We investigate the subadditivity of the bipartite entanglement entropy (EE) of many-particle states, represented by Slater determinants, with respect to single particle excitations. By quantifying this subadditivity we identify sets of single particle states that yield zero EE if jointly excited. Such states we dub entanglement erasing partner states (EEPS). By this we have identified a mechanism that allows to disentangle two subspaces of a Hilbert space by exciting additional states. We illustrate the entanglement erasure in Anderson insulators, where we identify the EEPS, and use the underlying mechanism to explain discrete entanglement bands in the clean tight binding model. Our results indicate a direct applicability to their interacting counterparts. We further discuss how EEPS impose a universal erasure of EE for randomly excited states - independent of the Hamiltonian of interest. This feature allows to compute many-particle EE by means of the associated single particle states and the filling ratio. This novel finding can be employed to drastically reduce the computational effort in free models.

The purely quantum phenomenon of entanglement has impacted the whole history of quantum mechanics and is to date under ongoing research. While entanglement first demonstrated the non-local nature of quantum physics, Bell’s inequalities later excluded a description of quantum mechanics by means of classical hidden variables. In addition, Bell’s work first quantified the quantum correlations between two subsystems that arise due their entanglement. Today many measures of entanglement are employed, among which entanglement entropy (EE) might be the most famous.

Quantum entanglement plays a key role in research fields ranging from the description of black holes over a plethora of aspects of quantum information to photosynthetic processes. EE further turned out to be extremely valuable for describing ground- and excited states of many-body systems and their behavior after quenches. A logarithmic growth of EE after a quench is even assumed to serve as a defining feature of the many-body localized phase.

Besides distinguishing states only by being a ground state or an excited state, the EE of many-body states turned out to show very different features depending on the number of (quasi) particles, both, in interacting and free models. Ongoing process tries to get new insights in the behavior of the EE of excited states within a quasi-particle picture. As we will show, many of these features can already be traced back to the fact that, even for free fermions, EE shows a subadditive behavior. That is, the EE of a many-particle state is in general less than the sum of the individual EE contributions of single particles. We will precise this statement below.

In this work, we give a constructive model-independent proof of the subadditive behavior of the EE for two arbitrary single particle states. This allows us to derive the exact amount of EE erasure and to study its dependency on the structure of the eigenstates in various models. In particular, we analytically demonstrate the subadditive behavior in the one-dimensional tight-binding model of free fermions. Our here computed band structure of the EE has recently also been observed in the interacting XXX spin 1/2 chain. We describe the conditions under which two states annihilate each others EE completely. This implies that quantum correlations between two bipartitions of a Hilbert space may totally be lifted simply by exciting additional states. In fact, our analysis suggests that for each single particle excitation, there is a model-dependent set of states such that, if totally excited, the EE is minimized or even totally erased. These sets we dub entanglement erasing partner states (EEPS). Such EEPS imply a universal erasure of EE for states that can be represented by a Slater determinant: We demonstrate that the relative reduction of EE of $N$ simultaneously excited single particle states chosen at random is almost model-independent but only depends on the filling factor. We indicate how this insight can be exhausted within numerical methods in order to reduce computation times drastically.

Subadditivity of entanglement entropy — We start by precisely our measure of entanglement and show its differences to other studies on entropy. Unlike other works, we do not study the entanglement between individual particles, but rather the entanglement between two bipartitions $\mathcal{H}_A, \mathcal{H}_B$ of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ that describes the particles. It is typically measured by means of the von Neumann entropy

$$S_E(\langle \psi \rangle) = -\text{tr} [\rho_A \log_2 \rho_A],$$

where $\rho_A = \text{tr}_B [\langle \psi \rangle \langle \psi \rangle^*]$, $\langle \psi \rangle \in \mathcal{H}$ is a single or many-particle state, and $\text{tr}_B$ is the partial trace over $\mathcal{H}_B$. The bipartitions often correspond to contiguous parts of a system in real space, but this is not required here. The von Neumann entropy is known for showing subadditivity $S(\langle \psi \rangle \langle \psi \rangle) \leq S(\rho_A) + S(\rho_B)$ with respect to the subspaces.

$$S_E(\langle \psi \rangle) = -\text{tr} [\rho_A \log_2 \rho_A],$$
$H_A, H_B$. In contrast, in this work, we will prove and employ a subadditivity with respect to fermionic single particle states $|i\rangle = \psi_i^\dagger |\rangle$

$$S_E(|1, 2\rangle) \leq S_E(|1\rangle) + S_E(|2\rangle),$$

(2)

where $|\rangle$ is the vacuum state of zero particles. That is, a Slater determinant $|1, 2\rangle = \psi_1^\dagger \psi_2^\dagger |\rangle$ yields a joint EE $S_E(|1, 2\rangle)$ between the two bipartitions $H_A, H_B$ that is lower or equal than the sum of the individual single particle contributions. As the proof of Eq. (2) is a straightforward calculation employing methods of Ref. 22, we position it in the supplementary material but focus on its intriguing consequences here.

Entanglement erasure — In Ref. 23, we show that two orthonormal fermionic states $|1\rangle, |2\rangle$ with $\psi_i = \sqrt{\lambda_i} \phi_i^A + \sqrt{1 - \lambda_i} \phi_i^B$ are normalized vectors and $\{|c_i\rangle\}_i$ forms a basis of $H$ that can be divided into a part corresponding to $H_A$ and $H_B$, respectively, yield a joint EE of

$$S_E(\psi_1^\dagger \psi_2^\dagger |\rangle) = s(\nu_1) + s(\nu_2),$$

(3)

where $s(x) = -x \ln x - (1-x) \ln(1-x)$ and

$$\nu_{1,2} = \lambda_{1,2} \pm 1/2 \left[ \sqrt{\left(\lambda_1 - \lambda_2\right)^2 + 4\lambda_1 \lambda_2 \left|\phi_1 \phi_2^\dagger\right|^2} - \left(\lambda_1 - \lambda_2\right) \right],$$

Thus, EE is only additive if $\phi_1 \phi_2^\dagger = 0$, which raises an additional orthogonality condition on the states $|1\rangle, |2\rangle$ within the subspaces $H_A$ and $H_B$. Similar orthogonality dependencies have been observed before in a situation where states are superposed instead of jointly excited.

Remarkably, EE is totally erased for $\phi_1 \phi_2^\dagger = 1$ and $\lambda_2 = 1 - \lambda_1$, i.e. if the two orthogonal states $|1\rangle, |2\rangle$ are equal within $H_A$. We illustrate this peculiar feature with a trivial example: Let $|a\rangle \in H_A$ and $|b\rangle \in H_B$. The two orthonormal states $|1\rangle = \sqrt{0.3}|a\rangle + \sqrt{0.7}|b\rangle$ and $|2\rangle = \sqrt{0.7}|a\rangle - \sqrt{0.3}|b\rangle$ both yield a finite entanglement between $H_A$ and $H_B$. However, the two particle state $|1, 2\rangle = |a\rangle \otimes |b\rangle$ is a product state with respect to $H_A$ and $H_B$, implying zero EE. This example is of course constructed in a way that maximizes the effect of entanglement erasure. However, as we show next, also eigenstates of important toy models for condensed matter theory show significant erasure and form EEPS.

EEPS in localized models — Free fermions constraint to a one-dimensional chain of sites $i$ with random potentials $h_i \in [-W, W]$, described by the Hamiltonian

$$H_{An} = -t \sum_{i=1}^L \left( c_i^\dagger c_{i+1} + \text{h.c.} \right) + \sum_{i=1}^L h_i c_i^\dagger c_i$$

(4)

are known to experience Anderson localization for any finite disorder strength $W > 0$. There, all eigenstates $|c_i^\dagger |\rangle$ are exponentially localized around site $i$. Thus, importantly, only the eigenstates that are localized near the cut between two halves of the system $H_A$ and $H_B$ contribute to entanglement between these subspaces. Exciting an extensive amount of randomly chosen eigenstates therefore yields an area law, e.g. $S_E(|\psi_1, \psi_2, \ldots, \psi_{N-2}\rangle) \sim L^0$. In order to demonstrate the effect of Eq. (3), we do not excite the eigenstates randomly but rather states that are neighbors in real space. This maximizes $|\phi_i \phi_j^\dagger|$ and thus erases most EE. In Fig. 1 we show that this procedure leads to a joint EE $S_E(N)$ of $N$ nearest neighbor states close to the cut, which decays exponentially in $N$. Remarkably, the two halves of a chain can be disentangled by simply exciting additional states. This mechanism of entanglement erasure is not limited to free particles only: If a density-density interaction term $V = 2t \sum_i c_i^\dagger c_i c_{i+1}^\dagger c_{i+1}$ is added to Eq. (4), a many-body localization transition occurs around $W_c \approx 3.5 \cdot 2^{26-29}$. The full Hamiltonian still conserves the total particle number, which allows us to study the EE for eigenstates of given particle number $N$. Analogously to the free model, our demand on the eigenstates is that the particles are localized near the cut. Again we find exponential

![Figure 1. Exponential erasure of EE by exciting additional states. Out of $L = 4096$ Anderson localized states, we excite $N$ states that are localized closest to the cut that defines the bipartic EE (see upper inset for $N \in \{2, 4, 8, 16\}$ and see 23). The data shows that the erasure of entanglement can dominate over the gain of entanglement if additional states are excited. In this model, the disorder strength $W$ determines the localization length and therefore the number of states within a set of EEPS. The black solid line is a guide for the eye and the black circles indicate data points that correspond to the upper inset. The lower inset shows the corresponding interacting model with $L = 16$. Erasure is effective within the MBL phase only.](image-url)
erasure of EE in the localized phase $W > W_c$, however, erasure is not effective for $W < W_c$, where the eigenstates are extended.

**Exact EE in the tight-binding model** Our analysis is not restricted to localized models. In fact it enables an exact computation of the EE of Slater determinants of two eigenstates in any model where the eigenstates are known. As an example, we show here new insights into the tight-binding model, described by the Hamiltonian in Eq. (4) for zero disorder $h_i = 0$. For periodic boundary conditions, the eigenstates, extended over the whole lattice, yield

$$|k⟩ = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ikj_0} c_j^\dagger. \quad (5)$$

Let us assume that here $\mathcal{H}_A$ is a contiguous part of the chain with length $L_A = lL$. Then, $\lambda_1 = \lambda_2 = l$. In the limit $L → ∞$, the overlap between eigenstates $|k_1⟩$, $|k_2⟩ = k_1 + n\frac{2\pi}{L}$ is $\phi_1\phi_2 = \sin \left[\frac{lnπ}{n}\right] / (lnπ)$. Employing Eq. (3) one gets

$$S_E^{th}(l, n) = s \left( l + \sin \left[\frac{lnπ}{n}\right] \right) + s \left( l - \sin \left[\frac{lnπ}{n}\right] \right), \quad (6)$$

which we illustrate in Fig. 2. Thus, $S_E^{th}(l, n)$ depends only on the relative momentum $\frac{2\pi}{L}n$, in agreement with previous observations, e.g. in Ref. 15. Let us now fix the relative size of $\mathcal{H}_A$ to the half the system size, $l = 1/2$. The maximum effect of erasure, and hence, the minimum value of a joint EE of two simultaneously excited momentum states, due to the quantization of momentum, yields

$$S_E^{th}(l = 0.5, n = 1) ≈ 1.36752 \text{ bit.} \quad (7)$$

Again our results hold surprisingly well for an interacting extension of this model: A similar value was observed for two magnon eigenstates of the XXX spin 1/2 chain. 16

Equation (6) further enables to quantify the EE erasure of states with big relative momentum. For $l = 1/2$ and $n \gg 1$ one finds

$$S_E^{th}(n) = 2 - \left( \frac{2\sin \left[\frac{π}{n}\right]}{nπ\sqrt{\log 2}} \right)^2 + O \left( \frac{1}{n^4} \right), \quad (8)$$

such that the erasure of EE decreases quadratically with $n$. These bands of allowed values of a two-particle EE (see Fig. 2) have been observed in the XXX chain, too16. Their origin is, as we have demonstrated, the quantization of momentum of single particle states, which shows a significant effect on the erasure of EE, even in the thermodynamic limit $L → ∞$.

**Universal erasure** — As we have illustrated, the amount of EE erasure depends on the structure of the excited states. Often, however, the EE of randomly drawn eigenstates is studied, for which we indentify a structure independent behavior of the entanglement erasure and derive a model independent application of this finding. Before we generalize, we illustrate the underlying mechanism by means of the following pairs of Bell states

$$|e_{2i}^{\text{Bell}}⟩ = \frac{1}{\sqrt{2}}(|2i⟩ + |2i + 1⟩) \quad (9)$$

$$|e_{2i+1}^{\text{Bell}}⟩ = \frac{1}{\sqrt{2}}(|2i⟩ - |2i + 1⟩), \quad (10)$$

which individually show maximum entanglement (1 bit) between the bipartitions $\mathcal{H}_A := \text{span}\{|i⟩ | i \text{ even}\}$ and $\mathcal{H}_B := \text{span}\{|i⟩ | i \text{ odd}\}$. Further, the two states $|e_{2i}^{\text{Bell}}⟩$ and $|e_{2i+1}^{\text{Bell}}⟩$ with same index $i$ form a EEPS set, i.e. they erase each others EE completely. Importantly, states with different index $i$ do not interfere with each other in case of joint excitation, as they are orthogonal within the bipartitions. For $N$ randomly exited states, the expectation value of the total EE is then simply given by the number of single-occupied pairs of states, i.e.:

$$E \left[S_B^{\text{Bell}}(N)\right] = \frac{(L - N)N}{L - 1} \text{ bit,} \quad (11)$$

where $L = |\mathcal{H}_A| + |\mathcal{H}_B|$ is the dimension of the full single-particle Hilbert space. For a fixed occupation ratio of $N/L$ and $L → ∞$ we quantify the effect of the erasure by

$$r_{\infty}^{\text{Bell}} = E \left[S_B^{\text{Bell}}(N)\right] / N \cdot 1 \text{ bit} = 1 - (N/L), \quad (12)$$

such that the relative amount of erasure depends only on the filling ratio $(N/L)$. This is natural: The more states are excited, the more likely it is to occupy entanglement erasing partner states simultaneously.

This idea can readily be generalized to arbitrary mod-
els. To this end, consider the erasure factor

\[ r_\infty := \frac{E[S_E(|e_1, e_2, \ldots, e_N|)]}{E[\sum_i^N S_E(|e_i|)]}, \]  

where the index \( \infty \) indicates that this quantity is to be evaluated for \( L \to \infty \). The expectation values are taken over different combinations of jointly excited eigenstates and, if disorder is present, disorder ensembles. Even though EE behaves extremely different for localized and extended states, as the above discussed examples show, the relative amount of erasure, expressed by \( r_\infty \), shows an almost model-independent behavior that is in agreement with the analytic result for the above constructed Bells states. This is illustrated in Fig. 3.

Beside the Anderson localized chain and the tight-binding model, we compare the results with a tight-binding Hamiltonian with staggered potentials, i.e. \( h_i \to (-\mu)^i \), which creates a gap in the spectrum. Also, a central site Hamiltonian\(^{14}\), where an additional site \( |0\rangle \) is coupled to each site of the above defined Anderson chain via the term \( A/\sqrt{L} \sum_i (c_i^\dagger c_0 + h.c.) \) shows surprisingly similar results. In this central site model, a single particle mobility edge exists and the eigenstates show multifractal properties\(^{14}\).

All studied single particle models, despite their significant differences in the structure of their eigenstates, show a universal behavior of the joint EE

\[ S_E(|e_1, e_2, \ldots, e_N|) \sim \left(1 - \frac{N}{L}\right) \sum_i^N S_E(|e_i|), \]  

Conveniently, the erasure compared to the bare sum of the single particle contributions is simply given by the occupation ratio \( N/L \).

This universal feature implies a direct application: In order to compute the total EE \( S_E(|e_1, e_2, \ldots, e_N|) \) of a Slater determinant, one is usually required to diagonalize a matrix of the size of the reduced Hilbert space, see Eq. (1). Our results in Fig. 3 suggest the approximation

\[ S_E(|e_1, e_2, \ldots, e_N|) = r_\infty \sum_i^N S_E(|e_i|) \]  

\[ \approx \frac{L - N}{L} \sum_i^N S_E(|e_i|) \]  

\[ = \frac{L - N}{L} \sum_i^N s(\lambda_i), \]  

where \( \lambda_i \) is the probability to find eigenstate \( |e_i| \) in \( \mathcal{H}_A \).

Thus, the universal behavior of entanglement erasure reduces the complexity of computing the EE of Slater determinants from performing diagonalizations to summations.

**Conclusion and Outlook** — We have shown that, albeit different single particle fermionic models obey significantly different structures of eigenstates, a universal mechanism of entanglement erasure can be employed in order to study the von Neumann entanglement entropy of the excited states between bipartitions. Counterintuitively, such bipartitions may be disentangled from each other, i.e. their quantum correlations may be destroyed, by simply exciting additional particles. Despite its simplicity, the described mechanism of entanglement erasure shows a surprisingly well agreement with many-particle eigenstates of interacting models. However, its precise range of applicability on interacting models, where the eigenstates are in general no Slater determinants, needs to be explored in a future work. Further, it would be interesting to understand why the tight-binding models slightly deviate from the proposed universal law. This could be due to a lack of disorder or due to the extended nature of eigenstates, which is impossible to separate in one dimensional systems. It could be therefore interesting to analyze the erasure of entanglement in three dimensional systems.

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**Appendix A: Proof of the subadditivity**

Here we prove the subadditivity of the entanglement entropy for two-particle excitations. In particular, given two orthonormal states \( |e_1^\dagger|, |e_2^\dagger| \), where \( e_i^\dagger \) excites a fermion in the state \( i \) and \( |\rangle \) is the vacuum state, the von Neumann entanglement entropy with respect to a bipartition \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) satisfies the inequality:

\[ S_E(e_1^\dagger e_2^\dagger |\rangle) \leq S_E(e_1^\dagger |\rangle) + S_E(e_2^\dagger |\rangle) \]  

(A1)
Proof — The von Neumann entanglement entropy of a state $|\psi\rangle$ is defined by
\[
S_E(|\psi\rangle) = -\text{tr} [\rho_A \ln \rho_A],
\]
where $\rho_A = \text{tr}_B [|\psi\rangle\langle\psi|]$ and $\text{tr}_B [\cdot]$ is the partial trace over $\mathcal{H}_B$. For Slater determinants $|\psi\rangle = e_1^i \ldots e_N^i \rangle$ of $N$ single particle states $S_E(|\psi\rangle)$ can be evaluated by\(^{22}\)
\[
S_E(|\psi\rangle) = -\text{tr} [C_A \ln C_A + (1 - C_A) \ln (1 - C_A)],
\]
where
\[
(C_A)_{ij} = \langle \psi | e_j^A \langle e_i^A | \psi \rangle
\]
is a correlation matrix with respect to a single particle basis $\{e_i^A \rangle\}$ that spans $\mathcal{H}_A$. In order to conduct the proof we first derive the following two lemmas.

**Lemma 1:** The correlation matrices are additive with respect to a joint excitation, i.e.
\[
C^A(|e_1e_2\rangle) = C^A(|e_1\rangle) + C^A(|e_2\rangle).
\]

**Lemma 2:** The correlation matrix of a single excitation $|e_n\rangle$ has at most one nonzero eigenvalue and can be expressed by
\[
C^A(|e_n\rangle) = \lambda_n \bar{\phi}_n (\bar{\phi}_n)^\dagger,
\]
where $\lambda_n$ is the probability of $|e_n\rangle = \sum_i U_{ni}|c_i\rangle$ to be in $\mathcal{H}_A$, and the components of $\bar{\phi}_n$ are given by
\[
(\bar{\phi}_n)_i = \frac{1}{\sqrt{\lambda_n}} U_{ni}.
\]

**Proof of Lemma 1:** We employ a single particle basis $\{e_i^A \rangle\}$ of $\mathcal{H}$ that can be divided into two sets $\{e_i^A \rangle\}$ and $\{e_j^B \rangle\}$ that span $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The two states $|e_1\rangle, |e_2\rangle$ can then be expressed as
\[
|e_i\rangle = \sum_k U_{ik}|c_k\rangle,
\]
where $U$ is a unitary matrix. This step requires the orthogonality of $|e_1\rangle$ and $|e_2\rangle$. Likewise $c_k = \sum_l (U^\dagger)_kl e_l$ and $c_k = \sum_l (U^\dagger)_kl e_l = \sum_l U_{lk}e_l$. Thus,
\[
C_{ij}(|e_1e_2\rangle) = \langle e_1e_2 | e_j^A \langle e_i^A | e_1e_2 \rangle
 = \sum_{kl} (U^\dagger)_{jk} e_1^i e_2^j e_k^e_l e_1^i e_2^e_l U_{ii}
 = \sum_k (U^\dagger)_{jk} \langle e_1e_2 | e_k^e_l e_l^e_k e_1e_2 \rangle U_{ki},
\]
i.e. $C(|e_1e_2\rangle) = U^\dagger DU$,
where $D$ is diagonal. Analogously
\[
C(|e_1\rangle) = U^\dagger D_1 U \quad \text{with} \quad D_1 = \text{diag}(1, 0, 0, 0, \ldots)
C(|e_1\rangle) = U^\dagger D_2 U \quad \text{with} \quad D_2 = \text{diag}(0, 1, 0, 0, \ldots)
\]
with the same unitary matrix $U$ and $D_1 + D_2 = D$. Hence,
\[
C(|e_1e_2\rangle) = U^\dagger (D_1 + D_2) U
 = U^\dagger D_1 U + U^\dagger D_2 U
 = C(|e_1\rangle) + C(|e_2\rangle),
\]
and thus $C_{ij}(|e_1e_2\rangle) = C_{ij}(|e_1\rangle) + C_{ij}(|e_2\rangle)$, which proves Lemma 1.

**Proof of Lemma 2:** Using the definition of the correlation matrix and $|e_n\rangle = \sum_i U_{ni}|c_i\rangle$, it is straightforward to show that
\[
C^A(|e_n\rangle) = U_{nj}^* U_{ni},
\]
with $i, j \in A$. We can thus define the normalized vector
\[
(\bar{\phi}_n)_i = \frac{U_{ni}}{\lambda_n^{1/2}} = \frac{U_{ni}}{\sqrt{\sum_{j \in A} |U_{nj}|^2}}
\]
and find Eq. (A6).

**Completion of the proof:** Using Lemma 1 and Lemma 2, the reduced correlation matrix of the joint state $|e_1e_2\rangle$ yields
\[
C^A(|e_1e_2\rangle) = \lambda_1 \bar{\phi}_1 (\bar{\phi}_1)^\dagger + \lambda_2 \bar{\phi}_2 (\bar{\phi}_2)^\dagger.
\]
Without loss of generality $\lambda_1 > \lambda_2$. Its eigenvalues are then
\[
\nu_1 = \lambda_1 + \delta/2,
\nu_2 = \lambda_2 - \delta/2 \quad \text{with} \quad \delta = \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2 \bar{\phi}_1^* \bar{\phi}_2^2},
\]
where $\delta = 0$ if the scalar product $\bar{\phi}_1 \bar{\phi}_2$ vanishes and $\delta > 0$ elsewhere. As the function
\[
s(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)
\]
is concave, $\delta > 0$ yields (see Fig. 4 for an illustration)
\[
S_E(|e_1e_2\rangle) = s(\nu_1) + s(\nu_2) < s(\lambda_1) + s(\lambda_2) = S_E(|e_1\rangle) + S_E(|e_2\rangle).
\]
The entanglement entropy is thus additive if and only if $\bar{\phi}_1 \bar{\phi}_2 = 0$, taking aside trivial cases in which one $\lambda_i$ is zero or unity. Herewith the theorem is proven.
the same side, a shift \( \nu \) experiences most erasure if the eigenstates are localized simultaneously excited states: The joint entanglement entropy is trivial. Bottom: For \( \lambda_1, \lambda_2 \) on the same side, a shift \( \nu_2 = \lambda_2 - \delta, \nu_1 = \lambda_1 + \delta \) always yields \( s(\nu_2) + s(\nu_1) < s(\lambda_2) + s(\lambda_1) \) due to the concavity of \( s(x) \).

Appendix B: Numerical methodology

1. Numerical data for Fig. 1 of the main part

The aim is to illustrate the existence of entanglement erasing partner states (EEPS) in Anderson- and many-body localized chains. The joint entanglement entropy (EE) of two simultaneously excited eigenstates depends on their overlap \( \phi_1 \phi_2 \) within a bipartition (see main or the above proof). Hence, we conjecture that within localized models, where particles are localized within a localization length of \( \xi \) that depends on the employed disorder strength, the erasure of entanglement is (on average) maximized for eigenstates that are localized on spatially adjacent sites. We infer this to be true for \( N \) simultaneously excited states: The joint entanglement entropy experiences most erasure if the eigenstates are localized on \( N \) adjacent sites in real space. Additionally, in order to see the effect, we study eigenstates that are localized as close as possible to the cut between the bipartitions. This is because only such states yield a finite contribution to the EE between both bipartitions.

Hence, for the non-interacting Anderson chain, we compute all single particle eigenstates \( |E_i\rangle \) by means of exact diagonalization. For a given particle number \( N \), we search the \( N/2 \) out of \( L \) states that have the largest overlap \( |\langle x_i | E_j \rangle|^2 \) with the \( N/2 \) lattice sites \( |x_i\rangle \) left and right of the cut and compute their Slater determinant. The particle density of this many-particle state is illustrated in the top right inset of Fig. 1 of the main part. The resulting many-particle EE is then readily derived by means of the above employed correlation matrix approach.\(^{22}\)

For the bottom right inset in Fig. 1 of the main text, we study the random field Heisenberg model that exhibits an MBL transition.\(^{29}\) This model corresponds to an interacting fermion model after a Jordan-Wigner transformation. As particle number is conserved, we are again able to search for such eigenstates that have most overlap with the many particle state

\[
|\psi\rangle = c_{-N/2}^\dagger c_{-N/2+1}^\dagger \cdots c_{-1}^\dagger c_0^\dagger \cdots c_{N/2-1}^\dagger |
\]

where \( c_i^\dagger \) creates a particle on the site \( i \) and negative and positive indices correspond to the different bipartitions \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively. This many-particle state is by definition the state with \( N \) particles as close as possible to the cut. By studying the properties of the many particle eigenstate \( |\phi_i\rangle \) that has most overlap \( |\langle \phi_i | \psi \rangle|^2 \) with \( |\psi\rangle \), we expect to see the eigenstate for which the effect of entanglement erasure is maximized. The entanglement entropy of \( |\phi_i\rangle \) is then computed regularly by tracing out one bipartition and evaluating the von Neumann entropy of the resulting reduced density matrix.

For both, the interacting and the free model, we employ many disorder ensembles over which we average our results.

2. Numerical data for Fig. 3 of the main part

In Fig. 3 of the main part, we follow the idea of randomly exciting \( N \) single particle states. Again, we solve the Hamiltonians under study by means of exact diagonalization. Then we compare the ratio between the sum of the single particle contributions and the joint entanglement entropy of the corresponding many-particle state. This we perform at various system sizes \( L \) and filling ratio \( N/L \). This allows us to conduct a finite size scaling and extrapolate our results to the thermodynamic limit.

Appendix C: Entanglement entropy for the constructed Bell states

For a given number of total states \( L \), i.e. for \( L/2 \) Bell-pairs, we excite \( N \) random states and ask in how many Bell-pairs \( s \) exactly one of the two states is excited. We assume \( N \) to be an even number henceforth. In the case of \( s \) single-occupied Bell-pairs, there exist \( \binom{L/2}{s} \) ways to choose them. Each of such configurations contributes with 2\( s \) ways to excite any of the two states of the \( s \) pairs. The remaining \( N - s \) states will form \( (N-s)/2 \) double-occupied pairs, which are distributed over the remaining \( L/2 - s \) not single-occupied pairs. For this, there are...
\((L/2-s)\) arrangements. In total, that gives

\[
n(s) = 2^n \left( \frac{L}{2s} \right) \left( \frac{L/2 - s}{(N-s)/2} \right)
\]

(C1)

possibilities. The expectation value for the number of single occupied states thus yields

\[
E[s] = \sum_{s \text{ even}} N \cdot 2^{n(s)} \left( \frac{L/2 - s}{(N-s)/2} \right)
\]

(C2)

\[
= \frac{(L-N)N}{L-1}.
\]

(C3)

where the evaluation of the sum is restricted to even values of \(s\).

1. A. Einstein, B. Podolsky, and N. Rosen, Physical Review, 47, 777 (1935).
2. J. S. Bell, Physics, 1, 195 (1964).
3. M. B. Plenio and S. Virmani, Quant. Inf. Comput., 7, 1 (2007).
4. L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D, 34, 373 (1986).
5. M. Srednicki, Phys. Rev. Lett., 71, 666 (1993).
6. M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information: 10th Anniversary Edition, 10th ed. (Cambridge University Press, New York, NY, USA, 2011) ISBN 1107002176, 9781107002173.
7. M. Sarovar, A. Ishizaki, G. R. Fleming, and K. B. Whaley, Nature Physics, 6, 462 (2010).
8. L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys., 80, 517 (2008).
9. V. Alba, M. Fagotti, and P. Calabrese, J. Stat. Mech. Theory Exp., 2009, P10020 (2009) ISSN 1742-5468.
10. J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys., 82, 277 (2010).
11. M. Žnidarič, T. Prosen, and P. Prelovšek, Phys. Rev. B, 77, 064426 (2008).
12. J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett., 109, 017202 (2012).
13. M. Serbyn, Z. Papič, and D. A. Abanin, Phys. Rev. B, 90, 174302 (2014).
14. D. Hetterich, M. Serbyn, F. Domínguez, F. Pollmann, and B. Trauzettel, Phys. Rev. B, 96, 104203 (2017) ISSN 2469-9950.
15. R. Berkovits, Phys. Rev. B, 87, 075141 (2013) ISSN 1098-0121.
16. J. Mölter, T. Barthel, U. Schollwöck, and V. Alba, J. Stat. Mech. Theory Exp., 2014, P10029 (2014) ISSN 1742-5468.
17. M. Storms and R. R. P. Singh, Phys. Rev. E, 89, 012125 (2014).
18. G. Ramírez, J. Rodríguez-Laguna, and G. Sierra, Journal of Statistical Mechanics: Theory and Experiment, 2014, P07003 (2014).
19. O. A. Castro-Alvaredo, C. De Fazio, B. Doyon, and I. M. Szécsényi, Phys. Rev. Lett., 121, 170602 (2018).
20. I. Pizorn, (2012), arXiv:1202.3336.
21. Y. Shi, Physical Review A, 67, 024301 (2003) ISSN 1050-2947.
22. I. Peschel and V. Eisler, Journal of Physics A: Mathematical and Theoretical, 42, 504003 (2009).
23. See supplementary material for the derivation of the proof, the derivation of the erasure, and details on the numerical procedures.
24. N. Linden, S. Popescu, and J. A. Smolin, Physical Review Letters, 97, 100502 (2006) ISSN 0031-9007.
25. P. W. Anderson, Phys. Rev., 109, 1492 (1958) ISSN 0031-899X.
26. D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys., 321, 1126 (2006) ISSN 0003-4916.
27. V. Oganesyan and D. A. Huse, Phys. Rev. B, 75, 155111 (2007) ISSN 1098-0121.
28. A. Pal and D. A. Huse, Phys. Rev. B, 82, 174411 (2010) ISSN 1098-0121.
29. D. J. Luitz, N. Laflorencie, and F. Alet, Phys. Rev. B, 91, 081103 (2015) ISSN 1098-0121.