LIMITING ABSORPTION PRINCIPLE
AND SINGULAR SPECTRUM

NURULLA AZAMOV

ABSTRACT. In this paper I give an explicit construction of an analogue of
eigenspace for points of singular spectrum of a self-adjoint operator. This con-
struction is based on an abstract version of homogeneous Lippmann-Schwinger
equation.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be (complex separable) Hilbert spaces, $H_0$ a self-adjoint operator
on $\mathcal{H}$ and $F: \mathcal{H} \to \mathcal{K}$ a bounded operator with zero kernel and co-kernel such that
the sandwiched resolvent
$$T_z(H_0) = F(H_0 - z)^{-1}F^*$$
is compact. One says that the limiting absorption principle (LAP) holds at a real
number $\lambda$ if the norm limit
$$T_{\lambda+i0}(H_0) := \lim_{y \to 0^+} T_{\lambda+iy}(H_0)$$
exists. Usually LAP means that such limit exists for a.e. $\lambda$ in some open interval,
but for the present purpose it suffices to consider it at a single point. It may well
happen that the limit (1) does not exist. In this case there are two scenarios: it is
possible that the limit
$$T_{\lambda+i0}(H_r),$$
exists for at least one real number $r \in \mathbb{R}$, or otherwise. In the first scenario the
limit exists for all real $r$ except some discrete set of values, called coupling resonance
points. In the first of these scenarios $\lambda$ is called a semi-regular point of the pair
$H_0, F$, and in the second $\lambda$ is called essentially singular. For a semi-regular point $\lambda$
the kernel, denoted
$$\Upsilon_{\lambda+i0}(H_r),$$
of the operator $1 - r T_{\lambda+i0}(H_r)$ is well-defined for non-resonance values of $r$ and in
that case it does not depend on the choice of such $r$. The aim of this paper is to
prove the following theorem. At the end of this introduction I make some remarks
explaining why such a theorem is interesting.

2000 Mathematics Subject Classification. Primary 47A40;
Key words and phrases. Lippmann-Schwinger equation, singular spectrum.
Theorem 1.1. Assume the above about $H_0$ and $F$. Suppose there exists a function $g$ from $L^1(\mathbb{R}, (1 + x^2)^{-1} \, dx)$ such that for a.e. $\lambda$

$$\sup_{y \in [0, 1]} \| T_{\lambda+iy}(H_0) \| \leq g(\lambda).$$

Then for a semi-regular point $\lambda$

$$\bigcap_{\lambda \in O} FEO(H_0)H = \Upsilon_{\lambda+i0}(H_0),$$

where the intersection is over all open neighbourhoods of $\lambda$.

Theorem 1.1 immediately implies the following theorem which gives a partial positive solution to [1, Subsection 15.9, Conjecture 7].

Theorem 1.2. Assume the premise of Theorem 1.1. If $\chi \in H$ obeys $F\chi \in \Upsilon_{\lambda+i0}(H_0)$ then $H_0 \chi = \lambda \chi$.

Thus, given the condition (4), Theorems 1.2 and [1, Theorem 4.1.1] assert that

$$FV(\lambda, H_0) = \Upsilon_{\lambda+i0}(H_0) \cap \text{ran}(F),$$

where $V(\lambda, H_0)$ is the eigenspace of $H_0$ corresponding to an eigenvalue $\lambda$.

Remark 1. Assuming that the rigging $F$ is bounded, the vector space $\text{ran}(F)$ endowed with the graph-norm, is a Hilbert space naturally isomorphic to $H$, the isomorphism being given by $F$ itself. The equality (5) asserts that the eigenvectors of $H_0$ corresponding to an eigenvalue $\lambda$ can be interpreted as those elements of $\Upsilon_{\lambda+i0}(H_0)$ which belong to the image of $F$. Vectors from $\Upsilon_{\lambda+i0}(H_0)$ which do not belong to $\text{ran}(F)$ can therefore be interpreted as generalised eigenvectors of $H_0$. Moreover, these generalised eigenvectors are $F$-images of elements of the singular subspace of $H_0$.

Remark 2. Instead of the straight line (2) we could have worked with the line $H_r = H_0 + rF^*JF$, where $J$ is any bounded self-adjoint operator on $K$ such that for some $r \in \mathbb{R}$ the limit

$$T_{\lambda+i0}(H_0 + rF^*JF)$$

exists (such operators $J$ are called regular directions), — proof is exactly the same. But as far as Theorem 1.1 is concerned, this makes no difference since the solution set (3) to the equation

$$(1 - rT_{\lambda+i0}(H_0 + rF^*JF))u = 0$$

does not depend on a choice of a regular direction $J$ and a non-resonant value of $r$, see [1]. Another reason for considering the direction $\text{Id}$ instead of an arbitrary $J$ is that if the limit (4) exists for some bounded $J$ then it also exists for the identity operator $J = \text{Id}$, see [2].

Remark 3. The equation (7) is nothing else but the homogeneous version of an abstract Lippmann-Schwinger equation, see e.g. [4, §4.3] or [7]. For a semi-regular energy $\lambda$, the limit $T_{\lambda+i0}(H_0)$ fails to exist if and only if the equation (7) has a non-zero solution. The solutions can be interpreted as bound states or meta-stable states (also called resonances) of $H_0$ with energy $\lambda$, where bound states correspond to elements of (5).

Remark 4. Theorem 1.1 is not unrelated to the well-known Simon-Wolff criterion [6], see also [5]. This relation will soon be discussed elsewhere.
2. Proof of Theorem

Lemma 2.1. Let $H_1 = H_0 + V$. For any $w \in \rho(H_0)$ and $z \in \rho(H_1)$

$$\tag{8} (w-z)R_w(H_0)R_z(H_1) = -R_z(H_1) + R_w(H_0) \left[ 1 - VR_z(H_1) \right].$$

Proof. Using the second resolvent identity

$$R_w(H_0) = (1 - R_w(H_1)V)^{-1}R_w(H_1)$$

we rewrite $R_w(H_0)$ in terms of $R_w(H_1)$ with the aim to use next the first resolvent identity:

$$(w-z)R_w(H_0)R_z(H_1) = (w-z)(1 - R_w(H_1)V)^{-1}R_w(H_1)R_z(H_1)$$

$$= (1 - R_w(H_1)V)^{-1} \left[ R_w(H_1) - R_z(H_1) \right]$$

$$= R_w(H_0) - (1 - R_w(H_1)V)^{-1}R_z(H_1).$$

Since $(1 - R_w(H_1)V)^{-1} = 1 + R_w(H_0)V$, this gives

$$(w-z)R_w(H_0)R_z(H_1) = R_w(H_0) - (1 + R_w(H_0)V)R_z(H_1)$$

$$= -R_z(H_1) + R_w(H_0) \left[ 1 - VR_z(H_1) \right].$$

\[\square\]

We only need to prove the inclusion

$$\tag{9} \mathcal{Y}_{\lambda+i0}(H_0) \subseteq \bigcap_{\lambda \in \mathcal{O}} \overline{FE_O(H_0)H},$$

since the other inclusion was proved in \[1\]. Let $u \in \mathcal{Y}_{\lambda+i0}(H_0)$, that is,

$$\tag{10} (1 - sT_{\lambda+i0}(H_s))u = 0.$$

Since solutions of \[10\] do not depend on the choice of a non-resonant value of $s$, without loss of generality we can assume that $s = 1$, in particular assuming that this value is non-resonant. Let

$$f_{\lambda+iy} := R_{\lambda+iy}(H_1)F^*u.$$

Our aim is to show that for small enough $y > 0$ the spectral representation of the vector $f_{\lambda+iy}$ with respect to $H_0$ is concentrated near $\lambda$.

Lemma 2.2. For $\lambda, x \in \mathbb{R}$ and $y > 0$ we have

$$\Im\langle f_{\lambda+iy}, R_{x+iy}(H_0) f_{\lambda+iy} \rangle$$

$$\tag{11} = (x - \lambda)^{-1} \Im \left( (x - \lambda - 2iy)^{-1} \left[ \ldots \left[ u, \left[ u, T_{\lambda+iy}(H_1)u \right] \right] \right] \right),$$

where

$$\ldots = -T_{\lambda+iy}(H_1) + T_{x-iy}(H_0)[1 - T_{\lambda+iy}(H_1)].$$
Proof. Using (8), we have
\[(x - \lambda)R_{x+iy}(H_0)\lambda+i\gamma = (x - \lambda)R_{x+iy}(H_0)R_{\lambda+i\gamma}(H_1)F^*u\]
\[= \left(-R_{\lambda+i\gamma}(H_1) + R_{x+iy}(H_0)\left[1 - F^*FR_{\lambda+i\gamma}(H_1)\right]\right)F^*u\]
\[= -f_{\lambda+i\gamma} + R_{x+iy}(H_0)F^*\left[u - T_{\lambda+i\gamma}(H_1)u\right].\]
Taking the scalar product of both sides of (12) with \(\langle f_{\lambda+i\gamma} \rangle\) and then taking the imaginary part of the resulting scalar products we get
\[(x - \lambda)\Im \langle f_{\lambda+i\gamma}, R_{x+iy}(H_0)f_{\lambda+i\gamma} \rangle = \Im \langle R_{\lambda+i\gamma}(H_0)f_{\lambda+i\gamma}, F^*\left[u - T_{\lambda+i\gamma}(H_1)u\right] \rangle.
Using (8) again, we transform the first argument of the last scalar product as follows:
\[R_{x-i\gamma}(H_0)f_{\lambda+i\gamma} = R_{x-i\gamma}(H_0)R_{\lambda+i\gamma}(H_1)F^*u\]
\[= (x - \lambda - 2i)\gamma^{-1} \times \left[-R_{\lambda+i\gamma}(H_1) + R_{x-i\gamma}(H_0)\left(1 - F^*FR_{\lambda+i\gamma}(H_1)\right)\right]F^*u.
Hence, denoting by \([\ldots]\) the expression in the last pair of square brackets, we get from (13)
\[(x - \lambda)\Im \langle f_{\lambda+i\gamma}, R_{x+i\gamma}(H_0)f_{\lambda+i\gamma} \rangle
= \Im \langle (x - \lambda - 2i)\gamma^{-1} [\ldots] F^*u, [u - T_{\lambda+i\gamma}(H_1)u] \rangle,
\]
as required.
\[\text{Lemma 2.3. Under the premise of Theorem 1.1 for any } \delta > 0 \]
\[\lim_{y \to 0^+} \int_{R \setminus (\lambda - \delta, \lambda + \delta)} \Im \langle f_{\lambda+i\gamma}, R_{x+i\gamma}(H_0)f_{\lambda+i\gamma} \rangle \, dx = 0.
Proof. We will use (11) for the integrand. The contribution of the summand 
\(-T_{\lambda+i\gamma}(H_1)\) in \([\ldots]\) to the limit (10) is clearly zero. Thus, introducing the notation
\[\chi_{\lambda+i\gamma} := \left[1 - T_{\lambda+i\gamma}(H_1)\right]u,
\]
it suffices to prove that the limit of the integral of
\[(x - \lambda)^{-1} \Im \langle (x - \lambda - 2i)\gamma^{-1} \langle x - i\gamma(H_0)\chi_{\lambda+i\gamma}, \chi_{\lambda+i\gamma} \rangle \rangle\]
over \(x \notin (\lambda - \delta, \lambda + \delta)\) goes to zero.
By (11), the vector \(\chi_{\lambda+i\gamma}\) converges to zero as \(y \to 0^+\). Thus, by the assumption (4), the integrand converges to zero for a.e. \(x \notin (\lambda - \delta, \lambda + \delta)\). Moreover, by the same assumption we can apply the Lebesgue Dominated Convergence Theorem to interchange the limit \(y \to 0^+\) with the integration.

Now we can complete proof of Theorem 1.1. By Stone’s formula, the integral of
\[\pi^{-1} \Im R_{x+i\gamma}(H_0)\]
on the complement of \((\lambda - \delta, \lambda + \delta)\) converges strongly to
\[E_{R \setminus (\lambda - \delta, \lambda + \delta)}(H_0) + \frac{1}{2} E_{(\lambda - \delta, \lambda + \delta)}(H_0),\]
as \( y \to 0^+ \). Combining this with (15) gives
\[
\text{(16)} \quad E_{\mathbb{R} \setminus (\lambda - \delta, \lambda + \delta)}(H_0)f_{\lambda + iy} \to 0
\]
as \( y \to 0^+ \).

In order to prove the inclusion (11), it suffices to show that for any \( \varepsilon > 0 \) and \( \delta > 0 \) there exists \( \psi \in E_{(\lambda - \delta, \lambda + \delta)}(H_0) \) such that the distance between \( u \) and \( F\psi \) is less than \( \varepsilon \). We claim that for small enough \( y > 0 \) the choice
\[
\psi = E_{(\lambda - \delta, \lambda + \delta)}f_{\lambda + iy}
\]
works. Indeed,
\[
\|u - F\psi\| = \|u - FE_{(\lambda - \delta, \lambda + \delta)}f_{\lambda + iy}\|
\leq \|u - Ff_{\lambda + iy}\| + \|Ff_{\lambda + iy} - FE_{(\lambda - \delta, \lambda + \delta)}f_{\lambda + iy}\|
\leq \|u - T_{\lambda + iy}(H_1)u\| + \|F\|\|f_{\lambda + iy} - E_{(\lambda - \delta, \lambda + \delta)}f_{\lambda + iy}\|.
\]
Since \( u \) is a solution to (10) (with \( s = 1 \)), for all small enough \( y > 0 \) the first summand is \( < \varepsilon/2 \). By (16), for all small enough \( y > 0 \) the second summand is also \( < \varepsilon/2 \).

Proof is complete.

Acknowledgements. I thank my wife for financial support during the work on this paper.

References

[1] N. A. Azamov, Spectral flow inside essential spectrum, Dissertationes Math. 518, 1-156 (2016)
[2] N. A. Azamov, Spectral flow inside essential spectrum IV: \( F^*F \) is a regular direction, arXiv: 2109.10545
[3] N. A. Azamov, Spectral flow inside essential spectrum VI: on essentially singular points, arXiv: 2110.08699
[4] F. A. Berezin, M. A. Shubin, Schrödinger equation, Dordrecht; Boston: Kluwer Academic Publishers, 1991
[5] B. Simon, Trace Ideals and their Applications, Second edition, Math. Surveys Monogr. (Amer. Math. Soc., 2005)
[6] B. Simon, T. Wolff, Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, Comm. Pure Appl. Math. 39 (1986), 75–90
[7] J. R. Taylor, Scattering theory, John Wiley & Sons, Inc. New York

Independent scholar, Adelaide, SA, Australia
Email address: azamovnurulla@gmail.com