Essential cohomology for elementary abelian $p$-groups

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For an odd prime $p$ the cohomology ring of an elementary abelian $p$-group is polynomial tensor exterior. We show that the ideal of essential classes is the Steenrod closure of the class generating the top exterior power. As a module over the polynomial algebra, the essential ideal is free on the set of M"{u}i invariants.

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1. Introduction

Let $G$ be a finite group and $k$ a field whose characteristic $p$ divides the order of $G$. A cohomology class $x \in H^n(G, k)$ is called essential if its restriction $\text{Res}_H(x)$ is zero for every proper subgroup $H$ of $G$. The essential classes form an ideal, called the essential ideal and denoted by $\text{Ess}(G)$. It is standard that restriction to a Sylow $p$-subgroup of $G$ is a split injection (see for example Theorem XII.10.1 of [1]), and so the essential ideal can only be non-zero if $G$ is a $p$-group. Many $p$-groups have non-zero essential ideal, for instance the quaternion group of order eight. The essential ideal plays an important role and has therefore been the subject of many studies: two such being Carlson’s work on the depth of a cohomology ring [2], and the cohomological characterization due to Adem and Karagueuzian of those $p$-groups whose order $p$ elements are all central [3].

The nature of the essential ideal depends crucially on whether or not the $p$-group $G$ is elementary abelian. If $G$ is not elementary abelian, then a celebrated result of Quillen (Theorem 7.1 of [4]) implies that $\text{Ess}(G)$ is a nilpotent ideal. By contrast, the essential ideal of an elementary abelian $p$-group contains non-nilpotent classes. Work to date on the essential ideal has concentrated on the non-elementary abelian case. In this paper we give a complete treatment of the outstanding elementary abelian case. As we shall recall in the next section, the case $p = 2$ is straightforward and well known. So we shall concentrate on the case of an odd prime $p$.

So let $p$ be an odd prime and $V$ a rank $n$ elementary abelian $p$-group. We may equally well view $V$ as an $n$-dimensional $\mathbb{F}_p$-vector space. Recall that the cohomology ring has the form

$$H^*(V, \mathbb{F}_p) \cong S(V^*) \otimes_{\mathbb{F}_p} A(V^*),$$

where the exterior copy of the dual space $V^*$ is $H^1(V, \mathbb{F}_p)$, and the polynomial copy lies in $H^2(V, \mathbb{F}_p)$: specifically, the polynomial copy is the image of the exterior copy under the Bockstein boundary map $\beta$. Our first result is as follows:

**Theorem 1.1.** Let $p$ be an odd prime and $V$ a rank $n$ elementary abelian $p$-group. Then the essential ideal $\text{Ess}(V)$ is the Steenrod closure of $\Lambda^n(V^*)$. That is, $\text{Ess}(V)$ is the smallest ideal in $H^*(V, \mathbb{F}_p)$ which contains the one-dimensional space $\Lambda^n(V^*) \subseteq H^n(V, \mathbb{F}_p)$ and is closed under the action of the Steenrod algebra.
Our second result concerns the structure of \( \text{Ess}(V) \) as a module over the polynomial subalgebra \( S(V^*) \) of \( H^*(V, \mathbb{F}_p) \). It was conjectured by Carlson (Question 5.4 in [5]) – and earlier in a less precise form by Mùi [6] – that the essential ideal of an arbitrary \( p \)-group is free and finitely generated as a module over a certain polynomial subalgebra of the cohomology ring. In [7], the second author demonstrated finite generation, and for most \( p \)-groups of a given order was able to prove freeness as well: specifically the method works provided the group is not a direct product in which one factor is elementary abelian of rank at least two. Our second result states that Carlson’s conjecture holds for elementary abelian \( p \)-groups too, and gives explicit free generators.

**Theorem 1.2.** Let \( p \) be an odd prime and \( V \) a rank \( n \) elementary abelian \( p \)-group. Then as a module over the polynomial part \( S(V^*) \) of the cohomology ring \( H^*(V, \mathbb{F}_p) \), the essential ideal \( \text{Ess}(V) \) is free on the set of Mùi invariants, as defined in **Definition 3.3**.

Structure of the paper. In Section 2 we briefly cover the well-known case \( p = 2 \). We introduce the Mùi invariants in Section 3. After proving **Theorem 1.2** in Section 4 we consider the action of the Steenrod algebra on the Mùi invariants in order to prove **Theorem 1.1** in Section 5.

### 2. Elementary abelian \( p \)-groups and the case \( p = 2 \)

The cohomology group \( H^1(G, \mathbb{F}_p) \) may be identified with the set of group homomorphisms \( \text{Hom}(G, \mathbb{F}_p) \). This set is an \( \mathbb{F}_p \)-vector space, and – assuming that \( G \) is a \( p \)-group – the maximal subgroups of \( G \) are in bijective correspondence with the one-dimensional subspaces: the maximal subgroup corresponding to \( \alpha : G \to \mathbb{F}_p \) being \( \ker(\alpha) \). Of course, the cohomology class \( \alpha \in H^1(G, \mathbb{F}_p) \) has zero restriction to the maximal subgroup \( \ker(\alpha) \). Note that in order to determine \( \text{Ess}(G) \) it suffices to consider restrictions to maximal subgroups.

**Definition.** Denote by \( L_n \) the polynomial

\[
L_n(x_1, \ldots, x_n) = \det \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^p & x_2^p & \cdots & x_n^p \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{vmatrix} \in \mathbb{F}_p[x_1, \ldots, x_n].
\]

There is a well-known alternative description of \( L_n \).

**Lemma 2.1.** \( L_n \) is the product of all monic linear forms in \( x_1, \ldots, x_n \). So for an \( n \)-dimensional \( \mathbb{F}_p \)-vector space \( V \) we may define \( L_n(V) \in S(V^*) \) up to a non-zero scalar multiple by

\[
L_n(V) = \prod_{[x] \in V^*} x.
\]

**Proof.** First part: Here we call a linear form monic if the first non-zero coefficient is one. The right hand side divides the left. Both sides have the same total degree. And the coefficient of \( x_1 x_2^p x_3^{p^2} \cdots x_n^{p^{n-1}} \) is \( +1 \) in both cases. The second part follows. □

Let \( V \) be an elementary abelian \( 2 \)-group. Then \( H^*(V, \mathbb{F}_2) \cong S(V^*) \), where the dual space \( V^* \) is identified with \( H^1(V, \mathbb{F}_2) \). Pick \( x_1, \ldots, x_n \) to be a basis for \( H^1(V, \mathbb{F}_2) \). The following is well-known:

**Lemma 2.2.** For an elementary abelian \( 2 \)-group \( V \), the essential ideal is the principal ideal in \( H^*(V, \mathbb{F}_2) \) generated by \( L_n(x_1, \ldots, x_n) \).

Moreover, \( \text{Ess}(V) \) is the free \( S(V^*) \)-module on \( L_n(V) \), and the Steenrod closure of this one generator.

**Proof.** \( L_n(V) \) is essential, because every non-zero linear form is a factor and every maximal subgroup is the kernel of a non-zero linear form. Now suppose that \( y \) is essential, and let \( x \in V^* \) be a non-zero linear form. Let \( U \subseteq V^* \) be a complement of the subspace spanned by \( x \). So \( y = y' + y'' \) with \( y' \in S(V^*) \) and \( y'' \in S(U) \). Hence \( \text{Res}_{H}(y'') = 0 \) for \( H = \ker(x) \), as \( y \) is essential and \( \text{Res}_{H}(x) = 0 \). But the map \( \text{Res}_{H} : V^* \to H^* \) satisfies \( \ker(\text{Res}_{H}) \cap U = 0 \), and so \( \text{Res}_{H} \) is injective on \( S(U) \). Hence \( y'' = 0 \), and \( x \) divides \( y \). By unique factorization in \( S(V^*) \) it follows that \( L_n(V) \) divides \( y \). So \( \text{Ess}(V) \) is the principal ideal generated by \( L_n(V) \), and the free module on this one generator. Finally, the definition of the essential ideal means that it is closed under the action of the Steenrod algebra. □

We finish off this section by recalling the action of the Steenrod algebra on the cohomology of an elementary abelian \( p \)-group in the case of an odd prime. So let \( p \) be an odd prime and \( V \) an elementary abelian \( p \)-group. Recall that the mod-\( p \)-cohomology ring is the free graded commutative algebra

\[
H^*(V, \mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n] \otimes_{\mathbb{F}_p} \Lambda(a_1, \ldots, a_n),
\]
where $a_i \in H^1(V, \mathbb{F}_p)$, $x_i \in H^2(V, \mathbb{F}_p)$, and $n$ is the rank of $V$. That is, $a_1, \ldots, a_n$ is a basis of the exterior copy of $V^*$, and $x_1, \ldots, x_n$ is a basis of the polynomial copy. The product $a_1 a_2 \cdots a_n \in H^n(V, \mathbb{F}_p)$ is a basis of the top exterior power $\Lambda^n(V^*)$. The Steenrod algebra $A$ acts on the cohomology ring, making it an unstable $A$-algebra with $\beta(a_i) = x_i$ and $s^1(x_i) = x_i^p$. Observe that $L_n(x_1, \ldots, x_n)$ is essential, for the same reason as in the case $p = 2$.

3. The Mùi invariants

Let $k$ be a finite field and $V$ a finite dimensional $k$-vector space. Consider the natural action of $GL(V)$ on $V^*$. The Dickson invariants generate the invariants for the induced action of $GL(V)$ on the polynomial algebra $S(V^*)$. But there is also an induced action on the polynomial tensor exterior algebra $S(V^*) \otimes_k A(V^*)$, and the Mùi invariants are $SL(V)$-invariants of this action: see Mùi’s original paper [8] as well as Crabb’s modern treatment [9].

We shall need several properties of the Mùi invariants. For the convenience of the reader, we rederive these from scratch: but see Mùi’s papers [8,10] and Sum’s work [11].

**Notation.** Often we shall work with the direct sum decomposition

$$H^*(V, \mathbb{F}_p) = \bigoplus_{r=0}^n N_r(V),$$

where $n$ is the rank of $V$ and we set

$$N_r(V) = S(V^*) \otimes_k A^r(V^*).$$

Observe that restriction to each subgroup respects this decomposition. This means that the essential ideal is well-behaved with respect to this decomposition:

$$\text{Ess}(V) = \bigoplus_{r=0}^n N_r(V) \cap \text{Ess}(V). \quad (3)$$

**Definition.** Recall that $L_n(x_1, \ldots, x_n)$ is the determinant of the $n \times n$-matrix

$$C = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix},$$

where $C_{s,i} = x_i^{p^s-1}$ for $1 \leq s \leq n$. For each such $s$, define $E(s)$ to be the matrix obtained from $C$ by deleting row $s$ and then prefixing $(a_1 \ a_2 \ \cdots \ a_n)$ as new first row: so

$$\det E(s) = \sum_{i=1}^n (-1)^{i+1} y_{s,i} a_i,$$

where $y_{s,i}$ is the determinant of the minor of $C$ obtained by removing row $s$ and column $i$.

Now define the Mùi invariant $M_{n,s} \in H^*(V, \mathbb{F}_p)$ by $M_{n,s} = \det E(s)$. Note that our indexing differs from Mùi’s: our $M_{n,s}$ is his $M_{n,s-1}$.

**Example.** So $M_{4,3} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix}$ and $y_{2,3} = \begin{vmatrix} x_1 & x_2 & x_4 \\ x_1^2 & x_2^2 & x_4^2 \\ x_1^3 & x_2^3 & x_4^3 \end{vmatrix}$.

**Lemma 3.1.** $M_{n,s} \in N_1(V) \cap \text{Ess}(V)$.

**Proof.** By construction $M_{n,s} \in N_1(V)$. Restricting to a maximal subgroup of $V$ involves killing a non-zero linear form on $V^*$: That is, one imposes a linear dependence on the $a_i$ and consequently the same linear dependence on the $x_i$. So one obtains a linear dependency between the columns of $E(s)$, meaning that restriction kills $M_{n,s} = \det E(s)$. \hfill $\Box$

**Lemma 3.2.** $\text{Ess}(V)^2 = L_n(V) \cdot \text{Ess}(V)$. 


Proof. As $L_n(V)$ is essential, the left hand side contains the right. Now let $H$ be a maximal subgroup of $V$. Then $H = \ker(a)$ for some non-zero $a \in H^1(V, \mathbb{F}_p)$. Let $x = \beta(a) \in H^2$. Observe that the kernel of restriction to $H$ is generated by $a, x$. Suppose that $f, g$ both lie in this kernel: then we may write $f = f' a + f'' x, g = g' a + g'' x$, and so $fg = (f' g' ± f'' g'') a x + f'' g'' x^2$, that is $fg = x h$ for $h = (f' g' ± f'' g'') a + f'' g'' x \in \ker \text{Res}_H$.

Since $H^2(V, \mathbb{F}_p)$ is a free module over the unique factorization ring $S(V^*)$, this means that $fg = L_n(V) \cdot y$ for some $y \in H^* (V, \mathbb{F}_p)$. So $h = \frac{L_n(V)}{x} \cdot y$. As $\text{Res}_H(h) = 0$ and $\text{Res}_H \left( \frac{L_n(V)}{x} \right)$ is a non-zero divisor, we deduce that $\text{Res}_H(y) = 0$. So $y \in \text{Ess}(V)$. □

Definition 3.3. Let $S = \{ s_1, \ldots, s_r \} \subseteq \{ 1, \ldots, n \}$ be a subset with $s_1 < s_2 < \cdots < s_r$. In view of Lemmas 3.1 and 3.2 we may define the M"{u}i invariant $M_{n,S} \in N_n(V) \cap \text{Ess}(V)$ by

$$M_{n,S} = \frac{1}{L_n(V)^{r-1}} M_{n,s_1} M_{n,s_2} \cdots M_{n,s_r}.$$ 

Note in particular that $M_{n,\emptyset} = L_n(V)$.

Remark. Observe that

$$M_{n,S} M_{n,T} = \begin{cases} \pm L_n(V) M_{n,S,T} & \text{if } S \cap T = \emptyset; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

4. Joint annihilators

In this section we study the joint annihilators of the $M_{n,S}$ with $|S| = r$ as a means to prove Theorem 1.2.

Lemma 4.1. The joint annihilator of $M_{n,1, \ldots, n}$ is $N_n(V)$.

Proof. The element $a_1 \cdots a_n$ is a basis for $A^n(V)$ and is clearly annihilated by each $M_{n,S}$. Conversely, suppose that $y \neq 0$ is annihilated by every $M_{n,S}$. As $M_{n,S} N_n(V) \subseteq N_{n+1}(V)$ we may assume without loss of generality that $y \in N_n(V)$ for some $r$. If $r \leq n - 1$, then for some $i$ we have $0 \neq a_i y \in N_{n+1}(V)$. So $a_i y$ also lies in the joint annihilator, it will suffice by iteration to eliminate the case $y \in N_{n-1}(V)$.

Suppose therefore that $0 \neq y \in N_{n-1}(V)$ lies in the joint annihilator. Denote by $K$ the field of fractions of $S(V^*)$, and let $W = K \otimes_k A^{n-1}(V^*)$. Each $M_{n,S}$ induces a linear form $\phi_S : W \rightarrow K$ given by $\phi_S(u) a_1 \cdots a_n = M_{n,S} u$. By assumption, $y \neq 0$ lies in the kernel of every $\phi_S$. A basis for $W$ consists of the elements $a_1 \cdots \hat{a}_r \cdots a_n$ for $1 \leq r \leq n$, where the hat denotes omission. Now

$$M_{n,S} \cdot a_1 \cdots \hat{a}_r \cdots a_n = (-1)^{r+1} y_{s,r} a_r \cdot a_1 \cdots \hat{a}_r \cdots a_n,$$

and so

$$\phi_S (a_1 \cdots \hat{a}_r \cdots a_n) = y_{s,r}.$$ 

Now consider the matrix $I^* \in M_n(K)$ given by $I_{s,r}^* = y_{s,r}$. If one transposes and then multiplies the $ith$ row by $(-1)^i$ and the $jth$ column by $(-1)^j$, then one obtains the adjugate matrix of $C$. As the determinant of $C$ is $L_n(V)$ and in particular non-zero, it follows that det $I^* \neq 0$.

So by construction of $I^*$, the $\phi_S$ form a basis of $W^*$. So their common kernel is zero, contradicting our assumption on $y$. □

Corollary 4.2. The joint annihilator of $\{ M_{n,S} : |S| = r \}$ is $\bigoplus_{s \geq n-r+1} N_s(V)$.

Proof. By induction on $r$, Lemma 4.1 being the case $r = 1$. As $M_{n,S} \in N_{s}(V)$ and $N_r(V) N_n(V) \subseteq N_{n+1}(V)$, the annihilator is at least as large as claimed. Now suppose that $y \in H^* (V, \mathbb{F}_p)$ does not lie in $\bigoplus_{s \geq n-r+1} N_s(V)$. We may therefore write

$$y = \sum_{s=0}^n y_s$$ 

with $y_s \in N_s(V)$, and we know that $s_0 \leq n - r$ for $s_0 = \min \{ s \mid y_s \neq 0 \}$. As $y_{s_0} \neq 0$ and $y_{s_0} \notin N_n(V)$, Lemma 4.1 tells us that $y_{s_0} M_{n,s_0} \neq 0$ for some $1 \leq t \leq n$. As $y_{s_0} M_{n,s_0} \in N_{s_0+1}(V)$, we conclude that $y M_{n,t}$ lies outside $\bigoplus_{s \geq n-r+2} N_s(V)$. So the inductive hypothesis means that there is some $T$ with $|T| = r - 1$ and $y M_{n,t} M_{n,T} \neq 0$. So $y M_{n,S} \neq 0$ for $S = T \cup \{ t \}$ and $|S| = r$. Note that $t \in T$ is impossible. □

Corollary 4.3. Every $M_{n,S}$ is non-zero. For $S = \emptyset = \{ 1, \ldots, n \}$ we have

$$M_{n,n}$$ 

is a non-zero scalar multiple of $a_1 a_2 \cdots a_n$. 

Proof. Observe that $M_{n,n}$ is a scalar multiple of $a_1 \cdots a_n$ for degree reasons. The case $r = n$ of Corollary 4.2 says that $1 \in N_0(V)$ does not annihilate $M_{n,n}$ and therefore $M_{n,n} \neq 0$. But from Eq. (4) we see that every $M_{n,S}$ divides $L_n(V)M_{n,n} \neq 0$. □

Proof of Theorem 1.2. In view of Eq. (3) it suffices to show that for each $r$ the M"u"h invariants $M_{n,S}$ with $|S| = r$ are a basis of the $S(V^*)$-module $N_r(V) \cap \text{Ess}(V)$. We observed in Definition 3.3 that these $M_{n,S}$ lie in this module.

So suppose that $y \in N_r(V) \cap \text{Ess}(V)$. We should like there to be $f_S \in S(V^*)$ such that

$$y = \sum_{|S|=r} f_SM_{n,S}. \quad (5)$$

Note that for $T = n - S$ we have $M_{n,S}M_{n,T} = \pm L_n(V)M_{n,n}$ by Eq. (4). Define $\varepsilon_S \in \{+1, -1\}$ by $M_{n,S}M_{n,T} = \varepsilon_S L_n(V)M_{n,n}$. So Eq. (5) implies that we should define $f_S$ by

$$f_S M_{n,n} = \frac{1}{L_n(V)} \varepsilon_S y M_{n,T},$$

since $T \cap S' \neq \emptyset$ and therefore $M_{n,S'}M_{n,T} = 0$ for all $S' \neq S$ with $|S| = r$. Note that this definition of $f_S$ makes sense, as $y M_{n,T}$ lies in both $N_r(V)N_{n-r}(V) = N_n(V)$ and $L_n(V) \text{Ess}(V)$, the latter inclusion coming from Lemma 3.2.

With this definition of $f_S$ we have

$$\left( y - \sum_{|S|=r} f_SM_{n,S} \right) M_{n,T} = 0$$

for every $|T| = n - r$. As $y - \sum_{|S|=r} f_SM_{n,S}$ lies in $N_r(V)$, this means that $y = \sum_{|S|=r} f_SM_{n,S}$ by Corollary 4.2.

Finally we show linear independence. Suppose that $g_S \in S(V^*)$ are such that $\sum_{|S|=r} g_SM_{n,S} = 0$. Pick one $S$ and set $T = n - S$. Multiplying by $M_{n,T}$ we deduce that $g_S = 0$. □

5. The action of the Steenrod algebra

To prepare for the proof of Theorem 1.1 we shall study the operation of the Steenrod algebra on the M"u"h invariants.

Lemma 5.1.

$$\beta(M_{n,S}) = \begin{cases} L_n(V) & s = 1 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \leq s \leq n - 2$ we have:

$$\mathcal{P}^s(M_{n,r}) = \begin{cases} M_{n,r-1} & r = s + 2 \\ 0 & \text{otherwise} \end{cases} \mathcal{P}^s(L_n(V)) = 0. \quad (7)$$

Proof. One sees Eq. (6) by inspecting the determinants in the definition of $M_{n,S}$ and $L_n(V)$. The proof of Eq. (7) is also based on an inspection of these determinants. Recall that $\mathcal{P}^m(a_i) = 0$ for every $m > 0$, and that $\mathcal{P}^m(x_i^p)$ is zero too except for $\mathcal{P}^p(x_i) = x_i^{p+1}$. We may use the Cartan formula

$$\mathcal{P}^m(xy) = \sum_{a+b = m} \mathcal{P}^a(x)\mathcal{P}^b(y)$$

to distribute $\mathcal{P}^p$ over the rows of the determinant. As $p^s$ cannot be expressed as a sum of distinct smaller powers of $p$, we only have to consider summands where all of $\mathcal{P}^p$ is applied to one row and the other rows are unchanged. This will result in two rows being equal unless it is the row consisting of the $x_i^{p+1}$ that is missing. □

Lemma 5.2. Let $S = \{s_1, \ldots, s_t\}$ with $1 \leq s_1 < s_2 < \cdots < s_t \leq n$.

1. Suppose that $1 \notin S$. Then $M_{n,S} = \beta(M_{n,S\cup\{1\}})$.
2. $L_n(V)^{r-1} \mathcal{P}^m(M_{n,S}) = \mathcal{P}^m(M_{n,s_1} \cdots M_{n,s_t})$ for each $m < p^{r-1}$.
3. For $2 \leq u \leq n$ set $X = \{s \in S \mid s \leq u\}$ and $Y = \{s \in S \mid s > u\}$. Then

$$L_n(V)\mathcal{P}^{p^{u-2}}(M_{n,S}) = \mathcal{P}^{p^{u-2}}(M_{n,X}) \cdot M_{n,Y}.$$ 
4. For $1 \leq r \leq n$ and $0 < m < p^{r-1}$ one has $\mathcal{P}^m(M_{n,\{1,\ldots,r\}}) = 0$.
5. For $2 \leq u \leq n$ one has $\mathcal{P}^{p^{u-2}}(M_{n,\{1,\ldots,u-2,u\}}) = M_{n,\{1,\ldots,u-1\}}$. 

Proof. Recall that
\[ L_i(V)M_{n,s} = L_i(V)M_{n,s_1} \cdots M_{n,s_r}. \] (8)
The first two parts follow by applying Eqs. (6) and (7).

Recall that by the Adem relations each \( \mathcal{P}^m \) may be expressed in terms of the \( \mathcal{P}^{p^i} \) with \( p^i \leq m \). So the third part follows from the second, since we deduce from Eq. (7) that \( \mathcal{P}^m(M_{n,s}) = 0 \) if \( 0 < m \leq p^{s+2} \) and \( s > u \).

Fourth part: By induction on \( r \). Follows for \( r = 1 \) from the Adem relations and Eq. (7). Inductive step: Enough to consider \( \mathcal{P}^p \) for \( 0 \leq s \leq n - 2 \). By the inductive hypothesis and a similar argument to the third part, deduce that
\[ L_i(V)\mathcal{P}^p(M_{n,1},...,r) = M_{n,1},...,r-1)\mathcal{P}^p(M_{n,r}). \]
But this is zero by Eq. (7), since \( M_{n,1},...,r-1)M_{n,r-1} = 0 \).

Fifth part: Using the fourth part and an argument similar to the third, deduce that
\[ L_i(V)\mathcal{P}^{p+2}(M_{n,1},...,u-2,u) = M_{n,1},...,u-2)\mathcal{P}^{p+2} (M_{n,u}) = M_{n,1},...,u-2)M_{n,u-1}: \]
but this is \( L_i(V)M_{n,1},...,u-1 \). \( \square \)

Proof of Theorem 1.1. We shall show that for every \( n \) there is an element \( \theta \) of the Steenrod algebra with \( M_{n,5} = \theta(M_{n,2}) \).

We do this by decreasing induction on \( r = |S| \). It is trivially true for \( r = n \), so assume now that \( r < n \). Amongst the S with \( |S| = r \) we shall proceed by induction over \( u \), the smallest element of \( n - S \). So \( S = \{ 1, \ldots, u - 1 \} \cup Y \) with \( s > u \) for every \( s \in Y \).

Part 1 of Lemma 5.2 covers the case \( u = 1 \), so assume that \( u > 2 \). Set \( T = \{ 1, \ldots, u - 2, u \} \). We complete the induction by showing that \( M_{n,5} = \mathcal{P}^{p+2}(M_{n,T,Y}) \). Part 3 of Lemma 5.2 gives us
\[ L_i(V)\mathcal{P}^{p+2}(M_{n,T,Y}) = \mathcal{P}^{p+2}(M_{n,Y}) \]
But \( \mathcal{P}^{p+2}(M_{n,Y}) = M_{n,1},...,u-1 \), by Part 5 of that lemma. So \( \mathcal{P}^{p+2}(M_{n,T,Y}) = M_{n,5} \), as claimed. \( \square \)

Remark. Theorem 1.2 shows that the \( S(V^*) \)-module generated by the Mui invariants \( M_{n,5} \) is the essential ideal and therefore closed under the action of the Steenrod algebra. One may however see more directly that this \( S(V^*) \)-module is Steenrod closed. This is observed for example in [11]. In view of Lemma 5.2 and Eqs. (6) and (7) it only remains to show that \( \mathcal{P}^{p+1}(M_{n,3}) \) lies in our \( S(V^*) \)-module. Now \( \mathcal{P}^{p+1}(M_{n,3}) = 0 \) by the unstable condition, so suppose \( s < n \). Recall that \( M_{n,3} \) is a determinant, the last row of the matrix having entries \( x_i^{p^{-1}} \). So applying \( \mathcal{P}^{p+1} \) replaces these entries by \( x_i^{p^{-1}} \). But it is well known that \( x_i^{p^{-1}} \) is an \( S(V^*) \)-linear combination of the \( x_i^{p^r} \) for \( r \leq n - 1 \), and that the coefficients are independent of \( i \): this is the “fundamental equation” in the sense of [12], and the coefficients are the Dickson invariants \( c_{n,r} \) in \( S(V^*) \). Applying \( S(V^*) \)-linearity of the determinant in the bottom row of the matrix, one deduces that \( \mathcal{P}^{p+1}(M_{n,3}) \) is an \( S(V^*) \)-linear combination of the \( M_{n,r} \).

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