Reflection, refraction, and transmission of SH waves at a micropolar layer separating two elastic media

K B Ghazaryan, A A Papyan, and S K Ohanyan
Institute of Mechanics of the National Academy of Sciences of Armenia, Yerevan, Armenia

Abstract. In this paper, we study the reflection, refraction, and transmission of SH-waves in media separated by a layer whose behavior is described by a simplified model of Cosserat (micropolar) elasticity. The effect of a micropolar medium on the reflection and refraction of SH-waves is obtained. The amplitude ratios and energy fluxes of reflected and transmitted waves are graphically represented.

1. Introduction
A simplification of the Cosserat elasticity model was first mentioned by Ugodchikov A. G. in [1], and it was used by Manookian V. F. to study the existence of surface shear waves in a micropolar medium [2]. The term simplified was first used by Belubekyan M. V. to study SH-waves in periodic stratified media [3]. Then a simplified model of the Cosserat continuum was considered by Avetisyan A. S. in [4]. We use a simplified model of Cosserat elasticity to describe the boundary layer separating two homogeneous elastic half-spaces described by the classical model of elasticity.

2. Formulation of the problem
The geometry of the problem under consideration is depicted in figure 1. Homogeneous elastic half-spaces with rigidity \( \mu_s \), mass density \( \rho_s \) (where \( s = 1, 2 \) denotes the lower and upper half-spaces, respectively) are separated by a micropolar layer with rigidity \( \mu_0 \), mass density \( \rho_0 \), rotational inertia \( J \), and thickness \( h \). The structure is referred to a rectangular coordinate system with positive \( Y \)-axis directed vertically upwards and origin at the middle of the micropolar layer. The SH-waves from the lower half-space reflect and transmit through the layer into the upper half-space.

At the interface \( Y = -h/2 \), the incoming SH-wave propagates back partially reflected as a SH-wave in the same medium and partially transmitted as a SH-wave into the upper media. Let the incident wave and the transmitted wave make respective angles \( \varphi_1 \) and \( \varphi_2 \) with the normal to the boundary.

By assumptions of small deformations and neglect of the body forces, the equation of motion for SH-wave is given by

\[
\frac{\partial \sigma_{xz}^{(s)}}{\partial x} + \frac{\partial \sigma_{yz}^{(s)}}{\partial x} = \rho \frac{\partial^2 u^{(s)}}{\partial t^2}, \quad s = 1, 2,
\]

where \( s \) denotes lower (1) and upper (2) half-spaces and \( u \equiv u(x, y, t) \) is the displacement component along \( Z \)-direction for anti-plane state of the problem.
2.1. Relations for the upper and lower half–spaces

The non-zero stress-strain relations are \( \sigma_{xz}^{(s)} = 2 \mu \varepsilon_{xz}^{(s)} \) and \( \sigma_{yz}^{(s)} = 2 \mu \varepsilon_{yz}^{(s)} \), where \( \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \), \( i, j = x, y \) are the strain tensor components. For half-spaces, the stress–displacement relations are

\[
\sigma_{xz}^{(s)} = \mu \frac{\partial u_{1}^{(s)}}{\partial y}, \\
\sigma_{yz}^{(s)} = \mu \frac{\partial u_{1}^{(s)}}{\partial x}.
\]

(2)

2.2. Relations for the micropolar layer

According to the simplified Cosserat elasticity model, the constitutive equations (generalized Hooke’s law) for bound layers are \( \sigma_{ij} = 2 \mu \varepsilon_{ij} + \delta_{ij} \lambda \varepsilon_{kk} + J \frac{\partial^2 \omega_{ij}}{\partial t^2} \), where \( \omega_{ij} = \frac{1}{2} (u_{j,i} - u_{i,j}) \) are components of the rotation tensor, \( \lambda \) and \( \mu \) are Lame’s coefficients, and \( J \) is a dynamic characteristic of the medium (rotational inertia).

The nonzero stress components for the layers are

\[
\sigma_{xz}^{(s)} = \mu \frac{\partial u_{1}^{(s)}}{\partial y} + J \frac{\partial^3 u_{1}^{(s)}}{\partial t^2 \partial y}, \\
\sigma_{yz}^{(s)} = \mu \frac{\partial u_{1}^{(s)}}{\partial x} + J \frac{\partial^3 u_{1}^{(s)}}{\partial t^2 \partial x}.
\]

(3)
3. Solutions of the equations of motion

After substituting stress-strain relations (2) and (3) in (1) we obtain

\[ \mu \Delta u + J \frac{\partial^2 \Delta u}{\partial t^2} = \rho \frac{\partial^2 u}{\partial t^2}, \]  
\[ \mu \Delta u(s) = \rho \frac{\partial^2 u(s)}{\partial t^2}, \]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the two-dimensional Laplace delta operator.

Substituting the solution \( u(x, y, t) = f(y)e^{i(\omega t - kx)} \) in (4), we obtain

\[ f'''(y) + p^2 f(y) = 0, \]

where \( p^2 = (\rho \omega^2 - \beta k^2)/\beta \) and \( \beta = \mu_0 - J \omega^2 \). The solution of differential equation (6) is \( f(y) = A_1 e^{ipy} + A_2 e^{-ipy}. \) For (5), the characteristic equation is

\[ g''_s(y) + q^2_s g_s(y) = 0, \quad s = 1, 2, \]

where \( q^2_s = \rho_s \omega^2/\mu_s - k^2 \) and \( g_s(y) \) is the amplitude function. At the half-spaces, the displacement components of incident, reflected, and refracted waves are

\[ u^{(1)} = B_1 e^{ipy} + B_2 e^{-ipy}, \]
\[ u^{(2)} = C e^{ipy}, \]

and the displacement and stress components for the micropolar layer are

\[ u(x, y, t) = (A_1 e^{ipy} + A_2 e^{-ipy}) e^{i(\omega t - kx)}, \]
\[ \sigma_{yz} = i \beta (A_1 e^{ipy} - A_2 e^{-ipy}) e^{i(\omega t - kx)}, \]

where \( A_1 \) and \( A_2 \) are the amplitudes of the incident and reflected waves, respectively.

4. Matrix manipulations

In matrix form, expressions (9) become

\[ \begin{pmatrix} u^{(0)} \\ \sigma^{(0)}_{yz} \end{pmatrix} = \begin{pmatrix} e^{ipy} & e^{-ipy} \\ ip\beta e^{ipy} & -ip\beta e^{-ipy} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i(\omega t - kx)}. \]

We rewrite (omitting \( e^{i(\omega t - kx)} \)) this equation as

\[ U^{(0)}(y) = M(y) A, \]

where

\[ U^{(0)}(y) = \begin{pmatrix} u^{(0)} \\ \sigma^{(0)}_{yz} \end{pmatrix}, \quad M(y) = \begin{pmatrix} e^{ipy} & e^{-ipy} \\ ip\beta e^{ipy} & -ip\beta e^{-ipy} \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \]

For two arbitrary points \( y_0 \) and \( y_1 \) within the layer, we have

\[ U^{(0)}(y_0) = M(y_0) A, \quad U^{(0)}(y_1) = M(y_1) A, \]

and subsequently, the stress strain state for these points can be represented by the relation

\[ U^{(0)}(y_0) = N(y_0, y_1) U^{(0)}(y_1), \]

where

\[ N(y_0, y_1) = M(y_0) M^{-1}(y_1) = \begin{pmatrix} \cos[p(y_0 - y_1)] & \sin[p(y_0 - y_1)]/p\beta \\ -p\beta \sin[p(y_0 - y_1)] & \cos[p(y_0 - y_1)] \end{pmatrix} \]

is the transfer matrix.

\[ \]
5. Boundary conditions
According to the boundary conditions $U^{(1)}(-h/2) = N(-h/2; h/2)U^{(2)}(h/2)$ at $(-h/2, h/2)$ of ideally contact, we obtain the following boundary relations for the displacement and stress components of the lower and upper half-spaces:

\[
\begin{align*}
u^{(1)}(-h/2) &= \cos(ph)u^{(2)}(h/2) - \frac{\sin(ph)\tau^{(2)}(h/2)}{\beta p}, \\
\tau^{(1)}(-h/2) &= \beta p\sin(ph)u^{(2)}(h/2) - \cos(ph)\tau^{(2)}(h/2).
\end{align*}
\]

After substituting displacements (8) in (10), we obtain

\[
B_1 + B_2e^{ihq_1} = C\frac{e^{ih(q_1+q_2)/2}}{p\beta}[p\beta \cos(ph) - i\mu_2q_2 \sin(ph)],
\]

\[
(B_1 + B_2)\sin(hq_1/2) + i(B_1 - B_2)\cos(hq_1/2) = C\frac{e^{ihq_2/2}}{\mu_1q_1}[i\mu_2q_2 \cos(ph) + p\beta \sin(ph)].
\]

6. Reflection and refraction coefficients
The reflection and refraction amplitude ratios from (11) are respectively $R = B_2/B_1$ and $T = C/B_1$:

\[
\begin{align*}
R &= \frac{e^{-ihq_1}[(\mu_1\mu_2q_1q_2 - p^2\beta^2)\cos(ph) - i\mu_2q_2 \sin(ph)]}{(\mu_1\mu_2q_1q_2 + p^2\beta^2)\cos(ph) + i\mu_2q_2 \sin(ph)}, \\
T &= \frac{2ie^{-ih(q_1+q_2)/2}\mu_1q_1p\beta}{(\mu_1\mu_2q_1q_2 + p^2\beta^2)\cos(ph) + i\mu_2q_2 \sin(ph)}.
\end{align*}
\]

where $q_s = k_s\sqrt{c_s^2/c_s^2 - 1} = k_s \tan(\varphi_s)$. The dependence of the angles of reflection and refraction is given by Snell’s law:

\[
\begin{align*}
c_1\sin \varphi_1 &= c_2\sin \varphi_2, \\
c_1\cos \varphi_1 &= c_2\cos \varphi_2.
\end{align*}
\]

7. Frequency ranges
From the existence condition $p^2 > 0$ for harmonic waves, we obtain $\mu_0k^2/(\rho_0 + Jk^2) < \omega^2 < \mu_0/J$; in this case, the solution of (6) is real and the condition $\mu_0k^2/(\rho_0 + Jk^2) < \mu_0/J$ is always true. For the elastic half-spaces (7), from the condition $q_s^2 > 0$, we obtain $\omega^2 > \mu_s k^2/\rho_s$. So we have the frequency range $\sqrt{\mu_0k^2/(\rho_0 + Jk^2)}$ < \omega < $\mu_0/J$, where $p^2 > 0$, and the solution is complex.

In the case where $\omega < \sqrt{\mu_0k^2/(\rho_0 + Jk^2)}$ or $\omega > \mu_0/J$, $p^2 < 0$, the solution is complex and can be represented as $iq_0$ for $p_0 > 0$.

The SH-wave propagates throughout the half-spaces and $q^2 > 0$, when

\[
\begin{align*}
a)\quad &\alpha_s < \alpha_l, \quad \begin{cases} p^2 > 0 & \text{if } \omega \in (\alpha_l, \alpha_u), \\
p^2 < 0 & \text{if } \omega \in (\alpha_s, \alpha_l) \cup (\alpha_u, +\infty), \end{cases} \\
b)\quad &\alpha_l < \alpha_s < \alpha_u, \quad \begin{cases} p^2 > 0 & \text{if } \omega \in (\alpha_s, \alpha_l), \\
p^2 < 0 & \text{if } \omega \in (\alpha_u, +\infty), \end{cases} \\
c)\quad &\alpha_u < \alpha_s, \quad p^2 < 0 \text{ if } \omega \in (\alpha_s, +\infty),
\end{align*}
\]

where $\alpha_s = \sqrt{\mu_s k^2/\rho_s}$ is the frequency limit for the half-spaces, $\alpha_l = \sqrt{\mu_0k^2/(\rho_0 + Jk^2)}$ and $\alpha_u = \sqrt{\mu_0/J}$ are the respective lower and upper limits for the frequency range of the micropolar layer. These cases are graphically presented in figure 2.
8. Particular case: half-spaces with identical material properties
There are two possible cases for reflection and refraction amplitude ratios (12): $p^2 > 0$ and $p^2 < 0$.

In the case $p^2 > 0$, $\omega \in (\alpha_l, \alpha_u)$ for $\alpha_s < \alpha_l$, and $\omega \in (\alpha_s, \alpha_u)$ for $\alpha_l < \alpha_s < \alpha_u$:

$$R = \frac{e^{-ihq}(q^2 \mu^2 - p^2 \beta^2) \sin(hp)}{2ipq \beta \mu \cos(hp) + (p^2 \beta^2 + q^2 \mu^2) \sin(hp)},$$
$$T = \frac{2ie^{-ihq}pq \beta \mu}{2ipq \beta \mu \cos(hp) + (p^2 \beta^2 + q^2 \mu^2) \sin(hp)}.$$ (13)

In the case $p^2 < 0$, $p = ip_0$, $p_0 > 0$, $\omega \in (\alpha_s, \alpha_l) \cup (\alpha_u, +\infty)$ for $\alpha_s < \alpha_l \omega \in (\alpha_u, +\infty)$ for $\alpha_l < \alpha_s < \alpha_u$, and $\omega \in (\alpha_s, +\infty)$ for $\alpha_u < \alpha_s$:

$$R = \frac{ie^{-ihq} \sinh(hp_0)(q^2 \mu^2 + \beta^2 p_0^2)}{-2q \beta \mu p_0 \cosh(hp_0) + i \sinh(hp_0)(q^2 \mu^2 - \beta^2 p_0^2)},$$
$$T = \frac{-2e^{-ihq}q \beta \mu p_0}{-2q \beta \mu p_0 \cosh(hp_0) + i \sinh(hp_0)(q^2 \mu^2 - \beta^2 p_0^2)}.$$ (14)

The energy flux conservation is then expressed via the reflection and transmission amplitudes by the algebraic identity $|R|^2 + q_2 \mu_2/(q_1 \mu_1)|T|^2 = 1$, in the case where the half-spaces are identical

$$|R|^2 + |T|^2 = 1,$$ (15)

for $p^2 > 0$,

$$|R|^2 = 1 - \frac{4p^2 q^2 \beta^2 \mu^2}{4p^2 q^2 \beta^2 \mu^2 \cos^2(hp) + (p^2 \beta^2 + q^2 \mu^2)^2 \sin^2(hp)},$$
$$|T|^2 = \frac{4p^2 q^2 \beta^2 \mu^2}{4p^2 q^2 \beta^2 \mu^2 \cos^2(hp) + (p^2 \beta^2 + q^2 \mu^2)^2 \sin^2(hp)}.$$ (16)
and for $p^2 < 0$,

$$|R|^2 = 1 - \frac{4q^2 \beta^2 \mu^2 p_0^2}{q^4 \mu^4 \sinh^2(h p_0) + q^2 \beta^2 \mu^2 [3 + \cosh^2(2h p_0)] p_0^2 + \beta^4 \sinh^2(h p_0) p_0^2},$$

$$|T|^2 = \frac{4q^2 \beta^2 \mu^2 p_0^2}{q^4 \mu^4 \sinh^2(h p_0) + q^2 \beta^2 \mu^2 [3 + \cosh^2(2h p_0)] p_0^2 + \beta^4 \sinh^2(h p_0) p_0^2},$$

where $q = k \tan \varphi$; we introduce the following notation to obtain the dimensionless parameters:

$p = k \sqrt{\eta/(1 - \gamma \eta) - 1}, \beta = \rho_0 c^2 (1/\eta - \gamma)$.

So for $p^2 > 0$,

$$|R|^2 = 1 - \frac{-4\zeta^2 a \tan^2 \varphi}{-4\zeta^2 a \cos^2 \theta \tan^2 \varphi + \sin^2 \theta (a - \zeta^2 \tan^2 \varphi)^2},$$

$$|T|^2 = \frac{-4\zeta^2 a \tan^2 \varphi}{-4\zeta^2 a \cos^2 \theta \tan^2 \varphi + \sin^2 \theta (a - \zeta^2 \tan^2 \varphi)^2},$$

and for $p^2 < 0$,

$$|R|^2 = 1 - \frac{-4a^4 \zeta^2 \tan^2 \varphi}{a^4 \sinh^2 \theta - \zeta^2 a^2 (3 + \cosh \theta) \tan^2 \varphi + \zeta^4 \sinh^2 \theta \tan^4 \varphi},$$

$$|T|^2 = \frac{-4a^4 \zeta^2 \tan^2 \varphi}{a^4 \sinh^2 \theta - \zeta^2 a^2 (3 + \cosh \theta) \tan^2 \varphi + \zeta^4 \sinh^2 \theta \tan^4 \varphi},$$

where $a = \eta^2(\gamma \eta - 1)(\gamma \eta + \eta - 1), \theta = \alpha \sqrt{\eta/(1 - \gamma \eta) - 1}, \alpha = h k, \eta = c/c_0, \gamma = J k^2/\rho_0$, and $\zeta = \mu/\mu_0$; here $\eta = c/c_0$ is dimensionless phase velocity.

9. Numerical calculations

The comparison of the micropolar elastic layer case, which is described by the simplified Cosserat model of elasticity, with classical case is presented in figure 3. In this case, the shear moduli ratio $\zeta$ and the dimensionless phase velocity $\eta$ are equal to 0.5, which means that the half-spaces are softer than the layer. Figure 3 shows that taking into account the effect of micro-rotations according to simplified Cosserat model of elasticity for a thin layer (there and below, thin means that the thickness of the layer is less than the wavelength) has a small impact on the dependence of the energy fluxes of reflected and transmitted waves on the angle of incidence. For the layer thickness equal to the wavelength of the incident wave, the influence of micro-rotations is sufficient, in the classical case, the wave is completely reflected for any angle of incidence, but in the micropolar layer case, the wave incident perpendicularly to the layer boundary is completely transmitted, and then the energy flux of the transmitted wave decreases until, at 30 degrees, the waves became almost completely reflected.

The dependence of energy fluxes and amplitude ratios on the angle of incidence are presented in figures 4–9 for $p^2 > 0$, and in figures 10–15 for $p^2 < 0$.

Figures 4–6 graphically present the effect of micropolar layer dependent on its thickness in the case where the shear moduli ratio $\zeta$ and dimensionless phase velocity $\eta$ are equal to 1. A thin layer does not influence the shear wave reflection and transmission, and waves are completely transmitted. But as the layer thickness increases to the wavelength dimensions (see figure 7), a significant impact is obtained, so that the changes in the shear moduli ratio (see figures 8 and 9) result in the complete reflection of waves.

The influence of increasing thickness of the micropolar layer described above is more visible in the complex case $p^2 < 0$ (see figures 10–12). Figures 13–15 show that the changes in rigidity ratio $\zeta$ make a smaller impact in the case $p^2 < 0$.

Graphs for the SH-waves represented by trigonometric functions are shown in figures 13–15.
Figure 3. Simplified Cosserat elastic layer case ((a) for $\alpha = 0.1$, $\eta = 0.5$, $\gamma = 1$, $\zeta = 0.5$ and (c) for $\alpha = 5$, $\eta = 0.5$, $\gamma = 1$, $\zeta = 0.5$) and the classical elastic layer case ((b) for $\alpha = 0.1$, $\eta = 0.5$, $\gamma = 0$, $\zeta = 0.5$ and (d) for $\alpha = 5$, $\eta = 0.5$, $\gamma = 0$, $\zeta = 0.5$).

Figure 4. Values of dimensionless parameters: $\alpha = 0.01$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 1$. The layer is thin enough to make no impact on the SH-wave transmission.

Conclusions
This paper presents an analytical solution to study of reflection and transmission of shear SH elastic waves in a multilayered structure composed of elastic micropolar layer sandwiched between two elastic materials on either side. The micropolar layer material obeys to the equation of the simplified model of Cosserat (micropolar) elasticity theory. The transfer matrix method
Figure 5. Values of dimensionless parameters: $\alpha = 0.1$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 1$. Here we can observe the appearance of a cross point at 85 degrees when only half the energy is transmitted, which is a result of increasing thickness of the layer.

Figure 6. Values of dimensionless parameters: $\alpha = 1$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 1$. In this figure, we can see a determined point at 60 degrees when half the energy of SH-waves is reflected and the other half is transmitted. The cusp at 15 degrees indicates the angle of incidence at which the SH-waves are completely transmitted.

Figure 7. Values of dimensionless parameters: $\alpha = 5$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 1$. The change in the thickness (equal to the wave number) results in two more visible points (7 and 37 degrees) at which the graphs for the transmitted and reflected waves intersect, so only half the energy is transmitted.
Figure 8. Values of dimensionless parameters: $\alpha = 5$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 0.1$. The changes in the rigidity ratio leads to changes in the position (degree) of cross points (and the cusp point). Subsequent decrease of rigidity ratio results to complete reflection.

Figure 9. Values of dimensionless parameters: $\alpha = 5$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 10$. The result of increase in the rigidity ratio is similar to that of decrease, so it leads to complete reflection.

Figure 10. Values of dimensionless parameters: $\alpha = 0.01$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 1$. As in figure 4, the micropolar layer is thin enough to make no changes in the SH-waves transmission.
Figure 11. Values of dimensionless parameters: $\alpha = 0.1, \eta = 1, \gamma = 0.1, \zeta = 1$. Here is no difference from figure 5, the cross point is at 85 degrees.

Figure 12. Values of dimensionless parameters: $\alpha = 1, \eta = 1, \gamma = 0.1, \zeta = 1$. The main difference from figure 6 is the absence of a cusp point (degree) at which the SH-waves are completely transmitted.

Figure 13. Values of dimensionless parameters: $\alpha = 5, \eta = 1, \gamma = 0.1, \zeta = 1$. This figure illustrates the difference from figure 7, there are no cross points for graphs and there is no point (degree) at which the SH-waves can be completely transmitted.
Values of dimensionless parameters: $\alpha = 5$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 0.1$. As in figure 8, the changes in the rigidity ratio lead to changes in the position (degree) of cusp point of the graphs. A subsequent decrease in the rigidity ratio results in the complete reflection.

Values of dimensionless parameters: $\alpha = 5$, $\eta = 1$, $\gamma = 0.1$, $\zeta = 10$. As in figure 9, the result of increase in the rigidity ratio is similar to that of decrease, so it leads to complete reflection.

is used to find the amplitudes and energy fluxes of reflected and transmitted waves in different frequency bands of incident wave. On the basis of analytical solutions it is concluded that micropolar properties of the layer produce a strong effect on the energy fluxes of reflected and transmitted waves when the layer thickness is compared with wavelength of the incident wave.

References
[1] Ugodchikov A G 1995 Torque dynamics of linear elastic bodies *Dokl. Akad. Nauk* **340** (1) 50–8
[2] Manukyan V F 1997 On the existence of surface shear waves in a micropolar medium *Izv. Nats. Akad. Nauk Armenii. Mekh.* **50** (2) 75–9
[3] Ambartsumian S A, Belubekyan M V, and Ghazaryan K B 2014 Shear elastic waves in a periodic medium with the Cosserat simplified model properties *Izv. Nats. Akad. Nauk Armenii. Mekh.* **67** (4) 3–9
[4] Ambartsunyan S A, Avetisyan A S, and Belubekyan M V 2017 Propagation of elastic waves in a plane waveguide layer on the basis of a simplified model of the Cosserat continuum *Izv. Nats. Akad. Nauk Armenii. Mekh.* **70** (2) 15–27