Exact solution for the time-dependent non-Hermitian generalized Swanson oscillator

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Abstract: We produce an exact solution of the Schrödinger equation for the generalized time-dependent Swanson oscillator. The system studied is a non-Hermitian setup characterized by time-dependent complex coefficients. The exact solution is obtained by applying two transformations, and under the right choice of the relevant parameters. Consequently, the model is reduced to a time-independent harmonic oscillator.

Keywords: PT-symmetry; Swanson oscillator; Non-Hermitian systems

1. Introduction

PT-symmetric non-Hermitian systems have gained a flurry of interest, due to their inherent potential of exhibiting a real spectrum, notwithstanding the non-hermiticity of the Hamiltonians. Historically, it was Bender’s [1–4] pioneering work, that marked the grounds for systems lacking the Hermiticity condition to acquire a new meaning with the constraint of PT-symmetry, i.e., the discrete space-time symmetries of parity (P) and time-reversal (T). Since then, many prototypical examples of the above symmetry-based scenario have been presented in the literature [5, 6], ranging from theoretical scrutiny in open quantum systems [7] or quantum optics [8] to powerful applications in optics, such as perfect laser absorbers [9], and spatial optical switches [10], among others. Nonetheless, only a few special classes of quantum PT-models are fully exactly solvable. The Swanson oscillator, for example, is a very popular PT-symmetric, and one of a handful of non-Hermitian systems that fulfills this requirement [11]; many features of this non-Hermitian quadratic Hamiltonian, as well as its basic algebraic properties, have been extensively undertaken in [11, 12]. Subsequently, studies related to the supersymmetric realization of the above non-Hermitian oscillator and q-deformation boson algebras were carried out in [13–17]. Going even further, several authors have found that the time-independent Swanson model can be mapped to a harmonic oscillator by performing a gauge-like transformation, Bogoliubov transformation or a non-unitary transformation [12, 18]. Although authors in [19] have demonstrated that the system is not necessarily isospectral to the harmonic oscillator under certain transformations. Naturally, the underlying physics behind the quadratic Hamiltonian in position and momentum operators has led to adjusting the model to time-dependent scenarios, where different perspectives have emerged to provide an exact solution for the system; some of them consist of the use of the Lewis–Riesenfeld invariant approach in conjunction with a time-dependent metric [20] or under appropriate time-dependent unitary transformations, where the Hamiltonian is written in a linear combination of SU(1,1) and SU(2) generators [21]. In the same way, a generalization of the Swanson model has inevitably followed the same direction; Zelaya et al. [22] have presented an extension of the Swanson Hamiltonian, under particular constraints linking the time-dependent real parameters on the Hamiltonian. There, the Schrödinger equation for a special case of this model produces a generalization of the Caldirola–Kanai oscillator, whose solution is obtained by using point transformations; in particular, the concrete generalization of the Swanson oscillator, provided in Reference [22], is missing a study of the non-Hermitian case defined by arbitrary time-dependent complex-valued functions. Although the above works have already led us to implement time-independent non-unitary transformations into the generalized Swanson Hamiltonian, where we have been able to find an exact solution to its Schrodinger equation [23]. Influenced by

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obtained. Section 3 is devoted to the scenario where the monic oscillator, whose exact solution can be easily system can be rendered to the non-Hermitian forced har-
we show that by a suitable choice of the coefficients, the
dependent or time-dependent. The layout of the paper is as comprehensive review; rather, we want to show that under an
partially our objective; we have solved the Swanson studied case; indeed, such non-unitary transformations are easy to use in practice for this particular case and work especially well without resorting to additional approaches, being a significant difference between the conventional methods that deal with the dynamics of the system with Lewis and Riesenfeld invariants in conjunction with time-dependent Dyson maps and metrics operators [24].

The main goal of our work is to solve the Swanson Hamiltonian in the general case, i.e., when the function coefficients are complex. The motivation to do that task is the great number of applications that this model has in physics, such as the design of electromagnetic traps in plasma physics [25, 26], and, in particular, in quantum optics, the study of pseudo-bosonic coherent states [27], bisqueezed states [28], and the boost of high interferometric sensitivity, where the treatments for time-dependent Hermitian quadratic Hamiltonians have proved the possibility of producing squeezed states with an infinite squeezing degree at a finite time interval [24]; this last problem is an interesting one that could be of great relevance to improve gravitational-wave interferometers within the LIGO and Virgo experiments [29]. Another attractive application involves many-body quantum systems, being the quadratic non-Hermitian interactions from \(N\) boson operators relevant in the context of metal-insulator transitions in type-II superconductors [30]. In this manuscript, we have achieved partially our objective; we have solved the Swanson oscillator for some special cases of the complex function coefficients. It is worth noting, however, that the main purpose of the present work is not to engage in a comprehensive review; rather, we want to show that under an adequate choice of complex time-dependent functions, it is possible to obtain an exact solution of the Hamiltonian; we present the model and the needed transformations for both scenarios where the Hamiltonian system may be time-independent or time-dependent. The layout of the paper is as follows: In Sect. 2, we emphasize solving the Schrödinger equation of the time-independent generalized Swanson oscillator, with real coefficients instead of complex ones; we show that by a suitable choice of the coefficients, the system can be rendered to the non-Hermitian forced harmonic oscillator, whose exact solution can be easily obtained. Section 3 is devoted to the scenario where the Hamiltonian is time-dependent on possessing complex coefficient functions; under the suitable selection of these functions, we present the exact solution of two specific cases concerning the non-Hermitian Caldirola–Kanai and the generalized Swanson system with time-dependent complex growing mass. Finally, conclusions and a discussion of the work are presented in Sect. 4. It is important to emphasize that the guiding thread of the entire paper is the solution of the Swanson oscillator for a particular case of certain complex coefficients, and for that, we are analyzing simpler cases until we reach the one that interests us.

2. Time-independent generalized Swanson oscillator

The starting point is the non-Hermitian time-independent Hamiltonian of the generalized Swanson oscillator defined by [22, 31]

\[
\frac{1}{\omega_0} H_{GSW} = \theta (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) + x_1 \hat{a}^\dagger + \beta_1 \hat{a} + x_2 \hat{a}^2 + \beta_2 \hat{a}^2,
\]

where \(\hat{a}\) and \(\hat{a}^\dagger\) are the bosonic annihilation and creation operators of the standard harmonic oscillator, with real time-independent functions \(\theta\) and \(x_j \neq \beta_j\) with \(j = 1, 2\), being \(\omega_0 > 0\) a constant with units of frequency. Note that the Hamiltonian describes a harmonic oscillator subjected to an unbalanced linear amplification process, apart from the parametric one, and such imbalance is determinate by \(\theta\), \(x_j\) and \(\beta_j\) in the interplay between bosonic annihilation and creation operators. In particular, using the well-known relations \(\hat{x} = \sqrt{\frac{m_0}{\hbar \omega_0}} (\hat{x} + i \frac{\hat{p}}{\hbar \omega_0})\) and \(\hat{p} = \sqrt{\frac{m_0 \omega_0}{\hbar}} (\hat{x} - i \frac{\hat{p}}{\hbar \omega_0})\), with constant mass \(m_0\), the Hamiltonian \(H_{GSW}\) can be cast in terms of position and momentum operators as

\[
H_{GSW} = v_1 \hat{p}^2 + v_2 \hat{x}^2 + iv_3 (\hat{x} \hat{p} + \hat{p} \hat{x}) + iv_4 \hat{p} + v_5 \hat{x},
\]

where

\[
\begin{align*}
 v_1 &= \frac{2\theta - (x_2 + \beta_2)}{2m_0}, \\
v_2 &= \frac{m_0 \omega_0^2}{2} (2\theta + x_2 + \beta_2), \\
v_3 &= \frac{\omega_0}{2} (\beta_2 - x_2), \\
v_4 &= \sqrt{\frac{\omega_0}{2m_0}} (\beta_1 - x_1), \\
v_5 &= \sqrt{\frac{m_0 \omega_0^2}{2}} (x_1 + \beta_1).
\end{align*}
\]

It is important to mention that the linear terms in \(\hat{x}\) and \(\hat{p}\) make this physical system non-\(\mathcal{PT}\)-symmetric; indeed, the usual form of parity and time reversal operators is \(\mathcal{P}\): \(\hat{x} \rightarrow -\hat{x}; \hat{p} \rightarrow -\hat{p};\mathcal{T}: \hat{x} \rightarrow \hat{x}; \hat{p} \rightarrow -\hat{p}; i \rightarrow -i\), and the Hamiltonian \(H_{GSW}\) is not invariant under them. On the other hand, when \(v_4 = v_5 = 0\), we return to the original \(\mathcal{PT}\)-symmetric Swanson oscillator, which describes the generalized quadratic time-dependent non-Hermitian harmonic oscillator. A subtle, but important point, is that one can link several models depending on the different elections of the time-independent functions of Eq. (1); indeed,
the Swanson Hamiltonian under appropriate transformation can be used to model systems such as the harmonic oscillator, or systems with instabilities, such as the repulsive harmonic oscillator [32].

2.1. Non-Hermitian forced harmonic oscillator

Before proceeding to solve exactly the Hamiltonian (2), we begin by discussing the particularly simple case when \( \alpha_2 = \beta_2 = 0 \), and which is commonly called the non-Hermitian and non-\( \mathcal{PT} \)-symmetric forced harmonic oscillator; this system is described by the Schrödinger equation

\[
i \frac{d}{dt} |\psi(t)\rangle = \left[ \frac{\theta}{m_0} \left( \p^2 + m_0^2 \omega_0^2 \phi^2 \right) + iv_4 \phi + v_3 \phi \right] |\psi(t)\rangle.
\]

If we introduce the non-unitary transformation \( |\psi(t)\rangle = \hat{\eta}|\phi(t)\rangle \) with \( \hat{\eta} = \exp \left\{ -\frac{[m_0 \omega_0 (x_2 - \beta_2) \phi - i (x_2 + \beta_2) \phi]}{2\theta} \right\} \) and use the formula \( e^{i\beta} e^{-A} = B + \left[ A, B \right] + \frac{1}{2} \left[ A, \left[ A, B \right] \right] + \ldots [33, 34] \), we get the new evolution equation

\[
i \frac{d}{dt} |\phi(t)\rangle = \left[ \frac{\theta}{m_0} \left( \p^2 + m_0^2 \omega_0^2 \phi^2 \right) - \frac{\omega_0 x_2 \phi_1}{2\theta} \right] |\phi(t)\rangle,
\]

which is nothing else than a harmonic oscillator displaced by the quantity \( -\frac{\omega_0 x_2 \phi_1}{2\theta} \), and with an energy \( E_n = \theta_0 (2n + 1) - \frac{\omega_0 x_2 \phi_1}{2\theta} \). Integrating the resulting expression over \( t \), and transforming back to the original representation \( |\psi(t)\rangle \), one finds the exact formal solution

\[
|\psi(t)\rangle = \exp \left\{ -i \frac{\theta}{m_0} \left( \p^2 + m_0^2 \omega_0^2 \phi^2 \right) t \right\} \hat{\eta}^{-1} |\psi(0)\rangle,
\]

where \( |\psi(0)\rangle \) is the initial state.

2.2. Exact solution for the generalized Swanson oscillator

In this section, we find the solution of the Schrödinger equation associated with the Hamiltonian \( \hat{H}_\text{GSW} \), expression (2). The exact solution can be obtained by performing two non-unitary transformations; first, we make \( |\psi(t)\rangle = \hat{\eta}_1 |\phi(t)\rangle \), where

\[
\hat{\eta}_1 = \exp \left\{ i \kappa \left[ \sqrt{\frac{m_0 \omega_0}{2}} \sin(\theta) \phi + \frac{\cos(\theta)}{\sqrt{2m_0 \omega_0}} \right] \right\},
\]

with

\[
\kappa = \sqrt{\frac{m_0 \omega_0}{2}} v_3 v_4 + v_1 v_5 \sqrt{1 - \frac{(v_2 v_4 - v_3 v_5)^2}{m_0^2 \omega_0^2 (v_3 v_4 + v_1 v_5)^2}},
\]

\[
\vartheta = i \arctanh \left[ \frac{v_3 v_4}{m_0 \omega_0 (v_3 + v_4)} \right],
\]

and we get the transformed Schrödinger equation

\[
i \frac{d}{dt} |\phi(t)\rangle = [v_4 \phi^2 + v_3 \phi^2 + iv_5 (\phi \p + \p \phi)] + \frac{v_2 v_4^2 - v_5 (v_1 v_5 + 2v_3 v_4)}{4(v_1 v_2 + v_3^2)} |\phi(t)\rangle.
\]

The transformation induced by (7) gets rid of the linear terms in \( \phi \) and \( \p \); in the absence of those terms, the Hamiltonian acquires the form of the \( \mathcal{PT} \)-symmetric Swanson oscillator. Secondly, we perform the transformation \( |\phi(t)\rangle = \hat{\eta}_2 |\chi(t)\rangle \), which is similar to the reported by [35], where

\[
\hat{\eta}_2 = \exp \left\{ \ln \left[ \frac{\sqrt{v_1 v_2 + v_3^2}}{\left( \frac{v_2}{2m_0 \omega_0} + \frac{v_1 v_5}{2m_0 \omega_0} \right) + \sqrt{\left( \frac{v_2}{2m_0 \omega_0} + \frac{v_1 v_5}{2m_0 \omega_0} \right)^2 - v_3^2}} \right] \right\} \times \left[ \frac{v_3}{2m_0 \omega_0} \phi^2 - \frac{\p^2}{2m_0 \omega_0} + i \left( \frac{v_2}{2m_0 \omega_0} + \frac{v_1 v_5}{2m_0 \omega_0} \right) (\p \phi + \phi \p) \right] \right\}
\]

and Eq. (9) is converted into

\[
i \frac{d}{dt} |\chi(t)\rangle = \hat{H} |\chi(t)\rangle = \left[ \frac{\tilde{\omega}}{2m_0 \omega_0} \left( \p^2 + m_0^2 \omega_0^2 \phi^2 \right) + \delta \right] |\chi(t)\rangle;
\]

hence, the above transformations, \( \eta_1 \) and \( \eta_2 \), lead us to the Schrödinger equation of a harmonic oscillator with

\[
\tilde{\omega} = 2 \sqrt{v_1 v_2 + v_3^2}, \quad \delta = \frac{v_2 v_4^2 - v_5 (v_1 v_5 + 2v_3 v_4)}{4(v_1 v_2 + v_3^2)}.
\]

The energy spectrum of the original Hamiltonian \( \hat{H}_\text{GSW} \) is obtained from the eigenequation of the Hamiltonian \( \hat{H} \), i.e., from \( \hat{H} |n\rangle = E_n |n\rangle \), where

\[
E_n = \sqrt{v_1 v_2 + v_3^2} (2n + 1) + \frac{v_2 v_4^2 - v_5 (v_1 v_5 + 2v_3 v_4)}{4(v_1 v_2 + v_3^2)}, \quad n = 0, 1, 2, \ldots.
\]

The eigenfunctions of \( \hat{H}_\text{GSW} \) can be derived from the
eigenfunctions of the harmonic oscillator via the association
\[
\hat{H}|n\rangle = \hat{n}_2^{-1}\hat{H}_{GSW}\hat{n}_1\hat{n}_2|n\rangle = \frac{1}{\omega_0}|n\rangle \Leftrightarrow \hat{H}_{GSW}|\tilde{n}\rangle = E_n|\tilde{n}\rangle,
\]
where \(|\tilde{n}\rangle = \hat{n}_1\hat{n}_2|n\rangle\). The above results successfully check the correctness of the results derived in [23].

Finally, the exact solution of the Schrödinger equation corresponding to \(\hat{H}_{GSW}\) can be written as
\[
|\psi(t)\rangle = \exp(-i\delta t)|\tilde{n}\rangle = \exp\left(-\frac{i\omega t^2}{2m_0\omega_0}\right)\hat{n}_2^{-1}\hat{n}_1^{-1}|\psi(0)\rangle.
\] (15)

3. Generalized Swanson Hamiltonian with time-dependent complex coefficients

In what follows, we focus on the time-dependent generalized Swanson Hamiltonian [22, 36, 37]
\[
\frac{1}{\omega_0}\hat{H}_{GSW}(t) = \theta(t)(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) + x_1(t)\hat{a}^\dagger
+ \beta_1(t)\hat{a} + x_2(t)\hat{a}^2 + \beta_2(t)\hat{a}^2 + V_0(t),
\] (16)
with complex time-dependent functions \(\theta(t), V_0(t), x_j(t)\) and \(\beta_j(t)\) with \(j = 1, 2\). Following the same procedure as in Sect. 2, and using the same notation reported in [22], we rewrite the Hamiltonian (16) in terms of momentum and position operators as
\[
\hat{H}_{GSW}(t) = \frac{\hat{p}^2}{2m(t)} + \frac{m(t)\omega^2(t)}{2}\hat{x}^2 + i\frac{\Omega(t)}{2}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right) + iv(t)\hat{p} + F(t)\hat{x} + \omega_0V_0(t),
\] (17)
where the new time-dependent \(m(t), \Omega(t), v(t)\) and \(F(t)\) functions are defined by
\[
m(t) = \frac{m_0}{2\theta(t) - x_2(t) + \beta_1(t)},
\]
\[
\omega^2(t) = \omega_0^2\left\{4\theta^2(t) - [x_2(t) + \beta_2(t)]^2\right\},
\]
\[
\Omega(t) = -\omega_0[x_2(t) - \beta_2(t)],
\]
\[
v(t) = -\sqrt{\frac{\omega_0}{2m_0}}[x_1(t) - \beta_1(t)],
\]
\[
F(t) = \sqrt{\frac{m_0\omega_0^2}{2}}[x_2(t) + \beta_1(t)].
\] (18)

3.1. Non-Hermitian Caldirola–Kanai case: \(x_2 = \beta_2\).

Clearly the time-dependent configuration of \(\hat{H}_{GSW}(t)\) is non-Hermitian when \(x_j(t) \neq \beta_j^*(t)\), and also it is not \(PT\)-
symmetric. Since the solution of the generalized Swanson oscillator is completely determined by the choice of the above time-dependent functions, it is interesting to consider some special cases for which the corresponding time-dependent Schrödinger equation has exact closed-form solutions. Let us consider a model generated by the time-dependent functions
\[
\theta(t) = \frac{1}{2}\cos(2\Gamma t),
\]
\[
\beta_1(t) = i\frac{2m_0}{\omega_0}\sin(\Gamma t),
\] (19)
which lead us to the non-Hermitian Caldirola–Kanai Hamiltonian system
\[
\hat{H}_{NC}(t) = \frac{\hat{p}^2}{2m_0} + \frac{m_0\omega_0^2}{2}\hat{x}^2 + i\frac{\Gamma}{2}(\hat{p}\hat{x} + \hat{x}\hat{p}) + iv_0\hat{p} + \nu_0\hat{x},
\] (20)
whose mass depends exponentially on time, i.e., \(m(t) = m_0\exp(2\Gamma t)\) [38, 39]. For the sake of simplicity, we have chosen \(V_0(t) = 0\).

In order to solve the corresponding Schrödinger equation of the above Hamiltonian, we consider the non-unitary transformation
\[
|\psi(t)\rangle = |\tilde{n}\rangle|\phi(t)\rangle = \exp\left[-\frac{\Gamma}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\right]|\phi(t)\rangle.
\]

The idea behind the transformation is to remove the temporal dependence of Hamiltonian (20), to get
\[
if \frac{d}{dt}|\phi(t)\rangle = \left[\frac{\hat{p}^2}{2m_0} + \frac{m_0\omega_0^2}{2}\hat{x}^2 + i\frac{\Gamma}{2}(\hat{p}\hat{x} + \hat{x}\hat{p}) + iv_0\hat{p} + \nu_0\hat{x}\right]|\phi(t)\rangle.
\]
\] (21)

Identifying \(\frac{1}{2m_0} \rightarrow v_1, \frac{m_0\omega_0^2}{2} \rightarrow v_2, \frac{\Gamma}{2} \rightarrow v_3, \nu_0 \rightarrow v_4, v_0\nu_0\omega_0 \rightarrow v_5\), and according to the procedure outlined in Sub sect. 2.2, we can write the exact solution as
\[
|\psi(t)\rangle = \exp\left[i\frac{\Gamma\nu_0\omega_0}{\omega_0^2 + \Gamma^2}t\right]|\tilde{n}\rangle|\tilde{n}\rangle \exp\left[-it\sqrt{\frac{\omega_0^2 + \Gamma^2}{2m_0\omega_0}}(\hat{x}^2 + m_0\omega_0^2\hat{x}) \right]\hat{n}_1^{-1}\hat{n}_2^{-1}|\psi(0)\rangle.
\] (22)

with the energy spectrum given by
\[
E_n = \sqrt{\omega_0^2 + \Gamma^2}\left(2n + 1\right) - \frac{\Gamma\nu_0\omega_0}{\omega_0^2 + \Gamma^2}.
\] (23)
3.2. Generalized Swanson Hamiltonian with complex mass growing with time: case \( x_2 \neq \beta_2 \)

In the following, we define the complex time-dependent functions

\[
\theta(t) = \frac{1 + (1 - 2it\Omega_0)^2}{2(1 - 2it\Omega_0)} ,
\]

\[
\alpha_1(t) = -\frac{v_0\sqrt{\frac{m_0\Omega_0^2}{2}}(1 + it\Omega_0 + 2t^2\omega_0\Omega_0)}{\sqrt{1 - 2it\Omega_0}} ,
\]

\[
\beta_1(t) = \frac{v_0\sqrt{\frac{m_0\Omega_0^2}{2}}(1 - it\Omega_0 - 2t^2\omega_0\Omega_0)}{\sqrt{1 - 2it\Omega_0}} ,
\]

\[
\alpha_2(t) = -\frac{\Omega_0(1 + 2it\Omega_0 + 2t^2\omega_0\Omega_0)}{2\omega_0(1 - 2it\Omega_0)} ,
\]

\[
\beta_2(t) = \frac{\Omega_0(1 - 2it\Omega_0 - 2t^2\omega_0\Omega_0)}{2\omega_0(1 - 2it\Omega_0)} ,
\]

\[
V_0(t) = -\frac{1}{2} \gamma^2(t)m_0\omega_0^2(1 - 2it\Omega_0),
\]

with \( \Omega_0 \geq 0 \) and \( v_0 \geq 0 \). Then the Hamiltonian (17) acquires the form

\[
\hat{H}_{GSW}(t) = \frac{\hat{p}^2}{2m_0(1 - 2it\Omega_0)} + \frac{m_0\omega_0^2(1 - 2it\Omega_0)}{2} \hat{x}^2
\]

\[
+ \frac{i}{2}\frac{\Omega_0}{1 - 2it\Omega_0}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{v_0}{\sqrt{1 - 2it\Omega_0}} \hat{p}
\]

\[
- itm_0v_0\omega_0\sqrt{1 - 2it\Omega_0}\hat{x}
\]

\[
- \frac{1}{2} \gamma^2(t)m_0\omega_0^2(1 - 2it\Omega_0).
\]

(25)

It is not difficult to infer that the set of complex functions (24) produces an oscillator with complex mass that increases with time; nevertheless, one can perform multiple configurations of the time-dependent functions to analyze different mass choices. In order to solve the Schrödinger equation associated with (24), we do two non-unitary transformations: First, consider \( |\psi(t)\rangle = \exp[\gamma(t)\hat{p}]|\phi(t)\rangle \) with \( \gamma(t) = \frac{v_0 t}{\sqrt{1 - 2it\Omega_0}} \); the main goal of this transformation is to remove the linear terms in \( \hat{x} \) and \( \hat{p} \), similarly to the time-independent case of Sub sect. 2.2; the transformed Schrödinger equation becomes

\[
i\frac{d}{dt} |\phi(t)\rangle = \left[ \frac{\hat{p}^2}{2m_0(1 - 2it\Omega_0)} + \frac{m_0\omega_0^2(1 - 2it\Omega_0)}{2} \hat{x}^2
\]

\[
+ \frac{i}{2}\frac{\Omega_0}{1 - 2it\Omega_0}(\hat{x}\hat{p} + \hat{p}\hat{x}) \right] |\phi(t)\rangle ;
\]

(26)

note that the non-Hermitian time-dependent Hamiltonian inside of brackets is \( PT \)-symmetric, since it is invariant under the combined operations of space-time inversion: \( P : \hat{x} \rightarrow -\hat{x}, \quad \hat{p} \rightarrow -\hat{p}; \quad T : \hat{x} \rightarrow \hat{x}; \quad \hat{p} \rightarrow -\hat{p}, t \rightarrow -t, i \rightarrow -i \[40-42\]. Secondly, we want to eliminate the time-dependence and get rid of the squeezing term \( (\hat{x}\hat{p} + \hat{p}\hat{x}) \); to do this, we consider a second and final transformation \( |\psi(t)\rangle = \exp\left[\frac{\delta(t)}{2}(\hat{x}\hat{p} + \hat{p}\hat{x})\right] |\chi(t)\rangle \) with \( \delta(t) = \frac{i}{2} \ln(1 - 2it\Omega_0) \), to get

\[
i\frac{d}{dt} |\chi(t)\rangle = \left( \frac{\hat{p}^2}{2m_0} + \frac{m_0\omega_0^2}{2} \hat{x}^2 \right) |\chi(t)\rangle .
\]

(27)

Evidently, our transformations map the time-dependent generalized harmonic oscillator to an ordinary time-independent harmonic oscillator.

Finally, the exact solution is obtained by using the inverse transformations previously established in this subsection,

\[
|\psi(t)\rangle = \exp\left(\frac{v_0 t}{\sqrt{1 - 2it\Omega_0}} \hat{p}\right) \exp\left[\frac{i}{4} \ln(1 - 2it\Omega_0)(\hat{x}\hat{p} + \hat{p}\hat{x})\right] \exp\left[\frac{it}{2m_0}(\hat{p}^2 + m_0\omega_0^2\hat{x}^2)\right] |\psi(0)\rangle .
\]

(28)

It is worth mentioning that the exact solution presented above is composed by the product of three exponential operators: the first one (from left to right) is a displacement-like operator, followed by a squeezed-like similarity operator, which acts over the propagator related to the harmonic oscillator; this factorization also takes place in the time-independent case in Eq. (15). In the case of the generalized Swanson model with a time-dependent complex mass, the above transformations can also be used to extend the study of the so-called bi-squeezed states [28], in the general form of position and momentum operators. This also opens the way toward interesting applications; for instance, the exploration of the squeezing degree under the linear and parametric amplification process in \( \hat{H}_{GSW}(t) \) to boost a high interferometric sensitivity; this requires a more careful analysis and it is beyond the scope of the present work. Moreover, one can use results from Ref. [43] to factorize the exponential operator of the harmonic oscillator and obtain the following form

\[
|\psi(t)\rangle = (1 - 2it\Omega_0)^{1/4} \exp\left(\frac{v_0 t}{\sqrt{1 - 2it\Omega_0}} \hat{p}\right) \exp\left[\frac{i}{2} \ln(1 - 2it\Omega_0)(\hat{x}\hat{p} + \hat{p}\hat{x})\right] \times \exp\left[\frac{-i}{2} \frac{\tan(\omega_0 t/2)}{2m_0\omega_0} \hat{p}^2\right] \exp\left[\frac{-i}{2} \frac{\tan(\omega_0 t/2)}{2m_0\omega_0} \hat{x}^2\right] |\psi(0)\rangle,
\]

(29)

which in the coordinate representation is

\[ \psi(x, t) = \langle x | |\psi(t)\rangle , \]

and reads as
\[ \psi(x, t) = (1 - 2\text{i}\Omega_0)^{1/4} \exp \left( -i \frac{v_0 t}{\sqrt{1 - 2\text{i}\Omega_0}} \frac{d}{dx} \right) \exp \left[ \frac{1}{2} \ln(1 - 2\text{i}\Omega_0) \frac{d}{dx} \right] \exp \left[ i \tan(\omega_0 t/2) \frac{d^2}{dx^2} \right] \exp \left[ \tan(\omega_0 t/2) \frac{d^2}{dx^2} \right] \psi(x, 0). \] (30)

The next step is to evaluate the exact solution with a given initial condition; we choose as an initial wave function a Gaussian \[ \psi(x, 0) = \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^{1/2} \exp \left[ -\frac{(x-x_0)^2}{4\sigma^2} \right], \] which is displaced a distance \( x_0 \) from the origin and with a width given by \( \sigma \). The product of the exponential operators of the harmonic oscillator acting on the initial state leads to

\[ \psi(x, t) = (1 - 2\text{i}\Omega_0)^{1/4} \exp \left( -i \frac{v_0 t}{\sqrt{1 - 2\text{i}\Omega_0}} \frac{d}{dx} \right) \exp \left[ \frac{1}{2} \ln(1 - 2\text{i}\Omega_0) \frac{d}{dx} \right] \exp \left[ i \frac{m_0 \omega_0}{2} \tan(\omega_0 t/2) \right] \exp \left[ -i \frac{m_0 \omega_0}{2} \tan(\omega_0 t/2) \right] \exp \left[ \tan(\omega_0 t/2) \right] \psi(x, 0). \] (31)

and using the known action of the dilatation operator, \[ \exp(\lambda \frac{d}{dx})f(x) = f(e^\lambda x), \] and of the translation operator, \[ \exp(\lambda \frac{d}{dx})f(x) = f(x + \lambda), \] acting on a given function \( f(x) \) \[ \psi(x, t) = (1 - 2\text{i}\Omega_0)^{1/4} \exp \left[ \frac{(m_0 \omega_0)^2}{2} \tan(\omega_0 t/2) + (1/4\sigma^2) \right] \exp \left[ -i \frac{m_0 \omega_0}{2} \tan(\omega_0 t/2) \right] \exp \left[ \tan(\omega_0 t/2) \right] \psi(x, 0). \] (32)

This wavefunction is a solution of the Schrödinger equation \[ i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{\text{GSW}}(t)|\psi(t)\rangle. \] In Fig. 1, we have plotted the probability density \( |\psi(x, t)|^2 \); the results depicted correspond to \( m_0 = 1, \omega_0 = 2, \sigma = 1/\sqrt{2} \) and \( x_0 = 2 \) with two choices of \( \Omega_0 \) and \( \nu_0 \), which are \( \Omega_0 = 0.015, \nu_0 = 0.00015 \) and \( \Omega_0 = 0.01 \) and \( \nu_0 = 0.015 \). As Fig. 1 reveals, the probability density amplitudes grow without bound as we increase the value of \( \Omega_0 \) and \( \nu_0 \) (see Fig. 1b); essentially, the influence of \( \Omega_0 \) and \( \nu_0 \) over \( t \) changes gradually the oscillating behavior of the wave packet as time increases.

4. Conclusions

In summary, we have studied the generalized version of the Swanson Hamiltonian in time-dependent and independent cases. Specifically, in the time-independent scenario, we notice that the energy spectrum and eigenfunctions of the non-Hermitian oscillator Hamiltonian \( \hat{H}_{\text{GSW}} \) may be mapped to a harmonic oscillator by non-unitary transformations. Meanwhile, in the time-dependent case, the degree of difficulty to determine the exact form of the solution of the

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Fig. 1 Amplitude probability distribution \( |\psi(x, t)|^2 \). The left graph (a) is plotted using the parameters \( m_0 = 1, \omega_0 = 2, \sigma = 1/\sqrt{2}, x_0 = 2, \Omega_0 = 0.015, \nu_0 = 0.00015 \). For (b) case, the set of values are \( m_0 = 1, \omega_0 = 2, \sigma = 1/\sqrt{2}, x_0 = 2, \Omega_0 = 0.01 \) and \( \nu_0 = 0.015 \).
Schrödinger equation for the Hamiltonian $\hat{H}_{\text{GSW}}$ depends on the special elections of the time-dependent functions involved; in particular, the judicious choice of the time-dependent or independent parameters lead to the Hamiltonian $\hat{H}_{\text{GSW}}$, which can be converted into relevant sub-classes of non-Hermitian Hamiltonians with real spectrum. For instance, in the time-independent case with real coefficients, we have demonstrated that $\hat{H}_{\text{GSW}}$ can then be transformed into the non-Hermitian forced harmonic oscillator. On the other hand, in the case of complex time-dependent functions, we have reached two models concerning the non-Hermitian Caldirola–Kanai and the $\hat{H}_{\text{GSW}}$ with a complex mass growing with time. In any of these cases, we found both systems become exactly solvable by directly applying time-dependent or independent parameters lead to the Hamiltonian $\hat{H}_{\text{GSW}}$, that allowed us to avoid the use of time-dependent invariants or point transformations; especially, the theoretical result of Eq. (28) can be useful as an alternative case for estimating the bi-squeezed excitations; besides, our results could be adapted to study the rich physics of a combined system of two coupled generalized Swanson-like oscillators.

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Declarations

Ethics approval This material is the author’s original work, which has not been previously published elsewhere. All authors have been personally and actively involved in substantial work leading to the paper and will take public responsibility for its content.

Conflict of interest The authors declare no competing interests regarding the publication of this manuscript.

References

[1] C M Bender, D C Brody and H F Jones Am. J. Phys. 71 1095 (2003)
[2] C M Bender, D C Brody and H F Jones Phys. Rev. Lett. 89 270401 (2002)
[3] C M Bender and A Turbiner Phys. Lett. A. 173 442 (1993)
[4] C M Bender and K A Milton Phys. Rev. D 55 R3255 (1997)
[5] C M Bender and S Boettcher Phys Rev Lett 80 5243 (1998)
[6] C M Bender Rept. Prog. Phys. 70 947 (2007)
[7] H Feshbach Ann. Phys. 5 357 (1958)
[8] M B Plenio and P L Knight Rev. Mod. Phys. 70 101 (1998)
[9] S Longhi Phys. Rev. A 82 031801 (2010)
[10] F Nazari, M Nazari and M K Moravvej-Farshi Opt. Lett. 36 4368 (2011)
[11] M S Swanson Jour. Math. Phys. 45 585 (2004)
[12] Z Ahmed Phys. Lett. A. 294 287 (2002)
[13] A Sinha and P Roy J. Phys. A: Math. Theor. 41 335306 (2008)
[14] B Bagchi and I Marquette Phys. Lett. A 379 1584 (2015)
[15] S Dey, A Fring and B Khantoul J. Phys. A: Math. Theor. 46 335304 (2013)
[16] S Dey, A Fring, and L Gouba, J. Phys. A: Math. Theor. 48 40FT01 (2015)
[17] F Bagarello and A Fring Int. J. of Mod. Phys. B 31 1750085 (2017)
[18] H B Geyer, F G Scholtz and I Snyman Czech. J. Phys. 54 1069 (2004)
[19] B Midya, P P Dube, R Roychoudhury, J. Phys. A Mathe. Theor. 44 062001 (2011)
[20] M Maamache, O K Djezgihiou, N Mana and W Koussa Eur. Phys. J. Plus 132 383 (2017)
[21] M Maamache Acta Polytechnica 57 424 (2017)
[22] K Zelaya and O Rosas-Ortiz Quantum Reports 3 458 (2021)
[23] B M Villegas-Martínez, F Soto-Eguibar, S A Hajjman, F A Asenjo and H M Moya-Cessa, [arXiv:2201.06536 [quant-ph]]
[24] M A de Ponte, F S Luiz, O S Duarte and M H Y Moussa Phys. Rev. A 100 012128 (2019)
[25] W Paul Electromagnetic traps for charged and neutral particles Rev. Mod. Phys 62 531 (1990)
[26] M A Combescure Quantum particle in a quadrupole radio-frequency trap Ann. Inst. Henri Poincare A 44 293 (1986)
[27] N Mana, O Zaidi and M Maamache J. Phys. Math. 61 102103 (2020). https://doi.org/10.1063/5.0013723
[28] F Bagarello, F Gargano and S Spagnolo Journal of Physics A: Mathematical and Theoretical 51 455204 (2018)
[29] B P Abbott et al Phys Rev. Lett. 116 061102 (2016)
[30] N Hatano and D R Nelson Phys. Rev. Lett. 77 570 (1997)
[31] B Bagchi and T Tanaka Phys. Lett. A. 372 5390 (2008)
[32] V Fernández, R Ramírez and M Reboiro Journal of Physics A: Mathematical and Theoretical 55 015303 (2022)
[33] B Hall Lie Groups, Lie Algebras, and Representatives: an elementary introduction (New York: Springer-Verlag) (2003)
[34] W Miller, Symmetry and separation of variables, Boston, M: Addison-Wesley (1977)
[35] H B Zhang, G Y Jiang and G C Wang J. Math. Phys. 56 072103 (2015)
[36] ŞA Büyükaşık and Z Çayič J. Math. Phys. 60 062104 (2019)
[37] J R Choi Pramana J. Phys. 61 7 (2003)
[38] P Caldirola Nuovo Cimento 18 393 (1941)
[39] F Bagarello and A Fring. (2017)
[40] N Moiseyev Phys. Rev. A 83 052125 (2011)
[41] X Luo, J Huang, H Zhong, X Qin, Q Xie, Y S Kivshar and C Lee Phys. Rev. Lett. 110 243902 (2013)
[42] J C Yuce Phys. Lett. A 336 290 (2005)
[43] P C García Quijas and L M Arévalo Aguilar, Phys. Scr.. 75 85 (2007)
[44] ŞA Büyükaşık and Z Çayič J. Math. Phys. 57 122107 (2016)

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