ON INJECTIVE HOMOMORPHISMS FOR PURE BRAID GROUPS, AND ASSOCIATED LIE ALGEBRAS

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Abstract. The purpose of this article is to record the center of the Lie algebra obtained from the descending central series of Artin’s pure braid group, a Lie algebra analyzed in work of Kohno [12, 13, 14], and Falk-Randell [9]. The structure of this center gives a Lie algebraic criterion for testing whether a homomorphism out of the classical pure braid group is faithful which is analogous to a criterion used to test whether certain morphisms out of free groups are faithful [3]. However, it is as unclear whether this criterion for faithfulness can be applied to any open cases concerning representations of $P_n$ such as the Gassner representation.

1. Introduction

A classical construction due to Philip Hall dating back to 1933 gave a Lie algebra associated to any discrete group $\pi$ which is obtained from filtration quotients of the descending central series of $\pi$. That Lie algebra has admitted applications to the structure of certain discrete groups such as Burnside groups, as well as applications to problems in topology. The purpose of this article is to record some additional structure for this Lie algebra in case $\pi$ is Artin’s pure braid group $P_n$ as described below.

That is, define the descending central series of a group $\pi$ inductively by $\{\Gamma^k(\pi)\}_{k \geq 1}$ with

1. $\Gamma^1(\pi) = \pi$,

2. $\Gamma^k(\pi)$ is the subgroup generated by commutators $[\cdots [\gamma_1, \gamma_2], \gamma_3, \cdots, \gamma_t]$ for $\gamma_i \in \pi$ with $t \geq k$,

3. $\Gamma^{k+1}(\pi)$ is a normal subgroup of $\Gamma^k(\pi)$,

4. $E^0_k(\pi) = \Gamma^k(\pi)/\Gamma^{k+1}(\pi)$, and

5. $E^*_0(\pi) = \bigoplus_{k \geq 1} \Gamma^k(\pi)/\Gamma^{k+1}(\pi)$.

There is a bilinear homomorphism

$$[-,-] : E^b_0(\pi) \otimes \mathbb{Z} E^q_0(\pi) \to E^{b+q}_0(\pi)$$

induced by the commutator map (not in general a homomorphism) $c : \pi \times \pi \to \pi$. Natural properties of the map $[-,-]$ due to P. Hall, and E. Witt give $E^*_0(\pi)$ the structure of a Lie algebra which was developed much further in work of W. Magnus, M. Lazard, A. I. Kostrikin, E. Zelmanov, T. Kohno [12, 13], M. Falk with R. Randell [9], D. Cohen [5], and others.

One standard notation for the Lie algebra attached to the descending central series is given by $gr_*(\pi)$. The notation $E^*_0(\pi) = gr_*(\pi)$ used below is adapted from the convention in [17] for the associated graded obtained from a decreasing filtration.

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Let $P_n$ denote the pure braid group on $n$ strands with $B_n$ the full braid group. A choice of generators for $P_n$ is $A_{i,j}$, $1 \leq i < j \leq n$, subject to relations given in [16]. Choices of braids which represent the $A_{i,j}$ are given by a full twist of strand $j$ around strand $i$. It is a classical fact using fibrations of Fadell-Neuwirth (8) that the choice of subgroup generated by $A_{i,n}$, $1 \leq i \leq n-1$, denoted $F_{n-1}$, is free, and is the kernel of the homomorphism obtained by “deleting the last strand”.

In the case of the pure braid group, the structure of the Lie algebra $E_0^*(P_n)$ is given in work of [12], [13], and subsequently in [7], [9]: this Lie algebra is generated by elements $B_{i,j}$ given by the classes of $A_{i,j}$ in $E_0^*(P_n)$, with $1 \leq i < j \leq n$. Since $E_0^*(P_n) = H_1(P_n)$ is an abelian group, the sum of all of the $B_{i,j}$ given by

$$\Delta(n) = \sum_{1 \leq i < j \leq n} B_{i,j}$$

is a well-defined element in $E_0^*(P_n)$. A complete set of relations for $E_0^*(P_n)$, the “infinitesimal braid relations”, are listed in section 3 here.

Properties required to state the main result are listed next. Consider the free group $F[S]$ generated by a set $S$ with $L[S]$ the free Lie algebra generated by the set $S$. A classical fact due to P. Hall [10] [18] is that the morphism of Lie algebras $e: L[S] \to E_0^*(F[S])$ which sends an element in $S$ to its equivalence class in $E_0^*(F[S]) = H_1(F[S])$ is an isomorphism of Lie algebras.

Restrict to the subgroup $F_{n-1}$ the free group generated by $A_{i,n}$ for $1 \leq i < n$. Let $L[V_n]$ denote the free Lie algebra generated by $B_{i,n}$ with $1 \leq i \leq n$. Thus there is a morphism of Lie algebras

$$\Theta_n : L[V_n] \to E_0^*(P_n)$$

which sends $B_{i,n}$ to the class of $A_{i,n}$ in $E_0^*(F_{n-1})$. One feature of $E_0^*(P_n)$ is that $\Theta_n$ is an isomorphism onto its image [13] [9]. From now on, $L[V_n]$ is identified with its image in $E_0^*(P_n)$.

Let $L$ denote a Lie algebra with Lie ideal $W$. The centralizer of $W$ in $L$ is defined by the equation

$$C_L(W) = \{x \in L \mid [x, B] = 0, \text{ for all } B \in W\}.$$ 

**Theorem 1.1.** If $n > 2$,

$$C_{E_0^*(P_n)}(L[V_n]) = C_{E_0^*(P_n)}(E_0^*(F_{n-1})) = L[\Delta(n)].$$

**Remark 1.2.** It is quite possible that Theorem 1.1 appears in the earlier work concerning the Lie algebra $E_0^*(P_n)$. The authors are unaware of a reference.

Several direct corollaries are listed next. Recall the classical construction of the adjoint representation

$$Ad : L \to Der^L_{\text{Lie}}(L)$$

of a graded Lie algebra $L$ for which $Der^L_{\text{Lie}}(L)$ denotes the graded Lie algebra of graded derivations of $L$. The map $Ad$ is defined by the equation $Ad(X)(Y) = [X, Y]$ for $X,$ and $Y$ in $L$. Regard $E_0^*(P_n)$ as a graded Lie algebra by the convention that $E_0^*(P_n)$ has degree $2q$. Restriction to the Lie ideal $L[V_n]$ gives an induced morphism of Lie algebras $Ad|_{L[V_n]} : E_0^*(P_n) \to Der^L_{\text{Lie}}(L[V_n])$ defined by $Ad|_{L[V_n]}(X)(Y) = [X, Y]$.

**Corollary 1.3.** The kernel of the adjoint representation $Ad : E_0^*(P_n) \to Der^L_{\text{Lie}}(E_0^*(P_n))$ as well as the kernel of the restriction of the adjoint representation $Ad|_{L[V_n]} : E_0^*(P_n) \to Der^L_{\text{Lie}}(L[V_n])$ is
given by the cyclic group generated by $\Delta(n)$ in $E^*_0(P_n)$. Thus there is a short exact sequence of Lie algebras

$$
0 \longrightarrow L[\Delta(n)] \longrightarrow E^*_0(P_n) \xrightarrow{Ad[L[V_n]]} Image(Ad[L[V_n]]) \longrightarrow 0.
$$

The construction of $E^*_0(\pi)$ is a functor from the category of discrete groups to the category of Lie algebras over $\mathbb{Z}$. An application here, a general method for possibly deciding whether certain homomorphisms are embeddings, arises from the observation that if $\pi$ is residually nilpotent, and a group homomorphism $f$ out of $\pi$ induces a Lie algebra monomorphism on the corresponding Lie algebras, then $f$ is a monomorphism (\cite{E}). This observation is applied to the case of $\pi = P_n$, the pure braid group on $n$ strands.

**Corollary 1.4.** Let $f : P_n \to G$ be a homomorphism. If the morphisms of Lie algebras

$E^*_0(f)|_{L[V_n]} : L[V_n] \to E^*_0(G)$, and $E^*_0(f)|_{L[\Delta(n)]} : L[\Delta(n)] \to E^*_0(G)$

are both monomorphisms, then $f$ is a monomorphism. In addition, the following two statements are equivalent:

1. The map $f : P_n \to G$ is faithful.
2. The maps of Lie algebras

$E^*_0(f)|_{L[V_n]} : L[V_n] \to E^*_0(f(P_n))$ and $E^*_0(f)|_{L[\Delta(n)]} : L[\Delta(n)] \to E^*_0(f(P_n))$

are both monomorphisms where $f(P_n)$ denotes the image of $f$.

The center $Z(n)$ of the braid group $B_n$, isomorphic to the integers with generator

$$(A_{1,2} \cdot (A_{1,3}A_{2,3}) \cdots (A_{1,n}A_{2,n} \cdots A_{n-1,n}))$$

\cite{E, F}, has image in $E^*_0(P_n)$ equal to $L[\Delta(n)]$. Let

$i : Z(n) \times F_{n-1} \to P_n$

denote the natural natural multiplication map. The following is a direct consequence of Corollary 1.4.

**Corollary 1.5.** If the composite

$$
Z(n) \times F_{n-1} \xrightarrow{i} P_n \xrightarrow{f} G
$$

induces a monomorphism of Lie algebras

$$
L[\Delta(n)] \oplus E^*_0(F_{n-1}) \xrightarrow{E^*_0(f \circ i)} E^*_0(G),
$$

then $f$ is a monomorphism.

**Remark 1.6.** Three remarks follow.

- A natural question raised by Corollary 1.5 is as follows. Does the assumption that $f \circ i$ is a monomorphism imply that $f$ is a monomorphism? This conclusion does not appear to follow from the techniques of this paper as the assumption that $f \circ i$ is a monomorphism does not directly imply that $E^*_0(f \circ i)$ is a monomorphism.
• The above Lie algebraic methods for testing whether a homomorphism out of a free group is a monomorphism was used in \([6]\) to describe certain natural free subgroups of \(P_n\) which imply that the \(n\)-th homotopy group of the two-sphere is a natural sub-quotient of \(P_n\). \([6, 1]\).

• It is natural to ask whether the above methods can be applied to various well-known representations such as the Gassner representation, the Burau representation for \(B_4\) \([4]\), or the Lawrence–Krammer representation \((3, 15)\). It is also natural to ask whether there are similar structure theorems for representations of other related discrete groups such as those in \([3]\), or variations which test whether a representation is both faithful as well as discrete.

The authors have attempted to use the above Lie algebraic methods above to test whether the classical Burau representation for \(B_4\) is faithful. Although there is substantial computer-based evidence that Corollary 1.5 is satisfied for the Burau representation of \(B_4\), the authors have been unable to verify this property in general.

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2. On the Lie algebra for the Pure Braid Group

The structure of \(E^\ast_0(P_n)\) is given in \([12, 13, 14]\), \([9]\), and \([7]\). Recall that \(L[S]\) denotes the free Lie algebra generated by a set \(S\). Then \(E^\ast_0(P_n)\) is the quotient of the free Lie algebra generated by \(B_{i,j}\) for \(1 \leq i < j \leq n\) modulo the infinitesimal braid relations (or horizontal 4T relations or Yang-Baxter-Lie relations)

\[
E^\ast_0(P_n) = \frac{L[B_{i,j} \mid 1 \leq i < j \leq n]}{I}
\]

where \(I\) denotes the 2-sided (Lie) ideal generated by the infinitesimal braid relations as listed next:

1. \([B_{i,j}, B_{s,t}] = 0\), if \(\{i, j\} \cap \{s, t\} = \emptyset\).
2. \([B_{i,j}, B_{i,s} + B_{s,t}] = 0\).
3. \([B_{i,j}, B_{i,t} + B_{j,t}] = 0\).
4. It follows from 2, and 3 above that \([B_{j,s}, B_{i,j} + B_{i,s}] = 0\).

In addition, it is convenient to introduce new generators \(B_{j,i}\) for \(i < j\) with the convention that

\[
B_{j,i} = B_{i,j}, \text{ for } i < j.
\]

Consider the abelianization homomorphism

\[
P_n \rightarrow P_n/[P_n, P_n] = E^\ast_0(P_n) = H_1(P_n)
\]

for which the image of \(A_{i,j}\) is denoted \(B_{i,j}\). The first homology group \(H_1(P_n)\) is isomorphic to \(\oplus_{(n-1)n/2}\mathbb{Z}\) with basis given by the \(B_{i,j}\), for \(1 \leq i < j \leq n\).

Furthermore, there is an induced split short exact sequence of Lie algebras

\[
0 \rightarrow E^\ast_0(F_{n-1}) \rightarrow E^\ast_0(P_n) \rightarrow E^\ast_0(P_{n-1}) \rightarrow 0.
\]

Thus for each \(i > 0\), there is a split short exact sequence of abelian groups

\[
0 \rightarrow E^b_0(F_{n-1}) \rightarrow E^b_0(P_n) \rightarrow E^b_0(P_{n-1}) \rightarrow 0,
\]

and \(E^b_0(P_n)\) is isomorphic, as an abelian group, to \(\oplus_{1 \leq j \leq n-1} E^b_0(F_j)\).
The structure of the Lie algebra $E_0^*(P_n)$ is given in more detail next via \cite{12 13 14 10}. Let $L[V_q]$ denote the free Lie algebra (over $\mathbb{Z}$) generated by the set $V_q$ with

$$V_q = \{B_{1,q}, B_{2,q}, \cdots, B_{q-1,q}\}, \quad \text{for } 2 \leq q \leq n.$$ 

Furthermore, there are morphisms of Lie algebras

$$\Theta_q : L[V_q] \to E_0^*(P_n) \quad \text{given by } \Theta_q(B_{j,q}) = B_{j,q}$$

for $1 \leq j < q$ such that the additive extension of the $\Theta_q$ to

$$\Theta : L[V_2] \oplus L[V_3] \oplus \cdots \oplus L[V_n] \to E_0^*(P_n)$$

is an isomorphism of graded abelian groups. That is if $a_j(q)$ is an element of $E_0^*(F_{j-1})$ with $E_0^*(F_{j-1}) = L[V_j]$ for $2 \leq j \leq n$ and

$$x(q) = a_2(q) + a_3(q) + \cdots + a_n(q),$$

then

$$\Theta(x(q)) = \Theta_2(a_2(q)) + \Theta_3(a_3(q)) + \cdots + \Theta_n(a_n(q)).$$

The elements $a_j(q)$ will be identified below with the image $\Theta_j(a_j(q))$ unless otherwise noted. The isomorphism of graded abelian groups $\Theta$ is not an isomorphism of Lie algebras, but restricts to a morphism of Lie algebras $\Theta_q : L[V_q] \to E_0^*(P_n)$ for each $q \geq 2$. The infinitesimal braid relations gives the “twisted” underlying Lie algebra structure of $E_0^*(P_n)$.

**Lemma 2.1.** If $i, j, s < n$, then,

$$[B_{i,j}, B_{s,n}] \in L[B_{1,n}, B_{2,n} \cdots B_{n-1,n}] = E_0^*(F_{n-1}).$$

Therefore, for each $X \in E_0^*(P_n)$, $[X, B_{s,n}] \in E_0^*(F_{n-1})$, and $E_0^*(F_{n-1})$ is a Lie ideal of $E_0^*(P_n)$.

**Proof.** This follows immediately from the infinitesimal braid relations. \qed

Centralizers in a free Lie algebra are the subject of the following exercise from Bourbaki (\cite{2}, Exercise 3, Chapter II, section 3).

**Lemma 2.2.** Let $L[S]$ be the free Lie algebra generated by a set $S$, and let $a$ be an element of $S$ with $S$ of cardinality at least 2. Then the centralizer of $a$ in $L[S]$ is the linear span of $a$.

**Proof.** Let $A_S$ denote the free abelian group generated by $S$ with $a \in S$, and $S$ of cardinality at least 2. The universal enveloping algebra of $L[S]$ is the tensor algebra $T[A_S]$ while the standard Lie algebra homomorphism

$$j : L[S] \to T[A_S]$$

is injective by the Poincaré-Birkhoff-Witt Theorem (\cite{2}, and \cite{11}, p. 168). Identify the elements of $L[S]$ with their images in $T[A_S]$. Thus if $x \in L[S]$ centralizes $a$, then $a$ commutes with all $x$ in $T[A_S]$.

Consider an element $x$ of $(A_S)^{\otimes n}$ such that $a \otimes x = x \otimes a$. Notice that $x = a \otimes x'$ for some element $x'$ in $(A_S)^{\otimes n-1}$. Thus by induction on $n$, $x$ is a scalar multiple of $a^{\otimes n}$, and so $x$ is in the subalgebra generated by $a$. The intersection of $L[S]$ with the subalgebra generated by $a$ is precisely the linear span of $a$, thus proving the lemma. \qed

The proof of Theorem \cite{11} is given next.
Proof. There are two parts to this proof. The first part is to show that the non-zero homogeneous elements of degree $q$ in $C_{E_0^i(P_n)}(L[V_n])$ are concentrated in degree $q = 1$. The second part of the proof is to show that the homogeneous elements of degree 1 in $C_{E_0^i(P_n)}(L[V_n])$ are precisely scalar multiples of $\Delta(n)$.

As described above, a restatement of results of Kohno [12] [13] and Falk-Randell [9] is that there is a splitting of $E_0^i(P_n)$ as an abelian group, for each $i > 0$:

$$E_0^i(P_n) = E_0^1(L[V_2]) \oplus E_0^1(L[V_3]) \oplus \ldots \oplus E_0^1(L[V_n])$$

where, for each $1 < m \leq n$, $V_m$ is the linear span of the set $\{B_{1,m}, B_{2,m}, \ldots, B_{m-1,m}\}$.

Let $x(q)$ denote an element in $E_0^i(P_n)$. Thus $x(q)$ is a linear combination given by $x(q) = a_2(q) + a_3(q) + \cdots + a_n(q)$, $a_j(q) \in E_0^1(L[V_j])$ for which all $a_j(q)$ have the same degree $q$.

Assume that $x(q)$ is in the centralizer of $L[V_n]$. Thus

$$[x(q), \Gamma] = 0, \quad \text{for all } \Gamma \in L[V_n].$$

It will be shown below by downward induction on $j$ that if $q > 1$, then $a_j(q) = 0$.

The first case to be checked is that the “top component” $a_n(q)$ vanishes for $q > 1$. Assume that $q > 1$. Let $B(n) = B_{1,n} + B_{2,n} + \cdots + B_{n-1,n}$. The infinitesimal braid relations

$$[B_{i,j}, B_{s,t}] = 0, \quad \text{if } \{i, j\} \cap \{s, t\} = \emptyset$$

and

$$[B_{i,n} + B_{j,n}, B_{i,j}] = 0$$

imply that, for $j < n$, $[a_j(q), B(n)] = 0$. It follows that

$$[x(q), B(n)] = [a_n(q), B(n)] = 0.$$ 

Thus $a_n(q)$ belongs to the centralizer of the element $B(n)$ and both are in $L[V_n]$, which is a free Lie algebra.

By a direct change of basis, there is an equality

$$L[V_n] = L[B(n), B_{2,n}, \ldots, B_{n-1,n}].$$

In addition, notice that Lemma [22] implies that $a_n(q)$ is a scalar multiple of $B(n)$ contradicting the assumption that $q > 1$, and $n > 2$.

Consider the action of the symmetric group on $n$-letters $\Sigma_n$ on the Lie algebra $E_0^*(P_n)$. This action arises from the classical action of the symmetric group on $P_n$, and thus induces automorphisms of the underlying Lie algebra $E_0^i(P_n)$. Note that this action does not preserve the top free Lie algebra. If $\sigma$ is an element in $\Sigma_n$, then

$$\sigma(B_{i,j}) = B_{\sigma(i),\sigma(j)} = B_{\sigma(j),\sigma(i)}.$$ 

By downward induction, assume that

$$a_{s+1}(q) = a_{s+2}(q) = \cdots = a_n(q) = 0.$$ 

Thus $x(q) = a_2(q) + a_3(q) + \cdots + a_s(q)$ for $s < n$. Then

$$0 = [x(q), B_{s,n}] = [a_s(q), B_{s,n}]$$

as $x(q)$ is assumed to be in the centralizer of $L[V_n]$, and $[a_i(q), B_{s,n}] = 0$ for $i < s$ by the infinitesimal braid relations.
Let $\tau_s$ denote the element in $\Sigma_n$ which interchanges $s$, and $n$ leaving the other points fixed. Regard $\tau_s$ as a Lie algebra automorphism applied to the previous equation to obtain

$$0 = [\tau_s(a_s(q)), \tau_s(B_{s,n})] = [\tau_s(a_s(q)), B_{s,n}].$$

Observe that $\tau_s(a_s(q))$ is an element of $L[V_n]$ as $s < n$, and $\tau_s(a_s(q))$ commutes with $B_{s,n}$ with $q > 1$. Hence $\tau_s(a_s(q)) = 0$ by Lemma 2.2. Thus, $a_s(q) = 0$ as $\tau_s$ is an automorphism of Lie algebras.

The second part of the proof is an inspection of the homogeneous elements of degree 1 in $C_{E_0^\ast(P_n)}(L[V_n])$, and consists of showing that these are precisely scalar multiples of $\Delta(n)$ as is given next. Consider the element $x(1) = a_2(1) + a_3(1) + \cdots + a_n(1)$ in $C_{E_0^\ast(P_n)}(L[V_n])$. Then

$$x(1) = \sum_{1 \leq i < j \leq n} \alpha_{i,j}B_{i,j},$$

where $a_m(1) = \sum_{1 \leq i < m} \alpha_{i,m}B_{i,m}$ for some choice of integers $\alpha_{i,j}$. Furthermore,

$$[x(1), B_{p,n}] = 0$$

for every $1 \leq p < n$ as $x(1)$ is in $C_{E_0^\ast(P_n)}(L[V_n])$. It will be checked below that $\alpha_{i,j} = \alpha_{s,t}$ for all $i < j$, and $s < t$, thus showing that $x(1)$ is a scalar multiple of $\Delta(n)$.

Notice that $[x(1), B_{p,n}]$ is equal to

$$\sum_{i \neq p, n} \alpha_{i,p}[B_{i,p}, B_{p,n}] + \sum_{i \neq p, n} \alpha_{i,n}[B_{i,n}, B_{p,n}] = \sum_{i \neq p, n} (-\alpha_{i,p})[B_{i,n}, B_{p,n}] + \sum_{i \neq p, n} \alpha_{i,n}[B_{i,n}, B_{p,n}]$$

by the infinitesimal braid relations, and the convention that $B_{i,j} = B_{j,i}$ for $i < j$. It follows that $\alpha_{i,n} = \alpha_{i,p}$ as $[B_{i,n}, B_{j,n}]$ for $i < j$ form a basis for the homogeneous elements of degree 2 in $L[V_n]$. A similar computation of $[x(1), B_{p,j}]$ gives $\alpha_{j,n} = \alpha_{p,j}$ for $p < j < n$.

Thus any element $x(1)$ in the centralizer of $L[V_n]$ is a scalar multiple of the element $\Delta(n)$. That $\Delta(n)$ centralizes $E_0^\ast(P_n)$ follows by inspection. Thus the centralizer of $L[V_n]$ is given by $L[\Delta(n)]$, the free Lie algebra generated by a single element $\Delta(n)$, a copy of $\mathbb{Z}$ in degree 1, and Theorem 1.1 follows. $\square$

3. Proof of Corollaries

The proof of Corollary 1.3 which gives the kernel of the adjoint representation follows next.

Proof. By definition, the kernel of $Ad|_{L[V_n]}$ is the centralizer of $L[V_n]$ in $E_0^\ast(P_n)$, $C_{E_0^\ast(P_n)}(L[V_n])$. Since $C_{E_0^\ast(P_n)}(L[V_n]) = C_{E_0^\ast(P_n)}(E_0^\ast(P_{n-1})) = L[\Delta(n)]$ by Theorem 1.1 the corollary follows. $\square$

The next proof is that of 1.4 which states that if $f : P_n \rightarrow G$ is a homomorphism such that the maps of Lie algebras

$$E_0^\ast(f)|_{L[V_n]} : L[V_n] \rightarrow E_0^\ast(G), \text{ and } E_0^\ast(f)|_{L[\Delta(n)]} : L[\Delta(n)] \rightarrow E_0^\ast(G)$$

are both monomorphisms, then $f$ is a monomorphism.

Proof. Since $P_n$ is residually nilpotent, it suffices to show that $E_0^\ast(f)$ is a monomorphism to conclude that $f$ is a monomorphism.

Let $x$ denote an element of least degree $p$ in $E_0^\ast(P_n)$ which is in the kernel of $E_0^\ast(f)$. Thus $E_0^\ast(f)([x,B]) = 0$ for every element $B$ in $E_0^\ast(P_n)$. Assume that $B$ is in $L[V_n]$. Then $[x,B]$ is in
Lemma 2.1. Since \( E_0^* (f) |_{L[V_n]} \) is a monomorphism, \([x, B] = 0\) for every \( B \) in \( L[V_n] \). Thus \( x \) centralizes \( L[V_n] \), and is a multiple of \( \Delta (n) \) by Theorem 1.1. Since \( E_0^* (f) |_{L[\Delta (n)]} \) is a monomorphism by hypothesis, \( x \) must be zero.

The final assertion of 1.4 is the statement that both \( E_0^* (f) |_{L[V_n]} \), and \( E_0^* (f) |_{L[\Delta (n)]} \) are monomorphisms is equivalent to the statement that \( f \) is a monomorphism. This follows directly from the previous step.

\[ \square \]

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