ARX modeling of unstable Box-Jenkins models *

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Abstract

The use of high-order polynomial models that are linear in the parameters is common in system identification to avoid the non-convexity of the prediction error method when applied to other model structures. A common and fairly general case is to use high-order ARX models to approximate Box-Jenkins structures. Then, a well known correspondence is made between the ARX polynomials and the plant and noise models in the Box-Jenkins structure. However, this commonly used result is only valid when the Box-Jenkins predictor is stable. In this contribution, we generalize these results to allow for unstable predictors due to an unstable plant. We show that high-order ARX models are appropriate for this situation as well. However, corrections must be made to correctly retrieve the noise model and noise variance.

Key words: System identification; polynomial models; closed-loop identification.

1 Introduction

System identification deals with obtaining mathematical models for systems using experimental data. To obtain such models, the prediction error method (PEM) is a benchmark in the field, since it provides asymptotically efficient estimates if the chosen model orders are correct [1]. The drawback with PEM is that, in general, it requires solving a non-convex optimization problem, and the global minimum may not be attained. A procedure to improve the convergence properties of PEM through input design has been presented in [2]. Moreover, for some particular model structures, minimizing the cost function of PEM consists simply in solving a linear regression problem. This is the case of autoregressive exogenous (ARX) models.

Despite the usefulness of ARX models due to the simplicity of estimation, they provide limited flexibility, not allowing the plant and the noise model to be parametrized independently. In this sense, Box-Jenkins (BJ) models are a more encompassing choice, as the plant and noise model are rational transfer functions parametrized independently. However, estimating BJ models with PEM suffers the drawbacks of solving a non-convex optimization problem.

Another advantage of ARX models is that, if the polynomial orders are allowed to increase, they can approximate a BJ model arbitrarily well. Their limitation is that, as the number of estimated parameters increases, so does the variance of the estimated model. Nevertheless, high-order ARX models are still useful: e.g., several methods use them as an intermediate step in the estimation algorithm [3–6]. Because, as the model order increases to infinity, the ARX model estimate is a sufficient statistic for the problem, it can replace the data and be used to obtain an estimate that is statistically efficient.

If the Box-Jenkins model has a stable predictor, it is well known that the plant and noise model can be recovered from the ARX polynomials [7]. In this paper, we generalize these results to unstable plants, with data obtained from a BJ system under stabilizing feedback. Due to the independent parametrization of the plant and noise model, these transfer functions are not required to share unstable poles. Although this is a non-standard case, because PEM has an unstable predictor, it makes sense: for example, if the noise model is used to model a sensor, restricting this model to contain eventual unstable plant dynamics is unreasonable from a physical perspective. To apply PEM in such a case, methods have been proposed to deal with unstable predictors [8, 9].

Our contributions are the following. First, we derive the ARX polynomials obtained asymptotically when the plant in a BJ system description is unstable. Second, we observe that appropriate corrections are required to obtain consistent estimates of the noise model and the noise variance, and illustrate how these can be applied. Third, although the variance of the estimated model increases with the number of estimated parameters, we observe that the variance of the unstable poles remains small.
2 Problem Statement

Consider that data is generated by the BJ system

\[ y_t = G(q)u_t + H(q)e_t, \quad (1) \]

where \( u_t \) is the plant input, \( e_t \) is Gaussian white noise with variance \( \lambda_e \), \( y_t \) is the output, and \( G(q) \) and \( H(q) \) are the true plant and noise models, respectively, which are rational transfer functions in the delay operator \( q^{-1} \), given by

\[ G(q) := \frac{L(q)}{F(q)} = \frac{l_1 q^{-1} + \ldots + l_m q^{-m_l}}{1 + f_1 q^{-1} + \ldots + f_m q^{-m_f}}, \]

\[ H(q) := \frac{C(q)}{D(q)} = \frac{1 + c_1 q^{-1} + \ldots + c_m q^{-m_c}}{1 + d_1 q^{-1} + \ldots + d_m q^{-m_d}}, \]

where \( m_l, m_f, m_c, \) and \( m_d \) are finite positive integers.

We impose that \( C(q) \) and \( D(q) \) are stable polynomials, i.e., all the roots lie inside the unit circle, and \( F(q) \) does not contain roots on the unit circle. Since \( F(q) \) is not required to be stable, we consider that the data is obtained with stabilizing feedback,

\[ y_t = G(q)S(q)r_t + H(q)S(q)e_t, \]

\[ u_t = S(q)r_t - K(q)H(q)S(q)e_t, \quad (2) \]

where \( r_t \) is a known external reference correlated with \( e_t \), \( S(q) = [1 + K(q)G(q)]^{-1} \) is the sensitivity function, and \( K(q) \) is a stabilizing regulator.

Consider also the ARX model

\[ A(q)y_t = B(q)u_t + e_t, \quad (3) \]

with infinite order polynomials

\[ A(q) = 1 + \sum_{k=1}^{\infty} a_k q^{-k}, \quad B(q) = \sum_{k=1}^{\infty} b_k q^{-k}. \]

Using a quadratic cost, the PEM estimate of the ARX model minimizes the cost function

\[ J := \mathbb{E}[\bar{A}y_t - \bar{B}u_t]^2, \quad (4) \]

as the number of samples tend to infinity (here, the argument \( q \) was dropped for notational simplicity). Because the data are generated by (2), the cost function can be expressed as

\[ J = \mathbb{E}[(AG - B)Sr_t + (A + KB)HSe_t]^2. \quad (5) \]

Then, the global minimizers of (5), \( \bar{A}(q) \) and \( \bar{B}(q) \), can be related to \( G(q) \) and \( H(q) \).

It is well known that, if the predictor

\[ \hat{y}_t = (1 - H^{-1})y_t + H^{-1}Gu_t, \quad (6) \]

is stable, (5) is minimized by [7]

\[ \bar{A}(q) = \frac{1}{H(q)}, \quad \bar{B}(q) = \frac{G(q)}{H(q)}, \quad (7) \]

with minimum

\[ J^* = \lambda_e. \quad (8) \]

Thus, an infinite order ARX model can be used to asymptotically recover a BJ model, as well as the noise variance \( \lambda_e \).

In this paper, we seek the minimizers \( \bar{A}(q) \) and \( \bar{B}(q) \) of (5) when the plant \( G(q) \) is unstable and the noise model \( H(q) \) is parametrized independently, as is the case with a BJ model. Although this makes the predictor (6) unstable in general, the cost function (5) will still have a minimum.

3 ARX Minimizer of Unstable BJ Model

First, we introduce a result that will be used to derive our main result.

**Theorem 1** Let \( X(q) \) and \( Z(q) \) be rational functions that can be expanded as infinite, monic polynomials. Let \( Z(q) \) be fixed, stable and inversely stable, and \( X(q) \) be a function of a free parameter vector \( \theta \), \( X(q, \theta) \). Then, the cost function

\[ J(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{i\omega}, \theta)|^2 |Z(e^{i\omega})|^2 d\omega \quad (9) \]

has the unique minimizer \( \hat{X}(q, \theta) = Z^{-1}(q) \), with minimum \( J^* = 1 \).

**PROOF.** This is a standard result, see e.g. Problem 3G.3 in [1]. A proof is included for completeness. The product \( X(q)Z(q) \) can be expanded as a polynomial,

\[ X(q)Z(q) = \sum_{i=0}^{\infty} g_i q^{-i}, \quad (10) \]

where \( g_0 = 1 \), since \( X(q) \) and \( Z(q) \) are both monic. Using Parseval’s identity on (9) together with (10) yields

\[ J = 1 + \sum_{i=1}^{\infty} |g_i|^2 \geq 1. \]

The minimum \( J^* = 1 \) is obtained for \( \hat{X}(q, \theta) = Z^{-1}(q) \), since \( Z(q) \) is inversely stable. Because the inverse is unique, the minimum will not be attained for any other \( X(q, \theta) \), since then at least one \( g_i \neq 0, i > 0 \).

Before stating the main result of the paper, we introduce the following definitions. Consider the factorization

\[ F(q) = F_s(q)F_a(q), \quad (11) \]
The following theorem states our main result.

**Theorem 2** Let $H(q)$ be stable and inversely stable, and factorize $F(q)$ according to (11). The asymptotic minimizers of (14) are given by

$$\bar{A}(q) = \frac{1}{H(q)} F_a(q), \quad \bar{B}(q) = \frac{1}{H(q)} F_l(q),$$

and the attained global minimum is

$$J^* = \left| \frac{F_a(e^{i\omega})}{F_a^*(e^{i\omega})} \right|^2 \lambda_e.$$  

PROOF. Using that $r_t$ and $e_t$ are uncorrelated and Parseval’s identity, (5) can be written as

$$J = J_r + J_e,$$

where

$$J_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert AG - B \rvert^2 |S|^2 \Phi_r \, d\omega,$$

$$J_e = \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert A + KB \rvert^2 |HS|^2 \lambda_e \, d\omega,$$

with $\Phi_r$ the spectrum of $r_t$. Let

$$\tilde{S}(q) := \tilde{S}(q) = \frac{F_a(q)}{F(q) + K(q)L(q)},$$

and re-write (16) as

$$J_e = \left| \frac{F_a(e^{i\omega})}{F_a^*(e^{i\omega})} \right|^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \lvert A + KB \rvert^2 |H\tilde{S}F_a^*|^2 \lambda_e \, d\omega,$$

where $|F_a(e^{i\omega})/F_a^*(e^{i\omega})|$ is moved outside the integral since it is an all-pass filter. Therefore, since $A(q) + K(q)B(q)$ is monic, and $H(q)\tilde{S}(q)F_a^*(q)$ is monic, stable, and inversely stable, we are in the situation of Theorem 1, and $J_r$ is minimized by

$$A(q) + K(q)B(q) = [H(q)\tilde{S}(q)F_a^*(q)]^{-1}. \quad (17)$$

If (15) can be made zero while satisfying (17), a global minimizer for $J$ has been found. This is done by setting $\bar{A}(q)$ and $\bar{B}(q)$ according to (12), with minimum (13).

Notice that, if the plant is stable, $F_a(q) = 1 = F_a^*(q)$ and $F_l(q) = F(q)$, and (12) reduces to (7). Moreover, using a similar approach to Theorem 2, it is straightforward to extend this result to the case with a non-minimum phase noise model $H(q)$. In this case, the noise model $H(q)$ in (12) will be replaced by its minimum phase equivalent. This corresponds to the well known result that PEM identifies an equivalent minimum phase noise model if the true noise model is non-minimum phase [8].

Recovering $G(q)$ from the asymptotic minimizers of the ARX model is straightforward, as

$$G(q) = \frac{\bar{B}(q)}{\bar{A}(q)}.$$  

The following corollary describes how the noise model and the variance $\lambda_e$ can be retrieved.

**Corollary 3** Let $J^*$ be the asymptotic minimum of (5), and $\bar{A}(q)$ the corresponding minimizer. Then, the noise model $H(q)$ and the noise variance $\lambda_e$ can be retrieved by

$$H(q) = \frac{1}{A(q)} \bar{A}_a(q) \quad (19)$$

and

$$\lambda_e = J^* \left| \frac{\bar{A}_a(e^{i\omega})}{\bar{A}_a(e^{i\omega})} \right|^2,$$  

respectively.

PROOF. The asymptotic minimizer $\bar{A}(q)$ can be factorized by one polynomial $F_a(q)$ with anti-stable roots and one polynomial with only stable roots corresponding to $1/H(q)F_a^*(q)$. Thus, $F_a(q)$ can be retrieved as the anti-stable roots of $\bar{A}(q)$,

$$F_a(q) = \bar{A}_a(q). \quad (21)$$

Then, (19) follows directly from (12) and (21), while (20) follows from (13) and (21). □

Comparing Corollary 3 to (7) and (8), it is observed that the noise model $H(q)$ and noise variance $\lambda_e$ will be wrongly estimated if the appropriate corrections due to the unstable plant are not made. In particular, we observe that $\bar{A}_a(q)/\bar{A}_a(q)$ is an amplifying all-pass filter. So, without this correction factor, the magnitude of the noise model is underestimated by a constant bias, and the noise variance is overestimated.

### 4 Practical Aspects

So far, we have only discussed consistency of the ARX model. For the consistency results to be valid, the ARX model has to be of infinite order. Otherwise, the system (1) is not in the model set defined by (3), and a bias is induced.
Consider the plant and noise models to our setting; however, due to the technical effort required, no conceptual reason limiting the extension of this theorem N the sample size cable to the setting of this paper, since such variance analysis class of models, encompassing both ARX models and Box-Jenkins models, it has been shown that the variance of any unstable root will converge to a finite limit (cf. Theorem 5.1 in [10]). However, Theorem 5.1 in [10] is not directly applicable to the setting of this paper, since such variance analysis requires that the model order tend to infinity as function of the sample size $N$, similarly to the approach in [7]. There is no conceptual reason limiting the extension of this theorem to our setting; however, due to the technical effort required, it will be considered in a separate contribution.

The inherent limitation of estimating a high order model is that the estimated model will have high variance. However, we observe that this does not apply to the unstable poles of the ARX model, as will be illustrated with a simulation in the next section. Thus, the estimate of $F_u(q)$ obtained from (21) will have high accuracy in comparison to the complete high order ARX-model estimate. In turn, this means that the noise variance can be estimated according to (20) with high accuracy.

There is theoretical support for the observation that the unstable roots of $A(q)$ have low variance. For a very general class of models, encompassing both ARX models and Box-Jenkins models, it has been shown that the variance of any unstable root will converge to a finite limit (cf. Theorem 5.1 in [10]). However, Theorem 5.1 in [10] is not directly applicable to the setting of this paper, since such variance analysis requires that the model order tend to infinity as function of the sample size $N$, similarly to the approach in [7]. There is no conceptual reason limiting the extension of this theorem to our setting; however, due to the technical effort required, it will be considered in a separate contribution.

5 Examples

Consider the plant and noise models

$$G(q) = \frac{1q^{-1} - 1.7q^{-2}}{1 - 2q^{-1} + 2q^{-2}}, \quad H(q) = \frac{1 + 0.2q^{-1}}{1 - 0.9q^{-1}},$$

which are used to generate data according to (2) with the controller $K(q) = 1$, where $r_t$ and $e_t$ are uncorrelated Gaussian white noise sequences with unit variance. Notice that the plant $G(q)$ has a pair of unstable complex poles at $1 \pm i$.

We use this system for two examples. First, we illustrate the limit properties of the ARX model that were shown in Section 3; then, we use different orders of the ARX model to illustrate the observation in Section 4 regarding the variance of the estimated unstable poles.

5.1 Limit Properties of the ARX Model

The objective of this example is to illustrate the result obtained in Theorem 2, and how Corollary 3 can be used to obtain estimates of $G(q)$ and $H(q)$ when the plant is unstable. Because these results concern the limit values, in both model order and sample size, of the estimates of $A(q)$ and $B(q)$, they can be more clearly illustrated if the estimation error is kept small. Thus, to minimize the bias error due to the ARX model truncation, we choose $G(q)$ and $H(q)$ such that the coefficients of $\hat{A}(q)$ and $\hat{B}(q)$ decay quickly, which allows us to use a relatively low order ($n_a = n_b = 15$). The low model order together with a large sample size ($N = 100000$) ensure that also the variance error will be small. Finally, we are also interested in estimating the noise variance, $\lambda_e$.

The procedure is as follows. First, the ARX polynomials $A(q)$ and $B(q)$ are estimated by minimizing the cost function

$$J = \frac{1}{N} \sum_{i=1}^{N} [A(q)r_t - B(q)u_t]^2,$$

which is a consistent estimate of (4) for finite sample size. This is a least-squares problem, and yields estimates $\hat{A}(q)$ and $\hat{B}(q)$, at which the minimum $J$ is obtained.

Then, we estimate the plant and noise model from the estimated ARX polynomials. Motivated by (18) and (19), we calculate

$$\hat{G}(q) = \frac{\hat{B}(q)}{\hat{A}(q)}, \quad \hat{H}(q) = \frac{1}{\hat{A}(q) \hat{A}_c(q)}.$$

In Fig. 1 and Fig. 2, the Bode plots of $G(q)$ and $H(q)$ are shown respectively together with their corresponding estimates, and, in the case of the noise model, also the estimate without the correction for the unstable plant is shown. Here, it is observed that the ARX model correctly captures the true system, according to (12). In particular, this illustrates the main result of this paper, that when the plant $G(q)$ has unstable poles and is parametrized independently of the noise model $H(q)$, a high-order ARX model is still appropriate to consistently model this system. However, while the standard result (7) still applies to consistently retrieve the plant, a consistent estimate of the noise model is obtained by using a correction factor according to (19).

Finally, we are interested in estimating the noise variance, $\lambda_e$. With that purpose, motivated by (20), we calculate

$$\hat{\lambda}_e = \hat{f} \frac{\hat{A}_e^*(q)}{\hat{A}_e(q)} \cdot \frac{(\lambda_e)}{\hat{A}_e(q)} \cdot \frac{(\lambda_e)}{\hat{A}_e(q)}.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} (22)

Recalling that $\lambda_e = 1$, the results obtained for this example were

$$\hat{f} = 3.9816, \quad \hat{\lambda}_e = 0.9988.$$

Again, we observe how the high-order ARX model can still be used to obtain a consistent estimate of the noise variance,
even in our generalized setting for unstable plants, as long as the appropriate correction (22) is made. Otherwise, taking the minimum \( \hat{J} \) as estimate for the noise variance, as in (8), would overestimate the noise variance by a factor of four.

5.2 Variance of the Estimated Unstable Poles

As ARX models typically need to be of high order to capture the dynamics of Box-Jenkins systems, the variance of the estimated model will be large. Although this intrinsic limitation was not evident in the previous example, since the dynamics of the considered system can be captured with relatively low orders of \( A(q) \) and \( B(q) \) (and a large sample size was used), it can be made clear by letting the order of the ARX polynomials increase.

As the variance of the estimated \( \hat{A}(q) \) will be large, also the variance of the estimated poles of \( G(q) \) should be large, since the poles of \( G(q) \) are obtained from the roots of \( \hat{A}(q) \). However, following the discussion in Section 4, we observe that this does not apply to the variance of the unstable poles.

To illustrate this, we perform a Monte Carlo simulation with 50 runs, where two ARX models with different orders are computed. The roots of \( \hat{A}(q) \) are plotted in Fig. 3 and Fig. 4, for \( n_a = n_b = 15 \) and \( n_a = n_b = 100 \), respectively. Here, it is clear that the variance of the unstable poles is small relative to the stable ones, and also that there is no apparent variance increase for the unstable poles when the number of estimated parameters increases.

6 Discussion

In this paper, we derived asymptotic results for the limit values of ARX models when used to model a BJ structure with an unstable plant. High-order ARX models can still be used to model the underlying system in this situation. However, while \( \frac{b(q)}{a(q)} \) still captures the plant, \( \frac{1}{a(q)} \) no longer corresponds to the noise model \( H(q) \), but will also depend on the unstable part of the plant \( G(q) \).

This result has also implications for modeling output-error (OE) structures with high-order models. Typically, in this case, a high-order finite impulse response model can be estimated instead of an ARX, since \( H(q) = 1 + A(q) \). However, that is only the case if \( G(q) \) is stable. When \( G(q) \) is unstable,
a finite-order polynomial $B(q)$ is not sufficient to approximate $G(q)$ arbitrarily well. Thus, even if the true system is OE, an ARX model is required to asymptotically capture the system dynamics in this case.

The results in this paper also have a correspondence to those in [9]. Therein, the authors derive modified, but asymptotically equivalent, versions of OE and BJ models that yield stable predictors in the case of unstable systems. The modified versions contain an additional factor $F^*(q)/F_a(q)$ to be estimated in the noise model, which corresponds to the same quantity appearing in the estimate of $1/A(q)$ here derived.

Finally, we have shown how the plant, noise model, and noise variance of a system of BJ structure can be obtained from an estimated ARX model, in the case of an unstable plant. Also, we note that the high variance inherent to the high order of the ARX model does not affect the estimation of the unstable poles nor the estimation of the noise variance.

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