Weighted Tutte–Grothendieck Polynomials of Graphs

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Abstract
In this paper, we introduce the notion of weighted chromatic polynomials of a graph associated to a weight function $f$ of a certain degree, and discuss some of its properties. As a generalization of this concept, we define the weighted Tutte–Grothendieck polynomials of graphs. When $f$ is harmonic, we notice that there is a correspondence between the weighted Tutte–Grothendieck polynomials of graphs and the weighted Tutte polynomials of matroids. Moreover, we present some constructions of the weighted Tutte–Grothendieck invariants for graphs as well as the weighted Tutte invariants for matroids. Finally, we give a remark on the categorification of the weighted chromatic polynomials of graphs and weighted Tutte polynomials of matroids.

Keywords Tutte polynomials · Chromatic polynomials · Matroids · Graphs · Discrete harmonic functions

Mathematics Subject Classification Primary 05B35; Secondary 94B05 · 11T71

1 Introduction
Motivated by the concept of the harmonic weight enumerator of a binary code introduced by Bachoc\cite{2} and the relation between the weight enumerator of a code and the Tutte polynomial of a matroid pointed out by Greene\cite{8}, Chakraborty, Miezaki, and
Oura [5] gave the notion of the harmonic Tutte polynomial of a matroid, and presented the Greene type relation between the harmonic weight enumerator of a code and the harmonic Tutte polynomial of a matroid. Tutte polynomials have been studied by many authors. We refer the readers to [15, 17] for more discussions on Tutte polynomials of a matroid. In the study of graph theory, the chromatic polynomial of a graph that counts the number of all proper colouring of a graph with a given number of colours is an invariant for graphs (see [16, 18]).

In this paper, we introduce some new polynomials associate to a graph and a weight function $f$ of a certain degree, specifically, the weighted chromatic polynomial and the weighted Tutte–Grothendieck polynomial. We show that a weighted chromatic polynomial is a particular case of a weighted Tutte–Grothendieck polynomial. Moreover, we present a Greene type relation between the weighted Tutte–Grothendieck polynomials of graphs and the weighted Tutte polynomials of matroids, provided the weight function $f$ is harmonic. Furthermore, we present the concept of weighted Tutte–Grothendieck invariants of graphs as well as weighted Tutte invariants of matroids, and give a connection between them. Finally, we give a remark on the categorification of the weighted chromatic polynomials of graphs and weighted Tutte polynomials of matroids.

This paper is organized as follows. In Sect. 2, we give the basic definitions and notations from graphs and matroids. We also give a brief discussion about (discrete) harmonic functions. In Sect. 3, we define the weighted chromatic polynomials of graphs and obtain a recurrence formula for it (Proposition 3.1). In Sect. 4, we give a recursive definition of weighted Tutte polynomials (Proposition 4.1). Moreover, if the weight function $f$ is harmonic, we establish a relation between the weighted chromatic polynomials of graphs and the weighted Tutte polynomials of matroids (Theorem 4.1). In Sect. 5, we define the weighted Tutte–Grothendieck polynomial of a graph, and give a generalization of Theorem 4.1 for this case (Theorem 5.1). In Sect. 6, we present the weighted Tutte–Grothendieck invariants for graphs (Theorem 6.1) and the weighted Tutte invariants for matroids (Theorem 6.2) and a correspondence between them (Theorem 6.3). In Sect. 7, we give categorifications to weighted chromatic polynomials (Theorem 7.1) and weighted Tutte polynomials (Theorem 7.2), respectively.

All computer calculations in this study were performed with the aid of Mathematica [19].

2 Preliminaries

In this section, we give a brief discussion on graphs and matroids including the basic definitions and notations. We review [1, 4, 13] for this discussion. We also recall [2, 7] for the definition and properties of the (discrete) homogenous functions and (discrete) harmonic functions.
2.1 Graphs

Let $G := (V, E)$ be a graph, where $V$ denotes the set of vertices, and $E$ is the set of edges. An edge is usually incident with two vertices. But if the edge incident with equal end vertices, the edge is called a loop. A graph is called simple if it has neither loops nor multiple edges. A path in $G$ is a sequence of edges $(e_1, e_2, \ldots, e_{m-1})$ having a sequence of vertices $(v_1, v_2, \ldots, v_n)$ satisfying $e_i \rightarrow \{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, m - 1$. A circuit in $G$ is a sequence of edges $(e_1, e_2, \ldots, e_{m-1})$ having a sequence of vertices $(v_1, v_2, \ldots, v_n)$ satisfying $e_i \rightarrow \{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, m - 1$ and $v_1 = v_n = v$. A circuit that does not repeat vertices is called cycle. The connected component of $G$ is a connected subgraph of $G$ which is not a proper subgraph of another connected subgraph of $G$. A bridge of $G$ is an edge whose removal increase the number of connected components of $G$ by 1. Throughout this note, we assume that the graphs are finite and not necessary to be simple.

2.2 Matroids

Let $E$ be a finite set of cardinality $n$ and $2^E$ denotes the set of all subsets of $E$. A (finite) matroid $M$ is an ordered pair $(E, \mathcal{I})$, where $\mathcal{I}$ is the collection of subsets of $E$ satisfying the following conditions:

(M1) $\emptyset \in \mathcal{I}$.
(M2) $I \in \mathcal{I}$ and $J \subseteq I$ implies $J \in \mathcal{I}$.
(M3) $I, J \in \mathcal{I}$ with $|I| < |J|$ then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the independent sets of $M$, and $E$ is called the ground set of $M$. A subset of the ground set $E$ that is not belongs to $\mathcal{I}$ is called dependent. For example, let $G = (V, E)$ be a finite graph. Let $\mathcal{I}$ be the set of all subsets $A$ of $E$ for which the subgraph $(V, A)$ contains no cycle. Then $G$ is a matroid (See [4, 17]). Such a matroid is called graphic matroid, and is denoted by $M_G$.

An independent set $I$ is called a maximal independent set if it becomes dependent on adding any element of $E \setminus I$. It follows from the axiom (M3) that the cardinality of all the maximal independent sets in a matroid $M$ is equal, called the rank of $M$. These maximal independent sets are called the bases of $M$. The rank $\rho(J)$ of an arbitrary subset $J$ of $E$ is the size of the largest independent subset of $J$. That is, $\rho(J) := \max_{I \subseteq J}(|I| : I \in \mathcal{I})$. This implies $\rho$ maps $2^E$ into $\mathbb{Z}$. This function $\rho$ is called the rank function of $M$. In particular, $\rho(\emptyset) = 0$. We shall denote $\rho(E)$ the rank of $M$. We refer the readers to [13] for detailed discussion.

2.3 Discrete Homogeneous and Harmonic Functions

Let $\Omega = \{1, 2, \ldots, n\}$ be a finite set. We define $\Omega_d := \{X \in 2^\Omega : |X| = d\}$ for $d = 0, 1, \ldots, n$. We denote by $\mathbb{R}^{2^\Omega}$, $\mathbb{R}\Omega_d$ the real vector spaces spanned by the elements of $2^\Omega$, $\Omega_d$, respectively. An element of $\mathbb{R}\Omega_d$ is denoted by

$$f := \sum_{Z \in \Omega_d} f(Z)Z \quad \text{(1)}$$
and is identified with the real-valued function on $\Omega_d$ given by $Z \mapsto f(Z)$. Such an element $f \in \mathbb{R}\Omega_d$ can be extended to an element $\tilde{f} \in \mathbb{R}2^\Omega$ by setting, for all $X \in 2^\Omega$,

$$\tilde{f}(X) := \sum_{Z \in \Omega_d, Z \subseteq X} f(Z).$$

(2)

If an element $g \in \mathbb{R}2^\Omega$ is equal to some $\tilde{f}$, for $f \in \mathbb{R}\Omega_d$, we say that $g$ has degree $d$. We call the vector space $\mathbb{R}\Omega_d$ the homogeneous space of degree $d$ and denote by $\text{Hom}_d(n)$. The differentiation $\gamma$ is the operator defined by the linear form

$$\gamma(Z) := \sum_{Y \in \Omega_{d-1}, Y \subseteq Z} Y$$

for all $Z \in \Omega_d$ and for all $d = 0, 1, \ldots, n$, and Harm$_d(n)$ is the kernel of $\gamma$:

$$\text{Harm}_d(n) := \ker\left(\gamma\big|_{\mathbb{R}\Omega_d}\right).$$

**Remark 2.1** ([2, 7]) Let $f \in \text{Harm}_d(n)$ and $J \subseteq \Omega$. Let

$$f^{(i)}(J) := \sum_{Z \in \Omega_d, \mid J \cap Z \mid = i} f(Z).$$

Then for all $0 \leq i \leq d$, $f^{(i)}(J) = (-1)^{d-i} \binom{d}{i} \tilde{f}(J)$.

**Remark 2.2** From the definition of $\tilde{f}$ for $f \in \text{Harm}_d(n)$, we have $\tilde{f}(J) = 0$ for any $J \in 2^\Omega$ such that $\mid J \mid < d$. Let $I, J \in 2^\Omega$ such that $I = \Omega \setminus J$. Then

$$\tilde{f}(J) = \sum_{Z \in \Omega_d, \mid Z \mid = 0} f(Z) = \sum_{Z \in \Omega_d, \mid Z \cap I \mid = 0} f(Z) = f^{(0)}(I) = (-1)^d \tilde{f}(\Omega \setminus J).$$

We have from the above equality that if $\mid J \mid > n - d$, then $\tilde{f}(J) = 0$.

For $X, Y \in 2^\Omega$, we introduce an operator $\circ$ on $\mathbb{R}2^\Omega$ as

$$\tilde{f}(X) \circ \tilde{f}(Y) := \tilde{f}(X \cup Y),$$

which is associative, and distributive with respect to addition. Then we have the following remark.
Remark 2.3 Let $I \subset \Omega$ and $J \subset \Omega \setminus I =: I^c$. Then for $f \in \text{Harm}_d(n)$ and by Remark 2.2, it is clear that

(i) $\tilde{f}(I) \circ \tilde{f}(I^c \setminus J) = \tilde{f}(\Omega \setminus (J \cup I)) = (-1)^d \tilde{f}(J)$,
(ii) $\tilde{f}(I^c \setminus J) = \tilde{f}(\Omega \setminus (J \cup I)) = (-1)^d \tilde{f}(I) \circ \tilde{f}(J)$.

3 Weighted Chromatic Polynomials

In this section, we discuss the weighted chromatic polynomial of a graph which is sometimes called the homogeneous chromatic polynomial of a graph. For the definitions and notations of the classical chromatic polynomials of graphs we refer the readers to [1, 4].

Let $G = (V, E)$ be a graph. We assume that the number of edges is $|E| = n$. Let $s$ be a bijective map from the edge set $E$ to $\Omega$, and $G(s)$ be a graph, where the edges are indexed by $s$. We call $G(s)$ the labelled graph and $s$ the label of the graph $G$. For $A \subset E$, we denote by $N_A(\lambda)$ the number of $\lambda$-colouring such that the vertices adjacent to $A$ have the same colour. Let $f$ be a homogeneous function of degree $d$. Then the weighted chromatic polynomial of $G$ associated with $f$ and $s$ is defined as:

$$\chi_f(G(s); \lambda) := \sum_{A \subset E} \tilde{f}(s(E \setminus A))(-1)^{|A|}N_A(\lambda).$$

Let $G_A := (V, A)$ be the subgraph of $G$ identified by $A$. We denote by $k(G_A)$ the number of connected components of $G_A$. Then $N_A(\lambda) = \lambda^{k(G_A)}$. Therefore the weighted chromatic polynomial $\chi_f(G(s); \lambda)$ can be written as:

$$\chi_f(G(s); \lambda) = \sum_{A \subset E} \tilde{f}(s(E \setminus A))(-1)^{|A|}\lambda^{k(G_A)}.$$

In the same manner as above, we can define the weighted chromatic polynomial of a graph related to any map $s$ instead of a bijective map. We will discuss the properties of such weighted chromatic polynomials in some subsequent papers [6]. In this paper, we only consider $s$ as bijective.

If $f$ is a harmonic function of degree $d$, we call $\chi_f(G(s); \lambda)$ the harmonic chromatic polynomial of $G$ associated with $f$ and $s$. When $f = 1$, we get the classical chromatic polynomial $\chi(G; \lambda)$ which is independent of the choice of $s$.

Remark 3.1 For a graph $G$ with no edge and $m$ vertices, we have

$$\chi_f(G(s); \lambda) = \tilde{f}(\emptyset)\lambda^m,$$

where $f \in \text{Hom}_d(n)$ and $s$ is any label of $G$.

Example 3.1 Let the graph $G$ given in Fig. 1. Let $\text{Hom}_1(4) \ni f = a_1[1] + a_2[2] + a_3[3] + a_4[4]$. Let us consider the following label $s$ for $G$: $\{v_1, v_2\} \mapsto 4, \{v_2, v_3\} \mapsto 1$. 

\[ \begin{array}{c}
\end{array} \]
\{v_2, v_4\} \mapsto 2, \{v_1, v_4\} \mapsto 3. Then we have

\[ \chi_f(G(s); \lambda) = (a_1 + a_2 + a_3 + a_4)(\lambda^4 - 3\lambda^3 + 3\lambda^2) - (a_1\lambda + a_2 + a_3 + a_4)\lambda. \]

For \( f \in \text{Harm}_1(4) \), we put \( a_1 + a_2 + a_3 + a_4 = 0. \)

Let \( G = (V, E) \) be a graph. For an edge \( e \) of \( G \), the deletion \( G \setminus e \) is the graph obtained by removing \( e \) from the edge set \( E \), and the contraction \( G/ e \) is the graph obtained identifying end vertices of \( e \) keeping all other adjacencies remain the same.

Now we give the following recurrence formula that can be used to calculate the weighted chromatic polynomials.

**Proposition 3.1** Let \( G = (V, E) \) be a graph and \( f \in \text{Hom}_d(n) \). Suppose \( e \) be an edge of \( G \). Then for a label \( s \) of \( G \), we have

\[ \chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) - \chi_f(G(s)/e; \lambda). \]

**Proof** Let \( e \) be an edge of \( G \). Then for a subset \( A \) of \( E \setminus \{e\} \), we denote by \( t \) the term \((-1)^{|A|}k(G_A)\). Also every subset \( A \) of \( E \setminus \{e\} \) corresponds to a pair of subsets \( A \) and \( A \cup \{e\} \) of \( E \). Let \( Q := A \cup \{e\} \). Then we have \( |E \setminus \{e\}| = |E| - 1 \) and \( |Q| = |A| + 1 \).

Suppose \( e \) is a loop. Then \( k(G_A) = k(G_Q) \). Clearly, \( \tilde{f}(s(E \setminus A)t) \) and \( \tilde{f}(s(E \setminus Q)t) \) be the terms in \( \chi_f(G(s) \setminus e; \lambda) \) and \( \chi_f(G(s)/e; \lambda) \), respectively corresponding to the set \( A \). Then the term in \( \chi_f(G(s); \lambda) \) corresponding to \( A \) is \( \tilde{f}(s(e)) \circ \tilde{f}(s(E \setminus A)t) \), while the term corresponding to \( Q \) is \((-1)\tilde{f}(s(E \setminus Q)t) \). So

\[ \chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) - \chi_f(G(s)/e; \lambda). \]

Now suppose that \( e \) is a bridge. Then \( k(G_A) = k(G_Q) \). It is immediate that \( \tilde{f}(s(E \setminus A)t) \) and \( \tilde{f}(s(E \setminus Q)t)/\lambda \) be the terms in \( \chi_f(G(s) \setminus e; \lambda) \) and \( \chi_f(G(s)/e; \lambda) \), respectively corresponding to the set \( A \). Then the term in \( \chi_f(G(s); \lambda) \) corresponding to \( A \) is \( \tilde{f}(s(e)) \circ \tilde{f}(s(E \setminus A)t) \), while the term corresponding to \( Q \) is \((-1)\tilde{f}(s(E \setminus Q)t)/\lambda \). So

\[ \chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) - \chi_f(G(s)/e; \lambda). \]

Finally, suppose \( e \) is neither loop nor bridge. Then we can write

\[ \chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \sum_{A \subseteq E \setminus e} (-1)^{|A|}k(G_A) + \sum_{A \subseteq E \setminus e} (-1)^{|Q|}k(G_Q). \]

Since \( k(G_A) = k(G_Q) \) for \( A \subseteq E \setminus e \), we complete the proof. \( \square \)
Now from the above proposition together with Remark 3.1 we have the following inductive formulation:

**Proposition 3.2** The weighted chromatic polynomials satisfies the following recursion:

(a) If $G$ has $m$ vertices and no edge, then $\chi_f(G(s); \lambda) = \tilde{f}(\emptyset)\lambda^m$.

(b) If $e$ is a loop, then

$$\chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \chi_f(G(s\setminus e); \lambda) - \chi_f(G(s\setminus e); \lambda).$$

(c) If $e$ is an edge which is not a loop, then

$$\chi_f(G(s); \lambda) = \tilde{f}(s(e)) \circ \chi_f(G(s\setminus e); \lambda) - \chi_f(G(s\setminus e); \lambda).$$

### 4 Weighted Tutte Polynomials

In this section, we present a correspondence between the harmonic chromatic polynomials of a graph associated with a harmonic function of degree $d$ and the harmonic Tutte polynomials of a matroid associated with a harmonic function of degree $d$. We refer the reader to [1, 4] for the basic definitions and notations of the usual Tutte polynomial of a matroid.

Let $M = (E, \mathcal{I})$ be a matroid with rank function $\rho$. We assume that the cardinality of $|E| = n$. Let $s$ be a bijective map from ground set $E$ to $\Omega$, and $M(s)$ be a matroid, where the elements of the ground set $E$ are indexed by $s$. We call $M(s)$ the *labelled matroid* and $s$ the *label* of the matroid $M$. Let $f$ be a homogeneous function of degree $d$. Then the *weighted Tutte polynomial* of $M$ associated with $f$ and $s$ is defined as follows:

$$T_f(M(s); x, y) := \sum_{A \subset E} \tilde{f}(s(A))(x - 1)^{\rho(E) - \rho(A)}(y - 1)^{|A| - \rho(A)}.$$

In a graphic matroid, $s$ is a label for the matroid if and only if $s$ is a label for the underlying graph.

In the definition of the weighted Tutte polynomial of a matroid, we can consider any map $s$ instead of only a bijective map in a similar way as above. We will discuss the properties of such weighted Tutte polynomials in some subsequent papers [6]. In this paper, we consider $s$ as a bijective mapping.

If $f$ is a harmonic function of degree $d$, then $T_f(M(s); x, y)$ is called the *harmonic Tutte polynomial* of $M$ associated to $f$ and $s$. If $f = 1$, then the weighted (harmonic) Tutte polynomials become the classical Tutte polynomial $T(M; x, y)$ independent of the label $s$. Let $G = (V, E)$ be graph. Then for any $A \subset E$, we define

$$\rho(A) := |V| - k(G_A).$$

Therefore, we find that $M_G$ is a matroid associated to $G$ having rank function $\rho$. 

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Example 4.1 From Example 3.1, we have the weighted Tutte polynomial as below:

\[ T_f(M_G(s); x, y) = (a_1 + a_2 + a_3 + a_4)(x - 1)^2 + (a_2 + a_3 + a_4)(x - 1)(y - 1) + 3(a_1 + a_2 + a_3 + a_4)(x - 1) + 2(a_1 + a_2 + a_3 + a_4) + a_1 \]

Remark 4.1 For \( A \subseteq E \), \( \rho(A) + k(G_A) = |V| = \rho(E) + k(G) \).

Before giving a recurrence formula for the calculation of the weighted Tutte polynomials, we recall [4] for the definitions of the loop, coloop, deletion and contraction from the matroid theoretical point of view.

Let \( M = (E, I) \) be a matroid. An element \( e \in E \) is called a loop if \( \{e\} \notin I \), or equivalently, if \( \rho(\{e\}) = 0 \). In a graphic matroid, \( e \) is a loop if and only if it is a loop of the underlying graph. An element \( e \in E \) is a coloop if it is contained in every basis of \( M \). This implies \( \rho(A \cup \{e\}) = \rho(A) + 1 \), whenever \( e \notin A \). In a graphic matroid, \( e \) is a coloop if and only if it is a bridge. The deletion of \( e \in E \) is a matroid \( M\setminus e \) on the set \( E \setminus \{e\} \) containing the independent sets of \( M \) which are contained in \( E \setminus \{e\} \). In a graphic matroid, deletion of \( e \) corresponds to deletion of the edge \( e \) from the graph. The contraction of \( e \) is the matroid \( M/e \) on the set \( E\setminus\{e\} \) in which a set \( A \) is independent if and only if \( A \cup \{e\} \) is independent in \( M \). In a graphic matroid, contraction of \( e \) corresponds to contraction of the edge \( e \).

Proposition 4.1 Let \( M = (E, I) \) be a matroid with rank function \( \rho \) and \( f \in \text{Hom}_d(n) \). Suppose \( e \in E \). Then for a label \( s \) of \( M \), we have

(a) \( T_f(\emptyset; x, y) = \tilde{f}(\emptyset) \), where \( \emptyset \) is the empty matroid.
(b) If \( e \) is a loop, then \( T_f(M(s); x, y) = T_f(M(s) \setminus e; x, y) + (y - 1)\tilde{f}(s(e)) \circ T_f(M(s) \setminus e; x, y) \).
(c) If \( e \) is a coloop, then \( T_f(M(s); x, y) = (x - 1)T_f(M(s)/e; x, y) + \tilde{f}(s(e)) \circ T_f(M(s)/e; x, y) \).
(d) If \( e \) is neither a loop nor a coloop, then

\[ T_f(M(s); x, y) = T_f(M(s) \setminus e; x, y) + \tilde{f}(s(e)) \circ T_f(M(s)/e; x, y) \].

Proof For \( e \in E \), every subset \( A \) of \( E \setminus \{e\} \) corresponds to a pair of subsets \( A \) and \( A \cup \{e\} \) of \( E \). Let \( \rho, \rho' \) and \( \rho'' \) denote the rank functions of \( M, M\setminus e \) and \( M/e \), respectively. Then (a) is trivial. Consider (b). Suppose \( e \) is a loop. Then the following hold:

(i) \( |E \setminus \{e\}| = |E| - 1 \);
(ii) \( \rho'(E \setminus \{e\}) = \rho(E) \);
(iii) \( |A \cup \{e\}| = |A| + 1 \);
(iv) \( \rho(A) = \rho(A \cup \{e\}) = \rho'(A) \).

Let \( \tilde{f}(s(A))t \) and \( \tilde{f}(s(A \cup \{e\}))t \) be the terms in \( T_f(M(s) \setminus e) \) and \( \tilde{f}(s(e)) \circ T_f(M(s) \setminus e) \), respectively corresponding to the set \( A \). Then the term in \( T_f(M(s)) \) corresponding to \( A \) is \( \tilde{f}(s(A))t \), while the term corresponding to \( A \cup \{e\} \) is \( (y - 1)\tilde{f}(s(A \cup \{e\}))t \). So,

\[ T_f(M(s); x, y) = T_f(M(s) \setminus e; x, y) + (y - 1)\tilde{f}(s(e)) \circ T_f(M(s)/e; x, y) \].
Now suppose that $e$ is a coloop. Then the following hold:

(i) $|E \setminus \{e\}| = |E| - 1$;
(ii) $\rho''(E \setminus \{e\}) = \rho(E) - 1$;
(iii) $|A \cup \{e\}| = |A| + 1$;
(iv) $\rho(A) = \rho(A \cup \{e\}) - 1 = \rho''(A)$.
(v) $\rho(E) - \rho(A) = \rho''(E \setminus \{e\}) - \rho''(A) + 1$.
(vi) $\rho(E) - \rho(A \cup \{e\}) = \rho''(E \setminus \{e\}) - \rho''(A)$.

Let $\tilde{f}(s(A))t$ and $\tilde{f}(s(A \cup \{e\}))t$ be the terms in $T_f(M(s)/e)$ and $T_f(M(s)/e)$, respectively corresponding to the set $A$. Then the term in $T_f(M(s))$ corresponding to $A$ is $(x - 1)\tilde{f}(s(A))t$, while the term corresponding to $A \cup \{e\}$ is $f(s(A \cup \{e\}))t$. Therefore,

$$T_f(M(s); x, y) = (x - 1)T_f(M(s)/e; x, y) + \tilde{f}(s(e)) \circ T_f(M(s)/e; x, y).$$

Finally, suppose $e$ is neither loop or coloop. Then we can write

$$T_f(M(s); x, y) = \sum_{A \subseteq E \setminus \{e\}} \tilde{f}(s(A))(x - 1)^{\rho(E) - \rho(A)}(y - 1)^{|A| - \rho(A)}$$
$$+ \sum_{A \subseteq E \setminus \{e\}} \tilde{f}(s(A \cup \{e\}))(x - 1)^{\rho(E) - \rho(A \cup \{e\})}(y - 1)^{|A \cup \{e\}| - \rho(A \cup \{e\})}.$$

Now for $A \subseteq E \setminus \{e\}$,

$$\rho(A \cup \{e\}) = \rho''(A) + 1 \text{ since } e \text{ is not a loop,}$$
$$\rho(A) = \rho(M/e) + 1 \text{ since } e \text{ is not a loop,}$$
$$\rho(A) = \rho'(A) \text{ since } e \text{ is not a coloop,}$$
$$\rho(E) = \rho(M \setminus e) \text{ since } e \text{ is not a coloop.}$$

This completes the proof.

The following result shows a correspondence between the harmonic chromatic polynomials and the harmonic Tutte polynomials.

**Theorem 4.1** Let $G = (V, E)$ be a graph with $|V| = m$ and $|E| = n$. Let $f \in \text{Harm}_d(n)$. Then for any label $s$ of $G$, we have

$$\chi_f(G(s); \lambda) = (-1)^{\rho(E) + d} \lambda^{k(G)} T_f(M_G(s); 1 - \lambda, 0).$$

**Proof** The matroid $M_G$ associated to the graph $G$ has the rank $\rho(J) = |V| - k(G_J)$ for $J \subseteq E$. Let $\lambda$ be a positive integer. Since $f \in \text{Harm}_d(n)$, therefore we have from Remark 2.2 that $f(J) = (-1)^d \tilde{f}(J)$ for any $J \subseteq E$. Now we need to show by induction that for $f \in \text{Harm}_d(n)$,

$$\chi_f(G(s); \lambda) = (-1)^{\rho(E) + d} \lambda^{k(G)} T_f(M_G(s); 1 - \lambda, 0). \quad (4)$$
Let \( \rho, \rho' \) and \( \rho'' \) be the rank functions of the matroids \( M_G, M_{G \setminus e} \) and \( M_{G/e} \), respectively.

Relation (4) is trivial when there is no edge since \( \rho(E) = 0, k(G) = m \) and \( T_f(M_G(s); 1 - \lambda, 0) = \tilde{f}(\emptyset) \).

Suppose \( e \) is a loop. Then \( k(G) = k(G \setminus e) \). Also \( \rho'(E \setminus \{e\}) = \rho(E) \). Now by Proposition 4.1(b), we have

\[
T_f(M_G(s); 1 - \lambda, 0) = T_f(M_{G \setminus e}(s); 1 - \lambda, 0) - \tilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0). \tag{5}
\]

Let \( A \subset E \setminus \{e\} \). Then \( T_f(M_{G \setminus e}(s); 1 - \lambda, 0) \) corresponds to the terms in \( T_f(M_G(s); 1 - \lambda, 0) \) involving \( A \), whereas \( \widetilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0) \) corresponds to the terms in \( T_f(M_G(s); 1 - \lambda, 0) \) involving \( A \cup \{e\} \). On the other hand, \( \tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) \) and \( \chi_f(G(s) \setminus e; \lambda) \) correspond to the term in \( \chi_f(G(s); \lambda) \) involving \( A \) and \( A \cup \{e\} \), respectively. Therefore, we can have

\[
\widetilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) = (-1)^{\rho(E) + d}k^{k(G)}T_f(M_{G \setminus e}(s); 1 - \lambda, 0). \tag{6}
\]

\[
\chi_f(G(s) \setminus e; \lambda) = (-1)^{\rho(E) + d}k^{k(G)}(\tilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0)). \tag{7}
\]

Combining (5), (6), (7), and using Proposition 3.2(b) we can have relation (4).

Suppose \( e \) is a bridge. Then \( k(G \setminus e) = k(G) + 1 \), and \( k(G/e) = k(G) \). Then \( \rho'(E \setminus \{e\}) = \rho''(E \setminus \{e\}) = \rho(E) - 1 \). By Proposition 4.1(c), we have

\[
T_f(M_G(s); 1 - \lambda, 0) = (-\lambda)T_f(M_{G \setminus e}(s); 1 - \lambda, 0) + \tilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0) \tag{8}
\]

Let \( A \subset E \setminus \{e\} \). Then \( (-\lambda)T_f(M_{G \setminus e}(s); 1 - \lambda, 0) \) corresponds to the terms in \( T_f(M_G(s); 1 - \lambda, 0) \) involving \( A \), whereas \( \tilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0) \) corresponds to the terms in \( T_f(M_G(s); 1 - \lambda, 0) \) involving \( A \cup \{e\} \). On the other hand, \( \tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) \) and \( \chi_f(G(s) \setminus e; \lambda) \) correspond to the term in \( \chi_f(G(s); \lambda) \) involving \( A \) and \( A \cup \{e\} \), respectively. Therefore, we can have

\[
\tilde{f}(s(e)) \circ \chi_f(G(s) \setminus e; \lambda) = (-1)^{\rho(E) - 1 + d}k^{k(G) + 1}T_f(M_{G \setminus e}(s); 1 - \lambda, 0). \tag{9}
\]

\[
\chi_f(G(s) \setminus e; \lambda) = (-1)^{\rho(E) - 1 + d}k^{k(G)}(\tilde{f}(s(e)) \circ T_f(M_{G \setminus e}(s); 1 - \lambda, 0)). \tag{10}
\]

Combining (8), (9), (10) and by using Proposition 3.2(c), we can have relation (4).

Finally, suppose that \( e \) is neither a loop nor a bridge. Then \( k(G) = k(G \setminus e) = k(G/e) \). Also \( \rho'(E \setminus \{e\}) = \rho(E) \), and \( \rho''(E \setminus \{e\}) = \rho(E) - 1 \). Therefore, by similar argument as above we can show relation (4) is true by using Proposition 3.2(c) together with Proposition 4.1(d).

\(\square\)
We can also give a direct proof of the above theorem with the help of Remark 4.1 as follows:

\[ (-1)^{\rho(E)} \lambda^{-k(G)} \chi_f(G(s); \lambda) \]

\[ = \sum_{A \subset E} \tilde{f}(s(E \setminus A))(-1)^{\rho(E)+|A|} \lambda^{k(G_A)-k(G)} \]

\[ = (-1)^d \sum_{A \subset E} \tilde{f}(s(A))(-1)^{\rho(E)+|A|} \lambda^{\rho(E)-\rho(A)} \]

\[ = (-1)^d \sum_{A \subset E} \tilde{f}(s(A))(-1)^{\rho(E)-\rho(A)}(-1)^{|A|-\rho(A)} \lambda^{\rho(E)-\rho(A)} \]

\[ = (-1)^d \sum_{A \subset E} \tilde{f}(s(A))(-1)^{|A|-\rho(A)}(-\lambda)^{\rho(E)-\rho(A)} \]

\[ = (-1)^d T_f(M_G(s); 1 - \lambda, 0). \]

Multiplying both sides by \((-1)^{\rho(E)} \lambda^{k(G)}\) we get the result.

## 5 Harmonic Tutte–Grothendieck Polynomials

In this section, we present the concept of harmonic Tutte–Grothendieck polynomial (harmonic T–G polynomial). Moreover, we give a generalization of the classical relation known as the recipe theorem between the Tutte–Grothendieck invariants and the Tutte polynomials to the case of harmonic T–G polynomials.

**Definition 5.1** Let \( G = (V, E) \) be a graph with number of edges \( n \). Let \( s \) be a label of \( G \), and \( e \) be an edge of \( G \). Let \( f \in \text{Hom}_d(n) \). Then the weighted Tutte-Grothendieck polynomial of \( G \) associated to \( f \) and \( s \) is denoted by \( \Phi_f(G(s)) := \Phi_f(G(s); X, Y, \alpha, \beta) \) and defined recursively for \( \alpha \neq 0 \) and \( \beta \neq 0 \) as follows:

(a) If \( G \) has no edge, then \( \Phi_f(G(s)) = \tilde{f}(\emptyset) \).

(b) If \( e \) is a loop, then

\[ \Phi_f(G(s)) = \alpha \tilde{f}(s(e)) \circ \Phi_f(G(s \setminus e) + (Y - \alpha) \Phi_f(G(s \setminus e). \]

(c) If \( e \) is a bridge, then

\[ \Phi_f(G(s)) = X'Y' \tilde{f}(s(e)) \circ \Phi_f(G(s \setminus e) + \beta Y' \Phi_f(G(s \setminus e), \]

where \( X' = X - \beta \) and \( Y' := \frac{Y - \alpha}{\alpha} \).

(d) If \( e \) is neither a loop nor a bridge, then

\[ \Phi_f(G(s)) = \alpha \tilde{f}(s(e)) \circ \Phi_f(G(s \setminus e) + \beta \Phi_f(G(s) / e). \]
If $f$ is a harmonic function of degree $d$, then we call $\Phi_f(G(s))$ the harmonic Tutte–Grothendieck polynomial of $G$ associated to $f$ and $s$.

**Remark 5.1** If $f \in \text{Harm}_d(n)$, then by Remark 2.3, we have instantly that $\Phi_f(G(s)) = (-1)^d P_f(G(s))$, where $P_f(G(s)) := P_f(G(s); X, Y, \alpha, \beta)$ is a polynomial of $G$ associated with $f$ and $s$ satisfying the following recurrence for $\alpha \neq 0$ and $\beta \neq 0$:

(a) If $G$ has no edge, then $P_f(G(s)) = \widetilde{f}(\emptyset)$.

(b) If $e$ is a loop, then

\[ P_f(G(s)) = \alpha P_f(G(s)\setminus e) + (Y - \alpha) \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e). \]

(c) If $e$ is a bridge, then

\[ P_f(G(s)) = X'Y' P_f(G(s)\setminus e) + \beta Y' \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e), \]

where $X' = X - \beta$ and $Y' = \frac{Y - \alpha}{\alpha}$.

(d) If $e$ is neither a loop nor a bridge, then

\[ P_f(G(s)) = \alpha P_f(G(s)\setminus e) + \beta \widetilde{f}(s(e)) \circ P_f(G(s)/e). \]

Now we give a generalization of the recipe theorem which presents a relation between the harmonic T–G polynomial and the harmonic Tutte polynomial as follows.

**Theorem 5.1** Let $G = (V, E)$ be a graph with $|E| = n$, and $M_G$ be a matroid associated with $G$. Let $\Phi_f(G(s); X, Y, \alpha, \beta)$ be a harmonic T–G polynomial of $G$ related to $f \in \text{Harm}_d(n)$ and a label $s$ of $G$. Then for $\alpha \neq 0$ and $\beta \neq 0$, we have

\[ \Phi_f(G(s); X, Y, \alpha, \beta) = (-1)^d \alpha^{\gamma} P_f(G(s)\setminus e) + \beta^{\gamma} \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e), \]

where $X' = X - \beta$ and $Y' = \frac{Y - \alpha}{\alpha}$.

**Proof** From Remark 5.1, it is sufficient to show that

\[ P_f(G(s); X, Y, \alpha, \beta) = \alpha^{\gamma} P_f(G(s)\setminus e) \circ P_f(G(s)\setminus e), \]

where $X' = X - \beta$ and $Y' = \frac{Y - \alpha}{\alpha}$.

If $G$ has no edge then the formula (12) is trivial and so is (11). Let $e$ be a loop. Then $k(G) = k(G\setminus e)$. Now we have

\[ P_f(G(s)) = \alpha P_f(G(s)\setminus e) + (Y - \alpha) \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e) \]

\[ = \alpha \alpha^{\gamma} \beta^{\gamma} \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e), \]

\[ + \alpha^{\gamma} \beta^{\gamma} \widetilde{f}(s(e)) \circ P_f(G(s)\setminus e). \]
Weighted T–G Invariants

Example 5.1 If we take $G$ of $T$–G invariant, in this section, we introduce the notion of weighted T–G invariant which was thoroughly developed and generalized by Brylawski [3]. Motivated by the concept of Tutte [15] pointed out an idea of graph invariant which is known as Tutte–Grothendieck invariant (T–G invariant) for all functions that satisfy deletion-contraction recurrence equation. Moreover, Theorem 5.1 implies Theorem 4.1 when $f \in \text{Harm}_{d}(n)$. Hence (12) as well as (11) is true when $e$ is a loop. Let $e$ be a bridge. Then $k(G) + 1 = k(G \setminus e)$ and we have

$$P_f(G(s)) = (X - \beta) \left(\frac{Y - \alpha}{\alpha}\right) P_f(G(s) \setminus e) + \beta \left(\frac{Y - \alpha}{\alpha}\right) \tilde{f}(s(e)) \circ P_f(G(s) \setminus e)$$

So, (11) is true when $e$ is a bridge. Now let $e$ be neither a loop nor a bridge. Then $k(G) = k(G \setminus e) = k(G / e)$ and similarly as above we find (11) is true.

Example 5.1 If we take $\Phi_f(G(s)) = \frac{x_f(G(s))}{\lambda_{k(G)}}$ with $f \in \text{Hom}_{d}(n)$, $X = \lambda - 1$, $Y = 0$, $\alpha = 1$ and $\beta = -1$, we immediately get the weighted chromatic polynomial. Moreover, Theorem 5.1 implies Theorem 4.1 when $f \in \text{Harm}_{d}(n)$.

6 Weighted Invariants for Graphs and Matroids

Tutte [15] pointed out an idea of graph invariant which is known as Tutte–Grothendieck invariant (T–G invariant) for all functions that satisfy deletion-contraction recurrence equation. The concept of Tutte–Grothendieck invariant was thoroughly developed and generalized by Brylawski [3]. Motivated by the concept of T–G invariant, in this section, we introduce the notion of weighted T–G invariant which is also called harmonic T–G invariant in some particular cases.

Weighted T–G Invariants

Let $G = (V, E)$ be a graph with $|E| = n$. Let $s$ be a label of $G$. We denote the set of all labels $s$ of $G$ by $S(G)$. It is easy to compute that $\sharp S(G) = n!$. Now we define

$$\hat{\Phi}_f(G) := \sum_{s \in S(G)} \Phi_f(G(s)). \quad (13)$$

Remark 6.1 For $f = 1$, $\hat{\Phi}_f(G) = n! \Phi(G)$, where $\Phi(G)$ is the classical T–G invariant.
We denote by $S_n$ the symmetric group of order $n$. Then the following result is immediate from the construction of $\Phi_f(G)$.

**Theorem 6.1** Let $G = (V, E)$ be a graph with $|E| = n$. Then for any $f \in \text{Hom}_d(n)$, we have

(i) $\hat{\Phi}_f(G)$ is an invariant for graphs.

(ii) Let $R(f) := \sum_{\sigma \in S_n} \sigma f$. Then $\Phi_{R(f)}(G)$ is an invariant for graphs, where $\Phi_{R(f)}(G)$ denotes the weighted $T$–$G$ polynomial of $G$ associated to $R(f)$ and independent of the choice of $s$. Moreover, $\hat{\Phi}_f(G) = \Phi_{R(f)}(G)$.

(iii) Let $\sigma f = f$ for all $\sigma \in S_n$. Then $\Phi_f(G)$ is an invariant for graphs. Also, $\hat{\Phi}_f(G) = n!\Phi_f(G)$.

We call $\hat{\Phi}_f(G)$ the weighted Tutte–Grothendieck invariant for graphs. If $f \in \text{Harm}_d(n)$, we call $\hat{\Phi}_f(G)$ the harmonic Tutte–Grothendieck invariant for graphs, which is zero for $d \neq 0$.

**Weighted Tutte Invariants**

Let $M$ be a matroid with ground set $E$ of cardinality $n$. Let $s$ be a label of $M$. Define

$$\hat{T}_f(M; x, y) := \sum_{s \in S(M)} T_f(M(s); x, y),$$

where $S(M)$ interprets a similar meaning as $S(G)$ for the case of graphs.

**Remark 6.2** For $f = 1$, $\hat{T}_f(M; x, y) = n!T(M; x, y)$.

**Theorem 6.2** Let $M = (E, \mathcal{I})$ be a matroid with $|E| = n$. Then for any $f \in \text{Hom}_d(n)$, we have

(i) $\hat{T}_f(M)$ is an invariant for graphs.

(ii) Let $R(f) := \sum_{\sigma \in S_n} \sigma f$. Then $T_{R(f)}(M)$ is an invariant for matroids, where $T_{R(f)}(M)$ denotes the weighted Tutte polynomial of $M$ associated to $R(f)$ and independent of the choice of label $s$. Moreover, $\hat{T}_f(M) = T_{R(f)}(M)$.

(iii) Let $\sigma f = f$ for all $\sigma \in S_n$. Then $T_f(M)$ is an invariant for matroids. Also, $\hat{T}_f(M) = n!T_f(M)$.

We call $\hat{T}_f(M; x, y)$ the weighted Tutte invariant for matroids. If $f \in \text{Harm}_d(n)$, we call $\hat{T}_f(M; x, y)$ the harmonic Tutte invariant for matroids which is zero for $d \neq 0$.

**Example 6.1** Let $M_1$ be a vector matroid over $\mathbb{F}_2$ and $M_2$ be a vector matroid over $\mathbb{F}_3$ as given below:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

We have checked numerically that for all $f \in \text{Hom}_d(7)$ ($1 \leq d \leq 7$), $\hat{T}_f(M_1, x, y) = \hat{T}_f(M_2, x, y)$. For the construction of vector matroids, we refer the readers to [4].
Problem 6.1 Find a non-isomorphic matroid pair \((M_1, M_2)\) having the same Tutte polynomial but the different weighted Tutte invariants.

The following result gives a correspondence between the weighted T–G invariant and the weighted Tutte invariant which is an analogue to Theorem 5.1.

**Theorem 6.3** Let \(\hat{\Phi}_f\) be a weighted T–G invariant associated to \(f \in \text{Hom}_d(n)\). Then for all graphs \(G = (V, E)\) we have

\[
\hat{\Phi}_f(G) = \alpha^{|E| - |V| + k(G)} \beta^{|V| - k(G)} \hat{T}_f\left(M_G; \frac{X}{\beta}, \frac{Y}{\alpha}\right),
\]

where \(X, Y, \alpha, \beta\) interpret the same meaning as in weighted T–G polynomials associated to \(f\).

**Proof** Let \(A\) be any subset of \(E\). Let \(\tilde{f}(s(A))t\) be the term in \(T_f(M_G(s))\) corresponding to the set \(A\) and label \(s\) of \(M_G\). Since \(s\) is a bijective map from \(E\) to \(\Omega\), therefore, \(\sum_{s \in S(M_G)} \tilde{f}(s(A))t = \sum_{s \in S(M_G)} \tilde{f}(s(E \setminus A))t\). This fact together with Proposition 4.1 and Definition 5.1 implies (14). \(\Box\)

**Weighted Chromatic Invariant**

Let \(s\) be a label of a graph \(G\) with \(n\) edges. Then from Example 5.1, if we take \(\Phi_f(G(s)) = \frac{\chi_f(G(s))}{\lambda^k(G)}\) with \(f \in \text{Hom}_d(n)\), \(X = \lambda - 1, Y = 0, \alpha = 1\) and \(\beta = -1\), we get from definition (13) that

\[
\hat{\chi}_f(G(s)) = \sum_{s \in S(G)} \chi_f(G(s)),
\]

which is from Theorem 6.1, an invariant for graphs. We call this invariant the *weighted chromatic invariant* for graphs. When \(f \in \text{Harm}_d(n)\), we call this invariant the *harmonic chromatic invariant* for graphs which is zero for \(d \neq 0\). Moreover, Theorem 6.3 yields therefore the following corollary.

**Corollary 6.1** We have

\[
\hat{\chi}_f(G; \lambda) = (-1)^{\rho(E)} \lambda^k(G) \hat{T}_f(M_G; 1 - \lambda, 0).
\]
Example 6.2 Let $G_1$ and $G_2$ be two non-isomorphic graphs shown in Fig. 2 having the same classical chromatic polynomials. Then we have $\Omega = \{1, 2, \ldots, 10\}$. Let $f = \sum_{X \in \Omega} X$. Then the weighted chromatic invariant for $G_1$ is

$$\hat{\chi}_f(G_1; \lambda) = 10! \times (210\lambda^6 - 1260\lambda^5 + 2975\lambda^4 - 3450\lambda^3 + 1960\lambda^2 - 435\lambda),$$

whereas the weighted chromatic invariant for $G_2$ is

$$\hat{\chi}_f(G_2; \lambda) = 10! \times (210\lambda^6 - 1260\lambda^5 + 2975\lambda^4 - 3434\lambda^3 + 1925\lambda^2 - 416\lambda).$$

Therefore, $\hat{\chi}_f(G_1; \lambda) \neq \hat{\chi}_f(G_2; \lambda)$. This concludes that the weighted chromatic invariants is stronger than the classical chromatic polynomials.

Now we recall [11] for the concept of log-concave sequences. A sequence of real numbers $a_0, a_1, \ldots, a_n$ is said to be log-concave if $a_i^2 - a_{i-1}a_{i+1} \leq 0$ for all $0 < i < n$. Observing the above example, we may conclude the following conjecture.

Conjecture 6.1 Let $\hat{\chi}_f(G; \lambda) = an\lambda^n - (n-1)\lambda^{n-1} + \cdots + (-1)^n a_0$ be a weighted chromatic invariant of a graph $G$ associated to $f \in \text{Hom}_d(n)$ such that $f$ is symmetric. Then the sequence $a_0, a_1, \ldots, a_n$ is log-concave.

7 Homological Categorifications of Weighted Polynomials

Originated from category theory, a categorification in homology theory usually means to categorify a quantity or a polynomial with some homology groups. For example, Khovanov homology is a categorification of Jone’s polynomial, Floer homology is a categorification of Alexander-Conway polynomial, and Magnitude homology is a categorification of magnitude.

In this section, we apply the method in [9, 10] to construct chain complexes and give categorifications to the weighted chromatic polynomial and weighted Tutte polynomial, respectively. We omit some basic facts of algebraic topology and homological algebra. For details, the readers may refer [12, 14].

We categorify the weighted polynomial by assigning a real-valued weight to the specially constructed chain complexes, and show the relationship between the chain complexes and our invariant in Sect. 6. We will write another paper to discuss the general weighted chain complex, and discuss only special cases in this paper.

7.1 A Categorification of Weighted Chromatic Polynomials

Let $\mathcal{M}$ be a free $\mathbb{Z}$-module with $\mathcal{M} = M_0 \oplus M_1$, where $M_0$ and $M_1$ are subgroups of $\mathcal{M}$ and are generated by basis element 1 and $x$, respectively. Hence $\mathcal{M}$ can be considered as a graded free $\mathbb{Z}$-module with non-trivial elements only at degree 0 and 1.

Definition 7.1 Let $G(s) = (V, E)$ be a labelled graph with $n$ edges. We use a vector $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0, 1\}^n$ to represent any given subgraph $G(s)'$ of $G(s)$ whose
vertices are $V$. Precisely, for $i = 1, 2, \ldots, n$, if the $i$-th edge exists in $G(s)'$, then we let $\epsilon_i = 1$; and $\epsilon_i = 0$ otherwise. We call $\epsilon$ the edge vector of $G(s)$. Given any edge vector, we also denote the corresponding subgraph $G_\epsilon$. We denote by $l(\epsilon)$ the number of 1s in edge vector $\epsilon$. Let $E^a(G(s))$ be the set of all edge vectors $\epsilon$ with $l(\epsilon) = a$, $a \in \mathbb{N}$.

It is easy to check that Definition 7.1 is well-defined, and the set of all subgraphs of $G$ whose vertices are $V$ is on 1-1 correspondence to the set of all edge vectors.

In order to construct a cochain complex with $M$, we first define the cochain group as follows.

**Definition 7.2** The $q$-th chromatic cochain group of a graph $G$ is defined as

$$C^q(G(s)) := \bigoplus_{l(\epsilon) = q} M^\otimes k(G_\epsilon),$$

where $\otimes$ denotes tensor product.

We also write

$$C^q(G(s)) = \bigoplus_{j \geq 0} C^q, j(G(s)),$$

where $C^q, j(G(s))$ denotes the elements of degree $j$ of $C^q(G(s))$. Hence $C^q(G(s))$ can also be considered as a bigraded module.

Let $f : 2^\Omega \rightarrow \mathbb{R}$ be a discrete function of degree $d$. Since each $M^\otimes k(G_\epsilon)$ corresponds to an edge vector $\epsilon$, by the definition, it also corresponds to an $f$-value. Therefore, besides the dimension of the graded module, we can assign a weighted dimension to the graded module, where the weight is given by $f$-values. We begin with the definition of weighted rank as follows.

**Definition 7.3** Let $G$ be a $\mathbb{Z}$-module with decomposition $G = \bigoplus_i G_i$. We suppose that for each $G_i$, there is a corresponding $f$-value $f(G_i)$. Then we define the $f$-rank of $G$ as

$$f\text{rank}(G) := \sum_i f(G_i) \cdot \dim(\mathbb{Q} G_i \otimes \mathbb{Q}).$$

Note that $\dim(\mathbb{Q} G_i \otimes \mathbb{Q})$ is known as the rank of $G_i$.

**Definition 7.4** Let $\mathcal{N} = \bigoplus_i N_i$ be a graded $\mathbb{Z}$-module with an $f$-value $f(\mathcal{N})$, where $N_i$ denotes the set of homogeneous elements of degree $i$. The graded dimension of $\mathcal{N}$ is defined as

$$q\text{dim}(\mathcal{N}) := \sum_i q^i \cdot \text{rank}(N_i),$$

and the $f$-value graded dimension of $\mathcal{N}$ is defined as

$$f\, q\text{dim}(\mathcal{N}) := \sum_i q^i \cdot f\, \text{rank}(N_i).$$
Lemma 7.1 Let \( \mathcal{N}, \mathcal{G}, \) and \( \mathcal{H} \) be graded modules with \( f \)-values and \( \mathcal{N} = \mathcal{G} \oplus \mathcal{H} \). Then

\[
\text{fqdim}(\mathcal{N}) = \text{fqdim}(\mathcal{G}) + \text{fqdim}(\mathcal{H}).
\]

Proof

\[
\text{fqdim}(\mathcal{N}) = \sum_i q^i \cdot \text{rank}(N_i) = \sum_i q^i \cdot \text{rank}(G_i \oplus H_i) = \sum_i q^i \cdot f(G_i) \cdot \text{rank}(G_i) + f(H_i) \cdot \text{rank}(H_i) = \text{fqdim}(\mathcal{G}) + \text{fqdim}(\mathcal{H}),
\]

where \( \text{rank}(G_i) = \dim \mathbb{Q}(G_i \otimes \mathbb{Q}) \).

According to the above lemma, we can calculate as an example that for two graded module \( M \) defined above, suppose that they correspond to \( f \)-values \( a_1 \) and \( a_2 \), then

\[
\text{fqdim}(M \oplus M) = (a_1 + a_2)(1 + q), \quad \text{and} \quad \text{fqdim}(M \otimes^m M \otimes^n M) = a_1(1 + q)^m + a_2(1 + q)^n.
\]

The desired chain complex is constructed by the cochain group above and a coboundary operator. A coboundary operator is a module homomorphism between two cochain groups whose dimensions differ by 1. By the correspondence between cochain groups and edge vectors, the coboundary operator is induced by a map that increases exactly one edge.

We define the coboundary operator by defining a map for each element module, and then linearly extend them. Let \( d_\epsilon : \mathcal{E}^a(G(s)) \to \mathcal{E}^{a+1}(G(s)) \) be a map that is identity on all elements but changes exact one 0 to 1. For an edge \( \tilde{e} \), if adding the edge does not change the number of connected components, then we define the element module map \( \tilde{d}_\epsilon : M \otimes k \to M \otimes k \) to be the identity map. On the other hand, if the added edge \( \tilde{e} \) joins two connected components, that is, the number of connected components decreases by one, then we define the element module map \( \tilde{d}_\epsilon : M \otimes^k M \otimes^{k-1} \) to be identity on the tensor elements which are irrelevant to this process. Moreover, for the two components that are connected, the map is defined to be \( M \otimes M \to M \) with \( m(1 \otimes 1) = 1, m(1 \otimes x) = m(x \otimes 1) = x, \) and \( m(x \otimes x) = 0 \).

Definition 7.5 (coboundary) The coboundary operator \( d^q : C^q(G(s)) \to C^{q+1}(G(s)) \) is defined as

\[
d^q := \sum_{l(\epsilon) = q} (-1)^{n(\epsilon)} \tilde{d}_\epsilon,
\]

where \( n(\epsilon) = \sum_{i=1}^{k_0} \epsilon_i \), and \( k_0 \) is the index of the new-added edge in the given order.
By definition and simple calculation, it is uncomplicated to verify the following statements.

**Lemma 7.2** The following statements are true.

1. \( d \) is degree preserving.
2. \( d \circ d = 0 \).

(2) of Lemma 7.2 implies that \((C^*, d^*)\) forms a cochain complex. We denote by \( H^i(G) := \ker(d^i)/\text{im}(d^{i-1}) \), the corresponding cohomology, where \( \ker(d^i) \) denotes the kernel of \( d^i \), and \( \text{im}(d^{i-1}) \) denotes the image of \( d^{i-1} \). It is shown in [9] that, the cochain complex does not depend on the choice of the order \( s \).

**Definition 7.6** The graded Euler number of a cochain complex \((C^*, d^*)\) is defined as \( \text{Euler}_i(C) := \sum (-1)^i \cdot q \dim(C^i(G)) \). Also, the weighted graded Euler number corresponding to a function \( f \) of \((C^*, d^*)\) is defined as \( w\text{Euler}_i(C)_f := \sum (-1)^i \cdot f q \dim(C^i) \).

By (1) of Lemma 7.2, the coboundary operator \( d \) is degree preserving, hence the (weighted) graded Euler number is also equal to the alternative sum of the \( f q \dim \) of cochain groups, those are \( \text{Euler}_i(C) = \sum (-1)^i q \dim(C^i) \), and \( w\text{Euler}_i(C) = \sum (-1)^i f q \dim(C^i) \).

When \( f \) is a discrete harmonic function it follows from the definition of harmonic polynomial, we have the following relationship between harmonic polynomial and the chain complex constructed. This relationship gives a weighted categorification to harmonic chromatic polynomials.

**Theorem 7.1** Let \( G(s) = (V, E) \) be a labelled graph and \( f \) a discrete harmonic function on \( E \), then

\[
\chi_f(G(s)) = \sum_i (-1)^{i+1} \cdot f q \dim(C^i(G(s))).
\]

Also,

\[
\hat{\chi}_f(G) = \sum_s \sum_i (-1)^{i+1} \cdot f q \dim(C^i(G(s))).
\]

**Example 7.1** We calculate the weighted chain complex and the weighted chromatic polynomial of graph \( G \) as an example. Let \( G = (V, E) \) be the following graph, and \( f : E \to \mathbb{R} \) be a discrete harmonic function with degree 1 on \( E \). The \( f \)-values of each edges are \( a_1, a_2, a_3, \) and \( a_4 \) as follows. The label is denoted by \( s \).

Thus the sequence of chromatic cochain group is

\[
0 \to C^0(G(s)) \to C^1(G(s)) \to C^2(G(s)) \to C^3(G(s)) \to C^4(G(s)) \to 0.
\]

By definition,

\[
C^0 \cong M^\otimes 4,
\]
Fig. 3 An example of harmonic chromatic chain complex

![Diagram of a graph with vertices labeled a1, a2, a3, and a4 connected by edges]

\[ C^1 \cong M^{\otimes 3} \oplus M^{\otimes 3} \oplus M^{\otimes 3} \oplus M^{\otimes 3}; \]
\[ C^2 \cong M^{\otimes 2} \oplus M^{\otimes 2} \oplus M^{\otimes 2} \oplus M^{\otimes 2} \oplus M^{\otimes 2}; \]
\[ C^3 \cong M^{\otimes 2} \oplus M \oplus M \oplus M; \]
\[ C^4 \cong M. \]

Then, we calculate the \(fq\)-dim of \(C^i(G(s))\) for \(i = 0, 1, 2, 3, 4\).

\[
\begin{align*}
fq \dim (C^0) &= 0 \cdot (1 + q)^4 = 0; \\
fq \dim (C^1) &= a_1(1 + q)^3 + a_2(1 + q)^3 + a_3(1 + q)^3 + a_4(1 + q)^3 \\
&= (a_1 + a_2 + a_3 + a_4)(1 + q)^3 = 0; \\
fq \dim (C^2) &= (a_1 + a_2)(1 + q)^2 + (a_1 + a_3)(1 + q)^2 + (a_1 + a_4)(1 + q)^2 \\
&+ (a_2 + a_3)(1 + q)^2 + (a_2 + a_4)(1 + q)^2 + (a_3 + a_4)(1 + q)^2 \\
&= 3(a_1 + a_2 + a_3 + a_4)(1 + q)^2 = 0; \\
fq \dim (C^3) &= (a_1 + a_2 + a_3)(1 + q)^2 + (a_1 + a_2 + a_4)(1 + q) \\
&+ (a_1 + a_3 + a_4)(1 + q) + (a_2 + a_3 + a_4)(1 + q) \\
&= -a_4(1 + q)^2 + a_4(1 + q); \\
fq \dim (C^4) &= (a_1 + a_2 + a_3 + a_4)(1 + q)^4 = 0.
\end{align*}
\]

Note that since \(f\) is a discrete harmonic function, \((a_1 + a_2 + a_3 + a_4) = 0\), and \((a_1 + a_2 + a_3) = -a_4\).

We suppose that \(\lambda = 1 + q\), then the weighted graded Euler number of \(C^\bullet(G(s))\) is \(a_4\lambda^2 - a_4\lambda\), which coincidences to the opposite of the weighted chromatic polynomial \(\chi_f(G(s))\) that is calculated in Example 3.1.

### 7.2 A Categorification of Weighted Tutte Polynomials

In this subsection, we apply the method in [10] to give a categorification to the Harmonic Tutte Polynomial defined in Sect. 4. Similar to the last subsection, we give a construction of a cochain complex and calculate its weighted alternative sum to categorify the harmonic Tutte polynomial. We omit some proofs in [10]. We use the...
notation \( TC^\bullet(G) \) and boundary operator \( td^\bullet \) in this section to distinguish the notations in this subsection and the last subsection.

The cochain group corresponding to the harmonic Tutte polynomial is constructed by a \( \mathbb{Z} \)-bigraded module. A \( \mathbb{Z} \)-bigraded module is a module \( A \) consisting of a decomposition \( A = \bigoplus_{(i,j) \in \mathbb{Z} \otimes \mathbb{Z}} A_{i,j} \). The graded dimension of \( A \) is defined as a two-variable power series \( q \dim A := \sum_{i,j} \dim_{\mathbb{Q}}(A_i \otimes \mathbb{Q}) \).

Let \( A \) and \( B \) be polynomial rings and \( A = \mathbb{Z}[x]/(x^2), B = \mathbb{Z}[y]/(y^2) \), where \( \deg x = (1,0), \deg y = (0,1) \). The degree of a polynomial is the largest natural number such that the coefficient is not zero. A simple calculation implies that \( q \dim A = 1 + x, q \dim B = 1 + y, \) and \( q \dim (A \otimes m \otimes B \otimes n) = (1 + x)^m(1 + y)^n \), for any \( n, m \in \mathbb{N} \).

Let \( G(s) = (V, E) \) be a labelled graph, and \( f : 2^\Omega \to \mathbb{R} \) be a discrete function of degree \( d \). Similar to the construction in the last subsection, we represent any subgraph \( G' \) of \( G \) whose vertices are \( V \) by the edge vector. We define the cochain group as following.

**Definition 7.7** The \( q \)-th Tutte chain group of \( G(s) \) is defined as

\[
TC^q(G(s)) := \bigoplus_{l(e) = q} A^{qk(G_e)} \otimes B^{q\beta_1(G_e)},
\]

where \( \beta_1(G_e) \) denotes the one-dimensional Betti number of graph \( (G_e) \).

Next, we define a Tutte differential operator \( td_q : TC^q(G) \to TC^{q+1}(G) \). We begin with a map from \( E^q(G) \) to \( E^{q+1}(G) \) that is identity on all elements but change exact one 0 to 1. That is, the map adds an edge into some subset \( D \subset E \). Known as a basic fact of algebraic topology, adding an edge \( \tilde{e} \) to \( D \subset E \) leads to only two cases, those are,

1. \( \tilde{e} \) joins two connected components of \( G_D \);
2. \( \tilde{e} \) forms a cycle in \( G_D \).

In the first case, \( k(G_{D_{\cup \tilde{e}}} = k(G_D) - 1 \), and \( \beta_1(G_{D_{\cup \tilde{e}}}) = \beta_1(G_D) \). In the second case, \( k(G_{D_{\cup \tilde{e}}} = k(G_D) \), and \( \beta_1(G_{D_{\cup \tilde{e}}}) = \beta_1(G_D) + 1 \).

For the first case, we define \( td^A_\tilde{e} : A^{e_1}(G) \to A^{e_2}(G) \) to be the identity map on all tensor factors other than those two tensor factors that are induced by two connected components joined. For the two joined tensor factors, we map \( a_1 \otimes a_2 \in A \otimes A \) to \( a_1a_2 \in A \). We also define \( td^B_\tilde{e} : B^{e_1}(G) \to B^{e_2}(G) \) to be the identity map.

For the second case, we define \( td^A_\tilde{e} : A^{e_1}(G) \to A^{e_2}(G) \) be the identity map, and \( td^B_\tilde{e} : B^{e_1}(G) \to B^{e_2}(G) \) be the module homomorphism mapping \( b \in B^{e_1}(G) \mapsto b \otimes 1 \in B^{e_2}(G) = B^{e_1}(G) \otimes B \).

Now by combining \( td^A_\tilde{e} \) and \( td^B_\tilde{e} \) in two different cases we can define the differential operator. More precisely,

**Definition 7.8** The differential operator

\[
Td^q : TC^q(G(s)) \to TC^{q+1}(G(s))
\]

is defined as \( Td^q = \sum_{l(e) = q} (-1)^{n(e)}(td^A_\tilde{e} \otimes td^B_\tilde{e}) \), where \( n(e) = \sum_{i=1}^{k_0} \epsilon_i \), and \( k_0 \) is the index of the new-added edge in the given order.
Lemma 7.3  The following statements are true.

(1) $td \circ td = 0$.

(2) $td$ is degree preserving.

The proof of $td \circ td = 0$ can be found in [10]. Since $td \circ td = 0$, $\{TC^\bullet(G), td^\bullet\}$ is a cochain complex. We denote by $TH^\bullet(G)$ for its cohomology. It is shown in [10] that, the cochain complex does not depend on the choice of the order $s$. The following theorem implies that the constructed cochain complex combined with a harmonic function is a weighted categorification to harmonic Tutte polynomial.

Theorem 7.2  Let $G = (V, E)$ be a graph with an order $s$ and $f$ a discrete harmonic function on $E$, then

$$T_f(M_G(s); x, y) = \sum_i (-1)^{i+1} \cdot f \dim(C^i(G(s))).$$

Also,

$$\hat{T}_f(M_G; x, y) = \sum_s \sum_i (-1)^{i+1} \cdot f \dim(C^i(G(s))).$$

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