MULTIPLIERS FOR \( p \)-BESSEL SEQUENCES IN BANACH SPACES

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Abstract. Multipliers have been recently introduced as operators for Bessel sequences and frames in Hilbert spaces. These operators are defined by a fixed multiplication pattern (the symbol) which is inserted between the analysis and synthesis operators. In this paper, we will generalize the concept of Bessel multipliers for \( p \)-Bessel and \( p \)-Riesz sequences in Banach spaces. It will be shown that bounded symbols lead to bounded operators. Symbols converging to zero induce compact operators. Furthermore, we will give sufficient conditions for multipliers to be nuclear operators. Finally, we will show the continuous dependency of the multipliers on their parameters.

1. Introduction and Preliminaries

1.1. Introduction. In [28], R. Schatten provided a detailed study of ideals of compact operators using their singular decomposition. He investigated the operators of the form \( \sum_k \lambda_k \varphi_k \otimes \overline{\psi_k} \) where \((\varphi_k)\) and \((\psi_k)\) are orthonormal families. In [4] the orthonormal families were replaced with Bessel and frame sequences to define Bessel and frame multipliers.

Definition 1.1. Let \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) be Hilbert spaces, let \((\psi_k) \subseteq \mathcal{H}_1\) and \((\phi_k) \subseteq \mathcal{H}_2\) be Bessel sequences. Fix \( m = (m_k) \in l^\infty(K)\). The operator

2000 Mathematics Subject Classification. Primary 42C40; Secondary 41A58, 47A58.

Key words and phrases. Multiplier operator, Bessel sequence, Frame, Schauder basis, \( p \)-frame, \((p, q)\)-Bessel multiplier, \( p \)-Schatten operator, Nuclear operator, \((r, p, q)\)-Nuclear operators.
The sequence \( m \) is called the symbol of \( M \).

Several basic properties of these operators were investigated in [4]. For a theoretical approach it is very natural to extend this notion and consider such operators in more general settings. For \( p \)-Bessel sequences in Banach spaces is leading to interesting results in functional analysis and operator theory.

We are going to show only theoretical properties. Nevertheless it should be mentioned, that multipliers are not only interesting from a theoretical point of view, see e.g. [5] [6] [13], but they are also used in applications, in particular in the fields of audio and acoustics. The first frame multipliers investigated were Gabor (frame) multipliers [14]. In signal processing they are used under the name 'Gabor filters' [20] as a particular choice to implement a time-variant filter. In computational auditory scene analysis they are known by the name 'time-frequency masks' [31] and are used to extract single sound source out of a mixture of sounds in a way linked to human auditory perception. In real-time implementations of filtering systems, they approximate time-invariant filters [8] as they are easily implementable. On the other hand, as a particular way to implement time-variant filters, they are used for example for sound morphing [12] or psychoacoustical modeling [9]. In general the idea for a Gabor (or wavelet) multiplier is to amplify or attenuate parts of audio signal, which can be separated in the time-frequency plane.

Clearly Banach space theory is right at the foundation of functional analysis and operator theory, and as such is relevant for theory. But it recently also has become more and more important for time-frequency analysis, see e.g. [16]. It is used in engineering applications in compressed sensing, refer e.g. to [25], as well as in audio or image sampling [2]. Applications in wireless communication can be envisioned [17] [21].

\[
M_{m,(\phi_k),(\psi_k)} : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \text{ defined by } \\
M_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \langle f, \psi_k \rangle \phi_k
\]
Therefore we hope that the results in this paper are not only interesting from a theoretical point of view but can be applied in the not-too-far future, for example by combining multipliers with the concept of sparsity and persistence [19].

In this paper, we define and investigate multipliers in Banach spaces. In Section 1.2, we will give the basic definitions and known results needed. In Section 2 we will give basic results for multipliers for $p$-Bessel sequences. In particular we will show that multipliers with bounded symbols are well-defined bounded operators with unconditional convergence and that symbols converging to zero correspond to compact operators. Section 3 will look at sufficient conditions for multipliers to be $(r,p,q)$-nuclear. Finally, in Section 4, we will look at how the multipliers depend on the given parameters, i.e. the analysis and synthesis sequences as well as the symbol. We will show that this dependence is continuous, using a similarity of sequences in an $l^p$ sense.

1.2. Preliminaries. We will only consider reflexive Banach spaces. We will assume that $p,q > 0$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For any separable Banach space, we can define $p$-frame and $p$-Bessel sequences [3, 11] as follows:

**Definition 1.2.** A countable family $(g_i)_{i \in I} \subseteq X^*$ is a $p$-frame for the Banach space $X$ ($1 < p < \infty$) if constants $A, B > 0$ exist such that

$$A\|f\|_X \leq \left( \sum_{i \in I} |g_i(f)|^p \right)^{\frac{1}{p}} \leq B\|f\|_X \quad \text{for all} \quad f \in X.$$ 

It is called a $p$-Bessel sequence with bound $B$ if the second inequality holds.

For $p$-Bessel sequences we can define the analysis operator $U : X \to l^p$ with $U(f) = (g_i(f))$. Following the definition we see that $\|U\| \leq B$. Furthermore, let $T : l^q \to X^*$ be the synthesis operator defined by $T((d_i)) = \sum_i d_i g_i$. 
Proposition 1.3. \([\mathbf{11}]\) \((g_i) \subseteq X^*\) is a \(p\)-Bessel sequence with bound \(B\) if and only if \(T\) is a well-defined (hence bounded) operator from \(l^q\) into \(X^*\) and \(\|T\| \leq B\). In this case \(T((d_i)) = \sum_{i} d_i g_i\) converges unconditionally.

Furthermore \(\|g_i\| \leq B\).

**Definition 1.4.** Let \(Y\) be a Banach space. A family \((g_i)_{i \in I} \subseteq Y\) is a \(q\)-Riesz sequence \((1 < q < \infty)\) for \(Y\) if constants \(A, B > 0\) exist such that for all finite scalar sequence \((d_i)\),

\[
A \left( \sum_{i \in I} |d_i|^q \right)^{\frac{1}{q}} \leq \left\| \sum_{i \in I} d_i g_i \right\|_Y \leq B \left( \sum_{i \in I} |d_i|^q \right)^{\frac{1}{q}}.
\]

The family is called a \(q\)-Riesz basis \((1 < q < \infty)\) for \(Y\) if it fulfills (1.1) and \(\text{span}\{g_i\}_{i \in I} = Y\).

It immediately follows from the definition that, if \((g_i)_{i \in I}\) is a \(q\)-Riesz basis then \(A \leq \|g_i\|_Y \leq B\) for all \(i \in I\). Any \(q\)-Riesz basis for \(X^*\) is a \(p\)-frame for \(X\) \([\mathbf{11}]\). The following proposition shows a connection between these two notions similar to the case for Hilbert spaces:

**Proposition 1.5.** \([\mathbf{11}]\) Let \((g_i)_{i \in I} \subseteq X^*\) be a \(p\)-frame for \(X\). Then the following are equivalent:

1. \((g_i)_{i \in I}\) is a \(q\)-Riesz basis for \(X^*\).
2. If \((d_i)_{i \in I} \in l^q\) and \(\sum d_i g_i = 0\), then \(d_i = 0\) for all \(i \in I\).
3. \((g_i)_{i \in I}\) has a biorthogonal sequence \((f_i)_{i \in I} \subseteq X\), i.e., a family for which \(g_i(f_j) = \delta_{i,j}\) (Kronecker delta), for all \(i, j \in I\).

**Theorem 1.6.** \([\mathbf{11}]\) Let \((g_i)_{i \in I} \subseteq X^*\) be a \(q\)-Riesz basis for \(X^*\) with bounds \(A, B\). Then there exists a unique \(p\)-Riesz basis \((f_i)_{i \in I} \subseteq X\) for which

\[
f = \sum_{i} g_i(f) f_i \quad \text{and} \quad g = \sum_{i} f_i(g) g_i
\]

for all \(f \in X\) and \(g \in X^*\). The bounds of \((f_i)_{i \in I}\) are \(1/B\) and \(1/A\), and it is biorthogonal to \((g_i)\).

**Definition 1.7.** We will call the unique sequence of Theorem 1.6 the dual of \((g_i)\) and denote it by \((\widetilde{g_i})\).
1.3. Perturbation of $p$-Bessel sequences. Similar to the case for Hilbert spaces perturbation results for Banach spaces are possible.

**Theorem 1.8.** [10] Let $U : X \to Y$ be a bounded operator, $X_0$ a dense subspace of $X$ and $V : X \to Y$ a linear mapping. If for $\lambda_1, \mu > 0$ and $0 \leq \lambda_2 < 1$

$$\|Ux - Vx\| \leq \lambda_1 \|Ux\| + \lambda_2 \|Vx\| + \mu \|x\|,$$

for all $x \in X_0$, then $V$ is a bounded linear operator.

**Corollary 1.9.** Let $(\psi_k) \subseteq X^*$ be a $p$-Bessel sequence.

1. If $(\phi_k) \subseteq X^*$ is a sequence with $\left( \sum_k \|\psi_k - \phi_k\|_{X^*}^p \right)^{1/p} < \mu < \infty$,
then $(\phi_k)$ is a $p$-Bessel sequences with bound $B + \mu$.

2. Let $(\phi_k^{(l)})$ be a sequence such that for all $\varepsilon$ there exists an $N_\varepsilon$ with

$$\left( \sum_k \|\psi_k - \phi_k^{(l)}\|_{X^*}^p \right)^{1/p} < \varepsilon$$

for all $l \geq N_\varepsilon$. Then the sequence $(\phi_k^{(l)})$ is a Bessel sequence and for all $l \geq N_\varepsilon$

$$\|U_{\phi_k^{(l)}} - U_{\psi_k}\|_{op} < \varepsilon \quad \text{and} \quad \|T_{\phi_k^{(l)}} - T_{\psi_k}\|_{op} < \varepsilon.$$

**Proof.** For any $c \in \ell^p$ with finite support, we have

$$\|T_{\Psi}c - T_{\Phi}c\| = \left\| \sum_i c_i (\psi_i - \phi_i) \right\|$$

$$\leq \sum_i |c_i| \|\psi_i - \phi_i\|$$

$$\leq \left( \sum_i |c_i|^q \right)^{1/q} \cdot \left( \sum_i \|\psi_i - \phi_i\|^p \right)^{1/p}$$

$$\leq \|c\|_q \mu.$$
Furthermore
\[ \|U_{\psi}f - U_{\Phi}f\| = \|(\psi_i(f) - \phi_i(f))\|_p \]
\[ = \left( \sum_i |\psi_i(f) - \phi_i(f)|^p \right)^{1/p} \]
\[ \leq \left( \sum_i \|\psi_i - \phi_i\|_{X^*}^p \right)^{1/p} \|f\|_X \]
\[ \leq \mu \|f\|_X. \]

We can apply Theorem 1.8 for \( \lambda_1 = \lambda_2 = 0 \) and use Proposition 1.3. Part (2) can be proved in an analogue way. \( \square \)

For a full treatment of perturbation for frames and Bessel sequences in Banach spaces refer to [29].

**Definition 1.10.** Let \( (\psi_k)_{k \in K} \subseteq X^* \) and \( (\psi^{(l)}_k)_{k \in K} \subseteq X^* \) be a sequence of elements for all \( l \in \mathbb{N} \). The sequences \( (\psi^{(l)}_k) \) are said to converge to \( (\psi_k) \) in an \( l^p \)-sense, denoted by \( (\psi^{(l)}_k) \rightarrow^{l^p} (\psi_k) \), if for any \( \varepsilon > 0 \) there exists \( N_\varepsilon > 0 \) such that \( \left( \sum_k \|\psi^{(l)}_k - \psi_k\|_{X^*}^p \right)^{1/p} < \varepsilon \), for all \( l \geq N_\varepsilon \).

This is related to the notions of ‘quadratic closeness’ [32], and ‘Bessel norm’ [4].

**2. Multipliers for \( p \)-Bessel sequences**

**Lemma 2.1.** Let \( (\psi_k) \subseteq X_1^* \) be a \( p \)-Bessel sequence for \( X_1 \) with bound \( B_1 \), let \( (\phi_k) \subseteq X_2 \) be a \( q \)-Bessel sequence for \( X_2^* \) with bound \( B_2 \), let \( m \in l^\infty \). The operator \( M_{m,(\phi_k),(\psi_k)} : X_1 \rightarrow X_2 \) defined by
\[ M_{m,(\phi_k),(\psi_k)}(f) = \sum_k m_k \psi_k(f) \phi_k. \]
is well defined. This sum converges unconditionally and
\[ \|M\|_{op} \leq B_2 B_1 \cdot \|m\|_\infty. \]

**Proof.** As \( \forall f \in X_1 \) \( (m_k \cdot \psi_k(f)) \in l^p \), \( M \) converges unconditionally and is well defined by Proposition 1.3.
For \( n > 0 \) we have
\[
\left\| \sum_{k=1}^{n} m_k \psi_k(f) \phi_k \right\|_{X_2} \leq \sum_{k=1}^{n} \left\| m_k \psi_k(f) \phi_k \right\|_{X_2}
\]
\[
\leq \|m\|_{\infty} \left( \sum_{k=1}^{n} |\psi_k(f)|^p \right)^{\frac{1}{p}} \sup_{\|h\| \leq 1} \left( \sum_{k=1}^{n} |\phi_k(h)|^q \right)^{\frac{1}{q}}
\]
\[
\leq \|m\|_{\infty} \cdot B_1 \|f\|_{X_1} \cdot \sup_{\|h\| = 1} (B_2 \|h\|_{X_2^*})\]
\[
= \|m\|_{\infty} \cdot B_1 \cdot B_2 \|f\|_{X_1}.
\]

So the multiplier is bounded with bound \( \|m\|_{\infty} \cdot B_1 \cdot B_2 \).

Using the representation \( \mathbf{M}_{m,(\phi_k),(\psi_k)} = T_{\phi_k} D_m U_{\psi_k} \) gives a more direct way to prove the above bound. Here \( D_m \) is the diagonal operator on \( \ell^\infty \) defined by \( D_m (\xi_i) = (m_i \xi_i) \).

Using the above Lemma, we can define:

**Definition 2.2.** Let \( (\psi_k) \subseteq X_1^* \) be a \( p \)-Bessel sequence for \( X_1 \) and let \( (\phi_k) \subseteq X_2 \) be a \( q \)-Bessel sequence for \( X_2^* \). Let \( m \in l^\infty \). The operator \( \mathbf{M}_{m,(\phi_k),(\psi_k)} : X_1 \rightarrow X_2 \), defined by
\[
\mathbf{M}_{m,(\phi_k),(\psi_k)}(f) = \sum_{k} m_k \psi_k(f) \phi_k
\]
is called \( (p,q) \)-Bessel multiplier. The sequence \( m \) is called the symbol of \( \mathbf{M} \).

**Proposition 2.3.** Let \( (\psi_k) \subseteq X_1^* \) be a \( p \)-Bessel sequence for \( X_1 \) with no zero elements, let \( (\phi_k) \subseteq X_2 \) be a \( p \)-Riesz sequence for \( X_2 \) and let \( m \in l^\infty \). Then the mapping
\[
m \rightarrow \mathbf{M}_{m,(\phi_k),(\psi_k)}
\]
is injective from \( l^\infty \) into \( \mathcal{B}(X_1,X_2) \).

**Proof.** Suppose \( \mathbf{M}_m = \mathbf{M}_{m'} \), then \( \sum_k m_k \psi_k(f) \phi_k = \sum_k m'_k \psi_k(f) \phi_k \) for all \( f \). As \( (\phi_k) \) is a \( p \)-Riesz basis for its span, \( m_k \psi_k(f) = m'_k \psi_k(f) \) for all \( f, k \). For every \( k \) there exists \( f \) such that \( \psi_k(f) \neq 0 \), which implies that \( m_k = m'_k \).
Proposition 2.4. Let \((\psi_k) \subseteq X_1^*\) be a \(q\)-Riesz basis for \(X_1^*\) with bounds \(A_1\) and \(B_1\), let \((\phi_k) \subseteq X_2\) be a \(q\)-frame for \(X_2^*\) with bounds \(A_2\) and \(B_2\) and let \(m \in l^\infty\). Then
\[
A_1A_2\|m\|_\infty \leq \|M_{m,(\phi_k),(\psi_k)}\|_{op} \leq B_1B_2\|m\|_\infty.
\]
Particularly \(M\) is bounded if and only if \(m\) is bounded.

Proof. Lemma 2.1 gives the upper bound.

Proposition 1.5 states that \((\psi_k)\) has a biorthogonal sequence \((f_i) \subseteq X_1\), i.e. \(\psi_k(f_i) = \delta_{k,i}\). \((f_i)\) is also a Riesz basis with bounds \(\frac{1}{B_1}, \frac{1}{A_1}\), and so \(\frac{1}{B_1} \leq \|f_i\| \leq \frac{1}{A_1}\) for all \(i \in I\). For arbitrary \(i \in I\), we have
\[
\|M\|_{op} = \sup_{f \in X_1} \frac{\|Mf\|}{\|f\|} \geq \sup_{i \in I} \|Mf_i\| \|f_i\|^{-1} = \frac{1}{\|f_i\|} \|f_i\| \|f_i\|^{-1} = \sup_{i \in I} |m_i| \|\phi_i\| \|\phi_i\|^{-1} \geq A_1A_2\|m\|_\infty.
\]
So \(A_1A_2\|m\|_\infty \leq \|M\|_{op}\).

The following proposition shows that under certain condition on \(m\) the multiplier can be invertible\(^1\), the inverse being the multiplier with the inverted symbol, similar to a result in \([7]\) for Hilbert spaces.

Proposition 2.5. Let \((\psi_k) \subseteq X_1^*\) be a \(q\)-Riesz basis for \(X_1^*\), \((\phi_k) \subseteq X_2\) be a \(p\)-Riesz basis for \(X_2^*\). Let \(m\) be semi-normalized (i.e. \(0 < \inf |m_k| \leq \sup |m_k| < +\infty\)). Then \(M_{m,(\phi_k),(\psi_k)}\) is invertible and
\[
(M_{m,(\phi_k),(\psi_k)})^{-1} = M_{\frac{1}{m_k},(\tilde{\phi}_k),(\tilde{\psi}_k)}.
\]

\(^1\) For a detailed study of invertible multiplier (on Hilbert spaces) see \([30]\).
Proof. It is clear that \( \frac{1}{m_k} \in \ell^\infty \) and thus \( M_{(\frac{1}{m_k});(\tilde{\psi}_k),(\tilde{\phi}_k)} \) is well-defined. For \( f \in X_1 \)

\[
M_{(\frac{1}{m_k});(\tilde{\psi}_k),(\tilde{\phi}_k)} \circ M_{m,(\phi_k),(\psi_k)} f = \sum_{i} \frac{1}{m_i} \tilde{\phi}_i (\sum_k m_k \psi_k(f) \phi_k) \tilde{\psi}_i
\]

\[
= \sum_{i} \frac{1}{m_i} \sum_k m_k \psi_k(f) \phi_k \tilde{\phi}_i \tilde{\psi}_i
\]

\[
= \sum_{i} \psi_i(f) \tilde{\psi}_i
\]

\[
= f.
\]

That \( M_{m,(\phi_k),(\psi_k)} \circ M_{(\frac{1}{m_k});(\tilde{\psi}_k),(\tilde{\phi}_k)} f = f \) for all \( f \in X_2 \) can be shown in an analogous way. Hence,

\[
(M_{m,(\phi_k),(\psi_k)})^{-1} = M_{(\frac{1}{m_k});(\tilde{\psi}_k),(\tilde{\phi}_k)}
\]

For Banach spaces it is well known that the limit of finite rank operators (in the operator norm) is a compact operator (although this is not an equivalent conditions as is the case for Hilbert spaces). We are using this property in:

Lemma 2.6. Let \((\psi_k) \subseteq X_1^\ast\) be a \(p\)-Bessel sequence for \(X_1\) with bound \(B_1\), let \((\phi_k) \subseteq X_2^\ast\) be a \(q\)-Bessel sequence for \(X_2^\ast\) with bound \(B_2\). If \(m \in c_0\) then \(M_{m,(\phi_k),(\psi_k)}\) is compact.

Proof. For a given \(m \in c_0\), let \(m^{(N)} = (m_0, m_1, \ldots, m_{N-1}, 0, 0, \ldots)\). The symbol \(m^{(N)}\) is converging to zero, so for all \(\epsilon > 0\) there is a \(N_\epsilon\) such that \(\|m - m^{(N)}\|_\infty \leq \epsilon\) for all \(N \geq N_\epsilon\). As \(m \in l^\infty\) by Lemma 2.1 we have for all \(N \geq N_\epsilon\)

\[
\|M_{m,(\phi_k),(\psi_k)} - M_{m^{(N)},(\phi_k),(\psi_k)}\|_{Op} = \|M_{(m - m^{(N)}),(\phi_k),(\psi_k)}\|_{Op}
\]

\[
\leq \|m - m^{(N)}\|_\infty \cdot B_2 \cdot B_1
\]

\[
\leq \epsilon \cdot B_2 \cdot B_1.
\]
Therefore $M^{(N), (\phi_k), (\psi_k)}_m$ is converging to $M^{(N), (\phi_k), (\psi_k)}_m$ in the operator norm. As $M^{(N), (\phi_k), (\psi_k)}_m$ is clearly a finite rank operator, we have shown the result.

For two normed spaces $X$ and $Y$, $\mathcal{B}(X, Y)$ denotes the set of all linear bounded operators from $X$ to $Y$. Let $W$, $X$, $Y$ and $Z$ be normed spaces. For elements $y \in Y$ and $\omega \in X^*$ define an operator $y \otimes \omega \in \mathcal{B}(X, Y)$ by

$$(y \otimes \omega)(z) = \omega(z)y \quad \text{for all} \quad z \in X.$$ 

For arbitrary $S \in \mathcal{B}(W, X)$, $T \in \mathcal{B}(Y, Z)$, $y \in Y$, $\omega \in X^*$, $z \in Z$ and $\tau \in Y^*$ these operators satisfy

$$(z \otimes \tau)(y \otimes \omega) = \tau(y) \cdot z \otimes \omega$$

$$T(y \otimes \omega) = T(y) \otimes \omega$$

$$(y \otimes \omega)S = y \otimes S^*(\omega)$$

$$(y \otimes \omega)^* = \omega \otimes \kappa(y)$$

$$\|y \otimes \omega\|_{op} = \|y\|_Y \|\omega\|_{X^*}$$

where $\kappa : Y \hookrightarrow Y^{**}$ is the canonical injection defined by

$$\kappa(y)(\eta) = \eta(y) \quad \text{for all} \quad y \in Y, \eta \in Y^*.$$ 

The above notations are borrowed from [22]. By using the above notations, we can write the $(p, q)$-Bessel multiplier in the form

$$M^{(N), (\phi_k), (\psi_k)}_m = \sum_k m_k \phi_k \otimes \psi_k.$$ 

It is easy to see that

$$M^{\ast, (\phi_k), (\psi_k)}_m = \sum_k m_k \psi_k \otimes \kappa(\phi_k) = M^{(N), (\psi_k), (\kappa(\phi_k))}.$$ 

Putting the above results together, we obtain the following theorem which is a generalization of one of the results in [4] for Banach spaces.

*Theorem 2.7.* Let $M = M^{(N), (\phi_k), (\psi_k)}_m$ be a $(p, q)$-Bessel multiplier for the $p$-Bessel sequence $(\psi_k) \subseteq X^*_1$, the $q$-Bessel sequence $(\phi_k) \subseteq X_2$ with bounds $B_1$ and $B_2$. Then, the following hold.
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1. If \( m \in l^\infty \), \( M \) is a well defined bounded operator with
\[
\|M\|_{op} \leq B_2 B_1 \cdot \|m\|_\infty.
\]
Furthermore, the sum \( \sum_k m_k \psi_k(f) \phi_k \) converges unconditionally for all \( f \in X_1 \).

2. \( M^* \cdot (m, (\psi_k)) = \sum_k m_k \psi_k \otimes \kappa(\phi_k) = M \cdot (m, (\psi_k)) \cdot \kappa(\phi_k) \).

3. If \( m \in c_0 \), \( M \) is a compact operator.

3. NUCLEAR OPERATORS IN BANACH SPACES

The theory of trace-class operators in Hilbert spaces was created in 1936 by J. Murray and J. Von Neumann. In the earlier Fifties, Alexander Grothendieck [18] and A. F. Ruston [26, 27] independently extended this concept to operators acting in Banach spaces. Trace-class operators on Banach spaces are called nuclear operators. This idea is generalized in [23]:

Let \( 0 < r \leq \infty \). A family \( \mathbf{x} = (x_i)_{i \in I} \subseteq X \), where \( x_i \in X \) for \( i \in I \), is called weakly \( p \)-summable if \( (x^*(x_i)) \in \ell^p(I) \) whenever \( x^* \in X^* \).

We put
\[
w_p(x_i) := \sup\{\|x^*(x_i)\|_p : \|x^*\| \leq 1\}.
\]
The class of all weakly \( p \)-summable sequences on \( X \) is denoted by \( W_p(X) \). Clearly \( w_p(x_i) < \infty \) (by Banach-Steinhaus Theorem).

From the above notations, it is clear that if \( (g_i)_{i \in I} \subseteq X^* \) is a \( p \)-Bessel sequence for \( X \) then \( (g_i) \in W_p(X^*) \).

**Definition 3.1.** [23] Let \( 0 < r \leq \infty \), \( 1 \leq p_1, q_1 \leq \infty \), and \( 1 + \frac{1}{r} \geq \frac{1}{p_1} + \frac{1}{q_1} \). An operator \( S \in \mathcal{B}(X, Y) \) is called \((r, p_1, q_1)\)-nuclear if
\[
S = \sum_{i=1}^\infty \sigma_i x_i^* \otimes y_i
\]
with \( (\sigma_i) \in \ell^r \), \( (x_i^*) \in W_{p_1}(X^*) \), and \( (y_i) \in W_{q_1}(Y) \) where \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{q} + \frac{1}{r} = 1 \). In the case \( r = \infty \) let us suppose that \( (\sigma_i) \in c_0 \). We put
\[
N_{(r, p_1, q_1)}(S) := \inf \{\|\sigma_i\|_r \cdot w_{p_1}(x_i^*) \cdot w_{q_1}(y_i)\},
\]
\[\text{As mentioned in the introduction we only consider reflexive Banach spaces}\]
where the infimum is taken over all so-called \((r, p_1, q_1)\)-nuclear representations described above.

**Theorem 3.2.** [23] An operator \(S \in \mathcal{B}(X, Y)\) is \((r, p_1, q_1)\)-nuclear if and only if there exist operators \(F, D\) and \(E\) with \(S = FDE\), such that \(D \sigma \in \mathcal{B}(\ell^{p_1}, \ell'\) if \(0 < r < \infty\) and \((\sigma_i) \in c_0\) if \(r = \infty\). Furthermore, \(E \in \mathcal{B}(X, \ell')\) and \(F \in \mathcal{B}(\ell^{p_1}, Y)\). In this case,

\[
N_{(r, p_1, q_1)}(S) := \inf \| E \| \| (\sigma_i) \|_r \| F \|
\]

where the infimum is taken over all possible factorizations.

From Theorem 3.2 and the above notations, we can easily conclude the next result for multipliers using

\[
M_{m, (\phi_k), (\psi_k)} = T_{\phi_k} D_m U_{\psi_k}
\]

as a decomposition of \(M\).

**Corollary 3.3.** Let \((\psi_k) \subseteq X_1^*\) be a \(p\)-Bessel sequence for \(X_1\) with bound \(B_1\), let \((\phi_k) \subseteq X_2\) be a \(q\)-Bessel sequence for \(X_2^*\) with bound \(B_2\). Let \(r > 0\) and \(m \in \ell^r\). Then \(M_{m, (\phi_k), (\psi_k)}\) is a \((r, p, q)\)-nuclear operator with

\[
N_{(r, p, q)}(M) \leq B_1 B_2 \| m \|_r.
\]

### 4. Changing the ingredients

Results from [4] can be generalized to \(p\)-Bessel sequences:

**Theorem 4.1.** Let \(M = M_{m, (\phi_k), (\psi_k)}\) be a \((p, q)\)-Bessel multiplier for the \(p\)-Bessel sequences \((\psi_k) \subseteq X_1^*\), the \(q\)-Bessel sequence \((\phi_k) \subseteq X_2\) with bounds \(B_1\) and \(B_2\). Let \(p_1, q_1 \geq 1\) be such that \(\frac{1}{p_1} + \frac{1}{q_1} = 1\) allowing \(p_1, q_1 = \infty\). Then the operator \(M\) depends continuously on \(m\), \((\psi_i)\) and \((\phi_i)\), in the following sense: Let \((\psi_i^{(l)}) \subseteq X_1^*\) and \((\phi_i^{(l)}) \subseteq X_2\) be \(p\)-Bessel sequences indexed by \(l \in I\).

1. Let \(m^{(l)} \to m\) in \(l^{p_1}\). Then \(\| M_{m^{(l)}, (\psi_i), (\phi_i)} - M_{m, (\psi_i), (\phi_i)} \|_{Op} \to 0\).

\(^3\)Please note that for a convergence of \(p\)-Bessel sequences in an \(l^p\)-sense we would get the Bessel property by Corollary 1.9 for big enough \(l\).
(2) Let \( m \in \ell^{p_1} \) and let the sequences \( (\psi^{(l)}_i) \) converge to \( (\psi_i) \) in an \( \ell^{q_1} \)-sense. Then for \( l \to \infty \)

\[
\left\| M_{m,(\psi^{(l)}_i),(\phi_i)} - M_{m,(\psi_i),(\phi_i)} \right\|_{O_p} \to 0.
\]

(3) Let \( m \in \ell^{p_1} \) and let the sequences \( (\phi^{(l)}_i) \) converge to \( (\phi_i) \) in an \( \ell^{q_1} \)-sense. Then for \( l \to \infty \)

\[
\left\| M_{m,(\psi_i),(\phi_i)} - M_{m,(\psi^{(l)}_i),(\phi^{(l)}_i)} \right\|_{O_p} \to 0.
\]

(4) Let \( m^{(l)} \to m \) in \( \ell^{p_1} \) and let the sequences \( (\psi^{(l)}_i) \) respectively \( (\phi^{(l)}_i) \) converge to \( (\psi_i) \) respectively \( (\phi_i) \) in an \( \ell^{q_1} \)-sense. Then for \( l \to \infty \)

\[
\left\| M_{m^{(l)},(\psi^{(l)}_i),(\phi^{(l)}_i)} - M_{m,(\psi_i),(\phi_i)} \right\|_{O_p} \to 0.
\]

Proof. (1) By Theorem 2.7

\[
\left\| M_{m^{(l)},(\psi^{(l)}_k),(\phi_k)} - M_{m,(\psi_k),(\phi_k)} \right\|_{O_p} = \left\| M_{m^{(l)}-m,(\psi_k),(\phi_k)} \right\|_{O_p} \\
\leq \|m^{(l)}-m\|_{\ell^{q_1}} \sqrt{B_1 B_2} \\
\leq \|m^{(l)}-m\|_{\ell^{p_1}} \sqrt{B_1 B_2} \\
\leq \epsilon \sqrt{B_1 B_2}.
\]

for \( l > N_\epsilon \).

(2) For \( l > N_\epsilon \)

\[
\left\| \sum m_k \psi^{(l)}_k \otimes_i \phi_k - \sum m_k \psi_k \otimes_i \phi_k \right\|_{O_p} = \left\| \sum m_k \left( \psi^{(l)}_k - \psi_k \right) \otimes_i \phi_k \right\|_{O_p} \\
\leq \sum_k |m_k| \left\| \psi^{(l)}_k - \psi_k \right\|_{X_i^*} \sqrt{B_2} \\
\leq \sqrt{B_2} \|m\|_{\ell^{p_1}} \left( \sum \left\| \psi^{(l)}_k - \psi_k \right\|_{X_i}^{q_1} \right)^{1/q_1} \\
\leq \sqrt{B_2} \|m\|_{\ell^{p_1}} \epsilon.
\]

(3) Use corresponding arguments as in (2).
\[ M_m(l) \leq M_m(l) \leq \epsilon \sqrt{B_1 B_2 + \|m\|_p} \sqrt{B_2} + \|m\| \sqrt{B_1} \epsilon \]

for \( l \) bigger than the maximum \( N \) needed for the convergence conditions.

\[ \Box \]

5. Outlook and Perspectives

We have shown that the concept of multipliers can be extended to \( p \)-frames in Banach spaces. It can also be done for other settings. For \( g \)-frames a paper was already accepted [24]. In the future we will consider to extend this notion to other setting, for example for Fréchet frames, matrix valued frames, \( pg \)-frames [1] or continuous frames.

In particular the last notion can be interesting also for application as in this setting the question, how continuous and discrete frame multipliers can be related, is of relevance. This would be an interesting result for the link of STFT and Gabor multipliers. Such a connection is particular interesting in relating a physical model, which normally is continuous, using multipliers to the implemented algorithm, which is discrete and finite-dimensional.

For the future work the relation of \( (p,q,r) \)-nuclear operators with Gelfand triple may be investigated.

For applications wavelet, Gabor and frames of translates are very important classes of frames. Most of these systems can be described as localized frames [15]. For multipliers of localized frames, which are currently investigated, the results of this paper are directly applicable and so can become more relevant for signal processing algorithms. We further hope that the results in this paper can be directly useful.
for applications in signal processing, as both Banach space methods and multipliers become more and more important for applications, as mentioned in the introduction.

Acknowledgment

Some of the results in this paper were obtained during the first author’s visit at the Acoustics Research Institute, Austrian Academy of Sciences, Austria. He thanks this institute for their hospitality.

This work was partly supported by the WWTF project MULAC (‘Frame Multipliers: Theory and Application in Acoustics; MA07-025).

The authors would like to thank Diana Stoeva for her discussions and comments. Also, the authors would like to thank referees for their suggestions.

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