Uniqueness Theorem of $\mathcal{W}$-Constraints for Simple Singularities

SI-QI LIU, DI YANG and YOUJIN ZHANG
Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.
e-mail: youjin@mail.tsinghua.edu.cn

Received: 15 May 2013 / Revised: 12 June 2013 / Accepted: 13 June 2013
Published online: 2 July 2013 – © Springer Science+Business Media Dordrecht 2013

Abstract. In a recent paper, Bakalov and Milanov (Compositio. Math. 149:840–888, 2013) proved that the total descendant potential of a simple singularity satisfies the $\mathcal{W}$-constraints, which come from the $\mathcal{W}$-algebra of the lattice vertex algebra associated with the root lattice of this singularity and a twisted module of the vertex algebra. In the present paper, we prove that the solution of these $\mathcal{W}$-constraints is unique up to a constant factor, as conjectured by Bakalov and Milanov in their paper.

Mathematics Subject Classification (2010). Primary 53D45; Secondary 17B69, 32S30, 81R10.

Keywords. $\mathcal{W}$-constraint, simple singularity, vertex algebra.

1. Introduction

Witten [36] proposed his celebrated conjecture on the relation between the partition function of the two-dimensional topological gravity and the Korteweg–de Vries (KdV) integrable hierarchy. This conjecture has two equivalent versions. Let us denote the partition function of the two-dimensional topological gravity by $\tau$, then the first version of the Witten conjecture states that $\tau$, which is a formal power series of variables $t^0, t^1, t^2, \ldots$, is uniquely determined by the following conditions:

- The string equation:
  \[ L_{-1} \tau = \sum_{p \geq 0} t^{p+1} \frac{\partial \tau}{\partial t^p} + \frac{(t^0)^2}{2} \tau - \frac{\partial \tau}{\partial t^0} = 0; \]

- The KdV hierarchy:
  \[ \frac{\partial U}{\partial t^p} = \partial_x R_p , \quad p = 0, 1, 2, \ldots, \]
where \( x = t^0, \ U = \partial^2_x \log \tau, \) and \( R_p \) are polynomials of \( U, U_x, \ldots, U_{(2p)x} \) which are determined by

\[
R_p(0) = 0, \quad R_0 = U,
\]

\[
(2p + 1) \partial_x R_p = \left( 2U \partial_x + U_x + \frac{1}{4} \partial_x^3 \right) R_{p-1}.
\]

Here, we use the following notations:

\[
\partial_x = \frac{\partial}{\partial x}, \quad U_x = \partial_x U, \quad \ldots, \quad U_{kx} = \partial_x^k U, \quad \ldots
\]

The second version of the Witten conjecture states that the partition function \( \tau \) is uniquely determined by a series of linear differential constraints

\[
L_m \tau = 0, \quad m \geq -1,
\]  

(1)

where \( L_{-1} \) is given above in the string equation, and \( L_m \ (m \geq 0) \) are given by

\[
L_m = \frac{1}{2} \sum_{p+q=m-1} \frac{(2p+1)!!(2q+1)!!}{2^{m+1}} \frac{\partial^2}{\partial t^p \partial t^q} + \sum_{p \geq 0} \frac{(2p+2m+1)!!}{2^{m+1}(2p+1)!!} \left( t^p - \delta_{p,1} \right) \frac{\partial}{\partial t^{p+m}} + \frac{1}{16} \delta_{m,0}.
\]  

(2)

Here, \( \delta_{i,j} \) is the Kronecker symbol

\[
\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Since the operators \( L_m \) satisfy the Virasoro commutation relations

\[
[L_m, L_n] = (m - n)L_{m+n}, \quad m, n \geq -1,
\]

the constraints (1) are called the Virasoro constraints of the two-dimensional gravity.

The Witten conjecture is proved by Kontsevich [29], it has inspired active researches on the following subjects:

1. The two-dimensional gravity can be regarded as a string theory in the zero-dimensional space-time, i.e. a point. By considering a similar theory in space-times with richer geometric structures such as Calabi–Yau threefolds, one obtains the Gromov–Witten (GW) invariants theory (see for example [4,30,34]). When the space-time is the complex projective line \( \mathbb{CP}^1 \), the analogue of the Witten conjecture holds true, i.e. the generating function of GW invariants is given by a tau function of the extended Toda integrable hierarchy, and it also satisfies the Virasoro constraints [10,11,23,31,32,40]. For more general space-times having certain nice properties (e.g. toric Fano), there are also some general results [7,9,24,35].
2. The two-dimensional gravity can be regarded as a field theory with spin two on Riemann surfaces. One can also consider fields with spin $n$, then the resulting theory is just the Landau–Ginzburg (LG) model for the $A_{n-1}$ singularity \cite{5,12,33,37}. By considering the LG model for general singularities, one obtains the Fan–Jarvis–Ruan–Witten (FJRW) invariants theory \cite{15,16}. When the singularities are of ADE type, analogues of the Witten conjecture also hold true, i.e. the generating functions of FJRW invariants are given by particular tau functions of the associated Drinfeld–Sokolov integrable hierarchies, and they also satisfy the Virasoro constraints \cite{14–16,19,38}.

In general, the first version of the Witten conjecture has the following analogue for the GW and FJRW invariants: the generating functions of the GW and FJRW invariants should be given by particular tau functions of certain hypothetical integrable hierarchies of KdV type. Assuming the semisimplicity of the underlining Frobenius manifolds and the validity of the Virasoro constraints for the partition functions, the hypothetical integrable hierarchies can be constructed in terms of the Frobenius manifold structures, as shown in \cite{7,9}. An important question is whether there is an analogue of the second version of the Witten conjecture for the GW and FJRW invariants, that is, whether there exist sufficiently many linear differential constraints which uniquely determine the generating functions of the GW and FJRW invariants.

The main result of the present paper is an affirmative answer to the question above for FJRW invariants of ADE singularities based on the result of Bakalov and Milanov \cite{3}. They proved that the partition function (also called the total descendant potential) associated with a simple singularity of type $X_\ell$ satisfies the $W$-constraints constructed from the $W$-algebra of the affine Lie algebra of type $X^{(1)}_\ell$ and a twisted module of the corresponding vertex algebra. Note that $W$-algebras were first introduced by Zamolodchikov \cite{39} and by Fateev and Lukyanov \cite{13} for the $A_n$ cases in the setting of conformal field theory. Then, Feigin and Frenkel \cite{17,18} defined $W$-algebras for any affine Lie algebra as the intersection of kernels of certain screening operators associated with the corresponding vertex algebra.

In this paper, we prove that the $W$-constraints constructed by Bakalov and Milanov for simple singularities uniquely determine the partition function up to a constant factor, as it is conjectured by Bakalov and Milanov in their paper. The answer for other cases is still unknown, and we will discuss this problem in subsequent publications.

Let us take the case of $A_1$ singularity for example to illuminate our method to prove the uniqueness of solution of the $W$-constraints. In this case, the $W$-constraints are just the Virasoro constraints of two-dimensional gravity. The uniqueness of solutions of these Virasoro constraints was proved by Dijkgraaf et al. \cite{6}, and by Kac and Schwarz \cite{28}. Our method follows the approach of Adler and van Moerbeke given in \cite{1}, where they proved the uniqueness of solutions of
\(W\)-constraints for \(A_n\) cases. Note that the Virasoro constraints (1) are linear PDEs, so when we say that their solution \(\tau\) is unique, we mean that \(\tau\) is uniquely determined up to a constant factor. To prove this assertion, we only need to show that if \(\tau\), as a formal power series of \(t^0, t^1, \ldots\), satisfies \(\tau(0) = 0\), then \(\tau\) vanishes itself. To this end we introduce a degree on the ring of formal power series of \(t^0, t^1, \ldots\)

\[
\deg t^p = p + \frac{1}{2}, \quad p \geq 0.
\]

If \(\tau\) does not vanish, then it must contain some nonzero monomials with the lowest degree. We denote one of them by

\[
c t^{p_1} \ldots t^{p_k},
\]

where \(c \neq 0\). The degree of this monomial is

\[
D = \left( p_1 + \frac{1}{2} \right) + \cdots + \left( p_k + \frac{1}{2} \right).
\]

Now, we take \(m = p_k - 1\) and denote the Virasoro constraint \(L_m \tau = 0\) as

\[
c_m \frac{\partial \tau}{\partial t^{m+1}} = \sum_{p=0}^{m-1} a_{p,m} \frac{\partial^2 \tau}{\partial t^p \partial t^{m-p}} + \sum_{p \geq 0} b_{p,m} t^p \frac{\partial \tau}{\partial t^{p+m}} + \frac{1}{16} \delta_{m,0} \tau + \frac{(t^0)^2}{2} \delta_{m,-1} \tau,
\]

where \(c_m = \frac{(2m+3)!!}{2^{m+1}}\), and \(a_{p,m}, b_{p,m}\) are some constants. Then, the monomials with the lowest degree in the left hand side of (3) must contain

\[
c_m \frac{\partial}{\partial t^{p_k}} (c t^{p_1} \ldots t^{p_k})
\]

which has degree

\[
d_1 = D - p_k - \frac{1}{2}.
\]

On the other hand, if we denote by \(d_2\) the degree of the lowest degree monomials of the right hand side of (3), then we have

\[
d_2 \geq D - p_k + 1 > d_1.
\]

So if the Virasoro constraint (3) holds true, then the coefficients of the monomials with degree \(d_1\) in the left hand side must be zero. Note that \(c_m\) is always nonzero, so we must have \(c = 0\), and we arrive at a contradiction.

In this proof, the dilaton shift \(t^1 \mapsto t^1 - 1\) and the fact that \(c_m \neq 0\) play crucial roles. In the general cases, we also have the dilaton shift, then the key step of our proof is to show that the coefficient of a term in the \(W\)-constraint with the lowest
degree does not vanish. To this end, one must check carefully the structure of these $W$-constraints, and find out this term.

The paper is organized as follows. In Section 2, we recall Bakalov and Milanov’s construction of $W$-constraints for simple singularities, and formulate the main theorem on the uniqueness of solutions of the $W$-constraints. In Section 3, we prove the main theorem. In the last section, we give an algorithm to obtain $\tau(t)$ from the $W$-constraints, and we take the $D_4$ singularity as an example to show the usage of this algorithm.

2. Lattice Vertex Algebras and Their Twisted Modules

In this section, we recall Bakalov and Milanov’s [3] construction of $W$-constraints of the total descendant potential associated with a simple singularity and formulate the uniqueness theorem of $W$-constraints.

Let $(Q, (\ | \ ))$ be the root lattice of a simple Lie algebra $\mathfrak{g}$ of ADE type. By definition, $Q$ is a free abelian group with generators $\alpha_1, \ldots, \alpha_\ell$, and $(\ | \ )$ is a symmetric positive definite bilinear form over $Q$ such that $a_{ij} = (\alpha_i | \alpha_j)$ give the entries of the Cartan matrix of $\mathfrak{g}$. We can define a vertex algebra associated with this lattice in the following way (see [27] for more details): Let us first choose a bimultiplicative function $\varepsilon : Q \times Q \to \{\pm 1\}$ such that

$$\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2(|\alpha|^2 + 1)/2},$$

where $|\alpha|^2 = (\alpha | \alpha)$. Note that there always exists such a function. For example, we can take

$$\varepsilon(\alpha, \beta) = (-1)^{(1-\sigma)^{-1} |\alpha| |\beta|},$$

where $\sigma$ is a Coxeter transformation of $Q$. Using the bimultiplicativity property

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma), \quad \varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma),$$

it is easy to see that $\varepsilon(\cdot, \cdot)$ satisfies the 2-cocycle condition (with trivial $Q$-action on $\{\pm 1\}$)

$$\varepsilon(\alpha, \beta)\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \beta + \gamma)\varepsilon(\beta, \gamma),$$

so we can introduce the twisted group algebra

$$\mathbb{C}_\varepsilon[Q] = \text{Span}_\mathbb{C}\{e^\alpha | \alpha \in Q\}$$

whose associative multiplication is defined by

$$e^\alpha e^\beta = \varepsilon(\alpha, \beta)e^{\alpha + \beta}.$$

Let us denote by $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ the complexification of $Q$, and extend the bilinear form $(\ | \ )$ to $\mathfrak{h}$ linearly. Define the current algebra $\hat{\mathfrak{h}}$ associated with $(\mathfrak{h}, (\ | \ ))$ by

$$\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K.$$
On $\hat{\mathfrak{h}}$, we have the following Lie algebra structure:
\[
[\phi t^m, \phi' t^n] = (\phi | \phi') m \delta_{m+n,0} K, \quad [\phi t^m, K] = 0,
\]
where $\phi, \phi' \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. Introduce the bosonic Fock space
\[
\mathcal{F} = S(\mathfrak{h}[t^{-1}]t^{-1}).
\]
Then, it is well known that $\mathcal{F}$ admits a level 1 irreducible $\hat{\mathfrak{h}}$-module structure which can be fixed by the following conditions:
\[
K \cdot 1 = 1, \quad \phi t^m \cdot 1 = 0 \quad (m \geq 0), \quad \phi t^m \cdot s = \phi t^m s \quad (m < 0, \ s \in \mathcal{F}).
\]
The lattice vertex algebra associated with $Q$ is defined by
\[
V_Q = \mathcal{F} \otimes \mathbb{C}[\epsilon][Q]
\]
with vacuum vector $1 \otimes e^0$. To define the state-field correspondence $Y$, we need to introduce some endomorphisms on $V_Q$ as follows:

- For $\phi t^m \in \hat{\mathfrak{h}}$, we define
  \[
  \phi t^m (s \otimes \epsilon^\alpha) = (\phi t^m \cdot s) \otimes \epsilon^\alpha + \delta_{m,0} (\phi | \alpha) s \otimes \epsilon^\alpha.
  \]
- For $\epsilon^\beta \in \mathbb{C}[\epsilon][Q]$, we define
  \[
  \epsilon^\beta (s \otimes \epsilon^\alpha) = \epsilon(\beta, \alpha) s \otimes \epsilon^{\beta+\alpha}.
  \]
- For $\gamma \in Q$ and an indeterminate $z$, we define
  \[
  z^\gamma (s \otimes \epsilon^\alpha) = z^{(\gamma | \alpha)} s \otimes \epsilon^\alpha.
  \]

Then, the state-field correspondence $Y$ of $V_Q$ is generated by the following vertex operators:
\[
\phi(z) = Y(\phi t^{-1} \otimes e^0, z) = \sum_{n \in \mathbb{Z}} (\phi t^n) z^{-n-1},
\]
\[
Y_\alpha(z) = Y(1 \otimes \epsilon^\alpha, z) = \epsilon^\alpha z^\alpha \exp \left( \sum_{n \geq 1} (\alpha t^{-n}) \frac{z^n}{n} \right) \exp \left( \sum_{n \geq 1} (\alpha t^n) \frac{z^{-n}}{-n} \right)
\]
for $\phi \in \mathfrak{h}, \alpha \in Q$. It is easy to see that $\mathcal{F}$ is a vertex subalgebra of $V_Q$.

**DEFINITION 2.1.** The $W$-algebra associated with $Q$ is the vertex subalgebra of $\mathcal{F}$ defined by
\[
W = \{ s \in \mathcal{F} | e^{\alpha_i}_{(0)}(s) = 0, \ i = 1, \ldots, \ell \},
\]
where $e^{\alpha_i}_{(0)} = \text{Res}_{z=0} Y_\alpha(z)$ is called the screening operator of $\alpha \in Q$. 
UNIQUENESS THEOREM OF W-CONSTRAINTS

Let us introduce a degree on $F$:

$$\deg \phi t^{-n} = n, \quad \forall \phi \in h, \quad n = 1, 2, \ldots,$$

then B. Feigin, E. Frenkel and I. Frenkel proved the following theorem.

**THEOREM 2.2** [18,20]. The $\mathcal{W}$-algebra is generated by some elements $w_1, \ldots, w_\ell$ of $F$ such that $\deg w_i = d_i = m_i + 1$ $(i = 1, \ldots, \ell)$, where $1 = m_1 \leq m_2 \leq \cdots \leq m_\ell$ are exponents of the Weyl group $W$ of $Q$. In particular, let $I_i(\alpha_1, \ldots, \alpha_n)$ $(i = 1, \ldots, \ell)$ be the generators of $S(h)^W$, then $w_i$ can be chosen as

$$w_i = I_i(\alpha_1 t^{-1}, \ldots, \alpha_\ell t^{-1}) + J_i,$$

where $J_i$ belong to the ideal in $F$ that is generated by $\phi t^{-n}$ for all $\phi \in h$ and $n \geq 2$.

We assume in what follows that the exponents $m_i$ and the generators $w_i$ are always given in the form of the above theorem.

To obtain the $W$-constraints from the $W$-algebra, we need to consider a twisted module $M$ of the vertex algebra $F$. Let $\sigma$ be a Coxeter transformation on $h$. We denote by $h$ the order of $\sigma$, which is called the Coxeter number of $W$. Then $\sigma$ has eigenvalues $\zeta^{m_j}$, where $\zeta = \exp(2\pi \sqrt{-1}/h)$, $j = 1, \ldots, \ell$. Suppose $\phi^i$ and $\phi^j$ are eigenvectors of $\sigma$ with eigenvalues $\zeta^{m_i}$ and $\zeta^{m_j}$, respectively, then we have

$$(\phi^i | \phi^j) = (\sigma(\phi^i) | \sigma(\phi^j)) = \zeta^{m_i + m_j}(\phi^i | \phi^j),$$

which implies that $(\phi^i | \phi^j) = 0$ unless $i + j = \ell + 1$. So we can choose a basis $\{\phi^i\}$ of $h$ consisting of eigenvectors of $\sigma$ such that $(\phi^i | \phi^j) = \delta_{i+j,\ell+1}$. We also use the notation $\phi_j = \phi^{\ell+1-j}$ to denote the dual basis $\{\phi_i\}$ of $\{\phi^i\}$, and we denote $\eta_{ij} = \delta_{i+j,\ell+1}$.

The twisted module $M$ is defined as the following polynomial ring

$$M = \mathbb{C}[q^i p | i = 1, \ldots, \ell; \quad p = 0, 1, 2, \ldots],$$

and the action of $F$ is given by the map

$$a \in F \mapsto Y^M(a, \lambda) = \sum_{n \in \frac{1}{h} \mathbb{Z}} a_{(n)} \lambda^{-n-1}, \quad \text{where } a_{(n)} \in \text{End}(M).$$

Here, the twisted state-field correspondence $Y^M$ is generated by the following simple operators and certain reconstruction theorems of twisted modules of vertex algebras (see [2,3] for more details):

- If $a = 1$, then $Y^M(1, \lambda) = \text{id}_M$.
- If $a = \phi^j t^{-1}$, then

$$Y^M(\phi^j t^{-1}, \lambda) = \sum_{p \in \mathbb{Z}} \phi^{j}_{(p+m_j)} \lambda^{-1-p-m_j},$$
where
\[ \phi_j^{(p+\frac{m_j}{p})} = \frac{\Gamma\left(\frac{m_j}{h} + p + 1\right)}{\Gamma\left(\frac{m_j}{h}\right)} \frac{\partial}{\partial q^{j:p}}, \]
\[ \phi_j^{(-p-\frac{m_j}{p})} = \frac{\Gamma\left(\frac{m_j}{h}\right)}{\Gamma\left(\frac{m_j}{h} + p\right)} q^{j:p}, \]
for \( p = 0, 1, 2, \ldots \), and \( j = 1, \ldots, \ell \).

To write down the vertex operators for other types of elements of \( \mathcal{F} \), we need to introduce some notations. First, for a twisted field \( A(\lambda) = \sum_{n \in \frac{1}{h} \mathbb{Z}} a(n) \lambda^{-n-1} \) we denote
\[ A(\lambda) = A(\lambda)_+ + A(\lambda)_-, \]
where
\[ A(\lambda)_+ = \sum_{n \in \frac{1}{h} \mathbb{Z}, n \geq 0} a(n) \lambda^{-n-1}, \quad A(\lambda)_- = \sum_{n \in \frac{1}{h} \mathbb{Z}, n < 0} a(n) \lambda^{-n-1}. \]
Then for two twisted fields \( A(\mu), B(\lambda) \), we define
\[ :A(\mu) B(\lambda): = A(\mu)_- B(\lambda) + B(\lambda) A(\mu)_+, \quad \langle A(\mu) B(\lambda) \rangle = [A(\mu)_+, B(\lambda)_-]. \]
For more fields, e.g. \( A, B, C \), we assume that
\[ :A B C: = :A (:B C:) : \]
and so on. Let us define
\[ P^{ij}(\mu, \lambda) = \langle Y^M(\phi^j t^{-1}, \mu)Y^M(\phi^i t^{-1}, \lambda) \rangle. \]
One can show that
\[ P^{ij}(\mu, \lambda) = \eta_{ij} \mu^{\frac{m_j}{h}} \lambda^{\frac{m_i}{h}} \frac{\Gamma\left(\frac{m_i}{h} + k + 1\right)}{k!(k+2)} \lambda^{-k-2}, \quad \text{for } |\lambda| < |\mu|. \]

Denote \( s = \mu - \lambda \), then it is easy to see that
\[ P^{ij}(\mu, \lambda) = \frac{\eta_{ij}}{s^2} + \sum_{k=0}^{\infty} P_k^{ij}(\lambda) s^k, \]
where
\[ P_k^{ij}(\lambda) = \eta_{ij} (-1)^k \left(1 - \frac{m_i}{h}\right) \frac{\Gamma\left(\frac{m_i}{h} + k + 1\right)}{k!(k+2)} \lambda^{-k-2}. \]

Now, we can define \( Y^M(a, \lambda) \) for a general \( a \in \mathcal{F} \). Suppose \( a \in \mathcal{F} \) is a monomial of the form
\[ a = \phi^{\sigma_1} t^{-k_1-1} \otimes \ldots \otimes \phi^{\sigma_r} t^{-k_r-1}, \]
then the vertex operator \( Y^M(a, \lambda) \) can be obtained from the Wick theorem as follows:

\[
Y^M(a, \lambda) = \sum_J \left( \prod_{(i,j) \in J} \partial^{(kj)}_{\lambda} P_{k_i}^{ij}(\lambda) \right) : \left( \prod_{l \in J'} \partial^{(kl)}_{\lambda} Y^M(\phi^{al}_l, \lambda) \right) :. \tag{8}
\]

Here, the summation is taken over all the collections \( J \) of disjoint ordered pairs \((i_1, j_1), \ldots, (i_s, j_s) \subset \{1, \ldots, r\} \) such that \( i_1 < \cdots < i_s \) and \( i_l < j_l \) for any \( l = 1, \ldots, s \), and \( J' = \{1, \ldots, r\} \setminus J \), \( \partial^{(kl)}_{\lambda} = \partial_{\lambda}^k/k! \). Note that the set \( J \) or \( J' \) can be an empty set, in such case the corresponding product is set to be 1.

Suppose \( w \) is an element of \( \mathcal{W} \), then using the fact that \( w \) is \( W \) invariant one can show that

\[
Y^M(w, \lambda) = \sum_{m \in \mathbb{Z}} w(m) \lambda^{-m-1}. \]

Let \( w_1, \ldots, w_\ell \) be a set of generators of \( \mathcal{W} \), we denote

\[
W_{i,m} = \text{Res}_{\lambda=0} \left( \lambda^m Y^M(w_i, \lambda) \right) , \quad \text{where } m \in \mathbb{Z}.
\]

These operators are called the \( \mathcal{W} \) operators associated with \( Q \).

We introduce the dilaton shift \( t^{i,p} = q^{i,p} + \delta_{i,1} \delta_{p,1} \), and complete \( M \) to \( \hat{M} = \mathbb{C}[t^{i,p} \mid i = 1, \ldots, \ell; \ p = 0, 1, 2, \ldots] \).

It is easy to see that \( \hat{M} \) is also a twisted module of \( \mathcal{F} \).

The main result of [3] can be stated as follows:

**THEOREM 2.3** [3]. Let \( \tau \) be the total descendant potential of the semisimple Frobenius manifold associated with a simple singularity of type \( X_\ell \), and \( W_{i,m} \ (i = 1, \ldots, \ell; \ m \in \mathbb{Z}) \) be the \( \mathcal{W} \) operators associated with the root lattice of type \( X_\ell \). Then as an element of \( \hat{M} \) the function \( \tau \) satisfies the \( \mathcal{W} \)-constraints

\[
W_{i,m} \tau = 0, \quad i = 1, \ldots, \ell, \quad m \geq 0. \tag{9}
\]

In this paper, we prove the following theorem.

**THEOREM 2.4** (Main Theorem). The solution to the \( \mathcal{W} \)-constraints given in (9) is unique up to a constant factor.

### 3. Proof of the Main Theorem

In this section, we prove the main theorem (Theorem 2.4) of this paper. Our method is similar to the one given in Section 1 for the \( A_1 \) singularity. We first assume \( \tau(0) = 0 \) and \( \tau \neq 0 \). Introduce a gradation on \( \hat{M} \) by defining

\[
\deg t^{i,p} = p + \frac{m_i}{h}, \tag{10}
\]
and consider the nonzero monomials of $\tau$ with the lowest degree. There may exist several monomials having the lowest degree. We choose an arbitrary one, say,

$$c t^{i_1 p_1} \ldots t^{i_k p_k}.$$  \hspace{1cm} (11)

Then, we will consider the equation $W_{i_k p_k} \tau = 0$. By separating the nonzero monomials of the left hand side of this equation with the lowest degree, we can show that $c = 0$ and thus arrive at a contradiction with the assumption $\tau \not\equiv 0$.

**Lemma 3.1.** If we assume

$$\deg q^{i p} = p + \frac{m_i}{h},$$  \hspace{1cm} (12)

then $W_{i, m}$, as an endomorphism of $M$, has degree $m_i - m$, i.e. $W_{i, m}(M^d) \subset M^{d + m_i - m}$, where $M^d$ is the homogeneous component of $M$ of degree $d$ with respect to the gradation (12).

**Proof.** By definition, it is easy to see that

$$\deg \phi_{(p + \frac{m_j}{h})}^i = -p - \frac{m_j}{h}, \quad \text{for } p \in \mathbb{Z},$$

so if we assume $\deg \lambda = -1$, then we have

$$\deg Y^M(\phi t^{-1}, \lambda) = 1, \quad \text{for } \phi \in \mathfrak{h},$$

which coincides with the degree of $\phi t^{-1}$ (see (4)).

Note that $\deg P_{ij}^{(k)}(\lambda) = k + 2$, so we have

$$\deg \partial_{\lambda}^{(k_j)} P_{ki}^{ij}(\lambda) = (k_i + 1) + (k_j + 1), \quad \deg \partial_{\lambda}^{(k_i)} Y^M(\phi t^{-1}, \lambda) = k_l + 1.$$  

Thus, it follows from the definition (8) that

$$\deg Y^M(a, \lambda) = \deg a, \quad \text{for all homogeneous } a \in \mathcal{F}.$$  

In particular,

$$\deg W_{i, m} = \deg Y^M(w_j, \lambda) - m - 1 = m_i - m.$$  

The lemma is proved. \hfill \square

The above lemma shows that if we do not perform the dilaton shift, then $W_{i, m}$ are homogeneous. On the other hand, the dilaton shift gives terms with lower degrees. Note that our aim is to find out the terms in $W_{i, m}$ which have the lowest degree, so we only need to consider the terms that contain $t^{1,1}$. 
LEMMA 3.2. The operator $W_{i,m}$ ($m \geq 0$) has the following expression:

$$W_{i,m} = \sum_{d=0}^{m_i} (t^{1,1} - 1)^d W_{i,m}^{(d)},$$

where $W_{i,m}^{(d)}$ are differential operators on $\hat{M}$ whose coefficients do not depend on $t^{1,1}$.

Proof. Every monomial of $W_{i,m}$ can be written as composition of bosons (6), (7), and the number of composed bosons is less or equal to the degree of $w_i$, i.e. $d_i = m_i + 1$. So we have the following expression

$$W_{i,m} = \sum_{d=0}^{d_i} (t^{1,1} - 1)^d W_{i,m}^{(d)},$$

where $W_{i,m}^{(d_i)}$ is a constant. If we replace $(t^{1,1} - 1)$ by $q^{1,1}$ and consider the gradation (12), then

$$\deg((t^{1,1} - 1)^d W_{i,m}^{(d_i)}) = (1 + \frac{1}{\hbar})(m_i + 1) > m_i - m = \deg W_{i,m},$$

which implies $W_{i,m}^{(d_i)} = 0$. The lemma is proved. \hfill \square

LEMMA 3.3. With an appropriate choice of the generators $w_1, \ldots, w_\ell$ of the $W$-algebra, we have

$$W_{i,m}^{(m_i)} = c_{i,m} \frac{\partial}{\partial q^{i,m}},$$

where $c_{i,m}$ are nonzero constants.

We assume henceforth that the generators $w_i$ are chosen such that (14) hold true. To prove the above lemma, we first need to prove the following lemma.

LEMMA 3.4. The generators $I_1, \ldots, I_\ell$ of $S(\mathfrak{h})^W$ can be chosen to have the form

$$I_i(\phi_1, \ldots, \phi_\ell) = \phi_1^{m_1} I_{i+1} + \sum_{d=0}^{m_i-1} \phi_1^d I_i^{(d)}(\phi_2, \ldots, \phi_\ell), \quad i = 1, \ldots, \ell.$$  

Here, the dual basis $\{\phi_1, \ldots, \phi_\ell\}$ are regarded as coordinates of $\mathfrak{h}$ with respect to the basis $\{\phi^1, \ldots, \phi^\ell\}$ (see Section 2), and the invariant polynomials $I_1, \ldots, I_\ell$ are represented as polynomials of $\phi_1, \ldots, \phi_\ell$.

Proof. We first take an arbitrary set $\{I_1, \ldots, I_\ell\}$ of generators of $S(\mathfrak{h})^W$. Let $d_i^j (1 \leq i \leq \ell)$ be the degree of $I_i$ which is not necessarily equal to $m_i + 1$. 
Let $J$ be the Jacobian determinant of $I_1, \ldots, I_\ell$ with respect to $\phi_1, \ldots, \phi_\ell$. Then $J$ is a constant multiple of the product of linear equations of all the walls of a Weyl chamber. Note that $\phi^1 = (1, 0, \ldots, 0)$, which is an eigenvector of the Coxeter transformation $\sigma$, does not lie on any wall of a Weyl chamber, so we have $J(1, 0, \ldots, 0) \neq 0$.

The determinant $J$ contains $\ell!$ summands, in which there is at least one that does not vanish at $(1, 0, \ldots, 0)$. We can re-number $I_1, \ldots, I_\ell$ such that this summand is given by

$$(-1)^{\ell(\ell-1)/2} \frac{\partial I_1}{\partial \phi_\ell} \frac{\partial I_2}{\partial \phi_{\ell-1}} \cdots \frac{\partial I_\ell}{\partial \phi_1},$$

then we have $\frac{\partial I_i}{\partial \phi_{\ell+1-i}}(1, 0, \ldots, 0) \neq 0$ for $i = 1, \ldots, \ell$.

Using the above fact, we can rescale $I_1, \ldots, I_\ell$ such that

$$I_i = \phi_1^{d'_i-1} \phi_{\ell+1-i} + \text{other terms}.$$

From the fact that $\sigma(\phi_i) = \zeta^{-m_i} \phi_i$, $m_1 = 1$ and $\sigma(I_i) = I_i$, it follows that $d'_i$ must be equal to $m_i + 1$, and the “other terms” in the above expression of $I_i$ only contains monomials in which the degree of $\phi_1$ is less than $d'_i - 1$ (in the $D_{2n}$ case, a linear recombination of $I_n$ and $I_{n+1}$ may be needed). The lemma is proved.

Remark 3.5. The proof of the above lemma follows the one given for Theorem 3.19 in [26].

Proof of Lemma 3.3. By counting degrees and the number of bosons, it is easy to see that $W_{i,m}^{(m_i)}$ must take the form (14) (in the $D_{2n}$ case, a linear recombination of $w_n$ and $w_{n+1}$ may be needed). We only need to prove that $c_{i,m} \neq 0$.

Let us call $(t^{i,1} - 1)^m w_{i,m}^{(m_i)}$ the leading term of $W_{i,m}$. This leading term is the composition of $d_i$ bosons. Note that if we split $w_i$ into $I_i + J_i$ (see (5)), then the leading term must come from $I_i$ since $J_i$ contains less bosons. Similarly, suppose $a$ is a monomial of $I_i$, we consider the operator $Y M(a, \lambda)$ (see (8)), then the leading term must come from the summands with $J = \emptyset$ since other summands contain less bosons. So to find out the leading term we only need to investigate the invariant polynomial $I_i$, and we can omit all terms that contain $P^{ij}_k(\lambda)$ in (8).

According to Lemma 3.4, the leading term of $W_{i,m}$ is same with the one of

$$\text{Res}_{\lambda=0} \left(\lambda^m Y M((\phi_1 t^{-1})^{m_i} (\phi^i t^{-1}), \lambda)\right).$$

By definition,

$$Y^M(\phi_1 t^{-1}, \lambda) = h(t^{i,1} - 1) t^{\frac{1}{\pi}} + \cdots,$$

$$Y^M(\phi^i t^{-1}, \lambda) = \frac{\Gamma(m_i + m + 1)}{\Gamma(m_i)} \frac{\partial}{\partial t^{i,m}} \lambda^{-1 - m - m_i \pi} + \cdots.$$
so the leading coefficient \( W_{i,m}^{(m_i)} \) reads
\[
W_{i,m}^{(m_i)} = h^{m_i} \frac{\Gamma \left( \frac{m_i}{h} + m + 1 \right)}{\Gamma \left( \frac{m_i}{h} \right)} \frac{\partial}{\partial t^{i,m}}.
\]
The lemma is proved. \( \Box \)

Now, we are ready to prove Theorem 2.4. From Lemmas 3.2 and 3.3, it follows that the lowest degree term in \( W_{i,m} \) is given by
\[
(-1)^{m_i} c_{i,m} \frac{\partial}{\partial t^{i,m}}.
\]
Its action on the term (11) implies
\[
(-1)^{m_k} c_{ik,p_k} \frac{\partial}{\partial t^{ik,p_k}} \left( c \tau^{i_1,p_1} \cdots \tau^{i_k,p_k} \right) = 0,
\]
so \( c = 0 \). The theorem is proved.

4. Applications of \( W \)-Constraints

If we begin with \( \tau(0) = 1 \) instead of \( \tau(0) = 0 \), then the proof given in the last section provides an algorithm to compute the partition function \( \tau(t) \).

**Step 1** Let
\[
\tau(t) = 1 + \sum_{k \geq 1} \sum_{1 \leq i_1, \ldots, i_k \leq \ell \atop p_1, \ldots, p_k \geq 0} \langle \tau_{i_1,p_1} \cdots \tau_{i_k,p_k} \rangle t^{i_1,p_1} \cdots t^{i_k,p_k} k!,
\]
and mark all coefficients \( \langle \cdots \rangle \) as unknown.

**Step 2** Find out the terms with the lowest degree with respect to (10) from the ones whose coefficients are marked as unknown.

**Step 3** Suppose the coefficient of one of the terms chosen in Step 2 is \( \langle \tau_{i_1,p_1} \cdots \tau_{i_k,p_k} \rangle \), then from the constraint \( W_{ik,p_k} \tau = 0 \) one can obtain this coefficient uniquely.

**Step 4** Mark \( \langle \tau_{i_1,p_1} \cdots \tau_{i_k,p_k} \rangle \) as known, then go back to Step 2.

When the singularity \( X_\ell \) is of A type, the explicit form of \( W \)-constraints was obtained in \([1,13,21,22,25]\), and the corresponding partition functions \( \tau \) can be obtained by solving these \( W \)-constraints recursively. In this section, we consider the \( D_4 \) singularity.

We take \( \mathfrak{h} = \mathbb{C}^4 \) with the natural basis \( e_1, \ldots, e_4 \) first. The metric (\( | \cdot | \)) is taken as the canonical one \( (e_i | e_j) = \delta_{i,j} \), and the generators \( \alpha_1, \ldots, \alpha_4 \) of the root lattice \( Q \) are chosen as
\[
\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4.
\]
Then, the Cartan matrix \( A = ((\alpha_i | \alpha_j)) \) reads

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix},
\]

and \( Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4 \) is a root lattice of \( D_4 \) type.

The Coxeter transformation is taken as \( \sigma = R_1 R_2 R_3 R_4 \), where \( R_i \) (\( i = 1, \ldots, 4 \)) is the reflection with respect to \( \alpha_i \). The order of \( \sigma \) is \( h = 6 \), and the exponents read \( m_1 = 1, \ m_2 = 3, \ m_3 = 3, \ m_4 = 5 \).

The eigenvectors \( \phi^1, \ldots, \phi^4 \) are chosen as

\[
\phi^1 = \overline{\phi^4} = \frac{1}{\sqrt{3}} \left( \zeta^2 e_1 + \zeta e_2 + e_3 \right),
\]

\[
\phi^2 = \overline{\phi^3} = \frac{1}{\sqrt{6}} \left( -e_1 + e_2 - e_3 + \sqrt{-3} e_4 \right),
\]

where \( \zeta = e^{\pi \sqrt{-1}/3} \), and \( \overline{\phi^3}, \overline{\phi^4} \) stand for the complex conjugations of \( \phi^3, \phi^4 \).

Bakalov and Milanov [3] showed that the generators of the corresponding \( \mathcal{W} \)-algebra can be chosen as

\[
\tilde{w}_i = \sum_{j=1}^{4} e^{e_j} (-m_i - 1) e^{-e_j} + \sum_{j=1}^{4} e^{-e_j} (-m_i - 1) e^{e_j}, \quad i = 1, 2, 4,
\]

\[
\tilde{w}_3 = (e_1 t^{-1}) (e_2 t^{-1}) (e_3 t^{-1}) (e_4 t^{-1}).
\]

These generators do not meet the requirement of Lemma 3.3, so we need to modify them to the following set of generators:

\[
w_1 = \tilde{w}_1, \quad w_2 = 3 \tilde{w}_2 + \sqrt{-3} \tilde{w}_3, \quad w_3 = 3 \tilde{w}_2 - \sqrt{-3} \tilde{w}_3, \quad w_4 = \tilde{w}_4.
\]

They have the explicit forms

\[
w_1 = \phi^\alpha \phi_\alpha,
\]

\[
w_k = 2\phi^\alpha \phi_\alpha^3 + \frac{3}{4} \phi^\alpha \phi_\alpha^2 \phi_\alpha + \frac{1}{8} w_1^2 + \phi_\alpha (\phi_k)^2 \phi^4 + \frac{1}{6} (\phi_k)^4 - \frac{1}{3} \frac{\phi_k (\phi_k)^3}{3} - \sqrt{2} \left( (\phi^1)^3 + (\phi^4)^3 \right) \phi^k \quad (k = 2, 3),
\]

\[
w_4 = \frac{2}{5} \phi^\alpha \phi_\alpha + \frac{1}{4} \phi^\alpha \phi_\alpha^2 + \frac{1}{9} \phi^\alpha \phi_\alpha^3 + \frac{\phi_\alpha (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2 + (\phi^4)^2 - 3240}{324 \sqrt{2}} - \frac{(\phi^1)^3 + (\phi^4)^3}{324 \sqrt{2}} \left( (\phi^2 + \phi^3)^3 + 3(\phi^2 + \phi^3) \phi^1 \phi^4 \right) + \frac{1}{432} \left( \phi^1 \phi^4 (\phi^2 + \phi^3)^4 + 6(\phi^1 \phi^4)^2 (\phi^2 + \phi^3)^2 + \frac{8}{3} (\phi^1 \phi^4)^3 \right)
\]
\[ + \phi^2 \phi^3 ((\phi^2)^4 - 2(\phi^2)^3 \phi^3 - 2\phi^2 (\phi^3)^3 + (\phi^3)^4) + \frac{10}{3} (\phi^2 \phi^3)^3 \]

\[ + \frac{1}{18} \left( (\phi^1 \phi^4 + \phi^3 \phi^1) \left( (\phi^2 + \phi^3)^2 + 2\phi^1 \phi^4 \right) \right) \]

\[ - \frac{1}{27\sqrt{2}} \left( ((\phi^1)^3 + (\phi^3)^2) (\phi^2 + \phi^3)^3 + 3(\phi^2 + \phi^3)((\phi^1)^2 \phi^1 + 3(\phi^2)^2 \phi^3) \right) \]

\[ + \sum_{k=2}^{3} \frac{\phi_k \phi_3}{18} \left( 2(\phi^k)^3 - (\phi_k)^3 \right) + \phi^2 \phi^3 (2\phi_k - \phi^k) + 2\phi^1 \phi^4 (\phi^2 + \phi^3) \]

where \( \phi^i, n = \phi^i t^{-n}, \phi^i, n = \phi^i t^{-n}, \phi^i = \phi^i, \phi_i = \phi^1, \) and summation with respect to the repeated upper and lower Greek index \( \alpha \) is assumed.

One can obtain the \( \mathcal{W} \) operators \( W_{i,m} \) from these generators, whose explicit form is omitted here. Using the \( \mathcal{W} \)-constraints \( W_{i,m} \tau = 0 \) \( (m \geq 0) \), we can obtain all Taylor coefficients of \( \tau (t) \). In particular, if we consider the genus expansion

\[ \mathcal{F}(t) = \log \tau (t) = \sum_{g=0}^{\infty} \mathcal{F}_g (t), \]

and restrict \( \mathcal{F}_0 (t) \) to the small phase space \( \text{Spec} \mathbb{C}[t^{1,0}, t^{2,0}, t^{3,0}, t^{4,0}] \), then we obtain the following potential of the Frobenius manifold associated with the \( D_4 \) singularity:

\[ F(v) = \frac{1}{2} (v^1)^2 v^4 + v^1 v^2 v^3 + \frac{((v^2)^3 + (v^3)^3) v^4}{18 \sqrt{2}} + \frac{v^2 v^3 (v^4)^3}{108} + \frac{(v^4)^7}{272160}, \]

where \( v^i = t^{i,0} \).

There are several degrees of freedom when we choose the basis \( \{ \phi^1, \ldots, \phi^4 \} \). If we choose a different basis, then the potential \( F(v) \) will be transformed to a different form via a linear coordinate transformation. For example, if we take

\[ v^1 \mapsto c^{-1} v^1, \quad v^4 \mapsto c v^4, \quad F \mapsto c^{-1} F \]

with \( c = \sqrt{18\sqrt{2}} \), then

\[ F = \frac{1}{2} (v^1)^2 v^4 + v^1 v^2 v^3 + (v^2)^3 v^4 + 6 v^2 v^3 (v^4)^3 + (v^3)^3 v^4 + \frac{54}{35} (v^4)^7 \]

which coincides with the potential derived by Dubrovin [8].

If we take

\[ v^1 = t^1, \quad v^2 = -\frac{tx - \sqrt{3} ty}{\sqrt{2}}, \quad v^3 = -\frac{tx + \sqrt{3} ty}{\sqrt{2}}, \quad v^4 = tx^2, \]

and rescale \( F \) to \( F/6 \), then

\[ F = \frac{1}{12} t^1 t^2 - \frac{1}{4} t^1 t^2 + \frac{1}{12} t^1 t^2 t^2 - \frac{1}{216} t^3 t^2 - \frac{1}{24} t^3 t^2 t^2 + \frac{1}{296} t^3 t^2 t^2 + \frac{1}{432} t^3 t^2 t^2 + \frac{1}{1632960} t^3 t^2 t^2 \]

which coincide with the potential derived by Fan et al. [14].
Acknowledgements

The authors thank Boris Dubrovin for his encouragement and many helpful discussions. This work is partially supported by the NSFC No. 11071135, No. 11171176 and No. 11222108, and by the Marie Curie IRSES project RIMMP.

References

1. Adler, M., van Moerbeke, P.: A matrix integral solution to two-dimensional $W_p$-gravity. Commun. Math. Phys. 147, 25–56 (1992)
2. Bakalov, B., Kac, V.: Twisted modules over lattice vertex algebras. Lie Theory and Its Applications in Physics V, pp. 3–26. World Scientific, River Edge (2004)
3. Bakalov, B., Milanov, T.: $W$-constraints for the total descendant potential of a simple singularity. Compositio. Math. 149, 840-888 (2013)
4. Behrend, K., Fantechi, B.: The intrinsic normal cone. Invent. Math. 128, 45–88 (1997)
5. Chiodo, A.: The Witten top Chern class via K-theory. J. Algebraic Geom. 15, 681–707 (2006)
6. Dijkgraaf, R., Verlinde, H., Verlinde, E.: Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity. Nucl. Phys. B 348, 435–456 (1991)
7. Dubrovin, B.: Geometry of 2D topological field theories. In: Integrable Systems and Quantum Groups (Montecatini Terme, 1993). Lecture Notes in Mathematics, vol. 1620, pp. 120–348. Springer, Berlin (1996)
8. Dubrovin, B.: Painlevé transcendents in two-dimensional topological field theory. The Painlevé property. CRM Series in Mathematical Physics. pp. 287–412. Springer, New York (1999)
9. Dubrovin, B., Zhang, Y.: Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants. eprint arXiv: math/0108160
10. Dubrovin, B., Zhang, Y.: Virasoro symmetries of the extended Toda hierarchy. Commun. Math. Phys. 250, 161–193 (2004)
11. Eguchi, T., Yang, S.-K.: The topological $\mathbb{CP}^1$ model and the large-N matrix integral. Mod. Phys. Lett. A 9, 2893–2902 (1994)
12. Faber, C., Shadrin, S., Zvonkine, D.: Tautological relations and the $r$-spin Witten conjecture. Ann. Sci. Éc. Norm. Supér 43, 621–658 (2010)
13. Fateev, V.A., Lukyanov, S.L.: The models of two-dimensional conformal quantum field theory with $Z_n$ symmetry. Int. J. Mod. Phys. A 3, 507–520 (1988)
14. Fan, H., Jarvis, T.J., Merrell, E., Ruan, Y.: Witten’s $D_4$ Integrable Hierarchies Conjecture. eprint arXiv: 1008.0927
15. Fan, H., Jarvis, T.J., Ruan, Y.: The Witten equation, mirror symmetry and quantum singularity theory. Ann. Math. 178, 1–106 (2013)
16. Fan, H., Jarvis, T.J., Ruan, Y.: The Witten equation and its virtual fundamental cycle. eprint arXiv: 0712.4025
17. Feigin, B., Frenkel, E.: Quantization of the Drinfeld–Sokolov reduction. Phys. Lett. B 246, 75–81 (1990)
18. Feigin, B., Frenkel, E.: Integrals of motion and quantum groups. Integrable systems and quantum groups (Montecatini Terme, 1993). Lecture Notes in Mathematics, vol. 1620, pp. 349–418. Springer, Berlin (1996)
19. Frenkel, E., Givental, A., Milanov, T.: Soliton equations, vertex operators, and simple singularities. Funct. Anal. Other Math. 3, 47–63 (2010)
20. Frenkel, I.B.: Representations of affine Lie algebras, Hecke modular forms and Korteweg-de Vries type equations. Lie algebras and related topics (New Brunswick,
UNIQUENESS THEOREM OF $W$-CONSTRAINTS

N.J., 1981). Lecture Notes in Mathematics, vol. 933, pp. 71–110. Springer, Berlin (1982)

21. Fukuma, M., Kawai, H., Nakayama, R.: Continuum Schwinger–Dyson equations and universal structures in two-dimensional quantum gravity. Int. J. Mod. Phys. A 6, 1385–1406 (1991)

22. Fukuma, M., Kawai, H., Nakayama, R.: Infinite dimensional Grassmannian structure of two-dimensional quantum gravity. Commun. Math. Phys. 143, 371–403 (1992)

23. Getzler, E.: The Toda conjecture. Symplectic geometry and mirror symmetry (Seoul, 2000). pp. 51–79. World Scientific, River Edge (2001)

24. Givental, A.B.: Gromov-Witten invariants and quantization of quadratic Hamiltonians. Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. Mosc. Math. J. 1, 551–568, 645 (2001)

25. Goeree, J.: $W$-constraints in 2D quantum gravity. Nucl. Phys. B 358, 737–757 (1991)

26. Humphreys, J.E.: Reflection groups and Coxeter groups. Cambridge Studies in Advanced Mathematics, 29. Cambridge University Press, Cambridge (1990)

27. Kac, V.: Vertex algebras for beginners, 2nd edn. University Lecture Series 10. American Mathematical Society, Providence (1998)

28. Kac, V., Schwarz, A.: Geometric interpretation of the partition function of 2D gravity. Phys. Lett. B 257, 329–334 (1991)

29. Kontsevich, M.: Intersection theory on the moduli space of curves and the matrix Airy function. Commun. Math. Phys. 147, 1–23 (1992)

30. Kontsevich, M., Manin, Yu.: Gromov-Witten classes, quantum cohomology, and enumerative geometry. Commun. Math. Phys. 164, 525–562 (1994)

31. Milanov, T.: Hirota quadratic equations for the extended Toda hierarchy. Duke Math. J. 138, 161–178 (2007)

32. Okounkov, A., Pandharipande, R.: The equivariant Gromov-Witten theory of $\mathbb{P}^1$. Ann. Math. 163, 561–605 (2006)

33. Polishchuk, A., Vaintrob, A.: Algebraic construction of Witten’s top Chern class. Advances in algebraic geometry motivated by physics (Lowell, MA, 2000). Contemporary Mathematics, vol. 276, pp. 229–249. American Mathematical Society, Providence (2001)

34. Ruan, Y., Tian, G.: A mathematical theory of quantum cohomology. J. Differ. Geom. 42, 259–367 (1995)

35. Teleman, C.: The structure of 2D semi-simple field theories. Invent. Math. 188, 525–588 (2012)

36. Witten, E.: Two-dimensional gravity and intersection theory on moduli space, Surveys in differential geometry (Cambridge, MA, 1990), pp. 243–310. Lehigh University, Bethlehem (1991)

37. Witten, E.: Algebraic geometry associated with matrix models of two-dimensional gravity. Topological methods in modern mathematics (Stony Brook, NY, 1991), pp. 235–269. Publish or Perish, Houston (1993)

38. Wu, C.-Z.: A remark on Kac–Wakimoto hierarchies of D-type. J. Phys. A 43, 035201 (2010)

39. Zamolodchikov, A.B.: Infinite extra symmetries in two-dimensional conformal quantum field theory. (Russian) Teoret. Mat. Fiz. 65, 347–359 (1985)

40. Zhang, Y.: On the $\mathbb{CP}^1$ topological sigma model and the Toda lattice hierarchy. J. Geom. Phys. 40, 215–232 (2002)