ON THE HASSE PRINCIPLE FOR ZERO-CYCLES ON SEVERI-BRAUER FIBRATIONS

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Abstract. Let $k$ be a number field, let $C$ be a smooth, projective and geometrically integral $k$-curve and let $\pi: X \to C$ be a Severi-Brauer fibration of squarefree index. Various authors have studied the cokernel of the natural map $CH_0(X/C) \to \bigoplus_v CH_0(X_v/C_v)$, where $CH_0(X/C) = \text{Ker}[\pi_*: CH_0(X) \to CH_0(C)]$. In this paper we study the kernel of the above map and find sufficient conditions for it to agree with the Tate-Shafarevich group of the Néron-Severi torus of $X$.

0. Introduction.

Let $k$ be a number field, let $C$ be a smooth, projective and geometrically integral $k$-curve and let $\pi: X \to C$ be a Severi-Brauer fibration of squarefree index. In the case that $C = \mathbb{P}^1_k$ and $X \to C$ is a conic bundle, J.-L. Colliot-Thélène and J.-J. Sansuc conjectured in 1981 [2] the existence of an exact sequence of torsion groups

$$0 \to \mathbb{H}^1(T) \to A_0(X) \to \bigoplus_v A_0(X_v) \to \text{Hom}(H^1(k, NS(X)), \mathbb{Q}/\mathbb{Z}),$$

(1)

where $T$ is the Néron-Severi torus of $X$ and $A_0(X)$ is the group of rational equivalence classes of 0-cycles of degree zero on $X$. This conjecture was proved in 1988 by P. Salberger, in his remarkable paper [10]. Then, in 1999 [3], J.-L. Colliot-Thélène partially extended Salberger’s proof to conic bundles $X \to C$ over a curve $C$ of arbitrary genus. Under certain extra assumptions which turn out to be unnecessary, this author obtained an exact sequence of torsion groups

$$CH_0(X/C) \to \bigoplus_v CH_0(X_v/C_v) \to \text{Hom}(\text{Br}X/\text{Br}C, \mathbb{Q}/\mathbb{Z}),$$

where $CH_0(X/C) = \text{Ker}[\pi_*: CH_0(X) \to CH_0(C)]$ is the relative 0-th Chow group of the fibration $X \to C$. On the other hand, in his 1990 ICM talk [1], S. Bloch put forth a general conjecture which, in the case of conic bundles over a curve $C$, asserts the equality of the groups

$$\mathbb{H}^1(CH_0(X/C)) = \text{Ker}\left[CH_0(X/C) \to \prod_v CH_0(X_v/C_v)\right].$$

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and $\mathbb{I}^1(T)$. This direct generalization of part of Salberger's result was shown to be incorrect by V. Suresh in his 1997 paper [11]. This author produced an example of a conic bundle over an elliptic curve over $\mathbb{Q}$ for which $\mathbb{I}^{1}(T) = 0$ but $\mathbb{I}(\text{CH}_0(X/C)) \neq 0$.

In this paper we study the group $\mathbb{I}(\text{CH}_0(X/C))$ for an arbitrary Severi-Brauer fibration $X \to C$ of squarefree index. Our main result is the following.

**Theorem 0.1.** Let $k$ be a number field, let $X \to C$ be a Severi-Brauer $k$-fibration of squarefree index and let $\Phi: \text{CH}_0(X/C) \to H^1(k, T)$ be the characteristic map of $X \to C$, where $T$ is the Néron-Severi torus of $X$. Then there exists a natural exact sequence

$$0 \to \mathbb{I}(\text{Ker}\Phi) \to \mathbb{I}(\text{CH}_0(X/C)) \to \mathbb{I}^{1}(T) \to 0.$$ 

In Section 2 we use a well-known Hasse-principle theorem due to K. Kato to obtain the following corollary of Theorem 0.1.

**Corollary 0.2.** Let $k$ be a number field and let $X \to C$ be a Severi-Brauer $k$-fibration of squarefree index $n$. Assume that the following conditions hold:

i) either $k$ is totally imaginary or $n$ is odd, and

ii) $C$ acquires good reduction at all primes over an extension of $k$ of degree prime to $n$.

Then there exists a canonical isomorphism

$$\mathbb{I}(\text{CH}_0(X/C)) = \mathbb{I}^{1}(T),$$

where $T$ is the Néron-Severi torus of $X$.

The key ingredient of the proof of Theorem 0.1 is an approximation lemma of E. Frossard [7, Lemma 3.3]. This fundamental result is used in the proofs of Lemmas 2.1 and 2.3 below, which form the technical center of the paper. These lemmas, together with some elementary homological algebra, yield Theorem 0.1 above.

Combining Corollary 0.2 and [7, Theorem 0.4], we obtain the following result.

**Corollary 0.3.** Let the hypotheses and notations be as in the statement of Corollary 0.2. Then there exists a natural exact sequence of torsion groups

$$0 \to \mathbb{I}(X) \to \text{CH}_0(X/C) \to \bigoplus_v \text{CH}_0(X_v/C_v) \to \text{Hom}(\text{Br}_X/\text{Br}_C, \mathbb{Q}/\mathbb{Z}).$$

We also obtain the following theorem, which generalizes Salberger's exact sequence (1)

**Theorem 0.4.** Let $k$ be a number field and let $X \to \mathbb{P}^1_k$ be a Severi-Brauer $k$-fibration of squarefree index. Assume that $X(k_v) \neq \emptyset$ for every real prime $v$ of $k$.

Then there exists a natural exact sequence

$$0 \to \mathbb{I}(X) \to A_0(X) \to \bigoplus_v A_0(X_v) \to \text{Hom}(H^1(k, \text{NS}(X)), \mathbb{Q}/\mathbb{Z}),$$

where $T$ is the Néron-Severi torus of $X$. \[ \square \]

\[ ^1 \] Suresh's base curve has multiplicative reduction at 3 and 7, so neither of the hypotheses of Corollary 0.2 is satisfied in his example.

\[ ^2 \] This result must be familiar to all specialists in this area. See Remark 4.9 of [5], where the existence of such an exact sequence is suggested (but not formally stated).
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I thank E. Frossard and J. van Hamel for sending me copies of their papers [6,7,12]. As mentioned above, the work of Frossard is crucial to this paper, and I wish to express my gratitude to her here for having laid the foundations on which this paper rests. Finally, I thank the referees for several helpful suggestions.

1. Preliminaries.

Let \( k \) be a field, fix a separable closure \( \overline{k} \) of \( k \) and let \( \Gamma = \text{Gal}(\overline{k}/k) \). If \( M \) is a discrete and continuous \( \Gamma \)-module, the \( i \)-th \( \Gamma \)-cohomology group of \( M \) will be denoted by \( H^i(k, M) \). Now assume that \( k \) is a number field. For each prime \( v \) of \( k \), we will write \( k_v \) for the completion of \( k \) at \( v \). We define

\[
\text{III}^i(M) = \ker \left[ H^i(k, M) \to \prod_{v} H^i(k_v, M) \right],
\]

where the map involved is the natural localization map.

Now let \( X \) be a smooth, projective and geometrically integral \( k \)-variety. We will write \( \mathcal{X} \) for the \( \overline{k} \)-variety \( X \otimes_k \overline{k} \) (resp. \( \overline{k}(X) \)) for the field of rational functions on \( X \) (resp. \( \mathcal{X} \)) and NS(\( \mathcal{X} \)) for the Néron-Severi group of \( \mathcal{X} \). In addition, the group of \( 0 \)-cycles on \( X \) will be denoted by \( Z_0(X) \) and \( CH_0(X) \) will denote its quotient by rational equivalence. Further, we will write \( A_0(X) \) for the kernel of the degree map \( \deg: CH_0(X) \to \mathbb{Z} \) and, for each prime \( v \) of \( k \), \( X_v \) will denote the \( k_v \)-variety \( X \otimes_k k_v \).

Let \( C \) be a smooth, projective and geometrically integral \( k \)-curve and let \( X \to C \) be a Severi-Brauer \( k \)-fibration, i.e., \( X \) is a smooth, projective and geometrically integral \( k \)-variety equipped with a proper and dominant \( k \)-morphism \( \pi: X \to C \) whose generic fiber \( X_\eta = X \times_C \text{Spec} k(C) \) is a Severi-Brauer \( k(C) \)-variety.

Given a central simple \( k(C) \)-algebra \( A \), there exists an Artin model \( X \) of \( A \) over \( C \) [6]. Thus \( X \) is a smooth, projective and geometrically integral \( k \)-variety equipped with a proper and dominant \( k \)-morphism \( \pi: X \to C \) whose generic fiber \( X_\eta = X \times_C \text{Spec} k(C) \) is the Severi-Brauer \( k(C) \)-variety associated to \( A \). We will write \( X_{k(\eta)} \) for the generic fiber of \( \pi: \mathcal{X} \to \overline{\mathcal{X}} \). Note that, since \( \mathcal{X}(C) \) is a \( C_1 \)-field, \( X_{k(\eta)} \) is a projective space and therefore \( \text{Pic} X_{k(\eta)} = \mathbb{Z} \). The index of such a fibration is defined to be the index of \( A \), i.e., the minimum degree of all extensions of \( k(C) \) which split \( A \) or, equivalently, the least positive degree of a 0-cycle on \( X_\eta \).

We are interested in the group

\[
\text{III} CH_0(X) = \ker \left[ CH_0(X) \to \prod_v CH_0(X_v) \right].
\]

(2)

Set

\[
CH_0(X/C) = \ker \left[ \pi_* : CH_0(X) \to CH_0(C) \right].
\]

Since the map \( CH_0(C) \to \prod CH_0(C_v) \) may be identified with the map \( \text{Pic} C \to \prod \text{Pic} C_v \), which is injective (see, e.g., [4], Proposition 1.1, p.3), we have

\[
\text{III} CH_0(X) = \text{III} CH_0(X/C),
\]
Thus the study of (2) is reduced to the study of (3). Assume now that \( X \to C \) has squarefree index. Then (2) (and hence also (3)) is a finite group [6, Théorème 4.7]. Further, \( CH_0(X_v/C_v) \) is zero for all but finitely many primes \( v \) of \( k \) [op.cit., Théorème 4.8]. We will write \( S \) for the set of primes \( v \) of \( k \) such that \( CH_0(X_v/C_v) \neq 0 \).

We now observe that, in studying the group (3), we may restrict our attention to the chosen model \( X \to C \) of \( A \), for if \( X' \to C \) is an arbitrary Severi-Brauer \( k \)-fibration whose generic fiber corresponds to \( A \), then \( X' \to C \) is birationally equivalent to \( X \) over \( C \) (as follows from [EGA IV, 8.8.2.5]), and the Chow group of dimension zero is a birational invariant of smooth projective varieties (see [8, 16.1.11, p.312], and note that the proof given there holds over any field).

We now write \( C_0 \) for the set of closed points of \( C \). For each \( P \in C_0 \), with residue field \( k(P) \), we let \( X_P = X \times_C \text{Spec} \, k(P) \) be the fiber of \( \pi \) over \( P \). We will write \( P = \{ P_i \}_{i \in I} \) for the finite set of closed points of \( C \) for which the fiber \( X_P \) is singular (i.e., not geometrically integral). Further, for each \( i \in I \), we will write \( k_i \) for \( k(P_i) \).

For each \( P \in C_0 \), there exists a natural residue homomorphism

\[
\text{res}_P : H^2(k(C), \mu_n) \to H^1(k(P), \mathbb{Z}/n\mathbb{Z}) = \text{Hom}(\text{Gal}(\overline{k}/k(P)), \mathbb{Z}/n\mathbb{Z}),
\]

where \( n \) is the index of \( A \) (see [6, pp.205-206]). The algebra \( A \) defines an element of \( H^2(k(C), \mu_n) \) and, for each \( i \in I \), we will write \( L_i \) for the cyclic extension of \( k_i \) determined by the residue \( \text{res}_P(A) \), i.e., \( L_i \) is the fixed field of the kernel of \( \text{res}_P(A) \). Let \( E = \prod_{i \in I} k_i, F = \prod_{i \in I} L_i, N_i = N_{L_i/k_i} \), and \( N = N_{F/E} \). The quotient of \( E^* / NF^* = \prod_{i \in I} k_i^* / N_i L_i^* \) by the image of the diagonal map \( k^* \to E^* \) will be denoted by \( E^* / k^* NF^* \). Clearly, \( E^* / k^* NF^* \) is annihilated by \( n \). Now, for each prime \( v \) of \( k \) (and each \( i \in I \)), we will write \( k_i,v \) (resp. \( L_i,v, E_v, F_v \)) for \( k_i \otimes_k k_v \) (resp. \( L_i \otimes_k k_v, E \otimes_k k_v, F \otimes_k k_v \)). Further, \( A_v \) will denote \( A \otimes_{k(C)} k_v(C) \).

There exists a natural exact sequence

\[
0 \to B_1 \to B_2 \oplus \mathbb{Z} \to \text{NS}(\overline{X}) \to \mathbb{Z} \to 0,
\]

where the map \( \text{NS}(\overline{X}) \to \mathbb{Z} \) is given by restricting divisors to the generic fiber, \( B_1 \) is the free group on the closed points \( y \) of \( \overline{X} \) for which the fiber of \( \overline{\pi} \) over \( y \), \( \overline{X}_y = \overline{X} \times_{\overline{\pi}} \text{Spec} \, k(y) \), is singular, \( B_2 \subset \text{Div} \, \overline{X} \) is the free abelian group on the irreducible components of the singular fibers of \( \overline{\pi} \) and the map \( B_1 \to B_2 \oplus \mathbb{Z} \) is induced by \( y \mapsto (\overline{X}_y, -1) \) (see [6, p.201] and note that \( \overline{\pi}^* \text{Pic}^0 \overline{C} = \text{Pic}^0 \overline{\overline{X}} \) [7]). The groups \( B_1 \) and \( B_2 \) are isomorphic to \( \bigoplus_{i \in I} \mathbb{Z}[\text{Gal}(k_i/k)] \) and \( \bigoplus_{i \in I} \mathbb{Z}[\text{Gal}(L_i/k)] \), respectively. It is not difficult to check that \( (B_2 \oplus \mathbb{Z}) / B_1 \) is torsion-free. Let \( T_0 \) denote the \( k \)-torus corresponding to \( (B_2 \oplus \mathbb{Z}) / B_1 \), and let \( T \) be the Néron-Severi torus of \( \overline{X} \). Then there exists a natural isomorphism

\[
\delta : E^* / k^* NF^* \to H^1(k, T_0).
\]

See [6, Lemma 3.5]. Further, (4) induces a natural exact sequence

\[
0 \to H^1(k, T) \to H^1(k, T_0) \to \text{Br} \, k.
\]
A similar exact sequence exists over $k_v$ for every prime $v$ of $k$, and we have a natural exact commutative diagram

$$
\begin{array}{c}
0 \longrightarrow H^1(k, T) \longrightarrow H^1(k, T_0) \longrightarrow \text{Br} k \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \bigoplus_v H^1(k_v, T) \longrightarrow \bigoplus_v H^1(k_v, T_0) \longrightarrow \bigoplus_v \text{Br} k_v,
\end{array}
$$

where the direct sums extend over all primes $v$ of $k$. Now the commutativity of the above diagram and the classical Brauer-Hasse-Noether theorem yield

**Lemma 1.1.** There exists a canonical isomorphism $\text{III}^1(T) = \text{III}^1(T_0)$.

Recall that $\text{III}^1(T)$ is a birational invariant of $X$.

We will need the following definition.

**Definition.** The group of *divisorial norms*, denoted $k(C)^*_\text{dn}$, is the subgroup of $k(C)^*$ consisting of functions which, at every point $P$ of $C$, can be written as the product of a unit at $P$ by an element of the reduced norm group $\text{Nrd} A^*$.

Equivalently

$$
k(C)^*_\text{dn} = \{ f \in k(C)^* : \text{div}_C f \in \pi_*(Z_0(X)) \}.
$$

There exists a canonical isomorphism

$$
\Psi : CH_0(X/C) \rightarrow k(C)^*_\text{dn}/k^*\text{Nrd} A^*
$$

which maps a cycle $z \in CH_0(X/C)$ to the function $f$ (defined up to multiplication by a constant) such that $\text{div}_C f = \pi_*(z)$. See [6, p.98]. Now, by evaluating functions at the points of $\mathcal{P}$, one obtains a natural “specialization map”

$$
\gamma : k(C)^*_\text{dn}/k^*\text{Nrd} A^* \rightarrow E^*/k^*NF^*,
$$

and E.Frossard [6] has defined a “characteristic” map

$$
\Phi_0 : CH_0(X/C) \rightarrow H^1(k, T_0).
$$

These functions (5)-(8) fit into a natural commutative diagram

$$
\begin{array}{c}
CH_0(X/C) \xrightarrow{\Psi} k(C)^*_\text{dn}/k^*\text{Nrd} A^* \\
\Phi_0 \downarrow \quad \gamma \downarrow \\
H^1(k, T_0) \xleftarrow{\delta} E^*/k^*NF^*.
\end{array}
$$

See [6, Proposition 3.8].

**Lemma 1.2.** Let $k$ be a number field and let $X \rightarrow C$ be an Artin model over $C$ of a central simple $k(C)$-algebra $A$ of squarefree index. Then the map (6) induces an isomorphism $\text{Ker} \Phi_0 = \text{Ker} \gamma$.

**Proof.** See Corollary 3.9 and Proposition 4.1 of [6]. □
2. The proofs.

In this Section we establish the main results of the paper. We keep the notations and hypotheses introduced in the previous Section.

Lemma 2.1. Let $A$ be a central simple $k(C)$-algebra of squarefree index, let $X \rightarrow C$ be an Artin model of $A$ over $C$ and let $\Phi_0$ be the characteristic map \((8)\). Then the natural map

$$\text{Ker} \Phi_0 \rightarrow \prod_v \text{Ker} \Phi_{0,v}$$

is surjective.

Proof. By Lemma 1.2, it suffices to show that the natural map $\text{Ker} \gamma \rightarrow \prod_v \text{Ker} \gamma_v$ is surjective. Using a transfer argument analogous to that used in [7, pp.94-95], we may reduce the proof to the case where the fibration $X \rightarrow C$ satisfies hypothesis 2.1 of [op.cit.]. Now recall the set $S$ of primes $v$ of $k$ such that $\text{CH}_0(X_v/C_v) \neq 0$. Clearly if $v \notin S$ then $\text{Ker} \gamma_v = \text{Ker} \Phi_{0,v} = 0$, whence $\prod_v \text{Ker} \gamma_v = \prod_{v \in S} \text{Ker} \gamma_v$. Let $(f_v)_{v \in S} \in \prod_{v \in S} \text{Ker} \gamma_v$. For each $v \in S$, lift $f_v$ to $f'_v \in k_v[C]^*_\text{dn}/k_\text{v}^*$ and choose $z_v \in Z_0(X_v)$ such that $(\pi_v)^*_v(z_v) = \text{div}_{C_v} f'_v$, where $\pi_v = \pi \otimes k_v$. Then Lemma 3.3 of [7] (which may be applied since $X \rightarrow C$ was assumed to satisfy hypothesis 2.1 of [op.cit.]), with $y = 0$, $\eta = 1 \in E^*/k^* NF^*$, $f_{z_v,y} = f'_v$ for $v \in S$ and $f_{z_v,y} = 1$ for $v \notin S$, asserts the existence of an element $g \in \text{Ker} \gamma$ which maps to $(f_v)_{v \in S}$, as desired (see Remark 2.2 below). \hfill \square

Remark 2.2. Regarding the proof of the lemma, the following comment is in order. The element $g \in k(C)^*/k^*$ whose existence is asserted by Lemma 3.3 of [7] may be chosen so that it specializes to the given $\eta \in E^*/k^* NF^*$. This follows from the fact that the function $h \in k(C)^*$ which appears in the proof of that lemma [op.cit.], p.94, line 16, and whose existence is asserted by Proposition 1.4 of [op.cit.], may be chosen so that it specializes to $\eta'$, as follows from the proof of Proposition 1.4 of [op.cit.]. See [op.cit.], p.90, line 9.

Lemma 2.3. Let $A$ be a central simple $k(C)$-algebra of squarefree index, let $X \rightarrow C$ be an Artin model of $A$ over $C$ and let $\Phi_0$ be the characteristic map \((8)\). Then the natural map

$$\text{Coker} \Phi_0 \rightarrow \prod_v \text{Coker} \Phi_{0,v}$$

is injective.

Proof. As in the proof of Lemma 2.1, a transfer argument analogous to that used in [7, pp.94-95] enables us to reduce the proof to the case where $X \rightarrow C$ satisfies hypothesis 2.1 of [7]\(^3\). Let $\alpha \in H^1(k, T_0)$ be such that $\text{res}_v(\alpha) \in \text{Im} \Phi_{0,v}$ for every prime $v$ of $k$, where $\text{res}_v : H^1(k, T_0) \rightarrow H^1(k_v, T_0)$ is the natural restriction map. For each $v$, choose $z_v \in \text{Ker}[(\pi_v)^*_v : Z_0(X_v) \rightarrow \text{CH}_0(C_v)]$ such that $\text{res}_v(\alpha) = \Phi_{0,v}([z_v])$.

\(^3\)With the notations of [op.cit.], the cokernel of $\text{Ker} \gamma \rightarrow \prod_v \text{Ker} \gamma_v$ is annihilated by multiplication by the index of $X \rightarrow C$ as well as by $[M_i : k]$ for each $i$ if the assertion of the lemma is true for the fibration $X_{M'_i} \rightarrow C_{M'_i}$ of [op.cit.], p.94, line 1. This immediately yields the surjectivity of $\text{Ker} \gamma \rightarrow \prod_v \text{Ker} \gamma_v$.

\(^4\)The essential fact again is that, for any (Galois) extension $M/k$, the kernel of the map $\text{Coker} \Phi_0 \rightarrow \text{Coker} \Phi_{0,M}$ induced by the restriction map is annihilated both by multiplication by $[M : k]$ and by multiplication by the index of $X \rightarrow C$. 


where \([z_v]\) denotes the class of \(z_v\) in \(CH_0(X_v/C_v)\). Further, let \(f_v \in k_v(C)_*/k_v^*\) be such that \((\pi_v)_*(z_v) = \text{div}_{C_v}(f_v)\). Now let \(\eta = \delta^{-1}(\alpha) \in E^*/k^*NF^*\), where \(\delta\) is the map (5). For each \(v\), the commutativity of diagram (9) (over \(k_v\)) shows that \(f_v\) specializes to \(\delta^{-1}(\alpha)\) in \(E^*/k^*NF^*_v\). Now Lemma 3.3 of [7], with \(y = 0\) and \(\eta\) as above, asserts the existence of elements \(g \in k(C)_*/k^*\) and \(z \in Z_0(X)\) such that \(g\) specializes to \(\eta\) (see Remark 2.2 above) and \(\Psi([z]) = g\), where \(\Psi\) is the map (6). It follows that \(\delta^{-1}(\alpha) = \delta^{-1}(\Phi_0([z]))\) (see diagram (9)), whence \(\alpha = \Phi_0([z])\). This completes the proof. □

Remark 2.4. In the case of conic bundles, Lemmas 2.1 and 2.3 are immediate consequences of Theorem 1.3 of [3] (to obtain Lemma 2.1 from loc.cit., set \(\alpha = 0\) there). This observation was the starting point of this paper.

We can now state our main result.

Theorem 2.5. Let \(k\) be a number field, let \(X \to C\) be a Severi-Brauer \(k\)-fibration of squarefree index and let \(\Phi_0\) be the characteristic map (8). Then there exists a natural exact sequence

\[0 \to \Pi_1(k) \to \Pi_0CH_0(X/C) \to \Pi_1(T) \to 0,\]

where \(T\) is the Néron-Severi torus of \(X\).

Proof. Assume first that \(X \to C\) is the Artin model of a central simple \(k(C)\)-algebra \(A\) of squarefree index. There exists a natural exact commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \ker \Phi_0 & \longrightarrow & CH_0(X/C) & \longrightarrow & \text{Im} \Phi_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_v \ker \Phi_0,v & \longrightarrow & \prod_v CH_0(X_v/C_v) & \longrightarrow & \prod_v \text{Im} \Phi_0,v & \longrightarrow & 0.
\end{array}
\]

Applying the snake lemma to the above diagram and using Lemma 2.1, we obtain an exact sequence \(0 \to \Pi_1(k) \to \Pi_0CH_0(X/C) \to \Pi_1(\text{Im} \Phi_0) \to 0\). On the other hand, the commutativity of the natural diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Im} \Phi_0 & \longrightarrow & H^1(k, T_0) & \longrightarrow & \text{Coker} \Phi_0 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_v \text{Im} \Phi_0,v & \longrightarrow & \prod_v H^1(k_v, T_0) & \longrightarrow & \prod_v \text{Coker} \Phi_0,v & \longrightarrow & 0
\end{array}
\]

together with Lemmas 2.3 and 1.1 yield a natural isomorphism \(\Pi_1(\text{Im} \Phi_0) = \Pi_1(T)\). This completes the proof when \(X \to C\) is an Artin model as above. The general case may be deduced from the preceding one by using the birational invariance of the groups \(\Pi_0CH_0(X/C)\) and \(\Pi_1(T)\). □

By an example of V.Suresh [11], the group \(\Pi_1(\ker \Phi_0)\) appearing in the statement of the theorem need not vanish. We will now study this group and find conditions under which it vanishes (see Theorem 2.8 below).
There exists a natural exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & k^*/k^* \cap \text{Nrd} A^* & \rightarrow & k(C)_{dn}^*/\text{Nrd} A^* & \rightarrow & k(C)_{dn}^*/k^* \text{Nrd} A^* & \rightarrow & 0 \\
\gamma_0 & \downarrow & \gamma & \downarrow & \gamma & \downarrow & \gamma & \downarrow & \gamma \\
0 & \rightarrow & k^*/k^* \cap \text{NFD}^* & \rightarrow & E^*/\text{NFD}^* & \rightarrow & E^*/k^* \text{NFD}^* & \rightarrow & 0,
\end{array}
\]

(10)

where $\gamma_0$ is induced by the identity on $k^*$ and $\gamma$ and $\gamma$ induced by specialization. Since $\gamma_0$ is surjective, the snake lemma applied to (10) yields a natural exact sequence

\[
0 \rightarrow \text{Ker} \gamma_0 \rightarrow \text{Ker} \gamma \rightarrow \text{Ker} \gamma \rightarrow 0. \quad (11)
\]

Note that $\text{Ker} \gamma_0$ is canonically isomorphic to $k^* \cap (\cap_{i \in I} N_i L_i^*)/k^* \cap \text{Nrd} A^*$.

Now there exist sequences analogous to (11) over $k_v$ for every prime $v$ of $k$, and we have a natural exact commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ker} \gamma_0 & \rightarrow & \text{Ker} \gamma & \rightarrow & \text{Ker} \gamma & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \prod_v \text{Ker} \gamma_0 \rightarrow & \prod_v \text{Ker} \gamma & \rightarrow & \prod_v \text{Ker} \gamma & \rightarrow & \prod_v \text{Ker} \gamma & \rightarrow & 0.
\end{array}
\]

(12)

We now have\footnote{One of the referees informed us that this result, as well as Proposition 2.7 and Theorem 2.9 below, have been known to the specialists in this area for quite some time. Unfortunately, we have been unable to find appropriate references for them and therefore provide the corresponding proofs.}

**Lemma 2.6.** The canonical map

\[
\text{Ker} \gamma \rightarrow \prod_v \text{Ker} \gamma_v
\]

is injective.

**Proof.** P. Salberger (see [6, Proposition 4.4]) has shown that, over any field $k$ of characteristic 0, there exists a canonical injection $\text{Ker} \gamma \hookrightarrow H^3_m(k(C)/C, \mu_n \otimes 2)$, where

\[
H^3_m(k(C)/C, \mu_n \otimes 2) = \text{Ker} \left[ H^3(k(C), \mu_n \otimes 2) \rightarrow \bigoplus_{P \in C_0} H^2(k(P), \mu_n) \right].
\]

Here, for each $P \in C_0$, $H^3(k(C), \mu_n \otimes 2) \rightarrow H^2(k(P), \mu_n)$ is the natural residue map. On the other hand, a well-known theorem of K. Kato [9, §4], asserts that the canonical map

\[
H^3_m(k(C)/C, \mu_n \otimes 2) \rightarrow \bigoplus_v H^3_m(k_v(C)/C_v, \mu_n \otimes 2)
\]

is an isomorphism, whence the lemma follows. \qed
We now apply the snake lemma to diagram (12) and use the preceding lemma together with Lemma 1.2⁶ to obtain the following result

**Proposition 2.7.** Let \( k \) be a number field, let \( X \to C \) be a Severi-Brauer \( k \)-fibration of squarefree index and let \( \Phi_0 \) be the characteristic map (8). Then there exists a canonical injection

\[
\text{III}(\text{Ker}\Phi_0) \hookrightarrow \text{Coker}\left[\text{Ker}\tilde{\gamma}_0 \to \prod_v \text{Ker}\tilde{\gamma}_{0,v}\right],
\]

where \( \tilde{\gamma}_0 \) is the map in diagram (10). \( \square \)

**Theorem 2.8.** Let \( k \) be a number field, let \( X \to C \) be a Severi-Brauer \( k \)-fibration of squarefree index with associated central simple \( k(C) \)-algebra \( A \), and let \( \Phi_0 \) be the characteristic map (8). Further, let \( S' \) be the set of primes of \( k \) consisting of the real primes and the primes of bad reduction for \( C \). Assume that

\[
k_v^* \cap (\cap_{i \in I} N_{i,v}L_{i,v}^*) = k_v^* \cap \text{Nrd} A_v^* \text{ for all } v \in S'.
\]

Then the natural map \( \text{Ker}\Phi_0 \to \prod_v \text{Ker}\Phi_{0,v} \) is injective. Consequently, there exists a canonical isomorphism

\[
\text{III}CH_0(X/C) = \text{III}^1(T).
\]

**Proof.** Let \( S' \) be as in the statement. Then \( H^3_{\text{nr}}(k_v(C)/C_v, \mu_n^{\otimes 2}) = 0 \) for \( v \notin S' \). See [9, Corollary 2.9] and note that \( k_v(C) \) has cohomological dimension 1 if \( v \) is complex. Now the proof of Lemma 2.6 shows that \( \text{Ker}\tilde{\gamma}_v \), and therefore also \( \text{Ker}\tilde{\gamma}_{0,v} \) (see (11)), vanishes if \( v \notin S' \). On the other hand, the hypothesis implies that \( \text{Ker}\tilde{\gamma}_{0,v} = 0 \) for \( v \in S' \). We conclude that \( \prod_v \text{Ker}\tilde{\gamma}_{0,v} = 0 \), and the theorem now follows from Theorem 2.4 and Proposition 2.7. \( \square \)

The next result gives conditions under which the full group \( \text{Ker}\Phi_0 \) (and not just its subgroup \( \text{III}(\text{Ker}\Phi_0) \)) vanishes.

**Theorem 2.9.** Let \( k \) be a number field, let \( A \) be a central simple \( k(C) \)-algebra of squarefree index \( n \) and let \( X \to C \) be an Artin model of \( A \) over \( C \). Assume that the following conditions hold:

i) either \( k \) is totally imaginary or \( n \) is odd, and

ii) \( C \) acquires good reduction at all primes over an extension of \( k \) of degree prime to \( n \).

Then the characteristic map \( \Phi_0 : CH_0(X/C) \to H^1(k, T_0) \) is injective.

**Proof.** Assume first that \( C \) has good reduction at all primes of \( k \). Then \( H^3_{\text{nr}}(k_v(C)/C_v, \mu_n^{\otimes 2}) = 0 \) for all non-archimedean \( v \). See [9, Corollary 2.9]. We conclude (cf. proof of Lemma 2.6) that \( \text{Ker}\tilde{\gamma}_v = 0 \) for all such \( v \). The same conclusion holds if \( v \) is a complex prime (cf. proof of Theorem 2.8). If \( k_v \simeq \mathbb{R} \), then \( H^3_{\text{nr}}(k_v(C)/C_v, \mu_n^{\otimes 2}) \) is annihilated by multiplication by 2 as well as by multiplication by \( n \), so (i) shows that this group is zero as well. We conclude that \( \text{Ker}\tilde{\gamma}_v = 0 \) for all \( v \), whence \( \text{Ker}\tilde{\gamma} = 0 \) by Lemma 2.6. Now (11) shows that \( \text{Ker}\gamma = 0 \), and

⁶As well as a “birational invariance” argument as in the proof of Theorem 2.4.
Lemma 1.2 completes the proof in this case. Now let $M$ be an extension of $k$ of degree prime to $n$ over which $C$ acquires good reduction at all primes. The first part of the proof shows that $H^3_{nr}(M_w(C)/C_w, \mu_n^\otimes) = 0$ for all primes $w$ of $M$, where $C_w = C \otimes_k M_w$. On the other hand, if $v$ is a prime of $k$ and $w$ is a prime of $M$ lying above $v$, the kernel of the restriction map

$$H^3_{nr}(k_v(C)/C_v, \mu_n^\otimes) \to H^3_{nr}(M_w(C)/C_w, \mu_n^\otimes)$$

is annihilated by multiplication by $[M:k]$ as well as by multiplication by $n$. We conclude that $H^3_{nr}(k_v(C)/C_v, \mu_n^\otimes) = 0$ for all $v$, and the proof goes through as before. □

Now let $Br_X/Br_C$ denote $Br_X/\pi_*Br_C$. Combining Theorem 0.4 of [7] (which yields the last three terms of the exact sequence appearing in the next Corollary) and Theorem 2.4, Theorem 2.9 and a “birational invariance” argument, we obtain

**Corollary 2.10.** Let $k$ be a number field and let $X \to C$ be a Severi-Brauer $k$-fibration of squarefree index $n$. Assume that the following conditions hold:

i) either $k$ is totally imaginary or $n$ is odd, and

ii) $C$ acquires good reduction at all primes over an extension of $k$ of degree prime to $n$.

Then there exists a natural exact sequence

$$0 \to III^1(T) \to CH_0(X/C) \to \bigoplus_v CH_0(X_v/C_v) \to Hom(Br_X/Br_C, \mathbb{Q}/\mathbb{Z}),$$

where $T$ is the Néron-Severi torus of $X$.

When $C = \mathbb{P}_k^1$, hypothesis (i) in the statement of Corollary 2.10 may be replaced by the condition: “$X(k_v) \neq \emptyset$ for every real prime $v$ of $k$” (see [6, Corollary 4.6]). Further, $Br_X/Br_C = H^1(k, NS(X))$ (see, for example, [3, p.116]). Thus, we have the following generalization of a well-known theorem of P.Salberger [10, Theorem 7.5, p.520].

**Corollary 2.11.** Let $k$ be a number field and let $X \to \mathbb{P}_k^1$ be a Severi-Brauer $k$-fibration of squarefree index. Assume that $X(k_v) \neq \emptyset$ for every real prime $v$ of $k$. Then there exists a natural exact sequence

$$0 \to III^1(T) \to A_0(X) \to \bigoplus_v A_0(X_v) \to Hom(H^1(k, NS(X)), \mathbb{Q}/\mathbb{Z}),$$

where $T$ is the Néron-Severi torus of $X$. □

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7See also the comments following (0.b) on p.83 of [7].
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