Scaling of decoherence and energy flow in interacting quantum spin systems

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We address the quantum dynamics of a system composed of a qubit globally coupled to a many-body system characterized by short-range interactions. We employ a dynamic finite-size scaling framework to investigate the out-of-equilibrium dynamics arising from the sudden variation (turning on) of the interaction between the qubit and the many-body system, in particular when the latter is in proximity of a quantum first-order or continuous phase transition. Although the approach is quite general, we consider $d$-dimensional quantum Ising spin models in the presence of transverse and longitudinal fields, as paradigmatic quantum many-body systems. To characterize the out-of-equilibrium dynamics, we focus on a number of quantum-information oriented properties of the model. Namely, we concentrate on the decoherence features of the qubit, the energy interchanges among the qubit and the many-body system during the out-of-equilibrium dynamics, and the work distribution associated with the quench. The scaling behaviors predicted by the dynamic finite-size scaling theory are verified through extensive numerical computations for the one-dimensional Ising model, which reveal a fast convergence to the expected asymptotic behavior with increasing the system size.

I. INTRODUCTION

The recent progress which has been achieved in the control and manipulation of complex systems at the nano scale has enabled a wealth of unprecedented possibilities aimed at addressing the unitary quantum evolution of many-body objects. These range from the (nearly) adiabatic dynamics induced by a slow change in time of one of the control parameters, to the deep out-of-equilibrium dynamics following an abrupt quench in the system [1–3]. In the latter scenario, several fundamental issues have been investigated, including the onset of thermalization at long times, quantum transport, and localization phenomena due to the mutual interplay between disorder and interactions [4–11]. All of them are eventually devoted to characterize the highly nonlinear response of the system to the drive, where nonequilibrium fluctuation relations may play a pivotal role [12–15].

Closely related to the scenario we are going to focus in the present paper, we also mention the raising interest in monitoring the coherent quantum dynamics of mutually coupled systems, with the purpose to address energy interchanges or the relative decoherence properties among the various subsystems [16]. This kind of study is relevant both to understand whether quantum mechanics can enhance the efficiency of energy conversion in complex networks [17, 18], and to devise novel quantum technologies which are able to optimize energy storage in subportions of the whole system [19–23]. We shall stress that energy flows are likely to be influenced by the different quantum phases of the system. Moreover one would expect an enhanced response in proximity of a quantum phase transition, which requires special attention [24].

The aim of this paper is to shed light on this latter issue. To this purpose, we consider the simplest scenario where to frame such analysis: a single qubit globally coupled to a $d$-dimensional quantum Ising spin model in a transverse and a longitudinal field. This spin model acts as a prototypical quantum many-body system since, when varying the intensity of the two external fields, it may undergo both first-order and continuous quantum transitions (FOQTs and CQTs, respectively). The composite setup belongs to the class of the so called central-spin models, where one (or few) qubit can be globally or locally coupled to the environmental system (see, e.g., Refs. [25–36]). We put forward a quantitative scaling theory which generalizes the results of Ref. [35] focused on the decoherence properties of the qubit, and meanwhile also carefully addresses the statistics of energy flows among the qubit and the many-body system. Specifically, we employ the finite-size scaling (FSS) framework, which has been shown to be able to predict the behavior of a system in proximity of either a CQT [37] or a FOQT [38], as well as in a dynamic context [39, 40], providing the asymptotic large-size scaling in a variety of situations. A consistent part of this work is devoted to a numerical validation of the dynamic FSS predictions through extensive simulations with exact diagonalization techniques, specialized to the one-dimensional case ($d = 1$).

In our setup, the global system is initialized in a state which is a product of pure states of the qubit $q$ and of the quantum Ising system $S$. The qubit $q$ is then suddenly coupled to all the $L^d$ spins of $S$, such that a nontrivial unitary dynamics sets in. Notice that we admit the possibility to have an interaction Hamiltonian which does not commute with the qubit Hamiltonian, so that the decoherent effect on the qubit is not only a pure dephasing. We focus on three quantum-information oriented properties of the model: the decoherence features of $q$, the statistics of the work distribution associated with the quench, and the statistics of the energy interchanges among $q$ and $S$ during the out-of-equilibrium dynamics. We show that such properties develop dynamic FSS behaviors when the system $S$ is close to quantum transitions. In particular, our numerics for the one-dimensional Ising systems show that the convergence to the dynamic
FSS behavior is remarkably fast, a fact which encourages a careful assessment of the role of criticality in near-future experiments of quantum transport in complex systems with few spins or particles (of the order of 10).

The paper is structured as follows. We start, in Sec. II, with defining the model and all the relevant quantities that will be analyzed. In Sec. III, we summarize the derivation of the dynamic FSS framework, which is capable to address the dynamics of a quantum many-body system, either at FOQTs or at CQTs. We also discuss differences which emerge when considering a disordered phase. The dynamic FSS is then specialized to our system of interest in Sec. IV, explicitly discussing the qubit phase. The dynamic FSS is then specialized to our system, either at FOQTs or at CQTs. We also discuss the support of the theory are presented (Sec. V), and along the FOQT line, where we also consider a two-level reduction of the many-body system (Sec. VI). Finally, Sec. VII is devoted to a summary and perspectives of this work. In the Appendices we provide some analytic insight for the special case in which the qubit Hamiltonian commutes with the qubit-system interaction term, and discuss more in detail some limitations emerging in the two-level approximation at the FOQT.

II. GENERAL SETTING OF THE PROBLEM

A. The model

Let us consider a d-dimensional quantum many-body system \( S \) of size \( L^d \), with Hamiltonian

\[
H_S(h) = H_c + H_h, \quad H_h = h P, \tag{1}
\]

where \( P \) is the spatial integral of local operators, such that \( [H_c, P] \neq 0 \), and the parameter \( h \) drives a quantum transition located at \( h = 0 \). As paradigm example, we shall focus on the quantum Ising model on a \( L^d \) lattice,

\[
H_c = H_{Is} = -J \sum_{(x,y)} \sigma_x^{(3)} \sigma_y^{(3)} - g \sum_x \sigma_x^{(1)}, \tag{2}
\]

where \( \sigma^{(k)} \) are the Pauli matrices, the first sum is over all bonds connecting nearest-neighbor sites \((x, y)\), while the other sum is over all sites. Hereafter we assume \( h = 1 \), \( J = 1 \), the lattice spacing \( a \) is \( 1 \), and \( g > 0 \). At \( g = g_c \) (in one dimension, \( g_c = 1 \)), the model undergoes a CQT belonging to the \((d+1)\)-dimensional Ising universality class \( 21 \) \( 11 \) \( 42 \), separating a disordered phase \((g > g_c)\) from an ordered one \((g < g_c)\). The presence of a homogeneous longitudinal external field is taken into account by adding the term

\[
h P = -h \sum_x \sigma_x^{(3)}. \tag{3}
\]

to the Hamiltonian \([2]\). The field \( h \) drives FOQTs along the \( h = 0 \) line for any \( g < g_c \). At the continuous transition, \( g = g_c \), such term is one of the relevant perturbations driving the critical behavior, the other one being the transverse field term, \((g - g_c) \sum_x \sigma_x^{(1)}\).

In addition, let us consider a qubit \((q)\) whose two-level Hamiltonian can be generally written as

\[
H_q = \sum_{a=\pm} \epsilon_a |a\rangle\langle a| = \alpha I_2 + \frac{1}{2} \Sigma^{(3)}, \tag{4}
\]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix, and the Pauli operator \( \Sigma^{(3)} \) is associated with the two states \(|\pm\rangle\) of the qubit, so that \( \Sigma^{(3)}|\pm\rangle = \pm|\pm\rangle \). Therefore

\[
\epsilon_{\pm} = \alpha \pm \frac{1}{2} \delta, \quad \delta = \epsilon_+ - \epsilon_. \tag{5}
\]

The qubit is globally and homogeneously coupled to the many-body system \( S \), through the Hamiltonian term

\[
H_{qS} = \left( u \Sigma^{(3)} + v \Sigma^{(1)} \right) P, \tag{6}
\]

where \( P \) is the operator appearing in Eq. (1), thus Eq. (3). Putting all the terms together, we obtain the global Hamiltonian

\[
H = H_S + H_q + H_{qS}. \tag{7}
\]

We are interested in the quantum evolution of the global system starting from the initial, \( t = 0 \), condition

\[
|\Psi_0\rangle = |q_0\rangle \otimes |0_h\rangle, \tag{8}
\]

where \(|q_0\rangle\) is a generic pure state of the qubit,

\[
|q_0\rangle = c_+ |+\rangle + c_- |\rangle, \quad |c_+|^2 + |c_-|^2 = 1, \tag{9}
\]

and \(|0_h\rangle\) is the ground state of the system with Hamiltonian \( H_S(h) \). The global wave function describing the quantum evolution for \( t > 0 \) must be solution of the Schrödinger equation

\[
i \frac{\partial}{\partial t} |\Psi(t)\rangle = H |\Psi(t)\rangle, \quad |\Psi(t = 0)\rangle = |\Psi_0\rangle. \tag{10}
\]

In particular, we consider a dynamic protocol arising from a sudden switching of the interaction \( H_{qS} \) between the qubit and the many-body system, at time \( t = 0 \), i.e., by quenching one or both of the control parameters \( u \) and \( v \) in Eq. (1) from zero to some finite value.

The above setting can be straightforwardly extended to \( N \)-level systems coupled to an environmental system \( S \), and also to the case the initial qubit state is mixed, thus described by a nontrivial density matrix. Most features addressed in the rest of the paper, in particular the dynamic scaling properties, can be straightforwardly extended as well.
B. Qubit decoherence, work, and qubit-system energy exchanges

1. Coherence of the qubit

An interesting issue arising from the dynamics of the problem outlined above concerns the coherence properties of the qubit during the global quantum evolution. Starting from a pure state, the interaction with the many-body system may give rise to a loss of coherence of the qubit, depending on the properties of its density matrix

\[ \rho_q(t) = \text{Tr}_S[\rho(t)] , \]

where \( \text{Tr}_S[\cdot] \) denotes the trace over the S-degrees-of-freedom of the global (pure) quantum state

\[ \rho(t) = |\Psi(t)\rangle\langle\Psi(t)| , \]

with \( |\Psi(t)\rangle \) given by the solution of Eq. (10). Of course, \( \text{Tr}[\rho_q(t)] = 1 \).

A good candidate to quantify the coherence properties of the qubit during its quantum evolution is provided by the so-called purity, that is, the trace of its square density matrix \( \rho_q \),

\[ \text{Tr} \left[ \rho_q(t)^2 \right] = 1 - D(t) , \]

where we have introduced the decoherence function \( D \), such that \( 0 \leq D \leq 1 \). This function measures the quantum decoherence, quantifying the departure from a pure state. Indeed \( D = 0 \) implies that the qubit is in a pure state, thus \( D(t = 0) = 0 \). The other extreme value \( D = 1 \) indicates that the qubit is totally unpolarized.

2. Quantum work associated with the initial quench

The initial quench, arising from turning on the interaction between the qubit and the many-body system \( S \), can be also characterized via the quantum work \( W \) done on the global system \( [14, 39] \). The work performed by quenching the control parameters \( u \) and \( v \) does not generally have a definite value, while it can be defined as the difference of two projective energy measurements \([13]\). The first one at \( t = 0 \) projects onto the eigenstates \( |m_i\rangle \) of the initial Hamiltonian \( H_I \) with a probability \( p_{i,m} \) given by the initial density matrix. The second energy measurement projects onto the eigenstates \( |n_l\rangle \) of the post-quench Hamiltonian \( H_I \). Since the energy is conserved after the quench, the latter measurement can be performed at any time \( t \) during the evolution, ruled by the unitary operator \( U(t,0) = e^{-iH_I t} \), without changing the distribution, in particular for \( t \to 0^+ \). The work probability distribution can be written as \([14, 44, 45]\):

\[ P(W) = \sum_{n,m} \delta[W - (E_{I,n} - E_{i,m})] \left| \langle n_l|m_i \rangle \right|^2 p_{i,m} . \]  

The average work and its higher moments are given by

\[ \langle W^k \rangle = \int dW W^k P(W) . \]  

As one can easily check, the average quantum work \( \langle W \rangle \) can be computed by taking the difference

\[ \langle W \rangle = \langle \Psi(t)|H|\Psi(t)\rangle - \langle \Psi_0|H_q + H_S|\Psi_0\rangle , \]

where

\[ \langle \Psi_0|H_q + H_S|\Psi_0\rangle = \langle q|H_q|q\rangle + \langle 0_h|H_S|0_h \rangle = E_{q0} + E_{S0}. \]

Since we are interested in a sudden quench at \( t = 0 \), we can obtain the average work from the difference of the expectation values of \( H_I = H \) and \( H_I = H_q + H_S \) on the initial state, obtaining

\[ \langle W \rangle = \langle \Psi_0|H_{qS}|\Psi_0\rangle . \]

An analogous expression can be derived for the average of the square work, obtaining

\[ \langle W^2 \rangle = \langle \Psi_0|H^2_{qS}|\Psi_0\rangle . \]

Note that the above relatively simple equations for the first two moments of the work distribution do not extend to higher moments, i.e. \( \langle W^k \rangle \neq \langle \Psi_0|H^k_{qS}|\Psi_0\rangle \) for \( k > 2 \). Their expressions are more complicated, requiring the computation of the whole spectrum, due to the fact that \( H_{qS} \) does not commute with the other Hamiltonian terms.

3. Energy-difference distributions

In order to study the qubit-system energy exchanges, we may consider the energy-difference distribution of the system \( S \) along the quantum evolution, associated with two energy measurements of \( S \), at \( t = 0 \) and at a generic time \( t \). We write it as

\[ P_S(U, t) = \sum_{n,a,b} \delta[U - (E_{Sn} - E_{S0})] \left| \langle n,b \rangle|e^{-iH_I t}|a,0 \rangle \right|^2 p_{a,b} , \]

where \( |n,b\rangle \equiv |b\rangle \otimes |n\rangle, |n\rangle \) and \( E_{Sn} \) indicate eigenstates and eigenvalues of the Hamiltonian \( H_S \) (we assume a discrete spectrum, as is generally appropriate for finite-size systems), \( |a\rangle, |b\rangle \) indicate the eigenstates \( |\pm\rangle \) of the qubit Hamiltonian, and \( p_{\pm} = |c_{\pm}|^2 \) are the probabilities of the initial qubit state at \( t = 0 \). One can check that

\[ \langle U \rangle = \int dy y P(y, t) = E_S(t) - E_{S0} . \]  

More general initial distributions may be also considered, for example associated with a bath at temperature \( T \), replacing the initial density matrix \( \rho_0 = |q,0\rangle\langle q,0| \) with the corresponding density matrices.
Of course one may also define an analogous energy-difference distribution associated with the qubit, obtained by a two-measurement procedure on it:

$$P_q(U, t) = \sum_{n,a,b} \delta(U - (E_{qa} - E_{qa})) \langle \{b, n|e^{-iHt}|a, 0\} \rangle^2 p_a.$$  \hspace{1cm} (22)

III. THE FINITE-SIZE SCALING FRAMEWORK

In this section we summarize the main features of the dynamic FSS framework that we will exploit to analyze the out-of-equilibrium quantum dynamics of the coupled qubit and many-body system, and in particular the quantities introduced in Sec. II B. The dynamic FSS framework has been recently developed to deal with dynamic behaviors of finite-size systems at quantum transitions \[35, 39, 46\], extending equilibrium FSS frameworks to study CQTs and FOQTs \[37, 47–50\]. This framework allows one to study the interplay among the Hamiltonian parameters, the finite linear size \(L\), and the finite temperature \(T\), assuming that \(T\) is sufficiently small, and that the Hamiltonian parameters keep the system \(S\) close to the transition point.

A. Scaling variables at quantum transitions

The scaling hypothesis for the many-body system described by the Hamiltonian \(H_S\) is based on the existence of a nontrivial large-volume limit, keeping the appropriate scaling variables fixed. At both CQTs and FOQTs, the FSS variable related to a relevant perturbation as \(H_h = hP\), cf. Eq. (1), can be generally written as the ratio

$$\kappa_h = E_h(L)/\Delta(L),$$  \hspace{1cm} (23)

between the energy variation associated with the \(H_h\) term (we assume \(E_h = 0\) at the transition point \(h = 0\)) and the energy difference of the lowest-energy states, \(\Delta(L) \equiv E_1 - E_0\), at the transition point \(h = 0\). Nonzero temperatures are taken into account by adding a further scaling variable

$$\tau = T/\Delta(L).$$  \hspace{1cm} (24)

More generally, any relevant low-energy energy scale \(\mathcal{E}\) is expected to behave as \(\mathcal{E} \sim \Delta(L)\) in the FSS limit. Dynamic behaviors, exhibiting nontrivial time dependencies, also require a scaling variable associated with the time variable, which is generally given by

$$\theta = \Delta(L) t.$$  \hspace{1cm} (25)

The equilibrium and dynamic FSS limits are defined as the large-size limit, keeping the above scaling variables fixed.

The outlined framework provides a unified picture of the FSS behaviors at quantum transitions, holding at both CQTs and FOQTs. Within the dynamic FSS framework, their differences are essentially related to the functional dependence of the above scaling variables on the size. Power laws generally arise at CQTs \[37\], while exponential laws emerge at FOQTs \[17\] \[18\], in particular when boundary conditions do not favor any particular phase.

B. First-order quantum transitions

As shown by earlier works \[38, 47–49\], the FSS behavior of isolated many-body systems at FOQTs turns out to be much depending on the type of boundary conditions, whether they favor one of the phases or they are neutral, giving rise to FSS characterized by exponential or power-law behaviors. To simplify our presentation, in the following, the system \(S\) will be taken as a quantum Ising model with boundary conditions that do not favor any of the two magnetized phases, such as periodic and open boundary conditions, which generally lead to exponential FSS laws.

The FOQT line for \(g < g_c\) is related to the level crossing of the two lowest states \(\uparrow\) and \(\downarrow\) for \(h = 0\), such that \(\langle \uparrow|\sigma_x^{(3)}|\uparrow\rangle = m_0\) and \(\langle \downarrow|\sigma_x^{(3)}|\downarrow\rangle = -m_0\) (independently of \(x\)), with \(m_0 > 0\). The degeneracy of these states is lifted by the longitudinal field \(h\). Therefore, \(h = 0\) is a FOQT point, where the longitudinal magnetization \(M = L^{-d} \sum_x M_x\), with \(M_x \equiv \langle \sigma_x^{(3)} \rangle\), becomes discontinuous in the infinite-volume limit. The transition separates two different phases characterized by opposite values of the magnetization \(m_0\), i.e.

$$\lim_{h \to 0^+} \lim_{L \to \infty} M = \pm m_0.$$  \hspace{1cm} (26)

For one-dimensional systems \[51\], \(m_0 = (1 - g^2)^{1/8}\).

In a finite system of size \(L\), the two lowest states are superpositions of two magnetized states \(\uparrow\) and \(\downarrow\) such that \(\langle \pm|\sigma_x^{(3)}|\pm\rangle = \pm m_0\) for all sites \(x\). Due to tunneling effects, the energy gap \(\Delta\) at \(h = 0\) vanishes exponentially as \(L\) increases \[38, 52\].

$$\Delta(L) \sim e^{-cL^d}, \hspace{1cm} (27)$$

apart from powers of \(L\). In particular, for the one-dimensional Ising system \[9\] at \(g < 1\), it is exponentially suppressed as \[51, 53\]

$$\Delta(L) \approx 2(1 - g^2)g^L [1 + O(g^{2L})]$$  \hspace{1cm} (28)

for open boundary conditions, and

$$\Delta(L) \approx 2 \sqrt{(1 - g^2)/(\pi L)} g^L$$  \hspace{1cm} (29)

for periodic boundary conditions. The differences \(E_i - E_0\) for the higher excited states \((i > 1)\) are finite for \(L \to \infty\).
Quantum Ising systems along the FOQT line develop FSS behaviors [35, 39, 40, 46], driven by the longitudinal field \( h \). Using Eq. (23), the corresponding scaling variable can be written as

\[
\kappa_h = \frac{2m_0 h L^d}{\Delta(L)},
\]

where \( 2m_0 h L^d \) approximately quantifies the energy associated with the corresponding longitudinal-field perturbation \( H_h \). For example, in the equilibrium FSS limit the magnetization is expected to behave as [47]

\[
M(h, L) = m_0 \mathcal{M}(\kappa_h),
\]

where \( \mathcal{M} \) is a FSS function.

Note that the FOQT scenario based on the avoided crossing of two levels is not realized for any boundary condition [38]: in some cases the energy difference \( \Delta(L) \) of the lowest levels may even display a power-law dependence on \( L \). However, the scaling variable \( \kappa_h \) obtained using the corresponding \( \Delta(L) \) turns out to be appropriate, as well [38].

### C. Continuous quantum transitions

FSS theories have been originally developed at continuous transitions, see e.g. Refs. 37-51 and references therein. The CQT of the Ising model, cf. Eqs. (2) and (3), is characterized by two relevant parameters, \( r = g - g_c \) and \( h \) (such that they vanish at the critical point), with renormalization-group dimension \( y_r \) and \( y_h \), respectively. The relevant FSS variables are

\[
\kappa_r = L^{y_r} r, \quad \kappa_h = L^{y_h} h.
\]

The FSS limit is obtained by taking \( L \to \infty \), keeping \( \kappa_r \) and \( \kappa_h \) fixed.

Note that the expression for \( \kappa_h \) in Eq. (31) can be obtained using the more general definition (23). Indeed, at CQTs the energy variation arising from the perturbation \( H_h = h \sum_x P_x \), with \( P_x = -\sigma_x^z \), is given by

\[
E_h(L) \sim h L^{d - y_r},
\]

where \( y_p \) is the renormalization-group critical dimension of the local operators \( P_x \) at the fixed point describing the quantum critical behavior. Moreover, we have

\[
\Delta(L) \sim L^{-z},
\]

where \( z \) is the universal dynamic exponent. Then, using the scaling relation among critical exponents [24, 37]

\[
y_h + y_p = d + z,
\]

where \( y_h \) is the RG dimension of the perturbation \( h \), we end up with the expression of \( \kappa_h \) reported in Eq. (31).

An analogous derivation can be obtained for \( \kappa_r \).

The equilibrium critical exponents \( y_r \) and \( y_h \) of the quantum Ising model are those of the \((d + 1)\)-dimensional Ising universality class [24, 41, 42]. Therefore, for one-dimensional systems they are \( y_r = 1/\nu = 1 \) and \( y_h = (d + 3 - \eta)/2 = (4 - \eta)/2 \) with \( \eta = 1/4 \). For two-dimensional models the critical exponents are not known exactly, but there are very accurate estimates (see, e.g., Refs. [50, 60]); in particular [59] \( y_r = 1/\nu \) with \( \nu = 0.629971(4) \) and \( y_h = (5 - \eta)/2 \) with \( \eta = 0.036298(2) \). For three-dimensional systems they assume mean-field values, \( y_r = 2 \) and \( y_h = 3 \), apart from logarithms. The temperature \( T \) gives rise to a relevant perturbation at CQTs, associated with the scaling variable \( \tau = t/T \), where \( z = 1 \) (for any spatial dimension) is the dynamic exponent characterizing the behavior of the energy differences of the lowest-energy states and, in particular, the gap \( \Delta \sim L^{-z} \).

A generic observable \( O \) in the FSS limit behaves as

\[
O(r, h, L) \approx L^{-y_o} \mathcal{O}(\kappa_r, \kappa_h),
\]

where the exponent \( y_o \) is the renormalization-group dimension associated with \( O \) and \( \mathcal{O} \) is a universal equilibrium FSS function. The approach to such an asymptotic behavior is characterized by power-law corrections, typically controlled by irrelevant perturbations at the corresponding fixed point [37]. The equilibrium FSS at quantum transitions has been also extended to quantum-information concepts [61-63], such as the ground-state fidelity and its susceptibility, which measure the change of the ground state when varying the Hamiltonian parameters around a quantum transition [50].

### D. The disordered phase

One may compare the above scaling behaviors with those expected when the system \( S \) is not close to a phase transition, for example for \( g > g_c \), in the case of quantum Ising models. In this region the system is in the disordered phase, where the length scale \( \xi \) of the correlations is finite. In particular, close to the transition point \( g_c \), it behaves as \( \xi \sim (g - g_c)^{-\nu} \). Thus the ratio \( L/\xi \) diverges in the large-\( L \) limit. The many-body system appears as effectively composed of \((L/\xi)^d\) uncorrelated subsystems. The gap \( \Delta(L) \) remains finite with increasing \( L \). Close to the CQT, i.e. for \( g \gg g_c \), it behaves as \( \Delta \sim \xi^{-z} \).

### IV. DYNAMIC FSS ANSATZ FOR THE QUBIT-SYSTEM SETUP

The dynamic processes arising from the instantaneous turning on of the interaction term \( H_{\text{int}} \) can be described within a dynamic FSS framework, extending the framework outlined in Sec. [11] to take into account the interaction of the many-body system \( S \) with the qubit \( q \).

Beside the scaling variable \( \kappa_h \), cf. Eq. (23), we also need to consider scaling variables associated with the other parameters of the global Hamiltonian \( H \), i.e., \( u, v \) and \( \delta \).
Since both the $u$- and $v$-term are coupled to the operator $P$ contained in the $h$-term of the global Hamiltonian $H_q$, we expect the corresponding scaling variables to scale analogously as $\kappa_h$. They can thus be obtained by replacing $h$ with $u$ (or $v$) in the definition of $\kappa_h$. Given that $\kappa_h$ is linear in $h$, one has

$$\kappa_u = u \kappa_h/h, \quad \kappa_v = v \kappa_h/h. \quad (36)$$

We must also associate a FSS variable with the energy difference $\delta$ of the eigenstates of the qubit Hamiltonian $H_q$,

$$\varepsilon_\delta = \delta/\Delta(L), \quad (37)$$

since $\delta$ is a further energy scale of the problem. Finally, the scaling variable $\theta = \Delta(L)t$ is associated with the time variable.

1. Qubit decoherence functions

Let us first address the issue of decoherence properties of the qubit along the global quantum evolution arising from the interaction with the system $S$. The dynamic scaling behavior of the decoherence function $D(t)$, cf. Eq. (13), as a function of time, size, and Hamiltonian parameters of the system $S$, has been discussed in Ref. [35] in the simplest case where $\delta = 0$ (i.e., the qubit Hamiltonian is trivial). As detailed in App. A, each time $[H_q, H_q S] = 0$ [as is the case for $v = 0$ in Eq. (6)], $D(t)$ does not depend on the qubit spectrum. We thus expect the same dynamic FSS behavior reported in Ref. [35],

$$D(u, h, L, t) = D(\kappa_u, \kappa_h, \theta), \quad (38)$$

which is independent of $\delta$, and therefore of $\varepsilon_\delta$. For small values of the coupling $u$, we have that

$$D(u, h, L, t) = \frac{1}{2} u^2 Q(h, L, t) + O(u^3), \quad (39)$$

where the growth-rate function $Q$ measures the sensitivity of the qubit coherence properties to the coupling $u$. Its scaling behavior can be derived by matching that of $D$ in Eq. (38), obtaining [35]

$$Q(h, L, t) \approx \left( \frac{\partial\kappa_u}{\partial u} \right)^2 Q(\kappa_h, \theta). \quad (40)$$

We now focus on the more general case $[H_q, H_q S] \neq 0$, restricting to the case $u = 0$ for simplicity, without loss of generality. The most natural working hypothesis is that analogous scaling behaviors develop, with a further dependence on the scaling variable $\varepsilon_\delta$ in Eq. (37), and replacing $\kappa_u$ with $\kappa_v$. One is then led to put forward the following FSS behavior:

$$D(\delta, v, h, L, t) = D(\varepsilon_\delta, \kappa_v, \kappa_h, \theta). \quad (41)$$

Assuming again analyticity at $v = 0$ and since $D \geq 0$, one expects an expansion analogous to Eq. (39),

$$D(\delta, v, h, L, t) = \frac{1}{2} v^2 Q(\delta, h, L, t) + O(v^3), \quad (42)$$

which can be matched to the scaling behavior in Eq. (41), to obtain

$$Q(\delta, h, L, t) \approx \left( \frac{\partial\kappa_v}{\partial v} \right)^2 Q(\varepsilon_\delta, \kappa_h, \theta). \quad (43)$$

Notice that the scaling relations for the growth-rate function $Q$ imply a power law for CQTS, i.e.

$$Q(h, L, t) \approx L^{2y_h} Q(\varepsilon_\delta, \kappa_h, \theta), \quad (44)$$

while exponential laws arise at FOQTs (when considering neutral boundary conditions), such as

$$Q(h, L, t) \approx \frac{L^{2d}}{\Delta(L)^2} Q(\varepsilon_\delta, \kappa_h, \theta), \quad (45)$$

thus increasing as $\sim \exp(bL^d)$.

One may compare the above scaling behaviors with those expected when the system $S$ is not close to a phase transition, for example for $g > g_c$ in the case of the quantum Ising models. Keeping into account that $Q$ is equivalent to a generalized susceptibility, the arguments reported in Sec. III D lead us to the expectation that the growth-rate function should increase as the volume of the system $S$, i.e.

$$Q \sim L^d. \quad (46)$$

The above scaling relations (44) and (45) demonstrate that the rate of the qubit decoherence gets enhanced when the system $S$ experiences a quantum transition. For example, we may compare the $L^d$ behavior expected in the disordered phase with the significantly faster increase $L^{2y_h}$ at $g = g_c$ where the system is critical, cf. Eq. (44), due to the fact that $(z = 1)$

$$2y_h = d + z + 2 - \eta = d + 3 - \eta > d. \quad (47)$$

We finally note that the scaling behavior in the critical region, $g - g_c \ll 1$, where $\xi \approx (g - g_c)^{-\nu} \gg 1$, can be inferred by standard scaling arguments. It essentially amounts to replace $L$ with $\xi$ in the dynamic FSS equations.

2. Quantum work associated with the initial quench

We now discuss the scaling behavior of the quantum work associated with the initial quench of the interaction term $H_q S$. We explicitly address the first and the second moment of the work probability distribution defined in Eq. (14), but similar arguments can be put forward for the higher moments, as defined in Eq. (15).

To begin with, we report the scaling ansatzes of the average work and average square work which are expected in the simplest case $[H_q, H_q S] = 0$ (for $v = 0$):

$$\langle W \rangle (\delta, u, h, L) \approx \Delta(L) W_1(\kappa_u, \kappa_h), \quad (48a)$$

$$\langle W^2 \rangle (\delta, u, h, L) \approx \Delta(L)^2 W_2(\kappa_u, \kappa_h). \quad (48b)$$
These are independent of the spectrum of the qubit, and in particular of \( \delta \) (analogous expressions hold for higher powers of the work). In the case of CQTs, this ansatz is supported by equilibrium FSS arguments, exploiting Eq. (A.11). Indeed, since

\[
\langle 0_h | P | 0_n \rangle \approx L^{d-yr} f_P(\kappa_h),
\]

it is easy to see that simple calculations lead to Eq. (48a) with \( W_1(\kappa_u, \kappa_v) \propto \kappa_u f_P(\kappa_h) \).

In the more general case \([H_q, H_{qS}] \neq 0\), we expect that the scaling variable \( \varepsilon_\delta \) associated with the gap \( \delta \) of the qubit Hamiltonian should also enter the dynamic FSS behavior. However, for the average and the average square work [cf. Eqs. (38) and (39), respectively], this is not the case:

\[
\langle W \rangle (\delta, u, v, h, L) \approx \Delta(L) W_1(\kappa_u, \kappa_v, \kappa_h),
\]

\[
\langle W^2 \rangle (\delta, u, v, h, L) \approx \Delta(L)^2 W_2(\kappa_u, \kappa_v, \kappa_h).
\]

Higher moments are expected to generally depend on \( \varepsilon_\delta \) too.

### 3. Time dependence of the energy exchanges

Coming to the energy distribution defined in Eq. (20), let us again begin with the simplest case \( v = 0 \). We expect the scaling behavior

\[
P_S(U, u, h, L, t) \approx \Delta(L)^{-1} P(v, \kappa_u, \kappa_h, \theta),
\]

where

\[
v = U/\Delta(L).
\]

Thus the average of \( U \) and its fluctuations \( \langle U^2 \rangle_c = \langle U^2 \rangle - \langle U \rangle^2 \) should scale respectively as

\[
\langle U \rangle (u, h, L, t) \approx \Delta(L) U_1(\kappa_u, \kappa_h, \theta),
\]

\[
\langle U^2 \rangle_c (u, h, L, t) \approx \Delta(L)^2 U_2(\kappa_u, \kappa_h, \theta).
\]

In the most general case \( v \neq 0 \), one should add a further dependence on the scaling variable \( \varepsilon_\delta \) associated with the gap of the qubit Hamiltonian. Therefore we obtain

\[
P_S(U, \delta, u, v, h, L, t) \approx \Delta(L)^{-1} P(v, \varepsilon_\delta, \kappa_u, \kappa_v, \kappa_h, \theta).
\]

### V. DYNAMIC BEHAVIOR AT THE CQT

We have seen that the dynamic FSS theory specialized to the qubit-system setup predicts a non trivial scaling limit for the different properties of the model. Here we focus on the behavior at the CQT of the one-dimensional Ising model, and present the results of numerical exact diagonalization simulations for the dynamics of a qubit, after it is suddenly and homogeneously coupled to an Ising ring with Hamiltonian \( \mathcal{H}_Q \). We shall remind the reader that, as detailed in App. [A] for \( v = 0 \) the qubit exhibits pure dephasing in time, while the system evolves in two independent branches with the same Hamiltonian form and different fields \( \mathcal{H}_V, \mathcal{H}_S \) [cf. Eq. (4.12)]. A dynamic FSS theory for the decoherence functions of the qubit coupled to a many-body system at a quantum transition has been already addressed in the literature, in this specific case and for \( \delta = 0 \) [35].

Here let us discuss the general case in which the Hamiltonian coupling term \( H_{qS} \) does not commute with the qubit Hamiltonian \( H_q \), i.e. \( v \neq 0 \) in Eq. (6). We simulated setups where the Ising ring is constituted of up to \( L = 24 \) sites for static calculations, while we limited ourselves to \( L = 16 \) sites for the dynamics of such systems. As we shall see below, the dynamic FSS scaling behavior turns out to emerge quite neatly already for these moderate lengths, thus making unnecessary, in practice, any further extensive check at larger size. A full exact diagonalization approach has been used for systems with \( L \leq 12 \), while a Lanczos diagonalization followed by a fourth-order Runge-Kutta integration of the unitary-evolution operator was employed for larger sizes \((13 \leq L \leq 16)\). We carefully checked that, for most of the simulations, a time step \( dt = 10^{-3} \) is sufficient to reach a high degree of convergence. As we shall explain below, for the calculation of second-order temporal fluctuations of the statistics of energy exchanges, at the largest considered size, a smaller time step \( dt = 10^{-4} \) turns out to be required.

We start with the analysis of the qubit decoherence function \( D \) defined in Eq. (11). Figure 1 displays the scaling behavior of \( D \) as a function of several different scaling variables, namely the rescaled time \( \theta \) [panel a], the qubit detuning \( \varepsilon_\delta \) [panel b], and the qubit-system coupling \( \kappa_v \) [panel c]. Remarkably, data collapse appears already for Ising-chain systems of \( L \lesssim 10 \) sites, as is evident from the figure. This validates the ansatz put forward in Eq. (11). We note that the strongly oscillating behavior which emerges as a function of \( \theta \), implying nearly perfect revivals of the coherence at short times (e.g., at \( \theta \approx 5.3, 6.6, 12.3 \) in the figure), is due to the fact that the dynamic FSS framework is probing the post-quench dynamics of a qubit coupled to a many-body system within the critical regime of a quantum transition. In particular, for a fixed value of the rescaled qubit-system coupling (e.g., \( \kappa_v = 1 \) in the figure) the corresponding coupling parameter \( v \) entering \( H_{qS} \) scales to zero polynomially with \( L \) [recall Eqs. (38) and (23)]. Therefore, as observed in Refs. [35, 39], one expects the global system not to thermalize, and thus a complete decoherence of the qubit not to occur, for finite and fixed rescaled coupling constants.

A similar reasoning can be drawn for the decoherence growth-rate function \( Q \), which is defined as the second derivative of \( D \) with respect to the qubit-system coupling parameter \( u \) or \( v \), cf. Eqs. (39) and (42), respectively. Specifically, we have numerically computed the second
Fig. 1: Decoherence function \(D\) for a qubit coupled to Ising spin-chain systems of different lengths \(L\) (see legend) at the CQT \((g = g_c = 1)\). All numerical data presented here, and in the following figures, are for a qubit-system coupling realized through \(u = 0\) and \(v \neq 0\) in Eq. (3), such that \([H_q, H_S] \neq 0\).

The three panels display the behavior of \(D\) as a function of various scaling variables, according to the following scheme:

(a) \(\varepsilon_\delta = 0.5, \kappa_v = 1\), varying \(\theta\);

(b) \(\kappa_v = 1, \theta = 10\), varying \(\varepsilon_\delta\);

(c) \(\varepsilon_\delta = 0.5, \theta = 10\), varying \(\kappa_v\).

In all simulations we fixed \(\kappa_h = 0.8\) and \(u = 0\), while the qubit was initialized with \(c_+ = \sqrt{2/3}\). To facilitate the readability, we kept the same scaling variables in all the panels, specified in the legend, except the one entering the \(x\) axis. The dark square in each panel corresponds to a common point in all plots.

With the decoherence properties of the qubit, numerical results indicate a nice scaling behavior, thus confirming the dynamic FSS ansatzes of Eqs. (50). A closer look at finite-size corrections for fixed \(\kappa_h\) reveals an approach to the asymptotic behavior which is characterized by \(O(L^{-2})\) and \(O(L^{-1})\) corrections, respectively (see the two insets). This reflects a slower approach to the expected asymptotic behavior of \(\langle W^2 \rangle_c\), rather than that of \(\langle W \rangle\), as is qualitatively visible by comparing the two main panels. These data suggest that the global convergence of full work statistics to its dynamic FSS may be \(O(L^{-1})\), as already pointed out in Ref. [46].

In an analogous spirit, it is eventually interesting to analyze the first two moments of the energy-difference distribution of the Ising system \(S\) along the dynamics at the CQT, cf. Eq. (20). Specifically, the behavior with respect to the rescaled time \(\theta\) of the average energy difference \(\langle U \rangle\) and of its fluctuations \(\langle U^2 \rangle_c = \langle U^2 \rangle - \langle U \rangle^2\) are shown in panels a) and b) of Fig. 3 respectively. We observe likewise a nice convergence to the dynamic FSS behavior predicted by Eqs. (23). We shall observe that the calculation of fluctuations is very sensitive to the numerical accuracy of the simulated dynamics, essentially because much larger precision is required when computing connected quantities, generally arising from large cancellations of their terms.
We point out that the numerical results presented in this section have been obtained by fixing the same initial state and adopting specific values for the various scaling variables (for details, see the insets of the various figures). However we have also performed simulations using other sets of parameters (not shown) and carefully checked that all our conclusions are not affected by such choices.

VI. DYNAMIC BEHAVIOR AT THE FOQT

A. Two-level reduction of the system $S$

We now turn to a situation where the Ising-chain system $S$ is along the FOQT line ($g < g_c$), and concentrate on boundary conditions which do not favor any particular phase. In such cases, one might be tempted to simplify the description of the global system by employing a two-level approximation for the system $S$, as was done in Refs. [23] [30], in different contexts. Under the assumption that only the lowest levels of the system $S$ are effectively involved by the dynamic behavior arising from the sudden quench of the qubit-system interaction in the dynamic FSS limit, we may consider the following two-level reduction of the Hamiltonian $H_S$:

$$H_{S_2}(h) = \frac{\beta}{2} \sigma^{(3)} + \frac{\gamma}{2} \sigma^{(1)},$$  \hspace{1cm} (55)

$$\beta = 2m_0 h L^d, \quad \gamma = \Delta(L), \quad \kappa_h = \beta/\gamma,$$

which acts on two-component wave functions, corresponding to the states $|+\rangle$ and $|-\rangle$, such that $\sigma^{(3)} |\pm\rangle = \pm |\pm\rangle$. Then, the qubit-system interaction term becomes

$$H_{qS_2} = -\frac{\zeta}{2} \sum (1) \sigma^{(3)} - \frac{\eta}{2} \sum (3) \sigma^{(3)} - \frac{\sigma (3)}{2} \Sigma (3),$$  \hspace{1cm} (56)

$$\eta = 2m_0 L^d \zeta, \quad \kappa_h = \eta/\gamma, \quad \zeta = 2m_0 L^d \zeta.$$

The above Hamiltonian terms are completed by the qubit Hamiltonian, which we can be written as

$$H_q = \frac{1}{2} \delta \Sigma (3), \quad \varepsilon = \delta/\gamma.$$

(57)

neglecting the irrelevant identity term of Eq. (4). Therefore, in this approximation, the global Hamiltonian is given by

$$H_2 = H_{S_2} + H_q + H_{qS_2},$$

(58)

to be compared with Eq. (7).

Within the two-level approximation for the system $S$ at a FOQT, we may thus write the global Hamiltonian as

$$\hat{H}_2 \equiv \frac{H_2}{\gamma} = -\kappa_h \sigma^{(3)} + \frac{1}{2} \sigma^{(1)} + \frac{\varepsilon}{2} \Sigma (3),$$

$$-\kappa_u \Sigma (3) \sigma^{(3)} - \kappa_v \Sigma (3) \sigma^{(3)},$$

(59)
or, using the bases where both $\Sigma^{(3)}$ and $q^{(3)}$ are diagonal over the qubit and $S$ states, we may write $H_2$ as the following $4 \times 4$ matrix:

$$
\frac{1}{2} \begin{pmatrix}
\varepsilon \delta - \kappa_{h+u} & 1 & -\kappa_{v} & 0 \\
1 & \varepsilon \delta + \kappa_{h+u} & 0 & \kappa_{v} \\
-\kappa_{v} & 0 & -\varepsilon \delta - \kappa_{h-u} & 1 \\
0 & \kappa_{v} & 1 & -\varepsilon \delta + \kappa_{h-u}
\end{pmatrix}
$$

with $\kappa_{h\pm u} \equiv \kappa_h \pm \kappa_u$. Note that the Hamiltonian (60) allows us to write the corresponding Schrödinger problem in terms of scaling variables only, i.e.

$$
\frac{i}{\theta} \frac{\partial}{\partial \theta} |\psi(\theta)\rangle = H_2 |\psi(\theta)\rangle , \quad \theta = \gamma t ,
$$

where $|\psi(\theta)\rangle$ denotes the wavefunction of the global system in the reduced four-dimensional Hilbert space. Equation (61) readily implies that, under the two-level reduction approximation for the system $S$, the dynamic FSS behavior put forward in Sec. IV is automatically guaranteed.

B. Numerical results

We now report a numerical verification of the dynamic FSS behavior outlined in Sec. IV within the one-dimensional Ising model along its FOQT line, and the comparison with the results of the two-level approximation of the system $S$.

We start comment on the simpler case $u \neq 0$ and $v = 0$, where the qubit Hamiltonian commutes with the interaction term. In such case, one can compute the corresponding FSS functions in an analytic form. Indeed the matrix representation (60) of $H_2$ reduces to a $2 \times 2$ block diagonal form, for which analytic expressions can be obtained. Appendix A1 reports the dynamic FSS functions of all the quantities defined in Eq. (1).

The analytic calculations for the less trivial case, $u = 0$ and $v \neq 0$, for which $[H_q, H_{qS_2}] \neq 0$, are more cumbersome. Indeed, the solution of the corresponding quantum problem requires the diagonalization of the full $4 \times 4$ matrix Hamiltonian (60) over the four-dimensional Hilbert space of the qubit and two levels associated with $S$. The results are not very illuminating, and for this reason we decided not to report them here. Conversely, we preferred to concentrate on a quantitative comparison between the outcomes of numerical exact diagonalization simulations for the full many-body Hamiltonian $H$ and those of the reduced $4 \times 4$ setup $H_2$.

The analysis of the dynamic FSS for the decoherence function $D$ as a function of the rescaled time $\theta$, when coupled to the Ising spin chain at the FOQT, is reported in Fig. 5, where we plotted the outcomes of the simulation of the full model. They demonstrate a clear qualitative accordance with those of the two-level approximation $D^{(2)}(\theta)$, obtained by solving Eq. (61) (thick dashed brown line). We notice the appearance of strong revivals, as is the case in proximity of a CQT, cf. Fig. 1. The convergence in $L$ to the analytic two-level approximation appears to be exponential, as shown by the inset of Fig. 5 for a fixed value of $\theta = 9$.

We now look at the statistics of energy exchanges. The first two moments of the statistics of the work, for the Ising-chain system at the FOQT, are reported in Fig. 6. Data collapse to a dynamic FSS behavior in the infinite volume volume is clearly evident. However, while the average work $\langle W \rangle$ nicely converges to the two-level prediction $W^{(2)}$, with an apparent exponential dependence on $L$, the work fluctuations $\langle W^2 \rangle - \langle W \rangle^2$ appear to be exponential as well, as shown by the inset of Fig. 6.

In particular, we display the temporal behavior of the average (panel a) and its fluctuations (panel b). The inset shows that the statistics of the energy-difference distribution in the system $S$ is automatically guaranteed.

Finally, in Fig. 7 we repeat the analysis of the statistics of the energy-difference distribution in the system $S$, where similar conclusions apply. In particular, we display the temporal behavior of the average (panel a) and of its fluctuations (panel b). Analogously to the outcomes we found at the CQT (cf. Fig. 1), we observe that the fluctuations $\langle W^2 \rangle - \langle W \rangle^2$ are very sensitive to the accuracy of the simulation. It is also worth noticing that the thick dashed curves which indicate the two-level predictions $U_1^{(2)}$ and $U_2^{(2)}$, respectively, follow the same behavior with $\theta$. More precisely, we can be shown that the ratio

$$
U_2^{(2)} / U_1^{(2)} = \sqrt{1 + \kappa_h^2}
$$

for a qubit coupled to an Ising spin chain at the FOQT ($g = 0.9$), as a function of the rescaled time $\theta$. We fixed $\kappa_u = 1$, $\varepsilon \delta = 0.5$, and set the initial state with $c_s = \sqrt{2/3}$, as in panel a) of Fig. 4. The inset shows numerical data for $\theta = 9$, supporting an exponential convergence to the prediction $D^{(2)}$ given by the two-level approximation for the Ising chain (thick dashed line in the main panel).
depends only on the value of $\kappa_h$ (in the figure we used $\kappa_h = 0.8$, therefore $\sqrt{1 + \kappa_h^2} \approx 1.28$). This comes in analogy with Eq. (12), which can be easily proven for a longitudinal qubit-system interaction ($v = 0$).

Before ending we stress that, similarly to the dynamic behavior at the CQT, analogous results have been obtained (and, in particular, the collapse of numerical data to the FSS behavior put forward in Sec. III B) for other values of scaling variables and initial states.

VII. CONCLUSIONS

We have addressed the quantum dynamics of a system composed of a qubit globally coupled to a many-body system characterized by short-range interactions. We employed a dynamic FSS framework to investigate the out-of-equilibrium dynamics arising from the sudden variation (turning on) of the interaction between the qubit and the many-body system, in particular when the latter is in proximity of a quantum first-order or a continuous phase transition. Although the approach is quite general, we considered $d$-dimensional quantum Ising spin models in the presence of transverse and longitudinal fields, as paradigmatic quantum many-body systems.

To characterize the out-of-equilibrium dynamics, we focused on a number of quantum-information oriented properties of the model. Generalizing the results of Ref. 35, we considered the information and energy flow among the various parts of the composite system: we studied the decoherence of the qubit and the statistics associated with the energy injected by switching on the qubit-system interaction and with the energy distribution of the system during the temporal evolution. When the many-body system $S$ is close at a quantum transition, either at a CQT or at a FOQT, we derived the asymptotic scaling behavior exploiting the dynamic FSS framework. The scaling behaviors of the above quantities were validated by means of extensive numerical simulations specialized to one-dimensional Ising systems. We always observed convergence to the expected asymptotic FSS behavior when the system $S$ is at both CQTs and FOQTs.

In the case of FOQTs, we also employed a two-level approximation for the many-body system $S$, to compute the dynamic FSS functions associated with the out-of-equilibrium quantum evolution. The agreement with the numerical results is satisfactory. Quantitative differences between the numerics and the two-level approximation emerge only when monitoring fluctuations and higher momenta of the various energy statistics.
It is however worth pointing out that, already with ten spins, it is possible to infer the asymptotic FSS behavior with a fair accuracy. This paves the way toward an experimental probe of the influence of criticality on the quantum transport properties, spurred on by the recent developments in quantum technologies with ultracold atoms and ions. Indeed they already demonstrated the capability to faithfully reproduce the unitary dynamics of quantum Ising-like chains with $\approx 10$ spins.

Finally we mention that it is possible to extend our dynamic FSS analysis to more general situations. Besides those based on sudden variations of the interactions between the qubit and the many-body system, one may consider other dynamic protocols, for example by taking the opposite limit of slow changes. This situation can be also analyzed within appropriate FSS frameworks, such as that considered in Ref. [10]. We may also devise extensions to more general models, where the qubit is replaced by a generic $N$-level quantum system, the environment is mapped in the continuum limit (or, more generally, can be modeled by a many-body system presenting CQTS or FOQTs), and the qubit-system coupling is not homogeneous or may have a different and more complicated shape.

Appendix A: The commutative case $[H_q, H_{qS}] = 0$

A particular case of the general problem outlined in Sec. II is realized when $v = 0$ in the qubit-system Hamiltonian (6), i.e.,

$$H_{qS} = u \Sigma^{(3)} P,$$  \hfill (A1)

which implies $[H_q, H_{qS}] = 0$. This condition allows us to write the time evolution of the global system in terms of dynamic evolutions of the system $S$ only. Indeed, one can easily prove that the solution of the corresponding Schrödinger problem is given by

$$|\Psi(t)\rangle = e^{-iE_t}\sum_{c_{\pm}} |h\rangle\otimes|\Phi_{h\pm u}(t)\rangle + e^{-iE_t'}\sum_{c_{\pm}} |h\rangle\otimes|\Phi_{h-\pm u}(t)\rangle,$$  \hfill (A2)

where

$$|\Phi_{h\pm u}(t)\rangle = e^{-iH_S(h\pm u)t}|0_h\rangle,$$  \hfill (A3)

i.e., they are solutions of the Schrödinger equations for the system $S$ only,

$$i\frac{\partial}{\partial t}|\Phi_{h\pm u}(t)\rangle = H_S(h \pm u)|\Phi_{h\pm u}(t)\rangle,$$  \hfill (A4)

with $|\Phi_{h\pm u}(t = 0)\rangle = |0_h\rangle$.

Notable relations can be obtained focusing on the evolution of the qubit only. The elements of its reduced density matrix, cf. Eq. (11), read:

$$\rho_{q,11}(t) = |c_+|^2, \quad \rho_{q,22}(t) = |c_-|^2,$$  \hfill (A5)

$$\rho_{q,21}(t) = e^{-i\delta t}c_+^*c_+\langle \Phi_{h-\pm u}(t)|\Phi_{h+\pm u}(t)\rangle = \rho_{q,21}(t)^*.$$
with \( H_{S_2}(h) \) as in Eq. (55). The initial condition is given by the ground state

\[
|\phi_{h\pm u}(\theta = 0)| = |0_h\rangle = \sin \left(\frac{\alpha_h}{2}\right) |+\rangle - \cos \left(\frac{\alpha_h}{2}\right) |+\rangle,
\]

(A14)

with \( \tan \alpha_h = \kappa_h^{-1} \), and \( \alpha_h \in (0, \pi) \). The quantum evolution described by Eq. (A12) can be easily obtained by diagonalizing the \( 2 \times 2 \) Hamiltonian \( H_{S_2}(h \pm u) \), whose eigenstates are

\[
|0_{h \pm u}\rangle = \sin \left(\frac{\alpha_{h \pm u}}{2}\right) |-\rangle - \cos \left(\frac{\alpha_{h \pm u}}{2}\right) |+\rangle,
\]

(A15a)

\[
|1_{h \pm u}\rangle = \cos \left(\frac{\alpha_{h \pm u}}{2}\right) |-\rangle + \sin \left(\frac{\alpha_{h \pm u}}{2}\right) |+\rangle.
\]

(A15b)

with \( \tan \alpha_{h \pm u} = \kappa_{h \pm u}^{-1} \) and \( \alpha_{h \pm u} \in (0, \pi) \). The corresponding energy eigenvalues are

\[
E_{0/1} = \Delta(L) \mathcal{E}_{0/1} = \mp \left[1 + \kappa_{h \pm u}^2\right].
\]

(A16)

The time-dependent state evolves as

\[
|\phi_{h\pm u}(\theta)| = e^{-i\mathcal{E}_0 \theta} \cos \left(\frac{\alpha_h - \alpha_{h \pm u}}{2}\right) |0_{h \pm u}\rangle + e^{-i\mathcal{E}_u \theta} \sin \left(\frac{\alpha_h - \alpha_{h \pm u}}{2}\right) |1_{h \pm u}\rangle.
\]

(A17)

Then, by rewriting them in terms of the original basis \( |\pm\rangle \), using Eqs. (A15), and replacing into Eq. (A2), with \( |\phi_{h \pm u}(\theta)| \) instead of \( |\Phi_{h \pm u}(\theta)| \), we obtain the solution of the dynamic problem within the two-level approximation of the system \( S \). This reads

\[
|\psi(\theta)| = e^{-i(\mathcal{E}_u \theta/2)} c_{+|} |\phi_{h + u}(\theta)| + e^{i(\mathcal{E}_u \theta/2)} c_{-|} |\phi_{h - u}(\theta)|.
\]

(A18)

This solution is already written in terms of the scaling variables, thus the scaling behaviors put forward in Sec. IV are fully confirmed. The corresponding FSS functions, thus the scaling behaviors put forward in Sec. IV are fully confirmed. This solution is already written in terms of the scal-

\( W^{(2)}_1(\kappa_u, \kappa_h) = -\frac{1}{2} \left( |c_{+}|^2 - |c_{-}|^2 \right) \kappa_u \left[ 1 - 2 \sin \left(\frac{\alpha_u}{2}\right)^2 \right].
\]

(A20)

On the other hand, for the second moment \( \langle W^2 \rangle \) we simply obtain

\[
W^{(2)}_2(\kappa_u, \kappa_h) = \frac{1}{4} \kappa_u^2.
\]

(A21)

For the energy fluctuations of the many-body system \( S \), we may obtain the time dependence of the average energy variation, cf. Eq. (21), using the formulas

\[
U^{(2)}_1(\theta) = \text{Tr} \left[ \hat{H}_{S_2}(h) \rho(\theta) \right] - \mathcal{E}_{0,0}, \quad \mathcal{E}_{0,0} = -\frac{1}{2} \sqrt{1 + \kappa_h^2}.
\]

(A22)

Then, using Eq. (A18), we may write it as

\[
U^{(2)}_1(\theta) = |c_{+}|^2 \langle \phi_{h + u}(\theta)| \hat{H}_{S_2}(h) |\phi_{h + u}(\theta)\rangle + |c_{-}|^2 \langle \phi_{h - u}(\theta)| \hat{H}_{S_2}(h) |\phi_{h - u}(\theta)\rangle.
\]

(A23)

where

\[
\hat{H}_{S_2}(h) = -\frac{\kappa_h}{2} \sigma_{(3)} + \frac{1}{2} \sigma_{(1)}.
\]

(A24)

Finally, for the average of the square energy variation we obtain

\[
U^{(2)}_2(\theta) = \text{Tr} \left[ \hat{H}_{S_2}(h)^2 \rho(\theta) \right] - 2 \mathcal{E}_{0,0} U^{(2)}_1(\theta) - \mathcal{E}_{0,0}^2
\]

(A25)

where we used the fact that \( [\hat{H}_{S_2}(h)]^2 = (1 + \kappa_h^2) I_2 \).

Appendix B: Accuracy of the two-level approximation at the FOQT

In Sec. VII B we observed that a two-level reduction of the many-body system to which the qubit is coupled, when the former is at a FOQT, is capable to accurately grasp the asymptotic FSS behavior of several properties of the global system, including the decoherence quantifiers for the qubit, the averages of the work done by the quench and of the energy pumped in the system \( S \). All these quantities are linear functionals of the Hamiltonian \( H \), for which the adiabatic theorem typically applies without any issue [10].

On the opposite hand, Figs. 3 and 7 have spotlighted clear discrepancies, when comparing fluctuations of the work and of the system energy (i.e., \( \langle W^2 \rangle \) and \( \langle \mathcal{L}^2 \rangle \), with numerical exact diagonalization results for the quantum Ising chains. We believe that such discrepancies are essentially related to the limited accuracy of the two-level approximations, which of course cannot capture the full complexity of a many-body quantum system. Below we provide evidence of this fact, focusing on a specific quantity. It would be tempting to further investigate the problem, in such a way to achieve a more exhaustive understanding of the accuracy of our approximation. This however lies outside the purposes of this paper and will be left for a future work.

Figure 8 compares the time behavior of the system energy and of its square value, evaluated either with the full Hamiltonian \( H_S \), or by keeping only its two lowest-energy levels \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \) (associated with the two energies \( \varepsilon_1 \) and \( \varepsilon_2 \)). Namely, \( \langle \Psi(\theta)| H_S |\Psi(\theta)\rangle \) or \( \langle \Psi(\theta)| H_S^{(2)} |\Psi(\theta)\rangle \)
respectively, with \(H_S^{(2)} = \varepsilon_1|\Phi_1\rangle + \varepsilon_2|\Phi_2\rangle\). We observe a general agreement between the two approaches [panels a) and c)], however a more detailed analysis reveals that the discrepancies among them are typically one order of magnitude larger for \(\langle H_S^2 \rangle\) [panel b)], rather than for \(\langle H_S \rangle\) [panel d)]. Moreover, while the data in panel b) suggest that such discrepancies systematically diminish with increasing the size, the situation in panel d) is less clear and fluctuations at \(L = 16\) are still quite large. Notice also the appearance of wiggles in panels a) and c), concerning the results obtained with a two-level truncation of the Hamiltonian spectrum.

The discrepancies highlighted above can be amplified when measuring fluctuations. Indeed, in Fig. 9 we show the ratio between the rescaled average energy of the system \(\langle U \rangle / \Delta\) and its rescaled fluctuations \(\langle U^2 \rangle_c / \Delta^2\). The two-level reduction of system \(S\) would predict a value for such ratio which depends only on \(\kappa_h\), since it can be shown that Eq. (A25) still holds if \(v \neq 0\). Conversely, as displayed in panel a), the full simulation shows a nontrivial dependence on \(\theta\), as well. The comparison between the numerical values averaged over \(\theta\) (horizontal dashed lines) and the analytic estimate \(\sim 1/\sqrt{1 + \kappa_h^2}\) given by Eq. (A25) is provided in panel b), as a function of \(\kappa_h\). Similarly to what observed in Fig. 6 b) and Fig. 7 b), we highlight the emergence of a discrepancy between the two approaches, which however cannot be interpreted as a simple offset independent of the value of \(\kappa_h\).

![Diagram](image-url)

FIG. 8: Comparison of the average system energy \(\langle H_S \rangle\) [panels a) and b)] and of its square value \(\langle H_S^2 \rangle\) [panels c) and d)], evaluated using the full many-body Hamiltonian and a truncation to its two lowest levels. The Ising system is at the FOQT with \(g = 0.9\), while all the other parameters are set as in Fig. 7. Panels a) and c) display the two evolutions, with respect to the rescaled time \(\theta\), for a fixed chain length \(L = 16\). Panels b) and d) highlight the absolute differences between the two cases, for various system sizes.

![Diagram](image-url)

FIG. 9: Panel a): ratio between the rescaled average energy of the system and its rescaled fluctuations, as a function of \(\theta\). The various curves are for different values of \(\kappa_h\), as indicated in the legend (from top to bottom, curves are for increasing \(\kappa_h\)). Panel b): comparison between numerical data averaged over the time [see horizontal dashed lines in panel a)] and the analytic estimate \(U_\theta / U_\Delta = 1/\sqrt{1 + \kappa_h^2}\), as given by Eq. (A25). Data are for an Ising-chain system with \(L = 12\) sites, while all the other parameters are set as in Fig. 8.

[1] J. Dziarmaga, Dynamics of a quantum phase transition and relaxation to a steady state, Adv. Phys. 59, 1063 (2010).
[2] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Colloquium: Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. 83, 863 (2011).
[3] R. Nandkishore and D. A. Huse, Many body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. 6, 15 (2015).
[4] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton’s cradle, Nature (London) 440, 900 (2006).
[5] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm, and J. Schmiedmayer, Non-equilibrium coherence dynamics in one-dimensional Bose gases, Nature (London) 449, 324 (2007).
[6] S. Trotzky, Y.-A. Chen, A. Flesch, I. P. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Probing the relaxation towards equilibrium in an isolated strongly correlated one-dimensional Bose gas, Nat. Phys. 8, 325 (2012).
[47] M. Campostrini, J. Nespolo, A. Pelissetto, and E. Vicari, Finite-size scaling at first-order quantum transitions of quantum Potts chains, Phys. Rev. E 91, 052103 (2015).

[48] A. Pelissetto, D. Rossini, and E. Vicari, Finite-size scaling at first-order quantum transitions when boundary conditions favor one of the two phases, Phys. Rev. E 98, 032124 (2018).

[49] M. Campostrini, A. Pelissetto, and E. Vicari, Quantum transitions driven by one-bond defects in quantum Ising rings, Phys. Rev. E 91, 042123 (2015); Quantum Ising chains with boundary terms, J. Stat. Mech. (2015) P11015.

[50] D. Rossini and E. Vicari, Ground-state fidelity at first-order quantum transitions, Phys. Rev. E 98, 062137 (2018).

[51] P. Pfeuty, The one-dimensional Ising model with a transverse field, Ann. Phys. 57, 79 (1970).

[52] V. Privman and M. E. Fisher, Finite-size effects at first-order transitions, J. Stat. Phys. 33, 385 (1983).

[53] G. G. Cabrera and R. Jullien, Universality of Finite-Size Scaling: Role of the Boundary Conditions, Phys. Rev. Lett. 57, 393 (1986); Role of the boundary conditions in the finite-size Ising model, Phys. Rev. B 35, 7062 (1987).

[54] M. N. Barber, Finite-size scaling, in Phase transitions and critical phenomena, vol. 8, page 145, C. Domb and J. L. Lebowitz eds. (Academic Press, London 1983).

[55] Finite Size Scaling and Numerical Simulations of Statistical Systems, ed. V. Privman (World Scientific, 1990).

[56] R. Guida and J. Zinn-Justin, Critical exponents of the N-vector model, J. Phys. A 31, 8103 (1998).

[57] M. Campostrini, A. Pelissetto, P. Rossi, and E. Vicari, 25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple-cubic lattice Phys. Rev. E 65, 066127 (2002).

[58] M. Hasenbusch, A finite size scaling study of lattice models in the three-dimensional Ising universality class, Phys. Rev. B 82, 174433 (2010).

[59] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, Precision islands in the Ising and O(N) models, J. High Energy Phys. 08, 036 (2016).

[60] M. V. Kompaniets and E. Panzer, Minimally subtracted six-loop renormalization of O(n)-symmetric $\varphi^4$ theory and critical exponents, Phys. Rev. D 96, 036016 (2017).

[61] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, Dynamics of Loschmidt echoes and fidelity decay, Phys. Rep. 435, 33 (2006).

[62] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Entanglement in many-body systems, Rev. Mod. Phys. 80, 517 (2008).

[63] S.-J. Gu, Fidelity approach to quantum phase transitions, Int. J. Mod. Phys. B 24, 437 (2010).

[64] D. Braun, G. Adesso, F. Benatti, R. Floreanini, U. Marzolino, M. W. Mitchell, and S. Pirandola, Quantum-enhanced measurements without entanglement, Rev. Mod. Phys. 90, 035006 (2018).

[65] G. De Chiara and A. Sanpera, Genuine quantum correlations in quantum many-body systems: a review of recent progress, Rep. Prog. Phys. 81, 074002 (2018).

[66] J. Simon, W. S. Bakr, R. Ma, M. E. Tai, P. M. Preiss, and M. Greiner, Quantum simulation of antiferromagnetic spin chains in an optical lattice, Nature 472, 307 (2011).

[67] E. E. Edwards, S. Korenblit, K. Kim, R. Islam, M.-S. Chang, J. K. Freericks, G.-D. Lin, L.-M. Duan, and C. Monroe, Quantum simulation and phase diagram of the transverse-field Ising model with three atomic spins, Phys. Rev. B 82, 060412(R) (2010).

[68] R. Islam, E. E. Edwards, K. Kim, S. Korenblit, C. Noh, H. Carmichael, G.-D. Lin, L.-M. Duan, C.-C. Joseph Wang, J. K. Freericks, and C. Monroe, Onset of a quantum phase transition with a trapped ion quantum simulator, Nat. Commun. 2, 377 (2011).

[69] G.-D. Lin, C. Monroe, and L.-M. Duan, Sharp Phase Transitions in a Small Frustrated Network of Trapped Ion Spins, Phys. Rev. Lett. 106, 230402 (2011).

[70] K. Kim, S. Korenblit, R. Islam, E. E. Edwards, M.-S. Chang, C. Noh, H. Carmichael, G.-D. Lin, L.-M. Duan, C. C. Joseph Wang, J. K. Freericks, and C. Monroe, Quantum simulation of the transverse Ising model with trapped ions, New J. Phys. 13, 105003 (2011).

[71] S. Debnath, N. M. Linke, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe, Demonstration of a small programmable quantum computer with atomic qubits, Nature 536, 63 (2016).

[72] H. Labuhn, D. Barredo, S. Ravets, S. de Leseleuc, T. Macrì, T. Lahaye, and A. Browaeys, Tunable two-dimensional arrays of single Rydberg atoms for realizing quantum Ising models, Nature 534, 667 (2016).