The $\chi$-part of the analytic class number formula, for global function fields.

Stéphane VIGUIÉ *

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Abstract

Let $F/k$ be a finite abelian extension of global function fields, totally split at a distinguished place $\infty$ of $k$. We show that a complex Gras conjecture holds for Stark units, and we derive a refined analytic class number formula.

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1 Introduction

Let $k$ be a global function field with constant field $\mathbb{F}_q$, and let $\infty$ be a distinguished place of $k$. We write $k_\infty$ for the completion of $k$ at $\infty$. For any finite abelian extension $K/k$, let $\mathcal{O}_K$ be the ring of functions of $k$ which are regular outside the places of $K$ sitting above $\infty$. We denote by $\text{Cl}(\mathcal{O}_K)$ the ideal-class group of $\mathcal{O}_K$. If $K \subseteq k_\infty$ we define in section 2 a group $\mathcal{E}_K$ of Stark units, which have finite index in $\mathcal{O}_K^\times$, the group of units of $\mathcal{O}_K$. This group $\mathcal{E}_K$ has already been studied in [3], [4].

We fix a finite abelian extension $F \subseteq k_\infty$ of $k$, with Galois group $G$, and degree $g$. In [9], for every nontrivial irreducible rational character $\psi$ of $G$, we established that

$$\# \left( \mathbb{Z} \otimes \mathbb{Z} \left( \mathcal{O}_F^\times / \mathcal{E}_F \right) \right)_\psi = \# \left( \mathbb{Z} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F) \right)_\psi,$$

(1.1)

where $\mathbb{Z} := \mathbb{Z} \left[ g^{-1} \right]$, and where the index $\psi$ means we take the $\psi$-parts. In [5] we used Euler systems to prove that $\mathcal{E}_F$ satisfies the Gras conjecture,

$$\# \left( \mathbb{Z} \otimes \mathbb{Z} \left( \mathcal{O}_F^\times / \mathcal{E}_F \right) \right)_\psi = \# \left( \mathbb{Z} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F) \right)_\psi,$$

for all prime number $p \nmid qg$ and all irreducible nontrivial $\mathbb{Q}_p$-character $\psi$ (but we were not able to prove the conjecture in the special case where the following conditions are simultaneously satisfied: $p \nmid \text{Cl}(\mathcal{O}_k)$, $\psi$ is a conjugate of the Teichmuller character, $\mu_p \nsubseteq k$ and $\mu_p \subseteq F$, where $\mu_p$ is the group of $p$-th roots of unity in the separable closure of $k$).

Let $\mu_g$ be the group of $g$-th roots of unity in the field of complex numbers. Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}(g)$ in $\mathbb{Q}(\mu_g)$. For any module $M$ over a commutative ring

* S.Viguié, Laboratoire de mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon cedex, France. e-mail: stephane.viguie@univ-fcomte.fr
A, we denote by $\text{Fit}_A(M)$ the Fitting ideal of $M$. Using the properties of Rubin-Stark units stated in [7] and [6], we prove here a complex version of the Gras conjecture for $\mathcal{E}_F$ (Theorem 4.1). More precisely, we prove that for every nontrivial complex character $\chi$ of $G$, we have

$$\text{Fit}_0 \left( \mathcal{O} \otimes \mathbb{Z} \left( \mathcal{O}_F^\chi / \mathcal{E}_F \right) \right)_\chi = \text{Fit}_0 \left( \mathcal{O} \otimes \mathbb{Z} \left( \text{Cl}(\mathcal{O}_F) \right) \right)_\chi,$$

which is a refinement of (1.1). From (1.2), one can easily deduce that the classical Gras conjecture holds for all prime number $p \nmid g$. Thus Theorem 4.1 expands the result obtained in [5], in particular it shows that the conjecture holds for $p$ equal to the characteristic of $k$, and also for the conjugates of the Teichmüller character in the above special case. Let us also mention the analogy between this theorem and a recent result of P. Buckingham (see [1, Theorem 7.1]), who is concerned with Rubin-Stark elements in cyclic extensions of totally real number fields.

Also we can combine this complex Gras conjecture (1.2) with the computations made in [9]. Thus we derive a «χ-part» version of the analytic class number formula (Theorem 4.2), connecting the Fitting ideal of the $\chi$-part of the $\mathcal{O}$-module $\mathcal{O} \otimes \mathbb{Z} \left( \text{Cl}(\mathcal{O}_F) \right)$ to the value at 0 of an $L$-function attached to $\overline{\chi}$, for any non trivial irreducible complex character $\chi$ of $G$.

### 2 Stark units in function fields.

Let $d$ be the degree of $\infty$ over $\mathbb{F}_q$. If $\mathfrak{m}$ is a nonzero ideal of $\mathcal{O}_k$, then we denote by $H_\mathfrak{m} \subseteq k_\infty$ the maximal abelian extension of $k$ contained in $k_\infty$, such that the conductor of $H_\mathfrak{m}/k$ divides $\mathfrak{m}$. The function field version of the abelian conjectures of Stark, proved by P. Deligne in [8] by using étale cohomology or by D. Hayes in [2] by using Drinfel’d modules, claims that, for any proper nonzero ideal $\mathfrak{m}$ of $\mathcal{O}_k$, there exists an element $\varepsilon_\mathfrak{m} \in H_\mathfrak{m}$, unique up to roots of unity, such that

1. The extension $H_\mathfrak{m} \left( \varepsilon_\mathfrak{m}^{1/w_\infty} \right)/k$ is abelian, where $w_\infty := q^d - 1$. Moreover, it is unramified outside $S_\mathfrak{m}$, where $S_\mathfrak{m}$ is the set containing $\infty$ and the places of $k$ which divide $\mathfrak{m}$.

2. If $\mathfrak{m}$ is divisible by two prime ideals then $\varepsilon_\mathfrak{m}$ is a unit of $\mathcal{O}_{H_\mathfrak{m}}$. If $\mathfrak{m} = q^e$, where $q$ is a prime ideal of $\mathcal{O}_k$ and $e$ is a positive integer, then

$$\varepsilon_\mathfrak{m} \mathcal{O}_{H_\mathfrak{m}} = (q)^{w_k} \left( \frac{w_k}{m} \right),$$

where $w_k := q - 1$ and $(q)_m$ is the product of the prime ideals of $\mathcal{O}_{H_\mathfrak{m}}$ which divide $q$.

3. We have

$$L_\mathfrak{m}(0, \chi) = \frac{1}{w_\infty} \sum_{\sigma \in \text{Gal}(H_\mathfrak{m}/k)} \chi(\sigma)v_\infty(\varepsilon_m^\sigma)^{-1}$$

for all complex irreducible characters of $\text{Gal}(H_\mathfrak{m}/k)$, where $v_\infty$ is the normalized valuation of $k_\infty$.

Let us recall that $s \mapsto L_\mathfrak{m}(s, \chi)$ is the $L$-function associated to $\chi$, defined for the complex numbers $s$ such that $\text{Re}(s) > 1$ by the Euler product

$$L_\mathfrak{m}(s, \chi) = \prod_{v \mid \mathfrak{m}} (1 - \chi(\sigma_v)N(v)^{-s})^{-1},$$

for all complex irreducible characters of $\text{Gal}(H_\mathfrak{m}/k)$, where $v_\infty$ is the normalized valuation of $k_\infty$.  

Let us also mention the analogy between this theorem and a recent result of P. Buckingham (see [1, Theorem 7.1]), who is concerned with Rubin-Stark elements in cyclic extensions of totally real number fields.
where \( v \) describes the set of places of \( k \) not dividing \( m \). For such a place, \( \sigma_v \) and \( N(v) \) are the Frobenius automorphism of \( H_m/k \) and the order of the residue field at \( v \) respectively.

Let us remark that \( \sigma_\infty = 1 \) and \( N(\infty) = q^d \).

For any finite abelian extension \( L \) of \( k \) we denote by \( \mathcal{J}_L \subseteq \mathbb{Z}[\text{Gal}(L/k)] \) the annihilator of \( \mu(L) \), the group of roots of unity in \( L \). The description of \( \mathcal{J}_L \) given in [2, Lemma 2.5] and the property \( i) \) of \( \varepsilon_m \) implies that for any \( \eta \in \mathcal{J}_{H_m} \) there exists \( \varepsilon_m(\eta) \in H_m \) such that

\[ \varepsilon_m(\eta)^{\infty} = \varepsilon_m^\eta. \quad (2.2) \]

**Definition 2.1** Let \( \mathcal{P}_F \) be the subgroup of \( F^\times \) generated by \( \mu(F) \) and by all the norms

\[ \varepsilon_{F, m}(\eta) := N_{H_m/H_m \cap F}(\varepsilon_m(\eta)), \]

where \( m \) is any nonzero proper ideal of \( \mathcal{O}_k \), and \( \eta \in \mathcal{J}_{H_m} \). Then we set

\[ \mathcal{E}_F := \mathcal{P}_F \cap \mathcal{O}_F^\times. \]

### 3 Preliminary lemmas.

For any finite group \( H \), let us denote by \( \hat{H} \) the group of complex irreducible characters of \( H \). Then for every \( \chi \in \hat{H} \) we set \( e_\chi := \frac{1}{\#H} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} \). In case \( H = G \), then \( e_\chi \) belongs to \( \mathcal{O}[G] \). Moreover, if \( \zeta \in \mu_g \) is such that \( \zeta \neq 1 \), then \( (1 - \zeta) \in \mathcal{O}^\times \), thanks to the formula

\[ g = \prod_{\substack{\zeta \in \mu_g \\
\zeta \neq 1}} (1 - \zeta). \]

For \( K/k \) a finite abelian extension, and \( S \neq \emptyset \) a finite set of places of \( k \), let \( \mathcal{O}_{K,S} \) be the Dedekind ring of the \( S \)-integers of \( K \), i.e the functions \( f \in K \) which only poles are at the places sitting over \( S \). Let \( \text{Cl}(\mathcal{O}_{K,S}) \) be the ideal class group of \( \mathcal{O}_{K,S} \).

**Lemma 3.1** Let \( m \) be a nonzero ideal of \( \mathcal{O}_k \), and let \( \chi \in \hat{G}, \chi \neq 1 \). Assume that for every prime ideal \( \mathfrak{p} \) of \( \mathcal{O}_k \) which divides \( m \), \( \chi \) is not trivial on the decomposition group \( D_{\mathfrak{p}} \) of \( \mathfrak{p} \) in \( F/k \). Let \( S_m \) be the set of places of \( k \) which contains \( \infty \) and all the prime divisors of \( m \). Then in the category of \( \mathcal{O}[G] \)-modules, we have

\[ (0 \otimes_{\mathbb{Z}} \mathcal{O}^\times_{F,S_m})_\chi = (0 \otimes_{\mathbb{Z}} \mathcal{O}^\times_F)_\chi \quad \text{and} \quad (0 \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_{F,S_m}))_\chi \simeq (0 \otimes_{\mathbb{Z}} \text{Cl}(\mathcal{O}_F))_\chi. \]

**Proof.** Let \( S \) be the \( \mathbb{Z}[G] \)-module generated by the prime ideals of \( \mathcal{O}_F \) dividing \( m \mathcal{O}_F \). Let \( \mathcal{S} \) be the image of \( S \) in \( \text{Cl}(\mathcal{O}_F) \). We have the following exact sequences:

\[
\begin{array}{c}
0 \longrightarrow \mathcal{S} \longrightarrow \text{Cl}(\mathcal{O}_F) \longrightarrow \text{Cl}(\mathcal{O}_{F,S_m}) \longrightarrow 0, \\
0 \longrightarrow \mathcal{O}_F^\times \longrightarrow \mathcal{O}^\times_{F,S_m} \longrightarrow \mathcal{S}, \end{array}
\]

where for all \( \mathfrak{p} \in S \), \( v_\mathfrak{p} \) is the normalized valuation at \( \mathfrak{p} \). Since \( \mathcal{O} \) is \( \mathbb{Z} \)-flat, all we have to show is \( (0 \otimes_{\mathbb{Z}} S)_\chi = 0 \). Let \( \mathfrak{p} \in S \). There is \( \gamma \in G \) such that \( \mathfrak{p}^\gamma = \mathfrak{p} \) and \( \chi(\gamma) \neq 1 \). Then \( (\chi(\gamma) - 1)e_\chi(1 \otimes \mathfrak{p}) = e_\chi(1 \otimes \mathfrak{p}^\gamma) = 0 \), with \( (\chi(\gamma) - 1) \in \mathbb{Z}^\times \), hence \( e_\chi(1 \otimes \mathfrak{p}) = 0 \). \( \square \)

For \( K \subseteq k_{\infty} \) a finite abelian extension of \( k \), let \( \ell_K : K^\times \rightarrow \mathbb{Z}[\text{Gal}(K/k)] \) be the \( \text{Gal}(K/k) \)-equivariant map defined by

\[
\ell_K(x) := \sum_{\sigma \in \text{Gal}(K/k)} v_\infty(x^\sigma) \sigma^{-1}.
\]
Let $\varepsilon$ be a nonzero ideal of $\mathcal{O}_k$, and $\alpha \in J_{H_m}$. We set $C_m := \text{Cl} (\mathcal{O}_{F \cap H_m, S_m})$. Then

$$\ell_F (\varepsilon_{F,m} (\alpha)) \in \text{Fit}_{Z_g} (C_m) \ell_F (O_{F \cap H_m, S_m}).$$

Proof. By the description of $J_{H_m}$ given in [2], Lemma 2.5, we know that there are a finite set $T$ of nonzero prime ideals of $\mathcal{O}_k$, and a family $(\alpha_p)_{p \in T} \in Z [\text{Gal} (H_m/k)]^T$, such that $S_m \cap T = \emptyset$ and $\alpha = \sum_{p \in T} \alpha_p \left(1 - N(p)\sigma_p^{-1}\right)$, where $\sigma_p$ is the Frobenius of $p$ in $H_m/k$. It suffices to show $\ell_F (\varepsilon_{F,m} (1 - N(p)\sigma_p^{-1})) \in \text{Fit}_{Z_g} (C_m) \ell_F (O_{F \cap H_m, S_m})$, for a fixed nonzero prime ideal $p$ of $\mathcal{O}_k$, with $p \notin S_m$.

For any abelian extension $K$ of $k$, we denote by $U_{K,m,p}$ the group of units of $\mathcal{O}_{K,S_m}$ which are congruent to 1, modulo all primes above $p$. From (2.2) we deduce that $\varepsilon_m (1 - N(p)\sigma_p^{-1})$ can be chosen such that $\varepsilon_m (1 - N(p)\sigma_p^{-1}) \in U_{H_m,m,p}$. For $\chi \in \text{Gal} (H_m/k)$, we define the meromorphic function $s \mapsto L_{m,p}(s, \chi)$ by

$$L_{m,p}(s, \chi) := (1 - N(\infty)^{-s}) \cdot L_m(s, \chi) \cdot (1 - N(p)^{-s} \chi(\sigma_p)).$$

Derivating, and using the property iii) of Stark units, we obtain

$$L'_m(0, \chi) = d \cdot \ln(q) L_m(0, \chi) \cdot (1 - N(p)\chi(\sigma_p^{-1})) = d \cdot \ln(q) \cdot \chi \left(\ell_{H_m} (\varepsilon_m (1 - N(p)\sigma_p^{-1}))\right),$$

where $\chi$ is extended to $\mathbb{C} [\text{Gal} (H_m/k)]$ by linearity. For $s \in \mathbb{C}$, we set

$$\Theta_{m,p}(s) = \sum_{\chi \in \text{Gal} (H_m/k)} L_{m,p}(s, \chi) \cdot e_{\chi} \quad \text{in} \quad \mathbb{C} [\text{Gal} (H_m/k)],$$

wherever it is defined. From (3.1), we have

$$\Theta'_{m,p}(0) = d \cdot \ln(q) \cdot \chi \left(\ell_{H_m} (\varepsilon_m (1 - N(p)\sigma_p^{-1}))\right) = - \sum_{\gamma \in \text{Gal} (H_m/k)} \ln \left(\left|\varepsilon_m (1 - N(p)\sigma_p^{-1})\gamma^{-1}\right|_\infty\right) \gamma.$$ 

We have just verified that $\varepsilon_m (1 - N(p)\sigma_p^{-1})$ satisfies [6, Theorem 0], for $(H_m, S_m, \{p\}, 1)$. Then $\varepsilon_{F,m} (1 - N(p)\sigma_p^{-1})$ satisfies [6, Theorem 0], for $(H_m \cap F, S_m, \{p\}, 1)$, and we have

$$\ell_F (\varepsilon_{F,m} (1 - N(p)\sigma_p^{-1})) \in \text{Fit}_{Z_g} (Z_g \otimes \mathbb{Z} \text{Cl} (\mathcal{O}_{F \cap H_m, S_m}, p)) \ell_F (U^\times_F \cap (\mathcal{O}_{F \cap H_m, S_m}, p)), $$

where $\text{Cl} (\mathcal{O}_{F \cap H_m, S_m}, p)$ is the quotient of fractional ideals of $\mathcal{O}_{F \cap H_m}$ by those principal fractional ideals, which are generated by an element congruent to 1 modulo $p$. But $Z_g \otimes \mathbb{Z} C_m$ is a quotient of $Z_g \otimes \mathbb{Z} \text{Cl} (\mathcal{O}_{F \cap H_m, S_m}, p)$, and $Z_g [G]$ is a finite product of Dedekind rings, so

$$\text{Fit}_{Z_g} (Z_g \otimes \mathbb{Z} \text{Cl} (\mathcal{O}_{F \cap H_m, S_m}, p)) \subseteq \text{Fit}_{Z_g} (Z_g \otimes \mathbb{Z} C_m),$$

and we are done. \hfill \square

**Definition 3.1** Let $\Omega$ be the $\mathbb{Z}[G]$-submodule of $F^\times$ generated by $\mu(F)$ and by the elements $\varepsilon_{F,m} := N_{H_m/F \cap H_m} (\varepsilon_m)$, where $m$ is any nonzero proper ideal of $\mathcal{O}_k$. 


For any $\chi \in \hat{G}$, let $F_\chi$ be the subfield of $F$ fixed by $\text{Ker}(\chi)$, and $f_\chi$ be the conductor of $F_\chi$. $\chi_{\text{pr}}$ denotes the character of $\text{Gal}(H_\chi/k)$ defined by $\chi$. Assume that $\chi$ is nontrivial. Then by [9, Proposition 3.1], we have

$$(\mathcal{O} \ell_F(\Omega))_\chi = \mathcal{O}w_{\infty}L_{f_\chi}(0, \chi_{\text{pr}})e_\chi.$$  \hspace{1cm} (3.2)

Moreover one can easily relate $\ell_F(\Omega)$ to $\ell_F(\mathcal{E}_F)$ (see [9, (3.13)]). If $\chi \neq 1$, we have

$$(\mathcal{O}\ell_F(\mathcal{E}_F))_\chi = (\mathcal{O}w_{\infty}^{-1}J_F\ell_F(\Omega))_\chi.$$  \hspace{1cm} (3.3)

**Lemma 3.3** Let $\chi \in \hat{G}$, $\chi \neq 1$. There is a nonzero ideal $m$ of $\mathcal{O}_k$, satisfying the following properties:

i) $m$ is divisible by at least two distinct prime ideals,

ii) $F_\chi \subseteq H_m$,

iii) for each prime ideal $p$ which divides $m$, $\chi$ is not trivial on the decomposition group $D_p$ of $p$ in $F/k$.

iv) As an $\mathcal{O}$-module, $(\mathcal{O} \otimes F, \chi)$ is generated by $(\mathcal{O} \otimes \mu(F))_\chi$ and by the elements $e_\chi(1 \otimes \mathcal{E}_m(\alpha))$, where $\alpha \in J_m$.

**Proof.** Since $\chi \neq 1$, we can find two distinct nonzero prime ideals $p$ and $q$ of $\mathcal{O}_k$, unramified in $F/k$, such that

$$\chi_{\text{pr}}(\sigma_p) \neq 1 \quad \text{and} \quad \chi_{\text{pr}}(\sigma_q) \neq 1.$$  

We set $m = f_\chi pq$. Obviously, conditions i), ii) and iii) are satisfied. As in the proof of [9, Proposition 3.1], from the property iii) of Stark units we obtain

$$\ell_F(\mathcal{E}_m) e_\chi = [F : F \cap H_m]w_{\infty}L_{f_\chi}(0, \chi_{\text{pr}}) (1 - \chi_{\text{pr}}(\sigma_p)) (1 - \chi_{\text{pr}}(\sigma_q)) e_\chi.$$  

Since $[F : F \cap H_m]$, $1 - \chi_{\text{pr}}(\sigma_p)$ and $1 - \chi_{\text{pr}}(\sigma_q)$ belong to $\mathcal{O}^\times$, and by (3.2), we have

$$\mathcal{O}\ell_F(\mathcal{E}_m) e_\chi = \mathcal{O}w_{\infty}L_{f_\chi}(0, \chi_{\text{pr}})e_\chi = \mathcal{O}\ell_F(\Omega)e_\chi.$$  \hspace{1cm} (3.4)

From (3.4) and (3.3), we deduce

$$\mathcal{O}J_Fw_{\infty}^{-1}\ell_F(\mathcal{E}_m) e_\chi = \mathcal{O}\ell_Fw_{\infty}^{-1}\ell_F(\Omega)e_\chi = \mathcal{O}\ell_F(\mathcal{E}_F)e_\chi,$$

and condition iv) follows. \hfill \Box

To go further we need some preliminary remarks.

**Remark 3.1** For any $\mathcal{O}[G]$-module $M$, we have $\text{Fit}_{\mathcal{O}[G]}(M) = \sum_{\chi \in \hat{G}} \text{Fit}_{\mathcal{O}}(M_\chi)e_\chi$.

**Remark 3.2** Let $H$ be a sub-group of $G$. Let $M$ and $N$ be two $G$-modules, and $\psi : M \to N$ a $G$-equivariant map. If $\text{Cok}(\Psi) := N/\text{Im}(\Psi)$ is annihilated by $\#(H)$ then we derive from $\Psi$ a surjective map

$$\Psi_\mathcal{O} : \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \otimes \mathcal{O} N.$$  

Let us assume, in addition, that $\text{Ker}(\Psi)$ is annihilated by $\Sigma\sigma$, $\sigma \in H$. Then, for every $\chi \in \hat{G}$ trivial on $H$, the restriction of $\Psi_\mathcal{O}$ gives an isomorphism

$$(\mathcal{O} \otimes \mathcal{O} M)_\chi \simeq (\mathcal{O} \otimes \mathcal{O} N)_\chi.$$
As a particular case, for any subextension \( K/k \) of \( F/k \) and \( H = \text{Gal}(F/K) \), any nonzero ideal \( m \) of \( \mathcal{O}_k \), the norm maps give isomorphisms

\[
(\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_{F,S}^m))_\chi \sim (\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{F,S}^m)_\chi \quad \text{and} \quad (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_{K,S}^m))_\chi \sim (\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{K,S}^m)_\chi,
\]

for any \( \chi \in \hat{G} \) which is trivial on \( H \). Since \( \#(H) \in \mathcal{O}^\times \), we deduce that for such a character, the canonical inclusion also gives an equality

\[
(\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{K,S}^m)_\chi = (\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{F,S}^m)_\chi.
\]

**Proposition 3.1** Let \( \chi \in \hat{G}, \chi \neq 1 \). We have

\[
(\mathcal{O} \ell_F (\epsilon_F))_\chi \subseteq \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_\chi \right) \left( (\mathcal{O} \ell_F (\mathcal{O}_F^m))_\chi \right).
\]

**Proof.** We choose an ideal \( m \) of \( \mathcal{O}_k \) satisfying the four conditions of Lemma 3.3. Because of Lemma 3.3, iv), it is sufficient to show that for \( \alpha \in \mathcal{J}_{H_m} \), we have

\[
\ell_F(\epsilon_{F,m}(\alpha)) e_\chi \in \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_\chi \right) \left( \ell_F (\mathcal{O}_F^m) \right)_\chi.
\]

By Lemma 3.2, we know that

\[
\ell_F(\epsilon_{F,m}(\alpha)) \in \text{Fit}_{\mathbb{Z}[G]} (C_m) \ell_F (\mathcal{O}_{F \cap H_m,S}^m).
\]

From Remark 3.1, we deduce

\[
\ell_F(\epsilon_{F,m}(\alpha)) e_\chi \in \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} C_m)_\chi \right) \left( \ell_F (\mathcal{O}_{F \cap H_m,S}^m) \right)_\chi.
\]

By Lemma 3.3, ii), and Remark 3.2, the norm map defines an isomorphism

\[
(\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_{F,S}^m))_\chi \simeq (\mathcal{O} \otimes \mathbb{Z} C_m)_\chi,
\]

and the canonical inclusion \( F \cap H_m \hookrightarrow F \) gives

\[
(\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{F \cap H_m,S}^m)_\chi = (\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{F,S}^m)_\chi.
\]

Then

\[
\ell_F(\epsilon_{F,m}(\alpha)) e_\chi \in \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_{F,S}^m))_\chi \right) \left( \ell_F (\mathcal{O}_{F,S}^m) \right)_\chi.
\]

From Lemma 3.1, and Lemma 3.3, iii), we know that

\[
\text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_{F,S}^m))_\chi \right) = \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_\chi \right),
\]

and

\[
(\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_{F,S}^m)_\chi = (\mathcal{O} \otimes \mathbb{Z} \mathcal{O}_F^m)_\chi.
\]

The proposition follows. \( \square \)
4 Statement and proof of the theorems.

Theorem 4.1 Let $\chi \in \hat{G}$, $\chi \neq 1$. We have

$$\text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F))_{\chi} \right) = \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\chi} \right).$$

Proof. We have

$$\prod_{\xi \in \hat{G}} \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F))_{\xi} \right) = \text{Fit}_0 \left( (1 - e_1) \mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F) \right)$$

$$= 0 \# \left( (1 - e_1) \mathcal{Z}(g) \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F) \right). \quad (4.1)$$

In the same way, we have

$$\prod_{\xi \in \hat{G}} \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\xi} \right) = \text{Fit}_0 \left( \mathcal{Z}(g) \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F) \right)$$

$$= 0 \left[ \mathcal{Z}(g) \otimes \mathcal{Z} \mathcal{O}_F^x : \mathcal{Z}(g) \otimes \mathcal{Z} \mathcal{E}_F \right]. \quad (4.2)$$

By (1.1), we know that

$$\# \left( (1 - e_1) \mathcal{Z}(g) \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F) \right) = \left[ \mathcal{Z}(g) \otimes \mathcal{Z} \mathcal{O}_F^x : \mathcal{Z}(g) \otimes \mathcal{Z} \mathcal{E}_F \right]. \quad (4.3)$$

Putting (4.1), (4.3) and (4.2) together, we obtain

$$\prod_{\xi \in \hat{G}} \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F))_{\xi} \right) = \prod_{\xi \in \hat{G}} \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\xi} \right).$$

From this equality, it follows that it is sufficient to show the divisibility

$$\text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F))_{\xi} \right) | \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\xi} \right), \quad (4.4)$$

for all $\xi \in \hat{G}$. Since $\mathcal{O}_F^x \cap \text{Ker}(\ell_F) = \mu(F)$, we have $\mathcal{O}_F^x / \mathcal{E}_F \simeq \ell_F (\mathcal{O}_F^x) / \ell_F (\mathcal{E}_F)$ and

$$(\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\xi} \simeq (\ell_F (\mathcal{O}_F^x) / (\ell_F (\mathcal{E}_F)))_{\xi}. \quad (4.5)$$

We set $\mathcal{F}_\xi := \text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} \text{Cl}(\mathcal{O}_F))_{\xi} \right)$ for convenience. From Proposition 3.1 we derive the tautological exact sequence

$$0 \to \mathcal{F}_\xi \left( \ell_F (\mathcal{O}_F^x) \right)_{\xi} \to \ell_F (\mathcal{O}_F^x)_{\xi} \to \mathcal{F}_\xi (\ell_F (\mathcal{O}_F^x))_{\xi} \to 0. \quad (4.6)$$

Since $\mathcal{O}$ is a Dedekind ring, we deduce from (4.6) and (4.5) that

$$\text{Fit}_0 \left( (\mathcal{O} \otimes \mathcal{Z} (\mathcal{O}_F^x / \mathcal{E}_F))_{\xi} \right) = \text{Fit}_0 \left( \frac{\mathcal{F}_\xi (\ell_F (\mathcal{O}_F^x))_{\xi}}{\ell_F (\mathcal{O}_F^x)_{\xi}} \right),$$

$$= \text{Fit}_0 \left( \frac{\mathcal{F}_\xi (\ell_F (\mathcal{O}_F^x))_{\xi}}{\ell_F (\mathcal{O}_F^x)_{\xi}} \right) \mathcal{F}_\xi. \quad (4.7)$$
Before we go further, we state here the definition and basic properties of index-modules, which we introduced in [9]. We refer the reader to [9] for the proofs.

Let $K$ be a commutative field, $A \subseteq K$ be a Dedekind ring and $V$ be a $K$-vector space. By an $A$-lattice of $V$, we mean a finitely generated $A$-submodule $R$ of $V$, such that the $K$-vector subspace of $V$ generated by $R$, denoted by $KR$, has dimension equal to the $A$-rank of $R$.

**Definition 4.1** Let $R \neq 0$ and $S$ be $A$-lattices of $V$. We call $A$-index-module of the couple $(R, S)$ the set

$$[R : S]_A := \{ \det(u) ; u \in \text{End}_K(V')/u(R) \subseteq S \},$$

where $V'$ is the $K$-subspace of $V$ generated by $R$ and $S$, $V' = KR + KS$.

In fact, $[R : S]_A$ is an $A$-submodule of $K$, and we have the following properties, for any $A$-lattices $R, S$, and $T$ of $V$, with $R \neq 0$.

i) $[R : R]_A = A$.

ii) If $KS \subseteq KR$, then $[R : S]_A$ is a finitely generated $A$-submodule of $K$. Moreover, its $A$-rank is 1 if $KR = KS$, and $[R : S]_A = 0$ if $KS \subsetneq KR$.

iii) Assume that $KR = KS$, and that $\mathfrak{r}$ and $\mathfrak{s}$ are two nonzero fractional ideals of $A$. Then $[\mathfrak{r}R : \mathfrak{s}S]_A = \mathfrak{s}^d \mathfrak{r}^{-d} [R : S]_A$, where $d$ is the common $A$-rank of $R$ and $S$.

iv) If $S \neq 0$, and $KT \subseteq KS \subseteq KR$, then $[R : T]_A = [R : S]_A [S : T]_A$.

v) If $S \subseteq R$, then $[R : S]_A = \text{Fit}_A(R/S)$.

In the sequel, we are concerned with the following situation. $A := \mathcal{O}$, $V := \mathbb{C}[G]$, and the $\mathcal{O}$-lattices are $(\mathcal{O} \ell_F(\mathcal{O}_F^\chi))_\chi$, $(\mathcal{O} \ell_F(\mathcal{E}_F))_\chi$, $(\mathcal{O} \ell_F(\Omega))_\chi$, ..., where $\chi$ is a nontrivial complex character of $G$. They are all of $\mathcal{O}$-rank 1.

**Lemma 4.1** Let $\chi \in \hat{G}$, $\chi \neq 1$. We have

$$\left[(\mathcal{O} \ell(\Omega))_\chi : (\mathcal{O} \ell(\mathcal{E}_F))_\chi \right]_\mathcal{O} = w^{-1}_\infty \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right),$$

where $\left[(\mathcal{O} \ell(\Omega))_\chi : (\mathcal{O} \ell(\mathcal{E}_F))_\chi \right]_\mathcal{O}$ and $w^{-1}_\infty \text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right)$ are viewed as $\mathcal{O}$-submodules of $\mathbb{Q}(\mu_g)$.

**Proof.** Let $\zeta$ be a primitive $w_F$-th root of unity in $F$, where $w_F := \#(\mu(F))$. We have an obvious exact sequence,

$$0 \rightarrow \mathcal{O} \mathcal{J}_F e_\chi \rightarrow \mathcal{O} e_\chi \rightarrow (\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \rightarrow 0,$$

$$\alpha e_\chi \rightarrow e_\chi (\alpha \otimes \zeta).$$

Using the property $v)$ of index-modules, we deduce

$$\text{Fit}_\mathcal{O} \left((\mathcal{O} \otimes_{\mathbb{Z}} \mu(F))_\chi \right) = [\mathcal{O} e_\chi : \mathcal{O} \mathcal{J}_F e_\chi]_\mathcal{O}. \quad (4.8)$$
Also, using (3.3), we have
\[
\left[ (\mathcal{O} \ell_F(\Omega))_x : (\mathcal{O} \ell_F(\mathcal{E}_F))_x \right]_0 = \left[ (\mathcal{O} \ell_F(\Omega))_x : (\mathcal{O} \mathcal{J}_F w^{-1} \ell_F(\Omega))_x \right]_0.
\]
By (3.2), the property \(iii\) of index-modules, and (4.8) for the last equality, we deduce
\[
\left[ (\mathcal{O} \ell_F(\Omega))_x : (\mathcal{O} \ell_F(\mathcal{E}_F))_x \right]_0 = \left[ \mathcal{O} w_\infty \mathcal{L}_F(0, \chi pr) e_x : (\mathcal{O} \mathcal{J}_F \mathcal{L}_F(0, \chi pr) e_x)_0 \right]
= w_\infty^{-1} \left[ \mathcal{O} e_x : (\mathcal{O} \mathcal{J}_F e_x)_0 \right]
= w_\infty^{-1} \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \mu(F))_x \right).
\]

The regulator \(\text{R}(\mathcal{O}_F)\) of \(\mathcal{O}_F\) is known to be equal to \(\mathbb{Z}[G]_0 : \ell_F(\mathcal{O}_F^\mathfrak{c})\), where \(\mathbb{Z}[G]_0\) is the augmentation ideal of \(\mathbb{Z}[G]\). Hence it is natural to take \(\text{R}(\mathcal{O}_F)_x := \left[ \mathcal{O} e_x : \ell_F(\mathcal{O}_F^\mathfrak{c}) \right]_0\) as a definition for the «\(\chi\)-part» of the regulator, for any nontrivial character \(\chi \in \hat{G}\).

**Theorem 4.2** Let \(\chi \in \hat{G}, \chi \neq 1\). Then we have
\[
\text{R}(\mathcal{O}_F)_x \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_x \right) = \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \mu(F))_x \right) \mathcal{L}_F(0, \chi pr),
\]
where \(\text{R}(\mathcal{O}_F)_x \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_x \right)\) and \(\text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \mu(F))_x \right) \mathcal{L}_F(0, \chi pr)\) are viewed as \(\mathcal{O}\)-submodules of \(\mathcal{Q}(\mu_g)\).

**Proof.** We keep the notation \(\mathcal{F}_x := \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_x \right)\). From Theorem 4.1, (4.5), and the property \(v\) of index-modules, we have
\[
\mathcal{F}_x = \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F^\mathfrak{c}))_x \right) = \left[ (\mathcal{O} \ell_F(\mathcal{O}_F))_x : (\mathcal{O} \ell_F(\mathcal{E}_F))_x \right]_0. \tag{4.9}
\]
By the property \(iv\) of index-modules, we deduce from (4.9) the decomposition
\[
\text{R}(\mathcal{O}_F)_x \mathcal{F}_x = \left[ \mathcal{O} e_x : (\mathcal{O} \ell_F(\Omega))_x \right]_0 \left[ (\mathcal{O} \ell_F(\Omega))_x : (\mathcal{O} \ell_F(\mathcal{E}_F))_x \right]_0. \tag{4.10}
\]
By (3.2), and the property \(iii\) of index-modules, we have
\[
\left[ \mathcal{O} e_x : (\mathcal{O} \ell_F(\Omega))_x \right]_0 = \mathcal{O} w_\infty \mathcal{L}_F(0, \chi pr). \tag{4.11}
\]
From (4.10), (4.11), and Lemma 4.1, we obtain
\[
\text{R}(\mathcal{O}_F)_x \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \text{Cl}(\mathcal{O}_F))_x \right) = \mathcal{L}_F(0, \chi pr) \text{Fit}_0 \left( (\mathcal{O} \otimes \mathbb{Z} \mu(F))_x \right). \]

□
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