The structure of finite groups and $\theta$-pairs of general subgroups

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Abstract: Using the concept of $\theta$-pairs of proper subgroups of a finite group, we obtain some critical conditions of the supersolvability and nilpotency of finite groups.

Keywords: Finite group, supersolvable group, Nilpotent group, $\theta$-pair

MSC: 20D10, 20D15

1 Introduction

In this paper, all groups considered are finite and $G$ stands for a finite group. Let $\pi(G)$ stand for the set of all prime divisors of $|G|$. We use “$M \vartriangleleft G$” to denote that $M$ is a maximal subgroup of $G$; We write “$N \text{ Char } G$” to mean that $N$ is a characteristic subgroup of $G$. The other notations and terminologies are standard (see [4]).

It is well-know that each maximal subgroup of a solvable group is complement of a chief factor of $G$. Taking this elementary fact as starting point, N. P. Mukherjee and P. Bhattacharya [7] introduced the concept of $\theta$-pairs of maximal subgroups and obtained some conditions for solvability, supersolvability or nilpotency of finite groups by using the properties of $\theta$-pairs of maximal subgroups of $G$. Since then, many interesting results have been subsequently obtained (see [1–3]). Following the idea of Mukherjee and Bhattacharya, X. H. Li and S. H. Li [5] introduced the concept of $\theta$-pairs of general subgroups. Let $H$ be a subgroup of $G$. We call $(A, B)$ a $\theta$-pair of $H$ in $G$ if (i) $A \leq G$, $(H, A) = G$ and $B = (A \cap H)_G$; (ii) If $A_1/B$ is a proper subgroup of $A/B$ and $A_1/B \vartriangleleft G/B$, then $G \neq (H, A_1)$. If $A \vartriangleleft G$, then $(A, B)$ is called a normal $\theta$-pair of $H$ in $G$. They gave sufficient conditions of supersolvability and nilpotence of a finite group by using the concept of $\theta$-pair of some maximal subgroups of Sylow subgroups of $G$.

Given $H \vartriangleleft G$, we denote by $\theta(H)$ the family of all $\theta$-pairs of $H$. A partial ordering $\leq$ on $\theta(H)$ is introduced as follows: $(A, B) \preceq (A', B')$ if $A \preceq A'$. Thus, $\theta(H)$ is a partial set. We call a maximal element with respect to $\leq$ a maximal $\theta$-pair (see[5]). In [10] and [11], Wang introduced the concept of $c$-normal subgroups ($c$-supplemented subgroup). A subgroup $H$ of a group $G$ is said to be $c$-normal ($c$-supplemented) in $G$ if $G$ has a normal subgroup $T$ (if $G$ has a subgroup $T$) such that $HT = G$ and $H \cap T \leq H_G$, where $H_G = \bigcap_{x \in G} H^x$ is the core of $H$ in $G$. Some authors have investigated the structure of a finite group $G$ under the assumption that some subgroups of prime power order of $G$ have those generalized normality in $G$, and obtained many results (see [8, 10] and [11]). By [5, Theorem 1], the definitions of $c$-normality etc. are developed by imposing some conditions on $\theta$-pairs. Hence, all results, obtainable by using a $c$-normal subgroup and $c$-supplemented subgroup of $G$, have analogs in the $\theta$-pairs of subgroups of $G$. Based on the fact that the general subgroup $H$ of $G$ has a maximal $\theta$-pair, then there exists a normal $\theta$-pair which satisfies some conditions (see Lemma 3 (2) in [5]). Thus it is enough to consider the normal $\theta$-pairs
of subgroups. In this paper, we investigate the supersolvability and nilpotency of a group by using the concept of normal $\theta$-pair of subgroups with order $p^k$ of a Sylow $p$-subgroup of a normal subgroup $N$ of $G$ for a given positive integer $k$. In comparison to the assumptions of [8, Theorem 0.1], we consider all cyclic subgroups of order 2 and not all cyclic subgroups of order 2 and 4 when $p = 2$. And we generalize the results in [6].

2 Preliminary results

Lemma 2.1 ([5, Lemma 3 (1)]). Let $H$ be a group of $G$. If $N \lhd G$ and $N \leq D$, then $(C, D)$ is a $\theta$-pair of $H$ in $G$ if and only if $(C/N, D/N)$ is a $\theta$-pair of $H/N$ in $G/N$.

Lemma 2.2 ([6, Lemma 2.2]). Let $N \lhd G$, $Q \leq G$, and $H = NQ < G$. Suppose that $(C, D)$ is a normal $\theta$-pair for $Q$ and $N \not\leq D$. Set $B := (CN \cap H)_G$ and suppose that $BQ < G$. Then there is a subgroup $A$ of $G$ such that $(A, B)$ is a normal $\theta$-pair for $H$ and $A/B$ is a section of $C/D$.

Lemma 2.3 ([5, Lemma 2 (2)]). If $(C, D)$ is a normal maximal $\theta$-pair of a subgroup $H$ of a group $G$, then $D = H_G$.

Lemma 2.4 ([6, Lemma 2.4]). Let $P$ be a $p$-subgroup of $G$ possessing a normal $\theta$-pair $(C, D)$ such that $C/D$ is supersolvable. Then $G$ is solvable.

Lemma 2.5 ([9, Lemma 2.2]). Let $G$ be a group, $p, q$ be different prime divisors of $|G|$, $P$ a non-cyclic Sylow $p$-subgroup of $G$ and $Q$ a Sylow $q$-subgroup of $G$. If any maximal subgroup of $P$ (except one) has a $q$-closed supplement in $G$, then $Q$ is normal in $G$.

Lemma 2.6. Assume that every maximal subgroup of Sylow subgroup $P$ of $G$ has a nilpotent supplement in $G$, then $G$ is nilpotent.

Proof. By [9, Theorem 1.4], $G$ is supersolvable. Let $q = \max 1, p \leq q$, then $Q \lhd G$, where $Q \in Syl_q(G)$. It is clear that $G/\Phi(Q)$ satisfies the hypothesis, then $\Phi(Q) = 1$ by the formation of nilpotent groups is saturated, so $Q$ is an elementary abelian $q$-subgroup. Since $G/Q$ is nilpotent, we have $G/Q = P_1Q/Q \times P_2Q/Q \times \cdots \times P_sQ/Q$, where $s = \pi(G) - 1$ and $i \leq s$. Let $Q_1 \lhd Q$, by the hypothesis, there is a nilpotent subgroup $M$ of $G$ such that $G = Q_1M$, so $P_iQ = P_iQ \cap Q_1M = Q_1(M \cap P_iQ)$, thus $Q_1$ has a nilpotent supplement $M \cap P_iQ$ in $P_iQ$. If $Q$ is cyclic, then $Q_1 = 1$, so $P_iQ = M \cap P_iQ$ is nilpotent, thus $P_i\text{Char}P_iQ$. If $Q$ is non-cyclic, then $P_i\text{Char}P_iQ$ by Lemma 2.5. We conclude that $P_i\text{Char}P_iQ \lhd G$, so $P_i \lhd G$, hence $G$ is nilpotent.

3 Main results

Theorem 3.1. Assume that every Sylow subgroup $P$ of a normal subgroup $N$ of $G$ has a subgroup $U$ with $1 < |U| < |P|$ such that every subgroup $H$ of $P$ of order $|U|$ has a normal $\theta$-pair $(C, D)$ with $C/D$ supersolvable, then $G$ is supersolvable.

Proof. Assume that the result is false and let $G$ be a counterexample with least $|G| + |N|$. By Lemma 2.4, $G$ is solvable.

Step 1. If $L$ is a minimal normal subgroup of $G$ contained in $P \in Syl_p(N)$, then $|L| \leq |U|$.

Suppose that $|L| > |U|$. Then every subgroup $H$ of $L$ of order $|U|$ has a normal $\theta$-pair $(C, D)$ such that $C/D$ is supersolvable. Thus $G = HC$ and $D = (H \cap C)G$. Since $D \leq H < L$, we have $D = 1$ by the minimal normality of $L$ in $G$. Obviously, $C \cap L \lhd G$, then $C \cap L = 1$ or $C \cap L = L$. If the former case is held, then $C \cap H = 1$, so
\[ |G| = |CH| = |CL|, \] that is, \(|H| = |L|\), a contradiction. Thus \( C \cap L = L, L \leq C \), hence \( G = CH = CL = C \).

By a contradiction, \( C/D = G \) is supersolvable, a contradiction.

**Step 2.** For every minimal normal subgroup \( L \) of \( G \) contained in \( N \), the factor group \( G/L \) is supersolvable.

Let \( p \) be a prime and \( |L| = p^a \). Let \( G \) be a \( p \)-Sylow subgroup of \( N \). Assume that \( |L| = |U| \). By the hypothesis, there is a normal \( \theta \)-pair \((C, D)\) for \( L \) such that \( C/D \) is supersolvable, then \( G = LC \) and \( D = (L \cap C)_G = L \cap C \).

By the minimal normality of \( L \) in \( G \), we have \( L \cap C = L \) or \( L \cap C = 1 \). If the former case is true, then \( L \leq C \), that is, \( G = C \) and \( C/D = G/L \) is supersolvable. If the latter case is hold, then \( C/D = C \) is supersolvable, so \( G/L = CL/L \cong C/C \cap L = C \) is supersolvable.

Assume that \( |L| < |U| \). If \( p = q \), then \( L < H \), where \( |H| = |U| \). By the hypothesis, there is a normal \( \theta \)-pair \((C, D)\) for \( H \) and \( C/D \) is supersolvable. Clearly, \( L \leq D \leq C \). Now Lemma 2.1 implies that \((\bar{C}, \bar{D}) \) is a normal \( \theta \)-pair for \( \bar{H} \) and \( \bar{C}/\bar{D} \cong C/D \) is supersolvable. If \( p \neq q \), then \( Q^* \) is a Sylow subgroup of \( N/L \), then by Schur-Zassenhaus theorem, it is easy to prove that \( Q^* = L \times Q \), where \( Q \in Syl_q(N) \). By the hypothesis, \( Q \) has a subgroup \( U \) such that \( 1 < |U| < |Q| \) and every subgroup \( H \) of \( Q \) of order \( |U| \) has a normal \( \theta \)-pair \((C, D)\) such that \( C/D \) is supersolvable. Set \( B := (CL \cap HL)_G \) and observe that \( H^* = HL < G, L \not\leq D \) and \( BH \leq L \), then there is a normal subgroup \( A \) of \( N \) minimal w.r.t. satisfying \( B \leq A \leq C \) and \( HA = N \).

As \( A/B \) is the homomorphic image of the subgroup \( A/H_G \) of the supersolvable group \( C/H_G \), we find that \((A, B)\) is a normal \( \theta \)-pair for \( H \) in \( N \) with \( A/B \) supersolvable. If \( N < G \), then \((N, N)\) satisfies the hypothesis, thus \( N \) is supersolvable by the minimality of \(|G| + |N| \), so \( N \) is \( q \)-closed. If \( N = G \), we can assume that \( \{G_r | r \in \pi(G)\} \) is a Sylow system of \( G \) and \( G = G_qG_r \) for any \( r \in \pi(G) \) with \( r \neq q \). By the similar discuss as above, the hypothesis is still true for \((K, K)\). If \( |\pi(G)| \geq 3 \), then \( N_q < K \), which implies that \( G_q \not< G \), a contradiction. Thus we may assume that \( |G| = |N| = p^aq^b \).

Let \( L \) be a minimal normal subgroup of \( G \), then \( G/L \) is \( q \)-closed by **Step 2**. Since \( q \)-closed is a saturated formation, we may assume that \( L \not< \Phi(G) \) and \( L \) is the only minimal normal subgroup of \( G \). If \( L \) is a \( q \)-group, then \( G_q \not< G \), a contradiction. Thus \( L \leq P \) and so \( L \leq O_p(G) \). We also get \( L \) is not cyclic by **Step 2**, so \( P \) is non-cyclic. Now we show that \( L = O_p(G) \). Let \( W \) be a maximal subgroup of \( G \) such that \( L \not< W \), then \( G = LW \) and \( L \cap W = 1 \). Since \( W \cong G/L, W \) is \( q \)-closed. By \( L \leq O_p(G), G = LW = O_p(G)W \).

From \( O_p(G) \leq F(G) \leq C_G(L), \) it is easy to see that \( L \) and \( W \) normalize \( O_p(G) \cap W, \) thus \( O_p(G) \cap W \not< \). So \( O_p(G) \cap W = 1 \) or \( L \leq O_p(G) \cap W \). If the later case happened, then \( L \leq W \), that is, \( G = L \times W \), a contradiction. So \( O_p(G) \cap W = 1 \), thus \( |O_p(G)| = |G : W| = |L|, \) hence \( L = O_p(G) \). By the hypothesis, \( P \) has a subgroup \( U \) with \( |U| \leq |P| \) such that every subgroup \( H \) of order \( |U| \) has a normal \( \theta \)-pair \((C, D)\) and \( C/D \) is supersolvable, then \( G = HC \) and \( D = (H \cap C)_G \). If \( L \leq H \), then \( G = HW \), so \( W \) is a \( q \)-closed supplement of \( H \) in \( G \). If \( L \not< H \), then \( D = 1 \) by the minimal normality of \( L \) in \( G \), so \( C \) is supersolvable, of course, is \( q \)-closed. Then every maximal subgroup of \( P \) has a \( q \)-closed supplement in \( G, \) so \( N = G \) is \( q \)-closed by Lemma 2.5, a contradiction.

**Step 3.** Let \( q = \max_\pi(N) \) and \( Q \) be a Sylow \( q \)-subgroup of \( N \). By **Step 3**, \( Q \) is normal in \( N \). By **Step 2**, we may assume that \( Q = N = P \). Let \( L \) be a minimal normal subgroup of \( G \) contained in \( N \). Then by the proof of **Step 3**, \( L = O_p(G) = P \). But by **Step 1**, \( |L| \leq |U| < |P| \), a contradiction. This contradiction completes the proof of this theorem.

\( \square \)
Corollary 3.2 ([6, Theorem 3.1]). Assume that every maximal subgroup of any Sylow subgroups of a normal subgroup $N$ of $G$ has a normal $\theta$-pair $(C, D)$ such that $C/D$ is supersolvable, then $G$ is supersolvable.

Theorem 3.3. Assume that every non-cyclic Sylow subgroup $P$ of $G$ has a subgroup $U$ with $1 < |U| < |P|$ such that every subgroup $H$ of $P$ of order $|U|$ has a normal $\theta$-pair $(C, D)$ with $C/D$ supersolvable, then $G$ is supersolvable.

Proof. The proof is similar to Theorem 3.1 and omitted here.

Corollary 3.4. Assume that every minimal subgroup of any non-cyclic Sylow subgroups of $G$ has a normal $\theta$-pair $(C, D)$ and $C/D$ is supersolvable, then $G$ is supersolvable.

Corollary 3.5. Assume that every maximal subgroup of any non-cyclic Sylow subgroups of $G$ has a normal $\theta$-pair $(C, D)$ and $C/D$ is supersolvable, then $G$ is supersolvable.

Theorem 3.6. Assume that every Sylow subgroup $P$ of a normal subgroup $N$ of $G$ has a subgroup $U$ with $1 < |U| < |P|$ such that every subgroup $H$ of $P$ of order $|U|$ has a normal $\theta$-pair $(C, D)$ with $C/D$ nilpotent, then $G$ is nilpotent.

Proof. Assume that the result is false and let $G$ be a counterexample with least $|G|$. By Theorem 3.1, $G$ is supersolvable and so is $N$. Let $p = \max \pi(N), P \in Syl_p(N)$, then $P Char N < G$, so $P < G$.

Lemma 2.6 and the arguments in the Step 2 of the proof of Theorem 3.1 show that there is the unique minimal normal subgroup $L$ of $G$ contained in $P$ such that $G/L$ is nilpotent. By the hypothesis, $P$ has a subgroup $U$ with $1 < |U| < |P|$ such that every subgroup $H$ of $P$ of order $|U|$ has a normal $\theta$-pair $(C, D)$ and $C/D$ is nilpotent. If $H_G = 1$, then $D = 1$. Thus $C$ is nilpotent, so is $G = PC$ by $C < G$ and $P < G$, a contradiction. Thus $H_G \neq 1$, so $L \leq H_G \leq H \leq P_1$, where $P_1 < P$, hence $L \leq \bigcap_{P_1 < P} P_1 = \Phi(P) \leq \Phi(G)$. Then $G/\Phi(G)$ is nilpotent. Since the formation of nilpotent groups is saturated, we have $G$ is nilpotent, a final contradiction.

Corollary 3.7 (see [6, Theorem 3.1]). Assume that every maximal subgroup of any Sylow subgroups of a normal subgroup $N$ of $G$ has a normal $\theta$-pair $(C, D)$ such that $C/D$ is nilpotent, then $G$ is nilpotent.

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