid \Gamma \in \mathbb{O}_N^{+} \} \right] \geq c N.
\]

Note that minimum of (2.1) is over all optimal path for \( t^+(\cdot) \).

We postpone the proof and first prove Theorem 1. By the definition of \( \mathbb{G} \)-turn, we have

\[
\beta \min \{ \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \mid \Gamma \in \mathbb{O}_N^{+} \} \leq t^+(0, N e_1) - t(0, N e_1) \\
\leq \beta \min \{ \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \mid \Gamma \in \mathbb{O}_N \}.
\]

Indeed, taking \( \Gamma \in \mathbb{O}_N^{+} \), which attains the minimum, by the definition of \( t^+(\cdot) \), we have

\[
t^+(0, N e_1) - t(0, N e_1) = \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \mid \Gamma \in \mathbb{O}_N^{+},
\]

which implies the first inequality. For the second inequality of (2.2), we only take \( \Gamma \in \mathbb{O}_N \) attaining the minimum and calculate \( t^+(\Gamma) - t(\Gamma) \geq t^+(0, N e_1) - t(0, N e_1) \).

By Kingman’s subadditive ergodic theorem, there exist \( \mu, \mu^+ \geq 0 \) such that almost surely,

\[
\lim_{N \to \infty} N^{-1}(t^+(0, N e_1) - t(0, N e_1)) = \lim_{N \to \infty} N^{-1}(\mathbb{E} t^+(0, N e_1) - \mathbb{E} t(0, N e_1)) = \mu^+ - \mu.
\]

On the other hand, by Lemma 1 and (2.2), we have \( \mu^+ - \mu \geq c \). This together with (2.2) leads to

\[
\min \{ \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \mid \Gamma \in \mathbb{O}_N \} \geq c N.
\]

Let us take an arbitrary optimal \( \Gamma = (x_i)_{i=1}^l \) path satisfying \( \mathbb{E}[z \{ x \in \Gamma \mid x \text{ is } \mathbb{G} \text{-turn for } \Gamma \}] \geq c N \). Let us define \( x_i^G \) as \( x_i^G = x_i^* \) if \( x_i \) is \( \mathbb{G} \)-turn and \( x_i^G = x_i \) otherwise. Note that for any choice \( y_i \in \{x_i, x_i^G\}, (y_i)_{i=1}^l \) is optimal path, i.e. \( t((y_i)) = t(0, N e_1) \). Although it may be not self-avoiding, since the number of overlaps at any vertex \( x \), i.e. \( \mathbb{E}[|y_i| \mid y_i^* = x] \), is at most \( 2d \), (2.3) yields \( \mathbb{E}[\mathbb{O}_N] \geq 2^{-N/2d} \) as desired.
Proof of Lemma 1. Recall that $F^+$ be the suprimum of the support of $F$. We take $n > 0$ sufficiently large depending on the distribution $F$ but not depending on $N$. We prepare three kinds of box whose notations are the same as in [2] (See Figure 2). First, define the hypercubes $S(l; n)$, for $l \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, by

$$S(l; n) = \{v \in \mathbb{Z}^d : nl \leq v_i < n(l + 1) \text{ for any } i\}.$$  

We call these hypercubes $n$-cubes. Second, we define the large $n$-cubes $T(l; n)$, for $l \in \mathbb{Z}^d$, by

$$T(l; n) = \{v \in \mathbb{Z}^d : nl - n \leq v_i \leq n(l + 2) \text{ for any } i\}.$$  

Finally, we define the $n$-boxes $B^j(l; n)$, for $l \in \mathbb{Z}^d$ and $j \in \{1, \cdots, d\}$, as

$$B^j(l; n) = T(l; n) \cap T(l + 2 \text{sgn}(j) e_j; n).$$

Note that $S(l; n) \subset T(l; n)$ and $B^j(l; n)$ is a box of size $3n \times \cdots \times 3n \times n \times \cdots \times n$. For the simplicity of notation, we set $B = B^j(l; n)$. We take sufficiently large $M > 0$ to be chosen later.

The following is the crucial property of useful distribution.

Lemma 2. If $F$ is useful, there exists $\delta_1 > 0$ and $D > 0$ such that for any $v, w \in \mathbb{Z}^d$,

$$P(t(v, w) < (F^+ + \delta_1)|v - w|_1) \leq e^{-D|v - w|_1}.$$  

For the proof of this lemma, see Lemma 5.5 in [2]. We take $\delta_1 > 0$ as in Lemma 2.

Definition 4. Let us consider the following conditions:

1. For any $v, w \in B^j(l; n)$ with $|v - w|_1 \geq n^{1/3}$,

$$t(v, w) \geq (F^+ + \delta_1)|v - w|_1,$$

where $\delta_1 > 0$ is as in Lemma 2.

2. For any $e \cap B \neq \emptyset$, $\tau_e \leq M$.

3. For any $e \cap B \neq \emptyset$, $\tau_e \leq F^+ - M^{-1}$.

An $n$-box $B$ is said to be black if

$$\begin{cases} F^+ = \infty \text{ and (1) and (2) hold,} \\ F^+ < \infty, \ P(\tau_e = F^+) = 0 \text{ and (1) and (3) hold, or} \\ F^+ < \infty, \ P(\tau_e = F^+) > 0 \text{ and (1) holds.} \end{cases}$$

It is easy to check that if $M = M(n)$ is sufficiently large, $P(B \text{ is black}) \to 1$ as $n \to \infty$. Indeed the first condition which appears in definition of blackness holds with high probability by using Lemma 2 (See also (5.5), (5.31) and (5.32) in [2]). Together with a similar argument (Peierls argument) of (5.2) in [2], the following lemma follows.

Figure 2.

Left: Boxes: $S, T, B$.
Right: $\bigcap_N$ crosses a $n$-box in the short direction.
Lemma 3. There exist $\epsilon, D, n_1, M_1 > 0$ such that for any $N \in \mathbb{N}$, $n \geq n_1$ and $M \geq M_1$,

$$\mathbb{P}(\exists \text{ path from } 0 \to Ne_1 \text{ which visits at most } \epsilon N \text{ distinct black } n\text{-cubes}) \leq e^{-DN}$$

A path which starts in $S(l; n)$ and ends outside of $T(l; N)$ must have a segment which lies entirely in one of the surrounding $n$-boxes, and which connects the two opposite large faces of that $n$-box. This means that this path crosses at least one $n$-box in the short direction (See Figure 2). Hereafter “crossing an $n$-box” means crossing in the short direction.

Definition 5. An $n$-box $B$ is said to be white if there exists $\Gamma \in O_N$ such that $\Gamma$ cross $B$.

An $n$-box $B$ is said to be gray if $B$ is black and white.

From these observations together with Lemma 3, we obtain

(2.4) \[ \mathbb{E}[\#\{\text{distinct gray } n\text{-box } B\}] \geq \epsilon N/2 \]

Definition 6. Define

$$F_M^+ = \begin{cases} M^2 & \text{if } F^+ = \infty, \\ F^+ - M^{-2} & \text{if } F^+ < \infty \text{ and } F(\{F^+\}) = 0, \\ F^+ & \text{if } F^+ < \infty \text{ and } F(\{F^+\}) > 0, \end{cases}$$

and

$$F_M^- = \begin{cases} F^- + M^{-2} & \text{if } F(\{F^-\}) = 0, \\ F^- & \text{if } F(\{F^-\}) > 0. \end{cases}$$

Note that if $M$ is sufficiently large,

(2.5) \[ F_M^- < F^- + \delta_1/2 < F_M^+ \text{ and } F_M^- \leq \alpha \leq F_M^+, \]

where $\alpha$ appears in the statement of Theorem 1.

Definition 7. $x \in \mathbb{Z}^d$ is said to be a turn for $\gamma = (x_0, \cdots, x_{|\gamma|})$ if there exists $i \in \{1, \cdots, l-1\}$ such that $x = x_i$, $x_i - x_{i-1}$ is perpendicular to $x_{i+1} - x_i$. Otherwise, we say that $x_i$ is flat.

A $n$-box $B$ is called $G$–turn if for any $\Gamma \in O_N$, there exists $x \in B$ such that $x$ is $G$–turn for $\Gamma$.

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**Figure 3.**

Left: rough boxes. Middle: Examples of $\tilde{\gamma}$. Right: Examples of $\gamma_{a,b}$.
Denote by $\partial^+ B$ the outer boundary of a $n$-box $B$. The following lemma will be used in Definition 8.

**Lemma 4.** Suppose that $F^+ < \infty$. If we take $n$ sufficiently large, for any $a, b \in \partial^+ B$ with $|a - b|_1 \geq \delta_1 n/(2F^+) := C(\delta_1, F^+) n$, there exists a self-avoiding path $\gamma_{a,b} = (x_0, \cdots, x_{|\gamma_{a,b}|})$ from $a$ to $b$ satisfying $\{x_i\}_{i=1}^{|\gamma_{a,b}|} \subset B$ such that the following hold:

\begin{align*}
(1) \quad |x_i - x_j|_1 = |i - j| \quad &\text{if } |i - j| \leq |a - b|_1/4, \quad [\tilde{x}_i, \tilde{x}_j]_1 = |i - j|, \quad \text{(b) if neither } \tilde{x}_i \text{ nor } \tilde{x}_j \text{ is flat for } \tilde{\gamma}, \quad [\tilde{x}_i, \tilde{x}_j]_1 \geq \sqrt{n}
\end{align*}

Then we consider rough boxes as in Figure 3. For the construction of $\gamma_{a,b}$, we attach rough box in the middle of the $\sqrt{n}$ successive flat points of $\tilde{\gamma}$ if they exist and we continue the procedure until they vanish. We take $\gamma_{a,b} = (x_0, \cdots, x_{|\gamma_{a,b}|})$ which was obtained by the above procedure eventually. Note that the number of rough boxes attached in this procedure is at most $|\gamma|/\sqrt{n}$ ($\leq 6d\sqrt{n}$). Thus, $|\gamma_{a,b}| \leq 5 \times 2 \times 6d\sqrt{n} + |\gamma| \leq |a - b|_1 + 100d\sqrt{n}$ which implies (5), (1), (3), (4), (6), (7) are trivial by the way of construction. Finally, we will prove (2). If $|i - j| \leq |a - b|_1/8$, then

\begin{align*}
|x_i - x_j|_1 \geq |i - j| - (5 \times 2 \times |i - j|/\sqrt{n}) \geq |i - j| - 20d\sqrt{|i - j|}.
\end{align*}

Otherwise, since $|i - j| - |x_i - x_j|_1 \leq |\gamma_{a,b}| - |a - b|_1 \leq 100d\sqrt{n}$ and $|a - b|_1 \geq C(\delta_1, F^+) n$, we have

\begin{align*}
|x_i - x_j|_1 \geq |i - j| - 20d\sqrt{n} \geq |i - j| - 800d_{\sqrt{C(\delta_1, F^+)}}/\sqrt{|i - j|}.
\end{align*}

\[ \square \]

**Definition 8.** For any $a, b \in \partial^+ B$ with $|a - b|_1 \geq \delta_1 n/(2F^+) + 1$, we take a self avoiding path $\gamma_{a,b} = (x_0, \cdots, x_{|\gamma_{a,b}|})$ from $a$ to $b$ with $\{x_i\}_{i=1}^{|\gamma_{a,b}|} \subset B$ so that one of the following holds:

(1) $F^+ = \infty$, (2) there exists at least one turn $x_i$ for $\gamma_{a,b}$ such that $d_{\infty}(x_i, B^c) \geq 2$, (3) for any distinct turn points $x, y$ for $\gamma_{a,b}$, $|x - y|_1 > 4$ and (4) $|a - b|_1 + 4d\sqrt{n} \geq |\gamma_{a,b}|$, or

(1) $F^+ < \infty$ and (2) **Lemma 4** holds.

In the case of $|a - b|_1 < \delta_1 n/(2F^+) + 1$, we take an arbitrary self avoiding path $\gamma_{a,b} \subset B \cap \partial^+ B$ from $a$ to $b$.

Given a path $\gamma = \gamma_{a,b} = (x_0, \cdots, x_{|\gamma|})$ and $n$-Box $B$, $\tau$ is said to be satisfied $(\gamma, B)$-condition if

(1) $\tau(x_{i-1}, x_i) = \tau(x_i, x_{i+1}) = \tau(x_{i+1}, x_i) = \tau(x_{i}, x_{i+1}) = \alpha$ if $x_i$ is turn for $\gamma$, (2) $\tau_e \leq F^+_M$ for $e \in \gamma$ with $e \notin \{\{x_{i-1}, x_i]\} x_i$ is turn for $\gamma \cup \{\{x_{i}, x_{i+1}\}\} x_i$ is turn for $\gamma$, (3) $\tau_e \geq F^+_M$ for other edges with $e \cap B \neq \emptyset$. Denote the independent copy of $\tau$ by $\tau^*$ and set $\tau^*_e = \tau^*_e$ if $e \cap B \neq \emptyset$, $\tau^*_e = \tau_e$ otherwise. Let $(\tilde{a}, \tilde{b})$ be random variable on $\partial^+ B \times \partial^+ B$ with $\tau_e^*$ distribution and its probability measure $P$. Given a path $\Gamma = (x_0, \cdots, x_t)$ and a $n$-box $B$, we set

\[ \text{st}(\Gamma, B) = \sum_{i=1}^t 1_{\{x_i \in \partial^+ B\}}, \quad \text{fin}(\Gamma, B) = \max_{i=1}^t 1_{\{x_i \in \partial^+ B\}}. \]

Note that if $\Gamma$ cross $B$ and $B$ is black, since $t^+(\text{st}(\Gamma, B), \text{fin}(\Gamma, B)) \geq (F^- + \delta_1)n$, we have

\[ |\text{st}(\Gamma, B) - \text{fin}(\Gamma, B)|_1 \geq \frac{(F^- + \delta_1)n}{2F^+} + 1. \]
Here the above inequality holds even if \( F^\ast = \infty \).

**Lemma 5.** We take \( \beta = M^{-2} \). If \( M \geq n^{2d} \) and \( n \) is sufficiently large, there exists \( c > 0 \) such that for any \( N \in \mathbb{N} \), unless \( 0 \in B \) or \( Ne_1 \in B \),

\[
\mathbb{P}( B \text{ is } \mathbb{G} \text{-turn for } \tau) = P \otimes \mathbb{P}( B \text{ is } \mathbb{G} \text{-turn for } \tau^B)
\]

\[
\geq P \otimes \mathbb{P}
\begin{cases}
B \text{ is gray for } \tau, \exists \Gamma \in \mathbb{O}_N^+ \text{ s.t. } \Gamma \text{ cross } B, \\
\quad (\tilde{a}, \tilde{b}) = (st(\Gamma, B), \text{fin}(\Gamma, B)) \text{ and } \tau^* \text{ satisfies } (\gamma_{\tilde{a}, \tilde{b}}, B) \text{-condition}
\end{cases}
\]

\[
(2.7) = \frac{1}{|\partial^+ B|^2} \sum_{(a, b)} \mathbb{P}
\begin{cases}
B \text{ is gray for } \tau, \exists \Gamma \in \mathbb{O}_N^+ \text{ s.t. } \Gamma \text{ cross } B, \\
\quad (a, b) = (st(\Gamma, B), \text{fin}(\Gamma, B)) \text{, } \tau^* \text{ satisfies } (\gamma_{a, b}, B) \text{-cond.}
\end{cases}
\]

\[
\geq c\mathbb{P}(B \text{ is gray}).
\]

**Proof.** We suppose that \( B \) is gray for \( \tau \), there exists \( \Gamma \in \mathbb{O}_N^+ \) such that \( (\tilde{a}, \tilde{b}) = (st(\Gamma, B), \text{fin}(\Gamma, B)) \) and \( \tau^* \) satisfy \( (\gamma_{\tilde{a}, \tilde{b}}, B) \)-condition. We write \( \gamma = (\gamma_{\tilde{a}, \tilde{b}} = (x_0, \cdots, x_\gamma)), a = \tilde{a} \) and \( b = \tilde{b} \) for simplicity. Our first goal is to prove that \( B \) is \( \mathbb{G} \)-turn for \( \tau^B \) under this condition. To this end, we take \( \Gamma^B \) to be an optimal path for the attached first passage time with respect to \( \tau^B \). Our first goal is to prove that \( B \) is \( \mathbb{G} \)-turn for \( \tau^B \).

First we consider the case \( F^+ = \infty \). The \((\gamma, B)\)-condition and blackness of \( B \) lead to

\[
t^{B, +}(0, N e_1) \leq t^{B, +}(\Gamma[0, a] \oplus \gamma \oplus \Gamma[b, N e_1])
\]

\[
\leq t^{B, +}(\Gamma[0, a] + (F^- + \delta_1/2)|a - b|_1 + t^{B, +}(\Gamma[b, N e_1]) + 2d\beta|B|
\]

\[
t^{B, +}(\Gamma[0, a] + (F^- + \delta_1)|a - b|_1 + t^{B, +}(\Gamma[b, N e_1]) \leq t^+(\Gamma),
\]

where \( t^{B, +} \) is attached first passage time with respect to \( \tau^B \). Therefore, \( \Gamma^B \) must enter \( B \), i.e. there exists \( e \in \Gamma^B \) such that \( e \cap B \neq \emptyset \). We will show that

\[
\Gamma^B \cap \{ e \in E^d \mid e \cap B \neq \emptyset, e \not\subseteq (\gamma \cup \{ x_i \mid i = 1, \cdots, |\gamma| - 1 \}) \} = \emptyset.
\]

In fact, if such an edge exists, by \((\gamma, B)\)-condition, \( \sum_{e \in E^d} \tau^B_e \geq M^2 \). On the other hand, since \( \sum_{e \in E^d} \tau^B_e \leq M^2/2 \), we derive a contradiction as follows:

\[
t^{+, B}(0, N e_1) = t^{+, B}(\Gamma^B)
\]

\[
\geq t^{+, B}(\Gamma^B[0, a] \cup \text{st}(\Gamma^B, B)) + M^2 + t^{+, B}(\Gamma[\text{fin}(\Gamma^B, B), N e_1])
\]

\[
\geq t^+(\Gamma^B[0, a] \cup \text{st}(\Gamma^B, B)) + 2d\beta|B| \geq t^+(\Gamma^B).
\]

Accordingly, since \( \Gamma^B \) is self-avoiding, \([2.9] \) implies that \( a, b \in \Gamma^B \) and there exists a path \( \{ z_i \}_{i=0}^{\gamma} \) with \( z_i \in \{ x_i, x_i^\ast \} \) such that \( (z_0, \cdots, z_\gamma) = \Gamma^B[a, b] \). Since \( \Gamma^B \) is arbitrary optimal path, it yields that \( B \) is \( \mathbb{G} \)-turn for \( \tau^B \).

Hereafter we suppose that \( F^+ < \infty \). Then since \( \Gamma \) cross \( B \) and \( B \) is black, we have \( |a - b|_1 \geq (F^- + \delta_1)n/(2F^++1) \) and \( t^+(a, b) \geq (F^- + \delta_1)|a - b|_1 \). By the \((\gamma, B)\)-condition and \((5) \) and \((7) \) of \([2.6] \), we have \( \sum_{e \in E^d} \tau^B_e \leq F_M^\ast(|a - b|_1 + 100d\sqrt{n}) + 2\alpha^2\{ x \in \gamma \mid x \text{ is turn for } \gamma \} \leq F_M^\ast(|a - b|_1 + 100d\sqrt{n}) + a\alpha^2|B| \). It follows that

\[
t^+(\Gamma^B) - t^{B, +}(\Gamma^B) + t^+(0, N e_1) - t^{B, +}(0, N e_1)
\]

\[
\geq t^+(\Gamma) - t^{B, +}(\Gamma[0, a] \cup \gamma \oplus \Gamma[b, N e_1])
\]

\[
\geq t^+(a, b) - \sum_{e \in \gamma} \tau^B_e - \beta(\{ x \in \gamma \mid x \text{ is turn for } \gamma \} + 2) \geq \delta_1 n/4.
\]

On the other hand, since \( \tau^B_e \leq \tau^B \) unless \( e \in \gamma \), \( e = (x_i, x_i^\ast) \), or \( e = (x_i^\ast, x_{i+1}) \) for any \( i \in \{ 1, \cdots, l - 1 \} \) such that \( x_i \) is turn for \( \gamma \),

\[
t^+(\Gamma^B) - t^{B, +}(\Gamma^B) + 2F^+ \cdot 16d\sqrt{n} + F^+ \cdot 2\{ i \in \{ 0, \cdots, |\gamma| - 1 \} \mid \{ x_i, x_{i+1} \} \in \gamma \cap \Gamma^B \} + 2d\beta|B|.
\]
Lemma 6. Consider $4d\sqrt{n} \leq p_1 < |\gamma| - 4d\sqrt{n}$ and $0 \leq q_1 < |\gamma|$. Then under the above condition, for any self avoiding path $(y_0, \ldots, y_l)$ which satisfies $y_0 = x_{p_1}, y_l = x_{q_1}, y_1 \notin (\gamma \cup \{x^* | x \in \gamma\})$ for $1 \leq i \leq l - 1$,

$$t^{B^+}(y_0, \ldots, y_l) > t^{B^+}(x_{p_1}, \ldots, x_{q_1}) + 2\beta,$$

where $2\beta$ in (2.12) is necessary later because $x_{p_1}$ and $x_{q_1}$ may be $G$-turn for $\Gamma^B$.

Proof. Without loss of generality, we suppose that $p_1 < q_1$. If $|p_1 - q_1| > 4dn^{1/3}$, by blackness of $B$ and (2) and (7) of (2.6), we have

$$t^{B^+}(x_{p_1}, \ldots, x_{q_1}) + 2\beta < 2(2\alpha + \beta)(|q_1 - p_1|) + F_M^+(q_1 - p_1) < (F^- + \delta_1)|x_{p_1} - x_{q_1}| \leq t^{B^+}(y_0, \ldots, y_l),$$

Next we assume $|p_1 - q_1| \leq 4dn^{1/3}$. Note that $l \geq 2$. If $l = 2$ and $x_{p_1+1}$ is turn for $\gamma$, $y_1$ must be an element of $\{x_{p_1+1}, x^*_1\}$, which is a contradiction. On the other hand, if $l = 2$ and $x_{p_1+1}$ is not turn for $\gamma$, we have

$$t^{B^+}(x_{p_1}, \ldots, x_{q_1}) + 2\beta \leq \alpha + F_M^+ + 2\beta < 2F_M^+ \leq t^{B^+}(y_0, \ldots, y_l),$$

as desired. We suppose that $l \geq 3$. Note that by (4), (6), (7) of (2.6),

$$(4dn^{1/3} \land l)F_M^+ \leq t^{B^+}(y_0, \ldots, y_l),$$

$$t^{B^+}(x_{p_1}, \ldots, x_{q_1}) \leq 2\alpha + \beta(q_1 - p_1) + F_M^+(q_1 - p_1 - 2) + 2(\alpha - F_M^+)\mathbb{E}_{l>4}.$$
Corollary 2. There exists \( c > 0 \) such that
\[
\mathbb{P}( \text{for any optimal path } \Gamma \in \mathcal{O}_N, \text{ the number of } \mathcal{G}-\text{turn points of } \Gamma \text{ is at least } cN ) \to 1.
\]

Proof of Theorem 2. We first define the event as
\[
A = \{ \forall \Gamma \in \mathcal{O}_N \text{ such that } \sharp \Gamma \leq KN \}.
\]
Then under the event \( A \), we have \( \sharp \mathcal{O}_N \leq (2d)^K N \), which yields that
\[
\frac{1}{N} \log \sharp \mathcal{O}_N \leq K \log 2d.
\]
On the other hand, by (4.16) below, there exists \( K > 0 \) such that for any \( N > 0 \)
\[
\mathbb{P}( \exists \Gamma \in \mathcal{O}_N \text{ such that } \sharp \Gamma > KN ) \leq KN^{-2d}.
\]
By the Borel–Cantelli lemma, we complete the proof of (1.3). \( \square \)

3. Proof of Theorem 3

We begin with the connection between the intersection and restricted union of optimal paths.

Lemma 7. For any \( \alpha > F^- \), there exists \( c > 0 \) such that for any \( N \in \mathbb{N} \),
\[
\mathbb{E} \left[ \sharp \left( \bigcap_{\Gamma \in \mathcal{O}_N} \Gamma \right) \right] \geq c \mathbb{E} \left[ \sharp \{ \eta \in E(\mathbb{Z}^d) | \exists \Gamma \in \mathcal{O}_N, \eta \in \Gamma, \tau_\eta > \alpha \} \right].
\]

Proof. Let \( \tau^* \) be an independent copy of \( \tau \). Given an edge \( \eta \in E(\mathbb{Z}^d) \), we set \( \tau^{(\eta)} \) as
\[
\tau^{(\eta)}_e = \begin{cases} \tau^*_e & \text{if } e = \eta, \\ \tau_e & \text{if } e \neq \eta. \end{cases}
\]
Note that for any edge \( \eta \in E(\mathbb{Z}^d) \), since \( \tau \) and \( \tau^{(\eta)} \) have same distributions,
\[
\mathbb{P} \left( \exists \Gamma \in \mathcal{O}_N, \eta \in \Gamma, \tau_\eta > \alpha \right) \mathbb{P} \left( \tau^{(\eta)}_\eta \leq \alpha \right)
= \mathbb{P} \left( \exists \Gamma \in \mathcal{O}_N, \eta \in \Gamma, \tau_\eta > \alpha, \tau^{(\eta)}_\eta \leq \alpha \right)
\leq \mathbb{P} \left( \forall \Gamma \in \mathcal{O}^{(\eta)}_N, \eta \in \Gamma \right) = \mathbb{P} \left( \forall \Gamma \in \mathcal{O}_N, \eta \in \Gamma \right),
\]
where \( \mathcal{O}^{(\eta)}_N \) is the set of all optimal paths from the origin to \( N \mathbf{e}_1 \) with respect to \( \tau^{(\eta)} \). Indeed, if there exists \( \Gamma \in \mathcal{O}_N \) such that \( \eta \in \Gamma, \tau_\eta > \alpha, \tau^{(\eta)}_\eta \leq \alpha \), since \( t^{(\eta)}(0, N \mathbf{e}_1) < t(0, N \mathbf{e}_1) \) and \( t(\eta)(\Gamma) < t(\Gamma) \) for any path \( \Gamma \) with \( e \notin \Gamma \), optimal paths for \( \tau^{(\eta)} \) must pass through \( \eta \). Therefore the inequality of (3.1) follows. Thus we have
\[
\mathbb{E} \left[ \sharp \left( \bigcap_{\Gamma \in \mathcal{O}_N} \Gamma \right) \right] \geq \mathbb{E} \left[ \sharp \{ \eta \in E(\mathbb{Z}^d) | \exists \Gamma \in \mathcal{O}_N, \eta \in \Gamma, \tau_\eta > \alpha \} \right] = F(\alpha) \mathbb{E} \left[ \sharp \{ \eta \in E(\mathbb{Z}^d) | \exists \Gamma \in \mathcal{O}_N, \eta \in \Gamma, \tau_\eta > \alpha \} \right],
\]
where \( F(\alpha) = \mathbb{P}(\tau_\epsilon \leq \alpha) \).

Next we show that there exist \( \alpha > F^- \) and \( c > 0 \) such that for any \( N \in \mathbb{N} \)
\[
\mathbb{E} \left[ \sharp \{ \eta \in E(\mathbb{Z}^d) | \exists \Gamma \in \mathcal{O}_N \text{ such that } \eta \in \Gamma, \tau_\eta > \alpha \} \right] \geq cN.
\]
In fact, if we take \( \alpha > F^- \) with \( \mathbb{P}(\tau_\epsilon > \alpha) > 0 \) and \( \check{\tau}_\epsilon = \tau_\epsilon + \mathbb{I}_{\{\tau_\epsilon > \alpha\}} \), the result of [2] leads us to that
\[
\lim_{N \to \infty} N^{-1} \mathbb{E}[t(0, N \mathbf{e}_1)] < \lim_{N \to \infty} N^{-1} \mathbb{E}[\check{t}(0, N \mathbf{e}_1)],
\]
where \( \check{t}(\cdot, \cdot) \) is the first passage time with respect to \( \check{\tau} \). Since
\[
\check{t}(0, N \mathbf{e}_1) \leq t(0, N \mathbf{e}_1) + \sharp \{ \eta \in E(\mathbb{Z}^d) | \exists \Gamma \in \mathcal{O}_N \text{ such that } \eta \in \Gamma, \tau_\eta > \alpha \},
\]
we have (3.3). Combining it with Lemma 7, we have Theorem 3.

4. Proof of Theorem 4

We take $\alpha > F^-$ arbitrary. By (3.2), we have

\[ \mathbb{E} \{ z \in E(\mathbb{Z}^d) \mid \exists \Gamma \in \mathcal{O}_N, \ e \in \Gamma, \ \tau_e > \alpha \} \leq F(\alpha)^{-1} \mathbb{E} [\min \{ \mathbb{E} \mid \Gamma \in \mathcal{O}_N \} ] \leq CN, \]

where we have used (4.17) below in the last inequality. The rest of the proof is similar to that of Theorem 2 in [12] except for Lemma 9. We take where we have used (4.17) below in the last inequality. The rest of the proof is similar to that

\[ \mathbb{E} z_R N \leq \mathbb{E} \{ z_R N; z_R N \geq M^z K_N \} + MCN \]

\[ \leq \mathbb{E} (\mathbb{E} z_R N)^{1/2} (\mathbb{P} (\mathbb{E} z_R N \geq M^z K_N))^{1/2} + MCN \]

Now we will estimate $\mathbb{P} (\mathbb{E} z_R N \geq M^z K_N)$. Given $k \in \mathbb{Z}$ and $u \in \mathbb{Z}^d$, we define the square $B_k(u)$ whose size is $k$ and corner is $ku$ and the fattened $\hat{R}_N$ by

\[ B_k(u) = \prod_{i=1}^d [ku_i, ku_i + k], \]

\[ \hat{R}_N(k) = \{ u \in \mathbb{Z}^d : B_k(u) \cap R_N \neq \emptyset \}. \]

Note that $\hat{R}_N(k)$ is connected and contains the origin. By our definition,

\[ \mathbb{E} \hat{R}_N(k) \geq \mathbb{E} \hat{R}_N(k)/k^d. \]

Since $\mathbb{E} R_N \geq N$ and (4.3),

\[ \mathbb{P} (\mathbb{E} z_R N \geq M^z K_N) = \sum_{m \geq N/k^d} \mathbb{P} (\mathbb{E} z_R N \geq M^z K_N, \ \mathbb{E} \hat{R}_N(k) = m). \]

Denote by $B_k(u)$ the vertex set of $B_k(u)$ and all of its neighbor cubes with respect to $| \cdot |_\infty$. A cube $B_k(u)$ is said to be bad if $B_k(u) \cap K_N \neq \emptyset$ and $u \in \hat{R}_N(k)$. Otherwise, the cube is said to be good. Let $B_k(u)$ be the event that $B_k(u)$ is bad and $D_N$ be the number of bad cubes $B_k(u)$ for $u \in \hat{R}_N(k)$. The following lemma corresponds to (5.7) of [12].

**Lemma 8.** On $\{ z_R N \geq M^z K_N, \ \mathbb{E} \hat{R}_N(k) = m \}$ for $m \geq N/k^d$, if $(8k)^d < M,$

\[ D_N \geq m/2 \]

**Proof.** If there are $m/2$ good cubes, $\hat{R}_N \geq m/2 \cdot 4^d \geq 8^{-d} m$, where $4^d$ appears because of the overlap of cubes. By (4.3), on $\{ z_R N \geq M^z K_N, \ \mathbb{E} \hat{R}_N(k) = m \}$,

\[ z_R N \geq M^z K_N \geq 8^{-d} M m = 8^{-d} M^z \hat{R}_N(k) > \hat{R}_N k^d, \]

which is a contradiction of (4.3). \[ \square \]

Thus we take $M > (8k)^d$. Then

\[ \mathbb{P} (\mathbb{E} z_R N \geq M^z K_N) = \sum_{m \geq N/k^d} \mathbb{P} (\mathbb{E} z_R N \geq M^z K_N, \ \mathbb{E} \hat{R}_N(k) = m, \ D_N \geq m/2) \]

\[ = \sum_{m \geq N/k^d} \sum_{\kappa_m} \mathbb{P} (\mathbb{E} z_R N \geq M^z K_N, \ \hat{R}_N(k) = \kappa_m, \ D_N \geq m/2), \]

where $\kappa_m$ is a connected subset of $\mathbb{Z}^d$ with $m$ vertices which contains the origin and the second sum is taken over all possible such $\kappa_m$.

**Lemma 9.** Suppose that $F$ is useful and there exists $\ell > 2(d-1)$ such that $\mathbb{E} [\tau_e^\ell] < \infty$. If $0 \notin \hat{B}_k(u)$ and $N e_1 \notin \hat{B}_k(u), \mathbb{P}(B_k(u)) \to 0$ as $k \to \infty$. 


Proof. It suffices to show that there exist a constant $C, \epsilon > 0$ such that for any $a, b \in \mathbb{Z}^d$ with $|a - b|_1 \geq k$,

(4.8) \[ \mathbb{P}(\exists \text{ optimal path } \Gamma \text{ for } t(a, b) \text{ such that } \forall e \in \Gamma, \tau_e \leq \alpha) \leq Ck^{-2(d-1) - \epsilon}. \]

Indeed, if the event $\mathcal{B}_k(u)$ occurs, there exist $a \in \{v \in B_k(u) \mid \exists w \notin B_k(u) \text{ s.t. } |v - w|_1 = 1\}$, $b \in \{v \notin \overline{B}_k(u) \mid \exists w \notin B_k(u) \text{ s.t. } |v - w|_1 = 1\}$, and an optimal path $\Gamma$ for $(a, b)$ such that $\forall e \in \Gamma, \tau_e \leq \alpha$. Since $\forall \{v \in B_k(u) \mid \exists w \notin B_k(u) \text{ s.t. } |v - w|_1 = 1\} \vee \{v \notin \overline{B}_k(u) \mid \exists w \notin B_k(u) \text{ s.t. } |v - w|_1 = 1\} \leq C'k^{d-1}$ with some constant $C' = C'(d)$,

\[ \mathbb{P}(\mathcal{B}_k(u)) \leq (C'k^{d-1})^2 Ck^{-2(d-1) - \epsilon} \to 0 \text{ as } k \to \infty. \]

We will show (4.8). From the result (or the same argument) of [2], if we set $\bar{\tau}_e = \tau_e + \mathbb{1}_{\{\tau_e > \alpha\}}$, there exists a constant $c > 0$ independent of $k$ such that $\mathbb{E}[[\hat{t}(a, b)] - \mathbb{E}[t(a, b)] \geq ck$. In addition, if we take $\epsilon > 0$ so that $2(d-1) + 2\epsilon < \ell$, from Theorem 1 of [13] with $m = 2(d-1) + \epsilon$, there exist $C_1, \tilde{C}_1 > 0$ such that

(4.9) \[ \mathbb{P}(|t(a, b) - \mathbb{E}[t(a, b)]| > ck/4) \leq (ck/4)^{-2m}\mathbb{E}[[t(a, b) - \mathbb{E}[t(a, b)]]^2] \leq C_1k^{-2(d-1) - 2\epsilon} \leq C_1k^{-2(d-1) - \epsilon}, \]

and

(4.10) \[ \mathbb{P}(|\hat{t}(a, b) - \mathbb{E}[\hat{t}(a, b)]| > ck/4) \leq \tilde{C}_1k^{-2(d-1) - \epsilon}. \]

Thus the left hand side of (4.8) can be bounded from above by

\[ \mathbb{P}(\hat{t}(a, b) = t(a, b)) \leq (C_1 + \tilde{C}_1)k^{-2(d-1) - \epsilon}. \]

\[ \square \]

Let $L > 0$ to be chosen later. The above lemma yields that if we take $k$ sufficiently large, we have $\mathbb{P}(\mathcal{B}_k(u)) \leq \exp(-4L)$ and in addition, if $m > 4^d + 2$,

(4.11) \[ \mathbb{P}(\hat{z}R_N \geq M^*_z K_N, \hat{R}_N(k) = \kappa_m, \hat{D}_N \geq m/2) \leq m \left( \frac{m}{m/2} \right) \exp(-4L(m/2 - 2 \cdot 4^d)) \leq m \left( \frac{m}{m/2} \right) \exp(-Lm), \]

where $2 \cdot 4^d$ appears because of the condition $0 \notin \overline{B}_k(u)$ and $Ne_1 \notin \overline{B}_k(u)$.

The following lemma appears in (4.24) in [8]. We skip the proof.

Lemma 10. There exists $C_1 > 0$ such that for any $m$,

(4.12) \[ \mathbb{P}(\kappa_m \subset \mathbb{Z}^d \mid 0 \in \kappa_m, \kappa_m \text{ is connected, } |\kappa| = m) \leq e^{C_1m}. \]

Thus if we take $L$ sufficiently large and thus $k$ as well, we have

(4.13) \[ \mathbb{P}(\hat{z}R_N \geq M^*_z K_N) \leq \sum_{m \geq N/k^d} m \left( \frac{m}{m/2} \right) \exp(-Lm + C_1m) \leq \sum_{m \geq N/k^d} \exp(-Lm/2) \leq C_2 \exp(-Lk^{-d}N/2), \]

with some constant $C_2 > 0$.

Let us move on to the estimate of $\mathbb{E}[(\hat{z}R_N)^2]$.

Lemma 11. Suppose that $\mathbb{E}[\tau_0^2] < \infty$ and $F$ is useful. Then there exists $C_3 > 0$ such that

\[ \mathbb{E}[(\hat{z}R_N)^2] \leq C_3 N^{2d}. \]
Proof. First we suppose that $F^- = 0$. From Proposition 5.8 in [10], there exist $A, B, C > 0$ such that for any $r > 0$

\[(4.14) \quad \mathbb{P}(\exists \text{ selfavoiding path } \Gamma \text{ from the origin with } |\Gamma| \geq r \text{ and } t(\Gamma) < Ar < B \exp(Cr)).\]

We take $K > 2\mathbb{E}[r_\epsilon]/A$. Then for any $s > K$,

\[(4.15) \quad \mathbb{P}(\exists \Gamma \in \mathcal{O}_N \text{ such that } |\Gamma| \geq sN) \leq \mathbb{P}(\exists \Gamma \in \mathcal{O}_N \text{ such that } |\Gamma| \geq sN \text{ and } t(0, Ne_1) < AsN) + \mathbb{P}(t(0, Ne_1) \geq AsN)
\leq B \exp(-CsN) + \mathbb{P}(t(0, Ne_1) \geq AsN).
\]

where we have used (4.14) in the second inequality. Now we consider $2d$ disjoint paths from the origin to $Ne_1$ so that $\max\{|r_i| : i = 1, \cdots, 2d\} \leq N + 8$ as in [10, p 135]. By the Chebyshev inequality, we have that there exists $D = D(d, F, A) > 0$ such that

\[(4.16) \quad \mathbb{P}(t(0, Ne_1) \geq AsN) \leq \prod_{i=1}^{2d} \mathbb{P}(t(r_i) \geq AsN)
\leq \prod_{i=1}^{2d} (AsN/2)^2 \leq Ds^{-4d}N^{-2d}.
\]

Thus we have for $s > K$,

\[(4.17) \quad \mathbb{P}(\exists \Gamma \in \mathcal{O}_N \text{ such that } |\Gamma| \geq sN) \leq 2Ds^{-4d}N^{-2d}.
\]

Since $\sharp R_N \leq 2d (\max_{\Gamma \in \mathcal{O}_N} \sharp \Gamma)^d$, there exists $C_3 > 0$ such that

\[(4.18) \quad \mathbb{E}(\sharp R_N^2) \leq (2d)^2 \mathbb{E}\left(\max_{\Gamma \in \mathcal{O}_N} \sharp \Gamma\right)^{2d} = (2d)^3 \int_0^\infty r^{2d-1} \mathbb{P}\left(\max_{\Gamma \in \mathcal{O}_N} \sharp \Gamma \geq r\right) dr
\leq (2d)^3 \left( (KN)^{2d} + \int_{KN}^\infty r^{2d-1} \mathbb{P}\left(\max_{\Gamma \in \mathcal{O}_N} \sharp \Gamma \geq r\right) dr \right)
\leq (2d)^3 \left( (KN)^{2d} + \int_{KN}^\infty r^{2d-1} \cdot 2D(r/N)^{-4d}N^{-2d} dr \right) \leq C_3 N^{2d}.
\]

When $F^- > 0$, since $\max_{\Gamma \in \mathcal{O}_N} \sharp \Gamma \leq t(0, Ne_1)/F^-$, the proof is completed as before. □

Combining (4.2), (4.13) and the above lemma, we complete the proof.

5. Proof of Corollary [1]

Proof of Corollary [1] As in the proof of Lemma [11] if we take $K > 0$ as in the proof of Lemma [11]

\[cN \leq \mathbb{E}\left(\sharp \left(\bigcap_{\Gamma \in \mathcal{O}_N} \Gamma\right)\right)
\leq \mathbb{E}\left(\left(\left(\bigcap_{\Gamma \in \mathcal{O}_N} \Gamma\right)^2\right)\mathbb{P}\left(\left(\bigcap_{\Gamma \in \mathcal{O}_N} \Gamma\right) > KN\right) + KN\mathbb{P}\left(cN/2 \leq \sharp \left(\bigcap_{\Gamma \in \mathcal{O}_N} \Gamma\right) \leq KN\right) + cN/2
\leq 1 + KN\mathbb{P}\left(cN/2 \leq \sharp \left(\bigcap_{\Gamma \in \mathcal{O}_N} \Gamma\right)\right) + cN/2.
\]

Rearranging this, we have the conclusion. □
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