A Non-Critical String Approach to Black Holes, Time and Quantum Dynamics

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Abstract

We review our approach to time and quantum dynamics based on non-critical string theory, developing its relationship to previous work on non-equilibrium quantum statistical mechanics and the microscopic arrow of time. We exhibit specific non-factorizing contributions to the $\mathcal{S}$ matrix associated with topological defects on the world sheet, explaining the rôles that the leakage of $W_\infty$ charges plays in the loss of quantum coherence. We stress the analogy with the quantum Hall effect, discuss the violation of $CPT$, and also apply our approach to cosmology.

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1 Introduction

String theory is widely heralded as a consistent quantum theory of gravity. As such, it should not only enable us to calculate meaningfully quantum-gravitational corrections to scattering processes in a fixed space-time background, but also to take into account quantum fluctuations in space and time themselves. The way to carry out the first part of this double programme in string theory is well known: calculate higher-genus effects in a given critical string vacuum that describes the appropriate classical background. The way to carry out the second part of the quantum gravity programme in string theory, namely to understand space-time foam, is less evident. One must master a multitude of string vacua and the quantum transitions between them, which necessarily involve non-critical string theory [1]. No-one can do this at present: the best one can do is study some tractable examples of non-critical string models [2] that one hopes are relevant and representative, and abstract from them features that may be generic.

It is to be expected that some shibboleths of conventional physics will be cast down when this programme is carried out. Certainly general relativity must be modified, and perhaps also special relativity and even quantum mechanics. We review in these lectures our work [3, 4, 5] indicating that our understandings of special relativity, quantum mechanics and quantum field theory should indeed be modified. We start from the string black hole solution of Witten [5], and abstract from it general features associated with a renormalization group analysis of the string effective action [6].

There is no arrow of time in conventional quantum field theory, nor in critical string theory. Indeed, it is even possible to formulate critical strings without introducing a time variable at all. However, the time we experience does have an arrow, both microscopically as codified in the second law of thermodynamics, and macroscopically as evidenced in the cosmological Hubble expansion. One of the main thrusts of our work has been to understand the arrow of time in the framework of non-critical string theory, and to relate its microscopic and macroscopic manifestations. As we shall see, an essential feature of this understanding is an apparent modification of quantum mechanics and quantum field theory, entailing the abandonment of the $S$-matrix description of scattering.

Some have long suspected that such a modification might be necessary, in view of the fact that black holes apparently behave thermodynamically in the context of local quantum field theory [7, 8, 9]. The appearance of an event horizon is accompanied by non-zero entropy proportional to its area, and a related non-zero temperature, properties that require a mixed-state quantum treatment. This entails use of the density-matrix formalism, and Hawking has suggested [7] that when space-time foam is taken into account scattering must be formulated as an asymptotic linear
transformation from incoming states $\rho_{in,B}^A$ to outgoing states $\rho_{out,D}^C$:

$$\rho_{out,D}^C = S_{DA}^{CB} \rho_{in,B}^A$$

(1)

where the superscattering matrix $S_{DA}^{CB}$ does not in general factorize as a product of $S$ and $S^\dagger$ matrix elements [7]

$$S_{DA}^{CB} \neq S_A^C(S^\dagger)_B^D$$

(2)

as in conventional quantum field theory. Correspondingly, the time-evolution of a quantum system cannot be governed simply by the Liouville equation, which integrates to yield just the conventional $S$ matrix, but there should be an extra term in the quantum Liouville equation due to space-time foam, which we may write in the form [10]

$$\partial_t \rho = i[\rho, H] + \delta H \rho$$

(3)

Such a modification of the quantum Liouville equation is characteristic of open quantum-mechanical systems, in which the observed (sub)system is in contact with an unobserved reservoir. It introduces a microscopic arrow of time, with in general dissipation, entropy increase and apparent wave-function collapse in the observed (sub)system. In the quantum-gravitational context discussed here, the unobserved states are associated with non-trivial microscopic event horizons, which are unobservable even in principle. We discuss below how non-trivial contributions to the $S$ matrix and to $\delta H$ may arise in string theory, using the general formalism of string backgrounds as $\sigma$-model field theories on the two-dimensional world-sheet.

We treat the two-dimensional (spherically-symmetric four-dimensional) string black hole background as an illustrative example in which specific calculations can be performed. We have derived an explicit general form for $\delta H$ in string theory, expressed in world-sheet $\sigma$-model notation and exemplified by the string black hole model. We have also shown explicitly how this modification of the quantum Liouville equation leads to decoherence and apparent collapse of the wave function, as suggested previously on the basis of more intuitive approaches to quantum gravity.

Before discussing this, however, we discuss in section 2 some relevant features of non-equilibrium quantum statistical mechanics, which is the appropriate framework for the modified quantum Liouville time-evolution equation (3). Specifically, we recall the general formalism of Misra and Prigogine [11, 12], as well as the so-called Lie-admissible approach of ref. [13], which are compatible under certain conditions [14]. In section 3 we recall [4] the general world-sheet $\sigma$-model derivation of the modified quantum Liouville equation (3), and show that it obeys the Lie-admissibility condition of ref. [14]. This follows from the existence of the Zamolodchikov metric in world-sheet $\sigma$-model coupling space. The arrow of time is associated with renormalization group flow in this space. Energy is conserved in the mean [15], as a consequence of renormalizable, which replaces the time translation invariance of conventional target-space field theory. Probability is also conserved, whilst entropy increases monotonically [4]. Section 4 contains a more detailed discussion of
time, which is interpreted in our approach as a renormalization group scale identified with a Liouville field \[2, 4, 16\]. Section 5 reviews the string black hole model and its interpretation in terms of monopoles \[17\], as well as instantons \[18\] in this model and their interpretation. Section 6 reviews our previous calculations of specific contributions to the $\mathcal{S}$ matrix and $\mathcal{H}$ due to monopoles and instantons on the world sheet associated with string black holes \[17\]. In section 7 we discuss in more detail the relation between our string black hole calculations and the approach of ref. \[14\], underlining in particular the role played by $W$ symmetries. Section 8 reviews the violation of $CPT$ in our formalism, relating it to the analysis of ref. \[19\] and discussing its possible manifestation in the neutral kaon system. In section 9 we mention applications of our approach to cosmology, with particular mention of the initial singularity, inflation, the time-dependences of the fundamental parameters, and the cosmological constant \[16\]. Finally, in section 10 we discuss the outlook for our approach.

2 Non-Equilibrium Quantum Statistical Mechanics for Pedestrians

In this section we review at an elementary level some relevant features of non-equilibrium quantum statistical mechanics, which we link in the next section with our string-modified quantum Liouville equation. The authors of \[11\] sought to include time irreversibility and the second law of thermodynamics into a description of microphysics based on the density matrix. It is clear that this aim entails modifications of conventional classical and quantum mechanics, that are to be regarded as incomplete in this context. As a first step in this programme, the authors of \[11\] introduced a dynamical transformation $\Lambda$, which is not unitary in general, that relates the full density matrix $\rho$ to that for the physically-relevant system, denoted by $\tilde{\rho}$:

$$\tilde{\rho} = \Lambda \rho$$

In our case, we shall consider as $\tilde{\rho}$ the density matrix for the low-mass light string states that can be measured locally, as distinct from extended solitonic string states whose properties can only be characterized by global measurements. We shall refer to $\tilde{\rho}$ as the locally-measurable density matrix. Misra and Prigogine \[11\] pointed out that the transformation $\Lambda$ should satisfy certain consistency conditions: it should preserve the positivity of $\rho$, it should obey the equations

$$\Lambda \cdot I = I$$

and

$$\int \Gamma d\mu Tr \Lambda \rho = \int \Gamma d\mu Tr \rho$$
where $\Gamma$ denotes the phase-space manifold, and the time-evolution operator $U_t = e^{-iLt}$, where $L$ is the Liouville operator, should have the intertwining property
\begin{equation}
\Lambda U_t = W_t^* \Lambda
\end{equation}
for $t \geq 0$, where $W_t^*$ is an adjoint strongly-irreversible Markov semigroup operator.

Two different possibilities for $\Lambda$ should be distinguished. One is that in which $\Lambda$ has an inverse, and the similarity relation
\begin{equation}
W_t^* = \Lambda U_t \Lambda^{-1}
\end{equation}
applies for $t \geq 0$. This mathematical invertibility does not mean, however, that physical information is retained, as we shall see later. In the other case, $\Lambda$ has no inverse, and can be regarded as a projection operator onto the physically-relevant states.

In the former case, which is the one that concerns us, as we show in the next section, the locally-measurable density matrix $\tilde{\rho}$ obeys the time-evolution equation
\begin{equation}
i\partial_t \tilde{\rho} = \Phi(L)\tilde{\rho}
\end{equation}
where
\begin{equation}
\Phi(L) = \Lambda^{-1}L\Lambda
\end{equation}
with $L$ the Liouvillian. The existence of a function in phase space which varies monotonically with time, called a Lyapounov function, is guaranteed if $\Phi(L)$ obeys the condition
\begin{equation}
i\Phi(L) - i[\Phi(L)]^\dagger \geq 0
\end{equation}
which holds if $\Lambda$ has the \textit{star hermiticity} property
\begin{equation}
\Lambda^{-1}(L) = \Lambda^\dagger(-L) \equiv [\Lambda(L)]^*
\end{equation}
However, this property may not hold in general.

Time is not necessarily irreversible in this framework. The mere existence of a Lyapounov function is not sufficient to guarantee physical time-reversal symmetry breaking. Generalizing equation (8), one can define distinct Markov semigroup operators
\begin{equation}
W_t^\pm = \Lambda_\pm U_t \Lambda_\mp^{-1}
\end{equation}
for $t \geq 0$ and $t \leq 0$, respectively. Thanks to the time-reversal invariance of the initial unitary evolution operator $U_t$, time is reversible if $\Lambda_+ = \Lambda_-$, but this is not the case in general. In most cases there are physical reasons for distinguishing $\Lambda_+$ and $\Lambda_-$, and thereby determining the arrow of time-reversal symmetry breaking, as is for instance the case in which the initial conditions in one of the Markov processes are
such that their preparation requires infinite entropy. It has been shown \[11\] that a dynamical system of the type described above admits an internal time variable, suitable for the discussion of aging, if $\Lambda$ obeys the conditions (5), (6) and (7), and is sufficiently large in the sense that for any given $\rho$ and any $\epsilon \geq 0$ there is another density matrix $\rho'$ with the property that

$$||\rho - U_\Lambda \rho'|| < \epsilon$$  \hspace{1cm} (14)

i.e., the subspace generated by $\Lambda$ evolves backward in time to generate arbitrarily good approximations to all possible states. In this case, it has been shown \[11\] that one can introduce a time-evolution operator $T$ conjugate to the Liouville operator $L$:

$$[L, T] = i\hbar$$  \hspace{1cm} (15)

The converse of this theorem is also true.

In such a dynamical system, a point in phase space, when evolved backwards in time, may become an arbitrarily complicated and non-local region of phase space, via a sort of inverse butterfly effect. It is not possible to reconstruct the past history of the system unless one measures all components of its final-state density matrix with arbitrarily high precision, which is impracticable. Thus, information is effectively lost during the time-evolution, and hence entropy increases. The system approaches an equilibrium state in which there is no memory of the initial state.

One example of such a system is provided by geodesics in an expanding Universe, described by a Robertson-Walker-Friedmann metric in $3 + 1$ dimensions:

$$ds^2 = dt^2 - R(t)^2 \sum_{i=1}^{3} (dx^i)^2$$  \hspace{1cm} (16)

Test particles move along four-dimensional geodesics that project onto geodesics in the three-dimensional hypersurfaces of simultaneity. Geodesic flow in four dimensions defines a corresponding geodesic flow in three dimensions, and the theorem of \[12\] tells us that an internal time $T$, or age, can be defined for the system, which is not the cosmological time, but is a monotonic function of it. It has the interesting property that

$$U_\lambda^* TU_\lambda = T + \lambda(t) I$$  \hspace{1cm} (17)

where

$$\lambda(t) = \lambda(t_0) + A \int_{t_0}^{t} ds \frac{1}{R(s)} \text{ for massless particles}$$  \hspace{1cm} (18)

is the affine parameter of the projected geodesic flow, and $t$ is the cosmic time. This tells us that the rate of change of $\lambda$ becomes very rapid at early times, and very slow at late times. Thus the rate of approach to equilibrium is huge close to the initial singularity, and very slow in an old Universe. This is an example of apparent indeterminacy in the \textit{a priori} deterministic theory of General Relativity. As we shall see later, a very similar situation arises in our Liouville approach to non-critical string theory.
Thus far, we have not considered in detail the form of the dynamical equations of motion describing the time-evolution of such a system. An appropriate framework is provided by the so-called Lie-admissible formulation of dissipative statistical mechanics, as reviewed in [13]. The starting point that describes the dissipative motion of a single particle is the open version of the Lagrange equation:

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i(t, q^i, \dot{q}^i) \quad (19) \]

The corresponding first-order Hamilton equations take the form

\[
\begin{align*}
\dot{q}^i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial q^i} + F_i : F_i(t, q^i, p_i) = F_i(t, q^i, \dot{q}^i) \quad (20)
\end{align*}
\]

The extension of this formulation to the statistical evolution of the phase-space density function \( \rho(q, p, t) \) entails a generalized Liouville equation

\[
\frac{\partial \rho}{\partial t} + \{\rho, H\} + F_i \frac{\partial \rho}{\partial p_i} + \rho \frac{\partial F_i}{\partial p_i} = 0 \quad (21)
\]

where \( \{,\} \) denotes a conventional Poisson bracket. As we shall see in section 3, in the string case of interest to us the last term in this equation is absent, whilst the previous term is very much present.

The treatment of such a system must answer the question how the physical energy operator \( E \) can be identified with the generator \( H \) of time translations, since \( \dot{E} \neq 0 \) at the operator level, due to dissipation or, more general, interactions with the “environment”, whereas

\[ \dot{H} = \{H, H\} = 0 \quad (22) \]

where \( \{,\} \) denotes a conventional Poisson bracket, which becomes a commutator \([,]\) in the quantum case. The answer presented in ref. [13] is to modify the Lie-algebraic structure, replacing \( \{,\} \) by an object \( \{\{,\}\} \) with the property that

\[ \dot{H} = \{\{H, H\}\} \neq 0 \quad (23) \]

and analogously \([,]\) becomes \((,\) in the quantum case.

A generalized product \((,\) is said to form a Lie-admissible algebra if it is linear, obeys the generalized Jacobi identity

\[
\begin{align*}
((A, B), C) &+ \text{cyclic permutations} + \\
(C, (B, A)) &+ \text{cyclic permutations} - \\
(A, (B, C)) &- \text{cyclic permutations} - \\
((C, B), A) &- \text{cyclic permutations} = 0
\end{align*}
\]

(24)
and the condition
\[(A, B) - (B, A) = 2[A, B]\] (25)
where \([,]\) denotes the conventional Lie product. As an example, consider a dynamical system with \(\frac{\partial H}{\partial p} \neq 0\), in which case
\[
\dot{A} = \frac{\partial A}{\partial \xi^i} \omega_{ij} \frac{\partial H}{\partial \xi^j} + \frac{\partial A}{\partial \xi_i} F_i
\] (26)
where \(\xi^i = (q^i, p_j)\), \(\omega_{ij} = -\omega_{ji}\) is a convenient notation for the Poisson bracket commutator, and
\[
F^i = T^{ij} \frac{\partial H}{\partial \xi^j}
\] (27)
with
\[
T^{ij} = \begin{pmatrix}
0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & s_{n \times n}
\end{pmatrix}
\]
and \(0_{n \times n}\) is an \(n \times n\) zero matrix. In this case, the time-evolution equation (26) can be written in the form
\[
\dot{A} = \frac{\partial A}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q^i} + \frac{\partial A}{\partial p_i} s_{ij} \frac{\partial H}{\partial p_j} = (A, H)
\] (29)
It is easy to check that the product (,) defined in this equation is of the Lie-admissible form defined earlier, and reduces to the conventional Lie product [.] if \(s_{ij} = 0\).

We are now in a position to compare this Lie-admissible formulation of dissipative statistical mechanics with the previous formulation of [11], based on a non-unitary time-evolution operator \(\Lambda\). We recall the form (9) of the time-evolution equation for the locally-measurable density matrix \(\tilde{\rho}\) in that formulation, and the star hermiticity condition (11,12). For the type of dynamical system described by (19,20, and 29), we have
\[
\Phi = i \sum_{i=1}^{n} \left( \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} \right) + \sum_{i,j=1}^{n} \frac{\partial H}{\partial p_i} s_{ij} \frac{\partial}{\partial p_j}
\] (30)
which is consistent with star-hermiticity if and only if
\[
\mathcal{F} = \mathcal{F}^\dagger \quad ; \quad \mathcal{F} = \sum_{i,j=1}^{n} \frac{\partial H}{\partial p_i} s_{ij} \frac{\partial}{\partial p_j}
\] (31)
A necessary and sufficient condition that this be satisfied is that the matrix \(s_{ij}\) be real and symmetric [14]. Dissipative statistical systems with this property provide examples of the \(\Lambda\)-transformation theory of [11]. As we shall see in subsequent sections, string theory is one such example.
3 Review of String Density Matrix Mechanics

In order to accommodate the mixed states inevitable in a quantum theory of gravity, one must use a density matrix formalism \([7, 10]\). The asymptotic \(S\) matrix may not have the familiar analyticity properties, and may only exist in a distribution-theoretic sense, as we shall see later. The evolution of the density matrix over finite times is given by a modification (3) of the quantum Liouville equation \([10]\). The commutator term in (3) would, when integrated alone over time, give the conventional \(S\) matrix. The extra term \(\delta H\) leads, when integrated over time, to the non-factorization property \(S \neq SS^\dagger\) As seen in section 2, the presence of such an extra term is characteristic of a dissipative quantum-mechanical system such as an open system interacting with an unobserved environmental reservoir \([20, 21, 22]\). In string density matrix mechanics this term would reflect the mixing of observable particles to unobservable quantum gravitational states, whose information is lost across microscopic event horizons. Such states are delocalized, and can be thought of as remnants of a symmetric (topological) phase of gravity \([23]\), whose breaking might result in the emergence of the ordinary space-time.

How should one set about modelling space-time foam and exploring these possibilities in string theory? In order for the background in which it propagates have a classical space-time interpretation, the string theory must be critical, i.e. must be characterized by a conformal field theory on the world sheet with central charge \(c = 26\) \((15)\) for a bosonic (supersymmetric) string. It is well known \([24]\) how to reproduce \(S\)-matrix elements via the operator product expansion for such a critical-string conformal field theory. One must look beyond this framework if one is to have any chance of locating a non-trivial contribution to the \(S\) matrix. This means looking at quantum fluctuations in the space-time background, and in particular transitions between them. The appropriate space of theories to discuss these is that of two-dimensional field theories on the world-sheet, but without the restriction that these be conformal. This is the space of generalized \(\sigma\)-models on the world sheet that we have used to derive \([3, 4]\) a form for \(\delta H\) in the framework of non-critical string theory \([1]\).

Such models are characterized by deformations of conformal points, described by vertex operators \(V_i\) associated with background fields \(g_i\) in the target space in which the string propagates. Mathematical consistency requires the restoration of criticality by turning a deformation into an exactly marginal one by Liouville dressing, thereby leading, in our interpretation, to a time-dependent background. To be more explicit, consider a two-dimensional critical conformal field theory model described by an action \(S_0(r)\) on the world sheet, where the \(\{r\}\) are matter fields spanning a \(D\)-dimensional target manifold of Euclidean signature, that we term “space”. Consider, now, a deformation

\[
S = S_0(r) + g \int d^2 z V_g(r)
\]  

(32)
Here $V_g$ is a $(1,1)$ operator, i.e. its anomalous dimension vanishes, but it is not \textit{exactly marginal} in the sense that the operator-product expansion coefficients $C_{ggg}$ of $V_g$ with itself are non-zero in any renormalization scheme, and hence ‘universal’ in the Wilsonian sense. The scaling dimension $\alpha_g$ of $V_g$ in the deformed theory (32) is, to $O(g)$ \[25\],

\[
\alpha_g = -gC_{ggg} + \ldots \tag{33}
\]

Liouville theory \[26\] requires that scale invariance of the theory (32) be preserved. The non-zero scaling dimension (33) would jeopardize this, but scale invariance is restored if one dresses $V_g$ gravitationally on the world-sheet as

\[
\int d^2 z g V_g(r) \rightarrow \int d^2 z g e^{\alpha_g \phi} V_g(r) = \int d^2 z g V_g(r) - \int d^2 z g^2 C_{ggg} V_g(r) \phi + \ldots \tag{34}
\]

where $\phi$ is the Liouville field. The latter acquires dynamics through integration over world-sheet covariant metrics $\gamma_{\alpha\beta}$ after conformal gauge fixing $\gamma_{\alpha\beta} = e^{\phi} \hat{\gamma}_{\alpha\beta}$ in the way discussed in \[28, 24\]. Scale invariance is guaranteed through the definition of renormalized couplings $g_R$, given in terms of $g$ through the relation

\[
g_R \equiv g - C_{ggg} \phi g^2 + \ldots \tag{35}
\]

This equation leads to the correct $\beta$-functions for $g_R$

\[
\beta_g = -C_{ggg} g_R^2 + \ldots \tag{36}
\]

implying a renormalization-group scale $\phi$-dependence of $g_R$. The reader might have noticed that above we viewed the Liouville field as a local scale on the world-sheet \[4, 1\]. Local world-sheet scales have been considered in the past \[28, 29\], but the crucial difference in our Liouville approach is that this scale is made dynamical by being integrated out in the path-integral. In this way, in Liouville strings the local dynamical scale acquires the interpretation of an additional target coordinate. If the central charge of the matter theory is $c_m > 25$, the signature of the kinetic term of the Liouville coordinate is opposite to that of the matter fields $r$, and thus the Liouville field is interpreted as Minkowski target time \[1, 30\], as we discuss in more detail in the next section and in ref. \[4, 16\].

The target-space density matrix in such a framework is viewed as a function of coordinates $g^i$ that parametrize the couplings of the generalized $\sigma$-models on the string world-sheet, and their conjugate momenta $p_i : \rho(g^i, p_i)$. The effective action functional for this dynamical system can be identified with the Zamolodchikov $C$-function \[31\] $C(g^i)$, whose gradient determines the rate of change of the coordinates (couplings) $g^i$:

\[
\dot{g}^i = \beta^i(g) \quad : \quad \frac{\delta C(g)}{\delta g^i} = G_{ij} \beta^j \tag{37}
\]

where

\[
C[g] = \int dt (p_i \dot{g}^i - E) \tag{38}
\]
where $E$ is the Hamiltonian and $G_{ij}$ is the metric in $g$-space,

$$G_{ij}[g] = 2|z|^4 <V_i(z)V_j(0)>$$

with $V_i$ the associated vertex operators corresponding to the background $g^i$ and $<\ldots>$ denotes a $\sigma$-model vacuum expectation value. Here and subsequently dots denote derivatives with respect to the renormalization scale. As described above and in ref. [10] and the next section, we identify the renormalization scale with a Liouville field, which has negative metric in target space because fluctuations make the string supercritical [1], and we identify it with the target time variable. This is the main reason for considering in (38) an ‘effective action’ and not simply a Lagrangian in target space. This is crucial in our formalism, as a result of the integration over the dynamical Liouville scale in a $\sigma$-model path integral. In this formalism, the simple gradient flow of the off-shell corollary of the $C$-theorem [31, 29, 32] is extended to a non-trivial functional derivative

$$\frac{\delta}{\delta g^i} C[g] = -\frac{d}{dt} \frac{\partial L(\{,\},\{,\})}{\partial \dot{g}^i} + \frac{\partial L}{\partial g^i} \neq 0 \quad (40)$$

which includes a generalized non-potential force term in the evolution equation for $g^i$ [2] derived from (38). The renormalizability of the world-sheet $\sigma$-model implies

$$\frac{d}{dt} \rho[g^i, p_i, t] = 0 = \frac{\partial}{\partial t} \rho + \dot{g}^i \frac{\partial}{\partial g^i} \rho + \dot{p}_i \frac{\partial}{\partial p_i} \rho \quad (41)$$

From (40) one can then derive straightforwardly a modified Liouville equation (3) with the explicit form [2]

$$\delta H \rho = G_{ij} \beta^j \frac{\partial \rho}{\partial p_i} = -\dot{\rho} G_{ij} \beta^j \quad (42)$$

where the second form holds in the quantum formulation.

We note here the fundamental point that this modification of the quantum Liouville equation obeys the Lie-admissibility condition of ref. [14], because the Zamolodchikov tensor (39) is real and symmetric, and hence defines a metric in coupling constant space. This property is non-trivial, since it does not hold in a more general renormalization scheme [29], in which the couplings $g^i$ have an arbitrary dependence on the world-sheet coordinates $g^i(\sigma, \tau)$ and not a simple local scale dependence [29]. In such a scheme there exists a local (in renormalization group space) function whose variations with respect to the couplings have off-shell relations with the $\beta$-functions given by matrices that have antisymmetric parts. More explicitly, one can prove the relation [23]:

$$\partial_i \tilde{\beta}^\Phi = \chi_{ij} \beta^j + \partial_j W_i \beta^j - \partial_i W_j \beta^j \quad (43)$$

where $\Phi$ is the dilaton background coupled to the world-sheet curvature, the $g^i$ denote the rest of the backgrounds, existing even on flat world sheets, and $\chi_{ij}$
is symmetric. It is related to divergences of the two-point functions of $V_i$, but its positivity is not evident in this approach. Finally, \( \tilde{\beta}^i \equiv \beta^i + W_i \beta^j \), where the $W_i$ are (computable) renormalization counterterms related to total world-sheet derivative terms in the expression for the trace of the stress-tensor. Such terms are crucial for the local scale invariance of the theory. In dimensional regularization the $W_i$ are non-trivial beyond three $\sigma$-model loop order \[29\]. Such off-shell relations appear only if the $g^i$ are allowed to depend arbitrarily on the world-sheet coordinates, which is not the case in the framework adopted above and in ref. \[28\] where the $g^i$ depend only on the renormalization scale that we identify with target time \[2, 4\].

The more general framework incorporates off-shell generalized forces that depend on the conjugate momenta $p_i$. It could be regarded as providing an ‘atlas’ relating ‘charts’ or ‘patches’, in each of which the $g^i$ space is torsion-free. In our interpretation, the Universe observable within our event horizon is contained within one patch, enabling physics everywhere within it to be described by the approach to a unique conformal field theory or critical string vacuum. There may well be other patches beyond our observable horizon, in which a different string vacuum is approached, with the generalized renormalization scheme \[29\], including torsion \[13\] describing transitions between the patches. This would provide a physical realization of the rather abstract analysis in ref. \[33\], where it was argued that the ‘classical’ coupling constant space should not be simply connected, with each component corresponding to one of our patches.

This ‘torsion-free’ modification of the quantum Liouville equation, that applies within our patch of the Universe, has other several important properties. One is that the total probability $P = \int dp_i dq_i \text{Tr}[\rho(g^i, p_j)]$ is conserved :

$$\dot{P} = \int dp_i dq_i \text{Tr} \left[ \frac{\partial}{\partial p_i} (G_{ij} \beta^j \rho) \right]$$

which can receive contributions only from the boundary of phase space, that must vanish for an isolated system. Secondly, energy is conserved on the average \[15\]. This can be seen by computing

$$\partial_t << E^n >> = n << (\partial_t E) E^{n-1} >> - i << \beta^i G_{ij} [g^j, E^n] >> =$$

$$n << (\partial_t E) E^{n-1} >> - i << \beta^i G_{ij} E[g^j, E^{n-1}] >> + << \beta^i G_{ij} \beta^j E^{n-1} >>$$

(45)

where $E$ is the Hamiltonian operator, and $<< \ldots >> \equiv \text{Tr}[\rho(\ldots)]$. In arriving at this result we took into account the quantization rules in coupling constant space discussed in ref. \[4\],

$$[g^i, g^j] = 0 \quad ; \quad [g^i, p^j] = - i \delta^{ij}$$

(46)

as well as the fact that in string $\sigma$-models the ‘quantum operators’ $\beta^i G_{ij}$ are functionals of the coordinates only $g^i$ and not of the generalized momenta $p^i$. For future
use we note that the total time derivative of an operator \( \hat{Q} \) is given as usual by

\[
\frac{d}{dt} \hat{Q} = -i[\hat{Q}, E]
\]  

(47)

We recall that total time derivatives incorporate both explicit and implicit (via running couplings) renormalization-scale dependence, whilst partial time derivatives incorporate only the explicit dependence.

For the energy conservation law we should take \( n = 1 \) in (45), in which case we find

\[
\frac{\partial}{\partial t} << E >> = \frac{\partial}{\partial t} Tr(E\rho) = << \partial_t (E - \beta^i G_{ij}\beta^j) >>
\]

(48)

Using the C-theorem results \([31, 15]\) and the formalism developed in ref. \([2]\) it is straightforward to arrive at

\[
\frac{\partial}{\partial t} << E >> = \frac{\partial}{\partial t} (p_i\beta^i) = 0
\]

(49)

due to the renormalizability of the stringy \( \sigma \)-model. The latter implies that any dependence on the renormalization group scale in the \( \beta^i \) functions is implicit through the renormalized couplings. Renormalizability replaces the time-translation invariance of conventional target-space field theory.

This conservation result does not generalize to quantum fluctuations in the energy \( \Delta E = << E^2 >> - (<< E >>)^2 \). To get the the energy fluctuations we set \( n = 2 \),

\[
\partial_t << E^2 >> = -i << [\beta^j, E]\beta^i G_{ij} >> = << \frac{d\beta^j}{dt}\beta^i G_{ij} >>
\]

(50)

which is non-zero for a non-critical string. This result implies that despite energy conservation the uncertainties in the energy \( \Delta E \) depend on the (Liouville) time \( t \). This point is relevant to the discussion of uncertainty in section 9 \([16]\).

The entropy \( S = -Tr(\rho ln\rho) \) is also not conserved:

\[
\dot{S} = (\beta^i G_{ij}\beta^j)S
\]

(51)

implying a monotonic increase for unitary theories for which \( G_{ij} \) is positive definite. We see from (51) that any running of any coupling will lead to an increase in entropy, and we have interpreted \([2]\) this behaviour in terms of quantum models of friction \([21]\). The increase (51) in the entropy corresponds to a loss of quantum coherence, which is also known in these models. Note that entropy increases within any ‘torsion-free’ cosmological ‘patch’ in coupling space: this is not in general true at the boundaries between patches, where ‘torsion’ may appear.
The final comment in this brief review of density matrix mechanics is that Ehrenfest’s theorem continues to hold. The time evolution of the expectation value of any observable \( O(g^i) \) that is a function of the coordinates alone, and not the momenta \( p_i \), is given by

\[
\frac{\partial}{\partial t} \langle\langle O(g^i) \rangle\rangle = \frac{\partial}{\partial t} \text{Tr}(O(g^i)\rho) = \text{Tr}(O(g^i)\dot{\rho})
\]

\[
=i\text{Tr}[O(g^i)[\rho, H]] + i\text{Tr}[g^i, O(g^i)\beta^i G_{ij}\rho] = i\text{Tr}\{O(g^i)[\rho, H]\}
\]

as usual.

It may be helpful to bear in mind a simple two-state system, whose \( 2 \times 2 \) density matrix can be decomposed with respect to the hermitian Pauli \( \sigma \)-matrix basis \( \{1, \sigma_x, \sigma_y, \sigma_z\} \), \( \rho = \rho_0 1 + \rho \sigma \). In conventional quantum mechanics with a Hamiltonian \( H = \Delta E\sigma_z \), a pure initial state \( \rho_{in} = \frac{1}{2}(|1> + |2>) <1| + <2| \) evolves unitarily:

\[
\rho(t) = \frac{1}{2} \begin{pmatrix} 1 & e^{i\Delta Et} \\ e^{-i\Delta Et} & 1 \end{pmatrix}
\]

A generic “open system” modification \( \delta H \) can be expressed as a \( 4 \times 4 \) matrix w.r.t. the coordinates \( \{0, \sigma_x, \sigma_y, \sigma_z\} \). The probability and energy conservation derived above (44,49) tell us that

\[
\delta H_{0\beta} = 0 = \delta H_{\beta 0}, \quad \delta H_{3\beta} = 0 = \delta H_{\beta 3}
\]

respectively. We can therefore write

\[
\delta H_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha & -\beta & 0 \\ 0 & -\beta & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

where positivity imposes \( \alpha, \gamma > 0, \alpha \gamma > \beta^2 \). It is easy to see that with the addition of such a term

\[
\rho(t) = \frac{1}{2} \begin{pmatrix} 1 & e^{-(\alpha+\gamma)t/2}e^{-i\Delta Et} \\ e^{-(\alpha+\gamma)t/2}e^{i\Delta Et} & 1 \end{pmatrix}
\]

which becomes asymptotically a completely mixed state

\[
\rho(\infty) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Such evolution towards a completely mixed state is the generic consequence of the monotonic increase (51) in the entropy. The completely-mixed form (57) of the density matrix corresponds to the advertised loss of coherence.
As discussed in more detail in section 8, the above formalism can be adapted, with conceptually minor modifications, to accommodate decays to the neutral $K^0 - \bar{K}^0$ system \cite{10, 15}, which is one of the best microscopic laboratories for testing quantum mechanics and its possible modification \cite{3}. The corresponding parameters $\alpha, \beta, \gamma$ \cite{55} violate CPT. String theory suggests that CPT violation should be considered a generic feature of our density matrix mechanics \cite{15}. The normal field-theoretical proof of the CPT theorem is based on locality, Lorentz invariance and unitarity. Clearly string theory is not local in space-time, and Lorentz invariance may be considered a derived property of critical string theory that does not hold in our treatment of time as a renormalization scale in non-critical string theory. Indeed, we have related CPT violation to charge non-conservation on the world-sheet associated with topological fluctuations such as monopoles whose appearance drives the string supercritical \cite{15, 4}.

4 Time and the Two-Dimensional String Black Hole Model

As an illustration of our approach to non-critical string theory, we now discuss the two-dimensional black hole model of ref. \cite{5}. We regard it as a toy laboratory that gives us insight into the nature of time in string theory and contributes to the physical effects mentioned in the previous section.

The action of the model is

$$S_0 = \frac{k}{2\pi} \int d^2 z [\partial r \bar{\partial} r - \tanh^2 r \partial t \bar{\partial} t] + \frac{1}{8\pi} \int d^2 z R^{(2)} \Phi(r)$$

(58)

where $r$ is a space-like coordinate and $t$ is time-like, $R^{(2)}$ is the scalar curvature, and $\Phi$ is the dilaton field. The customary interpretation of (58) is as a string model with $c = 1$ matter, represented by the $t$ field, interacting with a Liouville mode, represented by the $r$ field, which has $c < 1$ and is correspondingly space-like \cite{1, 30}. As an illustration of the approach outlined in the previous section, however, we re-interpret (58) as a fixed point of the renormalization group flow in the local scale variable $t$. In our interpretation, the “matter” sector is defined by the spatial coordinate $r$, and has central charge $c_m = 25$ when $k = 9/4$ \cite{5}. Thus the model (58) describes a critical string in a dilaton/graviton background. The fact that this is static, i.e. independent of $t$, reflects the fact that one is at a fixed point of the renormalization group flow \cite{4, 16, 34}.

We now outline how one can use the machinery of the renormalization group in curved space, with $t$ introduced as a local renormalization scale on the world sheet,
to derive the model (58). A detailed technical description is given in [34, 1]. There are two contributions to the kinetic term for $t$ in this approach, one associated with the Jacobian of the path integration over the world-sheet metrics, and the other with fluctuations in the background metric.

To exhibit the former, we first choose the conformal gauge $\gamma_{\alpha\beta} = e^{\rho} \tilde{\gamma}_{\alpha\beta}$ [26, 27], where $\rho$ represents the Liouville mode. We will later identify $\rho$ with an appropriate function of $\phi$, thereby making the local scale $\phi$ a dynamical $\sigma$-model field. Ref. [27] contains an explicit computation of the Jacobian using heat-kernel regularization, which yields

$$\frac{1}{48\pi^2} \left[ \frac{1}{2} \partial_\alpha \rho \partial^\alpha \rho + R^{(2)} \rho + \frac{\mu}{\epsilon} e^\rho + S'_G \right]$$  (59)

where the counterterms $S'_G$ are needed to remove the non-logarithmic divergences associated with the induced world-sheet cosmological constant term $\frac{\mu}{\epsilon} e^\rho$, and depend on the background fields. This procedure reproduces the critical string results of ref. [5] when one identifies the Liouville field $\rho$ with $2\alpha' \phi$. Equation (59) contains a negative (time-like) contribution to the kinetic term for the Liouville (time) field, but this is not the only such contribution, as we now show.

We recall that the renormalization of composite operators in $\sigma$-models formulated on curved world sheets is achieved by allowing an arbitrary dependence of the couplings $g^i$ on the world-sheet variables $z, \bar{z}$ [28, 29]. This induces counterterms of "tachyonic" form, which take the following form in dimensional regularization with $d = 2 - \epsilon$ [29]:

$$\int d^2 z \Lambda_0$$  (60)

where

$$\Lambda_0 = \mu^{-\epsilon}(Z(g)\Lambda + Y(g))$$  (61)

Here $Z(g)$ is a common wave function renormalization that maps target scalars into scalars, $\Lambda$ is a residual renormalization factor, and the remaining counterterms $Y(g)$ can be expanded as power series in $1/\epsilon$, with the the one-loop result giving a simple pole. Simple power-counting yields the following form for $Y(g)$:

$$Y(g) = \partial_\alpha g^i G_{ij} \partial^\alpha g^j$$  (62)

where $G_{ij}$ is the analogue of the Zamolodchikov metric [31] in this formalism, which is positive for unitary theories. It is related to the divergent part of the two-point function $\langle V_i V_j \rangle$ [29] that cannot be absorbed in the conventional renormalization of the operators $V_i$. We need to consider a $\sigma$-model propagating in a graviton background $G_{MN}$, in which case a standard one-loop computation [28] yields the following result for the simple $\epsilon$-pole in $Y$:

$$Y^{(1)} = \frac{\lambda}{16\pi\epsilon} \partial_\alpha G_{MN} \partial^\alpha G^{MN}$$  (63)
where \( \lambda \equiv 4\pi\alpha' \) is a loop-counting parameter. We note that the wave-function renormalization \( Z(g) \) vanishes at one-loop. In ref. \([23]\) \( G_{MN} \) was allowed to depend arbitrarily on the world-sheet variables, and all world-sheet derivatives of the couplings were set to zero at the end of the calculation. In our Liouville mode interpretation, we assume that such dependence occurs only through the local scale \( \mu(z, \bar{z}) \), so that
\[
\partial_\alpha g^i = \hat{\beta}^i \partial_\alpha \phi(z, \bar{z})
\]
(64)
where \( \hat{\beta}^i = \epsilon g^i + \beta^i(g) \) and \( \phi = \ln \mu(z, \bar{z}) \). Taking the \( \epsilon \rightarrow 0 \) limit, and separating the finite and \( O(\frac{1}{\epsilon}) \) terms, we obtain for the former
\[
O(1) - \text{terms} : \quad \text{Res} Y^{(1)}(\epsilon) = \alpha' R \partial_\alpha \phi \partial^\alpha \phi
\]
(65)
where \( R \) is the scalar curvature in target space, and we have used the fact that the one-loop graviton \( \beta \)-function is
\[
\beta^G_{MN} = \frac{\lambda}{2\pi} R_{MN}
\]
(66)
The terms without logarithmic divergences,
\[
\frac{1}{\epsilon} \beta^i G^{(1)}_{ij} \beta^j
\]
(67)
do not contribute to the renormalization group, and can be removed explicitly by target-space metric counterterms
\[
S_G = \frac{1}{\epsilon} G_{\phi\phi} \partial_\alpha \phi \partial^\alpha \phi + \delta S(\phi, r)
\]
(68)
where the coefficients \( G_{\phi\phi} \) are fixed by the requirement of cancelling the \( \frac{1}{\epsilon} \) terms. The \( \delta S \) denotes arbitrary finite counterterms, which are invariant under the simultaneous conformal rescalings of the fiducial world-sheet metric, \( \hat{\gamma} \rightarrow e^\sigma \hat{\gamma} \), and local shifts of the scale \( \phi \rightarrow \phi - \sigma \). This last requirement arises as in the conventional approach to Liouville gravity \([26, 27]\), where the local renormalization scale \( \phi \) is identified with the Liouville mode \( \rho \), after appropriate normalization. In our interpretation one is forced to treat the scale \( \phi \) simultaneously as the target time coordinate.

In the case of the Minkowski black hole model of ref. \([3]\), the Lorentzian curvature is
\[
R = \frac{4}{\cosh^2 r} = 4 - 4\tanh^2 r,
\]
(69)
which we substitute into equation (65) to obtain the form of the second contribution to the kinetic term for the Liouville field \( \phi \). Combining the world-sheet metric Jacobian term in (59) with the background fluctuation term (65, 69), we finally obtain the following terms in the effective action
\[
\frac{1}{4\pi\alpha'} \int d^2z [\partial_\alpha r \partial^\alpha r - \tanh^2 r \partial_\alpha \phi \partial^\alpha \phi + \text{dilaton - terms}]
\]
(70)
Thus we recover the critical string $\sigma$-model action (58) for the Minkowski black-hole. Dilaton counterterms are incorporated in a similar way, yielding the dilaton background of [5]. In addition, as standard in stringy $\sigma$-models, one also obtains the necessary counterterms that guarantee target-space diffeomorphism invariance of the Weyl-anomaly coefficients [28]. Details are given in ref. [34].

It should be noticed that the renormalization group yields automatically the Minkowski signature, due to the $c_m = 25$ value of the matter central charge [1, 30]. However, as we remarked in ref. [34, 4], one can also switch over to the Euclidean black hole model, and still maintain the identification of the compact time with some appropriate function of the Liouville scale $\phi$ that takes into account the compactness of $t$ in that case. The formalism of exactly-marginal deformations that turn on matter in the model (58) is better studied in this Euclidean version [35]. In ref. [35] it was argued that the exactly-marginal deformation that turned on a static tachyon background for the black hole of ref. [5] necessarily involved the higher-level topological string modes, that are non-propagating delocalized states, which are interrelated by an infinite-dimensional $W$ symmetry. This is a consequence of the operator product expansion of the tachyon zero-mode operator $F_{-\frac{1}{2},0}$ [35]:

$$F_{-\frac{1}{2},0} \circ F_{-\frac{1}{2},0} = F_{-\frac{1}{2},0} + W_{-1,0}^h + W_{-1,0}^l + \ldots$$

(71)

where we only exhibit the appropriate holomorphic part for reasons of economy of space. The $W$ operators and the . . . denote level-one and higher string states. The corresponding exactly-marginal deformation, constructed by tensoring holomorphic and antiholomorphic parts, is given by [35]

$$L_0 L_0^\dagger \propto F_{-\frac{1}{2},0}^{c-c} + i(\psi^{++} - \psi^{--}) + \ldots$$

(72)

where the $\psi$ denote higher-string-level operators [35], and the ‘tachyon’ operator is given by

$$F_{-\frac{1}{2},0}^{c-c} (r) = \frac{1}{\cosh r} F\left(\frac{1}{2}, \frac{1}{2}; 1, \tanh^2 r\right)$$

(73)

with

$$F\left(\frac{1}{2}, \frac{1}{2}; 1, \tanh^2 r\right) \approx \frac{1}{\Gamma^2\left(\frac{1}{2}\right)} \sum_{n=0}^{\infty} (\frac{1}{2})_n (\frac{1}{2})_n [2\psi(n + 1) - 2\psi(n + \frac{1}{2}) +$$

$$+ \ln(1 + |w|^2)](\sqrt{1 + |w|^2})^{-n}$$

(74)

There is an additional marginal deformation, dictated by the $SL(2, R)$ symmetry structure [35], which consists of topological string modes only. At large $k$, this operator rescales the black hole metric, as can be seen from its contribution to the

\footnote{The elevation of this symmetry to target space-time is discussed in more detail in section 7.}
action of the deformed Wess-Zumino $\sigma$-model after the gauge field integration \[35\],

\[
gL_0^2T_0^2 \ni \int d^2 z \{ \partial r \bar{r} (1 - 2 g \text{csch}^2 r - 2 g \text{sech}^2 r) + \\
\partial \theta \bar{\theta} (\sinh^2 r + 2 g - \frac{(\sinh^2 r + 2 g)^2}{\cosh^2 r + 2 g}) \}
\]

(75)

Changing variables $\cosh^2 r + 2 g \rightarrow \cosh^2 r$ in (75) one finds that to $O(g)$ the target space metric is rescaled by an overall constant.

The topological (higher-level) string modes cannot be detected in local scattering experiments, due to their delocalized character. From a formal field-theoretic point of view, such states cannot exist as asymptotic states to define scattering, and also cannot be integrated out in a local path-integral. An ‘experimentalist’ therefore sees necessarily a truncated matter theory, where the only deformation is the tachyon $F_{c, 0}^{-1, 0}$, which is a $(1,1)$ operator in the black hole $\sigma$-model (58), but is not exactly marginal. This truncated theory is non-critical, and hence Liouville dressing in the sense of (34) is essential, thereby implying time-dependence of the matter background. Due to the fact that the appropriate exactly-marginal deformation associated with the tachyon in these models involves all the higher-level string states that are interrelated by $W$-symmetry, one can conclude that in this picture the ensuing non-equilibrium time-dependent backgrounds are a consequence of information carried off by the unobserved topological string modes. These states are delocalized modes with definite (target-space) energies and momenta, so a low-energy scattering process involving propagating string degrees of freedom will not have any observable energy violations. This is an apparent physical explanation in this case for the general result (49) of energy conservation on the average in density matrix mechanics. The rôle of the space-time singularity\(^2\) was crucial for this argument. Indeed, in flat target-space matrix models [37] the tachyon zero-mode operator $F$ is exactly marginal. As we shall argue later on, these flat models can be regarded as an asymptotic ultraviolet limit in time of the Wess-Zumino black hole. Hence, any time-dependence of the matter disappears in the vacuum, leading to equilibrium.

The above ‘truncation’ procedure can be compared with the $\Lambda$ transformation theory of Misra and Prigogine [11], and its Lie-admissible formulation [13], discussed in section 2. It has been argued in ref. 38 that the ‘topological’ modes can in principle be observed by either Aharonov-Bohm (global) scattering experiments, in four-dimensional string theories, or via selection rules characterizing tachyon scattering off black-hole backgrounds in the (effective) two-dimensional

\(^2\)We would like to stress that the notion of ‘singularity’ is clearly a low-energy effective-theory concept. The existence of infinite-dimensional stringy symmetries associated with higher-level string states ($W_\infty$-symmetries [36]) ‘smooth out’ the singularity, and render the full string theory finite.
(s-wave four-dimensional) string theory, in an analogous fashion to fermion scattering on a monopole in the Callan-Rubakov effect [39]. This, as well as the coherence-maintaining property of the associated $W_\infty$-algebras that characterize the two-dimensional string theory [36], imply that no information is lost in the $\Lambda$ transformation and the associated $\Lambda$ operator is therefore invertible, as we discuss in more detail in section 7. The Lie-admissible structure then is evident from the work of [14] and the reality of the Zamolodchikov metric tensor $G_{ij}$ [33]. The modified Liouville equation of string density matrix mechanics (42) is to be compared to the (quantum version) of the generalized Liouville equation (21) or the equivalent Misra-Prigogine form (30). The $C$-theorem of Zamolodchikov [31], as extended in section 3 to incorporate Liouville strings, provides us with a natural Lyapunov function in the coupling constant (background field) phase space, as discussed in sections 2 and 3.

Having such a Lyapunov function, it is natural to enquire into the irreversibility in target time of the effective theory of locally-measurable observables. To this end, we recall that in Liouville theory a correlation function of $(1, 1)$ matter deformations $V_i$ is given by [40]

$$< V_{i_1} \ldots V_{i_n} >_{\mu} = \Gamma(-s) < V_{i_1} \ldots V_{i_n} >_{\mu=0}$$  \hspace{1cm} (76)

where $s$ is the sum of the appropriate Liouville energies, and $< \ldots >_{\mu}$ denotes a $\sigma$-model average in the presence of an appropriate cosmological constant $\mu$ deformation on the world-sheet [4]. The important point for our discussion is the $\Gamma$-factor $\Gamma(-s)$. For the interesting case of matter scattering off a two-dimensional (s-wave four-dimensional) string black hole, the latter is excited to a ‘massive’ (topological) string state [38] corresponding to a positive integer value for $s = n^+ \in \mathbb{Z}^+$ [4]. In this case, the expression (76) needs regularization. By employing the ‘fixed area constraint’ [26] one can use an integral representation for $\Gamma(-s)$

$$\Gamma(-s) = \int dA e^{-A} A^{-s-1}$$  \hspace{1cm} (77)

where $A$ is the covariant area of the world-sheet. In the case $s = n^+ \in \mathbb{Z}^+$ one can then employ a regularization by analytic continuation, replacing (77) by a contour integral as shown in fig. 1 [12, 4, 8]. This is a well-known method of regularization in conventional field theory, where integrals of form similar to (77) appear in terms of Feynman parameters. We note that it is the same regularization which was also used to prove the equivalence of the Bogolubov-Parasiuk-Hepp-Zimmerman renormalization prescription to the dimensional regularization of ‘t Hooft [43]. One result of such an analytic continuation is the appearance of imaginary parts in the respective correlation functions, which in our case are interpreted [12, 4, 8] as renormalization group instabilities of the system.

\footnote{In the case of a black-hole coset model this operator is a ‘modified cosmological constant’ involving some mixing with appropriate ghost fields parametrising the $SL(2, R)$ string [41].}
Figure 1: - Contour of integration in the analytically-continued (regularized) version of $\Gamma(-s)$ for $s \in \mathbb{Z}^+$. This is known in the literature as the Saalschutz contour, and has been used in conventional quantum field theory to relate dimensional regularization to the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization method.

Figure 2: - Schematic representation of the evolution of the world-sheet area as the renormalization group scale moves along the contour of fig. 1.
Interpreting the latter as an actual time flow, we then interpret the contour of fig. 1 as implying evolution of the world-sheet area in both (negative and positive) directions of time (c.f. fig. 2), i.e.

\[ \text{Infrared fixed point} \rightarrow \text{Ultraviolet fixed point} \rightarrow \text{Infrared fixed point} \quad (78) \]

In each half of the world-sheet diagram of fig. 2, the Zamolodchikov $C$-theorem tells us that we have an irreversible Markov process. According to the analysis of section 2, the physical system will be time-irreversible if the associated $\Lambda$ transformations are not equivalent. It has been argued in ref. [23] that a highly-symmetrical phase of the two-dimensional world-sheet renormalization group flow. At that point, the associated $\sigma$-model is a topological theory constructed by twisting [5, 44] an appropriate $N = 2$ supersymmetric black-hole $SL(2, R)$ Wess-Zumino $\sigma$-model. The singularity of a stringy black hole, then, describes a topological degree of freedom. The highly-symmetric phase is interpreted as the state with the most ‘appropriate’ initial conditions, whose preparation requires finite entropy. This in turn implies a ‘bounce’ interpretation of the renormalization group flow of fig. 2, in which the infrared fixed point is a ‘bounce’ point, similar to the corresponding picture in point-like field theory [45]. Thus, the “physical” flow of time is taken to be opposite to the conventional renormalization group flow, i.e. from the infrared to the ultraviolet fixed point on the world sheet. In the next section we shall see this explicitly, by using world-sheet instanton calculus to represent, at least qualitatively, the renormalization group flow of the effective target-space theory, providing a concise expression for the effects of the topological modes that are linked to the tachyon modes by $W$ symmetries.

5 The String Black Hole Model and its World-Sheet Instantons

In this section we review some calculations we have made of specific contributions to the $S$ matrix due to topologically non-trivial world-sheet configurations. The action of $SL(2, R)/U(1)$ coset Wess-Zumino model [3] describing a Euclidean black hole can be written in the form

\[ S = \frac{k}{4\pi} \int d^2z \frac{1}{1 + |w|^2} \partial_{\mu}w \partial^{\mu}w + \ldots \quad (79) \]

where the conventional radial and angular coordinates $(r, \theta)$ are given by $w = sinhre^{-i\theta}$ and the target space $(r, \theta)$ line element is

\[ ds^2 = \frac{dwd\bar{w}}{1 + w\bar{w}} = dr^2 + tanh^2r d\theta^2 \quad (80) \]
The Euclidean black hole can be written as a vortex-antivortex pair [17], which is a solution of the following Green function equations on a spherical world sheet:

\[ \partial_z \partial_{\bar{z}} X_v = i \pi \frac{q_v}{2} [\delta(z - z_1) - \delta(z - z_2)] \] (81)

The world-sheet can also accommodate monopole-antimonopole pairs [17], which are solutions of:

\[ \partial_z \partial_{\bar{z}} X_m = -\frac{q_m \pi}{2} [\delta(z - z_1) - \delta(z - z_2)] \] (82)

These are related to Minkowski black holes with masses \( \propto q_m \). Vortex and monopole configurations can both be regarded as sine-Gordon deformations of the effective action for the field \( X \equiv \beta^{-\frac{3}{2}} \tilde{X} \), where \( \beta^{-1} \) is an effective ‘pseudo-temperature’:

\[ \beta = \frac{3}{\pi(C-25)} \] in Liouville theory. The partition function [46]

\[ Z = \int D\tilde{X} \exp(-\beta S_{eff}(\tilde{X})) \]

\[ \beta S_{eff} = \int d^2z [2\partial \tilde{X} \bar{\partial} \tilde{X} + \frac{1}{4\pi} [\gamma_v \omega^2 - 2(2\sqrt{|g(z)|})^{1+\frac{3}{4}}] : \cos(2\pi \alpha [\tilde{X}(z) + \tilde{X}(\bar{z})]) : ] + (\gamma_v, \alpha, \tilde{X}(z) + \tilde{X}(\bar{z})) \to (\gamma_m, \alpha', \tilde{X}(z) - \tilde{X}(\bar{z})) \] (83)

requires for its specification an angular ultraviolet cut-off \( \omega \) on the world-sheet. Here \( \gamma_v, m \) are the fugacities for vortices and spikes respectively, and \( \alpha^4 \) is the conformal dimension \( \Delta \). This deformed sine-Gordon theory has a low-temperature phase in which monopole-antimonopole pairs are bound in dipoles as irrelevant deformations with the conformal dimension

\[ \Delta_m = \frac{\alpha_m}{4} = \frac{\pi \beta}{2} q_m^2 > 1 \] (84)

Monopole-antimonopole pairs correspond to the creation and annihilation of a microscopic black hole in the space-time foam.

As shown in ref. [18], the \( SL(2, \mathbb{R})/U(1) \) Wess-Zumino coset model describing a Euclidean black hole also has instantons given by the holomorphic function

\[ w(z) = \frac{\rho}{z - z_0} \] (85)

with topological charge

\[ Q = \frac{1}{\pi} \int d^2z \frac{1}{1 + |w|^2} [\overline{\partial \rho \partial w - h.c.}] = -2\ln(a) + \text{const} \] (86)

where \( a \) is an ultraviolet cut-off discussed later. The instanton action on the world-sheet also depends logarithmically on the ultraviolet cut-off. As in the case of the more familiar vortex configuration in the Kosterlitz-Thouless model, this logarithmic
divergence does not prevent the instanton from having important dynamical effects. The instanton-anti-instanton vertices take the form

\[ V_{IT} \propto -\frac{d}{2\pi} \int d^2 z \frac{d^2 \rho}{|\rho|^4} e^{-S_0} \left( e^{(k_0\rho^2 + h.c.)/f(|w|)} + e^{(k_0|\rho w + h.c.|/f(|w|)} \right) \]  

(87)

introducing a new term into the effective action. Making a derivative expansion of the instanton vertex and taking the large-\( k \) limit, i.e. restricting our attention to instanton sizes \( \rho \approx a \), this new term has the same form as the kinetic term in (79), and hence corresponds to a renormalization of the effective level parameter in the large \( k \) limit:

\[ k \to k - 2\pi k^2 d' \quad : \quad d' \equiv d \int \frac{d|\rho|}{|\rho|^3} \left( \frac{a^2}{(|\rho/a|^2 + 1)^{\frac{3}{2}}} \right) \]

(88)

If other perturbations are ignored, the instantons are irrelevant deformations and conformal invariance is maintained. However, in the presence of “tachyon” deformations, \( T_0 \int d^2 z \mathcal{F}^{c,c}_{-\frac{1}{2},0,0} \) in the \( SL(2,R) \) notation of ref. [35], there are extra logarithmic infinities in the shift (88), that are visible in the dilute gas and weak-“tachyon”-field approximations. In this case, there is a contribution to the effective action of the form

\[ T_0 \int d^2 z d^2 z' \mathcal{F}^{c,c}_{-\frac{1}{2},0,0}(z,\bar{z}) V_{IT}(z',\bar{z'}) > \]

(89)

Using the explicit form of the “tachyon” vertex \( \mathcal{F}^{c,c}_{-\frac{1}{2},0,0} \) given by \( SL(2,R) \) symmetry [35], it is straightforward to isolate a logarithmically-infinite contribution to the kinetic term in (79), associated with infrared infinities on the world-sheet expressible in terms of the world-sheet area \( \Omega/a^2 \) [1, 34],

\[ gT_0 \int d^2 z' \int d\rho \frac{a^2}{(a^2 + \rho^2)^{\frac{3}{2}}} \int d^2 z \frac{1}{|z - z'|^2} \frac{1}{1 + |w|^2} \partial_z w(z') \partial_{\bar{z}} \bar{w}(z') + \ldots \]

\[ \propto gT_0 \ln \frac{\Omega}{a^2} \int d^2 z' \frac{1}{1 + |w|^2} \partial_z w(z') \partial_{\bar{z}} \bar{w}(z') \]

(90)

Such covariant-scale-dependent contributions can be attributed to Liouville field dynamics, through the “fixed-area constraint” in the Liouville path integral [26, 47]. The zero-mode part can be absorbed in a scale-dependent shift of \( k \) [1], for which large \( k >> 1 \) may be assumed to exponentiate:

\[ k_R \propto \left( \frac{\Omega}{a^2} \right)^{(const).\beta^I T_0} \]

(91)

where \( \beta^I \) is the instanton \( \beta \)-function [18]. In ref. [1] we gave general arguments and verified to lowest order that instantons represent massive mode effects, enabling us to identify \( \beta^I = -\beta^T \), where \( \beta^T \) is the renormalization-group \( \beta \)-function of a matter deformation of the black hole [1]. Notice that in (91) both the infrared and

\[^4\text{Notice that this implies that the matter } \beta \text{-function has to be computed in a non-perturbative way, which is consistent with the exact conformal field theory analysis of ref. [35].}\]
the ultraviolet cut-off scales enter. In the following we shall not distinguish between infrared and ultraviolet cut-offs. The physical scale of the system, which varies along a renormalization group trajectory, is the dimensionless ratio of the two, which is identified with the Liouville field.

The change in $k$ and the associated change in the central charge $c = \frac{3k}{k - 2} - 1$ and the black-hole mass $M_{bh} \propto (k - 2)^{-\frac{1}{2}}$ do not conflict with any general theorems. An analogous instanton renormalization of $\theta$ (c.f. $k$) has been demonstrated \[18\] in related $\sigma$-models that describe the Integer Quantum Hall Effect (IQHE), discussed further in section 7. Instanton renormalization of $k$ can also be seen in the Minkowski black hole model of ref. \[3\], defined on a non-compact manifold $SL(2, R)/O(1,1)$ \[5\]. In our case, as we have seen, the instantons reflect a shift of the central charge between the matter and background sectors of a combined matter + black hole theory, in which the total central charge is unchanged. They correspond to a combination of world-sheet deformation operators in the Wess-Zumino model \[3\]: the exactly marginal operator $L_0^2\mathcal{T}_0$ and the irrelevant part of the exactly-marginal deformation $L_0\mathcal{T}_0$, which involves an infinite sum of massive string operators \[33\], as we saw in section 4 [see equations (72), (73)]. The fact that the $L_0^2\mathcal{T}_0$ operator rescales the target-space metric by an overall constant, implies that such perturbations have the same effect as the instanton. Thus the instanton represents the effects of massive string modes that are related to each other and to massless excitations by a $W$ symmetry. Matrix elements of the full exactly marginal light matter + instanton operator have no dependence on the ultraviolet cut-off $a$, but the separate matter and instanton parts do depend on $a$, as we have seen above.

Since instantons rescale the target-space metric and the black hole mass, they may also be used to represent black hole decay. This is higher-genus effect in string theory \[50\], so one should expect that instantons could reflect the contributions of higher genera. This expectation is indeed supported by an explicit computation of instanton effects in a dilute-gas approximation in the presence of dilatons. It is well known \[21\] in $\sigma$-model perturbation theory that modular infinities of resummed world-sheet surfaces may be absorbed as an effective renormalization of lower-genus Riemann surfaces, above and beyond the local renormalization effects at fixed genus. For example, modular infinities for the sphere and torus may be summarized by a logarithmically-divergent contribution to the dilaton background $\Phi$ \[52\]

$$\Phi = \Phi_R + \left( \frac{d - 26}{6} + \frac{d + 2}{4} \right) \ln |\Lambda/a| + O\left( \frac{1}{a^2} \right)$$

The $\sigma$-model action of such a theory contains \[49\], in addition to the action (79), a total-derivative $\theta$-term which can be thought of as a deformation of the black hole by an “antisymmetric tensor” background, which in two dimensions is a discrete mode as a result of the abelian gauge symmetry. Its Euclideanized version has also instanton solutions of the form \[83\], but with finite action, which induce “Liouville”-time-dependent shifts to $k$, prior to matter couplings.
for a string propagating in a $d$-dimensional target space. The first term $(d-26)/6$ in the coefficient of the logarithmic divergence is the world-sheet sphere contribution to the conformal anomaly, whilst the second term is due to the torus. It has the effect of increasing the effective central charge, driving a critical model supercritical, where it is subject to the renormalization group instability discussed in section 4. There is a similar instanton effect in the string black hole model when one considers the dilaton term, which takes the form

$$\exp\left(\frac{1}{8\pi} \int d^2z R^{(2)} \ln (1 + |w|^2) + \ldots\right)$$

where $R^{(2)}$ is the world sheet curvature. It is convenient for technical reasons to use conformal invariance to concentrate the world-sheet curvature at a point $z^*$

$$R^{(2)} = 4\pi \delta^{(2)}(z - z^*)$$

where $\chi = 2$ for a world sheet with spherical topology. Taking the limit $z^* \to \infty$, and integrating over the location $z_0$ of the instanton one finds the following contribution to the partition function

$$\int d^2\rho \left(\frac{a^2}{\rho^4}\right)^z \left(\frac{\rho^2}{\Lambda^2}\right)$$

where $\Lambda$ is the infrared cutoff. Since the leading instanton contributions to the path-integral come from $\rho = O(a)$, the ultraviolet cut-off, we find that instantons contribute

$$\int d^2z R^{(2)} \ln |\Lambda/a|$$

to the world-sheet effective action. This reproduces the above mention higher-genus effects. Such shifts in the dilaton are essential for the consistency of the full string theory within the interpretation of target time as a local renormalization scale.

We conclude this section by discussing the rôle of instantons in the renormalization group flow, in particular to justify the bounce picture discussed in section 4. Close to the infrared fixed point, the system is believed to be topological both on the world sheet and in target space-time. A topological $\sigma$-model is described by an appropriate twist of the $N = 2$ supersymmetric $SL(2, \mathbb{R})/U(1)$ Wess-Zumino model, under which the fermions of the $N = 2$ model become ghosts [5, 44, 23]. The topological model possesses an enhanced symmetry which includes a bosonic $W_\infty \otimes W_\infty$ symmetry. The breaking of $W_\infty \otimes W_\infty$ down to a single $W_\infty$ generates space-time dynamically [23]. This breaking of twisted supersymmetry (or topological BRST symmetry) arises from instanton effects [49], associated with the appearance in the presence of such configurations of logarithmic divergences in correlation functions, whose form has been computed in the dilute-gas approximation. It is essential for the discussion of this effect to include a constant $u$ in the definition of the instanton field [49]

$$w = u + \frac{\rho}{z - z_0}$$

(97)
The rôle of the field \( u \) is similar to that of the Higgs field in supersymmetric models \[53\] where the instantons break supersymmetry dynamically. In that case the field \( u \) labels vacua of the theory. In the limit where the infrared cutoff \( \Lambda \) is large: \( \Lambda / a \gg u \), in which case the relevant correlator has a double-logarithmic divergence \[49\]:

\[
<O(x_1)O(x_2)> = -8\pi^2 g^I \ln \left( \frac{|x_1 - x_2|}{a} \right) \ln(\Lambda |x_1 - x_2|) + \ldots \quad (98)
\]

where the \( \ldots \) denote subleading terms in the infrared limit. The dependence of the correlator \((98)\) on the distance between points on the world sheet indicates that the world-sheet topological symmetry is broken, with the appearance of a metric. On the other hand, the same correlator vanishes in the ultraviolet limit \( \Lambda \simeq a \) and the world-sheet topological symmetry is restored. The ultraviolet fixed point of the flow is a stable conformally-invariant background for the \( c = 1 \) string, which can be regarded as an appropriate mixing of the \( SL(2, R)/U(1) \) coset with ghost fields \[54, 55\]. In our picture, instantons provide a qualitative description of this mixing.

This picture is supported by a computation of the vacuum energy associated with instanton-anti-instanton configurations in the toy topological model described above. Identifying the vacuum energy with the one-point function of the anti-instanton vertex in an instanton background:

\[
E^{I\bar{I}}_{\text{vac}} = <V_{\bar{I}}> = -g^{I}\partial_{x_1} <O(x_1)O(x_2)> \Big|_{x_1 \rightarrow x_2} \quad (99)
\]

and recalling that the dominant anti-instanton configurations have sizes \( \rho \simeq a \), we can use \((98)\) to estimate that in the infrared limit when \( \Lambda / a \gg u \)

\[
E_{\text{vac}} = 16\pi^2 g^I g^\bar{J} V^{(2)} \frac{1}{a^2} [\ln(\Lambda / a) + O(1)] \quad (100)
\]

where \( V^{(2)} \) is the world-sheet volume. This logarithmic divergence will be removed in the full theory when all topological string modes are taken into account, but the vacuum energy of the effective theory of locally-measurable observables will be non-zero. On the other hand, the limit \( u \rightarrow \infty \) yields zero vacuum energy \[49\], as expected for a theory with unbroken BRST symmetry. According to our previous discussion this limit coincides with the critical \( c = 1 \) string. Indeed, it has been argued in \[55\] that the \( c = 1 \) string theory could be topological. In this case, the topological nature pertains only to the world-sheet, the target space theory being described by a flat spacetime, over which the ‘tachyon’ matter field propagates \[55\]. Thus the \( c = 1 \) model should be regarded as the stable ground state, to which the false string vacua with broken BRST symmetry flow, justifying the bounce interpretation given in section 4, according to which temporal flow is opposite to the renormalization flow, i.e., from the infrared fixed point to the ultraviolet.

\[6\] The topological world-sheet character allows for a formal resummation of the world-sheet genera in the \( c = 1 \) string case \[55\], and should be contrasted to the situation at the infrared fixed point, where both the world-sheet and the target space theories are argued to be topological \[23\].
6 Valley Contributions to the $S$ Matrix

We now consider the contributions of monopoles and instantons to $S$ matrix elements giving transitions between a generic initial-state density matrix $\rho_B^{(in)}$ and final-state density matrix $\rho_D^{(out)}$. This is described by an absorptive part of a world-sheet correlation function

$$\sum_{X_{out}} \langle D, X \rangle_{out} |\langle X, C |B \rangle_{in}| = \sum_{X_{out}} \langle 0 |T(\phi(z_A)\phi(z_D))|X \rangle_{out} \langle X|T(\phi(z_C)\phi(z_B))|0 \rangle_{in} = \langle 0 |T(\phi(z_A)\phi(z_D))T(\phi(z_C)\phi(z_B))|0 \rangle_{in}$$

(101)

Here we have used the optical theorem [56] on the world sheet, which is valid because conventional quantum field theory, and indeed quantum mechanics, remain valid on the world sheet, to replace the sum over unseen states $X$ by unity. Next, we use dilute-gas approximations to estimate the leading monopole-antimonopole and instanton-anti-instanton contributions to the absorptive part (101). We expect these to be dominated [57] by valley configurations in the Euclidean functional integral, so that in a semi-classical approximation

$$S \propto \text{Abs} \int D\phi_c \exp(-S_v(\phi_c))F_{\text{kin}}$$

(102)

where the integral is over the collective coordinates $\phi_c$ of the valley, whose action is $S_v(\phi_c)$. The function $F_{\text{kin}}$ depends on kinematic factors, taking generically the form

$$F_{\text{kin}} = \exp(E\Delta R)$$

(103)

in the case of a four-point function for large $E\Delta R$, where $E$ is the centre-of-mass energy and $\Delta R$ is the valley separation parameter. This enables us to make a saddle-point approximation to the integral (102), which we then continue back to Minkowski space.

Valley trajectories $\psi_v$ have a homotopic parameter $\mu$ and obey an equation of the form

$$\frac{\partial S_0}{\partial \psi}|_{\psi=\psi_v} = W_\psi(\mu) \frac{\partial \psi_v}{\partial \mu}$$

(104)

where $W_\psi(\mu)$ is a weight function that is positive definite and decays rapidly at large distances [58, 57]. We adapt techniques used in the $O(3) \sigma$-model [59, 60] to find the monopole-antimonopole and instanton-anti-instanton valleys in a reduced version of the $SL(2,R)/U(1)$ model. The separations of the topological defects and anti-defects are well-defined in the presence of conformal symmetry breaking, which is provided in our case by the dilaton field [4]. Valleys can be found by using the analogy [59] between $\mu$ and a ‘time’ variable for defect-anti-defect scattering. We do not discuss here the details of their construction, but record the results.
The monopole-antimonopole valley function, expressed in terms of the original world-sheet variables, reads

\[
w(z, \bar{z}) = \frac{(v - 1/v)\bar{z}}{1 + |z|^2}
\]

where \( v \) denotes the separation in the \( \sigma \)-model framework. Eq. (105) represents a concentric valley, which can then be mapped into an ordinary valley by applying appropriate conformal transformations. The function (105) interpolates between a far-separated monopole-antimonopole pair \( (v \to \infty) \) and the trivial vacuum \( (v = 1) \). For large but finite separations the corresponding valley action leads to the action of a monopole-antimonopole pair interacting via dipole interactions. The action of the monopole-antimonopole valley depends on the angular ultraviolet cut-off \( w \) introduced in section 5:

\[
S_m = 8\pi q^2 \ln(2) \sqrt{2} e^\gamma + 2\pi q^2 \ln \frac{2R}{\omega} + 2\pi q^2 \ln \left[ \frac{|z_1 - z_2|}{(4R^2 + |z_1|^2)^{1/2}} \left( \frac{4}{(4R^2 + |z_2|^2)^{1/2}} \right) \right]
\]

for a monopole and antimonopole pair of equal and opposite charges \( q \), which we treat as a collective coordinate over which we must integrate, where \( \gamma \) is Euler’s constant, the second term in (106) is a logarithmically-divergent self-energy term on a spherical world sheet of radius \( R \), and the last term in (106) is a dipole interaction energy. For finite separations \( 0 < |z_1 - z_2| < \infty \) and very small world-sheets \( R = O(a \to 0) \), the action (106) yields

\[
S_m = 2\pi q^2 \ln \frac{a}{\omega} + \text{finite parts}
\]

where the ultraviolet cut-off dependence is apparent.

To construct the instanton valley, we notice that in the reduced model used for the construction of the monopole valley (105) the solution for an instanton-anti-instanton pair is derived from the corresponding monopole case via a conformal transformation in the \( (\mu, \ln|z|) \)-plane. In ref. [4] we give arguments why this construction is true for finite separations as well, thereby leading to an expression of the instanton valley as an (approximate) conformal transform of the monopole valley (105). The action of the instanton-anti-instanton valley in the large-separation limit of the dilute-gas approximation is

\[
S_{IT} = k\ln(1 + |\rho|^2/a^2) + O\left( \frac{\rho^7}{(\Delta R)^2} \right)
\]

where \( \Delta R \) is the separation of an instanton of size \( \rho \) and an anti-instanton of size \( \bar{\rho} \), and we find a dependence on the ultraviolet cut-off \( a \). The actions (107,108) substituted into the general expression (102) make non-trivial contributions to the \( S \) matrix that do not factorize as a product of \( S \) and \( S^\dagger \) matrix elements, as we shall now see.
In the dilute-gas approximation introduced in section 5, the topologically trivial zero monopole-antimonopole, zero instanton-anti-instanton sector in the unitary sum in (101) provides the usual $S$-matrix description of scattering in a fixed background, with no back reaction of the light matter on the metric. This result is well-known in the conformal field theory approach to critical string theory, and is discussed explicitly in the present context in section 6 of ref. [4]. This $S$-matrix contribution corresponds to the usual Hamiltonian description of quantum mechanics, via the representation $S = 1 + iT : T = \int_{-\infty}^{\infty} dt H(t)$. Any topologically non-trivial contribution to the unitarity sum in (101) goes beyond the usual treatment of conformal field theory in critical strings, and makes a contribution to the non-factorization of the $S$-matrix: $S = SS^\dagger + \ldots$. Two such contributions that we have identified above come from the monopole-antimonopole and instanton-anti-instanton sectors discussed above, which we expect to be dominated by the valley actions (106) and (108) respectively.

The dependence of the monopole-antimonopole valley action (106) on the ultra-violet cutoff $\omega$, which we identify with the target-space time $t = -\ln \omega$, and of the instanton-anti-instanton valley action (108) on the local scale-dependent level parameter $k$ ([88, 91]) where $t = -\ln a$, tell us that both valleys contribute to the non-Hamiltonian term in the modified quantum Liouville equation (3), that are proportional to the anomalous dimensions $(\Delta_m - 1)$ ([84]) of an irrelevant dipole-like monopole-antimonopole pair and $\gamma_0$ ([91]) of a matter deformation respectively. Integrating up the corresponding modified quantum Liouville equation (3), we find that a generic $S$ matrix element, defined at finite time $t$ by $\rho(t) = S(t) \rho(0)$, contains a factor

$$S \simeq e^{-2(\Delta_m - 1)t + \ldots}$$  \hspace{1cm} (109)$$

associated with an irrelevant dipole-like monopole-antimonopole pair, and

$$S \simeq e^{-2\gamma_0 t + \ldots}$$  \hspace{1cm} (110)$$

from the instanton-anti-instanton valley. Both of these time-dependences apply in limits of far-separated defect and anti-defect.

Before discussing the rôle of topologically non-trivial world-sheet configurations in the suppression of coherence at large times, we review a similar phenomenon in Hall conductors, namely the suppression of spatial correlations by “de-phasons” ([61]). As we discuss in more detail in section 7, the two-dimensional black-hole model is analogous to a fractional Hall conductor ([7]), with the Wess-Zumino level parameter $k$ corresponding to the transverse conductivity. Hall systems generally are described by appropriate $\sigma$-models with Wess-Zumino $\theta$-terms, defined on the two-dimensional space of electron motion ([61]). The fields of such $\sigma$-models, which are space-time coordinates in the black-hole case, correspond to electrons propagating in the plane, with the transverse and longitudinal conductivities $\sigma_{\mu\nu}$ corresponding
to background fields in the black-hole case. The Wess-Zumino terms are associated with instantons that renormalize non-perturbatively these conductivities:

\[ \beta_{\mu\nu} = \frac{d\sigma_{\mu\nu}}{d\ln L} \neq 0 \]  \hspace{1cm} (111)

where \( L \) is an infrared cut-off on the instanton size that serves as a renormalization group scale. The effects of these instantons are seen clearly in the case of a \( \sigma \)-model defined on a compact manifold \( U(m+n)/U(m) \otimes U(n) \), where \( m,n \) are electron field replicas with the physical case corresponding to \( m = n \rightarrow 0 \). This limit is equivalent to the corresponding limit of the non-compact models \( U(m, n)/U(m) \otimes U(n) \), where the corresponding instantons have infinite action when \( m, n \neq 0 \), and might naïvely be thought unimportant. However, the physical limit of \( m \) and \( n \rightarrow 0 \) is sensitive to instanton effects. Another example of the importance of infinite-action topological solutions is provided by three-dimensional anyonic Chern-Simons theories, which are relevant to the fractional quantum Hall effect (FQHE).

We believe that localization in Hall systems is directly related to our problem of quantum coherence. In the IQHE model of ref. [61], impurities are responsible for the localization of the electron wave function in the plane. The localization is achieved formally by representing collectively the effects of impurities on electron trajectories via extended, static scattering centres termed “de-phasons”, which trap the electron waves into localized states with sizes \( O(1/\sqrt{\rho}) \), where \( \rho \) is the de-phason density. As a result, the electron correlation functions are suppressed at large spatial separations:

\[ \propto \exp[-(x-y)^2/\rho] \]  \hspace{1cm} (112)

at zero magnetic field (\( \theta = 0 \)). As the magnetic field is varied so that the transverse conductivity becomes a half-integer (in units of \( e^2/h \)), corresponding to a discrete value of the instanton angle \( \theta = \pi \), the property of the de-phasons to destroy phase coherence becomes lost. Quantitatively, the expectation value of an electron loop that encircles a de-phason, in the presence of a magnetic field, is proportional to \( e^{-(x-y)^2 \rho \cos \theta/2} \). Thus, for \( \theta = \pi \) the “effective de-phason density” \( \rho \cos(\theta/2) \) vanishes, and the electrons delocalize implying a non-zero longitudinal conductivity. This delocalization property is responsible for the transition between two adjacent plateaux of the transverse conductivity in the Hall conductivity diagram [61]. These ideas can be extended to the FQHE via the three-dimensional anyonic Chern-Simons theories mentioned above, which are closer to our black-hole interests. In our case, the massive modes of the \( SL(2, R)/U(1) \) black-hole model are the analogues of the de-phasons. As discussed earlier in this section, the instantons in this model renormalize the Wess-Zumino level-parameter \( k \) (c.f. \( \theta \)), changing the mass and size of the black hole. The delocalized phase at \( \theta = \pi \) may be identified with the “topological” phase at

\[^7\] In fact, it appears to be the zero-field Hall effect that describes physics at the space-time singularity [17].
the space-time singularity \cite{23}, which is an infrared fixed point. The propagating “tachyon” mixes in this limit, as we have discussed above, with the delocalized topological modes of the string that are analogous to the de-phasons. The localization properties are consistent with shrinking of the world-sheet as one approaches the ultraviolet fixed point that corresponds to a flat target space-time where the tachyons are normal localized fields that do not mix with topological modes.

Our formalism for the time evolution of the density matrix is analogous to the Drude model of quantum friction \cite{20, 21}, with the massive string modes playing the rôles of ‘environmental oscillators’. In the language of world-sheet \(\sigma\)-model couplings \(\{g\}\), the reduced density matrix of the observable states is given, relative to that evaluated in conventional Schrödinger quantum mechanics, by an expression of the form

\[
\rho(g, g', t) / \rho_S(g, g', t) \simeq e^{-\eta D t (g-g')^2 + \ldots}
\]

where \(\eta\) is a calculable proportionality coefficient, and \(G_{ij}\) is the Zamolodchikov metric \cite{31} in the space of couplings. In string theory, the identification of the target-space action with the Zamolodchikov \(C\)-function \(C(\{g\})\) \cite{31} enables the Drude exponent to be written in the form

\[
\beta^i G_{ij} \beta^j = \partial_t C(\{g\})
\]

which also determines the rate of increase of entropy

\[
\dot{S} = \beta^i G_{ij} \beta^j S
\]

In the string analogue \cite{114} of the Drude model \cite{113} the rôle of the coordinates in (real) space is played by the \(\sigma\)-model couplings \(g^i\) that are target-space background fields. Relevant for us is the tachyon field \(T(X)\), leading us to interpret \((g-g')^2\) in \cite{113} as

\[
(g-g')^2 = (T-T')^2 \simeq (\nabla T)^2 (X-X')^2
\]

for small target separations \((X, X')\). Equation \cite{116} substituted into \cite{113} gives us a suppression very similar to the IQHE case \cite{112}.

The effect of the time-dependences \cite{109, 110, 113, 116} is to suppress off-diagonal elements in the target configuration space representation \cite{64} of the out-state density matrix:

\[
\rho_{\text{out}}(x, x') = \hat{\rho}(x)\delta(x-x')
\]

This behaviour can be understood intuitively \cite{4, 3} as being related to the apparent shrinking of the string world sheet in target space, which destroys interferences between strings localized at different points in target configuration space, c.f. the de-phasons in the Hall model \cite{61}. This behaviour is generic for string contributions to the space-time foam, which make the theory supercritical locally, inducing renormalization group (target time) flow. The two specific contributions \cite{109, 110} to
this suppression of space-time coherence that we have identified in this paper correspond to microscopic black hole formation and to the back-reaction of matter on a microscopic black hole, entailing in each case information loss across an event horizon.

7 $W$ Symmetries and Non-Hamiltonian Time Evolution

At various places in earlier sections, we have mentioned the infinite-dimensional $W$ algebra underlying the string black hole model, and its rôle in interrelating different solitonic states in the spectrum of the model. In earlier papers we have stressed the rôle of $W$ symmetry in preserving quantum coherence when back-reaction is neglected. Subsequently, we have argued that the coherence of the effective theory of light states is suppressed when back-reaction is taken into account, in particular via the monopole-antimonopole and instanton-anti-instanton contributions discussed in the previous section. In this section we discuss the $W$-transformation properties of these explicit contributions to $\delta S$ and $\delta H$, and relate them to the formalisms of refs and . In this way, we link explicitly the loss of coherence to the leakage of $W$ quantum numbers.

As a warm-up, we first present an analogous phenomenon in the theory of the Quantum Hall Effect . The ground state of an integer Quantum Hall conductor with filling fraction $\nu = 1$ is represented by a non-singular wave function $\Psi_0(z_1, z_2, ..., z_N)$ for a system of non-interacting electrons. On the other hand, the ground state of a fractional Quantum Hall conductor is singular:

$$\Psi(z_1, z_2, ..., z_N) = S_p \Psi_0(z_1, z_2, ..., z_N)$$ (118)

where $p$ is related to the filling fraction $\nu$ by $\nu = 1/(2p + 1)$ and

$$S_p = \Pi_{k<l}(z_k - z_l)^{2p}$$ (119)

The prefactor $S_p$ can be regarded as the matrix element of an operator that creates monopoles and vortices on the world sheet:

$$S_p \propto exp\left(\sum_{k<l}[2pReln(z_k - z_l) + i2pImln(z_k - z_l)]\right)$$ (120)

This representation realizes the plasma picture of , and is in direct analogy with the representation of Minkowski and Euclidean black holes as monopoles and vortices on the world sheet which we introduced in section 5.
The important aspect of this analogy for the purposes of this section is the ob-
servation that the relation (118) between the IQHE and FQHE ground states can
be regarded as an invertible but non-unitary relation between the corresponding
density matrices:
\[
\rho_p = S_p S_p^\dagger \rho_0
\]
(121)
The relationship between the density matrices \(\rho_0\) and \(\rho_p\) corresponds to that between
the \(\rho\) and \(\tilde{\rho}\) of [11], as we shall see in more detail shortly.

It has been pointed out that the IQHE and FQHE systems both possess an infinite-
dimensional \(W\) symmetry associated with the presence of incompressible quantum
electron fluids [67], in which the quantum deformation parameter \(k\) is related
to the filling fraction \(\nu\) by
\[
\nu = \frac{1}{k}
\]
(122)
Thus the operator \(S_p\) can be regarded as inducing a quantum deformation of the
\(W\) algebra that does not change the classical \(W\) charges. The operator \(S_p\) induces
\(\delta\)-function Schwinger terms in the operator product expansion, associated with the
singularities in \(S_p\). They are related to the corresponding term in the operator product
expansion for two energy-momentum tensors, which is a measure of the central
charge \(c\) in the Virasoro algebra, which is related in turn to the level parameter \(k\)
and hence to the filling fraction \(\nu\):
\[
c = \frac{3k}{k-2} = \frac{3}{1-2\nu}
\]
(123)
As we shall see shortly, the world-sheet monopoles and instantons discussed in sec-
tion 5 play a similar rôle in the black-hole case.

In the case of a string black hole contribution to space-time foam, the rôle of
the density matrix \(\rho\) of [11] is played by the exact density matrix for external
tachyon fields dressed by higher-level operators as discussed in sections 4 and 5: As is well known [69], there is a well-defined \(S\)-matrix for the scattering of these
dressed tachyons off a string black hole, and hence a well-defined Hamiltonian \(H\),
and the time evolution of the density matrix is given by the conventional Liouville
equation using this Hamiltonian.

However, as we have mentioned previously, a realistic scattering experiment does
not measure the non-local topological solitonic states created by the operators \(L^0_0 L^1_0\)
etc., and hence deals only with bare tachyonic operators \(T\). As we have described in
section 5, the scattering of these bare operators exhibits additional renormalization
scale (i.e., Liouville field \(\phi\), i.e., time) dependence that cannot be absorbed within
the usual Hamiltonian/\(S\)-matrix description. It is the density matrix of this \((T, \phi)\)
system that we interpret as the \( \tilde{\rho} \) of [11]:

\[
\tilde{\rho} = \frac{e^{-\beta H([F_{c-c}^{-c} - \frac{1}{2}, 0, 0], r, \phi)}}{Tr[e^{-\beta H([F_{c-c}^{-c} - \frac{1}{2}, 0, 0], r, \phi)}]} \tag{124}
\]

where \( F_{-\frac{1}{2}, 0, 0} \) is the ‘tachyon’ deformation defined in (73), and \( \ldots \phi \) denotes the appropriate Liouville dressing. It is because of the extra time dependence mentioned above that this density matrix \( \tilde{\rho} \) obeys the modified Liouville equation (3) derived in section 3 [2].

The monopole-antimonopole pairs discussed in the previous section contribute to the relation between \( \rho \) (72) to \( \tilde{\rho} \) (124), by representing the creation and annihilation of black holes, and the instanton-anti-instanton pairs make a contribution to this relation that is associated with rescaling of the target-space metric due to changes in the black hole mass. According to the analysis of section 2, this relation is associated with a loss of quantum coherence, and we have exhibited just such a loss in section 5.

In order to understand the physical origin of this loss of coherence, it is instructive to consider the \( W \)-transformation properties of these topological defects on the world sheet. To do this, we express the string black hole action in terms of target-space Kruskal-Szekeres coordinates \( u \equiv sinh re^{it} \), \( \bar{u} = sinh re^{-it} \):

\[
S = \frac{k}{4\pi} \int d^2z \frac{\partial u \partial \bar{u} + \partial \bar{u} \partial u}{1 - \bar{u} u} \tag{125}
\]

Monopoles generate transformations of the form

\[
u \rightarrow u e^{i\alpha}, \bar{u} \rightarrow \bar{u} e^{-i\alpha} \tag{126}
\]

It is useful to construct parafermion operators \( \psi_{+, -} \) by attaching Dirac strings to these monopoles [70]:

\[
\psi_{+} = \frac{\partial u}{\sqrt{1 - \bar{u} u}} V_{+}, \quad \psi_{-} = \frac{\partial \bar{u}}{\sqrt{1 - \bar{u} u}} V_{-}
\]

\[
V_{\pm} = exp[\pm \frac{1}{2} \int_{C} (dz A + d\bar{z} \bar{A})]\]

\[
A = \frac{u \partial \bar{u} - \bar{u} \partial u}{1 - \bar{u} u} \quad ; \quad \bar{A} = \frac{\bar{u} \partial u - u \partial \bar{u}}{1 - \bar{u} u} \tag{127}
\]

The \( W \) currents are then given in terms of these constructs by

\[
W_{s} = \sum_{k=1}^{s-1} (-1)^{s-k-1} A_{k} \partial^{s-k-1} \psi_{+} \partial^{s-k-1} \psi_{-} \tag{128}
\]
where $s$ denotes the conformal spin, and
\[
A_k^s = \frac{1}{s-1} \left( \begin{array}{c} s-1 \\ k \end{array} \right) \left( \begin{array}{c} s-1 \\ s-k \end{array} \right)
\] (129)

We note that all these $W_\infty$ transformations on the world sheet are generated by (1,0) or (0,1) currents, so the corresponding stress-tensor deformations are (1,1), and hence can surely be elevated to target space \[30\]. This elevation has been worked out in \[20\], where it was shown that this elevation preserves the $W_\infty$ structure. One considers variations of the action $S$ under infinitesimal transformations $\delta u$, $\delta \bar{u}$:
\[
\delta S = \int \left( \delta^s u \frac{\delta S}{\delta u} + \delta^s \bar{u} \frac{\delta S}{\delta \bar{u}} \right)
\] (130)
where
\[
\frac{\delta S}{\delta u} = \frac{\partial \bar{u} \partial u}{1 - u \bar{u}} + \frac{\pi \partial u \partial \bar{u}}{(1 - u \bar{u})^2}
\]
\[
\frac{\delta S}{\delta \bar{u}} = \frac{\partial \bar{u} \partial u}{1 - u \bar{u}} + \frac{u \partial u \partial \bar{u}}{(1 - u \bar{u})^2}
\] (131)

Knowing that the $W_\infty$ transformations are generated by chiral fields, we compare (130) with the Ansatz \[70\]
\[
\delta S = \int d^2z \epsilon \bar{W}_s
\] (132)
to find the following infinite set of infinitesimal transformations:
\[
\delta^{s'}_s u = \sum_{i=0}^{s-2} B^s_i \epsilon \sqrt{1 - u \bar{u}} V \partial^{s-i-2} \psi_+ + \sum_{k=1}^{s-1} \sum_{l=0}^{k-2} (-1)^{l} A_k^s \left( \begin{array}{c} k-1 \\ l \end{array} \right) u \partial^{k-2-l} \epsilon \partial^l \psi_- \partial^{s-k-1} \psi_- - (-1)^s \epsilon \bar{\partial} \psi_+ \partial^{s-k-1} \psi_-
\] (133)
with
\[
B^s_i = \frac{1}{s-1} \left( \begin{array}{c} 2s-l-2 \\ s \end{array} \right) \left( \begin{array}{c} s-1 \\ l \end{array} \right)
\] (134)

The corresponding transformation for $\bar{u}$ is obtained by interchanging $u \leftrightarrow \bar{u}$ and multiplying by a factor $(-1)^s$.

It is easy to verify directly that these variations satisfy the $W_\infty$ algebra:
\[
[\delta^{s}_c, \delta^{s'}_d] = \delta^{s+s'-2}_{(s'-1)c (s-l) (s-1)e e' d} + \ldots
\] (135)
where the ... denote lower-spin terms. Hence the $W$ charges
\[
Q^s = \int_C dz W^s(z)
\] (136)
commute, as they constitute an infinite-dimensional Cartan subalgebra of $W_\infty$. It is easy to check that these charges are non-zero in general if $u$ and $\bar{u}$ are not holomorphic functions of the world-sheet coordinates: $u \neq u(z)$, etc., which is the case for the string black hole background. Therefore, their values make available an infinite set of quantum numbers (hair) to specify the black hole state. However, the instantons discussed in section 5 have vanishing $W$ quantum numbers, because they are holomorphic maps $u = u(z)$. One finds that

$$\delta^s \epsilon u = \sum_{l=0}^{s-2} B^s_l \partial^l \epsilon \partial^{s-l-1} u \quad ; \quad \delta^s \pi = 0 \quad (137)$$

Note that instantons and anti-instantons behave differently under the $W_\infty$ transformations.

Now we are ready to discuss in the light of these results the monopole and instanton contributions to the loss of coherence. As has already been mentioned, monopoles make contributions to the functional integral representation of $\tilde{\rho}$ that are identical in form to the prefactor $S_p$ in the relation (121) between the IQHE and FQHE ground states, and play the same rôle as the similarity transformation of [11]. We saw in the previous section [see equation (106)] that the action of a monopole-antimonopole pair depends logarithmically on the ultraviolet cutoff, which translates into a time-dependence, and hence a contribution to $\delta H$ that tends to suppress quantum coherence. We have seen in this section that monopoles possess $W$ hair, by virtue of being non-holomorphic maps of the world sheet into target space. Thus they carry off $W$ quantum numbers, which entails information loss and hence explains the loss of coherence. The action of the instanton-anti-instanton valley exhibits a logarithmic dependence on the ultraviolet cutoff (and hence time-dependence, a contribution to $\delta H$ and a loss of coherence) only when there is a finite separation between the instanton and anti-instanton. As we have discussed just above, an isolated instanton is represented by a holomorphic map from the world sheet into target space, hence carries no $W$ quantum numbers and does not contribute to decoherence. However, once an instanton and anti-instanton are superposed at finite separation, their configuration can no longer be represented by such a holomorphic map, and they will carry $W$ quantum numbers in general, explaining this loss of coherence.

Thus we have a complete understanding of the relation between our non-critical string formalism and the general similarity transformation theory of [11], including a physical understanding of the resulting loss of coherence due to string black holes in terms of a leakage of $W$ quantum numbers. We expect that a similar discussion could be given for any other topologically non-trivial contributions to the space-time foam in string theory, in which other string symmetries would play the rôle of the black hole’s $W_\infty$ algebra, which is but a small part of the enormous full local symmetry of string.
8 CPT Violation

It is a basic theorem [71] of quantum field theory that $CPT$ should not be violated, as a consequence of locality, Lorentz invariance and unitarity. String theory is of course based on local quantum field theory on the world sheet, but this is not equivalent to locality in target space-time. Lorentz invariance is a property of critical string theory, but requires re-examination in the context of the non-critical string approach to time that we have espoused here. Thus, even though we do not challenge the unitarity of the effective low-energy theory, not all the conditions needed to derive the $CPT$ theorem are satisfied in our string framework, and it is appropriate to re-open the possibility of $CPT$ violation.

This has often been discussed [19] in the general context of quantum gravity, motivated largely by the likelihood of non-locality, and also specifically in the context of string theory [72, 73, 74]. In this latter case, it was shown that $CPT$ violation was in principle possible if certain world-sheet charges were not conserved [72], but it was also shown that this could not occur in a string model with a flat target space-time [74].

We have taken this analysis one step further, pointing out that space-times that appear singular from the point of view of conventional general relativity, such as a black hole, can be described by topological defects on the world sheet, such as monopoles or vortices in the black-hole case [17]. Moreover, it is well-known that quantum numbers carried by fermionic fields may no longer be conserved in the presence of topological defects, c.f., monopoles and instantons in conventional three-dimensional space. To demonstrate that $CPT$ is violated in any specific elementary-particle system, such as the neutral kaon system, would require a complete string-derived model in which the world-sheet fermion contents of all quarks and leptons were known. Would that we had such a complete model! However, even in its absence, we [15, 75] and Huet and Peskin [76] have argued that the likelihood of $CPT$ violation in string theory makes it worthwhile to re-examine the traditional phenomenology of the neutral kaon and other systems, parametrizing possible $CPT$-violating effects and constraining their magnitudes. Before reviewing that work here, we would like to make contact with other work [19] on $CPT$ violation in the general framework of quantum gravity.

The conventional, strong form of $CPT$ invariance proved in local quantum field theory implies the existence of a $CPT$ operator $\Theta$ that transforms any initial density matrix $\rho_{in}$ into some final-state density matrix $\rho_{out}$:

$$\rho_{out} = \Theta \rho_{in}$$

(138)
and correspondingly
\[ \rho_{in}' = \Theta^{-1}\rho_{out} \]  
(139)
where \( \rho_{out} \) is related to \( \rho_{in} \) by the familiar \$ matrix:
\[ \rho_{out} = \$\rho_{in} \]  
(140)
and likewise for \( \rho_{out}' \) and \( \rho_{in}' \):
\[ \rho_{out}' = \$\rho_{in}' \]  
(141)
The following relation is a trivial consequence of equations (138) to (141):
\[ \Theta = \$\Theta^{-1}\$ \]  
(142)
Again, it is a trivial consequence of (142) that the \$ matrix must have an inverse:
\[ \$^{-1} = \Theta^{-1}\$\Theta^{-1} \]  
(143)
However, there can be no inverse of the superscattering matrix \$ in any framework that allows pure states to evolve into mixed states, as has been argued to be a general necessity in any quantum theory of gravity, and as we have found specifically in our non-critical string approach. Thus, there cannot be a CPT operator with the properties assumed above, and the CPT theorem cannot hold in its strong field-theoretical form.

A weaker form of CPT invariance has been proposed [19], according to which the probability \( P(\phi \to \psi) \) that a pure initial state \( \phi \) will be observed to become a given pure final state \( \psi \) is equal to the probability for the CPT-reversed process:
\[ P(\psi \to \phi) = P(\theta^{-1}\phi \to \theta\psi) \]  
(144)
This is automatically true in any theory in which the final-state density matrix is proportional to the unit matrix, as was the case in our simple two-state model in section 3. However, we will not address here the interesting question whether this or some other weak form of CPT invariance holds in string theory. For now, we simply observe that there is no reason for the strong CPT theorem to hold in string theory. To substantiate this claim we review briefly the arguments presented in ref. [14] concerning CPT-violation in our string model. Following [72] we consider the target-space CPT operator \( \Theta \) (138) as being derived from an appropriate worldsheet operator \( \Theta_w \). Any state in target-space can be considered as an eigenstate of the \( \sigma \)-model Hamiltonian \( E \) with eigenvalue \( m_i \), the mass of the corresponding particle, i.e.
\[ E|m_i, Q_i >= m_i|m_i, Q_i > \]  
(145)
where \( \{Q_i\} \) denotes a set of conserved worldsheet charges that can be elevated to target space,
\[ [E, \hat{Q}_i] = 0 \quad ; \quad \hat{Q}_i|m_i, Q_i >= Q_i|m_i, Q_i > \]  
(146)
The $\hat{A}$ denote quantum-mechanical operators on the world-sheet. World-sheet $CPT$ invariance is guaranteed if and only if
\[
[E, \Theta_w] = 0 \quad ; \quad \hat{Q}_i \Theta_w + \Theta_w \hat{Q}_i = 0
\] (147)

This implies $CPT$-invariance in target space in the following sense [72]: the $CPT$ transform of a state of mass $m_i$ and ‘charge’ $Q_i$, is $\Theta_w |m_i, Q_i >$. Using (147) we can readily see that it will be an eigenstate of $E$ with mass $m_i$ and ‘charge’ $-Q_i$. In our case, the existence of valleys of topological defects on the world-sheet spoils the conservation of $Q_i$, and thus (147), as a result of logarithmic divergences, as we have discussed in sections 5 and 6. These imply temporal dependences of the ‘charges’ $Q_i$, and hence their conservation is spoiled. As a consequence, the above ‘proof’ of $CPT$ invariance in target space fails.

Although the above picture is rather heuristic, and much more work is required to define the elevation process of the $CPT$ operation from the world sheet to target space in a mathematical rigorous way, however it is certainly suggestive of the kind of $CPT$ violation one should expect in string theory formulated in highly curved space-time backgrounds. It is therefore worthwhile to explore the possibility of its violation, though we also cannot exclude the possibility that the strong $CPT$ theorem might not be violated detectably in any given experiment.

We now describe briefly the formalism [10, 15] for describing the possible modification of quantum mechanics and violation of $CPT$ in the neutral kaon system, which is among the most sensitive microscopic laboratories for studying these possibilities. In the normal quantum-mechanical formalism, the time-evolution of a neutral kaon density matrix is given by
\[
\partial_t \rho = -i(H\rho - \rho H^\dagger)
\] (148)
where the Hamiltonian takes the following form in the $(K^0, \bar{K}^0)$ basis:
\[
H = \begin{pmatrix}
(M + \frac{1}{2} \Delta M) - \frac{1}{2}i(\Gamma + \frac{1}{2} \Delta \Gamma) & M_{12} - \frac{1}{2}i\Gamma_{12} \\
M_{12} - \frac{1}{2}i\Gamma_{12} & (M - \frac{1}{2} \Delta M) + \frac{1}{2}i(\Gamma - \frac{1}{2} \Delta \Gamma)
\end{pmatrix}
\] (149)
The non-hermiticity of $H$ reflects the process of $K$ decay: an initially-pure state evolving according to (148) and (149) remains pure.

In order to discuss the possible modification of this normal quantum-mechanical evolution, and allow for the possibility of $CPT$ violation, it is convenient to rewrite (148) and (149) in a Pauli $\sigma$-matrix basis [10], introducing components $\rho_\alpha$ of the density matrix:
\[
\rho = 1/2 \rho_\alpha \sigma_\alpha
\] (150)
which evolves according to
\[
\partial_t \rho_\alpha = h_{\alpha\beta} \rho_\beta
\] (151)
with
\[
h_{\alpha \beta} \equiv \begin{pmatrix}
Imh_0 & Imh_1 & Imh_2 & Imh_3 \\
Imh_1 & Imh_0 & -Reh_3 & Reh_2 \\
Imh_2 & Reh_3 & Imh_0 & -Reh_1 \\
Imh_3 & -Reh_2 & Reh_1 & Imh_0
\end{pmatrix}
\] (152)

It is easy to check that at large times \( \rho \) takes the form
\[
\rho \simeq e^{-\Gamma_L t} \begin{pmatrix} 1 & \epsilon^* \\ \epsilon & |\epsilon|^2 \end{pmatrix}
\] (153)
where \( \epsilon \) is given by
\[
\epsilon = \frac{\frac{1}{2}iIm\Gamma_{12} - ImM_{12}}{\frac{1}{2}\Delta\Gamma - i\Delta M}
\] (154)
in the usual way.

A modification of quantum mechanics of the form discussed in section 3 can be introduced by modifying equation (151) to become
\[
\partial_t \rho_\alpha = h_{\alpha \beta} \rho_\beta + \dot{h}_{\alpha \beta} \rho_\beta
\] (155)
The form of \( \dot{h}_{\alpha \beta} \) is determined if we assume probability and energy conservation, as proved in the string context in section 3, and that the leading modification conserves strangeness:
\[
\dot{h}_{\alpha \beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2\alpha & -2\beta \\
0 & 0 & -2\beta & -2\gamma
\end{pmatrix}
\] (156)
It is easy to solve the 4 \times 4 linear matrix equation (155) in the limits of large time:
\[
\rho_L \propto \begin{pmatrix} 1 \\
\frac{-\frac{1}{2}i(i(Im\Gamma_{12}+2\beta)-ImM_{12})}{\frac{1}{2}\Delta\Gamma + i\Delta M} \\
\frac{-\frac{1}{2}i\Delta M}{\frac{1}{2}\Delta\Gamma + i\Delta M} \\
\frac{-\frac{1}{2}i\Delta \Gamma + i\Delta M}{\frac{1}{2}\Delta\Gamma + i\Delta M} \\
\end{pmatrix}
\] (157)
and of short time:
\[
\rho_S \propto \begin{pmatrix} |\epsilon|^2 + \frac{\gamma}{\Delta\Gamma} & \frac{-4\beta ImM_{12}(\Delta M/\Delta\Gamma) + \beta^2}{\frac{1}{2}\Delta\Gamma^2 + \Delta M^2} \\
\frac{-4\beta ImM_{12}(\Delta M/\Delta\Gamma) + \beta^2}{\frac{1}{2}\Delta\Gamma^2 + \Delta M^2} & \frac{i\beta}{\frac{1}{2}\Delta\Gamma^2 + \Delta M^2}
\end{pmatrix}
\] (158)
We note that the density matrix (157) for \( K_L \) is mixed to the extent that the parameters \( \beta \) and \( \gamma \) are non-zero. It is also easy to check [13] that the parameters \( \alpha \), \( \beta \) and \( \gamma \) all violate CPT, in accord with the general argument of [13], and consistent with the string analysis mentioned earlier in this section.
Experimental observables $O$ can be introduced into this framework as matrices, with their measured values being given by

$$<O> = \text{Tr}(O\rho)$$ (159)

Examples are the $K$ to $2\pi$ and $3\pi$ decay observables

$$O_{2\pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad O_{3\pi} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ (160)

and the semileptonic decay observables

$$O_{\pi^-l^+\nu} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$O_{\pi^+l^-\bar{\nu}} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$ (161)

A quantity of interest is the difference between the $K_L$ to $2\pi$ and $K_S$ to $3\pi$ decay rates

$$\delta R \equiv R_{2\pi} - R_{3\pi} = \frac{8\beta}{|\Delta\Gamma|} |\varepsilon| \sin\phi_e$$ (162)

where $R_{2\pi}^L \equiv \text{Tr}(O_{2\pi}\rho_L)$, and $R_{3\pi}^S \equiv \text{Tr}(O_{3\pi}\rho_S)/0.22$, and the prefactors are determined by the measured branching ratio for $K_L \to 3\pi^0$. (Strictly speaking, there should be a corresponding prefactor of 0.998 in the formula (160) for the $O_{2\pi}$ observable.)

Using (161), one can calculate the semileptonic decay asymmetry

$$\delta \equiv \frac{\Gamma(\pi^-l^+\nu) - \Gamma(\pi^+l^-\bar{\nu})}{\Gamma(\pi^-l^+\nu) + \Gamma(\pi^+l^-\bar{\nu})}$$ (163)

in the long- and short-lifetime limits:

$$\delta_L = 2Re[\varepsilon(1 - \frac{i\beta}{ImM_{12}})]$$
$$\delta_S = 2Re[\varepsilon(1 + \frac{i\beta}{ImM_{12}})]$$ (164)

The difference between these two values

$$\delta\delta \equiv \delta_L - \delta_S = -\frac{8\beta}{|\Delta\Gamma|} \sin\phi_e - \frac{8\beta}{|\Delta\Gamma|} \sin\phi_e \cos\phi_e$$ (165)

with $\tan\phi_e = (2\Delta M)/\Delta\Gamma$, is a signature of $CPT$ violation that can be explored at the CPLEAR and DA$\phi$NE facilities.
Figure 3: - The \((\beta, \gamma)\) plane on a logarithmic scale for \(\beta > 0\). We plot contours of the conventional \(CP\)-violating parameter \(|\epsilon|\), evaluated from the \(K_L \to 2\pi\) decay rate. The dashed-double-dotted band is that allowed at the one-standard-deviation level by the comparison between measurements of the \(K_L \to 2\pi\) decay rate and the \(K_L\) semileptonic decay asymmetry \(\delta_L\). The dashed line delineates the boundary of the region allowed by the present experimental upper limit on \(K_S \to 3\pi^0\) decays \((R^L_{2\pi} - R^S_{3\pi})\) and a solid line delineates the boundary of the region allowed by a recent preliminary measurement of the \(K_S\) semileptonic decay asymmetry \(\delta_S\). A wavy line bounds approximately the region of \(|\beta|\) which may be prohibited by intermediate-time measurements of \(K \to 2\pi\) decays.

We have used [15] the latest experimental values of \(R_{2\pi}\) and \(R_{3\pi}\) to bound \(\delta R\), and the latest experimental values of \(\delta_{L,S}\) to bound \(\delta \delta\), expressing the results as contours in the \((\beta, \gamma)\) plane as seen in figure 3. Also shown there are contours of the usual \(CP\)-violating parameter \(\epsilon\), which is given in our case by [15]

\[
|\epsilon| = -\frac{2\beta}{|\Delta \Gamma|} \sin \phi_\epsilon + \sqrt{\frac{4\beta^2}{|\Delta \Gamma|^2} - \frac{\gamma}{|\Delta \Gamma|}} + R^L_{2\pi} \tag{166}
\]

On the basis of this preliminary analysis, it is safe to conclude that

\[
\frac{\beta}{\Delta \Gamma} \lesssim 10^{-4} \text{ to } 10^{-3} \quad ; \quad \frac{\gamma}{\Delta \Gamma} \lesssim 10^{-6} \text{ to } 10^{-5} \tag{167}
\]

In addition to more precise experimental data, what is also needed is a more complete global fit to all the available experimental data, including those at intermediate times, which are essential for bounding \(\alpha\), and may improve our bounds [167] on \(\beta\) and \(\gamma\) [15, 17, 18].
Figure 4: As in Fig. 3, on a logarithmic scale for $\beta < 0$.

Figure 5: As in Fig. 3, on a linear scale for the neighborhood of $\beta = 0$. 
We cannot resist pointing out that the bounds (167) are quite close to

\[ O(\Lambda_{QCD}/M_P) m_K \approx 10^{-19} \text{GeV} \]  (168)

which is perhaps the largest magnitude that any such CPT- and quantum-mechanics-violating parameters could conceivably have. Since any such effects are associated with topological string states that have masses of order \( M_P \), we expect them to be suppressed by some power of \( 1/M_P \). This expectation is supported by the analogy with the Feynman-Vernon model of quantum friction [20], in which coherence is suppressed by some power of the unobserved oscillator mass or frequency. If the CPT- and quantum-mechanics-violating parameters discussed in this section are suppressed by just one power of \( M_P \), they may be accessible to the next round of experiments with CPLEAR and/or DA\( \phi \)NE.

9 Connection with Cosmology

We complete this review with a discussion of cosmology in the context of our non-critical approach to string theory, commenting on inflation, entropy generation, variations in fundamental parameters, and the cosmological constant [16]. The first explicit cosmological string theory with a time-dependent background was that discovered in [1]. It has a dilaton field that depends linearly on the time variable \( t \), and the string black hole can be regarded as a Minkowski rotation of this model, in which the \( t \)-dependence of the dilaton field is replaced by the corresponding dependence on the radial coordinate \( r \). It was suggested in [1] that the Universe might make quantum transitions between models with different values of the central charge associated with the dilaton/time variable, but this suggestion was not worked out in detail. The mechanism for such quantum transitions via instantons has now been worked out in the black hole case [11, 12, 13]. As discussed in section 5, it leads (171) to a scale-dependent value of the level parameter \( k \), corresponding in turn to a time-dependence

\[ k(t) \approx ke^{-4\pi \beta I T_0 t} \]  (169)

in the dilute-gas approximation, where we recall that the corresponding instanton \( \beta \)-function

\[ \beta I = -(k/2)g^I \]  (170)

in the large-\( k \) limit is negative, leading to an increase in the effective level parameter \( k \), and hence an approach to the flat limit \( k \to \infty \). In the cosmological context, this implies a slowing down in the rate of expansion of the Universe. According to the analogy with the quantum Hall effect [17], this process is analogous to a series of transitions between different conductivity plateaux.
Several questions arise in this picture, including the following. What was the initial state of the Universe? Is there any analogue of inflation in this approach, particularly as regards entropy generation, which is an essential feature of our non-critical treatment of string theory, as seen in equation (51)? Do fundamental physics parameters such as the velocity of light $c$ and Planck’s constant $\hbar$ vary during the cosmological expansion [16]? How does the effective cosmological constant relax to zero, as it should do in the flat-space limit?

In partial answer to the first of these questions, we recall that the central charge $c = 2(k + 1)/(k - 2)$ becomes infinite in the limit $k \to 2$. The dilute gas approximation is not reliable here, but is suggestive that the Universe started from such a limit. In the picture of section 4, as shown in figure 2, this limit would correspond to the infrared limit of the renormalization group flow. In the case of the string black hole, it corresponds to the region close to the core where there is an appropriate description [23] in terms of a twisted $N = 2$ supersymmetric theory, which is equivalent to a topological field theory in which the concepts of space and time break down. Thus we are led to the conclusions that the origin of the Universe is presumably also described by such a topological field theory, and that the concepts of space and time also break down at the beginning of the Universe.

Our simple cosmological scenario provides a qualitative picture of the entropy production rate in the Universe. In our framework, the rate of entropy increase with time is given by [4, 16]

$$\partial_t S = \beta^i G_{ij} \beta^j S ; \quad G_{ij} = 2|z|^4 <V_i V_j>$$

(172)

where the unitarity requirement of the world-sheet theory implies the positivity of the Zamolodchikov metric [31] $G_{ij} > 0$. Using the C-theorem [31], especially in its string formulation [78] on the fiducial-metric world-sheet, one may write

$$\beta^i G_{ij} \beta^j = \partial_t C(g) ; \quad C(g) = -\frac{1}{12} \int d^D y \sqrt{G} e^{-2\Phi} <TT> + \ldots$$

(173)

In this expression, the $y$ denote target spatial coordinates, $\Phi$ is the dilaton field, and $T \equiv T_{zz}$ is a component of the world-sheet stress tensor. The ... denote the remaining two-point functions that appear in the Zamolodchikov C-function [31], which involve the trace $\Theta$ of the stress tensor, i.e. $<T \Theta>$ and $<\Theta \Theta>$. Taking into account the off-shell corollary of the C-theorem, $\frac{\delta C(g)}{\delta g^i} = G_{ij} \beta^j$, it can readily be shown [79] that such terms can always be removed by an appropriate renormalization-scheme choice, that is by appropriate redefinitions of the renormalized couplings $g^i$, and hence play no rôle in the physics. Thus, one can solve (172) for the entropy $S$ in terms of the Zamolodchikov C-function

$$S(t) = S_0 e^{-\frac{1}{12} \int_0^t \int d^D y \sqrt{G} e^{-2\Phi} <TT> + \ldots}$$

(174)
where the minus sign in the exponent indicates the opposite flow of the time $t$ with respect to the renormalization-group flow. Expression (174) reduces a complicated target-space computation of entropy production in an inflationary scenario to a conformal-field-theory computation of two-point functions involving components of the stress tensor of a first-quantized string. We observe from (172) that the rate of entropy increase is maximized on the maximum-$\beta^i$ surface in coupling constant space. At late stages of the inflationary era, i.e. close to the ultraviolet fixed point, the rate of change of $S$ is strongly suppressed, due to the smallness of the $\beta^i$.

In order to discuss the possible variation of fundamental physical parameters during the expansion of the Universe [16], we first recall the relation between a string black hole mass and the level parameter $k$:

$$M/M_{\text{Planck}} = \sqrt{\frac{1}{k(t) - 2}} e^{\text{const}}$$  \hspace{1cm} (175)

It is well known that light cones are distorted in the presence of a black hole. Specifically, the exact space-time background metric of the black hole Wess-Zumino model has the following asymptotic form for large $r$:

$$ds^2 = 2(k(t) - 2)(dr^2 - \frac{k(t)}{k(t) - 2} dt^2)$$  \hspace{1cm} (176)

This implies a $k$-dependence of the apparent velocity of light, which becomes a time-dependence as a result of equation (169):

$$c_q = c \sqrt{\frac{k(t)}{k(t) - 2}}$$  \hspace{1cm} (177)

where $c$ is the usual flat-space velocity. We note that the fact that $c_q \to \infty$ as $k \to 2$, corresponding to the broadening out of the effective light-cone, is consistent with the suggestion made above that the concepts of space and time break down in this limit. Specifically, in a Robertson-Walker-Friedmann universe the horizon distance $d$ in co-moving coordinates over which an observer can look back is [80]

$$d = \int c_q(t) = \int dt \sqrt{\frac{k(t)}{k(t) - 2}}$$  \hspace{1cm} (178)

which is larger than the naive estimate $d = ct$. Indeed, the horizon distance could even become infinite if $k(t) \to 2$ in a suitable way as $t \to 0$, but this conjecture takes us beyond the dilute-gas approximation where we can compute reliably.

The time-dependence of string physics is also reflected in a computation of the string position-momentum uncertainty relation. Defined appropriately to incorporate curved gravitational backgrounds, this uncertainty can be expressed as [16]

$$(\Delta X \Delta P)_{\text{min}} \equiv \hbar_{\text{eff}}(t) = \hbar(1 + O(\frac{1}{k(t)}))$$  \hspace{1cm} (179)
where $\Delta A = ( <A^2> - <A>^2)^{\frac{1}{2}}$, $< \ldots >$ denotes a $\sigma$-model vacuum expectation value, and $\hbar$ is the critical-string Planck’s constant. The string uncertainty relation introduces a minimum length $\lambda_s$, that in our case also decreases with time [10]:

$$\lambda_s(t) \equiv \left( \frac{\hbar_{\text{eff}}(t)\alpha'(t)}{c_q(t)^2} \right)^{\frac{1}{2}} = \lambda_s^0(1 + O(\frac{1}{k(t)}))$$ (180)

We mention in passing that the effective string Regge slope is also time-dependent [10]:

$$\frac{\alpha'(t)}{\alpha'^0} = \frac{c_q(t)^2}{c^2}$$ (181)

which stems from the relation between $k$ and $\alpha'$ in this model. It is also worth mentioning that this cosmological model exhibits certain Jeans-like instabilities [31] leading to the exponential growth of low-energy string modes at finite $k$ [16], thereby providing a scenario for string-sustained inflation [32].

The non-critical string scenario for the expanding Universe described in the preceding paragraphs offers the prospect of solving the three basic problems of the standard-model cosmology in a manner reminiscent of conventional inflation [33]. The horizon problem could be solved by the enhanced look-back distance [178], and/or the breakdown of the normal concepts of space and time in a transition to a topological phase close to the infrared fixed point. The flatness problem could be solved by an epoch of exponential expansion, induced by a Jeans-like instability [10]. The entropy problem could be solved by the enhanced rate of entropy production (174) at early times. However, the crucial difference in our approach is that the fundamental scalar field, usually termed the inflaton, is replaced by a world-sheet field, the Liouville mode, in our approach. Fluctuations of this field create the renormalization group flow of the system that leads to the generation of propagating matter, in the way described above and in previous works [17, 4]. Of course, this mode is associated with the appearance of a target space scalar, the dilaton, but the latter is part of the metric background. This can be seen clearly in the two-dimensional Wess-Zumino string theory, which may be considered as a prototype for the description of a spherically-symmetric (s-wave) four-dimensional Universe [34]. In this model the dilaton belongs to the graviton level-one string multiplet, which is a non-propagating (discrete) string mode, and as such can only exist as a background, in contrast to a massless ‘tachyon’ mode, which propagates and scatters.

In our approach to the cosmological constant question, we start by considering the following one-loop results for the dilaton and graviton $\beta$-functions in bosonic $\sigma$-models [35, 29]:

$$\beta^\Phi \equiv \frac{d\Phi}{d\phi} = -\frac{2}{\alpha'} \frac{\delta_C}{3} + \nabla^2 \Phi - (\nabla \Phi)^2$$

$$\beta^G_{MN} \equiv \frac{dG_{MN}}{d\phi} = -R_{MN} - 2\nabla_M \nabla_N \Phi$$ (182)
where $\Phi$ is the dilaton field and $\phi$ denotes our covariant Liouville cutoff (c.f. the relative minus sign compared with the notation of ref. [85, 29] where the cutoff is defined with the dimensions of mass), and $\delta c = C - 26$, the 26 coming from the space-time reparametrization ghosts. If the central charge of the theory is not 26, as is the case of non-critical bosonic strings, then a cosmological constant term appears in the target space effective action. The form of this target-space action, whose variations yield the $\beta$-functions (182), reads:

$$I = \frac{2}{\alpha'} \int d^D y \sqrt{G} e^{-\Phi} \left\{ \frac{1}{3} \delta c - \alpha' \left( R + 4(\nabla \Phi)^2 + \ldots \right) \right\}$$  \hspace{1cm} (183)

where the $\ldots$ denote other fields in the theory that we shall not use explicitly.

We now notice that the effects of the tachyons in our two-dimensional target-space string model amount to a shift of the level parameter $k(\phi)$ with the renormalization group scale. This is the result of the combined effects of tachyon and instanton deformations, the latter representing higher-genus instabilities [50, 52, 16, 4]. The instantons alone, as irrelevant deformations, produce an initial instability by inducing an increase of the central charge, which then flows downhill towards 26 in the presence of relevant matter (tachyon) couplings. Hence there is a running central charge $c(\phi) > 26$, according to the C-theorem [31], that will, in general, imply a non-vanishing, time-varying (running), positive cosmological constant, $\Lambda(\phi)$, for the background of (176). Its precise form is determined by consistency with the equations (182).

For simplicity, we assume that the only effect of the dilaton is a constant contribution to the scale anomaly, which is certainly the case of interest. This allows one to decouple $\Phi$ in the field equations obtained from (183). Then the latter read

$$\frac{\delta I}{\delta \Phi} = \Lambda(\phi) - R$$
$$\frac{\delta I}{\delta G_{MN}} = -R_{MN} + \frac{1}{2} G_{MN} R$$  \hspace{1cm} (184)

In two dimensions the second equation is satisfied identically. Decoupling of the dilaton field also implies that the first equation yields

$$R = \Lambda(\phi)$$  \hspace{1cm} (185)

The metric background (176) has a maximal symmetry in its space part. To make the analysis more general, we extend the background to $d = 2 + \epsilon$ dimensions, keeping the maximal symmetry in the spatial part of the metric [1]. Relating time to the Liouville field as we have discussed earlier, we find the following solution for

\footnote{The $G_{00}$-component depends at most on the time $\phi$ and can be absorbed in a redefinition of the time variable. It will not be of interest to us here.}
\( \Lambda(\phi) \)

\[
\Lambda(t) = \frac{\Lambda(0)}{1 + t \frac{\Lambda(0)}{\alpha' - 1}} ; \quad t \equiv -\phi > 0
\]

which for positive \( \Lambda \) implies an asymptotically-free cosmological constant \( \beta \)-function, thereby leading to a vanishing cosmological constant at the ultraviolet fixed point on the world-sheet.

The rate of the decrease of \( \Lambda(\phi) \) is determined by its initial value at the infrared fixed point, where we conjectured that the theory makes a transition to a topological (twisted \( N = 2 \) supersymmetric) \( \sigma \)-model. It is of great interest to estimate this value in our two-dimensional model. This can be done by noticing that

\[
\Lambda(0) = \frac{2}{\alpha'(0)} \frac{c(0) - 26}{3}
\]

where \( c(0) - 26 = \frac{3k(0)}{k(0)-2} - 27 \approx \frac{3k(0)}{k(0)-2} \), given that \( k(\phi) \to 2 \) as \( \phi \to 0 \). Thus, taking into account (187) one observes that \( \Lambda(0) \) is determined by the critical string tension, as it should be, given that \( \alpha'_0 \) is the only scale in the problem (or equivalently the minimal string length); the result is

\[
\Lambda(0) = \frac{2}{\alpha'_0}
\]

The latter result implies a really fast decay of the cosmological constant in this model. Notice that the finite initial value of \( \Lambda(0) \) implies from (185) no curvature singularity in the Euclidean model at the origin of target space \( r = 0 \), as is indeed the case of the two-dimensional black-hole model of ref. [5], given that this point is a pure coordinate singularity. The above analysis for the cosmological constant, therefore, applies most likely to singularity-free inflationary universes [87]. It is understood that until the precise behaviour of the running couplings near the infrared fixed point is found, there will always be uncertainties in the above estimates. String perturbation theory is not applicable near the topological phase transition, and the infinities we get in the various running couplings constitute an indication of this. In the complete theory, these infinities should be absent.

10 Outlook

We have outlined in these lectures an approach to non-equilibrium quantum statistical mechanics, black holes, time, quantum mechanics and cosmology that is based

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\footnote{We remark that a similar equation has also been considered in ref. [86], but the flow of time in that reference coincides with the renormalization group flow. In such a case, one gets sensible results only for negative initial values of the cosmological constant, contrary to our case where we have a vanishing cosmological constant asymptotically, starting from positive initial values.}
on non-critical string theory, with time described by a Liouville field. Our basic aim has been to understand some of the qualitative features of quantum fluctuations in the structure of space-time, and their physical consequences, i.e., to understand foam. We have tried to short-circuit the general ignorance of string field theory by using a sort of mini-superspace approach, exploiting our knowledge of one particular class of such fluctuations, namely string black holes, and arguing that their consequences are likely to be quite general. Specifically, we expect that many other types of space-time fluctuation will tend to suppress quantum coherence in the manner discussed here for the black hole case.

We are aware that many details of these ideas remain to be worked out, and that many questions could be raised. However, we believe that our approach manifests such a high degree of internal consistency, and brings together so many apparently unconnected features of different areas of physics, that it is worthy of further constructive study.

There are several aspects of our work that we have not discussed here in any detail. These include the possible rôle of the type of decoherence discussed here in the transition between microscopic quantum physics and macroscopic classical physics. We have already given some discussion of this, and see a connection with ideas of Penrose [88] that we plan to discuss elsewhere. It also seems that our approach can put the problem of measurement in a new light. These are just some examples of basic physical problems where others have conjectured that quantum gravity may play a rôle. In string theory, we have for the first time a consistent quantum theory of gravity in which these questions can be addressed in a meaningful way. We have started to provide some answers. Some of them may be incomplete or wrong in detail, but we believe we have put our fingers on some important aspects of the truth. We urge the reader to examine our approach with a cool head.

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Figure Captions

Figure 1 - Contour of integration in the analytically-continued (regularized) version of $\Gamma(-s)$ for $s \in \mathbb{Z}^+$. This is known in the literature as the Saalschutz contour, and has been used in conventional quantum field theory to relate dimensional regularization to the Bogoliubov-Parasiuk-Hepp-Zimmermann renormalization method.

Figure 2 - Schematic representation of the evolution of the world-sheet area as the renormalization group scale moves along the contour of fig. 1.

Figure 3 - The $(\beta, \gamma)$ plane on a logarithmic scale for $\beta > 0$. We plot contours of the conventional $CP$-violating parameter $|\epsilon|$, evaluated from the $K_L \to 2\pi$ decay rate. The dashed-double-dotted band is that allowed at the one-standard-deviation level by the comparison between measurements of the $K_L \to 2\pi$ decay rate and the $K_L$ semileptonic decay asymmetry $\delta_L$. The dashed line delineates the boundary of the region allowed by the present experimental upper limit on $K_S \to 3\pi^0$ decays ($R_{2\pi}^L - R_{3\pi}^S$) and a solid line delineates the boundary of the region allowed by a recent preliminary measurement of the $K_S$ semileptonic decay asymmetry $\delta_S$. A wavy line bounds approximately the region of $|\beta|$ which may be prohibited by intermediate-time measurements of $K \to 2\pi$ decays.

Figure 4 - As in Fig. 3, on a logarithmic scale for $\beta < 0$.

Figure 5 - As in Fig. 3, on a linear scale for the neighborhood of $\beta = 0$. 