Energy Gaps and Roton Structure above the $\nu = 1/2$ Laughlin State of a Rotating Dilute Bose-Einstein Condensate

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Exact diagonalization study of a rotating dilute Bose-Einstein condensate reveals that as the first vortex enters the system the degeneracy of the low-energy yrast spectrum is lifted and a large energy gap emerges. As more vortices enter with faster rotation, the energy gap decreases towards zero, but eventually the spectrum exhibits a rotonlike structure above the $\nu = 1/2$ Laughlin state without having a phonon branch despite the short-range nature of the interaction.

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Rotating dilute Bose-Einstein condensates (BECs) have attracted considerable interest in recent years [1, 2, 3, 4]. When the single-particle energy-level spacing is much larger than the interaction energy per particle, the system becomes highly degenerate with increasing the angular momentum (AM) $L$. Interactions then play a pivotal role in lifting the degeneracy and determining the many-body ground state [5], in close analogy with the physics of the fractional quantum Hall effect [6, 7].

Mottelson [8], on the other hand, pointed out the relevance of the quasi-degenerate yrast spectrum that is dominated by collective multipolar excitations [8]. An exact diagonalization study [13] and subsequent analytical studies [14, 15, 16] have demonstrated the existence of the quasi-degenerate yrast spectrum that is dominated by interactions between octupole modes. The purpose of this Letter is to report the results of our exact diagonalization study on fast rotating BECs. Our primary findings are two distinct kinds of energy gaps: one above the single-vortex state and the other associated with vortex-antivortex pair excitations above the $\nu = 1/2$ Laughlin state. We also find that higher excited states with $L \lesssim N$ form linear energy bands.

Consider a weakly interacting $N$-boson system trapped in an axisymmetric parabolic confining potential. The potential is assumed to be isotropic in the radial direction but in the axial direction it is assumed to be so tightly confined that all particles occupy the lowest-energy state in that direction. The problem is thus essentially two-dimensional with projected AM, $L$, on the symmetry axis being conserved. The many-body Hamiltonian of this system can be written as $H = \sum_i h_i + V$, where $h_i \equiv -\hbar^2 \nabla_i^2 / 2M + M\omega^2 r_i^2 / 2$ denotes the single-particle Hamiltonian for the $i$-th particle and $V = (4\pi \hbar g / M \omega) \sum_{i < j} \delta(r_i - r_j)$ describes the contact interaction between bosons having mass $M$. Here $g$ characterizes the strength of interaction between bosons, $g / \hbar \omega = a_s / (2\pi \hbar / M \omega)^{1/2}$, $\omega$ and $\omega_z$ are trap frequencies in the radial and axial directions, respectively, and $a_s$ is the $s$-wave scattering length. Throughout this Letter we shall assume weak repulsive interactions such that $\hbar \omega \gg g N > 0$.

The single-particle state is characterized by the radial quantum number $n$ and the AM quantum number $m$, with the corresponding eigenenergy given by $\hbar \omega (2n + |m| + 1)$. For a given AM $L > 0$, it is energetically favorable to assume $n = 0$ and let all particles have nonnegative AM, i.e., $m \geq 0$. In this lowest-Landau-level approximation, the noninteracting part of the Hamiltonian contributes a constant term $\hbar \omega(L + N)$, and we shall henceforth ignore this constant part and focus on the interaction Hamiltonian. It is convenient to take as a basis set of states single-particle states described by $\phi_m(z) = (z^m / \sqrt{m!}) \exp(-|z|^2 / 2)$, where $z \equiv x + iy$ and the lengths are measured in units of $(\hbar / M \omega)^{1/2}$. Expanding the field operator as $\Psi(z) = \sum_m \hat{b}_m \phi_m(z)$, the second-quantized form of the contact interaction Hamiltonian is written as $V = g \sum_{m_1, \ldots, m_4} V_{m_1,m_2,m_3,m_4} \hat{b}_{m_1}^\dagger \hat{b}_{m_2}^\dagger \hat{b}_{m_3} \hat{b}_{m_4}$, where the matrix elements $V_{m_1,m_2,m_3,m_4}$ are given by $V_{m_1,m_2,m_3,m_4} = \delta_{m_1 + m_2, m_3 + m_4} (m_1 + m_2) / (2^{m_1 + m_2} \sqrt{m_1! m_2! m_3! m_4!})$. Our task thus reduces to finding the energy spectrum of $V$ under the restrictions of a fixed particle number $\sum_m \hat{b}_m^\dagger \hat{b}_m = N$ and a fixed AM $\sum_m m \hat{b}_m^\dagger \hat{b}_m = L$. 


Figure 1 shows the yrast spectrum for $0 \leq L \leq 35$ with $N = 25$, where the energy is measured in units of $gN$ from the yrast line $E = gN(N - 1 - L/2)$ [11]. For $L \ll N$, there are quasi-degenerate excited states arising from pairwise repulsive interaction between octupole modes [13]. However, the excitation energies are very small, of the order of $g$. This quasi-degeneracy may be regarded as a precursor for spontaneous symmetry breaking of the axisymmetry associated with the entrance of the first vortex. Remarkably, the quasi-degeneracy for $L \ll N$ is lifted with increasing $L$ and that a large energy gap of the order of $gN$ appears as the first vortex enters the system. The energy gap implies that the many-body wave function responds to external rotation in such a manner that a single vortex state becomes stabilized.

The emergence of the energy gap can be inferred from the fact that excitations associated with the single-particle state $|\phi_m\rangle$ ($m \geq 3$) become increasingly massive, as the AM per particle $L/N$ approaches unity. As shown in Table I, the excitation energy increases by an amount of the order of $gN$ when the condensate develops from the nonvortex state at $L = 0$ to the single-vortex state at $L = N$, and an increase in excitation energy is larger for smaller $m$. In particular, excitations of the $m = 3$ mode, which are nearly gapless for $L/N \ll 1$, become increasingly massive as the vortex enters the system. The energy gap continues to increase till $L = N + m - 1$ for each $m$. We also note that the energy bands (shown in Fig. 1 as dash-dotted lines) for $m = 4, 5, 6, \ldots$ are almost linear. When $L/N \gtrsim 1$, a regular structure of rotational bands is stabilized by the energy gaps associated with the single-particle states $\phi_m$. In Figure 1 we show three rotational bands by linking the collective excitations $Q_{\lambda}(L = N)$, $Q_{\lambda}(L = N - 1)$, and $Q_{\lambda}(L = N - 2)$ ($\lambda \leq 7$) with solid, dashed, and dotted curves, respectively. Here $Q_{\lambda} = (1/\sqrt{N L!}) \sum_{\mu=1}^{N} \gamma_{\mu}^{\lambda}$ describes the collective multipolar excitation of order $\lambda$, and $|L = N\rangle$, $|L = N - 1\rangle$, and $|L = N - 2\rangle$ denotes the yrast states for $L = N$, $L = N - 1$, and $L = N - 2$, respectively.

The regularity of rotational bands for $L/N \gtrsim 1$ is more clearly seen in Fig. 2 where the energy is measured from the line $E = E_{L=N} - 5gN(L/N)/32$ [12]. As in Fig. 1 the collective excitations, $Q_{\lambda}(L = N)$, $Q_{\lambda}(L = N - 1)$, and $Q_{\lambda}(L = N - 2)$ ($\lambda \leq 7$), are linked by solid, dashed, and dotted curves, respectively. Those states that contain excitations of the center-of-mass motion are not shown except for the states $Q_{1}(L = N)$, $Q_{1}(L = N - 1)$, and $Q_{1}(L = N - 2)$ (shown by squares). Other low-lying excitations can be understood as (multiple) excitations of these collective modes.

When $L \geq N(N - 1)$, the system can respond in such a manner that the interaction energy becomes zero. In fact, the $\nu = 1/2$ Laughlin state [18], $\Psi_{\nu=1/2} = \prod_{\mu} e^{-i\mu_{z}/2} \prod_{i<j}(z_{i} - z_{j})^{2}$, appears at $L = N(N - 1)$ as the first zero-interaction-energy state; the contact interaction does not affect this state because the minimum of the relative AM between particles is two for this state. In such a high AM regime, we find a remarkable energy gap to appear above the $\nu = 1/2$ Laughlin state as shown in Fig. 3.

To investigate long-wavelength collective excitations while keeping the average density unaltered (i.e., neutral excitations), it is convenient to use the spherical geometry [19] rather than the disk geometry which is used thus far. The two geometries are related with each other by stereographic mapping. The $z$-component of the total AM in the spherical geometry is given by $L_{z} = L - N\bar{S}$, where $2\bar{S} + 1$ is the number of single-particle states on the sphere and $2\bar{S}$ is chosen to be $2(N - 1)$ in order to

| $|\phi_m\rangle$ | $|\phi_1\rangle$ | $|\phi_4\rangle$ | $|\phi_5\rangle$ | $|\phi_6\rangle$ | $|\phi_7\rangle$ | $|\phi_8\rangle$ |
|---|---|---|---|---|---|---|
| $\Delta E/gN$ | 1 | 7/8 | 3/4 | 21/32 | 19/32 | 71/128 |

TABLE I: The increase in energy of the excitation associated with the single-particle state $|\phi_m\rangle$ ($m \geq 3$) when the system develops from the nonvortex condensate with $L = 0$ into the single-vortex condensate with $L = N$. 

![Diagram](image-url)
study the \( \nu = 1/2 \) Laughlin state.  

Figure 2 shows the excitation spectrum calculated in the spherical geometry for \( 0 \leq L \leq 56 \) with \( N = 8 \). The \( \nu = 1/2 \) Laughlin state appears at \( L = N(N - 1) = 56 \) (i.e., \( L_z = 0 \)) as the zero-interaction-energy state (i.e., the lowest energy state) for the contact repulsive interaction \((4\pi g/R^2)\sum_{i<j}\delta(\Omega_i - \Omega_j)\), where \( \Omega_i \) is a unit vector that specifies the location of the \( i \)-th boson on the sphere and the radius of the sphere is given by \( R = \sqrt{S/2} \). The total AM, \( L_{\text{tot}} \), of this state is zero and the eigenstates with \( L_{\text{tot}} = 0 \) are plotted for \( L = NS \). The eigenstates with non-zero total AM are plotted only for \( L = NS - L_{\text{tot}} \) without showing their \( (2L_{\text{tot}} + 1) \)-fold degeneracy.  

The inset of Fig. 3 shows the excitation spectrum above the Laughlin state as a function of \( L_{\text{tot}} \) in the spherical system. A rotonlike structure manifests itself as the lowest-energy excitation branch. We have confirmed that the excitation energy remains finite \((1.229 \pm 0.038\) in units of \( g \)) in the thermodynamic limit \( 2S \geq 2N-1 \). We have also confirmed that the feature of the rotonlike minimum becomes more pronounced if we use long-range interactions such as the Coulomb interaction, \( \sum_{i<j}|\Omega_i - \Omega_j|^{-1} \). The two-roton continuum is also seen to exist above the single-roton branch.  

To understand the physics underlying these gapful excitations, let us recall that it is possible to produce quasi-particle excitations upon the Laughlin state \( 1 \). One of them is the quasi-hole (vortex) excitation, \( \prod_p [e^{-|z_p|^2/2}(z_p - z_0)] \prod_{i<j}(z_i - z_j)^2 \), and another one is the ‘quasi-boson’ (anti-vortex) excitation, \( \prod_p [e^{-|z_p|^2/2}(z_p - z_0)] \prod_{i<j}(z_i - z_j)^2 \), where \( z_0 \) denotes the location of the vortex or that of the anti-vortex. Then the lowest-lying neutral excitations are considered to be the bound states of a vortex-antivortex pair.  

By investigating a system with \( 2S = 2N - 3 \) or \( 2S = 2N - 1 \), we find that in the thermodynamic limit the quasi-boson creation energy remains finite \( (1.248 \pm 0.007\) in units of \( g \)) \( 2, \) while the quasi-hole creation energy is zero for repulsive contact interaction. Since each of quasiboson and quasihole excitations has AM \( N/2 \), the branch composed of a quasiboson-quasihole pair is expected to end at \( L_{\text{tot}} = N \). This is indeed the case in the lowest-lying excitation branch as shown in Fig. 4, where the branches for system size \( N = 4, 5, 6, 7 \) and 8 are plotted against “wave number” \( k = L_{\text{tot}}/R \).  

In Fig. 4 it is seen that the minimum of the neutral-excitation energy (indicated by closed symbols) is realized at \( k_{\text{min}} \approx 3(M\omega/\hbar)^{1/2} \). For \( k \geq k_{\text{min}} \), the excitation energy increases with increasing \( k \) and approaches the quasi-boson creation energy (indicated on the right side of the figure by a line for each \( N \)). As shown in the
received that the finite gap in the long-wavelength limit of the excitations between composite particles is different from that in the quantum Hall system, it has been discussed that interaction does not fit into this category. In the composite-particle picture for the fractional quantum Hall system, the interaction does not fit into this category. In the composite-boson picture, the contact interaction does not fit into this category. In the composite-particle picture for the quantum Hall system, it has been discussed that interactions between composite particles are different from those between original particles. It might be conceived that the finite gap in the long-wavelength limit is related with momentum-dependent long-range interactions introduced on the composite-boson transformation (without being induced by a laser). However, the understanding of the gap formation found in this Letter remains to be clarified and merits further investigation.

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