0. Introduction

Hyperkähler manifolds occupy a special position at the intersection of Riemannian, symplectic and algebraic geometry. A hyperkähler structure involves a Riemannian metric, as well as a triple of complex structures satisfying the quaternionic relations. Moreover we require that the metric is Kähler with respect to each complex structure, so we have a triple (in fact a whole two-sphere) of symplectic forms. Of course, there is no Darboux theorem in hyperkähler geometry because the metric contains local information. However, many of the constructions and results of symplectic geometry, especially those related to moment maps, do have analogues in the hyperkähler world. The prototype is the hyperkähler quotient construction [15], and more recent examples include hypertoric varieties [3] and cutting [9].

In this article we shall explore a hyperkähler analogue of Guillemin, Jeffrey and Sjamaar’s construction of symplectic implosion [13]. This may be viewed as an abelianisation procedure: given a symplectic manifold \( M \) with a Hamiltonian action of a compact group \( K \), the implosion \( M_{\text{impl}} \) is a new symplectic space with an action of the maximal torus \( T \) of \( K \), such that the symplectic reductions of \( M_{\text{impl}} \) by \( T \) agree with the reductions of \( M \) by \( K \). However the implosion is usually not smooth but is a singular space with a stratified symplectic structure. The implosion of the cotangent bundle \( T^*K \) acts as a universal object here; implosions of general Hamiltonian \( K \)-manifolds may be defined using the symplectic implosion \( (T^*K)_{\text{impl}} \). This space also has an algebro-geometric description as the geometric invariant theory quotient of \( K_\mathbb{C} \) by a maximal unipotent subgroup \( N \).

In [7] we introduced a hyperkähler analogue of the universal implosion in the case of \( SU(n) \) actions. The construction proceeds via quiver diagrams, and produces a stratified hyperkähler space \( Q \). The hyperkähler strata can be described in terms of open sets in complex symplectic quotients of the cotangent bundle of \( K_\mathbb{C} = SL(n, \mathbb{C}) \) by subgroups containing commutators of parabolic subgroups. There is a maximal torus action, and hyperkähler quotients by this action yield
not single complex coadjoint orbits but rather their canonical affine completions which are Kostant varieties.

In this article, we shall develop some of the ideas of [7], focusing on some aspects, such as toric geometry and gauge theory constructions, which may generalise to the case of an arbitrary compact group $K$. In particular, we shall show the existence in the case $K = SU(n)$ of a hypertoric variety inside the implosion, which has a natural description in terms of quivers. This is a hyperkähler analogue of the result of [13] that the universal symplectic implosion $(T^*K)^{\text{impl}}$ naturally contains the toric variety associated to a positive Weyl chamber for $K$.

The layout of the paper is as follows. In §1 we review the theory of symplectic implosion described in [13], and in §2 we recall how hyperkähler implosion for $K = SU(n)$ is introduced in [7]. In §3 we recall some of the theory of hypertoric varieties and describe a hypertoric variety which maps naturally to the universal hyperkähler implosion $Q$ for $K = SU(n)$. In §4 we recall the stratification given in [7] of $Q$ into strata which are hyperkähler manifolds, and in §5 we refine this stratification to obtain strata $Q/\sim_C$ which are not hyperkähler but which reflect the group structure of $K = SU(n)$ and can be indexed in terms of Levi subgroups and nilpotent orbits in the complexification $K_C$ of $K$. In §6, §7 and §8 we use Jordan canonical form to describe open subsets of the refined strata by putting their quivers into standard forms. Finally in §9 we explore briefly the relationship between the finite-dimensional picture of the universal hyperkähler implosion $Q$ for $K = SU(n)$ and an infinite-dimensional point of view involving the Nahm equations.

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1. Symplectic implosion

Our study of hyperkähler implosion in [7] was motivated by the theory of symplectic implosion, due to Guillemin, Jeffrey and Sjamaar [13]. For this we start with a symplectic manifold $M$ with a Hamiltonian symplectic action of a compact Lie group $K$ with maximal torus $T$. If $\lambda$ is a central element of $\mathfrak{t}^*$ the symplectic reduction $M/\mu^{-1}(\lambda)K$ is $\mu^{-1}(\lambda)/K$ where $\mu : M \rightarrow \mathfrak{t}^*$ is the moment map for the action of $K$ on $M$. For a general element $\lambda \in \mathfrak{t}^*$, we define the symplectic reduction $M/\mu^{-1}(\lambda)K$ to be the space $(M \times \mathbb{O}_{-\lambda})/\mu^{-1}(\lambda)K$, where $\mathbb{O}_{-\lambda}$ is the coadjoint orbit of $K$ through $\lambda$ with the standard Kirillov-Kostant-Souriau symplectic structure. This reduction may be identified with $\mu^{-1}(\lambda)/\text{Stab}_K(\lambda)$ where $\text{Stab}_K(\lambda)$ is the stabiliser of $\lambda$ under the coadjoint action of $K$.
The imploded space $M_{\text{impl}}$ is a stratified symplectic space with a Hamiltonian action of the maximal torus $T$ of $K$, such that

\[(1.1) \quad M \sslash^s K = M_{\text{impl}} \sslash^s T \]

for all $\lambda$ in the closure $t^*_+$ of a fixed positive Weyl chamber in $t^*$.

The key example is the implosion of the cotangent bundle $T^*K$. Now $T^*K$ carries a $K \times K$ action, which we can think of as commuting left and right actions of $K$. The left action is $(k, \xi) \mapsto (hk, \xi)$ while the right action is $(k, \xi) \mapsto (kh^{-1}, \text{Ad}(h) \xi)$. The moment maps for the left and right actions are $(k, \xi) \mapsto -\text{Ad}(k) \xi$ and $(k, \xi) \mapsto \xi$ respectively. We shall implode $T^*K$ with respect to the right action.

Explicitly, $(T^*K)_{\text{impl}}$ is obtained from $K \times t^*_+$, by identifying $(k_1, \xi)$ with $(k_2, \xi)$ if $k_1, k_2$ are related by the action of an element of the commutator subgroup of $\text{Stab}_K(\xi)$. Thus if $\xi$ is in the interior of the chamber, its stabiliser is a torus and no collapsing occurs, and an open dense subset of $(T^*K)_{\text{impl}}$ is just the product of $K$ with the interior of the Weyl chamber. Now symplectic reduction by the right action of $T$ at level $\lambda$ (in the closed positive Weyl chamber) will fix $\xi$ to be $\lambda$, and collapse by the product of $T$ with the commutator subgroup of $\text{Stab}_K(\lambda)$, which is equivalent to collapsing by $\text{Stab}_K(\lambda)$. Now we have

\[(T^*K)_{\text{impl}} \sslash^s T = K / \text{Stab}_K(\lambda) = O_\lambda = (T^*K) \sslash^s_k K\]

as required. $(T^*K)_{\text{impl}}$ inherits a Hamiltonian $K \times T$-action from the Hamiltonian $K \times K$-action on $T^*K$. This gives us a universal implosion, in the sense that the implosion $M_{\text{impl}}$ of a general symplectic manifold $M$ with a Hamiltonian $K$-action can be obtained as the symplectic reduction $(M \times (T^*K)_{\text{impl}}) \sslash^s T$.

It is also shown in [13] that the implosion $(T^*K)_{\text{impl}}$ may be embedded in the complex affine space $E = \oplus V_\varpi$, where $V_\varpi$ is the $K$-module with highest weight $\varpi$. and we take the sum over a minimal generating set for the monoid of dominant weights. We denote a highest weight vector of $V_\varpi$ by $v_\varpi$. In this picture, the symplectic implosion may be realised as the closure $K_{Cv}$, where $v = \sum v_\varpi$ is the sum of the highest weight vectors, and $K_C$ denotes the complexification of $K$.

In terms of the Iwasawa decomposition $K_C = KAN$ we have that the maximal unipotent subgroup $N$ is the stabiliser of $v$, so an open dense set in the implosion is $K_{Cv} = K_C/N$. Taking the closure gives lower-dimensional strata in the implosion, which may be identified with quotients $K_C/[P,P]$ where $P$ ranges over parabolic subgroups of $K_C$. Of course, taking $P$ to be the Borel $B$ gives the top stratum $K_C/N = K_C/[B,B]$. In fact the full implosion may be identified with
the Geometric Invariant Theory (GIT) quotient of \( K_C \) by the nonreductive group \( N \):

\[
K_C // N = \text{Spec}(\mathcal{O}(K_C)^N),
\]

This may also be viewed as the canonical affine completion of the quasi-affine variety \( K_C // N \). (We refer to [11] for background on nonreductive GIT quotients).

Using the Iwasawa decomposition as above, and recalling that \( T_C = TA \), we see that \( K_C v = KAv = K(T_Cv) \), the sweep under the compact group \( K \) of a toric variety \( X = T_Cv \). As \( T_C \) normalises \( N \), we have that \( N \) stabilises every point in \( X \); in fact \( X \) is the fixed point set \( E_N \) for the action of \( N \) on the vector space \( E \). The action of the compact torus \( T \) defines a moment map \( \mu_T : X \to t^* \) whose image is (minus) \( t^*_+ \), so \( -t^*_+ \) is the Delzant polytope for the toric variety \( X \). Equation (6.6) in [13] defines a \( T \)-equivariant map \( s : t^*_+ \to X \) which is a section for \( -\mu_T \). The map \( s \) extends to a \( K \times T \)-equivariant map \( K \times t^*_+ \to KX \), which induces a homeomorphism from \( (T^*K)_{\text{impl}} \) onto \( K_C v \).

Recall that the moment map for the left \( K \) action on \( T^*K \) is

\[
\mu_K : (k, \xi) \mapsto -\text{Ad}(k)\xi
\]

Note that two points \( (k_1, \xi), (k_2, \xi) \) in \( T^*K \) with the same \( k^* \) coordinate have the same image under \( \mu_K \) if and only if \( k_1 k_2^{-1} \in \text{Stab}_K(\xi) \). In particular two points of \( K \times t^* \) which are identified in the implosion will have the same image under \( \mu_K \), so this map descends to the implosion.

We have a commutative diagram

\[
\begin{array}{ccc}
(T^*K)_{\text{impl}} & \overset{\mu_K}{\longrightarrow} & t^*/K \\
\downarrow & & \parallel \\
X & \overset{-\mu_T}{\longrightarrow} & t^*/W
\end{array}
\]

where the left vertical arrow is induced by \( (k, \xi) \mapsto s(\xi) \), and the rightmost arrow in each row is the obvious quotient map.

In [11] we introduced a new model for the symplectic implosion for \( K = SU(n) \), in terms of symplectic quivers. These are diagrams

\[
(1.2) \quad 0 = V_0 \overset{\alpha_0}{\longrightarrow} V_1 \overset{\alpha_1}{\longrightarrow} V_2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{r-2}}{\longrightarrow} V_{r-1} \overset{\alpha_{r-1}}{\longrightarrow} V_r = \mathbb{C}^n.
\]

where \( V_i \) is a vector space of dimension \( n_i \). The group \( \prod_{i=1}^{r-1} \text{SL}(V_i) \) acts on quivers by

\[
\alpha_i \mapsto g_{i+1} \alpha_i g_{i}^{-1} \quad (i = 1, \ldots, r - 2),
\]

\[
\alpha_{r-1} \mapsto \alpha_{r-1} g_{r-1}^{-1}.
\]

There is also of course a commuting action of \( GL(n, \mathbb{C}) = GL(V_r) \) by left multiplication of \( \alpha_{r-1} \). We considered the GIT quotient of the space of quivers by \( \prod_{i=1}^{r-1} \text{SL}(V_i) \), focusing particularly on the full flag case when \( n_i = i \) for all \( i \). It turns out that such a quiver lies in a closed orbit if and only if, for each \( i \) we have
(i) \( \alpha_i \) is injective, or
(ii) \( V_i = \text{im} \alpha_{i-1} \oplus \ker \alpha_i \).

We may now decompose \( C_i = \ker \alpha_i \oplus C^{m_i} \), where \( C^{m_i} = \C^i \) if \( \alpha_i \) is injective and we take \( C^{m_i} = \text{im} \alpha_{i-1} \) otherwise. This defines a decomposition of the quiver into two subquivers; for one subquiver the maps are all injective while for the other they are all zero. We may therefore focus on the injective quiver. As explained in §4 of [7], we may contract any edges of this quiver where the maps are isomorphisms. More precisely, if \( m_i = m_{i-1} \) then we have \( m_i \leq i - 1 < i \), so we actually have a \( GL(m_i) \) action on \( C^{m_i} \) and the isomorphism \( C^{m_i - 1} \to C^{m_i} \) may be set to be the identity, so this edge of the quiver may be removed. After this process the dimensions of the spaces in the injective quiver are given by a strictly increasing sequence of integers ending with \( n \).

The upshot is that we have a stratification of the GIT quotient by \( \prod_{i=2}^{n-1} SL(i) \) of the space of full flag quivers. There are \( 2^{n-1} \) strata, indexed by the strictly increasing sequences of positive integers ending with \( n \), or equivalently by the ordered partitions of \( n \). Moreover, the injectivity property makes it easy to analyse the structure of each stratum. For we may now use the action of \( \prod_{i=2}^{n-1} SL(i) \) and \( SL(n) \) to put the \( \alpha_i \) into a standard form where all entries are zero except for the \((j, j)\) entries \((j = 1, \ldots, m_i)\), which equal 1. The freedom involved in putting the \( \alpha_i \) into this standard form is exactly an element of the commutator \([P, P]\), where \( P \) is the parabolic subgroup of \( SL(n) \) corresponding to the ordered partition of \( n \). We conclude that the strata can be identified with \( SL(n)/[P, P] \). In fact, the full GIT quotient may be identified with the symplectic implosion for \( SU(n) \) and the strata are just the strata of the implosion discussed above.

We may also realise the toric structure discussed above in this model. For we can instead put \( \alpha_i \) into a slightly different standard form where the \((j, j)\) entries \((j = 1, \ldots, m_i)\) can now equal a nonzero scalar \( \sigma_i \), not necessarily 1. This standard form is now preserved by an element of the parabolic \( P \), and the \( \sigma_i \) define an algebraic torus of dimension given by the length of the injective quiver (equivalently, we are considering the fibration \( T_{\C}^{n-1} \to SL(n)/[P, P] \to SL(n)/P \) for each stratum). These tori fit together to form the toric variety. Now the generalised flag variety \( SL(n)/P \) is a homogeneous space for the compact group \( SU(n) \), so we see again that the sweep of the toric variety under the \( SU(n) \) action is the full implosion. Alternatively, we may allow the entries \( \sigma_i \) to depend on \( j \) as well as \( i \). Now we have an action on such configurations of the product of the maximal tori of \( SL(m_i) \), and this action may be used to bring the quiver into the form above where \( \sigma_i \) depends only on \( i \).
2. Hyperkähler implosion

In [7] a hyperkähler analogue of the symplectic implosion was introduced for the group $K = SU(n)$. Motivated by the quiver model for symplectic implosion described in the preceding section, we look at quiver diagrams of the following form:

\[
0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{r-2}} V_{r-1} \xrightarrow{\alpha_{r-1}} V_r = \mathbb{C}^n
\]

where $V_i$ is a complex vector space of complex dimension $n_i$ and $\alpha_0 = \beta_0 = 0$. The space $M$ of quivers for fixed dimension vector $(n_1, \ldots, n_r)$ is a flat hyperkähler vector space.

There is a hyperkähler action of $U(n_1) \times \cdots \times U(n_r)$ on this space given by

\[
\alpha_i \mapsto g_{i+1} \alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \quad (i = 1, \ldots, r - 1),
\]

with $g_i \in U(n_i)$ for $i = 1, \ldots, r$.

Let $H$ be the subgroup, isomorphic to $U(n_1) \times \cdots \times U(n_{r-1})$, given by setting $g_r = 1$, and let $M/H$ be the residual action of $M/H$. In [7] a hyperkähler analogue of the symplectic implosion was introduced for the group $K = SU(n)$. Motivated by the quiver model for symplectic implosion described in the preceding section, we look at quiver diagrams of the following form:

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There is a hyperkähler action of $U(n_1) \times \cdots \times U(n_r)$ on this space given by

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\alpha_i \mapsto g_{i+1} \alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \quad (i = 1, \ldots, r - 1),
\]

with $g_i \in U(n_i)$ for $i = 1, \ldots, r$.

Let $H$ be the subgroup, isomorphic to $U(n_1) \times \cdots \times U(n_{r-1})$, given by setting $g_r = 1$, and let $H = SU(n_1) \times \cdots \times SU(n_{r-1}) \leq H$.

**Definition 2.1.** The universal hyperkähler implosion for $SU(n)$ will be the hyperkähler quotient $Q = M/H$, where $M/H$ are as above with $r = n$ and $n_j = j$, $(j = 1, \ldots, n)$, (that is, the case of full flag quivers).

The hyperkähler moment map equations for the $H$-action are (in the full flag case)

\[
(2.2) \quad \alpha_i \beta_i - \beta_{i+1} \alpha_{i+1} = \lambda_i^C I \quad (0 \leq i \leq n - 2)
\]

where $\lambda_i^C \in \mathbb{C}$ for $1 \leq i \leq n - 1$, and

\[
(2.3) \quad \alpha_i \alpha_i^* - \beta_i^* \beta_i + \beta_{i+1} \beta_{i+1} - \alpha_{i+1} \alpha_{i+1} = \lambda_i^R I \quad (0 \leq i \leq n - 2),
\]

where $\lambda_i^R \in \mathbb{R}$ for $1 \leq i \leq n - 1$. Now $Q$ has a residual action of $(S^1)^{n-1} = H/H$ as well as an action of $SU(n_r) = SU(n)$. In §4 we will identify $(S^1)^{n-1}$ with $T$, the maximal torus of $SU(n)$. There is also an $Sp(1) = SU(2)$ action which is not hyperkähler but rotates the complex structures. Using the standard theory relating symplectic and GIT quotients, we have a description of $Q = M/H$, as the quotient (in the GIT sense) of the subvariety defined by the complex moment map equations (2.2) by the action of

\[
H_C = \prod_{i=1}^{n-1} SL(n_i, \mathbb{C})
\]

\[
(2.4) \quad \alpha_i \mapsto g_{i+1} \alpha_i g_i^{-1}, \quad \beta_i \mapsto g_i \beta_i g_{i+1}^{-1} \quad (i = 1, \ldots, n - 2),
\]

\[
(2.5) \quad \alpha_{n-1} \mapsto \alpha_{n-1} g_{n-1}^{-1}, \quad \beta_{n-1} \mapsto g_{n-1} \beta_{n-1},
\]

where $g_i \in SL(n_i, \mathbb{C})$. 
We introduce the element \( X = \alpha_{n-1}\beta_{n-1} \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \), which is invariant under the action of \( \prod_{i=1}^{n-1} \text{GL}(n_i, \mathbb{C}) \) and transforms by conjugation under the residual \( \text{SL}(n, \mathbb{C}) = \text{SL}(n_r, \mathbb{C}) \) action on \( Q \). We thus have a \( T_C \)-invariant and \( \text{SL}(n, \mathbb{C}) \)-equivariant map \( Q \to \mathfrak{sl}(n, \mathbb{C}) \) given by:

\[
(\alpha, \beta) \mapsto X - \frac{1}{n} \text{tr}(X) I_n
\]

where \( I_n \) is the \( n \times n \)-identity matrix. In fact this is the complex moment map for the residual \( SU(n) \) action on \( Q \). It is shown in [7] that \( X \) satisfies an equation \( X(X + \nu_1) \ldots (X + \nu_{n-1}) = 0 \) where \( \nu_i = \sum_{j=i}^{n-1} \lambda_j \).

This generalises the equation \( X^n = 0 \) in the quiver construction of the nilpotent variety in [17].

In general it is useful to compare our construction with that in [17]. There one performs a hyperkähler quotient by \( \tilde{H} \), rather than \( H \), so all \( \lambda_i \) are zero. In our situation the \( \lambda_i \) are not constrained to be zero, and in fact give the value of the complex moment map for the residual \( T \) action on \( Q \).

In [7] we first analysed the points of the implosion that give closed orbits for the \( T_C \) action, or equivalently, quivers that satisfy the equations (2.2) and give closed orbits for the action of \( \tilde{H}_C \) as well as \( H_C \). Such quivers can be split into a sum of a quiver with \( \alpha_i \) injective and \( \beta_i \) surjective, and a collection of quivers where the non-zero maps are isomorphisms.

In general one must consider quivers that satisfy (2.2) and give a closed orbit for the \( H_C \) action but not necessarily for the \( \tilde{H}_C \) action. However for each such quiver we may rotate complex structures so that the closed orbit condition is actually satisfied for a larger subgroup of \( \tilde{H}_C \). In this way we obtain a stratification for the implosion.

Using the methods that appeared in the analysis of the symplectic implosion, we described in [7] §7 the strata for the hyperkähler implosion in terms of complex-symplectic quotients of \( T^* \text{SL}(n, \mathbb{C}) \) by extensions of abelian groups by commutators of parabolics. In more detail, we can follow the argument in the symplectic case to standardise the surjective maps \( \beta_i \) as \( (0 | I) \). The equations (2.2) now enable us to find \( \alpha_i \) in terms of \( \alpha_{i+1} \) and \( \lambda^{C}_{i+1} \). Now knowledge of (the tracefree part of) \( X = \alpha_{n-1}\beta_{n-1} \), together with the equations (2.2), enables us to work down the quiver inductively determining all the \( \alpha_i \). Further details of some of these arguments are given in §5 and §6, as well as in [7].

The universal hyperkähler implosion \( Q \) contains an open set which may be identified with \( \text{SL}(n, \mathbb{C}) \times_N \mathfrak{b} \), the complex-symplectic quotient of \( T^* \text{SL}(n, \mathbb{C}) \) by the maximal unipotent \( N \). This arises as the locus of full flag quivers with all \( \beta_i \) surjective. The full implosion \( Q \) may in fact be identified with the non-reductive GIT quotient \( (\text{SL}(n, \mathbb{C}) \times \mathfrak{b})//N \). The hyperkähler torus quotients of \( Q \) can be identified for any fixed complex structure with the complex-symplectic reductions of \( Q \) by the
complexified torus $T_C$ in the sense of GIT. That is, we take the GIT quotients with respect to $T_C$ of the level sets of the complex moment map for the action of $T$ on $Q$. These complex-symplectic reductions give us the Kostant varieties which are the subsets of $\mathfrak{sl}(n, \mathbb{C})$ obtained by fixing the eigenvalues. In particular torus reduction at level 0 gives the nilpotent variety. If, by contrast, we take the geometric (rather than GIT) complex-symplectic reduction at level 0 of $SL(n, \mathbb{C}) \times_N \mathfrak{h}$, we obtain $(SL(n, \mathbb{C}) \times_N \mathfrak{n})/T_C$, which is the Springer resolution $SL(n, \mathbb{C}) \times_R \mathfrak{n}$ of the nilpotent variety.

As in the symplectic case, the hyperkähler implosion is usually a singular stratified space. In fact the symplectic implosion may be realised as the fixed point set of a circle action on the hyperkähler implosion, so if the latter is smooth then so is the former, which implies by results of [13] that $K$ is, up to covers, a product of copies of $SU(2)$. If $K = SU(2)$ the implosion is just flat $\mathbb{H}^2$.

3. Hypertoric varieties

Classical toric varieties arise as symplectic quotients of $\mathbb{C}^d$ by a subtorus of $(S^1)^d$, and have a symplectic action of a compact torus $T$ whose real dimension is half that of the toric variety [12], [10]. The image of the toric variety under the associated moment map is called the Delzant polytope, and the toric variety is determined up to $T$-equivariant isomorphism by $T$ and this polytope in $\mathfrak{t}^*$. We recall that a hypertoric (or toric hyperkähler) variety is, by analogy, obtained as a hyperkähler quotient of flat quaternionic space $\mathbb{H}^d$ by a subtorus $N$ of $T^d$. If the subtorus is of codimension $n$ in $T^d$, the associated hypertoric variety $\mathbb{H}^d//N$ has real dimension $4n$ and has a hyperkähler action of $T^n \cong T^d//N$. The hyperkähler moment map for this action is a surjection onto $\mathbb{R}^{3n}$ and much of the geometry of the hypertoric is encoded in a collection of codimension 3 affine subspaces (the flats) in $\mathbb{R}^{3n}$. These play in some respects a role analogous to that of the hyperplanes giving the faces of the Delzant polytope for classical toric varieties. In particular the fibre of the moment map over a point in $\mathbb{R}^{3n}$ is a torus determined by the collection of flats passing through that point. We refer the reader to [3], [14] for further background on hypertorics. We want to relate the hyperkähler implosion $Q$ to the hypertoric variety associated to the arrangement of flats induced by the hyperplane arrangement given by the root planes in the Lie algebra $\mathfrak{t}$ of the maximal torus $T$ of $K = SU(n)$. 
**Definition 3.1.** Let $M_T$ be the subset of $M$ consisting of all hyperkähler quivers of the form

\[
\alpha_k = \begin{pmatrix} \nu^k_1 & 0 & 0 & \cdots & 0 \\ 0 & \nu^k_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \nu^k_k & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}
\]

and

\[
\beta_k = \begin{pmatrix} \mu^k_1 & 0 & 0 & \cdots & 0 \\ 0 & \mu^k_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & \mu^k_k & 0 \end{pmatrix}
\]

for some $\nu^k_i, \mu^k_i \in \mathbb{C}$. Recall from [7] that we use $M_{\text{hks}}$ to denote the set of hyperkähler stable quivers, that is, those that after an appropriate rotation of complex structures have all $\alpha_i$ injective and all $\beta_i$ surjective. Let $M_{\text{hks}}^T = M_{\text{hks}} \cap M_T$ be the subset of $M_{\text{hks}}$ consisting of all hyperkähler quivers of the form above; thus $M_{\text{hks}}^T$ consists of all quivers of the form above such that $\mu^k_i$ and $\nu^k_i$ are not simultaneously zero for any pair $(i, k)$ with $1 \leq i \leq k < n$.

Note that each of the compositions $\alpha_k \beta_k$, $\beta_k \alpha_k$, $\alpha_k^* \alpha_k$, $\alpha_k^* \beta_k$, $\beta_k^* \alpha_k$, and $\beta_k^* \beta_k$ is a diagonal matrix, so that for quivers of this form the hyperkähler moment map equations for the action of $H = \prod_{k=1}^{n-1} SU(k)$ reduce to the hyperkähler moment map equations for the action of its maximal torus

\[ T_H = \prod_{k=1}^{n-1} T_k \]

where $T_k$ is the standard maximal torus in $SU(k)$. Moreover two hyperkähler stable quivers of this form satisfying the hyperkähler moment map equations lie in the same orbit for the action of $H$ if and only if they lie in the same orbit for the action of its maximal torus $T_H$. Thus we get a natural map

\[ \iota : M_T \sslash T_H \to Q = M \sslash H \]

which restricts to an embedding

\[ \iota : Q_{\text{hks}}^T \to Q \]

where $Q_{\text{hks}}^T = M_{\text{hks}}^T \sslash T_H$.

**Remark 3.2.** Note that $M_T = \bigoplus_{k=1}^{n-1} \mathbb{H}^k = \mathbb{H}^{n(n-1)/2}$ is a (flat) hypertoric variety with respect to the action of the standard maximal torus $T_H = (S^1)^{n(n-1)/2}$ of $H = \prod_{k=1}^{n-1} U(k)$. The associated arrangement of flats in $\mathbb{R}^{3n(n-1)/2} = \mathbb{R}^3 \otimes \mathbb{R}^{n(n-1)/2}$ is just that induced by the hyperplane arrangement given by the coordinate hyperplanes in
\[ t_H = \mathbb{R}^{n(n-1)/2}. \] Thus \( M_T \parallel T_H \) is a hypertoric variety for the induced action of

\[
T_H / T_H \cong \prod_{k=1}^{n-1} U(k) / SU(k) = (S^1)^{n-1}.
\]

Moreover we can identify \( T_H / T_H \) with the standard maximal torus \( T \) of \( K = SU(n) \) in such a way that the induced action of \( T_H / T_H \) on \( Q_T^{\text{hks}} \) coincides with the restriction to \( T \) of the action of \( K \) on \( Q_T^{\text{hks}} \) embedded in \( Q = M \parallel H \) as above, since

\[
\begin{pmatrix}
t_1^k & 0 & 0 & \cdots & 0 \\
0 & t_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & t_k^k
\end{pmatrix}
\begin{pmatrix}
\nu_1^k & 0 & 0 & \cdots & 0 \\
0 & \nu_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & \nu_k^k
\end{pmatrix}
= 
\begin{pmatrix}
\mu_1^k & 0 & 0 & \cdots & 0 \\
0 & \mu_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & \mu_k^k
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
t_1^k & 0 & 0 & \cdots & 0 \\
0 & t_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & t_k^k
\end{pmatrix}
\begin{pmatrix}
\mu_1^k & 0 & 0 & \cdots & 0 \\
0 & \mu_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & \mu_k^k
\end{pmatrix}
= 
\begin{pmatrix}
\mu_1^k & 0 & 0 & \cdots & 0 \\
0 & \mu_2^k & 0 & \cdots & 0 \\
\vdots & & & & \\
0 & \cdots & 0 & 0 & \mu_k^k
\end{pmatrix}
\]

for any \( \nu_i^k, \mu_i^k \) and \( t_i^k \) in \( \mathbb{C} \). Here if \( (s_1, \ldots, s_{n-1}) \) are the standard coordinates on the Lie algebra \( \mathbb{R}^{n-1} \) of \( (S^1)^{n-1} \) and \( (\tau_1, \ldots, \tau_n) \) are the standard coordinates of the Lie algebra \( \mathbb{R}^n \) of the maximal torus of \( U(n) \) consisting of the diagonal matrices, then we identify \( \mathbb{R}^{n-1} \) with the subspace of \( \mathbb{R}^n \) defined by \( \tau_1 + \cdots + \tau_n = 0 \) via the relationship \( s_j = \tau_{j+1} + \cdots + \tau_n \) for \( 1 \leq j \leq n-1 \). With respect to this identification, \( M_T \parallel H \) becomes the hypertoric variety for \( T \) associated to the hyperplane arrangement in its Lie algebra \( \mathfrak{t} \) given by the root planes.

**Remark 3.3.** The action of \( T = (S^1)^n \) on \( M_T \parallel H \) extends naturally to the ‘quaternionification’ \( (\mathbb{H}^*)^{n-1} \) of \( (S^1)^{n-1} \). The open subset \( Q_T^{\text{hks}} = M_T^{\text{hks}} \parallel H \) is then a single \( (\mathbb{H}^*)^{n-1} \)-orbit in \( M_T \parallel H \).
Remark 3.4. Recall from §1 that the universal symplectic implosion \((T^*K)_{\text{impl}}\) for \(K = SU(n)\) can be embedded in the representation

\[ E = \bigoplus_{j=1}^{n-1} \wedge^j \mathbb{C}^n \]

of \(K \times T\) as the closure \(K_C v\) of the \(K_C\)-orbit of

\[ v = \sum_{j=1}^{n-1} v_j \in E \]

where \(v_j\) is a highest weight vector in the irreducible representation \(\wedge^j \mathbb{C}^n\) of \(K\). Moreover \(K_C v = K(T_C v)\) is the \(K\)-sweep of the toric variety \(T_C v\) whose \(T\)-moment map image (or Delzant polytope) is the positive Weyl chamber \(t^* + t\sim = t\). In the hyperkähler situation we have observed in the last remark that the open subset \(Q^{hks}_T\) of the hypertoric variety \(M_T \sslash H\) can be embedded in \(Q\) as a single orbit \(T\mathbb{H} w\) of the ‘quaternionification’ \(T\mathbb{H} = (\mathbb{H}^*)^{n-1}\) of the maximal torus \(T = (S^1)^{n-1}\) of \(K = SU(n)\). We cannot similarly extend the action of the non-abelian group \(K\) to the action of a ‘quaternionification’ in an obvious way, so we obtain no exact parallel of the description of the universal symplectic implosion \((T^*K)_{\text{impl}}\) as \(K_C v\). However it will follow from the standard forms for quivers in \(Q\) which will be described in §7 below that, just as \((T^*K)_{\text{impl}} = K(T_C v)\) has no proper \(K\)-invariant closed subset which contains the \(T_C\)-orbit \(T_C v\), so the universal hyperkähler implosion \(Q\) for \(K = SU(n)\) has no proper closed subset which contains the \(T\mathbb{H}\)-orbit \(T\mathbb{H} w\) and is invariant under the induced action for \(K_C\) for every choice of complex structure on \(Q\).

4. Stratifying the universal hyperkähler implosion into hyperkähler strata

The universal hyperkähler implosion \(Q = M \sslash H\) for \(K = SU(n)\) is a singular space with a stratification into locally closed hyperkähler submanifolds \(Q_{(S,\delta)}\) (cf. [7] Theorem 6.15). These strata \(Q_{(S,\delta)}\) can be indexed by subsets

\[ S = \{(i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p)\} \]

of \(\{1, \ldots, n\} \times \{1, \ldots, n\}\) with \(i_1, \ldots, i_p\) distinct and \(j_1 < j_2 < \cdots < j_p\), and sequences \(\delta = (d_1, \ldots, d_p)\) of strictly positive integers such that if \(1 \leq k \leq n\) then

\[ m_k = k - \sum_{\substack{1 \leq h \leq p \\text{ such that } i_h \leq k < j_h \\text{ and } i_h \leq k < j_h}} d_h \]
satisfies $0 = m_0 \leq m_1 \leq \cdots \leq m_n = n$. The open stratum $Q_{(\emptyset, \emptyset)} = Q^{hks}$, which is indexed by the empty set $S = \emptyset$ and the empty sequence $\delta = \emptyset$, consists of those elements of $Q = M//H$ represented by hyperkähler stable quivers.

More generally, for any $S$ and $\delta$ as above, the stratum $Q_{(S, \delta)}$ is the image of a hyperkähler embedding into $Q$ of a hyperkähler modification $\hat{Q}^{hks}_1$ (in the sense of Definition 4.3 below, following [9]) of the open subset $Q^{hks}_1$ represented by hyperkähler quivers in the hyperkähler quotient

$$Q_1 = M_1//H_S$$

where $M_1$ is the space of quivers of the form

$$(4.1) \quad 0 \xrightarrow{\alpha_0} \mathbb{C}^{m_1} \xrightarrow{\alpha_1} \mathbb{C}^{m_2} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} \mathbb{C}^{m_{n-1}} \xrightarrow{\alpha_{n-1}} \mathbb{C}^{m_n} = \mathbb{C}^n$$

and $H_S$ is the subgroup of

$$\prod_{k=1}^{n-1} U(m_k) \subseteq \tilde{H} = \prod_{k=1}^{n-1} U(k)$$

defined as follows.

**Definition 4.2.** To any set $S$ of pairs $(i, j)$ with $i, j \in \{1, \ldots, n\}$ we can associate a subtorus $T_S$ of $T = (S^1)^{n-1}$ such that the Lie algebra of $T_S$ is generated by the vectors $e_{ij} = (0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0)$, which have 1 in places $i, \ldots, j - 1$ and zero elsewhere, where $i, j$ range over all pairs $(i, j) \in S$ with $i < j$. Now consider the short exact sequence

$$1 \to H = \prod_{k=1}^{n-1} SU(m_k) \to \prod_{k=1}^{n-1} U(m_k) \xrightarrow{\phi} T \to 1,$$

where $\phi$ is the obvious product of determinant maps, and define $H_S$ to be the preimage

$$H_S = \phi^{-1}(T_S)$$

of $T_S$ in $\prod_{k=1}^{n-1} U(m_k)$.

**Definition 4.3.** Motivated by the concept of hyperkähler modification introduced in [9], we define $\hat{Q}^{hks}_1$ as

$$\hat{Q}^{hks}_1 = (Q^{hks}_1 \times (\mathbb{H} \setminus \{0\})^\ell)//(S^1)^\ell.$$ 

Here $\ell = |L|$ is the size of the set

$$L = \{(h, k) : 1 \leq h \leq p, \ i_h \leq k < j_h - 1\}.$$

The action of $(S^1)^\ell$ on $(\mathbb{H} \setminus \{0\})^\ell$ is the standard one, while the action of $(S^1)^\ell$ on $Q_1$ is given by the homomorphism

$$(S^1)^\ell \to T = (S^1)^{n-1}$$
whose restriction to the copy of $S^1$ in $(S^1)^l$ labelled by $(h, k) \in L$ sends the standard generator of the Lie algebra of $S^1$ to the vector

$$e_{k+1,jh} = (0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0)$$

in the Lie algebra of $T = (S^1)^{n-1}$ which has 1 in places $k+1$ to $jh-1$ and 0 elsewhere.

The stratum $Q_{(S, \delta)}$ is the image of a hyperkähler embedding

$$\hat{Q}^{h_{ks}}_l \to Q$$

which is SU(2)-equivariant and is defined as follows. Consider a quiver $Q$ together with an element $(\gamma^{(h)})$ of $\mathbb{H}^l$ such that $\gamma^{(h)}_k = \alpha^{(h)}_k + j\beta^{(h)}_k$ for $1 \leq h \leq p$ and $i_h \leq k < j_h - 1$, satisfying the $H$-hyperkähler moment map equations

$$\alpha_i \beta_i - \beta_{i+1} \alpha_{i+1} = \lambda^{C}_{i+1} I,$$

$$\alpha_i \alpha^*_i - \beta^*_i \beta_i + \beta_{i+1} \beta^*_{i+1} - \alpha^*_i \alpha_{i+1} = \lambda^{R}_{i+1} I,$$

for $1 \leq i \leq n-1$, where $\sum_{k \neq i_h}^{j_h} \lambda^C_k = 0$ and $\sum_{k \neq i_h}^{j_h} \lambda^R_k = 0$ for $1 \leq h \leq q$, and the $(S^1)^l$-hyperkähler moment map equations

$$\alpha^{(h)}_k \beta^{(h)}_k = \lambda^C_{k+1} + \lambda^C_{k+2} + \cdots + \lambda^C_{j_h-1} \text{ and}$$

$$|\alpha^{(h)}_k|^2 - |\beta^{(h)}_k|^2 = \lambda^R_{k+1} + \lambda^R_{k+2} + \cdots + \lambda^R_{j_h-1}$$

for $1 \leq h \leq p$ and $i_h \leq k < j_h - 1$. Our embedding takes the $H \times (S^1)^l$-orbit of this configuration to the $H$-orbit of the quiver

$$(4.4) \quad 0 \delta_{\beta_0} \cdots \cong C^{m_k} \oplus \bigoplus_{h: i_h \leq k < j_h} C^{d_h} \cong \cdots \cong C^n$$

which is the orthogonal direct sum of $Q_{(S, \delta)}$ with the quivers given for $1 \leq h \leq p$ by

$$C^{d_h} \cong C^{d_h} \cdots \cong C^{d_h} \cong C^{d_h} \cong C^{d_h} \cong C^{d_h}$$

in the places $i_h, i_h + 1, \ldots, j_h - 1$. Here the maps $\alpha^{(h)}_k, \beta^{(h)}_k$, for $i_h \leq k < j_h - 1$, are multiplication by the complex scalars, also denoted by $\alpha^{(h)}_k, \beta^{(h)}_k$, that satisfy $\alpha^{(h)}_k + j\beta^{(h)}_k = \gamma^{(h)}_k$.

**Remark 4.5.** Note that the stabiliser in $T = T_H / T_H$ of any $q \in Q_{(S, \delta)}$ is the subtorus $T_S$ of $T$ defined in Definition (4.2) which is the product $(S^1)^p$ of $p$ copies of $S^1$ where the $j$th copy of $S^1$ acts by scalar multiplication on the summand

$$C^{d_h} \cong C^{d_h} \cdots \cong C^{d_h} \cong C^{d_h} \cong C^{d_h}$$

of the quiver $q$. 
5. A REFINED STRATIFICATION OF THE UNIVERSAL HYPERKÄHLER IMPLOSION

In the last section we recalled the stratification of the universal hyperkähler implosion $Q = M/\!\!/H$ for $K = SU(n)$ into hyperkähler strata $Q_{(S,\delta)}$. In this section we will refine this stratification to obtain strata which are not in general hyperkähler but which reflect the structure of the group $K = SU(n)$; in particular we would like to find a description of the universal hyperkähler implosion which permits generalisation to other compact groups. First let us consider the hyperkähler moment map

$$\mu_{(S^1)^{n-1}} : Q \to (\mathbb{R}^3)^{n-1}$$

for the induced action of

$$T = (S^1)^{n-1} = \prod_{k=1}^{n-1} U(k)/SU(k) = \tilde{H}/H$$

on $Q = M/\!\!/H$. We are abusing notation slightly here by using the same symbol $T$ to denote both $(S^1)^{n-1}$ and the standard maximal torus of $K = SU(n)$. These tori are of course isomorphic; we will always make the particular choice of identification given in Remark 3.2, so that the restriction to $Q_{(S,\delta)}^{\hks}$ of the action of $T$ as a subgroup of $K$ agrees with the restriction of the action of $T$ identified with $(S^1)^{n-1}$. This hyperkähler moment map takes a quiver which satisfies the equations (2.2), (2.3) to the element of $t \otimes \mathbb{R}^3 = (\mathbb{R}^3)^{n-1} = (\mathbb{C} \oplus \mathbb{R})^{n-1}$ given by

$$(\lambda_1, \ldots, \lambda_{n-1}) = (\lambda_1^C, \lambda_1^R, \ldots, \lambda_{n-1}^C, \lambda_{n-1}^R).$$

We will define a stratification of $(\mathbb{R}^3)^{n-1}$ which we can pull back via the restriction of $\mu_{(S^1)^{n-1}}$ to each hyperkähler stratum $Q_{(S,\delta)}$ of $Q$.

**Definition 5.1.** If $(\lambda_1, \ldots, \lambda_{n-1}) \in (\mathbb{R}^3)^{n-1}$ there is an associated equivalence relation $\sim$ on $\{1, \ldots, n\}$ such that if $1 \leq i < j \leq n$ then

$$i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3.$$

There is thus a stratification of $(\mathbb{R}^3)^{n-1} = t \otimes \mathbb{R}^3$ into strata $(\mathbb{R}^3)_\sim^{n-1} = (t \otimes \mathbb{R}^3)_\sim$, indexed by the set of equivalence relations $\sim$ on $\{1, \ldots, n\}$, where

$$(\mathbb{R}^3)_\sim^{n-1} = \{(\lambda_1, \ldots, \lambda_{n-1}) \in (\mathbb{R}^3)^{n-1} : \text{ if } 1 \leq i < j \leq n \text{ then } i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3\}.$$
Remark 5.2. Under the identification of $T$ with $(S^1)^{n-1}$ given in Remark 3.2, this stratification of $(\mathbb{R}^3)^{n-1} = t \otimes \mathbb{R}^3$ is the tensor product with $\mathbb{R}^3$ of the stratification of $t$ associated to the hyperplane arrangement given by the root planes in $t$. Note also that an equivalence relation $\sim$ on $\{1, \ldots, n\}$ determines and is determined by a subgroup $K_\sim$ of $K = SU(n)$, where $K_\sim$ is the stabiliser in $K$ of any $(\lambda_1, \ldots, \lambda_{n-1}) \in t \otimes \mathbb{R}^3$ which lies in the stratum $(\mathbb{R}^3)^{n-1}_\sim$ of $(\mathbb{R}^3)^{n-1}$ identified with $t \otimes \mathbb{R}^3$ as in Remark 3.2.

Observe that each stratum $(\mathbb{R}^3)^{n-1}_\sim$ is an open subset of a linear subspace of the real vector space $(\mathbb{R}^3)^{n-1}$.

**Definition 5.3.** Given a hyperkähler stratum $Q_{(S, \delta)}$ of $Q$ as in §4, together with an equivalence relation $\sim$ on $\{1, \ldots, n\}$, define

$$Q_{(S, \delta, \sim)} = Q_{(S, \delta)} \cap \mu_{(S^1)^{n-1}}^{-1}((\mathbb{R}^3)^{n-1}_\sim),$$

that is, the inverse image of the stratum $(\mathbb{R}^3)^{n-1}_\sim$ in $(\mathbb{R}^3)^{n-1}$ under the restriction to $Q_{(S, \delta)}$ of the hyperkähler moment map $\mu_{(S^1)^{n-1}} : Q \to (\mathbb{R}^3)^{n-1}$.

**Remark 5.4.** We recall from [7] that, given a quiver which satisfies the complex moment map equations (2.2), we may decompose each space in the quiver into generalised eigenspaces $\ker(\alpha_i \beta_i - \tau I)^m$ of $\alpha_i \beta_i$. We showed that $\beta_i$ restricts to a map

$$(5.5) \quad \beta_i : \ker(\alpha_i \beta_i - \tau I)^m \to \ker(\alpha_{i-1} \beta_{i-1} - (\lambda_1^C + \tau) I)^m.$$  

Similarly $\alpha_i$ restricts to a map

$$(5.6) \quad \alpha_i : \ker(\alpha_{i-1} \beta_{i-1} - (\lambda_i^C + \tau) I)^m \to \ker(\alpha_i \beta_i - \tau I)^m.$$  

Moreover we showed the maps (5.5) and (5.6) are bijective unless $\tau = 0$. It follows that $\tau \neq 0$ is an eigenvalue of $\alpha_i \beta_i$ if and only if $\tau + \lambda_i^C \neq \lambda_j^C$ is an eigenvalue of $\alpha_{i-1} \beta_{i-1}$. Moreover $\alpha_i \beta_i$ has zero as an eigenvalue and $\alpha_i, \beta_i$ restrict to maps between the associated generalised eigenspace with eigenvalue 0 and the generalised eigenspace for $\alpha_{i-1} \beta_{i-1}$ associated to $\lambda_i^C$. One can deduce (cf. Lemma 5.14 of [7]) that

$$\alpha_{n-1} \beta_{n-1} - \frac{1}{n} \text{tr}(\alpha_{n-1} \beta_{n-1}) I_n \in \mathfrak{sl}(n, \mathbb{C})$$

now has eigenvalues $\kappa_1, \ldots, \kappa_n$, where

$$\kappa_j = \frac{1}{n} \left( \lambda_j^C + 2\lambda_j^C + \cdots + (j-1)\lambda_j^C - (n-j)\lambda_j^C - (n-j)\lambda_{j+1}^C - \cdots - \lambda_{n-1}^C \right).$$

In particular if $i < j$ then

$$\kappa_j - \kappa_i = \lambda_j^C + \lambda_{j+1}^C + \cdots + \lambda_{j-1}^C.$$  

We deduce that if $i \sim j$ then we have equality of the eigenvalues $\kappa_i$ and $\kappa_j$. 
We would like to find an indexing set for the subsets \( Q_{(S,\delta,\sim)} \) which reflects the group theoretic structure of \( K \). As we observed in Remark 5.2 the choice of \( \sim \) corresponds to the choice of a subgroup \( K_\sim \) of \( K \) which is the compact real form of a Levi subgroup of \( K_C \); this subgroup \( K_\sim \) is the centraliser of \( \mu_{(S^1)^{n-1}}(q) \in t \otimes \mathbb{R}^3 \) for any \( q \in Q_{(S,\delta,\sim)} \). Our next aim is to show that once \( \sim \) or equivalently \( K_\sim \) is chosen, the choice of \( (S,\delta) \) corresponds to the choice of a nilpotent adjoint orbit \( O \subseteq (\mathfrak{k}_\sim)_C \). We will see that if \( q \in Q_{(S,\delta,\sim)} \) then, for a generic choice of complex structure, when we decompose \( \mathbb{C}^n \) as the direct sum of the generalised eigenspaces of \( \alpha_{n-1} \beta_{n-1} - \frac{1}{n} \text{tr}(\alpha_{n-1} \beta_{n-1}) I_n \in \mathfrak{t}_C \), then the subgroup of \( K_C \) preserving this decomposition is conjugate to \( (K_\sim)_C \). In fact there is some \( g \in K_C \) such that this subgroup is \( g(K_\sim)_C g^{-1} \) and when we write the element

\[
\alpha_{n-1} \beta_{n-1} - \frac{1}{n} \text{tr}(\alpha_{n-1} \beta_{n-1}) I_n
\]

of \( \mathfrak{t}_C \) as the sum of commuting nilpotent and semisimple elements of \( \mathfrak{t} \), the semisimple element is the conjugate by \( g \) of \( \mu_{(S^1)^{n-1}}^C(q) \) and the nilpotent element lies in the conjugate by \( g \) of the \( (K_\sim)_C \)-orbit \( O \). To see this, we need to recall from [7] more about the hyperkähler strata \( Q_{(S,\delta)} \). So suppose that a quiver satisfies the hyperkähler moment map equations (2.2) and (2.3), and lies in the subset \( Q_{(S,\delta,\sim)} \), so that it lies in \( Q_{(S,\delta)} \) and

\[
i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3.
\]

Notice that \( Q_{(S,\delta,\sim)} \) is preserved by the rotation action of \( SU(2) \) on \( \mathbb{R}^3 \), and that given

\[
(\lambda_1, \ldots, \lambda_{n-1}) = (\lambda_1^C, \lambda_1^R, \ldots, \lambda_{n-1}^C, \lambda_{n-1}^R) \in (\mathbb{R}^3)^{n-1},
\]

by applying a generic element of \( SU(2) \) to rotate the complex structures and hence the decomposition \( \mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R} \), we can assume that if \( 1 \leq i < j \leq n \) then

\[
\sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^C = 0 \text{ in } \mathbb{C}.
\]

(5.8)

Thus

\[
Q_{(S,\delta,\sim)} = SU(2)Q^0_{(S,\delta,\sim)}
\]

where \( Q^0_{(S,\delta,\sim)} \) is the open subset of \( Q_{(S,\delta,\sim)} \) represented by those quivers in \( Q_{(S,\delta,\sim)} \) such that

\[
i \sim j \iff \sum_{k=i}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^C = 0 \text{ in } \mathbb{C}.
\]
Remark 5.9. It now follows immediately from (5.7) that for a quiver $q$ in $Q^0_{(S,\delta,\sim)}$ we have equality of the eigenvalues $\kappa_i$ and $\kappa_j$ of
\[
\alpha_{n-1}\beta_{n-1} = \frac{1}{n} \text{tr}(\alpha_{n-1}\beta_{n-1})I_n \in \mathfrak{sl}(n, \mathbb{C})
\]
if and only if $i \sim j$. In particular if $q \in Q^0_{(S,\delta,\sim)}$ then $(K_\mathbb{C})$ is the subgroup of $K_\mathbb{C}$ which preserves the decomposition of $q$ into the subquivers determined by the generalised eigenspaces of the compositions $\alpha_i\beta_i$ as in Remark 5.4 above.

Let us suppose now that our quiver $q$ lies in $Q^0_{(S,\delta,\sim)}$. As in Remark 5.4 we can decompose it into a direct sum of subquivers
\[
\cdots V^j_i \overset{\alpha_{i,j}}{\underset{\beta_{i,j}}{\rightleftharpoons}} V^j_{i+1} \cdots
\]
determined by the generalised eigenspaces (with eigenvalues $\tau_{i+1,j}$) of the compositions $\alpha_i\beta_i$, such that
\[
\alpha_{i,j}\beta_{i,j} - \beta_{i+1,j}\alpha_{i+1,j} = \lambda_{i+1}^C
\]
and $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms unless $\tau_{i+1,j} = 0$. If for some $j$ we have that $\alpha_{k,j}, \beta_{k,j}$ are isomorphisms for $i + 1 \leq k < s$ but not for $k = i, s$, then it follows that $\tau_{i+1,j} = \tau_{s+1,j} = 0$, hence $\sum_{k=i+1}^s \lambda_k^C = 0$, and so since the quiver lies in $Q^0_{(S,\delta,\sim)}$ we have
\[
\sum_{k=i+1}^s \lambda_k = 0 \in \mathbb{R}^3.
\]
This means that the quiver satisfies the hyperkähler moment map equations not just for $H = \prod_{k=1}^{n-1} SU(k)$ but for its extension
\[
H_{\{(i,s)\}}
\]
by $S^1$ in the sense of Definition 4.2. In particular it satisfies the real moment map equations for this subgroup of $\tilde{H} = \prod_{k=1}^{n-1} U(k)$, and thus its orbit under the complexification $(H_{\{(i,s)\}})_\mathbb{C}$ of $H_{\{(i,s)\}}$ is closed.

Remark 5.10. Note that $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms when $\tau_{i+1,j}$ is non-zero, and hence for each $i$ there is exactly one $j$ such that $\tau_{i+1,j} = 0$ and then
\[
\dim V^j_{i+1} = 1 + \dim V^j_i.
\]
In the case when $\tau_{i+1,j}$ is non-zero, and hence $\alpha_{i,j}$ and $\beta_{i,j}$ are isomorphisms, we may contract the subquiver
\[
\cdots V^j_i \overset{\alpha_{i,j}}{\underset{\beta_{i,j}}{\rightleftharpoons}} V^j_{i+1} \cdots
\]
by replacing
\[
V^j_{i-1} \overset{\alpha_{i-1,j}}{\underset{\beta_{i-1,j}}{\rightleftharpoons}} V^j_i \overset{\alpha_{i,j}}{\underset{\beta_{i,j}}{\rightleftharpoons}} V^j_{i+1} \overset{\alpha_{i+1,j}}{\underset{\beta_{i+1,j}}{\rightleftharpoons}} V^j_{i+2}
\]
with
\[ V^j_{i-1} \overset{\alpha_{i-1,j}}{\longrightarrow} V^j_i \overset{\alpha_{i+1,j}}{\longrightarrow} \cdots \overset{\alpha_{i+1,j}}{\longrightarrow} V^j_{i+2}, \]
and then the complex moment map equations are satisfied with
\[ \alpha_{i-1,j} \beta_{i-1,j} - (\alpha_{i,j})^{-1} \beta_{i+1,j} \alpha_{i+1,j} \alpha_{i,j} = \lambda_{j-1}^c + \lambda_j^c. \]
Moreover if we choose an identification of \( V^j_{i+1} \) with \( V^j_i \) and apply the action of \( SL(V_{i,j}) \) to set \( \alpha_{i,j} \) to be a non-zero scalar multiple \( aI \) of the identity, then \( \beta_{i,j} \) is determined by \( \alpha_{i-1,j}, \alpha_{i+1,j}, \beta_{i-1,j}, \beta_{i+1,j} \) via the equations \( \text{(2.2)} \) once we know the scalars \( a \) and \( \lambda_k^c \). We refer the reader to \( [7] \) for more details on contraction. After performing such contractions whenever \( \tau_{i+1,j} \) is non-zero, we obtain contracted quivers
\[ \cdots \overset{\alpha_{s-2,j}, \cdots, \alpha_{s-1,j}}{\longrightarrow} V^j_s \overset{\alpha_{s-1,j}, \cdots, \alpha_{s,j}}{\longrightarrow} \cdots \]
where
\[ V^j_s \cong V^j_{s+1} \cong \cdots \cong V^j_{s+1} \]
and \( \dim V^j_{s+1} = \dim V^j_s - 1 \). Moreover each of these contracted quivers satisfies the complex moment map equations for the induced action of
\[ \prod_{i: \dim V^j_{i+1} < \dim V^j_i} GL(V^j_i) \]
and its orbit under the action of this complex group is closed. It then follows from \( [17] \) Theorem 2.1 (cf. \( [7] \) Proposition 5.16) that each contracted subquiver is the direct sum of a quiver of the form
\[ 0 = V^0(s) \overset{\alpha^0(s)}{\longrightarrow} V^1(s) \overset{\alpha^1(s)}{\longrightarrow} V^2(s) \overset{\alpha^2(s)}{\longrightarrow} \cdots \overset{\alpha^{n-2}(s)}{\longrightarrow} V^{n-1}(s) \overset{\alpha^{n-1}(s)}{\longrightarrow} V^n(s) \leq \mathbb{C}^n, \]
where \( V^j(s) = 0 \) for \( 0 \leq j \leq k \) and \( \alpha^j(s) \) is injective and \( \beta^j(s) \) is surjective for \( k < j < n \), and a quiver
\[ 0 = V^0(0) \overset{\alpha^0(0)}{\longrightarrow} V^1(0) \overset{\alpha^1(0)}{\longrightarrow} V^2(0) \overset{\alpha^2(0)}{\longrightarrow} \cdots \overset{\alpha^{n-2}(0)}{\longrightarrow} V^{n-1}(0) \overset{\alpha^{n-1}(0)}{\longrightarrow} V^n(0) = \{0\} \]
in which all maps are 0. It also follows from the same theorem that the direct sum of the contracted subquivers is completely determined (modulo the action of \( \prod_{i: \dim V^j_{i+1} < \dim V^j_i} GL(V^j_i) \)) by the nilpotent element of \( (\mathfrak{k}_\infty)_{\mathbb{C}} \) given by the sum of the complex moment maps \( \alpha^0(s) \beta^0(s) \). Furthermore, given \( \sim \), the adjoint orbit of this nilpotent element in \( (\mathfrak{k}_\infty)_{\mathbb{C}} \) corresponds precisely to determining the dimensions of the various vector spaces \( V^j(s) \) and \( V^j(0) \). To see this, observe first that by Remarks \( 5.9 \) and \( 5.10 \) the equivalence relation \( \sim \) determines the dimensions of the generalised eigenspaces of the compositions \( \alpha_i \beta_i \) and determines how the corresponding subquivers are contracted. Also the nilpotent cone for \( (K_\infty)_{\mathbb{C}} \) in \( (\mathfrak{k}_\infty)_{\mathbb{C}} \) is the nilpotent cone for the product
[(K−)C, (K−)C] of special linear groups. Since (K−)C is the product of its centre Z((K−)C) and its commutator subgroup [(K−)C, (K−)C], the nilpotent orbits for (K−)C are the same as the nilpotent orbits for [(K−)C, (K−)C], and thus are given by products of nilpotent orbits in the special linear groups corresponding to the equivalence classes of −.

These nilpotent orbits in special linear groups are determined in turn by their Jordan types, which correspond exactly to the data given by the dimensions of the kernels of their powers. The contracted quivers satisfy αi(s)βj(s) = βi+1(s)αi+1(s) for all i, and so

\[(α_{n-1}β_{n-1})^s = α_{n-1}^sα_{n-2}^s...α_n^sβ_n^s...β_1^sβ_0^s.\]

Since the αi(s) are injective and the βj(s) are surjective this composition has rank \(\dim V^s\) and nullity \(\dim V^s\) − \(\dim V^s\). Finally note that the sums \(\dim V^s + \dim V^0\) are determined by the dimensions of the vector spaces in the contracted subquiver.

Remark 5.11. Recall that since our quiver \(q\) lies in \(Q_{(S,δ)}\) it is the direct sum of a hyperkähler stable quiver of the form

\[\begin{align*}
0 &\overset{α_0}{\rightarrow} \mathbb{C}m_1 &\overset{α_1}{\rightarrow} \mathbb{C}m_2 &\overset{α_2}{\rightarrow} \ldots &\overset{α_{n-2}}{\rightarrow} \mathbb{C}m_{n-1} &\overset{α_{n-1}}{\rightarrow} \mathbb{C}m_n = \mathbb{C}^n
\end{align*}\]

with quivers given for \(1 ≤ h ≤ p\) by

\[\begin{align*}
\mathbb{C}d_h &\overset{α_{ih}}{\rightarrow} \mathbb{C}d_h &\overset{α_{ih}^2}{\rightarrow} \mathbb{C}d_h &\ldots &\overset{α_{ih}^n}{\rightarrow} \mathbb{C}d_h &\mathbb{C}d_h
\end{align*}\]

in the places \(i_h, i_h + 1, \ldots, j_h - 1\), where the maps \(α_{ih}^{(h)}, β_{ih}^{(h)}\), for \(i_h ≤ k < j_h - 1\), are multiplication by complex scalars. The latter correspond in the description above to the zero summands of the contracted subquivers, while the former is the direct sum of the summands of the form

\[0 = V_j^s \overset{α_j^s}{\rightarrow} V_1^s \overset{α_1^s}{\rightarrow} V_2^s \overset{α_2^s}{\rightarrow} \ldots \overset{α_{n-2}^s}{\rightarrow} V_{n-1}^s \overset{α_{n-1}^s}{\rightarrow} V_n^s ≤ \mathbb{C}^n,\]

where \(V_j^s = 0\) for \(0 ≤ j ≤ k\) and \(α_j^s\) is injective and \(β_j^s\) is surjective for \(k < j < n\).

Remark 5.13. It follows that once the equivalence relation ~ or its corresponding subgroup \(K_−\) of \(K\) is fixed, the choice of index \((S, δ)\) such that \(Q_{(S,δ,∼)}\) is nonempty corresponds exactly to a nilpotent adjoint orbit \(O \subseteq (t_−)C\) for the complexification \((K−)C\) of \(K_−\). Thus we can make the following definition.

Definition 5.14. Let ~ be an equivalence relation on \(\{1, \ldots, n\}\) and let \(O\) be a nilpotent adjoint orbit for \((K−)C\). Then we will denote by
Q_{[\sim,O]} the subset $Q_{(S,\delta,\sim)}$ of $Q$ indexed by the corresponding $(S,\delta,\sim)$, and we will denote by $Q^\circ_{[\sim,O]}$ its open subset $Q^\circ_{(S,\delta,\sim)}$.

Remark 5.15. We also remark that the above analysis shows that the subset $Q_{(S,\delta,\sim)}$ of $Q$ is empty unless the subset $S$ of $\{1,\ldots,n\} \times \{1,\ldots,n\}$ is contained in $\sim$ (where the equivalence relation $\sim$ on $\{1,\ldots,n\}$ is formally identified with the subset
\[(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,n\} : i \sim j\]
of $\{1,\ldots,n\} \times \{1,\ldots,n\}$). Thus $Q_{(S,\delta)}$ is the disjoint union
\[Q_{(S,\delta)} = \coprod_{\sim} Q_{(S,\delta,\sim)}\]
over all the equivalence relations $\sim$ containing $S$, and

(5.16) \[Q = \coprod_{S,\delta,\sim} Q_{(S,\delta,\sim)}\]
is the disjoint union over all choices of $S$ and $\delta$ and equivalence relations $\sim$ containing $S$. Equivalently $Q$ is the disjoint union
\[Q = \coprod_{\sim,O} Q_{[\sim,O]}\]
over all equivalence relations $\sim$ on $\{1,\ldots,n\}$ and all nilpotent adjoint orbits $O$ in $(\mathfrak{k}_\sim)_\mathbb{C}$.

Remark 5.17. Notice that the values at a point in $Q$ of the hyperkähler moment maps for the actions on $Q$ of $K = SU(n)$ and $T = (S^1)^{n-1}$ determine the stratum $Q_{(S,\delta,\sim)}$ (or equivalently $Q_{[\sim,O]}$) to which the quiver belongs. For the value $(\lambda_1,\ldots,\lambda_{n-1})$ of $\mu_{(S^1)^{n-1}}$ determines the equivalence relation $\sim$ and also the generic choices of complex structures for which
\[\sum_{k=1}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3 \iff \sum_{k=i}^{j-1} \lambda_k^C = 0 \text{ in } \mathbb{C}.\]
Moreover for such choices of complex structures the quiver decomposes as a direct sum of subquivers determined by the generalised eigenspaces of the composition $\alpha_{n-1}\beta_{n-1}$ (which is given by the complex moment map for the action of $K$), and it follows that the Jordan type of $\alpha_{n-1}\beta_{n-1}$ determines the nilpotent orbit $O$ in $(\mathfrak{k}_\sim)_\mathbb{C}$, or equivalently as above the data $S$ and $\delta$.

6. Using Jordan Canonical Form

In this section we will use Jordan canonical form to study hyperkähler quivers as at (5.12) in order to find standard forms in the next section for quivers in a corresponding stratum $Q_{[\sim,O]}$. We have the following description of such quivers from [7] Proposition 7.2.
Proposition 6.1. Let $0 = m_0 \leq m_1 \leq \cdots \leq m_n = n$ and let $V_i = \mathbb{C}^{m_i}$ for $0 \leq i \leq n$. Consider quivers of the form

$$0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} V_{n-1} \xrightarrow{\alpha_{n-1}} V_n = \mathbb{C}^n,$$

where each $\beta_j$ is surjective and the complex moment map equations for $H = \prod_{i=1}^{n-1} SU(m_i)$ are satisfied. The set of such quivers modulo the action of $H_C = \prod_{i=1}^{n-1} SL(m_i, \mathbb{C})$ may be identified with

$$K_C \times [P, P]^\circ$$

where $P$ is the parabolic subgroup of $K_C = SL(n, \mathbb{C})$ associated to the flag $(m_1, \ldots, m_n = n)$, and $[P, P]^\circ$ is the annihilator of the commutator subgroup of $P$. The same is true if we replace the assumption that each $\beta_j$ is surjective with the assumption that each $\alpha_j$ is injective. When $m_i = i$ for all $i$ we have the space

$$K_C \times_N b$$

where $N$ is a maximal unipotent subgroup of $K_C = SL(n, \mathbb{C})$ and $b = n^o$ is a Borel subalgebra.

We can modify this result as follows to describe the subset of quivers for which each $\alpha_j$ is injective and each $\beta_j$ is surjective.

Proposition 6.2. Let $0 = m_0 \leq m_1 \leq \cdots \leq m_n = n$ and let $V_i = \mathbb{C}^{m_i}$ for $0 \leq i \leq n$. Consider quivers of the form

$$0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} V_{n-1} \xrightarrow{\alpha_{n-1}} V_n = \mathbb{C}^n,$$

where each $\alpha_j$ is injective and each $\beta_j$ is surjective and the complex moment map equations for $H = \prod_{i=1}^{n-1} SU(m_i)$ are satisfied. The set of such quivers modulo the action of $H_C = \prod_{i=1}^{n-1} SL(m_i, \mathbb{C})$ may be identified with

$$K_C \times [P, P]^\circ$$

where $[P, P]^\circ$ is the open subset of $[P, P]^\circ$ consisting of those $X \in [P, P]^\circ$ such that

$$X_i - \frac{\text{tr}X_{ii}}{k_i} \left( \begin{array}{c} 0_{k_i \times m_i} \\ I_{m_i \times m_i} \end{array} \right)$$

has maximal rank $m_i$ for each $i$ where $k_i = m_{i+1} - m_i$. Here $X_i$ is the bottom right $m_{i+1} \times m_i$ block in $X$ and $X_{ii}$ is its $i$th diagonal block (of size $k_i \times k_i$).

**Proof.** We first recall from [7] the proof of Proposition 6.1 above. Given a quiver of the form

$$0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} V_{n-1} \xrightarrow{\alpha_{n-1}} V_n = \mathbb{C}^n,$$
where each $\beta_j$ is surjective and the complex moment map equations for $H = \prod_{i=1}^{n-1} SU(m_i)$ are satisfied, it follows from an easy inductive argument that the vector spaces $V_i$ with $k < i \leq n$ have bases so that

$$\beta_i = (0_{m_i \times k_i} \ | \ I_{m_i \times m_i})$$

where $m_i = \dim V_i$ and $k_i = m_{i+1} - m_i$ is the dimension of the kernel of $\beta_i$. We have thus used the action of $H_C \times K_C = \prod_{i=1}^{n} SL(m_i, \mathbb{C})$ to put the maps $\beta_i$ in standard form.

This standard form is preserved by transformations satisfying

$$g_{i+1} = \begin{pmatrix} * & * \\ 0 & g_i \end{pmatrix},$$

where the top left block is $k_i \times k_i$, the bottom right block is $m_i \times m_i$ and $g_{k+1}$ is an arbitrary element of $SL(m_{k+1}, \mathbb{C})$. The freedom in $SL(n, \mathbb{C})$ is therefore the commutator of the parabolic group $P$ associated to the flag of dimensions $(m_{k+1}, m_{k+2}, \ldots, m_n = n)$ in $\mathbb{C}^n$.

With respect to bases chosen as above, the matrix of $\alpha_i \beta_i$ for $k < i < n$ is

$$\begin{pmatrix} 0_{k_i \times k_i} & D_{k_i \times m_i} \\ 0_{m_i \times k_i} & -\lambda_i^c I_{m_i} + \alpha_{i-1} \beta_{i-1} \end{pmatrix}$$

for some $k_i \times m_i$ matrix $D$.

We can use the pairing $(A, B) \mapsto \text{tr}(AB)$ to identify $\mathfrak{g}$ and $\mathfrak{g}_C$ with their duals. It now follows inductively that $X = \alpha_{n-1} \beta_{n-1}$ lies in the annihilator of the Lie algebra of the commutator $[P, P]$ of the parabolic determined by the integers $k_j$, and the diagonal entries of $X$ are 0 ($k_{n-1}$ times), $-\lambda_{n-1}^c$ ($k_{n-2}$ times), $\ldots$, $-\lambda_{n-1}^c + \cdots + \lambda_i^c$ ($k_i$ times) $\ldots$ Moreover one can show that any such $X$ occurs for a solution of the complex moment map equations, and that the trace-free part of $X$ determines all the $\alpha_i$ and hence the entire quiver modulo the action of $H_C$. Note that if $X \in [\mathfrak{p}, \mathfrak{p}]^o$ then the corresponding quiver

$$0 = V_0 \overset{\alpha_0}{\underset{\beta_0}{\Rightarrow}} V_1 \overset{\alpha_1}{\underset{\beta_1}{\Rightarrow}} V_2 \overset{\alpha_2}{\underset{\beta_2}{\Rightarrow}} \cdots \overset{\alpha_{n-2}}{\underset{\beta_{n-2}}{\Rightarrow}} V_{n-1} \overset{\alpha_{n-1}}{\underset{\beta_{n-1}}{\Rightarrow}} V_n = \mathbb{C}^n,$$

with each $\beta_i$ in standardised form

$$\beta_i = (0_{n_i \times k_i} \ | \ I_{m_i \times m_i}),$$

has

$$\alpha_i = X_i - \frac{\text{tr} X_i}{k_i} \left( 0_{k_i \times m_i} \right.$$}

$$\left. \ | \ I_{m_i \times m_i} \right)$$

where $X_i$ is the bottom right $m_{i+1} \times m_i$ block in $X$ and $X_{ii}$ is its $i$th diagonal block (of size $k_i \times k_i$). If we sweep out the space $[\mathfrak{p}, \mathfrak{p}]^o$ of quivers of this form with each $\alpha_j$ injective by the action of the torus $T_C = (\mathbb{C}^*)^{n-1}$ then we obtain the space $T_C[\mathfrak{p}, \mathfrak{p}]^o$ of quivers for which $X = \alpha_{n-1} \beta_{n-1} \in [\mathfrak{p}, \mathfrak{p}]^o$. Here each $\beta_k$ can be put in the form

$$\beta_i = (0_{m_i \times k_i} \ | \ a_i I_{m_i \times m_i}),$$
for some nonzero scalar $a_i$ by using the action of $H_\mathbb{C} = \prod_{i=k+1}^{n-1} SL(m_i, \mathbb{C})$ but not the action of $K_\mathbb{C} = SL(n, \mathbb{C})$. Then the set of all quivers of the form

$$0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} V_{n-1} \xrightarrow{\alpha_{n-1}} V_n = \mathbb{C}^n,$$

where each $\beta_j$ is surjective and each $\alpha_j$ is injective and the complex moment map equations for $H = \prod_{i=k+1}^{n-1} SU(m_i)$ are satisfied, modulo the action of $H_\mathbb{C} = \prod_{i=k+1}^{n-1} SL(m_i, \mathbb{C})$, is identified with

$$K_\mathbb{C} \times [P,P] [p,p]_\circ \cong K_\mathbb{C} \times_P (T_\mathbb{C}[p,p]_\circ)$$

since $P = T_\mathbb{C}[P,P]$. This completes the proof. \qed

**Remark 6.4.** Let

$$0 = V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} V_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} V_{n-1} \xrightarrow{\alpha_{n-1}} V_n = \mathbb{C}^n,$$

be a quiver such that each $\alpha_j$ is injective and each $\beta_j$ is surjective and the complex moment map equations for $H = \prod_{i=k+1}^{n-1} SU(m_i)$ are satisfied. Then $\ker \alpha_j \beta_j = \ker \beta_j$ and $\im \alpha_j \beta_j = \im \alpha_j$ for each $j$, and we can inductively choose coordinates on $V_1, \ldots, V_n$, starting with $V_n = \mathbb{C}^n$, so that each $\alpha_j \beta_j$ (and hence each $\beta_j \alpha_j$) is in Jordan canonical form while each $\beta_j$ is the direct sum over all the Jordan blocks of matrices in the standardised form

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
& & & & \cdots & \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

More precisely, we first choose a basis $\{e_1, \ldots, e_n\}$ for $V_n = \mathbb{C}^n$ so that $\alpha_{n-1} \beta_{n-1}$ is in Jordan canonical form. Since $\beta_{n-1} : V_n \to V_{n-1}$ is surjective and $\ker \beta_{n-1} = \ker \alpha_{n-1} \beta_{n-1}$, it follows that

$$\{\beta_{n-1}(e_i) : e_i \not\in \ker \beta_{n-1}\}$$

is a basis for $V_{n-1}$ such that $\beta_{n-1}$ is in the required form. It is now easy to check that $\beta_{n-1} \alpha_{n-1}$ is in Jordan canonical form, and it follows immediately from the complex moment map equations that the same is true of $\alpha_{n-2} \beta_{n-2}$, so that we can repeat the argument with the basis

$$\{\beta_{n-2} \beta_{n-1}(e_i) : e_i \not\in \ker \beta_{n-2} \beta_{n-1}\}$$

for $V_{n-2}$. Alternatively we can choose coordinates so that each $\alpha_j \beta_j$ and $\beta_j \alpha_j$ is in Jordan canonical form while each $\alpha_j$ is the direct sum
over all the Jordan blocks of matrices in the standardised form

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots \\
0 & \cdots & 0 & 1 \\
0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}
\]

Note that if we write elements \( \zeta = \zeta_s + \zeta_n \) where \( \zeta_s \) is semisimple and \( \zeta_n \) is nilpotent and \([\zeta_s, \zeta_n] = 0\) (cf. \([6] \) §1.1), then \((\alpha_{n-1} \beta_{n-1})_n \in [p, p]_s\) and the Jordan type of \(\alpha_{n-1} \beta_{n-1}\) determines, and is determined by, the flag \((m_1, \ldots, m_n = n)\) (or equivalently the parabolic subgroup \(P \) of \(K_C = SL(n, \mathbb{C})\)) together with the nilpotent orbit containing \((\alpha_{n-1} \beta_{n-1})_n\). Note also that the fact that \(\alpha_{n-1} \beta_{n-1}\) lies in the open subset \([p, p]_s\) of \([p, p]\) tells us that \(P\) is the Jacobson-Morozov parabolic of \((\alpha_{n-1} \beta_{n-1})_n\) (cf. \([6] \) §3.8). The choice of coordinates needed to put the quiver into the standard form above amounts to using the action of the group \(K_C \times \prod_{j=1}^{n-1} GL(m_j, \mathbb{C})\) to standardise the quiver.

Equivalently, once we have quotiented by the action of \(\prod_{j=1}^{n-1} SL(m_j, \mathbb{C})\), it amounts to using the action of \(K_C \times (\mathbb{C}^*)^{n-1} = K_C \times T_C\) to standardise the point of \(Q\) represented by the quiver.

7. Standard forms for quivers

Let \(\sim\) be an equivalence relation on \(\{1, \ldots, n\}\), and let \(\mathcal{O}\) be a nilpotent adjoint orbit in \((\mathfrak{t}, \mathfrak{t})_C\). Recall that \(Q_{\sim, \mathcal{O}}\) is the subset \(Q_{(S, \delta, \sim)}\) of \(Q\) indexed by the corresponding \((S, \delta, \sim)\) as in Definition \([5, 14]\). Similarly we will write \(Q_{\sim, \mathcal{O}}^p\) for the open subset \(Q_{(S, \delta, \sim), p}\) of \(Q_{(S, \delta, \sim)}\).

Putting the results of the last two sections together we find that given any quiver \(q\) in \(Q_{\sim, \mathcal{O}}^p\) we can choose coordinates on \(\mathbb{C}, \mathbb{C}^2, \ldots, \mathbb{C}^n\) to put the quiver into standard form as follows. Firstly, as in Remark \([5, 4]\) we can decompose \(q\) into a direct sum of subquivers

\[
\cdots V_{i}^j \overset{\alpha_{i,j}}{\underset{\beta_{i,j}}{\sim}} V_{i+1}^j \cdots
\]

determined by the generalised eigenspaces of the compositions \(\alpha_i \beta_i\).

Since \(q\) lies in \(Q_{\sim, \mathcal{O}}^p\) each such subquiver is the direct sum of a quiver \(q^{[j]}\) of the form

\[
(7.1) \quad 0 \overset{\alpha_0^{[j]}}{\underset{\beta_0^{[j]}}{\sim}} \mathbb{C}^{m_1} \overset{\alpha_1^{[j]}}{\underset{\beta_1^{[j]}}{\sim}} \mathbb{C}^{m_2} \overset{\alpha_2^{[j]}}{\underset{\beta_2^{[j]}}{\sim}} \cdots \overset{\alpha_{n-2}^{[j]}}{\underset{\beta_{n-2}^{[j]}}{\sim}} \mathbb{C}^{m_{n-1}} \overset{\alpha_{n-1}^{[j]}}{\underset{\beta_{n-1}^{[j]}}{\sim}} \mathbb{C}^{m_n}
\]
where the maps $\alpha_k^{[j]}$ for $1 \leq k \leq n - 1$ are injective and the maps $\beta_k^{[j]}$ for $1 \leq k \leq n - 1$ are surjective, with quivers given for $1 \leq h \leq p$ by

$$
\mathbb{C}^d_h \xrightarrow{\alpha^{(h)}_{j_h}} \mathbb{C}^d_{h-1} \xrightarrow{\alpha^{(h)}_{j_h-2}} \cdots \xrightarrow{\alpha^{(h)}_{j_2}} \mathbb{C}^d_1 \xrightarrow{\beta^{(h)}_{j_1}} \mathbb{C}^d_h
$$

in the places $i_h, i_h+1, \ldots, j_h-1$, where the maps $\alpha^{(h)}_k, \beta^{(h)}_k$, for $i_h \leq k < j_h - 1$, are multiplication by complex scalars such that $\gamma^{(h)}_k = \alpha^{(h)}_k + j \beta^{(h)}_k \in \mathbb{H} \setminus \{0\}$ (cf. Remark 5.11). Moreover the combinatorial data here and the Jordan type of $\alpha_{n-1}^\gamma \beta_{n-1}^\gamma$ for each summand (7.1) is determined by the pair $(\sim, \mathcal{O})$ (see Remarks 5.13 and 6.4). Now if we allow any complex linear changes of coordinates in $K \times H = \prod_{k=1}^n SL(k, \mathbb{C})$ then using Remark 6.4 we can put $\alpha_{n-1}^\gamma \beta_{n-1}^\gamma$ into Jordan canonical form and then decompose the quiver (7.1) into a direct sum of quivers determined by the Jordan blocks of $\alpha_{n-1}^{[j]} \beta_{n-1}^{[j]}$. Now $\alpha_k^{[j]}$ is a direct sum over the set $B_j$ of Jordan blocks for $\alpha_{n-1}^{[j]} \beta_{n-1}^{[j]}$ of matrices of the form

$$
\begin{pmatrix}
\nu_{1}^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \nu_{2}^{bjk} & 0 & \cdots & 0 \\
\cdots & & & & \\
0 & \cdots & 0 & \nu_{\ell_b-n+k}^{bjk} \\
0 & \cdots & 0 & 0 & \\
\end{pmatrix}
$$

(7.2)

for some $\nu_{i}^{bjk} \in \mathbb{C}^*$ where $\ell_b$ is the size of the Jordan block $b \in B_j$, and $\beta_k^{[j]}$ is a corresponding direct sum over $b \in B_j$ of matrices of the form

$$
\begin{pmatrix}
\mu_{1}^{bjk} & \xi_{b_{1}}^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \mu_{2}^{bjk} & \xi_{b_{2}}^{bjk} & 0 & \cdots & 0 \\
\cdots & & & & & \\
0 & \cdots & 0 & \mu_{\ell_{b_{n+k-1}}}^{bjk} & \xi_{b_{\ell_{b_{n+k-1}}}^{bjk}} & 0 \\
0 & \cdots & 0 & \mu_{b_{\ell_{b_{n+k}}}^{bjk}} & \xi_{\ell_{b_{n+k}}}^{bjk} & 0 \\
\end{pmatrix}
$$

for some $\mu_{i}^{bjk}, \xi_{i}^{bjk} \in \mathbb{C}^*$ satisfying the complex moment map equations (2.2). The resulting direct sum over all the Jordan blocks $\bigcup B_j$ for $\alpha_{n-1} \beta_{n-1}$ has closed $(H_S)_C$-orbit. Let $Q^\circ_{[\sim, \mathcal{O}]}$ be the subset of $Q^\circ_{[\sim, \mathcal{O}]}$ representing quivers of this form, where $\alpha_{n-1} \beta_{n-1}$ is in Jordan canonical form and the summands of the quiver corresponding to generalised eigenspaces of the compositions $\alpha_i \beta_i$ (and thus by Remark 5.9 to equivalence classes for $\sim$) are ordered according to the usual ordering on the minimal elements of the equivalence classes, and the Jordan blocks for each equivalence class are ordered by size. Then we have

$$
Q^\circ_{[\sim, \mathcal{O}]} = \sum_{d_{\gamma}}^\circ K \mathbb{C}^\circ_{[\sim, \mathcal{O}]}.
$$
Note that if we allow any complex linear changes of coordinates in $K_\mathbb{C} \times \hat{H}_\mathbb{C} = \prod_{k=1}^{\alpha} GL(k, \mathbb{C})$ (or equivalently allow the action of $K_\mathbb{C} \times T_\mathbb{C}$ on $Q_{(\sim, \mathcal{O})}$), then as in Remark 6.4, we can put our quiver into a more restricted form which is completely determined by $\alpha_{n-1} \beta_{n-1}$ and $(\lambda_1^C, \ldots, \lambda_{n-1}^C)$, and thus by the values of the complex moment maps for the actions of $K$ and $T$ on $Q$. Hence the fibres of the complex moment map
\[ Q_{(\sim, \mathcal{O})}^0 \to \mathfrak{t}_\mathbb{C} \oplus \mathfrak{t}_\mathbb{C} \]
for the action of $K \times T$ are contained in single $K_\mathbb{C} \times T_\mathbb{C}$-orbits.

Remark 7.3. Let us consider the stabiliser in $K_\mathbb{C} \times T_\mathbb{C}$ of a quiver $q \in Q_{(\sim, \mathcal{O})}^0$. We may assume that $q$ is in the standard form described above, so that $q \in Q_{(\sim, \mathcal{O})}^{0,JCF}$. We also want to consider how much of the $K_\mathbb{C} \times T_\mathbb{C}$-orbit of $q$ lies in $Q_{(\sim, \mathcal{O})}^{0,JCF}$.

We know from Remark 5.3 that the decomposition of $q$ into a direct sum of subquivers given by the generalised eigenspaces of the compositions $\alpha \beta_i$ is determined by the equivalence relation $\sim$, and since this decomposition is canonical it follows that the stabiliser of $q$ in $K_\mathbb{C} \times T_\mathbb{C}$ is a subgroup of $(K_\infty)_\mathbb{C} \times T_\mathbb{C}$, and indeed that elements of $K_\mathbb{C} \times T_\mathbb{C}$ which preserve the standard form all lie in $(K_\infty)_\mathbb{C} \times T_\mathbb{C}$. Now applying the proof of Proposition 6.2 and Remark 6.4 to the summands in this decomposition, it follows that elements of $K_\mathbb{C} \times T_\mathbb{C}$ which preserve the standard form are contained in $P \times T_\mathbb{C}$ where $P$ is the parabolic subgroup of $(K_\infty)_\mathbb{C}$ which is the Jacobson–Morozov parabolic of the element of the nilpotent orbit $\mathcal{O}$ for $(K_\infty)_\mathbb{C}$ given by the nilpotent component of
\[ \alpha_{n-1} \beta_{n-1} - \frac{1}{n} \text{tr}(\alpha_{n-1} \beta_{n-1})I_n \in (\mathfrak{t}_\infty)_\mathbb{C}. \]

Indeed, the elements which preserve the standard form must lie in $R_{(\sim, \mathcal{O})} \times T_\mathbb{C}$ where $R_{(\sim, \mathcal{O})}$ is the centraliser in $P$ of this nilpotent element of $[p, p]$. In particular the stabiliser of $q$ in $K_\mathbb{C} \times T_\mathbb{C}$ is contained in $R_{(\sim, \mathcal{O})} \times T_\mathbb{C}$. Conversely, it is easy to see that both the centre $Z((K_\infty)_\mathbb{C})$ of $(K_\infty)_\mathbb{C}$, embedded diagonally in $T_\mathbb{C} \times T_\mathbb{C}$, and the intersection $[P, P] \cap R_{(\sim, \mathcal{O})}$, embedded in $K_\mathbb{C} \times \{1\}$, stabilise the quiver $q \in Q_{(\sim, \mathcal{O})}^{0,JCF}$. This quiver is also stabilised by the complexification of the subgroup $S_3$ defined at Definition 4.2. Moreover since $Q_{(\sim, \mathcal{O})}^{0,JCF}$ is $R_{(\sim, \mathcal{O})} \times T_\mathbb{C}$-invariant it follows that an element of $K_\mathbb{C} \times T_\mathbb{C}$ preserves the standard form if and only if it lies in $R_{(\sim, \mathcal{O})} \times T_\mathbb{C}$, and so
\[ (7.4) \quad Q_{(\sim, \mathcal{O})}^0 \cong (K_\mathbb{C} \times T_\mathbb{C}) \times (R_{(\sim, \mathcal{O})} \times T_\mathbb{C}) Q_{(\sim, \mathcal{O})}^{0,JCF} \cong K_\mathbb{C} \times R_{(\sim, \mathcal{O})} Q_{(\sim, \mathcal{O})}^{0,JCF}. \]

Furthermore, to determine the stabiliser of $q$ in $K_\mathbb{C} \times T_\mathbb{C}$ we should first consider its intersection with $(T_{(\sim, \mathcal{O})})_\mathbb{C} \times T_\mathbb{C}$, where $T_{(\sim, \mathcal{O})}$ is the intersection of $T$ with $R_{(\sim, \mathcal{O})}$. This intersection contains the product $T([\sim, \mathcal{O}])_\mathbb{C}$ of the subgroups $Z((K_\infty)_\mathbb{C})$, $Z_P \cap R_{(\sim, \mathcal{O})}$ and $(T_S)_\mathbb{C}$ of $(T_{(\sim, \mathcal{O})})_\mathbb{C} \times T_\mathbb{C}$. 

Lemma 7.5. The non-empty fibres of the restriction
\[ Q^0_{\sim, \mathcal{O}} \rightarrow \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \]
to \( Q^0_{\sim, \mathcal{O}} \) of the complex moment map for the action of \( K \times T \) on \( Q \) are single \((T_{\sim, \mathcal{O}})_{\mathbb{C}} \times T_{\mathbb{C}}\)-orbits, where \( T_{\sim, \mathcal{O}} \) is the intersection of \( T \) with \( R_{\sim, \mathcal{O}} \).

Proof. We observed just before Remark 7.3 that the fibres of the complex moment map
\[ Q^0_{\sim, \mathcal{O}} \rightarrow \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \]
for the action of \( K \times T \) are contained in single \( K_{\mathbb{C}} \times T_{\mathbb{C}}\)-orbits. By (7.4) above, each \( K_{\mathbb{C}} \times T_{\mathbb{C}}\)-orbit which meets \( Q^0_{\sim, \mathcal{O}} \) meets it in a single \( R_{\sim, \mathcal{O}} \times T_{\mathbb{C}}\)-orbit where
\[ R_{\sim, \mathcal{O}} = (T_{\sim, \mathcal{O}})_{\mathbb{C}} \cap ([P, P] \cap R_{\sim, \mathcal{O}}) \]
and \([P, P] \cap R_{\sim, \mathcal{O}}\) stabilises the quiver \( q \in Q^0_{\sim, \mathcal{O}} \). Thus each fibre of the restriction to \( Q^0_{\sim, \mathcal{O}} \) of the complex moment map for the action of \( K \times T \) is contained in a single \((T_{\sim, \mathcal{O}})_{\mathbb{C}} \times T_{\mathbb{C}}\)-orbit. Since this complex moment map is \( K_{\mathbb{C}} \times T_{\mathbb{C}}\)-equivariant and \((T_{\sim, \mathcal{O}})_{\mathbb{C}} \times T_{\mathbb{C}}\) fixes the image in \( \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} \) of any element of \( Q^0_{\sim, \mathcal{O}} \), the result follows.

Remark 7.6. Recall that the subgroup \([P, P] \cap (T_{\sim, \mathcal{O}})_{\mathbb{C}}\) acts trivially on \( Q^0_{\sim, \mathcal{O}} \). In fact \((T_{\sim, \mathcal{O}})_{\mathbb{C}}/[P, P] \cap (T_{\sim, \mathcal{O}})_{\mathbb{C}}\) acts freely on \( Q^0_{\sim, \mathcal{O}} \). To see this, consider a quiver which is a direct sum over the set \( \bigcup_{j} B_j \) of Jordan blocks for \( \alpha_{n-1} \beta_{n-1} \) of quivers of the form described in (7.2), together with quivers given for \( 1 \leq h \leq p \) by
\[ \mathbb{C}^{d_h} \xrightarrow{\alpha_h^{(k)}} \mathbb{C}^{d_h} \supseteq \ldots \supseteq \mathbb{C}^{d_h} \xrightarrow{\alpha_{h-2}^{(k)}} \mathbb{C}^{d_h} \xrightarrow{\beta_{h-2}^{(k)}} \]
in the places \( i_h, i_h + 1, \ldots, j_h - 1 \), as in Remark 5.11. The centraliser in \( T_{\mathbb{C}} \) of \( \alpha_{n-1} \beta_{n-1} \) consists of matrices in \( T_{\mathbb{C}} \) which are themselves direct sums over all the Jordan blocks of diagonal matrices with diagonal entries \((r_1^{bjk}, \ldots, r_1^{bjk}, r_2^{bjk})\) for some \( r_1^{bjk}, r_2^{bjk} \in \mathbb{C}^* \). If such a matrix sends the quiver to an element of its \( H\)-orbit, then \( r_1^{bjk} = r_2^{bjk} \) for all \( b, j, k \), and so the matrix lies in \([P, P] \cap (T_{\sim, \mathcal{O}})_{\mathbb{C}}\).

Remark 7.7. The image in \( \mathfrak{t}_{\mathbb{C}} \) of \( q \in Q^0_{\sim, \mathcal{O}} \) under the complex moment map for \( K \) is the sum of an element \( \zeta \) of \( \mathfrak{t}_{\mathbb{C}} \) with centraliser \( K_\zeta \) in \( K \) and an element \( \xi \) of the nilpotent orbit \( O \) in \( (\mathfrak{k})_{\mathbb{C}} \), and its image in \( \mathfrak{t}_{\mathbb{C}} \) under the complex moment map for \( T \) is equal to \( \zeta \) by Remark 5.9. Thus the image of \( Q^0_{\sim, \mathcal{O}} \) under the complex moment map for \( K \times T \) is
\[ \Delta(t_\zeta) \oplus O = \{ (\zeta + \xi, \zeta) \in \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{t}_{\mathbb{C}} : \zeta \in (t_\zeta)_{\sim} \text{ and } \xi \in O \} \]
where \((t_C)_{\sim}\) consists of the elements of \(t_C\) with centraliser \(K_{\sim}\) in \(K\), while the image of \(Q^{\circ,JC_F}_{[\sim, O]}\) is

\[
\Delta(t_C)_{\sim} \oplus \xi_0 = \{(\zeta + \xi_0, \zeta) \in t_C \oplus t_C : \zeta \in (t_C)_{\sim}\}
\]

where \(\xi_0 \in O\) is the element of the nilpotent orbit \(O\) in Jordan canonical form with the Jordan blocks within each generalised eigenspace for \(\alpha_{n-1} \beta_{n-1}\) ordered by size and the generalised eigenspaces themselves ordered using the equivalence relation \(\sim\).

**Remark 7.8.** Notice that if a quiver is of the form

\[
\alpha_k^{bj} = \begin{pmatrix}
\nu_1^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \nu_2^{bjk} & 0 & \cdots & 0 \\
& & \ddots & & \\
0 & \cdots & 0 & \nu_{\ell_k-n+k}^{bjk}
\end{pmatrix}
\]

and

\[
\beta_k^{bj} = \begin{pmatrix}
\mu_1^{bjk} & \xi_1^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \mu_2^{bjk} & \xi_2^{bjk} & 0 & \cdots & 0 \\
& & \ddots & & & & \\
0 & \cdots & 0 & \mu_{\ell_k-n+k-1}^{bjk} & \xi_{\ell_k-n+k-1}^{bjk} & 0 \\
0 & \cdots & 0 & 0 & \mu_{\ell_k-n+k}^{bjk} & \xi_{\ell-k+n+k}
\end{pmatrix}
\]

for some \(\nu_1^{bjk}, \nu_2^{bjk}, \ldots, \nu_{\ell_k-n+k}^{bjk}, \xi_1^{bjk}, \xi_2^{bjk}, \ldots, \xi_{\ell_k-n+k-1}^{bjk} \in \mathbb{C}^*\) as at (7.2) above, then \(\alpha_k^{bj} \beta_k^{bj}\) and \(\beta_k^{bj} \alpha_k^{bj}\) are upper triangular matrices with diagonal entries

\[
\mu_1^{bjk}, \nu_1^{bjk}, \ldots, \mu_{\ell_k-n+k}^{bjk}, \nu_{\ell_k-n+k}^{bjk}, 0
\]

and

\[
\mu_1^{bjk}, \nu_1^{bjk}, \ldots, \mu_{\ell_k-n+k}^{bjk}, \nu_{\ell_k-n+k}^{bjk}, 0
\]

respectively. It follows that the complex moment map equations are also satisfied by the quiver given by replacing \(\alpha_k^{bj}\) and \(\beta_k^{bj}\) with

\[
(7.9) \quad \alpha_k^{bj,T} = \begin{pmatrix}
\nu_1^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \nu_2^{bjk} & 0 & \cdots & 0 \\
& & \ddots & & \\
0 & \cdots & 0 & \nu_{\ell_k-n+k}^{bjk}
\end{pmatrix}
\]

and

\[
(7.9) \quad \beta_k^{bj,T} = \begin{pmatrix}
\mu_1^{bjk} & 0 & 0 & \cdots & 0 \\
0 & \mu_2^{bjk} & 0 & \cdots & 0 \\
& & \ddots & & \\
0 & \cdots & 0 & \mu_{\ell_k-n+k}^{bjk}
\end{pmatrix}
\]

Similarly if \(q\) is any quiver representing a point in \(Q^{\circ,JC_F}_{[\sim, O]}\) whose Jordan blocks are of the form given by \(\alpha_k\) and \(\beta_k\) as above, then the quiver \(q^T\) obtained from \(q\) by replacing each such Jordan block with the quiver
given by $\alpha_k^T$ and $\beta_k^T$ satisfies the complex moment map equations for the action of $H$, or equivalently the complex moment map equations for the maximal torus $T_H$ of $H$.

Recall from Definition 3.1 the definition of $M_T$ and $\iota : M_T \sslash T_H \to Q$ inducing an identification of the open subset $Q^{\text{hks}}_T = M^{\text{hks}}_T \sslash T_H$ of the hypertoric variety $M_T \sslash T_H$ with its image in $Q$.

**Definition 7.10.** For any $(\sim, \mathcal{O})$ we can consider an open subset of a hyperkähler modification

$$((M_{\sim,\mathcal{O}} \sslash T_H \times \mathbb{H}^\ell) \sslash (S^1)^{\ell})$$

(cf. Definition 4.3) of the hyperkähler quotient $M_{\sim,\mathcal{O}} \sslash T_H$ of the space $M_{\sim,\mathcal{O}}$ of quivers which are direct sums of quivers of the form

$$\alpha_k^T = \left( \begin{array}{cccc} \nu_1^k & 0 & 0 & \cdots & 0 \\ 0 & \nu_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \nu_k^k & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array} \right)$$

and

$$\beta_k^T = \left( \begin{array}{cccc} \mu_1^k & 0 & 0 & \cdots & 0 \\ 0 & \mu_2^k & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \mu_k^k & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array} \right),$$

for $\nu_i^k, \mu_i^k \in \mathbb{C}$, with one such summand for every Jordan block of the canonical representative $\xi_i$ of the nilpotent orbit $\mathcal{O}$ in $(\mathfrak{t}_\sim)_\mathbb{C}$ as in Remark 7.8. If $M^o_{\sim,\mathcal{O}}$ is the open subset of $M_{\sim,\mathcal{O}}$ where all $\nu_i^k$ and $\mu_i^k$ are nonzero, then let

$$Q^o_{\sim,\mathcal{O}} = ((M^o_{\sim,\mathcal{O}} \sslash T_H \times \mathbb{H} \setminus \{0\})^\ell) \sslash (S^1)^{\ell}.$$  

Equivalently $Q^o_{\sim,\mathcal{O}}$ can be identified with an open subset of the hyperkähler quotient by $T_H$ of the space of quivers which, like $q^T$ in Remark 7.8 are direct sums of quivers of the form above and summands as in Remark 4.5.

The space $M_{\sim,\mathcal{O}}$ is a flat hypertoric variety with respect to the action of a quotient of $T_H$. Thus

$$((M_{\sim,\mathcal{O}} \sslash T_H) \times \mathbb{H}^\ell) \sslash (S^1)^{\ell}$$

is also hypertoric, for the action of a quotient of $T = T_H / T_H$, and it contains $Q^o_{\sim,\mathcal{O}}$ as an open subset.

We define $Q_{\sim,\mathcal{O}}$ to be the open subset $SU(2)Q^o_{\sim,\mathcal{O}}$ of the hypertoric variety $((M_{\sim,\mathcal{O}} \sslash T_H) \times \mathbb{H}^\ell) \sslash (S^1)^{\ell}$.

**Definition 7.12.** Let $\psi : Q_{\sim,\mathcal{O}}^{\text{JCF}} \to Q^o_{\sim,\mathcal{O}}$ be the map which associates to a quiver $q \in Q_{\sim,\mathcal{O}}^{\text{JCF}}$ the quiver $q^T$ described in Remark 7.8.
Note that $\psi$ is well defined, since any quiver which has the same form as that for $q$ described in Remarks 7.11 and 7.12 and which represents the same point in $Q^0_{[\sim, O]} \times T$ (that is, lies in the same $H$-orbit) actually lies in the same $T_H$-orbit.

**Lemma 7.13.** $\psi : Q^0_{[\sim, O]} \rightarrow Q^0_{[\sim, O], T}$ is a bijection.

**Proof.** Since $q \in Q^0_{[\sim, O]}$, satisfies the complex moment map equations for $H$ and $\alpha_{\beta, n-1}$ is in Jordan canonical form, the entries $\xi_i \in \mathbb{C}^*$ of the Jordan blocks of $q$ are uniquely determined by the entries $\nu_i, \mu_i \in \mathbb{C}^*$ of the corresponding blocks of $q^T$.

Recall from Remark 7.3 that $R_{[\sim, O]}$ is the centraliser of the canonical representative $\xi_0$ of the nilpotent orbit $O$ in its Jacobson–Morozov parabolic $P$ in $(K_\sim)_C$, while $[P, P] \cap R_{[\sim, O]}$ stabilises each point of $Q^0_{[\sim, O]}$. Thus $P = Z_P[P, P]$ and $R_{[\sim, O]} = (T_{[\sim, O]})_C([P, P] \cap R_{[\sim, O]})$ where $Z_P$ is the centre of the standard Levi subgroup of $P$ and $(T_{[\sim, O]})_C = Z_P \cap R_{[\sim, O]}$. Note also that $R_{[\sim, O]}$ contains the centre $Z((K_\sim)_C)$ of $(K_\sim)_C$, which acts trivially on $Q^0_{[\sim, O]}$ and on $Q^0_{[\sim, O], T}$, and that $(K_\sim)_C = Z((K_\sim)_C)((K_\sim)_C, (K_\sim)_C)$. We obtain an immediate corollary as follows.

**Corollary 7.14.**

$Q^0_{[\sim, O]} \cong K_C \times R_{[\sim, O]} \times (T_{[\sim, O]})_C Q^0_{[\sim, O], T}$

$\cong K_C \times (K_\sim)_C \left( \left. ([K_\sim)_C, (K_\sim)_C]/[P, P] \cap R_{[\sim, O]} \right) \times (T^*_{[\sim, O]})_C Q^0_{[\sim, O], T} \right)$

where $Q^0_{[\sim, O], T}$ is an open subset of a hypertoric variety and $(T^*_{[\sim, O]})_C = (T_{[\sim, O]})_C \cap [(K_\sim)_C, (K_\sim)_C]$.

**Remark 7.15.** The quotient of $K_C/[P, P] \cap R_{[\sim, O]}$ by $(T_{[\sim, O]})_C$ is of course the nilpotent orbit $K_C/R_{[\sim, O]}$ in $\mathfrak{k}_C$ which contains the nilpotent orbit

$O = (K_\sim)_C/R_{[\sim, O]} = [(K_\sim)_C, (K_\sim)_C]/[P, P] \cap R_{[\sim, O]} \cap [(K_\sim)_C, (K_\sim)_C]$ in $(\mathfrak{k}_\sim)_C$ or equivalently in $[(\mathfrak{k}_\sim)_C, (\mathfrak{k}_\sim)_C]$, which is itself the quotient of $[(K_\sim)_C, (K_\sim)_C]/[P, P] \cap R_{[\sim, O]}$ by $(T_{[\sim, O]})_C$. This nilpotent orbit for a product of special linear groups is an open subset of a hyperkähler quotient of a flat hyperkähler space of quivers (cf. [17]), and $K_C/[P, P] \cap R_{[\sim, O]}$ itself can likewise be described inductively in terms of the hyperkähler implosion of the corresponding product of special unitary groups.

**Remark 7.16.** The hyperkähler moment map $\mathbb{H} \rightarrow \mathbb{R}^3$ for the standard $S^1$-action on $\mathbb{H}$ restricts to a locally trivial fibration

$\mathbb{H} \setminus \{0\} \rightarrow \mathbb{R}^3 \setminus \{0\}$

with fibre $S^1$. Recall the definition of the hypertoric variety $M_T$ from §3. We can, as in [3], stratify $M_T = \mathbb{H}^{n(n-1)/2}$ using the quaternionic coordinate hyperplanes, each stratum corresponding to fixing the subset
of \( \{1, \ldots, n(n-1)/2\} \) indexing the quaternionic hyperplanes containing the points of the stratum. Then the hyperkähler moment map \( M_T \to (\mathbb{R}^3)^n(n-1)/2 \) restricted to a stratum is a locally trivial fibration with fibre

\[
T_H/(S^1)^{E[1]}
\]

where \((S^1)^{E[1]}\) is the subtorus of \( T_H = (S^1)^n(n-1)/2 \) whose Lie algebra is generated by the basis vectors indexed by the elements of \( E \).

Similarly it follows from Definition \[ \text{5.10} \] and Remark \[ \text{7.6} \] that that

\[
\mu_{(S^1)^{n-1}} : Q_{[\sim, \mathcal{O}], T} \to (\mathbb{R}^3)^{n-1} = (t \otimes \mathbb{R}^3)^{[\sim]} \text{ is a locally trivial fibration}
\]

with fibre the quotient \( T/[P, P] \cap T_{[\sim, \mathcal{O}]}, \) of \( T = T_H/T_H \).

8. The refined strata

Recall from \[ \text{5.10} \] that the universal hyperkähler implosion \( Q = M//H \) for \( K = SU(n) \) is a disjoint union

\[
Q = \bigsqcup_{S, \delta, \sim} Q_{(S, \delta, \sim)} = \bigsqcup_{\sim, \mathcal{O}} Q_{[\sim, \mathcal{O}]}
\]

of subsets indexed by \((S, \delta, \sim)\) or equivalently, as discussed immediately before Definition \[ \text{5.13} \] by pairs \((\sim, \mathcal{O})\) where \( \sim \) is an equivalence relation on \( \{1, \ldots, n\} \) and \( \mathcal{O} \) is a nilpotent adjoint orbit in \((\mathfrak{f}_\sim)_C \). Here

\[
Q_{[\sim, \mathcal{O}]} = Q_{(S, \delta, \sim)} = Q_{(S, \delta)} \cap \mu_{(S^1)^{n-1}}^{-1}((\mathbb{R}^3)^{n-1})
\]

as in Definition \[ \text{5.3} \]. We have

\[
Q_{[\sim, \mathcal{O}]} = SU(2)Q^o_{[\sim, \mathcal{O}]}
\]

where \( Q^o_{[\sim, \mathcal{O}]} \) is the open subset of \( Q_{[\sim, \mathcal{O}]} \) which is its intersection with

\[
\mu_{(S^1)^{n-1}}^{-1}((\{\lambda_1, \ldots, \lambda_{n-1}\} \in (\mathbb{R}^3)^{n-1} : \sum_{k=1}^{j-1} \lambda_k = 0 \text{ in } \mathbb{R}^3)
\]

\[
\iff \sum_{k=1}^{j-1} \lambda_k^C = 0 \text{ in } \mathbb{C} \}.
\]

Recall also from Corollary \[ \text{7.14} \] and Remark \[ \text{7.15} \] that

\[
Q^o_{[\sim, \mathcal{O}], T} = K_C \times_{R_{[\sim, \mathcal{O}]}} Q^{o,\text{JCF}}_{[\sim, \mathcal{O}], T} \cong (K_C/[P, P] \cap R_{[\sim, \mathcal{O}]})_C Q^o_{[\sim, \mathcal{O}], T}
\]

where \( Q^o_{[\sim, \mathcal{O}], T} \) is an open subset of a hypertoric variety, and \( R_{[\sim, \mathcal{O}]} \) is the centraliser in \((K_C)_C\) of the standard representative \( \xi_0 \) in Jordan canonical form of the nilpotent orbit \( \mathcal{O} \) in \((\mathfrak{f}_\sim)_C \) (cf. Remark \[ \text{7.7} \]), while \((T_{[\sim, \mathcal{O}]})_C = T_C \cap R_{[\sim, \mathcal{O}]}\). Moreover \([P, P] \cap R_{[\sim, \mathcal{O}]} \) acts trivially on \( Q^{o,\text{JCF}}_{[\sim, \mathcal{O}], T} \) and \((T_{[\sim, \mathcal{O}]})_C/[P, P] \cap (T_{[\sim, \mathcal{O}]})_C \) acts freely on \( Q^{o,\text{JCF}}_{[\sim, \mathcal{O}], T} \) by Remark \[ \text{7.6} \]. In addition \( K_C/[P, P] \cap R_{[\sim, \mathcal{O}]} \) can be described inductively in terms of the hyperkähler implosions of the special unitary groups whose product is
Recall finally from Lemma 7.5 and Remark 7.6 that the non-empty fibres of the restriction

\[ Q^o_{[\sim,\mathcal{O}]} \to \mathfrak{k} \oplus \mathfrak{t} \]

to \( Q^o_{[\sim,\mathcal{O}]} \) of the complex moment map for the action of \( K \times T \) on \( Q \) are single \( (T_{[\sim,\mathcal{O}]}(\mathbb{C}) \times T_{\mathcal{O}} \text{-orbits, and its image is} \)

\[ \Delta(t_{\mathcal{C}}) \oplus \xi_0 = \{ (\zeta + \xi_0, \zeta) \in \mathfrak{k} \oplus \mathfrak{t} : \zeta \in (t_{\mathcal{C}})_{[\sim]} \}. \]

Putting this all together we obtain the following theorem.

**Theorem 8.1.** For each equivalence relation \( \sim \) on \( \{1, \ldots, n\} \) and nilpotent adjoint orbit \( \mathcal{O} \) for \( (K_{[\sim]}(\mathbb{C})) \), the stratum \( Q_{[\sim,\mathcal{O}]}^o \) is the union over \( s \in SU(2) \) of its open subsets \( s Q_{[\sim,\mathcal{O}]}^o \), and

\[ Q_{[\sim,\mathcal{O}]}^o = K_{[\sim]}(\mathbb{C}) \times_{R_{[\sim,\mathcal{O}]}} Q_{[\sim,\mathcal{O}]}^o \]

where \( R_{[\sim,\mathcal{O}]} \) is the centraliser in \( (K_{[\sim]}(\mathbb{C})) \) of the standard representative \( \xi_0 \) in Jordan canonical form of the nilpotent orbit \( \mathcal{O} \) in \( (\mathfrak{t}_{[\sim]}(\mathbb{C})) \), and \( Q_{[\sim,\mathcal{O}]}^o \) can be identified with an open subset of a hypertoric variety.

The image of the restriction

\[ Q_{[\sim,\mathcal{O}]}^o \to \mathfrak{k} \]

of the complex moment map for the action of \( K \) on \( Q \) is \( K_{[\sim]}(\mathbb{C})(t_{\mathcal{C}})_{[\sim]} \oplus \mathcal{O} \cong K_{[\sim]}(\mathbb{C}) \times (t_{\mathcal{C}})_{[\sim]} \oplus \mathcal{O} \) and its fibres are single \( (T_{[\sim,\mathcal{O}]}(\mathbb{C}) \times T_{\mathcal{O}} \text{-orbits, where} \)

\( (T_{[\sim,\mathcal{O}]}(\mathbb{C}) = T_{\mathcal{C}} \cap R_{[\sim,\mathcal{O}]} \) and \( (T_{[\sim,\mathcal{O}]}(\mathbb{C}) = [P, P] \cap (T_{[\sim,\mathcal{O}]}(\mathbb{C}) \text{ acts freely on } Q_{[\sim,\mathcal{O}]}^o. \)

**Remark 8.2.** Recall that the symplectic implosion \( X_{\text{impl}} \) of a symplectic manifold \( X \) with a Hamiltonian action of a compact group \( K \) with moment map \( \mu_X \) is the disjoint union over the faces \( \sigma \) of a positive Weyl chamber \( t_+ \) of the strata

\[ \mu_X^{-1}(\sigma)/[K_\sigma, K_\sigma] \]

where \( K_\sigma \) is the stabiliser in \( K \) of any element of the face \( \sigma \) of \( t_+ \) and \([K_\sigma, K_\sigma] \) is its commutator subgroup. Similarly if \( X \) is a hyperkähler manifold with a hyperkähler action of \( K = SU(n) \) and hyperkähler moment map \( \mu_X : X \to \mathfrak{k} \otimes \mathbb{R}^3 \) and complex moment map its projection \( \mu_{X,\mathcal{C}} \) to \( \mathfrak{k} \), then the hyperkähler implosion \( X_{\text{h kidnapped}} \) of \( X \) is defined to be the hyperkähler quotient of \( X \times Q \) by the diagonal action of \( K \). In the light of Theorem 8.1 and Remark 7.6, we expect to have a description of \( X_{\text{h kidnapped}} \) as follows, at least when \( X \) is an affine variety with respect to all its complex structures so that symplectic quotients can be identified with GIT quotients. \( X_{\text{h kidnapped}} \) should be the disjoint union over all equivalence relations \( \sim \) on \( \{1, \ldots, n\} \) and nilpotent adjoint orbits in \( (\mathfrak{t}_{[\sim]}(\mathbb{C})) \) of its subsets \( X_{\text{h kidnapped}[\sim,\mathcal{O}]} \), which are themselves the unions of open subsets \( s X_{\text{h kidnapped}[\sim,\mathcal{O}]}^o \) for \( s \in SU(2) \), such that

\[ X_{\text{h kidnapped}[\sim,\mathcal{O}]}^o = \mu_{X,\mathcal{C}}^{-1}((t_{\mathcal{C}})_{[\sim]} \oplus \xi_0)/[P, P] \cap R_{[\sim,\mathcal{O}]} \]
where $\xi_0 \in \mathcal{O}$ is the canonical representative of the nilpotent orbit $\mathcal{O} \subseteq (\mathfrak{t}_\mathbb{C})_\mathbb{C}$ in Jordan canonical form as in Remark 7.6, and $(\mathfrak{t}_\mathbb{C})_\sim$ is the set of elements of $\mathfrak{t}_\mathbb{C}$ with centraliser $K_\sim$ in $K$, while $P$ is the Jacobson-Morozov parabolic of $\xi_0$ in $(K_\sim)_\mathbb{C}$ and $R_{(\sim,\mathcal{O})}$ is the centraliser of $\xi_0$ in $(K_\sim)_\mathbb{C}$.

9. An approach via Nahm’s equations

The results so far have been proved for the case $K = SU(n)$, where from [7] we have a finite-dimensional description of the universal hyperkähler implosion in terms of quivers. However in the current paper we have tried to formulate many of our results in a way that could potentially admit generalisation to other compact groups. In this final section we sketch a gauge-theoretic approach, involving Nahm’s equations, which could provide another means of attacking this problem. We recall that the Nahm equations are the system

$$\frac{dT_i}{dt} + [T_0, T_i] = [T_j, T_k], \quad (ijk) \text{ cyclic permutation of (123)},$$

where $T_i$ take values in $\mathfrak{k}$ and are smooth on some specified interval $I$. Moduli spaces of solutions to the Nahm equations are obtained by quotienting by the gauge action

$$T_0 \mapsto gT_0g^{-1} - gg^{-1}, \quad T_i \mapsto gT_ig^{-1} \quad (i = 1, 2, 3),$$

where $g: I \mapsto K$, subject to appropriate constraints on $g$. The Nahm equations may be interpreted as the vanishing condition for a hyperkähler moment map for the action (9.2) of the group of gauge transformations on an infinite-dimensional flat quaternionic space of $\mathfrak{k}$-valued functions on the interval $I$. In this way Nahm moduli spaces can acquire a hyperkähler structure. In particular Kronheimer [21], Biquard [1] and Kovalev [19] have shown that coadjoint orbits of $K_\mathbb{C}$ may be given hyperkähler structures as moduli spaces of Nahm data on the half-line $[0, \infty)$, while Kronheimer [20] has shown that the cotangent bundle of $K_\mathbb{C}$ may be given a hyperkähler structure as a moduli space of Nahm data on the interval $[0, 1]$. Let us fix a Cartan algebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{k}$ of the compact group $K$. We can consider quadruples $(T_0, T_1, T_2, T_3)$, where each $T_i$ takes values in $\mathfrak{t}$, satisfying the Nahm equations and defined on the half line $[0, \infty)$. We recall from [3] that such a solution has asymptotics

$$T_i = \tau_i + \frac{\sigma_i}{t} + \cdots \quad (i = 1, 2, 3),$$

where $\tau = (\tau_1, \tau_2, \tau_3)$ is a commuting triple, which we shall take to lie in the fixed Cartan algebra $\mathfrak{t}$. Also $\sigma_i = \rho(e_i)$ where $e_1, e_2, e_3$ is a standard basis for $\mathfrak{su}(2)$ and $\rho: \mathfrak{su}(2) \to \mathfrak{k}$ is a Lie algebra homomorphism, so $[\sigma_1, \sigma_2] = \sigma_3$ etc. Moreover we must have $[\tau_i, \sigma_j] = 0$ for $i, j =
1, 2, 3; equivalently, $\rho$ takes values in the Lie algebra $\mathfrak{c}$ of the common centraliser $C(\tau_1, \tau_2, \tau_3)$ of the triple $(\tau_1, \tau_2, \tau_3)$.

We factor out by gauge transformations equal to the identity at $0, \infty$. In addition, we have an action of $T$ by gauge transformations that are the identity at $t = 0$ and take values in $T$ at $t = \infty$. If the common centraliser $C(\tau_1, \tau_2, \tau_3)$ of the triple $(\tau_1, \tau_2, \tau_3)$ is just the maximal torus $T$, then in fact $\rho$ and hence the $\sigma_i$ are zero. The resulting Nahm moduli space, where we quotient by $T$, is exactly Kronheimer’s description of a semisimple orbit as a moduli space [21].

If the centraliser of the triple is larger, we must consider Nahm data asymptotic to the triple, but with various possible $\sigma_i$ terms. The various coadjoint orbits are obtained by choosing $\sigma_i$ and factoring out by gauge transformations that are $I$ at $t = 0$ and lie in $C(\tau_1, \tau_2, \tau_3) \cap C(\sigma_1, \sigma_2, \sigma_3)$ at infinity. The coadjoint orbits will fit together to form the Kostant variety corresponding to our choice of $\tau$. (We recall that the Kostant varieties are the varieties obtained by fixing the values of a generating set of invariant polynomials [18]). The semisimple stratum will correspond to $\sigma_i = 0$.

The universal hyperkähler implosion for an arbitrary compact group $K$ is expected to be a space, which, as in the finite-dimensional $SU(n)$ quiver picture of the preceding section, admits a hyperkähler torus action, with hyperkähler reductions giving the Kostant varieties.

In terms of Nahm data, this should mean that to obtain the universal hyperkähler implosion we do not fix triples $\tau$ but allow them to vary in a fixed Cartan algebra, and that we should factor out only gauge transformations asymptotic to the identity as $t$ tends to $\infty$, so the above $T$ action remains. The moment map for this should formally be evaluation of $(T_1, T_2, T_3)$ at $\infty$; that is, it should give the triple $(\tau_1, \tau_2, \tau_3)$.

Then the hyperkähler quotient by $T$ would be the space of Nahm data on $[0, \infty)$ asymptotic to a fixed triple $(\tau_1, \tau_2, \tau_3)$, modulo gauge transformations equal to the identity at $t = 0$ and $T$-valued at infinity. As mentioned above, if the triple has common centraliser $T$, this exactly gives the corresponding Kostant variety, which in this case is just the regular semisimple orbit.

For a general commuting triple $\tau = (\tau_1, \tau_2, \tau_3)$, the Kostant variety is stratified by different orbits, which as above are obtained by fixing $\tau$ and $\rho$ (or equivalently $\sigma$) and factoring by gauge transformations that take values in the common centraliser $C(\tau, \sigma)$ of $\tau$ and $\sigma$ at infinity. To obtain this Kostant variety via hyperkähler reduction by $T$, we need therefore to perform further collapsings on the moduli space. More precisely, we should collapse by factoring out gauge transformations that are the identity at $t = 0$ but take values at $t = \infty$ in the commutator $[C, C]$, where $C = C(\tau_1, \tau_2, \tau_3)$. So on the open dense set of the moduli space where the triple $(\tau_1, \tau_2, \tau_3)$ has centraliser $T$, no
collapsing occurs. In general, reducing by $T$ will lead to quotienting by $C = C(\tau_1, \tau_2, \tau_3)$. The action of $C$ can be used to bring the $\sigma_i$ into one of a finite list of standard forms (one for each stratum of the Kostant variety corresponding to the choice of $\tau$), and then the remaining freedom lies in $C(\tau, \sigma)$, which is what we need to factor out by to get the coadjoint orbit.

We denote the space obtained via this collapsing by $Q$, and this is a candidate for the universal hyperkähler implosion for the general compact group $K$. We can stratify $Q$ by the centraliser $C(\tau_1, \tau_2, \tau_3)$ of the triple $(\tau_1, \tau_2, \tau_3)$. That is, for each compact subgroup $C$ of $K$, we consider $Q_C$, the space of Nahm triples with $C(\tau_1, \tau_2, \tau_3) = C$, modulo gauge transformations which are the identity at $t = 0$ and take values in $[C, C]$ at $t = \infty$. The top stratum is then the open dense set where the centraliser of the triple is the maximal torus. This stratification agrees in the case $K = SU(n)$ with the stratification by $K_\omega$ in the quiver picture $Q = M \sslash H$, as in Remark 5.2 if we stratify further using $\sigma$ we can obtain strata corresponding to the subsets $Q_{[\sim, O]}$ of $Q$. Notice that the regularity condition (5.8) (which may always be achieved by a generic $SU(2)$ rotation) is the condition of Biquard regularity [4] for the triple $(\tau_1, \tau_2, \tau_3)$, that is $C(\tau_1, \tau_2, \tau_3) = C(\tau_2, \tau_3)$.

**Remark 9.3.** This is analogous to the symplectic case, where we obtain the implosion by taking $K \times t^*_+ \times C$ and collapsing by commutators of points in the Weyl chamber $t^*_+, \circledcirc$ so that in the interior of the chamber no collapsing takes place.

In the symplectic case each stratum can be viewed as a symplectic quotient of a suitable space by the commutator, and hence is itself symplectic. There is an analogous statement in the hyperkähler setting. For we have a decomposition 

$$c^* = \mathfrak{z} + \mathfrak{z}$$

where $c$ is the Lie algebra of the common centraliser $C$ of the triple $(\tau_1, \tau_2, \tau_3)$. Now consider the space of Nahm solutions with $T_i(\infty) \in \mathfrak{z}$ (for $i = 0, 1, 2, 3$), modulo gauge transformations equal to the identity at $0, \infty$. There is a $C$ action on this space by gauge transformations equal to the identity at $t = 0$ and lying in $C$ at $t = \infty$. The formal moment map for this action is evaluation at $\infty$. So the moment map for the $[C, C]$ action is evaluation at $\infty$ followed by projection onto $[\mathfrak{z}, \mathfrak{z}]$. So the formal hyperkähler quotient by $[C, C]$ at level zero is the set of Nahm matrices with $T_i(\infty) \in \mathfrak{z}$ : $i = 1, 2, 3$, modulo the action of $[C, C]$. The stratum $Q_C$, that is, the quotient by $[C, C]$ of the set of Nahm matrices with common centraliser of $T_1(\infty), T_2(\infty), T_3(\infty)$ equal to $C$, is then an open dense subset of this quotient.

It should also be possible to construct a hypertoric variety by considering the sweep under the $T$ action of the set of constant solutions.
(0, c_1, c_2, c_3) : c_i \in \mathfrak{t} to the Nahm equations. On the open subset where no collapsing takes place the $T$ action is free.

The above statements are formal — in order to carry out the programme mentioned at the beginning of this section we need to provide an analytical framework. In particular we need to work out a suitable stratified hyperkähler metric. We conclude by discussing some of the issues in defining such a metric.

As above, we fix a Cartan algebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{k}$ of the compact group $K$. We consider quadruples $(T_0, T_1, T_2, T_3)$, where each $T_i$ takes values in $\mathfrak{k}$, satisfying the Nahm equations and defined on the half line $[0, \infty)$. We form a moduli space $\tilde{M}$ from the above data by quotienting out by gauge transformations that are the identity at $t = 0, \infty$.

The standard $L^2$ metric on Nahm moduli spaces over the interval $[0, \infty)$ is given by

$$\| (X_0, X_1, X_2, X_3) \|^2 = \int_0^\infty \sum_{i=0}^3 \langle X_i, X_i \rangle \, dt$$

for tangent vectors $X = (X_0, \ldots, X_3)$, where $\langle \cdot, \cdot \rangle$ denotes the Killing form on $\mathfrak{k}$.

Write our tangent vector as

$$X_i = \delta_i + \frac{\epsilon_i}{t} + \ldots \quad (i = 1, 2, 3),$$

so

$$\langle X_i, X_i \rangle = \langle \delta_i, \delta_i \rangle + 2\frac{\langle \delta_i, \epsilon_i \rangle}{t} + O\left(\frac{1}{t^r}\right) \quad (r > 1)$$

The $L^2$ metric will not be finite except in tangent directions where $\delta_i = 0$. Such directions correspond to those tangent to the Nahm matrices with a fixed commuting triple $(\tau_1, \tau_2, \tau_3)$.

We may, however, modify our metric following Bielawski [1] thus:

$$| (X_0, X_1, X_2, X_3) |^2 = \int_0^\infty \sum_{i=0}^3 \left( \langle X_i, X_i \rangle - \langle X_i(\infty), X_i(\infty) \rangle \right) dt$$

$$+ c \langle X_i(\infty), X_i(\infty) \rangle,$$

where $c$ is a constant. This defines a symmetric bilinear form, though definiteness and nondegeneracy properties remain unclear. Now $X_i(\infty) = \delta_i$, so we see that the Bielawski pseudometric is finite in directions such that $\langle \delta_i, \epsilon_i \rangle = 0$ for all $i$. So it is finite even in certain directions that correspond to infinitesimally changing the $\tau_i$.

In particular, it is finite on the open dense set of $\tilde{M}$ consisting of all Nahm matrices where the triple $(\tau_1, \tau_2, \tau_3)$ is regular, since, as remarked above, the $\epsilon_i$ terms will be zero for directions tangent to this region.
To analyse which directions are finite in general we need to use the relation $[\tau_i, \sigma_j] = 0$, $(i, j = 1, 2, 3)$. Differentiated, this gives the relation

\[ (9.4) \quad [\delta_i, \sigma_j] + [\tau_i, \epsilon_j] = 0 \quad (i, j = 1, 2, 3) \]
on tangent vectors.

Suppose we stratify Nahm data by the centraliser $C$ of $(\tau_1, \tau_2, \tau_3)$. Let us consider a tangent vector to a stratum.

For all elements in the stratum, and for all $h \in c = \text{Lie}(C)$, we have $[\tau_i, h] = 0$. It follows that for our tangent vector,

\[ (9.5) \quad [\delta_i, h] = 0 \quad \forall h \in c \]

In particular,

\[ [\delta_i, \sigma_j] = 0 \quad (i, j = 1, 2, 3). \]

Now our above relation (9.4) shows

\[ [\tau_i, \epsilon_j] = 0 \quad (i, j = 1, 2, 3) \]

so $\epsilon_j \in c$. If $\epsilon_i$ is a commutator $[\xi, \eta]$, where $\xi, \eta \in c$, we have:

\[
\begin{align*}
\text{tr} \delta_i \epsilon_i &= \text{tr} \delta_i \xi \eta - \text{tr} \delta_i \eta \xi \\
&= \text{tr} \delta_i \xi \eta - \text{tr} \xi \delta_i \eta \\
&= 0
\end{align*}
\]

where in the last step we have used the above observation that $[\xi, \delta_i] = 0$ since $\xi \in c$. By linearity, this also holds if $\epsilon_i$ is a sum of commutators in $c$.

But recall that $[\sigma_1, \sigma_2] = \sigma_3$ etc. So, passing to tangent vectors,

\[ [\sigma_1, \epsilon_2] + [\epsilon_1, \sigma_2] = \epsilon_3 \]

and cyclically. Each $\epsilon_i$ is indeed therefore a sum of commutators in $c$, and hence we see $\langle \delta_i, \epsilon_i \rangle = 0$. We deduce

**Theorem 9.6.** *The Bielawski pseudometric is finite in directions tangent to the set of Nahm matrices with fixed centraliser of $(\tau_1, \tau_2, \tau_3)$.***

**Remark 9.7.** If we stratify Nahm matrices by centraliser of the triple, we therefore obtain a metric in the stratified sense. Note that the top stratum for this stratification is just the set of Nahm matrices where the centraliser of the triple is the maximal torus.

**Remark 9.8.** In fact, the above calculation works provided $[\delta_i, \sigma_j] = 0$ for all $i, j$, as we know from above that $\epsilon_i$ is a sum of commutators $[\xi_k, \eta_k]$ with $\xi_k = \sigma_k$. In particular if $\tau_i = 0$ for all $i$ then by (9.4) this condition holds, so the metric is finite on all tangent vectors, not just those tangent to the stratum.

If $K = SU(2)$ we only have two strata, the regular one and the stratum where all $\tau_i$ are zero. It follows that the metric is finite in this case, which checks as the implosion is now flat $\mathbb{H}^2$.
Remark 9.9. Here we have been considering the Nahm equations on the half-line [0, ∞), motivated by the constructions of coadjoint orbits of $K_C$ in Kronheimer [21], Biquard [4] and Kovalev [19]. However we expect the universal hyperkähler implosion for $K$ to be the complex-symplectic GIT reduction by the maximal unipotent group $N$ of the cotangent bundle $T^*K_C$ of $K_C$, and in [20] $T^*K_C$ is given a hyperkähler structure as a moduli space of Nahm data on the interval [0, 1]. Comparison of this construction with the description in the last section of a hyperkähler implosion $X_{hkimpl}$ when $K = SU(n)$ suggests a formal picture of the universal hyperkähler implosion for $K$ which is similar to that above but uses Nahm data on the interval [0, 1] instead of [0, ∞).

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(Dancer) Jesus College, Oxford, OX1 3DW, United Kingdom
E-mail address: dancer@maths.ox.ac.uk

(Kirwan) Balliol College, Oxford, OX1 3BJ, United Kingdom
E-mail address: kirwan@maths.ox.ac.uk

(Swann) Department of Mathematics, Aarhus University, Ny Munkegade 118, Bldg 1530, DK-8000 Aarhus C, Denmark, and, CP3-Origins, Centre of Excellence for Cosmology and Particle Physics Phenomenology, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark
E-mail address: swann@imf.au.dk