Abstract. In this article we introduce a new type of nonlocal operators and study the Cauchy problem for certain parabolic-type pseudodifferential equations naturally associated to these operators. Some of these equations are the $p$-adic master equations of certain models of complex systems introduced by Avetisov et al. The fundamental solutions of these parabolic-type equations are transition functions of random walks on the $n$-dimensional vector space over the field of $p$-adic numbers. We study some properties of these random walks, including the first passage time.

1. Introduction

During the last twenty-five years there have been a strong interest on random walks on ultrametric spaces mainly due its connections with models of complex systems, such as glasses and proteins. Random walks on ultrametric spaces are very convenient for describing phenomena whose space of states display a hierarchical structure, see e.g. [2]-[7], [9], [12], [14], [15], [19], [20], [25], [26], [27], and the references therein. In the middle of the eighties G. Frauenfelder, G. Parisi, D. Stain, among others, proposed using ultrametric spaces to describe the states of complex biological systems, which possess a natural hierarchical organization, see e.g. [19], [20]. Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes, see [2]-[7]. These models can be applied, among other things, to the study the relaxation of biological complex systems [5]. From a mathematical point view, in these models the time-evolution of a complex system is described by a $p$-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

This article continues and extends some of the mathematical results given in [5], [3]. We introduced a new class of nonlocal operators which includes the Vladimirov operator. These operators are determined by a radial function which determines the structure of the energy landscape being used. In [5] several solvable models for degenerated landscapes of types linear, logarithmic and exponential were studied. In this article we study a large class of solvable models, we have called them polynomial and exponential landscapes, which includes the linear and exponential landscapes considered in [5], see Section 3.2. We attach to each of these operators

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a Markov process (random walk) which is bounded and has no discontinuities other than jumps, see Theorems 4.3, 5.3. We also solve the Cauchy problem for the master equations attached to these operators, see Theorem 6.5. Finally, we study the first passage time problem for random walks attached to polynomial landscapes, see Theorem 7.7. All the results are formulated in arbitrary dimension.

The article is organized as follows. In section 2 we review the basic notions of $p$-adic analysis. In Section 3 we introduce a new type of nonlocal operators, these operators encode the underlying energy landscapes studied here. We show that these operators are pseudodifferential and give some properties of their symbols. In Sections 4-5 we study the Markov processes attached to the operators introduced in Section 3. In Section 6 we study the Cauchy problem for the master equations attached to these operators, see Theorem 6.5. Finally, we study the first passage time problem for the random walks attached to polynomial landscapes, see Theorems 4.3-5.3. We also solve the Cauchy problem for the master equations which are pseudodifferential equations of parabolic-type. Finally, in Section 7 we consider the problem of the first passage time for random walks on $p$-adic spaces. A particular case of this problem was considered earlier by Avetisov, Bikulov and Zubarev in [3]. In the present article more general evolutions are considered and the regimes of recurrent and transient random walks are investigated.

2. Preliminaries

In this section we fix the notation and collect some basic results on $p$-adic analysis that we will use through the article. For a detailed exposition on $p$-adic analysis the reader may consult [1], [23], [26].

2.1. The field of $p$-adic numbers. Along this article $p$ will denote a prime number different from 2. The field of $p$–adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$–adic norm $| \cdot |_p$, which is defined as

$$
|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b} 
\end{cases}
$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$–adic order of $x$. We extend the $p$–adic norm to $\mathbb{Q}_p^n$ by taking

$$
||x||_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n.
$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $||x||_p = p^{-\text{ord}(x)}$. Any $p$–adic number $x \neq 0$ has a unique expansion $x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \ldots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the fractional part of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$
\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{-\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0.
\end{cases}
$$

For $\gamma \in \mathbb{Z}$, denote by $B_p^n(\gamma) = \{x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^\gamma\}$ the ball of radius $p^\gamma$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_p^n(0) := B_p^n$. Note that $B_p^n(\gamma) = B_1(a_1) \times \cdots \times B_1(a_n)$, where $B_1(a_i) := \{x \in \mathbb{Q}_p : ||x - a_i||_p \leq p^\gamma\}$ is the one-dimensional ball of radius $p^\gamma$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_p^n(0)$ is equals the product of $n$ copies of $B_0(0) := \mathbb{Z}_p$, the ring of $p$–adic integers. We denote by $\Omega(||x||_p)$ the characteristic function of $B_p^n(0)$. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of $A$. 

2.2. The Bruhat-Schwartz space. A complex-valued function \( \varphi \) defined on \( \mathbb{Q}_p^n \) is called locally constant if for any \( x \in \mathbb{Q}_p^n \) there exist an integer \( l(x) \in \mathbb{Z} \) such that

\[
\varphi(x + x') = \varphi(x) \text{ for } x' \in B_l(x).
\]

A function \( \varphi : \mathbb{Q}_p^n \to \mathbb{C} \) is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. The \( \mathbb{C} \)-vector space of Bruhat-Schwartz functions is denoted by \( S(\mathbb{Q}_p^n) := S \). For \( \varphi \in S(\mathbb{Q}_p^n) \), the largest of such number \( l = l(\varphi) \) satisfying \( \varphi(x + x') = \varphi(x) \) is called the exponent of local constancy of \( \varphi \).

Let \( S'(\mathbb{Q}_p^n) := S' \) denote the set of all functionals (distributions) on \( S(\mathbb{Q}_p^n) \). All functionals on \( S(\mathbb{Q}_p^n) \) are continuous.

Set \( \Psi(y) = \exp(2\pi i \{y\}_p) \) for \( y \in \mathbb{Q}_p \). The map \( \Psi(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuous map from \( \mathbb{Q}_p \) into the unit circle satisfying \( \Psi(y_0 + y_1) = \Psi(y_0)\Psi(y_1), y_0, y_1 \in \mathbb{Q}_p \).

Given \( \xi = (\xi_1, \ldots, \xi_n) \) and \( x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n \), we set \( \xi \cdot x := \sum_{j=1}^n \xi_j x_j \).

The Fourier transform of \( \varphi \in S(\mathbb{Q}_p^n) \) is defined as

\[
(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \cdot x) \varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,
\]

where \( d^n x \) is the Haar measure on \( \mathbb{Q}_p^n \) normalized by the condition \( \text{vol}(B_1) = 1 \). The Fourier transform is a linear isomorphism from \( S(\mathbb{Q}_p^n) \) onto itself satisfying \( \mathcal{F}(\mathcal{F}\varphi)(\xi) = \varphi(-\xi) \). We will also use the notation \( \mathcal{F}_{x \rightarrow \xi} \varphi \) and \( \hat{\varphi} \) for the Fourier transform of \( \varphi \).

2.2.1. Fourier transform. The Fourier transform \( \mathcal{F}[f] \) of a distribution \( f \in S'(\mathbb{Q}_p^n) \) is defined by

\[
(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in S(\mathbb{Q}_p^n).
\]

The Fourier transform \( f \mapsto \mathcal{F}[f] \) is a linear isomorphism from \( S'(\mathbb{Q}_p^n) \) onto \( S'(\mathbb{Q}_p^n) \).

Furthermore, \( f = \mathcal{F}[\mathcal{F}[f](-\xi)] \).

3. A New Class of Nonlocal Operators

Take \( \mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\} \), and fix a function

\[
w : \mathbb{Q}_p^n \to \mathbb{R}_+
\]

satisfying the following properties:

(i) \( w(y) \) is a radial (i.e. \( w(y) = w(\|y\|_p) \)) and continuous function;

(ii) \( w(y) = 0 \) if and only if \( y = 0 \);

(iii) there exists constants \( C_0 > 0, M \in \mathbb{Z}, \) and \( \alpha_1 > n \) such that

\[
C_0 \|y\|_p^\alpha_1 \leq w(\|y\|_p), \text{ for } \|y\|_p \geq p^M.
\]

Note that condition (iii) implies that

\[
\int_{\|y\|_p \geq p^M} \frac{d^n y}{w(\|y\|_p)} < \infty.
\]

In addition, since \( w(y) \) is a continuous function, \( \text{(3.1)} \) holds for any \( M \in \mathbb{Z} \). Convergence conditions for integral kernels of type \( \text{(3.1)} \) were considered in [15], [16] and [17].

We define
(3.2) \( (W\varphi)(x) = \kappa \int_{Q^n_p} \frac{\varphi(x - y) - \varphi(x)}{w(y)} d^n y \), for \( \varphi \in S \),

where \( \kappa \) is a positive constant.

**Lemma 3.1.** For \( 1 \leq \rho \leq \infty \),

\[
S(Q^n_p) \rightarrow L^\rho(Q^n_p)
\]

is a well-defined linear operator. Furthermore,

\[
(3.3) \quad F[W\varphi](\xi) = -\kappa \left( \int_{Q^n_p} \frac{1 - \Psi(-y \cdot \xi)}{w(y)} d^n y \right) F[\varphi](\xi).
\]

**Proof.** Note that

\[
(3.4) \quad (W\varphi)(x) = \kappa \frac{1_{Q^n_p \setminus B^n_{pM}}(x)}{w(x)} * \varphi(x) - \kappa \varphi(x) \left( \int_{\|y\| \geq pM} \frac{d^n y}{w(y)} \right),
\]

for some constant \( M = M(\varphi) \). If \( \varphi \in S \subset L^\rho \), for \( 1 \leq \rho \leq \infty \), then the Young inequality implies that the first term on the right-hand side of (3.4) belongs to \( L^\rho \) for \( 1 \leq \rho \leq \infty \), and by (3.1) the second term in (3.4) also belongs to \( L^\rho \) for \( 1 \leq \rho \leq \infty \). Finally, formula (3.3) follows from Fubini’s theorem, since

\[
\left| \frac{\varphi(x - y) - \varphi(x)}{w(y)} \right| \in L^1(Q^n_p \times Q^n_p, d^n x d^n y).
\]

We set

\[
A_w(\xi) := \int_{Q^n_p} \frac{1 - \Psi(-y \cdot \xi)}{w(y)} d^n y.
\]

**Lemma 3.2.** The function \( A_w(\xi) \) has the following properties:

(i) for \( \|\xi\|_p = p^{-\gamma} \neq 0 \), with \( \gamma = \text{ord}(\xi) \),

\[
(3.5) \quad A_w(p^{-\gamma}) = (1 - p^{-\gamma}) \sum_{j=\gamma+1}^{\infty} \frac{p^{nj}}{w(p^j)} + \frac{p^{n\gamma}}{w(p^{j+1})};
\]

(ii) it is radial, positive, continuous, and \( A_w(0) = 0 \), (iii) \( A_w(p^{-\text{ord}(\xi)}) \) is a decreasing function of \( \text{ord}(\xi) \).

**Proof.** We write \( \xi = p^{-\gamma}\xi_0 \), with \( \gamma = \text{ord}(\xi) \) and \( \|\xi_0\|_p = 1 \). Then

\[
(3.6) \quad A_w(\xi) = \int_{Q^n_p} \frac{1 - \Psi(-y \cdot \xi_0)}{w(\|y\|_p)} d^n y = p^{n\gamma} \int_{Q^n_p} \frac{1 - \Psi(-z \cdot \xi_0)}{w(p^n \|z\|_p)} d^n z.
\]

We now note that

\[
Q^n_p \setminus \{0\} = \bigcup_{j \in \mathbb{Z}} p^j U
\]
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\[ U := \left\{ y \in \mathbb{Q}_p^n : \|y\|_p = 1 \right\} \]

By using this partition and (3.6), we have

\[
A_w(\xi) = \sum_{j \in \mathbb{Z}} p^{jn+\gamma} \int \frac{1 - \Psi(-z \cdot \xi_0)}{w(p^n \|z\|_p)} d^n z
\]

\[
= \sum_{j \in \mathbb{Z}} \frac{p^{-jn+\gamma}}{w(p^{-j+\gamma})} \left\{ (1 - p^n) - \int_U \Psi(-p^j y \cdot \xi_0) d^n y \right\}.
\]

By using the formula

\[
\int_U \Psi(-p^j y \cdot \xi_0) d^n y = \begin{cases} 
1 - p^n & \text{if } j \geq 0 \\
-p^n & \text{if } j = -1 \\
0 & \text{if } j < -1,
\end{cases}
\]

we get

\[
A_w(\xi) = (1 - p^n) \sum_{j=2}^{\infty} \frac{p^{n(\gamma+j)}}{w(p^{n+j})} + \frac{p^{n\gamma}}{w(p^{n+1})} = (1 - p^n) \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{w(p^j)} + \frac{p^{n\gamma}}{w(p^{n+1})}.
\]

From (3.8) follows that \( A_w(\xi) \) is radial, positive, continuous outside of the origin, and that \( A_w(p^{-\text{ord}(\xi)}) \) is a decreasing function of \( \text{ord}(\xi) \). To show the continuity at origin, we proceed as follows. Since \( \sum_{j=M}^{\infty} \frac{p^{nj}}{w(p^j)} < \infty \), c.f. (3.1),

\[
A_w(0) := \lim_{\gamma \to \infty} (1 - p^n) \sum_{j=\gamma+2}^{\infty} \frac{p^{nj}}{w(p^j)} + \lim_{\gamma \to \infty} \frac{p^{n\gamma}}{w(p^{n+1})} = 0.
\]

Proposition 3.3. (i) \((W \varphi)(x) = -\kappa \mathcal{F}_{\xi \to x}^{-1} \left( A_w(\|\xi\|_p) F_{x \to \xi} \varphi \right) \) for \( \varphi \in S(\mathbb{Q}_p^n) \), and \( W \varphi \in C(\mathbb{Q}_p^n) \cap L^p(\mathbb{Q}_p^n) \), for \( 1 \leq p \leq \infty \). The Operator \( W \) extends to an unbounded and densely defined operator in \( L^2(\mathbb{Q}_p^n) \) with domain

\[
\text{Dom}(W) = \left\{ \varphi \in L^2 ; A_w(\|\xi\|_p) F \varphi \in L^2 \right\}.
\]

(ii) \((-W, \text{Dom}(W))\) is self-adjoint and positive operator.

(iii) \( W \) is the infinitesimal generator of a contraction \( C_0 \) semigroup \((T(t))_{t \geq 0}\). Moreover, the semigroup \((T(t))_{t \geq 0}\) is bounded holomorphic with angle \( \pi/2 \).

Proof. (i) It follows from Lemma 3.1 and the fact that \( A_w(\|\xi\|_p) \) is continuous, c.f. Lemma 3.2. (ii) follows from the fact that \( W \) is a pseudodifferential operator and that the Fourier transform preserves the inner product of \( L^2 \). (iii) It follows of well-known results, see e.g. [11, Chap. 2, Sect. 3] or [8]. For the property of the semigroup of being holomorphic, see e.g. [11, Chap. 2, Sect. 4.7].
3.1. Some additional results.

**Lemma 3.4.** Assume that there exist positive constants $\alpha_1, \alpha_2, C_0, C_1$, with $\alpha_1 > n$, $\alpha_2 > n$, and $\alpha_3 \geq 0$, such that

\[(3.10) \quad C_0 \|\xi\|_p^{\alpha_1} \leq w(\|\xi\|_p) \leq C_1 \|\xi\|_p^{\alpha_2} e^{\alpha_3 \|\xi\|_p}, \] for any $\xi \in \mathbb{Q}_p^n$.

Then there exist positive constants $C_2, C_3$, such that

\[C_2 \|\xi\|_p^{\alpha_2} e^{-\alpha_3 \|\xi\|_p} \leq A_w(\|\xi\|_p) \leq C_3 \|\xi\|_p^{\alpha_1 - \alpha_2 + \alpha_3} \]

for any $\xi \in \mathbb{Q}_p^n$, with the convention that $e^{-\alpha_3 \|\xi\|_p} := \lim_{\|\xi\|_p \to 0} e^{-\alpha_3 \|\xi\|_p} = 0$.

Furthermore, if $\alpha_3 > 0$, then $\alpha_1 \geq \alpha_2$, and if $\alpha_3 = 0$, then $\alpha_1 = \alpha_2$.

**Proof.** By using the lower bound for $w$ given in (3.10), and $\|\xi\|_p = p^{-\gamma}$,

\[A_w(\|\xi\|_p) \leq \frac{(1 - p^{-\gamma})}{C_0} \sum_{j=1}^{\infty} \frac{p^{\alpha_3}}{p^{\alpha_1} + p^{\alpha_2}} \leq C_3 \|\xi\|_p^{\alpha_1 - \alpha_2 + \alpha_3}.
\]

On the other hand, $A_w \left( \|\xi\|_p \right) \geq \frac{p^{\alpha_3}}{w(p^{\alpha_1 + 1})}$, and by using the upper bound for $w$ given in (3.10),

\[A_w \left( \|\xi\|_p \right) \geq \frac{p^{\alpha_3}}{w(p^{\alpha_1 + 1})} \geq \frac{p^{\alpha_3}}{C_1 p^{\alpha_2 (\gamma + 1)} e^{\alpha_3 \|\xi\|_p}} \geq C_2 \|\xi\|_p^{\alpha_2 - \alpha_1 + \alpha_3} e^{-\alpha_3 \|\xi\|_p}.
\]

\[\square\]

**Definition 3.5.** We say that $W$ (or $A_w$) is of exponential type if inequality (3.10) is only possible for $\alpha_3 > 0$ with $\alpha_1, \alpha_2, C_0, C_1$ positive constants and $\alpha_1 > n$, $\alpha_2 > n$.

If (3.10) holds for $\alpha_3 = 0$ with $\alpha_1, \alpha_2, C_0, C_1$ positive constants and $\alpha_1 > n$, $\alpha_2 > n$, we say that $W$ (or $A_w$) is of polynomial type.

We note that if $W$ is of polynomial type then $\alpha_1 = \alpha_2 > n$ and $C_0, C_1$ are positive constants with $C_1 \geq C_0$.

**Lemma 3.6.** With the hypotheses of Lemma 3.4

\[e^{-t\kappa A_w(\|\xi\|_p)} \in L^p(\mathbb{Q}_p^n) \text{ for } 1 \leq p < \infty \text{ and } t > 0.
\]

**Proof.** Since $e^{-t\kappa A_w(\|\xi\|_p)}$ is a continuous function, it is sufficient to show that there exists $M \in \mathbb{N}$ such that

\[I_M(t) := \int_{\|\xi\|_p > p^M} e^{-\rho t A_w(\|\xi\|_p)} d\xi < \infty, \text{ for } t > 0.
\]

Take $M \in \mathbb{N}$, by Lemma 3.4, we have

\[C_2 \|\xi\|_p^{\alpha_2 - \alpha_1} e^{-\alpha_3 \|\xi\|_p} \geq C_2 \|\xi\|_p^{\alpha_2 - \alpha_1} e^{-\alpha_3 p^{M+1}} \text{ for } \|\xi\|_p > p^M,
\]

and (with $B = C_2 e^{-\alpha_3 p^{M+1}}$),

\[I_M(t) \leq \int_{\|\xi\|_p > p^M} e^{-t(B \|\xi\|_p^{\alpha_2 - \alpha_1})} d\xi \leq C(M, \kappa, \rho) t^{\frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}}, \text{ for } t > 0.
\]

\[\square\]
3.2. $p$-adic description of characteristic relaxation in complex systems.
In [5], Avetisov et al. developed a new approach to the description of relaxation processes in complex systems (such as glasses, macromolecules and proteins) on the basis of $p$-adic analysis. The dynamics of a complex system is described by a random walk in the space of configurational states, which is approximated by an ultrametric space ($\mathbb{Q}_p$). Mathematically speaking, the time-evolution of the system is controlled by a master equation of the form
\begin{equation}
\frac{\partial f(x,t)}{\partial t} = \int_{\mathbb{Q}_p} \{v(y|y)f(y,t) - v(y|x)f(x,t)\} \, dy, \quad x \in \mathbb{Q}_p, \quad t \in \mathbb{R}_+,
\end{equation}
where the function $f(x,t) : \mathbb{Q}_p \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a probability density distribution, and the function $v(y|x) : \mathbb{Q}_p \times \mathbb{Q}_p \rightarrow \mathbb{R}_+$ is the probability of transition from state $y$ to the state $x$ per unit time. The transition from a state $y$ to a state $x$ can be perceived as overcoming the energy barrier separating these states. In [5] an Arrhenius type relation was used:
\[ v(y|x) \sim A(T) \exp \left\{ -\frac{U(x|y)}{kT} \right\}, \]
where $U(x|y)$ is the height of the activation barrier for the transition from the state $y$ to state $x$, $k$ is the Boltzmann constant and $T$ is the temperature. This formula establishes a relation between the structure of the energy landscape $U(x|y)$ and the transition function $v(y|x)$. The case $v(y|x) = v(y|x)$ corresponds to a degenerate energy landscape. In this case the master equation (3.11) takes the form
\begin{equation}
\frac{\partial f(x,t)}{\partial t} = \int_{\mathbb{Q}_p} v(|x-y|_p) \{f(y,t) - f(x,t)\} \, dy,
\end{equation}
where $v(|x-y|_p) = A(T) \exp \left\{ -\frac{U(|x-y|_p)}{kT} \right\}$. By choosing $U$ conveniently, several energy landscapes can be obtained. Following [5], there are three basic landscapes: (i) (logarithmic) $v(|x-y|_p) = \frac{1}{|x-y|_p} \log(1+|x-y|_p)$, $\alpha > 1$ (ii) (linear) $v(|x-y|_p) = \frac{1}{|x-y|_p^\alpha}$, $\alpha > 0$, (iii) (exponential) $v(|x-y|_p) = \frac{e^{-\alpha|x-y|_p}}{|x-y|_p}$, $\alpha > 0$.

Thus, it is natural to study the following Cauchy problem:
\begin{equation}
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \kappa \int_{\mathbb{Q}_p^n} \frac{u(x-y,t)-u(x,t)}{w(y)} \, d^n y, \quad x \in \mathbb{Q}_p^n, \quad t \in \mathbb{R}_+, \\
u(x,0) &= \varphi \in S(\mathbb{Q}_p^n),
\end{aligned}
\end{equation}
where $w(y)$ is a radial function belonging to a class of functions that contains functions like:
(i) $w(||y||_p) = \Gamma_p^n(-\alpha) ||y||_p^{\alpha+n}$, here $\Gamma_p^n(\cdot)$ is the $n$-dimensional $p$-adic Gamma function, and $\alpha > 0$;
(ii) $w(||y||_p) = ||y||_p^2 e^{\alpha||y||}$, $\alpha > 0$.

We recall that operator $W$ corresponding to case (i) is the Taibleson operator which is a generalization of the Vladimirov operator, see [21].

By imposing condition (3.10) to $w$, we include the basic energies landscapes in our study. Take $w(||y||_p)$ satisfying (3.10) and take $f(||y||_p)$ a continuous function.
such that
\[ 0 < \sup_{y \in \mathbb{Q}_p^n} f \left( \|y\|_p \right) < \infty \quad \text{and} \quad 0 < \inf_{y \in \mathbb{Q}_p^n} f \left( \|y\|_p \right) < \infty. \]

Then \( f \left( \|y\|_p \right) w(\|y\|_p) \) satisfies (3.10).

On the other hand, take \( P(\|y\|_p) \) to be a polynomial in \( \|y\|_p \) with real positive coefficients and nonzero constant term, thus \( \inf_{y \in \mathbb{Q}_p^n} P \left( \|y\|_p \right) = P(0) > 0 \), and take \( w(\|y\|_p) = \|y\|_p^\beta e^{\alpha\|y\|_p} \) satisfying (3.10), then \( P \left( \|y\|_p \right) w(\|y\|_p) \) also satisfies (3.10).

Finally we note that \( \|y\|_p^\beta \ln^{\alpha}(1 + \|y\|_p), \beta > n, \alpha \in \mathbb{N}, \) does not satisfies \( \|y\|_p^{\alpha_1} \leq \|y\|_p^\beta \ln^{\alpha}(1 + \|y\|_p) \) for any \( y \in \mathbb{Q}_p^n \), and hence our results do not include the case of logarithmic landscapes.

4. Heat Kernels

In this section we assume that function \( w \) satisfies conditions (3.10).

We define
\[ Z(x, t; w, \kappa) := Z(x, t) = \int_{\mathbb{Q}_p^n} e^{-\kappa A_w(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi \] for \( t > 0 \) and \( x \in \mathbb{Q}_p^n \).

Note that by Lemma 3.6, \( Z(x, t) = \mathcal{F}_{\xi \to x}^{-1}[e^{-\kappa A_w(\|\xi\|_p)}] \in L^1 \cap L^2 \) for \( t > 0 \). We call such a function a heat kernel. When considering \( Z(x, t) \) as a function of \( x \) for \( t \) fixed we will write \( Z_t(x) \).

**Lemma 4.1.** There exists a positive constant \( C \), such that
\[ Z(x, t) < Ct \|x\|_p^{-\alpha_1}, \] for \( x \in \mathbb{Q}_p^n \setminus \{0\} \) and \( t > 0 \).

**Proof.** Let \( \|x\|_p = p^\beta \). Since \( Z(x, t) \in L^1(\mathbb{Q}_p^n) \) for \( t > 0 \), by applying the formula for the Fourier transform of radial function, we get
\[ Z(x, t) = \|x\|_p^{-n} \left( 1 - p^{-n} \sum_{j=0}^{\infty} e^{-\kappa A_w(p^{-\beta-j})t} p^{-n j} - e^{-\kappa A_w(p^{-\beta+1})t} \right). \]

By using that \( e^{-\kappa A_w(p^{-\beta-j})t} \leq 1 \) for \( j \in \mathbb{N} \), we have
\[ Z(x, t) \leq \|x\|_p^{-n} \left[ 1 - e^{-\kappa A_w(p^{-\beta+1})t} \right]. \]

We now apply the Mean Value Theorem to the real function \( f(u) = e^{-\kappa A_w(p^{-\beta+1})u} \) on \([0, t]\) with \( t > 0 \), and Lemma 3.3
\[ Z(x, t) \leq C_0 \|x\|_p^{-n} t A_w(p^{-\beta+1}) \leq Ct \|x\|_p^{-\alpha_1}. \]

**Lemma 4.2.** \( Z(x, t) \geq 0 \), for \( x \in \mathbb{Q}_p^n \) and \( t > 0 \).

**Proof.** Since \( e^{-t A_w(\|\xi\|_p)} \) is integrable for \( t > 0 \) and radial, we have
The function $Z(x,t)$ has the following properties:
(i) $Z(x,t) \geq 0$ for any $t > 0$;
(ii) $\int Z(x,t) d^n x = 1$ for any $t > 0$;
(iii) $Z_t(x) \in C(Q^n_p, \mathbb{R}) \cap L^1(Q^n_p) \cap L^2(Q^n_p)$ for any $t > 0$;
(iv) $Z_t(x) * Z_t'(x) = Z_{t+t'}(x)$ for any $t, t' > 0$;
(v) $\lim_{t \to 0^+} Z(x,t) = \delta(x)$ in $S'(Q^n_p)$.

Proof. (i) It follows from Lemma 3.2. (ii) For any $t > 0$ the function $e^{-\kappa t A_w(\|\|_p)}$ is continuous at $\xi = 0$ and by Lemma 3.6 we have $e^{-\kappa t A_w(\|\|_p)} \in L^1 \cap L^2$ for $t > 0$, then $Z_t(x) \in L^1 \cap L^2$ for $t > 0$. Now the statement follows from the inversion formula for the Fourier transform. (iii) From Lemma 3.6 with $\rho = 1, 2$, we have $Z_t(x) \in C(Q^n_p, \mathbb{R}) \cap L^1(Q^n_p), t > 0$, and by (i) and (ii), $Z_t(x) \in L^2(Q^n_p)$. (iv) By the previous property $Z_t(x) \in L^1$ for any $t > 0$, then

$$Z_t(x) * Z_t'(x) = \mathcal{F}^{-1}_{\xi \to x} \left( e^{-\kappa t A_w(\|\|_p)} e^{-\kappa t' A_w(\|\|_p)} \right) = \mathcal{F}^{-1}_{\xi \to x} \left( e^{-\kappa (t+t') A_w(\|\|_p)} \right) = Z_{t+t'}(x).$$

(v) Since we have $e^{-\kappa t A_w(\|\|_p)} \in C(Q^n_p, \mathbb{R}) \cap L^1$ for $t > 0$, c.f. Lemma 3.6 the inner product

$$\left\langle e^{-\kappa t A_w(\|\|_p)}, \phi \right\rangle = \int_{Q^n_p} e^{-\kappa t A_w(\|\|_p)} \overline{\phi(\xi)} d^n \xi$$

defines a distribution on $Q^n_p$, then, by the Dominated Converge Theorem,

$$\lim_{t \to 0^+} \left\langle e^{-\kappa t A_w(\|\|_p)}, \phi \right\rangle = \langle 1, \phi \rangle$$

and thus

$$\lim_{t \to 0^+} \langle Z(x,t), \phi \rangle = \lim_{t \to 0^+} \left\langle e^{-\kappa t A_w(\|\|_p)}, \mathcal{F}^{-1} \phi \right\rangle = \langle 1, \mathcal{F}^{-1} \phi \rangle = \langle \delta, \phi \rangle.$$

□

5. Markov Processes over $Q^n_p$

Along this section we consider $\left(Q^n_p, \|\|_p\right)$ as complete non-Archimedean metric space and use the terminology and results of [10, Chapters Two, Three]. Let $\mathcal{B}$ denote the Borel $\sigma$–algebra of $Q^n_p$. Thus $(Q^n_p, \mathcal{B}, d^n x)$ is a measure space. We set

$$p(t, x, y) := Z(x - y, t) \text{ for } t > 0, x, y \in Q^n_p,$$
and
\[ P(t, x, B) = \begin{cases} \int_B p(t, y, x) \, d^n y & \text{for } t > 0, \ x \in \mathbb{Q}_p^n, \ B \in \mathcal{B} \\ 1_B(x) & \text{for } t = 0. \end{cases} \]

**Lemma 5.1.** With the above notation the following assertions hold:
(i) \( p(t, x, y) \) is a normal transition density;
(ii) \( P(t, x, B) \) is a normal transition function.

**Proof.** The result follows from Theorem 4.3, see [10, Section 2.1] for further details. \( \square \)

**Lemma 5.2.** The transition function \( P(t, x, B) \) satisfies the following two conditions:
(i) for each \( u \geq 0 \) and compact \( B \)
\[ \lim_{x \to \infty} \sup_{t \leq u} P(t, x, B) = 0 \ [\text{Condition } L(B)]. \]
(ii) for each \( \epsilon > 0 \) and compact \( B \)
\[ \lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p^n \setminus B^n_\epsilon(x)) = 0 \ [\text{Condition } M(B)]. \]

**Proof.** (i) By Lemma 4.1 and the fact that \( \| \cdot \|_p \) is an ultranorm, we have
\[ P(t, x, B) \leq Ct \int_B \| x - y \|_p^{-\alpha_1} \, d^n y = tC \| x \|_p^{-\alpha_1} \, \text{vol}(B) \text{ for } x \in \mathbb{Q}_p^n \setminus B. \]
Therefore \( \lim_{x \to \infty} \sup_{t \leq u} P(t, x, B) = 0. \)

(ii) Again, by Lemma 4.1, the fact that \( \| \cdot \|_p \) is an ultranorm, and \( \alpha_1 > n \), we have
\[ P(t, x, \mathbb{Q}_p^n \setminus B^n_\epsilon(x)) \leq Ct \int_{\| x - y \|_p > \epsilon} \| x - y \|_p^{-\alpha_1} \, d^n y = C t \int_{\| z \|_p > \epsilon} \| z \|_p^{-\alpha_1} \, d^n z = C' (\alpha_1, \epsilon, n) t. \]
Therefore
\[ \lim_{t \to 0^+} \sup_{x \in B} P(t, x, \mathbb{Q}_p^n \setminus B^n_\epsilon(x)) = \lim_{t \to 0^+} C' (\alpha_1, \epsilon, n) t = 0. \]
\( \square \)

**Theorem 5.3.** \( Z(x, t) \) is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.

**Proof.** The result follows from [10, Theorem 3.6] by using that \( (\mathbb{Q}_p^n, \| x \|_p) \) is semi-compact space, i.e. a locally compact Hausdorff space with a countable base, and \( P(t, x, B) \) is a normal transition function satisfying conditions \( L(B) \) and \( M(B) \), c.f. Lemmas 5.1, 5.2. \( \square \)
6. The Cauchy Problem

Consider the following Cauchy problem:

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - W u(x, t) &= 0, \quad x \in \mathbb{Q}_p^n, \ t \in [0, \infty), \\
u(x, 0) &= u_0(x), \quad u_0(x) \in \text{Dom}(W),
\end{align*}
\]

where \((W \phi)(x) = -\kappa F^{-1}_{\xi \rightarrow x} (A_w(\|w\|_p) \mathcal{F}_{\xi \rightarrow \xi} \phi)\) for \(\phi \in \text{Dom}(W)\), see (6.3), and \(u : \mathbb{Q}_p^n \times [0, \infty) \rightarrow \mathbb{C}\) is an unknown function. We say that a function \(u(x, t)\) is a solution of (6.1), if \(u(x, t) \in C([0, \infty), \text{Dom}(W)) \cap C^1([0, \infty), L^2(\mathbb{Q}_p^n))\) and \(u\) satisfies (6.1) for all \(t \geq 0\).

In this section, we understand the notions of continuity in \(t\), differentiability in \(t\) and equalities in the \(L^2(\mathbb{Q}_p^n)\) sense, as it is customary in the semigroup theory.

We know from Proposition 3.3 that the operator \(W\) generates a \(C_0\) semigroup \((T(t))_{t \geq 0}\), then Cauchy problem (6.1) is well-posed, i.e. it is uniquely solvable with the solution continuously dependent on the initial data, and its solution is given by \(u(x, t) = T(t)u_0(x)\), for \(t \geq 0\), see e.g. [8] Theorem 3.1.1. However the general theory does not give an explicit formula for the semigroup \((T(t))_{t \geq 0}\). We show that the operator \(T(t)\) for \(t > 0\) coincides with the operator of convolution with the heat kernel \(Z_t * .\). In order to prove this, we first construct a solution of Cauchy problem (6.1) with the initial value from \(S\) without using the semigroup theory. Then we extend the result to all initial values from \(\text{Dom}(W)\), see Propositions 6.2-6.4.

6.1. Homogeneous equations with initial values in \(S\). To simplify the notation, set \(Z_t * u_0 = (Z_t(x) \ast u_0(x)) \mid_{t=0} = u_0.\) We define the function

\[
u(x, t) = Z_t(x) \ast u_0(x), \quad t \geq 0.
\]

Since \(Z_t(x) \in L^1\) for \(t > 0\) and \(u_0 \in \mathcal{S}^{\infty}(\mathbb{Q}_p^n) \subset L^{\infty}(\mathbb{Q}_p^n)\), the convolution exists and is a continuous function, see e.g. [22] Theorem 1.1.6.

**Lemma 6.1.** Take \(u_0 \in \mathcal{S}\) with the support of \(\hat{u}_0\) contained in \(B_R\), and \(u(x, t), t \geq 0\) defined as in (6.2). Then the following assertions hold:

(i) \(u(x, t)\) is continuously differentiable in time for \(t \geq 0\) and the derivative is given by

\[
\frac{\partial u(x, t)}{\partial t} = -\kappa F^{-1}_{\xi \rightarrow x} (e^{-\kappa t A_w(\|\xi\|_p)} A_w(\|\xi\|_p) 1_{B_R}(\xi)) \ast u_0(x);
\]

(ii) \(u(x, t) \in \text{Dom}(W)\) for any \(t \geq 0\) and

\[
(W u)(x, t) = -\kappa F^{-1}_{\xi \rightarrow x} (e^{-\kappa t A_w(\|\xi\|_p)} A_w(\|\xi\|_p) 1_{B_R}(\xi)) \ast u_0(x).
\]

**Proof.** (i) The proof is similar to the one given for Lemma 7.1 in [24]. (ii) Note that \(e^{-\kappa t A_w(\|\xi\|_p)} \hat{u}_0(\xi) \in \mathcal{S}\) for \(t \geq 0\), \(A_w(\|\xi\|_p) e^{-\kappa t A_w(\|\xi\|_p)} \hat{u}_0(\xi) \in \mathcal{S} \subset L^2\) for \(t \geq 0\).
i.e. \( u(x, t) \in \text{Dom}(W) \) for \( t \geq 0 \). Now
\[
(Wu)(x, t) = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left( A_w(\|\xi\|_\rho) \mathcal{F}_{\xi \rightarrow x} (u(x, t)) \right)
\]
\[
= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left( A_w(\|\xi\|_\rho) e^{-tA_w(\|\xi\|_\rho)} \hat{u}_0 (\xi) \right)
\]
\[
= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left( A_w(\|\xi\|_\rho) e^{-tA_w(\|\xi\|_\rho)} 1_{B_R}(\xi) \hat{u}_0 (\xi) \right)
\]
\[
= -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} \left( e^{-\kappa tA_w(\|\xi\|_\rho)} A_w(\|\xi\|_\rho) 1_{B_R}(\xi) \right) * u_0(x).
\]

As a direct consequence of Lemma 6.1 we obtain the following result.

**Proposition 6.2.** Assume that \( u_0 \in S \). Then function \( u(x, t) \) defined in (6.2) is a solution of Cauchy problem (6.1).

6.2. Homogeneous equations with initial values in \( L^2 \). We define
\[
T(t)u = \begin{cases} 
Z_t * u, & t > 0 \\
u, & t = 0,
\end{cases}
\]
for \( u \in L^2 \).

**Lemma 6.3.** The operator \( T(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) is bounded for any fixed \( t \geq 0 \).

**Proof.** For \( t > 0 \), the result follows from the Young inequality by using the fact that \( Z_t \in L^1 \), c.f. Theorem 4.(iii).

**Proposition 6.4.** The following assertions hold.
(i) The operator \( W \) generates a \( C_0 \) semigroup \( (T(t))_{t \geq 0} \). The operator \( T(t) \) coincides for each \( t \geq 0 \) with the operator \( T(t) \) given by (6.3).
(ii) Cauchy problem (6.1) is well-posed and its solution is given by \( u(x, t) = Z_t * u_0 \), \( t \geq 0 \).

**Proof.** (i) By Proposition 5.3 (iii) the operator \( W \) generates a \( C_0 \) semigroup \( (T(t))_{t \geq 0} \). Hence Cauchy problem (6.1) is well-posed, see e.g. [8, Theorem 3.1.1]. By Proposition 6.2 \( T(t)|_S = T(t)|_S \) and both operators \( T(t) \) and \( T(t) \) are defined on the whole \( L^2 \) and bounded, c.f. Lemma 6.3 By the continuity we conclude that \( T(t)|_S = T(t) \) on \( L^2 \). Now the statements follow from well-known results of the semigroup theory, see e.g. [8, Theorem 3.1.1], [11] Chap. 2, Proposition 6.2.

6.3. Non-homogeneous equations. Consider the following Cauchy problem:

\[
\begin{align*}
\frac{\partial u}{\partial t}(x, t) - W u(x, t) &= g(x, t), & x \in \mathbb{R}^n, t \in [0, T], T > 0, \\
u(x, 0) &= u_0(x), & u_0(x) \in \text{Dom}(W).
\end{align*}
\]

We say that a function \( u(x, t) \) is a solution of (6.4), if \( u(x, t) \in C([0, T], \text{Dom}(W)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \) and if \( u(x, t) \) satisfies equation (6.4) for \( t \in [0, T] \).
Theorem 6.5. Assume that $u_0 \in \text{Dom}(W)$ and that $g \in C \left( [0, \infty), L^2(Q^p_n) \right) \cap L^1 \left( [0, \infty), \text{Dom}(W) \right)$. Then Cauchy problem (6.4) has a unique solution given by

$$u(x, t) = \int_{Q^p_n} Z(x - \xi, t) u_0(\xi) d^n\xi + \int_{0}^{t} \int_{Q^p_n} Z(x - \xi, t - \theta) g(\xi, \theta) d^n\xi d\theta.$$ 

Proof. The result follows from Proposition 6.4 by using some well-known results of the semigroup theory, see e.g. [8, Proposition 4.1.6]. □

Remark 6.6. In [1, Sect. 10.3] a general theory for Cauchy Problems involving pseudodifferential operators on Lizorkin spaces was developed. By applying Proposition 3.3 (i) and Theorem 10.3.1 in [1] we can solve Cauchy problem (6.4) when $u_0(x)$ and $g(x, t)$ belong to a certain Lizorkin space. However the results obtained by using this approach are not sufficient for the purposes of application to stochastic processes, in particular we need several properties of the semigroup associated with operator $W$.

7. First Passage Time Problem

Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial \varphi(x, t)}{\partial t} = W \varphi(x, t), & t > 0, \ x \in Q^p_n, \\ \varphi(x, t) = \Omega \left( \|x\|_p \right). \end{cases} \tag{7.1}$$

In this section we assume that $\kappa$ satisfies

$$\kappa \int_{\|y\|_p > 1} \frac{d^n y}{w(\|y\|_p)} \leq 1. \tag{7.2}$$

By Theorem 6.5 the solution of (7.1) is $\varphi(x, t) = Z_t(x) \ast \Omega \left( \|x\|_p \right)$, and by Theorem 5.3 $Z_t$ is transition density of a time and space homogeneous Markov process $X(t, \omega)$ whose paths has only first class discontinuities.

This section is dedicated to the study of the following random variable.

Definition 7.1. The random variable $\tau_{Z^p_n}(\omega) := \tau(\omega) : Q^p_n \rightarrow \mathbb{R}^+$ defined by

$$\inf \{ t > 0; X(t, \omega) \in Z^p_n \ | \ there \ exists \ t' \ such \ that \ 0 < t' < t \ and \ X(t', \omega) \notin Z^p_n \}$$

is called the first passage time of a path of the random process $X(t, \omega)$ entering the domain $Z^p_n$.

Note that the initial condition in (7.1) implies that

$$\text{Pr} \left( \{ \omega \in Q^p_n; X(0, \omega) \in Z^p_n \} \right) = 1.$$

Definition 7.2. We say that $X(t, \omega)$ is recurrent with respect to $Z^p_n$ if

$$\text{Pr} \left( \{ \omega \in Q^p_n; \tau(\omega) < \infty \} \right) = 1. \tag{7.3}$$

Otherwise we say that $X(t, \omega)$ is transient with respect to $Z^p_n$.

The meaning of (7.3) is that every path of $X(t, \omega)$ is sure to return to $Z^p_n$. If (7.3) does not hold, then there exist paths of $X(t, \omega)$ that abandon $Z^p_n$ and never go back.
The function $\varphi(x, t) = Z_t(\|x\|) \Omega(\|x\|)$ is infinitely differentiable in the time $t \geq 0$ and its derivative is given by

$$
\frac{\partial^m}{\partial t^m} \varphi(x, t) = (-\kappa)^m \int_{Q^n_p} A(\|x\|) \varphi(x, t) e^{-\kappa A(\|x\|)} d^n x \text{ for } m \in \mathbb{N}.
$$

Proof. Note that for $t \geq 0$ and $m \in \mathbb{N}$, $A(\|x\|) \Omega(\|x\|) e^{-\kappa A(\|x\|)} \in L^1(Q^n_p)$. The announced formula is obtained by induction on $m$ by applying the Lebesgue Dominated Convergence Theorem.

The probability density function for a path of $X(t, \omega)$ to enter into $\mathbb{Z}^n_p$ at the instant of time $t$, with the condition that $X(0, \omega) \in \mathbb{Z}^n_p$ is given by

$$
g(t) = \kappa \int_{\mathbb{Z}^n_p \setminus \mathbb{Z}^n_p} \frac{\varphi(x, t)}{w(\|x\|)} d^n x.
$$

Proof. The survival probability, by definition

$$
S(t) := S_{\mathbb{Z}^n_p}(t) = \int_{\mathbb{Z}^n_p} \varphi(x, t) d^n x,
$$

is the probability that a path of $X(t, \omega)$ remains in $\mathbb{Z}^n_p$ at the time $t$. Because there are no external forces acting on the random walk, we have

$$
S^*(t) = \text{Probability that a path of } X(t, \omega) \text{ goes back to } \mathbb{Z}^n_p \text{ at the time } t - \text{Probability that a path of } X(t, \omega) \text{ exits } \mathbb{Z}^n_p \text{ at the time } t
$$

$$
g(t) = - C \cdot S(t) \text{ with } 0 < C \leq 1.
$$

By using Lemma 7.3 and the definition of $W$, we have

$$
S^*(t) = \int_{\mathbb{Z}^n_p} \frac{\partial \varphi(x, t)}{\partial t} d^n x = \kappa \int_{\mathbb{Z}^n_p} \varphi(x + y, t) - \varphi(x, t) d^n y d^n x
$$

$$
= \kappa \int_{\mathbb{Z}^n_p} \varphi(x + y, t) - \varphi(x, t) d^n y d^n x + \kappa \int_{\mathbb{Q}^n_p \setminus \mathbb{Z}^n_p} \frac{\varphi(x + y, t) - \varphi(x, t)}{w(\|y\|)} d^n y d^n x
$$

$$
= \kappa \int_{\mathbb{Z}^n_p} \varphi(x, t) d^n x - \kappa \int_{\mathbb{Q}^n_p \setminus \mathbb{Z}^n_p} \frac{\varphi(x, t)}{w(\|y\|)} d^n y d^n x
$$

$$
= \kappa \int_{\mathbb{Z}^n_p} \frac{\varphi(z, t)}{w(\|z\|)} d^n z d^n x - \int_{\mathbb{Z}^n_p} \varphi(x, t) d^n x \int_{\mathbb{Q}^n_p \setminus \mathbb{Z}^n_p} \frac{\kappa}{w(\|y\|)} d^n y
$$

$$
= \kappa \int_{\mathbb{Q}^n_p \setminus \mathbb{Z}^n_p} \frac{\varphi(z, t)}{w(\|z\|)} d^n z - \left( \int_{\mathbb{Q}^n_p \setminus \mathbb{Z}^n_p} \frac{\kappa}{w(\|y\|)} d^n y \right) S(t).
$$
Take $C = \int_{\mathbb{Q}_{p}^{n} \setminus \mathbb{Z}_{p}^{n}} \frac{\kappa}{n w(\|x\|_{p})} d^{n}y \leq 1$, c.f. (7.2). Finally, by using (7.6), one gets

$$g(t) = \kappa \int_{\mathbb{Q}_{p}^{n} \setminus \mathbb{Z}_{p}^{n}} \frac{\varphi(x, t)}{w(\|x\|_{p})} d^{n}x.$$ \hfill \Box

**Proposition 7.5.** The probability density function $f(t)$ of the random variable $	au(\omega)$ satisfies the non-homogeneous Volterra equation of second kind

$$(7.7) \quad g(t) = \int_{0}^{\infty} g(t-\tau)f(\tau)d\tau + f(t).$$

**Proof.** The result follows from Lemma 7.4 by using the argument given in the proof of Theorem 1 in $\mathbb{E}$. \hfill \Box

**Proposition 7.6.** The Laplace transform $G(s)$ of $g(t)$ is given by

$$(7.8) \quad G(s) = \kappa^{2}(1-p^{-n}) \sum_{i=1}^{\infty} \frac{p^{i}n}{w(p^{i})} \sum_{j=1}^{\infty} \frac{p^{i-2}n}{w(p^{j})} + \frac{p^{i-n}}{w(p^{j})} (s + \kappa A_{w}(p^{j})) (s + \kappa A_{w}(p^{j+1})) \quad \text{for } \text{Re}(s) > 0.$$

**Proof.** We first note that

$$(7.9) \quad e^{-st}e^{-\kappa A_{w}(\|\xi\|_{p})} \frac{\Omega(\|\xi\|_{p})}{w(\|x\|_{p})} \in L^{1}((0, \infty) \times \mathbb{Q}_{p}^{n} \times \mathbb{Q}_{p}^{n} \setminus \mathbb{Z}_{p}^{n}, dt d^{n} \xi d^{n}x) \quad \text{for } \text{Re}(s) > 0.$$

We compute the Laplace transform $G(s)$ of $g(t)$ by replacing

$$\varphi(x, t) = \int_{\mathbb{Q}_{p}^{n}} e^{-\kappa A_{w}(\|\xi\|_{p})} \Omega(\|\xi\|_{p}) \Psi(\xi \cdot x) d^{n}\xi$$

in (7.5) and interchanging the iterated integrals in a suitable form, which is allowed by (7.3) via Fubini’s Theorem, in this way one gets

$$G(s) = \kappa \int_{\mathbb{Q}_{p}^{n} \setminus \mathbb{Z}_{p}^{n}} \frac{\Omega(\|\xi\|_{p}) \Psi(\xi \cdot x)}{w(\|x\|_{p})} d^{n} \xi d^{n}x \quad \text{for } \text{Re}(s) > 0.$$

We now assert that

$$\frac{1}{s + \kappa A_{w}(\|\xi\|_{p})} \in L^{1}(\mathbb{Q}_{p}^{n} \setminus \mathbb{Z}_{p}^{n} \times \mathbb{Z}_{p}^{n}, d^{n}xd^{n}\xi) \quad \text{for } \text{Re}(s) > 0.$$

Indeed, since

$$(7.10) \quad \left|s + \kappa A_{w}(\|\xi\|_{p})\right| \geq \text{Re}(s) + \kappa A_{w}(\|\xi\|_{p}) > \kappa A_{w}(\|\xi\|_{p}) \quad \text{for } \text{Re}(s) > 0,$$

we have

$$\frac{1}{s + \kappa A_{w}(\|\xi\|_{p}) w(\|x\|_{p})} \leq \frac{1}{\kappa A_{w}(\|\xi\|_{p}) w(\|x\|_{p})} \leq \frac{C}{\|\xi\|_{p}^{\alpha_2-n} w(\|x\|_{p})} \leq \frac{C'}{\|\xi\|_{p}^{\alpha_2-n} \|x\|_{p}},$$
which is an integrable function for \( x \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n, \xi \in \mathbb{Z}_p^n \) and \( \text{Re} \, (s) > 0 \), c.f. Lemma 3.3

In order to calculate an explicit formula for \( G(s) \) for \( \text{Re} \, (s) > 0 \) we proceed as follows. We take \( U = \{ y \in \mathbb{Q}_p^n, \| y \|_p = 1 \} \) as before, then \( \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n = \bigcup_{t \in \mathbb{N} \setminus \{0\}} p^{-t}U, \mathbb{Z}_p^n \setminus \{0\} = \bigcup_{j \in \mathbb{N}pU, \text{and}} \]
\[
G(s) = \kappa \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \int_{p^{-1}U} \int_{p^{i}U} \frac{\Psi(\xi \cdot x)}{s + \kappa A_w(\|\xi\|_p)} \, d^n x \, d^n x
\]
for \( \text{Re} \, (s) > 0 \).

We now use the following change of variables:

\[
p^{-i}U \times p^iU \to U \times U
\]

\[
(x, \xi) \to (y', y)
\]

with \( x = p^{-i}y' \) for \( i \in \mathbb{N} \setminus \{0\}, \xi = p^{j}y \) for \( j \in \mathbb{N} \). Furthermore, \( d^n x d^n y = p^{-\alpha_n + \alpha} d^n y' \) for \( j \in \mathbb{N} \) and \( i \in \mathbb{N} \setminus \{0\} \). Then

\[
G(s) = \kappa \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} p^{in} \int_{U} \int_{U} \frac{\Psi(p^{-i}y' \cdot y')}{s + \kappa A_w(p^{-j})} \, d^n y \, d^n y'
\]

\[
= \kappa \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{p^{in}}{w(p^i)} \int_{U} \int_{U} \frac{\Psi(p^{-i}y' \cdot y')}{s + \kappa A_w(p^{-j})} \, d^n y \, d^n y'
\]

\[
= \kappa(1 - p^{-\alpha_n}) \sum_{i=1}^{\infty} \frac{p^{in}}{w(p^i)} \left( \sum_{j=0}^{\infty} \frac{1 - p^{-n(p^{-j})}}{s + \kappa A_w(p^{-j})} - \frac{p^{-n}p^{-n(j-1)}}{s + \kappa A_w(p^{-j})} \right)
\]

\[
= \kappa^2(1 - p^{-\alpha_n}) \sum_{i=1}^{\infty} \frac{p^{in}}{w(p^i)} \sum_{j=1}^{\infty} \frac{p^{-jn}}{s + \kappa A_w(p^{-j})} \frac{p^{-n}p^{-n(j-1)}}{s + \kappa A_w(p^{-j+1})}
\]

\[
\square
\]

**Theorem 7.7.** (i) If \( W \) is of polynomial type \((\alpha_3 = 0, \alpha_1 = \alpha_2 = \alpha > n)\) and \( \alpha \geq 2n \), then \( X(t, \omega; W) \) is transient with respect to \( \mathbb{Z}_p^n \).

(ii) If \( W \) is of polynomial type \((\alpha_3 = 0, \alpha_1 = \alpha_2 = \alpha > n)\) and \( n < \alpha < 2n \), then \( X(t, \omega; W) \) is transient with respect to \( \mathbb{Z}_p^n \).

**Proof.** By Proposition 7.5 the Laplace transform of \( F(s) \) of \( f(t) \) equals \( \frac{G(s)}{1+G(s)} \), where \( G(s) \) is the Laplace transform of \( g(t) \), and thus \( F(0) = \int_{0}^{\infty} f(t) \, dt = 1 - \frac{1}{1+G(0)} \). Hence in order to prove that \( X(t, \omega; W) \) is recurrent is sufficient to show that \( G(0) = \lim_{s \to 0} G(s) = \infty \), and to prove that it is transient that \( G(0) = \lim_{s \to 0} G(s) < \infty \).

(i) Take \( s \in \mathbb{R}, s > 0 \) and \( \alpha_3 = 0, \alpha_1 = \alpha_2 = \alpha > n \). First we note that

\[
C_2p^{-(\alpha-n)} \leq A_w(p^{-j}) \leq A_w(p^{-j-1}) \leq C_3p^{-(j-1)(\alpha-n)}
\]

c.f. Lemma 3.3 We take \( s \in \mathbb{R} \) with \( s > 0 \) and set

\[
s = C_3p^{-(\alpha_3(s)-1)(\alpha-n)}
\]
Note that $s \to 0^+ \Leftrightarrow j_0 := j_0(s) \to \infty$. Then
\[(s + \kappa A_w(p^{-j})) (s + \kappa A_w(p^{-j+1})) \leq (1 + \kappa)^2 s^2 \text{ for } j \geq j_0\]
because $A_w(p^{-\gamma})$ is a decreasing function of $\gamma$.

By (7.13),
\[
G(s) > \kappa^2 (1 - p^{-n}) \frac{p^n}{w(p)} \sum_{j=1}^{\infty} \frac{p_{n-2}^{w(p)}}{w(p^{j+1})} + \frac{p_{n}^{w(p)}}{w(p^{j})} \leq \kappa^2 (1 - p^{-n}) \frac{p^n}{w(p)} \sum_{j=0}^{\infty} \frac{p_{n-2}^{w(p)}}{w(p^{j+1})} + \frac{p_{n}^{w(p)}}{w(p^{j})}
\]
with $j_0 \in \mathbb{N}$. Now by (3.10) with $\alpha_3 = 0$, $\alpha_1 = \alpha_2 = \alpha$,
\[
\frac{p_{n-2}^{w(p+1)}}{w(p^{j+1})} + \frac{p_{n}^{w(p)}}{w(p^{j})} \geq \left( \frac{p_{n-2}^{w(p)}}{C_1 p} + \frac{p_{n}^{w(p)}}{C_1} \right) p^{-j\alpha} \text{ for } j \in \mathbb{N}.
\]
By (3.11) and (7.13), we have
\[
G(s) > \frac{C}{s^2} p^{-j_0\alpha} = C' \frac{p^{-j_0\alpha}}{p^{-2j_0(\alpha-2n)}}
\]
Hence, if $\alpha > 2n$,
\[
\lim_{s \to 0^+} G(s) > C' \lim_{s \to 0^+} p^{j_0(\alpha-2n)} = C' \lim_{j_0 \to \infty} p^{j_0(\alpha-2n)} = \infty.
\]
Now if $\alpha = 2n$ and $s \in \mathbb{R}$, $s > 0$, we have
\[
\lim_{j \to \infty} \frac{p_{n-2}^{w(p^{j+1})} + \frac{p_{n}^{w(p)}}{w(p^{j})}}{(s + \kappa A_w(p^{-j})) (s + \kappa A_w(p^{-j+1}))} \geq C \lim_{j \to \infty} p^{j(\alpha-2n)} = C
\]
where $C$ is a positive constant. Hence $\lim_{s \to 0^+} G(s) = \infty$.

(ii) In this case $\alpha_3 = 0$, $\alpha_1 = \alpha_2 = \alpha > n$, and $s \in \mathbb{C}$ with $\Re(s) > 0$. By (3.10),
\[
\frac{p_{n-2}^{w(p^{j+1})} + \frac{p_{n}^{w(p)}}{w(p^{j})}}{(s + \kappa A_w(p^{-j})) (s + \kappa A_w(p^{-j+1}))} \leq \left( \frac{p_{n-2}^{w(p)}}{C_0 p} + \frac{p_{n}^{w(p)}}{C_0} \right) p^{-j\alpha}
\]
for $j \in \mathbb{N}$. In addition, by Lemma 3.4
\[
\frac{1}{(s + \kappa A_w(p^{-j}))(s + \kappa A_w(p^{-j+1}))} \leq \frac{1}{|\kappa A_w(p^{-j})|^2} \leq \frac{1}{\kappa^2 C_2^2 p^{-2j(\alpha-\alpha)}}
\]
for $j \in \mathbb{N}$, c.f. (7.10).

Hence, by (7.3),
\[
G(s) \leq C \sum_{i=1}^{\infty} p^{-i(\alpha-n)} \sum_{j=1}^{\infty} p^{-j\alpha+2i(\alpha-n)} = C \sum_{i=1}^{\infty} p^{-i(\alpha-n)} \sum_{j=1}^{\infty} p^{-j(2n-\alpha)}
\]
\[
= C' \sum_{i=1}^{\infty} p^{-i(n)} < \infty \text{ if } 2n > \alpha.
\]
\[\square\]
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