ON THE POSITIVE MASS THEOREM FOR MANIFOLDS WITH CORNERS

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Abstract. We study the positive mass theorem for certain non-smooth metrics following P. Miao’s work. Our approach is to smooth the metric using the Ricci flow. As well as improving some previous results on the behaviour of the ADM mass under the Ricci flow, we extend the analysis of the zero mass case to higher dimensions.

1. Introduction

The positive mass theorem, roughly speaking, says that an asymptotically flat manifold with non-negative scalar curvature has non-negative ADM mass. This was proved by Schoen-Yau [13] for manifolds of dimension at most 7, and by Witten [19] (see also Bartnik [2]) for spin manifolds of arbitrary dimension. The ADM mass is an invariant of the manifold introduced by Arnowitt-Deser-Misner [1], depending on the geometry at infinity. For precise definitions see Section 2.

It is a natural question to generalise the positive mass theorem to non-smooth Riemannian metrics. Part of the problem is to define what non-negative scalar curvature means when the metric is not sufficiently regular for the scalar curvature to be defined in the usual way.

We study this problem for a special type of singular metric, following P. Miao [11]. He considers a smooth manifold $M$ with a compact domain $\Omega \subset M$ such that $M \setminus \Omega$ is diffeomorphic to $\mathbb{R}^n$ minus a ball, and $\Sigma = \partial \Omega$ is a smooth hypersurface. The metric $g$ on $M$ is assumed to be Lipschitz, such that its restriction to $\Omega$ and $M \setminus \overline{\Omega}$ is $C^{2,\alpha}$. Away from $\Sigma$ the scalar curvature of $g$ can be defined as usual. On $\Sigma$ write $H(\Sigma_-)$ for the mean curvature of $\Sigma$ in $(\overline{\Omega}, g)$, and $H(\Sigma_+)$ for the mean curvature of $\Sigma$ in $(M \setminus \Omega, g)$, in both cases with respect to the normal vectors pointing out of $\Omega$. As explained in [11], the condition

$$H(\Sigma_-) \geq H(\Sigma_+)$$

(1)

can be thought of as non-negativity of the scalar curvature along $\Sigma$ in a distributional sense. The main theorem in [11] is

Theorem 1 (Miao [11]). Suppose that $M$ is a manifold for which the positive mass theorem of Schoen-Yau [13] and Witten [19] holds for smooth asymptotically flat metrics. Suppose now that $g$ is a metric on $M$ with corners along a hypersurface $\Sigma$ as described above, and that $g$ is asymptotically flat
in $C^2_\delta$ where $\delta > (n - 2)/2$. In addition, suppose that the scalar curvature of $g$ is non-negative and integrable away from $\Sigma$ and in addition the condition (7) holds. Then the ADM mass $m(g) \geq 0$.

In this paper we will first give a slightly different proof of this result, although some ingredients will be used from [11]. The basic idea in [11] is to first obtain smoothings $g_\varepsilon$ of the metric $g$ using mollifiers, and then show that for small $\varepsilon$ one can conformally change $g_\varepsilon$ to make the scalar curvature non-negative. One then applies the usual positive mass theorem. Instead we use the Ricci flow introduced by Hamilton [8] to smooth the metric, which at least heuristically should preserve the non-negative scalar curvature condition. It was shown by Simon [16] that the Ricci flow can be started with a $C^0$ initial metric, and the construction relies on smoothings $g_\varepsilon$ as above. We then use the bounds on the scalar curvature of $g_\varepsilon$ obtained by Miao to show that the metrics along the Ricci flow starting at the singular metric $g$ have non-negative scalar curvature.

When the mass is zero, the corresponding rigidity result was shown by Miao for dimension 3 using the results of Bray-Finster [3]. We can extend this to any $M$ for which the smooth positive mass theorem holds.

**Theorem 2.** Under the assumptions of Theorem 1 suppose that $m(g) = 0$. Then there exists a $C^{1,\alpha}$ diffeomorphism $\phi : M \to \mathbb{R}^n$, such that $g = \phi^* g_{\text{Eucl}}$, where $g_{\text{Eucl}}$ is the standard Euclidean metric.

The result shows that when the mass is zero, then the metric $g$ is the flat metric, written in terms of a possibly non-smooth coordinate system. The zero mass case for Lipschitz metrics was studied also by Shi-Tam [14] and Eichmair-Miao-Wang [6] in higher dimensions, but there the metrics $g$ dealt with are assumed to satisfy equality in (11). It seems likely that their methods extend to the case where strict inequality holds in (11) at least on spin manifolds. Note that Theorem 2 answers a question of Bray-Lee [4], showing that one can remove the spin assumption in the equality case of their Theorem 1.4.

The main advantage of using the Ricci flow in proving Theorem 2 as opposed to the method of Miao [11], is that when we smooth out the metric using the Ricci flow, the mass can not increase (with more regularity assumptions as in Theorem 3 below, the mass is constant, but when the initial metric is singular we could only show that the mass does not increase). In contrast in Miao’s method, when we apply a conformal change to the smoothings $g_\varepsilon$ to make the scalar curvature non-negative, the mass will only change slightly, but it can increase. This is enough to prove non-negativity of the mass of the singular metric, but it makes analyzing the zero mass case harder.

The behaviour of the ADM mass under the Ricci flow has already been studied by Dai-Ma [5] and also briefly by Oliynyk-Woolgar [12] in their study of the Ricci flow on asymptotically flat manifolds. In these works, however,
less than optimal decay conditions are assumed for the metric. We therefore show the following.

**Theorem 3.** If the metric $g$ is asymptotically flat in $C^2_\delta$ for $\delta > (n - 2)/2$, and the scalar curvature $R(g)$ is integrable, then $R(g(t)) \in L^1$ for $t > 0$ and the ADM mass is constant along the Ricci flow. The short time solution of the Ricci flow used here is constructed by Shi [15].

In Section 2 we will recall the basic definitions of asymptotically flat manifolds and the ADM mass, as well as Simon’s construction [16] of the Ricci flow with $C^0$ initial data. In Section 3 we prove Theorem 3 as well as some other results that we need later. In Section 4 we study how the mass behaves under taking a limit of a sequence of metrics. Finally, in Section 5 we give the proof of Theorem 2. The proof of a technical lemma is given in the Appendix.

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## 2. Preliminaries

### 2.1. The ADM mass.

Suppose that $M$ is a non-compact manifold such that there exists a compact subset $K \subset M$ and functions $x^1, \ldots, x^n : M \setminus K \rightarrow \mathbb{R}^n$, which give a diffeomorphism

$$(x^1, \ldots, x^n) : M \setminus K \rightarrow \mathbb{R}^n \setminus B$$

for some ball $B \in \mathbb{R}^n$, which for simplicity we can take to be the unit ball (after rescaling the metric if necessary). For $\delta \in \mathbb{R}$ we define the weighted space $C^{k,\alpha}_\delta$ as follows (see Bartnik [2]). Fix a metric $h$ on $M$ which is the Euclidean metric outside $K$, and let $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function equal to $|x|$ outside the ball of radius 2, and equal to 1 on the ball of radius 1. Through the asymptotic coordinates $x^i$ we can think of $\rho$ as a function on $M$. Then the weighted Hölder norms are defined by

$$\|f\|_{C^{k,\alpha}_\delta} = \sup_M \rho^\delta |f| + \sup_M \rho^{\delta+1} |\nabla f| + \ldots + \sup_M \rho^{\delta+k} |\nabla^k f|$$

$$+ \sup_{x,y \in M} \min\{\rho(x), \rho(y)\}^{\delta+k+\alpha} \frac{|\nabla^k f(x) - \nabla^k f(y)|}{d(x,y)^\alpha},$$

where the derivatives and norms are taken with respect to $h$. A Riemannian metric $g$ on $M$ is defined to be asymptotically flat in $C^{k,\alpha}_\delta$ if $\delta > 0$ and we have

$$g_{ij} - h_{ij} \in C^{k,\alpha}_\delta (M \setminus K).$$
In particular if \( g \) is asymptotically flat in \( C^{1,\alpha}_\delta \), then there is some constant \( C \) such that on \( M \setminus K \) we have
\[
|g_{ij} - h_{ij}| < C|x|^{-\delta}
\]
\[
|\nabla g_{ij}| < C|x|^{-\delta-1},
\]
in terms of the asymptotic coordinates.

If \( g \) is a smooth asymptotically flat metric in \( C^{1,\alpha}_\delta \) for \( \delta > (n - 2)/2 \), and the scalar curvature \( R(g) \in L^1 \), then the ADM mass (see Arnowitt-Deser-Misner \cite{arnowitt1959} and Bartnik \cite{bartnik1986}) is defined by
\[
m(g) = \lim_{r \to \infty} \int_{\partial B_r} g_{ij,j} - g_{jj,i} \, dS^i,
\]
where the derivatives are taken with respect to the asymptotic coordinates \( x^i \). The positive mass theorem is then the following.

**Theorem 4** (Schoen-Yau \cite{shoen1981}, Witten \cite{witten1981}). Suppose that either \( \dim M \leq 7 \) or that \( M \) is a spin manifold. Let \( g \) be a smooth asymptotically flat metric of order \( \delta > (n - 2)/2 \) in \( C^{1,\alpha}_\delta \), and suppose that the scalar curvature of \( g \) is integrable, and non-negative. Then \( m(g) \geq 0 \). Furthermore, if \( m(g) = 0 \) then \( M \cong \mathbb{R}^n \) and \( g \) is flat.

When studying the mass, the following formulas will be useful (see Bartnik \cite{bartnik1986}). For any metric \( g \) we have
\[
R(g) = |g|^{-1/2} \partial_i \left( |g|^{1/2} g^{ij} \left( \Gamma_j - \frac{1}{2} \partial_j (\log |g|) \right) \right) - \frac{1}{2} g^{ij} \Gamma_i \partial_j (\log |g|) + g^{ij} g^{kl} \Gamma_{ikp} \Gamma_{jql},
\]
where \( |g| \) is the determinant, \( \Gamma_{kl} \) are the Christoffel symbols and \( \Gamma^i = g^{kl} \Gamma_i^{kl} \). For an asymptotically flat metric in \( C^{1,\alpha}_\delta \) for \( \delta > (n - 2)/2 \), we have
\[
|g|^{1/2} g^{ij} \left( \Gamma_j - \frac{1}{2} \partial_j (\log |g|) \right) = g_{ij,j} - g_{jj,i} + O(|x|^{-1 - 2\delta}),
\]
where the constant in the error term only depends on \( C \). Using this and integrating \( 4 \) by parts on the region \( M \setminus B_r \), we get
\[
m(g) = \int_{M\setminus B_r} R(g) \, dV + \int_{\partial B_r} |g|^{1/2} g^{ij} \left( \Gamma_j - \frac{1}{2} \partial_j (\log |g|) \right) \, dS^i
\]
\[
- \int_{M\setminus B_r} \frac{1}{2} g^{ij} \Gamma_i \partial_j (\log |g|) + g^{ij} g^{kl} \Gamma_{ikp} \Gamma_{jql} \, dV
\]
\[
= \int_{M\setminus B_r} R(g) \, dV + \int_{\partial B_r} g_{ij,j} - g_{jj,i} \, dS^i + O(r^{-\lambda})
\]
for some \( \lambda > 0 \). The constant in the error term \( O(r^{-\lambda}) \) only depends on the \( C^{1,\alpha}_\delta \) norm of \( g - h \).
2.2. Ricci flow on asymptotically flat manifolds. The Ricci flow is the evolution equation

\[ \frac{\partial}{\partial t} g_{ij} = -2R_{ij} \]

introduced by Hamilton [8], where \( R_{ij} \) is the Ricci tensor of the time-dependent metric \( g_{ij} \). It was studied recently on asymptotically flat manifolds by Oliynyk-Woolgar [12] and Dai-Ma [5]. Using the results of Shi [15] it is clear that a solution of the Ricci flow exists for a short time with an asymptotically flat initial metric, but the question is whether it remains asymptotically flat for positive time. The result that we need can be stated as follows.

**Theorem 5.** Suppose that \( g \) is an asymptotically flat metric of order \( \delta > 0 \) in \( C^2_\delta \). Suppose that \( g(t) \) solves the Ricci flow for \( t \in [0, T] \) with \( g(0) = g \). Then \( g(t) \) is asymptotically flat in \( C^2_\delta \) of the same order \( \delta \). More precisely there is a constant \( C \) depending on \( g \) and \( T \), such that

\[ \| g(t) - h \|_{C^2_\delta} < C \]

for all \( t \in [0, T] \).

This result could either be proved by systematically working in weighted spaces as was done in [12], or by using the maximum principle as in [5]. We will use the maximum principle argument to prove a slightly different version of this result in Lemma 16.

Suppose now that the metric \( g \) is only \( C^0 \) on a compact set \( K \subset M \). Using the results of M. Simon [16] we can find a short time solution to the Ricci flow with such an initial metric, using smooth approximations to \( g \).

**Definition 6.** Given a constant \( \delta \geq 1 \), a metric \( h \) is \( \delta \)-fair to \( g \) if the curvature of \( h \) is uniformly bounded, and

\[ \frac{1}{\delta} h \leq g \leq \delta h \text{ on } M. \]

M. Simon has shown that there exists a solution to the Ricci flow for a short time, starting with the singular metric \( \hat{g} \), if there exists a smooth metric \( h \) which is \( 1 + \varepsilon(n) \)-fair to \( \hat{g} \) for some universal constant \( \varepsilon(n) > 0 \) only depending on the dimension. More precisely we have

**Theorem 7.** Fix a metric \( h \) which is \( 1 + \varepsilon(n) \)-fair to \( g \). Then there exists a family of metrics \( g(t) \) for \( t \in (0, T] \) with \( T > 0 \), which solves the \( h \)-flow on this time interval. In addition \( h \) is \( 1 + 2\varepsilon(n) \) fair to \( g(t) \) for \( t \in (0, T] \) and \( g(t) \) converges to \( g \) as \( t \to 0 \), uniformly on compact sets.

The \( h \)-flow is the Hamilton-DeTurk flow with background metric \( h \). More precisely \( g(t) \) satisfies the equation

\[ \frac{d}{dt} g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i, \]
where
\[ W_j = g_{jk} g^{pq} (\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}). \]
The $\tilde{\Gamma}$ are the Christoffel symbols of the metric $h$. This flow is equivalent to the Ricci flow (7) modulo the action of diffeomorphisms. The advantage of the $h$-flow is that it is parabolic, while the unmodified Ricci flow (7) is only weakly parabolic.

The way the solution $g(t)$ is constructed is by first taking a sequence of smoothings $g_{\varepsilon}$ converging to $g$. Then we solve the $h$-flow for a short time with initial condition $g_{\varepsilon}$ for each small $\varepsilon$, obtaining families of metrics $g_{\varepsilon}(t)$. Then the key point is to show that there exists a $T > 0$ independent of $\varepsilon$, such that for $t \in (0, T]$ one obtains uniform bounds on all derivatives of the metrics $g_{\varepsilon}(t)$ of the form
\[
|\nabla^k g_{\varepsilon}(t)| \leq \frac{C_k}{t^{k/2}} \text{ for } t \in (0, T].
\]
Here the covariant derivative and norm is taken with respect to the metric $h$, and the constants $C_k$ are independent of $\varepsilon$. Then the solution $g(t)$ is extracted as a limit of a subsequence as $\varepsilon \to 0$, converging in $C^k$ for all $k$ on compact sets.

In our application the metric $g$ is asymptotically flat, and so we can choose $h$ to be flat outside a sufficiently large ball. If the metric $g$ is Lipschitz, then the estimates on the derivatives were improved in M. Simon [17]. Namely from [17] Lemma 2.1 we have
\[
|\nabla g_{\varepsilon}(t)| \leq C_1, \hspace{1cm} |\nabla^2 g_{\varepsilon}(t)| \leq \frac{C_2}{\sqrt{t}}, \text{ for } t \in (0, T],
\]
with $C_1, C_2$ independent of $\varepsilon$. Note that here we are applying Lemma 2.1 from [17] to the smooth initial metrics $g_{\varepsilon}$ and if $g$ is Lipschitz, then their smoothings $g_{\varepsilon}$ satisfy a uniform $C^1$ bound, independent of $\varepsilon$.

3. The mass is constant along the Ricci flow

In this section we will consider a solution of the Ricci flow given by Theorem 5 starting with an asymptotically flat metric $\hat{g}$ in $C^2_{\delta}$, which has integrable scalar curvature. So we have $g(t)$ for $t \in [0, T]$ such that $g(0) = \hat{g}$. By choosing $T$ smaller if necessary, we can assume that $T \leq 1$, and also that outside a ball $B \subset \mathbb{R}^n$ we have
\[
\frac{1}{2} \delta_{ij} < g_{ij}(t) < 2\delta_{ij}
\]
for $t \in [0, T]$. Without loss of generality we can take $B$ to be the unit ball. In addition we have constants $\kappa > 0$ and $\delta > (n - 2)/2$ such that on $\mathbb{R}^n \setminus B$
we have
\[ |g_{ij}(t) - \delta_{ij}| < \kappa|x|^{-\delta} \]
\[ |\nabla g_{ij}(t)| < \kappa|x|^{-\delta - 1} \]
\[ |\nabla^2 g_{ij}(t)| < \kappa|x|^{-\delta - 2}. \]

Finally, we will assume that we have a uniform lower bound on the scalar curvature of $\hat{g}$, and a bound for its $L^1$ norm, i.e. for some $K > 0$ and the $\kappa$ from above, we have
\[ R(\hat{g}) > -K \]
\[ \int_M |R(\hat{g})| dV < \kappa. \]

The constants below will depend on $\kappa, K$ and $\delta$. It will be important that the constant in Lemma 13 below only depends on $K$. The evolution equation of the scalar curvature is (see Hamilton [8])
\[ \frac{\partial}{\partial t} R(g) = \Delta R + 2 |\text{Ric}|^2. \]

Since $R(g(t))$ decays at infinity, we can use the maximum principle to see that $R(g(t)) > -K$ for all $t$.

We first construct some cutoff functions that we will use later.

**Lemma 8.** For any $r_1, r_2$ with $1 < r_1$ and $2r_1 < r_2$, there exists a function $f_{r_1, r_2} : M \to [0, 2]$ such that
\[ f_{r_1, r_2}(x) = 1/r_1^2, \text{ if } |x| < r_1, \]
\[ f_{r_1, r_2}(x) = 1, \text{ if } 2r_1 \leq |x| \leq r_2, \]
\[ f_{r_1, r_2}(x) \leq |x|^{-n - 1}, \text{ for } |x| > 2r_2. \]

In addition there is a constant $C$ independent of $r_1, r_2$ such that
\[ \Delta f_{r_1, r_2} \leq Cf_{r_1, r_2}, \]
where $\Delta$ is the Laplacian with respect to any of the metrics $g(t), t \in [0, T]$.

**Proof.** Let us begin by letting $\phi : [0, \infty) \to [1, 2]$ be a smooth function such that $\phi(x) = 1$ for $x < 1$ and $\phi(x) = 2$ for $x > 2$. Then define the function $g : \mathbb{R}^n \to [1, 2]$ by letting $g(x) = \phi(|x|)$. This function satisfies $|\nabla^2 g| \leq C$ for some $C$. Now define
\[ g_{r_1}(x) = g(r_1^{-1}x) - 1 + \frac{1}{r_1^2}. \]

Then $|\nabla^2 g_{r_1}| \leq r_1^{-2}C$ and $g_{r_1} \geq r_1^{-2}$ everywhere, so these functions satisfy
\[ |\nabla^2 g_{r_1}| \leq C g_{r_1}. \]

We will now define another set of functions $h_{r_2}$ on $\mathbb{R}^n$. First we define $h : \mathbb{R}^n \to [0, 1]$ to be such that $h(x) = 1$ for $|x| < 1$ and $h(x) = |x|^{-n - 1}$ for $|x| > 2$. Then we can check that $|\nabla^2 h| \leq Ch$ for some $C$. Now define
\[ h_{r_2}(x) = h(r_2^{-1}x). \]
from which it follows that as long as \( r_1 \geq 1 \), we have
\[
|\nabla^2 h_{r_2}| \leq Ch_{r_2}.
\]

Finally, if \( r_1 > 1 \) and \( r_2 > 2r_1 \), we define \( f_{r_1,r_2} : \mathbf{R}^n \to [0,2] \) to be
\[
f_{r_1,r_2}(x) = g_{r_1}(x)h_{r_2}(x).
\]
Then \( f_{r_1,r_2}(x) = g_{r_1}(x) \) for \( |x| < r_2 \), and \( f_{r_1,r_2}(x) = (1 + r_1^{-2})h_{r_2}(x) \) for \( |x| > 2r_1 \). It follows that the inequality
\[
|\nabla^2 f_{r_1,r_2}| \leq Cf_{r_1,r_2}
\]
holds on \( \mathbf{R}^n \).

We now simply transfer this function to the asymptotically flat manifold using the asymptotic coordinates. This is possible since \( f_{r_1,r_2} \) is constant in the ball of radius \( r_1 \). The inequality
\[
\Delta f_{r_1,r_2} \leq Cf_{r_1,r_2}
\]
(with a larger choice of \( C \)) follows because in the asymptotic coordinates the metric \( g(t) \) is uniformly equivalent to the Euclidean metric. \( \square \)

We will also use the following simple ODE result several times.

**Lemma 9.** Suppose that the function \( F(t) \) satisfies
\[
\frac{d}{dt}F(t) \leq AF(t) + B,
\]
for some constants \( A, B \geq 0 \). Then for \( t \in [0,1] \) we have
\[
F(t) \leq e^A F(0) + Be^A.
\]

We now study the integrability of the scalar curvature along the Ricci flow. The following lemma will also be useful when looking at the evolution of the ADM mass.

**Lemma 10.** Suppose that
\[
\int_{M \setminus B_r} |R(\hat{g})|dV < \eta(r)
\]
with \( \eta(r) \) going to zero as \( r \to \infty \). Then there exists a function \( \tilde{\eta}(r) \) depending on \( \eta, \kappa, K \) and \( \delta \), but not on \( t \), such that for \( t \in [0,T] \) we have
\[
\int_{M \setminus B_r} |R(g(t))|dV < \tilde{\eta}(r),
\]
and \( \tilde{\eta}(r) \) goes to zero as \( r \to \infty \). In particular \( R(g(t)) \in L^1 \) for \( t \in [0,T] \).

**Proof.** Let us write \( \hat{R} \) for \( R(\hat{g}) \) and \( R \) for \( R(g(t)) \). For any \( \varepsilon > 0 \) define the function (following [20])
\[
u = \sqrt{R^2 + \varepsilon}.
\]
We can compute that along the Ricci flow
\[
\frac{\partial u}{\partial t} = u^{-1} R(\Delta R + 2|Ric|^2)
\]
\[
\Delta u = -u^{-3} R^2 |\nabla R|^2 + u^{-1} |\nabla R|^2 + u^{-1} R\Delta R \geq u^{-1} R\Delta R,
\]
where we used that $u^{-2} R^2 \leq 1$. It follows that
\[
(10) \quad \frac{\partial u}{\partial t} \leq \Delta u + 2|Ric|^2.
\]

We will now use the cutoff functions $f_{r_1, r_2}$ from Lemma 8. Since $u$ is bounded, the decay of $f_{r_1, r_2}$ at infinity implies that $f_{r_1, r_2} u$ is integrable for all $t$. We can compute
\[
\frac{d}{dt} \int_M f_{r_1, r_2} u \, dV \leq \int_M f_{r_1, r_2} (\Delta u + 2|Ric|^2 - uR) \, dV.
\]
We have that $R > -K$ for all $t$, and in addition the decay condition on the metric implies that outside the ball $B$ we have $|Ric|^2 < C_0|x|^{-2\delta-4} < C_0|x|^{-n-2}$ for some constant $C_0$. The integral of $|Ric|^2$ outside $B_{r_1}$ is therefore of the order $r_1^{-2}$. It follows that we have
\[
(11) \quad \frac{d}{dt} \int_M f_{r_1, r_2} u \, dV \leq \int_M f_{r_1, r_2} \Delta u + C f_{r_1, r_2} u \, dV + C_1 r_1^{-2} + 4 \int_{M \setminus B_{r_1}} |Ric|^2 \, dV
\]
\[
\leq \int_M u \Delta f_{r_1, r_2} + C f_{r_1, r_2} u \, dV + C_2 r_1^{-2}
\]
\[
\leq 2C \int_M f_{r_1, r_2} u \, dV + C_2 r_1^{-2}.
\]
It follows using Lemma 9 that
\[
\int_M f_{r_1, r_2} u \, dV \leq C_3 \int_M f_{r_1, r_2} \sqrt{\hat{R}^2 + \varepsilon} \, dV + C_4 r_1^{-2}.
\]
Letting $\varepsilon \to 0$, this implies
\[
\int_M f_{r_1, r_2} |R| \, dV \leq C_3 \int_M f_{r_1, r_2} |\hat{R}| \, dV + C_4 r_1^{-2}.
\]
Finally, noticing that
\[
\int_{B_{r_2} \setminus B_{r_1}} |R| \, dV \leq \int_M f_{r_1, r_2} |R| \, dV,
\]
we let $r_2 \to \infty$ to get
\[
\int_{M \setminus B_{r_1}} |R| \, dV \leq C_3 \int_{B_{r_1}} \frac{1}{r_1^2} |\hat{R}| \, dV + 2C_3 \int_{M \setminus B_{r_1}} |\hat{R}| \, dV + C_4 r_1^{-2}
\]
\[
\leq C \left( r_1^{-2} + \eta(r_1) \right).
\]
This proves the result we want, with $\tilde{\eta}(2r) = C(r^{-2} + \eta(r))$. \qed
Remark. The fact that the scalar curvature remains integrable for positive time along the Ricci flow was shown under stronger decay conditions on the metric by Dai-Ma [5] and Oliynyk-Woolgar [12]. Also, if we only wanted to show that $R(g(t)) \in L^1$ for $t > 0$, then we could use a simpler argument using the cutoff functions $h_r$ from the proof of Lemma 8.

Lemma 11. For any $t > 0$ we have

$$\lim_{r \to \infty} \int_{\partial B_r} |\nabla R| dS = 0.$$ 

Moreover if $t_0 > 0$ then this convergence is uniform for $t \in [t_0, T]$.

Proof. We first use the local maximum principle for parabolic equations (Theorem 7.36 in Lieberman [10]) to obtain pointwise bounds for $R$ depending on the $L^1$ bound. We could also use the local estimate obtained by Moser iteration [10, Theorem 6.17]. We will use the equation

$$\frac{\partial}{\partial t} R = \Delta R + 2|Ric|^2.$$ 

In addition we work outside the ball $B$, where the asymptotic coordinates $x_i$ are defined. As long as $t \in [t_0/2, T]$, the metrics along the flow are all uniformly equivalent to the Euclidean metric outside $B$, and also their derivatives are controlled. This means that the ellipticity of $\Delta$ and the Christoffel symbols (which appear in the first order derivative term in $\Delta R$) are controlled uniformly. Let $\tau \in [t_0, T]$ and $p \in M \setminus B$, and choose $r$ with $0 < r \leq \sqrt{\tau - t_0/2}$. Then we apply the local maximum principle to the parabolic cylinder

$$Q(r) = \{(x, t) : |x - p| < r, \tau - r^2 < t < \tau\},$$ 

so we obtain

$$\sup_{Q(r/2)} |R| \leq C_r \left( \int_{Q(r)} |R| dV + \sup_{Q(r)} |Ric|^2 \right),$$ 

where $C_r$ is a constant depending on $r$.

We now apply local $L^p$ estimates and the Sobolev inequalities on the parabolic cylinder $Q(r/2)$. Note that on this cylinder as above, we control all derivatives of the metric. It follows that we get

$$\sup_{Q(r/4)} |\nabla R| \leq C'_r \left( \sup_{Q(r/2)} |R| + \sup_{Q(r/2)} |Ric|^2 \right),$$ 

where again $C'_r$ depends on $r$. In sum we obtain that

$$\sup_{Q(r/4)} |\nabla R| \leq C(r) \left( \int_{Q(r)} |R| dV + \sup_{Q(r)} |Ric|^2 \right).$$
Now we cover the sphere $\partial B_a$ with balls $B_i$ of radius $\tau/4$. This can be achieved so that each point is covered by at most $c(n)$ of the balls $4B_i$, where $c(n)$ depends on the dimension. Applying (12) and integrating, we find that
\[
\int_{\partial B_a} |\nabla R(\tau)|\ dS \leq C(n,r) \left( \int_{A(a,r)} |R|\ dV + \sup_{A(a,r)}|\operatorname{Ric}|^2 \right),
\]
where $A(a,r)$ is the annulus
\[A(a,r) = \{(x,t) : d(x,\partial B_a) < r, \tau - r^2 < t < \tau\}.
\]
The key point is that $C(n,\tau)$ does not depend on $a$ so we can let $a \to \infty$. The integrability result Lemma 10, and the fact that the metric is asymptotically flat in $C^2_\delta$ for some $\delta > (n-2)/2$, now implies the result we want.

We can now show that the mass is constant along the Ricci flow. This was shown under stronger decay conditions by Oliynyk-Woolgar [12] and Dai-Ma [5].

**Corollary 12.** The ADM mass is preserved under the Ricci flow.

**Proof.** Using the integration by parts formula (6) we have
\[
m(g(t)) = \int_{M \setminus B_r} R(g(t))\ dV + \int_{\partial B_r} g(t)_{ij,j} - g(t)_{jj,i}\ dS^i + O(r^{-\lambda}),
\]
for some $\lambda > 0$, where the constant in $O(r^{-\lambda})$ is independent of $t$. Let us write
\[
m_r(g(t)) = \int_{\partial B_r} g_{ij,j} - g_{jj,i}\ dS^i.
\]
We can use the surface measure $dS^i$ with respect to the fixed metric $\hat{g}$ (or even the Euclidean metric), since the metrics $g(t)$ are all asymptotically equal. Then
\[
\frac{d}{dt} m_r(g(t)) = \int_{\partial B_r} -2R_{ij,ij} + 2R_{ii}\ dS^i.
\]
The derivative $R_{ij,ij}$ differs from the covariant derivative $R_{ij,ij}$ by terms of the form $P^p_{ij}R_{pj}$, which are of order $|x|^{-2\delta - 3}$, so integrating over $\partial B_r$ contributes $O(r^{-\lambda})$. Using this, together with the contracted Bianchi identity $2R_{ij,ij} = R_{ij}$ we have
\[
\left| \frac{d}{dt} m_r(g(t)) \right| \leq \int_{\partial B_r} |\nabla R|\ dS + Cr^{-\lambda}.
\]
Now let $\varepsilon > 0$. As long as $t \in [t_0, T]$ for some $t_0 > 0$ we can choose $r \gg 1$ (using Lemma 11) such that
\[
\left| \frac{d}{dt} m_r(g(t)) \right| \leq \varepsilon,
\]
and also, using Lemma 10 together with Equation (13), we can ensure that $|m(g(t)) - m_r(g(t))| < \varepsilon$. These two bounds imply that for $s, t \in [t_0, T]$ we have $|m(g(t)) - m(g(s))| < (2 + T)\varepsilon$. Since this is true for any $\varepsilon > 0$, and
we can also choose $t_0 > 0$ arbitrarily, this shows that $m(g(t))$ is constant for $t > 0$.

What remains is to show that $\lim_{t \to 0} m(g(t)) = m(g(0))$. This follows from Equation (13) applied to $g(0)$ and $g(t)$. Indeed, using Lemma 10 we have a function $\varepsilon(r)$ which goes to zero as $r \to \infty$ (which absorbs the $O(r^{-\lambda})$ as well), such that

$$|m(g(t)) - m(g(0))| < \varepsilon(r) + \left| \int_{\partial B_r} g(t)_{ij,j} - g(0)_{ij,j} \, dS^i \right| + \left| \int_{\partial B_r} g(t)_{jj,i} - g(0)_{jj,i} \, dS^i \right|.$$

Letting $t \to 0$ and then $r \to \infty$, we get $\lim_{t \to 0} m(g(t)) = m(g(0))$. This completes the proof. □

We will need the following lemma when showing that the scalar curvature is non-negative for positive time, when starting the flow with the singular metric.

**Lemma 13.** For any $t \in [0, T]$ we have

$$\int_{\{R(g(t)) < 0\}} |R(g(t))| \, dV_t \leq e^K \int_{\{R(\hat{g}) < 0\}} |R(\hat{g})| \, dV,$$

where $-K$ is the lower bound for $R(\hat{g})$.

**Proof.** Let us simply write $R$ for $R(g(t))$. For $\delta > 0$ define the function

$$v = \sqrt{R^2 + \delta} - R,$$

which is a smoothing of $2 \max\{0, -R\}$. From Equation (10) we have that

$$\frac{\partial v}{\partial t} \leq \Delta v.$$

Now compute

$$\frac{d}{dt} \int_{B_r} v \, dV \leq \int_{B_r} \Delta v - vR \, dV \leq \int_{\partial B_r} |\nabla v| \, dS + K \int_{B_r} v \, dV,$$

where we used that the lower bound $R(\hat{g}) > -K$ is preserved along the flow. At the same time $|\nabla v| \leq 2 |\nabla R|$, so we have

$$\frac{d}{dt} \int_{B_r} v \, dV \leq 2 \int_{\partial B_r} |\nabla R| \, dS + K \int_{B_r} v \, dV.$$

We know from Lemma 11 that for each $t > 0$ the integral of $|\nabla R|$ on $\partial B_r$ goes to zero as $r \to \infty$, and that this convergence is uniform as long as
$t \in [\tau, T]$ for some $\tau > 0$. So for any $\eta, \tau > 0$ we can choose $r(\eta)$ such that for $r > r(\eta)$ and $t \in [\tau, T]$ we have

$$2 \int_{\partial B_r} |\nabla R| dS < \eta.$$  

It follows then using Lemma 9 that for $t \in [\tau, T]$ and $r > r(\eta)$,

$$\int_{B_r} v(t) dV_t \leq e^K \int_{B_r} v(\tau) dV_\tau + e^K \eta.$$  

We can now let $\delta \to 0$ in the definition of $v$, so that $v \to 2 \max\{0, -R\}$. It follows that

$$\int_{B_r \cap \{R(t) < 0\}} |R(t)| dV_t \leq e^K \int_{B_r \cap \{R(\tau) < 0\}} |R(\tau)| dV_\tau + e^K \eta.$$  

Now we let $r \to \infty$ and $\eta \to 0$, using the fact that $|R|$ is integrable on $M$ by Lemma 10. We get

$$\int_{\{R(t) < 0\}} |R(t)| dV_t \leq e^K \int_{\{R(\tau) < 0\}} |R(\tau)| dV_\tau.$$  

Since we can do this for any $\tau > 0$, we have

$$\int_{\{R(t) < 0\}} |R(t)| dV_t \leq e^K \int_{\{R(\hat{g}) < 0\}} |R(\hat{g})| dV,$$  

which is what we wanted to prove. $\square$

4. The mass of a limit metric

In this section we will study what we can say about the mass of a metric $g$, if a sequence of asymptotically flat metrics $g_\varepsilon$ converge to $g$ locally uniformly. Under convergence in a suitable topology it is known that $m(g) = \lim m(g_\varepsilon)$ (see for example Lee-Parker [9, Lemma 9.4]). Under weaker assumptions we will show that $m(g) \leq \lim \inf \varepsilon \to 0 m(g_\varepsilon)$.

**Theorem 14.** Suppose that $g_\varepsilon \to g$ locally uniformly in $C^2$ and that for some $\kappa > 0$ we have the asymptotic decay conditions

$$\|g_{\varepsilon, ij} - \delta_{ij}\|_{C^0_\varepsilon(M \setminus K)} < \kappa$$

for some $\delta > (n-2)/2$. In addition we require that $R(g_\varepsilon) \in L^1$ for all $\varepsilon > 0$, and the scalar curvature of $g_\varepsilon$ is almost non-negative in the following sense:

$$\int_{\{R(g_\varepsilon) < 0\}} |R(g_\varepsilon)| dV < \varepsilon.$$  

Then $R(g) \geq 0$, $m(g)$ is defined and

$$m(g) \leq \lim \inf \varepsilon \to 0 m(g_\varepsilon),$$

assuming that the limit is finite.

**Lemma 15.** The scalar curvature of $g$ is non-negative and integrable.
Proof. The fact that \( R(g) \geq 0 \) follows easily by taking the limit of Inequality (16).

Integrating Equation (4) by parts, and using (5), we get that for some \( \lambda > 0 \) (depending on \( \delta > (n - 2)/2 \))

\[
\int_{B_1 \setminus B_r} R(g) \, dV = \int_{\partial B_r} g_{ij,j} - g_{jj,i} \, dS^i + O(r^{-\lambda})
\]

(17)

\[
- \int_{\partial B_1} |g|^{1/2} g^{ij} \left( \Gamma_j - \frac{1}{2} \partial_j (\log |g|) \right) \, dS^i
\]

\[
- \int_{B_1 \setminus B_r} \frac{1}{2} g^{ij} \Gamma_i \partial_j (\log |g|) + g^{ij} g^{kl} g_{ikp} \Gamma_{jql} \, dV.
\]

The constant in \( O(r^{-\lambda}) \) only depends on \( \kappa \) in the decay bounds (15). Using these decay bounds again, and the bound on the negative part of \( R(g_\varepsilon) \) in Equation (17) we get

\[
\int_{B_1 \setminus B_r} |R(g_\varepsilon)| \, dV \leq \int_{\partial B_r} (g_\varepsilon)_{ij,j} - (g_\varepsilon)_{jj,i} \, dS^i + C,
\]

where \( C \) is independent of \( \varepsilon \) and \( r \). Taking \( r \to \infty \) we have

\[
\int_{M \setminus B_1} |R(g_\varepsilon)| \, dV \leq m(g_\varepsilon) + C.
\]

Since \( |R(g(\varepsilon(t)))| \to |R(g(t))| \) pointwise (and the volume forms also converge pointwise), we find that \( R(g(t)) \) is integrable, using Fatou’s Lemma, as long as \( \lim \inf m(g_\varepsilon) \) is finite. \( \square \)

**Proof of Theorem 14.** By Lemma 15 and the decay conditions (15) we know that the mass of \( g(t) \) is defined. From the integration by parts formula (6) we have

\[
m(g) = \int_{M \setminus B_r} R(g) \, dV + \int_{\partial B_r} g_{ij,j} - g_{jj,i} \, dS^i + O(r^{-\lambda}),
\]

Using the same formula again with \( g_\varepsilon \) instead of \( g \), we get

\[
m(g) = m(g_\varepsilon) + O(r^{-\lambda})
\]

\[
+ \int_{M \setminus B_r} R(g) \, dV - \int_{M \setminus B_r} R(g_\varepsilon) \, dV
\]

\[
+ \int_{\partial B_r} g_{ij,j} - g_{jj,i} \, dS^i - \int_{\partial B_r} (g_\varepsilon)_{ij,j} - (g_\varepsilon)_{jj,i} \, dS^i_\varepsilon.
\]

(18)

Note that for a fixed \( r > 0 \), the metrics \( g_\varepsilon \) converge uniformly in \( C^2 \) to \( g \) on \( \partial B_r \) as \( \varepsilon \to 0 \). At the same time, using (16) together with Fatou’s Lemma we have

\[
\int_{M \setminus B_r} R(g) \, dV \leq \lim \inf_{\varepsilon \to 0} \int_{M \setminus B_r} R(g_\varepsilon) \, dV.
\]
Therefore taking the limit as $\varepsilon \to 0$ in (18) for a fixed $r$, we get

$$m(g) \leq \liminf_{\varepsilon \to 0} m(g_\varepsilon) + O(r^{-\lambda}).$$

Now taking $r \to \infty$ we get the required result. $\square$

5. Proof of the main theorem

We now let $\tilde{g}$ be a metric with corners across a hypersurface, following Miao [11]. For the definition see the Introduction. We assume that $\tilde{g}$ is asymptotically flat in $C^2_\delta$ for some $\delta > (n-2)/2$, but note that $\tilde{g}$ only has to be $C^2$ outside a compact set. The important result for us is that the smoothing procedure in [11, Proposition 3.1] gives us metrics $\tilde{g}_\varepsilon$ which satisfy the following conditions, with a fixed $K > 0$:

$$\tilde{g}_\varepsilon = \tilde{g} \text{ outside } B,$$

$$(1 - \varepsilon)\tilde{g} \leq \tilde{g}_\varepsilon \leq (1 + \varepsilon)\tilde{g},$$

$$R(\tilde{g}_\varepsilon) > -K,$$

$$\int \{|R(\tilde{g}_\varepsilon)| \, dV_\varepsilon | < \varepsilon.$$

The last condition holds since the metrics $\tilde{g}_\varepsilon$ can be constructed such that the set $\{R(\tilde{g}_\varepsilon) < 0\}$ has measure less than $K^{-1}\varepsilon$ and at the same time $R(\tilde{g}_\varepsilon) > -K$ everywhere for some uniform constant $K$.

Let $g(t)$ be the solution of the Hamilton-DeTurk flow with initial metric $\tilde{g}$, and background metric $h = \tilde{g}_\varepsilon$ for sufficiently small $\varepsilon$, constructed by Simon [16]. In fact we can modify $h$ to be equal to the Euclidean metric outside a compact set, and it will still be $1 + \varepsilon(n)$-fair to $\tilde{g}$. Recall that the $h$-flow with initial metric $\tilde{g}$ is obtained by taking the limit of the $h$-flows with initial metrics $\tilde{g}_\varepsilon$. Let us write $g_\varepsilon(t)$ for the $h$-flow with initial metric $\tilde{g}_\varepsilon$.

We first want to show that the metrics $g(t)$ are asymptotically flat for $t > 0$. We are not able to show that $g(t)$ is asymptotically flat in $C^2_\delta$, but only in $C^{1,\alpha}_\delta$ with $\alpha > 0$ small, and $\delta' = \delta - \alpha$. This is enough for the positive mass theorem to hold (see e.g. Lee-Parker [9]).

Lemma 16. There is a $T > 0$ independent of $\varepsilon$, and for any $t_0 \in (0, T]$ there is a constant $\kappa > 0$ such that for $t \in [t_0, T]$ and $\varepsilon > 0$ we have

$$\|g_\varepsilon(t) - h\|_{C^{1,\alpha}_{\delta'-\alpha}} \leq \kappa.$$

By taking $\varepsilon \to 0$ the same estimate holds for $g(t)$. In particular if $\delta > (n-2)/2$ and $\alpha$ is small enough, then $\delta' = \delta - \alpha$ satisfies $\delta' > (n-2)/2$, and $g(t)$ is asymptotically flat in $C^{1,\alpha}_{\delta'}$.

This is proved using standard techniques, using the maximum principle. We give the proof in the Appendix.

Next we check that the other conditions of Theorem [14] are satisfied.
Lemma 17. For a fixed $t > 0$, the metrics $g_{\varepsilon}(t)$ satisfy the hypotheses of Theorem 14.

Proof. Each $\hat{g}_{\varepsilon}$ is asymptotically flat, so we can apply the results of Section 3 to the Ricci flow starting at $\hat{g}_{\varepsilon}$. Lemma 10 implies that $g_{\varepsilon}(t)$ has integrable scalar curvature. In addition we can apply Lemma 13, and we find that

$$\int_{\{R(g_{\varepsilon}(t)) < 0\}} |R(g_{\varepsilon}(t))| dV_t \leq e^K \int_{\{R(\hat{g}_{\varepsilon}) < 0\}} |R(\hat{g}_{\varepsilon})| dV_{\varepsilon} < e^K \varepsilon.$$ 

Finally, we give the proofs of Theorems 1 and 2.

Theorem 18. The $C^0$ metric $\hat{g}$ has $m(\hat{g}) \geq 0$. If $m(\hat{g}) = 0$, then $\hat{g}$ is the Euclidean metric up to a $C^{1,\alpha}$ change of coordinates.

Proof. Consider the solution $g(t)$ to the $h$-flow as above. Note that $m(\hat{g}_{\varepsilon}) = m(\hat{g})$ for all $\varepsilon$ since we are only changing the metric outside a ball. For each $\varepsilon > 0$, there are diffeomorphisms $\phi_{\varepsilon,t}$ such that $\phi_{\varepsilon,t}^* g_{\varepsilon}(t)$ is a solution of the Ricci flow, and $\phi_{\varepsilon,0}$ is the identity. Applying the results of Section 3, Corollary 12 implies that $m(\phi_{\varepsilon,t}^* g_{\varepsilon}(t)) = m(\hat{g}_{\varepsilon}) = m(\hat{g})$ for $t > 0$. Since under our decay conditions the mass is independent of the choice of asymptotic coordinates (see Bartnik [2, Theorem 4.2]), we also have $m(g_{\varepsilon}(t)) = m(\hat{g})$. Finally, we use that $g(t) = \lim g_{\varepsilon}(t)$, and Theorem 14 implies that $m(g(t)) \leq m(\hat{g})$.

By Theorem 14 the metrics $g(t)$ have non-negative scalar curvature for $t > 0$ and they are asymptotically flat in $C^{1,\alpha}$, for some $\alpha' > (n-2)/2$, so by the positive mass theorem, $m(g(t)) \geq 0$. We therefore have $m(\hat{g}) \geq 0$.

Suppose now that $m(\hat{g}) = 0$. Then necessarily $m(g(t)) = 0$ for $t > 0$. From the equality case of the positive mass theorem, $g(t)$ is flat for $t > 0$. This means that the $h$-flow is just acting by diffeomorphisms, according to the equations

$$\frac{\partial}{\partial t} g_{ij} = \nabla_i W_j + \nabla_j W_i$$
$$W_j = g_{jk} g^{pq} (\tilde{\Gamma}_p^k - \tilde{\Gamma}_q^k),$$

where $\tilde{\Gamma}$ are the Christoffel symbols of $h$. From the ODE

$$\frac{\partial}{\partial t} (\phi_t(p)) = W(\phi_t(p), t);$$
$$\phi_T(p) = \text{Id}(p),$$

we obtain a family of diffeomorphisms $\phi_t$ for $t > 0$ such that $g(t) = \phi_t^* g(T)$. We can think of $\phi_t$ as a perturbation of the identity given by $\phi_t(p) = \text{Id}(p) + \psi_t(p)$. Since the metric $g$ we started with was Lipschitz, the bound (9) applies, so $|\nabla g(t)| < C_1$ for $t > 0$. Since the $g(t)$ are bounded in $C^1$, the $\psi_t$ are bounded in $C^2$. This can be seen by the method of Taylor [18], in particular Equation (2.13), which expresses the second derivatives of $\psi_t$ in...
terms of its first derivatives and the Christoffel symbols of $g(t)$ and $g(T)$. The formula is
\[
\frac{\partial^2 \phi^m}{\partial x^i \partial x^j} = \Gamma^k_{ij} \frac{\partial \phi^m}{\partial x^k} - \hat{\Gamma}^m_{kl} \frac{\partial \phi^k}{\partial x^i} \frac{\partial \phi^l}{\partial x^j},
\]
where $\phi^m_i$ are the components of $\phi_t$, and $\hat{\Gamma}$ are the Christoffel symbols of $g(T)$. The same formula holds for the second derivatives of $\psi_t$ (in terms of the first derivatives of $\phi_t$). Moreover we can bound $\psi_t$ in $C_0$, using Equation (22) together with the fact that the vector field $W$ is bounded in $C_0$ from its definition (21). So the $\psi_t$ are bounded in $C^2$, and we can extract a subsequence converging in $C_1$ to some $\psi_0$. It follows that $g = \phi_0^* g(T)$, where $\phi_0 = \text{Id} + \psi_0$, and $g(T)$ is the flat Euclidean metric on $\mathbb{R}^n$. So up to a $C_1$ change of coordinates, $g$ is the flat metric. 

6. Appendix - Proof of Lemma 16

In this section we give the proof of Lemma 16. Recall that we have a smooth approximation $\hat{g}_\epsilon$ to our singular metric $\hat{g}$, and $g_\epsilon(t)$ is the solution of the $h$-flow with initial metric $\hat{g}_\epsilon$. The key point in this lemma is that since $\hat{g}$ is Lipschitz, the smoothings satisfy a uniform weighted $C_1$-bound $\|\hat{g}_\epsilon - h\|_{C^1_\delta} < C$, where $C$ is independent of $\epsilon$. We will show that this bound (with a larger $C$) is preserved for positive time along the flow, and in addition if $t \geq t_0 > 0$, then $|\nabla^2 g_\epsilon(t)| < C' \rho^{-\delta-1}$. Note that we do not obtain the natural decay of $\rho^{-\delta-2}$ for the second derivatives (see the remark after the following proof).

Proof. In this proof we will simply write $g$ instead of $g_\epsilon(t)$. Let
\[
\eta_{ij} = g_{ij} - h_{ij},
\]
where $h$ is our background reference metric. From [16] we know that $g$ is uniformly equivalent to $h$ for $t \in [0,T]$ and also we have the estimate (9), so for some constant $c_0$ we have $|\nabla \eta| < c_0$ for $t \in [0,T]$. Let us first show that $|\eta|$ has the required decay. The evolution equation for $\eta$ is given by (see [16] Equation (1.5))
\[
\frac{\partial}{\partial t} \eta_{ab} = \Delta \eta_{ab} - g^{cd} g_{ap} h^{pq} \hat{R}_{bcqd} - g^{cd} g_{bp} h^{pq} \hat{R}_{acqd} + \frac{1}{2} g^{cd} g^{pq} \left( \nabla_a \eta_{pc} \nabla_b \eta_{qd} + 2 \nabla_c \eta_{ap} \nabla_q \eta_{bd} - 2 \nabla_c \eta_{ap} \nabla_d \eta_{bq} - 4 \nabla_a \eta_{pc} \nabla_d \eta_{bq} \right),
\]
(23)
where all the derivatives are with respect to $h$, $\hat{R}$ is the curvature of $h$ and $\Delta$ is the operator $g^{cd} \nabla_c \nabla_d$. From this we obtain the following schematic equation.
\[
\frac{\partial}{\partial t} |\eta|^2 = \Delta |\eta|^2 + \eta \star \hat{R} + \eta \star \nabla \eta \star \nabla \eta - 2 h^{ik} h^{jl} g^{pq} \nabla_p \eta_{ij} \nabla_q \eta_{kl},
\]
where ∗ means an algebraic operation involving contractions with respect to h and g and the norms are computed with respect to h. Since g is ε(n)-fair to g (see Definition 6) for some small ε(n), we have
\[
h^{ik}h^{jl}g^{pq}\nabla_{p[h_j}\nabla_{q\eta_{kl}} \geq \frac{3}{4}|\nabla\eta|^2,
\]
where the norm is measured using h. In addition we can bound
\[
|\eta \ast \nabla\eta \ast \nabla\eta| \leq C_1|\eta|^2 + \frac{1}{2}|\nabla\eta|^2,
\]
using that |∇η| is uniformly bounded. It follows that
\[
\frac{\partial}{\partial t}|\eta|^2 \leq \Delta|\eta|^2 + C_1|\eta|^2 + C_2|\tilde{\eta}|^2 - |\nabla\eta|^2.
\]
We can assume that h is asymptotically flat in \(C_5^k\) for any k, so in particular \(|\tilde{\eta}| < C\rho^{-\delta - 2}\). We obtain
\[
\frac{\partial}{\partial t}|\eta|^2 \leq \Delta|\eta|^2 + C_1|\eta|^2 + C_3\rho^{-2\delta - 4}.
\]
Using the formulas
\[
\Delta(\rho^{2\delta}|\eta|^2) = \rho^{2\delta}\Delta|\eta|^2 + 4\delta\rho^{2\delta - 1}\nabla\rho \cdot \nabla|\eta|^2 + \Delta(\rho^{2\delta})|\eta|^2
\]
\[
\rho^{-1}\nabla\rho \cdot \nabla(\rho^{2\delta}|\eta|^2) = 2\delta\rho^{2\delta - 2}|\nabla\rho|^2 + \rho^{2\delta - 1}\nabla\rho \cdot \nabla|\eta|^2,
\]
together with bounds on \(\rho, \nabla\rho, \Delta\rho\), we get
\[
\frac{\partial}{\partial t}\rho^{2\delta}|\eta|^2 \leq \Delta(\rho^{2\delta}|\eta|^2) - 4\delta\rho^{-1}\nabla\rho \cdot \nabla(\rho^{2\delta}|\eta|^2) + C_4\rho^{2\delta}|\eta|^2 + C_3.
\]
Since \(\rho^{2\delta}|\eta|^2 < \kappa_0\) at \(t = 0\), the maximum principle (the version on non-compact manifolds due to Ecker-Huisken [7] for example) implies that for some \(\kappa\) we have \(\rho^{2\delta}|\eta|^2 < \kappa\) for \(t \in [0, T]\).

Let us now consider |∇\eta|^2. From (23) we obtain
\[
\frac{\partial}{\partial t}|\nabla\eta|^2 = \Delta|\nabla\eta|^2 + \nabla\eta \ast \nabla\eta \ast \nabla^2\eta + \nabla\eta \ast \nabla\eta \ast \nabla\eta \ast \nabla\eta
\]
\[
+ \nabla\eta \ast \nabla\eta \ast \tilde{\eta} + \nabla\eta \ast \nabla\tilde{\eta}
\]
\[
- 2g^{cd}h^{ik}h^{jl}h^{pq}\nabla_{c}\nabla_{p[h_j}\nabla_{d}\nabla_{q\eta_{kl}},
\]
where several terms come from commuting derivatives, and also from differentiating \(g^{ab}\) in \(\Delta = g^{ab}\nabla_a \nabla_b\). With similar arguments as above, we obtain
\[
\frac{\partial}{\partial t}|\nabla\eta|^2 \leq \Delta|\nabla\eta|^2 + C_1|\nabla\eta|^2 + C_2\rho^{-2\delta - 4} - |\nabla^2\eta|^2.
\]
We will use the negative squared term below, but for now repeating the arguments above, and using that at \(t = 0\) we have \(\rho^{2\delta + 2}|\nabla\eta|^2 < \kappa_0\), we again find that for some \(\kappa > 0\) we have \(\rho^{2\delta + 2}|\nabla\eta|^2 < \kappa\) for \(t \in [0, T]\).
Finally, consider $|\nabla^2 \eta|^2$. Once again from (23), using similar arguments, we get

$$
\frac{\partial}{\partial t} |\nabla^2 \eta|^2 = \Delta |\nabla^2 \eta|^2 + \nabla^2 \eta \ast \nabla \eta \ast \nabla^3 \eta + \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla^2 \eta
$$

$$
+ \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta
$$

$$
+ \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta
$$

$$
+ \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla^2 \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta \ast \Delta \eta \ast \nabla \eta
$$

From this we obtain

$$
\frac{\partial}{\partial t} |\nabla^2 \eta|^2 \leq \Delta |\nabla^2 \eta|^2 + C_1 |\nabla^2 \eta|^3 + C_2 |\nabla^2 \eta|^2 + C_3 \rho^{-2\delta -2},
$$

where we also used our previous estimate on $|\nabla \eta|$. Now computing, using (25), we find that if we let $f = (|\nabla \eta|^2 + 1)|\nabla^2 \eta|^2$ then

$$
\frac{\partial}{\partial t} f \leq \Delta f + C_4 |\nabla^2 \eta|^2 + C_5 \rho^{-2\delta -2}.
$$

Now using (25) and the bound we already have for $|\nabla \eta|$ we get

$$
\frac{\partial}{\partial t} (C_6 |\nabla \eta|^2 + tf) \leq \Delta (C_6 |\nabla \eta|^2 + tf) + C_7 \rho^{-2\delta -2}.
$$

It follows that if we write $F = C_6 |\nabla \eta|^2 + tf - C t \rho^{-2\delta -2}$ for some constant $C$, then

$$
\frac{\partial}{\partial t} F \leq \Delta F + C t \Delta (\rho^{-2\delta -2}) - C \rho^{-2\delta -2} + C_7 \rho^{-2\delta -2}.
$$

Now just as in the proof of Lemma 5 there is a constant $C_8$ such that $\Delta (\rho^{-2\delta -2}) < C_8 \rho^{-2\delta -2}$, using that the metrics along the flow are asymptotically flat in $C_5^1$, by our estimates on $\eta$ and $\nabla \eta$. So we have

$$
\frac{\partial}{\partial t} F \leq \Delta F + C (C_8 t - 1) \rho^{-2\delta -2} + C_7 \rho^{-2\delta -2}.
$$

As long as $t < (2C_8)^{-1}$, we can choose $C$ sufficiently large, so that

$$
\frac{\partial}{\partial t} F \leq \Delta F.
$$

Using the maximum principle as before, we obtain that for $t \geq t_0 > 0$ and as long as $t < (2C_8)^{-1}$, there is some constant $C_9$, for which $f < C_9 \rho^{-2\delta -2}$. This implies that $|\nabla^2 \eta| < C_{10} \rho^{-\delta -1}$ for some $C_{10}$. Since $|\nabla \eta|$ satisfies the same bound, we get the required weighted $C^{1,\alpha}$ bound on $\eta$.

\[\square\]

\textbf{Remark.} It is natural to expect that for the second derivatives of $\eta$ we can obtain the improved decay $|\nabla^2 \eta| < C_\rho^{-\delta -2}$ for $t \geq t_0 > 0$. This does not seem to be the case however, since even for the linear heat equation on $\mathbf{R}$ the analogous statement does not hold. One can check that if $f(x, t)$ is the bounded solution of $\partial_t f = \partial_x^2 f$ on $\mathbf{R}$, with

$$
f(x, 0) = \frac{1}{1 + x^2} \sin(x),
$$

then $f(x, t)$ is not bounded.
then for each fixed \( t_0 \geq 0 \) and \( k \) we have \( |\partial^k_x f(x,t_0)| = O(|x|^{-2}) \), and the decay cannot be improved even for positive time. This is why we work in the weighted \( C^{1,\alpha} \) spaces with sufficiently small \( \alpha > 0 \).

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