Asymptotics of a vanishing period: the quotient themes of a given fresco.

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Abstract

In this paper we introduce the word ”fresco” to denote a $[\lambda]-$primitive monogenic geometric $(a,b)$-module. The study of this ”basic object” (generalized Brieskorn module with one generator) which corresponds to the minimal filtered (regular) differential equation satisfied by a relative de Rham cohomology class, began in [B.09] where the first structure theorems are proved. Then in [B.10] we introduced the notion of theme which corresponds in the $[\lambda]-$primitive case to frescos having a unique Jordan-Hölder sequence. Themes correspond to asymptotic expansion of a given vanishing period, so to the image of a fresco in the module of asymptotic expansions. For a fixed relative de Rham cohomology class (for instance given by a smooth differential form $d-$closed and $df-$closed) each choice of a vanishing cycle in the spectral eigenspace of the monodromy for the eigenvalue $\exp(-2i\pi.\lambda)$ produces a $[\lambda]-$primitive theme, which is a quotient of the fresco associated to the given relative de Rham class itself. So the problem to determine which theme is a quotient of a given fresco is important to deduce possible asymptotic expansions of the various vanishing period integrals associated to a given relative de Rham class when we change the choice of the vanishing cycle.

In the appendix we prove a general existence result which naturally associate a fresco to any relative de Rham cohomology class of a proper holomorphic function of a complex manifold onto a disc.

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1 Introduction

Let $f : X \to D$ be an holomorphic function on a connected complex manifold. Assume that $\{df = 0\} \subset \{f = 0\} := X_0$. We consider $X$ as a degenerating family of complex manifolds parametrized by $D^* := D \setminus \{0\}$ with a singular member $X_0$ at the origin of $D$. Let $\omega$ be a smooth $(p + 1)$–differential form on $X$ satisfying $d\omega = 0 = df \wedge \omega$. Then in many interesting cases (see for instance [JAG.08] , [B.III] for the case of a function with 1-dimensional singular set and the Appendix for the proper case) the relative family of de Rham cohomology classes induced on the fibers $X_s, s \in D^*$ of $f$ by $\omega/df$ is solution of a minimal filtered differential equation defined from the Gauss-Manin connexion of $f$. This object, called a fresco is a monogenic regular (a,b)-module satisfying an extra condition, called ”geometric”, which encodes simultaneously the regularity at 0 of the Gauss-Manin connexion, the monodromy theorem and B. Malgrange’s positivity theorem.
We study the structure of such an object in order to determine the possible quotient themes of a given fresco. Such a theme corresponds to a possible asymptotic expansion of vanishing periods constructed from $\omega$ by choosing a vanishing cycle $\gamma \in H_p(X_{s_0}, \mathbb{C})$ and putting

$$F_\gamma(s) := \int_{\gamma_s} \omega/df$$

where $\gamma_s$ is the (multivalued) horizontal family of cycles defined from $\gamma$ in the fibers of $f$ (see [M.74]).

We obtain a rather precise description of the themes of these vanishing periods in term of the structure of the fresco associated to $\omega$.

We give in the Appendix the existence theorem of the fresco associated to a smooth $d$-closed and $df$-closed form in the case of a proper holomorphic function on a complex manifold. This key result was written in the preprint [B.08] and was not yet published.

It is also interesting to indicate to the reader that he may find some "algebraic" explicit computations in [B.09] of the fresco generated some monomial in the Brieskorn module of the isolated singularity $x^5 + y^5 + x^2.y^2$ which is one of the simplest example with a not semi-simple monodromy.

2 Some known facts.

2.1 Regular and geometric (a,b)-modules.

The main purpose of this paper is to give a precise description of the structure of a $[\lambda]-$primitive monogenic and geometric (a,b)-module ; we shall call such an object a $[\lambda]-$primitive fresco. It corresponds to the $[\lambda]-$primitive part of the minimal filtered differential equation satisfied by a relative de Rham cohomology class as indicate in the introduction.

Let us first recall the definition of an (a,b)-module$^1$.

Définition 2.1.1 An (a,b)-module $E$ is a free finite rank $\mathbb{C}[[b]]$-module endowed with an $\mathbb{C}$-linear map $a : E \rightarrow E$ which satisfies the following two conditions :

- The commutation relation $a.b - b.a = b^2$.
- The map $a$ is continuous for the $b$-adic topology of $E$.

Remark that these two conditions imply that for any $S \in \mathbb{C}[[b]]$ we have

$$a.S(b) = S(b).a + b^2.S'(b)$$

$^1$For more details on these basic facts, see [B.93]
where \( S' \) is defined via the usual derivation on \( \mathbb{C}[[b]] \). For a given free rank \( k \) \( \mathbb{C}[[b]] \)-module with basis \( e_1, \ldots, e_k \), to define a structure of \((a,b)\)-module it is enough to prescribe (arbitrarily) the values of \( a \) on \( e_1, \ldots, e_k \).

An alternative way to define \((a,b)\)-modules is to consider the \( \mathbb{C} \)-algebra

\[
\tilde{A} := \left\{ \sum_{\nu=0}^{\infty} P_\nu(a).b^\nu \right\}
\]

where the \( P_\nu \) are polynomials in \( \mathbb{C}[z] \) and where the product by \( a \) is left and right continuous for the \( b \)-adic filtration and satisfies the commutation relation \( a.b - b.a = b^2 \).

Then a left \( \tilde{A} \)-module which is free and finite rank on the subalgebra \( \mathbb{C}[[b]] \subset \tilde{A} \) is an \((a,b)\)-module and conversely.

An \((a,b)\)-module \( E \) has a **simple pole** when we have \( a.E \subset b.E \) and it is **regular** when it is contained in a simple pole \((a,b)\)-module. The regularity is equivalent to the finiteness on \( \mathbb{C}[[b]] \) of the saturation \( E^2 \) of \( E \) by \( b^{-1}.a \) in \( E \otimes_{\mathbb{C}[[b]]} \mathbb{C}[[b]][b^{-1}] \).

- Submodules and quotients of regular \((a,b)\)-modules are regular.

Another important property of regular \((a,b)\)-module is the existence of Jordan-Hölder sequences (J-H. sequences for short).

Recall first that any regular rank 1 \((a,b)\)-module is characterized up to isomorphism, by a complex number \( \lambda \) and the corresponding isomorphy class is represented by the \((a,b)\)-module \( E_\lambda := \mathbb{C}[[b]].e_\lambda \) where \( a.e_\lambda = \lambda.b.e_\lambda \), which is isomorphic to the \( \tilde{A} \)-module \( \tilde{A}/\tilde{A}.(a - \lambda.b) \).

Recall also that a submodule \( F \) of the \((a,b)\)-module \( E \) is called **normal** when \( F \cap b.E = b.F \). Normality is a necessary and sufficient condition in order that the quotient \( E/F \) is again an \((a,b)\)-module.

A **Jordan-Hölder sequence** for the rank \( k \) regular \((a,b)\)-module \( E \) is a sequence of normal submodules \( \{0\} = F_0 \subset F_1 \subset \ldots F_{k-1} \subset F_k = E \) such that the quotients \( F_j/F_{j-1} \) for \( j \in [1,k] \) are rank 1 \((a,b)\)-modules. So, to each J-H. sequence of \( E \), we may associate an ordered sequence of complex numbers \( \lambda_1, \ldots, \lambda_k \) such \( F_j/F_{j-1} \simeq E_{\lambda_j} \) for each \( j \in [1,k] \).

Existence of J-H. sequence for any regular \((a,b)\)-module and also the following lemma are proved in [B.93].

**Lemma 2.1.2** Let \( E \) be a regular \((a,b)\)-module of rank \( k \). Up to a permutation, the set \( \{\exp(-2\pi.\lambda_j), j \in [1,k]\} \) is independant of the choice of the J-H. sequence of \( E \). Moreover, the sum \( \sum_{j=1}^{k} \lambda_j \) is also independant of the choice of the J-H. sequence of \( E \).

\(^2\)For \( G \subset F \subset E \) submodules with \( F \) normal in \( E \), the normality of \( G \) in \( F \) is equivalent to the normality of \( G \) in \( E \).
The Bernstein polynomial of a regular \((a, b)\)-module \(E\) of rank \(k\) is defined as the minimal polynomial of \(-b^{-1}.a\) acting on the \(k\)-dimensional \(\mathbb{C}\)-vector space \(E^2/b.E^2\). Of course, when \(E\) is the \(b\)-completion of the Brieskorn module of a non constant germ \(f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)\) of holomorphic function with an isolated singularity, we find the "usual" (reduced) Bernstein polynomial of \(f\) (see for instance [K.76] or [Bj.93]).

We say that a regular \((a, b)\)-module \(E\) is geometric when all roots of its Bernstein polynomial are negative rational numbers. This condition which correspond to M. Kashiwara theorem [K.76], encodes the monodromy theorem and the positivity theorem of B. Malgrange (see [M.75] or the appendix of [B.84]) extending the situation of \((a, b)\)-modules deduced from the Gauss-Manin connection of an holomorphic function.

Recall that the tensor product of two \((a, b)\)-modules \(E\) and \(F\) (see [B.I]) is defined as the \(\mathbb{C}[[b]]\)-module \(E \otimes_{\mathbb{C}[[b]]} F\) with the \(\mathbb{C}\)-linear endomorphism defined by the rule \(a. (x \otimes y) = (a. x) \otimes y + x \otimes (a. y)\). The tensor product by a fix \((a, b)\)-module preserves short exact sequences of \((a, b)\)-modules and \(E_{\lambda} \otimes E_{\mu} \simeq E_{\lambda+\mu}\). So the tensor product of two regular \((a, b)\)-modules is again regular.

Définition 2.1.3 Let \(E\) be a regular \((a, b)\)-module. The dual \(E^*\) of \(E\) is defined as the \(\mathbb{C}[[b]]\)-module \(\text{Hom}_{\mathbb{C}[[b]]}(E, E_0)\) with the \(\mathbb{C}\)-linear map given by

\[
(a.\varphi)(x) = a.\varphi(x) - \varphi(a.x)
\]

where \(E_0 := \mathcal{A}/\mathcal{A}.a \simeq \mathbb{C}[[b]].e_0\) with \(a.e_0 = 0\).

It is an easy exercice to see that \(a\) acts and satisfies the identity \(a.b - b.a = b^2\) on \(E^*\) with the previous definition. We have \(E^*_\lambda \simeq E_{-\lambda}\) and the duality transforms a short exact sequence of \((a, b)\)-modules in a short exact sequence. So the dual of a regular \((a, b)\)-module is again regular. But the dual of a geometric \((a, b)\)-module is almost never geometric. To use duality in the geometric case we shall combine it with tensor product with \(E_N\) where \(N\) is a big enough rational number. Then \(E^* \otimes E_N\) is geometric and \((E^* \otimes E_N)^* \otimes E_N \simeq E\). We shall refer to this process as "twisted duality".

Define now the left \(\mathcal{A}\)-module of "formal multivalued expansions"

\[
\Xi := \bigoplus_{\lambda \in \mathbb{Q} \cap [0, 1]} \Xi_\lambda \quad \text{with} \quad \Xi_\lambda := \bigoplus_{j \in \mathbb{N}} \mathbb{C}[[b]].s^{\lambda-1}.\frac{(\text{Log } s)^j}{j!}
\]

with the action of \(a\) given by

\[
a. (s^{\lambda-1}.\frac{(\text{Log } s)^j}{j!}) = \lambda.b.s^{\lambda-1}.\frac{(\text{Log } s)^j}{j!} + b.s^{\lambda-1}.\frac{(\text{Log } s)^{j-1}}{(j-1)!}
\]

for \(j \geq 1\) and \(a.s^{\lambda-1} = \lambda.b(s^{\lambda-1})\), with, of course, the commutation relations \(a.S(b) = S(b).a + b^2.S'(b)\) for \(S \in \mathbb{C}[[b]]\).
For any geometric (a,b)-module of rank k, the vector space $\text{Hom}_{\tilde{A}}(E, \Xi)$ is of dimension k and this functor transforms short exact sequences of geometric (a,b)-modules in short exact sequences of finite dimensional vector spaces (see [B.05] for a proof).

In the case of the Brieskorn module of an isolated singularity germ of an holomorphic function f at the origin of $C^{n+1}$ this vector space may be identified with the n-th homology group (with complex coefficients) of the Milnor’s fiber of f (see [B.05]). The correspondance is given by associating to a (vanishing) cycle $\gamma$ the $\tilde{A}$-linear map $[\omega] \mapsto \left\lbrack \int_{\gamma_s} \omega / df \right\rbrack \in \Xi$ where $\omega \in \Omega^{n+1}_0$, $\gamma_s$ is the multivalued horizontal family of n-cycles defined by $\gamma$ in the fibers of $f$, and $[g]$ denotes the formal asymptotic expansion at $s = 0$ of the multivalued holomorphic function $g$.

**Définition 2.1.4** A regular (a,b)-module is $[\lambda]$–primitive (resp. $[\Lambda]$–primitive), where $[\lambda]$ is an element (resp. a subset) in $\mathbb{C}/\mathbb{Z}$, if all roots of its Bernstein polynomial are in $[-\lambda]$ (resp. in $[-\Lambda]$).

If we have a short exact sequence of (a,b)-modules

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

with $E$ regular (resp. geometric, resp. $[\lambda]$–primitive) then $F$ and $G$ are regular (resp. geometric, resp. $[\lambda]$–primitive).

Conversely if $F$ and $G$ are regular (resp. geometric, resp. $[\lambda]$–primitive) then $E$ is regular (resp. geometric, resp. $[\lambda]$–primitive).

This implies that $E$ is $[\lambda]$–primitive if and only if it admits a J-H. sequence such that all numbers $\lambda_1, \ldots, \lambda_k$ are in $[\lambda]$. And then any J-H. sequence of $E$ has this property.

The following proposition is proved in [B.09] section 1.3.

**Proposition 2.1.5** Let $E$ be a regular (a,b)-module and fix a subset $\Lambda$ in $\mathbb{C}/\mathbb{Z}$. Then there exists a maximal submodule $E[\Lambda]$ in $E$ which is $[\Lambda]$–primitive. This submodule is normal in $E$.

If $\{[\lambda_1], \ldots, [\lambda_d]\}$ is the image in $\mathbb{C}/\mathbb{Z}$ of the set of the opposite of roots of the Bernstein polynomial of $E$, given with an arbitrary order, there exists a unique sequence $0 = F_0 \subset F_1 \subset F_2 \subset F_d = E$ of normal submodules of $E$ such that $F_j/F_{j-1}$ is $[\lambda_j]$–primitive for each $j \in [1,d]$.

We call $E[\Lambda]$ the $[\Lambda]$–primitive part of $E$.
Thanks to this result, to understand what are the possible \([\lambda]\)–primitive themes which are quotient of a given fresco, it will be enough to work with \([\lambda]\)–primitive frescos.

Remark also that, in the geometric situation, the choice of a vanishing cycle which belongs to the generalized eigenspace of the monodromy for the eigenvalue \(\exp(-2i\pi.\lambda)\) produces vanishing periods with \([\lambda]\)–primitive themes.

### 2.2 Frescos and themes.

Here we recall some results from [B.09] and [B.10].

#### Définition 2.2.1
We shall call a fresco a geometric \((a,b)\)-module which is generated by one element as an \(\tilde{\mathbb{A}}\)–module.

#### Définition 2.2.2
We shall call a theme a fresco which is a submodule of \(\Xi\).

Recall that a normal submodule and a quotient by a normal submodule of a fresco (resp. of a theme) is a fresco (resp. is a theme).

A regular rank 1 \((a,b)\)-module is a fresco if and only if it is isomorphic to \(E_{\lambda}\) for some \(\lambda \in \mathbb{Q}^{+\ast}\). All rank 1 frescos are themes. The classification of rank 2 regular \((a,b)\)-modules given in [B.93] gives the list of \([\lambda]\)–primitive rank 2 frescos which is the following, where \(\lambda_1 > 1\) is a rational number:

\[
E = E \simeq \tilde{\mathbb{A}}/\tilde{\mathbb{A}}.(a-\lambda_1.b).(a-(\lambda_1-1).b) \quad (1)
\]
\[
E \simeq \tilde{\mathbb{A}}/\tilde{\mathbb{A}}.(a-\lambda_1.b).\left(1+\alpha.b^p\right)^{-1}.(a-(\lambda_1+p-1).b) \quad (2)
\]

where \(p \in \mathbb{N} \setminus \{0\}\) and \(\alpha \in \mathbb{C}\).

The themes in this list are these in (1) and these in (2) with \(\alpha \neq 0\). For a \([\lambda]\)–primitive theme in case (2) the number \(\alpha \neq 0\) will be called the parameter of the theme.

For frescos we have a more precise result on the numbers associated to a J-H. sequence:

#### Proposition 2.2.3
Let \(E\) be a \([\lambda]\)–primitive fresco and \(\lambda_1, \ldots, \lambda_k\) be the numbers associated to a J-H. sequence of \(E\). Then, up to a permutation, the numbers \(\lambda_j + j, j \in [1,k]\) are independent of the choice of the J-H. sequence.

The following structure theorem for frescos will be useful in the sequel.

#### Théorème 2.2.4
(see [B.09] th.3.4.1) Let \(E\) be a fresco of rank \(k\). Then there exists an element \(P \in \tilde{\mathbb{A}}\) which may be written as

\[
P := (a-\lambda_1.b)S_{-1}^{-1}.(a-\lambda_2.b) \ldots S_{k-1}^{-1}.(a-\lambda_k.b)
\]
such that $E$ is isomorphic to $\bar{A}/\bar{A}.P$. Here $S_1, \ldots, S_{k-1}$ are elements in $\mathbb{C}[b]$ such that $S_j(0) = 1$ for each $j \in [1, k-1]$. The element $P_E := (a - \lambda_1.b) \ldots (a - \lambda_k.b)$ of $\bar{A}$ is homogeneous in $(a,b)$ and gives the Bernstein polynomial $B_E$ of $E$ via the formula

$$(-b)^k.P_E = B_E(-b^{-1}.a).$$

So $P_E \in \bar{A}$ depends only on the isomorphism class of $E$.

Note that in the case of a fresco the Bernstein polynomial of $E$ is equal to the characteristic polynomial of the action of $-b^{-1}.a$ on $E\sharp$. This allows a nice formula for a short exact sequence of frescos:

**Proposition 2.2.5** (see [B.09] prop.3.4.4) Let $0 \to F \to E \to G \to 0$ be a short exact sequence of frescos. Then we have the equality in $\bar{A}$:

$$P_E = P_F.P_G$$

which is equivalent to $B_E(x) = B_F(x - rk(G)).B_G(x)$.

The situation for a $[\lambda]-$primitive theme is more rigid:

**Proposition 2.2.6** A fresco $E$ is a $[\lambda]-$primitive theme if and only if it admits a unique normal rank 1 submodule. In this case the J-H. sequence is unique and contains all normal submodules of $E$. The corresponding numbers $\lambda_1, \ldots, \lambda_k$ are such that the sequence $\lambda_j + j$ is increasing (may-be not strictly).

### 3 Commutation in Jordan-Hölder sequences.

In this section we shall study the possible different J-H. sequences of a given $[\lambda]-$primitive fresco. Thanks to proposition 2.1.5 it is easy to see that the $[\lambda]-$primitive assumption does not reduce the generality of this study.

#### 3.1 The principal Jordan-Hölder sequence.

**Définition 3.1.1** Let $E$ be a $[\lambda]-$primitive fresco of rank $k$ and let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_k = E$$

be a J-H. sequence of $E$. Then for each $j \in [1, k]$ we have $F_j/F_{j-1} \simeq E_{\lambda_j}$, where $\lambda_1, \ldots, \lambda_k$ are in $\lambda + \mathbb{N}$. We shall say that such a J-H. sequence is **principal** when the sequence $[1, k] \ni j \mapsto \lambda_j + j$ is increasing.

It is proved in [B.09] prop. 3.5.2 that such a principal J-H. sequence exists for any $[\lambda]-$primitive fresco. Moreover, the corresponding sequence $\lambda_1, \ldots, \lambda_k$ is unique. The following proposition shows much more.
Proposition 3.1.2 Let $E$ be a $[\lambda]$–primitive fresco. Then its principal J-H. sequence is unique.

We shall prove the uniqueness by induction on the rank $k$ of $E$.

We begin by the case of rank 2.

Lemma 3.1.3 Let $E$ be a rank 2 $[\lambda]$–primitive fresco and let $\lambda_1, \lambda_2$ the numbers corresponding to a principal J-H. sequence of $E$ (so $\lambda_1 + 1 \leq \lambda_2 + 2$). Then the normal rank 1 submodule of $E$ isomorphic to $E_{\lambda_1}$ is unique.

Proof. The case $\lambda_1 + 1 = \lambda_2 + 2$ is obvious because then $E$ is a $[\lambda]$–primitive theme (see [B.10] corollary 2.1.7). So we may assume that $\lambda_2 = \lambda_1 + p_1 - 1$ with $p_1 \geq 1$ and that $E$ is the quotient $E \simeq \tilde{A}/\tilde{A}.(a - \lambda_1.b).(a - \lambda_2.b)$ (see the classification of rank 2 frescos with $e_2 = [1]$ and $e_1 = (a - \lambda_2.b).e_2$). Let look for $x := U.e_2 + V.e_1$ such that $(a - \lambda_1.b).x = 0$. Then we obtain

$$b^2.U'.e_2 + U.(a - \lambda_2.b).e_2 + (\lambda_2 - \lambda_1).b.U.e_2 + b^2.V'.e_1 = 0$$

which is equivalent to the two equations:

$$b^2.U' + (p_1 - 1).b.U = 0 \quad \text{and} \quad U + b^2.V' = 0$$

The first equation gives $U = 0$ for $p_1 \geq 2$ and $U \in \mathbb{C}$ for $p_1 = 1$. As the second equation implies $U(0) = 0$, in all cases $U = 0$ and $V \in \mathbb{C}$. So the solutions are in $\mathbb{C}.e_1$. ■

Remark that in the previous lemma, if we assume $p_1 \geq 1$ and $E$ is not a theme, it may exist infinitely many different normal (rank 1) submodules isomorphic to $E_{\lambda_2+1}$. But then, $\lambda_2 + 2 > \lambda_1 + 1$. See remark 2 following 3.2.1.

Proof of Proposition 3.1.2. As the result is obvious for $k = 1$, we may assume $k \geq 2$ and the result proved in rank $\leq k - 1$. Let $F_j, j \in [1, k]$ and $G_j, j \in [1, k]$ two J-H. principal sequences for $E$. As the sequences $\lambda_j + j$ and $\mu_j + j$ coincide up to the order and are both increasing, they coincide. Now let $j_0$ be the first integer in $[1, k]$ such that $F_{j_0} \neq G_{j_0}$. If $j_0 \geq 2$ applying the induction hypothesis to $E/F_{j_0-1}$ gives $F_{j_0}/F_{j_0-1} = G_{j_0}/F_{j_0-1}$ and so $F_{j_0} = G_{j_0}$.

So we may assume that $j_0 = 1$. Let $H$ be the normalization of $F_1 + G_1$. As $F_1$ is the smallest normal submodule containing $F_1 + G_1$; it has the same rank than $F_1 + G_1$.\[\text{3}\]
and $G_1$ are normal rank 1 and distinct, then $H$ is a rank 2 normal submodule. It is a $[\lambda]-$primitive fresco of rank 2 with two normal rank 1 sub-modules which are isomorphic as $\lambda_1 = \mu_1$. Moreover the principal J-H. sequence of $H$ begins by a normal submodule isomorphic to $E_{\lambda_1}$. So the previous lemma implies $F_1 = G_1$. So for any $j \in [1, k]$ we have $F_j = G_j$. \hfill \blacksquare

**Définition 3.1.4** Let $E$ be a $[\lambda]-$primitive fresco and consider a J-H. sequence $F_j, j \in [1, k]$ of $E$. Put $F_j/F_{j-1} \simeq E_{\lambda_j}$ for $j \in [1, k]$ (with $F_0 = \{0\}$). We shall call fundamental invariants of $E$ the (unordered) k-tuple $\{\lambda_j + j, j \in [1, k]\}$.

Of course this definition makes sens because we know that this (unordered) k-tuple is independant of the choice of the J-H. of $E$.

Note that when $E$ is a theme, the uniqueness of the J-H. gives a natural order on this k-tuple. So in the case of a theme the fundamental invariants will be an ordered k-tuple. With this convention, this is compatible with the definition of the fundamental invariants of a $[\lambda]-$primitive theme given in [B.10], up to a shift.

In the opposite direction, when $E$ is a semi-simple $[\lambda]-$primitive fresco we shall see (in section 4) that any order of this k-tuple may be realized by a J-H. sequence of $E$.

### 3.2 Commuting in $\tilde{A}$.

Let $E$ be a $[\lambda]-$primitive fresco. Any isomorphism $E \simeq \tilde{A}/\tilde{A}P$ where $P \in \tilde{A}$ is given by

$$P := (a - \lambda_1.b)S_1^{-1}(a - \lambda_2.b)\ldots S_{k-1}^{-1}(a - \lambda_k.b)$$

determines a J-H. sequence for $E$ associated to the $\mathbb{C}[[b]]$-basis $e_1, \ldots, e_k$ such that the relations $(a - \lambda_j.b).e_j = S_{j-1}.e_{j-1}$ hold for $j \in [1, k]$ with the convention $e_0 = 0$, and the fact that $e_k$ corresponds, via the prescribed isomorphism, to the class of 1 modulo $\tilde{A}P$.

**Lemme 3.2.1** Let $p_1$ and $p_2$ be two positive integers and let $\lambda_1 \in \lambda + 2 + \mathbb{N}$, where $\lambda \in [0, 1] \cap \mathbb{Q}$. Define $P \in \tilde{A}$ as

$$P := (a - \lambda_1.b)S_1^{-1}(a - \lambda_2.b)S_2^{-1}(a - \lambda_3.b)$$

where $\lambda_{j+1} := \lambda_j + p_j - 1$ for $j = 1, 2$, and where $S_1, S_2$ lie in $\mathbb{C}[[b]]$ and satisfy $S_1(0) = S_2(0) = 1$. We assume that the coefficient of $b^{p_1}$ in $S_1$ vanishes and that the coefficient of $b^{p_2}$ of $S_2$ is $\alpha \neq 0$.

Then if $U \in \mathbb{C}[[b]]$ is any solution of the differential equation $b.U' = p_1.(U - S_1)$, we have

$$P = U^{-1}.(a - (\lambda_2 + 1).b).(S_1.U^{-2})^{-1}.(a - (\lambda_1 - 1).b).(U.S_2)^{-1}.(a - \lambda_3.b).$$

Moreover, there exists an unique choice of $U$ such that the coefficient of $b^{p_1+p_2}$ in $U.S_2$ vanishes.
Moreover, the coefficient of \( b \) in the situation of the previous lemma choose Corollaire 3.2.2

\[
V \text{ denote by } P
\]

Then \( P \) is proved in [B.09] lemma 3.5.1. We use here the case \( \delta := \lambda - \mu = \lambda_2 + 1 - \lambda_1 = p_1 \) and the fact that the coefficient of \( b \) in \( S_1 \) vanishes.

As the solution \( U \) is unique up to \( \mathbb{C}.b^{p_1} \), to prove the second assertion let \( U_0 \) be the solution with no term in \( b^{p_1} \). Now the coefficient \( \beta(\rho) \) of \( b^{p_1+p_2} \) in \( S_2.U \) where \( U := U_0 + \rho.b^{p_1} \), is \( \beta(\rho) = \beta(0) + \rho.a. \) As we assumed that \( \alpha \neq 0 \) there exists an unique choice of \( \rho \) for which \( \beta(\rho) = 0. \)

Remarks.

1. In the situation of the previous lemma the rank 3 fresco \( E := \tilde{A}/\tilde{A}.P \) is an extension
\[
0 \to E_{\lambda_1} \to E \to T_{\lambda_2,p_2}(\alpha) \to 0
\]
where \( T_{\lambda_2,p_2}(\alpha) \) is the rank 2 theme with fundamental invariants \( (\lambda_2, p_2) \) and parameter \( \alpha \) (see the definition [2.2.2]), so
\[
T_{\lambda_2,p_2}(\alpha) \simeq \tilde{A}/\tilde{A}.(a - \lambda_2,b).(1 + \alpha.b^{p_2})^{-1}.(a - (\lambda_2 + p_2 - 1).b).
\]

2. Let \( \xi \) be in \( \mathbb{C}^* \) and choose \( \rho := (\xi - \beta_0)/\alpha \) in the previous proof. Then \( E \) is a extension
\[
0 \to E_{\lambda_2+1} \to E \to T_{\lambda_1-1,p_1+p_2}(\xi) \to 0
\]
where \( T_{\lambda_1-1,p_1+p_2}(\xi) \) is the rank 2 theme with fundamental invariants \( (\lambda_1-1, p_1+p_2) \) and parameter \( \xi \). This shows that we may have infinitely many non isomorphic rank 2 themes as quotients of a given rank 3 \([\lambda]\)-primitive fresco \( E \). We have also infinitely many different J-H. sequences with the same quotients: \( (\lambda_2 + 1, \lambda_1 - 1, \lambda_3) \).

Note that in this situation we have\(^4\) \( \dim_{\mathbb{C}}[Ker(a - (\lambda_2 + 1).b)] = 2. \)

Corollaire 3.2.2 In the situation of the previous lemma choose \( \rho = -\beta(0).\alpha \) and denote by \( V \) a solution\(^5\) of the differential equation \( b.V' = (p_1 + p_2).V - U.S_2. \)
Then \( P \) is equal to
\[
U^{-1}.(a - (\lambda_2 + 1).b).S_1^{-1}.U^2.V^{-1}.(a - (\lambda_3 + 1).b).(US_2V^{-2})^{-1}.(a - (\lambda_1 - 2).b).V^{-1}. \quad(\@)
\]
Moreover, the coefficient of \( b^{p_2} \) in \( S_1.U^{-2}.V \) is \( (p_1 + p_2).\alpha/p_1. \)

\(^4\)It is easy to see that if \( Z \) is a solution in \( \mathbb{C}[[b]] \) of the differential equation \( b.Z' - p_1.Z + S_1 = 0, \)
then
\[
Ker(a - (\lambda_2 + 1).b) = \{(r,s) \in \mathbb{C}^2 \mid r.(Z.e_1 + b.e_2) + s.b^{p_1}.e_1\}.\]

\(^5\)Note that as \( U.S_2 \) has no term in \( b^{p_1+p_2} \) with our choice of \( \rho \), such a solution exists in \( \mathbb{C}[[b]] \). Moreover \( V(0) = (U.S_2)(0) = 1 \) because \( U(0) = S_1(0) = 1 = S_2(0). \)
Proof. Of course the choice of $\rho$ allows to apply again the lemma 3.5.1. of [B.09], with now $\delta = \lambda_3 + 1 - (\lambda_1 - 1) = p_1 + p_2$. This gives (\@). Using $b.U' = p_1.(U - S_1)$ we get
\[
\begin{align*}
b.U'.U^{-2} &= p_1.(U^{-1} - S_1.U^{-2}) \quad \text{and with } Z := U^{-1} \\
b.Z' &= -p_1.(Z - S_1.U^{-2}) \quad \text{and then} \\
b.Z'.V &= -p_1.(Z.V - S_1.U^{-2}.V) \quad (\@@)
\end{align*}
\]
But using also $b.V' = (p_1 + p_2).(V - U.S_2)$ we get
\[
b.V'.Z = (p_1 + p_2).(V.Z - S_2).
\]
Adding with (\@@) gives
\[
b.(V.Z)' - p_2.V.Z = p_1.S_1.U^{-2}.V - (p_1 + p_2).S_2
\]
which leads to the result, because the left handside has no term in $b^{p_2}$.
\[\blacksquare\]

An obvious consequence of this corollary is that there exists in $E$ a normal sub-theme isomorphic to $T_{\lambda_2 + 1, p_2}((1 + p_2/p_1).\alpha)$, so with fundamental invariants $(\lambda_2 + 1, p_2)$ and with parameter $(1 + p_2/p_1).\alpha$.

Recall that $T_{\lambda_2, p_2}(\alpha)$ was the rank 2 quotient theme which appears in the principal J-H. of $E$.

### 3.3 Some examples.

We shall give here some examples, showing the complexity of the non commutative structure of the algebra $\tilde{A}$.

**Lemme 3.3.1** Let $x, y, z$ non zero complex numbers, $\lambda_1$ a rational number bigger than 3 and $p_1, p_2, p_3$ three positive distinct integers. Assume that $p_3$ is not a multiple of $p_2$, and define $\lambda_{j+1} := \lambda_j + p_j - 1$ for $j = 1, 2, 3$. Put
\[
\begin{align*}
R_1 &= 1 + x.b^{p_1}, \\
R_3 &= 1 + y.b^{p_3} + z.b^{p_2 + p_3}, \\
U &= 1 - \frac{z}{y}.b^{p_2},
\end{align*}
\]
\[
S := R_1.U \quad \text{and } T := U.R_3.
\]
Then $T$ has no term in $b^{p_2 + p_3}$ and there exists a solution $V \in \mathbb{C}[[b]]$ of the differential equation $b.V' = (p_2 + p_3).(V - T)$.

Then the element $P := (a - \lambda_1.b).R_1^{-1}.(a - \lambda_2.b).(a - \lambda_3.b).R_3^{-1}.(a - \lambda_4.b)$ in $\tilde{A}$ is equal to
\[
(a - \lambda_1.b).S^{-1}.(a - (\lambda_3 + 1).b).U^2.V^{-1}.(a - (\lambda_4 + 1).b).T^{-1}.V^2.(a - (\lambda_2 - 2).b).V^{-1}. \quad (1)
\]
\textbf{Proof.} A simple computation gives
\[ T = 1 - \frac{z}{y} b^{p_2} + y b^{p_3} - \frac{z^2}{y} b^{2p_2+p_3} \]
and
\[ V = 1 - \frac{z p_2 + p_3}{y} b^{p_2} + \frac{p_2 + p_3}{p_2} y b^{p_3} + \rho b^{p_2+p_3} + p_2 \frac{z^2}{y} b^{2p_2+p_3} \]
where \( \rho \) is an arbitrary complex number.

Using the lemma 3.5.1. of [B.09] and the fact that \( U \) satisfies \( b.U' = p_2(U - 1) \) we get
\[ P = (a - \lambda_1 b).S^{-1}.(a - (\lambda_3 + 1) b).U^2.(a - (\lambda_2 - 1) b).T^{-1}.(a - \lambda_4 b). \]
As \( \lambda_4 = (\lambda_2 - 1) + p_2 + p_3 - 1 \) and \( T \) has no term in \( b^{p_2+p_3} \), we obtain, using again the lemma of \textit{loc. cit.}
\( (a - (\lambda_2 - 1) b).T^{-1}.(a - \lambda_4 b) = V^{-1}.(a - (\lambda_4 + 1) b)T^{-1}.V^2.(a - (\lambda_2 - 2) b) \)
if \( V \) is a solution of \( b.V' = (p_2 + p_3).(V - T) \); this implies (1).

\textbf{Lemme 3.3.2} In the situation of the previous lemma the rank 4 fresco given by \( E := \tilde{\mathcal{A}} \backslash \mathcal{A}.P \) is not a theme, but we have the following exact sequences:
\[ 0 \rightarrow T_1 \rightarrow E \rightarrow T_2 \rightarrow 0 \]
\[ 0 \rightarrow T_3 \rightarrow E \rightarrow E_{\lambda_2-2} \rightarrow 0. \]
where \( T_1 \) and \( T_2 \) are rank 2 themes and \( T_3 \) a rank 3 theme.

\textbf{Proof.} The first exact sequence is consequence of the definition of \( P \), and the rank 2 theme \( T_1 \) has \( (\lambda_1, p_1) \) as fundamental invariants and \( x \) as parameter; the rank 2 theme \( T_2 \) has \( (\lambda_3, p_3) \) as fundamental invariants and \( y \) as parameter.
Let \( e \) be a generator of \( E \) whose annihilator is \( \tilde{\mathcal{A}}.P \). Then the relation (1) shows that \( \varepsilon := T^{-1}.V^2.(a - (\lambda_2 - 2) b).V^{-1}.e \) in \( E \) is annihilated by
\[ Q := (a - \lambda_1 b).S^{-1}.(a - (\lambda_3 + 1) b).U^2.V^{-1}.(a - (\lambda_4 + 1) b). \]
So \( \tilde{\mathcal{A}}.\varepsilon \) has rank 3 and is normal because \( E \backslash \tilde{\mathcal{A}}.\varepsilon \simeq E_{\lambda_2-2} \). We shall prove that \( \tilde{\mathcal{A}}.\varepsilon \) is a theme. As \( \lambda_3 + 1 = \lambda_1 + p_1 + p_2 - 1 \) and \( \lambda_4 + 1 = (\lambda_3 + 1) + p_3 - 1 \), it is enough to check that the coefficient of \( b^{p_1+p_2} \) in \( S \) and the coefficient in \( b^{p_3} \) in \( U^{-2}V \) do not vanish. As we have
\[
\begin{align*}
\text{i)} & \quad S = 1 + x.b^{p_1} - \frac{z}{y} b^{p_2} - \frac{z^2}{y} b^{p_1+p_2} \\
\text{ii)} & \quad V = 1 - \frac{z}{y} \frac{p_2+p_3}{p_2} b^{p_2} + \frac{p_2+p_3}{p_2} y b^{p_3} + \rho b^{p_2+p_3} + p_2 \frac{z^2}{y} b^{2p_2+p_3} \\
\text{iii)} & \quad U^{-2} = \sum_{n=1}^{\infty} n.\left(\frac{z}{y} b^{p_2}\right)^{n-1}
\end{align*}
\]
these coefficients are respectively equal to \( -\frac{x}{z} \) and \( \frac{p_2+p_3}{p_2} y \) using the fact that \( p_3 \) is not a multiple of \( p_2 \).
Remark. Choosing for instance $U = 1$ gives

$$P = (a - \lambda_1.b).R_1^{-1} \cdot (a - (\lambda_3 + 1).b). (a - (\lambda_2 - 1).b). R_3^{-1}.(a - \lambda_4.b)$$

and then if $W$ is a solution of the differential equation $b.W' = (p_1 + p_2).(W - R_1)$ we obtain

$$P = W^{-1}.(a-(\lambda_3+2).b). (R_1.W^{-2})^{-1}.(a-(\lambda_1-1).b).W^{-1}.(a-(\lambda_2-1).b).R_3^{-1}.(a-\lambda_4.b)$$

and we have an exact sequence

$$0 \to E_{\lambda_3+2} \to E \to T_4 \to 0$$

where $T_4$ is a rank 3 theme and where the corresponding J-H. sequence associated to this exact sequence satifies $F_2 = S_1(E)$ where $S_1(E)$ is the maximal semi-simple normal submodule of $E$ (see section 4).

Note that the first exact sequence corresponds to the principal J-H. of $E$.

The second gives a J-H. sequence such that its quotients correspond to the order $\lambda_1 + 1, \lambda_3 + 3, \lambda_4 + 4, \lambda_2 + 2$ of the increasing sequence $\lambda_j + j, j \in [1, 4]$. The last sequence above corresponds to the order $\lambda_3 + 3, \lambda_1 + 1, \lambda_2 + 2, \lambda_4 + 4$. In this example the semi-simple depth $d(E)$ of $E$ (see section 4) is equal to 3.

Exemple. We give here an example a $[\lambda]-$primitive fresco of rank 4 with a J-H. sequence having no non commuting index but which is not semi-simple (see the section 4 below).

Let $\lambda_1 > 4$ be a rational number and $p_1, p_2, p_3$ be strictly positive integers. Then consider the fresco

$$E := \tilde{A}/(a - \lambda_1.b).(a - \lambda_2.b).(1 + b^{p_2+p_3})^{-1}.(a - \lambda_3.b).(a - \lambda_4.b)$$

where we define $\lambda_{j+1} = \lambda_j + p_j - 1$ for $j = 1, 2, 3$. Then it is clear that all indices of the principal J-H. sequence of $E$ are commuting indices: for $i = 1$ and $i = 3$ this is obvious, for $i = 2$ this results from the commuting lemma 3.5.1 of [B.09] and the fact that $p_2 + p_3 > p_3$ as we assume $p_2 \geq 1$. Now we have the equality in $\tilde{A}$:

$$(a - \lambda_1.b).(a - \lambda_2.b).(1 + b^{p_1+p_2})^{-1}.(a - \lambda_3.b).(a - \lambda_4.b) =$$

$$(a - \lambda_1.b).(a - \lambda_2.b).(1 + b^{p_1+p_2})^{-1}.(a - (\lambda_4 + 1).b).(a - (\lambda_3 - 1).b).$$

This shows, because $\lambda_4 + 1 = \lambda_2 + p_2 + p_3 - 1$, that there exists a subquotient of rank 2 of $E$ which is a theme; so $E$ is not semi-simple. In fact, we produce another J-H. sequence with one non commuting index!
4 Semi-simple frescos.

4.1 The semi-simple filtration.

Définition 4.1.1 We shall say that a fresco $E$ is semi-simple if any quotient of $E$ which is a $[\lambda]-$primitive theme for some $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ is of rank $\leq 1$.

Remarks.

1. A $[\lambda]-$primitive theme is semi-simple if and only if it has rank $\leq 1$.

2. An equivalent definition of a semi-simple fresco is to ask that any $\tilde{A}$-linear map

$$\varphi : E \rightarrow \Xi_{\lambda}$$

for some $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ has rank $\leq 1$. This is a necessary condition because $\varphi(E)$ is a $[\lambda]-$primitive theme which is a quotient of $E$. The converse comes from the fact that any $[\lambda]-$primitive theme admits an injective $\tilde{A}$-linear map in $\Xi_{\lambda}$.

3. A fresco is semi-simple if and only if for each $[\lambda]$ its $[\lambda]-$primitive part (see the proposition 2.1.5) is semi-simple: if $E[\lambda]$ is the $[\lambda]-$primitive part of $E$ the restriction map $\text{Hom}_{\tilde{A}}(E, \Xi_{\lambda}) \rightarrow \text{Hom}_{\tilde{A}}(E[\lambda], \Xi_{\lambda})$ is an isomorphism. For instance, a theme with only rank $\leq 1$ $[\lambda]-$primitive part for each $[\lambda] \in \mathbb{C}/\mathbb{Z}$ is semi-simple.

Lemme 4.1.2 For any $F \subset E$ a normal submodule of a semi-simple fresco $E$, $F$ and $E/F$ are semi-simple frescos. So any sub-quotient$^6$ and of a semi-simple fresco is again a semi-simple fresco.

Proof. As any $\tilde{A}$-linear map $\psi : F \rightarrow \Xi_{\lambda}$ extends to a $\tilde{A}$-linear map $\varphi : E \rightarrow \Xi_{\lambda}$ (see section 2 or [B.05]) the semi-simplicity of $E$ implies the semi-simplicity of $F$. The semi-simplicity of $E/F$ is obvious. $\blacksquare$

Corollaire 4.1.3 Let $E$ be a semi-simple fresco with rank $k$ and let $\lambda_1, \ldots, \lambda_k$ be the numbers associated to a J-H. sequence of $E$. Let $\mu_1, \ldots, \mu_k$ be a twisted permutation$^7$ of $\lambda_1, \ldots, \lambda_k$. Then there exists a J-H. sequence for $E$ with quotients corresponding to $\mu_1, \ldots, \mu_k$.

$^6$By a subquotient $H$ we mean that there exists $G \subset F$ normal submodules in $E$ such that $H := F/G$. Remark that $H$ is a quotient of a normal submodule but also a submodule of a quotient of $E$, as $F/G \subset E/G$.

$^7$This means that the sequence $\mu_j + j, j \in [1, k]$ is a permutation (in the usual sens) of $\lambda_j + j, j \in [1, k]$. 

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Proof. As the symmetric group \( \mathfrak{S}_k \) is generated by the transpositions \( t_{j,j+1} \) for \( j \in [1,k-1] \), it is enough to show that, if \( E \) has a J-H. sequence with quotients given by the numbers \( \lambda_1, \ldots, \lambda_k \) then there exists a J-H. sequence for \( E \) with quotients \( \lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1} + 1, \lambda_j - 1, \lambda_{j+2}, \ldots, \lambda_k \) for \( j \in [1,k-1] \). But \( G := F_{j+1}/F_{j-1} \) is a rank 2 sub-quotient of \( E \) with an exact sequence

\[
0 \to E_{\lambda_j} \to G \to E_{\lambda_{j+1}} \to 0.
\]

As \( G \) is a rank 2 semi-simple fresco, it admits also an exact sequence

\[
0 \to G_1 \to G \to G/G_1 \to 0
\]

with \( G_1 \simeq E_{\lambda_{j+1}} \) and \( G/G_1 \simeq E_{\lambda_j-1} \). Let \( q : F_{j+1} \to G \) be the quotient map. Now the J-H. sequence for \( E \) given by

\[
F_1, \ldots, F_{j-1}, q^{-1}(G_1), F_{j+1}, \ldots, F_k = E
\]

satisfies our requirement. \( \Box \)

Proposition 4.1.4 Let \( E \) be a \([\lambda]\)-primitive fresco. A necessary and sufficient condition in order that \( E \) is semi-simple is that it admits a J-H. sequence with quotient corresponding to \( \mu_1, \ldots, \mu_k \) such that the sequence \( \mu_j \) is strictly decreasing.

Remarks.

1. As a fresco is semi-simple if and only if for each \([\lambda]\) its \([\lambda]\)-primitive part is semi-simple, this proposition gives also a criterium to semi-simplicity for any fresco.

2. This criterium is a very efficient tool to produce easily examples of semi-simple frescos.

Proof. Remark first that if we have, for a \([\lambda]\)-primitive fresco \( E \), a J-H. sequence \( F_j, j \in [1,k] \) such that \( \lambda_j + j = \lambda_{j+1} + j + 1 \) for some \( j \in [1,k-1] \), then \( F_{j+1}/F_{j-1} \) is a sub-quotient of \( E \) which is a \([\lambda]\)-primitive theme of rank 2. So \( E \) is not semi-simple, thanks to the previous corollary. So when a \([\lambda]\)-primitive fresco \( E \) is semi-simple the principal J-H. sequence corresponds to a strictly increasing sequence \( \lambda_j + j \). Now, thanks again to the previous corollary we may find a J-H. sequence for \( E \) corresponding to the strictly decreasing order for the sequence \( \lambda_j + j \).

No let us prove the converse. We shall use the following lemma.

Lemme 4.1.5 Let \( F \) be a rank \( k \) semi-simple \([\lambda]\)-primitive fresco and let \( \lambda_j + j \) the strictly increasing sequence corresponding to its principal J-H. sequence. Let \( \mu \in [\lambda] \) such that \( 0 < \mu + k + 1 < \lambda_1 + 1 \). Then any fresco \( E \) in an exact sequence

\[
0 \to F \to E \to E_\mu \to 0
\]

is semi-simple (and \([\lambda]\)-primitive).
PROOF. Assume that we have a rank 2 quotient \( \varphi : E \to T \) where \( T \) is a \([\lambda]-\)primitive theme. Then \( \text{Ker} \varphi \cap F \) is a normal submodule of \( F \) of rank \( k-2 \) or \( k-3 \). If \( \text{Ker} \varphi \cap F \) is of rank \( k-3 \), the rank of \( F/\text{Ker} \varphi \cap F \) is 2 and it injects in \( T \) via \( \varphi \). So \( F/\text{Ker} \varphi \cap F \) is a rank 2 \([\lambda]-\)primitive theme. As it is semi-simple, because \( F \) is semi-simple, we get a contradiction.

So the rank of \( F/\text{Ker} \varphi \cap F \) is 1 and we have an exact sequence

\[
0 \to F/\text{Ker} \varphi \cap F \to T \to E/F \to 0.
\]

Put \( F/\text{Ker} \varphi \cap F \simeq E_\lambda \). Because \( T \) is a \([\lambda]-\)primitive theme, we have the inequality \( \lambda + 1 \leq \mu + 2 \). But we know that \( \lambda_1 + 1 \leq \lambda + k \) because \( \lambda + k \) is in the set \( \{\lambda_j + j, j \in [1, k]\} \) and \( \lambda_1 + 1 \) is the infimum of this set. So \( \lambda_1 + 1 \leq \mu + k + 1 \) contradicting our assumption that \( \mu + k + 1 < \lambda_1 + 1 \).

END OF PROOF OF THE PROPOSITION [4.1.4]. Now we shall prove by induction on the rank of a \([\lambda]\)-primitive fresco \( E \) that if it admits a J-H. sequence corresponding to a strictly decreasing sequence \( \mu_j + j \), it is semi-simple. As the result is obvious in rank 1, we may assume \( k \geq 1 \) and the result proved for \( k \). So let \( E \) be a fresco of rank \( k + 1 \) and let \( F_j, j \in [1, k + 1] \) a J-H. sequence for \( E \) corresponding to the strictly decreasing sequence \( \mu_j + j, j \in [1, k + 1] \). Put \( F_j/F_{j-1} \simeq E_{\mu_j} \) for all \( j \in [1, k + 1] \), define \( F := F_k \), and \( \mu := \mu_{k+1} \); then the induction hypothesis gives that \( F \) is semi-simple and we may apply the previous lemma.

**Proposition 4.1.6** Let \( E \) be a fresco. There exists a unique maximal normal semi-simple submodule \( S_1(E) \) in \( E \). It contains any (normal) submodule of rank 1 contained in \( E \). Moreover, if \( S_1(E) \) is of rank 1, then \( E \) is a \([\lambda]-\)primitive theme.

PROOF. For any \( \lambda \) and any non-zero \( \varphi \in \text{Hom}_A(E, \Xi_\lambda) \) let \( F_1(\varphi) \) be the rank 1 submodule of the \([\lambda]-\)primitive theme \( \varphi(E) \). Now put

\[
S_1(E) := \cap_\lambda \cap_{\varphi \in \text{Hom}_A(E, \Xi_\lambda) \setminus \{0\}} [\varphi^{-1}(F_1(\varphi))].
\]

Let us prove that \( S_1(E) \) is a normal semi-simple submodule. Normality is obvious as it is an intersection of normal submodules. To prove semi-simplicity, let \( \psi : S_1(E) \to \Xi_\lambda \) be a \( \mathcal{A} \)-linear map. Using the surjectivity of the restriction \( \varphi \in \text{Hom}_A(E, \Xi_\lambda) \to \varphi|_{S_1(E)} \in \text{Hom}_A(S_1(E), \Xi_\lambda) \), (see section 2.1 or [B.05]), we see immediately that \( \psi \) has rank \( \leq 1 \). So \( S_1(E) \) is semi-simple.

Now consider a semi-simple normal submodule \( S \) in \( E \). For any \( \varphi \in \text{Hom}_A(E, \Xi_\lambda) \) the restriction of \( \varphi \) to \( S \) has rank \( \leq 1 \). So \( \varphi(S) \) is contained in the normal rank 1 submodule \( F_1(\varphi) \) of the \([\lambda]-\)primitive theme \( \varphi(E) \). So \( S \) is contained in \( \varphi^{-1}(F_1(\varphi)) \) for each \( \varphi \). Then \( S \subset S_1(E) \), and this proves the maximality of \( S_1(E) \).

Consider now any rank 1 normal submodule \( F \) of \( E \). As \( F \) is semi-simple and
normal in $E$, we have $F \subset S_1(E)$. If $S_1(E)$ is rank 1, there exists an unique rank 1 normal submodule in $E$. Then $E$ is a $[\lambda]$–primitive theme, thanks to [B.10] theorem 2.1.6.

The following interesting corollary is an obvious consequence of the previous proposition.

**Corollaire 4.1.7** Let $E$ be a fresco and let $\lambda_1, \ldots, \lambda_k$ be the numbers associated to any J-H. sequence of $E$. Let $\mu_1, \ldots, \mu_d$ be the numbers associated to any J-H. sequence of $S_1(E)$. Then, for $j \in [1, k]$, there exists a rank 1 normal submodule of $E$ isomorphic to $E_{\lambda_j + j - 1}$ if and only if there exists $i \in [1, d]$ such that we have $\lambda_j + j - 1 = \mu_i + i - 1$.

Of course, this gives the list of all isomorphy classes of rank 1 normal submodules contained in $E$. So, using shifted duality, we get also the list of all isomorphy classes of rank 1 quotients of $E$.

**Définition 4.1.8** Let $E$ be a fresco. Define inductively the increasing sequence $S_j(E), j \geq 0$ of normal submodules of $E$ by putting $S_0(E) := \{0\}$ and for $j \geq 1$ $S_j(E)/S_{j-1}(E) := S_1(E/S_{j-1}(E))$. We shall call $S_j(E), j \geq 0$ the semi-simple filtration of $E$. We shall call semi-simple-depth of $E$ (ss-depth for short) the first integer $d = d(E) \geq 0$ such that $E = S_d(E)$.

**Example.** In the example of lemma [3.2.1] let $F_2$ the second step of the principal J-H. sequence of $E$. Then $F_2 = S_1(E)$ is the maximal semi-simple normal submodule of $E$. This is a consequence of the fact that $E$ is not semi-simple, $F_2$ admits a J-H. sequence with quotients $E_{\lambda_2 + 1}, E_{\lambda_1 - 1}$ with $\lambda_2 + 2 > \lambda_1 + 1$, so we may apply proposition [4.1.4].

**Proposition 4.1.9** Let $E$ be a fresco. Then we have the following properties :

i) Any $[\lambda]$–primitive sub-theme $T$ in $E$ of rank $j$ is contained in $S_j(E)$.

ii) Any $[\lambda]$–primitive quotient theme $T$ of $S_j(E)$ has rank $\leq j$.

iii) For any $j \in \mathbb{N}$ we have

$$S_j(E) = \bigcap_{\varphi \in \text{Hom}_J(E, \Xi_\lambda)} [\varphi^{-1}(F_j(\varphi))]$$

where $F_j(\varphi)$ is the normal submodule of rank $j$ of the $[\lambda]$–primitive theme $\varphi(E)$, with the convention that $F_j(\varphi) = \varphi(E)$ when the rank of $\varphi$ is $\leq j$.

iv) The ss-depth of $E$ is equal to $d$ if and only if $d$ is the maximal rank of a $[\lambda]$–primitive quotient theme of $E$.

v) The ss-depth of $E$ is equal to $d$ if and only if $d$ is the maximal rank of a normal $[\lambda]$–primitive sub-theme of $E$.  

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Remarks.

1. By definition of the ss-depth $d(E)$ of $E$ the semi-simple filtration is strictly increasing for $j \in [0, d(E)]$.

2. Let $E$ be a fresco and $N \in \mathbb{Z}$ such that $E \otimes E_N$ is geometric (so is again a fresco). Then $E$ is semi-simple (resp. a theme) if and only if $E \otimes E_N$ is semi-simple (resp. a theme). Moreover, in this situation we have $d(E) = d(E \otimes E_N)$.

3. Let $F$ be a submodule in a fresco $E$, and denote $\tilde{F}$ its normalization. Then $\tilde{F}$ is monogenic (being normal in a monogenic) and geometric. As there exists $N \in \mathbb{N}$ such that $b^N \tilde{F} \subset F$, $\tilde{F}$ is a theme for $F$ a theme. The analog result is also true for a semi-simple $F$: if $\varphi: \tilde{F} \rightarrow \Xi_\lambda$ has rank $\geq 2$, as $F$ has finite codimension in $\tilde{F}$, the restriction of $\varphi$ to $F$ has also rank $\geq 2$ which contradicts the semi-simplicity of $F$.

So we have proved the following two assertions:

- If $T \subset E$ is a theme in a fresco $E$, its normalization is also a theme (of same rank than $T$).
- If $S \subset E$ is a semi-simple fresco in a fresco $E$, its normalization is also a semi-simple fresco.

Proof of proposition [4.1.9] Let us prove i) by induction on $j$. As the case $j = 1$ is obvious, let us assume that $j \geq 2$ and that the result is proved for $j - 1$.

Let $T$ a $[\lambda]$–primitive theme in $E$, and let $F_{j-1}(T)$ be its normal submodule of rank $j - 1$ (equal to $T$ if the rank of $T$ is less than $j - 1$). Then by the induction hypothesis, we have $F_{j-1}(T) \subset S_{j-1}(E)$. Then we have a $\mathcal{A}$–linear map $T/F_{j-1}(T) \rightarrow E/S_{j-1}(E)$. If the rank of $T$ is at most $j$, then $T/F_{j-1}(T)$ has rank at most 1 and its image is in $S_1(E/S_{j-1}(E))$. So $T \subset S_j(E)$.

To prove ii) we also make an induction on $j$. The case $j = 1$ is obvious. So we may assume $j \geq 2$ and the result proved for $j - 1$. Let $\varphi: S_j(E) \rightarrow T$ a surjective map on a $[\lambda]$–primitive theme $T$. By the inductive hypothesis we have $\varphi(S_{j-1}(E)) \subset F_{j-1}(T)$. So we have an induced surjective map

$$\tilde{\varphi}: S_j(E)/S_{j-1}(E) \rightarrow T/F_{j-1}(T).$$

As $S_j(E)/S_{j-1}(E)$ is semi-simple, the image of $\tilde{\varphi}$ has rank $\leq 1$. It shows that $T$ has rank $\leq j$.

To prove iii) consider first a $\mathcal{A}$–linear map $\varphi: E \rightarrow \Xi_\lambda$. As $\varphi(E)$ is a $[\lambda]$–primitive theme, $\varphi(S_j(E))$ is a $[\lambda]$–primitive theme quotient of $S_j(E)$. So its rank is $\leq j$ and we have $\varphi(S_j(E)) \subset F_j(\varphi)$.

Conversely, for any $\mathcal{A}$–linear map $\varphi: E \rightarrow \Xi_\lambda$, the image $\varphi(S_j(E))$ is a $[\lambda]$–primitive quotient theme of $S_j(E)$. So its rank is $\leq j$ and it is contained in $F_j(\varphi)$. 

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Let us prove iv). If \( S_d(E) = E \) then any \([\lambda]\)-primitive sub-theme in \( E \) has rank \( \leq d \) thanks to ii). Conversely, assume that for any \([\lambda]\) any \([\lambda]\)-primitive sub-theme of \( E \) has rank \( \leq d - 1 \) and \( S_{d-1}(E) \neq E \). Then choose a \( \mathcal{A}\)-linear map \( \varphi : E \to \Xi_{\lambda} \) such that \( \varphi^{-1}(F_{d-1}(\varphi)) \neq E \). Then \( \varphi(E) \) is a \([\lambda]\)-primitive theme of rank \( d \) which is a quotient of \( E \), thanks to the following lemma. To prove v) let us show that if \( E \) is a fresco and \( N \gg 1 \) an integer, then \( E^* \otimes E_N \) is again a fresco and that we have the inequality \( d(E^* \otimes E_N) \geq d(E) \).

The fact that for \( N \) a large enough integer \( E^* \otimes E_N \) is again a fresco is clear. Now, as \( E \) has a \([\lambda]\)-primitive quotient theme of rank \( d \), then \( E^* \otimes E_N \) has a \([-\lambda]\)-primitive sub-theme of rank \( d \).

So we obtain the inequality \( d(E^* \otimes E_N) \geq d(E) \) from i). Now, using again duality and the fact that \([\lambda]\)-primitive themes are preserved by \( \otimes E_N \) where \( N \) is a natural integer, we conclude that \( d(E) = d(E^* \otimes E_N) \). Then \( E \) admits a \([\lambda]\)-primitive normal sub-theme of rank \( d \).

Conversely, if \( d \) is the maximal rank of a (normal) \([\lambda]\)-primitive sub-theme of \( E \), then we have \( d(E^* \otimes E_N) = d \) and \( d(E) = d \).

**Lemma 4.1.10** Let \( E \) be a rank \( k \) \([\lambda]\)-primitive theme and denote by \( F_j \) its normal rank \( j \) submodule. Let \( x \in E \setminus F_{k-1} \). Then the \((a,b)\)-module \( \tilde{A}.x \subset E \) is a rank \( k \) theme.

**Proof.** We may assume \( E \subset \Xi^{(k-1)}_{\lambda} \) and then (see [B.10]) we have the equality \( F_{k-1} = E \cap \Xi^{(k-2)}_{\lambda} \). So \( x \) contains a non zero term with \((\log s)^{k-1}\) and then the result is clear.

Our next lemma shows that the semi-simple filtration of a normal submodule of a fresco \( E \) is the trace on this submodule of the semi-simple filtration of \( E \).

**Lemma 4.1.11** Let \( E \) be a fresco and \( F \) any normal submodule of \( E \). Then for any \( j \in \mathbb{N} \) we have \( S_j(E) \cap F = S_j(F) \).

**Proof.** By induction on \( j \geq 1 \). First \( S_1(E) \cap F \) is semi-simple in \( F \) so contained in \( S_1(F) \) by definition. But conversely, \( S_1(F) \) is semi-simple, so contained in \( S_1(E) \) and also in \( F \).

Let assume now that \( j \geq 2 \) and that the result is proved for \( j - 1 \). Consider now the quotient \( E/S_{j-1}(E) \). As \( S_{j-1}(E) \cap F = S_{j-1}(F) \), \( E/S_{j-1}(F) \) is a submodule of \( E/S_{j-1}(E) \). Now by the case \( j = 1 \) \( S_j(F)/S_{j-1}(F) \) which is, by definition, \( S_1(F/S_{j-1}(F)) \) is equal to \( S_1(E/S_{j-1}(E)) \cap (F/S_{j-1}(F)) \). So we obtain

\[
S_j(F)/S_{j-1}(F) = (S_j(E)/S_{j-1}(E)) \cap (F/S_{j-1}(F)).
\]

This implies the equality \( S_j(F) = S_j(E) \cap F \).
Lemma 4.1.12 Let \( 0 \to F \to E \to G \to 0 \) be a short exact sequence of frescos. Then we have the inequalities
\[
\operatorname{sup}\{d(F), d(G)\} \leq d(E) \leq d(F) + d(G).
\]

**Proof.** The inequality \( d(F) \leq d(E) \) is obvious from the previous lemma. The inequality \( d(G) \leq d(E) \) is then a consequence of the property iv) in proposition 4.1.9.

Now let \( \varphi : E \to \Xi_{\lambda} \) be a \( \tilde{\mathcal{A}} \)-linear map with rank \( \delta \). Then the restriction of \( \varphi \) to \( F \) has rank \( \leq d(F) \). So \( \varphi(F) \) is contained in \( T_d \), the normal sub-theme of \( \varphi(E) \) of rank \( d = d(F) \). The map \( \tilde{\varphi} : E/F \to \Xi_{\lambda} \) defined by composition of \( \varphi \) with an injection of the theme \( \varphi(E)/T_d \) in \( \Xi_{\lambda} \) has rank \( \delta - d \leq d(E/F) \). So the inequality \( \delta \leq d(F) + d(E/F) \) is proved. \( \blacksquare \)

### 4.2 Co-semi-simple filtration.

**Lemma 4.2.1** Let \( E \) be a fresco. Then there exists a normal submodule \( \Sigma^1(E) \) which is the minimal normal submodule \( \Sigma \) such that \( E/\Sigma \) is semi-simple.

**Proof.** First recall that if \( T \) is a theme and \( T \otimes E_{\delta} \) is geometric for some \( \delta \in \mathbb{Q} \), then \( T \otimes E_{\delta} \) is again a theme.

We shall prove that if \( E \) is a fresco and if \( N \in \mathbb{Z} \) is such that \( E \otimes E_N \) is again a fresco, we have the equality of submodules in \( E \otimes E_N \):
\[
S_1(E \otimes E_N) = S_1(E) \otimes E_N. \tag{\@}
\]

As \( S_1(E) \otimes E_N \) is a normal semi-simple submodule of \( E \otimes E_N \) the inclusion \( \supset \) in (\@) is clear.

Conversely, \( S_1(E \otimes E_N) \otimes E_{-N} \) is a semi-simple submodule of \( E \simeq E \otimes E_N \otimes E_{-N} \).

So we obtain \( S_1(E \otimes E_N) \otimes E_{-N} \subset S_1(E) \) and we conclude by tensoring by \( E_N \).

Now we shall prove that \( S_1(E^* \otimes E_N)^* \otimes E_N \) is a fresco and does not depend of \( N \), large enough.

Let \( \lambda_j + j \in [1, k] \) the sequence corresponding to the quotient of a J-H. of \( E \). Then let \( q \in \mathbb{N} \) such that \( \lambda_j + j \in ]k, k+q[ \) for all \( j \in [1, k] \). The corresponding J-H. for \( E^* \otimes E_N \) has quotients associated to the numbers \( -(\lambda_j + j) + k + N \) which are in \( ]N-q, N[ \cap ]k, +\infty[ \) for \( N > k+q \). So \( E^* \otimes E_N \) is a fresco. Then \( S_1(E^* \otimes E_N) \) is also a fresco and has a J-H. sequence corresponding to numbers in a subset of the previous ones. Dualizing again, we obtain that \( S_1(E^* \otimes E_N)^* \otimes E_N \) has a J-H. sequence with corresponding numbers \( -\mu_j + j + k + N \) with \( \mu_j + j \in ]N-q, N[ \).

So \( S_1(E^* \otimes E_N)^* \otimes E_N \) is a fresco which is a quotient of \( E \).

We want to show that this quotient is independant of the choice of \( N \) large enough. This is consequence of the fact that
\[
S_1(E \otimes E_{N+1})^* = (S_1(E \otimes E_N) \otimes E_1)^*
\]
\[
\quad = S_1(E \otimes E_N)^* \otimes E_{-1}
\]

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and so
$$S_1(E \otimes E_{N+1})^* \otimes E_{N+1} = S_1(E \otimes E_N)^* \otimes E_N.$$  
As $S_1(E^* \otimes E_N)$ is the maximal semi-simple submodule in $E^* \otimes E_N$, we conclude that $S_1(E^* \otimes E_N)^* \otimes E_N$ is the maximal quotient of $E$ which is semi-simple. So we have $\Sigma^1(E) = \left( E^* \otimes E_N / S_1(E^* \otimes E_N) \right)^* \otimes E_N \subset E$.  

Définition 4.2.2 Let $E$ be a fresco and define inductively the normal submodules $\Sigma^j(E)$ as follows : $\Sigma^0(E) := E$ and $\Sigma^{j+1}(E) := \Sigma^j(E)$. We call $\Sigma^j(E), j \geq 0$ the co-semi-simple filtration of $E$.

Note that $\Sigma^j/\Sigma^{j+1}$ is the maximal semi-simple quotient of $\Sigma^j$ for each $j$.

Lemme 4.2.3 Let $E$ be a fresco. The normal submodules $\Sigma^j(E)$ satisfies the following properties:

i) For any $j \in [0, d(E) - 1]$ we have $\Sigma^{j+1}(E) \subset \Sigma^j(E) \cap S_{d-j-1}(E)$ where $\nu := d(E)$ is the ss-depth of $E$.

ii) For any $[\lambda]$—primitive sub-theme $T$ of rank $t$ in $\Sigma^j(E)$ we have the inclusion $F_{t-p}(T) \subset \Sigma^{j+p}$, where $p \in [0, t]$ and $F_{t-p}(T)$ is the rank $t-p$ normal sub-theme of $T$.

iii) Put $\nu := d(E)$. Then we have $d(\Sigma^j(E)) = \nu - j$ for each $j \in [0, \nu]$. This implies that $\Sigma^\nu(E) = \{0\}$ and that $\Sigma^{\nu-1}(E) \neq \{0\}$ is semi-simple.

iv) For any normal submodule $F \subset E$ we have $\Sigma^j(F) \subset \Sigma^j(E) \cap \Sigma^{j-1}(F)$.

Remarks.

1. The inclusion in i) implies $\Sigma^j(E) \subset S_{d-j}(E)$ $\forall j \in [0, d(E)]$.

2. The filtration $\Sigma^j, j \in [0, \nu]$, is strictly decreasing because of iii).

Proof. Let us prove i) by induction on $j \in [0, d(E) - 1]$. As i) is obvious for $j = 0$ assume $j \geq 1$ and i) proved for $j-1$. So we know that $\Sigma^j(E) \subset S_{d-j}(E)$. The quotient $S_{d-j}(E)/S_{d-j-1}(E)$ is semi-simple, by definition of $S_{d-j}(E)$, and so is its submodule $\Sigma^j(E)/\Sigma^{j+1}(E) \cap S_{d-j-1}(E)$. The definition of $\Sigma^{j+1}(E)$ implies then that we have $\Sigma^{j+1}(E) \subset \Sigma^j(E) \cap S_{d-j-1}(E)$. So i) is proved.

To prove ii) it is enough to show it for $p = 1$, by an obvious iteration. By definition $\Sigma^j(E)/\Sigma^{j+1}(E)$ is semi-simple. So is the submodule $T/T \cap \Sigma^{j+1}(E)$. As it is also a $[\lambda]$—primitive theme, its rank is $\leq 1$ showing that $T \cap \Sigma^{j+1}(E)$ contains the corank 1 normal submodule $F_{t-1}(T)$ of $T$.

To prove iii) let $\nu := d(E)$ and let $T$ a sub-theme in $E$ of rank $\nu$. Then, thanks to ii) with $j = 0$, $\Sigma^1(E) \cap T$ contains a sub-theme of rank $\geq \nu - 1$. So $d(\Sigma^1(E)) \geq \nu - 1$. Assume that $d(\Sigma^1(E)) = \nu$, then we obtain, thanks to
iteration of the previous inequality, that \(d(\Sigma^{d-1}(E)) \geq 2\). But from i) we know that \(\Sigma^{d-1}(E) \subset S_1(E)\) is semi-simple. This is a contradiction. So we obtain \(d(\Sigma^{1}(E)) = d - 1\) and then \(d(\Sigma^{j}(E)) = d - j\) for each \(j \in [0, d]\).

To prove iv) we shall make an induction on \(j \in [0, d(E)]\). As the case \(j = 0\) is obvious, assume \(j \geq 1\) and the case \(j - 1\) proved. As \(\Sigma^{j}(E) \cap F/\Sigma^{j+1}(E) \cap F\) is a submodule of \(\Sigma^{j}(E)/\Sigma^{j+1}(E)\) which is semi-simple by definition, it is semi-simple and so we have \(\Sigma^{j}(\Sigma^{j}(E) \cap F) \subset \Sigma^{j+1}(E) \cap F\). Now to conclude, as we know that \(\Sigma^{j}(F) \subset \Sigma^{j}(E)\), it is enough to remark that for \(G \subset H\) we have \(\Sigma^{1}(G) \subset \Sigma^{1}(H) \cap G\): as \(H/\Sigma^{1}(H)\) is semi-simple, its submodule \(G/G \cap \Sigma^{1}(H)\) is also semi-simple, and so \(\Sigma^{1}(G)\) is contained in \(\Sigma^{1}(H) \cap G\).

\[\blacksquare\]

**Remark.** We shall prove in section 5 that \(E/S_1(E)\) and \(\Sigma^{1}(E)\) are rank \(d(E) - 1\) themes and that any normal rank \(d(E)\) theme in \(E\) contains \(\Sigma^{1}(E)\).

### 4.3 Computation of the ss-depth.

**Définition 4.3.1** Let \(E\) be a rank \(k\) \([\lambda]\)-primitive fresco and consider \([F] := \{F_j, j \in [1, k]\}\) be any J.H. sequence of \(E\). We shall say that \(j \in [1, k - 1]\) is a non commuting index for \([F]\) if the quotient \(F_{j+1}/F_{j-1}\) is a theme. If it is not the case we shall say that \(j\) is a commuting index. Note that in this case the quotient \(F_{j+1}/F_{j-1}\) is semi-simple.

**Lemme 4.3.2** Let \(E\) be a rank \(k \geq 2\) \([\lambda]\)-primitive fresco and let \(F_j, j \in [1, k]\) be a J.H. sequence of \(E\). Assume that the \(\mathcal{A}\)-linear map \(\varphi : F_{k-1} \rightarrow \Xi_\lambda\) has rank \(\delta \geq 1\) and that \(E/F_{k-2}\) is a theme. Then any \(\tilde{\varphi} : E \rightarrow \Xi_\lambda\) extending \(\varphi\) has rank \(\delta + 1\).

**Proof.** Let \(e\) be a generator of \(E\) such that \((a - \lambda_{k-1}b).S_{k-1}^{-1}.(a - \lambda_kb).e\) is in \(F_{k-2}\). So, if \(\lambda_k = \lambda_{k-1} + p_{k-1} - 1\) we have either \(p_{k-1} = 0\) or \(p_{k-1} \geq 1\) and the coefficient of \(b^{p_{k-1}}\) in \(S_{k-1}\) does not vanish. Put \(\varepsilon := S_{k-1}^{-1}.(a - \lambda_kb).e\); it is a generator of \(F_{k-1}\). Up to a non zero constant, we may assume that

\[
\varphi(\varepsilon) - s^{(k-1)} \cdot (\log s)^{(\delta-1)}/(\delta - 1)! \in \Xi^{(\delta-2)}_\lambda.
\]

Now we want to define \(\tilde{\varphi}(e) = x\) where \(x\) is a solution in \(\Xi_\lambda\) of the equation

\[(a - \lambda_kb).x = S_{k-1}.\varphi(\varepsilon).
\]

Then it is easy to find that, because the coefficient of \(b^{p_{k-1}}\) in \(S_{k-1}\) is not zero, we have

\[x - s^{(k-1)} \cdot (\log s)^\delta/\delta! \in \Xi^{(\delta-1)}_\lambda.
\]

Now the degree in \(\log s\) gives our assertion.  

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\(^{8}\)with the convention \(\Sigma^{-1}(G) := G\).
Corollaire 4.3.3 In the situation of the lemma 4.3.2 we have the equality \( d(E) = d(F_{k-1}) + 1 \).

Lemme 4.3.4 Let \( E \) be a rank \( k \) \( [\lambda] \)-primitive fresco and \( [F] := \{F_j, j \in [1, k]\} \), be any J-H. sequence of \( E \). Let \( nci(F) \) be the number of non commuting indices for the J-H. sequence \( [F] \). Then we have the inequality \( d(E) \geq nci(F) + 1 \).

Proof. We shall prove this by induction on the rank of \( E \). The cases of rank 1 and 2 are clear. Let assume \( k \geq 3 \) and the inequality proved in rank \( \leq k - 1 \). Consider a J-H. sequence \( F_j, j \in [1, k] \), for \( E \) and assume first that \( E/F_{k-2} \) is semi-simple. Then we have, denoting \( [G] \) the J-H. sequence \( \{F_j, j \in [1, k-1]\} \), for \( F_{k-1} \):

\[
nci(F) = nci(G) \leq d(F_{k-1}) - 1 \leq d(E) - 1
\]

using the induction hypothesis and lemma 4.1.2; it concludes this case. Assume now that \( E/F_{k-2} \) is a theme. Then using corollary 4.3.3 we have \( d(E) = d(F_{k-1}) + 1 \). So we get using again the inductive hypothesis:

\[
nci(F) = nci(G) + 1 \leq d(F_{k-1}) = d(E) - 1
\]

which concludes the proof.

Remarks. 

1. This inequality may be strict for several J-H. sequences, including the principal one : there are examples of rank 3 fresco with a principal J-H. sequence \( [F] \) such that \( nci(F) = 0 \) which are not semi-simple (see 3.3 ).

2. We shall see using the corollary of the theorem 4.4.1 (see the remark following 5.1.2) that for any fresco \( E \) there always exists a J-H. sequence \( [F] \) for which we have the equality \( d(E) = nci(F) + 1 \).

4.4 Embedding for a semi-simple fresco.

The aim of this paragraph is to prove the following embedding theorem for semi-simple \( [\lambda] \)-primitive frescos.

Proposition 4.4.1 Let \( E \) be a rank \( k \) semi-simple \( [\lambda] \)-primitive fresco. Then there exists an \( \tilde{A} \)-linear injective map \( \varphi : E \rightarrow \Xi_\lambda \otimes \mathbb{C}^l \) if and only if \( l \geq k \).
Proof. To show that the existence of \( \varphi : E \to \Xi \otimes \mathbb{C}^l \) implies \( l \geq k \), remark that for any linear form \( \alpha : \mathbb{C}^l \to \mathbb{C} \) the composed map \( (1 \otimes \alpha) \circ \varphi \) has rank at most 1. So the inequality \( l \geq k \) is clear. To prove that there exists an \( \mathcal{A} \)-linear injective map from \( E \) to \( \Xi \otimes \mathbb{C}^k \) we shall use the following lemma.

Lemme 4.4.2 Let \( \lambda_1, \ldots, \lambda_k, k \geq 2 \), be numbers in \( [\lambda] \in \mathbb{Q}/\mathbb{Z} \) such that \( \lambda_{j+1} = \lambda_j + p_j - 1 \) for each \( j \in [1, k-1] \) with \( p_j < 0 \). Put

\[
Q := (a - \lambda_2.b).S_2^{-1} \ldots S_{k-1}^{-1}(a - \lambda_k.b) \quad \text{and} \quad P := (a - \lambda_1.b).S_1^{-1}.Q
\]

where \( S_j, j \in [1, k-1] \) are invertible elements in \( \mathbb{C}[[b]] \). Assume also that \( \lambda_1 > k - 1 \). Then there exists an unique element \( T \in \mathbb{C}[[b]] \) which satisfies

\[
Q.T.s^{\lambda_1-k} = S_1.s^{\lambda_1-1}.
\]

Moreover \( T \) is invertible in \( \mathbb{C}[[b]] \).

Proof. We begin by the proof of the case \( k = 2 \). Then we look for \( T \in \mathbb{C}[[b]] \) such that \( (a - \lambda_2.b).T.s^{\lambda_1-2} = S_1.s^{\lambda_1-1} \). This equation is equivalent to the differential equation

\[
b.T' - p_1.T = (\lambda_1 - 1).S_1
\]

which has an unique solution in \( \mathbb{C}[[b]] \) for any \( S_1 \in \mathbb{C}[[b]] \) because \( p_1 < 0 \). Moreover, we have \( -p_1.T(0) = (\lambda_1 - 1).S_1(0) \), so \( T \) is invertible as \( S_1 \) is invertible.

Let now prove the lemma by induction on \( k \geq 2 \). We may assume \( k \geq 3 \) and the lemma proved for \( k - 1 \). Put \( Q = (a - \lambda_2.b).S_2^{-1}.R \). Our equation is

\[
(a - \lambda_2.b).S_2^{-1}.R.T.s^{\lambda_1-k} = S_1.s^{\lambda_1-1}.
\]

Remark that \( S_2^{-1}.R.T.s^{\lambda_1-k} = V.s^{\lambda_1-2} \) for some \( V \in \mathbb{C}[[b]] \). So, let \( U \in \mathbb{C}[[b]] \) the unique solution of the equation

\[
(a - \lambda_2.b).U.s^{\lambda_1-2} = S_1.s^{\lambda_1-1}
\]

and consider now the equation in \( T \in \mathbb{C}[[b]] : \)

\[
R.T.s^{\lambda_1-k} = S_2.U.s^{\lambda_1-2}.
\]

The inductive hypothesis shows that there exists an unique invertible \( T \in \mathbb{C}[[b]] \) which is solution of \( (@@) \). Then it satisfies \( (@) \). The uniqueness of the solution \( T \) of \( (@) \) is consequence of the uniqueness of \( U \) and uniqueness in the inductive hypothesis.

\[\blacksquare\]
5 Quotient themes of a $[\lambda]-$primitive fresco.

5.1 Structure theorem for $[\lambda]-$primitive frescos.

Now we are ready to describe the precise structure of a $[\lambda]-$primitive fresco. We shall then deduce the possible ( $[\lambda]-$primitive) quotient themes of a any given $[\lambda]-$primitive fresco.

Théorème 5.1.1 Let $E$ be a $[\lambda]-$primitive fresco and let $d := d(E)$ be its ss-depth. Then there exists a J-H. sequence $G_j, j \in [1, k]$ for $E$ with the following properties :

i) The quotient $E/G_{k-d}$ is a theme (with rank $d$).

ii) We have the equality $G_{k-d+1} = S_1(E)$.

Remark. By twisted duality we obtain also a J-H. sequence $G'_j, j \in [1, k]$, such that $G'_{d-1} = \Sigma^1(E)$ and $G'_d$ is a theme ; then $E/G'_{d-1}$ is the maximal semi-simple quotient of $E$.

Proof. Let $\varphi : E \to \Xi_\lambda$ an $\tilde{A}$–linear map with rank $d$. Denote $F_j(\varphi)$ the J-H. sequence of the $[\lambda]-$primitive theme $\varphi(E)$. Put $H_j := \varphi^{-1}(F_j(\varphi))$ for $j \in [0, d]$. Note that $H_0 = \text{Ker} \varphi$. We shall show that $H_1 = S_1(E)$.
To show that $H_1$ is semi-simple, assume that we have a rank 2 $\tilde{A}$-linear map $	heta : H_1 \to \Xi$. Then $E / Ker \theta$ has the following J-H. sequence

$$0 \subset \theta^{-1}(F_1(\theta)) / Ker \theta \subset H_1 / Ker \theta \subset \cdots \subset H_d / Ker \theta.$$ 

As $H_1 / Ker \theta$ and $(H_{j+1} / Ker \theta) / (H_{j-1} / Ker \theta) \simeq H_{j+1} / H_{j-1}$ are $[\lambda]$-primitive rank 2 themes for $j$ in $[1, d-1]$, this would imply that $E / Ker \theta$ is a rank $d+1$ $[\lambda]$-primitive theme, contradicting the definition of $d$.

Then $H_1$ is semi-simple. But as $S_1(E)$ is contained in $H_1 := \varphi^{-1}(F_1(\varphi))$ the equality $H_1 = S_1(E)$ is proved.

Define now $G_j := H_j$ for $j \in [0, d]$, and complete the J-H. sequence $G_j, j \in [1, k]$ of $E$ by choosing a J-H. sequence $G_j, j \in [1, k-d]$, for $H_0$. ■

We have the following easy consequences of this theorem.

**Corollaire 5.1.2** Let $E$ be a $[\lambda]$-primitive fresco and let $d := d(E)$ be its ss-depth. Then $E / S_1(E)$ and $\Sigma^1(E)$ are $[\lambda]$-primitive themes of rank $d - 1$ and we have

$$rk(S_1(E)) + d(E) = rk(E) + 1. \quad \text{and} \quad rk(\Sigma^1(E)) = d(E) - 1.$$ 

For each $j \in [2, d]$ the rank of $S_j(E) / S_{j-1}(E)$ is 1. Moreover, any rank $d$ quotient theme $T$ of $E$ satisfies $E / S_1(E) \simeq T / F_1(T)$. Dualy, any rank $d$ theme contains $\Sigma^1(E)$.

**Proof.** With the notations of the theorem, let $T := E / G_{k-d}$. Then $G_{k-d+1} / G_{k-d}$ is $F_1(T)$ the unique rank 1 normal submodule of the $[\lambda]$-primitive theme $T$. Then we obtain that $E / S_1(E) \simeq E / G_{k-d+1} \simeq T / F_1(T)$ proving our first assertion. The computation of the rank of $S_1(E)$ follows.

Consider now the exact sequence

$$0 \to S_1(E) \to E \to T(E) \to 0$$

where $T(E)$ is the $d - 1$ $[\lambda]$-primitive theme $E / S_1(E)$. Dualizing and tensoring by $E_N$ for $N$ a large enough integer gives the exact sequence

$$0 \to T(E)^* \otimes E_N \to E^* \otimes E_N \to S_1(E)^* \otimes E_N \to 0$$

where $S_1(E)^* \otimes E_N$ is semi-simple. This implies that $\Sigma^1(E^* \otimes E_N) \subset T(E)^* \otimes E_N$.

And we have equality because we know that $E^* \otimes E_N$ contains a rank $d$ theme, so the dimension of $\Sigma^1(E^* \otimes E_N)$ is at most $d(E) - 1$. Then we conclude that $\Sigma^1(E)$ is a theme and that its rank is $d(E) - 1$.

We know that $E / S_1(E)$ is a theme, so $S_1(E / S_1(E)) = F_1(E / S_1(E))$ is rank 1 for $d(E) \geq 2$. A similar argument shows that $S_j(E) / S_{j-1}(E)$ for each $j \in [2, d]$. In fact it is naturally isomorphic to $F_{j-1}(T) / F_{j-2}(T)$ where $T := T(E)$. 27
Let $T'$ any rank $d$ quotient theme of $E$. As $E$ is $[\lambda]$–primitive, so is $T'$ and we may assume that $T' := \varphi(E)$ where $\varphi \in Hom_{\tilde{A}}(E, \Xi_{\lambda})$. But now $S_1(E)$ is in $\varphi^{-1}(F_1(T'))$, so we have a surjective map, induced by $\varphi$:

$$\tilde{\varphi} : E/S_1(E) \to T'/F_1(T')$$

between two $[\lambda]$–primitive themes of the same rank $d - 1$. This must be an isomorphism. ■

**Remark.** Building a J-H. sequence of $E$ via the exact sequence

$$0 \to S_1(E) \to E \to T(E) \to 0$$

we find that the number of non commuting indices in such a J-H. sequence is exactly $d(E) - 1$. Of course we may put the non commuting indices at the beginning by using the exact sequence

$$0 \Sigma^1(E) \to E \to E/\Sigma^1(E) \to 0.$$ 

### 5.2 Embedding dimension for a $[\lambda]$–primitive fresco.

From the embedding result in the semi-simple case 4.1.1 and the structure theorem 5.1.1 we shall deduce precise embedding theorem for $[\lambda]$–primitive frescos.

**Proposition 5.2.1** Let $E$ be a $[\lambda]$–primitive fresco. Then there exists an injective $\tilde{A}$–linear map $\varphi : E \to \Xi_{\lambda} \otimes \mathbb{C}^d$ if and only if $l \geq rk(E) - d(E) + 1$.

**Proof.** If we have such a $\varphi$ its restriction to $S_1(E)$ is an embedding and so $l \geq rk(S_1(E)) = rk(E) - d(E) + 1$.

Conversely, we shall prove that there exists and embedding of $E$ in $\Xi_{\lambda} \otimes \mathbb{C}^d$ with $l = rk(E) - d(E) + 1$. By the proposition 4.1.1 we may begin with an injective $\tilde{A}$–linear map $\varphi : S_1(E) \to \Xi_{\lambda} \otimes \mathbb{C}^d$ with $l := rk(E) - d(E) + 1$. Put $\varphi := \bigoplus_{i=1}^l \varphi_i$ where $\varphi_i : S_1(E) \to \Xi_{\lambda}$. Now, for each $i \in [1, l]$ we may find an extension $\Phi_i : E \to \Xi_{\lambda}$ to $\varphi_i$ thanks to the surjectivity of $Hom_{\tilde{A}}(E, \Xi_{\lambda}) \to Hom_{\tilde{A}}(S_1(E), \Xi_{\lambda})$ (see section 2.1). Then we may define $\Phi := \bigoplus_{i \in [1, l]} \Phi_i : E \to \Xi_{\lambda} \otimes \mathbb{C}^d$ which is an extension of $\varphi$ to $E$. Moreover, such an extension is injective because its kernel cannot meet non trivially $S_1(E)$ and so does not contain any normal rank 1 submodule of $E$. So the kernel has to be $\{0\}$. ■

### 5.3 Quotient themes of a $[\lambda]$–primitive fresco.

Now we shall describe all quotient themes of a given $[\lambda]$–primitive fresco. We begin by the description of quotient themes of maximal rank. 

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**Proposition 5.3.1** Let \( E \) be a \([\lambda]\)-primitive fresco of rank \( k \). Then any rank \( d := d(E) \) quotient theme of \( E \) is obtained as follows: let \( K \) be a corank 1 normal submodule of \( S_1(E) \) and assume that \( K \cap L(E) = \{0\} \) where \( L(E) := \Sigma^1(S_2(E)) \). Then \( E/K \) is a rank \( d \) theme.

**Proof.** Consider \( K \subset S_1(E) \) a corank 1 normal submodule in \( S_1(E) \) such that \( K \cap L(E) = \{0\} \). By definition of \( L(E) \) the quotient \( S_2(E)/K \) is rank 2 and not semi-simple. So it is a theme. We have the following Jordan-Hölder sequence for \( E/K \):

\[
0 \subset S_1(E)/K \subset S_2(E)/K \subset \cdots \subset S_d(E)/K = E/K \quad (\@)
\]

But \( S_2(E)/K \) and each \( S_{j+2}(E)/S_j(E) \) for \( j \in [1, d-2] \) is a theme of rank 2. So from [B.10] we conclude that \( E/K \) is a rank \( d \) theme.

Conversely, if \( E/K \) is a rank \( d \) theme, consider \( S_1(E)/S_1(E) \cap K \hookrightarrow E/K \). As \( S_1(E)/S_1(E) \cap K \) is semi-simple and \( E/K \) is a theme, the rank of \( S_1(E)/S_1(E) \cap K \) is at most 1. It is not 0 because \( K \) has rank \( k-d \) and \( S_1(E) \) has rank \( k-d+1 \). So \( K \) is contained in \( S_1(E) \) and has corank 1 in it. If \( K \) contains \( L(E) \) then \( S_2(E)/K \) is semi-simple and of rank 2, if we assume \( d \geq 2 \). But it is contained in \( E/K \) which is a theme, so we get a contradiction.

For \( d = 1 \) in the previous proposition (so \( E \) semi-simple) we have \( S_2(E) = E \) so \( L(E) = \{0\} \) and any corank 1 normal submodule of \( S_1(E) = E \) gives a rank 1 quotient which is, of course a rank 1 theme.

In the statement of the theorem we shall denote by \( L_j \) for \( j \in [1, d-1] \) the rank \( j \) theme defined as \( \Sigma^1(S_{j+1}(E)) \). So, by definition, \( S_{j+1}(E)/\Sigma_j \) is semi-simple, and \( L_j \) is a normal submodule which is minimal for this property. We have seen that \( L_j \) is then a theme with rank \( d(S_{j+1}(E)) - 1 \); as we know that \( d(S_{j+1}(E)) = j + 1 \), the rank of \( L_j \) is \( j \). In fact we have \( L_j = F_j(\Sigma^1(E)) \) for each \( j \in [0, d-1] \) and so \( L_1 = L(E) \).

**Théorème 5.3.2** Let \( E \) be a \([\lambda]\)-primitive fresco of rank \( k \). Put \( d := d(E) \) and denote by \( S_{j, j} \in [1, d] \) the semi-simple filtration of \( E \). Assume that \( d \geq 2 \) and let \( \Sigma_j \) be the first term of the co-semi-simple filtration of \( S_{j+1} \).

Let \( K \subset E \) be a normal submodule such that \( E/K \) is a theme. Then we have the following possibilities:

1. If \( K \) contains \( S_1(E) \), then \( E/K \) is a quotient theme of the rank \( d - 1 \) theme \( E/S_1(E) \) and we have exactly one such quotient for each rank in \([1, d-1] \).

2. If \( K \cap L_1 = \{0\} \), then \( E/K \) is a quotient of the rank \( d \) theme \( E/K \cap S_1(E) \) which belongs to the quotient themes described in the previous proposition.

3. If \( K \) contains \( L_{j_0} \) but not \( L_{j_0 + 1} \) (so \( K \cap \Sigma^1(E) \) has rank \( j_0 \)), we may apply the previous case to \( E' := E/L_{j_0} \) and \( K' := K/L_{j_0} \) and find that \( E/K = E'/K' \) is a quotient of the rank \( d - j_0 \) quotient theme \( E'/K' \cap S_1(E') \).
In this situation we have $S_1(E') = S_{j_0+1}(E)/L_{j_0}$ and $d(E') = d - j_0$, with $\text{rk}(E') = \text{rk}(E) - j_0$. So the rank of $E/K$ is at most $d - j_0$.

**Remarks.**

i) The case 2 of the previous theorem is the case 3 with $j_0 = 0$. We emphasis on this case because the rank $d$ quotients themes is the most interesting case.

ii) Let $\varphi : E \rightarrow T$ be a surjective $\mathbb{A}$--linear map on a rank $\delta \geq 2$ theme $T$. Then $S_1(E) \subset \varphi^{-1}(F_1(T))$ and so the map $\varphi$ induces a surjection $E/S_1(E) \rightarrow T/F_1(T)$. As $E/S_1(E)$ is a $[\lambda]-$primitive theme, it has an unique quotient of rank $\delta - 1$. So the quotient theme $T/F_1(T)$ depends only on $\delta$ and $E$, not on $T$. In the case $\delta = d(E)$ we find that $T/F_1(T) \simeq E/S_1(E)$ for any choice of $T$.

iii) The previous theorem gives very few information on the rank 1 quotients, because any corank 1 normal submodule contains $\Sigma^1(E)$. They will be described in the next proposition.

**Proof.** The first case is clear.

Assume that $L_1 \cap K = \{0\}$; then $K$ does not contain $S_1$. But $S_1/K \cap S_1$ is semi-simple and is contained in the theme $E/K$. So it has rank $\leq 1$. As we know that $S_1$ is not contained in $K$, the rank is exactly 1, and $K \cap S_1$ has corank 1 in $S_1$. As $K \cap L_1 = \{0\}$ the previous proposition shows that $E/K \cap S_1$ is a rank $d$ theme. So the case 2 is proved.

For the proof of the case 3 it is enough to prove the equalities

$$S_1(E') = S_{j_0+1}(E)/L_{j_0}, \quad L_1(E') = L_{j_0+1}/L_{j_0}, \quad \text{and} \quad d(E') = d - j_0, \quad \text{rk}(E') = \text{rk}(E) - j_0.$$

As we know that $L_j$ is a theme of rank $j$, because $d(S_{j+1}) = j + 1$, the rank of $E'$ is $\text{rk}(E) - j_0$. The equality $d(L_{j_0}) = j_0$ is then clear.

As $S_{j_0+1}(E)/L_{j_0}$ is semi-simple, we have $S_{j_0+1}(E)/L_{j_0} \subset S_1(E')$. But now we know that they have same rank because

$$\text{rk}(S_1(E')) = \text{rk}(E') - d(E') + 1 = \text{rk}(E) - j_0 - (d - j_0) + 1 = \text{rk}(E) - d + 1.$$ 

Also $L_1(E')$ and $L_{j_0+1}/L_{j_0}$ are rank 1 and normal submodules of $E'$. And as $S_2(E') = S_{j_0+2}/L_{j_0}$, we have $S_2(E')/\left[L_{j_0+1}/L_{j_0}\right] = S_{j_0+2}/S_{j_0+1}$ is semi-simple (rank 1), this gives the inclusion

$$L_1(E') := \Sigma^1(S_2(E')) \subset L_{j_0+1}/L_{j_0}$$

and so the equality $L_1(E') = L_{j_0+1}/L_{j_0}$ is proved. \[\blacksquare\]
Proposition 5.3.3 (The rank 1 quotients) Let $E$ be a $[\lambda]-$primitive rank $k$ fresco. Put $d := d(E)$. Then any rank 1 quotient of $E$ is a rank 1 quotient of $E/\Sigma^1(E)$. As $E/\Sigma^1(E)$ is semi-simple of rank $k - d + 1$ it shows that there are exactly $k - d + 1$ isomorphism classes of such a rank 1 quotient and they correspond to the fundamental invariants of $E/\Sigma^1(E)$ as follows: if $\lambda_1, \ldots, \lambda_{k-d+1}$ are numbers associated to any J-H. sequence of $E/\Sigma^1(E)$, the isomorphism classes of rank 1 quotients of $E$ are given by

$$\lambda_1 - k + d, \ldots, \lambda_2 - k + d - 1, \ldots, \lambda_{k+d-1}.$$ 

PROOF. Let $H$ be a normal co-rank 1 submodule of $E$. As $E/H$ is semi-simple, $H$ contains $\Sigma^1(E)$ so $E/H$ is a rank 1 quotient of $E/\Sigma^1(E)$. The converse is obvious.

REMARK. Assume $d(E) \geq 2$. With the exception of the (unique) rank 1 quotient of $E/S_1(E)$ which is $E/S_{d-1}(E)$, no rank 1 quotient of $E$ may be the rank 1 quotient of a quotient theme of rank $\geq 2$ of $E$. Another way to say that is the following: for any rank $r \geq 2$ quotient theme $T$ of $E$ we have $T/F_{r-1}(T) \simeq E/S_{d-1}(E)$.

Exemple. Let $E$ a rank 3 $[\lambda]-$primitive fresco with $d(E) = 2$ (so $E$ is not semi-simple and is not a theme). Then there exists $k - d + 1 = 2$ isomorphism classes of rank 1 quotients of $E$.

For instance assume that we have a J-H. sequence $0 \subset F_1 \subset F_2 \subset F_3 = E$, with $F_2 = S_1(E)$ and such $E/F_1$ is a rank 2 theme.

If $E \simeq (a - \lambda_1,b).S_1^{-1}.(a - \lambda_2,b).S_2^{-1}.(a - \lambda_3,b)$ this means, with $\lambda_{j+1} = \lambda_j + p_j - 1$ for $j = 1,2$, that $p_1 < 0$ or $p_1 \geq 1$ and no term in $b^p$ in $S_1$ and $p_2 \geq 0$ with a non zero term in $b^p$ in $S_2$.

So we put $F_j/F_{j-1} \simeq E_{\lambda_j}$ for $j \in [1,3]$; then the rank 1 quotients are isomorphic to $E_{\lambda_j} \simeq E/S_j(E)$ or $E_{\lambda_{j-2}}$, because using the computations of section 3.3 we see that $\Sigma^1(E) \simeq E_{\lambda_2+1}$, and so $E_{\lambda_{j-2}}$ is a rank 1 quotient of $E/\Sigma^1(E)$ and a fortiori of $E$.

To conclude we give a method to compute $L(E)$ in many cases.

Lemme 5.3.4 Let $E$ be a $[\lambda]-$primitive fresco and assume that $E \simeq \tilde{A}/\tilde{A}.P$ where

$$P := (a - \lambda_1,b).S_1^{-1} \ldots S_{k-1}^{-1}.(a - \lambda_k,b)$$

where $\lambda_j + j$ is an increasing sequence. Assume that the first non commuting index is $h \in [1,k-1]$. Then $E$ has a normal sub-theme of rank 2 with fundamental invariants $\lambda_1, \lambda_{h+1} + h$.

PROOF. It is a simple application of the corollary 3.2.3 which also allow to compute the parameter of the rank 2 obtain by commuting in $P$ from the parameter of the rank 2 theme $Fh + 1/F_{h-1}$ and the integers $p_1, \ldots, p_h$. ■
Remarks.

i) The hypothesis of the lemma means that \( p_1, \ldots, p_{h-1} \) are positive and that for each \( j \in [1, h-1] \) the coefficient of \( b^{p_j} \) in \( S_j \) is zero. But the coefficient of \( b^{p_h} \) in \( S_h \) is not zero (and \( p_h = 0 \) is allowed).

ii) The lemma implies that \( L(E) \simeq E_{\lambda_1} \) is the first term of the principal J-H. sequence of \( E \).

6 Appendix : the existence theorem.

The aim of this appendix is to prove the following existence theorem for the fresco associated to a relative de Rham cohomology class:

**Théorème 6.0.5** Let \( X \) be a connected complex manifold of dimension \( n+1 \) where \( n \) is a natural integer, and let \( f : X \to D \) be an non constant proper holomorphic function on an open disc \( D \) in \( \mathbb{C} \) with center 0. Let us assume that \( df \) is nowhere vanishing outside of \( X_0 := f^{-1}(0) \).

Let \( \omega \) be a \( C^\infty - (p+1) \)-differential form on \( X \) such that \( d\omega = 0 = df \wedge \omega \). Denote by \( E \) the geometric \( \mathcal{A}- \)module \( \mathbb{H}^{p+1}(X, (\hat{K}^\bullet, d^\bullet)) \) and \( [\omega] \) the image of \( \omega \) in \( E/B(E) \). Then \( \hat{\mathcal{A}}[\omega] \subset E/B(E) \) is a fresco.

Note that this result is an obvious consequence of the finiteness theorem [6.3.4] that we shall prove below. It gives the fact that \( E \) is naturally an \( \hat{\mathcal{A}} \)-module which is of finite type over the subalgebra \( \mathbb{C}[[b]] \) of \( \hat{\mathcal{A}} \), and so its \( b \)-torsion \( B(E) \) is a finite dimensional \( \mathbb{C} \)-vector space. Moreover, the finiteness theorem asserts that \( E/B(E) \) is a geometric \( (a,b) \)-module.

6.1 Preliminaries.

Here we shall complete and precise the results of the section 2 of [B.II]. The situation we shall consider is the following: let \( X \) be a connected complex manifold of dimension \( n+1 \) and \( f : X \to \mathbb{C} \) a non constant holomorphic function such that \( \{x \in X/ df = 0\} \subset f^{-1}(0) \). We introduce the following complexes of sheaves supported by \( X_0 := f^{-1}(0) \)

1. The formal completion ”in \( f \)” \((\hat{\Omega}^\bullet, d^\bullet)\) of the usual holomorphic de Rham complex of \( X \).

2. The sub-complexes \((\hat{K}^\bullet, d^\bullet)\) and \((\hat{I}^\bullet, d^\bullet)\) of \((\hat{\Omega}^\bullet, d^\bullet)\) where the subsheaves \( \hat{K}^p \) and \( \hat{I}^{p+1} \) are defined for each \( p \in \mathbb{N} \) respectively as the kernel and the image of the map

\[ \wedge df : \hat{\Omega}^p \to \hat{\Omega}^{p+1} \]

given by exterior multiplication by \( df \). We have the exact sequence

\[ 0 \to (\hat{K}^\bullet, d^\bullet) \to (\hat{\Omega}^\bullet, d^\bullet) \to (\hat{I}^\bullet, d^\bullet)[+1] \to 0. \] (1)
Note that $\hat{K}^0$ and $\hat{i}^0$ are zero by definition.

3. The natural inclusions $\hat{I}^p \subset \hat{K}^p$ for all $p \geq 0$ are compatible with the differential $d$. This leads to an exact sequence of complexes

$$0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0. \quad (2)$$

4. We have a natural inclusion $f^*(\hat{\Omega}_C^1) \subset \hat{K}^1 \cap Ker d$, and this gives a subcomplex (with zero differential) of $(\hat{K}^\bullet, d^\bullet)$. As in [B.07], we shall consider also the complex $(\hat{K}^\bullet, d^\bullet)$ quotient. So we have the exact sequence

$$0 \to f^*(\hat{\Omega}_C^1) \to (\hat{K}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to 0. \quad (3)$$

We do not make the assumption here that $f = 0$ is a reduced equation of $X_0$, and we do not assume that $n \geq 2$, so the cohomology sheaf in degree 1 of the complex $(\hat{K}^\bullet, d^\bullet)$, which is equal to $\hat{K}^1 \cap Ker d$ does not coincide, in general with $f^*(\hat{\Omega}_C^1)$. So the complex $(\hat{K}^\bullet, d^\bullet)$ may have a non zero cohomology sheaf in degree 1.

Recall now that we have on the cohomology sheaves of the following complexes $(\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)$ and $f^*(\hat{\Omega}_C^1), (\hat{K}^\bullet, d^\bullet)$ natural operations $a$ and $b$ with the relation $a.b - b.a = b^2$. They are defined in a naïve way by

$$a := \times f \quad \text{and} \quad b := \wedge df \circ d^{-1}.$$  

The definition of $a$ makes sens obviously. Let me precise the definition of $b$ first in the case of $H^p(\hat{K}^\bullet, d^\bullet)$ with $p \geq 2$ : if $x \in \hat{K}^p \cap Ker d$ write $x = d\xi$ with $\xi \in \hat{\Omega}^{p-1}$ and let $b[x] := [df \wedge \xi]$. The reader will check easily that this makes sens. For $p = 1$ we shall choose $\xi \in \hat{\Omega}^0$ with the extra condition that $\xi = 0$ on the smooth part of $X_0$ (set theoretically). This is possible because the condition $df \wedge d\xi = 0$ allows such a choice : near a smooth point of $X_0$ we can choose coordinates such $f = x_0^k$ and the condition on $\xi$ means independance of $x_1, \ldots, x_n$. Then $\xi$ has to be (set theoretically) locally constant on $X_0$ which is locally connected. So we may kill the value of such a $\xi$ along $X_0$.

The case of the complex $(\hat{I}^\bullet, d^\bullet)$ will be reduced to the previous one using the next lemma.

**Lemma 6.1.1** For each $p \geq 0$ there is a natural injective map

$$\tilde{b} : H^p(\hat{K}^\bullet, d^\bullet) \to H^p(\hat{I}^\bullet, d^\bullet)$$

which satisfies the relation $a.\tilde{b} = \tilde{b} (b + a)$. For $p \neq 1$ this map is bijective.
Proof. Let \( x \in \hat{K}^p \cap \text{Ker} \, d \) and write \( x = d\xi \) where \( x \in \hat{\Omega}^{p-1} \) (with \( \xi = 0 \) on \( X_0 \) if \( p = 1 \), and set \( \tilde{b}([x]) := [df \wedge \xi] \in \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \). This is independent on the choice of \( \xi \) because, for \( p \geq 2 \), adding \( d\eta \) to \( \xi \) does not modify the result as \([df \wedge d\eta] = 0\). For \( p = 1 \) remark that our choice of \( \xi \) is unique.

This is also independent of the the choice of \( x \) in \([x] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)\) because adding \( \theta \in \hat{K}^{p-1} \) to \( \xi \) does not change \([df \wedge \xi]\).

Assume \( \tilde{b}([x]) = 0 \) in \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \); this means that we may find \( \alpha \in \hat{\Omega}^{p-2} \) such \( df \wedge \xi = df \wedge d\alpha \). But then, \( \xi - d\alpha \) lies in \( \hat{K}^{p-1} \) and \( x = d(\xi - d\alpha) \) shows that \([x] = 0\). So \( \tilde{b} \) is injective.

Assume now \( p \geq 2 \). If \( df \wedge \eta \) is in \( \hat{I}^p \cap \text{Ker} \, d \), then \( df \wedge d\eta = 0 \) and \( y := d\eta \) lies in \( \hat{K}^p \cap \text{Ker} \, d \) and defines a class \([y] \in \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)\) whose image by \( \tilde{b} \) is \([df \wedge \eta]\).

This shows the surjectivity of \( \tilde{b} \) for \( p \geq 2 \).

For \( p = 1 \) the map \( \tilde{b} \) is not surjective (see the remark below).

To finish the proof let us compute \( \tilde{b}(a[x] + b[x]) \). Writing again \( x = d\xi \), we get
\[
a[x] + b[x] = [f.d\xi + df \wedge \xi] = [d(f.\xi)]
\]
and so
\[
\tilde{b}(a[x] + b[x]) = [df \wedge f.\xi] = a.\tilde{b}([x])
\]
which concludes the proof. \( \blacksquare \)

Denote by \( i : (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \) the natural inclusion and define the action of \( b \) on \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \) by \( b := \tilde{b} \circ \mathcal{H}^p(i) \). As \( i \) is \( a \)-linear, we deduce the relation \( a.b - b.a = b^2 \) on \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \) from the relation of the previous lemma.

The action of \( a \) on the complex \( ([\hat{K}/\hat{I}^\bullet, d^\bullet]) \) is obvious and the action of \( b \) is zero.

The action of \( a \) and \( b \) on \( f^*(\hat{\Omega}^{\bullet}_\mathbb{C}) \simeq E_1 \otimes \mathbb{C}_{X_0} \) are the obvious one, where \( E_1 \) is the rank 1 \((a,b)\)-module with generator \( e_1 \) satisfying \( a.e_1 = b.e_1 \) (or, equivalently, \( E_1 := \mathbb{C}[[z]] \) with \( a := \times z \), \( b := \int_0^z \) and \( e_1 := 1 \)).

Remark that the natural inclusion \( f^*(\hat{\Omega}^{\bullet}_\mathbb{C}) \hookrightarrow (\hat{K}^\bullet, d^\bullet) \) is compatible with the actions of \( a \) and \( b \). The actions of \( a \) and \( b \) on \( \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \) are simply induced by the corresponding actions on \( \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \).

Remark. The exact sequence of complexes (1) induces for any \( p \geq 2 \) a bijection
\[
\partial^p : \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \to \mathcal{H}^p(\hat{K}^\bullet, d^\bullet)
\]
and a short exact sequence
\[
0 \to \mathbb{C}_{X_0} \to \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \to \mathcal{H}^1(\hat{K}^\bullet, d^\bullet) \to 0 \tag{\@}
\]
because of the de Rham lemma. Let us check that for \( p \geq 2 \) we have \( \partial^p = (\tilde{b})^{-1} \) and that for \( p = 1 \) we have \( \partial^1 \circ \tilde{b} = \text{Id} \). If \( x = d\xi \in \hat{K}^p \cap \text{Ker} \, d \) then \( \tilde{b}([x]) = [df \wedge \xi] \) and \( \partial^p[df \wedge \xi] = [d\xi] \). So \( \partial^p \circ \tilde{b} = \text{Id} \) \( \forall p \geq 0 \). For \( p \geq 2 \) and
df ∧ α ∈ \hat{I}^p ∩ Ker d we have ∂^p df ∧ α = [dα] and \tilde{b}[dα] = [df ∧ α], so \tilde{b} ∘ ∂^p = Id. For p = 1 we have \tilde{b}[dα] = [df ∧ (α - α_0)] where α_0 ∈ C is such that α|_X_0 = α_0. This shows that in degree 1 \tilde{b} gives a canonical splitting of the exact sequence (\@).

6.2 \tilde{A}–structures.

Let us consider now the C–algebra

\[ \tilde{A} := \{ \sum_{\nu \geq 0} P_\nu(a) b^\nu \} \]

where \( P_\nu \in \mathbb{C}[z] \), and the commutation relation \( a.b - b.a = b^2 \), assuming that left and right multiplications by \( a \) are continuous for the \( b \)–adic topology of \( \tilde{A} \).

Define the following complexes of sheaves of left \( \tilde{A} \)–modules on \( X \):

\[ (\Omega^{\bullet}[[b]], D^\bullet) \quad \text{and} \quad (\Omega'^{\bullet}[[b]], D^\bullet) \quad \text{where} \]

\[ \Omega^{\bullet}[[b]] := \sum_{j=0}^{+\infty} b^j \omega_j \quad \text{with} \quad \omega_0 \in \hat{K}^p \]

\[ \Omega'^{\bullet}[[b]] := \sum_{j=0}^{+\infty} b^j \omega_j \quad \text{with} \quad \omega_0 \in \hat{I}^p \]

\[ D(\sum_{j=0}^{+\infty} b^j \omega_j) = \sum_{j=0}^{+\infty} b^j.(d\omega_j - df ∧ \omega_{j+1}) \]

\[ a. \sum_{j=0}^{+\infty} b^j \omega_j = \sum_{j=0}^{+\infty} b^j.(f.\omega_j + (j - 1).\omega_{j-1}) \quad \text{with the convention} \quad \omega_{-1} = 0 \]

\[ b. \sum_{j=0}^{+\infty} b^j \omega_j = \sum_{j=1}^{+\infty} b^j \omega_{j-1} \]

It is easy to check that \( D \) is \( \tilde{A} \)–linear and that \( D^2 = 0 \). We have a natural inclusion of complexes of left \( \tilde{A} \)–modules

\[ \tilde{i} : (\Omega'^{\bullet}[[b]], D^\bullet) \to (\Omega^{\bullet}[[b]], D^\bullet). \]

Remark that we have natural morphisms of complexes

\[ u : (\hat{I}^{\bullet}, d^\bullet) \to (\Omega'^{\bullet}[[b]], D^\bullet) \]
\[ v : (\hat{K}^{\bullet}, d^\bullet) \to (\Omega^{\bullet}[[b]], D^\bullet) \]

and that these morphisms are compatible with \( \tilde{i} \). More precisely, this means that we have the commutative diagram of complexes

\[ \begin{array}{ccc} (\hat{I}^{\bullet}, d^\bullet) & \xrightarrow{u} & (\Omega'^{\bullet}[[b]], D^\bullet) \\ \downarrow \tilde{i} & & \downarrow \tilde{i} \\ (\hat{K}^{\bullet}, d^\bullet) & \xrightarrow{v} & (\Omega^{\bullet}[[b]], D^\bullet) \end{array} \]
The following theorem is a variant of theorem 2.2.1. of [B.II].

**Théorème 6.2.1** Let \( X \) be a connected complex manifold of dimension \( n + 1 \) and \( f : X \to \mathbb{C} \) a non constant holomorphic function such that

\[
\{x \in X/ \ df = 0\} \subset f^{-1}(0).
\]

Then the morphisms of complexes \( u \) and \( v \) introduced above are quasi-isomorphisms. Moreover, the isomorphisms that they induce on the cohomology sheaves of these complexes are equivalent to short exact sequences of complexes of left \( \hat{A}-\)modules on each of the complex \((\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet), ([\hat{K}/\hat{I}]^\bullet, d^\bullet)\) and \( f^*(\hat{\Omega}_\mathbb{C}^1), (\hat{K}^\bullet, d^\bullet)\) in the derived category of bounded complexes of sheaves of \( \mathbb{C}-\)vector spaces on \( X \).

Moreover the short exact sequences

\[
0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K}/\hat{I}]^\bullet, d^\bullet) \to 0
\]

\[
0 \to f^*(\hat{\Omega}_\mathbb{C}^1) \to (\hat{K}^\bullet, d^\bullet), (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to 0
\]

are equivalent to short exact sequences of complexes of left \( \hat{A}-\)modules in the derived category.

**Proof.** We have to prove that for any \( p \geq 0 \) the maps \( \mathcal{H}^p(u) \) and \( \mathcal{H}^p(v) \) are bijective and compatible with the actions of \( a \) and \( b \). The case of \( \mathcal{H}^p(v) \) is handled (at least for \( n \geq 2 \) and \( f \) reduced) in prop. 2.3.1. of [B.II]. To seek for completeness and for the convenience of the reader we shall treat here the case of \( \mathcal{H}^p(u) \).

First we shall prove the injectivity of \( \mathcal{H}^p(u) \). Let \( \alpha = df \wedge \beta \in \hat{I}^p \cap \ker d \) and assume that we can find \( U = \sum_{j=0}^{+\infty} b^j u_j \in \Omega^{p-1}[[b]] \) with \( \alpha = DU \). Then we have the following relations

\[
u_0 = df \wedge \zeta, \quad \alpha = du_0 - df \wedge u_1 \quad \text{and} \quad du_j = df \wedge u_{j+1} \forall j \geq 1.
\]

For \( j \geq 1 \) we have \( [du_j] = b[du_{j+1}] \) in \( \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \); using corollary 2.2. of [B.II] which gives the \( b\)-separation of \( \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \), this implies \( [du_j] = 0, \forall j \geq 1 \) in \( \mathcal{H}^p(\hat{K}^\bullet, d^\bullet) \). For instance we can find \( \beta_1 \in \hat{K}^{p-1} \) such that \( du_1 = d\beta_1 \). Now, by de Rham, we can write \( u_1 = \beta_1 + d\xi_1 \) for \( p \geq 2 \), where \( \xi_1 \in \hat{\Omega}^{p-2} \). Then we conclude that \( \alpha = -df \wedge d(\xi_1 + \zeta) \) and \( \alpha = 0 \) in \( \mathcal{H}^p(\hat{I}^\bullet, d^\bullet) \).

For \( p = 1 \) we have \( u_1 = 0 \) and \( \alpha = [-df \wedge d\xi_1] = 0 \) in \( \mathcal{H}^1(\hat{I}^\bullet, d^\bullet) \).

We shall show now that the image of \( \mathcal{H}^p(u) \) is dense in \( \mathcal{H}^p(\Omega^{\nu}[[b]], D^\bullet) \) for its \( b\)-adic topology. Let \( \Omega := \sum_{j=0}^{+\infty} b^j \omega_j \in \Omega^{p-1}[[b]] \) such that \( D\Omega = 0 \). The following relations holds \( d\omega_j = df \wedge \omega_{j+1} \forall j \geq 0 \) and \( \omega_0 \in \hat{I}^p \). The corollary 2.2. of [B.II] again allows to find \( \beta_j \in \hat{K}^{p-1} \) for any \( j \geq 0 \) such that \( d\omega_j = d\beta_j \). Fix \( N \in \mathbb{N}^* \).

We have

\[
D(\sum_{j=0}^{N} b^j \omega_j) = b^N d\omega_N = D(b^N \beta_N)
\]
and \( \Omega_N := \sum_{j=0}^{N} b^j \omega_j - b^N \beta_N \) is \( D \)-closed and in \( \Omega^{mp}[[b]] \). And we have 
\( \Omega - \Omega_N \in b^N \mathcal{H}^p(\Omega''^*[b]), D^* \), so the sequence \( (\Omega_N)_{N \geq 1} \) converges to \( \Omega \) in 
\( \mathcal{H}^p(\Omega''^*[b]), D^* \) for its \( b \)-adic topology. Let us show that each \( \Omega_N \) is in the image of \( \mathcal{H}^p(u) \).

Write \( \Omega_N := \sum_{j=0}^{N} b^j w_j \). The condition \( D \Omega_N = 0 \) implies \( dw_N = 0 \) and 
\( dw_{N-1} = df \wedge w_N = 0 \). If we write \( w_N = dv_N \) we obtain \( d(w_{N-1} + df \wedge v_N) = 0 \) and 
\( \Omega_N - D(b^N, v_N) \) is of degree \( N-1 \) in \( b \). For \( N = 1 \) we are left with 
\( w_0 + b.w_1 - (df \wedge v_1 + b.dv_1) = w_0 + df \wedge v_1 \) which is in \( \hat{I} \cap Ker \) because 
\( dw_0 = df \wedge dv_1 \).

To conclude it is enough to know the following two facts

i) The fact that \( \mathcal{H}^p(\hat{I}, d^*) \) is complete for its \( b \)-adic topology.

ii) The fact that \( Im(\mathcal{H}^p(u)) \cap b^N \mathcal{H}^p(\Omega''^*[b]), D^* \) \( \subset Im(\mathcal{H}^p(u) \circ b^N) \) \( \forall N \geq 1 \).

Let us first conclude the proof of the surjectivity of \( \mathcal{H}^p(u) \) assuming i) and ii).

For any \( [\Omega] \in \mathcal{H}^p(\Omega''^*[b]), D^* \) we know that there exists a sequence \( (\alpha_N)_{N \geq 1} \) in 
\( \mathcal{H}^p(\hat{I}, d^*) \) with \( \Omega - \mathcal{H}^p(u)(\alpha_N) \in b^N \mathcal{H}^p(\Omega''^*[b]), D^* \). Now the property ii) implies that we may choose the sequence \( (\alpha_N)_{N \geq 1} \) such that \( [\alpha_{N+1}] - [\alpha_N] \) lies in 
\( b^N \mathcal{H}^p(\hat{I}, d^*) \). So the property i) implies that the Cauchy sequence \( ([\alpha_N])_{N \geq 1} \) converges to \([\alpha] \in \mathcal{H}^p(\hat{I}, d^*) \). Then the continuity of \( \mathcal{H}^p(u) \) for the \( b \)-adic topologies coming from its \( b \)-linearity, implies \( \mathcal{H}^p(u)([\alpha]) = [\Omega] \).

The compatibility with \( a \) and \( b \) of the maps \( \mathcal{H}^p(u) \) and \( \mathcal{H}^p(v) \) is an easy exercise.

Let us now prove properties i) and ii).

The property i) is a direct consequence of the completion of \( \mathcal{H}^p(\hat{K}, d^*) \) for its 
\( b \)-adic topology given by the corollary 2.2. of [B.II] and the \( b \)-linear isomorphism \( \tilde{b} \) between 
\( \mathcal{H}^p(\hat{K}, d^*) \) and \( \mathcal{H}^p(\hat{I}, d^*) \) constructed in the lemma 2.1.1. above.

To prove ii) let \( \alpha \in \hat{I} \cap Ker d \) and \( N \geq 1 \) such that 
\[
\alpha = b^N . \Omega + DU
\]
where \( \Omega \in \Omega^{mp}[[b]] \) satisfies \( D \Omega = 0 \) and where \( U \in \Omega^{mp-1}[[b]] \). With obvious notations we have 
\[
\alpha = du_0 - df \wedge u_1 \\
\ldots \\
0 = du_j - df \wedge u_{j+1} \quad \forall j \in [1, N-1] \\
\ldots \\
0 = \omega_0 + du_N - df \wedge u_{N+1}
\]
which implies \( D(u_0 + b.u_1 + \cdots + b^N . u_N) = \alpha + b^N . du_N \) and the fact that \( du_N \) lies in \( \hat{I} \cap Ker d \). So we conclude that \( [\alpha] + b^N.[du_N] \) is in the kernel of \( \mathcal{H}^p(u) \) which is \( 0 \). Then \([\alpha] \in b^N \mathcal{H}^p(\hat{I}, d^*) \).\( \blacksquare \)
Remark. The map

$$\beta : (\Omega'[b]^*, D^*) \to (\Omega''[b]^*, D^*)$$

defined by $\beta(\Omega) = b.\Omega$ commutes to the differentials and with the action of $b$. It induces the isomorphism $\tilde{b}$ of the lemma 6.1.1 on the cohomology sheaves. So it is a quasi-isomorphism of complexes of $C[[b]]$-modules.

To prove this fact, it is enough to verify that the diagram

$$
\begin{array}{ccc}
(K^*, d^*) & \xrightarrow{u} & (\Omega'[b]^*, D^*) \\
\downarrow{\tilde{b}} & & \downarrow{\beta} \\
(I^*, d^*) & \xrightarrow{u} & (\Omega''[b]^*, D^*)
\end{array}
$$

induces commutative diagrams on the cohomology sheaves.

But this is clear because if $\alpha = d\xi$ lies in $K^p \cap Ker d$ we have $D(b.\xi) = b.d\xi - df \wedge \xi$ so $H^p(\beta \circ H^p(v)([\alpha]) = H^p(u) \circ H^p(\tilde{b})([\alpha])$ in $H^p(\Omega''[[b]]^*, D^*)$.

6.3 The finiteness theorem.

Let us recall some basic definitions on the left modules over the algebra $\tilde{A}$.

Now let $E$ be any left $\tilde{A}$–module, and define $B(E)$ as the $b$–torsion of $E$. that is to say

$$B(E) := \{ x \in E / \exists N \ b^N.x = 0 \}.$$  

Define $A(E)$ as the $a$–torsion of $E$ and

$$A(E) := \{ x \in E / C[[b]].x \subset A(E) \}.$$  

Remark that $B(E)$ and $A(E)$ are sub-$\tilde{A}$–modules of $E$ but that $A(E)$ is not stable by $b$.

Définition 6.3.1 A left $\tilde{A}$–module $E$ is called small when the following conditions hold

1. $E$ is a finite type $C[[b]]$–module;
2. $B(E) \subset A(E)$;
3. $\exists N / a^N.\tilde{A}(E) = 0$;

Recall that for $E$ small we have always the equality $B(E) = A(E)$ (see [B.I] lemme 2.1.2) and that this complex vector space is finite dimensional. The quotient $E/B(E)$ is an $(a,b)$-module called the associate $(a,b)$-module to $E$.

Conversely, any left $\tilde{A}$–module $E$ such that $B(E)$ is a finite dimensional $C$–vector space and such that $E/B(E)$ is an $(a,b)$-module is small.

The following easy criterium to be small will be used later:
Lemme 6.3.2 A left $\tilde{\mathbb{A}}$–module $E$ is small if and only if the following conditions hold:

1. $\exists N / a^N \hat{\mathbb{A}}(E) = 0$ ;
2. $B(E) \subset \hat{\mathbb{A}}(E)$ ;
3. $\cap_{m \geq 0} b^m \cdot E \subset \hat{\mathbb{A}}(E)$ ;
4. $\text{Ker} \thinspace b$ and $\text{Coker} \thinspace b$ are finite dimensional complex vector spaces.

As the condition 3 in the previous lemma has been omitted in [B.II] (but this does not affect this article because this lemma was used only in a case were this condition 3 was satisfied, thanks to proposition 2.2.1. of loc. cit.), we shall give the (easy) proof.

Proof. First the conditions 1 to 4 are obviously necessary. Conversely, assume that $E$ satisfies these four conditions. Then condition 2 implies that the action of $b$ on $\hat{\mathbb{A}}(E)/B(E)$ is injective. But the condition 1 implies that $b^{2N} = 0$ on $\hat{\mathbb{A}}(E)$ (see [B.I]). So we conclude that $\hat{\mathbb{A}}(E) = B(E) \subset \text{Ker} \thinspace b^{2N}$ which is a finite dimensional complex vector space using condition 4 and an easy induction. Now $E/B(E)$ is a $\mathbb{C}[[b]]$–module which is separated for its $b$–adic topology. The finitness of $\text{Coker} \thinspace b$ now shows that it is a free finite type $\mathbb{C}[[b]]$–module concluding the proof. ■

Définition 6.3.3 We shall say that a left $\tilde{\mathbb{A}}$–module $E$ is geometric when $E$ is small and when it associated $(a,b)$–module $E/B(E)$ is geometric.

The main result of this section is the following theorem, which shows that the Gauss-Manin connection of a proper holomorphic function produces geometric $\tilde{\mathbb{A}}$–modules associated to vanishing cycles and nearby cycles.

Théorème 6.3.4 Let $X$ be a connected complex manifold of dimension $n + 1$ where $n$ is a natural integer, and let $f : X \to D$ be an non constant proper holomorphic function on an open disc $D$ in $\mathbb{C}$ with center 0. Let us assume that $df$ is nowhere vanishing outside of $X_0 : = f^{-1}(0)$. Then the $\tilde{\mathbb{A}}$–modules

$$H^j(X, (\hat{K}^\bullet, d^\bullet))$$

and

$$H^j(X, (\hat{I}^\bullet, d^\bullet))$$

are geometric for any $j \geq 0$.

In the proof we shall use the $\mathcal{C}^\infty$ version of the complex $(\hat{K}^\bullet, d^\bullet)$. We define $K^p_\infty$ as the kernel of $\wedge df : \mathcal{C}^{\infty, p} \to \mathcal{C}^{\infty, p+1}$ where $\mathcal{C}^{\infty, j}$ denote the sheaf of $\mathcal{C}^\infty$– forms on $X$ of degree $p$, let $\hat{K}^\bullet_\infty$ be the $f$–completion and $(\hat{K}^\bullet_\infty, d^\bullet)$ the corresponding de Rham complex.

The next lemma is proved in [B.II] (lemma 6.1.1.)
Lemme 6.3.5 The natural inclusion

\[(\hat{K}^\bullet, d^\bullet) \hookrightarrow (\hat{K}_\infty^\bullet, d^\bullet)\]

induce a quasi-isomorphism.

Remark. As the sheaves \(\hat{K}_\infty^\bullet\) are fine, we have a natural isomorphism

\[H^p(X, (\hat{K}^\bullet, d^\bullet)) \simeq H^p(\Gamma(X, \hat{K}_\infty^\bullet), d^\bullet).\]

Let us denote by \(X_1\) the generic fiber of \(f\). Then \(X_1\) is a smooth compact complex manifold of dimension \(n\) and the restriction of \(f\) to \(f^{-1}(D^*)\) is a locally trivial \(\mathcal{C}^\infty\) bundle with typical fiber \(X_1\) on \(D^* = D \setminus \{0\}\), if the disc \(D\) is small enough around \(0\). Fix now \(\gamma \in H_p(X_1, \mathbb{C})\) and let \((\gamma_s)_{s \in D^*}\) the corresponding multivalued horizontal family of \(p\)-cycles \(\gamma_s \in H_p(X_s, \mathbb{C})\). Then, for \(\omega \in \Gamma(X, \hat{K}_\infty^p \cap \text{Ker} \ d)\), define the multivalued holomorphic function

\[F_\omega(s) := \int_{\gamma_s} \frac{\omega}{df}.\]

Let now

\[\Xi := \bigoplus_{\alpha \in \mathbb{Q}\cap[-1,0], j \in [0,n]} \mathbb{C}[[s]] \cdot s^\alpha \cdot \frac{(\text{Logs})^j}{j!}.\]

This is an \(\hat{A}\)–modules with \(a\) acting as multiplication by \(s\) and \(b\) as the primitive in \(s\) without constant. Now if \(\hat{F}_\omega\) is the asymptotic expansion at \(0\) of \(F_\omega\), it is an element in \(\Xi\), and we obtain in this way an \(\hat{A}\)–linear map

\[\text{Int} : H^p(X, (\hat{K}^\bullet, d^\bullet)) \to H^p(X_1, \mathbb{C}) \otimes_{\mathbb{C}} \Xi.\]

To simplify notations, let \(E := H^p(X, (\hat{K}^\bullet, d^\bullet))\). Now using Grothendieck theorem \([G.65]\), there exists \(N \in \mathbb{N}\) such that \(\text{Int}(\omega) \equiv 0\), implies \(a^N \cdot [\omega] = 0\) in \(E\). As the converse is clear we conclude that \(\hat{A}(E) = \text{Ker}(\text{Int})\). It is also clear that \(B(E) \subset \text{Ker}(\text{Int})\) because \(\Xi\) has no \(b\)–torsion. So we conclude that \(E\) satisfies properties 1 and 2 of the lemma \([6.3.2]\). The property 3 is also true because of the regularity of the Gauss-Manin connection of \(f\).

End of the proof of theorem \([6.3.4]\). To show that \(E := H^p(X, (\hat{K}^\bullet, d^\bullet))\) is small, it is enough to prove that \(E\) satisfies the condition 4 of the lemma \([6.3.2]\). Consider now the long exact sequence of hypercohomology of the exact sequence of complexes

\[0 \to (\hat{I}^\bullet, d^\bullet) \to (\hat{K}^\bullet, d^\bullet) \to ([\hat{K} / \hat{I}]^\bullet, d^\bullet) \to 0.\]

It contains the exact sequence

\[H^{p-1}(X, ([\hat{K} / \hat{I}]^\bullet, d^\bullet)) \to H^p(X, (\hat{I}^\bullet, d^\bullet)) \xrightarrow{\text{Hyp}(i)} H^p(X, (\hat{K}^\bullet, d^\bullet)) \to H^p(X, ([\hat{K} / \hat{I}]^\bullet, d^\bullet)).\]
and we know that $b$ is induced on the complex of $\tilde{A}$–modules quasi-isomorphic to $(\hat{K}^\bullet, d^\bullet)$ by the composition $i \circ \tilde{b}$ where $\tilde{b}$ is a quasi-isomorphism of complexes of $\mathbb{C}[[b]]$–modules. This implies that the kernel and the cokernel of $H^p(i)$ are isomorphic (as $\mathbb{C}$–vector spaces) to $\ker b$ and $\text{Coker} b$ respectively. Now to prove that $E$ satisfies condition 4 of the lemma [6.3.2] it is enough to prove finite dimensionality for the vector spaces $H^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$ for all $j \geq 0$.

But the sheaves $[\hat{K}/\hat{I}]^j \simeq \ker d \frac{df}{Im df}$ are coherent on $X$ and supported in $X_0$. The spectral sequence $E_{pq}^2 := H^q(H^p(X, [\hat{K}/\hat{I}]^\bullet), d^\bullet)$ which converges to $H^j(X, ([\hat{K}/\hat{I}]^\bullet, d^\bullet))$ is a bounded complex of finite dimensional vector spaces by Cartan-Serre. This gives the desired finite dimensionality.

To conclude the proof, we want to show that $E/B(E)$ is geometric. But this is an easy consequence of the regularity of the Gauss-Manin connexion of $f$ and of the Monodromy theorem, which are already coded in the definition of $\Xi$: the injectivity on $E/B(E)$ of the $\tilde{A}$–linear map $\text{Int}$ implies that $E/B(E)$ is geometric.

Remark now that the piece of exact sequence above gives also the fact that $H^p(X, (\hat{I}^\bullet, d^\bullet))$ is geometric, because it is an exact sequence of $\tilde{A}$–modules.

Remark. It is easy to see that the properness assumption on $f$ is only used for two purposes:

– To have a (global) $C^\infty$ Milnor fibration on a small punctured disc around $0$, with a finite dimensional cohomology for the Milnor fiber.

– To have compactness of the singular set $\{ df = 0 \}$.

This allows to give with the same proof an analogous finiteness result in many other situations.

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