Upper Bounds on the Feedback Error Exponent of Channels With States and Memory

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Abstract
As a class of state-dependent channels, Markov channels have been long studied in information theory for characterizing the feedback capacity and error exponent. This paper studies a more general variant of such channels where the state evolves via a general stochastic process, not necessarily Markov or ergodic. The states are assumed to be unknown to the transmitter and the receiver, but the underlying probability distributions are known. For this setup, we derive an upper bound on the feedback error exponent and the feedback capacity with variable-length codes. The bounds are expressed in terms of the directed mutual information and directed relative entropy. The bounds on the error exponent are simplified to Burnashev’s expression for discrete memoryless channels. Our method relies on tools from the theory of martingales to analyze a stochastic process defined based on the entropy of the message given the past channel’s outputs.

I. INTRODUCTION

Communications over channels with feedback has been a longstanding problem in information theory literature. The early works on discrete memoryless channels (DMCs) pointed to negative answer as to whether feedback can increase the capacity [1]. Feedback, though, improves the channel’s error exponent — the maximum attainable exponential rate of decay of the error probability. The improvements are obtained using variable length codes (VLCs), where the communication length depends on the channel’s realizations. In a seminal work, Burnashev [2] completely characterized the error exponent of DMCs with noiseless and causal feedback. This characterization has a simple, yet intuitive, form:

\[ E(R) = C_1(1 - \frac{R}{C}), \]

where \( R \) is the (average) rate of transmission, \( C \) is the capacity of the channel, and \( C_1 \) is the maximum exponent for binary hypothesis testing over the channel. It is equal to the maximal relative entropy between conditional output distributions. The Burnashev’s exponent can significantly exceed the sphere-packing exponent, for no-feedback communications, as it approaches capacity with nonzero slope. The use of VLCs is shown to be essential to establish these results, as no improvements is gained using fixed-length codes [3]–[5].

This result led to the question as to whether the feedback improves capacity or error exponent of more general channels, modeling non-traditional communications involving memory and intersymbol interference (ISI). Among such models are channels with states where the transition probability of the channel varies depending on its state which itself evolves based on the past inputs and state realizations. Depending on the variants of this formulation, the agents may have no knowledge about the state (e.g. arbitrarily varying channels) or the may exactly know the state [6]. When state is known at the transmitter and the receiver, feedback can improve the error exponent. Particularly, Como, et al, [7] extended Burnashev-type exponent to finite-state ergodic Markov channels with known state and derived a similar form as in (1), under some ergodicity assumptions. The error exponent for channels with more general state evolution is still unknown. Only the feedback capacity of such channels when restricted to fixed-length codes is known [8].

This papers studies the feedback error exponent for channels with more general state evolution and allowing VLCs. More precisely, we study discrete channels with states where the state evolves as an arbitrary stochastic process (not necessarily ergodic or Markov) depending on the past realizations. Furthermore, the realization of the states are assumed to be unknown but the transmitter or the receiver may know the underlying probability distribution governing the evolution of the state. However, noiseless output is available at the transmitter with one unit of delay. The main contributions are two fold. First, we prove an upper bound on the error exponent of such channels which has the familiar form

\[ E(R) \leq \sup_{N \geq 0} \sup_{P^N \in \mathcal{P}^N} D(P^N)(1 - \frac{R}{I(P^N)}), \]

where \( D \) is the directed relative entropy, \( I \) is the directed mutual information, and \( \mathcal{P}^N \) is a collection of “feasible” probability distributions. As a special case, the bound simplifies to the Burnashev’s expression when the channel is DMC. Second, we

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introduce an upper bound on the feedback capacity of VLCs for communications over these channels with stochastic states. This upper bound generalizes the results of Tatikonda and Mitter [8], and Purmuter et al. [9] where fixed-length codes are studied. Our approach relies on analysis of the entropy of the stochastic process defined based on entropy of the message given the past channel’s output. We analyze the drift of the entropy via tools from the theory of martingales.

Related works on the capacity and error exponent of channels with feedback are extensive. Starting with DMCs with feedback, Yamamoto and Itoh [10] introduced a two-phase iterative for achieving the Burnashev exponent. Also, error exponent of DMCs with feedback and cost constraints is studied in [11]. Also channels with state and feedback has been studied under various frameworks on the evolution model of the states and whether they are known at the transmitter or the receiver. On one exeterm of such models are arbitrarily varying channels [12]. The feedback capacity these channels for fixed-length codes is derived in [8]. Tchamkerten and Telatar [13] studied the universality of Burnashev error exponent. They considered communication setups where the parties have no exact knowledge of the statistics of the channel but know it belongs to a certain class of DMCs. The authors proved that no zero-rate coding scheme achieves the Burnashev’s exponent simultaneously for all the DMC’s in the class. However, they showed positive results for two families of such channels (e.g., binary symmetric and Z) [14]. Another class of channels with state are Markov channels that has been studied extensively for deriving their capacity [6], [15], [16] and error exponent using fixed-length codes [8]. A lower bound on the error exponent of unifilar channels is derived [17], where the states is a deterministic function of the previous ones. Other variants of this problem have been studied, including continuous-alphabet channels [18], [19], and multi-user channels [20], [21].

II. PROBLEM FORMULATION AND DEFINITIONS

The formal definitions are presented in this section. For short hand, we use \([1 : M]\) to denote \(\{1, 2, ..., M\}\).

A discrete channel with stochastic state has three finite sets \(\mathcal{X}, \mathcal{Y}, \mathcal{S}\) representing the input, output, and state of the channel, respectively. Consider a collection of channels \(\mathcal{Q} := \{Q(\cdot, \cdot, s) : s \in \mathcal{S}\}\), indexed by \(s \in \mathcal{S}\), where each element \(Q(\cdot, \cdot, s) : \mathcal{X} \to \mathcal{P}(\mathcal{Y})\) is the transition probability of the channel at state \(s\). The states \(\{S_t\}_{t \geq 0}\), evolve according to a conditional probability distribution \(P_t(s_t | s_t^{-1}, x_t^{-1}, y_t^{-1})\), \(t > 0\) depending on the past inputs and state realizations. As a result, after \(t\) uses of the channel with \(x_t^{-1}, s_t^{-1}, y_t^{-1}\) being the channels input, state and output, the next output is given by

\[
P_t(s_t, y_t | x_t^{-1}, s_t^{-1}, y_t^{-1}) = P_t(s_t | s_t^{-1}, x_t^{-1})Q(y_t | x_t, s_t).
\]

Such evolution of the states induces memory over the time as it depends on past inputs.

After each use of the channel, the output of the channel \(y_t\) is available at the transmitter with one unit of delay. Moreover, we allow VLCs for communications, where where both the transmitter and the receiver do not know the state of the channel. More precisely, the setup is defined as follows.

**Definition 1.** An \((M, N)\)-VLC for communications over a channel \(\mathcal{Q}\) with states and feedback is defined by

- A message \(W\) with uniform distribution over \([1 : M]\).
- Encoding functions \(e_t : [1 : M] \times \mathcal{Y}^{t-1} \to \mathcal{X}, t \in \mathbb{N}\).
- Decoding functions \(d_t : \mathcal{Y}^t \to [1 : M], t \in \mathbb{N}\).
- A stopping time \(T\) with respect to \((\text{w.r.t})\) the filtration \(\mathcal{F}_t\) defined as the \(\sigma\)-algebra of \(\mathcal{Y}^t\) for \(t \in \mathbb{N}\). Furthermore, it is assumed that \(T\) is almost surely bounded as \(T \leq N\).

For technical reasons, we study a class of \((M, N)\)-VLCs for which the parameter \(N\) grows sub-exponentially with \(\log M\), that is \(N \leq (\log M)^m\) for some fixed number \(m\). An example is the sequence \((M^{(n)}, N^{(n)})\)-VLCs, \(n \geq 1\), where \(M^{(n)} = 2^{nr_1}, N^{(n)} = n^m\), with \(r_1, r_2, m > 0\) being fixed parameters.

In what follows, for any \((M, N)\)-VLC, we define average rate, error probability, and error exponent. Given a message \(W\), the \(t\)-th output of the transmitter is denoted by \(X_t = e_t(W, Y_t^{-1})\), where \(Y_t^{-1}\) is the noiseless feedback up to time \(t\). Let \(W_t = d_t(Y_t^t)\) represent the estimate of the decoder about the message. Then, at the end of the stopping time \(T\), the decoder declares \(\hat{W}_T\) as the decoded message. The average rate and (average) probability of error for a VLC are defined as

\[
R \triangleq \frac{\log_2 M}{E[T]}, \quad P_e \triangleq \mathbb{P}\{\hat{W}_T \neq W\}.
\]

**Definition 2.** A rate \(R\) is achievable for a given channel with stochastic states, if there exists a sequence of \((M^{(n)}, N^{(n)})\)-VLCs such that

\[
\lim_{n \to \infty} \sup P_e^{(n)} = 0, \quad \lim_{n \to \infty} \frac{\log M^{(n)}}{E[T^{(n)}]} \geq R,
\]

and \(N^{(n)} \leq (n)^m, \forall n > 1\), where \(m\) is fixed. The feedback capacity, \(C_F^{VLC}\), is the convex closure of all achievable rates.

Naturally, the error exponent of a VLC with probability of error \(P_e\) and stopping time \(T\) is defined as \(E \triangleq \frac{\log P_e}{2T}\). The following definition formalizes this notion.
Definition 3. An error exponent function $E(R)$ is said to be achievable for a given channel, if for any rate $R > 0$ there exists a sequence of $\left(M^{(n)}, N^{(n)}\right)$-VLCs such that

$$\liminf_{n \to \infty} -\frac{\log P_e^{(n)}}{E[T^{(n)}]} \geq E(R),$$

$$\limsup_{n \to \infty} -\frac{\log M^{(n)}}{E[T^{(n)}]} \leq R,$$

and $\limsup_{n \to \infty} M^{(n)} = \infty$ with $N^{(n)} \leq (n)^m, \forall n > 1$, where $m$ is fixed. The reliability function is the supremum of all achievable reliability functions $E(R)$.

III. Main Results

We start with deriving an upper bound on the feedback capacity of channels with stochastic states and allowing VLCs. The expressions are based on the directed information as introduced in [22] and defined as

$$I(X^n \to Y^n) \triangleq \sum_{i=1}^{n} I(X_i; Y_i | Y^{i-1}).$$

(2)

We further extend this notion to variable-length sequences. Consider a stochastic process $\{(X_t, Y_t)\}_{t \geq 0}$ and let $T$ be a (bounded) stopping time w.r.t an induced filtration $\mathcal{F}_t, t > 0$. Then, the directed mutual information is defined as

$$I(X^T \to Y^T) \triangleq \mathbb{E} \left[ \sum_{t=1}^{T} I(X_t; Y_t \mid \mathcal{F}_{t-1}) \right].$$

(3)

Now, we are ready for an upper bound on the feedback capacity. For any integer $N$, let $\mathcal{P}^N$ be the set of all $N$-letter distributions $P_{X^N,S^N,Y^N}$ on $X^N \times S^N \times Y^N$ that factor as

$$\prod_{t=1}^{N} P_{t,X}(x_t|x_{t-1}, y_{t-1})P_{t,S}(s_t|x_{t-1}, y_{t-1})Q(y_t|x_t, s_t).$$

(4)

Next, we have the following result on the capacity with the proof in Appendix A.

Theorem 1. The feedback capacity of a channel with stochastic states is bounded as

$$C_{F}^{VLC} \leq \sup_{N \geq 0} \sup_{P^N \in \mathcal{P}^N} \sup_{T: T \leq N} \frac{1}{E[T]} I(X^T \to Y^T),$$

where $T$ is a stopping time with respect to $\mathcal{F}_t, t > 0$.

Observe that for a trivial stopping time $T = N$, the bound simplifies to that for fixed-length codes as given in [8].

A. Upper Bound on the Error Exponent

We need a notation to proceed. Consider a pair of random sequences $(X^n, Y^n) \sim P_{X \to Y^n}$. Let $X^*_r$ be the MAP estimation of $X_r$ from observation $Y^{r-1}$, that is $X^*_r = \arg\max_{x_r} \mathbb{P} \{ X_r = x_r \mid Y^{r-1} = y^{r-1} \}$. Also, let $\tilde{Q}_r = P_{Y_r | X_r, Y^{r-1}}$ which is the effective channel (averaged over possible states) from the transmitter’s perspective at time $r$. With this notation, we define the directed KL-divergence as

$$D(X^n \to Y^n) \triangleq \max_{x^n} \sum_{r=1}^{n} D_{KL} \left( \tilde{Q}_r(\cdot | X^*_r, Y^{r-1}) \mid | \tilde{Q}_r(\cdot | x_r, Y^{r-1}) \right).$$

Intuitively, $D(X^n \to Y^n)$ measures the sum of the expected “distance” between the channels probability distribution conditioned on the MAP symbol versus the worst symbol, across different times $r \in [1 : n]$.

Theorem 2. The error exponent of a channel with stochastic states is bounded as

$$E(R) \leq \sup_{N \in \mathbb{N}} \sup_{P^N \in \mathcal{P}^N} \sup_{T: T_1 \leq T \leq N} \frac{D(P^N)}{I(P^N)} \left(1 - \frac{R}{I(P^N)}\right),$$

where $T, T_1$ are stopping times, and

$$I(P^N) = \frac{1}{E[T_1]} I(X^{T_1} \to Y^{T_1}),$$

$$D(P^N) = \frac{1}{E[T - T_1]} D(X^{T_1} \to Y^{T_1})_1.$$

In the next section, we present our proof techniques.
IV. PROOF OF THEOREM 2

The proof follows by a careful study of the drift of the entropy of the message $W$ conditioned on the channel’s output at each time $t$. Define the following random process:

$$H_t = H(W|F_t), t > 0,$$

(5)

where $F_t$ is the $\sigma$-algebra of $Y^t$. We show that $H_t$ drifts in three phases: (i) linear drift (data phase) until reaching a small value ($\epsilon$); (ii) fluctuation phase with values around $\epsilon$; and (iii) logarithmic drift (hypothesis testing phase) till the end. We derive bounds on the expected slope of the drifts and prove that the length of the fluctuation phase is asymptotically negligible as compared to the overall communication length (Fig. 1).

![Fig. 1](image)

More precisely, we have the following argument by defining a pruned time random process $\{t_n\}_{n>0}$. First, for any $\epsilon \geq 0$ and $N \in \mathbb{N}$ define the following random variables

$$\tau_\epsilon \triangleq \inf \{ t > 0 : H_t \leq \epsilon \} \wedge N$$

(6)

$$\tau^\epsilon \triangleq \sup \{ t > 0 : H_{t-1} \geq \epsilon \} \wedge N$$

(7)

Then the pruned time process is defined as

$$t_n \triangleq \begin{cases} n & \text{if } n < \tau_\epsilon \\ n \wedge \tau^\epsilon & \text{if } \tau_\epsilon \leq n \leq N \\ N & \text{if } n > N \end{cases}$$

(8)

Note that $\tau_\epsilon$ is a stopping time with respect to $\{H_t\}_{t>0}$ but this is not the case for $\tau^\epsilon$.

**Lemma 1.** Suppose a non-negative random process $\{H_r\}_{r>0}$ has the following properties w.r.t a filtration $F_r, r > 0$,

$$\mathbb{E}[H_{r+1} - H_r|F_r] \geq -k_{1,r+1}, \quad \text{if } H_r \geq \epsilon,$$

(9a)

$$\mathbb{E}[\log H_{r+1} - \log H_r|F_r] \geq -k_{2,r+1} \quad \text{if } H_r < \epsilon,$$

(9b)

$$|\log H_{r+1} - \log H_r| \leq k_3$$

(9c)

$$|H_{r+1} - H_r| \leq k_4$$

(9d)

where $k_{1,r}, k_{2,r}, k_3, k_4$ are non-negative numbers and $k_{1,r} \leq k_{2,r}$ for all $r > 0$. Given $\epsilon \in (0,1)$, and $D \geq I > 0$, let

$$Z_t \triangleq \frac{H_t - \epsilon}{I} \mathbb{I}\{H_t \geq \epsilon\} + \left(\frac{\log H_t}{D} + f(\log \frac{H_t}{\epsilon})\right) \mathbb{I}\{H_t < \epsilon\},$$

where $f(y) = \frac{1-e^{\lambda y}}{\lambda D}$ with $\lambda > 0$. Further define $\{S_t\}_{t>0}$ as

$$S_t \triangleq \sum_{r=1}^{t \wedge \tau_\epsilon} \frac{k_{1,r}}{I} + \sum_{r=t \wedge \tau_\epsilon}^{t \wedge \tau^\epsilon} \frac{k_4}{T} \mathbb{I}\{H_{r-1} \geq \sqrt{\epsilon}\} + \sum_{r=t \wedge \tau^\epsilon}^{t} \frac{k_{2,r}}{D} + \sqrt{\epsilon} \frac{N}{T} \mathbb{I}\{t \geq \tau^\epsilon\}.$$

Let $\{l_n\}_{n>0}$ be as in (8) but w.r.t $\{H_r\}_{r>0}$. Lastly define the random process $\{L_n\}_{n>0}$ as $L_n \triangleq Z_{t_n} + S_{t_n}$. Then, for small enough $\lambda > 0$ the process $\{L_n\}_{n>0}$ is a sub-martingale with respect to the time pruned filtration $F_{t_n}, n > 0$.

**Proof:** The objective is to prove $\mathbb{E}[L_{n+1} - L_n|y^n] \geq 0$ almost surely for all $n \geq 1$ and $y^n$. We prove the lemma by considering three cases depending on $n$.  

Case (a), \( n < \tau_e - 1 \): From the definition of \( t_n \) in (8), in this case \( t_n = n \) and \( t_{n+1} = n + 1 \). Also, as the time did not reach \( \tau_e \), then \( H_n > \epsilon \) and \( H_{n+1} > \epsilon \). Therefore, in this case, the random process of interest equals to

\[
L_n = Z_{t_n} + S_{t_n} = Z_n + S_n = \frac{H_n - \epsilon}{I} + \sum_{r=1}^{n} \frac{k_{1,r}}{I}
\]

\[
L_{n+1} = Z_{t_{n+1}} + S_{t_{n+1}} = Z_{n+1} + S_{n+1} = \frac{H_{n+1} - \epsilon}{I} + \sum_{r=1}^{n+1} \frac{k_{1,r}}{I}.
\]

As a result, the difference between \( L_n \) and \( L_{n+1} \) satisfies the following

\[
\mathbb{E}[ (L_{n+1} - L_n) \mathbb{I}\{ n < \tau_e - 1 \} | y^n ] = \mathbb{E}[ (L_{n+1} - L_n) \mathbb{I}\{ n < \tau_e - 1 \} | y^n ] = \mathbb{E}[ (L_{n+1} - L_n) | y^n ] \mathbb{I}\{ n < \tau_e - 1 \},
\]

where the first equality holds as \( t_n = n \) and the second equality holds as \( \tau_e \) is a stopping time which implies that \( \mathbb{I}\{ n < \tau_e - 1 \} \) is a function of \( y^n \). Next, from (10), the difference term above is bounded as

\[
\mathbb{E}[ L_{n+1} - L_n | y^n ] = \mathbb{E}[ (H_{n+1} - H_n) \mathbb{I}\{ H_{n+1} | y^n \}, r + \log \frac{H_{n+1} - H_n}{I} + \frac{k_{1,n+1}}{I} ] \geq 0,
\]

where the last inequality follows from (9a). As a result, we proved that \( \mathbb{E}[ (L_{n+1} - L_n) \mathbb{I}\{ n < \tau_e - 1 \} | y^n ] \geq 0 \).

Case (b), \( n = \tau_e - 1 \): In this case, \( t_n = n \) implying that \( H_n > \epsilon \) and \( t_{n+1} = (n+1) \vee \tau_e \). Furthermore, since, \( n+1 = \tau_e \leq \tau_e \), then \( t_{n+1} = \tau_e \). Consequently, the random process equals to

\[
L_n = Z_n + S_n = \frac{H_n - \epsilon}{I} + \sum_{r=1}^{n} \frac{k_{1,r}}{I}
\]

\[
L_{n+1} = Z_{\tau_e} + S_{\tau_e} = \left( \frac{H_{\tau_e} - \epsilon}{I} \right) \mathbb{I}\{ H_{\tau_e} \geq \epsilon \} + \left( \log \frac{H_{\tau_e} - \epsilon}{D} + \frac{f (\log \frac{H_{\tau_e}}{\epsilon})}{1} \right) \mathbb{I}\{ H_{\tau_e} < \epsilon \} + \sum_{r=1}^{\tau_e} \frac{k_{1,r}}{I} + \sum_{r=\tau_e+1}^{\tau_e} \frac{k_4}{I} \mathbb{I}\{ H_{r-1} \geq \sqrt{\epsilon} \} + \sqrt{\frac{N}{T}}.
\]

Note that \( Z_{\tau_e} \) does not necessarily equal to the logarithmic part. The reason is that \( \tau_e \) is pruned by \( N \) as in (7). Thus, \( H_{\tau_e} \) can be greater than \( \epsilon \) when \( \tau_e = N \). We proceed by bounding \( Z_{\tau_e} \). Note that, for small enough \( \lambda \) the following inequality holds

\[
\frac{\epsilon}{I} (e^y - 1) - \frac{y}{D} < f(y), \quad -k_3 < y < 0.
\]

Applying inequality (11) with \( y = \log \frac{H_{\tau_e}}{\epsilon} \), we can write that

\[
Z_{\tau_e} > \left( \frac{H_{\tau_e} - \epsilon}{I} \right) \mathbb{I}\{ H_{\tau_e} \geq \epsilon \} + \left( \frac{H_{\tau_e} - \epsilon}{I} \right) \mathbb{I}\{ H_{\tau_e} < \epsilon \} = \frac{H_{\tau_e} - \epsilon}{I}.
\]

Consequently, the difference \( L_{n+1} - L_n \) satisfies the following

\[
\mathbb{E}[ (L_{n+1} - L_n) \mathbb{I}\{ n = \tau_e - 1 \} | y^n ] = \mathbb{E}[ (L_{n+1} - L_n) | y^n ] \mathbb{I}\{ n = \tau_e - 1 \}
\]

\[
\geq \mathbb{E} \left[ \left( \frac{H_{\tau_e} - H_n}{I} + \frac{k_{1,\tau_e}}{I} \mathbb{I}\{ H_{\tau_e} \geq \sqrt{\epsilon} \} + \sqrt{\frac{N}{T}} \right) \mathbb{I}\{ n = \tau_e - 1 \} \right] (13)
\]

Next, we bound the first term above as

\[
H_{\tau_e} - H_n = H_{n+1} - H_n + \sum_{r=\tau_{e+1}}^{\tau_e} (H_r - H_{r-1}),
\]
where in the first equality, we add and subtract the intermediate terms $H_r, n + 1 \leq r \leq \tau^ε - 1$. Next, we substitute the above terms in the right-hand side of (13). As $n + 2 = \tau_e + 1$, then we obtain that

$$ (13) = E \left[ \frac{H_{n+1} - H_n}{I} + \frac{k_1}{I} + \sum_{r=\tau_e+1}^{\tau^ε} \left( \frac{H_r - H_{r-1}}{I} + \frac{k_4}{I} \mathbb{1}\{H_{r-1} \geq \sqrt{\epsilon}\} \right) + \sqrt{\epsilon} N \mathbb{1}\{n = \tau_e - 1\} \right] $$

$$ \geq E \left[ \sum_{r=\tau_e+1}^{\tau^ε} \left( \frac{H_r - H_{r-1}}{I} + \frac{k_4}{I} \mathbb{1}\{H_{r-1} \geq \sqrt{\epsilon}\} \right) + \sqrt{\epsilon} N \mathbb{1}\{n = \tau_e - 1\} \right] $$

where the inequality holds from (9a) and the fact that $n + 1 = \tau_e$. Next, by factoring $I$ and the indicator function inside the expectation, we have the following chain of inequalities

$$ (14) = \frac{1}{I} E \left[ \sum_{r=\tau_e+1}^{\tau^ε} \left( (H_r - H_{r-1}) + k_4 \right) \mathbb{1}\{H_{r-1} \geq \sqrt{\epsilon}\} + \left( (H_r - H_{r-1}) \mathbb{1}\{H_{r-1} < \sqrt{\epsilon}\} \right) \right] \mathbb{1}\{n = \tau_e - 1\} $$

$$ \geq \frac{1}{I} E \left[ \sum_{r=\tau_e+1}^{\tau^ε} \left( (H_r - H_{r-1}) \mathbb{1}\{H_{r-1} < \sqrt{\epsilon}\} \right) + \sqrt{\epsilon} N \mathbb{1}\{n = \tau_e - 1\} \right] $$

$$ \geq 0, $$

where (a) is due to (9d), inequality (b) holds as $H_r \geq 0$, inequality (c) holds as $H_{r-1} \mathbb{1}\{H_{r-1} < \epsilon\} < \epsilon$, and lastly (d) holds as $\tau^ε \leq N$. To sum up, we proved that

$$ E[(L_{n+1} - L_n) \mathbb{1}\{n = \tau_e - 1\}]y^n \geq 0. $$

**Case (c).** $n \geq \tau_e$: This is the last case. Note that if $n < \tau^ε$, then $t_n = t_{n+1} = \tau^ε$. Thus, immediately, $L_{n+1} - L_n = 0$ almost surely. Otherwise, if $n \geq \tau^ε$ and $\tau^ε = N$ or if $n \geq N$, then $t_n = t_{n+1} = N$ and hence $L_{n+1} - L_n = 0$. Therefore, it remains to consider the case that $\tau^ε < N$ and $\tau^ε \leq n < N$. Therefore, $t_n = n$ and $t_{n+1} = n + 1$. Furthermore, as $n + 1 > n \geq \tau^ε$ and $\tau^ε < N$, then $H_n < \epsilon$ and $H_{n+1} < \epsilon$, implying that we are in the logarithmic drift. Therefore, we have that

$$ L_n = Z_n = \log \frac{H_n - \epsilon}{D} + f(\log \frac{H_n}{\epsilon}) + S_n $$

$$ L_{n+1} = Z_{n+1} = \log \frac{H_{n+1} - \epsilon}{D} + f(\log \frac{H_{n+1}}{\epsilon}) + S_{n+1}. $$

Hence, to sum up the above sub-cases, we conclude that when $n \geq \tau_e$, then

$$ L_{n+1} - L_n = \frac{\log H_{n+1} - \log H_n}{D} + f(\log \frac{H_{n+1}}{\epsilon}) - f(\log \frac{H_n}{\epsilon}) + S_{n+1} - S_n. $$

Note that from (9b), the following inequality holds

$$ E \left[ \frac{\log H_{n+1} - \log H_n}{D} + S_{n+1} - S_n \right] y^n \geq 0. $$

Therefore, the difference $L_{n+1} - L_n$ satisfies the following

$$ E[(L_{n+1} - L_n) \mathbb{1}\{n \geq \tau_e\}]y^n $$

$$ = E[(L_{n+1} - L_n)]y^n \mathbb{1}\{n \geq \tau_e\} $$

$$ \geq E \left[ f(\log \frac{H_{n+1}}{\epsilon}) - f(\log \frac{H_n}{\epsilon}) \right] y^n \mathbb{1}\{n \geq \tau_e\}. $$

Next, we provide an argument similar to Point-to-Point (PtP) case. That is, we use the Taylor’s theorem for $f$. We only need to consider the case that $\tau^ε < N$ and $\tau^ε \leq n < N$ implying that $t_n = n$ and $t_{n+1} = n + 1$. Using the Taylor’s theorem we can write

$$ f(\log \frac{H_{n+1}}{\epsilon}) = f(\log \frac{H_n}{\epsilon}) + \frac{\partial f}{\partial y} \bigg|_{y = \log \frac{H_n}{\epsilon}} (\log H_{n+1} - \log H_n) + \frac{\partial^2 f}{\partial y^2} \bigg|_{y = \log \frac{H_n}{\epsilon}} (\log H_{n+1})^2, $$
where \( \zeta \) is between \( \log \frac{H_{n+1}}{\epsilon} \) and \( \log \frac{H_n}{\epsilon} \) and
\[
\frac{\partial f}{\partial y} \bigg|_{y = \log \frac{H_n}{\epsilon}} = -\frac{e^{\lambda \log \frac{H_n}{\epsilon}}}{I}, \quad \frac{\partial^2 f}{\partial y^2} \bigg|_{y = \zeta} = -\frac{\lambda}{I} e^{\lambda \zeta}.
\]
As a result, we have that
\[
\mathbb{E} \left[ f(\log \frac{H_{n+1}}{\epsilon}) - f(\log \frac{H_n}{\epsilon}) \right] = \mathbb{E} \left[ -\frac{e^{\lambda \log \frac{H_n}{\epsilon}}}{I} \left( \log \frac{H_{n+1}}{H_n} \right) - \frac{\lambda}{I} e^{\lambda \zeta} \left( \log \frac{H_{n+1}}{H_n} \right)^2 \right] \]
\[
\geq \mathbb{E} \left[ -\frac{e^{\lambda \log \frac{H_n}{\epsilon}}}{I} \left( \log \frac{H_{n+1}}{H_n} \right) - \frac{\lambda k_3^2}{I} e^{\lambda \zeta} \left( \log \frac{H_{n+1}}{H_n} \right)^2 \right] \]
\[
= \left( \frac{k_3}{I} - \frac{\lambda k_3^2}{I} e^{\lambda \zeta} \right) \geq 0,
\]
where inequality (a) holds as \( |\zeta - \log \frac{H_n}{\epsilon}| \leq |\log \frac{H_{n+1}}{\epsilon} - \log \frac{H_n}{\epsilon}| \leq k_3 \). The last inequality holds for sufficiently small \( \lambda > 0 \).

Lastly, combining all cases from (a) to (c), we prove that \( \mathbb{E} \left[ L_{n+1} - L_n \right] \geq 0 \) which completes the proof. \( \blacksquare \)

Now, we show that \( \{H_t\}_{t \geq 0} \) as in (5) has the conditions in Lemma 1. First (9a) holds because of the following lemma.

**Lemma 2.** Given any \((M, N)\)-VLC, the following inequality holds almost surely for \( 1 \leq r \leq N \)
\[
\mathbb{E}[H_r - H_{r-1} | \mathcal{F}_{r-1}] = -J_r,
\]
where \( J_r \triangleq I(X_r; Y_r | \mathcal{F}_{r-1}) \) with the induced \( P_{X_r, Y_r} \in \mathcal{P}^N \).

**Proof:** For any \( y^{r-1} \), we have that
\[
\mathbb{E}[H_r - H_{r-1} | y^{r-1}] = H(W | Y_r, y^{r-1}) - H(W | y^{r-1})
\]
\[
= -I(W; Y_r | y^{r-1})
\]
\[
= -H(Y_r | y^{r-1}) + H(Y_r | W, X_r, y^{r-1})
\]
\[
= -H(Y_r | y^{r-1}) + H(Y_r | X_r)
\]
\[
= -J_r.
\]
Hence the lemma is proved. \( \blacksquare \)

Condition (9b) holds as a result of the following lemma that is given in Appendix B.

**Lemma 3.** For any \((M, N)\)-VLC and \( \epsilon \in [0, \frac{1}{2}] \), if \( H_r < \epsilon \), then the following inequality holds almost surely
\[
\mathbb{E}[\log H_r - \log H_{r-1} | \mathcal{F}_{r-1}] \geq -D_r + O(h_r^{-1}(\epsilon)),
\]
where and \( D_r \) is a function of \( y^{r-1} \) and is defined as
\[
D_r \triangleq \max_{x \in A} D_{KL}(\hat{Q}_r \cdot | x^r, y^{r-1}) \parallel \hat{Q}_r \cdot | x, y^{r-1}),
\]
where \( \bar{Q} = P_{Y^2|X,Y^1} \) is the average channel from the transmitter's perspective, and \( x^* \) is the MAP input symbol given by

\[
x^*_r = \arg \max_x P \{ X = x | Y^r = y^r-1 \}.
\]

Condition (9c) is a direct consequence of Lemma 4 in [2]:

**Remark 1.** If \( Q(\cdot, \cdot) \) are positive everywhere then \( |\log H_r - \log H_{r-1}| \leq \eta \), where

\[
\eta \overset{\triangle}{=} \max_{x_1, x_2 \in X} \max_{s_1, s_2 \in S} \max_{y \in Y} \frac{Q(y|x_1, s_1)}{Q(y|x_2, s_2)}.
\]

Lastly, (9d) holds as \( H_r \leq \log M \) which implies that

\[
|H_r - H_{r-1}| \leq \max \{ H_r, H_{r-1} \} \leq \log M.
\]

Thus, we apply Lemma 1 on \( \{H_t \}_{t \geq 0} \) with

\[
k_1, r = J_r, \quad k_2, r = D_r, \quad k_3 = \eta, \quad k_4 = \log M,
\]

and constants \( I, D \) to be specified later. Therefore, \( \{L_n \}_{n \geq 0} \) as in the lemma is a sub-martingale w.r.t \( \mathcal{F}_{t_n}, n > 0 \).

**A. Connection to the error exponent**

Since \( \{L_n \}_{n \geq 0} \) is a sub-martingale, then \( L_0 \leq E[I_{T_{V^T}}] \), where \( T \) is the stopping time used in the VLC and \( \tau^* \) is as in (7). Note that \( L_0 = \log M \). In what follows, we analyze \( E[I_{T_{V^T}}] \).

By definition \( L_n = Z_{t_n} + S_{t_n} \). Since, \( T \leq N \), then from (8) we have that \( t_{V^T} = (T \lor \tau^*) \lor \tau^* = T \lor \tau^* \). Therefore,

\[
\frac{\log M}{I} \leq \mathbb{E}[I_{T_{V^T}}] = \mathbb{E}[\log \left( \frac{H_{T_{V^T}} - \epsilon}{I} \right) + \log \left( \frac{H_{T_{V^T}}}{I} \right) + f(\log \left( \frac{H_{T_{V^T}}}{\epsilon} \right)) + \mathbb{E}[S_{T_{V^T}}] - \log \epsilon] \leq \mathbb{E}[H_{T_{V^T}}] = \mathbb{E}[H(T\lor \tau^*)] \leq \mathbb{E}[H(Y^T)] = H_T,
\]

where inequality \((a)\) holds as \( Y^T \) is a function of \( Y\). Then, \( \alpha(P_e) = h_e(P_e) + P_e \log(M) \) is the Fano's expression. Therefore, from (19), we obtain that

\[
\frac{\log M}{I} \leq \frac{\alpha(P_e)}{I} + \frac{\log \alpha(P_e)}{D} + \frac{1}{\lambda D} + \mathbb{E}[S_{T_{V^T}}] - \log \epsilon.
\]

Rearranging the terms gives the following inequality

\[
-\log \alpha(P_e) \leq \frac{\alpha(P_e)}{I} + \frac{-\log \epsilon}{D} + \frac{1}{\lambda D} + \mathbb{E}[S_{T_{V^T}}] - \log \epsilon.
\]

Therefore, multiplying by \( D \) and dividing by \( \mathbb{E}[T] \) give the following

\[
\frac{-\log \alpha(P_e)}{\mathbb{E}[T]} \leq D \left( \frac{\mathbb{E}[S_{T_{V^T}}]}{\mathbb{E}[T]} - \frac{R}{T} \right) + U(P_e, M, \epsilon),
\]

where we used the fact that \( \frac{\log M}{\mathbb{E}[T]} \geq R \), and that

\[
U(P_e, M, \epsilon) = R \left( \frac{\alpha(P_e)}{I \log M} + \frac{-\log \epsilon}{D \log M} + \frac{1}{\lambda D \log M} \right).
\]
Next, for the left hand side of (20), we can write that

\[-\log \alpha(P_e) = -\log P_e - \log \frac{\alpha(P_e)}{P_e} = -\log P_e - \log \left(1 - \frac{1}{P_e} \right) \log(1 - P_e) + \log M \geq -\log P_e - \log \left(\frac{1}{P_e} \log(1 - P_e) + \log M \right) \geq -\log P_e - \log \left(\log P_e + 2 + \log M \right),\]

where the last inequality follows because \(\log(x) \geq 1 - \frac{1}{x}\) for \(x > 0\) implying that \(\log(1 - P_e) \geq 1 - \frac{1}{1 - P_e} = \frac{P_e}{1 - P_e}\); and hence, \(\frac{1}{P_e} \log(1 - P_e) \leq \frac{P_e}{1 - P_e} \leq 2\) as \(P_e \leq \frac{1}{2}\). Therefore, by factoring \(-\log P_e\) we have that

\[-\log \alpha(P_e) \geq (-\log P_e)(1 - \Delta),\]

where

\[\Delta = \frac{\log \left(\log P_e + 2 + \log M \right)}{-\log P_e} \tag{22}\]

Therefore, from (20) we get the following bound on the error exponent

\[-\frac{\log P_e}{E[T]} \leq \frac{D}{1 - \Delta} \left(\frac{E[S_{T+\tau}]}{E[T]} - \frac{R}{T} + U(P_e, M, \epsilon)\right) \tag{23}\]

Next, we find appropriate \(I\) and \(D\) so that \(E[S_{T+\tau}] \approx E[T]\). Further, we show that \(\Delta\) and \(U(P_e, M, \epsilon)\) converge to zero for any sequence of VLCs satisfying Definition 3.

We proceed with the following lemma that is proved in Appendix C.

**Lemma 4.** Given \(\epsilon > \alpha(P_e)\) and with

\[I = \frac{1}{E[\tau_r]} \mathbb{E} \left[ \sum_{r=1}^{\tau} J_r \right], \quad D = \frac{1}{E[T - \tau]} \mathbb{E} \left[ \sum_{r=\tau+1}^{T} D_r \right],\]

the inequality \(\mathbb{E}[S_{T+\tau}] \leq \mathbb{E}[T](1 + V(\epsilon, N))\) holds, where \(V(\epsilon, N) = \frac{E[\epsilon]}{\sqrt{\epsilon}} \mathbb{E}[J] + \mathbb{E}[\mathbb{V}[N]]\).

Therefore, with (23), we get the desired upper bound by appropriately setting \(I\) and \(D\) as in the lemma. Hence, we get

\[-\frac{\log P_e}{E[T]} \leq \frac{D}{1 - \Delta} \left(1 - \frac{R}{T} + U(P_e, M, \epsilon) + V(\epsilon, N)\right) \tag{24}\]

We show that for any \((M^{(n)}, N^{(n)})\)-VLCs as in Definition 3 the residual terms \(U, V, \Delta\) converge to zero as \(n \to \infty\). It is easy to see that \(\epsilon\) as in (22) converges to zero as \(P_e^{(n)} \to 0\). Further, by setting \(\epsilon^{(n)} = \left(\frac{1}{N^{(n)}}\right)^3\), we can check that \(\lim_{n \to \infty} V(\epsilon^{(n)}, N^{(n)}) = 0\). It remains to show the convergence of \(U(\cdot)\) as in (21). The convergence of the third term in (21) follows as \(\lim_{n \to \infty} \frac{1}{M^{(n)}} = 0\). For the second term, as \(\epsilon^{(n)} = \left(\frac{1}{N^{(n)}}\right)^3\) then we have that

\[\lim_{n \to \infty} -\frac{\log \epsilon^{(n)}}{\log M^{(n)}} = \lim_{n \to \infty} \frac{3 \log N^{(n)}}{\log M^{(n)}} = 0,\]

where the last equality holds as \(N^{(n)}\) grows sub-exponentially with \(n\). The convergence of the first term also follows from the fact that \(\lim_{n \to \infty} \alpha(P_e^{(n)}) = 0\), as \(P_e^{(n)}\) converges exponentially fast.\(^1\)

Hence, by maximizing over all distributions and from Definition 3, we get the desired upper bound on the error exponent.

\[\limsup_{n \to \infty} -\frac{\log P_e^{(n)}}{E[T^{(n)}]} \leq \sup_{N \in \mathcal{N}} \sup_{P_N \in \mathcal{P}_N} \sup_{T \leq N} \left\{ D(P_N^{(n)}) \left(1 - \frac{R}{T(P_N^{(n)})}\right) \right\}, \tag{25}\]

where the maximizations are taken over all distributions \(P_N^{(n)} \in \mathcal{P}_N\). Further, \(I(P_N)\) and \(D(P_N)\) are defined as in Lemma 4 with the distribution \(P_N\).

**Conclusion**

This paper presents an upper bound on the feedback error exponent and feedback capacity of channels with stochastic states, where the states evolve according to a general stochastic process. The results are based on the analysis of the drift of the entropy of the message as a random process.

\(^1\)The exponential convergence of \(P_e^{(n)}\) holds because otherwise the error exponent is zero.
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From the chain rule, we have that
\[ P = 0, \] the message \( W \) has the uniform distribution, then we have that
\[ \log M = H(W) = I(W; Y^T) + H(W|Y^T) \leq I(W; Y^T) + \alpha(P_e). \] (26)

We proceed by showing that
\[ I(W; Y^T) \leq I(X^T \rightarrow Y^T). \]

We first pad \( Y^T \) to make it a sequence of length \( N \). Let \( \xi \) be an auxiliary symbol and define
\[ Y^N = (Y_1, Y_2, \ldots, Y_T, \xi, \xi, \ldots, \xi). \]

Similarly, we extend the encoding functions and the channel’s transition probability to include \( \xi \). Specifically, after the stopping time \( T \), the encoders send the constant symbol \( \xi \) and the channel outputs \( \xi \) to the receiver. More precisely,
\[ e(W, Y^n) = \xi, \quad \forall n \geq T, \quad Q(y|\xi, s) = 1 \{ y = \xi \}, \quad \forall x, y. \]

This auxiliary adjustment is only for tractability of the analysis as it does not affect the performance of the code. Specifically, the mutual information stays the same by replacing \( Y^T \) with \( Y^N \):
\[ I(W; Y^N) = I(W; Y^T) + I(W; Y^N|Y^T) = I(W; Y^T) + I(W; \xi^N_{T+1}|Y^T) = I(W; Y^T). \] (27)

From the chain rule, we have that
\[ I(W; Y^N) = \sum_{r=1}^{N} I(W; Y_r|Y^{r-1}) = \sum_{r=1}^{N} H(Y_r|Y^{r-1}) - H(Y_r|W, Y^{r-1}) = \sum_{r=1}^{N} H(Y_r|Y^{r-1}) - H(Y_r|W, X_r, Y^{r-1}) = \sum_{r=1}^{N} H(Y_r|Y^{r-1}) - H(Y_r|X_r, Y^{r-1}) = \sum_{r=1}^{N} I(X_{1,r}; Y_r|Y^{r-1}) = I(X^N \rightarrow Y^N). \] (28)

Next, we show that the directed mutual information above equals to the following:
\[ I(X^N \rightarrow Y^N) = I(X^T \rightarrow Y^T \| X^T) = E \left[ \sum_{r=1}^{T} I(X_r; Y_r|F_{r-1}) \right], \]
where the second equality is due to the definition given in (3). Note that \( I(X_{1,r}; Y_r | F_{r-1}) = 0 \) almost surely for any \( r > T \) as \( Y_r = \xi \). Therefore, we have that

\[
I(X^T \to Y^T) = E \left[ \sum_{r=1}^{N} I(X_r; Y_r | F_{r-1}) \right] = \sum_{r=1}^{N} E[I(X_r; Y_r | F_{r-1})] = \sum_{r=1}^{N} I(X_r; Y_r | Y^{r-1}) = I(X^N \to Y^N). \tag{29}
\]

Therefore, combining (26)-(29) gives an upper bound on \( \log M \). Dividing both sides by \( E[T] \) gives the following upper bound on \( R \)

\[
R \leq \frac{\log M}{E[T]} \leq \frac{1}{E[T]} I(X^T \to Y^T) + \frac{\alpha(P_e)}{E[T]} \leq \sup_{N > 0} \sup_{P \in \mathcal{P}_N} \sup_{\text{stop time } T : T \leq N} \frac{1}{E[T]} I(X^T \to Y^T) + \frac{\alpha(P_e)}{E[T]} \tag{30}
\]

The first term above is the desired expression. The second term is vanishing as \( P_e \to 0 \). Hence the proof is complete.

**APPENDIX B**

**PROOF OF LEMMA 3**

**Proof:** Define the following quantities

\[
\mu(w) = \mathbb{P}\left\{ W = w | Y^{r-1} = y^{r-1} \right\}, \tag{31a}
\]

\[
\mu(w,y_r) = \mathbb{P}\left\{ W = w | Y^{r-1} = y^{r-1}, Y_r = y_r \right\}, \tag{31b}
\]

\[
Q_w(y_r) = \mathbb{P}\left\{ Y_r = y_r | W = w, Y^{r-1} = y^{r-1} \right\}, \tag{31c}
\]

where \( i \in [1:M], y_{t+1} \in \mathcal{Y} \). Let \( w^*_r \in [1:M] \) be the most likely message condition on \( y^{r-1} \). That is \( w^*_r = \max_{w \in [1:M]} \mu(w) \).

First, we show that having \( H_{r-1} < \epsilon, \epsilon \in (0,1) \), we conclude that \( \mu(w^*_r) = 1 - \eta_1(\epsilon) \) with \( \eta_1 \) being a function satisfying \( \lim_{\epsilon \to 0} \eta_1(\epsilon) = 0 \). The argument is as follows:

Using the grouping axiom we have

\[
H_{r-1} = H(W | y^{r-1}) = h_b(\mu(w^*_r)) + (1 - \mu(w^*_r)) H(\hat{W}), \tag{32}
\]

where \( \hat{W} \) is a random variable with probability distribution \( P(\hat{W} = w) = \frac{\mu(w)}{1 - \mu(w^*_r)}, w \in [1:M], w \neq w^*_r \). Hence, having \( H_{r-1} \leq \epsilon \) implies that \( h_b(\mu(w^*_r)) \leq \epsilon \). Taking the inverse image of \( h_b \) implies that either \( \mu(w^*_r) \geq 1 - h_b^{-1}(\epsilon) \) or \( \mu(w^*_r) \leq h_b^{-1}(\epsilon) \), where \( h_b^{-1} : [0,1] \to [0,\frac{1}{2}] \) is the lower-half inverse function of \( h_b \). We show that the second case is not feasible. For this purpose, we show that the inequality \( \mu(w^*_r) \leq h_b^{-1}(\epsilon) \leq \frac{1}{2} \) implies that \( H_{r-1} \geq 1 \) which is a contradiction with the original assumption \( H_{r-1} \leq \epsilon < 1 \). This statement is proved in the following proposition. With this argument, we conclude that \( H_{r-1} \leq \epsilon \) implies that \( \mu(w^*_r) \geq 1 - \eta_1(\epsilon) \), where \( \eta_1(\cdot) = h_b^{-1}(\cdot) \).

**Proposition 1.** Let \( W \) be a random variable taking values from a finite set \( W \). Suppose that \( P_W(w) \leq \frac{1}{2} \) for all \( w \in W \). Then \( H(W) \geq 1 \).

**Proof:** The proof follows from an induction on \( |W| \). For \( |W| = 2 \) the condition in the statement implies that \( W \) has uniform distribution and hence \( H(W) = 1 \) trivially. Suppose the statement holds for \( |W| = n - 1 \). Then, we prove it for \( |W| = n \). Sort elements of \( W \) in an descending order according to \( P_W(w) \), from the most likely (denoted by \( W \)) to the least likely (\( w_n \)). If \( P_W(w_n) = 0 \), then the statement holds trivially from the induction’s hypothesis. Suppose \( P_W(w_n) > 0 \). In this case, we can reduce \( H(W) \) by increasing \( P_W(W) \) and decreasing \( P_W(w_n) \) so that \( P_W(W) + P_W(w_n) \) remains constant. In that case, either \( P_W(w_n) \) becomes zero or \( P_W(W) \) reaches the limit \( \frac{1}{2} \). The first case happens if \( P_W(W) + P_W(w_n) \leq \frac{1}{2} \). For that, the statement \( H(W) \geq 1 \) follows from the induction’s hypothesis, as there are only \( (n-1) \) elements with non-zero probability. It remains to consider the second case in which \( P_W(W) = \frac{1}{2} \) and \( P_W(w_n) > 0 \). Again, we can further reduce the entropy by increasing \( P_W(w_2) \) and decreasing \( P_W(w_n) \) while \( P_W(w_2) + P_W(w_n) \) remains constant. Observe that \( P_W(w_2) + P_W(w_n) \leq \frac{1}{2} \) as \( \sum_{i=2}^{n} P_W(w_i) = 1 - P_W(W) = \frac{1}{2} \) and \( p_W(w_i) > 0 \). Hence, after this redistribution process \( P_W(w_n) \) becomes zero. Then, the statement \( H(W) \geq 1 \) follows from the induction’s hypothesis, as there are only \( (n-1) \) elements with non-zero probability.
We proceed with the proof of the lemma by applying Lemma 7 in [2]:

**Lemma 7 ([2]).** For any non-negative sequence of numbers \( p_\ell, \mu_\ell, \) and \( \beta_{i,\ell}, \ell \in [1 : L], i \in [1 : N] \) the following inequality holds

\[
\sum_{\ell = 1}^{L} p_\ell \log \left( \frac{\sum_{i=1}^{N} \mu_\ell}{\sum_{i=1}^{N} \beta_{i,\ell}} \right) \leq \max_i \sum_{\ell = 1}^{L} p_\ell \log \frac{\mu_\ell}{\beta_{i,\ell}}.
\]

As a result,

\[
\mathbb{E}[\log H_r - \log H_r | F_{r-1}]
\]

\[= \sum_{y_r} P(y_r | y_r^{-1}) \log \left( -\frac{-\sum_{w} \mu(w) \log \mu(w)}{\sum_{w} \mu(w, y_r) \log \mu(w, y_r)} \right) \leq \max \Gamma(w),
\]

where

\[
\Gamma(w) = \sum_{y_r} P(y_r | y_r^{-1}) \log \frac{-\mu(w) \log \mu(w)}{-\mu(w, y_r) \log \mu(w, y_r)}.
\]

Note that

\[
\mu(w, y_r) = \frac{\mu(w)Q_w(y_r)}{P(y_r | y_r^{-1})},
\]

Therefore, for a fixed \( w \neq w_* \) we have that

\[
-\Gamma(w)
\]

\[= \sum_{y_r} P(y_r | y_r^{-1}) \log \left[ \frac{Q_w(y_r)}{P(y_r | y_r^{-1})} \left( 1 + \frac{\log \frac{P(y_r | y_r^{-1})}{Q_w(y_r)}}{-\log \mu(w)} \right) \right]
\]

\[= \sum_{y_r} P(y_r | y_r^{-1}) \log \frac{Q_w(y_r)}{P(y_r | y_r^{-1})} + \log \left( 1 + \frac{\log \frac{P(y_r | y_r^{-1})}{Q_w(y_r)}}{-\log \mu(w)} \right)
\]

\[\equiv -D_{KL} \left( P(\cdot | y_r^{-1}) \mid Q_w \right) + \sum_{y_r} P(y_r | y_r^{-1}) \log \left( 1 + \frac{\log \frac{P(y_r | y_r^{-1})}{Q_w(y_r)}}{-\log \mu(w)} \right).
\]

The summation in the last equality is bounded using the inequality \( \log(1 + x) \geq x \) for all \( x \geq -1 \). Hence we get that

\[ -\Gamma(w) \geq -D_{KL} \left( P(\cdot | y_r^{-1}) \mid Q_w \right) + \sum_{y_r} P(y_r | y_r^{-1}) \log \frac{P(y_r | y_r^{-1})}{Q_w(y_r)} \frac{1}{-\log \mu(w)}
\]

\[= -D_{KL} \left( P(\cdot | y_r^{-1}) \mid Q_w \right) \left( 1 - \frac{1}{-\log \mu(w)} \right).
\]

Having \( \mu(w) \leq \eta_1(\epsilon) \) for all \( w \neq w_* \), we have that

\[
\Gamma(w) \leq D_{KL} \left( P(\cdot | y_r^{-1}) \mid Q_w \right) \left( 1 - \frac{1}{-\log \eta_1(\epsilon)} \right)
\]

\[\leq D_{KL} \left( P(\cdot | y_r^{-1}) \mid Q_w \right),
\]

where the last inequality follows as \( -\log \eta_1(\epsilon) \leq 0 \).

Next we consider the case \( w = w_* \). We use the Taylor’s theorem for the function \( f(x) = -x \log x \) around \( x = 1 \). With that, \( f(x) = (1 - x) - \zeta(1 - x)^2 \) for some \( \zeta \) between \( x \) and 1. Hence, with \( x = \mu(w_*) \), we have that

\[
-\mu(w_*) \log \mu(w_*) \leq (1 - \mu(w_*)) - \zeta(1 - \mu(w_*))^2 \leq (1 - \mu(w_*)).
\]
Next, from the inequality \( \log x \leq x - 1, \forall x > 0 \), we have that
\[
-\mu(w^*_r, y_r) \log \mu(w^*_r, y_r) \geq \mu(w^*_r, y_r)(1 - \mu(w^*_r, y_r)).
\]
As a result of these inequalities, we have that
\[
\Gamma(w^*_r) = \sum_{y_r} P(y_r|y^{r-1}) \log \frac{-\mu(w^*_r) \log \mu(w^*_r) - \mu(w^*_r, y_r) \log \mu(w^*_r, y_r)}{(1 - \mu(w^*_r)) \mu(w^*_r, y_r)(1 - \mu(w^*_r, y_r))} \\
\leq \sum_{y_r} P(y_r|y^{r-1}) \log \frac{1 - \mu(w^*_r)}{1 - \mu(w^*_r, y_r)} \\
= \sum_{y_r} P(y_r|y^{r-1}) \log \frac{1 - \mu(w^*_r)}{1 - \mu(w^*_r, y_r)} \\
- \sum_{y_r} P(y_r|y^{r-1}) \log \mu(w^*_r, y_r).
\] (35)

We proceed with simplifying the first summation above. From (37), we have that
\[
\mu(w^*_r, y_r) = \frac{\mu(w^*_r) Q_w^r(y_r)}{P(y_r|y^{r-1})}. 
\] (36)

Therefore,
\[
(1 - \mu(w^*_r, y_r)) = (1 - \mu(w^*_r)) \frac{\sum_{w \notin w^*_r} \mu(w^*_r)}{P(y_r|y^{r-1})} \frac{Q_w(y_r)}{P(y_r|y^{r-1})}. 
\] (37)

Thus, the first summation on the right-hand side of (35) is simplified as
\[
\sum_{y_r} P(y_r|y^{r-1}) \log \frac{P(y_r|y^{r-1})}{\sum_{w \notin w^*_r} \mu(w^*_r)^{1 - \mu(w^*_r)} Q_w(y_r)} \\
= D_{KL}(P(\cdot|y^{r-1}) \| Q_{\sim w^*_r}),
\]
where \( Q_{\sim w^*_r}(y_r) = \sum_{w \notin w^*_r} \mu(w^*_r)^{1 - \mu(w^*_r)} Q_w(y_r) \) for all \( y_r \in \mathcal{Y} \). Next, we bound the second summation in (35). Using (36), we have that
\[
- \sum_{y_r} P(y_r|y^{r-1}) \log \mu(w^*_r, y_r) \\
= - \log \mu(w^*_r) - \sum_{y_r} P(y_r|y^{r-1}) \log \frac{Q_w(y_r)}{P(y_r|y^{r-1})} \\
= - \log \mu(w^*_r) + D_{KL}(P(\cdot|y^{r-1}) \| Q_{w^*_r}) \\
\leq 2\eta_1(\epsilon) + D_{KL}(P(\cdot|y^{r-1}) \| Q_{w^*_r}),
\] (38)
where the last inequality holds from the fact that \(- \log(1 - x) \leq \frac{x}{1 - x} \) and that \( \mu(w^*_r) \geq 1 - \eta_1(\epsilon) \), implying \(- \log \mu(w^*_r) \leq - \log((1 - \eta_1(\epsilon)) \leq \frac{\eta_1(\epsilon)}{1 - \eta_1(\epsilon)} \leq 2\eta_1(\epsilon) \) which holds as \( \eta_1(\epsilon) \leq \frac{1}{2} \). Note that \( P(\cdot|y^{r-1}) = \mu(w^*_r) Q_{w^*_r} + (1 - \mu(w^*_r)) Q_{\sim w^*_r} \). Therefore, from the convexity of the relative entropy the right-hand side of (38) is bounded by
\[
(38) \leq 2\eta_1(\epsilon) + \mu(w^*_r) D_{KL}(Q_{w^*_r} \| Q_{w^*_r}) \\
+ (1 - \mu(w^*_r)) D_{KL}(Q_{\sim w^*_r} \| Q_{w^*_r}) \\
\leq 2\eta_1(\epsilon) + \eta_1(\epsilon) D_{KL}(Q_{\sim w^*_r} \| Q_{w^*_r}) \\
\leq (2 + d_{\text{max}}) \eta_1(\epsilon),
\]
where the last inequality is from the definition of \( d^{\text{max}} \) being the maximum relative entropy of the channel. As a result of the above argument, we have that

\[
\Gamma(w^*_r) \leq D_{KL}\left(P(\cdot|y^{r-1}) \parallel Q_{\sim w^*_r}\right) + (2 + d^{\text{max}})\eta_1(\epsilon)
\]

\[
\leq \sum_{w \not\in w^*_r} \frac{\mu(w)}{1 - \mu(w^*_r)} D_{KL}\left(P(\cdot|y^{r-1}) \parallel Q_w\right)
\]

\[
+ (2 + d^{\text{max}})\eta_1(\epsilon)
\]

\[
\leq \max_{w \not\in w^*_r} D_{KL}\left(P(\cdot|y^{r-1}) \parallel Q_w\right) + (2 + d^{\text{max}})\eta_1(\epsilon)
\]

where (a) is due to the convexity of the relative entropy and the definition of \( Q_{\sim w^*_r} \). Inequality (b) follows as \( \frac{\mu(w)}{1 - \mu(w^*_r)} \) form a probability distribution on \( w \neq w^*_r \).

Note that the right-hand side of (34) and (39) depends on the messages. In what follows we remove this dependency. Note that the convexity of the relative entropy gives

\[
D_{KL}\left(P(\cdot|y^{r-1}) \parallel Q_w\right) \leq \mu(w^*_r) D_{KL}\left(Q_{w^*_r} \parallel Q_w\right)
\]

\[
+ (1 - \mu(w^*_r)) D_{KL}\left(Q_{\sim w^*_r} \parallel Q_w\right)
\]

\[
\leq D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) + \eta_1(\epsilon) d^{\text{max}}.
\]

As a result the bound in (39) is simplified to the following

\[
\Gamma(w^*_r) \leq \max_w D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) + 2(1 + d^{\text{max}})\eta_1(\epsilon)
\]

(40)

Similarly, the bound on \( \Gamma(w) \) in (34) is simplified to

\[
\Gamma(w) \leq D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) + \eta_1(\epsilon) d^{\text{max}}
\]

(41)

Combining the two bounds above, we finally can bound the logarithmic drift as

\[
\mathbb{E}[\log H_{r-1} - \log H_r | \mathcal{F}_{r-1}] \leq \max_w \Gamma(w)
\]

\[
\leq \max_w D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) + 2(1 + d^{\text{max}})\eta_1(\epsilon).
\]

We proceed by bounding the relative entropy between \( Q_{w^*_r} \) and \( Q_w \) for all \( w \neq w^*_r \). Let \( Q_r = P_{Y_r|X_r, Y^{r-1}} \) which is the effective channel (averaged over possible states) from the transmitter’s perspective at time \( r \). Also let \( x_r^* = e(w^*_r, y^{r-1}) \). Then we have the following lemma.

**Lemma 5.** Given an AVC, let \( Q_w \) be as in (31c). Then, if \( H_{r-1} < \epsilon \), the following inequality holds

\[
\max_w D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) \leq \max_x D_{KL}\left(Q_r(\cdot|x^*_r, y^{r-1}) \parallel Q_r(\cdot|x, y^{r-1})\right) + O(\eta_1(\epsilon)),
\]

(42)

where \( Q_r = P_{Y_r|X_r, Y^{r-1}} \) is the average channel at time \( r \), and \( \eta_1(\epsilon) = h^{-1}_b(\epsilon) \).

With this lemma, we get the desired bound on the logarithmic drift of the entropy

\[
\mathbb{E}[\log H_{r-1} - \log H_r | \mathcal{F}_{r-1}] \leq \max D_{KL}\left(Q_r(\cdot|x^*_r, y^{r-1}) \parallel Q_r(\cdot|x, y^{r-1})\right) + O(\eta_1(\epsilon)).
\]

Therefore, the main lemma is proved. It remains to prove (42).

**Proof of Lemma 5:** From the convexity of the relative entropy, the left-hand side term in (42) equals to

\[
\max_w D_{KL}\left(Q_{w^*_r} \parallel Q_w\right) = \sup_{\mu(\cdot)} \sum_{w \not\in w^*_r} \frac{\mu(w)}{1 - \mu(w^*_r)} D_{KL}\left(Q_{w^*_r} \parallel Q_w\right),
\]

(43)

where the supremum is taken over all distributions \( \mu(\cdot) \) on all \( w \neq w^*_r \) satisfying \( \mu(w) \geq 0 \) and \( \sum_{w \neq w^*_r} \mu(w) = 1 - \mu(w^*_r) \).

We upper bound the right-hand side of (43) by approximating \( Q_{w^*_r}(y_r) \) and bounding \( \sum_{w \neq w^*_r} \frac{\mu(w)}{1 - \mu(w^*_r)} Q_w \) from below.
We start with lower-bounding the second term. For any \( x \in \mathcal{X} \), define \( \nu(x) \triangleq \mathbb{P}\{X_r = x|y^{r-1}\} \). Then, for any choice of \( \mu(w), w \neq w^*_r \), we have that

\[
\sum_{w \notin w^*_r} \frac{\mu(w)}{1 - \mu(w^*_r)} \mathbb{P}_w(y_r) = \frac{1}{1 - \mu(w^*_r)} \sum_{w \notin w^*_r} \mathbb{P}\{W = w, Y_r = y_r \mid Y_r = y_r\}
\]

\[
\geq (a) \frac{1}{1 - \mu(w^*_r)} \sum_{x \notin x^*_r, s \in S} \mathbb{P}\{W = w, X_r = x, S_r = s, Y_r = y_r \mid Y_r = y_r\}
\]

\[
= (b) \frac{1}{1 - \mu(w^*_r)} \sum_{x \notin x^*_r, s \in S} \mathbb{P}\{X_r = x, S_r = s, Y_r = y_r \mid Y_r = y_r\}
\]

\[
= (c) \frac{1}{1 - \mu(w^*_r)} \sum_{x \notin x^*_r} \mathbb{P}\{X_r = x \mid Y_r = y_r\} \sum_{s \in S} \mathbb{P}\{S_r = s \mid Y_r = y_r, X_r = x\} \mathbb{P}\{y_r|s, x^*_r\}
\]

\[
= (d) \frac{1}{1 - \mu(w^*_r)} \sum_{x \notin x^*_r} \nu(x) \mathbb{P}\{y_r|X_r, y^{r-1}\}
\]

\[
= (e) \frac{1}{1 - \nu(x^*_r)} \sum_{x \notin x^*_r} \nu(x) \mathbb{P}\{y_r|X_r, y^{r-1}\},
\]

where (a) holds by summing over \( x \) and \( s \). Note that this is an inequality because the summation is not over all values of \( x \).

Equality (b) holds by removing the condition \( w \neq w^*_r \). This is an equality, because the probability of the event \( \{W = w^*_r, X_r \neq x^*_r\} \) is zero. Equality (c) follows by breaking the joint probability on \( \{X_r, S_r\} \) and further moving the summation on \( s \).

Equality (d) is due to the definition of \( \nu \) and \( Q_r \). Lastly, (e) is due to the fact that \( \nu(x^*_r) \approx \mu(w^*_r) \approx 1 - \eta_1(\epsilon) \).

Next, we approximate \( Q_{w^*_r}(y_r) \) by deriving a lower bound and an upper bound that are converging to each other. We start with the lower bound on \( Q_{w^*_r}(y_r) \). Since \( X_r = e(W, y^{r-1}) \), we have that

\[
Q_{w^*_r}(y_r) = \mathbb{P}\{Y_r = y_r \mid W = w^*_r, Y_r = y_r, X_r = x^*_r\}
\]

\[
= \sum_{s \in S} \mathbb{P}\{Y_r = y_r, S_r = s \mid W = w^*_r, Y_r = y_r, X_r = x^*_r\}
\]

\[
= \sum_{s \in S} \mathbb{P}\{S_r = s \mid W = w^*_r, Y_r = y_r, X_r = x^*_r\} \mathbb{P}\{y_r|s, x^*_r\}
\]

where the last equality holds because of the channel’s probability rules. Next, we bound the conditional probability on \( S_r \) in the above summation. This quantity equals to

\[
\mathbb{P}\{S_r = s \mid Y_r = y_r, X_r = x^*_r\} \mathbb{P}\{W = w^*_r | Y_r = y_r, X_r = x^*_r, S_r = s\} \mathbb{P}\{X_r = x^*_r | Y_r = y_r\}
\]

\[
\geq (1 - \eta_1(\epsilon)) \mathbb{P}\{S_r = s \mid Y_r = y_r, X_r = x^*_r\}
\]

The denominator is greater than \((1 - \eta_1(\epsilon))\) because of the following argument:

\[
\mathbb{P}\{W = w^*_r | Y_r = y_r, X_r = x^*_r\} = \frac{\mathbb{P}\{W = w^*_r, X_r = x^*_r | Y_r = y_r\}}{\mathbb{P}\{X_r = x^*_r | Y_r = y_r\}}
\]

\[
= \frac{\mathbb{P}\{W = w^*_r, X_r = x^*_r, Y_r = y_r\}}{\mu(w^*_r)}
\]

\[
= (1 - \eta_1(\epsilon)),
\]

where the last inequality holds as the denominator is less than one and that \( \mu(w^*_r) \geq 1 - \eta_1(\epsilon) \).

As the nominator in (46) is less than one, then we get that

\[
(46) \leq \frac{1}{1 - \eta_1(\epsilon)} \mathbb{P}\{S_r = s \mid Y_r = y_r, X_r = x^*_r\}
\]
Hence, from (45), we have that

\[
Q_{w_r}(y_r) \leq \frac{1}{1 - \eta_1(\epsilon)} \sum_{s \in S} \mathbb{P}\{S_r = s \mid Y_r^{r-1} = y_r^{r-1}, X_r = x_r^* \} Q(y_r | s, x_r^*)
\]

\[
= \frac{1}{1 - \eta_1(\epsilon)} Q_r(y_r | x_r^*, y_r^{r-1}).
\]

With the bounds in (47) and (44), the right-hand side of (43) is bounded as

\[
D_{KL}\left( Q_{w_r} \parallel \sum_{w \neq w_r} \frac{\mu(w)}{1 - \mu(w)} Q_w \right) = \sum_{y_r} Q_{w_r}(y_r) \log \frac{Q_{w_r}(y_r)}{\sum_{w \neq w_r} \frac{\mu(w)}{1 - \mu(w)} Q_w(y_r)} \\
\leq (a) \sum_{y_r} Q_{w_r}(y_r) \log \frac{1}{1 - \eta_1(\epsilon)} Q_r(y_r | x_r^*, y_r^{r-1}) \\
= (b) \sum_{y_r} \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) \log \frac{Q_r(y_r | x_r^*, y_r^{r-1})}{\sum_{x \neq x_r^*} \frac{\nu_1(x)}{1 - \nu_1(x)} \bar{Q}_r(y_r | x, y_r^{r-1})} - \log(1 - \eta_1(\epsilon)) \\
= D_{KL}\left( \bar{Q}_r(\cdot | x_r^*, y_r^{r-1}) \parallel \sum_{x \neq x_r^*} \frac{\nu_1(x)}{1 - \nu_1(x)} \bar{Q}_r(\cdot | x, y_r^{r-1}) + O(\eta_1(\epsilon)), \right)
\]

where (a) follows from (47) and (44), and (b) follows from the following argument for bounding \(Q_{w_r}(y_r)\) from below:

\[
Q_{w_r}(y_r) = \mathbb{P}\{ Y_r = y_r \mid W = w_r^*, Y_r^{r-1} = y_r^{r-1}, X_r = x_r^* \} \\
\geq \mathbb{P}\{ Y_r = y_r \mid W = w_r^* \mid Y_r^{r-1} = y_r^{r-1}, X_r = x_r^* \} \mathbb{P}\{ W = w_r^* \mid Y_r^{r-1} = y_r^{r-1}, X_r = x_r^*, Y_r = y_r \} \\
= \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) \mathbb{P}\{ W = w_r^* \mid Y_r^{r-1} = y_r^{r-1}, X_r = x_r^*, Y_r = y_r \} \mathbb{P}\{ X_r = x_r^* \mid Y_r^{r-1} = y_r^{r-1}, Y_r = y_r \} \\
\leq (a) \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) \mathbb{P}\{ W = w_r^* \mid Y_r^{r-1} = y_r^{r-1}, X_r = x_r^* \} \mathbb{P}\{ X_r = x_r^* \mid Y_r^{r-1} = y_r^{r-1}, Y_r = y_r \} \\
\leq (b) \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) \mathbb{P}\{ W = w_r^* \mid Y_r^{r-1} = y_r^{r-1}, Y_r = y_r \} \\
\leq (c) \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) \mu(w_r^*, y_r) \\
\leq (d) \bar{Q}_r(y_r | x_r^*, y_r^{r-1}) (1 - \eta_2(\epsilon))
\]

where (a) holds as \(X_r\) is a function of \((W, Y_r^{r-1})\), (b) holds as the denominator is less than one, (c) is due to the definition of \(\mu(w, y_r)\), and (d) holds from (37) and by defining

\[
\eta_2(\epsilon) = \eta_1(\epsilon) \sum_{w \neq w_r} \frac{\mu(w)}{1 - \mu(w)} Q_w(y_r) \mathbb{P}\{ y_r \mid y_r^{r-1} \}.
\]

Lastly, by taking the supremum of (48) over \(\nu(x), x \neq x_r^*\), we obtain the desired bound in (42). With that the proof is complete.

\[\square\]

**APPENDIX C**

**PROOF OF LEMMA 4**

From the definition of \(\{S_t\}_{t \geq 0}\), we have that

\[
S_t = \sum_{r=1}^{t \wedge \tau_e} \frac{J_r}{T} + \sum_{r=t \wedge \tau_e+1}^{t \wedge \tau^*} \frac{\log M}{T} \mathbb{1}\{H_r - 1 \geq \sqrt{\epsilon}\} + \sum_{r=t \wedge \tau^*+1}^{T \wedge \tau^*} \frac{D_r}{D} + \sqrt{\epsilon} \frac{N}{T} \mathbb{1}\{t \geq \tau^*\}
\]

Therefore, as \((T \lor \tau^*) \geq \tau^* \geq \tau_e\), then

\[
E[S_{T \lor \tau^*}] = E\left[ \sum_{r=1}^{\tau_e} \frac{J_r}{T} + \sum_{r=\tau_e+1}^{\tau^*} \frac{\log M}{T} \mathbb{1}\{H_r - 1 \geq \sqrt{\epsilon}\} + \sum_{r=\tau^*+1}^{T \wedge \tau^*} \frac{D_r}{D} + \sqrt{\epsilon} \frac{N}{T} \mathbb{1}\{t \geq \tau^*\} \right]
\]

\[
= \sum_{r=1}^{\tau_e} \frac{J_r}{T} + \sum_{r=\tau_e+1}^{\tau^*} \frac{\log M}{T} \mathbb{1}\{H_r - 1 \geq \sqrt{\epsilon}\} + \sum_{r=\tau^*+1}^{T \wedge \tau^*} \frac{D_r}{D} + \sqrt{\epsilon} \frac{N}{T} \mathbb{1}\{t \geq \tau^*\}
\]
As for the first summation, after multiplying and dividing by \( E[\tau] \), we have that
\[
E\left[ \sum_{r=1}^{\tau} \frac{J_r}{I} \right] = \frac{1}{E[\tau]} E\left[ \sum_{r=1}^{\tau} J_r \right] = E[\tau],
\]
where the last equality follows by setting \( I \) as in the statement of the lemma. Similarly, the third summation is bounded as in the following
\[
E\left[ \sum_{r=1}^{T_{\tau^*}} D_r \right] = \frac{1}{E[T-\tau]} E\left[ \sum_{r=1}^{T_{\tau^*}} D_r \right] = E[T-\tau],
\]
where the first equality holds after multiplying and dividing by \( E[T-\tau] \), and the second equality follows by setting \( D \) as in the statement of the lemma. As a result,
\[
E[S_{T\wedge \tau^*}] \leq E[T] + \frac{\log M}{T} E\left[ \sum_{r=1}^{\tau^*} \mathbb{1}\{H_{r-1} \geq \sqrt{c}\} \right] + \sqrt{c} \frac{N}{T}
\]
\[
\leq E[T] + \frac{\log M}{T} E\left[ \sum_{r=1}^{N} \mathbb{1}\{H_{r-1} \geq \sqrt{c}\} \right] + \sqrt{\tau} \frac{N}{T}
\]
(50)
where the inequality follows as \( \tau^* \leq N \). Next, we bound the remaining summation. By iterative expectation we have that
\[
E\left[ \sum_{r=1}^{N} \mathbb{1}\{H_{r-1} \geq \sqrt{c}\} \right] = E_{\tau}\left[ \sum_{r=1}^{N} E\left[ \mathbb{1}\{H_{r-1} \geq \sqrt{c}\} \mid \tau \right] \right]
\]
\[
= E_{\tau}\left[ \sum_{r=1}^{N} P\left(H_{r-1} \geq \sqrt{c} \mid \tau \right) \right]
\]
(51)
where (a) follows from taking the supremum over all \( H_{r-1} \) appearing in the summation. Inequality (b) follows as the summation is less than \( N - \tau \) which is smaller than \( N \). Lastly, (c) holds by taking the expectation of the conditional probability. We proceed with the following lemma which is a variant of Doob's maximal inequality for super-martingales.

**Lemma 6 (Maximal Inequality for Supermartingales).** Let \( \{M_t\}_{t>0} \) be a non-negative supermartingale w.r.t a filtration \( \{\mathcal{F}_t\}_{t>0} \). If \( \tau \) is a bounded stopping time w.r.t this filtration, then the following inequality holds for any constant \( c > 0 \)
\[
P\left( \sup_{t \geq \tau} M_t > c \right) \leq \frac{E[M_\tau]}{c}
\]

**Proof:** Define \( S \triangleq \inf \{ t > 0 : t \geq \tau, M_t > c \} \). Note that \( S \) is a stopping time. Since \( \{M_t\}_{t>0} \) is non-negative, then for any fixed \( n \in \mathbb{N} \), we have that
\[
M_{\tau \wedge S} \geq c \mathbb{1}\{ \sup_{\tau \leq t \leq n} M_t > c \}.
\]
Therefore, taking the expectation of both sides and rearranging the terms gives the following inequality
\[
P\left( \sup_{\tau \leq t \leq n} M_t > c \right) \leq \frac{E[M_{\tau \wedge S}]}{c}.
\]
(52)
Since \( \{M_t\}_{t>0} \) is a super-martingale and that \( \tau \leq S \), then \( E[M_{\tau \wedge \tau}] \geq E[M_{\tau \wedge S}] \). Therefore, we can write
\[
P\left( \sup_{\tau \leq t \leq n} M_t > c \right) \leq \frac{E[M_\tau]}{c}.
\]
This is because if \( n < \tau \) then the left-hand side is zero and the inequality holds trivially. When \( n \geq \tau \), using the above argument, the right-hand side of (52) is less than \( \frac{E[M_{\tau-1}]}{c} \). Next, taking the limit \( n \to \infty \) and from monotone convergence theorem we get that
\[
\mathbb{P}\left\{ \sup_{\tau \leq t} M_t > c \right\} = \mathbb{P}\left\{ \bigcup_{n > 0} \left\{ \sup_{\tau \leq t \leq n} M_t > c \right\} \right\} = \lim_{n \to \infty} \mathbb{P}\left\{ \sup_{\tau \leq t \leq n} M_t > c \right\} \leq \frac{E[M_{\tau}]}{c},
\]
where the second equality follows from the continuity of the probability measure. With that the proof is complete.

Note that \( \{H_t\}_{t>0} \) is a super martingale. Therefore, from Lemma 6, we have that
\[
\mathbb{P}\left( \sup_{\tau_e \leq t \leq N-1} H_t \geq \sqrt{\epsilon} \right) \leq \frac{\mathbb{E}[H_{\tau_e}]}{\sqrt{\epsilon}}.
\]
If \( \tau_e < N \), then by definition of this stopping time \( H_{\tau_e} \leq \epsilon \); otherwise \( \tau_e = N \) which implies that \( H_{\tau_e} = H_N \). However, as \( T \leq N \), then
\[
\mathbb{E}[H_N] \leq \mathbb{E}[H_T] \leq h_0(P_\epsilon) + P_\epsilon \log M_1 M_2 \leq \epsilon,
\]
where the second inequality follows from Fano’s and the last inequality holds as \( P_\epsilon \ll \epsilon \). Consequently,
\[
\mathbb{P}\left( \sup_{\tau_e \leq t \leq N-1} H_t \geq \sqrt{\epsilon} \right) \leq \frac{\epsilon}{\sqrt{\epsilon}} = \sqrt{\epsilon}.
\]
Therefore, using this inequality in (51) we obtain that
\[
\mathbb{E}\left[ \sum_{r=\tau_e+1}^{N} \mathbb{I}\{H_{r-1} \geq \sqrt{\epsilon}\} \right] \leq N \sqrt{\epsilon}.
\]
Thus, from (50), we obtain that
\[
\mathbb{E}[S_{T\vee \tau_e}] \leq \mathbb{E}[T] + \frac{\log M}{T} \left( \sqrt{\epsilon N} \right) + \sqrt{\epsilon \frac{N}{T}}.
\]
Hence, factoring \( \mathbb{E}[T] \) gives the following inequality
\[
\mathbb{E}[S_{T\vee \tau_e}] \leq \mathbb{E}[T] \left( 1 + V(\epsilon, N) \right),
\]
where \( V(\epsilon, N) = \frac{h_0}{T}\sqrt{\epsilon N} + \sqrt{\epsilon \frac{N}{\mathbb{E}[T]}} \). Hence, the proof is complete.