Simulation of Dynamical Quantum Phase Transition of the 1D Transverse Ising Model with Neural Bosons in a Double-chain Tilted Lattice

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A spinless Bose-Hubbard model in an one-dimensional (1D) double-chain tilted lattice is analysed numerically at unit filling per cell. A subspace of this model can be faithfully mapped to the 1D transverse Ising model in terms of second-order perturbation theory. At a parameter regime where the second-order perturbative superexchange interaction interaction dominates, numerical results show good agreement of these two models both on energy spectrums and correlation functions. To simulate the dynamical quantum phase transition of the 1D transverse Ising model, we calculate the rate function of the recurrence probability of the double-chain Bose-Hubbard model after quenching from an initial equivalent ferromagnetic state. The rate function shows the same nonanalyticity at periodic time points as theory predicts. Our results may give some inspirations on exploring weak magnetic orders induced by superexchange interaction through dynamical quantum phase transition in experiment.

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Recently, enormous progress in simulating various kinds of quantum magnetism in cold atom systems opens new fascinating prospects for studying many long-standing unsolved problems and unconfirmed properties of various magnetic models. However, experimental simulation of such magnetic models strongly depends on how the desired magnetic models are constructed in experiment. In a system with long-range interaction, the major difficulty lies on how spins are represented and how to manipulate the magnetic interaction between these spins precisely. And it is reported that the transverse Ising model has been successfully implemented with ion trap[1–5], Rydberg atom[6–8] and superconducting quantum circuits[9–12]. And XXZ model has been realized with ultracold dipolar gases[13, 14]. While in neural atom systems, it is primarily hindered by the magnetic interaction. In such systems it usually requires a special design of experimental conditions to realize localized spin representation and a strong enough magnetic interaction simultaneously. Several designs have been carried out in a two-component Fermi-Hubbard model [15–19], a spinless Bose-Hubbard model in a tilted chain utilizing hopping constraints [20] and in a triangular lattice with lattice shaking [21] according to the propositions [22–24] respectively.

Despite these remarkable achievements, simulation of Ising model through superexchange interaction in neural atom systems has not been reported. The difficulty lies on the contradiction of realizing an Ising-like magnetic interaction and the requirement of localized spin representation[25]. In this letter, we propose a model in a tilted double-chain lattice to solve this problem. Another concern is that superexchange interaction is usually very weak comparing with typical accessible temperature in cold atom experiments. This limit makes it hard to observe magnetic orders and quantum phase transitions induced by superexchange interaction. However, dynamical quantum phase transition (DQPT) provides another tool for investigating such weak magnetic orders from a dynamical perspective. For transverse Ising model, DQPT requires a minimal magnetic interaction in the order of \( J_z \cdot t \sim 1 \) (we set \( h = 1 \)), where \( t \) is the evolving time after quenching [26]. For a typical superexchange interaction of tens of or hundreds of Hertz, it is feasible to satisfy this requirement in experiment. Therefore, it is possible to explore weak magnetic orders induced by superexchange interaction through DQPT in neural atom systems.

**Model and Methods.** — Our model begins with localized spin representation of the lattice model. As shown in FIG. 1(a), spin up or down is represented by the occupation of a spinless boson at upper or lower site in each cell. Every atom is localized in one single cell (FIG. 1(b)) because of the energy gap between nearest sites along x-axis when \( \Delta - U \gg t_x \). Here \( U \) is the on-site interaction, \( U_{t} \) is an introduced intersite interaction between the two sites in each cell, \( \Delta \) and \( t_x \) are the energy gap and the tunneling energy between two nearest sites along x-axis respectively. All the states with only one atom per cell in the tilted 1D lattice form a subspace (denoted as \( \mathcal{H}_{\text{one}} \) hereafter) which can be mapped to the Hilbert space of a spin-1/2 Ising model \( \mathcal{H}_{\text{Ising}} \).

Assuming a single-band case, this model can be described by a Bose-Hubbard model \( \hat{H} = \hat{H}_0 + \hat{V} \),

\[
\hat{H}_0 = \sum_{i,\sigma=\uparrow,\downarrow} \left[ \frac{U}{2} \hat{n}_{i\sigma} (\hat{n}_{i\sigma} - 1) + i \Delta \hat{n}_{i\sigma} \right] + U_{t} \sum_{i} \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \]

\[
\hat{V} = -t_x \sum_{i,\sigma=\uparrow,\downarrow} (\hat{c}^\dagger_{i\sigma} \hat{c}_{i+1\sigma} + h.c.) - t_{t} \sum_{i} \hat{c}^\dagger_{i\uparrow} \hat{c}_{i\downarrow} + h.c. \]

\[
+ \delta \sum_{i} (\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow}) \]

(1)
To find the subspace corresponding to $\mathcal{H}_{\text{one}}$, we calculate the expectation value of $\langle \hat{n}_{\uparrow i} + \hat{n}_{\downarrow i} \rangle$ of each cell for each eigenstate and select those eigenstates satisfying $\sum_{i=1}^{L} |\langle \hat{n}_{\uparrow i} + \hat{n}_{\downarrow i} \rangle - 1|/L < \epsilon$ where $\epsilon$ is a small quantity and is usually set to 0.05 in our calculation. If all the parameters are set properly, the number of selected eigenstates $N_S$ is usually $2^L$ which is just the size of the spin-1/2 model’s Hilbert space. To make quantitative comparison between these two models, the energy spectrum and correlation function are calculated exactly. For a lattice with finite size $L$, the correlation function $\langle \hat{S}^z_i \hat{S}^z_{i+d} \rangle$ for the $k$-th selected eigenstate $|\phi_k\rangle$ is defined as

$$C_d^{(k)} = \frac{\sum_{i=1}^{L-d} \langle \hat{S}^z_i - \hat{n}_{\uparrow i} \rangle \langle \hat{n}_{\uparrow i+d} - \hat{n}_{\uparrow i+d} \rangle |\phi_k\rangle}{4(L-d)}.$$ (4)

The results are depicted in FIG. 2(a) which shows very good consistency of these two models. And FIG. 2(b) reveals antiferromagnetism of the selected eigenstate with lowest energy.

As demonstrated above, the validity of mapping the double-chain Bose-Hubbard model to the Ising model depends on the equivalence of $\mathcal{H}_{\text{one}}$ and $\mathcal{H}_{\text{Ising}}$. However,

where $t_{in}$ is a small tunneling energy along z-axis and $\delta_z$ is a small energy offset between the upper site and the lower site in each cell. Here $\uparrow$ or $\downarrow$ represents particle occupation at the upper or lower site, not real spin. This model is isomorphic to a two-component Bose-Hubbard model in a single-chain tilted lattice with $U_{\uparrow\uparrow} = U_{\downarrow\downarrow} = U$. Applying the second-order perturbation theory, the effective Hamiltonian in $\mathcal{H}_{\text{one}}$ can be written as

$$H_{\text{eff}} = \sum_i J_z \hat{S}^z_i \hat{S}^z_{i+1} + J_x (\hat{S}^x_i \hat{S}^x_{i+1} + \hat{S}^y_i \hat{S}^y_{i+1}) + h_x \hat{S}^x_i + h_z \hat{S}^z_i$$

with

$$J_z = \frac{8U t_x^2}{\Delta^2 - U^2} - \frac{4U_{\uparrow\uparrow} t_{in}^2}{\Delta^2 - U_{\uparrow\uparrow}^2}, J_x = \frac{4U_{\uparrow\downarrow} t_{in}^2}{\Delta^2 - U_{\uparrow\downarrow}^2}.$$ (3)

where $\hat{S}^\alpha_i = \frac{1}{2} (\hat{c}^\dagger_i \sigma^\alpha \hat{c}_i + \hat{c}^\dagger_i \sigma^\alpha \hat{c}_i)$, $\sigma^\alpha$ are the Pauli matrices. And $h_x = -2t_{in}, h_z = \delta_z$ are equivalent magnetic fields. Above deduction is under the approximation $\langle \hat{n}_{\uparrow i} + \hat{n}_{\downarrow i} \rangle = 1$ for each cell corresponding to $\mathcal{H}_{\text{one}}$. As explained in FIG. 1(c), the above XXZ model will change into an Ising model when $U_{\uparrow\downarrow} = 0$.

**Validation of mapping to the 1D transverse Ising model.** To verify the validity of the effective Ising model at $U_{\uparrow\downarrow} = 0$, an exact-diagonalization calculation is performed on the upper lattice model with 6 bosons in a $2 \times 6$ lattice on account of the limit of our computer.

FIG. 1. (a) Spin mapping of the half-filled Bose-Hubbard model in a double-chain tilted lattice. Spin up or down is represented by the occupation of a spinless boson at the upper or lower site in each cell. (b) Localization of atoms in each cell is provided by the nearest lattice gap when $|\Delta - U| \gg t_x, |\Delta - U_{\uparrow\downarrow}| \gg t_x, \Delta \gg t_x$. But this kind of localization will be destroyed at some resonant points, such as the second-order resonant point $\Delta = \Delta/2$. (c) When $U_{\uparrow\downarrow} = 0$, the second-order superexchange interaction from the paired cases as above cancels out completely, leaving only a $\hat{S}^z_i \hat{S}^z_{i+1}$ superexchange interaction in the subspace $\mathcal{H}_{\text{one}}$.

FIG. 2. Validation of the equivalence between the double-chain Bose-Hubbard model and the XXZ model. (a) Energy spectrum and nearest correlation function of the selected eigenstates are both consistent with the equivalent XXZ model. The parameters are set to $U = 1$, $\Delta = 1.5$, $t_x = 0.04$, $U_{\uparrow\downarrow} = 0.03$, $t_{in} = 0.04$, $\delta_z = 0.01$. (b) $C_d$ of the ground state in (a) at different $t_{in}$. The other parameters are the same as those in (a). It reveals antiferromagnetism of this model at specified parameters. (c)-(d) Number of selected eigenstates $N_S$ with respect to $\Delta$ and $t_x$ at $U = 1$, $U_{\uparrow\downarrow} = 0.01$, $t_{in} = 0$ and $\delta_z = 0$. The valid region is in dark blue where $N_S = 2^L$. Those peaks centered at $\Delta/U = 1/n(n = 1, 2, 3, \cdots)$ reveals $n^{th}$-order resonant points where the equivalence between these two models fails.
there exist some resonant points where some states in $\mathcal{H}_{\text{one}}$ are resonantly coupled with other states outside $\mathcal{H}_{\text{one}}$, resulting in a number of selected eigenstates less than $2^L$. These resonant points can be determined by $a\Delta + bU + cU_{1\downarrow} = 0$ where $a, b, c$ are small integers. As shown in FIG. 2(c), the number of selected eigenstates $N_{S}$ regarding $\Delta$ and $t_x$ is calculated when $U_{1\downarrow} = 0.01$. The $n$-th resonant points at $\Delta = U/n(n = 1, 2, 3, \ldots)$ can be determined by those peaks where $N_{S}$ drops below $2^L$. Specifically, the second-order resonant point at $U_{1\downarrow} = 0$ will disappear due to the total cancellation of resonant coupling in the context of a tilted lattice, similar to the case in FIG. 1(c). But we usually set a tiny nonzero $U_{1\downarrow}$ to avoid computing error (see Supplementary Materials). The validity of $U_{1\downarrow} = 0$ means that the 1D Ising model can be fully achieved in this way.

**Simulation of dynamical quantum phase transition of the 1D transverse Ising model.** — With the upper model, we demonstrate that the dynamical quantum phase transition (DQPT) of the 1D transverse Ising model can be simulated. The typical characteristic of DQPT is the emergence of periodic nonanalytic points of a rate function when the system quenches across a quantum phase transition point from an initial ground state [26]. This phenomenon has been observed in an ion trap system [27] and in a superconducting qubit circuit [28] by directly simulating a transverse Ising model. And in a topological system, DQPT appears as a sudden creation or annihilation of vortex pairs at critical times [29]. But DQPT by simulating a 1D transverse Ising model in ultracold neutral atom systems has not been reported yet.

For dilute ultracold gases in optical lattice, it can be regarded as an isolated system if there is no obvious heating and atom loss within the time period of experiment. Thus, varying of system parameters preserves entropy and atom number. To simulate the process of DQPT of the above model, we assume the initial state is a Mott insulator prepared in the upper double-chain lattice with a very large negative $\delta_z$ with $\Delta = 0$. At this time, $\langle \hat{n}_{\uparrow\uparrow} \rangle \approx 1$ and $\langle \hat{n}_{\downarrow\downarrow} \rangle \approx 0$ for each cell. Then it follows a procedure shown in FIG. 3(a) to produce DQPT of the 1D transverse Ising model. We assume the initial Mott insulator is in thermal equilibrium with a certain entropy which keeps constant throughout the evolving process. The system state can be described by a density operator $\hat{\rho}(t)$ which follows the equation $i\hbar \frac{\partial}{\partial t} [\hat{H}(t), \hat{\rho}] = [e^{-i\hat{H}(0)/k_{\text{B}}T}, \hat{\rho}]$, where the temperature $T$ is determined by the designated total entropy $S/N_{k_{\text{B}}} = -\text{tr}(\hat{\rho}(0) \ln(\hat{\rho}(0)))/N$. $\hat{\rho}(t)$ can be decomposed as

$$\hat{\rho}(t) = \hat{\rho}_{\text{one}}(t) + \hat{\rho}_{\text{other}}(t)$$

$$\hat{\rho}_{\text{one}}(t) = P_+(t)|+\rangle\langle+| + P_-(t)|-\rangle\langle-| + \hat{\rho}_{\text{one}, \text{else}}$$

where $|+\rangle(|-\rangle)$ is the state with every atom on the upper(lower) site in each cell. $\hat{\rho}_{\text{one}}$ is the density operator in the subspace $\mathcal{H}_{\text{one}}$. $|+\rangle$ and $|-\rangle$ are two degenerate ground states of the effective Ising model when $J_z < 0, h_z = 0, h_x = 0$. Denoting $g = 2|h_z|/J_z$, when $g$ is quenched from $g_0 = 0$ to a designated $g_1$, the rate function for such a small system is introduced as [30, 31]

$$\lambda(t) = \frac{1}{L} \min(-\ln P_+(t), -\ln P_-(t)).$$

With proper approximation of the evolving process (see Supplementary Materials), the results are shown in FIG. 3(b) and FIG. 3(c). We can see the periodic nonanalytic behaviors of $\lambda(t)$ at certain times $t^*_n(g_0, g_1) = (n + 1/2)t^{\ast}$. $t^{\ast}$ is in good agreement with the theoretical result $t^{\ast}(g_0, g_1) = 2\pi/|J_z|\sqrt{g_1^2 - 1}$ ($g_1 > 1$) for $L \rightarrow \infty$ [26], as shown of the black dashed lines in FIG. 3(b). And only when $g_1$ is quenched across the quantum phase transition point $g_1 = 1$, there are periodic nonanalytical points of $\lambda(t)$. And as depicted in the inset of FIG. 3(c), the magnetization $|s_z(t)| = |\text{tr}(\hat{\rho}(t)\sum_i \hat{S}^z_i/L)|$ becomes zero at $t^*_n$ when the entropy is small. These are all in accordance with the DQPT of 1D transverse Ising model. The deviation from the ideal case is due to an open boundary condition and a small lattice size. The influence of the total entropy of the initial Mott insula-

![FIG. 3. Simulation of the DQPT of the effective 1D transverse Ising model from the double-chain Bose-Hubbard model.](image-url)
tor is shown in FIG. 3(c). It can be noticed that the total entropy has little effect on $\lambda(t)$ except for its magnitude, but has a noticeable effect on $|s_{z}(t)|$. This is because the subspace $\mathcal{H}_{\text{one}}$ is decoupled with the other subspaces in terms of second-order perturbation at the valid parameter regime. Thus, the evolution of $\hat{\rho}_{\text{one}}(t)$ after quenching is mainly governed by $\hat{\rho}_{\text{one}}(t = 0)$ and $\hat{H}_{\text{eff}}$ defined in Eqn. (2), producing a similar $\lambda(t)$ as long as $\hat{\rho}_{\text{one}}(t = 0) \approx P_{+}(t = 0)\{+\}\{+\}$ regardless of the initial total entropy. While $|s_{z}(t)|$ is $|s_{z}(t)|_{\text{one}} + |s_{z}(t)|_{\text{other}}$, $|s_{z}(t)| \approx |s_{z}(t)|_{\text{one}}$ is only established when $\hat{\rho}(t = 0) \approx \hat{\rho}_{\text{one}}(t = 0)$, in other words, the initial total entropy is small.

**Summary and outlook.**—In summary, we have proposed a double-chain Bose-Hubbard model in an 1D tilted lattice. The low-energy spectrum of a subspace $\mathcal{H}_{\text{one}}$ can faithfully simulate an 1D transverse Ising model at a validated parameter regime. Meanwhile, we design a process of simulating the dynamical quantum phase transition of the 1D transverse Ising model from a Mott insulator of the double-chain lattice. The non-analytical points show good agreement with theoretical predictions. And we expect that the initial entropy of the Mott insulator should not block the dynamical quantum phase transition of effective transverse Ising model. Recently, simulation of a two-component Bose-Hubbard model in a single titled chain with $^7\text{Li}$ atom has been reported [32]. This experiment reveals the superexchange interaction in a tilted lattice, which strongly supports the feasibility of realizing the above model with neutral bosonic $^7\text{Li}$ atom. Our results may give some inspirations of following experiments.

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Supplementary Materials

EFFECTIVE HAMILTONIAN

To derive the effective Hamiltonian $\hat{H}_{\text{eff}}$ (Eqn. (2) in the main text) on the subspace $\mathcal{H}_{\text{one}}$, we rewrite the Hamiltonian of the double-chain Bose-Hubbard model as $H = H_0 + \delta H + V_i$, where

$$\hat{H}_0 = \sum_{i \sigma=\uparrow,\downarrow} \left[ \frac{U}{2} \hat{n}_{i\sigma}(\hat{n}_{i\sigma} - 1) + i \Delta \hat{n}_{i\sigma} \right] + U_{\uparrow\downarrow} \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

$$\delta \hat{H} = - t_{in} \sum_i (\hat{c}_{i\uparrow} \hat{c}_{i+\downarrow} + h.c.) + \frac{\delta_z}{2} \sum_i (\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow})$$

$$\hat{V}_i = - t_x \sum_{i \sigma=\uparrow,\downarrow} (\hat{c}_{i\sigma} \hat{c}_{i+1\sigma} + h.c.)$$

The second-order perturbative Hamiltonian on the subspace $P_0$ is $H_{\text{eff}} = H_0 + P_0[\delta \hat{H} + \hat{V}_i(E_0 - H_0)^{-1}\hat{V}_i]P_0$, where $E_0 = \Delta(L+1)/2$ and $P_0$ is a projection operator on the subspace which is composed of eigenstates of $H_0$ with an eigenenergy $E_0$. $P_0 = P_{\text{one}} + P_1$, where $P_{\text{one}}$ is the projection operator on the subspace $\mathcal{H}_{\text{one}}$. If there are no resonant hoppings, $P_{\text{one}}$ is decoupled with $P_1$ in terms of second-order perturbation. $H_{\text{eff}}$ can be written as $H_{\text{eff}} = H_{\text{eff}}^{\text{one}} + H_{\text{eff}}$. Thus, in the subspace $\mathcal{H}_{\text{one}}$, the leading second-order Hamiltonian is

$$\hat{H}_{\text{eff}}^{\text{one}} = P_{\text{one}}[\delta \hat{H} + \hat{V}_i(E_0 - H_0)^{-1}\hat{V}_i]P_{\text{one}}$$

With $\hat{S}_i^\alpha = \frac{1}{2}(\hat{c}_{i\uparrow} \hat{c}_{i\downarrow} \hat{c}_{i\downarrow} \hat{c}_{i\uparrow})^T$, $P_{\text{one}}\delta \hat{H}P_{\text{one}}$ is mapped to $\sum_i (-2t_{in}\hat{S}_i^\gamma + \delta_z \hat{S}_i^\gamma)$. And the second term can be expanded as

$$P_{\text{one}}\hat{V}_i(E_0 - \hat{H}_0)^{-1}\hat{V}_iP_{\text{one}} = \sum_{\alpha,\beta,\gamma} |\alpha\rangle\langle\beta| \frac{\langle \alpha | \hat{V} | \gamma \rangle \langle \gamma | \hat{V} | \beta \rangle}{E_0 - E_{0\gamma}}$$

with $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_{\text{one}}$. These terms can be interpreted as two-step hopping processes similar to Fig. S1(b), and they give rise to a superexchange interaction $\sum_i (\frac{8U}{\Delta - U} - \frac{4U_{\uparrow\downarrow}^2}{\Delta - U}) |S_i^{\uparrow\downarrow} |S_{i+1}^{\uparrow\downarrow} + \frac{4U_{\uparrow\downarrow}^2}{\Delta - U} (|S_i^{\uparrow\downarrow} |S_{i+1}^{\uparrow\downarrow} + |S_{i+1}^{\uparrow\downarrow} |S_i^{\uparrow\downarrow})$ given the restriction of $\hat{n}_{i\uparrow} + \hat{n}_{i\downarrow} = 1$ for $\mathcal{H}_{\text{one}}$.

RESONANT HOPPING POINTS

The above deduction of $\hat{H}_{\text{eff}}^{\text{one}}$ will be invalid at some resonant hopping points. At these points, $P_{\text{one}}$ and $P_1$ are coupled through $n^{\text{th}}$-order ($n = 1, 2, 3, \cdots$) hopping. Thus, when solving for eigenstates, the number $N_S$ of the selected eigenstates with $|\hat{n}_{i\uparrow} + \hat{n}_{i\downarrow}| \approx 1$ for each cell will be less than $2^L$. In Fig. S1(a), we could clearly infer the resonant points from those lines where $N_S < 2^L$. And these resonant points satisfy the relation $a\Delta + bU + cU_{\uparrow\downarrow} = 0$, where $a, b, c$ are small integers. These resonant points give us a picture of how to select $U, U_{\uparrow\downarrow}$, $\Delta$ properly. And usually those resonant points with an order higher than three have negligible influence on $N_S$ when $|\Delta - U| \gg t_x, |\Delta - U_{\uparrow\downarrow}| \gg t_{\uparrow\downarrow}, \Delta \gg t_x$, thus they can be ignored. And the second-order resonant point at $U_{\uparrow\downarrow} = 0$ disappear due to the total cancellation of the coupling from the paired virtual hopping processes (Fig. S1(b)). However, we usually set a tiny nonzero $U_{\uparrow\downarrow}$ to avoid generating some eigenstates containing terms like $u|\alpha\rangle + v|\beta\rangle, |\alpha\rangle, |\beta\rangle \in \mathcal{H}_{\text{one}}, |\beta\rangle \in \mathcal{H}_{\text{one}}$ where $|\alpha\rangle, |\beta\rangle$ are degenerate in terms of $\hat{H}_0$ but not coupled through second-order perturbation. This computing error at $U_{\uparrow\downarrow} = 0$ sometimes occurs in our program, resulting in a $N_S < 2^L$, as shown of Fig. S1(a).

PARAMETER SETTING OF THE DYNAMICAL EVOLVING PROCESS

In the beginning, we assume the initial Mott insulator is prepared in thermal equilibrium in a flat lattice at a
parameter $\Delta = 0, U = 1, U_{t_z} = 0.01, t_x = 0, t_{in} = 0, \delta_z = 10$. At this time, all atoms are staying on the upper chain when $k_B T \ll |\delta_z|$. $\hat{\rho}(t = 0)$ is set by the initial temperaure derived from a designated total entropy.

Then $\Delta$ is increased to $\Delta = 0.7$ and $\delta_z$ is lowered to $\delta_z = 0$. Next, $t_x$ is increased to $t_x = 0.04$ rapidly followed by quenching of $t_{in}$. For simplicity of calculation, we assume varying of $\Delta, \delta_z, t_x$ and quenching of $t_{in}$ are all so quick that $\hat{\rho}(t_i) \approx \hat{\rho}(0)$ where $t_i$ is the time point after quenching. Basically it is enough that the varying time is much shorter than the tunneling time $\hbar/t_x$ and $\hbar/t_{in}$, keeping $\hat{\rho}_{one}(t_i) \propto |+\rangle\langle +|$. After quenching, the Hamiltonian does not change any more and controls the evolution of $\hat{\rho}(t)$ by $i\hbar \frac{\partial}{\partial t} = [\hat{H}, \hat{\rho}]$. Then $P_+(t) = \langle +|\hat{\rho}(t)|+\rangle$ and $P_-(t) = \langle -|\hat{\rho}(t)|-\rangle$ can be derived. In above deduction we have assumed there is no obvious heating and various forms of noise within the time period of evolving process.