Combined internal energy minimizing planar cubic Hermite curve

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Abstract
Internal energy minimization has been used to select the two free parameters of the planar cubic Hermite curve. In this paper, we consider to combine the three well known internal energies and select the two free parameters of the planar cubic Hermite curve by minimizing the combined internal energy. The proposed minimization is a generalization of the single internal energy minimizations. Some numerical examples show that the proposed method can synchronously make the three internal energies of the curve as small as possible.

Keywords: Hermite curve, Internal energy, Combined energy, Free parameter, Minimization

1. Introduction

The stretch energy, the strain energy and the curvature variation energy are three well known internal energies of the planar curve, they respectively represent length, curvature and variation of curvature that are the typical geometric characteristics of the curve (see Veltkamp and Wesselink, 1995; Xu et al, 2011a). Although there is no recognized way to describe the smoothness of a planar curve, a curve constructed by minimizing one of the internal energies is generally considered smooth. Some examples are the construction of energy-minimizing Bézier curve (see Ahn et al, 2014; Eriskin and Yücesan, 2016; Xu et al, 2011a), the construction of energy-minimizing B-spline curve (see Vassilev, 1996; Xu et al, 2019b), and the construction of energy-minimizing Hermite curve (see Jaklič and Žagar, 2011a; Jaklič and Žagar, 2011b; Li, 2019a; Li and Zhang, 2020b; Li et al, 2012; Lu, 2015a; Lu, 2015b; Lu et al, 2017c; Yong and Cheng, 2004). In this paper, we focus on constructing the planar cubic Hermite curve by energy minimization.

As we know, the curves with free parameters, such as the Bézier-like curves with free parameters, have been widely concerned (see Bashir et al, 2013a; Bashir et al, 2013b; Bibi et al, 2020; Li et al, 2020; Maqsood et al, 2020; Usman et al, 2020). In like wise, the cubic Hermite curve has two free parameters when the two points and the two associated unit tangent directions are fixed. For constructing the smooth planar cubic Hermite curve, many researchers used the energy minimizations to select the two free parameters of the planar cubic Hermite curve. Among these methods, the strain energy minimization and curvature variation energy minimization are two commonly used methods (see Jaklič and Žagar, 2011a; Jaklič and Žagar, 2011b; Li and Zhang, 2020b; Li et al, 2012; Lu, 2015a; Lu et al, 2017c; Yong and Cheng, 2004). It may not be possible to evaluate which of the energy minimizations is better for constructing the smooth planar cubic Hermite curve, since there is no recognized way to describe the smoothness. Thereby a natural idea arises: maybe we can select the two free parameters of the planar cubic Hermite curve by minimizing the combined internal energy.

In this paper, we introduce the combined internal energy of the planar cubic Hermite curve, and then present the combined internal energy minimization. The combined internal energy minimization is a generalized method, because it unifies the stretch energy minimization, the strain energy minimization and the curvature variation energy minimization. The rest of this paper is organized as follows. In Section 2 the combined internal energy is introduced. In Section 3 the combined internal energy minimization is described. In Section 4 some numerical examples are shown. Finally, a short conclusion is given in Section 5.
2. The combined internal energy

Given two points \( p_0, \ p_1 \) together with the associated unit tangent directions \( d_0, \ d_1 \), the two-point cubic Hermite curve can be described by (see Farin, 2002)

\[
b(t) = p_0 B_0^3(t) + \left(p_0 + \frac{1}{3} \alpha_0 d_0\right) B_1^3(t) + \left(p_1 - \frac{1}{3} \alpha_1 d_1\right) B_2^3(t) + p_1 B_3^3(t),
\]

(1)

where \( B_i^3(t) = \frac{3!}{k!(3-k)!} t^k (1-t)^{3-k} \) \((k = 0, 1, 2, 3)\), and \( \alpha_0, \alpha_1 \) are positive free parameters.

Three well known internal energies of planar curve are the stretch energy, the strain energy and the curvature variation energy (Veltkamp and Wesselink, 1995; Xu et al, 2011a). The stretch energy of the planar cubic Hermite curve \( b(t) \) can be described by

\[
E_1 = \int_{b_0}^{b_1} \|b'(t)\| \, dt,
\]

(2)

and the stretch energy (2) is often approximately expressed as

\[
\hat{E}_1 = \int_{b_0}^{b_1} \|b'(t)\|^2 \, dt.
\]

(3)

Let \( \kappa(t) \) be the curvature of the planar cubic Hermite curve \( b(t) \), the strain energy of the curve can be described by (see Jaklič and Žagar, 2011a)

\[
E_2 = \int_{b_0}^{b_1} (\kappa(t))^2 \, dt,
\]

(4)

and the curvature variation energy of the curve can be described by (see Jaklič and Žagar, 2011b)

\[
E_3 = \int_{b_0}^{b_1} (\kappa'(t))^2 \, dt.
\]

(5)

Suppose the curve \( b(t) \) is parameterized by arc length, the curvature would be simply expressed by \( \kappa(t) = \|b'(t)\| \). Then the strain energy (4) could be approximately expressed as (see Jaklič and Žagar, 2011a)

\[
\hat{E}_2 = \int_{b_0}^{b_1} \|b'(t)\| \, dt,
\]

(6)

and the curvature variation energy (5) could be approximately expressed as (see Lu, 2015a)

\[
\hat{E}_3 = \int_{b_0}^{b_1} \|b'(t)\|^2 \, dt.
\]

(7)

Then we approximately describe the combined internal energy of the planar cubic Hermite curve \( b(t) \) as

\[
\hat{E} = \lambda_1 \hat{E}_1 + \lambda_2 \hat{E}_2 + \lambda_3 \hat{E}_3,
\]

(8)

where \( \lambda_i \) \((i = 1, 2, 3)\) are the weights, \( 0 \leq \lambda_i \leq 1 \) \((i = 1, 2, 3)\) and \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \).

No doubt that the combined internal energy unifies the three well known internal energies of the planar cubic Hermite curve.

3. Minimization method

By computing from (1), the approximate stretch energy functional, the approximate strain energy functional and the approximate curvature variation energy functional of \( b(t) \) become
\[
\hat{E}_i(\alpha_o, \alpha_i) = \frac{1}{15} \left( 2\alpha_o^2 + 2\alpha_i^2 - \alpha_o \alpha_i (d_o \cdot d_i) - 3\alpha_o (\Delta p_o \cdot d_o) - 3\alpha_i (\Delta p_o \cdot d_i) + 18 \left\| \Delta p_o \right\| \right),
\]
\[
\hat{E}_i(\alpha_o, \alpha_i) = 4 \left( \alpha_o^2 + \alpha_i^2 + \alpha_o \alpha_i (d_o \cdot d_i) - 3\alpha_o (\Delta p_o \cdot d_o) - 3\alpha_i (\Delta p_o \cdot d_i) + 3 \left\| \Delta p_o \right\| \right),
\]
\[
\hat{E}_i(\alpha_o, \alpha_i) = 36 \left( \alpha_o^2 + \alpha_i^2 + 2\alpha_o \alpha_i (d_o \cdot d_i) - 4\alpha_o (\Delta p_o \cdot d_o) - 4\alpha_i (\Delta p_o \cdot d_i) + 4 \left\| \Delta p_o \right\| \right),
\]
where \( \Delta p_o \equiv p_o - p_0 \).

From (8), the combined internal energy functional of \( b(t) \) becomes
\[
\hat{E}(\alpha_o, \alpha_i) = \hat{E}_i(\alpha_o, \alpha_i) + \frac{\partial E}{\partial \alpha_o} + \frac{\partial E}{\partial \alpha_i} + \frac{\partial E}{\partial \alpha_i} + \frac{\partial E}{\partial \alpha_i}.
\]

Since \( \alpha_o, \alpha_i \) are positive free parameters, we should apply a feasible region \( D \equiv \{ (\alpha_o, \alpha_i) \in \mathbb{R}^2 : \alpha_o > 0, \alpha_i > 0 \} \).

Then we need to solve the combined internal energy minimization problem
\[
\min_{(\alpha_o, \alpha_i) \in D} \hat{E}(\alpha_o, \alpha_i).
\]

**Theorem 1** Let \( \theta_0 \equiv \angle(d_o, \Delta p_o), \theta_1 \equiv \angle(d_i, \Delta p_o) \), where \( \angle(a, b) \) represents the angle between \( a \) and \( b \). Assume that \( 0 < \theta_0 < \pi/2, 0 < \theta_1 < \pi/2 \), \( d_o \) and \( d_i \) point to the two sides of \( \Delta p_o \) respectively, the combined internal energy minimization problem (13) has a unique solution
\[
\alpha_o = s \cdot (4\lambda_0 + 120\lambda_2 + 1080\lambda_4) \cos \theta_0 - (\lambda_0 + 60\lambda_2 + 1080\lambda_4) \cos \theta_1 \cos (\theta_0 + \theta_1),
\]
\[
\alpha_i = s \cdot (4\lambda_0 + 120\lambda_2 + 1080\lambda_4) \cos \theta_1 - (\lambda_0 + 60\lambda_2 + 1080\lambda_4) \cos \theta_0 \cos (\theta_0 + \theta_1),
\]
where
\[
s = \frac{(3\lambda_0 + 180\lambda_2 + 2160\lambda_4) \left\| \Delta p_o \right\|}{(4\lambda_0 + 120\lambda_2 + 1080\lambda_4)^2 - (\lambda_0 + 60\lambda_2 + 1080\lambda_4)^2 \cos^2 (\theta_0 + \theta_1)}.\]

**Proof.** Since \( 0 < \theta_0 < \pi/2, 0 < \theta_1 < \pi/2 \), \( d_o \) and \( d_i \) point to the two the sides of \( \Delta p_o \) respectively, then \( \angle(d_o, d_i) = \theta_0 + \theta_1, \) \( 0 < \theta_0 + \theta_1 < \pi \). See Fig. 1 for an illustration.

![Fig. 1 An illustration of the two-point data](image)

By computing from (12), the gradients of \( \hat{E}(\alpha_o, \alpha_i) \) can be expressed by
\[
\frac{\partial \hat{E}}{\partial \alpha_o} = \frac{4}{15} \lambda_0 + 8\lambda_2 + 72\lambda_4 \alpha_0 + (\frac{1}{15} \lambda_0 + 4\lambda_2 + 72\lambda_4) \alpha_i \cos (\theta_0 + \theta_1) - (\frac{1}{5} \lambda_0 + 12\lambda_2 + 144\lambda_4) \left\| \Delta p_o \right\| \cos \theta_0,
\]
\[
\frac{\partial \hat{E}}{\partial \alpha_i} = (-\frac{1}{15} \lambda_0 + 4\lambda_2 + 72\lambda_4) \alpha_0 \cos (\theta_0 + \theta_1) + (\frac{4}{15} \lambda_0 + 8\lambda_2 + 72\lambda_4) \alpha_i - (\frac{1}{5} \lambda_0 + 12\lambda_2 + 144\lambda_4) \left\| \Delta p_o \right\| \cos \theta_1.
\]

The Hessian matrix of \( \hat{E}(\alpha_o, \alpha_i) \) is given by
\[
H = \begin{pmatrix}
\frac{4}{15} \lambda_1 + 8 \lambda_2 + 72 \lambda_3 & (-\frac{1}{15} \lambda_1 + 4 \lambda_2 + 72 \lambda_3) \cos(\theta_0 + \theta_1) \\
(-\frac{1}{15} \lambda_1 + 4 \lambda_2 + 72 \lambda_3) \cos(\theta_0 + \theta_1) & \frac{4}{15} \lambda_1 + 8 \lambda_2 + 72 \lambda_3 \\
\end{pmatrix}.
\]

Since \(0 < \theta_0 + \theta_1 < \pi\), \(0 \leq \lambda_1 \leq 1\) \((i = 1, 2, 3)\), \(\lambda_1 + \lambda_2 + \lambda_3 = 1\), then \(\frac{4}{15} \lambda_1 + 8 \lambda_2 + 72 \lambda_3 > 0\), and
\[
\det(H) = \left(\frac{4}{15} \lambda_1 + 8 \lambda_2 + 72 \lambda_3\right)^2 - \left(-\frac{1}{15} \lambda_1 + 4 \lambda_2 + 72 \lambda_3\right)^2 \cos^2(\theta_0 + \theta_1)
\]
\[
> \left(\frac{4}{15} \lambda_1 + 8 \lambda_2 + 72 \lambda_3\right)^2 - \left(-\frac{1}{15} \lambda_1 + 4 \lambda_2 + 72 \lambda_3\right)^2 = \left(\frac{3}{15} \lambda_1 + 12 \lambda_2 + 144 \lambda_3\right)\left(\frac{1}{3} \lambda_1 + 4 \lambda_2\right) \geq 0.
\]

Thus the matrix \(H\) is symmetric positive definite. That means \(\hat{E}(\alpha_d, \alpha_i)\) is strictly convex, and it has a unique global minimum. Then the unique minimum of \(\hat{E}(\alpha_d, \alpha_i)\) expressed in (14) and (15) can be obtained by solving from
\[
\frac{\partial \hat{E}}{\partial \alpha_d} = 0 \quad \text{and} \quad \frac{\partial \hat{E}}{\partial \alpha_i} = 0.
\]

Because \(\lambda_1 + \lambda_2 + \lambda_3 = 1\), then
\[
a = (4 \lambda_1 + 120 \lambda_2 + 1080 \lambda_3) \cos \theta_0 - (-\lambda_1 + 60 \lambda_2 + 1080 \lambda_3) \cos \theta_1 \cos(\theta_0 + \theta_1)
\]
\[
= (1080 - 1076 \lambda_1 + 960 \lambda_2 + 1080 \lambda_3) \cos \theta_0 - (1080 - 1080 \lambda_2 - 1020 \lambda_3) \cos \theta_1 \cos(\theta_0 + \theta_1).
\]

Recall that \(0 < \theta_0 < \pi/2\), \(0 < \theta_1 < \pi/2\), \(0 \leq \lambda_i \leq 1\) \((i = 1, 2, 3)\). If \(0 < \lambda_1, \lambda_2 < 1\), we have
\[
a > (1080 - 1080 \lambda_2 - 1020 \lambda_3) (\cos \theta_0 - \cos \theta_1 \cos(\theta_0 + \theta_1)) > 1080 (1 - \lambda_1 - \lambda_2) (\cos \theta_0 - \cos \theta_1 \cos(\theta_0 + \theta_1))
\]
\[
= 1080 \lambda_3 \sin \theta_0 \sin(\theta_0 + \theta_1) > 0.
\]

If \(\lambda_1 = \lambda_2 = 0\), we have
\[
a = 1080 (\cos \theta_0 - \cos \theta_1 \cos(\theta_0 + \theta_1)) = 1080 \sin \theta_0 \sin(\theta_0 + \theta_1) > 0.
\]

That means \(\alpha_d\) expressed in (14) is always positive. Similarly, \(\alpha_i\) expressed in (15) is also positive. Thus we have proved the theorem.

By setting \(\lambda_2 = \lambda_3 = 0\) and \(\lambda_1 = 1\), the combined internal energy minimization problem (13) would degenerate to be the stretch energy minimization problem
\[
\min_{(\alpha_d, \alpha_i) \in \mathcal{D}} \hat{E}_s(\alpha_d, \alpha_i).
\]

By setting \(\lambda_1 = \lambda_2 = 0\) and \(\lambda_3 = 1\), the combined internal energy minimization problem (13) would degenerate to be the strain energy minimization problem
\[
\min_{(\alpha_d, \alpha_i) \in \mathcal{D}} \hat{E}_s(\alpha_d, \alpha_i).
\]

By setting \(\lambda_1 = \lambda_2 = 0\) and \(\lambda_3 = 1\), the combined internal energy minimization problem (13) would degenerate to be the curvature variation energy minimization problem
\[
\min_{(\alpha_d, \alpha_i) \in \mathcal{D}} \hat{E}_s(\alpha_d, \alpha_i).
\]

Thus the combined internal energy minimization unifies the stretch energy minimization, the strain energy minimization and the curvature variation energy minimization. Then we could derive the following three corollaries from Theorem 1.

**Corollary 1** Assume that \(0 < \theta_0 < \pi/2\), \(0 < \theta_1 < \pi/2\), \(d_0\) and \(d_1\) point to two sides of \(\Delta p_0\) respectively, the stretch energy minimization problem (16) has a unique solution
\[
\alpha_d^{(1)} = \frac{3 \Delta p_0 \left((4 \cos \theta_0 + \cos \theta_1 \cos(\theta_0 + \theta_1)) \right)}{16 - \cos^2(\theta_0 + \theta_1)}, \quad \alpha_i^{(1)} = \frac{3 \Delta p_0 \left((4 \cos \theta_1 + \cos \theta_0 \cos(\theta_0 + \theta_1)) \right)}{16 - \cos^2(\theta_0 + \theta_1)}.
\]

**Corollary 2** Assume that \(0 < \theta_0 < \pi/2\), \(0 < \theta_1 < \pi/2\), \(d_0\) and \(d_1\) point to two sides of \(\Delta p_0\) respectively, the strain energy minimization problem (17) has a unique solution
\[
\alpha^{(2)}_0 = \frac{3}{4} \left\| \Delta \mathbf{p}_ \right\| \left( \cos \theta_0 + \sin \theta_0 \sin \left( \theta_0 + \theta_1 \right) \right), \quad \alpha^{(2)}_i = \frac{3}{4} \left\| \Delta \mathbf{p}_ \right\| \left( \cos \theta_0 + \sin \theta_0 \sin \left( \theta_0 + \theta_i \right) \right).
\]

**Corollary 3**  Assume that \(0 < \theta_0 < \pi/2, \quad 0 < \theta_i < \pi/2, \quad \mathbf{d}_0 \) and \( \mathbf{d}_i \) point to two sides of \( \Delta \mathbf{p}_ \) respectively, the curvature variation energy minimization problem (17) has a unique solution

\[
\alpha^{(3)}_0 = \frac{2}{\sin \left( \theta_0 + \theta_1 \right)} \left\| \Delta \mathbf{p}_ \right\| \sin \theta_0, \quad \alpha^{(3)}_i = \frac{2}{\sin \left( \theta_0 + \theta_i \right)} \left\| \Delta \mathbf{p}_ \right\| \sin \theta_0.
\]

Now, the remaining task is to select the value of the weights \( \lambda_i (i = 1, 2, 3) \). Here, we use the ranking algorithm in multi-objective optimization (see Lin and Dong, 1995) to select the value of the weights \( \lambda_i (i = 1, 2, 3) \). The algorithm is described as follows,

**Step 1**  Compute the deviations \( \delta^{(i)}_{ij} = |\widetilde{E}_i(\alpha^{(i)}_0, \alpha^{(i)}_1) - \widetilde{E}_j(\alpha^{(i)}_0, \alpha^{(i)}_1)|, \quad i, j = 1, 2, 3 \).

**Step 2**  Compute the average deviations \( m_i = \sum_{j=1}^{3} \delta^{(i)}_{ij}, \quad i = 1, 2, 3 \).

**Step 3**  Compute the initial weights \( \lambda_i = m_i / (m_1 + m_2 + m_3), \quad i = 1, 2, 3 \).

**Step 4**  If \( m_i = \max \{m_1, m_2, m_3\} \), set \( \lambda_i = \min \{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\} \); elseif \( m_i = \min \{m_1, m_2, m_3\} \), set \( \lambda_i = \max \{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\} \); else, set \( \lambda_i = 1 - \max \{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\} - \min \{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\}, \quad i = 1, 2, 3 \).

4. Numerical examples

In this section, we compare the combined internal energy minimization with the stretch energy minimization, the strain energy minimization and the curvature variation energy minimization through some numerical examples.

For convenience, let us consider the following two-point data,

\( \mathbf{p}_0 = (0, 0), \quad \mathbf{p}_1 = (1, 0), \quad \mathbf{d}_0 = (\cos \theta_0, \sin \theta_0), \quad \mathbf{d}_1 = (\cos \theta_1, -\sin \theta_1), \)

where \(0 < \theta_0 < \pi/2, \quad 0 < \theta_1 < \pi/2\). It implies \( \| \mathbf{d}_0 \| = \| \mathbf{d}_1 \| = 1, \quad \angle(\mathbf{d}_0, \Delta \mathbf{p}_0) = \theta_0, \quad \angle(\mathbf{d}_1, \Delta \mathbf{p}_0) = \theta_1 \), and \( \mathbf{d}_0, \mathbf{d}_1 \) point to two sides of \( \Delta \mathbf{p}_0 \) respectively. Fig. 2 shows the curves and the corresponding curvature plots obtained by different energy minimizations of four examples.
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Fig. 2  The curves (left) and the corresponding curvature plots (right) obtained by the stretch energy minimization (dark), the strain energy minimization (green), the curvature variation energy minimization (blue) and the combined internal energy minimization (red)

Fig. 2 shows that the curve obtained by the combined internal energy minimization lies among the curves obtained by the single internal energy minimizations, and the variation of the curvature obtained by the combined internal energy minimization is also among the variation of the curvature obtained by the single internal energy minimizations. That means the combined internal energy minimization can synchronously make the three internal energies of the curve as small as possible. The computational results of different energy minimizations could be seen in Table 1.

| Fig. | Stretch energy minimization | Strain energy minimization | Curvature variation energy minimization | Combined internal energy minimization |
|-----|-----------------------------|-----------------------------|---------------------------------------|--------------------------------------|
|     | $E_1$                       | $E_2$                       | $E_3$                                 | $E_4$                                |
| (a) | 1.026                       | 9.470                       | 2.431                                 | 1.028                                |
| (b) | 1.122                       | 5.059                       | 53.875                                | 1.194                                |
| (c) | 1.255                       | 5.250                       | 56.250                                | 1.200                                |
| (d) | 1.277                       | 9.718                       | 113.855                               | 1.310                                |

Because the combined internal energy minimization unifies the single internal energy minimizations, the combined internal energy minimization is a generalization of the single internal energy minimizations. We could replace the single internal energy minimizations with the combined internal energy minimization, since the single internal energy minimizations have been successfully applied to construct the smooth planar Hermite curve.

5. Conclusion

In this paper, we introduce the combined internal energy and present the combined internal energy minimization for selecting the two free parameters of the planar cubic Hermite curve. The combined internal energy minimization is a generalized method, because it unifies the single internal energy minimizations. Since the single internal energy
minimizations have been successfully applied to deal with the problem of planar cubic Hermite curve, the proposed combined internal energy minimization may be helpful to handle this problem. In addition, the proposed combined internal energy minimization may also be used to select the free parameters contained in other types of curves.

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