NEW REGULARITY ESTIMATES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We establish new quantitative Hessian integrability estimates for viscosity supersolutions of fully nonlinear elliptic operators. As a corollary, we show that the optimal Hessian power integrability $\varepsilon = \varepsilon(\lambda, \Lambda, n)$ in the celebrated $W^{2,\varepsilon}$-regularity estimate satisfies

$$
\left(1 + \frac{2}{n} \left(1 - \frac{1}{\Lambda} \right)\right)^{\frac{n-1}{n}} \left(\frac{\Lambda}{\Lambda} \right)^{\frac{n-1}{n}} \leq \varepsilon \leq \frac{n\lambda}{(n-1)\Lambda + \lambda},
$$

where $n \geq 3$ is the dimension and $0 < \lambda < \Lambda$ are the ellipticity constants. In particular, $\left(\frac{2}{n}\right)^{n-1} \varepsilon(\lambda, \Lambda, n)$ blows-up, as $n \to \infty$; previous results yielded fast decay of such a quantity. The upper estimate improves the one obtained by Armstrong, Silvestre, and Smart in [1].

Keywords: Fully nonlinear elliptic PDEs, Viscosity supersolutions, Hessian integrability, Quantitative regularity estimates.

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1. Introduction

That solutions of the uniformly elliptic inequality

\[ a_{ij}(x) \partial_{ij} u(x) \leq 0 \quad \text{in} \quad B_1 \subset \mathbb{R}^n \]

belong to the Sobolev space \( W^{2,\varepsilon} \) for some \( 0 < \varepsilon(n, \lambda, \Lambda) \leq 1 \) is a foundational result in the theory of elliptic PDEs. It was originally proven independently by Fang-Hua Lin [11] and Lawrence C. Evans [7], and later extended to the fully nonlinear framework by Luis Caffarelli [4], see also the book by Caffarelli and Cabre [5].

A careful look at the original constructions reveals that the exponent \( \varepsilon(n, \lambda, \Lambda) \) obtained, as a function of the ellipticity ratio, \( \lambda/\Lambda \), and the dimension, decays to zero very rapidly, i.e. exponentially fast. Improvements in such a decay have been recently obtained by Nam Le in [10] and by Connor Mooney in [13] by means of powerful ideas \( \text{ala} \) Cabre in [6] and Savin in [19], viz. the sliding paraboloids method to bypass the Aleksandrov-Bakelman-Pucci (ABP) estimate. This in turn improves the measure estimate of the contact points, yielding sharper control on the Hessian estimate.

Indeed, the measure estimate obtained by Le in [10] implies that \( \varepsilon(n, \lambda, \Lambda) \geq c_0(n) \left( \frac{1}{\Lambda} \right)^{(n+1)} \) where \( c_0(n) \sim n^{n/2} \). Mooney in [13] improves the degree of the polynomial decay, however, the dimensional constant dependence is smaller, viz. \( \varepsilon(n, \lambda, \Lambda) \geq c_1(n) \left( \frac{1}{\Lambda} \right)^{n-1} \), where \( c_1(n) \sim n^{-n} \).

Universal Hessian estimates, such as in the \( W^{2,\varepsilon} \)-regularity theory, yield compactness properties of solutions which can be used to obtain further, refined regularity of solutions in more organized media, e.g. [18, 20, 21] and references therein.

Quantitative estimates for fully nonlinear elliptic equations involving a large number of independent variables have become a prominent field of research as they naturally appear in models from financial engineering, machine learning and deep learning, among others, see for instance [2, 8], and references therein. In particular, improved Hessian exponent integrability for viscosity supersolutions, in very high dimensional spaces, is a relevant endeavor from both pure and applied viewpoints.

From the theoretical aspect of the theory, quantitative estimates upon \( \varepsilon \) relate to the Hausdorff measure of eventual singular sets of solutions to fully nonlinear equations of the form

\[ F(D^2 u) = 0, \quad B_1 \subset \mathbb{R}^n. \]

Indeed, since the program carried out by Nikolai Nadirashvili and Serge Vladuț, [14, 15, 16, 17] it has been known that viscosity solutions to (1.2) may fail to be twice differentiable and that \( C^{1,\alpha} \) estimates are, in fact, optimal. Under \( C^1 \) regularity of \( F \), Armstrong, Silvestre, and Smart in [1] proved \( C^2 \) differentiability of viscosity solutions to (1.2) in the complement of a closed subset \( \Sigma \). The authors actually manage to estimate the Hausdorff dimension of the singular set \( \Sigma \) by \( n - \varepsilon \), where \( \varepsilon > 0 \) is precisely the exponent of the \( W^{2,\varepsilon} \)-regularity theory. In [1] the authors also show that \( 2/(\Lambda/\lambda + 1) \) is a universal upper bound for the exponent \( \varepsilon \). Upon some heuristics inferences, the authors are then led to conjecture that

\[ \varepsilon_* = \frac{2}{(\Lambda/\lambda) + 1} \]

is the (universal) optimal exponent in the \( W^{2,\varepsilon} \) estimates. This has been an important, influential conjecture since then; see e.g. the introduction of [10] and [22] for further details.
In this paper we continue the program of obtaining improved bounds for the optimal Hessian exponent integrability, $\varepsilon(n, \lambda, \Lambda)$, for a viscosity supersolution of fully nonlinear elliptic equations. We are particularly interested in problems modeled in high-dimensional spaces and our main theorem states as follows:

**Theorem 1.** Let $u$ be viscosity solution of
\begin{equation}
\mathcal{M}_{\lambda, \Lambda}(D^2 u) \leq 0 \quad \text{in} \quad B_1 \subset \mathbb{R}^n,
\end{equation}
where $\mathcal{M}_{\lambda, \Lambda} : S(n) \to \mathbb{R}$ denotes the Pucci extremal operator and $n \geq 3$. Then, $u \in W^{2, \varepsilon}(B_{1/2})$, with universal estimates, for an optimal exponent $\varepsilon = \varepsilon(n, \lambda, \Lambda)$ satisfying
\begin{equation}
\mu_n \frac{\left(1 + \left(1 - \frac{\lambda}{\Lambda}\right)\left(1 - \frac{1}{n}\right)\right)^{n-1}}{\ln n} \cdot \left(\frac{\Lambda}{\lambda}\right)^{n-1} \leq \varepsilon \leq \frac{n \lambda}{(n-1)\Lambda + \lambda},
\end{equation}
for a computable, increasing sequence of positive numbers, $\frac{1}{2} < \mu_n \to 1 - e^{-1}$.

We actually prove a much stronger result as byproduct of the new ingredients introduced in this paper. For now, one should note that the upper bound in (1.5) solves, in the negative, the Armstrong-Silvestre-Smart conjecture, (1.3). The lower bound, on the other hand, provides a considerable improvement on the dimensional dependence, viz. the constants $c_0(n) \sim n^{-n/2}$ and $c_1(n) \sim n^{-n}$ from Le’s and Mooney’s theorems respectively. In particular, if $\lambda < \Lambda$,
\begin{equation}
\lim_{n \to +\infty} \frac{\varepsilon(n, \lambda, \Lambda)}{\left(\frac{\Lambda}{\lambda}\right)^{n-1}} = +\infty.
\end{equation}
Thus, our result suggests that the sharp polynomial decay on $\varepsilon(n, \lambda, \Lambda)$, as a function of $\frac{\lambda}{\Lambda}$, should likely be less than $(n-1)$.

We conclude this introduction by discussing the main new ideas involved in the proof of Theorem 1. As usual, the lower bound is obtained from an improved $L^\varepsilon$ estimate of the type:
\begin{equation}
\left|\left\{ \Theta_{\varepsilon} > t \right\} \cap B_{1/2} \right| \leq C t^{-\varepsilon}
\end{equation}
for a universal constant $C > 0$, where $\Theta_{\varepsilon}(x)$ measures the maximum aperture of concave paraboloids touching $u$ by below at $x$, see (2.2). We follow [13]; however the decision upon the dyadic iteration is delegated to an optimization process; a sort of intrinsic scaling for the problem. More precisely, for a new parameter $\delta > 0$, to be later selected by an optimization routine, we estimate the measure decay of the set of points where $u$ is above its lower envelope of paraboloids with opening $-(1 + \delta)^k$, for $k \in \mathbb{N}$. By choosing “the best possible” $\delta > 0$ and changing variables, we are able to show that the sharp $\varepsilon$ in the $W^{2, \varepsilon}$-regularity theory satisfies:
\begin{equation}
\varepsilon \geq \sup_{(0,1)} \frac{\ln(1 - cy^n)}{\ln(1 - \gamma)},
\end{equation}
for a carefully crafted constant $0 < c(n, \lambda, \Lambda) < 1$. The bigger the $c$, the bigger the above supremum. Hence, we decide on the “best possible” value for $c$ through yet another optimization problem which takes into account the number of possible negative eigenvalues of $D^2 u$. Since viscosity supersolutions must have at least one negative eigenvalue, we show
\begin{equation}
c(n, \lambda, \Lambda) \geq \left(1 + \left(\frac{\Lambda}{\lambda} - 1\right)\left(\frac{1}{n-1}\right)^{1-n}\right)^{n-1},
\end{equation}
and this combined with (1.6) yields
\begin{equation}
\varepsilon \geq \mu_n \frac{\left(1 + \left(1 - \frac{\lambda}{\Lambda}\right)\left(1 - \frac{1}{n}\right)\right)^{n-1}}{\ln n} \cdot \left(\frac{\Lambda}{\lambda}\right)^{n-1},
\end{equation}
which is the lower bound stated in Theorem 1.

The upper bound is obtained by carefully extending Armstrong-Silvestre-Smart’s original example; the analysis though is considerably more involved. As a consequence of our upper estimate, if it is possible to establish an adimensional \( W^{2,\infty} \) regularity theory, as suggested in [1], then such an \( \epsilon \) must be less than or equal to \( \frac{d}{N} \).

The rest of the paper is organized as follows: In Section 2 we collect preliminaries definitions and results that will assist the analysis throughout the paper. In Section 3, we establish a new key lemma which yields faster geometric measure decay of sets. In Section 4 we obtain improved \( W^{2,\infty} \) estimates and in Section 5 we discuss how our analysis, combined with the arguments from [13], yield improved global Hessian integrability. Finally in Section 6 we craft a viscosity supersolution whose Hessian does not belong to \( L^\infty \), fostering a the new upper bound of the theory.

2. Preliminaries

For a positive integer \( n \geq 2 \), let \( S_n \) be the set of all real \( n \times n \) symmetric matrices. We define for \( 0 < \lambda \leq \Lambda \) and \( M \in S_n \) the Pucci’s extremal operators by

\[
\begin{align*}
\mathcal{M}^{-\lambda}_{\lambda}(M) &= \inf_{\lambda \Id_n \leq A \leq \lambda \Id_n} \text{trace}(AM) \quad \text{and} \quad \mathcal{M}^{+\lambda}_{\lambda}(M) = \sup_{\lambda \Id_n \leq A \leq \lambda \Id_n} \text{trace}(AM)
\end{align*}
\]

where the inf and sup are taken over all symmetric matrices \( A \) whose eigenvalues belong to \([\lambda, \Lambda] \). Given a domain \( \Omega \subset \mathbb{R}^n \) and a function \( u \in C(\bar{\Omega}) \) we define the function \( \Theta(u, \Omega)(x_0) \) as

\[
\inf \left\{ \lambda > 0 \mid u(x) \geq u(x_0) + \lambda \cdot (x - x_0) - \frac{\lambda}{2} |x - x_0|^2 \text{ in } \Omega, \text{ for some } y \in \mathbb{R}^n \right\}.
\]

If there is no tangent paraboloid from below at \( x_0 \) then we say \( \Theta(u, \Omega)(x_0) = +\infty \). For \( a > 0 \) and \( L(x) \) an affine function, we say

\[
P^{L}_v(x) = -\frac{a}{2}|x|^2 + L(x),
\]

is a paraboloid of opening \(-a\). Following [13], if \( \Omega \) is a bounded, strictly convex domain and \( v \in C(\bar{\Omega}) \), we define the \( a \)-convex envelope \( \Gamma^a_v \) on \( \bar{\Omega} \) as

\[
\Gamma^a_v(x) := \sup_{L} \{ P^L_v(x) : P^L_v \leq v \text{ in } \bar{\Omega} \}.
\]

By convexity of \( \Omega \), one verifies that \( \Gamma^a_v = v \) on \( \partial \Omega \). Also, taking \( a = 0 \), one recovers the usual notion of convex envelopes by affine functions. Next we define

\[
A_a(v) := \left\{ x \in \Omega \mid v(x) = \Gamma^a_v(x) \right\}.
\]

That is, \( A_a(v) \) is the set of points in \( \Omega \) where \( v \) has a tangent paraboloid of opening \(-a\) from below in \( \Omega \). One can check that \( A_a(v) \subset A_b(v) \) if \( a \leq b \). Also, it is easy to verify that for any \( \lambda, \gamma \in \mathbb{R} \) and \( \beta > 0 \), there holds:

\[
\Gamma^a_{\beta \nu + \gamma |x|^2} = \beta \Gamma^a_{\nu} + \frac{\gamma}{2} |x|^2 \quad \text{and} \quad A_{\lambda} \left( \beta \nu + \frac{\gamma}{2} |x|^2 \right) = A_{\lambda \nu}(v).
\]

Throughout the paper, \( \Omega \) denotes a bounded, strictly convex domain in \( \mathbb{R}^n \). The next two lemmas have been established by C. Mooney in [13] and they are the starting point of our analysis:

**Lemma 1** (Mooney [13, Lemma 3.1]). Assume that \( v \in C(\bar{\Omega}) \) and \( \mathcal{M}^{-\lambda}_{\lambda}(D^2 v) \leq K < \infty \). Let \( P \) be a paraboloid of opening \(-a < 0 \) tangent from below to \( \Gamma^0_v \) at \( x_0 \in \bar{\Omega} \setminus A_0(v) \). Slide \( P \) up until \( P + t, \ t > 0 \), touches \( v \) from below at \( x_1 \in \bar{\Omega} \). Then, \( x_1 \in \bar{\Omega} \setminus A_0(v) \).
Lemma 2 (Mooney [13, Lemma 3.2]). Assume that $v$ is convex. For any measurable set $F \subset \Omega$, let $V$ denote the set of vertices of all tangents paraboloids of opening $-a < 0$ to $v$ at points in $F$. Then, $|V| \geq |F|$.

We conclude this preliminary section with a few concepts and definitions.

**Definition 1.** Let $M \in S_n$ be a symmetric matrix. We denote by $\text{Spec}(M) := \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\}$ the eigenvalues of $M$ (counting multiplicity) and

$$s^-(M) := \# \{ \lambda \in \text{Spec}(M) \mid \lambda \leq 0 \}.$$ 

**Definition 2.** Let $u \in C(\overline{\Omega})$. We say $D^2 u(x)$ has at least $k$ nonpositive eigenvalues (in the viscosity sense) if for all $\varphi \in C^2(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, then

$$(2.4) \quad s^-(D^2 \varphi(x_0)) \geq k.$$ 

The biggest such $k$ that makes (2.4) true for all $x_0 \in \Omega$ is called the minimal number of nonpositive eigenvalues of $D^2 u(x)$ in the viscosity sense.

We note that if $u \in C(\overline{\Omega})$ is a viscosity supersolution of $M_{\lambda, \Lambda}(D^2 u) \leq 0$ in $\Omega$, then the minimal number of nonpositive eigenvalues of $D^2 u(x)$ is at least 1.

We finish this preliminary section with a quick discussion upon the classical Lambert $W$ function, as it will play an important role in section 4. For a complex number $z$ the Lambert $W$ function is defined as the function that solves the equation

$$(2.5) \quad W(z)e^{W(z)} = z.$$ 

This equation has infinitely many solutions given by $W_k(z)$, $k \in \mathbb{Z}$, called the $k^{\text{th}}$-branch of $W$. Its branches are defined through the equations

$$W_k(z) \exp(W_k(z)) = z, \quad W_k(z) \sim \ln_k(z), \quad |z| \to \infty$$

and $\ln_k(z) = \ln |z| + 2\pi ik$ is the $k^{\text{th}}$-branch of the complex natural logarithm, see for instance [3]. When dealing with real numbers, we use the real-valued branches of the Lambert $W$ function, namely $W_0$ and $W_{-1}$. $W_0$ is called the principal branch and $W_{-1}$ is the only other branch that takes real values.

![Figure 1. Real branches for the Lambert W function](image)

As the range of the function $y \mapsto ye^y$ is $[-e^{-1}, \infty)$, for real numbers $x$ and $y$, the equation $ye^y = x$
has a solution if, and only if \( x \geq -e^{-1} \). For \( x \geq 0 \), the solution is \( y = W_0(x) \) and for \(-e^{-1} \leq x < 0 \) we have two solutions given by \( y = W_0(x) \) and \( y = W_{-1}(x) \). From the defining equation of \( W \), it is easy to obtain a few special values such as:

\[
W_0(0) = 0, \quad W_{-1}(0) = \lim_{x \to 0} W_{-1}(x) = -\infty \quad \text{and} \quad W_{-1}(-e^{-1}) = W_0(-e^{-1}) = -1.
\]

From the definition we have the following identities:

\[ W_0(xe^x) = x, \quad \text{for} \quad x \geq -1, \]

and

\[ W_{-1}(xe^x) = x, \quad \text{for} \quad x \leq -1, \]

Note, though, that since \( f(x) = xe^x \) is not injective, then \( W(f(x)) = x \) might not always hold. In fact, for a fixed \( x < 0 \) and \( x \neq -1 \), the equation \( xe^x = ye^y \) has two real solutions in \( y \). One of which is of course \( y = x \). And the other solution is

\[
y = \begin{cases} W_0(x), & x < -1 \\ W_{-1}(x), & -1 < x < 0 \end{cases}
\]

Differentiating (2.5) and solving for \( W'(z) \) one easily gets:

\[
\frac{dW}{dz} = \frac{1}{z + e^{W(z)}}, \quad z \neq -1.
\]

In addition, if \( z \neq 0 \), there holds

\[
\frac{dW}{dz} = \frac{W(z)}{z(1 + W(z))},
\]

and since \( W(0) = 0 \), it follows \( W'(0) = 1 \).

We now present lower and upper inequalities for the branch \( W_{-1} \) that will be useful for us in Section 4. These inequalities can be derived easily from the elementary properties of the Lambert \( W \) function, but we decided to include a proof as a courtesy to the readers.

**Lemma 3.** For any \( u \geq 0 \), the following sharp inequalities hold:

\[
-\frac{e}{e-1}(u+1) \leq W_{-1}(-e^{-(u+1)}) \leq -(u+1).
\]

**Proof.** First we note that, from the definition

\[
W_{-1}(-e^{-(u+1)}) \exp(W_{-1}(-e^{-(u+1)})) = -e^{-(u+1)}.
\]

Multiplying by \(-1\) and applying the logarithm function to both sides we obtain,

\[
W_{-1}(-e^{-(u+1)}) = -u - 1 - \ln(-W_{-1}(-e^{-(u+1)})).
\]

since \( W_{-1}(-e^{-(u+1)}) \leq -1 \), for all \( u \geq 0 \) then \( \ln(-W_{-1}(-e^{-(u+1)})) \geq 0 \). Therefore,

\[
W_{-1}(-e^{-(u+1)}) \leq -u - 1.
\]

Next, consider the function \( f(u, a) = W_{-1}(-e^{-(u+1)}) + a(u + 1) \) for \( u \geq 0 \) and \( a \in \mathbb{R} \). Note that \( f_2(u, a) = u + 1 > 0 \) for all \( u \geq 0 \) which implies that \( a \mapsto f(u, a) \) is an increasing function. Moreover, since \( f(u, 1) = W_{-1}(-e^{-(u+1)}) + u + 1 \leq 0 \) by (2.6), and \( \lim_{u \to \infty} f(u, a) = \infty \), then for each fixed \( u \geq 0 \), there exists \( a(u) \) such that \( f(u, a(u)) = 0 \). That is,

\[
a(u) = \frac{W_{-1}(-e^{-(u+1)})}{u + 1}.
\]

Note next that

\[
a'(u) = \frac{W_{-1}(-e^{-(u+1)})}{(1 + W_{-1}(-e^{-(u+1)}))(u + 1)^2} \left(u + 2 + W_{-1}(-e^{-(u+1)})\right).
\]
The sign of \(a'(u)\) is dictated by the sign of the term \(u + 2 + W_1(-e^{-(u+1)})\). Since \(u + 2 + W_1(-e^{-(u+1)}) \geq 0\) if and only if \(W_1(-e^{-(u+1)}) \geq -(u + 2)\), and the function \(xe^x\) is decreasing if \(x \leq -1\) then, the last inequality above holds if and only if
\[
W_1(-e^{-(u+1)}) \exp(W_1(-e^{-(u+1)})) \leq -(u + 2)e^{-(u+2)}
\]
i.e.,
\[
-e^{-(u+1)} \leq -(u + 2)e^{-(u+2)},
\]
which holds if and only if \(e-2 \geq u\). Therefore, \(a(u)\) is increasing in \((0, e-2)\) and decreasing in \((e-2, \infty)\). Moreover, since
\[
a(0) = \lim_{u \to \infty} a(u) = 1
\]
it follows that
\[
1 \leq a(u) \leq a(e-2) = e/(e - 1),
\]
for all \(u \geq 0\). Hence we can conclude
\[
\frac{e}{e-1}(u + 1) \leq W_1(-e^{-(u+1)}) \leq -(u + 1).
\]

The Lambert \(W\) function, also known as the \(ProductLog\) function, has many other applications in pure and applied mathematics, we refer the interested readers to [3] and [12] for further reading on this theme.

3. A NEW LEMMA

In this section we establish our first new key lemma in the endeavor of improving the \(W^{2,\varepsilon}\) regularity estimate. The proof follows a line of reasoning similar to [13]; the key main novelty though is the introduction of the new parameter \(\delta\), which yields enhanced measure estimates by means of an optimization problem, to be discussed in the next section. We also carry out a more general analysis, yielding an arbitrary number of nonpositive eigenvalues.

Lemma 4. Let \(u \in C(\bar{\Omega})\) and assume \(M_{1,\Lambda}(D^2 u) \leq 0\) in \(\Omega\). Let \(k \in [1, n]\) be the minimal number of nonpositive eigenvalues of \(D^2 u(x)\). Given \(a, \delta > 0\) and a measurable set \(F \subset \{u > \Gamma_a^u\}\), take the paraboloids of opening \(-(1 + \delta)a\), tangents from below to \(\Gamma_a^u\) on \(F\), and slide them up until they touch \(u\) on a set \(E\). If \(E \Subset \Omega\) then,
\[
|A_{(1+\delta)a}(u) \setminus A_a(u)| \geq c \left(1 + \frac{1}{\delta}\right)^{-n} |F|,
\]
where \(c = c(n, \lambda, \Lambda, k)\) is given by:
\[
c := \left(1 + \left(\frac{\Lambda}{\lambda} - 1\right) \frac{k}{n-k}\right)^{k-n}.
\]
Proof. We start off by defining the function
\[
v := \frac{1}{a} u + \frac{1}{2} |x|^2
\]
and noting that, in view of (2.3), there holds:
\[
\Gamma_a^u = a \Gamma_v^0 - \frac{a}{2} |x|^2, \quad A_a(u) = A_0(v), \quad \text{and} \quad A_{(1+\delta)a}(u) = A_{\delta}(v).
\]
In particular, $F \subset \{ u > \Gamma_u \}$ implies $F \subset \bar{\Omega} \setminus A_0(v)$. Next, if $P$ is a paraboloid of opening $-(1+\delta)a < 0$, tangent to $\Gamma_u$ at $x_0 \in F$, then $P$ is also tangent to $a \Gamma_v - \frac{a}{2}|x|^2$ at $x_0 \in F$. Since there exists an affine function $L(x)$ such that $P(x) = \frac{1}{2}(1+\delta)a|x|^2 + L(x) \leq a \Gamma_v - \frac{a}{2}|x|^2$ with equality at $x_0$, we conclude

$$\frac{-\delta}{2}|x|^2 + \frac{1}{a}L(x) \leq \Gamma_v$$

with equality at $x = x_0$. That is, there exists a paraboloid, $P_L^\delta$, of opening $-\delta < 0$, tangent from below to $\Gamma_v$ at $x_0 \in F$.

Next, we slide $P_L^\delta$ up until it touches $v$ from below at a point $x_1 \in \bar{\Omega}$. Lemma 1 yields $x_1 \in \bar{\Omega} \setminus A_0(v)$ and hence, $x_1 \in A_\delta(v) \setminus A_0(v)$. Collecting all such points $x_1$ in a new contact set $E \subset \bar{\Omega}$, we have,

$$E \subset A_\delta(v) \setminus A_0(v) = A_{(1+\delta)a}(u) \setminus A_a(u).$$

If $V_F$ is the set of vertices of all tangent paraboloids of opening $-\delta$ from below to $\Gamma_v$ in $F$, then by Lemma 2,

$$|V_F| \geq |F|.$$  

(3.3)

Moreover, the vertex of the paraboloid $P_L^\delta$ remains the same after we slide it up, so $V_F = V_E$.

where $V_E$ is the set of vertices of paraboloids tangent to $v$ on the set $E$.

For the time being, let us suppose that $u$ is semi-concave, and thus twice differentiable a.e in $E$. For such points $x \in E$, we have that the correspondent vertex is given by

$$\Phi(x) := x_v = x + \frac{1}{\delta} \nabla v(x).$$

Since $\Phi(x)$ is a Lipschitz map, we can apply the area formula,

$$|V_E| = |\Phi(E)| \leq \int_E |\det(D\Phi)|dx$$  

(3.4)

By the definition of $\Phi$ (and of $v$), the eigenvalues of $D\Phi$ are of the form

$$\lambda_\Phi^i = \frac{1}{\delta} + \frac{1}{\delta a} \lambda_u^i, \quad i = 1, \cdots n,$$

where $\lambda_u^i$ are the eigenvalues of $D^2u$. Now, because $x \in E \subset A_{(1+\delta)a}(u) \setminus A_a(u)$ we have

$$D^2u(x) \succeq -(1+\delta)a \cdot \text{Id}.$$ 

Thus, the eigenvalues of $D\Phi(x)$ are all non-negative. Next, since $M^\delta(\lambda)(D^2u) \leq 0$, for some $1 \leq k \leq n$, the Hessian of $u$ has $k$ nonpositive eigenvalues, i.e.

$$\lambda_i^u \leq 0 \quad \text{for} \quad i = 1, \cdots, k.$$
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for some $1 \leq k = k(x) \leq n$. Since $M_{\lambda, \Lambda}(D^2 u(x)) \leq 0$, we can estimate:

$$\sum_{i=1}^{n} \lambda_i^\Phi = n \left( 1 + \frac{1}{\delta} \right) + \frac{1}{\delta a} \sum_{i=1}^{n} \lambda_i^\mu \leq n \left( 1 + \frac{1}{\delta} \right) + \frac{1}{\delta a} \left( \sum_{i=1}^{k} \lambda_i^\mu + \sum_{i=k+1}^{n} \lambda_i^\mu \right) \leq n \left( 1 + \frac{1}{\delta} \right) + \frac{1}{\delta a} \left( 1 - \frac{\Lambda}{\lambda} \right) \sum_{i=1}^{k} \lambda_i^\mu \leq n \left( 1 + \frac{1}{\delta} \right) + \frac{1}{\delta a} \left( \frac{\Lambda}{\lambda} - 1 \right) \sum_{i=1}^{k} (-\lambda_i^\mu).$$

Moreover, as $\lambda_i^\mu \geq -(1 + \delta)a$, for all $i = 1, \ldots, n$ we have

$$\sum_{i=1}^{n} \lambda_i^\Phi \leq \left( 1 + \frac{1}{\delta} \right) \left[ n + \left( \frac{\Lambda}{\lambda} - 1 \right) k \right].$$

Next we estimate,

$$\det(D\Phi) = \prod_{i=1}^{n} \lambda_i^\Phi \leq \lambda_1^\Phi \cdots \lambda_k^\Phi \left( 1 + \frac{1}{\delta} \right) \left[ n + \left( \frac{\Lambda}{\lambda} - 1 \right) k \right]^{n-k} \leq \lambda_1^\Phi \cdots \lambda_k^\Phi \left( 1 + \frac{1}{\delta} \right) \left[ n + \left( \frac{\Lambda}{\lambda} - 1 \right) k \right]^{n-k} \leq \lambda_1^\Phi \cdots \lambda_k^\Phi \left( 1 + \frac{1}{\delta} \right) \left[ n + \left( \frac{\Lambda}{\lambda} - 1 \right) k \right]^{n-k} \left( k(\lambda_1^\Phi \cdots \lambda_k^\Phi)^{1/k} \right)^{n-k}$$

we have used the classical mean inequality in the last step. We note that the right hand side of (3.5) is an increasing function of $t = \lambda_1^\Phi \cdots \lambda_k^\Phi$ in the interval $[0, (1 + \delta^{-1})^k]$, and thus we can further estimate:

$$\det(D\Phi) \leq \left( 1 + \frac{1}{\delta} \right)^n \left( 1 + \left( \frac{\Lambda}{\lambda} - 1 \right) \frac{k}{n-k} \right)^{n-k} = c^{-1} \left( 1 + \frac{1}{\delta} \right)^n,$$

where $c = c(n, \lambda, \Lambda, k)$ is from (3.2). Finally we can estimate:

$$|F| \leq |V_E| \leq \int_E \det(D\Phi(x)) \leq c^{-1} \left( 1 + \frac{1}{\delta} \right)^n |E| \leq c^{-1} \left( 1 + \frac{1}{\delta} \right)^n |A_\delta(v) \setminus A_0(v)|$$

and the lemma is proved for semi-concave functions.

For the general case, we reduce it to the previous one by using the inf-convolution

$$u_m := \inf_{y \in \Omega} \{ u(y) + m|y - x|^2 \}.$$
Indeed, one can show, see e.g. proof of Lemma 2.1 in [19], that $u_m$ are semi-concave and converge locally uniformly to $u$ in $\Omega$ as $m \to \infty$. Furthermore $u_m$ is a viscosity supersolution of the same equation in $\Omega' \Subset \Omega$ and by the first part of the proof,

$$|E_m| \geq \beta |F|,$$

where $E_m$ is the correspondent touching set for $u_m$. One can easily check that

$$\limsup_{m \to \infty} E_m = \bigcap_{m \geq 1} \bigcup_{i \geq m} E_i \subset E,$$

and thus,

$$|A(1+\delta)a(u) \setminus A_a(u)| \geq \beta |F|,$$

and the proof of Lemma is complete.

\[ \square \]

Remark 3.1. As we shall discuss in the next Section, the constant $c = c(n, \lambda, \Lambda, k)$ defined in (3.2) plays a critical role in the endeavor of establishing quantitative lower bounds for the $W^{2,\varepsilon}$ regularity theory. The bigger the $c(n, \lambda, \Lambda, k)$ the better the lower estimate upon Hessian integrability exponent, viz. $\varepsilon$ in the $W^{2,\varepsilon}$ regularity theory.

Viscosity supersolutions to $M_{-\lambda, \Lambda}(D^2u) \leq 0$ must have at least one nonpositive eigenvalue, that is, one is always entitled to take $k = 1$. Geometric restrictions on the problem may impose a higher number of nonpositive eigenvalues for $D^2u$. In such situations, one can take $k$ to be greater than 1, but it is not obvious that this would increase the value of $c$. If not, one could simply disregard this extra information. That is, under the hypotheses of Lemma 4, its thesis, i.e. (3.1) actually holds if one substitutes $c(n, \lambda, \Lambda, k)$ by

(3.7) $$c_* := \max \{c(n, \lambda, 1), c(n, \lambda, 2), \cdots, c(n, \lambda, k)\}.$$

Motivated by Remark 3.1 next proposition discusses the best choice for $c_*$.  

Proposition 1. Let $t_0$ be the point of maximum in $[1, n)$ of the function

$$f(t) = \left(1 + \left(\frac{\Lambda}{\lambda} - 1\right) \frac{t}{n - t}\right)^{n-1}.$$

Then $t_0$ is a continuously decreasing function of the ellipticity ratio $\varrho := \Lambda/\lambda$ and, moreover, for any $\beta \in (1, n)$, there exists $1 < \varrho_{\beta}$ such that

$$t_0(\varrho_{\beta}) = \frac{n}{\beta}.$$

Proof. For $\lambda, \Lambda$ and $n$ fixed, by making the change of variables

$$x = 1 + \left(\frac{\Lambda}{\lambda} - 1\right) \frac{t}{n - t},$$

the maximum of $f$ in $[1, n)$ is attained at

$$t_0 = t_0\left(\frac{\Lambda}{\lambda}\right) = \frac{n(x_0 - 1)}{(\Lambda/\lambda) - 2 + x_0},$$

where $x_0$ verifies

(3.8) $$\ln(x_0) = \frac{(\Lambda/\lambda) - 2 + x_0}{x_0}.$$

That is,

$$x_0 = x_0\left(\frac{\Lambda}{\lambda}\right) = \frac{\Lambda - 2}{W_0\left((\Lambda/2)e^{-1}\right)} \in (1, \infty)$$
where $W_0$ denoted the principal branch of the Lambert function. Since $x_0$ solves (3.8) we can write

$$t_0 = \frac{n(x_0 - 1)}{x_0 \ln(x_0)}.$$  

Next, we note that $t_0(1^+) = n$ and $t_0$ is a continuous, decreasing function of $x_0$. Furthermore,

$$x_0\left(\frac{\Lambda}{\lambda}\right) = \frac{W_0 - e^{-1}(\Lambda/\lambda - 2)W_0'}{W_0^2}$$

$$= \frac{1}{1 + W_0}.$$  

(3.9)

By the classical properties of the Lambert function:

$$W'(z) = \frac{W(z)}{z(1 + W(z))}, \quad \text{for all } z \neq 0, -e^{-1}.$$  

In our case, (3.9) is valid for all $\Lambda/\lambda \neq 1, 2$. However, since

$$\lim_{\Lambda/\lambda \to 2} x_0'\left(\frac{\Lambda}{\lambda}\right) = 1$$  

and $x_0'$ is decreasing, we may define $x_0'(2) = 1$. Furthermore, since $x_0'\left(\frac{\Lambda}{\lambda}\right) > 0$ for all $\Lambda/\lambda > 1$ then, chain rule yields

$$t_0'\left(\frac{\Lambda}{\lambda}\right) = t_0'\left(x_0\left(\frac{\Lambda}{\lambda}\right)\right) x_0'\left(\frac{\Lambda}{\lambda}\right) \leq 0$$

for all $\Lambda/\lambda > 1$. Therefore, $t_0$ is decreasing with respect to $\Lambda/\lambda$ in $(1, \infty)$.

Next, given $1 < \beta \leq n$, we are now interested in finding $\lambda, \Lambda$ such that $t_0(\Lambda/\lambda) = \beta^{-1}n$. We then set

$$\frac{n}{\beta} = \frac{n(x_0 - 1)}{x_0 \ln(x_0)},$$

which holds if and only if

$$\ln(x_0) = \beta\frac{x_0 - 1}{x_0},$$

(3.10)

which is equivalent to

$$-\beta e^{-\beta} = -\frac{\beta}{x_0} e^{-\frac{\beta}{x_0}}.$$  

The above equation has two solutions, one is of course $x_0(\Lambda/\lambda) = 1$ and the second is

$$x_0(\Lambda/\lambda) = \frac{-\beta}{W_0(-\beta e^{-\beta})}.$$  

(3.11)

Since $x_0(\Lambda/\lambda) = 1$ only if $\Lambda = \lambda$ and $t_0(1^+) = n$, we are seeking $0 < \lambda < \Lambda$ such that $x_0(\Lambda/\lambda)$ is given by (3.11) and solves both (3.8) and (3.10). We readily obtain

$$\frac{(\Lambda/\lambda) - 2 + x_0}{x_0} = \beta\frac{x_0 - 1}{x_0} \implies \Lambda/\lambda = 2 - \beta + (\beta - 1)x_0,$$

and thus, using (3.11), we have

$$\Lambda = 2 - \beta + (\beta - 1)\frac{-\beta}{W_0(-\beta e^{-\beta})}.$$  

\[\square\]
In particular, Proposition 1 informs that for any $\lambda, \Lambda$ such that $1 \leq (\Lambda/\lambda) \leq (\Lambda/\lambda)$ we have

$$t_0 (\Lambda/\lambda) \geq t_0 (\Lambda/\lambda) = \frac{n}{\beta}.$$

4. IMPROVED $W^{2,\varepsilon}$ ESTIMATES

In this section we establish improved quantitative $W^{2,\varepsilon}$ regularity estimates for viscosity supersolutions of fully nonlinear, uniformly elliptic equations.

**Theorem 2.** (Interior $W^{2,\varepsilon}$ estimate) Let $u \in C(\bar{B}_1)$ such that $M_{\alpha, \Lambda}(D^2 u) \leq 0$ in $B_1$. Assume $D^2 u(x)$ has at least $k$ nonpositive eigenvalues, for some $k \in [1, n]$. There exists universal numbers $\gamma_0, \varepsilon \in (0, 1)$ such that for all $0 < \alpha < \varepsilon$,

$$\tag{4.1} \left\{ \Theta_{\alpha} > \frac{1 - \gamma_0}{\gamma_0} 2^n \|u\|_{C^0} \right\} \cap B_{1/2} \leq |B_1| \varepsilon^{-n}$$

holds for $t \geq (1 - \gamma_0)^{-n} \varepsilon$ where $\varepsilon = \min\{j \in \mathbb{N} \mid (\varepsilon - \alpha)^{-1} \leq j \}$. In addition, $\gamma_0, \varepsilon \in (0, 1)$ are related by the following formula:

$$\tag{4.2} \varepsilon = \gamma_0(\varepsilon) := \sup_{(0,1)} \frac{\ln(1-c\alpha^n)}{\ln(1-\gamma)} = \frac{\ln(1-c\gamma_0^n)}{\ln(1-\gamma_0)}$$

where $c = c_*(n, \Lambda, \Lambda, k)$ is the constant discussed in Remark 3.1, i.e.

$$c = \max\{c(n, \Lambda, 1), c(n, \Lambda, 2), \ldots, c(n, \Lambda, k)\},$$

for $c(n, \Lambda, k) = \left(1 + \left(\frac{\Lambda}{\alpha} - 1\right) \frac{i}{\pi^2}\right)^{-i}.$

**Proof.** We start off by noticing that since $\alpha < \varepsilon$ then there exists a natural number $j$ such that

$$\alpha \leq \frac{j}{1 + j} \varepsilon$$

and hence the set $\{ j \in \mathbb{N} \mid (\varepsilon - \alpha)^{-1} \leq j \}$ is nonempty. Next, for convenience, we set $\delta_0 = \frac{\gamma_0}{1 - \gamma_0} > 0$ and assume, with no loss of generality, that $0 < \alpha < \delta_0 2^{-2}$. Consider the extension

$$\tilde{u} = \begin{cases} \min(u, \frac{\delta_0}{2}(1 - |x|^2)), & \text{in } B_1 \\ \frac{\delta_0}{2}(1 - |x|^2), & \text{in } B_R \setminus B_1 \end{cases}$$

for $R$ large. Note that $\tilde{u} \in C(\bar{B}_R)$ with $\tilde{u} = u$ in $B_{\sqrt{3}/2}$ and $B_R \setminus A_{\delta_0}(\tilde{u}) \subset B_1$. Next we verify that

$$\tag{4.3} |B_1 \setminus A_{(1+\delta_0)^{2j}}(\tilde{u})| \leq \left| 1 - c \left( 1 + \frac{1}{\delta_0} \right)^{-n} \right| |B_1|,$$

for all $j \geq 0$. We argue by induction. The case $j = 0$ follows easily. Assume (4.3) has been checked for $j$ and define

$$F_j := B_1 \setminus A_{(1+\delta_0)^{2j}}(\tilde{u})$$

and consider the paraboloids of opening $-(1 + \delta_0)^{2j+1}$ tangents from below to $\Gamma_{\tilde{u}}^{(1+\delta_0)^{2j}}$ in $F_j$. Slide them up until they touch $\tilde{u}$ in a set $E_j$. From Lemma 1 and the very definition of $E_j$, we have

$E_j \subset A_{(1+\delta_0)^{2j+1}}(\tilde{u}) \setminus A_{(1+\delta_0)^{2j}}(\tilde{u}) \subset \tilde{B}_1 \subset B_R.$

From Lemma 4 we obtain,

$$|B_1 \setminus A_{(1+\delta_0)^{2j}}(u)| \leq c \left( 1 + \frac{1}{\delta_0} \right)^{-n} |A_{(1+\delta_0)^{2j+1}}(u) \setminus A_{(1+\delta_0)^{2j}}(u)|.$$
Since
\[ |B_1 \setminus A_{(1+\delta_0)^{j+1}}(\tilde{u})| = |B_1 \setminus A_{(1+\delta_0)^j}(\tilde{u})| - |A_{(1+\delta_0)^{j+1}} \setminus A_{(1+\delta_0)^j}(\tilde{u})| \leq \left( \frac{1}{1 - c} \left( 1 + \frac{1}{\delta_0} \right)^n \right)^{j+1} |B_1|, \]
by the induction hypothesis we have
\[ |B_1 \setminus A_{(1+\delta_0)^{j+1}}(\tilde{u})| \leq \left( 1 - c \left( 1 + \frac{1}{\delta_0} \right)^n \right)^{j+1} |B_1|, \]
and (4.3) is proved.

Continuing with the reasoning, since \( \tilde{u} \leq u \) and both functions agree in \( B_{\sqrt{3}/2} \), for \( j \geq 0 \) there holds
\[ \{ \Theta_u > (1 + \delta_0)^j \} \cap B_{1/2} \subset B_1 \setminus A_{(1+\delta_0)^j}(u). \]
To conclude, we note that for any \( t \geq (1 + \delta_0)^{j+\gamma(\alpha)} \) there exists \( j \in \mathbb{N} \) such that
\[ (1 + \delta_0)^{j+\gamma(\alpha)} \leq t \leq (1 + \delta_0)^{j+\gamma(\alpha)+1}, \]
and thus the following inclusions hold:
\[ \{ \Theta_u > t \} \subset \{ \Theta_u > (1 + \delta_0)^{j+\gamma(\alpha)} \} \subset B_1 \setminus A_{(1+\delta_0)^{j+\gamma(\alpha)}}(u). \]
We have proven that
\[ \{ \Theta_u > t \} \cap B_{1/2} \leq \left( 1 - c \left( 1 + \frac{1}{\delta_0} \right)^n \right)^{j+\gamma(\alpha)} |B_1|, \]
which, in view of (4.4), yields the aimed estimate as long as we take
\[ \alpha \leq -\frac{(j + j(\alpha)) \ln(1 - c \left( 1 + \frac{1}{\delta_0} \right)^n)}{(j + j(\alpha) + 1) \ln(1 + \delta_0)} = \frac{j + j(\alpha) \ln(1 - c \gamma_0^n)}{j + j(\alpha) + 1 \ln(1 - \gamma_0)}, \]
for all \( j \geq 1 \), which is guaranteed by the definition of \( j(\alpha) \).

We conclude the proof of Theorem 2 by commenting that condition \( 0 \leq u < \delta_0 2^{-\gamma} \) is not restrictive. Indeed, given a generic supersolution \( u \), consider
\[ v = \delta_0 \frac{u + ||u||_{\infty}}{2^\gamma ||u||_{\infty}} \]
and note that \( 0 \leq v < \delta_0 2^{-\gamma} \), \( M_{\varphi,v}^{-} (D^2 v) \leq 0 \) in \( B_1 \). Moreover,
\[ \{ \Theta_u > t \} = \{ \Theta_u > \delta_0^{-1} 2^\gamma ||u||_{\infty} t \} = \left\{ \Theta_u > \left( \frac{1 - \gamma_0}{\gamma_0} \right) 2^\gamma ||u||_{\infty} t \right\} \]
\[ \square \]

The function defining \( \varphi \) in (4.2), i.e. \( \varphi: (0, 1) \to \mathbb{R} \) given as
\[ \varphi(\gamma) := \frac{\ln(1 - c \gamma^n)}{\ln(1 - \gamma)}, \]
can be explored to provide lower bounds for \( \varepsilon \). The critical point \( \gamma_0 \), used in the proof of Theorem 2, satisfies
\[ n c \gamma_0^{n-1} \frac{1}{(1 - c \gamma_0^n)} (1 - \gamma_0) = \varepsilon(\gamma_0). \]
Such an equation comes from simplifying \( \varepsilon'(\gamma_0) = 0 \) and can be numerically solved for chosen values of \( n \) and \( \lambda \) and \( \Lambda \).
We further note that
\[-\ln(1 - cy^n) = -c \ln((1 - cy^n)^{1/c}) \geq cy^n,\]
with asymptotic equality as \(0 < c \ll 1\). Thus, we can write down the pointwise inequality
\[\varphi(\gamma) \geq \frac{cy^n}{-\ln(1 - \gamma)} =: f(\gamma).\]

The new function \(f(\gamma)\) can be used as lower bounds for \(\varepsilon\) and it is a slightly easier function to analyze. In particular, \(\varepsilon(\gamma_0) \geq \varepsilon(\gamma_*),\) where \(\gamma_*\) is the critical point for \(f(\gamma)\) in \((0, 1)\).

**Proposition 2** (Improved bound in the plane). Let \(u \in C(\bar{B}_1)\) satisfy \(M - \lambda, \Lambda(D^2u) \leq 0\) in \(B_1 \subset \mathbb{R}^2\). Then \(u \in W^{2,\varepsilon}_{loc}(B_1),\) with universal estimates, for an optimal \(\varepsilon > 0\) satisfying:
\[0.407 \frac{\lambda}{\Lambda} \leq \varepsilon \leq \frac{2}{2 + 1}.\]

**Proof.** The upper bound is the one from [1]. As for the lower bound, we initially note that \(c = c(2, \Lambda, \Lambda, 1),\) as defined in (3.2), equals \(\frac{1}{\Lambda}\). Hence, from the above discussion
\[\varepsilon \geq \frac{\lambda}{\Lambda - \ln(1 - \gamma)},\]
for all \(\gamma \in (0, 1)\). Evaluating at \(\gamma = 0.715\) yields the result. \(\Box\)

The analysis for higher dimensions is a bit more involved and it is the content of our next proposition:

**Proposition 3.** Let \(\varepsilon(\gamma_0)\) be defined as in Theorem 2. There holds:
\[\varepsilon(\gamma_0) \geq \frac{n(e - 1)}{1 + ne \ln(n)} \left( \frac{n \ln(n)}{1 + n \ln(n)} \right)^n c,\]
for all \(n \geq 3\).

**Proof.** By simplifying the equation \(f'(\gamma) = 0,\) we see that the critical point of \(f\) in \((0, 1),\) \(\gamma_*\), satisfies the equation
\[\gamma \frac{1}{1 - \gamma} = -n \ln(1 - \gamma)\]
which can be analytically solved in \((0, 1)\) by using the Lambert \(W\) function. Indeed, (4.7) is equivalent to
\[-\frac{1}{n(1 - \gamma)} \exp \left( -\frac{1}{n(1 - \gamma)} \right) = \frac{1}{n} \exp \left( -\frac{1}{n} \right)\]
whose solutions are \(\gamma = 0\) and
\[-\frac{1}{n(1 - \gamma)} = W_{-1} \left( \frac{1}{n} e^{-1/n} \right).\]
Therefore, in \((0, 1)\) the critical point is
\[\gamma_* := 1 + \frac{1}{nW_{-1}(\frac{1}{n} e^{-1/n})}.\]
From (4.7) and the definition of $\gamma_*$, we then have:

$$
\begin{align*}
\varphi(\gamma_*) &= \frac{-\ln(1 - c\gamma_*)}{-\ln(1 - \gamma_*)} \\
&= (-\ln(1 - c\gamma_*)\gamma_*)^{\frac{n(1 - \gamma_*)}{\gamma_*}} \\
&= (-\ln(1 - c\gamma_*)\gamma_*)^{\frac{-1}{w_n + \frac{1}{n}}},
\end{align*}
$$

where we have set $w_n := W_{-1}(-\frac{1}{n} e^{-1/n})$.

Now, from Lemma 3, the branch $W_{-1}$ of the Lambert W function satisfies the following inequalities:

$$
-\frac{e}{e - 1}(u + 1) \leq W_{-1}(-e^{-u-1}) < -(u + 1)
$$

for all $u > 0$. Letting $u = \ln(n) - \left(\frac{n - 1}{n}\right)$, for $n \geq 2$, in (4.9) we reach

$$
\frac{-e}{e - 1} \left(\ln(n) + \frac{1}{n}\right) \leq w_n < -\ln(n) - \frac{1}{n}.
$$

Add $1/n$ to both sides of the above inequalities:

$$
\frac{-1}{w_n + 1/n} \geq \frac{n(e - 1)}{en \ln(n) + 1}.
$$

The above inequality and (4.8) yields

$$
\begin{align*}
\varphi(\gamma_*) &\geq (-\ln(1 - c\gamma_*)\gamma_*)^{\frac{n(e - 1)}{1 + ne \ln(n)}} \\
&\geq c\gamma_*^{n(\frac{n \ln(n)}{1 + n \ln(n)})^n},
\end{align*}
$$

for all $n \geq 2$ and hence

$$
(4.11) \quad \varepsilon(\gamma_0) > \varphi(\gamma_*) \geq \frac{n(e - 1)}{1 + n \ln(n)} \left(\frac{n \ln(n)}{1 + n \ln(n)}\right)^n c,
$$

for all $n \geq 2$, and the proposition is proven. \hfill \Box

**Corollary 1.** Let $\varepsilon(\gamma_0)$ be defined as in Theorem 2. Then,

$$
\varepsilon(\gamma_0) \geq \frac{\tau_n}{\ln n} c,
$$

for an increasing sequence of positive numbers, $\frac{1}{3} < \tau_n \to 1 - e^{-1}$, $n \geq 3$. In particular, $\varepsilon(\gamma_0) > \frac{c}{\ln n}$, for all $n \geq 3$. 

Proof. It follows from (4.11) that
\[
\varepsilon(\gamma_0) > \varphi(\gamma_0) \geq \frac{n(e - 1)}{1 + ne \ln(n)} \left( \frac{n \ln(n)}{1 + n \ln(n)} \right)^n \ln n \cdot \frac{c}{\ln n}.
\]
Easily one sees that
\[
\tau_n := \frac{n(e - 1) \ln n}{1 + ne \ln(n)} \left( \frac{n \ln(n)}{1 + n \ln(n)} \right)^n
\]
is an increasing sequence converging to $1 - e^{-1}$. Direct calculation yields:
\[
\tau_3 \approx 0.2568 > \frac{1}{4},
\]
and the Corollary is proven. \qed

We now turn our attention to asymptotic lower bounds for the constant $c(n, \lambda, \Lambda, k)$ defined in (3.2). The case $k = \frac{n}{2}$ divides the theory. Indeed, it readily follows that:
\[
c(n, \lambda, \Lambda, \frac{n}{2}) = \left( \frac{\lambda}{\Lambda} \right)^{\frac{n}{2}},
\]
and thus, Theorem 2 along with Corollary 1 implies that if $u \in C(\bar{B}_1)$ such that $\mathcal{M}_{\lambda, \Lambda}(D^2u) \leq 0$ in $B_1$ and $D^2u(x)$ has at least $\frac{n}{2} \leq k$ nonpositive eigenvalues, then $u \in W^{2,\alpha}(B_{1/2})$, with universal estimates, for all
\[
\alpha < \frac{1}{\ln n^2} \left( \frac{\lambda}{\Lambda} \right)^{\frac{n}{2}}.
\]

In the case $\frac{n}{2} < k < n$, for $n$ fixed, Theorem 2 along with Corollary 1 yields the existence of a constant $g_n > 0$ such that for $u \in C(\bar{B}_1)$ with $\mathcal{M}_{\lambda, \Lambda}(D^2u) \leq 0$ in $B_1$ and $D^2u(x)$ has at least $k$ nonpositive eigenvalues, then $u \in W^{2,\alpha}(B_{1/2})$, with universal estimates, for all
\[
\alpha < g_n \cdot \left( \frac{\lambda}{\Lambda} \right)^{n-k}.
\]

The constant $g_n$, however, goes to zero as $n \to \infty$. On the other hand, in view of Remark 3.1, in the case $\frac{n}{2} < k < n$, one can select
\[
c = \max_{i=1-k} \left\{ 1 + \left( \frac{\lambda}{\Lambda} - 1 \right) \left( \frac{k}{n-i} \right)^{i-k} \right\}.
\]
Proposition 1 informs how to optimize such a choice.

We now turn our attention to the important case $1 \leq k < \frac{n}{2}$; recall for viscosity supersolutions, with no further information, one can always take $k = 1$ in the analysis.

Proposition 4. Let $c(n, \lambda, \Lambda, k) = \left( 1 + \left( \frac{\lambda}{\Lambda} - 1 \right) \frac{k}{n-k} \right)^{n-k}$ and assume $k < \frac{n}{2}$. Then
\[
c = c(n, \lambda, \Lambda, k) \geq \left( \frac{\lambda}{\Lambda} \right)^{k-n} \cdot \left( 1 + \frac{n-2k}{n-k} \right)^{n-k}. \]

Proof. We readily obtain
\[
c := \left( 1 + \left( \frac{\lambda}{\Lambda} - 1 \right) \frac{k}{n-k} \right)^{k-n} = \left( \frac{\lambda}{\Lambda} \right)^{k-n} \cdot \left( \frac{\lambda}{\Lambda} + \left( 1 - \frac{\lambda}{\Lambda} \right) \frac{k}{n-k} \right)^{k-n}.\]
In particular, if 
\[ \varepsilon \in u \]

and the proof is completed.

Let \( u \geq 0 \). We can further estimate:

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Finally, combining Theorem 2, Proposition 3, and Proposition 4 we obtain:

**Theorem 3.** Let \( u \in C(\tilde{B}_1) \) such that \( M^\varepsilon_{\lambda, \Lambda}(D^2 u) \leq 0 \) in \( B_1 \). Let \( k \in [1, n] \) be the minimal number of nonpositive eigenvalues of \( D^2 u \). Assume \( n \geq 3 \) and \( 1 \leq k < \frac{n}{2} \). Then \( u \in W^{2, \alpha}(B_{1/2}) \) with universal estimates, for all \( \alpha < \left( \frac{\ln n}{\ln k} \right)^{n-k} \).

In particular, if \( \varepsilon(n, \lambda, \Lambda, k) \) is the quantity \( \varepsilon(\gamma_0) \) defined in Theorem 2, with \( \lambda < \Lambda \), then:

In the case \( k = 1 \), that is, with no further assumption other than \( M^\varepsilon_{\lambda, \Lambda}(D^2 u) \leq 0 \), the last information provided by Theorem 3 seems to suggest that the sharp polynomial decay for Hessian integrability of \( u \), as a function of \( \frac{5}{2} \), should indeed be smaller than \( n - 1 \). In [1], Armstrong, Silvestre, and Smart conjectured that the decay should be linear. We shall discuss this important conjecture in Section 6.

**5. Global \( W^{2, \varepsilon} \) estimates**

In this intermediary section we comment on global \( W^{2, \varepsilon} \) estimates. We follow closely [13, Section 5]; the main difference is the advent of a new intrinsic optimization procedure, introduced in the analysis, as to accelerate the geometric measure decay.

**Theorem 4.** (Global \( W^{2, \varepsilon} \) estimate) Assume \( u \in C(\tilde{B}_1) \) and \( M^\varepsilon_{\lambda, \Lambda}(D^2 u) \leq 0 \) in \( B_1 \). Let \( k \in [1, n] \) be the minimal number of nonpositive eigenvalues of \( D^2 u \). Then, for all \( 0 < \alpha < \varepsilon^G \)

\[
\left\{ \varepsilon > \frac{n^2(3 + \sqrt{5})^3\|u\|_{W^{1, \infty}}}{2^4(1 + \sqrt{5})^4} \right\} \leq t^{-2} |B_1|
\]

for \( t \geq (2^{-1}(3 + \sqrt{5}))^{1+j(\alpha)} \), where \( j(\alpha) = \min\{j \in \mathbb{N} \mid \alpha(\varepsilon - \alpha)^{-1} \leq j \} \) and

\[
\varepsilon^G = \frac{\ln \left( 1 - 2c \left( \frac{1 + \sqrt{5}}{3 + \sqrt{5}} \right)^{\alpha} \right)}{\ln \left( 2(3 + \sqrt{5})^{-1} \right)},
\]
where $c$ is the same constant defined in (3.2).

**Proof.** First, assume that

\begin{equation}
0 < u < \frac{\delta^4}{n^2(1 + \delta)^8} \leq (1 + \delta^2)\rho_j^2
\end{equation}

for all $j \geq 0$, where $\delta > 0$ and $0 < \rho_j < (1 + \delta)^{-2}$ are to be determined over the course of the proof. Let

\[ P(x) = -\frac{(1 + \delta)^{j+1}}{2}|x - x_0|^2 + y \cdot (x - x_0) + \Gamma_u^{(1+\delta)^j}(x_0) \]

be a paraboloid of opening $-(1 + \delta)^{j+1}$ tangent to $\Gamma_u^{(1+\delta)^j}$ at $x_0 \in B_{1-(1+\delta)^2\rho_j}$. Since the vertex of $P$ can be written as $x_v = x_0 + \frac{1}{(1 + \delta)^{j+1}}y$, and $0 < \Gamma_u^{(1+\delta)^j}$, we have that

\[ \|x_v - x_0\|^2 \leq P(x_v) \leq \Gamma_u^{(1+\delta)^j}(x_v) \leq u(x_v) < (1 + \delta^2)\rho_j^2 \]

and then,

\[ \|x_v - x_0\|^2 \leq \sqrt{2}(1 + \delta)^{-1/2}\rho_j. \]

Therefore,

\[ \|x_1\| \leq 1 - (1 + \delta^2)\rho_j + \sqrt{2}(1 + \delta)^{-1/2}\rho_j. \]

After sliding $P$ up until it touches $u$, the new contact point, $x_1$ will then satisfies

\[ \|x_1\| \leq \|x_1 - x_v\| + \|x_v\| \leq 1 - \left[(1 + \delta^2)^2 - 2 \cdot \sqrt{2}(1 + \delta)^{-1/2}\right]\rho_j. \]

To ensure that $E_j \Subset B_1$, it suffices to have

\[ 0 < \left[(1 + \delta^2)^2 - 2 \cdot \sqrt{2}(1 + \delta)^{-1/2}\right]\rho_j < 1. \]

Since $0 < \rho_j < (1 + \delta)^{-2}$, then the above inequalities hold if

\begin{equation}
(1 + \delta^2)^2 - 2 \cdot \sqrt{2}(1 + \delta)^{-1/2} > 0 \iff \delta > 2^{3/5} - 1. \tag{5.3}
\end{equation}

For values of $\delta$ in this range, we have, by Lemma (4) that,

\[ |A_{(1+\delta)^j} \setminus A_{(1+\delta)^{j-1}}| \geq c \left(1 + \frac{1}{\delta}\right)^{-n} |F_j|, \]

where $F_j := B_{1-(1+\delta)^2\rho_j} \setminus A_{(1+\delta)^j}$. Our next goal is then to show that

\begin{equation}
|B_1 \setminus A_{(1+\delta)^j}| \leq \left(1 - c \frac{\delta^n}{(1 + \delta)^{n+1}}\right)^j |B_1| \tag{5.4}
\end{equation}

for all $j \geq 0$. As in the interior estimate, we argue by induction. The case $k = 0$ is easy to see. For the induction step we consider two cases:
Case I: $|F_j| \geq \frac{1}{1+\delta} |B_1 \setminus A_{1+(\delta)^j}|$. In this case, it follows that

\[
|B_1 \setminus A_{1+(\delta)^j}| = |B_1 \setminus A_{1+(\delta)^j}| - |A_{1+(\delta)^j} \setminus A_{1+(\delta)^j}| \\
\leq \left(1 - c \frac{\delta^n}{(1+\delta)^{n+1}}\right) |B_1 \setminus A_{1+(\delta)^j}| \\
\leq \left(1 - c \frac{\delta^n}{(1+\delta)^{n+1}}\right)^{j+1} |B_1|,
\]

(5.5)

using the inductive hypothesis.

Case II: $|F_j| < \frac{1}{1+\delta} |B_1 \setminus A_{1+(\delta)^j}|$. In this case,

\[
|B_1 \setminus A_{1+(\delta)^j}| \leq |B_1 \setminus A_{1+(\delta)^j}| \\
\leq |F_j| + |B_1 \setminus B_{1-(\delta)^j}\rho_j| \\
< \frac{1}{1+\delta} |B_1 \setminus A_{1+(\delta)^j}| + |B_1 \setminus B_{1-(\delta)^j}\rho_j|,
\]

and therefore,

\[
|B_1 \setminus A_{1+(\delta)^j}| \leq \left(\frac{1+\delta}{\delta}\right) |B_1 \setminus B_{1-(\delta)^j}\rho_j| \\
\leq \left(\frac{1+\delta}{\delta}\right) \left[1 - \left(1 + (\delta)^2\rho_j\right)^n\right] |B_1| \\
\leq \left(\frac{1+\delta}{\delta}\right) n(1 + (\delta)^2\rho_j) |B_1|
\]

where the last inequality is obtained by using Bernoulli’s inequality.

Now, we choose $\rho_j$ such that

\[
\left(\frac{1+\delta}{\delta}\right) n(1 + (\delta)^2\rho_j) = \left(1 - c \frac{\delta^n}{(1+\delta)^{n+1}}\right)^{j+1}
\]

(5.6)

and hence,

\[
|B_1 \setminus A_{1+(\delta)^j}| \leq \left(1 - c \frac{\delta^n}{(1+\delta)^{n+1}}\right)^{j+1} |B_1|.
\]

With this choice of $\rho_j$, the induction step is satisfied in both cases and therefore, (5.4) is proved.

Note that equation (5.6) completely determines $\rho_j$ and $0 < \rho_j < (1+\delta)^{-2}$ holds for such a $\rho_j$. Moreover,

\[
\frac{\delta^4}{n^2(1+\delta)^2} \leq (1+\delta)^j \rho_j^2
\]

for all $j \geq 0$, if we take $\delta = (1 + \sqrt{5})/2$.

To conclude the proof, we now argue as in Theorem 2. For any $t \geq (1 + \delta)^{1+\hat{\theta}(\alpha)}$, there exists $j \geq 1$ such that

\[
(1+\delta)^{j+\hat{\theta}(\alpha)} \leq t < (1+\delta)^{j+\hat{\theta}(\alpha)+1},
\]

(5.7)

and thus the following inclusions hold:

\[
\Theta_j > t \subset \Theta_j > (1+\delta)^{j+\hat{\theta}(\alpha)} \subset B_1 \setminus A_{1+(\delta)^{j+\hat{\theta}(\alpha)}}(u)
\]

(5.8)

which in view of 5.4 yields the aimed estimate if
\[ \alpha \leq \frac{-(j + j(\alpha)) \ln \left(1 - c \cdot \frac{\delta}{(1 + \delta)^r}\right)}{(j + j(\alpha) + 1) \ln (1 + \delta)} \]

\[ = \frac{j + j(\alpha) \ln \left(1 - 2c \cdot \frac{\delta}{(3 + \sqrt{5})^{r+1}}\right)}{j + j(\alpha) + 1 \ln \left(2(3 + \sqrt{5})^{-1}\right)}. \]

And the above inequality holds for any \( j \geq 1 \) by the definition of \( j(\alpha) \).

Finally, as in Theorem 2, the condition (5.2) on \( u \) is not restrictive. If \( u \) is a generic supersolution, we consider

\[ v = \delta^4 \frac{u + \|u\|_\infty}{2n^2(1 + \delta)^9\|u\|_\infty} = \frac{2^5(1 + \sqrt{5})^4}{n^2(3 + \sqrt{5})^5}(u + \|u\|_\infty) \]

and note that (5.2) holds for \( v \), \( M_{\lambda A}(D^2v) \leq 0 \) in \( B_1 \), and

\[ \Theta > t = \left\{ \Theta > \frac{n^2(3 + \sqrt{5})^5\|u\|_\infty}{2^5(1 + \sqrt{5})^4 t} \right\}. \]

The proof is complete. \( \square \)

**Remark 5.1.** Again, the bigger the constant \( c \) in (5.1) the sharper the lower bound for \( \varepsilon^G \).

Hence, as in the interior case, one is entitled to choose \( c \) as

\[ \max_{i=1,\ldots,k} \left\{ 1 + \left( \frac{\Lambda}{\lambda} - 1 \right) \frac{k}{n-i} \right\}^{i-n} \]

Proposition 1 informs about such an optimization problem.

### 6. New Upper Bound

In this section we discuss an upper bound for the Hessian integrability of viscosity supersolutions of fully nonlinear, uniformly elliptic equations. For each \( n \geq 3 \), we shall craft a supersolution \( \bar{v} \) in \( \mathbb{R}^n \) showing that the exponent \( \varepsilon \) as stated in Theorem 2 cannot be larger than \( \frac{\Lambda}{\lambda} \). This solves, in the negative, Armstrong-Silvestre-Smart’s conjecture, [1, Conjecture 3.1].

Let \( \alpha, R > 0 \) be given. We consider the function \( u = u_{\alpha,R} \) defined over the \( \mathbb{R}^n \setminus \{0\} \) by

\[ u(x) = \begin{cases} R^{\alpha+2}|x|^{-\alpha} + \frac{\alpha}{2}|x|^2 - (1 + \frac{\alpha}{2})R^2 & \text{for } 0 < |x| < R, \\ 0 & \text{for } R \leq |x| < \infty. \end{cases} \]

Clearly \( u \in C^1(\mathbb{R}^n \setminus \{0\}) \) and it is a non-negative function. Chain rule yields,

\[ \partial_i u = -\alpha R^{\alpha+2}|x|^{-\alpha-2}x_i + \alpha x_i, \text{ for } i = 1, \ldots, n \]

and for all \( i, j = 1, \ldots, n \)

\[ \partial_{ij} u = \alpha(\alpha + 2)R^{\alpha+2}|x|^{-\alpha-4}(x_ix_j) - \left(a R^{\alpha+2}|x|^{-\alpha-2} - \alpha \right) \delta_{ij}, \]

where \( \delta_{ij} \) is the Kronecker delta. Therefore,

\[ D^2u(x) = \alpha(\alpha + 2)R^{\alpha+2}|x|^{-\alpha-4}x \otimes x - \alpha|x|^{-\alpha-2} \left(R^{\alpha+2} - |x|^{\alpha+2} \right) I_n \]
for all \(0 < |x| < R\), where \(x \otimes x^T\) is the vector direct product of \(x\) and itself. Therefore the eigenvalues of \(D^2u(x)\) are

\[
\lambda_1 = \cdots = \lambda_{n-1} = -a|x|^{-a-2}\left(R^{a+2} - |x|^{a+2}\right) \quad \text{and} \\
\lambda_n = (\alpha(\alpha + 1)R^{a+2}|x|^{-a-2} + \alpha).
\]

in \(0 < |x| < R\). Thus, we can estimate,

\[
\begin{align*}
M_{\Lambda, \Lambda} \left(D^2u\right) &= \Lambda \cdot \left(\sum_{i=1}^{n-1} \lambda_i\right) + \lambda \cdot \lambda_n \\
&\leq -\Lambda(n - 1)|x|^{-a-2}\left(R^{a+2} - |x|^{a+2}\right) + \lambda \left(\alpha(\alpha + 1)R^{a+2}|x|^{-a-2} + \alpha\right) \\
&= -a|x|^{-a-2}R^{a+2} (\Lambda(n - 1) - \lambda(\alpha + 1)) + \alpha\lambda(n - 1) + \lambda\alpha \\
&\leq \Lambda n\alpha
\end{align*}
\]

provided that

\[
\Lambda(n - 1) - \lambda(\alpha + 1) \geq 0.
\]

That is,

\[
0 < \alpha \leq (n - 1)\frac{\Lambda}{\lambda} - 1
\]

Since \(u \equiv 0\) in \(\mathbb{R}^n \setminus B(0, R)\), then (6.2) actually holds (in the viscosity sense) in the entire set \(\mathbb{R}^n \setminus \{0\}\).

Our example is now constructed, as in [13], as follows: For \(R > 0\) small enough, define the function \(v = v_{\alpha, R}\) by

\[
v(x) = -|x|^2 + \sum_y \min\left(1, \frac{\lambda}{\Lambda\alpha}u(x - 2Ry)\right)
\]

where \(B(2Ry, R)\) are disjoint balls and \(y \in \mathbb{Z}^n\).

One can easily check that

\[
\sup_{B_i} |v| \leq 1 \quad \text{and} \quad M_{\Lambda, \Lambda} \left(D^2v\right) \leq 0 \quad \text{in} \quad \mathbb{R}^n.
\]

Let \(N\) denote a generic open neighborhood of \(x \in B(0, R) \setminus \{0\}\). In \(N\), we can estimate:

\[
\Theta(u, N)(x) \geq -\lambda_1 = a|x|^{-a-2}\left(R^{a+2} - |x|^{a+2}\right)
\]

Therefore, we conclude that if \(x \in B(0, R/2) \setminus \{0\}\)

\[
\Theta(u, N)(x) \geq c\alpha R^{a+2}|x|^{-a-2}
\]

for \(c\) given by

\[
c = \left(1 - \frac{1}{2\alpha+2}\right).
\]

By choosing \(R > 0\) small enough, we can assure,

\[
\min\left(1, \frac{\lambda}{\Lambda\alpha}u(x - 2Ry)\right) = \frac{\lambda}{\Lambda\alpha}u(x - 2Ry)
\]

whenever \(|x - 2Ry| \geq \tilde{c}R^{(a+2)/\alpha}\), where \(\tilde{c} = (\lambda/\Lambda\alpha)^{1/\alpha}\).
Moreover, for any neighborhood \( N \) of \( x \) in \( \{ 2R^{(\alpha+2)/\alpha} \leq |x - 2Ry| \leq R/2 \} \) we have, as in (6.5),

\[
\Theta(v, N)(x) \geq 2 - \frac{\lambda}{\Lambda} \lambda_n^2 \\
\geq c R^{(\alpha+2)} |x - 2Ry|^{-\alpha - 2}
\]

(6.6)

where \( \lambda_n^2 \) is a negative eigenvalue of \( D^2 u \).

Next, if we take \( \varepsilon > 0 \) such that

\[
(n - 1) \frac{\Lambda}{\Lambda} + 1 > \frac{n}{\varepsilon} - 2.
\]

(6.7)

we can choose \( \alpha > 0 \) such that

\[
(n - 1) \frac{\Lambda}{\Lambda} - 1 \geq \alpha > \frac{n}{\varepsilon} - 2.
\]

(6.8)

Furthermore, for any ball \( B(2Ry, R) \subset B_{1/2} \), owing to (6.6) we can estimate

\[
\int_{B(2Ry, R)} |\Theta(v, B_1)(x)|^\varepsilon \, dx \geq c R^{(\alpha+2)\varepsilon} \int_{B(2Ry) \setminus B_{2R^{(\alpha+2)/\alpha}}(2Ry)} (|x - 2Ry|)^{-2(\alpha+2)\varepsilon} \, dx
\]

\[
= c R^{(\alpha+2)\varepsilon} \int_{B(2Ry) \setminus B_{2R^{(\alpha+2)/\alpha}}(2Ry)} (|x|^{-(\alpha+2)\varepsilon} \, dx
\]

\[
= |\partial B_1| c R^{(\alpha+2)\varepsilon} \int_{R/2}^{R} t^{-(\alpha+2)\varepsilon + n - 1} \, dt,
\]

(6.9)

The idea now is to estimate \( ||\Theta(v, B_1)||_{L^\infty(B_{1/2})} \) by counting the number of disjoint balls \( B(2Ry, R) \) that fits inside \( B_{1/2} \). Let \( R > 0 \) be such that

\[
\frac{1}{2} \sqrt{n} < \frac{1}{8 \sqrt{nR}} = m + \frac{1}{2} \sqrt{n},
\]

for some \( m \in \mathbb{N} \) large. Then, if we choose \( y = (y_1, \ldots, y_n) \in \mathbb{Z}^n \) such that

\[
0 < |y_j| \leq \frac{1 - 4R}{8 \sqrt{nR}} = m, \quad j = 1, \ldots, n
\]

then \( B(2Ry, R) \) is entirely inside \( B(0, 1/2) \), and with this choice of \( R \) and \( y \in \mathbb{Z}^n \), there exist at least \( \left( \frac{1 - 4R}{8 \sqrt{nR}} \right)^n \) disjoint balls of the form \( B(2Ry, R) \), with \( y \in \mathbb{Z}^n \) inside \( B(0, 1/2) \).

Finally we estimate:

\[
\int_{B_{1/2}} |\Theta(v, B_1)|^\varepsilon \, dx \geq C \left( \frac{1 - 4R}{8 \sqrt{nR}} \right)^n \left[ c R^{(\alpha+2)(n-2\varepsilon)/\alpha} - \frac{R^n}{2n-(\alpha+2)\varepsilon} \right]
\]

\[
= C(1 - 4R)^n \left[ c R^{(n-2\varepsilon)/\alpha} - \frac{2}{n-(\alpha+2)\varepsilon} \right],
\]

and since, again in view of (6.8), the exponent \( 2(n - (\alpha + 2)\varepsilon)/\alpha \) is negative, we conclude

\[
||\Theta(v, B_1)||_{L^\infty(B_{1/2})} \rightarrow \infty \quad \text{as} \quad R \rightarrow 0.
\]
In conclusion, as the decay proven in (4.1) implies \( \Theta(\mathbf{u}, B_1) \in L^{\hat{\epsilon}}(B_{1/2}) \), for any \( 0 < \hat{\epsilon} < \epsilon \), the example we just crafted show that, in dimension \( n \geq 3 \), the \( W^{2,\epsilon} \) estimates as stated in Theorem 2 cannot hold (universally) for any \( \epsilon > 0 \) large enough so that

\[
(n - 1)\Lambda/\lambda + 1 < \epsilon.
\]

That is, the \( W^{2,\epsilon} \) regularity theory for viscosity supersolutions requires

\[
\epsilon \leq \frac{n}{(n - 1)\Lambda/\lambda + 1},
\]

for any \( n \geq 3 \). This quantity is strictly less than the one Conjectured in [1].

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