Lexicographic Logic: a Many-valued Logic for Preference Representation

Angelos Charalambidis, Giorgos Papadimitriou, Panos Rondogiannis, and Antonis Troumpoukis
University of Athens
{achara, gspapajim, prondo, antru}@di.uoa.gr

Abstract. Logical formalisms provide a natural and concise means for specifying and reasoning about preferences. In this paper, we propose lexicographic logic, an extension of classical propositional logic that can express a variety of preferences, most notably lexicographic ones. The proposed logic supports a simple new connective whose semantics can be defined in terms of finite lists of truth values. We demonstrate that, despite the well-known theoretical limitations that pose barriers to the quantitative representation of lexicographic preferences, there exists a subset of the rational numbers over which the proposed new connective can be naturally defined. Lexicographic logic can be used to define in a simple way some well-known preferential operators, like “A and if possible B”, and “A or failing that B”. Moreover, many other hierarchical preferential operators can be defined using a systematic approach. We argue that the new logic is an effective formalism for ranking query results according to the satisfaction level of user preferences.

1 Introduction

During the past three decades, many formalisms have been developed for representing preferences, both in artificial intelligence [9] as well as in databases [22]. Of particular interest are the logical approaches in which the specification of preferences is performed using operators that implicitly manipulate the underlying preference values [5,7,2,16,10,19,8]. Such formalisms are usually declarative, concise, and easy to understand and reason about.

In this paper, we develop lexicographic logic, a simple extension of classical propositional logic that can express a variety of preferences, most notably lexicographic ones. The proposed logic adds only one formation rule to the syntax of propositional logic: if $\phi_1$ and $\phi_2$ are formulas, then so is $(\phi_1 \gg \phi_2)$. The formula $(\phi_1 \gg \phi_2)$ can be read “$\phi_1$ has a lexicographic priority over $\phi_2$” (a notion that will be explained in detail in Section 2). The semantics of “$\gg$” can be defined in terms of finite lists of truth values of a three-valued logic. Actually, we demonstrate that such lists have a natural mapping to rational numbers in the interval $[-1,1]$, and the meaning of “$\gg$” can then also be understood as a function that maps pairs of rational numbers to rational numbers. This interpretation of “$\gg$” allows the evaluation of lexicographic logic formulas using standard floating-point arithmetic. Apart from its simplicity, an advantage of lexicographic logic is that it can be used to represent concisely well-known preferential operators as well as to define new ones. The main contributions of the paper can be summarized as follows:

- We define a novel, non-classical, propositional logic for expressing preferences. The primary connective of this logic expresses lexicographic priority, which
is known to be non-trivial to specify from a quantitative point of view [12].
We demonstrate that the semantics of this new connective can be specified quantitatively as a function over rational numbers. The main theorem of the paper (Theorem 2) asserts that “≫” ensures strict monotonicity with respect to lexicographic comparison.

– We present the properties of lexicographic logic and investigate its connections with alternative operators that have been proposed in the literature. We demonstrate that some well-known and useful such operators can be succinctly represented using simple formulas of lexicographic logic. Additionally, we propose a systematic approach for representing new hierarchical preferential operators as derived formulas of lexicographic logic.

The rest of the paper is organized as follows. Section 2 provides the motivation and intuition behind the proposed approach. Section 3 presents the syntax and semantics of lexicographic logic. Section 4 establishes properties of the proposed logic, and Section 5 demonstrates that the semantics of the logic can alternatively be defined in a numerical way. In Section 6 it is argued that formulas of the proposed logic can be used to define a variety of hierarchical preferential operators. Section 7 presents related work on preference representation formalisms and Section 8 concludes the paper giving pointers to future work.

2 Motivation and Intuition

In this paper, we contribute to the logical specification of preferences by introducing lexicographic logic, an extension of propositional logic for describing lexicographic (as-well-as many other) preferences. More specifically, we add to the syntax of propositional logic the new connective “≫”. The formula (φ₁ ≫ φ₂) is read “φ₁ has a lexicographic priority over φ₂”. Intuitively, lexicographic priority can be explained using levels of preference satisfaction, as follows:

– if both φ₁ and φ₂ are true, then we are completely satisfied;
– if φ₁ is true and φ₂ is false, then we are satisfied but not entirely;
– if φ₁ is false and φ₂ is true, we are dissatisfied, but not strongly so;
– finally, if both φ₁ and φ₂ are false, then we are completely dissatisfied.

The following example motivates the above ideas.

Example 1. Consider the specification of our preferences for buying a new car. The formula (electric ≫ fast) means that if we buy a car that is both electric and fast, we will be completely satisfied; if we buy one that is electric but not fast, then we will not be entirely satisfied; if we buy a car that is only fast, then we will be dissatisfied (but not entirely); and if none of our preferences is satisfied, then this will be our least preferred state of affairs.

To specify the formal semantics of (φ₁ ≫ φ₂), we use a many-valued logic to express the different levels of preference satisfaction discussed above. In particular, if F,T denote the classical false and true values, we expect that:

\[(F ≫ F) < (F ≫ T) < (T ≫ F) < (T ≫ T)\]

where “<” can be read as “less preferred than”. This suggests that we probably need a four-valued logic to express these four truth levels, which we could call
“false”, “less false”, “less true”, and “true”. However, it turns out that four truth levels are not enough. Imagine, for example, that we apply “≫” on a pair consisting of the “false” and the “less false” truth values. This will give a new truth value between “false” and “less false”. Actually, one can verify that we need an infinity of truth values if we want to be able to specify formulas with an arbitrary number of occurrences of “≫”. The question we have to answer is how this set of truth values is exactly structured and how are its elements ordered.

To understand the difficulties of defining such a set, we need to become slightly more formal. Let us denote by $\mathbb{V}$ this (yet unknown) set and let “<” be the (yet undefined) ordering relation on the elements of $\mathbb{V}$. Let $u_1, u_2, v_1, v_2 \in \mathbb{V}$ be truth values. We define the “lexicographically smaller” relation $<_L$ on $\mathbb{V} \times \mathbb{V}$, as follows:

$$(u_1, u_2) <_L (v_1, v_2) \iff (u_1 < v_1) \lor ((u_1 = v_1) \land (u_2 < v_2))$$

The meaning of “≫” should be a function $f$ that respects the $<_L$ ordering. This means that for all truth values $u_1, u_2, v_1, v_2$ it must hold:

$$(u_1, u_2) <_L (v_1, v_2) \iff f(u_1, u_2) < f(v_1, v_2)$$

One could view truth values as real numbers and attempt to define $f$ as a function $f : (X, X) \to X$ where $X$ is a subset of the set $\mathbb{R}$ of real numbers. For example, one could view the classical truth values $F$ and $T$ as the real numbers $-1$ and $1$ respectively, and try to define “≫” as a function $f : \{[-1, 1], [-1, 1]\} \to [-1, 1]$. However, there is a well-known obstacle in such an approach, which is described by the following folk theorem (see for example [12, p. 363] for a proof):

**Theorem 1.** There does not exist a function $f : (\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ such that $(u_1, u_2) <_L (v_1, v_2) \iff f(u_1, u_2) < f(v_1, v_2)$.

The proof of the above theorem can easily transfer to the case where we replace $\mathbb{R}$ with the closed interval $[-1, 1]$. Therefore, we conclude that if we would like to specify the semantics of the binary operator “≫”, we would have to focus on more refined and carefully constructed truth domains than the set of the real numbers (or intervals of it).

To understand how we bypass the above problem, we return to Example 1. Our first idea (which we will subsequently slightly refine) is that we can express the different levels of preferences by using a truth domain whose elements are *lists* of classical truth values $F$ and $T$. The semantics of the “≫” operator can then be defined as the concatenation of such lists. In our example, if both *electric* and *fast* are true, we assign to the formula the value $[T, T]$; if *electric* is true and *fast* is false, we assign the value $[T, F]$; if *electric* is false and *fast* is true, we assign the value $[F, T]$; if both atoms are false, we assign the value $[F, F]$. Notice that these lists, when viewed as words compared lexicographically over the alphabet $\{F, T\}$, where $F < T$, express precisely the four different levels of truth (“false”, “less false”, “less true”, and “true”) that we desire for this formula, namely:

$$[F, F] < [F, T] < [T, F] < [T, T]$$

In the following, even the classical truth values $F$ and $T$ will be written in list form, namely $[F]$ and $[T]$. Notice also that we expect $[T, T]$ to be equal to $[T]$, because this is a situation where all our preferences are satisfied; similarly, we expect $[F, F]$ to be equal to $[F]$. 

There is, however, a further refinement of the above scheme that is required. The basic complication that needs to be addressed is that the operator \( \gg \) should not be associative. This issue is illustrated by the following example.

**Example 2.** Consider the formulas \( \text{electric} \gg (\text{fast} \gg \text{blue}) \) and \( (\text{electric} \gg \text{fast}) \gg \text{blue} \). We claim that these two formulas should not be semantically equivalent. To see this, consider a truth assignment that assigns to \( \text{electric} \) the value \([T]\), to \( \text{fast} \) the value \([F]\), and to \( \text{blue} \) the value \([F]\). Intuitively speaking, the first formula evaluates to \([T] \gg [F,F]\) while the second one to \([T,F] \gg [F]\). Comparing these two values, we see that in the former one, our primary requirement is fully satisfied (truth value \([T]\)), while in the latter one, our primary requirement is only partially satisfied (truth value \([T,F]\)). In other words, under this truth assignment, the first formula is “more satisfied” than the second one.

The above discussion suggests that the meaning of \( \gg \) should not be associative, and therefore it should not be defined as just list-concatenation (which is an associative operator). To properly define the semantics of \( \gg \), we will use one extra truth value, namely 0. By prefixing each concatenation operation with a 0, formulas such as the ones that appear in the above example will be discriminated. As William W. Wadge observed\(^1\), our use of the zeros inside the lists essentially mimics the Polish notation of the parenthesized expressions. In this way, different lists are created for different parenthesizations of an expression, and this ensures non-associativity. Moreover, as we will demonstrate shortly (see Theorem 2) this simple operation ensures preservation of the lexicographic property. More specifically, we demonstrate that \( \gg \), when viewed as a binary infix function, satisfies the property:

\[(u_1, u_2) <_L (v_1, v_2) \iff (u_1 \gg u_2) < (v_1 \gg v_2)\]

for all \( u_1, u_2, v_1, v_2 \in V \). Therefore, our definition of \( \gg \) bypasses the restriction posed by Theorem 1, by relying on a carefully selected truth domain.

Actually, we demonstrate that the truth domain \( V \) corresponds to a subset of the set \( \mathbb{Q} \) of rational numbers: the three-valued lists of \( V \) can be viewed as numbers in the balanced-ternary number system \([15]\) and can be transformed into standard decimal rational numbers in the interval \([-1, 1]\). Therefore, lexicographic logic formulas can be easily evaluated using standard floating-point arithmetic.

### 3 Lexicographic Logic: Syntax and Semantics

In this section, we introduce the syntax and the semantics of lexicographic logic and motivate it with examples. The syntax of the proposed logic extends that of propositional logic with a new formation rule:

**Definition 1.** Let \( \mathcal{A} \) be a set of propositional atoms. The set of well-formed formulas of lexicographic logic is inductively defined as follows:

- Every element of \( \mathcal{A} \) is a well-formed formula,
- If \( \phi_1 \) and \( \phi_2 \) are well-formed formulas, then \( (\phi_1 \land \phi_2) \), \( (\phi_1 \lor \phi_2) \), \( (\neg \phi_1) \), and \( (\phi_1 \gg \phi_2) \), are well-formed formulas.

---

\(^1\) Personal communication.
For simplicity reasons, we will often omit the outermost parentheses from formulas. Moreover, to simplify the readability of expressions that contain multiple occurrences of “$\gg$”, we will assume that this operator associates to the right. So, for example, $x \gg (y \gg z)$ will be written as $x \gg y \gg z$.

The truth domain of lexicographic logic is denoted by $V$, and consists of lists of the truth values $F$, $0$, and $T$. By overloading notation, we will use the symbol “$\gg$” to also denote an operation on lists that corresponds to the meaning of the syntactic element “$\gg$”. More formally:

**Definition 2.** Let $u,v$ be lists of the elements $F$, $T$, and $0$. We define:

$$(u \gg v) = \begin{cases} [F], & \text{if } u = v = [F] \\ [T], & \text{if } u = v = [T] \\ [0] + u + v, & \text{otherwise} \end{cases}$$

where $+$ is the list concatenation operation.

We can now precisely define the truth domain of lexicographic logic:

**Definition 3.** The set $V$ of truth values is the set inductively defined as follows:

- $[F] \in V$ and $[T] \in V$.
- If $u,v \in V$, then $(u \gg v) \in V$.

Notice that due to Definition 2, lists of the form $[0,T,T]$ and $[0,F,F]$ are not allowed (because they are considered identical to $[T]$ and $[F]$ respectively).

Each element of $V$ represents some degree of “true” or “false”. To understand whether a given element is true or false, it suffices to look at its sign:

**Definition 4.** For every $v \in V$ we define $\text{sign}(v)$ to be the leftmost non-zero element of $v$.

Therefore, $[0,F,0,F,T]$ is a false value while $[0,T,F]$ a true one.

Given an arbitrary element $v \in V$, we denote with $\overline{v}$ the list that results from $v$ by inverting each $F$ to $T$ and each $T$ to $F$. For example, $[0,F,0,T,F] = [0,T,0,F,T]$.

**Lemma 1.** For every $v \in V$, $\overline{v} \in V$.

*Proof.* By a straightforward structural induction.

The following property of the elements of $V$ is easy to establish:

**Lemma 2.** For every $v \in V$, the length of $v$ is odd.

*Proof.* By a straightforward structural induction.

As it turns out, no element of $V$ is a proper prefix of another element. We will use this property in the proof of the main theorem of the paper (Theorem 2).

**Lemma 3.** Let $u,v \in V$. If $u$ is a prefix of $v$, then $u = v$.

*Proof.* By induction on the length of $u$. If $|u| = 1$ then, $u = [F]$ or $u = [T]$, and it must also be $v = [F]$ or $v = [T]$, respectively. Assume the lemma holds for all elements of $V$ of length $\leq k$. We demonstrate the statement for elements of length $k + 2$. Assume $u = [0] + u_1 + u_2$ and $v = [0] + v_1 + v_2$. Since $u$ is a prefix of $v$, $u_1 + u_2$ is a prefix of $v_1 + v_2$, which implies that either $u_1$ is a prefix of $v_1$ or $v_1$ is a prefix of $u_1$. By the induction hypothesis, $u_1 = v_1$. This implies that $u_2 = v_2$ and therefore $u = v$. 
The meaning of a formula with respect to \( I \)

**Definition 7.** Lexicographic logic is defined as follows:

The meaning of a formula is an element of \( \mathbb{V} \).

Notice that, due to Lemma 3, for any \( u, v \) we will write \( u \ll v \) if there exists \( 1 \leq r \leq \min\{k, m\} \) such that \( u_1 = v_1, \ldots, u_{r-1} = v_{r-1} \) and \( u_r < v_r \). We will write \( u \ll v \) if either \( u = v \) or \( u < v \).

Notice that, due to Lemma 3, for any \( u, v \in \mathbb{V} \), it will either be \( u \ll v \) or \( v \ll u \). It is easy to see that the lexicographic ordering \( \ll \) of Definition 5 is a total ordering on \( \mathbb{V} \).

The notion of truth assignment can be generalized as follows:

**Definition 6.** A truth assignment is a function from the set \( \mathcal{A} \) of propositional atoms to the set \( \mathbb{V} \) of truth values.

The meaning of a formula is an element of \( \mathbb{V} \). More specifically, the semantics of lexicographic logic is defined as follows:

**Definition 7.** Let \( \phi_1, \phi_2 \), and \( \phi \) be formulas and let \( I \) be a truth assignment. The meaning of a formula with respect to \( I \) is recursively defined as follows:

- \( [p](I) = I(p) \), where \( p \in \mathcal{A} \)
- \( [[\phi_1 \land \phi_2]](I) = \min\{[[\phi_1]](I), [[\phi_2]](I)\} \)
- \( [[\phi_1 \lor \phi_2]](I) = \max\{[[\phi_1]](I), [[\phi_2]](I)\} \)
- \( [[\lnot \phi]](I) = \lnot [[\phi]](I) \)
- \( [[\phi_1 \lhd \phi_2]](I) = [[\phi_1]](I) \lhd [[\phi_2]](I) \)

where \( \min \) and \( \max \) are defined with respect to the ordering of Definition 5.

Using the above semantics, Table 1 presents the possible values that the expressions \( (x \gg y) \gg z \) and \( x \gg (y \gg z) \) can take when the propositional atoms \( x, y, z \) receive the standard truth values \( [F] \) and \( [T] \). Table 2 presents the same values as that of Table 1 ordered from the lowest to the highest. The table is divided

| \( x \) | \( y \) | \( z \) | \( (x \gg y) \gg z \) | \( x \gg (y \gg z) \) |
|---|---|---|---|---|
| \( [F] \) | \( [F] \) | \( [F] \) | \( [F] \) | \( [F] \) |
| \( [F] \) | \( [F] \) | \( [T] \) | \( 0, F, 0, F, T \) | \( [F] \gg [T] \gg [F] \) |
| \( [F] \) | \( [F] \) | \( [0, F, T, F] \) | \( 0, F, 0, F, T \) | \( [F] \gg [F] \gg [F] \) |
| \( [F] \) | \( [T] \) | \( 0, F, T, F \) | \( [F] \gg [T] \gg [F] \) | \( [F] \gg [0, F, F, T] \) |
| \( [T] \) | \( [F] \) | \( 0, 0, T, F, F \) | \( 0, F, T, F \) | \( [T] \gg [F] \gg [F] \) |
| \( [T] \) | \( [T] \) | \( 0, 0, T, F, T \) | \( 0, T, 0, F, T \) | \( [T] \gg [F] \gg [T] \) |
| \( [T] \) | \( [T] \) | \( 0, T, F \) | \( 0, T, 0, F, T \) | \( [T] \gg [T] \gg [F] \) |
| \( [T] \) | \( [T] \) | \( [T] \) | \( [T] \) | \( [T] \gg [T] \gg [T] \) |

Table 1: Evaluation of \( (x \gg y) \gg z \) and \( x \gg (y \gg z) \) under different truth assignments.

| Expression | List |
|---|---|
| \([F] \gg [F] \) | \([F] \gg [F] \) |
| \([F] \gg [F] \gg [F] \) | \([F] \gg [F] \gg [F] \) |
| \([F] \gg [F] \gg [T] \) | \([0, F, 0, F, T] \) |
| \([F] \gg [T] \gg [F] \) | \([0, F, 0, T, F] \) |
| \([F] \gg [T] \gg [T] \) | \([0, F, T] \) |
| \([F] \gg [F] \gg [T] \) | \([0, F, T] \) |
| \([F] \gg [T] \gg [T] \) | \([0, F, T] \) |
| \([F] \gg [T] \gg [T] \) | \([0, 0, F, T, F] \) |
| \([F] \gg [T] \gg [T] \) | \([0, 0, F, T, F] \) |
| \([T] \gg [F] \gg [F] \) | \([0, 0, T, F, F] \) |
| \([T] \gg [F] \gg [T] \) | \([0, 0, T, F, F] \) |
| \([T] \gg [T] \gg [F] \) | \([0, T, F, F] \) |
| \([T] \gg [T] \gg [T] \) | \([0, T, F, F] \) |
| \([T] \gg [T] \gg [T] \) | \([0, T, F, F] \) |

Table 2: Ordered values for the expressions of Table 1.
in two layers. The upper layer (first 8 entries) contains values that correspond to different degrees of false (i.e., that have a sign equal to $F$), while the lower layer (next 8 entries) correspond to different degrees of true (have a sign equal to $T$). Given a formula $\phi$ and a set of different truth assignments, we can find the most preferable truth assignments for $\phi$ by calculating the meaning of $\phi$ under all these assignments, and comparing the results.

**Definition 8.** Let $\phi$ be a formula and consider truth assignments $I_1, I_2$. We will say that $I_2$ is preferable to $I_1$ with respect to formula $\phi$ if $[\phi](I_1) < [\phi](I_2)$.

**Example 3.** Let $\phi$ be the formula $\text{electric} \gg (\text{fast} \gg \text{blue})$ of Example 2. Consider the truth assignments:

$I_1 = \{(\text{electric},[T]),(\text{fast},[T]),(\text{blue},[T])\}$

$I_2 = \{(\text{electric},[T]),(\text{fast},[T]),(\text{blue},[F])\}$

$I_3 = \{(\text{electric},[F]),(\text{fast},[T]),(\text{blue},[T])\}$

Using Table 1 we get that $I_1(\phi) = T$, $I_2(\phi) = 0,T,0,T,F$ and $I_3(\phi) = 0,F,T$. By lexicographically comparing the corresponding lists, we get that with respect to formula $\phi$, the truth assignment $I_1$ is preferable to $I_2$ which is preferable to $I_3$.

**Example 4.** Let $\phi$ be the formula:

$$(\text{electric} \oplus \text{diesel}) \gg (\text{fast} \land \neg \text{expensive})$$

where $\oplus$ is the usual exclusive-or operation. One can easily verify that both of the interpretations:

$I_1 = \{(\text{electric},[T]),(\text{diesel},[F]),(\text{fast},[T]),(\text{expensive},[F])\}$

$I_2 = \{(\text{electric},[F]),(\text{diesel},[T]),(\text{fast},[T]),(\text{expensive},[F])\}$

are maximally preferable because they both assign to $\phi$ the truth value $[T]$.

One could define a consequence relation for lexicographic logic based on maximally preferred truth assignments (see, for example, the corresponding such notion for QCL in [5, Definition 6]). Due to space restrictions, we do not pursue this direction in the present paper, apart from a slightly more extended discussion at the end of Section 7.

### 4 Some Properties of Lexicographic Logic

One key property that needs to be established for lexicographic logic, is that the “$\gg$” operator indeed implements lexicographic priority. Intuitively speaking, this means that when we apply “$\gg$” on two distinct pairs of arguments, and the first pair is lexicographically smaller than the second one, then the first result is lexicographically smaller than the second one. More formally, we define the lexicographic ordering on pairs, as follows:

**Definition 9.** For all $u_1, u_2, v_1, v_2 \in V$, we write $(u_1,u_2) <_L (v_1,v_2)$ if either $u_1 < v_1$, or $u_1 = v_1$ and $u_2 < v_2$.

The above property is expressed by the following theorem:
Lemma 5. Easily established:

Theorem 2. Let $u_1, u_2, v_1, v_2 \in V$. Then, $(u_1, u_2) <_L (v_1, v_2)$ iff $(u_1 \gg u_2) < (v_1 \gg v_2)$.

Proof. Consider first the left to right implication. We examine cases with respect to $(u_1, u_2)$. If $(u_1, u_2) = ([F], [F])$ then $(v_1, v_2) \neq ([F], [F])$. But then, $(u_1 \gg u_2) = [F] < (v_1 \gg v_2)$, because the first element of $(v_1 \gg v_2)$ will either be 0 or $T$. Notice also that it can not be $(u_1, u_2) = ([T], [T])$, because in this case there do not exist $v_1, v_2$ such that $(u_1, u_2) <_L (v_1, v_2)$. If $(u_1, u_2) \neq ([F], [F])$ and $(u_1, u_2) \neq ([T], [T])$, then, by the definition of $\gg$, the first element of $(u_1 \gg u_2)$ will be 0. If $(v_1, v_2) = ([T], [T])$ then $(v_1 \gg v_2) = [T]$ and $(u_1 \gg u_2) < (v_1 \gg v_2)$ obviously holds. Assume that $(v_1, v_2) \neq ([T], [T])$. Then, by the definition of $\gg$, the first element of $(u_1 \gg u_2)$ will also be 0. Since $(u_1, u_2) <_L (v_1, v_2)$, it will either be $(u_1 < v_1$ or $u_1 = v_1$ and $u_2 < v_2$. If $u_1 < v_1$ then $[0] + u_1 + u_2 < [0] + v_1 + v_2$ and therefore $(u_1 \gg u_2) < (v_1 \gg v_2)$. If $u_1 = v_1$ and $u_2 < v_2$, then again $[0] + u_1 + u_2 < [0] + v_1 + v_2$ and therefore $(u_1 \gg u_2) < (v_1 \gg v_2)$.

Consider now the right to left implication. If $(u_1, u_2) = ([F], [F])$ then $u_1 \gg u_2 = [F]$ and it has to be $(v_1 \gg v_2) \neq [F]$, which implies that $(v_1, v_2) \neq ([F], [F])$ and therefore $(u_1, u_2) <_L (v_1, v_2)$. If $(u_1, u_2) = ([T], [T])$ then there can not exist $v_1, v_2$ such that $[T] < (v_1 \gg v_2)$. Consider now the case $(u_1, u_2) \neq ([F], [F])$ and $(u_1, u_2) \neq ([T], [T])$. Then, $(u_1 \gg u_2) = [0] + u_1 + u_2$. If $(v_1, v_2) = ([T], [T])$, then, obviously, $(u_1, u_2) <_L (v_1, v_2)$. Otherwise, $(v_1 \gg v_2) = [0] + v_1 + v_2$. Let $u = u_1 + u_2$ and $v = v_1 + v_2$, and let $u[i]$ and $v[i]$ denote the $i$th element of $u$ and $v$ respectively. Since $(u_1 \gg u_2) < (v_1 \gg v_2)$, there exists $i$ such that $u[i] < v[i]$. If $i \leq |u_1|$ and $i \leq |v_1|$, then $u_1 < v_1$ and therefore $(u_1, u_2) <_L (v_1, v_2)$. The case $i \leq |u_1|$ and $i > |v_1|$ is not applicable because then $v_1$ would be a proper prefix of $u_1$, which is impossible from Lemma 3. Similarly if $i > |u_1|$ and $i \leq |v_1|$. Finally, if $i > |u_1|$ and $i > |v_1|$, then $u_1 = v_1$. Since $(u_1 \gg u_2) < (v_1 \gg v_2)$, we get $u_2 < v_2$, which implies that $(u_1, u_2) <_L (v_1, v_2)$.

Two formulas $\phi_1$ and $\phi_2$ are semantically equivalent (denoted by $\phi_1 \equiv \phi_2$) iff for every truth assignment $I$, $[\phi_1](I) = [\phi_2](I)$. One basic property that lexicographic logic inherits from propositional logic, is the substitutivity of logically equivalent formulas (see, for example, [13, pp. 20-21]). This property holds due to the compositional semantics of lexicographic logic (Definition 7), and can be established by structural induction.

Lemma 4. Let $\phi$ be a formula of lexicographic logic, $\psi$ be a subformula of $\phi$ and $\psi'$ be a formula such that $\psi \equiv \psi'$. Then, $\phi \equiv \phi[\psi \leftarrow \psi']$, where $\phi[\psi \leftarrow \psi']$ is the formula that results from $\phi$ by replacing the subformula $\psi$ with $\psi'$.

The following are some basic properties of lexicographic logic that can be easily established:

Lemma 5. For all formulas $\phi_1, \phi_2, \phi_3, \phi$, the following equivalences hold:

- $(\phi_1 \lor \phi_2) \gg \phi_3 \equiv (\phi_1 \gg \phi_3) \lor (\phi_2 \gg \phi_3)$
- $\phi_1 \gg (\phi_2 \lor \phi_3) \equiv (\phi_1 \gg \phi_2) \lor (\phi_1 \gg \phi_3)$
- $(\phi_1 \land \phi_2) \gg \phi_3 \equiv (\phi_1 \gg \phi_3) \land (\phi_2 \gg \phi_3)$
- $\phi_1 \gg (\phi_2 \land \phi_3) \equiv (\phi_1 \gg \phi_2) \land (\phi_1 \gg \phi_3)$
- $\neg (\phi_1 \gg \phi_2) \equiv (\neg \phi_1) \gg (\neg \phi_2)$
- $\neg (\neg \phi) \equiv \phi$
Proof. We prove the first statement. The remaining ones are established in a similar way. Let $I$ be an arbitrary truth assignment. We show that:

$$
[(\phi_1 \lor \phi_2) \gg \phi_3](I) = [(\phi_1 \gg \phi_3) \lor (\phi_2 \gg \phi_3)](I)
$$

Using Definition 7, the left hand side evaluates to:

$$\max\{[\phi_1](I), [\phi_2](I)\} \gg [\phi_3](I)$$

while the right hand side evaluates to:

$$\max\{[\phi_1](I) \gg [\phi_3](I), [\phi_2](I) \gg [\phi_3](I)\}$$

If $[\phi_1](I) = [\phi_2](I)$, then the equality of the above two statements obviously holds. Otherwise, assume without loss of generality that $[\phi_1](I) > [\phi_2](I)$. By Theorem 2, it is:

$$([\phi_1](I) \gg [\phi_3](I)) > ([\phi_2](I) \gg [\phi_3](I))$$

and the equality of the two statements again holds.

5 Numerical Representation

In this section, we demonstrate that the elements of $\mathbb{V}$ can be mapped to rational numbers so as that their ordering is preserved. More formally, we demonstrate that there exists a function $val : \mathbb{V} \to \mathbb{Q}$ such that for all $u, v \in \mathbb{V}$, if $u < v$ then $val(u) < val(v)$. The key idea of defining $val$ is that the elements of $\mathbb{V}$ can be considered as numbers in the interval $[-1, 1]$ written in the balanced ternary number system [15, p. 207]. As Donald Knuth poses it, balanced ternary is “perhaps the prettiest number system of all”. Balanced ternary is a ternary number system in which the coefficients are the numbers $-1, 0, 1$ (instead of 0, 1, and 2, as it happens in standard ternary notation). It is called “balanced ternary” because the coefficients are symmetrical around zero.

We consider the elements of $\mathbb{V}$ as representing balanced ternary numbers in the interval $[-1, 1]$. More specifically, $val(F) = -1$, $val(T) = 1$, $val(0) = 0$, and for every other element $u = [u_1, \ldots, u_n] \in \mathbb{V}$, we derive a rational number in the interval $(-1, 1)$ by viewing $u$ as a balanced ternary number. Since we want the value of the number to belong in the interval $(-1, 1)$, we calculate its value by using the powers of $\frac{1}{3}$ (in the same way that the value of the fractional part of a decimal number can be calculated using the powers of $\frac{1}{10}$). Formally:

$$val([u_1, \ldots, u_n]) = \sum_{i=1}^{n} val(u_i) \cdot \frac{1}{3^{i-1}}$$

Table 3 depicts the numerical values that correspond to all elements of $\mathbb{V}$ of size less than or equal to 5. The following lemma justifies why we can use the numerical representation of the elements of $\mathbb{V}$ as an equivalent alternative:

Lemma 6. For all $u, v \in \mathbb{V}$, if $u < v$, then $val(u) < val(v)$. 

\[
\begin{array}{c|c}
\forall \in \mathcal{V} & \text{val}(\forall) \\
[F] & -1 \\
[0, F, 0, F, T] & -29/31 \\
[0, F, 0, T, F] & -23/31 \\
[0, F, T] & -2/9 \\
[0, 0, F, T, F] & -7/31 \\
[0, 0, F, T, T] & -5/31 \\
[0, 0, T, F, F] & 5/31 \\
[0, 0, T, F, T] & 7/31 \\
[0, T, F] & 2/9 \\
[0, T, 0, F, T] & 25/31 \\
[0, T, 0, T, F] & 29/31 \\
[T] & 1 \\
\end{array}
\]

Table 3: Rational values for elements of \(\mathcal{V}\).

**Proof.** If \(u = [F]\) and \(v = [T]\) the result obviously holds. If \(u = [F]\) and \(v \neq [T]\)
then \(v = [v_1, \ldots, v_m]\), where \(v_1 = 0\). In the extreme case where for all \(i \geq 2, v_i = F\), we get:

\[
\text{val}(v) = \sum_{i=2}^{m} \text{val}(v_i) \cdot \frac{1}{3^{i-1}} \geq -\frac{1}{2}
\]

Therefore, in this case \(\text{val}(u) < \text{val}(v)\).

Consider now the case where \(u = [u_1, \ldots, u_m]\) and \(v = [v_1, \ldots, v_n]\), with \(u_1 = v_1 = 0\). Since \(u < v\), there exists \(k\) with \(2 \leq k \leq \min\{m, n\}\) such that for all \(i < k, u_i = v_i, \) and \(u_k < v_k\). We show that \(\text{val}(u) < \text{val}(v)\), or equivalently that \(\text{val}(u) - \text{val}(v) < 0\). We have:

\[
\text{val}(u) - \text{val}(v) = \sum_{i=2}^{n} \text{val}(u_i) \cdot \frac{1}{3^{i-1}} - \sum_{i=2}^{m} \text{val}(v_i) \cdot \frac{1}{3^{i-1}}
\]

Notice now that \(\text{val}(u_k) \cdot \frac{1}{3^{k-1}} - \text{val}(v_k) \cdot \frac{1}{3^{k-1}} \leq -\frac{1}{3^{k-1}}\) (when \(u_k = 0\) and \(v_k = T\),
or when \(u_k = F\) and \(v_k = 0\)). We demonstrate that no matter what the remaining
digits of \(u\) and \(v\) are, they can not “close the gap” of \(-\frac{1}{3^{k-1}}\). Take the extreme
case where all the \(u_i, i > k,\) are equal to \(T\) and all the \(v_i, i > k,\) are equal to \(F\). Then:

\[
\sum_{i=k+1}^{n} \text{val}(u_i) \cdot \frac{1}{3^{i-1}} = \sum_{i=k+1}^{m} \text{val}(v_i) \cdot \frac{1}{3^{i-1}} + \sum_{i=k+1}^{m} \text{val}(v_i) \cdot \frac{1}{3^{i-1}} < \sum_{i=k+1}^{\infty} \frac{1}{3^{i-1}} + \sum_{i=k+1}^{\infty} \frac{1}{3^{i-1}} = 2 \cdot \sum_{i=k+1}^{\infty} \frac{1}{3^{i-1}} = \frac{1}{3^{k-1}}
\]

Thus, the gap can not close and \(\text{val}(u) - \text{val}(v) < 0\).

The above discussion suggests that the semantics of lexicographic logic can
be equivalently expressed using a special subset of the set \(\mathbb{Q}\) of rational numbers.
In a potential implementation of a query system based on lexicographic logic,
numerical values that rank query results would convey a much better intuition
than the lists of truth values.
6 Modeling Alternative Preferential Operators

In this section, we demonstrate that we can use “≫” as a primitive operator in order to define other interesting connectives that express levels of preference. We start with two well-known such operators [10], namely “and if possible” and “or at least”. We will denote the former by “&” and the latter by “×”.

6.1 Modeling the “and if possible” operator

The intuitive meaning of (x & y) is “I want x and if possible additionally y”. The behaviour we expect from “&” when applied to classical truth values, is depicted in Table 4. The intuitive reading of the table is as follows: if the first argument of “&” is false, then the result is false (no matter what the second argument is). If both arguments are true, then the result is true. If, however, the first argument is true and the second is false, then we are partially satisfied; this is expressed by the value [F] ≫ [F] which is less than the absolute true value [T]. It turns out that “&” can be described as a derived operator using “≫”. Formally:

\[ x \land y = x \gg (x \land y) \]

It can easily be verified that the above definition produces the values of Table 4. Notice also that the above definition is meaningful even if “&” is applied on non-classical elements of V, i.e., elements that are different from [F] or [T]. In this case, the intuition of the above definition is that “&” puts a strong emphasis on the value of its first argument (x in our case). In particular, if x has a false value (of whatever degree), then even if y has a true value, this value will be degraded (through the ∧ operation).

We can easily verify that our definition satisfies an intuitive equivalence observed in [10], namely that “x and if possible y” is equivalent to “x and if possible x ∧ y”. Indeed:

\[ x \land (x \land y) = x \gg (x \land y) = x \gg (x \land y) = x \land y \]

As we are going to see in the next subsection, our definition of “&” satisfies some intuitive and expected properties with respect to the “or at least” operator.
6.2 Modeling the “or at least” operator

The intuitive meaning of \((x \times y)\) is “I want \(x\), or failing that, \(y\)”. The behaviour we expect from “\(\times\)” when applied to classical truth values, is depicted in Table 5. The intuitive reading of the table is as follows: if both arguments of “\(\times\)” are false, then the result is false. If the first argument is true, then the result is true (no matter what the second argument is). If, however, the first argument is false and the second is true, then we are still satisfied, but not as much as if the first argument was true; this is expressed by the value \([T] \gg [F]\) which is not absolutely true (but is still true). We can express “\(\times\)” using “\(\gg\)”, as follows:

\[
x \times y = (x \lor y) \gg x
\]

The above definition is meaningful even for values of \(V\) that are non-classical. If either \(x\) or \(y\) have a true value (of whatever form), then the result will also be a true value of some form. In particular, if \(x\) has some false value, we degrade the overall result because we are not entirely satisfied (this is achieved by the occurrence of \(x\) in the right of “\(\gg\”)).

Using our definition of “\(\times\)”, we can easily verify that, as observed in [10], “\(x\) or at least \(y\)” is equivalent to “\(x\) or at least \(x \lor y\)”. Indeed:

\[
x \times (x \lor y) = (x \lor (x \lor y)) \gg x = (x \lor y) \gg x = x \times y
\]

Notice also that there is a connection between the “\(\times\)” and “&” operators, observed in [10], which the following lemma demonstrates.

**Lemma 7.** For all \(x, y \in V\), it holds:

\[- x \times y = (x \lor y) \& x
\]
\[- x \& y = (x \land y) \times x.
\]

**Proof.** We have:

\[
(x \lor y) \& x = (x \lor y) \gg ((x \lor y) \land x)
= (x \lor y) \gg x
= x \times y
\]

Also:

\[
(x \land y) \times x = ((x \land y) \lor x) \gg (x \land y)
= x \gg (x \land y)
= x \& y
\]

This completes the proof of the lemma.

6.3 Modeling Hierarchical Operators

As it turns out, we can use “\(\gg\)” to model preference operators that are “hierarchical” in nature. Consider, for example, the “\(x\) or at least \(y\)” operator and assume for simplicity that \(x\) and \(y\) are standard truth values, namely \([F]\) and \([T]\). The output of this operator can belong to three distinct levels of a hierarchy: the top-level value is attained when the input values are \(x = [T]\) and \(y = [T]\), or when \(x = [T]\) and \(y = [F]\); the next level is attained by \(x = [F]\) and \(y = [T]\); finally, for \(x = [F]\) and \(y = [F]\) we obtain the lowest value. A similar hierarchy exists in the definition of the “\(\&\)” operator.
The above ideas can be generalized as follows. Assume we want to construct an operator \( f \) on \( n \) input arguments \( x_1, \ldots, x_n \). Moreover, assume that we want the value of \( f(x_1, \ldots, x_n) \) to belong to \( m \leq 2^n \) distinct levels of a hierarchy. We mark all those inputs for which we want \( f \) to attain the largest possible value. For all those inputs, we define \( f \) to return the top possible truth value, namely \([T]\). We then mark all those inputs for which we want \( f \) to return the immediately lower truth value. For all those inputs, we define \( f \) to return a truth value below \([T]\), for example \([T] \gg \cdots \gg [T] \gg [F]\) (where “\( \gg \)” appears \( n-1 \) times). The next level will receive an even smaller value, and so on. We continue this process until all levels of the hierarchy are exhausted. Notice that the choice of the exact truth value that we assign to each level, is not important, as long as “a more important level” receives a higher truth value than the “less important” ones. This process helps us create a truth table, as illustrated by the following example.

Example 5. Assume we want to construct a ternary operator \( \text{more}(x_1, x_2, x_3) \), which is interpreted as “the more the better”. We would like \( \text{more}(x_1, x_2, x_3) \) to attain its highest value when all of its inputs are \([T]\). Additionally, we want \( \text{more}(x_1, x_2, x_3) \) to obtain a lower truth value when exactly two of its arguments are \([T]\). Finally, when exactly one of the arguments is equal to \([T]\), we want \( \text{more}(x_1, x_2, x_3) \) to return an even lower truth value. We create the truth table shown in Table 6. For symmetry reasons we represent \([T]\) by \([T] \gg [T] \gg [T]\). The second truth value that we use is \([T] \gg [T] \gg [F]\) and the third one \([T] \gg [F] \gg [F]\).

| \( x_1 \) | \( x_2 \) | \( x_3 \) | \( \text{more}(x_1, x_2, x_3) \) |
|---|---|---|---|
| \( F \) | \( F \) | \( F \) | \( F \gg F \gg F \) |
| \( F \) | \( F \) | \( T \) | \( F \gg T \gg F \) |
| \( F \) | \( T \) | \( F \) | \( F \gg T \gg F \) |
| \( F \) | \( T \) | \( T \) | \( F \gg T \gg T \) |
| \( T \) | \( F \) | \( F \) | \( F \gg F \gg F \) |
| \( T \) | \( F \) | \( T \) | \( F \gg F \gg T \) |
| \( T \) | \( T \) | \( F \) | \( F \gg T \gg F \) |
| \( T \) | \( T \) | \( T \) | \( T \gg T \gg T \) |

Table 6: Truth table for \( \text{more} \).

Let us call the last column of the truth table (ie., the one that contains the desired truth values for the function \( f \)), the assignment column. We create the function definition:

\[
f(x_1, \ldots, x_n) = E_1 \gg \cdots \gg E_n
\]

where the \( E_i \) are standard propositional formulas (ie., they do not contain the “\( \gg \)” operator) constructed using the variables \( x_1, \ldots, x_n \). Each \( E_i \) is constructed as the disjunction of a set of conjunctive terms; each such conjunctive term corresponds to a row of the truth table which has in the \( i \)’th position of the assignment column the value \([T]\).

Example 6. We construct a function \( f(x_1, x_2, x_3) = E_1 \gg E_2 \gg E_3 \) for the truth table given in Table 6. First we construct the disjunction of terms for \( E_1 \). We notice that the rows 2-8 of Table 6 have the value \([T]\) in the first position of the assignment column. Therefore, \( E_1 \) is equal to:

\[
(x_1 \land \overline{x_2} \land x_3) \lor (\overline{x_1} \land x_2 \land \overline{x_3}) \lor (\overline{x_1} \land x_2 \land x_3) \lor (x_1 \land \overline{x_2} \land x_3) \lor (x_1 \land x_2 \land \overline{x_3}) \lor (x_1 \land x_2 \land x_3)
\]
The above easily simplifies to \( E_1 = x_1 \lor x_2 \lor x_3 \). To construct the expression for \( E_2 \) we observe that the rows 4, 6, 7 and 8 of Table 6 have the value \([T]\) in the second position of the assignment column. Therefore, \( E_2 \) is equal to:

\[
(x_1 \land x_2 \land x_3) \lor (x_1 \land x_2 \land x_3) \lor (x_1 \land x_2 \land x_3) \lor (x_1 \land x_2 \land x_3)
\]

This simplifies to \( E_2 = (x_1 \land x_2) \lor (x_2 \land x_3) \lor (x_1 \land x_3) \). Finally, to construct the expression for \( E_3 \) we observe that only the 8th row of Table 6 has the value \([T]\) in the third position of the assignment column. Therefore, \( E_3 = (x_1 \land x_2 \land x_3) \).

Overall, the expression for \( \text{more}(x_1, x_2, x_3) \) is:

\[
(x_1 \lor x_2 \lor x_3) \gg ((x_1 \land x_2) \lor (x_2 \land x_3) \lor (x_1 \land x_3)) \gg (x_1 \land x_2 \land x_3)
\]

One can verify that the above expression produces the values of Table 6.

It is worth noting that the definitions of “&” and “×” given in Subsections 6.1 and 6.2 respectively, were both derived using the systematic procedure presented in the subsection.

7 Related Work

There exists a great variety of preference representation formalisms that have been developed mainly in the areas of artificial intelligence, databases, and logic programming. Nice surveys that outline the main approaches in these three areas are [9], [22], and [21], respectively. In general, preference representation formalisms are classified [22] as either “quantitative or “qualitative”. In the quantitative approach, numerical values are used in the syntax of the preference specification language in order to express degrees of preference. On the other hand, in the qualitative approach, preferences are expressed by implicitly establishing a preference ordering relation between the objects under consideration. Lexicographic logic falls somewhere in between: its syntax is qualitative because preferences are expressed implicitly through the “\( \gg \)” operator; however, its semantics is quantitative because, as shown in Section 5, formulas essentially receive rational number values when evaluated.

The work proposed in this paper suggests the use of a high-level logic syntax in order to represent preferences. The research works that we are aware of and that are closer to our contribution, are the following:

- The use of negation-as-failure in logic programming which acts as a quantitative preferential operator.
- The Qualitative Choice Logic (QCL) approach proposed in [5] and extended in the context of logic programming in [6].
- The flexible queries approach described in [10] that is based on possibilistic logic [11].
- The infinite-valued approach proposed in [1,2] and extended in the context of logic programming in [19].

In the following, we give a brief comparison of our approach with each one of the above formalisms.

Negation-as-failure: As remarked in [21], stratified logic programs [3] introduce a form of preferences in logic programming. A very concise discussion of this issue
can also be found in [18] where it is argued that in a logic program with negation, the negative literals in the body of a rule have a higher priority for falsification in the intended meaning of the program than the atom in the head of the same rule. In other words, the well-known “not” operator of logic programming is an implicit preferential operator that plays a role similar to that of the “≫” operator in lexicographic logic. Of course, there exists a significant difference between the two operators: negation-as-failure is used to express a very specific type of preference related to the falsification of atoms, while “≫” can be used to express more general types of preferences.

**Qualitative Choice Logic (QCL):** In [5], propositional logic is extended with the new connective “×”. Intuitively, \(A \times B\) is read “if possible \(A\), but if \(A\) is impossible then at least \(B\)”. QCL has a similar philosophy as lexicographic logic in the sense that preferences are represented implicitly using a new operator. However, there are important differences. QCL is built on the classical boolean truth domain while lexicographic logic uses a many-valued one. It has been remarked that the semantics of QCL lead to certain limitations [4]. Moreover, some intuitive tautologies, such as \(\neg(\neg\phi) \equiv \phi\) do not hold, and the notion of logical equivalence between QCL formulas is defined in a non-standard way (see [5][page 209]). Additionally, the semantics of QCL can be used to prove that the operator “×” of QCL is associative, something that does not hold for the operator “×” that we defined in Subsection 6.2. It is also worth noting that QCL is a non-monotonic logic, while Lexicographic Logic (in its present form) has not been extended with a non-monotonic consequence relation; a more general remark regarding this issue is given at the end of this section. Finally, as shown in Section 6, lexicographic logic can be used to define a variety of operators, something which does not appear to be the case for QCL.

**Flexible Queries:** The operators “\(A\) and if possible \(B\)” and “\(A\) or at least \(B\)” have been studied as different forms of bipolar constraints for flexible querying in [10]. The semantics of these operators has been modelled using possibilistic logic [10,11]. It is however unclear whether a lexicographic priority operator like “≫” can be encoded in the framework of [10] and whether arbitrary nestings of such operators can be supported. Moreover, it is not obvious whether alternative hierarchical preferential operators can be described. Overall, we believe that Lexicographic Logic provides a simpler and more natural means for encoding such operators.

**Infinite-valued Logic:** In [2,1] a propositional query language is developed that supports two operators which can express preferences of the form “\(A\) and optionally \(B\)” and “\(A\) or alternatively \(B\)”. The semantics of this query language is based on the infinite-valued logic introduced in [20]. The language was subsequently extended with recursion obtaining the logic programming language PrefLog [19]. The infinite-valued approach shares a similar philosophy with lexicographic logic: preferences are expressed implicitly using operators in the context of many-valued logics. However, there is a significant difference between our approach and the one developed in [1,2]: as it was demonstrated in [17], the infinite valued logic of [20] is not sufficient to express lexicographic preferences (the proof uses similar arguments to the ones used in the proof of Theorem 1, see for example [12, p. 363]). Therefore, our present approach is more powerful than the one developed in [1,2] because it can express the “optional” and “alternative” preferences of the latter, and additionally it can also express lexicographic and other hierarchical ones.
Other Approaches: There are many other logical formalisms that have been defined for representing preferences; however, to the best of our knowledge, most approaches are not directly related to lexicographic logic. An (unavoidably) incomplete list of such approaches can be found in [7,16,9].

Many logical approaches to preference representation are non-monotonic. Intuitively, this means that the logic provides a mechanism to distinguish among the models of a formula and to produce only the most preferred ones. One of the most widely used non-monotonic formalisms are extended logic programs [14] which have formed the basis for various preferential formalisms [7].

Non-monotonicity is quite indispensable if we want to distinguish among the models. However, when a logical formalism is used as a query language, we often expect to see even models that are not necessarily optimal in terms of our preferences. For example, when searching for hotel accommodation, we often want to also view results that are not optimal in terms of our preferences. In its present form, Lexicographic logic does not have a consequence relation based on maximally preferred models. Of course, it is conceivable that it can be extended to a non-monotonic setting but this is outside the scope of the present paper.

8 Conclusions

We have introduced lexicographic logic, a propositional, many-valued logic for preference representation. Lexicographic logic has a simple semantics based on sequences of the truth values $F$, 0, and $T$, which can be alternatively defined using a subset of the rational numbers in the interval $[-1, 1]$. Apart from representing lexicographic preferences (which are non-trivial to handle from a quantitative point of view), the proposed logic can be used to define in a simple way, alternative preferential operators.

The next step in the development of lexicographic logic, is the introduction of a deductive calculus for the logic. Such an investigation could include an inference procedure, investigation of completeness issues (with respect to the model-theoretic semantics), and a corresponding complexity-theoretic study. Another promising direction for future work would be the extension of lexicographic logic to the first-order case. The semantics of universal quantification would most probably need to be defined as the least upper bound of (a possibly infinite) set of elements of $V$. Since the least upper bound of such a set can be an infinite list, we would need to extend $V$ to contain lists of countably infinite length; moreover, the concatenation operation would also have to be extended to apply to such lists. We are currently investigating issues such as the above ones.

References

1. Agarwal, R.: A Framework for Expressing Prioritized Constraints Using Infinitesimal Logic. Master’s thesis, University of Victoria, Canada (2005)
2. Agarwal, R., Wadge, W.W.: The lazy evaluation of infinitesimal logic expressions. In: Arabnia, H.R. (ed.) Proceedings of The 2005 International Conference on Programming Languages and Compilers, PLC 2005, Las Vegas, Nevada, USA, June 27-30, 2005. pp. 3–7. CSREA Press (2005)
3. Apt, K.R., Blair, H.A., Walker, A.: Towards a theory of declarative knowledge. In: Minker, J. (ed.) Foundations of Deductive Databases and Logic Programming, pp. 89–148. Morgan Kaufmann (1988). https://doi.org/10.1016/b978-0-934613-40-8.50006-3
4. Benferhat, S., Sedki, K.: A revised qualitative choice logic for handling prioritized preferences. In: Mellouli, K. (ed.) Symbolic and Quantitative Approaches to Reasoning with Uncertainty, 9th European Conference, ECSQARU 2007, Hammamet, Tunisia, October 31 - November 2, 2007, Proceedings. Lecture Notes in Computer Science, vol. 4724, pp. 635–647. Springer (2007). https://doi.org/10.1007/978-3-540-75256-1_56

5. Brewka, G., Benferhat, S., Berre, D.L.: Qualitative choice logic. Artificial Intelligence 157(1-2), 203–237 (2004). https://doi.org/10.1016/j.artint.2004.04.006

6. Brewka, G., Niemelä, I., Syriänen, T.: Logic programs with ordered disjunction. Computational Intelligence 20(2), 335–357 (2004). https://doi.org/10.1111/j.0824-7935.2004.00241.x

7. Brewka, G., Niemelä, I., Truszczynski, M.: Preferences and nonmonotonic reasoning. AI Magazine 29(4), 69–78 (2008)

8. Charalambidis, A., Rondogiannis, P., Troumpoukis, A.: Higher-order logic programming: An expressive language for representing qualitative preferences. Science of Computer Programming 155, 173–197 (2018). https://doi.org/10.1016/j.scico.2017.09.002

9. Domshlak, C., Hüllermeier, E., Kaci, S., Prade, H.: Preferences in AI: an overview. Artificial Intelligence 175(7-8), 1037–1052 (2011). https://doi.org/10.1016/j.artint.2011.03.004

10. Dubois, D., Prade, H.: Modeling “and if possible” and “or at least”: Different forms of bipolarity in flexible querying. In: Pivert, O., Zadrozny, S. (eds.) Flexible Approaches in Data, Information and Knowledge Management, Studies in Computational Intelligence, vol. 497, pp. 3–19. Springer (2013). https://doi.org/10.1007/978-3-319-00954-4_1

11. Dubois, D., Prade, H.: Possibilistic logic - an overview. In: Siekmann, J.H. (ed.) Computational Logic, Handbook of the History of Logic, vol. 9, pp. 283–342. Elsevier (2014). https://doi.org/10.1016/B978-0-444-51624-4.50007-1

12. Fishburn, P.C.: Preference structures and their numerical representations. Theoretical Computer Science 217(2), 359–383 (1999). https://doi.org/10.1016/S0304-3975(98)00277-1

13. Fitting, M.: First-Order Logic and Automated Theorem Proving, Second Edition. Graduate Texts in Computer Science, Springer (1996). https://doi.org/10.1007/978-1-4612-2360-3

14. Gelfond, M., Lifschitz, V.: The stable model semantics for logic programming. In: Kowalski, R.A., Bowen, K.A. (eds.) Logic Programming, Proceedings of the Fifth International Conference and Symposium, Seattle, Washington, USA, August 15-19, 1988 (2 Volumes). pp. 1070–1080. MIT Press (1988)

15. Knuth, D.E.: The art of computer programming, vol. 2. Addison-Wesley Longman Publishing Co., Boston, MA, USA, 3rd edn. (1998)

16. Lang, J.: Logical representation of preferences. In: Bouyssou, D., Dubois, D., Pirlot, M., Prade, H. (eds.) Decision-making Process, pp. 321–363. Wiley (2009). https://doi.org/10.1002/9780470611876.ch7

17. Papadimitriou, G.: A Logic Query Language for Lexicographic Preferences. Master’s thesis, National and Kapodistrian University of Athens, Greece (2017)

18. Przymusinska, H., Przymusinski, T.: Semantic issues in deductive databases and logic programs. In: Formal Techniques in Artificial Intelligence. pp. 321–367. North-Holland (1990)

19. Rondogiannis, P., Troumpoukis, A.: Expressing preferences in logic programming using an infinite-valued logic. In: Falaschi, M., Albert, E. (eds.) Proceedings of the 17th International Symposium on Principles and Practice of Declarative Programming, Siena, Italy, July 14-16, 2015. pp. 208–219. ACM (2015). https://doi.org/10.1145/2790449.2790511

20. Rondogiannis, P., Wadge, W.W.: Minimum model semantics for logic programs with negation-as-failure. ACM Trans. Comput. Log. 6(2), 441–467 (2005). https://doi.org/10.1145/1055686.1055694
21. Sakama, C., Inoue, K.: Prioritized logic programming and its application to commonsense reasoning. Artificial Intelligence 123(1-2), 185–222 (2000). https://doi.org/10.1016/S0004-3702(00)00054-0

22. Stefanidis, K., Koutrika, G., Pitoura, E.: A survey on representation, composition and application of preferences in database systems. ACM Transactions on Database Systems 36(3), 19:1–19:45 (2011). https://doi.org/10.1145/2000824.2000829