Lower Bound and Exact Values for the Boundary Independence Broadcast Number of a Tree

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Abstract

A broadcast on a nontrivial connected graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of $v$) for all $v \in V$. The weight of $f$ is $\sigma(f) = \sum_{v \in V} f(v)$. A vertex $u$ hears $f$ from $v$ if $f(v) > 0$ and $d(u, v) \leq f(v)$.

A broadcast $f$ is boundary independent if, for any vertex $w$ that hears $f$ from vertices $v_1, \ldots, v_k$, $k \geq 2$, we have that $d(w, v_i) = f(v_i)$ for each $i$. The maximum weight of a boundary independent broadcast on $G$ is denoted by $\alpha_{\text{bn}}(G)$. We prove a sharp lower bound on $\alpha_{\text{bn}}(T)$ for a tree $T$. Combined with a previously determined upper bound, this gives exact values of $\alpha_{\text{bn}}(T)$ for some classes of trees $T$. We also determine $\alpha_{\text{bn}}(T)$ for trees with exactly two branch vertices and use this result to demonstrate the existence of trees for which $\alpha_{\text{bn}}$ lies strictly between the lower and upper bounds.

Keywords: broadcast domination; broadcast independence; hearing independence; boundary independence

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1 Introduction

An independent set $X$ of vertices in a graph $G$, when considered from the perspective of its vertices, has the property that no vertex in $X$ belongs to the neighbourhood of any other vertex

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in $X$ (i.e., vertices in $X$ are nonadjacent). Seen from the perspective of the edges of $G$, it has the property that no edge is covered by more than one vertex in $X$, that is, the neighbourhoods of vertices in $X$ overlap only on their boundaries. In generalizing independent sets to independent broadcasts, Erwin [11] used the former property, and defined a broadcast to be independent if no broadcasting vertex belongs to the neighbourhood of, or hears, another broadcasting vertex. We refer to this type of independent broadcast as being hearing independent. Focussing instead on the latter property, Mynhardt and Neilson [16] and Neilson [19] defined boundary independent broadcasts as broadcasts in which any edge is covered by at most one broadcasting vertex, that is, a vertex belongs to the broadcasting neighbourhoods of two or more broadcasting vertices only if it belongs to the boundaries of all such vertices.

Our goal here is to prove Theorem 2.1, which gives a tight lower bound on the boundary independence number of a tree. Together with an upper bound proved by Mynhardt and Neilson [18], this gives exact values of the boundary independence numbers of some classes of trees. We also determine an exact formula in Theorem 2.6 for boundary independence numbers of trees with exactly two branch vertices, and use this result to show that there exist trees whose boundary independence numbers lie strictly between the above-mentioned lower and upper bounds.

After giving the necessary definitions in Sections 1.1 and 1.2, we state the lower bound and the formula in Section 2. The proofs are given in Sections 4 and 5 respectively. For comparison with the lower bound we also give the upper bound from [18] in Section 2 and discuss some corollaries of the bounds. Section 3 contains previous results and lemmas required for the proofs of the main results. We conclude by listing open problems in Section 6. For undefined concepts we refer the reader to [9].

1.1 Broadcast definitions

As this paper is a sequel to the paper [18] by the same authors, the definitions and background information provided in this and the next subsections are essentially the same as those in [18]. A broadcast on a nontrivial connected graph $G = (V, E)$, as introduced by Erwin [11, 12], is a function $f : V \rightarrow \{0, 1, \ldots, \text{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of $v$) for all $v \in V$. If $G$ is disconnected, a broadcast on $G$ is the union of broadcasts on its components. The weight of $f$ is $\sigma(f) = \sum_{v \in V} f(v)$. Define $V_f^+ = \{v \in V : f(v) > 0\}$ and partition $V_f^+$ into the two sets $V_f^1 = \{v \in V : f(v) = 1\}$ and $V_f^{++} = V_f^+ - V_f^1$. A vertex in $V_f^+$ is called a broadcasting vertex. A vertex $u$ hears $f$ from $v \in V_f^+$, and $v$ $f$-dominates $u$, if the distance $d(u, v) \leq f(v)$. If $d(u, v) < f(v)$, we also say that $v$ overdominates $u$. Denote the set of all vertices that do not hear $f$ by $U_f$. A broadcast $f$ is dominating if $U_f = \emptyset$. If $f$ is a broadcast such that every vertex $x$ that hears more than one broadcasting vertex also satisfies $d(x, u) \geq f(u)$ for all $u \in V_f^+$, we say that the broadcast only overlaps in boundaries. If $uv \in E(G)$ and $u, v \in N_f(x)$ for some $x \in V_f^+$ such that at least one of $u$ and $v$ does not belong to $B_f(x)$, we say that the edge $uv$ is covered in $f$, or $f$-covered, by $x$. If $uv$ is not covered by any $x \in V_f^+$, we say that $uv$ is uncovered by $f$ or $f$-uncovered. We denote the set of $f$-uncovered edges by $U_f^E$.

If $f$ and $g$ are broadcasts on $G$ such that $g(v) \leq f(v)$ for each $v \in V$, we write $g \leq f$. If in addition $g(v) < f(v)$ for at least one $v \in V$, we write $g < f$. A dominating broadcast $f$ on $G$ is a minimal dominating broadcast if no broadcast $g < f$ is dominating. The upper broadcast
number of $G$ is $\Gamma_b(G) = \max \{\sigma(f) : f$ is a minimal dominating broadcast of $G\}$. First defined by Erwin [11], the upper broadcast number was also studied by, for example, Ahmadi, Fricke, Schroeder, Hedetniemi and Laskar [1], Bouchemakh and Fergani [6], Bouchouika, Bouchemakh and Sopena [8], Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [10], and Mynhardt and Schroeder, Hedetniemi and Laskar [1], Bouchemakh and Fergani [6], Bouchouika, Bouchemakh by Erwin [11], the upper broadcast number was also studied by, for example, Ahmadi, Fricke, Schroeder, Hedetniemi and Laskar [1], Bouchemakh and Fergani [6], Bouchouika, Bouchemakh and Sopena [8], Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [10], and Mynhardt and Roux [15].

We denote the independence number of $G$ by $\alpha(G)$. If $f$ is characteristic function of an independent set of $G$, then no vertex in $V_f^+$ hears $f$ from any other vertex. To generalize the concept of independent sets, Erwin [11] defined a broadcast $f$ to be independent, or, for our purposes, hearing independent, if no vertex $u \in V_f^+$ hears $f$ from any other vertex $v \in V_f^+$; that is, broadcasting vertices only hear themselves. We denote the maximum weight of a hearing independent broadcast on $G$ by $\alpha_h(G)$. This version of broadcast independence was also considered by, among others, Ahmane, Bouchemakh and Sopena [2, 3], Bessy and Rautenbach [4, 5], Bouchemakh and Zemir [7], Bouchouika et al. [8] and Dunbar et al. [10]. For a survey of broadcasts in graphs, see the chapter by Henning, MacGillivray and Yang [13].

For a broadcast $f$ on a graph $G$ and $v \in V_f^+$, we define the

\[ f\text{-neighbourhood} \begin{cases} N_f(v) & = \{u \in V : d(u, v) \leq f(v)\} \\ B_f(v) & = \{u \in V : d(u, v) = f(v)\} \end{cases} \]

\[ f\text{-boundary} \begin{cases} PN_f(v) & = \{u \in N_f(v) : u \notin N_f(w) \text{ for all } w \in V_f^+-\{v\}\} \\ PB_f(v) & = \{u \in N_f(v) : u \text{ is not dominated by } (f-\{(v, f(v))\}) \cup \{(v, f(v)-1)\} \} \end{cases} \]

If $v \in V_f^1$ and $v$ does not hear $f$ from any vertex $u \in V_f^+ - \{v\}$, then $v \in PB_f(v)$; and if $v \in V_f^{++}$, then $PB_f(v) = B_f(v) \cap PN_f(v)$. Also note that $f$ is a broadcast that overlaps only in boundaries if and only if $N_f(u) \cap N_f(v) \subseteq B_f(u) \cap B_f(v)$ for all distinct $u, v \in V_f^+$. The characteristic function of an independent set also has the feature that it only overlaps in boundaries. To generalize this property, we define a broadcast to be boundary independent, abbreviated to bn-independent, if it overlaps only in boundaries. The maximum weight of a bn-independent broadcast on $G$ is the boundary independence number $\alpha_{bn}(G)$; such a broadcast is called an $\alpha_{bn}(G)$-broadcast, often abbreviated to $\alpha_{bn}$-broadcast if the graph $G$ is clear. The respective definitions imply that $\alpha(G) \leq \alpha_{bn}(G) \leq \alpha_h(G)$ for all graphs $G$. Boundary independent broadcasts were introduced by Neilsen [19] and Mynhardt and Neilsen [16], and also studied in [14, 17, 18]. For example, it was shown in [16] that $\alpha_h(G)/\alpha_{bn}(G) < 2$ for all graphs $G$, and the bound is asymptotically best possible. In [17] it was shown that $\alpha_{bn}$ and $\Gamma_b$ are not comparable, and that $\alpha_{bn}(G)/\Gamma_b(G) < 2$ for all graphs $G$, while $\Gamma_b(G)/\alpha_{bn}(G)$ is unbounded. A sharp upper bound for $\alpha_{bn}(T)$, where $T$ is a tree, was proved in [18] (see Theorem 2.2).

### 1.2 Definitions for trees

The statements of the upper bound in [18] for the boundary independence number of trees, and the lower bound in this paper, require some definitions of concepts pertaining to trees. A vertex of a tree $T$ of degree 3 or more is called a branch vertex. We denote the set of leaves of $T$ by $L(T)$, the set of branch vertices by $B(T)$ and the set of vertices of degree 2 by $W(T)$. The unique neighbour of a leaf is called a stem.
Clearly, a tree which has exactly one branch vertex is obtained by deleting all leaves of $T$. Equivalently, $\rho(T) = 1$. The branch vertex $b_1$ has leaf set $L(b_1) = \{l_1, l_2, l_3\}$.

For integers $k \geq 3$ and $n_i \geq 1, i \in \{1, \ldots, k\}$, the (generalized) spider $\text{Sp}(n_1, \ldots, n_k)$ is the tree which has exactly one branch vertex $b$, called the head, with $\text{deg}(b) = k$, and for which the $k$ components of $\text{Sp}(n_1, \ldots, n_k) - b$ are paths of lengths $n_1 - 1, \ldots, n_k - 1$, respectively. The legs $L_1, \ldots, L_k$ of the spider are the paths from $b$ to the leaves. Let $t_i$ be the leaf of $L_i$, $i = 1, \ldots, k$. If $n_i = r$ for each $i$, we write $\text{Sp}(n_1, \ldots, n_k) = \text{Sp}(r^k)$. A caterpillar of length $k \geq 0$ is a tree such that removing all leaves produces a path of length $k$, called the spine.

The next few concepts are illustrated in Figure 1. The branch-leaf representation $BL(T)$ of $T$ is the tree obtained by suppressing all vertices $v$ with $\text{deg}(v) = 2$, and the branch representation $B(T)$ of a tree $T$ with at least one branch vertex is obtained by deleting all leaves of $BL(T)$. Thus, $V(B(T)) = B(T)$, and two vertices $b_1, b_2 \in B(T)$ are adjacent in $B(T)$ if and only if the $b_1 - b_2$ path in $T$ contains no other branch vertices. We denote $|B(T)|$ by $b(T)$.

An endpath in a tree is a path ending in a leaf and having all internal vertices (if any) of degree 2. If there exists a $v - l$ endpath, where $v \in B(T)$ and $l \in L(T)$, then $v$ and $l$ are adjacent in $BL(T)$; we also say that $l$ belongs to $L(v)$, the leaf set of $v$, and we refer to $l$ as a leaf of $v$ (even though $l$ is not necessarily adjacent to $v$ in $T$). Since $BL(T)$ is unique, we can talk about $L(v)$ for any branch vertex $v$, where the reference to $BL(T)$ is implied but not specifically mentioned. Let $R(T)$ be the set of all branch vertices $w$ of $T$ such that $|L(w)| \leq 1$ and define $\rho(T) = |R(T)|$. Equivalently, $\rho(T)$ is the number of branch vertices of $T$ with at most one leaf, that is, the branch vertices which belong to at most one endpath.

We define subsets $B_i(T)$ and $B_{\geq i}(T)$ of $B(T)$ by

$$B_i(T) = \{v \in B(T) : |L(v)| = i\} \text{ and } B_{\geq i}(T) = \{v \in B(T) : |L(v)| \geq i\}.$$  

Clearly, $B_0(T) \cup B_1(T) \cup B_{\geq 2}(T)$ is a partition of $B(T)$ while $B_0(T) \cup B_1(T)$ is a partition of $R(T)$. See Figure 2. We also partition the set $W(T)$ of vertices of degree 2 into two subsets,
Figure 2: A tree $T$ with the vertices of $W_{\text{int}}(T)$ coloured blue, $B_0(T)$ coloured yellow, $B_1(T)$ coloured red, $B_{\geq 2}(T)$ coloured black, and $W_{\text{ext}}(T)$ and the leaves uncoloured. The subgraph $\text{Int}(T)$ is induced by the square vertices; $\{w_1, w_2, b_3, b_4\}$ is a maximal independent set of $\text{Int}(T)$, and $R(T) = \{b_2, b_3, b_4\}$.

$W_{\text{ext}}(T)$ for the \textit{external vertices of degree 2}, and $W_{\text{int}}(T)$ for the \textit{internal vertices of degree 2}, as follows:

$$W_{\text{ext}}(T) = \{u \in W(T) : u \text{ lies on an endpath} \} \text{ and } W_{\text{int}}(T) = W(T) - W_{\text{ext}}(T).$$

The subgraph of $T$ induced by $B_0(T) \cup B_1(T) \cup W_{\text{int}}(T)$ is called the \textit{interior subgraph} of $T$, denoted by $\text{Int}(T)$. Note that $\text{Int}(T)$ is not necessarily connected.

For a branch vertex $b$, let $T_b$ be the subtree of $T$ induced by all the $b-l$ paths from $b$ to leaves $l \in L(b)$. Then $T_b = K_1$ if $b \in B_0(T)$, $T_b$ is a path of length $d(l, b)$ if $b \in B_1(T)$ and $L(b) = \{l\}$, $T_b$ is a path of length $d(l_1, b) + d(l_2, b)$ if $L(b) = \{l_1, l_2\}$, and $T_b$ is a generalized spider $S(d(l_1, b), \ldots, d(l_k, b))$ if $L(b) = \{l_1, \ldots, l_k\}$, $k \geq 3$. Observe that $\text{Int}(T)$ can also be obtained by deleting the vertices of $T_b$ for each $b \in B_{\geq 2}(T)$, and the $b-l$ path, except for $b$, if $b \in B_1(T)$ and $L(b) = \{l\}$. For a tree $T$ with $b(T) \geq 1$, the

$$\begin{align*}
\max \text{ leaf value} \quad \sum \text{ loss}
\end{align*}$$

of a branch vertex $v$ is \begin{align*} 
\max(v) &= \max\{d_T(v, x) : x \in L(v)\}, \\
\sum(v) &= \sum_{x \in L(v)} d_T(v, x), \\
\text{loss}(v) &= \sum(v) - \max(v). 
\end{align*}

For the tree $T$ in Figure 2, $\sum(b_1) = 8$, $\max(b_1) = 4$ and $\text{loss}(b_1) = 4$.

2 Statement of main results and their corollaries

With the necessary definitions in place, we now state the lower bound for $\alpha_{bn}(T)$, where $T$ is a tree, as well as the upper bound in [18] for comparison. We defer the proof of the lower bound to Section 4 after stating some known results required for the proof.

**Theorem 2.1** If $T$ is a tree such that $b(T) \geq 1$, then

$$\alpha_{bn}(T) \geq n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)).$$

**Theorem 2.2** [18, Theorem 1.1] For any tree $T$ of order $n$, $\alpha_{bn}(T) \leq n - b(T) + \rho(T)$.
Figure 3: A tree $T$ of order 18 such that $W_{\text{int}}(T) = \emptyset$, $B_0(T) = \{v\}$, $B_1(T) = \{u\}$, $R(T) = \{u, v\}$, and $\alpha_{bn}(T) = n - b(T) + \alpha(T[R(T)]) = n - b(T) + \rho(T) = 14$.

This bound is sharp for generalized spiders (see Proposition 3.3) and for some caterpillars (Corollary 3.4). Combining Theorems 2.1 and 2.2 gives the following result.

**Theorem 2.3** For any tree $T$ of order $n$ that is not a path,

$$n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T).$$

When the tree $T$ has no internal vertices of degree 2 (i.e., the only vertices of degree 2 lie on endpaths), we have the following corollary.

**Corollary 2.4** If $T$ is a tree of order $n$ such that $b(T) \geq 1$ and $W_{\text{int}}(T) = \emptyset$, then

$$n - b(T) + \alpha(T[R(T)]) \leq \alpha_{bn}(T) \leq n - b(T) + \rho(T).$$

**Proof.** If $W_{\text{int}}(T) = \emptyset$, then $\text{Int}(T)$ is the subgraph of $T$ induced by $R(T)$ and the result follows from Theorem 2.3. ■

Note that the difference between the upper and lower bounds in Corollary 2.4 equals $|R(T)| - \alpha(T[R(T)])$. The following question was posed in [18]:

**Question 1** [18] Can the upper bound in Theorem 2.2 be improved to $\alpha_{bn}(T) \leq |V(T)| - b(T) + \alpha(T[R(T)])$?

If the answer to Question 1 turns out to be yes, Corollary 2.4 would give us an exact formula for $\alpha_{bn}$ for trees with branch vertices but no internal degree 2 vertices, namely $\alpha_{bn}(T) = |V(T)| - b(T) + \alpha(T[R(T)])$. As it stands, Corollary 2.4 gives us an exact formula for a subclass of these trees, an example of which is depicted in Figure 3.

**Corollary 2.5** If $T$ is a tree of order $n$ such that $b(T) \geq 1$, $W_{\text{int}}(T) = \emptyset$, and $R(T)$ is either empty or an independent set, then

$$\alpha_{bn}(T) = n - b(T) + \rho(T).$$
Proof. If $R(T)$ is empty or independent, then $\alpha(T[R(T)]) = |R(T)| = \rho(T)$ and the result follows from Corollary 2.4.

We next state a formula for $\alpha_{bn}(T)$, where $T$ is a tree with exactly two branch vertices; its proof can be found in Section 5. We use this result to show that there exist trees for which $\alpha_{bn}$ lies strictly between the bounds given in Theorem 2.3. All of these trees support a positive answer to Question 1.

Theorem 2.6 If $T$ is a tree of order $n$ such that $B(T) = \{b_1, b_2\}$, then

$$\alpha_{bn}(T) = n - 1 - \min\{\left\lceil \frac{1}{2}d(b_1, b_2) \right\rceil, \text{loss}(b_1), \text{loss}(b_2)\}.$$

3 Known results

In this section we present known results that are of interest or will be used later on. It is often useful to know when a $bn$-independent broadcast $f$ is maximal $bn$-independent, that is, there does not exist a $bn$-independent broadcast $g$ such that $f < g$.

Proposition 3.1 (i) [16, Proposition 2.2] A $bn$-independent broadcast $f$ on a graph $G$ is maximal $bn$-independent if and only if it is dominating and either $V_f^+ = \{v\}$ or $B_f(v) - PB_f(v) \neq \emptyset$ for each $v \in V_f^+$.

(ii) [14, Proposition 3.1] Let $f$ be a $bn$-independent broadcast on a connected graph $G$ such that $|V_f^+| \geq 2$. Then $f$ is maximal $bn$-independent if and only if each component of $G - U_f$ contains at least two broadcasting vertices.

Suppose $f$ is a $bn$-independent broadcast on $G$ and an edge $uv$ of $G$ is covered by vertices $x, y \in V_f^+$. By the definition of covered, $\{u, v\} \notin B_f(x)$ and $\{u, v\} \subseteq N_f(x) \cap N_f(y)$. This violates the $bn$-independence of $f$. Hence we have the following observation.

Observation 3.2 [16, Observation 2.3] If $f$ is a $bn$-independent broadcast on a graph $G$, then each edge of $G$ is covered by at most one vertex in $V_f^+$.

The following bound on $\alpha_{bn}(G)$ was proved in [16, 19].

Proposition 3.3 [16, Corollaries 2.6, 2.7] For any connected graph $G$ of order $n$ and any spanning tree $T$ of $G$, $\alpha_{bn}(G) \leq \alpha_{bn}(T) \leq n - 1$. Moreover, $\alpha_{bn}(G) = n - 1$ if and only if $G$ is a path or a generalized spider.

Consider caterpillars $T$ such that $W_{\text{int}}(T) = \emptyset$, i.e., there are no vertices of degree 2 between the first and last branch vertices on the spine, and such that $B_1(T)$ is either empty or an independent set. For such a caterpillar $T$, let $b_1, \ldots, b_k$ be the branch vertices of $T$, labelled from left to right on the spine. Note that $b_1, b_k \in B_{\geq 2}(T)$, i.e., $b_1, b_k \notin R(T)$. If $v$ is a branch vertex of a caterpillar $T$, then $v$ is a stem, hence $B_0(T) = \emptyset$ and $R(T) = B_1(T)$. The next result, which was obtained in [18] by explicitly defining a $bn$-independent broadcast of the appropriate weight, now follows from Corollary 2.5.
Corollary 3.4 [18, Corollary 5.1] If \( T \) is a caterpillar whose branch vertices induce a path \( P = (b_1, ..., b_k) \), and \( R(T) \) (i.e., the branch vertices among \( b_2, ..., b_{k-1} \) that are adjacent to exactly one leaf) is either empty or an independent set, then \( \alpha_{bn}(T) = |V(T)| - b(T) + \rho(T) \).

To establish the upper bound in Theorem 2.2 several lemmas were proved in [18]. For simplicity we state those that we need here as separate items of a single lemma.

Lemma 3.5 (i) [18, Lemma 3.1] For any tree \( T \) and any \( \alpha_{bn}(T) \)-broadcast \( f \), no leaf of \( T \) hears \( f \) from any non-leaf vertex.

(ii) [18, Lemma 3.2] For any tree \( T \) there exists an \( \alpha_{bn}(T) \)-broadcast \( f \) such that \( f(v) = 1 \) whenever \( v \in V_f^+ - L(T) \).

(iii) [18, Lemma 3.4] Let \( f \) be an \( \alpha_{bn}(T) \)-broadcast on a tree \( T \) such that a leaf \( l \) \( f \)-dominates a branch vertex \( w \). If \( l' \) is a leaf in \( L(w) \) that does not hear \( f \) from \( l \), then \( l' \in V_f^+ \), the \( l' - w \) path contains a vertex \( b \in B_f(l) \), and \( f(l') = d(b, l') \).

The final lemma demonstrates the importance of the branch vertices of a tree \( T \) in determining the broadcast values which may be assigned to its leaves. Informally, Lemma 3.6 states that in an \( \alpha_{bn}(T) \)-broadcast, a leaf \( l \) never overdominates a branch vertex \( b \) by exactly 2. Either \( l \) overdominates a branch vertex \( b \) by exactly 1 and \( b \) has no leaves except possibly \( l \), or \( l \) overdominates \( b \) by at least 3 and has exactly one vertex in its boundary, this vertex being not on a \( b' - l' \) path for any \( b' \in B(T) \) and \( l' \in L(b') \). In addition to \( b \), \( l \) may overdominate any number of branch vertices by 3 or more as long as it also overdominates all of their leaves. Although a more general result was proved in [18, Lemma 3.6], we state Lemma 3.6 for broadcasts in which only leaves broadcast with strength greater than 1 as this is all that we need here.

Lemma 3.6 [18, Corollary 3.7] Let \( T \) be a tree with \( b(T) \geq 2 \) and \( F' \) the set of all \( \alpha_{bn}(T) \)-broadcasts in which only leaves broadcast with strength greater than 1. Let \( F \) be the set of broadcasts in \( F' \) with the minimum number of overdominated branch vertices. Then there exists a broadcast \( f \in F \) that satisfies the following statement:

For any leaf \( l \), let \( X \) be the set of all branch vertices overdominated by \( l \). If \( X \neq \emptyset \) and \( v \in B_f(l) \), then \( v \) is neither the leaf nor an internal vertex on any endpath of \( T \). Moreover,

(i) there exists \( w \in X \) such that \( f(l) = d(l, w) + 1 \), and either \( L(w) = \{l\} \) and \( X = \{w\} \), or \( L(w) = \emptyset \) and \( f(l) \geq d(l, w') + 3 \) for all \( w' \in X - \{w\} \), or

(ii) \( f(l) \geq d(l, w) + 3 \) for all \( w \in X \) and \(|B_f(l)| = 1\).

4 Proof of Theorem 2.1

We are now ready to prove Theorem 2.1 which we restate for convenience.

Theorem 2.1 If \( T \) is a tree such that \( b(T) \geq 1 \), then

\[ \alpha_{bn}(T) \geq n - b(T) - |W_{int}(T)| + \alpha(\text{Int}(T)). \]
Figure 4: A tree $T$ with a $bn$-independent broadcast $f$ as described in the proof of Theorem 2.1. Note that $\sigma(f) = n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)) = 26 - 6 - 4 + 4 = 20 < 23 = n - b(T) + \rho(T) = 26 - 6 + 3$.

**Proof.** It is sufficient to define a $bn$-independent broadcast $f$ on $T$ such that $\sigma(f) = n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T))$. To do this, let $X$ be a maximum independent set of $\text{Int}(T)$ and let $Y = B_1(T) - X$. The definition of $f$ below is illustrated in Figure 4 where $X = \{w_1, w_2, b_3, b_4\}$.

(i) For each vertex $b \in B_{\geq 2}(T) \cup Y$ and each leaf $l \in L(b)$, let $f(l) = d(b, l)$.

Then $\bigcup_{l \in L(b)} N_f(l) = V(T_b)$ and $\sum_{l \in L(b)} f(l) = |V(T_b)| - 1$.

(ii) For each vertex $b \in X \cap B_1(T)$ and the leaf $l \in L(b)$, let $f(l) = d(b, l) + 1$.

Then $N_f(l) = V(T_b) \cup N(b)$ and $f(l) = |V(T_b)|$.

(iii) For each vertex $v \in X \cap (B_0(T) \cup W_{\text{int}}(T))$, let $f(v) = 1$.

(iv) Otherwise let $f(v) = 0$.

The weight of $f$ is

\[
\sigma(f) = \sum_{b \in B_{\geq 2}(T) \cup Y} (|V(T_b)| - 1) + \sum_{b \in X \cap B_1(T)} |V(T_b)| + |X \cap (B_0(T) \cup W_{\text{int}}(T))| \\
= \sum_{b \in B_{\geq 1}(T)} (|V(T_b)| - 1) + |X \cap B_1(T)| + |X \cap (B_0(T) \cup W_{\text{int}}(T))| \\
= \sum_{b \in B_{\geq 1}(T)} (|V(T_b)| - 1) + |X|.
\]

Since the expression $\sum_{b \in B_{\geq 1}(T)} (|V(T_b)| - 1)$ counts all vertices in the subtrees $T_b$ except the branch vertex itself, and $T_b = K_1$ if $b \in B_0(T)$,

$\sum_{b \in B_{\geq 1}(T)} (|V(T_b)| - 1) = n - b(T) - |W_{\text{int}}(T)|$,

from which we obtain

\[
\sigma(f) = n - b(T) - |W_{\text{int}}(T)| + |X| \\
= n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)).
\]
We show that $f$ is bn-independent. Suppose this is not the case. Then there are two vertices $u, v \in V^+_f$ and an edge $xy \in E(T)$ such that the broadcasts from $u$ and $v$ overlap on $xy$, that is, $\{x, y\} \subseteq N_f(u) \cap N_f(v)$ but $\{x, y\} \not\subseteq B_f(u) \cap B_f(v)$. By the definition of $f$, $u$ satisfies (i), (ii) or (iii). We consider the three cases separately and indicate the end of the proof of each case by a solid diamond ($\blacklozenge$).

**Case 1:** $u$ satisfies the conditions of (i). Then there exists a vertex $b \in B_{21}(T) - X$ such that $u \in L(b)$ and $f(u) = d(u, b)$. Therefore both $x$ and $y$ lie on the $u - b$ path in $T$ and at least one of $x$ and $y$ is an internal vertex of this path. Since $v$ $f$-dominates $x$ and $y$, $f(v) \geq d(v, b) + 1$. We show that this is not possible. First suppose that $v \in L(b')$ for some $b' \in B_{21}(T) - X$. If $b' = b$, then $f(v) = d(v, b') < d(v, b) + 1$, and if $b' \neq b$, then $f(v) = d(v, b') < d(v, b)$. Suppose next that $v \in L(b')$ for some $b' \in X \cap B_1(T)$. Then $b \neq b'$. By the definition of $f$, $f(v) = d(v, b') + 1 \leq d(v, b) < d(v, b) + 1$. Finally, suppose $v \in X \cap (B_0(T) \cup W_{\text{int}}(T))$. Then $v \neq b$ and $f(v) = 1$, hence $f(v) \leq d(v, b) < d(v, b) + 1$. Each case produces a contradiction. $\blacklozenge$

**Case 2:** $u$ satisfies the conditions of (ii). Then $u \in L(b)$, where $b \in X \cap B_1(T)$ and $f(u) = d(u, b) + 1$, and at least one of $x$ and $y$ lies on the $u - b$ path in $T$. Hence $f(v) \geq d(v, b)$. We showed in Case 1 that $v$ does not satisfy (i). Hence either $v \in L(b')$, where $b' \in X \cap B_1(T)$ and $f(v) = d(v, b') + 1$, or $f(v) = 1$ and $v \in X \cap (B_0(T) \cup W_{\text{int}}(T))$. Since $f(v) \geq d(v, b)$, the former condition implies that $b$ and $b'$ are adjacent (note that $b \neq b'$ because $u \neq v$ and $b, b' \in B_1(T)$), while the latter condition implies that $v$ and $b$ are adjacent. This contradicts the independence of $X$. $\blacklozenge$

**Case 3:** $u, v \in X \cap (B_0(T) \cup W_{\text{int}}(T))$. In this case $f(u) = f(v) = 1$, hence $d(u, v) = 1$. But $u, v \in X$, an independent set, and we again have a contradiction. $\blacklozenge$

This exhausts all possible cases, hence $f$ is bn-independent. We deduce that $\alpha_{\text{bn}}(T) \geq \sigma(f) = n - b(T) - \lvert W_{\text{int}}(T) \rvert + \alpha(\text{Int}(T))$. $\blacksquare$

## 5 Proof of Theorem 2.6

In this section we first prove Theorem 2.6, then we use it to show that there exist trees for which $\alpha_{\text{bn}}$ lies strictly between the bounds in Theorem 2.3. We again restate the theorem for convenience.

**Theorem 2.6** If $T$ is a tree of order $n$ such that $B(T) = \{b_1, b_2\}$, then

$$\alpha_{\text{bn}}(T) = n - 1 - \min\{\lceil\frac{1}{2}d(b_1, b_2)\rceil, \text{loss}(b_1), \text{loss}(b_2)\}.$$  

**Proof.** Let $F'$ be the set of all $\alpha_{\text{bn}}(T)$-broadcasts in which only leaves broadcast with strength greater than 1. By Lemma 3.5(ii), $F' \neq \emptyset$. Let $F$ be the set of broadcasts in $F'$ with the minimum number of overdominated branch vertices and choose a broadcast $f \in F$ such that Lemma 3.6 applies. By Proposition 3.1(i), all vertices of $T$ are $f$-dominated. By the choice of $F'$, each non-leaf vertex of $T$ is $f$-dominated by a leaf or by a broadcast of strength 1, and (by Lemma 3.5(i)) each leaf is $f$-dominated by a leaf.

Let $P = (b_1 = v_0, v_1, ..., v_k = b_2)$ be the $b_1 - b_2$ path in $T$. Notice that $d(b_1, b_2) = k$. Suppose $k = 1$. Then there are no internal vertices on $P$ and $R(T) = \emptyset$. Also,

$$\min\{\lceil\frac{1}{2}d(b_1, b_2)\rceil, \text{loss}(b_1), \text{loss}(b_2)\} = \lceil\frac{1}{2}d(b_1, b_2)\rceil = 1.$$
By Corollary 3.5
\[ \alpha_{bn}(T) = n - b(T) = n - 2 = n - 1 - \min\{\frac{1}{2}d(b_1, b_2), \text{loss}(b_1), \text{loss}(b_2)\} \]
as required. We therefore assume that \( k \geq 2 \) and examine the way in which the internal vertices of \( P \) are dominated. Let \( P_{\text{int}} : v_1, ..., v_{k-1} \) be the path induced by the internal vertices of \( P \).

There are three possibilities for dominating \( P_{\text{int}} \). We indicate the end of the proof of each case by a solid diamond (\( \bigdiamond \)).

**Case 1:** No vertex on \( P_{\text{int}} \) is \( f \)-dominated by a leaf. We show that \( \sigma(f) = n - 1 - \left\lceil \frac{1}{2}d(b_1, b_2) \right\rceil \).

The vertices on \( P_{\text{int}} \) are \( f \)-dominated by a set \( D \) of vertices on \( P_{\text{int}} \). By Lemma 3.5(iii), \( f(x) = d(x, b_1) \) for every leaf \( x \in L(b_1) \). Similarly, for every leaf \( y \in L(b_2) \), we have \( f(y) = d(y, b_2) \). For \( i \in \{1, 2\} \), let \( f_i \) be the restriction of \( f \) to \( T_{b_i} \) (the subtree induced by the vertices on the \( b_i \)-l path for all \( l \in L(b_i) \)). Notice that \( \sigma(f_i) = |V(T_i)| - 1 \) for \( i = 1, 2 \); by Proposition 3.3, \( f_i \) is an \( \alpha_{bn}(T_{b_i}) \)-broadcast. By the choice of \( f \in F^f \) and Lemma 3.5(ii), \( f(x) = 1 \) for each \( x \in D \). Since \( f \) is a maximum \( bn \)-independent broadcast, \( D \) is a maximum independent set of \( P_{\text{int}} \), hence \( |D| = \alpha(P_{\text{int}}) = \left\lceil \frac{1}{2}(k - 1) \right\rceil \). It follows that
\[
\sigma(f) = |V(T_{b_1})| - 1 + |V(T_{b_2})| - 1 + \left\lceil \frac{1}{2}(k - 1) \right\rceil \\
= n - 2 - \left\lceil \frac{1}{2}(k - 1) \right\rceil \\
= n - 1 - \left\lceil \frac{1}{2}d(b_1, b_2) \right\rceil,
\]
which is what we wanted to show. \( \bigdiamond \)

**Case 2:** A leaf \( x \in L(b_1) \) \( f \)-dominates some or all of the vertices on \( P_{\text{int}} \) and thus overdominates \( b_1 \). We show that \( \sigma(f) = n - 1 - \text{loss}(b_1) \).

Since \( |L(b_1)| \geq 2 \), Lemma 3.6(i) does not apply to \( x \). Thus Lemma 3.6(ii) applies and \( B_f(x) = \{v\} \), where \( v \) is a vertex on \( P - b_1 \). Hence \( x \) dominates \( T_{b_1} \) and overdominates \( L(b_1) \) (otherwise a vertex on a \( b_1 \)-leaf path belongs to \( B_f(x) = \{v\} \), which is impossible).

Let \( v_j \) be the vertex on the path \( P \) such that \( B_f(x) = \{v_j\} \). We state and prove two claims, and indicate the end of the proof of each claim by an open diamond (\( \bigtriangleup \)).

**Claim 5.1** \( B_f(x) = \{b_2\} \), i.e., \( v_j = v_k = b_2 \), hence \( f(x) = d(x, b_2) \).

**Proof of Claim 5.1** Suppose first that \( j = k - 1 \). Then \( b_2 \) does not overdominate \( f \) from \( x \); hence some vertex \( y \in V(T_{b_2}) \) broadcasts to \( b_2 \). Suppose \( y \in L(b_2) \). Since \( |L(b_2)| \geq 2 \), Lemma 3.6 implies that \( y \) does not overdominate \( b_2 \) by exactly 1. Since \( f \) is \( bn \)-independent, \( y \) does not broadcast to \( v_{k-2} \). Therefore \( f(y) = d(y, b_2) \) and \( b_2 \in B_f(y) \). But then \( \{v_{k-1}\} = \text{PB}_f(x) = B_f(x) \), and by Proposition 3.1(i), \( f \) is not maximal \( bn \)-independent. Suppose \( y = b_2 \). By Lemma 3.5(ii), \( f(b_2) = 1 \). For each \( l \in L(b_2) \), let \( Q'_l \) be the \( b_2 \)-leaf path and \( Q_l = Q'_l - b_2 \). Since \( f(b_2) = 1 \), \( \sum_{v \in V(Q'_l)} f(v) \leq d(l, b_2) - 1 \). Define the broadcast \( g_1 \) by
\[
g_1(u) = \begin{cases} 
0 & \text{if } u = b_2, \\
0 & \text{if } u \text{ is a degree 2 vertex on } Q_l \text{ for } l \in L(b_2), \\
d(u, b_2) & \text{if } u \in L(b_2), \\
f(u) & \text{otherwise.}
\end{cases}
\]
Then \( g_1 \) is \( bn \)-independent and, since \( |L(b_2)| \geq 2 \), \( \sigma(g_1) > \sigma(f) \), which is impossible. We obtain a similar contradiction if \( y \) is an internal vertex on \( Q_l \) for some \( l \in L(b_2) \). Therefore \( j \neq k - 1 \).

Suppose next that \( j < k - 1 \). By our choice of \( f \) and because \( f \) is dominating, \( v_{j+1} \) is \( f \)-dominated either by \( t \in \{v_{j+1}, v_{j+2}\} \), where \( f(t) = 1 \), or by a leaf \( l \in L(b_2) \). In the former case, the broadcast \( g_2 \) with \( g_2(x) = d(x, t) + 1 \), \( g_2(t) = 0 \) and \( g_2(u) = f(u) \) otherwise, has greater weight than \( f \), and \( N_{g_2}(x) \subseteq N_f(t) \cup N_f(x) \). Hence \( g_2 \) is \( bn \)-independent and violates the maximality of \( f \). In the latter case, Lemma 3.1(ii) applies to \( l \), hence \( l \) overdominates \( L(b_2) \) and \( f(l) = d(l, v_j) \), \( f(x) = d(x, v_j) \) and \( f(u) = 0 \) otherwise.

Define a new broadcast \( g_3 \) by \( g_3(u) = d(u, b_2) \) if \( u = x \) or \( u \in L(b_2) \), and \( g_3(u) = 0 \) otherwise. For all \( \{u, u'\} \in V_{g_3}^+ \) such that \( u \neq u' \), \( N_{g_3}(u) \cap N_{g_3}(u') = \{b_2\} = B_{g_3}(u) \cap B_{g_3}(u') \), hence \( g_3 \) is \( bn \)-independent. Since \( g_3(x) + g_3(l) = f(x) + f(y) \) and \( |L(b_2)| \geq 2 \), \( \sigma(g_3) > \sigma(f) \). Thus, \( g_3 \) violates the maximality of \( f \). We deduce that \( f(x) = d(x, b_2) \), that is, \( B_f(x) = \{b_2\} \). ♦

Proceeding with the proof of Case 2, we point out that \( b_2 \) is the only vertex of \( T_{b_2} \) that is \( f \)-dominated by \( x \). As in Case 1, let \( f_2 \) be the restriction of \( f \) to \( T_{b_2} \). Then \( f_2 \) dominates all vertices of \( T_{b_2} \) except possibly \( b_2 \). By Proposition 3.3, \( \alpha_{bn}(T_{b_2}) = |V(T_{b_2})| - 1 \) and thus \( \sigma(f_2) \leq |V(T_{b_2})| - 1 \). Moreover, the broadcast \( h \) on \( T_{b_2} \) defined by \( h(l) = d(l, b_2) \) for each \( l \in L(b_2) \) is a \( bn \)-independent broadcast of weight \( \sigma(h) = |V(T_{b_2})| - 1 \) such that \( (f - f_2) \cup h \) is a \( bn \)-independent broadcast on \( T \) with weight at least the weight of \( f \). The maximality of \( f \) now implies that

\[
\sigma(f_2) = |V(T_{b_2})| - 1.
\]

We state and prove our second claim.

**Claim 5.2** \( d(x, b_1) = \max(b_1) \).

**Proof of Claim 5.2** Suppose to the contrary that \( d(x, b_1) < \max(b_1) \) and let \( x' \) be a leaf such that \( d(x', b_1) = \max(b_1) \). Create a new broadcast \( g_4 \) with \( g_4(x) = 0 \), \( g_4(x') = d(x', b_2) \) and \( g_4(u) = g_4(u) \) otherwise. Since \( d(x, b_1) < d(x', b_1) \), \( \sigma(g_4) > \sigma(g_4) \). Notice that \( N_{g_4}(x') = N_{g_4}(x) \) and all other boundaries are unchanged. Hence \( g_4 \) is \( bn \)-independent and contradicts the maximality of \( f \). ♦

Recall that by definition, \( \text{loss}(b_1) = \text{sum}(b_1) - \max(b_1) = \sum_{l \in L(b_1)} d_T(b_1, l) - \max(b_1) \), hence \( \max(b_1) = |V(T_{b_1})| - 1 - \text{loss}(b_1) \). By (1) and Claims 5.1 and 5.2 we now have that

\[
\sigma(f) = f(x) + \sigma(f_2) = d(x, b_1) + d(b_1, b_2) + \sigma(f_2) = \max(b_1) + d(b_1, b_2) + \sigma(f_2) = |V(T_{b_1})| - 1 - \text{loss}(b_1) + k + |V(T_{b_2})| - 1.
\]

Since \( V(T) \) is the disjoint union of \( V(T_{b_1}) - \{b_1\}, V(P) \) and \( V(T_{b_2}) - \{b_2\} \), and \( P \) is a path of order \( k + 1 \), it follows that \( \sigma(f) = n - 1 - \text{loss}(b_1) \). ♦

**Case 3:** A leaf \( x \in L(b_2) \) \( f \)-dominates some or all of the vertices on \( P_{int} \) and thus overdominates \( b_2 \). Exactly as in Case 2, \( \sigma(f) = n - 1 - \text{loss}(b_2) \). ♦

Since \( f \) is a \( bn \)-independent broadcast of maximum weight, it follows from Cases 1, 2 and 3 that \( \alpha_{bn}(T) = \sigma(f) = n - 1 - \min\{\frac{1}{2}d(b_1, b_2), \text{loss}(b_1), \text{loss}(b_2)\} \).

Theorem 2.6 allows us to determine a special class of trees \( T \) for which \( \alpha_{bn}(T) \) lies strictly in between our upper and lower bounds, an example of which is given in Figure 5.
Figure 5: A tree $T$ with $b(T) = 2$, $R(T) = \emptyset$, $|W_{\text{int}}(T)| = 4$ and $\alpha(\text{Int}(T)) = 2$. Hence $n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)) = 10 < \alpha_{bn}(T) = 11 < n - 2 = 12$.

**Corollary 5.3** Let $T$ be a tree with branch set $B(T) = \{b_1, b_2\}$ such that $2 \leq \text{loss}(b_1) \leq \text{loss}(b_2) < \left\lceil \frac{d(b_1, b_2)}{2} \right\rceil$. Then

$$n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)) < \alpha_{bn}(T) < n - b(T) - \rho(T).$$

**Proof.** For a tree $T$ with branch number $b(T) = 2$, $B_0(T) \cup B_1(T) = \emptyset$, hence $\rho(T) = 0$. Therefore $n - b(T) - \rho(T) = n - 2$. Also, $\text{Int}(T)$ is the subgraph of $T$ induced by $W_{\text{int}}(T)$, which, in this case, is the path $P_{\text{int}}$ of order $k - 1 = d(b_1, b_2) - 1$ as described above in the proof of Theorem 2.6. Hence

$$n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)) = n - 2 - (k - 1) + \left\lceil \frac{k - 1}{2} \right\rceil$$

$$= n - 2 - \left\lceil \frac{k - 1}{2} \right\rceil = n - 2 - \left\lceil \frac{d(b_1, b_2)}{2} \right\rceil.$$ 

By Theorem 2.6 and the choice of $T$,

$$n - 2 - \left\lceil \frac{d(b_1, b_2)}{2} \right\rceil < n - 1 - \text{loss}(b_1) = \alpha_{bn}(T) < n - 2. \quad \blacksquare$$

We close this section by remarking that strict inequality in the upper bound suggested in Question 1 holds for the trees described in the statement of Corollary 5.3 as well.

### 6 Open problems

In this final section we mention some open problems for future research, beginning by repeating Question 1 for the sake of completeness.

**Question 1** [18] Can the upper bound in Theorem 2.2 be improved to $\alpha_{bn}(T) \leq |V(T)| - b(T) + \alpha(T[R(T)])$?

It is well known that $\alpha(G) \leq n - \delta(G)$ and, when $G$ is connected, $\text{diam}(G) \leq n - \delta(G)$.

**Question 2** [17] Is it true that $\alpha_{bn}(G) \leq n - \delta(G)$ for all graphs $G$?

**Problem 1** Characterize trees $T$ such that

(i) [18] $\alpha_{bn}(T) = |V(T)| - b(T) + \rho(T)$, i.e., equality holds in the upper bound in Theorem 2.6.
(ii) \( \alpha_{bn}(T) = n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)), \) i.e., equality holds in the lower bound in Theorem 2.3.

(iii) \( \alpha_{bn}(T) = |V(T)| - b(T) + \rho(T) = n - b(T) - |W_{\text{int}}(T)| + \alpha(\text{Int}(T)), \) i.e., the bounds in Theorem 2.3 coincide.

By definition, \( \alpha(G) \leq \alpha_{bn}(G) \leq \alpha_h(G) \) for all graphs \( G \). Mynhardt and Neilson [16] showed that \( \alpha(G) = \alpha_{bn}(G) \) when \( G \) is a 2-connected bipartite graph. The spiders \( Sp(2^k) \), for which \( \alpha_{bn}(Sp(2^k)) = 2^k \) and \( \alpha(Sp(2^k)) = k + 1 \), demonstrate that this result does not hold for trees.

**Problem 2** Investigate the ratio \( \alpha_{bn}(G)/\alpha(G) \) when \( G \) is (i) a tree, (ii) a cyclic bipartite graph of connectivity 1, (iii) general graphs.

**Problem 3** [18] Characterize trees \( T \) such that (i) \( \alpha_{bn}(T) = \alpha_h(T) \) or (ii) \( \alpha_{bn}(T) = \alpha(T) \).

**Problem 4** [18] Characterize caterpillars \( T \) such that (i) \( \alpha_{bn}(T) = \alpha_h(T) \) or (ii) \( \alpha_{bn}(T) = \alpha(T) \).

**Problem 5** [18] Determine \( \alpha_{bn}(T) \) for all caterpillars \( T \).

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