ON POSITIVE SOLUTIONS TO SEMI-LINEAR
CONFORMALLY INVARIANT EQUATIONS ON
LOCALLY CONFORMALLY FLAT MANIFOLDS

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ABSTRACT. In this paper we study the existence and compactness of positive solutions to a family of conformally invariant equations on closed locally conformally flat manifolds. The family of conformally covariant operators $P_\alpha$ were introduced via the scattering theory for Poincaré metrics associated with a conformal manifold $(M^n, [g])$. We prove that, on a closed and locally conformally flat manifold with Poincaré exponent less than $\frac{n-\alpha}{2}$ for some $\alpha \in [2, n)$, the set of positive smooth solutions to the equation

$$P_\alpha u = u^{\frac{n+\alpha}{n-\alpha}}$$

is compact in the $C^\infty$ topology. Therefore the existence of positive solutions follows from the existence of Yamabe metrics and a degree theory.

1. Introduction

In a recent paper of Graham and Zworski [GZ] a meromorphic family of conformally covariant operators associated with a Riemannian manifold $(M^n, g)$ was introduced via scattering theory for a Poincaré metric associated with $(M^n, [g])$. For a metric $g \in [g]$ on $M^n$, the scattering operator $S(z)$ is a meromorphic family of pseudo-differential operators on $M$ for $\text{Re}(z) > n/2$ of order $2\text{Re}(z) - n$ with the principal symbol

$$\sigma(S(z)) = 2^{n-2z} \frac{\Gamma(n/2 - z)}{\Gamma(z - n/2)} \sigma((\Delta_g)^{z-\frac{n}{2}}).$$

The scattering operator is conformally covariant in the sense that

$$S(z)[e^{2\Upsilon}g] = e^{-z\Upsilon}S(z)[g]e^{(n-z)\Upsilon}.$$

At the special values $z = (n/2) + k$,

$$\text{Res}_{z = \frac{n}{2} + k} S(z) = c_k P_{2k},$$

where $c_k = (-1)^{k+1}[2^{2k-1}k!(k-1)!]^{-1}$ and $P_{2k}$ is the conformally invariant powers of the Laplacian introduced in [GJMS]. For instance,

$$P_2[g] = -\Delta_g + \frac{n-2}{4(n-1)} R[g]$$

(1.1)
is the conformal Laplacian and

\[
(1.2) \quad P_4[g] = (-\Delta)^2 - \text{div}_g\left(\frac{(n-2)^2 + 4}{2(n-1)(n-2)} R[g] - \frac{4}{n-2} \text{Ric}[g] d\right) + \frac{n-4}{2} Q[g]
\]

is the Paneitz operator \([P]\), where \(R[g]\) is the scalar curvature, \(\text{Ric}[g]\) is the Ricci curvature, and

\[
(1.3) \quad Q[g] = -\frac{1}{2(n-1)} \Delta R + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2 - \frac{4}{(n-2)^2} |\text{Ric}|^2
\]

is the so-called \(Q\)-curvature.

The conformal Laplacian governs the transformation of the scalar curvature under the conformal change of metrics

\[
(1.4) \quad P_2[g] u = \frac{n-2}{4(n-1)} R[u^{\frac{4}{n-2}} g] u^{\frac{n+2}{n-2}};
\]

while the Paneitz operator \(P_4\) governs the transformation of \(Q\) curvature

\[
(1.5) \quad P_4[g] u = \frac{n-4}{2} Q[u^{\frac{4}{n-4}} g] u^{\frac{n+4}{n-4}}.
\]

The well-known Yamabe problem in differential geometry is to find a metric of constant scalar curvature in a given class of conformal metrics, that is, to solve the Yamabe equation

\[
(1.6) \quad P_2[g] u = u^{\frac{n+2}{n-2}}
\]

for some positive function \(u\) on a given manifold \((M^n, g)\) \((n \geq 3)\). The affirmative resolution to the Yamabe problem was given in \([Sc1]\) after other notable works \([Ya]\) \([Tr]\) \([Au]\). Recent developments in the study of conformal geometry and conformally invariant partial differential equations have created increasingly interests in the higher order generalizations of Yamabe problem as are described in \([CY1]\) (references therein). One may ask whether there is a metric of constant \(Q\) curvature in a given conformal class of metrics, that is, to solve the Paneitz-Branson equation

\[
(1.7) \quad P_4[g] u = u^{\frac{n+4}{n-4}}
\]

for some positive smooth function \(u\) on a given manifold \((M^n, g)\) \((n \geq 5)\) \([P]\) \([Br]\) \([DHL]\) \([DMA]\).

In our previous paper \([QR]\), we considered the Paneitz-Branson equation (1.7) and were able to obtain the compactness of metrics of constant \(Q\) curvature in a given conformal class on a locally conformally flat manifold. But it has been a very challenging problem to find a positive solution to the higher order Paneitz-Branson equation. We discovered it is useful to consider a smooth family of equations

\[
(1.8) \quad P_\alpha[g] u = u^{\frac{n+\alpha}{n-\alpha}}
\]
where

\[(1.9) \quad P_\alpha[g] = 2^\alpha \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} S\left(\frac{n + \alpha}{2}\right)[g]\]

and \(\alpha \in [2, n)\).

On a locally conformally flat manifold \((M^n, g)\) with positive Yamabe constant, there is the family of conformally covariant operators \(P_\alpha\) associated with the hyperbolic metric whose conformal infinity is \((M^n, [g])\). Readers are referred to a paper of Patterson and Perry [PP] for more detailed discussions on the scattering theory for the conformally compact hyperbolic manifolds. If the Poincaré exponent of the holonomy representation of the fundamental group \(\pi_1(M)\) is less than \(n - \alpha^2\), then there is a conformally covariant integral operator \(I_\alpha\), which is a right inverse to \(P_\alpha\). The existence of such integral operators allow us to consider the conformally invariant integral equation

\[(1.10) \quad u = I_\alpha[g](u^{\frac{n+\alpha}{n}}).\]

We observed that the new moving plane method introduced by Chen, Li and Ou in [CLO] (please also see [CY2]) can be adopted to study the conformally invariant integral equation (1.10) and allows us to prove the following

**Theorem 1.1.** Suppose that \((M^n, g)\) is a locally conformally flat manifold with positive Yamabe constant and Poincaré exponent less than \(\frac{n-2\alpha}{2}\) for some \(\alpha \in [2, n)\). And suppose that \((M^n, g)\) is not conformally equivalent to the standard round sphere. Then there exists a constant \(C = C(n, \alpha, k)\) such that, for any smooth positive solution \(u\) to (1.10), we have

\[(1.11) \quad \|u\|_{C^k(M)} + \frac{1}{u}\|u\|_{C^k(M)} \leq C.\]

Consequently, based on the degree theory described in [Sc2], we can prove the existence of positive solutions to both equations (1.8) and (1.10).

**Theorem 1.2.** Suppose that \((M^n, g)\) is a locally conformally flat manifold with positive Yamabe constant and Poincaré exponent less than \(\frac{n-2\alpha_0}{2}\) for some \(\alpha_0 \in [2, n)\), then, for any \(\alpha \in [2, \alpha_0]\), there exists a positive smooth function \(u\), which solves both equations (1.8) and (1.10).

In particular,

**Theorem 1.3.** Suppose that \((M^n, g)\) (\(n \geq 5\)) is a locally conformally flat manifold with positive Yamabe constant and Poincaré exponent less than \(\frac{n-4}{2}\), then there exists a positive smooth solution \(u\) to the Paneitz-Branson equation (1.7).

The organization of this paper is as follows. We will first describe in Section 2 the conformally covariant integral operator \(I_\alpha\). Then we will discuss its relation to the conformally covariant pseudo-differential operator \(P_\alpha\) in Section 3. In Section 4 we adopt the new moving plane method to derive an apriori estimate in the form of a convexity theorem for solutions to (1.10). In Section 5 we will introduce the rescaling method for the conformally invariant integral equations (1.10), obtain the compactness and prove the existence.
2. CONFORMALLY COVARIANT OPERATORS ON LCF MANIFOLDS

We start with a discussion of the scattering operator and an introduction of the integral operators on locally conformally flat manifolds. We first recall from Proposition 4.1 in [PP] that on the hyperbolic space $H^{n+1}$ in the half space model the scattering operator is

$$S^0(z) = 2^{n-2z} \frac{\Gamma\left(\frac{n}{2} - z\right)}{\Gamma(z - \frac{n}{2})} (-\Delta)^{2z-n}.$$  \hspace{1cm} (2.1)

Hence

$$P_0^\alpha = 2^\alpha \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)} S^0\left(\frac{\alpha + n}{2}\right) = (-\Delta)^\frac{\alpha}{2}. \hspace{1cm} (2.2)$$

Suppose that $(M^n, g)$ is a closed locally conformally flat manifold with positive Yamabe constant. According to Schoen-Yau [SY], the developing map from the universal cover of $M^n$ into the round sphere $(S^n, g_1)$ is injective. The deck transformation group of this covering becomes a Kleinian group $\Gamma$. The image of the developing map is the set $\Omega(\Gamma)$ of ordinary points for the Kleinian group $\Gamma$, and $(M^n, [g])$ is conformally equivalent to $\Omega(\Gamma)/\Gamma$. Therefore we may consider the universal covering

$$\phi: (\Omega(\Gamma), \tilde{g}) \rightarrow (M, g) \hspace{1cm} (2.3)$$

and write $\tilde{g} = \phi^* g = \tilde{\eta}^2 g_1$. Using stereographic projection

$$\psi: \tilde{\Omega} \subset \mathbb{R}^n \rightarrow \Omega(\Gamma) \subset S^n \hspace{1cm} (2.4)$$

with respect to some point in $\Omega(\Gamma) \subset S^n$, we may write $\tilde{g} = \psi^* \tilde{g} = \tilde{\eta}^2 g_0$, where $g_0$ is the Euclidean metric and $\tilde{\eta} = (\tilde{\eta} \circ \psi)(\frac{2}{1 + |x|^2})$. Thus we unfolded a locally conformally flat manifold $(M^n, g)$ into a conformally flat manifold $(\tilde{\Omega}, \tilde{g})$.

One way to understand the scattering operator for the conformally compact hyperbolic manifold $H^{n+1}/\Gamma$ is to lift everything to the hyperbolic space $H^{n+1}$. Upon using the geodesic defining function corresponding to the choice of the metric $g \in [g]$ on $M$, we can equivalently consider the lifting from $M$ to $\tilde{\Omega} \subset \mathbb{R}^n$. Let $u$ be a function on $M$ and $\tilde{u} = u \circ \phi \circ \psi$. Then

$$P_\alpha[g] u = \hat{P}_\alpha[\tilde{g}] \tilde{u} |_F, \hspace{1cm} (2.5)$$

where $F$ is a fundamental domain for $\Gamma$. By the conformally covariant property, we have

$$\hat{P}_\alpha[\tilde{g}] \tilde{u} = \tilde{\eta}^{-\frac{n+\alpha}{2}} \hat{P}_\alpha[g_0] \tilde{\eta}^{-\frac{n-\alpha}{2}} \tilde{u}, \hspace{1cm} (2.6)$$

where clearly $\hat{P}_\alpha[g_0] = P_\alpha^0 = (-\Delta)^\frac{\alpha}{2}$ due to the uniqueness of solutions to the Poisson equations. Notice that $\tilde{\eta}^{-\frac{n-\alpha}{2}} \tilde{u} \in L^1(\mathbb{R}^n)$ (cf. Corollary 2.2 in [QR]).
We now turn to consider an integral operator on the Euclidean space $\mathbb{R}^n$

$$\hat{I}_{\alpha}(\hat{u})(x) = \int_{\mathbb{R}^n} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \hat{u}(y) dy, \quad \alpha \in (0, n),$$

for a function $\hat{u}$ on $\mathbb{R}^n$, where

$$c(n, \alpha) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{\frac{n-\alpha}{2}} \Gamma\left(\frac{n}{2}\right)}.$$  

It is well known that

$$(-\Delta)^{\frac{\alpha}{2}} \hat{I}_{\alpha}(\hat{u}) = \hat{u}, \quad \alpha \in (0, n).$$

Then, by the conformally covariant property, we have the integral operator, on $(\Omega(\Gamma), \tilde{g})$

$$\tilde{I}_{\alpha}[\tilde{g}](\tilde{u})(\tilde{x}) = \int_{S^n} \tilde{K}(\tilde{x}, \tilde{y})\tilde{u}(\tilde{y}) dV_{\tilde{g}}(\tilde{y}),$$

where

$$\tilde{K}(\tilde{x}, \tilde{y}) = \tilde{\eta}^{-\frac{n-\alpha}{2}}(\tilde{x})\left(\frac{1+|x|^2}{2}\right)^{-\frac{n-\alpha}{2}} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \left(\frac{1+|y|^2}{2}\right)^{-\frac{n-\alpha}{2}} \tilde{\eta}^{-\frac{n-\alpha}{2}}(\tilde{y}).$$

Notice that $S^n \setminus \Omega(\Gamma)$ is of Hausdorff dimension less than $\frac{n-2}{2}$ for a locally conformally flat manifold $(M^n, g)$ with positive Yamabe constant. Hence we have

$$\tilde{I}_{\alpha}[\tilde{g}](\tilde{u})(\tilde{x}) = \int_{\Omega(\Gamma)} \tilde{K}(\tilde{x}, \tilde{y})\tilde{u}(\tilde{y}) dV_{\tilde{g}}(\tilde{y}).$$

To push down the operator to $(M^n, g)$ we simply notice that we can always lift a function on $M$ to a periodic function on its covering $\Omega(\Gamma)$, which is,

$$\tilde{u}(\tilde{x}) = u(\phi(\tilde{x})).$$

So we define, for a fundamental domain $F \subset \Omega(\Gamma)$ and $\tilde{x} \in F$,

$$I_{\alpha}(u)(p) = \int_{\Omega(\Gamma)} \tilde{K}(\tilde{x}, \tilde{y})\tilde{u}(\tilde{y}) dV_{\tilde{g}}(\tilde{y}) = \sum_{\gamma \in \Gamma} \int_{\gamma F} \tilde{K}(\tilde{x}, \tilde{y}) u(q) dV_{\tilde{g}}(\tilde{y})$$

$$= \sum_{\gamma \in \Gamma} \int_{F} \tilde{K}(\tilde{x}, \gamma \tilde{y}) u(q) dV_{\tilde{g}}(\tilde{y}) = \int_{M} K(p, q) u(q) dV_{g}(q),$$

where $\phi(\tilde{x}) = p$, $\phi(\gamma \tilde{y}) = q$ and

$$K(p, q) = \sum_{\gamma \in \Gamma} \tilde{K}(\tilde{x}, \gamma \tilde{y}).$$
**Theorem 2.1.** Suppose that \((M^n, [g])\) is a locally conformally flat manifold with positive Yamabe constant and that the Poincaré exponent of the holonomy representation is less than \(\frac{n-\alpha}{2}\). Then the integral operator \(I_\alpha\) given in the above is a well-defined conformally covariant operator of bi-degree \((\frac{n-\alpha}{2}, \frac{n+\alpha}{2})\) on the space \(C^\infty(M)\). Moreover

\[(2.13)\quad P_\alpha[g] I_\alpha[g] u = u,\]

for any function \(u\) on \(M\).

**Proof.** Following the above discussion we need to show that the kernel \(K(p, q)\) is well-defined and that \(I_\alpha\) is independent of the choice of a fundamental domain \(F\). From (2.9) and

\[(2.14)\quad \eta(\gamma \bar{x}) = |\gamma'(\bar{x})|^{-1} \eta(\bar{x}).\]

we then have

\[
\sum_{\gamma \in \Gamma} \bar{K}(\bar{x}, \gamma \bar{y}) = \bar{\eta}^{-\frac{n-\alpha}{2}}(\bar{x}) \left(\frac{1 + |\bar{x}|^2}{2}\right)^{\frac{n-\alpha}{2}} \bar{\eta}^{-\frac{n+\alpha}{2}}(\bar{y}) \sum_{\gamma \in \Gamma} \frac{c(n, \alpha)}{|x - \gamma y|^{n-\alpha}} \left(\frac{1 + |\gamma y|^2}{2}\right)^{\frac{n+\alpha}{2}} |\gamma'|^{\frac{n+\alpha}{2}}(\bar{y}).
\]

Notice that there is some constant \(K > 0\) such that

\[(2.15)\quad \frac{c(n, \alpha)}{|x - \gamma y|^{n-\alpha}} \left(\frac{1 + |\gamma y|^2}{2}\right)^{\frac{n-\alpha}{2}} \leq K, \forall \gamma \in \Gamma \setminus \{1\}.
\]

Therefore, the convergence of \(K(p, q)\) depends only on the convergence of the Poincaré series

\[(2.16)\quad \sum_{\gamma \in \Gamma} |\gamma'|^{\frac{n+\alpha}{2}}.
\]

We know the above Poincaré series is convergent since the Poincaré exponent is less than \(\frac{n-\alpha}{2}\). To prove that the kernel function \(K(x, y)\) does not depends on \(F\), we only need to show that

\[(2.17)\quad \bar{K}(\gamma \bar{x}, \gamma \bar{y}) = \bar{K}(\bar{x}, \bar{y}).\]

We recall (1.3.2) from [Ni] that

\[(2.18)\quad |\gamma x - \gamma y| = |\gamma'(\bar{x})|^{\frac{1}{2}} |x - y||\gamma'(\bar{y})|^{\frac{1}{2}},\]

where \(|\gamma'|_e\) is the norm under the Euclidean metric. Then the above (2.17) follows from (2.18) and the following

\[(2.19)\quad |\gamma'(\bar{x})| = \frac{1 + |x|^2}{1 + |\gamma x|^2} |\gamma'(x)|_e.
\]
One may find a similar calculation in the proof of Proposition 1.1 in [CQY].

To prove (2.13) we simply calculate, for a function $u \in C^\infty(M)$ and $v = I_\alpha[g](u)$, in the light of (2.5),

$$P_\alpha[g]v = \hat{\eta}^{-\frac{n+\alpha}{2}}(-\Delta)^{-\frac{\alpha}{2}}\hat{\eta}^{-\frac{n-\alpha}{2}} \hat{v}$$

where, by (2.11) and (2.9),

$$\hat{\eta}^{-\frac{n+\alpha}{2}} \hat{v} = \int_{R^n} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}(1+|y|^2)^{n+\alpha}} \hat{\eta}^{-\frac{n-\alpha}{2}}(\psi(y))\hat{u}(y)dy.$$

Thus the proof is completed.

3. Conformally invariant equations

Suppose that $(M^n, g)$ is a locally conformally flat manifold with positive Yamabe constant, and that the Poincaré exponent of the holonomy representation of the fundamental group $\pi_1(M)$ is less than $\frac{n-\alpha}{2}$ for some $\alpha \in [2, n)$. We consider the conformally invariant integral equation

$$(3.1) \quad u(p) = I_\alpha(u^{\frac{n+\alpha}{n-\alpha}}) = \int_M K(p, q)u^{\frac{n+\alpha}{n-\alpha}}(q)dV_g(q)$$

for a positive function $u(p) \in C^\infty(M)$, where $K(p, q)$ is define as in Theorem 2.1 in the previous section. Let

$$(3.2) \quad \hat{v}(x) = u(\phi(\psi(x)))\hat{\eta}^{\frac{n-\alpha}{2}}(\psi(x))(\frac{2}{1+|x|^2})^{\frac{n-\alpha}{2}} \in C^\infty(\hat{\Omega}).$$

Then we have

$$(3.3) \quad \hat{v}(x) = \int_{\hat{\Omega}} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \hat{v}^{\frac{n+\alpha}{n-\alpha}}(y)dy.$$

Equation (2.13) shows that a positive solution $u$ to the integral equation (3.1) solves the pseudo-differential equation

$$(3.4) \quad P_\alpha u = u^{\frac{n+\alpha}{n-\alpha}} \quad \text{on } M$$

and a positive solution to (3.4) solves the integral equation (3.1) when $P_\alpha$ has a trivial kernel. Thus we have

Theorem 3.1. Suppose that $(M^n, [g])$ is a closed locally conformally flat manifold with positive Yamabe constant, and that the Poincaré exponent of the holonomy representation of the fundamental group $\pi_1(M)$ is less than $\frac{n-\alpha}{2}$ for some $\alpha \in [2, n)$. Then a positive smooth function $u$ solves the pseudo-differential equation

$$P_\alpha u = u^{\frac{n+\alpha}{n-\alpha}} \quad \text{on } M,$$
if and only if it solves the integral equation

\[ u = I_\alpha(u^{\frac{n+\alpha}{n-\alpha}}) \text{ on } M. \]

**Proof.** We need to prove that, when the Hausdorff dimension of the limit set of the holonomy representation of the fundamental group \( \pi_1(M) \) is less than \( \frac{n-\alpha}{2} \), the operator \( P_\alpha \) has a trivial kernel. (2.13) says \( P_\alpha \) has a right inverse \( I_\alpha \) when the Poincaré exponent of the holonomy representation of the fundamental group \( \pi_1(M) \) is less than \( \frac{n-\alpha}{2} \). Hence \( P_\alpha \) is surjective and its co-kernel is trivial. Therefore \( P_\alpha \) has a trivial kernel since \( P_\alpha \) for \( \alpha \in [2, n) \) is self-adjoint from [GZ] (see also an easier proof in [FG]).

For the scalar curvature and conformal Laplacian \( P_2 \), the Yamabe constant is defined as

\[
Y(M, [g]) = \inf \left\{ \int_M \phi P_2[g] \phi dV_g : \phi \in C^\infty(M), \phi > 0 \right\},
\]

which is a conformal invariant of \( (M, [g]) \). We may similarly define the Yamabe constant of order \( \alpha \) as follows:

\[
Y_\alpha(M, [g]) = \inf \left\{ \int_M \phi P_\alpha[g] \phi dV_g : \phi \in C^\infty(M), \phi > 0 \right\}.
\]

This is a family of conformal invariants of \( (M, [g]) \) since

\[
\frac{\int_M \phi P_\alpha[g] \phi dV_g}{(\int_M \phi^{\frac{2n}{n-\alpha}} dV_g)^{\frac{n-\alpha}{n}}} = \frac{\int_M (e^{-\frac{n-\alpha}{2} \phi} P_\alpha[e^{2\phi} g] e^{-\frac{n-\alpha}{2} \phi} \phi) dV_{e^{2\phi} g}}{(\int_M (e^{-\frac{n-\alpha}{2} \phi} \phi^{\frac{2n}{n-\alpha}} dV_{e^{2\phi} g})^{\frac{n-\alpha}{n}}}.
\]

We know that the first eigenvalue \( \lambda_1(P_2) \) of \( P_2 \) is positive if and only if the Yamabe constant is positive. We observed

**Theorem 3.2.** Suppose that \( (M^n, g) \) is a locally conformally flat manifold with positive Yamabe constant. Suppose that the Poincaré exponent of the holonomy representation of the fundamental group \( \pi_1(M) \) is less than \( \frac{n-\alpha}{2} \) for some \( \alpha \in [2, n) \). Then the Yamabe constant of order \( \alpha \) is positive.

**Proof.** We first claim that the first eigenvalue \( \lambda_1(P_\beta) \) for any \( \beta \in [2, \alpha] \) of \( P_\beta \) is positive. This is because we know \( \lambda_1(P_2) \) is positive. Then, due to the continuity of the operator \( P_\beta \) with respect to \( \beta \), \( \lambda_1(P_\beta) \) has to be zero before getting to negative. But, from the proof of the previous Theorem 3.1, we know \( P_\beta \) is injective for all \( \beta \in [2, \alpha] \). Hence \( \lambda_1(P_\beta) \) remains positive in \( [2, \alpha] \) from \( \beta = 2 \). Now, we simply apply the Sobolev inequality

\[
\|u\|_{L^{\frac{2n}{n-\alpha}}(M)} \leq C\|u\|_{W^{\frac{2n}{n-\alpha}, 2}(M)}
\]

to conclude \( Y_\alpha > 0 \) since \( \int_M uP_\alpha udV \) is equivalent to \( \|u\|_{W^{\frac{2n}{n-\alpha}, 2}(M)} \) by the definition of \( P_\alpha \) in this case.
4. Convexity

To obtain a priori estimates for positive solutions to the integral equation (3.1), we use the same approach as in [Sc2] [QR]. We will first establish a convexity theorem. The new moving plane method introduced by Chen, Li and Ou in [CLO], which was motivated by a work in [CY2], is what it takes for us to study the solutions to the class of integral equations (3.1). Let us state and prove the following convexity theorem.

**Theorem 4.1.** Suppose that \((M^n, [g])\) is a locally conformally flat manifold with positive Yamabe constant, and that the Hausdorff dimension of the limit set of the holonomy representation of the fundamental group \(\pi_1(M)\) is less than \(n-\alpha\) for some \(\alpha \in [2, n)\). In addition we assume that \((M^n, [g])\) is not conformally equivalent to the standard round sphere. Let \(u\) be a positive smooth solution to (3.1). Then every round ball in \((M, [g])\) is geodesically convex with respect to the metric \(u^{-\frac{4}{n-\alpha}}g\).

**Proof.** Fix any point \(\tilde{x}_0 \in \partial B \subset \Omega(\Gamma)\) and consider the stereographic projection with respect to the antipode of \(\tilde{x}_0\) on \(\partial B\). Hence

\[
\psi^{-1}(\partial B) = \{x \in \mathbb{R}^n : x_n = 0\}. \tag{4.1}
\]

It suffices to prove that, with respect to each \(\tilde{x}_0 \in \partial B\), the hyperplane \(\{x_n = 0\}\) is geodesically convex in the metric \(\hat{u}^{-\frac{4}{n-\alpha}}g_0\), where

\[
\hat{v}(x) = u \circ \phi \circ \psi(x)\eta\nabla^{\frac{n-\alpha}{\alpha}}(\psi(x))(\frac{2}{1 + |x|^2})^{\frac{n-\alpha}{\alpha}} \tag{4.2}
\]

is a positive solution to

\[
\hat{v}(x) = \int_{\Omega} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \frac{\hat{v}\nabla^{\frac{n+\alpha}{\alpha}}(y)}{\hat{v}\nabla^{\frac{n+\alpha}{\alpha}}(y)} dy. \tag{4.3}
\]

Now let us start the moving plane method introduced in [CLO]. Let

\[
\Sigma_\lambda = \{x \in \mathbb{R}^n : x_n > \lambda\} \quad \text{and} \quad S_\lambda = \{x \in \mathbb{R}^n : x_n = \lambda\}.
\]

We consider the reflection with respect to the hyperplane \(S_\lambda\)

\[
x_\lambda = (x_1, x_2, \cdots, x_{n-1}, 2\lambda - x_n), \forall x \in \mathbb{R}^n \tag{4.4}
\]

and define

\[
\hat{v}_\lambda(x) = \hat{v}(x_\lambda). \tag{4.5}
\]

Then, following a simple calculation, we have

\[
\hat{v}(x) - \hat{v}_\lambda(x) = c(n, \alpha) \int_{\Sigma_\lambda} (|x-y|^{-n+\alpha} - |x-y_\lambda|^{-n+\alpha})(\hat{v}\nabla^{\frac{n+\alpha}{\alpha}}(y) - \hat{v}_\lambda^{\nabla^{\frac{n+\alpha}{\alpha}}}(y)) dy. \tag{4.6}
\]
Hence it follows from Lemma 2.1 in [CLO] that, for some constant $C$,
\[(4.7) \hat{u}(x) - \hat{u}_\lambda(x) < C \int_{\Sigma_\lambda^-} |x - y|^{-n + \alpha} \hat{u}_{\alpha - \alpha}^\alpha(y)(\hat{u}(y) - \hat{u}_\lambda(y))dy,\]
where
\[(4.8) \Sigma_\lambda^- = \{ x \in \Sigma_\lambda : \hat{v}_\lambda(x) < \hat{v}(x) \} \]

We recall the classic Hardy-Littlewood-Sobolev inequality
\[(4.9) \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)|x - y|^{\alpha - n} g(y)dydx \right| \leq c(n, \alpha, s) \| f \|_{L^r(\mathbb{R}^n)} \| g \|_{L^s(\mathbb{R}^n)},\]
where
\[\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{n},\]
f $\in L^r(\mathbb{R}^n)$ and $g \in L^s(\mathbb{R}^n)$. Therefore, if let
\[(4.10) f = (\hat{v} - \hat{v}_\lambda)^{\mu - 1} \chi_{\Sigma^-_\lambda}, \quad g = \hat{v}^{\frac{2\alpha}{n - \alpha}}(\hat{v} - \hat{v}_\lambda)\chi_{\Sigma^-_\lambda},\]
\[\nu = \frac{n}{\alpha}, \quad \mu > \frac{n}{n - \alpha}, \quad \frac{1}{s} = \frac{1}{\nu} + \frac{1}{\mu}, \quad \frac{1}{r} + \frac{1}{\mu} = 1,\]
from (4.7) we have
\[(4.11) \| \hat{v} - \hat{v}_\lambda \|_{L^\mu(\Sigma^-_\lambda)} \leq c(n, \alpha, \mu)C \| \hat{v} - \hat{v}_\lambda \|_{L^\mu(\Sigma^-_\lambda)} \| \hat{v} \|_{L^{\frac{2\alpha}{n - \alpha}}(\Sigma^-_\lambda)},\]
which implies
\[(4.12) c(n, \alpha, \mu)C \| \hat{v} \|_{L^{\frac{2\alpha}{n - \alpha}}(\Sigma^-_\lambda)} \geq 1,\]
if
\[(4.13) \hat{v} - \hat{v}_\lambda \in L^\mu(\Sigma^-_\lambda).\]

We observe that
\[(4.14) \hat{v} \in L^{\frac{n + \alpha}{n - \alpha}}(\mathbb{R}^n),\]
which can be proven in a way similar to Corollary 2.2 in [QR]. (4.14) implies (4.13) since $\hat{L}$ is a compact subset of $\mathbb{R}^n$.

We notice that, when $\lambda$ is very large, $\Sigma_\lambda$ corresponds to a very small round ball $B_\epsilon \subset B \subset \Omega(\Gamma) \subset S^n$. Therefore, when $\lambda$ is very large,
\[|\Sigma^-_\lambda| = 0,\]
which implies $\Sigma_\lambda^-$ is empty when $\lambda$ is very large. Thus we have

$$\hat{v}(x) \leq \hat{v}_\lambda(x), \quad \forall x \in \Sigma_\lambda,$$

when $\lambda$ is sufficiently large. This gets the moving plane started. Next we show that we can move the hyperplane down as long as it does not touch the singular set $\hat{L}$, which is the image of the limit set under the stereographic projection $\psi$. Suppose that

$$\lambda_0 = \inf\{\lambda : \hat{v}(x) \leq \nu_\lambda(x), \quad \forall x \in \Sigma_\lambda\}$$

and

$$\Sigma_{\lambda_0} \cap \hat{L} = \emptyset.$$

Notice that $\hat{L}$ is not empty because $(M^n, [g])$ is not conformally equivalent to the round sphere. Then, by (4.15), we know

$$\hat{v}(x) < \hat{v}_{\lambda_0}(x), \forall x \in \Sigma_{\lambda_0}.$$

Hence, as observed in [CLO],

$$\lim_{\lambda \to \lambda_0} |\Sigma_\lambda^-| = 0.$$

Therefore there is some small number $\epsilon > 0$, such that

$$\Sigma_{\lambda_0 - \epsilon} \cap \hat{L} = \emptyset$$

and

$$c(n, \alpha, \mu) C\|\hat{v}\|_L^{\frac{\alpha}{n-2\alpha}}(\Sigma_\lambda^-) < \frac{1}{2}, \quad \forall \lambda_0 - \epsilon \leq \lambda \leq \lambda.$$

Thus $\Sigma_\lambda^-$ has to be empty and

$$\hat{v}(x) \leq \hat{v}_\lambda(x), \forall x \in \Sigma_\lambda$$

for $\lambda \geq \lambda_0 - \epsilon$, which contradicts with the definition of $\lambda_0$. Particularly we have, for $\lambda = 0$,

$$\hat{v}(x) < \hat{v}_0(x), \forall x \in \Sigma_0$$

and

$$\frac{\partial \hat{v}}{\partial x_n}|_{x_n=0} = (n - \alpha) c(n, \alpha) \int_{\Sigma_0} \frac{y_n (\hat{u}_{\frac{n+\alpha}{n-\alpha}} - \hat{u}_{\frac{n-\alpha}{n+\alpha}}(y))}{|x-y|^{n-\alpha+2}} dy < 0.$$

So the convexity is proven.
5. A priori estimate and existence

In this section we derive a priori estimates for solutions to the integral equation (3.1) and pseudo-differential equation (3.4) on a locally conformally flat manifold \((M^n, [g])\), when the Poincaré exponent of the holonomy representation of its fundamental group \(\pi_1(M)\) is less than \(\frac{n-\alpha}{2}\). The idea will be the same as in [Sc2] and [QR], which is to use the convexity to eliminate possible blow-ups. Due to the global nature of the equations we describe, for instance, the rescaling method for integral equation (3.1). For that, we first establish the \(C^{k,\theta}\) estimates based on the \(L^\infty\) bounds.

**Lemma 5.1.** Suppose that \((M^n, [g])\) is a locally conformally flat manifold \((M^n, [g])\) such that the Poincaré exponent of the holonomy representation of its fundamental group \(\pi_1(M)\) is less than \(n-\alpha\) for some \(\alpha \in [2, n)\). And suppose that \(u\) is a positive smooth solution to the equation (3.1) on \(M\). Then, for \(\theta \in (0, 1)\),

\[
\|u\|_{C^{k,\theta}(M)} \leq C(\|u\|_{L^\infty(M)}).
\]

**Proof.** We may unfold the function \(u\) and consider \(\hat{v}\) on \(\hat{\Omega}\), which satisfies

\[
\hat{v} = \int_{\mathbb{R}^n} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \hat{v}^{\frac{n+\alpha}{n-\alpha}}(y) dy.
\]

Then the estimate follows from standard elliptic theory.

Next suppose that \(u\) is a positive smooth solution to (3.1). Let \(p_0 \in M\) be a fixed point in \(M\) and let

\[
\zeta(x) = \phi(\psi(\eta^\frac{n-\alpha}{n-2}(0)x)) : B_{2\delta}(0) \to M, \text{ and } \zeta_\lambda(x) = \zeta(\frac{x}{\lambda^\frac{2}{n-\alpha}}).
\]

Then if let

\[
v_\lambda(x) = \frac{1}{\lambda} u(\zeta_\lambda(x)),
\]

we have, for \(x \in B_{\lambda^\frac{n-\alpha}{n-\alpha} \delta}\)

\[
\begin{align*}
\int_{M \setminus \zeta(B_{\delta})} & K(\zeta_\lambda(x), q) u^{\frac{n+\alpha}{n-\alpha}} dV_g
\end{align*}
\]

\[
= \int_{B_{\lambda^\frac{n-\alpha}{n-\alpha} \delta}} \frac{1}{\lambda^2} K(\zeta_\lambda(x), \zeta_\lambda(y)) v_\lambda^{\frac{n+\alpha}{n-\alpha}}(y) dV_g + \frac{1}{\lambda} \int_{M \setminus \zeta(B_{\delta})} K(\zeta_\lambda(x), q) u^{\frac{n+\alpha}{n-\alpha}} dV_g,
\]

where \(g_\lambda = \lambda^{-\frac{4}{n-\alpha}} g\). Meanwhile

\[
\hat{v}_\lambda(x) = \int_{\mathbb{R}^n} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} \hat{v}_\lambda^{\frac{n+\alpha}{n-\alpha}}(y) dy.
\]

Therefore, from (2.9), we have
Lemma 5.2. In the above, for any fixed $\Lambda > 0$,

\[
(5.5) \quad \begin{cases} 
\frac{1}{\lambda^2} K(\zeta(\frac{x}{\lambda^{n-\alpha}}), \zeta(\frac{y}{\lambda^{n-\alpha}})) \Rightarrow \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} & \text{on } B_\Lambda(0) \\
\delta V_g(y) \Rightarrow |dy|^2,
\end{cases}
\]

as $\lambda \to \infty$.

On the other hand, there is a priori integral bound for any positive smooth solutions as follows:

Lemma 5.3. Suppose $u$ is a positive smooth solution to the equation (3.4). Then, for a constant $C = C(M, g)$,

\[
(5.6) \quad \int_M u^{\frac{n+\alpha}{n-\alpha}} dv_g < C(M, g).
\]

Proof. One simply integrates the equation (3.4) and gets

\[
(5.7) \quad \int_M u^{\frac{n+\alpha}{n-\alpha}} dv_g = \int_M P_\alpha udv_g = \int_M u P_\alpha dv_g
\]
due to the fact that $P_\alpha$ is self-adjoint (cf. [GZ] [FG]). Hence

\[
(5.8) \quad \int_M u^{\frac{n+\alpha}{n-\alpha}} dv_g \leq \max_M |P_\alpha[g]|1(\int_M u^{\frac{n+\alpha}{n-\alpha}} dv_g)^{\frac{n-\alpha}{n+\alpha}} \text{vol}_g(M)^{\frac{2\alpha}{n+\alpha}}.
\]

Therefore

\[
(5.9) \quad \int_M u^{\frac{n+\alpha}{n-\alpha}} dv_g \leq \max_M |P_\alpha[g]|1^{\frac{n+\alpha}{2\alpha}} \text{vol}_g(M).
\]

Now, let us consider a sequence of smooth solutions $\{u_k\}$ to (3.1) such that

\[
\lambda_k = \max_M u_k = u_k(p_k) \to \infty.
\]

Let $v_k$ be the rescaled function as defined in the above. Then we have, for any $\Lambda > 0$, via a priori estimates in Lemma 5.1 and (5.5),

\[
(5.10) \quad v_k(x) \Rightarrow v(x) \quad \text{in } C^1(B_\Lambda),
\]

where $v \in C^1_{loc}(\mathbb{R}^n)$, at least for some subsequence of $\{u_k\}$. Moreover it follows from the above Lemma 5.3 that the second term in (5.4) always converges to zero as $\lambda_k \to \infty$. Hence

\[
(5.11) \quad v(x) = \int_{\mathbb{R}^n} \frac{c(n, \alpha)}{|x-y|^{n-\alpha}} v^{\frac{n+\alpha}{n-\alpha}}(y) dy.
\]

Therefore we have
Theorem 5.4. Suppose that $(M^n, [g])$ is a locally conformally flat manifold with positive Yamabe constant such that the Hausdorff dimension of the limit set of the holonomy representation of its fundamental group is less than $\frac{n-\alpha}{2}$ and $(M, [g])$ is not conformally equivalent to the standard round sphere. Then there exists a constant $C = C(n, \alpha, k)$ such that, for any positive smooth solution $u$ to the integral equation (3.1),

$$\|u\|_{C^k(M)} + \|\frac{1}{u}\|_{C^k(M)} \leq C(n, \alpha, k).$$

Proof. We first use the above rescaling method to derive a priori $L^{\infty}$-bound. Assume otherwise there are a sequence of smooth solutions as the above sequence $\{u_k\}$. Then we obtain a smooth solution $v$ to the integral equation on $\mathbb{R}^n$ as in (5.11). The classification given in [CLO], which generalizes the result in [Ln] and [WX], shows $v^{\frac{n}{n-\alpha}}|dx|^2$ is isometric to the standard round sphere (may not be the unit round sphere though). Hence the sufficiently large ball $B_K = \{x \in \mathbb{R}^n : |x| < K\}$ has a concave boundary. Therefore, in the light of (5.10), such $v$ could not exist because of the convexity theorem in the previous section (see similar arguments in [Sc2] [QR]). Higher order estimates follows from standard elliptic theory.

Next we use the fact that the Yamabe constant of order $\alpha$ is positive by Theorem 3.2 to prove the positive lower bound. Assume otherwise, there is a sequence of smooth solutions $\{u_k\}$ such that

$$\inf_M u_k = u_k(p_k) \to 0.$$  

Then, because of the $L^{\infty}$ estimates in the first step, there is a subsequence $\{u_k\}$ converging strongly, say in $C^2(M)$, to a solution $u$ with $u(p_0) = 0$ for some $p_0 \in M$, which implies that $u \equiv 0$. But,

$$\frac{\int_M u_k P_\alpha u_k dV_g}{(\int_M u_k^{\frac{2n}{n-\alpha}} dV_g)^{\frac{n-\alpha}{n}}} = (\int_M u_k^{\frac{2n}{n-\alpha}} dV_g)^{\frac{n}{n-\alpha}} \to 0,$$

which contradicts with the assumption that $Y_\alpha(M, [g]) > 0$. Thus the proof is completed.

Finally we adopt the degree theory approach from [Sc2] to prove the existence of solutions to (3.1), therefore existence of solutions to (3.4). We consider, for $\theta \in (0, 1)$,

$$\Omega_\Lambda = \{u \in C^{2, \theta}(M) : \|u\|_{C^{2, \theta}(M)} + \|\frac{1}{u}\|_{C^{2, \theta}(M)} < \Lambda, u > 0\}$$

and the map

$$F_\alpha = u - I_\alpha(u^{\frac{n+\alpha}{n-\alpha}}) : \Omega_\Lambda \to C^{2, \theta}(M).$$

From the elliptic theory we know that $F_\alpha = Id + \text{compact}$ and we may define the Leray-Schauder degree (cf. [N]) of $F_\alpha$ in the region $\Omega_\Lambda$ with respect to $0 \in C^{2, \theta}(M)$, denoted by $\text{deg}(F_\alpha, \Omega_\Lambda, 0)$, provided that $0 \notin F_\alpha(\partial \Omega_\Lambda)$. 

Theorem 5.5. Suppose that \((M^n, g)\) is a locally conformally flat manifold with positive Yamabe constant. Suppose that the Hausdorff dimension of the limit set of the holonomy representation of the fundamental group \(\pi_1(M)\) is less than \(\frac{n-\alpha_0}{2}\) for some \(\alpha_0 \in [2, n)\). Then, for each \(\alpha \in [2, \alpha_0]\), there exists a positive smooth solution to both the integral equation (3.1) and the pseudo-differential equation (3.4).

Proof. Suppose that \((M^n, g)\) is not conformally equivalent to the standard round sphere. Otherwise we know a standard round metric is the solution to both (3.1) and (3.4). Then, by Theorem 5.4 in the above, there is a number \(\Lambda > 0\) such that \(0 \not\in F_\alpha(\partial \Omega_\Lambda)\) for any \(\alpha \in [2, \alpha_0]\). And the homotopy invariance of the degree tells us that

\[
\deg(F_\alpha, \Omega_\Lambda, 0) = \deg(F_2, \Omega_\Lambda, 0).
\]

for all \(\alpha \in [2, \alpha_0]\). Hence, due to [Sc2],

\[
(5.15) \quad \deg(F_\alpha, \Omega_\Lambda, 0) = \deg(F_2, \Omega_\Lambda, 0) = -1.
\]

Therefore there is a positive solution \(u \in C^{2,\beta}(M)\) to the integral equation (3.1). Thus, by elliptic regularity theory, \(u\) is smooth and \(u\) also solves the pseudo-differential equation (3.4).

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