ASSOCIAHEDRA MINIMIZE $f$-VECTORS OF SECONDARY POLYTOPES OF PLANAR POINT SETS

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Abstract. Kupavskii, Volostnov, and Yarovikov have recently shown that any set of $n$ points in general position in the plane has at least as many (partial) triangulations as the convex $n$-gon. We generalize this in two directions: we show that regular triangulations are enough, and we extend the result to all regular subdivisions, graded by the dimension of their corresponding face in the secondary polytope.

1. Introduction

Let $\mathcal{A} = \{P_1, \ldots, P_n\} \subset \mathbb{R}^d$ be a point configuration of size $n$, that is, a set of $n$ points from $\mathbb{R}^n$. A polyedral subdivision $T$ of $\mathcal{A}$ is, loosely speaking, a decomposition of the convex hull $\text{conv}(\mathcal{A})$ into convex polytopes that intersect properly, meaning that the intersection of any two of them is a face of both, and with the property that all vertices of all these polytopes are taken from $\mathcal{A}$. The individual polytopes of the decomposition are called the cells of $T$.

Observe that we do not require that all the points of $\mathcal{A}$ are vertices of a cell. These makes some authors (e.g., [KVY21]) to call “partial subdivisions” what we simply call “subdivisions”.

For reasons that will become apparent when we speak of regular subdivisions, it is convenient to partition the set of points that are not vertices of any cell into some that are considered used (that is, part of the cell containing them) and some that are not. This leads to the following more combinatorial definition, where a subdivision is not a collection of geometric cells (polytopes) but of “combinatorial cells” (subsets of $\mathcal{A}$):

**Definition 1.1** (Polyhedral subdivision [DRS10, Theorem 4.55]). Let $\mathcal{A} \subset \mathbb{R}^d$ be a point configuration. A (polyhedral) subdivision of $\mathcal{A}$ is a collection $T = \{C_1, \ldots, C_k\}$ of subsets of $\mathcal{A}$, called cells of $T$ with the following properties:

1. Each $C_i$ is full dimensional. That is, $\dim(\text{conv}(C_i)) = d$.
2. The $C_i$ cover $\text{conv}(\mathcal{A})$. That is,
   $$\cup_i \text{conv}(C_i) = \text{conv}(\mathcal{A}).$$

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(3) The $C_i$ intersect properly. That is, for every $i, j$ we have that $F := \text{conv}(C_i) \cap \text{conv}(C_j)$ is a common face of $\text{conv}(C_i)$ and $\text{conv}(C_j)$ and, moreover,

$$F \cap C_i = F \cap C_j.$$  

Subdivisions of $A$ form a poset under refinement, where $T$ refines $T'$ if every cell of $T$ is contained in a cell of $T'$. The unique maximal element in this poset is the trivial subdivision $\{A\}$ and the minimal elements are the triangulations of $A$: the subdivisions in which all cells are affinely independent.

A subdivision $T$ of $A = \{P_1, \ldots, P_n\}$ is regular if it can be obtained from a lifting vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ as follows: Consider the lifted point configuration $
\tilde{A} := \{(P_i, \omega_i) : i = 1, \ldots, n\} \subset \mathbb{R}^{d+1}\n$ and take as cells of the regular subdivision of $A$ induced by $\omega$ the projections in $A$ of the lower facets of $\tilde{A}$. Here, a facet of $\tilde{A}$ is called lower if the hyperplane containing that facet is not vertical and lies below $\text{conv}(\tilde{A})$. Observe that this definition makes some points of $A$ not part of any cell (those that are not in the lower hull of $\text{conv}(\tilde{A})$) but it also makes some points to be part of a cell and not vertices of it (those that are in the lower hull of $\text{conv}(\tilde{A})$ but are not vertices of $\text{conv}(\tilde{A})$).

For more details on the theory of regular subdivisions we refer to [DRS10]. Perhaps the fundamental theorem in the theory is:

**Theorem 1.2** (Gelfand, Kapranov and Zelevinsky [GKZ90, GKZ94], see also [DRS10, Thms. 5.1.9 and 5.2.16]). The decomposition of the space $\mathbb{R}^n$ of lifting vectors $\omega$ according to what regular subdivision of $A$ they produce is a complete polyhedral fan called the secondary fan of $A$ and it is the normal fan of a polytope $\Sigma(A)$ of dimension $n-3$, called the secondary polytope of $A$.

That is to say, the poset of regular subdivisions of $A$ under refinement is isomorphic to the lattice of (non-empty) faces of the secondary polytope of $A$, of dimension $n-3$. In particular, we have a well-defined dimension associated to each regular subdivision $T$, the dimension of the face of $\Sigma(A)$ corresponding to $T$. Regular triangulations have dimension zero (they biject to vertices of $\Sigma(A)$) and the trivial subdivision has dimension $n-3$ (it corresponds to $\Sigma(A)$ considered as a face of itself). To construct regular subdivisions in a controlled way we use several times the following lemma:

**Lemma 1.3** (Regular refinement [DRS10 Lemma 2.3.16]). Let $S$ be a regular subdivision of $A$, obtained for a certain lifting vector $\alpha \in \mathbb{R}^n$. Let $\omega \in \mathbb{R}^n$ be another height vector. Then, for any sufficiently small $\epsilon > 0$, the regular subdivision of $A$ for the height vector $\alpha + \epsilon \omega$ equals the refinement of $S$ obtained subdividing each cell $C$ of $S$ as a regular subdivision given with the heights $\omega|_C$. 
Example 1.4 (The associahedron). Let $A$ consist of $n$ points in convex position in the plane, that is, the vertices of a convex $n$-gon. Every subdivision of $A$ is regular, so the face poset of the secondary polytope equals the poset of all polyhedral subdivisions of $A$. In turn, polyhedral subdivisions of $A$ are in bijection to noncrossing sets of diagonals, so that they form a simplicial complex of dimension $n - 4$, independent of which particular set of points (in convex position) we started with. Theorem 1.2 tells us that this simplicial complex is dual to a simple $(n - 3)$-polytope, called the $(n - 3)$-associahedron [Lee91] (see also [DRS10, Sect. 1.1] or [Zie94, pp. 18, 306]). Subdivisions using $k$ diagonals of the $n$-gon have dimension $n - 3 - k$ in the associahedron.

It is well-known that a convex $n$-gon has exactly $C_{n-2}$ triangulations, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n$-th Catalan number. That is, the associahedron has $C_{n-2}$ vertices. There is also a closed formula for the number $C^k_n$ of $k$-dimensional faces of the associahedron of dimension $n - 1$. Namely

$$C^k_n = \frac{1}{n+1} \binom{n-1}{k} \binom{2n-k}{n} = \frac{n-k}{n(n+1)} \cdot \frac{(2n-k)!}{k!(n-k)!(n-k)!}$$

E.g. $C^0_n = C_n$, $C^{n-1}_n = 1$.

The numbers $C^k_n$ form a triangle that appears as sequence A033282 in the Online Encyclopedia of Integer Sequences [OEIS] and are related to the better known Narayana numbers, which give (among other combinatorial interpretations) the $h$-vector of the associahedron.

In the recent paper [KVY21] the authors show that every point configuration of size $n$ in general position in the plane has at least as many triangulations as the convex $n$-gon, that is, the Catalan number. In Section 2 we review their proof and check that the Catalan-many triangulations so obtained can be taken regular, thus proving that the secondary polytope of $n$ points in general position in the plane has at least as many vertices as the associahedron. In Section 3 we extend the ideas in the proof to also cover subdivisions, thus showing that:

**Theorem 1.5** (Main theorem). For every point configuration $A$ of size $n$ in general position in the plane, and for every $k \in \{0, \ldots, n-3\}$, the secondary polytope $\Sigma(A)$ has at least as many faces of dimension $k$ as the secondary polytope of the $n$-gon (the $(n - 3)$-associahedron).

**Remark 1.6** (General position). As customary, we say that a point configuration in $\mathbb{R}^d$ is in general position if no hyperplane contains more than $d$ of the points.

General position is needed in the Theorem 1.5 (and already in [KVY21]) since, for example, if $A$ consists of $n - 1$ collinear points plus an extra non-collinear one then it has exactly $2^{n-3}$ triangulations: each of the non-extremal points along the line can be used or not-used in a particular triangulation, and the choice of points to be used completely determines the
triangulation. This is much smaller than the Catalan number, which is in \( \Theta(4^n n^{-3/2}) \).

One can ask whether there is an analogue of Theorem 1.5 in higher dimension. We include some remarks regarding this question in Section 4. For example, we show how the problem, restricted to configuration with four more points than their dimension, is very much connected to the geodesic crossing number of the complete graph in the 2-sphere.

2. THE NUMBER OF REGULAR TRIANGULATIONS

Except for the regularity part (the proof of which we partially defer to the next section) in this section we are only rewriting the proof of the main result from [KVY21]. Strictly speaking we could omit this section, since it is nothing but a special case of the result proved in the next one, but it is worth doing the case of triangulations first, since it is simpler and serves as a warm-up. Also, the way we write it is intended to serve as a preparation for the more general case in the next section.

Throughout the paper we let \( \mathcal{A} \) be a point configuration of size \( n \) in general position in the plane. Let \( A \in \mathcal{A} \) be a vertex of \( \text{conv}(A) \) and let \( B \) and \( C \) be its adjacent vertices. We denote \( P_0 = B, P_1, \ldots, P_{n-2} = C \) the \( n-1 \) points of \( \mathcal{A} \setminus \{A\} \), ordered counter-clockwise as seen from \( A \).

Any polygonal line from \( B = P_0 \) to \( C = P_{n-3} \), and with vertex set an ordered (as seen from \( A \)) subset of \( \mathcal{A} \setminus \{A\} \) will be called an \( A \)-monotone polygonal line in \( \mathcal{A} \), or a monotone polyline for short. Since every subset of \( \mathcal{A} \setminus \{A\} \) containing \( B \) and \( C \) is the vertex set of a unique such polyline, there are exactly \( 2^{n-3} \) of them.

Following [KVY21], to each monotone polyline \( L \) we associate a signature \( \sigma_L \in \{-1,1\}^{n-3} \) (that is, we define a sign for each point \( P_i \in \mathcal{A} \setminus \{A,B,C\} \), in the following way. Write \( L \) as a list of its vertices, namely \( L = P_{a_0} P_{a_1} \cdots P_{a_{n-3}} P_{a_s} \), where \( P_{a_0} = B \) and \( P_{a_s} = C \). Given a point \( P_i, 1 \leq i \leq n-3 \), let \( P_{a_l} \) and \( P_{a_r} \) be the two points of \( L \) with \( a_l < i < a_r \) such that \( r-l \) is minimal. Then,

- if \( P_i \) is a point in \( L \), we set \( \sigma_L(P_i) = 1 \) if the segments \( AP_i \) and \( P_{a_l}P_{a_r} \) intersect, and \( \sigma_L(P_i) = -1 \) otherwise;
- if \( P_i \) is not in \( L \), we set \( \sigma_L(P_i) = -1 \) if the segments \( AP_i \) and \( P_{a_l}P_{a_r} \) intersect, and \( \sigma_L(P_i) = 1 \) otherwise.

We say that the points of \( \mathcal{A} \setminus \{A\} \) with negative signature are below the polyline \( L \) and those with positive signature are above.

The following result is [KVY21] Lemma 2:

**Lemma 2.1.** The map so obtained is a bijection between the \( A \)-monotone polygonal lines and the set \( \{-1,1\}^{n-3} \).

The following intuitive idea behind this construction can be considered an informal proof of the lemma: think of the points of \( \mathcal{A} \) as nails on a board and of \( L \) as a rubberband that goes below some nails and above some others.
Positive points are those that are either strictly above \( L \) or where \( L \) bends upwards (implying that the rubberband needs to go below the corresponding nails) and negative points are those below \( L \) or where \( L \) bends downwards.

**Example 2.2.** When \( A \) is in convex position, the positive entries in \( \sigma_L \) are the internal vertices of \( L \) and negative entries are the points not in \( L \).

We can apply this construction to triangulations of \( A \). If \( T \) is a triangulation, the link of \( A \) in \( T \) (that is, the sequence of segments that form a triangle with \( A \)) is an \( A \)-monotone polyline \( L \). We set \( \sigma_T = \sigma_L \) and call this sign vector the **link signature** of \( T \) from \( A \).

The main result in this section is the following. Let \( B = \{ A, Q_0, Q_1, \ldots, Q_{n-3}, Q_{n-2} \} \) be a point configuration in convex position and such that \( P_i \) lies in the segment \( AQ_i \) for every \( i \). (That is, obtain each point \( Q_i \) by moving each \( P_i \) radially from \( A \) until it becomes a vertex of the configuration). Then:

**Theorem 2.3.** \( A \) has at least as many regular triangulations as \( B \).

We will show this result by induction on the number of points of \( A \). In fact, the proof is stratified by signature. That is, we prove the following statement, which trivially implies 2.3:

**Lemma 2.4.** Let \( \sigma \in \{-1, +1\}^{[n-3]} \) be a signature. Then, \( A \) has at least as many regular triangulations with link signature equal to \( \sigma \) as \( B \).

In the proof we will need the following inequality between Catalan numbers. The inequality follows from the definition of \( C_n \), or from the fact that a convex \((k_1 + \cdots + k_m + 2)\)-gon can be subdivided into \( m \) convex polygons of sizes \( k_1 + 2, \ldots, k_m + 2 \), and then these polygons can be triangulated independently:

**Lemma 2.5** ([KVY21, Corollary 1]). For integers \( k_1, \ldots, k_m \), we have

\[
C_{k_1} \cdots C_{k_m} \leq C_{k_1 + \cdots + k_m}.
\]

**Proof of Lemma 2.4.** Let \( L = P_{a_0}P_{a_1} \cdots P_{a_{s-1}}P_{a_s} \) be the polyline associated to the signature \( \sigma \) in \( A \), with \( B = P_{a_0} \) and \( C = P_{a_s} \). A **negative interval** of \( \sigma \) (with respect to \( A \)) is each subset

\[
\{i, j\} \cup ([i, j] \cap \sigma^-),
\]

where \( a_i \) and \( a_j \) are two consecutive points with non-negative signature along \( L \). For the purpose of this definition, \( B = P_{a_0} \) and \( C = P_{a_s} = P_{n-2} \) are considered non-negative points in \( \sigma \), so that the first negative interval starts at 0 and the last one ends at \( n-2 \). We call length of a negative interval its number of negative points, that is, the cardinality of \([i, j] \cap \sigma^-\). A negative interval of length \( l \) has \( l + 2 \) points, where the two extra points are the end-points \( i \) and \( j \).

Observe that each negative interval corresponds to a maximal concave (as seen from \( A \)) chain in \( L \). Here we call a chain \( P_{a_1} \cdots P_{a_s} \) in \( L \) concave if for each intermediate point \( P_{a_k} \) in the chain we have that \( P_{a_k} \) is inside the
triangle $AP_{a_{k-1}}P_{a_{k+1}}$ or, equivalently, if $L$ bends downwards at $P_{ak}$. Being maximal implies that $P_{ai}$ and $P_{aj}$ are non-negative while the rest of the points in the chain are negative, and the negative interval corresponding to the chain is

$$\{i,j\} \cup ([i,j] \cap \sigma^-).$$

Let $\mathcal{P}_1, \ldots, \mathcal{P}_l$ be the subconfigurations of $\mathcal{A}$ consisting of the points in each negative interval. Also, for each segment $P_{a_{i-1}}P_{ai}$ in $L$ ($i = 1, \ldots, s$) let $\mathcal{R}_j$ be the triangle $\{A, P_{a_{i-1}}, P_{ai}\}$. The following claim is a particular case of Lemma 3.6 in the next section. See Figure 1 for an illustration:

**Claim:** There is a regular subdivision $S$ of $\mathcal{A}$ containing as cells all the negative intervals $\mathcal{P}_i$ and triangles $\mathcal{R}_j$.

Let $\alpha \in \mathbb{R}^n$ be a height function producing $S$ as a regular subdivision of $\mathcal{A}$.

![Figure 1. Illustration of the proof of Lemma 2.4 with $\sigma$ positive at 2, 6, 13 and 18, and negative in the rest of the $P_i$. The link $L$ of $\mathcal{A}$ is marked thicker. The white polygons above and below $L$ are the $\mathcal{R}_i$ and $\mathcal{P}_j$ respectively, and they are part of the regular subdivision $S$ constructed in the proof. The way in which the shaded regions are subdivided in $S$ is not determined.](image)

By inductive hypothesis, we assume that each polygon $\mathcal{P}_i$ has at least $C_{n_i}$ regular triangulations, where $n_i$ is its length. Fix one such regular triangulation $T_i$ for each $\mathcal{P}_i$. We can assume that the height function producing it is 0 for the first and last point in $\mathcal{P}_i$, which are the only points it has in common with the rest of $\mathcal{P}_j$'s. Thus, there is a global height function $\omega \in \mathbb{R}^n$ that restricted to each $\mathcal{P}_i$ produces the regular triangulation $T_i$.

Then, by Lemma 1.3, we can choose independently a regular triangulation for each of the polygons $\mathcal{P}_i$ to obtain many different regular triangulations
of $A$ with the given signature. Since each $P_i$ has at least $C_{n_i}$ triangulations (by inductive hypothesis), we get $\prod_i C_{n_i}$ regular triangulations of $A$ with signature $\sigma$, where $n_1, \ldots, n_k$ are the lengths of the negative intervals of $\sigma$ in $A$.

Now we look at $B$. We can consider the polyline $L'$ induced by $\sigma$ in $B$, and its negative intervals. Let $m_1, \ldots, m_\ell$ be their lengths. As above, we have that $B$ has at least $\prod_j C_{m_j}$ triangulations with signature $\sigma$; but we can now also argue that the count is exact. Indeed, above the polyline all triangulations with a given signature are the same. Below the polyline what we have are convex polygons of sizes $m_1, \ldots, m_\ell$, so the number of ways of refining them to triangulations is exactly $\prod_j C_{m_j}$.

Thus, $A$ has at least $\prod_i C_{n_i}$ regular triangulations of signature $\sigma$ and $B$ has exactly $\prod_j C_{m_j}$ of them. What remains to be shown is that

$$\prod_i C_{n_i} \geq \prod_j C_{m_j}.$$  

This inequality follows from the following remark: the negative intervals of $\sigma$ in $B$ are simply the “negative intervals of $\sigma$” in the standard sense; that is, each one starts and ends with a pair of consecutive non-negative entries of $\sigma$. Put differently, the negative intervals with respect to $B$, considered as an ordered partition of $|\sigma^-|$, form a refinement of the negative intervals with respect to $A$. In the refinement process the lengths of intervals in $A$ are decomposed as sums of lengths of intervals in $B$. Thus, the inequality (1) follows from applying Lemma 2.5 to each negative interval of $A$.  

\[\Box\]

3. The number of regular subdivisions

3.1. Extended signatures and extended stars. In order to prove Theorem 1.5 we need to extend to arbitrary subdivisions the formalism of link signatures, and introduce several additional concepts.

As in the previous section, we let $A$ be an arbitrary point configuration in general position in the plane and with $n$ points, let $B, A, C \in A$ be three consecutive vertices of its boundary and we let $P_0 = B, \ldots, P_{n-2} = C$ be the list of points in $A \setminus \{A\}$, ordered as seen from $A$. We also let $B$ be a configuration in convex position obtained moving each point $P_i$ to a new position $Q_i$ along the ray from $A$ through $P_i$.

Let $T$ be an arbitrary subdivision of $A$ or $B$. We define its link signature $\sigma_T \in \{0, +1, -1\}^{[n-3]}$ modifying the definition above as follows: first, the link $L$ of $A$ in $T$ divides the $n-3$ points into points “above” and “below” $L$, exactly as in Section 2. The points below the link get negative sign in $\sigma_T$. However, the points above can get either positive or zero sign, depending on the following:

- We give positive sign to the points that either form an edge with $A$ or are not used in the subdivision $T$.  

• We give zero sign to the points that are used but do not form an edge with \( A \).

**Example 3.1.** If \( T \) is a triangulation then all points along but “above” \( L \) form edges with \( A \) and all points strictly above \( L \) are unused in \( T \), so \( \sigma_T \) has no zero entries.

That is, the generalized definition of link signature for subdivisions is consistent with the one for triangulations in the previous section.

**Example 3.2.** For points in convex position, the points above \( L \) that are used are exactly the ones that form an edge with \( A \). That is, in this case, each point \( P_1, \ldots, P_{n-3} \) receives a positive sign if \( AP_i \) is an edge in \( T \); 0 if \( P_i \) is in some cell of \( T \) containing \( A \) but \( AP_i \) is not an edge; and negative sign if \( P_i \) is not in a cell with \( A \).

In the following statement, we call **star** of \( A \) in a subdivision \( T \) the collection of cells of \( T \) that contain \( A \).

**Lemma 3.3.** The above rules provide a bijection between signatures \( \sigma \in \{0, +1, -1\}^{[n-3]} \) and possible stars of \( A \) in subdivisions of \( A \).

**Proof.** We know how to construct the link signature from the subdivision; let us see how to recover the star of \( A \) in a subdivision \( T \) knowing only the link signature \( \sigma_T \in \{0, +1, -1\}^{[n-3]} \).

Taking all zeroes as if they were +1 gives us the polyline \( L \) from \( \sigma_T \). This polyline is the boundary of the star of \( A \) and the only extra information that we need is (a) how to partition \( L \) into sub-polylines corresponding to the individual cells in the star and (b) which points strictly above the polyline are used or not used in \( T \). Both pieces of information are contained in \( \sigma_T \): for (a) we only need to know which vertices of \( L \) are joined to \( A \) by edges, and these are precisely the positive points along \( L \). For (b), the link signature tells us which points strictly above the polyline are to be used (points with zero signature) or not used (points with positive signature) in the subdivision. \( \square \)

Notice how, both in the general and in the convex configurations, each 0 in the signature acts like a +1 in terms of the polyline, but it increases by one the dimension of the subdivision in the secondary polytope (provided that the subdivision is regular). Thus, a regular subdivision with signature \( \sigma \) has dimension at least \( |\sigma^0| \) as a face of the secondary polytope. Here and elsewhere we denote \( \sigma^0, \sigma^+ \) and \( \sigma^- \) the zero, positive, and negative parts of \( \sigma \). That is,

\[
\sigma^\varepsilon := \{ i \in [n-3] : \sigma(i) = \varepsilon \}.
\]

**Figure 2** shows the nine coarse subdivisions of a hexagon (corresponding to the nine facets of the three-dimensional associahedron) each with its signature.
Figure 2. The nine coarse subdivisions of a hexagon, each with its extended signature. The point $A$ is the top one, displayed with a thicker dot.

Figure 3 shows the ten regular coarse subdivisions of the so-called “mother of all examples”, consisting of the vertices of two concentric parallel triangles, each with its extended signature.

Our proof of Theorem 1.5 will be stratified by signature. That is, we will show that $A$ has at least as many regular subdivisions as $B$ for each possible dimension (as a face of the secondary polytope) and signature $\sigma \in \{-1, 0, 1\}^{n-3}$. This can be seen in Figures 2 and 3: the former contains nine subdivisions with nine different signatures and the latter contains these same nine plus an additional tenth signature.
In fact, the proof will be stratified by extended star according to the following definitions.

**Definition 3.4 (Negative intervals, extended star).** Let \( \sigma \in \{0, +1, -1\}^{[n-3]} \) be a signature and \( L \) its corresponding polyline in a configuration \( A \).

We call negative intervals of \( \sigma \) (with respect to \( A \)) the subsets

\[
\{i, j\} \cup [i, j] \cap \sigma^-,
\]
where \(i\) and \(j\) are two consecutive non-negative entries of \(\sigma\) along the polyline.

The extended star induced by \(\sigma\) in \(A\) consists of the following two types of cells:

1. The cells in the star of \(A\), as given by the bijection of Lemma 3.3. We call these the cells above \(L\).
2. The negative intervals. We call these the cells below \(L\).

Remark 3.5. The cells of the extended star above \(L\) could well be called “zero intervals”, since each of them consists of the zero entries in \(\sigma\) between any two consecutive points \(P_i\) and \(P_j\) in \(L\) with non-zero signature (including \(A\) and \(P_i\) and \(P_j\) as elements of the interval).

If \(T\) is a triangulation, these points include all the points along \(L\), so the cells above \(L\) are exactly the triangles \(R_j\) in the proof of Lemma 2.4.

Observe that the two types of cells in the extended star form two ordered sequences as seen from \(A\). In the cells above \(L\), each cell shares with the next one an edge of the form \(AP_i\). In the cells below \(L\), each cell shares with the next one a single point (the last point of a negative interval, which coincides with the first point of the next).

For a configuration in convex position (say \(B\)) the cells in the extended star cover the whole convex hull. That is to say, they form a polyhedral subdivision of the configuration. For a general configuration this is not the case. Still:

**Lemma 3.6.** Let \(A\) be a configuration in general position and let \(\sigma \in \{0, +1, -1\}^{n-3}\) be a signature. Then, the extended star \(S\) induced by \(\sigma\) in \(A\) can be extended to a regular subdivision (that is, there is a regular subdivision of \(A\) containing \(S\)).

**Proof.** As usual, let \(B\) and \(C\) be the vertices of \(\text{conv}(A)\) adjacent to \(A\). We are going to assume that there is a point \(D\) in the plane such that the quadrilateral \(ABDC\) is convex (with vertices in this order) and contains \(\text{conv}(A)\). Calling \(B'\) and \(C'\) the vertices of \(\text{conv}(A)\) past \(B\) and \(C\), the existence of \(D\) as above is equivalent to the lines \(BB'\) and \(CC'\) meeting on the side of \(\text{conv}(A)\) opposite to \(A\). This is no loss of generality: if such a \(D\) does not exist let \(D'\) be the point where the lines \(BB'\) and \(CC'\) meet, and perform a projective transformation that sends to infinity any line separating \(D'\) from \(\text{conv}(A)\). This makes the image of \(D'\) ends in the opposite side, as we want (see Figure 4 for an illustration). Such projective transformations do not change the subdivisions of \(A\) or their regularity.
In these conditions, let $R_1, \ldots, R_\ell$ be the cells in $S$ above $L$ and $P_1, \ldots, P_k$ those below $L$, in their circular order as seen from $A$. We extend the latter by the point $D$, that is, we let $S'$ consist of the cells $R_i$ and $P'_j$ where $P'_j = P_j \cup \{D\}$. We are going to show that $S'$ is a regular subdivision of $A' = A \cup \{D\}$. If this happens, any lifting vector producing $S'$ as a regular subdivision of $A'$, restricted to $A$, produces a regular subdivision of $A$ containing all the cells of $S$.

For the time being we assume that $\sigma$ has no positive entry, that is, it uses only 0 or $-1$. On $S'$ this has the effect that every internal vertex of the polyline $L$ belongs to exactly three cells, two above $L$ and one below if the point has negative signature, and one above and two below if it has positive signature. Also, this allows as to give a linear order on the cells of $S'$. If, as in the previous section, we denote $P_{a_0} P_{a_1} \cdots P_{a_{s-1}} P_{a_s}$ the points along $L$, in order, we have that the first cell in the extended str is the one meeting $L$ only on the edge $P_{a_0} P_{a_1}$, the second one is the one containing (at least) $P_{a_0} P_{a_1} P_{a_2}$ and, in general, the $i$-th cell ($i = 2, \ldots, s$) is the cell containing $P_{a_{i-1}} P_{a_i} P_{a_{i+1}}$. There is a last and $(s + 1)$-th cell meeting $L$ only on the edge $P_{a_{s-1}} P_{a_s}$. (In fact, this ordering is a shelling of $S'$). See Figure 5.

To show that $S'$ is regular, assign height zero to $A$, $B$ and $D$, and an arbitrary negative height to the first internal vertex $P_{a_1}$ along $L$. After this is done there is a unique way of assigning heights that lift all cells in $S$ coplanar: process the cells in their linear order, observe that when processing the $i$-th cell exactly three points of it have been given a height (namely $P_{a_{i-1}}$, $P_{a_i}$, and $A$ or $D$ depending on whether the cell is above or below $L$), and extend those three heights coplanarly to the rest of points in the cell.

It remains to show that this (essentially unique, the only choice was the height of $P_{a_1}$) height vector not only lifts the cells coplanar, but it lifts them as lower facets of the lifted configuration. This is equivalent to checking that for every edge separating two cells, the two cells incident to it are lifted.
convexly. Now, since every interior vertex of $S'$ has degree three, if the convex lifting property holds on one edge incident to a vertex $P$ then it holds also for the other two edges. Thus, the fact that the initial edge $BP_{a_1}$ of $L$ was lifted convex (by our giving negative height to $P_{a_1}$), propagates to the whole of $S'$.

This finishes the proof of the Lemma under the assumption that $\sigma$ has no positive entries. The modification for positive entries is easy:

- Positive points of $\sigma$ that are strictly above $L$ correspond to unused points in the extended star. Such points do not affect regularity, since we can give them any sufficiently big positive height.
- Positive points of $\sigma$ that lie in $L$ correspond to edges incident to $A$. In a first phase take those signs as if they were zero and construct the heights for a regular subdivision extending $S$ as explained above. In a second phase, iteratively perturb the chosen heights as follow, processing the positive points along $L$ in an arbitrary order: For each such point $P$, let $r_P$ be the straight line containing $AP$ and consider the lifting height $\omega_P$ that lifts each point of $A$ to height equal to its distance to $r_P$. Since $P$ has positive sign, all negative cells lie completely on one side of $r_P$ and, thus, they are lifted planarly. Among the positive cells, only the one containing the segment $AP$ crosses $r_P$, so that this one is the only cell of $S$ refined by the perturbation, and its refinement is precisely what we want: the edge $AP$ is introduced in the subdivision. □
3.2. **Well-formed subdivisions.** We now want to take dimension of a regular subdivision (as a face of the secondary polytope) into consideration. We first analyze the case of points in convex position.

Let \( T \) be a subdivision of \( B \). Let \( \sigma \in \{0, +1, -1\}^{n-3} \) be its link signature. As mentioned above, the extended star \( S \) of \( \sigma \) in \( B \) is a polyhedral subdivision and moreover:

- \( T \) refines \( S \), and
- \( T \) and \( S \) coincide above the link \( L \) of \( A \).

Let \( \ell \) be the number of cells in \( S \) below \( L \) and call \( P_1, \ldots, P_\ell \) those cells (observe that, as sets of labels, they are simply the negative intervals in \( \sigma \)). Let \( m_i \) be the length of the \( i \)-th negative interval, so that \( |P_i| = m_i + 2 \).

In \( T \), each \( P_i \) gets subdivided into a subdivision \( T_i \). Since \( P_i \) is a convex polygon, \( T_i \) is a regular subdivision and it has a well-defined dimension \( \delta_i \).

**Definition 3.7.** We call \( (\sigma, \delta) \) (with \( \delta = (\delta_1, \ldots, \delta_\ell) \)) the extended signature of the subdivision \( T \) of \( B \).

The possible extended signatures corresponding to a signature \( \sigma \) are easy to characterize. Since \( \delta_i \) needs to be the dimension of a subdivision of the \( m_i + 2 \)-gon, the only condition on it is that \( \delta_i \in [0, 1, \ldots, m_i - 1] \). Observe that the sequence \( m_1, \ldots, m_\ell \) can be derived from \( \sigma \) alone, since we are looking at a configuration in convex position.

**Lemma 3.8.** Let \( (\sigma, \delta) \) be an extended signature of length \( n - 3 \). Let \( m_1, \ldots, m_\ell \) be the list of lengths of the negative intervals in \( \sigma \). Then,

1. The number of subdivisions of the convex \( n \)-gon with that extended signature equals
   \[
   \prod_{i=1}^\ell C_{m_i}^{\delta_i},
   \]

2. All such subdivisions correspond to faces of dimension \( \sum_i \delta_i + |\sigma^0| \) in the secondary polytope.

**Proof.** The extended star covers the whole \( \text{conv}(B) \), and all subdivisions of \( \text{conv}(B) \) are regular. Moreover, the dimension of a subdivision \( T \) in the associahedron equals the sum of dimensions of the individual cells.

Thus, both parts of the statement follow from counting in how many ways we can independently subdivide the cells in the extended star. For the cells above \( L \) we have no choice (since \( (\sigma, \delta) \) fixes the star of \( A \)), and their combined dimension equals \( |\sigma^0| \). The \( i \)-th cell below is a convex \((m_i + 2)\)-gon, which has exactly \( C_{m_i}^{\delta_i} \) subdivisions of dimension \( \delta_i \).

For later use we mention the following inequality among face numbers of associahedra, which follow from partitioning the \((m_1 + \ldots + m_\ell + 2)\)-gon into \( m \) convex polygons of sizes \( m_1 + 2, \ldots, m_\ell + 2 \) and then subdividing these independently:
Lemma 3.9. For nonnegative integers \( m_1, \ldots, m_\ell \) and \( d \), we have
\[
C_{m_1 + \cdots + m_\ell}^d \geq \sum_{d_1 + \cdots + d_\ell = d} C_{m_1}^{d_1} \cdots C_{m_\ell}^{d_\ell}
\]

We now want to extend this formalism of extended signatures (that is, the vector \( \delta \) of dimensions associated to the negative intervals in \( \sigma \)) to some regular subdivisions of an arbitrary point configuration \( \mathcal{A} \) that we call well-formed. The fact that we do not extend it to all subdivisions is not a loss of generality, since what we show is that the well-formed subdivisions are sufficiently many to prove Theorem 1.5.

Definition 3.10. Let \( T \) be a regular subdivision of \( \mathcal{A} \). Let \( \sigma \) be its signature and \( S \) be the corresponding extended star, with \( \mathcal{P}_1, \ldots, \mathcal{P}_k \) the cells of \( S \) below the polyline \( L_\sigma \). Remember that \( T \) and \( S \) coincide above \( L_\sigma \). We say that \( T \) is well-formed if it satisfies the following conditions:

1. \( T \) refines the extended star \( S \) below the polyline; that is, \( T \) contains a polyhedral subdivision \( T_i \) of each cell \( \mathcal{P}_i \).

2. The dimension of \( T \) (as a regular subdivision of \( \mathcal{A} \)) equals \( |\sigma^0| + \sum_i \delta_i \), where \( \delta_i \) is the dimension of \( T_i \).

In these conditions, we call \((\sigma, \delta)\) the extended signature of \( T \).

Remark 3.11. For a given \( \sigma \), the possible dimension vectors \((\delta_1, \ldots, \delta_k)\) for the extended signatures are those with \( \delta_i < n_i \), where \((n_1, \ldots, n_k)\) is the sequence of lengths of negative intervals of \( \sigma \) in \( \mathcal{A} \). We emphasize that the latter depend not only on \( \sigma \), but also on the configuration \( \mathcal{A} \), as already happened for triangulations.

More precisely, if \((n_1, \ldots, n_k)\) and \((m_1, \ldots, m_\ell)\) are the sequences of negative intervals of \( \sigma \) in \( \mathcal{A} \) and \( \mathcal{B} \) respectively, we have that both sequences are ordered partitions of \(|\sigma^-|\) and the latter refines the former.

The key statement that we want to prove to derive Theorem 1.5 is:

Lemma 3.12. Let \( \sigma \in \{0, +1, -1\}^{[n-3]} \) be an extended signature for \( \mathcal{A} \), let \( S \) be the extended star induced by it, let \( \mathcal{P}_1, \ldots, \mathcal{P}_k \) be the cells of \( S \) below \( L \), and let \( \delta = (\delta_1, \ldots, \delta_k) \) be a valid vector of dimensions (that is, \( \delta_i < n_i \) for every \( i \), where \( n_i = |\mathcal{P}_i| - 2 \)).

For each \( \mathcal{P}_i \), let \( T_i \) be a regular subdivision of \( \mathcal{P}_i \) of dimension \( \delta_i \). Then, there is a well-formed regular subdivision \( T \) of \( \mathcal{A} \) with signature \((\sigma, \delta)\) and such that \( T \) restricted to \( \mathcal{P}_i \) equals \( T_i \).

We postpone the proof of Lemma 3.12 and first show how to derive Theorem 1.5 from it.

Corollary 3.13. With the same notation as in Lemma 3.12, the number of well-formed subdivisions of \( \mathcal{A} \) with extended signature \((\sigma, \delta)\) is at least
\[
\prod_{i=1}^{k} C_{n_i}^{\delta_i}.
\]
Proof of Corollary 3.13 and Theorem 1.5. The proof is by induction. We first prove Corollary 3.13 for configurations of size \( n \) assuming Theorem 1.5 for smaller configurations, and then derive from it Theorem 1.5 for size \( n \).

The first part is easy. Lemma 3.12 says that there are as many well-formed subdivisions of \( \mathcal{A} \) with extended signature \( (\sigma, \delta) \) as there are choices of regular polyhedral subdivisions of dimensions \( \delta_i \) for the cells \( P_i \) of \( S \) below \( L \). The inductive hypothesis says that the latter are at least \( C_{\delta_i}^{n_i} \) for each \( i \).

To derive Theorem 1.5, remember that there is a well-defined map that sends each extended signature \( (\sigma, \delta') \) in \( \mathcal{B} \) to an extended signature \( (\sigma, \delta) \) in \( \mathcal{A} \), obtained by grouping together in \( \delta' \) the entries of \( \delta' \) that correspond to the same negative interval of \( \sigma \) in \( \mathcal{A} \). To emphasize this map let us slightly change the notation used so far. Let \( (n_1, \ldots, n_k) \) be the length-sequence of negative intervals of \( \sigma \) in \( \mathcal{A} \), and denote \( (m_1, \ldots, m_1, \ldots, m_k, \ldots, m_k) \) the sequence in \( \mathcal{B} \), so that

\[
n_i = \sum_{j=1}^{\ell_i} m_i^j \quad \text{for each} \quad i = 1, \ldots, k.
\]

Similarly, we use the notation \( (\gamma_1, \ldots, \gamma_1, \ldots, \gamma_k, \ldots, \gamma_k) \) for a dimension sequence in \( \mathcal{B} \) compatible with \( (m_1, \ldots, m_1, \ldots, m_k, \ldots, m_k) \).

Then, by Lemma 3.8 the number of subdivisions of \( \mathcal{B} \) whose extended signature maps to a fixed extended signature \( (\sigma, \delta) \) of \( \mathcal{A} \) equals

\[
\prod_{i=1}^{k} \left( \sum_{\sum_{j=1}^{\ell_i} \gamma_i^j = \delta_i} \left( \prod_{j=1}^{\ell_i} C_{m_i^j}^{\gamma_i^j} \right) \right),
\]

where the sum in the middle is over all non-negative tuples \( (\gamma_1, \ldots, \gamma_i) \) adding up to \( \delta_i \).

By Lemma 3.9 this is less or equal than

\[
\prod_{i=1}^{k} C_{\delta_i}^{\sum_{j=1}^{\ell_i} m_i^j} = \prod_{i=1}^{k} C_{n_i}^{\delta_i}
\]

and, by Corollary 3.13 this is less or equal than the number of regular subdivisions of \( \mathcal{A} \) with signature \( (\sigma, \delta) \).

To complete our job we need to prove Lemma 3.12.

Proof of Lemma 3.12. We assume that \( S \) uses all the points in \( \mathcal{A} \). This is no loss of generality because unused points do not affect regularity and each of them adds one both to the count \(|\sigma^0| + \sum_i \delta_i\) and to the dimension of the corresponding regular subdivision of \( \mathcal{A} \).

Let \( T_0 \) be a regular subdivision that extends \( S \), which exists by Lemma 3.6 and let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be its corresponding height vector. By Lemma 1.3 for any choice of a perturbing height vector \( \omega = (\omega_1, \ldots, \omega_n) \) the subdivision
obtained refining $T_0$ as prescribed by $\omega$ (restricted to each individual cell) is regular.

Thus, we need to show that there exists an $\omega$ that restricted to each $P_i$ produces the regular subdivision $T_i$. This follows from the same arguments as in Lemma 3.6 as we process the cells in the extended star in their monotone order and assign heights $\omega_i$ to the points, each cell has at most three points in common with the previously processed ones, so that any regular subdivision of the cell can be realized by $\omega$.

The only thing that needs to be shown is that the regular subdivision $T$ so obtained has exactly dimension $|\sigma^0| + \sum_i \delta_i$ in the secondary polytope. Observe that actually $T$ is not uniquely defined. Since $S$ does not cover $\text{conv}(A)$, different choices of $\omega$ may produce different subdivisions in those uncovered regions. Our claim is that as long as $\omega$ is chosen sufficiently generic (among the possible $\omega$'s with $T|_{P_i} = T_i$), we get that $T$ has the desired dimension.

More precisely: let $\{R_1, \ldots, R_l\}$ be the cells of $S$ above $L$, in order. As we process each cell $C$ (be it a $P_i$ or a $R_i$) in the order specified in Lemma 3.6 there is a certain set $\omega_J$ ($J \subset [n]$) of coordinates that we are going to fix in this step, and our constraint is that $\omega_J$ needs to lie in the corresponding secondary cone of the configuration $C$. The dimension of this cone is $|C| - \delta - k$, where $\delta$ is the dimension, in the secondary polytope of $C$, of the subdivision that we want (which is the trivial subdivision for the cells above $L$, and the subdivision $T_i$ for the cells below) and $k$ is the number of points of $C$ whose $\omega$ was already fixed by the previous cells. Let us count these parameters in each case:

1. There is an initial cone (in fact a linear plane) of dimension two corresponding to the fact that we can choose the heights of $A$ and $B$ arbitrarily, before processing any cell.
2. If $C = R_i$ is a cell above $L$, then we want the trivial subdivision in it, whose dimension in the secondary polytope is $|R_i| - 3$. The number of points of $R_i$ that were already processed was exactly 3, so that the dimension of the cone we are looking at is zero. This simply reflects the fact that we have no choice for the lift of $R_i$, as happened in Lemma 3.6.
3. If $C = P_i$ is a cell below $L$, then the dimension of the subdivision $T_i$ equals $\delta_i$, $|P_i| = n_i + 2$, and $k$ equals 1 or 2. More precisely, we have $k = 1$ if and only if the initial point in $P_i$ forms an edge with $A$; that is, if and only if it has positive signature. Thus, the dimension of the cone equals $n_i - \delta_i + 1$ if $P_i$ starts with a positive point and it equals $n_i - \delta_i$ if $P_i$ starts with a zero point. In this count we need to consider $B$ as an extra positive point, since $P_1$ has only one point in common (the point $B$) with the part that is processed before it (the points $A$ and $B$).
This implies that the global contribution of all cells comes from the two initial points and the cells below $L$, and it equals
\[2 + \sum n_i - \sum \delta_i + |\sigma^+| + 1\]
\[= 2 + |\sigma^-| - \sum \delta_i + |\sigma^+| + 1\]
\[= 2 + (n - 3 - |\sigma^0|) - \sum \delta_i + 1\]
\[= n - (|\sigma^0| + \sum \delta_i).\]

That is, if a $\omega$ is sufficiently generic among the ones that refine $S$ in the way we want, then the secondary cone of the subdivision so obtained has dimension $n - (|\sigma^0| + \sum \delta_i)$. (Observe here that $\omega$ being sufficiently generic implies it to be in the relative interior of the secondary cone of $T$). Since the dimension of a regular subdivision of $\mathcal{A}$ equals $n$ minus the dimension of the corresponding secondary cone. This finishes the proof. 

4. SOME REMARKS IN HIGHER DIMENSION

For a given dimension $d$ and number of points $n$, what is the $d$-dimensional configuration of size $n$ minimizing the number of triangulations?

Although this question is probably too difficult to be answered explicitly for every $d$ and $n$, we here include several remarks regarding it.

Regular versus non-regular triangulations. In two dimensions, the configuration minimizing the set of all triangulations is the same as the one minimizing the number of regular ones and, in fact, it is a configuration that has only regular triangulations.

Moreover, the numbers of regular and non-regular triangulations in the plane for a particular configuration (or for arbitrary configurations) are not that different. They both have upper and lower bounds of the type $k^n$, for constants $k > 1$.

In higher dimension several things change drastically:

(1) For every $d \geq 3$ there is a constant $N$ such that every configuration of size $N$ or more in general position in $\mathbb{R}^d$ has non-regular triangulations. This follows from the combination of two facts: the cyclic polytope $C_d(n)$ has non-regular triangulations for every $n \geq \max d + 6, 9$ [AD+00, Theorem 4.1], and there is a constant $N = N(n, d)$ such that every configuration of more than $N$ points in general position in $\mathbb{R}^d$ contains a subconfiguration isomorphic to the vertex set of $C_d(n)$ (this is called the “higher-dimensional Erdős-Szekeres Theorem” in [BL+92, Proposition 9.4.7]).

(2) The number of regular triangulations of any configuration is bounded above by $2^{O(n \log n)}$ [DRS10, Theorem 8.4.2], while the number of non-regular ones can be much higher; for $C_d(n)$ it is bounded below by $2^{\Omega(n^{d/2})}$ [DRS10, Theorem 6.1.22 and Theorem 8.4.3].
Cyclic polytopes. The natural candidate generalizing the convex \( n \)-gon to dimension \( d \) is the vertex set of a cyclic \( d \)-polytope with \( n \) vertices, mentioned above. Recall that the cyclic polytope \( C_d(n) \) is defined as the convex hull of \( n \) arbitrary points along the \( d \)-dimensional moment curve. The number of triangulations of it is independent of the points chosen (since the oriented matroid is fixed) but the number of regular triangulations is not. See, for example, [AS02], where these numbers are computed quite explicitly for the case \( n = d + 4 \):

- The total number of triangulations of \( C_{n-4}(n) \) is in \( \Theta(n2^n) \) [AS02, Theorem 1].
- The number of regular ones is in the order of \( \frac{n^4}{64} \pm \Theta(n^3) \), with the cubic term depending on the specific realization [AS02, Theorem 4.3 and Remark 4.4].

However, cyclic polytopes are typically used as examples of polytopes with many triangulations or subdivisions; as we already noted, they have \( 2^{\Omega(n^{d/2})} \) of them. This, looked from the distance, does not seem too far from the upper bound of \( 2^{O(n^{d/2} \log n)} \) that we know for the number of triangulations of any point configuration [DRS10, Theorem 8.4.2.1].

Thus, it would be extremely surprising if cyclic polytopes turn out to minimize the number of triangulations, as the \( n \)-gon does in the plane. They might, however, minimize the number of regular ones, as we now see in a particular case.

The case \( n = d + 4 \). It is easy to show that every configuration of \( n = d + 2 \) points in general position has exactly two triangulations and with \( n = d + 3 \) it has \( n \) of them, all regular. The secondary polytopes are, respectively, a segment and an \( n \)-gon [DRS10, Section 5.5.1].

The next case, configurations of size \( n = d + 4 \), is more complicated but still tractable via Gale transforms [Zie94, Sect. 6.4].

For the purposes of this discussion, the Gale transform of a configuration \( \mathcal{A} \) of \( n \) points in dimension \( d \) is a configuration \( \mathcal{A}^* \) of \( n \) points in the sphere of dimension \( n - d - 2 \); that is, in the 2-sphere for \( n = d + 4 \). For example, the Gale transform of the cyclic polytope \( C_{n-4}(n) \) can be realized by placing \( \lfloor n/2 \rfloor \) points in a small circle around the north pole and the other \( \lceil n/2 \rceil \) in a small circle around the south pole, in a regular manner (see, e.g., [Zie94, Ex. 6.13] and [DRS10, p. 262]).

In the case \( n = d + 4 \) there is an easy recipe to compute or count regular triangulations of \( \mathcal{A} \) from its Gale transform \( \mathcal{A}^* \): draw the \( \left( \begin{array}{c} n \\left( \begin{array}{c} 2 \end{array} \right) \right. \) (shorter) geodesic arcs joining the \( n \) points of \( \mathcal{A}^* \) in the sphere \( S^2 \), and the regular triangulations of \( \mathcal{A} \) turn out to be in bijection with the 2-dimensional regions cut by these geodesics. This is an instance of [DRS10, Corollary 5.4.9]; the cell decomposition of the 2-sphere produced by the \( \left( \begin{array}{c} n \\left( \begin{array}{c} 2 \end{array} \right) \right. \) arcs is the chamber complex of \( \mathcal{A}^* \).
Let us make the simplifying assumption that the \(^{\binom{n}{2}}\) geodesic crossings do not produce triple crossings. This is a genericity condition that can always be attained via a small perturbation of the points in \(A^*\) (or, equivalently, of the points in \(A\), since \(A\) and \(A^*\) depend continuously on one another). Then, a simple application of Euler's formula gives the following relation between the number \(t\) of regions in the chamber complex of \(A^*\) (that is, the number of regular triangulations of \(A\)) and the number \(c\) of crossings among the arcs in the Gale transform (see Lemma 4.1 in \[AS02\]):

\[
\begin{align*}
t &= c + \binom{n}{2} - n + 2.
\end{align*}
\]

Thus, deciding what is the configuration of \(n\) points in dimension \(n - 4\) that minimizes (under our genericity assumption) the number of regular triangulations is equivalent to answering the following question:

**Question 4.1** (Spherical crossing number of \(K_n\)). *What is the geodesic embedding of the complete graph \(K_n\) in the 2-sphere that minimizes the number \(c\) of pairs of edges that cross each other?*

This is a classical question in geometric/topological graph theory, for which we do not have the complete answer despite considerable efforts. What we know, however is:

1. It is conjectured that the minimum value of \(c\) that can be attained is

\[
Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,
\]

not only in geodesic drawings but actually in any topological drawing of \(K_n\) in the 2-sphere (or, equivalently, in the plane). This was originally conjectured by Hill and popularized by Guy \[Guy60, HH63\].

2. Several embeddings attaining precisely that number are known, and one of them happens to be the geodesic embedding with points in two opposite circles, that is, the Gale transform of the cyclic polytope. This is one of the embeddings originally found by Hill, see e.g., \[HH63, Figure 5\]; among other places, its number of crossings is also computed, in connection to regular triangulations of \(C_{n-4}(n)\), in \[AS02, Proposition 4.2\].

See \[BW10\] for an account on the early history of the crossing number problem and its variations, and \[Sch21\] for a comprehensive survey.

As a conclusion we have that:

**Corollary 4.2.** *If Hill’s Conjecture holds then the minimum number of regular triangulations among all generic configurations of size \(n\) and dimension \(n - 4\) is attained by the vertex set of a (generic) cyclic polytope \(C_{n-4}(n)\). The number is \(Z(n) + \binom{n}{2} - n + 2\).*

Here “generic” is stronger than general position; it can be defined via the property that no three geodesic arcs in the Gale transform \(A^*\) meet at
a point or, also, calling a configuration $A$ generic if any sufficiently small perturbation preserves its set of regular triangulations.

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