Duo and Projective Gamma Acts

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Abstract.

Let M be a Γ-monoid and A a unitary right M_Γ-act, two concepts projective and duo on gamma acts have been introduced. Several characterizations of projective and duo are given. We prove projectivity in gamma acts has dual basis using to prove relationship between duo and Γ-multiplication.

Abstract.

Let M be a Γ-monoid and A a unitary right M_Γ-act, two concepts projective and duo on gamma acts have been introduced. Several characterizations of projective and duo are given. We prove projectivity in gamma acts has dual basis using to prove relationship between duo and Γ-multiplication.

1. Introduction

Let M and Γ be nonempty sets. Then M is called a Γ-monoid, if there is a mapping $M \times \Gamma \times M \rightarrow M$ defined by $(m, \alpha, n) \mapsto m \alpha n$ which is satisfying the condition $(1) \ (m \alpha n) \beta m' = m \alpha (n \beta m')$ for all $m, n, m' \in M$ and $\alpha, \beta \in \Gamma$. (2) For each $m \in M$ and $\beta \in \Gamma$ there is an element $e \in M$ such that $e \beta m = m \beta e = m$. Sen. 4. Madhusudhana. 6 A Γ-monoid M is said to be commutative provided $m_1 \alpha m_2 = m_2 \gamma m_1$ for all $m_1, m_2 \in M$ and $\alpha, \gamma \in \Gamma$. This definition is similar as in Madhusudhana 4, since $(m_1 \alpha m_2 = m_2 \gamma (1 \alpha m_1) = (m_2 \gamma (1 \alpha m_1)) \alpha m_1 = m_2 \gamma (1 \alpha m_1))$ for all $\alpha, \gamma \in \Gamma$ and $m_1, m_2 \in M = S^1$ be the Γ-semigroup S with an identity adjoined usually denoted by the symbol 1), Madhusudhana 4. Let M be Γ-monoid. A nonempty set A is called right M_Γ-act, if there is a mapping $A \times \Gamma \times M \rightarrow A$ written $(a, \alpha, m) \mapsto a \alpha m$ such that the following condition is satisfied $a \beta (m_1 \alpha m_2) = (a \beta m_1) \alpha m_2$ for all $m_1, m_2 \in M$, $\alpha, \beta \in \Gamma$ and $a \in A$. Let A and A′ be two M_Γ-acts. A mapping $g: A \rightarrow A'$ is called M_Γ-homomorphism if $g(a \alpha m) = g(a) \alpha m$, for all $m \in M$, $\alpha \in \Gamma$ and $a \in A$. Let A be an M_Γ-act and U a nonempty subset of A. Then $[U]_A = \bigcup_{u \in U} u \Gamma M$ where $u \Gamma M = \{u \alpha m \mid m \in M \text{ and } \alpha \in \Gamma\}$, Abbas. 1. A non-empty subset U of M_Γ-act A is said to be a set of generating elements or generating set of A if $A = [U]_A$. We say that A is finitely
generated if \( [U]_A = A \) for some subset \( U \) of \( A \) which \( |U| < \infty \). And \( A \) is a cyclic if \( A = \{ [u] \}_A \) for some \( u \in A \). Let \( A \) be an \( M_\Gamma \)-act. Then \( A \) is called decomposable, if there are two gamma subacts \( L, K \) of \( A \) such that \( A = L \cup K \) which \( |U| \). And \( A \) is a cyclic if \( A = \{ u \} \) for some \( u \in A \). Let \( A \) be an \( M_\Gamma \)-act. Then \( A \) is called decomposable, if there are two gamma subacts \( L, K \) of \( A \) such that \( A = L \cup K \) and \( L \cap K = \emptyset \), otherwise \( A \) is call indecomposable. Every cyclic \( M_\Gamma \)-act is indecomposable.

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2. Projective Gamma Acts.

The following definition was mentioned as condition, see Proposition (2.10) in Abbas.

2.1. Definition: An \( M_\Gamma \)-act \( A \) is projective, if every \( M_\Gamma \)-epimorphism \( f: C \to B \) and any \( M_\Gamma \)-homomorphism \( g: A \to B \), where \( C, B \) are two \( M_\Gamma \)-acts, there is an \( M_\Gamma \)-homomorphism \( h: A \to C \) such that \( f h = g \).

2.2. Definition: An \( M_\Gamma \)-act \( A \) is a retract(co-retract) of an \( M_\Gamma \)-act \( B \), if there exist \( M_\Gamma \)-homomorphisms \( f: A \to B \) and \( g: B \to A \) such that \( gf = I_A \) and \( fg = I_B \) where \( I_A \) (\( I_B \)) is the identity mapping from \( A \) to \( A \) (\( B \) to \( B \)).

The following proposition describes a retraction in items of an \( M_\Gamma \)-subact.

2.3. Proposition: Let \( A \) and \( B \) be \( M_\Gamma \)-acts. Then \( B \) is a retract of \( A \), if and only if, there exist an \( M_\Gamma \)-subact \( A' \) of \( A \) and an \( M_\Gamma \)-epimorphism \( h: A \to A' \) such that \( B \cong A' \) and \( h(a') = a' \) for each \( a' \in A' \).

Proof: Necessity. Assume that \( B \) is a retract of \( A \). Then \( fg = I_B \). Set \( A' = g(B) \). Then, clearly \( A' \) is an \( M_\Gamma \)-subact of \( A \) and \( B \cong A' \). Define \( h: A \to A' \) by \( h(a) = (gf)(a) \) for every \( a \in A \). Then \( h \) is an \( M_\Gamma \)-epimorphism. Let \( a' \in A' \) and \( b \in B \) such that \( g(b) = a' \). Then \( h(a') = (gf)(a') = (gf)(g(b)) = (g(fg))(b) = g(b) = a' \).
Sufficiency. Let $B$ be an $M_\Gamma$-subact of $A$ such that there is an $M_\Gamma$-epimorphism $h: A \to B$ with $h(b) = b$ for each $b \in B$. Then $hi = iB$, for the natural embedding $i: B \to A$. Hence $B$ is a retract of $A$.

2.4. Proposition: Let $S$, $M$ be a $\Gamma$-monoids and $e$ a $\Gamma$-idempotent. If $B$ is a $(M - S)_\Gamma$-biact, then $\text{Hom}(e\Gamma S, MB_S)$ is a $(M - e\Gamma S)e_\Gamma$-biact by the mapping $\text{Hom}(e\Gamma S, MB_S) \times \Gamma \times e\Gamma S \Gamma e \to \text{Hom}(e\Gamma S, MB_S)$ definitions $(f\alpha u)(x) = f(ux)$ and $(\alpha\beta t)(x) = \alpha f(x)$, $u \in e\Gamma S\Gamma e$, $\alpha, \beta \in \Gamma$, $x \in e\Gamma S$, $t \in M$ and $f \in \text{Hom}(e\Gamma S, MB_S)$. The mapping $g: \text{Hom}(e\Gamma S, MB_S) \to \text{Hom}(e\Gamma S, MB_S)$, $g(f) = f(e)$ is an $\text{Sr}$-biisomorphism.

In particular $\text{Hom}(S, MB_S) \cong MB_S$ as $(M - S)_\Gamma$-biacts. Moreover $\text{End}(e\Gamma S) \cong e\Gamma S\Gamma e$ and as $(M - S)_\Gamma$-biacts, and in particular $\text{End}(S)_{SF} \cong S$ as $\Gamma$-monoids.

Proof: Let the mapping $\text{Hom}(e\Gamma S, MB_S) \times \Gamma \times e\Gamma S \Gamma e \to \text{Hom}(e\Gamma S, MB_S)$ definitions $(f\alpha u)(x) = f(ux)$, $u \in e\Gamma S\Gamma e$, $\alpha, \beta \in \Gamma$, $x \in e\Gamma S$ and $f \in \text{Hom}(e\Gamma S, MB_S)$. Let $u, v \in e\Gamma S\Gamma e$, $\alpha, \beta, \gamma \in \Gamma$, $x \in e\Gamma S$, $(f(u\beta v)\alpha)(x) = f((u\beta v)\alpha(x)) = f((u\beta v)\alpha x) = (f\beta u)(\alpha x) = (f\alpha)(u\beta v x) = ((f\alpha)\beta v)(x)$. Hence $\text{Hom}(e\Gamma S, MB_S)$ is an $(e\Gamma S\Gamma e)_\Gamma$-act. Let the mapping $M \times \Gamma \times \text{Hom}(e\Gamma S, MB_S) \to \text{Hom}(e\Gamma S, MB_S)$ definitions $(\alpha\beta t)(x) = \alpha f(x)$, $t \in M$, $\alpha \in \Gamma$, $x \in e\Gamma S$ and $f \in \text{Hom}(e\Gamma S, MB_S)$. Let $t, s \in M$, $\alpha, \beta, \gamma \in \Gamma$, $x \in e\Gamma S$, $(\alpha\beta\gamma)(x) = (\alpha\beta)(\alpha\beta\gamma)(x)) = (\alpha\beta)(\alpha\beta\gamma)(x)$. Hence $\text{Hom}(e\Gamma S, MB_S)$ is an $M_\Gamma$-act. $g$ is an $(e\Gamma S\Gamma e)_\Gamma$-homomorphism, since $g(f\alpha) = (f\alpha)(e) = f(ue\alpha) = f(e\alpha u) = g(f)\alpha u$, for each $f \in \text{Hom}(e\Gamma S, MB_S)$, $ue\Gamma S\Gamma e$ and $\alpha \in \Gamma$. $g$ is a $M_\Gamma$-homomorphism, since $g(f\alpha) = (f\alpha)(e) = f(e\alpha)\gamma e\alpha = h(e)\alpha e\alpha = h(e\alpha)$, for each $e\Gamma S$ and $\alpha \in \Gamma$ and thus $f = h$. Hence $g$ is injective. If $b \in B$, then $h: e\Gamma S \to B$ with $h(e\alpha) = h(b\alpha e\alpha)$, for each $\beta$ and $\alpha \in \Gamma$. $h$ is well-defined and $S_\Gamma$-homomorphism $h(e\alpha\beta\gamma)(e) = h(e\alpha\beta\gamma)(e)$, for each $\gamma, \alpha, \beta \in \Gamma$ and $t \in S$. And $h(e) = h(e) = b\alpha e\alpha$. Hence $g$ is surjective. Taking $e = 1$ one obtains that $\text{Hom}(S, MB_S) \cong MB_S$ as $(M - S)_\Gamma$-biacts. If $M = \{1\}$ and $B = e\Gamma S$ then for $f, h \in \text{End}(e\Gamma S)_{SF}$, we get $g(f\alpha) = (f\alpha)(e) = (f(h(e)\alpha 1) = (f(h(e)\beta h(e)\alpha 1) = (f(h(e))\beta h(e)) = (f(e)(\beta)(h(e)) = (f(e))(\beta)(h(e))$ and since $g(id_{e\Gamma S}) = e$, this shows that $g$ is a $\Gamma$-monoids. Hence $\text{End}(e\Gamma S)_{SF} \cong e\Gamma S\Gamma e$ as $\Gamma$-monoids. Taking $e = 1$ one obtains that $\text{End}(S)_{SF} \cong S_\Gamma$ as $\Gamma$-monoids.

Let $A$ be an $M_\Gamma$-act. A generating set $U$ of $A$ is said to be a basis of $A$, if every element $a \in A$ can be uniquely presented in the form $a = \alpha \gamma m$ for some $m \in M$, $\alpha \in \Gamma$ and $u \in U$, that is $a = a_1\alpha_1m_1 = a_2\alpha_2m_2$ if and only if $m_1 = m_2$, $\alpha_1 = \alpha_2$ and $a_1 = a_2$. If an $M_\Gamma$-act $A$ has a basis $U$, then it is called a free $M_\Gamma$-act. Let $A$ be right $M_\Gamma$-act. An element $\Theta \in A$ is called a zero of $A$, if $\Theta m = \Theta$ for all $m \in M$ and $\theta \in \Gamma$, Abbass.

2.5. Remarks:

(1) Every retract of projective $M_\Gamma$-act is projective.

(2) $M$ is projective (free) $M_\Gamma$-act.

(4) Every $M_\Gamma$-isomorphic to projective $M_\Gamma$-act is projective.
(5) The one element \( M_\Gamma \)-act \( \Theta \) is projective \( M_\Gamma \)-act if and only if \( M \) contains a left \( \Gamma \)-zero.

(6) Every free \( M_\Gamma \)-act is projective, Abbas \(^2\).

The following proposition gives a characterization of Projectivity.

2.6. **Proposition:** The following are equivalent for an \( M_\Gamma \)-act \( A \).

1. \( A \) is projective.
2. \( A \) is a retract of a free \( M_\Gamma \)-act.

**Proof:** (1)\(\Rightarrow\)(2) Let \( A \) be an \( M_\Gamma \)-act. There is a free \( M_\Gamma \)-act \( F \), and an \( M_\Gamma \)-epimorphism \( f: F \to A \), Abbas \(^2\). By (1), there is an \( M_\Gamma \)-homomorphism \( h: A \to F \) such that \( fh = I_A \). Hence \( A \) is retract of \( F \).

(2)\(\Rightarrow\)(1) It follows Remarks (2.5), (1),(6).

Let \( A \) be an \( M_\Gamma \)-act and \( M^A=\text{Hom}(A, M) \) under the mapping \( (f, \alpha, m) \mapsto f\alpha m \) where \( (f\alpha m)(a) = f(\alpha m) \) then \( M^A \) is an \( M_\Gamma \)-act.

2.7. **Definition:** Let \( A \) be an \( M_\Gamma \)-act and \( a \in A \). Then an \( M_\Gamma \)-homomorphism \( \lambda^\Gamma_A: M \to A \) is called annihilator congruence of \( a \), defined by \( \lambda^\Gamma_A(m) = \alpha m \) for each \( m \in M \) and \( \alpha \in \Gamma \).

2.8. **Definition:** Let \( A \) be an \( M_\Gamma \)-act and \( a \in A \). Then an \( M_\Gamma \)-homomorphism \( \lambda^\Gamma_A: M \to A \) is called dual basis for \( A \), if for every \( a \in A \), there is exactly one \( (a_i, \alpha, f_i(a)) \in \Gamma \) such that \( a = a_i\alpha f_i(a) \).

2.9. **Proposition:** Let \( a \in A \) and \( A = a\Gamma M \) be a cyclic \( M_\Gamma \)-act. Then \( a\Gamma M \cong \frac{M}{\ker(\lambda^\Gamma_A)} \).

**Proof:** By Proposition (4.12) in Abbas\(^1\), we have \( \frac{M}{\ker(\lambda^\Gamma_A)} \cong \lambda^\Gamma_A(M) = a\Gamma M \) for each \( m \in M \) and \( \alpha \in \Gamma \).

2.10. **Proposition:** Let \( P_i \) be an \( M_\Gamma \)-act, for each \( i \in I \). Then \( P = \bigsqcup_{i \in I} P_i \) is Projective, if and only if, \( P_i \) is Projective.

**Proof:** Necessity. Let \( f:A \to B \) and \( g: P_i \to B \) be two \( M_\Gamma \)-homomorphisms with \( f \) is \( M_\Gamma \)-epimorphism. Let \( f': A \to B \) be defined by \( f'(|\theta| = \theta, f'\big|_A = f \) and \( g': P \to B \) by \( g'|_{P_i} = g \) \( g'(p) = \theta \) for each \( p \in P \) and \( p \not\in P_i \). Since \( P \) is projective, there exists an \( M_\Gamma \)-homomorphism \( h:P \to A \) such that \( g' = f'h \). Then \( h(p_1) \subseteq A \) and \( h|_{P_i}: P_i \to A \) is an \( M_\Gamma \)-homomorphism, for which \( g = f'h \). Thus \( P_i \) is projective.

Sufficiency. Let \( f:A \to B \) and \( g: P \to B \) be two \( M_\Gamma \)-homomorphisms with \( f \) is \( M_\Gamma \)-epimorphism. Since \( P_i \) is Projective, there is an \( M_\Gamma \)-homomorphism \( g_i: P_i \to A \) such that \( g_i = g_i|_{P_i} \) where the inclusion mapping \( i_{P_i}: P_i \to P \). By Definition(2.1.9),(2) in Mati\(^8\), there is an \( M_\Gamma \)-homomorphism \( h: P \to A \) such that \( g_i = h|_{P_i} \) implies that \( f g_i = (fh)|_{P_i} = g_i|_{P_i} \). Since \( (i_{P_i})_{i \in I} \) is epimorphic family, (Definition (2.1.12) & Proposition(2.1.13)) in Mati\(^7\). Hence \( fh = g \).
2.11. Proposition: Let $A$ be an indecomposable $M_\Gamma$-act and $f: A \to A'$ an $M_\Gamma$-homomorphism. Then $f(A)$ is an indecomposable $M_\Gamma$-subact of $A'$.

Proof: Assume that $f(A) = A_1' \cup A_2'$ is an $M_\Gamma$-subact of $A'$. Then $A = f^{-1}(A_1' \cup A_2') = f^{-1}(A_1') \cup f^{-1}(A_2')$ where $f^{-1}(A_1') \cap f^{-1}(A_2') \neq \emptyset$, since $A$ is an indecomposable. Hence $\emptyset \neq f(f^{-1}(A_1') \cap f^{-1}(A_2')) \subseteq A_1' \cap A_2'$, contradiction.

2.12. Theorem. Every $M_\Gamma$-act $A$ has a unique decomposition into indecomposable $M_\Gamma$-subacts.

The following statement is structure of projective $M_\Gamma$-acts.

2.13. Proposition: Any indecomposable projective $M_\Gamma$-act $A$ is cyclic and $A \cong e\Gamma M$, for some $e \in M$ where $e = e\alpha e$ for each $\alpha \in \Gamma$.

Proof: By Projectivity of $A$, we have a retraction $f:F \to A$ with coretraction $g$ such that $fg=i_A$, for free $M_\Gamma$-act $F \cong \bigsqcup_{i \in I} M_i$, $M_i \cong M$, for each $i \in I$, Abbas. $^2$. We have $A = fg(A)$. By (2.11), $g(A)$ is a subset of $M_j$, for some $j \in J$. Then $A = fg(A) \subseteq f(M_j) \subseteq A$, since $f$ is $M_\Gamma$-epimorphism and $M_j \subseteq F$. Then $A = f(1)\Gamma M_j \cong f(1)\Gamma M$. Then $g(A) = g(f(1))\Gamma M_j \cong g(f(1))\Gamma M$ since $g:A \to F$ is an $M_\Gamma$-monomorphism, implies that $A \cong g(A) = g(f(1))\Gamma M$. Set $e = g(f(1))$, then for each $\alpha \in \Gamma$, $\epsilon \alpha e = g(f(1))\alpha e = g(f(e)) = g(f(g(f(1)))) = g(f(1)) = e$ and thus $A \cong e\Gamma M$.

The following proposition gives some source of projective gamma act.

2.14. Proposition: The principal $\Gamma$-ideal $e\Gamma M$ of $\Gamma$-monoid $M$ is (retract) projective $M_\Gamma$-act for any $e \in M$ where $e = e\alpha e$ for each $\alpha \in \Gamma$, that is, $e$ is an $\Gamma$-idempotent in $M$.

Proof: It follows (2.5),(2), $M$ is free $M_\Gamma$-act and define $h:M \to e\Gamma M$ by $h(m) = e\alpha m$, for each $m \in M$. It is easy to show $h$ is an $M_\Gamma$-epimorphism and $hi = i_{e\Gamma M}$. Hence $e\Gamma M$ is projective.

The following proposition gives the structure of an arbitrary projective $M_\Gamma$-act.

2.15. Proposition: An $M_\Gamma$-act $A$ is projective, if and only if, $A = \bigsqcup_{i \in I} A_i$ where $A_i \cong e_i\Gamma M$, for any $e_i \in M$ where $e_i = e_i\alpha e_i$ for each $\alpha \in \Gamma$ and $i \in I$.

Proof: Necessity. Let $A$ be projective $M_\Gamma$-act. Then $A = \bigsqcup_{i \in I} A_i$ where $A_i$ are indecomposable, by (2.12). By (2.10), $A_i$ is Projective and we have $A_i \cong e_i\Gamma M$, for some $\Gamma$-idempotent $e_i \in M$, by (2.13).

Sufficiency. By (2.14), any principal $\Gamma$-ideal $e\Gamma M$ where $e$ is a $\Gamma$-idempotent in $M$. Hence $e\Gamma M$ is projective. By (2.5),(4), $A_i$ is projective and (2.10), we have $A$ is projective.

The following corollary shows characterization of cyclic projectivity.

2.16. Proposition: Let $\delta$ be a congruence on $M$. Then the following as serration are equivalent:
(1) $\frac{M}{\delta}$ is a projective $M_1$-act. (2) $\frac{M}{\delta} \cong e \Gamma M$ for some $\Gamma$-idempotent $e \in M$. (3) There is a $\Gamma$-idempotent $e \in M$ such that $1 \delta e$ for each $\delta$ is an equal or less than $\text{Ker}(\lambda^\delta_e)$. (4) There is a $\Gamma$-idempotent $e \in M$ such that $1 \delta e$ for each $\delta = \text{Ker}(\lambda^\delta_e)$. (5) $\frac{M}{\delta}$ is a retract of $M$.

Proof:(1)$\implies$(2): Let $M$ be indecomposable, Abbas $^2\text{.}$ There is a natural $M_1$-epimorphism $\pi_\delta : M \to \frac{M}{\delta}$. By (2.11), $\pi_\delta(M) = \frac{M}{\delta}$ is indecomposable and (1). Then $\frac{M}{\delta} \cong e \Gamma M$, for some $\Gamma$-idempotent $e \in M$, by (2.13).

(2)$\implies$(3): Let $e$ be an $\Gamma$-idempotent in $M$, $f : e \Gamma M \to \frac{M}{\delta}$ an $M_1$-isomorphism and let $u, v \in M$ such that $f(e) = [u]_\delta$ and $f(e\alpha v) = [1]_\delta$, for some $\alpha \in \Gamma$. Now from $f(e) = [u]_\delta = [1]_\delta \alpha u = f(e\alpha v) \beta u = f(e\alpha \beta u)$, since $f$ is $M_1$-monomorphism, we get $e = e\alpha \beta u$. $[u]_\delta \beta e = f(e) \beta e = f(e\beta e) = f(e) = [u]_\delta$. Set $t = u \beta e\alpha v$. Then $[1]_\delta = f(e\alpha v) = f(e)\alpha v = [u]_\delta \alpha v = [u]_\delta \beta e\alpha v = [u \beta e\alpha v]_\delta = [t]_\delta$. Hence $1 \delta t$. Since $\Delta(t \alpha u) = u \beta (e \alpha \nu) \beta e\alpha v = u \beta (e \alpha \nu) \beta e\alpha v = u \beta (e \alpha \nu) \beta e\alpha v = t$. Hence $t$ is $\Gamma$-idempotent. Let $x \beta y, x, y \in M$. Then $[x]_\delta = [y]_\delta$ which yield $f(e\alpha) \beta x = f(e\alpha) \beta y$. Since $f$ is $M_1$-monomorphism, $e\alpha \beta x = e\alpha \beta y$. Then $t \beta x = u \beta e\alpha v \beta x = u \beta e\alpha v \beta y = t \beta y$. Hence $\delta$ is an equal or less than $\text{Ker}(\lambda^\delta_e)$. (3)$\implies$(4) Obvious. (4)$\implies$(5): By (4), $\frac{M}{\delta} = \frac{M}{\text{Ker}(\lambda^\delta_e)}$. Then $e \Gamma M \cong \frac{M}{\text{Ker}(\lambda^\delta_e)}$ by (1.9). There is an $M_1$-epimorphism $h : M \to e \Gamma M$ by $h(m) = e \alpha m$ for each $m \in M$, such that $h(n) = n$, for each $n \in e \Gamma M$. Then $\frac{M}{\delta}$ is a retract of $M$, by (2.3).

(5)$\implies$(1): By (2.5)(2) & (1).

2.17. Proposition: Let $A$ be an $M_1$-act and $a \in A$. Then $a \Gamma M$ is projective if and only if, there is an $M_1$-homomorphism $g : a \Gamma M \to M$ such that $a \alpha g(a) = a$.

Proof: Necessity. Let $a \Gamma M$ be a projective $M_1$-act. Since $a \Gamma M \cong \frac{M}{\text{Ker}(\lambda^\delta_e)}$. By (2.16), there is an $\Gamma$-idempotent $e \in M$ such that $\text{Ker}(\lambda^\delta_e) = \text{Ker}(\lambda^\delta_e)$. Since $e \alpha e = e \alpha 1$, then $(e, 1) \in \text{Ker}(\lambda^\delta_e)$, the previous equality implies that $(e, 1) \in \text{Ker}(\lambda^\delta_e)$, thus $e \alpha e = e \alpha 1 = a$. Define $f : a \Gamma M \to M$ by $f(a \alpha m) = e \alpha m$ for each $m \in M$. Let $a \alpha m_1 = a \alpha m_2$ for each $m_1, m_2 \in M$ and $\alpha, \beta \in \Gamma$, then $(m_1, \alpha) \in \text{Ker}(\lambda^\delta_e) = \text{Ker}(\lambda^\delta_e)$ implies that $e \alpha m_1 = e \alpha m_2$. Hence $f$ is a well-defined. $f$ is $M_1$-homomorphism, since $f(a \alpha m \beta m') = (e \alpha m \beta m') = f(a \alpha m) \beta m'$, for each $m' \in M$ and $\alpha, \beta \in \Gamma$. Then $a \alpha f(a) = a \alpha f(a \beta 1) = a \alpha (e \beta 1) = a \alpha e = a$.

Sufficiency. Let $f : a \Gamma M \to M$ be an $M_1$-homomorphism such that $a \alpha f(a) = a$ and $e = f(a)$. Then $e = f(e) = f(a \alpha f(a)) = (a)\alpha f(a) = e \alpha e$. Now $\text{Ker}(\lambda^\delta_e) = \text{Ker}(\lambda^\delta_e)$, since $(x, y) \in \text{Ker}(\lambda^\delta_e) = \text{Ker}(\lambda^\delta_e)$, $a \alpha x = a \alpha y$ implies that $e \alpha x = f(a) \alpha x = f(a \alpha x) = f(a \alpha y) = f(a) \alpha y = e \alpha y$. On the other hand $e \alpha x = e \alpha y$ implies that $a \alpha x = a \alpha f(a) \alpha x = a \alpha e \alpha x = a \alpha e \alpha y = a \alpha f(a) \alpha y = a \alpha y$. Therefore $\frac{M}{\text{Ker}(\lambda^\delta_e)}$ is projective, (2.16), (2), hence $a \Gamma M$ is projective, (2.16), (4).

2.18. Theorem: Let $A$ be an $M_1$-act. Then $A$ is projective, if and only if, $A$ has a dual basis.
Proof: **Necessity.** Let A be a projective $M_{r}$-act, then $A = \bigsqcup_{i \in I} P_{i} \cong \bigsqcup_{j \in J} e_{j}GM$ for some $\Gamma$-idempotent $e_{j} \in M$ by (2.15). There exists an $M_{r}$-isomorphism $f_{i}: e_{i}GM \rightarrow P_{i}$ such that $P_{i} = f_{i}(e_{i})GM$. By (2.17), there is $g_{i} \in \text{Hom}(f_{i}(e_{i})GM, M)$ such that $f_{i}(e_{i}) = f_{i}(e_{i})\alpha g_{i}(f_{i}(e_{i})\alpha)$. We show that the set $T_{\Gamma} = \{(f_{i}(e_{i}), \alpha_{g_{i}})\}_{i \in I}$ is dual basis for A. Let $a \in A$, then there is an $i \in I$ such that $a \in P_{i}$. Hence $a = f_{i}(e_{i})\alpha g_{i}$ for some $m_{i} \in M$ and $\alpha \in \Gamma$, thus $a = f_{i}(e_{i})\alpha g_{i}(f_{i}(e_{i}))\alpha m_{i} = f_{i}(e_{i})\alpha g_{i}(a)$. We define $g: F \rightarrow A$ such that $g((a_{1}, \alpha_{1})) = a_{1}$. Also we define $h: A \rightarrow F$ such that $h(a) = (a_{1}, \alpha_{1})\alpha f_{i}(a)$. We have $g(h(a)) = g((a_{1}, \alpha_{1}))\alpha f_{i}(a) = a$. Hence A is a retract of F. Hence A is projective, by (2.6).

**Sufficiency.** Let A have a dual basis as $T = \{(a_{i}, \alpha_{i}, \beta_{i})\}_{i \in I}$, i.e. if $a \in A$ then there exists $i \in I$ and $\alpha \in \Gamma$ such that $a = a_{i}\alpha f_{i}(a)$. Now let F be the free $M_{r}$-act with basis $\{(a_{i}, \alpha, 1)\}_{i \in I}$. We define $g: F \rightarrow A$ such that $g((a_{i}, \alpha, 1)) = a_{i}$. Also we define $h: A \rightarrow F$ such that $h(a) = (a_{i}, \alpha, 1)\alpha f_{i}(a)$. We have $gh(a) = g(h(a)) = g((a_{i}, \alpha, 1))\alpha f_{i}(a) = a$. Hence A is a retract of F. Hence A is projective, by (2.6).

3. **Duo Gamma Acts.**

**3.1. Definition:** An $M_{r}$-subact $B$ of $M_{r}$-act $A$ is said to be a fully invariant, if $f(B)$ is a subset of $B$, for every $M_{r}$-endomorphism $f$ of $A$. Clearly $A$ and $\emptyset$ are fully invariant $M_{r}$-subacts of $A$.

**Note that:** Every $Z_{H}$-subact $nHZ$ of $Z_{H}$-act $Z$ (where $H$ is $H$-submonoid of $Z$ and $n$ is positive integer) is fully invariant. Since $f(nHZ)$ is a subset of $nHZ$, for every $Z_{H}$-endomorphism $f$ of $Z$.

**3.2. Proposition:** Let $A$ be an $M_{r}$-act with $B_{i}$, $B$, $B'$ are $M_{r}$-subacts of $A$, $i = 1, 2, 3, \ldots, n$.

(1) If $B_{i}$ is a fully invariant $M_{r}$-subact of $B'$ and $B'$ is a fully invariant $M_{r}$-subact of $A$, then $B$ is a fully invariant $M_{r}$-subact of $A$.

(2) If $B$ is a fully invariant $M_{r}$-subact of $A$ and $B'$ is a retract $M_{r}$-subact of $A$, then $B$ is a fully invariant $M_{r}$-subact of $B'$.

(3) If $B_{i}$ is a fully invariant $M_{r}$-subact of $A$, $i = 1, 2, 3, \ldots, n$, then $\bigcap_{i=1}^{n} B_{i}$ is a fully invariant $M_{r}$-subact of $A$.

(4) If $B_{i}$ is a fully invariant $M_{r}$-subact of $A$, $i = 1, 2, 3, \ldots, n$ then $\bigcup_{i=1}^{n} B_{i}$ is a fully invariant $M_{r}$-subact of $A$.

(5) Every stable $M_{r}$-subact of $A$ is fully invariant. But these notions coincide for retract $M_{r}$-subact of $A$.

**Proof:** (1) Let $f$ be an $M_{r}$-endomorphism of $A$. Since $B'$ is a fully invariant $M_{r}$-subact of $A$, then $f(B') \subseteq B'$. Thus $f$ is an $M_{r}$-endomorphism of $B'$, then $f(B) \subseteq B$, by $B$ is a fully invariant $M_{r}$-subact of $B'$.

(2) Let $B$ be an $M_{r}$-subact of $B'$ and $f$ an $M_{r}$-endomorphism of $B'$. Since $B'$ is a retract $M_{r}$-subact of $A$, there is an epimorphism $h: A \rightarrow B'$. We have $i_{B'}: f: A \rightarrow A$ and $\alpha \in \Gamma$. By fully invariant of $B'$, $f(B') = f(h(B')\alpha 1) = i_{B'} f dh(B') \subseteq B'$. 


(3) Let $f$ be an $M_\Gamma$-endomorphism of $A$. Since $B_i$ is a fully invariant $M_\Gamma$-subact of $A$, $i = 1, 2, 3, \ldots, n$, then $f(\bigcap_{i=1}^n B_i) \subseteq f(B_i) \subseteq B_i$, for each $i = 1, 2, 3, \ldots, n$. Thus $f(\bigcap_{i=1}^n B_i) \subseteq \bigcap_{i=1}^n B_i$.

(4) Let $f$ be an $M_\Gamma$-endomorphism of $A$. Since $B_i$ is a fully invariant $M_\Gamma$-subact of $A$, $i = 1, 2, 3, \ldots, n$, then $f(\bigcup_{i=1}^n B_i) \subseteq \bigcup_{i=1}^n f(B_i) \subseteq \bigcup_{i=1}^n B_i$.

3.3. Proposition: Let $A$ be a projective $M_\Gamma$-act. If $C$ is a fully invariant $M_\Gamma$-subact of $A$, then $\frac{C}{\delta}$ is a fully invariant $M_\Gamma$-subact of $A$, where $\delta = \delta \cap (C \times C)$, for each congruence $\delta$ on $A$.

Proof: Let $\frac{C}{\delta}$ be an $M_\Gamma$-subact of $\frac{A}{\delta}$ and $g: \frac{A}{\delta} \rightarrow \frac{A}{\delta}$ an $M_\Gamma$-endomorphism. Since $A$ is a projective $M_\Gamma$-act, there exists $g': A \rightarrow A$ such that $\pi_\delta g' = g \pi_\delta \text{ ------(i)}$. Where $\pi_\delta: A \rightarrow \frac{A}{\delta}$ is the natural mapping.

Since $C$ is a fully invariant $M_\Gamma$-subact of $A$, then $g'(C) \subseteq C$ implies $\pi_\delta g'(C) \subseteq \pi_\delta(C)$, by using (i), $g(\frac{C}{\delta}) = g \pi_\delta(C) = \pi_\delta g'(C) \subseteq \pi_\delta(C) = \frac{C}{\delta}$.

3.4. Definition: An $M_\Gamma$-act $A$ is said to be duo, if every $M_\Gamma$-subact of $A$ is fully invariant. A $\Gamma$-monoid $M$ is called right (left) duo, if $M$ as a right (left) $M_\Gamma$-act is duo, Madhusudhana.

In the following, we give a simpler form of duo $M_\Gamma$-act which is more unable than the definition.

3.5. Lemma: The following statements are equivalents for an $M_\Gamma$-act $A$.

(1) $A$ is duo. (2) Every indecomposable $M_\Gamma$-subact of $A$ is a fully invariant.

(3) For each $a \in A$, $a \Gamma M$ is a fully invariant $M_\Gamma$-subact of $A$. (4) For each $M_\Gamma$-endomorphism $f: A \rightarrow A$ and $a \in A$, $f(a) = a \beta t$, for some $\beta \in \Gamma$ and $t \in M$.

Proof: (1) $\Rightarrow$ (2) It is clear. (2) $\Rightarrow$ (3) By (2) and Every cyclic $M_\Gamma$-act is indecomposable, Proposition (5.15) in Abbas. (3) $\Rightarrow$ (4) Since $f(a \Gamma M) \subseteq a \Gamma M$, for each $a \in A$, thus there are $\beta \in \Gamma$ and $t \in M$ such that $f(a) = a \beta t$. (4) $\Rightarrow$ (1) Let $B$ be an $M_\Gamma$-subact of $A$. For each $b \in B$, $f(b) \in f(B)$, then $f(b) = b \beta t \in B$ for some $\beta \in M$ and $t \in \Gamma$. Hence $f(B) \subseteq B$.

3.6. Corollary: The following statements are equivalents for an $M_\Gamma$-act $A$ with $M_\Gamma$-endomorphism $\Gamma$-monoid $T$ of $A$.

(1) $A$ is duo. (2) $T \Delta a \subseteq a \Gamma M$, for each $a \in A$.

Proof: (1) $\Rightarrow$ (2) Let $f \alpha a \in T \Delta a$, for each $a \in A$. By (3.5),(4), $f \alpha a = f(a \alpha t) = f(a) = a \alpha t \in a \Gamma M$, for some $t \in M$ and $a \in \Gamma$. (2) $\Rightarrow$ (1) For each $M_\Gamma$-endomorphism $f: A \rightarrow A$ and for each $a \in A$, $f(a) = f(a \alpha t) = f \alpha a \in T \Delta a$. By (2), $f \alpha a \in a \Gamma M$, implies that $f(a) = a \beta t$ for some $\beta \in \Gamma$ and $t \in M$.

3.7. Corollary: The following statements are equivalents for a commutative $\Gamma$-monoid $M$ with $M_\Gamma$-endomorphism $\Gamma$-monoid $T$.

(1) $A$ is a duo $M_\Gamma$-act. (2) $T a \alpha = a \Gamma M$, for each $a \in A$. 

Proof: (2)⇒(1) It follows (3.6). (1)⇒(2) Firstly, let \( a \alpha m \in a \Gamma M \), for each \( \alpha \in \Gamma \) and \( m \in M \). Define mapping \( \lambda^\Gamma_m : A \rightarrow A \) by \( \lambda^\Gamma_m(a) = a \alpha m \). By commutative \( \Gamma \)-monoid of \( M \), \( f \) is a well-defined and \( \lambda^\Gamma_m \) is \( M_f \)-endomorphism. Then, \( \lambda^\Gamma_m(a \alpha 1) = \lambda^\Gamma_m(\alpha m) \in T \alpha a \). Hence, \( a \Gamma M \subseteq T \Gamma a \). By (3.6), (2), \( T \Gamma a = a \Gamma M \).

3.8. Corollary: The following statements are equivalents for an \( M_f \)-act \( M \) and \( T = \text{End}_{M_f}(M) \).

(1) \( M \) is duo. (2) \( T \Gamma m \subseteq m \Gamma M \), for each \( m \in M \). (3) \( M \Gamma m \subseteq m \Gamma M \), for each \( m \in M \). (4) Every right \( \Gamma \)-ideal \( I \) of \( M \) is two sided \( \Gamma \)-ideal.

Proof: (1)⇒(2) It follows (3.6), (2). (2)⇒(3) Since \( T \Gamma m \subseteq m \Gamma M \). (3)⇒(4) By Madhusudhana.

3.9. Proposition: Let \( A \) be a duo \( M_f \)-act. Then \( f(A) \) is an essential \( M_f \)-subact in \( A \), for every \( M_f \)-monomorphism \( f \in \text{End}_{M_f}(A) \).

Proof: Let \( B \) be an \( M_f \)-subact in \( A \) such that \( B \cap f(A) = \emptyset \). Since \( B \) is fully invariant, \( f(B) \subseteq B \). Then \( f(B) = f(A) \cap f(B) \subseteq B \cap f(A) = \emptyset \), since \( f(B) \subseteq f(A) \) and hence \( B = \emptyset \). Thus \( f(A) \) is an essential \( M_f \)-subact in \( A \).

3.10. Examples and Remarks:

(1) Every simple \( M_f \)-act is duo.

(2) A cyclic \( M_f \)-act may be not duo, in general \( Q \) as \( (Z_H \text{-act}) \) is not duo, for any \( \Gamma \)-submonoid \( H \) of \( Z \).

For define \( f : Q \rightarrow Q \), by \( f(q) = \frac{1}{b}mq \), for each \( m \in H \) and \( q \in Q \). But \( f(Z) \not\subseteq Z \), since \( f(q) = \frac{1}{b}mq \), take \( q=m=1 \), \( f(1) = \frac{1}{b} \in Z \), contradiction.

(3) \( Z_4 \cup Z_4 \) as \( (Z_H \text{-act}) \) is not duo, for any \( \Gamma \)-submonoid \( H \) of \( Z \). Since there is \( Z_H \)-homomorphism \( f : Z_4 \cup Z_4 \rightarrow Z_4 \cup Z_4 \) defined by \( f(z_1, z_2) = (z_2, z_1) \), for each \( (z_1, z_2) \in Z_4 \cup Z_4 \). But \( f(Z_4 \cup \emptyset) = \emptyset \cup Z_4 \not\subseteq Z_4 \cup \emptyset \).

(4) Let \( N \) be a proper \( \Gamma \)-submonoid of a \( \Gamma \)-monoid \( M \). Then \( M \) is not duo \( N_f \)-act.

Proof: Define \( f : M \rightarrow M \) by \( f(m) = m \alpha m \) for each \( \alpha \in \Gamma \) and \( m \in M \), but \( m' \not\in N \). It is trivial to see \( f \) is well-defined and \( M_f \)-homomorphism. Then \( f(N) \not\subseteq N \), since \( 1 \in N \), but \( f(1) = m \alpha 1 \not\in N \). Hence \( N \)-act \( M \) is not Duo.

(5) Every retract of duo \( M_f \)-act is duo.

(6) An \( M_f \)-isomorphic to duo \( M_f \)-act is duo and the proof is a routine matter.

3.11. Proposition: Let \( A \) be a duo \( M_f \)-act and projective. Then every \( M_f \)-homomorphism image of \( A \) is a duo \( M_f \)-act.

Proof: By (3.3).

\( Z \) is Duo \( Z_f \)-act and projective, by (3.11), every \( Z_f \)-subact of \( Z \) is Duo.

3.12. Corollary: Let \( M \) be a duo \( M_f \)-act. Then every \( M_f \)-homomorphism image of \( M \) is a duo \( M_f \)-act.
An $M_\Gamma$-act $A$ is said to be a $\Gamma$-multiplication, if each $M_\Gamma$-subact of $A$ is of the form $A\Gamma I$, for some $\Gamma$-ideal $I$ of $M$.

3.13. **Proposition:** Let $A$ be an $M_\Gamma$-act. Then, the following are equivalent.

(1) $A$ is $\Gamma$-multiplication. (2) For each $M_\Gamma$-subact $B$ of $A$, $B = A\Gamma[B:A]$ (3) Every cyclic $M_\Gamma$-subact of $A$ is of the form $A\Gamma I$, for some $\Gamma$-ideal $I$ of $M$.

**Proof:** (1)$\implies$(2): Let $B$ be an $M_\Gamma$-subact of $A$. Since $A$ is $\Gamma$-multiplication, $B = A\Gamma I$, for some $\Gamma$-ideal $I$ of $M$. In particular $A\Gamma m \subseteq B$ for all $m \in I$. Hence, $m \in [B:A]$, thus $b = am \in A\Gamma[B:A]$, for some $a \in A$, $\alpha \in \Gamma$ and $m \in I$ implies $B \subseteq A\Gamma[B:A]$. The other side. Let there exists an element $m \in [B:A]$, then, $A\Gamma m \subseteq B$. Hence, $A\Gamma[B:A] \subseteq B$ and thus, $B = A\Gamma[B:A]$. (2)$\implies$(3) Obvious. (3)$\implies$(1): Let $B$ be an $M_\Gamma$-subact of $A$. Hence, $B = \cup_{b \in B} b\Gamma M = \cup_{b \in B}(A\Gamma I_b)$ for some $\Gamma$-ideal $I_b$ of $M$. Thus, $B = \cup_{b \in B}(A\Gamma I_b) = A\Gamma(\cup_{b \in B} I_b)$.

The following proposition gives some source of duo gamma acts.

3.14. **Proposition:** Every $\Gamma$-multiplication $M_\Gamma$-act $A$ is duo.

**Proof:** Let $B$ be an $M_\Gamma$-subact of $A$ and an $\Gamma$-endomorphism $f : A \to A$. By $\Gamma$-multiplication of $A$, there exists a $\Gamma$-ideal $I$ of $M$, such that $B = A\Gamma I$, therefore $f(B) = f(A\Gamma I) = f(A\Gamma) = f(A)\Gamma \subseteq A\Gamma I = B$.

But converse of (3.14) isn't true in general, for example the set $Z_{P\infty}$ as $Z_{G}$-act is duo but not $\Gamma$-multiplication.

3.15. **Corollary:** Let $M$ be a commutative $\Gamma$-monoid. Every cyclic $M_\Gamma$-act $A$ is duo.

3.16. **Theorem:** Let $A$ has a dual basis. Then $A$ is duo $M_\Gamma$-act, if and only if, $A$ is $\Gamma$-multiplication $M_\Gamma$-act.

**Proof:** **Necessity.** Let $A$ be a duo $M_\Gamma$-act, $B$ an $M_\Gamma$-subact of $A$ and $M^A = \text{Hom}(A, M)$. Since $A$ has dual basis, there exists a subset $T_\Gamma = \{(a_j, \alpha, f_j), j \in J\}$ of $A \times \Gamma \times M^A$ such that for each $a \in A$, $a = a_j\alpha f_j(a)$, where $(a_j, \alpha, f_j) \in T_\Gamma$. Let $I$ be a $\Gamma$-ideal generated by $f_j(a)$, for $a \in B$ and $j \in J$. We claim that $B = A\Gamma I$. If $a \in B \subseteq A$, then $a = a_j\alpha f_j(a)$ for some $j \in J$ and $(a_j, \alpha, f_j) \in T_\Gamma$ and hence $a \in A\Gamma I$. Let $b \in B$, $a \in A$ and $j \in J$. Thus $h_a : M \to A$ is defined by $h_a(m) = am$, for each $m \in M$ and $\alpha \in \Gamma$. Then $a\alpha f_j(b)\equiv h_a(f_j(b)) \in h_a(f_j(b)\Gamma M) \subseteq b\Gamma M$, since duo property of $A$. Hence $A\Gamma I \subseteq B$ and so $A\Gamma I = B$. Thus $A$ is $\Gamma$-multiplication $M_\Gamma$-act. **Sufficiency.** By (3.14).

The following proposition, we characterize duo $\Gamma$-monoids in terms of their gamma acts.

3.17. **Proposition:** The following statements are equivalent for a $\Gamma$-monoid $M$.

(1) $M$ is duo. (2) Every cyclic $M_\Gamma$-act $A$ is duo.
Proof: (1)⇒(2) Let \( a\Gamma M \) be a cyclic \( M_\Gamma \)-act. Then \( a\Gamma M \cong \frac{M}{\delta} \) for some congruence \( \delta \) on \( M \). Hence \( a\Gamma M \) is duo \( M_\Gamma \)-act, by (3.12) & (1). Hence \( a\Gamma M \) is duo, by (3.10) & (6). (2)⇒(1) Obvious.

3.18. Proposition: The following statements are equivalent for a \( \Gamma \)-monoid \( M \).

(1) \( M \) is duo. (2) \( M \) is \( \Gamma \)-multiplication. (3) Every cyclic \( M_\Gamma \)-act is \( \Gamma \)-multiplication. (4) Every cyclic \( M_\Gamma \)-act is duo.

Proof: (1)⇒(2) By (2.5), (2) & (3.16). (2)⇒(3) Let \( a\Gamma M \) be a cyclic \( M_\Gamma \)-act. Then \( a\Gamma M \cong \frac{M}{\delta} \) for some \( \Gamma \)-congruence \( \delta \) on \( M \). \( \frac{M}{\delta} \) is a \( \Gamma \)-multiplication \( M_\Gamma \)-act. Hence \( a\Gamma M \) is \( \Gamma \)-multiplication. (1)⇒(4) By (3.17).

3.19. Proposition: Let \( A \) be a duo \( M_\Gamma \)-act where \( A=\bigcup_{i\in I} A_i \). Then \( B=\bigcup_{i\in I} (B \cap A_i) \) for each \( M_\Gamma \)-subact \( B \) of \( A \).

Proof: Let \( B \) be an \( M_\Gamma \)-subact of \( A \) and let projection mapping \( f_i:A\rightarrow A_i \). Since \( B \) is a fully invariant \( M_\Gamma \)-subact of \( A \), we have \( f_i(B)\subseteq B \) for all \( i \in I \). Therefore, \( B \subseteq \bigcup_{i\in I} f_i(B) \), since any \( b \in B \), implies that \( b \in A=\bigcup_{i\in I} A_i \), then \( b = a_i \), for some \( i \in I \). Thus \( f_i(b) = a_i \in B \cap A_i \), by \( f_i(B) \subseteq B \) for all \( i \in I \). Then \( B \subseteq \bigcup_{i\in I} (B \cap A_i) \subseteq B \) and \( \cap_{i\in I} (B \cap A_i) = \emptyset \), since \( \cap_{i\in I} (A_i) = \emptyset \). Hence \( B = \bigcup_{i\in I} (B \cap A_i) \).

4. Conclusion and Future Works.

In the present paper, we have introduced the concepts of duo and projective on gamma acts and investigated some of their applications and essential properties. We think this work would enhance the scope for further study in this field of duo and projective acts. It is our hope that this work is going to impact the upcoming research works in this field of gamma acts with a new horizon of interest and innovation such full stability and cyclic quasi-injectivity. These new notions will depend on duo gamma acts and hence we can study their properties using the structure of full stability and cyclic quasi-injectivity on acts.

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