Donovan’s conjecture and extensions by the centralizer of a defect group∗

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Abstract

We consider Donovan’s conjecture in the context of blocks of groups $G$ with defect group $D$ and normal subgroups $N < G$ such that $G = C_D(D \cap N)N$, extending similar results for blocks with abelian defect groups. As an application we show that Donovan’s conjecture holds for blocks with defect groups of the form $Q_8 \times C_2^n$ or $Q_8 \times Q_8$ defined over a discrete valuation ring.

1 Introduction

Let $p$ be a prime and $k := \mathbb{F}_p$. Let $(K, \mathcal{O}, k)$ be a $p$-modular system, so $\mathcal{O}$ is a complete discrete valuation ring with residue field $k$. Donovan’s conjecture states that for a given finite $p$-group $P$, there are only finitely many Morita equivalence classes amongst blocks of finite groups with defect groups isomorphic to $P$ (this may be stated over $k$ or $\mathcal{O}$). In reducing Donovan’s conjecture to quasisimple groups, we must inevitably compare blocks $B$ of finite groups $G$ with those of normal subgroups $N$. The case where $N$ contains a defect group $D$ of $B$ was treated by Külshammer in [25] for $k$-blocks, and by Eisele in [16] for $\mathcal{O}$-blocks. This paper concerns the problematic case of normal subgroups of index a power of $p$, where it suffices to assume $G = ND$. The subcase that $D$ is abelian was first considered, for $k$-blocks and with an additional splitting condition, in [23]. In [13] the full $D$ abelian case was treated by introducing strong Frobenius numbers, related to the Morita Frobenius numbers introduced in [21]. The approach taken here also involves strong Frobenius numbers.

The purpose of this paper is to extend the reduction result above to the case that $G = C_D(D \cap N)N$. As an application, we show that Donovan’s conjecture with respect to $\mathcal{O}$ holds when $D \cong Q_8 \times C_2^n$ or $Q_8 \times Q_8$ for some $n$. Blocks with defect group $Q_2^m \times C_2^n$ were studied by Sambale in [29] and the number of irreducible characters and Brauer characters computed. Donovan’s conjecture for $\mathcal{O}$-blocks with defect group $Q_8$ was proved by Eisele in [15].

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In order to make canonical choices of $k$ and $\mathcal{O}$, we choose $k$ to be the algebraic closure of the field $\mathbb{F}_p$ of $p$-elements and $\mathcal{O}$ to be the ring of Witt vectors for $k$. There is a discussion of this in [14]. When we need to differentiate the versions of Donovan conjecture, the $R$-Donovan conjecture will relate to blocks defined with respect to the ring $R$, where $R$ may be $k$ or $\mathcal{O}$.

We also consider the following, based on a question of Brauer, which is now often referred to as the weak Donovan conjecture:

**Conjecture 1.1** Let $P$ be a finite $p$-group. There is $c \in \mathbb{N}$ such that for all blocks $B$ of finite groups $G$ with defect groups isomorphic to $P$, the entries of the Cartan matrix of $B$ are at most $c$.

In [21] Kessar showed that the $k$-Donovan conjecture is equivalent to showing Conjecture 1.1 and that the Morita Frobenius number (defined in [22]) of a block is bounded in terms of the order of the defect groups. Variations on the Morita Frobenius number for blocks defined over $\mathcal{O}$ were given in [13], including the strong $\mathcal{O}$-Frobenius number $\text{sf}_\mathcal{O}(B)$, and in [14] the analogue of Kessar’s result was shown for blocks defined over $\mathcal{O}$. In [11] Düvel reduced Conjecture 1.1 to quasisimple groups (although the result in [11] is not quite strong enough for our purposes as stated). Our first reduction result concerns the other half of the problem, i.e., bounding the strong $\mathcal{O}$-Frobenius numbers in terms of the defect groups:

**Theorem 1.2** Let $G$ be a finite group and $B$ be a block of $\mathcal{O}G$ with defect group $D$. In addition, let $N \triangleleft G$ such that $G = C_D(D \cap N)N$ and let $b$ be a block of $\mathcal{O}N$ covered by $B$. Then $\text{sf}_\mathcal{O}(B) \leq \text{sf}_\mathcal{O}(b)$.

Combining with Lemma 3.1 we obtain the following:

**Theorem 1.3** Let $P$ be a finite $p$-group. In order to verify Donovan’s conjecture for $P$ for $\mathcal{O}$-blocks, it suffices to check it for blocks of finite groups $G$ with defect group $D$ isomorphic to a subgroup of $P$ and no proper normal subgroup $N \triangleleft G$ such that $G = C_D(D \cap N)N$.

After analysing the blocks of quasisimple groups with defect groups $Q_8 \times C_{2^n}$ or $Q_8 \times Q_8$ in Section 4 using [2] (in the latter case there are none), we show:

**Theorem 1.4** There are only finitely many Morita equivalence classes of $\mathcal{O}$-blocks of finite groups with defect groups isomorphic to $Q_8 \times C_{2^n}$ or $Q_8 \times Q_8$, where $n \geq 0$.

The reader might ask what happens for some similar $p$-groups. We note that $Q_8 \times C_{2^n}$ and $Q_8 \times Q_8$ have particularly restricted subgroup structures, meaning that covered blocks of normal subgroups are either amenable to the application of Theorem 1.2 or are easily dealt with by other methods. If, for example, the defect group were $Q_{16} \times C_{2^n}$, then we would have to consider the case of a normal subgroup with defect group $Q_8 \times C_{2^n}$, a case which does arise and to which Theorem 1.2 would not apply. Similarly, $D_8 \times C_2$ is problematic as it contains a normal subgroup $(C_2)^3$. New methods will be needed for these cases.
The structure of the paper is as follows. In Section 2 we treat strong Frobenius numbers and prove Theorem 1.2. We prove Theorem 1.3 in Section 3. In Section 4 we show that there are no blocks of quasisimple groups with defect group $Q_8 \times Q_8$, and few with defect group $Q_8 \times C_{2n}$ for $n \geq 1$. We give some preliminary reductions and results about blocks with the above defect groups in Section 5 and the proof that Donovan’s conjecture holds for these blocks in Section 6.

2 Strong Frobenius numbers and extensions by the centralizer of a defect group

Throughout this section, let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$. We denote by $\text{Irr}(G)$ the set of irreducible characters of $G$ and $\text{Irr}(B)$ the subset of $\text{Irr}(G)$ of irreducible characters lying in the block $B$. We write $kB$ for the block of $kG$ corresponding to $B$. We denote by $e_B \in OG$ the block idempotent for $B$ and by $e_\chi \in Q_cG$ the character idempotent for $\chi \in \text{Irr}(G)$, where $Q_c$ is the universal cyclotomic extension of $\mathbb{Q}$. If $A_1$ and $A_2$ are finitely generated $k$-algebras (respectively $O$-algebras), we write $A_1 \sim_{\text{Mor}} A_2$ if $A_1$ and $A_2$ are Morita equivalent as $k$-algebras (respectively $O$-algebras). We quote the following definition from [13, Definition 3.2].

**Definition 2.1** Let $q$ be a, possibly zero or negative, power of $p$. We denote by $-^{(q)} : k \to k$ the field automorphism given by $\lambda \mapsto \lambda^{q}$. Let $A$ be a $k$-algebra. We define $A^{(q)}$ to be the $k$-algebra with the same underlying ring structure as $A$ but with a new action of the scalars given by $\lambda a = \lambda^{(q)} a$, for all $\lambda \in k$ and $a \in A$. For $a \in A$ we define $a^{(q)}$ to be the element of $A$ associated to $a$ through the ring isomorphism between $A$ and $A^{(q)}$. Note that we have $kG \cong kG^{(q)}$ as we can identify $-^{(q)} : kG \to kG$ with the ring isomorphism:

$$-^{(q)} : kG \to kG$$

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} (\alpha_g)^q g.$$

From now on, we identify $(kB)^{(q)}$ with the image of $kB$ under the above isomorphism. We define $B^{(q)}$ to be the unique block of $OG$ satisfying $k(B^{(q)}) = (kB)^{(q)}$.

By an abuse of notation, we also use $-^{(q)}$ to denote the field automorphism of $Q_c$ defined by $\omega_p \omega_p^{q} \mapsto \omega_p^{q} \omega_p^{q}$, for all $p^{th}$-power roots of unity $\omega_p$ and $p^{th}$ roots of unity $\omega_p^{q}$, and also the ring automorphism

$$-^{(q)} : Q_cG \to Q_cG$$

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} (\alpha_g)^{(q-1)} g.$$

If $\chi \in \text{Irr}(G)$, then we define $\chi^{(q)} \in \text{Irr}(G)$ to be given by $\chi^{(q)}(g) = \chi(g)^{(q-1)}$, for all $g \in G$. Note that if $\chi \in \text{Irr}(B)$, then $(e^{(q)}_\chi) = e^{(q)}_\chi$ and $\chi^{(q)} \in \text{Irr}(B^{(q)})$. 

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For \( R \in \{ \mathbb{Z}, \mathbb{Q}_c, \mathcal{O} \} \), we define the \( R \)-linear map \( \zeta_G : RG \to RG, g \mapsto g_pg'_p \), where \( g_p \) and \( g'_p \) are the \( p \)-part and \( p' \)-part of \( g \) respectively. In general, the relevant ring \( R \) should always be clear from the context. The following was proved in [13, Proposition 3.5].

**Proposition 2.2** \( \zeta_G \) restricted to \( \mathbb{Z}(\mathbb{Z}G) \subseteq \mathbb{Z}G \) induces a \( \mathbb{Z} \)-algebra automorphism of \( \mathbb{Z}(\mathbb{Z}G) \). Furthermore the algebra automorphism induced on \( \mathbb{Z}(\mathbb{Q}_cG) \cong \mathbb{Q}_c \otimes_{\mathbb{Z}} \mathbb{Z}(\mathbb{Z}G) \) sends \( e_\chi \) to \( e_{\chi(p)} \), for all \( \chi \in \text{Irr}(G) \).

We now generalise the above Proposition to deal with normal subgroups of index a power of \( p \). In what follows, \( A^H \) will denote the set of fixed points in \( A \) under the action of \( H \), where \( A \) is an algebra with an action of a group \( H \). In practice, \( A \) will always be the group algebra of \( G \) or one of its blocks with the natural conjugation action of \( H \leq G \). For each \( n \in \mathbb{N} \), we use \( \omega_n \in \mathbb{Q}_c \) to denote some fixed primitive \( n \)-th root of unity.

**Proposition 2.3** Let \( N \triangleleft G \) be of index a power of \( p \). Then \( \zeta_G \) induces a \( \mathbb{Z} \)-algebra automorphism of \( (\mathbb{Z}G)^N \).

**Proof.** As noted in the proof of [13, Proposition 3.5], \( \zeta_G \) commutes with conjugation by any \( g \in G \). Therefore, since \( (\mathbb{Z}G)^N \) has a \( \mathbb{Z} \)-basis consisting of the \( N \)-conjugacy class sums of \( G \), \( \zeta_G \) maps \( (\mathbb{Z}G)^N \) to itself. Hence, it is sufficient to prove that \( \zeta_G \) induces a \( \mathbb{Q}_c \)-algebra automorphism of \( (\mathbb{Q}_cG)^N \cong \mathbb{Q}_c \otimes_{\mathbb{Z}} (\mathbb{Z}G)^N \). We do this by showing that \( \zeta_G \) induces an isomorphism

\[
(\mathbb{Q}_cG)^N e_\chi \rightarrow (\mathbb{Q}_cG)^Ne_{\chi(p)},
\]

for all \( \chi \in \text{Irr}(N) \). Note that \( \text{Stab}_G(\chi) = \text{Stab}_G(\chi^{(p)}) \) and if \( g \notin \text{Stab}_G(\chi) \), then

\[
C_g^N e_\chi = e_\chi C_g^N = C_g^N e_\chi e_\chi = 0,
\]

where \( C_g^N \in \mathbb{Z}G \) is the sum of the elements in the \( N \)-conjugacy class containing \( g \). We may, therefore, assume that \( G = \text{Stab}_G(\chi) \). For each \( g \in G \) we define

\[
e_{\chi,gN} := \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i e_{\chi',\lambda^i},
\]

where \( \chi' \) is an extension of \( \chi \) to \( \langle g \rangle N \) and \( \lambda \) is the linear character of \( \langle g \rangle N \) given by

\[
\lambda : \langle g \rangle N \rightarrow \mathbb{Q}_c
\]

\[
g^iN \mapsto \omega_{\text{ord}(gN)}^i.
\]

If we define the \( \mathbb{Q}_c \)-algebra automorphism

\[
\tilde{\lambda} : \mathbb{Q}_c(\langle g \rangle N) \rightarrow \mathbb{Q}_c(\langle g \rangle N)
\]

\[
h \mapsto \lambda(h)h,
\]

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for all \( h \in (g)N \), then

\[
\tilde{\lambda}(e_{\chi,gN}) = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i \tilde{\lambda}(e_{\chi',\lambda_i}) = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i e_{\chi',\lambda_i-1} = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i e_{\chi,gN}.
\]

(1)

Therefore, \( e_{\chi,gN} \in g(Q_cNe_\chi) \). Since we are making a choice of extension \( \chi' \) of \( \chi \), \( e_{\chi,gN} \) is only defined uniquely up to multiplication by some power of \( \omega_{\text{ord}(gN)} \). With this in mind, we introduce the notation of \( \alpha \approx \beta \), for \( \alpha, \beta \in (Q_cG)^N \), if \( \alpha = \mu \beta \) for some \( p^\text{th} \)-power root of unity \( \mu \in Q_c \). Note that if when defining \( e_{\chi,g^{-1}N} \) we choose the same extension \( \chi' \) of \( \chi \) as when defining \( e_{\chi,gN} \), i.e.,

\[
e_{\chi,g^{-1}N} := \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i e_{\chi',\lambda_i} = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^{-i} e_{\chi',\lambda_i},
\]

then

\[
e_{\chi,gN} e_{\chi,g^{-1}N} = \sum_{i=1}^{\text{ord}(gN)} e_{\chi',\lambda_i} = e_\chi.
\]

Therefore, for any choice of \( e_{\chi,g^{-1}N}, e_{\chi,gN} e_{\chi,g^{-1}N} \approx e_\chi \) and so we have shown that \((Q_cG)^Ne_\chi \) is a crossed product of \( G/N \) with \( Z(Q_cNe_\chi) = Q_c e_\chi \) in the sense of Külshammer [25]. In other words, \((Q_cG)^Ne_\chi = \bigoplus_{gN \in G/N} Q_c e_{\chi,gN} \). Now

\[
\zeta_G(e_{\chi,gN}) = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i \zeta_G(e_{\chi',\lambda_i}) = \sum_{i=1}^{\text{ord}(gN)} \omega_{\text{ord}(gN)}^i e_{(\chi',\lambda_i)^{(p)}}
\]

(2)

where the second equality follows from Proposition [2.2]. Therefore, \( \zeta_G \) induces a \( Q_c \)-vector space isomorphism between \((Q_cG)^Ne_\chi \) and \((Q_cG)^Ne_{\chi,(p)} \).

Before proceeding we note that, for all \( g, h \in G \), \((e_{\chi,gN})^{\text{ord}(gN)} = e_\chi \). Also, as noted just after (1), \( e_{\chi,gN} = gxe_\chi \), for some \( x \in (Q_cNe_\chi)^X \) and so \( e_{\chi,gN} e_{hN} e_{\chi,gN}^{-1} = g e_{\chi,N} g^{-1} \approx e_{\chi,gN} \). We use both these facts below in (3).

It remains to show that for all \( g, h \in G \),

\[
\zeta_G(e_{\chi,gN} e_{\chi,hN}) = \zeta_G(e_{\chi,gN}) \zeta_G(e_{\chi,hN}).
\]

By (2), \( \zeta_G(e_{\chi,gN}) = e_{\chi,gN}^{(p)} \) and \( \zeta_G(e_{\chi,hN}) = e_{\chi,hN}^{(p)} \) and so we need only show that \( \zeta_G(e_{\chi,gN} e_{\chi,hN}) = (e_{\chi,gN} e_{\chi,hN})^{(p)} \). Since \( \zeta_G \) fixes coefficients that are \( p^\text{th} \)-power roots of unity, it suffices in turn to prove that \( e_{\chi,gN} e_{\chi,hN} \approx e_{\chi,gN} \). We prove this last statement via induction on the lowest layer of \( gN, hN \) or \( ghN \) in the upper central series of \( G/N \). Note that

\[
e_{\chi,gN} e_{\chi,hN} \approx e_{\chi,hN} \Leftrightarrow e_{\chi,hN} e_{\chi,(gh)^{-1}N} \approx e_{\chi,g^{-1}N} \Leftrightarrow e_{\chi,(gh)^{-1}N} e_{\chi,gN} \approx e_{\chi,h^{-1}N}
\]
and so we may assume that \( hN \) is in the lowest layer. If \( h \in N \), then \( e_{\chi, hN} = e_{\chi} \) and so \( e_{\chi, gN}e_{\chi, hN} = e_{\chi, ghN} \). Now let \( g, h \) be arbitrary elements of \( G \) and set \( p^n := \max\{\text{ord}(gN), \text{ord}(hN), \text{ord}(ghN)\} \). Then

\[
(e_{\chi, gN}e_{\chi, hN})^{p^n} \approx (e_{\chi, gN})^{p^n}(e_{\chi, hN}^{p^n-1} \cdots e_{\chi, hN}) \approx (e_{\chi, hN}^{p^n-1} \cdots e_{\chi, hN}) (e_{\chi, hN})^{p^n-1} \\
\approx (e_{\chi, hN}^{p^n-1} \cdots e_{\chi, hN})(e_{\chi, hN}^{p^n-2} \cdots e_{\chi, hN}) \\
\approx e_{\chi, [g^{-(p^n-1)}, h]N} \cdots e_{\chi, [h^{p^n-2}, g^{-1}, h]N} \approx e_{\chi, [g^{-(p^n-1)}, h] \cdots [h^{p^n-2}, g^{-1}, h]N} \\
= e_{\chi, (gh)^{p^n}N} = e_{\chi},
\]

where the fourth and fifth relations follow from the inductive hypothesis. Now, since \( e_{\chi, gN}e_{\chi, hN} \in gh(\mathbb{Q}_c N e_{\chi}) \cap (\mathbb{Q}_c G)^N e_{\chi} = \mathbb{Q}_c e_{\chi, ghN} \), we have \( e_{\chi, gN}e_{\chi, hN} \approx e_{\chi, ghN} \). \( \square \)

For the following definitions see [21] and [13].

Definition 2.4 The Morita Frobenius number \( mf(A) \) of a finite dimensional \( k \)-algebra \( A \) is the smallest integer \( n \) such that \( A \approx_{\text{Mor}} A^{(p^n)} \) as \( k \)-algebras. The \( \mathcal{O} \)-Morita Frobenius number \( mf_{\mathcal{O}}(B) \) of \( B \) is the smallest integer \( n \) such that \( B \approx_{\text{Mor}} B^{(p^n)} \) as \( \mathcal{O} \)-algebras. The strong \( \mathcal{O} \)-Frobenius number \( sf_{\mathcal{O}}(B) \) of \( B \) is the smallest integer \( n \) such that there exists an \( \mathcal{O} \)-algebra isomorphism \( \phi : B \to B^{(p^n)} \) with the induced bijection of characters given by \( \chi \mapsto \chi^{(p^n)} \), for all \( \chi \in \text{Irr}(B) \) or, equivalently, that \( \phi \) restricted to \( Z(B) \) coincides with \( \zeta^n \). Such an isomorphism \( \phi \) is called a strong Frobenius isomorphism of degree \( n \).

Before the next lemma we need to give a brief overview of Picard groups of blocks. For a more detailed discussion see [3, §1].

Let \( A \) be an \( \mathcal{O} \)-algebra. The Picard group \( \text{Pic}(A) \) of \( A \) consists of isomorphism classes of \( A \)-\( A \)-bimodules which induce \( \mathcal{O} \)-linear Morita auto-equivalences of \( A \). \( \text{Pic}(A) \) forms a group with the group multiplication given by tensoring over \( A \). For each \( \alpha \in \text{Aut}(A) \) we define \( A^\alpha \) to be the \( A \)-\( A \)-bimodule with the canonical right action of \( A \) and the left action of \( A \) given via \( \alpha \). Moreover, \( A^\alpha \cong A \) as bimodules if and only if \( \alpha \in \text{Inn}(A) \). In other words, we can view \( \text{Out}(A) = \text{Aut}(A)/\text{Inn}(A) \) as a subgroup of \( \text{Pic}(A) \).

Now set \( A := iBi \), where \( i \in B^D \) is a source idempotent of \( B \). We identify \( D \) with its image in \( A \) and denote by \( \text{Aut}_D(A) \) the group of \( \mathcal{O} \)-algebra automorphisms of \( A \) which fix \( D \) pointwise. In addition, we set \( \text{Out}_D(A) \) to be the image of \( \text{Aut}_D(A) \) in \( \text{Out}(A) \), in other words \( \text{Out}_D(A) \) is the quotient of \( \text{Aut}_D(A) \) by the subgroup of inner automorphisms induced by conjugation with elements in \( (A^D)^\times \). We set \( \mathcal{T}(B) \) to be the subgroup of bimodules in \( \text{Pic}(B) \) with trivial source, when considered as \( \mathcal{O}(G \times G) \)-modules. Through the natural Morita equivalence between \( B \) and \( A \), we may identify \( \text{Out}_D(A) \) with the subgroup of \( \mathcal{T}(B) \) consisting of bimodules with vertex \( \Delta D \).

The following Lemma and Theorem were shown for \( D \) abelian in [13] Theorems 3.15, 3.16:

Lemma 2.5 Let \( N \triangleleft G \) such that \( G = C_D(D \cap N)N \) and \( b \) a block of \( ON \) covered by a block \( B \) of \( OG \) with defect group \( D \). Then \( e_B = e_b \) and \( b \) has defect group \( D \cap N \).

Viewing \( b \) as a subalgebra of \( B \), there is a choice of \( a_{gN} \in (gb)^\times \) for each \( gN \in G/N \) such that \( B^N = \bigoplus_{gN \in G/N} a_{gN}Z(b) \).
PROOF. By [27, Corollary 5.5.6], $b$ is the unique block of $OG$ covering $b$ and so $e_b$ is the sum of $e_b$ and its $G$-conjugates. However, [27, Theorem 5.5.10(v)] gives that $b$ is $G$-stable, so that $e_b = e_b$ and $D \cap N$ is a defect group for $b$.

Since $B^N$ is $G/N$-graded, it remains to find a unit $a_{gN}$ in the $gN$-graded component of $B^N$, for each $gN \in G/N$. Let $g \in C_D(D \cap N)$ and consider $c_g \in \text{Aut}(B)$ given by conjugation by $g$. Now $c_g$ induces the element

$$M := \mathcal{O}_{\Delta g} \uparrow^{N \times N} e_b \in \text{Pic}(b),$$

where $\Delta g = \{(h, c_g(h))| h \in N\} \leq N \times N$ and $\mathcal{O}_{\Delta g}$ is the trivial $\mathcal{O}(\Delta g)$-module. In particular, $M \in T(b)$ and so, by [26, Theorem 1.1(i)], $M$ has vertex $\{(h, c_g(h))| h \in D \cap N\} = \Delta(D \cap N)$. Therefore, by the comments preceding the lemma, $iMi \in \text{Out}_{D \cap N}(ibi)$, where $i$ is a source idempotent for $b$. Now by [28, 14.5, Proposition 14.9], $\text{Out}_{D \cap N}(ibi)$ is a $p'$-subgroup of $T(b)$. In particular, $iMi$ has $p'$-order in $\text{Out}(ibi)$ or, equivalently, $M$ has $p'$-order in $\text{Out}(b)$. However, since $g \in D$, $c_g$ has order a power of $p$ meaning $M$ induces the trivial auto-equivalence and $c_g \in \text{Inn}(b)$.

Let $gN$ be a left coset of $N$ in $G$, where we choose coset representative $g \in C_D(D \cap N)$. Set $a_g \in b^c$ such that $c_g$ is given by conjugation by $a_g$. We set $a_{gN} := g(a_g)^{-1}$. \hfill \qedsymbol

Note that, if we assume the stronger condition $D = (D \cap N)Z(D)$, then $B^N$ can be replaced with $Z(B)$ in Lemma 2.5. The result would then be proved via induction on $|G/N|$, since when $G/N$ is cyclic the $a_{gN}$'s constructed above are in $Z(B)$. We now prove Theorem 1.2

PROOF OF THEOREM 1.2. Set $n := sf_{\mathcal{O}}(b)$ and let $\phi : b \to b^{(p^n)}$ be a strong $\mathcal{O}$-Frobenius isomorphism of degree $n$. Then, by Proposition 2.3, we can extend $\phi$ to an isomorphism $\tilde{\phi} : B \to B^{(p^n)}$ by sending $a_{gN}$ to $\zeta_G^{n}(a_{gN})$, for each left coset $gN$ of $N$ in $G$ and the $a_{gN}$'s are as in Lemma 2.5. By construction, $\tilde{\phi}$ agrees with $\zeta_G^{n}$ on $B^N$ so certainly it does on $Z(B)$ and we have $sf_{\mathcal{O}}(B) \leq n$.

In the case $D = (D \cap N)Z(D)$, the proof of Theorem 1.2 can be greatly shortened as we may avoid reference to Proposition 2.3. Indeed, given the comments following Lemma 2.5, we only need that $\zeta_G$ induces an $\mathcal{O}$-algebra automorphism between $Z(B)$ and $Z(B^{(p)})$. The proof then follows in very much the same vein as the abelian defect group case in [13, Theorems 3.16]. \hfill \qedsymbol

3 A reduction theorem for Donovan’s conjecture

The following reduction result for Cartan invariants is presumably well-known, but we provide a proof. Write $c(B)$ for the largest entry of the Cartan matrix of a block $B$.

Lemma 3.1 Let $G$ be a finite group and $B$ a block of $kG$.

(i) Let $N \triangleleft G$ have index $p^r$ and suppose $B$ covers a block $b$ of $kN$. Then $c(B) \leq p^rc(b)$.

(ii) Let $Z \leq Z(G)$ be a 2-group and $\mathcal{B}$ the corresponding block of $G/Z$. Then $c(\mathcal{B}) \leq c(B)$. 

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Proof.

(i) Let $B'$ be the unique block of $k\text{Stab}_G(b)$ covering $b$. Then, by [21, Theorem C], $B' \sim_{\text{Mor}} B$ and so we may assume that $b$ is $G$-stable and, as noted in the proof of Lemma 2.5, that $e_B = e_b$. Therefore, by Green’s Indecomposability Theorem, for every projective, indecomposable $b$-module $M$, the induced module $M \uparrow^G$ is a projective, indecomposable $B$-module. Furthermore, since every projective $B$-module $L$ is a summand of $e_B(L \downarrow_N \uparrow^G) = (e_bL \downarrow_N) \uparrow^G$, in fact every projective, indecomposable $B$-module is isomorphic to $M \uparrow^G$ for some projective, indecomposable $b$-module $M$.

Now fix some projective, indecomposable $b$-module $M$ and consider the composition factors of $M \uparrow^G$. Let $S$ be a simple $B$-module and $T$ a simple $b$-module appearing as a composition factor in $S \downarrow_N$. Certainly the multiplicity of $S$ among the composition factors of $M \uparrow^G$ is at most the multiplicity of $T$ among the composition factors of $(M \uparrow^G) \downarrow_N$. However, $(M \uparrow^G) \downarrow_N$ is the direct sum of $p'$ projective, indecomposable $b$-modules, the $G$-conjugates of $M$. The claim follows.

(ii) For a $kG$-module $M$ we define the $k(G/Z)$-module $ZM$ to be the fixed points of $M$ under left multiplication by $Z$. Note that $Z(kG) \cong k(G/Z)$. Therefore, if $M$ is the projective cover of the simple $B$-module $S$, then $ZM$ is the projective cover of the simple $B$-module $ZS = S$. In particular, $c(B) \leq c(B')$.

We now prove Theorem 1.3.

Proof of Theorem 1.3. Fix a finite $p$-group $P$.

Let $\mathcal{X}$ be the class of blocks $C$ of $O_H$ for some finite group $H$ with defect group $Q$ isomorphic to a subgroup of $P$ and no normal subgroup $M < H$ such that $H = C_Q(Q \cap M)M$. Suppose that there are only finitely many Morita equivalence classes amongst the members of $\mathcal{X}$. Then there is a largest Cartan invariant $c$ and a largest strong $O$-Frobenius number $s$ amongst blocks in $\mathcal{X}$.

Let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$ isomorphic to a subgroup of $P$. We claim that $c(B) \leq |D|c$ and $sf_O(B) \leq s$. Suppose that $c(B) > |D|c$ and that $|G|$ is minimal with respect to these conditions. By the definition of the constant $c$, $B$ is not in $\mathcal{X}$ and so there is a proper subgroup $N < G$ with $G = C_D(D \cap N)N$. Let $b$ be a block of $ON$ covered by $B$. Note that by Lemma 2.5, $b$ has defect group $D \cap N$. Then by Lemma 3.1(i)

$$c(b) \geq [D : D \cap N]^{-1}c(B) > [D \cap N]c,$$

contradicting the minimality of $G$. A similar argument using Theorem 1.2 shows the bound on strong $O$-Frobenius numbers. We have shown that the Cartan invariants and the strong $O$-Frobenius numbers of all blocks with defect group isomorphic to $P$ are bounded, and so the result follows by [14, Corollary 3.11].

$\square$
4 Blocks of quasisimple groups with defect groups $Q_8 \times C_{2^n}$ and $Q_8 \times Q_8$

Let $G$ be a block of a finite group $G$ with defect group $D$ and maximal $B$-subpair $(D, b_D)$. Recall that $B$ is controlled if for all $B$-subpairs $(Q, b_Q) \leq (D, b_D)$ and $g \in G$ with $(Q, b_Q)^g \leq (D, b_D)$, there are $c \in C_G(Q)$ and $n \in N_G(D, b_D)$ such that $g = cn$. We will use [31, Theorem 4.8] to observe that every block with defect group $Q_8^m \times A$ for $m \geq 0$ and $A$ an abelian 2-group is controlled, and apply the classification of controlled blocks of quasisimple groups given in [2]. To do so we first review some notation.

Let $F$ be a saturated fusion system on a $p$-group $D$. A subgroup $Q \leq D$ is weakly $F$-closed if for any $\phi \in \text{Hom}_F(Q, D)$ we have $\phi(Q) = Q$, and $Q$ is strongly $F$-closed if for any $P \leq Q$ and any $\phi \in \text{Hom}_F(P, D)$ we have $\phi(P) \leq Q$.

The normalizer $N_F(D)$ is the fusion subsystem of $F$ on $D$ such that for all $P, Q \leq D$, the morphisms $\text{Hom}_F(P, Q)$ are those $\phi \in \text{Hom}_F(P, Q)$ such that there is $\bar{\phi} \in \text{Hom}_F(D, D)$ extending $\phi$.

A $p$-group $D$ is called resistant if $F = N_F(D)$ whenever $F$ is a saturated fusion system on $D$.

**Proposition 4.1** Let $D = Q_8^m \times A$, where $m \geq 0$ and $A$ is an abelian 2-group. Then $D$ is resistant.

**Proof.** Let $F$ be a saturated fusion system on $D$. By [31, Theorem 4.8] $F = N_F(D)$ if and only if there is a central series $D = Q_n \geq Q_{n-1} \geq \cdots \geq Q_1 \geq 1$ with each $Q_i$ weakly $F$-closed and $D$ strongly $F$-closed.

Now let $Q_1 = \Omega_1(D)$, the (unique) largest elementary abelian subgroup of $D$, and $Q_2 = D$, so $Q_2 \geq Q_1 \geq 1$ forms a central series. Since $Q_1$ is the unique elementary abelian subgroup of $D$ of maximal rank, it must be weakly $F$-closed. Also $D$ is automatically strongly $F$-closed, so the result follows. \hfill $\Box$

Now let $F$ be the fusion system on $D$ afforded by $B$ and $(D, b_D)$, sometimes written $F_{(D, b_D)}(G)$. By Alperin’s fusion theorem $B$ is controlled if and only if $N_G(D, b_D)$ controls strong fusion in $D$ (see [1, Proposition 4.24]). This is equivalent to $F = N_F(D)$. We conclude that every block with defect group $D \cong Q_8^m \times A$, where $m \geq 0$ and $A$ is an abelian 2-group, is controlled.

**Proposition 4.2** Let $G$ be a finite quasisimple group. Let $m \geq 1$ and $A$ be a finite abelian 2-group. If $m > 1$, then there is no block of $G$ with defect groups isomorphic to $(Q_8)^m \times A$. If $B$ is a block of $G$ with defect group $Q_8 \times A$ where $A$ is nontrivial, then $G$ is a quotient of a classical group of Lie type not of type $A$ or $2A$, defined over a field of order $q$ a power of an odd prime and $B$ corresponds to a non-quasi-isolated block of the corresponding group of Lie type. Furthermore, if $m = 1$ and $A \cong C_4$, then $G \cong Sp_{2r}(q)$ for some $r$.

**Proof.** By the discussion above, all blocks with these defect groups are controlled. The result follows directly from [2, Theorem 1.1] and its proof. \hfill $\Box$
When we come to consider blocks of arbitrary groups with defect group \(Q_8 \times Q_8\), we must deal with the case of blocks \(b\) of \(Sp_{2r}(q)\) with defect group \(Q_8 \times C_4\). We show that there can be no overgroup of \(Sp_{2r}(q)\) in \(\text{Aut}(Sp_{2r}(q))\) possessing a block covering \(b\) with defect group \(Q_8 \times Q_8\).

In what follows, for \(r \in \mathbb{N}\) and \(q\) a power of a prime, we set \(I_r \in GL_r(q)\) to be the \(r \times r\) identity matrix,

\[
J_r := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in GL_r(q)
\]

and

\[
\Omega_{2r} := \begin{pmatrix} 0 & J_r \\ -J_r & 0 \end{pmatrix} \in GL_{2r}(q).
\]

We define

\[
Sp_{2r}(q) := \{ x \in GL_{2r}(q) | x\Omega_{2r}x^T = \Omega_{2r}\},
\]

\[
CSp_{2r}(q) := \{ x \in GL_{2r}(q) | x\Omega_{2r}x^T = \lambda\Omega_{2r} \text{ for some } \lambda \in \mathbb{F}_q^\times \}.
\]

**Lemma 4.3** Let \(H := Sp_{2r}(q)\), where \(4 \mid (q - 1)\) but \(8 \nmid (q - 1)\) and \(b\) a block of \(\mathcal{O}H\) with defect group \(P \cong Q_8 \times C_4\) labelled by some \(s \in H^* = SO_{2r+1}(q)\) such that \(m_{X-1}(s) = 3\) and other \(m_F(s) \leq 1\). Furthermore, let \(H \triangleleft G\) such that \(C_G(H) \leq H\) and \(B\) a block of \(\mathcal{O}G\) lying above \(b\) with defect group \(D\). Then \(D \cong Q_8 \times Q_8\).

**Proof.** We first describe \(P \leq H\) (see [18] for a description of defect groups of finite classical groups). We denote by \(i \in \mathbb{F}_q^\times\) a primitive 4th root of unity. Then \(P = P_1 \times P_2 \leq Sp_{2}(q) \times Sp_{2(r-1)}(q) \leq Sp_{2r}(q)\), where \(P_1 = \text{Syl}_2(Sp_2(q)) \cong Q_8\) is generated by

\[
\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

and \(P_2 \cong C_4\).

Since \(C_G(H) \leq H\), \(G/H \rightarrow \text{Out}(H)\) and as \(8 \nmid (q - 1)\), \(q\) is not a square and so \(H\) has no field automorphisms of order 2. It follows that \(\text{Out}(H)\) has a normal Sylow 2-subgroup of order 2 generated by the diagonal automorphism induced by

\[
\begin{pmatrix} iI_r & 0 \\ 0 & I_r \end{pmatrix}.
\]

Furthermore, this automorphism restricted to \(Sp_{2}(q) \times Sp_{2(r-1)}(q)\) is induced by

\[
(g_1, g_2) := \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} iI_{r-1} & 0 \\ 0 & I_{r-1} \end{pmatrix} \in GL_2(q) \times GL_{2(r-1)}(q).
\]
Suppose $D \cong Q_8 \times Q_8$. Then we may choose $D$ such that the unique block of $\mathcal{O}HD$ covering $b$ has defect group $D$. Moreover, the image of $D$ in $\text{Out}(H)$ is generated by the non-trivial diagonal automorphism.

We will have reached our desired contradiction once we have proved that no $h \in D \setminus P$ commutes with $P_1$. Since

$$C_{\text{Sp}_2(q)}(Z(P_1)) = C_{\text{Sp}_2(q)}\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} I_{r-1} & 0 \\ 0 & I_{r-1} \end{pmatrix}\right) = \text{Sp}_2(q) \times \text{Sp}_{2(r-1)}(q),$$

it is enough to show there exists no $g \in \text{Sp}_2(q)$ such that $g_1g^{-1} \in C_{\text{Sp}_2(q)}(P_1)$. This follows from the fact that $C_{\text{Sp}_2(q)}(P_1) = Z(\text{Sp}_2(q))$ and that conjugation by $g_1$ is not an inner automorphism of $\text{Sp}_2(q)$. These two facts can be readily checked. \qed

## 5 Blocks with defect group $Q_8 \times C_{2^n}$ or $Q_8 \times Q_8$

We begin by gathering together some information on subgroups and automorphism groups of these 2-groups, easily verified by the reader. Blocks with defect group $Q_8 \times C_{2^n}$ were studied in [29], where many of their numerical invariants were computed.

**Lemma 5.1** (a) Let $P \cong Q_8 \times C_{2^n}$, where $n \geq 1$. Let $Q \leq P$ with $[P : Q] = 2$.

(i) If $n = 1$, then $Q \cong Q_8$ or $C_4 \times C_2$.

(ii) If $n \geq 2$, then $Q \cong C_4 \times C_{2^n}$, $Q_8 \times C_{2^{n-1}}$ or $C_4 \times C_{2^n}$.

In particular, every subgroup of $P$ has the form $1$, $C_2$, $C_2 \times C_2$, $C_4 \times C_2$, $C_4 \times C_4$, $Q_8$, $Q_8 \times C_2$, $C_4 \times C_4$, $Q_8 \times C_4$, $C_4 \times C_4$ and $C_4 \times Q_8$.

(b) The proper subgroups of $Q_8 \times Q_8$ are isomorphic to the following: $1$, $C_2$, $C_2 \times C_2$, $C_4 \times C_2$, $C_4 \times C_4$, $Q_8$, $Q_8 \times Q_8$, $C_4 \times C_4$, $C_4 \times Q_8$, $Q_8 \times Q_8$, $C_4 \times Q_8$ and $Q_8 \times Q_8$.

**Lemma 5.2** (i) $\text{Aut}(C_{2^n} \times C_{2^m})$ is a 2-group if $m \neq n$, and is a $\{2, 3\}$-group with Sylow 3-subgroup of order 3 if $m = n$.

(ii) $\text{Aut}(Q_8 \times C_{2^m})$ is a $\{2, 3\}$-group with Sylow 3-subgroup of order 3.

(iii) $\text{Aut}(Q_8 \times Q_8)$ is a $\{2, 3\}$-group with Sylow 3-subgroup $C_3 \times C_3$.

(iv) $\text{Aut}(C_4 \times C_{2^m})$ is a 2-group.

(v) $\text{Aut}(C_4 \times Q_8)$ is a 2-group.

The key to our treatment of blocks with defect group $Q_8 \times C_{2^n}$ or $Q_8 \times Q_8$ is that in most cases covered blocks of normal subgroups of index 2 are nilpotent. This is covered in the following lemma.

**Lemma 5.3** Let $B$ be a 2-block of a finite group $G$ with defect groups isomorphic to

(i) $C_{2^m} \times C_{2^m}$ for $m \neq n$, (ii) $C_4 \times C_{2^n}$ for $n \geq 2$, or (iii) $C_4 \times Q_8$. Then $B$ is nilpotent.
Proof. Let $D$ be a defect group for $B$.

(i) Since $D$ is abelian, it suffices to observe that $\text{Aut}(D)$ is a 2-group.

(ii) Since $C_4 \rtimes C_{2^n}$ is metacyclic, by [10, Theorem 3.7] there is only one saturated fusion system on this 2-group, and so $B$ must be nilpotent.

(iii) There is only one saturated fusion system on $C_4 \rtimes Q_8$ by [30, Table 13.1].  

We summarize the results of [29] that we need here:

Proposition 5.4 ([29]) Let $B$ be a block with defect group $Q_8 \times C_{2^n}$ for some $n$. Then one of the following occurs:

(i) $k(B) = 2^n \cdot 7$ and $l(B) = 3$;

(ii) $B$ is nilpotent, $k(B) = 2^n \cdot 5$ and $l(B) = 1$.

Proof. This follows from [29, Lemma 2.2, Theorem 2.7].  

The next result will be used frequently without reference throughout the remainder of the article.

Proposition 5.5 Let $G$ be a finite group and $N \triangleleft G$ with $G/N$ a $p$-group. Let $b$ be a block of $\mathcal{O}_N$ and $B$ the unique block of $\mathcal{O}_G$ covering $b$. Then $B$ is nilpotent if and only if $b$ is.

Proof. That $B$ nilpotent implies $b$ nilpotent is clear. The other direction is [8, Theorem 2].  

Lemma 5.6 Let $G$ be a finite group and $N \triangleleft G$ such that $G/N$ is supersolvable of odd order. Let $B$ be a block of $\mathcal{O}_G$ with defect group $Q_8 \times C_{2^n}$ for some $n$ and let $b$ be a block of $\mathcal{O}_N$ covered by $B$. Suppose that $B$ covers no nilpotent block of any normal subgroup containing $N$. Then $B$ and $b$ are Morita equivalent.

Proof. Note that $B$ and $b$ must share a defect group, and that for any $M \triangleleft G$ with $N \leq M$ and any block $C$ of $\mathcal{O}_M$ covered by $B$ we have $l(C) = 3$ by Proposition [5.4]. By considering a chief series between $N$ and $G$ with prime factors it suffices to consider the case that $[G : N]$ is an odd prime $w$. If $b$ is not $G$-stable, then we are done. Suppose that $b$ is $G$-stable. By [22, Proposition 2.2] if $G$ acts as inner automorphisms on $b$, then $B$ and $b$ are Morita equivalent. By [22, Proposition 2.3] if $G$ does not act as inner automorphisms on $b$, then $B$ is the unique block of $\mathcal{O}_G$ covering $b$. Consider the action of $G$ on the three irreducible Brauer characters of $b$. If $w \geq 5$, then $G$ must fix every such Brauer character, and $l(B) = w l(b) = 3w$, contradicting $l(B) = 3$. If $w = 3$, then either $G$ fixes each irreducible Brauer character, again a contradiction, or $G$ permutes the irreducible Brauer characters of $b$ transitively and $l(B) = 1$, contradicting our assumption that $B$ is not nilpotent (using Proposition [5.4]).  

Lemma 5.7 Let $G$ be a finite group and $N \triangleleft G$ with $[G : N] = 2$. Let $b$ be a non-nilpotent block of $\mathcal{O}_N$ with defect groups isomorphic to $C_4 \times C_4$. Then the unique block $B$ of $\mathcal{O}_G$ covering $b$ cannot have defect group $Q_8 \times C_4$.  

Proof. We assume the contrary. By Proposition 5.4 we have \( k(B) \geq 20 \) and by [12, Theorem 1.1] \( k(b) = 8 \). But by Clifford theory the number of irreducible characters of \( G \) lying over irreducible characters of \( b \) is at most 16, and the result follows. \( \square \)

A block of a finite group \( G \) is described as quasiprimitive if every covered block of a normal subgroup of \( G \) is \( G \)-stable. The following is proved in [3, Lemma 2.4].

**Lemma 5.8** Let \( G \) be a finite group and \( B \) be a quasiprimitive block of \( \mathcal{O}_G \) with defect group \( D \). Let \( N \trianglelefteq G \) such that \( G/N \) is solvable and \( b \) the unique block of \( \mathcal{O}_N \) covered by \( B \). Then \( ND/N \in \text{Syl}_p(G/N) \).

Recall that a \( p \)-solvable group \( G \) has \( p \)-length one if there are normal subgroups \( N, M \trianglelefteq G \), with \( N \leq M \), such that \( N \) and \( G/M \) are \( p' \)-groups and \( M/N \) is a \( p \)-group. The abelian Sylow 2-subgroup case of the following is well-known, and possibly also the \( Q_8 \) case, but since we do not know a reference we include a proof.

**Lemma 5.9** Let \( G \) be a solvable group with Sylow 2-subgroups which are abelian or \( Q_8 \). Then \( G \) has 2-length one. If further \( G \) has cyclic 2-subgroups, then it is 2-nilpotent.

**Proof.** We may assume that \( O_2(G) = 1 \), so that \( O_2(G) \neq 1 \) and \( C_G(O_2(G)) \leq O_2(G) \). Let \( P \in \text{Syl}_2(G) \). If \( P \) is abelian, then \( P \leq C_G(O_2(G)) \leq O_2(G) \) and we are done. Suppose that \( P \cong Q_8 \). Since \( Z(P) \) is the unique subgroup of \( P \) of order two, we have \( Z(P) \leq O_2(G) \). If \( O_2(G) = Z(P) \) or \( O_2(G) = P \), then we are done. If \( O_2(G) \cong C_4 \), then \( G/O_2(G) \) is a 2-group and again we are done.

In the case that \( P \) is cyclic, the fact that \( G \) is 2-nilpotent follows from \( \text{Aut}(P) \) being a 2-group. \( \square \)

The following is by now a standard reduction when treating Donovan’s conjecture, using Fong reductions and [26].

**Proposition 5.10** Let \( G \) be a finite group and let \( B \) be a block of \( \mathcal{O}_G \) with defect group \( D \). There is a finite group \( H \) with \( [H : O_{p'}(Z(H))] \leq [G : O_{p'}(Z(G))] \) and a block \( C \) of \( \mathcal{O}_H \) with defect group \( P \cong D \) such that \( B \) is Morita equivalent to \( C \) and the following are satisfied:

1. \( C \) is quasiprimitive;
2. If \( N \trianglelefteq H \) and \( C \) covers a nilpotent block \( c \) of \( \mathcal{O}_N \), then \( N \leq Z(H)O_{p'}(H) \);
3. \( O_{p'}(Z(H)) \leq [H, H] \).

**Proof.** See the first part of the proof of [14, Proposition 4.3]. \( \square \)

For the purpose of this article, we call the pair \((H, C)\), where \( C \) is a block of \( \mathcal{O}_H \), **reduced** if it satisfies conditions (R1), (R2) and (R3) of Proposition 5.10. If the group is clear, then we just say \( C \) is reduced. The property of being reduced is very restrictive in our situation:

**Proposition 5.11** Let \((G, B)\) be a reduced pair, where \( B \) has defect group \( D \).
(a) If \( D \cong Q_8 \times C_{2^n} \), then either:

(i) there is \( N \triangleleft G \) such that \( G = ND \) and \( D \cap N \cong Q_8 \) (so \( G/N \) is cyclic); or

(ii) there is \( N \triangleleft G \), a quasisimple group that is a quotient of a group of classical Lie type other than \( A_2 \) or \( 2A_2 \), such that \( G/N \) is 2-nilpotent with cyclic Sylow 2-subgroup \( ND/N \) and \( O_2(G/N) \) is supersolvable. The unique block of \( O_N \) covered by \( B \) is not quasi-isolated and \( D \cap N \) has a subgroup isomorphic to \( Q_8 \).

(b) If \( D \cong Q_8 \times Q_8 \), then either:

(i) there are normal subgroups \( N \triangleleft H \triangleleft G \) such that \( H = ND \), \( D \cap N \cong Q_8 \) or \( Q_8 \times C_2 \), and \( [G : H] \) is odd; or

(ii) there are commuting, normal subgroups \( N_1, N_2 \triangleleft G \) such that \( N_1 \cap N_2 \leq Z(G) \), \( D \cap N_1, D \cap N_2 \cong Q_8 \), or \( Q_8 \times C_2 \), and \( [G : N_1N_2] \) is odd.

**Proof.** Let \( G \) be a finite group and \( B \) a block of \( OG \) with defect group \( D \cong Q_8 \times C_{2^n} \) or \( Q_8 \times Q_8 \) for some \( n \geq 1 \), and suppose that \((G,B)\) is reduced. Write \( E(G) \) for the layer of \( G \), the central product of all of the components of \( G \), and \( F^*(G) \) for the generalized Fitting subgroup (see [4]). Then \( F^*(G) = E(G)Z(G)O_2(G) \) by our assumption, and \( C_G(F^*(G)) \leq F^*(G) \) because of general properties of \( F^*(G) \). Write \( E(G) \) for the central product \( L_1 \circ \cdots \circ L_t \), where the \( L_i \) are the components of \( G \). Write \( B_E \) for the unique block of \( OE(G) \) covered by \( B \). Then (using the fact that \( B_E \) is \( G \)-stable) \( D_E := D \cap E(G) \) is a defect group for \( B_E \) and \( D_i := D \cap L_i \) is a defect group for the unique block \( b_i \) of \( O_L \) covered by \( B_E \). By Lemma 5.1 \( O_2(G), D_E \) and each \( D_i \) is of the form \( 1, C_2^m, C_2 \times C_2, C_4 \times C_2^m, Q_8, Q_8 \times C_2, Q_8 \times Q_8, C_4 \times C_2^m \) or \( C_4 \times Q_8 \) for some \( m \) (with \( m \geq 2 \) in the eighth case). Write \( Z_E := O_2(E(G)) \). Note that \( Z_E \) is central in \( E(G) \). Now the unique block \( \overline{B}_E \) of \( O(E(G)/Z_E) \) corresponding to \( B_E \) has defect group \( \overline{D}_E := D_E/Z_E \cong D_1/Z_E/\cdots \times D_t/Z_E \) and has 2-rank at most four (we are using that \( \overline{B}_E \) corresponds to a block of a direct product of quasisimple groups of which \( E(G)/Z_E \) is a quotient by a 2'-group).

Suppose \( t \geq 3 \), or \( t = 2 \) in case (a). Then at least one \( D_iZ_E/Z_E \) is a cyclic 2-group, so that the unique block \( \overline{L}_i \) of \( L_iZ_E/Z_E \) corresponding to \( b_i \) is nilpotent. Let \( M \) be the product of those \( L_i \) such that \( b_i \) is nilpotent. Then \( M \) must be a normal subgroup (since \( G \) permutes the components). Let \( B_M \) be the unique block of \( OM \) covered by \( B \) and \( \overline{B}_M \) be the unique block of \( MZ_E/Z_E \) corresponding to \( B_M \). Now \( \overline{B}_M \) is isomorphic to a product of nilpotent blocks and so is itself nilpotent (\( \overline{B}_M \) corresponds to a block of a direct product of quasisimple groups of which \( MZ_E/Z_E \) is a quotient by a 2'-group).

By Lemma 5.2 \( B_M \) is then nilpotent, a contradiction to our assumption that \( B \) is reduced. Hence \( t \leq 2 \), with equality only if \( D \cong Q_8 \times Q_8 \). It follows from Schreier’s conjecture and Lemmas 5.1 and 5.2 that \( G/F^*(G) \) has a solvable subgroup of index at most two (namely the subgroup stabilizing the components if \( t \geq 1 \), and \( G/F^*(G) \) itself if there are no components) and so \( G/F^*(G) \) is itself solvable. Similarly \( G/E(G) \) is solvable.

By Lemma 5.8 \( DF^*(G)/F^*(G) \) is a Sylow 2-subgroup of \( G/F^*(G) \) and \( DE(G)/E(G) \) is a Sylow 2-subgroup of \( G/E(G) \), facts we will use frequently.

Suppose that \( t = 2 \), in which case \( D \cong Q_8 \times Q_8 \). Note also that each component is normal in \( G \) since \( G/E(G) \) is of odd order (i.e., there is no involution permuting
the two components). By considering all of the possible expressions of $Q_8 \times Q_8$ as a central product of two groups in Lemma 5.1, we must have (without loss of generality) $D_1$ and $D_2 \cong Q_8$ or $Q_8 \times C_2$, otherwise $B$ covers a block of a component with cyclic defect group, which forces a contradiction as in the previous paragraph. Note that $O_2(G) \leq Z(G)$ in this case. We are now in case (b)(ii) of the statement, with $N_i = L_i$.

Suppose that $t = 0$. Then $F^*(G) = O_2(G)Z(G)$. Since $C_G(F^*(G)) \leq F^*(G)$, we have $G/F^*(G) \leq \text{Out}(O_2(G))$. In particular $O_2(G)$ is a self-centralizing subgroup of $D$. Note that if $D = O_2(G)$, then $G \cong D \times E$ for some $2'$-group $E$. It follows from Lemma 5.2 that $G/C_E(D)$ is a subgroup of $SL_2(3) \times C_{2^n}$ or $SL_2(3) \times SL_2(3)$ and we must be in case (a)(i) or (b)(i) respectively. Suppose $O_2(G)$ is a proper subgroup of $D$. Note that, by Lemma 5.2, all self-centralizing subgroups of $D$ have solvable automorphism group. Suppose first that $D \cong Q_8 \times C_{2^n}$. By Lemma 5.1 the only proper self-centralizing normal subgroup of $D$ is $C_4 \times C_{2^n}$ and we need only consider the case $O_2(G) \cong C_4 \times C_4$ as in the other cases the automorphism group is a 2-group. In this case we must have that $G/O_2(G) \cong S_3$ and $B$ covers a non-nilpotent block of $O_2(G)$. However this cannot happen by Lemma 5.1. Now suppose that $D \cong Q_8 \times Q_8$. By Lemma 5.1 the proper self-centralizing normal subgroups of $D$ are $C_4 \times C_4$, $Q_8 \times Q_4$ and $C_4 \times Q_8$, of which we need only consider $C_4 \times C_4$ and $Q_8 \times C_4$ since $C_4 \times Q_8$ has automorphism group a 2-group. If $O_2(G) \cong C_4 \times C_4$, then $G/F^*(G)$ has order 12 and so has a non-trivial normal 2-subgroup, a contradiction. If $O_2(G) \cong Q_8 \times C_4$, then $G/F^*(G) \cong S_3$. This forces $G \cong SL_2(3) \times Q_8$ and we are in case (a)(i).

Now suppose that $t = 1$, so $F^*(G) = O_2(G)Z(G)L_1$. Since, by Lemma 5.3, every block with defect group $C_4 \times Q_8$, $C_4 \times C_{2m}$ or $C_4 \times C_{2^n}$ is nilpotent for $m \geq 2$ and $s \neq 2$, we may assume, by Lemma 5.1 that $D_1 \cong Q_8 \times Q_8$, $Q_8 \times C_{2m}$, $Q_8$, $C_4 \times C_4$ or $C_2 \times C_2$ for some $m \geq 1$. We treat each of these cases in turn.

By Proposition 4.2 we cannot have $D_1 \cong Q_8 \times Q_8$. Suppose that $D_1 \cong Q_8 \times C_{2m}$ for $m \geq 1$. Consider first the case $D \cong Q_8 \times C_{2m}$. By Proposition 4.2, $L_1$ is of classical type other than $A$ or $2A$ as in case (a)(ii) of the statement with $N = L_1$. That $G/N$ has cyclic Sylow 2-subgroup $ND/N$ is immediate, and so by Lemma 5.9 $G/N$ is 2-nilpotent. Considering the outer automorphism groups of such quasisimple groups we have that $O_2(G/L_1)$ is supersolvable as required. As a note to this last calculation, observe that unless $L_1$ has type $D_4$ the only odd order elements of the outer automorphism group are field automorphisms, so that $O_2(G/L_1)$ is cyclic. In the case of type $D_4$ we may either analyse [2, Theorem 1.1] a little more deeply and observe that this case does not after all occur, or observe that a Hall 2'-subgroup of $\text{Out}(L_1)$ is a subgroup of $C_3 \times Y$ where $Y$ is cyclic, implying that $O_2(G/L_1)$ is supersolvable.

Now suppose that $D_1 \cong Q_8 \times C_{2m}$ for $m \geq 1$ and $D \cong Q_8 \times Q_8$. Then $1 \leq m \leq 2$. The case $m = 2$ is ruled out by Proposition 4.2 and Lemma 4.3. If $m = 1$, then we are in case (b)(i).

Suppose that $D_1 \cong Q_8$. Then since $G/L_1$ is a solvable group which has cyclic or $Q_8$ Sylow 2-subgroup, by Lemma 5.9, it has 2-length one. Let $N$ be the preimage in $G$ of $O_2(G/L_1)$. Then the unique block $B_N$ of $O\,N$ covered by $B$ has defect group $Q_8$. If $D/D_1$ is cyclic, then by Lemma 5.9 $G/L_1$ is 2-nilpotent and we are in case (a)(i) of the statement. If $D/D_1 \cong Q_8$, then let $H = O^2(G)$ and we are in case (b)(i) of the statement.
Finally we claim that we cannot have $D_1 \cong C_4 \times C_4$ or $C_2 \times C_2$. Suppose that we do and so $D/D_1$ is an abelian. Let $M$ be the preimage in $G$ of $O_{2'}(G/L_1)$, and let $B_M$ be the unique block of $OM$ covered by $B$. Then $B_M$ has defect group $D_1$. Now there is a subgroup $M_1$ of $G$ containing $M$ with $[M_1 : M] = 2$. The unique block $B_{M_1}$ of $OM_1$ covering $B_M$ has defect group $Q_8 \times C_4$ or $C_4 \times Q_8$ (if $D_1 \cong C_4 \times C_4$) or $C_4 \times C_2$ (if $D_1 \cong C_2 \times C_2$), since it is a subgroup of $D$. In the $Q_8 \times C_4$ case we obtain a contradiction by Lemma 5.4 in the $C_1 \times Q_8$ and $C_4 \times C_2$ cases by Lemma 5.3 $B_M$, and so by Proposition 5.11 $b_1$, must be nilpotent, a contradiction.

\[\]

**Corollary 5.12** Let $G$ be a finite group and $B$ a block of $OG$ with defect group $D$ isomorphic to $Q_8 \times C_{2^n}$ for $n \geq 0$. Then $\text{sf}_G(B) \leq (|D|^2)!$ and $c(B) \leq 2^{n+3}$.

**Proof.** Morita equivalence preserves the Cartan invariants, and by [13] Proposition 3.12] Morita equivalence of $O$-blocks preserves $\text{sf}_G$, so by Proposition 5.10 it suffices to consider reduced blocks. Apply Proposition 5.11.

First let $(G, B)$ be a reduced pair satisfying condition (a)(ii) of Proposition 5.11. Let $B'$ be the unique block of $O\Omega^2(G)$ covered by $B$. By Lemma 5.11(i) $c(B) \leq [G : O^2(G)]c(B')$ and by Theorem 1.2 $\text{sf}_G(B) \leq \text{sf}_G(B')$. By considering the outer automorphism groups of the classical groups of Lie type (see for example [9, Table 5]) we see that every normal subgroup of $O^2(G)$ is also normal in $G$, so $B'$ is quasiprimitive. Hence we see that $B'$ is also a reduced block satisfying (a)(ii). We may now assume that $O^2(G) = G$. Now let $B_N$ be the unique block of the quasisimple group $N$ covered by $B$. By Lemma 5.6 $B$ is Morita equivalent to $B_N$, and note that they share a defect group. Hence we may assume that $G = N$ since Morita equivalence preserves both of these invariants (see [13, Proposition 3.12] for the latter).

We make use of [7], to which we refer for notation. Assume for the moment that $G$ is a group of Lie type, i.e., the centre is largest possible. We note that $Z(G)$ is a 2-group for the groups we are considering. Here the identity element is the only quasi-isolated element (see for example [6, Table 2]) and so the principal block is the only quasi-isolated block. However, for groups of these types the principal block of $G$ (or that of any quotient of $G$) cannot have the given defect groups, so we may assume our block is not quasi-isolated. We may now apply the Bonnafé-Dat-Rouquier correspondence [7, Theorem 7.7], so that $B$ is Morita equivalent to a block $C_1$ (with isomorphic defect group) of a proper subgroup $H_1$ of $G$. We note that the well-known error in [7] does not apply in our situation, since we are working with 2-blocks and the centre of $G$ is a 2-group. If $G$ is not a group of Lie type (i.e., the centre is not largest possible), then we note that by, for example, [12, Proposition 4.1] the Bonnafé-Dat-Rouquier correspondence induces a Morita equivalence modulo central 2-subgroups and we may apply the same argument.

Applying Proposition 5.10 to $C_1$, it is Morita equivalent to a block $C_2$ in a reduced pair $(H_2, C_2)$ with isomorphic defect groups, where $[H_2 : O_{2'}(Z(H_2))] \leq [H_1 : O_{2'}(Z(H_1))]$. We have $c(B) = c(C_1)$ and $\text{sf}_G(B) = \text{sf}_G(C_1)$. Now apply Proposition 5.11 to $(H_2, C_2)$. Either we are in case (a)(i) of Proposition 5.11 or we may repeat the above argument. Since the index of the $2'$-part of the centre strictly
decreases each time we apply the Bonnafé-Dat-Rouquier correspondence, repetition of this process must eventually end in case (a)(i) of Proposition 5.11.

Now let \((G, B)\) be a reduced pair satisfying condition (a)(i), so there is \(N \triangleleft G\) such that \(G = ND\) and \(D \cap N \cong Q_8\). Let \(b\) be the unique block of \(\mathcal{O}N\) covered by \(B\), noting that this has defect group \(Q_8\). By 13 there is a unique Morita equivalence class of blocks with defect group \(Q_8\) and a given Cartan matrix, so \(\text{mf}_\mathcal{O}(b) = 1\). By Theorem 4.2 and Corollary 3.11 \(\text{sf}_\mathcal{O}(B) \leq \text{sf}_\mathcal{O}(b) \leq |Q_8|^2\). Considering Cartan invariants, \(c(b) \leq 8\) and so, by Lemma 3.1(i), \(c(B) \leq 2^{n+3}\).

We remark that we cannot at present so easily obtain a similar bound on the strong \(\mathcal{O}\)-Frobenius number for blocks with defect group \(Q_8 \times Q_8\), since we do not know how this invariant behaves with respect to normal subgroups of \(p'\)-index.

We further remark that in order to bound only the strong Frobenius number of a quasisimple group we could have used [17].

6 Donovan’s conjecture for blocks with defect group \(Q_8 \times C_{2^n}\) or \(Q_8 \times Q_8\)

We are now in a position to verify Donovan’s conjecture for \(Q_8 \times C_{2^n}\) and \(Q_8 \times Q_8\).

**Proof of Theorem 1.4**

By Proposition 5.10 in verifying Donovan’s conjecture it suffices to consider reduced blocks.

Consider first \(Q_8 \times C_{2^n}\). From 14 Corollary 3.11 we need only bound strong \(\mathcal{O}\)-Frobenius numbers and Cartan invariants for reduced blocks with defect groups isomorphic to \(Q_8 \times C_{2^n}\), hence the result follows in this case by Corollary 5.12.

Now consider blocks with defect groups \(Q_8 \times Q_8\). By Proposition 5.11 either: (i) there are normal subgroups \(N \triangleleft H \triangleleft G\) with \(H = ND\) and \(D \cap N \cong Q_8\) or \(Q_8 \times C_2\), and \([G : H]\) odd; or (ii) there are commuting \(N_1, N_2 \triangleleft G\) with \(N_1 \cap N_2 \leq \text{Z}(G)\) such that \(D \cap N_1, D \cap N_2 \cong Q_8\) or \(Q_8 \times C_2\), and \([G : N_1N_2]\) is odd.

By 16 Corollary 4.18 it suffices to show that there are only finitely many possibilities for the Morita equivalence class of the unique block of \(\mathcal{O}H\) or \(\mathcal{O}N_1N_2\) covered by \(B\). Hence we may assume that \(G = H\) in case (i) and \(G = N_1N_2\) in case (ii).

As above, from 14 Corollary 3.11 we need only bound strong \(\mathcal{O}\)-Frobenius numbers and Cartan invariants for such blocks.

In case (i) we have \(\text{sf}_\mathcal{O}(B) \leq 16^2!\) and \(c(B) \leq 2^6\) using arguments as in Corollary 5.12 noting that \(D \cong Q_8 \times Q_8\) with \(D \cap N \cong Q_8\) or \(Q_8 \times C_2\) satisfies the conditions of Theorem 1.2 Suppose that we are in case (ii). Note that \(G \cong (N_1 \times N_2)/W\) for some group \(W \leq \text{Z}(N_1 \times N_2)\). Now \(B\) corresponds to a block \(A\) of \(N_1 \times N_2\) with \(O_2\text{-index}\) in its kernel and defect group \(Q_8 \times Q_8, Q_8 \times Q_8 \times C_2\) or \(Q_8 \times Q_8 \times C_2 \times C_2\). Write \(A_i\) for the block of \(N_i\) covered by \(A\), with defect group \(Q_8\) or \(Q_8 \times C_2\). As above we have \(\text{sf}_\mathcal{O}(A_i) \leq 16^2!,\) for \(i = 1, 2\). Hence \(\text{sf}_\mathcal{O}(A) \leq (16^2!)^2\). By 13 Proposition 3.17] we have \(\text{sf}_\mathcal{O}(B) \leq \text{sf}_\mathcal{O}(A)\). Finally we have \(c(A_i) \leq 16\), for \(i = 1, 2\), so by Lemma 3.1(ii) \(c(B) \leq c(A) \leq 16^2\). \qed
The authors are aware of only three Morita equivalence classes of blocks with defect group $Q_8 \times C_{2^n}$ for each $n \geq 0$, namely those with representatives the principal blocks of $\mathcal{O}(Q_8 \times C_{2^n})$, $\mathcal{O}(SL_2(3) \times C_{2^n})$ and $\mathcal{O}(SL_2(5) \times C_{2^n})$. Similarly the known Morita equivalence classes of blocks with defect group $Q_8 \times Q_8$ have representatives the principal blocks of $\mathcal{O}(Q_8 \times Q_8)$, $\mathcal{O}(Q_8 \times SL_2(3))$, $\mathcal{O}((Q_8 \times Q_8) \rtimes C_3)$ (SmallGroup(192,1022), where $C_3$ acts on $(Q_8 \times Q_8)/Z(Q_8 \times Q_8)$ with only one fixed point), $\mathcal{O}(Q_8 \times SL_2(5))$, $\mathcal{O}(SL_2(3) \times SL_2(3))$, $\mathcal{O}(SL_2(3) \times SL_2(5))$ and $\mathcal{O}(SL_2(5) \times SL_2(5))$, and a non-principal block of $(Q_8 \times Q_8) \rtimes 3^{1+2}$, where the centre of $3^{1+2}$ acts trivially.

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