The conservation of the Hamiltonian structures in Whitham’s method of averaging.

A. Ya. Maltsev.

L.D.Landau Institute for Theoretical Physics, Kosygina 2, Moscow 117940, maltsev@itp.ac.ru

Abstract

The work is devoted to the proof of the conservation of local field-theoretical Hamiltonian structures in Whitham’s method of averaging. The consideration is based on the procedure of averaging of local Poisson bracket, proposed by B.A.Dubrovin and S.P.Novikov. Using the Dirac procedure of restriction of the Poisson bracket on the submanifold in the functional space, it is shown in the generic case that the Poisson bracket, constructed by method of Dubrovin and Novikov, satisfies the Jacobi identity. Besides that, the invariance of this bracket with respect to the choice of the set of local conservation laws, used in this procedure, is proved.

Introduction.

This work is devoted to Whitham’s method of averaging, which permits to explore the evolution of slow modulated m - phase solutions of nonlinear systems of equations ( [1], see also [2], [4], [5], [6]). The slow modulated parameters of m - phase solutions (for example “running waves” if m = 1) satisfy in this approach to quasilinear homogeneous evolution system of type:

\[ U_T^i = V_j^i(U)U_X^j, \quad i, j = 1, \ldots, N, \]

\[ U = (U^1, \ldots, U^N). \]

For the exploring of systems of this kind it appears to be important to explore them on the subject of being Hamiltonian with respect to local
Poisson brackets of hydrodynamic type (see [3], [4], [5]). The theory of these brackets, constructed by B.A.Dubrovin and S.P.Novikov ([3], [4], [5]), has been used by S.P.Tsarev (see [7]) for integration of systems (1), having Hamiltonian structure and reducible to the diagonal form. The investigation of Hamiltonian systems which do not satisfy to the last condition, was made by E.V.Ferapontov (see [8], [9]).

In the work [4] the procedure of constructing of Hamiltonian structure of the required type for Whitham’s system of equations (1), under the condition that the initial system is Hamiltonian with respect to local field-theoretical Poisson brackets was proposed. However, the proof of the Jacobi identity for constructed by this way brackets was absent (see [10]). In this work we shall prove the Jacobi identity for constructed by Dubrovin-Novikov’s method brackets in the case of ”generic situation” with the aid of Dirac’s procedure of restriction of Poisson brackets on the submanifold in the space of functions. So that, the results of this work permit to make a statement about the conservation of local Hamiltonian structures in Whitham’s method of averaging.

1. General constructions.

Consider the evolution system on the space of fields $\varphi = (\varphi^1, \ldots, \varphi^n)$ of type:

$$\varphi^i_t = Q^i(\varphi, \varphi_x, \varphi_{xx}, \ldots), \quad i = 1, \ldots, n,$$

(2)

which is Hamiltonian with respect to local field-theoretical Poisson bracket of type:

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B^{ij}_k(\varphi, \varphi_x, \ldots)\delta^{(k)}(x - y)$$

(3)

(there is a finite number of terms in the sum) with the Hamiltonian:

$$H = \int P_H(\varphi, \varphi_x, \ldots)dx.$$

(4)

The bracket (3) can be compatible with the operator of translation, that is, there may exist the local functional:

$$P = \int P_P(\varphi, \varphi_x, \ldots)dx,$$

(5)
(the momentum operator) such that: $\{\varphi^i(x), P\} = \varphi_x^i$. (This is not a necessary condition and as was pointed to the author by O.I. Mokhov, there is a special class of local Poisson brackets which satisfy it, see [11], [12]. However, in the Hamiltonian structures, connected with the “physical” systems, this property, as a rule, is present.) Besides that, we shall admit that bracket (3) can have a finite number of annihilators, that is, functionals (not necessary having the form (4), (5)) $N_1, \ldots, N_p$, such that: $\{\varphi^i(x), N_q\} = 0$.

Definition. Let we are given a function of m variables $\Phi(\theta) = (\Phi^i(\theta_1, \ldots, \theta_m)), i = 1, \ldots, n, 2\pi$-periodic with respect to each of the arguments, and corresponding to it m-vectors: $\omega = (\omega^1, \ldots, \omega^m), k = (k^1, \ldots, k^m)$, such that:

$$\omega^\alpha \Phi^i_{\theta^\alpha} = Q^i(\Phi, k^\alpha \Phi_{\theta^\alpha}, \ldots)$$

(6)

then the corresponding to them function:

$$\varphi(x, t) = \Phi(kx + \omega t)$$

(7)

we shall call the m-phase quasiperiodic solution of the system (2).

Note that the existence of m-phase solutions with $m > 1$ presents as a rule only for “integrable” systems like KdV.

System (2) is a system of differential equations in partial derivatives on functions $\Phi(\theta) (\theta = (\theta^1, \ldots, \theta^m))$, and its $2\pi$-periodic with respect to all variables solutions (if they exist) at all possible $k$ and $\omega$ give the full family of m-phase solutions of system (2).

We shall suppose that at any $k$ and $\omega$ in some open set (we suppose that system (2) is not linear) the system (2) has some family of the required solutions, which can be parametrized by the initial phase shifts $\theta_0^\alpha$, and may be also by some set of additional parameters $r^1, \ldots, r^g$, (the number of which is constant at all $\omega$ and $k$), which do not change under the variation of $\theta_0^\alpha$. So that, the m-phase solutions of (2) will be given by the values $\omega = (\omega^1, \ldots, \omega^m), k = (k^1, \ldots, k^m), r^1, \ldots, r^g$ and $\theta_0^1, \ldots, \theta_0^m$.

Let now the system (2) has $N = 2m + g$ translation-invariant functionals:

$$I^\nu = \int \mathcal{P}^\nu(\varphi, \varphi_x, \ldots) dx, \quad \nu = 1, \ldots, N,$$

(8)

commuting with the Hamiltonian (3) and being in the involution with each other with respect to the bracket (3):

$$\{I^\nu, I^\mu\} = 0.$$  

(9)
Among the integrals (8) there may be Hamiltonian (4), functional of momentum and annihilators of brackets (3) having the form (8).

We shall assume that parameters $k, \omega$ and $r = (r^1, \ldots, r^g)$ are independent on the family of m - phase solutions of (2) and, besides that, may be expressed in terms of the values of integrals (8) on the functions of family (7), that is $k = k(U), \omega = \omega(U), r = r(U), U = (U^1, \ldots, U^N)$, where

$$U^n = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \mathcal{P}^n(\varphi, \varphi_x, \ldots) dx = \frac{1}{(2\pi)^m} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^n(\Phi, k^\alpha \Phi_{\theta^\alpha}, \ldots) d^m \theta$$

(it can be easily seen that values in (10) do not depend upon the initial phase shifts $\theta_0^\alpha$). So that, after the choice at each value of parameters $U$ on $\mathcal{M}$ of some definite function $\Phi_m(\theta, U)$ as having zero initials phases $\theta_0$, we can put values $U^n$ and $\theta_0^\alpha$ to be the parameters on the family of m - phase solutions of (2).

Each of the functionals (8) generates the Hamiltonian flow of the form:

$$\varphi^i_{\tau^n} = Q^i_{\tau^n}(\varphi, \varphi_x, \ldots).$$

(11)

All these flows are commutative with each other and with (4), and leave the family of m - phase solutions of (2) invariant, generating the linear dependence of the initial phases upon the ”times” $\tau^n$ on it and leaving constant all the other parameters. It means that there exist the functions $\omega_0^\alpha(U)$, such that for all $\Phi(\theta + \theta_0, U)$, corresponding to the family of m - phase solution (2), we have:

$$\omega_0^\alpha(U) \Phi^i_{\theta^\alpha} = Q^i_{\tau^n}(\Phi, k^\alpha \Phi_{\theta^\alpha}, \ldots)$$

(12)

From (4) we can conclude that all evolution systems (11), generated by integrals $I^n$, have the same properties that the system (2), and all the statements, proved for system (2), are valid for systems (11) (besides zero flows and operator of shift of $x$, we shall speak about this in the consideration of Whitham’s method in which the nonlinearity of the system is essential).

System (2) at each $\omega$ and $k$ defines some ”submanifold” (let denote it by $\mathcal{M}_{\omega,k}$, in the space of $2\pi$ – periodic with respect to each $\theta^\alpha$ functions. Functionals

$$F_{\omega,k}^i[\Phi](\theta) = \omega^\alpha \Phi^i_{\theta^\alpha} - Q^i(\Phi, k^\alpha \Phi_{\theta^\alpha}, \ldots)$$

(13)

are being the constraints, defining these submanifolds.
We shall assume that, if the operator $I^\nu$ is not the operator of momentum or annihilator of (3), the corresponding systems (12) at all $\omega^\nu$ and $k$ define (as well as (3)) the full family of $m$-phase solutions of (2), that is, the constraints:

$$F_{i(\nu)}^{(\nu)}[\Phi](\theta) = \omega^\alpha_{\nu}\Phi^i_{\theta^\alpha} - Q^i_{\nu}(\Phi, k^\alpha\Phi^j_{\theta^\alpha}, \ldots) = 0,$$

being consider at some definite $\omega^\nu$ and $k$, define some $m + g$-parametric family $M_{\omega^\nu,k}$ of $2\pi$-periodic with respect to each $\theta^\alpha$ solutions, parametrized by initial phase shifts and some additional parameters the number of which is $g$, so that the joint of all these $M_{\omega^\nu,k}$ at all possible $\omega^\nu$ and $k$ gives the full family of $m$-phase solutions of (2) (it is so in many known examples).

We shall also assume in our situation (it is also valid in the examples), that the number $g$ of parameters $r^1, \ldots, r^g$ is equal to the number $p$ of annihilators of (3), for which there exist the following motivations:

any of $m + 1$ flows (11), generated by arbitrary $m + 1$ integrals (8), are linearly dependent because of (12) on the family of $m$-phase solutions of (2), that is, there exist $\lambda^1(U), \ldots, \lambda^{m+1}(U)$, such that $\sum_{j=1}^{m+1}(\lambda^j)^2 = 1$ and $\sum_{j=1}^{m+1}\lambda^j(U)\omega^\nu_{\nu_1}(U) = 0, \forall \alpha$. From this we can conclude that $m$-phase solutions of the system (2) can be also defined by the relation:

$$\delta \sum_{j=1}^{m+1}\lambda^jI^\nu = \delta \sum_{q=1}^{p}\mu^qN_q,$$

(where $N^q$ - are the annihilators of (3), since the functional $\sum_{j=1}^{m+1}\lambda^j(U)I^\nu$ generates the zero flow on the corresponding $m$-phase solutions), on the functions:

$$\varphi(x) = \Phi(kx),$$

where $\Phi(\theta) = \Phi(\theta^1, \ldots, \theta^m)$ is $2\pi$-periodic function of $\theta$. Supposing that at any $k = (k^1, \ldots, k^m), \mu^1, \ldots, \mu^p$ and $\lambda^1, \ldots, \lambda^{m+1}$, satisfying the relation $\sum_{j=1}^{m+1}(\lambda^j)^2 = 1$, the functional $\sum_{j=1}^{m+1}\lambda^jI^\nu - \sum_{q=1}^{p}\mu^qN_q$ has on the functions (15) the only extremum modulo the initial phase shifts, we have that all $m$-phase solutions of (2) can be characterized by $2m + p$ independent parameters (except the initial phase shifts), and, so that, we obtain the formulated statement.

As in the finite-dimensional situation we shall require the submanifolds $M_{\omega,k}$ and $M_{(\nu)\omega^\nu,k}$ (for $I^\nu$ which are not annihilators of (3) or momentum
operator) to satisfy some property of regularity, analogous to the maximality of rank of matrix of constraints derivatives in the finite-dimensional situation. Namely, let us linearize the functional (13) \( F_{\omega,k}(\theta) = (F_{\omega,k}^1(\theta), \ldots, F_{\omega,k}^m(\theta)) \) (respectively any of functionals (14)) on the solution of system (12), \( \Phi_{\nu}(\theta + \theta_0, U) \) (it will be also a solution of (14)), that is introduce the operator \( \hat{L}_{U,\theta_0} \) (respectively \( \hat{L}_{U,\theta_0}^{(\nu)} \)) with the kernel \( L_{ij}^{(\nu)}(\theta + \theta_0, \theta' + \theta_0) \) \( (L_{ij}^{(\nu)}(\theta + \theta_0, \theta' + \theta_0)) \), such that:

\[
\delta F_{\omega(U),k(U)}^i(\theta) = \\
= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_j \left( L_{ij}^{(\nu)}(\theta + \theta_0, \theta' + \theta_0)(\Phi'(\theta') - \Phi'(\theta' + \theta_0, U)) \right) \theta = \\
= (\hat{L}_{U,\theta_0} \delta \Phi)^i(\theta) \quad (16)
\]

(similar for the constraints \( F_{\omega,v,k}^{(\nu)}[\Phi](\theta) \)).

Defined by such a way \( \hat{L}_{U,\theta_0} \) (\( \hat{L}_{U,\theta_0}^{(\nu)} \)) are differential with respect to \( \theta \) operators with the periodic coefficients in the space of \( 2\pi \)-periodic functions \( \Phi(\theta), i = 1, \ldots, n \). We shall require the following conditions:

A) For all \( U \) and \( \theta_0 \) the kernel of operator (16) \( \hat{L}_{U,\theta_0} \) consists of vectors tangential to the submanifold \( M_{\omega(U),k(U)} \), defined by system (12), that is, the functions \( \Phi_{\nu}(\theta, k, \omega, r) \) and \( \Phi_{\nu}(\theta, k, \omega, r) \) give the basis in the space of solutions of system

\[
(\hat{L}_{U,\theta_0} \delta \Phi)(\theta) = 0,
\]

and the same property is valid for operators \( \hat{L}_{U,\theta_0}^{(\nu)} \), corresponding to \( I^{\nu} \), which are not annihilators of (3) and operator of momentum.

B) The co-dimension of the images of the operators \( \hat{L}_{U,\theta_0} \), \( \hat{L}_{U,\theta_0}^{(\nu)} \) in the space of \( 2\pi \)-periodic with respect to \( \theta \) functions is equal to the dimension of their kernels, that is, all \( \hat{L}_{U,\theta_0}, \hat{L}_{U,\theta_0}^{(\nu)} \) have exactly \( m + g \) (\( g = p \)) left eigen vectors (let denote them \( \kappa_{iU,\theta_0}^{(s)} \), \( \kappa_{U,\theta_0}^{(s),(\nu)} \)), that is \( 2\pi \)-periodic with respect to each \( \theta^a \) functions \( \kappa_{iU}^{(s)}(\theta + \theta_0), j = 1, \ldots, n, s = 1, \ldots, m + g \) and \( \kappa_{iU}^{(s),(\nu)}(\theta + \theta_0), \) such that:

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \kappa_{iU}^{(s)}(\theta) L_{ij}^{(\nu)}(\theta, \theta' \theta) d^m \theta \equiv 0 \quad (17)
\]

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \kappa_{iU}^{(s),(\nu)}(\theta) L_{ij}^{(\nu)}(\theta, \theta' \theta) d^m \theta \equiv 0. \quad (18)
\]
If all these conditions are satisfied, we say that submanifolds \( M_{\omega,k} \) and \( M'_{\omega',k} \) in the space of \( 2\pi \)-periodic functions of \( \theta \) have the property of regularity. Besides that, under the assumption that vectors \( \Phi_{\theta^\alpha}(\theta, k, \omega, r), \Phi_{r^\alpha}(\theta, k, \omega, r), \Phi_{k^\alpha}(\theta, k, \omega, r) \) and \( \Phi_{\omega^\beta}(\theta, k, \omega, r) \) are linearly independent at all values of parameters \((\theta_0, k, \omega, r)\) (it is essential requirement for the following consideration), the joint of described above submanifolds \( M_{\omega,k} \) (or \( M'_{\omega',k} \)) gives \( 3m + g = N + m \) dimensional submanifold \( M \) in the space of \( 2\pi \)-periodic with respect to each \( \theta^\alpha \) functions, corresponding to the full family of \( m \)-phase solutions of \( (2) \).

2. Method of Whitham.

The constructions described above are closely connected with Whitham’s method of averaging for nonlinear systems of differential equations in partial derivatives (it can not be applied to the flows, generated by annihilators of \( (3) \) or momentum operator), which is the following procedure: introduced the small parameter \( \epsilon \), we put \( T = \epsilon t, X = \epsilon x \) and rewrite the system \((2)\) on the fields \( \varphi^i(X, T) \) in the form:

\[
\epsilon \varphi^i_T = Q^i(\varphi, \epsilon \varphi_X, \epsilon^2 \varphi_{XX}, \ldots).
\]  

(19)

Let now consider the system \((19)\) on the space of functions \( \varphi(\theta, X, T), \theta = (\theta^1, \ldots, \theta^m) \), \( 2\pi \)-periodic with respect to each of variables \( \theta^\alpha \). In method of Whitham we try to find the functions:

\[
S(X, T) = (S^1(X, T), \ldots, S^m(X, T)),
\]

and \( 2\pi \)-periodic with respect to all \( \theta^\alpha \) functions:

\[
\Phi(\theta, X, T, \epsilon) = (\Phi^i(\theta, X, T, \epsilon)),
\]

represented by asymptotical series when \( \epsilon \to 0 \):

\[
\Phi^i(\theta, X, T, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Phi^i_{(n)}(\theta, X, T),
\]

(20)

such that the function:

\[
\varphi(\theta, X, T, \epsilon) = \Phi(\theta + \frac{S(X, T)}{\epsilon}, X, T, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \Phi_{(n)}(\theta + \frac{S(X, T)}{\epsilon}, X, T),
\]

(21)
satisfies the system (19) at all θ and ε, (ε → 0).

It can be easily seen that the substitution of (21) into the system (19) gives the following equations in the zero order of ε

\[ S^0_T \Phi^i_0(\theta, X, T) = Q^i(\Phi_0), S^0_X \Phi_0, \ldots, \] (22)

that is, at any X and T, \( \Phi_0(\theta, X, T) \), as a function of \( \theta \), represents one of the function from family \( M \) described above, and the functions \( \varphi(x, t, \epsilon) = \Phi(\theta_0 + \frac{1}{\epsilon} S(\epsilon x, \epsilon t), \epsilon) \), obtained from (21) after the replacement of X and T by \( \epsilon x \) and \( \epsilon t \) respectively, tend at small \( \epsilon \) to the slow modulated m - phase solutions of (2).

Besides that, from (22) we obtain:

\[ k(U) = S_X, \quad \omega(U) = S_T, \] (23)

where \( U, k \) and \( \omega \) - are parameters on \( M \).

Terms with \( \epsilon^k, k > 0 \), give the relations:

\[ (\hat{L}_{U, \theta_0}(X, T) \Phi(k))^i(\theta, X, T) = f^i_k(\Phi(0), \Phi(1), \ldots, \Phi(k-1), S_X, S_T, \ldots) \] (24)

- where \( \hat{L}_{U, \theta_0} \) - is described in (16) linear differential (with respect to \( \theta \)) operator with \( 2\pi \) - periodic with respect to \( \theta \) coefficients (expressed in terms of the \( \Phi(0)(\theta, \theta_0(X, T), U(X, T)) \) and its derivatives with respect to \( \theta^\alpha \)), \( f_k \) is discrepancy, which depends upon the previous functions \( \Phi(j)(\theta, X, T) \) and being of order of \( k \), regarding that functions \( \Phi(j) \) have order \( j \), functions \( S \) - order \( -1 \), multiplication of functions adds their orders and differentiation with respect to \( T \) and \( X \) adds 1 to the order of function.

The system (24) has solutions in the class of \( 2\pi \) - periodic with respect to \( \theta \) functions if and only if at any X and T the vector \( f_k(\theta, X, T) \) is orthogonal for all defined in (17) left eigen vectors of operator \( \hat{L}_{U(X, T), \theta_0(X, T)} \), that is:

\[ \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \kappa_{i(U, X, T)}^{(s)}(\theta + \theta_0) f^i_k(\theta, X, T) d^m \theta \equiv 0, \] (25)

which gives on the function \( f_k(\theta, X, T) \) at any X and T \( m + g = N - m \) independent relations according to the number of left eigen vectors of the operator \( \hat{L}_{U(X, T), \theta_0(X, T)} \).
Taking $\Phi(0)$ in the form: $\Phi(0) = \Phi_{in}(\theta + \theta_0(X, T), U(X, T))$, we can easily see from the statements formulated above that in the first order of $\epsilon$ equations (24) have the form:

$$(\hat{L}[U(X, T), \theta_0(X, T)] \Phi_1)(\theta) = \sum_n \alpha_{j(n)}^i (S_X, \Phi(0), \Phi(0)\theta^\alpha, \Phi(0)\theta^\alpha \theta^\beta, \ldots) \times$$

$$\times \left( \Phi_{(0)n\theta,U\nu}^j X^\nu + \Phi_{(0)n\theta,\theta^\alpha} \theta_0^\alpha X \right) + \beta^i_\alpha (S_X, \Phi(0), \Phi(0)\theta^\alpha, \Phi(0)\theta^\alpha \theta^\beta, \ldots) S_{XX} -$$

$$- \left( \Phi_{(0)U\nu,U_T'} + \Phi_{(0)\theta^\alpha \theta_0^\alpha T} \right), \tag{26}$$

where $n = (n_1, \ldots, n_m)$ is an integral $m$-vector with nonnegative components, $n\theta$ denotes here $(n_1\theta^1 \ldots n_m\theta^m)$, $\alpha_{j(n)}^i$ $\beta^i_\alpha$ - some definite functions, values $S_X$ are $S_T$ are connected with the parameters $U(X)$, corresponding to $\Phi(0)(\theta, X)$, by the relations (23). So that the condition (23) gives on the functions $U(X, T)$ and $\theta_0(X, T)$ $N - m$ equations of the form:

$$A^\zeta_\mu(U) U''_T + B^\zeta_\mu(U) U'_X + A^\zeta_\alpha(U) \theta_0^\alpha T + B^\zeta_\alpha(U) \theta_0^\alpha X = 0, \ \zeta = 1, \ldots, N - m. \tag{27}$$

Adding to them $m$ equations:

$$k^\zeta_T = \omega^\zeta_X, \tag{28}$$

following from (23), we obtain the system of $N$ quasilinear equations on $N + m$ functions $U''(X, T), \theta_0^\alpha(X, T)$.

If the conditions (27) and (28) are satisfied, the function $\Phi_{(1)}(\theta, X, T)$ can be found at any $X$ and $T$ from the differential with respect to $\theta$ equation modulo the arbitrary linear combination of $N - m$ functions described in (A) and lying in the kernel of the operator $\hat{L}[U(X, T), \theta_0(X, T)]$ (let here denote them as $\xi_{U,\theta_0}$), that is, modulo the function of type: $\sum_{s=1}^{N-m} C_s(X, T) \xi_{U,\theta_0}^{(s)}(X, T) - \xi_{U,\theta_0}(X, T)$.

The values of the coefficients $C_s(X, T)$ can be found from the condition of the solvability of system (24) in the next order of $\epsilon$, where we have the same situation and, so that, if (27) and (28) are satisfied, we can obtain sequentially all the terms of series (20), which permits to find the required functions $\Phi(\theta, X, T, \epsilon)$. Function $S(X, T)$ can be constructed using the values $k(U)$ and $\omega(U)$.

Lemma 1.
Under the conditions formulated above, that is, in the presence of $N$ local translation-invariant integrals \((8)\), such that the evolution of the densities $P^\nu(\varphi, \varphi_x, \ldots)$ according to the flow \((\mathbf{2})\) has the form:

$$P^\nu_t = R^\nu_x(\varphi, \varphi_x, \ldots) \quad (29)$$

with some $R^\nu$, the system \((\mathbf{27})\) does not contain $\theta_0(X, T)$ and provides the relations only for $U(X, T)$, that is, all the terms containing $\theta_0X$ and $\theta_0T$ are orthogonal to the left eigen vectors of the operator $\hat{L}$ independently upon the values of parameters $U(X, T)$ and $\theta_0(X, T)$, so that $A^\nu_\mu \equiv 0, B^\nu_\mu \equiv 0$. Besides that, in method of Whitham for any of the systems \((\mathbf{11})\), which is not the operator of translation or trivial flow, we have the same situation.

Proof.

If \((\mathbf{27})\) and \((\mathbf{28})\) are valid, there exist the solutions of \((\mathbf{19})\) in form of asymptotic series \((\mathbf{20})\), the substitution of which into \((\mathbf{29})\) and integration with respect to $\theta$ gives in the first order of $\epsilon$:

$$U^\nu_T = \partial_X \langle R^\nu \rangle(U), \quad (30)$$

- where $\langle \ldots \rangle$ means the averaging on the family $M$ defined by the formula:

$$\langle F(\varphi, \varphi_x, \ldots) \rangle(U) \equiv \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} F(\Phi, k^\alpha \Phi_{\theta^\alpha}, k^\alpha k^\beta \Phi_{\theta^\alpha \theta^\beta}, \ldots) d^m \theta, \quad (31)$$

if $\varphi(x) = \Phi_m(kx + \theta_0, U)$.

So that, from the system \((\mathbf{27})-(\mathbf{28})\) follows the system \((\mathbf{30})\), which is a system of $N$ independent in the general case equations on functions $U(X, T)$, having the same form as \((\mathbf{27})-(\mathbf{28})\), and, so that, \((\mathbf{30})\) is equivalent to \((\mathbf{27})-(\mathbf{28})\). Since these conclusions may be applied to any of systems \((\mathbf{11})\), which is not translation or trivial flow, the Lemma is proved.

Let us note that in the presence of the additional conservation laws of form \((\mathbf{29})\) their averaging gives the equations which are the corollaries of \((\mathbf{30})\), the independence of \((\mathbf{30})\), which is the system of $N$ equations on $N$ parameters $U^\nu(X)$, is the property of generic situation.

System \((\mathbf{30})\) (or equivalent to it \((\mathbf{27})-(\mathbf{28})\)) giving the evolution of slow modulated parameters $U(X)$ of m - phase solutions of \((\mathbf{3})\) is called the system of equations of Whitham.

System \((\mathbf{30})\) has the form:

$$U^\nu_T = V^\nu_\mu(U)U^\mu_X, \quad \nu, \mu = 1, \ldots, N, \quad (32)$$

10
that is, it refers to evolution systems of hydrodynamic type. For smooth initial dates $U(X)$ system (32) has in general case smooth solution up to some moment $T_0$ depending upon the initial dates and after that the solution will be broken. So that we can use the solutions of Whitham up to the definite point of time, after which they are not defined.

Let us point also that usually we consider just the finite number of terms of asymptotic series (20) when we give us solutions of initial system (2) modulo terms of higher order of $\epsilon$. In particular it is possible to consider just the first term of (20), which gives the evolution of slow-modulated m-phase solutions of (2) provided that the Whitham equations on parameters $U$ hold and the next term in (20) is uniformly bounded. We shall consider all arising here asymptotic series from this point of view, in particular, we shall not interest in there convergency regions since we consider the relations on the first (say k) terms of such series modulo the higher orders of $\epsilon$.

### 3. The Poisson brackets of the hydrodynamic type.

Among the systems (32) there is an especial class of them which are Hamiltonian with respect to Poisson brackets of type:

$$\{ U^\nu(X), U^\mu(Y) \} = g^{\nu\mu}(U(X)) \delta'(X - Y) + b^{\nu\mu}(U(X)) U^\lambda \delta(X - Y)$$

(33)

with local Hamiltonian of hydrodynamic type:

$$H = \int h(U(X)) dx,$$

(34)

which plays an important role in their integrability (see [7], [8], [9]).

The theory of brackets (33) with nondegenerated $g^{\nu\mu}(U)$, constructed by B.A.Dubrovin and S.P. Novikov (see [3], [4], [5]), is closely connected with Riemannian geometry. In particular, from their skew-symmetry follows:

$$g^{\nu\mu} = g^{\mu\nu}, \quad b^{\nu\mu} + b^{\mu\nu} = \frac{\partial g^{\nu\mu}}{\partial U^\lambda};$$

(35)

Leibnitz identity leads to the fact that under the transformations of coordinates $U^\nu \rightarrow \tilde{U}^\nu(U)$ functions $g^{\nu\mu}$ transform as the contravariant components.

---

1 Author is grateful to I.M.Krichever for fruitful discussions of these questions.
of a metric tensor, whereas the functions $\Gamma^{\nu}_{\mu\lambda} = -g_{\nu\tau} b^{\nu}_{\lambda\tau}$ transform as coefficients of connection, consistent with metric view (35). The Jacobi identity for (33) in the case of nondegenerated metric $g^{\nu\mu}$ is equivalent to the symmetry of connection $\Gamma^{\nu}_{\mu\lambda}$ and zero curvature of the metric: $R^{\nu}_{\mu\lambda\tau} \equiv 0$.

The theory of brackets (33) with degenerated $g^{\nu\mu}(U)$ is more complicated, but also has a nice geometrical form, see [13].

In [4] B.A.Dubrovin and S.P.Novikov proposed also the method of constructing of the brackets (33) for Whitham’s system of equations (32), starting from the Hamiltonian structure (3) for the initial system (2) under the condition that we have the necessary number of commuting integrals (8). Namely, let us calculate the brackets of densities of integrals (8) in the form:

$$\{P^{\nu}(\varphi(x), \varphi_x, \ldots), P^{\mu}(\varphi(y), \varphi_y, \ldots)\} = \sum_{k \geq 0} A^{\mu\nu}_k(\varphi(x), \varphi_x, \ldots) \delta^k(x - y),$$

(there is a finite number of terms in the sum, $\nu, \mu = 1, \ldots, N$).

View (3) we can conclude that

$$A^{\mu\nu}_0(\varphi, \varphi_x, \ldots) \equiv \partial_x Q^{\nu\mu}(\varphi, \varphi_x, \ldots)$$

for some $Q^{\nu\mu}$. In the described above coordinates $U = (U^1, \ldots, U^N)$ the brackets of Dubrovin-Novikov have the form:

$$\{U^{\nu}(X), U^{\mu}(Y)\} = \langle A^{\nu\mu}_1(U(X)) \delta(X - Y) + \frac{\partial(Q^{\nu\mu})(U(X))}{\partial X} \delta(X - Y) \rangle$$

(38)

-where $\langle \ldots \rangle$, as previously, denotes the averaging on the m - phase solutions of (2), defined by formula (31).

However, the proof of the fact that constructed by such a way brackets satisfy the Jacobi identity, was absent in [4] (see [10]). The main purpose of this paper is to prove the fact that the procedure (36) - (38) of averaging of brackets (3) really gives the Poisson brackets of type (33), satisfying the Jacobi identity.

We shall need for the further purposes the Dirac procedure of restriction of Poisson bracket on the submanifold. Let us describe here this procedure in the notations of spaces of finite dimension.

Let in the space $V$ with the coordinates $x = (x^1, \ldots, x^l)$ and the Poisson bracket:

$$\{x^g, x^p\} = J^{gp}(x),$$

(39)
be a submanifold $N$, defined with the aid of constraints $g^1, \ldots, g^s$ by the equations:

$$g^1(x) = 0, \ldots, g^s(x) = 0. \quad (40)$$

Let us suppose also that some $l-s$ functions $f^1(x), \ldots, f^{l-s}(x)$, defined in the vicinity of $N$ in the space $V$, give the coordinate system on $N$ after the restriction on it. All the redefinitions of the functions $f^\lambda(x)$, having the form

$$\tilde{f}^\lambda(x) = f^\lambda(x) + \sum_{\zeta=1}^{s} \tau^\lambda_\zeta(x)g^\zeta(x)$$

with arbitrary $\tau^\lambda_\zeta(x)$, do not change this coordinate system. Let us suppose that we can find functions $\tau^\lambda_\zeta(x)$, such that the submanifold $N$ is invariant under Hamiltonian flows generated by all the functions $\tilde{f}^\lambda(x)$ (in the case of nondegeneracy of the matrix $\{g^\zeta(x), g^\xi(x)\}$ on $N$ it is always possible), that is: $\{\tilde{f}^\lambda(x), g^\zeta(x)\} \equiv 0$ if $g(x) = 0$. Then we can define the Dirac restriction of bracket (39) on the submanifold (40) by the formula:

$$\{f^\lambda, f^\mu\}^* = \{\tilde{f}^\lambda(x), \tilde{f}^\mu(x)\}|_N(f) \quad (41)$$

The bracket (41) satisfies automatically all the necessary identities. It can be easily verified that with the functions $\tau^\lambda_\zeta(x)$, founded by such a way, the bracket (41) may be also written in the form:

$$\{f^\lambda, f^\mu\}^* = \{f^\lambda(x), f^\mu(x)\}|_N(f) - \left(\tau^\lambda_\zeta(x)\tau^\mu_\xi(x)\{g^\zeta(x), g^\xi(x)\}\right)|_N(f). \quad (42)$$

The infinite dimensional form of (42) will be very convenient in the further analysis.

4. The coordinates in the vicinity of submanifold corresponding to the full family of m-phase solutions.

Let us now return to the constructions connected with Whitham’s method of averaging. After the substitution $X = \varepsilon x$ the bracket (3) has the form:

$$\{\varphi^i(X), \varphi^j(Y)\} = \sum_{k \geq 0} B^{ij}_k(\varphi, \varepsilon\varphi_{XX}, \varepsilon^2\varphi_{XXX}, \ldots)\varepsilon^k \delta^{(k)}(X - Y), \quad (43)$$

integrals (8) become:

$$I^\nu = \varepsilon^{-1} \int \mathcal{P}(\varphi, \varepsilon\varphi_X, \ldots) dX \quad (44)$$
Let us consider the space of functions $\varphi(\theta^1, \ldots, \theta^m, X)$, $2\pi$-periodic with respect to each of $\theta^\alpha$ and define at all $\epsilon$ the Poisson bracket on it according to the formula:

$$\{\varphi^i(\theta, X), \varphi^j(\theta', Y)\} = \sum_{k \geq 0} B_{ij}^k(\varphi, \epsilon \varphi_X, \epsilon^2 \varphi_{XX}, \ldots) \epsilon^k \delta^{(k)}(X - Y) \delta(\theta - \theta')$$  \hspace{1cm} (45)

(where $\delta(\theta - \theta')$ is the $\delta$-function in $m$-dimensional space).

Consider in the space of $2\pi$-periodic with respect to $\theta$ functions the submanifold $M'$ of functions $\varphi(\theta, X)$, such that at any $X \varphi(\theta, X)$ as a function of $\theta$ lies in $M$. The functions $U^\nu(X)$ and $\theta^\alpha_0(X)$ can be taken as the coordinates on the submanifold $M'$, so that the functions $\varphi(\theta, X)$ from $M'$ will be represented by the formula:

$$\varphi(\theta, X) = \Phi_{in}(\theta + \theta_0(X), U(X)).$$ \hspace{1cm} (46)

After the prolongation of coordinates $U(X)$ and $\theta_0(X)$ by the independent upon $\epsilon$ way in the vicinity of $M'$ (let us denote it by $\Delta_\delta$) in the functional space of $2\pi$-periodic with respect to $\theta$ functions, where these functions satisfy the conditions:

$$\varphi(\theta, X) \in \Delta_\delta \iff \exists \varphi_0(\theta, X) \in M': ||\varphi(\theta, X) - \varphi_0(\theta, X)|| \equiv$$

$$\equiv \sum_i (max|\varphi^i(\theta, X) - \varphi_0^i(\theta, X)| + \frac{1}{1!} (\sum_\alpha max|\varphi_\theta^\alpha - \varphi_0^\alpha| + max|\varphi_X^i - \varphi_0^i|) +$$

$$+ \frac{1}{2!} (\sum_{\alpha,\beta} max|\varphi_{\theta^\alpha \theta^\beta} - \varphi_0^\alpha \varphi_0^\beta| + \sum_\alpha max|\varphi_{\theta^\alpha X}^i - \varphi_0^\alpha X^i| + max|\varphi_{XX}^i - \varphi_0^i|) +$$

$$+ \frac{1}{3!} (\ldots + \ldots) < \delta$$ \hspace{1cm} (47)

(that is we imply that there exists a function $\varphi_0(\theta, X)$ from $M'$ such that all the terms of the series (47) exist, the series (47) is convergent and satisfies to the formulated condition), we can define the submanifold $M'$ with the aid of constraints:

$$F^i(\theta, X)[\Phi] = \omega^\alpha(U[\varphi](X))\varphi_{\theta^\alpha}^i - Q^i(\varphi, k^\alpha(U[\varphi](X))) \varphi_{\theta^\alpha}, \ldots) = 0. \hspace{1cm} (48)$$

The difference between the systems (6) and (48) is that the system (48) does not depend upon the parameters $k$ and $\omega$, since they are now determined
functions of \( \varphi(\theta, X) \) (in the vicinity of \( \mathcal{M}' \)), the values of which on the \( \mathcal{M}' \) coincide with the corresponding parameters of the family. It can be easily seen that only the functions from \( \mathcal{M}' \) satisfy to (48).

The functionals \( U(X) \) and \( \theta_0(X) \) in \( \Delta_\delta \) can be defined for instance by the following way: let introduce in \( \Delta_\delta \) \( N \) functionals of the form

\[
a^\nu[\varphi](X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} A^\nu(\varphi(\theta, X), \varphi_{\theta^0}(\theta, X), \ldots) d^m\theta
\]

with some functions \( A^\nu \), such that on the \( \mathcal{M}' \) the values \( a^\nu \) are functionally independent. On the \( \mathcal{M}' \) the values \( a^\nu \), as can be easily seen, can be expressed in terms of the values \( U(X) \) and do not depend upon \( \theta_0(X) \). So that, dividing \( \mathcal{M}' \) into the "maps" in which \( |U^\nu(1)(X) - U^\nu(2)(X)| < \delta' \), we can express \( U(X) \) in terms of \( a^\nu(X) \) in each of these maps in the form \( U^\nu(X) = f^\nu(a(X)) \) and then, using the definition (49) of \( a(X) \) in \( \Delta_\delta \), we can extend \( U^\nu(X) \) into the vicinity of each map (this can be done at each \( X \) independently). Then, after the definition of the functionals \( U^\nu(X) \) in \( \Delta_\delta \), let consider in \( \Delta_\delta \) the functionals:

\[
\vartheta^\alpha(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_i \varphi^i(\theta, X) \Phi^i_{\nu\theta^0}(\theta, U(\varphi(X))) d^m\theta,
\]

where \( \Phi^i_{\nu\theta^0}(\theta, U) \) are introduced in (46) functions from \( \mathcal{M}' \).

If \( \varphi(\theta, X) \in \mathcal{M}' \) and \( \theta_0(X) \equiv 0 \), then \( \vartheta^\alpha(X) \equiv 0 \) and, in the generic situation, at small \( \theta_0^\alpha(X) \) the values \( \{\theta_0^\alpha(X)\} \) can be expressed in terms of \( \{\vartheta^\alpha(X)\} \) on \( \mathcal{M}' \) in the form: \( \theta_0^\alpha(X) = \tau^\alpha(\vartheta(X)) \). After that, by the same way, dividing, if it is necessary, each of the maps described above into the parts in which: \( |\theta_0^\alpha(1)(X) - \theta_0^\alpha(2)(X)| < \delta'^\alpha \) (independently at each \( X \)), and expressing by such formulas \( \theta_0^\alpha(X) \) in terms of the \( \vartheta(X) \), we can extend \( \theta_0^\alpha(X) \) into \( \Delta_\delta \) using the functionals \( \vartheta(X) \).

It can be easily checked that after such definition of \( U(X) \) and \( \theta_0(X) \) in \( \Delta_\delta \) we have for two functions \( \varphi_{(1)}(\theta, X) \) and \( \varphi_{(2)}(\theta, X) \) from \( \Delta_\delta \), satisfying the condition of type (47), that is \( \|\varphi_{(1)}(\theta, X) - \varphi_{(2)}(\theta, X)\| < \delta \), where \( \delta \) is small, the relation: \( |U^\nu_{(1)}(X) - U^\nu_{(2)}(X)| < C\delta, |\theta_0^\alpha(1)(X) - \theta_0^\alpha(2)(X)| < C\delta \), where \( C \) is constant. The analogous relations will be also satisfied for the variational derivatives of \( U(X) \) and \( \theta_0(X) \).

We shall suppose that after the prolongation described above the submanifold \( \mathcal{M}' \), given by the system (48), possesses the property of regularity,
analogous to the regularity of $\mathcal{M}\omega_{\mathbf{k}}$. Namely, it is clear that the vectors: $\Phi_{\mu\alpha}(\theta + \theta_0(X), U(X))$ and $\Phi_{\mu\nu}(\theta + \theta_0(X), U(X))$ (let us denote this set as $\{\tilde{\xi}_{[U_0]}(X), q = 1, \ldots, N + m\}$) lie at all $X$ at the kernel of linearized on the function from $\mathcal{M}'$ (characterized by the corresponding values $U(X)$ and $\theta_0(X)$) functional $F^i(\theta, X)[\varphi]$, introduced in [18]:

$$
\delta F^i(\theta, X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}_j[\theta_0(X), \theta + \theta_0(X), X] \varphi^j(\theta, X) d^m \theta =
$$

$$
= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left[ (\omega^\alpha(U(X)) \delta^j_\alpha \delta_{\theta^0}(\theta - \theta') - \frac{\partial Q^i_{j\alpha}}{\partial \varphi^j(\theta, X)} \right]_{\text{const}} \delta(\theta - \theta') - \frac{\partial Q^i_{j\alpha}}{\partial \varphi^j(\theta, X)} \right]_{\text{const}} \delta(\theta - \theta') - \ldots +
$$

$$
+ \left( \frac{\partial \omega_{\mu
u}(X)}{\partial U_{\nu\nu}(X)} \delta U^\nu_{j\alpha}(X) \delta_{\theta^0}(\theta, X, X) \varphi^j_{\theta^0}(\theta, X) - \frac{\partial Q^i_{j\alpha}}{\partial \varphi^j(\theta, X)} \right]_{\text{const}} \delta(\theta - \theta') - \ldots +
$$

$$
+ \left( \frac{\partial \omega_{\mu
u}(X)}{\partial U_{\nu\nu}(X)} \delta U^\nu_{j\alpha}(X) \delta_{\theta^0}(\theta, X, X) \varphi^j_{\theta^0}(\theta, X) - \frac{\partial Q^i_{j\alpha}}{\partial \varphi^j(\theta, X)} \right]_{\text{const}} \delta(\theta - \theta') - \ldots +
$$

$$
+ (\varphi^j_{\theta^0}(\theta, X, X) \varphi^j_{\theta^0}(\theta, X)) \delta(\theta - \theta') - \ldots +
$$

$$
= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{\xi}_{[U_0]}(\theta, X) \tilde{L}_j[\theta_0(X), \theta + \theta_0(X), X] d^m \theta \equiv 0.
$$

(52)

We shall assume that there is no other vectors possessing this property and, besides that, at all points of $\mathcal{M}'$ there exist at all $X$ exactly $N + m$ of "left eigen vectors" $\tilde{\kappa}_{[U_0]}(\theta, X)$ for the operator $\tilde{L}_{[U, \theta]}$ with zero eigen values, that is such linearly independent at each $X$ $2\pi-$ periodic functions of $\theta$: $\tilde{\kappa}_{[U]}(\theta + \theta_0, X), \ i = 1, \ldots, n$, that:

$$
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{\kappa}_{[U]}(\theta, X) \tilde{L}_j[\theta, \theta', X] d^m \theta \equiv 0.
$$

(52)

Note that unlike $\{\tilde{\xi}_{[U_0]}(X)\}$ the form of functions $\{\tilde{\kappa}_{[U_0]}(X)\}$ depends upon the manner of prolongation of $U(X)$ and $\theta_0(X)$ into the $\Delta_\delta$, since $\tilde{\xi}$ from the kernel of $\tilde{L}$ correspond to the variations of functions $\varphi(\theta, X)$ along the $\mathcal{M}'$, on which $U(X)$ and $\theta_0(X)$ are defined initially, whereas the $\tilde{\kappa}(\theta, X)$ are connected with the image of the operator $\tilde{L}$, for finding of which it is necessary to know the change of $U(X)$ and $\theta_0(X)$ at all variations of $\varphi(\theta, X)$.

With the aid of the constructions described above we can introduce another system of coordinates in $\Delta_\delta$ instead of the standard system, given by
the functionals $\varphi^i(\theta, X)$. Namely, we take as the coordinates in $\Delta_\delta$ the functionals $\{ U^\nu(X), \theta_0^\alpha(X), \nu = 1, \ldots, N, \alpha = 1, \ldots, m \}$, and

$$G_i^{[U, \theta_0]}[\varphi](\theta, X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{L}_j^{[U]}(\theta + \theta_0(X), \theta' + \theta_0(X), X) \times$$

$$\times (\varphi^j(\theta', X) - \Phi^j_{in}(\theta' + \theta_0(X), U(X))) d^m \theta',$$  \hspace{1cm} (53)$$

where $\hat{L}_j^{[U]}$ are functionals from (51), appeared in the linearization of the constraints $F^i(\theta, X)$ on the submanifold $\mathcal{M}'$.

So that, the functions $\varphi(\theta, X)$ from $\Delta_\delta$ will be characterized by the set of $N + m$ smooth functions $U^\nu(X), \theta_0^\alpha(X)$, and by the functions $G^i(\theta, X), i = 1, \ldots, n$, taking the values at the space of $2\pi$-periodic with respect to $\theta$ functions, satisfying at each $X$ (and given $U(X)$ and $\theta_0(X)$) to $N + m$ linear integral conditions of type:

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{\kappa}^{(q)}_{i[\theta]}(\theta + \theta_0(X), X) G^i(\theta, X) d^m \theta = 0$$ \hspace{1cm} (54)$$

(The conditions of such type are not customary in usual systems of coordinates, however, they can be encountered in the theory of stratifications).

Lemma 2.
At small enough $\delta$ the values of functionals $U^\nu(X), \theta_0^\alpha(X)$ and $G^i(\theta, X)$ in $\Delta_\delta$, satisfying (53), give uniquely the values $\varphi^i(\theta, X)$.

Proof.
Indeed, the system:

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{L}_j^{[U]}(\theta + \theta_0(X), \theta' + \theta_0(X), X) \times$$

$$\times (\varphi^j(\theta', X) - \Phi^j_{in}(\theta' + \theta_0(X), U(X))) d^m \theta' = G^i(\theta, X)$$

has, view (54), a solution, defined modulo the linear combination of the vectors $\{ \xi^{(q)}_{i[\theta]}(X) \}$, tangential to $\mathcal{M}'$ and corresponding to variations of parameters $U(X)$ and $\theta_0^\alpha(X)$; so that, their coefficients in small enough $\Delta_\delta$ may be defined by the values $U^\nu(X)$ and $\theta_0^\alpha(X)$.

We shall also need another system of coordinates in $\Delta_\delta$, which connected with the system described above by the transformation depending upon $\epsilon$. 

17
Namely, consider the functionals:

$$J^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\nu(\varphi(\theta, X), \epsilon \varphi_X(\theta, X), \ldots) d^m \theta, \ \nu = 1, \ldots, N,$$

(55)

and

$$\theta^*_0(X) = \theta_0(X) - \frac{1}{\epsilon} \int_{X_0}^X k(J(X')) dX'$$

(56)

(for some initial point $X_0$). As a system of coordinates in the vicinity of $\mathcal{M}'$ we take the functionals $\{J^\nu(X), \theta^*_0(X)\}$, and also the set of constraints $G^i(\theta, X)$ satisfying to (54).

The transition from $(U^\nu(X), \theta^0_\alpha(X))$ to $(J^\nu(X), \theta^*_0(X))$, as can be easily seen from (56), is defined at $\epsilon \neq 0$. On the submanifold $\mathcal{M}'$ (that is at $G^q(\theta, X) \equiv 0$) at $\epsilon \to 0$ we have:

$$J^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\nu(\Phi_{in}(\theta + \theta_0(X), U(X))),$$

$$\epsilon (\theta^0_X \Phi_{in\theta^0} + U^{\nu}_X \Phi_{inU^{\nu}}, \ldots) d^m \theta =$$

$$\sum_{k \geq 0} \epsilon^k J^\nu_{(k)}(U, U_X, \ldots, U_{kX}, \theta_0X, \ldots, \theta_{0kX}),$$

where $J^\nu_{(k)}$ represents the averaged with respect to $\theta$ the $k$th Teilour term of the expansion of $P^\nu$ at $\epsilon \to 0$ on the functions from $\mathcal{M}'$. There is a finite number of terms in the sum, there introduced the notations: $U_{kX} \equiv \partial^k U / \partial X^k, \theta_0X \equiv \partial^k \theta_0 / \partial X^k$. The expression for $\theta^*_0(X)$ in terms of $U(X)$ and $\theta_0(X)$ on the $\mathcal{M}'$ can be obtained by the substitution of these expansions for $J^\nu(X)$ into (56).

We shall need also the inverse transformation from $J^\nu(X)$ and $\theta^*_0(X)$ to $U(X), \theta_0(X)$ at $G^q(\theta, X) \equiv 0$ (that is on the $\mathcal{M}'$). Let us note that the values $J^\nu(X), \theta^*_0$ and $U^\nu(X)$ are connected on $\mathcal{M}'$ by the relations (the definition of $J^\nu(X)$):

$$J^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\nu(\Phi_{in}(\theta + \theta^*_0(X)) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X')) dX', U(X)),$$

$$\epsilon \partial_X \Phi_{in}(\theta + \theta^*_0(X) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X')) dX', U(X)), \ldots) d^m \theta =$$

18
where \( P(\Phi_{in}(\theta + \theta^*_0(X)) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', U(X)) \) and their derivatives with respect to \( U^\nu \) and \( \theta^\alpha \) with the coefficients of type: \( U_X(X), U_{XX}(X), \ldots, k(J), \partial_X k(J), \partial^2_X k(J), \ldots, \) and \( \theta^*_{0X}(X), \theta^*_{0XX}(X), \ldots, \) given by the collecting together these terms, having the general multiplier \( \epsilon^k \). The term, corresponding to the zero power of \( \epsilon \), is written separately.

After the integration with respect to \( \theta \), which removes the singular at \( \epsilon \to 0 \) phase shift \( \theta_0 \) in the argument \( \Phi_{in} \), we obtain on \( M' \):

\[
J^\nu(X) = \zeta^\nu(J, U) + \sum_{k \geq 1} \epsilon^k \zeta^\nu_{(k)}(U, U_X, \ldots, U_{kX}, J, J_X, \ldots, J_{kX}, \theta^*_{0X}, \ldots, \theta^*_{0kX}).
\]

The sum in (57) contains the finite number of terms, functions \( \zeta^\nu \) and \( \zeta^\nu_{(k)} \) are integrated with respect to \( \theta \) functions \( P^\nu_{(k)} \) and \( P^\nu \) respectively. Since

\[
\zeta^\nu(J, U) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} P^\nu(\Phi_{in}(\theta, U(X)), k^\alpha(J) \partial_{\theta^\alpha} \Phi_{in}(\theta, U(X))),
\]

\[
k^\alpha(J) k^\beta(J) \partial_{\theta^\alpha} \partial_{\theta^\beta} \Phi_{in}(\theta, U(X)), \ldots) d^m \theta
\]

(in the integration with respect to \( \theta \) we omitted the unessential phase shift \( \Phi_{in} \)) we obtain that the system:

\[
J^\nu(X) = \zeta^\nu(J(X), U(X))
\]

is satisfied by the solution \( J^\nu(X) \equiv U^\nu(X) \), according to the definition (14) of parameters \( U(X) \) on the family \( M' \). Since we suppose that the system (58) is of general form, we shall assume that it is equivalent to the system \( J^\nu(X) = U^\nu(X) \).

Taking this into account, we can resolve the system (57) by the iterations, taking on the initial step \( U^\nu(X) = J^\nu(X) \); for \( U^\nu(X) \) we shall obtain the
expression of the form:

\[
U^\nu(X) = J^\nu(X) + \sum_{k \geq 1} \epsilon^k U^\nu_{(k)}(J, J_X, \ldots, J_k X, \theta^*_0 X, \ldots, \theta^*_{0k} X). \tag{59}
\]

The substitution of (59) into (57), under the condition of nonsingularity of matrix \(\left| \frac{\partial \xi^\nu(J, U)}{\partial U^\mu} \right|_{U=J} \), will uniquely define the functions \(U^\nu_{(k)}\). The value \(\theta_0(X)\) is expressed in terms of \(J(X)\) and \(\theta^*_0(X)\) according to the formula (56).

Later it will be convenient to write the expressions of type \(\theta^0[J, \theta^*_0, \epsilon]\) and \(U^\nu[J, \theta^*_0, \epsilon]\), assuming the expressions (56) and (59) (not necessary on the \(M'\)).

The functionals (55) are well defined on the full space of functions \(\varphi(\theta, X)\) and the Hamiltonian flows, generated with the aid of (45) by the integrals:

\[
\int q(X) J^\nu(X) dX = \int q(X) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\varphi, \epsilon \varphi_X, \ldots) d^n \theta, \tag{60}
\]

have for all smooth functions \(q(X)\) the form:

\[
\varphi^i = q(X) Q^i_\nu(\varphi, \epsilon \varphi_X, \ldots) + \epsilon q_X \tilde{Q}^i_\nu(\varphi, \epsilon \varphi_X, \ldots) + \epsilon^2 q_{XX} \tilde{\tilde{Q}}^i_\nu(\varphi, \epsilon \varphi_X, \ldots) + \ldots, \tag{61}
\]

where \(Q^i_\nu\) are introduced in (11) flows generated by the functionals \(I^\nu, \tilde{Q}^i_\nu, \tilde{\tilde{Q}}^i_\nu, \ldots\) are some functions. (The derivatives of \(q(X)\) with respect to \(X\) arise in the calculation of the variational derivative and also as a result of action of bracket (13). As can be easily seen, each differentiation of \(q(X)\) with respect to \(X\) appears with the multiplier \(\epsilon\).)

5. **The restriction of the Poisson bracket on the submanifold corresponding to the full family of m-phase solutions.**

For the Dirac restriction of bracket (13) on the submanifold \(\mathcal{M}'\) in the coordinates \(J(X), \theta^*_0(X)\) and \(G(\theta, X) = (G^i(\theta, X))\) we shall need their Poisson brackets with each other (on the \(\mathcal{M}'\)).

We shall begin with the brackets of type: \(\{ J^\nu(X), J^\mu(Y) \} \). View (30), these brackets have the form:

\[
\{ J^\nu(X), J^\mu(Y) \} =
\]
higher order of $\epsilon$

family of $m$-phase solutions of (2), defined by the formula (31).

ing to the system (61), generated by the functional $M$
at the points of $M$

depend the values $\phi$

At the points of the submanifold $M$

this type will arise later and have the form of regular at $\epsilon$

series of type:

be easily seen from (36), has the form:

where view (37):

\[
A^{\nu\mu}_0(\varphi, \epsilon \varphi_X, \epsilon^2 \varphi_{XX}, \ldots) \equiv \epsilon \partial_X Q^{\nu\mu}(\varphi, \epsilon \varphi_X, \epsilon^2 \varphi_{XX}, \ldots). \tag{63}
\]

At the points of the submanifold $M'$ the function $\varphi(\theta, X)$ has the form:

$\varphi^i(\theta, X) = \Phi^i_{in}(\theta + \theta^*_0(X) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', U[J, \theta^*_0, \epsilon](X))$ and after the substitution of it into (62) we can obtain the brackets $\{J^\nu(X), J^\mu(Y)\}$ at the points of $M'$ (characterized by coordinates $J(X), \theta^*_0(X)$) in form of regular at $\epsilon \to 0$ series, since the integration with respect to $\theta$ removes the irregularity at $\epsilon \to 0$ in the argument of function $\Phi_{in}$. View (63), it can be easily seen that the zero terms are absent in these series, so that they can be written in the form:

\[
\{J^\nu(X), J^\mu(Y)\}|_{g(\theta, X)=0} = \\
= \epsilon \left( \langle A^\nu_{1}(J(X)) \delta'(X-Y) + \frac{\partial \langle Q^{\nu\mu}(J(X)) \rangle}{\partial X} \delta(X-Y) \right) + \epsilon^2 J^{\nu\mu}[J, \theta^*_0, \epsilon], \tag{64}
\]

where $J^{\nu\mu}$ are regular at $\epsilon \to 0$ functionals of $J$ and $\theta^*_0$. The functionals of this type will arise later and have the form of regular at $\epsilon \to 0$ asymptotic series of type:

\[
C[J, \theta^*_0, \epsilon] = \sum_{k \geq 0} C(k)(J, J_X, \ldots, J_{kX}, \theta^*_{0X}, \ldots, \theta^*_{0kX})\epsilon^k.
\]

Here we used the relation (33) giving $U(X)$ in terms of $J(X)$ and $\theta^*_0(X)$ at the points of $M'$, according to which the arguments $U^\nu(X)$, on which depend the values $\langle A^\nu_{1}(\epsilon) \rangle$ and $\langle Q^{\nu\mu}(\epsilon) \rangle$ on $M'$, are replaced modulo the terms of higher order of $\epsilon$ by $J^\nu(X)$, $\langle \ldots \rangle$ means, as previously, the averaging on the family of $m$-phase solutions of (2), defined by the formula (31).

The evolution of the densities of conservation laws $P^\nu(\varphi, \epsilon \varphi_X, \ldots)$ according to the system (61), generated by the functional $\int q(X) J^\mu(X)dX$, as can be easily seen from (66), has the form:

\[
P^\nu_{\tau_k}(X) = \sum_{k \geq 0} A^\nu_{k}(\varphi, \epsilon \varphi_{XX}, \epsilon^2 \varphi_{XX}, \ldots)\epsilon^k q_{kX}(X). \tag{65}
\]
The evolution of \( J^\nu(X) \) at the points of \( \mathcal{M}' \), characterized by the coordinates \( J(X), \theta^*_0(X) \), will be written in the form:

\[
J^\nu_{\tau\mu}(X) = \epsilon \left( \langle A^\nu_{1\mu}(J(X)) \rangle q_X(X) + \frac{\partial (Q^\nu_{\mu}(J(X)))}{\partial X} q(X) \right) + \\
+ \epsilon^2 f^\nu_{\mu}[J, \theta^*_0, q(X), \epsilon], \ \nu, \mu = 1, \ldots, N,
\]

(66)

where \( f^\nu_{\mu} \) are regular at \( \epsilon \to 0 \) asymptotic series with respect to \( \epsilon^k \), the coefficients of which are local functionals of \( q(X), J(X) \) and \( \theta^*_0(X) \).

Lemma 3.

From the system (66) \((\nu = 1, \ldots, N, \mu - \text{is fixed})\), giving the evolution of \( J(X) \) according to the system (61), at the points of \( \mathcal{M}' \) follow the relations:

\[
k^\alpha_{\tau\mu}(J(X)) = \epsilon \left( q(X) \omega^\alpha_{\mu}(J(X)) \right)_X + \epsilon^2 \tilde{k}^\alpha_{\mu}[J, \theta^*_0, q(X), \epsilon],
\]

(67)

- where \( \omega^\nu = (\omega^1, \ldots, \omega^m) \) are introduced in (12) functions, \( \tilde{k}^\alpha_{\mu} \) are regular at \( \epsilon \to 0 \) functionals of \( J, \theta^*_0 \) and \( q(X) \) (linear with respect to \( q(X) \)).

Proof.

System (61) admits the construction of the family of solutions similar to the described in Whitham’s method ones, that is the family of solutions of type:

\[
\varphi = \varphi \left( \frac{\sigma(X, T)}{\epsilon} + \theta, X, T, \epsilon \right) = \sum_{k \geq 0} \varphi_k \left( \frac{\sigma(X, T)}{\epsilon} + \theta, X, T \right) \epsilon^k,
\]

(68)

\( T = \epsilon \tau^\mu \).

By the substitution of (68) into (61) we obtain in the zero order that \( \varphi_0(\theta, X, T) \) at each \( T \) is the function from the \( \mathcal{M}' \), since it satisfies to one of the systems (12) (we assume here that \( I^\mu \) is not the momentum operator or annihilator of bracket (3), for which the statement of Lemma is evident), besides that:

\[
\sigma^\alpha_T = q(X) \omega^\alpha_{\mu}(U(X)), \ \sigma^\alpha_X = k^\alpha(U(X)).
\]

(69)

We assume, as it was declared previously, that the systems (11), generated by the functionals \( I^\mu \) (except the annihilators and momentum operator), and corresponding to them submanifolds \( \mathcal{M}_{\omega_{\mu,k}} \) satisfy to the nondegeneracy conditions (A) and (B), which permit to construct the asymptotic series (68). The conditions of the solvability of system on \( \varphi_1(\theta, X, T) \), that is \( N_m \)
equations of orthogonality of the discrepancy $f_1$ to the left eigen vectors of
the corresponding operator $\hat{L}_{[U(X), \theta_0(X)]}$, together with the equations

$$k(U)_T = (q(X)\omega_\mu(U))_X,$$  \hspace{1cm} (70)

following from (69), give, as previously, the system of equations on $U(X)$
and $\theta_0(X)$.

It can be easily seen (view (69)), that the system on $\Phi_1(\theta, X, T)$ has
now the form:

$$q(X) \left( \hat{L}^\mu_{[U(X), \theta_0(X)]} \Phi_1 \right) (\theta, X, T) = q(X)\sum_n \alpha^{i(\mu)}_n(\sigma_X, \Phi(0), \Phi(0)\theta^\alpha, \ldots)\times$$

$$\times \left( \Phi(0)_{n\theta, U^\nu} U^\nu_X + \Phi(0)_{n\theta, X} \theta^\alpha_0 \right) + \beta^{i(\mu)}_\alpha(\sigma_X, \Phi(0), \Phi(0)\theta^\alpha, \ldots)\sigma_X + q_X(X) \tilde{Q}^i(\Phi(0), \sigma_X \Phi(0)\theta^\alpha, \ldots) - \left( \Phi(0)_{0, U^\nu} U^\nu_T + \Phi(0)_{0, \theta^\alpha} \theta^\alpha_T \right),$$

all the notations are the same that the notations in (26), the values $\sigma_T$ and
$\sigma_X$ are connected with the parameters $U(X)$, corresponding to $\Phi(0)(\theta, X, T)$,
by the relations (69), the functions $\alpha^{i(\mu)}_n$ and $\beta^{i(\mu)}_\alpha$ are the same that in
Whitham’s method for the evolution system, generated by the operator $I^\nu$.
The last property permits to conclude that, similar to Whitham’s situation,
the conditions of orthogonality of discrepancy $f_1$ to the left eigen vectors of
$\hat{L}$ do not impose any restriction on the parameters $\theta_0(X, T)$ and have a form:

$$q(X) A^\mu_\nu(U) U^\nu_X + B^\nu_\mu(U) U^\nu_T + q_X(X) C^\zeta(U) = 0, \ \zeta = 1, \ldots, N - m. \hspace{1cm} (71)$$

Together with the equations (70), the system (71) gives $N$ equations on the functions $U(X)$.

At given initial dates the system (70)-(71) has the smooth solution at
$T < T_0$ (up to the moment of destruction of waves), and in this region, as
in Whitham’s method, we can, using the function $\varphi(0)(\theta, X, T)$ from $\mathcal{M}'$
characterized by the values $U(X, T)$ and $\theta_0(X, T)$, construct uniquely the
next terms $\varphi(k)(\theta, X, T)$ of the series (18), which will be the functionals of
$U(X, T)$ and $\theta_0(X, T)$. Taking this into account, and using the definition of
$J(X)$ (23) and the parameters $U(X)$ (14), we can obtain on the family (68)
of solutions of (11) the relations:

$$J^\nu(X, T) = U^\nu(X, T) + \sum_{k \geq 1} \epsilon^k \tilde{J}^\nu_{(k)}[U, \theta_0],$$  \hspace{1cm} (72)

23
- where \( U(X, T) \) and \( \theta_0(X, T) \) are parameters, connected with the function \( \varphi_{(0)}(\theta, X, T) \) from \( M' \), \( J(X) \) are values of the corresponding functionals on the solutions (68), corresponding to the given \( U(X, T) \) and \( \theta_0(X, T) \).

After the substitution of \( \varphi(\theta, X, T) \) in form of (68) into the expression (65) and the integration with respect to \( \theta \) we obtain the expression for the evolution of \( J(X) \) on the solutions (68), which, as can be easily seen, has the form:

\[
J_{\tau\nu}(X) = \epsilon (q_X(X) \langle A^\nu_{\mu} \rangle(U(X)) + q(X) \partial_X \langle Q^\nu_{\mu} \rangle(U(X))) + \epsilon^2 \tilde{f}^\nu_{\mu}[U, \theta_0, \epsilon].
\]

Comparing (73) and (72) we can conclude that the system

\[
U^\nu_T = q_X(X) \langle A^\nu_{\mu} \rangle(U(X)) + q(X) \partial_X \langle Q^\nu_{\mu} \rangle(U(X)), \quad (T = \epsilon \tau^\mu),
\]

is equivalent to (70,71), and so that, the relations (67) follow from (66). The Lemma is proved.

Corollary.

On the functions of family \( M' \), characterized by coordinates \( J(X), \theta^*_0(X) \) at \( \epsilon \to 0 \), take place the following relations:

\[
\{ k^\alpha(X), J^\nu(Y) \} = \epsilon \left( \omega^\alpha_{\nu}(J(X)) \delta'(X - Y) + \frac{\partial \omega^\alpha_{\nu}(J(X))}{\partial X} \delta(X - Y) \right) +
\]

\[
+ \epsilon^2 k^\alpha_{\nu}[J, \theta^*_0, \epsilon],
\]

where \( k^\alpha_{\nu} \) are regular at \( \epsilon \to 0 \) functionals of \( J(X) \) and \( \theta^*_0(X) \).

Lemma 4.

Under the formulated previously assumption that the number of parameters \( U^\nu \) is equal to \( 2m + p \), where \( p \) is the number of the annihilators of the bracket (6), the following conditions take place:

\[
\sum_{\nu=1}^{N} \frac{\partial k^\alpha(U)}{U^\nu} \omega^\beta_{\nu}(U) \equiv 0, \quad \forall \alpha, \beta.
\]

Proof.

As was formulated previously, any of \( m+1 \) flows, generated by \( m+1 \) integrals from the set (8), are linearly dependent on the family of \( m - \) phase solutions.
of system (2), and for some \( \lambda^\nu_s(U) \), \( \nu = 1, \ldots, N, s = 1, \ldots, N - m \) we have on \( \mathcal{M}' \) \( (N - m) \) independent relations:

\[
\sum_{\nu=1}^{N} \lambda^\nu_s(U) \omega^\alpha_\nu(U) = 0, \quad \alpha = 1, \ldots, m,
\]  

(76)

- where \( \omega^\alpha_\nu(U) \) are introduced in (12) frequencies, corresponding to the integrals \( I^\nu \). It follows from this that on \( m - \) phase solutions of (2), characterized by the parameters \( U(X) \), for some functions \( \mu^q_s(U) \) take place the relations:

\[
\sum_{\nu=1}^{N} \lambda^\nu_s(U) \delta I[\varphi] = \sum_{q=1}^{p} \mu^q_s(U, \theta_0) \delta N_q, 
\]  

(77)

where \( N_q \) are annihilators of bracket (3). Let us denote the values of the annihilators \( N_q \) on \( m - \) phase solutions of (2) by \( n^q_s(U) \) (the dependence upon \( \theta_0 \) is absent because of the commutation of \( N_q \) with \( I^\nu \), generating the linear dependence of the initial phases upon the time). The values of the functionals \( I^\nu \) on \( m - \) phase solutions of (2) are the parameters \( U^\nu \), and for infinitely small shift on the \( \mathcal{M} \) along the direction \( \vec{\xi} \) (in the space of parameters on the \( \mathcal{M} \)), tangential to the submanifold \( k(U) = \text{const}, (k = (k^1, \ldots, k^m)) \), we obtain the uniformly bounded variations of functions \( \varphi^i(x) \), so that, view (77), we obtain on \( \mathcal{M} \):

\[
\sum_{\nu=1}^{N} \lambda^\nu_s(U)dU^\nu - \sum_{q=1}^{p} \mu^q_s(U, \theta_0)dn^q_s(U) = 0, 
\]  

(79)

if \( dk^\alpha = 0 \).

From this we can conclude that \( \mu(U, \theta_0) = \mu(U) \) (the dependence upon \( \theta_0 \) is absent), and for some functions \( \tau^\alpha_s(U) \), \( \alpha = 1, \ldots, m, s = 1, \ldots, N - m \), take place the relations:

\[
\sum_{\nu=1}^{N} \lambda^\nu_s(U)dU^\nu = \sum_{q=1}^{p} \mu^q_s(U)dn^q_s(U) + \sum_{\alpha=1}^{m} \tau^\alpha_s(U)dk^\alpha(U). 
\]  

(78)

After the expression of \( N - m = m + p \) differentials \( dn^q_s(U) \) and \( dk^\alpha(U) \) from \( N - m \) equations (78) (we assume that in the generic situation this can be done) we obtain the relations:

\[
dk^\alpha(U) = \sum_{s=1}^{N-m} a^{\alpha s}(U) \sum_{\nu=1}^{N} \lambda^\nu_s(U)dU^\nu, 
\]  

(79)
$$dn_q(U) = \sum_{s=1}^{N-m} b_s^q(U) \sum_{\nu=1}^N \lambda^\nu_q(U) dU^\nu$$  \hspace{1cm} (80)

for some functions $a^{\alpha_s}(U)$ and $b^q_s(U)$. After that the statement of the Lemma follows immediately from (76). Lemma is proved.

It can be easily seen that, under the conditions formulated in Lemma 4, for the functions $n_q(U)$, view (76) and (80), also take place the relations similar to (75), that is:

Lemma 5.

Under the formulated in Lemma 4 conditions: $N = 2m + p$, where $p$ is the number of the annihilators of bracket (3), on the submanifold $\mathcal{M}$, connected with the full family of $m$ - phase solutions of (2), take place the relations:

$$\sum_{\nu=1}^N \frac{\partial n_q(U)}{\partial U^\nu} \omega^\beta_\nu \equiv 0.$$  \hspace{1cm} (81)

Corollary.

On the submanifold $\mathcal{M}'$ with the coordinates $(J(X), \theta^*_0(Y))$ take place the relations:

$$\{ k^\alpha(J(X)), k^\beta(J(Y)) \} = \epsilon^2 \rho^{\alpha\beta}[J, \theta^*_0, \epsilon],$$  \hspace{1cm} (82)

$$\{ k^\alpha(J(X)), n_q(J(Y)) \} = \epsilon^2 v^\alpha_q[J, \theta^*_0, \epsilon],$$  \hspace{1cm} (83)

where $\rho^{\alpha\beta}, v^\alpha_q$ are regular at $\epsilon \rightarrow 0$ functionals of $J(X)$ and $\theta^*_0(X)$.

Indeed, using (74) and (75), we obtain:

$$\{ k^\alpha(J(X)), k^\beta(J(Y)) \} = \epsilon \left( \sum_{\nu=1}^N \omega^\alpha_\nu(J(X)) \frac{\partial k^\beta(J)}{\partial J^\nu}(X) \delta'(X - Y) + \right.$$  

$$+ \left( \omega^\alpha_\nu(J(X)) \frac{\partial k^\beta(J)}{\partial J^\nu}(X) \right)_X \delta(X - Y) \bigg) + \epsilon^2 k^\alpha[J, \theta^*_0, \epsilon] \frac{\partial k^\beta(J)}{\partial J^\nu}(Y) =$$

$$= \epsilon^2 k^\alpha[J, \theta^*_0, \epsilon] \frac{\partial k^\beta(J)}{\partial J^\nu}(Y).$$

The relation (83) is proved by the same way.

Now we shall calculate the brackets $\{ J^\nu(X), \theta^*_0(\alpha(Y)) \}$ and $\{ \theta^*_0(\alpha(X), \theta^*_0(\beta(Y)) \}$ on $\mathcal{M}'$. As was mentioned above, the functionals of type $\int q(X) J^\nu(X) dX$ generate the local flows of form (74). The corresponding evolution of the
densities $A^\nu(\varphi, \varphi_\alpha, \ldots)$ of integrals $\alpha^\nu(X)$, introduced in (19) for the definition of coordinates $U^\nu(X)$ in the vicinity of $\mathcal{M}'$, also have the form similar to (61), that is

$$A^\nu_{\mu}(\theta, X) = q(X)S^\nu_{\mu}(\varphi, \varphi_\alpha, \ldots, \epsilon \varphi_X, \epsilon \varphi_\theta X, \ldots) +$$

$$+ \epsilon q_X S^\nu_{\mu}(\varphi, \theta_\alpha, \ldots, \epsilon \varphi_X, \epsilon \varphi_\theta X, \ldots) + \ldots .$$

After the substitution of functions $\varphi(\theta, X)$ in the form:

$$\varphi(\theta, X) = \Phi_{in}(\theta + \theta_0^*(X) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X'))dX', U[J, \theta_0^*, \epsilon])$$

and integration with respect to $\theta$, we obtain the expression for the evolution of $a^\nu(X)$ on the functions from $\mathcal{M}'$, characterized by the coordinates $J(X)$ and $\theta_0^*(X)$, in the form:

$$a^\nu_{\tau \mu}(X) = \sum_{k \geq 0} \bar{r}^\nu_{\mu(k)}[J, \theta_0^*, q(X)](X)_k.$$  

So that, the analogous expressions will be for the evolution of the functionals $U(X)$ on the $\mathcal{M}'$, since they are expressed in terms of $\alpha(X)$ by the point substitutions. Besides that, since the system (61) generates at zero order of $\epsilon$ on the functions from $\mathcal{M}'$ (having the form (84)) the shift of the initial phases $\theta_0^*$ and does not touch the other parameters, the zero terms do not present in this series. So that, the Poisson brackets between $U^\nu(X)$ and $J^\mu(Y)$ on the $\mathcal{M}'$ will (as functions of coordinates $J(X)$ and $\theta_0^*(X)$) have the form:

$$\{U^\nu(X), J^\mu(Y)\}|_{G(\theta, X)=0} = \sum_{k \geq 1} r^\nu_{\mu(k)}[J, \theta_0^*, \epsilon] \epsilon^k$$

(that is $\{U^\nu(X), J^\mu(Y)\}|_{\mathcal{M}'} = O(\epsilon)$ at $\epsilon \to 0$ in the coordinates $(J(X), \theta_0^*(X))$).

Lemma 6.

At $Y \neq X_0$ we have on the functions from $\mathcal{M}'$, characterized by the coordinates $J(X), \theta_0^*(X)$, the relations:

$$\{\theta_0^{\alpha*}(X), J^\nu(Y)\} = \epsilon \sigma^{\alpha \beta}[J, \theta_0^*, \epsilon],$$

where $\sigma^{\alpha \beta}$ are regular at $\epsilon \to 0$ functionals of $J$ and $\theta_0^*$.

Proof.

As can be easily seen, the functionals $\theta_0^{\alpha*}(X)$ are expressed, according to their
definition (see (56)), by the same (independent upon \(\epsilon\)) functions \(\tau^\alpha(\vartheta^*(X))\) in terms of the functionals \(\vartheta^{*\alpha}(X)\), introduced by the formula:

\[
\vartheta^{*\alpha}(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_i \varphi^i(\theta, X) \times
\]

\[
\times \Phi_{in\theta^\alpha}^i(\theta + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', U[\varphi](X))d^m\theta,
\]

(87)

by which the functionals \(\theta_0^{\alpha}(X)\) are expressed in terms of \(\vartheta_0^*(X)\), introduced in (50). The formulas

\[
\theta_0^{\alpha}(X) = \tau^\alpha(\vartheta^*(X))
\]

(88)
take place in the map on the \(M'\), in which \(\theta^*_{0\alpha}(X)\) takes at all \(\epsilon\) the same (that is independent upon \(\epsilon\)) values, that \(\theta_{0}(X)\) takes in the map corresponding to the functions \(\tau^\alpha\) in the definition of \(\theta_0^{\alpha}(X)\) in terms of \(\vartheta_0^*(X)\).

The evolution of the functionals (87) according to the flows (61), generated by the functionals of type \(\int q(Y')J^\mu(Y')dY'\), is determined by the evolution of \(\varphi(\theta, X)\), given by the system (61), and, besides that, by the evolution of \(J(X)\) and \(U[\varphi](X)\), which presents in \(\Phi_{in}\) and in the coordinates \(J(X)\) and \(\theta^*_{0\alpha}\) on the \(M'\) is given by the formulas (66) and (85). After the substitution of \(\varphi(\theta, X)\) in the form (84) into the system (61) and integration with respect to \(\theta\), removing the singularity at \(\epsilon \to 0\) in the shift of \(\theta\), we can, as in the case with \(U(Y)\), conclude that the Poisson brackets of \(\theta^{*\alpha}(X)\) with \(J^\mu(Y)\) on the \(M'\) are regular (view (88)) at \(\epsilon \to 0\) functionals of \(J(X)\) and \(\theta^*_{0\alpha}(X)\). That is

\[
\{\theta^{*\alpha}(X), J^\mu(Y)\} = \sum_{k \geq 0} \epsilon^k \chi^{\alpha\mu}[J, \theta^*_{0\alpha}].
\]

Let now \(Y \neq X_0\). Consider the functionals of type \(\int q(Y')J^\mu(Y')dY'\) with the functions \(q(Y')\), having the support in the vicinity of \(Y\), such that \(q(X_0) = 0\). Taking into account the statement (57) of Lemma 3, it can be easily seen, that the evolution of the functionals (87) according to the corresponding flows has on the functions from the \(M'\), characterized by the coordinates \(J\) and \(\theta^*_{0\alpha}\), in the zero order of \(\epsilon\) at \(\epsilon \to 0\) the following form:

\[
\vartheta^{*\alpha}_{\tau\nu}(X)|_{\epsilon \to 0} = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} [q(X)Q_\mu(\Phi_{in}, k^\beta(J)\Phi_{in\theta^\beta}, \ldots)\Phi_{in\theta^\alpha} +
\]

\[
+ \Phi_{in} \int_{X_0}^X (\omega^\beta_{\mu}(J(X'))q(X'))_{X'}dX' \times \Phi_{in\theta^\alpha\theta^\beta}\]d^m\theta =
\]

28
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} q(X) \omega_\mu^\beta(J(X)) \left[ \Phi_{in\theta\beta} \Phi_{in\theta\alpha} + \Phi_{in} \Phi_{in\theta\alpha \beta} \right] d^n\theta = 0
\]

(after the substitution of \(\varphi(\theta, X)\) in form of (84) we use the relation (12) for the functions \(Q^\mu_\alpha\)). The statement of the Lemma immediately follows from this.

For the \(J(X_0)\) (at the point \(X_0\) the depending upon \(\epsilon\) initial phase shift \(\theta_0\) is absent) we can obtain by the similar way on the \(M'\) (in the coordinates \(J(X), \theta_0^*(X)\)):

\[
\{\theta^{\alpha}(X), J^\mu(X_0)\} = \omega_\mu^\alpha(J(X_0)) \delta(X - X_0) + O(\epsilon), \tag{89}
\]

(since at the zero order of \(\epsilon\) the functionals \(\int q(X)J^\mu(X) dX\) generate the linear dependence of the initial phases \(\theta_0(X)\) upon the time with the frequencies \(q(X)\omega_\mu(X)\) and at the point \(X_0\): \(\theta_0^*(X_0) \equiv \theta_0(X_0)\)). For the functionals \(k^\alpha(J(X))\), view (75), we have in the coordinates \(J(X), \theta_0^*(X)\) on the \(M'\) at all \(Y\), including \(X_0\), the relations:

\[
\{\theta_0^*(X), k^\alpha(J(Y))\}|_{M'} = O(\epsilon), \quad \epsilon \to 0. \tag{90}
\]

For the functionals \(\vartheta^{\alpha}(X)\), introduced in (87), we obtain also, using (82) and (90), that their Poisson brackets with each other are regular on \(M'\) in the coordinates \(J(X), \theta_0^*(X)\) at \(\epsilon \to 0\), and, so that, the same property is valid for the brackets of type: \(\{\theta_0^{\alpha}(X), \theta_0^{\beta}(Y)\}\), that is

\[
\{\theta_0^{\alpha}(X), \theta_0^{\beta}(Y)\}|_{M'} = \gamma^{\alpha\beta}[J, \theta_0^*, \epsilon](X, Y), \tag{91}
\]

where \(\gamma^{\alpha\beta}(X, Y)\) are regular at \(\epsilon \to 0\) functional of \(J, \theta_0^*\). We shall not need for the more precise information about the brackets of this kind.

Let now consider the Dirac procedure of restriction of the bracket (45) on the submanifold \(M'\), using the coordinates \(J(X), \theta_0^*(X)\) and \(G^i(\theta, X)\) in \(\Delta_\delta\).

For the Dirac restriction of the bracket (45) on \(M'\) we must find for the functionals \(J^\nu(X)\) and \(\theta_0^{\nu\alpha}(X)\) the additions of type:

\[
V^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} v^\nu_i(X, Y, \theta, \epsilon) G^i(\theta, Y) d^m\theta dY, \tag{92}
\]

\[
W^\alpha(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \ldots \int_0^{2\pi} w^\alpha_i(X, Y, \theta, \epsilon) G^i(\theta, Y) d^m\theta dY, \tag{93}
\]
such that the flows, generated by the functionals $\tilde{\mathcal{J}}^\nu(X) = J^\nu(X) + V^\nu(X)$ and $\tilde{\theta}_0^\alpha(X) = \theta_0^\alpha(X) + W^\alpha(X)$, leave $\mathcal{M}'$ invariant. After that we must put on $\mathcal{M}'$:

$$\{J^\nu(X), J^\mu(Y)\}^* = \{\tilde{J}^\nu(X), \tilde{J}^\mu(Y)\}|_{\mathcal{M}'}(\mathbf{J}, \theta_0^\alpha),$$

$$\{\theta_0^\alpha(X), \theta_0^\beta(Y)\}^* = \{\tilde{\theta}_0^\alpha(X), \tilde{\theta}_0^\beta(Y)\}|_{\mathcal{M}'}(\mathbf{J}, \theta_0^\alpha),$$

$$\{J^\nu(X), \theta_0^\alpha(Y)\}^* = \{\tilde{J}^\nu(X), \tilde{\theta}_0^\alpha(Y)\}|_{\mathcal{M}'}(\mathbf{J}, \theta_0^\alpha).$$

On the functions $v_i^j(X, Y, \theta, \epsilon)$ and $w_i^\alpha(X, Y, \theta, \epsilon)$ we obtain, respectively, the relations:

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \int v_j^i(Y, Z, \theta', \epsilon) \times$$

$$\times \{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha(Y)\} d^m\theta' dZ|_{\mathcal{M}'} =$$

$$= -\{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), J^\nu[\varphi](Y)\}|_{\mathcal{M}'}$$

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \int w_j^\alpha(Y, Z, \theta', \epsilon) \times$$

$$\times \{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha(Y)\} d^m\theta' dZ|_{\mathcal{M}'} =$$

$$= -\{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha[\varphi](Y)\}|_{\mathcal{M}'}$$

In the calculation of the Poisson bracket of constraints $G^i(\theta, X)$ with any functional on $\mathcal{M}'$ we can use the property, that on $\mathcal{M}'$ the values standing in the brackets in the definition of constraints $G^i(\theta, X)$ in terms of $\varphi(X)$ (see (53)) become zeros and, so that, in the calculation of the brackets of type $\{G^i(\theta, X), C(\theta', Z)\}$ on $\mathcal{M}'$ we can omit the brackets of the kernels of the operators $\tilde{\mathbf{L}}$, presenting in (53), with $C(\theta', Z)$ and replace the kernels of $\tilde{\mathbf{L}}$ from the Poisson brackets in the form of multipliers according to the Leibnitz identity.

Regarding also the relations (90), (83), and (64), (86), (89), (91) for the functionals $\mathbf{J}$ and $\theta_0^\alpha$, presenting in (53), we can see that the Poisson brackets of type $\{G^i(\theta, X), C^j(\theta', Z)\}$, $\{G^i(\theta, X), J^\nu(Y)\}$ and $\{G^i(\theta, X), \theta_0^\alpha(Y)\}$ can on the submanifold $\mathcal{M}'$ with the coordinates $\mathbf{J}(X), \theta_0^\alpha(X)$ be represented in the most general form at $\epsilon \to 0$ by the asymptotic series of type:

$$\{G^i(\theta, X), G^j(\theta', Z)\}|_{\mathcal{M}'} =$$

$$= \frac{1}{(2\pi)^{2m}} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}^i_{s[\mathbf{J}, \theta_0^\alpha, \epsilon]} \left( \theta + \theta_0^\alpha(X) + \frac{s(X)}{\epsilon}, \tau + \theta_0^\alpha(X) + \frac{s(X)}{\epsilon}, X \right) \times$$

$$\times \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \int v_j^i(Y, Z, \theta', \epsilon) \times$$

$$\times \{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha(Y)\} d^m\theta' dZ|_{\mathcal{M}'}$$

$$= -\{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), J^\nu[\varphi](Y)\}|_{\mathcal{M}'}$$

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \int w_j^\alpha(Y, Z, \theta', \epsilon) \times$$

$$\times \{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha(Y)\} d^m\theta' dZ|_{\mathcal{M}'} =$$

$$= -\{G^i(U[\varphi](X), \varphi(\theta, X), \ldots), \theta_0^\alpha[\varphi](Y)\}|_{\mathcal{M}'}$$
\[ \times \left( \sum_{n,k \geq p \geq 0} M_{(k),(p),n}^{\text{shift}}(\tau + \theta_0^*(X)) + \frac{s(X)}{\epsilon}, [J, \theta_0^*] \right) e^{k\delta(p)(X - Z)}\delta_{\text{shift}}(\tau - \zeta) \times \]

\[ \times \tilde{L}_{i[U,J,\theta_0^*,\epsilon]}^i \left( \theta^* + \theta_0^*(Z) + \frac{s(Z)}{\epsilon}, \zeta + \theta_0^*(Z) + \frac{s(Z)}{\epsilon}, Z \right) d^m\tau d^m\zeta, \quad (99) \]

\(s(X)\) and \(s(Z)\) denote the integrals \(\int_X k(J(X'))dX'\) and \(\int_Z k(J(X'))dX'\), and also:

\[ \{G^i(\theta, X), J^\nu(Y)\}|_{M'} = \]

\[= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}_{i[U,J,\theta_0^*,\epsilon]} \left( \theta^* + \theta_0^*(X) + \frac{s(X)}{\epsilon}, \tau + \theta_0^*(X) + \frac{s(X)}{\epsilon}, X \right) \times \]

\[\times \left( \sum_{k \geq p \geq 0} S_{(k),(p)}^{sa}(\tau + \theta_0^*(X)) + \frac{s(X)}{\epsilon}, [J, \theta_0^*] \right) e^{k\delta(p)(X - Y)} \right) d^m\tau, \quad (100) \]

\[ \{G^i(\theta, X), \theta^*\alpha(Y)\}|_{M'} = \]

\[= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}_{i[U,J,\theta_0^*,\epsilon]} \left( \theta^* + \theta_0^*(X) + \frac{s(X)}{\epsilon}, \tau + \theta_0^*(X) + \frac{s(X)}{\epsilon}, X \right) \times \]

\[\times \left( \sum_{k \geq p \geq 0} T_{(k),(p)}^{sa}(\tau + \theta_0^*(X)) + \frac{s(X)}{\epsilon}, [J, \theta_0^*] \right) e^{k\delta(p)(X - Y)} \right) d^m\tau. \quad (101) \]

(All the functions in the sums within brackets, depending upon \(Z\) and \(Y\), are replaced by the functions, depending upon \(X\), according to the formulas of type: \(\delta'(X - Z) f(Z) = \delta f(X) \delta'(X - Z) + \delta f_X(X) \delta(X - Z)\). As can be easily seen, all differentiations with respect to \(X\) appear with the multiplier \(\epsilon\).)

The functions

\[ \tilde{\kappa}^q_{[U,J,\theta_0^*,\epsilon]}(\theta + \theta_0^*(X)) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', X) \quad (102) \]

and

\[ \tilde{\kappa}^q_{[U,J,\theta_0^*,\epsilon]}(\theta + \theta_0^*(Z)) + \frac{1}{\epsilon} \int_{X_0}^Z k(J(X'))dX', Z) \quad (103) \]

are respectively left and right eigen vectors on the \(M'\) of the linear operator in the space of \(2\pi\)-periodic with respect to \(\theta\) functions with the kernel \(\{G^i(\theta, X), G^j(\theta', Z)\}\) and correspond to zero eigen values. The functions \(\kappa_{[U,J,\theta_0^*,\epsilon]}(\theta + \theta_0^*(X)) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', X)\), as can be easily seen from
and (101), are also orthogonal (at all \(\epsilon\)) to the right parts of (97), (98), and, so that, the systems (97), (98) are resolvable in the generic case. The solutions \(v^\nu_j(X,Y,\theta)\) and \(w^\alpha_j(X,Y,\theta)\) are defined modulo the arbitrary linear combination of vectors \(\kappa_{[U,J,\theta_0,\epsilon]]}(\theta + \theta_0^*(Y) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(X'))dX',Y)\), but it does not influence, view (54), neither on the form of the additions \(V^\nu\) and \(W^\alpha\), nor on the Dirac restriction of the bracket (45) on \(\mathcal{M}'\) according to the formulas (94)-(96), since in the calculation of the brackets of \(V^\nu\) and \(W^\alpha\) with any functionals on \(\mathcal{M}'\) the contraction of this linear combination with the kernel of corresponding \(L^i_{[U,J,\theta_0,\epsilon]]}(\ldots)\) will be replaced (according to the said above) from the Poisson bracket as a multiplier which is equal to zero on \(\mathcal{M}'\).

For the unique determination of the functions:

\[
v^\nu_j(X,Y,\theta,\epsilon) = \bar{v}^\nu_j(X,Y,\theta + \theta_0^*(Y)) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(X'))dX',\epsilon)
\]

and

\[
w^\alpha_j(X,Y,\theta,\epsilon) = \bar{w}^\alpha_j(X,Y,\theta + \theta_0^*(Y)) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(X'))dX',\epsilon)
\]

in the coordinates \(J(X)\) and \(\theta_0^*(X)\) on \(\mathcal{M}'\) we put \(N + m\) additional relations on them, demanding from them to be also orthogonal at all \(X\) and \(Y\) to the corresponding functions:

\[
\tilde{\kappa}_{[U,J,\theta_0,\epsilon]]}^q(\theta + \theta_0^*(Y)) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(X'))dX',Y,\epsilon,
\]

that is

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_j \bar{v}^\nu_j(X,Y,\theta,\epsilon)\tilde{\kappa}_{[U,J,\theta_0,\epsilon]]}^q(\theta, Y,\epsilon) d^m\theta = 0,
\]

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_j \bar{w}^\alpha_j(X,Y,\theta,\epsilon)\tilde{\kappa}_{[U,J,\theta_0,\epsilon]]}^q(\theta, Y,\epsilon) d^m\theta = 0,
\]

at all \(X,Y,\epsilon\) on \(\mathcal{M}'\).

After the substitution of values \(v^\nu_j(X,Y,\theta,\epsilon)\) and \(w^\alpha_j(X,Y,\theta,\epsilon)\) in the form (104),(105) into the systems (17), (18), where \(\{G^i(\theta,X), G^j(\theta',Z)\}, \{G^i(\theta,X), J^j(Y)\}\) and \(\{G^i(\theta,X), \theta_0^\alpha(Y)\}\) are taken in the form (99), (100)
and (101) respectively, on the functions $\bar{v}_j^\nu$ and $\bar{w}_j^\alpha$ (after all differentiations with respect to $X$ in (97),(98), appearing in every case with the multiplier $\epsilon$, the singular at $\epsilon \to 0$ phase shift $\theta_0^\alpha(X) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX'$, which presents in all functions, depending upon $\theta$, can be omitted) we obtain the linear nonhomogeneous systems, which can be represented in form of regular at $\epsilon \to 0$ asymptotic series with respect to $\epsilon$. As was mentioned above, in the generic case corresponding to the nonsingularity of the matrix of Poisson brackets of constraints in the finite-dimensional case, these systems are resolvable at all $\epsilon$, and, in the presence of the additional relations (107),(108), the functions $\bar{v}(X,Y,\theta,\epsilon)$ and $\bar{w}(X,Y,\theta,\epsilon)$ can be uniquely determined on the submanifold $\mathcal{M}'$ with the coordinates $J(X),\theta_0^\alpha(X)$ (the systems (97),(98) depend of them as of parameters) in the form of regular at $\epsilon \to 0$ asymptotic series with respect to $\epsilon$, that is

\begin{align}
\bar{v}_j^\nu(X,Y,\theta,[J,\theta_0^\alpha],\epsilon) &= \sum_{k \geq 0} \bar{v}_j^{\nu(k)}(X,Y,\theta,[J,\theta_0^\alpha])\epsilon^k, \\
\bar{w}_j^\alpha(X,Y,\theta,[J,\theta_0^\alpha],\epsilon) &= \sum_{k \geq 0} \bar{w}_j^{\alpha(k)}(X,Y,\theta,[J,\theta_0^\alpha])\epsilon^k.
\end{align}

Remark. 

Very frequently the Poisson brackets (3) for the initial system (2), such as the Gardner - Zakharov - Faddeev bracket:

$$\{\varphi(x), \varphi(y)\} = \delta'(x - y),$$

or Magri - Lenard bracket:

$$\{\varphi(x), \varphi(y)\}_{ML} = -\frac{1}{2}\delta''(x - y) + (\varphi(x) + \varphi(y))\delta'(x - y),$$

have such a form, that the corresponding to them bracket (43) is degenerated at zero order of $\epsilon$ in the coordinates $\varphi(\theta, X)$. This degeneracy arises because of the presence of the derivatives of $\delta -$ functions with respect to $x$ (that is the operators $\partial/\partial x$) or the derivatives with respect to $x$ of functions $\varphi(x)$ in every term of such brackets, which arise in (43) with the multipliers $\epsilon$. However, in the introduced above coordinates $J^\nu(X), \theta_0^\alpha(X)$ and $G^i(\theta, X)$ in the vicinity of $\mathcal{M}'$ the operators $\epsilon \partial/\partial X$, being applied to the functions (73) on the $\mathcal{M}'$, contain the nonvanishing at $\epsilon \to 0$ value of type $k^\alpha(J(X))\frac{\partial}{\partial \theta^\alpha}$, which permits to assume in the generic case (for the generic constraints) that
the systems on the functions $\bar{v}^\nu_j$ and $\bar{w}^\alpha_j$ are nondegenerate in the zero order of $\epsilon$, and to write for the functions $\bar{v}^\nu_j$ and $\bar{w}^\alpha_j$ the regular at $\epsilon \to 0$ asymptotic expansions (109) and (110).

More precisely, the expansions (109) and (110) have the form:

$$\bar{v}^\nu_j(X, Y, \theta, \epsilon) = \sum_{k \geq p \geq 0} \bar{v}^\nu_{j(k), (p)}(X, \theta, [J, \theta^*_0]) \epsilon^k \delta^{(p)}(X - Y), \quad (111)$$

$$\bar{w}^\alpha_j(X, Y, \theta, \epsilon) = \sum_{k \geq p \geq 0} \bar{w}^\alpha_{j(k), (p)}(X, \theta, [J, \theta^*_0]) \epsilon^k \delta^{(p)}(X - Y) \quad (112)$$

(here at every given $k$ $p$ runs the finite number of values $p \leq k$), the values $V^\nu(X)$ and $W^\alpha(X)$, as can be easily seen, do not contain the generalized functions. The substitution of (111) and (112) in the corresponding systems (97), (98) gives in the k-th order of $\epsilon$ and at any given $X$ and $p$ (the coefficients with the derivatives of $\delta-$ functions must be equal to each other similar to the coefficients of powers of $\epsilon$) the linear (differential with respect to $\theta$) nonhomogeneous systems on the functions $\bar{v}^\nu_{j(k), (p)}(X, \theta)$ and $\bar{w}^\alpha_{j(k), (p)}(X, \theta)$, depending of $J$ and $\theta^*_0$ as of the parameters. The right sides of these systems depend (in the linear manner) upon the previous $\bar{v}^{(k')}, (q)$ and $\bar{w}^{(k'), (q)} \quad k' < k$. Under the assumption made above about the unique resolvability of the systems (97), (98) at all $\epsilon$ in the presence of the additional conditions (107) and (108), corresponding to the nonsingularity of the matrix of Poisson brackets of constraints on the $M'$, the system described now will be uniquely resolvable under the additional conditions (107) and (108) in the corresponding order of $\epsilon$, and so that, the series (111), (112) may be constructed by successive approximations.

Besides that, as was mentioned above, the flows (11), generated by the functionals $\int q(X)J^\mu(X) dX$ on the functions of form (84), leave invariant the submanifold $M'$ at zero order of $\epsilon$, generating on it the linear dependence of the initial phases upon the times $\tau^\mu$, and, so that, in the expression (100) and, respectively, in the right side of the system (97) the zero term of $\epsilon$ is absent ($S_{0^\nu}^0 \equiv 0$). From this we may conclude that, under the assumption of the unique resolvability of the systems on the functions $\bar{v}^\nu_j$ at zero order of $\epsilon$ (in the coordinates $J, \theta^*_0$ on $M'$): $\bar{v}^\nu_{j(0), (p)} \equiv 0$, that is for the functions $v^\nu_j$ and $w^\alpha_j$ we may write:

$$v^\nu_j(X, Y, \theta, [J, \theta^*_0], \epsilon) =$$
The more precise information about the functions $v^\nu_j$ and $w^\alpha_j$ will be not necessary for us.

Let us now formulate the main result of this paper.

Theorem 1.

Suppose that for the system (2), Hamiltonian with respect to bracket (3) and having the family of $m$ - phase solutions and the sufficient number of conservation laws (8), take place all the described above properties of generic situation, concerning the functional independence of the parameters $U^\nu$ on this family and the possibility of the expression of parameters $k, \omega$ and $r$ in terms of them, and, besides that, takes place the relation, connecting the number of the annihilators of bracket (3) with the number of introduced above additional parameters $r_1, \ldots, r_g (g = p)$. Then, under the assumption of regularity ((A) and (B)) of the introduced previously submanifolds $M^\nu_\nu, k$ (that is the possibility of constructing for any system (61), generated by the functional $\int q(X)J^\nu(X)dX$, of the asymptotical solutions (68)), and the analogous regularity of $M$ with the nonsingularity of matrix of Poisson brackets of constraints $\{G^i(\theta, X), G^j(\theta', Z)\}$ on $M'$ at zero order of $\epsilon$ in the coordinates $J, \theta^*_0$ (that is the possibility of constructing of functions $v^\nu_j$ and $w^\alpha_j$ in form of the asymptotical series (113),(114):

1) The Dubrovin - Novikov bracket, defined by the formula (38):

$$\{U^\nu(X), U^\mu(Y)\} = \langle A_1^\nu\rangle(U(X))\delta'(X - Y) + \frac{\partial(Q^\nu\mu)}{\partial U^\lambda}U^\lambda U^\mu \delta(X - Y),$$

satisfies to the Jacobi identity.

2) For the bracket (38) take place the following relations:

$$\{k^\alpha(U(X)), k^\beta(U(Y))\} = 0, \quad \{k^\alpha(U(X)), n_q(U(Y))\} = 0, \quad \{k^\alpha(U(X)), U^\mu(Y)\} = \omega^\alpha_\nu(U(X))\delta'(X - Y) + \omega^\alpha_\nu(U(X))\delta(X - Y).$$
Proof.
If the conditions, formulated above, are valid, we can restrict the Poisson bracket (45) on the submanifold \( \mathcal{M}' \) with the coordinates \( J(X), \theta^*_{\alpha}(X) \) according to the described procedure. According to the formula (42), for the restricted on the \( \mathcal{M}' \) bracket we shall have the following formulas:

\[
\{ J^\nu(X), J^\mu(Y) \}^* = \{ J^\nu(X), J^\mu(Y) \}|_{\mathcal{M}'}(J, \theta^*) - \frac{1}{(2\pi)^2} \int_0^{2\pi} \cdots \int_0^{2\pi} \int v_i^\nu(X, \theta, \epsilon) \times \{ G^i(\theta, X), G^j(\theta, \bar{Y}) \} \times \\
v_j^\mu(Y, \theta', \epsilon) dX dY d\theta d\theta',
\]

and the analogous formulas for \( \{ J^\nu(X), \theta^*_{\alpha}(Y) \}^* \) and \( \{ \theta^*_{\alpha}(X), \theta^*_{\beta}(Y) \}^* \) (with the replacing of functions \( v_i^\nu, v_j^\mu \) to \( w_i^\alpha, w_j^\beta \) for the functions \( \theta^*_{\alpha}(X) \) and \( \theta^*_{\beta}(Y) \)). As previously, it can be easily seen that in the integration of functions (113), (114) and (99) the singular at \( \epsilon \to 0 \) phase shift \( \theta^*_{\alpha}(X) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX' \), which presents in all functions, depending upon \( \theta \) and \( \theta' \) (after the integration with respect to \( d\bar{X} \) and \( d\bar{Y} \)), is unessential, and the Poisson bracket \( \{ \ldots, \ldots \}^* \) on \( \mathcal{M}' \) is regular at \( \epsilon \to 0 \) in the coordinates \( J(X), \theta^*_{\alpha}(X) \). Using the relations (64), (86), (91), and (113), (114), we obtain for the bracket \( \{ \ldots, \ldots \}^* \) in the coordinates \( J, \theta^*_{\alpha} \):

\[
\{ J^\nu(X), J^\mu(Y) \}^* = \\
\epsilon \left( \langle A^\nu_{\mu i}(J(X)) \rangle \right) \delta'(X - Y) + \left( \frac{\partial}{\partial X} \langle Q^\mu_{\nu i}(J(X)) \rangle \right) \delta(X - Y) + O(\epsilon^2),
\]

\[
\{ J^\nu(X), \theta^*_{\alpha}(Y) \}^* = O(\epsilon)
\]

(117)

at \( X \neq X_0 \),

\[
\{ \theta^*_{\alpha}(X), \theta^*_{\beta}(Y) \}^* = O(1)
\]

(118)

at \( \epsilon \to 0 \).

From this it can be seen that the Jacobi identities, containing only the functionals \( J \) at the points \( X, Y, Z \), which do not coincide with \( X_0 \), coincide in the first nonvanishing order of \( \epsilon \) (at \( \epsilon^2 \)) with the corresponding Jacobi identities for the bracket (58) on the space of fields \( U^\nu(X) \). Regarding now that the point \( X_0 \) is arbitrary, and the expression (58) does not depend upon
we obtain that the bracket (38) satisfies the Jacobi identity. Its skew-
symmetry is a trivial corollary from the skew-symmetry of the bracket (3).

The relations (115), (116) now follow from (82), (83) and (74) respectively.

Theorem is proved.

Theorem 2.

Suppose now that under the assumptions formulated in Theorem 1 we have
two different sets \( \{ I_1, \ldots, I_N \} \), \( \{ \bar{I}_1, \ldots, \bar{I}_N \} \)
of form (3), (some of the integrals of these two sets may coincide with
each other). Then, the Dubrovin - Novikov brackets obtained with the aid of
these two sets are coincide. That is, if \( U^\nu = \langle P^\nu \rangle, \bar{U}^\nu = \langle \bar{P}^\nu \rangle \),
- are the densities of the integrals of the first and second sets respectively,
- \( \langle \ldots \rangle \) is the averaging on the family of m - phase solutions of (2),
then the transformation of bracket (38), obtained with the aid of the set \( \{ I_1, \ldots, I_N \} \),
to the coordinates \( \bar{U}^\nu(X) \) on the \( \mathcal{M} \), expressed by the point substitutions:

\[
\bar{U}^\nu = \bar{u}^\nu(U)
\]  

in terms of \( U(X) \), gives the Dubrovin - Novikov bracket, obtained with the
aid of the set \( \{ \bar{I}_1, \ldots, \bar{I}_N \} \).

Proof.

In the case of the generic situation under consideration, the Dirac restriction
of the bracket (15) on the submanifold \( \mathcal{M}' \) is uniquely determined, that is,
the brackets (17) - (19), obtained in the coordinates \( (J, \theta^*_0) \) and \( (\bar{J}, \theta^*_0) \)
(corresponding to the first and second sets of integrals respectively) in the
vicinity of \( \mathcal{M}' \), transform into each other under the corresponding transfor-
mations of the coordinates \( (J, \theta^*_0) \) and \( (\bar{J}, \theta^*_0) \) on the \( \mathcal{M}' \). Regarding (54)
and the similar relation for \( \bar{U} \) and \( (\bar{J}, \theta^*_0) \), we can conclude that the transition
from the coordinates \( J(X) \) to \( \bar{J}(X) \) has the form:

\[
\bar{J}^\nu(X) = \bar{u}^\nu(J(X)) + \sum_{k \geq 1} e^k \bar{J}^\nu_{(k)}(J, J_X, \ldots, J_{kX}, \theta^*_{0X}, \ldots, \theta^*_{0kX}),
\]

and, so that, the transition from the brackets (17) to \( \{ \bar{J}^\nu(X), \bar{J}^\mu(Y) \} \)
coincides for such transformation (if \( X, Y \neq X_0 \)) in the first nonvanishing order
of \( \epsilon \) (at zero power of \( \epsilon \)) with the corresponding transformation of Dubrovin
- Novikov bracket under the substitution (20), which proves the Theorem.

The evolution of densities \( P^\nu(x) \) according to the flows (11), generated
by the integrals \( I^\mu \), as can be easily seen from (36), has the form:

\[
P^\nu_{\tau\mu}(\varphi, \varphi_x, \ldots) = \partial_x Q^\nu(\varphi, \varphi_x, \ldots).
\]  

37
The flows, generated by the functionals $\int U^\mu(X) dX$ on the space of fields $U(X)$ with the aid of Dubrovin - Novikov bracket, has the form:

$$U^\nu_{T^\mu} = \partial_X \langle Q^{\nu\mu}(U) \rangle,$$ (122)

and, by such a way, represent (see (30)) the Whitham’s equations for $m$-phase solutions of the Hamiltonian system, generated by the functional $I^\mu$ (if it is not the momentum operator or annihilator of bracket (3), let us remind that all these systems have the common family of $m$-phase solutions).

All these flows commute with each other because of the commutation of functionals $\int J^\mu(X) dX$ with respect to Dubrovin - Novikov bracket, and, besides that, the same is valid for the flows generated by the integrals with respect to $X$ of the averaged densities of all functionals, having the form (3) and commuting with the Hamiltonian and integrals $I^\nu$, since any of these functionals can be included into the set $\{I^\nu\}$ instead of any of presenting there integrals $I^\nu$, and, according to Theorem 2, this will not change the bracket (38). The integrals with respect to $X$ of the averaged density of momentum operator (5) and the annihilators of bracket (3), having the form (3), generate in the bracket (38) the shift with respect to $X$ and zero flows respectively. The integral with respect to $X$ of the averaged density of the Hamiltonian (4) generates Whitham’s equations for $m$-phase solutions of system (2).

In closing, the author expresses his gratitude to S.P.Novikov, who suggested the above problem, for his attention to the work and also to V.L.Alexeev, O.I.Mokhov, M.V.Pavlov and E.V.Ferapontov for fruitful discussions.

The research was supported by Russian Foundation for Fundamental Research, grant 96-01-01623, grant INTAS N 96-0770 and the Landau Scholarship awarded by KFA Forschungszentrum Jülich GmbH.

References

[1] G. Whitham. Linear and Nonlinear Waves, Wiley, New York (1974).

[2] Luke J.C. A perturbation method for nonlinear dispersive wave problems. Proc. Roy. Soc. London Ser. A, 292, No. 1430, 403-412 (1966).

[3] B.A.Dubrovin and S.P.Novikov. “Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov -
Whitham averaging method.” Dokl. Akad. Nauk SSSR, 270, No. 4, 781-785 (1983).

[4] B.A.Dubrovin and S.P.Novikov. ”Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory.” Uspekh Mat.Nauk, 44, No. 6 (270), 29-98 (1989).

[5] B.A.Dubrovin and S.P.Novikov. Hydrodynamics of soliton lattices. Sov. Sci. Rev. C, Math. Phys. 1993, V.9. part 4. P. 1-136.

[6] S.Yu. Dobrokhotov and V.P.Maslov. Finite-Gap Almost Periodic Solutions in the WKB Approximation. Contemporary Problems in Mathematics [in Russian], Vol. 15, Itogi Nauki i Tekhniki, VINITI, Moscow (1980).

[7] S.P.Tsarev. "On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type." Dokl. Akad. Nauk SSSR, 282, No. 3, 534-537 (1985).

[8] E.V.Ferapontov. On integrability of $3 \times 3$ semi-Hamiltonian hydrodynamic type systems $u^i_t = v^j_i(u)u^j_x$ which do not possess Riemann invariants. Physica D 63 (1993) 50-70 North-Holland.

[9] E.V.Ferapontov. On the matrix Hopf equation and integrable Hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants. Physics Letters A 179 (1993) 391-397 North-Holland.

[10] S.P.Novikov and A.Ya.Maltsev. ”The Liouville form of averaged Poisson brackets." Uspekhi Mat. Nauk, 48, No. 1 (289), 155-156 (1993).

[11] O.I.Mokhov. Uspekhi Mat. Nauk. Vol.40, N 5, 257-258 (1985).

[12] O.I.Mokhov. Funk. Analiz i ego pril. Vol. 21, N 3, 53-60 (1987).

[13] N.I.Grinberg. Uspekhi Mat. Nauk. Vol.40, N 4, 217-218 (1985).

[14] A.Ya.Maltsev and M.V.Pavlov. ”On Whitham’s Averaging Method.” Functional Analysis and Its Applications, Vol. 29, No. 1, 7-24 (1995).