OPERATOR HOLES AND EXTENSIONS OF SECTORIAL OPERATORS
AND DUAL PAIRS OF CONTRACTIONS

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Abstract. A description of the set of \(m\)-sectorial extensions of a dual pair \(\{A_1, A_2\}\) of nonnegative operators is obtained. Some classes of nonaccretive extensions of the dual pair \(\{A_1, A_2\}\) are described too. Both problems are reduced to similar problems for a dual pair \(\{T_1, T_2\}\) of nondensely defined symmetric contractions \(T_j = (I - A_j)(I + A_j)^{-1}, \ j \in \{1, 2\}\). In turn these problems are reduced to the investigation of the corresponding operator “holes”. A complete description of the set of all proper and improper extensions of a nonnegative operator is obtained too.

1. Introduction

In the theory of extensions of a nonnegative operator \(A(\subset A^*)\) in a Hilbert space \(\mathcal{H}\) to a selfadjoint or \(m\)-sectorial \(23\) operator there are two well-known approaches in which extensions \(\tilde{A} \supset A\) in various classes are described in diverse forms. One of these, proposed by M. G. Krein in \(25\) (see also \(11\ 51\)) uses the linear fractional transformation \(T_1 = (I - A)(I + A)^{-1}\) to reduce the problem to the description of various classes of extensions \(T \supset T_1\) of a nondensely defined (on the subspace \(\mathcal{H}_1 = (I + A)\mathcal{H}\)) symmetric contraction \(T_1\).

The other approach to the description of proper extensions \(\tilde{A}\) of an operator \(A > 0\) was proposed by Vishik \(39\) and Birman \(9\). They associate with each extension \(\tilde{A} \supset A\) (not necessarily selfadjoint) a ”boundary” operator \(B\) acting in an auxiliary space \(\mathcal{H}(dim\mathcal{H} = dim(A^* - i)\mathcal{H})\), and they describe the properties of the extension \(\tilde{A} = \tilde{A}_B\) in terms of the operator \(B\), i.e. essentially in terms of the boundary conditions if \(A\) is a differential operator. This approach was subsequently formalized in the concept of a ”boundary triplet’ and was developed in later papers by many authors (see for instance \(19\ 13\) and and references therein).

We remark that the methods used in these approaches are essentially different, as are the descriptions obtained with their help.

Recall that a closed densely defined operator \(A\) in \(\mathcal{H}\) is called sectorial with a half-angle \(\varphi \in (0, \pi/2]\) if

\[
Re(Af, f) \geq \cot \varphi \cdot |Im(Af, f)|, \quad f \in \text{dom } A.
\]

It is called a maximal sectorial (\(m\)-sectorial) and is put in class \(S_\varphi(\varphi)\) if additionally \(\rho(A) \neq 0\). If \(\varphi = \pi/2\) inequality \(1.1\) turns into the inequality \(Re(Af, f) \geq 0\) and the class \(S_{\pi/2}(\pi/2)\) is the class of maximal accretive operators. Denote also by \(S_{\varphi}(0)\) the class of nonnegative selfadjoint operators in \(\mathcal{H}\) and note that \(S_{\varphi}(0) = \cap_{\varphi > \pi/2} S_\varphi(\varphi)\).

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In this paper we solve among others the following two problems.

**Problem 1S.** Given a closed nonnegative symmetric operator A ≥ 0 in $\mathcal{H}$. Describe the set $\operatorname{Ext}_A(\varphi)$ of all proper and improper $S_0(\varphi)$-extensions of A with $\varphi \in [0, \pi/2]$.

**Problem 2S.** Given a dual pair $\{A_1, A_2\}$ of closed nonnegative symmetric operators in $\mathcal{H}$. Find necessary and sufficient conditions for $\{A_1, A_2\}$ to admit an extension $\tilde{A}$ ($A_1 \subset \tilde{A} \subset A_2^*$) of the class $S_0(\varphi)$ with $\varphi \in [0, \pi/2]$ and describe the set $\operatorname{Ext}_{\{A_1, A_2\}}(\varphi)$ of such extensions.

Note, that Problem 1S is solvable for any $\varphi \in [0, \pi/2]$. Indeed, it is known (see [22, 1, 34]) that any symmetric operator $A \geq 0$ admits a selfadjoint extension $\tilde{A} \geq 0$, say the Friedrichs extension $A_F$. In other words, $\operatorname{Ext}_A(0) \neq \emptyset$, hence $\operatorname{Ext}_A(\varphi) \neq \emptyset$ for any $\varphi \in (0, \pi/2]$.

A complete description of the set $\operatorname{Ext}_A(0)$ in terms of “boundary” operators have been obtained in [3] in the case of a positive definite operator A. The set $\operatorname{Extp}_A(\varphi) := \operatorname{Ext}_{\{A, A\}}(\varphi)$, $\varphi \in [0, \pi/2]$, of all proper $m$-sectorial extensions of an operator $A \geq 0$ with zero lower bound was described via boundary triplets and Weyl functions in [24] and [15]. Another description in the framework of Krein’s approach has been obtained in [3] [6].

On the other hand, even a solvability criterion of Problem 2S was unknown. We will show below that Problem 2S is not necessary solvable for any $\varphi \in [0, \pi/2]$. It may even happen that it is solvable only with $\varphi = \pi/2$.

We will also discuss the following more general problems.

**Problem 3S.** Given a sectorial operator A with a half-angle $\varphi_0 \in [0, \pi/2)$. Describe the set $\operatorname{Ext}_A(\varphi)$ of all $S_0(\varphi)$-extensions of A with $\varphi \geq \varphi_0$.

**Problem 4S.** Given a dual pair $\{A_1, A_2\}$ of sectorial operators in $\mathcal{H}$. Find necessary and sufficient conditions for $\{A_1, A_2\}$ to have an extension $\tilde{A}$ ($A_1 \subset \tilde{A} \subset A_2^*$) belonging to the class $S_0(\varphi)$ with $\varphi \geq \varphi_0$ and describe the set $\operatorname{Ext}_{\{A_1, A_2\}}(\varphi)$ of all such extensions.

By the Kato-Schecter theorem (see [23]) any sectorial operator A obeying (1.1) with $\varphi_0 \in (0, \pi/2]$ admits $m$-sectorial extension, say the Friedrichs extension $A_F$. In other words, $\operatorname{Ext}_A(\varphi_0) \neq \emptyset$, hence $\operatorname{Ext}_A(\varphi) \neq \emptyset$ for $\varphi \geq \varphi_0$. Thus, Problem 3S is solvable for any $\varphi \geq \varphi_0$.

Note, that even a criterion of solvability of Problem 4S is unknown.

In accordance with Krein’s approach we consider a linear fractional transformation $T_1 = (I - A)(I + A)^{-1}$ of a sectorial operator A. It is clear that $T_1$ is a nondensely defined contraction, $T_1(\in [\mathcal{H}_1, \mathcal{H}])$, obeying the following condition

$$\|T_1 \sin \varphi \pm i \cos \varphi \cdot I\| \leq 1 \quad \text{dom} \ T_1 = \mathcal{H}_1 := \text{ran} \ (I + A).$$

We put an operator $T$ in the class $C_0(\varphi)$ with $\varphi \in (0, \pi/2]$ if $\text{dom} \ T = \mathcal{H}$ and inequality (1.2) holds with $T$ in place of $T_1$. Note that $C_0(\pi/2)$ is the class of all contractions in $\mathcal{H}$ and denote by $C_0(0)$ the class of all selfadjoint contractions in $\mathcal{H}$.

Now we can reformulate Problems 1S—4S in the following way.

**Problem 1C.** Given a nondensely defined symmetric operator $T_1 \in [\mathcal{H}_1, \mathcal{H}]$. Describe the set of all proper and improper $C_0(\varphi)$-extensions of $T$ with $\varphi \geq \varphi_0$.

**Problem 2C.** Given a dual pair $\{T_1, T_2\}$ of nondensely defined symmetric contractions. Find necessary and sufficient conditions for $\{T_1, T_2\}$ to admit an extension $T \in C_0(\varphi)$ with $\varphi \geq \varphi_0$ and describe the set $\operatorname{Ext}_{\{T_1, T_2\}}(\varphi)$ of all such extensions.

**Problem 3C.** Given a nondensely defined operator $T_1 \in [\mathcal{H}_1, \mathcal{H}]$ obeying (1.2) with $\varphi = \varphi_0$. Describe the set $\operatorname{Ext}_{T_1}(\varphi)$ of all $C_0(\varphi)$-extensions of $T$ with $\varphi \geq \varphi_0$.

**Problem 4C.** Given a dual pair $\{T_1, T_2\}$ of nondensely defined contractions, obeying condition (1.2) with $\varphi = \varphi_0(\in [0, \pi/2])$. Find necessary and sufficient conditions for $\{T_1, T_2\}$
to have an extension \( T \in C_S(\varphi) \) with \( \varphi \geq \varphi_0 \) and describe the set \( \text{Ext}(T_1, T_2)(\varphi) \) of all such extensions.

It is convenient to regard Problems 1C and 3C as a problem on the \"completion\" of a contractive operator matrix \( T_1 = (T_{i1}) \) to form a matrix \( T = (T_{jk})_{j,k=1}^2 \) which is connected in a natural way with the problem of extending of a dual pair of contractions to operators in various classes. A description is given in terms of operator balls, \"holes\", and objects close to them.

Starting point of our investigation is a description of the set of all contractive extensions of a dual pair of contractions \( \{T_1 = (T_{i1}), T_2 = (T_{i2})\} \) or what is the same a description of all \"completions\" of a matrix

\[
T_0 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & * \end{pmatrix} = \begin{pmatrix} T_{11} & D_{T_{11}}U \\ VD_{T_{11}} & * \end{pmatrix}
\]

to form a contractive matrix \( T = (T_{ij})_{i,j=1}^2 \).

It has been shown in \( \{7, 11, 12, 13\} \) that all missing blocks \( T_{22} \) in (1.3) form an operator ball \( B(-VT_{11}U; D_{V^*}, D_U) : \)

\[
T_{22} = -VT_{11}^*U + D_{V^*}KD_U, \quad ||K|| \leq 1.
\]

Our approach to Problems 1C–4C is essentially based on the solution to the following

**Problem 5.** Given two operator balls

\[
B(C_\pm, R^\pm_1, R^\pm_2) = \{Z \in [H] : Z = C_\pm + R^\pm_1KR^\pm_2, \quad ||K|| \leq 1\}. \tag{1.5}
\]

Find a criterion for an operator \"hole\" (\"loone")

\[
L := B(C_+; R^+_1, R^+_2) \cap B(C_-; R^-_1, R^-_2)
\]

to be nonempty and obtain a parametrization of \( L \).

Problem 5 naturally arises in different areas and is of interest itself. We will show here that all Problems 1C–4C are reduced to Problem 5. Analysis of operator holes \( \{1.5\} \) corresponding to Problems 1C–4C shows that degree of difficulty of any Problem \( j \)C with \( j \in \{1, ..., 4\} \), can be characterized by means of the corresponding radii \( R^\pm_1 \) and \( R^\pm_2 \). From this point of view Problem 2C with \( T_1 = T_2(\iff)T_{11} = T_{11}^*, T_{21} = T_{12}^* \) is the simplest one. It is reduced to Problem 5 with four equal radii \( R^\pm_1 = R^\pm_2 = D_U \). This problem is always solvable and it is equivalent to a description of the set \( \text{Ext}_{D_U}(\varphi) \) of proper \( C(\varphi) \)-extensions of a symmetric contraction \( T_1 \), which has been solved in \( \{5, 6\} \) by different method.

Next, a solution to Problem 2C is equivalent to a description of missing blocks \( T_{22} \) in matrix \( \{1.3\} \) (with \( T_{11} = T_{11}^* \)) such that \( T = (T_{ij}) \in C_S(\varphi) \), that is \( T \sin \varphi \pm i \cos \varphi \cdot I \in C_S(\pi/2) \). Due to (1.4) this problem is reduced to Problem 5 with \( R^+_1 = R^-_1 = D_{V^*} \) and \( R^+_2 = R^-_2 = D_U \). It is not always solvable in general (see below).

Further, Problem 3C with \( \varphi_0 > 0 \) is reduced to Problem 5 (see \( \{30\} \) and Remark \( \{3.18\} \)) with different left radii \( R^+_1 \neq R^-_1 \) and equal right radii \( R^+_2 = R^-_2 \), while it is always solvable.

Finally, the most difficult Problem 4C with \( \varphi_0 > 0 \) is reduced to Problem 5 with different left radii \( R^+_1 \neq R^-_1 \) and different right radii \( R^+_2 \neq R^-_2 \) (see Proposition \( \{3.17\} \)).

I don’t know a criterion of solvability of Problem 5 if either \( R^+_1 \neq R^-_1 \) or \( R^+_2 \neq R^-_2 \), while a parametrization of the hole \( L \) can be easily obtained if at least one of its elements is known (see \( \{24, 30\} \)). However a solution to Problem 5 with \( R^+_1 = R^-_1 \) and \( R^+_2 = R^-_2 \) is rather simple and is contained in Lemma \( \{3.3\} \).
The paper is organized as follows.

In Section 2 we summarize some definitions and statements which are necessary in the sequel.

In Section 3 we present a solution to Problem 2C (see Theorem 3.4) based on Lemma 3.3 on a parametrization of an operator hole (1.5) with \( R^+_l = R^-_l \) and \( R^+_r = R^-_r \). It is worth to note that though Ext \( \{ T_1, T_2 \} (\varphi/2) \neq \emptyset \), it may happen that Ext \( \{ T_1, T_2 \} (\varphi) = \emptyset \) for any \( \varphi \in [\varphi_0, \pi/2) \). The solvability of Problem 2C depends on the operator

\[
Q_0 = D_U^{-1} (I - VU) D_U^{-1}.
\]

More precisely, Ext \( \{ T_1, T_2 \} (\varphi) \neq \emptyset \) if and only if \( \varphi \in [\varphi_1, \pi/2) \) where \( \varphi_1 = \arccos(\|Q_0\|^{-1}) \). In particular, Ext \( \{ T_1, T_2 \} (\varphi) = \emptyset \) for any \( \varphi \in (0, \pi/2) \) if and only if \( Q_0 \) is unbounded.

Further, in Section 3 we present a description of the set Ext \( \{ T_1 \} (\varphi) \) of all (proper and improper) extensions of a symmetric contraction \( T_1(\in [\mathcal{H}_1, \mathcal{H}]) \) (see Theorem 3.4). This result gives a complete solution to Problem 1C.

We also present here (see Propositions 3.6 and 3.8) a partial description of the set Ext \( \{ T_1, T_2 \} (\varphi) \) of extreme points of the set Ext \( \{ T_1, T_2 \} (\varphi) \). It is interesting to note that even in a finite dimensional case (\( \dim \mathcal{H} = n < \infty \)) the set \( C^*_\mathcal{H}(\varphi) \) of extreme points of the operator loone \( C_\mathcal{H}(\varphi) \) with \( \varphi \in (0, \pi/2) \) essentially differs from the set \( C^*_\mathcal{H}(\pi/2) \) of extreme points of the operator ball in \( \mathbb{C}^n \). Namely, though the set \( C^*_\mathcal{H}(\pi/2) \) consists of unitary matrices, the set \( C^*_\mathcal{H}(\varphi) \) in addition to normal matrices with “boundary spectrum” contains continuum nonnormal matrices with “nonboundary” spectrum.

Finally, in Propositions 3.17 we discuss a reduction of Problem 4C to Problem 5.

In Section 4 we investigate noncontractive extensions of a dual pair \( \{ T_1, T_2 \} \) of symmetric contractions. Namely, we consider a (not necessary contractive) extension \( T_K \) of the form (1.3), (1.4) and calculate the Schur complement of any of the operators

\[
G^\pm := (G^\pm_{_{ij}})_{i,j=1} := I - T_K T_K^* \pm i \cot \varphi (T_K - T_K^*), \quad \varphi \in (0, \pi/2).
\]

More precisely, assuming (for simplicity) that \( 0 \in \rho(G_{_{11}}) \) we prove (see Theorem 4.1) the following identities

\[
(1.6) \quad \sin^2 \varphi \cdot (G^\pm_{22} - G^\pm_{21} G_{11}^{-1} G^\pm_{12}) = D_U \cdot [I - (K^* \sin \varphi \mp iQ^*)(K \sin \varphi \pm iQ)] \cdot D_U,
\]

where \( Q \) is the closure of \( Q_0 \).

Using (1.6) we describe the classes \( C^*_\mathcal{H}(\varphi; \pm) \) of operators \( T_K \) obeying conditions \( \dim \text{ran} (G^\pm) = \varphi \pm \), where \( \pm \) stands for the ”negative” part of the operator \( G = G^* \). Some applications of this result to the boundary value problems can be found in [29]. Moreover, formula (1.6) makes it possible to give another solution to Problem 2C as well as to obtain some complements to Theorem 3.4.

In Section 5 we investigate completions of an incomplete matrix \( T'_0 = \begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix} \). Namely, in Proposition 5.2 we describe the set of some classes of noncontractive completions of \( T'_0 \). This result complements and generalizes the result of Nagy and Foias [35].

Moreover, in Proposition 5.5 we describe the sets of \( C^*_\mathcal{H}(\varphi) \)-completions of \( T'_0 \), giving an answer to Yu. L. Shmul’yan’s question. This description is given in terms of operator holes.

Some results of the paper have been announced in [28] and partially published (with proofs) in [24].
Notations. By $\mathfrak{H}$ and $\mathcal{H}$ we denote separable Hilbert spaces; $[\mathfrak{H}_1, \mathfrak{H}_2]$ stands for the set of all bounded linear operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$; $[\mathfrak{H}] := [\mathfrak{H}, \mathfrak{H}]$; $C(\mathfrak{H})$ stands for the set of closed operators in $\mathfrak{H}$. We denote by $\rho(T)$, $\sigma(T)$ and $\sigma_p(T)$ the resolvent set, the spectrum and the purely point spectrum of $T (\in C(\mathfrak{H}))$ respectively; $\sigma_p(T)$ stands for the set of eigenvalues of $T$; $\text{dom} T$ and $\text{ran} T$ stand for the domain of definition and the range of the operator $T$ respectively. As usual $E_T(\cdot)$ stands for the spectral measure (resolution of the identity) of a self-adjoint operator $T \in C(\mathfrak{H})$; $T_- := TE_T(0, \infty)$.

2. Preliminaries

2.1. Dual pairs of contractions. We recall a definition of a dual pair of bounded operators.

Definition 2.1. Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \mathfrak{H}'_1 \oplus \mathfrak{H}'_2$ be orthogonal decompositions of the Hilbert space $H$. Operators $T_1 \in [\mathfrak{H}_1, \mathfrak{H}]$, $T_2 \in [\mathfrak{H}'_1, \mathfrak{H}]$ are said to form a dual pair of bounded operators if
\begin{equation}
(T_1 f, g) = (f, T_2 g), \quad f \in \mathfrak{H}_1, \; g \in \mathfrak{H}'_1.
\end{equation}

An operator $T (\in [\mathfrak{H}])$ is termed an extension of the dual pair $\{T_1, T_2\}$ if
\begin{equation*}
T[\mathfrak{H}_1 = T_1 \quad \text{and} \quad T^* [\mathfrak{H}_2 = T_2.
\end{equation*}

The set of all extensions of a dual pair $\{T_1, T_2\}$ is denoted by $\text{Ext} \{T_1, T_2\}$.

When rewritten in the block-matrix representation with respect to the pointed out decompositions of the space $\mathfrak{H}$, the operators $T_1$ and $T_2$ form a dual pair if and only if
\begin{equation}
T_1 = \begin{pmatrix} T_{11} \\ T_{21} \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_{11}^* \\ T_{21}^* \end{pmatrix}
\end{equation}
with $T_{11} \in [\mathfrak{H}_1, \mathfrak{H}'_1]$, $T_{21} \in [\mathfrak{H}_1, \mathfrak{H}'_2]$, $T_{21}^* \in [\mathfrak{H}'_1, \mathfrak{H}_2]$. Setting $T_{12} = (T_{21}^*)^*$, an extension $T$ of the DP $\{T_1, T_2\}$ can be rewritten in the form
\begin{equation}
T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad \text{with} \quad T_{22} \in [\mathfrak{H}_2, \mathfrak{H}'_2].
\end{equation}

In this case the problem of description of a certain class $X$ of extensions of the dual pair $\{T_1, T_2\}$ is equivalent to the problem of completing an incomplete block-matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & * \end{pmatrix}$ with respect to the matrix $T$ of the form (2.3) and such that $T \in X$.

In what follows we consider contractive extensions of a dual pair of contractions $\{T_1, T_2\}$. The union of all such extensions will be denoted by $\text{Ext} \{T_1, T_2\}(\pi/2)$.

The set $\text{Ext} \{T_1, T_2\}(\pi/2)$ turns out to be an operator ball in the sense of the following definition.

Definition 2.2. The totality of the operators $Z \in [\mathfrak{H}]$ of the form
\begin{equation}
Z = C_0 + R_l KR_r, \quad \|K\| \leq 1
\end{equation}
is referred to as an operator ball $B(C_0; R_l, R_r)$.

Here $C_0$ is called the center of the ball, and $R_l = R_l^* \geq 0$ and $R_r = R_r^* \geq 0$ are called left and right radii respectively.

We will use the following simple and known result.
Lemma 2.3. Let \( Q_j \in [\mathcal{H}], j \in \{1, 2, 3\} \), \( Q_3 = Q_3^\ast \), \( Q_1 > 0 \) and \( 0 \in \rho(Q_1) \). Then the inequality
\[
Z^*Q_1Z + Z^*Q_2 + Q_2Z + Q_3 \leq 0
\]
has a solution if and only if
\[
Q_3^\ast Q_1^{-1}Q_2 - Q_3 \geq 0.
\]
Under this condition the set of the solutions of the inequality (2.5) makes up an operator ball \( B(C_0; R_l, R_r) \) of the form (2.4) with
\[
C_0 = -Q_1^{-1}Q_2, \quad R_l = Q_1^{-1/2} \quad \text{and} \quad R_r = (Q_2^*Q_1^{-1}Q_2 - Q_3)^{1/2}.
\]

2.2. The operators \( T_1 \) and \( T_2 \) of the form (2.2) are contractive if and only if
\[
T_{11}T_{11} + T_{21}^*T_{21} \leq I \iff T_{21}^*T_{21} \leq D_{T_{11}} := I - T_{11}^*T_{11},
\]
\[
T_{11}T_{11}^* + T_{12}^*T_{12} \leq I \iff T_{12}^*T_{12} \leq D_{T_{11}^*} := I - T_{11}^*T_{11}.
\]
It is known (and it is obvious) that these relations are equivalent to the following ones
\[
T_{21} = VD_{T_{11}}, \quad T_{12} = D_{T_{11}^*}U
\]
with contractions \( V \) and \( U \) (\( V \in [\mathcal{H}_1, \mathcal{H}_2], U \in [\mathcal{H}_2, \mathcal{H}_1'] \)), which are uniquely determined provided that \( \ker V \supset \ker D_{T_{11}} \) and \( \ker U^* \supset \ker D_{T_{11}^*} \), that is \( V^* = D_{T_{11}^*}^*T_{21}^* \) and \( U = D_{T_{11}^*}^{-1}T_{12} \).

A complete description of the set \( \text{Ext}\{T_1, T_2\}(\pi/2) \) is contained in the following theorem. We will essentially use it in the sequel.

Theorem 2.4. Let \( \{T_1, T_2\} \) be a dual pair of contractions,
\[
T_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ VD_{T_{11}} \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_{11}^* & T_{12}^* \\ U^*D_{T_{11}^*} \end{pmatrix},
\]
\[
\mathcal{H}_1 = \overline{\text{ran}}(U) \quad \text{and} \quad \mathcal{H}_2 = \overline{\text{ran}}(V^*).
\]
Then the formula
\[
T := T_K = \begin{pmatrix} T_{11} & D_{T_{11}^*}U \\ VD_{T_{11}} & T_{22} \end{pmatrix}, \quad T_{22} = -VT_{11}^*U + D_{V^*}KD_U,
\]
establishes a bijective correspondence between all contractive extensions \( T := T_K = (T_{ij}) \in \text{Ext}\{T_1, T_2\}(\pi/2) \) and all contractions \( K \in [\mathcal{H}_1, \mathcal{H}_2] \).

Thus, the set \( \text{Ext}\{T_1, T_2\}(\pi/2) \) forms an operator ball \( B(C_0; R_l, R_r) \) with the center \( C_0 = -VT_{11}^*U \) and left and right radii \( R_l = D_{V^*} \) and \( R_r = D_U \) respectively.

Remark 2.5. Let us make some historical remarks concerning Theorem 2.4. The case \( T_{11} = T_{11}^*, T_{12} = T_{22}^* \) was considered by M.G. Krein in connection with selfadjoint extensions of positive unbounded operators while his description of the class \( \text{Ext}\{T_1, T_2\}(0) \) differs from that followed from Theorem 2.4. The existence of contractive extensions of a DPC \( \{T_1, T_2\} \) (that is the fact \( \text{Ext}\{T_1, T_2\}(\pi/2) \neq \emptyset \)) was first established by B.S. Nagy and C. Foias, p.190, by means of a corresponding generalization of the Krein’s method. Note also that the claim \( \text{Ext}\{T_1, T_2\}(\pi/2) \neq \emptyset \) is implicitly contained in [32] [33]. Another proof of the existence part of Theorem 2.4 has also been obtained by S. Parrot.

The complete description of the set \( \text{Ext}\{T_1, T_2\}(\pi/2) \), i.e., Theorem 2.4 was obtained in [7, 11, 12, 38]. In the special case \( T_{21} = 0 \) Theorem 2.4 has been obtained by B.Sz.-Nagy and C.Foias much earlier. Several other proofs of Theorem 2.4 based on different ideas,
can also be found in [17] [24] [30]. In particular the proof of C. Foias and A.E. Frazho [17] is based on Redheffer’s products, the author’s proof in [30] is based on Lemma 2.3.

2.3. Extreme points of the unit ball.
Recall the following

Definition 2.6. Let $G$ be a closed convex set in a Banach space $X$. A point $f \in G$ is called an extreme point of $G$ if it does not admit a representation $f = f_1 + (1-t)f_2$ with $f_1, f_2 \in G$, $f_1 \neq f_2$ and $t \in (0, 1)$.

Let $H_1, H_2$ be Hilbert spaces. An operator $T(\in [H_1, H_2])$ is called a partial isometry if $T^*T = P$, where $P$ is an orthoprojection in $H_1$. An operator $T(\in [H_1, H_2])$ is a maximal partial isometry if either $T$ or $T^*$ is an isometry from $H_1$ to $H_2$, that is, if either $T^*T = I_{H_1}$ or $TT^* = I_{H_2}$.

In the sequel we need the following result which is well known in the case $H_1 = H_2$ (see [20]). The general case can be easily derived from the known one.

Proposition 2.7. The set $\mathcal{R}_C$ of extreme points of the unit ball $\mathcal{R}_C := \{T : T \in [H_1, H_2], \|T\| \leq 1\}$ in $[H_1, H_2]$ consists of maximal partial isometries from $H_1$ to $H_2$.

2.4. Sectorial operators and $C(\varphi)$-contractions.

Definition 2.8. [23]. A closed linear operator $A$ in a Hilbert space $H$ is called sectorial with vertex zero and half-angle $\varphi \in (0; \pi/2)$ if its numerical range is contained in sector $G_\varphi = \{z \in \mathbb{C} : |\arg z| \leq \varphi < \pi/2\}$, that is

$$\cot \varphi \cdot |\text{Im}(Af)| \leq \text{Re}(Af), \quad f \in \text{dom} A.$$

(2.13)

If in addition $A$ has no sectorial extensions ($\iff \rho(A) \neq \emptyset$) it is called a $m$-sectorial operator and is put in the class $S_\varphi(H)$.

Further by $S_\varphi(\pi/2)$ we denote the class of $m$-accretive operators in $H$, i.e. $A \in S_\varphi(\pi/2)$ if $\text{Re}(Af, f) \geq 0$ for all $f \in \text{dom} A$ and $\rho(A) \neq \emptyset$.

Finally, $S_\varphi(0)$ stands for the set of all nonnegative selfadjoint linear operators in $H$.

Following [27] an operator $A$ with $\text{dom}(A) = H$ is called regularly dissipative if $-A$ is $m$-sectorial.

Let $A$ be a closed sectorial closed operator in $H$. In the framework of the approach accepted in this paper with each $A$ it is connected a linear transformation

$$T_1 = X(A) := -I + 2(I + A)^{-1},$$

(2.14)

being a contraction with a nondense in $H$ domain of the definition $\mathcal{R}_1 := \text{dom}(T_1) = (I + A)\text{dom} A$. In so doing condition (2.13) is transformed to the following one

$$2\cot \varphi \cdot |\text{Im}(T_1f, f)| \leq \|(I - T_1^*T_1)f, f\| = \|D_{T_1}f\|^2, \quad f \in \text{dom} T_1.$$

(2.15)

The following definition naturally arises from what has been said.

Definition 2.9. We put an operator $T \in [H]$ in the class $C_\varphi(H)$ if

$$\|T \sin \varphi \pm i \cos \varphi \cdot I\| \leq 1, \quad (\varphi \in (0, \pi/2])$$

(2.16)

and in the class $C_\varphi(0)$ if $T = T^*$ and $\|T\| \leq 1$. 


It is clear that if \( T \in C_S(\varphi) \), then \( \sigma(T) \subset L_\varphi \) where
\[
L_\varphi := \{ z \in \mathbb{D} : |z \sin \varphi \pm i \cos \varphi| \leq 1 \}.
\]

**Lemma 2.10.** Let \( T \in [\mathfrak{H}] \), \( \varphi \in (0, \pi/2] \). Then the following properties of the operator \( T \) are equivalent:

1. \( T \in C_S(\varphi) \);
2. \( 2 \cot \varphi |\text{Im}(Tf, f)| \leq \|D_Tf\|^2 \), \( f \in \mathfrak{H} \);
3. \( 2 \cot \varphi |(Tf, g)| \leq \|D_Tf\| \cdot \|D_Tg\| \), \( f, g \in \mathfrak{H} \).

It follows from Lemma 2.10 that
\[
C_S(0) = \cap_{\varphi \in (0, \pi/2]} C_S(\varphi).
\]

**Definition 2.11.** Let \( A \) be a closed sectorial operator in \( \mathfrak{H} \) with vertex zero and half-angle \( \varphi_0 \in [0; \pi/2) \) and let \( T_1 \) be a contraction obeying (2.15) with \( \varphi = \varphi_0 \). Denote by \( \text{Ext}_A(\varphi) \) the class of all \( m \)-sectorial extensions \( \tilde{A}(\in C(\mathfrak{H})) \) of \( A \) with vertex zero and the half-angle \( \varphi \in [\varphi_0; \pi/2) \) and by \( \text{Ext}_{T_1}(\varphi) \) the class of all extensions \( T \in [\mathfrak{H}] \) of \( T_1 \) obeying (2.15) with \( \varphi \in [\varphi_0; \pi/2) \).

**Lemma 2.12.** Let \( A \) be a densely defined sectorial operator in \( \mathfrak{H} \) with a semiangle \( \varphi \). The linear fractional transformation (2.14) establishes the bijective correspondence
\[
\tilde{A} \rightarrow T = X(\tilde{A}) = -I + 2(I + \tilde{A})^{-1}, \quad T \rightarrow \tilde{A} = X^{-1}(T) = -I + 2(I + T)^{-1},
\]

between the set \( \text{Ext}_A(\varphi) \) and the subset \( \text{Ext}_{T_1}(\varphi) = \{ T : T \in \text{Ext}_{T_1}(\varphi), -1 \not\in \rho(T) \} \) of \( \text{Ext}_{T_1}(\varphi) \).

If \( A \) is a nondensely defined sectorial operator, then the set \( \text{Ext}_A(\varphi) \) contains \( m \)-sectorial linear relations too. Lemma 2.12 remains valid in this case if we replace \( \text{Ext}_{T_1}(\varphi) \) by \( \text{Ext}_{T_1}(\varphi) \).

### 3. Some classes of contractive extensions of dual pairs of Hermitian contractions

#### 3.1. A parametrization of the operator loone in the special case.

In this subsection we present an elementary result (see Lemma 3.3) on parametrization of an operator hole \( L = B_1 \cap B_2 \) in the case of operator balls \( B_1 = B(Z_1; R_l, R_r) \) and \( B_2 = B(Z_2; R_l, R_r) \) with equal left radii and right radii. This lemma gives a partial solution to Problem 5 mentioned in the Introduction.

We start with the following simple lemma.

**Lemma 3.1.** Let \( R_1 = R_1^* \geq 0 \), \( R_2 = R_2^* \geq 0 \), \( R_i \in [\mathfrak{H}] \), \( j \in \{1, 2\} \), and let \( A \in [\mathfrak{H}] \). Then the following conditions are equivalent:

\[
\begin{align*}
(i) \quad & A = R_2BR_1, \quad B \in C_S(\pi/2); \\
(ii) \quad & 2 |(Af, g)| \leq (R_1^2 f, f) + (R_2^2 g, g), \quad f, g \in \mathfrak{H}.
\end{align*}
\]

**Proof.** Inequality (3.2) is equivalent to the inequality
\[
|(Af, g)| \leq \|R_1 f\| \cdot \|R_2 g\|, \quad f, g \in \mathfrak{H}.
\]
Hence $\ker A \supset \ker R_1$ and $\ker A^* \supset \ker R_2$. Letting $R_1 f := h_1$, $R_2 g := h_2$ one rewrites (3.3) in the form

$$|(AR_1^{-1}h_1, R_2^{-1}h_2)| \leq \|h_1\| \cdot \|h_2\|, \quad h_1 \in \ran R_1, \; h_2 \in \ran R_2.$$ 

It follows that the bilinear form $(AR_1^{-1}h_1, R_2^{-1}h_2)$ may be continuedly extended to a bounded bilinear form $t(h_1, h_2)$ on $\mathcal{H}_1 \times \mathcal{H}_2$, with $\mathcal{H}_j = \ran R_j$, $j \in \{1, 2\}$. Hence $t(h_1, h_2) = B(h_1, h_2)$, where $B \in [\mathcal{H}_1, \mathcal{H}_2]$ and $\|B\| \leq 1$. Since $(AR_1^{-1}h_1, R_2^{-1}h_2) = (Bh_1, h_2)$, $h_j \in \mathcal{H}_j$, $j \in \{1, 2\}$, then $AR_1^{-1}h_1 \in \dom (R^{-1}_2)$ and $R_2^{-1}AR_1^{-1}h_1 = Bh_1$, $h_1 \in \dom (R^{-1}_1)$. Thus, $B$ is the closure of the operator $R_2^{-1}AR_1^{-1}$. Hence $A = R_2BKR_1$ and the implication (ii)$\implies$(i) is proved.

The converse implication (i)$\implies$(ii) is clear.

The following statement easily follows from Lemma 3.1.

**Lemma 3.2.** Let $R_j = R_j^* \geq 0$, $R_j \in [\mathcal{S}]$, $j \in \{1, 2\}$ and $A \in [\mathcal{S}]$. Suppose additionally that $\ker R_1 = \ker R_2$ and $\mathcal{H} := \mathcal{S} \ominus \ker R_1$. Then $A$ admits a representation $A = R_2KR_1$ with $K \in C_{\mathcal{H}}(\phi)$, $\phi \in [0, \pi/2]$, if and only if

$$2Re((\sin \phi \cdot A + i \cos \phi \cdot R_2 \cdot R_1)f, g) \leq (R_2^2f, f) + (R_2^2g, g), \quad f, g \in \mathcal{H}. \tag{3.4}$$

**Proof.** Necessity is immediately implied by Lemma 3.1.

Sufficiency. Suppose that (3.4) is satisfied. Assume that $\phi > 0$, since the case $\phi = 0$ is trivial. Then (3.4) yields

$$2|(Af, g)| \leq (\sin \phi)^{-1} \cdot (\|R_1f\|^2 + \|R_2g\|^2), \quad f, g \in \mathcal{S}.$$ 

By Lemma 3.1 $A$ admits a representation $A = R_2KR_1$ with $K \in [\mathcal{H}], \|K\| \leq 1/\sin \phi$. Substituting this expression for $A$ in (3.4), we get the required. \qed

The following lemma, being a partial solution to Problem 5, gives a parametrization of the operator loone $L := B_1 \cap B_2$ in the case of operator balls $B_1$ and $B_2$ in $[\mathcal{S}]$ with equal left and right radii and, in particular, it gives a criterion of nonemptyness of the loone $L$.

**Lemma 3.3.** Let $B_1 = B(C_1; R_l, R_r)$ and $B_2 = B(C_2; R_l, R_r)$ be two operator balls in $[\mathcal{S}]$ with equal left and right radii and $\mathcal{H}_1 := \mathcal{S} \ominus \ker R_r$, $\mathcal{H}_2 := \mathcal{S} \ominus \ker R_l$. Then

(i) their intersection $L := B_1 \cap B_2$ is nonempty if and only if one of the following (equivalent) conditions is satisfied:

$$\begin{align*}
(3.5) \quad & (a) \quad |(C_1 - C_2)f, g)| \leq 2^{-1}[(R_l^2f, f) + (R_r^2g, g)]; \\
(3.6) \quad & (b) \quad |(C_1 - C_2)f, g)| \leq \|R_l f\| \cdot \|R_r g\|, \quad f, g \in \mathcal{S}; \\
(3.7) \quad & (c) \quad C_1 - C_2 = 2R_lQR_r \quad \text{with} \quad Q \in C(\pi/2), \; Q \in [\mathcal{H}_1, \mathcal{H}_2];
\end{align*}$$

(c') the operator $Q_0 := 2^{-1}R_l^{-1}(C_1 - C_2)^2R_l^{-1}$ is bounded and its closure $Q := Q_0(\in [\mathcal{H}_1, \mathcal{H}_2])$ is a contraction.

(ii) If any of the conditions (a), (b), (c) is satisfied, then the operator loone $L$ admits the following parameter representation

$$\begin{align*}
(3.8) \quad & T \in L = B_1 \cap B_2 \iff T = T_K := 2^{-1}(C_1 + C_2) + R_lKR_r \quad \text{with} \quad K \pm Q \in C(\pi/2). \\
(iii) \quad & L \text{ consists of one element, } L = \{2^{-1}(C_1 + C_2)\}, \text{ if and only if at least one of the following three conditions holds}
\end{align*}$$

$$\begin{align*}
(1) \quad & R_l = 0; \quad (2) \quad R_r = 0; \quad (3) \quad Q \text{ is a maximal partial isometry.}
\end{align*}$$
Proof. (i), (ii). Equivalence of the conditions (a)-(c) is implied by Lemma 3.1. It remains to show, for example, that equality (3.7) is equivalent to the condition \( L \neq \emptyset \). Let \( T \in L = B_1 \cap B_2 \), that is

\[
T = C_1 + R_lK_1R_r = C_2 + R_lK_2R_r.
\]

Then setting

\[
K := 2^{-1}(K_1 + K_2) (\in C(\pi/2)) \quad \text{and} \quad Q := 2^{-1}(K_2 - K_1) (\in C(\pi/2)),
\]

we deduce

\[
T = 2^{-1}(C_1 + C_2) + R_lKR_r, \quad \text{where} \quad K \pm Q \in C(\pi/2).
\]

Thus, conditions (3.7) and (3.8) are satisfied.

Conversely, suppose that (3.7) is valid. Then setting \( K_1 := -Q, \ K_2 := +Q \) and \( T := C_1 + R_lK_1R_r \) we get

\[
T = C_1 + R_lK_1R_r = C_2 + R_lK_2R_r.
\]

Hence \( T \in L = B_1 \cap B_2 \).

(iii) Let \( T \in L \) and \( T \neq 2^{-1}(C_1 + C_2) \). Then according to (3.8) \( R_l \neq 0, \ R_r \neq 0 \) and there exists \( K \in C(\pi/2) \setminus \{0\} \) such that \( K_\pm := Q \pm K \in C(\pi/2) \). Hence \( Q = (K_+ + K_-)/2 \neq K_+ \neq K_- \) is not an extreme point of the unit ball in \([\mathcal{H}_1, \mathcal{H}_2] \). By Proposition 2.7 \( Q \) is not a maximal isometry.

Conversely, suppose that \( R_l \neq 0, \ R_r \neq 0 \) and \( Q \) is not a maximal isometry. By Proposition 2.7 \( Q = (K_+ + K_-)/2 \) where \( K_\pm \in [\mathcal{H}_1, \mathcal{H}_2] \), \( K_\pm \in C(\pi/2) \) and \( K_+ \neq K_- \). Setting \( K_j := (K_+ - K_-)/2 \) and \( K_j := -K_1 \) we easily get that \( K_j \pm Q \in C(\pi/2) \), \( j \in \{1,2\} \). Hence by (3.8) \( T_{K_j} := 2^{-1}(C_1 + C_2) + R_lK_jR_r \in L, \ j \in \{1,2\} \). Since \( T_{K_j} \neq 2^{-1}(C_1 + C_2) \), we get the required. □

### 3.2. A description of the class of \( C_{\mathcal{H}}(\varphi) \)-extensions of a dual pair of symmetric contractions.

In this subsection we present a solution to the Problem 2C with \( \varphi_0 = 0 \).

More precisely, let \( \{T_1, T_2\} \) be a dual pair of symmetric contractions in \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_1' \oplus \mathcal{H}_2' \). Due to (2.11) the operators \( T_1 \) and \( T_2 \) admit the following block-matrix representations

\[
(3.9) \quad T_1 = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & VD_{T_{11}} \end{pmatrix}, \quad T_2 = \begin{pmatrix} T_{11} & T_{12}' \\ T_{21}' & U^*D_{T_{11}}' \end{pmatrix},
\]

since \( T_{11} = T_{11}' \). In particular, in this case \( \mathcal{H}_1 = \mathcal{H}_1' \) and \( \mathcal{H}_2 = \mathcal{H}_2' \).

Let

\[
(3.10) \quad \text{Ext}_{\{T_1, T_2\}}(\varphi) := \text{Ext}_{\{T_1, T_2\} \cap C_{\mathcal{H}}(\varphi)} \quad \varphi \in [0, \pi/2],
\]

stand for the set of \( C_{\mathcal{H}}(\varphi) \)-extensions of the dual pair of symmetric contractions \( \{T_1, T_2\} \).

By Theorem 2.4 \( \text{Ext}_{\{T_1, T_2\} \cap \mathcal{H}}(\varphi/2) \neq \emptyset \), that is there always exists an extension \( T \in C_{\mathcal{H}}(\pi/2) \) of the dual pair \( \{T_1, T_2\} \). It turns out that it is not the case for the classes \( C_{\mathcal{H}}(\varphi) \) and \( C_{\mathcal{H}}(\varphi'; \nu) \) with \( \varphi < \pi/2 \). The solvability of both problems depends on the properties of the operator

\[
(3.11) \quad Q_0 := D_{T_{11}}^*(I-VU)D_{T_{11}}^{-1}.
\]

Moreover, we show that if the operator \( Q_0 \) is unbounded the Problem 2C has a solution only with \( \varphi_1 = \pi/2 \), that is, \( \text{Ext}_{\{T_1, T_2\}}(\varphi) \neq \emptyset \) iff \( \varphi = \varphi_1 = \pi/2 \).
Theorem 3.4. Let \( \{T_1, T_2\} \) be a dual pair of symmetric contractions in \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) of the form (3.3), \( \varphi \in (0, \pi/2] \) and let \( \mathcal{H}_1 := \overline{\text{ran}}(D_U), \mathcal{H}_2 := \overline{\text{ran}}(D_V) \). Then:

(i) the set \( \text{Ext}_{\{T_1, T_2\}}(\varphi) \) is nonempty if and only if \( \varphi \in [\varphi_1, \pi/2] \), where

\[
\varphi_1 := \arccos(\|D_V^{-1}(I - VU)D_U^{-1}\|^{-1}) \tag{3.12}
\]

(ii) for any \( \varphi \in [\varphi_1, \pi/2] \) the following equivalence holds:

\[
T_K = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \text{Ext}_{\{T_1, T_2\}}(\varphi) \iff T_{22} = -VT_{11}U + D_VKDU, \quad K \cdot \sin \varphi \pm iQ \cos \varphi \in C(\pi/2),
\]

where \( Q := \overline{Q_0} \in \mathcal{H}_1, \mathcal{H}_2) \) is the closure of the operator \( Q_0 \) of the form (3.11) and \( K \in \mathcal{H}_1, \mathcal{H}_2) \);

(iii) the set \( \text{Ext}_{\{T_1, T_2\}}(\varphi) \) consists of one element if at least one of the following three conditions is satisfied:

(a) \( D_V = 0 \);
(b) \( D_U = 0 \);
(c) \( Q \cos \varphi \) is a maximal partial isometry.

Proof. (i) Let \( T = (T_{ij})_{i,j=1}^2 \) be a contractive extension of the dual pair \( \{T_1, T_2\} \). Suppose for the begining that \( \varphi > 0 \). In this case the inclusion \( T \in C_\Sigma(\varphi) \) means that

\[
T_\pm := T \sin \varphi \pm i \cos \varphi \cdot I \in C_\Sigma(\pi/2),
\]

that is,

\[
T_\pm := \begin{pmatrix} B_{11} \pm B_{12} & B_{21} \pm B_{22} \\ B_{21} \pm B_{22} & B_{21} \pm B_{22} \end{pmatrix} = \begin{pmatrix} \sin \varphi \cdot T_{11} \pm i \cos \varphi \cdot I & T_{21} \sin \varphi \\ T_{21} \sin \varphi & \sin \varphi \cdot T_{22} \pm i \cos \varphi \cdot I \end{pmatrix} \in C_\Sigma(\pi/2).
\]

It is easily seen that

\[
D_{B_{11}}^2 = I - (\sin \varphi \cdot T_{11} - i \cos \varphi \cdot I)(\sin \varphi \cdot T_{11} + i \cos \varphi \cdot I) = \sin^2 \varphi \cdot D_{T_{11}}^2,
\]

\[
D_{B_{11}}^2 = I - (\sin \varphi \cdot T_{11} + i \cos \varphi \cdot I)(\sin \varphi \cdot T_{11} - i \cos \varphi \cdot I) = \sin^2 \varphi \cdot D_{T_{11}}^2.
\]

Therefore \( D_{B_{11}} = D_{B_{11}} = \sin \varphi \cdot D_{T_{11}} \) and consequently

\[
B_{12} = \sin \varphi T_{12} = \sin \varphi \cdot D_{T_{11}}U, \quad B_{21} = \sin \varphi T_{21} = \sin \varphi \cdot VD_{T_{11}} = VD_{B_{11}}.
\]

Thus the contractions \( T_\pm \) have the form

\[
T_\pm = \begin{pmatrix} B_{11} \pm B_{12} & B_{21} \pm B_{22} \\ B_{21} \pm B_{22} & B_{21} \pm B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} \pm B_{12} & D_{B_{11}}U \\ VD_{B_{11}} & B_{22} \pm B_{22} \end{pmatrix}
\]

with \( B_{11} = T_{11} \sin \varphi \pm i \cos \varphi \cdot I \) and \( B_{22} = T_{22} \sin \varphi \pm i \cos \varphi \cdot I \). According to Theorem 2.4 the equivalences

\[
T_\pm \in C(\pi/2) \iff B_{22} = T_{22} \sin \varphi \pm i \cos \varphi \cdot I = C_\pm + D_V \cdot K_\pm DU
\]

hold true with some contractions \( K_\pm \) and operators \( C_\pm \) defined by

\[
C_\pm := -V(B_{11})^*U = -V(\sin \varphi \cdot T_{11}^* \mp i \cos \varphi)U.
\]

Setting

\[
C_1 := C_+ - i \cos \varphi \cdot I, \quad C_2 := C_- + i \cos \varphi \cdot I,
\]
we rewrite equivalences (3.16) in the form
\[(3.19) \quad T \in C_\mathcal{Y}(\varphi) \iff T_{22}\sin \varphi = C_1 + D_V \cdot K_+ D_U = C_2 + D_V \cdot K_- D_U, \quad K_\pm \in C(\pi/2).\]

Thus, \(T \in C_\mathcal{Y}(\varphi)\) if and only if the operator \(\sin \varphi \cdot T_{22}\) belongs to the intersection of the operator balls \(B_1 := B(C_1; D_V \cdot ; D_U)\) and \(B_2 := B(C_2; D_V \cdot , D_U),\) that is, \(\sin \varphi \cdot T_{22} \subseteq L := B_1 \cap B_2.\) By Lemma 3.3 with account of (3.17) and (3.18) the condition \(L = B_1 \cap B_2 \neq \emptyset\) amounts to saying that the operator \(C_0 := 2^{-1}(C_1 - C_2) = -i \cos \varphi (I - VU)\)

admits the representation \(C_0 = D_V \cdot K D_U\) with \(K \in C(\pi/2)\) or, what is the same, the operator \(Q_0 \cdot \cos \varphi\) is contractive where the operator \(Q_0\) is of the form (3.11). This proves the first assertion.

(ii) Suppose that condition (3.12) is satisfied. To obtain a parametrization of the hole \(L\) we note that by (3.17) and (3.18)

\[
2^{-1}(C_1 + C_2) = -\sin \varphi \cdot VT_{11}^* U.
\]

Now Lemma 3.3 yields the equivalence
\[(3.21) \quad T \in L(= B_1 \cap B_2) \iff \sin \varphi \cdot T_{22} = -\sin \varphi \cdot VT_{11}^* U + D_V \cdot \tilde{K} D_U,\]

where \(\tilde{K} \pm iQ \cos \varphi \in C(\pi/2).\) Setting in (3.21) \(K := \tilde{K} / \sin \varphi\) we arrive at (3.13).

It remains to consider the case \(T \in \text{Ext}_{\{T_1, T_2\}}(0).\) This inclusion means that \(T\) is a self-adjoint contraction in \(\mathcal{Y},\) that is \(T_{21} = T_{12}^*\) and \(U = V^*.\) Hence \(Q_0 = I\) and \(\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}.\) Therefore equivalence (3.13) with \(\varphi \in (0, \pi/2)\) takes the form
\[(3.22) \quad T_K \in C_\mathcal{Y}(\varphi) \iff K \cdot \sin \varphi \pm i \cos \varphi \cdot I \in C_\mathcal{H}(\pi/2).\]

The desired equivalence
\[(3.23) \quad T_K \in C_\mathcal{Y}(0) \iff K = K^* \in C_\mathcal{H}(\pi/2)\]

is implied now by (3.18).

(iii) This assertion is immediately implied by the statement (iii) of Lemma 3.3 \(\square\)

Remark 3.5. Comparison of condition (3.12) with the obvious criterion \(U = V^*\) for the existence of \(T = T^* \in \text{Ext}_{\{T_1, T_2\}}(\pi/2)\) yields a curious fact:
\[(3.24) \quad U, \ V \in C_\mathcal{Y}(\pi/2), \quad |((I - VU)f, g)| \leq \|D_U f\| \cdot \|D_V g\| \iff U = V^*.\]

I don’t know the direct proof if this equivalence.

### 3.3. Extreme points of the set Ext\(_{\{T_1, T_2\}}(\varphi)\)

Denote by \(\text{Ext}^e_{\{T_1, T_2\}}(\varphi)\) the set of extreme points of the closed convex set \(\text{Ext}_{\{T_1, T_2\}}(\varphi)\). Theorem 3.4 makes it possible to describe a part of the set \(\text{Ext}^e_{\{T_1, T_2\}}(\varphi)\). To this end for any operator \(Q(\in \mathcal{H}_1, \mathcal{H}_2)\) we introduce the operator loones
\[(3.25) \quad L(Q; \varphi) := \{K \in [\mathcal{H}_1, \mathcal{H}_2]: \ K \sin \varphi \pm iQ \cos \varphi \in C(\pi/2)\}, \quad \varphi \in (0, \pi/2),\]

and denote by \(L^e(Q; \varphi)\) the set of its extreme points.

**Proposition 3.6.** Let \(Q \in [\mathcal{H}_1, \mathcal{H}_2], \ \varphi_1 := \arccos(\|Q\|^{-1}) > 0\) and \(\varphi \in [\varphi_1, \pi/2].\) Then

(i) the following equivalence holds
\[(3.26) \quad K \in L(Q; \varphi) \iff \sin 2\varphi \cdot (K^* Q)_1 = D_{K,Q} C D_{K,Q}, \quad C = C^* \in C_\mathcal{H}_1(0),\]
where $(K^*Q)_I := (2i)^{-1}(K^*Q - Q^*K)$ and
\begin{equation}
D_{K,Q} := (I - K^*K \sin^2 \varphi - Q^*Q \cos^2 \varphi)^{1/2} \geq 0.
\end{equation}

(ii) If additionally $\text{ran} D_{K,Q}$ is closed, that is $\text{ran} D_{K,Q} = \overline{\text{ran}} D_{K,Q}$, then the following implication holds
\begin{equation}
\sigma(C) \subset \{\pm 1\} \implies K \in L^\circ(Q; \varphi).
\end{equation}

Proof. (i) By definition $K \in L(Q; \varphi)$ iff
\[(I - K^* \sin \varphi \mp iQ^* \cos \varphi)(K \sin \varphi \pm iQ \cos \varphi) \geq 0.
\]
With account of definition (3.27) this inequality may be rewritten as
\[
\pm \sin 2\varphi (K^*Q)_I \leq D_{K,Q}^2.
\]
By Lemma 3.1, this inequality is equivalent to representation (3.26) with some selfadjoint contraction $C$.

(ii) The proof of this statement is similar to that of Proposition 3.18 from [30]. Suppose the contrary, that is $K \notin L^\circ(Q; \varphi)$. Then $2K = K_1 + K_2$ where $K_j \in L(Q; \varphi)$, $j \in \{1, 2\}$ and $K_1 \neq K$. For any $f \in \ker D_{K,Q}$ we have
\[
4\|f\|^2 = \sin^2 \varphi \cdot \|2K\|^2 + 4 \cos^2 \varphi \cdot \|Qf\|^2 = \sin^2 \varphi \cdot \|K_1f + K_2f\|^2 + 4 \cos^2 \varphi \cdot \|Qf\|^2
\]
\[
\leq 2\sin^2 \varphi \cdot \|K_1f\|^2 + 4 \cos^2 \varphi \cdot \|Qf\|^2 + 2\sin^2 \varphi \cdot 2\cos^2 \varphi \cdot \|Qf\|^2 \leq 4\|f\|^2.
\]
Hence
\[
\sin^2 \varphi \cdot \|K_jf\|^2 + \cos^2 \varphi \cdot \|Qf\|^2 = \|f\|^2, \quad j \in \{1, 2\}.
\]
Thus, $\|K_1f\| = \|K_2f\| = \|Kf\|$ and $\|K_1f + K_2f\| = \|K_1f\| + \|K_2f\|$. In view of strict convexity of the unit ball in $\mathfrak{H}$ we get
\begin{equation}
K_1f = K_2f = Kf, \quad f \in \ker D_{K,Q}.
\end{equation}

Further, setting $K_{\pm} := K \sin \varphi \pm iQ \cos \varphi$ and using representation (3.26) we obtain
\begin{equation}
D_{K_{\pm}}^2 = D_{K,Q}^2 \pm \sin 2\varphi (K^*Q)_I = D_{K,Q}(I \pm C)D_{K,Q}.
\end{equation}
Suppose that $f \in (\ker D_{K,Q})^\perp$ and $D_{K,Q}f \in \ker (I + C)$. Then, it follows from (3.30) that $D_{K_{\pm}}^2 f = 0$, that is $\|K_{\pm}f\| = \|f\|$. Setting
\[
K_{J_{\pm}} := K_j \sin \varphi + iQ \cos \varphi (\in C(\pi/2)), \quad j \in \{1, 2\},
\]
and noting that $2K_{\pm} = K_1 + K_2 \mp C$ we easily get
\[
2\|f\| = 2\|K_{\pm}f\| = \|(K_{\pm} + K_{\mp})f\| \leq 2\|f\|.
\]
Hence $\|K_{\pm}f\| = \|K_{\mp}f\| = \|K_{\pm}f\| = \|f\|$ and
\[
\|K_{\pm}f + K_{\mp}f\| = \|K_{\pm}f\| + \|K_{\pm}f\|.
\]
In view of strict convexity of the unit ball in $\mathfrak{H}$ we get $K_{\pm}f = K_{\mp}f = K_{\pm}f$, that is $K_1f = K_2f = Kf$ for any $f$ obeying $D_{K,Q}f \in \ker (I + C)$. Similarly we obtain that $K_1f = K_2f = Kf$ for any $f$ such that $D_{K,Q}f \in \ker (I - Q)$. Taking into account the hypothesis of proposition we get
\begin{equation}
K_1f = K_2f = Kf, \quad f \in \text{ran} D_{K,Q} = \ker (I + C) \oplus \ker (I - C).
\end{equation}
Combining (3.28) with (3.31) we get $K = K_1 = K_2$. This contradicts the assumption that $K_1 \neq K$. □
Remark 3.7. (a) Closability of the linear manifolds \( \text{ran} D_{K,Q} \) in Proposition 3.6 may be replaced by \( \text{ran} D_{K,Q} \cap \mathcal{H}_+ = \mathcal{H}_+ \) where \( \mathcal{H}_+ := \ker (I \pm C) \), which are, for example, valid if either dimension \( \mathcal{H}_+ < \infty \) or dimension \( \mathcal{H}_- < \infty \).

(b) Note that \( \text{ran} D_{K,Q} \) is closed if both \( K \) and \( Q \) are compact operators, \( K, Q \in \mathcal{S}_\infty \).

Next we clarify and complement Proposition 3.6 in the case \( \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H} \) and \( Q = I_\mathcal{H} \).

Now \( L(Q; \varphi) = L(I_\mathcal{H}; \varphi) = C_\mathcal{H}(\varphi) \). Denote by \( C_\mathcal{H}(\varphi) := L^c(I_\mathcal{H}; \varphi) \) the set of extreme points of the set \( C_\mathcal{H}(\varphi) \) and by

\[
(3.32) \quad \partial L_{\varphi} := \partial L^+_\varphi \cup \partial L^-_{\varphi}, \quad \text{where} \quad \partial L^\mu_{\varphi} := \{ z \in D : |z \sin \varphi \pm i \cos \varphi| = 1 \},
\]

the (topological) boundary of the hole \( (2.17) \). Note that \( \partial L_{\varphi} \) is at the same time the set of extreme points of the hole \( (2.17) \).

Proposition 3.8. Let \( \varphi \in (0, \pi/2) \) and \( K \in C_\mathcal{H}(\varphi) \). Then

(i) there exists a contraction \( C = C^* \) such that

\[
(3.33) \quad 2K_I = \tan \varphi \cdot D_K C D_K, \quad C \in C_\mathcal{H}(0).
\]

Conversely, if \( K \in C_\mathcal{H}(\pi/2) \) and \( (3.33) \) holds then \( K \in C_\mathcal{H}(\varphi) \);

(ii) the following implication holds

\[
(3.34) \quad \sigma(C) \subset \{ \pm 1 \} \quad \text{and} \quad \text{ran} D_{K,Q} \cap \mathcal{H}_+ = \mathcal{H}_+ := \ker (I \pm C) \implies K \in C_\mathcal{H}(\varphi);
\]

(iii) if \( K \) is a normal operator, \( KK^* = K^* K \), and \( \sigma(K) \subset \partial L_{\varphi} \) then \( K \in C_\mathcal{H}^e(\varphi) \);

(iv) if \( K \in C_\mathcal{H}(\varphi) \), \( \sigma(K) \subset \partial L_{\varphi} \) and the spectrum \( \sigma(K) \) is purely point, then \( K \) is normal, hence \( K \in C_\mathcal{H}^e(\varphi) \);

(v) the set \( C_\mathcal{H}^e(\varphi) \) contains continuum (nonnormal) operators \( K \) with \( \sigma(K) = 0 \).

Proof. (i) If \( Q = I_\mathcal{H} \) then \( D_{K,Q} = D_{K,I} = \sin \varphi \cdot D_K \) and the statement is implied by Proposition 3.6 (i).

(ii) This statement is implied by Proposition 3.6 (ii).

(iii) Assume for brevity that \( \pm 1 \notin \sigma_p(K) \). Then starting with \( (3.33) \) and applying Spectral theorem we get

\[
C = \cot \varphi \cdot D_K^{-1}(2K_I)D_K^{-1} = \cot \varphi \cdot \int_{\partial L_\varphi} \frac{\lambda - \bar{\lambda}}{i \sqrt{1 - |\lambda|^2}} dE_K(\lambda) = \cot \varphi \cdot \int_{\partial L^+_\varphi} \frac{\lambda - \bar{\lambda}}{i \sqrt{1 - |\lambda|^2}} dE_K(\lambda) + \cot \varphi \cdot \int_{\partial L^-_{\varphi}} \frac{\lambda - \bar{\lambda}}{i \sqrt{1 - |\lambda|^2}} dE_K(\lambda) = \int_{\partial L^+_\varphi} dE_K(\lambda) - \int_{\partial L^-_{\varphi}} dE_K(\lambda) =: P_+ - P_-
\]

Here \( E_K(\cdot) \) is the spectral measure of \( K \), and \( P_\pm \) are the corresponding spectral projections. Since \( P_+ + P_- = I \), we have \( \sigma(C) \subset \{ \pm 1 \} \). Moreover, \( P_\pm \text{ran} D_K \) is dense in \( \mathcal{H}_\pm := P_\pm \mathcal{H} \). Hence by statement (ii) \( K \in C_\mathcal{H}^e(\varphi) \).

(iv) Let us set \( K_\pm := K \sin \varphi \pm i \cos \varphi \). If \( \lambda_j \in \partial L_{\varphi} \cap \sigma_p(K) \) and \( \mathcal{H}_j := \ker (K - \lambda_j)(\neq 0) \), then either \( \mu^+_j := \lambda_j \sin \varphi + i \cos \varphi \in \sigma_p(K^+_{\mathcal{H}}) \cap \mathbb{T} \) or \( \mu^-_j := \lambda_j \sin \varphi - i \cos \varphi \in \sigma_p(K^-_{\mathcal{H}}) \cap \mathbb{T} \) where \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \). Thus, the subspace \( \mathcal{H}_j \) reduces the operator \( K \) for any \( j \in \mathbb{Z}_+ \) since either \( \mathcal{H}_j = \ker (K^+_j - \mu^+_j) \) or \( \mathcal{H}_j = \ker (K^-_j - \mu^-_j) \), and both \( K^+_j \) and \( K^-_j \) are contractions. Since the spectrum \( \sigma(K) \) is purely point, then \( K = \oplus_{j=1}^{\infty} \lambda_j I_{\mathcal{H}_j} \) and \( K \) is normal.
(v) First we consider the case $\mathcal{H} = \mathbb{C}^2$. We let
\[
K(\theta) := e^{i\theta} \begin{pmatrix} 0 & \sin \varphi \\ 0 & 0 \end{pmatrix}, \quad \theta \in [0, 2\pi].
\]
Then
\[
D_{K(\theta)} = \begin{pmatrix} 1 & 0 \\ 0 & \cos \varphi \end{pmatrix}, \quad K(\theta)_I = i \sin \varphi \begin{pmatrix} 0 & -e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}, \quad C(\theta) = i \begin{pmatrix} 0 & -e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}.
\]
Hence $\sigma(C(\theta)) = \{\pm 1\}$ and by statement (ii) $K(\theta) \in C^e_H(\varphi)$.

**Remark 3.9.** (i) Another proof of statement (iii) is contained in [30]. The proof of statement (iv) is borrowed from [30] and it is presented for the sake of completeness.

(ii) Note that while a complete description of the set $C^e_H(\varphi)$ is unknown, it essentially differs from that of the sets $C^e_H(0)$ and $C^e_H(\pi/2)$ even in the case $\dim \mathcal{H} < \infty$. Indeed, if $\dim \mathcal{H} < \infty$ then by Proposition 2.7 both $C^e_H(\pi/2)$ and $C^e_H(0)$ consist of normal matrices with "boundary spectrum", that is, $C^e(\pi/2)$ (resp. $C^e_H(0)$) is the set of unitary (resp. unitary selfadjoint) matrices.

On the other hand, the sets $C_0(\varphi), \varphi \in (0, \pi)$ may be considered as "interpolation sets" between $C_0(0)$ and $C_0(\pi/2)$. This observation makes natural the following hypothesis: for any $\varphi \in (0, \pi)$ the set $C_0(\varphi)$ consists of normal matrices with "boundary spectrum". However Proposition 3.8 shows that this hypothesis is false to be true, since the set $C^e_H(\varphi)$ contains continuum nonnormal matrices in addition to the set of normal matrices with spectrum lying on $\partial \mathcal{L}_\varphi$.

Combining Theorem 3.4 with Proposition 3.6 we arrive at the following result.

**Corollary 3.10.** Suppose that conditions of Theorem 3.4 are satisfied, $Q_0$ and $\varphi_1$ are defined by (3.11) and (3.12) respectively, and $T_K \in \text{Ext}_{\{T_1, T_2\}}(\varphi)$. Then

(i) for any $\varphi \in [\varphi_1, \pi/2]$ the following equivalence holds
\[
T_K \in \text{Ext}^e_{\{T_1, T_2\}}(\varphi) \iff K \in L^e(Q; \varphi);
\]

(ii) there exists a selfadjoint contraction $C \in C_{\mathcal{H}_1}(0)$ such that
\[
\sin 2\varphi \cdot (K^*Q)_I = D_{K,Q} C D_{K,Q};
\]

(iii) the following implication holds
\[
\sigma(C) = \{\pm 1\} \quad \text{and} \quad \text{ran} D_{K,Q} = \text{ran} D_{K,Q} \implies T_K \in \text{Ext}^e_{\{T_1, T_2\}}(\varphi);
\]

(iv) $T_K \in \text{Ext}^e_{\{T_1, T_2\}}(\varphi)$ if at least one of the following identities holds
\[
D^2_{K,Q} \pm \sin 2\varphi \cdot (K^*Q)_I = 0, \quad D^2_{K,Q} \pm \sin 2\varphi (K^*Q)_I = 0.
\]

### 3.4. Proper $C_0(\varphi)$-extensions of symmetric contractions.

Here we apply Theorem 3.4 to the case of a dual pair $\{T_1, T_1\}$.

Let $T_1 \in [\mathfrak{S}_1, \mathfrak{S}_2]$ be a nondensely defined symmetric contraction in $\mathfrak{S} = \mathfrak{S}_1 \oplus \mathfrak{S}_2$. As usual $\text{Ext}_{T_1}$ stands for the set of all proper extensions of $T_1$, that is $T \in \text{Ext}_{T_1}$ iff $T \supset T_1$ and $T^* \supset T_1$. Denote by
\[
(3.35) \quad \text{Extp}_{T_1}(\varphi) := \text{Ext}_{T_1} \cap C_0(\varphi), \quad \varphi \in [0, \pi/2],
\]
the set of all proper $C_0(\varphi)$-extensions of the symmetric contraction $T_1$. 

By Definition 2.11, \( \text{Ext}\{T_1, T_1\} = \text{Ext}\ T_1 \). Moreover, it follows from (3.10) and (3.12) that \( \text{Ext}\{T_1, T_1\}(\varphi) = \text{Extp}^{T_1}(\varphi) \).

**Corollary 3.11.** Let \( T_1 \in [\mathcal{H}, \mathcal{H}] \) be a nondensely defined symmetric contraction in \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \). Then \( \text{Extp}^{T_1}(\varphi) = \text{Ext}\{T_1, T_1\}(\varphi) \neq \emptyset \) for any \( \varphi \in [0, \pi/2] \). Moreover, the following equivalence holds

\[
T := T_K \in \text{Extp}^{T_1}(\varphi) \iff T_{22} = -U^*T_{11}U + D_U KD_U, \quad K \in C_H(\varphi),
\]

where \( \mathcal{H} := \mathcal{H}_2 \oplus \ker D_U \).

**Proof.** According to (3.10) \( T_1 = T_2 \) if and only if \( T_{12} = T_{21}^* \), that is iff \( V^* = U \). Therefore the operator \( Q_0 \) defined by (3.11) takes the form \( Q_0 = I_H \), where \( I_H \) is the identical operator in \( \mathcal{H} \). Thus \( \varphi_1 = \arccos(\|Q_0\|^{-1}) = \arccos 1 = 0 \) and \( \text{Extp}^{T_1}(\varphi) \neq \emptyset \) for any \( \varphi \in [0, \pi/2] \). Moreover, now equivalence (3.13) takes the form (3.36).

**Remark 3.12.** In the case \( T_1 = T_2 \) both left and right radii of the balls \( B_1 \) and \( B_2 \) are equal: \( R_l = D_{V^*} = D_U = R_r \).

According to (3.13), the set \( \text{Ext}\{T_1, 0\} \) of selfadjoint contractive extensions of \( T_1 \) forms an operator segment (“the self-adjoint part” of the operator ball \( B(-U^*T_{11}U; D_U, D_U) \)) which is parametrized by the operator segment \( \{K \in [\mathcal{H}]: -I_H \leq K \leq I_H\} \).

Consider the extremal selfadjoint contractive extensions \( T_m := T_{\min} \) and \( T_M := T_{\max} \) of the operator \( T_1 \). It is clear that \( T_m := T_{-I} \) and \( T_M := T_{I} \) are the extreme points of the segment \( \text{Extp}^{T_1}(0) \), corresponding to the operators \( K = -I_H \) and \( K = I_H \) respectively. Their block-matrix representations are of the form

\[
T_m = \begin{pmatrix} T_{11} & D_{T_{11}} U \\ U^* D_{T_{11}} & -I + U^* (I - T_{11}) U \end{pmatrix}, \quad T_M = \begin{pmatrix} T_{11} & T_{D_{T_{11}}} U \\ U^* D_{T_{11}} & I - U^* (I + T_{11}) U \end{pmatrix}.
\]

Using representations (3.37) we rewrite description (3.36) as

\[
T_K \in \text{Extp}^{T_1}(\varphi) \iff 2T_K = (T_M + T_m) + (T_M - T_m)^{1/2} K (T_M - T_m)^{1/2}, \quad K \in C_H(\varphi).
\]

Note that this description of the class \( \text{Extp}^{T_1}(0) \) has been obtained by M.G. Krein [25] (see also [1], [26]). Other proofs are contained in [10], [24]. A generalization of the Krein result to the case of \( C_H(\varphi) \)-contractions, that is a description of the class \( \text{Extp}^{T_1}(\varphi) \) in the form (3.38) has been obtained in [5, 6] (see also [24], [14], [30] for other proofs).

**3.5. A description of the set of all proper and improper \( C_H(\varphi) \)-extensions of symmetric contractions.**

Let \( A \) be a closed densely defined symmetric operator in \( \mathcal{H} \). It is known, that any \( m \)-dissipative (in particular selfadjoint) extension \( \tilde{A} \) of \( A \) is a proper extension \( \widetilde{A} \in \text{Extp}_A \), that is \( A \subset \tilde{A} \subset A^* \). It is not the case for \( m \)-sectorial extensions of a nonnegative operator \( A \geq 0 \).

Therefore we clarify Definition 2.11 for the case of a nonnegative operator.

**Definition 3.13.** Let \( A(\geq 0) \) be a closed densely defined nonnegative operator in \( \mathcal{H} \). Denote by \( \text{Extp}_A([0, \infty); \varphi) \) the class of all proper \( m \)-sectorial extensions of \( A \) with vertex zero and half-angle \( \varphi \in (0; \pi/2) \). The class of all (proper and improper) \( m \)-sectorial extensions \( \tilde{A}(\in \mathcal{C}(\mathcal{H})) \) of \( A \) will be denoted by \( \text{Extp}_A([0, \infty); \varphi) \).
Here we present a description of the set $\text{Ext}_A((0, \infty); \varphi)$. In accordance with the approach accepted in this paper (cf. Lemma 2.12) it suffices to describe the set $\text{Ext}_{T_1}(\varphi)$ of all the extensions of the class $C_{T_1}(\varphi)$ of a nondensely defined Hermitian contraction $T_1 := (I - A)(I + A)^{-1}(\in [\mathcal{F}_1, \mathcal{F}_2])$ where $\mathcal{F}_1 := \text{ran} (I - A)$. In turn, considering the block-matrix representation of $T_1$ with respect to the orthogonal decomposition $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, one reduces the problem to the problem of a description of all the "completions" of a contractive operator-matrix $T_1 = (\frac{T_{tt}}{T_{t1}})$ to form a matrix $T = (T_{ij})_{i,j=1}^2 \in C\mathcal{F}(\varphi)$.

Note that $A \in \text{Ext}_p A$ iff the entries $T_{12}$ and $T_{21}$ of $(T_{ij})_{i,j=1}^2 := T := (I - \tilde{A})(I + \tilde{A})^{-1}$ are connected by $T_{21} = T_{12}^{-1}$.

**Theorem 3.14**. Let $T_1 = (T_{11})_{i,j=1}^2 = (\frac{T_{tt}}{T_{t1}})$ be a symmetric contraction in $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, $T := (T_{ij})_{i,j=1}^2 \in [\mathcal{F}]$, $\mathcal{H}_1 := \text{ran} D_{V^*}$, $\mathcal{H}_2 := \text{ran} D_V$, and $\varphi \in [0, \pi/2]$. Then

(i) $T \in \text{Ext}_{T_1}(\pi/2)$, i.e. $T(\in C\mathcal{F}(\pi/2))$ is a contractive extension of $T_1$ if and only if it is of the form (2.12), that is

$$
T_{12} = D_{T_{11}}, \quad T_{22} = -VT_{11}^* + D_{V} \cdot K D_{U}, \quad U, K \in C(\pi/2).
$$

(ii) $T \in \text{Ext}_{T_1}(\varphi) := \text{Ext}_{T_1}(\pi/2) \cap C\mathcal{F}(\varphi)$ if and only if $U$ "runs through" the operator ball of the form

$$
U = \sin \varphi (\sin^2 \varphi D_{V^*}^2 + \cos^2 \varphi)^{-1/2} D_{V} M D_{V^*} (\sin^2 \varphi D_{V^*}^2 + \cos^2 \varphi)^{-1/2} + \cos^2 \varphi (\sin^2 \varphi D_{V^*}^2 + \cos^2 \varphi)^{-1} V^*,
$$

$$
M \in C(\pi/2) \cap [\mathcal{H}_2, \mathcal{H}_2],
$$

and $T_{22}$ (for fixed $U$) "runs through" the operator "hole"

$$
T_{22} = -VT_{11}^* U + D_{V} \cdot K D_{U}, \quad K \sin \varphi \pm i Q \cos \varphi \in C(\pi/2) \cap [\mathcal{H}_1, \mathcal{H}_2].
$$

Here $\mathcal{H}_1 := \mathcal{H}_1(U) := \text{ran} D_{U}$, $Q = \overline{Q}_0(\in [\mathcal{H}_1, \mathcal{H}_2])$ and $Q_0$ is defined by (3.11).

**Proof.** Equality (3.39) is implied by Theorem 2.11.

Let further $T_{12} = D_{T_{11}} U$, $T_2 = (\frac{T_{tt}}{U^* D_{T_{11}}})$. Then $\{T_1, T_2\}$ is a dual pair of contractions and according to Theorem 3.4 the condition $\text{Ext}_{\{T_1, T_2\}}(\varphi) \neq \emptyset$ is equivalent to the contractibility of the operator $Q_0 \cos \varphi$, where $Q_0$ is defined by (3.11), i.e. to the inequality

$$
\cos \varphi \cdot \|D_{V^*}^{-1}(I - VU)f\| \leq \|D_{U}f\|, \quad f \in \mathcal{F}_2.
$$

Supposing first that $0 \in \rho(D_{V^*})$, we rewrite inequality (3.42) in the equivalent form

$$
\cos^2 \varphi[D_{V^*}^{-2} - U^* V^* D_{V^*}^{-2} - D_{V^*}^{-2} VU + U^* V^* D_{V^*}^{-2} VU] \leq I - U^* U
$$

or

$$
U^*(1 + \cos^2 \varphi V^* D_{V^*}^{-2} VU - \cos^2 \varphi (U^* D_{V^*}^{-2} V^* + V D_{V^*}^{-2} U))
$$

$$
+ \cos^2 \varphi D_{V^*}^{-2} - I \leq 0.
$$

Since inequality (3.43) is equivalent to (3.42), the set of its solutions is nonempty for any fixed $V$. By Lemma 2.13 for any fixed $V$ the set of solutions of inequality (3.43), that is the set of operators $U$ obeying (3.43), forms an operator ball $B(C_0; R_l, R_r)$. Applying Lemma 2.13 we find its center and radii. We have

$$
R_l = Q_1^{-1/2} = (I + \cos^2 \varphi V^* D_{V^*}^{-2} V^{-1/2} = (I + \cos^2 \varphi V^* V D_{V^*}^{-2})^{-1/2}
$$

$$
= [\cos^2 \varphi(D_{V^*}^{-2} - I) + I]^{-1/2} = [D_{V^*}^{-2}(\sin^2 \varphi D_{V^*}^2 + \cos^2 \varphi)]^{-1/2}
$$

$$
= (\sin^2 \varphi D_{V^*}^2 + \cos^2 \varphi)^{-1/2} D_{V^*}.
$$
To complete the proof it remains to apply both Proposition 2.7 and Corollary 3.10.

It follows from (3.39) and (3.40) that the mapping

\[ R^2_v = Q^2_v Q_1^{-1} Q_2 - Q_3 = -Q^*_2 C_0 - Q_3 \]

\[ = \cos^4 \varphi V^*(\sin^2 \varphi D^2_{V*} + \cos^2 \varphi)^{-1} I - \cos^2 \varphi D^2_{V*} \]

\[ = (\sin^2 \varphi D^2_{V*} + \cos^2 \varphi)^{-1} \]

\[ \cdot [\cos^4 \varphi D^2_{V*} - \cos^4 \varphi \cdot I + (I - \cos^2 \varphi D^2_{V*})(\sin^2 \varphi D^2_{V*} + \cos^2 \varphi)] \]

\[ = (\sin^2 \varphi D^2_{V*} + \cos^2 \varphi)^{-1} [\sin^2 \varphi D^2_{V*} + \cos^2 \varphi (1 - \cos^2 \varphi - \sin^2 \varphi)] \]

\[ = \sin^2 \varphi (\sin^2 \varphi D_{V*} + \cos^2 \varphi \cdot I)^{-1} D^2_{V*}. \]

Thus,

\[ R_v = \sin \varphi (\sin^2 \varphi D_{V*} + \cos^2 \varphi \cdot I)^{-1/2} D_{V*}. \]

Applying Lemma 2.3 and taking relations (3.44)-(3.46) into account we get that inequality (3.43) or, what is the same, inequality (3.42) is satisfied iff \( U \) admits a representation (3.40) with some \( M \in C(\pi/2) \). Thus, we proved (3.40) under the additional assumption \( 0 \in \rho(D_{V*}) \) (\( \Longleftrightarrow 0 \in \rho(D_V) \)).

Next, we may easily free ourselves of the additional assumption \( 0 \in \rho(D_{V*}) \) by passing to the limit. Actually since \( D_{V*} > D_{V*}^{-1}, \quad r \in (0, 1) \), inequality (3.42) takes place if and only if for any \( r \in (0, 1) \) the inequality

\[ \cos \varphi \| D_{V*}^{-1} (I - V) f \| \leq \| D_U f \|, \quad f \in \mathcal{D}, \]

holds true. Since \( 0 \in \rho(D_{rV}) \), then in accordance with what has been proved in the previous step inequality (3.47) (for fixed \( r < 1 \)) is equivalent to equality (3.40) with \( D_V \) and \( D_{V*} \) replaced by \( D_{rV} \) and \( D_{rV*} \) respectively. In these equalities it is possible to pass to the limit as \( r \uparrow 1 \) (in the sense of strong convergence).

Now the relations (3.41) follow from Theorem 3.4. □

According to Theorem 3.14 (see formulas (3.39)-(3.41)) any extension \( T \in \text{Ext}_{\chi}(\varphi) \) is uniquely determined by a pair \( \{M, K\} \) of ”free” parameters. Denote the corresponding extension \( T \) by \( T_{M,K} \).

Next we denote by \( \text{Ext}^c_{\chi}(\varphi) \) the set of extreme points of \( \text{Ext}_{\chi}(\varphi) \).

Corollary 3.15. Let \( T = T_{M,K} \in \text{Ext}_{\chi}(\varphi) \). Then

(i) \( T_{M,K} \in \text{Ext}^c_{\chi}(\varphi) \) if and only if \( M \) is a maximal partial isometry from \( \mathcal{H}_2 \) to \( \mathcal{H}_2 \) and \( K \in L^*(Q; \varphi) \), where \( L(Q; \varphi) \) is defined by (3.25);

(ii) if \( M \) is a maximal partial isometry, then the following implication holds

\[ \sigma(C) = \{\pm 1\} \quad \text{and} \quad \text{ran} D_{K,Q} = \text{ran} D_{K,Q} \Longrightarrow T_{M,K} \in \text{Ext}^c_{\chi}(\varphi). \]

Proof. It follows from (3.39) and (3.40) that the mapping \( \{M, K\} \rightarrow T_{M,K} \) preserves convexity: if the ”free” parameters \( \{M_j, K_j\}, \ j \in \{1, 2\} \) and \( \{M, K\} \) are connected by \( M = tM_j + (1-t)M_2 \) and \( K = tK_1 + (1-t)K_2 \) with \( t \in (0, 1) \), then \( T_{M,K} = tT_{M_1,K_1} + (1-t)T_{M_2,K_2} \). To complete the proof it remains to apply both Proposition 2.7 and Corollary 3.10. □
Remark 3.16. (i) The set $\text{Ext}_{T_1}(\varphi)$ of all (proper and improper) $C_\delta(\varphi)$-extensions of $T_1$ admits a representation $\text{Ext}_{T_1}(\varphi) = \cup_{T_2} \text{Ext}_{(T_1,T_2)}(\varphi)$ where $T_2 = (\psi_{T_{11}}^{*}, u_{D_{T_{11}}})$ and $U$ "runs through" the operator ball (3.40). Note that $T$ is a proper $C_\delta(\varphi)$-extension of $T_1$ iff $U = V^*$. In this case $Q = I$ and (3.41) turns into (3.36).

(ii) Theorem [3.14] has been proved by the author together with V. Kolmanovich in [24] in a different but equivalent form.

3.6. $C_\delta(\varphi)$-extensions of a dual pair of $C(\varphi)$-contractions. Here we consider a dual pair $\{T_1, T_2\}$ of $C(\varphi)$-contractions of the form

\[(3.48) \quad T_1 = \left( \begin{array}{c} T_{11} \\ T_{21} \end{array} \right) = \left( \begin{array}{c} T_{11} \\ V D_{T_{11}} \end{array} \right), \quad T_2 = \left( \begin{array}{c} T_{11}^* \\ U^* D_{T_{11}} \end{array} \right),\]

and show that the Problem 2C mentioned in the Introduction is reduced to the Problem 3 with different left $R^+_T \neq R^-_T$ and right $R^+_T \neq R^-_T$ radii.

Proposition 3.17. Let $\{T_1, T_2\}$ be a dual pair of contractions in $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Suppose that both $T_1$ and $T_2$ obey condition (2.13) with $\varphi = \varphi_0$. Then

(i) for any $\varphi \in (\varphi_0, \pi/2)$ the following relations hold

\[(3.49) \quad 2(T_{11}^*)_1 = \tan \varphi \cdot D_{T_{11}} C(\varphi) D_{T_{11}} = \tan \varphi \cdot D_{T_{11}} C_2(\varphi) D_{T_{11}}, \]

where $C_j(\varphi) := C_j \cdot \tan \varphi_0 / \tan \varphi$, $j \in \{1, 2\}$, is a selfadjoint contraction and

\[(3.50) \quad C_1 = D_V C' D_V, \quad C_2 = D_U C'' D_U, \quad -I \leq C', \quad C'' \leq +I; \]

(ii) for any $\varphi \in (\varphi_0, \pi/2)$ the set $\text{Ext}_{\{T_1, T_2\}}(\varphi)$ forms an operator hole:

\[(3.51) \quad T \in \text{Ext}_{\{T_1, T_2\}}(\varphi) \iff T_{22} \sin \varphi \in L(\varphi) := B(C_+; R^+_T, R^+_T) \cap B(C_-; R^-_T, R^-_T), \]

where

\[(3.52) \quad C_\pm := \mp i \cos \varphi \cdot I - V_\pm (T_{11}^* \sin \varphi \pm i \cos \varphi) U_\pm, \quad R^+_T = D_V^+, \quad R^-_T = D_U^+, \]

and $V_\pm, U_\pm$ are contractions of the form

\[(3.53) \quad V^*_\pm := V^{*}_{1\pm} (I \mp C_1(\varphi))^{-1/2} V^*, \quad U_\pm := U_{1\pm} (I \pm C_2(\varphi))^{-1/2} U, \]

and $V_\pm, U_\pm$ are (uniquely determined) partial isometries.

In particular, $\text{Ext}_{\{T_1, T_2\}}(\varphi) \neq \emptyset$ if and only if $L(\varphi) \neq \emptyset$.

Proof. The inclusion $T = (T_{ij})^2_{i,j=1} \in C_\delta(\varphi)$ means that

\[(3.54) \quad T_\pm := \left( \begin{array}{cc} B_{11}^\pm & B_{12} \\ B_{21} & B_{22}^\pm \end{array} \right) := \left( \begin{array}{cc} \sin \varphi \cdot T_{11} \pm i \cos \varphi \cdot I & T_{21} \sin \varphi \\ T_{21} \sin \varphi & \sin \varphi \cdot T_{22} \pm i \cos \varphi \cdot I \end{array} \right) \in C_\delta(\pi/2). \]

First we note that

\[D^2_{T_1} = I - T^*_1 T_{11} - T^*_{21} T_{21} = D^2_{T_{11}} - D_{T_{11}} V^* V D_{T_{11}} = D_{T_{11}} D^2_V D_{T_{11}}, \]

and $D^2_{T_2} = D_{T_{11}} D^2_U D_{T_{11}}$. Combining these relations with (2.13) and applying Lemma 3.1 we obtain (3.49).

Next, starting with (3.54) and taking (3.49) into account we get

\[(3.55) \quad D^2_{T_{11}} = I - (\sin \varphi \cdot T_{11}^* \mp i \cos \varphi) (\sin \varphi \cdot T_{11} \mp i \cos \varphi) = \sin^2 \varphi \cdot D^2_{T_{11}} \mp \sin 2\varphi \cdot (T_{11})_I = \sin^2 \varphi \cdot D_{T_{11}} (I \pm C_1(\varphi)) D_{T_{11}}. \]
Hence there exist partial isometries $V_{1\pm}(\varphi)$ such that

$$
(3.56) \quad D_{B_{1i}^\pm} = \sin \varphi \cdot D_{T_{1i}} (I \pm C_1(\varphi))^{1/2} V_{1\pm}(\varphi) = \sin \varphi \cdot V_{1\pm}^*(\varphi)(I \pm C_1)^{1/2} D_{T_{1i}}.
$$

Combining (3.54), (3.55) and (3.56) we derive

$$
B_{21} = T_{21} \sin \varphi = \sin \varphi \cdot V D_{T_{1i}} = V_\pm D_{B_{11}^\pm} = \sin \varphi \cdot V_\pm V_{1\pm}^*(\varphi)(I \pm C_1)^{1/2} \cdot D_{T_{1i}}.
$$

It follows that $V = V_\pm \cdot V_{1\pm}^*(\varphi) (I \pm C_1(\varphi))^{1/2}$, which yields the first of relations (3.53).

Similarly we get

$$
(3.57) \quad D_{B_{11}^\pm}^2 = \sin^2 \varphi \cdot D_{T_{11}}^2 (I \pm C_2(\varphi)) \cdot D_{T_{11}}.
$$

According to polar decomposition we have $\sin \varphi \cdot (I \pm C_2(\varphi))^{1/2} D_{T_{11}} = U_{1\pm} D_{B_{11}^\pm}$, with some partial isometries $U_{1\pm}$. These representations imply

$$
(3.58) \quad B_{12} = T_{12} \sin \varphi = \sin \varphi \cdot D_{T_{1i}} U = D_{B_{11}^\pm} U_{1\pm} = \sin \varphi \cdot D_{T_{1i}} (I \pm C_2(\varphi))^{1/2} U_{1\pm} U_{1\pm}.
$$

Hence $U = (I \pm C_2(\varphi))^{1/2} U_{1\pm} U_{1\pm}$. This equality yields the second relation in (3.53).

By Theorem 2.4 $T_{1\pm} \in C_\beta(\pi/2)$ if and only if

$$
(3.59) \quad T_{22} \sin \varphi \pm i \cos \varphi \cdot I = C_\pm + D_{V_{1\pm}^*} K_{U_{1\pm}},
$$

where $K_{U_{1\pm}}$ are contractions and $C_\pm$ are defined by

$$
(3.60) \quad C_\pm = -V_{\pm} (B_{11}^*)^* U_{\pm} = -V_{\pm} (T_{11}^* \sin \varphi \pm i \cos \varphi \cdot I) U_{\pm}.
$$

to complete the proof it suffices to set $C_{\pm} = C_\pm \mp i \cos \varphi \cdot I$. \hfill \Box

**Remark 3.18.** Let $T_1 = (T_{11})_{T_{12}} \in C(\varphi)$ and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. It is shown in [30], Theorem 4.11, that Problem 3C is reduced to Problem 5 mentioned in the Introduction. Namely, it is proved in [30] that $T \in \text{Ext}_{T_1}(\varphi)$ iff

$$
(3.61) \quad TP_2 \in L := B_+ \cap B_-,
$$

where

$$
B_\pm = B(C_\pm; D_{S_{\pm}^*}/\sin \varphi, P_2),
$$

and $S_{\pm} = T_{1i} \sin \varphi \pm i \cos \varphi \cdot I$, $C_{\pm} = \mp \cot \varphi \cdot P_2$. Thus, the set $\text{Ext}_{T_1}(\varphi)$ forms an operator hole of the form (1.3) with $R_{\pm} = D_{S_{\pm}^*}/\sin \varphi$, $R_{\pm}^* = P_2$, and $C_{\pm} = \mp \cot \varphi \cdot P_2$.

4. Noncontractive extensions of dual pair of symmetric contractions.

**4.1. Schur complements.** In this section we investigate some spectral properties of contractive and noncontractive extensions of a dual pair $\{T_1, T_2\}$ of symmetric contractions using their block-matrix representations (3.3). Throug this section we keep a notation $T_K$ for any (not necessary contractive) extension of the dual pair $\{T_1, T_2\}$ having the form (2.12) with a bounded operator $K \in [\mathcal{H}_1, \mathcal{H}_2]$. Observe that any bounded extension $T \in \text{Ext}_{\{T_1, T_2\}}$ has such a form iff $T_1$ and $T_2$ are transversal, that is $0 \in \rho(D_U) \cap \rho(D_{V^*})$. Note also that in the nonsingular case $\mathcal{H}_1 = \mathcal{H}_1$ and $\mathcal{H}_2 = \mathcal{H}_2$.

We investigate some spectral properties of extensions $T_K(\in \text{Ext}_{\{T_1, T_2\}})$ in terms of ”boundary” operators $K$. In particular we obtain descriptions of the classes $C_\beta(\varphi, \mathcal{S}^\pm)$ and $C_\beta(\varphi, \mathcal{S}^\pm)$.

As well as in Theorem 3.12 these descriptions essentially depend on the operator

$$
(4.1) \quad Q_0 := D_{V^*}^{-1} (I - VU) D_{U}^{-1}.
$$

In the following theorem which is the main result of the section we calculate Schur complement of the operator block-matrices $I - T_K T_K^*$ and $\cos \varphi \cdot (T_K - T_K^*)$.
Theorem 4.1. Let \( \{T_1, T_2\} \) be a dual pair of symmetric contractions of the form \((3.3)\) in \( \mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2 \), \( T_K \in \text{Ext} \{T_1, T_2\} \) and let \( \varphi \in [\varphi_1, \pi/2] \) where \( \varphi_1 := \arccos(\|Q\|^{-1}), \ Q := Q_0 \) and \( Q_0 \) be defined by \((4.1)\). Let further

\[
S^\pm := (S^\pm_{ij})^2_{i,j=1} := I - T_K T_K^* \pm \cot \varphi \cdot (T_K - T_K^*),
\]

\[
G^\pm := (G^\pm_{ij})^2_{i,j=1} := I - T_K^* T_K \pm \cot \varphi \cdot (T_K - T_K^*),
\]

with \( S^\pm_{ij}, G^\pm_{ij} \in [\mathcal{S}_1, \mathcal{S}_2] \). Then

(i) \( \text{ran} (S^{1/2}_{11}) \supset \text{ran} (S^{1/2}_{12}), \text{ran} (G^{1/2}_{11}) \supset \text{ran} (G^{1/2}_{12}) \) and, consequently the operators \( S^{1/2}_{11} S^{1/2}_{12} \) and \( G^{1/2}_{11} G^{1/2}_{12} \) are well defined and bounded, where \( S_{11} := S^+_1 = S^+_1; \)

(ii) the identities

\[
\sin^2 \varphi \cdot [S^\pm_{22} - (S^{1/2}_{11} S^{1/2}_{12})(S^{1/2}_{11} S^{1/2}_{12})^*] = D_V \cdot [I - (K \sin \varphi \mp iQ \cos \varphi)(K^* \sin \varphi \mp iQ^* \cos \varphi)] \cdot D_V^*,
\]

\[
\sin^2 \varphi \cdot [G^\pm_{22} - (G^{1/2}_{11} G^{1/2}_{12})(G^{1/2}_{11} G^{1/2}_{12})^*] = D_U \cdot [I - (K \sin \varphi \mp iQ^* \cos \varphi)(K \sin \varphi \pm iQ \cos \varphi)] \cdot D_U,
\]

hold true.

Proof. (i) We let \((G_{ij})^2_{i,j=1} := I - T_K^* T_K\). Then

\[
(G^\pm_{ij})^2_{i,j=1} = (G_{ij})^2_{i,j=1} \pm i \cot \varphi \left( V^* - U^* \right) D_{T_{11}} \left( T_{22} - T_{22}^* \right).
\]

By definition the operator \( T_K (\in \text{Ext} \{T_1, T_2\}) \) is of the form \((2.12)\) with \( T_{22} = -V T_{11} U + D_V \cdot K D_U \) and \( K \in [\mathcal{H}_1, \mathcal{H}_2] \). Therefore taking into account \((2.12)\) and \((4.6)\) we get

\[
G^{1/2}_{11} = G_{11} = D_{T_{11}} D_V^2 D_{T_{11}},
\]

and

\[
-G^{1/2}_{21} = -G_{12})^* = (U^* T_{11} D_V + D_U K^* V) D_V D_{T_{11}} \mp i \cot \varphi \cdot (U^* - V) D_{T_{11}}.
\]

Since \( \varphi \in [\varphi_1, \pi/2] \), then according to Theorem \(3.13\) the operators \( U \) and \( V \) are connected by equality \((3.40)\). Setting

\[
Y := (\sin^2 \varphi D_V^2 + \cos^2 \varphi)^{-1/2} \quad \text{and} \quad Y_* := (\sin^2 \varphi D_V^2 + \cos^2 \varphi)^{-1/2},
\]

and taking into account the identity \( V D_V = D_V \cdot V \) we rewrite \((3.40)\) in the form

\[
U^* - V = \sin \varphi D_V \cdot Y_* \cdot (M^* - \sin \varphi \cdot V) \cdot Y D_V.
\]

Let, further

\[
X_{\pm} := U^* T_{11} D_V + D_U K^* V \mp i \cos \varphi \cdot D_V \cdot Y_* \cdot (M^* - \sin \varphi \cdot V) \cdot Y.
\]

Now relations \((4.8)-(4.11)\) yield \(-G^{1/2}_{21} = X_{\pm} D_V D_{T_{11}}\). Combining this equality with \((4.7)\) we easily get

\[
\|G^{1/2}_{12} f\|^2 \leq \|X_{\pm}\|^2 \cdot (G_{11} f, f) = \|X_{\pm}\|^2 \cdot \|(G^{1/2}_{11} f\|^2, f \in \mathcal{S}_1.
\]

This inequality yields the inclusion \( \text{ran} (G^{1/2}_{12}) \subset \text{ran} (G^{1/2}_{11}) \), that is the second of the required inclusions.
The proof of the first inclusion \( \text{ran}(S_{12}^{+}) \supset \text{ran}(S_{12}^{+}) \) can be obtained in just the same way. It is suffices to use the equalities \( S_{11}^{+} = S_{11} = D_{T11} D_{U}^{2}, D_{T11} \) and \( S_{21}^{+} = X_{T11} D_{U}^{*} D_{T11} \) in place of (4.7) and (4.14) respectively, and the relation
\[
(4.13) \quad V - U^* = \sin \varphi \cdot D_{U} (\sin^2 \varphi D_{U}^{2} + \cos^2 \varphi)^{-1/2}.
\]
in place of (4.10).

\( (i1) \) Let us prove equality (4.5) assuming at the beginning that \( 0 \in \rho(D_{T11}) \cap \rho(D_{V}) \). In this case setting \( Z := T_{22} \), we obtain from (4.4) - (4.8) that
\[
G_{22}^{+} - (G_{12}^{+})^{*} G_{11}^{-1} G_{12}^{+} = I - U^* D_{T11} U - Z^* Z \pm i \cot \varphi \cdot (Z - Z^*)
\]
\[
- [U^*(T_{11} \pm i \cot \varphi) \pm (Z^* \mp i \cot \varphi)V] \cdot D_{V}^{-2} \cdot [(T_{11} \mp i \cot \varphi)U + V^*(Z \pm i \cot \varphi)]
\]
\[
= (1 + \cot^2 \varphi) \cdot I - U^* D_{T11} U - (Z^* \mp i \cot \varphi \cdot I) \cdot (Z \pm i \cot \varphi \cdot I)
\]
\[
- (Z^* \mp i \cot \varphi) \cdot V D_{V}^{-2} V^* \cdot (Z \pm i \cot \varphi) - U^*(T_{11} \pm i \cot \varphi) \cdot D_{V}^{-2} V^* \cdot (Z \pm i \cot \varphi)
\]
\[
- (Z^* \mp i \cot \varphi) \cdot V D_{V}^{-2} \cdot (T_{11} \mp i \cot \varphi)U - U^*(T_{11} \pm i \cot \varphi) \cdot D_{V}^{-2} \cdot (T_{11} \mp i \cot \varphi)U
\]
\[
= D_{V}^{2} + \cot^2 \varphi \cdot I + U^* T_{11} U - (Z^* \mp i \cot \varphi \cdot D_{V}^{-2} \cdot (Z \pm i \cot \varphi \cdot I)
\]
\[
- U^*(T_{11} \pm i \cot \varphi) \cdot D_{V}^{-2} V^* \cdot (Z \pm i \cot \varphi) - (Z^* \mp i \cot \varphi) \cdot V D_{V}^{-2} \cdot (T_{11} \mp i \cot \varphi)U
\]
\[
- U^*(T_{11} \pm i \cot \varphi) \cdot D_{V}^{-2} \cdot (T_{11} \mp i \cot \varphi)U.
\]

On the other hand, combining (4.1) with the equality \( Z := T_{22} = -VT_{11} U + D_{V} \cdot K D_{U} \), we get
\[
(4.15) \quad QD_{U} = D_{V}^{-1} (I - VU) \quad \text{and} \quad KD_{U} = D_{V}^{-1} (Z + V T_{11} U).
\]
Inserting these relations in the right-hand side of (4.5) we deduce
\[
(4.16) \quad A_{\mp} := D_{U} \cdot \left[ \frac{1}{\sin^2 \varphi} - (K^* \pm i Q^* \cot \varphi) \cdot (K \mp i Q \cot \varphi) \right] \cdot D_{U}
\]
\[
= \frac{D_{U}^{2}}{\sin^2 \varphi} \cdot \left[ (Z^* + U^* T_{11} V^*) \mp i \cot \varphi (I - U^* V^*) \right] \cdot D_{V}^{-2} \cdot \left[ (Z + VT_{11} U) \mp i \cot \varphi (I - VU) \right]
\]
\[
= \frac{D_{U}^{2}}{\sin^2 \varphi} \cdot \left[ (Z^* \mp i \cot \varphi) + U^* (T_{11} \pm i \cot \varphi) V^* \right] \cdot D_{V}^{-2} \cdot \left[ (Z \pm i \cot \varphi) + V (T_{11} \mp i \cot \varphi) U \right]
\]
\[
= \frac{D_{U}^{2}}{\sin^2 \varphi} \cdot \left[ (Z^* \mp i \cot \varphi) \cdot D_{V}^{-2} \cdot (Z \pm i \cot \varphi) - (Z^* \mp i \cot \varphi) \cdot D_{V}^{-2} V \cdot (T_{11} \mp i \cot \varphi) U
\]
\[- U^*(T_{11} \pm i \cot \varphi) \cdot V^* D_{V}^{-2} \cdot (Z \pm i \cot \varphi) - U^*(T_{11} \pm i \cot \varphi) \cdot V^* D_{V}^{-2} V \cdot (T_{11} \mp i \cot \varphi) U.
\]
Since \( V^* D_{V}^{-2} V = D_{V}^{-2} V^* V = D_{V}^{-2} - I \), then the last term in (4.16) is transformed as follows:
\[
(4.17) \quad -U^*(T_{11} \pm i \cot \varphi) \cdot V^* D_{V}^{-2} V \cdot (T_{11} \mp i \cot \varphi) U = U^* T_{11}^{2} U
\]
\[
+ \cot^2 \varphi \cdot U^* U - U^*(T_{11} - i \cot \varphi) \cdot D_{V}^{-2} \cdot (T_{11} + i \cot \varphi) U.
\]
Comparing (4.14) with (4.16) and noting that
\[
\frac{1}{\sin^2 \varphi} \cdot D^2_U + \cot^2 \varphi \cdot U^* U = D^2_U + \cot^2 \varphi \cdot (D^2_U + U^* U) = D^2_U + \cot^2 \varphi \cdot I,
\]
we arrive at the equality
\[
A_\mp = G_{22}^\pm - (G_{11}^{-1/2} G_{12}^\pm)^* (G_{11}^{-1/2} G_{12}^\pm),
\]
coinciding with (4.15).

\((ii_2)\) Now we free ourselves of the additional restriction \(0 \in \rho(D_{V^*}) \cap \rho(D_{T_{11}})\). Consider the strict contractions \(rT_1 = (r_{T_{11}} T_{21}), \ r \in (0, 1)\). We have \(rT_21 = V(r) D_{rT_{11}}, \ where\)
\[
V(r) := rV D_{T_{11}} D_{rT_{11}}^{-1} = rV(I - T_{11}^* T_{11})^{1/2} \cdot (I - r^2 T_{11}^* T_{11})^{-1/2}.
\]
Let us define the operator \(U(r)\) by (3.30) with \(V\) replaced by \(V(r)\), but not replacing \(M\). Then the operator
\[
Q_0(r) := D_{V^*(r)}^{-1} \cdot (I - V(r)U(r)) \cdot D_{U(r)}^{-1}
\]
is bounded and \(Q_0(r) \cos \varphi\) is contractive. We set \(Q(r) := \overline{Q_0(r)}\) and note that \(Q(r) \cos \varphi \in C(\pi/2)\).

Next, starting with \(U(r)\) we define a dual pair of Hermitian contractions \(\{rT_1, T_2(r)\}\) by setting
\[
T_{21}(r) := D_{rT_{11}} U(r) \quad and \quad T_2(r) := \begin{pmatrix} rT_{11} \\ T_{21}(r) \end{pmatrix}.
\]
Denote by \(T_K(r) \in \text{Ext}_{\{rT_1, T_2(r)\}}\) the extension of \(\{T_1, T_2\}\) defined by the same operator \(K\), as the extension \(T_K \in \text{Ext}_{\{r, T_2\}}\), that is
\[
T_K(r) := \begin{pmatrix} rT_{11} \\ T_{21} \\ T_{22}(r) \end{pmatrix}, \quad T_{22}(r) := -rV(r) T_{11} U(r) + D_{V^*(r)} K D_{U(r)}.
\]
Since \(0 \in \rho(D_{rT_{11}}) \cap \rho(D_{V(r)})\) and \(Q(r) \cos \varphi \in C(\pi/2)\), then for the operator-matrix
\[
(G_{ij}^\pm(r))_{i,j=1} := I - T_K^* T_K(r) \pm i \cos \varphi (T_K(r) - T_K^*(r))
\]
equality (4.3) is already proved in the previous step, that is
\[
\sin^2 \varphi \cdot [G_{22}^\pm(r) - (G_{12}^\pm(r))^* G_{11}^{-1}(r) G_{12}(r)]
= D_{U(r)} \cdot [I - (K^* \sin \varphi \pm i Q(r) \cos \varphi) \cdot (K \sin \varphi \mp i Q(r) \cos \varphi)] \cdot D_{U(r)}.
\]

It remains to justify the possibility to pass to the limit in (4.22) as \(r \to 1\). We may assume without restriction of generality that \(\ker G_{11} = \{0\}\). Then, as it follows from (4.7), (4.8) and (4.11),
\[
G_{11}^{1/2} = U_1 D_V D_{T_{11}} = D_{T_{11}} D_V U_1^*, \quad G_{11}^{-1/2} G_{12}^\pm = -U_1 X_\pm, \quad (G_{11}^{1/2} G_{12}^\pm)^* = -X_\pm U_1^*,
\]
where the operator \(U_1\) is unitary.

Further, introducing the operators
\[
Y(r) := (\sin^2 \varphi \cdot D_{V^*(r)}^2 + \cos^2 \varphi)^{-1/2} \quad and \quad Y_\pm(r) := (\sin^2 \varphi \cdot D_{V^*(r)} + \cos^2 \varphi)^{-1/2},
\]
one derives from the definition of the operator \(U^*(r) (r < 1)\) that
\[
U^*(r) = V(r) = \sin \varphi D_{V^*(r)} Y_\pm(r) \cdot (M^* - r \sin \varphi \cdot V(r)) \cdot Y(r) D_{V(r)}.
\]
Next, we define $G_{11}^{1/2}(r)$ and $G_{21}^\pm(r)$ by (4.24) and (4.28) with $V(r)$, $U(r)$ and $rT_{11}$ in place of $V$, $U$ and $T_{11}$ respectively. Further, similarly to definition (4.11) of $X_\pm$ we set

$$X_\pm(r) := rU^*(r)T_{11}D_V(r) + D_{V(r)}K^*V(r) \mp i \cos \varphi \cdot Y_\pm(r)D_{V^*(r)}(M^* - r \sin \varphi \cdot V(r))Y(r).$$

Combining these definitions we arrive at the relations

$$G_{11}^{1/2}(r) = U_1(r)D_V(r)D_{rT_{11}} \text{ and } G_{21}^\pm(r) = X_\pm(r)D_{V(r)}D_{rT_{11}},$$

which are analogous to that of (4.23). Here $U_1(r)$, $r \in (0, 1)$, is a family of unitary operators. Hence

$$-(G_{12}^\pm(r))^*G_{11}^{-1/2}(r) = X_\pm(r)U^*_1(r), \quad -G_{11}^{-1/2}G_{12}^\pm(r) = U_1(r)X_\pm(r).$$

It follows from (4.18) that $s - \lim_{r \to 1} V(r) = V$ and $s - \lim_{r \to 1} V^*(r) = V^*$. Hence and taking into account (4.24) we get

$$s - \lim_{r \to 1} D_V(r) = D_V, \quad s - \lim_{r \to 1} D_{V^*(r)} = D_{V^*},$$

$$s - \lim_{r \to 1} Y(r) = Y, \quad s - \lim_{r \to 1} Y_\pm(r) = Y_\pm.$$

Relations (4.25), (4.29) and (4.10) yield

$$s - \lim_{r \to 1} U(r) = U, \quad s - \lim_{r \to 1} U^*(r) = U^*, \quad s - \lim_{r \to 1} D_U(r) = D_U.$$

It follows from (4.20) and (4.21) that

$$s - \lim_{r \to 1} G_{22}(r) = G_{22}.$$

Further, (4.26) and (4.11) yield $s - \lim_{r \to 1} X_\pm(r) = X_\pm$ and $s - \lim_{r \to 1} X^*_\pm(r) = X^*_\pm$. Therefore combining relations (4.22) with (4.28) and taking into account the obvious identities $U^*_1(r)U_1(r) = U^*_1U_1 = I$ we arrive at

$$s - \lim_{r \to 1}(G_{12}^\pm(r))^*G_{11}^{-1}(r)G_{12}^\pm(r) = s - \lim_{r \to 1} X_\pm(r)X^*_\pm(r) = X_\pm X^*_\pm = (G_{11}^{-1/2}G_{12}^\pm)^*G_{11}^{-1/2}G_{12}^\pm.$$

Relations (4.31) and (4.32) allow us to pass to the limit in left-hand side of (4.22) as $r \to 1$. So, it remains to justify passage to the limit in the right-hand side of (4.22). In turn it suffices to prove the relations

$$s - \lim_{r \to 1} Q(r)D_U(r) = QD_U \quad \text{and} \quad s - \lim_{r \to 1} D_{U^*(r)}Q^*(r) = D_UQ^*.$$ 

We derive from (4.25) and (4.13) that

$$Q(r)D_{U(r)} = D_{V^*(r)}^{-1}(I - V(r)U(r))$$

$$= \{I - \sin \varphi Y(r)V(r)(M - r \sin \varphi \cdot V^*(r))Y_\pm(r)\}D_{V^*(r)}.$$

It follows from (4.29) that there exists the limit of the right-hand side of (4.34) as $r \to 1$. Hence there exist the limit of the left-hand side of (4.34) as $r \to 1$. Moreover, the first of relations (4.33) is now implied by (4.34) and similar formula for $QD_U$ which follows from (4.10). The second formula in (4.33) may be proved similarly.

Finally, passing to the limit in (4.22) as $r \to 1$ and taking into account (4.31), (4.32) and (4.33) we arrive at (4.5). Relation (4.14) may be proved in just the same way. □
4.2. Descriptions of the classes \( C(\pi/2; \kappa_{\pm}) \) and \( T \in C(\pi/2; \mathcal{G}^{\pm}) \)

Here we present some corollaries from Theorem 4.1. To formulate them we need some definitions and an elementary lemma.

Let \( \kappa(t) \) be the number of negative squares of the symmetric quadratic form \( t \), that is, the maximum dimensions of the "negative" linear manifolds

\[
L_{-} = \{ f \in \mathcal{D}(t) \setminus \{0\} : t[f] < 0 \} \cup \{0\}.
\]

For any selfadjoint operator \( T = T^{*} \in \mathcal{C}(\mathcal{H}) \) with the resolution of the identity \( E_{T}() \) we let \( T_{-} := E_{T}(-\infty, 0)T \) and \( \kappa(T) := \dim(\text{ran}T_{-}) = \dim E_{T}(-\infty, 0)\mathcal{H} \). If the form \( t \) is closed and \( T \) is the operator associated with it, \( t = t_{T} \), (see \[23\]) then by virtue of the minimax principle \( \kappa(t) = \kappa(T) \).

Next we define the classes \( C(\pi/2; \kappa_{\pm}) \) and \( T \in C(\pi/2; \mathcal{G}^{\pm}) \).

**Definition 4.2.** Let \( \kappa \in \mathbb{Z}_{+} \) and \( \mathcal{G} \) be a two-sided ideal in \([\mathcal{H}]\). We write

(a) \( T \in C(\pi/2; \kappa) \) if \( T \in [\mathcal{H}, \mathcal{H}] \) and \( \kappa(I - T^{*}T) = \kappa \);

(b) \( T \in C(\pi/2; \mathcal{G}) \) if \( T \in [\mathcal{G}, \mathcal{H}] \) and \( (I - T^{*}T)_{-} \in \mathcal{G} \).

**Definition 4.3.** Let \( \varphi \in [0, \pi/2], \kappa_{\pm} \in \mathbb{Z}_{+} \), and let \( \mathcal{G}^{\pm} \) be two-sided ideals in \([\mathcal{H}]\). An operator \( T(\in [\mathcal{H}]) \) is put

(a) in the class \( C_{\mathcal{H}}(\varphi; \kappa_{\pm}) \) with \( \varphi \in (0, \pi/2] \), if

\[
T \sin \varphi \pm i \cos \varphi \cdot I \in C_{\mathcal{H}}(\pi/2; \kappa_{\pm});
\]

(b) in the class \( C_{\mathcal{H}}(\varphi; \mathcal{G}^{\pm}) \) with \( \varphi \in (0, \pi/2] \), if

\[
T \sin \varphi \pm i \cos \varphi \cdot I \in C_{\mathcal{H}}(\pi/2; \mathcal{G}^{\pm}).
\]

(c) in the class \( C_{\mathcal{H}}(0; \varphi) \ (C_{\mathcal{H}}(0; \mathcal{G})) \), if \( T = T^{*} \) and \( \kappa(T) = \kappa \) \((T_{-} \in \mathcal{G})\).

We write \( C_{\mathcal{H}}(\varphi; \kappa) \) and \( C_{\mathcal{H}}(\varphi; \mathcal{G}) \) in place of \( C_{\mathcal{H}}(\varphi; \kappa_{\pm}) \) and \( C_{\mathcal{H}}(\varphi; \mathcal{G}^{\pm}) \) respectively if \( \kappa := \kappa_{+} = \kappa_{-} \) and \( \mathcal{G} := \mathcal{G}^{+} = \mathcal{G}^{-} \).

Observe that the class \( C_{\mathcal{H}}(\pi/2; \kappa_{\pm}) \) is not empty only if \( \kappa_{+} = \kappa_{-} \). Some properties of the class \( C(\pi/2; \mathcal{G}_{\infty}) \) can be found in \[30\].

**Lemma 4.4.** \[23\] \[21\] Let \( T_{1} = \begin{pmatrix} T_{11}^{+} \hfill T_{12}^{+} \\ T_{12}^{+} \hfill T_{11}^{-} \end{pmatrix} \in [\mathcal{H}_{1}, \mathcal{H}_{2}] \) be a nonnegative symmetric operator \( \iff T_{11} \geq 0 \) admitting a bounded nonnegative selfadjoint extension and let \( T(\in [\mathcal{H}]) \) be any selfadjoint extension of \( T_{1} \) with the block-matrix representation \( T = T^{*} = (T_{ij})_{i,j=1}^{2} \) with respect to the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_{1} \oplus \mathcal{H}_{2} \). Then

(i) \( \mathcal{R}(T_{11}^{1/2}) \supset \mathcal{R}(T_{12}) \) and the operator \( S := T_{11}^{-1/2}T_{12} \) is well-defined and bounded;

(ii) \( \kappa_{-(T)} = \kappa_{-(T_{22} - S^{*}S)} \). In particular, \( T \geq 0 \) iff \( T_{22} - S^{*}S \geq 0 \).

Now we are ready to present the corollaries.

**Corollary 4.5.** Let \( \{T_{1}, T_{2}\} \) be a dual pair of symmetric contractions, \( T_{K} \in \text{Ext}_{\{T_{1}, T_{2}\}} \), \( \varphi \in [\varphi_{1}, \pi/2] \) and \( \varphi_{1} > 0 \). Then the following equivalences are valid:

\[
T_{K} \in C_{\mathcal{H}}(\varphi; \kappa_{\pm}) \iff K \sin \varphi \mp iQ \cos \varphi \in C(\pi/2; \kappa_{\pm}).
\]

**Proof.** Let as in Theorem 4.1

\[
(G^{\pm})_{i,j=1}^{2} := G^{\pm} := I - T_{K}^{*}T_{K} \pm i \cot \varphi(T_{K} - T_{K}^{*}).
\]
The operators \( G_0^\pm : = \begin{pmatrix} G_{11}^{1/2} & 0 \\ G_{21}^{1/2} & 0 \end{pmatrix} \) are nonnegative \( (\iff G_{11} \geq 0) \). Moreover, both of them admit bounded nonnegative selfadjoint extensions. For example, the operator

\[
\begin{pmatrix}
G_{11}^{1/2} & 0 \\
G_{21}^{1/2} & 0 
\end{pmatrix}
\begin{pmatrix}
G_{11}^{1/2} & B_+ \\
0 & 0 
\end{pmatrix}
\begin{pmatrix}
G_{11}^+ & G_{12}^+ \\
G_{21}^+ & G_{22}^+ 
\end{pmatrix}
\geq 0
\]

with a bounded \( B_+ = G_{11}^{1/2}G_{12}^+ \) is a nonnegative extension of \( G_0^+ \). Therefore combining Lemma 4.4 with Theorem 4.1 (see equality (4.5)) we get

\[
\kappa(I - \rho^{*}TK) = \kappa(G_{22}^+ - (G_{11}^{1/2}G_{12}^+)\ast(G_{11}^{1/2}G_{12}^+)) = \kappa(I - K_+K_-),
\]

where \( K_\pm : = K\sin\phi \pm iQ\cos\phi \).

**Corollary 4.6.** Let \( \{T_1, T_2\} \) be a dual pair of symmetric contractions and \( 0 \in \rho(D_{T_1}) \cap \rho(D_{T_2}) \). Suppose additionally that \( \varphi \in [\varphi_1, \pi/2] \) and \( \varphi_1 > 0 \). Then the following implications hold

(4.37) \( K \sin \varphi \pm Q \cos \varphi \in C(\pi/2; \mathcal{G}^\pm) \implies T_K \in C_\delta(\varphi; \mathcal{G}^\pm) \).

If additionally \( 0 \in \rho(D_U) \) then implications (4.37) turns into the equivalences.

**Proof.** The required assertion immediately follows from (4.5) and the identity

\[
\begin{pmatrix}
I & 0 \\
-G_{21}^+G_{11}^{-1} & I 
\end{pmatrix}
\begin{pmatrix}
G_{11}^+ & G_{12}^+ \\
G_{21}^+ & G_{22}^+ 
\end{pmatrix}
\begin{pmatrix}
I & -G_{11}^{-1}G_{12}^+ \\
0 & I 
\end{pmatrix}
\begin{pmatrix}
G_{11}^+ & 0 \\
G_{12}^+ & 0 
\end{pmatrix}
\geq 0
\]

\( \Box \)

**Remark 4.7.** (i) Let \( \varphi = \pi/2 \). Then both relations (4.36) and (4.37) are simplified and take the form

(4.38) \( K \in C(\pi/2; \kappa) \iff T_K \in C(\pi/2; \kappa) \),

(4.39) \( (I - K^*K)_- \in \mathcal{G} \implies (I - T_K^*T_K)_- \in \mathcal{G} \).

Both relations have been established in [28, 30] for any (not necessarily symmetric) dual pair of contractions.

(ii) Let \( \varphi_1 = \arccos(\|Q_0\|^{-1}) = 0 \). Then \( \|Q_0\| = 1 \) and by Remark 3.3 (see (3.24)) \( U = V^* \), that is \( Q = I \) and \( T_1 = T_2 \). In this case description of the sets \( \text{Ext}_{T_1}(\varphi; \kappa) = \text{Ext}_{\{T_1,T_1\}}(\varphi; \kappa) \) and \( \text{Ext}_{T_2}(\varphi; \mathcal{G}) = \text{Ext}_{\{T_1,T_1\}}(\varphi; \mathcal{G}) \), \( \varphi \in [0, \pi/2] \), can easily be derived from Corollaries 4.5 and 4.6. Now in place of relations (4.36) and (4.37) we have

(4.40) \( K \in C_H(\varphi; \kappa^\pm) \iff T_K \in C_\delta(\varphi; \kappa^\pm) \),

(4.41) \( K \in C_H(\varphi; \mathcal{G}^\pm) \implies T_K \in C_\delta(\varphi; \mathcal{G}^\pm) \),

where \( H \subseteq \overline{\text{ran}} D_U = \overline{\text{ran}} D_{V^*} \). Both formulas have earlier been obtained in [28, 30]. Note also, that if \( \kappa^\pm = 0 \) then formula (4.40) gives one more proof of Corollary 3.11.
Definition 4.8. Let \( \varphi \in [0, \pi/2) \), \( \mathcal{Z}^{\pm} \subseteq \mathbb{Z}_{+} \), \( \mathcal{G}^{\pm} \) two-sided ideals in \( \mathfrak{H} \), and \( B \in \mathcal{C}(\mathfrak{H}) \). Let further, the quadratic forms
\[
t_{\pm}[f] = \text{Re}(Bf, f) \pm \cot \varphi \cdot \text{Im}(Bf, f), \quad f \in \text{dom} \, B,
\]
be semibounded below and \( B^{\pm} \) the linear operators associated with their closures (the closability of the form \( t_{\pm} \) is a consequence of their semiboundedness (see (2.14)). We write
\[
(a) \quad B \in S_{\mathcal{B}}(\varphi; \mathcal{Z}^{\pm}), \text{ if } \rho(B) \notin \mathcal{G} \text{ and } \mathcal{Z}(B^{\pm}) = \mathcal{Z}^{\pm}; \\
(b) \quad B \in S_{\mathcal{B}}(\varphi; \mathcal{G}^{\pm}), \text{ if } (B^{\pm})_{-} \in \mathcal{G}^{\pm} \text{ and } \rho(B) \notin \mathcal{G}.
\]
A closed linear relation \( \theta \) in \( \mathfrak{H} \) is also put in the class \( S_{\mathcal{B}}(\varphi; \mathcal{Z}^{\pm}) \) \( (S_{\mathcal{B}}(\varphi; \mathcal{G}^{\pm})) \) if \( \text{Re}(f, f') \geq \beta \|f\|^2 \) for all \( \{f, f'\} \in \theta \) (with some \( \beta \in \mathbb{R} \) and its operator part is in \( S_{\mathcal{B}}(\varphi; \mathcal{Z}^{\pm}) \) \( (S_{\mathcal{B}}(\varphi; \mathcal{G}^{\pm})) \).

It is clear that the classes \( C_{\mathcal{B}}(\varphi; \mathcal{Z}^{\pm}) \) and \( S_{\mathcal{B}}(\varphi; \mathcal{Z}^{\pm}) \) are connected by means of the linear fractional transformation (2.14). The same is also true for the classes \( C_{\mathcal{B}}(\varphi; \mathcal{G}^{\pm}) \) and \( S_{\mathcal{B}}(\varphi; \mathcal{G}^{\pm}) \).

4.3. Shorted operators. Here we present two additional corollaries from Theorem 4.1 complementing Theorem 4.1. For this purpose we recall some well-known results and the definition of a shorted operator.

Definition 4.9. \((25)\) For any nonnegative operator \( A(\in [\mathfrak{H}]) \) and a subspace \( \mathfrak{M}(\subset \mathfrak{H}) \) there exists the largest element in the set of all bounded operators not exceeding \( A \) and annihilating \( \mathfrak{M}^{\perp} = \mathfrak{H} \ominus \mathfrak{M} \). This element, is denoted by \( A_{\mathfrak{M}} \) and is called the shorted to \( \mathfrak{M} \) operator.

The transformation \( A \rightarrow A_{\mathfrak{M}} \) is called the Krein transformation.

Lemma 4.10. \((25, 37, 21)\). Let \( A = (A_{ij})_{i,j=1}^{2} \) be a block-matrix representation of an operator \( A \geq 0 (A \in [\mathfrak{H}] \) with respect to the decomposition \( \mathfrak{H} = \mathfrak{H}_{1} \oplus \mathfrak{M} \). Then the shorted to \( \mathfrak{M} \) operator \( A_{\mathfrak{M}} \) is of the form
\[
A_{\mathfrak{M}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{22} - S^{*}S \end{pmatrix}, \quad S = A_{11}^{-1/2}A_{12}.
\]
If in addition \( 0 \in \rho(A_{11}) \) then \( S^{*}S = A_{21}A_{11}^{-1}A_{12} \).

Corollary 4.11. \((25, 26, 37)\). Let \( \mathfrak{H} = \mathfrak{H}_{1} \oplus \mathfrak{M} \), \( A \in [\mathfrak{H}] \) and \( A \geq 0 \). Then
\[
\inf_{g \in \mathfrak{H}_{1}} (A(f - g), f - g) = (A_{\mathfrak{M}}f, f), \quad f \in \mathfrak{H}.
\]

Corollary 4.12. Let \( \{T_{1}, T_{2}\} \) be a dual pair of symmetric contractions in \( \mathfrak{H} = \mathfrak{H}_{1} \oplus \mathfrak{H}_{2} \), \( \varphi \in [\varphi_{1}, \pi/2] \) where \( \varphi_{1} := \arccos(||Q_{0}||^{-1}) \) and \( T_{K} \in \text{Ext} \{T_{1}, T_{2}\}(\varphi) \). Then the shorted to \( \mathfrak{M} := \mathfrak{H}_{2} \) operators
\[
G^{\pm} = I - T_{K}^{*}T_{K} \pm i \cot \varphi(T_{K} - T_{K}^{*}) \quad \text{and} \quad S^{\pm} = I - T_{K}T_{K}^{*} \pm i \cot \varphi(T_{K} - T_{K}^{*})
\]
have the following form
\[
(G^{\pm})_{\mathfrak{M}} = \begin{pmatrix} 0 & 0 \\ 0 & \sin^{-2} \varphi \cdot D_{U}[I - (K^{*} \sin \varphi \pm iQ^{*} \cos \varphi)(K \sin \varphi \mp iQ \cos \varphi)]D_{U} \end{pmatrix}
\]
and
\[
(S^{\pm})_{\mathfrak{M}} = \begin{pmatrix} 0 & 0 \\ 0 & \sin^{-2} \varphi \cdot D_{V^{*}}[I - (K \sin \varphi \mp iQ \cos \varphi)(K^{*} \sin \varphi \pm iQ^{*} \cos \varphi)]D_{V^{*}} \end{pmatrix}.
\]

Proof. One deduces the proof combining Theorem 4.1 with Lemma 4.10. \( \square \)
Corollary 4.13. Let \( \{T_1, T_2\} \) be a dual pair of Hermitian contractions in \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \), and \( T_K \in \text{Ext} \{T_1, T_2\}(\varphi) \). Suppose additionally that \( \varphi_1 = \arccos(\|Q_0\|^{-1}) < \pi/2 \), \( \varphi \in [\varphi_1, \pi/2] \) and \( \mathcal{G}^\pm \) are two-sided ideals in \( \mathcal{H} \). Then

(i) \( \text{Ext} \{T_1, T_2\}(\varphi) \neq \emptyset \) if and only if \( \varphi \in [\varphi_1, \pi/2] \);

(ii) The following equivalence holds

\[
T_K \in \text{Ext} \{T_1, T_2\}(\varphi) \iff K_\pm := K \sin \varphi \pm iQ \cos \varphi \in C(\pi/2);
\]

(iii) The following implications hold with \( \mathcal{R} := \mathcal{H}_2 \)

\[
D_{K_\pm}^2 \in \mathcal{G}^\pm \implies (G^\pm)_\mathcal{R} \in \mathcal{G}^\pm,
\]

\[
D_{K_\pm}^2 \in \mathcal{G}^\pm \implies (S^\pm)_\mathcal{R} \in \mathcal{G}^\pm.
\]

If additionally \( 0 \in \rho(D_U) \) then implications \((4.48)\) turn into the equivalences.

Proof. (i)-(ii) By definition \( T_K \in C_\mathcal{H}(\varphi) \) if and only if \( G^\pm \in C_\mathcal{H}(\pi/2) \), where \( G^\pm \) are defined by \((4.44)\). Note that \( G^\pm_{11} = G_{11} = I - T_{11}^* T_{11} - T_{21}^* T_{21} \geq 0 \) since \( T_i \) is a contraction. Therefore by Sylvester criterion (see Lemma 4.4) \( G^\pm \geq 0 \) iff \((G^\pm)_\mathcal{R} \geq 0 \) with \( \mathcal{R} = \mathcal{H}_2 \). Combining this inequality with \( (4.43) \) we arrive at equivalence \((4.47)\).

Hence, if \( \text{Ext} \{T_1, T_2\}(\varphi) \neq \emptyset \) then \( Q \cos \varphi \in C(\pi/2) \), that is \( \varphi \in [\varphi_1, \pi/2] \). Conversely, if \( \varphi \in [\varphi_1, \pi/2] \) then \( Q \cos \varphi \in C(\pi/2) \) and the operator \( T_0 \), that is \( T_K \) with \( K = 0 \), belongs to \( \text{Ext} \{T_1, T_2\}(\varphi) \).

(iii) This statement is immediately implied by formulas \((4.45)\) and \((4.46)\). \( \square \)

Remark 4.14. (i) Suppose that in Corollary 4.12 \( \varphi = \pi/2 \). Then \( G^\pm = I - T_{rK}^* T_K = D_{rK}^2 \) and \( S^\pm = I - T_K T_{rK}^* = D_{T_K}^2 \). Now formulas \((4.45)\) and \((4.46)\) are simplified and take the form

\[
(D_{T_K}^2)_{\mathcal{R}} = \begin{pmatrix} 0 & 0 \\ D_U D_{T_K}^2 D_U & 0 \end{pmatrix}, \quad (D_{T_K}^2)_{\mathcal{R}'} = \begin{pmatrix} 0 & 0 \\ D_{V'} D_{T_K}^2 D_{V'} & 0 \end{pmatrix}.
\]

where \( \mathcal{R} := \mathcal{H}_2 \) and \( \mathcal{R}' := \mathcal{H}'_2 \). Both formulas have earlier been obtained in \([28, 30]\) for any (not necessary symmetric) dual pair of contractions.

(ii) Corollary \((4.13)\) (iii) complements Theorem \(3.4\). Moreover, Corollary \((4.13)\) gives another proof of Theorem \(3.4\). Indeed, in the case \( 0 \in \rho(D_U) \cap \rho(D_{V'}) \) the proof of Theorem \(3.4\) does not depend on Theorems \(3.4\) and \(3.14\). The proof of equivalence \((4.47)\) without the additional assumption \( 0 \in \rho(D_U) \cap \rho(D_{V'}) \) can easily be obtained by considering the family \( \{rT_1, rT_2\}, \ r \in (0, 1) \), of dual pairs of contractions and passage to the limit as \( r \to 1 \) (cf. the proof of Theorem \(3.14\)).

5. Completions of a Special Triangular Operator-Matrix.

5.1. A Complement to the S. Nagy and C. Foias Result. Here we describe the operators \( T_{12} \) completing the incomplete contractive operator block-matrix

\[
\begin{pmatrix} T_{11} & * \\ 0 & T_{22} \end{pmatrix}
\]

to form an operator matrix of some class.

We start with the following S. Nagy and C. Foias result.
Corollary 5.3. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}'_1 \oplus \mathcal{H}'_2$ and let $T_{jj}(\in [\mathcal{H}_j, \mathcal{H}'_j])$ be a contraction, $j \in \{1, 2\}$. Then the family of operators $T_{12} \in [\mathcal{H}_2, \mathcal{H}_1]$, completing the block-matrix $(5.1)$ to a contractive matrix $T = (T_{ij})_{i,j=1}^2 \in [\mathcal{H}]$, forms an operator ball $B(0; D_{T_{11}}, D_{T_{22}})$, that is

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \in C_\mathcal{H}^\pi/2 \iff T_{12} = D_{T_{11}}K D_{T_{22}}, \quad \|K\| \leq 1, \quad K \in [\mathcal{H}_2, \mathcal{H}_1].$$

Proof. Let at first $0 \in \rho(D_{T_{12}})$. Then according to the Sylvester criterion the equivalence

$$I - T^*T \geq 0 \iff D_{T_{22}}^2 - Z^*(I + T_{11}D_{T_{11}}^2 T_{11}^*)Z \geq 0,$$

holds true with $Z := T_{12}$. By Lemma 2.3 the set of solutions of $(5.2)$ forms an operator ball. Observing that $I + T_{11}D_{T_{11}}^{-2} T_{11}^* = D_{T_{11}}^{-2}$ and applying Lemma 2.3 to $(5.2)$ we arrive at the required relation

$$T_{12} = Z = D_{T_{11}}K D_{T_{22}}, \quad \|K\| \leq 1, \quad K \in [\mathcal{H}_2, \mathcal{H}_1].$$

We may easily free ourselves of the condition $0 \in \rho(D_{T_{12}})$ by virtue of passage to the limit. \□

Thus the contractive ”completions” of the matrix $(5.1)$ are of the form

$$(5.3) \quad T_K = \begin{pmatrix} T_{11} & D_{T_{11}}K D_{T_{22}} \\ 0 & T_{22} \end{pmatrix}$$

with $\|K\| \leq 1, \quad K \in [\mathcal{H}_2, \mathcal{H}_1]$. Let us now consider ”completions” of the incomplete block-matrix $(5.1)$ of the form $(5.3)$, not assuming the operator $K$ to be a contraction.

Proposition 5.2. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}'_1 \oplus \mathcal{H}'_2$, $T_{jj} \in [\mathcal{H}_j, \mathcal{H}'_j]$, and $\|T_{jj}\| \leq 1, \quad j \in \{1, 2\}$. Assume that $T_K$ is an operator matrix of the form $(5.3)$ with $K \in [\mathcal{H}_2, \mathcal{H}_1]$ and put $G := (G_{ij})^2_{i,j=1} = I - T_K^*T_K$ and $S := (S_{ij})^2_{i,j=1} = I - T_K^*T_K^*$. Then

(i) $\text{ran}(G_{12}) \subset \text{ran}(G_{11}^{1/2})$ and $\text{ran}(S_{12}) \subset \text{ran}(S_{11}^{1/2})$, hence the operators $G_{11}^{-1/2}G_{12}$ and $S_{11}^{-1/2}S_{12}$ are well defined and bounded;

(ii) the following identities are valid:

$$G_{22} - (G_{11}^{-1/2}G_{12})^*(G_{11}^{-1/2}G_{12}) = D_{T_{22}}(I - K^*K) D_{T_{22}},$$

$$S_{22} - (S_{11}^{-1/2}S_{12})^*(S_{11}^{-1/2}S_{12}) = D_{T_{11}}(I - KK^*) D_{T_{11}}.$$

Proof. Imposing the condition $0 \in \rho(D_{T_{11}})$ we have

$$G_{22} - G_{12}^*G_{11}^{-1}G_{12} = D_{T_{22}}(I - K^*D_{T_{11}}^*K) D_{T_{22}} - D_{T_{22}}K^*D_{T_{11}}^*T_{11}D_{T_{11}}^2 T_{11}^*D_{T_{11}} K D_{T_{22}} = D_{T_{22}}(I - K^*K + K^*T_{11}T_{11}^*K - K^*T_{11}T_{11}^*K) D_{T_{22}} = D_{T_{22}}D_{K}^2 D_{T_{22}}.$$ We may free ourselves of the condition $0 \in \rho(D_{T_{11}})$ by passage to the limit just like it was done in the proof of Theorem 3.14. Equality $(5.5)$ may be proved similarly. \□

Corollary 5.3. Suppose that conditions of Proposition 5.2 are satisfied. Then

(i) the following equivalence holds

$$K \in C(\pi/2; \mathcal{H}) \cap [\mathcal{H}_2, \mathcal{H}_1] \iff T_K \in C_\mathcal{H}^\pi/2;$$
(ii) if in addition \(0 \in \rho(G_{11})\), then for any two-sided ideal \(\mathcal{G}\) in \([\mathcal{H}]\) the following implication holds

\[ K \in C(\pi/2; \mathcal{G}) \implies T_K \in C_{S_2}(\pi/2; \mathcal{G}). \]

This implication turns into the equivalence if additionally \(0 \in \rho(D_{T_{22}})\).

**Corollary 5.4.** Let \(T_K\) be a contraction of the form \(5.3\). Then the operators \(G := D^2_{T_K}\) and \(S := D^2_{T'_K}\) shorted to \(\mathcal{M} := S_2\) and \(\mathcal{M}' := S'_2\) respectively have the form

\[
G_{\mathcal{M}} = \begin{pmatrix}
0 & 0 \\
D_{T_{22}} D^2_{T_K} D_{T_{22}} & 0
\end{pmatrix} \quad \text{and} \quad S_{\mathcal{M}'} = \begin{pmatrix}
0 & 0 \\
D_{T_{11}} D^2_{T_K} D_{T'_{11}} & 0
\end{pmatrix}.
\]

Corollaries 5.3 and 5.4 may be derived from Proposition 5.2 just like Corollaries 4.5, 4.6 and 4.12 from Theorem 4.1.

**5.2. A solution to Yu. L. Shmul’yan’s problem.**

The following proposition provides an answer to the Yu. L. Shmul’yan question.

**Proposition 5.5.** Let \(\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2\), \(T_{jj} \in C_{S_j}(\varphi)\), \(j \in \{1, 2\}\), and \(\varphi \in (0, \pi/2)\). Then

1. there exist contractions \(U_\varphi = U_\varphi^* \in C_{S_j}(\pi/2)\) and \(V_\varphi = V_\varphi^* \in C_{S_j}(\pi/2)\) such that

\[
2 \cot \varphi(\text{Im } T_{11}) = D_{T_{11}^*} U_{\varphi} D_{T_{11}^*} \quad \text{and} \quad 2 \cot \varphi(\text{Im } T_{22}) = D_{T_{22}} V_{\varphi} D_{T_{22}};
\]

2. the following equivalence holds true

\[
T = \begin{pmatrix}
T_{11} & T_{12} \\
0 & T_{22}
\end{pmatrix} \in C_{S_j}(\varphi) \iff
T_{12} = \sin \varphi D_{T_{11}^*} K D_{T_{22}}, \quad (I \pm U_{\varphi})^{-1/2} K (I \pm V_{\varphi})^{-1/2} \in C(\pi/2), \quad K \in [\mathcal{H}_2, \mathcal{H}_1].
\]

**Proof.** (i) Equalities \((5.6)\) have already been proved in Proposition 3.8.

(ii) The inclusion \(T \in C_{S_j}(\varphi)\) means that \(T \sin \varphi \pm i \cos \varphi \cdot I \in C_{S_j}(\pi/2)\). Since \(T_{jj} \sin \varphi \pm i \cos \varphi \cdot I \in C_{S_j}(\pi/2)\), \(j \in \{1, 2\}\), then by Proposition 5.1 the equivalences

\[
T \sin \varphi \pm i \cos \varphi \cdot I \in C(\pi/2) \iff T_{12} \sin \varphi = R_{T}^\pm K_{T}^\pm R_{\varphi}^\pm, \quad ||K_{T}^\pm|| \leq 1,
\]

hold true. Here \(K_{T}^\pm \in [\mathcal{H}_T^\pm, \mathcal{H}_T^\pm]\), \(\mathcal{H}_T^\pm = \text{ran}(R_{T}^\pm), \quad \mathcal{H}_T^\pm = \text{ran}(R_{\varphi}^\pm)\), and the operators \(R_{T}^\pm\) and \(R_{\varphi}^\pm\) are defined by

\[
(R_{T}^\pm)^2 = \sin^2 \varphi D^2_{T_{11}} \pm \sin 2\varphi \cdot (\text{Im } T_{11}) = \sin^2 \varphi \cdot D_{T_{11}} (I \pm U_{\varphi}) D_{T_{11}},
\]

\[
(R_{\varphi}^\pm)^2 = \sin^2 \varphi D^2_{T_{22}} \pm \sin 2\varphi \cdot (\text{Im } T_{22}) = \sin^2 \varphi \cdot D_{T_{22}} (I \pm V_{\varphi}) D_{T_{22}}.
\]

It is clear that

\[
R_T^2 := (R_T^\pm)^2 + (R_T^-)^2 = 2 \sin^2 \varphi D_{T_{11}}^2, \quad R_{\varphi}^2 := (R_{\varphi}^\pm)^2 + (R_{\varphi}^-)^2 = 2 \sin^2 \varphi D_{T_{22}}^2.
\]

Further, relations \((5.9)\) yield polar representations for the operators \(\sin \varphi \cdot (I \pm U_{\varphi})^{1/2} D_{T_{11}}\) and \(\sin \varphi \cdot (I \pm V_{\varphi})^{1/2} D_{T_{22}}\). Namely, we have

\[
\sin \varphi \cdot (I \pm U_{\varphi})^{1/2} D_{T_{11}} = U_{\pm} R_{T}^\pm \quad \text{and} \quad \sin \varphi \cdot (I \pm V_{\varphi})^{1/2} D_{T_{22}} = V_{\pm} R_{\varphi}^\pm,
\]

where \(U_{\pm}\) and \(V_{\pm}\) are partial isometries with initial spaces \(\mathcal{H}_T^\pm\) and \(\mathcal{H}_{\varphi}^\pm\) respectively. We deduce the following equalities from \((5.10)\) and \((5.11)\):

\[
R_{T}^\pm = \frac{1}{\sqrt{2}} R_{T}(I \pm U_{\varphi})^{1/2} U_{\pm}, \quad R_{\varphi}^\pm = \frac{1}{\sqrt{2}} V_{\pm}^*(I \pm V_{\varphi})^{1/2} R_{\varphi}.
\]
Taking \((5.12)\) into account we rewrite expression \((5.8)\) for \(T_{12} \sin \varphi\) in the form
\[(5.13)\]
\[
T_{12} \sin \varphi = \frac{1}{2} R_l (I \pm U_\varphi)^{1/2} U_\pm K_\pm V_\pm^* (I \pm V_\varphi)^{1/2} R_r.
\]
It follows from \((5.13)\) that
\[(5.14)\]
\[
(I + U_\varphi)^{1/2} U_+ K_+ V_+^* (I + V_\varphi)^{1/2} = (I - U_\varphi)^{1/2} U_- K_- V_-^* (I - V_\varphi)^{1/2}.
\]
Denoting the operator in the left-hand side of \((5.14)\) by \(K\) and taking into account \((5.10)\) we arrive at the following formula for \(T_{12}\):
\[
T_{12} = \frac{1}{2 \sin \varphi} R_l K R_r = \sin \varphi D T_{11} K D T_{22},
\]
with \((I \pm U_\varphi)^{-1/2} K (I \pm V_\varphi)^{-1/2} \in C(\pi/2)\). Here we have made use of the obvious equivalences
\[
U_\pm K_\pm V_\pm^* \in C(\pi/2) \iff K_\pm \in C(\pi/2).
\]

\( \square \)

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