Varieties of CM-type
by
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We will introduce the notion of a variety (or more generally a motive) of CM-type which generalises the well known notion of abelian variety of CM-type. Just as in that particular case it will turn out that the cohomology of the variety is determined by purely combinatorial data; the type of the variety. As applications we will show that the ℓ-adic representations are given by algebraic Hecke characters whose algebraic parts are determined by the type and give a method for computing the discriminant of the Néron-Severi group of super-singular Fermat surfaces.

1. Preliminaries.
To begin with let us recall the following facts from category theory. If \( \mathcal{A} \) is an additive category all of whose idempotents have kernels and \( R \) is a ring, then for a finitely right resp. left projective \( R \)-module \( P \) resp. \( Q \) and a left \( R \)-object \( M \) in \( \mathcal{A} \) we can define objects \( P \otimes_R M \) resp. \( \text{Hom}_R(Q, M) \) of \( \mathcal{A} \) characterised by

\[
\text{Hom}_\mathcal{A}(P \otimes_R M, N) = \text{Hom}_R(P, \text{Hom}_\mathcal{A}(M, N))
\]

resp.

\[
\text{Hom}_\mathcal{A}(N, \text{Hom}_R(Q, M)) = \text{Hom}_R(Q, \text{Hom}_\mathcal{A}(N, M)).
\]

We always have a natural \( R \)-morphism \( ev: P \to \text{Hom}_{\text{End}_R(P)}(\text{Hom}_R(P, M), M) \), the evaluation map, obtained by interpreting an element \( p \in P \) as an \( R \)-morphism \( R \to P \) and using \( M = \text{Hom}_R(R, M) \). If \( R = \bigoplus P_i^{r_i} \) and \( \text{Hom}_R(P_i, P_j) = 0 \) for \( i \neq j \) then for any \( R \)-object \( M \) in \( \mathcal{A} \) we have

\[
M = \bigoplus P_i \otimes_{S_i} \text{Hom}_R(P_i, M),
\]

where \( S_i := \text{End}_R(P_i) \) and the map is defined using the evaluation maps. To see this we first note that \( P_i = \text{Hom}_R(R, P_i) \cong (S_i)^{r_i} \) so that \( P_i \) is \( S_i \)-projective and then the desired equivalence follows by decomposing the two \( R \)-factors of \( M = R \otimes R \text{Hom}_R(R, M) \).

The following setup will be with us during the rest of the paper: We let \( k \) be a perfect field of characteristic \( p \geq 0 \), \( X \) a proper, smooth variety over \( k \) (alternatively a motive) and \( S \) a set of \( k \)-correspondences of \( X \). Furthermore, \( H^*(X, r) \), \( r \) prime, will denote the \( \ell \)-adic cohomology of \( X_k \), where \( k \) is a fixed algebraic closure of \( k \), when \( r \neq p \) and the crystalline cohomology of \( X/k \) when \( r = p \). Recall that when \( r \neq p \) \( H^*(X, r) \) is a graded \( \mathbb{Z}_r \)-algebra, finitely generated as \( \mathbb{Z}_r \)-module, having a continuous action of \( \text{Gal}(k/k) \) and that when \( r = p \) \( H^*(X, r) \) is a graded \( \mathbb{W}(k) \)-algebra, finitely generated as \( \mathbb{W}(k) \)-module having a \( \sigma \)-linear endomorphism \( F \). Here \( \mathbb{W}(k) \) is the ring of Witt vectors of \( k \) and \( \sigma \) sends a Witt vector \( (x_i) \) to \( (x_i^p) \). We let \( L_r \) be an algebraically closed field containing \( \mathbb{Z}_r \) resp. \( \mathbb{W}(k) \). Furthermore we will denote by \( K \) the fraction field of \( \mathbb{W}(k) \).

Finally, if \( p > 0 \) we will have need of the following technical condition. There is a scheme \( T \) of finite type over \( \mathbb{F}_p \), a smooth and proper morphism \( \mathcal{X} \to T \) and a cartesian diagram

\[
\begin{array}{ccc}
X & \to & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec } k & \to & T
\end{array}
\]

such that for every closed point \( t \in T \) the eigenvalues of the Frobenius with respect to \( k(t) \) on \( H^i(X_t, p) \) are algebraic integers all of whose archimedean absolute values are \(|k(t)|^{1/2} \).
This condition is fulfilled when $X$ (possibly over $\bar{k}$) is the image of a smooth and projective variety ([K-M]) and that this is always the case has recently been verified by J. de Jong ([Jo]).

For a field $L$ and a set $R$ let $K(R, L)$ be the Grothendieck group of the category of finite dimensional representations (i.e. maps of $R$ into the set of endomorphisms) of $R$. Then $K(R, L)$ is a functor in $R$ and $L$; contravariant in $R$ and covariant in $L$. If $M(R)$ is the free monoid generated by $R$ then the trace map gives an additive map $\text{Tr} : K(R, L) \to L^{M(R)}$.

Lemma 1.2. Let $L$ be algebraically closed of characteristic 0.

i) $\text{Tr} : K(R, L) \to L^{M(R)}$ is injective.

ii) Let $L' \subseteq L$ be a subfield of $L$ and $N$ a semi-simple $L$-representation of $R$ s.t. for all $r \in M(R)$ $\text{Tr}_N(r) \in L'$. If $L'(R)$ is the free associative $L'$-algebra on $R$ then $I := \ker(L'(R) \to \text{End}_L(N))$ depends only on the function $\text{Tr}_N : M(R) \to L'$ and $L'(R)/I \otimes_{L'} L \to \text{End}_L(N)$ is an injection. In particular, if $L'$ is algebraically closed $N$ is isomorphic to the scalar extension of some $L'$-representation of $R$.

iii) If $L'$ is an algebraically closed field and $L' \to L$ a field homomorphism, then the following diagram

$$
\begin{array}{ccc}
K(R, L') & \to & K(R, L) \\
\downarrow & & \downarrow \\
L'^{M(R)} & \to & L^{M(R)}
\end{array}
$$

is cartesian.

Proof: Let us begin with ii). Note first that as $\text{im}(L(R) \to \text{End}_L(N))$ is semi-simple and that for a finite dimensional semi-simple $L$-algebra $M$ and a faithful finite dimensional $L$-representation $V$, the linear form

$$
M \times M \to L \\
(m, m') \mapsto \text{Tr}_V(mm')
$$

is non-degenerate. Hence $t \in L(R)$ acts as zero on $N$ iff $\text{Tr}_N(rt) = 0$ for all $r \in M(R)$. If $t = \sum_{r \in M(R)} \lambda_r r$ then this is a set of linear conditions on the $\lambda_r$ with coefficients in $L'$ depending only on $\text{Tr}_N : M(R) \to L'$. Furthermore, $L'(R)/I \otimes_{L'} L \to \text{End}_L(N)$ is injective iff whenever there is an $L$-linear relation in $\text{End}_L(N)$ between elements in $L'(R)$ there is also an $L'$-linear relation. This also follows from the fact that the above conditions have $L'$-coefficients. Now iii) follows immediately from i) and ii) whereas i) is well known (cf. [C-R:Thm. 30.12]).

If $L$ is a field of characteristic zero and $L'$ an algebraic closure of $L$, we put $\overline{K}(R, L) := K(R, L') \cap L^{M(R)}$. This is clearly independent of the choice of $L'$ and we have $K(R, L) \subseteq \overline{K}(R, L)$. If $N$ is a representative over $L'$ of an $n \in \overline{K}(R, L)$ then we can construct the $L(R)/I$ of Lemma 1.1, which depends only on $n$. It is a semi-simple $L$-algebra as its scalar extension to $L'$ is, and we will denote it $A_L(n)$. In case $L = Q$ then we put $A(n) := \text{im}(Z(R) \to A_Q(n))$, where $Z(R)$ denotes the free associative algebra on $R$. If $M$ is an over-field of $L$ then we say that $n$ is realisable over $M$ if the induced element in $\overline{K}(R, M)$ belongs to $K(R, M)$. This is equivalent to $N$ being realisable by an $A_L(n) \otimes_{L'} LM$-representation. For any $n \in K(R, L')$ we let $\text{Irr}(n) \subseteq K(R, L')$ be the set of irreducible constituents of $n$. If now $n \in \overline{K}(R, L)$ then $\text{Irr}(n)$ is a finite set stable under the action of $\text{Gal}(Q/Q)$. Under the correspondence of Galois theory, $\text{Irr}(n)$ then corresponds the étale $L$-algebra $Z(A_L(n))$. If we return to the situation at hand we have elements $H^i(X, r) \otimes_{Z, L_r} (\text{resp. } H^i(X, p) \otimes_{W(k)} L_p)$ in $K(S, L_r)$. Let $Q$ be an algebraic closure of $Q$.

Lemma 1.3. Let $K_r$ denote $Q_r$ when $r \neq p$ and $K_r$ when it isn’t. There exists a unique element $[H^i(X)] \in \overline{K}(S, Q)$ whose image in $K(S, K_r)$ coincides with $[H^i(X, r)]$. Furthermore, $A([H_i(X)])$ is finitely generated as $Z$-module and $A \otimes K_r$ equals $A_r/\text{rad}(A_r)$ where

$$
A_r := \text{Im}(Q_r(S) \to \text{End}_Q_r(H^i(X, r)) \quad (\text{resp. } \ldots).
$$

Proof: I first claim that, for every $s \in M(S)$, $\text{Tr}(s, H_i(X, r))$ is a rational number independent of $r$. By standard specialisation arguments we reduce to $k$ being a finite field where it is [K-M:Thm 2] (supplemented by [Gr] for the definition of the cycle map in crystalline cohomology). Note that if $p = 0$, using Chow’s lemma and resolution of singularities we can get a reduction for which our
technical condition is fulfilled. In this case, a transcendental argument can also be used. This already, using (1.2 ii), gives the existence of \([H_i(X)]\) and that \(A \otimes L_r = A_r/rad(A_r)\). Hence \(A \otimes \mathbb{Q}\) is a finite dimensional semi-simple \(\mathbb{Q}\)-algebra. For any \(t \in \mathbb{Z}(S)\) the characteristic polynomial of \(t\) on \(H_i(X,r) \otimes L_r\) is independent of \(r\) and has rational coefficients by [loc. cit.]. As \(t\) stabilises a \(\mathbb{Z}_\ell\)- (resp. \(\mathbb{W}(k)\)-) lattice in \(H_i(X,r) \otimes L_r\), those coefficients are \(r\)-integral for all \(r\) and so integral. By the Cayley-Hamilton theorem the image of \(t\) in \(A \otimes \mathbb{Q}\) is integral over \(\mathbb{Z}\) and so \(A\), being equal to \(\text{im} : \mathbb{Z}(S) \rightarrow A \otimes \mathbb{Q}\), is finitely generated as it is contained in the different ideal of any order containing it.

If still \(L\) is algebraically closed of characteristic 0 and \(R\) and \(T\) are two sets, then we let \(K(R,T,L)\) denote the Grothendieck group of the category of finite dimensional \(L\)-representations of \(R \cup T\) such that every element of \(R\) commutes with every element of \(T\). It is easy to see that every simple object of this category is a tensor product of an irreducible representation of \(R\) and one of \(T\) and that the two factors are well-determined up to isomorphism. Hence \(K(R,T,L) = K(R,L) \otimes K(T,L)\). Furthermore, as \(K(R,L)\) has a canonical base consisting of irreducible representations we get a canonical pairing

\[
K(R,L) \otimes K(R,L) \rightarrow \mathbb{Z}
\]

where the canonical base is orthonormal. Using this we get a mapping

\[
K(R,L) \otimes K(R,T,L) = K(R,L) \otimes K(R,L) \otimes K(T,L) \rightarrow K(T,L)
\]

and so for each \(N \in K(R,T,L)\) a mapping

\[
(1.4) \quad N \cap : K(R,L) \rightarrow K(T,L).
\]

This mapping is compatible, in the obvious way, with the mappings obtained from homomorphisms \(L \rightarrow L'\) of algebraically closed fields. Hence we get

**Corollary 1.5.** Let \(S'\) be a set of \(k\)-correspondences of \(X\) and suppose that every element of \(S\) commutes up to homological equivalence with \(S'\). Then (1.4) gives a \(Gal(\mathbb{Q}/\mathbb{Q})\)-equivariant homomorphism

\[
(1.6) \quad [H^i(X)] \cap : K(S, \bar{\mathbb{Q}}) \rightarrow K(S', \bar{\mathbb{Q}})
\]

which equals, for each \(r\), the restriction of \([H^i(X,r) \otimes L_r] \cap\) to \(K(S,\bar{\mathbb{Q}})\).

**PROOF:**

2. Varieties of CM-type.

Let us fix an \(n \in \mathbb{N}\) and assume, for simplicity, that

\[
b_n(X) = \sum_{i+j=n} \dim_k H^i(X, \Omega^j_{X/k}),
\]

where \(b_n(X) := \dim_{\mathbb{Q}_p} H^n(X,r)\) for any \(r\). (This is of course always true when \(p = 0\).) If \(p > 0\) this implies (cf. [Ek: IV, 1.2] or [B-Og: 8]) that \(H^n_{DR}(X/k) = H^n(X,p)/pH^n(X,p)\) and that if

\[
M^i := \text{im}(F^{-1}^{-p}H^n(X,p) \rightarrow H^n(X,p)/pH^n(X,p))
\]

then

\[
M^i/M^{i+1} = H^{n-i}(X, \Omega^i_{X/k}).
\]

Hence, no matter the value of \(p\), \(H^n_{DR}(X/k)\) has a Hodge filtration with the \(H^{n-i}(X, \Omega^i_{X/k})\) as successive quotients.
Definition 2.1. $(X, S)$ is said to be of separable CM-type in degree $n$ if the $A_i$ of (1.3) has the property that $A^\otimes \mathbb{Z}_p$ is a separable (cf. [D-I,IL1]) $\mathbb{Z}_p$-algebra and for every $0 \leq i \neq j \leq n$, $H^{n-i}(X, \Omega^i_{X/k})$ and $H^{n-j}(X, \Omega^j_{X/k})$ are disjoint $S$-modules (i.e. they have no common composition factors).

Remark: As the quotient of a separable algebra is separable it suffices to verify that $\mathbb{Z}_p\langle S \rangle$ factored by some known relations is separable.

Example: i) If $X$ is an abelian variety of CM-type in the usual sense and $End_k(X)$ is separable at $p$, which is always true if $p = 0$, then $(X, End_k(X))$ is of CM-type in degree 1 (and in fact in all other degrees).

ii) Kummer surfaces associated to abelian surfaces of CM-type are of CM-type in degree 2. Hence, by [S-I], K3-surfaces in characteristic 0 for which the rank of the Néron-Severi group is 20 are of CM-type in degree 2.

iii) (Fermat hyper-surfaces, diagonal automorphisms). This is well known (cf. e.g. [Ka:Sect. 6]).

Lemma 2.2. Suppose $(X, S)$ is of separable CM-type. If $p = 0$ then $[H^n(X)]$ is realisable over $\mathbb{Q}$ and if $p > 0$ then $A^\otimes \mathbb{Z}_p$ is unramified (i.e. a product of matrix algebras over unramified extensions of $\mathbb{Z}_p$) and in particular $[H^n(X)]$ is realisable over $\mathbb{Q}_p$.

Proof: The case $p = 0$ follows by transcendental methods, in fact $[H_n(X)]$ is realised by singular cohomology, and the $p > 0$ is well-known (use the fact that the Brauer group of a finite field is trivial and lift an idempotent).

Remark: Is it possible to give an algebraic proof of the first part of the lemma? The existence of $\ell$-adic cohomology implies that it suffices to prove realisability over $\mathbb{R}$.

Suppose now that $(X, S)$ is of separable CM-type and let $M$ be an irreducible component of $\{H_n(X)\}$. If $p = 0$ there is then an irreducible $S$-module $N$ such that $M \otimes \mathbb{Q} \overline{k}$ is a factor of $M \otimes \mathbb{Q} \overline{k}$ for an embedding of $\mathbb{Q}$ in $\overline{k}$ and $N$ is a sub-quotient of $H^*_D(X/k)$. (Note that the base extension of $[H^n(X)]$ to $k$ equals the extension of $[H^n_D(X/k)]$ in $K(S,k)$ to $\overline{k}$, which is seen by either using a constructibility argument to reduce to (1.3) or a transcendental argument.) By assumption there is a unique $i$, $0 \leq i \leq n$, such that $N$ occurs as a sub-quotient of $H^{n-i}(X, \Omega^i_{X/k})$ and this $i$ depends only on $M$. If $p > 0$ we get in the same way an irreducible $K(S)$-module $\tau(N, K := W(k) \otimes \mathbb{Q}$, such that $M \otimes \mathbb{Q} \overline{k}$ occurs in $N \otimes \mathbb{Q} \overline{k} \overline{K}$ for an algebraic closure $\overline{k}$ of $K$ and an embedding of $\mathbb{Q}$ in $\overline{k}$ and $N$ occurs in $H^{n}(X,p) \otimes W(k)$ $K$. As $A \otimes W(k)$ is separable, there is a unique, up to isomorphism, $A \otimes W(k)$-lattice $N'$ with $N' \otimes \mathbb{Q} \overline{k} = N$ (this follows from (2.2)) and $N' \otimes \mathbb{Q} \overline{k}$ is an irreducible $A \otimes \mathbb{Q}$-module and we see that $N' \otimes \mathbb{Q} \overline{k}$ is an irreducible sub-quotient of $H^{n}_D(X/k)$. By assumption there is then a unique $i$, $0 \leq i \leq n$, such that $N \otimes \mathbb{Q} \overline{k}$ occurs in a $H^{n-i}(X, \Omega^i_{X/k})$ and this $i$ depends only on $M$. In both cases we put $\tau(M) := i$.

In conclusion we have obtained a mapping

$$\tau : \text{Irr}([H^n(X)]) \to n + 1 := \{0, 1, \ldots, n\}.$$  

Note that the action of $Gal(\overline{k}/k)$ on $\text{Irr}([H^n(X)])$ obtained through the action of $Gal(\mathbb{Q}/\mathbb{Q})$ on it and the induced map $Gal(\overline{k}/k) \to Gal(\mathbb{Q}/\mathbb{Q})$ (resp. $Gal(\overline{k}/k) \to Gal(\mathbb{W}(\overline{k})/\mathbb{W}(k))$) preserves the fibers of $\tau$ by construction.

Definition 2.3. Under the assumption of (2.1) the type of $(X, S)$ is the pair $([H^n(X)], \tau)$.

Finally, when $p > 0$ the condition that $A$ is separable at $p$ implies that the action of $Gal(\mathbb{Q}/\mathbb{Q})$ on $\text{Irr}([H^n(X)])$ is unramified at $p$ so that we may unambiguously speak about the action of the Frobenius morphism on $\text{Irr}([H^n(X)])$ having once and for all chosen an embedding of $\mathbb{Q}$ in $\mathbb{L}_p$. This permutation of $\text{Irr}([H^n(X)])$ we will denote $\sigma$.

We have now come to the main result of the present paper. Before we formulate it we will need to introduce some constructions. Let $T$ be a set, $M \subseteq \overline{K}(T, Q)$ and $\tau : \text{Irr}(M) \to n + 1$ a function.

Let us choose an embedding of $\mathbb{Q}$ into $\mathbb{C}$ and let $i \in Gal(\mathbb{Q}/\mathbb{Q})$ be the element corresponding through this embedding to complex conjugation. Suppose that for every $\rho \in \text{Irr}(M)$, $\tau(i(\rho)) = n - \tau(\rho)$ and also that $M$ is realisable over $\mathbb{Q}$ by a module $V$. For each simple factor $A_r$ of $A_{\mathbb{Q}}$
we let $V_r$ be an irreducible $A_r$-module. We then put a rational Hodge structure on $V_r$, of weight $n$, as follows: For $\rho \in \text{Irr}(M) \cap \text{Irr}(V_r \otimes \mathbb{C})$ we let $V_{r,\rho}$ be the $\rho$-isotypical component of $V_r \otimes \mathbb{C}$ and then we put $(V_r \otimes \mathbb{C})^{\dagger, n-i} := \sum_{\tau(\rho) = i} V_{r,\rho}$. We then put a Hodge structure on $V_r$ by forcing $V = \bigoplus V_i \otimes_{\text{End}(V_i)} \text{Hom}(V_i, V)$ to be an isomorphism of Hodge structures. By construction $T$ acts as morphisms of Hodge structures. However, the Hodge structure itself depends only on the action of $Z(A_{\mathbb{Q}}(M))$ on $V$ so that an alternative method of construction is to start with the set $\text{Irr}(M)$ with its action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, let $K$ be the associated étale $\mathbb{Q}$-algebra, let $V$ be the $K$-module of dimension specified by $M$ and then let $(V \otimes \mathbb{C})^{\dagger, n-i} := \sum_{\tau(\rho) = i} V_{r,\rho}$, where $\rho$ runs over the $\mathbb{Q}$-algebra homomorphisms $K \to \mathbb{C}$. In this way it is seen that $V$ as a rational Hodge structure depends only on the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set $\text{Irr}(M)$ and two functions $\tau: \text{Irr}(M) \to n+1$ and $\text{dim}: \text{Irr}(M) \to \mathbb{N}$, where $\text{dim}$ is defined by $\text{dim}(n) = \text{dim}_{Z(A_{\mathbb{Q}}(M))} \mathbb{N}$ for a representative of $N$ ($\mathbb{C}$ can of course be replaced by any algebraically closed field). If $k$ is a subfield of $\mathbb{C}$ such that the action of $\text{Gal}(\overline{k}/k)$ on $\text{Irr}(M)$ preserves the fibers of $\tau$ then for a choice of descent of the Hodge filtration on each $V_r \otimes \mathbb{C}$ to $V_r \otimes k$ for each we get a descent of the Hodge filtration on $V \otimes \mathbb{C}$ again by forcing the isomorphism above to preserve the descent.

If $p$ is a prime such that $A(M) \otimes \mathbb{Z}_{[\rho]}$ is finitely generated as $\mathbb{Z}_{(\rho)}$-module and separable we associate, in a similar way, an $F$-crystal to $(M, \tau)$: Suppose that the action of $\text{Gal}(\overline{k}/k)$ on $\text{Irr}(M)$ preserves the fibers of $\tau$. We know that $M$ is realisable over $\mathbb{Q}_p$ and there is, up to isomorphism, a unique $A(M) \otimes \mathbb{Z}_p$-lattice $V$ such that $V \otimes \mathbb{Q}$ is such a realisation. We also get analogous $V_r$. Further, $V_r \otimes \mathbb{Z}_p, W(k)$ is the sum of its isotypical components $(V_r \otimes \mathbb{Z}_p, W(k))_\rho$.

The $\sigma$-linear isomorphism $1 \otimes \sigma$ takes $(V_r \otimes \mathbb{Z}_p, W(k))_\rho$ to $(V_r \otimes \mathbb{Z}_p, W(k))_{\sigma(\rho)}$ and we define the structure of an $F$-crystal on $V_r \otimes \mathbb{Z}_p, W(k)$ by $F = p^{r(i)}(1 \otimes \sigma): (V_r \otimes \mathbb{Z}_p, W(k))_\rho \to (V_r \otimes \mathbb{Z}_p, W(k))_{\sigma(\rho)}$. The $F$-crystal structure on $V \otimes \mathbb{W}(k)$ is constructed as before. Again $T$ acts by endomorphisms and there is an alternative way of constructing the $F$-crystal if one is prepared to forget the $T$-action. Indeed, consider the set $\text{Irr}(M)$ with its action of $\sigma$ and the two functions $\tau$ and $\text{dim}$. We let $R$ be the set containing for each $n \in \text{Irr}(M)$, $\text{dim}(n)$ copies of $n$ with $\sigma$ and $\tau$ extended in the obvious way. We then consider $\mathbb{W}(k)[R]$, the free $\mathbb{W}(k)$-module on $R$, and define the Frobenius map by $F[\tau] = p^{r(\tau)}[\sigma(\tau)]$.

Also if we have for each $V_i$ an $\text{End}(V_i)$-representation $\rho$ of $\text{Gal}(\overline{k}/k)$ on $V_i$, then we can twist by this by letting $F$ act by $p^{r(i)}(\rho \otimes \sigma)$.

Finally, we would like to associate to $(M, \tau)$ the $\ell$-adic analogue of this, that is a Hecke character. We will be able to associate to our data the algebraic part of a Hecke character but a problem arises as there is no canonical choice for a Hecke character with a given algebraic part, indeed such a character may exist only after an extension of the coefficient field. This will have as a consequence that our description of the $\ell$-adic cohomology will not be as satisfactory as the description of the Hodge structure or $F$-crystal of a variety of CM-type. In case $[H_n(X)]$ is multiplicity free we will be able to do better however. In any case the algebraic part (cf. [De:5.3]) can be associated to our data as follows. Assume that for any embedding of $\mathbb{Q}$ in $\mathbb{C}$ we have $\tau(\iota(\rho)) = n - \tau(\rho)$ as above for the corresponding $\iota$. Assume also that $k$ is a number field for which the action of $\text{Gal}(\overline{k}/k)$ on $\text{Irr}(M)$ stabilises the fibers of $\tau$. If again $K$ is the $\mathbb{Q}$-algebra corresponding to $\text{Irr}(M)$, then we can define a multiplicative map

$$k^\times \to K^\times$$

$$\lambda \mapsto \prod_{\rho \in \text{Irr}(M)} \rho(N_{k/\mathbb{Q}}(\lambda))^{\tau(\rho)}.$$ 

By the assumption on $k$ this is well-defined and by the assumption on $\tau$ the projection onto each simple factor of $K$ fulfills the conditions for being the algebraic part of a Hecke character of weight $n$ so we obtain in this way a set of algebraic parts of Hecke characters.

**Theorem 2.4.** Let $(X, S)$ be of separable CM-type in degree $n$.

i) If $k \subseteq \mathbb{C}$ then $H^n_{\text{alg}}(X(k), \mathbb{Q})$ is isomorphic as a Hodge structure with $S$-action to the one associated to the type of $(X, S)$ with a descent of the Hodge filtration to $k$ of the sort described.

ii) If $p > 0$ then $H^n(X, p)$ is isomorphic as $F$-crystal to the $F$-crystal associated to the type of $(X, S)$ and a representation of $\text{Gal}(\overline{k}/k)$.

iii) After a finite extension of $k$ the $\text{Gal}(\overline{k}/k)$-representation on $H_n(X, r), (r \neq p)$ factors through the Galois group of the algebraic closure $K$ of the prime field in (the finite extension of) $k$. If $p = 0$
this representation is given, on the Galois group of a finite extension of $K$, by a direct sum of algebraic Hecke characters whose algebraic parts are the ones associated to the type of $(X, S)$ and with multiplicities given by dim.

iv) If $\text{Irr}(M)$ is multiplicity free (i.e. every irreducible representation of $S$ occurs at most once in $\text{Irr}(M)$) and $k$ is a number field then the $\text{Gal}(k/k)$-representation on $H^n_{\text{sing}}(X, k)$ is given by a direct sum of algebraic Hecke characters with values in the simple components of $Z(\Delta)$.

**Proof:** To begin with let $R$ be $Q$, $Z_p$ resp. $Q_r$. Then $S$ generates an $R$-subalgebra $B$ of the algebra of endomorphisms of $H^n_{\text{sing}}(X(k), Q)$, $H^n(X, p)$ resp. $H^n(X, r) \otimes Z, Q_r$, such that

$$B/(\text{maximal nilpotent ideal}) = A^\otimes Z R$$

(the $A^\otimes$ being that of (2.1)). By [C-R:Thm. 72.19] $B \rightarrow A^\otimes R$ splits as an algebra map and so we can assume that $A \otimes R$ acts on $H^n_{\text{sing}}(X(k), Q)$, $H^n(X, p)$ resp. $H^n(X, r) \otimes Z, Q_r$. Using (1.1) we reduce to the case when $A \otimes R$ is a division algebra or, in the case of ii), isomorphic to $W(F)$ for a finite field $F$. The proof of i) is then easy: By the comparison theorem $H^n_{\text{sing}}(X(k), Q)$ is a representative of $[H^n(X)]$ and as the Hodge decomposition on $H^n_{\text{sing}}(X(k), Q) \otimes Q C$ is stable under $A \otimes C$, the assumption of CM-type forces the Hodge decomposition to be obtained as the lumping together of isotypical components.

For ii), the fact that $A \otimes W(k)$ is separable implies that we have a unique isotypical decomposition $H^n(X, p) = \bigoplus M_{\rho}$, where $\rho$ runs over the irreducible $A \otimes K$-modules $(K := W(k) \otimes Z Q)$ occurring in $H^n(X, p) \otimes Q$. As $F$ is $\Sigma$-linear and commutes with $A$ it maps $M_{\rho}$ to $M_{\Sigma(\rho)}$. If $(p^\rho_1, p^\rho_2, \ldots, p^\rho_k)$ are the elementary divisors of the linear mapping $F: M_{\rho} \rightarrow \Sigma_{\rho} M_{\Sigma(\rho)}$, the characterisation of the Hodge filtration recalled at the beginning of this section shows that $M_{\rho}/p^i M_{\rho}$ is non-disjoint from $H^{n-i}(X, \Omega^j_{X/k})$ exactly when $i$ equals some $n_j$. Hence, by assumption, all the $n_j$ equal $\tau(\Lambda)$ where $\Lambda$ is any component of $[H_n(X)]$ which occurs in $\rho \otimes K \bar{K}$. We will denote, by abuse, this common value $\tau(\rho)$. Thus $F: M_{\rho} \rightarrow \Sigma_{\rho} M_{\Sigma(\rho)}$ is $p^{\tau(\Lambda)}$ times an isomorphism. Hence if we define $F': H^n(X, p) \rightarrow H^n(X, p)$ as being $p^{-\tau(\rho)} F$ on $M_{\rho}$, $H_n(X, p)$ becomes a unit root crystal and is hence described by the $\text{Gal}(\bar{k}/k)$-representation on the fixed points of $F$ (over $k$). This action commutes with $A \otimes Z R$, and so gives the desired description.

Let us now turn to iii) and let us begin with the case $p = 0$. We may assume that $k$ is a finitely generated field. We have a representation $\phi: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_Z \otimes Q, (H^n(X, r) \otimes Q)$ and after possibly enlarging $Z := \mathbb{Z}(\text{Irr}(M)))$ and $k$ we may assume that there exists an algebraic Hecke character $I_m(k') \rightarrow Z'$, where $k'$ is the algebraic closure of $Q$ in $k$, whose algebraic part is the one coming from the type of $(X, S)$ (using that the condition on $\tau$ is fulfilled by the transcendental theory). Twist $\phi$ by this character, considered as a character of $\text{Gal}(\bar{k}/k)$ through the morphism $\text{Gal}(\bar{k}/k) \rightarrow \text{Gal}(\bar{k}/k')$ and $(\text{Se:II,2.7})$. What we now need to prove is that this twist $\phi'$ has finite image (cf. [De:Thm. 5.10]).

Let us first show that if $\Sigma \in \text{Gal}(\bar{k}/k)$ then $\phi'(\Sigma)$ is quasi-unipotent. The possible orders for the eigenvalues of a quasi-unipotent matrix over $Q_r$ of given order is bounded as the degrees of the extensions of $Q_r$ obtained by adjoining an $m$th root of unity goes to infinity with $m$. It is therefore enough to verify the quasi-unipotence on a dense set of Frobenius elements. By the Ceborotiev density theorem it suffices to check quasi-unipotence for the Frobenius elements corresponding to maximal ideals for some thickening of Speck over which $A_n$ is separable, $X$ is smooth and

$$b_n(X) = \sum_{i+j=n} \text{dim}_h H^i(X, \Omega^j_{X/k}),$$

where $h$ is the residue field. If $F_m$ is the Frobenius element of $\text{Gal}(\bar{k}/k)$ we then want to show that all the eigenvalues of $\phi'(F_m)$ are roots of unity or, as they are all algebraic numbers, that all their absolute values are equal to 1. For the infinite primes we use the Riemann hypothesis for $X$. At finite places away from $q := \text{char } h$ there is no problem. Let us therefore consider the places over $q$. Pick a place $v$ of $Z$ lying over $q$ normalised so that $v(|h|) = 1$. By definition $v(\phi(F_m))$ equals the average of $\tau$ over the orbit of $v$ of the action of $\Sigma$ on $\text{Irr}([H^n(X)])$, where $v$ is seen as a homomorphism $Z \rightarrow L_q$ and thus giving an element of $\text{Irr}([H^n(X)])$ (recall that $L_q$ is an algebraic closure of $Q_q$). Hence we
want to show that any eigenvalue of the action of $F_m$ on the $v$-isotypical part of $H^n(X, q)$ has the same valuation. By construction $(X_h, S)$ is of separable CM-type in degree $n$ and the eigenvalues of $F_m$ are of course the same for $X_k$ and $X_h$ so we may replace $k$ by $h$. Applying (1.5) to $S$ and $\{F_h\}$ we see, as $F_m = F_h^*$ on $H^n(X, r)$, that the eigenvalues of $F_m$ on the $v$-isotypical part of of $H^n(X, r)$ are the same as the the eigenvalues of $F_h^*$ on $H^n(X, q)$. By ii), if $u$ is the length of the $\Sigma$-orbit of $v$ then $F^u$ is divisible exactly by $p^t$ on the $v$-isotypical component of $H^n(X, q)$, where $t$ is the sum of the values of $v$ over the $\Sigma$-orbit of $v$. As $F_h^* = F^*$, where $|h| = p^r$, we immediately get what we want.

Now again as the orders of the eigenvalues of the elements of $\text{Gal}(k/k)$ are bounded after replacing $k$ by a finite extension we may assume that the image of $\phi'$ consists entirely of unipotent matrices and so by Engel’s theorem $\phi'$ is a unipotent representation. We aim to show that it is in fact trivial. As $G := \phi'(\text{Gal}(\bar{k}/k))$ is a compact $r$-adic Lie group the closed subgroup of $G$ generated by $r$th powers is of finite index in $G$ and by the Frattini lemma any closed subgroup mapping surjectively onto the quotient of $G$ by this subgroup equals all of $G$. We may then apply the Hilbert irreducibility theorem to get a number field specialisation $k''$ of $k$ such that $X$ has good reduction at $k''$ and that the composed map $\text{Gal}(k''/k) \to \text{Gal}(k/k) \to G$ is surjective. Hence we may assume that $k$ is a number field. I claim that for each prime of $k$ over $r$, the inertia group of that prime has finite image in $G$. Indeed, $\phi'$ is Hodge-Tate as an algebraic Hecke character is always Hodge-Tate and by [F], the finiteness then follows from ([S1:1.4,Cor. 3]) as the unipotence implies that the Hodge-Tate weight is zero. As $\phi'$ is unipotent this implies that $\phi'$ is unramified over $r$. The other monodromy groups automatically have finite, and therefore trivial, images. The triviality of $G$ then follows from the finiteness of the Hilbert class field of $k$.

We have therefore proved iii) when $p = 0$. The case $p > 0$ is similar up to the point where we have arrived at a unipotent representation. Any homomorphism $\text{Gal}(S, \bar{s}) \to \mathbb{Z}_r$ for a finitely generated $\mathbb{F}_p$-scheme $S$ is geometrically trivial, by [K-L:Thm 1], so by thickening $k$ we finish.

As for iv) we start as above so that we have an action of $A := A^n \otimes \mathbb{Q}$ on $H^n(X)$ by correspondences. Note that the assumption of multiplicity freeness implies that the commutant of $A$ in $\text{End}(H^n(X, r))$ equals $\mathbb{Z} \otimes \mathbb{Q}_r$. Let $v$ be a place of $k$ at which $X$ has good reduction with fiber $X_v$ over the residue field $F_v$. Apply the construction of a semi-simple algebra of correspondences to all correspondences so as to get $B$. Then $B$ contains the Frobenius correspondence in its center as well as the subalgebra $A$. Let $C$ be the commutant of $A$ in $B$ so that $B = AC$. By the observation just made $C \otimes \mathbb{Q}_r = Z \otimes \mathbb{Q}_r \subseteq B \otimes \mathbb{Q}_r$ and so $C \subseteq B$ and therefore $A = B$. Hence the Frobenius correspondence $F_v$ lies in $Z$. Therefore we have associated to every place $v$ of $k$ outside a finite set an element $F_v$ of $Z$. Extending by multiplicativity we get a homomorphism $I_m(k) \to Z^\times$ for a suitable $m$. As in the proof of iii) we show that the projections onto the simple factors of $Z$ are algebraic Hecke characters with algebraic parts given by the type of $(X, S)$.

**Remark:** i) It is probably true that in ii) we also get the conclusion that $[H^n(X, p)]$ is geometrically constant. This would follow from a good theory of over-convergent $F$-crystals.

ii) Can one find a good extension of iii) that would contain iv) as a special case?

iii) An example showing that there are problems in the $\ell$-adic case is obtained as follows. Pick an imaginary quadratic field $K$ with class number greater than one. There is an elliptic curve $E$ with complex multiplication by the ring of integers $R$ of $K$ defined over the Hilbert class field $H$ of $K$. The pair $(R_{H/K}E, R)$ is of CM-type in degree 1 over $K$ yet there is no algebraic Hecke character whose algebraic part is that obtained from the type of $(R_{H/K}E, R)$.

As will come as no particular surprise, for abelian varieties our notion coincides with the traditional one.

**Proposition 2.5.** Suppose $(X, S)$ is of separable CM-type in degree 1. Then its Albanese variety is of CM-type in the usual sense possibly after a finite extension of $k$.

**Proof:** This follows from (2.5) and the Tate conjecture for homomorphisms between abelian varieties. Another proof is for $p = 0$ to note that (2.3 i) says that the degree 1 Hodge structure of $X$ is visibly of CM-type and for $p > 0$ that $(\text{Alb}X, S)$ is rigid by definition $\text{Hom}_S(H^0(X, \mathcal{O}_{X/k}), H^1(X, \mathcal{O}_X))$ is equal to 0 and so after a finite extension of $k$, $\text{Alb}X$ can be defined over a finite field and is hence of CM-type.
3. Hereditary CM-type

Theorem 2.3 suffers somewhat on the $p$-adic side as the very natural example of $(E, \text{End} E)$ where $E$ is a supersingular elliptic curve is not of separable CM-type; $\text{End} E$ is not separable at $p$. It is possible to give a result which in that case specialises to a satisfactory answer. In this section we will give a generalisation of the previous results that will cover this case. The maximal possible generality would seem to be to assume that $A^n \otimes \mathbb{Z}_p$ should be a hereditary order which means that any $A^n$-splitting of crystalline cohomology tensored with $\mathbb{Q}$ comes from an $A^n$-splitting of crystalline cohomology itself. Let us recall that an order is hereditary if each lattice over it is projective.

**Remark:** The meaning of the term differs somewhat in various areas of the literature as hereditary sometimes means just that a submodule of a projective module is projective. The definition used here means that the base extension of the order to the fraction field of its base ring is semi-simple together with the fact that every sub-module of a projective module is projective. We will want this extra condition and hence adopt the current definition (which is to be found for instance in [Re]).

On the other hand, the example of an automorphism of order $p$ acting (non-trivially) on a curve of genus $(p - 1)/2$ shows that the condition that different Hodge pieces be disjoint is not reasonable as the cyclic group of order $p$ has only one irreducible representation mod $p$. The situation will no longer be as simple as in the separable case. It is still true that one to any irreducible $A^n \otimes k$-module can associate an irreducible $A^n \otimes K$-module but this map is no longer injective (though surjective).

We will use [Re:Ch. 9] as a general reference to the theory of hereditary orders. For the reader’s convenience we repeat the salient facts in the following proposition as well as adding a result – a weak version of the elementary divisor theorem – which is not to be found in [loc. cit.] (but no doubt is not new).

**Proposition 3.1.** Let $A$ be a hereditary order over a henselian discrete valuation ring $R$ with fraction field $K$.

i) Any submodule of an $A$-lattice is a submodule of finite colength of a direct factor of the lattice.

ii) In every indecomposable $A$-lattice there is exactly one submodule of a given colength.

iii) If $M$ is an indecomposable $A$-lattice and $M \hookrightarrow N_i$ two inclusions of finite colength. Then one of these inclusions is contained in the other.

iv) Every indecomposable finitely generated torsion $A$-module is a quotient of an indecomposable $A$-lattice.

v) Let $M$ be an $A$-lattice and $N$ a sub-lattice of it. Then there is a decomposition of $M$ as a direct sum of indecomposable submodules whose intersection with $N$ also gives a decomposition of $N$ into a direct sum of indecomposable submodules.

**Proof:** For i) we take the saturation of the submodule. The quotient of the lattice by that saturation is torsion-free and hence projective and the saturation is therefore a direct factor. For ii) we notice that by i) any submodule of the lattice is also indecomposable so we may assume by induction that the given colength is 1. However, the lattice being projective is the projective hull of its co-socle (the maximal semi-simple quotient) and so being indecomposable the co-socle is simple which means that there is a unique submodule of colength 1; the radical. For iii) we note that $M \hookrightarrow N_i$ are included in a common inclusion of finite colength (being of finite colength). We then apply ii).

As for iv) we use induction on the length of the module $M$. We therefore find a simple quotient $S$ of $M$ and apply the induction hypothesis to the kernel $M'$ of this map. We will temporarily (and improperly) call a torsion quotient of an indecomposable lattice a cyclic module. Thus we may assume that $M$ is an extension of a sum of cyclic modules by the simple module $S$. This extension is the sum, as extension, of the extension of the cyclic summands by $S$. Let us first study the latter extensions and let us denote by $P$ the projective hull of $S$, by $Q$ its radical, by $V$ the cyclic summand and by $R$ its projective hull. Then every extension of $V$ by $S$ comes from pushout by a map from $Q$ to $V$, the same extensions being obtained if the difference of two morphism extends to a map from $P$ to $S$. Now, I claim that all non-surjective maps $Q \to S$ so extend. In fact the map lifts to a map $Q \to R$ which necessarily is injective as $Q$ is indecomposable. If the map $Q \to S$ is not surjective then the map $Q \to R$ is neither. By applying iii) we see that $P$ must be isomorphic to the unique submodule of $R$ containing $Q$ as a submodule of colength 1 and thus the original map lifts to $P$.  
This result shows that if \( V \) is not a quotient of \( Q \) then any extension of \( V \) by \( S \) is trivial and if it is, then the group of extensions can be identified with maps from the co-socle of \( Q \) to the co-socle of \( V \) which are isomorphic to simple modules. We will now show that, after possibly changing the direct sum decomposition of \( M' \) we may assume that all but one of the extension classes of direct summands by \( S \) are trivial. This will clearly show iv). For this we may immediately discard summands of \( M' \) which are not quotients of \( Q \) as their extension classes have just been shown to be trivial. Furthermore, we may use induction on the number of non-trivial extension classes. Note now that if \( V_1 \) and \( V_2 \) are summands then for any map \( \phi: V_1 \to V_2 \) we may consider the automorphism of \( V \) which maps \( v \in V_1 \) to \( v + \phi v \) and acts as the identity on all other factors. If \( e_1 \) are the extension classes then all of them but \( e_2 \) are unchanged and \( e_2 \) is changed into \( e_2 + \phi e_1 \). If we identify extension classes of \( V \) with homomorphisms from the co-socle of \( Q \) to that of \( V \) then \( \phi \) is just composition by the map on co-socles induced by \( \phi \). As \( V_1 \) and \( V_2 \) are both both quotients of \( Q \), by ii) on is a quotient by the other and we may assume that \( V_2 \) is a quotient of \( V_1 \). In that case, any endomorphism of \( Q \) induces a morphism \( V_1 \to V_2 \) and by the proetivity of \( Q \), any map from the co-socle of \( V_1 \) to that of \( V_2 \) is induced by an endomorphism of \( Q \). Putting this together we see that any extension class is of the form \( \phi e_1 \) so that the we may choose \( \phi \) so that \( \phi e_1 = -e_2 \) which allows us to decrease the number of non-zero extension classes.

To finally prove v) we consider the module \( M/N \). This is a direct sum of a lattice and a torsion module and the lattice may be split off from \( M \) without changing \( N \). Thus we may assume that \( M/N \) is torsion. We then use iv) to write that quotient as a direct sum of quotients of indecomposable projective modules. The sum of the projective hulls of each summand is a projective hull of the sum. That projective hull is a direct summand of the map \( M \to M/N \). This immediately gives the pair \( (M,N) \) as a direct sum of of pairs \( (P_i,P'_i) \), where \( P_i \) is indecomposable and a factor \( (M',M') \). As \( M' \) is a sum of indecomposables, v) follows.

We will need a definition which is very special to the situation at hand.

**Definition 3.2.** Let \( A \) be a hereditary order over a henselian mixed characteristic discrete valuation ring \( \hat{R} \) with positive residue field characteristic \( p \) and let \( M \) be a finitely generated torsion module killed by \( p \). By the complementary module to \( M \) we mean the torsion module (defined up to isomorphism) obtained as \( P/pP' \), where \( P/pP' \) is a projective hull of \( M \) and \( P' \) is the kernel of the natural map \( P \to M \).

As there can be, as opposed to the separable case, non-trivial extensions of modules we also will need to recall the definition of block, well-known in the theory of general orders,

**Definition 3.3.** Let \( A \) be a hereditary order over a henselian discrete valuation ring \( R \) with fraction field \( K \). Two indecomposable (finitely generated) \( A \)-modules belong to the same block if there is a non-zero morphism from the projective hull of one to the other. This is equivalent to the two hulls tensored with \( K \) being isomorphic. If \( M \) is a finitely generated \( A \)-module then the \( B \)-component of \( M \) is the sum of all indecomposable factors belonging to the block \( B \). (It is clear that any finitely generated \( A \)-module is the direct sum of its components associated to different blocks.)

What is different with the hereditary case as opposed to the separable case is that we may have non-semisimple (f.g.) modules killed by \( p \). This will imply that to define CM-type it is not enough to look at what simple modules occur in which Hodge piece; the more precise module structure needs to be taken into account. As we will see this forces certain relations between Hodge pieces. Our results will be purely algebraic so we will, rather than sticking to the notation of this article as a whole, use the following notation: \( A \) will be a hereditary \( \mathbb{Z}_p \)-order and \( M \) will be an \( F \)-crystal with an action of \( A \). We define the Hodge filtration on \( M/pM \) by \( M^i := F^{-i}pM/pM \) and the Hodge modules \( H^i := M^i/M^{i+1} \) (which may be considered as \( A \otimes W \)-modules).

**Definition-Lemma 3.4.** We define the \( A \)-primitive part of \( H^i \) as the direct factor (defined up to isomorphism only) by induction on \( i \). For \( i = 0 \) we let the primitive part be all of \( H^i \). For \( i > 0 \) the complement of the primitive part of \( H^{i-1} \) is a direct factor of \( H^i \) and we let the primitive part be a complementary factor of it.

**Proof:** What is to be proven is the statement about the complement of the primitive part being a direct summand. We will give another description of the primitive part which will make this obvious.
Consider therefore the Frobenius map as a \( W \)-linear map \( \sigma^* M \to M \), which then also is a \( A \otimes W \)-linear. This is a hereditary order so we may by (3.1 v) split this map up in indecomposable factors. Using Mazur-Ogus' characterisation ([B-Og]) of the Hodge filtration and the fact that submodules are linearly ordered we immediately see that each indecomposable factor will contribute a cyclic module to one Hodge piece and its complement to the next.

We are now ready to define what we mean by CM-type in the context of actions of hereditary orders.

**Definition 3.5.** The pair \((M, A)\) is of hereditary CM-type if \( A \otimes \mathbb{Z}_p \) is a hereditary order and for each block \( B \), the \( B \)-component of the primitive part of \( H^1 \) is non-zero for at most one \( i \) and all indecomposable factors of that \( B \)-component have the same length.

We have now set up our definitions so that we may carry through the same analysis as in the separable case (it should be noted that in the case that \( \sigma \) is isomorphic as \( B \)-module to \( B \) itself. As the extension \( \mathbb{Z}_p \to W \) is faithfully flat this means that such a \( B \)-module is projective. Hence it is determined up to isomorphism by its co-socle and to prove the lemma it is enough to show that if we have two semi-simple \( B \)-modules which become isomorphic under extension of scalars to \( W \) are isomorphic. This however is obvious (using for instance the independence of central characters of a semi-simple algebra).

**Theorem 3.7.** Suppose \((M, A)\) is of hereditary CM-type and that \( k \) is algebraically closed. Then it is determined up to isomorphism by which blocks appear in the primitive part of which Hodge-modules and the common length of indecomposable factors of each such block.

**Proof:** Note first that we can make \( \sigma \) act on the blocks of \( A \otimes W \) by the condition that \( N \) belongs to the block \( B \) iff \( \sigma^* N \) belongs to \( \sigma^* B \). If we now split up \( M \) in blocks, \( M = \bigoplus B M_B \), then it is clear that \( F \), considered as a map \( \sigma^* M \to M \) is a sum of maps \( \sigma^* M_B \to M_{\sigma^* B} \). Now, for an indecomposable \( B \otimes W \)-lattice \( N \) the length of \( N/pN \) depends on which block \( N \) belongs to. Indeed, any two indecomposable \( B \otimes W \)-lattices in the same block are contained in each other with quotient of finite length, the kernel and cokernel of the map induced by reduction modulo \( p \) then has the same length. We now consider the component of \( F \), \( \sigma^* M_B \to M_{\sigma^* B} \), as a \( W \)-linear map and split it up into indecomposable pieces according to lemma 3.1. Looking at each indecomposable piece we see that if \( B \) appears in the \( i \)th Hodge piece of \( M \) then \( F \) maps \( \sigma^* M_B \) into \( p^i M_{\sigma^* B} \), the image contains \( p^i+1 M_{\sigma^* B} \) and the length of each indecomposable factor of \( p^i M_{\sigma^* B} \). Indeed, \( M \) has the same length, as each such length added to the common length of the indecomposables of the primitive \( B \)-part of the Hodge piece adds up to the common length of an indecomposable lattice in \( B \) modulo \( p \). This means that any indecomposable factor of \( M_{\sigma^* B} \) has the same length, which is the same as saying that \( M' = \text{rad}^m M_{\sigma^* B} \) for a suitable \( m \), where \( \text{rad}(\cdot) \) is the radical functor. We may thus use \( F \) to identify \( \sigma^* M_B \) with \( \text{rad} M_{\sigma^* B} \), where \( m \) is determined by \( i \) and the common length of indecomposables of the primitive part belonging to the block \( B \). If \( n \) is the smallest positive integer for which \( \sigma^n B = B \), then \( F^m \) maps \( M_B \) onto \( \text{rad}^k M_{\sigma^* B} \) for a suitable \( k \). It is then enough to show that all such maps are conjugate under automorphisms of \( M_B \). Fix one such map \( \phi \). Now, \( I \) claim that the relation \( \phi \circ f^\sigma = g \circ \phi \) defines an automorphism \( f \to g \) of \( \text{End}(M_{\sigma^* B}) \). Indeed, for any \( g \) there is an \( f \) fulfilling that relation as the image of \( \phi \) is equal to \( \text{rad}^m M_{\sigma^* B} \). Conversely, the inverse image of \( M_{\sigma^* B} \) in \( \sigma^* M_B \otimes K \) under \( \phi \) is equal to its largest sub-lattice for which the quotient by \( \sigma^* M_B \) has all its indecomposable components of length less than or equal to \( m \) which shows that to any \( f \) there is a \( g \). As any map fulfilling the conditions imposed on \( \phi \) differs from it by an automorphism of \( \sigma^* M_B \) we can apply lemma 3.6 to include that there is, up to isomorphism, only one \( F \).
I would also like to record that, just as in the separable case, multiplicity freeness implies CM-type.

**Proposition 3.8.** Suppose that $(M, A)$ is multiplicity free in the sense that an irreducible $A \otimes K$-module appears at most once in $M \otimes K$. Then $(M, A)$ is of hereditary CM-type.

**Proof:** The condition implies that for any block, the component of $M$ in that block is indecomposable. That immediately implies that a given block appears in the primitive part of just a single Hodge piece and that part is indecomposable so the condition on length is fulfilled. 

We finish this section with some examples.

**Example:** i) Consider a supersingular elliptic curve $E$ and its ring of endomorphisms $A$, which is a maximal order in a division ring and hence hereditary. The action of $A$ on the first crystalline cohomology group is multiplicity free and hence of hereditary CM-type. More precisely, $H^1(E, p)$ is an indecomposable $A \otimes W$-lattice and $\sigma H^1(E, p)$ is the other indecomposable $A \otimes W$-lattice – both are in the same block. Hence, $H^1(E, \Omega^1)$ is one irreducible $A \otimes W$-module and $H^1(E, \mathcal{O}_E)$ the other. The image of $\sigma H^1(E, p)$ under $F$ is the maximal proper submodule of $H^1(E, p)$.

ii) Let $C$ be the projective, smooth completion of the curve $y^p - y = x^2, p \neq 2$, and consider the action of $\mathbb{Z}/p$ on $y$ by $y \mapsto y + \alpha$. The action of the group algebra of $\mathbb{Z}/p$ on $H^1(C, p)$ factors through the quotient $A$ that is the ring of $p$th roots of unity. $H^1(C, p)$ is then a free $A \otimes W$-module of rank 1 and hence $(C, A)$ is of hereditary CM-type. This time the situation is simpler as the ring is commutative and to prove the classification theorem we could simply have divided $F$ by $(\zeta - 1)^{(p-1)/2}$ to obtain a unit root crystal.

### 4. Applications to the Néron-Severi group.

In this section we will suppose that $X$ is a surface and that $(X, S)$ is of separable type in degree 2. Then $S$ acts on the Néron-Severi group $NS$ of $X$. If $p = 0$, $(2.5)$ and the Lefschetz theorem on $(1,1)$-classes shows that $[NS \otimes \mathbb{Q}] \in K(S, \mathbb{Q})$ equals the sum of all irreducible $\rho$ in $[H^2(X)]$ such that $\tau(\sigma(\rho)) = 1$ for all $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ and the Tate conjectures implies this in all characteristics. However, $(2.3)$ can be used to obtain further information on $NS$. To illustrate this let us suppose that $p > 0$ and that $NS = b_2(X)$. By possibly extending $k$ we may assume that $NS$ is defined over $k$. As $c_1 : NS \otimes \mathbb{Z} \rightarrow H^2(X, \mathbb{Z}) (\ell \neq p)$ has torsion free cokernel (cf. [Gro:8.7]) and the two modules have the same rank, $c_1$ is an isomorphism. By Poincaré duality the intersection pairing is perfect at $\ell$. By [IlI:II,5.8.5.5,20] the image of $c_1 : NS \otimes \mathbb{W}(k) \rightarrow H^2(X, p)$ is the largest sub-$F$-crystal in which $F$ is divisible by $p$ and, again by Poincaré duality, if $\sigma_0$ is the $\mathbb{W}(k)$-length of the cokernel, then $p^\tau(\sigma_0)$ is the exact power of $p$ dividing $\text{disc}(NS)$. Hence by the Hodge index theorem $\text{disc}(NS) = (-1)^{b_2 - 1}p^{2\tau(\sigma_0)}$.

As the whole $H^2(X, p)$ is determined by the type of $(X, S)$, $\sigma_0$ is as well and we will now see how this can be done explicitly. By $(2.3)$ $M := H^2(X, p) = \bigoplus_{\rho \in \text{Irr}([H^2(X)])}M_{\rho}$ and $F : M_{\rho} \rightarrow M_{\sigma(\rho)}$ is $p^{\tau(\rho)}$ times an isomorphism. Let $N \subseteq M$ be the maximal sub-$F$-crystal on which $F$ is divisible by $p$. Consider $T := \text{Irr}([H^2(X)])$ with the functions $\tau$ and $\text{dim}$ and the action of $\sigma$. We shall now describe an algorithm for computing $\sigma_0$. To do this we start by by considering $M$ with $F' := p^{-1}F$ as a virtual $F$-crystal i.e. $p^{-1}F$ takes $M$ into $M \otimes \mathbb{Q}$ rather than into $M$ itself. Now $N$ can then be characterised as the maximal sub-$F$-virtual-crystal which is actually a crystal. As it is unique it is a sub-representation and so it is the direct sum of the $N_{\rho}$. We will now concentrate on one specific $\sigma$-orbit on $T$ and assume that $M$ is in fact the $F$-crystal associated to it. Pick one $\rho$ in this orbit. As $M_{\rho}$ is of rank 1 $N_{\rho}$ is equal to $p_nM_{\rho}$ for some $n$. All powers of $F'$ must take $N_{\rho}$ to $M$ which means that $n + \sum_{j=0}^{k}(\sigma(\rho) - 1)$ is greater than or equal to 0 for all $k$. Hence if we put $n$ equal to $-\min_k\sum_{j=0}^{k}(\sigma(\rho) - 1)$ and define $N'$ as $\bigoplus p^n'\rho M$, where $m_{\rho'} := n + \sum_{j=0}^{k}(\sigma(\rho) - 1)$ with $\rho' = \sigma^k\rho$ we have a sub-$F$-virtual-crystal of $M$ which clearly is an actual $F$-crystal (here we use the fact that the sum of $\tau - 1$ over the orbit is 0). We have also seen that $N_{\rho} \subseteq N_{\rho'}'$ and as $N$ is the maximal sub-$F$-crystal we have equality. Finally, again using that the sum of $\tau - 1$ over the orbit is 0 it is immediately realised that $N'$ is independent of the choice $\rho$ and so has to be equal to $N$. In particular we see that the contribution of this orbit to $\sigma_0$ equals the multiplicity of the orbit times the sum of the $m_{\rho'}$. 

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Example: We consider one orbit for $\sigma$ and describe such an orbit by $(\tau(t), \tau(\sigma(t)), \ldots, \tau(\sigma^{h-1}(t)))$, where $h$ is the length of the orbit. We also assume that the starting point $\rho$ is the first element of this list.

i) $(0, 2)$ gives partial sums $(-1, 0)$ and so $n = 1$ and the list of the $m_{\rho'}$ is $(0, 1)$ and finally the contribution to $\sigma_0$ is 1.

ii) $(0, 2, 1)$ gives partial sums $(-1, 0, 0)$, $ms (0, 1, 1)$ and $\sigma_0 = 2$.

iii) $(2, 1, 1, 1, 1, 0)$ gives partial sums $(1, 1, 1, 1, 1, 0)$, $ms (1, 1, 1, 1, 1, 0)$ and so $\sigma_0 = 3$.

As a geometric example let us first consider the Fermat surface $X_m = \{X_0^m + X_1^m + X_2^m + X_3^m = 0\}$ and the group of diagonal automorphisms $G_m = \mu_m^d/(scalars)$. The irreducible representations of this group are the elements of the dual group

$$G_m := \{(b_0, b_1, b_2, b_3) \in (\mathbb{Z}/m)^3 : \sum_{i=0}^3 b_i = 0\}$$

and it is well known (cf. [Ka:Sect. 6]) that each character occurs at most once in $H^2(X_m)$ and those that occur are exactly those in the set $T := \{(b_0, b_1, b_2) \in G : \forall i : i \neq 0\} \cup \{(0, 0, 0, 0)\}$. Furthermore, if we for $b \in \mathbb{Z}/m$ let $\langle b \rangle$ be the unique integer s.t. $b \in \mathbb{Z}/m$ and $0 \leq \langle b \rangle < m$ then $\tau(\langle b \rangle) = 1/m \sum_{i=0}^3 b_i$ if $\langle b \rangle \neq 0$ and 1 if $\langle b \rangle = 0$.

Finally, the action by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is given by $F_p(\langle b \rangle) = \langle pb \rangle$ for a prime $p \nmid m$.

Remark: The proof of this in [Ka:Sect. 6] uses transcendental methods. A purely algebraic proof can be given by tracing the action of $G$ through the calculations of [SGA7:Exp. XI].

The Fermat surfaces verify the Tate conjecture ([S-K]) so $rkNS = b_2$ over a field of positive characteristic iff the average of $\tau$ over any $\sigma$-orbit equals 1. Now complex conjugations in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ exchanges the values 0 and 2 and fixes 1 so we see that if the subgroup generated by $\sigma$ contains complex conjugation this is always the case. Hence if $-1 \in \langle p \rangle \subseteq (\mathbb{Z}/m)^\times$ then $rkNS(X_m) = b_2(X_m)$ in characteristic $p$.

Example: i) $p \equiv -1 \pmod{m}$. Then all the orbits are of type $(1, 1, \ldots, 1)$ or $(0, 2)$ giving a contribution of 0 resp. 1 to $\sigma_0$.

ii) $m=5$, $p \equiv 2, 3 \pmod{5}$. Then there are four orbits of type $(0, 1, 2, 1)$ and the rest are of type $(1, 1, \ldots, 1)$. Now the algorithm applied to $(0, 1, 2, 1)$ gives partial sums $(-1, -1, 0, 0)$ and a contribution of 2 to $\sigma_0$ for each copy of this orbit and hence $discNS = p^{16}$ whereas $p_g = 4$ so that we get a higher power of $p$ than is guaranteed by $p^{2p_g}/discNS$.

Proposition 4.1. Suppose that $X$ is a smooth surface over $k$ and that $p > 0$. Suppose that $G$ is a finite group of order prime to $p$ acting on $X$. Let $\tilde{X}$ be a minimal resolution of $X/G$. Then $H^2(\tilde{X}, p) = H^2(\tilde{X}, p)^G \perp E$, where $E$ is the $W$-module spanned by the Chern classes of the exceptional curves of $\tilde{X} \to X/G$ and orthogonality is wrt the cup product. Furthermore, the cup product pairing restricted to $E$ is perfect.

PROOF: Let $\pi: X' \to X$ be a $G$-equivariant blowing up of $X$ such that we have a map $\rho: X' \to \tilde{X}$ covering the quotient map $X \to X/G$. The cup product pairing on $E$ is perfect because the cokernel of $E \to E$ equals $E$ tensored with the sum of the local Picard groups of the singularities of $X/G$ (cf. [Li:14.4]) and these are killed by the order of $G$ by the existence of a norm map. Hence we may write $H^2(\tilde{X}, p) = V \perp E$. Now $\rho_\rho^* = [G]$ so $\rho^*$ is injective on $H^2(\tilde{X}, p)$ and the image is contained in the $G$-invariants and is a direct factor. Furthermore, by the projection formula, $\rho^*V$ is orthogonal to the submodule of $H^2(X', p)$ spanned by the curves exceptional for $\pi$. Therefore $\rho^*V \subseteq \pi^*H^2(\tilde{X}, p)^G$ and we are finished if we can show that this is an equality. First, we show this for the $p$-torsion. Indeed, consider the slope spectral sequence for $X'$ and $\tilde{X}$ (cf. [Ill:II.3]). By duality (cf. [Ekl1]) $\rho_\rho$ is defined as a map of spectral sequences and we still have $\rho_\rho^* = [G]$. Furthermore, $\rho^*$ is an isomorphism on $H^*(\tilde{X}, W\Omega^1\tilde{X}) \to H^*(X', W\Omega^1X')$ as $H^*(X', W\Omega^1X')^G = H^*(X, W\Omega^1X)$ is torsion free as $W\Omega^1X$ is the last as the singularities are rational. Hence as $H^0(-, W\Omega^1_X)$ is torsion free as $W\Omega^1_X$ is we see that we have equality on torsion groups if we have equality for the torsion of $H^1(-, W\Omega^1_X)$.

The nilpotent torsion (cf. [Ekl1:IV,3.3.13]) of it is dual to the nilpotent torsion of $H^2(-, W\Omega^1_\cdot)$.
(loc. cit.) and is hence taken care of, whereas the semi-simple torsion (loc. cit.:IV,3.4) comes from $H^2(−,\mathbb{Z}_p(1))$ which in turn comes from the Néron-Severi group [III:II.5.8.5] which is taken care of by noting that $\Pi^{cris}(\tilde{X}) = \Pi^{cris}(X/G)$ as the singularities are rational and $\Pi^{cris}(X/G) = \Pi^{cris}(X)^G$ outside of the order of $G$. Hence, as $p^*V$ is a direct factor of $\pi^*H^2(X,p)^G$, it suffices to show that that they have the same rank. As the rank of $V$ is the rank of $H^2(\tilde{X},p)$ minus the number of exceptional curves we may replace $p$ by $\ell$ and then $H^2(X,\ell)^G = H^2(X/G,\ell)$ and the latter space is isomorphic to the orthogonal complement of the exceptional curves of $\tilde{X} \to X/G$ by the Leray spectral sequence. □

Using the proposition we get a description of the crystalline cohomology of the minimal resolution of the quotient of $X_m$ by any subgroup of $G$.

We can also compute other invariants of Fermat surfaces and their quotients. Consider for instance the formal Brauer group of a surface $X$ which is Mazur-Ogus (cf. [Ek:IV,1.1]) (in positive characteristic) or rather $H^2(X,\mathcal{O}_X)$ the knowledge of which is equivalent to knowing the formal Brauer group. It follows from [loc. cit.:III,Thm 4.3] that $H^2(X,\mathcal{O}_X)$ is the quotient of $H^2_{cris}(X/W)(\otimes_{\mathbb{Z}[\ell]})D$, where $D$ is the Dieudonné-ring (with power series in $V$), by the submodule generated by $m \otimes 1 - V(n \otimes 1)$ for all $m,n \in H^2_{cris}(X/W)$ for which $Fm = pn$. Hence if $X$ is of CM-type in degree 2 we get a description of $H^2(X,\mathcal{O}_X)$.

**Example:**

i) $(0,1,0,2,1,2)$ gives a $D$-module with generators $a$ and $b$ and relations $Fa = Vb$ and $Fb = 0$. This is the Dieudonné-module of a 2-dimensional formal group isogenous but not isomorphic to $W_2$. For $p \equiv 3 \pmod{7}$ this appears in the cohomology of the Fermat surface of degree 7 (the orbit of $(1,1,1,4) \in (\mathbb{Z}/7)^4$).

ii) $(0,0,2,2)$ gives a $D$-module with generator $a$ and relation $F^2a = 0$. This is the Dieudonné-module of $W_2$. For $p \equiv 5 \pmod{7}$ this appears in the cohomology of the Fermat surface of degree 13 (the orbit of $(3,3,3,4) \in (\mathbb{Z}/7)^4$).

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