On the Structure of the Observable Algebra of QCD on the Lattice

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Abstract

The structure of the observable algebra $\mathcal{O}_\Lambda$ of lattice QCD in the Hamiltonian approach is investigated. As was shown earlier, $\mathcal{O}_\Lambda$ is isomorphic to the tensor product of a gluonic $C^*$-subalgebra, built from gauge fields and a hadronic subalgebra constructed from gauge invariant combinations of quark fields. The gluonic component is isomorphic to a standard CCR algebra over the group manifold $SU(3)$. The structure of the hadronic part, as presented in terms of a number of generators and relations, is studied in detail. It is shown that its irreducible representations are classified by triality. Using this, it is proved that the hadronic algebra is isomorphic to the commutant of the triality operator in the enveloping algebra of the Lie super algebra $sl(1/n)$ (factorized by a certain ideal).
1 Introduction

This paper is a continuation of [1] and [2], where we have investigated quantum chromodynamics (QCD) on a finite lattice in the Hamiltonian approach. In [1] we have analyzed the structure of the field algebra of QCD and the Gauss law. Concerning the latter, there is a remarkable difference comparing with quantum electrodynamics (QED). In QED, we have a local Gauss law, which is built from gauge invariant operators and which is linear. Thus, one can “sum up” the local Gauss laws over all points of a given (spacelike) hyper-plane in space time yielding the following gauge invariant conservation law: The global electric charge is equal to the electric flux through a 2-sphere at infinity. In QCD the local Gauss law is neither built from gauge invariant operators nor is it linear, but it is possible to extract a gauge invariant, additive law for operators with eigenvalues in the dual of the center of $SU(3)$, which is identified with $\mathbb{Z}_3$. This implies – as in QED – a gauge invariant conservation law: The global $\mathbb{Z}_3$-valued colour charge (triality) is equal to a $\mathbb{Z}_3$-valued gauge invariant quantity obtained from the colour electric flux at infinity. We stress that the notion of triality occurred in the literature a long time ago. On the level of lattice gauge theories, this notion is already implicitly contained in a paper by Kogut and Susskind, see [3]. In particular, Mack [4] used it to propose a certain (heuristic) scheme of colour screening and quark confinement, based upon a dynamical Higgs mechanism with Higgs fields built from gluons. For similar ideas we also refer to papers by ‘t Hooft, see [5] and references therein. This concept was also used in a paper by Borgs and Seiler [6], where the confinement problem for Yang-Mills theories with static quark sources at nonzero temperature was discussed. In this context, also the Gauss law for colour charge was analyzed.

In [2] we have analyzed the observable algebra of QCD in the above context. The observable algebra $\mathfrak{O}_\Lambda$ is obtained by imposing gauge invariance and the local Gauss law. It turns out that $\mathfrak{O}_\Lambda$ is isomorphic to the tensor product of a gluonic $C^*$-subalgebra, built from lattice gauge fields and a hadronic subalgebra constructed from gauge invariant combinations of quark fields. The gluonic component is isomorphic to a standard CCR algebra over the group manifold $SU(3)$, whereas the hadronic subalgebra $\mathfrak{O}_T^{\text{mat}}$ is built from bilinear and trilinear gauge invariant combinations of the quark fields. We show that it is isomorphic to the commutant of the triality operator in the CAR-algebra generated by $n = 12N$ creation and annihilation operators. Here, $N$ is the number of lattice sites. This fact enables us, to prove in an elegant way that irreducible representations of $\mathfrak{O}_\Lambda$ are labelled by triality. Moreover, we show that $\mathfrak{O}_T^{\text{mat}}$ is isomorphic to the commutant of the triality operator in the enveloping algebra of the Lie superalgebra $\mathfrak{sl}(1/\mathbb{N})$, factorized by a certain ideal. In this language, the 3 inequivalent representations naturally arise via a standard Kac module construction. This classification result confirms the classification of irreducible representations of $\mathfrak{O}_\Lambda$ obtained in [2] by a completely different method. We believe that the various presentations of $\mathfrak{O}_T^{\text{mat}}$ found here will be crucial in future investigations of dynamical problems of QCD in terms of observables.
The presentation of $\mathfrak{O}_\Lambda$ used in this paper is based upon a certain gauge fixing procedure, which works well on the generic stratum of the action of the gauge group on the underlying classical configuration space. First steps towards including non-generic strata have been made as well [7]. It is worthwhile to try to omit the gauge fixing philosophy and to analyze $\mathfrak{O}_\Lambda$ in more intrinsic terms. This leads to polynomial super algebras, see [8].

We stress that a similar analysis has been performed for (spinorial and scalar) QED, see [9], [10] and [11]. There, the matter field part of the observable algebra is generated by an ordinary Lie algebra. For QCD, we get a Lie superalgebra, because here we have additionally trilinear gauge invariant operators (of baryonic type) built from matter fields.

Finally, we note that standard methods from algebraic quantum field theory for models which do not contain massless particles, see [12], do not apply here. For an analysis of problems with massless particles within this approach we refer to [13], [14], [15], [16], [17], [18] and further references therein. For basic notions concerning lattice gauge theories (including fermions), see [19] and references therein.

Our paper is organized as follows: to keep the paper self-contained, in Sections 2, 3 and 4 we briefly summarize the results of our previous papers. In Subsection 5.1 we present a systematic study of $\mathfrak{O}_{\text{mat}}^T$ in terms of generators and relations. Next, in Subsection 5.2 we reduce the set of relations to a certain minimal set and in Subsection 5.3 we present a classification of irreducible representations of $\mathfrak{O}_{\text{mat}}^T$. Finally, in Subsection 5.4 the above mentioned super Lie structure is discussed.

2 The Field Algebra

Here, we briefly recall the structure of the field algebra of lattice QCD, for details we refer to [11].

We consider QCD in the Hamiltonian framework on a finite regular cubic lattice $\Lambda \subset \mathbb{Z}^3$, with $\mathbb{Z}^3$ being the infinite regular lattice in 3 dimensions. We denote the lattice boundary by $\partial \Lambda$ and the set of oriented, $j$-dimensional elements of $\Lambda$, respectively $\partial \Lambda$, by $\Lambda^j$, respectively $\partial \Lambda^j$, where $j = 0, 1, 2, 3$. Such elements are (in increasing order of $j$) called sites, links, plaquettes and cubes. Moreover, we denote the set of external links connecting boundary sites of $\Lambda$ with “the rest of the world” by $\Lambda^1_{\infty}$ and the set of endpoints of external links at infinity by $\Lambda^0_{\infty}$. For the purposes of this paper, we may assume that for each boundary site there is exactly one link with infinity. Then, external links are labelled by boundary sites and we can denote them by $(x, \infty)$ with $x \in \partial \Lambda^0$. The set of non-oriented $j$-dimensional elements will be denoted by $|\Lambda|^j$. If, for instance, $(x, y) \in \Lambda^1$ is an oriented link, then by $|(x, y)| \in |\Lambda|^1$ we mean the corresponding non-oriented link. The same notation applies to $\partial \Lambda$ and $\Lambda_{\infty}$.

The basic fields of lattice QCD are quarks living at lattice sites and gluons living on links, including links connecting the lattice under consideration with “infinity”. The field
algebra is thus, by definition, the tensor product of fermionic and bosonic algebras:

$$\mathcal{A}_\Lambda := \mathcal{F}_\Lambda \otimes \mathcal{B}_\Lambda ,$$  \hspace{1cm} (2.1)

with

$$\mathcal{F}_\Lambda := \bigotimes_{x \in |\Lambda|^0} \mathcal{F}_x$$  \hspace{1cm} (2.2)

and

$$\mathcal{B}_\Lambda := \mathcal{B}_\Lambda^i \otimes \mathcal{B}_\Lambda^b = \bigotimes_{|(x,y)| \in |\Lambda|^1} \mathcal{B}_{|(x,y)|} \bigotimes_{x \in \partial \Lambda} \mathcal{B}_{|x,\infty|} .$$  \hspace{1cm} (2.3)

Here, $\mathcal{B}_\Lambda^i$ and $\mathcal{B}_\Lambda^b$ are the internal and boundary bosonic algebras respectively. We impose locality of the lattice quantum fields by postulating that the algebras corresponding to different elements of $\Lambda$ commute with each other.

The fermionic field algebra $\mathcal{F}_x$ associated with the lattice site $x$ is the algebra of canonical anticommutation relations (CAR) of quarks at $x$. The quark field generators are denoted by

$$|\Lambda|^0 \ni x \rightarrow \psi^{aA}(x) \in \mathcal{F}_x,$$  \hspace{1cm} (2.4)

where $a$ stands for bispinorial and (possibly) flavour degrees of freedom and $A = 1, 2, 3$ is the colour index corresponding to the fundamental representation of the gauge group $G = SU(3)$. (In what follows, writing $G$ we have in mind $SU(3)$, but essentially our discussion can be extended to arbitrary compact groups and their representations.) The conjugate quark field is denoted by $\psi^{*aA}(x)$. The only nontrivial canonical anti-commutation relations for generators of $\mathcal{F}_x$ read:

$$[\psi^{*aA}(x), \psi^{bB}(x)]_+ = \delta^B_A \delta^b_a .$$  \hspace{1cm} (2.5)

The bosonic field algebra $\mathcal{B}_{|(x,y)|}$ associated with the non-oriented link $|(x, y)|$, (where $y$ also stands for $\infty$), is given in terms of its isomorphic copies $\mathcal{B}_{(x,y)}$ and $\mathcal{B}_{(y,x)}$, corresponding to the two orientations of the link $(x, y)$. The algebra $\mathcal{B}_{(x,y)}$ is generated by matrix elements of the gluonic gauge potential on the link $(x, y)$,

$$A^1 \ni (x, y) \rightarrow U^{A_B}(x, y) \in \mathcal{C}_{(x,y)} ,$$  \hspace{1cm} (2.6)

with $\mathcal{C}_{(x,y)} \cong C(G)$ being the commutative $C^*$-algebra of continuous functions on $G$ and $A, B = 1, 2, 3$ denoting colour indices, and by colour electric fields,

$$A^1 \ni (x, y) \rightarrow E^{A_B}(x, y) \in \mathfrak{g}_{(x,y)} ,$$  \hspace{1cm} (2.7)

spanning the Lie algebra $\mathfrak{g}_{(x,y)} \cong su(3)$. These elements generate, in the sense of Woronowicz [20], the $C^*$-subalgebra $\mathcal{P}_{(x,y)} \cong C^*(G) \subset \mathcal{B}_{(x,y)}$.

Observe that $G$ acts on $C(G)$ naturally by left translations,

$$\alpha_g(u)(g') := u(g^{-1}g') , \quad u \in C(G).$$  \hspace{1cm} (2.8)

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Differentiating this relation, we get an action of $e \in \mathfrak{su}(3)$ on $u \in C^\infty(G)$ by the corresponding right invariant vector field $e^R$. Thus, we have a natural commutator between generators of $\mathfrak{P}_{(x,y)}$ and smooth elements of $\mathfrak{C}_{(x,y)}$:

$$i [e, u] := e^R(u).$$

(2.9)

To summarize, we have a $C^*$-dynamical system $(\mathfrak{C}_{(x,y)}, G, \alpha)$, with automorphism $\alpha$ given by the left action \[23\]. The field algebra $\mathfrak{B}_{(x,y)}$ is, by definition, the corresponding crossed product $C^*$-algebra,

$$\mathfrak{B}_{(x,y)} := \mathfrak{C}_{(x,y)} \otimes_\alpha G.$$  

(2.10)

We refer to \[21, 22\] for these notions.

The transformation law of elements of $\mathfrak{B}_{(x,y)}$ under the change of the link orientation is derived from the fact that the (classical) $G$-valued parallel transporter $g(x, y)$ on $(x, y)$ transforms to $g^{-1}(x, y)$ under the change of orientation. This transformation lifts naturally to an isomorphism

$$\mathcal{I}_{(x,y)} : \mathfrak{B}_{(x,y)} \to \mathfrak{B}_{(y,x)}$$

(2.11)

of field algebras, defined by:

$$\mathcal{I}_{(x,y)}(e, f) := (\hat{e}, \hat{f}) ,$$

(2.12)

where $\hat{f}(g) := f(g^{-1})$ and $\hat{e}$ is the left invariant vector field on $G$, generated by $-e$. The bosonic field algebra $\mathfrak{B}_{(x,y)}$ is obtained from $\mathfrak{B}_{(x,y)}$ and $\mathfrak{B}_{(y,x)}$ by identifying them via $\mathcal{I}_{(x,y)}$.

Next, we give a full list of relations satisfied by generators of $\mathfrak{B}_{(x,y)}$. Being functions on $SU(3)$, the generators of $\mathfrak{C}_{(x,y)}$ have to fulfill the following conditions:

$$(U^A_B(x, y))^* U^C_A(x, y) = \delta^C_B 1 ,$$

(2.13)

$$\epsilon_{ABC} U^A_D(x, y) U^B_E(x, y) U^C_F(x, y) = \epsilon_{DEF} 1 .$$

(2.14)

The entries of the colour electric field obviously fulfill

$$(E^A_B(x, y))^* = E^A_B(x, y) .$$

(2.15)

The transformation law \[2,12\] reads

$$U^A_B(y, x) = \tilde{U}^A_B(y, x) = (U^A_B(x, y))^* ,$$

(2.16)

$$E^A_B(y, x) = \tilde{E}^A_B(y, x) = -U^A_D(y, x) U^C_B(x, y) E^D_C(x, y)$$

(2.17)

and the $su(3)$-commutation relations take the form

$$[E^A_B(x, y), E^C_D(u, z)] = \delta_{xu} \delta_{yz} (\delta^C_B E^A_D(x, y) - \delta^A_D E^C_B(x, y)) .$$

(2.18)
Finally, the generalized canonical commutation relations (2.9) are given by:

\[ i \left[ E^A_B(x, y), U^C_D(u, z) \right] = +\delta_{xu}\delta_{yz}\left( \delta^C_B U^A_D(x, y) - \frac{1}{3} \delta^A_B U^C_D(x, y) \right) \]

\[-\delta_{xz}\delta_{yu}\left( \delta^A_D U^C_B(y, x) - \frac{1}{3} \delta^A_B U^C_D(y, x) \right). \tag{2.19} \]

To summarize, the field algebra \( \mathfrak{A}_\Lambda \), given by (2.1) – (2.3), is a \( C^* \)-algebra, generated by elements \( \{ \psi^aA(x), \psi^{a\ast}A(x), U^A_B(x, y), E^A_B(x, y) \} \), fulfilling relations (2.13) – (2.17), together with canonical (anti-) commutation relations (2.5), (2.18) and (2.19).

Using standard arguments, one can prove that the field algebra \( \mathfrak{A}_\Lambda \) has a unique (up to unitary equivalence) irreducible representation. This representation is obtained as follows, for details see [2]:

1. Take the representation \( \pi \) of the commutative \( C^* \)-algebra \( C(G) \) given by multiplication with elements of \( C(G) \).

2. Consider the left regular (unitary) representation \( \hat{\pi} \) of \( G \) on \( L^2(G, \mu) \),

\[ (\hat{\pi}(g)\xi)(g') := \xi(g^{-1}g'), \xi \in L^2(G, \mu). \tag{2.21} \]

One easily calculates

\[ \hat{\pi}(g) \circ \pi(u) \circ \hat{\pi}(g^{-1}) = \pi(\alpha_g(u)), \tag{2.22} \]

showing that the pair \( (\pi, \hat{\pi}) \) defines a covariant representation of \( (C(G), G, \alpha) \) on \( L^2(G, \mu) \). Differentiating (2.22) yields the generalized canonical commutation relations (2.9).

By the Gelfand-Najmark theorem for commutative \( C^* \)-algebras, we have a spectral measure \( dE \) on \( G \), such that \( \pi(u) = \int u(g) \, dE(g) \), for \( u \in C(G) \). Next, equation (2.22) implies

\[ L_g \circ dE(g') \circ L_{g^{-1}} = dE(gg'), \tag{2.23} \]

showing that the spectral measure \( dE \) defines a transitive system of imprimitivity for the representation \( \hat{\pi} \) of \( G \) based on the group manifold \( G \). Then, the imprimitivity theorem, see [23, 24], yields uniqueness, up to unitary equivalence. Finally, by the one-one-correspondence between covariant representations of \( C^* \)-dynamical systems and non-degenerate representations of the corresponding crossed products we get a unique irreducible representation of \( C(G) \otimes_\alpha G \). Disregarding the \( C^* \)-context, this statement is a classical result of Mackey, see [24] and references therein. It generalizes the classical uniqueness theorem by von Neumann to the case of commutation relations (2.9).
We take the tensor product of the above irreducible representations over all links:

\[ \bigotimes_{(x,y) \in \Lambda^1} L^2(C_{(x,y)}, \mu) \bigotimes_{x \in \partial \Lambda^0} L^2(C_{(x,\infty)}, \mu) \cong L^2(\mathcal{C}, \mu), \tag{2.24} \]

where

\[ \mathcal{C} := \prod_{(x,y) \in \Lambda^1} C_{(x,y)} \prod_{x \in \partial \Lambda^0} C_{(x,\infty)}, \]

and each space \( C_{(x,y)} \) being diffeomorphic to the group space \( G \). This is the unique representation space of the gluonic field algebra \( \mathfrak{B}_\Lambda \). Moreover, using the classical uniqueness theorem for CAR-representations by Jordan and Wigner, any representation of fermionic fields is equivalent to the fermionic Fock representation. Finally, we have the following isomorphism of \( C^* \)-algebras:

\[ C(G) \otimes \alpha G \cong \mathfrak{R}(L^2(G, \mu)), \]

where \( \mathfrak{R}(L^2(G, \mu)) \) denotes the algebra of compact operators on \( L^2(G, \mu) \), for the proof see [2]. This implies that the field algebra \( \mathfrak{A}_\Lambda \) can be identified with the algebra \( \mathfrak{R}(H_\Lambda) \) of compact operators on the Hilbert space

\[ H_\Lambda = \mathcal{F}(\mathbb{C}^{12N}) \otimes L^2(\mathcal{C}, \mu), \tag{2.25} \]

with \( \mathcal{F}(\mathbb{C}^{12N}) \) denoting the fermionic Fock space generated by \( 12N \) anti-commuting pairs of quark fields.

### 3 Gauge Transformations and Gauss Law

The group \( G_\Lambda \) of local gauge transformations related to the lattice \( \Lambda \) consists of mappings

\[ \Lambda^0 \ni x \to g(x) \in G, \]

which represent internal gauge transformations, and of gauge transformations at infinity,

\[ \Lambda_\infty^0 \ni z \to g(z) \in G. \]

Thus,

\[ G_\Lambda := G_i \times G_\infty = \prod_{x \in \Lambda^0} G_x \prod_{z \in \Lambda_\infty^0} G_z, \tag{3.1} \]

with \( G_y \cong SU(3) \), for every \( y \).

The group \( G_\Lambda \) acts on the classical configuration space \( \mathcal{C} \) as follows:

\[ \mathcal{C}_{(x,y)} \ni g(x,y) \to g(x)g(x,y)g(y)^{-1} \in \mathcal{C}_{(x,y)}, \]
with \( g(x) \in G_x \) and \( g(y) \in G_y \). This action lifts naturally to functions on \( \mathcal{C} \). Moreover, we have an action of \( G_x \) on itself by inner automorphisms. This yields an action of \( G_\Lambda \) by automorphisms on each \( C^* \)-dynamical system \((\mathcal{C}_{(x,y)}, G, \alpha)\) and, therefore, on the gluonic field algebra \( \mathfrak{B}_\Lambda \). For generators of \( \mathfrak{B}_{(x,y)} \subset \mathfrak{B}_\Lambda \), this action is given by

\[
U^A_{B}(x,y) \rightarrow g^C_A(x)U^D_C(x,y)(g^{-1})^D_B(y), \quad (3.2)
\]

\[
E^A_{B}(x,y) \rightarrow g^C_A(x)E^C_D(x,y)(g^{-1})^D_B(x), \quad (3.3)
\]

with \( y \) standing also for \( \infty \). Fermionic generators transform under the fundamental representation:

\[
\psi^{aA}(x) \rightarrow g^A_B(x)\psi^{aB}(x). \quad (3.4)
\]

To summarize, the group of local gauge transformation \( G_\Lambda \) acts on the field algebra \( \mathfrak{A}_\Lambda \) in a natural way by automorphisms.

The local Gauss law at \( x \in \Lambda^0 \) reads

\[
\sum_{y+x} E^A_{B}(x,y) = \rho^A_B(x), \quad (3.5)
\]

where

\[
\rho^A_B(x) = \sum_a \left( \psi^{*aA}(x)\psi^{aB}(x) - \frac{1}{3} \delta^A_B \psi^{*aC}(x)\psi^{aC}(x) \right) \quad (3.6)
\]

is the local matter charge density, fulfilling \( \rho^A_A(x) = 0 \).

In [1] we have analyzed the Gauss law in detail. Here, we briefly recall the main result. Consider any integrable representation \( F \) of the Lie algebra \( su(3) \) on a Hilbert space \( \mathcal{H} \), i.e. a collection of operators \( F^A_B \) in \( \mathcal{H} \), fulfilling \( F^A_A = 0 \), \( (F^A_B)^* = F^B_A \) and

\[
[F^A_B, F^C_D] = \delta^C_B F^A_D - \delta^A_D F^C_B. \quad (3.7)
\]

By (2.18), (3.6) and (2.5), the operators \( E^A_B(x,y) \) and \( \rho^A_B(x) \), occurring on both sides of the local Gauss law, are of this type. Integrability means that for each \( F \) there exists a unitary representation \( SU(3) \ni g \rightarrow \tilde{F}(g) \in B(\mathcal{H}) \) of the group \( SU(3) \). If \( F \) and \( G \) are two commuting (integrable) representations of \( su(3) \), then so is \( F + G \). Moreover, \( -F^* \) is also a representation of \( su(3) \). Such a collection of operators is an operator domain in the sense of Woronowicz (see [25]).

We define an operator function on this domain, i.e. a mapping \( F \rightarrow \varphi(F) \), which satisfies \( \varphi(UFU^{-1}) = U\varphi(F)U^{-1} \) for an arbitrary isometry \( U \), as follows: For any integrable representation \( F \) of \( su(3) \), consider the corresponding representation \( \tilde{F} \) of \( SU(3) \). Its restriction to the center \( Z \) of \( SU(3) \) acts as a multiple of the identity on each irreducible subspace \( \mathcal{H}_\alpha \) of \( \tilde{F} \),

\[
\tilde{F}(c)|_{\mathcal{H}_\alpha} = \chi^\alpha_f(c) \cdot 1_{\mathcal{H}_\alpha}, \quad c \in Z.
\]
Obviously, $\chi_{\bar{F}}^\alpha$ is a character on $\mathbb{Z}$ and, therefore, $(\chi_{\bar{F}}^\alpha(c))^3 = 1$. Since the group of characters on $\mathbb{Z} = \{\zeta \cdot 1_3 \mid \zeta^3 = 1, \, \zeta \in \mathbb{C}\}$ is isomorphic to the additive group $\mathbb{Z}_3 \cong \{-1,0,1\}$, there exists a $\mathbb{Z}_3$-valued operator function $F \rightarrow \varphi(F)$, defined by

$$\zeta \varphi_{\alpha}(F) = \chi_{\bar{F}}^\alpha(\zeta \cdot 1_3), \quad \varphi(F) = \sum_{\alpha} \varphi_{\alpha}(F) 1_{H_{\alpha}}. \quad (3.8)$$

Since $\chi_{\bar{F}}^\alpha$ are characters, we have

$$\varphi(F + G) = \varphi(F) + \varphi(G), \quad (3.9)$$

for $F$ and $G$ commuting. Now, using the equivalence of the irreducible representation $\bar{F}$ of $SU(3)$ with highest weight $(m, n)$ with the tensor representation in the space $T_{m,n}(\mathbb{C}^3)$ of $m$-contravariant, $n$-covariant, completely symmetric and traceless tensors over $\mathbb{C}^3$, we get

$$\chi_{\bar{F}}^\alpha(z) = \zeta^{m(\alpha) - n(\alpha)}, \quad (3.10)$$

for $z = \zeta \cdot 1_3 \in \mathbb{Z}$. Thus, we have

$$\varphi_{\alpha}(F) = (m(\alpha) - n(\alpha)) \mod 3 \quad (3.11)$$

for every irreducible highest weight representation $(m(\alpha), n(\alpha))$. In [1] we have given an explicit construction of $\varphi(F)$ in terms of Casimir operators of $F$.

Applying $\varphi$ to the local Gauss law (3.5) and using additivity (3.9) we obtain a gauge invariant equation for operators with eigenvalues in $\mathbb{Z}_3$:

$$\sum_y \varphi(E(x,y)) = \varphi(\rho(x)) \quad (3.12)$$

valid at every lattice site $x$. The quantity on the right hand side is the (gauge invariant) local colour charge density carried by the quark field. Using the transformation law (2.17) for $E(x,y)$ under the change of the link orientation and additivity (3.9) of $\varphi$, one can show that

$$\varphi(E(x,y)) + \varphi(E(y,x)) = 0 \quad (3.13)$$

for every lattice bond $(x,y)$. Now we take the sum of equations (3.12) over all lattice sites $x \in \Lambda$. Due to the above identity, all terms on the left hand side cancel, except for contributions coming from the boundary. This way we obtain the total flux through the boundary $\partial \Lambda$ of $\Lambda$:

$$\Phi_{\partial \Lambda} := \sum_{x \in \partial \Lambda} \varphi(E(x,\infty)) \quad (3.14)$$

On the right hand side we get the (gauge invariant) global colour charge (triality), carried by the matter field

$$t_\Lambda := \sum_{x \in \Lambda} \varphi(\rho(x)) \quad (3.15)$$
Both quantities appearing in the global Gauss law
\[ \Phi_{\partial \Lambda} = t_{\Lambda}, \]
(3.16)
take values in the center \( \mathbb{Z}_3 \) of \( SU(3) \).

4 The Observable Algebra

4.1 The Algebra of Internal Observables

Physical observables, internal relative to \( \Lambda \) are, by definition, gauge invariant fields, respecting the Gauss law. Hence, we proceed as follows:

1. Take the subalgebra \( \mathfrak{A}^{G_{\Lambda}} \subset \mathfrak{A}_{\Lambda} \) of \( G_{\Lambda} \)-invariant elements of \( \mathfrak{A}_{\Lambda} \).

2. Require vanishing of the ideal \( \mathfrak{I}_{\Lambda}^{i} \cap \mathfrak{A}^{G_{\Lambda}} \), generated by local Gauss laws at all lattice sites.

Then, the algebra of internal observables, relative to \( \Lambda \), is given by:
\[ \mathcal{O}_{\Lambda}^{i} = \mathfrak{A}^{G_{\Lambda}} / \{ \mathfrak{I}_{\Lambda}^{i} \cap \mathfrak{A}^{G_{\Lambda}} \}. \]

Using the identification \( \mathfrak{A}_{\Lambda} \cong \mathcal{R}(H_{\Lambda}) \) yields a unitary representation of \( G_{\Lambda} \) in \( \mathcal{R}(H_{\Lambda}) \). Thus, the subalgebra \( \mathfrak{A}^{G_{\Lambda}} \) can be viewed as the commutant \( (G_{\Lambda})' \) of this representation in \( \mathcal{R}(H_{\Lambda}) \). Consider the \( G_{\Lambda} \)-invariant subspace
\[ \mathcal{H}_{\Lambda} := \{ h \in H_{\Lambda} \mid G_{\Lambda} h = h \} . \]

**Theorem 4.1.** The algebra of internal observables is canonically isomorphic with the algebra of those compact operators on the Hilbert space \( \mathcal{H}_{\Lambda} \), which commute with the action of the group \( G_{\Lambda}^{\infty} \),
\[ \mathcal{O}_{\Lambda}^{i} \cong \mathcal{R}(\mathcal{H}_{\Lambda}) \cap (G_{\Lambda}^{\infty})'. \]

For the proof see [2]. Next, we want to classify the irreducible representations of \( \mathcal{O}_{\Lambda}^{i} \): Note that the restriction of the action of \( G_{\Lambda}^{\infty} \) to \( \mathcal{H}_{\Lambda} \) is not irreducible. Thus, \( \mathcal{H}_{\Lambda} \) splits into the direct sum of irreducible subspaces of \( G_{\Lambda}^{\infty} \). These are labelled by sequences of highest weights,
\[ (m, n) = (m_{z_1}, \ldots, m_{z_M}; n_{z_1}, \ldots, n_{z_M}) , \quad z_{i} \in \Lambda^0_{\infty} , \]
describing the boundary flux distributions, carried by the gluonic field. We decompose
\[ \mathcal{H}_{\Lambda} = \bigoplus \mathcal{H}_{\Lambda}^{(m, n)}, \]
with \( \mathcal{H}_{\Lambda}^{(m, n)} \) denoting the sum of all irreducible subspaces with respect to the action of \( G_{\Lambda}^{\infty} \), carrying the same type \( (m, n) \). Then we have
\[ \mathcal{O}_{\Lambda}^{i} \mathcal{H}_{\Lambda}^{(m, n)} \subset \mathcal{H}_{\Lambda}^{(m, n)} . \]
Theorem 4.2. The irreducible representations of $\mathfrak{O}_i^\Lambda$ are labelled by highest weight representations $(m,n)$ of $G_\Lambda^\infty$. For any $(m,n)$, the corresponding irreducible representation of $\mathfrak{O}_i^\Lambda$ coincides with the algebra of those compact operators on $\mathcal{H}_\Lambda^{(m,n)}$, which commute with the action of the group $G_\Lambda^\infty$.

For the proof see [2].

4.2 The Full Algebra of Observables

For constructing the thermodynamical limit of finite lattice QCD, one has to take into account “correlations with the rest of the world”: Consider two lattices $\Lambda_1$ and $\Lambda_2$, having a common wall and denote $\tilde{\Lambda} = \Lambda_1 \cup \Lambda_2$. If $u \in \Lambda_1$ and $v \in \Lambda_2$ are adjacent points in $\tilde{\Lambda}$, we identify their infinities. Imagine this joint infinity $z$ as the middle point of the connecting link $(u,v)$. We put:

$$U_{AB}^A(u,v) := U_{AC}^A(u,z)U_{CB}^B(z,v).$$

Now, typical observables describing correlations between $\Lambda_1$ and $\Lambda_2$ are:

$$J_{\gamma}^{ab}(x,y) := \psi^a_A(x)U_{\gamma B}^A\psi^b_B(y), \quad U_{\tau} := U_{\tau A}^A,$$

with $\gamma$ being a path starting at some point $x \in \Lambda_1$ and ending at the point $y \in \Lambda_2$, and $\tau$ being a closed path running partially through $\Lambda_1$ and partially through $\Lambda_2$. (Strictly speaking, these are elements of the multiplier algebra of the observable algebra $\mathfrak{O}_i^{\tilde{\Lambda}}$.)

In order to construct observables of this type, we have to admit “charge carrying” fields, “having free tensor indices at infinity”, like $\psi^a_A(x)U_{\gamma C}^A(x,z)$ and $U_{\gamma B}^C(z,y)\psi^b_B(y)$, with $z$ being a joint infinity point.

Thus, we are naturally led to extend the Hilbert space $\mathcal{H}_\Lambda$ to $T_\infty \otimes \mathcal{H}_\Lambda$ by tensorising it with

$$T_\infty := \bigotimes_{z \in \Lambda_0} T(z), \quad T(z) := \bigoplus_{(m,n)} T^m_n(z).$$

Moreover, we extend both the action of $G_\Lambda^\infty$ on $\mathcal{H}_\Lambda$,

$$T(g)(t \otimes \Psi) := t \otimes (g \cdot \Psi),$$

and the action of $G_\Lambda^\infty$ on $T_\infty$,

$$R(g)(t \otimes \Psi) := (g \cdot t) \otimes \Psi.$$
Consequently, the full algebra of observables is:

$$\mathcal{O}_\Lambda := \mathcal{H}(H_\Lambda) \cap (G_\Lambda^\infty)' .$$

(4.1)

Decompose

$$H_\Lambda = \bigoplus_{(m,n)} H^{(m,n)}_\Lambda ,$$

with $H^{(m,n)}_\Lambda$ denoting the intersection of $H_\Lambda$ with $T^{(m,n)}_\infty \otimes H_\Lambda$ and

$$T^{(m,n)}_\infty := \bigotimes_{z \in \Lambda_0^\infty} T^m_{n_z}(z) .$$

We denote by $\mathcal{O}_\Lambda^\infty$ the algebra of compact operators acting on $T_\infty$, invariant with respect to the action of $G_\Lambda^\infty$. By classical invariant theory, this algebra is generated (in the sense of Woronowicz) by operations of tensorizing or contracting with $SU(3)$-invariant tensors $\delta^A_B$, $\epsilon^{ABC}$ and $\epsilon_{ABC}$, and by projection operators $P^{(m,n)}$ onto $T^{(m,n)}_\infty \subset T_\infty$. One shows:

$$\mathcal{O}_\Lambda \cong \mathcal{O}_\Lambda^i \otimes \mathcal{O}_\Lambda^\infty .$$

The action of $\mathcal{O}_\Lambda^\infty$ on $T_\infty$ is not irreducible, the image of $t(z) \in T^m_{n_z}(z)$ under this action is a sum of components belonging to $T^k_{n_z}(z)$, with $(k,l)$ fulfilling

$$(m - n) \mod 3 = (k - l) \mod 3 .$$

Thus, the $\mathbb{Z}_3$-valued flux

$$\Phi(z) := (m_z - n_z) \mod 3$$

through each external link $(x,z), z \in \Lambda_0^\infty$, is conserved under the action of $\mathcal{O}_\Lambda^\infty$. We denote the sequence of $\mathbb{Z}_3$-valued fluxes assigned to all boundary points by:

$$\Phi := (\Phi(z_1), \Phi(z_2), \ldots)$$

and put

$$H^{\Phi}_\Lambda := \bigoplus_{m(z_i) - n(z_i) \mod 3 = \Phi(z_i)} H^{(m,n)}_\Lambda .$$

Then, obviously

$$H_\Lambda = \bigoplus_{\Phi} H^{\Phi}_\Lambda .$$

**Lemma 4.3.** *The spaces $H^{\Phi}_\Lambda$ provide all the irreducible representations of the algebra of observables $\mathcal{O}_\Lambda$.***
The global flux associated with a given boundary flux distribution $\Phi$:

$$\Phi_{\partial \Lambda} := \sum_{z \in \Lambda^0_\infty} \Phi(z) \mod 3.$$ 

If we denote the total number of gluonic and antigluonic flux lines running through the boundary by

$$m := \sum_{z_i \in \Lambda^0_\infty} m(z_i), \quad n := \sum_{z_i \in \Lambda^0_\infty} n(z_i),$$

then we have:

$$\Phi_{\partial \Lambda} = (m - n) \mod 3.$$ 

**Lemma 4.4.** The irreducible representations of $\mathfrak{O}_\Lambda$ in $H^{\Phi}_\Lambda$ and in $H^{\Phi'}_{\Lambda}$ are unitarily equivalent, if and only if $\Phi$ and $\Phi'$ carry the same global flux,

$$\Phi_{\partial \Lambda} = \Phi'_{\partial \Lambda}.$$ 

This yields the following classification of irreducible representations:

**Theorem 4.5.** There are three inequivalent representations of $\mathfrak{O}_\Lambda$ labelled by values of the global flux $\Phi_{\partial \Lambda}$. Consequently, the space $H_\Lambda$ splits into the sum of three eigenspaces of $\Phi_{\partial \Lambda}$

$$H_\Lambda = \bigoplus_{\lambda = -1,0,1} H_{\Lambda}^{\lambda}.$$ 

Each of the spaces $H_{\Lambda}^{\lambda}$ is a sum of superselection sectors $H^{\Phi}_{\Lambda}$ corresponding to all possible distributions $\Phi$ of the global flux $\lambda$. They carry equivalent representations of $\mathfrak{O}_\Lambda$.

Finally, by the global Gauss law, we get

**Corollary 4.6.** The inequivalent representations of $\mathfrak{O}_\Lambda$ are labelled by eigenvalues of global colour charge $t_\Lambda$.

For the proof of the above statements we refer to [2].

### 4.3 Generators and relations

We recall the decomposition (4.1),

$$\mathfrak{O}_\Lambda \cong \mathfrak{O}^i_\Lambda \otimes \mathfrak{O}^\infty_\Lambda.$$ 

As already mentioned above, $\mathfrak{O}^\infty_\Lambda$ is generated by operations of tensorizing or contracting with $SU(3)$-invariant tensors $\delta^{AB}$, $\epsilon^{ABC}$ and $\epsilon_{ABC}$, and by projection operators $P^{(m,n)}_{\infty}$ onto $T^{(m,n)}_{\infty} \subset T_\infty$. Thus, one has to find a complete set of generators of $\mathfrak{O}^i_\Lambda$. In [2] we have shown the following
Theorem 4.7. The observable algebra \( \mathfrak{O}_\Lambda \) is generated by the following gauge invariant elements (together with their conjugates):

\[
U_\gamma := U^A_\gamma A \quad (4.3)
\]

\[
E_\gamma(x, y) := U^A_\gamma B E^B_A(x, y) \quad (4.4)
\]

\[
J^{ab}_{\gamma}(x, y) := \psi^a A(x) U^A_\gamma B \psi^b B(y) \quad (4.5)
\]

\[
W^{abc}_{\alpha, \beta, \gamma}(x, y, z) := \frac{1}{6} \epsilon_{ABC} U^A_\alpha D U^B_\beta E U^C_\gamma F \psi^a D(x) \psi^b E(y) \psi^c F(z), \quad (4.6)
\]

with \( \gamma \) denoting an arbitrary closed lattice path in formula (4.3), a closed lattice path starting and ending at \( x \) in (4.4) and a path from \( x \) to \( y \) in (4.5). In formula (4.6), \( \alpha, \beta \) and \( \gamma \) are paths starting at some reference point \( t \) and ending at \( x, y \) and \( z \), respectively. In formula (4.4), both \( x \) and \( y \) stand also for \( \infty \).

Note that the observables \( J^{ab}_{\gamma} \) and \( W^{abc}_{\alpha, \beta, \gamma} \) represent hadronic matter of mesonic and baryonic type.

The above set of generators is, however, highly redundant. In a first step, it can be reduced by using the concept of a lattice tree. As a result, one obtains a presentation of the observable algebra in terms of tree data, which are still subject to gauge transformations at the (arbitrarily chosen) tree root. Finally, this gauge freedom has to be removed. This reduction procedure has been discussed in detail in \([2]\). The second step leads to delicate problems (Gribov problem and the occurrence of nongeneric strata), which suggest that one should investigate the stratified structure of the underlying classical configuration (resp. phase) space in more detail. See \([7]\) for first results.

As a result of this reduction, the observable algebra \( \mathfrak{O}_\Lambda \) is obtained as

\[
\mathfrak{O}_\Lambda = \mathfrak{O}_\Lambda^{glu} \otimes \mathfrak{O}_\Lambda^{mat} \otimes \mathfrak{O}_\Lambda^b \otimes \mathfrak{O}_\Lambda^\infty. \quad (4.7)
\]

Here, the gluonic component \( \mathfrak{O}_\Lambda^{glu} \) is generated by reduced gluonic tree data \((u_i, e_i), i = 0, \ldots, K-2\), with \( K \) denoting the number of off-tree lattice links. These bosonic generators satisfy the generalized canonical commutation relations over \( G \):

\[
\begin{align*}
[e^r_{i,s}, e^p_{j,q}] &= \delta_{ij} (\delta^p_s e^r_{i,q} - \delta^r_s e^p_{i,q}) , \quad (4.8) \\
[e^r_{i,s}, u^p_{j,q}] &= \delta_{ij} (\delta^p_s u^r_{i,q} - \frac{1}{3} \delta^r_s u^p_{i,q}) , \quad (4.9) \\
[u^r_{i,s}, u^p_{j,q}] &= 0 , \quad (4.10)
\end{align*}
\]

with \( r, s, \cdots = 1, 2, 3 \). The generators \((u_i, e_i)\) are subject to a certain discrete symmetry described in \([2]\), (which, however, is not a remainder of the gauge symmetry).

Applying the above gauge fixing procedure to the fermionic matter field, we obtain fermionic operators \( \phi_k \). Here, \( k = (a, r, x) \) is a multi-index running from 1 to \( 12N \), with \( N \) being the number of lattice sites. These quantities fulfil the canonical anti-commutation relations

\[
[\phi^* k, \phi_l]_+ = \delta^k_l . \quad (4.11)
\]
(We stress that in [2], the generators $\phi_k$ were denoted by $a_k$.) Again, an additional discrete symmetry arises, because the gauge-fixing is defined only up to the stabilizer $Z_3$ of the generic stratum of the underlying classical configuration space. Thus, strictly speaking, the generators $\phi_k$ are not observables, whereas the bosonic quantities $u$ and $e$ are, because they are not affected by this ambiguity. It is clear from classical invariant theory that only the following combinations of $\phi^*k$ and $\phi_k$ (together with functions built from them) are observables:

\begin{align*}
  j^k_l &= \phi^*k \phi_l, \quad (4.12) \\
  i^k_l &= \phi_l \phi^*k, \quad (4.13) \\
  w_{pqr} &= \phi_p \phi_q \phi_r, \quad (4.14) \\
  w^{*ijk} &= \phi^*k \phi^*j \phi^*i. \quad (4.15)
\end{align*}

We have, of course,

\begin{align*}
  i^k_l = \delta^k_l 1 - j^k_l. \quad (4.16)
\end{align*}

Thus, the matter field component $O^\text{mat}_T$ is generated by the set \{\(j, w, w^*\)\}, together with the unit element 1. These generators are observables of hadronic type. Thus, in what follows we call $O_T^\text{mat}$ hadronic component of the observable algebra, or simply hadronic subalgebra.

Finally, $O^b_\Lambda$ is generated by (gauge invariant) color electric boundary fluxes and the generators of $O^\infty_\Lambda$ have been already given above.

By the uniqueness theorem for generalized CCR, fulfilled by generators \((u, e)\), of the gluonic subalgebra $O^\text{glu}_T$, the problem of classifying irreducible representations of $O_\Lambda$ is reduced to classifying irreducible representations of the hadronic subalgebra $O_T^\text{mat}$. For that purpose, the structure of this algebra will be investigated in the sequel.

\section{5 Structure of the Hadronic Subalgebra}

\subsection{5.1 Generators and Relations}

We start analyzing $O_T^\text{mat}$ by listing relations implied from definitions \((4.12) - (4.15)\).

First, note that $O_T^\text{mat}$ is a unital $*$-algebra, with unit element 1 and $*$-operation given by

\begin{align*}
  (j^k_l)^* &= j^l_k, \quad (5.1) \\
  (w_{pqr})^* &= w^{*qp}, \quad (5.2) \\
  (w^{*ijk})^* &= w_{ijk}, \quad (5.3)
\end{align*}

where the generators $w$ and $w^*$ are totally antisymmetric in their indices:

\begin{align*}
  w_{mkn} = w_{knm} = w_{nmk} = -w_{kmn}. \quad (5.4)
\end{align*}
Next, the anticommutation relations \((4.11)\) immediately yield:

\[
\left[ j^k_l, j^m_n \right] = \delta^m_l j^k_n - \delta^k_n j^m_l , \tag{5.5}
\]

\[
\left[ j^i_k, w_{lmn} \right] = -\delta^i_l w_{kmn} - \delta^i_m w_{lkn} - \delta^i_n w_{lmk} , \tag{5.6}
\]

and, consequently,

\[
\left[ j^k_i, w^*_{lmn} \right] = \delta^i_l w^*_{kmn} + \delta^m_i w^*_{lkn} + \delta^n_i w^*_{lmk} . \tag{5.7}
\]

Observe that (5.5) are the commutation relations of \(gl(n, \mathbb{C})\), with \(n = 12N\). As a direct consequence of \((4.12)\), these generators fulfill a number of additional quadratic relations:

First, the diagonal generators \(j^k_k\) are idempotent

\[
\left( j^k_k \right)^2 = j^k_k . \tag{5.8}
\]

Because of the Hermiticity condition (5.1) they are, thus, projectors. Commutation relations (5.5) implies that they all commute with each other.

Finally, products of \(w\) and \(w^*\) can be expressed in terms of \(j\)’s:

\[
w^*_{k^m n} w_{k^m n} = j^k_m j^m_n , \quad \text{for different } k, m, n , \tag{5.9}
\]

\[
w_{k^m n} w^*_{k^m n} = i^k_m i^m_n , \quad \text{for different } k, m, n . \tag{5.10}
\]

We recall that

\[
i^k_k = 1 - j^k_k , \tag{5.11}
\]

see (4.16). In the next subsection, we are going to prove that the above properties uniquely characterize the algebra \(O_{mat}\).

We show that the triality operator belongs to the center of the algebra. For this purpose, note that \(j^k_k\) is the particle number operator at position \(k\). Thus,

\[
n = \sum_{k=1}^n j^k_k \tag{5.12}
\]

is the total particle number operator. By definition (3.15) of \(t_\Lambda\) we have

\[
t_\Lambda = \varphi(n) . \tag{5.13}
\]

This means that, in any representation, \(t_\Lambda\) is equal to the particle number, modulo 3. As a direct consequence of (5.5), (5.6) and (5.7), we obtain

\[
\left[ n, j^k_l \right] = 0 \tag{5.14}
\]

\[
\left[ n, w^{*ijk} \right] = 3 w^{*ijk} , \tag{5.15}
\]

\[
\left[ n, w_{pqr} \right] = -3 w_{pqr} . \tag{5.16}
\]
Together with (5.13), these relations imply that all generators and, thus, all hadronic observables commute with the triality operator $t_\Lambda$.

**Remark:**

Due to (5.6), the whole set of baryonic invariants $w_{lmn}$ can be generated from one chosen $w_{lmn0}$ by successively taking commutators with $j$’s. Indeed, for $i \neq l$, commutation relation (5.6) reduces to the identity $[j^l_k, w_{lmn}] = -w_{kmn}$ which enables us to “flip” the multi-index $(l, m, n)$ to any other position. All relations for the remaining $w$’s then follow from the relations for the single selected element $w_{lmn0}$.

### 5.2 Axiomatic Description

Consider now the abstract, unital $*$-algebra $A$, generated by abstract elements $j$, $w$ and $w^*$, which fulfil relations (5.1) – (5.10). In the sequel, we shall prove that these relations define the algebra uniquely, i.e. $A$ is identical with the previously defined algebra $O_{T^{mat}}$. For this purpose, we derive from the defining relations of $A$ a number of additional identities.

**Theorem 5.1.** The defining relations (5.1) – (5.10) of $A$ imply the following additional identities:

1. 
   
   $j^m_l j^k_l = \delta^k_l j^m_l$, \hspace{1cm} (5.17)
   
   $j^k_l j^k_n = \delta^{k_l} j^k_n$. \hspace{1cm} (5.18)

   Thus, in particular, the off-diagonal generators are nilpotent:

   \((j^k_l)^2 = 0 \text{ for } k \neq l\). \hspace{1cm} (5.19)

2. 

   \begin{align*}
   j^k_l w_{lmn} &= -j^k_l w_{kmn} = -j^i_m w_{lkn} = -j^i_n w_{lmk}, \\
   w^*_{lmn} j^k_i &= -w^*_{kmn} j^k_i = -w^*_{lkn} j^m_l = -w^*_{lmk} j^n_l.
   \end{align*}

   In particular, $j^k_l$ multiplied with $w$ vanishes, if at least one of the indices of $w$ coincides with $k$, e.g.

   \(j^k_l w_{lmn} = 0\), \hspace{1cm} w^*_{lmn} j^l_k = 0. \hspace{1cm} (5.22)

3. The following identities hold:

   \begin{align*}
   j^i_l w_{lmn} &= 0 = w_{lmm} j^i_l, \hspace{1cm} (5.23) \\
   i^i_l w_{lmn} &= w_{lmm} = w_{lmm} j^i_l, \hspace{1cm} (5.24) \\
   w^*_{lmn} j^l_i &= 0 = i^i_l w^*_{lmn}, \hspace{1cm} (5.25) \\
   w^*_{lmn} i^i_l &= w^*_{lmm} = i^i_l w^*_{lmn}. \hspace{1cm} (5.26)
   \end{align*}
4. The generators \( \mathbf{w} \) and \( \mathbf{w}^* \) are nilpotent:

\[
(\mathbf{w}^{*ijk})^2 = 0 , \quad (\mathbf{w}_{pqr})^2 = 0 .
\]

Proof:

1. To show relations (5.17), we have to prove

\[
j^m j^k = 0 , \quad \text{for} \quad k \neq l ,
\]

\[
j^m j^l = j^m .
\]

From (5.5) we get

\[
j^k = [j^k, j^l] = j^k j^l - j^l j^k .
\]

Multiplying this relation by \( j^l \) to the left and using (5.8) yields

\[
j^l j^k = j^l j^k j^l - j^l j^l ,
\]

or, inserting expression (5.31) for \( j^k \),

\[
2j^l j^k = j^l j^k j^l = j^l (j^k j^l - j^l j^k) j^l = 0 .
\]

This proves (5.29) for the case \( m = l \). Consequently, (5.31) reduces to

\[
j^k = j^k j^l .
\]

This proves relation (5.30). Finally, multiplying (5.30) by \( j^k \) to the left and using \( j^l j^k = 0 \) yields (5.29) for \( m \neq l \). The proof of relations (5.18) is completely analogous and, therefore, we omit it here.

2. We show the first relation in (5.20),

\[
j^k \mathbf{w}_{lmn} = -j^l \mathbf{w}_{kmn} .
\]

First, the commutation relations (5.6) imply:

\[
j^l \mathbf{w}_{lmn} - \mathbf{w}_{lmn} j^l = -\mathbf{w}_{lmn} .
\]

Multiplying this equation from both sides by \( j^l \) we obtain

\[
j^l \mathbf{w}_{lmn} j^l = 0 .
\]

Hence, multiplying (5.33) from the left by \( j^l \) yields

\[
j^l \mathbf{w}_{lmn} = -j^l \mathbf{w}_{lmn} ,
\]
or \( j^I_l w_{lmn} = 0 \). Multiplying this relation to the left by \( j^k_I \) and using (5.34) yields

\[
j^k_I w_{lmn} = 0 ,
\]

showing the special case (5.22). Now, multiplying the commutation relations (5.6) to the left by \( j^I_l \) and using (5.18) together with (5.34) gives \( \delta^i_I j^j_I w_{lmn} = -\delta^i_I j^j_I w_{kmn} \) or, equivalently,

\[
j^j_I w_{kmn} = -j^j_I w_{kmn} .
\]

Finally, multiplying this equation by \( j^j_I \) and using (5.6), (5.34) and (5.30) yields the proof of the statement. The proof of the remaining equations contained in (5.20) is identical.

3. First, observe that by (5.34), the auxiliary identity (5.33) reduces to

\[
-w_{lmn} j^I_l = -w_{lmn} ,
\]

and, hence, we have \( w_{lmn} j^I_l = 0 \). This way (5.23) and (5.24) are proved. Acting with the operator * on both sides we obtain the remaining identities (5.25) and (5.26).

4. Identity (5.23) and (5.24) imply nilpotency of \( w \):

\[
w_{lmn} w_{lmn} = w_{lmn} (j^I_l w_{lmn}) = (w_{lmn} j^I_l) w_{lmn} = 0 .
\]

Similarly, nilpotency of \( w^* \) follows from the remaining two identities.

Finally, observe that the idempotency and nilpotency properties (5.8) and (5.19) render \( \mathcal{A} \) finite–dimensional. To summarize, \( \mathcal{A} \) is a (finite-dimensional) associative unital *-algebra. It is obtained from the free algebra, generated by elements \( \{ j, w, w^*, 1 \} \), by factorizing with respect to the relations listed above.

For the sake of completeness, we have listed additional interesting identities, see Appendix [A] which have to be taken into account, if one wants to build arbitrary monomials in the generators.

### 5.3 Irreducible Representations

**Lemma 5.2.** There is at least one nontrivial, faithful irreducible representation of \( \mathcal{A} \), for each eigenvalue \(-1, 0, 1\) of the triality operator \( t_\Lambda \).

**Proof:** Take the CAR–algebra \( \mathcal{C} \) given by

\[
\{ a^{*k}, a_k \mid k = 1, 2, \ldots, n \} ,
\]

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and fulfilling canonical anticommutation relations,

\[ [a^*_k, a_l]_+ = \delta^k_l. \] (5.38)

Denote its unique Hilbert representation space by \( H \) and define

\[ j^k_l = a^*_k a_l, \] (5.39)
\[ w_{pqr} = a_p a_q a_r, \] (5.40)
\[ w^{*ijk} = a^*_k a^*_j a^*_i. \] (5.41)

These abstract elements fulfill, of course, all relations of \( \mathcal{A} \) listed above. Thus, \( H \) carries a representation of \( \mathcal{A} \). Since \( t_\Lambda \) commutes with all \( j \)'s, \( w \)'s and \( w^* \)'s, \( H \) decomposes into superselection sectors,

\[ H = H_{-1} \oplus H_0 \oplus H_1, \]

corresponding to different eigenvalues of \( t_\Lambda \). Each of these subspaces is invariant under the action of \( j \)'s, \( w \)'s and \( w^* \)'s, providing a nontrivial, faithful and irreducible representation of \( \mathcal{A} \).

**Theorem 5.3.** Any irreducible, nontrivial representation of \( \mathcal{A} \) is equivalent to one of the three irreducible representations provided by Lemma 5.2.

For purposes of the proof, let us denote:

\[ E_{\nu_1, \nu_2, \ldots, \nu_n} := (j^1_1)^{\nu_1} (i^1_1)^{\nu_1+1} (j^2_2)^{\nu_2} (i^2_2)^{\nu_2+1} \cdots (j^n_n)^{\nu_n} (i^n_n)^{\nu_n+1}, \] (5.42)

where all indices \( \nu_k \) assume values 0 or 1 and the summation is meant modulo 2. Since the \( i \)'s and \( j \)'s are Hermitean, orthogonal and commuting projectors, \( \{E_{\nu_1, \nu_2, \ldots, \nu_n}\} \) is a family of Hermitean orthogonal and commuting projectors, too. Moreover, we have an obvious

**Corollary 5.4.** The above projectors sum up to the unit element:

\[ \bigoplus E_{\nu_1, \nu_2, \ldots, \nu_n} = 1. \] (5.43)

**Lemma 5.5.** The following relations hold for arbitrary \( k \neq l \):

1. \( j^k_l \cdot E_{\nu_1, \nu_2, \ldots, \nu_n} = 0 \) unless \( \nu_k = 0 \) and \( \nu_l = 1 \).
2. \( E_{\nu_1, \nu_2, \ldots, \nu_n} \cdot j^k_l = 0 \) unless \( \nu_k = 1 \) and \( \nu_l = 0 \).
3. For \( \nu_k = 0 \) and \( \nu_l = 1 \) we have \( j^k_l \cdot E_{\nu_1, \nu_k, \nu_2, \ldots, \nu_n} = E_{\nu_1, \nu_k+1, \nu_2, \ldots, \nu_n} \cdot j^k_l \).
Proof: The proof follows by direct inspection from the following identities, (which are all simple consequences of (5.17) and (5.18)):

\[ j^k j^l k^k = 0 , \quad j^k j^l i^k = j^l , \quad j^k i^l j^k = 0 , \quad j^k i^l i^k = 0 \]

and

\[ j^l i^k k^l = 0 , \quad i^l i^k j^l = 0 , \quad i^l i^k i^l = j^l , \quad i^l i^k i^l = 0 . \]

Proof of the theorem: Take such an irreducible representation. Since the triality operator \( t_\Lambda \) lies in the center of \( \mathcal{A} \), it corresponds to a fixed value of triality. Take any other two irreducible representations, corresponding to the remaining values of triality. Let us denote these three representations by \( \mathcal{H}_t \), with \( t = -1, 0, 1 \). We are going to prove that there exist isomorphisms

\[ U_t : \mathcal{H}_t \to \mathcal{H}_t \]  

(5.44)

intertwining the representations \( \mathcal{H}_t \) with the three CAR-representations \( \mathcal{H}_t \), defined in Lemma 5.2. This will be accomplished by defining operators

\[ c^*: \mathcal{H}_t \to \mathcal{H}_{t+1} \]

and

\[ c_1 : \mathcal{H}_t \to \mathcal{H}_{t-1} \]

(with summation modulo 3), fulfilling the CAR and such that equations (5.39) – (5.41) are satisfied with \( a \)'s replaced by \( c \)'s. Then, the statement of the theorem is a consequence of the classical uniqueness theorem for CAR-representations.

Let us denote

\[ \mathcal{H}_{\nu_1, \nu_2, \ldots, \nu_n} := E_{\nu_1, \nu_2, \ldots, \nu_n} \mathcal{H}_t . \]

(5.45)

We obviously have

\[ t_\Lambda E_{\nu_1, \nu_2, \ldots, \nu_n} = E_{\nu_1, \nu_2, \ldots, \nu_n} t_\Lambda = t E_{\nu_1, \nu_2, \ldots, \nu_n} . \]

On the other hand, formula (5.13) implies

\[ t_\Lambda E_{\nu_1, \nu_2, \ldots, \nu_n} = \left( \sum_{i=1}^n \nu_i \text{ mod } 3 \right) E_{\nu_1, \nu_2, \ldots, \nu_n} . \]

Thus, the only non-trivial subspaces are those fulfilling the condition

\[ t = \sum_{i=1}^n \nu_i \text{ mod } 3 . \]

(5.46)

This fact, together with (5.38), implies

\[ \mathcal{H}_t = \bigoplus_{t = \sum_{i=1}^n \nu_i \text{ mod } 3} \mathcal{H}_{\nu_1, \nu_2, \ldots, \nu_n} . \]

(5.47)
Now, Lemma 5.5 implies that \( j^k \mathcal{H}_{\nu_1, \nu_2, \ldots, \nu_n} = 0 \), unless \( \nu_k = 0 \) and \( \nu_l = 1 \) and that, in the latter case, \( j^k \) maps \( \mathcal{H}_{\nu_1, \ldots, \nu_k, \ldots, \nu_n} \) onto \( \mathcal{H}_{\nu_1, \ldots, \nu_k+1, \ldots, \nu_n-1, \ldots, \nu_n} \). Observe, that \( j^k \) is an isomorphism of these two Hilbert spaces, with the inverse given by \( j^l \). Indeed, we have:

\[
(j^k)^* j^k = j^l j^k = j^l - j^k + j^l j^k.
\]

By Lemma 5.5 this gives, for \( \nu_k = 0 \) and \( \nu_l = 1 \),

\[
(j^k)^* j^k E_{\nu_1, \ldots, \nu_k, \ldots, \nu_n} = E_{\nu_1, \ldots, \nu_{k+1}, \ldots, \nu_n}.
\]

Similarly, relations 5.23 imply that \( \mathbf{w}_{lmn} \mathcal{H}_{\nu_1, \ldots, \nu_m, \ldots, \nu_n} = 0 \), unless \( \nu_l = \nu_m = \nu_n = 1 \) and, in the latter case, it maps \( \mathcal{H}_{\ldots, \nu_l, \ldots, \nu_m, \ldots, \nu_n} \) onto \( \mathcal{H}_{\ldots, \nu_l-1, \ldots, \nu_m-1, \ldots, \nu_n-1, \ldots} \). Observe that, according to (5.9) and (5.10), \( \mathbf{w}_{lmn} \) is an isomorphism of these two Hilbert spaces, with the inverse given by \( \mathbf{w}^{lmn} \).

Since the representations \( \mathcal{H}_t \) are non-trivial, there is at least one non-vanishing vector in at least one of the subspaces \( \mathcal{H}_{\nu_1, \ldots, \nu_n} \), for every \( \mathcal{H}_t \). Thus, let us choose three such normalized vectors and denote them by

\[
|\nu_1(t), \nu_2(t) \ldots \nu_n(t) > \in \mathcal{H}_{\nu_1(t), \nu_2(t) \ldots \nu_n(t)}.
\]

where \( \sum_{i=1}^n \nu_i(t) \mod 3 = t, t = -1, 0, 1 \). Acting with operators \( j^k \), \( \mathbf{w}_{lmn} \) and \( \mathbf{w}^{lmn} \) on each of these three vectors, we obtain, for every \( t \), a normalized vector, say \( |\nu_1, \nu_2 \ldots \nu_n > \), in each of the subspaces \( \mathcal{H}_{\ldots, \nu_l, \ldots, \nu_m, \ldots, \nu_n} \). (The information about \( t \) is encoded implicitly, see equation (5.49).) Moreover, we can label the vectors in the representation spaces \( \mathcal{H}_{\nu_1, \nu_2, \ldots, \nu_n} \) in such a way that the following relations are fulfilled:

\[
\begin{align*}
\begin{cases}
  j^k |\nu_k, \ldots, \nu_l, \ldots, \nu_n > = \begin{cases}
    \sigma(k, l) \cdot |\nu_k + 1, \ldots, \nu_l - 1, \ldots > & \text{if } \nu_k = 0, \nu_l = 1, \\
    0 & \text{otherwise},
  \end{cases} \\
  \mathbf{w}_{lmn} |\nu_l, \ldots, \nu_m, \ldots, \nu_n > = \begin{cases}
    \sigma(l, m, n) \cdot |\nu_l - 1, \ldots, \nu_m - 1, \ldots, \nu_n - 1, \ldots > & \text{if } \nu_l = \nu_m = \nu_n = 1, \\
    0 & \text{otherwise},
  \end{cases} \\
  \mathbf{w}^{lmn} |\nu_l, \ldots, \nu_m, \ldots, \nu_n > = \begin{cases}
    -\sigma(l, m, n) \cdot |\nu_l + 1, \ldots, \nu_m + 1, \ldots, \nu_n + 1, \ldots > & \text{if } \nu_l = \nu_m = \nu_n = 0, \\
    0 & \text{otherwise},
  \end{cases}
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
\sigma(k, l) &= (-1)^{s(k) - s(l)}, \\
\sigma(l, m, n) &= s(l, m, n)(-1)^{s(l) + s(m) + s(n)},
\end{align*}
\]

\[
\begin{align*}
s(k) &= \sum_{i<k} \nu_i, \\
\sum_{i<k} \nu_i.
\end{align*}
\]
and \( s(l, m, n) \) is the sign of the permutation which is necessary to sort the triple \((l, m, n)\) in growing order (i.e. \( s(l, m, n) = 1 \) if \( l < m < n \), \( s(l, m, n) = -1 \) if \( l < n < m \) etc.).

We show that this labelling is possible, indeed: We start with three arbitrarily chosen vectors given by sequences of \( \nu_i = \nu_i(t) \), for \( t = -1, 0, 1 \). Next, we apply operators \( j^k_l \), \( w_{lmn} \) and \( w^{*lmn} \) to these vectors and use the above formulae as the definition of the corresponding vectors on the right hand side. Now, it remains to prove that this definition does not depend upon the order of these operations. For this purpose, we use the commutation rules (5.5) and (5.6). As far as the commutation relations \([w, w]\), \([w^*, w]\) and \([w^*, w^*]\) are concerned, we can use relations (5.21) – (5.22) to flip the indices of occurring \( w \)'s and \( w^* \)'s in such a way that, whenever these objects meet, they have always the same indices. Then, we use relations (5.9) and (5.10) together with nilpotency properties (5.27) and (5.28). Having done this, the formula may be checked by inspection.

Because of the irreducibility of the representations \( \mathcal{H}_t \), the vectors \( |\nu_1(t), \nu_2(t) \ldots \nu_n(t) > \) form (orthogonal) bases in each \( \mathcal{H}_t \). Hence, we define the intertwining operator \( U \) putting:

\[
U|\nu_1, \nu_2 \ldots \nu_n > := (a^*)^\nu_1 (a^2)^{\nu_2} \cdots (a^{*n})^{\nu_n} |0 > ,
\]

where \( a^* \)'s are the CAR-creation operators from Lemma 5.2 and \( |0 > \in H \) is the Fock vacuum. (The label \( t \) has been omitted.) Then, the operators

\[
c^* := U^{-1} a^* U
\]

and

\[
c := U^{-1} a U
\]

satisfy the CAR. It is easy to check that they fulfill equations (5.39) – (5.41), with \( a \) replaced by \( c \). This ends the proof.

This theorem shows that any algebra \( \mathcal{A} \) generated by abstract elements \( j, w \) and \( w^* \), fulfilling relations (5.1) – (5.10), is isomorphic to the commutant of the triality operator

\[
t = \varphi \left( \sum_k a^{*k} a_k \right)
\]

in \( \mathcal{C} \),

\[
\mathcal{A} \cong t'(\mathcal{C}) \subset \mathcal{C} .
\]

This implies the following

**Corollary 5.6.** The algebras \( \mathcal{A} \) and \( \mathcal{O}^{\text{mat}}_T \) are isomorphic.

### 5.4 Super Lie Structure

Formula (5.49) provides us with a simple and nice algebraic characterization of \( \mathcal{O}^{\text{mat}}_T \). Nonetheless, since in the case of lattice QED, we have found a Lie algebraic characterization of the matter field part \[10, 11\], it is worthwhile to ask, whether a similar characterization is possible in QCD as well. The answer is affirmative, as we show now.
Using an idea of Palev [26], see also Dondi and Jarvis [27], we define the following operators:

\[ b^k := \phi^k \sqrt{p-n}, \]  
\[ b_k := \sqrt{p-n} \phi_k, \]  
with \( p \) being a positive integer. In what follows we use the following obvious formulae:

\[ \phi_k f(n) = f(n+1) \phi_k, \]  
\[ \phi^k f(n) = f(n-1) \phi^k, \]  
for any operator function \( f \).

In terms of the \( b \)-operators, the (anti-)commutation relations take the following form:

\[ [j^k_l, j^m_n] = \delta^m_l j^k_n - \delta^k_n j^m_l, \]  
\[ [j^k_l, b^k_i] = \delta^i_l b^k_j, \]  
\[ [j^k_l, b_i] = -\delta^k_l b_i, \]  
\[ [b^k, b]_+ = (p-n) \delta^k_l + j^k_l. \]  

This shows that

\[ \mathfrak{A} := lin.env. \{ b^k, b_k, j^k_l \ | \ k, l = 1, 2, \ldots, n \} \]  
is isomorphic to the Lie superalgebra \( \text{sl}(1/n) \). In more detail, identifying

\[ e^k_l = j^k_l - \frac{1}{N} \delta^k_l n, \quad e^0_0 = \frac{N}{N-1} p - n, \quad e^k_0 = b^k, \quad e^0_k = b_k, \]  
we obtain the standard (anti-)commutation relations for \( \text{sl}(1/n) \):

\[ [e^k_l, e^m_n] = \delta^m_l e^k_n - \delta^k_n e^m_l, \]  
\[ [e^i_0, e^i_0] = -e^i_0, \]  
\[ [e^0_0, e^i_0] = e^0_i, \]  
\[ [e^k_l, e^k_0] = \delta^k_l e^k_0 - \frac{1}{N} \delta^k_l e^0_0, \]  
\[ [e^k_l, e^0_i] = -\delta^k_i e^0_l + \frac{1}{N} \delta^k_l e^0_i, \]  
\[ [e^k_0, e^0_l]_+ = e^k_l + \frac{N-1}{N} \delta^k_t e^0_0. \]  

The even part is isomorphic to \( \text{gl}(n, \mathbb{C}) \),

\[ \text{sl}(1/n)_0 = \text{gl}(n, \mathbb{C}) = lin.env. \{ e^k_l, e^0_0 \ | \ k, l = 1, 2, \ldots, n \}, \]  
and the odd part is given by

\[ \text{sl}(1/n)_1 = lin.env. \{ e^0_0, e^k_l \ | \ k = 1, 2, \ldots, n \}. \]
Next, observe that
\[ b_i b_j b_k = \sqrt{F(n)} w_{ijk} \, , \tag{5.60} \]
with
\[ F(n) = (p - n) (p - 1 - n) (p - 2 - n) \, . \]
From now on we assume
\[ p = n + 3 \, . \]
Then \( F(n) \) is a positive operator in every representation. Thus, in every representation we can express the baryonic invariants \( w \) in terms of the fermionic operators \( b \):
\[ w_{ijk} = F(n)^{-\frac{1}{2}} b_i b_j b_k \, . \tag{5.61} \]
We denote
\[ \tilde{w}_{ijk} = b_i b_j b_k \, , \tag{5.62} \]
and
\[ \tilde{w}^{*ijk} = b^* k b^* j b^* i \, . \tag{5.63} \]
For the bosonic part, we implement relations (5.1) and (5.8)
\[ (j^k_i)^* = j^i_k \, , \tag{5.64} \]
\[ (j^k_k)^2 = j^k_k \, . \tag{5.65} \]
see Subsection 5.1. These relations define a Lie ideal \( \mathcal{I} \) in the enveloping algebra
\[ \mathcal{U}(gl(n, \mathbb{C})) \subset \mathcal{U}(sl(1/n)) \, , \]
by which we factorize. Moreover, we implement that every observable has to commute with the triality operator. Thus, we have to take the commutant of \( t \) in this factor algebra, which we denote by
\[ \mathcal{L} := t'(\mathcal{U}(sl(1/n)) / \mathcal{I}) \, . \tag{5.66} \]

**Theorem 5.7.** The associative unital *-algebras \( \mathcal{A} \) and \( \mathcal{L} \) are isomorphic.

**Proof:**
First, observe that the operations of taking the commutant and of factorizing with respect to \( \mathcal{I} \) commute, because \( t \) commutes with every \( j^k_k \).
To prove the above isomorphism, we show that \( \mathcal{A} \) and \( \mathcal{L} \) have exactly the same irreducible representations. For that purpose, recall that \( sl(1/n) \) is a basic Lie superalgebra of type I, which means
\[ sl(1/n)_\perp = sl(1/n)_{-1} \oplus sl(1/n)_{+1} \, , \tag{5.67} \]
with \( sl(1/n)_{-1} \) and \( sl(1/n)_{+1} \) being two irreducible modules of \( sl(1/n)\perp \cong gl(n, \mathbb{C}) \), in terms of our generators spanned by \( \{ b_k \} \) and \( \{ b^* k \} \) respectively. It follows from general
representation theory, see [28], that any finite dimensional irreducible representation of a basic Lie superalgebra $\mathfrak{g}$ is obtained from a Kac module. For superalgebras of type I, every Kac module $V(\lambda)$ is induced from a highest weight module $V_0(\lambda)$ of the even part $\mathfrak{g}_0$:

$$V(\lambda) = \text{Ind}_{\mathfrak{k}}^{\mathfrak{g}} V_0(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} V_0(\lambda),$$

(5.68)

where

$$\mathfrak{k} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

(5.69)

and $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k})$ denote the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$ respectively. Formula (5.68) has to be understood as follows: The $\mathfrak{g}_0$-module $V_0(\lambda)$ has been extended to a $\mathfrak{k}$-module by putting

$$\mathfrak{g}_1 V_0(\lambda) = 0$$

and one has to identify elements

$$k \otimes v = 1 \otimes k(v),$$

for $k \in \mathfrak{k}$ and $v \in V_0(\lambda)$. Then the induced representation of $\mathfrak{g}$ is defined by

$$g(u \otimes v) := gu \otimes v,$$

(5.70)

for $g \in \mathfrak{g}$, $u \in \mathcal{U}(\mathfrak{g})$ and $v \in V_0(\lambda)$. We stress that $V(\lambda)$ is not always simple. In that case, one has to factorize by a certain maximal submodule, to obtain an irreducible representation.

Now, let $V_0(\lambda)$ be a highest weight module of $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{C})$. Since in our case $[\mathfrak{g}_-, \mathfrak{g}_-]_+ = 0$, we have

$$V(\lambda) \cong \Lambda (\mathfrak{g}_-) \otimes V_0(\lambda),$$

(5.71)

with

$$\Lambda (\mathfrak{g}_-) = \bigoplus_{k=0}^{n} \Lambda^k (\mathfrak{g}_-)$$

denoting the exterior algebra of $\mathfrak{g}_1$. Thus, in terms of generators, we have

$$V(\lambda) \cong \bigoplus_{1<k_1<\cdots<k_n \leq n} b_{k_1} \ldots b_{k_n} V_0(\lambda).$$

(5.72)

We show that taking the above commutant and factorizing with respect to $\mathfrak{z}$ reduces the set of irreducible representations to three inequivalent representations labelled by triality.

First, in the commutant of $\mathfrak{t}$, only monomials in $\mathfrak{b}$ and $\mathfrak{b}^*$ built from $\mathfrak{w}$ and $\mathfrak{w}^*$ can occur. Thus, $V(\lambda)$ takes the form:

$$V(\lambda) \cong \bigoplus_{1<i_1<j_1<k_1<\cdots<i_n<j_n<k_n \leq n} \mathfrak{w}_{i_1j_1k_1} \ldots \mathfrak{w}_{i_nj_nk_n} V_0(\lambda).$$

(5.73)
Since the $\tilde{w}$'s act transitively on this direct sum, $V(\lambda)$ is an irreducible module. Moreover, as a direct consequence of the commutation relations we have

$$[\mathbf{n}, j^k_l] = 0,$$

$$[\mathbf{n}, \tilde{w}^{ijk}] = 3 \tilde{w}^{ijk},$$

$$[\mathbf{n}, \tilde{w}_{pq}] = -3 \tilde{w}_{pq}. \quad (5.74, 5.75, 5.76)$$

Thus, in any representation, $\tilde{w}_{pq}$ lowers the particle number by 3, whereas $\tilde{w}^{ijk}$ raises it by 3.

Next, by (5.65), the particle number operator $j^k_k$ at position $k$ can take only eigenvalues 0 and 1, on any highest weight module $V_0(\lambda)$ of $\mathfrak{gl}(n, \mathbb{C})$. Every highest weight module of $\mathfrak{gl}(n, \mathbb{C})$ is built – by taking tensor products – from fundamental representations, which in turn are all isomorphic to some exterior product $\Lambda^l(\mathbb{C}^n)$. But, whenever we take a tensor product of such exterior products, which is not antisymmetric, there exists a vector, for which $j^k_k$ has an eigenvalues greater than one. Thus, (5.65) reduces the admissible highest weight modules to the fundamental ones. Since the operators $\tilde{w}$ lower the particle number by 3, the lowest weight component of (5.73) can have particle numbers 0, 1 or 2, only. Using the canonical basis of $\mathbb{C}^n$, an explicit isomorphism intertwining these 3 representations with the representations $H_t$ can be written down, as in the proof of Theorem 5.3.

\section*{6 Discussion}

1. The generators $J_{\gamma}^{ab}(x, y)$ and $W_{\alpha\beta\gamma}^{abc}(x, y, z)$ (see (4.5) and (4.6)) are difficult to handle. This is why we have replaced them by generators $j^k_l$ and $w_{pq}$ (see (4.12) and (4.14)), fulfilling much simpler relations. To define these observables we have used a gauge fixing procedure based upon the choice of a tree. However, it is obvious that the specific gauge we have chosen is irrelevant for the structure of the algebra, defined by relations (5.1) – (5.10). Changing the gauge condition does not affect these relations. Thus, there should exist another, more intrinsic procedure for obtaining this algebra, which does not rely on gauge fixing.

To make this transparent, assume that we have chosen a tree. Now, instead of fixing the gauge, we rewrite the operators $J$ and $W$ in terms of fermionic operators parallel-transported to the tree root $x_0$. Denoting these transported operators by $\tilde{\psi}$, we get:

$$J_{\gamma}^{ab}(x, y) = \tilde{\psi}_{A}^{*a}(x_0)U_{A}^{B}B^{b}B^{b}(x_0), \quad (6.1)$$

with

$$\sigma = \beta \circ \gamma \circ \alpha^{-1}$$

being the closed path uniquely defined by this parallel transport, ($\alpha$ and $\beta$ are the unique on-tree paths from $x$ resp. $y$ to $x_0$.) Thus, the operators $J$ acquire a
labelling by (unparameterized) closed paths. Collecting the spinorial index and the point \( x \in \Lambda \) into a single index \( u = (a, x) \), we get a mapping \( \sigma \mapsto J^u_v(\sigma) \). It can be easily checked that the commutation relations for the quantities \( J \) then take the following form:

\[
[J^u_v(\beta), J^w_t(\gamma)] = \delta^u_v J^w_t(\beta \circ \gamma) - \delta^w_t J^u_v(\gamma \circ \beta),
\]

where \( \circ \) denotes the natural multiplication in the group of (unparameterized) closed paths. Similarly, the baryonic operators \( W \) and \( W^* \) can be rewritten, acquiring a labelling by closed paths, \( \sigma \mapsto W^{uvw}(\sigma) \), \( \sigma \mapsto W^{*uvw}(\sigma) \). The anticommutation relations for \( W \) and \( W^* \) can be easily worked out, but we omit them here.

2. Now, let us restrict ourselves to on–tree paths in \( J \) and \( W \) only. It is clear from (6.1) that for them all on-tree parallel transporters \( U_\sigma \) are equal to 1, yielding quantities \( J^u_v, W^{uvw}, W^{*uvw} \). Thus the algebra labelled by closed paths descends to an algebra defined by the following (anti-)commutation relations:

\[
[J^u_v, J^w_t] = \delta^w_t J^u_v - \delta^u_v J^w_t,
\]

\[
\{W^{(uvw)}, W_{(pqr)}\} = \frac{1}{2}(\partial \cdot \beta)(uvw)_{(pqr)} + 2(\partial \cdot \beta)(uvw)_{(pqr)} - 6\delta^{(uvw)}_{(pqr)},
\]

\[
\{W^{*(uvw)}, W^{*(pqr)}\} = 0, \quad \{W_{(uvw)}, W_{(pqr)}\} = 0,
\]

\[
[J^s_t, W^{*(uvw)}] = \delta^s_t W^{*(uvw)} + \delta^s_t W^{*(usw)} + \delta^t_w W^{*(usv)},
\]

\[
[J^s_t, W_{(pqr)}] = -\delta^s_p W_{(tqr)} - \delta^s_q W_{(ptr)} - \delta^s_r W_{(pqt)},
\]

Here \( J \cdot \beta \cdot \beta \) and \( J \cdot \beta \cdot \beta \) are the appropriate totally symmetric combinations. Allowing for cyclic permutations on \( uvw \) and \( pqr \), \( J \cdot \beta \cdot \beta \) contains a total of 4 \( \times \) 9 = 36 terms, \( J \cdot \beta \cdot \beta \) contains 9 \( \times \) 2 = 18 terms, and \( \beta \cdot \beta \cdot \beta \) just 3 \( \times \) 2 = 6 terms:

\[
(J \cdot \beta)(uvw)_{(pqr)} = (J^u_p J^v_q + J^v_p J^u_q + J^u_q J^v_p + J^v_q J^u_p)\delta^{wq}_{pr} + \ldots,
\]

\[
(J \cdot \beta)(uvw)_{(pqr)} = J^u_p (\delta^v_q \delta^{wq} r + \delta^v_q \delta^{qw} r) + \ldots,
\]

\[
\delta^{(uvw)}_{(pqr)} = \delta^u_p (\delta^v_q \delta^{wq} r + \delta^v_q \delta^{qw} r) + \ldots.
\]

Obviously, equations (6.8) are the commutation relations of \( \text{gl}(4N, \mathbb{C}) \). The anticommutator (6.4) closes on a quadratic polynomial in the enveloping algebra of the even (Lie) subalgebra \( \text{gl}(4N, \mathbb{C}) \). Thus we have identified the \( W \) and \( W^* \) as odd generators of a supersymmetry algebra belonging to a class of ‘polynomial’ superalgebras. Such ‘nonlinear’ extensions of Lie algebras and superalgebras have been recognised in other contexts in recent literature. An initial investigation of them in the case of generalisations of \( \text{gl}(4N/1) \) (or more generally of type I Lie superalgebras) has been given in [3] (see also the related remarks in the appendix).

We stress that, again by formula (6.1), the full set of operators \( J \) and \( W \) can be reconstructed, knowing the generators \( J^u_v, W^{uvw}, W^{*uvw} \), together with the Wilson loops \( U_\sigma \).
3. Clearly, the quantities $J_{uv}$, $W_{uvw}$, $W^{*uvw}$ constructed under point 2 can be viewed as obtained from on-tree gauge fixing (putting the parallel transporter on every on-tree link equal to 1). If we remove the residual gauge freedom (at the root), we can pass to the quantities $j^k_l$, $w_{ijk}$, $w^{*pqr}$ used in this paper. In the case of a generic orbit we have proved (see [2]) that the representation $J_{uv}^{\beta}$ is “sufficiently non-degenerate”, and we may reduce it to $j^k_l$. Actually, this non-degeneracy follows from the non-degeneracy of the representation of the electric fluxes $E_\gamma(x, y)$ – see (4.4). We expect that there exist “degenerate” representations, related to non–generic orbits, which do not allow to extract the representation of the full $\text{gl}(n, \mathbb{C})$ Lie algebra. Indeed, the impossibility to fix the gauge completely on a non–generic orbit (having a non–trivial stabilizer) implies the impossibility of reconstructing the quantities $j^k_l$, because they are not invariant with respect to the stabilizer. In this case, we expect that the fermions $a_k$ carrying the representation (see formulae (5.37) – (5.40)) will be replaced by some “anyons”, satisfying (possibly) a different statistics. A consistent mathematical analysis of such representations of the observable algebra (if they do exist) together with their physical implications will be one of our next goals.

A  Additional Relations

Here, we list additional relations, also following from relations (5.1) – (5.10).

First, we have the following so-called characteristic identities:

\[ j^k_l j^l_m = (n + 1 - n) j^k_m, \quad (A.1) \]

(with the sum taken over all $l$.) Next, one can analyze arbitrary higher order monomials, built from $w$ and $w^*$. For that purpose, let us introduce the following tensor operators (totally antisymmetric in both upper and lower indices) built from $j$'s:

\[ X^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} sgn(\rho) sgn(\sigma) j^{i_{\rho_1} p_{\rho_1}} j^{i_{\rho_2} p_{\rho_2}} j^{i_{\rho_3} p_{\rho_3}}, \quad (A.2) \]
\[ Y^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} sgn(\rho) sgn(\sigma) j^{i_{\rho_1} p_{\rho_1}} j^{i_{\rho_2} p_{\rho_2}} \delta^{i_{\rho_3} p_{\rho_3}}, \quad (A.3) \]
\[ Z^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} sgn(\rho) sgn(\sigma) j^{i_{\rho_1} p_{\rho_1}} \delta^{i_{\rho_2} p_{\rho_2}} \delta^{i_{\rho_3} p_{\rho_3}}, \quad (A.4) \]
\[ D^{i_1 i_2 i_3}_{p_1 p_2 p_3} = \sum_{\rho, \sigma} sgn(\rho) sgn(\sigma) \delta^{i_{\rho_1} p_{\rho_1}} \delta^{i_{\rho_2} p_{\rho_2}} \delta^{i_{\rho_3} p_{\rho_3}}, \quad (A.5) \]

with sums running over all permutations $\rho$ and $\sigma$. Using (A.4) and (A.5), a lengthy but simple calculation yields:

\[ 36 w^{*ijk} w_{pqr} = X^{ijk}_{pqr} + 3 Y^{ijk}_{pqr} + 2 Z^{ijk}_{pqr}, \quad (A.6) \]
\[ 36 w_{pqr} w^{*ijk} = -X^{ijk}_{pqr} + 6 Y^{ijk}_{pqr} - 11 Z^{ijk}_{pqr} + 6 D^{ijk}_{pqr}. \quad (A.7) \]
Using these relations and keeping in mind the nilpotency properties, one can calculate arbitrary (even order) polynomials in \( w \) and \( w^* \) in terms of the above tensor operators. In particular, taking the sum of these two relations, we get the following anticommutator for the baryonic observables:

\[
[w^{*ijk}, w_{pqr}]_+ = \frac{1}{4} \left( Y^{ijk}_{pqr} - Z^{ijk}_{pqr} + \frac{2}{3} D^{ijk}_{pqr} \right).
\]  

(A.8)

In fact, these relations (A.8) together with (5.5), (5.6), (5.7) and the mutual anticommutativity of \( w \) and \( w^* \) can again be taken as the defining relations for a type of polynomial superalgebra generalising \( gl(12N/1) \), this time with odd generators of antisymmetric type (see [8] for details, and also the discussion above).

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