On fractional-order maps and their synchronization

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Abstract

We study the stability of linear fractional order maps. We show that in the stable region, the evolution is described by Mittag-Leffler functions and a well defined effective Lyapunov exponent can be obtained in these cases. For one-dimensional systems, this exponent can be related to the corresponding fractional differential equation. A fractional equivalent of map \( f(x) = ax \) is stable for \( a_c(\alpha) < a < 1 \) where \( \alpha \) is a fractional order parameter and \( a_c(\alpha) \approx -\alpha \). For coupled linear fractional maps, we can obtain ‘normal modes’ and reduce the evolution to effectively one-dimensional system. If the eigenvalues are real the stability of the coupled system is dictated by the stability of effectively one-dimensional normal modes. For complex eigenvalues, we obtain a much richer picture. However, in the stable region, the evolution of modulus is dictated by Mittag-Leffler function and the effective Lyapunov exponent is determined by modulus of eigenvalues. We extend these studies to synchronized fixed points of fractional nonlinear maps.

1 Introduction

After May’s influential paper [1], the difference equations gained wide popularity in all walks of science. They were studied from the viewpoint of nonlinear dynamics and chaos which was ubiquitous in disciplines ranging from ecology to economics. Mathematicians studied it as an interesting object in its right [2]. The studies on maps are computationally less intensive. In certain cases, it was easier to track them analytically. Feigenbaum’s study of period-doubling bifurcations in logistic maps is an example [3,4]. Lower order difference equations can show the same phenomena as higher-order differential equations. For example, we can observe chaos in a logistic map that has a single variable while we need
at least three first-order autonomous differential equations to observe chaos. Insights gained from studies in difference equations were often (though not always) useful in studies in differential equations and vice versa. From control problems to synchronization schemes, many theoretical ideas have found applications in both difference and differential equations \[5\]. In this work, we study fractional linear difference equations and coupled fractional linear difference equations. We show that the analysis of linear fractional difference equations has remarkable similarities with corresponding fractional differential equations. For coupled difference equations, the Jacobian, its eigenvalues and eigenvectors play a central role. We show that similar concepts can be very useful for coupled fractional difference equations.

Synchronization of dynamical systems has been extensively studied in the past three decades both theoretically as well as experimentally. Exact synchronization for master-slave type systems as well as synchronization for mutually coupled systems has been investigated extensively. Apart from exact synchronization, several other types such as anti-synchronization, anticipated synchronization, lag synchronization, phase synchronization, generalized synchronization, etc. have been studied. For spatially extended systems, the connectivity matrix of underlying topology plays an important role in synchronization. For exact synchronization, we have a clear mathematical formalism for finding necessary and sufficient conditions for synchronization.

Ordinary differential equations have successfully described a variety of physical system and found countless applications since Newton and Leibniz introduced them. However, memory plays an important role in many physical systems. Apart from a mathematical curiosity, such systems are modeled by fractional-order differential equations from viewpoint of applications. For fractional-order systems, fractional order maps have been introduced recently. These systems are not very well investigated yet. In a previous work \[6\], a coupled map lattice model of fractional order maps was studied. It was found that it is possible to have synchronization even in the thermodynamic limit in these systems. However, the error reduced as a power-law. The exponent is the same as the fractional order of maps. We give certain pointers to understand these findings analytically in this work.

As in differential equations, we start by studying the stability of a fixed point in linear systems. We study two coupled fractional maps and derive conditions for synchronization analytically. For simple linear maps the bounds of synchronization and its relation to Mittag-Leffler function can be shown analytically. Asymptotically, Mittag-Leffler function behaves as a power-law and this could be the reason for the observation in previous work. For a linear function with constant slope, this relation can be shown very clearly. We will be studying symmetric coupling. However, most of the studies can be easily extended to asymmetric coupling. We give conditions for the stability of the synchronized state and demonstrate it with certain examples. We propose that the usual definition of Lyapunov exponent using logarithm (which is inverse of exponential) should be appropriately modified to obtain an accurate quantifier that describes the convergence of trajectories in a stable regime. (Mittag-Leffler function is a power-law asymptotically which is slower decay than exponential. Thus the Lyapunov exponent will always be zero if we fit it with an exponential.) However, the divergence of trajectories in the unstable region is still exponential and the usual definition of Lyapunov exponent may hold in this case.
There are several definitions of fractional differential equations and the same is true for fractional difference equations. We will study the definition obtained by Gejji and Deshpande [7]. The evolution depends on the value at all previous time-steps. The weight of previous values decays as a power-law and there is a long term memory built-in in the system. It is not surprising that the fluctuations also decay extremely slowly and decay can be approximated by power-law with power related to the order of fractional difference equation. This is in turn related to the properties of Mittag-Leffler function which plays an important role in fractional differential equations and plays a similar role here. In difference equations, the concept of coupled maps was introduced by Kapral, Kuznetsov, and Kaneko. Kaneko can be credited for making it popular.

The fractional difference equations have been studied only recently. In 1989, Miller and Ross began this investigations [8]. Some of the studies in fractional difference equations are due to Atici and coworkers [9,10], Holm [11], and others [12,13]. We extend this definition to the coupled fractional difference equations. While exploratory works on fractional equivalents of known nonlinear maps can yield useful insights, we focus on linear systems in this work. Linearizations of nonlinear systems are a standard tool in difference equations as well as coupled difference equations. Most of analytic work in these systems is dependent on linearization. Thus understanding linear systems and coupled linear systems is extremely important in nonlinear dynamics. We believe that studies in linear systems can be equally useful in fractional difference equations.

2 Effective Lyapunov Exponent

In this section, we introduce an alternative viz. Effective Lyapunov Exponent (ELE) for the classical Lyapunov exponent. We will follow the notation and definition used by Deshpande and Gejji [7]. They define the fractional equivalent for the $x(n + 1) = f(x(n))$ in the following manner. They construct an discrete Caputo-type fractional difference operator and define $u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \gamma_{\alpha}(t-j) (f(j-1, u(j-1)) - u(j-1))$ in general. We assume that the function $f$ does not depend on time. We define $\gamma_{\alpha}(k) = \frac{\Gamma(k+\alpha)}{\Gamma(k+1)}$ and alternatively write above expression as $u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \gamma_{\alpha}(t-j) (f(u(j-1)) - u(j-1))$.

Few one-dimensional maps such as the Gaussian map and Bernoulli map have been studied in this work. Liu has numerically investigated coupled fractional Henon map [14]. Henon map is a two-dimensional map and Liu introduces memory in only one of the variables. We will call such systems ‘fractional difference equations of inhomogeneous order’. On the other hand, we will investigate maps of homogeneous order.

Consider $f(x) = rx$ where $r \in \mathcal{R}$

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} \gamma_{\alpha}(t-j) (r - 1) u(j-1) \tag{1}$$

can be identified with the continuous-time system

$$D^\alpha u(t) = (r - 1) u(t) \tag{2}$$

for sufficiently small values of coefficient.
The continuous-time system has exact solution in terms of Mittag-Leffler function as below

\[ u(t) = u(0) E_\alpha ((r - 1)t^\alpha). \]  

This can be alternatively written as

\[ u(t) = u(0) E_\alpha (\lambda t^\alpha). \]  

where \( \lambda_e \) is effective Lyapunov exponent. For a linear first order ordinary differential equation \( x'(t) = \lambda x(t) \), the solution would be \( x(t) = x(0) \exp(\lambda t) \) where \( \lambda \) is Lyapunov exponent. When it is negative, the system goes to absorbing state. Our formulation could be considered as generalization

\[ \lambda_e = \lim_{t \to \infty} t^{-\alpha} E^{-1}_\alpha \left( \frac{u(t)}{u(0)} \right). \]  

This formulation is very similar to standard definition of Lyapunov exponent for \( \alpha = 1 \) where \( E_\alpha(x) = \exp(x) \). We will demonstrate that this quantity is a well-defined quantity which indeed converges in the stable regime. On the other hand, if we insist on using the definition of Lyapunov exponent used for ordinary differential equations, it will lead to zero value. The reason is that Mittag-Leffler function is a power-law asymptotically which is slower than exponential. Like Lyapunov exponent, the effective Lyapunov exponent is negative in the absorbing state. In Fig. 2 we sketch the numerically computed \( \lambda_e \) for various values of \( r \). It is clear that \( \lambda_e = r - 1 \).

The system is unstable for \( r > 1 \) for any \( \alpha \). However, lower bound \( a_c(\alpha) \) depends on \( \alpha \). Lower bound \( a_c(\alpha) \to -1 \) as \( \alpha \to 1 \). This is an expected limit for integer order difference equation.

### 3 Fractional order coupled maps

We define two coupled maps in this setting and define.

\[
\begin{align*}
    x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t-j)G(x(j-1), y(t-j)), \\
    y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t-j)G(y(j-1), x(t-j))
\end{align*}
\]
Figure 2: Computation of $\lambda_e$ for few values of $r$. The value obtained is $r - 1$.

$$G(a,b) = \delta f(a) + q f(b) - a.$$

**Case 1: Real ‘Normal Modes’**

First we consider the case $f(x) = x$ for which the coefficient matrix of the system has real eigenvalues viz. $\delta + q$ and $\delta - q$. Now we consider two new variables $u(t) = x(t) + y(t)$ and $v(t) = x(t) - y(t)$ and obtain

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t-j)((\delta + q - 1)u(j-1)),$$  \hspace{1cm} (7)

$$v(t) = v_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t-j)((\delta - q - 1)v(j-1)).$$  \hspace{1cm} (8)

This is a simple decoupled system of linear difference equations.

Again, the discrete-time equations above can be identified with the continuous-time system

$$D^\alpha u(t) = (\delta + q - 1)u(t)$$  \hspace{1cm} (9)

$$D^\alpha v(t) = (\delta - q - 1)v(t)$$  \hspace{1cm} (10)

for sufficiently small values of coefficient and exact solution in terms of Mittag-Leffler function is given as

$$u(t) = u(0)E_\alpha((\delta + q - 1)t^\alpha).$$  \hspace{1cm} (11)

$$v(t) = v(0)E_\alpha((\delta - q - 1)t^\alpha).$$  \hspace{1cm} (12)

Thus the effective Lyapunov exponents are given by $\delta + q - 1$ and $\delta - q - 1$ for the above decoupled system. We compute the system for $T = 8 \times 10^4 - 10^5$ time-steps. The decay is extremely slow for smaller $\alpha$. After discarding first 2000 time-steps, we check if the distance of $(x(t),y(t))$ from origin every 100 time-steps and the trajectory is stable if the distance does not increase till time $T$.

The stability regime for the above system is plotted as a function of $\delta$ and $q$ in Fig. 3. The stability region is a rhombus bounded by lines parallel to $\delta = q$.
and $\delta = -q$. For all values of $\alpha$, the system is unstable for $\delta + q - 1 > 0$ and $\delta - q - 1 > 0$. The other two bounding lines change with $\alpha$. For $q = 0$, we have an effectively one-dimensional system. The bounds for $q = 0$ are $\delta = 1$ and $\delta = a_c(\alpha)$. As $\alpha \to 1$, $a_c(\alpha) \to -1$. The lines enclosing stability region are given by $\delta = 1$ and $\delta + q = a_c(\alpha)$. It is clear from $a_c(\alpha)$ are close to $-\alpha$. This explains the rhombus structure in Fig. 3. (For $f(x) = rx$, the phase diagram will be similar except that the values of $\delta$ and $q$ will be scaled to $\frac{\delta}{r}$ and $\frac{q}{r}$.)

**Synchronization and antisynchronization**

We consider the linear system (6) with real normal modes. As shown in (7)–(8), the sum and difference variables $u(t) = x(t) + y(t)$ and $v(t) = x(t) - y(t)$ are effectively decoupled. We can also say the $u$ corresponds to $(1, 1)$ mode and $v$ corresponds to $(1, -1)$ mode. If the effective Lyapunov exponent corresponding to variable $u$ is in the unstable region while the one corresponding to $v$ is in stable region, we will find that $v(t) \to 0$. Thus $x(t) \to y(t)$ implying synchronization. On the other hand if $\lambda_e$ corresponding to $v(t)$ is in unstable region and corresponding to $u(t)$ is in the stable region we will observe that $v(t) \to 0$. This phenomenon is termed as antisynchronization. Fig. 4 shows both these phenomena for different values of parameters. We also find that the decaying mode decays slower than exponential making it necessary to give a new definition for Lyapunov exponent. However, growing mode increases exponentially. The reason may be that approximating fractional difference equation by fractional differential equation may not be valid for large values. In coupled map lattices, it is known that the condition for chaotic synchronization is that all modes except one corresponding to the mean, i.e. eigenmode $(1, 1, \ldots, 1)$ should decay. We can obtain normal modes in fractional system in a similar manner and find conditions for chaotic synchronization.

**Case 2: Complex ‘Normal Modes’**

Now let us consider another coupled system

\begin{align*}
  x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t - j)(\delta x(j - 1) + q y(t - j) - x(j - 1)), \quad (13) \\
  y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t - j)(\delta y(j - 1) - q x(t - j) - y(j - 1)). \quad (14)
\end{align*}
The eigenvalues of coefficient matrix of the system (13)–(14) are $\delta \pm iq$.

We have plotted stability region in $\delta - q$ space for various values of $\alpha$ in Fig. [5]. This structure is far richer than one obtained in Fig. [3]. As $\alpha \to 1$, the stability region tends to the unit circle in the complex plane which is a stability region for integer-order difference maps in two dimensions.

As $\alpha$ decreases, the stability region gradually deforms from an unit circle to a non-convex shape.

Consider $z(t) = x(t) + iy(t)$.

Thus

$$z(t) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t - j) (((\delta - 1) - iq)x(j - 1) + i(\delta - 1 - iq)y(j - 1)). \quad (15)$$

Similarly, if we define $\bar{z}(t) = x(t) - iy(t)$, we get

$$\bar{z}(t) = \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{t} g_\alpha(t - j)((\delta - 1) + iq)\bar{z}(j - 1). \quad (17)$$

The variables $z$ and $\bar{z}$ can be associated to ordered pairs $(x, y)$ and $(x, -y)$ describing a complex number and its conjugate in complex plane.

The equivalent continuous-time system of (13)–(14) is given by

$$D^\alpha x = (\delta - 1)x + qy \quad (18)$$
$$D^\alpha y = -qx + (\delta - 1)y. \quad (19)$$

The general solution of system (18)–(19) is

$$x(t) + iy(t) = E_\alpha ((\delta - 1 - iq)t^\alpha) (x(0) + iy(0)) \quad (20)$$

If we define $z(t) = x(t) + iy(t)$ and $\lambda'_c = (\delta - 1 - iq)$, we have
Figure 5: Stability region for coupled maps with antisymmetric coupling. It is clear that the stability region approaches unit circle as $\alpha \to 1$ while it is significantly different for small $\alpha$. The stability region is shown for different values of $\alpha$ in ascending order in a) and in descending order in b).

\[ z(t) = E_\alpha (\lambda'_e t^\alpha) z(0). \]  

(21)

This motivates us to define the effective Lyapunov exponent $\lambda'_e$ of the above system as

\[ \lambda'_e = \lim_{t \to \infty} t^{-\alpha} E^{-1}_\alpha \left( \frac{z(t)}{z(0)} \right). \]  

(22)

This is a well-defined quantity and we always find $\lambda'_e = \delta - 1 - \iota q$ numerically. This is a complex number. On the other hand, Lyapunov exponents for dynamical systems have real Lyapunov exponents. We believe that a single real number can determine stability for difference equation because the stability condition for integer-order differential equation is that the real part of eigenvalues is less than zero. This is a single condition. On the other, for fractional order differential equations, stability condition is given by two lines. Thus a single number may not be enough to determine stability of fractional equations. However, if we insist that the effective Lyapunov exponents should be real, we can take $-|\lambda'_e|$ as an effective Lyapunov exponent. Alternatively, we may define the effective Lyapunov exponent $\lambda_c$ of system (18)–(19) as

\[ \lambda_c = \lim_{t \to \infty} \left| t^{-\alpha} E^{-1}_\alpha \left( \frac{x(t) + \iota y(t)}{x(0) + \iota y(0)} \right) \right| = \sqrt{(\delta - 1)^2 + q^2}. \]  

(23)

Note that, $\sqrt{x(t)^2 + y(t)^2} = |E_\alpha ((\delta - 1 - \iota q)t^\alpha)|$. In general, it is not possible
Figure 6: Effective Lyapunov exponent $\lambda_e$ is plotted as function of expression $\sqrt{(\delta - 1)^2 + q^2}$.

In Fig. 6 we have plotted effective Lyapunov found numerically for various values of $\delta$, $q$ and $\alpha$ as a function of above quantity and it is clear that there is an excellent match.

4 Generalization to nonlinear case

Let us try to extend the analysis to nonlinear systems. Let us consider stability of the fixed point of a nonlinear system. Nonlinear system can be linearized by

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \ldots$$

We define $y = x - x^*$. Now $x_{n+1} = f(x_n)$ is equivalent to $y_{n+1} = x_{n+1} - x^* = f(x_n) - x^* = f'(x^*)y_n = ry_n$ where $r = f'(x^*)$.

Thus we can conjecture that the stability regime for the fixed point is given by

$$a_c(\alpha) < f'(x^*) < 1.$$  

Interestingly, this conjecture works. Consider the cases of Bernoulli map and Gauss map considered by Deshpande and Gejji [7].

For Bernoulli map $f(x) = rx \mod 1, r > 0$, we can guess that for $r < 1$, the fixed point is stable for any value of $\alpha$. This is precisely what Deshpande and Gejji [7] find. In the case of Gauss map given by $f(x) = \exp(-7.5x^2) + \beta$, the fixed point can be found numerically using bisection method or other methods.

The fixed point is close to $\beta$. The slope is negative and the stability regime is dependent on value of $\alpha$. One expects this fixed point to be more stable if $\alpha$ increases. It can be checked that $a_c(0.4) = -0.3132$, $a_c(0.6) = -0.5085$ and $a_c(0.8) = -0.7328$. The critical points corespond to values of $\beta$ at which the $f'(x^*)$ matches with these values. Thus we expect the fixed point to be stable for $\beta > 0.49$ for $\alpha = 0.4$, for $\beta > 0.57$ for $\alpha = 0.6$ and $\beta > 0.65$ for $\alpha = 0.4$ (cf. Fig. 7).

This can be confirmed from the above paper [7] as well as independent simulations. We start with $\beta$ as an initial condition.
5 Discussion and Conclusions

Analytic studies in nonlinear dynamics or coupled nonlinear systems are often based on local linearization of dynamics. Thus linear systems serve as a basis which helps understanding dynamics in these systems. We have studied the fractional equivalent of linear maps, which are not studied before to our knowledge. Our studies indicate that the results obtained are useful in studies of nonlinear systems as well. For coupled systems, we have studied the stability of the fixed point. We also find that the conditions for chaotic synchronization and antisynchronization and find that the conditions are very similar to those obtained for coupled integer order maps.

For one-dimensional $f(x) = rx$, we find that the stability regime is given by $a_c(\alpha) < r < 1$. In the stable regime the dynamics is governed by Mittag-Leffler function. We also define the effective Lyapunov exponent and find that $r - 1$ is effective Lyapunov exponent in this case.

For two coupled linear systems, the behavior is different for symmetric and antisymmetric coupling. The analysis is motivated by study of linear difference equations which is essentially the theory of matrices. We can reduce dynamics to ‘normal modes’ which could be real or complex. We can find effective Lyapunov exponents in these cases as well. When the normal modes are real, the stability condition is the same as the condition for a single linear map for each mode. A much richer picture is observed for complex normal modes. The effective Lyapunov exponents, in this case, are complex which is not entirely unexpected.

The stability region of a continuous-time fractional-order dynamical system is a superset of that of classical integer-order one. As we increase the fractional
order to 1, the cone-like stability region of the fractional case gets contracted to the left-half complex plane and we get the usual region of stability of classical case. In this article, we showed that the stability properties of discrete-time systems are different. The cardioid-like stability region of fractional order system gets deformed and converted to the unit disc as we increase the order to 1.

We also extend this work to fixed points of nonlinear maps and confirm that a similar criterion holds. This work can be extended in many directions. We can try to find the stability of periodic orbits of the higher period and find routes to chaos in low-dimensional fractional difference equations.

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