Some Noncommutative Constructions and Their Associated NCCW Complexes

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Abstract

In this article some noncommutative topological objects such as NC mapping cone and NC mapping cylinder are introduced. We will see that these objects are equipped with the NCCW complex structure of \([P]\). As a generalization we introduce the notions of NC mapping cylindrical and conical telescope. Their relations with NC mappings cone and cylinder are studied. Some results on their \(K_0\) and \(K_1\) groups are obtained and the cyclic six term exact sequence theorem for their \(k\)-groups are proved. Finally we explain their NCCW complex structure and the conditions in which these objects admit NCCW complex structures.

1 Introduction

In topology, the mapping cylinder and mapping cone for a continuous map \(f : X \to Y\) between topological spaces are defined by quotient. These two constructions together the cone and suspension for a topological space, are
important concepts in classical algebraic topology (especially homotopy theory and CW complexes).

The analog versions of the above constructions in noncommutative case, are defined for C*-morphisms and C*-algebras. We review this concepts from [W], and study some results about their related NCCW complex structure, the notion which was introduced by Pedersen in [P].

Another construction which is studied in algebraic topology, is the mapping telescope for a $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ of continuous maps between topological spaces. We define two noncommutative version of this: NC mapping cylindrical and conical telescope.

This paper contains five sections. In sections 2 and 3 we study NCCW complexes and simplicial morphisms, and prove some results which we need in the other sections. In section 4 we discus NC mapping cone and cylinder. Finally in section 5 we introduce the NC mapping cylindrical and conical telescope.

## 2 NCCW Complexes

In this section we explain the notions of NCCW complexes from [P]. To this regard we express some basic definitions from [W].

**Definition 2.1.** Let $A$ and $C$ be two C*-algebras. An *extension* for $A$ with respect to $C$ is a C*-algebra $B$ together with two morphisms $\alpha$ and $\beta$ for which the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$  

**Definition 2.2.** A *pullback* for the C*-algebra $C$ via C*-morphisms $\alpha_1 :$
A_1 \to C \text{ and } \alpha_2 : A_2 \to C \text{ is the } C^*-\text{subalgebra } PB \text{ of } A_1 \oplus A_2 \text{ defined by}

\[
PB := \{a_1 \oplus a_2 \in A_1 \oplus A_2 \mid \alpha_1(a_1) = \alpha_2(a_2)\}
\]

From now on the pullback decomposition notation \( PB := A_1 \bigoplus_C A_2 \) is used all throughout this paper.

**Remark 2.3.** Since \( \alpha_1 \) and \( \alpha_2 \) are continuous maps, \( PB \) is closed in \( A_1 \oplus A_2 \) and so it is a \( C^* \)-algebra.

**Remark 2.4.** From the above definition it follows that the pullback satisfies the following universality property; i.e.

i) It commutes the following diagram

\[
\begin{array}{ccc}
PB & \xrightarrow{\pi_2} & A_2 \\
\downarrow{\pi_1} & & \downarrow{\alpha_2} \\
A_1 & \xrightarrow{\alpha_1} & C
\end{array}
\]

(\( \pi_1 \) and \( \pi_2 \) are projections onto the first and second coordinates)

ii) For any \( C^* \)-algebra \( D \) and any two \( C^* \)-morphisms \( \delta_1 : D \to A_1 \) and \( \delta_2 : D \to A_2 \) for which the following diagram commutes,

\[
\begin{array}{ccc}
D & \xrightarrow{\delta_2} & A_2 \\
\downarrow{\delta_1} & & \downarrow{\alpha_2} \\
A_1 & \xrightarrow{\alpha_1} & C
\end{array}
\]

there exists a unique \( C^* \)-morphism \( \Delta : D \to PB \) such that the following diagram commutes.
In what follows we refer to [P].

**Notations.** Set $\mathbb{I} = [0, 1]$, $\Pi^n = [0, 1]^n$, $\mathbb{I}_0^n = (0, 1)^n$ and for any C*-algebra $A$,

\[
\mathbb{I}A = C(\mathbb{I} \to A), \quad \Pi^n A = C(\Pi^n \to A)
\]

\[
\mathbb{I}_0^n A = C_0(\mathbb{I}_0^n \to A), \quad S^n A = C(S^n \to A)
\]

Here $\Pi^{n+1}/\mathbb{I}_0^{n+1}$ is identified with the sphere $S^n$.

All the above sets together with the usual pointwise operations, and supremum norm are C*-algebras.

**Definition 2.5.** The NCCW complexes are defined by induction on their dimension as follows.

A NCCW complex of dimension zero is defined to be a finite linear dimensional C*-algebra $A_0$ corresponding to the decomposition $A_0 = \bigoplus_k M_{n(k)}$ of finite dimensional matrix algebras.

In dimension $n$, a NCCW complex is defined as a sequence of C*-algebras $\{A_0, A_1, \ldots, A_n\}$, where each $A_k$ obtained inductively from the previous one
by the following pullback construction.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I_k^k F_k & \longrightarrow & A_k & \longrightarrow & A_{k-1} & \longrightarrow & 0 \\
\| & & \| & & \rho_k & & \sigma_k & & \\
0 & \longrightarrow & I_k^k F_k & \longrightarrow & I_k^0 F_k & \overset{\partial}{\longrightarrow} & S^{k-1} F_k & \longrightarrow & 0 \\
\end{array}
\]

In the above diagram, the rows are extensions, \(F_k\) is some \(C^*\)-algebra of finite dimension, \(\delta\) - the boundary map - is the restriction morphism, \(\sigma_k\) the connecting morphism can be any morphism, and finally, \(\rho_k\) and \(\pi\) are projections onto the first and second factors in the pullback decomposition

\[A_k = I_k^k F_k \bigoplus_{S^{k-1} F_k} A_{k-1} .\]

**Remark 2.6.** From the above recursive definition it follows that for each \(n\)-dimensional NCCW complex \(A_n\), there corresponds a decreasing family of closed ideals, called canonical ideals,

\[A_n = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{n-1} \supseteq I_n \neq 0\]

where \(I_n = I_0^k F_n\) and for each \(k \geq 1\), \(I_k/I_{k+1} = I_k^k F_k\). Moreover for each \(0 \leq k \leq n - 1\), \(A_n/I_{k+1}\) is a \(k\)-dimensional NCCW complex.

**Example 2.7.** \(C(\mathbb{I})\) is a 1-dimensional NCCW complexes. For see this, let \(F_1 = \mathbb{C}, A_0 = \mathbb{C} \oplus \mathbb{C}\) and \(A_1 = C(\mathbb{I})\), then we will have \(I_0^1 F_1 = C_0((0, 1))\), \(I_1^1 F_1 = C(\mathbb{I})\), and \(S^0 F_1 = \mathbb{C} \oplus \mathbb{C}\). And the pullback construction diagram becomes

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C_0((0, 1)) & \longrightarrow & C(\mathbb{I}) & \longrightarrow & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & 0 \\
\| & & \| & & \rho_1 & & \| & & \\
0 & \longrightarrow & C_0((0, 1)) & \longrightarrow & C(\mathbb{I}) & \overset{\partial}{\longrightarrow} & \mathbb{C} \oplus \mathbb{C} & \longrightarrow & 0 \\
\end{array}
\]
where $C(I)$ is identified with

$$C(I) \simeq \{ f \oplus (\lambda \oplus \mu) \in C(I) \oplus (C \oplus C) \mid f(0) = \lambda, f(1) = \mu \}$$

$$= C(I) \bigoplus_{C \oplus C} (C \oplus C)$$

And for each $f \in C(I)$ and $\lambda, \mu \in C$,

$$\pi(f \oplus (\lambda \oplus \mu)) = \lambda \oplus \mu$$

$$\rho_1(f \oplus (\lambda \oplus \mu)) = f$$

$$\partial f = f(0) \oplus f(1)$$

Now the sequence $\{A_0 = C \oplus C, A_1 = C(I)\}$ makes $C(I)$ into a 1-dimensional NCCW complex. The canonical ideals for $C(I)$ are

$$C(I) = I_0 \supset I_1 = \mathbb{I}_0^1 F_1 = C_0((0, 1))$$

In a similar way we can see that both $C_0((0, 1])$ and $C_0((0, 1))$ are 1-dimensional NCCW complexes. In 3.2 we describe it in another way.

### 3 Simplicial morphisms

Simplicial morphisms are the most important morphisms in the category of NCCW complexes. They are in fact the NC analogue of simplicial map on CW complex.

**Definition 3.1.** A *simplicial morphism* from the $n$-dimensional NCCW complex $A_n$ into the $m$-dimensional NCCW complex $B_m$ is a mapping $\alpha : A_n \to B_m$ satisfying the following two conditions:

i) If

$$A_n = I_0 \supset I_1 \supset \cdots \supset I_{n-1} \supset I_n \neq 0$$

$$B_m = J_0 \supset J_1 \supset \cdots \supset J_{m-1} \supset J_m \neq 0$$
be the sequences of canonical ideals for $A_n$ and $B_m$, then $\alpha(I_k) \subset J_k$ for all $k$. Particularly $\alpha(I_k) = 0$ for $k > m$.

ii) for $0 \leq k \leq n$, if $I_k/I_{k-1} = \mathbb{I}^k_0 F_k$, $J_k/J_{k-1} = \mathbb{I}^k_0 G_k$ and $\tilde{\alpha}_k : \mathbb{I}^k_0 F_k \to \mathbb{I}^k_0 G_k$ be the homomorphism induced by $\alpha$, then there exists a morphism $\varphi_k : F_k \to G_k$ and a homeomorphism $i_k$ of $\mathbb{I}^k$ such that $\tilde{\alpha}_k = i_k^* \otimes \varphi_k$, where $i_k^*: C_0(\mathbb{I}^k) \to C_0(\mathbb{I}^k)$ is induced by $i_k$. Here $\mathbb{I}^k_0 F_k$ is identified with $C_0(\mathbb{I}^k) \otimes F_k$ and the same for $\mathbb{I}^k_0 G_k$.

Here we state some properties of simplicial morphisms from [P]. (In the following propositions $A_n$ and $B_m$ are NCCW complex of dimensions $n,m$):

**Proposition (i).** The kernel and the image of a simplicial morphism are NCCW complexes.

**Proposition (ii).** The pullback of an NCCW complex $C$ via simplicial morphisms $\alpha : A_n \to C$ and $\beta : B_m \to C$ is an NCCW complex of dimension $\max\{n,m\}$.

**Proposition (iii).** The tensor product $A_n \otimes B_m$ of NCCW complexes is again an NCCW complex of dimension $n + m$.

**Example 3.2.** We show that $C_0((0,1])$, $C_0([0,1))$, and $C_0((0,1))$ are 1-dimensional NCCW complexes. By example 2.7 $C(\mathbb{I})$ is an NCCW complex of dimension 1. Now $C_0((0,1])$ is the kernel of the map

$$\alpha : C(\mathbb{I}) \to \mathbb{C}$$

$$f \mapsto f(0)$$

which is a simplicial morphism. (We note that since $\mathbb{C}$ is a zero dimensional NCCW complex, with the only nonzero ideal $\mathbb{C}$, so $\alpha$ satisfies the two conditions of being a simplicial morphism.) So $C_0((0,1])$ is an NCCW complex. Its dimension is one, because it is not of finite linear dimension (and so it is not a
0-dimensional NCCW complex). Also $C_0([0, 1))$ being identical to $C_0((0, 1])$ is an NCCW complex of dimension one. In a similar way, $C_0((0, 1))$ as the kernel of the simplicial morphism

$$\beta : C_0((0, 1]) \to \mathbb{C}$$

$$f \mapsto f(1)$$

is an 1-dimensional NCCW complex.

**Lemma 3.3.** For each NCCW complex $A$ of dimension $n$, $I_A$ is an NCCW complex of dimension $n + 1$.

**Proof.** It is obvious from the proposition (iii) and the fact that $I_A$ can be identified with the tensor product $C(I) \otimes A$. \qed

**Proposition 3.4.** For any NCCW complex $A$, the evaluation map $ev(1) : I_A \to A$ defined by $f \mapsto f(1)$ is a simplicial morphism.

**Proof.** Since both $I_A$ and $A$ are NCCW complexes, and $A$ is embedded in $I_A$, so we can regard $A$ as an NCCW subcomplex of $I_A$. Now by [P, 11.14] the quotient map $I_A \to A$ is a simplicial morphism. So $ev(1)$ (and also $ev(t_0)$ for each $t_0 \in I$) is simplicial, since it is a quotient map. \qed

### 4 NC constructions

Following [W] in this section the notions of cone and suspension for an arbitrary C*-algebra and also the NC mapping cone and NC mapping cylinder for C*-morphisms are reviewed. We also study their associated NCCW complexes.
**Definition 4.1.** For a C*-algebra $A$, the *NC cone* over $A$, $CA$, and the *NC suspension* of $A$, $SA$ are defined respectively by

$$CA := \{ f \in \mathbb{P}A | f(0) = 0 \}$$

$$SA := \{ f \in \mathbb{P}A | f(0) = f(1) = 0 \}$$

**Remark 4.2.** From the above definition we have the inclusions $SA \subset CA \subset \mathbb{P}A$. The following facts are stated from [W]:

i) $CA$ is contractible.

ii) $SA$ is contractible only if $A$ is contractible.

**Proposition 4.3.** For every NCCW complex $A$ of dimension $n$, the cone $CA$ and the suspension $SA$ are NCCW complexes of dimension $n + 1$.

**Proof.** This follows from example 3.2 proposition(iii), and the following identifications:

$$CA = C_0((0,1] \to A) \simeq C_0([0,1]) \otimes A$$

$$SA = C_0((0,1) \to A) \simeq C_0((0,1)) \otimes A.$$

\[ \square \]

**Definition 4.4.** For a C*-morphism $\alpha : A \to B$ the *NC mapping cone* is defined by

$$Cone(\alpha) := \{ a \oplus f \in A \oplus CB | f(1) = \alpha(a) \}$$

**Remark 4.5.** $Cone(\alpha)$ satisfies the following pullback diagram

$$
\begin{array}{ccc}
Cone(\alpha) & \rightarrow & CB \\
\downarrow{\pi_1} & & \downarrow{ev(1)} \\
A & \alpha \rightarrow & B
\end{array}
$$

where $\pi_1$ and $\pi_2$ are projections onto the first and second coordinates and $ev(1)$ is the evaluation map.

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Proposition 4.6. If \( \alpha : A_n \to B_m \) is a simplicial morphism between NCCW complexes of dimension \( n \) and \( m \), then \( \text{Cone}(\alpha) \) is an NCCW complex of dimension \( \max\{n, m + 1\} \).

Proof. Since \( CB_m \) is an NCCW complex of dimension \( m + 1 \), and \( ev(1) \) and \( \alpha \) are simplicial morphisms in the pullback diagram for \( \text{Cone}(\alpha) \), from proposition (ii) it follows that \( \text{Cone}(\alpha) \) is an NCCW complex of dimension \( \max\{n, m + 1\} \). \( \square \)

Definition 4.7. For a C*-morphism \( \alpha : A \to B \), the NC mapping cylinder is defined by

\[
\text{Cyl}(\alpha) := \{ a \oplus f \in A \oplus IB | f(1) = \alpha(a) \}.
\]

As in \( \text{Cone}(\alpha) \) for \( \text{Cyl}(\alpha) \) we have the following pullback diagram:

\[
\begin{array}{ccc}
\text{Cyl}(\alpha) & \xrightarrow{\pi_2} & IB \\
\downarrow{\pi_1} & & \downarrow{ev(1)} \\
A & \xrightarrow{\alpha} & B
\end{array}
\]

Remark 4.8. Since \( CB \subset IB \), so for any C*-morphism \( \alpha \) we have the inclusion \( \text{Cone}(\alpha) \subset \text{Cyl}(\alpha) \). Also for the zero morphism \( 0 : A \to B \) we have:

\[
\text{Cone}(0) = \{ a \oplus f \in A \oplus CB | f(1) = 0 \} = A \oplus C_0((0, 1) \to b) = A \oplus SB
\]

\[
\text{Cyl}(0) = \{ a \oplus f \in A \oplus IB | f(1) = 0 \} = A \oplus C_0([0, 1) \to B) \simeq A \oplus CB.
\]

As in the case \( \text{Cone}(\alpha) \) we have the following proposition for \( \text{Cyl}(\alpha) \):

Proposition 4.9. For a simplicial morphism \( \alpha : A_n \to B_m \) between NCCW complexes of dimension \( n \) and \( m \), the \( \text{Cyl}(\alpha) \) is an NCCW complex of dimension \( \max\{n, m + 1\} \).
Proof. We know that $\mathbb{I}B_m$ is an NCCW complex of dimension $m+1$. The rest of proof is similar to 4.6.

In the next section we will apply the following fact.

**Proposition 4.10.** If $\alpha : A \to B$ is a $C^*$-morphism, then $A$ is a deformation retract of $Cyl(\alpha)$.

**Proof.** Let $\pi : Cyl(\alpha) \to A$ be given by $\pi(a \oplus f) := a$ and $\eta : A \to Cyl(\alpha)$ by $\eta(a) := a \oplus \alpha(a)$, where $\alpha(a)$ denotes the constant map $t \mapsto \alpha(a)$. Then $\pi \circ \eta = id_A$. Now we define the $C^*$-morphism $\psi : Cyl(\alpha) \to \mathbb{I}Cyl(\alpha)$ by

\[
\psi(a \oplus f) : \mathbb{I} \to Cyl(\alpha)
\]

\[
t \mapsto a \oplus f_t
\]

where $f_t : \mathbb{I} \to B$ is defined as $f_t(s) = f((1-s)t + s)$ for each $s \in \mathbb{I}$. Then we can see that

\[
\psi(a \oplus f)(0) = a \oplus f = id_{Cyl(\alpha)}(a \oplus f)
\]

\[
\psi(a \oplus f)(1) = a \oplus f(1) = a \oplus \alpha(a) = \eta \circ \pi(a \oplus f))
\]

and so $\eta \circ \pi$ and $id_{Cyl(\alpha)}$ are homotopic.

5 NC mapping telescope

In this section we generalize the notions of NC mapping cylinder and NC mapping cone. Their related exact sequences are studied, and their $K_0$ and $K_1$ groups are obtained and the conditions for their NCCW complex structure are specified.
Definition 5.1. For a sequence of length $n$ of C*-algebras

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \quad (1)$$

the **NC mapping cylindrical telescope** is defined by

$$T_n := \{ a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus IA_2 \oplus \cdots \oplus IA_{n+1} | f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n \}$$

Since each $\alpha_k$ is continuous, $T_n$ is a closed subalgebra of the C*-algebra $A_1 \oplus IA_2 \oplus \cdots \oplus IA_{n+1}$, and so it is a C*-algebra. Also we note that for $n = 1$, $T_1 = Cyl(\alpha_1)$.

**Proposition 5.2.** For each $n \geq 2$, $T_{n-1}$ is a deformation retract of $T_n$.

**Proof.** The proof is done by induction on $n$. For $n = 1$, let $\beta : T_1 \to A_3$ be defined by $\beta_2(a \oplus f_2) = \alpha_2 \circ \alpha_1(a)$. Then $\beta_2$ is a C*-morphism and from proposition 4.10, $T_1$ is a deformation retract of $Cyl(\beta_2)$. But $Cyl(\beta_2) = T_2$, since

$$Cyl(\beta_2) = \{(a \oplus f_2) \oplus f_3 \in T_1 \oplus IA_3 | \beta_2(a \oplus f_2) = f_3(1)\}$$

$$= \{ a \oplus f_2 \oplus f_3 \in A_1 \oplus IA_2 \oplus IA_3 | f_2(1) = \alpha_1(a), f_3(1) = \alpha_2 \circ \alpha_1(a) \}$$

$$= T_2.$$

Now by induction if we define $\beta_n : T_{n-1} \to A_{n+1}$ by $\beta_n(a \oplus f_2 \oplus \cdots \oplus f_n) = \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a)$, we see that $T_{n-1}$ is a deformation retract of $T_n$. \qed

**Corollary 5.3.** For the sequence (1), $A_1$ is a deformation retract of $T_n$.

**Proof.** From the previous proposition, $T_n$ deformation retract to $T_1 = Cyl(\alpha_1)$ and since by 4.10 $Cyl(\alpha_1)$ deformation retracts to $A_1$, so $A_1$ is a deformation retract of $T_n$. \qed

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Corollary 5.4. For the sequence $(1)$, $K_0(A_1) = K_0(T_1) = \cdots = K_0(T_n)$.

Proposition 5.5. For the sequence $(1)$, if $\alpha_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_{m-k} = 0$ for some $k < m \leq n$, then
\[ T_n \simeq T_{m-1} \oplus \left( \bigoplus_{i=m+1}^{n+1} CA_i \right) \]

Proof.
\[ T_n = \{ a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus I A_2 \oplus \cdots \oplus I A_{n+1} \mid f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n \} \]
\[ = \{ a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus I A_2 \oplus \cdots \oplus I A_{n+1} \mid f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), 1 \leq k < m; f_{k+1}(1) = 0, m \leq k \leq m \} \]
\[ \simeq T_{m-1} \oplus CA_{m+1} \oplus \cdots \oplus CA_{n+1}. \]

\[ \square \]

Corollary 5.6. For the exact sequence $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ of length $n$, $T_n \simeq T_{n-1} \oplus CA_{n+1}$ and in particular
\[ T_n \simeq Cyl(\alpha_1) \oplus \left( \bigoplus_{i=3}^{n+1} CA_i \right). \]

Proposition 5.7. If $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ is a sequence of simplicial morphism between NCCW complexes of dimensions $m_1, m_2, \ldots, m_{n+1}$ then $T_n$ is an NCCW complex of dimension $\max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$.

Proof. Since $\alpha_1 : A_1 \to A_2$ is a simplicial morphism, by 4.9, $T_1 = Cyl(\alpha_1)$ is an NCCW complex of dimension $\max\{m_1, 1 + m_2\}$. Let $\beta_2 : T_1 \to A_3$ be the C*-morphisms of proposition 5.2. Since $\alpha_1$ and $\alpha_2$ are simplicial morphism then so is $\beta_2$, and $T_2 = Cyl(\beta_2)$ is an NCCW complex of dimension
max\{max\{m_1, 1 + m_2\}, 1 + m_3\} = max\{m_1, 1 + m_2, 1 + m_3\}$. Inductively, the morphism
\[
\beta_n : T_{n-1} \longrightarrow A_{n+1}
\]
\[
a \oplus f_2 \oplus \cdots \oplus f_n \mapsto \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a)
\]
is simplicial and so $T_n = Cyl(\beta_n)$ is an NCCW complex of dimension $max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$.

**Definition 5.8.** Let $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1}$ be a sequence of C*-morphisms. The NC mapping conical telescope is defined as
\[
T_n C := \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} | f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n\}
\]
As in the case $T_n$, for $n = 1$, $T_1 C = Cone(\alpha_1)$.

**Proposition 5.9.** For the sequence (1), if $\alpha_m \circ \alpha_{m-1} \circ \cdots \circ \alpha_{m-k} = 0$ for some $k < m \leq n$, then
\[
T_n C = T_{m-1} C \oplus ( \bigoplus_{i=m+1}^{n+1} SA_i)
\]

**Proof.**
\[
T_n C = \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} | f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), k = 1, 2, \ldots, n\}
\]
\[
= \{a \oplus f_2 \oplus \cdots \oplus f_{n+1} \in A_1 \oplus CA_2 \oplus \cdots \oplus CA_{n+1} | f_{k+1}(1) = \alpha_k \circ \alpha_{k-1} \circ \cdots \circ \alpha_1(a), 1 \leq k < m; f_{k+1}(1) = 0, m \leq k \leq m\}
\]
\[
= T_{m-1} C \oplus SA_{m+1} \oplus \cdots \oplus SA_{n+1}.
\]
Corollary 5.10. If the sequence (1) is exact, then \( T_nC = T_{n-1} \oplus SA_{n+1} \), and in particular
\[
T_nC = \text{Cone}(\alpha_1) \oplus \bigoplus_{i=3}^{n+1} SA_i.
\]

Proposition 5.11. For each sequence of length \( n > 1 \), there is an exact sequence
\[
0 \rightarrow SA_{n+1} \rightarrow T_nC \rightarrow T_{n-1} \rightarrow 0.
\]

Proof. Let \( i : SA_{n+1} \rightarrow T_nC \) be the inclusion morphism \( f \mapsto 0 \oplus 0 \oplus \cdots \oplus f \) and \( \pi : T_nC \rightarrow T_{n-1}C \) be the projection \( a \oplus f_2 \oplus \cdots \oplus f_{n+1} \mapsto a \oplus f_2 \oplus \cdots \oplus f_n \), then \( \ker \pi = i(SA_{n+1}) \).

Proposition 5.12. For each sequence of length \( n \), there exists an exact sequence
\[
0 \rightarrow T_nC \rightarrow T_n \rightarrow \bigoplus_{k=2}^{n+1} A_k \rightarrow 0.
\]

Proof. Let \( i : T_nC \rightarrow T_n \) be the obvious inclusion and \( \pi : T_n \rightarrow \bigoplus_{k=2}^{n+1} A_k \) be defined by \( \pi(a \oplus f_2 \oplus \cdots \oplus f_{n+1}) := f_2(0) \oplus \cdots \oplus f_{n+1}(0) \), Then \( \ker \pi = T_nC \).

Proposition 5.13. For each sequence of length \( n \), there exists a cyclic six term exact sequence,
\[
\begin{array}{cccccc}
K_0(T_nC) & \rightarrow & K_0(T_n) & \rightarrow & \bigoplus_{i=2}^{n+1} K_0(A_i) & \\
\uparrow & & & & \downarrow & \\
\bigoplus_{i=2}^{n+1} K_1(A_i) & \leftarrow & K_1(T_n) & \leftarrow & K_1(T_nC)
\end{array}
\]

Proof. Since \( T_nC \) is an ideal in \( T_n \), and \( T_n/T_nC = \bigoplus_{i=2}^{n+1} A_i \), and \( K_0(\bigoplus_{i=2}^{n+1} A_i) = \bigoplus_{i=2}^{n+1} K_0(A_i) \) and \( K_1(\bigoplus_{i=2}^{n+1} A_i) = \bigoplus_{i=2}^{n+1} K_1(A_i) \), the exactness of the six term sequence follows from [W,9.3.2].
Proposition 5.14. If $A_1 \overset{\alpha_1}{\longrightarrow} A_2 \overset{\alpha_2}{\longrightarrow} \cdots \overset{\alpha_{n-1}}{\longrightarrow} A_n \overset{\alpha_n}{\longrightarrow} A_{n+1}$ is a sequence of simplicial morphisms between NCCW complexes of dimensions $m_1, m_2, \ldots, m_{n+1}$, then $T_n C$ is an NCCW complex of dimension $\max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$.

Proof. Since $\alpha_1 : A_1 \to A_2$ is a simplicial morphism, by [4.6], $T_1 C = Cone(\alpha_1)$ is an NCCW complex of dimension $\max\{m_1, 1 + m_2\}$. Let $\beta_2 : T_1 C \to A_3$ be defined as $\beta(a \oplus f_2) = \alpha_2 \circ \alpha_1(a)$. Since $\alpha_1$ and $\alpha_2$ are simplicial morphism then so is $\beta_2$, and $T_2 C = Cone(\beta_2)$ is an NCCW complex of dimension $\max\{\max\{m_1, 1 + m_2\}, 1 + m_3\} = \max\{m_1, 1 + m_2, 1 + m_3\}$. Inductively, the morphism

\[ \beta_n : T_{n-1} C \longrightarrow A_{n+1} \]

\[ a \oplus f_2 \oplus \cdots \oplus f_n \mapsto \alpha_n \circ \alpha_{n-1} \circ \cdots \circ \alpha_1(a) \]

is simplicial and so $T_n C = Cone(\beta_n)$ is an NCCW complex of dimension $\max\{m_1, 1 + m_2, 1 + m_3, \ldots, 1 + m_{n+1}\}$.

References

[GVF] J.M.GRACIA-BONDIA, J.C.VARILLY and H.FIGUEROA, Elements of Noncommutative Geometry, Birkhauser, Boston, 2001.

[H] A.HATCHER, Algebraic topology, Cambridge University Press, 2002.

[P] G. K. PEDERSEN, Pullback and pushout constructions in C*-algebras theory, J.Funct.Analysis 167(1999), 243-344.

[W] N.E.WEGGE-OLSEN, K-theory and C*-Algebras : a Friendly Approach, Oxford University Press, 1993.