Supersymmetry approach in the field theory of ergodicity breaking transitions

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Abstract

The supersymmetry (SUSY) self–consistent approximation for the model of non–equilibrium thermodynamic system with quenched disorder is derived from the dynamical action calculated by means of generalized second Legendre transformation technique. The equations for adiabatic and isothermal susceptibilities, memory and field induced parameters are obtained on the basis of asymptotic analysis of dynamical Dyson equations. It is shown that the marginal stability condition that defines the critical point is governed by fluctuations violating fluctuation–dissipation theorem (FDT). The temperature of ergodicity breaking transition is calculated as a function of quenched disorder intensities. Transformation of superfields related to the mapping between an instanton process and the corresponding causal solution is discussed.

Key words: Supersymmetry; Disorder; Ergodicity; Legendre transformation;
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1 Introduction

In recent years significant study has been given to the microscopic theory of non–equilibrium thermodynamic systems with quenched disorder that reveal non–ergodic behavior and exhibit memory effects. Spin glasses [1,2] and random heteropolymers [3], that received the most of attention, provide the well known examples of such systems. Procedure of the averaging over disorder is at the heart of theoretical approaches developed for the description of the systems.

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Thus the bulk of static theories are based on the replica method firstly introduced in [4] or employ the methods of random matrix theory (RMT) (see [5] for review). In addition, the powerful supersymmetry approach by Efetov [6], where generating functional describing statistics of the density of states is mapped on the nonlinear supermatrix $\sigma$–model, has had RMT as a key ingredient of its development.

The starting point of dynamical theories is stochastic dynamics defined in terms of the corresponding stochastic equations. A wide range of problems can be formulated in such a way: kinetics of ordering [7], non–equilibrium dynamics of spin–glass systems [2,8,9] and so on. When the stochastic dynamics is governed by Langevin equations, the generating functional of the stochastic problem can be written as a functional integral [10,11] (MRS formalism) thus allowing for the averaging over disorder at the very beginning of calculations. It has led to the field theoretic formulation of the problem, so one could take advantage of using the machinery of the field theory. The theory is known can represented in the supersymmetric form that reveals hidden supersymmetry (SUSY) of the stochastic problem [12,22].

In this paper SUSY formalism in the superfield representation is employed to study ergodicity breaking transition in the model thermodynamic system with quenched disorder. It implies that generating functional of Langevin dynamical system is represented as functional integral over superfields with Euclidean action by means of introducing Grassmann anticommuting variables. These variables and their products serve as a basis for superfields which components include Grassmann fields in addition to the usual real–valued fields. The correlation functions of the superfields (supercorrelators) then encode physically relevant information on observables in components of the correlators that are the autocorrelator of the order parameter field $C$ and the response functions $G_\pm$ (advanced and retarded Green functions). The analysis is based on a system of dynamical Dyson equations supplemented with the equations for averaged order parameter provided the distribution of quenched random variables is not symmetric. It was pointed out in [14] that, in the mode–coupling approximation, the Dyson equations can be regarded as Euler–Lagrange equations for the functional of supercorrelators known as dynamical action and this functional bears striking similarity with the replica expression for the free energy.

In our study the dynamical action is shown to be a second Legendre transformation [15] of the free energy. So, it depends on both averaged order parameter and two–time correlation functions and can be calculated on the basis of suitably defined diagrammatics. On the technical side, since algebraic structure of the superfield representation is directly related to the underlying symmetry, we have the reduction of the number of diagrams to be taken into consideration in the perturbation theory. From the other hand, SUSY formalism allows ergodicity breaking transition be interpreted as a dynamical symmetry breaking transition. Indeed, in the SUSY language, the well–known ”causality” condition and fluctuation–dissipation theorem (FDT) immediately follow from the corresponding Ward identities [9,16].
Below the critical temperature FDT and time translational invariance are dynamically violated that result in the appearance of anomalous solutions to the Dyson equations. In particular, the latter include the case, where the system being in a non–ergodic state is characterized by a very slow relaxation. This phenomenon is known as aging and, as it has been demonstrated for a number of solvable models [17–19], the effect is a consequence of trapping in metastable attractors.

In this article, leaving aside detailed study of aging, we shall perform asymptotic and stability analysis of dynamical Dyson equations to characterize the transition in terms of asymptotic quantities such as adiabatic and isothermal susceptibilities, memory and field induced parameters. This approach is applied to the simple model of thermodynamic system with quenched external field and two body interaction. Note that the second Legendre transformation technique plays the unifying role in this theory, so that the SUSY based theory can be directly employed for the study of ergodicity breaking transitions in different disordered systems such as random heteropolymers [3] and filled nematic liquid crystals [20].

Layout of the paper is as follows.

In Sec. 2 the formalism of SUSY approach is briefly outlined. Second Legendre transformation for disordered systems is introduced in Sec. 3. In Sec. 4 the model of non–equilibrium thermodynamic system with quenched disorder is studied on the basis of asymptotic analysis of the Dyson equations. In high temperature region it is found that the relevant parameters are the static susceptibility $\chi$ and the field induced parameter $q_h$. The latter is due to the presence of random field that affects asymptotics of the autocorrelator $C(t)$. Equations for $\chi$ and $q_h$ combined with the marginal stability condition define the temperature of ergodicity breaking $T_c$. Dependencies of $T_c$ on the quenched disorder intensities are calculated. It is shown that the low temperature region $T < T_c$ can be described in terms of adiabatic $\chi_a$ and isothermal $\chi$ susceptibilities, the dynamical Edwards–Anderson memory parameter $q$ and the field induced parameter $q_h$, so that the role of an order (non–ergodicity) parameter plays the difference $\Delta q = q - q_h$. Discussion of numerical results and concluding remarks are given in Sec. 5. Appendix A details the remark that the superfield representation induced by the shift in time, $t \rightarrow t - \bar{\theta} \theta$, leads to the mapping between an instanton process and normal downhill motion.
order parameter at the instant of time $t$.

Relaxational dynamics of the order–parameter field in the presence of thermal noise is governed by the Langevin equation:

$$\dot{\eta}_i = -\frac{\delta V}{\delta \eta_i} + \zeta_i(t),$$  \hspace{1cm} (1)

where the relaxation constant is absorbed by suitable rescaling of time and $\zeta$. The thermodynamic potential $V$ is assumed to be a $t$–local functional

$$V\{\eta\} = \int dt \, V(\eta),$$  \hspace{1cm} (2)

and $\zeta_i(t)$ are Gaussian stochastic functions subjected to the white noise conditions:

$$\langle \zeta_i(t)\zeta_j(0) \rangle_\zeta = 2T\delta_{ij}\delta(t), \quad \langle \zeta_i \rangle_\zeta = 0,$$  \hspace{1cm} (3)

where $T$ is the temperature.

2.1 Generating functional in superfield representation

The starting point of the MRS formalism [10,11] is the generating functional for correlation functions of the stochastic problem written in the form:

$$Z\{u\} = \left\langle \prod_i D\eta_i \det\left( \frac{\delta L(\eta)}{\delta \eta} \right) \delta(L(\eta)) \exp\left( \int dt u_i \eta_i \right) \right\rangle_\zeta,$$  \hspace{1cm} (4)

where $\delta(\cdot)$ is the delta function, $L(\eta) = -\partial_t \eta - \delta V/\delta \eta + \zeta$ and $\langle \cdot \rangle_\zeta$ denotes averaging over noise realizations. (Hereafter the functional notations will be adopted assuming summation over repeated indexes and integration over repeated non-discrete arguments.)

Exponentiating the delta function through 'Lagrange multipliers' $\varphi_i(t)$ and the Jacobian functional determinant through Grassmann fields (ghosts) $\psi_i(t)$, $\bar{\psi}_i(t)$ and averaging away the noise we obtain functional integral representation for the generating functional (see [21] for recent discussion of the corresponding steps)

$$Z\{u\} = \int \prod_i D\eta_i D\varphi_i D\psi_i D\bar{\psi}_i \exp\{-S + \int dt u_i \eta_i\}$$  \hspace{1cm} (5)

where the action reads

$$S = \int dt L,$$  \hspace{1cm} (6)
\[ L = -T \phi_i^2 + \varphi_i \left\{ \ddot{\eta}_i + \frac{\delta V}{\delta \eta_i} \right\} - \bar{\psi}_i \left[ \delta_{ij} \frac{\partial}{\partial t} + \frac{\delta^2 V}{\delta \eta_i \delta \eta_j} \right] \psi_j \]  \hspace{1cm} (7)

Introducing the superfields

\[ \phi_i(z) \equiv \phi_i = \eta_i + \bar{\theta} \psi_i + \bar{\psi}_i \theta + \bar{\theta} \theta \phi_i, \quad z \equiv \{ t, \bar{\theta}, \theta \} \equiv \{ t, \theta \}, \]  \hspace{1cm} (8)

where \( \bar{\theta} \) and \( \theta \) are anticommuting Grassmann variables, and substituting \( \{ \bar{\theta}, \theta \} \to \{ T^{-1/2} \bar{\theta}, T^{-1/2} \theta \} \) (or, alternatively, \( \{ \bar{\psi}_i, \psi_i \} \to \{ T^{1/2} \bar{\psi}_i, T^{1/2} \psi_i \} \), \( \phi_i \to T \phi_i \)) we derive the action as a functional of superfields

\[ S = \frac{1}{T} \int dz L = \frac{1}{T} \int dt d^2 \theta L, \]  \hspace{1cm} (9)

\[ L = \bar{D} \phi_i D \phi_i + V(\phi), \]  \hspace{1cm} (10)

where \( d^2 \theta \equiv d\theta d\bar{\theta} \) and

\[ \bar{D} = \frac{\partial}{\partial \bar{\theta}}, \quad D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial t}. \]  \hspace{1cm} (11)

It is not difficult to show that the operators \( \bar{D} \) and \( D \) enjoy the following properties:

\[ D^2 = D^2 = 0, \quad \{ D, \bar{D} \} = -\frac{\partial}{\partial t}, \quad [D, \bar{D}]^2 = \frac{\partial^2}{\partial \theta \partial \bar{\theta}}, \]  \hspace{1cm} (12)

where the curly brackets stand for anticommutator, and the kinetic term in the action \( \int dz \bar{D} \phi_i D \phi_i \) can be written in the form

\[ \frac{1}{2} \int dz \phi_i D^{(2)} \phi_i, \quad D^{(2)} \equiv [D, \bar{D}] = -2 \frac{\partial^2}{\partial \theta \partial \bar{\theta}} + \left( 1 - 2\theta \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial t}. \]

Note that the form of superfield representation (8) is fixed up to the change of the basis in the space of superfields, so that the corresponding transformation would give another representation of the SUSY group. Interestingly, as it is shown in Appendix A, alternative superfield representation, generated by the transformation: \( \phi(t, \theta) \to \phi(t - \theta \bar{\theta}, \theta) \), can be used to construct the mapping between an instanton process and the corresponding causal solution inverted in time.
2.2 Symmetries and Ward identities for 2–point functions

The action by Eqs. (9,10) is invariant under the action of the group of supersymmetry [9,14] with generators given by

\[ \bar{D}'_i = \frac{\partial}{\partial \bar{\theta}_i}, \quad D'_i = \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial t_i}. \]  

(13)

As a consequence, the correlation function \( G(z_1, z_2) = \langle \phi(z_1) \phi(z_2) \rangle \) (for brevity, the indexes of superfields are suppressed) meets a set of Ward identities provided that the supersymmetry is not broken [9]. Clearly, the invariance under translations in time implies that \( G(z_1, z_2) \) depends only on \( t = t_1 - t_2 \). Another two identities are of special interest to us:

\[ (\bar{D}'_1 + \bar{D}'_2) G(z_1, z_2) = 0, \]  

(14)

\[ (D'_1 + D'_2) G(z_1, z_2) = 0, \]  

(15)

where

\[ \bar{D}'_i = \frac{\partial}{\partial \bar{\theta}_i}, \quad D'_i = \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial t_i}. \]

Eq. (14), known as "causality condition", implies that the correlator is of the following form

\[ G(z_1, z_2) = C(t_1, t_2) + \left( \bar{\theta}_1 - \bar{\theta}_2 \right) \left( G_+(t_1, t_2) \theta_1 - G_-(t_1, t_2) \theta_2 \right), \]  

(16)

where

\[ C(t_1, t_2) = \langle \eta(t_1) \eta(t_2) \rangle, \]  

(17a)

\[ G_+(t_1, t_2) = \langle \varphi(t_1) \eta(t_2) \rangle = \langle \bar{\psi}(t_1) \psi(t_2) \rangle, \]  

(17b)

\[ G_-(t_1, t_2) = \langle \eta(t_1) \varphi(t_2) \rangle = -\langle \psi(t_1) \bar{\psi}(t_2) \rangle. \]  

(17c)

Thus the identity (14) allows the correlators of Grassmann fields \( \bar{\psi} \) and \( \psi \) be expressed in terms of response functions.

Algebraic structure of the correlator can be conveniently emphasized by introducing a set of operators \( \{ T, A_+, A_- \} \)

\[ A_+(\theta, \theta') = (\bar{\theta} - \bar{\theta}') \theta, \]  

(18a)

\[ A_-(\theta, \theta') = A_+(\theta', \theta) = -(\bar{\theta} - \bar{\theta}') \theta', \]  

(18b)

\[ T(\theta, \theta') = 1, \]  

(18c)
equipped with the operator product

\[ A_1 \cdot A_2(\theta, \theta') = \int d^2\theta'' A_1(\theta', \theta'') A_2(\theta'', \theta'), \quad (19) \]

so that

\[ G = CT + G_+A_+ + G_-A_- . \quad (20) \]

It is not difficult to verify that the above operators form the basis of algebra with respect to the operator product

\[ A_\pm \cdot A_\pm = A_\pm, \quad T \cdot A_+ = A_- \cdot T = T, \quad (21a) \]
\[ A_\pm \cdot A_\mp = A_+ \cdot T = T \cdot A_- = T \cdot T = 0. \quad (21b) \]

In particular, Eqs. (21) ease finding of inversion formulae for superoperators. For example, we can derive the expression for \( G^{-1} \):

\[ G^{-1} = G_+^{-1}A_+ + G_-^{-1}A_- - G_-^{-1}C \cdot G_+^{-1}T, \quad (22) \]

so that

\[ G \cdot G^{-1} = \delta(\theta - \theta') \equiv (\bar{\theta} - \bar{\theta}')(\theta - \theta') = A_+ + A_- . \quad (23) \]

Note that expansion of the bare correlation function \( G^{(0)} \) over the above basis is

\[ G^{(0)} = \{ D^{(2)} + m \} \delta(\theta - \theta')\delta_{ij} = \{-2T + (m - \partial_t)A_+ + (m + \partial_t)A_- \} \delta_{ij}. \quad (24) \]

The second identity Eq. (15) provides the relation between the autocorrelator of the order parameter \( C(t, t') \) and the response functions \( G_-(t, t') \equiv G(t, t') \) and \( G_+(t, t') = G(t', t) \) (retarded and advanced Green functions) known as fluctuation-dissipation theorem (FDT):

\[ \frac{\partial}{\partial t} C(t, t') = -\theta(t-t')G(t, t') + \theta(t'-t)G(t', t) \quad (25a) \]
\[ \frac{\partial}{\partial t'} C(t, t') = \theta(t-t')G(t, t') - \theta(t'-t)G(t', t) \quad (25b) \]

where \( \theta(t) \) is the step function.
It is convenient to reformulate FDT (25) in terms of time–dependent susceptibility \(\chi(t, t')\)

\[
\chi(t, t') = \int_{t'}^{t} d\tau' G(t, \tau')
\]  

(26)

that gives the response to an applied field held constant from time \(t'\) up to \(t\). (Hereafter it is assumed that \(t \geq t'\).) Since FDT implies the time translational invariance, \(G, C\) and \(\chi\) depend only on the time separation \(\tau \equiv t - t'\). Then integrating Eq. (25b) and taking the limit \(\tau \to \infty\) yields the required form of FDT:

\[
\chi = \int_{0}^{\infty} d\tau' G(\tau') = q_0 - q_h ,
\]  

(27)

where \(\chi\) is the static susceptibility, \(q_0 = C(t, t) = C(0)\) and \(q_h = C(\infty)\).

Analogously, Ward identities for proper vertices lead to the same relations for the mass operator (self–energy),

\[
\Sigma(z_1, z_2) = \Sigma(t_1, t_2)T + \Sigma_+(t_1, t_2)A_+ + \Sigma_-(t_1, t_2)A_- ,
\]  

(28)

that enter the Dyson equation.

### 3 Dynamical action for system with quenched disorder: second Legendre transformation technique

In this section we derive the dynamical action as a second Legendre transformation of the free energy functional. Since it is straightforward to generalize the subsequent considerations, for the sake of simplicity, the technique will be employed to study the system with the thermodynamic potential \(V(\phi)\) of the following form

\[
V(\phi) = \sum_i \{U(\phi_i(z)) + H_i \phi_i(z)\} - \sum_{ij} W_{ij} \phi_i(z) \phi_j(z) ,
\]  

(29a)

\[
U(\phi) = \frac{m}{2!} \phi^2 + U_{anh}(\phi) = \frac{m}{2!} \phi^2 + \frac{\lambda}{4!} \phi^4 ,
\]  

(29b)

where \(m = \mu T\). Under suitable assumptions, it can be regarded as a discrete version of the well–known Landau–Ginzburg free energy functional for coarse–grained scalar order parameter field [7]. We consider the case where the couplings \(W_{ij}\) and the field \(H_i\) correspond to quenched degrees of freedom and are independent Gaussian variables:

\[
\mathbb{W}_{mn} = 0, \quad \mathbb{W}_{mn} \mathbb{W}_{m'n'} = \delta_{mm'} \delta_{nn'} w_{mn}, \quad \mathbb{H}_m = 0, \quad \mathbb{H}_m \mathbb{H}_{m'} = \delta_{mm'} h^2 .
\]

Averaging away the quenched variables gives
\[ \exp\{-S\} = \exp\{-T^{-1} \int dz dz' L_{eff}\}\]  
(30a)

\[ L_{eff} = \sum_i \left\{ \frac{1}{2} \phi_i(z) K(z, z') \phi_i(z') + \delta(z - z') U_{anh}(\phi_i(z)) \right\} - (2T)^{-1} \sum_{ij} w_{ij} \phi_i(z) \phi_j(z) \phi_i(z') \phi_j(z') \]  
(30b)

where (see Eq. (24))

\[ K(z, z') = \left[ -\frac{h^2}{4T} + (2T + (m - \partial_t)A_+ + (m + \partial_t)A_-) \delta(t - t') \right] \delta_{ij}. \]  
(31)

Note that, by contrast with the real two–particle interaction, the term generated by the averaging is non–local in time.

At this stage we can use functional methods of the field theory to derive equations of motion for the one– and two–point correlators, \( \langle \phi_i(z) \rangle \) and \( \langle \phi_i(z) \phi_j(z') \rangle \). To this end let us write the effective action (30) in slightly generalized form:

\[ S_{eff} = \sum_i S_0(\phi_i|A^{(i)}) + \frac{1}{2} \int dz_1 dz_2 \sum_{ij} \bar{w}_{ij} \phi_i(z_1) \phi_j(z_2) \phi_i(z_1') \phi_j(z_2'), \]  
(32)

where

\[ S_0(\phi_i|A^{(i)}) = \sum_{n=1}^4 \frac{1}{n!} \int dz_1 \ldots dz_n A^{(i)}_n(z_1, \ldots z_n) \phi_i(z_1) \ldots \phi_i(z_n) \]  
(33)

and \( A^{(i)} \equiv \{A^{(i)}_1, A^{(i)}_2, A^{(i)}_3, A^{(i)}_4\} \) stands for a set of ”superpotentials” that define the one–particle action \( S_0 \) and are assumed to be symmetric functions of arguments. In our case \( A^{(i)}_1 = A^{(i)}_4 = 0, A^{(i)}_2 = -K(z_1, z_2)/T \) and \( A^{(i)}_3 = -\lambda/T \delta(z_1 - z_2) \). For convenience, we will keep the generalized denotions and the above \( A^{(i)}_k \) will be inserted into the final equations. Note that \( A_1 \) is commonly referred to as an external field and the inverse of operator with the kernel \(-A_2(z_1, z_2)\) is proportional to the bare correlation function.

The effective action \( S_{eff} \) defines the partition function \( Z(A) \) and the free energy \( F(A) \) as generating functionals of \( n \)–point correlators without vacuum loops and connected \( n \)–point correlators, respectively:
\[ Z(A) = \exp \{ F(A) \} = N^{-1} \int \prod \phi_i \exp \{ S_{eff}(\phi | A) \} , \quad (34a) \]

\[
\frac{\delta^n Z(A)}{\delta A^{(i)}_1(z_1) \ldots \delta A^{(i)}_n(z_n)} = \langle \phi_i(z_1) \ldots \phi_i(z_n) \rangle \equiv n! \alpha_n^{(i)}(z_1 \ldots z_n) , \quad (34b) \]

\[
\frac{\delta^n F(A)}{\delta A^{(i)}_1(z_1) \ldots \delta A^{(i)}_n(z_n)} = \langle \phi_i(z_1) \ldots \phi_i(z_n) \rangle_c \equiv \beta_n^{(i)}(z_1 \ldots z_n) , \quad (34c) \]

\[
\frac{\delta F(A)}{\delta A^{(i)}_2(z_1, z_2)} = \alpha_2^{(i)}(z_1, z_2) . \quad (34d) \]

After introducing auxiliary field \( X_i(z_1, z_2) \) and performing the Hubbard–Stratonovich transformation

\[
\exp \left[ \frac{1}{2} \int d z_1 d z_2 \sum_{ij} \bar{w}_{ij} \phi_i(z_1) \phi_i(z_2) \phi_j(z_1) \phi_j(z_2) \right] \propto \int D X_i \exp \left[ - \frac{1}{2} \int d z_1 d z_2 \sum_{ij} \bar{w}_{ij}^{-1} X_i(z_1, z_2) X_j(z_1, z_2) + \sum_i X_i(z_1, z_2) \phi_i(z_1) \phi_i(z_2) \right] \quad (35) \]

the partition function assumes the following form

\[
Z(A) = \int D X_i \exp \left[ - \frac{1}{2} \int d z_1 d z_2 \sum_{ij} \bar{w}_{ij}^{-1} X_i(z_1, z_2) X_j(z_1, z_2) + \sum_i F_0(A^{(i)}_1, A^{(i)}_2 + 2 X_i, A^{(i)}_3, A^{(i)}_4) \right] , \quad (36) \]

where \( F_0(A^{(i)}) \) is the generating functional of connected correlators for the one–particle action \( S_0 : \)

\[
\exp \{ F_0(A^{(i)}) \} = \int D \phi_i \exp \{ S_0(\phi_i | A^{(i)}) \} . \quad (37) \]

After changing variables \( \tilde{A}^{(i)}_2 = A^{(i)}_2 + 2 X_i \) in Eq. (36) it is ready to derive the equations of the saddle–point approximation:

\[
\frac{\delta F_0}{\delta \tilde{A}^{(i)}_2} = \alpha_2^{(i)} = \frac{1}{4} \sum_j \bar{w}_{ij}^{-1} (\tilde{A}^{(i)}_2 - A^{(i)}_2) . \quad (38) \]

It is known that the mean field approximation is given by the leading order in steepest descent calculations of such kind [16]. The only difference from the standard
mean field theory is that the interaction term in Eq. (36) contains $A^{(i)}_2$ instead of $A^{(i)}_1$. So it is natural to define second Legendre transformation for $F$ as a generalized Legendre transformation with respect to the first two ”superpotentials” $A_1$ and $A_2$ [15] :

$$\Gamma(\alpha_1, \alpha_2 | A_3, A_4) = F(A) - A^{(i)}_1 \frac{\delta F}{\delta A^{(i)}_1} - A^{(i)}_2 \frac{\delta F}{\delta A^{(i)}_2} =$$

$$= F(A) - A_1\alpha_1 - A_2\alpha_2 , \quad (39)$$

where

$$\alpha^{(i)}_1(z_1) = \beta^{(i)}_1(z_1) = \langle \phi_i(z_1) \rangle , \quad 2\alpha^{(i)}_2(z_1, z_2) = \beta^{(i)}_2(z_1, z_2) + \beta^{(i)}_1(z_1)\beta^{(i)}_1(z_2)$$

and universal functional notations used in (39) imply that

$$A_1\alpha_1 \equiv \sum_i \int dz_1 A^{(i)}_1(z_1)\alpha^{(i)}_1(z_1) , \quad A_2\alpha_2 \equiv \sum_i \int dz_1dz_2 A^{(i)}_2(z_1, z_2)\alpha^{(i)}_2(z_1, z_2) .$$

Similar to the standard Legendre transformation, given the functional $\Gamma$ with $A_1$ and $A_2$, the correlators $\alpha_1$ and $\alpha_2$ are defined as solutions to the equations

$$\frac{\delta \Gamma}{\delta \alpha^{(i)}_1(z_1)} = -A^{(i)}_1(z_1) , \quad (40a)$$

$$\frac{\delta \Gamma}{\delta \alpha^{(i)}_2(z_1, z_2)} = -A^{(i)}_2(z_1, z_2) . \quad (40b)$$

Eqs. (40) are Euler–Lagrange equations for the functional

$$\Phi(\alpha_1, \alpha_2 | A) = \Gamma(\alpha_1, \alpha_2 | A_3, A_4) + \alpha_1 A_1 + \alpha_2 A_2 , \quad (41)$$

so that substituting of extremals from Eqs. (40) into $\Phi$ provides the corresponding values of the free energy $F$. It follows that $\Phi$ has the meaning of dynamical action.

In the mean field approximation, from Eq. (38) the expression for $\Gamma$ is

$$\Gamma = \sum_i \Gamma_0(\alpha^{(i)}_1, \alpha^{(i)}_2 | A_3, A_4) + 2 \sum_{ij} \bar{w}_{ij} \int dz_1dz_2 \alpha^{(i)}_2(z_1, z_2)\alpha^{(j)}_2(z_1, z_2) , \quad (42)$$

where $\Gamma_0$ is the Legendre transform of the one–particle free energy. As a functional of $\beta_1$ and $\beta_2$, $\Gamma_0$ is given by [15]

$$\Gamma_0 = \frac{1}{2} \text{Tr} \ln \beta_2 + \bar{\Gamma}_0 . \quad (43)$$

where $\bar{\Gamma}_0$ can be calculated as a sum of two–particle irreducible diagrams (it cannot be made disconnected by cutting off two lines) with ”dressed” internal lines ($\beta_2$ is
the propagator). Up to the second order of the perturbation theory with $A_3 = 0$, we have

$$\bar{\Gamma}_0 = \frac{1}{4!} A_4 \left( \beta_1^4 + 6 \beta_2 \beta_1^2 + 3 \beta_2^2 \right) + \frac{1}{48} A_4 \beta_2 A_4 + \frac{1}{12} (A_4 \beta_1) \beta_2^2 (A_4 \beta_1). \quad (44)$$

The diagrammatic representation of the terms on the right hand side of Eq. (44) is

$$\frac{1}{24} \quad + \quad \frac{1}{4} \quad + \quad \frac{1}{8} \quad + \quad + \quad \frac{1}{48} \quad + \quad \frac{1}{12} \quad .$$

In the diagrams zigzag lines correspond to $\beta_1$–lines directly joined to the vertices. The equations of motion in new variables are

$$\frac{\delta \Gamma}{\delta \beta_1(z_1)} - 2 \frac{\delta \Gamma}{\delta \beta_2(z_1, z_2)} \beta_1(z_2) = -A_1(z_1) \quad (45a)$$

$$2 \frac{\delta \Gamma}{\delta \beta_2(z_1, z_2)} = -A_2(z_1, z_2). \quad (45b)$$

Note that, owing to the identity

$$\frac{\delta}{\delta \beta_2(z_1, z_2)} \text{Tr} \ln \beta_2 = \beta_2^{-1}(z_1, z_2),$$

Eq. (45b) is the Dyson equation with suitably defined mass operator.

In what follows it is supposed that the system under consideration is spatially homogeneous, so that the correlators $\beta_1(z) \equiv \langle \phi \rangle$ and $\beta_2(z, z') \equiv G(z, z')$ do not depend on the site index. Moreover, since Eq. (45a) has a trivial solution $\langle \phi \rangle = 0$ at $A_1 = 0$ and ergodicity breaking transition is related to the anomaly in $G$ when FDT is violated, we eliminate $\langle \phi \rangle$ from the consideration by putting $\langle \phi \rangle = 0$. Eqs. (41-44) then provide the expression for the dynamical action:

$$2\Phi = \text{Tr} \ln[G] + \frac{\lambda^2}{24T^2} \int dz_1 dz_2 G^4(z_1, z_2) +$$

$$+ T^{-1} \int dz_1 dz_2 \left[ \frac{w}{2T} G^2(z_1, z_2) - K(z_1, z_2) G(z_1, z_2) \right] \quad (46)$$
where \( w = \sum_{ij} w_{ij} \). The corresponding Dyson equation follows from the stationarity condition \( \frac{\delta \Phi}{\delta G(z, z')} = 0 \):

\[
G^{-1}(z, z') = \frac{1}{T} K(z, z') - \frac{w}{T^2} G(z, z') - \Sigma(z, z'). \tag{47}
\]

The mass operator \( \Sigma(z, z') = \lambda^2 / (6T^2) G^3(z, z') \) has the following components

\[
\Sigma(t, t') = \Sigma(C) = \frac{\lambda^2}{6T^2} C^3(t, t'), \quad \Sigma_\pm(t, t') = \Sigma'(C) G_\pm,
\tag{48}
\]

where \( \Sigma' = \frac{\partial \Sigma}{\partial C} \). Clearly, \( \Sigma \) satisfies the FDT: \( \partial_t \Sigma = -\partial_t \Sigma = \Sigma_- \).

By using the identity (22), it is not difficult to derive the components of Eq. (47). So the Dyson equation can be represented as a system of nonlinear integro–differential equations:

\[
G^{-1}_\pm = \frac{1}{T} \delta(t - t')(m \mp \partial_t) - \frac{w}{T^2} G_\pm - \Sigma_\pm.
\tag{49}
\]

\[
G^{-1}_- \cdot C \cdot G^{-1}_+ = 2 \frac{T}{T^2} \delta(t - t') + \frac{h^2}{T^2} + \frac{w}{T^2} C + \Sigma.
\tag{50}
\]

Recall that the dot “.” denotes the operator product.

Finally, we arrive at the explicit form of the dynamical Dyson equations for the autocorrelator \( C(t, t') \) and the response (Green) function \( G(t, t') \):

\[
\frac{1}{T}(m + \partial_t) G(t, t') = \delta(t - t') + \frac{w}{T^2} \int_{t'}^t d\tau' G(t, \tau') G(\tau', t') + \\
+ \int_{t'}^t d\tau' \Sigma_-(t, \tau') G(\tau', t')
\tag{51}
\]

\[
\frac{1}{T}(m + \partial_t) C(t, t') = \frac{2}{T} G(t', t) + \frac{h^2}{T^2} \int_{t'}^t d\tau' G(t', \tau') + \\
+ \frac{w}{T^2} \left[ \int_{t'}^t d\tau' C(t, \tau') G(t', \tau') + \int_{-\infty}^{t'} d\tau' G(t, \tau') C(\tau', t') \right] + \\
+ \int_{-\infty}^{t'} d\tau' \Sigma(t, \tau') G(t', \tau') + \int_{-\infty}^{t'} d\tau' \Sigma_-(t, \tau') C(\tau', t')
\tag{52}
\]
4 Ergodicity breaking transition

In this section we concentrate on asymptotic analysis of the Dyson equations (51,52). Our primary purpose is to describe the behavior of the system in terms of static susceptibilities and memory parameters characterizing asymptotics of the autocorrelator \( C(t, t') \) in the limit of large time separation, \( \tau = t - t' \to \infty \).

The first step is to transform Eq. (52) in such a way that FDT is not broken explicitly. To this end we assume that interaction is adiabatically switched \( (C(-\infty) = 0) \) and apply FDT on the interval of time from \(-\infty \) to \( t' \). The result reads

\[
\frac{1}{T} (m + \partial_t) C(t, t') = q_0 \left\{ \frac{h^2}{T^2} + \frac{w}{T^2} C(t, t') + \Sigma(t, t') \right\} + \\
+ \frac{w}{T^2} \int_{t'}^t d\tau' G(t, \tau') C(\tau', t') + \int_{t'}^t d\tau' \Sigma_-(t, \tau') C(\tau', t'),
\]

where \( q_0 \) is the value of equal time autocorrelator \( C(t', t') \). Thus the starting point for subsequent analysis is the Dyson equations (51) and (53).

4.1 High temperature region

Since in this region the time translational invariance is unbroken, Eqs. (51,53) can be easily analyzed by making use of Laplace transformation. The resulting system for the Laplace transforms \( \hat{C}(p) \) and \( \hat{G}(p) \) is given by

\[
\frac{1}{T} (m + p) \hat{C}(p) = 1 + \frac{w}{T^2} \hat{G}^2(p) + \hat{\Sigma}_-(p) \hat{G}(p),
\]

\[
\frac{1}{T} [(m + p) \hat{C}(p) - q_0] = \frac{h^2}{p T^2} \hat{C}(p) + \frac{w}{T^2} \hat{\Sigma}(p) + \left( \frac{w}{T^2} \hat{G}(p) + \hat{\Sigma}_-(p) \right).
\]

By using the relations: \( \chi = \hat{G}(0) \), \( q_0 = \lim_{p \to \infty} p \hat{C}(p) \), \( q_h = \lim_{p \to 0} p \hat{C}(p) \) and \( \Sigma(q_h) = \lim_{p \to 0} p \hat{\Sigma}(p) \), from Eq. (54a) and Eq. (54b) multiplied by \( p \) the following system for susceptibilities and the field induced parameter can be deduced.
\[
\frac{m}{T} \chi = 1 + \frac{w}{T^2} \chi^2 + \hat{\Sigma}_-(0) \chi, \tag{55a}
\]
\[
\frac{m}{T} \chi_a = 1 + \frac{w}{T^2} \chi_a^2 + \chi_a \left\{ \Sigma(\chi_a + q_h) - \Sigma(q_h) \right\} - q_h \left\{ \frac{w}{T^2} \Delta \chi + \Delta \hat{\Sigma}_-(0) \right\}, \tag{55b}
\]
\[
q_h = \chi(\chi_a + q_h) \left\{ \frac{h^2}{T^2} + \frac{w}{T^2} q_h + \Sigma(q_h) \right\}, \tag{55c}
\]

where \( \chi_a = q_0 - q_h, \Delta \chi = \chi - \chi_a \) and \( \Delta \hat{\Sigma}_-(0) = \hat{\Sigma}_-(0) + \Sigma(q_h) - \Sigma(\chi_a + q_h) \). It should be stressed that, except for the identity \( \lim_{p \to \infty} p(C(p) - q_0) = -\lim_{p \to \infty} p \hat{G}(p) = -T \) that yield the first term on the right hand side of Eq. (55b), FDT have not been used in the derivation of the above equations.

In the FDT regime \( \Delta \chi = 0 \) and \( \Delta \hat{\Sigma}_-(0) = 0 \). So the equations for the susceptibility \( \chi \) and the field induced parameter \( q_h \) in the high temperature region are

\[
\frac{m}{T} \chi = 1 + \frac{w}{T^2} \chi^2 + \chi \left\{ \Sigma(\chi + q_h) - \Sigma(q_h) \right\}, \tag{56a}
\]
\[
q_h = \chi(\chi + q_h) \left\{ \frac{h^2}{T^2} + \frac{w}{T^2} q_h + \Sigma(q_h) \right\}. \tag{56b}
\]

Notice that, in the case of \( h^2 = 0 \), the temperature dependence of \( \chi \) is defined by Eq. (56a) with \( q_h = 0 \). This is why the parameter \( q_h \) is referred to as a field induced parameter (there are no memory effects in the absence of random field at \( T > T_c \)).

### 4.2 Critical point

Ergodicity breaking transition is determined by the point where the FDT compliant solution with \( C_{\text{FDT}}(t) \) and \( G_{\text{FDT}}(t) \) becomes dynamically unstable. In order to study stability let us first differentiate Eq. (53) with respect to time and then insert the perturbed solution \( \partial_t C = \partial_t C_{\text{FDT}} + \delta D \) and \( G = G_{\text{FDT}} + \delta G \) into the resulting system. As far as stability analysis is concerned only linear part of the system is relevant. After rather straightforward calculations, it can be derived in the form:

\[
\begin{align*}
\left[ \frac{1}{T} \partial_t + \chi^{-1} - \chi \left( \frac{w}{T^2} + \Sigma'(q_h) \right) \right] \delta G &= \ldots, \tag{57a}
\left[ \frac{1}{T} \partial_t + \chi^{-1} - (\chi + q_h) \left( \frac{w}{T^2} + \Sigma'(q_h) \right) \right] \delta D - q_h \left( \frac{w}{T^2} + \Sigma'(q_h) \right) \delta G &= \ldots, \tag{57b}
\end{align*}
\]

where " \ldots " stands for the terms that are nonlinear in \( \delta D \) and \( \delta G \).
As a result, the marginal stability condition reads

\[ 1 = \chi(\chi + q_h) \left\{ \frac{w}{T^2} + \Sigma'(q_h) \right\}. \]  

(58)

It should be emphasized that fluctuations do not need to obey FDT, \( \delta D + \delta G \neq 0 \). As a consequence, at \( q_h \neq 0 \) Eq. (58) is stronger than the condition

\[ 1 = \chi^2 \left\{ \frac{w}{T^2} + \Sigma'(q_h) \right\}. \]  

(59)

obtained under the assumption that fluctuations do not violate FDT (\( \delta D + \delta G = 0 \)).

Eqs. (56,58) yield the temperature of ergodicity breaking transition, \( T_c \), and the value of \( q_h \) at the critical point, \( q_c \). The latter can be easily obtained from Eqs. (56b,58):

\[ q_c = \left( \frac{3h^2}{\lambda^2} \right)^{1/3}. \]

4.3 **Behavior at the limit** \( T \to T_c - 0 \)

Despite the detailed analysis of low temperature region is beyond the scope of this paper it is instructive to see how equation that define the limiting value of susceptibility when approaching \( T_c \) from below is related to the marginal stability condition.

Let us first consider Eqs. (55) and suppose that near \( T_c \) FDT is modified as follows:

\[-\frac{d}{dt} C(t) = G(t) - \Delta G(t), \quad \Delta G(t) = -\frac{d}{dt} \Psi(C(t)),\]  

(60)

where \( \Psi \) is a nondecreasing function which derivative \( \Psi' \) vanishes outside the interval \([q_1, q_2]\) with \( q_2 < q_0 \). Obviously, these assumptions ensure the validity of Eq. (55) and lead to the following results:

\[ \Delta \chi = \Psi(q_2) - \Psi(q_1), \]  

(61)

\[ \Delta \Sigma_-(0) = \int_{q_1}^{q_2} dq \Sigma'(q)\Psi'(q) = \Sigma'(q_m)\Delta \chi, \]  

(62)

where \( q_m \in (q_1, q_2) \) is the middle point.

Below \( T_c \) \( \Delta \chi \neq 0 \), so subtracting Eq. (55b) from Eq. (55a) and dividing the result by \( \Delta \chi \) gives

\[ 1 = \chi(\chi + q_h) \left\{ \frac{w}{T^2} + \Sigma'(q_m) \right\}. \]  

(63)
where the terms proportional to $\Delta \chi$ are neglected. From Eqs. (55c,63) $q_m \to q_h$ at $T \to T_c - 0$ and we recover the marginal stability condition as equation for the limiting value of susceptibilities.

Clearly, aging cannot be taken into account provided the time translational invariance is unbroken, so the above consideration is closely related to the regime known as true ergodicity breaking [22,23]. In this case the system equilibrates in a separate ergodic component. Alternatively, according to the concept of weak ergodicity breaking [24], the system does not equilibrate and asymptotically $(t, t' \gg 1)$ FDT is violated at large separation times, $\tau \sim t'$, whereas the system reveals quasi-equilibrium behavior at sufficiently small $\tau$. It implies that the generalized form of FDT breaking contributions to the correlation functions is [17]

$$C(t, t') = C_0(t - t') + C_a(t'/t), \quad G(t, t') = G_0(t - t') + t^{-1}G_a(t'/t), \quad (64)$$

and the modification of FDT is given by the following relations

$$\partial_{t'} C_0(t - t') = G_0(t - t'), \quad x\partial_z C_a(z) = G_a(z), \quad (65)$$

where $z \equiv t'/t$ and $x$ parameterizes the violation of FDT. The time-dependent susceptibility $\chi(t, t')$ (see Eq. (26)) now depends on both the waiting time $t'$ and the separation time $\tau$. So the value of static susceptibility is different depending on the order in which limits $t' \to \infty$ and $\tau \to \infty$ are taken

$$\lim_{\tau \to \infty} \lim_{t' \to \infty} \chi(t' + \tau, t') = \chi_a = q_0 - q, \quad \lim_{\tau \to \infty} \chi(t' + \tau, t') = \chi = \chi_a + x\Delta q, \quad (66)$$

where $q_0 \equiv C(t, t) = C_0(0) + C_a(1)$, $\Delta q \equiv q - q_h = C_a(1) - C_a(0)$ and

$$q = \lim_{\tau \to \infty} \lim_{t' \to \infty} C(t' + \tau, t') = C_0(\infty) + C_a(1), \quad q_h = \lim_{\tau \to \infty} C(t' + \tau, t') = C_0(\infty) + C_a(0). \quad (67)$$

In the sense of Cubo [25], $\chi_a$ and $\chi$ can be referred to as adiabatic (thermodynamic) and isothermal susceptibilities, respectively; $q$ is the dynamical Edwards–Anderson parameter. Inserting Eq. (64) into Eq. (51) integrated over the second argument of $G$ gives the system

$$\frac{m}{T} \chi_a = 1 + \frac{w}{T^2} \chi_a^2 + \chi_a \left\{ \Sigma(\chi_a + q) - \Sigma(q) \right\}, \quad (68a)$$

$$\frac{m}{T} \chi = 1 + \frac{w}{T^2} \chi^2 + \chi \left\{ \Sigma(\chi_a + q) - \Sigma(q_h) + (x - 1)\Delta q \Sigma'(q_h) \right\}. \quad (68b)$$

In exactly the same manner as the marginal stability condition (58) was recovered from Eqs. (55,61,62), Eqs. (68) yield the condition (59) in the limit $\Delta q \to 0$ at
\( T \to T_c - 0 \). Interestingly, it follows that, despite the susceptibility \( \chi \) and non-ergodicity parameter \( \Delta q \) are continuous at the critical point, \( q \) can be discontinuous.

So, in both cases \( \Delta \chi \) plays the role of order parameter, but, in general, equations for the limiting values of susceptibilities are different depending on whether the aging is taken into consideration. Assuming the time homogeneity implies that the system can be described by Eqs. (55) below \( T_c \), but it can be shown that in this case we would encounter discontinuity of the susceptibility at the critical point. Thus we arrive at the conclusion that aging plays an important part in the problem under consideration.

5 Numerical results and discussion

In the above section the ergodicity breaking transition is treated on the basis of the dynamical approach. The critical temperature is defined as the point of marginal stability for FDT complaint solution. At \( h^2 = 0 \) Eqs. (56a,58) are easy to solve, so the critical temperature \( T_c \) is given by

\[
T_c = \frac{4 w^{3/2}}{\mu w_c} \left( 1 - \sqrt{1 - w_c/w} \right), \quad w_c = 3 \left[ \frac{2\lambda}{3\mu} \right]^2.
\] (69)

From Eq. (69) it is clear that, in order for the transition to occur, the disorder intensity \( w \) must exceed its critical \( w_c \). Dependence of \( w_c \) on the intensity of random field is depicted in Fig. 1. It is seen that there is no threshold for \( w \) at sufficiently large intensities of random field, whereas \( w_c \) is an increasing function of \( h^2 \) in the range of small intensities. Note that, in order to simplify analysis, the relevant quantities are rescaled as follows:

\[
\chi \to \mu \chi, \quad q \to \mu q, \quad w \to 6\mu^2 \lambda^{-2} w, \quad h^2 \to 6\mu^3 \lambda^{-2} w, \quad T \to \sqrt{6\mu^2 \lambda^{-1} T},
\]

so the system for \( T_c \) and \( \chi_c \) at the critical point takes the form:

\[
(\chi_c - 1) T_c^2 = w \chi_c^2 + \chi_c \left( (\chi_c + q_c)^3 - q_c^3 \right), \quad (70a)
\]

\[
T_c^2 = \chi_c (\chi_c + q_c) \left[ w + 3q_c^2 \right], \quad (70b)
\]

where \( q_c = h^{2/3}/2 \). Dependencies of \( T_c \) and the temperature of freezing transition, where the derivative of \( \chi \) with respect to temperature diverges, \( T_f \) on the disorder intensity \( w \) at \( h^2 = 0 \) are shown in Fig. 2. As is seen, the difference \( T_c - T_f \) goes to zero when \( w \) increases. The latter is not very surprising, for \( T_c = T_f \) at \( \lambda = 0 \) and the rescaled intensity \( w \) increases indefinitely when \( \lambda \to 0 \). As is shown in Fig. 3, dependence of \( T_c \) on \( q_c \), that is proportional to \( h^{2/3} \), reveals more complicated non-monotonous behavior at \( w \) just above its critical value. The critical temperature becomes an increasing function of the random field intensity at sufficiently large \( w \).
In solving the equations (56,58) one has to handle nonuniqueness of the solutions. For example, changing the sign of the radical in the expression for the critical temperature at \( h^2 = 0 \) gives another solution that corresponds to the nonphysical branch and has wrong behavior, \( T_c \to \infty \), in the limit \( \lambda \to 0 \). Note that choosing this branch gives another physically absurd result that \( q_h \) grows as temperature increases at \( T > T_c \).

Let us summarize the results of the paper. It is shown that the dynamical action of the thermodynamic system with quenched disorder can be calculated as a second Legendre transformation of the effective free energy functional. Despite the technique was employed to study the disordered system in the mean field approximation, it can be equally applied for consideration of fluctuational effects when going beyond the scope of the mean field theory. From the other hand, owing to the reflection symmetry of the system, \( \phi \to -\phi \), we have eliminated \( \langle \phi \rangle \) from the considerations concerning the ergodicity breaking transitions in the case of symmetric distribution of quenched variables. Clearly, when the distribution is nonsymmetric or \( A_3 \neq 0 \) (the presence of cubic anharmonicity), \( \langle \phi \rangle \neq 0 \) and the dynamical action contains the terms dependent on the averaged order parameter. Eqs. (45) then give coupled equations for the order parameter and correlation functions.

The method is applied for the study of \( \phi^4 \) model of thermodynamic system with quenched couplings and external field written in the site representation. Asymptotics of correlator is found to be affected by the random static field. It is characterized by the field induced parameter \( q_h \) which is a decreasing function of temperature at \( T > T_c \). Discussion at the end of Sec. 4 led us to the conclusion that the difference between the memory parameter \( q \) and \( q_h \) plays the role of an order parameter, so that adiabatic and isothermal susceptibilities differ if \( \Delta q = q - q_h \neq 0 \). Numerical analysis reveals that using the system (55) below \( T_c \) would predict discontinuity of the susceptibility at the critical point. Much more reasonable results can be obtained in the case of weak ergodicity breaking. So aging plays an important part in dynamics of the system below critical point. Recently reported results [27] for heteropolymers support the conclusion.
Appendix A

In the bulk of the paper the solutions under investigation are casual. In general, this is not the case. An instanton motion, where both initial and final boundary conditions need to be fixed, provides an important example of this kind. In this Appendix we show that the mapping, recently introduced in [26], between the uphill motion related to an instanton and the corresponding downhill motion going back in time can be constructed by making use of SUSY formalism.

Let us define the transformation of superfields as follows:

\[ \phi(z) \rightarrow \tilde{\phi}(z) = T_-(z) \phi(z), \quad \tilde{\phi}(z) \rightarrow \phi(z) = T_+(z) \tilde{\phi}(z), \]  
(A.1)

where

\[ T_\pm(z) \equiv \exp(\pm \bar{\theta} \theta \partial), \]  
(A.2)

Inserting Eq. (A.1) into the action (9,10) integrated from \( t = t_i \) to \( t = t_f \) gives the expression in terms of superfields

\[ S = \tilde{S} + (V_f - V_i)/T, \quad \tilde{L} = \tilde{D}_- \tilde{\phi} D_\phi \tilde{\phi} + V(\tilde{\phi}), \]  
(A.3a)

\[ \tilde{D}_- = T_-(z) \tilde{D} T_+(z) = \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial t} \equiv D'_{t \rightarrow -t}, \]  
(A.3b)

\[ D_- = T_-(z) D T_+(z) = \frac{\partial}{\partial \bar{\theta}} \equiv D'_{t \rightarrow -t}, \]  
(A.3c)

where \( V_i \equiv V(\eta(t_i)) \) and \( V_f \equiv V(\eta(t_f)) \). From Eq. (A.1) it is clear that

\[ G(z_1, z_2) = T_+(z_1) T_+(z_2) \tilde{G}(z_1, z_2), \quad \tilde{G}(z_1, z_2) = \langle \tilde{\phi}(z_1) \tilde{\phi}(z_2) \rangle. \]  
(A.4)

The latter follows the relations:

\[ \langle \varphi(t_1) \varphi(t_2) \rangle = \frac{\partial^2}{\partial t_1 \partial t_2} \tilde{C}(t_1, t_2) + \frac{\partial}{\partial t_1} \tilde{G}(t_1, t_2) + \frac{\partial}{\partial t_2} \tilde{G}(t_2, t_1), \]  
(A.5a)

\[ G(t_1, t_2) = \tilde{G}(t_1, t_2) + \frac{\partial}{\partial t_2} \tilde{C}(t_1, t_2), \]  
(A.5b)

\[ \tilde{C}(t_1, t_2) = \tilde{\tilde{C}}(t_1, t_2). \]  
(A.5c)

It remains to notice that FDT is invariant under the action of the transformation (A.1) followed by the inversion of time \( t \rightarrow -t \), so the Lagrangian \( \tilde{L} \) describes normal downhill motion formally inverted in time. The resulting relations (A.5) are identical with those derived in [26] and enable calculating of the Green function for the instanton process provided that the corresponding causal solutions are known. More details on the subject will be published elsewhere.
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FIGURE CAPTIONS

Figure 1. Dependence of the critical value of quenched disorder intensity $w_c$ on the random field $q_c = [h^2/2]^{1/3}$. ($w_c$ and $h^2$ are calculated in units of $\lambda^2/(6\mu^2)$ and $\lambda^2/(6\mu^3)$, respectively.)

Figure 2. Dependencies of the critical temperatures $T_c$ (ergodicity breaking transition) and $T_f$ (freezing transition) on the quenched disorder intensity $w$ in the absence of random field, $h^2 = 0$. ($w$ and the temperatures are calculated in units of $\lambda^2/(6\mu^2)$ and $\mu^2/(\sqrt{6}\lambda)$, respectively.)

Figure 3. Temperature of ergodicity breaking transition $T_c$ versus the intensity of random field $q_c = [h^2/2]^{1/3}$ at various values of $w$. 


FIGURE 1.
FIGURE 2.
FIGURE 3.