PENCILS OF PLANE CURVES AND CHARACTERISTIC VARIETIES

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ABSTRACT. We give a geometric approach to the relation between the irreducible components of the characteristic varieties of local systems on a plane curve arrangement complement and the associated pencils of plane curves. In the case of line arrangements, this relation was recently discovered by M. Falk and S. Yuzvinsky [15] and the geometric point of view was already hinted at by A. Libgober and S. Yuzvinsky, see [20], Section 7. Our study yields new geometric insight on the translated components of the characteristic varieties relating them to the multiplicities of curves in the associated pencil, in a close analogy to the compact situation treated by A. Beauville [4].

1. INTRODUCTION

Let \( \mathcal{A} \) be a line arrangement in the complex projective plane \( \mathbb{P}^2 \) and let \( M(\mathcal{A}) \) be the corresponding complement. The characteristic varieties \( \mathcal{V}_m(M(\mathcal{A})) \) (resp. the resonance varieties \( \mathcal{R}_m(M(\mathcal{A})) \)) describe the jumping loci for the dimension of the twisted cohomology group \( H^1(M(\mathcal{A}), \mathcal{L}) \), with \( \mathcal{L} \) a rank one local system on the complement \( M(\mathcal{A}) \), (resp. a rank one local system \( \mathcal{L} \) close to the trivial local system), see for details section 3 below.

Recently M. Falk and S. Yuzvinsky [15] have shown that the existence of a global \( d \)-dimensional irreducible component \( E \) in \( \mathcal{R}_1(M(\mathcal{A})) \) with \( d \geq 2 \) is equivalent to the existence of a pencil \( \mathcal{C} \) of plane curves on \( \mathbb{P}^2 \) with an irreducible generic member such that

(i) the pencil \( \mathcal{C} \) has \( d + 1 \) fibers \( \mathcal{C}_b \), for \( b \in B \) a finite subset of \( \mathbb{P}^1 \) with \( |B| = d + 1 \), each one of them being the union of lines in \( \mathcal{A} \) (possibly with some multiplicities);

(ii) these \( d + 1 \) degenerate fibers \( \mathcal{C}_b \) correspond to a partition of the set of lines in \( \mathcal{A} \).

We say that an arrangement of this type is minimal with respect to the pencil \( \mathcal{C} \) and the set \( B \subset \mathbb{P}^1 \), see Definition 2.3 below.

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This surprising equivalence is established in [15] via a combinatorial approach, based on the description of the irreducible components of the resonant variety \( R_1(M(A)) \) in terms of generalized Cartan matrices obtained by Libgober and Yuzvinsky [20].

In this paper, the first aim is to reprove this result in the more general setting of curve arrangement complements \( M = \mathbb{P}^2 \setminus C \), which allows us to grasp the main features of the situation, and, in the end, to better understand even the special case of line arrangements. Indeed, on one hand, a general fiber of a pencil \( C \) associated to a line arrangement is a curve. On the other hand this setting is more flexible, and we can construct easily vivid examples which are hard to find in the class of line arrangements, see Example 6.11 at the end.

The main technical tool is provided by Arapura’s results on the irreducible components of the characteristic variety \( V_1(M) \), see [1], section V and Theorem 3.9 below. To pass from the irreducible components of the characteristic variety to the irreducible components of the resonance variety, we use one of the main result in [14] (see D. Cohen and A. Suciu [8] in the case of line arrangements). Though our Theorem 4.1 is not as precise as the description of the line arrangements in [15], we feel that our geometric approach brings light to what would be otherwise a mysterious property. In fact, we can recover most of the additional results in [15] concerning the combinatoric of the arrangement under the additional hypothesis that all the irreducible curves in our arrangement are smooth and intersecting transversally. In the general case, at each base point one has a pencil of plane curve singularities which can be studied, see for instance [17] and [10].

The second and main aim of this paper is to study the translated components of the characteristic varieties \( V_1(M) \). According to Arapura’s results, such a component \( W \) is described by a pair \((f, \rho)\) where

(a) \( f \) is a surjective morphism \( M \to S = \mathbb{P}^1 \setminus B \), which is nothing else but a not necessarily minimal arrangement with respect to a given pencil \( C \) and the set \( B \), see Proposition 2.2.

(b) \( \rho \) is a torsion character such that \( W \) is the translate by \( \rho \) of a subtorus constructed via \( f \).

Our results can be described briefly as follows:

(A) The set of mappings \( f \) arising in (a) above are parametrized by the (rationally defined) maximal isotropic linear subspaces \( E \subset H^1(M, \mathbb{C}) \). In fact, in the case \( \dim E = 1 \), not all rationally defined maximal isotropic linear subspaces yield components in \( V_1(M) \), see Remark 3.22 (ii).

When \( \dim E \geq 2 \), then this maximal isotropy condition is equivalent to asking \( E \) to be an irreducible component of the resonance variety \( R_1(M) \), see Corollary 3.15, Corollary 3.17, Proposition 3.18 and Corollary 3.21. Moreover in this case the rationality condition is automatically fulfilled, see Remark 3.16.
If the arrangement \( C \) is given (e.g. the equations \( f_j = 0 \) for the components \( C_j \) of \( C \) are known), then the map \( f \) associated to a (rationally defined) maximal isotropic linear subspace \( E \subset H^1(M, \mathbb{C}) \) can be constructed explicitly, see Propositions 3.24 and 3.25. These results can be regarded as a non-proper Castelnuovo-De Franchis Lemma, see [4], [6]. However, it is not clear whether this construction is combinatorial in the case of a line arrangement.

(B) The characters \( \rho \) arising in (b) above for a given map \( f \) are parametrized by the Pontrjagin dual \( T(f)^* = \text{Hom}(T(f), \mathbb{C}^*) \) of a finite group \( T(f) \) defined in terms of the topology of the mapping \( f \). This group depends on the multiple fibers in the pencil associated to \( f \), see the formulas (6.4), (6.7), Theorem 6.3 and Corollary 6.6. For components of dimension at least 2, any character in \( T(f)^* \) actually gives rise to a component, see Proposition 3.18 while for 1-dimensional components one should discard the trivial character in \( T(f)^* \), see Proposition 6.7 which covers a rather general situation.

The 1-dimensional case is the most mysterious, and Suciu’s example of such a component for the deleted \( B_3 \)-arrangement given in [23], [24] played a key role in our understanding of this question. We consider this component in detail in Example 3.11 and at the end of the paper, together with its generalization given by the \( A_m \)-arrangements discussed in [7] and [9], as a good test for our results.

Completely similar results on the characteristic varieties of rank 1 local systems on a compact Kähler manifold were obtained by A. Beauville in [4]. The techniques of proof are rather different and it does not seem easy to obtain our results by using Beauville’s.

In section 2 we collect some basic facts on rational mappings \( f : \mathbb{P}^2 \to \mathbb{P}^1 \) and the associated pencils. Lemma 2.5 intends to clarify the key notion of admissible map used by Arapura in [1].

In section 3 we give the main definitions and properties of characteristic and resonant varieties. Theorem 3.9 collects some (more or less known) facts on the irreducible components of the characteristic varieties, which are derived by a careful reading of Arapura’s paper [1]. We also prove the major part of the claims in (A).

In section 4, Theorem 4.1 is our analog of the main results in [15]. As a consequence, we obtain a necessary numerical condition involving self-intersection numbers for the existence of an essential positive dimensional irreducible component of a characteristic variety. This is practically the same condition as that in Theorem 4.1.1 in Libgober [18], which was established via the use of adjunction ideals.

In section 5 we discuss the complements \( M \) which are in an obvious way \( K(\pi, 1) \)-spaces, i.e. for which the mapping \( f : M \to S \) considered above is a fibration, and we conclude with Example 5.3 where several of the above features are clearly illustrated. In particular we point out several differences with the case of line arrangements.
In the final section we associate to a map $f : M \to S$ as above a finite abelian group $T(f)$, such that the torsion character $\rho$ is determined by a character $\tilde{\rho}$ of $T(f)$, see formula (6.7). Then we compute this group $T(f)$ in terms of the multiplicities of some special fibers of the pencil associated to $f$, see Theorem 6.3. This result explains why usually $T(f) = 1$ and hence there are no translated components associated to $f$. Several key examples are also included here.

A list of very interesting related open questions may be found in Libgober [19].

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2. On rational maps from $\mathbb{P}^2$ to $\mathbb{P}^1$

Let $f : \mathbb{P}^2 \to \mathbb{P}^1$ be a rational map. Then there is a minimal non-empty finite set $A \subset \mathbb{P}^2$ such that $f$ is defined on $U = \mathbb{P}^2 \setminus A$. We recall the following basic fact.

**Proposition 2.1.** Any morphism $f : U \to \mathbb{P}^1$ is given by a pencil $\mathcal{C} : \alpha_1 P_1 + \alpha_2 P_2$ of plane curves having the base locus $V(P_1, P_2)$ the minimal finite set $A$. This pencil is unique up-to an isomorphism of $\mathbb{P}^1$.

**Proof.** We have to show the existence of two homogeneous polynomials $P_1, P_2 \in \mathbb{C}[X, Y, Z]$ of same degree $D$, called the degree of the pencil, such that

1. $V(P_1, P_2) = \{x \in \mathbb{P}^2 \mid P_1(x) = P_2(x) = 0\} = A$ (in particular, these polynomials have no common factor), and
2. for any point $x \in U$, one has $f(x) = (P_1(x) : P_2(x))$.

It is well known, see for instance [16], p. 150, that a morphism $f : U \to \mathbb{P}^1$ is given by a line bundle $\mathcal{L} \in \text{Pic}(U)$ and two sections $s_1, s_2 \in \Gamma(U, \mathcal{L})$ which do not vanish both at any point in $U$. In fact $\mathcal{L} = f^*(\mathcal{O}(1))$ and $s_i = f^*(y_i)$, with $y_1, y_2$ a system of homogeneous coordinates on $\mathbb{P}^1$. With this notation, one has $f(x) = [a : b]$ where $[a : b] \in \mathbb{P}^1$ is such that $as_2(x) - bs_1(x) = 0$.

Since $U$ is smooth, we have $\text{Pic}(U) = \text{Cl}(U)$ and similarly $\text{Pic}(\mathbb{P}^2) = \text{Cl}(\mathbb{P}^2)$, see for instance [16], p. 145. On the other hand, the inclusion $j : U \to \mathbb{P}^2$ induces an isomorphism $j^* : \text{Cl}(\mathbb{P}^2) \to \text{Cl}(U)$, as codim $A = 2$, see [16], p. 133. It follows that $j^* : \text{Pic}(\mathbb{P}^2) \to \text{Pic}(U)$ is also an isomorphism, i.e. any line bundle $\mathcal{L} \in \text{Pic}(U)$ is the restriction to $U$ of a line bundle $\mathcal{O}(D)$ and the global sections of $\mathcal{L}$ are nothing else but the restrictions of global sections of the line bundle $\mathcal{O}(D)$, which are the degree $D$ homogeneous polynomials.

Let $C \subset \mathbb{P}^2$ be a reduced curve such that $C = \bigcup_{j=1,r} C_j$, with $C_j$ irreducible curve of degree $d_j$. We set $M = \mathbb{P}^2 \setminus C$. 

Remark 2.4. If $C$ is a reduced curve.

Lemma 2.5. Let $X$ and $S$ be smooth irreducible algebraic varieties, $\dim S = 1$ and let $f : X \to S$ be a non-constant morphism. Then for any compactification $f' : X' \to S'$ of $f$ with $X'$, $S'$ smooth, the following are equivalent.
(i) The generic fiber $F$ of $f$ is connected.
(ii) The generic fiber $F'$ of $f'$ is connected.
(iii) All the fibers of $f'$ are connected.

If these equivalent conditions hold, then $f'_* : \pi_1(X) \to \pi_1(S)$ and $f'^*_* : \pi_1(X') \to \pi_1(S')$ are surjective.

Proof. Note that $D = X' \setminus X$ is a proper subvariety (not necessarily a normal crossing divisor) with finitely many irreducible components $D_m$. For each such component $D_m$, either $f'_*(D_m)$ is a point, or $f' : D_m \to S'$ is surjective. In this latter case, it follows that $\dim(F' \cap D_m) < \dim D_m \leq \dim F'$. Since $F'$ is smooth of pure dimension, it follows that $F'$ is connected if and only if $F = F' \setminus \cup_m(D_m \cap F')$ is connected. To show that (ii) implies (iii) it is enough to use the Stein factorization theorem, see for instance [16], p. 280, and the fact that a morphism between two smooth projective curves which is of degree one (i.e. generically injective) is in fact an isomorphism.

To prove the last claim for $f$, note that there is a Zariski open and dense subset $S_0 \subset S$ such that $f$ induces a locally trivial topological fibration $f : X_0 = f^{-1}(S_0) \to S_0$ with fiber type $F$. Since $F$ is connected, we get an epimorphism $f'_* : \pi_1(X_0) \to \pi_1(S_0)$. The inclusion of $S_0$ into $S$ induces an epimorphism as well at the level of fundamental groups. Let $j : X_0 \to X$ be the inclusion. Then we have seen that $f \circ j$ induces an epimorphism as well at the level of fundamental groups. Therefore the same is true for $f$. The proof for $f'$ is completely similar. 

\[\square\]

3. Local systems, characteristic varieties, and resonance varieties

3.1. Local systems on $S$. Here we return to the notation $S = \mathbb{P}^1 \setminus B$, with $B = \{b_1, ..., b_k\}$ a finite set of cardinal $|B| = k > 0$. If $\delta_i$ denotes an elementary loop based at some base point $b \in B$ and turning once around the point $b_i$, then using the usual choices, the fundamental group of $S$ is given by

\[(3.1) \quad \pi_1(S) = \langle \delta_1, ..., \delta_k \mid \delta_1 \cdots \delta_k = 1 \rangle.\]

It follows that the first integral homology group is given by

\[(3.2) \quad H_1(S) = \mathbb{Z} < \delta_1, ..., \delta_k > / < \delta_1 + \ldots + \delta_k = 0 >.\]

Therefore, the rank one local systems on $S$ are parametrized by the $(k-1)$-dimensional algebraic torus

\[(3.3) \quad \mathbb{T}(S) = \text{Hom}(H_1(S), \mathbb{C}^*) = \{\lambda = (\lambda_1, ..., \lambda_k) \in (\mathbb{C}^*)^k \mid \lambda_1 \cdots \lambda_k = 1\}.\]
Here $\lambda_j \in \mathbb{C}^*$ is the monodromy about the point $b_j \in B$. For $\lambda \in \mathbb{T}(S)$, we denote by $\mathcal{L}_\lambda$ the corresponding rank one local system on $S$.

The twisted cohomology groups $H^m(S, \mathcal{L}_\lambda)$ are easy to compute. There are two cases.

Case 1 ($\mathcal{L}_\lambda = \mathbb{C}_S$). Then we get the usual cohomology groups of $S$, namely we have $\dim H^0(S, \mathcal{L}_\lambda) = 1$, $\dim H^1(S, \mathcal{L}_\lambda) = k - 1$ and $H^m(S, \mathcal{L}_\lambda) = 0$ for $m \geq 2$.

Case 2 ($\mathcal{L}_\lambda$ is nontrivial). This case corresponds to the case when at least one monodromy $\lambda_j$ is not 1. Then we have $\dim H^0(S, \mathcal{L}_\lambda) = 0$, $\dim H^1(S, \mathcal{L}_\lambda) = k - 2$ and $H^m(S, \mathcal{L}_\lambda) = 0$ for $m \geq 2$.

3.2. Local systems on $M$. Let $\gamma_j$ be an elementary loop around the irreducible component $C_j$, for $j = 1, ..., r$. Then it is known, see for instance [11], p. 102, that

$$H_1(M) = \mathbb{Z} \langle \gamma_1, ..., \gamma_r \rangle / \langle d_1\gamma_1 + ... + d_r\gamma_r = 0 \rangle$$

where $d_j$ is the degree of the component $C_j$. It follows that the rank one local systems on $M$ are parametrized by the algebraic group

$$T(M) = \text{Hom}(H_1(S), \mathbb{C}^*) = \{\rho = (\rho_1, ..., \rho_r) \in (\mathbb{C}^*)^r \mid \rho_1^{d_1} \cdots \rho_r^{d_r} = 1\}.$$ 

The connected component $T^0(M)$ of the unit element $1 \in T(M)$ is the $(r - 1)$-dimensional torus given by

$$T^0(M) = \{\rho = (\rho_1, ..., \rho_r) \in (\mathbb{C}^*)^r \mid \rho_1^{d_1} \cdots \rho_r^{d_r} = 1\}$$

with $D = G.C.D.(d_1, ..., d_r)$ and $e_j = d_j / D$ for $j = 1, ..., r$.

Remark 3.3. If $d_1 = 1$, then $\{\gamma_2, ..., \gamma_r\}$ is a basis for $H_1(M)$ and the torus $T(M)$ can be identified to $(\mathbb{C}^*)^{r-1}$ under the projection $\rho \mapsto (\rho_2, ..., \rho_r)$.

Now the computation of the twisted cohomology groups $H^m(M, \mathcal{L}_\rho)$ is one of the major problems. The case when $\mathcal{L}_\rho = \mathbb{C}_M$ is easy, and the result depends on the local singularities of the plane curve $C$. In fact $\dim H^0(M, \mathbb{C}) = 1$, $\dim H^1(M, \mathbb{C}) = r - 1$ and $H^m(M, \mathbb{C}) = 0$ for $m \geq 3$. To determine the remaining Betti number $b_2(M) = \dim H^2(M, \mathbb{C})$, the same as determining the Euler characteristic $\chi(M) = 3 - \chi(C)$ and this can be done, e.g. by using the formula for $\chi(C)$ given in [11], p. 162.

In the sequel we concentrate on the case $\mathcal{L}_\rho \neq \mathbb{C}_M$ and assume $\chi(M)$ known. Then we have $H^m(M, \mathcal{L}_\rho) = 0$ for $m = 0$ and $m \geq 2$, and $\dim H^2(M, \mathcal{L}_\rho) = \chi(M)$, see for instance [12], p. 49. To study these cohomology groups, one idea is to study the characteristic varieties

$$\mathcal{V}_m(M) = \{\rho = (\rho_1, ..., \rho_r) \in (\mathbb{C}^*)^r \mid \dim H^1(M, \mathcal{L}_\rho) \geq m\}.$$ 

Definition 3.4. An irreducible component $W$ of such an $m$-th characteristic variety $\mathcal{V}_m(M)$ is called a coordinate component (resp. a translated coordinate component) if $W$ is contained in a subgroup $T_j$ of $T(M)$ defined by an equality $\rho_j = 1$ for some $j$ (resp. there is a torsion character $\rho \in T(M)$ such that $W \subset \rho T_j$ for some $j$). An
irreducible component \( W \) which is not a translated coordinate component is called a global component.

Note that if \( 1 \in W \), then \( W \) is a coordinate component if and only if \( W \) is a translated coordinate component.

Let \( C(j) \) be the plane curve obtained from \( C \) by discarding the \( j \)-th component \( C_j \). Let \( M(j) = \mathbb{P}^2 \setminus C(j) \) be the corresponding complement. Then the inclusion \( \iota_j : M \to M(j) \) induces an epimorphism \( H_1(M) \to H_1(M(j)) \) and hence an embedding \( \iota_j^* : \mathbb{T}(M(j)) \to \mathbb{T}(M) \).

**Definition 3.5.** An irreducible component \( W \) of the \( m \)-th characteristic variety \( V_m(M) \) is called a non-essential component, or a pull-back component if \( W = \iota_j^*(W_j) \) for some \( j \) and some irreducible component \( W_j \) of the \( m \)-th characteristic variety \( V_m(M(j)) \), see [2], [15], [18]. An irreducible component \( W \) which is not non-essential is called an essential component.

**Remark 3.6.** The notions of coordinate and (non-)essential component depend on the curve arrangement \( C = \cup C_i \), i.e. on the chosen embedding of \( M \) into \( \mathbb{P}^2 \). So they are not invariants of the surface \( M \). For more on this see [2].

Assume given a surjective morphism \( f : M \to S \) such that \( f^* : \pi_1(M) \to \pi_1(S) \) is surjective. This gives rise to an embedding \( f^* : \mathbb{T}(S) \to \mathbb{T}(M) \), which implies in particular \( k \leq r \). More precisely, if we start with \( \mathcal{L}_\lambda \in \mathbb{T}(S) \), then the monodromy \( \rho_j \) of the pull-back local system \( f^*\mathcal{L}_\lambda = \mathcal{L}_\rho \) is given by

(i) \( \rho_j = 1 \) if the component \( C_j \) is not in the first case of Proposition 2.2 and by

(ii) \( \rho_j = \lambda_i^{m_j} \) if the component \( C_j \) is in the first case of Proposition 2.2, i.e. \( g_1(C_j) = b_i \) in the notation from the proof of Proposition 2.2. Recall that \( m_j \) is the multiplicity of \( C_j \) in \( f^{-1}(b_i) \).

**Corollary 3.7.** With the above notation, the pull-back local system \( f^*\mathcal{L}_\lambda = \mathcal{L}_\rho \) satisfies \( \rho_j \neq 1 \) for all \( j = 1, ..., r \) if and only if

(i) The curve \( C \) consists exactly of the fibers of the associated pencil \( \mathcal{C} \) corresponding to the points in \( B \).
(ii) For all \( j = 1, ..., r \), if we set \( g_1(C_j) = b_{i(j)} \), then \( \lambda_{i(j)}^{m_{ij}} \neq 1 \).

### 3.8. Arapura’s results

We recall here some of the main results from [1], applied to the rank one local systems on \( M \), with some additions from [18], [14] and some new consequences.

**Theorem 3.9.** Let \( W \) be an irreducible component of \( V_1(M) \) and assume that \( d_W := \dim W \geq 1 \). Then there is a surjective morphism \( f_W : M \to S_W \) with connected generic fiber \( F(f_W) \), and a torsion character \( \rho_W \in \mathbb{T}(M) \) such that

\[
W = \rho_W \otimes f_W^*(\mathbb{T}(S_W)).
\]
More precisely, the following hold.

(i) \( S_W = \mathbb{P}^1 \setminus B_W \), with \( B_W \) a finite set satisfying \( k_W := |B_W| = d_W + 1 \).

(ii) For any local system \( \mathcal{L} \in W \), the restriction \( \mathcal{L}|_{F(f_W)} \) of \( \mathcal{L} \) to the generic fiber of \( f_W \) is trivial, i.e. \( \mathcal{L}|_{F(f_W)} = \mathbb{C}_F(f_W) \).

(iii) If \( N_W \) is the order of the character \( \rho_W \), then there is a commutative diagram

\[
\begin{array}{ccc}
M' & \xrightarrow{p} & M \\
\downarrow{f'_W} & & \downarrow{f_W} \\
S'_W & \xrightarrow{q} & S_W
\end{array}
\]

where \( p \) is a unramified \( N_W \)-cyclic Galois covering, \( q \) is possibly ramified \( N_W \)-cyclic Galois covering, \( f'_W \) is \( \mu_{N_W} \)-equivariant in the obvious sense, has a generic fiber \( F(f'_W) \) isomorphic to the generic fiber \( F(f_W) \) of \( f_W \), and \( p^*\rho_W \) is trivial. Here \( \mu_{N_W} \) denotes the cyclic group of the \( N_W \)-th roots of unity.

(iv) If \( 1 \in W \) and \( \mathcal{L} \in W \), then \( \dim H^1(M, \mathcal{L}) \geq -\chi(S) = d_W - 1 \) and equality holds with finitely many exceptions.

(v) If \( 1 \in W \), then \( d_W \geq 2 \).

(vi) If \( 1 \notin W \) and \( d_W \geq 2 \), then the subtorus \( W' = f_W^*(\mathbb{T}(S_W)) \) is another irreducible component of \( V_1(M) \). In this situation, \( W' \) is a coordinate component if and only if \( W \) is a translated coordinate component.

**Proof.** The first claim is just Thm. 1.6 in [1], section V.

Now we prove the claims (i) and (vi). The fact that \( S_W \) has to be rational in this situation is noted in [18], and it follows from the fact that a compactification \( \overline{M} \) of \( M \) is simply-connected and hence, via Lemma 2.3, we get that a compactification \( \overline{S_W} \) of \( S_W \) is simply-connected as well. A different proof follows from Proposition 5.10 (2) in [14]. When \( 1 \in W \), the equality \( k_W = d_W + 1 \) was noted in [18], see also Proposition 6.3 in [14].

Consider now the situation \( 1 \notin W \). Then there are two cases.

**Case 1.** \( k_W = 2 \). Then \( S_W = \mathbb{C}^* \) and hence \( 1 \leq d_W \leq \dim \mathbb{T}(S_W) = 1 \).

**Case 2.** \( k_W \geq 3 \). Then \( \chi(S_W) < 0 \) and \( W' = f_W^*(\mathbb{T}(S_W)) \) is another irreducible component of \( V_1(M) \) by Prop. 1.7 in [1]. Since \( 1 \in W' \), \( d_{W'} = d_W \), we get the equality \( k_W = d_W + 1 \). Moreover, this implies \( d_W \geq 2 \), i.e. we get the claim (vi) as well.

Now we prove the claim (ii). Since \( W = \rho_W \otimes f_W^*(\mathbb{T}(S_W)) \), it is enough to prove that \( \rho_W|_{F(f_W)} = \mathbb{C}_F(f_W) \). And this is proved in the final part of the proof of Prop. 1.3 in [1]. Just note that on the last line of this proof, one should replace “which forces \( \psi \) to be trivial” by “which forces \( \psi|(F \cap X) \) to be trivial”. (This is due to
the fact that $F$ in $[1]$ denotes the compactification of our $F = F(f_W)$, and $X$ in $[1]$ corresponds to our $M$.)

The claim (iii) is just the “untwisting” part of the proof of Thm. 1.6 in $[1]$. The existence of the diagram is explained there via the Stein factorization for $f_W \circ p$. However, the fact that the morphism $q$ has degree $N_W$ depends on the previous claim (ii), and this key point is not mentioned in $[1]$.

The proof of the claim (iv) is more technical. Using the Projection Formula

$$p_*(C_{M'}) \otimes \mathcal{L} \simeq p_*(p^*(\mathcal{L}))$$

for $\mathcal{L} \in W$, see for instance $[12]$, p.42 and then the Leray Spectral Sequence for $p$, see for instance $[12]$, p. 33, one gets an isomorphism of $\mu_{N_W}$-representations

$$H^1(M', p^*\mathcal{L}) = H^1(M, p_*(C_{M'}) \otimes \mathcal{L}).$$

Following the argument in the proof of Thm. 1.6 in $[1]$, we get the following

$$\dim H^1(M, \mathcal{L}) \geq -\chi(S_W) = k_W - 2 = d_W - 1.$$  

The only point which deserves some attention is the fact that $S_W$ and $S'_W$ do not admit finite triangulations as claimed in $[1]$, since they are not compact. However, we can replace them by finite simplicial complexes without changing the topology, e.g. $S'_W$ can be replaced by the compact Riemann surface with boundary obtained from $\mathbb{P}^1$ by deleting small open discs centered at the points in $B_W$.

The fact that there are only finitely many local systems $\mathcal{L} \in W$ such that $\dim H^1(M, \mathcal{L}) \geq d_W$ follows by an argument similar to the end of the proof of Prop. 1.7 in $[1]$, section V, see for details Remark 3.20 below.

Finally, the claim (v) is well-known, see for instance $[20]$ or Cor. 6.4 in $[14]$. 

Note that the claim (vi) above is obviously false for $d_W = 1$ by (v). A deeper fact is that (iv) is false when 1 $\notin W$, see Corollary 6.8.

Remark 3.10. Conversely, if $f : M \to S$ is a morphism with a generic connected fiber and with $\chi(S) < 0$, then $W_f = f^*(\mathbb{T}(S))$ is an irreducible component in $\mathcal{V}_1(M)$ such that $1 \in W_f$ and $\dim W_f \geq 2$, see $[1]$, Section V, Prop. 1.7. Some basic situations of this general construction are the following.

(i) The local components, see for instance $[23]$, subsection (2.3) in the case of line arrangements. The case of curve arrangements runs as follows. Let $p \in \mathbb{P}^2$ be a point such that there is a degree $d_p$ and an integer $k_p > 2$ such that

1. the set $A_p = \{ j \mid p \in C_j and \ \deg C_j = d_p \}$ has cardinality $k_p$;
2. $\dim < f_j \mid j \in A_p >= 2$, with $f_j = 0$ being an equation for $C_j$. 
If \( \{P, Q\} \) is a basis of this 2-dimensional vector space, then the associated pencil induces a map
\[
f_p : M \to S_p
\]
where \( S_p \) is obtained from \( \mathbb{P}^1 \) by deleting the \( k_p \) points corresponding to the curves \( C_j \), for \( j \in A_p \). In this way, the point \( p \) produces an irreducible component in \( \mathcal{V}_1(M) \), namely
\[
W_p = f_p^*(\mathcal{T}(S_p))
\]
of dimension \( k_p - 1 \), and which is called local because it depends only on the chosen point \( p \). Note that in the case of line arrangements \( p \) can be chosen to be any point of multiplicity at least 3.

(ii) The components associated to neighborly partitions, see [20], corresponds exactly to pencils associated to the line arrangement, as remarked in [15], see the proof of Theorem 2.4.

All these points are illustrated by the following.

**Example 3.11.** This is a key example discovered by A. Suciu, see Example 4.1 in [23] and Example 10.6 in [24]. Consider the line arrangement in \( \mathbb{P}^2 \) given by the equation
\[
xyz(x - y)(x - z)(y - z)(x - y - z)(x - y + z) = 0.
\]

We number the lines of the associated affine arrangement in \( \mathbb{C}^2 \) (obtained by setting \( z = 1 \)) as follows: \( L_1 : x = 0, L_2 : x - 1 = 0, L_3 : y = 0, L_4 : y - 1 = 0, L_5 : x - y - 1 = 0, L_6 : x - y = 0 \) and \( L_7 : x - y + 1 = 0 \), see the pictures in Example 4.1 in [23] and Example 10.6 in [24]. As stated in Example 4.1 in [23], there are

(i) Seven local components: six of dimension 2, corresponding to the triple points, and one of dimension 3, for the quadruple point.

(ii) Five components of dimension 2, passing through 1, coming from the following neighborly partitions (of braid subarrangements): \((15|26|38), (28|36|45), (14|23|68), (16|27|48)\) and \((18|37|46)\). For instance, the pencil corresponding to the first partition is given by \( P = L_1L_5 = x(x - y - z) \) and \( Q = L_2L_6 = (x - z)(x - y) \). Note that \( L_3L_6 = yz = Q - P \), a fiber in this pencil.

(iii) Finally, there is a 1-dimensional component \( W \) in \( \mathcal{V}_1(M) \) with
\[
\rho_W = (1, -1, -1, 1, 1, -1, 1, -1) \in \mathcal{T}(M) \subset (\mathbb{C}^*)^8
\]
and \( f_W : M \to \mathbb{C}^* \) given by
\[
f_W(x : y : z) = \frac{x(y - z)(x - y - z)^2}{(x - z)y(x - y + z)^2}
\]
or, in affine coordinates
\[ f_W(x, y) = \frac{x(y - 1)(x - y - 1)^2}{(x - 1)y(x - y + 1)^2}. \]

Then \( W \subset V_1(M) \) and \( W \cap V_2(M) \) consists of two characters, \( \rho_W \) above and \( \rho'_W = (-1, 1, 1, -1, 1, -1) \).

Note that this component \( W \) is a translated coordinate component. This is related to the fact that the associated pencil is special. More on this aspect at the end of the paper.

It is clear that any non-essential component is a coordinate component. The following converse result on positive dimensional coordinate components \( W \) of \( V_m(M) \) was obtained by Libgober [18]. For reader’s convenience we include a proof.

**Proposition 3.12.** Any positive dimensional translated coordinate component of \( V_m(M) \) is non-essential.

**Proof.** Let \( W = \rho_W \otimes f_W^*(T(S_W)) \) be a positive dimensional irreducible component of \( V_m(M) \). Assume \( W \) is contained in the subtorus of \( \mathbb{T}^r \) given by \( \rho_j = 1 \). It follows that the corresponding component \( \rho_{W,j} \) of the character \( \rho \) is 1, and that the torus \( \mathbb{T}_W = f_W^*(T(S_W)) \) is also contained in the same subtorus. The discussion before Corollary 3.7 implies that the corresponding component \( C_j \) of \( C \) is not in the first case of Proposition 2.2. This in turn implies the existence of an extension \( f(j) : M(j) \to S_W \), whose generic fibers are still connected (being obtained from those of \( f \) by adding at most finitely many points). It follows that \( W = \iota_j^*(W_j) \), with \( \iota_j : M \to M(j) \) the inclusion and \( W_j = \rho_j \otimes f(j)^*(T(S_W)) \), where the character \( \rho_j \) is obtained from \( \rho_W \) by discarding the \( j \)-th component. To show that \( W_j \) is an irreducible component in \( V_m(M(j)) \), we can use Proposition 3.18 in the case \( \chi(S_W) < 0 \) and Corollary 3.21 in the case \( S_W = \mathbb{C}^* \) (the fact that \( K_{f(j)} = (\iota_j^*)^{-1}(K_f) \) is a rationally defined maximal isotropic subspace is obvious). In the both cases we have to use in addition the equality

\[ \dim H^1(M, \mathcal{L}_1 \otimes f_W^*\mathcal{L}_2) = \dim H^1(S_W, R^0 f_W^*\mathcal{L}_1 \otimes \mathcal{L}_2) = \]

\[ = \dim H^1(S_W, R^0 f(j)^*\mathcal{L}_1' \otimes \mathcal{L}_2) = \dim H^1(M(j), \mathcal{L}_1' \otimes (f(j)^*\mathcal{L}_2)). \]

Here \( \mathcal{L}_1 \in \mathcal{T}(M) \) (resp. \( \mathcal{L}_1' \in \mathcal{T}(M(j)) \) is the local system corresponding to \( \rho_W \) (resp. \( \rho_j \)), the first and the third equalities come from Remark 3.20 while the middle equality comes from \( \mathcal{L}_1' = \iota_j^*\mathcal{L}_1 \) and \( f(j) \circ \iota_j = f_W \).

\[ \square \]

In view of this result, it is natural to study first the non-coordinate positive dimensional components. Indeed, the other components come from simpler arrangements,
involving fewer components $C_j$’s. The situation of translated components is different, e.g. the component $W$ studied in Example 3.11 is NOT coming from a simpler arrangement. The case of 0-dimensional components is very interesting as well, see [2], [23], [24].

3.13. Resonance varieties. Let $H^*(M, \mathbb{C})$ be the cohomology algebra of the surface $M$ with $\mathbb{C}$-coefficients. Right multiplication by an element $z \in H^1(M, \mathbb{C})$ yields a cochain complex $(H^*(M, \mathbb{C}), \mu_z)$. The resonance varieties of $M$ are the jumping loci for the degree one cohomology of this complex:

\begin{equation}
R_m(M) = \{ z \in H^1(M, \mathbb{C}) \mid \dim H^1(H^*(M, \mathbb{C}), \mu_z) \geq m \}.
\end{equation}

One of the main results in [14] gives the following. For the case of hyperplane arrangements see [8].

**Theorem 3.14.** The exponential map $\exp : H^1(M, \mathbb{C}) \rightarrow T^0(M)$ induces for any $m \geq 1$ an isomorphism of analytic germs

$$(R_m(M), 0) \simeq (V_m(M), 1).$$

The following easy consequence will play a key role.

**Corollary 3.15.** The irreducible components of $R_1(M)$ are precisely the maximal linear subspaces $E \subset H^1(M, \mathbb{C})$, isotropic with respect to the cup product on $M$

$$\cup : H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \rightarrow H^2(M, \mathbb{C})$$

and such that $\dim E \geq 2$.

**Proof.** Let $E$ be a component of $R_1(M)$. By the above Theorem there is a component $W$ in $V_1(M)$ such that $1 \in W$ and $T_1 W = E$. By Theorem 3.9 we can write $W = f^*(T(S))$, and hence $T_1 W = f^*(H^1(S, \mathbb{C}))$ is isotropic with respect to the cup product, since the cup product on $H^1(S, \mathbb{C})$ is trivial. Maximality of $E$ comes from the fact that $E$ is a component of $R_1(M)$. The restriction $\dim E \geq 2$ comes from Theorem 3.9 (v).

\[\square\]

**Remark 3.16.** It follows from the proof of Corollary 3.15 that any maximal isotropic linear subspaces $E \subset H^1(M, \mathbb{C})$ is rationally defined, i.e. there is a linear subspace $E_\mathbb{Q} \subset H^1(M, \mathbb{Q})$ such that $E = E_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$ under the identification $H^1(M, \mathbb{C}) = H^1(M, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$. Indeed, one can take $E_\mathbb{Q} = f^*(H^1(S, \mathbb{Q}))$.

We can restate the above Corollary as follows.

**Corollary 3.17.** If $f : M \rightarrow S$ is a surjective morphism with connected generic fiber $F$ and $\rho \in T(M)$ is a torsion character such that $W = \rho \otimes f^*(T(S))$ is an irreducible component of $V_1(M)$ with $\dim W \geq 2$, then $K_f = f^*(H^1(S, \mathbb{C}))$ is a (rationally defined) maximal isotropic subspace in $H^1(M, \mathbb{C})$ with respect to the cup-product.
Proof. Using Theorem 3.9 (vi), we can take \( \rho = 1 \). Then \( K_f \) is exactly the tangent space at \( 1 \in W \), and, by Theorem 3.14, an irreducible component of \( R_1(M) \). In addition, \( K_f \) is obviously an isotropic subspace in \( H^1(M, \mathbb{C}) \) with respect to the cup-product. It should be maximal, since any strictly larger isotropic subspace would contradict the fact that \( K_f \) is an irreducible component of \( R_1(M) \).

\[ \square \]

Now we can state the following key result, which can be regarded as a strengthening of Remark 3.10.

**Proposition 3.18.** Let \( f : M \to S \) be a surjective morphism with a generic connected fiber, such that \( S = \mathbb{P}^1 \setminus B \) with \( \chi(S) < 0 \). Then \( K_f = f^*(H^1(S, \mathbb{C})) \) is a (rationally defined) maximal isotropic subspace in \( H^1(M, \mathbb{C}) \) with respect to the cup-product and for any character \( \rho \in \mathbb{T}(M) \) with \( L_\rho|_F = \mathbb{C}_F \) for a generic fiber \( F \) of \( f \), the translate subtorus

\[ W_{f,\rho} = \rho \otimes f^*(\mathbb{T}(S)) \]

is an irreducible component in \( V_1(M) \) such that \( \dim W_{f,\rho} = -\chi(S) + 1 \geq 2 \).

In the proof, we use the following version of projection formula, which is used very often, e.g. [1], [18], but for which I was not able to find a reference.

**Lemma 3.19.** For any local system \( L_1 \) on \( M \) and any local system \( L_2 \) on \( S \), one has

\[ (Rf_1^* L_1) \otimes L_2 = Rf_1^* (L_1 \otimes f^{-1} L_2). \]

**Proof.** To prove this Lemma, we start with the usual projection formula, i.e. with the above notation

\[ (Rf_1^* L_1) \otimes L_2 = Rf_1^* (L_1 \otimes f^{-1} L_2) \]

see Thm. 2.3.29, p.42 in [12]. Let \( Z \) be a connected smooth complex algebraic variety of dimension \( m \). Then the dualizing sheaf \( \omega_Z \) is just \( \mathbb{C}_Z[2m] \) and \( D_Z L = L^\vee[2m] \) for any local system \( L \) on \( Z \), see Example 3.3.8, p.69 in [12]. Note also that for two bounded constructible complexes \( A^\bullet \) and \( B^\bullet \) in \( D_b^c(Z, \mathbb{C}) \) we have the isomorphisms

\[ D_Z A^\bullet \otimes B^\bullet = R\text{Hom}(A^\bullet, \omega_Z) \otimes B^\bullet = R\text{Hom}(A^\bullet, \omega_Z \otimes B^\bullet) = R\text{Hom}(A^\bullet, B^\bullet)[2m]. \]

It follows that

\[ D_Z (A^\bullet \otimes B^\bullet) = R\text{Hom}(A^\bullet \otimes B^\bullet, \omega_Z) = R\text{Hom}(A^\bullet, R\text{Hom}(B^\bullet, \omega_Z)) \]

\[ = D_Z A^\bullet \otimes D_Z B^\bullet[-2m]. \]

For the second isomorphism here we refer to Prop. 10.23, p.175 in [3]. Apply now the duality functor \( D_S \) to the projection formula (3.11). In the left hand side we
get \( D_S((Rf_2\mathcal{L}_1) \otimes \mathcal{L}_2) = D_S(Rf_2\mathcal{L}_1) \otimes D_S(\mathcal{L}_2)[2] = Rf_2(D_M\mathcal{L}_1) \otimes D_S(\mathcal{L}_2)[2] = Rf_2(\mathcal{L}_1') \otimes \mathcal{L}_2'^2 \)[4]. Except the isomorphisms explained above we have used here the isomorphism \( D_S Rf = Rf D_M \), see Cor. 4.1.17, p.90 in [12]. Similarly, the in the right hand side we get \( D_S Rf_1(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = Rf_2 D_M(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = Rf_2(\mathcal{L}_1' \otimes (f^{-1}\mathcal{L}_2)^{\vee})[4] \). Since \((f^{-1}\mathcal{L}_2)^{\vee} = f^{-1}(\mathcal{L}_2^2) \) and since any local system is the dual of its own dual, the proof is completed.

\[ \square \]

**Proof.** Now we prove Proposition 3.18. The first claim follows from Remark 3.10 and Corollary 3.17. Let \( \mathcal{L}_1 = \mathcal{L}_p \) and \( \mathcal{L}_2 \) be any rank 1 local system on \( S \). To estimate \( \dim H^1(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \), we use the Leray spectral sequence

\[
E_2^{p,q} = H^p(S, R^q f_*(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2))
\]

converging to \( H^{p+q}(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \). This spectral sequence degenerates at \( E_2 \) since \( E_2^{p,q} = 0 \) for \( p \notin \{0, 1\} \) by Artin Theorem, see Thm.4.1.26, p.95 in [12]. By Lemma 3.19 we have

\[
R^q f_*(\mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) = R^q f_*(\mathcal{L}_1) \otimes \mathcal{L}_2.
\]

In particular, in this way, the above spectral sequence yields the following exact sequence

\[
(3.14)
0 \to H^1(S, R^0 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2) \to H^1(M, \mathcal{L}_1 \otimes f^{-1}\mathcal{L}_2) \to H^0(S, R^1 f_*(\mathcal{L}_1) \otimes \mathcal{L}_2) \to 0.
\]

Note that \( \mathcal{F} = R^0 f_*(\mathcal{L}_1) \) is in general no longer a local system on \( S \), but a *constructible sheaf*. By definition, it exists a minimal finite set \( \Sigma \subset S \) such that \( \mathcal{F}|(S \setminus \Sigma) \) is a local system. If \( S' \subset S \) is a Zariski open subset such that the restriction \( f': M' \to S' \) with \( M' = f^{-1}(S') \), is a topologically locally trivial fibration, it follows that \( \mathcal{F}|S' \) is a local system of rank 1. Indeed, for \( s \in S' \) we have

\[
\mathcal{F}_s = \lim_{s \in D} \mathcal{F}(D) = \lim_{s \in D} H^0(f^{-1}(D), \mathcal{L}_1) = \mathbb{C}.
\]

Here the limit is taken over all the sufficiently small open discs \( D \) in \( s \) centered at \( s \), and the last equality comes from the fact that the inclusion \( F_s = f^{-1}(s) \to f^{-1}(D) \) is a homotopy equivalence and \( \mathcal{L}_1|F_s = \mathbb{C}_{F_s} \) (recall that \( F_s \) is connected). In particular \( \Sigma \subset S \setminus S' \). The above argument shows also that \( \dim \mathcal{F}_s \leq 1 \) for any \( s \in \Sigma \) (since \( f^{-1}(D) \) is connected as well).

To estimate \( \dim H^1(S, \mathcal{F} \otimes \mathcal{L}_2) \) we compute

\[
\chi(S, \mathcal{F} \otimes \mathcal{L}_2) = \dim H^0(S, \mathcal{F} \otimes \mathcal{L}_2) - \dim H^1(S, \mathcal{F} \otimes \mathcal{L}_2)
\]

using Thm. 4.1.22, p.93 in [12]. We get

\[
\chi(S, \mathcal{F} \otimes \mathcal{L}_2) = \chi(S \setminus \Sigma) + \sum_{s \in \Sigma} \dim \mathcal{F}_s.
\]
It follows that
\[
\dim H^1(S, F \otimes L_2) = \dim H^0(S, F \otimes L_2) - \chi(S) + \sum_{s \in \Sigma} (1 - \dim F_s) \geq -\chi(S).
\]

It follows that the translate torus \( W_{f,\rho} = \rho \otimes f^* (\mathbb{T}(S)) \) is contained in some irreducible component \( W = \rho_1 \otimes f^* (\mathbb{T}(S_1)) \) of \( V_1(M) \). Moreover, we can take \( \rho = \rho_1 \). Then, taking the tangent spaces of \( W_{f,\rho} \) and of \( W \) at the common point \( \rho \) we get
\[
K_f \subset f^* (H^1(S_1, \mathbb{C})).
\]
But \( f^* (H^1(S_1, \mathbb{C})) \) is an isotropic subspace, and \( K_f \) is maximal with this property, hence \( K_f = f^* (H^1(S_1, \mathbb{C})) \), and therefore \( W_{f,\rho} = W \).

**Remark 3.20.** In the proof of Proposition 1.7 in Arapura [1], section V, it is shown that for any \( L_1 \in \mathbb{T}(M) \) with trivial restriction to generic fibers of \( f \), one has
\[
H^0(S, R^1 f_*(L_1) \otimes L_2) = 0
\]
for all but finitely many local systems \( L_2 \in \mathbb{T}(S) \). Actually the proof in Arapura [1] is given in the case \( L_1 = \mathbb{C}_M \), but the same approach using properties of constructible sheaves yields the general case stated above.

Using this, the exact sequence (3.14) yields
\[
H^1(M, L_1 \otimes f^{-1} L_2) = H^1(S, R^0 f_*(L_1) \otimes L_2)
\]
for all but finitely many local systems \( L_2 \in \mathbb{T}(S) \). In particular, when \( L_1 = \mathbb{C}_M \), then \( R^0 f_*(L_1) = \mathbb{C}_S \) and the formula (3.15) implies that
\[
\dim H^1(M, f^{-1} L_2) = -\chi(S)
\]
for all but finitely many local systems \( L_2 \in \mathbb{T}(S) \).

**Corollary 3.21.** If \( f : M \to \mathbb{C}^* \) is a surjective morphism with connected generic fiber \( F \) and \( \rho \in \mathbb{T}(M) \) is a torsion character such that \( W = \rho \otimes f^* (\mathbb{T}(\mathbb{C}^*)) \) is an irreducible component of \( V_1(M) \) with \( \dim W = 1 \), then \( K_f = f^* (H^1(\mathbb{C}^*, \mathbb{C})) \) is a rationally defined maximal isotropic subspace in \( H^1(M, \mathbb{C}) \) with respect to the cup-product.

Conversely, let \( f : M \to \mathbb{C}^* \) be a morphism with a generic connected fiber. Assume that \( K_f = f^* (H^1(\mathbb{C}^*, \mathbb{C})) \) is a rationally defined maximal isotropic subspace in \( H^1(M, \mathbb{C}) \) with respect to the cup-product. Then for any character \( \rho \in \mathbb{T}(M) \) with \( \mathcal{L}_\rho | F = \mathbb{C}_F \) for a generic fiber \( F \) of \( f \) and \( \rho \notin f^* (\mathbb{T}(\mathbb{C}^*)) \), the translated subtorus
\[
W_{f,\rho} = \rho \otimes f^* (\mathbb{T}(\mathbb{C}^*))
\]
is either an irreducible component in \( V_1(M) \) such that \( \dim W_{f,\rho} = 1 \), or \( H^1(M, \mathcal{L}) = 0 \) for \( \mathcal{L} \in W_{f,\rho} \) with finitely many exceptions.
Proof. Assume that $K_f$ is not maximal, and let $K \supset K_f$ be a maximal isotropic subspace in $H^1(M, \mathbb{C})$ with respect to the cup-product. Then $\dim K \geq 2$ and there is a morphism $f_1 : M \to S_1$ surjective, with connected generic fiber such that $K = K_{f_1}$. But then $W$ is strictly contained in $W_1 = \rho \otimes f_1^*(T(S_1))$, which is a component of $\mathcal{V}_1(M)$ by Proposition 3.18, a contradiction.

For the converse part, just note that, using the same argument as above, $W_{f, \rho}$ cannot be strictly contained in a component of $\mathcal{V}_1(M)$.

□

Remark 3.22. (i) Unlike the case of maximal isotropic subspaces in $H^1(M, \mathbb{C})$ of dimension at least two which are automatically rationally defined, see Remark 3.16, there are a lot of non rationally defined maximal isotropic subspaces in $H^1(M, \mathbb{C})$ of dimension 1 as soon as a rationally defined one exists. To see this, use the semi-continuity of the dimension of $H^1(H^*(M, \mathbb{C}), \mu_z)$ with respect to $z \in H^1(M, \mathbb{C})$.

(ii) If $E$ is a maximal isotropic subspace in $H^1(M, \mathbb{C})$ of dimension at least two, then it has at least one associated component $W_E$ in $\mathcal{V}_1(M)$ corresponding to $E$ under the bijection in Theorem 3.14. On the other hand, if $E$ is a rationally defined maximal isotropic subspace in $H^1(M, \mathbb{C})$ of dimension 1, it is quite possible that there is no associated component in $\mathcal{V}_1(M)$. As an explicit example, consider the case of the line arrangement $xyz = 0$ in $\mathbb{P}^2$. Then $M = (\mathbb{C}^*)^2$ and any 1-dimensional subspace $E_\mathbb{Q} \subset H^1(M, \mathbb{Q})$ gives rise to a rationally defined maximal isotropic subspace in $H^1(M, \mathbb{C})$ of dimension 1. However, it is well known that $\mathcal{V}_1(M) = \{1\}$ in this case.

We say that an irreducible component $E$ of some $R_m(M)$ (which is a vector subspace in $H^1(M)$) is a coordinate component, resp. a non-essential component, if it corresponds under the above isomorphism to a coordinate (resp. non-essential) component of $\mathcal{V}_m(M)$. Proposition 3.12 can be reformulated as follows.

Proposition 3.23. An irreducible component of $R_m(M)$ is non-essential if and only if it is a coordinate component.

An irreducible component $E$ of some $R_m(M)$ which is not a coordinate component is called a global component in [15]. This is compatible with our Definition 3.3 above.

Let us consider the component $W_E$ introduced in Remark 3.22 in the case $\dim E \geq 2$. Then $W_E$ corresponds to a mapping $f_E : M \to S_E$ as in Theorem 3.9. We have the following result.

Proposition 3.24. If the arrangement $C$ is given, then the mapping $f_E$ is determined by the vector subspace $E \subset H^1(M, \mathbb{C})$.

Proof. First we know that $S_E$ is obtained from $\mathbb{P}^1$ by deleting a subset $B$, with $|B| - 1 = \dim E$. Let $k := \dim E$ and assume that the points in $B$ are $(0 : 1)$ and
(1 : bj), for some bj ∈ C, j = 1, ..., k. If (u : v) are the homogeneous coordinates on \( \mathbb{P}^1 \), then the cohomology group \( H^1(S_E, \mathbb{C}) \) has a basis given by

\[
\omega_j = \frac{d(v - b_ju)}{v - b_ju} - \frac{du}{u}
\]

where j = 1, ..., k. As explained in Proposition 2.2, the mapping \( f_E \) corresponds to a pencil \( (P, Q) \), where \( P \) and \( Q \) are homogeneous polynomials in \( \mathbb{C}[X, Y, Z] \), of the same degree and without common factors. In terms of this pencil, one has

\[
f_E^*(\omega_j) = \frac{d(Q - b_jP)}{Q - b_jP} - \frac{dP}{P}
\]

So, in down-to-earth terms, the question is: how to determine the pencil \( (P, Q) \) from the vector space of 1-forms with logarithmic poles

\[
E = \{ f_E^*(\omega_j) \mid j = 1, ..., k \}
\]

Using only logarithmic poles allows us to work with rational differential forms rather than cohomology classes and this is essential for this proof. Start with the curve \( C_1 \) in the curve arrangement \( C \) and consider the subset

\[
E_1 = \{ \omega \in E \mid \int_{\gamma_1} \omega = 0 \}.
\]

Two cases may occur.

Case 1. \( (E_1 = E) \) This case occur exactly when \( C_1 \) is not a connected component in any of the \( (k + 1) \) special fibers of the pencil \( (P, Q) \) corresponding to the set \( B \). If this happens, we discard the curve \( C_1 \) and test the next curve \( C_2 \) and so on.

Case 2. \( (E_1 \neq E) \) Then \( C_1 \) is a component of a special fiber of the pencil, say of the fiber \( C_{b_1} : Q - b_1P = 0 \). Note that any form \( \omega \in E \) can be written as a sum

\[
\omega = \sum_j a_j f_E^*(\omega_j)
\]

and, with this notation, one has

\[
\int_{\gamma_1} \omega = 2\pi i a_1 m(C_1)
\]

where \( i^2 = -1 \) and \( m(C_1) \) is the multiplicity of the irreducible curve \( C_1 \) in the divisor \( C_{b_1} \). It follows that \( \omega \notin E_1 \) if and only if \( a_1 \neq 0 \). This shows that \( Q - b_1P \) is the G.C.D. of the denominators of the forms \( \omega \in E \setminus E_1 \). After we have determined the polynomial \( Q - b_1P \) as above, we discard all the curves \( C_j \) in the arrangement
which are contained in the support of the divisor \( C_{b_1} \). From the remaining curves in \( C \), we can find a new curve, say \( C_2 \), such that
\[
E_2 := \{ \omega \in E \mid \int_{\gamma_2} \omega = 0 \} \neq E.
\]
(such a curve exists since \(|B| \geq 3\)). Then \( C_2 \) is a component in a new fiber of the pencil \((P,Q)\), say of the fiber \( C_{b_2} : Q - b_2P = 0 \) and hence \( Q - b_2P \) is the G.C.D. of the denominators of the forms \( \omega \in E \setminus E_2 \). The two homogeneous polynomials \( Q - b_1P \) and \( Q - b_2P \) span the same vector space as the polynomials \( P \) and \( Q \), i.e. they determine the same pencil up to an automorphism of \( \mathbb{P}^1 \).

Now we treat the special case of rationally defined maximal isotropic subspace in \( H^1(M, \mathbb{C}) \) of dimension 1.

**Proposition 3.25.** Let \( E \) be a rationally defined maximal isotropic subspace in \( H^1(M, \mathbb{C}) \) of dimension 1. Then there is a surjective mapping \( f_E : M \rightarrow \mathbb{C}^* \) with connected generic fiber such that \( E = f_E^*(H^1(\mathbb{C}^*, \mathbb{C})) \).

**Proof.** Let \( f_j = 0 \) be a homogeneous reduced equation for the component \( C_j \) in the curve arrangement \( C \), for \( j = 1, ..., r \). Assume that \( \deg f_j = d_j \). A basis of the \((r-1)\)-dimensional vector space \( H^1(M, \mathbb{Q}) \) is given by the 1-forms
\[
\eta_k = \frac{1}{d_k f_k} df_k - \frac{1}{d_{k+1} f_{k+1}} df_{k+1}
\]
for \( k = 1, ..., r-1 \). Using this, we see that any 1-dimensional subspace in \( H^1(M, \mathbb{Q}) \) has a unique generator (up to a ±-sign) of the form
\[
\eta = \sum_{j=1,r} m_j \frac{df_j}{f_j}
\]
where \( m_j \) are relatively prime integers, i.e. G.C.D.\((m_1, ..., m_r) = 1\), such that
\[
\sum_{j=1,r} d_j m_j = 0.
\]
It follows that the rational fraction
\[
f = \prod_{j=1,r} f_j^{m_j}
\]
is homogeneous of degree 0 and hence induces a morphism \( f : M \rightarrow \mathbb{C}^* \). The fact that \( f \) is surjective is obvious, while the connectivity of the generic fiber follows from Bertini’s Theorem, see [21], p. 79, using the condition G.C.D.\((m_1, ..., m_r) = 1\). The equality \( E = f^*(H^1(\mathbb{C}^*, \mathbb{C})) \) is obvious by taking \( \frac{d}{z} \) as a basis of \( H^1(\mathbb{C}^*, \mathbb{C}) \). \( \square \)
4. Minimal arrangements

In this section we prove the following result which applies to an arbitrary plane curve arrangement.

**Theorem 4.1.** Let \( C = \cup_{i=1}^{r} C_i \) be a plane curve arrangement in \( \mathbb{P}^2 \), having \( r \) irreducible components \( C_i \), for \( i = 1, r \). Let \( M = \mathbb{P}^2 \setminus C \) be the corresponding complement. Then, for \( d \geq 2 \), the following are equivalent.

(i) there is a global \( d \)-dimensional irreducible component \( E \) in the resonance variety \( \mathcal{R}_1(M) \);

(ii) there is a global \( d \)-dimensional irreducible component \( W \) in the characteristic variety \( \mathcal{V}_1(M) \);

(iii) there is a pencil \( C \) of plane curves on \( \mathbb{P}^2 \) with an irreducible generic member and having \( d+1 \) fibers \( C_i \) whose reduced supports form a partition of the set of irreducible components \( C_i \), for \( i = 1, r \).

Moreover, for \( d = 1 \), (i) always fails and (ii) implies (iii).

**Proof.** For \( d \geq 2 \), the equivalence between (i) and (ii) follows from Theorem 3.14 and Theorem 3.9 claim (vi). And the equivalence between (ii) and (iii) follows from Arapura’s results recalled in subsection 3.8 combined with Proposition 2.2 and Corollary 3.7. The case \( d = 1 \) follows from Theorem 3.9 claim (i).

Note that the condition (ii) above can be reformulated as the existence of a global irreducible component \( W \) in the characteristic variety \( \mathcal{V}_1(M) \) such that \( 1 \in W \) and \( \dim H^1(M, \mathcal{L}) = d - 1 \geq 1 \) for a generic local system \( \mathcal{L} \in W \).

The following result is similar to Theorem 4.1.1 in Libgober [18] and closely related to the discussion in [20], just before Proposition 7.2.

**Corollary 4.2.** Assume the equivalent statements in Theorem 4.1 above hold. Let \( f : \mathbb{P}^2 \to \mathbb{P}^1 \) be the rational morphism associated to the pencil \( C \). Let \( \pi : X \to \mathbb{P}^2 \) be a sequence of blowing-ups such that \( g = f \circ \pi \) is a regular morphism on \( X \). If \( \tilde{C} \) denotes the proper transform of the (reduced) curve \( C \) under \( \pi \), then the self intersection number of \( \tilde{C} \) is non-positive, i.e.

\[ \tilde{C} \cdot \tilde{C} \leq 0. \]

**Proof.** There is a partition of \( \tilde{C} = \cup_{i=1}^{d+1} \tilde{C}_i \) of \( \tilde{C} \) as a union of disjoint curves \( \tilde{C}_i \), such that \( g(\tilde{C}_i) = b_i \) for \( i = 1, ..., d+1 \). It follows that

\[ \tilde{C} \cdot \tilde{C} = \sum_{i=1,d+1} \tilde{C}_i \cdot \tilde{C}_i. \]
Each curve is contained in the support of the positive divisor \( g^{-1}(b_i) \), and hence by Zariski’s Lemma, see [3], p. 90, we get

\[ \tilde{C}_i \cdot \tilde{C}_i \leq 0. \]

\[ \square \]

**Example 4.3.** Assume that \( C \) is a line arrangement, i.e. \( d_j = 1 \) for all \( j = 1, \ldots, r \). Let \( \mathcal{X} \) be the base locus of a pencil as in Corollary 4.2, \( k = d + 1 \) the number of sets in the associated partition \( (\tilde{C}_i)_i \) of the set of lines in \( C \). Then it is shown in [15] that the following hold.

(i) For each base point \( p \in \mathcal{X} \), the multiplicity

\[ n_p = \text{mult}_p(C_b) \]

is independent of \( b \in B \).

(ii) \( \sum_{p \in \mathcal{X}} n_p = D^2 \), where \( D \) is the degree of the pencil.

(iii) \( \sum_{j=1}^{r} m(C_j) = kD \), where \( m(C_j) \geq 1 \) is the multiplicity with which \( C_j \) occurs in the corresponding fiber of the pencil.

To resolve the indeterminacy points of the associated pencil (i.e. to determine the map \( \pi : X \to \mathbb{P}^2 \)), one has in this case just to blow-up once the points in the base locus \( \mathcal{X} \). This is a direct consequence of the property (i) above.

Assume moreover that \( m(C_j) = 1 \) for all \( j = 1, r \), i.e. all \( C_b \) for \( b \in B \) are reduced.

Then, again by (i) above, it follows that \( \text{mult}_p(C) = kn_p \) for any \( p \in \mathcal{X} \). On the other hand, by (iii) we get \( \deg C = kD \). Finally, in this very special case we get

\[ \tilde{C} \cdot \tilde{C} = C \cdot C - \sum_{p \in \mathcal{X}} \text{mult}_p(C)^2 = k^2D^2 - k^2D^2 = 0 \]

since \( C \cdot C = (\deg C)^2 \). This happens for instance in Example 3.4 in [15]: the Ceva arrangement given by the pencil \( ax^d + by^d + cz^d = 0 \) with \( (a : b : c) \in \mathbb{P}^2 \) satisfying \( a + b + c = 0 \). There are 3 special fibers, corresponding to \( x^d - y^d, y^d - z^d, z^d - x^d \).

In the case of a general line arrangement, the condition \( \tilde{C} \cdot \tilde{C} \leq 0 \) may bring new non-trivial information on the arrangement. In particular it can be used as a test for candidates to the base locus \( \mathcal{X} \) of a pencil associated to a given arrangement. In Example 3.6 in [15] the \( B_3 \)-arrangement consists of 9 lines, and the base locus \( \mathcal{X} \) consists of 3 points of multiplicity 4 and 4 other points of multiplicity 3. As a result we have

\[ \tilde{C} \cdot \tilde{C} = C \cdot C - \sum_{p \in \mathcal{X}} \text{mult}_p(C)^2 = 81 - 3 \times 16 - 4 \times 9 = -3. \]

This latter arrangement is associated to the pencil \( (x^2 - y^2)z^2 : (y^2 - z^2)x^2 \) which has again 3 special fibers (this time non-reduced!), corresponding to \( (x^2 - y^2)z^2, \)
\[(y^2 - z^2)x^2, (x^2 - z^2)y^2.\]  Up to a linear change of coordinates, this is the same \(B_3\)-arrangement as in the final subsection (6.7) below.

5. Fibered complements and \(K(\pi, 1)\)-spaces

Let \(f : M \to S\) be a morphism associated to a plane curve pencil \(C\) with base locus \(X\), as in Theorem 4.1 and Corollary 4.2 above. Consider a fiber \(C_s\) of the pencil \(C\) corresponding to \(s \in S = \mathbb{P}^1 \setminus B\). We say that \(C_s\) is a special fiber of \(C\) if either \(C_s \setminus X\) is singular, or if \(C_s \setminus X\) is smooth and exists a point \(p \in X\) such that

\[\mu(C_s, p) > \min_{t \in S} \mu(C_t, p),\]

where \(\mu\) denotes the Milnor number of an isolated singularity.

Let \(C_{\text{spec}}\) be the union of all the special fibers in the pencil \(C\). (There is a finite number of such fibers, and they are easy to identify, see [17], [10]). Let \(B'\) be the union of \(B\) and the set of all \(s \in S\) such that \(C_s\) is a special fiber. We call \(C' = C \cup C_{\text{spec}}\) the extended plane curve arrangement associated to the plane curve arrangement \(C\) and denote by \(M'\) the corresponding complement. We set \(S' = \mathbb{P}^1 \setminus B'\). With this notation, we have the following result.

**Proposition 5.1.** The restriction \(f' : M' \to S'\) is a locally topologically trivial fibration. The plane curve arrangement complement \(M'\) is a \(K(\pi, 1)\)-space.

**Proof.** Note that for any \(p \in X\), the family of plane curve isolated singularities \((C_s, p)\) for \(s \in S'\) is a \(\mu\)-constant family. Using the relation between \(\mu\)-constant families and Whitney regular stratifications, as well as Thom’s First Isotopy Lemma, see for instance [11], pp. 11-16 and especially the proof of Proposition (1.4.1) on p. 20, we get the first claim above. The second claim is an obvious consequence, as explained already in [15].

**Remark 5.2.** With the notation from Theorem 3.9, Libgober has remarked in the proof of Lemma 1.4.3 in [18] that

\[\dim H^1(M, \mathcal{L}_{W} \otimes f^*_W(\mathcal{L})) = \dim H^1(S, f_W^*(\mathcal{L}_{W}) \otimes \mathcal{L})\]

for almost all local systems \(\mathcal{L} \in \mathcal{T}(S_W)\) (this comes from the exact sequence (3.14) by proving the vanishing of the last term). This implies that a morphism \(f_W : M \to S_W\) as in Theorem 3.9 is never a locally trivial fibration in the case \(S_W = \mathbb{C}^*\) (this is related to the case \(\Sigma = 0\) in formula (3.15)). By definition, \(\dim H^1(M, \mathcal{L}_{W} \otimes f^*_W(\mathcal{L})) \neq 0\) while the second dimension would vanish in such a situation. Indeed then \(f_W^*(\mathcal{L}_{W})\) would be a local system on \(\mathbb{C}^*\), given by an automorphism \(A\) of some \(\mathbb{C}^n\) and we can take \(\mathcal{L}\) to have a monodromy different from the eigenvalues of \(A\).
Example 5.3. We discuss now the following curve arrangement, considered already in Example 4.8 in [13]. The curve $C$ consists of the following: three lines $C_1 : x = 0$, $C_2 : y = 0$ and $C_3 : z = 0$ and a conic $C_4 : x^2 - yz = 0$. The corresponding pencil can be chosen to be $f = (x^2 : yz)$, and the set $B = \{b_1, b_2, b_3\}$ is given by $b_1 = (0 : 1), C_1 = 2C_1, b_2 = (1 : 0), C_2 = C_2 \cup C_3$, and $b_3 = (1 : -1), C_3 = C_4$.

The base locus of this pencil is $X = \{(p_1 = (0 : 0 : 1), p_2 = (0 : 1 : 0)\}$ and it is easy to check that there no special fibers. It follows that $f : M \to S$ is a locally topologically trivial fibration with fiber $C^* (a smooth conic minus two points)$. Hence the complement $M$ of this curve arrangement is a $K(\pi, 1)$-space, where the group $\pi$ fits into an exact sequence
\[ 1 \to \mathbb{Z} \to \pi \to \mathbb{F}_2 \to 1 \]
with $\mathbb{F}_2$ denoting the free group on two generators.

Note also that $\text{mult}_{p_1} C_1 = 2 > 1 = \text{mult}_{p_1} C_2$, hence the property (i) in Example 4.3 does not hold for arbitrary curve arrangements.

A local system $L$ on $S$ is given by a triple $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1$. With this notation, the pull-back local system $f^* L$ is given by $(\rho_1, \rho_2, \rho_3, \rho_4)$, where $\rho_1 = \lambda_1^2$, $\rho_2 = \rho_3 = \lambda_2$ and $\rho_4 = \lambda_3$ (recall the discussion before Corollary 3.7). With this, it is easy to check that the irreducible component given by $f^* T(S)$ coincide with the irreducible component obtained in Example 4.8 in [13] by computation using the associated integrable connections.

Indeed, the $\rho_i$’s satisfy the equation $\rho_1 \cdot \rho_2 \cdot \rho_3 \cdot \rho_4^2 = 1$, see Equation (3.5), and hence we can use $(\rho_1, \rho_2, \rho_4)$ to parametrize the torus $T(M)$. It follows that the irreducible component given by $W = f^* T(S)$ is parametrized by $\rho_1 = \lambda_1^2$, $\rho_2 = \lambda_2$ and $\rho_4 = \lambda_3$, i.e. $W$ is given by the equation
\[ \rho_1 \cdot \rho_2^2 \cdot \rho_4^2 = 1 \]
which appears in Example 4.8 in [13] with slightly different notation.

Let $\tilde{C}$ be the proper transform on $C$ under the blowing-up $\tilde{\pi} : \tilde{X} \to \mathbb{P}^2$ of the two points in $X$. Then
\[ \tilde{C} \cdot \tilde{C} = C \cdot C - 3^2 - 3^2 = 7 > 0. \]
This is not in contradiction with Corollary 4.2 since in order to resolve the indeterminacy points of $f$ in this case we have to blow the points $\tilde{p}_1, \tilde{p}_2 \in \tilde{X}$ corresponding to the tangents of the conic $C_4$ at $p_1$ and $p_2$. The new multiplicities are
\[ \text{mult}_{\tilde{p}_1} \tilde{C} = \text{mult}_{\tilde{p}_2} \tilde{C} = 3 \]
and hence for the new proper transform $\tilde{C}$ we get
\[ \tilde{C} \cdot \tilde{C} = \tilde{C} \cdot C - 3^2 - 3^2 = -11. \]

6. On the translated components of \( V_1(M) \)

Let \( W \) be a translated irreducible component of \( V_1(M) \), i.e. \( 1 \notin W \). Then, as in Theorem 3.9, there is a torsion character \( \rho \in \mathbb{T}(M) \) and a surjective morphism \( f : M \to S \) with connected generic fiber \( F \) such that

\[ W = \rho f^*(\mathbb{T}(S)) \]

We say in this situation that the component \( W \) is associated to the mapping \( f \). In this section we give detailed information on the torsion character \( \rho \in \mathbb{T}(M) \) in terms of the geometry of the associated mapping \( f : M \to S \).

6.1. The general setting. Let \( F \) be the generic fiber of the mapping \( f : M \to S \), i.e. \( F \) is the fiber of the topologically locally trivial fibration \( f' : M' \to S' \) associated to \( f \) as in the previous section. Then, we have an exact sequence

\[ H_1(F) \xrightarrow{i_\ast} H_1(M') \xrightarrow{f'_\ast} H_1(S') \to 0 \]

as well as a sequence

\[ H_1(F) \xrightarrow{i_\ast} H_1(M) \xrightarrow{f_\ast} H_1(S) \to 0 \]

which is not necessarily exact in the middle, i.e. the group

\[ T(f) = \frac{\ker f_\ast}{\text{im } i_\ast} \]

is in general non-trivial. Here \( i : F \to M \) and \( i' : F \to M' \) denote the inclusions, and homology is taken with \( \mathbb{Z} \)-coefficients if not stated otherwise. The group \( T(f) \) is a finite abelian group according to Theorem 6.3, an apparently not obvious fact.

This group was studied in a compact (proper) setting by Serrano, see [22], but no relation to local systems was considered there. On the other hand, this compact situation was also studied by A. Beauville in [4], with essentially the same aims as ours. However, the actual results are distinct, because of the key role played in our case by the partially deleted fibers, corresponding to the two types of terms in the sum in formula (6.9) below.

The second sequence induces an obvious exact sequence

\[ 0 \to T(f) \to \frac{H_1(M)}{\text{im } i_\ast} \xrightarrow{f_\ast} H_1(S) \to 0. \]

Since \( H_1(S) \) is a free \( \mathbb{Z} \)-module, applying the functor \( \text{Hom}(\cdot, \mathbb{C}^*) \) to the exact sequence (6.5), we get a new exact sequence

\[ 1 \to \mathbb{T}(S) \to \mathbb{T}(M)_{F} \to \text{Hom}(T(f), \mathbb{C}^*) \to 1. \]
Here $\mathbb{T}(M)_F$ is the subgroup in $\mathbb{T}(M)$ formed by all character $\chi : H_1(M) \to \mathbb{C}^*$ such that $\chi \circ \iota_* = 0$. This means exactly that the associated local system $L_\chi$ by restriction to $F$ yields the trivial local system $\mathbb{C}_F$.

The torsion character $\rho \in \mathbb{T}(M)$ which occurs in 6.1 is in this subgroup $\mathbb{T}(M)_F$, see Theorem 3.9 (ii). Moreover, this character $\rho$ is not unique, but its class

$$\bar{\rho} \in \frac{\mathbb{T}(M)_F}{\mathbb{T}(S)} \approx \text{Hom}(T(f), \mathbb{C}^*)$$

is uniquely determined. From now on, we will regard $\bar{\rho} \in \text{Hom}(T(f), \mathbb{C}^*)$. Hence, to understand the possible choices for $\bar{\rho}$, we have to study the group $\mathbb{T}(f)$.

6.2. The computation of the group $T(f)$. In order to simplify the presentation, we assume in this subsection that at least one of the curves $C_j$ in the curve arrangement $C$ is a line. This covers the case of line arrangements and of curve arrangements in the affine plane $\mathbb{C}^2$. More specifically, we assume that $C_1$ is the line at infinity in $\mathbb{P}^2$ and hence $M = \mathbb{C}^2 \setminus \bigcup_j (\mathbb{C}^2 \cap C_j)$. We assume that $\infty \in B$ and set $B_1 = B \setminus \{\infty\} \subset \mathbb{C}$.

For each $b \in B$, consider the following divisor on $\mathbb{C}^2$

$$D_b = g_1^{-1}(b) \cap \mathbb{C}^2 = \sum_a m_{ba} C_{ba}$$

where $g_1$ is the extension of $f$ from the proof of Proposition 2.2, $m_{ba} \geq 1$ are integers and $C_{ba}$ are irreducible curves in the arrangement $C$.

Recall the larger set $B' \supset B$ obtained from $B$ by adding the bifurcation points of $f : M \to S$. Set $C(f) = B' \setminus B$ and assume from now on that $C(f)$ is nonempty. (Otherwise $f$ is a fibration and hence $T(f) = 1$). For each $c \in C(f)$, consider the following divisor on $\mathbb{C}^2$

$$D_c = g_1^{-1}(c) \cap \mathbb{C}^2 = \sum_{a'} m_{ca'}'C_{ca'}' + \sum_{a''} m_{ca''}''C_{ca''}''$$

where $g_1$ is the extension of $f$ as above, $m_{ca'}' \geq 1$ and $m_{ca''}'' \geq 1$ are integers, the irreducible curves $C_{ca'}'$ are curves in our arrangement (corresponding to case (2) in Proposition 2.2) and $C_{ca''}''$ are the new curves to be deleted from $M$ in order to obtain $M'$. Since $f : M \to S$ is a surjection, it follows that there is at least one term in the second sum in the equality (6.9). At least some of the first type sums are non-trivial if and only if the arrangement $C$ is special.

Note that $M$ is obtained from $\mathbb{C}^2$ by deleting the curves $C_{ba}$, $C_{ca'}'$ and possibly some horizontal components $C_h$. For each irreducible curve $Z$ in $\mathbb{C}^2$ we denote by $\gamma(Z)$ the elementary oriented loop associated to $Z$. It follows that $H_1(M)$ is a free $\mathbb{Z}$-module with a basis given by

$$\gamma(C_{ba}), \gamma(C_{ca'}'), \gamma(C_h)$$
Similarly, $H_1(S)$ is a free $\mathbb{Z}$-module with a basis given by $\delta_b$ for $b \in B_1$, where $\delta_b$ is an elementary loop based at $b$ as subsection (3.1). In term of these bases, the morphism $f_* : H_1(M) \to H_1(S)$ is described as follows. Let $\alpha \in H_1(M)$ be given by

\begin{equation}
\alpha = \sum \alpha_{ba} \gamma(C_{ba}) + \sum \alpha_{ca'} \gamma(C_{ca'}) + \sum \alpha_h \gamma(C_h).
\end{equation}

Here $a \in A_b$, an index set depending on $b$. This is not written explicitly, in order to keep the notation simpler. Similar remarks apply to the indices $a'$ and $a''$ below. Then

\begin{equation}
f_*(\alpha) = \sum_{b \in B_1} \left( \sum_a m_{ba} \alpha_{ba} - \sum_a m_{\infty a} \alpha_{\infty a} \right) \delta_b.
\end{equation}

In particular $\alpha \in \ker f_*$ if and only if

\begin{equation}
\sum_a m_{ba} \alpha_{ba} = \sum_a m_{\infty a} \alpha_{\infty a}
\end{equation}

for all $b \in B_1$. Next, $M'$ is obtained from $M$ by deleting the curves $C^\prime_{ca'}$. Hence, it follows that $H_1(M')$ is a free $\mathbb{Z}$-module with a basis given by

\begin{equation}
(\gamma(C_{ba}), \gamma(C_{ca'}), \gamma(C_h), \gamma(C^\prime_{ca'})).
\end{equation}

The inclusion $j : M' \to M$ induces a morphism $j_* : H_1(M') \to H_1(M)$ which, at coordinate level, is just the obvious projection. Similarly, $H_1(S')$ is a free $\mathbb{Z}$-module with a basis given by $\delta_b$ for $b \in B_1$, and $\delta_c$ for $c \in C(f)$.

In term of these bases, the morphism $f'_* : H_1(M') \to H_1(S')$ is described as follows. Let $\beta \in H_1(M')$ be given by

\begin{equation}
\beta = \sum \beta_{ba} \gamma(C_{ba}) + \sum \beta_{ca'} \gamma(C^\prime_{ca'}) + \sum \beta_h \gamma(C_h) + \sum \beta_{ca''} \gamma(C^\prime_{ca''}).
\end{equation}

Then $f'_*(\beta) = E_1 + E_2$, where

\begin{equation}E_1 = \sum_{b \in B_1} \left( \sum_a m_{ba} \beta_{ba} - \sum_a m_{\infty a} \beta_{\infty a} \right) \delta_b.
\end{equation}

and

\begin{equation}E_2 = \sum_{c \in C(f)} \left( \sum_{a'} m_{ca'} \beta_{ca'} + \sum_{a''} m_{ca''} \beta_{ca''} - \sum_a m_{\infty a} \beta_{\infty a} \right) \delta_c.
\end{equation}

The exact sequence (6.12) yields $\ker i'_* = \ker f'_*$. On the other hand $\im i_* = j_*(\im i'_*)$. It follows that $\alpha \in H_1(M)$ as above is in $\im i_*$ if and only if there is a $\beta \in H_1(M')$ such that

(i) $\beta$ is a lifting of $\alpha$, i.e. $j_*(\beta) = \alpha$;

(ii) $f'_*(\beta) = 0$.

In terms of our bases this means that
(i') $\beta_{ba} = \alpha_{ba}, \beta_{ca'} = \alpha_{ca'}, \beta_h = \alpha_h$;

(ii') The coordinates of $\alpha$ satisfy the equation \((6.13)\) and there is a choice of coordinates $\beta_{ca''}$ such that one has

\[(6.18) \sum_{a''} m''_{ca''} \beta_{ca''} = \sum_a m_{\infty a} \alpha_{\infty a} - \sum_{a'} m'_{ca'} \alpha_{ca'}\]

for each $c \in C(f)$. Let $m''(c) = G.C.D\{m''_{ca''}\}$ where $a''$ takes all the possible values (this set of indices is nonempty). The above equation has a solution if and only if the right hand side is divisible by $m''(c)$. This explains the following construction. Let $G(f) = \oplus_{c \in C(f)} \mathbb{Z}/m''(c)\mathbb{Z}$ and let

$$\theta : \ker f \rightarrow G(f)$$

be the morphism sending $\alpha \in \ker f$ to the element in $G(f)$ having as its $c$-coordinate the class of

$$\sum_{a} m_{\infty a} \alpha_{\infty a} - \sum_{a'} m'_{ca'} \alpha_{ca'}$$

modulo $m''(c)$. Let $m'(c) = G.C.D\{m'_{ca'}\}$ where $a'$ takes all the possible values (this set can be empty and in this case we set $m'(c) = 0$). The above discussion is summarized in the following.

**Theorem 6.3.** With the above notation, there is an isomorphism $T(f) \simeq \text{im } \theta$. In particular, if for all $c \in C(f)$, one has $G.C.D(m'(c), m''(c)) = 1$, i.e. none of the fibers $D_c$ is multiple, then

$$T(f) \simeq G(f).$$

Moreover, if all the fibers $D_c$ are reduced (i.e. all components occur with multiplicity 1), then $T(f) = 1$, and hence there are no translated components in $\mathcal{V}_1(M)$ associated to $f$ in this case.

**Remark 6.4.** The horizontal components $C_h$ play no role in the computation of $T(f)$, since they are obviously in $\text{im } i_s$.

The following consequence does not rule out the possibility of a translated global component, but explains in a sense why they should be quite exceptional. Recall that by Theorem 4.1, a translated global component corresponds to a minimal arrangement. Conversely, a translated coordinate component is more likely to occur, and then it is related to a special arrangement, as in the deleted $B_3$-arrangement revisited below.

**Corollary 6.5.** Assume that $C$ is a minimal arrangement with respect to $f : M \rightarrow S$. For $b \in B$, set $m(b) = G.C.D\{m_{ba}\}$ and then $m(f) = L.C.M\{m(b) \mid b \in B\}$. Then $T(f)$ is isomorphic to the cyclic subgroup in $G(f)$ spanned by

$$(m(f), m(f), ..., m(f)).$$
In particular, if \( m''(c)|m(f) \) for all \( c \in C(f) \), then \( T(f) = 1 \).

**Proof.** If \( C \) is a minimal arrangement, then there are no indices of type \( a' \), so the morphism \( \theta \) has a simpler form. The fact that \( \text{im} \theta \) is the cyclic subgroup in \( G(f) \) spanned by \( (m(f), m(f), ..., m(f)) \) follows from the equations \( 6.13 \).

**Corollary 6.6.** If \( f : M \to S \) has no multiple fibers, then there are no translated components in \( V_1(M) \) associated to \( f \).

The following result clarifies to a certain extent the case of 1-dimensional translate components.

**Proposition 6.7.** Let \( E \subset H^1(M, \mathbb{C}) \) be a maximal isotropic subspace with respect to the cup product such that \( \dim E = 1 \). Let \( f : M \to \mathbb{C}^* \) be the corresponding surjective morphism, with connected generic fiber \( F \), i.e. \( E = f^*(H^1(\mathbb{C}^*)) \). Assume that \( m'(c) = 1 \) for all \( c \in C \). Then, for any nontrivial character \( \tilde{\rho} : T(f) \to \mathbb{C}^* \), the associated 1-dimensional translated subtorus

\[
W_{f,\rho} = \rho \otimes f^*(\mathbb{T}(\mathbb{C}^*))
\]

is a component in \( V_1(M) \).

**Proof.** Since \( m'(c) = 1 \) for all \( c \in C \), we obtain a system of generators for the group \( T(f) \) by taking \( \theta(\gamma(C'_{ca})) \), for all \( c \in C \) and all possible values of \( a' \). Since \( \tilde{\rho} \) is non-trivial, it exists at least one point \( c \in C \) and one value for \( a' \) such that

\[
\rho(\gamma(C'_{ca})) = \tilde{\rho}(\theta(\gamma(C'_{ca}))) \neq 1.
\]

If \( D \) is any small disc containing \( c \), it follows that

\[
H^0(f^{-1}(D), L_{\rho}) = 0.
\]

In the notation of the proof of Proposition \( 3.18 \) we get \( F_c = 0 \), and hence \( c \in \Sigma \). The formula \( 3.15 \) then implies that \( \dim H^1(S, \mathcal{F} \otimes L_2) \geq 1 \) for all \( L_2 \in \mathcal{T}(\mathbb{C}^*) \). We conclude by applying Corollary \( 3.21 \).

The same proof as above yields the following result, to be compared with Theorem \( 3.9 \) (iv).

**Corollary 6.8.** Let \( f : M \to S \) be a surjective morphism, with connected generic fiber \( F \), such that \( \chi(S) < 0 \). Assume that \( m'(c) = 1 \) for all \( c \in C \). Then, for any nontrivial character \( \tilde{\rho} : T(f) \to \mathbb{C}^* \), one has

\[
\dim H^1(M, L_{\rho} \otimes f^* L_2) \geq -\chi(S) + 1
\]

for any local system \( L_2 \in \mathcal{T}(S) \).
Theorem 6.3 implies that

\[ \rho \in \text{ker} \gamma \]

\( \rho \) is well-defined on \( \text{ker} \gamma \) since it is multiplicative. It follows that \( \rho \gamma = 1 \) and that the associated torus is \( \mathbb{T}(\mathbb{C}^*) \). Indeed, \( \rho \gamma = 1 \).

Example 6.10. (A more general example: the \( \mathcal{A}_m \)-arrangement)

Let \( \mathcal{A}_m \) be the line arrangement in \( \mathbb{P}^2 \) defined by the equation

\[ x_1 x_2 (x_1^m - x_2^m)(x_1^m - x_3^m)(x_2^m - x_3^m) = 0. \]

This arrangement is obtained by deleting the line \( x_3 = 0 \) from the complex reflection arrangement associated to the full monomial group \( G(3,1,m) \) and was studied in [7] and in [9]. The deleted \( B_3 \)-arrangement studied above is obtained by taking \( m = 2 \).

Consider the associated pencil

\[ (P,Q) = (x_1^m(x_2^m - x_3^m), x_2^m(x_1^m - x_3^m)). \]

Then the set \( B \) consists of two points, namely \( (0 : 1) \) and \( (1 : 0) \), and the set \( C \) is the singleton \( (1 : 1) \), see for instance [15], Example 4.6. It follows that \( m'(c) = 1 \), \( m''(c) = m \) and hence via Theorem 6.3 we get

\[ T(f) = \frac{\mathbb{Z}}{m\mathbb{Z}}. \]
Using Proposition 6.7 we expect \((m - 1)\) 1-dimensional components in \(\mathcal{V}_1(M)\), and this is precisely what has been proved in \([7]\), or in Thm. 5.7 in \([9]\). There are \(r = 2 + 3m\) lines in the arrangement, and to describe these components we use the coordinates
\[
(z_1, z_2, z_{12:1}, \ldots, z_{12:2}, z_{13:1}, \ldots, z_{13:2}, z_{23:1}, \ldots, z_{23:2})
\]
on the torus \((\mathbb{C}^*)^r\) containing \(\mathbb{T}(M)\). Here \(z_j\) is associated to the line \(x_j = 0\), for \(j = 1, 2\), and \(z_{ij:k}\) is associated to the line \(x_i - w^k x_j\), where \(i, j = 1, 3, k = 1, \ldots, m\), and \(w = \exp(2\pi \sqrt{-1}/m)\). All the above 1-dimensional components have the same associated 1-dimensional subtorus
\[
\mathbb{T} = f^*(\mathbb{T}(\mathbb{C}^*)) = \{(u^m, u^{-m}, 1, \ldots, 1, u^{-1}, \ldots, u^{-1}, u, \ldots, u) \mid u \in \mathbb{C}^*\}
\]
where \(f : M \to \mathbb{C}^*\) is the morphism associated to the pencil \((P, Q)\), and each element \(1, u^{-1}\) and \(u\) is repeated \(m\) times. The associated maximal isotropic subspace \(E\) in \(H^1(M, \mathbb{C})\) is spanned by the 1-form
\[
\omega = \frac{m}{x_1} dx_1 - \frac{m}{x_2} dx_2 - \sum_{k=1,m} \frac{dx_1 - w^k dx_3}{x_1 - w^k x_3} + \sum_{k=1,m} \frac{dx_2 - w^k dx_3}{x_2 - w^k x_3}.
\]
The patient reader may check that for any \(\alpha \in H^1(M, \mathbb{C})\), the vanishing \(\alpha \land \omega = 0\) in \(H^2(M, \mathbb{C})\) implies that \(\alpha\) is a multiple of \(\omega\) (this is the maximality condition in this 1-dimensional case). Let \(\gamma_c\) be an elementary loop about one line \(L\) in the fiber \(\mathcal{C}_c\), with multiplicity 1, e.g. \(L : x_1 - x_2 = 0\). Similarly, let \(\gamma_b\) be an elementary loop about one line \(L'\) in the fiber \(\mathcal{C}_b\), with multiplicity 1, where \(b = \infty = (0 : 1)\), e.g. \(L' : x_2 - x_3 = 0\). And let \(\gamma_0\) be an elementary loop about one line \(L_0\) in the fiber \(\mathcal{C}_0\), with multiplicity 1, where \(0 = (1 : 0)\), e.g. \(L_0 : x_1 - x_3 = 0\). One can show easily that

(i) the classes \([\gamma_c]\) and \([\gamma_b + \gamma_0]\) in the group \(T(f)\) are independent of the choices made;

(ii) \([\gamma_c] = -[\gamma_b + \gamma_0]\) is a generator of \(T(f)\).

It follows that a torsion character \(\rho \in \mathbb{T}(M)\) such that \(\mathcal{L}_\rho|F = \mathbb{C}_F\) and inducing a nontrivial character \(\tilde{\rho} : T(f) \to \mathbb{C}^*\) is given by
\[
\rho = (1, 1, w^k, \ldots, w^k, w^{-k}, \ldots, w^{-k}, 1, \ldots, 1)
\]
for \(k = 1, \ldots, m - 1\). Here \(\tilde{\rho}([\gamma_c]) = w^k\) and \(\rho\) is normalized by setting the last \(m\) components equal to 1.

**Example 6.11.** (A non-linear arrangement)

Consider again the pencil \(\mathcal{C} : (P, Q) = (x_1^m (x_2^m - x_3^m), x_2^m (x_1^m - x_3^m))\) associated above to the \(\mathcal{A}_m\)-arrangement, for \(m \geq 2\). We introduce the following new notation: \(C = \{(0 : 1), (1 : 0), (1 : 1)\}\). Let \(B \subset \mathbb{P}^1\) be a finite set such that \(|B| = k \geq 2\) and
Consider the curve arrangement in $\mathbb{P}^2$ obtained by taking the union of the $3m$ lines given by
\[(x_1^m - x_2^m)(x_1^m - x_3^m)(x_2^m - x_3^m) = 0\]
with the $k$ fibers $C_b$ for $b \in B$. Let $M$ be the corresponding complement and $f : M \to S := \mathbb{P}^1 \setminus B$ be the map induced by the pencil $\mathcal{C}$. Then one has the following.

(i) $T(f) = \frac{\mathbb{Z}}{m\mathbb{Z}} \oplus \frac{\mathbb{Z}}{m\mathbb{Z}} \oplus \frac{\mathbb{Z}}{m\mathbb{Z}}$. Let $e_j$ for $j = 1, 2, 3$ denote the canonical basis of $T(f)$ as a $\frac{\mathbb{Z}}{m\mathbb{Z}}$-module.

(ii) For a character $\tilde{\rho} : T(f) \to \mathbb{C}^*$, let $W_\rho = \mathcal{L}_\rho \otimes f^*(\mathbb{T}(S))$ be the associated component. Then $\dim W_\rho = k - 1$ and for a local system $\mathcal{L} \in W_\rho$ one has
\[\dim H^1(M, \mathcal{L}) \geq k - 2 + \epsilon\]
where equality holds for all but finitely many $\mathcal{L} \in W_\rho$ and
\[\epsilon = |\{j \mid \tilde{\rho}(e_j) \neq 1\}|.\]

This shows that the various translates of the subtorus $T_W = f^*(\mathbb{T}(S))$ have all the same dimension, but they are irreducible components of various characteristic varieties $\mathcal{V}_m(M)$, a fact not noticed before.

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