Dirac structures on the space of connections

Yuji HIROT A*  
Azabu University

Tosiaki KORI†
Waseda University

Abstract

We shall give a twisted Dirac structure on the space of irreducible connections on a $SU(n)$-bundle over a three-manifold, and give a family of twisted Dirac structures on the space of irreducible connections on the trivial $SU(n)$-bundle over a four-manifold. The twist is described by the Cartan 3-form on the space of connections. It vanishes over the subspace of flat connections. So the spaces of flat connections are endowed with (non-twisted) Dirac structures. The Dirac structure on the space of flat connections over the three-manifold is obtained as the boundary restriction of a corresponding Dirac structure over the four-manifold. We discuss also the action of the group of gauge transformations over these Dirac structures.

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1 Introduction

Let $X$ be a four-manifold with the boundary three-manifold $M$. Let $\mathcal{A}_X$ and $\mathcal{A}_M$ be the spaces of irreducible connections on the principal bundles $X \times SU(n)$ and $M \times SU(n)$ respectively. In physics language these are spaces of gauge fields. We shall investigate twisted Dirac structures on the spaces $\mathcal{A}_X$ and $\mathcal{A}_M$. These twisted Dirac structures are affected by the presence of closed 3-forms. Twisted Poisson structures arose from the study of topological sigma models and play an important role in string theory [5, 7]. P. Ševera and A. Weinstein, [9], investigated how Poisson geometry on a manifold is affected by the presence of a closed 3-form, and they found that the notions of Courant algebroids and Dirac structures provide a framework where one can carry out computations in Poisson geometry in the presence of background 3-forms.

In [6] one of the author proved that there is a pre-symplectic structure on $\mathcal{A}_X$ that is induced from the canonical symplectic structure on the cotangent bundle $T^*\mathcal{A}_X$ by a generating function $CS : \mathcal{A}_X \to T^*\mathcal{A}_X$ which is given by the Chern-Simons form. Let $\omega$ be the boundary reduction of this pre-symplectic form to $\mathcal{A}_M$. $\omega$ is no longer pre-symplectic but twisted by the Cartan 3-form on $\mathcal{A}_M$. Associated to the 2-form $\omega$, we have the correspondence:

$$\omega_A : T_A\mathcal{A}_M \ni a \mapsto \omega_A(a, \cdot) \in T^*_A\mathcal{A}_M,$$

*hirota@azabu-u.ac.jp
†kori@waseda.jp
at each \( A \in \mathcal{A}_M \). Then we have the subbundle 
\[
\mathcal{D}_M = \{ a \oplus \omega_A(a) \in T\mathcal{A}_X \oplus T^*\mathcal{A}_X \mid a \in T_A\mathcal{A}_M, A \in \mathcal{A}_M \}.
\]

\( \mathcal{D}_M \) gives a twisted Dirac structure of the standard Courant algebroid \( E_0(M) = T\mathcal{A}_M \oplus T^*\mathcal{A}_M \). The 3-form \( \kappa = d\omega \) which describes the twist is given by the Cartan 3-form on \( \mathcal{A}_M \):
\[
\kappa_A(a, b, c) = \frac{1}{8\pi^3} \int_M \text{tr} [abc - bac], \quad a, b, c \in T_A\mathcal{A}_M, \ A \in \mathcal{A}_M,
\]
where \( abc = a \wedge b \wedge c \) etc.

On the space of connections \( \mathcal{A}_X \) over a four-manifold \( X \) we consider the correspondence
\[
\phi_A : T_A\mathcal{A}_X \ni a \mapsto F_Aa + aF_A \in T_A^*\mathcal{A}_X, \quad A \in \mathcal{A}_X,
\]
where \( F_A \) is the curvature of \( A \). Then we have the following subbundle of the standard Courant algebroid \( E_0(X) = T\mathcal{A}_X \oplus T^*\mathcal{A}_X \):
\[
\mathcal{D}_X^\phi = \{ a \oplus \phi_A(a) \in E_0(X) \mid a \in T_A\mathcal{A}_X, A \in \mathcal{A}_X \}.
\]

\( \mathcal{D}_X^\phi \) gives a twisted Dirac subbundle of \( E_0(X) \). The twist in this case is given by the following 3-form \( \kappa \) on \( \mathcal{A}_X \):
\[
\kappa_A(a, b, c) = \kappa(\tilde{a}, \tilde{b}, \tilde{c}), \quad a, b, c \in T_A\mathcal{A}_X, \ A \in \mathcal{A}_X,
\]
where \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \) indicate the restriction of \( a \), \( b \) and \( c \) respectively to the boundary \( M \), and the right-hand side is the 3-form \( \kappa \) on \( \mathcal{A}_M \) described above.

Moreover if we deal with the correspondence from \( T_A\mathcal{A}_X \) to \( T_A^*\mathcal{A}_X \) given by
\[
\gamma'_A(a) = (F_A + tA^2) a + a(F_A + tA^2), \quad t \in \mathbb{R},
\]
then we have the \( \kappa \)-twisted Dirac structure
\[
\mathcal{D}_X^\kappa = \{ a \oplus \gamma'_A(a) \in E_0(X) \mid a \in T_A\mathcal{A}_X, A \in \mathcal{A}_X \}.
\]

As for the pre-symplectic structure on the space of connections \( \mathcal{A}_X \) we have a family of closed 2-forms on \( \mathcal{A}_X \) given by
\[
\Omega^i_A(a, b) = \frac{1}{24\pi^3} \int_X \text{tr} \left[ (ab - ba) \left( 3F_A - (t - 1)A^2 \right) \right] - \frac{1}{24\pi^3} \int_M \text{tr} \left[ (ab - ba)A \right],
\]
for \( a, b \in T_A\mathcal{A}_X \). \( \Omega^i \) is the pre-symplectic form discussed in [6].

If we restrict ourselves to the spaces of flat connections we will have (non-twisted) Dirac structures. In fact we have the following Dirac structures
\[
\mathcal{D}_M^\phi = \{ a \oplus \omega_A(a) \in E_0(M) \mid a \in T_A\mathcal{A}_M^0, A \in \mathcal{A}_M^0 \},
\]
\[
\mathcal{D}_X^\phi = \{ a \oplus \gamma'_A(a) \in E_0(X) \mid a \in T_A\mathcal{A}_X^0, A \in \mathcal{A}_X^0 \},
\]
where \( \mathcal{A}_M^0 \) and \( \mathcal{A}_X^0 \) are flat connections in \( \mathcal{A}_M \) and \( \mathcal{A}_X \) respectively. By the boundary restriction from \( X \) to \( M \) we have the correspondence \( r : \mathcal{A}_X^0 \to \mathcal{A}_M^0 \) and we find that the image of \( r \) consists of those flat connections \( A \in \mathcal{A}_M^0 \) with degree 0: \( \deg A = \int_M \text{tr} [A^1] = 0 \). Then \( \mathcal{D}_M^{\phi, \deg 0} = \{ a \oplus \omega_A(a) \in \mathcal{D}_M^\phi \mid \deg A = 0 \} \) is a Dirac substructure of \( \mathcal{D}_M^\phi \) and there is an isomorphism of Dirac structures between \( \mathcal{D}_X^\phi \) and \( \mathcal{D}_M^{\phi, \deg 0} \) induced by the restriction to the boundary.
2 Preliminaries

2.1 Differential calculus on the space of connections

Let $M$ be a compact, connected and oriented $m$-dimensional Riemannian manifold possibly with boundary $\partial M$. Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$. We shall denote by $\Gamma(M, E)$ the space of smooth sections of a smooth vector bundle $E \to M$. Especially, if $E = TM$ we write $\text{Vect}(M)$ for $\Gamma(M, TM)$, and if $E = \wedge^k T^* M$ we write $\Omega^k(M)$ for $\Gamma(M, \wedge^k T^* M)$. $\Omega^2(M, E)$ denotes the space $\Gamma(M, \wedge^2 T^* M \otimes E)$ of $E$-valued $k$-forms on $M$.

Let $P$ be a principal $G$-bundle on $M$. A connection 1-form on $P$ is a $\mathfrak{g}$-valued 1-form on $P$ that is invariant under $G$, acting by a combination of the action on $P$ and the adjoint action on $\mathfrak{g}$. Let $\mathcal{A}_M$ be the space of irreducible connections on $P$. The space $\mathcal{A}_M$ is an affine space modeled on the vector space $\Omega^1(M, \text{ad} P)$; the space of 1-forms with values in the adjoint bundle $\text{ad} P$ over $M$. For the trivial principal bundle $P = M \times G$ it is merely the space of $\mathfrak{g}$-valued differential 1-forms on $M$. So the tangent space at $A \in \mathcal{A}_M$ is

$$T_A \mathcal{A}_M = \Omega^1(M, \mathfrak{g}).$$

The cotangent space at $A \in \mathcal{A}_M$ is

$$T^*_A \mathcal{A}_M = \Omega^{m-1}(M, \mathfrak{g}).$$

The dual pairing of $T_A \mathcal{A}_M = \Omega^1(M, \mathfrak{g})$ and $T^*_A \mathcal{A}_M = \Omega^{m-1}(M, \mathfrak{g})$ is given by

$$\langle \alpha, \alpha \rangle_A = \int_M \text{tr} (\alpha \wedge \alpha), \quad \alpha \in T_A \mathcal{A}_M, \quad \alpha \in T^*_A \mathcal{A}_M.$$  

For a function $H = H(A)$ on $\mathcal{A}_M$ with values in a vector space $V$, the directional derivative $(\partial_a H)(A)$ at $A \in \mathcal{A}_M$ to the direction $a \in T_A \mathcal{A}_M$ is defined by

$$(\partial_a H)(A) := \lim_{t \to 0} \frac{1}{t} [H(A + ta) - H(A)].$$

For example, the directional derivative of the identity map $\text{Id} : A \mapsto A$ on $\mathcal{A}_M$ is $(\partial_a \text{Id})(A) = a$. The curvature 2-form of $A \in \mathcal{A}_M$: $F_A = dA + \frac{1}{2}[A \wedge A] = dA + A \wedge A$ is viewed as a function $F$ over $\mathcal{A}_M$ with values in $\Omega^2(M, \mathfrak{g})$. Since $F_{A+tA} - F_A = t (da + a \wedge A + A \wedge a) + t^2 a \wedge a$ we have $(\partial_a F)(A) = da + a \wedge A + A \wedge a$.

Let $V$ be a vector field over $\mathcal{A}_M$. The directional derivative $\partial_a V$ of $V$ to the direction $a$ at $A \in \mathcal{A}_M$ is defined by taking the directional derivative of the coefficients of $V$. Namely, if $V$ is written locally as $V = \sum_i \xi_i \partial_i$ with coefficients $\xi_i$, the directional derivative $\partial_a V$ to the direction $a \in T_A \mathcal{A}_M$ is given by

$$(\partial_a V)(A) := \sum_i (\partial_a \xi_i)(A) \partial_i.$$  

Then the Lie bracket of vector fields $V$ and $W$ on $\mathcal{A}_M$ is given by

$$[V, W](A) = (\partial_{V(A)} W)(A) - (\partial_{W(A)} V)(A), \quad A \in \mathcal{A}_M. \quad (2.1)$$

Let $\theta$ be a $k$-form ($k \geq 1$) on the connection space $\mathcal{A}_M$ and $X$ a vector field on $\mathcal{A}_M$. The directional derivative $\partial_X \theta$ of $\theta$ to the direction of the vector field $X$ is a $k$-form that is obtained by the directional derivative of the component functions of $\theta$ to the direction $X(A)$ at each
Let \( \phi \in \mathcal{A}_M \). That is, if \( \phi \) is written locally in the form \( \phi = \sum_i f_i \varepsilon_i \) with component functions \( f_i \) and local frames \( \{ \varepsilon_i \} \) of cotangent bundle \( T^* \mathcal{A}_M \), \( \partial_X \phi \) is given by

\[
(\partial_X \phi)_A := \sum_i (\partial_X(\varepsilon_i)) f_i(A) \varepsilon_i.
\]

Let \( \langle \theta \mid V \rangle \) denote the evaluation of a 1-form \( \theta \) and a vector field \( V \). Then it holds that \( \partial_X(\theta \mid V) = \langle \partial_X \theta \mid V \rangle + \langle \theta \mid \partial_X V \rangle \). \hfill (2.2)

The exterior derivative of a \( k \)-form \( \theta \) on \( \mathcal{A}_M \) is the \( (k + 1) \)-form \( \tilde{\dd} \theta \) that is given by

\[
(\tilde{\dd} \theta)_A(V_1(A), \ldots, V_{k+1}(A)) := \sum_{i=1}^{k+1} (-1)^{i+1}(\partial_{V_i} \theta(V_1, \ldots, \hat{V}_i, \ldots, V_{k+1}))(A)
+ \sum_{i<j} (-1)^{i+j} \theta_A([V_i, V_j](A), V_1(A), \ldots, \hat{V}_i(A), \ldots, V_j(A), \ldots, V_{k+1}(A)),
\]

for any vector fields \( V_1, \ldots, V_{k+1} \) on \( \mathcal{A}_M \). It can be shown that \( \tilde{\dd} \circ \tilde{\dd} = 0 \). In particular the exterior derivative of a 1-form \( \theta \) becomes

\[
(\tilde{\dd} \theta)(V_1, V_2) = \langle \partial_{V_1} \theta \mid V_2 \rangle - \langle \partial_{V_2} \theta \mid V_1 \rangle \hfill (2.3)
\]

by (2.1) and (2.2). The exterior derivative \( \tilde{\dd} \varphi \) of a 2-form \( \varphi \) is given by

\[
(\tilde{\dd} \varphi)_A(V_1, V_2, V_3) = \langle \partial_{V_1} \varphi(V_2, V_3) \rangle + \langle \partial_{V_2} \varphi(V_3, V_1) \rangle + \langle \partial_{V_3} \varphi(V_1, V_2) \rangle. \hfill (2.4)
\]

The Lie derivative is also defined by the same manner as in the case of finite dimensional smooth manifolds. Let \( \phi \) be a \( k \)-form and \( \mathbf{V} \) be a vector field on \( \mathcal{A}_M \), the Lie derivative \( \mathcal{L}_V \phi \) of \( \phi \) by \( \mathbf{V} \) is a \( k \)-form on \( \mathcal{A}_M \) defined by

\[
(\mathcal{L}_V \phi)_A(V_1(A), \ldots, V_k(A)) := \langle \partial_{V} \phi(V_1, \ldots, V_k) \rangle(A)
- \sum_i \phi([V_i, V_j](A), V_1(A), \ldots, \hat{V}_i(A), \ldots, V_j(A) \ldots, V_k(A)).
\]

Especially, for \( k = 1 \), we have

\[
(\mathcal{L}_V \phi)(\mathbf{W}) = \partial_V(\theta \mid \mathbf{W}) - \langle \theta \mid [V, \mathbf{W}] \rangle. \hfill (2.5)
\]

For further details of differential calculus on Banach space we refer the readers to [2, 4, 10].

### 2.2 Courant algebroids and Dirac structures

The notions of Courant algebroid and Dirac structure are developed in many forms since T. Courant’s work in 1990 [3]. P. Ševera and A. Weinstein showed that Courant algebroid and Dirac structure provide a framework to carry out computations in Poisson geometry in the presence of a background 3-form. Poisson structures on a manifold \( M \) may be identified with certain Dirac structures in the standard Courant algebroid \( E_0 = TM \oplus T^* M \), and a closed 3-form \( \phi \) on \( M \) may be used to modify the bracket on \( E_0 \), yielding a new Courant algebroid \( E_0 \). Here we shall give a explanatory introduction of these subjects after [8, 9].
Definition 2.1. A Courant algebroid over a manifold \( M \) is a vector bundle \( E \rightarrow M \) equipped with a field of nondegenerate symmetric bilinear forms \((\cdot, \cdot)\) on the fibers, an \( \mathbb{R}\)-bilinear bracket \([\cdot, \cdot]\) : \( \Gamma(M, E) \times \Gamma(M, E) \rightarrow \Gamma(M, E) \) on the space of sections on \( E \), and a bundle map \( \rho : E \rightarrow TM \); the anchor, such that the following properties are satisfied:

1. for any \( e_1, e_2, e_3 \in \Gamma(M, E) \), \([e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]\);
2. for any \( e_1, e_2 \in \Gamma(M, E) \), \( \rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]\);
3. for any \( e_1, e_2 \in \Gamma(M, E) \) and \( f \in C^\infty(M) \), \([e_1, fe_2] = fe_1 + (\rho(e_1)f)e_2\);
4. for any \( e, h_1, h_2 \in \Gamma(M, E) \), \( \rho(e)(h_1, h_2) = ([e, h_1], h_2) + (h_1, [e, h_2])\);
5. for any \( e \in \Gamma(M, E) \), \([e, e] = \mathbb{D}(e, e)\),

where \( \mathbb{D} = \frac{1}{2} \beta^{-1} \beta \cdot \mathbb{D} \) and \( \beta \) is the isomorphism between \( E \) and \( E^* \) given by the bilinear form: \((\beta x)(y) = (x, y)\). That is, \( (\mathbb{D} f, e) = \frac{1}{2} \beta (\rho e)f\).

The assertion 5 says that there is a linear map \( Z : \Gamma(E) \ni e \mapsto Z_e \in \text{Vect}(E) \) such that \( Z_e \) is a lift of \( \rho(e) \in \text{Vect}(M) \), and the first four axioms say that the flow of \( Z_e \) preserves the structure of \( E \). The bracket \([e_1, e_2]\) is the Lie derivative of \( e_2 \) by \( Z_{e_1}\).

Definition 2.2. A Dirac structure in \( E \) is a maximal isotropic subbundle \( \mathcal{D} \) of \( E \) whose sections are closed under the bracket, i.e., which is preserved by the flow of \( Z_e \) for every \( e \in \Gamma(\mathcal{D}) \).

The restriction of the bracket and anchor to any Dirac structure \( \mathcal{D} \) forms a Lie algebroid structure on \( \mathcal{D} \).

On any manifold we have the standard Courant algebroid \( E_0 = TM \oplus T^*M \) with bilinear form \( (X_1 \oplus \xi_1, X_2 \oplus \xi_2) = \frac{1}{2} \left[ \xi_1(X_2) + \xi_2(X_1) \right] \), the anchor \( \rho(X \oplus \xi) = X \) and the bracket

\[ [X_1 \oplus \xi_1, X_2 \oplus \xi_2] = [X_1, X_2] \oplus (\mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1). \]

(2.6)

Now let \( \phi \) be a 3-form on \( M \). We define a new bracket on \( E_0 \) by adding the term \( \phi(X_1, X_2, \cdot) \) to the right-hand side of (2.6):

\[ [X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{\phi} = [X_1, X_2] \oplus (\mathcal{L}_{X_1}\xi_2 - i_{X_2}d\xi_1 - i_{X_1\wedge X_2}\phi). \]

(2.7)

A simple computation shows that the new bracket together with the original bilinear form and anchor constitute a Courant algebroid structure on \( E_0 = TM \oplus T^*M \) if and only if \( d\phi = 0 \). We denote this modified Courant algebroid by \( E_{\phi} \). A maximal isotropic subbundle \( \mathcal{D} \) of \( E_{\phi} \) whose sections are closed under the bracket \([\cdot, \cdot]_{\phi}\) is called a \( \phi \)-twisted Dirac structure.

We endow the following canonical skew symmetric form on the standard Courant algebroid \( E_0 = TM \oplus T^*M \):

\[ \Lambda(a \oplus \alpha \mid b \oplus \beta) := \frac{1}{2} \{ \langle \alpha \mid b \rangle - \langle \beta \mid a \rangle \}, \]

(2.8)

for \( a \oplus \alpha, b \oplus \beta \in E_0 \).

3 Dirac structures on the space of connections

We shall introduce several Dirac structures on the space of connections over the manifolds of dimension 3 and 4. First we give an explanation about the related pre-symplectic structures.
3.1 Pre-symplectic structures on the space of connections  

Let \( X \) be a four-manifold with the boundary three-manifold \( M \). Let \( \mathcal{A}_X \) and \( \mathcal{A}_M \) be the spaces of irreducible connections on the principal bundles \( X \times SU(n) \) and \( M \times SU(n) \) respectively. The symplectic structure on the space of connections over a Riemann surface was introduced in 1983 by M. Atiyah and L. Bott in their study of the geometry and topology of moduli spaces of gauge fields (see [1]). In [6], we introduced a pre-symplectic structure on the space \( \mathcal{A}_X \) of irreducible connections over a four-manifold \( X \). That was given by the 2-form:

\[
\Omega_A(a, b) = \frac{1}{8\pi^2} \int_X \text{tr} [(ab - ba)F_A] - \frac{1}{24\pi^2} \int_M \text{tr} [(ab - ba)A] 
\]  

(3.1)

for \( a, b \in T_\mathcal{A}_X = \Omega^1(X, g) \). We abbreviate often the exterior product of differential forms \( a \wedge b \) to \( ab \). Let \( \theta \) be the the canonical 1-form on the cotangent bundle \( T^*\mathcal{A}_X \), and let \( \sigma = \tilde{d}\theta \) be the canonical 2-form. A 1-form \( \varphi \) on \( \mathcal{A}_X \) gives a tautological section of the cotangent bundle \( T^*\mathcal{A}_X \) so that the pullback \( \theta^\varphi \) of \( \theta \) by \( \varphi \) becomes \( \varphi \) itself: \( \theta^\varphi = \varphi \), this is the characteristic property of the canonical 1-form \( \theta \). The pullback \( \sigma^\varphi \) of the canonical 2-form \( \sigma \) by \( \varphi \) is a closed 2-form on \( \mathcal{A}_X \). In particular if we take the 1-form on \( \mathcal{A}_X \) given by

\[
CS(A) = q \left( AF_A + F_A A - \frac{1}{2} A^2 \right), 
\]  

(3.2)

where \( q = \frac{1}{2\pi^2} \), then we see that the pullback \( \Omega = \sigma^{CS} \) is given by the equation (3.1) (the notation \( CS \) comes from the Chern-Simons function). Thus, for a four-manifold \( X \) there is a pre-symplectic form on \( \mathcal{A}_X \) that is induced from the canonical symplectic form on the cotangent bundle \( T^*\mathcal{A}_X \) by the generating function \( CS : \mathcal{A}_X \to T^*\mathcal{A}_X \). The quantity \( \int_M \text{tr}[A^3] \) for a connection \( A \in \mathcal{A}_X \) plays an analogous role of winding number. In fact, when \( A \) is a pure gauge: \( A = f^{-1} \tilde{d}f \), it is equal to the degree of \( f \). We shall change the ratio of counting this number and look for the family of pre-symplectic structures affected by it.

Put

\[
\Theta_t^\varphi(a) = q \int_X \text{tr} \left[ (AF_A + F_A A - t A^2) a \right], \quad (t \in \mathbb{R}).
\]

By the same observation as above the pullback of the canonical 2-form \( \sigma \) on the cotangent space \( T^*\mathcal{A}_X \) by the generating function \( \Theta_t \) provides a pre-symplectic structure \( \Omega_t^\varphi \) on \( \mathcal{A}_X \). It holds that

\[
\Omega_t^\varphi(a, b) = (\tilde{d}\Theta_t^\varphi)_\Lambda(a, b) = \langle (\tilde{d}_a \Theta_t^\varphi)_\Lambda \mid b \rangle - \langle (\tilde{d}_b \Theta_t^\varphi)_\Lambda \mid a \rangle 
\]

\[
= q \int_X \text{tr} \left[ 2(ab - ba) F_A - t (ab - ba) A^2 
\right.
\]

\[
- (d_A a \wedge b - a \wedge d_A b - d_A b \wedge a + b \wedge d_A a) \wedge A \right] 
\]

\[
= q \int_X \text{tr} \left[ 3(ab - ba) F_A - (t - 1)(ab - ba) A^2 \right] - q \int_M \text{tr} [(ab - ba) A], 
\]

because

\[
\text{tr} \left[-(d_A a \wedge b - a \wedge d_A b - d_A b \wedge a + b \wedge d_A a) \wedge A \right] 
\]

\[= \text{tr} \left[(ab - ba) F_A + (ab - ba)A^2 \right] - d \text{tr} [(ab - ba)A].
\]

**Theorem 3.1.** We have a family of pre-symplectic structures on the space \( \mathcal{A}_X \) parametrized by \( t \in \mathbb{R} \):

\[
\Omega_A^t(a, b) = \Omega_A(a, b) - (t - 1) \gamma_A^t(a, b),
\]

with \( \gamma_A^t(a, b) = q \int_X \text{tr} [(ab - ba) A^2] \).
The case for $t = 1$ was discussed in [6].

Every principal $G$-bundle over a three-manifold $M$ is extended to a principal $G$-bundle over a four-manifold $X$ that cobords $M$, and for a connection $A \in \mathcal{A}_M$ there is a connection $A \in \mathcal{A}_X$ that extends $A$. So we detect a pre-symplectic structure on $\mathcal{A}_M$ as the boundary restriction of the pre-symplectic structure $\Omega$ on $\mathcal{A}_X$. The boundary 2-form $\omega$ is given by

$$
\omega_A(a, b) := \frac{1}{24\pi^3} \int_M \text{tr} [(ab - ba)A], \quad A \in \mathcal{A}_M, \ a, b \in T_A\mathcal{A}_M. \quad (3.3)
$$

But it is not a closed form and does not give a presymplectic structure on $\Omega$. Instead it gives the following 3-form on $\mathcal{A}_M$ that is called the Cartan 3-form. Let $\kappa$ be the 3-form on $\mathcal{A}_M$ defined by

$$
\kappa_A(a, b, c) = \frac{1}{8\pi^3} \int_M \text{tr} [(ab - ba)c] \quad (3.4)
$$

for any $a, b, c \in T_A\mathcal{A}_M$. It holds that $\kappa_A(a, b, c) = \kappa_A(b, c, a) = \kappa_A(c, a, b)$. We have then

$$
\tilde{d}\omega = \kappa. \quad (3.5)
$$

In fact, (2.4) yields

$$
(\tilde{d}\omega)_A(a, b, c) = (\partial_a\kappa)_A(b, c) + (\partial_b\kappa)_A(c, a) + (\partial_c\kappa)_A(a, b)
$$

$$
= \frac{1}{24\pi^3} \int_M \text{tr} [(b \wedge c - c \wedge b) \wedge a] + \frac{1}{24\pi^3} \int_M \text{tr} [(c \wedge a - a \wedge c) \wedge b]
$$

$$
+ \frac{1}{24\pi^3} \int_M \text{tr} [(a \wedge b - b \wedge a) \wedge c],
$$

The 2-form (3.3) gives a pre-symplectic structure twisted by the Cartan 3-form $\kappa$.

In the next section we shall describe twisted Dirac structures that are associated to the twisted pre-symplectic structures discussed above.

### 3.2 Twisted Dirac structure on the space of connections on a 3-manifold $M$

In this section $M$ is a compact connected oriented Riemannian 3-manifold and $G = SU(n)$ with $n \geq 2$. We shall write $\mathfrak{g} = \mathfrak{su}(n)$. Let $P$ be a $G$-principal bundle. Any principal bundle over a 3-manifold has a trivialization so that we may assume $P = M \times G$. Let $\Omega^r(M, \mathfrak{g})$ be the Sobolev space of $L^2_r$-sections of $\mathfrak{g}$-valued $r$-forms on $M$. It is the completion of $\{ \phi \otimes X | \phi \in \Omega^r(M), X \in \mathfrak{g} \}$ by the $L^2_r$-norms. $\Omega^1_{-1}(M, \mathfrak{g})$ is a Hilbert space by the inner product

$$
(\xi | \eta) := \int_M \langle \xi, \eta \rangle \text{vol}_M \quad (3.6)
$$

for any $\xi, \eta \in \Omega^1_{-1}(M, \mathfrak{g})$. Here, $\text{vol}_M$ stands for a volume form on $M$, and $\langle \cdot, \cdot \rangle$ is the inner product induced from the Riemannian metric on $M$ and the Killing form on $\mathfrak{g}$.  

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We write $\mathcal{A}_M$ the space of irreducible connections over $P$. $\mathcal{A}_M$ is an affine space modeled by the vector space $\Omega^1_{\mathfrak{g},-1}(M, g)$. So the tangent space at $A \in \mathcal{A}_M$ is

$$T_A\mathcal{A}_M = \Omega^1_{\mathfrak{g},-1}(M, g),$$

that is, the space of differential 1-forms with values in $\mathfrak{su}(n)$ whose component functions are of class $L^2_{\mathfrak{g},-1}$. The dual pairing of a 2-form $\alpha \in \Omega^2_{\mathfrak{g},-1}(M, g)$ and a 1-form $a \in \Omega^1_{\mathfrak{g},-1}(M, g)$ is given by

$$\langle \alpha | a \rangle_M := q \int_M \text{tr} (\alpha \wedge a), \quad q = \frac{1}{24\pi^3},$$

which yields the identification of the cotangent space $T^*_A\mathcal{A}_M$ and $\Omega^2_{\mathfrak{g},-1}(M, g)$.

Let $E_0(M) = T\mathcal{A}_M := T\mathcal{A}_M \oplus T^*\mathcal{A}_M$ be the standard Courant algebroid with the bilinear form

$$\langle a \oplus \alpha, b \oplus \beta \rangle_\omega := \frac{1}{2} \left( \langle \alpha | b \rangle_M + \langle \beta | a \rangle_M \right), \quad (3.7)$$

the anchor $\rho(a \oplus \alpha) = a$, and the bracket $\llbracket \cdot, \cdot \rrbracket$ for sections defined by

$$\llbracket a \oplus \alpha, b \oplus \beta \rrbracket := [a, b] \oplus (L_a\beta - i_b\tilde{\alpha}). \quad (3.8)$$

We take the Cartan 3-form $\kappa$ on $\mathcal{A}_M$ of (3.4). Since $\kappa = \tilde{\partial}_a \omega$ is a closed 2-form we have the Courant algebroid $E_\kappa(M)$ modified by the bracket

$$\llbracket a \oplus \alpha, b \oplus \beta \rrbracket_\kappa := [a, b] \oplus (L_a\beta - i_b\tilde{\alpha} - i_{a \wedge b} \kappa). \quad (3.9)$$

Note that the bilinear form and the anchor of $E_\kappa(M)$ are the same ones as for the standard Courant algebroid $E_0(M)$.

We define a vector subbundle $\mathcal{D}_M$ of $E_\kappa(M)$ by

$$\mathcal{D}_M := \bigsqcup_{A \in \mathcal{A}_M} \left\{ a \oplus (A \wedge a - a \wedge A) \mid a \in T_A\mathcal{A}_M \right\}. \quad (3.10)$$

Let $\omega : T\mathcal{A}_M \longrightarrow T^*\mathcal{A}_M$ be a bundle homomorphism given by

$$\omega_a(a) := A \wedge a - a \wedge A, \quad a \in T_A\mathcal{A}_M, A \in \mathcal{A}_M. \quad (3.11)$$

Any vector field $a : \mathcal{A}_M \rightarrow T\mathcal{A}_M$ gives rise to a section $\tilde{a}$ of $\mathcal{D}_M$ by $\tilde{a} = a \oplus \omega(a)$, where $\omega(a)$ denotes the section of $T^*\mathcal{A}_M$: $\omega(a)(A) = \omega_a(a(A))$.

**Lemma 3.2.** $\langle \tilde{a}, \tilde{b} \rangle_\kappa = 0$ for any section $\tilde{a}, \tilde{b}$ of $\mathcal{D}_M$.

In fact, for $a = a(A), b = b(A)$, we have

$$\langle a \oplus \omega(a), b \oplus \omega(b) \rangle_\kappa(A) = \frac{1}{2} \left\{ \langle \omega_a(a) | b \rangle_M + \langle \omega_a(b) | a \rangle_M \right\}$$

$$= \frac{q}{2} \int_M \text{tr} [(A \wedge a - a \wedge A) \wedge b] + \frac{1}{2} \int_M \text{tr} [(A \wedge b - b \wedge A) \wedge a]$$

$$= 0.$$

The 3-form $\kappa$ which describes the twist on $E_0(M)$ is defined by

$$\kappa_a(a, b, c) = \frac{1}{8\pi^3} \int_M \text{tr} [abc - bac], \quad a, b, c \in T_A\mathcal{A}_M, A \in \mathcal{A}_M,$$

where $abc = a \wedge b \wedge c$ etc..
Lemma 3.3. The space of sections \( \Gamma(\mathcal{A}_M, D_M) \) of \( D_M \) is closed under the bracket (3.9):

\[
\mathcal{L}_{a} \omega(b) - i_{b} \tilde{\omega}(a) - i_{a \wedge b} \kappa = \omega([a, b])
\]

for any vector field \( a, b \) on \( \mathcal{A}_M \). Let \( a, b \) and \( c \) be vector fields on \( \mathcal{A}_M \) and put \( a = a(A), b = b(A) \) and \( c = c(A) \). From (2.5) and (2.2)

\[
(\mathcal{L}_{a} \omega(b))_{A}(e(A)) = \mathcal{L}_{a}(A \wedge b - b \wedge A)(c)
\]

\[
= \partial_{a} \langle A \wedge b - b \wedge A \mid c \rangle_{M}(A) - \langle A \wedge b - b \wedge A \mid [a, c](A) \rangle_{M}
\]

\[
= \langle a \wedge b + A \wedge (\partial_{a} b) - (\partial_{a} b) \wedge A - b \wedge a \mid c \rangle_{M} + \langle A \wedge b - b \wedge A \mid \partial_{a} a \rangle_{M}.
\]

(3.12)

On the other hand, we see from the formula (2.3) that \( i_{b} \tilde{\omega}(a)(c) \) is equal to

\[
\tilde{\omega}(\omega(a))_{A}(b, c) = \langle \partial_{b} \omega_{A}(a) \mid c \rangle_{M} - \langle \partial_{a} \omega_{A}(a) \mid b \rangle_{M}
\]

\[
= \langle b \wedge a + A \wedge (\partial_{b} a) - (\partial_{b} a) \wedge A - a \wedge b \mid c \rangle_{M}
\]

\[
- \langle c \wedge a + A \wedge (\partial_{a} a) - (\partial_{a} a) \wedge A - a \wedge c \mid b \rangle_{M}
\]

\[
= \langle b \wedge a + A \wedge (\partial_{a} a) - (\partial_{a} a) \wedge A - a \wedge b \mid c \rangle_{M}
\]

\[
- \langle a \wedge b - b \wedge a \mid c \rangle_{M} - \langle b \wedge A - A \wedge b \mid \partial_{a} a \rangle_{M}.
\]

(3.13)

By Eqs.(3.12) and (3.13),

\[
\mathcal{L}_{a} \omega(b))_{A}(c) - \tilde{\omega}(\omega(a))_{A}(b, c)
\]

\[
= \langle A \wedge (\partial_{b} a) - (\partial_{a} b) \wedge A - A \wedge (\partial_{a} a) + (\partial_{a} a) \wedge A \mid c \rangle_{M} + 3 \langle a \wedge b - b \wedge a \mid c \rangle_{M}
\]

\[
= \langle A \wedge [a, b] - [a, b] \wedge A \mid c \rangle_{M} + 3 \langle a \wedge b - b \wedge a \mid c \rangle_{M}
\]

\[
= \langle \omega([a, b]) \mid c \rangle_{A} + (i_{a \wedge b} \kappa)_{A}(c),
\]

which proves the lemma. \( \square \)

The definition of twisted Dirac structure in (2.7) together with Lemmas 3.2 and 3.3 yield the following theorem.

Theorem 3.4. \( D_M \) is a k-twisted Dirac structure on \( \mathcal{A}_M \).

3.3 Dirac structures on the space of connections on a 4-manifold \( X \)

3.3.1 The twisted Dirac structure induced from the curvature form

Let \( X \) be a four-manifold with the boundary \( \partial X = M \) which may be empty. We denote by \( \mathcal{A}_X \) the connection space for the trivial bundle \( X \times G \) over \( X \) with \( G = SU(n) \). We denote \( \mathfrak{g} = su(n) \) for \( n \geq 2 \). The tangent space \( T_{A} \mathcal{A}_X \) at \( A \in \mathcal{A}_X \) is identified with the space \( \Omega_{s-\frac{1}{2}}^{1}(X, \mathfrak{g}) \) of \( \mathfrak{g} \)-valued 1-forms, and the cotangent space \( T_{A}^{*} \mathcal{A}_X \) is identified with \( \Omega_{s-\frac{1}{2}}^{1}(X, \mathfrak{g}) \). The pairing of \( T_{A} \mathcal{A}_X \) and \( T_{A}^{*} \mathcal{A}_X \) is given by

\[
\langle \alpha \mid a \rangle_{X} := \frac{1}{8\pi^{2}} \int_{X} \text{tr} (\alpha \wedge a), \quad \text{for } \alpha \in T_{A}^{*} \mathcal{A}_X, \ a \in T_{A} \mathcal{A}_X.
\]
Let $E_0(X) = T\mathcal{A}_X$ be the standard Courant algebroid over $\mathcal{A}_X$. We define a twist $\kappa$ on the Courant algebroid $E_0(X)$ by the following 3-form:

$$\kappa_A(a, b, c) = \kappa_A(\tilde{a}, \tilde{b}, \tilde{c}), \quad \text{for } a, b, c \in T_A\mathcal{A}_X, A \in \mathcal{A}_X,$$  \hspace{1cm} (3.14)

where $\tilde{a}, \tilde{b}$ and $\tilde{c}$ indicate the restriction of $a, b$ and $c$ respectively to the boundary $M$, and the right-hand side is the 3-form $\kappa$ on $\mathcal{A}_M$ described in 3.2.

**Lemma 3.5.** $\kappa$ can be represented in the form

$$\kappa_A(a, b, c) = \langle d_Aa \wedge b + b \wedge d_Aa \mid c \rangle_X + \langle d_Ab \wedge c + c \wedge d_Ab \mid a \rangle_X + \langle d_Ac \wedge a + a \wedge d_Ac \mid b \rangle_X,$$

for $a, b, c \in T_A\mathcal{A}_X$.

**Proof.** By the Stokes theorem

$$\kappa_A(a, b, c) = \frac{1}{8\pi^3} \int_M \text{tr} [(ab - ba) \wedge c] = \frac{1}{8\pi^3} \int_X d\text{tr} [(ab - ba) \wedge c].$$

Since

$$d\text{tr} [(ab - ba)c] = \text{tr} [(d_Aa \wedge b - a \wedge d_Ab - d_Ab \wedge a + b \wedge d_Aa) \wedge c + (a \wedge b - b \wedge a) \wedge d_Ac]$$

$$= \text{tr} [(d_Aa \wedge b + b \wedge d_Aa) \wedge c] - \text{tr} [(a \wedge d_Ab + d_Ab \wedge a) \wedge c]$$

$$+ \text{tr} [(a \wedge b - b \wedge a) \wedge d_Ac],$$  \hspace{1cm} (3.15)

the lemma is proved. \hfill \Box

Let $\phi : T\mathcal{A}_X \to T^*\mathcal{A}_X$ be a bundle homomorphism defined by

$$\phi_A(a) := F_A \wedge a + a \wedge F_A, \quad \text{for } a \in T_A\mathcal{A}_X, A \in \mathcal{A}_X,$$  \hspace{1cm} (3.16)

where $F_A$ is the curvature form of $A \in \mathcal{A}_X$. Let $D^\phi_X$ be the vector subbundle of $E_0(X)$ defined by

$$D^\phi_X := \bigcap_{A \in \mathcal{A}_X} \{ a \oplus \phi_A(a) \mid a \in T_A\mathcal{A}_X \}.$$  \hspace{1cm} (3.17)

We shall show that $D^\phi_X$ gives a $\kappa$-twisted Dirac structure on $\mathcal{A}_X$.

First, for vector fields $a$ and $b$ on $\mathcal{A}_X$, we have

$$\langle \phi(a) \mid b \rangle_X(A) = \frac{1}{8\pi^3} \int_X \text{tr}(\phi_A(a) \wedge b) = \frac{1}{8\pi^3} \int_X \text{tr}(F_A \wedge a \wedge b + a \wedge F_A \wedge b)$$

$$= \frac{1}{8\pi^3} \int_X \text{tr} [(a \wedge b - b \wedge a) \wedge F_A],$$

where $a = a(A)$ and $b = b(A)$. Then we have the following:

**Proposition 3.6.** It holds that

$$\langle \phi(a) \mid b \rangle_X + \langle \phi(b) \mid a \rangle_X = 0$$  \hspace{1cm} (3.18)

for any vector fields $a, b$ on $\mathcal{A}_X$. \hfill 10
Next, we shall show that the sections of $\mathcal{D}^\phi_{X}$ is closed under the bracket $[\cdot, \cdot]_\kappa$.

**Lemma 3.7.** Let $a, b$ and $c$ be vector fields on $\mathcal{A}_X$ and put $a = a(A), b = b(A), c = c(A)$ for $A \in \mathcal{A}_X$. We have the following formulas:

\[
(L_a \phi(b))_X(a) = \langle \phi_A(\partial_a b) | c \rangle_X + \langle d_A a \wedge b + b \wedge d_A a | c \rangle_X + \langle \phi_A(b) | (\partial_a a) \rangle_X,
\]
and

\[
(i_b \tilde{d} \phi(a))_X(a) = \langle \phi_A(\partial_b a) | c \rangle_X + \langle \phi_A(b) | (\partial_a a) \rangle_X
- \langle d_A a \wedge b + b \wedge d_A a | a \rangle_X - \langle a \wedge d_A c + d_A c \wedge a | b \rangle_X.
\]

**Proof.** Since $\partial_a F_A = d_A a$ for the curvature $F_A$ of $A$. We have from (2.2) and (2.5),

\[
(L_a \phi(b))_X(a) = L_a(F_A \wedge b + b \wedge F_A(c)) \quad (3.19)
\]

\[
= \partial_a \langle F_A \wedge b + b \wedge F_A | c \rangle_X - \langle F_A \wedge b + b \wedge F_A | [a, c] \rangle_X
\]

\[
= \langle d_A a \wedge b + F_A \wedge \partial_b b + \partial_a b \wedge F_A + b \wedge d_A a | c \rangle_X + \langle F_A \wedge b + b \wedge F_A | d_A a \rangle_X - \langle F_A \wedge b + b \wedge F_A | \partial_a c \rangle_X
\]

\[
= \langle F_A \wedge \partial_a b + \partial_b b \wedge F_A | c \rangle_X + \langle d_A a \wedge b + b \wedge d_A a | c \rangle_X + \langle F_A \wedge b + b \wedge F_A | \partial_c a \rangle_X,
\]

which proves the first formula (3.19). Next, we have

\[
(i_b \tilde{d} \phi(a))_X(a) \quad (3.20)
\]

\[
= \partial_b \langle F_A \wedge a + a \wedge F_A | c \rangle_X - \partial_b \langle F_A \wedge a + a \wedge F_A | b \rangle_X - \langle a \wedge F_A + F_A \wedge a | b \rangle_X
\]

\[
= \langle F_A \wedge \partial_b a + \partial_b a \wedge F_A + a \wedge d_A b + d_A b \wedge a | c \rangle_X + \langle a \wedge F_A + F_A \wedge a | \partial_b c \rangle
- \langle a \wedge F_A + F_A \wedge a | \partial_b c - \partial_c b \rangle_X
\]

\[
= \langle F_A \wedge \partial_b a + \partial_b a \wedge F_A | c \rangle_X + \langle F_A \wedge b + b \wedge F_A | \partial_c a \rangle_X
- \langle c \wedge d_A b + d_A b \wedge c | a \rangle_X - \langle a \wedge d_A c + d_A c \wedge a | b \rangle_X,
\]

where we used $\langle a \wedge d_A b + d_A b \wedge a | c \rangle_X = -\langle c \wedge d_A b + d_A b \wedge c | a \rangle_X$. The second formula (3.20) is proved. \hfill \Box

Lemmas 3.5 and 3.7 yield the following:

**Proposition 3.8.** For any vector field $a, b$ on $\mathcal{A}_X$, $L_a \phi(b) - i_b \tilde{d} \phi(a) - i_{a+b} \kappa = \phi([a, b])$.

Propositions 3.6 and 3.8 yield the following:

**Theorem 3.9.**

\[
\mathcal{D}^\phi_{X} := \bigcap_{A \in \mathcal{A}_X} \left\{ a \oplus \phi_A(a) \mid a \in T_A \mathcal{A}_X \right\}
\]

is a $\kappa$-twisted Dirac structure.

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3.3.2 A non-twisted Dirac structure on $\mathcal{A}_X$

On the subspace of flat connections $\mathcal{A}_X^0 \subset \mathcal{A}_X$, the subspace $\mathcal{D}_X^0 \subset E_0(X)$ reduces to the subspace $T\mathcal{A}_X^0 \oplus \{0\}$. That implies no knowledge about the Courant algebroid structure on $T\mathcal{A}_X^0 \oplus T\mathcal{A}_X^0$. In the sequel we shall introduce a more precise Dirac structure over $\mathcal{A}_X$ that relates to the winding number of flat connections.

We introduce the following bundle homomorphism:

$$\gamma_A(a) = A^2 \wedge a + a \wedge A^2, \quad A \in \mathcal{A}_X,$$

(3.22)

where $A^2 = A \wedge A$.

Theorem 3.10. The subbundle of $E_0(X)$ given by

$$\mathcal{D}_X^0 := \bigcup_{A \in \mathcal{A}_X} \{ a \oplus \gamma_A(a) \mid a \in T_A\mathcal{A}_X \}$$

is a Dirac structure.

For the proof we verify the following two conditions:

1. $\langle \bar{a}, \bar{b} \rangle_X = 0$,

2. $\mathcal{L}_a \gamma'(b) - i_b \tilde{\partial} \gamma'(a) = \gamma'([a, b])$

for $\bar{a} = a \oplus \gamma'(a)$ and $\bar{b} = b \oplus \gamma'(b)$. It is easy to check (1) since

$$\langle \gamma'(a) \mid b \rangle_X = \frac{1}{8\pi^2} \int_X \text{tr} (A^2 \wedge a + a \wedge A^2) \wedge b = \frac{1}{8\pi^2} \int_X \text{tr} [(a \wedge b - b \wedge a) \wedge A^2].$$

As for (2), the Lie derivative $\mathcal{L}_a \gamma'(b)$ is calculated to be

$$\langle \bar{a}, \bar{b} \rangle_X = \mathcal{L}_a(A^2 \wedge b + b \wedge A^2)(c)$$

$$= \partial_a \langle A^2 \wedge b + b \wedge A^2 \mid c \rangle_X - \langle A^2 \wedge b + b \wedge A^2 \mid [a, c](A) \rangle_X$$

$$= \langle (a \wedge A + A \wedge a) \wedge b + b \wedge (a \wedge A + A \wedge a) + A^2 \wedge \partial_a b + \partial_a b \wedge A^2 \mid c \rangle_X$$

$$+ \langle A^2 \wedge b + b \wedge A^2 \mid \partial_a c - \partial_a a \rangle_X - \langle A^2 \wedge b + b \wedge A^2 \mid \partial_a c - \partial_a a \rangle_X$$

$$= \langle (a \wedge A + A \wedge a) \wedge b \mid c \rangle_X + \langle b \wedge (a \wedge A + A \wedge a) \mid c \rangle_X$$

$$+ \langle A^2 \wedge \partial_a b + \partial_a b \wedge A^2 \mid c \rangle_X + \langle A^2 \wedge b + b \wedge A^2 \mid \partial_a a \rangle_X.$$

On the other hand, we have by (2.3) the following:

$$\langle i_b \tilde{\partial} \gamma'(a) \rangle_X(c)$$

$$= \langle \tilde{\partial} (A^2 \wedge a + a \wedge A^2) \rangle_X(b, c)$$

$$= \langle \tilde{\partial} (A^2 \wedge a + a \wedge A^2) \mid c \rangle_X - \langle \tilde{\partial} (A^2 \wedge a + a \wedge A^2) \mid b \rangle_X$$

$$= \langle (b \wedge A + A \wedge b) \wedge a + A^2 \wedge \partial_a a + \partial_a a \wedge A^2 + a \wedge (b \wedge A + A \wedge b) \mid c \rangle_X$$

$$- \langle (c \wedge A + A \wedge c) \wedge a + A^2 \wedge \partial_a a + \partial_a a \wedge A^2 + a \wedge (c \wedge A + A \wedge c) \mid b \rangle_X$$

$$= \langle (a \wedge A + A \wedge a) \wedge b \mid c \rangle_X + \langle (b \wedge (a \wedge A + A \wedge a) \mid c \rangle_X$$

$$+ \langle A^2 \wedge \partial_a a + \partial_a a \wedge A^2 \mid c \rangle_X + \langle A^2 \wedge b + b \wedge A^2 \mid \partial_a a \rangle_X.$$

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Therefore,
\[
(L_a \gamma'(b))_A(c) - (i_b \tilde{d} \gamma'(a))_A(c) = \langle A^2 \wedge (\partial_a b - \partial_b a) + (\partial_a b - \partial_b a) \wedge A^2 | c \rangle_X
\]
\[
= \langle A^2 \wedge [a, b](A) + [a, b](A) \wedge A^2 | c \rangle_X
\]
\[
= \langle \gamma'_A([a, b](A)) | c \rangle_X ,
\]
which proves (2).

3.3.3 A family of $\kappa$-twisted Dirac structures on $\mathcal{A}_X$

Now we discuss the perturbation of the $\kappa$-twisted Dirac structure $\mathcal{D}_X^\kappa$ by the (non-twisted) Dirac structure $\mathcal{D}_X$. We consider the bundle homomorphism $\gamma' : T\mathcal{A}_X \rightarrow T^*\mathcal{A}_X$ given by
\[
\gamma'_A(a) := (F_A + t A^2) \wedge a + a \wedge (F_A + t A^2), \quad t \in \mathbb{R}.
\]
(3.23)
We have $\gamma' = \phi + t \gamma'$.

The following theorem is a consequence of Theorems 3.9 and 3.10

**Theorem 3.11.** For $t \in \mathbb{R}$, put
\[
\mathcal{D}_X^t := \bigsqcup_{A \in \mathcal{A}_X} \{ a \oplus \gamma'_A(a) | a \in T_A \mathcal{A}_X \}.
\]
(3.24)
Then $\{\mathcal{D}_X^t\}_{t \in \mathbb{R}}$ gives a family of $\kappa$-twisted Dirac structures.

3.3.4 Dirac structure over the space of flat connections

Let $\mathcal{A}_X^\flat$ be the space of flat connections:
\[
\mathcal{A}_X^\flat = \{ A \in \mathcal{A}_X | F_A = 0 \}.
\]
We have
\[
T_A \mathcal{A}_X^\flat = \{ a \in \Omega^1_{\text{flat}}(X, \mathfrak{g}) | d_A a = 0 \}.
\]
The subbundle $T \mathcal{A}_X^\flat$ is integrable, that is, if $a$ and $b$ are flat vector fields; $a(A), b(A) \in T_A \mathcal{A}_X^\flat$ for $\forall A \in \mathcal{A}_X^\flat$, then $[a, b]$ is a flat vector field. In fact, this follows from the formula $d_A [a, b] = [d_A a, b] + [a, d_A b]$.

Let
\[
\mathcal{D}_X^\flat := \bigsqcup_{A \in \mathcal{A}_X} \{ a \oplus \gamma'_A(a) | a \in T_A \mathcal{A}_X^\flat \},
\]
(3.25)
with
\[
\gamma'_A(a) = \gamma'_A(a) := A^2 \wedge a + a \wedge A^2, \quad A \in \mathcal{A}_X^\flat.
\]
(3.26)
Lemma 3.5 shows that the Cartan 3-form $\kappa$ on $\mathcal{A}_X^\flat$ vanishes. Therefore we have the following:

**Theorem 3.12.** $\mathcal{D}_X^\flat$ is a Dirac structure over $\mathcal{A}_X^\flat \subset \mathcal{A}_X$. 

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We look at the boundary restriction of the Dirac manifold $\mathcal{D}_X^b$ to $M$. Let $\mathcal{A}_M^b$ be the space of flat connections; 
\[ \mathcal{A}_M^b = \{ A \in \mathcal{A}_M \mid F_A = 0 \}. \]

The tangent space of $\mathcal{A}_M^b$ at $A \in \mathcal{A}_M^b$ is given by 
\[ T_A \mathcal{A}_M^b = \{ a \in \Omega^1_{\chi^{-1}}(M, g) \mid d_A a = 0 \}. \]

Any principal $G$-bundle $P$ over a 3-manifold $M$ is extended to a principal $G$-bundle $\mathcal{P}$ over $X$, and any connection $A$ on $P$ has an extension to a connection $\mathcal{A}$ on $\mathcal{P}$. Then the boundary restriction map $r : \mathcal{A}_X \longrightarrow \mathcal{A}_M$ is well defined and surjective. The tangent map of $r$ at $A \in \mathcal{A}_X$ is also given by the restriction to the boundary: 
\[ (\partial r)_A : T_A \mathcal{A}_X = \Omega^1_{\chi^{-1}}(X, g) \longrightarrow T_A \mathcal{A}_M = \Omega^1_{\chi^{-1}}(M, g), \quad A = r(A). \]

In [6], it is proved that the space of flat connections $\mathcal{A}_X^b$ is mapped by $r$ onto the space of flat connections over $M$ that are of degree 0:
\[ \mathcal{A}_M^{b, \text{deg} 0} = \left\{ A \in \mathcal{A}_M^b \mid \int_M \text{tr} A^3 = 0 \right\}. \]

Let $\mathcal{D}_M^b$ and $\mathcal{D}_M^{b, \text{deg} 0}$ be the subbundles of $E^b(M)$ that are defined by 
\[ \mathcal{D}_M^b = \left\{ a \oplus \omega_A(a) \mid a \in T_A \mathcal{A}_M^b, A \in \mathcal{A}_M^b \right\}, \]
and 
\[ \mathcal{D}_M^{b, \text{deg} 0} = \left\{ a \oplus \omega_A(a) \mid a \in T_A \mathcal{A}_M^{b, \text{deg} 0}, A \in \mathcal{A}_M^{b, \text{deg} 0} \right\}. \]

**Lemma 3.13.** For any $a \in T_A \mathcal{A}_X^b$ we have 
\[ \gamma_A^b(a) = \omega_{r_A}(\partial r_A(a)). \]  
\[ \tag{3.27} \]

**Proof.** Let $A \in \mathcal{A}_X^b$ and $a, b \in T_A \mathcal{A}_X^b$. Put $A' = r(A) \in \mathcal{A}_M^b$, $\bar{a} = \partial r(a) \in T_A \mathcal{A}_M^b$, and $\bar{b} = \partial r(b) \in T_{A'} \mathcal{A}_M^b$. We have 
\[ \langle \omega_A(\bar{a}) \mid \bar{b} \rangle = q \int_X \text{tr} [(\bar{a} \bar{b} - \bar{b} \bar{a}) A'] = q \int_X d \text{tr} [(ab - ba) A]. \]

Since $F_A = 0$ and $d_A a = d_A b = 0$ for $a, b \in T_A \mathcal{A}_X^b$, by a similar calculation as in (3.15) we find that the last formula is equal to 
\[ q \int_X \text{tr} [(ab - ba) d_A A] = q \int_X \text{tr} [(ab - ba) A^2] = \langle \gamma_A^b(a) \mid b \rangle. \]

\[ \square \]

From the discussion hitherto we are convinced to have the following consequences.

Let $E^b(M) = T \mathcal{A}_M^b$ be the standard Courant algebroid over the space of flat connections $\mathcal{A}_M^b$.

**Theorem 3.14.**

1. $\mathcal{D}_M^b$ is a Dirac structure over $\mathcal{A}_M^b$.

2. $\mathcal{D}_M^{b, \text{deg} 0}$ is a Dirac structure over $\mathcal{A}_M^{b, \text{deg} 0}$.

3. The boundary restriction map $r$ implies an isomorphism between the Dirac structures $\mathcal{D}_X^b$ and $\mathcal{D}_M^{b, \text{deg} 0}$.
4 The action of the group of gauge transformations on Dirac spaces

Let $G = SU(n)$ and $P \to N$ be a $G$-principal bundle over a $n$-dimensional Riemannian manifold $N$ as in subsection 2.1. Let $\text{Ad}P$ be the fiber bundle $\text{Ad}P = P \times_G G \to N$. The group of smooth sections of $\text{Ad}P$ under the fiber-wise multiplication is called the group of gauge transformations. We denote by $\mathcal{G}'_N$ the group of $L_i^2$-gauge transformations, i.e., $\mathcal{G}'_N = \Omega_i^0(N, \text{Ad}P)$. $\mathcal{G}'_N$ acts on $\mathcal{A}_N$ by

$$g \cdot A = g^{-1}dg + g^{-1}Ag = A + g^{-1}dAg, \quad g \in \mathcal{G}'_N.$$ 

$\mathcal{G}'_N$ is a Hilbert Lie Group and its action is a smooth map of Hilbert manifolds. The action of $g \in \mathcal{G}'_N$ on the tangent space $T_A\mathcal{A}_N$ is given by

$$\text{Ad}_{g^{-1}} : T_A\mathcal{A}_N \ni a \mapsto \text{Ad}_{g^{-1}}a = g^{-1}ag \in T_A\mathcal{A}_N.$$ 

We choose a fixed point $p_0 \in N$ and deal with the group of gauge transformations that are identity at $p_0$:

$$\mathcal{G}_N = \{g \in \mathcal{G}'_N \mid g(p_0) = 1\}.$$ 

$\mathcal{G}_N$ acts freely on $\mathcal{A}_N$. We have $\text{Lie}\mathcal{G}_N = \Omega_i^0(N, \text{ad}P)$. The action of Lie $\mathcal{G}_N$ on $\mathcal{A}_N$ is given by the covariant exterior derivative:

$$d_A = d + [A \wedge \cdot] : \text{Lie}\mathcal{G} = \Omega_i^0(N, \text{ad}P) \ni \xi \mapsto d_A\xi \in \Omega^1_{\mathcal{A}_N}(N, \text{ad}P) = T_A\mathcal{A}_N.$$ 

So, the fundamental vector field on $\mathcal{A}_N$ corresponding to $\xi \in \text{Lie}\mathcal{G}_N$ is given by

$$d_A\xi = \frac{d}{dt}{\bigg|}_{t=0}(\exp{t\xi}) \cdot A.$$ 

ker $d_A = 0$ because $A \in \mathcal{A}_N$ is irreducible by our assumption. The tangent space to the orbit at $A \in \mathcal{A}_N$ is

$$T_A(\mathcal{G}_N \cdot A) = \{d_A\xi \mid \xi \in \Omega_i^0(N, \text{ad}P)\}.$$ 

Moreover, $\{d_A\xi \mid \xi \in \text{Lie}\mathcal{G}_N\}$ is tangent to the space of flat connections $\mathcal{A}_N^0$, because $d_A(d_A\xi) = \{F_A, \xi\} = 0$ on $\mathcal{A}_N^0$. Evidently, the condition $F_A = 0$ is $\mathcal{G}_N$-invariant, so $\mathcal{A}_N^0$ is also $\mathcal{G}_N$-invariant manifold.

Now we restrict ourselves to the case treated previously, that is, $N = X$ is a four-manifold or $N = M$ is the boundary three-manifold $M = \partial X$.

4.1 $\mathcal{D}_X^\phi$ under the action of $\mathcal{G}_X$

Let $\phi_A : T_A\mathcal{A}_X \to T^*_A\mathcal{A}_X$ be the bundle homomorphism $\phi_A(a) = F_A \wedge a + a \wedge F_A$ in (3.16). Since $F_{gA} = g^{-1}F_Ag$, we have

$$\phi_{gA} = g^{-1}\phi_A g.$$ 

(4.1)

We also see that $\kappa$ is invariant under $\mathcal{G}_X$. Therefore we have the following;

**Proposition 4.1.** The $\kappa$-twisted Dirac manifold $\mathcal{D}_X^\phi$ is invariant under the action of $\mathcal{G}_X$. 

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4.2 \( \mathcal{D}_M \) under the infinitesimal action of \( \mathcal{G}_M \)

In [6] we saw that the action of \( \mathcal{G}_M \) on the space of flat connections \( \mathcal{A}_M^b \) is infinitesimally pre-symplectic, that is, the Lie derivative of the 2-form \( \omega \) in (3.3) by the fundamental vector field \( d_A \xi \) vanishes:

\[
\mathcal{L}_{d_A \xi} \omega = 0, \quad A \in \mathcal{A}_M^b.
\]

The counterpart of this assertion for the infinitesimal action of \( \mathcal{G}_M \) on the Dirac structure \( \mathcal{D}_M^b \) holds as well. We shall explain it in the following.

We have the canonical skew symmetric form \( \Lambda_0 \) on \( E_0(M) = T \mathcal{A}_M \oplus T^* \mathcal{A}_M \) :

\[
\Lambda_0(a \oplus \alpha | b \oplus \beta) := \frac{1}{2} \{ \langle \alpha | b \rangle_M - \langle \beta | a \rangle_M \},
\]

(see (2.8)). We know that the space of connections \( \mathcal{A}_M \) admits the Dirac structure

\[
\mathcal{D}_M = \bigoplus_{A \in \mathcal{A}_M} \{ a \oplus \omega_A(a) | a \in T_A \mathcal{A}_M \}, \quad \omega_A(a) = Aa - aA.
\]

whose anchor map is \( \rho : E_0(M) \rightarrow T \mathcal{A}_M \) (see (3.10)).

**Lemma 4.2.** Restricted to \( \mathcal{D}_M \) the canonical 2-form \( \Lambda_0 \) is the pull back of the pre-symplectic form \( \omega \) by \( \rho : \Lambda_0|\mathcal{D}_M = \rho^* \omega. \)

**Proof.** For any section \( a \oplus \omega(a), b \oplus \omega(b) \) of \( \mathcal{D}_M \), we have

\[
\Lambda_0 \left( a \oplus \omega_A(a) \mid b \oplus \omega_A(b) \right) = \frac{1}{2} \{ \langle Aa - aA \mid b \rangle - \langle Ab - bA \mid a \rangle \}
\]

\[
= \frac{1}{2} \int_M \text{tr} [(ab - ba) A] = \omega_A(a, b),
\]

which is equal to \( (\rho^* \omega)(a \oplus \omega_A(a), b \oplus \omega_A(b)) \). \( \square \)

Lie \( \mathcal{G}_M \) acts on \( \mathcal{D}_M \subseteq E_0(M) \) by the fundamental vector field

\[
v_\xi = d_A \xi \oplus \omega_A(d_A \xi), \quad \xi \in \text{Lie } \mathcal{G}_M, A \in \mathcal{A}_M.
\]

The derivation of the 2-form \( \Lambda_0 \) along the orbit of Lie \( \mathcal{G}_M \) is given by the Lie derivation \( \mathcal{L}_{v_\xi(A)} \Lambda_0 \). Then, we have the following result.

**Proposition 4.3.** It holds that \( \mathcal{L}_{v_\xi(A)} \Lambda_0 = 0 \) at each \( A \in \mathcal{A}_M^b \). Hence the action of \( \mathcal{G}_M \) on the Dirac manifold \( \mathcal{D}_M^b \) is infinitesimally symplectic.

**Proof.** Since \( \rho(v_\xi(A)) = \rho(d_A \xi \oplus \omega_A(d_A \xi)) = d_A \xi \), Lemma 4.2 and the equation (4.2) imply

\[
\mathcal{L}_{v_\xi(A)} \Lambda_0 = \mathcal{L}_{d_A \xi @ \omega_A(d_A \xi)} (\rho^* \omega) = \rho^* (\mathcal{L}_{d_A \xi} \omega) = 0.
\]

\( \square \)
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