Comments to the article ”The polarizability of the pion: no conflict between dispersion theory and chiral perturbation theory”

L.V. Fil’kov and V.L. Kashevarov

Lebedev Physical Institute, Leninsky Prospect 53, Moscow 119991, Russia.
E-mail: filkov@sci.lebedev.ru

Abstract

The statement of the authors of the article [1], that spurious singularities occur in the dispersion relation approach of the works [2–5], is analyzed. It has been shown that there are not any additional singularities in this approach and the disagreement between the predictions of the dispersion relations and ChPT for \((\alpha_1 - \beta_1)_{\pi^\pm}\) remains. Reasons for the negative results of the description of the process \(\gamma\gamma \to \pi^0\pi^0\) in [1] are also discussed.

PACS. 13.60.Fz Elastic and Compton scattering - 11.55.Fv Dispersion relations - 14.40.-n Mesons

In the paper of B. Pasquini, D. Drechsel, and S. Scherer [1] the authors attempt to eliminate discrepancies between predictions of the dispersion relations (DR) in [2–5] and the results obtained in the framework of chiral perturbation theory (ChPT) [6–8]. The authors claimed that, as the absorptive part of the Compton amplitudes in [3–5] is expressed by Breit-Wigner poles with coupling constants and decay widths dependent on energy, there must appear additional spurious singularities. As a result, the values of the polarizabilities obtained in [3–5] have to be modified essentially.

The process \(\gamma\gamma \to \pi\pi\) is described by the following invariant variables:

\[
t = (k_1 + k_2)^2, \quad s = (q_1 - k_1)^2, \quad u = (q_1 - k_2)^2,
\]

(1)

where \(q_1(q_2)\) and \(k_1(k_2)\) are the pion and photon four-momenta, respectively.

In order to analyze this process, DRs at fixed \(t\) with one subtraction at \(s = \mu^2\) (where \(\mu\) is the pion mass) have been constructed [4, 5] for the helicity amplitudes \(M_{++}\) and \(M_{+-}\) of this process. Via the cross symmetry these DRs are identical to DRs with two subtractions. The subtraction functions were determined with the help of DRs at fixed \(s = \mu^2\) with two subtractions and the subtraction constants were expressed through the sum and the difference of the electric and magnetic dipole and quadrupole pion polarizabilities. It is worth to note that these DRs do not have any expansions and so they can be used for the determination of the polarizabilities in the region of both low and intermediate energies.

Besides, dispersion sum rules (DSR) have been constructed [3, 4] for the dipole and quadrupole pion polarizabilities using DRs at fixed \(u = \mu^2\) without and with one subtraction, respectively. The DSR for the difference of the electric and magnetic dipole polarizabilities reads

\[
(\alpha_1 - \beta_1) = \frac{1}{2\pi^2\mu} \left\{ \int_{\mu^2}^\infty \frac{IM_{++}(t', u = \mu^2)}{t'} dt' + \int_{\mu^2}^\infty \frac{IM_{++}(s', u = \mu^2)}{s' - \mu^2} ds' \right\}.
\]

(2)
Table 1: The DSR predictions for the dipole polarizabilities of the charged pions in units of $10^{-4}$ fm$^3$.

| ($\alpha_1 - \beta_1$)$_{full}$ | $\rho$ | $b_1$ | $a_1$ | $a_2$ | $f_0$ | $f'_0$ | $\sigma$ | $\Sigma$ | $\Delta\Sigma$ |
|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|---------|
| ($\alpha_1 - \beta_1$)$_{z-w}$ | -1.11 | 0.85  | 3.39  | 1.51  | 0.59  | 0.03  | 9.45  | 13.70 | -        |
| ($\alpha_1 + \beta_1$)$_{full}$ | 0.063 | 0.021 | 0.051 | 0.031 | -     | -     | 0.166 | 0.024 | 0.188   |
| ($\alpha_1 + \beta_1$)$_{z-w}$ | 0.072 | 0.022 | 0.062 | 0.032 | -     | -     | 0.02  | 0.166 | 0.024   |

Table 2: The DSR predictions for the dipole polarizabilities of the $\pi^0$ meson.

| ($\alpha_1 - \beta_1$)$_{full}$ | $\rho$ | $\omega$ | $\phi$ | $f_0$ | $f'_0$ | $\sigma$ | $\Sigma$ | $\Delta\Sigma$ |
|-----------------------------|-------|---------|-------|-------|-------|-------|-------|---------|
| ($\alpha_1 - \beta_1$)$_{z-w}$ | -1.58 | -12.56  | -0.04 | 0.60  | 0.02  | 10.07 | -3.49 | 2.13    |
| ($\alpha_1 + \beta_1$)$_{full}$ | 0.080 | 0.721   | 0.001 | -     | -     | 0.802 | 0.035 |         |
| ($\alpha_1 + \beta_1$)$_{z-w}$ | 0.122 | 0.703   | 0.001 | -     | -     | 0.826 |         |         |

The imaginary parts of the amplitudes in these DRs and DSRs are saturated by the contributions of meson resonances by using Breit-Wigner expressions. It should be noted that these expressions are used to calculate the imaginary parts of the amplitude only. For example, the contribution of the vector and axial-vector mesons are calculated with the help of the expression

$$ImM_{++}^{(V)}(s,t) = \mp s \text{Im}M_{+-}^{(V)}(s,t) = \pm 4g_V^2 s \left( \frac{\Gamma_0}{(M_V^2 - s)^2 + \Gamma_V^2} \right),$$

where $M_V$ is the vector meson mass, the sign ”+” corresponds to the contribution of the $a_1$ and $b_1$ mesons and

$$g_V^2 = 6\pi \sqrt{\frac{M_V^2}{s}} \left( \frac{M_V}{M_V^2 - \mu^2} \right)^3 \Gamma_{V\rightarrow\gamma\pi},$$

$$\Gamma_0 = \left( \frac{s - 4\mu^2}{M_V^2 - 4\mu^2} \right)^3 M_V \Gamma_V.$$  (4)

Here $\Gamma_V$ and $\Gamma_{V\rightarrow\gamma\pi}$ are the full width and the decay width into $\gamma\pi$ of these mesons, respectively. A dependence of the width on the energy is conditioned by the threshold behaviour. The energy dependence of the constant $g_V$ appears via an expression of the total cross section of the process $\gamma\gamma \rightarrow \gamma\pi$ through the vector meson contribution. Moreover, a similar dependence of the imaginary part of the amplitude on $1/\sqrt{s}$ is caused by the unitarity condition and does not lead to any additional singularity.

In order to check the possibility of the appearance of additional spurious singularities, we calculated the contribution of all mesons, except $\sigma$, to the DSR (2) in the zero-width approximation. The results of these calculations (($\alpha_1 - \beta_1$)$_{z-w}$ and ($\alpha_1 + \beta_1$)$_{z-w}$) are listed in Table 1 and Table 2 together with the full calculations obtained in Ref. [4].

As was noted in [3], we consider the $\sigma$ meson as an effective description of the strong $S$-wave $\pi\pi$ interaction using the broad Breit-Wigner resonance expression for the imaginary part of the amplitude. The parameters of such a sigma meson were found from the fit to the experimental data [10] for the process $\gamma\gamma \rightarrow \pi^0\pi^0$ in the energy region of $\sqrt{t} = 270 \div 825$ MeV.
Therefore, such a "σ meson" describes the contribution of the full S-wave ππ interaction to the process $\gamma\gamma \rightarrow \pi^0\pi^0$ and its decay width could be bigger than for a real σ meson. If we use energy independent values of the decay width and the coupling constant of the σ meson, then the results of the calculations are not changed essentially. For example, the value of $(\alpha_1 - \beta_1)_{\pi^\pm(z-v)}$ would be equal to 13.1.

As seen from these Tables the results of the calculations in the zero-width approximation practically coincide with the calculations performed in Ref. [4] beyond such an approximation. This result is evidence of the absence of any additional spurious singularities in the approach which was used in [3–5].

It is worth noting that the calculation of $(\alpha_1 - \beta_1)_{\pi^\pm}$ with the help of the DSR at finite energy [9], which takes into account the s-channel contributions and Regge asymptotic only, yielded $(\alpha_1 - \beta_1)_{\pi^\pm} = 10.3 \pm 1.3$. This value practically coincides with the result of the calculation in [4] (see Table 1) and also confirms the absence of additional singularities in the works under consideration.

By the way, if one follows the statement of the authors of [1], spurious singularities could also appear in their work [11] via the kinematical coefficients, for example, in the expressions (40) for $Im \phi_A(t, t')$.

The authors of the paper [1] calculated the dipole pion polarizabilities by using the expressions:

$$(\alpha_1 - \beta_1) = -\frac{1}{4\pi\mu^2} A_V^I(\mu^2, 0) - \frac{1}{\pi} \int_{4\mu^2}^\infty dt' \frac{H_{00}(t') Im \left[ (\Omega(t')^{-1}(t') \right]}{t'},$$

$$(\alpha_1 + \beta_1) = -\frac{1}{4\pi\mu^2} \left( A_V^I(\mu^2, 0) + A_V^I(\mu^2, 0) \right).$$

where $\Omega(t)$ is the Omnès function. The function $H_{00}(t)$ is expressed by means of the contribution of the vector mesons, which is assumed in the form

$$A_V^I(s, t) = -2e^2 R_V \left[ \frac{s}{s - M_V^2} + \frac{u}{u - M_V^2} \right],$$

$$A_V^I(s, t) = 2e^2 R_V \left[ \frac{1}{s - M_V^2} + \frac{1}{u - M_V^2} \right],$$

where

$$R_V = \frac{24\pi M_V^3 \Gamma(V \rightarrow \pi\gamma)}{e^2 (M_V^2 - \mu^2)^3}.$$  

However, Eq.(5) is a solution of the DR for the amplitude $A_1(s, t)$ in the work [1]. Therefore, the contributions of the vector mesons should be taken into account as poles in the s and u channels:

$$A_V^I(s, t) = -2e^2 R_V \left[ \frac{M_V^2}{s - M_V^2} + \frac{M_V^2}{u - M_V^2} \right].$$

Use of the expression (7) in Eq.(5) in [1] is the main reason of the unsatisfactory description of the process $\gamma\gamma \rightarrow \pi^0\pi^0$ and the erroneous value of $(\alpha_1 - \beta_1)_{\pi^\pi} = 6.62$ obtained in this work. This value differs from the ChPT result $(\alpha_1 - \beta_1)_{\pi^\pi} = -1.90$ both in the quantity and in the sign.
In the work [1] the contribution of the $S$-wave $\pi\pi$ interaction is determined by integrating Eq.(5) and a corresponding expression for the DRs from the threshold up to 800 MeV. However, the $S$-wave contribution is not limited to this interval of energy and strongly depends on the upper integration limit ($\Lambda$). As was shown in Ref. [5], the results of integration over $t'$ in the DRs are not changed for $\Lambda$ greater than $(5 \text{ GeV})^2$ only. Such an extension of the integration region will lead to an essential increase of the contribution of the $S$-wave $\pi\pi$ interaction to $(\alpha_1 - \beta_1)$.

For the reaction $\gamma\gamma \to \pi^0\pi^0$ the Born term is equal to zero and the main contribution to this process in the energy region up to 800 MeV is given by $S$-wave $\pi\pi$ interaction. Therefore, an analysis of this process in this energy region allows a determination of the parameters of the $S$-wave with sufficient accuracy. The unsatisfactory description of this process in the energy region under consideration in the work [1] is probably connected, in particular, with the incorrect determination of the $S$-wave contribution in this work.

In conclusion, we showed that there are not any additional spurious singularities in the dispersion approach of [2–5]. Unfortunately, the difference between the predictions of the DSRs [4, 9] and ChPT [6, 8] for $(\alpha_1 - \beta_1)_{\pi\pm}$ remains. This discrepancy is connected with a different account of the contribution of the $\sigma$ meson and the vector mesons in the DSR and the ChPT calculations (see [12]). In the present ChPT the $\sigma$ meson is taken into account only partially through the two-loop calculations. Moreover, the contributions of the vector mesons are considered as poles at $s = M_V^2$ in DSRs and as the Born terms in ChPT. It results in the difference of the predictions for $M_{1+}^{(V)}$ in these calculations by a factor $M_V^2/\mu^2$.

The authors thank A.I. L’vov for useful discussions. This research is part of the EU integrated initiative hadron physics project under contract number RII3-CT-2004-506078 and was supported in part by the Russian Foundation for Basic Research (Grant No. 05-02-04014).

References

[1] B. Pasquini, D. Drechsel, and S. Scherer, arXiv:0805.0213 [hep-ph].
[2] L.V. Fil’kov, I. Guiasu and E.E. Radescu, Phys. Rev. D 26, 3146 (1982).
[3] L.V. Fil’kov and V.L. Kashevarov, Eur.Phys.J. A 5, 285 (1999).
[4] L.V. Fil’kov and V.L. Kashevarov, Phys. Rev. C 72, 035211 (2005).
[5] L.V. Fil’kov and V.L. Kashevarov, Phys. Rev. C 73, 035210 (2006).
[6] U. Bürgi, Nucl. Phys. B 479, 392 (1997).
[7] J. Gasser, M.A. Ivanov, and M.E. Sainio, Nucl. Phys. B 728, 31 (2005).
[8] J. Gasser, M.A. Ivanov, and N.E. Sainio, Nucl. Phys. B 745, 84 (2006).
[9] V.A. Petrun’kin, Sov. J. Part. Nucl. D12, 278 (1981); A.I. L’vov and V.A. Petrun’kin, Sov. Phys.-Lebedev Inst. Rep. 12, 39 (1985).
[10] H. Marsiske, D. Antreasyan, H.W. Bartels et al., Phys. Rev. D 41, 3324 (1990).
[11] D. Drechsel, et al., Phys. Rev. C 61, 015204 (1999).

[12] L.V. Fil’kov and V.L. Kashevarov, arXiv:0802.0965 (nucl-th); Proceedings of ”NSTAR 2007”, Bonn, Germany, 05-08 September (2007).