Free energy of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables

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Abstract. We obtain an exact formula for the equilibrium free energy of a charged quantum particle moving in a harmonic potential in the presence of a uniform external magnetic field and linearly coupled to a heat bath of independent quantum harmonic oscillators through the momentum variables. We show that the free energy has a different expression than that for the coordinate–coordinate coupling between the particle and the heat-bath oscillators. For an illustrative heat-bath spectrum, we evaluate the free energy in the low-temperature limit, thereby showing that the entropy of the charged particle vanishes at zero temperature, in agreement with the third law of thermodynamics.

Keywords: rigorous results in statistical mechanics
1. Introduction

Consider the dissipative dynamics of a charged quantum particle moving in a binding potential in the presence of a magnetic field and coupled to the external environment. Such a situation is frequently encountered in typical experiments designed to study the magnetic response of a charged particle in the context of, e.g., Landau diamagnetism [1], quantum Hall effect [2], and high-temperature superconductivity [3].

In a recent paper [4], we analyzed the dynamics of the charged particle by regarding the environment as a quantum mechanical heat bath or reservoir, and by invoking a gauge-invariant system-plus-reservoir model. The heat bath was taken to consist of a collection of independent quantum harmonic oscillators, while its interaction with the charged particle was modeled in terms of bilinear coupling between the momentum variables of the particle and the oscillators. We considered a harmonic binding potential and a magnetic field which is uniform in space. The Hamiltonian of the system is

\[ H_0 = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + \frac{1}{2} m \omega_0^2 \mathbf{r}^2 + \sum_{j=1}^{N} \left[ \frac{1}{2m_j} \left( \mathbf{p}_j - g_j \mathbf{p} + \frac{g_j e}{c} \mathbf{A} \right)^2 + \frac{1}{2} m_j \omega_j^2 \mathbf{q}_j^2 \right], \tag{1} \]

where \( e, m, \mathbf{p}, \mathbf{r} \) are, respectively, the charge, the mass, the momentum operator and the coordinate operator of the particle, while \( \omega_0 \) is the frequency characterizing its motion in the harmonic potential. The \( j \)th heat-bath oscillator has mass \( m_j \), frequency \( \omega_j \), coordinate operator \( \mathbf{q}_j \), and momentum operator \( \mathbf{p}_j \). The dimensionless parameter \( g_j \) describes the coupling between the particle and the \( j \)th oscillator. The speed of light

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Free energy of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables in vacuum is denoted by $c$. The vector potential $A = A(r)$ is related to the uniform external magnetic field $B = (B_x, B_y, B_z)$ through $B = \nabla \times A(r)$. The field has the magnitude $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$. The commutation relations for the different coordinate and momentum operators are

$$[r_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}, [q_{j\alpha}, p_{k\beta}] = i\hbar \delta_{jk} \delta_{\alpha\beta}, \quad (2)$$

while all other commutators vanish. In the above equation, $\delta_{jk}$ denotes the Kronecker delta function. Throughout this paper, Greek indices $(\alpha, \beta, \ldots)$ refer to the three spatial directions, while Roman indices $(i, j, k, \ldots)$ represent the heat-bath oscillators. Moreover, we use the Einstein summation convention for the Greek indices. Let us remark that momentum–momentum coupling has been considered earlier in the literature [5], and, in particular, to model the physical situation of a single Josephson junction interacting with the blackbody electromagnetic field in the dipole approximation [6, 7]. Our model Hamiltonian is similar to that considered in [6, 7]; the additional interesting feature that we consider here is the inclusion of the effects of an external magnetic field.

In [4], we derived a quantum Langevin equation (QLE) satisfied by the coordinate operator of the charged particle. In this equation, coupling to the bath is described solely by (i) an operator-valued random force, and (ii) a mean force characterized by a memory function. We showed that, similar to the case of coordinate–coordinate coupling between the charged particle and the heat-bath oscillators, the QLE involves a quantum-generalized Lorentz force term, and also that the random force does not depend on the magnetic field. This latter force, nevertheless, has a modified form, with symmetric correlation and unequal time commutator different from their corresponding expressions in the case of coordinate–coordinate coupling. Other differences include (i) the memory function has an explicit dependence on the magnetic field, and (ii) the inertial term and the harmonic potential term in the QLE get renormalized by the coupling constants.

In this paper, we further our study of the system (1) by deriving an exact formula for the mean internal energy, and thence the free energy of the charged particle in thermal equilibrium. Knowing either the mean internal energy or the free energy, one can compute by taking suitable derivatives a host of thermodynamic functions, e.g., specific heat, susceptibility, etc [8]–[11]. Our result shows important differences in the form of the free energy with respect to that for the coordinate–coordinate coupling [12]. For an illustrative heat-bath spectrum, we evaluate the free energy in the low-temperature limit, thereby showing that the entropy of the charged particle vanishes at zero temperature, in conformity with the third law of thermodynamics.

The paper is structured as follows. In the following section, we write down the equations of motion for the charged particle and the heat-bath oscillators. In section 3, we present a detailed derivation of the equilibrium free energy of the charged particle, and point out its differences from the case of coordinate–coordinate coupling. In the section that follows, we consider an example of a heat-bath spectrum and analyze the free energy of the charged particle in the limit of low temperatures. We draw our conclusions in section 5.
2. Equations of motion

For system (1), the Heisenberg equations of motion for the charged particle are

\[ \dot{r} = \frac{1}{i\hbar} [r, H_0] = \frac{1}{m} \left( p - \frac{e}{c} A \right) - \sum_{j=1}^{N} \frac{g_j}{m_j} \left( p_j - g_j p + \frac{g_j e}{c} A \right), \quad (3) \]

and

\[ \dot{p} = \frac{1}{i\hbar} [p, H_0] = \frac{e}{c} (\dot{r} \times B) + \frac{e}{c} (\dot{r} \cdot \nabla) A + \frac{i\hbar e}{2m_c} \nabla (\nabla \cdot A) - m_0^2 \dot{r}, \quad (4) \]

where \( m_r \) is the ‘renormalized mass’,

\[ m_r \equiv m \left/ \left[ 1 + \sum_{j=1}^{N} \frac{g_j^2 m}{m_j} \right] \right., \quad (5) \]

and dots denote differentiation with respect to time. The Heisenberg equations for the heat-bath oscillators are

\[ \dot{q}_j = \frac{1}{i\hbar} [q_j, H_0] = \frac{1}{m_j} \left( p_j - g_j p + \frac{g_j e}{c} A \right), \quad (6) \]

and

\[ \dot{p}_j = \frac{1}{i\hbar} [p_j, H_0] = -m_j \omega_j^2 q_j. \quad (7) \]

These equations lead to

\[ m_r \ddot{r} = -m_0^2 \ddot{r} + \frac{e}{c} (\dot{r} \times B) + \sum_{j=1}^{N} g_j m_r \omega_j^2 q_j, \quad (8) \]

and

\[ m_j \ddot{q}_j = -m_j \omega_j^2 q_j + g_j m_0^2 \dot{r} - \frac{g_j e}{c} (\dot{r} \times B). \quad (9) \]

For details on the derivation of equations (3)–(9), see [4].

3. Equilibrium free energy

In this section, we present our computation of the equilibrium free energy of the charged particle. The steps of computation are as follows: we assume that the system (1) is in weak contact with a super-bath that allows the system to attain thermal equilibrium at temperature \( T \). In such an equilibrium state, we obtain the mean internal energy \( U_0(T, B) \) of the particle, defined as the equilibrium mean internal energy of the system of the charged particle interacting with the heat bath, given by \( \langle H_0 \rangle \), minus that of the heat bath in the absence of coupling with the particle. In our case of a heat bath comprising independent quantum harmonic oscillators, this latter quantity is given by

\[ U_B(T) = \sum_{j=1}^{N} \frac{3\hbar \omega_j}{2} \coth \frac{\hbar \omega_j}{2k_B T}. \quad (10) \]

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Once $U_0(T, B)$ has been computed, the free energy $F_0(T, B)$ of the charged particle is obtained by using the usual relationship between the two: $U_0(T, B) = F_0(T, B) - T(\partial F_0(T, B) / \partial T)$.

Let us start by computing $\langle H_0 \rangle$. Using equation (3) and (6), we get

$$\langle H_0 \rangle = \left[ \frac{1}{2} m \langle \dot{r}^2 \rangle + \frac{1}{2} m \omega_0^2 \langle r^2 \rangle \right] + \frac{m}{2} \sum_{j=1}^{N} g_j \langle \dot{r} \cdot \dot{q}_j + \dot{q}_j \cdot \dot{r} \rangle$$

\[ + \sum_{j,k=1}^{N} g_j g_k \langle \dot{q}_j \cdot \dot{q}_k \rangle \] 

Then, to find $\langle H_0 \rangle$, we need equilibrium averages, such as $\langle r^2 \rangle$ and $\langle \dot{r}^2 \rangle$, which may be obtained by knowing the equilibrium autocorrelation function of the position of the charged particle, defined as

$$\psi_{\rho \sigma}^{\rho \sigma}(t - t') \equiv \frac{1}{2} \langle r_\rho(t) r_\rho(t') + r_\sigma(t') r_\rho(t) \rangle.$$ (12)

Similarly, $\langle q_j^2 \rangle$, $\langle \dot{q}_j \rangle$, and $\langle \dot{q}_j \cdot \dot{q}_k \rangle$ are obtained from the position autocorrelation function of the heat-bath oscillators, defined as

$$\psi_{jk,\rho\sigma}^{\rho \sigma}(t - t') \equiv \frac{1}{2} \langle q_{j \rho}(t) q_{k \sigma}(t') + q_{k \sigma}(t') q_{j \rho}(t) \rangle,$$ (13)

while $\langle \dot{r} \cdot \dot{q}_j + \dot{q}_j \cdot \dot{r} \rangle$ may be computed from the correlations

$$\psi_{j,\rho \sigma}^{\rho \sigma}(t - t') \equiv \frac{1}{2} \langle r_\rho(t) q_{j \sigma}(t') + q_{j \sigma}(t') r_\rho(t) \rangle,$$ (14)

and

$$\psi_{j,\rho \sigma}^{\sigma \rho}(t - t') \equiv \frac{1}{2} \langle q_{j \rho}(t) r_\sigma(t') + r_\sigma(t') q_{j \rho}(t) \rangle.$$ (15)

Correlations such as in equations (12)–(15) are computed conveniently by invoking the fluctuation–dissipation theorem that relates such equilibrium correlations to response of the system to small external perturbations [13]. To this end, consider a weak external force $f(t)$ to act on the charged particle and another set of weak external forces $\{f_j(t), j = 1, 2, \ldots, N\}$ to act on the oscillators. Here, $f(t)$ and $f_j(t)$ are c-number functions of time. We take the perturbed Hamiltonian to be of the form

$$H = H_0 - \mathbf{r} \cdot \mathbf{f}(t) - \sum_{j=1}^{N} \mathbf{q}_j \cdot \mathbf{f}_j(t).$$ (16)

In the presence of these external forces, the Heisenberg equations of motion (3) and (6) remain the same, while those given by equations (4) and (7), respectively, are modified to

$$\dot{\mathbf{p}} = \frac{e}{c} (\mathbf{r} \times \mathbf{B}) + \frac{e}{c} (\mathbf{r} \cdot \nabla) \mathbf{A} + \frac{i \hbar e}{2m_e c} \nabla (\nabla \cdot \mathbf{A}) - m \omega_0^2 \mathbf{r} + \mathbf{f},$$ (17)

and

$$\dot{\mathbf{p}}_j = -m_j \omega_j^2 \mathbf{q}_j + \mathbf{f}_j.$$ (18)

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Consequently, equations (8) and (9), respectively, are modified to

\[ m_r \ddot{r} = -m_0 \omega^2 r + \frac{e}{c} (\dot{r} \times B) + \sum_{j=1}^{N} g_j m_j \omega_j^2 q_j - \sum_{j=1}^{N} \frac{g_j m_j}{m_j} f_j + f, \]  

(19)

and

\[ m_j \ddot{q}_j = -m_j \omega_j^2 q_j + g_j m_0 \omega^2 r - \frac{g_j e}{c} (\dot{r} \times B) + f_j - g_j f. \]  

(20)

Taking the Fourier transform of equations (19) and (20), we get

\[ \delta_{\rho \sigma} (-m_0 \omega^2 + m_0 \omega_0^2) \tilde{r}_\sigma - \sum_{j=1}^{N} g_j m_j \omega_j^2 \tilde{q}_{j \rho} = - \sum_{j=1}^{N} \frac{g_j m_j}{m_j} \tilde{f}_{j \rho} + \tilde{f}_\rho, \]  

(21)

and

\[ (-m_j \omega^2 + m_j \omega_j^2) \tilde{q}_{j \rho} - \left( g_j m_0 \omega_0^2 \delta_{\rho \sigma} + \frac{i \omega e}{c} \epsilon_{\rho \sigma \eta} B_\eta \right) \tilde{r}_\sigma = \tilde{f}_{j \rho} - g_j \tilde{f}_\rho, \]  

(22)

where \( \epsilon_{\rho \sigma \eta} \) is the Levi-Civita symbol. Equations (21) and (22) give

\[ D_{\rho \sigma} (\omega) \tilde{r}_\sigma = G(\omega) \tilde{f}_\rho + \sum_{j=1}^{N} \frac{g_j m_j \omega^2}{m_j (\omega_j^2 - \omega^2)} \tilde{f}_{j \rho}, \]  

(23)

where

\[ G(\omega) = 1 - \sum_{j=1}^{N} \frac{(g_j)^2 m_j \omega_j^2}{m_j (\omega_j^2 - \omega^2)}, \]  

(24)

\[ D_{\rho \sigma} (\omega) = \lambda(\omega) \delta_{\rho \sigma} + \frac{i \omega e G(\omega)}{c} \epsilon_{\rho \sigma \eta} B_\eta, \]  

(25)

and

\[ \lambda(\omega) = -m_0 \omega^2 + m_0 \omega_0^2 G(\omega). \]  

(26)

From equation (23), we get

\[ \tilde{r}_\rho = G(\omega) \alpha_{\rho \gamma}(\omega) \tilde{f}_\gamma + \sum_{j=1}^{N} \beta_{j, \rho \gamma}(\omega) \tilde{f}_{j \gamma}, \]  

(27)

where

\[ \alpha_{\rho \gamma}(\omega) = [D(\omega)^{-1}]_{\rho \gamma} \]  

\[ = \left[ (\lambda(\omega))^2 \delta_{\rho \gamma} - \left( \frac{\omega e G(\omega)}{c} \right)^2 B_\rho B_\gamma - \frac{i \omega \lambda(\omega) e G(\omega)}{c} \epsilon_{\rho \eta \gamma} B_\eta \right] / \text{Det} \ D(\omega), \]  

(28)

\[ \text{Det} \ D(\omega) = \lambda(\omega) \left[ (\lambda(\omega))^2 - \left( \frac{\omega B e G(\omega)}{c} \right)^2 \right], \]  

(29)

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Free energy of a charged oscillator in a magnetic field and coupled to a heat bath through momentum variables and

$$\beta_{j,\rho\gamma}(\omega) = \frac{g_j m \omega^2}{m_j (\omega_j^2 - \omega^2)} \alpha_{\rho\gamma}(\omega).$$  \hfill (30)

Using equation (27) in either equation (21) or (22) gives

$$\tilde{q}_{jp} = \Delta_{j,\rho\gamma}(\omega) \tilde{f}_\gamma + \sum_{k=1}^{N} \gamma_{jk,\rho\gamma}(\omega) \tilde{f}_{k\gamma},$$  \hfill (31)

where

$$\Delta_{j,\rho\gamma}(\omega) = \frac{g_j m G(\omega)}{m_j (\omega_j^2 - \omega^2)} \left( \omega_0^2 \delta_{\rho\sigma} + \frac{i \omega e}{mc \epsilon_{\rho\eta} B_\eta} \right) \alpha_{\sigma\gamma}(\omega) - \frac{g_j}{m_j (\omega_j^2 - \omega^2)} \delta_{\rho\gamma},$$  \hfill (32)

and

$$\gamma_{jk,\rho\gamma}(\omega) = \frac{g_j g_k m \omega^2}{m_j m_k (\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2)} \left( \omega_0^2 \delta_{\rho\sigma} + \frac{i \omega e}{mc \epsilon_{\rho\eta} B_\eta} \right) \alpha_{\sigma\gamma}(\omega) + \frac{\delta_{jk} \delta_{\rho\gamma}}{m_k (\omega_k^2 - \omega^2)}. \hfill (33)$$

The functions $\alpha_{\rho\gamma}(\omega), \beta_{j,\rho\gamma}(\omega), \Delta_{j,\rho\gamma}(\omega)$, and $\gamma_{jk,\rho\gamma}(\omega)$ are coefficient matrices of the response of the system to the external perturbations, $f(t)$ and $f_j(t)$, and may be interpreted as generalized susceptibilities.

We may now use the fluctuation–dissipation theorem, which relates the Fourier transform $\tilde{\psi}_{\rho\sigma}^\alpha(\omega)$ of the position autocorrelation function $\psi_{\rho\sigma}^\alpha(t - t')$ to $G(\omega)\alpha_{\rho\sigma}(\omega)$:

$$\tilde{\psi}_{\rho\sigma}^\alpha(\omega) = \frac{\hbar}{2i} \coth \left[ \frac{\hbar \omega}{2kB_T} \right] \left[ G(\omega) \alpha_{\rho\sigma}(\omega) - G^*(\omega) \alpha^*_{\rho\sigma}(\omega) \right].$$  \hfill (34)

where * denotes complex conjugation. Using equation (24), we see that $G(\omega)$ is real, so

$$\tilde{\psi}_{\rho\sigma}^\alpha(\omega) = \frac{\hbar}{2i} \coth \left[ \frac{\hbar \omega}{2kB_T} \right] G(\omega) [\alpha_{\rho\sigma}(\omega) - \alpha^*_{\rho\sigma}(\omega)].$$  \hfill (35)

Similarly, one has

$$\tilde{\psi}_{jk,\rho\sigma}^{qq}(\omega) = \frac{\hbar}{2i} \coth \left[ \frac{\hbar \omega}{2kB_T} \right] [\gamma_{jk,\rho\sigma}(\omega) - \gamma^*_{jk,\sigma\rho}(\omega)],$$  \hfill (36)

$$\tilde{\psi}_{j,\rho\sigma}^{rr}(\omega) = \frac{\hbar}{2i} \coth \left[ \frac{\hbar \omega}{2kB_T} \right] [\beta_{j,\rho\sigma}(\omega) - \beta^*_{j,\sigma\rho}(\omega)],$$  \hfill (37)

and

$$\tilde{\psi}_{j,\rho\sigma}^{ar}(\omega) = \frac{\hbar}{2i} \coth \left[ \frac{\hbar \omega}{2kB_T} \right] [\Delta_{j,\rho\sigma}(\omega) - \Delta^*_{j,\sigma\rho}(\omega)].$$  \hfill (38)

In passing, we note that

$$\alpha^*_{\rho\sigma}(\omega) = \alpha_{\sigma\rho}(-\omega),$$  \hfill (39)

which implies that $\text{Im}[\alpha_{\sigma\rho}(\omega)]$ is an odd function of $\omega$, while $\text{Re}[\alpha_{\sigma\rho}(\omega)]$ is an even function of $\omega$. These properties are also shared by $\beta_{j,\sigma\rho}(\omega), \Delta_{j,\sigma\rho}(\omega)$, and $\gamma_{jk,\sigma\rho}(\omega)$.

In the following, we use equations (35)–(38) to evaluate the various averages involved in computing $\langle H_0 \rangle$.
3.1. Computation of various averages

3.1.1. Averages involving $r$. We start by decomposing $\alpha_{\rho\sigma}$ into symmetric and antisymmetric parts as

$$\alpha_{\rho\sigma}(\omega) = \alpha^s_{\rho\sigma}(\omega) + \alpha^a_{\rho\sigma}(\omega).$$

We then have

$$\alpha_{\rho\sigma}(\omega) - \alpha^*_{\rho\sigma}(\omega) = [\alpha^s_{\rho\sigma}(\omega) - \alpha^s_{\rho\sigma}(\omega)^*] + [\alpha^a_{\rho\sigma}(\omega) + \alpha^a_{\rho\sigma}(\omega)^*] = 2\text{Im}[\alpha^s_{\rho\sigma}(\omega)] + 2\text{Re}[\alpha^a_{\rho\sigma}(\omega)].$$

Using equations (12), (35) and (41), and the properties that Im$[\alpha_{\rho\sigma}(\omega)]$ is an odd function of $\omega$, while both $G(\omega)$ and Re$[\alpha_{\rho\sigma}(\omega)]$ are even functions of $\omega$, we get

$$\frac{1}{2}\langle r_\rho(t)r_\sigma(t') + r_\sigma(t')r_\rho(t) \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ G(\omega)\text{Im}[\alpha^s_{\rho\sigma}(\omega)] \coth \left[ \frac{\hbar \omega}{2k_BT} \right] \cos[\omega(t - t')]$$

$$- \frac{\hbar}{\pi} \int_0^\infty d\omega \ G(\omega)\text{Re}[\alpha^a_{\rho\sigma}(\omega)] \coth \left[ \frac{\hbar \omega}{2k_BT} \right] \sin[\omega(t - t')],$$

which for $\rho = \sigma$ gives

$$\frac{1}{2}\langle r(t) \cdot r(t') + r(t') \cdot r(t) \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ G(\omega)\text{Im}[\alpha_{pp}(\omega)] \coth \left[ \frac{\hbar \omega}{2k_BT} \right] \cos[\omega(t - t')].$$

Then, putting $t = t'$, we get

$$\langle r^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ G(\omega)\text{Im}[\alpha_{pp}(\omega)] \coth \left[ \frac{\hbar \omega}{2k_BT} \right],$$

where from equation (28), we have

$$\alpha_{pp}(\omega) = \left( \frac{\lambda(\omega)^2}{\omega c^2} - \left( \frac{\omega G(\omega)}{c} \right)^2 B_\rho B_\rho \right) / \text{Det} D(\omega).$$

Differentiating equation (43) successively with respect to $t$ and $t'$, and finally putting $t = t'$, we get

$$\langle r^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ G(\omega)\text{Im}[\alpha_{pp}(\omega)] \coth \left[ \frac{\hbar \omega}{2k_BT} \right] \omega^2.$$
Adding equations (53) to (52), we get
\[ \langle q_j^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ \text{Im}[\gamma_{j,j',p}(\omega)] \coth \left( \frac{\hbar \omega}{2k_B T} \right), \] (49)
\[ \langle q_j^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ \text{Im}[\gamma_{j,j',p}(\omega)] \coth \left( \frac{\hbar \omega}{2k_B T} \right) \omega^2, \] (50)
and
\[ \langle \dot{q}_j \cdot \dot{q}_k \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ \text{Im}[\gamma_{j,j',p}(\omega)] \coth \left( \frac{\hbar \omega}{2k_B T} \right) \omega^2. \] (51)

3.1.3. Averages involving \( r \) and \( q_j \). We have
\[ \frac{1}{2} \langle r_\rho(t)q_{j,\sigma}(t') + q_{j,\sigma}(t')r_\rho(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\omega(t-t')} \tilde{\psi}_{r_\rho q_{j,\sigma}}(\omega), \] (52)
and
\[ \frac{1}{2} \langle q_{j,\sigma}(t')r_\rho(t) + r_\rho(t)q_{j,\sigma}(t') \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ e^{-i\omega(t-t')} \tilde{\psi}_{q_{j,\sigma}r_\rho}(\omega). \] (53)

Adding equations (53) to (52), we get
\[ \langle r_\rho(t)q_{j,\sigma}(t') + q_{j,\sigma}(t')r_\rho(t) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ [e^{-i\omega(t-t')} \tilde{\psi}_{r_\rho q_{j,\sigma}}(\omega) + e^{-i\omega(t-t')} \tilde{\psi}_{q_{j,\sigma}r_\rho}(\omega)]. \] (54)

Differentiating equation (54) successively with respect to \( t \) and \( t' \), and then putting \( t = t' \), we get
\[ \langle \dot{r}_\rho \dot{q}_{j,\sigma} + \dot{q}_{j,\sigma} \dot{r}_\rho \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ \omega^2 \left[ \tilde{\psi}_{r_\rho q_{j,\sigma}}(\omega) + \tilde{\psi}_{q_{j,\sigma}r_\rho}(\omega) \right]. \] (55)

Using equations (37) and (38) for \( \tilde{\psi}_{r_\rho q_{j,\sigma}}(\omega) \) and \( \tilde{\psi}_{q_{j,\sigma}r_\rho}(\omega) \), respectively, and noting that the imaginary part of both \( \gamma_{j,j',\rho}(\omega) \) and \( \Delta_{j,j',\rho}(\omega) \) is an odd function of \( \omega \), while their real parts are even functions of \( \omega \), we get
\[ \langle \dot{q}_j \cdot \dot{r} + \dot{r} \cdot \dot{q}_j \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ \omega \ coth \left[ \frac{\hbar \omega}{2k_B T} \right] \omega^2 [\text{Im}[\gamma_{j,j',\rho}(\omega)] + \text{Im}[\Delta_{j,j',\rho}(\omega)]], \] (56)
where from equation (32) we get
\[ \Delta_{j,j',\rho}(\omega) = \frac{g_j m G(\omega) \omega_0^2}{m_j (\omega_j^2 - \omega^2)} \alpha_{j,j',\rho}(\omega) - \frac{g_j (G(\omega))^2 \lambda(\omega) \omega^2 e^2}{m_j (\omega_j^2 - \omega^2) e^2 \text{Det} D(\omega)} (\delta_{j,j'} B_0^2 - B_\rho B_\rho) \]
\[ - \frac{g_j}{m_j (\omega_j^2 - \omega^2)} \delta_{j,j'}. \] (57)

3.2. Mean internal energy
Using equations (44), (46), (49), (50), (51), and (56) in equation (11), we get
\[ \langle H_0 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ coth \left[ \frac{\hbar \omega}{2k_B T} \right] \left[ \frac{1}{2} m (\omega^2 + \omega_0^2) G(\omega) \text{Im}[\alpha_{j,j',\rho}(\omega)] \right]. \]
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\[ + \sum_{j=1}^{N} \frac{mg_j\omega_j^2}{2} \left( \text{Im}[\beta_{j,pp}(\omega)] + \text{Im}[\Delta_{j,pp}(\omega)] \right) \]

\[ + \sum_{j,k=1}^{N} \frac{mg_jg_k\omega_j^2}{2} \text{Im}[\gamma_{jk,pp}(\omega)] + \sum_{j=1}^{N} \frac{1}{2} m_j(\omega^2 + \omega_j^2) \text{Im}[\gamma_{jj,pp}(\omega)] \] .

(58)

The mean internal energy of the heat bath is

\[ \langle H_B \rangle = \sum_{j=1}^{N} \frac{1}{2} m_j \omega_j^2 \langle q_j^2 \rangle + \sum_{j=1}^{N} \frac{1}{2} m_j \langle q_j^2 \rangle \]

\[ = \sum_{j=1}^{N} \frac{m_j}{2} \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \text{Im}[\gamma_{jj,pp}(\omega)](\omega^2 + \omega_j^2) \right] \]

\[ = \sum_{j=1}^{N} \frac{3\hbar}{2} \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) \left[ \frac{\hbar \omega}{2} \right] \sum_{j=1}^{N} m_j(\omega^2 + \omega_j^2) \]

\[ \times \text{Im} \left[ \frac{(g_j)^2 m_j \omega_j^2 \omega_0^2}{(m_j)^2(\omega_j^2 - \omega^2)^2} \alpha_{pp}(\omega) - \frac{2(g_j)^2 m_j \omega^4 \lambda(\omega) B^2 e^2 G(\omega)}{(m_j)^2(\omega_j^2 - \omega^2)^2} \right] \]

\[ = \sum_{j=1}^{N} \frac{3\hbar \omega_j}{2} \coth \left( \frac{\hbar \omega_j}{2k_B T} \right) \]

\[ + \frac{\hbar}{2} \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right) \sum_{j=1}^{N} m_j(\omega^2 + \omega_j^2) \]

\[ \times \text{Im} \left[ \frac{(g_j)^2 m_j \omega_j^2 \omega_0^2}{(m_j)^2(\omega_j^2 - \omega^2)^2} \alpha_{pp}(\omega) - \frac{2(g_j)^2 m_j \omega^4 \lambda(\omega) B^2 e^2 G(\omega)}{(m_j)^2(\omega_j^2 - \omega^2)^2} \right] . \]

(59)

Here, in obtaining equation (59), we have used equation (48). To arrive at equation (60), we have performed the first integral on the right hand side of equation (59) by noting that \( \omega \) in the integral is approached from above the real axis, \( \omega \to \omega + i0^+ \), and by using the result that

\[ \frac{1}{\omega - \omega_j + i0^+} = P \left[ \frac{1}{\omega - \omega_j} \right] - i\pi \delta(\omega - \omega_j), \]

with \( P \) denoting the principal value. Note that the first term on the right hand side of equation (60) is the mean internal energy \( U_B(T) \) of the heat bath in the absence of any coupling with the charged particle:

\[ U_B(T) = \sum_{j=1}^{N} \frac{3\hbar \omega_j}{2} \coth \left( \frac{\hbar \omega_j}{2k_B T} \right) . \]

(60)

We now obtain the mean internal energy \( U_0(T, B) \) of the particle as the mean internal energy of the system of the charged particle interacting with the heat bath minus the
mean internal energy of the heat bath in the absence of coupling with the particle:

\[ U_0(T, B) = \langle H_0 \rangle - U_B(T) \]

Following [12], we write

\[ U = \frac{h}{\pi} \int_0^\infty d\omega \coth \left( \frac{h\omega}{2k_BT} \right) \left[ \frac{1}{2} m(\omega^2 + \omega_0^2)G(\omega) \Im \alpha_{\rho\rho}(\omega) \right] \]

\[ + \sum_{j=1}^N \frac{mg_j\omega^2}{2} \left( \Im \left[ \frac{g_jm_j\omega^2}{m_j(\omega_j^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \right) \]

\[ + \Im \left[ \frac{g_jm_j\omega^2}{m_j(\omega_j^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \]

\[ + \sum_{j,k=1}^N \frac{mg_jg_k\omega^2}{2} \left( \Im \left[ \frac{g_jg_km_j\omega^2}{m_jm_k(\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \right) \]

\[ - \frac{2g_jg_km_j\omega^4\lambda(\omega)\omega^2G(\omega)B^2}{m_jm_k(\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2)c^2 \det D(\omega)} + \frac{3\delta_{jk}}{m_k(\omega_k^2 - \omega^2)} \]

\[ + \sum_{j=1}^N \frac{1}{2} m_j(\omega_j^2 + \omega^2) \Im \left[ \frac{(g_j)^2mm_j\omega^2}{(m_j)^2(\omega_j^2 - \omega^2)^2} \alpha_{\rho\rho}(\omega) \right] - \frac{2(g_j)^2m_j\omega^4\lambda(\omega)B^2\omega^2G(\omega)}{(m_j)^2(\omega_j^2 - \omega^2)^2c^2 \det D(\omega)} \right] , \]

(63)

where we have used equations (30), (48), and (57). After simplification, we get

\[ U_0(T, B) = \frac{h}{\pi} \int_0^\infty d\omega \coth \left( \frac{h\omega}{2k_BT} \right) \left[ \frac{1}{2} m(\omega^2 + \omega_0^2)G(\omega) \Im \alpha_{\rho\rho}(\omega) \right] \]

\[ + \sum_{j=1}^N \frac{mg_j\omega^2}{2} \left( \Im \left[ \frac{g_jm_j\omega^2}{m_j(\omega_j^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \right) \]

\[ + \Im \left[ \frac{g_jm_j\omega^2}{m_j(\omega_j^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \]

\[ + \sum_{j,k=1}^N \frac{mg_jg_k\omega^2}{2} \left( \Im \left[ \frac{g_jg_km_j\omega^2}{m_jm_k(\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2)} \alpha_{\rho\rho}(\omega) \right] \right) \]

\[ - \frac{2g_jg_km_j\omega^4\lambda(\omega)\omega^2G(\omega)B^2}{m_jm_k(\omega_j^2 - \omega^2)(\omega_k^2 - \omega^2)c^2 \det D(\omega)} + \frac{3\delta_{jk}}{m_k(\omega_k^2 - \omega^2)} \]

\[ + \sum_{j=1}^N \frac{1}{2} m_j(\omega_j^2 + \omega^2) \Im \left[ \frac{(g_j)^2mm_j\omega^2}{(m_j)^2(\omega_j^2 - \omega^2)^2} \alpha_{\rho\rho}(\omega) \right] - \frac{2(g_j)^2m_j\omega^4\lambda(\omega)B^2\omega^2G(\omega)}{(m_j)^2(\omega_j^2 - \omega^2)^2c^2 \det D(\omega)} \right] , \]

(64)

From equations (28) and (29), we have

\[ \det \alpha(\omega) = [\det D(\omega)]^{-1} = \left[ \lambda(\omega) \left( \frac{\omega eB}{c} \right)^2 \frac{d(G(\omega))^2}{d\omega} - \frac{\omega}{2\det D(\omega)} \right] \]

(65)

while the trace of \( \alpha(\omega) \) is

\[ \alpha_{\rho\rho}(\omega) = \left[ 3(\lambda(\omega))^2 - \left( \frac{\omega eG(\omega)}{c} \right)^2 B^2 \right] / \det D(\omega) . \]

(66)

Following [12], we write

\[ \omega \frac{d}{d\omega} \ln[\det \alpha(\omega)] = -\omega \left\{ \frac{d\lambda(\omega)}{d\omega} \left[ 3(\lambda(\omega))^2 - \left( \frac{\omega eB}{c} \right)^2 - 2\omega \lambda(\omega) \left( \frac{B eG(\omega)}{c} \right)^2 \right] \right\} / \det D(\omega) \]

(67)

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From equation (26), we get
\[ \lambda(\omega) - \omega \frac{d\lambda(\omega)}{d\omega} = m_r(\omega^2 + \omega_0^2) + \sum_{j=1}^{N} \frac{(g_j)^2 m m_r^2 \omega_0^2 (\omega_j^2 + \omega^2)}{m_j(\omega_j^2 - \omega^2)^2}. \] (68)

Now, using equations (65), (67) and (68) in equation (64), we get
\[ U_0(T, B) = \frac{1}{\pi} \int_0^\infty d\omega \ u(\omega, T) \times \text{Im} \left[ \frac{d}{d\omega} \ln[\text{Det} \ \alpha(\omega)] + \lambda(\omega) \left( \frac{\omega e B}{c} \right)^2 \left( \frac{d (G(\omega))^2}{d\omega} \right) \text{Det} \ \alpha(\omega) \right], \] (69)

where \( u(\omega, T) \) is the Planck energy of a free oscillator of frequency \( \omega \):
\[ u(\omega, T) = \frac{\hbar \omega}{2} \coth \left[ \frac{\hbar \omega}{2 k_B T} \right]. \] (70)

### 3.3. Free energy

The free energy \( F_0(T, B) \) of the charged particle is obtained from equation (69) for the mean internal energy, since the two quantities are related as \( U_0(T, B) = F_0(T, B) - T(\partial F_0(T, B)/\partial T) \). We get
\[ F_0(T, B) = \frac{1}{\pi} \int_0^\infty d\omega \ f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln[\text{Det} \ \alpha(\omega)] 
+ \lambda(\omega) \left( \frac{\omega e B}{c} \right)^2 \left( \frac{d (G(\omega))^2}{d\omega} \right) \text{Det} \ \alpha(\omega) \right], \] (71)

where \( f(\omega, T) \) is the free energy of a free oscillator of frequency \( \omega \)
\[ f(\omega, T) = k_B T \ln \left[ 2 \sinh \left( \frac{\hbar \omega}{2 k_B T} \right) \right] = \frac{\hbar \omega}{2} + k_B T \ln \left[ 1 - \exp \left( \frac{-\hbar \omega}{k_B T} \right) \right], \] (72)
and where in the last equality we have separated out the contribution from the zero-point energy.

Equation (71) is the central result of the paper. To make explicit the contribution of the external magnetic field to the free energy, we write equation (65) as
\[ \text{Det} \ \alpha(\omega) = [\alpha^{(0)}(\omega)]^3 \left[ 1 - \left( \frac{\omega e G(\omega)}{c} \right)^2 [\alpha^{(0)}(\omega)]^2 \right]^{-1}, \] (73)
where
\[ \alpha^{(0)}(\omega) = 1/\lambda(\omega) \] (74)
is the susceptibility in the absence of the magnetic field. Using equation (73) in equation (71), we get
\[ F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B) + \Delta_2 F_0(T, B), \] (75)
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where

\[
F_0(T, 0) = \frac{3}{\pi} \int_0^\infty d\omega \, f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln \alpha^{(0)}(\omega) \right]
\]  

(76)

is the free energy of the charged particle in the absence of the magnetic field. The contribution from the field is contained in the two terms \(\Delta_1 F_0(T, B)\) and \(\Delta_2 F_0(T, B)\), given by

\[
\Delta_1 F_0(T, B) = -\frac{1}{\pi} \int_0^\infty d\omega \, f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln \left\{ 1 - \left( \frac{G(\omega)}{c} \right)^2 \left( \frac{\omega Be}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right\} \right],
\]  

(77)

and

\[
\Delta_2 F_0(T, B) = \frac{1}{\pi} \int_0^\infty d\omega \, f(\omega, T) \text{Im} \left[ \left( \frac{\alpha^{(0)}(\omega)}{c} \right)^2 \left( \frac{\omega Be}{c} \right)^2 \right]
\]

\[
\times \left\{ \left( \frac{dG(\omega)}{d\omega} \right)^2 \left[ 1 - \left( \frac{\omega Be G(\omega)}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right] \right\}.
\]  

(78)

Let us now comment on the form of the free energy (75) with respect to that for coordinate–coordinate coupling between the particle and the heat-bath oscillators, obtained in [12]. In the latter case, the free energy is given by

\[
F_0(T, B) = F_0(T, 0) + \Delta_1 F_0(T, B),
\]  

(79)

where \(F_0(T, 0)\) has the same form as in equation (76), while \(\Delta_1 F_0(T, B)\) is given by

\[
\Delta_1 F_0(T, B) = -\frac{1}{\pi} \int_0^\infty d\omega \, f(\omega, T) \text{Im} \left[ \frac{d}{d\omega} \ln \left\{ 1 - \left( \frac{\omega Be}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right\} \right].
\]  

(80)

We thus see that, with respect to coordinate–coordinate coupling, the free energy has two differences, namely, (i) the appearance of the extra factor \((G(\omega))^2\) in the term \(\Delta_1 F_0(T, B)\), and (ii) the presence of the additional term \(\Delta_2 F_0(T, B)\). With respect to the two schemes of coupling, the differences observed here in the form of the thermodynamic potential, i.e., the free energy, add to the ones noted in the reduced dynamical description of the charged particle by means of the quantum Langevin equation [4].

4. Explicit free energy for an illustrative heat-bath spectrum

In this section, we utilize the results obtained in the previous section to obtain and analyze the low-temperature thermodynamic behavior of the charged particle. To proceed, we first express the integrands in equations (76), (77), and (78) in terms of the Laplace transform of the diagonal part of the memory function, given by [4]

\[
\tilde{\mu}_d(\omega) = \sum_{j=1}^{N} \frac{g_j^2 m_m \omega_j^2 \omega}{m_j (w^2 - \omega_j^2)};
\]  

(81)

one has

\[
\alpha^{(0)}(\omega) = \frac{1}{m \omega_0 \omega^2 - \omega^2 - k \tilde{\mu}_d(\omega)}.
\]  

(82)
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\[ G(\omega) = \frac{m_r}{m} - \frac{i\omega \bar{\mu}_d(\omega)}{m_0^2}, \]  

so that we get

\[ F_0(T, 0) = \frac{3}{\pi} \int_0^\infty d\omega \ f(\omega, T) I_1, \]  

where

\[ I_1 = \text{Im} \left[ \frac{d}{d\omega} \ln \alpha^{(0)}(\omega) \right] = \frac{m_r (\omega^2 + \omega_0^2) \bar{\mu}_d(\omega) + \omega(-\omega^2 + \omega_0^2)(d\bar{\mu}_d(\omega)/d\omega)}{m_r^2(\omega^2 - \omega_0^2)^2 + \omega^2 \bar{\mu}_d^2(\omega)}. \]  

Also, we have

\[ \Delta_1 F_0(T, B) = -\frac{1}{\pi} \int_0^\infty d\omega \ f(\omega, T) I_2, \]  

where

\[ I_2 = \text{Im} \left[ \frac{d}{d\omega} \ln \left\{ 1 - (G(\omega))^2 \left( \frac{eB\omega}{c} \right)^2 \left[ \alpha^{(0)}(\omega) \right]^2 \right\} \right] = \frac{A}{B}; \]  

\[ A = 2m_r \omega^4 \omega_c^2 \left[-m_r^4 \omega_0^6 (\omega^6 + 5\omega_0^6 + \omega^4(3\omega_0^2 + \omega_0^2) - 3\omega^2 \omega_0^2(3\omega_0^2 + \omega_0^2)) \bar{\mu}_d(\omega) + m_r^2 \omega^2 \omega_0^2 (-8\omega_0^6 + \omega^4(\omega_0^2 - \omega_0^2) + \omega^2(9\omega_0^4 + 4\omega_0^2 \omega_0^2)) \bar{\mu}_d^2(\omega) + (-3\omega^4 \omega_0^4 + \omega^6 \omega_0^2) \bar{\mu}_d^4(\omega) + m_r^4 \omega_0^6 (\omega^2 - \omega_0^2) (\omega^4 + \omega_0^4 - \omega^2(2\omega_0^2 + \omega_0^2)) \frac{d\bar{\mu}_d(\omega)}{d\omega} \right] \]  

\[ -m_r^2 \omega^2 \omega_0^2 (-3\omega_0^4 + \omega^2(3\omega_0^2 + \omega_0^2)) \bar{\mu}_d^2(\omega) \frac{d\bar{\mu}_d(\omega)}{d\omega} + \omega^5 (\omega_0^4 - \omega^2 \omega_0^2) \bar{\mu}_d^4(\omega) \frac{d\bar{\mu}_d(\omega)}{d\omega} \right]; \]  

\[ B = (m_r^2(\omega^2 - \omega_0^2)^2 + \omega^2 \bar{\mu}_d^2(\omega)) \left[m_r^4 \omega_0^6 (\omega^4 + \omega_0^4 - \omega^2(2\omega_0^2 + \omega_0^2))^2 + 2m_r^2 \omega^2 \omega_0^4 (\omega_0^6 + \omega^2 \omega_0^2 - 2\omega^2 \omega_0^2(\omega_0^2 + \omega_0^2) + \omega^4(\omega_0^2 + \omega_0^2)^2) \bar{\mu}_d^2(\omega) + \omega^4(\omega_0^4 - \omega^2 \omega_0^2)^2 \bar{\mu}_d^4(\omega) \right], \]  

and

\[ \omega_c = \frac{eB}{mc}, \]  

is the cyclotron frequency. Similarly, equation (78) may be rewritten as

\[ \Delta_2 F_0(T, B) = \frac{1}{\pi} \int_0^\infty d\omega \ f(\omega, T) I_3, \]

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where

\[
I_3 = \text{Im} \left[ \alpha^{(0)}(\omega)^2 \left( \frac{\omega e B}{c} \right)^2 \left( \frac{d(G(\omega))^2}{d\omega} \right) \left\{ 1 - \left( \frac{\omega Be G(\omega)}{c} \right)^2 [\alpha^{(0)}(\omega)]^2 \right\}^{-1} \right]
\]

\[
= \left[ -2m_i\omega^2\omega_0^2\omega_c^2 \left\{ m_i^2\omega_0^2 (\omega^4 + \omega_0^4 - \omega^2(2\omega_0^2 + \omega_c^2)) \\
+ \omega^2 (\omega_0^4 - \omega^2(2\omega_0^2 + \omega_c^2)) \tilde{\mu}_d(\omega) \right\} \\
\times \left( \tilde{\mu}_d(\omega) + \omega \frac{d\tilde{\mu}_d(\omega)}{d\omega} \right) \left[ m_i^4\omega_0^8 (\omega^4 + \omega_0^4 - \omega^2(2\omega_0^2 + \omega_c^2))^2 \\
+ 2m_i^2\omega^2\omega_0^4 (\omega_0^8 + \omega_0^6\omega_c^2 - 2\omega_0^2\omega_c^2(\omega_0^4 + \omega_c^4) + \omega^4(\omega_0^2 + \omega_c^2)^2) \tilde{\mu}_d^3(\omega) \\
+ \omega^4(\omega_0^4 - \omega^2\omega_c^2)^2\tilde{\mu}_d^3(\omega) \right]^{-1} \right)
\]

(92)

One can check that \( I_1, I_2, \) and \( I_3 \) all have the dimensions of \( 1/\omega \) as required.

Now, \( f(\omega, T) \) vanishes exponentially for frequencies \( \omega \gg k_B T/\hbar \). Then, in order to evaluate the free energy of the charged particle at low temperatures, we need to consider only low-\( \omega \) contributions in evaluating the integrals in equations (84), (86) and (91). To proceed, let us consider a heat bath for which the memory function for small \( \omega \) has the form

\[
\tilde{\mu}_d(\omega) = m_i b^{1-\nu}(-i\omega)^\nu; \quad -1 < \nu < 1,
\]

(93)

where \( b \) is a positive constant with the dimension of frequency and \( \nu \) is within the indicated range so that \( \tilde{\mu}_d(\omega) \) is a positive real function, as required [14, 15]. The Ohmic, sub-Ohmic and super-Ohmic heat-bath spectra are obtained by considering \( \nu = 0, -1 < \nu < 0, \) and \( 0 < \nu < 1, \) respectively. Using equation (93) in equations (85), (87) and (92), we find that to lowest order in \( \omega \) one has

\[
I_1 = \frac{(1+\nu)b^{1-\nu}}{\omega_0^2} \cos \left( \frac{\nu\pi}{2} \right) \omega^\nu,
\]

(94)

while to the same order in \( \omega \) the quantities \( I_2 \) and \( I_3 \) have no contributions. We use the above expression for \( I_1 \) in equation (84), and the result

\[
\int_0^\infty dy \ y^\nu \log(1 - e^{-y}) = -\Gamma(\nu + 1)\zeta(\nu + 2),
\]

(95)

where \( \Gamma(z) \) is the gamma function, while \( \zeta(z) \) is the Riemann zeta function,

\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},
\]

(96)

to finally get that at low temperatures,

\[
F_0(T, B) = F_0(T, 0) = -3\Gamma(\nu + 2)\zeta(\nu + 2) \cos \left( \frac{\nu\pi}{2} \right) \frac{\hbar b^3}{\pi^2\omega_0^2} \left( \frac{k_B T}{\hbar b} \right)^{\nu+2}.
\]

(97)

It is readily seen that the entropy \( S = -\partial F/\partial T \) approaches zero as \( T \to 0 \), in agreement with the third law of thermodynamics [16].

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5. Conclusions

In this paper, we have derived an exact formula for the equilibrium free energy of a charged quantum particle moving in a harmonic potential in the presence of a uniform magnetic field and linearly coupled to a heat bath of independent quantum harmonic oscillators through the momentum variables. The free energy has an expression which is different from that for the case of coordinate–coordinate coupling between the particle and the heat-bath oscillators. For an illustrative heat-bath spectrum, we evaluated the free energy for non-zero magnetic fields in the low-temperature limit, showing thereby that the entropy of the charged particle vanishes at zero temperature, in conformity with the third law of thermodynamics.

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