Orbital stability of bound states of nonlinear Schrödinger equations with linear and nonlinear optical lattices

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We study the orbital stability and instability of single-spike bound states of semiclassical nonlinear Schrödinger (NLS) equations with critical exponent, linear and nonlinear optical lattices (OLs). These equations may model two-dimensional Bose-Einstein condensates in linear and nonlinear OLs. When linear OLs are switched off, we derive the asymptotic expansion formulas and obtain necessary conditions for the orbital stability and instability of single-spike bound states, respectively. When linear OLs are turned on, we consider three different conditions of linear and nonlinear OLs to develop mathematical theorems which are most general on the orbital stability problem.

1 Introduction

Recently, optical lattices have created many interesting phenomena in Bose-Einstein condensates (BECs) and attracted a great deal of attention. Two types of optical lattices are considered: a linear optical lattice (OL) (cf. [28]) and a nonlinear OL (cf. [1] and [35]). A linear OL is a series of potential wells having a periodic (in space) intensity pattern which may confine atoms of BECs in the potential minima. A nonlinear OL can be obtained by inducing a periodic spatial variation of the atomic scattering length, leading to a periodic space modulation of the nonlinear coefficient in the Gross-Pitaevskii equation (GPE) governing the dynamics of BECs. The GPE is a nonlinear Schrödinger (NLS) equation in the presence of the Kerr nonlinearity describing a BEC in a linear and a nonlinear OL given by

$$-i\frac{\partial \psi}{\partial t} = D\Delta \psi - V_{\text{trap}} \psi - g|\psi|^2\psi,$$  

(1.1)

for $x \in \mathbb{R}^N$, $N \leq 3$ and $t > 0$. Here $\psi = \psi(x,t) \in \mathbb{C}$ is the wavefunction, $D$ is the diffraction (or dispersion) coefficient, and $V_{\text{trap}}$ is the potential of the linear lattice. Besides, $g = \mu m(x) \sim a$
characterizes the nonlinear lattice, where \( a \) denotes the spatially modulated scattering length, \( \mu \) is a nonzero constant and \( m(x) = m(x_1, \cdots, x_N) > 0 \) is a function depending on spatial variables (transverse coordinates) \( x_1, \cdots, x_N \) (cf. [2], [6]).

The underlying dynamics of (1.1) is dominated by the interplay between adjacent potential wells of linear OLs and nonlinearity of nonlinear OLs. When the nonlinearity is self-focusing i.e. \( D > 0 \) and \( \mu < 0 \), a balance between these two effects may resist collapse or decay and result in bright solitons. Experimentally, bright solitons can be observed in linear and nonlinear OLs, respectively. One may find stable bright solitons in three-dimensional linear OLs (cf. [7]). On the other hand, two-dimensional bright solitons can also be investigated in two-dimensional nonlinear OLs (cf. [13]). Consequently, under the influence of linear and nonlinear OLs, two-dimensional bright solitons must have suitable stability for experimental observations. However, most theoretical results (e.g. [10] and [11]) focus on the orbital (dynamical) stability of only one-dimensional single-spike bound states which are steady state bright solitons in one-dimensional nonlinear OLs without the effect of linear OLs. To see how linear and nonlinear OLs affect the stability of two-dimensional single-spike bound states, we develop mathematical theorems for the orbital stability and instability of two-dimensional single-spike bound states of (1.1) under different conditions of linear and nonlinear OLs.

To get two-dimensional single-spike bound states of (1.1), we may assume \( N = 2, D > 0 \) and the scattering length \( a \), i.e., \( \mu \) is negative and large due to the Feshbach resonance (cf. [9]). Setting \( h^2 = D/(−\mu) \), \( V(x) = V_{\text{trap}}(x)/(−\mu) \) and suitable time scale, the equation (1.1) with negative and large \( \mu \) can be equivalent to a semi-classical nonlinear Schrödinger equation (NLS) given by

\[
-ih \frac{\partial \psi}{\partial t} = h^2 \Delta \psi - V \psi + m |\psi|^2 \psi, \quad x \in \mathbb{R}^2, t > 0,
\]

(1.2)

where \( 0 < h \ll 1 \) is a small parameter, \( V = V(x) \) is a smooth nonnegative function and \( m = m(x) \) is a smooth positive function. For the spatial dimension \( N \geq 1 \), we may generalize the equation (1.2) to a NLS having the following form

\[
-ih \frac{\partial \psi}{\partial t} = h^2 \Delta \psi - V \psi + m |\psi|^{p-1} \psi, \quad x \in \mathbb{R}^N, t > 0,
\]

(1.3)

with critical exponent

\[
p = 1 + \frac{4}{N}, \quad N \geq 1.
\]

(1.4)

In particular, when \( N = 2 \), the equation (1.3) with (1.4) is exactly same as (1.2).

Single-spike bound states of (1.3) are of the form \( \psi(x, t) = e^{i\lambda t/h}u(x) \), where \( \lambda \) is a positive constant and \( u = u(x) \) is a positive solution of the following nonlinear elliptic equation

\[
h^2 \Delta u - (V + \lambda) u + m u^p = 0, \quad u \in H^1(\mathbb{R}^N),
\]

(1.5)

with zero Dirichlet boundary condition, i.e., \( u(x) \to 0 \) as \( |x| \to \infty \). When \( V \equiv 0 \) and \( m \equiv 1 \), problem (1.5) admits a unique radially symmetric ground state which is stable for any \( \lambda > 0 \) if \( p < 1 + \frac{4}{N} \), and unstable for any \( \lambda > 0 \) if \( p \geq 1 + \frac{4}{N} \) (cf. [4], [8] and [43]). For \( V \neq 0 \) or \( m \neq 1 \), there exists \( u_h \) a single-spike solution of (1.5), provided both \( V \) and \( m \) are bounded and
satisfy another conditions, for example, conditions in the following Theorem 1.1-1.4 (cf. [20]). For other other nonlinearity in the possibly degenerate setting, see [3], [14], [19], [31], [32], [37], [39], [40], [41] and reference therein. Hereafter, we set \( \psi_h(x, t) := e^{iMt/h}u_h(x) \) as a single-spike bound state of (1.3), where \( u_h \) is the single-spike solution of (1.5).

In this paper, we want to study the orbital stability of the bound state \( \psi_h \) for the equation (1.3) with critical exponent (1.4). One may regard the bound state \( \psi_h \) as an orbit of (1.3). From [17], the orbital stability of \( \psi_h \) is defined as follows: For all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| \psi_0 - u_h \|_{H^1} < \delta \) and \( \psi \) is a solution of (1.3) in some interval \([0, t_0]\) with \( \psi|_{t=0} = \psi_0 \), then \( \psi(\cdot, t) \) can be extended to a solution in \( 0 \leq t < \infty \) and \( \sup_{0 \leq t < \infty} \inf_{s \in \mathbb{R}} \| \psi(\cdot, t) - \psi_h(\cdot, s) \|_{H^1} < \epsilon \). Otherwise, the orbit \( \psi_h \) is called orbital unstable.

The functions \( V = V(x) \) and \( m = m(x) \) may play a crucial role on the orbital stability of \( \psi_h \). When \( m \equiv 1 \) and \( V \) is of class \((V)_a\) and fulfills other conditions in [29]-30, the orbital stability and instability of \( \psi_h \) for the equation (1.3) was established by Lin and Wei [25] if \( V \) has non-degenerate critical points. Under different conditions, e.g., \( h = 1 \) and \( \lambda \) is large, results of the orbital stability problem can be found in [15]. One may also remark that the orbital stability problem of NLS with inhomogeneous nonlinearity has been investigated in [5] but only for the subcritical case, i.e., \( 1 < p < 1 + \frac{4}{N} \).

To state our main results, we need to introduce some notations. It is well-known that the positive solution of

\[
\begin{align*}
\Delta w - w + w^p &= 0 \quad \text{in } \mathbb{R}^N, \\
w(0) &= \max_{y \in \mathbb{R}^N} w(y), \quad w(y) \to 0 \text{ as } |y| \to +\infty.
\end{align*}
\]  

(1.6)

is radial [16] and unique [24]. We denote the solution and its linearized operator as \( w = w(r) \) and

\[
L_0 := \Delta - 1 + pw^{p-1},
\]  

(1.7)

respectively. For the orbital stability of \( \psi_h \), we set

\[
L_h := h^2\Delta - (V + \lambda) + mpw^{p-1},
\]  

(1.8)

as the linearized operator of (1.5) with respect to \( u_h \) and

\[
\begin{align*}
d(\lambda) &= \int_{\mathbb{R}^N} \left[ \frac{h^2}{2} |\nabla u_h|^2 + \frac{1}{2} (V + \lambda) u_h^2 - \frac{1}{p+1} m u_h^{p+1} \right] dx,
\end{align*}
\]  

(1.9)

as the energy of \( u_h \). Observe that \( u_h \) may depend on the variable \( \lambda \). Assume that \( d(\lambda) \) is non-degenerate, i.e., \( d''(\lambda) \neq 0 \). Let \( p(d'') = 1 \) if \( d'' > 0 \); \( p(d'') = 0 \) if \( d'' < 0 \), and \( n(L_h) \) be the number of positive eigenvalues of \( L_h \). According to general theory of orbital stability of bound states (cf. [17], [18]), \( \psi_h \) is orbital stable if \( n(L_h) = p(d'') \), and orbital unstable if \( n(L_h) - p(d'') \) is odd (see page 309 of [18]). It is remarkable that if both \( V \) and \( m \) are constant and \( p = 1 + \frac{4}{N} \), then \( d''(\lambda) = 0 \). Consequently, from now on, we consider the critical exponent \( p = 1 + \frac{4}{N} \) and assume the point \( x_0 \) as a non-degenerate critical point of the function \( G \) defined by (cf. [20], [39])

\[
G(x) := [V(x) + \lambda] m^{-N/2}(x), \quad \forall x \in \mathbb{R}^N,
\]  

(1.10)

provided \( V \neq 0 \) and \( m > 0 \) in \( \mathbb{R}^N \). When \( V \equiv 0 \) in \( \mathbb{R}^N \), \( x_0 \) is set as a non-degenerate critical point of the function \( m \).

For simplicity, we firstly switch off the potential \( V \) and obtain the following result.
Theorem 1.1. Let $N \leq 3$ be a positive integer, $p = 1 + \frac{4}{N}$ and the potential $V \equiv 0$. Assume the function $m = m(x)$ satisfies

$$m \in C^4(\mathbb{R}^N); 0 < m_0 \leq m(x) \leq m_1 < \infty; |m^{(i)}(x)| \leq C\exp(\gamma|x|), \ i = 1, 2, 3, 4,$$  \hspace{1cm} (1.11)

where $m_0, m_1, \gamma$ and $C$ are positive constants, and $m^{(i)}(x)$ are the $i$-th derivatives of $m(x)$. Suppose also that $x_0$ be a non-degenerate critical point of $m(x)$ ($x_0$ is independent of $\lambda$). Let $\psi_h(x,t) := e^{i\lambda t/h}u_h(x)$ be a bound state of (1.2), where $u_h$ is a single-spike solution of (1.3) concentrating at $x_0$. Assume also

$$m(x_0)\Delta^2 m(x_0) < C_{N,1}|\Delta m(x_0)|^2 + C_{N,2}\left[N\|\nabla^2 m(x_0)\|_2^2 - |\Delta m(x_0)|^2\right] + C_{N,3}m(x_0)\nabla(\Delta m)(x_0) \cdot \left[\nabla^2 m(x_0)\right]^{-1}\nabla(\Delta m)(x_0),$$  \hspace{1cm} (1.12)

where

$$C_{N,1} = \frac{2(N+2)^2}{N^2} \int_0^\infty r^{N+1}w^p L_0^{-1}(r^2w^p)dr,$$  \hspace{1cm} (1.13)

$$C_{N,2} = \frac{4(N+2)}{N^2} \int_0^\infty r^{N+1}w^p \Phi_0 dr,$$  \hspace{1cm} (1.14)

$$C_{N,3} = \frac{(N+2)(\int_0^\infty r^{N+1}w^p dr)^2}{N \int_0^\infty r^{N-1}w^p dr \int_0^\infty r^{N+3}w^p dr},$$  \hspace{1cm} (1.15)

are constants depending only on $N$. Here $\Phi_0 = \Phi_0(r)$ satisfies

$$\begin{cases}
\Phi''_0 + \frac{N-1}{r} \Phi'_0 - \Phi_0 + pw^{p-1}\Phi_0 - \frac{2N}{r^2}\Phi_0 - r^2w^p = 0, \ r = |x| \in (0, \infty), \\
\Phi_0(0) = \Phi'_0(0) = 0.
\end{cases}$$  \hspace{1cm} (1.16)

where $L_0$ is defined in (1.4). Then for any $\lambda > 0$, $\psi_h$ is orbitally stable if $h$ is sufficiently small and $x_0$ is a non-degenerate local maximum point of the function $m$. Furthermore, for any $\lambda > 0$, $\psi_h$ is orbitally unstable if $h$ is sufficiently small and the number of positive eigenvalues of the Hessian matrix $\nabla^2 m(x_0)$ is odd.

Remark 1: When $N = 1$, $x_0 = 0$ and the function $m$ satisfies $m'''(x_0) = 0$, (see (C.2) of [10]), the condition (1.12) of Theorem 1.1 is exactly same as the condition (4.14) of [10]. For $N \geq 2$, G. Fibich and X.-P. Wang (cf. [12]) considered the function $m$ with radial symmetry, i.e., $m = m(r), r = |x|$ and $m'''(0) = 0$, and studied the orbital stability problem only for radial perturbations. Here we may include the case that the function $m$ is not radially symmetric and the third order derivatives of the function $m$ at $x_0$ can be nonzero. Moreover, we study the orbital stability problem for general perturbations including the non-radial perturbations.
Consequently, Theorem 1.1 can be regarded as the most general theorem on the orbital stability problem of semiclassical NLS equations with critical exponent and nonlinear OLS.

When the potential $V$ is turned on, we may generalize the argument of Theorem 1.1 to obtain three theorems as follows:

**Theorem 1.2.** Let $N \leq 3$ be a positive integer, $p = 1 + \frac{4}{N}$. Assume both the potential $V = V(x)$ and the function $m = m(x)$ satisfy the following conditions: there exist positive constants $V_0, V_1, m_0, m_1, \gamma$ and $C$ such that

$$ V \in C^2(\mathbb{R}^N); 0 < V_0 \leq V(x) \leq V_1 < \infty; \quad |V^{(i)}(x)| \leq C \exp(\gamma|x|), \quad i = 1, 2, \quad (1.17) $$

and

$$ m \in C^2(\mathbb{R}^N); 0 < m_0 \leq m(x) \leq m_1 < \infty; \quad |m^{(i)}(x)| \leq C \exp(\gamma|x|), \quad i = 1, 2, \quad (1.18) $$

where $V^{(i)}(x), m^{(i)}(x)$ are the $i$-th derivatives of $V(x), m(x)$, respectively. Suppose also that $x_0$ is a non-degenerate critical point of the function $G$ defined in (1.10) for fixed $\lambda > 0$ ($x_0$ may depend on $\lambda$). Let \( \psi_h(x, t) := e^{i\lambda t/h}u_h(x) \) be a bound state of (1.3), where $u_h$ is a single-spike solution of (1.3) concentrating at $x_0$. Then $\psi_h$ is orbitally unstable if $h$ is sufficiently small and $x_0$ is a non-degenerate local minimum point of $G$ such that $\nabla V(x_0) \neq 0$.

**Theorem 1.3.** Under the same hypotheses of Theorem 1.2 assume also that $\nabla V(x_0) = 0$ and $\Delta V(x_0) \neq 0$ (thus $x_0$ may be independent of $\lambda$). Let $n$ be the number of negative eigenvalues of the matrix $\nabla^2 G(x_0)$. Then $\psi_h$ is orbitally stable if $h$ is sufficiently small and $x_0$ is a non-degenerate local minimum point of $G$ with $\Delta V(x_0) > 0$. Furthermore, $\psi_h$ is orbitally unstable if $h$ is sufficiently small and $n - \frac{1}{2} \left(1 + \frac{\Delta V(x_0)}{|\Delta V(x_0)|}\right)$ is even.

**Theorem 1.4.** Under the same hypotheses of Theorem 1.2 assume also that $\nabla V(x_0) = 0$, $\Delta V(x_0) = 0$ and (1.17) holds for both $V$ and $m$. Let $n$ be the number of negative eigenvalues of the matrix $\nabla^2 G(x_0)$. Suppose also that $H(x_0) > 0$, where $H(x_0)$ defined in (4.35) involves the $i$-th derivatives (for $0 \leq i \leq 4$) of $V$ and $m$ at $x_0$. Then $\psi_h$ is orbitally stable if $h$ is sufficiently small and $x_0$ is a non-degenerate local minimum point of $G$. Furthermore, $\psi_h$ is orbitally unstable if $n$ is odd.

**Remark 2:** Theorem 1.2, 1.4 may include all the cases of values $\nabla V(x_0)$ and $\Delta V(x_0)$ for the orbital stability problem of (1.3) with critical exponent (1.4). Theorem 1.3 may generalize the main result of [25] to the case that the function $m$ is a positive and nonconstant function. As $V \equiv 0$, Theorem 1.4 coincides with Theorem 1.1 because of

$$ \nabla^2 G(x_0) = m(x_0)^{-\frac{N}{2}} - 1 \left[ m(x_0) \nabla^2 V(x_0) - \frac{N}{2} \left[ V(x_0) + \lambda \right] \nabla^2 m(x_0) \right]. $$

**Remark 3:** In the following we give examples in dimension $N = 2$. Similar examples in dimension $N = 1$ and 3 can also be given. First for $x \in \mathbb{R}$ we define

$$ X_1(x) = \sin x + \frac{1}{6} \sin^3 x = \frac{9}{8} \sin x - \frac{1}{24} \sin(3x), $$

$$ X_2(x) = 2(1 - \cos x) + \frac{1}{3} (1 - \cos x)^2 = \frac{5}{2} \cos x + \frac{1}{6} \cos(2x), $$

$$ X_3(x) = \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin(3x), $$

$$ X_4(x) = 4(1 - \cos x)^2 = 6 - 8 \cos x + 2 \cos(2x), $$

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respectively. Then $X_1, X_2, X_3$ and $X_4$ satisfy

$$|X_1| \leq \frac{7}{6}, \quad X_1'(0) = 1, \quad X_1^{(j)}(0) = 0, \text{ for } 2 \leq j \leq 4,$$

$$0 \leq X_2 \leq \frac{16}{3}, \quad X_2''(0) = 1, \quad X_2^{(j)}(0) = 0, \text{ for } j = 1, 3, 4,$$

$$|X_3| \leq 1, \quad X_3^{(3)}(0) = 1, \quad X_3^{(j)}(0) = 0, \text{ for } j = 1, 2, 4,$$

$$0 \leq X_4 \leq 16, \quad X_4^{(4)}(0) = 1, \quad X_4^{(j)}(0) = 0, \text{ for } j = 1, 2, 3.$$

Next for $(x, y) \in \mathbb{R}^2$ we set

$$V(x, y) = a_0 + \sum_{i=1}^{4} a_i X_i(x) + \sum_{i=1}^{4} b_i X_i(y), \quad (1.19)$$

and

$$m(x, y) = c_0 + \sum_{i=1}^{4} c_i X_i(x) + \sum_{i=1}^{4} d_i X_i(y), \quad (1.20)$$

where $a_i, b_i, c_i,$ and $d_i$ are constants. By the properties of $X_1, X_2, X_3$ and $X_4$, the $i$-th derivatives of $V$ and $m$ at $x_0 = (0, 0)$ depend only on $a_i, b_i$ and $c_i, d_i$ respectively for $1 \leq i \leq 4$. Recall that $G(x, y) = [V(x, y) + \lambda] m^{-1}(x, y)$ for $N = 2$, we have

$$\nabla G(0) = c_0^{-2} \left( c_0 a_1 - (a_0 + \lambda)c_1, \quad c_0 b_1 - (a_0 + \lambda)d_1 \right)^T,$$

and then if $\nabla G(0) = 0$,

$$\nabla^2 G(0) = c_0^{-2} \begin{pmatrix} c_0 a_2 - (a_0 + \lambda)c_2 & 0 \\ 0 & c_0 b_2 - (a_0 + \lambda)d_2 \end{pmatrix}.$$

Now we can give examples for the potentials $V$ and $m$ which satisfy the assumptions in Theorems 1.2-1.4.

(I) (Examples for Theorem 1.2) $N = 2, x_0 = (0, 0), V$ and $m$ given in $(1.19)$ and $(1.20)$ and $a_i, b_i, c_i, d_i$ satisfy

$$c_0 = a_0 + \lambda, \quad (a_1, b_1) = (c_1, d_1) \neq 0, \quad a_2 > c_2 > 0, \quad b_2 > d_2 > 0,$$

and $c_0 > \frac{7}{6}(|a_1| + |b_1|), \quad a_0 > \frac{7}{6}(|c_1| + |d_1|), \quad a_i = b_i = c_i = d_i = 0 \quad \text{for} \quad i = 3, 4,$$

(II) (Examples for Theorem 1.3) First a special case for Theorem 1.3 is that $\nabla m(x_0) = 0, \nabla^2 m(x_0) = 0$ and $x_0$ is a non-degenerate critical point of $V(x)$. Here we give another examples. The first one is in the stability case and the second is in the instability case.

(a) (Stability) $N = 2, x_0 = (0, 0), V$ and $m$ given in $(1.19)$ and $(1.20)$ and $a_i, b_i, c_i, d_i$ satisfy

$$a_0 > 0, \quad c_0 > -\frac{32}{3} c_2 > 0, \quad c_0 > -\frac{32}{3} d_2 > 0, \quad a_2 > 0, \quad b_2 > 0, \quad a_i = b_i = c_i = d_i = 0 \quad \text{for} \quad i = 1, 3, 4,$$

then for any $\lambda > 0$, the conditions in Theorem 1.3 for orbital stability will be satisfied.
(b) (Instability) \( N = 2, x_0 = (0, 0), V \) and \( m \) given in (1.19) and (1.20) and \( a_i, b_i, c_i, d_i \) satisfy
\[
a_0 > -\frac{16}{3}b_2 > 0, c_0 > -\frac{16}{3}c_2 > 0, a_2 + b_2 > 0, d_2 > 0,
\]
and \( a_i = b_i = c_i = d_i = 0 \) for \( i = 1, 3, 4 \),
then for any \( \lambda > 0 \), the conditions in Theorem 1.3 for orbital instability will be satisfied.

(III) (Examples for Theorem 1.4) Here we give two different examples. First we give examples in the case of \( a_4 = b_4 = 0 \). Specially, Theorem 1.4 is in this case.

(a) (Stability) \( N = 2, x_0 = (0, 0), V \) and \( m \) given in (1.19) and (1.20) and \( a_i, b_i, c_i, d_i \) satisfy
\[
a_0 > 0, c_0 > -\frac{32}{3}c_2 > 0, d_2 > 0, c_0 > -\frac{32}{3}d_2 > 0, |c_2|, |d_2| \text{ small, or } c_0, |c_4| \text{ large,}
\]
and \( a_i = b_i = 0 = c_3 = d_3 = d_4 \) for \( i = 1, 2, 3, 4 \),
then for any \( \lambda > 0 \), the conditions in Theorem 1.4 for orbital stability will be satisfied. Here \( |c_2|, |d_2| \text{ small or } c_0, |c_4| \text{ large are independent on } \lambda \).

(b) (Instability) \( N = 2, x_0 = (0, 0), V \) and \( m \) given in (1.19) and (1.20) and \( a_i, b_i, c_i, d_i \) satisfy
\[
a_0 > 0, c_0 > -\frac{16}{3}c_2 > 0, d_2 > 0, c_0 > -16c_4 > 0, |c_2|, |d_2| \text{ small, or } c_0, |c_4| \text{ large,}
\]
and \( a_i = b_i = 0 = c_3 = d_3 = d_4 \) for \( i = 1, 2, 3, 4 \),
then for any \( \lambda > 0 \), the conditions in Theorem 1.4 for orbital instability will be satisfied. Here \( |c_2|, |d_2| \text{ small or } c_0, |c_4| \text{ large are independent on } \lambda \).

Second we give examples in the case of \( a_4 + b_4 \neq 0 \).

1 (Stability) \( N = 2, x_0 = (0, 0), V \) and \( m \) given in (1.19) and (1.20) and \( a_i, b_i, c_i, d_i \) satisfy
\[
a_0 > 0, c_0 > -\frac{32}{3}c_2 > 0, d_2 > 0, c_0 > -\frac{32}{3}d_2 > 0, a_4 > 0, b_4 > 0, (a_4 + b_4) \text{ large,}
\]
and \( a_i = b_i = 0 = c_3 = c_4 = d_3 = d_4 \) for \( i = 1, 2, 3 \),
then for fixed \( \lambda > 0 \), the conditions in Theorem 1.4 for orbital stability will be satisfied. Here \( (a_4 + b_4) \text{ large may depend on } \lambda \).

2 (Instability) \( N = 2, x_0 = (0, 0), V \) and \( m \) given in (1.19) and (1.20) and \( a_i, b_i, c_i, d_i \) satisfy
\[
a_0 > 0, c_0 > -\frac{16}{3}c_2 > 0, d_2 > 0, a_4 > 0, b_4 > 0, (a_4 + b_4) \text{ large,}
\]
and \( a_i = b_i = 0 = c_3 = c_4 = d_3 = d_4 \) for \( i = 1, 2, 3, 4 \),
then for fixed \( \lambda > 0 \), the conditions in Theorem 1.4 for orbital instability will be satisfied. Here \( (a_4 + b_4) \text{ large may depend on } \lambda \).
The rest of this paper is organized as follows: In Section 2, we show the properties of $u_h$. Then we state the proof of Theorem 1.1 in Section 3. Theorem 1.2-1.4 are proved in Section 4.

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2 Preliminaries

In this section, we study the properties of $u_h$ a single-spike bound state of (1.5) concentrated at a non-degenerate critical point of $G(x) := [V(x) + \lambda] m^{-N/2}(x)$ (cf. [20], [39]). Let $x_h$ be the unique local maximum point of $u_h$. So $x_h \to x_0$ as $h \to 0$.

Let $v_h(y) := u_h(hy + x_h)$ for all $y \in \mathbb{R}^N$. Then by (1.5), $v_h$ is a positive solution of

$$\Delta v - [V(hy + x_h) + \lambda] v + m(hy + x_h)v^p = 0. \quad (2.1)$$

For notation convenience, we still denote

$$L_h := \Delta - [V(hy + x_h) + \lambda] + m(hy + x_h)pv_h^{p-1} \quad (2.2)$$

as the linearized operator of the equation (2.1) with respect to the solution $v_h$. As the result of [39], $v_h$ can be written as $v_h = w_{x_h} + \phi_h$, where $w_{x_h}$ is the unique positive solution of

$$\left\{ \begin{array}{ll}
\Delta w - [V(x_h) + \lambda] w + m(x_h)w^p = 0 & \text{in } \mathbb{R}^N, \\
w(0) = \max_{y \in \mathbb{R}^N} w(y), \quad w(y) \to 0 & \text{as } |y| \to +\infty,
\end{array} \right. \quad (2.3)$$

and

$$\|\phi_h\|_\infty \to 0 \quad \text{as } h \to 0. \quad (2.4)$$

Moreover,

$$v_h(y) \leq C|y|^{1-N} \exp\left( -V^{1/2}|y| \right), \quad \forall y \in \mathbb{R}^N, \quad (2.5)$$

where $V := \inf_{\mathbb{R}^N} [V(x) + \lambda]$. From (2.3), it is easy to check that

$$w_{x_h}(y) = [V(x_h) + \lambda]^{-\frac{1}{p-1}} m(x_h)^{-\frac{1}{p-1}} w(\sqrt{V(x_h) + \lambda y}), \quad (2.6)$$

where $w$ is the positive solution of (1.6).

For the single-spike solution of (1.5), we recall the following result from [38] and [39]:

Lemma 2.1. Assume that there are positive constants $\gamma$ and $C$ such that

$$|\nabla V(x)|, |\nabla m(x)| \leq C \exp(\gamma |x|), \quad \forall x \in \mathbb{R}^N. \quad (2.7)$$

Then

$$\int_{\mathbb{R}^N} \left[ \frac{1}{p+1} \nabla m(hy + x_h)v_h^{p+1} - \frac{1}{2} \nabla V(hy + x_h)v_h^2 \right] dy = 0 \quad (2.8)$$

for $0 < h < h_0$, where $h_0$ is a positive constant depending on $\gamma$ and $\lambda$.
In the rest of this section, for simplicity, we switch off the potential $V$, i.e., set $V \equiv 0$. Then by Lemma 2.1 we obtain the uniqueness of $u_h$ as follows:

**Lemma 2.2.** Suppose (2.7) holds, $V \equiv 0$ and $x_0$ is a non-degenerate critical point of $m$. Then $u_h$ is unique.

**Proof.** Suppose $u_{h,1}$ and $u_{h,2}$ are different single-spike solutions of (1.5) concentrating at the same point $x_0$. Let $v_1(y) := u_{h,1}(hy + x_0)$ and $v_2(y) := u_{h,2}(hy + x_0)$. Then both $v_1$ and $v_2$ satisfy
\[
\Delta v - \lambda v + m(hy + x_0)v^p = 0, \quad \text{for } y \in \mathbb{R}^N,
\]
and $v_1, v_2 \to w_{x_0}$ uniformly on $\mathbb{R}^N$ as $h \to 0$. Due to $v_1 \neq v_2$, we may set
\[
\tilde{v}_h := \frac{v_1 - v_2}{\|v_1 - v_2\|_{\infty}},
\]
and then $\tilde{v}_h$ satisfies
\[
\Delta \tilde{v}_h - \lambda \tilde{v}_h + m(x_0)pw_{x_0}^{p-1} \tilde{v}_h + [m(hy + x_0) - m(x_0)]pw_{x_0}^{p-1} \tilde{v}_h + N(\tilde{v}_h) = 0, \quad (2.9)
\]
where $N(\tilde{v}_h) = m(hy + x_0)[v_1^p - v_2^p - pw_{x_0}^{p-1}(v_1 - v_2)]/\|v_1 - v_2\|_{\infty}$. Hence by the standard elliptic PDE theorems on the equation (2.9), we may take a subsequence $\tilde{v}_h \to \tilde{v}_0$, where $\tilde{v}_0$ solves
\[
\Delta \tilde{v}_0 - \tilde{v}_0 + m(x_0)pw_{x_0}^{p-1} \tilde{v}_0 = 0.
\]
Consequently, there exist constants $c_j$’s such that
\[
\tilde{v}_0 = \sum_{j=1}^{N} c_j \partial_j w_{x_0}. \quad (2.10)
\]

Let $y_h$ be such that $\tilde{v}_h(y_h) = \|\tilde{v}_h\|_{\infty} = 1$ (the same proof applies if $\tilde{v}_h(y_h) = -1$). Then by the Maximum Principle, we have $|y_h| \leq C$. On the other hand, as (2.8), we may obtain
\[
\int_{\mathbb{R}^N} \nabla m(hy + x_0)v_1^{p+1} dy = 0 = \int_{\mathbb{R}^N} \nabla m(hy + x_0)v_2^{p+1} dy.
\]
Thus
\[
\int_{\mathbb{R}^N} \nabla m(hy + x_0) \left( \frac{v_1^{p+1} - v_2^{p+1}}{v_1 - v_2} \right) \tilde{v}_h dy = 0. \quad (2.11)
\]
Note that for all $i = 1, \ldots, N$, as $h \to 0$,
\[
\partial_i m(hy + x_0) = h \sum_{k=1}^{N} \partial_i k m(x_0) y_k + o(h), \quad \text{and} \quad \frac{v_1^{p+1} - v_2^{p+1}}{v_1 - v_2} = (p + 1)w_{x_0}^{p} + o(1). \]
Hence from (2.10) and (2.11), we may obtain
\[
0 = \int_{\mathbb{R}^N} \left[ h \sum_{k=1}^{N} \partial_i k m(x_0) y_k \right] (p + 1)w_{x_0}^{p} \left( \sum_{j=1}^{N} c_j \partial_j w_{x_0} \right) dy + o(h)
\]
\[
= -h \sum_{j=1}^{N} \partial_i j m(x_0) c_j \int_{\mathbb{R}^N} w_{x_0}^{p+1} dy + o(h). \]
Hence by the assumption that $\nabla^2 m(x_0)$ is non-degenerate, $c_j = 0$ for $j = 1, \cdots, N$, i.e., $\tilde{v}_0 \equiv 0$. This may contradict to the fact that $1 = \tilde{v}_h(y_h) \to \tilde{v}_0(y_0)$ for some $y_0 \in \mathbb{R}^N$. Therefore, we may complete the proof of Lemma 2.2.

By Lemma 2.1 we may simplify the proof of [21] and get a shorter proof of the asymptotic behavior of $x_h$'s as follows:

**Lemma 2.3.** Under the same hypotheses of Lemma 2.2

$$x_h = x_0 + o(h) \quad \text{as} \quad h \to 0.$$ \quad (2.12)

**Proof.** Fix $i \in \{1, \cdots, N\}$ arbitrarily. By Taylor's expansion of $\partial_i m(x)$ and $\nabla m(x_0) = 0$, we obtain

$$\partial_i m(hy + x_h) = \sum_{j=1}^N \partial_{ij} m(x_0) (hy_j + x_{h,j} - x_{0,j}) + o(h) + o(|x_h - x_0|).$$

Hence by Lemma 2.1 and $v_h = w_{x_0} + o(1)$, we have

$$0 = \int_{\mathbb{R}^N} \partial_i m(hy + x_h) v_h^{p+1} \, dy$$

$$= \sum_{j=1}^N \partial_{ij} m(x_0) (x_{h,j} - x_{0,j}) \int_{\mathbb{R}^N} w_{x_0}^{p+1} \, dy + o(h) + o(|x_h - x_0|).$$

Here we have used the fact that $\int_{\mathbb{R}^N} y_j w_{x_0}^{p+1} \, dy = 0$ for $j = 1, \cdots, N$. Using the assumption that $\nabla^2 m(x_0)$ is non-degenerate, we obtain (2.12). \qed

Following the idea of [25], we may use Lemma 2.3 to show the asymptotic behavior of $v_h$ as follows:

**Lemma 2.4.** Under the same hypotheses of Lemma 2.2

$$v_h = w_{x_h} + h^2 \phi_2 + o(h^2), \quad \text{as} \quad h \to 0,$$ \quad (2.13)

where $\phi_2$ satisfies

$$\Delta \phi_2 - \lambda \phi_2 + m(x_h) p w_{x_h}^{p-1} \phi_2 + \frac{1}{2} \sum_{i,j=1}^N \partial_{ij} m(x_0) y_i y_j w_{x_h}^p = 0, \quad \text{and} \quad \nabla \phi_2(0) = 0.$$ \quad (2.14)

**Proof.** Let $\phi_h = v_h - w_{x_h}$. Then it is easy to check that $|\phi_h| \to 0$ uniformly, and $\phi_h$ satisfies

$$\Delta \phi_h - \lambda \phi_h + m(hy + x_h) p w_{x_h}^{p-1} \phi_h + N(\phi_h) + R(\phi_h) = 0, \quad \text{and} \quad \nabla \phi_h(0) = 0,$$ \quad (2.15)

where

$$N(\phi_h) = m(hy + x_h) \left[ (w_{x_h} + \phi_h)^p - w_{x_h}^p - p w_{x_h}^{p-1} \phi_h \right],$$

and

$$R(\phi_h) = \left[ m(hy + x_h) - m(x_h) \right] w_{x_h}^p.$$
Note that by Lemma 2.3 and $\nabla m(x_0) = 0$, 
\[
m(hy + x_h) - m(x) = hy \cdot \nabla m(x) + \frac{h^2}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_h) y_i y_j + o(h^2) 
\]
\[
= \frac{h^2}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j + o(h^2). 
\] (2.16)

Now we claim that $|\phi_h| \leq c h^2$ by contradiction. Suppose that $h^{-2}\|\phi_h\|_\infty \to \infty$. Let $
abla^\dagger = \phi_h / \|\phi_h\|_\infty$. Then $
abla^\dagger$ satisfies
\[
\Delta \nabla^\dagger - \lambda \nabla^\dagger + m(hy + x_h)pw_{x_h}^{-1} \nabla^\dagger + \frac{N(\phi_h)}{\|\phi_h\|_\infty} + \frac{R(\phi_h)}{\|\phi_h\|_\infty} = 0. 
\] (2.17)

Note that by (2.16),
\[
\frac{R(\phi_h)}{\|\phi_h\|_\infty} \leq C \frac{h^2}{\|\phi_h\|_\infty}. 
\] (2.18)

Let $y_h$ be such that $\nabla^\dagger(y_h) = \|\nabla^\dagger\|_\infty = 1$ (the same proof applies if $\nabla^\dagger(y_h) = -1$). Then by (2.17) - (2.18) and the Maximum Principle, we have $|y_h| \leq C$. On the other hand, by the usual elliptic regularity theory, we may take a subsequence $\nabla^\dagger \to \nabla_0$, where $\nabla_0$ satisfies
\[
\Delta \nabla_0 - \lambda \nabla_0 + m(x_0)pw_{x_0}^{-1} \nabla_0 = 0, \text{ and } \nabla \nabla_0(0) = 0. 
\]

Hence $\nabla_0 \equiv 0$. This may contradict to the fact that $1 = \nabla_h(y_h) \to \nabla_0(y_0)$ for some $y_0$. Therefore, we may complete the claim that $|\phi_h| \leq c h^2$.

Now we set $\phi_{h,2} = \phi_h - h^2 \phi_2$. Then $\phi_{h,2} = O(h^2)$ and satisfies
\[
\Delta \phi_{h,2} - \lambda \phi_{h,2} + m(hy + x_h)pw_{x_h}^{-1} \phi_{h,2} + N(\phi_{h,2}) + R(\phi_{h,2}) = 0, \text{ and } \nabla \phi_{h,2}(0) = 0 
\]
where
\[
N(\phi_{h,2}) = m(hy + x_h) \left[ (w_{x_h} + h^2 \phi_2 + \phi_{h,2})^p - w_{x_h}^p - pw_{x_h}^{-1}(h^2 \phi_2 + \phi_{h,2}) \right], 
\]
and
\[
R(\phi_{h,2}) = \left[ m(hy + x_h) - m(x_h) - \frac{h^2}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j \right] w_{x_h}^p + h^2 \left[ m(hy + x_h) - m(x_h) \right] pw_{x_h}^{-1} \phi_2. 
\]

Thus as for previous argument, we may have $\phi_{h,2} = o(h^2)$ and complete the proof of Lemma 2.4.

As for Proposition 3.1 of [23], one may get two lemmas as follows:

**Lemma 2.5.** For $h$ small enough, the maps
\[
L_{x_h}\phi := \Delta \phi - [V(x_h) + \lambda] \phi + m(x_h)pw_{x_h}^{-1}\phi 
\]

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are uniformly invertible from $K_{x_0}^\perp$ to $C_{x_0}^\perp$, where

$$K_{x_0}^\perp = \left\{ \phi \in H^2(\mathbb{R}^N) \middle| \int_{\mathbb{R}^N} \phi \partial_j w_{x_0} dy = 0, j = 1, \ldots, N \right\} \subset H^2(\mathbb{R}^N),$$

$$C_{x_0}^\perp = \left\{ \phi \in L^2(\mathbb{R}^N) \middle| \int_{\mathbb{R}^N} \phi \partial_j w_{x_0} dy = 0, j = 1, \ldots, N \right\} \subset L^2(\mathbb{R}^N).$$

**Lemma 2.6.** The map

$$L_{x_0} \phi := \Delta \phi - [V(x_0) + \lambda] \phi + m(x_0) p w_{x_0}^{-1} \phi$$

has eigenvalues $\mu_j, j = 1, \ldots, N + 2$ satisfying

$$\mu_1 > 0 = \mu_2 = \cdots = \mu_{N+1} > \mu_{N+2} \geq \cdots,$$

where the kernel of $L_{x_0}$ is spanned by $\partial_j w_{x_0}, j = 1, \ldots, N$ and $\mu_1$ is simple.

In this section, our main result is the small eigenvalue estimates of $L_h$ given by

**Theorem 2.7.** Under the same hypotheses of Lemma 2.6 for $h$ small enough, the eigenvalue problem

$$L_h \varphi_h = \mu_h \varphi_h \quad (2.19)$$

has exactly $N$ eigenvalues $\mu^j, j = 1, \ldots, N$, in the interval $\left(\frac{1}{2} \mu_1, \frac{1}{2} \mu_{N+2}\right]$, which satisfy

$$\frac{\mu^j_h}{h^2} = c_0 \nu_j, \quad \text{as } h \to 0, \quad \text{for } j = 1, \ldots, N, \quad (2.20)$$

where $\mu_1$ and $\mu_{N+2}$ are defined in Lemma 2.6, $\nu_j$'s are the eigenvalues of the Hessian matrix $\nabla^2 m(x_0)$ and $c_0 = \frac{N}{2m(x_0)}$ is a positive constant. Furthermore, the corresponding eigenfunctions $\varphi^j_h$ satisfy

$$\varphi^j_h = \sum_{i=1}^N \left[ a_{ij} + o(1) \right] \partial_i w_{x_h} + O(h^2), \quad j = 1, \ldots, N, \quad (2.21)$$

where $a_j = (a_{1j}, \ldots, a_{Nj})^T$ is the eigenvector associated with $\nu_j$, namely,

$$\nabla^2 m(x_0) a_j = \nu_j a_j. \quad (2.22)$$

Here $o(1)$ is a small quantity tending to zero and $O(1)$ is a bounded quantity as $h$ goes to zero.

**Remark 4:** (1) Since $L_h$ converges to $L_{x_0}$ in the strong resolvent sense, in the interval $\left(\frac{1}{2} \mu_1, \infty\right)$ $L_h$ has only one positive eigenvalues $\mu^0_h$, which is simple and goes to $\mu_1$ as $h$ goes to 0.

(2) After changing variables $t \mapsto t/h, y = (x - x_0)/h$, $L_h$ becomes $-R_h$, which is the notation used in page 190 of [17]. Thus the number of negative eigenvalues of $R_h$ equals the number of positive eigenvalues of $L_h$, which we denote by $n(L_h)$.

(3) By (2.20), the sign of small eigenvalue $\mu^0_h$ of $L_h$ is the same as the one of eigenvalue $\nu_j$ of $\nabla^2 m(x_0)$. If we denote the number of positive eigenvalues of $\nabla^2 m(x_0)$ by $n$, then the number of positive eigenvalues of $L_h$ in the interval $\left[\frac{1}{2} \mu_1, \frac{1}{2} \mu_{N+2}\right]$ equals $n$. Adding another one in the interval $\left(\frac{1}{2} \mu_1, \infty\right)$, the number of positive eigenvalues of $L_h$ equals $n + 1$. In particular, if $\nabla^2 m(x_0)$ is negative definite, then $n = 0$ and thus $n(L_h) = 1$. 

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Proof. We may follow the arguments given in Section 5 of [42]. Assume that \( \| \varphi_h \|_{L^2} = 1 \). By Lemma (2.6) it is easy to see that \( \mu_h \to 0 \) as \( h \to 0 \), where \( \mu_h \in \{ \mu_1^h, \ldots, \mu_N^h \} \). Then the corresponding eigenfunctions \( \varphi_h \)'s can be written as

\[
\varphi_h = \sum_{j=1}^{N} a_j^h \partial_j w_{xh} + \varphi^\perp_h,
\]

where \( \varphi^\perp_h \in K^\perp_{xh} \). Hence by (2.19) and (2.23), \( \varphi^\perp_h \) satisfies

\[
\Delta \varphi^\perp_h - \lambda \varphi^\perp_h + m(x_h)pw^{p-1}_{xh} \varphi^\perp_h + R(\varphi^\perp_h) + \sum_{j=1}^{N} a_j^h L_h \partial_j w_{xh} = \mu_h \left( \sum_{j=1}^{N} a_j^h \partial_j w_{xh} + \varphi^\perp_h \right),
\]

where

\[
R(\varphi^\perp_h) = m(hy + x_h)p(v_{xh}^{p-1} - w_{xh}^{p-1}) \varphi^\perp_h + \left[ m(hy + x_h) - m(x_h) \right] pw^{p-1}_{xh} \varphi^\perp_h.
\]

Using (2.16) and Lemma 2.4, we have

\[
L_h \partial_j w_{xh} = m(hy + x_h)p(v_{xh}^{p-1} - w_{xh}^{p-1}) \partial_j w_{xh} + \left[ m(hy + x_h) - m(x_h) \right] pw^{p-1}_{xh} \partial_j w_{xh} = O(h^2).
\]

From Lemma 2.5, the map \( L_{xh} = \Delta - \lambda + m(x_h)pw^{p-1}_{xh} \) is uniformly invertible in the space \( K^\perp_{xh} \). Thus by (2.25) and \( \mu_h \to 0 \), we have

\[
\| \varphi^\perp_h \|_{H^2} \leq c(h^2 + |\mu_h|) \sum_{j=1}^{N} |a_j^h|.
\]

To estimate \( \mu_h \) and \( a_j^h \)'s, multiplying (2.24) by \( \partial_k w_{xh} \) and integrating over \( \mathbb{R}^N \), we may obtain

\[
\int_{\mathbb{R}^N} (L_h \varphi^\perp_h) \partial_k w_{xh} \, dy + \sum_{j=1}^{N} a_j^h \int_{\mathbb{R}^N} (L_h \partial_j w_{xh}) \partial_k w_{xh} \, dy = \mu_h \sum_{j=1}^{N} a_j^h \int_{\mathbb{R}^N} \partial_j w_{xh} \partial_k w_{xh} \, dy.
\]

Here we have used the fact that \( \varphi^\perp_h \in K^\perp_{xh} \). Using (2.25), (2.26), \( \mu_h = o(1) \) and integration by parts, we obtain

\[
\int_{\mathbb{R}^N} (L_h \varphi^\perp_h) \partial_k w_{xh} \, dy = \int_{\mathbb{R}^N} \varphi^\perp_h L_h \partial_k w_{xh} \, dy = o(h^2),
\]

and

\[
\int_{\mathbb{R}^N} (L_h \partial_j w_{xh}) \partial_k w_{xh} \, dy = \frac{h^2}{p + 1} \int_{\mathbb{R}^N} w^{p+1}_{xh} \, dy \partial k \partial m(x_0) + o(h^2),
\]

which we have proved in Appendix A. Substituting (2.28) and (2.29) into (2.27), we may obtain

\[
\frac{1}{p + 1} \int_{\mathbb{R}^N} \frac{w^{p+1}_{xh} \partial_j m(x_0) a_k}{h^2} = \frac{\mu_h}{h^2} \sum_{j=1}^{N} a_j^h \int_{\mathbb{R}^N} (\partial_k w_{xh})^2 \, dy + o(1).
\]
Since \( \| \varphi_h \|_{L^2} = 1 \), (2.23) implies that \( a_h := (a_1^h, \cdots, a_N^h)^T \) is bound. Moreover, by (2.26), \( a_h \) does not converge to 0. Thus \( \frac{\rho_h}{h^2} \to c_0 \nu_j \) for \( j = 1, \cdots, N \) and \( a_h \to a_j \), where

\[
c_0 = \frac{N \int_{\mathbb{R}^N} w^{p+1}_x \, dy}{(p+1) \int_{\mathbb{R}^N} |\nabla w_{x_0}|^2 \, dy} = \frac{N}{2m(x_0)},
\]

and \( a_j \) is the eigenvector corresponding to \( \nu_j \). Here we have use the fact that\( \int_{\mathbb{R}^N} |\nabla w_{x_0}|^2 \, dy = \frac{N}{N+2} m(x_0) \int_{\mathbb{R}^N} w^{p+1}_x \, dy, \)

which can be proved by Pohozeve identity. The rest of the proof follows from a perturbation result, similar to page 1473-1474 of [42]. We may omit the details here.

\[ \square \]

3 Proof of Theorem 1.1

In this Section, we firstly study the asymptotic expansion of \( d''(\lambda) \) as \( h \to 0 \), and then complete the proof of Theorem 1.1. To drive the \( O(h^4) \) order terms of \( d''(\lambda)/h^N \), we need the following lemma:

**Lemma 3.1.** Under the same hypotheses of Lemma 2.2,

\[
x_h = x_0 + h^2 x_1 + O(h^3), \quad \text{as} \quad h \to 0,
\]

where \( x_1 \in \mathbb{R}^N \) satisfies

\[
\nabla^2 m(x_0) x_1 = -\frac{1}{2N} \frac{\int_{\mathbb{R}^N} |y|^2 w^{p+1}_x \, dy}{\int_{\mathbb{R}^N} w^{p+1}_x \, dy} \nabla (\Delta m)(x_0). \tag{3.2}
\]

**Proof.** By Lemma 2.3 and \( \nabla m(x_0) = 0 \), for all \( i = 1, \cdots, N \), we have

\[
\partial_i m(hy + x_h) = \sum_{j=1}^N \partial_{ij} m(x_0) (hy_j + x_{h,j} - x_{0,j}) + O(h^2). \tag{3.3}
\]

Then by (2.8), (3.3) and Lemma 2.4, we have

\[
0 = \int_{\mathbb{R}^N} \partial_i m(hy + x_h) v^{p+1}_h \, dy = \sum_{j=1}^N \partial_{ij} m(x_0) \left( \int_{\mathbb{R}^N} (hy_j + x_{h,j} - x_{0,j}) \left[ w^{p+1}_{x_h} + O(h) \right] \, dy + O(h^2) \right)
\]

\[
= \sum_{j=1}^N \partial_{ij} m(x_0) (x_{h,j} - x_{0,j}) \int_{\mathbb{R}^N} w^{p+1}_{x_0} \, dy + O(h^2).
\]
Here we have used the fact that \( \int_{\mathbb{R}^N} y_jw_{x_h}^{p+1} dy = 0 \) for \( j = 1, \cdots, N \). Thus \( x_h = x_0 + O(h^2) \). Consequently, we may set \( x_h = x_0 + h^2 \bar{x}_h \). Then \( \bar{x}_h = O(1) \) and by Taylor’s formula of \( \partial_i m(x) \), we have

\[
\partial_i m(hy + x_h) = \sum_{j=1}^{N} \partial_{ij} m(x_0) (hy_j + h^2 \bar{x}_j) + \frac{h^2}{2} \sum_{j,k=1}^{N} \partial_{ijk} m(x_0) y_j y_k + O(h^3). \tag{3.4}
\]

Hence by (2.8), (3.4) and Lemma 2.4, we may obtain

\[
0 = h^2 \sum_{j=1}^{N} \partial_{ij} m(x_0) \bar{x}_{x,j} \int_{\mathbb{R}^N} w_{x_0}^{p+1} dy + \frac{h^2}{2} \sum_{j,k=1}^{N} \partial_{ijk} m(x_0) \int_{\mathbb{R}^N} y_j y_k w_{x_h}^{p+1} dy + O(h^3)
\]

\[
= h^2 \sum_{j=1}^{N} \partial_{ij} m(x_0) \bar{x}_{x,j} \int_{\mathbb{R}^N} w_{x_0}^{p+1} dy + \frac{h^2}{2N} \sum_{k=1}^{N} \partial_{kkk} m(x_0) \int_{\mathbb{R}^N} |y|^2 w_{x_0}^{p+1} dy + O(h^3).
\]

Here we have used the fact that

\[
\left\{ \begin{array}{l}
\int_{\mathbb{R}^N} y_j w_{x_0}^{p+1} dy = 0, \quad \forall j = 1, \cdots, N, \\
\int_{\mathbb{R}^N} y_j y_k w_{x_0}^{p+1} = \frac{\delta_{jk}}{N} \int_{\mathbb{R}^N} |y|^2 w_{x_0}^{p+1} dy, \quad \forall j, k = 1, \cdots, N.
\end{array} \right.
\]

Therefore, we may complete the proof because

\[
w_{x_0}(y) = \lambda^{N/4} m(x_0)^{-N/4} w(\sqrt{\lambda} y).
\]

\[ \Box \]

From Lemma 2.4 and 3.1 we may deduce that

**Theorem 3.2.** Under the same hypotheses of Lemma 2.2, for \( h \) small enough, \( u_h \) is smooth on \( \lambda \). Let \( R_h := \frac{\partial u_h}{\partial \lambda}(hy + x_h) \). Then

\[
L_h R_h - v_h = 0, \tag{3.5}
\]

and

\[
R_h = R_0 + \sum_{j=1}^{N} c^j_h \partial_j w_{x_h} + h^2 R_1 + R_h^1, \tag{3.6}
\]

where \( R_0 = \lambda^{-1}\left( \frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h \right) \), \( c^j_h = O(h) \), \( R_h^1 = O(h^3) \) and \( R_1 \) satisfies

\[
\Delta R_1 - \lambda R_1 + m(x_h) p w_{x_h}^{p-1} R_1 - \frac{1}{2\lambda} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w_{x_h}^{p} = 0. \tag{3.7}
\]

Furthermore,

\[
\nabla^2 m(x_0) (h^{-1} c_h) \rightarrow -\frac{\int_{\mathbb{R}^N} |y|^2 w^{p+1} dy}{2N\lambda^2 \int_{\mathbb{R}^N} w^{p+1} dy} \nabla(\Delta m)(x_0), \quad \text{as} \ h \to 0, \tag{3.8}
\]

where \( c_h := (c^1_h, \cdots, c^N_h)^T \).
Proof. By Lemma 2.2 and Theorem 2.7, \( u_h \) is unique and non-degenerate. Consequently, \( u_h \) is smooth on \( \lambda \) and \( R_h \) satisfies (3.5). Now we decompose \( R_h \) as

\[
R_h = R_0 + \sum_{j=1}^{N} c_j^h \partial_j w_{x_h} + h^2 R_1 + R_h^\perp,
\]

where \( R_h^\perp \in K_{x_h}^\perp \). Then \( R_h^\perp \) satisfies

\[
L_h R_h^\perp + \left[ L_h R_0 + h^2 L_h R_1 - v_h \right] + \sum_{j=1}^{N} c_j^h L_h \partial_j w_{x_h} = 0. \tag{3.9}
\]

As for the proof of Theorem 2.7, we have

\[
\| R_h^\perp \|_{H^2} \leq c \left( \| L_h R_0 + h^2 L_h R_1 - v_h \|_{L^2} + \sum_{j=1}^{N} |c_j^h| h^2 \right). \tag{3.10}
\]

It is easy to check

\[
L_h R_0 = v_h - \frac{h^3}{2\lambda} y \cdot \nabla m(hy + x_h) v_h^p. \tag{3.11}
\]

Hence by Lemma 2.4, 3.1, (3.7) and (3.11), we obtain

\[
L_h R_0 + h^2 L_h R_1 - v_h
\]

\[
= - \frac{h^3}{2\lambda} \sum_{i,j=1}^{N} \partial_{ij} m(x_0)x_{1,i}y_{j} + \frac{1}{2} \sum_{i,j,k=1}^{N} \partial_{ijk} m(x_0)y_{i}y_{j}y_{k} w_{x_h}^p + O(h^4). \tag{3.12}
\]

Consequently, by (3.10),

\[
\| R_h^\perp \|_{H^2} \leq c \left( h^3 + \sum_{j=1}^{N} |c_j^h| h^2 \right). \tag{3.13}
\]

To estimate \( c_j^h \)'s, we may multiply (3.9) by \( \partial_k w_{x_h} \) and integrate over \( \mathbb{R}^N \). Then

\[
\int_{\mathbb{R}^N} (L_h R_h^\perp) \partial_k w_{x_h} \, dy + \int_{\mathbb{R}^N} \left[ L_h R_0 + h^2 L_h R_1 - v_h \right] \partial_k w_{x_h} \, dy
\]

\[
+ \sum_{j=1}^{N} c_j^h \int_{\mathbb{R}^N} (L_h \partial_j w_{x_h}) \partial_k w_{x_h} \, dy = 0. \tag{3.14}
\]

Hence by (2.29), (3.14) may imply

\[
|c_j^h| \leq \frac{C}{h^2} \left[ \left| \int_{\mathbb{R}^N} (L_h R_h^\perp) \partial_k w_{x_h} \, dy \right| + \left| \int_{\mathbb{R}^N} \left[ L_h R_0 + h^2 L_h R_1 - v_h \right] \partial_k w_{x_h} \, dy \right| \right]. \tag{3.15}
\]

Using integration by parts and (2.25), we have

\[
\int_{\mathbb{R}^N} (L_h R_h^\perp) \partial_k w_{x_h} \, dy = \int_{\mathbb{R}^N} R_h^\perp L_h \partial_k w_{x_h} \, dy = \| R_h^\perp \|_{L^2} O(h^2). \tag{3.16}
\]
Therefore, by (3.12), (3.13), (3.15) and (3.16), we may obtain $|c^j_h| = O(h)$. Consequently, by (3.13), $R_h^2 = O(h^3)$. Thus by (3.16),

$$
\int_{\mathbb{R}^N} (L_h R_h^2) \partial_k w_{x_h} dy = O(h^5).
$$

(3.17)

Hence by (2.29), (3.12) and (3.17), (3.14) gives

$$
\frac{1}{p + 1} \int_{\mathbb{R}^N} w_{x_0}^{p+1} dy \sum_{j=1}^N \partial_j \gamma_m(x_0) \left( h^{-1} c^j_h \right)
$$

$$
= -\frac{1}{2\lambda} \int_{\mathbb{R}^N} \left[ \sum_{i,j=1}^N \partial_{ij} \gamma_m(x_0) x_i y_j + \frac{1}{2} \sum_{i,j,l=1}^N \partial_{ijkl} \gamma_m(x_0) y_i y_j y_l \right] w_{x_h}^p \partial_k w_{x_h} dy + o(1).
$$

(3.18)

Using integration by parts, we obtain

$$
\left\{ \begin{array}{l}
\int_{\mathbb{R}^N} y_j w_{x_h}^{p} \partial_k w_{x_h} dy = -\delta_{jk} \int_{\mathbb{R}^N} w_{x_h}^{p+1} dy, \\
\int_{\mathbb{R}^N} y_j y_j w_{x_h}^{p} \partial_k w_{x_h} dy = -\delta_{jk} \delta_{ij} + \delta_{jk} \delta_{il} + \delta_{ik} \delta_{jl} \frac{1}{N(p+1)} \int_{\mathbb{R}^N} |y|^2 w_{x_h}^{p+1} dy,
\end{array} \right.
$$

where $\delta$ is the Kronecker symbol. Hence by (3.18), $|c^j_h| = O(h)$ for $j = 1, \ldots, N$. Moreover, by (3.2), we obtain (3.8) and complete the proof.

Let us now compute $d''(\lambda)$. From (1.3), it is easy to get

$$
d'(\lambda) = \frac{1}{2} \int_{\mathbb{R}^N} u_h^2 dx
$$

and hence

$$
d''(\lambda) = \int_{\mathbb{R}^N} u_h \frac{\partial u_h}{\partial \lambda} dx = h^N \int_{\mathbb{R}^N} v_h R_h dy.
$$

(3.19)

Using integration by parts and (3.5), we have

$$
\int_{\mathbb{R}^N} v_h R_0 dy = \int_{\mathbb{R}^N} v_h \lambda \left( \frac{1}{p - 1} v_h + \frac{1}{2} y \cdot \nabla v_h \right) dy = \lambda^{-1} \left( \frac{1}{p - 1} - \frac{N}{4} \right) \int_{\mathbb{R}^N} v_h^2 dy = 0,
$$

(3.20)
since $p = 1 + \frac{4}{N}$. Hence, by (3.19) and Theorem 3.2, we have

$$
\frac{d''(\lambda)}{h^N} = \int_{\mathbb{R}^N} v_h \left[ R_0 + \sum_{j=1}^N c^j_h \partial_j w_{x_h} + h^2 R_1 + R_h^2 \right] dy
$$

$$
= \int_{\mathbb{R}^N} v_h \left[ \sum_{j=1}^N c^j_h \partial_j w_{x_h} + h^2 R_1 + R_h^2 \right] dy \quad \text{(because } \int_{\mathbb{R}^N} v_h R_0 dy = 0) \]

$$
= \int_{\mathbb{R}^N} R_h \left[ \sum_{j=1}^N c^j_h L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^2 \right] dy \quad \text{(because } L_h R_h = v_h) \]

$$
= \int_{\mathbb{R}^N} \left[ R_0 + \sum_{j=1}^N c^j_h \partial_j w_{x_h} + h^2 R_1 + R_h^2 \right] \left[ \sum_{j=1}^N c^j_h L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^2 \right] dy.
$$
Therefore, by (2.26), (3.9) and $c^j_h = O(h)$,
\[
\frac{d^n}{h^N} = \int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy + \sum_{j,k=1}^N c^j_h c^k_h \int_{\mathbb{R}^N} \partial_k w_{x_h} (L_h \partial_j w_{x_h}) dy \\
+ h^4 \int_{\mathbb{R}^N} R_1 (L_h R_1) dy + O(h^5) .
\] (3.21)

For the integral $\int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy$, by (3.11) and using integration by parts, we have
\[
\int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy = \int_{\mathbb{R}^N} \lambda^{-1} \left( \frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h \right) \left[ \frac{h}{2\lambda} y \cdot \nabla m(hy + x_h) v_h^p \right] dy
\]
\[
- \frac{1}{2\lambda^2} \int_{\mathbb{R}^N} \frac{N}{4(N+2)} \left[ h y \cdot \nabla m(hy + x_h) - h^2 \sum_{i,j=1}^N \partial_{ij} m(hy + x_h) y_i y_j \right] v_h^{p+1} dy.
\]
Note that by Lemma 2.4, 3.1 and Theorem 3.2, we have
\[
hy \cdot \nabla m(hy + x_h) - h^2 \sum_{i,j=1}^N \partial_{ij} m(hy + x_h) y_i y_j
\]
\[
= hy \cdot \nabla m(x_h) - \frac{h^3}{2} \sum_{i,j,k=1}^N \partial_{ijk} m(x_h) y_i y_j y_k - \frac{h^4}{3} \sum_{i,j,k,l=1}^N \partial_{ijkl} m(x_h) y_i y_j y_k y_l + o(h^4),
\]
and
\[
v_h^p = w_{x_h}^p + h^2 p w_{x_h}^{p-1} \phi_2 + O(h^3) .
\] (3.22)

Hence
\[
\int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy = \frac{N}{8(N+2)} \lambda^{-2} \int_{\mathbb{R}^N} \left[ - \frac{h^4}{3} \sum_{i,j,k,l=1}^N \partial_{ijkl} m(x_h) y_i y_j y_k y_l \right] w_{x_h}^{p+1} dy + o(h^4)
\]
\[
= - \frac{h^4}{8(N+2)^2} \lambda^{-3} \int_{\mathbb{R}^N} |y|^{4} w_{x_h}^{p+1} dy \Delta^2 m(x_0) + o(h^4)
\]
\[
= - \frac{h^4}{8(N+2)^2} \lambda^{-3} m(x_0)^{-\frac{N-1}{2}} \int_{\mathbb{R}^N} |y|^{4} w_{x_h}^{p+1} dy \Delta^2 m(x_0) + o(h^4) .
\] (3.23)

Here we have used the following identities:
\[
\begin{align*}
\int_{\mathbb{R}^N} y_i w_{x_h}^{p+1} dy &= \int_{\mathbb{R}^N} y_i y_j y_k w_{x_h}^{p+1} dy = 0 , \quad \text{for all } i, j, k = 1, \ldots, N ; \\
\int_{\mathbb{R}^N} y_i y_j y_k y_l w_{x_h}^{p+1} dy &= 0 , \quad \text{if } y_i y_j y_k y_l \text{ is an odd function on one of its variate} ; \\
\int_{\mathbb{R}^N} y_i^4 w_{x_h}^{p+1} dy &= \frac{3}{N(N+2)} \int_{\mathbb{R}^N} |y|^{4} w_{x_h}^{p+1} dy , \quad \text{for all } i = 1, \ldots, N ; \\
\int_{\mathbb{R}^N} y_i^2 y_j^2 w_{x_h}^{p+1} dy &= \frac{1}{N(N+2)} \int_{\mathbb{R}^N} |y|^{4} w_{x_h}^{p+1} dy , \quad \text{for all } i \neq j ,
\end{align*}
\]
which can be proved by polar coordinates.

For the sum \( \sum_{j,k=1}^{N} c_j c_h \int_{\mathbb{R}^N} \partial_j w_{x_h} (L_h \partial_j w_{x_h}) dy \), we may use (2.29) and (3.8) to get

\[
\sum_{j,k=1}^{N} c_j c_h \int_{\mathbb{R}^N} \partial_j w_{x_h} (L_h \partial_j w_{x_h}) dy = \frac{h^4}{p + 1} \int_{\mathbb{R}^N} w_{x_h}^{p+1} dy \sum_{j,k=1}^{N} (h^{-1} c_j h^{-1} c_k) \partial_j m(x_0) + o(h^4)
\]

\[
= \frac{h^4}{8N(N+2)} \lambda^{-3} m(x_0)^{-\frac{N}{2}-1} \left( \frac{\int_{\mathbb{R}^N} |y|^2 w^{p+1} dy}{\int_{\mathbb{R}^N} w^{p+1} dy} \right)^2 \nabla(\Delta m)(x_0) \cdot [\nabla^2 m(x_0)]^{-1} \nabla(\Delta m)(x_0) + o(h^4).
\]

(3.24)

For the integral \( h^4 \int_{\mathbb{R}^N} R_1(L_h R_1) dy \), by (3.7), it is obvious that \( R_1(\lambda^{-\frac{1}{2}} y) \) satisfies

\[
\Delta R - R + pw^{p-1} R - \frac{1}{2} \lambda^{-2} m(x_0)^{-\frac{N}{2}-1} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w = 0.
\]

(3.25)

Hence

\[
h^4 \int_{\mathbb{R}^N} R_1(L_h R_1) dy = h^4 \int_{\mathbb{R}^N} R_1(L_{x_h} R_1) dy + O(h^6)
\]

\[
= \frac{h^4}{4} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} \sum_{i,j,k,l=1}^{N} \partial_{ij} m(x_0) \partial_{kl} m(x_0) \int_{\mathbb{R}^N} y_i y_l w^{p} L_0^{-1}(y_k y_l w^{p}) dy + O(h^6)
\]

\[
= \frac{h^4}{4N^2} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} |\Delta m(x_0)|^2 \int_{\mathbb{R}^N} r^2 w_p L_0^{-1}(r^2 w_p) dy
\]

\[
+ \frac{h^4}{2N(N+2)} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} \|\nabla^2 m(x_0)\|_2^2 \int_{\mathbb{R}^N} r^2 w_p \Phi_0(r) dy
\]

\[
- \frac{h^4}{2N^2(N+2)} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} |\Delta m(x_0)|^2 \int_{\mathbb{R}^N} r^2 w_p \Phi_0(r) dy + O(h^6).
\]

(3.26)
Here $\|\nabla^2 m(x_0)\|^2 = \sum_{i,j=1}^{N} m_{ij}^2(x_0)$ and we have used the following identities:

$$
\int_{\mathbb{R}^N} y_N^2 w^p L_0^{-1}(y_N^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy + \frac{2(N-1)}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, 
$$

$$
\int_{\mathbb{R}^N} y_{N-1}^2 w^p L_0^{-1}(y_{N-1}^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy - \frac{2}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, 
$$

$$
\int_{\mathbb{R}^N} y_{N-1} y_N w^p L_0^{-1}(y_{N-1} y_N w^p) dy = \frac{1}{N(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, 
$$

where $\Phi_0$ satisfies (1.16), which we have proved in Appendix B.

Therefore, combining (3.21), (3.23), (3.24) and (3.26), we obtain

$$
\frac{d''(\lambda)}{h^N} + o(h^4) = -\frac{h^4}{8(N+2)^2} \lambda^{-3} m(x_0)^{-\frac{N}{2}-1} \int_{\mathbb{R}^N} |y|^4 w^{p+1} dy \Delta^2 m(x_0) \cdot \nabla (\Delta m)(x_0)
$$

$$
+ \frac{h^4}{8N(N+2)} \lambda^{-3} m(x_0)^{-\frac{N}{2}-1} \left( \int_{\mathbb{R}^N} |y|^2 w^{p+1} dy \right)^2 \nabla (\Delta m)(x_0) \cdot \nabla (\Delta^2 m)(x_0)
$$

$$
+ \frac{h^4}{4N^2} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} |\Delta m(x_0)|^2 \int_{\mathbb{R}^N} |y|^2 w^p L_0^{-1}(|y|^2 w^p) dy
$$

$$
+ \frac{h^4}{2N^2(N+2)} \lambda^{-3} m(x_0)^{-\frac{N}{2}-2} \left[ N \|\nabla^2 m(x_0)\|^2 - |\Delta m(x_0)|^2 \right] \int_{\mathbb{R}^N} |y|^2 w^p \Phi_0(|y|) dy.
$$

Consequently,

$$
\frac{8(N+2)^2 m(x_0)^{\frac{N}{2}+2} \lambda^3}{h^{N+4}} \int_{\mathbb{R}^N} |y|^4 w^{p+1} dy \cdot \frac{d''(\lambda)}{d\lambda} = C_{N,1} |\Delta m(x_0)|^2 + C_{N,2} \left( N \|\nabla^2 m(x_0)\|^2 - |\Delta m(x_0)|^2 \right)
$$

$$
+ C_{N,3} m(x_0) \left[ \nabla (\Delta m)(x_0) \cdot \nabla (\Delta^2 m)(x_0) \right]^{-1} \nabla (\Delta m)(x_0)
$$

$$
- m(x_0) \Delta^2 m(x_0) + o(1),
$$

where $C_{N,1}, C_{N,2}, C_{N,3}$ are constants given by (1.13), (1.14), (1.15), respectively.

Now we may prove Theorem 1.1 as follows: Suppose that $x_0$ is a non-degenerate local maximum point of the function $m(x)$, then the Hessian matrix $\nabla^2 m(x_0)$ of $m$ at $x_0$ is negative definite. By Theorem 2.7, we have $n(L_h) = 1$. On the other hand, we have $p(d'') = 1$. Thus $\psi_h$ is orbital stable by the orbital stability criteria of [17]-[18]. For orbital instability, we denote the number of positive eigenvalues of the Hessian matrix $\nabla^2 m(x_0)$ by $n$. Then by Theorem 2.7, we obtain $n(L_h) = n + 1$. On the other hand, we have $p(d'') = 1$. Thus by the instability criteria of [18], we conclude that $\psi_h$ is orbitally unstable if $n$ is odd. This may complete the proof of Theorem 1.1.
4 Proof of Theorem 1.2-1.4

In this section, we may generalize the argument of Section 2 and 3 to prove Theorem 1.2-1.4. Let \( v_h(y) := u_h(hy + x_h) \), where \( u_h \) is a single-spike bound state of (1.5) with a unique local maximum point at \( x_h \). Then \( v_h \) satisfies

\[
\Delta v_h - \left[ V(hy + x_h) + \lambda \right] v_h + m(hy + x_h)v_h^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{4.1}
\]

Suppose (2.7) hold. By (2.8) and (39), we have

\[
m(x_0)\nabla V(x_0) = \frac{N}{2}[V(x_0) + \lambda]\nabla m(x_0), \tag{4.2}
\]

so \( x_0 \) may depend on \( \lambda \). Note that by (4.2), \( \nabla m(x_0) = 0 \) if and only if \( \nabla V(x_0) = 0 \). By direct computation on the function \( G \),

\[
\partial_{ij}G(x_0) = m(x_0)^{-\frac{N}{2}-1}\left[m(x_0)\partial_{ij}V(x_0) + (1 - \frac{N}{2})\partial_i V(x_0)\partial_j m(x_0) - \frac{N}{2}[V(x_0) + \lambda]\partial_{ij}m(x_0)\right].
\]

In particular, if \( \nabla m(x_0) = 0 \), then

\[
\nabla^2 G(x_0) = m(x_0)^{-\frac{N}{2}-1}\left[m(x_0)\nabla^2 V(x_0) - \frac{N}{2}[V(x_0) + \lambda]\nabla^2 m(x_0)\right].
\]

Using the identity (2.8), one may follow the arguments of Lemma 2.2-2.4 to get the uniqueness of \( u_h \) and

\[
\begin{align*}
x_h &= x_0 + o(h); \tag{4.3} \\
v_h &= w_{x_h} + h\phi_1 + h^2\phi_2 + o(h^2), \tag{4.4}
\end{align*}
\]

where \( \phi_1 \) and \( \phi_2 \) satisfy \( \nabla \phi_1(0) = \nabla \phi_2(0) = 0 \),

\[
\Delta \phi_1 - [V(x_0) + \lambda] \phi_1 + m(x_0)p w_{x_0}^{p-1} \phi_1 - y \cdot \nabla V(x_0)w_{x_0} + y \cdot \nabla m(x_0)w_{x_0}^p = 0, \tag{4.5}
\]

and

\[
\begin{align*}
\Delta \phi_2 - [V(x_h) + \lambda] \phi_2 + m(x_h)p w_{x_h}^{p-1} \phi_2 - y \cdot \nabla V(x_h)\phi_1 - \frac{1}{2}\sum_{i,j=1}^{N} \partial_{ij}V(x_h)y_iy_j w_{x_h} \\
+ y \cdot \nabla m(x_h)w_{x_h}^{p-1} \phi_1 + \frac{1}{2}\sum_{i,j=1}^{N} \partial_{ij}m(x_h)y_iy_j w_{x_h}^p + \frac{1}{2}m(x_0)p(p-1)w_{x_h}^{p-2} \phi_1^2 = 0. \tag{4.6}
\end{align*}
\]

Here we have used the hypothesis that \( x_0 \) is a non-degenerate point of the function \( G \). And the only difference in the proof is that we need to estimate the term

\[
\frac{1}{p+1}\nabla m(x_0) \int_{\mathbb{R}^N} v_h^{p+1} dy - \frac{1}{2}\nabla V(x_0) \int_{\mathbb{R}^N} v_h^2 dy,
\]

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to estimate which one may use the following Pohozaev identity (cf. [34])
\[
\int_{\mathbb{R}^N} \left[ \frac{2}{N+2} m(hy + x_h) + \frac{h}{p+1} y \cdot \nabla m(hy + x_h) \right] v_h^{p+1} dy \\
= \int_{\mathbb{R}^N} \left[ V(hy + x_h) + \frac{h}{2} y \cdot \nabla V(hy + x_h) \right] v_h^2 dy.
\]

For the small eigenvalue estimates of $L_h$, one may generalize the idea of Theorem 2.7 to get

**Theorem 4.1.** For $h$ small enough, the eigenvalue problem
\[
L_h \varphi_h = \mu_h \varphi_h
\]
has exactly $N$ eigenvalues $\mu_j^h$, $j = 1, \ldots, N$, in the interval $[\frac{1}{2}\mu_1, \frac{1}{2}\mu_{N+2}]$, which satisfy and
\[
\frac{\mu_j^h}{h^2} \to c_0 \nu_j, \quad \text{as} \quad h \to 0, \quad \text{for} \quad j = 1, \ldots, N,
\]
where $\mu_1$ and $\mu_{N+2}$ are defined Lemma 2.4, $\nu_j$'s are the eigenvalues of the Hessian matrix $\nabla^2 G(x_0)$, and $c_0 = -\frac{m(x_0)^N}{V(x_0) + \lambda} = -G(x_0)^{-1}$ is a negative constant. Furthermore, the corresponding eigenfunctions $\varphi_j^h$'s satisfy
\[
\varphi_j^h = \sum_{i=1}^{N} \left[ a_{ij} + o(1) \right] (\partial_i w_{x_h} + h \psi_i) + O(h^2), \quad j = 1, \ldots, N,
\]
where each $\psi_i$ is the solution of
\[
\Delta \psi_i - \left[ V(x_h) + \lambda \right] \psi_i + m(x_h) p w_{x_h}^{p-1} \psi_i \\
+ \left[ -y \cdot \nabla V(x_h) + y \cdot \nabla m(x_h) p w_{x_h}^{p-1} + m(x_h) p(p-1) w_{x_h}^{p-2} \phi_1 \right] \partial_i w_{x_h} = 0,
\]
and $a_j = (a_{1j}, \ldots, a_{Nj})^T$ is the eigenvector corresponding to $\nu_j$, namely,
\[
\nabla^2 G(x_0) a_j = \nu_j a_j.
\]

**Remark 5:** (1) To prove it, one may follow the arguments in the proof of Theorem 2.7 and use the following identity
\[
\int_{\mathbb{R}^N} \partial_k w_{x_h} L_h (\partial_j w_{x_h} + h \psi_j) dy = -\frac{h^2}{N+2} \int_{\mathbb{R}^N} w^{p+1} dy \partial_{jk} G(x_0) + o(h^2),
\]
to replace (2.29) (see Appendix C). The main difference between Theorem 2.7 and 4.1 is that (4.9) has the solution $\psi_i$ of (4.10) which comes from
\[
L_h \partial_i w_{x_h} = h \left[ -y \cdot \nabla V(x_0) + y \cdot \nabla m(x_0) p w_{x_h}^{p-1} \\
+ m(x_0) p(p-1) w_{x_h}^{p-2} \phi_1 \right] \partial_i w_{x_h} + O(h^2).
\]
(2) Let $n$ be the number of negative eigenvalues of the matrix $\delta^2 G(x_0)$, then similar to the Remark 4(3), the number of positive eigenvalues of $L_h$ equals $n + 1$, i.e., $n(L_h) = n + 1$.

Since the potential function $V$ is nonzero, then $x_0$ may depend on $\lambda$ and the asymptotic expansion of $d''(\lambda)$ becomes more complicated. Indeed, when $m \equiv 1$ and $\Delta V(x_0) \neq 0$, the result in [25] shows that the effect of potential function $V$ on $d''(\lambda)$ is $O(h^2)$. On the other hand, when $V \equiv 0$ and condition (1.12) holds, the effect of $m$ on $d''(\lambda)$ is $O(h^4)$ (see Section 3). Generally, when both $m$ and $V$ are not constant, we may show

(I) The effect of $V$ and $m$ on $d''(\lambda)$ is $O(1)$ if $\nabla V(x_0) \neq 0$ (see Theorem 1.2);

(II) The effect of $V$ and $m$ on $d''(\lambda)$ is $O(h^2)$ if $\nabla V(x_0) = 0$ and $\Delta V(x_0) \neq 0$ (see Theorem 1.3);

(III) The effect of $V$ and $m$ on $d''(\lambda)$ is $O(h^4)$ if $\nabla V(x_0) = 0$, $\Delta V(x_0) = 0$ and some local condition hold (see Theorem 1.4).

Now we divide three cases to prove these results.

**Case I:** $\nabla V(x_0) \neq 0$.

Let $R_h := \frac{\partial h}{\partial x}(hy + x_h)$. Then (3.5) and (3.20) hold. Hence one may apply the idea of Theorem 3.2 to get

$$R_h = \sum_{i=1}^{N} c_h^i (\partial_i w_{x_h} + h \psi_i) + R_0 + R_h^\perp,$$

where as $h \to 0$, $c_h = (c_h^1, \ldots, c_h^N)$ satisfies

$$\nabla^2 G(x_0)(h c_h) \to -\frac{N}{2} m(x_0)^{-\frac{N}{2} - 1} \nabla m(x_0),$$

and

$$R_0 = [V(x_h) + \lambda]^{-1} \left( \frac{1}{p - 1} v_h + \frac{1}{2} y \cdot \nabla v_h \right), \quad R_h^\perp = O(h).$$

Thus

$$\frac{d''(\lambda)}{h^N} = \int_{\mathbb{R}^N} v_h R_h dy = \int_{\mathbb{R}^N} v_h \left[ \sum_{i=1}^{N} c_h^i (\partial_i w_{x_h} + h \psi_i) + R_0 + R_h^\perp \right] dy$$

$$\quad = \int_{\mathbb{R}^N} v_h \sum_{i=1}^{N} c_h^i (\partial_i w_{x_h} + h \psi_i) dy + O(h) \quad \text{(because } \int_{\mathbb{R}^N} v_h R_0 dy = 0)$$

$$\quad = \int_{\mathbb{R}^N} R_h \sum_{i=1}^{N} c_h^i L_h (\partial_i w_{x_h} + h \psi_i) dy + O(h) \quad \text{(because } L_h R_h = v_h)$$

$$\quad = \int_{\mathbb{R}^N} \left[ \sum_{k=1}^{N} c_h^k (\partial_k w_{x_h} + h \psi_k) + R_0 + R_h^\perp \right] \sum_{i=1}^{N} c_h^i L_h (\partial_i w_{x_h} + h \psi_i) dy + O(h).$$
Therefore, by (4.10), (4.13), (4.12), (4.15) and (4.16), we obtain
\[
\frac{d^n(\lambda)}{h^N} = -\frac{N^2}{4(N+2)}m(x_0)^{-N-2}\int_{\mathbb{R}^N} w^{p+1}dy \nabla m(x_0) \cdot \left[\nabla^2 G(x_0)\right]^{-1} \nabla m(x_0) + O(h). \tag{4.17}
\]

Consequently, if \(x_0\) is a non-degenerate local minimum point of \(G\), then the Hessian matrix \(\nabla^2 G(x_0)\) is positive definite. By Theorem 4.1 we have \(n(L_h) = 1\). On the other hand, by (4.17), we have \(p(d^n) = 0\). Thus we complete the proof of Theorem 1.2 by the orbital instability criteria of [17]-[18].

**Case II:** \(\nabla V(x_0) = 0\) and \(\Delta V(x_0) \neq 0\).

Firstly, note that in this case, \(\phi_1 \equiv 0\) and \(\psi_i \equiv 0\). Then one may apply the idea of Lemma 3.1 and Theorem 3.2 to obtain
\[
x_h = x_0 + h^2 x_1 + O(h^3); \tag{4.18}
\]
\[
R_h = R_0 + \sum_{j=1}^{N} c_j^h \partial_j w_{x_h} + h^2 R_1 + R_h^1, \tag{4.19}
\]
where \(x_1 \in \mathbb{R}^N\) satisfies
\[
\nabla^2 G(x_0)x_1 = -\frac{N+2}{4N} [V(x_0) + \lambda]^{-1} m(x_0)^{-\frac{N}{2}} \left(\int_{\mathbb{R}^N} |y|^2 w^2 dy \right) \nabla(\Delta V)(x_0)
+ \frac{1}{4} m(x_0)^{-\frac{N}{2}} \left( \int_{\mathbb{R}^N} w h^{p+1} dy \right) \nabla(\Delta m)(x_0), \tag{4.20}
\]
\(R_1\) satisfies
\[
\Delta R_1 - [V(x_h) + \lambda] R_1 + m(x_h) p w_{x_h}^{p-1} R_1
+ [V(x_h) + \lambda]^{-1} \left[ \sum_{i,j=1}^{N} \partial_{ij} V(x_0) y_i y_j w_{x_h} - \frac{1}{2} \sum_{i,j=1}^{N} \partial_{ij} m(x_0) y_i y_j w_{x_h}^p \right] = 0, \tag{4.21}
\]
\(R_h^1 = O(h^3)\) and \(c_j^h = O(h)\) for \(j = 1, \cdots, N\). Moreover, \(c_h := (c_1^h, \cdots, c_N^h)\) satisfies
\[
\nabla^2 G(x_0)(h^{-1} c_h) = c_0 + o(1), \tag{4.22}
\]
where
\[
c_0 = - [V(x_0) + \lambda]^{-1} m(x_0)^{-\frac{N}{2}} \nabla^2 V(x_0) x_1
- \frac{N+2}{2N} [V(x_0) + \lambda]^{-2} m(x_0)^{-\frac{N}{2}} \left( \int_{\mathbb{R}^N} |y|^2 w^2 dy \right) \nabla(\Delta V)(x_0)
+ \frac{1}{4} [V(x_0) + \lambda]^{-1} m(x_0)^{-\frac{N}{2}-1} \left( \int_{\mathbb{R}^N} w h^{p+1} dy \right) \nabla(\Delta m)(x_0). \tag{4.23}
\]
Hence

\[
\frac{d''(\lambda)}{h^N} = \int_{\mathbb{R}^N} v_hR_h dy = \int_{\mathbb{R}^N} v_h \left[ R_0 + \sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^4 \right] dy
\]

\[
= \int_{\mathbb{R}^N} v_h \left[ \sum_{j=1}^N c_h^j \partial_j w_{x_h} + h^2 R_1 + R_h^4 \right] dy \quad \text{(because } \int_{\mathbb{R}^N} v_h R_0 dy = 0)\]

\[
= \int_{\mathbb{R}^N} R_h \left[ \sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^4 \right] dy \quad \text{(because } L_h R_h = v_h)\]

\[
= \int_{\mathbb{R}^N} \left[ R_0 + \sum_{k=1}^N c_h^k \partial_k w_{x_h} + h^2 R_1 + R_h^4 \right] \left[ \sum_{j=1}^N c_h^j L_h \partial_j w_{x_h} + h^2 L_h R_1 + L_h R_h^4 \right] dy.
\]

Therefore, by (4.10), (4.13) and (4.19), we obtain

\[
\frac{d''(\lambda)}{h^N} = \int_{\mathbb{R}^N} R_0 [v_h - L_h R_0] dy + \sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} L_h (\partial_j w_{x_h}) dy
\]

\[+ h^4 \int_{\mathbb{R}^N} R_1 (L_h R_1) dy + O(h^5). \quad (4.24)\]

For the integral \( \int_{\mathbb{R}^N} R_0 [v_h - L_h R_0] dy \), by direct computation, we have

\[
v_h - L_h R_0 = - \left[ V(x_h) + \lambda \right]^{-1} \left[ V(hy + x_h) - V(x_h) + \frac{h}{2} y \cdot \nabla V(hy + x_h) \right] v_h
\]

\[+ \frac{h}{2} \left[ V(x_h) + \lambda \right]^{-1} y \cdot \nabla m(hy + x_h) v_h^p. \quad (4.25)\]

Thus by (4.4), (4.18) and (2.6), we obtain

\[
\int_{\mathbb{R}^N} R_0 [v_h - L_h R_0] dy = \frac{h^2}{2N} [V(x_0) + \lambda]^{-3} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^2 w^2 dy \Delta V(x_0) + O(h^4). \quad (4.26)
\]

For the sum \( \sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} (L_h \partial_j w_{x_h}) dy \), by (4.10), (4.13) and \( c_h^j = O(h) \) for \( j = 1, \ldots, N \), we have

\[
\sum_{j,k=1}^N c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} (L_h \partial_j w_{x_h}) dy = O(h^4). \quad (4.27)
\]

Combining (4.26), (4.27) and (4.24), we obtain

\[
\frac{d''(\lambda)}{h^N} = \frac{h^2}{2N} [V(x_0) + \lambda]^{-3} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^2 w^2 dy \Delta V(x_0) + O(h^4). \quad (4.28)
\]
Consequently, by (4.28), we have \( p(d'') = \frac{1}{2}(1 + \frac{\Delta V(x_0)}{|\Delta V(x_0)|}) \). On the other hand, by Theorem 4.1 we have \( n(L_h) = n + 1 \). Thus we complete the proof of Theorem 1.3 by the orbital stability and instability criteria of [17]-[18].

**Case III:** \( \nabla V(x_0) = 0, \Delta V(x_0) = 0 \).

In this case, we shall use (4.23), (4.20) and (4.24) to compute the \( O(h^4) \) term of \( d''(\lambda)/h^N \). For the integral \( \int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy \), by (4.25) and integration by parts, we obtain

\[
\int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy = - [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left( \frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h \right) [V(hy + x_h) - V(x_h) + \frac{h}{2} y \cdot \nabla V(hy + x_h)] v_h dy
\]

\[+ [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left( \frac{1}{p-1} v_h + \frac{1}{2} y \cdot \nabla v_h \right) \frac{h}{2} y \cdot \nabla m(hy + x_h) v_h^p dy\]

\[= \frac{1}{8} [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left[ 3 h y \cdot \nabla V(hy + x_h) + h^2 \sum_{i,j=1}^{N} \partial_{ij} V(hy + x_h) y_i y_j \right] v_h^2 dy \]

\[+ \frac{N}{8(N+2)} [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left[ h y \cdot \nabla m(hy + x_h) - h^2 \sum_{i,j=1}^{N} \partial_{ij} m(hy + x_h) y_i y_j \right] v_h^{p+1} dy.\]

Hence by (4.18), (4.19) and Taylor’s formulas of \( V \) and \( m \), we have

\[
\int_{\mathbb{R}^N} R_0 \left[ v_h - L_h R_0 \right] dy
\]

\[= \frac{1}{8} [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left[ 4 h^2 \sum_{i,j=1}^{N} \partial_{ij} V(x_0) y_i y_j w_{x_h}^2 + 8 h^4 \sum_{i,j=1}^{N} \partial_{ij} V(x_0) y_i y_j w_{x_h} \phi_2 \right. \]

\[+ 4 h^4 \sum_{i,j,k=1}^{N} \partial_{ijk} V(x_0) x_{1,i} y_{j} y_k w_{x_h}^2 + h^4 \sum_{i,j,k,l=1}^{N} \partial_{ijkl} V(x_0) y_i y_j y_k y_l w_{x_h}^2 \] \[\left. \right] dy \]

\[+ \frac{N}{8(N+2)} [V(x_h) + \lambda]^{-2} \int_{\mathbb{R}^N} \left[ - \frac{h^4}{3} \sum_{i,j,k,l=1}^{N} \partial_{ijkl} m(x_0) y_i y_j y_k y_l \right] w_{x_h}^{p+1} dy + o(h^4). \quad (4.29)\]

For the sum \( \sum_{j,k=1}^{N} c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} (L_h \partial_j w_{x_h}) dy \), by (4.12) and (4.22), we obtain

\[
\sum_{j,k=1}^{N} c_h^j c_h^k \int_{\mathbb{R}^N} \partial_k w_{x_h} (L_h \partial_j w_{x_h}) dy = - \frac{h^4}{N + 2} \int_{\mathbb{R}^N} w^{p+1} dy \nabla^2 G(x_0) c_0 \cdot c_0 + o(h^4). \quad (4.30)\]
For the integral $\int_{\mathbb{R}^N} R_1(L_h R_1) dy$, by (4.21), $R_1(\frac{y}{\sqrt{V(x_h) + \lambda}})$ satisfies

$$\Delta R - R + pw^{p-1}R + [V(x_h) + \lambda]^{-\frac{N}{2}} \sum_{i,j=1}^{N} \partial_{ij} V(x) y_i y_j w = 0.$$  

Hence

$$\int_{\mathbb{R}^N} R_1(L_h R_1) dy = \int_{\mathbb{R}^N} R_1(L_{x_h} R_1) dy + O(h^2)$$

$$= [V(x_h) + \lambda]^{-\frac{3}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij} V(x) \partial_{kl} V(x) \int_{\mathbb{R}^N} y_i y_j w L_0^{-1}(y_k y_l w) dy$$

$$- [V(x_h) + \lambda]^{-\frac{1}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij} V(x) \partial_{kl} m(x) \int_{\mathbb{R}^N} y_i y_j w L_0^{-1}(y_k y_l w) dy$$

$$+ \frac{1}{4} [V(x_h) + \lambda]^{-\frac{1}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij} m(x) \partial_{kl} m(x) \int_{\mathbb{R}^N} y_i y_j w^p L_0^{-1}(y_k y_l w) dy + O(h^2).$$

As in Section 3, we have used the following identities:

$$\sum_{i,j=1}^{N} \partial_{ij} V(x) \int_{\mathbb{R}^N} y_i y_j w_{x,h}^2 dy = \frac{1}{N} \int_{\mathbb{R}^N} |y|^2 w_{x,h}^2 dy \Delta V(x) = 0,$$

$$\sum_{i,j=1}^{N} \partial_{ij} V(x) \int_{\mathbb{R}^N} y_i y_j w_{x,h} \phi_2 dy$$

$$= \frac{1}{2} [V(x_h) + \lambda]^{-\frac{3}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij} V(x) \partial_{kl} V(x) \int_{\mathbb{R}^N} y_i y_j w L_0^{-1}(y_k y_l w) dy$$

$$- \frac{1}{2} [V(x_h) + \lambda]^{-\frac{1}{2}} \sum_{i,j,k,l=1}^{N} \partial_{ij} V(x) \partial_{kl} m(x) \int_{\mathbb{R}^N} y_i y_j w L_0^{-1}(y_k y_l w) dy,$$

$$\left\{ \begin{array}{l}
\sum_{i,j,k,l=1}^{N} \partial_{ijk} V(x) y_{x,h} \int_{\mathbb{R}^N} y_j y_k y_{x,h} w^2 dy = \frac{1}{N} \int_{\mathbb{R}^N} |y|^2 w_{x,h}^2 dy \nabla(\Delta V)(x_h) \cdot x_1 , \\
\sum_{i,j,k,l=1}^{N} \partial_{ijk} V(x) \int_{\mathbb{R}^N} y_j y_k y_{x,h} w_{x,h}^2 dy = \frac{3}{N(N+2)} \int_{\mathbb{R}^N} |y|^4 w_{x,h}^2 dy \Delta^2 V(x) , \\
\sum_{i,j,k,l=1}^{N} \partial_{ijk} m(x) \int_{\mathbb{R}^N} y_j y_k y_{x,h} w_{x,h}^{p+1} dy = \frac{3}{N(N+2)} \int_{\mathbb{R}^N} |y|^4 w_{x,h}^{p+1} dy \Delta^2 m(x) , \\
\end{array} \right.$$
where $\Phi$ which can be proved as in Appendix B.

Therefore, combining (4.24), (4.29), (4.30) and (4.32), we obtain

$$
\begin{align*}
\int_{\mathbb{R}^N} y_N^2 w L_0^{-1}(y_N^2 w) dy &= \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w L_0^{-1}(r^2 w) dy + \frac{2(N-1)}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w \Phi_1(r) dy, \\
\int_{\mathbb{R}^N} y_{N-1}^2 w L_0^{-1}(y_{N-1}^2 w) dy &= \frac{1}{N(N+2)} \int_{\mathbb{R}^N} r^2 w \Phi_1(r) dy, \\
\int_{\mathbb{R}^N} y_{N-1} y_N w L_0^{-1}(y_{N-1} y_N w) dy &= \frac{1}{N(N+2)} \int_{\mathbb{R}^N} r^2 w \Phi_1(r) dy, \\
\end{align*}
$$

where $\Phi_0, \Phi_1$ satisfy

$$
\begin{align*}
\Phi''_0 + \frac{N-1}{r} \Phi_0 - \Phi_0 + p w^{-1} \Phi_0 - \frac{2N}{r^2} \Phi_0 - r^2 w = 0, & \quad r \in (0, \infty), \\
\Phi_0(0) = \Phi_0'(0) = 0, 
\end{align*}
$$

and

$$
\begin{align*}
\Phi''_1 + \frac{N-1}{r} \Phi_1' - \Phi_0 + p w^{-1} \Phi_1 - \frac{2N}{r^2} \Phi_1 - r^2 w = 0, & \quad r \in (0, \infty), \\
\Phi_1(0) = \Phi_1'(0) = 0, 
\end{align*}
$$

which can be proved as in Appendix B.

Therefore, combining (4.24), (4.29), (4.30) and (4.32), we obtain

$$
d''(\lambda) + o(1) = H_2(x_0) + H_3(x_0) + H_4(x_0) \equiv H(x_0),
$$

where

$$
H_2(x_0) = \frac{3}{N(N+2)} \left[ V(x_0) + \lambda \right]^{-5} m(x_0) - \frac{N}{N+2} \int_{\mathbb{R}^N} |y|^2 w \Phi_1(|y|) dy \| \nabla^2 V(x_0) \|_2^2 \\
- \frac{3}{N(N+2)} \left[ V(x_0) + \lambda \right]^{-4} m(x_0) - \frac{N}{N+2} \int_{\mathbb{R}^N} |y|^2 w \Phi_0(|y|) \nabla^2 V(x_0) \cdot \nabla^2 m(x_0) \\
+ \frac{1}{4N^2} \left[ V(x_0) + \lambda \right]^{-3} m(x_0) - \frac{N}{N+2} \int_{\mathbb{R}^N} |y|^2 w \Phi_0(|y|) \| \nabla^2 V(x_0) \|_2^2 \\
+ \frac{1}{2N(N+2)} \left[ V(x_0) + \lambda \right]^{-3} m(x_0) - \frac{N}{N+2} \int_{\mathbb{R}^N} |y|^2 w \Phi_0(|y|) \| \nabla^2 m(x_0) \|_2^2 \\
- \frac{1}{2N^2(N+2)} \left[ V(x_0) + \lambda \right]^{-3} m(x_0) - \frac{N}{N+2} \int_{\mathbb{R}^N} |y|^2 w \Phi_0(|y|) \| \Delta m(x_0) \|_2^2, \tag{4.34}
$$

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$$H_3(x_0) = \frac{1}{2N} [V(x_0) + \lambda]^{-3} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^2 w^2 dy \nabla (\Delta m)(x_0) \cdot x_1$$

$$- \frac{1}{N+2} \int_{\mathbb{R}^N} w^{p+1} dy c_0 \cdot [\nabla^2 G(x_0)]^{-1} c_0,$$

$$H_4(x_0) = \frac{3}{8N(N+2)} [V(x_0) + \lambda]^{-4} m(x_0)^{-\frac{N}{2}} \int_{\mathbb{R}^N} |y|^4 w^2 dy \Delta^2 V(x_0)$$

$$- \frac{1}{8(N+2)^2} [V(x_0) + \lambda]^{-3} m(x_0)^{-\frac{N}{2}-1} \int_{\mathbb{R}^N} |y|^4 w^{p+1} dy \Delta^2 m(x_0).$$

Consequently, $p(d^n) = 1$ if $H(x_0) > 0$, where $H(x_0)$ defined in (4.33) involves the $i$-th derivatives (for $0 \leq i \leq 4$) of $V$ and $m$ at $x_0$. On the other hand, by Theorem 4.1, we have $n(L_h) = n + 1$. Thus we complete the proof of Theorem 4.4 by the orbital stability and instability criteria of [17-18].

5 Appendix A

In this section, we want to prove (2.29) of Section 2, i.e.

$$\int_{\mathbb{R}^N} (L_h \partial_j w_{xh}) \partial_k w_{xh} dy = \frac{h^2}{p+1} \int_{\mathbb{R}^N} w^{p+1} dy \partial_j m(x_0) + o(h^2). \quad (5.1)$$

Proof. Note that by Lemma 2.3 and 2.4 we obtain

$$L_h \partial_j w_{xh} = [m(hy + x_h) - m(x_h)] p w^{p-1}_{xh} \partial_j w_{xh} + m(hy + x_h) p (v_{xh}^{p-1} - w^{p-1}_{xh}) \partial_j w_{xh}$$

$$= h^2 \frac{1}{p+1} \sum_{i,l=1}^{N} \partial_i m(x_0) y_i y_l p w^{p-1}_{xh} \partial_j w_{xh} + h^2 m(x_h) p (p-1) w^{p-2}_{xh} \phi_2 \partial_j w_{xh} + o(h^2).$$

Hence we may write the integral $\int_{\mathbb{R}^N} (L_h \partial_j w_{xh}) \partial_k w_{xh} dy$ as follows:

$$\int_{\mathbb{R}^N} (L_h \partial_j w_{xh}) \partial_k w_{xh} dy = I_1 + I_2 + o(h^2), \quad (5.2)$$

where

$$I_1 = h^2 \frac{1}{p+1} \sum_{i,l=1}^{N} \partial_i m(x_0) \int_{\mathbb{R}^N} y_i y_l p w^{p-1}_{xh} \partial_j w_{xh} \partial_k w_{xh} dy,$$

$$I_2 = h^2 \int_{\mathbb{R}^N} m(x_h) p (p-1) w^{p-2}_{xh} \phi_2 \partial_j w_{xh} \partial_k w_{xh} dy. \quad (5.4)$$

Note that from (2.3), we have

$$\left[ \Delta - \lambda + m(x_h) p w^{p-1}_{xh} \right] \partial_j k w_{xh} + m(x_h) p (p-1) w^{p-2}_{xh} \partial_j w_{xh} \partial_k w_{xh} = 0. \quad (5.5)$$
Hence by (2.14), (5.4) and (5.5), we may use integration by parts to get

\[
I_2 = -h^2 \int_{\mathbb{R}^N} \phi_2 \left[ \Delta - \lambda + m(x_h)pw_{x_h}^{p-1} \right] \partial_{jk} w_{x_h} dy
\]

\[
= -h^2 \int_{\mathbb{R}^N} \partial_{jk} w_{x_h} \left[ \Delta - \lambda + m(x_h)pw_{x_h}^{p-1} \right] \phi_2 dy
\]

\[
= \frac{h^2}{2} \sum_{i,t=1}^{N} \partial_t m(x_0) \int_{\mathbb{R}^N} y_i y_t w_{x_h} \partial_{jk} w_{x_h} dy
\]

\[
= -\frac{h^2}{2} \sum_{i,t=1}^{N} \partial_t m(x_0) \int_{\mathbb{R}^N} \frac{\partial (y_i y_t w_{x_h})}{\partial y_j} \partial_k w_{x_h} dy
\]

\[
= -\frac{h^2}{2} \sum_{i,t=1}^{N} \partial_t m(x_0) \int_{\mathbb{R}^N} y_i y_t w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy - h^2 \partial_{jk} m(x_0) \int_{\mathbb{R}^N} y_k w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy
\]

\[
= -\frac{h^2}{2} \sum_{i,t=1}^{N} \partial_t m(x_0) \int_{\mathbb{R}^N} y_i y_t w_{x_h}^{p-1} \partial_j w_{x_h} \partial_k w_{x_h} dy + \frac{h^2}{p+1} \partial_{jk} m(x_0) \int_{\mathbb{R}^N} w_{x_h}^{p+1} dy. \tag{5.6}
\]

Combining (5.2), (5.3) and (5.6), we obtain (5.1). \qed

6 Appendix B

In this section, we prove (3.27), (3.28) and (3.29) of Section 3, i.e.

\[
\int_{\mathbb{R}^N} y_i^2 w^p L_0^{-1}(y_N^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy + \frac{2(N-1)}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, \tag{6.1}
\]

\[
\int_{\mathbb{R}^N} y_{N-1}^2 w^p L_0^{-1}(y_N^2 w^p) dy = \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1}(r^2 w^p) dy - \frac{2}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, \tag{6.2}
\]

\[
\int_{\mathbb{R}^N} y_{N-1} y_N w^p L_0^{-1}(y_{N-1} y_N w^p) dy = \frac{1}{N(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy, \tag{6.3}
\]

where \( r := |y| \) and \( \Phi_0 \) satisfies

\[
\begin{cases}
\Phi''_0 + \frac{N-1}{r} \Phi'_0 - \Phi_0 + pw^{p-1} \Phi_0 - \frac{2N}{r^2} \Phi_0 - r^2 w^p = 0, \quad r \in (0, \infty), \\
\Phi_0(0) = \Phi'_0(0) = 0.
\end{cases} \tag{6.4}
\]

Proof. From (6.4), it is easy to check that

\[
L_0 \left[ \Phi_0 \frac{y_N^2}{r^2} + \frac{1}{N} L_0^{-1}(r^2 w^p) - \frac{1}{N} \Phi_0 \right] = y_N^2 w^p, \quad \text{and} \quad L_0 \left[ \Phi_0 \frac{y_{N-1} y_N}{r^2} \right] = y_{N-1} y_N w^p. \tag{6.5}
\]
Then using the polar coordinate, we obtain
\[
\int_{\mathbb{R}^N} y_N^2 w^p L_0^{-1} (y_N^2 w^p) dy \\
= \int_{\mathbb{R}^N} y_N^2 w^p x_0 \left[ \Phi_0(r) \frac{y_N^2}{r^2} - \frac{1}{N} \Phi_0(r) + \frac{1}{N} L_0^{-1} (r^2 w^p) \right] dy \\
= \int_{\mathbb{R}^N} r^2 \cos^2 \theta_{N-1} w^p \left[ \Phi_0(r) \frac{r^2 \cos^2 \theta_{N-1}}{r^2} - \frac{1}{N} \Phi_0(r) + \frac{1}{N} L_0^{-1} (r^2 w^p) \right] dy \\
\int_{0}^{\pi} \cos^4 \theta_{N-1} \sin^{N-2} \theta_{N-1} d\theta_{N-1} \\
= \int_{0}^{\pi} \sin^{N-2} \theta_{N-1} d\theta_{N-1} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy \\
= \frac{3}{N(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy + \frac{1}{N} \int_{\mathbb{R}^N} r^2 w^p \left[ - \frac{1}{N} \Phi_0(r) + \frac{1}{N} L_0^{-1} (r^2 w^p) \right] dy \\
= \frac{1}{N^2} \int_{\mathbb{R}^N} r^2 w^p L_0^{-1} (r^2 w^p) dy + \frac{2(N-1)}{N^2(N+2)} \int_{\mathbb{R}^N} r^2 w^p \Phi_0(r) dy.
\]

This completes the proof of (6.1). Similarly, one may obtain (6.2) and (6.3), respectively.

7 Appendix C

In this section, we prove (4.12) of Section 4, i.e.
\[
\int_{\mathbb{R}^N} \partial_k w_j x_h (\partial_j w_{x_h} + h \psi_j) dy = - \frac{h^2}{N+2} \int_{\mathbb{R}^N} w^{p+1} dy \partial_k G(x_0) + o(h^2). \quad (7.1)
\]
Proof. Note that by (4.3), (4.4) and (4.10), we obtain

\[
L_h \partial_j w_{xh} = L_{xh} \partial_j w_{xh} + \left[ m(hy + x_h) - m(x_h) \right] pw_{xh}^p \partial_j w_{xh} + m(hy + x_h)p(v_{xh}^p - w_{xh}^p) \partial_j w_{xh} - \left[ V(hy + x_h) - V(x_h) \right] \partial_j w_{xh} = \]

\[
= h \left[ y \cdot \nabla m(x_h)pw_{xh}^p + m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] \partial_j w_{xh} + h^2 \left[ \frac{1}{2} \sum_{i,l} \partial_i m(x_h)y_iy_lpw_{xh}^p + y \cdot \nabla m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 + m(x_h)p(p - 1)w_{xh}^{p-2} \phi_2 \right] + \frac{1}{2} m(x_h)p(p - 1)(p - 2)w_{xh}^{p-3} \phi_1 - \frac{1}{2} \sum_{i,l} \partial_i V(x_h)y_iy_l \partial_j w_{xh} + o(h^2),
\]

and

\[
L_h \psi_j = L_{xh} \psi_j + \left[ m(hy + x_h) - m(x_h) \right] pw_{xh}^p \psi_j + m(hy + x_h)p(v_{xh}^p - w_{xh}^p) \psi_j - \left[ V(hy + x_h) - V(x_h) \right] \psi_j = \]

\[
= - \left[ y \cdot \nabla m(x_h)pw_{xh}^p + m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] \partial_j w_{xh} + h \left[ y \cdot \nabla m(x_h)pw_{xh}^p + m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] \psi_j + O(h^2).
\]

Hence we may write the integral \( \int_{\mathbb{R}^N} \partial_k w_{xh} L_h (\partial_j w_{xh} + h \psi_j) dy \) as follows:

\[
\int_{\mathbb{R}^N} \partial_k w_{xh} L_h (\partial_j w_{xh} + h \psi_j) dy = I_0 + I_1 + I_2 + o(h^2), \quad (7.2)
\]

where

\[
I_0 = h^2 \int_{\mathbb{R}^N} \left[ y \cdot \nabla m(x_h)pw_{xh}^p + m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] \psi_j \partial_k w_{xh} dy, \quad (7.3)
\]

\[
I_1 = h^2 \int_{\mathbb{R}^N} \left[ \frac{1}{2} \sum_{i,l} \partial_i m(x_h)y_iy_lpw_{xh}^p + y \cdot \nabla m(x_h)p(p - 1)w_{xh}^{p-2} \phi_1 \right] + \frac{1}{2} m(x_h)p(p - 1)(p - 2)w_{xh}^{p-3} \phi_1 - \frac{1}{2} \sum_{i,l} \partial_i V(x_h)y_iy_l \partial_j w_{xh} \partial_k w_{xh} dy, \quad (7.4)
\]

\[
I_2 = h^2 \int_{\mathbb{R}^N} m(x_h)p(p - 1)w_{xh}^{p-2} \phi_2 \partial_j w_{xh} \partial_k w_{xh} dy. \quad (7.5)
\]

Note that from [23], we have

\[
\left[ \Delta - (V(x_h) + \lambda) + m(x_h)pw_{xh}^p \right] \partial_{jk} w_{xh} + m(x_h)p(p - 1)w_{xh}^{p-2} \partial_j w_{xh} \partial_k w_{xh} = 0. \quad (7.6)
\]
Hence by (4.6), (7.4) and (7.5), we may use integration by parts to get

\[ I_2 = -h^2 \int_{\mathbb{R}^N} \phi_2 \left[ \Delta - (V(x_h) + \lambda) + m(x_h)pw_{x_h}^{p-1} \right] \partial_j w_{x_h} dy \]

\[ = -h^2 \int_{\mathbb{R}^N} \partial_j w_{x_h} \left[ \Delta - (V(x_h) + \lambda) + m(x_h)pw_{x_h}^{p-1} \right] \phi_2 dy \]

\[ = h^2 \int_{\mathbb{R}^N} \left[ -y \cdot \nabla V(x_h) \phi_1 - \frac{1}{2} \sum_{i=1}^N \partial_i V(x_h) y_i y_i w_{x_h} + y \cdot \nabla m(x_h)pw_{x_h}^{p-1} \phi_1 \right. \]

\[ + \frac{1}{2} \sum_{i=1}^N \partial_i m(x_h) y_i y_i w_{x_h}^{p} + \frac{1}{2} m(x_h)p(p-1)w_{x_h}^{p-2} \phi_1^2 \]

\[ \left. \partial_j w_{x_h} \right] dy \]

\[ = h^2 \int_{\mathbb{R}^N} \left[ \partial_j V(x_h) \phi_1 + y \cdot \nabla V(x_h) \partial_j \phi_1 + \frac{1}{2} \sum_{i=1}^N \partial_i V(x_h) y_i y_i \partial_j w_{x_h} + \partial_jk V(x_h) y_k w_{x_h} \right. \]

\[ - \partial_j m(x_h)pw_{x_h}^{p-1} \phi_1 - y \cdot \nabla m(x_h)p(p-1)w_{x_h}^{p-2} \phi_1 \partial_j w_{x_h} - y \cdot \nabla m(x_h)pw_{x_h}^{p-1} \partial_j \phi_1 \]

\[ - \frac{1}{2} \sum_{i=1}^N \partial_i m(x_h) y_i y_i pw_{x_h}^{p-1} \partial_j w_{x_h} - \partial_jk m(x_h) y_k w_{x_h}^p \]

\[ - \frac{1}{2} m(x_h)p(p-1)(p-2)w_{x_h}^{p-3} \phi_1^2 \partial_j w_{x_h} - m(x_h)p(p-1)w_{x_h}^{p-2} \phi_1 \partial_j \phi_1 \]

\[ \left. \partial_k w_{x_h} \right] dy \]

\[ = -I_1 - h^2 \int_{\mathbb{R}^N} \left[ y \cdot \nabla m(x_h)pw_{x_h}^{p-1} + m(x_h)p(p-1)w_{x_h}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] \partial_j \phi_1 \partial_k w_{x_h} dy \]

\[ + h^2 \int_{\mathbb{R}^N} \left[ \partial_j V(x_h) \phi_1 + \partial_jk V(x_h) y_k w_{x_h} - \partial_j m(x_h)pw_{x_h}^{p-1} \phi_1 - \partial_jk m(x_h) y_k w_{x_h}^p \right] \partial_k w_{x_h} dy . \]

(7.7)

Note that from (4.5), we have

\[ \Delta(\partial_j \phi_1) - [V(x_0) + \lambda] \partial_j \phi_1 + m(x_0)pw_{x_0}^{p-1} \partial_j \phi_1 + m(x_0)p(p-1)w_{x_0}^{p-2} \phi_1 \partial_j w_{x_0} \]

\[ - y \cdot \nabla V(x_0) \partial_j w_{x_0} - \partial_j V(x_0) w_{x_0} + y \cdot \nabla m(x_0)pw_{x_0}^{p-1} \partial_j w_{x_0} + \partial_j m(x_0)w_{x_0}^p = 0 , \]

(7.8)

and by direct computation,

\[ \left\{ \begin{array}{l}
 L_{x_0} w_{x_0} = (p-1)m(x_0)w_{x_0}^p , \\
 L_{x_0}(\frac{1}{p-1}w_{x_0} + \frac{1}{2} y \cdot \nabla w_{x_0}) = [V(x_0) + \lambda] w_{x_0} .
 \end{array} \right. \]

(7.9)
Recall that

\[ \partial_{ij} G(x_0) = m(x_0) \cdot \frac{N}{2 - p - 1} \left[ m(x_0) \partial_{ij} V(x_0) + (1 - \frac{N}{2}) \partial_i V(x_0) \partial_j m(x_0) - \frac{N}{2} [V(x_0) + \lambda \partial_{ij} m(x_0)] \right], \]

\[
\begin{align*}
\{ & w_{x_0}(y) = [V(x_0) + \lambda]^{p-1} m(x_0)^{-\frac{p-1}{2}} w(\sqrt{V(x_0)} + \lambda y), \\
& m(x_0) \nabla V(x_0) = \frac{N}{2} [V(x_0) + \lambda] \nabla m(x_0), \\
& \partial_{ij} G(x_0) = m(x_0)^{-\frac{N}{2-1}} \left[ m(x_0) \partial_{ij} V(x_0) + (1 - \frac{N}{2}) \partial_i V(x_0) \partial_j m(x_0) - \frac{N}{2} [V(x_0) + \lambda \partial_{ij} m(x_0)] \right].
\end{align*}
\]

Thus we may use \( \{7.3\}-\{7.9\} \) and integration by parts to get

\[
I_0 + I_1 + I_2 = h^2 \int_{R^N} \left[ y \cdot \nabla m(x_h) w_{x_h}^{p-1} + m(x_h) p(p-1) w_{x_h}^{p-2} \phi_1 - y \cdot \nabla V(x_h) \right] (\psi_j - \partial_j \phi_1) \partial_k w_{x_h} dy \\
+ h^2 \int_{R^N} \left[ \partial_j V(x_h) \phi_1 + \partial_{jk} V(x_h) y_k w_{x_h} - \partial_j m(x_h) w_{x_h}^{p-1} \phi_1 - \partial_{jk} m(x_h) w_{x_h}^{p} \right] \partial_k w_{x_h} dy \\
= - h^2 \int_{R^N} (\psi_j - \partial_j \phi_1) L_{x_h} \psi_k dy + h^2 \int_{R^N} \left[ \partial_j V(x_h) - \partial_j m(x_h) w_{x_h}^{p-1} \right] \phi_1 \partial_k w_{x_h} dy \\
+ h^2 \int_{R^N} \left[ \partial_{jk} V(x_h) y_k w_{x_h} - \partial_{jk} m(x_h) w_{x_h}^{p} \right] \partial_k w_{x_h} dy \\
= h^2 \int_{R^N} \left[ \partial_j V(x_0) w_{x_0} - \partial_j m(x_0) w_{x_0}^{p} \right] \psi_k dy - h^2 \int_{R^N} \left[ \partial_j V(x_h) w_{x_h} - \partial_j m(x_h) w_{x_h}^{p} \right] \partial_k \phi_1 dy \\
- h^2 \int_{R^N} \left[ \frac{1}{2} \partial_{jk} V(x_h) w_{x_h}^{2} - \frac{1}{p+1} \partial_{jk} m(x_h) w_{x_h}^{p+1} \right] dy + o(h^2) \\
= h^2 \int_{R^N} \left[ \partial_j V(x_0) (V(x_0) + \lambda)^{-1} \left( \frac{1}{p-1} w_{x_0} + \frac{1}{2} y \cdot \nabla w_{x_0} \right) \\
- \frac{1}{p-1} m(x_0)^{-1} \partial_j m(x_0) w_{x_0} \right] L_{x_0} \left( \psi_k - \partial_k \phi_1 \right) dy \\
- h^2 \int_{R^N} \left[ \frac{1}{2} \partial_{jk} V(x_0) w_{x_0}^{2} - \frac{1}{p+1} \partial_{jk} m(x_0) w_{x_0}^{p+1} \right] dy + o(h^2) \\
= h^2 \int_{R^N} \left[ \partial_j V(x_0) (V(x_0) + \lambda)^{-1} \left( \frac{1}{p-1} w_{x_0} + \frac{1}{2} y \cdot \nabla w_{x_0} \right) \\
- \frac{1}{p-1} m(x_0)^{-1} \partial_j m(x_0) w_{x_0} \right] \partial_k V(x_0) w_{x_0} - \partial_k m(x_0) w_{x_0}^{p} \right] dy. \\
- h^2 \int_{R^N} \left[ \frac{1}{2} \partial_{jk} V(x_0) w_{x_0}^{2} - \frac{1}{p+1} \partial_{jk} m(x_0) w_{x_0}^{p+1} \right] dy + o(h^2) \\
= - h^2 \left[ \left( \frac{1}{p-1} - \frac{N}{4} \right) [V(x_0) + \lambda]^{-1} \partial_j V(x_0) \partial_k V(x_0) \\
- \frac{1}{p-1} m(x_0)^{-1} \partial_j m(x_0) \partial_k V(x_0) \right] \int_{R^N} w_{x_0}^{2} dy \\
+ h^2 \left[ \left( \frac{1}{p-1} - \frac{N}{2p+1} \right) [V(x_0) + \lambda]^{-1} \partial_j V(x_0) \partial_k m(x_0) \\
- \frac{1}{p-1} m(x_0)^{-1} \partial_j m(x_0) \partial_k m(x_0) \right] \int_{R^N} w_{x_0}^{p+1} dy \\
- h^2 \int_{R^N} \left[ \frac{1}{2} \partial_{jk} V(x_0) w_{x_0}^{2} - \frac{1}{p+1} \partial_{jk} m(x_0) w_{x_0}^{p+1} \right] dy + o(h^2) \right). \tag{7.10}
and the integral identity

\[
V(x_0) + \lambda \int_{\mathbb{R}^N} w^2 \, dy = \frac{2}{N + 2} m(x_0) \int_{\mathbb{R}^N} w^{p+1} \, dy.
\]

Combining (7.2) and (7.10), we obtain (7.1).  

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