Thermodynamics of Einstein-Born-Infeld black holes with negative cosmological constant

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Abstract

We study the thermodynamics associated to topological black hole solutions of AdS gravity coupled to nonlinear electrodynamics (Born-Infeld) in any dimension, using a background-independent regularization prescription for the Euclidean action given by boundary terms which explicitly depend on the extrinsic curvature (Kounterterms series). A finite action principle leads to the correct definition of thermodynamic variables as Noether charges, which satisfy a Smarr-like relation. In particular, for the odd-dimensional case, a consistent thermodynamic description is achieved if the internal energy of the system includes the vacuum energy for AdS spacetime.

1 Introduction

A remarkable feature of nonlinear Born-Infeld electrodynamics \cite{1} is that it is able to describe a classical theory of charged particles with finite self-energy and an electric field that is regular at the origin. Due to a screening effect on the electric field for small distances, its source can be interpreted as an extended object with an effective radius $r_{\text{eff}} = q^2/b^2$, where $q$ is related to the electric charge of the particle and $b$ is the Born-Infeld coupling.

The regulated behavior of this electromagnetic theory makes a Born-Infeld-like action a sensible candidate for a gravitational theory \cite{2, 3}. Indeed, such action arises naturally in string theory as it governs the dynamics of D-branes \cite{4, 5, 6, 7, 8}.

When Born-Infeld electrodynamics is coupled to anti-de Sitter (AdS) gravity, charged black holes exhibit thermodynamic properties similar to the ones of Reissner-Nordström AdS solutions \cite{9, 10, 11}.

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showing a phase structure isomorphic to van der Waals-Maxwell liquid-gas system (for the four-
dimensional Einstein-Born-Infeld (EBI) AdS case, see [12, 13]). Asymptotically (A)dS black hole
configurations in this theory were found in an arbitrary dimension and their thermodynamics was
discussed in Refs. [14, 15]. Electrically charged rotating black branes in EBI AdS gravity were studied
in [16]. Solutions of nonlinear BI like (Hoffman-Infeld) electromagnetism coupled to Lovelock gravity
were found in [17]. Finally, the attractor mechanism for spherically symmetric extremal black holes in
EBI-dilaton theory of gravity in four dimensions with cosmological constant was investigated in [18].

As in Einstein-Maxwell gravity, the black hole entropy $S$ is a quarter of the horizon’s area (in fun-
damental units), and the variations of the solution parameters obey the First Law of thermodynamics

$$\beta dU = dS + \beta \Phi dQ,$$

where $U$ is the internal energy, $\beta$ is the inverse of the black hole temperature, $\Phi$ is the gauge potential
measured at infinity respect to the horizon and $Q$ is the \textit{thermodynamic} electric charge.

In general, it is possible to show that the first law holds simply assuming that the thermodynamic
variables whose variations appear in Eq.(1) are the conserved charges of the theory. However, the
thermal properties of black holes can only be understood when one identifies the Euclidean path
integral with the thermal partition function. In the semiclassical approximation, the partition function
is given by the exponential of the classical Euclidean action $I_{clas}^{E}$,

$$Z = e^{-I_{clas}^{E}},$$

and therefore, for spacetimes with AdS asymptotics, $I_{clas}^{E}$ needs to be regulated to cancel the diver-
gences that appear in the asymptotic region. In doing so, the thermodynamic information is encoded
in the finite part the Euclidean action $I^{E}$, which obeys the Smarr relation

$$I^{E} = \beta U - \beta \Phi Q - S.$$  (3)

Background-substraction methods for asymptotically AdS spacetimes are useful to extract a finite
value from $I^{E}$, considering the Euclidean action for the black hole minus the same functional evaluated
for the corresponding vacuum solution. The result satisfies Eq.(3) and it is equivalent to integrate
out the first law for global AdS ($U = Q = 0$) as the background. The procedure, however, does not
guarantee the existence of such background for a complex enough gravitational configuration because
it is not always possible to embed an arbitrary solution into a given reference spacetime.

The identification of the thermodynamic variables as asymptotic charges may be more subtle
in the context of the AdS/CFT correspondence [19]. In this framework, the standard regularization
method consists on the addition of counterterms which are covariant functionals of the boundary metric.
and intrinsic curvature, constructed by a systematic procedure known as holographic renormalization \[20, 21\]. The counterterms cancel the divergences in both conserved charges and Euclidean action in a background-independent fashion \[22, 23\]. The novel feature is the appearance of a nonvanishing zero-point energy $E_{\text{vac}}$ for global AdS spacetime in the odd-dimensional case, which matches the Casimir energy of a boundary CFT, and it is helpful to realize explicit examples of the gravity/gauge duality.

It is clear that the first law (1) is insensitive to the existence of a vacuum energy for asymptotically AdS spacetimes. What is far from evident is whether the Smarr relation is still valid for an internal energy corresponding to the total energy of the system in odd-dimensional gravity with negative cosmological constant. Employing Dirichlet counterterms, one is able to check, on a rather case-by-case basis, the consistency of the formula (3), only if the internal energy is shifted as $U = M + E_{\text{vac}}$ with respect to the Hamiltonian mass $M$. That implies that the regulated Euclidean action also appears shifted accordingly, respect to the value obtained in a background-subtraction procedure.

An explicit proof of the Smarr formula in an arbitrary dimension faces the problem of finding a general counterterm action, which is still unknown for a high enough dimension \[24\]. It also presupposes that one can isolate the contribution from the stress tensor to $E_{\text{vac}}$ so as to write down a covariant formula only for that part. However, the fact that such formula has not been found prevents from casting the shift in $I^E$ as $\beta E_{\text{vac}}$ for an arbitrary asymptotically AdS solution.

An alternative regularization scheme for gravity with AdS asymptotics, known as Kounterterms method, has been recently proposed \[25, 26\]. This prescription considers supplementing the action by boundary terms that are a given polynomial of the intrinsic and extrinsic curvatures. A remarkable feature of this approach is its universality as –for a given dimension– the Kounterterms series preserve its form for Einstein-Gauss-Bonnet AdS \[27\] and even for any Lovelock gravity with AdS asymptotics \[28\]. Furthermore, because of a profound relation to topological invariants and Chern-Simons forms, the explicit form of the boundary terms can be written down in any dimension. As a consequence, background-independent formulas for the conserved charges which, on the contrary to what happens in the standard counterterm procedure, can be shown in all dimensions. Background-independence is particularly relevant when global AdS spacetime possesses a non-vanishing energy. It has been shown in Ref.\[26\] that a covariant formula for the vacuum energy in an arbitrary odd dimension can indeed be obtained using Kounterterms method.

In this paper, we study black hole thermodynamics in EBI AdS gravity as an application of the regularization prescription described above. First, we are able to identify the thermodynamic variables of the system as Noether charges, which are rendered finite employing Kounterterm series. Second, the direct evaluation of the Euclidean action allows us to recognize the contribution from radial infinity as the corresponding conserved quantities and to show that they verify a Smarr-type formula. Finally, in the odd-dimensional case, we show that this picture is consistent only if the internal energy $U$ includes
the vacuum (Casimir) energy for AdS spacetime.

2 Action and equations of motion

We consider the Einstein-Born-Infeld gravity with negative cosmological constant in $D = d + 1$ dimensions,

$$I_{\text{reg}} = \int_M d^D x \sqrt{-g} L + c_d \int_{\partial M} d^d x B_d,$$

where the bulk Lagrangian density has the form

$$L = -\frac{1}{16\pi G} \left[ \hat{R} - 2\Lambda + 4b^2 \left(1 - \sqrt{1 + \frac{F^2}{2b^2}}\right) \right],$$

and the boundary term $B_d$ shall be discussed below. Hatted curvatures refer to the full spacetime, which is endowed with the metric tensor $g_{\mu\nu}$. The first two terms in (5) correspond to the Einstein-Hilbert action with negative cosmological constant (for conventions see Appendix A),

$$\Lambda = -\frac{(D - 1)(D - 2)}{2\ell^2},$$

and $G$ is the gravitational constant. The second part is the Born-Infeld term, where the parameter $b$ (with dimension of mass) is related to the string tension $\alpha'$ as $b = \frac{1}{2\pi\alpha'}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength associated to the Abelian gauge field $A_\mu$. We also denote $F^2 = g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho}$. In the limit $b \to \infty$, the Born-Infeld term recovers Maxwell electrodynamics, whereas in the limit $b \to 0$, it vanishes.

In $D = 4$, apart from $F^2$, one can construct another quadratic invariant, $\tilde{F}^{\mu\nu} F_{\mu\nu}$, which involves the dual of the field strength $\tilde{F}$. This term, when added to the action, is locally a boundary term and thus, it does not change the bulk dynamics. However, nonlinear electrodynamics action may include this quadratic invariant in a nontrivial form, on the condition that the Maxwell limit is recovered [29, 30, 31]. Unfortunately, because of the dependence on $\tilde{F}$, such action cannot be extended to higher dimensions than four.

Varying the action with respect to the metric $g_{\mu\nu}$ and the gauge field $A_\mu$ produces the equations of motion plus a surface term,

$$\delta I_{\text{reg}} = \frac{1}{16\pi G} \int_M d^D x \sqrt{-g} \left( (g^{-1} \delta g)^{\nu}_{\mu} \mathcal{E}^\mu_{\nu} + \delta A_\mu \mathcal{E}^\mu \right) + \int_{\partial M} d^d x \Theta,$$
where

\[ \mathcal{E}_\mu^\nu \equiv \hat{R}_\nu^\mu - \frac{1}{2} \delta_\nu^\mu \hat{R} + \Lambda \delta_\nu^\mu - T_\nu^\mu, \quad (8) \]

\[ \mathcal{E}_\mu \equiv -\hat{\nabla}_\nu \left( \frac{4F^{\mu\nu}}{\sqrt{1 + \frac{F^2}{2b^2}}} \right). \quad (9) \]

Here \( \hat{\nabla}_\mu \) denotes covariant derivative with respect to the Christoffel connection \( \hat{\Gamma}_\alpha^{\mu\nu} \), whereas the Born-Infeld electromagnetic stress tensor is given by

\[ T_\nu^\mu = 2b^2 \delta_\nu^\mu \left( 1 - \sqrt{1 + \frac{F^2}{2b^2}} \right) + \frac{2F^{\mu\lambda}F_{\nu\lambda}}{\sqrt{1 + \frac{F^2}{2b^2}}}. \quad (10) \]

For a large value of the parameter \( b \), we have

\[ T_\nu^\mu = 2F^{\mu\lambda}F_{\nu\lambda} - \frac{1}{2} \delta_\nu^\mu F^2 + O\left( \frac{1}{b^2} \right), \quad (11) \]

such that, as we mentioned above, the limit \( b \to \infty \) reproduces the stress tensor for Maxwell electrodynamics.

Using the Stokes’ theorem, the surface term in the variation (17) can be written as

\[ \Theta = \frac{1}{16\pi G} \sqrt{-h} n_\mu \left( \delta^{[\mu}_{[\alpha} g^{\nu\beta]} \delta \hat{\Gamma}_\gamma^\beta \gamma^\nu + \frac{4F^{\mu\nu} A_\nu}{\sqrt{1 + \frac{F^2}{2b^2}}} \right) + c_d \delta B_d. \quad (12) \]

We consider a manifold with a single boundary at radial infinity such that we can choose a radial foliation in Gaussian form,

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = N^2 (r) \, dr^2 + h_{ij}(r, x) \, dx^i dx^j, \quad \sqrt{-g} = N \sqrt{-h}. \quad (13) \]

Here \( h_{ij} \) describes the intrinsic geometry of the spacelike boundary \( \partial M \) parameterized by the coordinate set \( \{ x^i \} \), whereas its extrinsic properties are given in terms of the outward-pointing unit normal \( n_\mu = (n_r, n_i) = (N, \vec{0}) \). When one projects the fully-covariant expression \( \Theta \) to the boundary, one must consider the components of the Christoffel symbol that are not expressible as tensorial quantities on \( \partial M \). However, the only relevant components of the connection \( \hat{\Gamma}_\mu^\alpha \) are indeed expressed in terms of the extrinsic curvature \( K_{ij} = -\frac{1}{2N} h_{ij}' \) as

\[ \hat{\Gamma}^r_{ij} = \frac{1}{N} K_{ij}, \quad \hat{\Gamma}^i_{rj} = -NK^i_j, \quad \hat{\Gamma}^r_r = \frac{N'}{N}, \quad (14) \]
where the prime denotes radial derivative. Thus, the surface term (12) projected to the boundary takes the form

\[ \Theta = - \frac{1}{16\pi G} \sqrt{-h} \left( (h^{-1}\delta h)^i_j K^j_i + 2\delta K^i_i - \frac{4NF'^{\mu i}\delta A^\mu_i}{\sqrt{1 + \frac{F^2}{2b^2}}} \right) + c_d \delta B_d. \]  

(15)

The radial foliation (13) implies the Gauss-Codazzi relations for the spacetime curvature, as well,

\[ \hat{R}^{ir}_{kl} = \frac{1}{N} \left( \nabla_l K^i_k - \nabla_k K^i_l \right), \]  

(16)

\[ \hat{R}^{ir}_{kr} = \frac{1}{N} \left( K^i_k \right)' - K^i_k K^l_k, \]  

(17)

\[ \hat{R}^{ij}_{kl} = R^{ij}_{kl}(h) - K^i_k K^j_l + K^i_l K^j_k \equiv R^{ij}_{kl} - K^{[i}_k K^{j]}_l, \]  

(18)

where \( \nabla_l = \nabla_l (\hat{\Gamma}^k_{ij}) \) is the covariant derivative defined in the Christoffel symbol of the boundary \( \hat{\Gamma}^{k}_{ij}(g) = \Gamma^k_{ij}(h) \).

3 Topological black hole solutions

A static black hole ansatz for the metric \( g_{\mu\nu} \) in the coordinate set \( x^\mu = (t, r, \varphi^m) \) is given by the line element

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2\gamma_{mn}(\varphi) d\varphi^m d\varphi^n, \quad \sqrt{-h} = r^{D-2} \sqrt{f} \sqrt{\gamma}. \]  

(19)

Here \( \gamma_{mn} \) is the metric of a \((D-2)\)-dimensional Riemann space \( \Sigma_{D-2} \) with curvature \( R^{m_1 n_1}_{m_2 n_2}(\gamma) = k \delta^{[m_1 n_1]}_{[m_2 n_2]}\), so that \( k = +1, 0 \) or \(-1\) describes a spherical, locally flat or hyperbolic transversal section, respectively. As a consequence, the horizon \( r = r_+ \) possess the same topology, and it is defined by the largest root of the equation \( f(r_+) = 0 \).

The extrinsic curvature of the boundary has the simple form

\[ K^i_j = -\frac{1}{2} \sqrt{f} h^{ik} h_{kj}' = \begin{pmatrix} -\frac{f'}{2\sqrt{f}} & 0 \\ 0 & -\frac{2f'}{r^2} \delta^m_n \end{pmatrix}. \]  

(20)

The non-vanishing components of the intrinsic curvature are

\[ R^{m_1 n_1}_{m_2 n_2}(h) = \frac{k}{r^2} \delta^{[m_1 n_1]}_{[m_2 n_2]}. \]  

(21)

We choose a gauge field with dependence on the radial coordinate

\[ A_\mu = \phi(r) \delta^\mu_\mu, \]  

(22)
with the associated field strength
\[ F_{\mu\nu} = E(r) \delta_{[\mu\nu]}^{[tr]} , \]
where the electric field is given by
\[ E(r) = -\phi'(r) . \]

The equation of motion \( \mathcal{E}^\mu = 0 \) determines the form of the electric field as
\[ E(r) = \frac{q}{\sqrt{\frac{q^2}{b^2} + r^{2D-4}}} , \]
where \( q \) is an integration constant related to the electric charge, since \( E \) behaves as \( \frac{q}{r^{D-2}} + \mathcal{O} \left( \frac{1}{r^7} \right) \).

The \( U(1) \) gauge potential measured with respect to the horizon is
\[ \phi(r) = -\int_{r_+}^r dv E(v) , \]
that, for EBI gravity, can be written down in terms of a hypergeometric function \( _2F_1(a, b; c; z) \) (see Appendix C) as \([16]\)
\[ \phi(r) = \frac{q}{(D-3)r^{D-3}} _2F_1 \left( \frac{1}{2}, \frac{D-3}{2D-4}; \frac{3D-7}{2D-4}; -\frac{q^2}{b^2r^{2D-4}} \right) - \Phi . \]

The integration constant \( \Phi \) is fixed by the condition \( \phi(r_+) = 0 \), that is,
\[ \Phi = \frac{q}{(D-3)r_+^{D-3}} _2F_1 \left( \frac{1}{2}, \frac{D-3}{2D-4}; \frac{3D-7}{2D-4}; -\frac{q^2}{b^2r_+^{2D-4}} \right) . \]

Note that the asymptotic value of the potential is \( \phi(\infty) = -\Phi \). As we will see below, the constant \( \Phi \) plays an important role in black hole thermodynamics as it is the conjugated variable to the \( U(1) \) charge for Born-Infeld theory.

A useful property of the hypergeometric function in (28) for later purposes is
\[ \frac{d}{dr} \left( \frac{2F_1}{(D-3)r^{D-3}} \right) = -\frac{1}{\sqrt{\frac{q^2}{b^2} + r^{2D-4}}} . \]

The equation of motion \( \mathcal{E}^t_t = 0 \) in the black hole ansatz (19), with the field strength as given by Eq. (23), adopts the form
\[ rf' + (D-3)(f-k) - (D-1) \frac{r^2}{f^2} - 4b^2r^2 \left( \frac{1}{\sqrt{1 - \frac{E^2}{b^2}}} \right) = 0 . \]
Plugging in the explicit form of the electric field \((25)\) into the above relation, the solution for the metric function is \([13, 14, 15]\)

\[
f(r) = k + \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{4b^2r^2}{(D-1)(D-2)} \left(1 - \sqrt{1 + \frac{q^2}{b^2r^{D-4}}}\right) + \frac{4q^2}{(D-1)(D-3)r^{2D-6}} \, _2F_1\left(1, \frac{1}{2}; \frac{D - 3}{2D - 4}; \frac{3D - 7}{2D - 4}; -\frac{q^2}{b^2r^{D-4}}\right),
\]

(31)

where \(\mu\) is a parameter of dimension [mass] × [Newton’s constant]. The remaining equations of motion are identically satisfied for the function \(f(r)\) given above. Then, the horizon radius \(r_+\) can be obtained from the equation

\[
0 = f(r_+) = k + \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{4b^2r_+^2}{(D-1)(D-2)} \left(1 - \sqrt{1 + \frac{q^2}{b^2r_+^{D-4}}}\right) + \frac{4q\Phi}{(D-1)r_+^{D-3}}.
\]

(32)

The expansion of \(f(r)\) goes as

\[
f(r) = k + \frac{r^2}{\ell^2} - \frac{\mu}{r^{D-3}} + \frac{2q^2}{(D-2)(D-3) r^{2D-6}} + \mathcal{O}\left(\frac{r^{-4D+10}}{\ell^2}\right),
\]

(33)

which, in the Maxwell’s limit, reduces to the metric function of the Reissner-Nordström-AdS black hole (see, e.g., [9]) and, for \(q = 0\), to the one of topological Schwarzschild-AdS black holes. Extreme black hole has a degenerate horizon at \(r_{\text{ext}}\), when both \(f(r_{\text{ext}})\) and \(f'(r_{\text{ext}})\) vanish. The mass for the extreme solution in terms of the extremal charge \(q_{\text{ext}}\) and the radius \(r_{\text{ext}}\) is given by

\[
\mu_{\text{ext}} = \frac{2k}{D-1} \frac{r_{\text{ext}}^{D-3}}{(D-1)(D-3)} + \frac{4q_{\text{ext}}^2}{r_{\text{ext}}^{D-3}} 2F_1\left(1, \frac{1}{2}; \frac{D - 3}{2D - 4}; \frac{3D - 7}{2D - 4}; -\frac{q_{\text{ext}}^2}{b^2r_{\text{ext}}^{2D-4}}\right),
\]

(34)

and, in turn, the extremal charge is given by

\[
q_{\text{ext}}^2 = \frac{1}{4} (D - 2) r_{\text{ext}}^{2D-4} \left(\frac{(D - 3) k}{r_{\text{ext}}^2} + \frac{D - 1}{\ell^2}\right) \left[2 + \frac{D - 2}{4b^2} \left(\frac{(D - 3) k}{r_{\text{ext}}^2} + \frac{D - 1}{\ell^2}\right)\right],
\]

(35)

such that the solution is characterized by a single parameter.

It can be easily proved that extremality condition for the Reissner-Nordström-AdS black holes

\[
\mu_{\text{ext}}^2 = \frac{8}{(D - 2)(D - 3)} q_{\text{ext}}^2 \left(\frac{k}{\ell^2} + \frac{(D - 2) q_{\text{ext}}^2}{(D - 3) b^4}\right)^2,
\]

(36)

is recovered combining Eqs. (34) and (35), and taking the suitable limit. In the latter relation, the vanishing cosmological constant limit (consistent only with \(k = +1\)) leads to an extremal mass proportional to the electric charge.
A solution with mass parameter $\mu < \mu_{\text{ext}}$ corresponds to a naked singularity. Depending on the parameters $\mu$ and $q$, Born-Infeld-AdS solutions with $\mu > \mu_{\text{ext}}$ may have two horizons (so-called RN-AdS type black hole) or one horizon (Schwarzschild-AdS type black hole). Thus, in general, a diagram $(q, \mu)$ contains the three types of solutions, which meet at the triple point, where all phases coexist \[15, 12, 14\]. This behavior resembles the one in liquid-gas-solid phase diagram, for RNAdS type, naked singularities and Schwarzschild-AdS type solutions, respectively. The nonlinear nature of the theory makes possible this richer phase structure, which does not appear in Reissner-Nordström-AdS case.

The interpretation of the parameters $\mu$ and $q$ in the general solution in terms of conservation laws associated to global symmetries confronts us with the long-standing problem of definition of conserved charges in AdS gravity. In the context of AdS/CFT correspondence, a finite, background-independent expression for the conserved quantities is achieved through the addition of standard counterterms to the bulk action. However, in this method, it is not possible to get a closed formula for the asymptotic charges in an arbitrary dimension, as the counterterms series itself has not been found yet for any $D$. In what follows, we show that the Kounterterms series is a suitable prescription to deal with a regularization problem of both conserved quantities and Euclidean action for EBI AdS black holes.

4 Black hole thermodynamics in even dimensions

4.1 Variational principle and regularization problem

Let us consider the pure AdS gravity action in four dimensions supplemented by the Gauss-Bonnet term

$$I_4 = - \int_M d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} \left( \hat{R} - 2\Lambda \right) + \alpha \left( \hat{R}^{\mu\nu\sigma\rho} \hat{R}_{\mu\nu\sigma\rho} - 4 \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \hat{R}^2 \right) \right],$$

where $\alpha$ is an arbitrary coupling constant. Because the Euler-Gauss-Bonnet term in four dimensions is a topological invariant, the bulk dynamics is not modified by this combination of quadratic-curvature terms.

It has been claimed that this freedom might shift the value of the entropy by a constant proportional to $\alpha$, as a direct application of the Wald’s formalism \[32\]. However, an arbitrary coupling of the Euler term is inconsistent from the point of view of the action principle. In fact, the theory has a well-posed variational principle only if the coupling constant is properly adjusted as $\alpha = \ell^2/(64\pi G)$, as shown in Ref.\[33\]. This reasoning results in the on-shell cancellation of divergences in the conserved quantities.

For an arbitrary $\alpha$, the Gauss-Bonnet invariant introduces additional divergent terms in the Euclidean action. It is, therefore, remarkable that fixing its coupling constant as above, also provides
a mechanism to regularize the Euclidean action for asymptotically AdS spacetimes and to reproduce the correct black hole thermodynamics [25]. Nevertheless, the entropy is still modified by an additive constant. In case of static black holes, this constant is given by $\ell^2 V(\Sigma_2)\chi(\Sigma_2)/8G$, where $V(\Sigma_2)$ stands for the volume of the two-dimensional transversal section, which has an Euler characteristic $\chi(\Sigma_2) = 2k$. Unfortunately, for topological Schwarzschild-AdS solutions with $k = -1$, this value leads to negative entropy for black holes with $r_+ < \ell$, what is clearly a drawback.

The way to avoid this inconsistency is to consider, instead, the closed form that is locally equivalent to the Gauss-Bonnet term, i.e., the second Chern form. Since for a given spacetime foliation this boundary term is written in terms of the extrinsic and intrinsic curvatures of the boundary, what we have is an alternative counterterm series, that does not depend only on intrinsic tensors on $\partial M$ [25].

In higher even dimensions ($D = 2n$), the situation is quite similar, only the Euler term is not longer quadratic in the curvature [34]. Taking then the $n$-th Chern form
\begin{equation}
B_{2n-1} = 2n\sqrt{-h} \int_0^1 dt \delta_{[j_1 \cdots j_{2n-1}]_{11 \cdots 2n-1}} K_{j_1}^{i_1} \times \left( \frac{1}{2} R_{j_1 j_3}^{i_2 i_3} - i^2 K_{j_2}^{i_2} K_{j_3}^{i_3} \right) \cdots \left( \frac{1}{2} R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - i^2 K_{j_{2n-2}}^{i_{2n-2}} K_{j_{2n-1}}^{i_{2n-1}} \right),
\end{equation}
the action (38) becomes finite if we adjust the coefficient of this boundary term as
\begin{equation}
c_{2n-1} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{n(2n - 2)}.
\end{equation}
One can also understand the Kounterterms for this case as the correction to the Euler characteristic due to the boundary in any even-dimensional manifold (a transgression form for the Lorentz group $SO(2n - 1, 1)$ [35]). Using the boundary formulation locally equivalent to bulk topological invariants would make easier a possible comparison with the standard regularization procedure.

The term (38) varies (up to a total derivative) as
\begin{equation}
\delta B_{2n-1} = \frac{n}{2n-1} \sqrt{-h} \delta_{[j_1 \cdots j_{2n-1}]_{11 \cdots 2n-1}} \left[ (h^{-1} \delta h)_k^{j_1} K_k^{j_1} + 2\delta K_{j_1}^{i_1} \right] \times \left( R_{j_1 j_3}^{i_2 i_3} - K_{j_2}^{i_2} K_{j_3}^{i_3} \right) \cdots \left( R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - K_{j_{2n-2}}^{i_{2n-2}} K_{j_{2n-1}}^{i_{2n-1}} \right),
\end{equation}
so that the complete action (41) on-shell changes under arbitrary variations in the following way,
\begin{equation}
\delta I_{2n} = \frac{n C_{2n-1}}{2n-1} \int_{\partial M} d^{2n-1}x \sqrt{-h} \delta_{[j_1 \cdots j_{2n-1}]_{11 \cdots 2n-1}} \left[ (h^{-1} \delta h)_k^{j_1} K_k^{j_1} + 2\delta K_{j_1}^{i_1} \right] \times \left( R_{j_1 j_3}^{i_2 i_3} - K_{j_2}^{i_2} K_{j_3}^{i_3} \right) \cdots \left( R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - K_{j_{2n-2}}^{i_{2n-2}} K_{j_{2n-1}}^{i_{2n-1}} \right) - (-1)^{n-1} \frac{1}{\ell^{2(n-1)}} \delta_{[i_1 \cdots i_{2n-2}]}^{[j_1 \cdots j_{2n-1}]} \delta_{[j_2 j_3] [j_{2n-2} j_{2n-1}]} \left( R_{j_1 j_3}^{i_2 i_3} - K_{j_2}^{i_2} K_{j_3}^{i_3} \right) \cdots \left( R_{j_{2n-2} j_{2n-1}}^{i_{2n-2} i_{2n-1}} - K_{j_{2n-2}}^{i_{2n-2}} K_{j_{2n-1}}^{i_{2n-1}} \right) + \frac{1}{4\pi G} \int_{\partial M} d^{2n-1}x \sqrt{-h} \frac{N F^{r_i} \delta A_i}{\sqrt{1 + F^2/2e^2}}. 
\end{equation}
Using the Gauss-Codazzi equation \((18)\), it is easy to prove that the second line in the above expression can be factorized by \(\hat{R}_{kl} + \frac{1}{\ell^2} \delta^{[ij]}_{[kl]}\) so that

\[
\delta I_{2n} = \frac{n (n - 1) c_{2n-1}}{2^{n-1}} \int d^d x \delta (g) \cdot \delta (\gamma) \cdot \left[ \frac{1}{\ell^2} \delta^{(ij)}_{(kl)} \left( (h - 1) \delta h + 2 \delta K \right) \right] \times \\
\times \left[ t \left( \hat{R}_{jkl} + \frac{1}{\ell^2} \delta \frac{\delta_{[ij]} \delta_{[kl]}}{[jk]} \right) \right] - \frac{1}{\ell^2 (n-1)} \delta \frac{\delta_{[ij]} \delta_{[kl]}}{[jk]} + \frac{1}{4\pi G} \int d^{n+1} x N \delta A_i \sqrt{1 + \frac{\hat{F}^2}{2\pi^2}}.
\]

Thus, in order to ensure that the action is stationary under arbitrary variations, we take the condition on the curvature

\[
\hat{R}_{\alpha\beta} + \frac{1}{\ell^2} \delta \frac{\delta_{[\alpha\beta]}}{[\alpha\beta]} = 0, \quad \text{at } \partial M,
\]

that, in particular, is valid for the boundary indices. The rest of the surface term is cancelled assuming that the transversal components of the gauge field satisfy

\[
\delta A_i = 0, \quad \text{at } \partial M.
\]

The condition \((43)\) simply means that the spacetime has constant curvature in the asymptotic region (asymptotically locally AdS).

### 4.2 Conserved quantities

The spacetime diffeomorphisms \(\delta x^\mu = \xi^\mu (x)\) generate the changes of the dynamical fields \(g_{\mu\nu}\) and \(A_\mu\), whose infinitesimal form is given in terms of the Lie derivative,

\[
\delta_{\xi} g_{\mu\nu} = \mathcal{L}_{\xi} g_{\mu\nu} = - \left( \hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu \right),
\]

\[
\delta_{\xi} A_\mu = \mathcal{L}_{\xi} A_\mu = - \partial_\mu (\xi^\nu A_\nu) + \xi^\nu F_{\mu\nu}.
\]

Then, the EBI action \((4)\) transforms under the diffeomorphisms according to

\[
\delta \xi I = \int_{\partial M} d^d x \left[ \mathcal{L}_{\xi} (\sqrt{-g} \mathcal{L}) + \partial_\mu (\sqrt{-g} \xi^\mu \mathcal{L}) \right] + c_d \int_{\partial M} d^d x \left[ \mathcal{L}_{\xi} B_d + \partial_\xi (\xi^i B_d) \right]
\]

\[
= \int_{\partial M} d^d x \sqrt{-h} n_\mu \left( \Theta^\mu (\xi) + \xi^\mu \mathcal{L} + \frac{1}{\sqrt{-h}} c_d n_\mu \partial_\xi (\xi^i B_d) \right) + \text{e.o.m.},
\]

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where \( \Theta = \sqrt{-h} n_\mu \Theta^\mu \). It has been used that the volume element, Jacobian and Lagrangian density transform as

\[
\delta \xi (d^D x) = \partial_\mu \xi^\mu d^D x, \\
\mathcal{L}_\xi \sqrt{-g} = -\sqrt{-g} \nabla_\mu \xi^\mu, \\
\delta \xi \mathcal{L} = \mathcal{L}_\xi \mathcal{L} + \xi^\mu \partial_\mu \mathcal{L},
\]

respectively. The Lie derivative acts on the Lagrangian as

\[
\mathcal{L}_\xi \mathcal{L} = \partial_\mu \mathcal{L} \partial_\mu \mathcal{L} \mathcal{L} + \xi^\mu \partial_\mu \mathcal{L} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \hat{\Gamma}^\beta_{\mu
u}} \mathcal{L} \hat{\Gamma}^\beta_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial A_\mu} \mathcal{L} \hat{A}_\mu.
\]

Similar expressions hold for the boundary term as well.

The diffeomorphic invariance \( (\delta \xi I = 0) \) defines the Noether current

\[
J^\mu = \Theta^\mu (\xi) + \xi^\mu \mathcal{L} + \frac{1}{\sqrt{-h}} c_d n^\mu \partial_i \left( \xi^i B_d \right).
\]

Using (12), we write out the surface term as

\[
\Theta(\xi) = \frac{1}{16\pi G} \sqrt{-h} n_\mu \left( \delta^{(\mu\nu)}_{(\alpha\beta)} g^{\alpha\gamma} \mathcal{L}_\xi \hat{\Gamma}^\beta_{\gamma\nu} + \frac{4 F^{\mu\nu} \mathcal{L}_\xi A_\nu}{\sqrt{1 + \frac{F^2}{2\sigma}}} \right) + c_d \mathcal{L} \xi B_d.
\]

The fact that the current (51) satisfies \( \partial_\mu (\sqrt{-g} J^\mu) = 0 \) means, by virtue of the Poincaré's lemma, that it can always be written locally as an exact form. However, only when the current can be cast as a total derivative globally at the boundary, the Noether charge can be directly read off from it. The radial foliation (13) defines a conservation law along the radial coordinate, such that the quantity

\[
Q[\xi] = \int_{\partial M} \sqrt{-g} J^r
\]

is a constant of motion.

We take a timelike ADM foliation for the line element on \( \partial M \) with the coordinates \( x^i = (t, y^m) \), as

\[
h_{ij} dx^i dx^j = -\tilde{N}^2 (t) dt^2 + \sigma_{mn} (dy^m + \tilde{N}^m dt)(dy^n + \tilde{N}^n dt), \quad \sqrt{-h} = \tilde{N} \sqrt{\sigma},
\]

that is generated by the outward-pointing unit normal vector \( u_i = (u_t, u_m) = (-\tilde{N}, \tilde{0}) \). \( \sigma_{mn} \) represents the metric of the boundary of spatial section at constant time \( \Sigma_\infty \).

If the radial component of the current adopts the form

\[
\sqrt{-g} J^r = \partial_j (\sqrt{-h} \xi^i q^j_i),
\]

the Noether theorem provides the conserved charges \( Q[\xi] \) of the theory as surface integrals on \( \Sigma_\infty \) as

\[
Q[\xi] = \int_{\Sigma_\infty} d^{2n-2} y \sqrt{\sigma} u_j \xi^i q^j_i,
\]

for a given set of asymptotic Killing vectors \( \{ \xi \} \).
Using Eq. (14), the action of the diffeomorphism on the extrinsic curvature can be worked out from the procedure defines the Noether charge as

\[ Q[\xi] = \frac{nc_{2n-1}}{2^{n-2}} \int_{S^{2n-2}} d^{2n-2}y \sqrt{\sigma} u_j \xi^i \xi^{i_1 \cdots i_{2n-1}} \hat{K}^{i_1 \cdots i_{2n-1}}_i \times \]

\[ \times \left( \hat{R}^{i_2 i_3} - K^{i_2 i_3} + \delta^{i_2 i_3} \right) \delta^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} - \frac{(-1)^{n-1}}{e^{2(n-1)}} \delta^{i_2 i_3} \delta^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} \right) + \frac{1}{16\pi G} \frac{4NF^i \xi A_i}{\sqrt{1 + \frac{e^2}{2\pi}}} \quad (56) \]

Using Eq. (14), the action of the diffeomorphism on the extrinsic curvature can be worked out from the corresponding components of the Christoffel symbol

\[ \mathcal{L}_\xi \hat{\Gamma}_{\mu \nu}^\alpha = \frac{1}{2} \left( \hat{R}_{\mu \nu \beta} + \hat{R}_{\nu \mu \beta} \xi^\alpha - \frac{1}{2} \left( \hat{\nabla}_\nu \hat{\nabla}_\mu \xi^\alpha + \hat{\nabla}_\nu \hat{\nabla}_\mu \xi^\alpha \right) \right) \quad (57) \]

The second line in the above expression can be always factorized by \( \left( \hat{R}^{i_2 i_3} + \frac{1}{e^{2(n-1)}} \delta^{i_2 i_3} \right) \), that means that the charge is identically zero for any global constant-curvature spacetime. The mass of the EBI AdS black hole solution \( \text{(19, 31)} \) is computed evaluating the formula \( (58) \) for the Killing vector \( \xi^i = (1, \vec{0}) \),

\[ Q[\partial_k] = -\frac{nc_{2n-1}}{2^{n-2}} \int_{S^{2n-2}} d^{2n-2}y \sqrt{\sigma} r^{2n-2} \delta^{[m_1 \cdots m_{2n-2}]}_{[n_1 \cdots n_{2n-2}]} K^t_i \times \]

\[ \times \left( \hat{R}^{m_1 m_2} \cdots \hat{R}^{m_{2n-3} m_{2n-2}} - \frac{(-1)^{n-1}}{e^{2(n-1)}} \delta^{n_1 n_2} \delta^{m_{2n-3} m_{2n-2}}_{m_2 m_{2n-2}} \right) \quad (59) \]

and we get

\[ Q[\partial_k] = \frac{V(\Sigma_{2n-2})}{16\pi G} \lim_{r \to \infty} f' \left( r^{2n-2} - e^{2n-2} (f - k)^{n-1} \right) \quad (60) \]

With the help of the identity \( a^{n-1} - b^{n-1} = (a - b) a^{n-2} \sum_{p=0}^{n-2} \left( \frac{k}{a} \right)^p \) and using the asymptotic expa-
we see that all divergences at radial infinity are cancelled out, such that the energy of the EBI AdS black hole in even dimensions is

\[ E = Q [\partial_t] = \frac{(D - 2) V(\Sigma_{D-2}) \mu}{16\pi G} \equiv M. \]  

It is straightforward to check that the contribution of the Born-Infeld electromagnetic term to the charge in any dimension \( D \) vanishes. In fact, it can be worked out from the general form of the surface term (52), that the extra piece in the charge is

\[ Q_{BI} [\xi] = \frac{1}{16\pi G} \int_{\Sigma_\infty} d^{D-1}y \sqrt{\sigma} u_{ij} \frac{N F^{rj} (\xi^i A_i)}{\sqrt{1 + F^2}}. \]  

Using Eq. (70) and asymptotic condition \( \phi(\infty) = -\Phi \), we find that

\[ Q_{BI} [\partial_t] = -Q\Phi \lim_{r \to \infty} \frac{1}{\sqrt{f}} = 0, \]  

since \( \frac{1}{\sqrt{f}} \) behaves as \( O \left( \frac{1}{r} \right) \) at large distances.

Next we calculate a Noether charge associated to a \( U(1) \) gauge transformation \( \delta_{\lambda} A_\mu = \partial_\mu \lambda, \delta_{\lambda} g_{\mu\nu} = 0 \). The EBI action in any dimension \( D \) changes by a boundary term,

\[ \delta_{\lambda} I = \frac{1}{4\pi G} \int_{\partial M} d^{D-1}x \sqrt{-h} \frac{n_\mu F^{\mu\nu} \partial_\nu \lambda}{\sqrt{1 + F^2}} , \]  

from where we can recognize a piece corresponding to the the equation of motion

\[ \sqrt{-h} n_\mu F^{\mu\nu} \partial_\nu \lambda = \partial_i \left( \sqrt{-h} \lambda N F^r i \right) + \frac{1}{4} \lambda N \sqrt{-h} \mathcal{E}^r , \]  

and project the total derivative on the surface \( \Sigma_\infty \). Assuming that \( \lambda \) is constant at the boundary, the \( U(1) \) charge is

\[ Q = \frac{1}{4\pi G} \int_{\Sigma_\infty} d^{D-1}y \sqrt{\sigma} u_i \frac{N F^{r i}}{\sqrt{1 + F^2}}, \]
that can also be written as

\[ Q = \frac{V(\Sigma D-2)}{4\pi G} \lim_{r \to \infty} \frac{r^{D-2} E}{\sqrt{1 - \frac{E^2}{r^2}}} \]  

(70)

Finally, evaluated for the black hole solution (19, 31), the electric charge is

\[ Q = \frac{V(\Sigma D-2) q}{4\pi G} \]  

(71)

4.3 Regularized Euclidean action and Smarr relation

The Euclidean continuation of the gravity action \( I^E = -iI \) considers a manifold that spans between the horizon \( r_+ \) and radial infinity. As the horizon is shrunk to a point, the avoidance of a conical singularity at the origin of the radial coordinate requires to identify the Euclidean time \( \tau = -it \) as \( \tau \sim \tau + \beta \), where the period \( \beta = \frac{1}{T} \) is the inverse of the Hawking temperature \( T \),

\[ T = \frac{1}{4\pi} \left. \frac{df(r)}{dr} \right|_{r=r_+}. \]  

(72)

We evaluate the temperature using Eq.(30) at the horizon, and we obtain

\[ T = \frac{1}{4\pi r_+} \left[ (D-3) k + \frac{(D-1) r_+^2}{\ell^2} + \frac{4b^2 r_+^2}{D-2} \left( 1 - \sqrt{1 + \frac{q^2}{b^2 r_+^{2D-4}}} \right) \right]. \]  

(73)

In what follows, we work in the grand canonical ensemble, where the natural variables are the temperature \( T \) and the electric potential \( \Phi \). The Gibbs free energy \( G(T, \Phi) = U - TS - Q\Phi \), that satisfies the differential equation \( dG = -SdT - Qd\Phi \), is represented by the Euclidean action,

\[ G = \frac{1}{\beta} I^E. \]  

(74)

The partition function in semiclassical approximation reads

\[ Z = e^{-I^E}, \]  

(75)

from where the gravitational entropy is obtained as

\[ S = \beta \left( \frac{\partial I^E}{\partial \beta} \right)_\Phi - I^E, \]  

(76)

the internal energy is

\[ U = \left( \frac{\partial I^E}{\partial \Phi} \right)_\beta - \frac{\Phi}{\beta} \left( \frac{\partial I^E}{\partial \Phi} \right)_\beta, \]  

(77)
and the thermodynamic charge is
\[ Q = -\frac{1}{\beta} \left( \frac{\partial I^E}{\partial \Phi} \right)_\beta. \] (78)

First we calculate the bulk Euclidean action (5) evaluated for the black hole solution. Since the solution is static and \( \Sigma_{D-2} \) is a maximally symmetric submanifold, the integrations along \( \tau \) and \( \varphi^m \) are trivial, leading to
\[ I^E_{\text{bulk}} = \frac{\beta V(\Sigma_{D-2})}{16\pi G} \int_{r_+}^{\infty} dr r^{D-2} \left[ f'' + 2 (D - 2) \frac{f'}{r} + (D - 2) (D - 3) \frac{(f - k)}{r^2} \right. \\
+ 2\Lambda - 4\beta^2 \left( 1 - \frac{1}{\sqrt{1 + b^2}} \right) \right] \] (79)

Using the equations of motion (9) and (30), the Euclidean action becomes a total derivative in the radial coordinate, so that
\[ I^E_{\text{bulk}} = \frac{\beta V(\Sigma_{D-2})}{16\pi G} \left( r^{D-2} f' + \frac{4 r^{D-2} E \phi}{\sqrt{1 - \frac{E^2}{b^2}}} \right) \left. \right|_{r_+}^{\infty}. \] (80)

In even dimensions \( D = 2n \), the Euclidean boundary term (38) evaluated on the black hole solution (19, 31) is
\[ \int_{\partial M} d^{2n-1} x B_{2n-1}^E = 2n \beta V(\Sigma_{2n-2}) \lim_{r \to \infty} r^{2n-2} \sqrt{f} \int_0^1 dt \left[ \frac{1}{2} P^{n_1 n_2}_{m_1 m_2} - (2n - 1) t^2 K^{n_1}_{m_1} K^{n_2}_{m_2} \right. \times \\
\left. \left( \frac{1}{2} P^{n_3 n_4}_{m_3 m_4} - t^2 K^{n_3}_{m_3} K^{n_4}_{m_4} \right) \cdots \left( \frac{1}{2} P^{n_{2n-3} n_{2n-2}}_{m_{2n-3} m_{2n-2}} - t^2 K^{n_{2n-3}}_{m_{2n-3}} K^{n_{2n-2}}_{m_{2n-2}} \right) \right]. \] (81)

Replacing Eqs.(20, 21) and using the integral
\[ \int_0^1 ds \left[ k - (2n - 1) t^2 f \right] (k - t^2 f)^{n-2} = (k - f)^{n-1}, \] (82)

the boundary term becomes
\[ c_{2n-1} \int_{\partial M} d^{2n-1} x B_{2n-1}^E = -\frac{\beta \ell^{2n-2} V(\Sigma_{2n-2})}{16\pi G} f' \left( f - k \right)^{n-1} \left. \right|_{r=\infty}. \] (83)
Therefore, the total Euclidean action $I_{2n}^E = I_{2n}^E + c_{2n-1} \int_{\partial M} d^{2n-1}x B_{2n-1}$ is

$$I_{2n}^E = \frac{\beta V(\Sigma_{D-2})}{16\pi G} \left[ \left( r^{2n-2} f' + \frac{4r^{2n-2} E \phi}{\sqrt{1 - \frac{E^2}{c^2}}} \right) \bigg|_{r_+}^{\infty} - \ell^{2n-2} f' (f - k)^{n-1} \bigg|_{r=\infty} \right]. \quad (84)$$

The contribution at infinity from the bulk action combines with the boundary one to produce $\beta$ times the Noether mass, as it may be recognized from Eq.(60). Also, it is possible to identify the second term in the above equation as $-\beta Q \Phi$ using Eq.(70). As Kounterterm series achieves the cancellation of divergences in the asymptotic charges, the finiteness of the Euclidean action is ensured for any static black hole.

Therefore, the term at the horizon corresponds to the black hole entropy

$$S = \frac{V(\Sigma_{D-2}) r_+^{D-2}}{4G} = \frac{\text{Area}}{4G}, \quad (85)$$

which satisfies the Smarr relation

$$I_{2n}^E = \beta \mathcal{E} - \beta Q \Phi - S. \quad (86)$$

The energy $\mathcal{E}$ and electric charge $Q$ obtained in Sec.4.2 as Noether charges can also be rederived thermodynamically in the grand canonical ensemble using the definitions (77) and (78). In order to carry out these calculations, we introduce a new variable

$$\eta = \frac{q}{b r_+^{D-2}}, \quad (87)$$

instead of $q$. The horizon radius is expressed in terms of the variable $\eta$ as

$$r_+ (\eta) = \frac{(D - 3) \Phi}{b \eta F_1 \left( \frac{1}{2}; \frac{D-3}{2D-4}; \frac{3D-7}{2D-4}; -\eta^2 \right)}, \quad (88)$$

and the energy and electric charge as

$$M (\eta) = \frac{V(\Sigma_{D-2})}{16\pi G} (D - 2) r_+^{D-3} \left[ k + \frac{r_+^2}{\ell^2} + \frac{4b^2 r_+^2 (1 - \sqrt{1 + \eta^2})}{(D - 1)(D - 2)} + \frac{4b r_+ \eta \Phi}{(D - 1)} \right],$$

$$Q (\eta) = \frac{V(\Sigma_{D-2}) b}{4\pi G} \eta r_+^{D-2}, \quad (89)$$

$$M (\eta) = \frac{V(\Sigma_{D-2})}{16\pi G} (D - 2) r_+^{D-3} \left[ k + \frac{r_+^2}{\ell^2} + \frac{4b^2 r_+^2 (1 - \sqrt{1 + \eta^2})}{(D - 1)(D - 2)} + \frac{4b r_+ \eta \Phi}{(D - 1)} \right],$$

$$Q (\eta) = \frac{V(\Sigma_{D-2}) b}{4\pi G} \eta r_+^{D-2}, \quad (90)$$

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while the potential is obtained as the largest root of Eq. (73),

\[ \Phi(\eta) = -\frac{2b\eta_2 F_1 \left( \frac{1}{2}; \frac{D-3}{2D-4}, \frac{3D-7}{2D-4}; \eta^2 \right)}{(D-3) \left[ \left(\frac{D-1}{\ell^2} \right) + \frac{4\ell^2}{D-2} \left( 1 - \sqrt{1 + \eta^2} \right) \right]} \times \]

\[ \times \left[ \frac{\pi}{\beta} + \frac{\pi^2}{\beta^2} - (D-3) \left( \frac{(D-1)k}{4\ell^2} + \frac{k b^2 \left( 1 - \sqrt{1 + \eta^2} \right)}{D-2} \right) \right]. \] (91)

After a straightforward calculation, using Eqs. (88–91), we find

\[ \left( \frac{\partial I_E}{\partial \beta} \right)_\Phi = \left( \frac{\partial I_E}{\partial \eta} \right)_\Phi = (E - Q \Phi), \] (92)

\[ \left( \frac{\partial I_E}{\partial \Phi} \right)_\beta = \left( \frac{\partial I_E}{\partial \eta} \right)_\beta = -\beta Q. \] (93)

Thus, from the above relations, it directly follows that internal energy (77) is

\[ U = E, \] (94)

whereas the thermodynamic charge (78) is given by

\[ Q = Q. \] (95)

It is reassuring to prove that the entropy (76) is

\[ S = \frac{\text{Area}}{4G}, \] (96)

as expected.

5 Black hole thermodynamics in odd dimensions

5.1 Variational principle and regularization problem

Three-dimensional AdS gravity can be written as a Chern-Simons density for the $SO(2,2)$ group. In this formulation, the bulk action comes naturally supplemented by a boundary term that is half of the Gibbons-Hawking term, which makes the action finite. This extrinsic regularization can be,
therefore, thought as built-in in a Chern-Simons AdS form \[36\]. In higher odd dimensions, the corresponding Chern-Simons form for the AdS group produces a gravity theory whose bulk action is given by Lovelock-type series with specific coefficients. A finite action principle for this theory is obtained adding a boundary term which is a given polynomial in the extrinsic and intrinsic curvatures \[37, 38\]. The same form of the boundary term solves the regularization problem for Einstein-Hilbert AdS gravity \[26\]. In this case, the Kounterterms series written in a concise form in terms of the parametric integrations

\[
B_{2n} = 2n\sqrt{-h} \int_0^1 dt \int_0^t ds \delta^{[j_1 \cdots j_{2n}]}_{[i_1 \cdots i_{2n}]} K_{j_1 j_2}^{i_1 i_2} \left( \frac{1}{2} R_{j_3 j_4}^{i_3 i_4} - t^2 K_{j_3 j_4}^{i_3 i_4} + \frac{s^2}{\ell^2} \delta^{i_3 i_4}_{j_3 j_4} \right) \times \\
\cdots \times \left( \frac{1}{2} R_{j_3 j_4}^{i_3 j_5 \cdots j_{2n}} - t^2 K_{j_3 j_4}^{i_3 j_5 \cdots j_{2n}} + \frac{s^2}{\ell^2} \delta^{i_3 j_5 \cdots j_{2n}}_{j_3 j_4} \right). \quad (97)
\]

With the coupling constant choice

\[
c_{2n} = -\frac{\ell^{2(n-1)}}{32\pi G n^2 (2n-1)!} \left[ \int_0^1 dt \int_0^t ds (s^2 - t^2)^{n-1} \right]^{-1} = \frac{1}{16\pi G 2^{2n-2} n (n-1)!^2}, \quad (98)
\]

the total action varies on-shell as

\[
\delta I_{2n+1} = -\frac{1}{2n-1} \int_{\partial M} d^{2n} x \sqrt{-h} \delta^{[j_1 \cdots j_{2n}]}_{[i_1 \cdots i_{2n}]} \left[ (h^{-1} \delta h)^{i_1}_{k} K^k_{j_1} + 2 \delta K^{i_1}_{j_1} \right] \delta^{i_2 j_2} \times \\
\quad \times \left[ \frac{1}{16\pi G (2n-1)!} \delta^{[i_3 i_4]}_{[j_3 j_4]} \cdots \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} + 
\quad + n c_{2n} \int_0^1 dt \left( \tilde{R}_{j_3 j_4}^{i_3 i_4} + \frac{t^2}{\ell^2} \delta^{[i_3 i_4]}_{[j_3 j_4]} \right) \cdots \left( \tilde{R}_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} + \frac{t^2}{\ell^2} \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} \right) \right] \\
\quad + \frac{n c_{2n}}{2n-1} \int_{\partial M} d^{2n} x \sqrt{-h} \delta^{[j_1 \cdots j_{2n}]}_{[i_1 \cdots i_{2n}]} \left[ (h^{-1} \delta h)^{i_1}_{k} K^k_{j_2} - \delta K^{i_1}_{j_2} \right] \times \\
\quad \times \left( R_{j_3 j_4}^{i_3 i_4} - t^2 K_{j_3 j_4}^{i_3 i_4} + \frac{t^2}{\ell^2} \delta^{[i_3 i_4]}_{[j_3 j_4]} \right) \cdots \left( R_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} - t^2 K_{j_{2n-1} j_{2n}}^{i_{2n-1} i_{2n}} + \frac{t^2}{\ell^2} \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} \right) \\
\quad + \frac{1}{4\pi G} \int_{\partial M} d^{2n} x \sqrt{-h} \frac{N F^{i_1} \delta A_i}{\sqrt{1 + \frac{\ell^2}{2r^2}}}. \quad (99)
\]
As usual, the surface term from the electromagnetic part vanishes fixing the gauge potential at the boundary, Eq. (44). The first two lines of the above relation can be factorized as

\[-\frac{nc_{2n}}{2^{n-1}} \int_{\partial M} d^{2n} x \sqrt{-h} \delta^{[i_1 \cdots i_{2n}]}_{[i_1 \cdots i_{2n}]} \left[ (h^{-1} \delta h)^{i_1}_{k} K_{j_1}^{k} + 2 \delta K_{j_1}^{i_1} \right] \delta^{i_2}_{j_2} \left( \hat{R}^{i_3 i_4 \cdots i_{2n}}_{j_3 j_4 \cdots j_{2n}} + \frac{1}{\ell^2} \delta^{[i_3 i_4]}_{[j_3 j_4]} \right) P^{i_5 \cdots i_{2n}}_{j_5 \cdots j_{2n}}(\hat{R}), \tag{100}\]

where $P^{i_5 \cdots i_{2n}}_{j_5 \cdots j_{2n}}(\hat{R})$ is a Lovelock-type polynomial of degree $(n - 2)$ in the Riemann tensor $\hat{R}_{ijkl}$ (for explicit expression, see [26]). Thus, this term is identically vanishing for the spacetimes with locally AdS asymptotics, i.e., satisfying the condition [43]. On the other hand, for any asymptotically AdS spacetime, the Fefferman-Graham theorem [39] ensures that there exists a regular expansion for the AdS asymptotics, i.e., satisfying the condition (43). On the other hand, for any asymptotically AdS spacetime, the Fefferman-Graham theorem [39] ensures that there exists a regular expansion for the asymptotic extrinsic curvature, $K_{j}^{i} = -\frac{1}{\ell} \delta_{j}^{i} + \cdots$, where the additional terms are negative powers of $r$ (in Schwarzschild-like coordinates). This justifies the choice of the asymptotic condition

$$K_{j}^{i} = -\frac{1}{\ell} \delta_{j}^{i}, \quad \text{at } \partial M, \tag{101}$$

which, on top of

$$\delta K_{j}^{i} = 0, \quad \text{at } \partial M, \tag{102}$$

guarantees a well-posed action principle for odd-dimensional AdS gravity [26].

Notice that the asymptotic conditions in both the extrinsic curvature and the spacetime Riemann tensor are not longer valid for EBI-dilaton theory, where the solutions are neither asymptotically flat nor (A)dS (see, e.g., [40]).

### 5.2 Conserved quantities

In odd dimensions, we use Eq. (99) and write the surface term $\Theta$ as

$$\Theta(\xi) = -\frac{1}{2^{n-1}} \sqrt{-h} \delta^{[i_1 \cdots i_{2n}]}_{[i_1 \cdots i_{2n}]} \left[ (h^{-1} L \xi h)^{i_1}_{k} K_{j_1}^{k} + 2 L \xi K_{j_1}^{i_1} \right] \delta^{i_2}_{j_2} \left[ \frac{1}{16 \pi G (2n - 1)!} \delta^{[i_3 i_4]}_{[j_3 j_4]} \cdots \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} \right]$$

$$+ nc_{2n} \int_{0}^{1} dt \left( \hat{R}^{i_3 i_4}_{j_3 j_4} + \frac{t^2}{\ell^2} \delta^{[i_3 i_4]}_{[j_3 j_4]} \right) \cdots \left( \hat{R}^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}} + \frac{t^2}{\ell^2} \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} \right) \right]$$

$$+ \frac{nc_{2n}}{2^{n-1}} \int_{\partial M} dt d^{2n} x \sqrt{-h} \delta^{[i_1 \cdots i_{2n}]}_{[i_1 \cdots i_{2n}]} \left[ (h^{-1} L \xi h)^{i_1}_{k} \left( K_{j_1}^{k} \delta_{j_2}^{i_2} - \delta_{j_1}^{i_2} K_{j_2}^{i_2} \right) + 2 L \xi K_{j_1}^{i_2} \delta_{j_2}^{i_2} \right] \times$$

$$\times \left( \hat{R}^{i_3 i_4}_{j_3 j_4} t^2 K^{[i_3}_{j_3} K^{i_4]}_{j_4} + \frac{t^2}{\ell^2} \delta^{[i_3 i_4]}_{[j_3 j_4]} \right) \cdots \left( \hat{R}^{i_{2n-1} i_{2n}}_{j_{2n-1} j_{2n}} t^2 K^{[i_{2n-1}}_{j_{2n-1}} K^{i_{2n}]}_{j_{2n}} + \frac{t^2}{\ell^2} \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]} \right)$$

$$+ \frac{1}{16 \pi G} \sqrt{-h} \frac{4NF^{ij} L \xi A_i}{\sqrt{1 + \frac{F^2}{2\ell^2}}} . \tag{103}$$

20
Proceeding similarly as in the even-dimensional case, it can be shown that the integrand in the Noether charge is split in two pieces, given by

\[
q^j_i = -\frac{1}{2^{n-2}} \delta^{[j_1\cdots j_{2n-1}]}_{[k_1\cdots k_{2n-1}]} K_i^k \delta^{i_1}_{j_1} \left[ \frac{1}{16\pi G (2n-1)!} \delta^{[i_2i_3]}_{[j_2j_3]} \cdots \delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} \right] \\
+ nc_{2n} \int_0^1 dt \left( \tilde{R}^{[j_1 j_2]}_{j_2j_3} + \frac{t^2}{\ell^2} \delta^{[i_2i_3]}_{[j_2j_3]} \right) \cdots \left( \tilde{R}^{[i_{2n-2}i_{2n-1}]}_{j_2j_3} + \frac{t^2}{\ell^2} \delta^{[i_{2n-2}i_{2n-1}]}_{[j_2j_3]} \right) \right] \\
+ \frac{1}{16\pi G} \frac{4NF^{j} \xi_k A_k}{\sqrt{1 + \frac{t^2}{\ell^2}}},
\]

(104)

and

\[
q^{(0)j}_i = \frac{nc_{2n}}{2^{n-2}} \delta^{[j_1\cdots j_{2n-1}]}_{[k_1\cdots k_{2n-1}]} \int_0^1 dt \left( K_i^k \delta^{i_1}_{j_1} + K_j^k \delta^{j_1}_{i_1} \right) \left( R^{[i_2i_3]}_{j_2j_3} - t^2 K^{[i_2i_3]}_{[j_2j_3]} + \frac{t^2}{\ell^2} \delta^{[i_2i_3]}_{[j_2j_3]} \right) \times \cdots \\
\times \left( R^{[i_{2n-2}i_{2n-1}]}_{j_{2n-2}j_{2n-1}} - t^2 K^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} + \frac{t^2}{\ell^2} \delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} \right) \right).
\]

(105)

Therefore, the Noether charge is in odd dimensions is

\[
Q[\xi] = q[\xi] + q(0)[\xi],
\]

(106)

where

\[
q[\xi] = \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} u_j \xi^i q^j_i,
\]

(107)

\[
q(0)[\xi] = \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} u_j \xi^i q^{(0)j}_i,
\]

(108)

and whose explicit expressions are

\[
q[\xi] = \frac{1}{2^{n-2}} \int_{\Sigma_\infty} d^{2n-1}y \sqrt{\sigma} \delta^{[j_1\cdots j_{2n-1}]}_{[i_1\cdots i_{2n-1}]} u_j \xi^i K_i^k \delta^{i_1}_{j_1} \left[ \frac{1}{16\pi G (2n-1)!} \delta^{[i_2i_3]}_{[j_2j_3]} \cdots \delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} \right] \\
+ nc_{2n} \int_0^1 dt \left( \tilde{R}^{[j_1 j_2]}_{j_2j_3} + \frac{t^2}{\ell^2} \delta^{[i_2i_3]}_{[j_2j_3]} \right) \cdots \left( \tilde{R}^{[i_{2n-2}i_{2n-1}]}_{j_{2n-2}j_{2n-1}} + \frac{t^2}{\ell^2} \delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]} \right),
\]

(109)
and
\[
q(0) [\xi] = -\frac{n c_{2n}}{2^{n-2}} \int_{\Sigma_{\infty}} d^{2n-1} y \sqrt{\sigma} \delta^{[i_{1} \ldots j_{2n-1}]}_{[i_{1} \ldots j_{2n-1}]} u_{j} \xi^{k} \left(K_{k}^{i} \delta_{j_{1}^{i}}^{i_{1}} + K_{j_{1}^{i}}^{i} \delta_{k}^{i_{1}}\right) \times \\
\int_{0}^{1} dt t \left(R_{j_{2} j_{3}}^{i_{2} i_{3}} - t^{2} K_{j_{2} j_{3}}^{i_{2} i_{3}} + \frac{t^{2}}{\ell^{2}} \delta_{[i_{2} i_{3}]^{i}_{[j_{2} j_{3}]}\right) \times \ldots \\
\ldots \times \left(R_{j_{2n-2} j_{2n-1}}^{i_{2} i_{3}} - t^{2} K_{j_{2n-2} j_{2n-1}}^{i_{2} i_{3}} + \frac{t^{2}}{\ell^{2}} \delta_{[i_{2} i_{3}]^{i}_{[j_{2} j_{3}]}\right) n^{-1}\right)] .
\tag{110}
\]

Once again, the electromagnetic part of the action does not contribute to the charge (106).

For \(\xi^{i} = (1, \vec{0})\), the charge (109) for topological static black holes (19, 31) becomes
\[
q [\partial_{t}] = \frac{V(\Sigma_{2n-1})}{16\pi G} \lim_{r \to \infty} r^{2n-1} f' \left[1 + 16\pi G n (2n-1)! c_{2n} \int_{0}^{1} dt \left(\frac{k - f}{r^{2}} + \frac{t^{2}}{\ell^{2}}\right)^{n-1}\right] .
\tag{111}
\]

For large \(r\), the above integral can be evaluated through the expansion
\[
\int_{0}^{1} dt \left(\frac{k - f}{r^{2}} + \frac{t^{2}}{\ell^{2}}\right)^{n-1} = \frac{(-1)^{n-1} 2^{n-1} (n-1)!}{\ell^{2n-2} (2n-1)!} \left[1 - \frac{(2n-1) \ell^{2} \mu}{2 r^{2n}}\right] + O(r^{-4n+2}) ,
\]
so the black hole mass is
\[
q [\partial_{t}] = M ,
\tag{112}
\]
given by the expression (63). On the other hand, the charge (110) is
\[
q(0) [\partial_{t}] = 2n (2n-1)! c_{2n} V(\Sigma_{2n-1}) \lim_{r \to \infty} \int_{0}^{1} dt t \left(f - \frac{r f'}{2}\right) \left[k + \left(\frac{r^{2}}{\ell^{2}} - f\right) t^{2}\right]^{n-1} ,
\tag{113}
\]
that, in the limit \(r \to \infty\), represents the vacuum energy
\[
E_{\text{vac}} = q(0) [\partial_{t}] = (-k)^{n} V(\Sigma_{2n-1}) \frac{8\pi G}{2^{n-2} (2n-1)!!} (2n)! .
\tag{114}
\]

This vacuum energy matches the one of Einstein-Hilbert AdS gravity found in Ref. [23]. In the next section, we show that the Smarr relation derived from black hole thermodynamics in EBI gravity is valid only if the internal energy corresponds to the total Noetherian energy
\[
\mathcal{E} = Q [\partial_{t}] = M + E_{\text{vac}} ,
\tag{115}
\]
which shifts the mass coming from background-dependent methods by a constant.

22
5.3 Regularized Euclidean action and Smarr relation

In odd dimensions $D = 2n + 1$, the Euclidean boundary term is obtained plugging in the static metric ansatz (19) into Eq. (97), that is

$$\int d^{2n} x B_{2n}^E = 2n \beta V (\Sigma_{2n-1}) \lim_{r \to \infty} r^{2n-1} \sqrt{f} \int_0^1 dt \int_0^t ds \, \delta^{[m_1 \ldots m_{2n-1}]}_{[n_1 \ldots n_{2n-1}]}$$

or explicitly, after using the expressions (20) and (21),

$$\int d^{2n} x B_{2n}^E = -2n (2n - 1)! \beta V (\Sigma_{2n-1}) \lim_{r \to \infty} \int_0^t ds \, \left( k - t^2 f + s^2 \frac{r^2}{\ell^2} \right)^{n-2} \times \left[ \frac{r f'}{2} (k - (2n - 1) t^2 f + s^2 \frac{r^2}{\ell^2}) + f \left( k - t^2 f + (2n - 1) s^2 \frac{r^2}{\ell^2} \right) \right].$$

Applying the integrals

$$\int_0^1 dt \int_0^t ds \, \left( k - t^2 f + s^2 \frac{r^2}{\ell^2} \right)^{n-2} \left( k - (2n - 1) t^2 f + s^2 \frac{r^2}{\ell^2} \right) = \int_0^1 dt \left( k - f + t^2 \frac{r^2}{\ell^2} \right)^{n-1} - \int_0^1 dt \left( k - t^2 f + t^2 \frac{r^2}{\ell^2} \right)^{n-1}$$

and

$$\int_0^1 dt \int_0^t ds \, \left( k - t^2 f + s^2 \frac{r^2}{\ell^2} \right)^{n-2} \left( k - t^2 f + (2n - 1) s^2 \frac{r^2}{\ell^2} \right) = \int_0^1 dt \left( k - t^2 f + t^2 \frac{r^2}{\ell^2} \right)^{n-1},$$

23
the surface term becomes

\[ \int_{\partial M} d^2 n x B_2^n = n (2n-1)! \beta V(\Sigma_{2n-1}) \lim_{r \to \infty} \left[ r^{2n-1} f' \int_0^1 dt \left( \frac{k - f}{r^2} + \frac{t^2}{r^2} \right)^{n-1} + 2 \left( f - \frac{r f'}{2} \right) \int_0^1 dt t \left( k - t^2 f + \frac{t^2 r^2}{r^2} \right)^{n-1} \right]. \]  

(120)

When the above boundary term is added to the bulk Euclidean action (80) with a suitable coupling constant, \( I_{2n+1}^E = I_{\text{bulk}}^E + c_{2n} \int_{\partial M} d^2 n x B_2^n \), the total contribution coming from radial infinity can be identified with \( \beta (M + E_{\text{vac}}) \), as seen from the two parts of Noether charge (111) and (113). The consistency of the black hole thermodynamics is therefore verified through the Smarr relation

\[ I_{2n+1}^E = \beta (M + E_{\text{vac}}) - \beta Q \Phi - \frac{\text{Area}}{4G}, \]

(121)

for a total energy which includes the zero-point energy \( E_{\text{vac}} \).

Proceeding in a similar way as in even dimensions, using the thermodynamic relations (87-91), it follows

\[ \left( \frac{\partial I^E}{\partial \beta} \right)_\Phi = M + E_{\text{vac}} - Q \Phi, \]

(122)

\[ \left( \frac{\partial I^E}{\partial \Phi} \right)_\beta = -\beta Q, \]

(123)

from where the internal energy (77) can be computed as

\[ U = M + E_{\text{vac}}, \]

(124)

and the thermodynamic charge (78) is simply

\[ Q = Q. \]

(125)

This computation verifies for the odd-dimensional case the consistency between the Noether charges and the corresponding extensive variables in black hole thermodynamics.

In the extremal case, with solution parameters related by eq. (34), the black hole temperature vanishes and thus, the Euclidean action becomes infinite. However, the Gibbs free energy defined in eq. (74) remains finite even when we take the extremal black hole limit

\[ G = E_{\text{vac}} + \frac{V(\Sigma_{D-2})}{16\pi G} \left( 2k \ell_{\text{ext}}^{D-3} - \mu_{\text{ext}} \right), \]

(126)

24
where $E_{\text{vac}}$ only appears in odd dimensions. Therefore, the analysis carried out above still holds, as the thermodynamics is described by continuous functions which are finite in the extremal case. In this respect, the EBI AdS system presents an analogous behavior as the one of RN-AdS black holes studied in Ref. [9]. In particular, the black hole entropy $S$ is still a quarter of the horizon’s area.

6 Conclusions

Black hole entropy is expected to be due to underlying microscopic degrees of freedom at the horizon. At a macroscopic level, $S$ comes from local properties of the horizon and can be simply computed, e.g., using Wald’s formalism for a given gravity theory [32]. However, the interplay between thermodynamic quantities and the conserved charges at infinity can be better understood directly from the evaluation of the Euclidean action.

In this spirit, we have checked the validity of the Smarr relation (3) for EBI AdS black holes in all dimensions using Kounterterms regularization of the action and the conserved charges. This is particularly important in the odd-dimensional case, where the method give rise to a non-vanishing vacuum (Casimir) energy $E_{\text{vac}}$ for AdS spacetime, what cannot be observed using any background-dependent definition of conserved quantities, and whose formula cannot be worked out in an arbitrary odd dimension in the standard counterterms approach.

Kounterterms prescription is interesting because it has been proved to be universal: the explicit form of the boundary terms which regularize Einstein-Hilbert AdS gravity remains the same when one couples the Gauss-Bonnet term in higher dimensions than four [27]. In this way, the correct black hole thermodynamics can be recovered from a finite Euclidean action. Remarkably, this is still true for any higher-curvature gravity theory of the Lovelock family, whenever an AdS branch can be defined.

There is some evidence that indicates that the full Dirichlet counterterm series should be generated from Kounterterms by a suitable expansion of the fields. This has been carried out explicitly for certain Lovelock theories where the symmetry enhancement permits the integration of intrinsic counterterms from the variation of the Dirichlet action [41].

This argument strongly supports the idea that a similar proof might be given for Einstein-Hilbert and Einstein-Gauss-Bonnet AdS gravity.

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A Conventions

The totally-antisymmetric Kronecker delta of rank $p$ is defined as the determinant

$$
\delta_{[\mu_1 \ldots \mu_p]}^{[\nu_1 \ldots \nu_p]} := \begin{vmatrix}
\delta_{\mu_1}^{\nu_1} & \delta_{\mu_1}^{\nu_2} & \cdots & \delta_{\mu_1}^{\nu_p} \\
\delta_{\mu_2}^{\nu_1} & \delta_{\mu_2}^{\nu_2} & \cdots & \delta_{\mu_2}^{\nu_p} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{\mu_p}^{\nu_1} & \delta_{\mu_p}^{\nu_2} & \cdots & \delta_{\mu_p}^{\nu_p}
\end{vmatrix}.
$$

(127)

A contraction of $k$ indices in the above Kronecker delta produces a delta of order $p - k$,

$$
\delta_{[\mu_1 \ldots \mu_p]}^{[\nu_1 \ldots \nu_p]} \delta_{\nu_1}^{\mu_1} \cdots \delta_{\nu_k}^{\mu_k} = \frac{(N - p + k)!}{(N - p)!} \delta_{[\mu_{k+1} \ldots \mu_p]}^{[\nu_{k+1} \ldots \nu_p]}, \quad (1 \leq k \leq p \leq N),
$$

(128)

where $N$ is the range of indices.

The Riemann tensor in our notation has the form

$$
\hat{R}^\alpha_{\beta\mu\nu} = \partial_\mu \hat{\Gamma}^\alpha_{\beta\nu} + \hat{\Gamma}^\alpha_{\sigma\mu} \hat{\Gamma}^\sigma_{\beta\nu} - \hat{\Gamma}^\sigma_{\beta\mu} \hat{\Gamma}^\alpha_{\sigma\nu},
$$

(129)

in terms of the Christoffel symbol $\hat{\Gamma}^\mu_{\alpha\beta}$. From the above relation it follows that its variation is

$$
\delta \hat{R}^\alpha_{\beta\mu\nu} = \nabla_\mu \delta \hat{\Gamma}^\alpha_{\beta\nu} - \nabla_\nu \delta \hat{\Gamma}^\alpha_{\beta\mu}.
$$

(130)

The scalar curvature can be conveniently written as

$$
\hat{R} = \frac{1}{2} \delta_{[\alpha\beta]} g^{\beta\gamma} \hat{R}^\alpha_{\gamma\mu\nu}.
$$

(131)

B Stokes’ theorem

The Stokes’ theorem states

$$
\int_M d\Omega = \int_{\partial M} \Omega,
$$

(132)

where $\Omega$ is a $d$-form defined on a $(d + 1)$-dimensional manifold $M$ parameterized by the coordinates $x^\mu$,

$$
\Omega = \frac{1}{d!} \Omega_{\mu_1 \ldots \mu_d} dx^{\mu_1} \cdots dx^{\mu_d}.
$$

(133)
In components, the Stokes’ theorem is (\(x^i\) are the coordinates on \(\partial M\))
\[
\int_M \partial_\mu \Omega_{\mu \nu \cdot \cdot \cdot} dx^\mu dx^{\mu_1} \cdot \cdot \cdot dx^{\mu_d} = \int_{\partial M} \Omega_{i_1 \cdot \cdot \cdot i_d} dx^{i_1} \cdot \cdot \cdot dx^{i_d} .
\] (134)

If we choose the metric \(g_{\mu \nu}\) with the signature (\(-, +, \cdot \cdot \cdot, +\)) in the following way,
\[
g_{\mu \nu} dx^\mu dx^{\nu} = g_{tt} dt^2 + h_{ij} dx^i dx^j ,
\] (135)
so that the boundary \(\partial M\) is placed at \(x^0 = t = Const.\), then the Stokes’ theorem (134) can be written as
\[
\int_M d^{d+1}x \partial_\mu \left( \sqrt{|g_{\mu \nu}|} V^\mu \right) = \int_{\partial M} d^d x \sqrt{|h_{ij}|} n_\mu V^\mu ,
\] (136)
where
\[
V^\mu = \frac{1}{d! \sqrt{|g_{\mu \nu}|}} \varepsilon_{\mu \nu \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot} \Omega_{\mu \nu \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot} ,
\] (137)
and the outward pointing unit vector normal to \(\partial M\) has components
\[
n_\mu = \left( n_t, \vec{0} \right) = \left( -\sqrt{-g_{tt}}, \vec{0} \right) , \quad n^2 = -1 .
\] (138)
The \(d\)-dimensional Levi-Civita symbol is defined as
\[
\varepsilon^{i_1 \cdot \cdot \cdot i_d} = -\varepsilon^{i_1 \cdot \cdot \cdot i_d} .
\] (139)

If, however, instead of (135) we choose the metric in the form where \(\partial M\) is placed at \(x^1 = r = const.\),
\[
g_{\mu \nu} dx^\mu dx^{\nu} = g_{rr} dr^2 + h_{ij} dx^i dx^j ,
\] (140)
then the outward pointing unit vector normal to \(\partial M\) in the Stokes’ theorem (136) is
\[
n_\mu = \left( n_r, \vec{0} \right) = \left( \sqrt{g_{rr}}, \vec{0} \right) , \quad n^2 = 1 .
\] (141)
The difference in sign is due to the \(d\)-dimensional Levi-Civita symbol defined as
\[
\varepsilon^{i_1 \cdot \cdot \cdot i_d} = \varepsilon^{r_1 \cdot \cdot \cdot i_d} .
\] (142)

### C Hypergeometric function

The hypergeometric series has the form
\[
_2F_1(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a + 1) b(b + 1)}{2c(c + 1)} z^2 + \cdot \cdot \cdot \\
= \sum_{p=0}^{\infty} \frac{(a)_p (b)_p}{(c)_p} \frac{z^p}{p!} ,
\] (143)
where \((a)_p\) is a Pochhammer symbol. It converges if \(c\) is not a negative integer \((i)\) for all \(|z| < 1\) and \((ii)\) on the unit circle \(|z| = 1\) if \(\Re(c - a - b) > 0\). An integral representation of the hypergeometric function is

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt \frac{t^{b-1}(1 - t)^{c-b-1}}{(1 - zt)^b}, \quad \Re(c) > \Re(b) > 0, \tag{144}
\]

and its derivative is

\[
\frac{d}{dz} 2F_1(a, b; c; z) = \frac{ab}{c} 2F_1(a + 1, b + 1; c + 1; z). \tag{145}
\]

In particular, the following integral is solved in the text,

\[
\int_0^1 dt \frac{u^{b-1}}{\sqrt{1 + zt}} = \frac{1}{b} 2F_1 \left( \frac{1}{2}, b; b + 1; -z \right), \quad b > 0. \tag{146}
\]

References

[1] M. Born and I. Infeld, \textit{Foundations of the new field theory}, Proc. R. Soc. \textbf{A144}, 425 (1934).

[2] S. Deser and G.W. Gibbons, \textit{Born-Infeld-Einstein actions?}, Class. Quantum Grav. \textbf{15}, L35 (1998). [arXiv: hep-th/9803049]

[3] D. Klemm, \textit{Black holes and singularities in string theory}. [arXiv: hep-th/0410040]

[4] E.S. Fradkin and A.A. Tseytlin, \textit{Nonlinear electrodynamics from quantized strings}, Phys. Lett. B \textbf{163}, 123 (1985).

[5] A.A. Tseytlin, \textit{Vector field effective action in the open superstring theory}, Nucl. Phys. B \textbf{276}, 391 (1986).

[6] R.G. Leigh, \textit{Dirac-Born-Infeld action from Dirichlet sigma model}, Mod. Phys. Lett. A \textbf{4}, 2767 (1989).

[7] C.G. Callan and J.M. Maldacena, \textit{Brane death and dynamics from the Born-Infeld action}, Nucl. Phys. B\textbf{513} 198 (1998). [arXiv: hep-th/9708147]

[8] G.W. Gibbons, \textit{Born-Infeld particles and Dirichlet p-branes}, Nucl. Phys. B\textbf{514} 603 (1998). [arXiv: hep-th/9709027]
[9] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, Phys. Rev. D 60, 104026 (1999). [arXiv: hep-th/9904197]; *Charged AdS black holes and catastrophic holography*, Phys. Rev. D 60, 064018 (1999). [arXiv: hep-th/9902170]

[10] C. Peca and J.P.S. Lemos, *Thermodynamics of Reissner-Nordstrom anti-de Sitter black holes in the grand canonical ensemble*, Phys. Rev. D 59, 124007 (1999). [arXiv: gr-qc/9805004]; *Thermodynamics of toroidal black holes*, J. Math. Physics 41, 4783 (2000). [arXiv: gr-qc/9809029]

[11] J.P.S. Lemos and V.T. Zanchin, *Rotating charged black string and three-dimensional black holes*, Phys. Rev. D 54, 3840 (1996). [arXiv: hep-th/9511188]

[12] S. Fernando, *Thermodynamics of Born-Infeld-anti-de Sitter black holes in the grand canonical ensemble*, Phys. Rev. D 74, 104032 (2006). [arXiv: hep-th/0608040]

[13] S. Fernando and D. Krug, *Charged black hole solutions in Einstein-Born-Infeld gravity with a cosmological constant*, Gen. Rel. Grav. 35, 129 (2003). [arXiv: hep-th/0306120]

[14] T.K. Dey, *Born-Infeld black holes in the presence of a cosmological constant*, Phys. Lett. B 595, 484 (2004). [arXiv: hep-th/0406169]

[15] R. Cai, D. Pang and A. Wang, *Born-Infeld black holes in (A)dS spaces*, Phys. Rev. D 70, 124034 (2004). [arXiv: hep-th/0410158]

[16] M.H. Dehghani and H.R. Sedehi, *Thermodynamics of rotating black branes in (n+1)-dimensional Einstein-Born-Infeld gravity*, Phys. Rev. D 74, 124018 (2006). [arXiv: hep-th/0610239]

[17] M. Aiello, R. Ferraro and G. Giribet, *Hoffmann-Infeld black hole solutions in Lovelock gravity*, Class. Quant. Grav. 22, 2579 (2005). [arXiv: gr-qc/0502069]

[18] X. Gao, *Non-supersymmetric Attractors in Born-Infeld Black Holes with a Cosmological Constant*, J. High Energy Phys. 11, 006 (2007). [arXiv: 0708.1226]

[19] J.M. Maldacena, *The large N limit of superconformal field theories*, Adv. Theor. Math. Phys. 2, 231 (1998); Int. J. Theor. Phys. 38, 1113 (1999). [arXiv: hep-th/9711200]; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semiclassical limit of the gauge string correspondence*, Nucl. Phys. B636, 99 (2002). [arXiv: hep-th/0204051]; E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2, 253 (1998). [arXiv: hep-th/9802150]

[20] M. Henningsson and K. Skenderis, *The holographic Weyl anomaly*, J. High Energy Phys. 07, 023 (1998). [arXiv: hep-th/9806087]
[21] S. de Haro, K. Skenderis and S. Solodukhin, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, Commun. Math. Phys. 217, 595 (2001). [arXiv: hep-th/0002230]

[22] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, Commun. Math. Phys. 208, 413 (1999). [arXiv: hep-th/9902121]

[23] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, Phys. Rev. D 60, 104001 (1999). [arXiv: hep-th/9903238]; R.B. Mann, *Misner string entropy*, Phys. Rev. D 60, 1040047 (1999). [arXiv: hep-th/9903229]

[24] Thermodynamics of asymptotically AdS spacetimes, where the regularization scheme is based on Dirichlet counterterms, was discussed in I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*. [arXiv: hep-th/0404116]; *Thermodynamics of asymptotically locally AdS spacetimes*, J. High Energy Phys. 08, 004 (2005). [arXiv: hep-th/0505190]

[25] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, J. High Energy Phys. 06, 023 (2005). [arXiv: hep-th/0504233]

[26] R. Olea, *Regularization of odd-dimensional AdS gravity: Kounterterms*, J. High Energy Phys. 04, 073 (2007). [arXiv: hep-th/0610230]

[27] G. Kofinas and R. Olea, *Vacuum energy in Einstein-Gauss-Bonnet AdS gravity*, Phys. Rev. D 74, 084035 (2006). [arXiv: hep-th/0606253]

[28] G. Kofinas and R. Olea, *Universal regularization prescription for Lovelock AdS gravity*, J. High Energy Phys. 11, 069 (2007). [arXiv: 0708.0782]

[29] D.L. Wiltshire, *Black holes in string generated gravity models*, Phys. Rev. D 38, 2445 (1988).

[30] D.A. Rasheed, *Nonlinear electrodynamics: Zeroth and first laws of black hole mechanics*. [arXiv: hep-th/9702087]

[31] N. Breton, *Smarr's formula for black holes with non-linear electrodynamics*, Gen. Rel. Grav. 37, 643 (2005). [arXiv: gr-qc/0405116]

[32] V. Iyer and R. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D 50, 846 (1994); R. Wald and A. Zouhas, *A General definition of 'conserved quantities' in general relativity and other theories of gravity*, Phys. Rev. D 61, 084027 (2000).
[33] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for gravity with locally AdS asymptotics*, Phys. Rev. Lett. **84**, 1647 (2000). [arXiv: gr-qc/9909015]

[34] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for even dimensional asymptotically AdS gravity theories*, Phys. Rev. D **62**, 044002 (2000). [arXiv: hep-th/9912045]

[35] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, Phys. Rept. **66**, 213 (1980).

[36] O. Mišković and R. Olea, *On boundary conditions in three-dimensional AdS gravity*, Phys. Lett. B **640**, 101 (2006). [arXiv: hep-th/0603092]

[37] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, J. High Energy Phys. **06**, 036 (2004). [arXiv: hep-th/0405267]

[38] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, J. High Energy Phys. **02** 067 (2006). [arXiv: hep-th/0601081]

[39] C. Fefferman and R. Graham, *Conformal invariants*, The mathematical heritage of Elie Cartan (Lyon 1984), Astérisque, 1985, Numero Hors Serie, 95.

[40] A. Sheykhi, *Thermodynamical properties of topological Born-Infeld-dilaton black holes*. [arXiv: 0801.4112]

[41] O. Mišković and R. Olea, *Counterterms in Dimensionally Continued AdS Gravity*, J. High Energy Phys. **10**, 028 (2007). [arXiv: 0706.4460]