ONE-DIMENSIONAL, FORWARD-FORWARD MEAN-FIELD GAMES WITH CONGESTION

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Abstract. Here, we consider one-dimensional forward-forward mean-field games (MFGs) with congestion, which were introduced to approximate stationary MFGs. We use methods from the theory of conservation laws to examine the qualitative properties of these games. First, by computing Riemann invariants and corresponding invariant regions, we develop a method to prove lower bounds for the density. Next, by combining the lower bound with an entropy function, we prove the existence of global solutions for parabolic forward-forward MFGs. Finally, we construct traveling-wave solutions, which settles in a negative way the convergence problem for forward-forward MFGs. A similar technique gives the existence of time-periodic solutions for non-monotonic MFGs.

1. Introduction. Mean-field games (MFGs) model competitive interactions in large populations in which the agents’ actions depend on statistical information about the population. These games are modeled by a Hamilton-Jacobi equation coupled with a Fokker-Planck equation. An important class of MFGs concerns congestion problems, where the agents’ motion is hampered in high-density areas.

In one-dimension, the congestion problem is the system

\[
\begin{cases}
-u_t + m^\alpha H \left( \frac{u}{m^\alpha} \right) = \varepsilon u_{xx} + g(m) \\
mt - \left( H'(m) \frac{u}{m^\alpha} \right)_x = \varepsilon m_{xx},
\end{cases}
\]

together with terminal-initial conditions; we prescribe the initial value of \( m \) at \( t = 0 \) and the terminal value of \( u \), at \( t = T \):

\[
\begin{cases}
 u(x, T) = u_T(x) \\
m(x, 0) = m_0(x).
\end{cases}
\]

Here, the Hamiltonian, \( H : \mathbb{R} \to \mathbb{R}, H \in C^2 \), is convex and bounded by below, the coupling, \( g : \mathbb{R}^+ \to \mathbb{R}, g \in C^1 \), is a monotone increasing function, the viscosity coefficient, \( \varepsilon \), is a non-negative parameter and \( \alpha, 0 < \alpha < 2 \) is the congestion exponent. The space variable, \( x \), lies on the one-dimensional torus, \( T \), identified with \( [0, 1] \); the time variable, \( t \), belongs to \( [0, T] \) for some terminal time, \( T > 0 \).

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The unknowns in the above system are \( u : \mathbb{T} \times [0, T] \to \mathbb{R} \) and \( m : \mathbb{T} \times [0, T] \to \mathbb{R}^+ \), where for each \( t > 0 \), \( m \) is a probability density; that is,

\[
\int_\mathbb{T} m(x, t) dx = 1.
\]

In MFGs, each agent follows a path determined by a control problem. For \( \varepsilon = 0 \), these paths are deterministic, whereas for \( \varepsilon > 0 \) they are subject to random noise. If \( \varepsilon = 0 \) then (1) is a first-order MFG otherwise the system is a parabolic MFG. For an agent at \( x \in \mathbb{R} \) at time \( t \), the quantity \( u(x, t) \) is the value function of a control problem in which the Hamiltonian, \( H \in C^2 \), determines the cost and \( g \in C^1 \) provides a coupling between each agent and the mean field, \( m \). The evolution of \( m \) through the Fokker-Planck equation, the second equation in (1), also depends on \( H \).

MFGs, introduced in [22], were investigated extensively in the last few years. Numerous results on the existence and uniqueness of solution are now available. For example, the parabolic problem was considered for strong solutions in [23] and subsequently in [17, 19, 15, 16] and weak solutions in [23, 28]. The first-order problem was tackled in [4, 5] and weak solutions are obtained. As for the stationary problem, classical and weak solution were sought in [9, 12, 13]. Some explicit examples were studied in [11, 10] in one dimension and in [8] with radial symmetry. In both time-dependent and stationary problems, the uniqueness of solution is guaranteed by the monotonicity of the coupling \( g \). The congestion problem was first addressed in [26] and later, other approaches such as density constraints and nonlinear mobilities were used in [3, 27, 29]. The existence of smooth solutions for a small terminal time was discussed in [18] and in [21].

Forward-forward MFGs result from reversing the time in the Hamilton-Jacobi equation in (1). Here, we focus on the forward-forward MFGs with congestion:

\[
\begin{align*}
&\begin{cases} 
  u_t + m^\alpha H \left( \frac{u_x}{m} \right) = \varepsilon u_{xx} + g(m) \\
  m_t - \left( \frac{H'}{m^\alpha} \right) m_x = \varepsilon m_{xx} 
\end{cases} \\
&\text{with the initial conditions:} \\
&\begin{cases} 
  u(x, 0) = u_0(x) \\
  m(x, 0) = m_0(x) 
\end{cases}
\end{align*}
\]

In [1], the authors propose the forward-forward MFG model and study numerically its convergence to a stationary MFG. This scheme relies on the parabolicity in (1) to force the long-time convergence to a stationary solution. In [14], the authors studied parabolic forward-forward MFGs and proved the existence of a solution. Next, in [20] using the entropy method, the convergence for one-dimensional forward-forward MFGs without congestion was proven. Additionally, there, the authors compute entropies for first-order problems and establish connections between these models and certain nonlinear wave equations from elastodynamics.

In [14], the authors proved the existence and regularity for parabolic forward-forward MFGs with subquadratic Hamiltonians. In [24], the authors investigated forward-forward MFGs with logarithmic coupling in the framework of eductive stability. The large-time convergence was studied in [20] for one-dimensional parabolic problems and numerical evidence from [2] and [6] suggests that convergence holds.

The main contributions of this paper are as follows. First, for quadratic Hamiltonians, we convert the forward-forward problem into a system of conservation laws
and compute new convex entropies (Lemma 3.1) and Riemann invariants (Lemmas 3.2 and 3.3). These Riemann invariants give lower bounds for the density, \(m\), and, for parabolic MFGs, these bounds combined with an entropy estimate gives the existence of a classical global solution (Theorem 4.3). Finally, by computing traveling-wave solutions, we prove that forward-forward MFGs may fail to converge to a stationary solution. With a similar method, we construct traveling waves for non-monotonic MFGs.

2. Preliminaries. We first recall a few well-known concepts on systems of conservation laws in one dimension; see for instance [7, 25] for further details. We consider a conservation law with the general form

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(t, x) \in U. \tag{5}
\]

where \(U\) is an open subset of \(\mathbb{R}^2\), \(F: U \to \mathbb{R}\) is the flux-function, differentiable on \(U\) and \(u = u(t, x)\) is the unknown. \((\eta, q)\) is an entropy/entropy-flux for (5) if

\[
D\eta(u)F(u) = Dq(u), \tag{6}
\]

for all \(u \in U\). In that case, the following holds:

\[
\eta(u)_t + (q(u))_x = 0 \tag{7}
\]

for all smooth solution \(u\) of (5). The conservation law (5) is hyperbolic if the Jacobian matrix \(DF(u)\) has two real eigenvalues \(\lambda_i(u)\), \(i = 1, 2\) for each \(u \in U\). If \(\lambda_1(u) \neq \lambda_2(u)\) for each \(u \in U\), (5) is strictly hyperbolic.

Assume (5) is hyperbolic and \(DF(u)\) has eigenvalues \(\lambda_1(u)\) and \(\lambda_2(u)\) with corresponding the right eigenvectors \(r_1(u)\) and \(r_2(u)\). A function \(w^i: U \to \mathbb{R}\), \(i = 1, 2\) is a Riemann invariant of the conservation law (5) corresponding to the eigenvalue \(\lambda_i\) if \(w^i\) is differentiable and

\[
Dw^i(u) \cdot r_i(u) = 0, \quad u \in U. \tag{8}
\]

The conservation law (5) is genuinely nonlinear if

\[
D\lambda_i(u) \cdot r_i(u) \neq 0 \quad u \in U \quad i = 1, 2.
\]

3. Systems of conservation laws and first-order, forward-forward MFGs. Here, we study the following forward-forward MFG with congestion and a quadratic Hamiltonian:

\[
\begin{align*}
\left\{ \begin{array}{ll}
u_t + \left(\frac{p+u_x}{2m}\right)_x = 0, \\
m_t - \left(\frac{p+u_x}{m}\right)_x = 0,
\end{array} \right. \tag{8}
\end{align*}
\]

with \(p \in \mathbb{R}\). Assuming enough regularity, we differentiate the first equation with respect to \(x\) and set \(v = p + u_x\). We thus obtain the following system of conservation laws:

\[
\begin{align*}
v_t + \left(\frac{v^2}{2m} - \frac{v}{m^{\alpha-1}}\right)_x = 0, \\
m_t - \left(\frac{v}{m^{\alpha-1}}\right)_x = 0.
\end{align*} \tag{9}
\]

3.1. Existence of convex entropies. First, we construct convex entropies for (9). We recall that \((\eta, q)\) is an entropy/entropy-flux pair for (9) if

\[
D\eta DF = Dq, \tag{10}
\]

where

\[
F(v, m) = \left(\frac{v^2}{2m^a}, -\frac{v}{m^{a-1}}\right)
\]
is the flux function in (9). A direct computation shows that $\eta$ solves (10) if and only if it satisfies:

$$\frac{\alpha v^2}{2m^{\alpha+1}} \eta_{vv} - \frac{1}{m^{\alpha-1}} \eta_{mm} + \frac{(2 - \alpha)v}{m^\alpha} \eta_{vm} = 0. \quad (11)$$

In the next lemma, we investigate the existence of entropies and determine conditions under which these entropies are convex.

**Lemma 3.1.** The system (9) has the following family of entropies:

$$\eta(v, m) = v^a m^b, \quad (12)$$

where $a, b$ satisfy

$$\alpha a(-a + 2b + 1) + 2b(-2a + b - 1) = 0. \quad (13)$$

Moreover, if, additionally $a$ is even and

$$a(a - 1) \geq 0 \quad \text{and} \quad ab(a + b - 1) \leq 0, \quad (14)$$

then $\eta$ as defined in (12) and (13) is convex.

**Proof.** By taking cross derivatives in (10), we see that $\eta$ as defined in (12) satisfies (11) if and only if (13) holds.

Note that

$$D^2 \eta = \begin{pmatrix} (a - 1)am_b v^{a-2} & abm_b v^{a-1} \\ abm_{b-1} v^{a-1} & (b - 1)bm_{b-2} v^a \end{pmatrix}. \quad (19)$$

Using Sylvester’s criterion, that $a$ is even, and that $m > 0$, we derive (14). $\square$

**Remark 1.** For $0 < \alpha < 2$, the conditions (13) and (14) hold if $a < 0$ or $a > 1$ and with $b$ given by

$$b = b(\alpha, a) := \frac{1 + 2a - a\alpha - \sqrt{1 + 4a - 4a\alpha + a^2(4 - 2\alpha + \alpha^2)}}{2}. \quad (15)$$

We note that if $a > 1$ then $b < 0$. Moreover, we have

$$\lim_{\alpha \to \infty} -\frac{b(\alpha, a)}{a} = \theta(\alpha), \quad (16)$$

with

$$\theta(\alpha) = \frac{1}{2} \left( \sqrt{\alpha^2 - 2\alpha + 4 + \alpha - 2} \right) \quad (17)$$

and, for $a > 1$,

$$\theta(\alpha) - \frac{b(\alpha, a)}{a} > 0. \quad (18)$$

### 3.2. Hyperbolicity and genuinely nonlinearity

Now, we show that (9) is a hyperbolic, genuinely nonlinear system of conservation laws. To that end, we compute the Jacobian of $F$ and get

$$DF(v, m) = \begin{pmatrix} m^{-\alpha}v & -1/2\alpha m^{-1-\alpha} v^2 \\ -m^{1-\alpha} & -(1-\alpha)m^{-\alpha}v \end{pmatrix}. \quad (19)$$

**Proposition 1.** The system (9) is strictly hyperbolic outside the set $\{v \neq 0\}$. More precisely, (19) has eigenvalues

$$\lambda_1 = \left( \alpha - \sqrt{4 - 2\alpha + \alpha^2} \right) \frac{v}{2m^\alpha} \quad (20)$$

and

$$\lambda_2 = \left( \alpha + \sqrt{4 - 2\alpha + \alpha^2} \right) \frac{v}{2m^\alpha} \quad (21)$$
with corresponding eigenvectors
\[ r_1 = \left( (2 - \alpha)v - \sqrt{4 - 2\alpha + \alpha^2} v, -2m \right) \] (22)
and
\[ r_2 = \left( (2 - \alpha)v + \sqrt{4 - 2\alpha + \alpha^2} v, -2m \right). \] (23)
Furthermore, for \( 0 < \alpha < 2 \), the system \((9)\) is genuinely nonlinear on
\[ \mathcal{A} = \{(v, m) \in \mathbb{R}^2 : v > 0, m > 0\}. \] (24)

**Remark 2.** The set \( \mathcal{A} \) corresponds to the case where \( px + u(x) \) is increasing. Analogous results hold for decreasing functions.

**Proof.** Simple computations show that \( DF \) has eigenvalues given by \((20)\) and \((21)\). These are distinct if \( v \neq 0 \).

Thus, \((8)\) is a strictly hyperbolic system of conservation laws if \( v \neq 0 \). Next, we find the right eigenvectors corresponding to \( \lambda_1 \) and \( \lambda_2 \). Accordingly, we determine \( r_i, i = 1, 2 \), such that
\[ DG^T r_i = \lambda_i r_i. \]
Again, straightforward computations ensure that \( r_1, r_2 \) can be chosen as in \((22)\) and \((23)\).

Next, we compute
\[ D\lambda_1 \cdot r_1 = \left( 2 + \alpha^2 - \sqrt{(\alpha - 2)\alpha + 4 - \alpha\sqrt{(\alpha - 2)\alpha + 4}} \right) vm^{-\alpha} \]
and
\[ D\lambda_2 \cdot r_2 = \left( 2 + \alpha^2 + \sqrt{(\alpha - 2)\alpha + 4 + \alpha\sqrt{(\alpha - 2)\alpha + 4}} \right) vm^{-\alpha}. \]
Finally, we observe that, for \( 0 < \alpha < 2 \), we have
\[ D\lambda_1 \cdot r_1 \leq 0. \] (25)
The equality in \((25)\) holds if and only if \( v = 0 \). On the other hand, for any \( \alpha \in \mathbb{R} \),
\[ D\lambda_2 \cdot r_2 \geq 0. \] (26)
Likewise, the equality in \((26)\) holds if and only if \( v = 0 \). Thus, for \( 0 < \alpha < 2 \), the system is genuinely nonlinear on \( \mathcal{A} \).

### 3.3. Riemann invariants

Now, we compute Riemann invariants for the above system. Later, we show that solutions whose initial values take values in \( \mathcal{A} \) remain in \( \mathcal{A} \). Set
\[ A(\alpha) = 2 - \alpha + \sqrt{4 - 2\alpha + \alpha^2} \quad \text{and} \quad B(\alpha) = 2 - \alpha - \sqrt{4 - 2\alpha + \alpha^2} \] (27)
and note that \( A(\alpha) \) is a positive for all \( \alpha \in \mathbb{R} \) whereas \( B(\alpha) \) is negative if and only \( \alpha > 0 \).

Moreover,
\[ B(\alpha) + 2 > 0 \iff \alpha < 2. \] (28)
Define
\[ S_0 := \{(s, \alpha) : s < 0, \ \alpha \geq 0\} \]
and
\[ S_1 := \left\{ (r, \alpha) : r < \frac{2B(\alpha)}{B(\alpha) + 2}, 0 < \alpha < 2 \right\}. \]
In the following Lemmas, we compute Riemann invariants for \((9)\).
Lemma 3.2. The family of functions
\[ z(v, m) = v^{\frac{1}{4s(\alpha)}} m^2, \]
s \in \mathbb{R} are Riemann invariants for (9). Moreover, if \((s, \alpha) \in S_0\) then \(z\) is convex.

Proof. We recall that \(z\) is a Riemann invariant corresponding to \(r_1\) if 
\[ Dz \cdot r_1 = 0. \]
This means that 
\[ A(\alpha)v \partial_v z - 2m \partial_m z = 0 \] (29)
for 
\[ A(\alpha) = 2 - \alpha + \sqrt{4 - 2\alpha + \alpha^2}. \]
Setting \(z(v, m) = a(v)b(m)\), (29) becomes 
\[ A(\alpha)v a'(v)b(m) = 2ma(v)b'(m), \]
or 
\[ A(\alpha)\frac{a'(v)}{a(v)} = 2m\frac{b'(m)}{b(m)}. \]
Thus, \(a(v) = v^{s(A(\alpha))}\) and \(b(m) = m s/2\) for \(s \in \mathbb{R}\). Therefore, 
\[ z(v, m) = v^{s(A(\alpha))} m^2. \]
To study the convexity of \(z\), we compute its Hessian:
\[ D^2z(v, m) = \begin{pmatrix}
s(-A(\alpha)+s)m s/2 v^{s(A(\alpha)) - 2} & s^2 m s/4 v^{s(A(\alpha)) - 1} \\
\frac{s^2 m s/4 v^{s(A(\alpha)) - 1}}{2A(\alpha)} & \frac{1}{4}(s - 2)s m s/2 v^{s(A(\alpha)) - 2}
\end{pmatrix}. \]
First, we observe that the trace of \(D^2z(v, m)\) has the sign of 
\[ s \left[ s \left( A^2(\alpha)v^2 + 4m^2 \right) - 2A^2(\alpha)v^2 - 4A(\alpha)m^2 \right]. \]
Thus, the trace is positive if only if 
\[ s < 0 \quad \text{or} \quad s > \frac{2A^2(\alpha)v^2 + 4A(\alpha)m^2}{A^2(\alpha)v^2 + 4m^2}. \] (30)
The determinant of \(D^2z(v, m)\) has the sign of 
\[ -s^2 \left[ (A(\alpha) + 2)s - 2A(\alpha) \right]. \] (31)
Hence, the determinant is positive if and only if 
\[ s < \frac{2A(\alpha)}{A(\alpha) + 2}. \] (32)
Observe that 
\[ 0 < \frac{2A(\alpha)}{A(\alpha) + 2} < \frac{2A^2(\alpha)v^2 + 4A(\alpha)m^2}{A^2(\alpha)v^2 + 4m^2} \] (33)
for all \(\alpha \in \mathbb{R}\). In view of (30), (32) and (33), if \((s, \alpha) \in S_0\), then \(z\) is convex. \(\Box\)

Lemma 3.3. Let \(0 < \alpha < 2\). The family of functions
\[ w(v, m) = v^{\frac{1}{4r}} m^2, \]
r \in \mathbb{R} are Riemann invariants for (9). Moreover, if \((r, \alpha) \in S_1\) then \(w\) is convex.
Proof. If \( w \) is a Riemann invariant corresponding to \( r_2 \) then

\[
\nabla w \cdot r_2 = 0.
\]

Furthermore, we have (29) for \( w(v, m) = a(v)b(m) \), with \( a(v) = v^{r/B(\alpha)} \), \( b(m) = m^{r/2} \), and \( r \in \mathbb{R} \).

Moreover,

\[
D^2w(v, m) = \begin{pmatrix}
\frac{r(r-B(\alpha))m^{r/2}w - \frac{m^{r+1}}{r}}{B(\alpha)} & \frac{r^2m^{r-1}w^{r+1}}{2B(\alpha)} \\
\frac{r^2m^{r-1}w^{r+1}}{2B(\alpha)} & \frac{1}{4}(r-2)r^2m^{r-2}w^{r+1}
\end{pmatrix}.
\]

Note that the first leading principal minor of \( D^2w(v, m) \) has the same sign as

\[
r(r-B(\alpha)),
\]

and is thus positive if only if

\[
r < B(\alpha) \quad \text{or} \quad r > 0.
\]

The determinant of \( D^2w(v, m) \) has the sign of

\[
-r^2[(B(\alpha) + 2)r - 2B(\alpha)].
\]

Thus, the determinant is positive if and only if

\[
r < \frac{2B(\alpha)}{B(\alpha) + 2}.
\]

If \((r, \alpha) \in S_1\) then all the principal minors of the Hessian \( D^2w(v, m) \) are positive. Accordingly, \( w \) is convex.

![Figure 1. Domain \( S_1 \).](image)

#### 4. Parabolic forward-forward MFGs.

Now, we consider the parabolic forward-forward MFG corresponding to (8):

\[
\begin{align*}
v_t + \left( \frac{v^2}{2m^r} \right)_x &= \varepsilon v_{xx}, \\
m_t - \left( \frac{v}{m^{r-1}} \right)_x &= \varepsilon m_{xx},
\end{align*}
\]

(37)
with \( \varepsilon > 0 \) and initial conditions
\[
\begin{aligned}
v(x,0) &= v_0(x) \\
m(x,0) &= m_0(x),
\end{aligned}
\tag{38}
\]
where \((v_0, m_0)\) takes values in a compact subset, \(\mathcal{K}\), of \(\mathcal{A}\), where, as before,
\[
\mathcal{A} = \{(v,m) \in \mathbb{R}^2 : v > 0, \ m > 0\}.
\]

Standard PDE theory guarantees that (37)-(38) has a unique classical solution \((v^\varepsilon, m^\varepsilon)\) on \(T \times [0, T_\infty)\) for some \(0 < T_\infty \leq \infty\). Our goal is to prove that the maximal existence time is \(T_\infty = \infty\).

We recall the Riemann invariants from Lemmas 3.2 and 3.3
\[
z(v,m) = v \sqrt{\frac{\varepsilon}{A}} m^x \quad \text{and} \quad w(v,m) = v \sqrt{\frac{\varepsilon}{B}} m^x,
\tag{39}
\]
where \(A, B\) are defined in (27). By the same Lemmas, \(w\) and \(z\) are convex if \((s, \alpha) \in \mathcal{S}_0\) and \((r, \alpha) \in \mathcal{S}_1\).

**Proposition 2.** Assume that \((s, \alpha) \in \mathcal{S}_0\) and \((r, \alpha) \in \mathcal{S}_1\) and that the initial conditions in (38) for (37) take values in a compact subset set, \(\mathcal{K}\), of \(\mathcal{A}\) and satisfy \(w(v_0, m_0) < M\) and \(z(v_0, m_0) < M\) for some \(M > 0\). Then, the solution \((v^\varepsilon, m^\varepsilon)\) of (37) satisfies
\[
w(v^\varepsilon, m^\varepsilon) < M \quad \text{and} \quad z(v^\varepsilon, m^\varepsilon) < M
\]
for all \(t \in [0, T_\infty)\). Moreover, there exists \(\tilde{c}_0\) such that \(m^\varepsilon(t, x) \geq \tilde{c}_0 > 0\) on \(T\) for \(t \in [0, T_\infty)\). Furthermore, \(v(x,t) > 0\) for \(t \in [0, T_\infty)\).

**Proof.** First, using (37), we get
\[
w(v,m)_t + \lambda_1 w(v,m)_x = \varepsilon \left[ (w(v,m)_{xx} - (v_x, m_x)^T D^2 w(v,m)(v_x, m_x)) \right]
\]
and
\[
z_t(v,m) + \lambda_2 z_x(v,m) = \varepsilon \left[ (z(v,m)_{xx} - (v_x, m_x)^T D^2 z(v,m)(v_x, m_x)) \right].
\]
Here, \(\lambda_1\) and \(\lambda_2\) are eigenvalues as obtained in Proposition 1. Since \(z\) and \(w\) are convex, we obtain
\[
w(v,m)_t + \lambda_1 w(v,m)_x \leq \varepsilon w(v,m)_{xx}
\]
and
\[
z(v,m)_t + \lambda_2 z(v,m)_x \leq \varepsilon z(v,m)_{xx}.
\]
Finally, we use the maximum principle to get that, if \(w(v_0, m_0) < M\) and \(z(v_0, m_0) < M\) for some \(M > 0\), then the solution, \((v^\varepsilon, m^\varepsilon)\), of (37) satisfies
\[
w(v^\varepsilon, m^\varepsilon) < M \quad \text{and} \quad z(v^\varepsilon, m^\varepsilon) < M.
\]
Next, we observe that \(z, w < M\), with \(z\) and \(w\) given in (39), implies that \(m\) is bounded by below, as can be seen from the level sets of \(z\) and \(m\) depicted in Figure 2.

In particular, \(m\) satisfies
\[
m > M \frac{A(s) - B(s)}{|A(s) - B(s)|},
\]
where the preceding lower bound is determined by the value of \(m\) corresponding to \(z = w = M\).

Finally, if \(v_0 > 0\), the condition \(z < M\) gives that \(v(x,t) > 0\) for all \(t\).

Next, we combine the lower bound from the preceding lemma with the entropy from Lemma 3.1 to improve the integrability of \(v\).
Lemma 4.1. Let \( v, m \in C^2(\mathbb{T} \times (0,T_\infty)) \cap C(\mathbb{T} \times (0,T_\infty)) \) be the classical solution for (37) and (38) (we drop the index \( \varepsilon \) for simplicity). Let \( \eta \) be a smooth function satisfying (11). Then,

\[
\frac{d}{dt} \int_\mathbb{T} \eta(v,m)dx = -\varepsilon \int_\mathbb{T} (v_x, m_x)^T D^2\eta(v,m)(v_x, m_x)dx.
\]  

Furthermore, assume that \( m(\cdot, t) \in L^p(\mathbb{T}) \) for \( 0 \leq t < T_\infty \) and that \( \eta(v,m) = v^a m^b \) with \( a > 1 \) and \( b \) given by (15). Then,

\[
\int_\mathbb{T} v^{\frac{ap}{p-b}}(x,t)dx \leq C
\]  

for \( 0 \leq t < T_\infty \).

Proof. Because \( \eta \) is an entropy for (9), we have (40). Moreover, according to Remark 1, \( \eta(v,m) = v^a m^b \) is convex. Thus, (40) implies that

\[
\int_\mathbb{T} v^a(x,t)m^b(x,t)dx \leq \int_\mathbb{T} v^a(x,0)m^b(x,0)dx.
\]  

Because \( m(x,t) \) is bounded away from 0 for all \( 0 \leq t < T_\infty \) and \( x \in \mathbb{T} \), by Proposition 2 and because \( b < 0 \), we have

\[
\left\| m^b(x,t) \right\|_{L^\infty(\mathbb{T})} \leq C_1.
\]  

By Hölder’s inequality,

\[
\int_\mathbb{T} v^{\frac{ap}{p-b}}(x,t)dx \leq \left( \int_\mathbb{T} v^a m^b dx \right)^{\frac{p-b}{p}} \left( \int_\mathbb{T} m^p(x,t)dx \right)^{\frac{1}{p-b}}.
\]

Finally, we combine (42), (43), and (44) to obtain (41). \( \square \)

By considering the limit \( a \to \infty \), we obtain the following corollary.

Corollary 1. If \( m(\cdot, t) \in L^p(\mathbb{T}) \) for \( 0 \leq t < T_\infty \) then

\[
\int_\mathbb{T} v^{\tilde{p}}(x,t)dx \leq C
\]  

for all \( \tilde{p} \leq \frac{p}{\theta(\alpha)} \), where \( \theta(\alpha) \) is given by (17).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Level sets of the Riemann invariants for (37) with \( \alpha = \frac{3}{2} \).}
\end{figure}
Remark 3. For $0 < \alpha < 2$, we have $0 < \theta(\alpha) < 1$. Hence, $\frac{p}{\theta(\alpha)} > p$.

Proof. The corollary follows by taking $a \to \infty$ and using (16). \qed

Lemma 4.2. Let $v, m \in C^2(T \times (0, T_\infty)) \cap C(T \times (0, T_\infty))$ be a classical solution of (37)-(38). Suppose the initial conditions in (38) for (37) take values in a compact subset set, $K$, of $A$. Then,

$$\frac{d}{dt} \int_T m^r(x, t) dx \leq C$$

for all $1 < r < \infty$ and all $0 \leq t < T_\infty$.

Proof. Using the second equation in (37), we have

$$\frac{d}{dt} \int_T m^r(x, t) dx = \int_T r m^{r-1} m_t dx$$

Integrating by parts, we get

$$\frac{d}{dt} \int_T m^r(x, t) dx = -r(r-1) \int_T m^{r-2} m_x (vm^{1-\alpha} + \varepsilon m_x) dx$$

It thus follows that

$$\frac{d}{dt} \frac{1}{r(r-1)} \int_T m^r(x, t) dx \leq \int_T |m^{r-2} m_x vm^{1-\alpha}| - \int_T \varepsilon m^{r-2} m_x^2 dx. \quad (47)$$

Next, by Cauchy inequality,

$$\int_T |m^{r-2} m_x vm^{1-\alpha}| dx = \int_T |m^{r/2-1} m_x vm^{r/2-\alpha}| dx \leq \frac{1}{2\varepsilon} \int_T m^{r-2} v^2 dx + \frac{1}{2} \int_T \varepsilon m^{r-2} m_x^2 dx. \quad (48)$$

We combine (47) and (48) to obtain

$$\frac{d}{dt} \frac{1}{r(r-1)} \int_T m^r(x, t) dx \leq C \int_T m^{r-2} v^2 dx - \frac{1}{2} \int_T \varepsilon m^{r-2} m_x^2 dx. \quad (49)$$

Now, for $a > 1$ and $b = b(\alpha, a)$ as in (15), we rewrite

$$\int_T m^{r-2} v^2 dx = \int_T m^{r-2} v^{r-2} \left( m^b v^a \right)^{\frac{2}{a}}$$

$$\leq \left( \int_T m^{(r-2)\frac{2}{a}} \right)^{\frac{a-2}{a}} \left( \int_T m^b v^a \right)^{\frac{2}{a}}$$

$$\leq C_a \left( \int_T m^{(r-2)\frac{2}{a}} \right)^{\frac{a-2}{a}}.$$

Next, we notice that as $a \to \infty$, we have

$$r - 2\alpha - \frac{b}{a} \frac{a}{a-2} \to r - 2\alpha + 2\theta(\alpha) < r.$$
Accordingly, for large enough $a$,
\[
\int_T m^{r-2a}v^2 \, dx \leq C \left( \int_T m^r \right)^\mu,
\]
for some $\mu < 1$.

Next, we use Morrey’s theorem to obtain that
\[
\|m\|_{L^\infty} \leq C \left( 1 + \int_T [(m^{r/2})_x]^2 \, dx \right).
\]
Because $\mu < 1$, Young’s inequality yields
\[
C^\mu \left( 1 + \int_T [(m^{r/2})_x]^2 \, dx \right) \leq \frac{\varepsilon^2}{2C^2} \int_T [(m^{r/2})_x]^2 \, dx + C_{\varepsilon,\mu}
\]
for some $C_{\varepsilon,\mu}$. We combine (50) with the preceding inequalities to get
\[
\int_T m^{(r-2\alpha)}v^2 \, dx \leq \frac{1}{4} \int_T \varepsilon m^{r-2}m_x^2 \, dx + C_{\varepsilon,\mu}.
\]
Finally, we use (49) and (51) to obtain
\[
\frac{d}{dt} \left( \frac{1}{r(r-1)} \int_T m^r (x,t) \, dx \right) \leq C_{\varepsilon,\mu} - \frac{1}{4} \int_T \varepsilon m^{r-2}m_x^2 \, dx.
\]
The estimate (46) follows from (52).

Finally, we prove our main result, the existence of a global solution for (37).

**Theorem 4.3.** Suppose the initial conditions in (38) for (37) take values in a compact subset set, $K$, of $A$ and satisfy $w(v_0, m_0) < M$ and $z(v_0, m_0) < M$ for some $M > 0$. Then, the maximal existence time $T_\infty$ is $T_\infty = +\infty$.

**Proof.** Suppose that the maximal existence time, $T_\infty$, satisfies $T_\infty < \infty$. First, we notice that by the conservation of mass, we have
\[
\sup_{0 \leq t < T_\infty} \|m\|_{L^1} = 1.
\]
Thus, using first Lemma 4.2, we obtain that
\[
\sup_{0 \leq t < T_\infty} \|m(\cdot, t)\|_{L^p(T)} < C
\]
for all $p < \infty$. Next, using Corollary 1, we obtain
\[
\sup_{0 \leq t < T_\infty} \|v(\cdot, t)\|_{L^q(T)} < C
\]
for all $q < \infty$. Thus, the solution $(v, m)$ is bounded in $L^p \times L^q$ uniformly in $t$ up to $T_\infty$. Finally, a simple regularity argument for parabolic equations gives that the solution is classical up $T_\infty$ and, thus, can be continued for $t > T_\infty$, which contradicts the maximality of $T_\infty$.

5. **Traveling waves.** In this final section, we compute traveling waves for forward-forward MFGs and for MFGs with congestion with an anti-monotonic coupling. For forward-forward MFGs, the existence of traveling waves shows that these PDEs may fail to converge to a stationary solution. Similarly, for MFGs with congestion, the existence of traveling waves indicates that without monotonicity, convergence to a stationary solution may as well not hold. As far as we know, these are the first examples of traveling waves in MFGs.
5.1. Traveling waves for forward-forward congestion MFGs. We consider the following forward-forward congestion problem:

\[
\begin{aligned}
&v_t + \left(\frac{v^2}{2m^\alpha} - K m^\alpha\right)_x = 0 \\
&m_t - (m^{1-\alpha} v)_x = 0,
\end{aligned}
\]

(53)

with $K > 0$. It is straightforward to check that for smooth initial data $m_0, v_0$ such that $v_0 = cm_0^\alpha$ with $c = \pm \sqrt{\frac{2K}{\alpha}}$, we have that

\[
m(x,t) = m_0(x + ct) \quad \text{and} \quad v(x,t) = cm^\alpha(x,t)
\]

solve (53).

**Remark 4.** In the presence of diffusion at least in models without congestion, the forward-forward model converges to a stationary solution as shown in [20]. We expect the same behavior in model with diffusion and congestion. To obtain a traveling wave, we use a special structure that is not present in model without congestion.

5.2. Traveling waves for non-monotonic MFGs with congestion. Now, we consider the following non-monotonic MFG with congestion:

\[
\begin{aligned}
&-v_t + \left(\frac{v^2}{2m^\alpha} + K m^\alpha\right)_x = 0 \\
&m_t - (m^{1-\alpha} v)_x = 0,
\end{aligned}
\]

(54)

where $K > 0$. When $\alpha = 1$, then

\[
\begin{aligned}
&v_t + \left(\frac{v^2}{2m} + K m\right)_x = 0 \\
&m_t - v_x = 0.
\end{aligned}
\]

(55)

This congestion problem admits traveling wave solutions:

\[
m(t, x) = m_0(x - \sqrt{2K}t) \quad \text{and} \quad v(t, x) = -v_0(x - \sqrt{2K}t)
\]

and

\[
m(t, x) = m_0(x + \sqrt{2K}t) \quad \text{and} \quad v(t, x) = v_0(x + \sqrt{2K}t),
\]

which solve (55) if the initial data $(m_0, v_0)$ is smooth and satisfies $v_0 = \sqrt{2K}m_0$.

5.3. Potentials. For $\alpha = 1$, the two preceding MFGs can be converted into a scalar equation by introducing a potential function. In this last section, we record these remarkable equations.

The second equation in (53) becomes

\[
m_t - v_x = 0.
\]

Thus, we introduce a potential, $q$, such that $m = q_x$ and $v = q_t$. Accordingly, the first equation in (53) becomes

\[
-K + \frac{q_t^2}{2q_x^2} q_{xx} + \frac{q_t}{q_x} q_{xt} + q_{tt} = 0,
\]

which is a wave-type equation for $q$.

For the non-monotonic MFG, (55) and $q$ such that $m = q_x$ and $v = q_t$, we have

\[
K - \frac{q_t^2}{2q_x^2} q_{xx} + \frac{q_t}{q_x} q_{xt} + q_{tt} = 0.
\]
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