CONTINUITY FOR THE ONE-DIMENSIONAL CENTERED HARDY-LITTLEWOOD MAXIMAL OPERATOR AT THE DERIVATIVE LEVEL

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Abstract. We prove the continuity of the map $f \mapsto (Mf)'$ from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$, where $M$ is the centered Hardy-Littlewood maximal operator. This solves a question posed by Carneiro, Madrid and Pierce.

1. Introduction

Maximal operators are central objects in analysis. The most classical of these operators is the centered Hardy-Littlewood maximal operator. This is defined as follows: for any $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$Mf(x) := \sup_{r>0} \frac{\int_{B(x,r)} |f|}{|B(x,r)|} =: \sup_{r>0} \frac{\int_{B(x,r)} f}{|B(x,r)|},$$

where $|X|$ is the Lebesgue measure of the measurable set $X \subset \mathbb{R}^d$. We define $\tilde{M}$ as its uncentered version, where the supremum is taken over all balls that contain $x$ but are not necessarily centered at $x$. The regularity theory for these operators started with Kinnunen [16], who proved that

$$f \mapsto Mf$$

is bounded from $W^{1,p}(\mathbb{R}^d)$ to itself when $p > 1$. The same result for $\tilde{M}$ follows by similar methods. The case $p = 1$ is much more delicate. Certainly, since $Mf \notin L^1(\mathbb{R}^d)$ for any non-trivial $f$, one cannot expect Kinnunen’s result to hold for $p = 1$. Nonetheless, the boundedness is true at the derivative level when $p = 1$ and $n = 1$, i.e. the map $f \mapsto (Mf)'$ is bounded from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. This was established by Kurka in [17], while the same result for $\tilde{M}$ was obtained by Tanaka [21] and sharpened by Aldaz and Pérez-Lázaro [1]. This boundedness has been also investigated in higher dimensions. Luiro [19] established the boundedness of the map $f \mapsto |\nabla \tilde{M} f|$ from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$, while Weigt [22] proved the boundedness when restricting the map to characteristic functions.

The continuity for these types of maps has been an active topic of research over the last years. According to [15] Question 3, it was asked by T. Iwaniec whether the map (1.1) is continuous from $W^{1,p}(\mathbb{R}^d)$ to itself, when $p > 1$. This question was answered affirmatively by Luiro in [18]. Again, the endpoint case $p = 1$ is significantly more involved. For the uncentered Hardy-Littlewood maximal operator, the continuity of the map

$$f \mapsto (\tilde{M} f)'$$

from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$ was proved by Carneiro, Madrid and Pierce in [10]. This was later generalized in [14] and [6] to the $BV(\mathbb{R})$ case and to the higher dimensional radial case, respectively. In the fractional version

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of this problem, based on previous developments made in \cite{3,20,23}, the general case was obtained in \cite{2}. However, the centered classical case is beyond the scope of the methods developed in these previous works.

Another family of operators was also object of study in this topic: maximal operators of convolution type associated to smooth kernels. We write $M_\phi$ for the centered maximal operator associated to a radially non-increasing kernel $\phi \in L^1(\mathbb{R}^d)$. The boundedness of the map $f \mapsto (M_\phi f)'$ from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$ was proved in \cite{11} and later in \cite{5} for certain smooth kernels $\phi$ related to partial differential equations that include the heat and Poisson kernels. A radial version of these results was achieved in \cite{7}. A step forward for the understanding of the continuity in the centered setting was made recently by the author in the work \cite{13}, where the continuity of some of these maps was established.

In the present manuscript, we establish the continuity for the centered Hardy-Littlewood maximal operator, solving a question posed by Carneiro, Madrid and Pierce in \cite[Question A]{10} and establishing, in the one-dimensional case, the endpoint version of \cite[Question 3]{15} at the derivative level.

**Theorem 1.** We have that the map $f \mapsto (Mf)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$.

We notice that the map considered here is well defined and bounded (see Lemma 3). We highlight that the methods developed in the aforementioned works \cite{6,10,13,14} are not enough to conclude our result. For instance, in the works \cite{6,10,14} it is important that the operator $\widetilde{M}$ has the flatness property; this is, that the maximal functions have a.e. zero derivative at the points where they coincide with the original function. In \cite{13}, the subharmonicity property, which the maximal functions considered there satisfy, plays a crucial role in the proof of the continuity. The centered Hardy-Littlewood maximal operator does not satisfy either of these properties, therefore, new insights are required in order to achieve our result. Our method is based on a decomposition of $M$ as a maximum of two operators $M_1$ and $M_2$, both of them depending on $f$ and on a simple function $g_\varepsilon$ that approximates $f'$ in $L^1(\mathbb{R})$. The operator $M_1$, the local one, is restricted to balls that are contained in the support of an interval determined by $g_\varepsilon$. On the other hand, the operator $M_2$, the global one, is restricted to balls that are not contained in any of these lines. The idea is that, since the operator $M_1$ is well behaved with respect to some lines, it is possible to conclude that $M_1f_j$ is close to $f_j$ at the derivative level, for any $j$ big enough. A different approach is needed in order to deal with the contribution of the operator $M_2$, for this we shall take advantage of the fact that the radii considered in $M_2$ are generally bounded by below. In essence, this yields a smoother nature to this operator that is helpful for our purposes.

Considering the progress made in this manuscript, we summarize the situation of the endpoint continuity program (originally proposed in \cite[Table 1]{10}) in the table below. The word YES in a box means that the continuity of the corresponding map has been proved, whereas the word NO means that it has been shown that it fails. We notice that after this work the only open problem in this program is to determine if the map $f \mapsto Mf$ is continuous from $BV(\mathbb{R})$ to itself, marked with OPEN in the table below.

|          | $W^{1,1}(\mathbb{R})$ | $L^1(\mathbb{R})$ |
|----------|------------------------|--------------------|
| YES      | YES                    | YES                |
| NO       | NO                     | NO                 |

2. Preliminaries

In this section we discuss some preliminary results for our purposes. Let us consider $f_j \to f$ in $W^{1,1}(\mathbb{R})$. In order to prove Theorem 1 by \cite[Lemma 14]{10} we may assume henceforth that $f_j, f \geq 0$. Also, since the case $f = 0$ of Theorem 1 follows by the boundedness, we assume that $f \neq 0$. We start with the well known Luiro’s formula.
Table 1. Endpoint continuity program

| Endpoint continuity program                  | $W^{1,1}$-continuity; continuous setting | $BV$-continuity; continuous setting | $W^{1,1}$-continuity; discrete setting | $BV$-continuity; discrete setting |
|---------------------------------------------|------------------------------------------|-------------------------------------|----------------------------------------|-----------------------------------|
| Centered classical maximal operator         | YES: Theorem [1]                         | OPEN                               | YES$^2$                                 | YES$^4$                           |
| Uncentered classical maximal operator       | YES$^1$                                  | YES$^6$                            | YES$^2$                                 | YES$^1$                           |
| Centered fractional maximal operator        | YES$^5$                                  | NO$^1$                             | YES$^3$                                 | NO$^1$                            |
| Uncentered fractional maximal operator      | YES$^4$                                  | NO$^1$                             | YES$^3$                                 | NO$^1$                            |

1 Result previously obtained in [10].
2 Result previously obtained in [8, Theorem 1].
3 Result previously obtained in [9, Theorem 3].
4 Result previously obtained in [20].
5 Result previously obtained in [4].
6 Result previously obtained in [14].

Proposition 2 (Case $p = 1$ of [18, Theorem 3.1]). Let us take $g \in W^{1,1}(\mathbb{R})$. Assume that $M g$ is differentiable at the point $x$, if $M g(x) = \int_{[x-r,x+r]} |g|$ with $r > 0$, we have that

$$(Mg)'(x) = \int_{[x-r,x+r]} |g'|.$$

Proof. This follows from [2, Proposition 2.4] and the remark thereafter. \(\square\)

The next result provide us with a local control for the variation of $M$. For any interval $I$ (not necessarily finite) and $g \in L^1(\mathbb{R})$ we define

$$M_If(x) := \sup_{[x-r,x+r] \subset I} \int_{[x-r,x+r]} |g|.$$  

Lemma 3. If $f \in W^{1,1}(I)$, we have that $M_If$ is absolutely continuous and that there exists an universal constant $C$, such that

$$\int_I |(M_If)'| \leq C \int_I |f'|.$$  

Proof. The absolutely continuity of $M_If$ can be concluded by following the reasoning in [17, Corollary 1.3]. The boundedness follows from [17, Remark 6.4]. \(\square\)

Also, we need the following uniform control near a finite number of points.

Lemma 4. Let $f_j \to f$ in $W^{1,1}(\mathbb{R})$. Let $\{p_1, \ldots, p_s\}$ be a finite set. For any $\varepsilon > 0$ there exists $\delta > 0$ such that, for $j$ big enough, we have

$$\sum_{i=1}^s \int_{[p_i-\delta,p_i+\delta]} |(Mf_j)'| < \varepsilon.$$  

Proof. This proof follows a similar path than the one presented originally in [12, Proposition 19]. It is enough to prove that there exists $\delta > 0$ such that

$$\int_{[p_i-\delta,p_i+\delta]} |(Mf_j)'| < \frac{\varepsilon}{s}.$$
for any fixed i and j big enough. Let us take \( \delta_i > 0 \) such that
\[
\int_{(a_i - \delta_i, a_i + \delta_i)} |f'| < \frac{\varepsilon}{2C^s},
\]
where \( C \) is the universal constant that appears in Lemma \([3]\). For \( j \) big enough we have
\[
\int_{(a_i - \delta_i, a_i + \delta_i)} |f_j'| < \frac{\varepsilon}{2C^s}.
\]
For any given \( \ell \in \mathbb{Z}_{>0} \) let us define
\[
A_{\ell,j}^1 := \left\{ x \in \left( a_i - \frac{\delta_i}{\ell}, a_i + \frac{\delta_i}{\ell} \right) : Mf_j(x) = M_{(a_i - \delta_i, a_i + \delta_i)}f_j(x) \right\}
\]
and
\[
A_{\ell,j}^2 = \left\{ x \in \left( a_i - \frac{\delta_i}{\ell}, a_i + \frac{\delta_i}{\ell} \right) : Mf_j(x) > M_{(a_i - \delta_i, a_i + \delta_i)}f_j(x) \right\}.
\]
Since \( Mf_j \geq M_{(a_i - \delta_i, a_i + \delta_i)}f_j \) we know that \( (Mf_j)' = (M_{(a_i - \delta_i, a_i + \delta_i)}f_j)' \) a.e. in \( A_{\ell,j}^1 \). Therefore
\[
\int_{A_{\ell,j}^1} |(Mf_j)'| \leq \int_{(a_i - \delta_i, a_i + \delta_i)} |(M_{(a_i - \delta_i, a_i + \delta_i)}f_j)'| \leq C \int_{(a_i - \delta_i, a_i + \delta_i)} |f_j'| \leq \frac{\varepsilon}{2\ell},
\]
Also, for a.e. \( x \in A_{\ell,j}^2 \), we have that there exists \( r_x \geq \delta_i - \frac{\delta_i}{\ell} = \frac{\delta_i(\ell-1)}{\ell} \) such that \( f_{[x-r_x, x+r_x]}f_j = Mf_j(x) \). Then, by Luiro’s formula (Proposition \([2]\)), we have that \( (Mf_j)'(x) = f_{[x-r_x, x+r_x]}f_j' \), and therefore
\[
|(Mf_j)'(x)| \leq \frac{1}{2}\|f_j'\|_1.
\]
Thus, for \( x \in A_{\ell,j}^2 \) we have
\[
|(Mf_j)'(x)| \leq \frac{\delta_i\ell}{2(\ell - 1)}\|f_j'\|_1.
\]
In consequence, we have
\[
\int_{A_{\ell,j}^2} |(Mf_j)'| \leq \int_{(a_i - \frac{\delta_i}{\ell}, a_i + \frac{\delta_i}{\ell})} \frac{\delta_i\ell}{2(\ell - 1)}\|f_j'\|_1 \leq \frac{\delta_i^2}{(\ell - 1)}\|f_j'\|_1.
\]
From here, we conclude our lemma by choosing \( \ell \) such that \( \frac{\delta_i^2}{(\ell - 1)} < \frac{\varepsilon}{\ell} \), \( \delta := \frac{\delta_i}{\ell} \) and by taking \( j \) big enough such that \( \frac{2\|f_j'\|_1}{\ell} \leq \|f_j'\|_1 \leq \frac{3\|f_j'\|_1}{2} \).

Also, we need the following uniform control near infinity.

**Lemma 5 (Proposition 4.11).** Let \( f_j \to f \) in \( W^{1,1}(\mathbb{R}) \) and \( \varepsilon > 0 \) be given. There exists \( K > 0 \) such that, for \( j \) big enough, we have
\[
\int_{(-K,K)} |(Mf_j)'| < \varepsilon.
\]

### 3. The auxiliary maximal operators

In this section we define the main objects of our work. Let us take \( \varepsilon > 0 \) and consider \( g_\varepsilon = \sum_{k=0}^{N} \alpha_k \chi_{(a_k, a_{k+1})} \) such that \( \|f' - g_\varepsilon\|_1 < \varepsilon \). That is, we approximate the derivative of our limit function by a simple function. We write \( a_0 = -\infty, a_{N+1} = \infty \) and \( \mathcal{P} := \{a_1, \ldots, a_N\} \). We assume that \( \mathcal{P} \) is non-empty. We observe that \( \alpha_0 = \alpha_n = 0 \). Now, we define our auxiliary maximal operators \( M_1, M_2 \) as follows: for any \( h \in L^1(\mathbb{R}) \) and \( x \in \mathbb{R} \) we set
\[
M_1h(x) := \sup_{r < d(x, \mathcal{P})} \int_{|x-r, x+r|} |h|,
\]
In the figure the scope of $L$ is $\alpha_i$ and $[x-r, x+r]$ is an admissible interval for $x$ and $M_1$.

and

$$M_2h(x) := \sup_{r \geq d(x,P)|x-r, x+r|} \int_{x-r, x+r} |h|.$$ 

We now state some basic results about our operators $M_1, M_2$.

**Lemma 6.** Let $f_j \to f$ in $W^{1,1}(\mathbb{R})$. We have

$$M_i f_j \to M_i f$$

uniformly, for $i = 1, 2$.

**Proof.** It follows from the fact that $|M_i f_j - M_i f| \leq |M_i(f_j - f)| \leq \|f_j - f\|_\infty$. □

3.1. **Properties of $M_2$.** For any $K, \delta > 0$ such that the intervals $(a_i - \delta, a_i + \delta)$ are pairwise disjoint, let us define $U_{\delta, K} = (-K, K) \setminus \bigcup_{i=1}^n (a_i - \delta, a_i + \delta)$. We observe that for any $x \in U_{\delta, K}$ and any $g \in W^{1,1}(\mathbb{R})$ there exists a radius $r_x \geq \delta$ such that $\int_{x-r_x, x+r_x} |g| = M_2 g(x)$. We have then the following.

**Lemma 7.** For any $g \in W^{1,1}(\mathbb{R})$ we have that $M_2 g$ is weakly differentiable in $U_{\delta, K}$.

**Proof.** For any $x, y \in U_{\delta, K}$ with $M_2 g(x) > M_2 g(y)$, we have

$$M_2 g(x) - M_2 g(y) = \int_{x-r_x, x+r_x} |g| - \int_{y-r_y, y+r_y} |g| \leq \int_{x-r_x, x+r_x} |g| - \int_{y+r_y, y-r_y} |g|$$

$$\leq \|g\|_1 \left( \frac{1}{2r_x} - \frac{1}{2r_y + 2|x-y|} \right) \leq \|g\|_1 C(\delta)|x-y|,$$

where $C(\delta)$ is the Lipschitz constant of the function $\frac{1}{2x}$ in the set $[\delta, \infty)$. Therefore, we have that $M_2 g$ is Lipschitz in this set, from where we conclude our lemma. □

In the next result we present a formula for the derivative of $M_2 g$ that has similarities with the one presented in [14, Lemma 10]. We use the notation $x \pm \infty = \pm \infty$. 

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**Figure 1.** In the figure the scope of $L$ is $\alpha_i$ and $[x-r, x+r]$ is an admissible interval for $x$ and $M_1$. 

[Figure of a graph with intervals and functions labeled $M_1 f(x)$ and $M_2 g(x)$]
Lemma 8. Let $g \in W^{1,1}(\mathbb{R})$. Let $x \in U_{\delta,K}$ be such that $M_2 g$ is differentiable at $x$, and let $r_x$ such that $M_2 g(x) = f_{[x-r_x,x+r_x]} \cdot |g|$ with $r_x \geq d(x, \mathcal{P})$. Assume that $\frac{a_i + a_{i+1}}{2} < x < a_{i+1}$. Then, we have

$$(M_2 g)'(x) = \frac{\int_{[x-r_x,x+r_x]} |g|}{2r_x^2} - \frac{|g|(x - r_x)}{r_x}.$$ 

Proof. Observe that, for $h > 0$, we have

$$\frac{M_2 g(x) - M_2 g(x-h)}{h} = \frac{\int_{x-r_x,x+r_x+|h|} |g|}{2r_x} - \frac{\int_{x-r_x,x+r_x} |g|}{2r_x} \leq \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x + 2h} - \frac{\int_{x-r_x,x+r_x} |g|}{2r_x}$$

$$= \frac{\int_{x-r_x,x+r_x} |g|}{2r_x} - \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x + 2h} - \frac{\int_{x-r_x,x+r_x} |g|}{2r_x} - \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x + 2h}$$

when $h \to 0$, where we use the continuity of $g$. Therefore $(M_2 g)'(x) \leq \frac{\int_{x-r_x,x+r_x+|h|} |g|}{2r_x} - \frac{|g|(x - r_x)}{r_x}$. Also, for $h > 0$, since $x < a_i \leq x + r_x$ (and hence the interval $[x - r_x + 2h, x + r_x]$ is admissible for $x + h$ for the operator $M_2$), we have

$$\frac{M_2 g(x+h) - M_2 g(x)}{h} \geq \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x} - \frac{\int_{x-r_x,x+r_x} |g|}{2r_x} \geq \int_{x-r_x,x+r_x} |g| \left( \frac{1}{2r_x + 2h} - \frac{1}{2r_x} \right) - \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x + 2h}$$

$$\int_{x-r_x,x+r_x} |g| \left( \frac{1}{2r_x} - \frac{1}{2r_x + 2h} \right) - \frac{\int_{x-r_x,x+r_x} |g|}{2r_x} - \frac{\int_{x-r_x,x+r_x+2h} |g|}{2r_x + 2h}$$

when $h \to 0$, and therefore $(M_2 g)'(x) \geq \frac{\int_{x-r_x,x+r_x} |g|}{2r_x^2} - \frac{|g|(x - r_x)}{r_x}$, from where we conclude our lemma. \hfill \Box

Lemma 9. Let $f_j \to f$ in $W^{1,1}(\mathbb{R})$. Let $x \notin \mathcal{P}$. Assume that $M_2 f_j(x) = f_{[x-r_{x,j},x+r_{x,j}]} \cdot |f_j|$ for some $r_{x,j} \geq d(x, \mathcal{P})$. If $r_{x,j} \to r$ then

$$M_2 f(x) = \int_{[x-r,x+r]} |f|.$$ 

Proof. This follows as [10, Lemma 12]. \hfill \Box

Now we can conclude the pointwise a.e convergence at the derivative level.

Lemma 10. Let $f_j \to f$ in $W^{1,1}(\mathbb{R})$. Then, for a.e $x \in U_{\delta,K}$, we have

$$(M_2 f_j)'(x) \to (M_2 f)'(x).$$

Proof. Let us assume that $x$ is such that $M_2 f_j$, for every $j$, and $M_2 f$ are differentiable at the point $x$ and $x \in \left( \frac{a_i + a_{i+1}}{2}, a_{i+1} \right)$ for some $i$. The other case follows analogously. Now, for every $j$, let us take $r_{x,j} \geq d(x, \mathcal{P})$ such that $M_2 f_j(x) = f_{[x-r_{x,j},x+r_{x,j}]} \cdot |f_j|$. By Lemma 8 we have that

$$(M_2 f_j)'(x) = \frac{\int_{[x-r_{x,j},x+r_{x,j}]} |f_j|}{2r_{x,j}^2} - \frac{|f_j|(x - r_{x,j})}{r_{x,j}}.$$
Assume that there exists a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that \( |(M_2 f_{j_k})'(x) - (M_2 f)'(x)| > \rho > 0 \). Let us take \( R > 0 \) such that \( \int_{[x-R,x+R]} |f_j| > \frac{\|f_j\|_1}{2} \). For \( j \) big enough we have that \( \int_{[x-R,x+R]} |f_j| > \frac{\|f_j\|_1}{2} \). Since
\[
\frac{\|f_j\|_1}{4R} < \frac{\int_{[x-R,x+R]} |f_j|}{2R} < \frac{\int_{[x-r_{x,j},x+r_{x,j}]} |f_j|}{2r_{x,j}} \leq \frac{\|f_j\|_1}{2r_{x,j}},
\]
we note that \( r_{x,j} \leq 2R \). Therefore, there exists a subsequence of \( \{j_k\}_{k \in \mathbb{N}} \) (that we keep calling \( \{j_k\}_{k \in \mathbb{N}} \) with a harmless abuse of notation) such that \( r_{x,j_k} \to r > 0 \). Thus, by Lemma 9 we have
\[
(M_2 f_{j_k})'(x) = \frac{\int_{[x-r_{x,j_k},x+r_{x,j_k}]} |f_{j_k}|}{2r^2_{x,j_k}} - \frac{|f_{j_k}|(x-r_{x,j_k})}{r_{x,j_k}} \to \frac{\int_{[x-r,x+r]} |f|}{2r^2} - \frac{f(x-r)}{r} = (M_2 f)'(x).
\]
From this we conclude our lemma. \( \square \)

We are now in position to conclude our desired \( L^1(U_{\delta,K}) \) convergence.

**Proposition 11.** We have \( (M_2 f_j)' \to (M_2 f)' \) in \( L^1(U_{\delta,K}) \).

**Proof.** Let us take \( x \in U_{\delta,K} \) with \( x \in \left( \frac{a_i+a_{i+1}}{2}, a_{i+1} \right) \) and such that \( M_2 f_j \), for every \( j \), and \( M_2 f \) are all differentiable at the point \( x \). The symmetric case follows similarly. By Lemma 8 we have that (using the notation of the previous lemma)
\[
|(M_2 f_j)'(x)| = \left| \frac{\int_{[x-r_{x,j},x+r_{x,j}]} |f_{j_k}|}{2r^2_{x,j}} - \frac{|f_{j_k}|(x-r_{x,j})}{r_{x,j}} \right| \leq \frac{\|f_j\|_1}{2r^2} + \frac{\|f_j\|_\infty}{\delta} \leq 2\|f\|_{1,1} \left( \frac{1}{2\delta^2} + \frac{1}{\delta} \right),
\]
for \( j \) big enough. Therefore, by combining the dominated convergence theorem with Lemma 10 we conclude our proposition. \( \square \)

3.2. **Properties of \( M_1 \).** About our local operator \( M_1 \), by Lemma 3 we have that \( M_1 \) is weakly differentiable in \( \mathbb{R} \setminus \mathcal{P} \). We now prove the following.

**Proposition 12.** Let \( f_j \to f \) in \( W^{1,1}(\mathbb{R}) \) (recall that we assume \( f_j, f \geq 0 \)). We have that, for \( j \) big enough
\[
\|(M_1 f_j)' - f_j'\|_1 \leq 2(C + 1) \varepsilon,
\]
where \( C \) is the universal constant appearing in Lemma 3.

**Proof.** Let \( L_i : (a_i, a_{i+1}) \to \mathbb{R} \) be a line such that \( L_i' = \alpha_i \) and \( L_i \leq 0 \) (since \( \alpha_0 = \alpha_n = 0 \), \( L_0 \) and \( L_n \) are constant). We observe that
\[
\int_{(a_i, a_{i+1})} |(M_1 f_j)' - (f_j)'| \leq \int_{(a_i, a_{i+1})} |(M_1 f_j)' - L_i'| + |L_i' - (f_j)'|.
\]
Let us notice that, for every \( x \in (a_i, a_{i+1}) \), we have
\[
M_1 f_j - L_i = \left( \sup_{r < d(x, \{a_i, a_{i+1}\}) \cap [x-r, x+r]} f_j \right) - L_i \\
= \sup_{r < d(x, \{a_i, a_{i+1}\}) \cap [x-r, x+r]} (f_j - L_i) = M_1 (f_j - L_i).
\]
Therefore, we have
\[
\int_{(a_i, a_{i+1})} |(M_1f_j)' - L_i'| = \int_{(a_i, a_{i+1})} |(M_1f_j - L_i)'| \\
= \int_{(a_i, a_{i+1})} |(M_1(f_j - L_i))'| \\
= C \int_{(a_i, a_{i+1})} |(f_j - L_i)|.
\]

Combining this with (3.1) we have that
\[
\int_{(a_i, a_{i+1})} |(M_1f_j)' - (f_j)'| \leq (C + 1) \int_{(a_i, a_{i+1})} |f_j' - \alpha_i| \leq (C + 1) \left( \int_{(a_i, a_{i+1})} |f' - \alpha_i| + |f' - f_j'| \right).
\]

Therefore, we have
\[
\|(M_1f_j)' - f_j'\|_1 \leq (C + 1) \left( \varepsilon + \|f' - f_j'\|_1 \right).
\]

Since \(\|f' - f_j'\|_1 < \varepsilon\) for \(j\) big enough, we conclude our proposition. \(\square\)

Analogously, we conclude that \(\|(M_1f)' - (f)'\|_1 \leq 2(C + 1)\varepsilon\), and therefore \(\|(M_1f_j)' - (M_1f)'\|_1 \leq (4C + 5)\varepsilon\), for \(j\) big enough.

4. Proof of Theorem

Now we are able to conclude our result.

Proof. By choosing \(K\) big enough and \(\delta\) small enough such that Lemmas 4 and 5 hold, we have that
\[
\int_{R \setminus U_{\delta,K}} |(M_1f_j)' - (f)'| < 2\varepsilon,
\]
for \(j\) big enough. Now we focus on \(U_{\delta,K}\). We follow a similar strategy than in [10, Lemma 11]. We observe that \(M = \max\{M_1, M_2\}\). Let us write \(X_j := \{x \in U_{\delta,K}; M_1f_j(x) > M_2f_j(x)\}\), \(Y_j := \{x \in U_{\delta,K}; M_1f_j(x) = M_2f_j(x)\}\) and \(Z_j := \{x \in U_{\delta,K}; M_1f_j(x) < M_2f_j(x)\}\). We define \(X, Y\) and \(Z\) analogously, but this time with respect to \(f\) instead of \(f_j\). We observe that \((M_1f_j)' = (M_1f_j)'\) a.e. in \(X_j\), \((M_1f_j)' = (M_1f_j)' = (M_2f_j)'

Figure 2. \(f_j\) and \(M_1f_j\) are close at the derivative level to \(L_i\) when \(j\) is big enough.
a.e in $Y$ and $(M_2f_2)'(x) = (M_2f_2)'(x)$ in $Z$. Analogous properties hold for $f$ in $X,Y$ and $Z$. Let us observe that

\[
\int_X |(M_2f_2)' - (Mf)'| = \int_{X \cap X_j} |(M_2f_2)' - (Mf)'| + \int_{X \cap Y_j} |(M_2f_2)' - (Mf)'| + \int_{X \cap Z_j} |(M_2f_2)' - (Mf)'|
\]

\[
\leq \int_{X \cap X_j} |(M_1f_1)' - (Mf)'| + \int_{X \cap Y_j} |(M_1f_1)' - (Mf)'| + \int_{X \cap Z_j} |(M_2f_2)' - (Mf)'| + \int_{X \cap Z_j} |(M_2f_2)' - (Mf)'|
\]

By Lemma \[6\] we have that $\chi_{X \cap Z_j} \to 0$ a.e., therefore by the dominated convergence theorem we have \[\int_{X \cap Z_j} |(M_2f)' - (Mf)'| < \varepsilon\] for $j$ big enough. Then, by combining Propositions \[11\] and \[12\] with this we have that there exists and universal constant $\tilde{C}$ such that

\[
\int_X |(Mf)' - (Mf)'| < \tilde{C}\varepsilon,
\]

for $j$ big enough. Similarly, we conclude an analogous statement about $Y$ and $Z$. Therefore, considering \[4.1\], we have that there exist an universal constant $\tilde{C}$ such that

\[
\|(Mf)' - (Mf_2)'\|_1 < \tilde{C}\varepsilon,
\]

for $j$ big enough. From this we conclude our result.

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