Abstract

Purpose – In this paper, the author presents a hybrid method along with its error analysis to solve (1+2)-dimensional non-linear time-space fractional partial differential equations (FPDEs).

Design/methodology/approach – The proposed method is a combination of Sumudu transform and a semi-analytic technique Daftardar-Gejji and Jafari method (DGJM).

Findings – The author solves various non-trivial examples using the proposed method. Moreover, the author obtained the solutions either in exact form or in a series that converges to a closed-form solution. The proposed method is a very good tool to solve this type of equations.

Originality/value – The present work is original. To the best of the author’s knowledge, this work is not done by anyone in the literature.

Keywords Caputo derivative, FPDEs, Error analysis, Sumudu transform, Daftardar-Gejji and Jafari method, Population model

Paper type Research paper

1. Introduction

In the past few decades, Fractional Calculus has drawn the attention of many researchers due to its wide applicability in all disciplines. In contrast to ordinary derivatives, fractional derivatives are non-local in nature and carry the past information [3–6]. Differential equations of fractional orders have become an essential tool to understand real-life problems. It has been established that fractional order partial differential equations (FPDEs) provide an appropriate framework for the description of anomalous and non-Brownian diffusion. They are more effective while developing processes having memory effects [7,8]. Recently, various models such as fractal foam drainage model [9], Klein-Gordon model [10], a fractal model for the soliton motion [11] and so on in micro-gravity space have been discussed. The fractal Schrödinger system using the fractal derivatives has been studied in references [12,13].

Several analytic approximate and numerical methods have been developed to solve fractional differential equations and FPDEs in the literature. For solving linear-differential equations transform methods such as Laplace transform [14], Fourier transform [15], Mellin transform [16,17], fractional Fourier transform [18], natural transform [19], Sumudu transform [1], Elzaki transform [20], Differential transform [21], Jafari transform [22] and so on are useful. Further, several decomposition/iterative methods such as Adomian

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Data Availability: The data that supports the findings of this study are available within the article.
decomposition [23], homotopy perturbation [24], Daftardar-Gejji and Jafari [2], variational iteration [25] and so on are developed to solve linear/non-linear FPDEs that give solutions in terms of convergent series. Moreover, these methods have become popular as they do not involve discretization and are free from rounding off errors. Furthermore, various hybrid methods i.e. combinations of integral transform and decomposition/iterative methods such as iterative Laplace transform [26], fractional Laplace homotopy perturbation transform [27], homotopy perturbation Sumudu transform [28], Sumudu decomposition [29], Laplace decomposition [30], Laplace homotopy analysis [31], homotopy perturbation transform [32], etc. have proven to be quite effective. In 2016, Wang and Liu [33] introduced a hybrid method known as Sumudu transform iterative method (STIM). STIM is a combination of Sumudu transform and the Daftardar-Gejji and Jafari method (DGJM). Besides, it is observed that DGJM with Sumudu transform requires less computational time to solve non-linear fractional models as compared to the traditional methods, and gives more accuracy. In this paper, we extend STIM along with its error analysis for solving general (1+2)-dimensional time-space FPDEs. Moreover, the utility and efficiency of STIM is demonstrated by solving various non-trivial examples.

The organization of this paper is as follows: In section 2, the author gives some basic definitions, properties of fractional calculus and Sumudu transform. In section 3, the author develops STIM for (1+2)-dimensional time-space FPDEs, whereas its error-analysis is presented in section 4. In section 5, the author solves various non-linear (1+2)-dimensional time-space FPDEs. Finally the author draws conclusions in section 6.

2. Preliminaries

Useful definitions and properties of fractional calculus and Sumudu transform are presented here.

**Definition 2.1.** [34] *Mittag-Leffler function with one parameter μ is defined as*

\[
E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)}, \quad \text{Re}(\mu) > 0, \quad z \in \mathbb{C}.
\]  

Whereas the Mittag-Leffler function with two parameters μ and ν is defined as [35]

\[
E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}, \quad \text{Re}(\mu) > 0, \quad z, \nu \in \mathbb{C}.
\]

**Definition 2.2.** [36] *Caputo partial fractional derivative of order μ > 0 with respect to the variable t, of w(t, x, y) is defined as*

\[
\frac{\partial^\mu w(t, x, y)}{\partial t^\mu} = \begin{cases} 
\frac{1}{\Gamma(n - \mu)} \int_0^t (t - \tau)^{n-\mu-1} \frac{\partial^n w(\tau, x, y)}{\partial \tau^n} d\tau, & n - 1 < \mu < n, \\
\frac{\partial^n w(t, x, y)}{\partial t^n}, & \mu = n \in \mathbb{N}.
\end{cases}
\]
Note:

(1) If $\mu = m$, where $m$ is a positive integer then

$$
\frac{d^\mu t^\mu}{dt^\mu} = \begin{cases} 
0, & \text{if } q \in \{0, 1, 2, \ldots, m - 1\}, \\
\Gamma(q + 1) t^\mu, & \text{if } q \in \mathbb{N} \text{ and } q \geq m, \text{ or } q \not\in \mathbb{N} \text{ and } q > m - 1.
\end{cases}
$$

(2) The Caputo derivative of one parameter Mittag-Leffler function is

$$
\frac{d^\mu E_\mu(kt^\mu)}{dt^\mu} = kE_\mu(kt^\mu), \quad \mu > 0, \quad k \in \mathbb{R}.
$$

Note that we consider the fractional partial derivatives $\frac{\partial^m w(t,x,y)}{\partial x^m}$ and $\frac{\partial^m w(t,x,y)}{\partial y^m}$, where $m, p \in \mathbb{N}$ as the sequential fractional derivatives [37,38] i.e.

$$
\frac{\partial^m w(t,x,y)}{\partial x^m} = \frac{\partial^\nu}{\partial x^\nu} \frac{\partial^\nu}{\partial x^\nu} \cdots \frac{\partial^\nu}{\partial x^\nu} w(t,x,y), \quad \frac{\partial^m w(t,x,y)}{\partial y^m} = \frac{\partial^\nu}{\partial y^\nu} \frac{\partial^\nu}{\partial y^\nu} \cdots \frac{\partial^\nu}{\partial y^\nu} w(t,x,y).
$$

**Definition 2.3.** [39] The Sumudu transform (ST) over the set of functions $D = \{ \phi(t) \mid \exists L, \theta_k > 0, k = 1, 2 \text{ such that } |\phi(t)| < L e^{(t/\theta)} \text{ if } t \in (-1)^k \times [0, \infty) \}$ is defined as

$$
S[\phi(t)](\rho) = F(\rho) = \int_0^\infty \frac{1}{\rho} e^{-\rho t} \phi(t) dt = \int_0^\infty e^{-\rho t} \phi(t) dt, \quad \rho \in (-\theta_1, \theta_2).
$$

The Sumudu transform of the power function $t^\mu$ is

$$
S[t^\mu](\rho) = \Gamma(\mu + 1)\rho^\mu, \quad \mu > -1.
$$

**Definition 2.4.** [39] The inverse Sumudu transform (IST) of $F(\rho)$ is denoted by $\phi(t)$, and is defined by the following integral:

$$
\phi(t) = S^{-1}[F(\rho)] = \frac{1}{2\pi i} \int_{z=\infty}^{z=\infty} \frac{1}{\rho} e^{\rho t} F(\rho) d\rho, \text{ where } \text{Re}(1/\rho) > z, \quad z \in \mathbb{C}.
$$

Inverse Sumudu transform of $\rho^\mu$ is

$$
S^{-1}[\rho^\mu] = \frac{t^\mu}{\Gamma(\mu + 1)}, \quad \mu > -1.
$$

**Definition 2.5.** [40] Sumudu transform of a time-fractional Caputo derivative of order $\mu > 0$ of a real valued function $w(t,x,y)$, where $m - 1 < \mu \leq m, \quad m \in \mathbb{N}$ is defined as

$$
S\left[ \frac{\partial^\mu w(t,x,y)}{\partial t^\mu} \right] = \rho^{-\mu} S[w(t,x,y)] - \sum_{k=0}^{m-1} \rho^{-\mu+k} S[w(0,x,y)] \frac{\partial^{\mu-k} w(0,x,y)}{\partial t^{\mu-k}}.
$$
3. Sumudu transform iterative method (STIM) for (1+2)-dimensional time and space FPDEs

In this section, the author extends STIM [33] to solve (1+2)-dimensional time-space FPDEs.

Consider the following general non-linear time-space FPDE in (1+2)-dimensional \((l, m, n, p, q \in \mathbb{N})\):

\[
\frac{\partial^\mu w(t, x, y)}{\partial t^\mu} = \mathcal{F}\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right),
\]

\[
\frac{\partial^k w(0, x, y)}{\partial t^k} = h_k(x, y), \quad k = 0, 1, 2, \ldots, l - 1,
\]

where \(\mu \in (l - 1, l], \nu \in (n - 1, n], \gamma \in (1 - 1, q] \) and \(w\) is a function of three variables \(t, x, y\) and \(\mathcal{F}\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right)\) a known non-linear operator. Taking ST on both sides of eqn (6) and simplifying, we get

\[
S[w(t, x, y)] = \sum_{k=0}^{l-1} \rho^k \frac{\partial^k w(0, x, y)}{\partial t^k} + \rho^\mu S\left[\mathcal{F}\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right)\right].
\]

Taking IST of eqn (7), we get

\[
w(t, x, y) = S^{-1}\left(\sum_{k=0}^{l-1} \rho^k h_k(x, y)\right) + S^{-1}\left[\rho^\mu S\left[\mathcal{F}\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right)\right]\right].
\]

Eqn (8) is of the following form

\[
w(t, x, y) = g(x, y) + N\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right),
\]

where

\[
g(x, y) = S^{-1}\left(\sum_{k=0}^{l-1} \rho^k h_k(x, y)\right),
\]

\[
N = S^{-1}\left[\rho^\mu S\left[\mathcal{F}\left(x, y, w, \frac{\partial^\nu w}{\partial x^\nu}, \ldots, \frac{\partial^{mu} w}{\partial x^{mu}}, \frac{\partial^p w}{\partial y^p}, \ldots, \frac{\partial^{pq} w}{\partial y^{pq}}\right)\right]\right],
\]

in which \(g(x, y)\) is a known function and \(N\) a known non-linear operator. The author applies DGJM to solve eqn (9), in which the solution is expressed in terms of the following infinite series

\[
w(t, x, y) = \sum_{i=0}^{\infty} w_i,
\]
where $w_i$ are calculated recursively. According to DGJM, the non-linear operator $N$ is decomposed as

\[
N = N\left(x, y, w_0, \frac{\partial^x w_0}{\partial x^x}, \ldots, \frac{\partial^m w_0}{\partial x^m}, \frac{\partial^y w_0}{\partial y^y}, \ldots, \frac{\partial^n w_0}{\partial y^n}\right)
\]

\[
+ \sum_{j=1}^{\infty} \left( N\left(x, y, \sum_{i=0}^{j-1} w_i, \frac{\partial^x \left( \sum_{i=0}^{j-1} w_i \right)}{\partial x^x}, \ldots, \frac{\partial^m \left( \sum_{i=0}^{j-1} w_i \right)}{\partial x^m}, \frac{\partial^y \left( \sum_{i=0}^{j-1} w_i \right)}{\partial y^y}, \ldots, \frac{\partial^n \left( \sum_{i=0}^{j-1} w_i \right)}{\partial y^n}\right) \right)
\]

\[
- \sum_{j=1}^{\infty} \left( N\left(x, y, \sum_{i=0}^{j-1} w_i, \frac{\partial^x \left( \sum_{i=0}^{j-1} w_i \right)}{\partial x^x}, \ldots, \frac{\partial^m \left( \sum_{i=0}^{j-1} w_i \right)}{\partial x^m}, \frac{\partial^y \left( \sum_{i=0}^{j-1} w_i \right)}{\partial y^y}, \ldots, \frac{\partial^n \left( \sum_{i=0}^{j-1} w_i \right)}{\partial y^n}\right) \right)
\]

(12)

\[
S^{-1} \left[ \rho^\mu S \left( \mathcal{F} \left(x, y, \sum_{i=0}^{\infty} w_i, \frac{\partial^x \left( \sum_{i=0}^{\infty} w_i \right)}{\partial x^x}, \ldots, \frac{\partial^m \left( \sum_{i=0}^{\infty} w_i \right)}{\partial x^m}, \frac{\partial^y \left( \sum_{i=0}^{\infty} w_i \right)}{\partial y^y}, \ldots, \frac{\partial^n \left( \sum_{i=0}^{\infty} w_i \right)}{\partial y^n}\right) \right) \right]
\]

(13)
Using eqns (11, 13) in eqn (9), we get
\[ \sum_{i=0}^{\infty} w_i = S^{-1} \left( \sum_{k=0}^{l-1} \left[ \rho^k h_k(x, y) \right] \right) + S^{-1} \left[ \rho^0 S \left( \mathcal{F} \left( x, y, w_0, \frac{\partial^r w_0}{\partial x^r}, \cdots, \frac{\partial^{mu} w_0}{\partial x^{mu}} \right) \right) \right] \]

Thus, the author defines the recursive relation to calculate \( w_i \), \( i \in \{0\} \cup \mathbb{N} \) as follows:

\[
\begin{align*}
    w_0 &= S^{-1} \left( \sum_{k=0}^{l-1} \left[ \rho^k h_k(x, y) \right] \right), \\
    w_1 &= S^{-1} \left[ \rho^0 S \left( \mathcal{F} \left( x, y, w_0, \frac{\partial^r w_0}{\partial x^r}, \cdots, \frac{\partial^{mu} w_0}{\partial x^{mu}} \right) \right) \right], \\
    w_{r+1} &= S^{-1} \left[ \rho^0 S \left( \mathcal{F} \left( x, y, \sum_{i=0}^{r} w_i, \frac{\partial^r \left( \sum_{i=0}^{r} w_i \right)}{\partial x^r}, \cdots, \frac{\partial^{mu} \left( \sum_{i=0}^{r} w_i \right)}{\partial x^{mu}} \right) \right) \right], \quad r \geq 1.
\end{align*}
\]

Note that the \( k \) – term STIM approximate solution of eqn (6) is: \( w(t, x, y) \approx w_0 + w_1 + \cdots + w_{k-1} \).
4. Error analysis
The author presents the error analysis of the proposed method by proving the following theorem.

**Theorem 4.1.** Let $N$ be a non-linear operator form a Hilbert space $\mathbb{H} \rightarrow \mathbb{H}$ and $w(t, x, y)$ be the solution of eqn (6). Then the series $\sum_{i=0}^{\infty} w_i$ converges to $w(t, x, y)$ if $\|w_m(t, x, y)\| \leq \mu \|w_m-1(t, x, y)\|$, $\forall m \in \mathbb{N}$ and $0 < \mu < 1$. Further, let $s_m = \sum_{h=0}^{m-1} w_h(t, x, y)$ be the $m$-term approximate solution of eqn (6). Then we have

$$\|w(t, x, y) - s_m\| \leq \frac{\mu^m}{(1 - \mu)} \|w_0(t, x, y)\|.$$

**Proof.** It is clear that

$$\|w(t, x, y) - s_m\| = \left\| w(t, x, y) - \sum_{h=0}^{m-1} w_h(t, x, y) \right\| = \left\| \sum_{h=m}^{\infty} w_h(t, x, y) \right\|$$

$$\leq \sum_{h=m}^{\infty} \|w_h(t, x, y)\| \leq \sum_{h=m}^{\infty} \mu^h \|w_0(t, x, y)\|$$

$$\leq (\mu^m + \mu^{m+1} + \cdots) \|w_0(t, x, y)\|$$

$$\leq \frac{\mu^m}{(1 - \mu)} \|w_0(t, x, y)\|,$$

which is the required result. □

5. Illustrative examples
5.1 Time-space fractional non-linear Boussinesq equation
Consider the following non-linear time-space fractional $(1+2)$-dimensional Boussinesq equation [41]:

$$\frac{\partial^\nu w}{\partial t^\nu} = \frac{\partial^\nu}{\partial x^\nu} \left( (k_1 w + k_2) \frac{\partial^\nu (k_1 w + k_2)}{\partial x^\nu} \right) + \frac{\partial^\nu}{\partial y^\nu} \left( (k_1 w + k_2) \frac{\partial^\nu (k_1 w + k_2)}{\partial y^\nu} \right), \quad (15)$$

along with the initial condition

$$w(0, x, y) = a + by^\gamma, \quad t > 0, \quad \mu, \nu, \gamma \in (0, 1), \quad (16)$$

where $k_1, k_2, a, b$ are arbitrary constants.

Taking ST on both sides of eqn (15) and using the property (5), we get

$$S[w(t, x, y)] = w(0, x, y) + \rho^\nu \left( S \left[ \frac{\partial^\nu}{\partial x^\nu} \left( (k_1 w + k_2) \frac{\partial^\nu (k_1 w + k_2)}{\partial x^\nu} \right) \right] + \frac{\partial^\nu}{\partial y^\nu} \left( (k_1 w + k_2) \frac{\partial^\nu (k_1 w + k_2)}{\partial y^\nu} \right) \right), \quad (17)$$
Taking the IST on both sides of eqn (17), we obtain
\[ w(t,x,y) = S^{-1}[w(0,x,y)] + S^{-1}\left[ \rho^\nu \left( S \left[ \frac{\partial^\nu}{\partial x^\nu} \left( (k_1w + k_2) \frac{\partial^\nu (k_1w + k_2)}{\partial x^\nu} \right) \right] + \frac{\partial^\nu}{\partial y^\nu} \left( (k_1w + k_2) \frac{\partial^\nu (k_1w + k_2)}{\partial y^\nu} \right) \right] \right]. \] (18)

Using the recurrence relation (14), we get
\begin{align*}
w_0 &= w(0,x,y) = a + by^\gamma, \\
w_1 &= S^{-1}\left[ \rho^\nu \left( S \left[ \frac{\partial^\nu}{\partial x^\nu} \left( (k_1w_0 + k_2) \frac{\partial^\nu (k_1w_0 + k_2)}{\partial x^\nu} \right) \right] + \frac{\partial^\nu}{\partial y^\nu} \left( (k_1w_0 + k_2) \frac{\partial^\nu (k_1w_0 + k_2)}{\partial y^\nu} \right) \right] \right], \\
&= \frac{bk_1^2 \Gamma(\gamma + 1)^2}{\Gamma(\mu + 1)} t^\mu, \\
w_j &= 0, \quad \forall j \geq 2.
\end{align*}

Hence, we obtain the following exact solution of (15-16)
\[ w(t,x,y) = a + by^\gamma + \frac{bk_1^2 \Gamma(\gamma + 1)^2}{\Gamma(\mu + 1)} t^\mu. \]

Note that for \( a = \frac{9}{5}, \ b = e^2 \) the STIM solution is same as obtained by invariant subspace method in reference [41].

5.2 Time-space fractional non-linear diffusion equation
Consider the following non-linear time-space fractional (1+2)-dimensional diffusion like FPDE:
\[ \frac{\partial^\mu w(t,x,y)}{\partial t^\mu} = \frac{1}{2} \left( y^\nu \frac{\partial^{2\nu} w(t,x,y)}{\partial x^{2\nu}} + x^\nu \frac{\partial^{2\nu} w(t,x,y)}{\partial y^{2\nu}} \right), \] (19)
along with the initial condition
\[ w(0,x,y) = y^{2\nu}, \quad t > 0, \quad \mu, \nu, \gamma \in (0,1). \] (20)

Taking ST and then IST on both sides of eqn (19), we obtain
\[ w(t,x,y) = S^{-1}[w(0,x,y)] + S^{-1}\left[ \rho^\nu \left( \frac{1}{2} y^{2\nu} \frac{\partial^{2\nu} w}{\partial x^{2\nu}} + x^{2\nu} \frac{\partial^{2\nu} w}{\partial y^{2\nu}} \right) \right]. \] (21)
In view of the recurrence relation (14), we obtain
\[ w_0 = S^{-1}[v(0, x, y)] = y^{2r}, \]
\[ w_1 = S^{-1}\left[ \rho^m S\left( \frac{1}{2} \left( y^{2r} \frac{\partial^2 w_0}{\partial x^{2r}} + x^{2r} \frac{\partial^2 w_0}{\partial y^{2r}} \right) \right) \right], \]
\[ = \frac{\Gamma(2r + 1) t^{2r} x^{2r}}{2\Gamma(\mu + 1)}, \]
\[ w_2 = S^{-1}\left[ \rho^m S\left( \frac{1}{2} \left( y^{2r} \frac{\partial^2 (w_0 + w_1)}{\partial x^{2r}} + x^{2r} \frac{\partial^2 (w_0 + w_1)}{\partial y^{2r}} \right) \right) \right], \]
\[ = \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} y^{2r}}{4\Gamma(2 \mu + 1)}, \]
\[ w_3 = \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} x^{2r}}{8\Gamma(3 \mu + 1)}, \]
\[ w_4 = \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} y^{2r}}{16\Gamma(4 \mu + 1)}, \]
\[ w_5 = \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} x^{2r}}{32\Gamma(5 \mu + 1)} \]
\[ \vdots \]

Hence, the series solution of (19-20) is
\[ w(t, x, y) = y^{2r} + \frac{\Gamma(2r + 1) \mu t^{2r} x^{2r}}{2\Gamma(\mu + 1)} + \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} y^{2r}}{4\Gamma(2 \mu + 1)} \]
\[ + \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} x^{2r}}{8\Gamma(3 \mu + 1)} + \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} y^{2r}}{16\Gamma(4 \mu + 1)} \]
\[ + \frac{\Gamma(2r + 1) \Gamma(2r + 1) t^{2r} x^{2r}}{32\Gamma(5 \mu + 1)} + \cdots, \]
which converges to
\[ w(t, x, y) = \lambda_1 \nu^2 E_{2\mu, \mu+1}(\lambda t^{2r})x^{2v} + (1 + \lambda t^{2r} E_{2\mu, 2\mu+1}(\lambda t^{2r}))y^{2v}, \]
where \( \lambda_1 = \frac{\Gamma(2r+1)}{2}, \lambda_2 = \frac{\Gamma(2r+1)}{2} \) and \( \lambda = \lambda_1 \lambda_2. \)

5.3 Time-space fractional equation: population model-I
Consider the following time-space fractional generalized biological population model equation [42]:
\[ \frac{\partial^\mu w(t, x, y)}{\partial t^\mu} = \frac{\partial^2 w}{\partial x^{2r}} + \frac{\partial^2 w}{\partial y^{2r}} + hw(1 - rw), \quad t > 0, \quad \mu, \nu, \gamma \in (0, 1), \] (22)
along with the initial condition
\[ w(0, x, y) = E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right). \]  
(23)

Taking ST and IST of eqn (22), we get
\[ w(t, x, y) = S^{-1}[w(0, x, y)] + S^{-1} \left[ \rho^\nu S \left( \frac{\partial^{2\nu} w}{\partial x^{2\nu}} + \frac{\partial^{2\nu} w}{\partial y^{2\gamma}} + hw(1 - rw) \right) \right]. \]

Using the recurrence relation (14), we obtain
\[
\begin{align*}
  w_0 &= S^{-1}[w(0, x, y)] = E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right), \\
  w_1 &= S^{-1} \left[ \rho^\nu S \left( \frac{\partial^{2\nu} w_0}{\partial x^{2\nu}} + \frac{\partial^{2\nu} w_0}{\partial x^{2\gamma}} + hw_0(1 - rw_0) \right) \right] \\
   &= E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right) \frac{ht}{\Gamma(\alpha + 1)}, \\
  w_2 &= E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right) \frac{h^2 t^{2\alpha}}{\Gamma(2\alpha + 1)}, \\
  w_3 &= E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right) \frac{h^3 t^{3\alpha}}{\Gamma(3\alpha + 1)}.
\end{align*}
\]

Hence, the series solution of eqns (22-23) is
\[ w(t, x, y) = E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right) \left( 1 + \frac{ht}{\Gamma(\alpha + 1)} + \frac{h^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots \right), \]  
(24)

which converges to the following closed form solution:
\[ w(t, x, y) = E_v \left( \frac{r}{8} x^\nu \right) E_\gamma \left( \frac{r}{8} y^\gamma \right) E_\mu (ht). \]  
(25)

Note that for \( \mu = \nu = \gamma = 1 \), we have \( w(t, x, y) = e^{\sqrt{2} \sqrt{h} (x^\nu + y^\nu)} \), which is the standard solution of the biological model given in reference [43].

5.4 Time-space fractional equation: population model-II

Consider the following non-linear time-space fractional order biological population model equation [44, 45]:
\[ \frac{\partial^\mu w(t, x, y)}{\partial t^\mu} = \frac{\partial^{2\nu} w}{\partial x^{2\nu}} + \frac{\partial^{2\nu} w}{\partial y^{2\gamma}} + hw, \quad t > 0, \quad \mu, \nu, \gamma \in (0, 1), \]  
(26)

along with the initial condition
\[ w(0, x, y) = k_1 + k_2(x^\nu y^\gamma)^{\frac{1}{2}}, \quad h, k_1, k_2 \in \mathbb{R}. \]  
(27)
Taking ST and IST of (26), we get

$$w(t, x, y) = S^{-1}[w(0, x, y)] + S^{-1} \left[ \rho^\mu S \left( \frac{\partial^2 w^2}{\partial x^2} + \frac{\partial^\gamma w^2}{\partial x^\gamma} + hw \right) \right].$$  \hspace{1cm} (28)

In view of the recurrence relation (14), we get

$$w_0 = S^{-1}[w(0, x, y)] = k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}},$$

$$w_1 = S^{-1} \left[ \rho^\mu S \left( \frac{\partial^2 w^2_0}{\partial x^2} + \frac{\partial^\gamma w^2_0}{\partial x^\gamma} + hw_0 \right) \right] = \left( k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}} \right) \frac{ht^\mu}{\Gamma(\mu + 1)},$$

$$w_2 = \left( k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}} \right) \frac{h^2 t^{2\mu}}{\Gamma(2\mu + 1)},$$

$$w_3 = \left( k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}} \right) \frac{h^3 t^{3\mu}}{\Gamma(3\mu + 1)},$$

\[ \vdots \]

Hence, the series solution of (26-27) is

$$w(t, x, y) = \left( k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}} \right) \left( 1 + \frac{ht^\mu}{\Gamma(\mu + 1)} + \frac{h^2 t^{2\mu}}{\Gamma(2\mu + 1)} + \frac{h^3 t^{3\mu}}{\Gamma(3\mu + 1)} + \cdots \right),$$

which converges to the following closed form

$$w(t, x, y) = \left( k_1 + k_2 (x^r y^\gamma)^{\frac{1}{2}} \right) E_\mu(ht^\mu).$$

6. Conclusions

The author presented a hybrid method along with its error-analysis for solving non-linear (1+2)-dimensional time-space FPDEs, where the derivatives are considered in Caputo sense. Moreover, the author solved various non-trivial examples of (1+2)-dimensional FPDEs to demonstrate the utility and efficiency of the proposed method. It has been observed that the proposed method is accurate and provides the solutions either in a convergent series or in exact form. Thus, the proposed method is a useful tool to solve non-linear (1+2)-dimensional time-space FPDEs.

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