Analog of modulus of convexity for Grand Lebesgue Spaces.

M.R.Formica, E.Ostrovsky, L.Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.

e-mail: mara.formica@uniparthenope.it

Israel, Bar - Ilan University, department of Mathematic and Statistics, 59200,

e-mails: eugostrovsky@list.ru

Abstract

We introduce and evaluate the degree of convexity of an unit ball, so - called, characteristic of convexity (COC) for the Grand Lebesgue Spaces, (GLS), which is a slight analog of the classical notion of the modulus of convexity (MOC).

Key words and phrases. Banach, Lebesgue - Riesz and Grand Lebesgue Spaces (GLS) and norms, triangle inequality, unit ball, embedding, modulus of convexity (MOC), weak characteristic of convexity (WCOC).

1 Notations. Definitions. Statement of problem.

Let \((X, \| \cdot \|)\) be Banach space, \(S\) be its unit sphere: \(S = \{ x, x \in X, \| x \| = 1 \}\) and \(B\) be its unit ball with the center in origin: \(B = \{ x, x \in X, \| x \| \leq 1 \}\).

Let also \(\epsilon\) be arbitrary number from the segment \([0, 2]\): \(0 \leq \epsilon \leq 2\). Recall the following very important in the geometrical theory of Banach spaces notion Modulus Of Convexity (MOC) \(\delta_X(\epsilon)\) for the space \((X, | \cdot |)\):

**Definition 1.1.** The Modulus Of Convexity (MOC) for the space \(X = (X, \| \cdot \|) = (X, || \cdot ||X)\), which is denoted by \(\delta_X(\epsilon)\) is defined as follows

\[
\delta_X(\epsilon) \overset{def}{=} \inf \left\{ 1 - \frac{\| x + y \|}{2} : x, y \in B; \| x - y \| \geq \epsilon \right\};
\]

the Ball definition; or equally
\[ \delta_X(\epsilon) \overset{\text{def}}{=} \inf \left\{ 1 - \frac{||x + y||}{2}; \ x, y \in S; \ ||x - y|| \geq \epsilon \right\}; \]  

spherical definition.

This important for Functional Analysis notion was introduced by O.Hanner (1956), see [22], and was investigated in many works, see e.g. [6], [8], [9], [17], [22], [34], [35] and so one.

The other application, indeed, in the theory of random fields, may be found in [29], chapter 3.

For example, let \( X \) be the classical Lebesgue - Riesz space \( L_p \), builded on certain atomless measure space; the correspondent Module Of Convexity will be denoted by \( \delta_p(\epsilon); \ p \in (1, \infty) \). If \( p \in (1, 2] \), then the function \( \delta_p(\epsilon) \) is an unique positive solution of an equation

\[
(1 - \delta_p(\epsilon) + 0.5\epsilon)^p + (1 - \delta_p(\epsilon) - 0.5\epsilon)^p = 2,
\]

so that when \( \epsilon \in [0, 2] \)

\[
\delta_p(\epsilon) \geq \frac{p - 1}{8} \epsilon^2. \tag{3}
\]

If now \( p \in (2, \infty) \), then

\[
\delta_p(\epsilon) = 1 - (1 - (0.5\epsilon)^p)^{1/p}, \tag{4}
\]

and when again \( \epsilon \in [0, 2] \)

\[
\delta_p(\epsilon) \geq \frac{\epsilon^p}{p 2^p}; \tag{5}
\]

see e.g. [8], [9], [17].

We intent in this short report to introduce some weak analog of modulus of convexity for Grand Lebesgue Spaces (GLS) and derive some its properties.

It follows from the definition 1.1. that for \( x, y \in B \)

\[
||x + y|| \leq 2 - 2\delta_X(||x - y||), \tag{6}
\]

a refined triangle inequality.

Leu us give some generalization of this notion MOC.

**Definition 1.2.** Let again \( (X, ||\cdot||X = ||\cdot||) \) be the Banach space. Suppose that there exists an another Banach space \( (Y, ||\cdot||Y = ||\cdot||) \) such that \( X \) is embedded in \( Y : \ X \subset Y \), and a non - negative numerical valued function
(functional!) \( \Delta[X,Y] = \Delta[X](u), \) \( u \in X, \) which is named as a weak characteristic of convexity, (WCOC), such that

\[ \Delta[X](u) = \Delta[X,Y](u) = 0 \iff u = 0, \]

and for which

\[ \forall x,y \in B \Rightarrow ||x + y|| \leq 2 - 2\Delta[X,Y](||x - y||). \] (7)

It is this inequality (7) for the Grand Lebesgue Spaces \( \mathcal{X}, \) that was applied in particular in the theory of random fields, see for example [29], chapter 3, sections 3.3 - 3.6.

To be more precisely, we want to prove the existence \( \Delta[G_\psi](\cdot) \) for the so-called Grand Lebesgue Spaces \( G_\psi \) and derive some its estimations.

**Brief note about Grand Lebesgue Spaces (GLS).**

We recall here for reader convenience some known definitions and facts about the theory of Grand Lebesgue Spaces (GLS) using in this article. Let \( (Z, M, \mu) \) be measurable space with non-trivial atomless measure \( \mu. \) The ordinary Lebesgue-Riesz norm \( ||f||_p \) for the numerical valued measurable function \( f : Z \to R \) is defined as ordinary

\[ ||f||_p := \left( \int_Z |f(z)|^p \mu(dz) \right)^{1/p}, \quad 1 \leq p < \infty; \]

and as ordinary \( L_p = L_p(Z, \mu) = \{ f : Z \to R, ||f||_p < \infty \}. \)

Further, let the numbers \( (a, b) \) be constants such that \( 1 \leq a < b \leq \infty; \) and let \( \psi = \psi(p) = \psi[a, b](p), \) \( p \in (a, b), \) be numerical valued strictly positive function not necessary to be finite in every point:

\[ \inf_{p \in (a,b)} \psi[a,b](p) > 0. \] (8)

In the case when \( b < \infty \) one can assume sometimes \( p \in (a, b). \)

For instance

\[ \psi_{(m)}(p) := p^{1/m}, \quad m = \text{const} > 0, \quad p \in [1, \infty) \]

or

\[ \psi^{(b, \beta)}(p) := (p - a)^{-\beta_1} \cdot (b - p)^{-\beta_2}, \quad p \in (a, b), \quad \beta_1, \beta_2 = \text{const} \geq 0. \]

The set of all such a functions will be denoted by \( \Psi = \Psi[a,b] = \{ \psi(\cdot) \}. \)
By definition, the (Banach) Grand Lebesgue Space (GLS) $G_\psi = G_\psi[a,b]$, consists of all the real (or complex) numerical valued measurable functions $f : Z \to R$ defined on the whole our space $Z$ and having a finite norm 

$$
\| f \|_{G_\psi} = \| f \|_{G_\psi[a,b]} \overset{def}{=} \sup_{p \in (a,b)} \left[ \frac{|f|_p}{\psi(p)} \right].
$$

(9)

The function $\psi = \psi(p) = \psi[a,b](p)$ is named as the generating function for this space.

If for instance

$$
\psi(p) = \psi^{(r)}(p) = 1, \ p = r; \ \psi^{(r)}(p) = +\infty, \ p \neq r,
$$

where $r = \text{const} \in [1, \infty)$, $C/\infty := 0$, $C \in R$, (an extremal case), then the correspondent $G^{(r)}_\psi(p)$ space coincides with the classical Lebesgue - Riesz space $L_r = L_r(Z, \mu)$.

Note that the introduced in [16], [18] etc. norms

$$
\| f \|_{b,\theta,G} \overset{def}{=} \sup_{0<\epsilon \leq b-1} \left[ \epsilon^{\theta/(b-\epsilon)} |f| b-\epsilon \right], \ \theta \geq 0
$$

quite coincides with appropriate ones up to norm equivalence $\| f \|_{G_\psi^{(b;\beta)}}$, $\beta = \theta/b$.

Let $f : Z \to R$ be certain measurable function such that

$$
\exists a, b, \ 1 \leq a < b \leq \infty \Rightarrow \forall p \in (a,b) \ |f|_p \ < \infty.
$$

The so-called natural function $\psi^{(f)}(p), \ p \in (a,b)$ for this function $f$ is defined as follows

$$
\psi^{(f)}(p) \overset{def}{=} |f|_p.
$$

Obviously,

$$
\| f \|_{G_\psi^{(f)}} = 1.
$$

These spaces are investigated in many works, e.g. in [10], [12], [13], [23], [24], [25], [26], [28], [29] - [33] etc. They are applied for example in the theory of Partial Differential Equations [12], [13], in the theory of Probability [7], [31] - [33], in Statistics [29], chapter 5, theory of random fields, [25], [26], [29], [32], in the Functional Analysis [29], [30], [32] and so one.

These spaces are rearrangement invariant (r.i.) Banach functional spaces; its fundamental function is considered in [32]. They do not coincide in general case with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc., see [28], [30].
2 Main result. The case of small values of parameters.

We consider in this section the case of Grand Lebesgue Spaces $G_{\psi}^{a,b}$, where $1 < a < b \leq 2$. As before, the measure $\mu$ is presumed to be atomless.

Introduce the following auxiliary function

$$\kappa[G_{\psi}^{a,b}](u) := \inf_{p \in (a,b)} \left\{ \frac{||u||^2_p}{\psi^2(p)} \right\}, \ u \in G_{\psi}^{a,b}.$$  

Evidently, this definition is correct also for the arbitrary elements from the space $L_p: \ u \in L_p, \ 1 \leq p < \infty$, but we will apply this notion only for the suitable GLS.

**Theorem 2.1.** In this case, i.e. when $1 < a < b \leq 2$, the weak characteristic of convexity (WCOC) for the space $G_{\psi} = G_{\psi}^{a,b}$ allows a following lower estimate:

$$\Delta[G_{\psi}^{a,b}](x - y) \geq \frac{a - 1}{4} \cdot \kappa[G_{\psi}^{a,b}](x - y), \ x, y \in B[G_{\psi}^{a,b}], \quad (10)$$

so that

$$||x + y||G_{\psi} \leq 2 - \frac{a - 1}{4} \kappa[G_{\psi}^{a,b}](x - y), \ x, y \in B[G_{\psi}^{a,b}].$$

**Proof.** Let $x, y \in B[G_{\psi}^{a,b}]$. We have from the direct definition of the norm in GLS

$$\frac{||x||_p}{\psi(p)} \leq 1, \quad \frac{||y||_p}{\psi(p)} \leq 1, \ p \in (a, b).$$

It follows on the basis of definition 1.1 being applied to the space $L_p$

$$\left|\frac{x}{\psi(p)} + \frac{y}{\psi(p)}\right|_p \leq 2 - 2\delta_{L_p} \left( \frac{||x - y||_p}{\psi(p)} \right).$$

We derive taking supremum over $p$ using the estimate

$$||x + y||G_{\psi}[a,b] \leq 2 - 2 \inf_{p \in (a,b)} \delta_{L_p} \left( \frac{||x - y||_p}{\psi(p)} \right). \quad (11)$$

Further, we will use in particular the relation (3); from one and (11) it follows

$$\delta_p(\epsilon) \geq \frac{a - 1}{8} \epsilon^2, \ \epsilon \in [0, 2]. \quad (12)$$

Therefore

$$\frac{||x + y||_p}{\psi(p)} \leq 2 - \frac{a - 1}{4} \cdot \frac{||x - y||^2_p}{\psi^2(p)}.$$
It remains to take the supremum over $p \in (a, b)$, using the direct definition of the norm in the Grand Lebesgue Spaces:

$$||x + y||_{G\psi} = \sup_{p \in (a, b)} \left\{ \frac{||x + y||_p}{\psi(p)} \right\} \leq \frac{a - 1}{4} \inf_{p \in (a, b)} \left\{ \frac{||x - y||_p^2}{\psi^2(p)} \right\} = 2 - \frac{a - 1}{4} \kappa[G\psi[a, b] ](x - y),$$

Q.E.D.

3 Main result. The case of great values of parameters.

We consider in this section the opposite case of Grand Lebesgue Spaces $G\psi_{a,b}$, where $2 < a < b < \infty$. The measure $\mu$ is again presumed to be atomless.

Introduce the following auxiliary function

$$\theta[G\psi[a, b] ](u) := \inf_{p \in (a, b)} \left\{ \frac{||u||_p}{p \cdot 2^p \cdot \psi^p(p)} \right\}, \quad u \in G\psi[a, b].$$

Obviously, the last definition is correct also for the space $L_p : \quad u \in L_p, \quad 1 \leq p < \infty$.

**Theorem 3.1.** In this case the weak characteristic of convexity (WCOC) for the space $G\psi = G\psi[a, b]$, where $2 < a < b < \infty$ obeys a following lower estimate:

$$\Delta[G\psi[a, b] ](x - y) \geq \theta[G\psi[a, b] ](x - y), \quad (13)$$

so that

$$||x + y||_{G\psi} \leq 2 - \theta[G\psi[a, b] ](x - y), \quad x, y \in B[G\psi[a, b]].$$

**Proof** is quite alike as before in the previous section, as well. We will use the relation (5) for the values $x, y \in B[G\psi[a, b]]$:

$$\frac{||x + y||_p}{\psi(p)} \leq 2 - \frac{||x - y||_p^2}{p \cdot 2^p \cdot \psi^p(p)}, \quad x, y \in B[G\psi[a, b]].$$

It remains to take the supremum over $p \in (a, b)$, using the direct definition of the norm in the Grand Lebesgue Spaces:

$$||x + y||_{G\psi} = \sup_{p \in (a, b)} \left\{ \frac{||x + y||_p}{\psi(p)} \right\} \leq \frac{a - 1}{4} \kappa[G\psi[a, b]](x - y),$$

Q.E.D.
\[ \leq 2 - \inf_{p \in (a,b)} \left\{ \frac{||x - y||_p^p}{p \cdot 2^p \cdot \psi(p)} \right\} = 2 - \theta[G\psi[a,b]](x - y), \]
Q.E.D.

4 Some examples.

**Example 1.** Suppose in addition to the of theorem 2.1., i.e. to the case of Grand Lebesgue Spaces \( G\psi[a,b] \), where \( 1 < a < b \leq 2 \), that the measure \( \mu \) is bounded: \( \mu(Z) = 1 \). Then one can apply the famous Liyapunov’s inequality:

\[ ||x||_p \geq ||x||_a, \quad p \in [a, b]. \]

Assume yet that the generating function \( \psi(\cdot) \) is upper bounded: \( \psi(p) \leq d = \text{const} \in (0, \infty) \). It follows from the proposition of theorem 2.1

\[ ||x + y||G\psi[a,b] \leq 2 - \frac{a - 1}{4} \frac{||x - y||_a^2}{d^2}, \quad x, y \in B[G\psi[a,b]]. \]

**Example 2.** We retain the restrictions \( \mu(Z) = 1 \) and \( \psi(p) \leq d = \text{const} \in (0, \infty) \); but let here \( 2 \leq a < b < \infty \). Then \( ||x||_p \leq ||x||_b, \quad p \leq b \); and we deduce under conditions and assertions of theorem 2.1

\[ ||x + y||G\psi[a,b] \leq 2 - \frac{||x - y||_a^a}{b \cdot 2^b \cdot d^b}, \quad x, y \in B[G\psi[a,b]]. \]

5 Concluding remarks.

It is interest in our opinion to study the Modulus Of Convexity for other Grand Lebesgue Spaces, in particular, which are mentioned in the first section; especially in the case when \( b = \infty \).

Open question: are the GLS \( G\psi[a, \infty] \), for instance, the Subgaussian Spaces with

\[ \psi(p) = \sqrt{p}, \quad p \in (1, \infty) \]

modulatively convex?

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References

[1] C. Bennett and R. Sharpley. *Interpolation of Operators*. Academic Press, New York, 1988.

[2] Buldygin V.V., Mushtary D.I., Ostrovsky E.I, Pushalsky M.I. *New Trends in Probability Theory and Statistics*. Mokslas, 1992, Amsterdam, New York, Tokyo.

[3] Capone C, Formica M.R, Giova R. *Grand Lebesgue spaces with respect to measurable functions*. Nonlinear Analysis 2013; 85: 125 - 131.

[4] Capone C, and Fiorenza A. *On small Lebesgue spaces*. Journal of function spaces and applications. 2005; 3; 73 - 89.

[5] C. Capone, M. R. Formica and R. Giova. *Grand Lebesgue spaces with respect to measurable functions*. Nonlinear Anal. 85 (2013), 125–131.

[6] Clarkson, James. (1936). *Uniformly convex spaces*. Trans. Amer. Math. Soc., American Mathematical Society, 40, (3): 396 - 414, doi:10.2307/1989630, JSTOR 1989630

[7] V. Ermakov, and E. I. Ostrovsky. *Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields*. Moscow, VINITY, (1986), (in Russian).

[8] T. Figiel. *Uniformly convex norms in spaces with unconditional basis*. in: Seminaire Maurey-Schwartz (1974-1975), Espaces $L_p$, applications radonifiantes et geometrie des espaces de Banach, Exp. No. XXIV, 975, pp. 11 pp. (erratum, p. 3).

[9] T Figel. *On the moduli of convexity and smoothness*. Studia Math., 56, (1976), pp. 121 - 155.

[10] Fiorenza A, and Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs*. Journal for Analysis and its Applications 2004; 23 (4) : 657 - 681.

[11] A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces*. Collect. Math. 51 (2000), no. 2, 131–148.

[12] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani and J. M. Rakotoson. *Characterization of interpolation between grand, small or classical Lebesgue spaces*. Preprint arXiv:1709.05892, Nonlinear Anal., to appear.
[13] Fiorenza A., and Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[14] A.Fiorenza, M.R.Formica, T.Roskovec and F.Soudsky. *Gagliardo - Nirenberg inequality for rearrangement invariant Banach functional spaces*. arXiv:1812.04295v1 [math.FA] 11 Dec 2018

[15] A.Fiorenza, M.R.Formica, T.Roskovec and F.Soudsky. *Detailed proof of classical Gagliardo - Nirenberg interpolation inequality with historical remarks*. arXiv:1812.04281v1 [math.FA] 11 Dec 2018

[16] M. R. Formica and R. Giova. *Boyd indices in generalized grand Lebesgue spaces and applications*. Mediterr. J. Math. 12 (2015), no. 3, 987–995.

[17] Gao J. *Modulus of Convexity in Banach Spaces*. Pergamon Applied Mathematics Letters, 16, (2003), 273 - 278; Applied Mathematics Letters. www.elsevier.com.

[18] Greco L, Iwaniec T, Sbordone C. *Inverting the p-harmonic operator*. Manuscripta Math.1997; 92: 259 - 272.

[19] Gurkanli A.T. *Inclusions and the approximate identities of the generalized grand Lebesgue spaces*. Turk J Math.2018; 42:, 3195 - 3203.

[20] Gurkanli A.T. *Multipliers of some Banach ideals and Wiener - Ditkin sets*. Mathematica Slovacia, 2005; 55: (2), 237 - 248.

[21] A.Turan Gürkanli. *Multipliers of grand and small Lebesgue Spaces*. arXiv:1903.06743v1 [math.FA] 15 Mar 2019

[22] Hanner O. *On the uniform convexity of Lp and lp*. Ark. Mat., 3, (1956), 239 - 244.

[23] Iwaniec T., and Sbordone C. *On the integrability of the Jacobian under minimal hypotheses*. Arch. Rat. Mech. Anal., 119, (1992), 129 - 143.

[24] Iwaniec T., P. Koskela P., and Onninen J. *Mapping of finite distortion: Monotonicity and Continuity*. Invent. Math. 144 (2001), 507 - 531.

[25] Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43 - 57.

[26] Kozachenko Yu.V., Ostrovsky E., Sirota L. *Relations between exponential tails, moments and moment generating functions for random variables and vectors*. arXiv:1701.01901v1 [math.FA] 8 Jan 2017
Krasnoselsky M.A., Routisky Ya. B. Convex Functions and Orlicz Spaces. P. Noordhoff Ltd, (1961), Groningen.

Liflyand E., Ostrovsky E., Sirota L. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. J. Math.; 34 (2010), 207 - 219.

Ostrovsky E.I. Exponential estimations for random fields. OINPE, Moscow - Obninsk, 1999, (in Russian).

Ostrovsky E. and Sirota L. Moment Banach spaces: theory and applications. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).

E.Ostrovsky, L.Sirota. Multidimensional Dilation Operators, Boyd and Shimogaki indices of Bilateral Weight Grand Lebesgue Spaces. arXiv:0809.3011[math.FA] 17 Sep 2008.

E.Ostrovsky, L.Sirota. Fundamental functions for Grand Lebesgue Spaces. arXiv:1509.03644v1 [math.FA] 11 Sep 2015

E.Ostrovsky, L.Sirota. Boundedness of operators in bilateral Grand Lebesgue Spaces, with exact and weakly exact constant calculation. arXiv:1104.2963v1 [math.FA] 15 Apr 2011

L.Pick, A.Kufner, O.John and S.Fucik. Function Spaces. Volume 1, 2nd Revised and Extended Edition, De Gruyter Series in Nonlinear Analysis and Applications 14, De Gruyter, Berlin 2013.

J.Reif. A characterization of (locally) uniformly convex spaces in terms of relative openness of quotient maps on the unit ball. J. Funct. Anal., 177, (2000), no. 1, 115.