S\textsuperscript{1}-Bundles and Gerbes over Differentiable Stacks

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Abstract. We study S\textsuperscript{1}-bundles and S\textsuperscript{1}-gerbes over differentiable stacks in terms of Lie groupoids, and construct Chern classes and Dixmier-Douady classes in terms of analogues of connections and curvature. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

S\textsuperscript{1}-Fibrés et Gerbes sur des Champs Différentiables

Résumé. On étudie les S\textsuperscript{1}-fibrés et les S\textsuperscript{1}-gerbes sur des champs différentiables en termes de groupoïdes de Lie et construit les classes de Chern et Dixmier-Douady en termes d’analogues aux connexions et courbure. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit \mathfrak{X} un champ différentiable et \mathcal{P} \to \mathfrak{X} un S\textsuperscript{1}-fibré sur \mathfrak{X}. Soit \Gamma \rightrightarrows M une présentation par un groupoïde de Lie pour \mathfrak{X}. Alors \mathcal{P} induit un S\textsuperscript{1}-fibré P sur M sur lequel agit \Gamma \rightrightarrows M. On réalise la classe de Chern de \mathcal{P} en termes de données de type connexion sur P et prouve l’existence des préquantifications. Plus précisément, Soit \theta \in \Omega\textsuperscript{1}(P) une pseudo-connexion, et \omega + \Omega \in Z\textsuperscript{2}_{DR}(\Gamma\textsuperscript{•}) sa pseudo-courbure.

Theorem 0.1. – La classe [\omega + \Omega] \in H\textsuperscript{2}_{DR}(\Gamma\textsuperscript{•}) est indépendante du choix de la pseudo-connexion \theta et correspond à la classe de Chern de P. Réciproquement, soit \omega + \Omega \in C\textsuperscript{2}_{DR}(\Gamma\textsuperscript{•}) un 2-cocycle entier. Alors il existe un S\textsuperscript{1}-fibré \mathcal{P} sur \Gamma \rightrightarrows M et une pseudo-connexion \theta \in \Omega\textsuperscript{1}(P) ayant \omega + \Omega pour pseudo-courbure. De plus, l’ensemble des classes d’isomorphisme de tous ces (\mathcal{P}, \theta) est un H\textsuperscript{1}(\Gamma\textsuperscript{•}, \mathbb{R}/\mathbb{Z})-ensemble.

Si \mathcal{G} est une S\textsuperscript{1}-gerbe sur \mathfrak{X}, et \mathcal{R} \rightrightarrows M une présentation du champ différentiable \mathcal{G} et soit \Gamma \rightrightarrows M le groupoïde de Lie défini par la présentation induite \mathcal{R} \to \mathfrak{X} de \mathfrak{X}. Alors \mathcal{R} est une S\textsuperscript{1}-extension centrale du groupoïde de Lie \Gamma \rightrightarrows M. Ainsi les S\textsuperscript{1}-extensions centrales de \Gamma \rightrightarrows M sont exactement les S\textsuperscript{1}-gerbes sur \mathfrak{X}, données d’une trivialisation sur M. A nouveau, on peut réaliser...
les classes caractéristiques de la gerbe (que nous appelons classes de Dixmier-Douady) en termes de données de type connexion et prouver l’existence de préquantifications. Plus précisément, soit \( \theta + B \in C^2_{DR}(R) \) une pseudo-connexion sur \( R \), et \( \theta + \omega + \Omega \in Z^3_{DR}(\Gamma) \) sa pseudo-courbure.

**Theorem 0.2.** — La classe \( [\eta + \omega + \Omega] \in H^3_{DR}(\Gamma) \) est indépendante du choix de la pseudo-connexion \( \theta + B \) sur \( R \), et correspond à la classe de Dixmier-Douady de \( R \). Réciproquement, pour tout 3-cocycle \( \eta + \omega + \Omega \in Z^3_{DR}(\Gamma) \) tel que \( [\eta + \omega + \Omega] \) est une classe entière et \( \Omega \) est exact, il existe une extension centrale \( \Gamma \Rightarrow M \) du groupoïde \( \Gamma \Rightarrow M \), et une pseudo-connexion \( \theta + B \in C^2_{DR}(R) \) sur \( R \) telle que \( \eta + \omega + \Omega \) soit la pseudo-courbure. Les paires \( (R, \theta, B) \) forment, à un isomorphisme près, un ensemble simplement transitif sous le groupe des extensions centrales plates.

Dans le cas s-connexe, on obtient un construction explicite de l’extension centrale avec pseudo-connexion. Cela donne également un critère pour qu’une classe dans \( H^2_{DR}(\Gamma) \) soit entière. Ce théorème généralise le résultat de [3].

**Theorem 0.3.** — Soit \( \Gamma \Rightarrow M \) un groupoïde de Lie s-connexe, et \( \eta + \omega \in C^2_{DR}(\Gamma) \) un 3-cocycle, où \( \eta \in \Omega^1(\Gamma_2) \) et \( \omega \in \Omega^2(\Gamma) \). Supposons que \( \omega \) représente une classe de cohomologie entière dans \( H^2_{DR}(\Gamma) \), de telle sorte qu’il existe un \( S^1 \)-fibré \( \pi : R \rightarrow \Gamma \) avec une connexion \( \theta \in \Omega^1(R) \), dont la courbure est \( \omega \). Supposons que \( \epsilon \in R \), doté d’une connexion plate \( \epsilon' \) et \( \epsilon \) soit sans holonomie. (Ici \( \epsilon : M \rightarrow \Gamma \) et \( \epsilon_2 : M \rightarrow \Gamma_2 \) sont les morphisme d’identité respectifs.) Alors \( \Gamma \Rightarrow M \) admet de façon naturelle une structure de groupoïde, telle que \( R \) soit une extension \( S^1 \)-cente de \( \Gamma \Rightarrow M \) et \( \eta + \omega \) la pseudo-courbure de \( \theta \).

Puisque les extensions centrales de groupoïdes décrivent les gerbes sur \( X \) avec des trivialisations données sur \( M \), on peut seulement décrire les gerbes qui sont effectivement triviales sur \( M \) en terme d’extensions centrales de groupoïdes de \( \Gamma \Rightarrow M \). Pour décrire toutes les gerbes sur \( X \), on doit passer en général à un groupoïde de Lie Morita-équivalent \( \Gamma' \Rightarrow M' \).

1. Introduction

We study \( S^1 \)-bundles and \( S^1 \)-gerbes over differentiable stacks in terms of Lie groupoids.

Let \( \mathcal{X} \) be a differentiable stack and \( \mathcal{P} \rightarrow \mathcal{X} \) an \( S^1 \)-bundle over \( \mathcal{X} \). Let \( \Gamma \Rightarrow M \) be a Lie groupoid presentation for \( \mathcal{X} \), i.e., \( \mathcal{X} \) is (isomorphic to) the stack of \( \Gamma \Rightarrow M \)-torsors. Then \( \mathcal{P} \) gives rise to an \( S^1 \)-bundle \( P \) over \( M \) on which \( \Gamma \Rightarrow M \) acts. We realize the Chern class of \( \mathcal{P} \) in terms of connection-like data on \( P \) and prove that prequantizations exist.

Note that \( H^2(\Gamma, \Omega^3) \) contains the obstructions to the existence of \( \mathcal{P} \) for an arbitrary integer cohomology class and \( H^1(\Gamma, \Omega^1) \) contains the obstructions to the existence of a connection on \( \mathcal{P} \) if \( \mathcal{P} \) exists. The possibility of non-vanishing of these cohomology groups distinguishes our case from the standard case of manifolds.

If \( \mathcal{G} \) is an \( S^1 \)-gerbe over \( \mathcal{X} \), and \( \Gamma \Rightarrow M \) a presentation for \( \mathcal{X} \) as above, then \( \mathcal{G} \) gives rise to a gerbe over \( M \). So we do not immediately get a description of \( \mathcal{G} \) in terms of groupoids. Instead, we can start with a presentation \( R \Rightarrow M \) of the differentiable stack \( \mathcal{G} \) and let \( \Gamma \Rightarrow M \) be the Lie groupoid defined by the induced presentation \( M \rightarrow \mathcal{X} \) of \( \mathcal{X} \), in other words, \( \Gamma = M \times_X M \). In this situation, we get a morphism of groupoids from \( R \Rightarrow M \) to \( \Gamma \Rightarrow M \), and, moreover, \( R \Rightarrow \Gamma \) is an \( S^1 \)-principal bundle. In fact, \( R \) is an \( S^1 \)-central extension of the Lie groupoid \( \Gamma \Rightarrow M \).

Thus the \( S^1 \)-central extensions of \( \Gamma \Rightarrow M \) are exactly the \( S^1 \)-gerbes over \( \mathcal{X} \), endowed with a trivialization over \( M \). Therefore, the central extension case is not entirely analogous to the bundle case.

Again, we can realize the characteristic class of the gerbe (which we call the Dixmier-Douady class) in terms of connection-like data and prove that prequantizations exist. Note that there are again obstructions to the existence of honest connective structures and curvings. More precisely,
2. Homology and cohomology

Let $\Gamma \rightrightarrows M$ be a Lie groupoid. Define $\Gamma_p = \Gamma \times_M \ldots \times_M \Gamma$, i.e., $\Gamma_p$ is the manifold of composable sequences of $p$ arrows in the groupoid $\Gamma \rightrightarrows M$. We have $p+1$ canonical maps $\Gamma_p \to \Gamma_{p-1}$ (each leaving out one of the $p+1$ objects involved a sequence of composable arrows), giving rise to a diagram

\[
\ldots \Gamma_2 \longrightarrow \Gamma_1 \longrightarrow \Gamma_0.
\]

In fact, $\Gamma_*$ is a simplicial manifold.

The piecewise differentiable chain complex of $\Gamma_*$ is the total complex associated to the double complex $C_*(\Gamma_*)$. Here $C_k(\Gamma_p)$ is the free abelian group generated by the piecewise differentiable maps $\Delta_k \to \Gamma_p$. Its homology groups $H_k(\Gamma_*, \mathbb{Z}) = H_k(C_*(\Gamma_*))$ are called the homology groups of $\Gamma \rightrightarrows M$.

We denote the dual of the double complex $C_*(\Gamma_*)$ by $C^*(\Gamma_*)$. Its total cohomology groups $H^k(\Gamma_*, \mathbb{Z}) = H^k(C^*(\Gamma_*))$ are called the integer cohomology groups of $\Gamma \rightrightarrows M$. In the case that $\Gamma \rightrightarrows M$ is a transformation groupoid $G \times M \rightrightarrows M$, these are the $G$-equivariant cohomology groups.

Finally, we introduce the double complex $\Omega^*(\Gamma_*)$. Its boundary maps are $d : \Omega^k(\Gamma_p) \to \Omega^{k+1}(\Gamma_p)$, the usual exterior derivative of differentiable forms and $\partial : \Omega^k(\Gamma_p) \to \Omega^k(\Gamma_{p+1})$, the alternating sum of the pull back maps of $[\Omega]$. We denote the total differential by $\delta = (-1)^d + \partial$. The total cohomology groups of $\Omega^*(\Gamma_*)$, $H^k_{DR}(\Gamma_*) = H^k(\Omega^*(\Gamma_*))$ are called the De Rham cohomology groups of $\Gamma \rightrightarrows M$.

Recall that a Morita morphism from the Lie groupoid $\Gamma' \rightrightarrows M'$ to $\Gamma \rightrightarrows M$ is a morphism of Lie groupoids satisfying the two conditions

1. the diagram
   
   \[
   \begin{array}{ccc}
   \Gamma' & \rightarrow & M' \times M' \\
   \downarrow & & \downarrow \\
   \Gamma & \rightarrow & M \times M
   \end{array}
   \]

   is cartesian, i.e., a pullback diagram,

2. $M' \rightarrow M$ is a surjective submersion.

Two Lie groupoids are Morita equivalent, if and only if there exist a third Lie groupoid together with a Morita morphism to each of them.

**Proposition 2.1.** Let $f : [\Gamma' \rightrightarrows M'] \rightarrow [\Gamma \rightrightarrows M]$ be a Morita morphism of Lie groupoids. Then we get induced isomorphisms $f^* : H^k(\Gamma, \mathbb{Z}) \rightarrow H^k(\Gamma', \mathbb{Z})$ and $f^* : H^k_{DR}(\Gamma) \rightarrow H^k_{DR}(\Gamma')$.

In particular, if $\Gamma \rightrightarrows M$ is a banal groupoid, i.e., there exists a surjective submersion $\pi : M \rightarrow X$, for some manifold $X$, and $\Gamma \rightrightarrows M$ is isomorphic to $M \times_X M \rightrightarrows M$, then we have canonical isomorphisms

\[
f^* : H^k(X, \mathbb{Z}) \rightarrow H^k(\Gamma, \mathbb{Z})
\]
Running authors

and

$$f^* : H^k_{DR}(X) \simto H^k_{DR}(\Gamma_\ast).$$

(2)

The canonical homomorphism $$\Omega^\ast(\Gamma_\ast) \to C^\ast(\Gamma_\ast) \otimes \mathbb{R}$$ induces isomorphisms

$$H^k_{DR}(\Gamma_\ast) \simto H^k(\Gamma_\ast, \mathbb{R})$$

(3)

and pairings

$$Z_k(\Gamma_\ast, \mathbb{Z}) \otimes Z^k_{DR}(\Gamma_\ast) \rightarrow \mathbb{R}; \; \gamma \otimes \omega \mapsto \int_\gamma \omega.$$

We call a De Rham cocycle an integer cocycle, if it maps under (3) into the image of the canonical map $$H^k(\Gamma_\ast, \mathbb{Z}) \to H^k(\Gamma_\ast, \mathbb{R}).$$

PROPOSITION 2.2. – Let $$\omega \in Z^k_{DR}(\Gamma_\ast)$$ be a De Rham cocycle. The following are equivalent:

1. $$\omega$$ is an integer cocycle,
2. $$\int_\gamma \omega \in \mathbb{Z},$$ for all $$\gamma \in Z_k(\Gamma_\ast, \mathbb{Z}).$$
3. for every closed surface $$S$$ and every $$\Gamma \rightrightarrows M$$-torsor $$T$$ over $$S,$$ giving rise to a morphism of groupoids $$g$$ from $$T \times_S T \rightarrow T \rightarrow \Gamma \rightrightarrows M,$$ we have $$\int_S g^* \omega \in \mathbb{Z}.$$ Here we use the isomorphism [4], to make sense of the integral.

(Recall that a $$\Gamma \rightrightarrows M$$-torsor over $$S$$ is a surjective submersion $$T \rightarrow S,$$ together with an action of $$\Gamma \rightrightarrows M$$ on $$T,$$ such that $$S$$ is the quotient of $$T$$ by this action.)

For any abelian sheaf $$F$$ on the category of differentiable manifolds, we have the cohomology groups $$H^k(\Gamma_\ast, F).$$ One way to define them is by choosing for every $$p$$ an injective resolution $$F_p \rightarrow I_p^\ast$$ of sheaves on $$\Gamma_p,$$ where $$F_p$$ is the sheaf induced by $$F$$ on $$\Gamma_p;$$ then choosing homomorphisms $$f^{-1}I_{p-1}^\ast \rightarrow I_p^\ast$$ for every map $$f : \Gamma_p \rightarrow \Gamma_{p-1}.$$ This gives rise to a double complex $$I^\ast(\Gamma_\ast),$$ whose total cohomology groups are the $$H^k(\Gamma_\ast, F).$$

Examples of abelian sheaves on the category of manifolds are: $$\mathbb{Z},$$ $$\mathbb{R},$$ $$\mathbb{R}/\mathbb{Z},$$ $$\Omega^k$$ and $$S^1.$$ The first three are sheaves of locally constant functions, $$S^1$$ is the sheaf of differentiable $$S^1$$-valued functions. With respect to the first three, the notation $$H^k(\Gamma, F)$$ does not conflict with the notation introduced before.

It is well-known that $$H^1(\Gamma_\ast, S^1)$$ classifies principal $$S^1$$-bundles over $$\Gamma_\ast,$$ whereas $$H^2(\Gamma_\ast, S^1)$$ classifies $$S^1$$-gerbes over $$\Gamma_\ast.$$

3. $$S^1$$-bundles

DEFINITION 3.1. – Let $$\Gamma \rightrightarrows M$$ be a Lie groupoid. A (right) $$S^1$$-bundle over $$\Gamma \rightrightarrows M$$ is a (right) $$S^1$$-bundle $$P$$ over $$M,$$ together with a (left) action of $$\Gamma$$ on $$P,$$ which respects the $$S^1$$-action, i.e. we have $$(\gamma \cdot x) \cdot t = \gamma \cdot (x \cdot t),$$ for all $$t \in S^1$$ and all compatible pairs $$(\gamma, x) \in \Gamma \times_{I, M} P.$$

Let $$Q = \Gamma \times_{I, M} P$$ be the manifold of compatible pairs. Action and projection form a diagram $$Q \rightrightarrows P$$ and it is easy to check that $$Q \rightrightarrows P$$ is in a natural way a groupoid (called the transformation groupoid of the $$\Gamma$$-action). Moreover, there is a natural morphism of groupoids $$\pi$$ from $$Q \rightrightarrows P$$ to $$\Gamma \rightrightarrows M.$$ Of course, $$Q$$ is an $$S^1$$-bundle over $$\Gamma.$$

More is true: the $$S^1$$-bundle $$P$$ over $$\Gamma \rightrightarrows M$$ gives rise to an $$S^1$$-bundle on the simplicial manifold $$\Gamma_\ast.$$ As such it has an associated class in $$H^1(\Gamma_\ast, S^1)$$ and, in fact, $$S^1$$-bundles over $$\Gamma \rightrightarrows M$$ are classified by $$H^1(\Gamma_\ast, S^1).$$ The exponential sequence $$\mathbb{Z} \rightarrow \Omega^0 \rightarrow S^1$$ induces a boundary map $$H^1(\Gamma_\ast, S^1) \rightarrow H^2(\Gamma_\ast, \mathbb{Z});$$ the image of the class of $$P$$ under this boundary map is called the Chern class of $$P.$$

Let $$\theta \in \Omega^1(P)$$ be a connection form for the $$S^1$$-bundle $$P \rightarrow M.$$ One checks that $$\delta \theta \in C^2_{DR}(\Gamma_\ast)$$ descends to $$C^2_{DR}(\Gamma_\ast).$$ In other words, there exist unique $$\omega \in \Omega^1(\Gamma)$$ and $$\Omega \in \Omega^2(M)$$ such that $$\pi^* (\omega + \Omega) = \delta \theta.$$

PROPOSITION 3.2. – The class $$\omega + \Omega \in H^2_{DR}(\Gamma_\ast)$$ is independent of the choice of the connection $$\theta$$ on $$P \rightarrow M.$$ Under the canonical homomorphism $$H^2(\Gamma_\ast, \mathbb{Z}) \rightarrow H^2_{DR}(\Gamma_\ast),$$ the Chern class of $$P$$ maps to $$\omega + \Omega.$$
Here is a converse.

**Proposition 3.3.** – Let \( \omega + \Omega \in C^2_{DR}(\Gamma, \mathbb{R}) \) as above be an integer 2-cocycle. Then there exists an \( S^1 \)-bundle \( P \) over \( \Gamma \rightrightarrows M \) and a connection form \( \theta \in \Omega^1(P) \) for the bundle \( P \to M \), such that \( \pi^*(\omega + \Omega) = \delta \theta \).

Moreover, the set of isomorphism classes of all such \((P, \theta)\) is a simply transitive \( H^1(\Gamma, \mathbb{R}/\mathbb{Z})\)-set. Here \((P, \theta)\) and \((P', \theta')\) are isomorphic if \( P \) and \( P' \) are isomorphic as \( S^1 \)-bundles over \( \Gamma \rightrightarrows M \) and under such an isomorphism \( \theta \) is identified with \( \theta' \).

These two propositions indicate that \( \theta \) can be thought of as an analogue of a connection on \( P \) and \( \omega + \Omega \) as an analogue of the curvature of this connection.

On the other hand, we do not call \( \theta \) a connection on the \( S^1 \)-bundle over \( \Gamma \rightrightarrows M \), because this term should be reserved for \( \theta \) satisfying \( \partial \theta = 0 \).

Thus we suggest the name pseudo-connection for a connection on the underlying bundle over \( M \). If \( \theta \) is such a pseudo-connection, we call \( \omega + \Omega \in \Omega^2_{DR}(\Gamma, \mathbb{R}) \) such that \( \pi^*(\omega + \Omega) = \delta \theta \) the pseudo-curvature of \( \theta \).

### 4. \( S^1 \)-central extensions

**Definition 4.1.** – Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. An \( S^1 \)-central extension of \( \Gamma \rightrightarrows M \) consists of

1. a Lie groupoid \( R \rightrightarrows M \), together with a morphism of Lie groupoids \((\pi, \text{id}) : [R \rightrightarrows M] \to [\Gamma \rightrightarrows M] \),

2. a left \( S^1 \)-action on \( R \), making \( \pi : R \to \Gamma \) a (left) principal \( S^1 \)-bundle. These two structures are compatible in the sense that \((s \cdot x)(t \cdot y) = st \cdot (xy)\), for all \( s, t \in S^1 \) and \( x, y \in R \times_M R \).

Since \( S^1 \) is abelian, any left principal \( S^1 \)-bundle is a right principal \( S^1 \)-bundle in a natural way. Thus, if \( R \) and \( R' \) are central extensions of \( \Gamma \rightrightarrows M \) as in the definition, we may form the associated bundle \( R \times_{S^1} R' \), which is again an \( S^1 \)-bundle over \( \Gamma \). It has a natural groupoid structure making it into another \( S^1 \)-central extension of \( \Gamma \rightrightarrows M \). We denote this central extension by \( R \otimes R' \). This operation turns the set of isomorphism classes of \( S^1 \)-central extensions into an abelian group.

Central extensions of groupoids pull back via morphisms of groupoids.

Groupoid central extensions of \( \Gamma \rightrightarrows M \) give rise to \( S^1 \)-gerbes over \( \Gamma \), which are trivialized over \( M \). Thus we have the

**Proposition 4.2.** – There is a natural exact sequence

\[
H^1(\Gamma, S^1) \to H^1(M, S^1) \to \{ S^1 \text{-central extensions of } \Gamma \rightrightarrows M \} \to H^2(\Gamma, S^1) \to H^2(M, S^1).
\]

Given a central extension \( R \) of \( \Gamma \rightrightarrows M \), then a connection form \( \theta \in \Omega^1(R) \) for the bundle \( R \to \Gamma \), such that \( \partial \theta = 0 \) is a connective structure on \( R \). Given \((R, \theta)\), a 2-form \( B \in \Omega^2(M) \), such that \( d\theta = dB \) is a curving on \( R \), and given \((R, \theta, B)\), the 3-form \( \Omega = dB \in H^0(\Gamma, \Omega^3) \subset \Omega^3(M) \) is called the curvature of \((R, \theta, B)\). If \( \Omega = 0 \), then \((R, \theta, B)\) is called a flat \( S^1 \)-central extension of \( \Gamma \rightrightarrows M \). Note that the flat central extensions form an abelian group.

**Proposition 4.3.** – There is a natural exact sequence

\[
H^1(\Gamma, \mathbb{R}/\mathbb{Z}) \to H^1(M, \mathbb{R}/\mathbb{Z}) \to \{ \text{flat } S^1 \text{-central extensions of } \Gamma \rightrightarrows M \} \to H^2(\Gamma, \mathbb{R}/\mathbb{Z}) \to H^2(M, \mathbb{R}/\mathbb{Z}).
\]

The exponential sequence gives rise to a homomorphism \( H^2(\Gamma, S^1) \to H^3(\Gamma, \mathbb{Z}) \). The image of a central extension \( R \) in \( H^3(\Gamma, \mathbb{Z}) \) is called the Dixmier-Douady class of \( R \). The Dixmier-Douady class behaves well with respect to pullbacks and the tensor operation.
Let $R$ be a central extension of $\Gamma \rightrightarrows M$. Choose a connection form $\theta \in \Omega^1(R)$ for the $S^1$-bundle $\pi : R \to \Gamma$. One checks that $\delta \theta \in Z^3_{DR}(R)$ descends to $Z^3_{DR}(\Gamma_\ast)$, i.e., there exist unique $\eta \in \Omega^1(\Gamma_2)$ and $\omega \in \Omega^2(\Gamma)$ such that $\pi^*(\eta + \omega) = \delta \theta$.

**Proposition 4.4.** The class $[\eta + \omega] \in H^3_{DR}(\Gamma_\ast)$ is independent of the choice of the connection $\theta$ on $R \to \Gamma$. Under the canonical homomorphism $H^3(\Gamma, \mathbb{Z}) \to H^3_{DR}(\Gamma_\ast)$, the Dixmier-Douady class of $R$ maps to $[\eta + \omega]$.

Since the class $[\eta + \omega]$ does not change by adding a coboundary, we may choose, in addition to $\theta$, any $B \in \Omega^2(M)$, and then the Dixmier-Douady class of $R$ is represented by $\eta + \omega + \Omega$, such that $\pi^*(\eta + \omega + \Omega) = \delta(\theta + B)$.

**Proposition 4.5.** Given any 3-cocycle $\eta + \omega + \Omega \in Z^3_{DR}(\Gamma_\ast)$, as above, satisfying
1. $[\eta + \omega + \Omega]$ is integer,
2. $\Omega$ is exact,
there exists a groupoid central extension $R \rightrightarrows M$ of the groupoid $\Gamma \rightrightarrows M$, a connection $\theta$ on the bundle $R \to \Gamma$ and a 2-form $B \in \Omega^2(M)$, such that $\delta(\theta + B) = \pi^*(\eta + \omega + \Omega)$. The pairs $(R, \theta, B)$ up to isomorphism form a simply transitive set under the group of flat central extensions.

Because of these propositions, $\theta + B$ plays a role similar to a connection (connective structure plus curving) on a gerbe over a manifold. We therefore call $\theta + B$ a pseudo-connection on $R$, and $\theta + \omega + \Omega$ its pseudo-curvature.

**Remark 1.** Given a 3-cocycle $\eta + \omega + \Omega$ of integer class, we may have to pass to a Morita equivalent groupoid via a Morita morphism $[\Gamma' \rightrightarrows M'] \to [\Gamma \rightrightarrows M]$, in order to realize the condition that $\Omega$ be exact. For example, if $\Gamma = M$ we may have to pass to an open cover $\{U_\alpha\}$ of $M$ to construct a groupoid central extension. In this case we use the Morita morphism $\coprod_{\alpha, \beta} U_{\alpha \beta} \rightrightarrows \coprod_{\alpha} U_\alpha \to [M \rightrightarrows M]$. See [1]. If $M$ is connected, another possibility is to pass to the (infinite dimensional) path space $PM \to M$ and use the Morita morphism $[LM \rightrightarrows PM] \to [M \rightrightarrows M]$, where $LM$ is the space of based loops. See [2].

We close with a theorem that gives an explicit construction of the central extension with pseudo-connection in the s-connected case. It also gives a criterion for a class in $H^3_{DR}(\Gamma_\ast)$ to be integer. This theorem generalizes the result of [2].

**Theorem 4.6.** Let $\Gamma \rightrightarrows M$ be an s-connected Lie groupoid, and $\eta + \omega \in C^3_{DR}(\Gamma_\ast)$ a 3-cocycle, where $\eta \in \Omega^1(\Gamma_2)$ and $\omega \in \Omega^2(\Gamma)$. Assume that $\omega$ represents an integer cohomology class in $H^2_{DR}(\Gamma)$, so that there exists an $S^1$-bundle $\pi : R \to \Gamma$ with a connection $\theta \in \Omega^1(R)$, whose curvature is $\omega$. Assume that $\epsilon^*R$ endowed with the flat connection $\epsilon^*\theta + \pi^*c^2_\eta$ is holonomy free. (Here $\epsilon : M \to \Gamma$ and $\epsilon_2 : M \to \Gamma_2$ are the respective identity morphisms.) Then $R \rightrightarrows M$ admits in a natural way the structure of a groupoid, such that $R$ becomes an $S^1$-central extension of $\Gamma \rightrightarrows M$ and $\eta + \omega$ the pseudo-curvature of $\theta$.

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