THE EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR THE CHOQUARD TYPE EQUATION ON GROUPS OF POLYNOMIAL GROWTH

RUOWEI LI

Abstract. In this paper, let $G$ be a Cayley graph of a discrete group of polynomial growth with homogeneous dimension $N \geq 3$. We study the Choquard type equation on $G$:

$$(0.1) \quad \Delta u + (R_\alpha * |u|^p)|u|^{p-2}u = 0,$$

where $\alpha \in (0, N)$, $p > \frac{N+\alpha}{N-2}$ and $R_\alpha$ stands for the Green’s function of the discrete fractional Laplace operator, which has same asymptotics as the Riesz potential. We prove the discrete Hardy-Littlewood-Sobolev inequality on such Cayley graphs, and by the discrete Concentration-Compactness principle we prove the existence of extremal functions for the corresponding Sobolev type inequalities in supercritical cases, which yields a positive ground state solution of (0.1). Moreover, we obtain positive ground state solutions of Choquard type equations with $p$-Laplace, biharmonic and $p$-biharmonic operators etc.

1. Introduction

The nonlinear Choquard equation

$$(1.1) \quad \Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u, \text{ in } \mathbb{R}^N$$

where $I_\alpha(x) = \frac{\Gamma(N/2)}{\Gamma(N/2 - \alpha/2) \pi^{N/2} (2\alpha/\pi)^{\alpha/2}} |x|^{N-\alpha}$ is the Riesz potential, has been extensively studied in the literature. In the physical case $N = 3$, $p = 2$ and $\alpha = 2$, in 1954 the problem appeared in a work by S. I. Pekar describing the quantum mechanics of a polaron at rest [58]. In 1976 P. Choquard used (1.1) to describe an electron trapped in its own hole, in a certain approximation to Hartree–Fock theory of one component plasma [38]. In 1996 R. Penrose proposed (1.1) as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon [51]. The equations of type (1.1) are usually called the nonlinear Schrödinger–Newton equation. If $u$ solves (1.1) then the function $\psi$ defined by $\psi(t, x) = e^{it}u(x)$ is a solitary wave of the focusing time-dependent Hartree equation

$$i\psi_t = -\Delta \psi - (I_\alpha * |\psi|^p)|\psi|^{p-2}\psi, \text{ in } \mathbb{R}_+ \times \mathbb{R}^N.$$
where the integer $N$ growth with the homogeneous dimension $N$ of $G$ and by growth estimate of nilpotent groups [5], for any group $G$. Different finite generating sets $D$ of the group $G$ of the set $S$ is the unit element of $G$, the volume $B_p^S(n)$ of $B_p^S(n)$ is called the growth function of the group, see [28, 31, 52]. As far as we know, there is no existence results for the nonlinear Choquard equation on graphs as the continuous setting. In this article, we prove the existence results on graphs by discrete Concentration-Compactness principle and generalize our previous results on the existence of solutions for $p$-Laplace equations and $p$-biharmonic equations [29, 30] to the Choquard type equations.

Let $(G, S)$ be a Cayley graph of a group $G$ with a finite symmetric generating set $S$, i.e. $S = S^{-1}$. There is a natural metric on $(G, S)$ called the word metric, denoted by $d^S$. Let $B_p^S(n) := \{ x \in G \mid d^S(p, x) \leq n \}$ denote the closed ball of radius $n$ centered at $p \in G$ and denote $| B_p^S(n) | := \# B_p^S(n)$ as the volume (i.e. cardinality) of the set $B_p^S(n)$. When $e$ is the unit element of $G$, the volume $\beta_S(n) := | B_p^S(n) |$ of $B_p^S(n)$ is called the growth function of the group, see [28, 31, 52]. A group $G$ is called of polynomial growth, or of polynomial volume growth, if $\beta_S(n) \leq C n^A$, for any $n \geq 1$ and some $A > 0$, which is independent of the choice of the generating set $S$ since the metrics $d^S$ and $d^{S_1}$ are bi-Lipschitz equivalent for different finite generating sets $S$ and $S_1$. By Gromov’s theorem and Bass’ volume growth estimate of nilpotent groups [5], for any group $G$ of polynomial growth there are constants $C_1(S), C_2(S)$ depending on $S$ and $N \in \mathbb{N}$ such that for any $n \geq 1$, 

$$C_1(S)n^N \leq \beta_S(n) \leq C_2(S)n^N,$$

where the integer $N$ is called the homogeneous dimension or the growth degree of $G$. Since $N$ is a sort of dimensional constant of $G$, we always omit the dependence of $N$ in various constants.

In this paper, we consider the Cayley graph $(G, S)$ of a group of polynomial growth with the homogeneous dimension $N \geq 3$. In particular, $\mathbb{Z}^N$ is a Cayley graph of a free abelian group. We denote by $l^p(G)$ the $l^p$-summable functions on $G$ and by $D_k^{l^p}(G)$ ($k = 1, 2$) the completion of finitely supported functions in the
\[D^{k,p}\] norm, where
\[
\|u\|_{D^{1,p}(G)} := \left\|\left|\nabla u\right|_{p}\right\|^{1/p}_{1(G)} = \left(\sum_{x \in G} \sum_{y \sim x} |\nabla_{xy} u|^p\right)^{1/p},\quad \nabla_{xy} u := u(y) - u(x),
\]

\[
\|u\|_{D^{2,p}(G)} := \|\Delta u\|_{\ell^p(G)},\quad \Delta u(x) := \sum_{y \sim x} \nabla_{xy} u,
\]

see Section 2 for details.

By a standard trick and the isoperimetric estimate [66, Theorem 4.18], the discrete Sobolev inequality (1.2) holds on \(G\), see [32, Theorem 3.6],

\[
\|u\|_{\ell^q} \leq C_p \|u\|_{D^{1,p}},\quad \forall u \in D_0^{1,p}(G),
\]

where \(N \geq 2, 1 \leq p < N, q = p^* := \frac{Np}{N-p}\). Since \(\ell^p(G)\) embeds into \(\ell^q(G)\) for any \(q > p\), see [33, Lemma 2.1], one verifies that the discrete Sobolev inequality (1.2) hold when \(q \geq p^*\). Recalling the continuous setting, it is called the subcritical for \(q < p^*\), critical for \(q = p^*\) and supercritical for \(q > p^*\) for the Sobolev inequality.

Therefore, (1.2) hold in both critical and supercritical cases on \(G\).

The discrete Hardy-Littlewood-Sobolev (HLS for abbreviation) inequality with the Riesz potential \(I_\alpha\) has been proved on \(\mathbb{Z}^N\) [36].

\[
\sum_{x,y \in \mathbb{Z}^N, x \neq y} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \leq C_{r,s,\alpha} \|f\|_{\ell^r} \|g\|_{\ell^s},\quad \forall f \in \ell^r(\mathbb{Z}^N), g \in \ell^s(\mathbb{Z}^N),
\]

where \(r, s > 1, 0 < \alpha < N, \frac{1}{r} + \frac{1}{s} + \frac{N-\alpha}{N} = 2\). Since the Riesz potential \(I_\alpha\) is exactly the Green’s function of the fractional Laplace on \(\mathbb{R}^N\), \(\alpha \in (0, N)\), it is natural to consider the discrete fractional Laplace \((-\Delta)^{\frac{\alpha}{2}}\) and its Green’s function \(R_\alpha\) on \(G\) (see [18] for \(\mathbb{Z}^N\)). The heat kernel \(k_t(x, y)\) of the Laplace operator \(\Delta\) on \(G\) has the Gaussian heat kernel bounds [31, Theorem 6.19]. By the method of subordination and Bochner’s functional calculus [59, 9], the fractional Laplace operator on \(G\) is defined as

\[(-\Delta)^{\frac{\alpha}{2}} u := \left|\Gamma\left(-\frac{\alpha}{2}\right)\right| \int_0^\infty (e^{t\Delta} u - u) t^{-1-\frac{\alpha}{2}} dt,
\]

where \(e^{t\Delta}\) is the semigroup of \(\Delta\), see [37]. And the Green’s function \(R_\alpha\) of the fractional Laplace on \(G\) is

\[R_\alpha(x, y) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty k_t(x, y) t^{-1+\frac{\alpha}{2}} dt,\quad x, y \in G,
\]

which has the asymptotic relation \(R_\alpha(x, y) \simeq (\delta^G(x, y))^{\alpha-N}\). By the Young’s inequality for weak type spaces [22, Theorem 1.4.25.], we get the discrete HLS inequality on \(G\), see Theorem 2.2.

\[
\sum_{x, y \in G, x \neq y} R_\alpha(x, y) f(x) g(y) \leq C_{r,s,\alpha} \|f\|_{\ell^r} \|g\|_{\ell^s},\quad \forall f \in \ell^r(G), g \in \ell^s(G),
\]

where \(r, s > 1, 0 < \alpha < N, \frac{1}{r} + \frac{1}{s} + \frac{N-\alpha}{N} = 2\), and an equivalent form of the HLS inequality is
We want to prove that the optimal constant is achieved by some extremal function, we consider a maximizing sequence \( \{ u_n \} \subset D^{1.2}(G) \) satisfying

\[
\| u_n \|_{D^{1.2}} = 1, \quad \sum_G (R_\alpha \ast |u|^p) \cdot |u|^p \to K, \quad n \to \infty.
\]

We want to prove \( u_n \to u \) strongly in \( D^{1.2}(G) \), which yields that \( u \) is a maximizer.

We prove the following main result.

**Theorem 1.1.** For \( N \geq 3 \), \( \alpha \in (0, N) \), \( p > \frac{N+\alpha}{N-2} \), let \( \{ u_n \} \subset D^{1.2}(G) \) be a maximizing sequence satisfying (1.8). Then there exists a sequence \( \{ x_n \} \subset G \) and \( v \in D^{1.2}(G) \) such that the sequence after translation \( \{ v_n(x) := u_n(x_n, x) \} \) contains a convergent subsequence that converges to \( v \) in \( D^{1.2}(G) \). And \( v \) is a maximizer for \( K \).

**Remark 1.2.** (1) This result implies that the best constant can be obtained in the supercritical case \( p > \frac{N+\alpha}{N-2} \).

(2) Different from the continuous setting, \( u \in D^{1.2}(G) \setminus \{0\} \), does not yield \( \sum (R_\alpha \ast |u|^p) \cdot |u|^p \neq 0 \), for example, \( u(e) = 1, u(x) = 0 \) for \( x \neq e \), where \( e \) is the unit element of \( G \). Hence the normalized condition \( \| u_n \|_{D^{1.2}} = 1 \), rather than \( \sum (R_\alpha \ast |u_n|^p) \cdot |u_n|^p = 1 \), in the variational problem helps to rule out the vanishing case, see Lemma 3.3.

We will provide two proofs for the main result. In the continuous setting, Lions proved the existence of extremal functions by the Concentration-Compactness principle [44, Theorem III.3.] and a rescaling trick [44, Theorem I.1, (17)]. And Lieb in [39] used a compactness technique and the rearrangement inequalities. Following Lions, the main idea of proof I is to prove a discrete analog of Concentration-Compactness principle, see Lemma 3.1. However, we don’t know proper notion of the rescaling trick on \( G \) to exclude the vanishing case of the limit function. Inspired by [36], for the supercritical case, we prove that the normalized maximizing sequence after translation has a uniform positive lower bound at the unit element,
see Lemma 3.3, which excludes the vanishing case. The idea of proof II is based on a compactness technique by Lieb [39, Lemma 2.7] and the nontrivial nonvanishing of the limit of the translation sequence.

**Conjecture 1.5.** For \( N \geq 3, \alpha \in (0, N), p > \frac{N+\alpha}{N-2}, \) there is a positive ground state solution of the equation

\[
\Delta u(x) + (R_\alpha \ast u^p(x))u^{p-1}(x) = 0, \quad x \in G,
\]

Equivalently, there exists a pair of positive solution \((u, v) = (u, R_\alpha \ast u^p)\) of the following system

\[
\begin{align*}
\Delta u + vu^{p-1} &= 0 \\
(-\Delta)^{\frac{N}{2}} v &= u^p.
\end{align*}
\]

**Remark 1.4.** (1) By the equivalent form of the HLS inequality (1.5) we obtain another inequality

\[
\|R_\alpha \ast |u|^p\|_{\infty, \omega} \lesssim \|u^p\|_{\infty, \omega} \lesssim \|u\|_{D^{1,2}, \omega}^p, \quad \forall u \in D^{1,2}(G),
\]

here we omit the constants in inequalities. Similarly, there exist positive solutions \(\tilde{u}\) and \((\tilde{u}, \tilde{v}) = (\tilde{u}, R_\alpha \ast \tilde{u}^p)\) of the following equation and system respectively,

\[
\begin{align*}
\Delta u + (R_\alpha \ast u^p)\frac{N+\alpha}{N-2}u^{p-1} &= 0, \\
\Delta u + v^\frac{N+\alpha}{2}u^{p-1} &= 0 \\
(-\Delta)^{\frac{N}{2}} v &= u^p.
\end{align*}
\]

(2) By the discrete HLS inequality (1.3) on \(\mathbb{Z}^N\), the above existence results can be obtained on \(\mathbb{Z}^N\) when replacing \(R_\alpha\) with \(I_\alpha\) in (1.9), (1.10), (1.11) and (1.12).

(3) The range of the parameter for the existence of a positive ground state solution \(p > \frac{N+\alpha}{N-2}\) is different from the continuous case \(\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}\), thanks to the discrete nature, that is, \(\ell^p(G)\) embeds into \(\ell^q(G)\) for any \(q > p\). Hence we don’t need the control of \(L^2\) norm in the right hand side of (1.6), and we can remove the \(u\) term in (1.9) on \(G\).

(4) In the continuous setting [52], the existence range of ground state solutions \(\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}\) is sharp, in the sense that if \(p \leq \frac{N+\alpha}{N}\) or \(p \geq \frac{N+\alpha}{N-2}\) problem (1.1) does not have any nontrivial variational solution. It follows from the Pohozaev identity and the scaling trick on the Euclidean space \(\mathbb{R}^N\), see [52, Theorem 2, Proposition 3.1], which are unknown on Cayley graphs \(G\). This leads to an open problem for the nonexistence of nontrivial solutions to the equation (1.3) on \(G\), see Conjecture 1.6.

(5) Since the nonvanishing of the limit of the translation sequence based on the fact that the parameter \(p > \frac{N+\alpha}{N-2}\) is supercritical, see Lemma 3.3, the existence of nontrivial solutions in the critical case \(p = \frac{N+\alpha}{N-2}\) is still an open problem, see Conjecture 1.5.

**Conjecture 1.5.** According to the results in continuous cases [44, Corollary I.2], we conjecture that (1.9), (1.10), (1.11) and (1.12) have positive solutions when \(p = \frac{N+\alpha}{N-2}\), and the non-negative solutions are trivial when \(p < \frac{N+\alpha}{N-2}\).

According to [33, 12], we define the \(p\)-Laplace of \(u\) for \(p > 1\) as

\[
\Delta_p u(x) := \sum_{y \sim x} |\nabla_x y u|^{p-2} \nabla_x y u,
\]
and for $p = 1$ as
\[
\Delta_1 u(x) := \left\{ \sum_{y \sim x} f_{xy} : f_{xy} = -f_{yx}, f_{xy} \in \text{Sgn}(\nabla_{xy} u) \right\}, \quad \text{Sgn}(t) = \begin{cases} [1], & t > 0, \\ [-1,1], & t = 0, \\ \{-1\}, & t < 0. 
\end{cases}
\]

And since the second-order Sobolev inequalities hold on $G$ \cite[Theorem 10]{30}, by the same argument we can prove the following results.

**Theorem 1.6.** For $N \geq 3$, $\alpha \in (0, N - 2)$, $1 \leq p < \frac{N + \alpha}{N - 4}$, there exists a positive ground state solution for $p > 1$ and a non-negative solution for $p = 1$ of the equation
\[
\Delta p u + (R_\alpha \ast u^p) u^{p-1} = 0.
\]

For $N \geq 5$, $\alpha \in (0, N)$, $p > \frac{N + \alpha}{N - 4}$, there exists a positive ground state solution of the equation
\[
\Delta^2 u - (R_\alpha \ast u^p) u^{p-1} = 0.
\]

For $N \geq 5$, $\alpha \in (0, N - 4)$, $1 < p < \frac{N - \alpha}{N - 4}$, there exists a positive ground state solution of the equation
\[
\Delta (|\Delta u|^{p-2} \Delta u) - (R_\alpha \ast u^p) u^{p-1} = 0.
\]

**Remark 1.7.** The systems of (1.13), (1.14) and (1.15) also have positive solutions.

**Conjecture 1.8.** Also we conjecture that (1.13), (1.14) and (1.15) have positive solutions when $p = \frac{N + \alpha}{N - 4}$, $p = \frac{N + \alpha}{N - 2}$ and $p = \frac{N - \alpha}{N - 4}$ respectively, and the non-negative solutions are trivial when $p > \frac{N + \alpha}{N - 4}$, $p < \frac{N + \alpha}{N - 2}$ and $p > \frac{N - \alpha}{N - 4}$ respectively.

The paper is organized as follows. In Section 2, we recall some basic facts and prove some useful lemmas. In Section 3, we prove the discrete Concentration-Compactness principle and a key lemma to exclude the vanishing case, see Lemma 3.1 and Lemma 3.3. In Section 4, we give two proofs for Theorem 1.1 and prove Corollary 4.3. In Section 5, we prove Theorem 1.5 which generalize the existence results to the Choquard type equations with $p$-Laplace, biharmonic and $p$-biharmonic operators.

## 2. Preliminaries

Let $G$ be a countable group. It is called finitely generated if it has a finite generating set $S$. We always assume that the generating set $S$ is symmetric, i.e. $S = S^{-1}$. The Cayley graph of $(G, S)$ is a graph structure $(V, E)$ with the set of vertices $V = G$ and the set of edges $E$ where for any $x, y \in G, xy \in E$ (also denoted by $x \sim y$) if $x = ys$ for some $s \in S$. The Cayley graph of $(G, S)$ is endowed with a natural metric, called the word metric \cite{11}: For any $x, y \in G$, the distance between them is defined as the length of the shortest path connecting $x$ and $y$ by assigning each edge of length one,
\[
d^S(x, y) = \inf\{k : x = x_0 \sim \cdots \sim x_k = y\}.
\]

One sees easily that for two generating sets $S$ and $S_1$ the metrics $d^S$ and $d^{S_1}$ are bi-Lipschitz equivalent, i.e. there exist two constants $C_1(S, S_1), C_2(S, S_1)$ such that for any $x, y \in G$
\[
C_1(S, S_1) d^{S_1}(x, y) \leq d^S(x, y) \leq C_2(S, S_1) d^{S_1}(x, y).
\]
Let $B^S_p(n) := \{ x \in G \mid d^S_p(x, n) \leq n \}$ denote the closed ball of radius $n$ centered at $p \in G$. By the group structure, it is obvious that $| B^S_p(n) | = | B^S_q(n) |$, for any $p, q \in G$. The growth function of $(G, S)$ is defined as $\beta_S(n) := | B^S_p(n) |$ where $e$ is the unit element of $G$. A group $G$ is called of polynomial growth if there exists a finite generating set $S$ such that $\beta_S(n) \leq Cn^A$ for some $C, A > 0$ and any $n \geq 1$. One checks that this definition is independent of the choice of the generating set $S$. Thus, the polynomial growth is indeed a property of the group $G$. In this paper, we consider the Cayley graph $(G, S)$ of a group of polynomial growth

$$C_1(S)n^N \leq \beta_S(n) \leq C_2(S)n^N,$$

for some $N \in \mathbb{N}$ and any $n \geq 1$, where $N$ is called the homogenous dimension of $G$.

We denote by $C(G)$ the space of functions on $G$. The support of $u \in C(G)$ is defined as $\text{supp}(u) := \{ x \in G : u(x) \neq 0 \}$. Let $C_0(G)$ be the set of all functions with finite support. For any $u \in C(G)$, the $\ell^p$ norm of $u$ is defined as

$$\| u \|_{\ell^p(G)} := \begin{cases} \left( \sum_{x \in G} |u(x)|^p \right)^{1/p} & 0 < p < \infty, \\
\sup_{x \in G} |u(x)| & p = \infty, \end{cases}$$

and we shall write $\| u \|_{\ell^p(G)}$ as $\| u \|_p$ for convenience. The $\ell^p(G)$ space is defined as

$$\ell^p(G) := \{ u \in C(G) : \| u \|_{\ell^p(G)} < \infty \}.$$ For any $u \in C(G)$, the difference operator is defined as for any $x \sim y$

$$\nabla_{xy} u = u(y) - u(x).$$

Let

$$|\nabla u(x)|_p := \left( \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p},$$

be the $p$-norm of the gradient of $u$ at $x$.

We define the Laplace operator as

$$\Delta u(x) := \sum_{y \sim x} (u(y) - u(x)).$$

The $D^{k,p}$ ($k = 1, 2$) norms of $u$ are given by

$$\| u \|_{D^{k,p}(G)} := \| |\nabla u|^p \|_{\ell^1(G)} = \left( \sum_{x \in G} \left( \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p} \right),$$

$$\| u \|_{D^{k,p}(G)} := \| \Delta u \|_{\ell^1(G)} = \left( \sum_{x \in G} \left( \sum_{y \sim x} |\nabla_{xy} u|^p \right)^{1/p} \right).$$

We define $D^{k,p}_0(G)$ ($k = 1, 2$) as the completion of $C_0(G)$ in the $D^{k,p}$ norm and define

$$D^{k,p}(G) := \{ u \in \ell^{\frac{Np}{k+p}}(G) : \| u \|_{D^{k,p}(G)} < \infty \}.$$ In fact $D^{k,p}_0(G) = D^{k,p}(G)$, see [30, Section 3] for a proof.

Let $\Omega$ be a subset of $G$. We denoted by

$$\delta \Omega := \{ x \in G \setminus \Omega : \exists y \in \Omega, \text{s.t. } x \sim y \}.$$
the vertex boundary of \( \Omega \), possibly an empty set. We set \( \bar{\Omega} := \Omega \cup \delta \Omega \).

Let \( c_0(G) \) be the completion of \( C_0(G) \) in \( \ell^\infty \) norm. It is well-known that \( \ell^1(G) = (c_0(G))^* \). We set

\[
\|\mu\| := \sup_{u \in c_0(G), \|u\|_\infty = 1} \langle \mu, u \rangle, \quad \forall \mu \in \ell^1(G).
\]

By definition,

\[
\mu_n \overset{w^*}{\rightharpoonup} \mu \text{ in } \ell^1(G) \text{ if and only if } \langle \mu_n, u \rangle \rightarrow \langle \mu, u \rangle, \forall u \in c_0(G).
\]

In the proof, we will use the following facts (see [15]).

**Fact 2.1.** (a) Every bounded sequence of \( \ell^1(G) \) contains a \( w^* \)-convergent subsequence.

(b) If \( \mu_n \overset{w^*}{\rightharpoonup} \mu \) in \( \ell^1(G) \), then \( \mu_n \) is bounded and

\[
\|\mu\| \leq \lim_{n \to \infty} \|\mu_n\|.
\]

(c) If \( \mu \in \ell^{1+}(G) := \{ \mu \in \ell^1(G) : \mu \geq 0 \} \), then

\[
\|\mu\| = \langle \mu, 1 \rangle.
\]

Next, we prove the discrete HLS inequalities on \( G \).

**Theorem 2.2.** For \( r, s > 1, 0 < \alpha < N, \frac{1}{s} + \frac{1}{r} = 2 \), we get the discrete HLS inequality

\[
\tag{2.1}
\sum_{x,y \in G \atop x \neq y} R_\alpha(x,y)f(x)g(y) \leq C_{r,s,\alpha} \|f\|_r \|g\|_r, \forall f \in \ell^r(G), g \in \ell^s(G),
\]

and an equivalent form is

\[
\tag{2.2}
\|R_\alpha * f\|_r \leq C_{r,s,\alpha} \|f\|_r, \forall f \in \ell^r(G),
\]

where \( 1 < r < \frac{N}{\alpha}, 0 < \alpha < N, t = \frac{N}{N-\alpha r} \).

**Proof.** By the method of subordination and Bochner’s functional calculus [59, 9], the fractional Laplace operator on \( G \) is defined as

\[
(-\Delta)^{\frac{\alpha}{2}} u := \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty (e^{t\Delta} u - u) t^{-1 - \frac{\alpha}{2}} \, dt,
\]

where \( e^{t\Delta} \) is the semigroup of \( \Delta \), see [37]. Since \( G \) with the homogenous dimension \( N \) satisfies the weak Poincaré inequality, the heat kernel \( k_t(x, y) \) of the Laplace operator on \( G \) has the Gaussian heat kernel bounds [31 Theorem 6.19]. Hence the Green’s function \( R_\alpha \) of the fractional Laplace is

\[
R_\alpha(x, y) = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty k_t(x, y) t^{-1 + \frac{\alpha}{2}} \, dt, \quad x, y \in G,
\]

which has the asymptotic relation \( R_\alpha(x, y) \sim (d^x(x, y))^{\alpha - N} \). Then by the Young’s inequality for weak type spaces [22 Theorem 1.4.25],

\[
\|R_\alpha * f\|_r \lesssim \|R_\alpha\|_{\ell^{\infty, \infty}} \|f\|_r \simeq \|f\|_r,
\]

where \( \|f\|_{r, \infty} := \sup_{t \geq 0} t^{\lambda_f^r} \lambda_f(t) \), \( \lambda_f(t) := \{ x : f(x) > t \} \).

And (2.1) follows from the Hölder inequality. \( \square \)
Then we introduce the classical Brézis-Lieb lemma \cite[Theorem 1]{10}. Consider a measure space \((\Omega, \Sigma, \mu)\), which consists of a set \(\Omega\) equipped with a \(\sigma\)-algebra \(\Sigma\) and a Borel measure \(\mu : \Sigma \to [0, \infty]\).

**Lemma 2.3. (Brézis-Lieb lemma)** Let \((\Omega, \Sigma, \mu)\) be a measure space, \(\{u_n\} \subset L^p(\Omega, \Sigma, \mu)\), and \(0 < p < \infty\). If

(a) \(\{u_n\}\) is uniformly bounded in \(L^p\),

(b) \(u_n \to u, n \to \infty\) \(\mu\)-almost everywhere in \(\Omega\), then

\[
\lim_{n \to \infty} (\|u_n\|_{L^p}^p - \|u_n - u\|_{L^p}^p) = \|u\|_{L^p}^p.
\]

**Remark 2.4.** (1) The preceding lemma is a refinement of the Fatou’s Lemma.

(2) Since \(\{u_n\}\) is uniformly bounded in \(L^p\), passing to a subsequence if necessary, we have

\[
\lim_{n \to \infty} \|u_n\|_{L^p}^p = \lim_{n \to \infty} \|u_n - u\|_{L^p}^p + \|u\|_{L^p}^p.
\]

(3) If \(\Omega\) is countable and \(\mu\) is a positive measure defined on \(\Omega\), then we get a discrete version of the classical Brézis-Lieb lemma.

(4) If \(p \geq 1\), for every \(q \in [1, p]\), \(\{u_n\} \subset L^p\) still satisfies conditions (a) and (b), then there exists an easy variant of the classical Brézis-Lieb lemma

\[
\lim_{n \to \infty} \int |u_n|^q - |u_n - u|^q - |u|^q \, d\mu = 0.
\]

(5) For \(p > 1\), (a) and (b) yield \(u_n \to u\) weakly in \(L^p(\Omega)\), that is, the pointwise convergence of a bounded sequence yields the weak convergence, see \cite[Proposition 4.7.12]{8}.

Now we are ready to prove three discrete versions of the Brézis-Lieb lemma.

**Lemma 2.5.** Let \(\Omega \subset G, \{u_n\} \subset D^{1,p}(G)\), and \(1 \leq p < \infty\). If

(a) \(\{u_n\}\) is uniformly bounded in \(D^{1,p}(G)\),

(b) \(u_n \to u, n \to \infty\) pointwise on \(G\), then

\[
\lim_{n \to \infty} \left( \sum_{x \in \Omega} \left| \nabla u_n(x) \right|_p^p - \sum_{x \in \Omega} \left| \nabla (u_n - u)(x) \right|_p^p \right) = \sum_{x \in \Omega} \left| \nabla u(x) \right|_p^p.
\]

**Proof.** We define two directed edge sets as follows

\[
E_1 := \{ e = (e_-, e_+) : e_\pm \in \Omega, e_- \sim e_+ \},
\]

\[
E_2 := \{ e = (e_-, e_+) : (e_- + e_+) \in \Omega \times \delta \Omega, e_- \sim e_+ \},
\]

where \(E_1\) is the set of internal edges of \(\Omega\), \(E_2\) is the set of edges that cross the boundary of \(\Omega\), and \(e_-\) and \(e_+\) are the initial and terminal endpoints of \(e\).

Set \(\tilde{E} := E_1 \cup E_2\). We define \(\pi, \mu : \tilde{E} \to \mathbb{R}, \pi(e) = u(e_+) - u(e_-), \mu(e) = 1\).

Then we get

\[
\sum_{x \in \Omega} \left| \nabla u_n(x) \right|_p^p = \sum_{E_1} \left| \pi_n(e) \right|_p^p + \sum_{E_2} \left| \pi_n(e) \right|_p^p = \|\pi_n\|_{L^p(\tilde{E}, \Sigma, \mu)}^p < \infty,
\]

\[
\pi_n \to \pi \text{ pointwise in } \tilde{E}.
\]

For the measure space \((\tilde{E}, \Sigma, \mu)\), by the Brézis-Lieb lemma we have

\[
\lim_{n \to \infty} \left( \|\pi_n\|_{L^p(\tilde{E}, \Sigma, \mu)}^p - \left| \pi_n - \pi \right|_{L^p(\tilde{E}, \Sigma, \mu)}^p \right) = \|\pi\|_{L^p(\tilde{E}, \Sigma, \mu)}^p,
\]

which is equivalent to the equation (2.3). \(\square\)
Lemma 2.6. Let \( \Omega \subset G \), \( \{ u_n \} \subset D^{2,p}(G) \), and \( 1 < p < \infty \). If
(a) \( \{ u_n \} \) is uniformly bounded in \( D^{2,p}(G) \),
(b) \( u_n \rightarrow u, n \rightarrow \infty \) pointwise on \( G \), then

\[
\lim_{n \rightarrow \infty} \left( \sum_{x \in \Omega} |\nabla u_n(x)|^p - \sum_{x \in \Omega} |\nabla (u_n - u)(x)|^p \right) = \sum_{x \in \Omega} |\nabla u(x)|^p.
\]

Proof. Since \( \{ \nabla u_n \} \) is uniformly bounded in \( \ell^p(G) \) and \( \nabla u_n \rightarrow \nabla u, n \rightarrow \infty \) pointwise on \( G \), by the Brézis-Lieb lemma we get

\[
\lim_{n \rightarrow \infty} (\|\nabla u_n\|_{\ell^p(\Omega)}^p - \|\nabla (u_n - u)\|_{\ell^p(\Omega)}^p) = \|\nabla u\|_{\ell^p(\Omega)}^p,
\]
which is equivalent to the equation (2.5).

Lemma 2.7. Let \( \Omega \subset G \), for \( N \geq 3 \), \( \alpha \in (0, N) \), \( p \geq 1 \), and \( \{ u_n \} \subset \ell^\frac{2N\alpha}{N-p} (G) \). If
(a) \( \{ u_n \} \) is uniformly bounded in \( \ell^\frac{2N\alpha}{N-p} (G) \),
(b) \( u_n \rightarrow u, n \rightarrow \infty \) pointwise on \( G \), then

\[
\lim_{n \rightarrow \infty} \left( \sum_{\Omega} (R_\alpha \ast |u_n|^p) |u_n|^p - \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n - u|^p \right) = \sum_{\Omega} (R_\alpha \ast |u|^p) |u|^p.
\]

Proof. First we have

\[
\sum_{\Omega} (R_\alpha \ast |u_n|^p) |u_n|^p - \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n - u|^p
= \sum_{\Omega} (R_\alpha \ast (|u_n|^p - |u_n - u|^p)) (|u_n|^p - |u_n - u|^p)
+ \sum_{\Omega} (R_\alpha \ast |u_n|^p) |u_n - u|^p + \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n|^p
- 2 \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n - u|^p
=: I + II,
\]

where

\[
I = \sum_{x \in \Omega} (R_\alpha \ast f_n(x)) f_n(x), \quad f_n(x) := |u_n(x)|^p - |u_n(x) - u(x)|^p
\]

and

\[
II = \sum_{\Omega} (R_\alpha \ast |u_n|^p) |u_n - u|^p + \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n|^p
- 2 \sum_{\Omega} (R_\alpha \ast |u_n - u|^p) |u_n - u|^p.
\]

For \( I \), by the HLS inequality (1.4) we get
\[ |I - \sum_{\Omega} (R_{\alpha} * |u|^p) |u|^p| \]
\[ \leq \left| \sum_{\Omega} (R_{\alpha} * (f_n - |u|^p)) f_n \right| + \sum_{\Omega} (R_{\alpha} * |u|^p) (f_n - |u|^p) \]
\[ \leq \left| \sum_{G} (R_{\alpha} * |f_n - |u|^p|) f_n \right| + \sum_{G} (R_{\alpha} * |u|^p) |f_n - |u|^p| \]
\[ \lesssim \|f_n - |u|^p\|_{\frac{2N}{N-\alpha}} \|f_n\|_{\frac{2N}{N-\alpha}} + \|f_n - |u|^p\|_{\frac{2N}{N-\alpha}} \|u\|_{\frac{2N}{N-\alpha}}^p. \]

Since \( \{u_n\} \) is uniformly bounded in \( \ell_{\frac{2N}{N-\alpha}} \), by Remark 2.4 (4) we get \( f_n \to |u|^p \) strongly in \( \ell_{\frac{2N}{N-\alpha}} \) as \( n \to \infty \). Hence by letting \( n \to \infty \),
\[ I \to \sum_{\Omega} (R_{\alpha} * |u|^p) |u|^p. \]

For II,
\[ |II| = \left| \sum_{\Omega} (R_{\alpha} * f_n) |u_n - u|^p \right| + \sum_{\Omega} (R_{\alpha} * |u_n - u|^p) f_n \]
\[ \leq \sum_{G} (R_{\alpha} * |f_n|) |u_n - u|^p + \sum_{G} (R_{\alpha} * |u_n - u|^p) |f_n| \]
\[ \leq 2 \sum_{G} (R_{\alpha} * |f_n|) |u_n - u|^p. \]

By the HLS inequality (1.5), Remark 2.4 (4) and (5),
\[ R_{\alpha} * |f_n| \to R_{\alpha} * |u|^p \text{ in } \ell_{\frac{2N}{N-\alpha}}, \]
\[ |u_n - u|^p \xrightarrow{w} 0 \text{ weakly in } \ell_{\frac{2N}{N-\alpha}}, \]
then we get \( II \to 0 \) as \( n \to \infty \). Hence we reach the conclusion. \( \square \)

**Remark 2.8.** Note that a counterpart of the Brézis-Lieb lemma holds for the non-local term of the Riesz potential \( I_{\alpha} \) on \( \mathbb{R}^N \) [68, Lemma 3.2][1, §5.1]: if the sequence \( \{u_n\} \) is uniformly bounded in \( L_{\frac{2N}{N-\alpha}} (\mathbb{R}^N) \) and \( u_n \to u, n \to \infty \) almost everywhere on \( \mathbb{R}^N \), then
\[ \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^p) |u_n|^p - \int_{\mathbb{R}^N} (I_{\alpha} * |u_n - u|^p) |u_n - u|^p \right) = \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p) |u|^p. \]

Here we prove a discrete version on \( \Omega \subset G \), which is necessary in the later proofs of the discrete Concentration-Compactness lemma.

### 3. Concentration-Compactness Principle

In this section, we will establish the discrete Concentration-Compactness principle and prove a key lemma to rule out the vanishing case of the limit function.
Lemma 3.1. (Discrete Concentration-Compactness lemma) For $N \geq 3$, $\alpha \in (0, N), p \geq \frac{N+\alpha}{N-2}$, if $\{u_n\}$ is uniformly bounded in $D^{1,2}(G)$. Then passing to a subsequence if necessary, still denoted as $\{u_n\}$, we have

\begin{equation}
(3.1) \quad u_n \to u \text{ pointwise on } G,
\end{equation}

\begin{equation}
(3.2) \quad \|\nabla u_n\|_2^2 \xrightarrow{w^*} \|\nabla u\|_2^2 \text{ in } \ell^1(G).
\end{equation}

And the following limits

\begin{equation}
\lim_{r \to \infty} \lim_{n \to \infty} \sum_{d^S(x,e) > r} |\nabla u_n(x)|_2^2 =: \mu_\infty, \lim_{r \to \infty} \lim_{n \to \infty} \sum_{d^S(x,e) > r} (R_\alpha \ast |u_n|^p)|u_n(x)|_p^p =: \nu_\infty
\end{equation}

exist. For the above $\{u_n\}$, we have

\begin{equation}
(3.3) \quad \|\nabla(u_n - u)\|_2^2 \xrightarrow{w^*} 0 \text{ in } \ell^1(G),
\end{equation}

\begin{equation}
(3.4) \quad (R_\alpha \ast |u_n - u|^p)|u_n - u|^p \xrightarrow{w^*} 0 \text{ in } \ell^1(G),
\end{equation}

\begin{equation}
(3.5) \quad \nu_\infty \leq K \mu_\infty^p,
\end{equation}

\begin{equation}
(3.6) \quad \lim_{n \to \infty} \|u_n\|_{D^{1,2}}^2 = \|u\|_{D^{1,2}}^2 + \mu_\infty,
\end{equation}

\begin{equation}
(3.7) \quad \lim_{n \to \infty} \sum_{d^S(x,e) > r} (R_\alpha \ast |u_n|^p)|u_n|^p = \sum (R_\alpha \ast |u|^p)|u|^p + \nu_\infty.
\end{equation}

Proof: Since $\{u_n\}$ is uniformly bounded in $\ell^{\frac{2Np}{2N - p}}(G)$, and hence in $\ell^\infty(G)$. By the diagonal principle, passing to a subsequence we get (3.1). Since $\{\|u_n\|_2^2\}$ is uniformly bounded in $\ell^1(G)$, we get (3.2) by the Banach-Alaoglu theorem and (3.1). For every $r \geq 1$, passing to a subsequence if necessary,

\begin{equation}
\lim_{n \to \infty} \sum_{d^S(x,e) > r} |\nabla u_n(x)|^2, \lim_{n \to \infty} \sum_{d^S(x,e) > r} (R_\alpha \ast |u_n|^p)|u_n(x)|^p
\end{equation}

exist, where $d^S$ is the word metric as defined in Section 2. Then we can define $\mu_\infty$, $\nu_\infty$ by the monotonicity in $R$.

Let $v_n := u_n - u$, then $v_n \to 0$ pointwise on $G$ and $\{|\nabla v_n|_2^2\}$ is uniformly bounded in $\ell^1(G)$. Then any subsequence of $\{|\nabla v_n|_2^2\}$ contains a subsequence (still denoted as $\{|\nabla v_n|_2^2\}$) that $w^*$-converges to 0 in $\ell^1(G)$, which follows from

\begin{equation}
\sum h|\nabla v_n|_2^2 \to 0, \forall h \in C_0(G).
\end{equation}

Hence we get (3.3). Similarly, we get (3.4).

For $r \geq 1$, let $\Psi_r \in C(G)$ such that $\Psi_r(x) = 1$ for $d^S(x,e) \geq r + 1$, $\Psi_r(x) = 0$ for $d^S(x,e) \leq r$. By the HLS inequality (1.4), we have

\begin{equation}
(3.8) \quad \left(\sum (R_\alpha \ast |v_n \Psi_r|^p)|v_n \Psi_r|^p\right) \leq K \left(\sum |\nabla(v_n \Psi_r)|^2\right)^p
\end{equation}

\begin{equation}
= K \left(\sum \sum |\nabla x_y \Psi_r v_n(y) + \Psi_r(x)\nabla x_y v_n|^2\right)^p.
\end{equation}
For the left side of (3.8), by the definition of $\Psi_r$, the H"{o}lder inequality and the HLS inequality, we have the following key observation

$$
\left| \sum_{x \neq y} (R_\alpha \ast |v_n \Psi_r|^p) \right| |v_n \Psi_r|^p - \sum (R_\alpha \ast |v_n|^p) |\Psi_r|^p |v_n \Psi_r|^p
\right.

\simeq \left| \sum_{x \neq y} \left( |\Psi_r(y)|^p - |\Psi_r(x)|^p \right) |v_n(y)|^p |v_n(x)|^p |\Psi_r(y)|^p |\Psi_r(x)|^p \right|

= \left| \sum_{d^2(x,e) \geq r + 1} \sum_{d^2(y,e) \leq r} \frac{(0 - 1)|v_n(y)|^p |v_n(x)|^p}{d^2(x,y)^{N-\alpha}} \right|

\lesssim \sum_{d^2(x,e) \leq r} \sum_{d^2(y,e) \leq r} \frac{|v_n(y)|^p |v_n(x)|^p}{d^2(x,y)^{N-\alpha}}

\lesssim \|v_n^p\|_{\ell^\infty(G)} \|v_n^p\|_{\ell^{\infty}(\{y: d^2(y,e) \leq r\})}

\lesssim \|v_n^p\|_{\ell^\infty(G)} \|v_n^p\|_{\ell^{\infty}(\{y: d^2(y,e) \leq r\})},

which goes to zero since $v_n \to 0$ pointwise on $G$. Thus we get

(3.9)

$$
\lim_{n \to \infty} \left( \sum (R_\alpha \ast |v_n \Psi_r|^p) |v_n \Psi_r|^p - \sum (R_\alpha \ast |v_n|^p) |\Psi_r|^p |v_n \Psi_r|^p \right) = 0.
$$

For the right side of (3.8), note that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|\nabla_x \Psi_r v_n(y) + \Psi_r(x) \nabla_x v_n)|^2 \leq C_\varepsilon |\nabla_x \Psi_r|^2 |v_n(y)|^2 + (1 + \varepsilon) |\nabla_x v_n|^2 |\Psi_r(x)|^2.
$$

Since $v_n \to 0$ pointwise on $G$, by (3.8), (3.9) and letting $\varepsilon \to 0^+$ we obtain

$$
\lim_{n \to \infty} \sum (R_\alpha \ast |v_n|^p) |\Psi_r|^p |v_n \Psi_r|^p \leq K \lim_{n \to \infty} \left( \sum |\nabla v_n|^2 \right)^2.
$$

From the definition of $\Psi_r$,

(3.10)

$$
\lim_{n \to \infty} \sum_{d^2(x,e) > r} (R_\alpha \ast |v_n|^p) |v_n|^p \leq K \lim_{n \to \infty} \left( \sum |\nabla v_n|^2 \right)^2.
$$

By Lemma 2.3, Lemma 2.7 and Remark 2.4 (2),

$$
\lim_{n \to \infty} \sum_{d^2(x,e) > r} |\nabla u_n(x)|^2 \leq \lim_{n \to \infty} \sum_{d^2(x,e) > r} |\nabla v_n(x)|^2 = \sum_{d^2(x,e) > r} |\nabla u(x)|^2,
$$

$$
\lim_{n \to \infty} \sum_{d^2(x,e) > r} (R_\alpha \ast |u_n|^p) |u_n|^p - \lim_{n \to \infty} \sum_{d^2(x,e) > r} (R_\alpha \ast |v_n|^p) |v_n|^p

= \sum_{d^2(x,e) > r} (R_\alpha \ast |u|^p) |u|^p.
$$

Hence by the above equalities, we get

(3.11)

$$
\lim_{r \to \infty} \lim_{n \to \infty} \sum_{d^2(x,e) > r} |\nabla v_n(x)|^2 = \mu_\infty,
$$

(3.12)

$$
\lim_{r \to \infty} \lim_{n \to \infty} \sum_{d^2(x,e) > r} (R_\alpha \ast |v_n|^p) |v_n|^p = \nu_\infty.
$$
Combining the equations (3.10), (3.11) and (3.12), we get
\[ \nu_{\infty} \leq K p_{\infty}. \]

Since \( u_n \to u \) pointwise on \( G \), for every \( r \geq 1 \), we have
\[
(3.13) \quad \lim_{n \to \infty} \sum |\nabla u_n|^2 = \lim_{n \to \infty} \left( \sum |\nabla \Psi_r|^2 + \sum (1 - \Psi_r)|\nabla u_n|^2 \right) \\
= \lim_{n \to \infty} \sum |\nabla u_n|^2 + \sum (1 - \Psi_r)|\nabla u|^2.
\]
Since \( \{ (R_\alpha \ast u_n)^p | u_n |^p \} \) is uniformly bounded in \( \ell^1(G) \) and by (3.1),
\[
(R_\alpha \ast u_n)^p | u_n |^p \overset{w_*}{\to} (R_\alpha \ast u)^p | u |^p \quad \text{in } \ell^1(G).
\]
Similarly,
\[
(3.14) \quad \lim_{n \to \infty} \sum (R_\alpha \ast u_n)^p | u_n |^p \\
= \lim_{n \to \infty} \left( \sum |\nabla \Psi_r (R_\alpha \ast u_n)^p | u_n |^p + \sum (1 - \Psi_r)(R_\alpha \ast u_n)^p | u_n |^p \right) \\
= \lim_{n \to \infty} \sum |\nabla u_n|^2 + \sum (1 - \Psi_r)(R_\alpha \ast u)^p | u |^p.
\]

Letting \( r \to \infty \) in (3.13) and (3.14), we obtain
\[
\lim_{n \to \infty} \sum |\nabla u_n|^2 = \mu_{\infty} + \sum |\nabla u|^2 = \mu_{\infty} + \|u\|_{D^{1,2}}^2,
\]
\[
\lim_{n \to \infty} \sum (R_\alpha \ast u_n)^p | u_n |^p = \nu_{\infty} + \sum (R_\alpha \ast u)^p | u |^p.
\]

\( \square \)

Remark 3.2. (1) In the continuous setting, P. L. Lions [44], Bianchi et al. [7] and Ben-Naoum et al. [6] proved that the mass of a weakly convergent sequence can be divided into three parts, i.e. the mass of the limit, the mass concentrated at finite points and the loss of mass of the sequence at infinity. The corresponding parts still satisfy the Sobolev inequalities, also see [65, Lemma 1.40] [17].

(2) The part of the mass concentrated at finite points vanish on \( G \), i.e. (3.3) and (5.4), which may be not true in the continuous setting. For example, consider the sequence of probability measures \( \{ \delta_n \} \) in \([0,1]\), where \( \delta_n(x) := n^2 \chi_{[0,1/n]}(x) \). Then \( \delta_n \to 0 \) almost everywhere in \([0,1]\). However, \( \delta_n \overset{w_*}{\rightharpoonup} \delta_0 \) in \( C([0,1])^* \) and the Dirac measure \( \delta_0 \) is non-zero. This is the advantage of the discrete setting.

(3) The part of the loss of mass at infinity satisfies the inequality (3.5) which may be not true in the continuous setting, since the claim (3.9) only holds for smooth compact support functions \( \Psi_r \in C_0^\infty (\mathbb{R}^N) \), see [17] (2.8). The key point is that the convergence pointwise in a bounded domain of \( G \) yields \( \ell^1 \) convergence which may be not true on \( \mathbb{R}^N \), and \( \delta_n(x) \) in (2) is a counter example.

Next, we prove that the maximizing sequence after translation has a uniform positive lower bound at the unit element \( e \in G \). This is crucial to rule out the vanishing case of the limit function in supercritical cases.

Lemma 3.3. For \( N \geq 3 \), \( \alpha \in (0, N) \), \( p > \frac{N+\alpha}{N-\alpha} \), let \( \{ u_n \} \subset \mathcal{D}^{1,2}(G) \) be a maximizing sequence satisfying (2.6). Then
\[ \lim_{n \to \infty} \|\nabla u_n\|_{L^\infty} = \lim_{n \to \infty} \sup_{x \sim y} \sum_{y \sim x} |\nabla u_n| > 0. \]
Proof. Since \( p > \frac{N+\alpha}{N-2} \) yields \( \frac{2Np}{N+\alpha} > 2^* \), let \( r^* := \frac{2Np}{N+\alpha} > 2^* \), hence \( r > 2 \) and \( r = \frac{2Np}{N+\alpha+2p} \in (\frac{2Np}{N+\alpha}, \frac{2Np}{N+2p}) \). Choosing \( q \) such that \( 2 < q < r < \infty \), by the interpolation inequality we have

\[
C_{\alpha,p,q} \left( \sum (R_{\alpha^*} | u_n |^p) | u_n |^p \right)^{\frac{1}{p}} \leq \| \nabla u_n \|_2^q \leq \| \nabla u_n \|_2^q \leq \| \nabla u_n \|_\infty^{q-2} \leq \| \nabla u_n \|_\infty^{q-2},
\]

where \( C_{\alpha,p,q} \) is the constant in the inequality (1.6).

By taking the limit, we obtain

\[
C_{\alpha,p,q} K^{\frac{q}{p}} \leq \lim_{n \to \infty} \| \nabla u_n \|_\infty^{q-2}.
\]

This proves the lemma.

Remark 3.4. The maximum of \( |\nabla u_n(x)| \) is attainable since \( \| \nabla u_n \|_2 = 1 \). Define \( v_n(x) := u_n(x_n) \), where \( |\nabla u_n(x_n)| = \max_x |\nabla u_n(x)| \). Then the translation sequence \( \{v_n\} \) is uniformly bounded in \( D^{1,2}(G) \), \( \| \nabla v_n \|_2 = 1 \) and \( |\nabla v_n(e)| = \| \nabla u_n \|_\infty \), where \( e \) is the unit element of \( G \). By Lemma 3.3 passing to a subsequence if necessary, we have

\[
v_n \to v \quad \text{pointwise on } G,
\]

\[
|\nabla v(e)| = \lim_{n \to \infty} \| \nabla u_n \|_\infty > 0.
\]

4. Proofs for Theorem 1.1 and Corollary 1.3

In this section, we will give two proofs for Theorem 1.1 and prove Corollary 1.3.

Proof I of Theorem 1.1 Let \( \{u_n\} \subset D^{1,2}(G) \) be a maximizing sequence satisfying (1.8). And the translation sequence \( \{v_n\} \) is defined in Remark 3.4.

By equalities (3.7) and (3.8) in Lemma 3.1 passing to a subsequence if necessary, we get

\[
1 = \lim_{n \to \infty} \| v_n \|_{D^{1,2}}^2 = \| v \|_{D^{1,2}}^2 + \mu_\infty,
\]

\[
K = \lim_{n \to \infty} \sum (R_{\alpha^*} | v_n |^p) | v_n |^p = \sum (R_{\alpha^*} | v |^p) | v |^p + \nu_\infty.
\]

By the Sobolev type inequality (1.6), (3.8) and the inequality

\[
(a+b)^p \geq a^p + b^p, \forall a, b \geq 0,
\]

we get

\[
K = \sum (R_{\alpha^*} | v |^p) | v |^p + \nu_\infty \leq K(\| v \|_{D^{1,2}}^p + \mu_\infty) \leq K(\| v \|_{D^{1,2}}^p + \mu_\infty)^p = K.
\]

Since \( (a+b)^p \geq a^p + b^p \) unless \( a = 0 \) or \( b = 0 \), we deduce from (3.15) that \( \| v \|_{D^{1,2}} = 1 \) and \( \mu_\infty = 0 \). Hence \( \nu_\infty = 0 \) which yields

\[
\sum (R_{\alpha^*} | v |^p) | v |^p = K.
\]

That is, \( v \) is a maximizer.

Next, we give another proof for Theorem 1.1 using the discrete Brézis-Lieb lemma.
Proof II of Theorem 1.1. Using Lemma 3.3 by the translation and taking a subsequence if necessary as before, we can get a maximizing sequence \( \{u_n\} \) satisfying (1.8), \( u_n \to u \) pointwise on \( G \), and \( |\nabla u(e)| > 0 \).

By Lemma 2.3, Lemma 2.7, the inequalities (4.1) and (1.6), then passing to a subsequence if necessary, we have

\[
K = \lim_{n \to \infty} \sum (R_\alpha | u_n |^p | u_n |^p) = \lim_{n \to \infty} \sum (R_\alpha | u_n |^p | u_n |^p) \leq \lim_{n \to \infty} \sum (R_\alpha | u_n - u |^p | u_n - u |^p + \sum (R_\alpha | u |^p | u |^p) \leq \lim_{n \to \infty} \frac{K|u_n - u|_{D^{1,2}}^2 + \sum (R_\alpha | u |^p | u |^p)}{\|u_n - u\|_{D^{1,2}}^2 + \|u\|_{D^{1,2}}^2}.
\]

Since \( \|u\|_{D^{1,2}} \neq 0 \), we have that

\[
\sum (R_\alpha | u |^p | u |^p) \geq K\|u\|_{D^{1,2}}^{2p},
\]

which yields

\[
\sum (R_\alpha | u |^p | u |^p) = K\|u\|_{D^{1,2}}^{2p}.
\]

By (4.2), passing to a subsequence, we get

\[
\lim_{n \to \infty} \sum (R_\alpha | u_n - u |^p | u_n - u |^p) = K\lim_{n \to \infty} \|u_n - u\|_{D^{1,2}}^{2p}.
\]

Since \( 0 < \|u\|_{D^{1,2}} \leq \lim_{n \to \infty} \|u_n\|_{D^{1,2}} = 1 \), it suffices to show that \( \|u\|_{D^{1,2}} = 1 \). Suppose that it is not true, i.e. \( 0 < \|u\|_{D^{1,2}} = D < 1 \), then by Lemma 2.5

\[
\lim_{n \to \infty} \|u_n - u\|_{D^{1,2}}^2 = \lim_{n \to \infty} \|u_n\|_{D^{1,2}}^2 - \|u\|_{D^{1,2}}^2 = 1 - D^2 > 0.
\]

However, \((a + b)^p > a^p + b^p\) if \(a, b > 0\). This yields a contradiction by (4.2). Thus, \( \|u\|_{D^{1,2}} = 1 \) and \( u \) is a maximizer.

Finally we prove Corollary 1.3.

Proof of Corollary 1.3. By Theorem 1.1 there exists a maximizer \( u \) for \( K \). Replacing \( u \) by \( |u| \), we know that \( |u| \) is still a maximizer. Therefore, we get a non-negative maximizer \( u \). Define \( I(u) := \sum (R_\alpha | u |^p | u |^p - \lambda \sum | \nabla u |^2 \), and it follows from the Lagrange multiplier that \( u \) is a non-negative solution of the equation (1.9). The maximum principle yields that \( u \) is positive.

Let \( v := R_\alpha | u |^p \), which satisfies \((-\Delta)^{\frac p2} v = u^p \). Hence \((u, v)\) is a positive solution of the system (1.10).

By the equivalent form of HLS inequality (1.5) we obtain another inequality

\[
\|R_\alpha | u |^p \|_{D^{1,2}} \lesssim \|u^p\|_{D^{1,2}} \lesssim \|\nabla u\|_{D^{1,2}}, \forall u \in D^{1,2}(G).
\]

Then define its variational problem, and by the same argument as before we can prove Remark 1.4.
5. Proof for Theorem 1.6

In this section, we generalize the existence results to the Choquard type equations with $p$-Laplace, biharmonic and $p$-biharmonic operators. By the Concentration-Compactness principle, we can prove the existence of the extremal functions using the idea of proof I. Here we omit the proof I and give the proof II below for brevity.

Proof of (1.13). For $N \geq 3$, $\alpha \in (0, N - 2)$, $1 \leq p < \frac{N - \alpha}{N + \alpha}$, then $2Np - p > N - p^*$, by the Sobolev inequality (1.2) and the HLS inequality (1.4) we get

$$\sum_{G} (R_{\alpha} \ast |u|^p) |u|^p \leq C_{p,\alpha} \left( \sum_{G} |u|^{2Np/(N+\alpha)} \right)^{N/(N+\alpha)} \leq C_{p,\alpha} \tilde{C}_{2Np/(N+\alpha),p} \left( \sum_{G} |\nabla u|^p \right)^2, \forall u \in D^{1,p}(G).$$

The optimal constant in the inequality (5.1) is given by

$$K_2 := \sup_{\|u\|_{D^{1,p}} = 1} \sum_{G} (R_{\alpha} \ast |u|^p) |u|^p.$$

Consider a maximizing sequence $\{u_n\} \subset D^{1,p}(G)$ satisfying

$$\|u_n\|_{D^{1,p}} = 1, \sum_{G} (R_{\alpha} \ast |u_n|^p) |u_n|^p \to K_2, n \to \infty.$$

Since $p < \frac{N - \alpha}{2}$ yields $\frac{2Np}{N + \alpha} =: r^* > p^*$, that is $r = \frac{Np}{N+\alpha} \in \left( \frac{Np}{2N+p}, \frac{Np}{N+p} \right)$, choose $q$ such that $p < q < r < \infty$. By the interpolation inequality we have

$$C_{\alpha,p,q} \left( \sum_{G} (R_{\alpha} \ast |u_n|^p) |u_n|^p \right)^\frac{q}{q-p} \leq \|\nabla u_n\|_q^q \leq \|\nabla u_n\|_p^p \|\nabla u_n\|^{q-p}_{\infty} \leq \|\nabla u_n\|_{\infty}^{q-p},$$

where $C_{\alpha,p,q}$ is the constant in the inequality (5.1).

Taking the limit, we obtain

$$C_{\alpha,p,q} K_2^\frac{q}{q-p} \leq \lim_{n \to \infty} \|\nabla u_n\|_{\infty}^{q-p}.$$

This proves $\lim_{n \to \infty} \|\nabla u_n\|_{\infty} > 0$.

By the translation and taking a subsequence if necessary as Remark 3.4, we can get a maximizing sequence $\{u_n\}$ satisfying (5.3), $u_n \to u$ pointwise on $G$, and $|\nabla u(e)| > 0$.  

By Lemma 2.5, Lemma 2.7, the inequalities (4.1) and (5.1), then passing to a subsequence if necessary, we have

\[ K_2 = \lim_{n \to \infty} \sum (R_{\alpha}^* | u_n |^p) | u_n |^p \]
\[ = \lim_{n \to \infty} \sum (R_{\alpha}^* | u_n |^p) | u_n |^p \]
\[ = \lim_{n \to \infty} \frac{\sum (R_{\alpha}^* | u_n - u |^p) | u_n - u |^p}{\left( \| u_n - u \|_{D^1,p}^p + \| u \|_{D^1,p}^p \right)^2} \]
\[ \leq \lim_{n \to \infty} \frac{\sum (R_{\alpha}^* | u_n - u |^p) | u_n - u |^p}{\left( \| u_n - u \|_{D^1,p}^p + \| u \|_{D^1,p}^p \right)^2} \]
\[ \leq \lim_{n \to \infty} \frac{K_2 \| u_n - u \|^2_{D^1,p} \| u \|^2_{D^1,p}}{\| u_n - u \|^2_{D^1,p} + \| u \|^2_{D^1,p}}. \]

Since \( \| u \|_{D^1,p} \neq 0 \), we have that

\[ \sum (R_{\alpha}^* | u |^p) | u |^p \geq K_2 \| u \|^2_{D^1,p}, \]
which yields

\[ \sum (R_{\alpha}^* | u |^p) | u |^p = K_2 \| u \|^2_{D^1,p}. \]

By (5.4), passing to a subsequence, we get

\[ \lim_{n \to \infty} \sum (R_{\alpha}^* | u_n - u |^p) | u_n - u |^p = K_2 \lim_{n \to \infty} \| u_n - u \|^2_{D^1,p}. \]

Since \( 0 < \| u \|_{D^1,p} \leq \lim_{n \to \infty} \| u_n \|_{D^1,p} = 1 \), it suffices to show that \( \| u \|_{D^1,p} = 1 \).

Suppose that it is not true, i.e. \( 0 < \| u \|_{D^1,p} = D < 1 \), then by Lemma 2.5

\[ \lim_{n \to \infty} \| u_n - u \|_{D^1,p} = \lim_{n \to \infty} \| u_n \|_{D^1,p} - \| u \|_{D^1,p} = 1 - D^p > 0. \]

However, \( (a + b)^p > a^p + b^p \) if \( a, b > 0 \). This yields a contradiction by (5.4).

Thus, \( \| u \|_{D^1,p} = 1 \) and \( u \) is a maximizer. Replacing \( u \) by \( | u | \), we know that \( | u | \) is still a maximizer. Therefore, we get a non-negative maximizer \( u \). It follows from the Lagrange multiplier that \( u \) is a non-negative solution of (1.13). Moreover, for \( p > 1 \), the maximum principle yields that \( u \) is positive.

In [30, Theorem 10], by the boundedness of Riesz transforms we prove the discrete second-order Sobolev inequality (5.5).

**Lemma 5.1.** For \( N \geq 3, 1 < p < \frac{N}{2}, P^* := \frac{Np}{N - 2p}, \)

\[ \| u \|_{P^{**}} \leq C_p \| u \|_{D^2,p}, \forall u \in D^2,p(G). \]

Then we are ready to prove (1.14).
Proof of (1.14). For $N \geq 5$, $\alpha \in (0, N)$, $p > \frac{N+\alpha}{N-4}$, then $\frac{2Np}{N+\alpha} > 2^{**} = \frac{2N}{N-4}$, by (5.5) and the HLS inequality (1.4) we get

$$\tag{5.6} \sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p \leq C_{p,\alpha} \left( \sum_{G} |u|^\frac{2Np}{N+\alpha} \right)^\frac{N+\alpha}{2Np} \leq C_{p,\alpha} \tilde{C} \frac{2Np}{N+\alpha} \left( \sum_{G} |\Delta u|^2 \right)^p, \quad \forall u \in D^{2,2}(G).$$

The optimal constant in the inequality (5.6) is given by

$$\tag{5.7} K_3 := \sup_{\|u\|_{D^{2,2}} = 1} \sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p.$$ 

And consider the maximizing sequence $\{u_n\} \subset D^{2,2}(G)$ satisfying

$$\tag{5.8} \|u_n\|_{D^{2,2}} = 1, \quad \sum_{G} \left( R_{\alpha} \ast |u_n|^p \right) |u_n|^p \to K_3, \quad n \to \infty.$$ 

Since $p > \frac{N+\alpha}{N-4}$ yields $\frac{2Np}{N+\alpha} = r^{**} > 2^{**}$, that is $r = \frac{2Np}{N+\alpha+4p} \in \left( \frac{2Np}{N+\alpha+4p}, \frac{2Np}{N+\alpha+2p} \right)$, we can choose $q$ such that $2 < q < r < \infty$. By the interpolation inequality we have

$$C_{\alpha,p,q} \left( \sum_{G} \left( R_{\alpha} \ast |u_n|^p \right) |u_n|^p \right)^\frac{q}{2p} \leq \|\Delta u_n\|_q^2 \leq \|\Delta u_n\|_2^2 \|\Delta u_n\|_\infty^{q-2} \leq \|\Delta u_n\|_\infty^{q-2},$$

where $C_{\alpha,p,q}$ is the constant in the inequality (5.6).

Taking the limit, we obtain

$$C_{\alpha,p,q} K_3^\frac{q}{2p} \leq \lim_{n \to \infty} \|\Delta u_n\|_\infty^{q-2}.$$ 

This proves $\lim_{n \to \infty} \|\Delta u_n\|_\infty > 0$.

By the translation and taking a subsequence if necessary as Remark 3.4, we can get a maximizing sequence $\{u_n\}$ satisfying (5.8), $u_n \to u$ pointwise on $G$, and $|\Delta u(e)| > 0$.

By Lemma 2.6, Lemma 2.7, the inequalities (4.1) and (5.6), then passing to a subsequence if necessary, we have

$$K_3 = \lim_{n \to \infty} \sum_{G} \left( R_{\alpha} \ast |u_n|^p \right) |u_n|^p$$

$$= \lim_{n \to \infty} \sum_{G} \left( R_{\alpha} \ast |u_n|^p \right) |u_n|^p$$

$$= \lim_{n \to \infty} \sum_{G} \left( R_{\alpha} \ast |u_n - u|^p \right) |u_n - u|^p + \sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p$$

$$\leq \lim_{n \to \infty} \sum_{G} \left( R_{\alpha} \ast |u_n - u|^p \right) |u_n - u|^p + \sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p$$

$$\leq \lim_{n \to \infty} \frac{K_3 |u_n - u|_{D^{2,2}}^{2p} + \sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p}{|u_n - u|_{D^{2,2}}^{2p} + |u|_{D^{2,2}}^{2p}}.$$

Since $\|u\|_{D^{2,2}} \neq 0$, we have that

$$\sum_{G} \left( R_{\alpha} \ast |u|^p \right) |u|^p \geq K_3 \|u\|_{D^{2,2}}^{2p},$$
which yields
\[ \sum (R_\alpha \ast |u|^p) |u|^p = K_3 |u|^{2p}_{D^{2,2}}. \]
By the same argument, \( |u|_{D^{2,2}} = 1 \) and \( u \) is a maximizer. Let \( v \) be a solution of
\[ -\Delta v = |-\Delta u| \in \ell^2(G). \]
Since the Laplace operator \( \Delta \) is an isometry from \( D^{2,2}(G) \) to \( \ell^2(G) \) (see [30, Theorem 12] for a proof), and by the inequality (5.6),
\[ \sum (R_\alpha \ast |v|^p) |v|^p \lesssim |\Delta v|^{2p}_{2} = |u|^{2p}_{D^{2,2}}. \]
Note that \( u \leq v \) by the maximum principle. Replacing \( u \) by \(-u\), we get \(-u \leq v\) similarly. Hence,
\[ 0 \leq |u| \leq v. \]
In particular we have \( |\Delta v|_2 = |\Delta u|_2 = 1 \) and
\[ \sum (R_\alpha \ast |u|^p) |u|^p \leq \sum (R_\alpha \ast v^p) v^p. \]
Therefore, we know that \( v \) is a non-negative maximizer. It follows from the Lagrange multiplier that \( v \) is a non-negative solution of (1.14). The maximum principle yields that it is positive. \( \Box \)

For \( N \geq 5, \alpha \in (0, N - 4), 1 < p < \frac{N - \alpha}{N + \alpha} \), then \( \frac{2Np}{N + \alpha} > p^* = \frac{Np}{N - 2p} \), hence by the second-order Sobolev inequality (5.5) and the HLS inequality (1.4) we get

\[ \sum_G (R_\alpha \ast |u|^p) |u|^p \leq C_{p,\alpha} \left( \sum_G |u|^{\frac{2Np}{N + \alpha}} \right)^{\frac{N + \alpha}{Np}} \]
\[ \leq C_{p,\alpha} \tilde{C}_{N,\alpha} \frac{2Np}{N + \alpha} \left( \sum_G |\Delta u|^p \right)^2, \forall u \in D^{2,p}(G). \]

Similarly, we can define the variational problem for (5.10) and get a positive solution of (1.15) as before.

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Ruowei Li: School of Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China; Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, People’s Republic of China

Email address: rwli19@fudan.edu.cn