COMBINATORIAL MODELS FOR SCHUBERT POLYNOMIALS

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Abstract. Schubert polynomials are a basis for the polynomial ring that represent Schubert classes for the flag manifold. In this paper, we introduce and develop several new combinatorial models for Schubert polynomials that relate them to other known bases including key polynomials and fundamental slide polynomials. We unify these and existing models by giving simple bijections between the combinatorial objects indexing each. In particular, we give a simple bijective proof that the balanced tableaux of Edelman and Greene enumerate reduced expressions and a direct combinatorial proof of Kohnert’s algorithm for computing Schubert polynomials. Further, we generalize the insertion algorithm of Edelman and Greene to give a bijection between reduced expressions and pairs of tableaux of the same key diagram shape and use this to give a simple formula, directly in terms of reduced expressions, for the key polynomial expansion of a Schubert polynomial.

1. Introduction

Schubert [Sch79] began asking questions in enumerative geometry, such as how many lines in space meet four given lines. He answered some of these questions using his principle of conservation of number to reduce to a situation that was easier to solve. Hilbert [Hil02], in his 15th problem, called for this principle to be made mathematically rigorous. The resolution to this is the theory of cohomology rings for manifolds. Varying the manifold in a continuous way leaves the cohomology class unchanged, and so the cohomology class is precisely what is preserved in the sense of Schubert.

For the flag manifold parameterizing complete flags of subspaces in complex affine n-space, Borel [Bor53] showed that the cohomology ring is naturally identified with the quotient of the polynomial ring in n variables by the ideal generated by positive degree elementary symmetric polynomials. The Schubert cell decomposition studied by Bernstein Gel’fand and Gel’fand [BGG73] and Demazure [Dem74] gives a geometrically important basis of the cohomology ring which is naturally identified with a distinguished linear basis indexed by permutations. This allows one to compute intersection numbers, such as the number of lines incident to four given lines, by carrying out a product in the cohomology ring and considering the Schubert decomposition of the result.

Schubert polynomials were introduced by Lascoux and Schützenberger [LS82] as polynomial representatives of Schubert classes for the cohomology of the flag manifold with nice algebraic and combinatorial properties. Schubert polynomials were originally defined using divided difference operators that act on a certain monomial associated to the long permutation according to a reduced expression for the given permutation. The geometric significance of Schubert polynomials was first established by Fulton [Ful92] who made connections between the divided difference operators and modern intersection theory. Knutson and Miller [KM05] found further geometric insights connecting the combinatorics of Schubert polynomials to the geometry of the flag manifold.

Surprisingly, at least from their definition, Schubert polynomials form an integral basis for the full polynomial ring, and their structure constants precisely give the Schubert cell decomposition for the corresponding product of Schubert classes. Therefore they give a way to avoid working modulo the ideal of symmetric polynomials in order to compute intersection numbers. In order to realize the advantage that Schubert polynomials present, we seek to find nice combinatorial models for Schubert polynomials that facilitate computations.
Billey, Jockusch, and Stanley \[BJS93\] gave a combinatorial definition for the monomial expansion of Schubert polynomials in terms of compatible sequences for reduced expressions. Assaf and Searles \[AS17\] refined this to give a combinatorial model for the expansion in terms of fundamental slide polynomials. The significance of the latter is that they have nonnegative structure constants. Demazure \[Dem74\] studied characters for certain general linear group modules that arise when considering generalized Schubert cells of the flag manifold. These characters coincide with key polynomials studied combinatorially by Lascoux and Schützenberger \[LS90\], who proved that Schubert polynomials expand non-negatively in the key polynomial basis. While geometrically significant, the structure constants for key polynomials are not nonnegative.

In this paper, we develop the combinatorics of Schubert polynomials in terms of monomials, fundamental slide polynomials, and key polynomials by presenting graphical representations for reduced expressions. We use these new models to connect together, via explicit bijections, various other models for Schubert polynomials that have combinatorial or computational significance.

This paper is structured as follows. In §2 we recall the combinatorial formula for Schubert polynomials of Billey, Jockusch, and Stanley \[BJS93\]. We re-express this in terms of the fundamental slide basis of Assaf and Searles \[AS17\] by developing a notion of weak descent compositions for reduced expressions. Therefore we have the following expansions,

\[
S_w = \sum_{\rho \in R(w)} x_{\alpha_1} \cdots x_{\alpha_{\ell(w)}} = \sum_{\rho \in R(w)} \mathcal{F}_{\text{des}}(\rho).
\]

We further develop the combinatorics of reduced expressions by realizing a ranked poset structure with cover relations given by the Coxeter relations for simple transpositions of the symmetric group. In fact, the structure is that of a join semi-lattice and can be embedded into weak Bruhat order, and we provide a metric on reduced expressions for a given permutation analogous to the Coxeter length for permutations. This framework facilitates the bijective proofs that follow.

In §3, we give a recipe for creating a reduced diagram from a reduced expression that allows one to read off the descent composition directly from the numbers of cells per row. We call this combinatorial model a diagram model because the cells move as opposed to a tableau model in which the entries within the cells change. In §4, we relabel the cells and push them back up to the Rothe diagram shape, recovering the balanced tableaux of Edelman and Greene \[EG87\] later generalized by Fomin, Greene, Reiner, and Shimozono \[FGRS97\]. Our simple bijection between certain reduced diagrams and balanced tableaux also proves that balanced tableaux are in bijection with reduced expressions. Thus the Schubert polynomial for \(w\) has the following nonnegative combinatorial expansions,

\[
S_w = \sum_{D \in RD(w)} x^{\text{wt}(D)} = \sum_{D \in QRD(w)} \mathcal{F}^{\text{wt}(D)} = \sum_{R \in SBT(w)} \mathcal{F}_{\text{des}}(R) = \sum_{R \in SSBT(w)} x^{\text{wt}(R)},
\]

**Figure 1.** Combinatorial models for the reduced expression \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\).
where RD denotes reduced diagrams, QRD denotes quasi-Yamanouchi reduced diagrams, SBT denotes standard balanced tableaux, and SSBT denotes semi-standard balanced tableaux.

In [15] we recall the insertion algorithm of Edelman and Greene [EG87] that takes a reduced expression and generates a pair of semi-standard Young tableaux of the same shape where the insertion tableau is increasing and the recording tableau is standard. Edelman and Greene used this to establish the Schur positivity of Stanley’s symmetric functions [Sta84]. This algorithm generalizes the insertion algorithm of Schensted [Sch61], based on work of Robinson [Rob38] and later generalized by Knuth [Knu70], that takes a permutation and generates a pair of standard Young tableaux of the same shape. We show that Edelman–Greene insertion has a natural generalization, which we call weak insertion, that takes a reduced expression and generates a pair of tableaux of the same key diagram shape where the insertion tableau is increasing and the recording tableau is a standard key tableau, introduced by Assaf [Ass17] to study key polynomials. We show that weak insertion gives rise to a simple Yamanouchi condition on reduced diagrams that establishes the following nonnegative combinatorial expansion of the Schubert polynomial for w as
\[
\mathfrak{S}_w = \sum_{D \in \text{YRD}(w)} \kappa_{\text{wt}(D)}.
\]

This expansion avoids the complicated left-nil key computation of Lascoux and Schützenberger [LS90] to get the key expansion of a Schubert polynomial.

Finally, in [11] we recall Kohnert’s algorithm [Koh91] for generating a key polynomial based on pushing down cells of the key diagram. Kohnert asserted that this rule also generates Schubert polynomials, and while there are claims at proofs of this in the literature due to Winkel [Win99, Win02], none is widely accepted given the intricate and opaque nature of the arguments. We use our new simple expansion of a Schubert polynomial in terms of key polynomials to give a simple, direct, bijective proof of this rule.

2. Reduced Expressions

A reduced expression is a sequence \( \rho = (i_k, \ldots, i_1) \) such that the permutation \( s_{i_k} \cdots s_{i_1} \) has \( k \) inversions, where \( s_i \) is the simple transposition that interchanges \( i \) and \( i + 1 \). Let \( R(w) \) denote the set of reduced expressions for \( w \). For example, the elements of \( R(42153) \) are shown in Figure 2.

\[
(4,2,1,2,3) \quad (4,1,2,1,3) \quad (4,1,2,3,1) \quad (2,4,1,2,3) \quad (2,1,4,2,3) \quad (2,1,2,4,3)
\]
\[
(1,4,2,3,1) \quad (1,2,4,3,1) \quad (1,4,2,1,3) \quad (1,2,4,1,3) \quad (1,2,1,4,3)
\]

**Figure 2.** The set of reduced expressions for 42153.

For \( \rho \in R(w) \), say that a strong composition \( \alpha \) is \( \rho \)-compatible if \( \alpha \) is weakly increasing with \( \alpha_j < \alpha_{j+1} \) whenever \( \rho_j < \rho_{j+1} \) and \( \alpha_j \leq \rho_j \).

For example, there are two compatible sequences for \( (4,2,1,2,3) \), namely \((1,1,1,2,4)\) and \((1,1,1,2,3)\), and there is one compatible sequence for \( (2,4,1,2,3) \), namely \((1,1,1,2,2)\). None of the other reduced expressions for 42153 has a compatible sequence.

**Definition 2.1** ([BJS93]). The Schubert polynomial \( \mathfrak{S}_w \) is given by
\[
\mathfrak{S}_w = \sum_{\rho \in R(w)} \sum_{\alpha \rho \text{-compatible}} x_{\alpha_1} \cdots x_{\alpha_{\ell(w)}},
\]
where the sum is over compatible sequences \( \alpha \) for reduced expressions \( \rho \).

For example, we can compute \( \mathfrak{S}_{42153} = x_1^3 x_2 x_4 + x_1^3 x_2 x_3 + x_1^3 x_2^2 \).
Let $1^m \times w$ denote the permutation obtained by adding $m$ to all values of $w$ in one-line notation and pre-pending $1, 2, \ldots, m$. Note that the reduced expressions for $1^m \times w$ are simply those for $w$ with each index increased by $m$. To make the example slightly more interesting, consider $1 \times 42153 = 153264$. Then seven reduced expressions contribute a total of 26 monomials to the Schubert polynomial, giving

$$
\mathcal{S}_{153264} = x_1^3 x_2^2 + 2x_1^3 x_2 x_3 + x_1^3 x_2 x_4 + x_1^3 x_2 x_5 + x_1^3 x_3 x_4 + x_1^3 x_3 x_5 + x_1^2 x_2^3 + 2x_1^2 x_2^3 x_3 + x_1^2 x_2 x_3 x_5 + 2x_1 x_2^3 x_3 + x_1 x_2^3 x_4 + x_1 x_2^3 x_5 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5 + x_1 x_2 x_4 x_5 + x_1 x_2 x_5 + x_1 x_3^2 x_5 + x_1 x_3^2 x_5 + x_1 x_3 x_4 + x_1 x_3 x_5 + x_1 x_4^2 x_5 + x_1 x_4^2 x_5 + x_1 x_4 x_5 + x_1 x_5^2 x_5 + x_1 x_5 x_5 + x_1 x_5 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6.
$$

We harness the power of the *fundamental slide polynomials* of Assaf and Searles [AS17] to re-express Schubert polynomials as the generating function for reduced expressions. Given a weak composition $a$, let $\text{flat}(a)$ denote the strong composition obtained by removing all zero parts.

**Definition 2.2 (AS17).** For a weak composition $a$ of length $n$, define the *fundamental slide polynomial* $\mathcal{F}_a = \mathcal{F}_a(x_1, \ldots, x_n)$ by

$$
\mathcal{F}_a = \sum_{b \geq a, \text{flat}(b) \text{ refines flat}(a)} x_1^{b_1} \cdots x_n^{b_n},
$$

where $b \geq a$ means $b_1 + \cdots + b_k \geq a_1 + \cdots + a_k$ for all $k = 1, \ldots, n$.

To facilitate virtual objects as defined below, we extend notation and set

$$
\mathcal{F}_\emptyset = 0.
$$

The *run decomposition* of a reduced expression $\rho$ partitions $\rho$ into increasing sequences of maximal length. We denote the run decomposition by $(\rho^{(i)} | \cdots | \rho^{(1)})$. For example, the run decomposition of $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$, a reduced expression for $41758236$, is $(563457314236)$.

**Definition 2.3.** For a reduced expression $\rho$ with run decomposition $(\rho^{(k)} | \cdots | \rho^{(1)})$, set $r_k = \rho^{(k)}_1$ and, for $i < k$, set $r_i = \min(\rho^{(i)}_1, r_{i+1} - 1)$. Define the *weak descent composition* of $\rho$, denoted by $\text{des}(\rho)$, by $\text{des}(\rho)_r = |\rho^{(i)}|$ and all other parts are zero if all $r_i > 0$ and $\text{des}(\rho) = \emptyset$ otherwise.

$$
(5, 3, 2, 3, 4) \quad (5, 2, 3, 2, 4) \quad (5, 2, 3, 4, 2) \quad (3, 5, 2, 3, 4) \quad (3, 2, 5, 3, 4) \quad (3, 2, 3, 5, 4) \quad (2, 3, 5, 2, 4)
$$

**Figure 3.** The set of non-virtual reduced expressions for 153264.

We say that $\rho$ is *virtual* if $\text{des}(\rho) = \emptyset$. For example, $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$ is virtual since $r_1 = 0$. Let $0^m \times a$ denote the weak composition obtained by pre-pending $m$ zeros to $a$. Then for $\rho \in R(w)$ non-virtual, the corresponding reduced expression for $R(1^m \times w)$ will have weak descent composition $0^m \times \text{des}(\rho)$. For example, the weak descent composition for $(6, 7, 4, 5, 6, 8, 4, 2, 5, 3, 4, 7)$, a reduced expression for $152869347$, is $(3, 2, 1, 4, 0, 2)$. Note the reversal from the run decomposition to the descent composition.

**Theorem 2.4.** For $w$ any permutation, we have

$$
\mathcal{S}_w = \sum_{\rho \in R(w)} \mathcal{F}_{\text{des}(\rho)},
$$

where the sum may be taken over non-virtual reduced expressions $\rho$. 

Proof. Map each compatible sequence $\alpha$ to the weak composition $a$ whose $i$th part is the number of $j$ such that $\alpha_j = i$. For example, the compatible sequence $(1, 1, 1, 2, 4)$ for the reduced expression $(4, 2, 1, 2, 3)$ maps to the weak composition $(3, 1, 0, 1)$. The greedy choice of a compatible sequence takes each $\alpha_i$ as large as possible. Under the correspondence, this precisely becomes $\text{des}(\rho)$ since the condition $\alpha_j < \alpha_{j+1}$ whenever $\rho_j < \rho_{j+1}$ corresponds precisely to taking $r_i < r_{i+1}$ and the conditions $\alpha_j \leq \rho_j$ and $\alpha_j \leq \alpha_{j+1}$ correspond precisely to $r_i \leq \rho_1^{(i)}$. Furthermore, $\text{des}(\rho) = \emptyset$ precisely when $\rho$ admits no compatible expressions.

Given a compatible sequence for $\rho$, we may decrement parts provided we maintain $\alpha_j < \alpha_{j+1}$ whenever $\rho_j < \rho_{j+1}$, and this corresponds precisely to sliding parts of the weak composition left, possibly breaking them into refined pieces. Every compatible sequence may be obtained from the greedy one in this way, just as every term in the monomial expansion of the fundamental slide polynomial arises in the analogous way. $\Box$

For example, from Figure 2 the two non-virtual reduced expressions give

$$G_{12153} = \mathfrak{S}_{(3,1,0,1)} + \mathfrak{S}_{(3,2,0,0)},$$

a slight savings over the monomial expansion. Bumping this example up, from Figure 3 we have

$$G_{153264} = \mathfrak{S}_{(0,3,1,0,1)} + \mathfrak{S}_{(2,2,0,0,1)} + \mathfrak{S}_{(1,3,0,0,1)} + \mathfrak{S}_{(0,3,2,0,0)} + \mathfrak{S}_{(2,2,1,0,0)} + \mathfrak{S}_{(1,3,1,0,0)} + \mathfrak{S}_{(2,3,0,0,0)},$$

which is considerably more compact than the 26-term monomial expansion. Furthermore, this paradigm shift to fundamental slide generating functions facilitates development of many more combinatorial models for Schubert polynomials, each of which has interesting applications.

Before moving to these models, we present the basic tool for the inductive proofs to follow: the rigid structure on the set of reduced expressions given by the defining relations for the symmetric group. The base case will be the super-Yamanouchi elements.

Definition 2.5. A reduced expression $\rho$ with run decomposition $(\rho^{(k)}|\cdots|\rho^{(1)})$ is super-Yamanouchi if each $\rho^{(i)}$ is an interval and $\rho_1^{(k)} > \cdots > \rho_1^{(1)}$.

For example, the unique super-Yamanouchi reduced expression for 41758236 decomposes as $(567|45|3456|123)$; see Figure 4 for its construction.

$$41758236 \rightarrow 417\overset{5}{\circ}\overset{236}{\circ} \rightarrow 417\overset{5}{\circ}236\overset{68}{\circ} \rightarrow 41\overset{7}{\circ}2356\overset{8}{\circ} \rightarrow \overset{4}{\circ}123\overset{5678}{\circ} \rightarrow \overset{123}{\circ} \rightarrow 12345678$$

Figure 4. Constructing the super-Yamanouchi element of $R(41758236)$.

Proposition 2.6. Given a permutation $w$, there exists a unique super-Yamanouchi reduced expression $\pi_w$. Moreover if $\rho$ is any other reduced expression for $w$, then we have $\text{des}(\rho) > \text{des}(\pi_w)$.

Proof. We construct $\pi = \pi_w$ as follows. Find the largest index $i$ for which $w_i > w_{i+1}$, and let $j > i$ be the smallest index such that $w_i < w_j$. Then set $\rho^{(k)} = i \cdots (j - 1)$. Repeat, necessarily with an index $h < i$, until the permutation is sorted to the identity. Since each step removes one inversion, this will give a reduced expression for $w$, and it satisfies the super-Yamanouchi conditions.

Given any other reduced expression $\rho$, there must be some largest block, say $j$, of the run decompositions of $\pi$ and $\rho$ diverge. Then $\text{des}(\pi)_i = \text{des}(\rho)_i$ for $i > r_j$, where $r_j$ is with respect to $\pi$. Then either $\rho_1^{(j)} < \pi_1^{(j)}$, since the construction for $\pi$ takes the latest descent, or $|\rho^{(j)}| < |\pi^{(j)}|$ since the construction for $\pi$ takes every possible descent thereafter. Thus $\text{des}(\pi)_j > \text{des}(\rho)_j$, or, equivalently, $\text{des}(\rho)_1 + \cdots + \text{des}(\rho)_{r_j-1} > \text{des}(\pi)_1 + \cdots + \text{des}(\pi)_{r_j-1}$. Uniqueness now follows. $\Box$

Definition 2.7. Given a permutation $w$ and an index $1 \leq i < \text{inv}(w)$, define $s_i$ on $R(w)$ by swapping $\rho_i$ and $\rho_{i+1}$ whenever $|\rho_i - \rho_{i+1}| > 1$ and the identity otherwise.
Definition 2.8. Given a permutation \( w \) and an index \( 1 < i < \text{inv}(w) \), define \( b_i \) on \( R(w) \) by braiding \( \rho_{i-1}\rho_i\rho_{i+1} \) to \( \rho_i\rho_{i±1}\rho_i \) whenever \( \rho_{i-1} = \rho_{i+1} = \rho_i ± 1 \) and the identity otherwise.

For examples of swaps and braids on reduced expressions, see Figure 5. Note that the left reduced expression is super-Yamanouchi, and the right is our running example.

\[
56745345123 \xleftarrow{b_6} 56734541236 \xleftarrow{b_8} 567345341236 \xleftarrow{b_6} 567345341236 \xleftarrow{b_8} 563457314236
\]

**Figure 5.** Examples of swaps and braids on \( R(41758236) \).

Proposition 2.9. The maps \( s_i \) and \( b_i \) are well-defined involutions on \( R(w) \).

Proof. Recall the defining relations on the simple transpositions \( s_i \) that generate the symmetric group: \( s_i \) corresponds to the commutativity relation \( s_k s_j = s_j s_k \) when \( |k-j| > 1 \) and \( b_i \) corresponds to the braid relation \( s_j^{-1} s_j s_{j-1} = s_j s_{j-1} s_j \). The maps only act non-trivially when the conditions for each relation are met, so we have \( s_i(\rho), b_i(\rho) \in R(w) \), and they are easily seen to be involutions.

Since the swaps and braids correspond to the defining relations on the simple transpositions \( s_i \), any two reduced expressions for \( w \) are equal as products in the symmetric group, and so can be transformed into one another by a sequence of swaps and braids. The following definition measures how many swaps and braids are needed to get from a given reduced expression to the super-Yamanouchi one.

Definition 2.10. Given \( \rho \in R(w) \), define the inversion number of \( \rho \) by

\[
\text{inv}(\rho) = \text{inv}(v) - \sum_i (\pi_i - \rho_i),
\]

where \( \pi \in R(w) \) is super-Yamanouchi and \( v = v(\rho) \) is the permutation of \( \rho \) constructed as follows: for \( i \) from 1 to \( \text{inv}(w) \), set \( k = \pi_i \) and for \( j \) from 1 to \( \text{inv}(w) \) if \( \rho_j \) is already paired then increment \( j \) to \( j+1 \); else if \( \rho_j = k \) then pair \( \rho_j \) and \( \pi_i \); else if \( \rho_j = k-1 \) then decrement \( k \) to \( k-1 \) and increment \( j \) to \( j+1 \); other wise increment \( j \) to \( j+1 \). Set \( v_i = j \) whenever \( \pi_i \) is paired with \( \rho_j \).

For example, \( v(56|3457|3|14|236) = 23518904671112 \) as illustrated in Figure 6. Therefore \( \text{inv}(56|3457|3|14|236) = 13 - 2 = 11 \). Observe from Figure 5 that

\[
(56|3457|3|14|236) = s_7 s_5 s_9 s_4 s_6 b_8 s_6 s_7 s_1 s_2 s_3 (567|45|3456|123),
\]

which is a sequence of 11 involutions, exactly 2 of which are braids.

**Figure 6.** Constructing the permutation for \( (56|3457|3|14|236) \).

Theorem 2.11. For \( \rho \in R(w) \), \( \text{inv}(\rho) \) is a well-defined nonnegative integer. Moreover, there exists a sequence \( f = f_{\text{inv}(\rho)} \cdots f_1 \) of swaps and braids, i.e. \( f_j = s_j \) or \( b_j \), such that \( f(\rho) \) is super-Yamanouchi, and for any sequence \( g = g_m \cdots g_1 \) of swaps and braids such that \( g(\rho) \) is super-Yamanouchi, we have \( m \geq \text{inv}(\rho) \).
Proof. We claim that the theorem holds for $\rho$ if and only if it holds for $s_i(\rho)$. This is vacuously true if $s_i$ acts trivially on $\rho$, so assume the action is nontrivial. If $\rho$ has permutation $v$, then $s_i(\rho)$ will have permutation $s_iv$, and, since the letters of $\rho$ and $s_i(\rho)$ are the same, we have

$$\text{inv}(s_i\rho) = \text{inv}(s_iv) - \sum_j (\pi_j - (s_i\rho)_j) = \text{inv}(v) \pm 1 - \sum_j (\pi_j - \rho_j) = \text{inv}(\rho) \pm 1,$$

and, furthermore, $\text{inv}(s_i\rho) = \text{inv}(\rho) + 1$ precisely when $i$ is left of $i+1$ in $v$.

Next we claim that the theorem holds for $\rho$ if and only if it holds for $b_i(\rho)$. If $b_i$ acts trivially on $\rho$, the claim is vacuously true, so assume the action is nontrivial. If $\rho$ has permutation $v$, then $b_i(\rho)$ will have permutation $s_i\rho s_{i-1}v$ or $s_{i-1}s_i\rho v$, the former when $\rho_{i\pm 1} = \rho_i + 1$ and the latter when $\rho_{i\pm 1} = \rho_i - 1$. Assuming the former, we have

$$\text{inv}(b_i\rho) = \text{inv}(s_i\rho s_{i-1}v) = \sum_j (\pi_j - (b_i\rho)_j) = \text{inv}(v) + 2 - (\sum_j (\pi_j - \rho_j) + 1) = \text{inv}(\rho) + 1,$$

and, by the same computation, $\text{inv}(b_i\rho) = \text{inv}(\rho) - 1$ in the latter case.

Recall from earlier that any two reduced expressions for $w$ can be transformed into one another by a sequence of swaps and braids. Let $m$ be the minimum number of braids and swaps needed to transform $\rho$ into the super-Yamanouchi reduced expression. If $m = 0$, then $\rho$ is super-Yamanouchi, in which case the permutation for $\rho$ is the identity and $\text{inv}(\rho) = 0$, so the theorem holds. Assume, for induction, that the theorem holds for any $n < m$, and suppose $\rho = f_m \cdots f_1 \pi$, where $\pi$ is super-Yamanouchi and $f_j$ is $s_i$ or $b_i$ for some $i$. By induction, the result holds for $f_{m-1} \cdots f_1 \pi = f_m \rho$, and so, by the claims, it holds for $\rho$ as well. \hfill \square

Corollary 2.12. For $w$ a permutation, the partial order on $R(w)$ given by the transitive closure of covering relations $\rho < s_i\rho$ if $\text{inv}(s_i\rho) = \text{inv}(\rho) + 1$ and $\rho < b_i\rho$ if $\text{inv}(b_i\rho) = \text{inv}(\rho) + 1$ makes $R(w)$ into a ranked partially ordered set where for any $\sigma, \tau \in R(w)$, $\sigma$ and $\tau$ have a least upper bound.

Remark 2.13. The inversion number can be generalized to give a metric on reduced expressions for a permutation. For $\sigma, \tau \in R(w)$, define $v(\sigma, \tau)$ to be the permutation constructed as described in Definition 2.10 with $\sigma$ in place of $\pi$ and $\tau$ in place of $\rho$. Then set

$$(2.6) \quad \text{inv}(\sigma, \tau) = \text{inv}(v(\sigma, \tau)) - \left| \sum_i (\sigma_i - \tau_i) \right|.$$

Clearly this reduces to $\text{inv}(\tau)$ whenever $\sigma$ is taken to be the super-Yamanouchi reduced expression. This inversion number denotes the minimum number of relations needed to change $\sigma$ to $\tau$, and the offset number $\left| \sum_i (\sigma_i - \tau_i) \right|$ tracks the number of braid relations used.

3. Reduced diagrams

A diagram is a finite collection of cells in $\mathbb{Z} \times \mathbb{Z}^+$. The weight of a diagram $D$, denoted by $\text{wt}(D)$, is the weak composition whose $i$th part is the number of cells in row $i$ of $D$ when all cells have positive row index and $\emptyset$ otherwise. A diagram $D$ is virtual if $\text{wt}(D) = \emptyset$.

![Figure 7](image_url)

Figure 7. Two diagrams with weight $(3,0,4,2,3)$.

For example, Figure 7 gives two diagrams with the same weight, where the left one if left-justified.
Definition 3.1. Given a labeled diagram \( D \) with strictly increasing rows, define the alignment of \( D \) to be the diagram obtained as follows. Group cells together: begin with highest (then smallest, if tied) ungrouped entry, say \( i \), search the next row down for \( i - 1 \) in which case you take it and continue, otherwise search for \( i \) in which case you end the group, otherwise continue to the next row down. Move cells right: maintaining the order within rows, push cells to the right until all entries in each group lie in the same column, where if two groups have a common value then the lower instance of the value is further right, and if two groups have no common values then the group with the larger entries is further right.

For an example of alignment, see Figure 8.

![Figure 8](image)

**Figure 8.** The diagram for the reduced expression \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\).

Proposition 3.2. Aligning a labeled diagram with strictly increasing rows is well-defined.

Proof. We begin by showing that groups are disjoint. Suppose two groups, say \( i_r \) in row \( r \) and \( j_s \) in row \( s \) both wish to add \( x \). Then we must have \( i_r = x + 1 = j_s \). Moreover, no row between \( r \) and the row of \( x \) nor between \( s \) and the row of \( x \) can have entry equal to \( x \) (lest that \( x \) would have been added to the group(s)) nor \( x + 1 \) (lest the group(s) would have terminated). Therefore row \( r = s \) has two equal entries, a contradiction to the increasing rows assumption. Therefore each entry of the diagram belongs to a unique group.

We next claim that groups do not cross: if \( i_r < j_r \) are in row \( r \) and \( i_s < j_s \) are in row \( s \), then we cannot have \( i_r, j_s \) in one group and \( j_r, i_s \) in another group. Assuming \( i_r < j_r \) are in row \( r \) and the group for \( i_r \) extends to \( j_s \) in row \( s < r \), the highest instance of \( i_r - 1 \) below row \( r \) must lie weakly above the highest instance of \( i_r \) below row \( r \) else the group of \( i_r \) would terminate at row \( r \). Continuing, the highest instance of \( i_r - k \leq j_s \) below the prior \( i_r \) group member must lie weakly above the highest instance of \( i_r - (k - 1) \) below the prior \( i_r \) group member. In particular, in row \( s \), if there is a group member of \( j_r \), it must be at least \( i_s + 1 \). Therefore groups do not cross. Moreover, if the two groups have a common value, this shows that the higher instance belongs to the left group, and if they have no common value, then the larger values belong to the right group. Therefore moving groups that share a row is well-defined and consistent.

Next we consider the case when two groups have no common row. If the two groups, say \( A \) and \( B \), have one common value, then clearly the process of moving the lower instance further right is well-defined. If \( A \) and \( B \) have two common values, say \( i \) and \( i + 1 \) (without loss of generality since groups are intervals), with \( (i + 1)_A \) above \( (i + 1)_B \), then \( i_A \) must be weakly above \( (i + 1)_B \) as well, else group \( A \) would end with \( i + 1 \), so \( i_A \) is above \( (i + 1)_B \) which is above \( i_B \). Therefore comparing rows of \( i \) or \( i + 1 \) lead to the same choice of which group moves further right. Finally, if \( A \) and \( B \) have no common row and no common value, then all entries of \( A \) are greater than all entries of \( B \), so again which group moves further right is well-defined.

Finally, we must show the ordering of groups is transitive, that is, if \( B \) should be right of \( A \) and \( C \) should be right of \( B \), then we must show that \( C \) should indeed be right of \( A \). This follows from the transitivity of the properties discussed above, namely that \( A \) left of \( B \) means for any common row the member of \( A \) is left of that for \( B \), for any common value the instance for \( A \) is higher than that for \( B \), and for no common row or common value \( A \) has smaller entries than \( B \).
The *row reading word* of a labeled diagram \( D \), denoted by \( \text{row}(D) \), is the word obtained by reading the cells of \( D \) from left to right, top to bottom. Given a reduced expression \( \rho \), we use alignment to construct a labeled diagram \( \mathbb{D}(\rho) \) such that \( \text{row}(\mathbb{D}(\rho)) = \rho \) and \( \text{wt}(\mathbb{D}(\rho)) = \text{des}(\rho) \).

**Definition 3.3.** The *diagram of a reduced expression* \( \rho \), denoted by \( \mathbb{D}(\rho) \), is the alignment of the diagram with the values of \( \rho^{(i)} \) placed consecutively from smallest to largest in row \( r_i \).

The following definition characterizes which diagrams arise from reduced expressions.

**Definition 3.4.** A *reduced diagram* is a positive integer filling of a diagram such that entries are at least the row index, the row reading word is reduced, and aligning preserves the diagram.

The *shape* of a reduced diagram \( D \), denoted by \( \text{sh}(D) \), is the permutation given by \( \text{sh}(D) = s_{i_k} \cdots s_{i_1} \) where \( \text{row}(D) = (i_1, \ldots, i_k) \). Denote the set of reduced diagrams for a permutation \( w \) by \( \text{RD}(w) \). For example, there are only 3 non-virtual reduced diagrams for 42153, and, for a more illustrative example, the 26 non-virtual reduced diagrams for 153264 are given in Figure 9. To anticipate the main result of this section, notice that

\[
\sum_{D \in \text{RD}(153264)} x^{\text{wt}(D)} = \mathcal{S}_{153264},
\]

where the sum may be taken over the 26 non-virtual reduced diagrams in Figure 9.

The row reading word gives a map from reduced diagrams of shape \( w \) to reduced expressions for \( w \). To make this map injective, we introduce the following condition on reduced diagrams.

**Definition 3.5.** A reduced diagram is *quasi-Yamanouchi* if the leftmost cell of a row has entry equal to its row index or has a cell immediately above and weakly right of it.

Denote the set of quasi-Yamanouchi reduced diagrams of shape \( w \) by \( \text{QRD}(w) \). For example, Figure 10 gives the quasi-Yamanouchi reduced diagrams for 42153. Notice that their row reading words are given by the reduced expressions in Figure 2 respectively.

**Lemma 3.6.** For \( \rho \in R(w) \), \( \mathbb{D}(\rho) \in \text{QRD}(w) \) and \( \text{des}(\rho) = \text{wt}(\mathbb{D}(\rho)) \).
Proof. The placement of cells into rows ensures $\text{des}(\rho) = \text{wt}(\mathbb{D}(\rho))$, with rows strictly increasing by definition. The definition of $\mathbb{D}(\rho)$ necessitates that alignment preserves it, and clearly the row $\text{row}(\mathbb{D}(\rho)) = \rho$ since alignment doesn’t change the order within rows. In particular, $\mathbb{D}(\rho)$ is a reduced diagram. To see the quasi-Yamanouchi condition, note that $r_i = \min(\rho_1^{(i)}, r_{i+1} - 1)$ and the left argument ensures an entry of the row equals the row index while the right argument ensures there is a nonempty row above and, since $\rho_1^{(i+1)} > \rho_1^{(i)}$, some entry of that row lies weakly right by the grouping procedure. □

Definition 3.7. The de-standardization of a reduced diagram $D$, denoted by $\text{dst}(D)$, is defined as follows. If there exists some row for which every cell both contains an entry strictly greater than the row index and lies strictly right of every cell in the row immediately above, then move the contents of this row up by one. Repeat until no such row exists.

For example, the reduced diagrams in Figure 11 are precisely those that de-standardize to the leftmost diagram, which is the unique quasi-Yamanouchi diagram among them.

Proposition 3.8. The de-standardization map is a well-defined map from $\text{RD}(w)$ to $\text{QRD}(w)$ that is the identity of $\text{QRD}(w)$. Moreover, for $C, D \in \text{RD}(w)$, we have $\text{row}(C) = \text{row}(D)$ if and only if $\text{dst}(C) = \text{dst}(D)$, and for any $C \in \text{QRD}(w)$ we have

$$\sum_{D \in \text{RD}(w) \atop \text{dst}(D) = C} x^{\text{wt}(D)} = \mathfrak{F}_{\text{wt}(C)}.$$  

Proof. The de-standardization map is independent of the order in which rows are moved up, since moving rows up only adds cells to right of existing cells. The de-standardization procedure is vacuous precisely when the quasi-Yamanouchi condition is met.

Note that if $\text{dst}(D) = C$, then $\text{wt}(D) \geq \text{wt}(C)$ and $\text{flat}(\text{wt}(D))$ refines $\text{flat}(\text{wt}(C))$ since $C$ is obtained by a sequence of moves in which all entries in row $i - 1$ ascend to row $i$. Conversely, we claim that given $C \in \text{QRD}(w)$, for every weak composition $b$ of length $n$ such that $b \geq \text{wt}(C)$ and $\text{flat}(b)$ refines $\text{flat}(\text{wt}(C))$, there is a unique $D \in \text{RD}(w)$ with $\text{wt}(D) = b$ such that $\text{dst}(D) = C$. The theorem then follows from the claim. To construct $D$ from $b$ and $C$, for $j = 1, \ldots, n$, if
wt(C)_j = b_{i_j-1+1} + \cdots + b_{i_j}, then, from east to west, push the first b_{i_j-1+1} entries down to row \( i_j-1 + 1 \), the next b_{i_j-1+2} entries down to row \( i_j-1 + 2 \), and so on. This pushing maintains the groups in the alignment process, so the result is a reduced diagram. Existence is now proved and uniqueness follows from the lack of choice at each step.

**Definition 3.9.** A reduced diagram \( D \) is super-Yamanouchi if each row is an increasing interval with leftmost entry equal to its row index.

For example, the unique super-Yamanouchi reduced diagram for 41758236 is shown in Figure 12. While multiple reduced diagrams can have super-Yamanouchi reading words, the super-Yamanouchi diagram is the only quasi-Yamanouchi reduced diagram with a super-Yamanouchi reading word.

![Figure 12. The super-Yamanouchi reduced diagram for 41758236.](image)

**Lemma 3.10.** For any \( w, \rho \in R(w) \) is super-Yamanouchi if and only if \( D(\rho) \) is super-Yamanouchi, and \( D \in QRD(w) \) is super-Yamanouchi if and only if \( \rho \) is super-Yamanouchi.

**Proof.** Let \( \rho \in R(w) \). If \( \rho \) is super-Yamanouchi, then by Lemma 3.6 \( D(\rho) \) is a (quasi-Yamanouchi) reduced diagram, and, since \( r_i = \rho^{(i)}_1 \) for all \( i \), \( D(\rho) \) will have \( \rho^{(i)}_1 \) in increasing order in row \( \rho^{(i)}_1 \) for all \( i \), therefore it is super-Yamanouchi. Conversely, if \( D(\rho) \) is super-Yamanouchi, then descents in the reading word occur precisely at row breaks, so \( \rho^{(i)}_1 \) is the interval given by the entries in row \( \rho^{(i)}_1 \). In particular, \( \rho^{(i)}_1 \) forms a decreasing sequence from top to bottom, so \( \rho \) is super-Yamanouchi.

Let \( D \in QRD(w) \). If \( D \) is super-Yamanouchi, then, as above, descents in the reading word occur precisely at row breaks, so \( \text{row}(D)^{(i)} \) is the interval given by the entries in row \( \text{row}(D)^{(i)}_1 \), ensuring that \( \text{row}(D) \) is super-Yamanouchi. Conversely, if \( \text{row}(D) \) is super-Yamanouchi, then there are no descents within a row. Furthermore, if nonempty rows \( r < s \) have only empty rows between them, then if \( row \ r \) has leftmost entry equal to \( r \), then there is a descent between the last entry of \( s \) and the first of \( r \) since entries of row \( s \) are at least \( s \). On the other hand, if the leftmost entry of row \( r \), say \( A \), lies weakly left of an entry, say \( B \), of row \( s = r + 1 \), then we must have \( A < B \) by the analysis of alignment in the proof of Proposition 3.2 so again there is a descent from the last entry of \( s \) to the first of \( r \). In particular, the rows of \( D \) give the run decomposition of \( \text{row}(D) \), and so, in particular, they form intervals. Take the highest row, say \( r \), for which the leftmost entry is greater than the row index. Then the quasi-Yamanouchi condition ensures that row \( r + 1 \) is nonempty, and by choice of \( r \), the leftmost entry of row \( r + 1 \) is \( r + 1 \). Since row \( (D) \) is super-Yamanouchi, row \( (D)^{(r+1)} = r + 1 \) and row \( (D)^{(r)} \) is nonempty row \( r \) must be at most \( r \) after all. In particular, \( D \) is super-Yamanouchi.

If we wish to generate all quasi-Yamanouchi reduced diagrams for \( w \), then we can use the following analogs of swaps and braids on reduced expressions.

**Definition 3.11.** Given a permutation \( w \) and an index \( 1 \leq i < \text{inv}(w) \), define a swap involution on \( QRD(w) \), denoted by \( s_i \), as follows: numbering cells of \( D \) in reverse reading order, if the \( ith \) and \( i + 1 \)st cells have consecutive entries, then set \( s_i(D) = D \); otherwise set \( s_i(D) \) to be the result of pushing the \( i + 1 \)st cell down one row or to the row of the \( ith \) cell (whichever is lower), pushing the 1st through \( i \) cells down one row, and then de-standardizing.
**Lemma 3.12.** The map \( s_i \) is a well-defined involution on QRD with \( \text{row}(s_i(D)) = s_i(\text{row}(D)) \).

*Proof.* Let \( p_{i+1} \) be the map that pushes all cells after the \( i \)-th down one row and pushes the \( i+1 \)-st cell down to the nearest empty position weakly below the row of the \( i \)-th cell. Then \( s_i = \text{dst} \circ p_{i+1} \).

To show that \( s_i \) is well-defined, it suffices to show that \( p_{i+1}(D) \) is a reduced diagram; to show that \( \text{row}(s_i(D)) = s_i(\text{row}(D)) \), by Proposition 3.8, \( \text{dst} \circ p_{i+1}(\text{row}(D)) = p_{i+1}(\text{row}(D)) \), so it is enough to show that \( \text{row}(p_{i+1}(D)) = s_i(\text{row}(D)) \); and to show that \( s_i \) is an involution, we will show that \( \text{dst} \circ p_{i+1} \circ \text{dst} \circ p_{i+1} = \text{dst} \circ p_{i+1} \circ p_{i+1} \) with the latter being the identity.

Let \( b \) be the entry of the \( i+1 \)-st cell in reverse reading order and \( a \) the entry immediately after, as depicted in Figure 13. Consider two cases: (i) \( b \) lies strictly above \( a \) as in the top row of Figure 13 or (ii) \( b \) lies in the same row as \( a \) as depicted in the bottom row of Figure 13. In both cases, there can be no cells after \( b \) and before \( a \) in reading order.

For case (i), the quasi-Yamanouchi condition ensures \( b \) lies weakly right of \( a \) since there are no cells left of \( a \) or right of \( b \). Further, since \( |b-a| > 1 \), \( b \) cannot be in the column of \( a \) since they would not be grouped consecutively during alignment. Moreover, the analysis of groups in the proof of Proposition 3.2 ensures that \( a < b \) since the group of \( a \) is left of the group of \( b \). Therefore \( p_{i+1}(D) \) has the same groups ordered in the same way, so it is a reduced diagram. Moreover, one easily sees that \( p_{i+1}(\text{row}(D)) = \text{row}(s_i(D)) \). Finally, one easily sees from Figure 13 that \( \text{dst} \circ p_{i+1} \circ p_{i+1} \) is the identity no matter which, if any, of the potential cells is present.

For case (ii), since \( a \) and \( b \) are in the same row with \( b \) left of \( a \), we must have \( b < a \). Since \( a \neq b+1 \), \( p_{i+1}(D) \) has the same groups as \( D \), ordered in the same way, so it is a reduced diagram. Once again, it is evident that \( p_{i+1}(\text{row}(D)) = \text{row}(s_i(D)) \), and, from Figure 13 that \( \text{dst} \circ p_{i+1} \circ p_{i+1} \) is the identity. \( \square \)

**Definition 3.13.** Given a permutation \( w \) and an index \( 1 < i < \text{inv}(w) \), define a braid involution on QRD\((w)\), denoted by \( b_i \), as follows: numbering cells of \( D \) in reverse reading order, if the \( i \)-1st and \( i+1 \)-st cells have equal entries that are within one of the \( i \)-th cell, then set \( b_i(D) \) to be the result of moving cells strictly below and weakly right of the \( i \)-1st cell or weakly above and weakly right of the \( i+1 \)-st cell to the right one column, pushing the \( i+1 \)-st cell down to below the \( i \)-1st cell and decrementing its value, pushing the 1st through \( i-2 \)nd cells down one row, realigning, and then de-standardizing.

**Lemma 3.14.** The map \( b_i \) is a well-defined involution on QRD with \( \text{row}(b_i(D)) = b_i(\text{row}(D)) \).

*Proof.* Let \( o_i \) be the map that pushes right all cells that are weakly above and weakly right of the \( i+1 \)-st cell or strictly below and weakly right of the \( i \)-1st cell. Let \( p_i \) be as in the proof of Lemma 3.12. Then \( b_i = \text{dst} \circ \text{align} \circ p_i \circ o_i \). To show \( b_i \) is well-defined, it suffices to show \( p_i \circ o_i(D) \) is a reduced diagram; to show that \( \text{row}(b_i(D)) = b_i(\text{row}(D)) \), by Proposition 3.8, it is enough to show \( \text{row}(p_i \circ o_i(D)) = b_i(\text{row}(D)) \); and to show \( b_i \) is an involution, we will show \( \text{dst} \circ \text{align} \circ p_i \circ o_i \circ \text{dst} \circ \text{align} \circ p_i \circ o_i = \text{dst} \circ \text{align} \circ p_i \circ o_i \circ p_i \circ o_i \) with the latter being the identity.
Let $b$ be the entry of the $i + 1$st cell in reverse reading order, $a$ be the entry immediately after, and $b$ the entry immediately after that, as depicted in Figure 14. Consider two cases: (i) $b$ lies strictly above $a, b$ as in the top row of Figure 14 or (ii) $b, a$ lies above $b$ as depicted in the bottom row of Figure 14. In both cases, there can be no cells between $b, a, b$ in reading order.

Consider case (i). The quasi-Yamanouchi condition ensures that the upper $b$ must be weakly right of $a$, and so by alignment we must have $b > a$, and since $|a - b| = 1$, we must have $b$ above $a$ in the same column. Similarly, the lower $b$ must be right of $a$ since it is larger, and the quasi-Yamanouchi condition ensures that it lies in the same row. Furthermore, we claim that no entries can lie strictly right of the upper $b$ and strictly higher and weakly left of the lower $b$. Indeed, the alignment forces the nearest $b + 1$ above these to be in the column of the upper $b$, and any other $b + 1$ must be below and right of this. A similar argument shows that there can be no entries below $a$ and the lower $b$ that lie strictly between them. Thus we have the situation depicted in the top row of Figure 14. Following the maps through the figure, note that the group of $a$ changes after applying $p_i$, and if there is a disjoint column left of this, alignment could push $a$ further left, either swapping it to the other side of the column or aligning it under the column. However, applying $p_i$ will bring it back after realigning, and the combined result is the identity.

Consider case (ii). Since $b$ and $a$ lie in the same row, we have $b = a - 1$, and so by alignment we must have the lower $b$ in the column of $a$. In this case, the upper $b$ can have grouped cells above it, but $p_i$ will split the group, and after another iteration of $p_i$, alignment will recombine them as described in the previous case. Thus once again, we follow the bottom row of Figure 14 to see that again the combined result is the identity.

For examples of swaps and braids on quasi-Yamanouchi reduced diagrams, see Figure 15.

**Figure 15.** Examples of swaps and braids on QRD(41758236).

**Theorem 3.15.** The row reading word is a bijection $\text{QRD}(w) \sim R(w)$ that takes weights to weak descent compositions.

**Proof.** By Lemma 3.10, the super-Yamanouchi terms correspond under row and $D$. Lemmas 3.12 and 3.14 show that $s_i$ and $b_i$ on QRD commute with row, and the definition of the involutions on...
diagrams show that they admit the same swap and braid moves as their reading words. Therefore the result follows from Theorem 2.11. □

Corollary 3.16. The Schubert polynomial $S_w$ is given by

$$S_w = \sum_{D \in \text{QRD}(w)} f_{\text{wt}(D)} = \sum_{D \in \text{RD}(w)} x^{\text{wt}(D)},$$

where the sum may be taken over non-virtual reduced diagrams for $w$.

Theorem 2.11 shows that one can generate all reduced expressions for $w$ by applying a sequence of swaps and braids to the super-Yamanouchi expression, and by Theorem 3.15 the same applies to quasi-Yamanouchi reduced diagrams. However, when doing this, often more terms are generated than are needed in the computation of a Schubert polynomial since many of the expressions and diagrams may be virtual. If one is contented to use the monomial expansion, then all of the non-virtual reduced diagrams for a Schubert polynomial may be generated by applying a sequence of pushes to the super-Yamanouchi diagram.

4. Balanced tableaux

In order to enumerate quasi-Yamanouchi reduced diagrams more easily, we transform this model for Schubert polynomials into bijective fillings of the Rothe diagram of a permutation by re-labeling cells consecutively in reverse reading order and then pushing cells back up to the Rothe diagram shape by reversing swaps and braids.

**Definition 4.1.** The Rothe diagram of a permutation $w$, denoted by $\mathbb{D}(w)$, is given by

$$\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+.$$ 

For example, see Figure 16. The number of cells in $\mathbb{D}(w)$ is simply $\text{inv}(w)$. Furthermore, the Rothe diagram for $w$ is precisely super-Yamanouchi reduced diagram for $w$ with entries deleted.

![Figure 16. The Rothe diagram for 41758236.](image)

**Definition 4.2.** The ascended tableau of a quasi-Yamanouchi reduced diagram $D$, denoted by $A(D)$ is obtained by labeling the cells of $D$ with $1, 2, \ldots, \text{inv}(w)$, where $w$ is the shape of $D$, in reverse reading order and applying any minimal length sequence $f = f_k \cdots f_1$ of swaps and braids such that $f(D)$ is super-Yamanouchi.

For example, Figure 17 constructs the ascended tableau for the quasi-Yamanouchi reduced diagram in Figure 8 using the sequence of braids and swaps given in Figure 15.

Edelman and Greene [EG87] introduced balanced labelings of Rothe diagrams in order to enumerate reduced expressions. We recall the notion here, but give independent, elementary proofs of their enumerative results by identifying standard balanced tableaux as precisely those fillings of a Rothe diagram that arise as ascended tableaux for quasi-Yamanouchi reduced diagrams.

**Definition 4.3 (EG87).** A standard balanced tableau is a bijective filling of a Rothe diagram with entries from $\{1, 2, \ldots, n\}$ such that for every entry of the diagram, the number of entries to its right that are greater is equal to the number of entries above it that are smaller.
COMBINATORIAL MODELS FOR SCHUBERT POLYNOMIALS

1. Constructing the ascended tableau of a reduced diagram for 41758236.

Given a cell $x$ in a tableau, define the arm of $x$, denoted by $\text{arm}(x)$, to be the set of cells to the right of $x$ and in the same row, and define the leg of $x$, denoted by $\text{leg}(x)$, to be the set of cells to the above $x$ and in the same column. The balanced condition can be restated as

$$\{y \in \text{arm}(x) \mid R(y) > R(x)\} = \{y \in \text{leg}(x) \mid R(y) < R(x)\}$$

for all $x \in R$. Denote the set of standard balanced tableaux on $D(w)$ by $\text{SBT}(w)$. For example, the elements of $\text{SBT}(42153)$ are shown in Figure 18.

2. The standard balanced tableaux for 42153.

To prove that standard balanced tableaux are in bijection with reduced expressions, observe first that there is a canonical super-Yamanouchi standard balanced tableau.

**Definition 4.4.** A standard balanced tableau $R$ is super-Yamanouchi if its reverse row reading word is the identity.

For example, the unique super-Yamanouchi balanced tableau for 41758236 is shown in Figure 19.

3. The super-Yamanouchi balanced tableau for 41758236.

Next we define analogs of swaps and braids for standard balanced tableaux.

**Definition 4.5.** Given a permutation $w$ and an index $1 \leq i < \text{inv}(w)$, let $s_i$ act on $\text{SBT}(w)$ by exchanging $i$ and $i+1$ if they are not in the same row or column and the identity otherwise.

**Definition 4.6.** Given a permutation $w$ and an index $1 < i < \text{inv}(w)$, let $b_i$ act on $\text{SBT}(w)$ by exchanging $i-1$ and $i+1$ if one is in the same column and above $i$ and the other is in the same row and right of $i$ in $R$ and the identity otherwise.

**Lemma 4.7.** The maps $s_i$ and $b_i$ are well-defined involutions on $\text{SBT}(w)$. 
Proof. For \( R \in \text{SBT}(w) \), if \( i \) and \( i + 1 \) are not in the same row or same column, then interchanging them cannot unbalance the tableau since all other entries compare the same with \( i \) and with \( i + 1 \). Thus \( s_i(R) \in \text{SBT}(w) \). If \( i + 1 \) is in the same row as \( i \) and \( i + 1 \) is in the same column, then swapping them maintains the balance since, again, every \( j \) less than \( i - 1 \) or greater than \( i + 1 \) compares with some with both, the two cannot be in the same row or same column as one another, and \( i \) has traded the two to maintain its balance.

For examples of swaps and braids on standard balanced tableaux, see Figure 20.

![Figure 20. Examples of swaps and braids on SBT(41758236).](image)

Parallel to the case of reduced expressions, we introduce a statistic on standard balanced tableaux that gives the minimum distance from a standard balanced tableau to the super-Yamanouchi one.

**Definition 4.8.** Given \( R \in \text{SBT}(w) \), define the inversion number of \( R \), denoted by \( \text{inv}(R) \), by

\[
\text{inv}(R) = |\{(i < j) \mid i \text{ lies in a strictly higher row and different column than } j\}|.
\]

We call such a pair an inversion of \( R \).

For example, the super-Yamanouchi balanced tableau in Figure 19 has no inversions, and the standard balanced tableaux in Figure 20 have 0, 4, 5, 6, 7, 11 inversions, respectively.

**Theorem 4.9.** Let \( P_w \in \text{SBT}(w) \) be the unique super-Yamanouchi tableau. Then for any \( R \in \text{SBT}(w) \), there exists a sequence \( f = f_{\text{inv}(R)} \cdots f_1 \) of swaps and braids such that \( f(P_w) = R \), and, for any sequence \( g = g_n \cdots g_1 \) of swaps and braids with \( g(P_w) = R \), we have \( m \geq \text{inv}(R) \).

Proof. We proceed by induction on \( \text{inv}(R) \). Clearly \( \text{inv}(P_w) = 0 \) since it is the unique balanced filling such that all larger entries occur weakly above smaller entries, and the result holds for this case. Moreover, if \( R \) has some \( i < j \) with \( i \) above \( j \) and in the same column, then the balanced condition ensures that there is some \( k > j \) in the same row as \( j \), and so \( i < k \) with \( i \) and \( k \) not in the same column. In particular, \( \text{inv}(R) > 0 \) for \( R \neq P_w \). This establishes the base case.

Let \( R \in \text{SBT}(w) \) with \( \text{inv}(R) > 0 \). We claim that there is a pair \((i, i + 1)\) with \( i \) above \( i + 1 \). If not, then for any pair \((i < j)\) with \( i \) above \( j \) (such a pair exists since \( \text{inv}(R) > 0 \)), there exists \( k \) with \( i < k < j \) and neither \((i < k)\) nor \((k < j)\) has the smaller strictly above the larger. Thus \( k \) is weakly above \( i \) and weakly below \( j \), an impossibility since \( i \) is strictly above \( j \). Therefore we may take \( i \) such that \( i + 1 \) lies in a strictly lower row. There are two cases to consider.

If \( i \) and \( i + 1 \) are not in the same column, then \( s_i \) acts non-trivially on \( R \). Furthermore, \( \text{inv}(s_i(R)) = \text{inv}(R) - 1 \) since the pair \((i, i + 1)\) is removed from the set of inversions and all other pairs remain but with \( i \) and \( i + 1 \) interchanged. By induction, the result holds for \( s_i(R) \), and so, too, for \( R \).

If \( i \) and \( i + 1 \) are in the same column for every pair with \( i \) above \( i + 1 \), then take \( i \) maximal among all such pairs. We claim that \( i + 2 \) must lie in the same row and to the right of \( i + 1 \). If not, then \( i + 2 \) must lie strictly above \( i + 1 \) and, by the choice of \( i \), \( k + 1 \) must lie weakly above \( k \) for all \( k > i + 2 \). However, this would mean no larger entry was in the row of \( i + 1 \), contracting the balanced condition since \( i \) is in the same column and above it. Therefore \( i + 2 \) does lie in the same row as \( i + 1 \), and so \( b_{i+1} \) acts non-trivially on \( R \) by interchanging \( i \) and \( i + 2 \). Furthermore, \( \text{inv}(b_{i+1}(R)) = \text{inv}(R) - 1 \) since the pair \((i, i + 2)\) is removed from the set of inversions and all other pairs remain but with \( i \) and \( i + 2 \) interchanged. By induction, the result holds for \( b_{i+1}(R) \), and so it holds for \( R \) as well. 

\[ \square \]
Beginning with the super-Yamanouchi elements, we can generate all of $R(w)$ and $\text{SBT}(w)$ using $s_i$ and $b_i$. Moreover, in doing so, we claim that we create a bijection between these two sets.

**Definition 4.10.** Given $R \in \text{SBT}(w)$, define the *permutation of $R$*, denoted by $v(R)$, to be the reverse reading word of $R$ with rows sorted to decreasing.

For example, the permutation of the standard balanced tableau in Figure 21 is $23518910467112$. Compare this with the permutation constructed for the reduced expression $\langle 563457314236 \rangle$ in Figure 6. Notice that while the standard balanced tableau has 11 inversions, its permutation has 13 inversions. The difference between the two is precisely the number of steps needed to sort the rows of the tableau. The following result shows that this holds in general.

\[
\begin{array}{|c|c|c|c|}
\hline
121 & 7 & \text{row sort} & 121 \\
6 & 4 & & 6 \\
9 & 8 & 10 & 1 \\
5 & 3 & 2 & \\
\hline
\end{array}
\quad \quad \quad
\begin{array}{|c|c|c|c|}
\hline
121 & 7 & \text{reverse} & 23518910467112 \\
6 & 4 & & \\
10 & 9 & 8 & 1 \\
5 & 3 & 2 & \\
\hline
\end{array}
\]

**Figure 21.** Constructing the permutation for an element of $\text{SBT}(41758236)$

**Proposition 4.11.** For $R \in \text{SBT}(w)$, let $v = v(R)$ be its permutation. Then

\[
\text{inv}(R) = \text{inv}(v) - \sum_r \text{coinv(row}_r(R)),
\]

where the sum is over co-inversions within the rows of $R$.

**Proof.** Let $\hat{\text{inv}}$ be defined by the right hand side of (4.4). Let $R \in \text{SBT}(w)$ and suppose $s_i$ acts non-trivially on $R$. Then $i$ and $i+1$ lie in different rows and different columns in $R$, so $\text{sort}(R)$ and $\text{sort}(s_iR)$ differ exactly in that $i$ and $i+1$ have been exchanged, and so $v(s_iR) = s_i v(R)$. Further, since all letters other than $i, i+1$ compare the same with $i$ and $i+1$, $R$ and $s_iR$ have the same number of row (co)inversions. In particular, we have

\[
\hat{\text{inv}}(s_iR) = \hat{\text{inv}}(s_i v(R)) = \sum_r \text{coinv(row}_r(R)) = \text{inv}(R) + 1,
\]

and, moreover, $\hat{\text{inv}}(s_iR) = \hat{\text{inv}}(R) + 1$ precisely when $i$ is left of $i+1$ in $v$.

Next suppose that $b_i$ acts non-trivially on $R$, exchanging $i-1$ and $i+1$ when $i$ lies directly below the one and directly left of the other. The permutation exchanging $i-1$ and $i+1$ is given by $s_{i-1}s_is_{i-1} = s_is_{i-1}s_i$, but since $i-1$ and $i+1$ compare differently with $i$, when the rows are sorted the one in the row of $i$ will flip to the other side of it. Therefore $v(b_iR) = s_{i-1}s_i v(R)$ if $i+1$ is above $i-1$, and $v(b_iR) = s_is_{i-1}v(R)$ otherwise, and in the former case we have

\[
\hat{\text{inv}}(b_iR) = \hat{\text{inv}}(s_{i-1}s_i v(R)) = \sum_r (\text{coinv(row}_r(R)) + 1) = \text{inv}(R) + 1,
\]

and, by the same computation, $\hat{\text{inv}}(b_iR) = \hat{\text{inv}}(R) - 1$ in the latter case.

By Theorem 4.9, $\text{inv}(R) = 0$ if and only if $R$ is super-Yamanouchi, in which case $v(R)$ is the identity and $R$ has decreasing rows, thus giving $\hat{\text{inv}}(R) = 0$ as well. Conversely, if we consider $\hat{v}$ to be the permutation obtained by following Definition 4.10 without first sorting the rows of $R$, then we have $\text{inv}(\hat{v}) = \text{inv}(v) + \sum_r \text{coinv(row}_r(R))$. In particular, $\hat{\text{inv}}(R) = 0$ if and only if $v$ is the identity, in which case $R$ is super-Yamanouchi. Therefore $\text{inv}(R) = \text{inv}(R)$ whenever either is 0. By Theorem 4.9, for any $R \in \text{SBT}(w)$, we may write $R = f_{\text{inv}(R)} \cdots f_1(P)$, where $P$ is super-Yamanouchi and each $f_i$ is a swap or a braid. The result for $R$ now follows from the analysis of swaps and braids above. \qed
Comparing Theorem 2.11 with Theorem 4.9, one can anticipate that there is a bijection between reduced expressions and standard balanced tableaux that preserves the permutation and inversion number. Indeed, given the permutation $v$, one can recover the row entries for the corresponding balanced tableau, if it exists. The following result shows that there is at most one balanced tableau with the given row entries.

**Proposition 4.12.** For $R, S \in \text{SBT}(w)$, if $R$ and $S$ row sort to the same tableau, then $R = S$.

**Proof.** Let $T$ be the sorted tableau. We will show there is at most one ordering on the rows of $T$ such that $T$ is balanced. Beginning with the top row, we must place entries in decreasing order from left to right. Assuming all higher rows have been uniquely balanced, begin balancing row $r$ from left to right. If the available entries for cell $x$ are $x_1 > \cdots > x_k$, then let $c_i$ be the number of cells above $x_i$ that are smaller than $x_i$ and let $r_i = i - 1$, which is the number of entries right of $x$ that will be greater than $x_i$ should it be placed into cell $x$. Note that $c_1 \geq \cdots \geq c_k$ and $r_1 < \cdots < r_k$. Thus there is at most one index $i$ for which $r_i = c_i$, i.e. there is at most one entry that can be placed into cell $x$ for which the resulting tableau will be balance. □

To prove that we indeed have a bijection, it is easier to describe the bijection between quasi-Yamanouchi reduced diagrams and standard balanced tableaux.

**Definition 4.13.** The ascended diagram of a quasi-Yamanouchi reduced diagram $D$, denoted by $\hat{A}(D)$ is obtained by labeling the cells of $D$ from 1 to $n$ in reverse row reading order and applying any minimal length sequence $f = f_k \cdots f_1$ of swaps and braids such that $f(\text{dst}(D))$ is super-Yamanouchi.

For more examples, Figure 22 gives the labeling of the quasi-Yamanouchi reduced diagrams in Figure 10. These ascend to the standard balanced tableaux in Figure 18, respectively.

![Figure 22. Descended diagrams for the standard balanced tableaux of shape 42153.](image)

**Theorem 4.14.** Ascended diagrams give a bijection $A : \text{QRD}(w) \sim \rightarrow \text{SBT}(w)$. Moreover, composing with $D : R(w) \sim \rightarrow \text{QRD}(w)$, for $\rho \in R(w)$, we have $v(\rho) = v(A D(\rho))$ and $\text{inv}(\rho) = \text{inv}(A D(\rho))$.

**Proof.** By Theorem 3.15 we may identifying quasi-Yamanouchi reduced diagrams with their reduced expressions. Therefore we will proceed by induction on $\text{inv}(D)$. By Theorem 2.11 $\text{inv}(D) = 0$ if and only if $D$ is super-Yamanouchi, in which case $A(D)$ is the super-Yamanouchi balanced tableau. Conversely, if $A(D)$ is super-Yamanouchi, then no cells moved up nor left, else the reverse reading word would no longer be the identity, so the minimal length sequence was the identity. In particular, $D$ was super-Yamanouchi. Therefore the theorem holds for $\text{inv} = 0$.

From Lemma 3.12 it follows that $s_i D$ precisely changes the orders of the $i$th and $i + 1$st cells. Moreover, this exchange does not alter where cells land in the ascension process, and so $A(s_i D) = s_i A(D)$. Similarly, from Lemma 3.14 we see that $b_i D$ either moves the $i + 1$st cell to position $i - 1$
or moves the $i - 1$st cell to position $i + 1$. Moreover, following the ascension process in this case shows that $A(b_i D) = b_i A(D)$. By Theorems 2.11 and 4.9 and induction on $\text{inv}$, the map $A$ gives the desired bijection that preserves $\text{inv}$. Moreover, the analysis of swaps and braids on permutations in Theorem 2.11 and Proposition 4.11 proves that the permutation is also preserved. □

As a consequence, we recover the following result of Edelman and Greene [EG87].

Corollary 4.15. The number of reduced expressions for $w$ is equal to the number of standard balanced tableaux of shape $D(w)$.

More generally, we may connect balanced tableaux with arbitrary reduced diagrams by generalizing the definition of the former as follows. Note that this is equivalent to the formulation given by Fomin, Greene, Reiner, and Shimozono [FGRS97].

Definition 4.16. A semi-standard balanced tableau is a positive integer filling of a Rothe diagram such that entries do not exceed their row index, columns have distinct values, and, for every entry of the diagram, the number of weakly larger entries to the right is at least as great as the number of smaller entries above, and the number of weakly smaller entries above is at least as great as the number of larger entries to the right.

Denote the set of semi-standard balanced tableaux for $w$ by $\text{SSBT}(w)$. For example, Figure 23 gives the elements of $\text{SSBT}(153264)$.

![Figure 23. The semi-standard balanced tableaux of shape 153264.](image)

Define arm and leg statistics on cells of a tableau $R$ by

$$
A(x) = \{y \in \text{arm}(x) \mid R(y) > R(x)\}, \\
L(x) = \{y \in \text{leg}(x) \mid R(y) < R(x)\}, \\
a(x) = \{y \in \text{arm}(x) \mid R(y) \geq R(x)\}, \\
l(x) = \{y \in \text{leg}(x) \mid R(y) \leq R(x)\}.
$$

(4.5) äëþáîðèéêàëüíå ñòàöèè îöåíêè íà ÷åëàêîâ íà ýëåêòðî è ëåêñèðîâàíèå.
The semi-standard balanced condition may be restated as
\[(4.7) \quad a(x) \geq L(x) \quad \text{and} \quad l(x) \geq A(x).\]
In the standard case, \(a(x) = A(x)\) and \(l(x) = L(x)\) for all \(x \in R\), and so (4.7) implies \(A(x) = L(x)\) for all \(x \in R\), which is precisely (4.2). In particular, a standard balanced tableau satisfies the semi-standard condition, except, perhaps, for the constraint on entries not exceeding their row index.

**Definition 4.17.** The *ascended tableau* of a reduced diagram \(D\), denoted by \(\hat{A}(D)\) is obtained by labeling the cells of \(D\) with their row index, de-standardizing and applying any minimal length sequence \(f = f_k \cdots f_1\) of swaps and braids such that \(f(\text{dst}(D))\) is super-Yamanouchi.

For example, the reduced diagrams in Figure 9 ascend to the semi-standard balanced tableaux in Figure 23, respectively. The *weight* of a semi-standard balanced tableau \(R\), denoted by \(\text{wt}(R)\), is the weak composition whose \(i\)th part is the number of \(i\)s that occur in \(R\).

**Theorem 4.18.** Ascended diagrams gives a weight-preserving bijection \(\hat{A} : \text{RD}(w) \sim \rightarrow \text{SSBT}(w)\).

**Proof.** Ascension on reduced diagrams is well-defined since de-standardization and ascension of quasi-Yamanouchi reduced diagrams are both well-defined. Comparing ascension for reduced diagrams with that for quasi-Yamanouchi reduced diagrams, we see that semi-standard balanced tableaux map to standard balanced tableaux by standardizing the reading word to a permutation. Doing so amounts to selecting a weak composition \(b\) that dominates and refines the descent composition of the standard balanced tableau, so the result follows from expanding fundamental slide polynomials into monomials. \(\square\)

We say that a semi-standard balanced tableau is *quasi-Yamanouchi* if it is the ascended tableau of a quasi-Yamanouchi reduced diagram. For example, see Figure 24.

\[
\begin{array}{cccccc}
5 & 5 & 5 & 3 & 2 & 2 \\
3 & 2 & 2 & 1 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 & 2
\end{array}
\]

**Figure 24.** The quasi-Yamanouchi balanced tableaux of shape 153264.

Note that quasi-Yamanouchi balanced tableaux naturally embed into standard balanced tableaux, but the former includes only images of the non-virtual quasi-Yamanouchi reduced diagrams. Therefore we have the following precise characterization of Schubert polynomials.

**Corollary 4.19.** The Schubert polynomial for a permutation \(w\) is given by
\[(4.8) \quad S_w = \sum_{R \in \text{QBT}(w)} \mathcal{F}_{\text{wt}(R)} = \sum_{R \in \text{SSBT}(w)} x^{\text{wt}(R)},\]
where both sums have only positive terms.

5. **Weak insertion**

Edelman and Greene defined an insertion algorithm that maps reduced expressions to pairs of Young tableaux where the left is row and column strict and the right is standard [EG87]. They used this to prove that the Stanley symmetric functions, introduced by Stanley to enumerate reduced expressions [Sta84], are Schur positive.
Definition 5.1. The Young diagram of a partition $\lambda$, denoted by $\mathbb{D}(\lambda)$, is given by
\[(5.1) \quad \mathbb{D}(\lambda) = \{(i, j) \mid 1 \leq i \leq \lambda_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+.
\]

For example, see Figure 25. A Young tableau is a filling of a Young diagram with positive integers. A tableau is increasing if it has strictly increasing rows (left to right) and columns (bottom to top). A tableau is standard if it uses the integers $1, 2, \ldots, n$ each exactly once. A standard Young tableau is an increasing, standard Young tableau.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{young_diagram.png}
\caption{The Young diagram for $(5, 4, 1)$.}
\end{figure}

Definition 5.2 ([EG87]). Let $P$ be a Young tableau, and let $x$ be a positive integer. Let $P_i$ be the $i$th lowest row of $P$. Define the Edelman-Greene insertion of $x$ into $P$, denoted by $P \leftarrow x$, as follows. Set $x_0 = x$ and for $i \geq 0$, insert $x_i$ into $P_{i+1}$ as follows: if $x_i \geq z$ for all $z \in P_{i+1}$, place $x_i$ at the end of $P_{i+1}$ and stop; otherwise, let $x_{i+1}$ denote the smallest element of $P_{i+1}$ such that $x_{i+1} > x_i$ (we say that $x_i$ bumps $x_{i+1}$ in row $i+1$), replace $x_{i+1}$ by $x_i$ in $P_{i+1}$ only if $x_{i+1} \neq x_i$ or $x_i$ is not already in $P_{i+1}$, and continue.

This algorithm generalizes the insertion algorithm of Schensted [Sch61], building on work of Robinson [Rob38]. Robinson-Schensted insertion becomes a bijective correspondence between permutations and pairs of standard Young tableaux by constructing a second tableau to track the order in which new cells are added. The pair is typically denoted by $(P, Q)$, where $P$ is called the insertion tableau, and $Q$ is called the recording tableau.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{edelman_greene_insertion.png}
\caption{The Edelman-Greene insertion tableau for the reduced expression $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$.}
\end{figure}

For example, Figure 26 shows the construction of the Edelman-Greene insertion tableau for the reduced expression $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$. In [EG87], Edelman and Greene derived many properties of this generalization, most important of which for our purposes is the following.

Theorem 5.3 ([EG87]). For $\rho \in R(w)$, the Edelman-Greene insertion tableau $P(\rho)$ defined by inserting $\rho_k, \ldots, \rho_1$ into the empty tableau is a well-defined increasing tableau whose row reading word is a reduced expression for $w$.

The key diagram of a weak composition $a$, denoted by $\mathbb{D}(a)$, is the diagram with $a_i$ cells left-justified in row $i$. For example, the left diagram in Figure 7 is the key diagram for $(3, 0, 4, 2, 3)$. A key tableau is a filling of a key diagram with positive integers, and we adopt the notions of increasing and standard for key tableaux as well. We generalize Edelman-Greene insertion to an insertion algorithm on reduced expressions that outputs a tableau of key shape, that is, a tableau whose cells form a key diagram.
**Definition 5.4.** For $P$ an increasing Young tableau, define the *lift* of $P$, denoted by $\text{lift}(P)$, to be the tableau of key shape obtained by raising each entry in the first column of $P$ until it equals its row index, and, once columns 1 through $c - 1$ have been lifted, raising entries in column $c$ from top to bottom, maintaining their relative order, placing each entry in the highest available row such that there is an entry in column $c - 1$ that is strictly smaller.

For examples, the Young tableaux in Figure 26 lift to the key tableaux given in Figure 27. Lifting of an increasing tableau, Young or key, is clearly well-defined since the first column has distinct entries, and later column entries lift at least as high as their original neighbors to the left.

For $P$ a key tableau, define the *drop* of $P$, denoted by $\text{drop}(P)$, to be the Young tableau defined by letting the entries of $P$ fall in their columns while maintaining their relative order. It is clear that $\text{drop}(\text{lift}(P)) = P$ for any $P$ of partition shape.

**Definition 5.5.** For $P$ a key tableau and $x$ a positive integer, define the *weak insertion of $x$ into $P$* by $\text{lift}(\text{drop}(P) \leftarrow x)$.

For example, Figure 27 shows the construction for the weak insertion tableau for the reduced expression $(5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$. Conflating notation, we denote the result of weak insertion by $P \leftarrow x$ whenever the shape of the output is clear from context.

Given any diagram, we can always push the cells left to form a key diagram. We call this the *left justification* of the diagram.

**Definition 5.6.** A reduced diagram $D$ is *Yamanouchi* if its left justification is increasing and $\text{lift}(D) = D$.

Denote the set of Yamanouchi reduced diagrams for $w$ by $\text{YRD}(w)$. For example, inspecting Figure 10 the Yamanouchi reduced diagrams for 42153, shown in Figure 28, correspond to the reduced expressions $(4, 2, 1, 2, 3)$ and $(2, 4, 1, 2, 3)$. Note that Yamanouchi reduced diagrams are quasi-Yamanouchi and are never virtual.

**Theorem 5.7.** A key tableau is a weak insertion tableau if and only if it is the left justification of a Yamanouchi reduced diagram.

**Proof.** Let $P$ be a weak insertion tableau. By Definition 5.5, $\text{drop}(P)$ is an Edelman-Greene insertion tableau. By Theorem 5.3, $\text{drop}(P)$ has increasing rows and columns and row($\text{drop}(P)$) is a reduced expression, say for some permutation $w$. The lifting algorithm will never lift $i$ in column
Definition 5.8. Given results from [EG87] to determine for which reduced expressions a given insertion tableau can arise. Theorem 5.9 ([EG87]). For reduced expressions \( \sigma, \tau \), we have \( P(\sigma) = P(\tau) \) if and only if \( \sigma \) and \( \tau \) are Coxeter-Knuth equivalent.

The recording tableau for both Robinson-Schensted and Edelman-Greene insertion places successively increasing entries into the new cell added at each step of the insertion process. For example, Figure 29 shows the construction of the Edelman-Greene recording tableau for the reduced expression \( (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6) \), parallel to the insertion tableau constructed in Figure 26.

![Figure 29. A Coxeter-Knuth equivalence class for \( R(42153) \).](image)

![Figure 30. The Edelman-Greene recording tableau for the reduced expression \( (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6) \).](image)

We define the analogous concept for weak insertion, though following the pattern for reduced expressions, we record successively decreasing entries. Recall the key tableaux developed by Assaf [Ass17] based on the Kohnert tableaux model of Assaf and Searles [AS16].

Definition 5.10 ([Ass17]). A standard key tableau is a bijective filling of a key diagram with \( \{1, 2, \ldots, n\} \) such that rows weakly decrease and if some entry \( i \) is above and in the same column as an entry \( k \) with \( i < k \), then there is an entry right of \( k \), say \( j \), and \( i < j \).
We denote the set of standard key tableaux of shape \(a\) by \(\SKT(a)\). For example, see Figure 31.

From [Ass17], there is a simple bijection between standard key tableaux of weak composition shape \(a\) and standard Young tableaux of partition shape \(\text{sort}(a)\) given by letting the cells fall, sorting columns, and replacing \(i\) with \(n-i+1\), where \(n\) is the number of cells. Using the inverse of this bijection, we can now define weak recording tableaux.

**Definition 5.11.** For a reduced expression \(\rho\), define weak recording tableau for \(\rho\), denoted by \(Q(\rho)\), to be the unique standard key tableaux of the same shape as \(P(\rho)\) such that letting the cells fall, sorting columns, and replacing \(i\) with \(n-i+1\) gives the Edelman-Greene recording tableau for \(\rho\).

For example, Figure 32 shows the construction of the weak recording tableau for the reduced expression \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\), parallel to the insertion tableau constructed in Figure 27.

The run decomposition of a standard key tableau \(T\) partitions the decreasing word \(n \cdots 21\) by separating \(i+1\) and \(i\) precisely when \(i+1\) lies weakly right of \(i\) in \(T\). In this case, we call \(i\) a descent of \(T\). For example, the run decompositions for the standard key tableaux in Figure 31 are \((54|321), (5|43|21), (5|432|1), (54|32|1), (543|21))\), respectively.

**Definition 5.12 ([Ass17]).** For a standard tableau \(T\), let \((\tau^{(k)}| \cdots |\tau^{(1)})\) be the run decomposition of \(T\). Set \(t_k = \text{row}(\tau^{(k)})\) and, for \(i < k\), set \(t_i = \min(\text{row}(\tau^{(i)}), t_{i+1} - 1)\). Define the weak descent composition of \(T\), denoted by \(\text{des}(T)\), by \(\text{des}(T)_{t_i} = |\tau^{(i)}|\) and all other parts are zero if all \(t_i > 0\) and \(\text{des}(T) = \emptyset\) otherwise.

For example, the recording tableau for \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\), constructed in Figure 32, is a standard key tableau. Though it is virtual, if we bump the example to \((6, 7, 4, 5, 6, 8, 4, 2, 5, 3, 4, 7)\), the weak descent composition of the standard key tableau becomes \((3, 2, 1, 4, 0, 2)\), which is precisely the weak descent composition of \((6, 7, 4, 5, 6, 8, 4, 2, 5, 3, 4, 7)\).

To connect insertion back to polynomials, we recall the key polynomial basis. The key polynomials first arose as Demazure characters for the general linear group [Dem74] and were later studied combinatorially by Lascoux and Schützenberger [LS90] who proved that Schubert polynomials expand non-negatively into the key basis. Weak insertion provides a new proof of this result along with a simple characterization of the coefficients.
Definition 5.13 ([Ass17]). The key polynomial for a weak composition \( a \) is given by
\[
\kappa_a = \sum_{T \in \text{SKT}(a)} F_{\text{des}(T)},
\]
where the sum may be taken over non-virtual standard key tableaux of shape \( a \).

For example, from Figure 31 we compute,
\[
\kappa_{(0,3,0,2)} = F_{(0,3,0,2)} + F_{(2,2,0,1)} + F_{(1,3,0,1)} + F_{(2,3,0,0)}.
\]

Notice that we have the following key expansion for our running example,
\[
\kappa_{(0,3,0,2)} = \kappa_{(3,0,0,0)} + \kappa_{(3,2,0,0)} = \kappa_{\text{des}(4,2,1,2,3)} + \kappa_{\text{des}(2,4,1,2,3)}.
\]

Observe from Figure 28 that this expansion precisely corresponds to the Yamanouchi terms. To prove that this happens in general, we recall from [Ass17] the weak dual equivalence involutions on standard key tableaux that play an analogous role to Coxeter-Knuth relations on reduced expressions.

Definition 5.14 ([Ass17]). Given \( T \in \text{SKT}(a) \) and \( 1 < i < |a| \), define \( d_i(T) \) as follows. Let \( b, c, d \) be the cells with entries \( i - 1, i, i + 1 \) taken in column reading order. Then
\[
d_i(T) = \begin{cases} 
  b_i(T) & \text{if } b, d \text{ are in the same row and } c \text{ is not}, \\
  s_{i-1}(T) & \text{else if } c \text{ has entry } i + 1, \\
  s_i(T) & \text{else if } c \text{ has entry } i - 1, \\
  T & \text{otherwise},
\end{cases}
\]
where \( b_j \) cycles \( j - 1, j, j + 1 \) so that \( j \) shares a row with \( j \pm 1 \) and \( s_j \) interchanges \( j \) and \( j + 1 \).

The involutions \( d_i \) are elementary weak dual equivalence relations, and any two standard key tableaux in the same equivalence class are called weak dual equivalent. For examples, Figure 33.

![Figure 33. The weak dual equivalence class for SKT(0,3,0,2).](image)

Lemma 5.15. For \( \rho \) a reduced expression, \( Q(d_i(\rho)) = d_i(Q(\rho)) \) and \( \text{des}(\rho) = \text{des}(Q(\rho)) \).

Proof. In [Ass17], it is shown that two standard key tableaux are weak dual equivalent if and only if they have the same shape. Moreover, the elementary involutions commute with the bijection between standard key tableaux and standard Young tableaux. Finally, the Coxeter-Knuth involutions give a weak dual equivalence, which implies there is a des-preserving bijection between Coxeter-Knuth equivalence classes of reduced expressions and weak dual equivalence classes of standard key tableaux. The result now follows.

By Lemma 5.15 we may use the Coxeter-Knuth involutions together with the weak dual equivalence involutions to show that Yamanouchi reduced expressions characterize the key expansion of Schubert polynomials.

Theorem 5.16. The weak insertion algorithm induces a des-preserving bijection
\[
R(w) \xrightarrow{\sim} \bigcup_{\rho \in YR(w)} \text{SKT}(\text{des}(\rho)),
\]
where the disjoint union is over Yamanouchi reduced expressions for \( w \).
Proof. By Theorem 5.9, $P(\sigma) = P(\tau)$ if and only if $\sigma$ and $\tau$ are Coxeter-Knuth equivalent, and this holds for both Edelman-Greene and weak insertion tableaux. If $\rho$ is Yamanouchi, then row$(P(\rho)) = \rho$, so each Coxeter-Knuth class has a unique Yamanouchi element. By Lemma 5.15 and the result that two standard key tableaux are weak dual equivalent if and only if they have the same shape [Ass17], the set of $Q(\rho)$ such that $P(\rho) = P$ is equal to SKT($a$) for some weak composition $a$. If $\rho$ is Yamanouchi, then $Q(\rho)$ is the Yamanouchi key tableau whose reverse row reading word is the identity. Therefore, for $\rho$ is Yamanouchi, the set of $Q(\sigma)$ such that $P(\sigma) = P(\rho)$ is equal to SKT(des($\rho$)). This establishes the bijection, and the preservation of weak descent compositions follows by Lemma 5.15. 

Corollary 5.17. The Schubert polynomial $G_w$ is given by
\begin{equation}
G_w = \sum_{D \in \text{YRD}(w)} \kappa_{\text{wt}(D)},
\end{equation}
where the sum is over Yamanouchi reduced diagrams $D$.

For a final example, from Figure 34 we can compute
\[ G_{41758236} = \kappa_{(3,0,4,2,3)} + \kappa_{(5,0,2,2,3)} + \kappa_{(4,0,4,2,2)}. \]

Note that the monomial expansion of $G_{41758236}$ contains 143 terms, the fundamental slide expansion contains 65 non-virtual terms, and the key expansion contains only 3 terms.

6. Kohnert diagrams

Kohnert [Koh91] defined a class of moves on diagrams that selects a nonempty row and pushes the rightmost cell of that row down to the first open position below it. We call such a move a Kohnert move. Given a diagram $D$, call the diagrams that result from a sequence of Kohnert moves on $D$ the Kohnert diagrams for $D$, and denote the set of them by $\text{KD}(D)$. For example, Figure 35 shows the Kohnert diagrams for the Rothe diagram of 153264.

For a weak composition $a$, denote the Kohnert diagrams for the key diagram of $a$ by $\text{KD}(a)$. Kohnert proved that $\text{KD}(a)$ precisely generates a key polynomial.

Theorem 6.1 ([Koh91]). The key polynomial $\kappa_a$ is given by
\begin{equation}
\kappa_a = \sum_{D \in \text{KD}(a)} x^{\text{wt}(D)},
\end{equation}
where $\text{KD}(a)$ denotes the set of non-virtual Kohnert diagrams for $a$.

For a permutation $w$, denote the Kohnert diagrams for the Rothe diagram of $w$ by $\text{KD}(w)$. Kohnert [Koh91] asserted the corresponding rule for computing the Schubert polynomial for $w$ from Kohnert diagrams for the Rothe diagram for $w$. For example, from Figure 35 we have
\[ G_{153264} = \sum_{D \in \text{KD}(153264)} x^{\text{wt}(D)}. \]
Kohnert attempted proofs of this rule, and Winkel [Win99, Win02] published two proofs, though neither of their proofs is widely accepted by the community. We present a simple, direct proof of this rule by giving a bijection between reduced diagrams and Kohnert diagrams. We do this in levels, beginning with the trivial case of the super-Yamanouchi reduced diagram. We work down through Yamanouchi reduced diagrams using Corollary 5.17 then quasi-Yamanouchi reduced diagrams using the weak recording tableaux, then general reduced diagrams via de-standardization.

**Definition 6.2.** Given a a Yamanouchi reduced expression $\rho$ for a permutation $w$, the Kohnert diagram of $\rho$, denoted by $K(\rho)$, is obtained by aligning $P(\rho)$ with respect to the super-Yamanouchi reduced expression for $w$. Call such a diagram a **Yamanouchi Kohnert diagram for** $w$.

Denote the set of Yamanouchi Kohnert diagrams for $w$ by YKD($w$). For example, Figure 34 shows YKD(41758236).

**Proposition 6.3.** Every Yamanouchi Kohnert diagram for $w$ is a Kohnert diagram for $w$.

**Proof.** Given a Yamanouchi reduced expression $\rho$ for a permutation $w$, take any sequence of braids and swaps from the super-Yamanouchi reduced expression for $w$ to $\rho$. Then we may obtain the Kohnert diagram of $\rho$ by applying $s_i$ as usual, and for $b_i$ we allow cells to jump without shifting themselves and other elements of their column to the right. Since the leftmost cell of each column remains fixed, this is always possible. Moreover, $s_i$ can be regarded as a sequence of Kohnert moves and their inverses, where cells after the $i+1$st and in the same row move down by Kohnert moves, then the $i$th cell moves down by a Kohnert move, then de-standardize, which itself is a sequence of reverse Kohnert moves. When moving down from the super-Yamanouchi reduced expression, braids will be of the form $(a+1)a(a+1)$ goes to $a(a+1)a$. Thus a modified braid move pushes cells after the $i+1$st and in the same row move down by Kohnert moves, then the $i-1$st cell is pushed, jumping over the $i$th cell, by a Kohnert move, then de-standardize by a sequence of reverse Kohnert moves. Therefore the Kohnert diagram for $\rho$ is indeed a Kohnert diagram for $w$. □

**Definition 6.4.** Given a reduced expression $\rho$ for a permutation $w$, the Kohnert diagram of $\rho$, denoted by $K(\rho)$, is obtained as follows: begin with the weak recording tableau for $\rho$; move cells right, maintaining their relative row order, until the positions of cells agree with those for the Yamanouchi Kohnert diagram corresponding to the weak insertion tableau for $\rho$; push cells minimally.
down until the reverse row reading word is the identity. Call such a diagram a *quasi-Yamanouchi Kohnert diagram for* \( w \).

Denote the set of quasi-Yamanouchi Kohnert diagrams for \( w \) by \( \text{QKD}(w) \). For example, the Kohnert diagram for the reduced expression \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\) is shown in Figure 36.

**Figure 36.** The Kohnert diagram for the reduced expression \((5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)\).

**Proposition 6.5.** Every quasi-Yamanouchi Kohnert diagram for \( w \) is a Kohnert diagram for \( w \).

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**Proof.** The only step to consider is the descended diagram. This can be done systematically by pushing the smallest labeled box down first, then continuing with the next highest box that needs to be pushed. Since rows weakly decrease for a key tableau, and since this property is maintained by this choice of cells to push, each move is a Kohnert move, beginning with the Yamanouchi Kohnert diagram. The result now follows from Proposition 6.3. \( \Box \)

Given a quasi-Yamanouchi Kohnert diagram \( D \), we may label all cells by their row index and consider certain Kohnert moves that do not allow cells with different labels in the same row. These are precisely the diagrams that de-standardize to \( D \). Therefore, by Proposition 3.8, the generating polynomial of these moves is the fundamental slide polynomial indexed by the weight of \( D \). This leads to the following definition.

**Definition 6.6.** Given a reduced diagram \( D \) for \( w \), the *Kohnert diagram of* \( D \), denoted by \( \mathbb{K}(D) \), is obtained as follows: let \( \rho \) be the reduced expression to which \( D \) de-standardizes, and set \( \mathbb{K}(D) \) to be the result of applying to \( \mathbb{K}(\rho) \) the reverse of the de-standardization moves that lift \( D \) to \( \mathbb{D}(\rho) \).

**Theorem 6.7.** The map \( \mathbb{K} \) is a \( \text{wt} \)-preserving bijection \( \text{RD}(w) \xrightarrow{\sim} \text{KD}(w) \).

---

**Proof.** By Proposition 6.5 and the earlier observation that de-standardization is a sequence of reverse Kohnert moves, we have that \( \mathbb{K}(D) \in \text{KD}(w) \) for any \( D \in \text{RD}(w) \). Injectivity follows from the fact that reduced diagrams are distinct, which is implied by Theorem 3.15. For surjectivity, we note that the subset of Kohnert moves that maintain the Yamanouchi-class is in the image, so it is enough to show that any Kohnert move that does not maintain the Yamanouchi-class gives a diagram that lies in another Yamanouchi-class for \( w \). Clearly \( \mathbb{D}(w) \) belongs to the Yamanouchi-class of the super-Yamanouchi reduced expression, so we may proceed by induction.

Using the bijection between the Yamanouchi-class for \( \rho \) and \( \text{KD}(\text{wt}(\rho)) \), the Kohnert moves that do not preserve the Yamanouchi-class are precisely those in which a cell lands in a row that, under the bijection with \( \text{KD}(\text{wt}(\rho)) \), is occupied by another cell. Using the labeling algorithm from [AS16] together with the bijection between the Yamanouchi-class for \( \rho \) and \( \text{KD}(\text{wt}(\rho)) \), we may uniquely identify each cell of \( D \) as originating in a given row of \( \mathbb{K}(\rho) \) (which coincides with the rows of \( \mathbb{D}(\rho) \)). Labeling cells in this way, the Kohnert moves that do not preserve the Yamanouchi-class are precisely those where a cell labeled \( i \) is pushed weakly below a cell labeled \( j \) and there are more \( j \)s left of the cell than \( i \)s. This means that part of the cells from row \( i \) have been pushed into the longer row \( j \). Relabeling the offending \( i \) and every \( i \) to its right as a \( j \), we may apply reverse Kohnert moves to push cells back into the rows from which they originated. The result will have the property that left justifying cells, those whose labels changed from \( i \) to \( j \) cannot raise up to row
i because that row is strictly shorter. That is, this is a Yamanouchi Kohnert diagram. Therefore $D$ lies in a Yamanouchi-class.

Theorems 6.1, 6.7, and Corollary 5.17 establish Kohnert’s rule for Schubert polynomials.

**Corollary 6.8** (Kohnert’s rule). The Schubert polynomial for $w$ is given by

$$S_w = \sum_{D \in KD(w)} x^{\text{wt}(D)},$$

where the sum may be taken over non-virtual Kohnert diagrams for $w$.

**References**

[AS16] Sami Assaf and Dominic Searles, *Key polynomials, quasi-key polynomials, and Kohnert tableaux*, arXiv:1609.03507, 2016.

[AS17] ———, *Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams*, Adv. in Math. 306 (2017), 89–122.

[Ass17] Sami H. Assaf, *Weak dual equivalence for polynomials*, arXiv:1702.04051, 2017.

[BGG73] I. N. Bernšteǐn, I. M. Gel’fand, and S. I. Gel’fand, *Schubert cells, and the cohomology of the spaces $G/P$*, Uspehi Mat. Nauk 28 (1973), no. 3(171), 3–26. MR 0429933

[BJS93] Sara C. Billey, William Jockusch, and Richard P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. 2 (1993), no. 4, 345–374. MR 1241505 (94m:05197)

[Bor53] Armand Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) 57 (1953), 115–207. MR 0051508

[Dem74] Michel Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. (2) 98 (1974), no. 3, 163–172. MR 0430001 (55 #3009)

[EG87] Paul Edelman and Curtis Greene, *Balanced tableaux*, Adv. in Math. 63 (1987), no. 1, 42–99. MR 871081 (88b:05012)

[FGRS97] Sergey Fomin, Curtis Greene, Victor Reiner, and Mark Shimozono, *Balanced labellings and Schubert polynomials*, European J. Combin. 18 (1997), no. 4, 373–389. MR 1444248

[Ful92] William Fulton, *Flags, Schubert polynomials, degeneracy loci, and determinantal formulas*, Duke Math. J. 65 (1992), no. 3, 381–420. MR 1154177

[HI02] David Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. 8 (1902), no. 10, 437–479. MR 1557926

[KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) 161 (2005), no. 3, 1245–1318. MR 2180402

[Knu70] Donald E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific J. Math. 34 (1970), 709–727. MR 0272654

[Koh91] Axel Kohnert, *Weintrauben, Polynome, Tableaux*, Bayreuth. Math. Schr. (1991), no. 38, 1–97, Dissertation, Universität Bayreuth, Bayreuth, 1990. MR 1132534

[LS82] Alain Lascoux and Marcel-Paul Schützenberger, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 13, 447–450. MR 660739 (83e:14039)

[LS90] ———, *Keys & standard bases*, Invariant theory and tableaux (Minneapolis, MN, 1988), IMA Vol. Math. Appl., vol. 19, Springer, New York, 1990, pp. 125–144. MR 1035493 (91c:05198)

[Rob38] G. de B. Robinson, *On the Representations of the Symmetric Group*, Amer. J. Math. 60 (1938), no. 3, 745–760. MR 01507943

[Sch61] C. Shensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. 13 (1961), 179–191. MR 0121305

[Sch79] Hermann Schubert, *Kalkül der abzählenden Geometrie*, Springer-Verlag, Berlin-New York, 1979, Reprint of the 1879 original, With an introduction by Steven L. Kleiman. MR 555576 (82c:01073)

[Sta84] Richard P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. 5 (1984), no. 4, 359–372. MR 782057 (86i:05011)

[Win99] Rudolf Winkel, *Diagram rules for the generation of Schubert polynomials*, J. Combin. Theory Ser. A 86 (1999), no. 1, 14–48. MR 1682961

[Win02] ———, *A derivation of Kohnert’s algorithm from Monk’s rule*, Sém. Lothar. Combin. 48 (2002), Art. B48f, 14. MR 1988615

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