The Asymmetric Index of a Graph

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Abstract

A graph G is asymmetric if its automorphism group of vertices is trivial. Asymmetric graphs were introduced by Erdős and Rényi [1] in 1963 where they measured the degree of asymmetry of an asymmetric graph. They proved that any asymmetric graph can be made non-asymmetric by removing some number r of edges and/or adding some number s of edges, and defined the degree of asymmetry of a graph to be the minimum value of r + s. In this paper, we define another property that how close a given non-asymmetric graph is to being asymmetric. We define the asymmetric index of a graph G, denoted ai(G), to be the minimum of r + s in order to change G into an asymmetric graph.

We investigate the symmetry asymmetric index of both connected and disconnected graphs and obtain precise values for paths, cycles, certain circulant graphs, Cartesian products involving paths and cycles, and bounds for complete graphs, split graphs, and stars.

1 Introduction

We consider undirected graphs without multiple edges or loops. A graph G is asymmetric if its automorphism group of vertices is trivial. To avoid confusion with symmetric graphs where the automorphism group of vertices is all permutations of vertices, a graph with a non-trivial automorphism group of vertices will be referred to as non-asymmetric. Asymmetric graphs were introduced by Erdős and Rényi [1] in 1963 where they measured the degree of asymmetry of an asymmetric graph. They proved that any asymmetric graph can be made non-asymmetric by removing some number r of edges and subsequently adding some number s of edges, and defined the degree of asymmetry of a graph to be the minimum of r + s. In this paper, we define a property that measures how close a non-asymmetric graph is to being asymmetric. We define the asymmetric index of a graph G, denoted ai(G), to be the minimum of r + s in order to transform G into an asymmetric graph. At first glance it might appear that calculating the asymmetric degree of a graph is the dual problem of calculating the asymmetry index - thinking that adding (removing) edges from an asymmetric graph to obtain a symmetric graph would be the same as removing (adding) edges in a non-asymmetric graph to obtain an asymmetric graph. However the key difference between the two properties is that the asymmetric degree only involves finding a single non-asymmetric graph that could result by removing or adding edges to an asymmetric graph, where as we see in the symmetric index, there are often many non-asymmetric graphs that can be transformed to the same asymmetric graph. This difference becomes more evident as we show in Proposition 1.2 that a graph and its complement have the same asymmetric index, a property that is not true for the degree of asymmetry. We will also give examples of graphs that are symmetric (vertex-transitive) that can be transformed into asymmetric graphs by adding only a pair of edges.

We will use n to denote the number of vertices in a graph. For a graph G we will use V(G) to denote the set of vertices, and E(G) to denote the set of edges. The edge between vertices u and v will be denoted

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Two graphs $G$ and $H$ are isomorphic if there is a bijection $f : G \to H$ where $uv \in E(G) \iff f(u)f(v) \in E(H)$. Recall that $f$ is an automorphism if it is an isomorphism from a graph to itself, and the set of all automorphisms of a graph form an algebraic group under function composition. We will use $\text{Aut}(H)$ to denote the automorphism group of a graph $H$. The complement of a graph $G$ will be denoted $\overline{G}$. We will use $K_n$ to denote a complete graph on $n$ vertices; $K_{s,t}$ to denote a complete bipartite graph where one part has $s$ vertices and the other part has $t$ vertices; $C_n$ will denote a cycle on $n$ vertices; $W_n$ will denote a wheel graph on $n$ vertices; and $P_n$ will denote a path on $n$ vertices. The degree of a vertex $v$ is the number of edges incident to $v$. The distance between two vertices $u$ and $v$ is the number of edges in a shortest path between $u$ and $v$ and will be denoted $d(u,v)$. For graphs $G$ and $H$, the join of $G$ and $H$ is denoted $G \vee H$ and is a graph with the vertices and edges of $G$ and $H$, along with edges between each vertex of $G$ and each vertex of $H$. We will denote the disjoint union of two graphs $G$ and $H$ with $G + H$. Given two graphs $H$ and $K$, with vertex sets $V(H)$ and $V(K)$ the Cartesian product $G = H \times K$ is a graph where $V(G) = \{(u_i,v_j)\mid u_i \in V(H) \text{ and } v_j \in V(K)\}$, and $E(G) = \{(u_i,v_j),(u_k,v_l)\}$ if and only if $i = k$ and $v_j$ and $v_l$ are adjacent in $K$ or $j = l$ and $u_i$ and $u_k$ are adjacent in $H$. For any undefined notation, please see the text [2] by West.

We begin by presenting an elementary fact about asymmetric graphs. For the sake of completeness we include the details.

**Proposition 1.1.** Given any graph $G$, $\text{Aut}(G) = \overline{\text{Aut}(G)}$.

*Proof.* Let $\sigma \in \text{Aut}(G)$. Then $e \in E(G) \Rightarrow e \notin E(G) \Rightarrow \sigma(e) \notin E(G) \Rightarrow \sigma(e) \in E(\overline{G})$. This implies $\sigma \in \text{Aut}(\overline{G})$ and hence $\text{Aut}(G) \subseteq \text{Aut}(\overline{G})$. Using a similar argument, we can show $\text{Aut}(\overline{G}) \subseteq \text{Aut}(G)$, which completes the proof. $\square$

As a consequence, if $G$ is an asymmetric graph, then the complementary graph $\overline{G}$ is also asymmetric. We next show that a graph and its complement have the same asymmetric index.

**Proposition 1.2.** Given any graph $G$, $\text{ai}(G) = \text{ai}(\overline{G})$.

*Proof.* Suppose $G$ can be made into an asymmetric graph by removing some set $R$ with $r$ edges and adding some set $S$ with $s$ edges. Then by definition of $\overline{G}$, if we now add those same $r$ edges in $R$ and remove the same $s$ edges in $S$ to $\overline{G}$, we produce an asymmetric graph. $\square$

We continue by presenting two elementary results involving the join and disjoint union of two non-isomorphic asymmetric graphs.

**Proposition 1.3.** If $G$ and $H$ are non-isomorphic asymmetric graphs then $G \vee H$ is asymmetric.

*Proof.* Since $G$ is an asymmetric graph, and in $G \vee H$, each vertex $u$ in $G$ has the same adjacencies to vertices in $H$, each vertex of $G$ will be unique in $G \vee H$. Similarly each vertex in $H$ will be unique in $G \vee H$. $\square$

**Proposition 1.4.** If $G$ and $H$ are non-isomorphic asymmetric graphs then $G + H$ is asymmetric.

*Proof.* By Proposition 1.1, if $G$ and $H$ are asymmetric then $\overline{G}$ and $\overline{H}$ are asymmetric. Then by Theorem 1.1, $\overline{G} \vee \overline{H}$ is asymmetric. Since $\overline{G} \vee \overline{H} = \overline{G + H}$, by Proposition 1.1, $G + H$ is asymmetric. $\square$

Other than the trivial case of a single vertex, the next smallest asymmetric graph has six vertices. Hence any graph with five or fewer vertices cannot be made asymmetric by removing or adding edges. Every graph on six or more vertices has a finite asymmetric index. To prove this we need the following lemma.

**Lemma 1.1.** Every asymmetric graph on $n \geq 6$ vertices can be extended to an asymmetric graph on $n + 1$ vertices by adding a single vertex and a single edge.

*Proof.* Let $G$ be an asymmetric graph without a pendant vertex. Let $G'$ be a graph obtained by adding a new vertex $u$ and an edge $uv$, where $v$ is a vertex of maximum degree in $G$. We claim that $G'$ is asymmetric. If $G'$ is not asymmetric then there exists an automorphism $f$ where two vertices in $G'$ can be transposed. We note that any automorphism of $V(G')$ must send $v$ to itself since it is the only vertex of degree $\Delta(G) + 1$.
and $f$ must send $u$ to itself since it is the only vertex of degree 1. Let $v_i$ and $v_j$ be two vertices that can be transposed by the automorphism $f$. Since removing the vertex $u$ will impact $v_i$ and $v_j$ in exactly the same way, then there exists an automorphism of $V(G)$ in which $v_i$ and $v_j$ could be switched. This would contradict the fact that $G$ is asymmetric.

If $G$ has a vertex of degree one, then we choose a vertex $u$ with degree one that has a greatest distance $d$ from a vertex of degree greater than or equal to 3. Then we can create a new graph $G^*$ where a vertex $z$ and edge $uz$ are added to $G$. We next show that $G^*$ is asymmetric. Since $G$ is asymmetric, each in $G - u$ has a property that each of the other vertices does not. The vertex $u$ is the only vertex in $G^*$ that has degree 2 and is adjacent to $z$ which has the greatest distance $d + 1$ to a vertex of degree of 3 or more. Hence $u$ and $z$ will both be unique in $G^*$, making $G^*$ asymmetric.

**Theorem 1.2.** For any graph $G$ with six or more vertices $ai(G)$ is finite.

**Proof.** For any graph $G$ with $n$ vertices we can simply remove all edges from $G$ and add back in edges to create an asymmetric graph on $n$ vertices.

In the next theorem we give general bounds for the asymmetric index of a graph. The lower bound is clear as asymmetric graphs have an asymmetric index of 0. We prove the upper bound in Theorem 2.7.

**Theorem 1.3.** For a given graph $G$, $0 \leq ai(G) \leq \frac{n(n-1)}{2} - (n - 2)$.

**Definition.** In a graph $G$ two vertices $u$ and $v$ can be transposed if there exists an automorphism $\sigma : G \rightarrow G$ in which $\sigma(u) = v$ and $\sigma(v) = u$.

**Lemma 1.4.** If a graph $G$ has a set of $t$ vertices where any pair can be transposed, then $ai(G) \geq \lfloor \frac{t-1}{2} \rfloor$.

**Proof.** Let $T$ be a set of $t$ vertices, any two of which can be transposed. To eliminate all symmetries in $G$ we must either add or remove and edge incident to each vertex in $T$. Since the addition of an edge can be incident to two vertices, the minimum number of edges that needs to be added or removed is equal to $\lfloor \frac{t-1}{2} \rfloor$.

## 2 Results for Connected Graphs

In this section we investigate the asymmetric index for connected graphs.

### 2.1 Paths

We consider $P_n$ where $n \geq 6$. Removing any number of edges will leave a non-asymmetric graph, either in the form of a shorter path or a graph with two or more isolated vertices. However we will show that the addition of a single edge can make the resulting graph asymmetric.

**Theorem 2.1.** For $n \geq 6$, $si(P_n) = 1$.

**Proof.** Consider a path on $n \geq 6$ vertices with consecutive labels $v_1, v_2, \ldots, v_n$. Adding the edge $v_2v_4$ will produce an asymmetric graph, which implies $ai(P_n) = 1$.

An example of this fact can be seen in Figure 1.

![Figure 1: A path on six vertices adding a single edge](image-url)

$si(P_n) = 1$. 

2.2 Cycles

A cycle $C_n$ is both vertex and edge transitive, meaning Aut($C_n$) is isomorphic to the symmetric group $S_n$. In other words, the cycle is "fully" symmetric. Following the result of Section 2.1, we consider $C_n$ where $n \geq 6$ and find that all symmetries can be removed by deleting one edge and adding another.

**Theorem 2.2.** For $n \geq 6$, $si(C_n) = 2$.

**Proof.** We first show that $ai(C_n) > 1$. Note that if we add any edge to $C_n$, the two vertices of degree three can be transposed. If we remove any edge from $C_n$, the two vertices of degree one could be transposed. Next we prove that $ai(C_n) \leq 2$. Since removing a single edge from a cycle results in a path, then using the result of Theorem 2.1 we have that $r = 1$ and $s = 1$. Therefore, $ai(C_n) = 2$. \qed

Interestingly, one could instead add two edges to a cycle and result in an asymmetric graph. An example of this is demonstrated in Figure 1 on a six-cycle, and we find there are numerous ways to construct such graphs.

![Asymmetric graph on six vertices](image.png)

Figure 2: Asymmetric graph on six vertices

The number of non-isomorphic ways to add two edges to a cycle and remove all symmetries is given in the following remark.

**Lemma 2.3.** The number of ways to partition an integer $i \geq 6$ into two distinct positive integers each greater than or equal to 3 is $\lfloor \frac{i-5}{2} \rfloor$.

**Proof.** We first note that the number of ways to partition an integer $i \geq 6$ into two positive integers is $\lfloor \frac{i}{2} \rfloor$. We can then remove the two cases when the partitions have one part equal to 1 or 2. Then the number of ways to partition an integer $i \geq 6$ into two positive integers each of which is greater than or equal to 3 is $\lfloor \frac{i}{2} \rfloor - 2 = \lfloor \frac{i-4}{2} \rfloor$. We note that when $i$ is even this will include one partition where the numbers are equal. Hence we need to remove 1 from this quantity when $i$ is even and leave the formula unchanged when $i$ is odd. The function $2 \left( \frac{1}{2} - \frac{1}{2} \right) - 1$ will equal $-1$ when $i$ is even and 0 when $i$ is odd. Hence, the number of ways to partition an integer $i \geq 6$ into two distinct positive integers each of which is greater than or equal to 3 is $\lfloor \frac{i-5}{2} \rfloor + 2 \left( \frac{1}{2} - \frac{1}{2} \right) - 1 = \lfloor \frac{i-5}{2} \rfloor$. To see this consider the cases where $i = 2k$ and $i = 2k + 1$ where $k$ is a positive integer. When $i = 2k$, $\lfloor \frac{i-4}{2} \rfloor + 2 \left( \frac{1}{2} - \frac{1}{2} \right) - 1 = \lfloor \frac{2k-4}{2} \rfloor + 2 \left( \frac{k}{2} - \frac{1}{2} \right) - 1 = 2k - 3 = i - 3 = \lfloor \frac{i-5}{2} \rfloor$. When $i = 2k + 1$, $\lfloor \frac{i-4}{2} \rfloor + 2 \left( \frac{k}{2} + 1 \right) - 1 = \left( \frac{k-3}{2} \right) = \lfloor \frac{2k-4}{2} \rfloor = \lfloor \frac{i-5}{2} \rfloor$. \qed

**Remark.** There are $\sum_{i=7}^{n+1} \left( \left\lfloor \frac{i-5}{2} \right\rfloor \cdot \left\lfloor \frac{n+i-3}{2} \right\rfloor \right)$ distinct ways of adding two edges to $C_n$ and result in an asymmetric graph.

In order to result in an asymmetric graph, we add two edges to a cycle so that the resulting graph has three distinct $C_n$ subgraphs. We can name each of these subgraphs $C_k$, $C_m$, and $C_l$, where $C_m$ is the subgraph between $C_l$ and $C_l$ and $k + m + l = n + 4$. For example in Figure 2, $C_k$ is the $C_3$ subgraph on the left, $C_m$ is the middle $C_3$, and $C_l$ is the $C_4$ subgraph. In order to ensure an asymmetric graph, the two edges we add to a cycle must produce subgraphs $C_k$, $C_m$, and $C_l$, where $2 < k < l$ and $m \geq 3$. If we let
\[i = k + l,\] we find that there are \(\sum_{i=7}^{n+1} \left(\left\lfloor \frac{i-5}{2} \right\rfloor \cdot \left\lfloor \frac{n-i+3}{2} \right\rfloor\right)\) distinct ways to add two edges to a cycle such that the resulting graph is asymmetric.

In general, this sum counts all possibilities of \(C_m\) and the number of ways to arrange \(C_l\) and \(C_k\) for each \(C_m\). The index of the summation accounts for all possible \(C_m\). We find that the sum must begin at \(i = 7\), as the smallest \(k + l\) can be is seven. We know \(k > 2\), so the smallest \(k\) can be is three. Since \(k \neq l\), the smallest \(l\) can be is four. Therefore, since \(i = k + l\), the lower limit of the summation is seven. The upper limit of the summation is \(n + 1\), as since \(m \geq 3\), the smallest \(m\) can be is three. If substitute \(m = 3\) into \(k + m + l = n + 4\), we find that \(k + l = n + 1\) and thus, the largest \(i\) can be is \(n + 1\).

In order to count all possibilities of \(C_k\) and \(C_l\) at each \(C_m\), we count partitions of \(k + l\). Since we know \(2 < k < l\), we are interested in partitions of \(k + l\) into two distinct parts, where both parts are bigger than three. These partitions of \(k + l\) for each possible \(C_m\) can be counted with \(\left\lfloor \frac{i-5}{2} \right\rfloor\). Not only must we count the different possibilities of \(C_k\) and \(C_l\), we must also count the number of non-isomorphic ways to arrange \(C_k\) and \(C_l\) around \(C_m\). This can be described by \(\left\lfloor \frac{m-1}{2} \right\rfloor\), which in terms of \(i\) is \(\left\lfloor \frac{n-i+3}{2} \right\rfloor\). In this way, the summation \(\sum_{i=7}^{n+1} \left(\left\lfloor \frac{i-5}{2} \right\rfloor \cdot \left\lfloor \frac{n-i+3}{2} \right\rfloor\right)\) counts the number of distinct ways to add two edges to \(C_n\), such that the resulting graph is asymmetric.

It is important to note that since the removal of any number of edges from a cycle will result in two or more paths, it is impossible to only remove edges from a cycle to obtain an asymmetric graph.

### 2.3 Wheel Graph

Clearly wheel graphs are non-asymmetric as any two degree three vertices could be transposed.

#### Theorem 2.4

For \(n \geq 6\), \(si(W_n) = 2\).

**Proof.** We will first show that \(ai(W_n) > 1\). If we remove an edge incident to the vertex of degree \(n - 1\), then this results in a vertex of degree two and the neighbors of this vertex can be transposed. Similarly, if we remove an edge whose end points are both degree three, this creates two vertices of degree two that can be transposed. Now if an edge is added to the graph, it must be added between two degree three vertices. These vertices are now degree four and can be transposed. It has now been established that \(ai(W_n)\) is at least two.

Now consider the case when a single edge was removed, resulting in two vertices of degree two. If an edge incident to the degree \(n\) vertex and one of these degree two vertices is now removed, then the resulting graph is asymmetric. \(\square\)

### 2.4 Circulant Graphs

#### Theorem 2.5

For \(n \geq 4\), \(ai(C_{n^2+1}(1,n)) = 2\).

We must remove at least two edges from \(C_{n^2+1}(1,n)\) to result in an asymmetric graph, as removing one edge results in a graph that is not asymmetric: We know that circulant graphs are regular and edge transitive. The graph \(C_{n^2+1}(1,n)\) will always be 4-regular. Let \(G\) be this circulant graph with the edge \(v_xv_y\). Removing the edge \(v_xv_y\) leaves vertices \(v_x\) and \(v_y\) with a degree of 3, and they can be transposed. Thus, the graph is symmetric. As a result, we must remove at least two edges to result in an asymmetric graph.
Similarly, adding the edge \( v_av_b \) leaves vertices \( v_a \) and \( v_b \) with a degree of 5, and they can be transposed. Thus, the graph is not asymmetric. As a result, we must add at least two edges to result in an asymmetric graph.

We remove the edges \( v_2v_3 \) and \( v_4v_{4+n} \). Interestingly, instead of removing \( v_4v_{4+n} \), we could remove the edge \( v_0v_{6+n} \) and the graph would still be asymmetric.

In order to show that \( C_{n^2 \pm 1}(1, n) = 2e \) is asymmetric, we must show that all vertices in the graph are unique.

The vertex \( v_{3+n} \) is unique from all other vertices in the graph because it is the only vertex with degree four that is adjacent to two different vertices of degree three. This is the special vertex.

The vertices \( v_2, v_3, v_4, \) and \( v_{4+n} \) are the only vertices in the graph with degree three. Vertices \( v_2 \) and \( v_4 \) are not adjacent to the special vertex \( v_{3+n} \), making them different from \( v_3 \) and \( v_{4+n} \). Vertex \( v_3 \) is different to \( v_{4+n} \) because \( v_3 \) is adjacent to \( v_2 \). Vertex \( v_2 \) and \( v_4 \) are different because \( v_2 \) is always contained in less \( n+1 \) cycles with the special vertex \( v_{3+n} \), than \( v_4 \) is. This makes the vertices \( v_2, v_3, v_4, v_{n+4} \) and \( v_{n+3} \) unique.

Since we know \( v_2, v_3, v_4, v_{n+4} \) and \( v_{n+3} \) are unique, let the vertices \( v_2, v_3, v_{n+4} \) and \( v_{n+3} \) be elements of the unique set \( U \). We begin at the vertex \( v_x \) and check to see if its adjacent vertices are elements of \( U \). If all of the vertices adjacent to \( v_x \) are unique, we can say that \( v_x \) is unique and conclude it is an element of \( U \).

Since the graph is a circulant graph, the vertex \( v_x \) is always adjacent to the vertices \( v_{x-1}, v_{x+1}, v_{x+n}, \) and \( v_{(x-n) \mod m} \), where \( m \) denotes the number of vertices in the graph, unless one of the edges removed was adjacent to \( v_x \).

We begin at the vertex \( v_4 \), which we know is unique, and visit its adjacent vertices, that is: \( v_{x-1}, v_{x+1}, \) and \( v_{(x-n) \mod m} \). We find that the vertex \( v_{x-1} \) is unique because it is the only vertex adjacent to the unique vertex \( v_4 \) that has a degree of three. When \( x = 4 \) we add \( v_{x-1} \) to \( U \).

For vertices \( v_x \), where \( 4 < x < n+1 \), the adjacent vertex \( v_{x-1} \) is an element of \( U \). The vertex \( v_{x+1} \) is always distance \( n-3 \) away from the vertex \( v_{3+n+(x-5)} \), which is a unique property so we add the vertex \( v_{x+1} \) to \( U \). The vertex \( v_{x+n} \) is always distance \( x-4 \) from the vertex \( v_{4+n} \), a unique property, so we add \( v_{x+n} \) to \( U \). We can deduce that the vertex \( v_{(x-n) \mod m} \) is unique because all other vertices adjacent to \( v_x \) are also unique, and thus different from \( v_{(x-n) \mod m} \). We add \( v_{(x-n) \mod m} \) to \( U \). All vertices adjacent to \( v_x \) are now elements of \( U \), consequently, \( v_x \) is unique and an element of \( U \).

For the vertex \( v_x \), where \( x = n+1 \), the adjacent vertex \( v_{x-1} \) is an element of \( U \). We find that the vertex \( v_{(x-n) \mod m} \) is the only vertex adjacent to both \( v_x \) and to the unique vertex \( v_2 \). Thus, we add \( v_{(x-n) \mod m} \) to \( U \). The vertex \( v_{x+1} \) is the only vertex adjacent to both \( v_x \) and to the unique vertex \( v_{n+3} \), so it is unique and we add it to \( U \). We can deduce that the vertex \( v_{x+n} \) is unique because all other vertices adjacent to \( v_x \) are also unique, and thus different from \( v_{x+n} \). We add \( v_{x+n} \) to \( U \). All vertices adjacent to \( v_x \) are now elements of \( U \), consequently, \( v_x \) is unique and an element of \( U \).

For all vertices \( v_x \), such that \( n+1 < x \leq m - n \), we find that the adjacent vertices \( v_{x-1}, v_{x+1} \) and \( v_{(x-n) \mod m} \) are elements of \( U \). Thus, we deduce that the vertex \( v_{x+n} \) must also be unique. All vertices adjacent to \( v_x \) are now elements of \( U \), hence, \( v_x \) is unique and an element of \( U \).

After this process, the cardinality of \( U \) is equal to the number of vertices in the graph, which implies the graph is asymmetric.

Thus, when removing edges from \( C_{n^2 \pm 1}(1, n), r = 2 \) and \( s = 0 \). Therefore \( r + s = 2 \) for circulant graphs.

We can also add the edges \( v_1v_3 \) and \( v_1v_4 \) to \( C_{n^2 \pm 1}(1, n) \) and result in an asymmetric graph. In this way, when we add edges to \( C_{n^2 \pm 1}(1, n), r = 0 \) and \( s = 2 \). Therefore \( r + s = 2 \) for circulant graphs.

Interestingly, a combination of adding the edge \( v_1v_3 \) and removing the edge \( v_4v_{4+n} \) results in an asymmetric graph. Here, \( r = 1 \) and \( s = 1 \). When we add the edge \( v_1v_3 \) and remove the edge \( v_4v_{4+n} \) we identify five vertices that are unique: \( v_1, v_3, v_4, \) and \( v_{4+n} \). We find that \( v_1 \) and \( v_3 \) are the only vertices in the graph that have degree five. Likewise, the vertices \( v_4 \) and \( v_{4+n} \) are the only vertices in the graph with degree three.

We distinguish between \( v_1 \) and \( v_3 \), as \( v_3 \) is adjacent to a vertex with degree three, while \( v_1 \) is not. Thus \( v_1 \) and \( v_3 \) are unique. Similarly, we distinguish between vertices \( v_4 \) and \( v_{4+n} \), as \( v_4 \) is adjacent to a vertex with degree five while, \( v_{4+n} \) is not. In this way, \( v_4 \) and \( v_{4+n} \) are unique. From this we can show that all other vertices in the graph are unique, by looking at their relationship to these four unique vertices.

In this way the symmetric index of a circulant graph is always two.
2.5 Stars and Complete Graphs

Theorem 2.6. For \( n \geq 6 \), \( \left\lfloor \frac{n-1}{7} \right\rfloor \leq ai(K_{1,n-1}) \leq n - 1 \).

Proof. The lower bound follows by Lemma 1.4. For the upper bound, note that a path \( P_{n-1} \) can be formed using each vertex of degree one, adding \( n - 2 \) edges to the graph. The resulting graph is one edge short of being a wheel graph, and by Theorem 2.3 we need only remove one additional edge to be asymmetric. \( \square \)

Of all graphs, complete graphs appear to have the highest symmetric index. It is not too difficult to establish an upper bound for \( ai(K_n) \). There exists an asymmetric tree \( H \) with seven vertices and six edges. By the extension lemma and the disjoint union lemma, \( H \) can be extended to an asymmetric graph \( H_n \) consisting of a tree with \( n - 1 \) vertices along with an isolated vertex. Then by the complement lemma, \( K_n - H_n \) will be asymmetric. We have shown that \( ai(K_n) \leq n - 2 \).

However in some cases this bound can be improved. Consider \( ai(K_{28}) \). It is known that there are three non-isomorphic trees on 9 vertices which we will refer to as \( T_1, T_2, \) and \( T_3 \). By the disjoint union lemma the graph \( K_1 + T_1 + T_2 + T_3 \) is symmetric. Then the graph \( K_{28} - (K_1 + T_1 + T_2 + T_3) \) is asymmetric. We have shown that \( ai(K_{28}) \leq n - 3 \). What led to this improvement is that there are multiple non-isomorphic asymmetric graphs of the same order. The more of these graphs the better the bound will be.

We can create a general lower bound using multiple copies of the asymmetric tree with seven vertices and six edges. We note that since these graphs are the same graph this bound can be improved. Let \( G \) be a graph with \( n \) vertices. We first isolate a single vertex. Then we construct \( \left\lfloor \frac{n-1}{7} \right\rfloor \) sets with seven vertices and one remaining set with \( n - 1 - 7 \left\lfloor \frac{n-1}{7} \right\rfloor \) vertices. From this we create \( \left\lfloor \frac{n-1}{7} \right\rfloor - 1 \) asymmetric trees with seven vertices and one asymmetric tree with \( n - 1 - 7 \left( \left\lfloor \frac{n-1}{7} \right\rfloor - 1 \right) \) vertices. The total number of edges in this graph will be \( 6 \left\lfloor \frac{n-1}{7} \right\rfloor - 1 + n - 1 - 7 \left( \left\lfloor \frac{n-1}{7} \right\rfloor - 1 \right) - 1 = n - \frac{1}{7}n - \frac{1}{7} + 4 \).

Hence we have proved the following general formula which for specific cases can be improved.

Theorem 2.7. For \( n = 6 \) or 7, \( ai(K_n) = 6 \). \( n \geq 8 \), \( n - \left\lfloor \frac{1}{7}n - \frac{1}{7} \right\rfloor + 4 \leq si(K_n) \leq \ n - 2 \).

Asymptotically this gives the bound \( 6 \left\lfloor \frac{n}{7} \right\rfloor \leq si(K_n) \leq n - 2 \). As a result the symmetric index of a general graph cannot be bounded from above by a fixed integer \( N \).

Using the technique of removing edges from a graph to leave an asymmetric graph can be used to establish general bounds for the symmetry index of a graph.

Theorem 2.8. For a given graph \( G \), \( 0 \leq si(G) \leq \frac{n(n-1)}{2} - (n-2) \).

Proof. The lower bound is clear as asymmetric graphs have a symmetric index of 0. For the upper bound note that for a given a graph \( G \), either \( G \) or \( \overline{G} \) has at most \( \frac{n(n-1)}{2} \) edges. Then we could delete all but \( n - 2 \) edges from either \( G \) or \( \overline{G} \) to obtain a symmetric graph. \( \square \)

2.6 Cartesian products of paths and cycles

Theorem 2.9. For all \( r, s \geq 2 \), \( ai(P_r \times P_s) = 1 \).

Proof. The graph is clearly not asymmetric, so \( ai(P_r \times P_s) > 0 \).

Let the vertices of the graph \( G = P_r \times P_s \) have Cartesian coordinates \((i,j)\) where \( 0 \leq i \leq r \) and \( 0 \leq j \leq s \). Let \( u \) be the vertex with coordinate \((0,0)\) and let \( v \) be the vertex with coordinate \((1,0)\). We will show that the graph \( G - uv \) is asymmetric. A vertex \( w \) with coordinate \((0,j)\) has the properties \( d(w,u) = j \) and \( d(w,v) = j + 1 \). A vertex \( z \) with coordinate \((i,j)\) with \( i > 0 \) has the properties \( d(z,u) = i + j \) and \( d(z,v) = i + j - 1 \) with \( i \) distinct paths of length \( i + j - 1 \) to \( v \). Since the only vertices \( t \) that will have the same distances from \( u \) and \( v \) will be on the diagonals, and the vertices on the diagonal have different numbers of shortest paths to \( v \), it follows that all vertices in \( G - uv \) are unique. \( \square \)

Theorem 2.10. For all \( r \geq 2 \) and \( s \geq 3 \), \( ai(P_r \times C_s) = 2 \).
Proof. For the lower bound note that if we remove a single edge $uv$ from $G$ then there exists an automorphism which transposes $u$ and $v$. Hence $ai(P_r \times C_s) \geq 2$.

For the upper bound we remove the edges $uv$ and $uw$ from $P_r \times C_s$. Then let $x$ and $y$ be two vertices in the resulting graph. Then note that if $d(x, u) = d(y, u)$ and $d(x, v) = d(y, v)$ then $x = y$.

\[\Box\]

**Theorem 2.11.** For all $r,s \geq 10$, $ai(C_r \times C_s) = 3$.

Proof. We first note that if two edges $uv$ and $wx$ are removed from a torus there exists an automorphism which transposes $u$ and $w$ or $v$ and $x$. Hence $ai(C_r \times C_s) \geq 3$. For the upper bound, consider a torus with three edges removed: $ab$, $cd$, and $ef$ (see Figure 3).

![Diagram](image)

Figure 3: Two vertices $x$ and $y$ that are each distance 3 from $a$, distance 4 from $d$, and distance 5 from $f$. Note that $x$ has two paths of length 3 to $a$, and $y$ only has one.

We first show that the six vertices incident to these edges are unique. The vertex $d$ is unique since it has degree 3, is one away from a vertex of degree 3 and is distance two away from another vertex of degree 3. The vertex $c$ is unique since it is the only vertex from this set that adjacent to $d$. Vertex $b$ is unique since it is the only vertex that is distance one away from $d$. Vertex $a$ is unique since it is the only vertex with a single distance two path from $d$. Vertex $e$ is unique since it is the only vertex that has two paths of length two to $d$. Vertex $f$ is unique since it is distance 3 from $d$.

Next we show that all of the other vertices in the graph are unique. It will be helpful to refer to Figure 3. We start with a vertex $x$ and calculate its distance from vertices $a,d$, and $f$. We can identify other vertices...
with the same distances to \(a, d,\) and \(f\) as \(x\) in the following manner. We use squares centered at \(a, d,\) and \(f\) to denote vertices that are at a particular taxicab metric distance from each of the centers. Two vertices \(x\) and \(y\) have the same distances from \(a, d,\) and \(f\) if and only if vertices must lie on the intersection of all three squares. The only way for two vertices \(x\) and \(y\) to have the same number of shortest paths \(a\) and \(d\) and to \(f\) is for \(a, d,\) and \(f\) to be colinear, which is not the case.

3 Disconnected Graphs

In this section we investigate the asymmetric index for graphs that are not connected, having more than one component. We first present an upper bound for the asymmetric index of a graph that is not connected.

**Theorem 3.1.** Let \(G\) be a graph with components \(G_1, G_2, ..., G_k\). Then \(\min \sum_{i=1}^{k} s_i(G_i) \leq s_i(G) \leq \sum_{i=1}^{k} s_i(G_i).

**Proof.** For the lower bound note that to make \(G\) asymmetric we need to make each \(G_i\) asymmetric. For the upper bound note that for each component \(G_i\) we can remove \(r_i\) edges and add \(s_i\) edges, where \(r_i + s_i\) is minimized, to make it asymmetric. This creates a set of \(\sum_{i=1}^{k} r_i\) edges that can be removed and \(\sum_{i=1}^{k} s_i\) edges can be added, where \(\sum_{i=1}^{k}(r_i + s_i)\) is minimized, to make \(G\) asymmetric.

However, surprisingly this upper bound is not tight, as shown in the next example.

**Example.** Let \(G\) consist of \(l + 1\) components including a cycle \(C_l\) and paths \(P_6, P_7, ..., P_{l+5}\). Since \(ai(C_i) = 2\) and \(ai(P_i) = 1\) for all \(i \geq 6\), \(ai(G) \leq l + 2\). However we can add \(l\) edges connecting a vertex of degree 1 in each of the paths with a different vertex on the cycle. To see that this graph is asymmetric, first note that all of the vertices on the cycle are unique since they are all incident to paths with different lengths. Next we show that all of the vertices in each of the paths are unique. We first consider vertices on the same pendant path. They have different distances to a vertex on a cycle. Two vertices on different paths are in different size components of \(G - C_l\). Hence all vertices in \(G\) are unique. Hence \(ai(G) \leq l\).

We next investigate the symmetry index of a split graph which is the disjoint union of a complete graph and a set of isolated vertices.

**Theorem 3.2.** For \(s \geq 8\) and \(t \geq 1\), we have \(ai(K_s + tK_1) \leq s - 2 + t - 1\).

**Proof.** We can remove \(s - 2\) edges to create an asymmetric graph \(H\) with two components: a tree on \(s - 1\) vertices and an isolated vertex. Then we can add \(t - 2\) edges to the set of \(t\) isolated vertices to create a path \(P_{t-1}\). Then by the extension lemma, we can add an edge joining a vertex of degree one in the path \(P_{t-1}\) to a vertex in \(H\). Hence, \(ai(K_s + tK_1) \leq s - 2 + t - 2 + 1 = s - 2 + t - 1\).

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