Discrete maximal regularity and the finite element method for parabolic equations

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Abstract Maximal regularity is a fundamental concept in the theory of partial differential equations. In this paper, we establish a fully discrete version of maximal regularity for a parabolic equation. We derive various stability results in \( \ell^p(0,T;\ell^q(\Omega)) \) norm, \( p, q \in (1,\infty) \) for the finite element approximation with the mass-lumping to the linear heat equation. Our method of analysis is an operator theoretical one using pure imaginary powers of operators and might be a discrete version of G. Dore and A. Venni (On the closedness of the sum of two closed operators. Math. Z., 196(2):189–201, 1987). As an application, optimal order error estimates in that norm are proved. Furthermore, we study the finite element approximation for semilinear heat equations with locally Lipschitz continuous nonlinearity and offer a new method for deriving optimal order error estimates. Some interesting auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented.

Keywords maximal regularity · parabolic equation · finite element method

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1 Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with the boundary \( \partial \Omega \). Let \( J_T = (0,T) \) be a time interval with \( T \in (0,\infty] \). We consider the finite element approximation
approximation of linear heat equation for the function $u = u(x,t)$ of $(x,t) \in \Omega \times [0,T)$:

\[
\begin{array}{ll}
\frac{\partial}{\partial t}u = \Delta u + g & \text{in } \Omega \times J_T, \\
u = 0 & \text{on } \partial \Omega \times J_T, \\
|\text{at } t=0 = u_0 & \text{on } \Omega,
\end{array}
\]  

where $\frac{\partial}{\partial t}u = \frac{du}{dt}$, $\Delta u = \sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}$, $g = g(x,t)$, and $u_0 = u_0(x)$; $g$ and $u_0$ are prescribed functions. All functions and function spaces considered in this paper are complex-valued.

The purpose of this paper is to derive various stability estimates in the $L^p(J_T;L^q(\Omega))$ norm

\[
||v||_{L^p(J_T;L^q(\Omega))} = \left[ \int_0^T \left( \int_\Omega |v(x,t)|^q \, dx \right)^{1/q} \, dt \right]^{1/p}
\]

and discrete $L^p(J_T;L^q(\Omega))$ norm defined as \cite{8} with $X = L^q(\Omega)$, where $p,q \in (1,\infty)$. As applications of those estimates, we also derive optimal order error estimates in those norms for the finite element approximations of \cite{1} and semilinear heat equation

\[
\begin{array}{ll}
\frac{\partial}{\partial t}u = \Delta u + f(u) & \text{in } \Omega \times J_T, \\
u = 0 & \text{on } \partial \Omega \times J_T, \\
|\text{at } t=0 = u_0 & \text{in } \Omega,
\end{array}
\]  

where $f : \mathbb{C} \to \mathbb{C}$ is a prescribed function. Particularly, we assume only a locally Lipschitz continuity and offer a new method of error analysis for \cite{2}.

In other words, we intend to develop a discrete version of theory of maximal regularity for evolution equations of parabolic type. To recall maximal regularity in a general context, let us consider an abstract Cauchy problem on a Banach space $X$ as

\[
\begin{array}{ll}
\frac{du}{dt}(t) = Au(t) + g(t), & t \in J_T, \\
u(0) = 0,
\end{array}
\]  

where $A$ is a densely defined closed operator on $X$ with the domain $D(A) \subset X$, $g : J_T \to X$ is a given function, $u : J_T \to X$ is an unknown function and $u'(t) = du(t)/dt$.

**Definition 1 (Maximal regularity, MR, CMR)** Let $p \in (1,\infty)$. The operator $A$ has maximal $L^p$-regularity ($L^p$-MR) on $J_T$, if and only if, for every $g \in L^p(J_T;X)$, there exists a unique solution $u \in W^{1,p}(J_T;X) \cap L^p(J_T;D(A))$ of \cite{3} satisfying

\[
||u||_{L^p(J_T;X)} + ||u'||_{L^p(J_T;X)} + ||Au||_{L^p(J_T;X)} \leq C_{\text{MR}}||g||_{L^p(J_T;X)},
\]  

where $C_{\text{MR}} > 0$ denotes a constant that is independent of $g$. We say that $A$ has maximal regularity (MR) if $A$ has maximal $L^p$-regularity for some $p \in (1,\infty)$.
posedness of (1) and (2). For example, assume \( g \) analytical semigroup theory, which is a powerful method to establish the well-
not a trivial fact. For comparison, we recall the solution obtained using the
cannot be in a better function space than \( g \) the same regularity as the right-hand side function
with \( u \) admits a unique solution
\( \sigma \) introduced later, we say that \( A \) has continuous maximal \( L^p \)-regularity (\( L^p\)-CMR) and continuous maximal regularity (CMR).

It is proved that the \( L^q(\Omega) \) realization \( A \) of \( \Delta \) with \( D(A) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \) has \( L^p \)-CMR for any \( p, q \in (1, \infty) \) (see [14][32]). The problem (1) admits a unique solution \( u \in W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A)) \) satisfying (4) with \( u_0 = 0 \). This result implies that \( \partial_t u \) and \( \Delta u \) are well defined and have
the same regularity as the right-hand side function \( g \). Moreover, \( \partial_t u \) and \( \Delta u \) cannot be in a better function space than \( g \), since \( g = \partial_t u - \Delta u \). This is not a trivial fact. For comparison, we recall the solution obtained using the
analytical semigroup theory, which is a powerful method to establish the well-
and CMR) introduced later, we say that \( A \) has continuous maximal \( L^p \)-regularity (\( L^p\)-CMR) and continuous maximal regularity (CMR).

Then, by application of the analytical semigroup theory, we can prove that
the problem (1) with \( u_0 = 0 \) admits a unique solution \( u \in C(J_T; X) \cap C(J_T; D(A)) \cap C^1(J_T; L^q(\Omega)) \); see [35] Theorems 4.3.2, 7.3.5]. However, we
are able to obtain slightly less regularity \( \partial_t u - \Delta u \in C(J_T; L^q(\Omega)) \) than \( g \). To obtain the same regularity \( \partial_t u - \Delta u \in C^0(\overline{J_T}; L^q(\Omega)) \), we must further assume \( g(x, 0) = 0 \) for all \( x \in \Omega \); see [35] Theorem 4.3.5]. Therefore,
\( W^{1,p}(J_T; L^q(\Omega)) \cap L^p(J_T; D(A)) \) is an appropriate function space to study
parabolic equations such as (1). Moreover, CMR is a “stronger” property than
the generation of analytical semigroup in the sense that, if \( A \) has CMR, then
\( A \) generates the analytical (bounded) semigroup (cf. [15]). Although CMR is
a concept for linear equations, it actually has many important applications
to nonlinear equations, as reported in the literature [32][40]. Moreover, the
analytic semigroup theory and its discrete counterparts play important roles
in construction and study of numerical schemes for parabolic equations (see
e.g. [18][20][21][37][38][46]). Therefore, it is natural to wonder whether a discrete
version of CMR is available.

This study has another motivation. Considering the problem (2) with
\( f(u) = u|u|^\alpha \) for \( \alpha > 0 \), then without loss of the generality, we assume \( u_0 \in \Omega \).
Let \( \lambda > 0 \). Then the function

\[ u_\lambda(x, t) = \lambda^{\frac{2}{\alpha}} u(\lambda x, \lambda^2 t) \]

also solves (2) where \( \Omega \) and \( J_T \) are replaced, respectively, by \( \Omega_\lambda = \{ \lambda^{-1} x \mid x \in \Omega \} \) and \( J_{T/\lambda^2} \). Moreover, if \( p, q \in (1, \infty) \) satisfy

\[ \frac{2}{\alpha} = \frac{d}{p} + \frac{2}{q}, \tag{5} \]

we have

\[ \| u_\lambda \|_{L^p(J_{T/\lambda^2}; L^q(\Omega_\lambda))} = \| u \|_{L^p(J_T; L^q(\Omega))} \]
for any $\lambda > 0$. Those $p, q$ are called the scale invariant exponents. The function space $L^p(J_T; L^q(\Omega))$ with $p,q$ satisfying (5) plays a crucially important role in the study of time-local and time-global well-posedness of (3). Furthermore, such a scaling argument is applied to deduce a novel numerical method for solving (2) (see [5]). Therefore, it would be interesting to derive stability and error estimates in those norms from the dual perspectives of numerical and theoretical analysis.

Based on those motivations, we studied a time discrete version of maximal regularity for (2) in an earlier study [29]. Let

$$N_T = \begin{cases} \lfloor T/\tau \rfloor & (T < \infty) \\ \infty & (T = \infty). \end{cases}$$  

(6)

We consider the implicit $\theta$ scheme for (3) given as

$$\begin{cases} u^{n+1} - u^n = Au^{n+\theta} + g^{n+\theta}, & n = 0, 1, \ldots, N_T - 1, \\ u^0 = 0, \end{cases}$$  

(7)

where $\tau > 0$ is the time increment, $\theta \in [0, 1]$, $g = (g^n)_{n=0}^{N_T}$ is a given $X^{N_T+1}$-valued function, and $u = (u^n)_{n=0}^{N_T}$ is an unknown $X^{N_T+1}$-valued function. Set

$$v^{n+\theta} = (1 - \theta)v^n + \theta u^{n+1}$$

for a sequence $v = (v^n)_n$. We moreover assume that $A$ is bounded when $\theta \neq 1$.

The function $u^n$ might be an approximation of $u(n\tau)$ for $n = 1, \ldots, N_T$.

We introduce the space $l^p(N; X)$ by setting

$$l^p(N; X) = \begin{cases} X^{N+1}, & N \in \mathbb{N}, \\ l^p(\mathbb{N}; X), & N = \infty, \end{cases}$$

and let

$$\|v\|_{l^p(N; X)} = \left( \sum_{n=0}^{N-1} \|v^n\|_{X^\tau}^p \right)^{1/p},$$  

(8)

$$D_{\tau}v = \left( \frac{v^{n+1} - v^n}{\tau} \right)_{n=0}^{N-1}, \quad Av = (Av^n)_{n=0}^N,$$

for $v = (v^n) \in l^p(N; X)$.

Discrete maximal regularity is then introduced as follows (see [29]).

**Definition 2 (Discrete maximal regularity, DMR)** Let $p \in (1, \infty)$. The operator $A$ has maximal $l^p$-regularity ($l^p$-DMR) on $J_T$ if and only if, for every $g \in l^p(N_T; X)$, there exists a unique solution $u \in X^{N_T}$ of (7) satisfying

$$\|u_0\|_{l^p(N_T; X)} + \|D_{\tau}u\|_{l^p(N_T; X)} + \|Au_0\|_{l^p(N_T; X)} \leq C_{\text{DMR}} \|g_0\|_{l^p(N_T; X)},$$  

(9)
uniformly with respect to $\tau$, where $C_{\text{DMR}} > 0$ is independent of $g$. We say that $A$ has discrete maximal regularity (DMR) if $A$ has $l^p$-DMR for some $p \in (1, \infty)$.

In [6], Blunck considered the forward Euler method ($\theta = 0$) and characterized DMR by developing a discrete version of the operator-valued Fourier multiplier theorem. However, the dependence of $\tau$ on DMR inequalities is not clear since only the case $\tau = 1$ is studied. The backward Euler method ($\theta = 1$) with an arbitrary time increment $\tau$ is discussed in [4]. Ashyralyev and Sobolevskii provided no reasonable sufficient conditions for DMR. Consequently, those results cannot be applied straightforwardly to numerical analysis. In contrast to those works, we gave sufficient conditions on $\tau, \theta, A$ for DMR to hold in [29].

We recall the statement below (see Lemma 6). Spatial discretization must be addressed next. We introduce the finite element approximation $L_h$ of $\Delta$ in $H_0^1(\Omega)$ and prove that $L_h$ has CMR. Herein, $h$ denotes the size parameter of a triangulation $T_h$. As a matter of fact, Geisert studied CMR for the finite element approximation of the second order parabolic equations in the divergence form in [22,23]. He considered a smooth convex domain $\Omega$ and triangulations defined on a polyhedral approximation $\Omega_h$ of $\Omega$. (For the Neumann boundary condition case, he considered the exactly fitted triangulation.) Therefore, combining those results with our Lemma 6, we are able to obtain DMR for the smooth domain case. In those works, the method of [39] and [42] for studying stability and analyticity in $L^\infty$ norm is applied. He first derived some estimates for the discrete Green function associated with the finite element operator in parabolic annuli. Then he obtained some estimates in the whole $\Omega$ by a dyadic decomposition technique. Consequently, the proofs are quite intricate. Moreover, he applied several kernel estimates for the Green function associated with a parabolic equation. Therefore, the domain and coefficients should be suitably smooth.

In the present paper, we take a completely different approach. We directly establish a discrete version of the method using pure imaginary powers of operators developed by [16]. To this end, we consider polyhedral domains and study the discrete Laplacian with mass-lumping $A_h$ instead of the standard discrete Laplacian since the positivity-preserving property of the semigroup generated by $A_h$ (see Lemma 9) plays an important role in our analysis. Actually, the standard discrete Laplacian has no such property (see [43]). It must be borne in mind that the $L^q$ theory for the discrete Laplacian with mass-lumping is of great use in study of nonlinear problems, such as the finite element and finite volume approximation of the Keller-Segel system modelling chemotaxis (see [37,35,16]).

After having established CMR and DMR for $A_h$ (see Theorems I, II, III and IV), we derive optimal order error estimates for the finite element approximations combined with the implicit $\theta$ method to (1) (see Theorem V). We address not only unconditionally stable cases ($\theta \in [1/2, 1]$), but also conditionally stable cases ($\theta \in [0, 1/2]$). For the latter case, we give a useful sufficient condition for the scheme to be stable. As a further application, we study the
finite element approximation for \([2]\) and prove optimal order error estimates (see Theorem \([VI]\)). Since nonlinearity \(f\) is assumed to be only locally Lipschitz continuous, the solution \(u\) might blow up in some sense. Our error estimate is valid as long as \(u\) exists in contrast to \([23]\). To achieve such an objective, we apply the fractional powers of \(-A_h\) and derive a sub-optimal error estimate in the \(L^\infty(\Omega \times (0,T))\) norm as an intermediate result (see Theorem \([VII]\)). Our proposed method is apparently new in the literature. Some auxiliary results including discrete Gagliardo-Nirenberg and Sobolev inequalities are also presented (see Lemmas \([24]\) and \([28]\)).

We learned about \([33,31]\) after completion of the present study. The paper \([33]\) specifically examined the time-discrete version of \(L^p-L^q\)-maximal regularity for arbitrary \(p, q \in [1, \infty]\), by discontinuous Galerkin time stepping (cf. \([41]\)) for parabolic problems. This result is valid for \(p, q = 1, \infty\). However, they did not consider the R-boundedness of sets of operators, which plays an important role in the theory of maximal regularity developed by Weis \([45]\). The main tools in \([33]\) were the smoothing properties of the continuous and discrete Laplace operators. Consequently, their estimate invariably contained the logarithmic term, so that the optimal error estimate is never obtained. It was established by a related work \([31]\) that arbitrary A-stable time-discretization preserves the time-discrete version of maximal \(L^p\)-regularity for abstract Cauchy problems and for \(p \in (1, \infty)\). These results were obtained via the theory of R-boundedness. It is therefore partially the same result of our previous work \([29]\). An optimal error estimate was established only for semi-discrete backward Euler scheme for a semilinear parabolic problem. In contrast to these works, we deal only with the finite difference scheme in time. However, our error estimate is optimal for fully discretized problems.

The plan of this paper is as follows. In Sec. 2, we introduce the notion of finite element approximation and state main results (Theorems \([I-VII]\)). We summarize some preliminary results used in the proofs of Theorems in Sec. 3. Some auxiliary lemmas related to MR, DMR and \(A_h\) are described there. A useful sufficient condition for DMR to hold is also described there (Lemma \([6]\)). In Sec. 4, we prove Theorems \([I-IV]\) by a discrete version of the method of \([16]\) using pure imaginary powers of operators. Auxiliary results, Lemmas \([15, 18]\) themselves are of interest. The proof of error estimate (Theorem \([V]\)) for the linear equation \([1]\) is described in Sec. 5. The semilinear equation \([2]\) is studied in Sec. 6. Therein, we also prove auxiliary results including discrete Gagliardo-Nirenberg, Sobolev inequalities and provide useful results related to the fractional powers of \(A_h\). Combining those results, we prove the final error estimate, Theorems \([VI]\) and \([VII]\).

2 Main results

Throughout this paper, \(\Omega\) is assumed to be a bounded polygonal or polyhedral domain in \(\mathbb{R}^d\), \(d = 2, 3\), with the boundary \(\partial \Omega\). We follow the notation of \([1]\). As an abbreviation, we write \(L^q = L^q(\Omega)\), \(W^{s,q} = W^{s,q}(\Omega)\) and \(H^s = W^{s,2}\) for
q ∈ [1, ∞] and s > 0. We use \( W^{1,q}_0 = \{ v \in W^{1,q} \mid v|_{\partial \Omega} = 0 \} \) and \( H^1_0 = W^{1,2}_0 \).

Generic positive constants which are independent of discretization parameters, \( h \) and \( \tau \), are denoted as \( C \). Their values might be different in each appearance.

Since the boundary \( \partial \Omega \) is not smooth, we make the following shape assumption on \( \Omega \).

**Assumption 1 (Shape assumption on \( \Omega \))** There exists \( \mu > d \) satisfying
\[
\|v\|_{W^{2,q}} \leq C \|\Delta v\|_{L^q}, \quad \forall v \in W^{2,q} \cap W^{1,q}_0,
\]
for \( q \in (1, \mu) \), where \( C > 0 \) depends only on \( \Omega \) and \( q \).

For example, if \( \Omega \) is a convex polygonal domain in \( \mathbb{R}^2 \), then one can find \( \mu > 2 \) satisfying Assumption 1 (see [24]).

Let \( T_h \) be a triangulation of \( \Omega \) with the granularity parameter \( h \) defined below. Hereinafter, a family \( T \) of triangles or tetrahedra is a triangulation of \( \Omega \) if and only if
1. each element of \( T \) is an open triangle or tetrahedron in \( \Omega \) and
   \[
   \Omega = \text{Int} \left( \bigcup_{K \in T} K \right),
   \]
   where \( \text{Int}(\cdot) \) is the interior part of a set,
2. any two elements of \( T \) meet only in entire common faces (when \( d = 3 \)), sides or vertices.

We use the following notations:
- \( h = \max_{K \in T_h} h_K \); \( h_K \) = the diameter of a triangle or tetrahedron \( K \);
- \( N_h \) = the number of nodes of \( T_h \); \( N_h \) = the number of interior nodes;
- \( \{P_j\}_{j=1}^{N_h} \) = the nodes of \( T_h \); \( \{P_j\}_{j=1}^{N_h} \) = the interior nodes.

We assume the following.

**Assumption 2 (Regularity of \( \{T_h\}_h \))** There exists \( \nu > 0 \) such that
\[
h_K \leq \nu \rho_K, \quad \forall K \in T_h, \quad \forall h > 0,
\]
where \( \rho_K \) denotes the radius of the inscribed circle or sphere of \( K \).

Here we consider the \( P_1 \) finite element. Let \( V_h \) be the space of continuous functions on \( \Omega \) which are affine in each element \( K \in T_h \). For every node \( P_j \) \( (j = 1, \ldots, N_h) \), \( \phi_j \) is the corresponding basis of \( V_h \), which satisfies \( \phi_j(P_i) = \delta_{ij} \), where \( \delta_{ij} \) is Kronecker’s delta. Namely, \( V_h \) is the linear space spanned by \( \{\phi_j\}_{j=1}^{N_h} \). We also set
\[
S_h = \{ v_h \in V_h \mid v_h|_{\partial \Omega} = 0 \} = \text{span}\{\phi_j\}_{j=1}^{N_h}.
\]

Moreover, we presume that \( \{T_h\}_h \) satisfies the following conditions if necessary.
(H1) (Inverse assumption) There exists $\gamma > 0$ such that
\[ h \leq \gamma h_K, \quad \forall K \in T_h, \quad \forall h > 0. \]

(H2) (Acuteness) For each $h > 0$ and for each $i, j \in \{1, 2, \ldots, N_h\}$ with $i \neq j$,
\[ \int \nabla \phi_i \cdot \nabla \phi_j \, dx \leq 0. \quad (11) \]

Remark 1 In the two-dimensional case, let $\sigma \subset \Omega$ be an edge of the triangulation $T_h$ and $K$ and $L$ be the triangles of $T_h$ which meet in $\sigma$. Assume that the nodes $P_i$ and $P_j$ be both endpoints of $\sigma$. We denote the interior angle of $K$ opposite to the edge $\sigma$ by $\alpha_{K}^{i,j}$. Then, the condition (11) is equivalent to the equation of $\alpha_{K}^{i,j} + \alpha_{L}^{i,j} \leq \pi$. See [30, Corollary 3.48] for the detail.

Remark 2 (Discrete maximum principle) The condition (H2) is equivalent to the discrete maximum principle, i.e., the following conditions are equivalent.

1. The triangulation $T_h$ fulfills the acuteness condition.
2. Let $u_h \in V_h$ be the solution of the following problem for $f \in L^2$ and $g_h \in V_h$:
\[
\begin{align*}
(\nabla u_h, \nabla v_h)_{L^2} &= (f, v_h)_{L^2}, \quad \forall v_h \in S_h, \\
u_h|_{\partial\Omega} &= g_h.
\end{align*}
\]

Then, $u_h \geq 0$ in $\Omega$ provided that $f \geq 0$ in $\Omega$ and $g_h \geq 0$ on $\partial\Omega$.

See [30, Theorem 3.49] for details.

Remark 3 When $q = 2$, $A_h$ is a self-adjoint operator in $X_{h,2}$. Therefore, (H2) is not required in the following discussion. However, the condition (H1) is required for the inverse inequality, which implies $H^1$-stability of the $L^2$-projection (the equation (29)) and the discrete Gagliardo-Nirenberg type inequality (Lemma 24). Therefore, this condition is imposed for the consequences of (29) and Lemma 24 for example, Theorems VII for even if $q = 2$.

We describe the method of mass-lumping. For a node $P_j$, we designate the corresponding barycentric domain as $\Lambda_j$; see Figure 1 for illustration and see [20,19] for the definition. We denote the characteristic function of $\Lambda_j$ by $\chi_j$ for $j = 1, \ldots, N_h$. Then, we set
\[ \overline{S}_h = \text{span}\{\chi_j\}_{j=1}^{N_h} \]
and define the lumping operator $M_h: S_h \to \overline{S}_h$ as
\[ M_h v_h = \sum_{j=1}^{N_h} v_h(P_j) \chi_j. \]

Moreover, we define $K_h = M_h^* M_h$, where $M_h^*$ is the adjoint operator of $M_h$ with respect to the $L^2$-inner product. As one might expect, $M_h$ is invertible.
and therefore $K_h$ is as well. We define the mesh-dependent norms and inner product as

$$\|v_h\|_{h,q} = \|M_h v_h\|_{L^q}, \quad (u_h, v_h)_h = (M_h u_h, M_h v_h), \quad u_h, v_h \in S_h$$

for $q \in [1, \infty]$. In fact, $\|\cdot\|_{h,q}$ is an equivalent norm to $\|\cdot\|_{L^q}$ in $S_h$ for each $q \in [1, \infty]$ (see Lemma 12).

At this stage, we introduce a discrete Laplacian as follows. Define the operator $A_h$ on $S_h$ as

$$ (A_h u_h, v_h)_h = - (\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h, $$

for $u_h \in S_h$. We designate $A_h$ the discrete Laplacian with mass-lumping. From the Poincaré inequality, $A_h$ is injective so that it is invertible due to $\dim S_h < \infty$.

We are now in a position to state the main results of this study. In the theorems below, we always presume that Assumptions 1 and 2 are satisfied, unless otherwise stated explicitly. The first one is about CMR for $A_h$.

**Theorem I (CMR for $A_h$)** Let $T \in (0, \infty]$, $p \in (1, \infty)$ and $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then, $A_h$ has $L^p$-CMR on $J_T$ in $X_{h,q}$ uniformly for $h > 0$. That is, there exists $C > 0$ independent of $h > 0$ satisfying

$$ \|u_h\|_{L^p(J_T; X_{h,q})} + \|u'_h\|_{L^p(J_T; X_{h,q})} + \|A_h u_h\|_{L^p(J_T; X_{h,q})} \leq C \|g_h\|_{L^p(J_T; X_{h,q})}, $$

where $g_h \in L^p(J_T; X_{h,q})$ and $u_h$ is the solution of

$$\begin{cases}
  u'_h(t) = A_h u_h(t) + g_h(t), & t \in J_T, \\
  u_h(0) = 0.
\end{cases}$$

(12)
Remark 4. Since (12) is a system of (inhomogeneous) linear ordinary differential equations, the unique existence of a solution follows immediately.

Next, we state results about DMR for $A_h$. To state them, we set
\[
\theta_q = \arccos \left| 1 - q/2 \right|, \quad \kappa_h = \min_{K \in T_h} \kappa_K,
\]
where $\kappa_K$ denotes the minimum length of perpendiculars of $K$.

**Theorem II (DMR for $A_h$ in $J_\infty$)** Let $p \in (1, \infty)$, $q \in (1, \mu)$ and $\theta \in [0, 1]$. Assume that (H1) and (H2) are satisfied when $\theta \neq 2$. We choose $\varepsilon$ and $\tau$ sufficiently small to satisfy
\[
\tau \frac{\kappa_h}{\varepsilon} \leq \frac{2 \sin \theta_q - \varepsilon}{(1 - 2\theta)(d + 1)^2},
\]
when $\theta \in [0, 1/2)$. Then, $A_h$ has $l^p$-DMR on $J_\infty$ in $X_{h,q}$ uniformly for $h > 0$. That is, there exists $C > 0$ independent of $h$ and $\tau$ satisfying
\[
\|u_h\|_{l^p(N, X_{h,q})} + \|D\tau u_h\|_{l^p(N, X_{h,q})} + \|A_h u_h\|_{l^p(N, X_{h,q})} \leq C\|g_h\|_{l^p(N, X_{h,q})},
\]
where $g_h = (g^{(n)}_h)_n \in l^p(N; X_{h,q})$ and $u_h = (u^{(n)}_h)_n$ is the solution of
\[
\begin{cases}
(D\tau u_h)^n = A_h u^{n+\theta}_h + g^{n+\theta}_h, & n \in \mathbb{N}, \\
u^0_h = 0.
\end{cases}
\]

**Theorem III (DMR for $A_h$ in $J_T$)** Let $p \in (1, \infty)$, $q \in (1, \mu)$ and $\theta \in [0, 1]$. Assume that (H1) and (H2) are satisfied when $\theta \neq 2$. Choose $\varepsilon$ and $\tau$ sufficiently small to satisfy (14), when $\theta \in [0, 1/2)$. Then, for every $T > 0$ and for every $g_h \in l^p(N_T - 1; X_{h,q})$, there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of
\[
\begin{cases}
(D\tau u_h)^n = A_h u^{n+\theta}_h + g^n_h, & n = 0, \ldots, N_T - 1, \\
u^0_h = 0,
\end{cases}
\]
and it satisfies
\[
\|u_h\|_{l^p(N_T, X_{h,q})} + \|D\tau u_h\|_{l^p(N_T, X_{h,q})} + \|Au_h\|_{l^p(N_T, X_{h,q})} \leq C\|g_h\|_{l^p(N_T, X_{h,q})},
\]
where $C > 0$ is independent of $g$, $T$, $h$, and $\tau$.

**Theorem IV (DMR for non-zero initial value)** Let $p \in (1, \infty)$ and $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $\theta \neq 2$. Then, for every $T > 0$, $g_h \in l^p(N_T; X_{h,q})$, and $u_{0,h} \in (X_{h,q}, D(A_h))_{1-1/p, p}$, there exists a unique solution $u_h \in l^p(N_T; X_{h,q})$ of
\[
\begin{cases}
(D\tau u_h)^n = A_h u^{n+1}_h + g^{n+1}_h, & n = 0, \ldots, N_T - 1, \\
u^0_h = u_{0,h},
\end{cases}
\]
which satisfies
\[ \|u_{h,1}\|_{L^2(N_T;X_{h,q})} + \|D_T u_{h}\|_{L^2(N_T;X_{h,q})} + \|A u_{h,1}\|_{L^2(N_T;X_{h,q})} \leq C \left[ \|g_{h,1}\|_{L^2(N_T;X_{h,q})} + \|u_{0,h}\|_{1-1/p,p} \right], \]
where \( C > 0 \) is independent of \( g, u_{0,h}, T, h, \) and \( \tau. \)

Therein, \( (X_{h,q}, D(A_h))_{1-1/p,p} \) and \( \|\cdot\|_{1-1/p,p} \) respectively denote the real interpolation space and its norm. (see Subsection 6.1 for related details.)

Those theorems are applicable for error analysis of the fully discretized finite element approximation for heat equations. First, we consider a linear interpolation space and its norm. Since \( A_h \) is invertible, there exists a unique solution of (17). We introduce \( P_h u_0 \) be the \( L^2 \)-projection onto \( S_h \) defined as
\[ (P_h u, v_h)_{L^2} = (v, v_h)_{L^2}, \quad \forall v_h \in S_h \]
for \( v \in L^1. \)

Then, (15) is equivalently written as
\[ \begin{aligned}
\{(D_T u_h)^n, v_h\}_h + \langle \nabla u_h^{n+\theta}, \nabla v_h \rangle_{L^2} &= (G^{n+\theta}, v_h)_{L^2}, \quad \forall v_h \in S_h, \\
(u_0^{n}, v_h)_{L^2} &= (u_0, v_h)_{L^2},
\end{aligned} \]
where \( \tau \in (0, 1), \theta \in [0, 1], t_n = n\tau, \) and \( G^n = g(\cdot, t_n) \). An alternative scheme is obtained with replacement \( (G^{n+\theta}, v_h)_{L^2} \) by \( (G^{n+\theta}, v_h)_h \). However, the resulting scheme has a shortcoming reported in Appendix B.

Let \( P_h \) be the \( L^2 \)-projection onto \( S_h \) defined as
\[ (P_h u, v_h)_{L^2} = (v, v_h)_{L^2}, \quad \forall v_h \in S_h \]
for \( v \in L^1. \)

Then, (15) is equivalently written as
\[ \begin{aligned}
\{(D_T u_h)^n, v_h\}_h &= A_h u_h^{n+\theta} + K_h^{-1} P_h G^{n+\theta}, \quad n = 0, 1, \ldots, N_T - 1, \\
u_0^n &= P_h u_0.
\end{aligned} \]
Since \( A_h \) is invertible, there exists a unique solution of (17). We introduce
\[ j_\theta = \begin{cases} 2, & \theta = 1/2, \\ 1, & \text{otherwise} \end{cases} \]
and \( \mu_d = \max\{\mu', d/2\} \). Since \( \mu' < d' \leq 2 \leq d < \mu \), it might be apparent that
\[ \mu_d = \begin{cases} \mu', & d = 2, \\ d/2 = 3/2, & d = 3. \end{cases} \]

**Theorem V (Error estimate for linear equation)** Let \( p \in (1, \infty) \) and \( q \in (\mu_d, \mu). \) Let \( u_h = (u^n_h)_{n \in \mathbb{N}} \in \ell^p(N_T; S_h) \) be the solution of (17) and \( u \) be that of (1). Assume \( u \in W^{1,p}(J_T; W^{2,q}) \cap W^{2,p}(J_T; W^{1,q}) \cap W^{1+\theta,p}(J_T; L^q) \) and set \( U^n = u(\cdot, t_n). \) Assume that (H1) and (H2) are satisfied. Moreover, we
choose $\varepsilon$ and $\tau$ sufficiently small to satisfy (14), when $\theta \in [0, 1/2)$. Then, there exists a positive constant $C$ such that
\[
\left( \sum_{n=0}^{N_T-1} \| u_h^{n+\theta} - U^n \|_{L^p(\Omega)}^p \right)^{1/p} \leq C(h^2 + \tau^{2\theta}).
\] (19)

The constant $C$ is taken as
\[ C = C'( \| u \|_{W^{1,q}(J_T; W^{1,q}(\Omega))} + \| \partial_t u \|_{W^{1,q}(J_T; W^{1,q}(\Omega))} + \| u \|_{W^{1+\gamma,p}(J_T; L^p(\Omega))} ), \]
where $C'$ depends only on $\Omega$, $p$, $q$, and $\theta$, but is independent of $h$ and $\tau$.

For $q \in (1, \infty)$, let $A_q$ be the realization of the Dirichlet Laplacian:
\[ D(A_q) = W^{2,q} \cap W^{1,q}_0, \quad A_q u = \Delta u. \] (20)

We are assuming Assumption [1]. We consider a semilinear heat equation (2) under the following basic assumptions:
\[ u_0 \in (L^q, D(A_q))_{1-1/p, p}, \]
\[ f : \mathbb{C} \to \mathbb{C} \text{ is locally Lipschitz continuous with } f(0) = 0. \] (22)

Herein, $(L^q, D(A_q))_{1-1/p, p}$ denotes the real interpolation space [23][44]. Restriction $f(0) = 0$ is set for simplicity. It is noteworthy that the solution $u$ of (2) might blow-up: let $T_\infty \in (0, \infty]$ be the life span of $u$ (the maximal existence time of $u$).

To avoid unnecessary difficulties, we restrict our consideration to a semi-implicit scheme for (2) given as
\[
\begin{cases}
(D_t u_h)^n = A_h u_h^{n+1} + K_h^{-1} P_h f(u_h^n), & n = 0, 1, \ldots, N_T - 1, \\
u_h^0 = P_h u_0,
\end{cases}
\] (23)

or, equivalently,
\[
\begin{cases}
((D_t u_h)^n, v_h)_h + (\nabla u_h^{n+1}, \nabla v_h)_{L^2} = (f(u_h^n), v_h)_{L^2}, & \forall v_h \in S_h, \\
(u_h^0, v_h)_{L^2} = (u_0, v_h)_{L^2}, & \forall v_h \in S_h.
\end{cases}
\]

Since $A_h$ is invertible, there exists a unique solution of (23). Our final theorem is the following error estimate for semilinear equation. Our error estimate remains valid as long as the solution of (2) exists and requires no size condition on $u_0$.

**Theorem VI (Error estimate for semilinear equation)** Let $p \in (1, \infty)$, $q \in (\mu_d, \mu)$ and $p > 2q/(2q - d)$. Assume that (H1) and (H2) are satisfied. Presuming that (2) admits a sufficiently smooth solution $u$ under the conditions [21] and [22], then, for every $T \in (0, T_\infty)$ and the solution $u_h = (u_h^n)_{n=0}^{N_T}$ of (23), we have
\[
\left( \sum_{n=1}^{N_T} \| u_h^n - U^n \|_{L^p(\Omega)}^p \right)^{1/p} \leq C(h^2 + \tau),
\]
where $U^n = u(\cdot, n\tau)$. 


In the proof of Theorem VI (Sec. 6), the following sub-optimal error estimate, which is worth stating separately, will be used.

**Theorem VII (L∞ error estimate for semilinear equation)** Under the same assumptions of Theorem VI, for every $\alpha \in (0, \alpha_{p,q,d})$ and $T \in (0, T_\infty)$, the following error estimate holds:

\[
\max_{0 \leq n \leq N_T} \|u_h^n - U^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau),
\]

where $\alpha_{p,q,d} = 1 - \frac{1}{p} - \frac{d}{2q}$ and $U^n = u(\cdot, n\tau)$.

### 3 Preliminaries

As explained in this section, we collect some preliminary results used for this study.

#### 3.1 Continuous maximal regularity

The definition of CMR in Definition 1 is the classical one. The weaker one is introduced in  [45, Definition 4.1], which requires the inequality

\[
\|u'\|_{L^p(J_T;X)} + \|Au\|_{L^p(J_T;X)} \leq C\|f\|_{L^p(J_T;X)}
\]

instead of (1). Also, CMR in this sense is characterized by operator-theoretical properties ([45, Theorem 4.2]). However, two inequalities (1) and (24) are equivalent if $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of $A$. Since the condition $0 \in \rho(A)$ is satisfied in our application, we ignore the differences between these definitions.

Conditions necessary for CMR to hold have been studied by many researchers (see e.g. [15, 45]). Among them, we review some sufficient conditions for CMR, which will be used for this study. For the detail, see [15] and references therein.

**Lemma 1** Let $T \in (0, \infty]$, $X$ be a Banach space and $A$ be a densely defined and closed operator on $X$. Assume that $A$ has $L^{p_0}$-CMR on $J_T$ for some $p_0 \in (1, \infty)$. Then, $A$ has $L^p$-CMR on $J_T$, for any $p \in (1, \infty)$.

**Lemma 2** Let $p \in (1, \infty)$, $X$ be a Banach space and let $A$ be a densely defined and closed operator on $X$. Assume that $A$ has $L^p$-CMR on $J_\infty$. Then, $A$ has $L^p$-CMR on $J_T$, for any $T > 0$.

The next lemma is the celebrated result of Dore and Venni [16, Theorem 3.2] (see also [3, Section III.4]).

**Lemma 3** Let $p \in (1, \infty)$, $X$ be a UMD space, and let $A$ be a densely defined and closed operator on $X$. Assume that $-A \in \mathcal{P}(X; K) \cap B\ell p(X; M, \theta)$ for some $K > 0$, $M \geq 1$, and $\theta \in [0, \pi/2)$. Then, $A$ has $L^p$-CMR on $J_T$, for any $T > 0$ and $T = \infty$. Moreover, the constant $C_{MR} > 0$ depends only on $X$, $K$, $M$, $\theta$, and $T$. 
Herein, the sets $\mathcal{P}(X; K)$ and $BIP(X; M, \theta)$ are defined as

$$
\mathcal{P}(X; K) = \{ A \in \mathcal{C}(X) \mid \rho(A) \subset (-\infty, 0] \} \quad \text{and} \quad \mathcal{P}(X; K) = \{ A \in \mathcal{C}(X) \mid (1 + \lambda)R(\lambda; A)\|_{\mathcal{L}(X)} \leq K, \forall \lambda \geq 0 \},
$$

$$
BIP(X; M, \theta) = \{ A \in \mathcal{P}(X) \mid A^{it} \in \mathcal{L}(X) \text{ and } \|A^{it}\|_{\mathcal{L}(X)} \leq M\theta^{[t]}, \forall t \in \mathbb{R}\},
$$

for $K > 0$, $M \geq 1$, and $\theta \geq 0$, where $\mathcal{C}(X)$ is the set of all closed linear operators on $X$ with dense domains, $\mathcal{P}(X) = \bigcup_{K>0} \mathcal{P}(X; K)$. The imaginary power $A^{it}$ is defined by $H^{\infty}$-functional calculus (see Appendix A).

The dependence of the constant $C_{MR}$ on the Banach space $X$ derives from the boundedness of imaginary powers of the time-differential operator on $L^p(J_T; X)$. See [3, Lemma III.4.10.5] for $T < \infty$ and [25, Corollary 8.5.3] for $T = \infty$. Chasing the constants appearing in the proofs, we can obtain the following property (see [25]).

**Lemma 4** Let $p \in (1, \infty)$, $X$ be a UMD space, $X_0 \subset X$ be a closed subspace, and $A$ be a densely defined and closed operator on $X_0$. Assume that $-A \in \mathcal{P}(X_0; K) \cap BIP(X_0; M, \theta)$ for some $K > 0$, $M \geq 1$, and $\theta \in [0, \pi/2]$. Then $A$ has $L^p$-CMR on $J_T$, for any $T > 0$ and $T = \infty$. Moreover, the constant $C_{MR} > 0$ depends only on $X$, $K$, $M$, $\theta$, and $T$, but is independent of $X_0$.

In the definition of CMR, we consider only the zero initial value. However, in general cases, particularly in the nonlinear cases, the choice of initial values is extremely important. Therefore, we now consider the following Cauchy problem:

$$
\begin{align*}
&u'(t) = Au(t) + g(t), \quad t \in J_T, \\
u(0) = u_0,
\end{align*}
$$

for $u_0 \in X$.

**Lemma 5** Let $p \in (1, \infty)$, $T \in (0, \infty]$, $X$ be a Banach space and $A$ be a densely defined and closed operator. Assume that $A$ has $L^p$-CMR on $J_T$. Then, for each $g \in L^p(J_T; X)$ and for each $u_0 \in (X, D(A))_{1-1/p, p}$, there exists a unique solution $u \in W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$ of (25) satisfying

$$
\|u\|_{L^p(J_T; X)} + \|u'\|_{L^p(J_T; X)} + \|Au\|_{L^p(J_T; X)} \leq C_{MR} \left( \|g\|_{L^p(J_T; X)} + \|u_0\|_{1-1/p, p} \right),
$$

where $C_{MR} > 0$ is independent of $g$ and $u_0$.

Herein, the norm $\| \cdot \|_{1-1/p, p}$ is the norm of the real interpolation space $(X, D(A))_{1-1/p, p}$.
3.2 Discrete maximal regularity

As in the CMR case, the weaker definition can be considered, which does not require that 0 ∈ ρ(A). Indeed, the weaker one is used in [6,29]. However, for the same reason as that presented in the previous subsection, we do not distinguish these two definitions.

We investigated a sufficient condition for DMR on \( J_\infty \), in the UMD case in [29]. More precisely, we proved the following result.

**Lemma 6** Let \( p \in (1, \infty) \), \( \theta \in [0, 1] \), \( X \) be a UMD space, \( X_0 \subset X \) be a closed subspace, and \( A \) be a bounded operator on \( X_0 \). Assume that \( A \) has \( L^p \)-CMR on \( J_\infty \) with the constant \( C_{\text{MR}} \). Furthermore, we suppose that the following conditions (condition \((\text{NR})_{\delta, \varepsilon}\)) are satisfied when \( \theta \in [0, 1/2) \):

\((\text{NR1})\) There exists \( \delta \in (0, \pi/2) \) such that \( S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2} \).

\((\text{NR2})\) There exists \( \varepsilon > 0 \) such that \( (1 - 2\theta)\tau r(A) + \varepsilon \leq 2\sin \delta \).

Then, \( A \) has \( l^p \)-DMR on \( J_\infty \). Moreover, the constant \( C_{\text{DMR}} \) depends only on \( p \), \( \theta \), \( \delta \), \( \varepsilon \), \( X \), and \( C_{\text{MR}} \), but is independent of \( X_0 \).

Herein, for \( \omega \in (0, \pi) \), the set \( \Sigma_\omega \) denotes the sector

\[ \Sigma_\omega = \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \omega \}. \]  

(26)

The set \( S(A) \subset \mathbb{C} \) is the numerical range of \( A \) defined as

\[ S(A) = \left\{ (x^*, Ax) \mid x \in D(X), \|x\| = 1, \right\} \]

\[ x^* \in X^*, \|x^*\| = 1, \langle x^*, x \rangle = 1, \right\}, \]

where \( \langle \cdot, \cdot \rangle \) is the duality pairing (20,35). We set

\[ r(A) = \max_{z \in S(A)} |z|. \]

Actually, DMR on finite intervals is obtainable from the infinite-interval case. The following lemma corresponds to Lemma 2. Although the inequality (28) below is slightly different from (9), it does not affect error analysis.

**Lemma 7** Let \( p \in (1, \infty) \), \( \theta \in [0, 1] \), \( X \) be a Banach space, and \( A \) be a bounded operator on \( X \). Assume that \( A \) has \( l^p \)-DMR on \( J_\infty \) with \( C_{\text{DMR}} = C_0 \). Then, for every \( T > 0 \) and for every \( g \in l^p(N_T-1; X) \), there exists a unique solution \( u \in l^p(N_T; X) \) of the equation

\[ \begin{cases} \frac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + g^n, & n = 0, 1, \ldots, N_T - 1, \\ u^0 = 0, \end{cases} \]  

(27)

and it satisfies

\[ \|u^\theta\|_{l^p(N_T; X)} + \|D_{\tau}u\|_{l^p(N_T; X)} + \|Au^\theta\|_{l^p(N_T; X)} \leq C_0 \|g\|_{l^p(N_T; X)}. \]  

(28)
Proof Fix $T > 0$, $\tau > 0$, and $g \in l^p(N_T - 1; X)$ arbitrarily. Define $\tilde{g} \in l^p(N; X)$ as
\[
\tilde{g}^n = \begin{cases} 
g^n, & n = 0, 1, \ldots, N_T - 1, \\
0, & n \geq N_T,
\end{cases}
\]
and consider the Cauchy problem
\[
\begin{cases}
\frac{\tilde{u}^{n+1} - \tilde{u}^n}{\tau} = A\tilde{u}^{n+\theta} + \tilde{g}^n, & n = 0, 1, \ldots,
\tilde{u}^0 = 0.
\end{cases}
\]
Since $A$ has $l^p$-DMR on $J_\infty$, we can find the corresponding solution $\tilde{u} = (\tilde{u}^n) \in X^N$ satisfying
\[
\|\tilde{u}_\theta\|_{l^p(N; X)} + \|D_\tau \tilde{u}\|_{l^p(N; X)} + \|A\tilde{u}_\theta\|_{l^p(N; X)} \leq C_{\text{DMR}} \|\tilde{g}\|_{l^p(N; X)}.
\]
Then, $u := (\tilde{u}^n)^{N_T}_{n=0} \in l^p(N_T; X)$ is a solution of (27), and fulfills
\[
\|u_\theta\|_{l^p(N_T; X)} + \|D_\tau u\|_{l^p(N_T; X)} + \|A u_\theta\|_{l^p(N_T; X)} \leq C_{\text{DMR}} \|\tilde{g}\|_{l^p(N; X)},
\]
which implies (28). The uniqueness of the solution might be readily apparent.

An a priori estimate with non-zero initial value is obtained only in the case where $\theta = 1$. See [4] for $T < \infty$ and [29] for $T = \infty$.

Lemma 8 Let $p \in (1, \infty)$, $\theta \in [0, 1]$, $T \in (0, \infty]$, $X$ be a UMD space, $X_0 \subset X$ be a closed subspace, and $A$ be a bounded operator on $X_0$. Assume that $A$ has $l^p$-DMR on $J_T$. Then, for each $g \in l^p(N_T; X_0)$ and for each $u_0 \in (X_0, D(A))_{1-1/p, p}$, there exists a unique solution $u \in l^p(N_T; X_0)$ of the equation
\[
\begin{cases}
\frac{u^{n+1} - u^n}{\tau} = Au^{n+1} + g^{n+1}, & n = 0, 1, \ldots, N_T - 1,
\end{cases}
\]
which satisfies
\[
\|u_1\|_{l^p(N_T; X_0)} + \|D_\tau u\|_{l^p(N_T; X_0)} + \|A u_1\|_{l^p(N_T; X_0)} \leq C_{\text{DMR}} \left( \|g_1\|_{l^p(N_T; X_0)} + \|u_0\|_{1-1/p, p} \right),
\]
where $C_{\text{DMR}} > 0$ is independent of $g$, $u_0$, and $X_0$. 
3.3 Operator-theoretical properties of $A_h$

A semigroup $T(t)$ on a Lebesgue space $X = L^q(\Omega, \mu)$ ($q \in [1, \infty]$) is said to be positivity-preserving if

$$u \geq 0 \text{ } \mu\text{-a.e. in } \Omega \implies T(t)u \geq 0 \text{ } \mu\text{-a.e. in } \Omega$$

for each $t > 0$ and $u \in X$. In the proofs of the following two lemmas, the discrete maximum principle (Remark 2) plays a crucially important role.

**Lemma 9** ([41, Theorem 15.5]) Let $t > 0$ and $u \in X$. Assume that the family of triangulations $\{T_h\}$ satisfies the acuteness condition (H2). Then, the semigroup $e^{tA_h}$ generated by $A_h$ is positivity-preserving in $X_{h,q}$.

**Lemma 10** ([13, Theorem 4.1]) Let $q \in [1, \infty]$. Assume that the family of triangulations $\{T_h\}$ satisfies the acuteness condition (H2). Then, $A_h$ generates an analytic and contraction semigroup on $X_{h,q}$. Moreover, if $q \in (1, \infty)$, then $A_h$ satisfies the condition (NR1) with the angle $\theta_q$ defined as [13].

We introduce several mesh-depending operators on $S_h$. The $L^2$ projection $P_h$ is defined as [16]. Let $R_h$ be the Ritz projection of $W^{1,1} \to S_h$ defined as

$$\langle \nabla R_h u, \nabla v_h \rangle_{L^2} = \langle \nabla u, \nabla v_h \rangle_{L^2}, \quad \forall v_h \in S_h$$

for $u \in W^{1,1}$. These operators have the following well-known properties. See [25][12][7] for the proofs.

**Lemma 11** Assume that $\{T_h\}_h$ satisfies (H1). Then, there exists $C > 0$ depending only on $\Omega$ and $q$ such that

$$\|P_h v\|_{L^q} \leq C \|v\|_{L^q}, \quad \forall v \in L^q, \quad \forall q \in [1, \infty],$$

$$\|P_h v\|_{W^{1,q}} \leq C \|v\|_{W^{1,q}}, \quad \forall v \in W^{1,q}, \quad \forall q \in [1, \infty],$$

$$\|R_h v\|_{W^{1,q}} \leq C \|v\|_{W^{1,q}}, \quad \forall v \in W^{1,q}, \quad \forall q \in [1, \infty],$$

$$\|v - P_h v\|_{L^q} \leq C h^2 \|v\|_{W^{2,q}}, \quad \forall v \in W^{2,q}, \quad \forall q \in (d/2, \infty],$$

$$\|v - R_h v\|_{L^q} \leq C h^2 \|v\|_{W^{2,q}}, \quad \forall v \in D(A_q), \quad \forall q \in (\mu', \infty),$$

where $\mu'$ is the Hölder conjugate of $\mu$. When $q \neq 2$, (H1) is not required for all inequalities above except for (29).

Mass-lumping operator $M_h$ and $K_h$ have the following properties. For the proof, see [20].

**Lemma 12** Let $q \in [1, \infty]$. Then, there exists $C > 0$ depending only on $q$ and $\Omega$ such that

$$C^{-1} \|v_h\|_{L^q} \leq \|M_h v_h\|_{L^q} \leq C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty].$$

Moreover, if $\{T_h\}_h$ satisfies (H1) when $q \neq 2$, then there exists $C > 0$ depending only on $q$ and $\Omega$ such that

$$C^{-1} \|v_h\|_{L^q} \leq \|K_h v_h\|_{L^q} \leq C \|v_h\|_{L^q}, \quad v_h \in S_h, \quad q \in [1, \infty].$$
We use the standard discrete Laplacian $L_h$ defined as

$$(L_h u_h, v_h) = - (\nabla u_h, \nabla v_h), \quad \forall v_h \in S_h,$$

for $u_h \in S_h$. We designate $L_h$ the discrete Laplacian without mass-lumping. From the Poincaré inequality, $L_h$ is injective. Consequently, it is invertible due to $\dim S_h < \infty$. Then, by the definitions given above, it is apparent that

$$L_h = K_h A_h, \quad R_h = L_h^{-1} P_h A. \quad (30)$$

From these relations, the following estimate is obtained.

**Lemma 13** Assume that $\{T_h\}_h$ satisfies (H1) when $q \neq 2$. Then, for $q \in (1, \mu)$, there exists $C > 0$ satisfying

$$\|v_h\|_{h,q} \leq C \|A_h v_h\|_{h,q}, \quad \forall v_h \in S_h,$$

where $C$ depends only on $\Omega$ and $q$.

**Proof** By (30) and Lemma 12, it suffices to show that

$$\|v_h\|_{L^q} \leq C \|L_h v_h\|_{L^q}$$

for all $v_h \in S_h$. Fix $v_h \in S_h$ arbitrarily and set $f_h = L_h v_h$ and $v = A^{-1} f_h \in D(A)$. Then, noting that $P_h f_h = f_h$ and from (30), one obtains

$$v_h = L_h^{-1} P_h f_h = L_h^{-1} P_h A v = R_h v.$$

Therefore, we have

$$\|v_h\|_{L^q} \leq \|R_h v\|_{W^{1,q}} \leq C \|v\|_{W^{1,q}} \leq C \|v\|_{W^{2,q}} \leq C \|A v\|_{L^q} = C \|L_h v_h\|_{L^q}$$

by Lemma 11 and (10). \qed

Furthermore, the following estimate holds. See [37, Lemma 4.6] for the proof.

**Lemma 14** Assume that $\{T_h\}_h$ satisfies (H1) when $q \neq 2$. Let $q \in (\mu', \mu)$. Then, there exists $C > 0$ depending only on $q$ and $\Omega$ such that

$$\|A_h^{-1} (I - K_h^{-1}) v_h\|_{h,q} \leq C h^2 \|\nabla v_h\|_{L^q}, \quad v_h \in S_h.$$
4 Proofs of Theorems I, II, III and IV

The aim of this section is to establish CMR and DMR for $A_h$. We first consider the continuous case via the method of imaginary powers of operators. Then, we obtain DMR for $A_h$ by our previous result (Lemma 6). We also present a useful criterion to check the condition $(\text{NR})_{h,\varepsilon}$.

In view of Lemma 3, it suffices to show that

$$-A_h \in \mathcal{P}(X_{h,q}; K) \cap \mathcal{BIP}(X_{h,q}; M, \theta)$$

for some $K > 0$, $M \geq 1$, and $\theta \in [0, \pi/2)$, uniformly with respect to $h$. We first show that $-A_h \in \mathcal{P}(X_{h,q}; K)$.

**Lemma 15** Let $q \in (1, \mu)$. Assume that the family $\{T_h\}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, there exists $K_q > 0$ satisfying

$$-A_h \in \mathcal{P}(X_{h,q}; K_q),$$

where $K_q$ is independent of $h > 0$.

**Proof** Let $T_h(t)$ be the semigroup $e^{tA_h}$ generated by $A_h$ in $X_{h,q}$. Then, by Lemma 10, $T_h(t)$ is an analytic and contraction semigroup. Since $T_h(t)$ is contraction semigroup, we have

$$\|R(\lambda; A_h)\|_{\mathcal{L}(X_{h,q})} \leq \frac{1}{\lambda}, \quad \forall \lambda > 0$$

for each $h > 0$. In addition, by virtue of Lemma 13 and analyticity of $T_h(t)$, we have

$$\|R(\lambda; A_h)f_h\|_{h,q} = \|A_h^{-1}[\lambda R(\lambda; A_h) - I]f_h\|_{h,q} \leq C\|f_h\|_{h,q}, \quad \forall f_h \in S_h$$

for all $\lambda > 0$ and $h > 0$, where $C > 0$ is independent of $h$. Therefore, we obtain $-A_h \in \mathcal{P}(X_{h,q}; K_q)$ with $K_q = C + 1$ since $R(\cdot; A_h,q) \in C([0, \infty); \mathcal{L}(X_{h,q}))$.

To show $-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta)$, we use Duong’s result, which is based on $H^\infty$-functional calculus. The imaginary power is understood as the special case of the function of operators. Let $X$ be a Banach space, $D \subset \mathbb{C}$ be a domain and $\mathcal{O}(D)$ be the space of holomorphic functions on $\mathbb{C}$. We set

$$H^\infty(D) = \mathcal{O}(D) \cap L^\infty(D; \mathbb{C}).$$

(31)

Then, for $A \in \mathcal{P}(X)$ and for $m \in H^\infty(\Sigma_q)$ with suitable $\theta$, $m(A)$ can be defined as a linear operator on $X$. When we take $m(z) = z^\theta$, the imaginary power $A^\theta$ is defined in this sense. The definition and details of the properties of $m(A)$ have been presented in the literature \cite{11} and in the Appendix A. We refer to \cite{17} for the proof of the following lemma (see also \cite{10}).
Lemma 16 ([17, Theorem 2]) Let $(\Omega, \mu)$ be a $\sigma$-finite measure space and let $A$ be a linear operator on $X = L^q(\Omega, \mu)$ for $q \in (1, \infty)$. Assume that $A \in \mathcal{P}(X)$ and that $-A$ generates a contraction semigroup $T(t)$ on $X$. Moreover, we suppose that $T(t)$ is positivity-preserving on $X$. Then, for each $\theta \in (\pi/2, \pi)$, there exists $M > 0$ satisfying

$$\|m(A)\|_{\mathcal{L}(X)} \leq M\|m\|_{L^\infty(\Sigma_\theta)}$$

for all $m \in H^\infty(\Sigma_\theta)$. Furthermore, $M$ depends only on $q$ and $\theta$, but is independent of $A$ and measure space $(\Omega, \mu)$.

Lemma 17 Let $X$ and $A$ be as in Lemma 16. Then, for each $\theta \in (\pi/2, \pi)$, there exists $M > 0$ such that $A \in \mathcal{BIP}(X; M, \theta)$.

Proof Let $m(z) = z^it$ for $z \in \Sigma_\theta$ and $t \in \mathbb{R}$. Here, $z^it$ is defined as

$$z^it = e^{it(\log|z| + i \arg z)}, \quad \arg z \in (-\pi, \pi)$$

for $z \in \Sigma_\pi$. Then, setting $z = |ze^{i\vartheta} \ (\vartheta \in (-\theta, \theta))$, one can readily obtain $|z^it| = e^{-t\vartheta}$. Therefore, we have

$$\|m\|_{L^\infty(\Sigma_\theta)} \leq e^{t\vartheta},$$

which yields $m \in H^\infty(\Sigma_\theta)$ and $A \in \mathcal{BIP}(M, \theta)$ for some $M > 0$ by Lemma 16. \qed

Now, we are ready to show the following lemma.

Lemma 18 (Imaginary powers of discrete Laplacian) Let $q \in (1, \mu)$. Assume that (H1) and (H2) are satisfied when $q \neq 2$. Then there exist $M_q > 0$ and $\theta_q \in (0, \pi/2)$ satisfying

$$-A_h \in \mathcal{BIP}(X_{h,q}; M_q, \theta_q),$$

where $M_q$ and $\theta_q$ are independent of $h > 0$.

Proof We begin by proving that $-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta)$ for each $\theta \in (\pi/2, \pi)$ and for suitable $M > 0$ independent of $h$. Let $T_h(t)$ be the semigroup $e^{tA_h}$ generated by $A_h$ in $X_{h,q}$. Then, by Lemma 9 and 15 we can apply Lemma 17. Therefore, for each $\theta \in (\pi/2, \pi)$, there exists $M > 0$ satisfying

$$-A_h \in \mathcal{BIP}(X_{h,q}; M, \theta). \quad (32)$$

Now, we show our assertion. We first assume that $q = 2$. In this case, $X_{h,2}$ is a Hilbert space and $-A_h$ is self-adjoint and positive definite without conditions on the triangulation by Poincaré inequality. Consequently, by Theorem 32 we have

$$\|(-A_h)^it\|_{\mathcal{L}(X_{2,h})} \leq \int_{0}^{\infty} dE_{-A_h}(\lambda) = 1$$
for all \( t \in \mathbb{R} \), which implies \( -A_h \in B\text{IP}(X_{2,h}; 1, 0) \). Here, \( E_{-A_h} \) is the spectral decomposition of \( -A_h \). Then we presume that \( q \neq 2 \). Set
\[
\theta_{q,r} = \frac{q^{-1} - 2^{-1}}{r^{-1} - 2^{-1}}
\]
for \( r \neq 2 \). Since \( q \neq 2 \), we can choose \( r \in (1, \infty) \) satisfying \( \theta_{q,r} \in (0, 1) \). Then, by the Riesz-Thorin theorem, we obtain
\[
\| (-A_h)^t \|_{X_{q,r}} \leq \| (-A_h)^t \|_{X_{2,2}}^{1 - \theta_{q,r}} \| (-A_h)^t \|_{X_{2,r}}^{\theta_{q,r}} \leq M^{\theta_{q,r}} e^{\theta_{q,r}|t|}
\]
for any \( t \in \mathbb{R} \) and \( \theta \in (\pi/2, \pi) \), where \( M > 0 \) is as in (32). Since \( \theta_{q,r} \in (0, 1) \), we can choose \( \theta \) as
\[
\frac{\pi}{2} < \theta < \frac{\pi}{2}\theta_{q,r},
\]
which implies
\[
-A_h \in B\text{IP}(X_{q,r}; M^{\theta_{q,r}}, \theta^{\theta_{q,r}})
\]
with \( \theta^{\theta_{q,r}} < \pi/2 \). This is the desired assertion. \( \square \)

Owing to Lemma 6 and Theorem 1, we are able to obtain DMR for \( A_h \). To apply Lemma 6, it is necessary to verify that the condition (NR) is satisfied. From Lemma 10, the condition (NR1) is always satisfied. Therefore, what is left is to check the condition (NR2). We begin with the following lemma, which is a generalization of \([19, \text{Lemma 2}]\). No condition on the triangulation is required.

**Lemma 19** Let \( r \in [1, \infty) \). Then, we have
\[
\| \nabla v_h \|_{L^r} \leq \frac{d + 1}{\kappa_h} \| v_h \|_{h,r,K}, \quad \forall v_h \in S_h.
\]

**Proof** Fix \( K \in T_h \) arbitrarily. Then it suffices to show that
\[
\| \nabla v_h \|_{L^r(K)} \leq \frac{d + 1}{\kappa_h} \| v_h \|_{h,r,K}, \quad \forall v_h \in S_h,
\]
where \( \| v_h \|_{h,r,K} = \| M_h v_h \|_{L^r(K)} \). Let \( Q_j \) (\( j = 0, \ldots, d \)) be the vertex of \( K \), \( \lambda_j \) be the corresponding barycentric coordinate in \( K \), and \( \kappa_j \) be the length of the perpendicular from \( P_j \) in \( K \). Then it is well-known that \( |\nabla \lambda_j| = 1/\kappa_j \). Take \( v_h \in S_h \) arbitrarily and set \( v_j = v_h(Q_j) \). Since \( v_h|_K = \sum_{j=0}^d v_j \lambda_j \), we have
\[
\| \nabla v_h \|_{L^r(K)} \leq \sum_{j=0}^d |v_j| \| \nabla \lambda_j \|_{L^r(K)} \leq \sum_{j=0}^d \frac{|u_j|}{\kappa_j} |K|^{1/r} \leq \left( \sum_{j=0}^d \frac{1}{\kappa_j} \right)^{1/r} \left( \sum_{j=0}^d |u_j|^r \right)^{1/r} |K|^{1/r} \leq \left( \frac{d + 1}{\kappa_h} \right)^{1/r} \left( \sum_{j=0}^d |u_j|^r \right)^{1/r} |K|^{1/r},
\]
(33)
where \( r' \) is the Hölder conjugate of \( r \). Moreover, it is readily apparent that
\[
\|v_h\|_{h,r,K} = \left( \frac{1}{d+1} |K| \sum_{j=0}^{d} |v_j|^r \right)^{1/r}.
\]

This, together with (33), implies that
\[
\|\nabla v_h\|_{L^{r'}(K)} \leq \frac{(d+1)^{1/r'+1/r}}{\kappa_h} \|v_h\|_{h,r,K} = \frac{d+1}{\kappa_h} \|v_h\|_{h,r,K}.
\]

Thereby we complete the proof. \( \square \)

Now, we describe a sufficient condition for (NR2) to hold.

**Lemma 20 (A sufficient condition for (NR)\(_{\theta,\varepsilon}\))** Assume \( \theta \in [0, 1/2) \) and \( q \in (1, \infty) \), and Let \( \theta_q = \arccos \left| 1 - \frac{2}{q} \right| \). If we choose \( \varepsilon \) and \( \tau \) sufficiently small so that \( A_h \) satisfies (14), for every \( h \), then the condition (NR)\(_{\theta,q,\varepsilon}\) is fulfilled.

**Proof** The numerical range of \( A_h \) is expressed as
\[
S(A_h) = \{(A_hv_h, v_h^*)_h \mid v_h \in S_h, \|v_h\|_{h,q} = 1\},
\]
where \( v_h^* \in S_h \) is defined as
\[
v_h^*(P) = |v_h(P)|^{q-2}v_h(P) \quad \text{for every node } P \text{ of } T_h
\]
for \( v_h \in S_h \). Therefore, by Lemma 19 we have
\[
|(A_hv_h, v_h^*)_h| \leq \|\nabla v_h\|_{L^q} \|\nabla v_h^*\|_{h,q'} \leq \frac{(d+1)^2}{\kappa_h^2} \|v_h\|_{h,q}\]
for all \( v_h \in S_h \). Hence we can deduce (NR2) from the assumption (14). \( \square \)

At this stage, we can state the following proofs.

**Proof (Proof of Theorem I)** It is a consequence of Lemmas 2, 4, and 18. \( \square \)

**Proof (Proof of Theorem II)** It is a consequence of Theorem I and Lemmas 6 and 20. \( \square \)

**Proof (Proof of Theorem III)** It is a consequence of Theorem II and Lemma 7. \( \square \)

**Proof (Proof of Theorem IV)** It is a consequence of Theorem II and Lemma 8. \( \square \)
5 Proof of Theorem [V]

This section is devoted to error analysis of the solution $u_h = (u^n_h) \in l^p(N_T; S_h)$ of (17). We begin by presenting some lemmas.

**Lemma 21** Let $X$ be a Banach space, $T > 0$, $p \in (1, \infty)$ and $\tau \in (0, 1)$. Set $t_n = n\tau$ for $n = 0, 1, \ldots, N_T$. Then, there exists $C_S > 0$ satisfying

$$\left( \sum_{n=0}^{N_T-1} \|v(t_n)\|_{X}^p \tau^\frac{1}{p} \right)^{1/p} + \left( \sum_{n=1}^{N_T} \|v(t_n)\|_{X}^p \tau^\frac{1}{p} \right)^{1/p} \leq C_S \|v\|_{W^{1,p}(J_T; X)} \quad (34)$$

for all $v \in W^{1,p}(J_T; X)$, where $C_S$ depends only on $p$, but is independent of $T$, $\tau$, and $X$.

**Proof** By the Sobolev embedding $W^{1,p}(0,1; X) \hookrightarrow L^\infty(0,1; X)$, there exists $C_1 > 0$ such that

$$\|v\|_{L^\infty(0,1; X)} \leq C_1 \|v\|_{W^{1,p}(0,1; X)}$$

for $v \in W^{1,p}(0,1; X)$. One can check that $C_1$ is independent of $X$. See the proof of [8, Theorem 8.8]. Then, setting $J_n = (t_n, t_{n+1})$ and considering the change of variables, we have

$$\|v(t_n)\|_X \leq \|v\|_{L^\infty(J_n, X)} \leq C_1 (1 + \tau)^{-1/p} \|v\|_{W^{1,p}(J_n, X)}$$

for each $n \in \mathbb{N}$. Therefore, we have (34) with $C_S = 2C_1$. $\square$

The next lemma is shown readily by Taylor’s theorem. Therefore, we skip the proof.

**Lemma 22** Let $X$ be a Banach space, $T > 0$, $p \in (1, \infty)$, $\theta \in [0,1]$ and $\tau \in (0,1)$. Set $t_n = n\tau$ for $n = 0, 1, \ldots, N_T$ and

$$r^n = \frac{v(t_{n+1}) - v(t_n)}{\tau} - \left[ (1 - \theta) \frac{dv}{dt}(t_n) + \theta \frac{dv}{dt}(t_{n+1}) \right]$$

for $v \in W^{j_\theta+1,p}(J_T; X)$, where $j_\theta$ is defined as (18). Then, there exists $C > 0$ such that

$$\left( \sum_{n=0}^{N_T-1} \|r^n\|_{X}^p \tau^\frac{1}{p} \right)^{1/p} \leq C \tau^{j_\theta} \|v\|_{W^{j_\theta+1,p}(J_T; X)},$$

where $C$ is independent of $\tau$ and $X$.

Now we can state the following proof.

**Proof (Proof of Theorem [V])** We set $e^n_h = u^n_h - P_h U^n$ so that

$$u^n_h - U^n = e^n_h + (P_h - I)U^n.$$
Then, by Lemmas 11 and 21, we have
\[
\sum_{n=0}^{N_T-1} \|(P_h - I)U^{n+\theta}\|_{L^q}^p \\
\leq Ch^2 \left[ (1 - \theta)^p \sum_{n=0}^{N_T-1} \|U^n\|_{W^{2,q}}^p + \theta^p \sum_{n=1}^{N_T} \|U^n\|_{W^{2,q}}^p \right] \\
\leq Ch^2 \|u\|_{W^{1,p}(J_T;\mathbb{R}^2)}^{1/p}. \tag{35}
\]

It remains to derive an estimation for \(e_h^n\). Set \(V^n = \partial_t u(\cdot,t_n)\) and
\[
r_h^{n,\theta} = (K^{-1}_h P_h A - A_h P_h)U^{n+\theta} + P_h \left( \frac{u(t_{n+1}) - u(t_n)}{\tau} \right) - K^{-1}_h P_h V^{n+\theta}.
\]

Then, by a simple computation, we have
\[
\left\{ \begin{array}{ll}
(D\tau e_h)^n = A_h e_h^{n+\theta} + r_h^{n,\theta}, & n = 0, 1, \ldots, N_T - 1, \\
e_h^0 = 0. & 
\end{array} \right.
\]

Therefore,
\[
\left\{ \begin{array}{ll}
(D\tau (A_h^{-1} e_h)) = A_h (A_h^{-1} e_h^{n+\theta}) + A_h^{-1} r_h^{n,\theta}, & n = 0, 1, \ldots, N_T - 1, \\
A_h^{-1} e_h^0 = 0. & 
\end{array} \right.
\]

Consequently, according to Theorem 11, we obtain
\[
\sum_{n=0}^{N_T-1} \|e_h^{n+\theta}\|_{L^q}^p = \sum_{n=0}^{N_T-1} \|A_h (A_h^{-1} e_h^{n+\theta})\|_{L^q}^p \leq C \sum_{n=0}^{N_T-1} \|A_h^{-1} r_h^{n,\theta}\|_{L^q}^p \tag{36}
\]

We divide \(r_h^{n,\theta}\) into two parts as
\[
r_h^{n,\theta} = r_{1,h}^{n,\theta} + r_{2,h}^{n,\theta},
\]
where
\[
r_{1,h}^{n,\theta} = (K^{-1}_h P_h A - A_h P_h)U^{n+\theta}, \quad r_{2,h}^{n,\theta} = P_h \left( \frac{u(t_{n+1}) - u(t_n)}{\tau} \right) - K^{-1}_h P_h V^{n+\theta}.
\]

We first estimate \(r_{1,h}^{n,\theta}\). Noting the relation (30), we have
\[
A_h^{-1} r_{1,h}^{n,\theta} = (R_h - P_h)U^{n+\theta},
\]
so that
\[
\left( \sum_{n=0}^{N_T-1} \|A_h^{-1} r_{1,h}^{n,\theta}\|_{L^q}^p \right)^{1/p} \leq Ch^2 \left( \sum_{n=0}^{N_T-1} \|U^{n+\theta}\|_{W^{2,q}}^p \right)^{1/p} \leq Ch^2 \|u\|_{W^{1,p}(J_T;\mathbb{R}^2)}\tag{37}
\]
by Lemma 11 and Lemma 21. Also, \( A_{h}^{-1}r_{n,h,2}^{\theta} \) is expressed as

\[
A_{h}^{-1}r_{n,h,2}^{\theta} = A_{h}^{-1}P_{h} \left[ \frac{u(t_{n+1}) - u(t_{n})}{\tau} - V^{n+\theta} \right] + A_{h}^{-1}(I - K_{h}^{-1})P_{h}V^{n+\theta}.
\]

According to Lemmas 11, 13, 14, 21, and 22, we have

\[
\left( \sum_{n=0}^{N_{T}-1} \|A_{h}^{-1}r_{n,h,2}^{\theta}\|_{L^{q}(\tau)}^{p} \right)^{1/p} \leq C_{\tau^{j}}\|u\|_{W^{j\theta}(J_{T}^{T};L^{q})} + Ch^{2} \left( \sum_{n=0}^{N_{T}-1} \|\nabla P_{h}V^{n+\theta}\|_{L^{q}(\tau)}^{p} \right)^{1/p}
\]

\[
\leq C_{\tau^{j}}\|u\|_{W^{j\theta+1,p}(J_{T};L^{q})} + Ch^{2} \left( \sum_{n=0}^{N_{T}-1} \|V^{n}\|_{W^{1,q}(\tau)}^{p} \right)^{1/p}
\]

Combining (35), (36), (37), and (38), we obtain the error estimate (19).

\( \square \)

6 Proofs of Theorems VI and VII

This section is devoted to analysis of semilinear problems (2) and (23). We first prove several auxiliary lemmas.

6.1 Embedding and trace theorems

For \( q \in (1, \infty) \), we recall that \( A_{q} \) denotes the realization of the Dirichlet Laplacian defined as (20). Let \( D(A_{q}) \) be a Banach space equipped with the norm \( \|A_{q}\cdot\|_{L^{q}} \). This is a norm if \( q \in (1, \mu) \) by the regularity assumption (10).

We also set \( D(A_{h,q}) = (S_{h}, \|A_{h}\cdot\|_{h,q}) \), which is a Banach space for \( q \in (1, \mu) \) by Lemma 13.

For \( N \in \mathbb{N} \cup \{\infty\} \) and \( v_{h} \in S_{h}^{N+1} \), we set

\[
\|v_{h}\|_{Y_{h,s,N}^{p,q}} = \|v_{h,1}\|_{L^{q}(N;X_{h,q})} + \|A_{h}v_{h,1}\|_{L^{q}(N;X_{h,q})} + \|D_{\tau}v_{h}\|_{L^{q}(N;X_{h,q})}
\]

and \( Y_{h,s,N}^{p,q} = \left( S_{h}^{N+1}, \|\cdot\|_{Y_{h,s,N}^{p,q}} \right) \). For abbreviation, we write \( Y_{h,s,N}^{p,q} = Y_{h,s,\infty}^{p,q} \) and

\[
\|v_{h}\|_{Y_{T}} = \|v_{h}\|_{Y_{h,s,\infty}^{p,q}}
\]

for \( T > 0 \), where \( N_{T} \) is defined as (6).

Then, we have the following embedding result.
Lemma 23 Let \( q \in (\mu_d, \mu) \) and \( p > 2q/(2q - d) \). Assume that the family \( \{T_h\}_h \) satisfies (H1) and (H2) when \( q \neq 2 \). Then, the embedding
\[
(X_{h,q}, D(A_{h,q}))_{1-1/p, p} \hookrightarrow L^\infty
\]
holds uniformly for \( h > 0 \).

To show Lemma 23, we prove the discrete Gagliardo-Nirenberg type inequality. The following result is the generalization of [26, Lemma 3.3], and that the proof is almost identical. However, for the reader’s convenience, we provide the proof.

Lemma 24 (Discrete Gagliardo-Nirenberg type inequality) Let \( q \in (\mu_d, \mu) \). Assume that the family \( \{T_h\}_h \) satisfies (H1) and (H2). Then, we have
\[
\|v_h\|_{L^\infty} \leq C\|A_h v_h\|_{L^\infty}^{\frac{d}{2}q_{h,q}} \|v_h\|_{W^{1, q}_{h,q}}^{1 - \frac{d}{2}q_{h,q}}, \quad \forall v_h \in S_h.
\]

Proof
It suffices to show that
\[
\|L^{-1}_h f_h\|_{L^\infty} \leq C\|f_h\|_{L^\infty}^{\frac{d}{2}q} \|L^{-1}_h f_h\|_{L^q}^{1 - \frac{d}{2}q},
\]
for every \( f_h \in S_h \). We decompose the left-hand side as
\[
\|L^{-1}_h f_h\|_{L^\infty} \leq \|(L^{-1}_h - P_h A^{-1}_q f_h) f_h\|_{L^\infty} + \|P_h A^{-1}_q f_h f_h\|_{L^\infty} =: a + b.
\]
From the usual Gagliardo-Nirenberg inequality [1, Theorem 5.9] and the regularity assumption (10), we have
\[
b \leq C\|A^{-1}_q f_h\|_{L^\infty} \leq C\|f_h\|_{L^\infty}^{\frac{d}{2}q} \|A^{-1}_q f_h\|_{L^q}^{1 - \frac{d}{2}q} \leq C\|f_h\|_{L^\infty}^{\frac{d}{2}q} \left( \|L^{-1}_h f_h\|_{L^q}^{1 - \frac{d}{2}q} + \|(A^{-1}_q - L^{-1}_h P_h) f_h\|_{L^q}^{1 - \frac{d}{2}q} \right).
\]
Setting \( u = A^{-1}_q f_h \in D(A_q) \), we have
\[
\|(A^{-1}_q - L^{-1}_h P_h) f_h\|_{L^q} = \|u - R_h u\|_{L^q} \leq Ch^2\|u\|_{W^{2,q}} \leq Ch^2\|f_h\|_{L^q},
\]
by Lemma 11 and (10). Since Lemma 19 and the inverse assumption imply
\[
\|L_h v_h\|_{L^q} \leq Ch^{-2}\|v_h\|_{L^q}, \quad \forall v_h \in S_h,
\]
we obtain
\[
\|(A^{-1}_q - L^{-1}_h P_h) f_h\|_{L^q} \leq C\|L^{-1}_h f_h\|_{L^q}.
\]
From (41) and (43), we have
\[
b \leq C\|f_h\|_{L^\infty}^{\frac{d}{2}q} \|L^{-1}_h f_h\|_{L^q}^{1 - \frac{d}{2}q}.
\]
We estimate $a$. The inverse assumption (H1) is well known to imply (see [9, theorem 3.2.6]) the inverse inequality
\[ \|v_h\|_{L^\infty} \leq C h^{-d/q} \|v_h\|_{L^q}, \quad \forall v_h \in S_h, \]
where $C > 0$ is independent of $h$. This, together with (42) and (43), implies
\[ a = \|P_h(L_h^{-1} P_h - A_h^{-1}) f_h\|_{L^\infty} \leq C h^{-d/q} \|P_h(L_h^{-1} P_h - A_h^{-1}) f_h\|_{L^q} \]
\[ \leq C \|f_h\|_{L^q} \|L_h^{-1} f_h\|_{L^q}. \]

Therefore, we can complete the proof. $\square$

**Proof (Proof of Lemma 25)** From the general theory of interpolation spaces, it is readily apparent that the embedding
\[ (X_{h,q}, D(A_{h,q}))^{1-1/p,p} \hookrightarrow (X_{h,q}, D(A_{h,q}))^{1-1/p-\varepsilon,1} \]
for $\varepsilon \in (0,1-1/p)$, uniformly with respect to $h$. Take $\varepsilon = 1 - 1/p - d/(2q)$ so that $1 - 1/p - \varepsilon = d/(2q)$. Then, the assumptions $q > d/2$ and $p > 2q/(2q - d)$ imply $\varepsilon \in (0,1-1/p)$. Therefore, we can obtain from Lemma 24 that the embedding
\[ (X_{h,q}, D(A_{h,q}))^{d/(2q),1} \hookrightarrow L^\infty, \]
holds uniformly with respect to $h$, by the same argument of the embedding theorem for the Besov spaces (see [1, Theorem 7.34]). $\square$

We next show the trace theorem for $Y_{h,r}^{p,q}$. The following result is the discrete version of the characterization of the real interpolation space via the analytic semigroup [34, Lemma 6.2].

**Lemma 25** Let $q \in (1,\mu)$ and $p \in (1,\infty)$. Assume that the family $\{T_h\}$ satisfies (H1) and (H2) when $q \neq 2$. Then there exists $C > 0$ depending only on $p$ such that
\[ \sup_{n \geq 1} ||v_h^n||_{1-1/p,p} \leq C ||v_h||_{Y_{h,r}^{p,q}}. \]
for every $v_h \in Y_{h,r}^{p,q}$.

**Proof** Fix $v_h \in Y_{h,r}^{p,q}$ arbitrarily. It suffices to show that
\[ ||v_h^n||_{1-1/p,p} \leq C ||v_h||_{Y_{h,r}^{p,q}} \quad (44) \]
by translation. Since
\[ v_h^n = - \sum_{j=1}^{n} (v_h^{j+1} - v_h^j) + v_h^{n+1} \]
for $n \geq 1$, we have
\[ K(t,v_h^n) \leq \sum_{j=1}^{n} ||v_h^{j+1} - v_h^j||_{h,q} + t ||A_h v_h^{n+1}||_{h,q} \quad (45) \]
for $t > 0$. Here, the function

$$K(t, w_h) = \inf \{ \| a_h \|_{h,q} + t \| A_h b_h \|_{h,q} \mid w_h = a_h + b_h, \ a_h, b_h \in X_{h,q} \},$$

t > 0, \ w_h \in X_{h,q}

is the $K$-function with respect to the interpolation pair $(X_{h,q}, D(A_{h,q}))$ (see [31] and [44]). Then, (45) implies that

$$\| v^1_h \|_{1-1/p,p}^p = \int_0^\infty \left| t^{-1+1/p} K(t, v^1_h) \right|^p \frac{dt}{t} \leq \int_0^\tau |t^{-1} K(t, v^1_h)|^p dt + 2^p \sum_{n=1}^\infty \int_{t_n}^{(n+1)/n} \left( \frac{1}{n} \sum_{j=1}^n \| v_{n+1}^j - v_{n}^j \|_{h,q} \right)^p dt + \tau \| A_h v_{n+1}^1 \|_{h,q}^p \leq 2^p \| A_h v^1_h \|_{1_2(X_{h,q},N)}^p + 2^p \sum_{n=1}^\infty I_n.$$  (46)

In the last step, we used the property $K(t, v^1_h) \leq t \| A_h v^1_h \|_{h,q}$ and we defined $I_n$ as

$$I_n = \int_{t_n}^{t_{n+1}} \left( \frac{1}{n} \sum_{j=1}^n \| v_{n+1}^j - v_{n}^j \|_{h,q} \right)^p dt$$

for $n \geq 1$. The term $I_n$ is bounded as

$$I_n \leq \int_{t_n}^{t_{n+1}} \left( \frac{1}{n} \sum_{j=1}^n \| v_{n+1}^j - v_{n}^j \|_{h,q} \right)^p dt = \tau \left( \frac{1}{n} \sum_{j=1}^n \| (D_\tau v_h)^j \|_{h,q} \right)^p.$$  (47)

Therefore, we can obtain

$$\sum_{n=1}^\infty I_n \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^\infty \| (D_\tau v_h)^j \|_{h,q}^p \tau$$

by the Hardy inequality [27], and inequalities (46) and (47) imply (44), with a constant $C$ depending only on $p$. \hfill \Box

For $Y_{h,\tau, N}^{p,q}$, we have the following trace theorem.

**Lemma 26** Let $N \in \mathbb{N}$, $q \in (1, \mu)$ and $p \in (1, \infty)$. Assume that the family $\{ T_h \}_h$ satisfies (H1) and (H2) when $q \neq 2$. Then, there exists $C > 0$ independent of $N$, $h$, and $\tau$ such that

$$\max_{0 \leq n \leq N} \| v^n_h \|_{1-1/p,p} \leq C \left( \| v_h \|_{Y_{h,\tau, N}^{p,q}} + \| v^0_h \|_{1-1/p,p} \right)$$

for every $v_h \in Y_{h,\tau, N}^{p,q}$. 
To prove this result, we need to extend each element of $Y_{p,q}^{h,N}$ to that of $Y_{p,q}^{h,\infty}$. First, we obtain the following extension lemma, which corresponds to [3, Lemma 7.2].

**Lemma 27** Let $X$ be a Banach space and $A$ be a linear operator which has discrete maximal regularity and which satisfies $0 \in \rho(A)$. Let $N \in \mathbb{N} \cup \{\infty\}$ and set

$$
\|v\|_{p,N} = \|v_1\|_{1;X} + \|Av_1\|_{1;X} + \|D_\tau v\|_{1;X}
$$

for $v \in X^{N+1}$ and $Y_p^N = \{v \in X^{N+1} \mid v^0 \in (X, D(A))_{1-1/p,p}^1, \ \|v\|_{p,N} < \infty\}$.

Then, for $M \in \mathbb{N}$ with $M < N$, there exists a map $\text{ext}_M : Y_M^p \to Y_N^p$ satisfying

$$(\text{ext}_M v)^n = v^n, \quad n = 0, \ldots, M,$$

and

$$
\|\text{ext}_M v\|_{p,N} \leq C (\|v\|_{p,M} + \|v^0\|_{1-1/p,p}),
$$

where $C$ is independent of $\tau$ and $M$.

**Proof** For $v \in Y_M^p$, we define $g \in Y_N^p$ as

$$
g^n = \begin{cases} 
(D_\tau v)^n - Av^{n+1}, & n = 0, \ldots, N - 1, \\
0, & \text{otherwise}.
\end{cases}
$$

Let $V$ be the solution of

$$
\begin{cases} 
(D_\tau V)^n = V^{n+1} + g^n, & n \in \mathbb{N}, \\
V^0 = v^0,
\end{cases}
$$

which is uniquely solvable by discrete maximal regularity of $A$. Then, if we set $\text{ext}_M v = V$, it satisfies the desired properties. Indeed, since $w^n = v^n - V^n$ satisfies

$$
\begin{cases} 
(D_\tau w)^n = Aw^n, & n = 0, \ldots, M - 1, \\
w^0 = 0,
\end{cases}
$$

we can obtain $w^n = (I - \tau A)^{-n}w^0 = 0$ for $n = 0, \ldots, M$. Moreover, by discrete maximal regularity, we have

$$
\|V\|_{p,N} \leq C (\|g\|_{1;X} + \|V^0\|_{1-1/p,p}) \leq C (\|v\|_{p,M} + \|v^0\|_{1-1/p,p}).
$$

\[\square\]

**Proof (Proof of Lemma 26)** Let $v_h \in Y_{p,q}^{h,\tau,N}$. Then, by Lemmas 25 and 27, we have

$$
\max_{0 \leq n \leq N} \|v_h^n\|_{1-1/p,p} \leq \sup_{n \in \mathbb{N}} \|(v_h^n)_{\text{ext}}\|_{1-1/p,p}
$$

$$
\leq C \left( \|v_h\|_{Y_{p,q}^{h,\tau,N}} + \|v_h^0\|_{1-1/p,p} \right)
$$

$$
\leq C \left( \|v_h\|_{Y_{p,q}^{h,\tau,N}} + \|v_h^0\|_{1-1/p,p} \right)
$$

\[\square\]
6.2 Fractional powers

We will use the fractional power \((-A_h)^z\) for \(z \in (0, 1)\) and \(z \in (-1, 0)\); see [35]. The negative powers are defined as

\[
(-A_h)^{-z}v_h = \frac{\sin(\pi z)}{\pi} \int_0^\infty t^{-z} R(t; A_h)v_h dt
\]  

(48)

for \(z \in (0, 1)\). Since \(-A_h\) is an operator of positive type, it is well-defined. One can check that \((-A_h)^{-z}\) is invertible. Consequently, the positive power \((-A_h)^z\) defined by the inverse operator of \((-A_h)^{-z}\) for \(z \in (0, 1)\). Fractional powers satisfy the following interpolation properties:

\[
\|(-A_h)^z v_h\|_{h,q} \leq C \|v_h\|_{h,q} \|A_h v_h\|_{\tilde{h},q},
\]

(49)

\[
\|(-A_h)^{-z} v_h\|_{h,q} \leq C \|v_h\|_{h,q} \|A_h^{-1} v_h\|_{\tilde{h},q},
\]

(50)

for each \(z \in (0, 1)\) and \(v_h \in S_h\), uniformly for \(h\). Consequently, we have

\[
\|(-A_h)^{-z} v_h\|_{h,q} \leq C \|v_h\|_{h,q}, \quad \forall v_h \in S_h
\]

(51)

uniformly for \(h\), because of Lemma [13]. Below we set \((-A_h)^0 = I\) and \((-A_h)^1 = -A_h\).

**Lemma 28 (Discrete Sobolev inequality)** Assume that the family \(\{T_h\}_h\) satisfies (H1) and (H2). For every \(q > d/2\) and \(\alpha \in (d/(2q), 1)\), there exists \(C > 0\) independent of \(h\), which fulfills the inequality

\[
\|v_h\|_{L^\infty} \leq C \|(-A_h)^\alpha v_h\|_{L^q},
\]

for all \(v_h \in S_h\).

**Proof** It suffices to show that

\[
\|(-A_h)^{-\alpha} f_h\|_{L^\infty} \leq C \|f_h\|_{h,q}, \quad \forall f_h \in S_h.
\]

(52)

By the definition \([48]\), it is necessary to estimate \(\|R(t; A_h)f_h\|_{L^\infty}\). Lemmas \([24]\) and \([15]\) imply

\[
\|R(t; A_h)f_h\|_{L^\infty} \leq C (1 + t)^{-1 + \frac{d}{2q}} \|f_h\|_{h,q}.
\]

Consequently,

\[
\|(-A_h)^{-\alpha} f_h\|_{L^\infty} \leq \frac{\sin(\pi \alpha)}{\pi} \int_0^\infty t^{-\alpha} \|R(t; A_h)f_h\|_{L^\infty} dt
\]

\[
\leq C \int_0^\infty t^{-\alpha} (1 + t)^{-1 + \frac{d}{2q}} dt \|f_h\|_{h,q}.
\]

(53)

Since \(\alpha \in (d/(2q), 1)\), the integral in the right-hand-side of \([53]\) is finite. Therefore, we can obtain the estimate \([52]\). \(\square\)
Lemma 29 Assume that the family \{T_h\}_h satisfies (H1) and (H2) when \( q \neq 2 \). For every \( \beta \in (0, 1 - 1/p) \), there exists \( C > 0 \) independent of \( h \), which satisfies
\[
\|((-A_h)\beta v_h)_{h,q}\| \leq C \|v_h\|_{1 - \frac{1}{p}, p},
\]
for all \( v_h \in S_h \). Here, the norm \( \| \cdot \|_{1 - \frac{1}{p}, p} \) is that of \((X_{h,q}, D(A_h))_{1 - \frac{1}{p}, p}\).

Proof By the general embedding theorem for positive operators [31 Proposition 4.7], we have
\[
(X_{h,q}, D(A_h))_{\beta, 1} \hookrightarrow D((-A_h)\beta).
\]

Moreover, \( \beta < 1 - 1/p \) implies
\[
(X_{h,q}, D(A_h))_{1 - \frac{1}{p}, p} \hookrightarrow (X_{h,q}, D(A_h))_{\beta, 1}.
\]

Chasing the constants in these proofs, one can show that both embedding properties are uniform for \( h \). Therefore, we can establish the desired estimate. \( \square \)

Lemma 30 Assume that the family \{T_h\}_h satisfies (H1) and (H2). For every \( \alpha \in (0, \alpha_{p,q,d}) \), there exists \( C > 0 \) independent of \( h \), which satisfies
\[
\max_{0 \leq n \leq N} \|v_h\|_{L^\infty} \leq C \left( \|(-A_h)^{-\alpha}v_h\|_{Y_{h,t,N}} + \|(-A_h)^{-\alpha}v_h^0\|_{1 - \frac{1}{p}, p} \right),
\]
for all \( N \in \mathbb{N} \) and \( v_h \in S_h^{N+1} \).

Proof Since \( \alpha + d/(2q) < 1 - 1/p \), we can find \( \beta \in (0, 1) \) that satisfies
\[
\frac{d}{2q} + \alpha < \beta < 1 + \alpha \quad \text{and} \quad 0 < \beta < 1 - \frac{1}{p}.
\]

\( \beta - \alpha \in (d/(2q), 1) \). Then, owing to Lemmas [28, 29] and [26] we have
\[
\|v_h^n\|_{L^\infty} \leq C \|(-A_h)^{\beta - \alpha}v_h^n\|_{h,q} \leq C \|(-A_h)^{-\alpha}v_h^n\|_{1 - \frac{1}{p}, p}
\]
\[
\leq C \left( \|(-A_h)^{-\alpha}v_h\|_{Y_{h,t,N}} + \|(-A_h)^{-\alpha}v_h^0\|_{1 - \frac{1}{p}, p} \right),
\]
for \( v_h = (v_h^n)^n \in S_h^{N+1} \) and \( n \in \mathbb{N} \). \( \square \)

6.3 Completion of the proofs of Theorems VI and VII

Let \( u \) and \( u_h = (u^n_h)_{n=0}^{N_T} \) be solutions of (2) and (23), respectively. Set \( U^n = u(n\tau) \). We consider the error \( e_h = (e_h^n)_{n=0}^{N_T} \in S_h^{N_T+1} \) defined as
\[
e_h^n = u^n - P_h U^n \quad (n = 0, 1, \ldots, N_T).
\]

We first state the sub-optimal error estimate for a globally Lipschitz nonlinearity \( f \). If \( f \) is a globally Lipschitz continuous function, then (2) admits a unique time-global solution and the solution of (23) is bounded from above uniformly in \( h \) and \( \tau \) (see Remark 3). Recall that \( \| \cdot \|_{Y_p} \) is defined as [30] and [40].
Lemma 31 In addition to hypotheses of Theorem [VI], we assume that \( f \) is a globally Lipschitz continuous function. Then, for every \( \alpha \in [0,1] \) and \( T \in (0,\infty) \),
\[
\|(-A_h)^{-\alpha}e_h\|_{Y_T} \leq C(h^{2\alpha} + \tau). \tag{54}
\]

Proof The proof is divided into two steps.

Step 1. We prove that there exists \( T_1 = T_1(u_0, T) \in (0,T) \) satisfying
\[
\|(-A_h)^{-\alpha}e_h\|_{Y_{T_1}} \leq C(h^{2\alpha} + \tau). \tag{55}
\]
The error \( e_h \) satisfies
\[
\begin{cases}
(D_x e_h)^n = A_h e_h^{n+1} + r_h^n, & n = 0, 1, \ldots, \\
e_0^0 = 0,
\end{cases}
\]
where \( r_h^n = F_h(u_h^n) - P_h(D_x U)^n + A_h P_h U^{n+1} \) and \( F_h = K_h^{-1} \circ P_h \circ f \). We decompose \( r_h^n \) into two parts:
\[
\begin{align*}
r_h^n &= r_{1,h}^n + r_{2,h}^n, \\
r_{1,h}^n &= F(u_h^n) - F(P_h U^n), \\
r_{2,h}^n &= r_h^n - r_{1,h}^n.
\end{align*}
\]
We perform an estimation for \( r_{2,h}^n \). Let \( V^n = \partial_t u(\cdot, nt) \). Noting that \( V^{n+1} = AU^{n+1} + f(U^{n+1}) \), the residual term \( r_{2,h}^n \) is can be decomposed as
\[
\begin{align*}
r_{2,h}^n &= R_{1,h}^n + R_{2,h}^n + R_{3,h}^n, \\
R_{1,h}^n &= A_h (P_h - R_h) U^{n+1}, \\
R_{2,h}^n &= (K_h^{-1} - I) P_h V^{n+1} + P_h (V^{n+1} - (D_x U)^n), \\
R_{3,h}^n &= [F_h(P_h U^n) - F_h(U^n)] + [F_h(U^n) - F_h(U^{n+1})].
\end{align*}
\]
From the interpolation property \([49]\) and the inverse inequality, we have
\[
\|(-A_h)^{-\alpha}v_h\|_{h,q} \leq Ch^{-2\gamma} \|v_h\|_{h,q},
\]
for \( \gamma \in (0,1) \). Therefore, the first term \( R_{1,h}^n \) is estimated as
\[
\begin{align*}
\|(-A_h)^{-\alpha}R_{1,h}^n\|_{h,q} &= \|(-A_h)^{1-\alpha}(P_h - R_h) U^{n+1}\|_{h,q} \\
&\leq Ch^{-2(1-\alpha)} \cdot h^2 \|U^{n+1}\|_{W^{2,q}} \\
&= Ch^{2\alpha} \|U^{n+1}\|_{W^{2,q}}. \tag{56}
\end{align*}
\]
Similarly, from \([50]\) and Lemmas \([11]\) and \([14]\) we have
\[
\|(-A_h)^{-\alpha}(K_h^{-1} - I) P_h V^{n+1}\|_{h,q} \leq Ch^{2\alpha} \|V^{n+1}\|_{W^{1,q}}.
\]
Combining this inequality with Lemma \([22]\) we have
\[
\left(\sum_{n=0}^{N-1} \|(-A_h)^{-\alpha}R_{2,h}^n\|_{h,q}^p\right)^{1/p} \leq C(h^{2\alpha} + \tau). \tag{57}
\]
Since \( f \) is globally Lipschitz continuous, we have by (51)
\[
\left( \sum_{n=0}^{N-1} \| (A_h)^{-\alpha} R_{3,h}^n \|_{h,q}^p \right)^{1/p} \leq CL \left( \sum_{n=0}^{N-1} \| (P_h - I) U^n \|_{h,q}^p \right)^{1/p} + \left( \sum_{n=0}^{N-1} \| U^{n+1} - U^n \|_{h,q}^p \right)^{1/p} \leq CL(h^2 + \tau),
\]
where \( L \) is the Lipschitz constant of \( f \). The equations (56), (57), and (58) yield
\[
\left( \sum_{n=0}^{N-1} \| (A_h)^{-\alpha} R_{2,h}^n \|_{h,q}^p \right)^{1/p} \leq C(h^{2\alpha} + \tau).
\]

Now, we are ready to show (55). We designate some constants appearing in this proof. Since \( A_h \) has discrete maximal regularity on \( J_\infty \) in \( X_{h,q} \) uniformly for \( h \), there exists \( C_{\text{DMR}} > 0 \) depending only on \( p, q, \Omega \) satisfying
\[
\| v_h \|_{Y_S} \leq C_{\text{DMR}} \left( \| g_{h,1} \|_{\ell^p(N_S;X_{h,q})} + \| x_h \|_{1-1/p,p} \right),
\]
for every \( g_h = (g^n_h) \in \ell^p(N_S;X_{h,q}) \) and \( x_h \in S_h \), where \( v_h = (v^n_h) \) is the solution of
\[
\begin{cases}
(D \tau v_h)^n = A_h v_h^{n+1} + g_h^{n+1}, & n = 0, \ldots, N_S - 1, \\
v_h^0 = x_h.
\end{cases}
\]

In view of (51) and the Lipschitz continuity of \( f \), we have
\[
C_{\text{Lip}} = \sup \left\{ \frac{\| (A_h)^{-\alpha} (F_h(v_h) - F_h(w_h)) \|_{h,q}}{\| v_h - w_h \|_{h,q}} \mid \begin{array}{l} h > 0, \ v_h, w_h \in S_h, \\
v_h \neq w_h \end{array} \right\} < \infty,
\]
which is the Lipschitz constant of \( (A_h)^{-\alpha} \circ F_h \). Finally, we set
\[
C_0 = C_{\text{DMR}} C_{\text{Lip}} |\Omega|^{1/q},
\]
where \( |\Omega| \) denotes the \( d \)-dimensional Lebesgue measure.

Let \( e_{j,h} \) \((j = 1, 2)\) be the solution of
\[
\begin{cases}
(D \tau e_{j,h})^n = A_h e_{j,h}^{n+1} + r_{j,h}^n, & n = 0, \ldots, N_T - 1, \\
e_{j,h}^0 = 0.
\end{cases}
\]

It is apparent that \( e_h = e_{1,h} + e_{2,h} \). Moreover, for every \( S < T_\infty \), one can obtain
\[
\| (A_h)^{-\alpha} e_{2,h} \|_{Y_S} \leq C(h^{2\alpha} + \tau)
\]
by (51) and (59).
Next, it is necessary to derive an estimation for $e_{1,h}$. Take $S < T$ arbitrarily. Since $e_{1,h}$ is the solution of (61), discrete maximal regularity (60) and Lemma 30 yield

$$\|(-A_h)^{-}\alpha e_{1,h}\|_{Y_S} \leq C_{DMR} \left( \sum_{n=0}^{N_\sigma - 1} \|F_h(u^n_h) - F_h(P_h U^n)\|_{p,q,\tau}^p \right)^{1/p}$$

$$\leq C_{DMR} \left[ C_{\text{Lip}} \left( \sum_{n=0}^{N_\sigma - 1} \|e^n_{1,h}\|_{h,q,\tau}^p \right)^{1/p} + C_{\text{Lip}} \left( \sum_{n=0}^{N_\sigma - 1} \|e^n_{2,h}\|_{h,q,\tau}^p \right)^{1/p} \right]$$

$$\leq C_{DMR} C_{\text{Lip}} \langle \Omega \rangle^{1/q} S^{1/p} \max_{0 \leq n \leq N_\sigma - 1} \|e^n_{1,h}\|_{L^\infty} + C(h^{2\alpha} + \tau)$$

$$\leq C_0 S^{1/p} \|(-A_h)^{-}\alpha e_{1,h}\|_{Y_S} + C(h^{2\alpha} + \tau).$$

Consequently, taking $S = (2C_0)^{-p}$, we obtain

$$\|(-A_h)^{-}\alpha e_{1,h}\|_{Y_{T_1}} \leq C(h^{2\alpha} + \tau) \quad (63)$$

with $T_1 = (2C_0)^{-p}$. This, together with (62), implies (55).

**Step 2.** We prove (54) for any $T \in (0, \infty)$. We denote the constants appearing in Lemma 26 by $C_{tr}$, and set

$$C_1 = C_0 C_{tr}, \quad C_2 = C_{DMR} C_{tr}.$$

Then we show that

$$\|(-A_h)^{-}\alpha e_{1,h}^{+N_\sigma}\|_{Y_S} \leq C \left( \|(-A_h)^{-}\alpha e_{1,h}\|_{Y_S} + h^{2\alpha} + \tau \right) \quad (64)$$

for all $S < T$ and $\sigma \leq \min\{T_1, T - S\}$. Take $S < T$ and $\sigma \leq \min\{T_1, T - S\}$ arbitrarily, and set $w^n_{j,h} = e^{n+\sigma}_{j,h}$ ($j = 1, 2$). Then, $w_{1,h}$ satisfies

$$\begin{cases}
(D_T w^n_{1,h})^n = A_h (u^n_{1,h} + F_h (u^{n+\sigma}_{1,h}) - F_h (P_h U^{n+\sigma})), \quad n = 0, \ldots, N_{T-S}, \\
  w^0_{1,h} = e^{N_\sigma}_{1,h}.
\end{cases}$$
Therefore, discrete maximal regularity (60), Lemmas 26, 30, and (63) yield

\[ \|(-A_h)^{-\alpha} w_{1,h}\|_{Y_1} \leq C_{DMR}\left[ (N_\sigma - 1) \sum_{n=0}^{N_\sigma - 1} \|w_{1,n,h}\|_{p,h_0,q}^p \right]^{1/p} + \|(-A_h)^{-\alpha} e_{1,n,h}\|_{1-1/p,p,p}^{1-1/p} \]

\[ \leq C_{DMR} C_{\text{Lip}} \left[ (N_\sigma - 1) \sum_{n=0}^{N_\sigma - 1} \|w_{2,n,h}\|_{p,h_0,q}^p \right]^{1/p} \]

\[ \leq C_0 \sigma^{1/p} \left( \|(-A_h)^{-\alpha} w_{1,h}\|_{Y_1} + \|(-A_h)^{-\alpha} e_{1,n,h}\|_{1-1/p,p,p} \right) + C(h^{2\alpha} + \tau) \]

\[ + C_{\text{tr}} \|(-A_h)^{-\alpha} e_{1,h}\|_{Y_2} + C_{\text{DMR}} C_{\text{tr}} \|(-A_h)^{-\alpha} e_{1,h}\|_{Y_2} \]

\[ \leq \frac{1}{2} \|(-A_h)^{-\alpha} w_{1,h}\|_{Y_1} + \left( \frac{C_{\text{tr}}}{2} + C_2 \right) \|(-A_h)^{-\alpha} e_{1,h}\|_{Y_2} + C(h^{2\alpha} + \tau), \]

since \( \sigma \leq T_1 = (2C_0)^{-\tau} \). Therefore, we obtain (64).

Noting that \( N_{S+\sigma} \leq N_S + N_\sigma \), one obtains

\[ \|v_h\|_{Y_{S+\sigma}} \leq \|v_h\|_{Y_S} + \|v_h^{+N_S}\|_{Y_\sigma} \]

for \( v_h \in l^p(N_S + N_\sigma; S_h) \) and \( S, \sigma > 0 \). Therefore, we can inductively establish (64) from (63) and (64). Now we can complete the proof owing to Lemma 30. \( \square \)

Finally, we state the following proof.

Proof (Proof of Theorems VI and VII) Observe that

\[ \|u^n_h - U^n\|_{L^q} \leq \|e^n_h\|_{L^q} + \|P_h U^n - U^n\|_{L^q} \leq \|e^n_h\|_{L^q} + Ch^2 \|U^n\|_{W^{2,q}} \]

by Lemma 11. Therefore, it suffices to prove

\[ \left( \sum_{n=1}^{N_T} \|e^n_h\|_{L^q}^{p} \right)^{1/p} \leq C(h^{2\alpha} + \tau) \]

and

\[ \max_{0 \leq n \leq N_T} \|e^n_h\|_{L^{\infty}} \leq C(h^{2\alpha} + \tau) \]

for \( \alpha \in (0, \alpha_{p,q,d}) \). To this end, let

\[ M = \|u\|_{L^\infty(\Omega \times (0,T))} + \sup_{h > 0} \|P_h u\|_{L^\infty(\Omega \times (0,T))} \]
for the solution $u$ of (2) and $T \in (0, T_\infty)$. It is apparent that $M$ is finite since the $L^2$-projection $P_h$ is stable in the $L^\infty$-norm (Lemma 11). We introduce

$$
\hat{f}(z) = \hat{f}_M(z) = \begin{cases} f(z), & |z| \leq M, \\
M \left( \frac{z}{M} \right), & |z| > M. 
\end{cases}
$$

Then, $\hat{f}$ is a globally Lipschitz continuous function. We consider the problems

(2) and (23) with replacement of $f$ by $\hat{f}$, and denote the corresponding solutions by $\tilde{u}$ and $\tilde{u}_h$, respectively. Moreover, we consider the error $\tilde{e}_h = (\tilde{e}_h^n)_{n=0}^{N_T} \in S_h^{N_T+1}$, where $\tilde{e}_h^n = \tilde{u}_h^n - P_h \hat{u}(n\tau)$.

In view of Lemma 31, the following error estimate holds:

$$
\|(-A_h)^{-\alpha} \tilde{e}_h\|_{Y_T} \leq C(h^{2\alpha} + \tau)
$$

for any $\alpha \in [0, 1]$. By setting $\alpha = 1$, we obtain

$$
\left( \sum_{n=1}^{N_T} \|\tilde{e}_h^n\|_{L^\infty}^p \right)^{1/p} \leq C(h^2 + \tau). \quad (55)
$$

Applying Lemma 30, we can deduce

$$
\max_{0 \leq n \leq N_T} \|\tilde{e}_h^n\|_{L^\infty} \leq C(h^{2\alpha} + \tau), \quad (56)
$$

for $\alpha \in (0, \alpha_{p,q,d})$.

At this stage, we have $\tilde{u} = u$ by the unique solvability of (2). Indeed, $\|u\|_{L^\infty(\Omega \times (0,T))} \leq M$ implies $\hat{f}(u(x,t)) = f(u(x,t))$ for every $(x,t) \in \Omega \times (0,T)$. Moreover, according to (56), we estimate as

$$
\max_{0 \leq n \leq N_T} \|\tilde{e}_h^n\|_{L^\infty} \leq \max_{0 \leq n \leq N_T} \|\tilde{e}_h^n\|_{L^\infty} + \max_{0 \leq n \leq N_T} \|P_h U^n\|_{L^\infty}
\leq C(h^{2\alpha} + \tau) + \sup_{h>0} \|P_h u\|_{L^\infty(\Omega \times (0,T))}
$$

for $\alpha \in (0, \alpha_{p,q,d})$. Therefore, there exist $h_0 > 0$ and $\tau_0 > 0$ such that

$$
\max_{0 \leq n \leq N_T} \|\tilde{e}_h^n\|_{L^\infty} \leq \max_{0 \leq n \leq N_T} \|\tilde{u}_h^n\|_{L^\infty} \leq M, \quad \forall h < h_0, \quad \forall \tau < \tau_0,
$$

which implies that $\hat{f}((\tilde{u}_h^n) = f((\tilde{u}_h^n)$. Again, the unique solvability of (23) yields $\tilde{u}_h = u_h$ for $h \leq h_0$ and $\tau \leq \tau_0$. Hence we can replace $\tilde{e}_h^n$ by $e_h^n$ in (55) and (56), which completes the proof of Theorems VI and VII. $\Box$

**Remark 5** Based on the same assumptions of Lemma 31, the solution $u_h = (u_h^n)_{n}$ of (23) admits

$$
\|u_h^n\|_{L^\infty} \leq C\|u_0\|_{L^\infty} e^{TL}.
$$

We briefly show this inequality. Let $T_h, \tau = (I - \tau A_h)^{-1}$ and $F_h = K_h^{-1} \circ P_h \circ f$. Then the first equation of (23) is equivalent to

$$
u^n_{\alpha} = T_{h,\tau} P_h u_0 + \tau \sum_{n=0}^{n-1} T_{h,\tau}^{n-j} F_h(u_j^n)$$
for \( n \in \mathbb{N} \). It follows from Lemma 10 and the Hille-Yosida theorem that
\[
\| \lambda^n R(\lambda; A_h) \|_{L(X_h, \infty)} \leq 1
\]
for all \( n \in \mathbb{N} \) and \( \lambda > 0 \). Particularly, we have
\[
\| T_{n, h, \tau} \|_{L(X_h, \infty)} \leq 1
\]
for all \( n \in \mathbb{N} \). Moreover, one can find \( L > 0 \), independent of \( h \), such that
\[
\| F_h(v_h) - F_h(w_h) \|_{L^\infty} \leq L \| v_h - w_h \|_{L^\infty}, \quad \forall h > 0
\]
for \( v_h, w_h \in S_h \) by the globally Lipschitz continuity of \( f \) and Lemmas 11 and 12. Then, we obtain
\[
\| u^n_h \|_{L^\infty} \leq C \| u_0 \|_{L^\infty} + \tau \sum_{j=0}^{n-1} L \| u^j_h \|_{L^\infty}.
\]
Therefore, the well-known discrete Gronwall lemma [36, Lemma 2.3] implies
\[
\| u^n_h \|_{L^\infty} \leq C \| u_0 \|_{L^\infty} e^{n\tau L} \leq C \| u_0 \|_{L^\infty} e^{TL}
\]
for \( n \in \mathbb{N} \).

A \( H^\infty \)-functional calculus

In this appendix, we review the notion of \( H^\infty \)-functional calculus. We present only the definition and the theorem used for this study. For relevant details, one can refer to [11] and references therein. Throughout this section, \( X \) denotes a Banach space and \( \Sigma_\omega \) is the sector defined as (26).

Definition 3 For \( \omega \in (0, \pi) \), a linear operator \( A \) is of type \( \omega \) if and only if
1. \( A \) is closed and densely defined,
2. \( \sigma(A) \subseteq \Sigma_\omega \),
3. for each \( \theta \in (\omega, \pi] \), there exists \( C_\theta > 0 \) satisfying
\[
\| R(z; A) \|_{L(X)} \leq C_\theta |z|^{-\theta}
\]
for all \( z \in \mathbb{C} \setminus \Sigma_\theta \) with \( z \neq 0 \).

Every positive type operator is of type \( \omega \) for some \( \omega \in (0, \pi/2) \). Now, we define the functions of operators of type \( \omega \). For \( \theta \in (0, \pi) \), we set
\[
\Psi(\Sigma_\theta) = \bigcup_{C \geq 0, s > 0} \left\{ f \in H^\infty(\Sigma_\theta) \mid |f(z)| \leq C \frac{|z|^s}{1 + |z|^2}, \quad \forall z \in \Sigma_\theta \right\},
\]
where \( H^\infty(\Sigma_\theta) \) is defined as (31). Let \( \Gamma_\theta = \{-te^{-i\theta} \mid -\infty < t < 0\} \cup \{te^{i\theta} \mid 0 \leq t < \infty\} \) be a contour for \( \theta \in (0, \pi) \), which is oriented so that the imaginary parts increase along \( \Gamma_\theta \).

Definition 4 Let \( A \in \mathcal{P}(X; K) \) for some \( K \geq 1 \). Assume that \( A \) is of type \( \omega \) and let \( \omega < \theta < \theta \). Then, we define the function of operator \( A \) as
\[
\psi(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} (A - zI)^{-1}\psi(z)\,dz
\]
for \( \psi \in \Psi(\Sigma_\theta) \). We also define \( m(A) \) for \( m \in H^\infty(\Sigma_\theta) \) as
\[
m(A) = \psi_0(A)^{-1}(\psi_0 m)(A),
\]
where \( \psi_0(z) = z/(1 + z)^2 \).
In the case in which $X$ is a Hilbert space and the operator $A$ is positive type and self-adjoint, we can define $m(A)$ for $m \in L^\infty(\mathbb{R}^+)$ by the spectral decomposition. It is natural to wonder whether these two definitions coincide. The answer is as follows. See for example Theorem 4.6.7 in Chapter III for the proof.

**Lemma 32** Let $X$ be a Hilbert space and $A \in \mathcal{P}(X)$. Assume that $A$ is self-adjoint and let $E_{\lambda}^A$ be its spectral decomposition. Then, we have

$$m(A) = \int_0^\infty m(\lambda) dE_{\lambda}^A$$

for $m \in H^\infty(\Sigma_{\theta})$.

**B Remark on the scheme (17)**

An alternate of the scheme (17) is given as

$$(D_\tau u_h)^n = A_h u_h^{n+\theta} + P_h G^{n+\theta}, \quad (67)$$

or, equivalently,

$$((D_\tau u_h)^n, v_h)_h = -(\nabla u_h^{n+\theta}, \nabla v_h)_L^2 + (P_h G^{n+\theta}, v_h)_h, \quad \forall v_h \in S_h.$$  

If taking (67) instead of the first equation of (17), we can only obtain the following error estimate:

$$\left(\sum_{n=0}^{N_T-1} \|u_h^n - u^n\|_{L^p}^p\right)^{1/p} \leq C(h + \tau^{2\theta}), \quad (68)$$

since Lemma 14 is not available. This shortcoming is confirmed by numerical examples as follows.

Let us consider the following two-dimensional heat equation in $\Omega = (0,1)^2$:

$$\begin{cases}
\frac{\partial u}{\partial t}(x,y,t) = \Delta u(x,y,t) + g(x,y,t), & (x,y) \in \Omega, 0 < t \leq T, \\
u(x,y,t) = 0, & (x,y) \in \partial\Omega, 0 < t \leq T, \\
u(x,y,0) = x^{5/2}(1-x)^{5/2}y(1-y), & (x,y) \in \Omega,
\end{cases} \quad (69)$$

where $T > 0$ and

$$g(x,y,t) = x^{1/2}(1-x)^{1/2}e^{t} \left[ x^2(1-x)^2y(1-y) - \frac{5}{4}(3-4x)(1-4x)y(1-y) + 2 \right].$$

The exact solution is $u(x,y,t) = x^{5/2}(1-x)^{5/2}y(1-y)e^t$. We approximate the equation (69) by the schemes (17) and (67) with meshes such as Figure 2 which satisfies the conditions (H1) and (H2).

We consider the case for $\theta = 0, 1/2$ and 1. When $\theta = 1/2$ and $\theta = 1$, we take $\tau$ as $\tau = h$. In the case for $\theta = 0$, $\tau$ should be chosen to satisfy the condition (14). We take $\varepsilon = \sin \theta q$ and

$$\tau = \frac{\sin \theta q}{(1-2\theta)(d+1)^2}h^2,$$

so that $\tau$ satisfies $\tau = O(h^2)$ by the inverse assumption. We set the parameters as follows:

- $(p,q) = (4,2)$
- $T = 0.1$ ($\theta = 0$) or $T = 0.5$ ($\theta = 1/2, 1$).
Behavior of the errors is shown in Figure 3. In these figures, cases 1–5 mean the following cases:

- **case 1**: \( \theta = 0 \) (\( \tau = O(h^2) \)),
- **case 2**: \( \theta = 1/2, \ \tau = h \),
- **case 3**: \( \theta = 1/2, \ \tau = h^2 \),
- **case 4**: \( \theta = 1, \ \tau = h \),
- **case 5**: \( \theta = 1, \ \tau = h^2 \).

Let us consider the order of the error. In case 4 with the scheme (17), for example, from Theorem V and \( \tau = h \), we have

\[
\text{(The error)} \leq C(h^2 + \tau) \leq Ch
\]

if \( h \) is sufficiently small. We summarize these theoretical orders and results in Table 1. When we use the scheme (17), the orders correspond to the theoretical bounds. In the case for the scheme (67), all orders are expected to be \( O(h) \). However, except for case 4, the orders are apparently \( O(h^2) \). It is of course no problem since the error estimate (68) is just an upper bound. In case 4, it also seems that the order is \( O(h^2) \). However, when we compute (67) in case 4 for smaller \( h \), the error decreases more slowly. It seems to approach \( O(h^{\alpha}) \) for some \( \alpha \in [1, 2) \): Figure 4. We leave more rigorous error estimates for the scheme (67) as a subject for future work.

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### Table 1: The convergence rates: theoretical bounds and results.

| Conditions | Bounds | Results |
|------------|--------|---------|
| \(\theta = 0\) | \(\tau \propto h^2\) | \(O(h^2)\) |
| \(\theta = 1/2\) | \(\tau = h\) | \(O(h^2)\) |
| \(\theta = h^2\) | \(O(h^2)\) | \(O(h^2)\) |

| Conditions | Bounds | Results |
|------------|--------|---------|
| \(\theta = 0\) | \(\tau \propto h^2\) | \(O(h)\) |
| \(\theta = 1/2\) | \(\tau = h\) | \(O(h)\) |
| \(\tau = h^2\) | \(O(h)\) | \(O(h^2)\) |

(a) Scheme (17)  
(b) Scheme (67)

Fig. 4: Behavior of \(L^4-L^2\)-errors in case 4 with scheme (17) for smaller \(h\).

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