Matrix Representation of Bi-Periodic Jacobsthal Sequence

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Authors’ contributions
This work was carried out in collaboration between both authors. Author SU designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author EO managed the analyses of the study. Author SU managed the literature searches. All authors read and approved the final manuscript.

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Abstract
In this paper, we bring into light the matrix representation of bi-periodic Jacobsthal sequence, which we shall call the bi-periodic Jacobsthal matrix sequence. We define it as

\[ J_n = \begin{cases} 
  b J_{n-1} + 2 J_{n-2}, & \text{if } n \text{ is even} \\
  a J_{n-1} + 2 J_{n-2}, & \text{if } n \text{ is odd}
\end{cases} \]

with initial conditions \( J_0 = I \) identity matrix, \( J_1 = \begin{pmatrix} b & 2b \\ 1 & 0 \end{pmatrix} \). We obtained the nth general term of this new matrix sequence. By studying the properties of this new matrix sequence, the well-known Cassini or Simpson’s formula was obtained. We then proceed to find its generating function as well as the Binet formula. Some new properties and two summation formulas for this new generalized matrix sequence were also given.

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1 Introduction

The increasing applications of integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal Lucas, Pell, etc. in the various fields of science and arts can not be overemphasized. For example, the ratio of two consecutive Fibonacci numbers converges to what is widely known as the Golden ratio whose applications appear in many research areas, particularly in Physics, Engineering, Architecture, Nature and Art [1].

The same can easily be said for Jacobsthal sequence. For instance, it is known that microcontrollers and other computers change the flow of execution of a program using conditional instructions. Along with branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction which boil down to being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 on 5 bits, 21 on 6 bits, 43 on 7 bits, 85 on 8 bits and continue in that order, which are exactly the Jacobsthal numbers.

Now, the classical Jacobsthal sequence \( \{j_n\}_{n=0}^{\infty} \) which was named after the German mathematician Ernst Jacobsthal is defined recursively by the relation

\[
j_n = j_{n-1} + 2j_{n-2}
\]

with initial conditions \( j_0 = 0, j_1 = 1 \). The other related sequence is the Jacobsthal Lucas sequence \( \{c_n\}_{n=0}^{\infty} \) which satisfies the same recurrence relation, that is \( c_n = c_{n-1} + 2c_{n-2} \) but with different initial conditions \( c_0 = 2, c_1 = 1 \) [2]. Applications of these two sequences to curves can be found in [3].

There are many studies in literature on the bi-periodic integer sequences. For example, in [4], [5], for any two non-zero real numbers \( a \) and \( b \), Edson and Yayenie defined the bi-periodic Fibonacci sequence as

\[
q_n = \begin{cases} 
    aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\
    bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}
\end{cases} \quad n \geq 2
\]

with initial conditions \( q_0 = 0, q_1 = 1 \). Then, Bilgici [6] also defined the bi-periodic Lucas sequence as

\[
l_n = \begin{cases} 
    al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \\
    bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even}
\end{cases} \quad n \geq 2
\]

with initial conditions \( l_0 = 2, l_1 = a \).

In [7], Uygun and Owusu defined the bi-periodic Jacobsthal sequence as

\[
j_n = \begin{cases} 
    a j_{n-1} + 2j_{n-2}, & \text{if } n \text{ is even} \\
    b j_{n-1} + 2j_{n-2}, & \text{if } n \text{ is odd}
\end{cases} \quad n \geq 2,
\]

with initial conditions \( j_0 = 0, j_1 = 1 \).

In [8], Coskun and Taskara carried out bi-periodic sequences to matrix theory and defined the bi-periodic Fibonacci matrix sequence as

\[
F_n (a, b) = \begin{cases} 
    aF_{n-1} (a, b) + 2F_{n-2} (a, b), & \text{if } n \text{ is even} \\
    bF_{n-1} (a, b) + 2F_{n-2} (a, b), & \text{if } n \text{ is odd}
\end{cases} \quad n \geq 2,
\]

with the initial conditions given as

\[
F_0 (a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_1 (a, b) = \begin{pmatrix} b & a \\ 1 & 0 \end{pmatrix}.
\]
Then they found the relation between the bi-periodic Fibonacci matrix sequence and number sequence as
\[
F_n = \begin{pmatrix} b & a \\ \varepsilon(n) b q_n & \varepsilon(n) q_n \end{pmatrix},
\]
where \( \varepsilon(m) \) is the parity function which is defined as
\[
\varepsilon(m) = \begin{cases} 0, & \text{if } m \text{ is even} \\ 1, & \text{if } m \text{ is odd} \end{cases}
\]
Soykan also used the matrix sequences of Tetranacci and Tetranacci-Lucas numbers in [9]. The authors in [10] investigated the relations of the bi-periodic Jacobsthal sequence. Uygun, Karatas similarly studied on the properties of the bi-periodic Pell-Lucas sequence in [11]. Gul in [12], studied on bi-periodic Jacobsthal and Jacobsthal-Lucas quaternions. Choo examined some identities on generalized bi-periodic Fibonacci sequences in [13]. The authors made a research on convolutions of the bi-periodic Fibonacci numbers in [14]. In [15], Uygun and Owusu defined the bi-periodic Jacobsthal Lucas sequence and gave some basic important properties of this sequence. The authors found some sum formulas for Pell and Jacobsthal numbers by using matrix methods in [16]. The authors studied on associated Hessenberg matrices with Jacobsthal numbers in [17].

In this paper, we bring into light the matrix representation of bi-periodic Jacobsthal sequence, which we shall call the bi-periodic Jacobsthal matrix sequence. We then proceed to obtain the \( n \)th general term of this new matrix sequence. By studying the algebraic properties of this new matrix sequence, the well-known Cassini formula is obtained. The generating function together with the Binet formula are given. Some new properties as well as some summation formulas for this new generalized matrix sequence are also investigated.

## 2 Main Results

**Definition 2.1.** Bi-periodic Jacobsthal matrix sequence denoted by \( \{ J_n(a, b) \}_{n=0}^{\infty} \) is defined recursively by
\[
J_n(a, b) = \begin{cases} a J_{n-1}(a, b) + 2 J_{n-2}(a, b), & \text{if } n \text{ is even} \\ b J_{n-1}(a, b) + 2 J_{n-2}(a, b), & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2,
\]
with the initial conditions given as
\[
J_0(a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_1(a, b) = \begin{pmatrix} b & 2a \\ 1 & 0 \end{pmatrix}.
\]

**Theorem 2.1.** For any integer \( n \geq 0 \), the \( n \)th element of the bi-periodic Jacobsthal matrix sequence is demonstrated by using bi-periodic Jacobsthal numbers as
\[
\begin{align*}
J_n &= \begin{pmatrix} (\frac{b}{x})^{\varepsilon(n)} j_{n+1} & 2 b J_n \\ 2 (\frac{b}{x})^{\varepsilon(n)} j_{n-1} & j_n \end{pmatrix}.
\end{align*}
\]

**Proof.** We obtain the proof by means of mathematical induction. We will start by noting from Definition (2.1) that \( j_0 = 0, j_1 = 1, j_{-1} = \frac{1}{2} \) and \( j_2 = a \). Hence, the induction for \( n = 0 \) is satisfied as
\[
J_0 = \begin{pmatrix} j_1 & 2 b j_0 \\ j_0 & 2 j_{-1} \end{pmatrix} = I.
\]
We now assume that the equation is true for \( n = k \), where \( k \) is any positive integer, that is;
\[
J_k = \begin{pmatrix} (\frac{b}{x})^{\varepsilon(k)} j_{k+1} & 2 b J_k \\ j_k & 2 (\frac{b}{x})^{\varepsilon(k)} j_{k-1} \end{pmatrix}
\]
For any integer $n \geq 0$, we obtain
\[
J_{2m} = (ab + 4)J_{2m-2} - 4J_{2m-4},
\]
\[
J_{2m+1} = (ab + 4)J_{2m-1} - 4J_{2m-3}.
\]

**Proof.** The proof can easily be obtained by using the definition of the bi-periodic Jacobsthal matrix sequence.

**Theorem 2.3.** For any positive integer $n$, we have
\[
\det [J_n] = 2^n \left( \frac{b}{a} \right)^{\varepsilon(n)}
\]

**Proof.** It is obtained by induction method.

**Theorem 2.4.** The generating function for the bi-periodic Jacobsthal matrix sequence is given by
\[
\sum_{m=0}^{\infty} J_m x^m = \frac{J_0 + J_1 x + [aJ_1 - (ab + 2)J_0] x^2 + [2bJ_0 - 2J_1] x^3}{1 - (ab + 4)x^2 + 4x^4}
\]
which is expressed in component form as
\[
\sum_{m=0}^{\infty} J_m x^m = \frac{1}{1 - (ab + 4)x^2 + 4x^4} \begin{pmatrix}
1 + bx - 2x^2 & 2b^2 x + 2bx^2 - 4b^2 x^3 \\
x + ax^2 - 2x^3 & 1 - (ab + 2)x^2 + 2bx^3
\end{pmatrix}.
\]

**Proof.** We divide the series into two parts
\[
J(x) = \sum_{m=0}^{\infty} J_m x^m = \sum_{m=0}^{\infty} J_{2m} x^{2m} + \sum_{m=0}^{\infty} J_{2m+1} x^{2m+1}.
\]
The even part of the above series is simplified as follows
\[
J_0(x) = \sum_{m=0}^{\infty} J_{2m} x^{2m} = J_0 + J_2 x^2 + \sum_{m=2}^{\infty} J_{2m} x^{2m}.
\]
By multiplying through by \((ab + 4)x^2\) and \(4x^4\) respectively, it’s obtained that

\[
(ab + 4)x^2 J_0(x) = (ab + 4)J_0x^2 + (ab + 4) \sum_{m=2}^{\infty} J_{2m-2}x^{2m}
\]

and

\[
4x^4 J_0(x) = 4 \sum_{m=2}^{\infty} J_{2m-4}x^{2m}.
\]

Hence, it follows that,

\[
\left[1 - (ab + 4)x^2 + 4x^4\right] J_0(x) = J_0 + J_2x^2 - (ab + 4)J_0x^2 + \sum_{m=2}^{\infty} [J_{2m} - (ab + 4)J_{2m-2} + 4J_{2m-4}]x^{2m}
\]

By using Lemma (2.2), we obtained that;

\[
J_0(x) = \frac{J_0 + J_2x^2 - (ab + 4)J_0x^2}{1 - (ab + 4)x^2 + 4x^4}.
\]

Similarly, the odd part of the above series is simplified as follows

\[
J_1(x) = \sum_{m=0}^{\infty} J_{2m+1}x^{2m+1} = \frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4}.
\]

By combining the two results \([J(x) = J_0(x) + J_1(x)]\), we have

\[
J(x) = \frac{J_0 + J_2x^2 - (ab + 4)J_0x^2 + J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4},
\]

which can be simplified using Definition (2.1) as

\[
J(x) = \frac{J_0 + J_1x + [aJ_1 - (ab + 2)J_0]x^2 + [2bJ_0 - 2J_1]x^3}{1 - (ab + 4)x^2 + 4x^4}.
\]

By substituting the matrix form definition of \(J_0\) and \(J_1\) and simplifying, we obtain

\[
J(x) = \frac{1}{1 - (ab + 4)x^2 + 4x^4} \left( 1 + bx - 2x^2 \quad 2x^2 - 2x^3 \quad 2x^3 - 4x^4 \right).
\]

which completes the proof.

(2) The proof can be made by using the recurrence relation (2.1) for the bi-periodic Jacobsthal matrix sequence. The generating function for \(J(x)\) is represented in power series by

\[
J(x) = \sum_{m=0}^{\infty} J_m x^m = J_0 + J_1x + \ldots + J_kx^k + \ldots
\]

By multiplying through this series by \(bx\) and \(2x^2\) respectively, it’s obtained that

\[
bxJ(x) = b \sum_{m=0}^{\infty} J_m x^{m+1} = b \sum_{m=1}^{\infty} J_{m-1}x^m
\]

and,

\[
2x^2 J(x) = 2 \sum_{m=0}^{\infty} J_m x^{m+2} = 2 \sum_{m=2}^{\infty} J_{m-2}x^m.
\]
\begin{align*}
(1 - bx - 2x^2)J(x) &= J_0 + xJ_1 - bxJ_0 \\
&\quad + \sum_{m=2}^{\infty} (J_m - bJ_{m-1} - 2J_{m-2})x^m \\
&= J_0 + xJ_1 - bxJ_0 \\
&\quad + \sum_{m=1}^{\infty} (J_{2m} - bJ_{2m-1} - 2J_{2m-2})x^{2m}. 
\end{align*}

It's denoted that by (2.1)

\begin{align*}
(1 - bx - 2x^2)J(x) &= J_0 + xJ_1 - bxJ_0 + \sum_{m=1}^{\infty} (a - b)J_{2m-1}x^{2m}. \\
(1 - bx - 2x^2)J(x) &= J_0 + xJ_1 - bxJ_0 + (a - b)\sum_{m=1}^{\infty} J_{2m-1}x^{2m-1}. 
\end{align*}

Now let's define \( j(x) \) as

\[ j(x) = \sum_{m=1}^{\infty} J_{2m-1}x^{2m-1}. \]

Simplying \( j(x) \) in the same way as above and using Lemma (2.2 gives;

\begin{align*}
(1 - (ab + 4)x^2 + 4x^4)j(x) &= \sum_{m=1}^{\infty} J_{2m-1}x^{2m-1} - (ab + 4)\sum_{m=2}^{\infty} J_{2m-3}x^{2m-1} \\
&\quad + 4\sum_{m=3}^{\infty} J_{2m-5}x^{2m-1} \\
&= (J_1x + J_3x^3) - (ab + 4)J_1x^3 \\
&\quad + \sum_{m=3}^{\infty} (J_{2m-1} - (ab + 4)J_{2m-3} + 4J_{2m-5})x^{2m-1} \\
&= J_1x + J_3x^3 - (ab + 4)J_1x^3 + 0. \\
\end{align*}

\[ j(x) = \frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4}. \]

Plugging \( j(x) \) into \( J(x) \) above gives

\begin{align*}
(1 - bx - 2x^2)J(x) &= J_0 + xJ_1 - bxJ_0 + \\
&\quad (a - b)x \left( \frac{J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4} \right). 
\end{align*}

Simplifying this using the basic rules and properties of algebra, the generating function is demonstrated by

\[ J(x) = \frac{J_0 + J_1x - (ab + 4)J_0x^2 + J_1x + J_3x^3 - (ab + 4)J_1x^3}{1 - (ab + 4)x^2 + 4x^4}. \]

The simplified form is given as

\[ J(x) = \frac{J_0 + J_1x + [aJ_1 - (ab + 2)J_0]x^2 + [2bJ_0 - 2J_1]x^3}{1 - (ab + 4)x^2 + 4x^4}. \]
Theorem 2.5. For every $n$ belonging to the set of natural numbers, the Binet formula for the bi-periodic Jacobsthal matrix sequence is given by

$$J_n = A \left( \alpha^n - \beta^n \right) + B \left( \alpha^{n+1} \frac{J_{n+1}}{2} - \beta^{n+1} \frac{J_{n+1}}{2} \right),$$

where

$$A = \frac{(J_1 - bJ_0)^{\binom{n}{2}} (\alpha J_1 - 2J_0 - abJ_0)^{1-\binom{n}{2}}}{(ab) \frac{n}{2} (\alpha - \beta)}$$

and

$$B = \frac{b^{\binom{n}{2}} J_0}{(ab) \frac{n}{2} + 1 (\alpha - \beta)}.$$

The matrices $A$ and $B$ are expressed in component form as

$$A = \frac{1}{(ab) \frac{n}{2} (\alpha - \beta)} \begin{pmatrix} 0 & 2a \binom{n}{2} \\ 1 & -b \binom{n}{2} \end{pmatrix} \begin{pmatrix} -2 & 2b \binom{n}{2} \\ a & -2 - ab \binom{n}{2} \end{pmatrix}$$

and

$$B = \frac{b^{\binom{n}{2}} J_0}{(ab) \frac{n}{2} + 1 (\alpha - \beta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof.

Theorem 2.5.

Using partial fraction decomposition, we split $J(x)$ as

$$J(x) = \frac{1}{1 - (ab + 4)x^2 + 4x^4} = \frac{1}{4} \left[ \frac{Ax + B}{x^2 - \frac{a+2}{4}} + \frac{Cx + D}{x^2 - \frac{a-2}{4}} \right].$$

By solving for the constants $A, B, C$ and $D$ above, we express $J(x)$ in partial fraction as

$$\frac{1}{4(\alpha - \beta)} \begin{pmatrix} x \{ 2\alpha [ bJ_0 - J_1] + 4bJ_0 \} \\ +\alpha \{ aJ_1 - 2J_0 - abJ_0 \} + 2aJ_1 - 2abJ_0 \end{pmatrix} + \begin{pmatrix} x \{ 2\beta [ J_1 - bJ_0] - 4bJ_0 \} + \beta \{ abJ_0 (a, b) + 2J_0 (a, b) - aJ_1 \} + 2abJ_0 - 2aJ_1 \end{pmatrix}. $$

The Maclaurin series expansion of the function $\frac{Ax + B}{x^2 - C}$ is expressed in the form

$$\frac{Ax + B}{x^2 - C} = -\sum_{n=0}^{\infty} AC^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty} BC^{-n-1} x^{2n}.$$

Hence $J(x)$ can be expanded and simplified as

$$J(x) = \frac{1}{4(\alpha - \beta)} \begin{pmatrix} -\sum_{n=0}^{\infty} \{ 2abJ_0 - J_1 + 4bJ_0 \} \left( \frac{a+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ \alpha [ aJ_1 - 2J_0 - abJ_0 ] \} \left( \frac{a+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ 2\beta [ J_1 - bJ_0 ] - 4bJ_0 \} \left( \frac{b+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ \beta \{ abJ_0 + 2J_0 \} - aJ_1 \} \left( \frac{b+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ \beta \{ aJ_1 + 2J_0 \} - bJ_0 \} \left( \frac{b+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ \beta \{ bJ_0 + 2J_0 \} - aJ_1 \} \left( \frac{b+2}{4} \right)^{-n-1} x^{2n+1} \\ -\sum_{n=0}^{\infty} \{ \beta \{ bJ_0 + 2J_0 \} - aJ_1 \} \left( \frac{b+2}{4} \right)^{-n-1} x^{2n+1} \end{pmatrix}.$$
By using the identity
\[ \alpha \]
The even part of \( J \) which can be simplified as
\[ \beta + 2 = \]
which can be simplified as
From the identity that \((\alpha + 2)(\beta + 2) = 4\), we have
\[ \frac{1}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} 2\beta (\beta + 2)^n [aJ_1 - 2J_0 - abJ_0] \\ - (\alpha + 2)^n [2aJ_1 - 2abJ_0] \\ 2a (\alpha + 2)^n [abJ_0 + 2J_0 - aJ_1] \\ - (\alpha + 2)^n [2abJ_0 - 2aJ_1] \end{array} \right\} x^{2n} \]
By using the identity \( \alpha + 2 = \frac{\alpha^2}{\alpha} \), we get
\[ 4(\alpha - \beta) \sum_{n=0}^{\infty} \left\{ \begin{array}{c} \frac{n+1}{ab} (aJ_1 - 2J_0 - abJ_0) \\ + J_0 \left\{ \frac{a^{2n+2} - \beta^{2n+2}}{(\beta - \alpha)} \right\} \end{array} \right\} x^{2n} \]
Also, making use of the identity \( ab = \alpha + \beta \) gives
\[ \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} aJ_1 - 2J_0 - abJ_0 \left\{ \frac{a^{2n} - \beta^{2n}}{(\beta - \alpha)} \right\} \\ + J_0 \left\{ \frac{a^{2n+2} - \beta^{2n+2}}{(\beta - \alpha)} \right\} \end{array} \right\} x^{2n} \]
In the same way, the odd part of \( J(x) \) is obtained as
\[ \frac{-4^{n+1}}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} 2a [bJ_0 - J_1] + 4bJ_0 \left\{ \frac{4^{n+1}}{(\alpha + 2)^{n+1} (\beta + 2)^{n+1}} \right\} \\ + 2\beta [J_1 - bJ_0] - 4bJ_0 \left\{ \frac{4^{n+1}}{(\alpha + 2)^{n+1} (\beta + 2)^{n+1}} \right\} \end{array} \right\} x^{2n+1} \]
which can be simplified as
\[ \frac{-4^{n+1}}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} (\beta + 2)^{n+1} \left\{ 2a [bJ_0 - J_1] + 4bJ_0 \right\} \\ + (\alpha + 2)^{n+1} \left\{ 2\beta [J_1 - bJ_0] - 4bJ_0 \right\} \end{array} \right\} x^{2n+1} \]
\( \beta + 2 = -\frac{\beta}{\alpha} \) simplifies the following result
\[ \frac{4}{4(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} [bJ_0 - J_1] \left\{ \alpha (\alpha + 2)^n - \beta (\beta + 2)^n \right\} + bJ_0 \left\{ (\beta + 2)^{n+1} - (\alpha + 2)^{n+1} \right\} \end{array} \right\} x^{2n+1} \]
With \( (\alpha + 2) = \frac{\alpha^2}{\alpha} \), we simplify the above expression as
\[ \frac{-1}{(\alpha - \beta)} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} \frac{n+1}{ab} \frac{ab [bJ_0 - J_1] \left( \alpha^{2n+1} - \beta^{2n+1} \right)}{-bJ_0 (\alpha^{2n+2} - \beta^{2n+2})} \end{array} \right\} x^{2n+1} \]
This can be further expanded and simplified as
\[
\sum_{n=0}^{\infty} \left( (J_1 - bJ_0) \left\{ \alpha^{2n+1} - \beta^{2n+1} \right\} + bJ_0 \left\{ \alpha^{2n+2} - \beta^{2n+2} \right\} \right) x^{2n+1}.
\]

Now the even and the odd expressions can be condensed by means of the parity function as
\[
J(x) = \sum_{n=0}^{\infty} \left\{ (J_1 - bJ_0)^{\varepsilon(n)} (aJ_1 - 2J_0 - abJ_0)^{1-\varepsilon(n)} \left\{ \frac{\alpha^n - \beta^n}{(ab)^n (\alpha - \beta)} \right\} \right\} x^n.
\]

Therefore compared with \( J(x) = \sum_{n=1}^{\infty} J_n x^n \), the Binet formula for the bi-periodic Jacobsthal matrix sequence is computed as
\[
J_n = A (\alpha^n - \beta^n) + B \left( \alpha^{2\lfloor \frac{n}{2} \rfloor + 1} - \beta^{2\lfloor \frac{n}{2} \rfloor + 1} \right)
\]
where
\[
A = \frac{(J_1 - bJ_0)^{\varepsilon(n)} (aJ_1 - 2J_0 - abJ_0)^{1-\varepsilon(n)} (ab)^{\lfloor \frac{n}{2} \rfloor} (\alpha - \beta)}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)} \quad \text{and} \quad B = \frac{b^{\varepsilon(n)} J_0}{(ab)^{\lfloor \frac{n}{2} \rfloor + 1} (\alpha - \beta)}.
\]

\[\square\]

**Theorem 2.6.** (SUMMATION FORMULAS) For any positive integer \( n \), and \( ab \neq 1 \), the sum of the first \( n \) terms of the bi-periodic Jacobsthal matrix sequence is computed as
\[
\sum_{k=0}^{n-1} J_k = \frac{J_n (1 - a^{n\varepsilon(n)}b^{1-\varepsilon(n)}) + 2J_{n-1} (1 - a^{1-\varepsilon(n)}b^{\varepsilon(n)})}{1 - ab}.
\]

Proof. If \( n \) is even, it’s obtained that by using Binet formula

\[
\sum_{k=0}^{n-1} J_k = \sum_{k=0}^{\frac{n-2}{2}} J_{2k} + \sum_{k=0}^{\frac{n-2}{2}} J_{2k+1}
\]

\[
= \sum_{k=0}^{\frac{n-2}{2}} \frac{aJ_1 - 2J_0 - abJ_0}{(ab)^{k}} \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{J_0}{(ab)^{k+1}} \frac{\alpha^{2k+2} - \beta^{2k+2}}{\alpha - \beta} + \sum_{k=0}^{\frac{n-2}{2}} \frac{bJ_0}{(ab)^{k+1}} \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta}.
\]

By using the sum of geometric series, it’s computed that

\[
\sum_{k=0}^{\frac{n-1}{2}} J_k = \frac{aJ_1 - 2J_0 - abJ_0}{(ab)^{\lfloor \frac{n}{2} \rfloor - 1} (\alpha - \beta)} \left[ \frac{\alpha^n - (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^n - (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right]
\]

\[
+ \frac{J_0}{(ab)^{\frac{n}{2}} (\alpha - \beta)} \left[ \frac{\alpha^{n+2} - \alpha^2 (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+2} - \beta^2 (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right]
\]

\[
+ \frac{bJ_0}{(ab)^{\frac{n}{2}} (\alpha - \beta)} \left[ \frac{\alpha^{n+1} - \alpha (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+1} - \beta (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right]
\]

\[
+ \frac{1}{(ab)^{\frac{n}{2}} (\alpha - \beta)} \left[ \frac{\alpha^{n+2} - \alpha^2 (ab)^{\frac{n}{2}}}{(\alpha^2 - ab)} - \frac{\beta^{n+2} - \beta^2 (ab)^{\frac{n}{2}}}{(\beta^2 - ab)} \right].
\]
After some algebraic operations, the following result is evaluated

\[
\begin{align*}
&= \frac{(aJ_1 - 2J_0 - abJ_0)}{(ab)^{2+1}(\alpha - \beta)(1 - ab)} \left[ 4a^2b^2(\alpha^{n-2} - \beta^{n-2}) - ab(\alpha^n - \beta^n) + (ab)^2(\alpha^2 - \beta^2) \right] \\
&\quad + \frac{J_0}{(ab)^{2+2}(\alpha - \beta)(1 - ab)} \left[ 4a^2b^2(\alpha^n - \beta^n) - ab(\alpha^{n+2} - \beta^{n+2}) \right] \\
&\quad + \frac{J_1 - bJ_0}{(ab)^{2+1}(\alpha - \beta)(1 - ab)} \left[ 4a^2b^2(\alpha^{n-1} - \beta^{n-1}) - ab(\alpha^{n+1} - \beta^{n+1}) \right] \\
&\quad + \frac{bJ_0}{(ab)^{2+2}(\alpha - \beta)(1 - ab)} \left[ 4a^2b^2(\alpha^n - \beta^n) - ab(\alpha^{n+2} - \beta^{n+2}) \right] \\
&= \frac{-n_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a - 1) + J_0(2b - ab - 1)}{1 - ab} \\
&= \frac{-n_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a - 1) + J_0(2b - ab - 1)}{1 - ab} \\
&= \frac{J_n(1 - b) + J_{n-1}(2 - 2a) + J_1(a - 1) + J_0(2b - ab - 1)}{1 - ab} 
\end{align*}
\]

Similarly if \( n \) is odd, we obtain

\[
\sum_{k=0}^{\frac{n-1}{2}} J_k = \sum_{k=0}^{\frac{n-1}{2}} J_{2k} + \sum_{k=0}^{\frac{n-3}{2}} J_{2k+1} \\
= \frac{-n_{n+1} - J_n + 4J_{n-1} + 4J_{n-2} + J_1(a - 1) + J_0(2b - ab - 1)}{1 - ab} \\
= \frac{J_n(1 - a) + J_{n-1}(2 - 2b) + J_1(a - 1) + J_0(2b - ab - 1)}{1 - ab} 
\]

Hence putting the two results together by means of the parity function gives

\[
\sum_{k=0}^{\frac{n-1}{2}} J_k = \frac{J_n(1 - a^{\xi(n)}b^{\frac{n-1}{2}}) + 2J_{n-1}(1 - a^{\frac{1-\xi(n)}{2}}b^{\frac{n-1}{2}})}{1 - ab}. 
\]

\[\square\]

**Theorem 2.7.** For any positive integer \( n \), we have

\[
\sum_{k=0}^{n-1} J_k x^k = \frac{J_n(2 - x - a^{\xi(n)}b^{1-\xi(n)}x) + 2J_{n-1}(2 - a^{\frac{1-\xi(n)}{2}}b^{\frac{n-1}{2}})x^2}{x^2 - (ab + 4)x + 4} + \frac{x^2(J_1 - bJ_0) + x(-2J_1 + 3bJ_0 + aJ_1 - J_0 - abJ_0)}{x^2 - (ab + 4)x + 4}. 
\]

**Proof.** The proof is obtained is a similar fashion as the above theorem. \[\square\]
3 Conclusions

In this study a new generalized number sequences called bi-periodic Jacobsthal sequence is carried out to matrix theory. Therefore we define bi-periodic Jacobsthal matrix sequence whose entries are bi-periodic Jacobsthal numbers. Then the generating function, Binet formula and some basic properties are investigated. New interesting properties and polynomial matrix sequences can also be examined.

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Competing Interests

Authors have declared that no competing interests exist.

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