QUANTUM-$s\ell(2)$ ACTION ON A DIVIDED-POWER QUANTUM PLANE AT EVEN ROOTS OF UNITY

A.M. SEMIKHATOV

ABSTRACT. We describe a nonstandard version of the quantum plane, the one in the basis of divided powers at an even root of unity $q = e^{i\pi/p}$. It can be regarded as an extension of the “nearly commutative” algebra $\mathbb{C}[X,Y]$ with $XY = (-1)^p YX$ by nilpotents. For this quantum plane, we construct a Wess–Zumino-type de Rham complex and find its decomposition into representations of the $2p^3$-dimensional quantum group $\overline{U}_q\mathfrak{s}\ell(2)$ and its Lusztig extension $U_q\mathfrak{s}\ell(2)$; the quantum group action is also defined on the algebra of quantum differential operators on the quantum plane.

1. INTRODUCTION

Recent studies of logarithmic conformal field theory models have shown the remarkable fact that a significant part of the structure of these models is captured by factorizable ribbon quantum groups at roots of unity [1, 2, 3, 4]. This fits the general context of the Kazhdan–Lusztig correspondence/duality between conformal field theories (vertex-operator algebras) and quantum groups [5], but in logarithmic conformal field theories this correspondence goes further, to the extent that the relevant quantum groups may be considered “hidden symmetries” of the corresponding logarithmic models (see [6] for a review and [7, 8, 9] for further development). This motivates further studies of the quantum groups that are Kazhdan–Lusztig-dual to logarithmic conformal field theories. For the class of $(p,1)$ logarithmic models (with $p = 2, 3, \ldots$), the dual quantum group is $\overline{U}_q\mathfrak{s}\ell(2)$ generated by $E$, $K$, and $F$ with the relations

\[ KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \]

\[ [E,F] = K - K^{-1}, \quad \frac{K - K^{-1}}{q - q^{-1}}, \]

\[ E^p = F^p = 0, \quad K^{2p} = 1 \]

(1.1)

(1.2)

at the even root of unity

\[ q = e^{i\pi/p}, \]

(1.3)

This $2p^3$-dimensional quantum group first appeared in [10] (also see [11,12]; a somewhat larger quantum group was studied in [13]).

In this paper, we study not the $\overline{U}_q\mathfrak{s}\ell(2)$ quantum group itself but an algebra carrying its action, a version of the quantum plane [14].

Quantum planes (which can be defined

\[ 1 \]Quantum planes and their relation to $s\ell(2)$ quantum groups is a “classic” subject discussed in too many papers to be mentioned here; in addition to [14], we just note [15,16], and [17]. In a context close to the one in this paper, the quantum plane was studied at the third root of unity in [18,19,20].
in slightly different versions at roots of unity, for example, infinite or finite) have the nice property of being $H$-module algebras for $H$ given by an appropriate version of the quantum $s\ell(2)$ (an $H$-module algebra is an associative algebra endowed with an $H$ action that “agrees” with the algebra multiplication in the sense that $X(\mu \nu) = \sum (X'u)(X''\nu)$ for all $X \in H$). Studying such algebras is necessary for extending the Kazhdan–Lusztig correspondence with logarithmic conformal field theory, specifically, extending it to the level of fields; module algebras are to provide a quantum-group counterpart of the algebra of fields in logarithmic models when these are described in a manifestly quantum-group invariant way. Another nice feature of quantum planes, in relation to the Kazhdan–Lusztig correspondence in particular, is that they allow a “covariant calculus” \cite{15}, i.e., a de Rham complex with the space of 0-forms being just the quantum plane and with a differential that commutes with the quantum-group action. (A “covariant calculus” on a differential $U_q\ell(2)$-module algebra was also considered in \cite{8} in a setting ideologically similar but distinct from a quantum plane.)

Here, we explore a possibility that allows extending the $U_q\ell(2)$-action on a module algebra to the action of the corresponding Lusztig quantum group $U_q\ell(2)$, with the additional $s\ell(2)$ generators conventionally written as $E = \frac{p}{[p]!}$ and $F = \frac{p}{[p]!}$. We use Lusztig’s trick of divided powers twice: to extend the $U_q\ell(2)$ quantum group and to “distort” the standard quantum plane $C_q[x,y]$, i.e., the associative algebra on $x$ and $y$ with the relation
\begin{equation}
xy = qyx,
\end{equation}
by passing to the divided powers
\begin{align*}
x^m &= \frac{x^m}{[m]!}, & y^m &= \frac{y^m}{[m]!}, & m \geq 0.
\end{align*}
Constructions with divided powers \cite{29,30} are interesting at roots of unity, of course; in our root-of-unity case (1.3), specifically, $[p] = 0$, and the above formulas cannot be viewed as a change of basis. We actually define the divided-power quantum plane $C_q[x,y]$ (or just $C_q$ for brevity) to be the span of $x^m y^n, m, n \geq 0$, with the (associative) multiplication
\begin{equation}
x^m y^n = \frac{m+n}{m} x^{m+n}, & y^m y^n = \frac{m+n}{m} y^{m+n}
\end{equation}
and relations
\begin{equation}
y^m x^m = q^{mn} x^m y^m.
\end{equation}

\textsuperscript{2}We use the standard notation $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, [n]! = [1][2] \ldots [n]$, and $\left[ \frac{m}{n} \right] = \frac{[m]!}{[m-n]! [n]!}$.

\textsuperscript{3}The counterpart of this $s\ell(2)$ Lie algebra on the “logarithmic” side of the Kazhdan–Lusztig correspondence, in particular, underlies the name “triplet” for the extended chiral algebra of the $(p,1)$ logarithmic models, introduced in \cite{21,22,23} (also see \cite{24}; the triplet algebra was shown to be defined in terms of the kernel of a screening for general $p$ in \cite{25} and was studied, in particular, in \cite{26,27,28}).
(Somewhat more formally, \((1.5)\) can be considered relations as well.) Divided-power quantum spaces were first considered in \([31]\), in fact, in a greater generality (in an arbitrary number of dimensions, correspondingly endowed with an \(\mathcal{U}_q\mathfrak{sl}(n)\) action)\footnote{I thank N. Hu for pointing Ref. \([31]\) out to me and for useful remarks.}. Our quantum plane carries a \(\mathcal{U}_q\mathfrak{sl}(2)\) action that makes it a \(\mathcal{U}_q\mathfrak{sl}(2)\)-module algebra (a particular case of the construction in \([31]\)),

\[
\begin{align*}
E(x^m y^n) &= [m+1]x^{m+1}y^{n-1}, \\
K(x^m y^n) &= q^{m-n}x^m y^n, \\
F(x^m y^n) &= [n+1]x^{m-1}y^{n+1},
\end{align*}
\]

with \(x^m = y^n\) for \(m < 0\). A useful way to look at \(\mathbb{C}_q\) is to consider the “almost commutative” polynomial ring \(\mathbb{C}_q[X, Y]\) with \(XY = \varepsilon YX\) for \(\varepsilon = (-1)^p\) and extend it with the algebra of “infinitesimals” \(x^m y^n, 0 \leqslant m, n \leqslant p-1\). Then, in particular, \(X\) and \(Y\) behave under the \(\mathfrak{sl}(2)\) algebra of \(\mathcal{E}\) and \(\mathcal{F}\) “almost” (modulo signs for odd \(p\)) as homogeneous coordinates on \(\mathbb{C}P^1\).

We extend \(\mathbb{C}_q\) to a differential algebra, the algebra of differential forms \(\Omega_q\), which can be considered a (Wess–Zumino-type) de Rham complex of \(\mathbb{C}_q\), and then describe the \(\mathcal{U}_q\mathfrak{sl}(2)\) action on \(\Omega_q\) and find how it decomposes into \(\mathcal{U}_q\mathfrak{sl}(2)\)-representations (Sec. \(\S\)). Further, we extend the quantum group action on the quantum plane and differential forms to a \(\mathcal{U}_q\mathfrak{sl}(2)\) action. In Sec. \(\S\) we introduce quantum differential operators on \(\mathbb{C}_q\), make them into a module algebra, and illustrate their use with a construction of the projective quantum-group module \(\mathcal{Q}^+\).

2. Divided-power quantum plane

The standard quantum plane \(\mathbb{C}_q[x, y]\) at a root of unity and the divided-power quantum plane \(\mathbb{C}_q\) at a root of unity can be considered two different root-of-unity limits of the “generic” quantum plane. The formulas below therefore follow from the “standard” ones (for example, \(E(x^m y^n) = [n]x^{m+1}y^{n-1}\) and \(F(x^m y^n) = [m]x^{m-1}y^{n+1}\) for the quantum-\(\mathfrak{sl}(2)\) action) by passing to the divided powers at a generic \(q\), when \(x^m = \frac{x^m}{[m]!}\) is a change of basis, and setting \(q\) equal to our \(q\) in \((1.3)\) in the end. In particular, we apply this simple strategy to the Wess–Zumino calculus on the quantum plane \([15]\).

2.1. De Rham complex of \(\mathbb{C}_q[x, y]\). We define \(\Omega_q\), the space of differential forms on \(\mathbb{C}_q\), as the differential algebra on the \(x^m y^n\) and \(\xi, \eta\) with the relations (in addition to those
in \((\mathbb{C}_q)\)
\[
\xi \eta = -q \eta \xi, \quad \xi^2 = 0, \quad \eta^2 = 0,
\]
and
\[
\xi x^n = q^{2n} x^n \xi, \quad \eta x^n = q^n x^n \eta + q^{2n-2}(q^2 - 1)x^{n-1} y \xi, \\
\xi y^n = q^n y^n \xi, \quad \eta y^n = q^{2n} y^n \eta
\]
and with a differential \(d (d^2 = 0)\) acting as
\[
d(x^m y^n) = q^{m+n-1} x^{m+1} y^{n+1} \xi + q^{n-1} x^m y^{n-1} \eta.
\]
In particular, \(dx = \xi\) and \(dy = \eta\).

The \(\mathcal{U}_q \mathfrak{sl}(2)\) action on 1-forms that commutes with the differential is given by
\[
E(x^m y^n \xi) = q[m + 1] x^{m+1} y^{n-1} \xi, \\
E(x^m y^n \eta) = q^{-1}[m + 1] x^{m+1} y^{n-1} \eta + x^m y^n \xi, \\
K(x^m y^n \xi) = q^{-m-n+1} x^m y^n \xi, \\
K(x^m y^n \eta) = q^{-m-n+1} x^m y^n \eta, \\
F(x^m y^n \xi) = [n + 1] x^{m-1} y^{n+1} \xi + q^{n-m} x^m y^n \eta, \\
F(x^m y^n \eta) = [n + 1] x^{m-1} y^{n+1} \eta
\]
(and on 2-forms, simply by \(E(x^m y^n \xi) = [m + 1] x^{m+1} y^{n-1} \xi \eta\) and \(F(x^m y^n \xi) = [n + 1] x^{m-1} y^{n+1} \xi \eta\)).

A feature of the divided-power quantum plane is that the monomials \(x^{pm-1} y^{pn-1}\) with \(m, n \geq 1\) are \(\mathcal{U}_q \mathfrak{sl}(2)\) invariants, although not “constants” with respect to \(d\). Also, the kernel of \(d\) in \(\mathbb{C}_q\) is given by \(\mathbb{C}1\), and the cohomology of \(d\) in \(\Omega^1 q\) is zero. A related, “compensating,” difference from the standard quantum plane at the same root of unity amounts to zeros occurring in the multiplication table: for \(1 \leq m, n \leq p - 1\), it follows that \(x^m x^n = 0\) whenever \(m + n \geq p\).

2.2. “Frobenius” basis. An ideal (the nilradical) is generated in \(\mathbb{C}_q\) from the monomials \(x^m y^n\) with \(1 \leq m, n \leq p - 1\). We now introduce a new basis by splitting the powers of \(x\) and \(y\) accordingly (actually, by passing to “nondivided” powers of \(x^p\) and \(y^p\)).

2.2.1. The divided-power quantum plane \(\mathbb{C}_q[x, y]\) can be equivalently viewed as the linear span of
\[
(2.1) \quad X^M Y^N x^m y^n, \quad 0 \leq m, n \leq p - 1, \quad M, N \geq 0,
\]
where \( X = x^p \) and \( Y = y^p \); the change of basis is explicitly given by:

\[
X^{M+m}Y^m = (-1)^{Mm}x^{M_p}x^m = (-1)^{Mm+\frac{M(M-1)}{2}}\frac{1}{M!} X^M x^m
\]

and similarly for \( Y^{N+p+n} \) (cf. Proposition 2.4 in [31]). Clearly, monomials in the right-hand side here are multiplied “componentwise,”

\[
(X^M x^m)(X^N x^m) = X^{M+N} \left[ \begin{array}{c} m+n \\ m \end{array} \right] x^{m+n}.
\]

The relations involving \( X \) and \( Y \) are

\[
XY = (-1)^p YX, \\
X^n Y^n = x^n X^n, \\
X^m Y^m = (-1)^n y^n X, \\
Y^n X^n = (-1)^n x^n Y^n, \\
Y^n Y^n = y^n Y^n.
\]

We continue writing \( \mathbb{C}_q \) for \( \mathbb{C}_q[X,Y,x,y] \) with all the relations understood. The quotient of \( \mathbb{C}_q \) by the nilradical is the “almost commutative” polynomial ring \( \mathbb{C}_\varepsilon[X,Y] \), where \( XY = \varepsilon YX \) for \( \varepsilon = (-1)^p \).

In \( \mathbb{C}_q \), the relations involving \( X \) and \( Y \) are

\[
\xi X = X \xi, \\
\eta X^M = (-1)^M X^M \eta + (-1)^M M(q^{2M} - 1) X^{M-1} x^{p-1} y \xi, \\
\xi Y^N = (-1)^N Y^N \xi, \\
\eta Y = Y \eta,
\]

and the differential is readily expressed as

\[
d(X^M Y^N x^m y^n) = q^{m+n-1} \begin{cases} 
X^M Y^N x^{\frac{m-1}{n-1}} y \xi, & m \neq 0, \\
(-1)^{NP} M X^{M-1} Y^N x^{\frac{p-1}{n-1}} y \xi, & m = 0 \\
+ q^{n-1} \begin{cases} 
X^M Y^N x^{\frac{m}{n-1}} \eta, & n \neq 0, \\
(-1)^{MN} X^M Y^{N-1} x^{\frac{m+1}{n-1}} y \eta, & n = 0.
\end{cases}
\end{cases}
\]

In particular, \( d(X^M Y^N) = (-1)^{NP} M q^{-1} X^{M-1} Y^{N-1} x^{\frac{p-1}{n-1}} \xi - N q^{-1} X^M Y^{N-1} y^{p-1} \eta \).

2.2.2. The \( \overline{U}_qsl(2) \) action in the new basis (2.1) becomes

\[
E(X^M Y^N x^m y^n) = \begin{cases} 
[m+1]X^M Y^N x^{\frac{m+1}{n-1}} y^{\frac{n-1}{n}} & n \neq 0, \\
(-1)^{MN} [m+1] X^M Y^{N-1} x^{\frac{m+1}{n-1}} y^{\frac{n-1}{n}} & n = 0,
\end{cases}
\]

\[
K(X^M Y^N x^m y^n) = (-1)^{M+N} q^{m-n} X^M Y^{N} x^{\frac{m}{n}} y^n,
\]

\[\text{and in what follows, we “resolve” the } q\text{-binomial coefficients using the general formula [30]}
\]

\[
\left[ \begin{array}{c} Mp + m \\ Np + n \end{array} \right] = (-1)^{(M-1)Ip+M-N-M} \binom{M}{N} \binom{m}{n}
\]

for \( 0 \leq m, n \leq p-1 \) and \( M \geq 1, N \geq 0 \).
\[ F(X^m Y^n, x^m y^n) = \begin{cases} (-1)^{M+N}[n+1]X^m Y^n x^{m-1} y^{n+1}, & m \neq 0, \\ M(-1)^{M-1+N(p-1)}[n+1]X^m Y^n x^{m-1} y^{n-1}, & m = 0, \end{cases} \]

and on 1-forms, accordingly,
\[ E(X^m Y^n, x^m y^n) = \begin{cases} q[m+1]X^m Y^n x^{m-1} y^{n-1}, & n \neq 0, \\ (-1)^m q[n+1]X^m Y^n x^{m-1} y^{n-1}, & n = 0, \end{cases} \]

F. We first decompose the space of 0-forms, the quantum plane \( \mathbb{C}_q \), into irreducible representations \( \mathcal{X}_r^\pm (1 \leq r \leq p) \), indecomposable representations \( \mathcal{W}_r^\pm (n) (1 \leq r \leq p-1, n \geq 2) \), and projective modules \( \mathcal{P}_{r^{-1}}^\pm \). All of these are described in [2]; we only briefly recall from [1, 2] that \( \mathbb{U}_q^{s\ell}(2) \) has 2p irreducible representations \( \mathcal{X}_r^\pm \), and projective modules \( \mathcal{P}_r^\pm \) [6] (and \( \mathcal{X}_r^\pm \) is the trivial representation). We also note that \( \dim \mathcal{P}_r^\pm = 2p \) if \( 1 \leq r \leq p-1 \).

We first decompose the space of 0-forms, the quantum plane \( \mathbb{C}_q \) itself; clearly, the \( \mathbb{U}_q^{s\ell}(2) \) action restricts to each graded subspace \( (\mathbb{C}_q)_i \) spanned by \( x^m y^n \) with \( m + n = i \), \( i \geq 0 \). To avoid an unnecessarily long list of formulas, we explicitly write the decompositions for \( (\mathbb{C}_q)_i \) spanned by \( x^m y^n \) with \( 0 \leq m + n \leq i \); the decomposition of each graded component is in fact easy to extract.

2.3. Module decomposition. We next find how \( \mathcal{O}_q \) decomposes as a \( \mathbb{U}_q^{s\ell}(2) \)-module. The \( \mathbb{U}_q^{s\ell}(2) \) representations encountered in what follows are the irreducible representations \( \mathcal{X}_r^\pm \), indecomposable representations \( \mathcal{W}_r^\pm (n) \), and indecomposable modules \( \mathcal{P}_r^\pm \). All of these are described in [2]; we only briefly recall from [1, 2] that \( \mathbb{U}_q^{s\ell}(2) \) has 2p irreducible representations \( \mathcal{X}_r^\pm \), and projective modules \( \mathcal{P}_r^\pm \) [6] (and \( \mathcal{X}_r^\pm \) is the trivial representation). We also note that \( \dim \mathcal{P}_r^\pm = 2p \) if \( 1 \leq r \leq p-1 \).

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2.3.1. Lemma. \( \mathbb{C}_q \) decomposes into \( \mathbb{U}_q^{s\ell}(2) \)-representations as follows:

\[ (\mathbb{C}_q)_{(p-1)} = \bigoplus_{r=1}^{p} \mathcal{X}_r^+ \] (2.2)

(\textit{where each} \( \mathcal{X}_r^+ \in (\mathbb{C}_q)_{r-1} \).

\[ (\mathbb{C}_q)_{(2p-1)} = (\mathbb{C}_q)_{(p-1)} \oplus \bigoplus_{r=1}^{p-1} \mathcal{W}_r^+(2) \oplus 2 \mathcal{X}_p^- \] (2.3)
(where each $\mathcal{W}_r^-(2) \in (\mathbb{C}_q)_{p+r-1}$ and $\mathcal{X}_r^- \in (\mathbb{C}_q)_{2p-1}$, and, in general,

\begin{equation}
(\mathbb{C}_q)_{(n-1)p-1} = (\mathbb{C}_q)_{(n-1)p-1} \oplus \bigoplus_{r=1}^{p-1} W_r^+(n) \oplus n\mathcal{X}_r^+ \tag{2.4}
\end{equation}

with the + sign for odd $n$ and − for even $n$.

The proof is by explicit construction and dimension counting; the general pattern of the construction is already clear from the lower grades. For $1 \leq r \leq p$, the irreducible representations $\mathcal{X}_r^+$ are realized as

$$x^{r-1} \xrightarrow{E} x^{-2} \xrightarrow{F} y \xrightarrow{E} y^{r-1}.$$

(the $E$ and $F$ arrows represent the respective maps up to nonzero factors). The $\mathcal{X}_r^+$ thus constructed exhaust the space $(\mathbb{C}_q)_{(p-1)}$, which gives (2.2). To pass to the next range of grades, Eq. (2.3), it is easiest to multiply the above monomials with $X$ or $Y$ and then note that (up to sign factors) this operation commutes with the $\mathbb{U}_q\mathfrak{sl}(2)$ action except “at the ends” in half the cases, as is expressed by the identities

$$XE(X^M Y^N x^m y^n) - E(X^{M+1} Y^N x^m y^n) = 0,$$

$$YE(X^M Y^N x^m y^n) - (−1)^{M+m} E(X^M Y^N x^m y^n) = −δ_{m,0}(−1)^{M+m} [m + 1]X^M Y^N x^{m+1} y^{p-1},$$

$$XF(X^M Y^N x^m y^n) + F(X^{M+1} Y^N x^m y^n) = δ_{m,0}(−1)^{M+N(p-1)} [n + 1]X^M Y^N x^{p-1} y^{n+1},$$

$$YF(X^M Y^N x^m y^n) + (−1)^{M+p} F(X^{M+1} Y^N x^m y^n) = 0.$$

It then immediately follows that for each $r = 1, \ldots, p − 1$, the states

\begin{equation}
(2.5)
\begin{array}{c}
XX x^{p-1} \xrightarrow{F} \ldots \xrightarrow{F} X Y x^{p-1} \\
\xrightarrow{E} \xrightarrow{E} \xrightarrow{E} \\
YX x^{p-1} \xrightarrow{F} \ldots \xrightarrow{F} Y Y x^{p-1}
\end{array}
\end{equation}

realize the representation $\mathcal{X}_r^-$

\begin{equation}
(2.6)
\mathcal{W}_r^-(2) = \mathcal{X}_r^- \xrightarrow{E} \mathcal{X}_r^-.
\end{equation}

These $\mathcal{W}_r^-(2)$ fill the space $(\mathbb{C}_q)_{(2p-2)}$. Next, in the subspace $(\mathbb{C}_q)_{2p-1}$, this picture degenerates into a sum of two $\mathcal{X}_r^-$, spanned by $X x^{p-1}, \ldots, X y^{p-1}$ and $Y x^{p-1}, \ldots, Y y^{p-1}$, with the result in (2.3).
The longer “snake” modules follow similarly; for example, in the $\mathcal{W}^+_r(3)$ module

$$\begin{align*}
\mathcal{X}^+_r & \rightarrow \mathcal{X}^+_p \rightarrow \mathcal{X}^+_r \\
\mathcal{X}^-_{p-r} & \rightarrow \mathcal{X}^-_{p-r} \rightarrow \mathcal{X}^-_r 
\end{align*}$$

the leftmost state in the left $\mathcal{X}^+_r$ is given by $X^2 x^r y^{r-1}$ and the rightmost state in the right $\mathcal{X}^+_r$ is $Y^2 y^{r-1}$. These $\mathcal{W}^+_r(3)$ with $1 \leq r \leq p-1$ exhaust $(C \mathbb{Q})_{(3p-2)}$, and for $r = p$, the snake degenerates into $3 \mathcal{X}^+_p$, and so on, yielding (2.4) in general.

Next, to describe the decomposition of the 1-forms $\Omega^1_q$, we use the grading by the subspaces $(\Omega^1_q)_i = (C \mathbb{Q})_i \xi + (C \mathbb{Q})_i \eta$, $i \geq 0$, and set $(\Omega^1_q)_i) = (C \mathbb{Q})_i \xi + (C \mathbb{Q})_i \eta$ accordingly. For those representations $\chi \in C \mathbb{Q}$ that are unchanged under the action of $d$, we write $d \chi$ for their isomorphic images in $\Omega^1_q$, to distinguish the $d$-exact representations in the decompositions that follow.

**2.3.2. Lemma.** The space of 1-forms $\Omega^1_q$ decomposes into $U_q\mathfrak{sl}(2)$-representations as follows:

(2.7) $$(\Omega^1_q)_{(p-1)} = \bigoplus_{r=2}^{p} d\mathcal{X}^+_r \oplus \bigoplus_{r=1}^{p-2} \mathcal{X}^+_r \oplus \mathcal{P}^+_{p-1}$$

(with $d\mathcal{X}^+_r \in (\Omega^1_q)_{r-2}$, $\mathcal{X}^+_r \in (\Omega^1_q)_r$, and $\mathcal{P}^+_{p-1} \in (\Omega^1_q)_{p-1}$).

(2.8) $$(\Omega^1_q)_{(2p-2)} = (\Omega^1_q)_{(p-1)} \oplus \bigoplus_{r=2}^{p-1} d\mathcal{W}^-_r(2) \oplus 2d\mathcal{X}^-_r \oplus \mathcal{X}^+_p \oplus \bigoplus_{r=1}^{p-2} \mathcal{W}^-_r(2) \oplus 2\mathcal{P}^-_{p-1}$$

(with $d\mathcal{W}^-_r(2) \in (\Omega^1_q)_{p+r-2}$, $d\mathcal{X}^-_r \in (\Omega^1_q)_{2p-2}$, $\mathcal{X}^+_p \in (\Omega^1_q)_p$, $\mathcal{W}^-_r(2) \in (\Omega^1_q)_{p+r}$, and $\mathcal{P}^-_{p-1} \in (\Omega^1_q)_{2p-1}$).

(2.9) $$(\Omega^1_q)_{(3p-1)} = (\Omega^1_q)_{(2p-1)} \oplus \bigoplus_{r=2}^{p-1} d\mathcal{W}^+_r(3) \oplus 3d\mathcal{X}^+_p \oplus 2\mathcal{X}^+_p \oplus \bigoplus_{r=1}^{p-2} \mathcal{W}^+_r(3) \oplus 3\mathcal{P}^+_{p-1}$$

(with $d\mathcal{W}^+_r(3) \in (\Omega^1_q)_{2p+r-2}$, $d\mathcal{X}^+_p \in (\Omega^1_q)_{3p-2}$, $\mathcal{X}^-_p \in (\Omega^1_q)_{2p}$, $\mathcal{W}^+_r(3) \in (\Omega^1_q)_{2p+r}$, and $\mathcal{P}^+_{p-1} \in (\Omega^1_q)_{3p-1}$, and, in general,

(2.10) $$(\Omega^1_q)_{(np-1)} = (\Omega^1_q)_{((n-1)p-1)} \oplus \bigoplus_{r=2}^{p-1} d\mathcal{W}^\pm_r(n) \oplus n d\mathcal{X}^\pm_p \oplus \bigoplus_{r=1}^{p-2} (n-1)\mathcal{X}^\pm_p \oplus \bigoplus_{r=1}^{p-2} \mathcal{W}^\pm_r(n) \oplus n\mathcal{P}^\pm_{p-1},$$

with the upper signs for odd $n$ and the lower signs for even $n$.

The proof is again by construction. To begin with $(\Omega^1_q)_{(p-1)}$ in (2.7), we first note that is contains $d\mathcal{X}^+_r \approx \mathcal{X}^+_r$ spanned by $dx^{r-1}$, $d(x^{r-2} y)$, $...$, $dy^{r-1}$ for each $r = 2, \ldots, p$ (clearly,
the singlet $X^+_1$ spanned by 1 for $r = 1$ is annihilated by $d$). Next, for each $r = 1, \ldots, p - 2$, there is an $X^+_r$ spanned by the 1-forms

$$x^{r-1} y \xi - q[r] x^r \eta, \ldots, [i + 1] x^{r-i-1} y^{i+1} \xi - [r-i] q^{i+1} x^{r-i} y^i \eta, \ldots, [r] y^{r} \xi - q^r x^{r-1} y \eta$$

(with just the singlet $y \xi - q^r \eta$ for $r = 1$); for $r = p - 1$, however, these states are in the image of $d$ and actually constitute the socle of the projective module $P^+_{p-1}$ realized as

(2.11)

This gives (2.7).

In the next grade, we have $(\Omega^1_q)_p = dW^+_2(2) \oplus X^+_p$, where $X^+_p$ is spanned by

$$x^{p-1} y \xi, \ldots, [i + 1] x^{p-i-1} y^{i+1} \xi - [r-i] q^{i+1} x^{p-i} y^i \eta, \ldots, x y^{p-1} \eta$$

(with $i = 0$ at the left end and $i = p - 1$ at the right end). Next,

$$\Omega^1_q(p-2) = (\Omega^1_q(p-1) \oplus X^+_p \oplus \bigoplus_{r=2}^{p-1} dW^+_r(2) \oplus 2dX^+_r \oplus \bigoplus_{r=1}^{p-2} W^+_r(2),$$

where the $W_r(2)$ modules with $r = 1, \ldots, p - 2$ are spanned by (with nonzero factors dropped)

(2.12)

In the next grade, the two $P^-_{p-1}$ modules occurring in (2.8) are given by

(2.13)
(the left–right arrows are again omitted for compactness). Next, we have
\[
(\Omega_q^1)_{(3p-2)} = (\Omega_q^1)_{(2p-1)} \oplus 2\chi_p^- \oplus \bigoplus_{r=2}^{p-1} d\mathcal{W}_r^+ (3) \oplus 3d\chi_p^+ \oplus \bigoplus_{r=1}^{p-2} \mathcal{W}_r^+ (3),
\]
where, for example, each \( \mathcal{W}_r^+ (3) \) is spanned by monomials that can be easily constructed starting with \( X^2 x^{r-1} y \xi - q^r X^2 x^{r-1} \eta \) in the top-left corner, and so on; the pattern readily extends to higher degrees by multiplying with powers of \( X \) and \( Y \), yielding (2.10).

### 2.4. Extending to Lusztig’s quantum group.

#### 2.4.1. The above formulas for the \( \mathcal{U}_q sl(2) \) action of course imply that \( E^p \) and \( F^p \) act on \( \Omega_q^1 \) by zero. But using Lusztig’s trick once again, we can define the action of Lusztig’s divided-power quantum group \( \mathcal{U}_q sl(2) \) that extends our \( \mathcal{U}_q sl(2) \). As before, temporarily taking the quantum group deformation parameter \( q \) generic, evaluating the action of \( E = \frac{1}{[p]!} E^p \) and \( F = \frac{1}{[p]!} F^p \), and finally setting \( q = q \) gives
\[
\mathcal{E}(X^M Y^N x^m y^n \xi) = N(-1)^{(N-1)p+n+m} X^{M+1} Y^{N-1} x^m y^n \xi,
\]
\[
\mathcal{F}(X^M Y^N x^m y^n \eta) = M(-1)^{(M-1)p} X^{M-1} Y^{N+1} x^m y^n \eta.
\]
Then, clearly, \( \mathcal{H} = [\mathcal{E}, \mathcal{F}] \) acts on \( \Omega_q^1 \)
\[
\mathcal{H}(X^M Y^N x^m y^n \xi) = (-1)^{(M+N-1)p+m+n} (M-N) X^M Y^N x^m y^n \xi.
\]
Modulo the “slight noncommutativity” of \( X \) and \( Y \) for odd \( p \), we here have the standard \( sl(2) \) action on functions of the homogeneous coordinates on \( \mathbb{CP}^1 \):
\[
\mathcal{E} f(X, Y, x, y) = f(X, Y, x, y) \frac{\partial}{\partial Y} X, \quad \mathcal{F} f(X, Y, x, y) = Y \frac{\partial}{\partial X} f(X, Y, x, y)
\]
From the standpoint of this \( \mathbb{CP}^1 \), the \( x^m \) and \( y^n \) are to be regarded as some kind of “infinitesimals,” almost (modulo signs) invisible to \( sl(2) \) generated by \( \mathcal{E} \) and \( \mathcal{F} \).

The \( \mathcal{E} \) and \( \mathcal{F} \) act on \( \Omega_q^1 \) as
\[
\mathcal{E}(X^M Y^N x^m y^n \xi) = -N(-1)^{(N-1)p+n+m} X^{M+1} Y^{N-1} x^m y^n \xi,
\]
\[
\mathcal{E}(X^M Y^N x^m y^n \eta) = -N(-1)^{(N-1)p+n+m} X^{M+1} Y^{N-1} x^m y^n \eta +
\]
\[
+ \delta_{m,0} \begin{cases} N(-1)^n X^M Y^{N-1} x^{p-1} y^{n+1} \xi, & 0 \leq n \leq p-2, \\ X^M Y^{N-1} x^{p-1} \xi, & n = p-1, \end{cases}
\]
\[
\mathcal{F}(X^M Y^N x^m y^n \xi) = M(-1)^{(M-1)p} X^{M-1} Y^{N+1} x^m y^n \xi,
\]
\[
+ \delta_{n,0} \begin{cases} M(-1)^{(M-1)p+m} q^{-m-1} X^{M-1} Y^{N} x^{p-1} y^{m+1} \eta, & 0 \leq m \leq p-2, \\ (-1)^{(M+N-1)p} X^M Y^{N} y^{p-1} \eta, & m = p-1, \end{cases}
\]
\[
\mathcal{F}(X^M Y^N x^m y^n \eta) = M(-1)^{(M-1)p} X^{M-1} Y^{N+1} x^m y^n \eta.
\]
2.4.2. \( \mathcal{U}_q^{\mathfrak{sl}(2)} \) representations. Using the above formulas, it is easy to see how the \( \mathcal{U}_q^{\mathfrak{sl}(2)} \) representations in (2.2)–(2.10) combine into \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representations. The simple result is that decomposition (2.4) rewrites in terms of \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representations as

\[
(\mathcal{C}_q)_{(n-1)p-1} = (\mathcal{C}_q)_{((n-1)p-1)} \oplus \bigoplus_{r=1}^{p-1} \mathcal{W}_r^\pm (n) \oplus \mathcal{X}_p^\pm,
\]

where the underbrackets indicate that each \( \mathcal{W}_r^\pm (n) \) becomes an (indecomposable) \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representation and the \( n \) copies of \( \mathcal{X}_p^\pm \) are combined into an (irreducible) \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representation. Similarly, (2.10) becomes

\[
(\mathcal{O}_q^1)_{(n-1)p-1} = (\mathcal{O}_q^1)_{((n-1)p-1)} \oplus \bigoplus_{r=2}^{p-1} d \mathcal{W}_r^\pm (n) \oplus n \mathcal{X}_p^\pm (n) \oplus \bigoplus_{r=1}^{p-2} \mathcal{W}_r^\pm (n) \oplus n \mathcal{D}_r^\pm (n-1),
\]

where \( (n-1)\mathcal{X}_p^\pm \) is an irreducible, \( \mathcal{W}_r^\pm (n) \) an indecomposable, and \( n \mathcal{D}_r^\pm (n-1) \) a projective \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-module.

To see this, we first note that \( \mathcal{E} \) and \( \mathcal{F} \) act trivially on (2.2); each of the \( \mathcal{X}_r^\pm \) representations of \( \mathcal{U}_q^{\mathfrak{sl}(2)} \) is also an irreducible representation of \( \mathcal{U}_q^{\mathfrak{sl}(2)} \). Next, in (2.3), for the \( \mathcal{W}_r^\pm (2) \) modules shown in (2.6), \( \mathcal{E} \) and \( \mathcal{F} \) map between the two \( \mathcal{X}_r^\pm \) and (in this lowest, \( n = 2 \), case of \( \mathcal{W}_r^\pm (n) \) modules) act trivially on the socle,

\[(\mathcal{E})_{\mathcal{X}_r^\pm} = \mathcal{X}_r^\pm, \quad (\mathcal{F})_{\mathcal{X}_r^\pm} = \mathcal{X}_r^\pm, \quad \mathcal{E}(\mathcal{X}_p^\pm (n)) = \mathcal{X}_p^\pm (n), \quad \mathcal{F}(\mathcal{X}_p^\pm (n)) = \mathcal{X}_p^\pm (n), \]

yielding an indecomposable representation of \( \mathcal{U}_q^{\mathfrak{sl}(2)} \). By the same pattern, the sum of two \( \mathcal{X}_p^\pm \) in (2.3) becomes an irreducible \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representation. This extends to the general case (2.4): \( \mathcal{E} \) and \( \mathcal{F} \) act horizontally on all the \( \mathcal{W}_r^\pm (n) \) modules and also on the \( n \) copies of \( \mathcal{X}_p^\pm, n \geq 2 \), making them into respectively indecomposable and irreducible \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-modules.

Passing to the space of 1-forms, each of the \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-representations in (2.7) remains an \( \mathcal{U}_q^{\mathfrak{sl}(2)} \) representation; \( \mathcal{E} \) and \( \mathcal{F} \) act trivially on the \( \mathcal{X}_r^\pm \) and interchange the “corners” of \( \mathcal{D}_p^\pm (n) \) shown in (2.11):

\[
\mathcal{F}(x^{p-1} \xi) = y^{p-1} \eta, \quad \mathcal{E}(y^{p-1} \eta) = x^{p-1} \xi,
\]

(\( \mathcal{D}_p^\pm (n) \) thus becomes a projective \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-module).

Next, in (2.8), \( \mathcal{X}_p^\pm \) remains an irreducible \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-module (with \( \mathcal{E} \) and \( \mathcal{F} \) acting trivially), and each of the \( \mathcal{W}_r^\pm (2) \) is an indecomposable \( \mathcal{U}_q^{\mathfrak{sl}(2)} \)-module; in (2.12), for ex-
ample, \( \mathcal{F}(Xx^{r-1}y\xi - q[r]Xx^{r}\eta) = Yx^{r-1}y\xi - q[r]Yx^{r}\eta \), etc. Further, the sum of two \( P_{p-1}^- \) in (2.8) becomes a projective \( \mathcal{U}_q\mathfrak{s}\ell(2) \)-module; there, the top floor (see (2.13)) is a doublet under the \( s\ell(2) \) Lie algebra of \( E \) and \( \mathcal{F} \), the middle floor is a singlet \((Xy^{p-1}\eta - (-1)^pYx^{p-1}\xi) + \) a triplet \((Xx^{p-1}\xi, XY^{p-1}\eta + (-1)^pYx^{p-1}\xi, Yx^{p-1}\eta) \), and the lowest floor is also a doublet.

The organization of \( \mathcal{U}_q\mathfrak{s}\ell(2) \) representations in the general case in (2.10) is now obvious; it largely follows from the picture in the lower degrees by multiplying with powers of \( X \) and \( Y \).

2.4.3. “Lusztig–Frobenius localization”. The formulas in 2.2 can be extended from nonnegative integer to integer powers of \( X \) and \( Y \). This means passing from \( \mathbb{C}_q = \mathbb{C}_q[X,Y,x,y] \) to \( \mathbb{C}_q[X,X^{-1},Y,Y^{-1},x,y] \), which immediately gives rise to infinite-dimensional \( \mathcal{U}_q\mathfrak{s}\ell(2) \) representations, as we now indicate very briefly.

Taking the picture in (2.5) (with \( 1 \leq r \leq p-1 \)) and multiplying all monomials by \( X^{-1} \) destroys the south-east \( F \) arrow and the \( \mathcal{F} \) arrows shown in (2.14), yielding

\[
\begin{align*}
X^{-1}Y^r &\implies \ldots \implies X^{-1}Yx^r \implies \ldots \implies X^{-1}Xy^r \\
X^{-1}X^{p-1}Y &\implies \ldots \implies X^{-1}x^{p-1}y \\
\end{align*}
\]

But this now extends to the right, indefinitely, producing a pattern shown in Fig. 1 (the top diagram), where the \( \mathcal{E} \) and \( \mathcal{F} \) generators map between the respective monomials in each block. A “mirror-reflected” picture follows by multiplying (2.6) by \( Y^{-1} \). Multiplying by \( X^{-1}Y^{-1} \) gives the bottom diagram in Fig. 1 (where the \( \implies \) arrows are omitted for brevity).

After the extension by \( X^{-1} \) and \( Y^{-1} \), the differential \( d \) acquires a nonzero cohomology in \( \Omega^1\mathbb{C}[X,X^{-1},Y,Y^{-1},x,y] \), represented by \( X^{-1}X^{p-1}Y \) and \( Y^{-1}X^{p-1}Y \).

There is a curious possibility to further extend \( \mathbb{C}[X,X^{-1},Y,Y^{-1},x,y] \) by adding a new element \( \log(XY) \) that commutes with \( x, y, X, \) and \( Y \) and on which the differential and the quantum group generators act, by definition, as follows:

\[
\begin{align*}
d\log(XY) &= -q^{-1}x^{-1}X^{p-1}Y^{p-1}Y^{-1}y^{p-1}\eta, \\
E\log(XY) &= Y^{-1}xy^{p-1}, \quad \mathcal{E}\log(XY) = (-1)^pXY^{-1}, \\
K\log(XY) &= \log(XY), \\
F\log(XY) &= -X^{-1}x^{p-1}y, \quad \mathcal{F}\log(XY) = (-1)^pX^{-1}Y.
\end{align*}
\]

We then have the diagram shown in Fig. 2. There, the \( \implies \ldots \implies \) arrows represent maps.
Figure 1. Two infinite-dimensional $U_q sl(2)$ representations realized on the quantum plane extended by negative powers of $X$ and $Y$.

$(-1)^p XY^{-1}$

$\log(XY)$

$(-1)^p X^{-1} Y$

Figure 2. A $U_q sl(2)$ representation involving $\log(XY)$ on the extended quantum plane, and its map by the differential.
modulo nonzero factors; for all the other arrows, the precise numerical factors are indicated. The pattern extends infinitely both left and right. In the central part, two more maps that did not fit the picture are

\[
\begin{array}{cccc}
X & Y & -1 \\
\Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow & \Rightarrow
\end{array}
\]

3. Quantum Differential Operators

We next consider differential operators on the divided-power quantum plane \( \mathbb{C}_q \).

3.1. We define \( \partial_x \) and \( \partial_y : \mathbb{C}_q \to \mathbb{C}_q \) standardly, in accordance with

\[
d f = \xi \partial_x f + \eta \partial_y f
\]

for any function \( f \) of divided powers of \( x \) and \( y \). It then follows that

\[
\partial_x \partial_y = q \partial_y \partial_x
\]

and

\[
\partial_x (x^m y^n) = q^{-m-2n+1} x^{m-1} y^n,
\]

\[
\partial_y (x^m y^n) = q^{-m-n+1} x^m y^{n-1}
\]

for \( m, n \in \mathbb{N} \); in terms of the basis in \( \ref{2.2.1} \) therefore,

\[
\partial_x (X^M Y^N x^m y^n) = \begin{cases} 
(-1)^N q^{-m-2n+1} X^M Y^N X^{m-1} y^n, & m \neq 0, \\
M(-1)^N (p-1) + q^{-2n+1} X^M Y^N x^{p-1} y^n, & m = 0,
\end{cases}
\]

\[
\partial_y (X^M Y^N x^m y^n) = \begin{cases} 
(-1)^M q^{-m-n+1} X^M Y^N x^m y^{n-1}, & n \neq 0, \\
N(-1)^M (-1)^{m-1} q^{-m+1} X^M Y^N x^m y^{p-1}, & n = 0
\end{cases}
\]

for \( 0 \leq m, n \leq p-1 \) and \( M, N \geq 0 \).

3.2. Let \( \mathcal{D} \) denote the linear span of \( x^m y^n \partial_x^a \partial_y^b \) with \( m, n, a, b \geq 0 \). The following commutation relations are easily verified (\( \ell \in \mathbb{N} \)):

\[
\partial_x^\ell x = q^{-2\ell} x \partial_x^\ell + q^{-\ell+1}[\ell] \partial_x^{\ell-1} - q^{-2\ell+1}[\ell](q - q^{-1}) y \partial_x^{\ell-1} \partial_y,
\]

(3.1)

\[
\partial_y^\ell y = q^{-2\ell} y \partial_y^\ell + q^{-\ell+1}[\ell] \partial_y^{\ell-1}.
\]

(Here and in similar relations below, \( x \) is to be understood as the operator of multiplication by \( x \), etc.) It then follows that \( \partial_x^\ell \) and \( \partial_y^\ell \) are somewhat special: they (anti)commute with all the divided powers \( x^m \) and \( y^n \) with \( n \leq p \). In general, we have

\[
\partial_x^\ell x^m y^n = q^{-2\ell m - \ell n} x^m y^n \partial_x^\ell
\]

\[\text{Footnote: The rule to commute} \partial_x \text{ and } \partial_y \text{ inherited from the Wess–Zumino-type complex is not the one in [31], as was also noted in that paper.}\]
for all nonnegative integers \( m, n \), and \( \ell \), and hence the commutation relations of \( \partial_x^p \) and \( \partial_y^p \) with elements of \( \mathbb{C}_q \) can be written as

\[
\partial_x^p X^{M}Y^{N}x^m y^n = (-1)^{Np+n}X^{M}Y^{N}x^m y^n \partial_x^p + Mq^{\frac{p(p-1)}{2}}X^{M-1}Y^{N}x^m y^n,
\]

\[
\partial_y^p X^{M}Y^{N}x^m y^n = (-1)^{Mp+m}X^{M}Y^{N}x^m y^n \partial_y^p + (-1)^{Mq}Nq^{\frac{p(p-1)}{2}}X^{Y^{N-1}}x^m y^n
\]

for \( M, N \geq 1 \) and \( 0 \leq m, n \leq p - 1 \); in other words, \( \partial_x^p \) and \( \partial_y^p \) can be represented as the (left) derivatives

\[
\partial_x = i^{1-p} \frac{\partial}{\partial x}, \quad \partial_y = i^{1-p} \frac{\partial}{\partial y}.
\]

(Evidently, \( \partial_x^p \partial_y^p = (-1)^p \partial_y \partial_x^p \).) We here used that \( q^{\frac{p(p-1)}{2}} = i^{p-1} \).

**3.3. Lemma.** The relations in \( \mathcal{D} \) are compatible with the \( \mathbb{U}_q \mathfrak{s}\ell(2) \)-module algebra structure if we define the \( \mathbb{U}_q \mathfrak{s}\ell(2) \) action as

\[
E \partial_x = -q \partial_y, \quad E \partial_y = 0, \\
K \partial_x = q^{1} \partial_x, \quad K \partial_y = q \partial_y, \\
F \partial_x = 0, \quad F \partial_y = -q^{-1} \partial_x.
\]

The proof amounts to verifying that the relations in 3.2 are mapped into one another under the \( \mathbb{U}_q \mathfrak{s}\ell(2) \) action. Then \( \mathcal{E} \) and \( \mathcal{F} \) act on the differential operators as

\[
\mathcal{E}(\partial_x^M \partial_y^N) = -(-1)^{(M+1)p+n} \partial_x^{M-1} \partial_y^{N+1}, \\
\mathcal{F}(\partial_x^M \partial_y^N) = -(-1)^{(M+1)p+m} \partial_x^{M} \partial_y^{N-1}.
\]

Modulo sign factors, this is the \( \mathfrak{s}\ell(2) \) action on \( \mathbb{C}[\partial_x^p, \partial_y^p] = \mathbb{C}[\partial_x, \partial_y] \).

**3.4. A projective \( \mathbb{U}_q \mathfrak{s}\ell(2) \) module in terms of quantum differential operators.** As an application of the \( q \)-differential operators on \( \mathbb{C}_q \), we construct the projective \( \mathbb{U}_q \mathfrak{s}\ell(2) \)-module \( \mathcal{D} \) (see 3.1) as follows:

\[
\lambda_{p-1} \equiv \ldots \equiv \lambda_1 \equiv \rho_1 \equiv \ldots \equiv \rho_{p-1} \equiv \tau
\]

where the horizontal \( \equiv \) arrows, as before, stand for the action by \( F \) and \( E \) up to nonzero factors, and the actual expressions for the module elements are as follows: first,
\[
\tau = -q^{\frac{p(p-1)}{2}+1}\sum_{i=1}^{p} a_i x^{\frac{p-i}{2}} y^{i-1}\partial_x^{p-i} \partial_y^{i-1}, \quad \text{where} \quad a_i = \alpha + \sum_{j=2}^{i} \frac{q^{1-j}}{j-1},
\]
then \( \lambda_i = E^i \tau \) and \( \rho_i = F^i \tau \), with
\[
\lambda_{p-1} = (-1)^{p-1}(p-1)! q^{\frac{p(p-1)}{2}+1} x^{\frac{p-1}{2}} \partial_y^{p-1}, \quad \lambda_1 = q^{\frac{p(p-1)}{2}} \sum_{i=1}^{p-1} q^{i+2} x^{\frac{p-i}{2}} y^{i-1} \partial_x^{p-i-1} \partial_y^i,
\]
\[
\rho_1 = -q^{\frac{p(p-1)}{2}} \sum_{i=1}^{p-1} q^{i+1} x^{\frac{i-1}{2}} y^{\frac{p-i}{2}} \partial_x^{i} \partial_y^{p-i-1-i}, \quad \rho_{p-1} = -[p-1]! q^{\frac{p(p-1)}{2}+2} y^{\frac{p-1}{2}} \partial_x^{p-1}
\]
in particular, and, finally, \( \beta = F \lambda_1 = E \rho_1 \) is
\[
\beta = -q^{\frac{p(p-1)}{2}+1} \sum_{i=1}^{p} x^{\frac{p-i}{2}} y^{i-1} \partial_x^{p-i} \partial_y^{i-1}.
\]
In the expression for \( \tau \), \( \alpha \) is an arbitrary constant (clearly, adding a constant to all the \( a_i \) amounts to redefining \( \tau \) by adding \( \beta \) times this constant). The normalization is here chosen such that \( \beta \) be a projector,
\[
\beta \beta = \beta.
\]
(We note the useful identity \([p-1]! (q^{-1})^p = p q^{\frac{p(p-1)}{2}}\).

The “wings” of the projective module are commutative,
\[
\lambda_i \lambda_j = \lambda_j \lambda_i, \quad \rho_i \rho_j = \rho_j \rho_i
\]
for all \( 0 \leq i, j \leq p-1 \), where \( \lambda_0 = \rho_0 = \beta \), and, moreover, \( \lambda_i \lambda_j = \rho_i \rho_j = 0 \) whenever \( i + j \geq p \).

A similar (but notably simpler) realization of \( \mathcal{P}_1^+ \) in a \( \mathfrak{gl}_q(2) \)-module algebra of quantum differential operators on a “quantum line” was given in [8].

4. Conclusion

Quantum planes provide a natural example of module algebras over \( \mathfrak{sl}(2) \) quantum groups (they do not allow realizing all of the quantum-\( \mathfrak{sl}(2) \) representations, but the corresponding quantum differential operators make up a module algebra containing the projective modules in particular). By [15], moreover, \( \mathfrak{gl}_q(2) \) can be characterized as the “quantum automorphism group” of the de Rham complex of the quantum plane. This is conducive to the occurrence of quantum planes in various situations where the \( \mathfrak{sl}(2) \) quantum groups play a role. From the standpoint of the Kazhdan–Lusztig correspondence, the old subject of a quantum \( \mathfrak{sl}(2) \) action on the quantum plane is interesting in the case of even roots of unity, which we detailed in this paper.

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LEBEDEV PHYSICS INSTITUTE

ams@sci.lebedev.ru