COMPUTING THE HOMOLOGY FUNCTOR ON SEMI-ALGEBRAIC MAPS AND DIAGRAMS

SAUGATA BASU AND NEGIN KARISANI

Abstract. Developing an algorithm for computing the Betti numbers of semi-algebraic sets with singly exponential complexity has been a holy grail in algorithmic semi-algebraic geometry and only partial results are known. In this paper we consider the more general problem of computing the image under the homology functor of a semi-algebraic map \( f : X \to Y \) between closed and bounded semi-algebraic sets. For every fixed \( \ell \geq 0 \) we give an algorithm with singly exponential complexity that computes bases of the homology groups \( H_i(X), H_i(Y) \) (with rational coefficients) and a matrix with respect to these bases of the induced linear maps \( H_i(f) : H_i(X) \to H_i(Y) \), \( 0 \leq i \leq \ell \). We generalize this algorithm to more general (zigzag) diagrams of maps between closed and bounded semi-algebraic sets and give a singly exponential algorithm for computing the homology functors on such diagrams. This allows us to give an algorithm with singly exponential complexity for computing barcodes of semi-algebraic zigzag persistent homology in small dimensions.

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1. **Introduction**

Let $R$ be a real closed field and $D$ an ordered domain contained in $R$.

The problem of effective computation of topological properties of semi-algebraic subsets of $R^k$ has a long history. Semi-algebraic subsets of $R^k$ are subsets defined by first-order formulas in the language of ordered fields (with parameters in $R$). Since the first-order theory of real closed fields admits quantifier-elimination, we can assume that each semi-algebraic subset $S \subset R^k$ is defined by some quantifier-free formula $\phi$. A quantifier-free formula $\phi(X_1, \ldots, X_k)$ in the language of ordered fields with parameters in $D$, is a formula with atoms of the form $P = 0, P > 0, P < 0, P \in D[X_1, \ldots, X_k]$.

Semi-algebraic subsets of $R^k$ have tame topology. In particular, closed and bounded semi-algebraic subsets of $R^k$ are semi-algebraically triangulable (see for example [3, Chapter 5]). This means that there exists a finite simplicial complex $K$, whose geometric realization, $|K|$, considered as a subset of $R^N$ for some $N > 0$, is semi-algebraically homeomorphic to $S$. The semi-algebraic homeomorphism $|K| \to S$ is called a semi-algebraic triangulation of $S$. All topological properties of $S$ are then encoded in the finite data of the simplicial complex $K$.

There exists a classical algorithm which takes as input a quantifier-free formula defining a semi-algebraic set $S$, and produces as output a semi-algebraic triangulation of $S$ (see for instance [3, Chapter 5]). However, this algorithm is based on the technique of cylindrical algebraic decomposition, and hence the complexity of this algorithm is
prohibitively expensive, being doubly exponential in $k$. More precisely, given a description by a quantifier-free formula involving $s$ polynomials of degree at most $d$, of a closed and bounded semi-algebraic subset of $S \subset \mathbb{R}^k$, there exists an algorithm computing a semi-algebraic triangulation of $h : |K| \to S$, whose complexity is bounded by $(sd)^{2^{O(k)}}$. Moreover, the size of the simplicial complex $K$ (measured by the number of simplices) is also bounded by $(sd)^{2^{O(k)}}$.

1.1. Doubly vs singly exponential. One can ask whether the doubly exponential behavior for the semi-algebraic triangulation problem is intrinsic to the problem. One reason to think that it is not so comes from the fact that the ranks of the homology groups of $S$ (following the same notation as in the previous paragraph), and so in particular those of the simplicial complex $K$, is bounded by $O(sd)^k$ (see for instance [3, Chapter 7]), which is singly exponential in $k$. So it is natural to ask if this singly exponential upper bound on rank($H_*(S)$) is “witnessed” by an efficient semi-algebraic triangulation of small (i.e. singly exponential) size. This is not known.

In fact, designing an algorithm with a singly exponential complexity for computing a semi-algebraic triangulation of a given semi-algebraic set has remained a holy grail in the field of algorithmic real algebraic geometry and little progress has been made over the last thirty years on this problem (at least for general semi-algebraic sets). We note here that designing algorithms with singly exponential complexity has being a leit motif in the research in algorithmic semi-algebraic geometry over the past decades – starting from the so called “critical-point method” which resulted in algorithms for testing emptiness, connectivity, computing the Euler-Poincaré characteristic, as well as for the first few Betti numbers of semi-algebraic sets (see [2] for a history of these developments and contributions of many authors). More recently, such algorithms have also been developed in other (more numerical) models of computations [11, 12, 13].

In [11, 12, 13], the authors take a different approach. Working over $\mathbb{R}$, and given a “well-conditioned” semi-algebraic subset $S \subset \mathbb{R}^k$, they compute a witness complex whose geometric realization is $k$-equivalent to $S$. The size of this witness complex as well the complexity of the algorithm is bounded singly exponentially in $k$, but also depends on a real parameter, namely the condition number of the input (and so this bound is not uniform). The algorithm will fail for ill-conditioned input when the condition number becomes infinite. This is unlike the kind of algorithms we consider in the current paper, which are supposed to
work for all inputs and with uniform complexity upper bounds. So these approaches are not comparable.

1.2. Homology as a functor. Homology is a functor from the category of topological spaces to $\mathbb{Z}$-modules. Restricted to the category of semi-algebraic sets and maps and considering homology groups with only rational coefficients, it is a functor from the category of semi-algebraic subsets of $\mathbb{R}^k, k > 0$ to finite dimensional $\mathbb{Q}$-vector spaces. The algorithms discussed in the previous section aimed only at computing the dimension of the homology groups. However, a very natural algorithmic question that arises is the following.

**Problem 1.** Given a first-order formula $\phi_f$ describing the graph of a semi-algebraic map $f : X \to Y$, compute with singly exponential complexity a description of the map $H_i(f) : H_i(X) \to H_i(Y)$ (i.e. compute a basis of of $H_i(X), H_i(Y)$ and the matrix corresponding to these bases of the linear map $H_i(f)$). More generally, given a diagram of semi-algebraic maps, compute with singly exponential complexity bases of the homology groups of the various semi-algebraic sets and matrices corresponding to the different maps. We will say that such an algorithm **computes the homology functor for semi-algebraic maps (or more generally diagram of maps) in dimension $i$.**

**Remark 1.1.** Studying the “functor complexity” of the homology functor was raised in [5, Section 7, Problem (4)] in the setting of categorical complexity. In this paper we initiate the study of this functor from the complexity point of view, though the definition of complexity that we use in this paper is the classical notion and not the categorical one introduced in [5].

**Remark 1.2.** One important point to note that is that semi-algebraic maps $f : X \to Y$ between closed and bounded semi-algebraic sets are not necessarily triangulable (unless $\dim Y \leq 1$). An easy example is the so called “blow-down” map.

**Example 1.1.** Let $S \subset \mathbb{R}^3$ be defined by the formula

$$(Y - ZX = 0) \land (X^2 + Y^2 - 1 \leq 0) \land (Y - X \leq 0) \land (X \geq 0),$$

$T$ the unit disk in $\mathbb{R}^2$, and $f : S \to T$ the projection map along the $Z$-coordinate. The map $f$ is easily seen to be not triangulable. More precisely, there are no semi-algebraic triangulations $h_S : |K_S| \to S, h_T : |K_T| \to T,$ and a simplicial map $F : K_S \to K_T$, such that $|F| \circ h_S = h_T \circ F.$
Thus, one cannot expect to solve Problem 1 by computing semi-algebraic triangulations of $h_X : |K_X| \to X$ and $h_Y : |K_Y| \to Y$, such that the induced map $h_Y^{-1} \circ f \circ h_X$ is simplicial.

Remark 1.3. It is not at all clear if the algorithms designed so far for computing Betti numbers of semi-algebraic sets with singly exponential complexity (both exact algorithms such as those in [1, 4] or numerical ones such as those in [11, 12, 13]) can extend to solve Problem 1.

1.3. Main contributions. The main contribution of this paper is a partial solution to Problem 1. We prove the following theorem which we state informally here (see Theorem 3 in Section 3 for a precise statement).

**Theorem** (Computing homology functor on semi-algebraic maps). For each fixed $\ell \geq 0$, there exists an algorithm with singly exponential complexity that computes the homology functor for semi-algebraic maps between closed and bounded semi-algebraic sets in each dimension $i, 0 \leq i \leq \ell$.

Remark 1.4. Note that up to isomorphism (in the category of vector spaces) a linear map $L : V \to W$ between finite dimensional vector spaces $V, W$ is determined by the numbers $\dim V, \dim W, \text{rank}(L)$. Thus, for a semi-algebraic map $f : X \to Y$, computing a description up to isomorphism of the linear map $H_i(f) : H_i(X) \to H_i(Y)$, amounts to computing $\dim H_i(X), \dim H_i(Y), \text{rank}(H_i(f))$. However, the isomorphism class of more general diagrams of vector spaces (such as zigzag diagrams in Theorem 5) is not determined just by the dimensions of the vector spaces and the ranks of the linear maps.

1.3.1. More general diagrams. Once we have an algorithm for computing the homology functor on semi-algebraic maps it is natural to try to extend it to more complicated diagrams of maps. As an example, in this paper we consider zigzag diagrams (see Notation 6.1 below) of semi-algebraic maps. We prove the following theorem (see Theorem 5 for a more precise statement).

**Theorem** (Computing homology functor on zigzag diagrams). For each fixed $\ell \geq 0$, there exists an algorithm with singly exponential complexity that computes the homology functor for diagrams of semi-algebraic maps of the zigzag type between closed and bounded semi-algebraic sets in each dimension $i, 0 \leq i \leq \ell$.

1.3.2. Computing semi-algebraic zigzag persistence. As an application of the previous theorem we consider the problem of computing zigzag
persistence modules for semi-algebraic maps. Persistent homology theory is foundational in the area of topological data analysis mainly in the context of finite simplicial complexes (see for example [17, 16] for background and many applications of persistent homology theory).

Persistence homology is defined for any filtration of topological spaces and its underlying module structure gives rise to barcodes via decomposition into irreducibles. The algorithmic study of persistent homology for filtrations of semi-algebraic sets by the sublevel sets of a polynomial function was initiated in [7] and we refer the reader to that paper for the basic definitions including that of barcodes. Although persistent homology was originally defined for filtrations of topological spaces, it has since been generalized to arbitrary diagrams $D : P \rightarrow \text{Top}$ (here $D$ is a functor from a poset category $P$ to the category of topological spaces) [10]. One particular class of diagrams that has been studied in the literature are zigzag diagrams. Zigzag persistence modules was introduced in [14] and was studied from the algebraic as well as algorithmic point of view. In particular, they showed that it is possible to associate barcodes to zigzag persistent homology as well and gave an algorithm to compute them. Zigzag persistence is a very active area of current research [9, 15].

The previous theorem allows to reduce the computation of the barcode of a zigzag diagram of semi-algebraic maps to that of finite dimensional vector spaces, and here we can use the algorithm in [14, Section 4.2]. In this way we obtain for each fixed $\ell$, a singly exponential algorithm for computing zigzag persistent module for semi-algebraic zigzag diagrams in dimensions $0$ to $\ell$. We have the following theorem.

**Theorem 1** (Computing semi-algebraic zigzag persistent modules). For each fixed $\ell \geq 0$, there exists an algorithm with singly exponential complexity that computes the barcode of diagrams of semi-algebraic maps of the zigzag type between closed and bounded semi-algebraic sets in each dimension $i, 0 \leq i \leq \ell$.

1.4. **Key ideas.** We summarize here the key ideas that go into the proofs of the theorems stated above.

1.4.1. *Replacing semi-algebraic maps by inclusions which are homologically equivalent.* Semi-algebraic maps between closed and bounded semi-algebraic sets which are inclusions can be treated much more easily than in the general case. Indeed, the main result proved in [6] implies that for each fixed $\ell \geq 0$, there exists an algorithm that given a semi-algebraic inclusion map $X \hookrightarrow Y$ between closed and bounded
semi-algebraic sets $X, Y$ as input, computes two finite simplicial complexes, $K$ and $L$ with $K \subset L$, such that the inclusion of the corresponding geometric realizations $|K| \hookrightarrow |L|$ is $\ell$-equivalent to the inclusion $X \hookrightarrow Y$ (see Definition 2.2). The key new idea in this paper is to replace an arbitrary semi-algebraic map between closed and bounded semi-algebraic sets, by an inclusion map which is equivalent to the given map up to homotopy.

1.4.2. *Realizing the mapping cylinder of a semi-algebraic map up to homotopy as a semi-algebraic set.* The standard tool for achieving this involves taking the mapping cylinder, $\text{cyl}(f)$, of the map $f$. The canonical inclusion $X \hookrightarrow \text{cyl}(f)$ is then homotopy equivalent to the $f$. However, the definition of mapping cylinder (see Definition 4.1 below) involves identification of certain points of a disjoint union (or equivalently passing to a quotient space). While quotients of semi-algebraic sets by proper equivalence relations are semi-algebraic (see for example [20]), no singly exponential algorithm for computing a semi-algebraic description of such a quotient is known.

We overcome this difficulty by modifying the construction of the mapping cylinder (see Section 4.1). We associated to the map $f$, a modified mapping cylinder of $f$, which we denote $\tilde{\text{cyl}}(f)$ (see Definition 4.2) which is a semi-algebraic set having similar properties as mapping cylinder of $f$ (see Proposition 4.3).

The definition of $\tilde{\text{cyl}}(f)$ does not involve taking quotients. However, it does involve quantifier elimination of an existential block of quantifiers (or equivalently taking the image under a linear projection map). This leads to a further technical complication. It is important for us in order to be able to apply the result of [6] that the semi-algebraic sets that we deal are not only closed, but are described by closed formulas (see Notation 2.1). While the image under projection of a closed and bounded semi-algebraic set is closed and bounded, the quantifier elimination algorithm that we use to obtain its description by a quantifier-free formula is not guaranteed to produce a closed description. Indeed no algorithm with singly exponential complexity is known for obtaining a closed description of a given closed semi-algebraic set and designing such an algorithm is considered to be a difficult open problem in algorithmic semi-algebraic geometry. Thus, we need an additional step.

1.4.3. *Replacing closed semi-algebraic set by ones described by closed formulas.* We replace (see Section 4.2) a closed semi-algebraic set by
another one, which is infinitesimally larger but has the same homotopy type, and moreover is described by a closed formula having size bounded linearly in the size of the original formula (see Notation 4.7 and Proposition 4.5 below). For this purpose, as usual in algorithmic semi-algebraic geometry we utilize extensions (obtained by adjoining infinitesimal elements) of the given real closed fields by fields of Puiseux series in these infinitesimals (see Section 4.3).

1.4.4. Mapping cylinder for diagrams. Finally, to extend our algorithm to zigzag diagrams we need to further generalize the definition of $\text{cyl}(f)$, so that every map in the diagram simultaneously becomes inclusions without changing the homological type of the diagram (see Section 6). We generalize the definition of $\text{cyl}(f)$ to define $\text{cyl}(D)$, the semi-algebraic mapping cylinder of a zigzag diagram $D$ (Definition 6.3). We prove that the $\text{cyl}(D)$ and $D$ are homologically equivalent (Proposition 6.1). Using similar techniques as in the case of maps (i.e. replacing by a set defined by closed formulas etc.) we are then able to extend the algorithm for maps to the case of zigzag diagrams, and ultimately give an algorithm to compute barcodes of semi-algebraic zigzag diagrams.

The rest of the paper is organized as follows. In Section 2 we fix notation and give precise definitions of complexity and topological equivalences. We also give the necessary background in real algebraic geometry to make the rest of the paper self-contained. In Section 3 we give precise statements of the theorems proved in this paper. The subsequent sections are devoted to the proofs of these theorems. In Section 4 we state and prove some mathematical results that play an important role in the algorithms described in this paper. In Section 4.1, we give the construction of the semi-algebraic mapping cylinder (i.e. of the semi-algebraic set $\text{cyl}(f)$ referred to in the previous paragraph) and prove its main properties. In Section 4.2 we give the procedure for replacing a given closed semi-algebraic set by one having the same homotopy type and which is described by a closed formula. In Section 5 we complete the proof of Theorem 3. Finally, in Section 6 we apply the ideas developed in the proof of Theorem 3 to develop an algorithm for computing semi-algebraic zigzag persistent barcodes. In Section 7 we state some open problems.
2. Preliminaries

2.1. Homological equivalence of semi-algebraic maps. We begin with the precise definitions of the two kinds of topological equivalence that we are going to use in this paper.

2.1.1. Homological equivalences.

**Definition 2.1** (Homological ℓ-equivalences). We say that a semi-algebraic map \( f : X \to Y \) between two semi-algebraic sets \( X, Y \) is a homological ℓ-equivalence, if the induced homomorphisms between the homology groups \( H_i(f) : H_i(X) \to H_i(Y) \) are isomorphisms for \( 0 \leq i \leq \ell \).

Given two semi-algebraic maps \( f : X \to Y, f' : X' \to Y' \), a homological ℓ-equivalence between \( f \) and \( f' \) is a pair of semi-algebraic maps \( F_X, F_Y \) such that \( f' \circ F_X = F_Y \circ f \), and \( F_X, F_Y \) are homological ℓ-equivalences.

The relation of homological ℓ-equivalence as defined above is not an equivalence relation since it is not symmetric. In order to make it symmetric one needs to “formally invert” homologically ℓ-equivalences.

**Definition 2.2** (Homologically ℓ-equivalent). We will say that \( X \) is homologically ℓ-equivalent to \( Y \) (denoted \( X \sim_{\ell} Y \)), if and only if there exists spaces, \( X = X_0, X_1, \ldots, X_n = Y \) and homological ℓ-equivalences \( f_1, \ldots, f_n \) as shown below:

\[
\begin{array}{cccccc}
X_0 & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \cdots & \xrightarrow{f_n} & X_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X_0 & & X_1 & & X_2 & & \cdots & & X_n
\end{array}
\]

Similarly, we say that a semi-algebraic map \( f : X \to X' \) is homologically ℓ-equivalent to the semi-algebraic map \( g : Y \to Y' \), if there exists maps \( f = f_0, f_1, \ldots, f_n = g \), and homological ℓ-equivalences, \( F_1, \ldots, F_n \), as below:

\[
\begin{array}{cccccc}
f_0 & \xleftarrow{F_1} & f_1 & \xleftarrow{F_2} & f_2 & \cdots & \xleftarrow{F_n} & f_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_0 & & F_1 & & F_2 & & \cdots & & F_n
\end{array}
\]

It is clear that \( \sim_{\ell} \) is an equivalence relation.

**Remark 2.1.** One main tool that we use is the Vietoris-Begle theorem. Since, there are many versions of the Vietoris-Begle theorem in the literature we make precise what we use below. If \( X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n \)
are compact semi-algebraic subsets (and so are locally contractible), and $f : X \to Y$ is a semi-algebraic continuous map such that $f^{-1}(y)$ is homologically $\ell$-connected for each $y \in Y$, then we can conclude that $f$ is a homological $\ell$-equivalence (see for example, the statement of the Vietoris-Begle theorem in [18]). This latter theorem is also valid for semi-algebraic maps between closed and bounded semi-algebraic sets over arbitrary real closed fields, once we know it for maps between compact semi-algebraic subsets over $\mathbb{R}$. This follows from a standard argument using the Tarski-Seidenberg transfer principle and the fact that homology groups of closed bounded semi-algebraic sets can be defined in terms of finite triangulations. We will refer to this version of the Vietoris-Begle theorem as the homological version of the Vietoris-Begle theorem.

2.2. Definition of complexity of algorithms. We will use the following notion of “complexity of an algorithm” in this paper. We follow the same definition as used in the book [3].

Definition 2.3 (Complexity of algorithms). In our algorithms we will take as input quantifier-free first order formulas whose terms are polynomials with coefficients belonging to an ordered domain $D$ contained in a real closed field $\mathbb{R}$. By complexity of an algorithm we will mean the number of arithmetic operations and comparisons in the domain $D$. If $D = \mathbb{R}$, then the complexity of our algorithm will agree with the Blum-Shub-Smale notion of real number complexity [8]. In case, $D = \mathbb{Z}$, then we are able to deduce the bit-complexity of our algorithms in terms of the bit-sizes of the coefficients of the input polynomials, and this will agree with the classical (Turing) notion of complexity.

2.3. Real algebraic preliminaries.

Notation 2.1 (Realizations, $P_-$, $P$-closed semi-algebraic sets). For any finite set of polynomials $P \subset \mathbb{R}[X_1, \ldots, X_s]$, we call any quantifier-free first order formula $\phi$ with atoms, $P = 0, P < 0, P > 0, P \in P$, to be a $P$-formula.

Given any semi-algebraic subset $Z \subset \mathbb{R}^k$, we call the realization of $\phi$ in $Z$, namely the semi-algebraic set

$$R(\phi, Z) := \{ x \in Z \mid \phi(x) \}$$

a $P$-semi-algebraic subset of $Z$.

If $Z = \mathbb{R}^k$, we will denote the realization of $\phi$ in $\mathbb{R}^k$ by $R(\phi)$.

We say that a quantifier-free formula $\phi$ is closed if it is a formula in disjunctive normal form with no negations, and with atoms of the form $P \geq 0, P \leq 0$ (resp. $P > 0, P < 0$), where $P \in \mathbb{R}[X_1, \ldots, X_k]$. 

If the set of polynomials appearing in a closed formula is contained in a finite set \( \mathcal{P} \), we will call such a formula a \( \mathcal{P} \)-closed formula, and we call the realization, \( \mathcal{R}(\phi) \), a \( \mathcal{P} \)-closed semi-algebraic set.

We now state precisely the main results proved in this paper.

## 3. Main Results

**Theorem 2.** For each \( \ell \geq 0 \), there is an algorithm that accepts as input
(a) finite sets of polynomials
\[
\mathcal{P}_S \subset \mathbb{D}[X_1, \ldots, X_k], \\
\mathcal{P}_T \subset \mathbb{D}[Y_1, \ldots, Y_m], \\
\mathcal{P}_f \subset \mathbb{D}[X_1, \ldots, X_k, Y_1, \ldots Y_m];
\]
(b) a \( \mathcal{P}_S \)-closed formula \( \phi_S \), a \( \mathcal{P}_T \)-closed formula \( \phi_T \), and a \( \mathcal{P}_f \)-closed formula \( \phi_f \), such that \( \mathcal{R}(\phi_S), \mathcal{R}(\phi_T) \) are bounded and \( \mathcal{R}(\phi_f, \mathbb{R}^k \times \mathbb{R}^m) \) is the graph of a semi-algebraic map \( f : S = \mathcal{R}(\phi_S) \to \mathcal{R}(\phi_T) = T \);

and produces as output for each \( i, 0 \leq i \leq \ell \):
(a) bases of \( H_i(S), H_i(T) \);
(b) the matrix corresponding to these bases of the linear map \( H_i(f) : H_i(X) \to H_i(Y) \).

The complexity of the algorithm is bounded by
\[
(sd)^{(k+m)O(\ell)},
\]
where
\[
s = \max(\text{card}(\mathcal{P}_S), \text{card}(\mathcal{P}_T), \text{card}(\mathcal{P}_f)),
\]
and
\[
d = \max_{P \in \mathcal{P}_S \cup \mathcal{P}_T \cup \mathcal{P}_f} \deg(P).
\]

Theorem 2 will follow (using standard linear algebra algorithms) from the following theorem.

**Theorem 3.** For each \( \ell \geq 0 \), there is an algorithm that accepts as input
(a) finite sets of polynomials
\[
\mathcal{P}_S \subset \mathbb{D}[X_1, \ldots, X_k], \\
\mathcal{P}_T \subset \mathbb{D}[Y_1, \ldots, Y_m], \\
\mathcal{P}_f \subset \mathbb{D}[X_1, \ldots, X_k, Y_1, \ldots Y_m];
\]
(b) a $\mathcal{P}_S$-closed formula $\phi_S$, a $\mathcal{P}_T$-closed formula $\phi_T$, and a $\mathcal{P}_f$-closed formula $\phi_f$, such that $\mathcal{R}(\phi_S), \mathcal{R}(\phi_T)$ are bounded and $\mathcal{R}(\phi_f, \mathbb{R}^k \times \mathbb{R}^m)$ is the graph of a semi-algebraic map $f : S = \mathcal{R}(\phi_S) \to \mathcal{R}(\phi_T) = T$;

and produces as output simplicial complexes $\Delta_S, \Delta_T, \Delta_S \subset \Delta_T$, such that $|\Delta_S| \leftrightarrow |\Delta_T|$ is homologically $\ell$-equivalent to $f : S \to T$.

The complexity of the algorithm is bounded by

$$(sd)^{(k+m)O(\ell)},$$

where

$$s = \max(\text{card}(\mathcal{P}_S), \text{card}(\mathcal{P}_T), \text{card}(\mathcal{P}_f)),$$

and

$$d = \max_{P \in \mathcal{P}_S \cup \mathcal{P}_T \cup \mathcal{P}_f} \deg(P).$$

We extend Theorem 2 to zigzag diagrams. We prove the following theorem.

**Theorem 4.** For each fixed $\ell \geq 0$, there exists an algorithm with the following properties. The algorithm takes the following input:

1. $R > 0$;
2. a tuple of closed formulas $\Phi = (\phi_0, \ldots, \phi_n)$, with $S_i = \mathcal{R}(\phi_i, B) \subset \mathbb{R}^k$ for $0 \leq i \leq n$, where $B = B_k(0, R)$;
3. a tuple of closed formulas $\Psi = (\psi_1, \ldots, \psi_n)$, such that $\mathcal{R}(\psi_i, B \times B)$ is the graph of a semi-algebraically continuous map $f_i : S_i \to S_{i-1}$ if $i$ is odd, and is the graph of a semi-algebraically continuous map $f_i : S_{i-1} \to S_i$ if $i$ is even.

The algorithm produces as output for each $i, 0 \leq i \leq \ell$:

(a) Bases of the homology groups $H_i(S_0), 1 \leq j \leq n$;
(b) Matrices of the maps $H_i(f_j) : H_i(S_{j-1}) \to H_i(S_j), 1 \leq j \leq n$.

The complexity of the algorithm is bounded by $(nsd)^{O(\ell)}$, where $s$ is the cardinality of the set of polynomials occurring in all the formulas in the input and $d$ their maximum degree.

4. **Mathematical Preliminaries**

In this section we state and prove some mathematical results that will play a role in the proofs of the main theorems.
4.1. **Replacing semi-algebraic maps by inclusion maps.** The key idea that goes into the proof of Theorem 3 below is a semi-algebraic adaptation of the classical topological notion of a mapping cylinder of a map $f : X \to Y$ which we recall now.

**Definition 4.1** (Mapping cylinder). Given a map $f : X \to Y$, the mapping cylinder $\text{cyl}(f)$ of $f$ is the space defined by

$$\text{cyl}(f) = \left( (X \times [0, 1]) \bigsqcup Y \right) / \sim,$$

where for each $x \in X$, $(x, 0) \sim f(x)$.

It is easy to prove that there exists a deformation retraction $p : \text{cyl}(f) \to Y$. Denoting by $i : X \to \text{cyl}(f)$, the inclusion $i(x) = (x, 1)$, we have a factorization $f = p \circ i$. Since, $p$ is a homotopy equivalence, one obtains that the inclusion $i : X \to \text{cyl}(f)$ is homologically equivalent to the map $f$ via the commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i} & \text{cyl}(f) \\
\downarrow\text{id} & & \downarrow p \\
X & \xrightarrow{f} & Y
\end{array}
$$

Now suppose that $f : S \to T$ is a semi-algebraic map between two closed and bounded semi-algebraic sets $S, T$. The mapping cylinder construction suggests a way to construct an inclusion map $i : S \hookrightarrow \text{cyl}(f)$ which is homologically equivalent to $f$. This is important for us since once we have replaced the given map $f$ by an inclusion, we can apply the main result in [6] to obtain a pair of simplicial complexes, $\Delta_1 \subset \Delta_2$ such that the inclusion $|\Delta_1| \hookrightarrow |\Delta_1|$ is homologically $\ell$-equivalent to $i : S \hookrightarrow \text{cyl}(f)$, and hence to $f$.

However, one obstruction to realizing the above goal is the fact that the definition of the mapping cylinder involves taking a quotient. It is true that the quotient of a closed and bounded semi-algebraic set by a proper equivalence relation is homeomorphic to a semi-algebraic set [20] – however, there is no algorithm known with a singly exponential complexity for obtaining a semi-algebraic description of this quotient.

We take a slightly different route. We define below a modification of the classical mapping cylinder of a map $f$, which we denote by $\widetilde{\text{cyl}}(f)$ (see Definition 4.2) which in the case where $f$ is a semi-algebraic map between closed and bounded semi-algebraic sets satisfies the same properties as the classical mapping cylinder i.e. $f$ factorizes through an inclusion $i : S \to \widetilde{\text{cyl}}(f)$ and a semi-algebraic homotopy equivalence $p : \widetilde{\text{cyl}}(f) \to T$, so that finally $f = p \circ i$, and the inclusion $i : S \to \widetilde{\text{cyl}}(f)$
is homologically equivalent to $f$. The main advantage of $\tilde{\text{cyl}}(f)$ is that, as a semi-algebraic set it is described by a (existentially) quantified formula (see Eqn.(3)) which is determined in a simple way from any first-order formulas defining the semi-algebraic sets $S, T$ and the graph of the map $f$. Using effective quantifier-elimination algorithms we can then obtain a quantifier-free formula defining $\tilde{\text{cyl}}(f)$.

There is one technical issue that creates complications in the above picture. If $S, T$ are closed and bounded semi-algebraic sets, then the semi-algebraic set $\tilde{\text{cyl}}(f)$ is obtained as the image under projection of a closed and bounded semi-algebraic set, and is thus known to be closed and bounded. However, even if we start with closed formulas defining $S, T$ and graph$(f)$, since the known effective quantifier-elimination algorithm with single exponential complexity that we use does not guarantee that the quantifier-free formula that we obtain describing $\tilde{\text{cyl}}(f)$ is closed. It is important for the algorithm downstream that we use for simplicial replacement that this description be closed. We deal with this technical issue in a subsequent section.

We now define $\tilde{\text{cyl}}(f)$.

**Definition 4.2** (Mapping cylinder for semi-algebraic maps). Let $S \subset \mathbb{R}^k, T \subset \mathbb{R}^m$ be semi-algebraic subsets and $f : S \rightarrow T$ a semi-algebraic map. We denote

$$\tilde{\text{cyl}}(f) = \{(\lambda \cdot x, f(x), \lambda) \mid x \in S, \lambda \in [0, 1]\} \cup \{(0, y, 0) \mid y \in T\},$$

With the above notation we have the following proposition.

**Proposition 4.1.** Suppose that $S$ is closed and bounded and let $r : \tilde{\text{cyl}}(f) \rightarrow T$ be the map defined by $r(x, y, \lambda) = y$. Then, $r$ is a homological equivalence.

**Proof.** It follows from Eqn. (1), that for $y \in T$,

$$r^{-1}(y) = \{(\lambda \cdot x, y, \lambda) \mid x \in S, f(x) = y, \lambda \in [0, 1]\} \cup \{(0, y, 0)\}.$$

There are two cases.

(A) If $y \in \text{Im}(f)$, then

$$r^{-1}(y) = \{(\lambda \cdot x, y, \lambda) \mid x \in S, f(x) = y, \lambda \in [0, 1]\},$$

which is semi-algebraically homeomorphic to the cone over $f^{-1}(y)$ and hence semi-algebraically contractible.

(B) If $y \notin \text{Im}(f)$, then

$$r^{-1}(y) = \{(0, y, 0)\}$$

and hence semi-algebraically contractible.
The proposition now follows from the homological version of the Vietoris-Begle theorem (see Remark 2.1).

\[ \square \]

**Proposition 4.2.** Let \( i : S \to \sim \text{yl}(f) \) be the injective map \( x \mapsto (x, f(x), 1) \). Then the following diagram is commutative.

\[
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow{\sim \text{yl}(f)} & & \downarrow{id} \\
\sim \text{yl}(f) & \xrightarrow{r} & T
\end{array}
\]

**Proof.** This is immediate from the definition of \( \sim \text{yl}(f) \) (Eqn. (1)) and the definition of the map \( r \).

\[ \square \]

**Proposition 4.3.** Suppose that \( S \) is closed and bounded. Then the inclusion map, \( i(S) \hookrightarrow \text{cyl}(f) \) is homologically equivalent to \( f : S \to T \).

**Proof.** Follows directly from Propositions 4.1 and 4.2.

\[ \square \]

**Proposition 4.4.** Let \( \phi_S(X_1, \ldots, X_k), \phi_T(Y_1, \ldots, Y_m), \phi_f(X_1, \ldots, X_k, Y_1, \ldots, Y_m) \) be first order formulas such that \( \mathcal{R}(\phi_f, \mathbb{R}^k \times \mathbb{R}^m) \) is the graph of a semi-algebraic map \( f : S = \mathcal{R}(\phi_S) \to \mathcal{R}(\phi_T) = T \). Let

\[
\Theta_{\phi_S, \psi_T, \phi_f}(\overline{X}, \overline{Y}, T) = \begin{cases} 
(T = 0) \land \overline{X} = \mathbf{0} \land \phi_T(\overline{Y}) & \lor \\
(0 \leq T \leq 1) \land \exists \overline{Z} \ (\overline{X} = T \overline{Z} \land \phi_S(\overline{Z})) \land \phi_f(\overline{Z}, \overline{Y}) \land \phi_T(\overline{Y}) &
\end{cases}
\]

Then,

\[ \mathcal{R}(\Theta_{\phi_S, \psi_T, \phi_f}) = \sim \text{yl}(f). \]

Moreover,

\[ \mathcal{R}(\Theta_{\phi_S, \psi_T, \phi_f} \land (T = 1)) = i(S) \hookrightarrow \sim \text{yl}(f). \]

**Proof.** Follows directly from the definition of \( \sim \text{yl}(f) \) (Eqn. (1)).

\[ \square \]

### 4.2. Replacing a closed semi-algebraic set by a semi-algebraic set defined by a closed formula

One basic open problem in algorithmic semi-algebraic geometry is to design an efficient algorithm which takes as input a quantifier-free formula \( \phi \) such that \( \mathcal{R}(\phi) \) is a closed semi-algebraic set \( S \subset \mathbb{R}^k \), and produces as output a finite set \( P \subset \mathbb{R}[X_1, \ldots, X_k] \) and a \( P \)-closed formula \( \psi \) such that \( \mathcal{R}(\phi) = \mathcal{R}(\psi) \).
No algorithm with a singly exponential complexity is known for this problem.

In the absence of an efficient algorithm for solving the above problem, we consider the following substitute that is often enough for application. A fundamental construction due to Gabrielov and Vorobjov [19] gives an efficient procedure to replace an arbitrary semi-algebraic set by a closed and bounded one having the same homotopy type. This homotopy equivalence is usually not a deformation retraction.

We describe below a construction similar to that in [19], when applied to a formula $\phi$ such that $R(\phi, B)$ is a closed semi-algebraic subset of a closed Euclidean ball $B \subseteq R^k$, produces a closed formula $\psi$ defined over a real closed extension $R'$ of $R$, such that the extension of $R(\phi, B)$ to $R^k$ is a semi-algebraic deformation retraction of $R(\psi, B)$.

But we first need to introduce some preliminary definitions and notation.

4.3. Real closed extensions and Puiseux series. We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [3] for further details.

**Notation 4.1.** For $R$ a real closed field we denote by $R\langle \varepsilon \rangle$ the real closed field of algebraic Puiseux series in $\varepsilon$ with coefficients in $R$. We use the notation $R\langle \varepsilon_1, \ldots, \varepsilon_m \rangle$ to denote the real closed field $R\langle \varepsilon_1 \rangle \langle \varepsilon_2 \rangle \cdots \langle \varepsilon_m \rangle$.

Note that in the unique ordering of the field $R\langle \varepsilon_1, \ldots, \varepsilon_m \rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$.

**Notation 4.2.** For elements $x \in R\langle \varepsilon \rangle$ which are bounded over $R$ we denote by $\lim_\varepsilon x$ to be the image in $R$ under the usual map that sets $\varepsilon$ to 0 in the Puiseux series $x$.

**Notation 4.3.** If $R'$ is a real closed extension of a real closed field $R$, and $S \subseteq R^k$ is a semi-algebraic set defined by a first-order formula with coefficients in $R$, then we will denote by $\text{Ext}(S, R') \subseteq R^k$ the semi-algebraic subset of $R^k$ defined by the same formula. It is well known that $\text{Ext}(S, R')$ does not depend on the choice of the formula defining $S$ [3, Proposition 2.87].

Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subseteq R[X_1, \ldots, X_k]$ be a finite set of polynomials, and let $B \subseteq R^k$ be a closed euclidean ball.

**Notation 4.4.** For $\sigma \in \{0, 1, -1\}^\mathcal{P}$, let

$$\text{level}(\sigma) = \text{card}(\{P \in \mathcal{P} \mid \sigma(P) = 0\}).$$
For $c, d \in \mathbb{R}, 0 < d < c$, and $\sigma \in \{0, 1, -1\}^P$, let $\sigma(c, d)$ denote the closed formula

$$\bigwedge_{\sigma(P) = 0} (-d \leq P \leq d) \land \bigwedge_{\sigma(P) = 1} (P \geq c) \land \bigwedge_{\sigma(P) = -1} (P \leq -c).$$

**Notation 4.5.** For a $\mathcal{P}$-formula $\phi$ we denote

$$\Sigma_\phi = \left\{ \sigma \in \{0, 1, -1\}^P \mid \left( \bigwedge_{P \in \mathcal{P}} (\text{sign}(P) = \sigma(P)) \right) \Rightarrow \phi \right\},$$

where “$\Rightarrow$” denotes logical implication.

Let

$$R' = R(\mu_s, \nu_s, \ldots, \mu_0, \nu_0) = R(\tilde{\eta}),$$

denoting by $\tilde{\eta}$ the sequence $\mu_s, \nu_s, \ldots, \mu_0, \nu_0$.

**Notation 4.6.** We denote

$$\mathcal{P}^*(\bar{\mu}, \bar{\nu}) = \bigcup_{P \in \mathcal{P}} \bigcup_{j=0}^s \{P \pm \mu_j, P \pm \nu_j\} \subset R'[X_1, \ldots, X_k].$$

Finally,

**Notation 4.7.** We denote by $\overline{\phi(\bar{\mu}, \bar{\nu})}$ the $\mathcal{P}^*(\bar{\mu}, \bar{\nu})$-closed formula

$$\bigvee_{\sigma \in \Sigma_\phi} \overline{\sigma(\mu_{\text{level}}(\sigma), \nu_{\text{level}}(\sigma))}$$

(see Notation 4.5).

Following the notation introduced above.

**Proposition 4.5.** Let $R > 0, B = B_k(0, R)$, and suppose that $S = \mathcal{R}(\phi, B)$ is closed. Then,

$$S' \subseteq S,$$

where $S' = \mathcal{R}(\phi(\tilde{\mu}, \tilde{\nu}), \text{Ext}(B, R'))$. In particular, $\text{Ext}(S, R')$ is a semi-algebraic deformation retract of $S'$.

**Proof.** See [6, Appendix].

---

5. **Proofs of the theorems**

5.1. **Algorithmic Preliminaries.** We recall the following definition from [6].
Notation 5.1 (Diagram of various unions of a finite number of subspaces). Let $J$ be a finite set, $A$ a topological space, and $\mathcal{A} = (A_j)_{j \in J}$ a tuple of subspaces of $A$ indexed by $J$.

For any subset $J' \subseteq J$, we denote
\[
\mathcal{A}^{J'} = \bigcup_{j' \in J'} A_{j'}, \\
\mathcal{A}_{J'} = \bigcap_{j' \in J'} A_{j'},
\]

We consider $2^J$ as a category whose objects are elements of $2^J$, and whose only morphisms are given by:
\[
2^J(J', J'') = \emptyset \quad \text{if} \quad J' \notin J'', \\
2^J(J', J'') = \{ i, j', j'' \} \quad \text{if} \quad J' \subseteq J''.
\]

We denote by $\text{Simp}^I(\mathcal{A}) : 2^J \to \text{Top}$ the functor (or the diagram) defined by
\[
\text{Simp}^I(\mathcal{A})(J') = \mathcal{A}^{J'}, J' \in 2^J,
\]
and $\text{Simp}^I(\mathcal{A})(i, J', J'')$ is the inclusion map $\mathcal{A}^{J'} \hookrightarrow \mathcal{A}^{J''}$.

We will use an algorithm whose existence is proved in [6, Theorem 1], and which we will refer to as Algorithm for computing simplicial replacement, that given a tuple of closed-formulas $\Phi = (\phi_0, \ldots, \phi_N)$, $R > 0$, and $\ell \geq 0$, produces as output a simplicial complex $K$ and subcomplexes $K_i, 0 \leq i \leq N$ of $K$, such that the diagram
\[
\text{Simp}^{[N]}(\overline{(\mathcal{R}(\phi_i, B_k(0, R)))_{i \in [N]}})
\]

is homologically $\ell$-equivalent ([6, Section 2.1.1]) to the diagram
\[
\text{Simp}^{[N]}((|K_i|)_{i \in [N]})
\]
(where $|K_i| \subset |K|$ is the geometric realization of $K_i$ and $[N] = \{0, \ldots, N\}$).

We refer the reader to [6] for the details.

The complexity of this algorithm, as well as the size of the output simplicial complex $\Delta$, are bounded by
\[
(Nsd)^{kO(m)},
\]
where $s = \text{card}(\mathcal{P})$, and $d = \max_{P \in \mathcal{P}} \deg(P)$. 
5.2. **Proofs of Theorems 2 and 3.** We first prove Theorem 3 by describing an algorithm (Algorithm 1 below) and proving its correctness and analyzing its complexity. Theorem 2 will then follow in a straightforward way.

**Algorithm 1** (Applying homology functor to semi-algebraic maps)

**Input:**
1. $\ell \geq 0$;
2. a finite sets of polynomials $\mathcal{P}_S \subset \mathbb{D}[X_1, \ldots, X_k], \mathcal{P}_T \subset \mathbb{D}[Y_1, \ldots, Y_m], \mathcal{P}_f \subset \mathbb{D}[X_1, \ldots, X_k, Y_1, \ldots Y_m]$;
3. $\mathcal{P}_S$-closed formulas $\phi_S$, $\mathcal{P}_T$-closed formula $\phi_T$, and $\mathcal{P}_f$-closed formula $\phi_f$, such that $\mathcal{R}(\phi_S), \mathcal{R}(\phi_T)$ are bounded and $\mathcal{R}(\phi_f, \mathbb{R}^k \times \mathbb{R}^m)$ is the graph of a semi-algebraic map $f : S = \mathcal{R}(\phi_S) \rightarrow \mathcal{R}(\phi_T) = T$.

**Output:**
Simplicial complexes $\Delta_S, \Delta_T, \Delta_S \subset \Delta_T$, such that $|\Delta_S| \hookrightarrow |\Delta_T|$ is homologically $\ell$-equivalent to $f : S \rightarrow T$.

**Procedure:**
1. Let $\varepsilon$ be an infinitesimal and $R \leftarrow \mathbb{R}[\varepsilon]$.
2. Call Algorithm 14.5 in [3] (Quantifier Elimination) with input the existentially quantified formula $\Theta_{\phi_S,\phi_T,\phi_f}$ to obtain a quantifier-free formula $\Theta'_{\phi_S,\phi_T,\phi_f}$ equivalent to $\Theta_{\phi_S,\phi_T,\phi_f}$.
3. $\Theta''_{\phi_S,\phi_T,\phi_f} \leftarrow \Theta'_{\phi_S,\phi_T,\phi_f} \wedge (T = 1)$.
4. $\mathcal{P} \leftarrow$ the set of polynomials appearing in the formula $\Theta''_{\phi_S,\phi_T,\phi_f}$.
5. Apply Algorithm for computing simplicial replacement, with input the pair of formulas $(\Theta''_{\phi_S,\phi_T,\phi_f}, \Theta'_{\phi_S,\phi_T,\phi_f})$ (recall Notation 4.7) and $R = \frac{1}{\varepsilon}$, and obtain as output a simplicial complex $K$, and subcomplexes $K_1, K_2$ of $K$.
6. $\Delta_S \leftarrow K_1, \Delta_T \leftarrow K_1 \cup K_2$.
7. Output the pair $(\Delta_S, \Delta_T)$.

**Complexity:** The complexity of the algorithm is bounded by

$$(sd)^{(k+m)O(\ell)},$$

where

$s = \max(\text{card}(\mathcal{P}_S), \text{card}(\mathcal{P}_T), \text{card}(\mathcal{P}_f)),$

and

$$d = \max_{P \in \mathcal{P}_S \cup \mathcal{P}_T \cup \mathcal{P}_f} \text{deg}(P).$$
Proof of correctness. The correctness of the algorithm follows from the correctness of Algorithm 14.5 in [3] (Quantifier Elimination), Propositions 4.5 and 4.3 as well as the correctness of Algorithm for computing simplicial replacements in [6]. □

Complexity analysis. The complexity of the algorithm follows from the complexity of Algorithm 14.5 in [3] (Quantifier Elimination), and the Algorithm for computing simplicial replacement. □

Proof of Theorem 3. The theorem follows from the correctness and the complexity analysis of Algorithm 1. □

Proof of Theorem 2. Theorem 2 follows from Theorem 3 after observing that once we have the finite simplicial complexes $\Delta_S, \Delta_T$ with $\Delta_S \subset \Delta_T$, then using standard algorithms from linear algebra (Gauss-Jordan elimination) one can compute bases of $H_i(\Delta_S)$ and $H_i(\Delta_T), 0 \leq i \leq \ell$, and the matrix for the map $H(j)_i$, where $j : \Delta_S \hookrightarrow \Delta_T$ is the inclusion map. We omit the details but it is clear that the complexity of this step is bounded polynomially in the size of $\Delta_T$. This proves Theorem 2. □

6. Application to semi-algebraic zigzag persistence

In this section we discuss one application of the main result of the paper. In the previous section we have designed an algorithm for effectively applying the homology functor, $H_i(\cdot)$, to semi-algebraic maps between closed and bounded semi-algebraic sets. A next step is to effectively apply the homology functor to more general diagrams (of semi-algebraic maps).

One important class of diagrams that has been studied previously from an effective homology computation from the point of view were diagrams of the form:

$$S_0 \to S_1 \to \cdots \to S_n,$$

where each $S_i$ is a closed and bounded semi-algebraic sets and all maps are inclusions. This is the setting of persistent homology. The problem of computing the persistent homology groups (and the associated barcode) of a filtration of a given semi-algebraic set by the sub-level sets of a semi-algebraic map was studied in [7]. It is proved in [7] that for each fixed $\ell > 0$, there exists an algorithm with singly exponential complexity that computes the first $\ell$-dimensional barcodes of such a filtration.

The ideas introduced in the previous section allow us now to consider more general diagrams. Indeed the notion of persistent homology has been generalized to arbitrary diagrams $D : P \to \textbf{Top}$ (here $D$ is a
functor from a poset category $P$ to the category of topological spaces) [10]. One particular class of poset diagrams that has been studied in the literature are the so called “zigzag” diagrams that we define below (see [14]).

6.1. Zigzag diagrams. We begin by defining precisely zigzag diagrams.

**Notation 6.1.** We denote by $Z_n$ the poset whose set of elements is $\mathbb{Z}_n$, and whose Hasse diagram is indicated in the following figure.

```
\begin{array}{ccccccc}
& & & & & & \\
& 1 & & 3 & & 5 & \cdots \\
0 & \downarrow & 2 & \downarrow & 4 & \downarrow & 6 & \cdots \\
& & & & & & \\
\end{array}
```

**Definition 6.1 (Zigzag diagrams).** We call a functor $D : Z_n \to \mathbf{SA}_R$ from the poset category $Z_n$ to the category of semi-algebraic sets and semi-algebraic maps a zigzag diagram.

**Remark 6.1.** The zigzag diagrams that we consider in this paper where the arrows (maps) alternate in directions are not the most general possible. A general zigzag diagram need not have this alternation. Applying the homology functor to general zigzag diagrams gives rise to zigzag persistence modules which are precisely the quiver representations of quivers of type $A$ (see [14]). We restrict to the case of alternating arrows (also called regular zigzag diagrams) for the ease of exposition and simplifying notation.

We prove below that for each fixed $\ell \geq 0$, there exists an algorithm that given a zigzag functor $D : Z_n \to \mathbf{SA}_R$ (i.e. given quantifier-free formulas describing the semi-algebraic sets $D(i)$ and the graphs of the various maps $D(i) \to D(i-1), D(i) \to D(i+1)$ for every odd $i, 0 \leq i \leq n$), computes a functor $D'$ from $Z_n$ to the category of finite simplicial complexes, such that the composition of $D'$ with the geometric realization functor $| \cdot |$ is homologically $\ell$-equivalent to $D$, and all the morphisms $D'(i) \to D'(i-1), D'(i) \to D'(i+1)$ are inclusions. Moreover, the complexity of the algorithm is bounded by $(nsd)^{\ell O(\ell)}$, where $s$ is the cardinality of the set of polynomials occurring in all the formulas in the input and $d$ their maximum degree.

The more precise statement is as follows.

**Theorem 5.** For each fixed $\ell \geq 0$, there exists an algorithm with the following properties. The algorithm takes the following input:

(a) $R > 0$;
(b) a tuple of closed formulas $\Phi = (\phi_0, \ldots, \phi_n)$, with $S_i = \mathcal{R}(\phi_i, B) \subset \mathbb{R}^k$ for $0 \leq i \leq n$, where $B = B_k(0, \mathbb{R})$;
(c) a tuple of closed formulas $\Psi = (\psi_1, \ldots, \psi_n)$, such that $\mathcal{R}(\Psi_i, B \times B)$ is the graph of a semi-algebraically continuous map $f_i : S_i \to S_{i-1}$ if $i$ is odd, and is the graph of a semi-algebraically continuous map $f_i : S_{i-1} \to S_i$ if $i$ is even.

The algorithm produces as output simplicial complexes $\Delta_0, \ldots, \Delta_n$, having the following properties:

(a) $\Delta_i$ is a subcomplex of $\Delta_{i-1}$ and $\Delta_{i+1}$ if $i$ is odd, and $\Delta_{i-1}, \Delta_{i+1}$ are subcomplexes of $\Delta_i$ if $i$ is even.
(b) The diagram

$$|\Delta_0| \leftrightarrow |\Delta_1| \leftrightarrow |\Delta_2| \cdots$$

is homologically $\ell$-equivalent to

$$S_0 \leftrightarrow S_1 \to S_2 \cdots$$

The complexity of the algorithm is bounded by $(nsd)^{O(\ell)}$, where $s$ is the cardinality of the set of polynomials occurring in all the formulas in the input and $d$ their maximum degree.

6.2. Proofs of Theorems 1, 4 and 5. We first prove Theorem 5 which is the main theorem of this section. Theorems 1 are 4 straightforward consequences.

In order to prove Theorem 5, we first introduce a more general mapping cylinder construction which is adapted to the zigzag diagrams. For a zigzag diagram consisting of just one zigzag

$$S_{i-1} \xrightarrow{f_i} S_i \xleftarrow{f_{i+1}} S_{i+1}$$

we would like to replace the diagram by the union of two mapping cylinders glued along $S_i$ as shown in Figure 1. The maps $f_i, f_{i+1}$ are replaced by inclusions of $S_{i-1}$ and $S_{i+1}$ into the mapping cylinders of $f_i$ and $f_{i+1}$, and $S_i$ is replaced by the union of the mapping cylinders of $f_i$ and $f_{i+1}$ (oriented in opposite directions and glued along $S_i$).
We now extend the above idea in two directions. We consider zigzag diagrams with a finite number of zigzags instead of just one, and instead of the classical mapping cylinder we use the semi-algebraic version introduced in Definition 4.2.

We first define directed versions of the semi-algebraic mapping cylinder construction.

**Definition 6.2** (Directed semi-algebraic mapping cylinder). Let $S \subset \mathbb{R}^k, T \subset \mathbb{R}^k$ be semi-algebraic subsets and $f : S \to T$ a semi-algebraic map, and $a, b \in \mathbb{R}, a < b$. We denote:

$$\tilde{\text{cyl}}(f)(a, b) = \{((\lambda - a)/(b - a) \cdot x, f(x), \lambda) \mid x \in S, \lambda \in [a, b]\} \cup \{(0, y, a) \mid y \in T\},$$

$$\tilde{\text{cyl}}(f)(a, b) = \{((b - \lambda)/(b - a) \cdot x, f(x), \lambda) \mid x \in S, \lambda \in [a, b]\} \cup \{(0, y, b) \mid y \in T\}.$$ (Note that $\tilde{\text{cyl}}(f) = \tilde{\text{cyl}}(f)(0, 1)$.)

We now define semi-algebraic mapping cylinders of zigzag diagrams.

**Definition 6.3** (Semi-algebraic mapping cylinder of zigzag diagrams). Let $D : \mathbb{Z}_n \to \text{SA}_{\mathbb{R}}$ be a zigzag diagram and for $0 \leq i \leq n$, let $S_i = D(i)$, and for $1 \leq i \leq n$, let $f_i$ denote the map $f_i : S_{i-1} \to S_i$.

For $0 < i \leq n$, with $i$ odd we define

$$\tilde{S}_i = \begin{cases} \{(x, h_i(x, \mu), \mu) \mid x \in S_i, \mu \in [i - \frac{1}{2}, i + \frac{1}{2}]\}, & i \neq n, \\ \{(x, (\mu - n + \frac{3}{2}) \cdot f_i(x), \mu) \mid x \in S_n, \mu \in [n - \frac{1}{2}, n]\} & i = n, \end{cases}$$

where

$$h_i(x, \mu) = (\mu - i + \frac{3}{2}) \cdot f_i(x) + (\mu - i + \frac{1}{2}) \cdot f_{i+1}(x).$$
For $0 \leq i \leq n$, with $i$ even, we define
\[
\tilde{S}_i = \begin{cases} 
\tilde{cyl}(f_i)(i - \frac{1}{2}, i) \cup \tilde{cyl}(f_{i+1})(i, i + \frac{1}{2}) \cup \tilde{S}_{i-1} \cup \tilde{S}_{i+1} & i \neq 0, n, \\
\tilde{cyl}(f_i)(0, \frac{1}{2}) \cup \tilde{S}_1 & i = 0, \\
\tilde{cyl}(f_n)(n - \frac{1}{2}, n) \cup \tilde{S}_{n-1} & i = n.
\end{cases}
\]

We denote the diagram
\[
\begin{array}{cccccc}
\tilde{S}_0 & \tilde{S}_1 & \tilde{S}_2 & \tilde{S}_3 & \tilde{S}_4 & \tilde{S}_5 & \cdots \\
\tilde{S}_1 & \tilde{S}_3 & \tilde{S}_4 & \tilde{S}_5 & \tilde{S}_6 & \cdots \\
\tilde{S}_2 & \tilde{S}_4 & \tilde{S}_5 & \tilde{S}_6 & \cdots \\
\tilde{S}_3 & \tilde{S}_5 & \tilde{S}_6 & \cdots \\
\tilde{S}_4 & \tilde{S}_6 & \cdots \\
\tilde{S}_5 & \cdots \\
\tilde{S}_6 & \cdots \\
\end{array}
\]

where the arrows denote inclusions, by $\tilde{cyl}(D)$.

Suppose in Definition 6.3 each $S_i$ is a closed and bounded semi-algebraic subset of $\mathbb{R}^k$. Notice the following (see also the schematic Figure 2).

(a) For each even $i$, $\tilde{S}_{i-1}, \tilde{S}_{i+1} \subset \tilde{S}_i$ (using the convention that $\tilde{S}_{-1} = \tilde{S}_{n+1} = \emptyset$).
(b) For each odd $i$, the map $\tilde{S}_i \to S_i, (x, y, \mu) \mapsto x$ is a homological equivalence using the homological version of the Vietoris-Begle theorem.
(c) The union $\bigcup_{0 \leq i \leq n} \tilde{S}_i$ is a closed and bounded semi-algebraic subset of $\mathbb{R}^k \times \mathbb{R}^k \times [0, n]$.
(d) For $T \subset \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}$, and $\mu \in \mathbb{R}$, let $T_\mu$ denote the subset of $T$ with last coordinate equal to $\mu$. Then for each even $i$, we have
\[
\begin{align*}
(\tilde{S}_{i-1})_{i-\frac{1}{2}} &= (\tilde{cyl}(f_i)(i - \frac{1}{2}, i))_{i-\frac{1}{2}}, \\
(\tilde{S}_{i+1})_{i+\frac{1}{2}} &= (\tilde{cyl}(f_{i+1})(i - \frac{1}{2}, i))_{i+\frac{1}{2}}, \\
(\tilde{S}_i)_i &= 0 \times \mathbb{S}_i \times \{i\} \cong \mathbb{S}_i.
\end{align*}
\]
The following proposition captures the key property of $\text{cyl}(D)$. We use the same notation introduced in Definition 6.3.

**Proposition 6.1.** Let $D : \mathbb{Z}_n \rightarrow \text{SA}_R$ be a zigzag diagram and for $0 \leq i \leq n$ $S_i = D(i)$ is a closed and bounded semi-algebraic subset of $\mathbb{R}^k$. Then the diagrams $D$ and $\text{cyl}(D)$ are homologically equivalent.

**Proof.** For $0 \leq j \leq n$, we define $g_j : \tilde{S}_j \rightarrow S_j$ be defined as follows.

$$g_j(x, y, \mu) = \begin{cases} 
  y & \text{if } j \text{ is even, } j - \frac{1}{2} \leq \mu \leq j + \frac{1}{2}, \\
  f_j(x) & \text{if } j \text{ is even, } j \neq 0, j - \frac{3}{2} \leq \mu \leq j - \frac{1}{2}, \\
  f_{j+1}(x) & \text{if } j \text{ is even, } j \neq n, j + \frac{1}{2} \leq \mu \leq j + \frac{3}{2}, \\
  x & \text{if } j \text{ is odd.}
\end{cases}$$

It is now easy to check that the for each (even) $j$ the following diagram commutes:

Finally, using the homological version of the Vietoris-Begle theorem it is an easy exercise to check that each $g_j, 0 \leq j \leq n$ is a homological equivalence. This proves the proposition. \qed
**Proof of Theorem 5.** Let D denote the zigzag diagram

```
/ \    / \    / \    / \    / \    / \   
S1  f1  S3  f3  S5  f5  ...  S0  f0  S2  f2  S4  f4  S6  f6  ...  
```

Using Algorithm 14.5 in [3] (Quantifier Elimination) and following Definition 6.3 we can compute using the input tuples of formulas $\Phi$ and $\Psi$ a tuple $\Phi = (\phi_0, ..., \phi_n)$ whose realization is $\text{cyl}(D)$. Finally we replace the tuple of formulas $\Phi$ by a tuple of closed formulas $\Phi$ $(\overline{\phi_0}, ..., \overline{\phi_n})$ (recall Notation 4.7). The number of polynomials and their degrees appearing in the $\Phi$ are all bounded singly exponentially. We then use the Algorithm for simplicial replacement to compute the simplicial complexes, $\Delta, \Delta_0, ..., \Delta_n$ having the required properties. 

**Proof of Theorem 4.** Theorem 4 follows from Theorem 5 using standard algorithms from linear algebra.

**Proof of Theorem 1.** Use Theorem 4 to reduce the computation of the barcode of a zigzag diagram of semi-algebraic maps to that of finite dimensional vector spaces, and then use the algorithm in [14, Section 4.2]. The complexity remains singly exponentially bounded.

---

7. **Conclusion and open problems**

In this paper we have described for each $\ell \geq 0$, algorithms with singly exponential complexity for computing the homology functor $H_i(\cdot)$ for $0 \leq i \leq \ell$ on semi-algebraic maps and zigzag diagrams on closed and bounded semi-algebraic sets.

We end with some open problems.

1. Is it possible to compute the homology functor with singly exponential complexity without having to restrict to only the first few dimensions?
2. Remove the assumption that all the semi-algebraic sets in the input are closed and bounded. One possible approach is to generalize the results of Gabrielov and Vorobjov [19] on replacing an arbitrary semi-algebraic set by a locally closed one without changing its homotopy type, to semi-algebraic maps.
3. Is it possible to extend the numerical algorithms mentioned in the Introduction for computing Betti numbers of semi-algebraic sets to
the functor setting? It will be necessary to study the condition number of semi-algebraic maps or more general diagrams?

4. Study the categorical complexity of the semi-algebraic homology functor, and prove a singly exponential bound on its functor complexity (as defined in [5]).

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Department of Mathematics, Purdue University, West Lafayette, IN 47906, U.S.A.

*Email address: sbasu@math.purdue.edu*

Department of Computer Science, Purdue University, West Lafayette, IN 47906, U.S.A.

*Email address: nkarisan@cs.purdue.edu*