SIMULTANEOUS RATIONAL PERIODIC POINTS OF DEGREE-2 RATIONAL MAPS

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Abstract. Let $S$ be the collection of quadratic polynomial maps, and degree 2-rational maps whose automorphism groups are isomorphic to $C_2$ defined over the rational field. Assuming standard conjectures of Poonen and Manes on the period length of a periodic point under the action of a map in $S$, we give a complete description of triples $(f_1, f_2, p)$ such that $p$ is a rational periodic point for both $f_i \in S, i = 1, 2$. We also show that no more than three quadratic polynomial maps can possess a common periodic point over the rational field. In addition, under these hypotheses we show that two nonzero rational numbers $a, b$ are periodic points of the map $\phi_{t_1, t_2}(z) = t_1 z + t_2/z$ for infinitely many nonzero rational pairs $(t_1, t_2)$ if and only if $a^2 = b^2$.

1. Introduction

A (discrete) dynamical system consists of a set $S$ and a map $\phi : S \rightarrow S$ that permits iteration

$$\phi^n = \phi \circ \phi \circ \ldots \circ \phi \text{ \makebox[n- times]}$$

For a given point $P \in S$, the orbit of $P$ is the set

$$O_\phi(P) = O(P) = \{\phi^n(P) : n \geq 0\}.$$ 

One of the main goals of dynamics is to classify the points $P$ in the set $S$ according to the size of their orbits $O_\phi(P)$. If $O_\phi(P)$ is infinite, $P$ is called a wandering point; otherwise $P$ is called a preperiodic point. In this paper, we focus on a special type of preperiodic points, the so-called periodic points. A point $P \in S$ is periodic if there exists an integer $n > 0$ such that $\phi^n(P) = P$, where $n$ is called the period of $P$. If $n$ is the smallest such integer, we say that $P$ has exact period $n$. In our work, we will take the set $S$ to be the rational field $\mathbb{Q}$ and the map $\phi$ to be a degree-2 rational map of either forms

$$f(z) = z^2 + c, \quad \phi_{k,b}(z) = kz + b/z,$$

where $k, b, c \in \mathbb{Q}$ with $k, b$ are nonzero. Due to the work of Northcott, [7], we know that these maps can have only finitely many rational periodic points.

A complete classification of rational quadratic polynomials $f(z)$ with periodic points of exact period 1, 2 or 3 can be found for example in [8,11]. A polynomial $f(z)$ cannot have a rational periodic point of exact period 4, [9], or exact period 5, [3]. Under the assumption of Birch-Swinnerton-Dyer conjecture, Stoll proved that a polynomial $f(z) = z^2 + c$ cannot have a rational periodic point of exact period 6, [10]. Poonen conjectured that a rational quadratic polynomial $f(z)$ cannot have a rational periodic point of exact period $\geq 6$, [8]. In this work, we assume that the latter conjecture holds.

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In \([5]\), a classification of rational maps of the form \( \phi_{k,b}(z) \) that possess rational periodic points of exact period 1, 2 or 4 is given. It was proved that \( \phi_{k,b}(z) \) cannot have a rational periodic point of exact period 3. It was also conjectured that \( \phi_{k,b}(z) \) cannot have a rational periodic point of exact period at least 5. Again, in this work we assume that the latter conjecture holds.

We classify all triples of rational numbers \((k, b, c)\) such that \( f(z) = z^2 + c \) and \( \phi_{k,b}(z) = kz + \frac{t}{z} \) possess a rational periodic point \( p \) under the assumption of Poonen and Manes’ conjectures. In addition, we find all such triples for which

\[
| \text{Orb}_f(p) \cap \text{Orb}_{\phi_{k,b}}(p) | \geq 2.
\]

We also prove that the size of the intersection of \( \text{Orb}_f(p) \) and \( \text{Orb}_{\phi_{k,b}}(p) \) cannot exceed 2.

Under the assumption of Manes’ conjecture, we list down the tuples \((k_1, b_1, k_2, b_2)\) such that the maps \( \phi_{k_1,b_1}(z) \) and \( \phi_{k_2,b_2}(z) \) have a simultaneous rational periodic point \( p \). We then illustrate which such tuples allow the following

\[
| \text{Orb}_{\phi_{k_1,k_2}}(p) \cap \text{Orb}_{\phi_{k_2,b_2}}(p) | \geq 2.
\]

Again, we show that the above intersection size cannot exceed 2 unless \( \phi_{k_1,b_1} = \pm \phi_{k_2,b_2} \).

In \([1]\), it was proved that for fixed \( c_1, c_2 \in \mathbb{C} \), the set of \( t \in \mathbb{C} \) such that both \( c_1 \) and \( c_2 \) are preperiodic for \( z^d + t \) is infinite if and only if \( c_1^d = c_2^d \) where \( d \) is greater than one. Assuming Manes’ conjecture, we prove a similar result over the rational field. Setting \( \phi_{t_1, t_2}(z) = t_1 z + \frac{t_2}{z} \), two rational numbers \( a, b \in \mathbb{Q}^\times \) are periodic points of \( \phi_{t_1, t_2}(z) \) for infinitely many nonzero rational pairs \((t_1, t_2)\) if and only if \( a^2 = b^2 \).

Although it is easily seen that given a nonzero rational number \( p \) we can find infinitely many nonzero rational pairs \((k, b)\) such that \( p \) is a periodic point of \( \phi_{k,b}(z) \), there are at most three quadratic polynomial maps sharing \( p \) as a common periodic point. The latter result is in the same realm of results on the size of sets \( S \) of quadratic polynomial maps for which \( p \) is preperiodic for any composition of maps in \( S \), \([4]\).

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### 2. Periodic points of degree-2 rational maps

In this section, we collect some preliminaries on dynamical systems and known results on periodic points of the quadratic rational maps that we deal with in this article.

#### 2.1. Preliminaries.** We start with recalling the definition of the dynatomic polynomial associated to a certain rational map defined over a field \( K \) with algebraic closure \( \overline{K} \).

**Definition 2.1.** \([9]\) Let \( K \) be a field. Let \( \phi(z) \in K(z) \) be a rational function of degree \( d \). We write

\[
\phi^n = [F_n(x, y), G_n(x, y)]
\]

where \( F_n, G_n \in K[x, y] \) are homogeneous polynomials of degree \( d^n \) for any \( n \geq 0 \). The \( n \)-period polynomial of \( \phi \) is the polynomial

\[
\Phi_{\phi,n}(x, y) = y \cdot F_n(x, y) - x \cdot G_n(x, y).
\]
The $n^{th}$ dynatomic polynomial of $\phi$ is the polynomial
\[
\Phi_{\phi,n}^*(x,y) = \prod_{k|n} (y \cdot F_k(x,y) - x \cdot G_k(x,y))^\mu(n/k) = \prod_{k|n} (\Phi_{\phi,k}(x,y))^\mu(n/k)
\]
where $\mu$ is the Möbius function, defined by $\mu(1) = 1$, $\mu(n) = (-1)^l$ if $n = p_1 p_2 \ldots p_l$ with $p_i$ distinct primes, and $\mu(n) = 0$ if $n$ is not square free. If $\phi$ is fixed, we write $\Phi_n$ and $\Phi_n^*$. We will write $\Phi_n(z)$ for $\Phi_n(z,1)$ and $\Phi_n^*(z)$ for $\Phi_n^*(z,1)$.

By definition, every exact period-$n$ point of $\phi$ is a root of $\Phi_{\phi,n}^*(x,y)$. It is important to notice that $\Phi_{\phi,n}^*(x,y)$ can have roots whose periods divide $n$ and strictly smaller than $n$.

Let $f \in \text{PGL}_2(\mathbb{K})$ act on the points of $\mathbb{P}^1$ as a fractional linear transformation in the usual way. Then we define the rational map $\phi^f = f^{-1} \phi f$. In fact, two rational maps $\phi$ and $\psi$ are said to be linearly conjugate if there is some $f \in \text{PGL}_2(\mathbb{K})$ such that $\phi^f = \psi$. They are linearly conjugate over $\mathbb{K}$ if there is some $f \in \text{PGL}_2(\mathbb{K})$ such that $\phi^f = \psi$.

We notice that if $P$ is a periodic point of exact period $n$ for $\phi$, then $f^{-1}(P)$ is a periodic point of exact period $n$ for $\phi^f$. One can argue similarly for preperiodic points of $\phi$ and $\phi^f$. In other words, linear conjugation preserves the dynamical behavior of points under rational maps. Moreover, if $\phi, f$ and $P$ are defined over $\mathbb{K}$ such that $\phi^f(P) = P$, then $\psi := \phi^f$ and $Q := f^{-1}(P)$ are defined over $\mathbb{K}$ with $\psi^n(Q) = Q$.

### 2.2. Quadratic polynomial maps.

Any quadratic polynomial map $\phi(z) = A z^2 + B z + C \in \mathbb{K}[z]$, $A \in \mathbb{K}^\times$, is linearly conjugate over $\mathbb{K}$ to a map of the form $\psi(z) = z^2 + c$ for some $c \in \mathbb{K}$. Therefore, we only consider quadratic polynomial maps of this form.

If $\mathbb{K}$ is chosen to be the rational field $\mathbb{Q}$, a complete classification of quadratic polynomial maps with periodic points of periods $1$, $2$, or $3$ was given in [11]. The following can be found for example as [3 Theorem 1].

**Proposition 2.2.** Let $f(z) = z^2 + c$ with $c \in \mathbb{Q}$. Then

1) $f(z)$ has a rational point of period $1$, i.e., a rational fixed point, if and only if $c = 1/4 - \rho^2$ for some $\rho \in \mathbb{Q}$. In this case, there are exactly two, $1/2 + \rho$ and $1/2 - \rho$, unless $\rho = 0$, in which case they coincide.

2) $f(z)$ has a rational point of period $2$ if and only if $c = -3/4 - \sigma^2$ for some $\sigma \in \mathbb{Q}$, $\sigma \neq 0$. In this case, there are exactly two, $-1/2 + \sigma$ and $-1/2 - \sigma$ (and these form a 2-cycle).

3) $f(z)$ has a rational point of period $3$ if and only if
\[
c = \frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau + 1)^2}
\]
for some $\tau \in \mathbb{Q}$, $\tau \neq -1,0$. In this case, there are exactly three,
\[
x_1 = \frac{\tau^3 + 2\tau^2 + \tau + 1}{2\tau(\tau + 1)}, \quad x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau + 1)}, \quad x_3 = \frac{-\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau + 1)}
\]
and these are cyclically permuted by $f(z)$.

The following conjecture can be found in [3, 8].

**Conjecture 2.3.** If $N \geq 4$, then there is no quadratic polynomial $f(z) \in \mathbb{Q}[z]$ with a rational point of exact period $N$.
The conjecture has been proved for $N = 4$, [6], for $N = 5$, [3], and conditionally for $N = 6$, [10].

2.3. Degree-2 rational maps. Given a rational map $\phi(z) \in K(z)$, we define the automorphism group of $\phi$, $\text{Aut}(\phi)$, to be $\{f \in \text{PGL}_2(K) : \phi^f = \phi\}$. Let $\phi$ be a degree-2 rational map defined over $K$, with $K \neq 2,3$. Then $\text{Aut}(\phi) \cong C_2$ if and only if $\phi$ is linearly conjugate over $K$ to a map of the form $\phi_{k,b}(z) = kz + b/z$, $k \notin \{0, -1/2\}$, $b \in K^\times$. In addition, two such maps $\phi_{k,b}$ and $\phi_{k',b'}$ are linearly conjugate over $K$ if and only if $k = k'$ and $b/b' \in (K^\times)^2$. The map $\phi_{k,b}$ has the automorphism $z \mapsto -z$, see [5, Lemma 1]. Since the homogeneous form of $\phi_{k,b}(z) = kz + b/z$ is given by $[kx^2 + by^2 : xy]$, it follows that the point at infinity is a $K$-rational fixed point for $\phi_{k,b}(z)$. Therefore, throughout the sequel when we mention periodic points of $\phi_{k,b}(z)$ we mean finite periodic points.

The following proposition describes $\mathbb{Q}$-rational periodic points of maps of the form $\phi_{k,b}(z) = kz + b/z$, $k, b \in \mathbb{Q}^\times$, see [5, §2].

**Proposition 2.4.** Let $\phi_{k,b}(z) = kz + b/z$, $k \in \mathbb{Q}^\times$, $b \in \mathbb{Q}^\times/((\mathbb{Q}^\times)^2)$.

1) $\phi_{k,b}(z)$ has a $\mathbb{Q}$-rational fixed point if and only if $b/(1-k) = m^2$ for some $m \in \mathbb{Q}^\times$, $k \neq 1$.

   In this case there are two finite $\mathbb{Q}$-rational fixed points, $\pm m$.

2) $\phi_{k,b}(z)$ has a $\mathbb{Q}$-rational periodic point of exact period 2 if and only if $b/(k+1) = -m^2$ for some $m \in \mathbb{Q}^\times$, $k \neq -1$.

   In this case there are exactly two such points, $\pm m$.

3) $\phi_{k,b}(z)$ has no $\mathbb{Q}$-rational periodic point of exact period 3.

4) $\phi_{k,b}(z)$ has a $\mathbb{Q}$-rational periodic point of exact period 4 if and only if

$$k = \frac{2m}{m^2 - 1}, \quad b = \frac{-m}{m^4 - 1} \quad \text{for some } m \in \mathbb{Q} \setminus \{0, \pm 1\}.$$

In this case, the rational periodic points are

$$x_1 = \frac{1}{m^2 + 1}, \quad x_2 = \frac{-m}{m^2 + 1}, \quad x_3 = \frac{-1}{m^2 + 1}, \quad x_4 = \frac{m}{m^2 + 1}.$$

The following conjecture is [5, Conjecture 1].

**Conjecture 2.5.** If $\phi(z) \in \mathbb{Q}(z)$ is a degree-2 rational map with $\text{Aut}(\phi) \cong C_2$, then $\phi$ has no rational point of exact period $N > 4$.

3. Simultaneous rational periodic points of $f(z)$ and $\phi_{k,b}(z)$

**Proposition 3.1.** Let $f(z) = z^2 + c$ and $\phi_{k,b}(z) = kz + b/z$, $c \in \mathbb{Q}, k, b \in \mathbb{Q}^\times$. Given $p \in \mathbb{Q}^\times$, the set of triples $(k, b, c)$ such that $p$ is a rational periodic point of exact period 1 (rational fixed point) of $f(z)$ and a rational periodic point of exact period $n$ of $\phi_{k,b}(z)$ is described as follows

$$(k, b, c) = \left(\frac{q \pm p}{p}, -qp, p - p^2\right), \quad q \in \mathbb{Q} \setminus \{0, -p\} \quad \text{if } n = 1$$

$$(k, b, c) = \left(\frac{q - p}{p}, -qp, p - p^2\right), \quad q \in \mathbb{Q} \setminus \{0, p\} \quad \text{if } n = 2$$

$$(k, b, c) = \left(\frac{2m}{m^2 - 1}, -\frac{p^2(m^2 + 1)}{m(m^2 - 1)}, p - p^2\right), \quad m \in \mathbb{Q} \setminus \{0, \pm 1\} \quad \text{if } n = 4$$

**Proof:** One may check easily that $p$ is a rational fixed point for both $f(z)$ and $\phi_{k,b}(z)$ if $k, b, c$ are as given in the statement of the proposition. Now, we know that $f(z)$ has a rational fixed point if and only if $c = 1/4 - p^2$ for some $p \in \mathbb{Q}$, see Proposition 2.2, where $1/2 \pm p$ are rational fixed points of $f(z)$. Suppose that $p = 1/2 + \rho$ is a rational fixed point of $\phi_{k,b}(z)$, it follows that $1/2 + \rho = \frac{\pm \sqrt{5-4k}}{1+k}$. 


where \( k \neq 1 \) and \( b - bk = q^2 \) for some \( q \in \mathbb{Q}^\times \). It follows that \( k = \frac{\pm 2r + 2p + 1}{2p + 1} \) and \( b = \mp(1 + 2p)q/2 \).

We obtain the same parametrization if we take \( p = 1/2 - \rho \) to be the common periodic point of \( f(z) \) and \( \phi_{k,b}(z) \).

For \( n = 2 \), the proof is similar.

As for \( n = 4 \), we recall that the 4th-dynatomic polynomial of \( \phi_{k,b}(z) \) factors as follows

\[
\Phi_4^*(z) = \Psi_4(z) \cdot \Lambda_4(z)
\]

where

\[
\Psi_4(z) = b^2 k + 2b z^2 + 2bk^2 z^2 + k z^4 + k^3 z^4,
\]

\[
\Lambda_4(z) = b^4 k^5 + b^3 z^2 + b^3 k^2 z^2 + 2b^3 k^4 z^2 + 4b^3 k^6 z^2 + b^2 k^4 z^4 + 3b^2 k^3 z^4 + 4b^2 k^5 z^4 + 6b^2 k^7 z^4 + bk^4 z^6 + 2bk^6 z^6 + 4bk^8 z^6 + k^9 z^6.
\]

Moreover, \( \Lambda_4(z) \) does not have a rational root, see [5] \( \S 2 \). If \( 1/2 + \rho \) is a rational periodic point of exact period 4 of \( \phi_{k,b}(z) \), then it is a root of \( \Psi_4(z) \), it follows that \( \rho \) is one of the following four expressions

\[
\frac{1}{2} \left( -1 \pm \sqrt{\frac{-4b \cdot (1 + k^2 \pm \sqrt{1 + k^2})}{k \cdot (1 + k^2)}} \right).
\]

The rationality of \( \rho \) implies that \( 1 + k^2 \) is a rational square, hence \( k = \frac{2m}{m^2 - 1} \) for some \( m \in \mathbb{Q} \setminus \{0, \pm 1\} \). Moreover,

\[
b = -\frac{s^2 \cdot (m^2 + 1)}{4m(m^2 - 1)}, \quad \text{where} \quad s = \sqrt{\frac{-4b \cdot (1 + k^2 \pm \sqrt{1 + k^2})}{k \cdot (1 + k^2)}}.
\]

Now, the expressions for \( b \) and \( c \) follow by noticing that \( p = 1/2 + \rho = \pm s/2 \). In fact, the cycle of periodic points of period 4 of \( \phi_{k,b}(z) \) is given by \( (p, p/m, -p, -p/m) \). \( \square \)

**Example 3.2.** The point \( p = 3/2 \) is a common fixed point of \( f(z) = z^2 - \frac{4}{9} \), and \( \phi(z) = \frac{5z}{3} - \frac{3}{2z} \) corresponding to \( q = 1 \) in Proposition [3.1]

Setting \( p = 3 \) and \( q = \frac{1}{2} \), we get that 3 is a rational fixed point of \( f(z) = z^2 - 6 \), whereas it is a periodic point of exact period 2 of \( \phi(z) = -\frac{5z}{6} - \frac{3}{2z} \).

Finally, taking \( m = 2 \) and \( p = 2 \) in Proposition [3.1] implies that 2 is a rational fixed point of \( f(z) = z^2 - 2 \), while it is a periodic point of exact period 4 for \( \phi(z) = \frac{4z}{3} - \frac{10}{3z} \).

**Proposition 3.3.** Let \( f(z) = z^2 + c \) and \( \phi_{k,b}(z) = k/z + b/z, c \in \mathbb{Q}, k, b \in \mathbb{Q}^\times \). Given \( p \in \mathbb{Q} \setminus \{0, -1/2\} \), the set of triples \((k, b, c)\) such that \( p \) is a rational periodic point of exact period 2 of \( f(z) \) and a rational periodic point of exact period \( n \) of \( \phi_{k,b}(z) \) is described as follows

\[
(k, b, c) = \begin{cases} 
\left( \frac{q + p}{p}, -q, -(p^2 + p + 1) \right), & q \in \mathbb{Q} \setminus \{0, -p\} \quad \text{if } n = 1 \\
\left( \frac{q - p}{p}, -q, -(p^2 + p + 1) \right), & q \in \mathbb{Q} \setminus \{0, p\} \quad \text{if } n = 2 \\
\left( \frac{2m + 1}{m^2 - 1}, \frac{p^2(m^2 + 1)}{m(m^2 - 1)}, -(p^2 + p + 1) \right), & m \in \mathbb{Q} \setminus \{0, \pm 1\} \quad \text{if } n = 4
\end{cases}
\]

**Proof:** The proof is similar to the proof of Proposition [3.1] using Propositions [2.2] and [2.4]. \( \square \)
Example 3.4. The point $p = 1/2$ is a periodic point of exact period 2 of $f(z) = z^2 - \frac{7}{4}$, and a fixed point of $\phi(z) = 3z - \frac{1}{2z}$ corresponding to $q = 1$ in Proposition 3.3.

Setting $p = 1$ and $q = -1$, we get that 1 is a periodic point of exact period 2 of $f(z) = z^2 - 3$ and $\phi(z) = -2z + \frac{1}{2}$.

Finally, taking $m = 3$ and $p = -1$ in Proposition 3.3 implies that $-1$ is a periodic point of exact period 2 of $f(z) = z^2 - 1$, while it is a periodic point of exact period 4 for $\phi(z) = \frac{3z}{4} - \frac{5}{12z}$.

Proposition 3.5. Let $f(z) = z^2 + c$ and $\phi_{k,b}(z) = kz + b/z$, $c \in \mathbb{Q}$, $k, b \in \mathbb{Q}^\times$. The set of triples $(k, b, c)$ such that $f(z)$ has a rational periodic point $x_i$ of exact period 3 which is a rational periodic point of exact period $n$ of $\phi_{k,b}(z)$ is described as follows

$$(k, b, c) = \left(1 - q, qx_i^2, \frac{-q^6 + 2q^5 + 4q^4 + 8q^3 + 9q^2 + 4q + 1}{4q^2(\tau + 1)^2}\right) \quad \text{if } n = 1$$

$$(k, b, c) = \left(q - 1, -qx_i^2, \frac{-q^6 + 2q^5 + 4q^4 + 8q^3 + 9q^2 + 4q + 1}{4q^2(\tau + 1)^2}\right) \quad \text{if } n = 2$$

where $q \in \mathbb{Q} \setminus \{0, 1\}$ and $\tau \in \mathbb{Q} \setminus \{-1, 0\}$, $i = 1, 2, 3$,

$$(k, b, c) = \left(\frac{2m}{m^2 - 1}, \frac{-\tau^2 + 2\tau + 1}{2\tau(\tau + 1)}, \frac{\tau^2 - \tau - 1}{2\tau(\tau + 1)}, \frac{-\tau^2 + 2\tau + 1}{2\tau(\tau + 1)}\right) \quad \text{if } n = 4$$

where $x_1 = \frac{-3 + 2\tau^2 + \tau + 1}{2\tau(\tau + 1)}$, $x_2 = \frac{\tau^3 - \tau - 1}{2\tau(\tau + 1)}$, $x_3 = \frac{-3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau + 1)}$, $\tau \in \mathbb{Q} \setminus \{0, -1\}$.

Proof: The cases $n = 1, 2$ can be treated similarly as in Propositions 3.1 and 3.3.

For $n = 4$, Proposition 2.2 asserts that $f(z)$ has a rational periodic point of exact period 3 if and only if $c$ and $x_i$ are given in the statement of the proposition. To force one of these points to be a periodic point of exact period 4 for $\phi_{k,b}$ for some rational numbers $k, b$, the point should be a root of $\Psi_4(z)$, see the proof of Proposition 3.1. The four roots of $\Psi_4(z)$ are given by

\[ \pm \sqrt{\frac{-b}{k} \left(1 \pm \frac{1}{\sqrt{1 + k^2}}\right)} \]

The rationality of the periodic points yields the result by proceeding as in the proof of Proposition 3.1. Notice that in this case the cycle of periodic points of exact period 4 of $\phi_{k,b}(z)$ is given by $(x_i, x_i/m, -x_i, -x_i/m)$.

Example 3.6. Fixing $\tau = 1$ and $q = 16$, then $x_2 = -1/4$ is a rational periodic point of exact period 3 of $f(z) = z^2 - 29/16$ and it is a rational fixed point of $\phi(z) = -15z + \frac{1}{z}$.

Setting $\tau = 1/2$ and $q = 9$, then $x_1 = 17/12$ is a rational periodic point of exact period 3 of $f(z) = z^2 - 421/144$ while it is a rational periodic point of exact period 2 of $\phi(z) = 8z - \frac{289}{16z}$.

If we set $\tau = -1/2$ and $m = 2$, then $x_3 = -1/4$ is a rational periodic point of exact period 3 of $f(z) = z^2 - 29/16$ and it is a rational periodic point of exact period 4 of $\phi(z) = \frac{41}{3} - \frac{5}{96z}$.

The propositions above imply the following result.

Theorem 3.7. Assuming Conjecture 2.3 and Conjecture 2.5, the following table contains all pairs $f(z) = z^2 + c$ and $\phi_{k,b}(z) = kz + b/z$, $c \in \mathbb{Q}$, $k, b \in \mathbb{Q}^\times$, such that some rational number $p$ is a periodic point for both maps.
| \( \phi_{k,b}(z) \) | Periodic Points | PL | \( \beta(z) \) | Periodic Points | PL |
|---|---|---|---|---|---|
| \( \frac{z^2+p}{p} \) | 1 | \( z^2+p-p^2 \) | \( p, -p \) | 1 |
| \( \frac{z^2-p}{p} \) | 2 | \( z^2+p-p^2 \) | \( p, -p \) | 1 |
| \( \frac{2q^2+1}{q} \) | 4 | \( z^2+p-p^2 \) | \( p, -p \) | 1 |
| \( \frac{2q^2-1}{q} \) | 4 | \( z^2+p-p^2 \) | \( p, -p \) | 1 |

PL represents the period length of the orbit of the periodic point with  
where  
and  

\( \mathcal{P} \) is the period length of the orbit of the periodic point with  
where  

In particular, examining the table of Theorem 3.7 gives the following result.

\[ |\text{Orb}_f(p) \cap \text{Orb}_{\phi_{k,b}}(p)| \geq 2 \]

In fact, examining the table of Theorem 3.7 gives the following result.

**Theorem 3.8.** Assuming Conjecture 2.8 and Conjecture 2.5 if there exists a rational periodic point  
then the triples  
are given as follows:

\[ (k,b,c) = \left( \frac{\pm 2p(p+1)}{2p+1}, \frac{p(p+1)(p^2+(p+1)^2)}{2p+1}, -(p^2+p+1) \right) \]

where  
and  
are given as follows:

where  

**Example 3.9.** The maps  
and  
have the following cycles  

and  

have the following cycles  

4. Simultaneous Rational Periodic Points of $\phi_{k_1,b_1}$ and $\phi_{k_2,b_2}$

The aim of this section is to characterize the tuples $(k_1, b_1, k_2, b_2)$ such that the maps $\phi_{k_1,b_1}(z) = k_1z + \frac{b_1}{z}$ and $\phi_{k_2,b_2}(z) = k_2z + \frac{b_2}{z}$ with $k_1, b_1, k_2, b_2 \in \mathbb{Q}^\times$ share a common rational periodic point. Furthermore, we will find the tuples $(k_1, b_1, k_2, b_2)$ such that $|\text{Orb}_{\phi_{k_1,b_1}}(p) \cap \text{Orb}_{\phi_{k_2,b_2}}(p)| \geq 2$ for some rational periodic point $p$.

A careful work out of the possibilities in Proposition 2.4 implies the following result.

**Proposition 4.1.** Assuming Conjecture 2.5, the tuples $(k_1, b_1, k_2, b_2)$ such that there is $p \in \mathbb{Q}^\times$ with $p$ a periodic point for the maps $\phi_{k_1,b_1}(z)$ and $\phi_{k_2,b_2}(z)$ are given as follows

| $(k_1, b_1)$ | Periodic Points | PL $(k_2, b_2)$ | Periodic Points | PL |
|-------------|----------------|----------------|----------------|----|
| $(1 - s_1, s_1 \cdot p^2)$ | $p, -p$ | 1 | $(1 - s_2, s_2 \cdot p^2)$ | $p, -p$ | 1 |
| $(s_1 - 1, -s_1 \cdot p^2)$ | $p, -p$ | 2 | $(s_2 - 1, -s_2 \cdot p^2)$ | $p, -p$ | 2 |
| $(\frac{2s_1}{s_1^2 - 1} - p^2(s_1 + 1)/s_1(s_1 - 1))$ | $p, \frac{p}{s_1}, -p, -\frac{p}{s_1}$ | 4 | $(\frac{2s_2}{s_2^2 - 1} - p^2(s_2 + 1)/s_2(s_2 - 1))$ | $p, \frac{p}{s_2}, -p, -\frac{p}{s_2}$ | 4 |
| $(1 - s_1, s_1 \cdot p^2)$ | $p, -p$ | 1 | $(s_2 - 1, -s_2 \cdot p^2)$ | $p, -p$ | 2 |
| $(1 - s_1, s_1 \cdot p^2)$ | $p, -p$ | 1 | $(\frac{2s_2}{s_2^2 - 1} - p^2(s_2 + 1)/s_2(s_2 - 1))$ | $p, \frac{p}{s_2}, -p, -\frac{p}{s_2}$ | 4 |
| $(s_1 - 1, -s_1 \cdot p^2)$ | $p, -p$ | 2 | $(\frac{2s_1}{s_1^2 - 1} - p^2(s_1 + 1)/s_1(s_1 - 1))$ | $p, \frac{p}{s_1}, -p, -\frac{p}{s_1}$ | 4 |

where PL is the period length of the orbit of the periodic point, $s_i \in \mathbb{Q} \setminus \{0, 1\}$ if PL is either 1 or 2; and $s_i \in \mathbb{Q} \setminus \{0, \pm 1\}$ if PL is 4.

Using Proposition 4.1, one obtains the following result.

**Theorem 4.2.** Assuming Conjecture 2.5 the quadruples $(k_1, b_1, k_2, b_2)$ such that there exists a periodic point $p \in \mathbb{Q}^\times$ and maps $\phi_{k_i,b_i}(z) = k_i z + \frac{b_i}{z}$, with $k_i, b_i \in \mathbb{Q}^\times$, $i = 1, 2$, satisfying

$$|\text{Orb}_{\phi_{k_1,b_1}}(p) \cap \text{Orb}_{\phi_{k_2,b_2}}(p)| \geq 2$$

are given as follows:

$$(k_1, b_1, k_2, b_2) = (s_1 - 1, -s_1 \cdot p^2, s_2 - 1, -s_2 \cdot p^2), \quad s_1, s_2 \in \mathbb{Q} \setminus \{0, 1\};$$

$$(k_1, b_1, k_2, b_2) = (\frac{2s_1}{s_1^2 - 1} - p^2(s_1 + 1)/s_1(s_1 - 1), s_2 - 1, -s_2 \cdot p^2), \quad s_1 \in \mathbb{Q} \setminus \{0, 1\}, s_2 \in \mathbb{Q} \setminus \{0, \pm 1\},$$

$$(k_1, b_1, k_2, b_2) = \left(\frac{2s_1}{s_1^2 - 1}, -p^2(s_1 + 1)/s_1(s_1 - 1), \frac{2s_2}{s_2^2 - 1} - p^2(s_2 + 1)/s_2(s_2 - 1)\right), \quad s_1 \neq \pm s_2, s_1, s_2 \in \mathbb{Q} \setminus \{0, \pm 1\}$$

where in these cases $\text{Orb}_{\phi_{k_1,b_1}}(p) \cap \text{Orb}_{\phi_{k_2,b_2}}(p) = \{p, -p\}$.

In particular, if $|\text{Orb}_{\phi_{k_1,b_1}}(p) \cap \text{Orb}_{\phi_{k_2,b_2}}(p)| = 4$, then $\phi_{k_1,b_1} = \pm \phi_{k_2,b_2}$.

**Example 4.3.** The maps $\phi(z) = \frac{4z}{3} - \frac{3}{10z}$ and $\phi(z) = -\frac{3z}{3} + \frac{27}{20z}$ have the following cycles $\left(\frac{3}{5}, \frac{3}{10}, -\frac{3}{5}, -\frac{3}{10}\right)$ and $\left(\frac{3}{5}, \frac{3}{10}, -\frac{3}{5}, -\frac{3}{10}\right)$, respectively.

We now recall the main theorem in [1].

**Theorem 4.4.** Let $d \geq 2$ be an integer. Fix $c_1, c_2 \in \mathbb{C}$. The set of $t \in \mathbb{C}$ such that both $c_1$ and $c_2$ are preperiodic for $z^d + t$ is infinite if and only if $c_1^d = c_2^d$.

Assuming Conjecture 2.5, we will prove the following theorem in this section.
Theorem 4.5. Assume Conjecture 2.5 holds. Let \(a, b \in \mathbb{Q}^\times\). There exists infinitely many rational pairs \((t_1, t_2) \in \mathbb{Q}^\times \times \mathbb{Q}^\times\) such that \(a\) and \(b\) are both rational periodic points of \(\phi_{t_1, t_2}(z) = t_1 \cdot z + \frac{t_2}{z}\) if and only if \(a^2 = b^2\).

Before we start proving Theorem 4.5 we give the following straightforward application of Proposition 2.4.

Lemma 4.6. If \(\phi_{k, b}(z) = kz + \frac{b}{z}\) has two distinct rational periodic points \(q_1, q_2 \in \mathbb{Q}^\times\) with exact periods 1 and 2, respectively, where \(k, b \in \mathbb{Q}^\times\), then

\[
\phi_{k, b}(z) = \frac{-q_2^2 - q_1^2}{q_2 - q_1} z + \frac{2q_1 q_2}{(q_2 - q_1)z}.
\]

The following Lemma is [5, Proposition 10].

Lemma 4.7. Let \(\phi_{k, b}(z) = kz + \frac{b}{z}\) where \(k \in \mathbb{Q} \setminus \{0, -1/2\}, b \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^2\). Then \(\phi_{k, b}(z)\) cannot have two periodic points with exact period \(n\) and exact period 4, \(n = 1, 2\).

Now we proceed with the proof of Theorem 4.5.

**Proof of Theorem 4.5** Suppose that there exists infinitely many rational pairs \((t_1, t_2)\) such that \(a\) and \(b\) are both rational periodic points of \(\phi_{t_1, t_2}(z)\). By Lemma 4.6 and Lemma 4.7, \(a\) and \(b\) cannot have orbits with different period lengths under the action of \(\phi_{t_1, t_2}(z)\). We may therefore assume that both \(a\) and \(b\) have period \(n\), where \(n\) is either 1, 2 or 4. In view of Proposition 4.1, we must have \(a = -b\). \(\square\)

5. **Simultaneous rational periodic points of two quadratic polynomials**

One remarks that a rational number \(p\) can be a periodic point for infinitely many non-linearly conjugate maps of the form \(\phi_{k, b}(z) = k z + b/z, k, b \in \mathbb{Q}^\times\), see Proposition 4.11. However, this is not the case for quadratic polynomial maps.

Theorem 5.1. Assume Conjecture 2.3 holds. If \(q \in \mathbb{Q}\) is a periodic point for every map \(f_i(z) = z^2 + c_i, i = 1, \ldots, m, c_i \in \mathbb{Q}^\times, c_i \neq c_j\) if \(i \neq j\), then \(m \leq 3\).

**Proof:** Suppose \(q\) is rational fixed point of \(f(z) = z^2 + c\). It follows that \(c = \frac{1}{4} - \rho^2\) for some \(\rho \in \mathbb{Q}\) where \(q\) is either 1/2 + \(\rho\) or 1/2 - \(\rho\), see Proposition 2.2 so \(\rho = \pm(q - 1/2)\), and \(c\) is determined uniquely.

Similarly, if \(q\) is a rational periodic point of exact period 2 of \(f(z) = z^2 + c\), then \(c = -\frac{3}{4} - \sigma^2\) for some \(\sigma \in \mathbb{Q}^\times\). In this case, \(q\) is either -1/2 + \(\sigma\) or -1/2 - \(\sigma\), see Proposition 2.2 so that \(\sigma = \pm(q + 1/2)\), and \(c\) is determined uniquely.

We now suppose \(q\) is rational periodic point of exact period 3 of \(f(z) = z^2 + c\). Then, \(c = -\frac{\tau^6 + 17\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau + 1)}\), for some \(\tau \in \mathbb{Q} \setminus \{ -1, 0\}\), and \(q\) is one of the \(x_i\)’s in Proposition 2.2.

Assume that \(q = x_1(\tau) = \frac{x^3 + 2\tau^2 + \tau + 1}{2\tau(\tau + 1)}\). This implies that \(\tau\) is a root of the following cubic polynomial equation

\[
x^3 + (2 - 2q)x^2 + (1 - 2q)x + 1 = 0.
\]

We now check whether the latter polynomial has a rational root other than \(\tau\). We know that for other roots \(x_1, x_2\) of this polynomial we have

\[
x_1 + x_2 = 2q - 2, \quad x_1 \cdot x_2 = -1/\tau.
\]
Therefore $x_1$ and $x_2$ satisfy the following quadratic polynomial equation
\[ x^2 + (\tau + 2 - 2q)x - \frac{1}{\tau} = 0. \]

This polynomial has rational roots if and only if the discriminant $(\tau + 2 - 2q)^2 + \frac{4}{\tau}$ is a square in $\mathbb{Q}$. Writing $q$ in terms of $\tau$, the latter condition on the discriminant is equivalent to the fact that the following expression $\frac{\tau^4 + 6\tau^3 + 7\tau^2 + 2\tau + 1}{\tau^2(\tau+1)^2}$ is a square in $\mathbb{Q}$. Equivalently, one needs to find a rational solution on the elliptic curve
\[ (5.1) \quad y^2 = \tau^4 + 6\tau^3 + 7\tau^2 + 2\tau + 1. \]

Using MAGMA, one can check $(0, 1), (0, -1), (-1, 1), (-1, -1)$, and the two points at infinity are the only rational points on this elliptic curve. Since $\tau \neq 0, -1$, the discriminant cannot be a square in $\mathbb{Q}$.

Arguing similarly, if $q = x_2(\tau) = \frac{\tau^3 - \tau - 1}{2\tau(\tau+1)}$, then $\tau$ is a root of the following polynomial $x^3 - 2qx^2 - (1 + 2q)x - 1$. The other two roots $x_1$ and $x_2$ satisfy the quadratic polynomial $x^2 + (\tau - 2q)x + \frac{1}{\tau}$. This polynomial has rational roots if and only if the discriminant $(\tau - 2q)^2 - \frac{4}{\tau}$ is a square in $\mathbb{Q}$. Writing $q$ in terms of $\tau$, the latter occurs if and only if $\frac{\tau^4 - 2\tau^3 - 5\tau^2 - 2\tau + 1}{\tau^2(\tau+1)^2}$ is a square in $\mathbb{Q}$. Knowing that the rational points on the elliptic curve defined by $y^2 = \tau^4 - 2\tau^3 - 5\tau^2 - 2\tau + 1$ are $(0, 1), (0, -1), (-1, 1), (-1, -1)$, and the two points at infinity, MAGMA, the discriminant cannot be a square in $\mathbb{Q}$.

Finally, assuming $q = x_3(\tau) = -\frac{\tau^3 + 2\tau^2 + 3\tau + 1}{2\tau(\tau+1)}$, for some $\tau \in \mathbb{Q} \setminus \{0, 1\}$, leads us to studying the rational points on the elliptic curve $y^2 = \tau^4 + 2\tau^3 + 7\tau^2 + 6\tau + 1$ which is $\mathbb{Q}$-birational to the elliptic curve in (5.1). Hence, we conclude that if $q = x_i(\tau)$, $i = 1, 2, 3$, for some $\tau \in \mathbb{Q} \setminus \{0, 1\}$, then $\tau$ is uniquely determined. This fact together with the fact that $x_1$, $x_2$, and $x_3$ are permuted under the action of the quadratic map imply that if $q$ is a rational periodic point of exact period $3$ of $f(z) = z^2 + c$, then both the orbit of $q$ and $c$ are determined uniquely. \hfill \Box

Remark 5.2. In the proof of Theorem 5.1, we showed that for distinct $c_1, c_2 \in \mathbb{Q}$, given a rational point $p$ such that $p$ is a periodic point of $z^2 + c_1$ and $z^2 + c_2$ of exact period $m \leq 3$ and $n \leq 3$, respectively, then $m \neq n$.

A rational point $p \in \mathbb{Q}$ is a periodic point of the maps $f(z) = z^2 + p - p^2$ and $f'(z) = z^2 - (p^2 + p + 1)$ of exact periods $1$ and $2$, respectively.

If $p = p_r \in \mathbb{Q}$ is a periodic point of exact period $3$ of the map $f_3(z) = z^2 + c_r$, then $p_r$ is a periodic point of exact period $1$ of $f_1(z) = z^2 + p_r - p_r^2$, and a periodic point of exact period $2$ of $f_2(z) = z^2 - (p_r^2 + p_r + 1)$.

Example 5.3. The maps $f_1(z) = z^2 - \frac{6161}{1000}$, $f_2(z) = z^2 - \frac{15841}{1000}$, and $f_3(z) = z^2 - \frac{7841}{1000}$ have the following cycles $(\frac{101}{40}, \frac{101}{40}, -\frac{141}{40})$, and $(\frac{101}{40}, \frac{59}{40}, -\frac{109}{40})$, respectively.

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