JEFFREY-KIRWAN-WITTEN LOCALIZATION FORMULA
FOR REDUCTIONS AT REGULAR CO-ADJOINT ORBITS

DO NGOC DIEP

Abstract. For Marsden-Weinstein reduction at the point 0 in $\mathfrak{g}^*$, the well-known Jeffrey-Kirwan-Witten localization formula was proven and then by M. Vergne modified. We prove in this paper the same kind formula for the reduction at regular co-adjoint orbits by using the universal orbital formula of characters.

1. Introduction and Statement of Results

Let $(M, \sigma)$ be a symplectic manifold, $G$ a compact Lie group acting on $M$ by an Hamiltonian action, $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra and $\mathfrak{g}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$. It is well-known the co-adjoint action of $G$ on $\mathfrak{g}^*$. Let us consider the moment map $\mu : M \to \mathfrak{g}^*$ defined by the following formula

$$\langle \mu(m), X \rangle = f_X(m),$$

where by definition $f_X$ is the function such that $\iota(X_M)\sigma = df_X$ and $X_M$ is the Hamiltonian field defined by the action of Lie group $G$ on $M$

$$X_M(m) := \frac{d}{dt}|_{t=0} \exp(-tX)m.$$

Let us consider a co-adjoint orbit $\mathcal{O} \in \mathfrak{g}^*/G$. Recall that $\mu^{-1}(\mathcal{O}) \to \mathcal{O} \subset \mathfrak{g}^*$ is a principal bundle with the structural group $G$. The quotient $M_{red}^{\mathcal{O}} := G \setminus \mu^{-1}(\mathcal{O})$ is known as the Marsden-Weinstein reduction of the symplectic manifold $(M, \sigma)$ at the orbit $\mathcal{O}$ with respect to the moment map $\mu$. Let us consider some 1-form $\mu(X)$ on $M$, defined by

$$\mu(X)(m) := \langle \mu(m), X \rangle.$$

One defines also an extended differential form $\sigma_{\mathcal{O}} := \mu(X) + \sigma$ and denotes the horizontal component of this form on $M_{red}^{\mathcal{O}}$ by $\sigma_{red}^{\mathcal{O}}$.

Assume that some open tubular neighborhood $M_0^{\mathcal{O}}$ of the orbit $\mathcal{O}$ is contained in the set of regular values of the moment maps. This is equivalent

1991 Mathematics Subject Classification. Primary 22E41, 19E20; Secondary 57T10.
Key words and phrases. de Rham cohomology, Chern-Weil homomorphism.

The author was supported in part by Vietnam National Research Programme in Fundamental Sciences and in part by the International Centre for Theoretical Physics,
also to the assumption that the action of $G$ on $M^0_0$ is free. Let us consider the function
\[
\mu_\mathcal{O} := \min_{\lambda \in \mathcal{O}} \|\mu(X) - \lambda\|_{\mathfrak{g}^*} = \text{dist}_{\mathfrak{g}^*}(\mu(X), \mathcal{O}).
\]
Following Witten, we consider also the form $\frac{1}{2}\|\mu_\mathcal{O}\|^2$, which is a $G$-invariant function. Choose on $M$ an $G$-invariant metrics $(\cdot, \cdot)$. Denote by $H_\mathcal{O}$ the corresponding Hamiltonian field, i.e. the symplectic gradient $\text{grad}(\frac{1}{2}\|\mu_\mathcal{O}\|^2)$ and by $\lambda^{M_0}_\mathcal{O}(\cdot) := (H_\mathcal{O}, \cdot)$ the corresponding $G$-equivariant differential 1-form. If $\alpha$ is some $G$-equivariant differential form, let us denote its horizontal component by $\alpha_{\text{red}}^\mathcal{O}$ and refer to it as its reduction on $M^0_0$. Consider $G$-equivariant differential form of type $e^{i\sigma(X)}\beta(X)$, where $\beta(X)$ is closed $G$-invariant differential form depending polynomially on $X \in \mathfrak{g}$, $\alpha_{\text{red}}^\mathcal{O} = e^{i\sigma_{\text{red}}^\mathcal{O}}\beta_{\text{red}}^\mathcal{O}$. Denote $d_X = d - \iota(X_M)$ the differential on $G$-equivariant differential forms. We decompose the manifold $M$ into a union $M = M^0_0 \cup (M \setminus M^0_0)$.

For each $t \in \mathbb{R}$, $X \in \mathfrak{g}$, consider the generalized function $\Theta(M, t)$ given by the integral
\[
\Theta(M, t)(X) := \int_M e^{-itd_X\lambda^{M_0}_\mathcal{O}} \alpha(X).
\]

**Theorem 1.1 (Main Result).** For every closed $G$-equivariant differential form $\alpha$, there exist limits in sense of the theory of generalized functions
\[
\Theta^\mathcal{O}_0 := \lim_{t \to \infty} \Theta(M^0_0, t),
\]
\[
\Theta^\mathcal{O}_{\text{out}} := \lim_{t \to \infty} \Theta(M \setminus M^0_0, t)
\]
and one has a decomposition of the integral $\int_M \alpha$ into their sum
\[
\int_M \alpha = \Theta^\mathcal{O}_0 + \Theta^\mathcal{O}_{\text{out}}
\]
also in the sense of generalized functions. For each test function $\Phi$, the first summand can be written as
\[
\int_\mathfrak{g} \Theta^\mathcal{O}_0(X)\Phi(X)dX = (2\pi i)^{\dim G} \int_{P_0^\mathcal{O}} \alpha_{\text{red}}^\mathcal{O}(\Omega) \wedge \text{vol}_\omega.
\]
This term can be also expressed by the Kirillov orbital formula for characters
\[
\int_\mathfrak{g} \Theta^\mathcal{O}_0(X)\Phi(X)dX =
\]
\[
(2\pi i)^n \int_{M^0_0_{\text{red}}} \alpha_{\text{red}}^\mathcal{O} dP(\mathcal{O}) \int_\mathfrak{g} e^{-i(\Omega, \xi)}(\int_\mathfrak{g} e^{i(\xi, X)} J_{\mathfrak{g}}^{-1/2}(X)\Phi(X)dX)d\beta(\xi) \text{vol}_\omega,
\]
where $\beta(\xi)$ is the Liouville measure on the co-adjoint orbit, $J_{\mathfrak{g}}^{-1/2}(X) := \left(\det \left(\frac{\text{sinh(ad } X/2)}{\text{ad } X/2}\right)\right)$ is the Kirillov factor for its universal character formula and $dP(\mathcal{O})$ is the Plancherèl measure.
Remark 1.2. The expression of the double integral inside the last formula is just the Kirillov universal formula for characters of unitary representations corresponding to the co-adjoint orbit $\mathcal{O}$.

Remark 1.3. The outer term is some integral over $M_{\text{out}}^\mathcal{O}$ and is given by integral also

\[
\Theta_{\text{out}}^\mathcal{O}(X) = \int_{M_{\mathcal{O},\text{out}}} e^{-id_X \lambda(x)} \alpha(X).
\]

These results play some important role in the Witten intersection theory of cohomology of $G$-equivariant differential forms.

Witten conjectured the following formula

Conjecture 1.4.

\[
\int_{\mathcal{O}} \Theta_0^\mathcal{O}(X) \Phi(X) dX = (2\pi i)^{\dim G} \text{vol}(G) \int_{M_{\text{red}}^\mathcal{O}} \alpha_{\text{red}}^\mathcal{O} W(\Phi),
\]

where $W : C^\infty(\mathfrak{g})^G \to H^*(M_{\text{red}}^\mathcal{O})$ is the Chern-Weil homomorphism, associated to the principal fibration $\mu^{-1}(\mathcal{O}) \to M_{\text{red}}^\mathcal{O}$.

Remark 1.5. The localization formula and, in particular, this conjecture were proved for $G = S^1$ by Kalkman \[4\] and Wu \[7\], for general $G$ and $\mathcal{O} = \{0\}$ by Jeffrey, Kirwan \[3\] and then modified by M. Vergne \[6\]. We prove this formula by using the universal orbital formula for characters by Kirillov \[5\].

2. Proof of Theorem 1.1

2.1. Local Fourier Transform. Let us recall the main moments in Vergne’s modification for Jeffrey-Kirwan-Witten localization theorem. Recall that the group $G$ acts on $M$ by a Hamiltonian action. For $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$ and $\Phi \in C^\infty(\mathfrak{g})$, one has the Fourier transform

\[
\mathcal{F}(\Phi)(\xi) := (2\pi)^{\dim G} \int_{\mathfrak{g}} \Phi(X) e^{-i\langle \xi, X \rangle} dX
\]

and by the well-known inverse Fourier transform, we have

\[
\Phi(X) = \int_{\mathfrak{g}^*} e^{i\langle \xi, X \rangle} \mathcal{F}(\Phi)(\xi) d\xi.
\]

We can identify each element $P$ from the symmetric algebra $S(\mathfrak{g}^*)$ with a polynomial function on $\mathfrak{g}$, $X \to P(X)$ and also with a differential operator $P(\partial_\xi)$ on $\mathfrak{g}^*$, defined by the property

\[
P(\partial_\xi)(e^{i\langle \xi, X \rangle}) = P(X)e^{i\langle \xi, X \rangle}.
\]

Similarly, we can identify the symmetric algebra $S(\mathfrak{g})$, either with the algebra of polynomial functions on $\mathfrak{g}^*$ or with the algebra of differential operators
on \( \mathfrak{g} \). In particular, to \( X \in \mathfrak{g} \) corresponds the Hamiltonian vector field \( X_M \) on \( M \).

Consider the algebra \( \mathcal{A}_G^\infty(\mathfrak{g}, M) := C^\infty(\mathfrak{g}, \mathcal{A}(M))^G \) of smooth \( G \)-equivariant functions \( \alpha, X \in \mathfrak{g} \mapsto \alpha(X) \). We refer to \( \mathcal{A}_G^\infty(M) \) as the space of smooth \( G \)-equivariant differential forms on \( M \), i.e.

\[
\alpha(g.X)(g.m) \equiv \alpha(X)(m), \forall X \in \mathfrak{g}.
\]

In particular, our moment maps \( \mu : M \to \mathfrak{g}^* \) defines the function \( \mu(X), \mu(X)(m) := f_X(m), X \in \mathfrak{g} \) as an element of \( \mathcal{A}(\mathfrak{g}, C^\infty(M)), C^\infty(M) = \mathcal{A}^0(M) \).

One defines \( G \)-equivariant co-boundary operator

\[
d_{\mathfrak{g}} : \mathcal{A}_G^\infty(\mathfrak{g}, M) \to \mathcal{A}_G^{\infty+1}(\mathfrak{g}, M)
\]

by the formula

\[
d_{\mathfrak{g}} \alpha(X) = d(\alpha(X)) - \iota(X_M)\alpha(X),
\]

where the second term is the contraction of Hamiltonian vector field \( X_M \) with the differential form \( \alpha(X) \). It is easy to check that

\[
d_{\mathfrak{g}} \alpha(X) = d^2_{\mathfrak{g}} \alpha(X) - \iota(X_M) d\alpha(X) - d(\iota(X_M) \alpha(X) + \iota(X_M) \iota(X_M) \alpha(X)) = 0,
\]

where the first and the last terms are 0 in virtue of properties of differential forms \( \alpha(X) \). The sum of two remain terms gives us the Lie covariant derivative \( \text{Lie}_{X_M} \alpha(X) \) of \( \alpha(X) \) along the vector field \( X_M \), which vanishes because \( \alpha \) is \( G \)-equivariant.

One denotes also

\[
d_X := d - \iota(X_M).
\]

One has therefore a complex \( G \)-equivariant differential forms

\[
0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}_G^0(M) \xrightarrow{d_{\mathfrak{g}}} \mathcal{A}_G^1(M) \xrightarrow{d_{\mathfrak{g}}} \mathcal{A}_G^2(M) \xrightarrow{d_{\mathfrak{g}}} \ldots
\]

It is easy to see that \( d_{\mathfrak{g}}^2 = 0 \) and one defines the \( G \)-equivariant cohomology as the cohomology of this complex

\[
\mathcal{H}_G^\infty(\mathfrak{g}, M) = \text{Ker} d_{\mathfrak{g}} / \text{Im} d_{\mathfrak{g}}.
\]

Suppose that \( M \) is oriented and \( \alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M) \) such that \( \forall X \in \mathfrak{g}, \text{supp} \alpha(X) \) is contained in a compact. One defines then

\[
\int_M \alpha(X) = \int_M \alpha(X)_{[n]},
\]

if \( n = \dim M \) and \( \alpha(X) \) is decomposed into a sum

\[
\alpha(X) = \alpha(X)_{[0]} + \alpha(X)_{[1]} + \cdots + \alpha(X)_{[n]}
\]

of homogeneous differential forms \( \alpha(X)_{[i]} \) of degree \( i \).
This means that one has a map
\[ \int_M : \mathcal{A}_G(\mathfrak{g}, M) \to C^\infty(\mathfrak{g})^G. \]
Suppose that our manifold \( M \) is closed, \( \partial M = \emptyset \). Then because \( \iota(X_M)\alpha(X) \) is of degree in \(-1\) lower and \( d\alpha(X) \) is of degree in \(+1\) upper than the degree of \( \alpha(X) \), we have
\[ \int_M d_\mathfrak{g}\alpha(X) = \int_M (d\alpha(X) - \iota(X_M)\alpha(X)) = \int_M d\alpha(X) = \int_{\partial M} \alpha(X) = 0. \]
Thus we obtain a well-defined map
\[ \int_M : \mathcal{H}^\infty_G(\mathfrak{g}, M) \to C^\infty(\mathfrak{g})^G. \]
Let us consider the case where \( M = \mathfrak{g}^* \). In this case we have element \( \xi \in \mathcal{A}^\infty_G(\mathfrak{g}, \mathfrak{g}^*) \), defined as \( X \mapsto (\xi, X) \) and if \( U \) is an open subset of \( \mathfrak{g}^* \) and \( \beta \in \mathcal{A}^\infty_G(\mathfrak{g}, U) \), we can consider the form \( \alpha(X) := e^{i(\xi,X)}\beta(X) \) and its differential
\[ (d_\mathfrak{g}\alpha)(X) = e^{i(\xi,X)}(i(d\xi, X) + (d_\mathfrak{g}\beta)(X))), \]
where \( d\xi = \sum d\xi^i e^i_1 \), if \( X = \sum x_i e^i \) in a basis \( E_1, \ldots, E_n \) of \( \mathfrak{g} \) and \( \xi = \sum \xi^i E^*_i \) in the corresponding dual basis \( E^*_1, \ldots, E^*_n \) of \( \mathfrak{g}^* \). If \( \beta \in \mathcal{A}^{pol}_G(\mathfrak{g}, U) \) is a polynomial function on \( X \), then \( d_\mathfrak{g}\alpha(X) = e^{i(\xi,X)}\gamma(X) \), where \( \gamma \) polynomially depend on \( X \). We can therefore consider the sub-complex
\[ \mathcal{A}^\mathcal{F}_G(\mathfrak{g}, U) := \{ \alpha(X) = e^{i(\xi,X)}\beta(X); \beta \in \mathcal{A}^{pol}_G(\mathfrak{g}, V) \} \]
and its cohomology is denoted as \( \mathcal{H}^\mathcal{F}_G(\mathfrak{g}, U) \). Choose an orientation on \( \mathfrak{g}^* \), we have
\[ \int_{\mathfrak{g}^*} \alpha(X) = \int_{\mathfrak{g}^*} \alpha(X)_{[n]}, \]
if for example, \( \alpha(X) \) is a rapidly decreasing \( C^\infty \)-functions on \( \mathfrak{g}^* \). The result is also a rapidly decreasing \( C^\infty \)-function on \( \mathfrak{g} \). We can therefore define also its Fourier transform. Suppose that
\[ \alpha(X)_{[n]} = \sum P_a(X)\alpha_a(\xi)d\xi, \]
where \( Pa \in S(\mathfrak{g}^*) \), \( \alpha_a(\xi) \in C^\infty(\mathfrak{g}^*) \). Then we have a formula
\[ \mathcal{F}(\int_{\mathfrak{g}^*} \alpha) = (\sum P_a(i\partial_\xi)\alpha_a(\xi))d\xi. \]
M. Vergne defined the local Fourier transform as the generalized function
\[ V(\alpha) = (\sum P_a(i\partial_\xi)\alpha_a(\xi))d\xi. \]
It was proven that if \( \beta \in \mathcal{A}^\mathcal{F}_G(\mathfrak{g}, V) \), then \( V(d_\mathfrak{g}\beta) = 0 \). The local Fourier transform is defined indeed on the cohomology groups
\[ V : \mathcal{H}^\mathcal{F}_G(\mathfrak{g}, U) \to \mathcal{A}^\kappa(U)^G. \]
Let us consider the moment map $\mu : M \to \mathfrak{g}^*$. For each element $m \in M$, $\mu(m)$ is a function on $\mathfrak{g}^*$, $(\mu(m), X) = f_X(m)$ and for each $X$, one has a function $(\mu(\cdot), X)$ on $M$. Thus one can define the function $X \mapsto e^{i(\mu, X)} \in \mathcal{A}^\mu_G(\mathfrak{g}, M)$ and for all $\beta \in \mathcal{A}^\text{pol}_G(\mathfrak{g}, M)$, $\alpha(X) = e^{i(\mu, X)} \beta(X) \in \mathcal{A}^\mu_G(\mathfrak{g}, M)$.

The subspaces
$$\mathcal{A}^\mu_G(\mathfrak{g}, M) := \{\alpha(X) = e^{i(\mu, X)} \beta(X); \beta \in \mathcal{A}^\text{pol}_G(\mathfrak{g}, M)\}$$
is stable under $d_{\mathfrak{g}}$. We have therefore a sub-complex $(\mathcal{A}^\mu_G(\mathfrak{g}, M), d_{\mathfrak{g}})$ and the corresponding cohomology is denoted by $\mathcal{H}^\mu_G(\mathfrak{g}, M)$.

**Theorem 2.1** ([6]). Assume that the manifold $M$ is oriented and $U$ is a $G$-invariant open set containing in the set of regular values of the moment map $\mu$. Let $\alpha \in \mathcal{A}^\mu_G(\mathfrak{g}, M)$. Then over $U$ one has
$$\mathcal{F}(\int_M^\mu \alpha) = V(\mu_\ast \alpha),$$
where $\mu_\ast \alpha$ is the push-forward of $\alpha$. If $\alpha$ is closed, $\mathcal{F}(\int_M^\mu \alpha)$ depend only on the cohomology class of $\alpha$ in $\mathcal{H}^\mu_G(\mathfrak{g}, \mu^{-1}(U))$. Thus for $\alpha \in \mathcal{H}^\mu_G(\mathfrak{g}, \mu^{-1}(U))$, in order to determine $\mathcal{F}(\int_M^\mu \alpha)$ near a regular value $F$ of $\mu$, we need only to determine the class of $\alpha$ in $\mathcal{H}^\mu_G(\mathfrak{g}, \mu^{-1}(U))$, where $U$ is a $G$-invariant tubular open neighborhood of the orbit $O$ of $F$.

Let us consider also push-forward $\mu_\ast((\omega_a)_{[\text{dim } M]})$, which is a Radon measure. We have by definition,
$$\int_M^\mu \omega = \int_{\mathfrak{g}^*}^\mu \mu_\ast \omega.$$
One has also
$$(\int_M^\mu \alpha)(X) = \sum_a P_a(X) \int_{\mathfrak{g}^*} e^{i(\mu, X)} \mu_\ast((\omega_a)_{[\text{dim } M]})$$
and
$$\mathcal{F}(\int_M^\mu \alpha) = \sum_a P_a(i \partial_\xi)(\mu_\ast(\omega_a)_{[\text{dim } M]}) = V(\mu_\ast \alpha).$$
From these, one has a result

**Theorem 2.2** (Berline-M. Vergne [3], 1982). Let $T \subset G$ be a maximal torus of $G$ and $M^T$ is the sub-manifold of fixed point under the action of the torus $T$, then $\int_M^\mu \alpha$ depend only on the restriction $\alpha|_{MT}$.

Let us now deduce an explicit formula for $V(\mu_\ast)$ near the point $F \in O \subset \mathfrak{g}^*$. Suppose that the action of $G$ on $\mu^{-1}(O)$ is locally free. Choose a $G$-invariant Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$ and $U$ a $G$-invariant open ball centered at $F \in \mathfrak{g}^*$, $N^O := P^O \times_G U$ is a fibration over $N^O_{\text{red}} := G \setminus P^O$. The map $\mu : N^O \to U$ is the projection on the second component. Let us consider a form $\alpha \in \mathcal{A}^\mu_G(\mathfrak{g}, N^O)$, the restriction $\alpha|_{P^O}$ is $G$-equivariant. Choose an orientation in
\[ P^O = \mu^{-1}(O) \] and a basis \( E_1, E_2, \ldots, E_n \) of \( g \) and the corresponding dual basis \( E^*_1, E^*_2, \ldots, E^*_n \) of \( g^* \). Choose a connection, i.e. a trivialization by open covering of the base \( N^O_G \). Let us write \( \omega \) a connection on the fibration \( P^O \to G \setminus P^O \),

\[ \omega = \sum \omega_k E_k, \]

and

\[ \Omega = \text{curv}(\omega) = \sum \Omega_k E_k \]

the curvature of \( \omega \) with values in \( g \). It is not hard to see that \( q^* : H^\infty_G(g, P^O) \cong H^*(G \setminus P^O) \). If \( \Phi \) is a polynomial function on \( g \) then \( \Phi(\Omega) \) is a form on \( P^O \) and if \( \alpha \) is closed \( G \)-equivariant, the horizontal component \( \alpha_{\text{red}} := h(\alpha(\Omega)) \) is a closed differential form on \( N^O_{\text{red}} \). The map \( \alpha \mapsto \alpha_{\text{red}} \) is just the inverse map

\[ (q^*)^{-1} : H^\infty_G(g, P^O) \to H^*_{\text{DR}}(G \setminus P^O). \]

one denote also \( (\Omega, \xi) = \sum \Omega_k \xi_k \) on \( P^O \) and \( \text{vol}^O = \omega_1 \wedge \cdots \wedge \omega_n \) be the vertical form on \( P^O \) of degree \( n = \dim G \). Let us denote also by coordinates

\[ dX = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \]

the volume form then \( E^*_1 \wedge E_1 \wedge \cdots \wedge E^*_n \wedge E_n = dX \wedge d\xi \), where \( d\xi = (-1)^{n(n+1)/2} d\xi_1 \wedge d\xi_2 \cdots \wedge d\xi_n \).

**Theorem 2.3** ([6]). If \( P^O \) is oriented with free \( G \)-action then for all \( \Phi \in S(g^*)^G \), one has

\[ ((V_\mu, \alpha) / d\xi, \Phi) = i^n \int_{P^O} \alpha_{\text{red}}^O \Phi(-i\Omega) \text{vol}_\omega = i^n \text{vol}(G) \int_{N^O_{\text{red}}} \alpha_{\text{red}}^O \Phi(-i\Omega). \]

**Theorem 2.4** ([Jeffrey-Kirwan][3]). For \( \alpha \in A^\mu_G(g, M) \),

\[ \mathcal{F}(\int_M \alpha) = i^n \int_{P^O} \alpha_{\text{red}} e^{-i(\xi, \Omega)} \text{vol}_\omega. \]

Witten defined the integral

\[ Z(\varepsilon) = \int_M \int_g e^{i\sigma_g(X)} \beta(X) e^{-\frac{\varepsilon \|X\|^2}{2}} dX. \]

**Theorem 2.5** ([Witten][6],[3]). Let \( (M, \sigma, \mu) \) be a symplectic manifold with a Hamiltonian action of \( G \), \( O \) a co-adjoint orbit of \( G \) in \( g^* \), \( \mu_O : \mu^{-1}(O) \to M^O_{\text{red}} = G \setminus \mu^{-1}(O) \), \( R \) the smallest critical value of \( \|\mu_O\|^2 \), \( r < R \). Then for each closed \( G \)-equivariant polynomially depend on \( X \), there exists a constant \( C \) such that

\[ Z(\varepsilon) = (2\pi i)^{\dim G} \int_{M^O_{\text{red}}} e^{i\sigma_{\text{red}} \beta_{\text{red}} \varepsilon \frac{\|\Omega\|^2}{2}} + N(\varepsilon), \]
where
\[ |N(\varepsilon)| \leq Ce^{-r/2\varepsilon}, \forall \varepsilon > 0. \]

2.2. Kirillov Universal Character Formula. Let us describe in this section the well-known Kirillov orbital formula for characters of representations.

Theorem 2.6 (Liouville Measure).
\[
\int_{M_F} \Phi d\beta_F = \prod_{\alpha \in P_F} \frac{\langle F, iH_\alpha \rangle}{2\pi} \int_{G/G_F} \Phi(gF) d\bar{g},
\]
where \( d\bar{g} \) is the quasi-invariant measure on the co-adjoint orbit \( O = G_F \setminus G \).

The theorem what follows describes the image of the Fourier transform of Liouville measure.

Theorem 2.7 (Harish-Chandra, see [1], Theorem 7.24 and Corollary 7.25).
Given \( \lambda \in \mathfrak{t}^* \), let \( W_\lambda = \{ w \in W | w\lambda = \lambda \} \) be the stabilizer of \( \lambda \) in the Weyl group \( W \). For \( X \in \mathfrak{t} \), \( X \) regular, the Fourier transform
\[
F_{M_\lambda}(X) = \int_{M_\lambda} e^{i\langle X \rangle} d\beta_\lambda(f)
\]
is given by the formula
\[
F_{M_\lambda}(X) = \sum_{w \in W/W_\lambda} \frac{e^{i\langle w\lambda, X \rangle}}{\prod_{\alpha \in P_\lambda} \langle w\alpha, X \rangle}. 
\]

If \( M_\lambda \) is a regular orbit of the co-adjoint representation, then for \( X \in \mathfrak{t} \), \( X \) regular,
\[
F_{M_\lambda}(X) = \prod_{\alpha \in P_\lambda} \langle \alpha, X \rangle^{-1} \sum_{w \in W} \varepsilon(w) e^{i\langle w\lambda, X \rangle}.
\]

Let us recall that if \( G \) is a connected compact Lie group, we denote by \( T \subseteq G \) a fixed maximal torus in \( G \),
\[
L_T := \{ X \in \mathfrak{t} | e^X = 1 \}
\]
the lattice of co-roots,
\[
L_T^* = \{ \lambda \in \mathfrak{t}^* | \lambda(X) \in 2\pi\mathbb{Z}, \forall X \in L_T \} \subseteq \mathfrak{t}^*
\]
the root lattice. One fixes some basis in this lattice and every element from lattice can be expressed as linear combination with integral coefficients. One fixes a system \( P \) of positive roots in \( \Delta \). Let us denote by \( \rho \) the half-sum of positive roots and
\[
X_G := i\rho + L_T^*.
\]
E. Cartan described the irreducible finite dimensional representations of \( G \) and H. Weyl described the characters of irreducible representations: There
is a bijective correspondence between regular elements $\lambda \in X_G$ and the irreducible representations, denoted by $T_\lambda$. Weyl character formula is
\[
\text{Tr}(T_\lambda(e^X)) = \frac{\sum_{w \in W} \varepsilon(w) e^{i(w,\lambda)X}}{\prod_{\alpha \in P_\lambda} (e^{\langle \alpha, X \rangle/2} - e^{-\langle \alpha, X \rangle/2})};
\]

Let us consider the function
\[
J_g(X) := \det \left( \frac{\sinh(\text{ad} X/2)}{\text{ad} X/2} \right).
\]
Then for all $X \in \mathfrak{t}$,
\[
J_g(X) = \prod_{\alpha \in \Delta} \frac{e^{\langle \alpha, X \rangle/2} - e^{-\langle \alpha, X \rangle/2}}{\langle \alpha, X \rangle} = \left( \prod_{\alpha \in \Delta} \frac{e^{\langle \alpha, X \rangle/2} - e^{-\langle \alpha, X \rangle/2}}{\langle \alpha, X \rangle} \right)^2.
\]

**Theorem 2.8** (Chevalley’s Theorem, see [1], Thm. 7.28). There are isomorphisms between $G$-invariants and $W$-invariants:
\[
C^\infty(\mathfrak{t})^W \cong C^\infty(\mathfrak{g})^G,
\]
\[
C^\omega(\mathfrak{t})^W \cong C^\omega(\mathfrak{g})^G,
\]
\[
C[[\mathfrak{t}]]^W \cong C[[\mathfrak{g}]]^G.
\]

The main reason for these isomorphisms is the correspondence $\Phi \mapsto c(\Phi)$, where
\[
c(\Phi) = \frac{1}{|W|} \int_G (\partial_w(\pi_{\mathfrak{g}/\mathfrak{t}}\Phi)(\text{pr}(g.X))dg,
\]
where $\partial_w := \prod_{\alpha \in P} \langle \rho, \alpha \rangle^{-1} \prod_{\alpha \in P} \partial_{H_{\alpha,}}$, $\pi_{\mathfrak{g}/\mathfrak{t}}(X) := \prod_{\alpha \in P} i\alpha(X) \in \mathbb{C}[t]$.

From this theorem, it is easy to see that there is a unique function
\[
J^{1/2}_g(X) := \prod_{\alpha \in P} \frac{\sinh(\langle \alpha, X \rangle/2)}{\langle \alpha, X \rangle/2}
\]
such that
\[
J^{1/2}_g(X) \text{ Tr}(T_\lambda(e^X)) = \sum_{\alpha \in \Delta} \varepsilon(\alpha) X^{-1} \sum_{w \in W} \varepsilon(w) e^{i(w,\lambda)X}
\]
\[
= F_{M_\lambda}(X) = \int_{M_\lambda} e^{i\beta(X)}d\beta(\lambda).
\]

The same formula for semi-simple Lie groups was obtained by Rossman:

**Theorem 2.9** (see [1], Theorem 7.29). Let $M$ be a closed co-adjoint orbit of a real semi-simple Lie group $G$ with non-empty discrete series, i.e. rank($G$) = rank($K$) and let $W = W(\mathfrak{k}_G, \mathfrak{t}_G)$ be the compact Weyl group. Then for regular element $X \in \mathfrak{t}$, we have the following results:
1. If \( M \cap \mathfrak{t}^* = \emptyset \), then \( F_M(X) = 0 \).
2. If \( M = G.\lambda \) with \( \lambda \in \mathfrak{t}^* \), and \( W_\lambda \) is the subgroup of \( W \) stabilizing \( \lambda \), then
\[
F_{M_\lambda}(X) = (-1)^{n(\lambda)} \sum_{W/W_\lambda} \frac{e^{i(w_\lambda, X)}}{\prod_{\alpha \in P_\lambda} \langle w_\alpha, X \rangle}.
\]

2.3. Witten Type Localization Theorem. Let \((M, \sigma, \mu)\) be a compact symplectic manifold with a Hamiltonian action of compact Lie group \( G \). Let us assume that the co-adjoint orbit \( \mathcal{O} \) is contained in the set of regular values of \( \mu \). Assume that the action of \( G \) is free in \( P^\mathcal{O} = \mu^*(\mathcal{O}) \). Let us denote \( \omega \) a connection form on \( P^\mathcal{O} \) with curvature \( \Omega \). We consider the so called Marsden-Weinstein reduction \( M^\mathcal{O}_{red} := G \setminus P^\mathcal{O} \) of \( M \). We denote \( \sigma^\mathcal{O}_{red} \) the de Rham cohomology class \((\alpha|_{P^\mathcal{O}})_{red}\) on \( M^\mathcal{O}_{red} \) determined by \( \alpha|_{P^\mathcal{O}} \). In particular \( (\sigma^\mathcal{O}_\mathfrak{g})_{red} \) is the symplectic form \( \sigma^\mathcal{O}_\mathfrak{g} \) on \( M^\mathcal{O}_{red} \).

Let us consider the function \( \mu_\mathcal{O} := \min_{x \in \mathcal{O}} \|\mu(x) - \lambda\| \), the distance from \( \mu(X) \) to the co-adjoint orbit \( \mathcal{O} \). Following Witten, we introduce the function \( \frac{1}{2}\|\mu_\mathcal{O}\|^2 \) and its Hamiltonian field \( H_\mathcal{O} \). This is an invariant vector field on \( M_\mathcal{O} \). Choose a \( G \)-invariant metric \( (.,.) \) on \( M_\mathcal{O} \) and put \( \lambda^\mathcal{O} := (H_\mathcal{O},.) \). Then \( \lambda^\mathcal{O}_M \) is a \( G \)-invariant 1-form on \( M_\mathcal{O} \). Let \( R \) be the smallest critical value of the function \( \|\mu_\mathcal{O}\|^2 \). Let \( r < R \) and let
\[
M^\mathcal{O}_0 = \{x \in M; \|\mu_\mathcal{O}(x)\|^2 < r\}, \quad M^\mathcal{O}_{out} = \{x \in M; \|\mu_\mathcal{O}(x)\|^2 > r\}.
\]

Let \( \alpha(X) \) be a closed \( G \)-invariant differential form on \( M \). Let us consider
\[
\Theta(M,t) = \int_M e^{itd_\mathcal{O}\lambda^M} \alpha(X),
\]
which is independent of \( t \) in virtue that \( \alpha \) is a closed form and \( e^{itd_\mathcal{O}\lambda^M} \) congruent to 1 in cohomology.

Let us consider the integrals
\[
\Theta(M^\mathcal{O}_0, t) = \int_{M^\mathcal{O}_0} e^{itd_\mathcal{O}\lambda^M} \alpha(X),
\]
\[
\Theta(M^\mathcal{O}_{out}, t) = \int_{M^\mathcal{O}_{out}} e^{itd_\mathcal{O}\lambda^M} \alpha(X).
\]

Then the values \( \Theta(M^\mathcal{O}_0, t)(X) \) and \( \Theta(M^\mathcal{O}_{out}, t)(X) \) \( C^\infty \)-smoothly depend on \( X \in \mathfrak{g} \).

**Theorem 2.10.** For every \( t \in \mathbb{R} \) and \( X \in \mathfrak{g} \), there is a decomposition
\[
\left( \int_M e^{itd_\mathcal{O}\lambda^M} \alpha \right)(X) = \Theta(M^\mathcal{O}_0, t)(X) + \Theta(M^\mathcal{O}_{out}, t)(X).
\]

There exist the limits \( \Theta^\mathcal{O}_0 = \lim_{t \to \infty} \Theta(M^\mathcal{O}_0, t) \) and \( \Theta^\mathcal{O}_{out} = \lim_{t \to \infty} \Theta(M^\mathcal{O}_{out}, t) \) in the sense of distributions such that
\[
\int_{\mathfrak{g}} \left( \int_M \alpha \right)(X)\Phi(X)dX = \int_{\mathfrak{g}} \Theta^\mathcal{O}_0(X)\Phi(X)dX + \int_{\mathfrak{g}} \Theta^\mathcal{O}_{out}(X)\Phi(X)dX,
\]
for every test function $\Phi$, where

$$
\int_\mathfrak{g} \Theta_0^\mathcal{O}(X)\Phi(X)dX = (2\pi i)^{\dim G} \int_{\mathcal{O}_\text{red}} \alpha^\mathcal{O}_\text{red} \Phi(\Omega) \wedge \text{vol}_\omega.
$$

**Proof.** The proof is the same as in the note of M. Vergne [6] with change everywhere $M_0, M_\text{out}, \ldots, \mu, \ldots$ by $M_0^\mathcal{O}, M_\text{out}^\mathcal{O}, \ldots, \mu^\mathcal{O}, \ldots$. We need:

- to prove the existence of limit $\Theta_0 = \lim_{t \to 0} \Theta(M_0, t)$ in the sense of generalized functions and
- to compute it.

Let us start now to do this. Let $E_1, \ldots, E_n$ be a basis of $\mathfrak{g}$, such that $E_1, \ldots, E_k$ is a basis of a stabilizer $\mathfrak{g}_F = \mathfrak{g}^\perp$ in $\mathfrak{g}$. This means that

$$
\mu^\mathcal{O} = \sum_{i=k}^n \mu(E_i)E_i
$$

and

$$
\frac{1}{2} \|\mu^\mathcal{O}\|^2 = \sum_{i=k}^n \mu(E_i)\mu(E_i) = \sum_{i=k}^n \mu(E_i)\langle (E_i)_M, \sigma\rangle.
$$

One deduces easily that

$$
\lambda^\mathcal{O} = (H^\mathcal{O}, \mu^\mathcal{O}) = \sum_{i=1}^m \mu(E_i)\omega^i_M,
$$

where $\omega^i_M = ((E_i)_M, \cdot)$ and $\omega_M = \sum_{i=1}^m \omega^i_ME_i$. We have therefore

$$
\lambda^\mathcal{O} = (\omega^\mathcal{O}_M, \mu^\mathcal{O}).
$$

On $M_0^\mathcal{O}$ the action of $G$ is locally free. Thus we an choose on $M_0^\mathcal{O}$ a metric $(\cdot, \cdot)$ such that $((E_i)_M, (E_j)_M) = \delta_{ij}$ on $M_0^\mathcal{O}$. One has hence

$$
\omega_M(X_M) = X, \forall X \in \mathfrak{g}.
$$

This means that $\omega_M$ is a connection form on $M_0^\mathcal{O}$ and

$$
\lambda^\mathcal{O}(X_M) = (H^\mathcal{O}, X_M) = \mu^\mathcal{O}(X).
$$

Let us consider the function $f_{\lambda^\mathcal{O}} : M \to \mathfrak{g}^*$, defined by the condition that

$$
f_{\lambda^\mathcal{O}}(X) = \lambda^\mathcal{O}(X_M).
$$

Then it is easy to see that

$$
\langle f_{\lambda^\mathcal{O}}, \mu^\mathcal{O} \rangle = \sum_{i=1}^m \mu^\mathcal{O}(E_i)\langle (E_i)_M, H^\mathcal{O}\rangle = (H^\mathcal{O}, H^\mathcal{O}) \geq 0.
$$

On $M_0^\mathcal{O}$ we have

$$
d_X \lambda^\mathcal{O} = -i(\mu^\mathcal{O}, X) + d\lambda^\mathcal{O} = -i(\mu^\mathcal{O}, X) + (\mu, d\omega_M) - (\omega_M, d\mu).
Thus we have
\[
\int g \Theta_0^O(X) \Phi(X) dX = \int_{M^0_\varepsilon} \int g \left( e^{it(\mu, X) - it(\mu, d\omega_M) + it(\omega_M, d\mu)} \alpha(X) \Phi(X) dX \right).
\]
Let \( \varepsilon > 0 \) be a small number. Let us consider a small tubular neighborhood \( M^0_{\varepsilon} \) of the orbit \( O \),
\[
M^0_{\varepsilon} = \{ x \in M^O; \| \mu_O(x) \| < \varepsilon \}
\]
and let
\[
\chi_O(m) := \begin{cases} 
1 & \forall m \in M^0_{\varepsilon/2} \\
0 & \forall m \notin M^0_{\varepsilon/2}
\end{cases}
\]
to be the cut-off function. Then
\[
\lim_{\varepsilon \to 0} \int_{M^0_{\varepsilon}} (1 - \chi_O(m))(\int g e^{itdX} \lambda^M \alpha(X) \Phi(X)) = 0.
\]
Let us denote \( \beta(X) := \alpha(X) \Phi(X) \in C^\infty_c(g, A(M^O)) \). Then we have
\[
\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) = \int g e^{it(\mu, X)} e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX = e^{-itdX} \hat{\beta}(t\mu_O),
\]
where \( \hat{\beta} \) is, by definition, the Fourier transform of \( \beta = \alpha \Phi \). Because \( \beta \) is of compact support, its Fourier transform \( \hat{\beta} \) is of Schwartz class and because \( e^{-itdX} \lambda^M \) is polynomial on \( t \), we have
\[
\lim_{t \to \infty} \int_{M^0_{\varepsilon}} (1 - \chi_O(m))(\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX) = 0.
\]
Because on the support of the function \( 1 - \chi_O \), one has the estimate \( \| \mu_O(m) \| \geq \frac{1}{2}\varepsilon \), then
\[
\lim_{t \to \infty} \int_{M^0_{\varepsilon}} ((\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX) =
\]
\[
= \lim_{t \to \infty} \int_{M^0_{\varepsilon}} \chi_O(m)(\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX).
\]
Choose \( N^O = P^O \times g^* \) and let \( \omega_O = \omega^O_M|_{P^O} \), then \( \omega_O \) is a connection form on \( P^O \). Choose \( \varepsilon \) sufficiently small, then \( M^0_{\varepsilon} \cong \) open set of \( N^O = P^O \times g^* \). Thus the map \( \mu_O : N^O \to g^* \) becomes the projection \( (x, \xi) \mapsto \xi \). This isomorphism is identity on \( P^O \). Because \( \chi_O \) has compact support contained in \( M^0_{\varepsilon} \), we can consider the integral
\[
\int_{M^0_{\varepsilon}} \chi_O(m)(\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX)
\]
as an integral over \( N^O \) and still denote \( \omega^M_O \) for the 1-form corresponding to \( \omega^M_O \) and \( \omega^M_O|_{P^O} = \omega_O \). We have thus
\[
\lim_{t \to \infty} \int_{M^0_{\varepsilon}} \chi_O(m)(\int g e^{-itdX} \lambda^M \alpha(X) \Phi(X) dX) =
\]
We denote \( \nu \) then for all \( \mu_k \) we have to study the integrals of type \( \int \alpha(X) \Phi(X) dX \).

By the same reason as above, there exist polynomials \( P_k(t\xi, t d\xi) \) in \( \xi^1, \ldots, \xi^m, \partial \xi^1, \ldots, \partial \xi^m \) such that

\[
e^{it(\omega^j_M, d\xi)-it(\xi, d\omega^j_M)} = \sum_{k=1}^n P_k(t\xi, t d\xi) \mu_k,
\]

where \( \mu_k \) are differential forms on \( N^\Omega \), independent of \( t \). Denote

\[
\nu_k := \chi_\Omega \mu_k \wedge \alpha(X) \Phi(X),
\]

we have to study the integrals of type

\[
\int_{N^\Omega} (\int_\mathfrak{g} e^{it(\xi, X) P_k(t\xi, t d\xi) \nu_k(X) dX}).
\]

We denote \( \nu_0 = \nu(X)|_{P^\Omega} \), then the map \( X \mapsto \nu_0(X) \) is an element in the class \( C^\infty(\mathfrak{g}, \mathcal{A}(P^\Omega)) \). Its Fourier transform \( \tilde{\nu}_0(\xi) = \nu_0(X)(\xi) \) as differential forms on \( N^\Omega = P^\Omega \times \mathfrak{g}^* \). It was shown that if \( G(\xi, d\xi) \) is a polynomial, then for all \( \nu \in C^\infty(\mathfrak{g}, \mathcal{A}(P^\Omega)) \),

\[
\lim_{t \to \infty} \int_{N^\Omega} (\int_\mathfrak{g} e^{it(\xi, X) G(t\xi, t d\xi)} \nu(X) dX = \int_{N^\Omega} G(\xi, d\xi) \tilde{\nu}_0(\xi).
\]

Because \( \chi_\Omega|_{P^\Omega} \equiv 0 \), \( \omega^j_M|_{P^\Omega} = \omega_\Omega \), we have

\[
\lim_{t \to \infty} \int_{N^\Omega} \chi_\Omega(m) (\int_\mathfrak{g} e^{it(\xi, X) e^{it(\omega^j_M, d\xi)-it(\xi, d\omega^j_M)} \alpha(X) \Phi(X) dX) =
\]

\[
= \int_{N^\Omega} e^{i(\omega, d\xi)-i(\xi, d\omega)} \Phi(\omega_\Omega) = \int_{N^\Omega} \int_\mathfrak{g} e^{-idX \lambda^M_\Omega} \alpha_0(X) \Phi(X) dX,
\]

see (1). Lemma 21). We have seen that

\[
\int_\mathfrak{g} \Theta\nu_0(X) \Phi(X) dX = \int_{N^\Omega} \int_\mathfrak{g} e^{-itdX \lambda^M_\Omega} \alpha_0(X) \Phi(X) dX.
\]

Let us now prove that

\[
\int_{N^\Omega} \int_\mathfrak{g} e^{-itdX \lambda^M_\Omega} \alpha_0(X) \Phi(X) dX = (2\pi i)^{\dim G} \int_{P^\Omega} \alpha^\Omega_{\text{red}} \int \Phi(\Omega) \wedge \operatorname{vol}_\omega.
\]

Indeed, first we have

\[
\int_{N^\Omega} \int_\mathfrak{g} e^{-itdX \lambda^M_\Omega} \alpha_0(X) \Phi(X) dX =
\]

\[
= \int_{P^\Omega} \alpha^\Omega_{\text{red}}(\int_{N^\Omega/P^\Omega} \int_\mathfrak{g} e^{-idX \lambda^M_\Omega} \alpha_0(X) \Phi(X) dX),
\]

with

\[
e^{-idX \lambda^M_\Omega} = e^{i(\xi, X)} e^{-i(\omega, \xi) + i(\omega, d\xi)}.
\]

Remark that its term of maximal degree in \( d\xi \) is \( cd\xi^1 \ldots d\xi^m \wedge \operatorname{vol}_\omega \), where \( c = i^n \varepsilon \) and \( \varepsilon = (-1)^{n(n+1)/2} \) is a sign. Recall also that \( \Omega = d\omega + \frac{1}{2}[\omega, \omega] \) is
the curvature of the connection form \( \omega = \omega^M \). Because \( \omega_i \wedge \text{vol}_\omega = 0, \forall i \), we have

\[
e^{-i(d_\omega, \xi)} \wedge \text{vol}_\omega = e^{-i(\Omega, \xi)} \wedge \text{vol}_\omega
\]

and then

\[
\int_{N^O/P^O} \int_{g \in G} e^{-itd_X \lambda^M_0} \alpha_0(X) \Phi(X) dX =
\]

\[
= c \int_{M^O/P^O} e^{-i(d_\omega, \xi)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right) d\xi \ \text{vol}_\omega
\]

\[
= c \int_{N^O/P^O} e^{-i(\Omega, \xi)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right) d\xi \ \text{vol}_\omega.
\]

Following the Fourier inverse formula, we have

\[
\int_{N^O/P^O} e^{-i(\Omega, \xi)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right) = (2\pi)^n \Phi(\Omega).
\]

The last integral is therefore equal

\[
(2\pi)^n \Phi(\Omega) \wedge \text{vol}_\omega.
\]

The proof is therefore achieved. \( \square \)

**Remark 2.11.** By the same way as preceded in [6], we can deduce the Jeffrey-Kirwan formula 2.4 and the outer term formula.

We finish the proof of our main theorem in this section by using together the local Fourier transform and the universal orbital formula for characters by Kirillov. By the Jeffrey-Kirwan-Witten localization theorem, we have

\[
\int_{g \in G} \Theta^O_0(X) \Phi(X) dX = (2\pi i)^{\dim G} \int_{P^O} \alpha^O_{\text{red}} \Phi(\Omega) \wedge \text{vol}_\omega
\]

\[
= i^n \int_{N^O/P^O} e^{-i(\xi, X)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right) d\xi \ \text{vol}_\omega.
\]

By the Kirillov universal trace formula, we have

\[
(2\pi)^n \Phi(\Omega) = \int_{N^O/P^O} e^{-i(\Omega, \xi)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right) = \int_{g \in g^*} e^{-i(\Omega, \xi)} \left( \int_{g \in G} e^{i(\xi, X)} \Phi(X) dX \right)
\]

\[
= \int_{g^*/G} dP(\mathcal{O}) \int_{\mathcal{O}} \left( \int_{g \in G} e^{i(\xi, X)} J^{-1/2}_g(X) \Phi(X) dX \right) d\beta_{\mathcal{O}}(\xi),
\]

where \( P(\mathcal{O}) \) is the Plancherel measure on \( g^*/G \) and \( \beta_{\mathcal{O}}(\xi) \) is the Liouville measure on the orbit \( \mathcal{O} \). The proof of the main theorem is therefore achieved.

Let us now apply the construction of Chern-Weil homomorphism to our case of \( M^O_0 \). Let \( \mathcal{A}(M^O_0) \) be the algebra of differential forms over \( B = M^O_0 \). There is the natural principal bundle \( G \rightarrow P^O \rightarrow M^O_0 \), with connection 1-form \( \omega = \sigma^O_\Omega \) with curvature \( \Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \mathcal{A}^2(P^O, g)^G \). The decomposition \( TP^O = T_h P^O + T_v P^O \) of the tangent bundle \( TP^O \) into the sum
of horizontal and vertical components, $T_h P^O = \ker(\omega)$. The vertical sub-bundle $T_v P^O$ determines a projection $h$ from $\mathcal{A}(P^O)$ onto the sub-algebra of horizontal forms

$$\mathcal{A}(P^O)_{\text{hor}} = \{ \alpha \in \mathcal{A}(P^O) \mid \iota(X)\alpha = 0, \forall X \in \mathfrak{g} \}.$$ 

With respect to a basis $E_1, \ldots, E_n$ of $\mathfrak{g}$, we can write

$$\omega = \sum_{i=1}^n \omega^i E_i \quad \text{and} \quad \Omega = \sum_{i=1}^n \Omega^i E_i,$$

where $\omega^i \in \mathcal{A}^1(P^O)_{\text{hor}}$ and $\Omega^i \in \mathcal{A}^2(P^O)_{\text{hor}}$ and the projector $h$ can be written as

$$h = \prod_{i=1}^n (I - \omega^i \iota(E_i)) = \sum_{1 \leq i_1 < \cdots < i_r} \omega^{i_1} \cdots \omega^{i_r} \iota(E_{i_1}) \cdots \iota(E_{i_r}),$$

see for example ([1], Lemma 7.30). If $V = \mathcal{E}_\rho(V) = P^O \times_G V$ is a vector bundle, associated with a representation $\rho$, with the associated affine connection, or equivalently, a covariant derivation $\nabla^V$, then

$$\nabla^V \alpha = d\alpha + \sum_{i=1}^n \omega^i \rho(E_i)\alpha.$$

It is easy to see that the basic differential forms over $P^O$ are just the ordinary differential forms on $B = M^O_0 = G \setminus P^O$, $\mathcal{A}(P^O) \cong \mathcal{A}(M^O_0)$, the same for $V$-valued differential forms

$$\mathcal{A}(P^O, V) = \mathcal{A}(P^O, V)_{\text{bas}} \cong \mathcal{A}(M^O_0) \otimes \mathbb{C} V.$$

If $\alpha \in \mathcal{A}(P^O, V)$ then $h\alpha = \alpha$. This means that $h$ acts as the identity operator on $\mathcal{A}(P^O, V) = \mathcal{A}(P^O, V)_{\text{bas}}$ and $d$ conserves this subspace of forms. We have

$$\iota(X) d\alpha = (\text{Lie}(X) \otimes I)\alpha = -\rho(X)\alpha,$$

then

$$\iota(E_i)\iota(E_j) d\alpha \equiv 0, \forall E_i, E_j.$$ 

Thus,

$$h(d\alpha) = h(d\alpha) = \nabla^V \alpha.$$

It is easy to see ([1], Proposition 7.32) that the covariant derivation $\nabla^V$ on $V$ coincides with the restriction of $D \otimes I$ on $\mathcal{A}(P^O, V)$ to the subspace $\mathcal{A}(P^O, V) := \mathcal{A}(P^O, V)_{\text{bas}}$, where

$$D := hdh = h(d - \sum_{i=1}^n \Omega^i \iota(E_i))$$

on $\mathcal{A}(P^O)$. 

If $M$ is a $G$-manifold, denote $\mathcal{M} = P^G \times_G M$ the associated fibration. One has

$$\mathcal{A}(\mathcal{M}) \cong \mathcal{A}(P^G \times_G M)_{bas} = \{ \alpha \in \mathcal{A}(P^G \times M) \mid \alpha \text{ is basic w.r.t. the action of } G \},$$

$$i(X) := i(X_{P^G \times M}) = i(X_P) + i(X_M),$$

$h : \mathcal{A}(P^G \times M) \to \mathcal{A}(P^G \times M)_{bas}$ is given by

$$h = \prod_{i=1}^{m} (I - \omega^i i(X_i)), $$

$$D = h.d.h|_{\mathcal{A}(\mathcal{M})} = h.d.h|_{\mathcal{A}(P^G \times M)_{bas}} = d_{\mathcal{M}}.$$

Let us consider the complex $(\mathbb{C}[g] \otimes \mathcal{A}(M), d_g)$. We correspond to each element $\alpha = f \otimes \beta$ the element $\alpha(\Omega) := f(\Omega) \otimes \beta \in \mathcal{A}(P^G) \otimes \mathcal{A}(M)$. One defines thus the so called Chern-Weil homomorphism

$$W = \Phi_\omega : \mathbb{C}[g] \otimes \mathcal{A}(M) \to \mathcal{A}(P^G \times M)_{hor},$$

$$W = \Phi_\omega(\alpha) := h(\alpha(\Omega)).$$

It was shown ([1], Theorem 7.34) that if $G$ is a Lie group, $G \to P^G \to M_0^G$ is the principal bundle with connection $\omega = \sigma^G_{red}$ and the curvature $\Omega = d\omega + \frac{1}{2} [\omega, \omega]$ and $M$ is a smooth $G$-manifold, then the Chern-Weil homomorphism induces a homomorphism of the complexes of differential graded algebras

$$(\mathcal{A}_G(M), d_g) = (\mathbb{C}[g] \otimes \mathcal{A}(M))^G, d_g) \to (\mathcal{A}(\mathcal{M}), d).$$

In particular, if $M = \{pt\}$ is a point, we obtain the classical Chern-Weil homomorphism

$$W = \Phi_\omega : \mathbb{C}[g]^G \to \mathcal{A}(M_0^G).$$

If $M = V$ is a vector $G$-space, we have Chern-Weil homomorphism for induced vector bundle

$$W = \Phi_\omega : f \otimes v \mapsto f(\Omega) \otimes v,$$

which is a homomorphism from $G$-equivariant K-groups to the even de Rham cohomology $\oplus_* H^*_{DR}(M_0^G, \mathbb{R})$.

We can now reformulate the main result as what follows.

**Theorem 2.12.** If $\Phi$ is an $G$-invariant, then

$$\int_{\mathbb{G}} \Theta^G_0(X) \Phi(X) dX = (2\pi i)^{\dim G} \text{vol}(G) \int_{M_0^G} \alpha_{\text{red}} W(\Phi),$$

where $W : C^\infty(g)^G \to H^*(M_0^G)$ is the Chern-Weil homomorphism, associated to the principal fibration $\mu^{-1}(O) \to M_0^G$. 
ACKNOWLEDGMENTS

The author would like to thank M. Vergne for the related reprints of her works she sent.

This work is supported in part by the International Centre for Theoretical Physics (Italy), and the Vietnam National Research Programme in Fundamental Sciences. The final version of this work was completed during a stay of the author at the UP Diliman, the Philippines. The author would like to thank Department of Mathematics and UP Diliman for hospitality and especially professor Milagros P. Navarro for the invitation, help and friendship.

REFERENCES

1. N. Berline, E. Getzler and Michèle Vergne Heat Kernels and Dirac Operators, Grundlehren der mathematischen Wissenschaften, No 298, Springer-Verlag, 1994; 2nd correcting printing 1996.
2. V. Guillemin and S. Sternberg, Symplectic Techniques in Physics, Cambridge University Press, Cambridge, 1984.
3. L. C. Jeffrey and F. C. Kirwan, Localization for nonabelian group action, Topology 34(1995) 291-268.
4. J. Kalkman, Cohomology rings of symplectic quotients, preprint, University Utrecht 1993.
5. A. A. Kirillov, Elements of the Theory of Representations, Springer-Verlag, Berlin-New York-Heidelberg, 1975.
6. M. Vergne, A note on the Jeffrey-Kirwan-Witten localisation formula, Topology, 35(1996), No 1, 243-266.
7. S. Wu, An integration formula for the square of moment maps of circle actions, preprint Hep-th/921207.

Institute of Mathematics, National Centre for Natural Science and Technology, P. O. Box 631, Bo Ho, VN-10.000 Hanoi, Vietnam
Current address: International Centre for Theoretical Physics, ICTP P. O. Box 586, 34100, Trieste, Italy
E-mail address: dndiep@thevinh.ncst.ac.vn