Separable \( d \)-Permutations and Guillotine Partitions

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Abstract. We characterize separable multidimensional permutations in terms of forbidden patterns and enumerate them by means of generating function, recursive formula, and explicit formula. We find a connection between multidimensional permutations and guillotine partitions of a box. In particular, a bijection between separable \( d \)-dimensional permutations and guillotine partitions of a \( 2d-1 \)-dimensional box is constructed. We also study enumerating problems related to guillotine partitions under certain restrictions revealing connections to other combinatorial structures. This allows us to obtain several results on patterns in permutations.

Keywords: \( d \)-permutations, separable permutations, patterns in permutations, guillotine partitions, binary trees, Schröder paths

1. Introduction

In the first part of this paper we study the multidimensional generalization of separable permutations. A permutation \( \pi \) of \( [n] = \{1, 2, \ldots, n\} \) is separable if either \( n = 1 \), or \( n > 1 \) and it is possible to split the graph of \( \pi \) by a horizontal and a vertical line into four parts so that two opposite parts are empty, and two other parts are non-empty and contain graphs of (smaller) separable permutations. Separable permutations may be also defined in terms of forbidden patterns as \((2413, 3142)\)-avoiding permutations.

A \( d \)-permutation is a sequence of \( d \) permutations, the first of them being the natural order permutation \( 12\cdots n \). The notion of separable permutations generalizes to that of separable \( d \)-permutations in a natural way. After formal definitions (Subsection 2.1), we find an equation satisfied by the generating function, a recursive formula and an explicit formula for the number of separable \( d \)-permutations of \( [n] \) (Subsection 2.2) and characterize them in terms of forbidden patterns (Subsection 2.3).

The second part of the paper is devoted to guillotine partitions of a \( d \)-dimensional box, i.e., recursive partitions of a \( d \)-dimensional box \( B \) by axis-aligned hyperplanes.

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Guillotine partitions were introduced in 1980s, and they have numerous applications in computational geometry, computer graphics, etc. Recently, Ackerman, Barequet, Pinter, and Romik studied the enumerative issues related to guillotine partitions [1,2]. We observe that the generating function for the number of separable \(d\)-permutations is identical to the generating function for the number of (structurally different) guillotine partitions of a \(2^{d-1}\)-dimensional box. Ackerman et al. constructed a bijection between these sets in the case \(d = 2\). In Section 3, we generalize this bijection to any \(d\), and find a subclass of separable \(d\)-permutations which correspond to guillotine partitions of a \(q\)-dimensional box where \(q\) is not necessarily a power of 2.

In Section 4 we deal with guillotine partitions with certain restrictions. In Subsections 4.1–4.3 we enumerate some classes of restricted guillotine partitions, in Subsection 4.4 we use these results and the correspondence between separable permutations and guillotine partitions to enumerate permutations avoiding certain patterns.

2. Separable \(d\)-Permutations

2.1. Notation and Convention

A \(d\)-permutation of \([n]\) is a sequence \(P = (p_1, p_2, \ldots, p_d)\) where each \(p_i\) is a permutation of \([n]\) and \(p_1\) is the natural-order permutation \(12\cdots n\). It may be represented as a \(d \times n\) matrix (also denoted by \(P\)) each row of which is a permutation of \([n]\), the first row being \(12\cdots n\). Thus, for \(1 \leq i \leq d\), \(p_i\) is a row of this matrix, and we shall denote by \(P^{(j)}\) its \(j\)th column \((1 \leq j \leq n)\). \(P_{i,j}\) will denote the \((i,j)\)-entry of the matrix \(P\).

A \(d\)-permutation \(P\) may be represented geometrically as a point set in \(\mathbb{R}^d\), columns of \(P\) being the coordinate vectors of the points. It is a subset of size \(n\) of the discrete cube \([n]^d\) such that each hyperplane \(x_i = j, 1 \leq i \leq d, 1 \leq j \leq n\), contains precisely one point. We shall refer to this geometric representation as the graph of \(P\).

It is clear that there are \((n!)^{d-1}\) \(d\)-permutations of \([n]\).

Remark 2.1. There is another definition of \(d\)-permutations to be found in the literature, which is equivalent to what we call \((d+1)\)-permutations. For example, in our definition usual permutations are \(2\)-permutations, while in the other they are \(1\)-permutations.

**Definition 2.2.** A \(d\)-permutation \(P\) of \([n]\) is separable if

- either \(n = 1\),
- or \(n > 1\) and there is a number \(\ell, 1 \leq \ell < n\), such that
  
  - for each \(1 \leq i \leq d\), either \(P_{i,j_1} < P_{i,j_2}\) for all \(1 \leq j_1 \leq \ell, \ell + 1 \leq j_2 \leq n\),
  - or \(P_{i,j_1} > P_{i,j_2}\) for all \(1 \leq j_1 \leq \ell, \ell + 1 \leq j_2 \leq n\),
  - the two \(d\)-permutations obtained from \(P\) by taking the first \(\ell\) columns and the last \(n - \ell\) columns and applying the order preserving relabeling so that they become \(d\)-permutations of \([\ell]\) and of \([n - \ell]\) respectively, are themselves separable.