Anderson localization in an interacting fermionic system

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In the present article, we theoretically and numerically study a system of interacting fermions in a one-dimensional disordered lattice, described by the Fermi Hubbard Hamiltonian, in presence of an on-site random potential. We show that, given the proper identification of the elementary excitations of the system described in terms of doublons and unpaired particles, the Anderson localization picture survives. Ensuing a “global quench”, we show that the system exhibits a rich localization scenario, which can be ascribed to the nearly-free dynamics of the elementary excitations of the Hubbard Hamiltonian.

Anderson localization (AL) represents one of the most prominent manifestations of the role of disorder in quantum mechanical systems [1]. Over the last 5 decades, the concept of AL has been successfully employed to describe the physics of localization in the most diverse contexts, ranging from the localization of optical [2, 3] and acoustical waves [4] to that of matter waves in solid-state systems [5] and ultracold gases [6, 8].

In its original formulation, AL predicts the absence of diffusion and exponential localization of a quantum particle due to coherent multiple scatterings with a random potential [1]. This picture allows the description of the role of disorder in a many-body situation when the interactions among constituents are absent, and thus the physics can be recast in terms of a one-body problem. Conversely, the description of the physics of localization in a genuinely many-body situation – i.e. when two-body interactions are present – has proven to be an outstanding challenge (see e.g. [9]), with different approaches providing different scenarios for the effect of interactions on localization [8, 10–15].

In the recent past ultracold atomic gases have become the ideal test-bed for models in condensed-matter physics owing to the wide tunability of their characteristic parameters, such as the possibility of controlling the particle-particle interaction and the system dimensionality. They thus represent optimal candidates for the investigation of AL, as attested by the experimental observation of AL in a Bose-Einstein condensate of non-interacting alkali atoms [6, 7]. These two experiments have provided evidence for AL in matter waves while in [8], the effect of a weak interacting potential on the localization scenario for a bosonic system was explored. Moreover, the effect of interaction has been recently investigated numerically and theoretically in [15], showing the persistence of AL in a one-dimensional bosonic system in presence of attractive interaction beyond a mean-field description.

In this article we theoretically and numerically study a “global quantum quench” [16] of an interacting fermionic cloud of ultracold atoms in a disordered lattice. Our investigation is focused on the dynamics generated by the one-dimensional Fermi-Hubbard model in presence of an on-site disordered potential on a two-component fermionic gas.

The relevance of this model in the context of the study of the combined effect of interaction and disorder lies in the fact that the Hubbard model constitutes the prototypical model for the study of interactions in a solid-state framework. Furthermore, our analysis is closely related to the current experimental trends in the field, represented by the experiments aimed at establishing the AL picture for non-interacting bosonic systems [6, 7], or by the expansion of a fermionic BI in an otherwise empty lattice in two spatial dimensions (in absence of disorder) [17]. Since the setup here considered represents a natural extension to disordered systems of the one considered in [18], we expect that it can be realized experimentally in the near future.

On general grounds, beyond any specific reference to a given model, we here point out how the correct identification of the elementary excitations within any genuinely many-body system constitutes the fundamental step in the definition of the framework for the investigation of the interaction/disorder interplay. In this same spirit, in [14], the localization properties of Bogoliubov quasiparticles in an interacting Bose gas was studied. Our goal is to illustrate here how AL survives for a fermionic system in presence of two-body interactions —either attractive or repulsive— and how the rich localization scenario exhibited by the system is associated to the collective character of its excitations.

Our numerical simulations are performed using a time-evolving block decimation algorithm (TEBD), which represents an essentially exact numerical technique for the simulation of the ground-state and the dynamical properties of one-dimensional quantum lattice systems [19]. In addition, the existence of the Bethe-ansatz solution for the 1D Hubbard chain [20] in absence of disorder, represents our starting point for the theoretical discussion about the disorder and interaction beyond mean-field and perturbative approaches.

In our analysis, we consider a system of \( N = N_\uparrow + N_\downarrow = 40 \) fermionic atoms with equal pseudospin populations \( (N_\uparrow = N_\downarrow = 20) \) in a one-dimensional lattice with \( L = 256 \) sites. The system is prepared at \( t = 0^- \) by loading...
a band insulator (BI) state in the central portion of the disordered lattice (see Fig. 1). At $t = 0^+$ the BI is allowed to expand in the empty lattice. The dynamics is described in terms of the Fermi-Hubbard Hamiltonian with a disordered on-site potential

$$ H = H_J + H_U + \sum_{i=1}^{L} V_i (n_{i\uparrow} + n_{i\downarrow}), $$

where $H_J = -J \sum_{i=1}^{L} \sigma_i \uparrow \sigma_{i+1} \uparrow + h.c.$, $H_U = U \sum_{i=1}^{L} n_{i\uparrow} n_{i\downarrow}$, $J$ is the hopping amplitude, $U$ is the on-site interaction, $V_i$ is the random on-site potential with probability distribution $P(V_i)$. In the following, we will focus on the case where $V_i$ is uniformly distributed between 0 and $V_0$. We expect that a different choice for the disorder distribution will alter the specific value of the localization length –see Eq. (2)–, but the qualitative content of our predictions is expected to be unchanged.

In all our simulations we have considered a Schmidt number $\chi = 160$, corresponding, in the worst-case scenario ($U/J = -5$, $V_0/J = 1$), to a discarded weight of $10^{-5}$ [19]. The results depicted in Figs. 2a,b are obtained averaging over 100 disorder realizations for the $U = 0$ case and 10 disorder realizations for the interacting cases. The crucial aspect allowing reliable simulations with the TEBD algorithm in our system, is that the entropy of entanglement, describing the degree of entanglement –as opposed to the on-site nature of the interaction in the Hubbard model– might lead to a different localization scenario, and to a different behaviour of the entropy of entanglement.

Firstly, we have compared, for a non-interacting system ($U = 0$), the expansion in a lattice where disorder is absent (Fig. 2a) with the expansion in presence of uniform disorder ($V_0 = 5J$, Fig. 2b). In the case of a non-interacting system, the presence of an on-site disordered potential prevents the fermionic cloud from expanding as it does in the case where disorder is absent, confirming the AL picture for non-interacting atoms [1]. Following [22], for non-interacting atoms in presence of uniformly distributed disorder, the localization length of an expanding cloud, initially prepared in a BI state is given by

$$ l_{loc}^{-1}(V_0/J) = \frac{1}{2} \ln \left( 1 + \frac{V_0^2/J^2}{16} \right) + \frac{4 \arctan \left( \frac{V_0}{4J} \right)}{V_0/J} - 1, $$

which represents the localization length of a particle lying at the band centre. It is possible to show that the overall localization length of the system of non-interacting particles is given by this length (see Supplementary Information).

The logarithmic plot of the particle distribution $\langle n_{i\sigma} \rangle$ ($\sigma = \uparrow, \downarrow$) at time $t = 60J^{-1}$, given in Fig. 2c, exhibits an exponential decay compatible with a localization length expressed by Eq. (2), confirming the expected effect of an on-site disordered potential.

![FIG. 2. a. Expectation value of the number of particles $\langle n_{i\sigma} \rangle(t)$ with spin $\sigma = \uparrow, \downarrow$, for $U = 0$, and $V_0 = 0$. It is possible to see how the particle wavefront propagates with velocity $v_0 = 2J$. b. $\langle n_{i\sigma} \rangle(t)$ for $U = 0$ and $V_0 = 5$. Particles are exponentially localized. c. Logarithmic plot of $\langle n_{i\sigma} \rangle(t) |_{t=60J^{-1}}$ (blue curve), compared with the theoretical value the exponential decay over the localization length $l_{loc}$ given by Eq. (2) (black dotted curve).](image-url)
excitations of the system in terms of unpaired particles and local pairs (doublons), whose local density is given by
\[ n_i^d = \langle \hat{n}_{i \uparrow} \hat{n}_{i \downarrow} \rangle, \] and \[ n_i^u = \langle \hat{n}_{i \sigma} \rangle - \langle \hat{n}_{i \uparrow} \hat{n}_{i \downarrow} \rangle \] respectively.

The description of the elementary excitations in the one-dimensional Hubbard model in terms of doublons and unpaired particles can be obtained considering the string hypothesis for the solution of the Lieb-Wu equations: in this framework, the doublons correspond to \( k - \Lambda \) strings, while the single \( k_0 \) correspond to unpaired particles (see Supplementary Information) \[20, 23\]. Following the string hypothesis, in the strong-coupling limit, the dispersion relation for unpaired particles and doublons read respectively

\[ \epsilon_u = -2J \cos(k_u), \] \[ \epsilon_d = -4J^2/|U| \cos(k_d). \]

Eqs. (3)\[4\] correspond to the dispersion relation for free particles with hopping parameter given respectively by \( J^u = J \) and \( J^d = 2J^2/|U| \), thus describing the free evolution of a (non-interacting) gas of unpaired particles and doublons. The dynamics at \( t = 0^+ \) can be intuitively described in terms of a local two-sites dynamics taking place at the edge of the BI cloud (see Fig. 1), this description is possible owing to the specific form of the initial state: sites away from the edge do not contribute to the initial dynamics, because they are either empty or completely Pauli blocked \[18\]. In this picture the initial state where a doublon is present on the edge of the cloud, will evolve into a superposition of a doublon (on either side of the edge of the BI cloud) and two unpaired particles

\[ U(t)|0, \uparrow \downarrow \rangle = \alpha(t)|0, \uparrow \downarrow \rangle + \beta(t)|\uparrow, \downarrow \rangle + \gamma_1(t)|\uparrow, \downarrow \rangle + \gamma_2(t)|\downarrow, \uparrow \rangle \] (5)

The ensuing dynamics of the many body-system can be approximately described as the independent dynamics of these two components, following the dispersion relation given by Eqs. (3) and (4) for unpaired particles and doublons respectively: and therefore describing the expansion of two fluids characterized by hopping parameters \( J^u \) and \( J^d \).

We are now in the position to translate the above considerations to the disordered potential. In this case as well, we recognize that the expansion of a BI in presence of a two-body interaction can be described in terms of the dynamics of a two-fluid system, constituted by local pairs (doublons) and unpaired particles. We are thus lead to infer that the Anderson localization scenario must remain valid, separately, for doublons and unpaired particles. Furthermore, we can determine how unpaired particles and doublons localize with different localization lengths \( l_u \) and \( l_d \), the former being directly given by (4), while the latter corresponds the the localization of a (composite) free particle whose dynamics is regulated by the effective hopping \( J^d = 2J^2/|U| \), and by a local potential

\[ V_i^d = 2V_i, \] namely

\[ l_u = l_{loc} \left( \frac{V_0}{J} \right) \] (6)

\[ l_d = l_{loc} \left( \frac{V_0^d}{J^d} \right) = l_{loc} \left( \frac{V_0 |U|}{J} \right). \] (7)

The numerical simulations of the exact many-body dynamics show the validity of the picture described above, according to which doublons and unpaired particles localize independently following the AL scenario. In Fig. 3, we have plotted the localization properties for three different setups \( (U = 0, V_0/J = 5, \) green curve; \( U/J = -5, V_0/J = 1, \) red curve; \( U/J = -10, V_0/J = 0.5, \) blue curve). The numerical results show that the three setups

FIG. 3. a. Numerical value of \( \langle n_{i \sigma} \rangle(t) \) \( (U = 0, V_0/J = 5, \) green curve; \( U/J = -5, V_0/J = 1, \) red curve; \( U/J = -10, V_0/J = 0.5, \) blue curve), compared with the theoretical result (black curve). In agreement with the two-fluid description, the three set of parameters exhibit the same localization length. b. Numerical value of \( \langle n_{i \sigma} \rangle(t^*) \) (blue curve) and \( \langle n_{i \sigma} \rangle(t^*) \) \( (U/J = -5, V_0/J = 1, t^* = 60J^{-1}) \) compared with the theoretical value of the localization length for doublons and unpaired particles. The central core exhibits a localization length characteristic of doublons localization, while the outer wings exhibit a localization length corresponding to the localization of unpaired particles.
exhibit almost identical localization lengths for unpaired particles ($U = 0, V_0/J = 5$) and doublons ($U/J = -5, V_0/J = 1; U/J = -10, V_0/J = 0.5$). These numerical results, in turn, are in agreement with the theoretical prediction given by Eqs. (6, 7), showing how the role of the interaction profoundly modifies the localization scenario, and that the correct identification of the elementary excitations in the system represents a crucial step in the correct definition of the localization properties of the system. The deviation of the outer tails from an exponential decay for $U/J = -10, V_0/J = 0.5$ is related to the slower propagation of the doublon wavefront, whose expansion velocity is given by $|v_u| = 4J^2/|U|$. Therefore, at $t = 60J^{-1}$, the propagating doublon wavefront is still visible in the plot (see Supplementary Information - animation and [LS]). In Fig. 3b, it is possible to see the results of the exact numerical simulations exhibiting two localization lengths for $U/J = -5, V_0/J = 1$: the central core is almost entirely constituted by doublons which exponentially localize over a localization length $l_d = l_{loc}(V_0 |U|/J)$, while the outer wings of the expanding cloud are constituted by unpaired particles which localize over a length $l_u = l_{loc} (2^\delta)$, in agreement with the predictions given by the two-fluid model.

Another aspect of the localization properties of this system is given by the independence of the doublon localization on the sign of the interaction, which is connected to the unitarity of the evolution considered here. The independence of $l_{loc}$ of the sign of the interaction in Eqs. (6, 7) is confirmed by the comparison of the numerical results for $U/J = -5, V_0/J = 1$ and $U/J = -5, V_0/J = 1$ (see Fig. 4).

In conclusion, we have studied the expansion dynamics of an interacting fermionic gas in a one-dimensional lattice, in presence of on-site disorder. We have found that the two-body interaction affects the localization properties of the system in a non-trivial way. We have shown how the identification of the elementary excitations of the system in terms of doublons and unpaired particles allows to recast the rich localization scenario observed in the numerical simulations in terms of localization of these elementary excitations. Moreover, we have shown that the localization length for this system does not depend on the sign of the two-body interaction.

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FIG. 4. BI expansion for $U = -5, V_0 = 0.4$ (blue curve), compared with the expansion for $U = 5, V_0 = 0.4$ (red curve). Both curves exhibit the same exponential decay in agreement with the theory (black dashed line).

[1] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[2] D. S. Wiersma, et al., Nature 390, 671 (1997).
[3] T. Schwartz, et al., Nature 446, 52 (2007).
[4] H. Hu, et al., Nat. Phys. 4, 945 (2008).
[5] P. Lee and T. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
[6] J. Billy, et al., Nature 453, 891 (2008).
[7] G. Roati, et al., Nature 453, 895 (2008).
[8] B. Deissler, et al., Nat. Phys. 6, 354 (2010).
[9] L. Sanchez-Palencia and M. Lewenstein, Nat. Phys. 6, 87 (2010).
[10] L. Sanchez-Palencia, et al., Phys. Rev. Lett. 98, 210401 (2007).
[11] T. Paul, et al., Phys. Rev. Lett. 98, 210602 (2007).
[12] A. Pikovsky and D. Shepelyansky, Phys. Rev. Lett. 100, 024101 (2008).
[13] G. Kopidakis, et al., Phys. Rev. Lett. 100, 084103 (2008).
[14] P. Lugan and L. Sanchez-Palencia, Phys. Rev. A 84, 013612 (2011).
[15] D. Delande, et al., New J Phys 15, 045021 (2013).
[16] P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).
[17] U. Schneider, et al., Nat. Phys. 8, 213 (2012).
[18] J. Kajala, F. Maszel, and P. Törmä, Phys. Rev. Lett. 106, 206401 (2011).
[19] G. Vidal, Phys. Rev. Lett. 93, 040502 (2004).
[20] F. H. L. Essler, et al., The One-Dimensional Hubbard Model (Cambridge University Press, 2005).
[21] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. 109, 017202 (2012).
[22] F. M. Izrailev, S. Ruffo, and L. Tessieri, J Phys A-Math Gen 31, 5263 (1999).
[23] F. Maszel, et al., New J Phys 15, 045018 (2013).