Classification and stability of simple homoclinic cycles in $\mathbb{R}^5$

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Abstract
Heteroclinic cycles, unions of equilibria and connection trajectories, can be structurally stable in a $\Gamma$-equivariant system due to the existence of invariant subspaces. A structurally stable heteroclinic cycle is called simple if all connecting trajectories are one-dimensional. Heteroclinic cycles where equilibria are related by a symmetry $\gamma \in \Gamma$ are called homoclinic. This paper presents a complete study of simple homoclinic cycles in $\mathbb{R}^5$. We find all symmetry groups $\Gamma$ such that a $\Gamma$-equivariant dynamical system in $\mathbb{R}^5$ can possess a simple homoclinic cycle. We introduce a classification of simple homoclinic cycles in $\mathbb{R}^n$ based on the action of the system symmetry group. For systems in $\mathbb{R}^5$, we list all classes of simple homoclinic cycles. For each class, we derive necessary and sufficient conditions for asymptotic stability and fragmentary asymptotic stability in terms of eigenvalues of linearization near the steady state involved in the cycle. For any action of the groups $\Gamma$ which can give rise to a simple homoclinic cycle, we list classes to which the respective homoclinic cycles belong, thus determining the conditions for the asymptotic stability of these cycles.

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1. Introduction

In a generic dynamical system, heteroclinic cycles are of codimension at least one, but they can be structurally stable in a system with a non-trivial symmetry group [16]. Hence, a classification of heteroclinic cycles in a symmetric system can be formulated in terms of symmetry groups and their actions on $\mathbb{R}^n$, for which heteroclinic cycles can exist.
In this paper, we consider structurally stable homoclinic cycles in a smooth dynamical system
\[ \dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n, \] (1)
equivariant under the action of a non-trivial finite symmetry group \( \Gamma \):
\[ f(\gamma x) = \gamma f(x), \quad \text{for all } \gamma \in \Gamma \subset O(n). \] (2)
Let \( \xi_1, \ldots, \xi_m \in \mathbb{R}^n \) be hyperbolic equilibria of (1) and \( \kappa_j : \xi_j \to \xi_{j+1}, j = 1, \ldots, m, \xi_{m+1} = \xi_1 \), be a set of trajectories from \( \xi_j \) to \( \xi_{j+1} \). The union of the equilibria and the connecting trajectories is called a heteroclinic cycle. A heteroclinic cycle is structurally stable (or robust) to symmetric perturbations, if, for each \( j \), \( \kappa_j \) belongs to the fixed-point subspace \( P_j \) of a non-trivial subgroup \( \Sigma_j \subset \Gamma \), and in this subspace \( \xi_{j+1} \) is a sink [2, 16, 31]. Cycles where all connections are one-dimensional are called simple heteroclinic cycles. If the equilibria are related by a symmetry \( \gamma \in \Gamma, \gamma \xi_j = \xi_{j+1} \), then the cycle is called homoclinic.

Heteroclinic cycles often arise in systems related to biology [14, 21], fluid dynamics [4, 13, 30] and game theory [3]. Heteroclinic cycles are often responsible for complex and intermittent behaviour [16]. They may have a simple geometric structure but complex local attraction properties—heteroclinic cycles which are not asymptotically stable can attract a positive measure set in its small neighbourhood [3, 5, 8, 15, 19, 28]. In [25] such cycles were called fragmentarily asymptotically stable.

Classifying heteroclinic cycles in \( \mathbb{R}^3 \) is straightforward. Heteroclinic cycles in \( \mathbb{R}^4 \) were categorized into classes A, B and C in [17–19]. The symmetry groups giving rise to heteroclinic cycles of types B and C are known, but this is not the case for cycles of type A. In [31, 32] a classification of simple homoclinic cycles such that \( \dim P_j = 2 \) was given for dynamical systems in \( \mathbb{R}^4 \) and \( \mathbb{R}^5 \). However, in these two papers only minimal admissible groups (i.e. the simplest groups for which homoclinic cycles exist) were found and only \( \Gamma \subset SO(5) \) were considered in \( \mathbb{R}^5 \).

The theory of asymptotic and fragmentary asymptotic stability is as yet incomplete, even for simple heteroclinic cycles. A sufficient condition for the asymptotic stability of heteroclinic cycles was presented in [17]. Necessary and sufficient conditions for the asymptotic stability of simple homoclinic and heteroclinic cycles in \( \mathbb{R}^4 \) were proven in [6, 9, 19] and for the fragmentary asymptotic stability of simple heteroclinic cycles in \( \mathbb{R}^4 \) in [26]. The asymptotic stability of heteroclinic cycles in some particular systems was studied in [3, 5, 6, 8, 9, 15, 18, 28, 29]. Necessary and sufficient conditions for the asymptotic stability of the so-called type A cycles were proven in [17] and for the asymptotic and fragmentary asymptotic stability of the so-called type Z cycles in [25].

We begin by recalling several examples of homoclinic cycles in \( \mathbb{R}^n \) (section 3) and complete the study of [31, 32] by listing all possible groups \( \Gamma \) acting on \( \mathbb{R}^4 \), for which simple homoclinic cycles can arise. Our arguments, based on the quaternion description of the group \( SO(4) \), are much shorter than the ones in [31, 32]. Using the exhaustive description [22] of finite subgroups of \( O(5) \), we find in section 4 all groups \( \Gamma \) acting on \( \mathbb{R}^5 \) that give rise to homoclinic cycles in a \( \Gamma \)-equivariant system. All homoclinic cycles found in section 4 are associated with the cycles listed in section 3. In section 5.1 we introduce a classification of simple homoclinic cycles in \( \mathbb{R}^5 \) based on the isotypic decomposition of \( P_j^\perp \) under the action of the symmetry group of \( P_j \). Application of this classification to cycles in \( \mathbb{R}^5 \) enables us to determine conditions for asymptotic stability and fragmentary asymptotic stability. Derivation of conditions for asymptotic stability for two classes of homoclinic cycles is given in appendices.
2. Definitions

2.1. Stability

Let us recall the definitions of various types of asymptotic stability of an invariant set of system (1). We denote by \( \Phi_t(x) \) the trajectory of (1) that starts at point \( x \). For a set \( X \) and a number \( \epsilon > 0 \), the \( \epsilon \)-neighbourhood of \( X \) is the set
\[
B_\epsilon(X) = \{ x \in \mathbb{R}^n : d(x, X) < \epsilon \}.
\]

(3)

Let \( X \) be a compact invariant set of system (1). We denote by \( B_\delta(X) \) its \( \delta \)-local basin of attraction:
\[
B_\delta(X) = \{ x \in \mathbb{R}^n : d(\Phi_t(x), X) < \delta \text{ for any } t \geq 0 \text{ and } \lim_{t \to \infty} d(\Phi_t(x), X) = 0 \}.
\]

(4)

**Definition 1.** A compact invariant set \( X \) is asymptotically stable, if for any \( \delta > 0 \) there exists \( \epsilon > 0 \) such that
\[
B_\epsilon(X) \subset B_\delta(X).
\]

**Definition 2.** A compact invariant set \( X \) is fragmentarily asymptotically stable, if for any \( \delta > 0 \)
\[
\mu(B_\delta(X)) > 0.
\]

(Here \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^n \).)

**Definition 3.** A compact invariant set \( X \) is completely unstable, if there exists \( \delta > 0 \) such that
\[
\mu(B_\delta(X)) = 0.
\]

We define now various types of stability of a fixed point \( x \) of a map \( g : \mathbb{R}^n \to \mathbb{R}^n \) (i.e. \( gx = x \)).

**Definition 4.** A fixed point \( x \in \mathbb{R}^n \) of a map \( g : \mathbb{R}^n \to \mathbb{R}^n \) is asymptotically stable, if for any \( \delta > 0 \) there exists \( \epsilon > 0 \), such that
\[
|y - x| < \epsilon \text{ implies } |g^k y - x| < \delta \text{ for all } k > 0 \text{ and } \lim_{k \to \infty} g^k y = x.
\]

**Definition 5** ([25], adapted). A fixed point \( x \in \mathbb{R}^n \) of a map \( g : \mathbb{R}^n \to \mathbb{R}^n \) is fragmentarily asymptotically stable, if for any \( \delta > 0 \) the measure of the set
\[
V_\delta(x) = \{ y \in \mathbb{R}^n : |g^k y - x| < \delta \text{ for all } k > 0 \text{ and } \lim_{k \to \infty} g^k y = x \}
\]
is positive.

**Definition 6.** A fixed point \( x \in \mathbb{R}^n \) of a map \( g : \mathbb{R}^n \to \mathbb{R}^n \) is completely unstable, if there exists \( \delta > 0 \) such that
\[
\mu(V_\delta(x)) = 0.
\]

2.2. Heteroclinic cycles

Let \( \xi_1, \ldots, \xi_m \) be hyperbolic equilibria of the system (1)–(2) with stable and unstable manifolds \( W^s(\xi_j) \) and \( W^u(\xi_j) \), respectively. Assuming \( \xi_{m+1} = \xi_1 \), we denote by \( \kappa_j, j = 1, \ldots, m \), the set of trajectories from \( \xi_j \) to \( \xi_{j+1} \), \( \kappa_j = W^u(\xi_j) \cap W^s(\xi_{j+1}) \neq \emptyset \).

**Definition 7.** A heteroclinic cycle is the union of equilibria \( \{\xi_1, \ldots, \xi_m\} \) and their connecting orbits \( \{\kappa_1, \ldots, \kappa_m\} \).
The isotropy group of a point \( x \in \mathbb{R}^n \) is the subgroup of \( \Gamma \) satisfying
\[
\Sigma_x = \{ \gamma \in \Gamma : \gamma x = x \}.
\]

The fixed-point subspace of a subgroup \( \Sigma \subset \Gamma \) is the subspace
\[
\text{Fix}(\Sigma) = \{ x \in \mathbb{R}^n : \sigma x = x \text{ for all } \sigma \in \Sigma \}.
\]

**Definition 8.** A heteroclinic cycle is structurally stable (or robust), if for any \( j, 1 \leq j \leq m \), there exist \( \Sigma_j \subset \Gamma \) and a fixed-point subspace \( P_j = \text{Fix}(\Sigma_j) \) such that
- \( \xi_{j+1} \) is a sink in \( P_j \);
- \( \kappa_j \subset P_j \).

We denote \( L_j = P_{j-1} \cap P_j \) by \( T_j \), the isotropy subgroup of \( L_j \) (evidently, \( \xi_j \in L_j \)), and by \( P_j^\perp \) the orthogonal complement to \( P_j \) in \( \mathbb{R}^n \).

For a structurally stable heteroclinic cycle, eigenvalues of \( df(\xi_j) \) can be divided into four sets [17–19]:
- Eigenvalues with the associated eigenvectors in \( L_j \) are called radial.
- Eigenvalues with the associated eigenvectors in \( P_{j-1} \ominus L_j \) are called contracting.
- Eigenvalues with the associated eigenvectors in \( P_j \ominus L_j \) are called expanding.
- Eigenvalues that do not belong to any of the three aforementioned groups are called transverse.

**Definition 9 ([19], adapted).** A robust heteroclinic cycle in \( \mathbb{R}^n \setminus \{0\} \) is simple, if for any \( j \)
\[ \dim(P_{j-1} \ominus L_j) = 1. \]

**Definition 10 ([25]).** A simple robust heteroclinic cycle is of type Z, if for any \( j \)
- \( \dim(P_j) = \dim(P_{j+1}) \);
- the isotropy subgroup of \( P_j, \Sigma_j \), decomposes \( P_j^\perp \) into one-dimensional isotypic components.

Note that the first condition in the definition [25] of type Z cycles is redundant, because it is implied by the simplicity of the cycles:

**Lemma 1.** Let \( X \) be a simple robust heteroclinic cycle, i.e. the respective fixed-point subspaces satisfy \( \dim(P_{j-1} \ominus L_j) = 1 \) for any \( j, 1 \leq j \leq m \). Then \( \dim(P_j) = \dim(P_{j+1}) \) for all \( j, 1 \leq j \leq m - 1 \).

**Proof.** The dimension of the expanding eigenspace for a steady state \( \xi_j \) involved in a heteroclinic cycle cannot be less than one and, therefore,
\[
\dim(P_j) \geq \dim(L_j) + 1 = \dim(P_{j-1}).
\]
This inequality applied for each \( m \) steady states yields
\[
\dim(P_1) \leq \dim(P_2) \leq \cdots \leq \dim(P_m) \leq \dim(P_1)
\]
(recall that the equilibria are cyclically connected, i.e., \( \xi_1 = \xi_{m+1} \)). Here, the leftmost and rightmost values coincide, and thus all terms in the inequality are equal. \( \square \)

Since \( \dim(P_{j-1} \ominus L_j) = 1 \) for a simple heteroclinic cycle, all expanding and contracting eigenspaces are one-dimensional.

**Definition 11.** A simple robust heteroclinic cycle is of type \( A' \), if for any \( j \) the isotypic decomposition of \( P_j^\perp \) under the action of \( \Sigma_j \) involves only one isotypic component.

Our \( A' \) type cycles are a subset of \( A \) type cycles:
Definition 12 ([17–19]). A simple robust heteroclinic cycle is of type A, if for any $j$

- all eigenvectors, associated with $\lambda_j^r$, $\lambda_j^l$, $\lambda_j^r_{j=1}$ and $\lambda_j^l_{j=1}$, belong to the same isotypic component in the decomposition of $P^+_j$ under $\Sigma_j$;

- all eigenvectors of $df(\xi_j)$, associated with transverse eigenvalues with positive real parts, belong to the same isotypic component in the decomposition of $P^+_j$ under $\Sigma_j$.

Here $\lambda_j^r$ and $\lambda_j^l$ denote the contracting and transverse eigenvalues of $df(\xi_j)$ with the minimum real parts, respectively, and $\lambda_j^r$ the expanding eigenvalue with the maximum real part.

Note that if a system depends on a parameter and the classification involves conditions for eigenvalues, then on variation of the parameter the type of the cycle can change without any qualitative change in the overall behaviour of the system. This is not the case, when the classification is based on the action of the symmetry group, as we propose here.

Definition 13. A heteroclinic cycle is called a homoclinic cycle, if there exists a symmetry $\gamma \in \Gamma$ such that for any $1 \leq j \leq m$

$$\gamma^j = \xi_j .$$

The symmetry $\gamma$ is called a twist [2, 31].

Definition 14. A homoclinic network is a connected component of the group orbit of a homoclinic cycle under the action of $\Gamma$.

In this paper we study the stability of simple homoclinic cycles, and we use the symbols $\xi$, $\kappa$, $P$ and $L$ without subscripts provided this does not create ambiguity. The radial eigenvalues of $df(\xi)$ are denoted by $-r = \{ -r_l \}$, $1 \leq l \leq n_r$, the contracting one by $-c$, the expanding one by $e$ and the transverse ones by $t = \{ t_l \}$, $1 \leq l \leq n_t$. Here $n_r$ and $n_t$ are the numbers of the radial and transverse eigenvalues, respectively.

3. Examples of homoclinic cycles

In this section we recall five examples of homoclinic cycles in $\mathbb{R}^n$ which will be used in the next section in the investigation of simple homoclinic cycles in $\mathbb{R}^5$. In the last section we find all subgroups $\Gamma \subset O(4)$ such that a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^4$ can possess a simple homoclinic cycle under the assumptions that (i) $\dim P = 2$ and (ii) the cycle does not belong to any proper subspace of $\mathbb{R}^4$. We use the homomorphism $\mathbb{Q} \times \mathbb{Q} \rightarrow SO(4)$, where $\mathbb{Q}$ is the multiplicative group of unit quaternions; these notions are reviewed in section 3.2. We end this section with a table summarizing the results of the first and the third subsections.

3.1. Three simple examples

Example 1. Suppose a system (1) in $\mathbb{R}^3$ is equivariant with respect to the group $D_4$ involving two reflections and rotation:

$$s_1 : (x_1, x_2, x_3) \rightarrow (x_1, -x_2, x_3),$$

$$s_2 : (x_1, x_2, x_3) \rightarrow (x_1, x_2, -x_3),$$

$$s_3 : (x_1, x_2, x_3) \rightarrow (-x_1, x_3, x_2).$$

Suppose the system possesses two equilibria in the $x_1$ axis, $\xi_1 = (a, 0, 0)$ and $\xi_2 = (-a, 0, 0)$, that are connected by a trajectory $k_1 : \xi_1 \rightarrow \xi_2$ lying in the plane $(x_1, x_2, 0)$. The symmetry $s_3$ permutes the equilibria and maps the trajectory $k_1$ to $k_2 : \xi_2 \rightarrow \xi_1$ in $(x_1, 0, x_3)$.

1 For instance, cycles of class $3-2-\{12\}[3]$ (see section 5.4 and table 4) are not of type $A'$, but for $t_1 < t_2$ they are of type $A$ as defined in [17]. Such a cycle is asymptotically stable for $c > e$ and $t_1, t_2 < 0$. When $t_1$ exceeds $t_2$, the cycle just ceases to be of type $A$, although no bifurcations take place and the stability of the cycle does not change.
Example 2. Let a system (1) in \(\mathbb{R}^n\) be equivariant with respect to the group \((\mathbb{Z}_2)^n \rtimes \mathbb{Z}_n\) acting on \(\mathbb{R}^n\) by inversion and cyclic permutation of coordinates. If the system possesses an equilibrium \(\xi_1 = (a, 0, \ldots)\), then it also has a set of equilibria \(((\pm a, 0, \ldots))\) (here double parentheses \(\cdot\) denote all cyclic permutations of the quantities listed in the parentheses). Existence of a connection \(\xi_2 : \xi_1 \rightarrow \xi_2 = (0, a, \ldots)\) implies existence of a homoclinic cycle connecting \(n\) equilibria \(((a, 0, \ldots))\), and also of cycles connecting \(2n\) equilibria \(((\pm a, 0, \ldots))\), where only some combinations of the signs \(\pm\) are present in an individual cycle.

Example 3. The subgroup of \((\mathbb{Z}_2)^n \rtimes \mathbb{Z}_n\) comprises orientation-preserving transformations of \(\mathbb{R}^n\) is \((\mathbb{Z}_2)^{n-1} \rtimes \mathbb{Z}_n\). A system (1) with such a symmetry group can possess a homoclinic cycle comprising \(2n\) equilibria \(((\pm a, 0, \ldots))\).

For a given twist \(\gamma\), the symmetry \(\gamma \sigma\) is also a twist for any \(\sigma \in T\). The new cycle linked with the twist \(\gamma \sigma\) belongs to the same homoclinic network as the cycle with the twist \(\gamma\), but it can involve a different number of equilibria.

3.2. Quaternions

We recall here some properties of quaternions [7, 10]. A real quaternion is a set of four real numbers, \(q = (q_1, q_2, q_3, q_4)\). Multiplication of quaternions is defined by the relation

\[
qw = (q_1 w_1 - q_2 w_2 - q_3 w_3 - q_4 w_4, q_1 w_2 + q_2 w_1 + q_3 w_4 - q_4 w_3, q_1 w_3 + q_2 w_4 + q_3 w_1 - q_4 w_2, q_1 w_4 + q_2 w_3 - q_3 w_2 + q_4 w_1).
\]

\(\tilde{q} = (q_1, -q_2, -q_3, -q_4)\) is the conjugate of \(q\), and \(|q|^2 = q\tilde{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2\) is the square of the norm of \(q\). \(q^{-1}\) is the inverse of a unit quaternion \(q\). We denote by \(Q\) the multiplicative group of unit quaternions; obviously, the unity in it is \((1, 0, 0, 0)\).

The four numbers \((q_1, q_2, q_3, q_4)\) can be regarded as Euclidean coordinates of a point in \(\mathbb{R}^4\). The transformation \(q \rightarrow lqr^{-1}\) is a rotation in \(\mathbb{R}^4\), i.e. an element of the rotation group \(SO(4)\). The direct product \(Q \times Q\) of two groups \(Q\) is the set of ordered pairs \((l, r)\) of elements of \(Q\) equipped with the multiplication \((l, r)(l', r') = (ll', rr')\). The mapping \(\Phi : Q \times Q \rightarrow SO(4)\) that relates the pair \((l, r)\) with the rotation \(q \rightarrow lqr^{-1}\) is a homomorphism onto, whose kernel consists of two elements, \((1, 1)\) and \((-1, -1)\); thus the homomorphism is two to one.

Therefore, a finite subgroup of \(SO(4)\) is a subgroup of a product of two finite subgroups of \(Q\). The group \(Q\) has six finite subgroups:

\[
\mathbb{Z}_n = \oplus_{r=0}^{n-1}(\cos 2\tau \pi/n, 0, 0, \sin 2\tau \pi/n),
\]

\[
\mathbb{D}_n = \mathbb{Z}_{2n} \oplus \oplus_{r=0}^{2n-1}(0, \cos r\pi/n, \sin r\pi/n, 0),
\]

\[
\mathbb{V} = ((\pm 1, 0, 0, 0))
\]

\[
T = \mathbb{V} \oplus \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).
\]

\[
O = T \oplus \sqrt{2}(\pm 1, \pm 1, 0, 0),
\]

\[
I = T \oplus \frac{1}{2}(\pm \tau, \pm 1, \pm \tau^{-1}, 0),
\]

where \(\tau = (\sqrt{5} + 1)/2\). A complete list of finite subgroups of \(SO(4)\) and \(O(4)\) in the quaternion form can be found in [10]. A detailed discussion of the application of quaternions for classification of heteroclinic cycles in \(\mathbb{R}^4\) will be given in [27]. The quaternion form of finite subgroups of \(O(4)\) was employed in [11, 20].
3.3. Homoclinic cycles in $\mathbb{R}^4$

Following [31, 32], we choose a basis in $\mathbb{R}^4$ such that $\xi = (0, a, 0, 0)$ and invariant planes, containing the trajectories that join the steady states involved in the cycle, are

$$P_1 = y^{-1}P = (e_1, e_2),$$
$$P_2 = P = (e_2, e_3),$$
$$P_3 = yP = (\cos te_2 + \sin te_3, \cos se_1 + \sin se_4).$$

Here $t$ is the angle between the fixed point subspaces of two consecutive equilibria and $s$ is the angle between the lines of intersections of $y^{-1}P$ and $yP$ with $P^\perp$. On this basis the matrix of the twist $\gamma : P_j \to P_{j+1}$ is

$$A = \begin{pmatrix}
0 & 0 & \cos s & -\sin s \\
-\alpha \sin t & \cos t & 0 & 0 \\
\alpha \cos t & \sin t & 0 & 0 \\
0 & 0 & \sin s & \cos s
\end{pmatrix},$$

(7)

where $\alpha = 1$ for orientation-preserving transformations and $\alpha = -1$ for orientation-reversing ones.

**Lemma 2.** Let $X$ be a simple robust homoclinic cycle in $\mathbb{R}^4$, $\dim P = 2$, $\xi$ is an equilibrium involved in the cycle and $T$ is the isotropy group of $\xi$. Then $T = \mathbb{Z}_2^l$ or $T = \mathbb{Z}_3^l$.

**Proof.** The condition $\dim P = 2$ implies that the radial, contracting and expanding subspaces, $V' = L, V$ and $V^r$, respectively, are one-dimensional. The plane $y^{-1}P = L \oplus V^r$ accommodates two incoming homoclinic trajectories, and the plane $P = L \oplus V^c$ two outcoming trajectories. If an element of the group $T$ is of order 3 or more, then $df(\xi)$ has an eigenspace of dimension 2 or higher. This eigenspace cannot be radial because $\dim P = 2$, and it cannot be transverse because the sum of all dimensions does not exceed four. If this eigenspace contains the contracting subspace, then more than two homoclinic trajectories approach $\xi$. Since in a homoclinic network the numbers of incoming and outcoming trajectories are equal, we then lack dimension(s) for outcoming trajectories. Hence, the multidimensional eigenspace does not contain the contracting subspace; similarly it does not contain the expanding one. Therefore, $T$ has no elements of order higher than two. The group $T$ has an element that acts as $I$ on $V^c$ and $-I$ on $V'$. It has another element that acts as $I$ on $V^c$ and $-I$ on $V^r$; this implies $T = \mathbb{Z}_2^l$ or $T = \mathbb{Z}_3^l$.

**QED**

**Lemma 3.** Let $s_1$ and $s_2$ be the reflections in $\mathbb{R}^4$ about the planes $N_1$ and $N_2$, respectively, $\Phi^{-1}s_1 = (l_1, r_1)$ $\Phi^{-1}s_2 = (l_2, r_2)$, and where $\Phi$ is the homomorphism defined in the previous section. Denote by $(l_1l_2)_1$ and $(r_1r_2)_1$ the first components of the respective quaternion products. The planes $N_1$ and $N_2$ intersect if and only if $(l_1l_2)_1 = (r_1r_2)_1$.

**Proof.** The planes $N_1$ and $N_2$ intersect if and only if the superposition $s_1s_2$ is a rotation about a two-dimensional plane by some angle $\alpha$. The rotation is conjugate to the rotation about the plane $(0, 0, x_3, x_4)$ whose preimage under $\Phi$ is

$$(l_\alpha, r_\alpha) = ((\cos \alpha, \sin \alpha, 0, 0), (\cos \alpha, -\sin \alpha, 0, 0)).$$

The symmetry $s_1s_2$ is conjugate to this rotation if and only if there exist quaternions $q$ and $w$ such that

$$\Phi^{-1}s_1s_2 = (qlq^{-1}, wrw^{-1}).$$
The quaternion $q\cdot q^{-1}$ takes the form $(\cos \alpha, 0, 0, 0) + \sin \alpha v$, where $v = (0, v_2, v_3, v_4)$ is a unit quaternion. On varying $q$, one encounters any such $v$ [10]. $wr_\alpha w^{-1}$ takes the same form. Therefore, $s_1s_2$ is a rotation by $\alpha$ about a two-dimensional plane if and only if $(t_1, t_2) = (r_1,r_2) = \cos \alpha$.

**Theorem 1.** Consider a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^4$ possessing a simple homoclinic cycle $X = \{\xi_1, \xi_2, \ldots, \xi_m\}$ linked with the twist $\gamma : \xi_i \mapsto \xi_{i+1}$. Suppose that $\dim P = 2$ and the cycle does not lie in any hyperplane of $\mathbb{R}^4$. Then $\Gamma$ is one of the following groups:

(i) $(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_4$ and $m = 4$ or 8;
(ii) $(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_4$ and $m = 8$;
(iii) $\mathbb{Z} \rtimes \mathbb{Z}_4$ and $m = 12$ or 24;
(iv) $\mathbb{D}_{2k} \rtimes \mathbb{Z}_k \cong \mathbb{Z}_2$ and $m = 4$ or 2k.

**Proof.** Lemma 2 implies that $\Sigma_j$ contains the reflection about the plane $P_j$; if $T = (\mathbb{Z}_2)^2$ then $\Sigma_j = \mathbb{Z}_2$ is comprised of this symmetry and the identity, otherwise $\Sigma_j = (\mathbb{Z}_2)^2$ and the reflection is its element.

The twist $\gamma$ is or is not a rotation. We consider the two possibilities separately.

First, suppose $\gamma \in \text{SO}(4)$. Denote by $s_j$ the reflection about the plane $P_j$ and by $(l_j, r_j)$ a preimage of $s_j$ under the homomorphism $\Phi$. It is easy to show that

$$
\Phi^{-1} s_1 = (l_1, r_1) = ((0, 1, 0, 0), (0, 1, 1, 0)),
\Phi^{-1} s_2 = (l_2, r_2) = ((0, 0, 0, 1), (0, 0, 0, 0)).
$$

Denote by $(g, h)$ a preimage of $\gamma$. The reflections $s_1$ and $s_2$ are conjugate by $\gamma$, i.e. $\gamma'^{-1} s_1 \gamma'^{-1} = s_2$, or

$$
g l_1 g^{-1} = l_2, \quad h r_1 h^{-1} = r_2.
$$

From these relations and the above expressions for $\Phi^{-1}s_i$ we find $(g, h) = ((a, b, -a, b), (c, d, c, -d))$ or $(g, h) = ((a, b, a, -b), (c, d, -c, d))$,

where $2a^2 + 2b^2 = 2c^2 + 2d^2 = 1$.

The group generated by $s_1$ and $\gamma$ is finite if and only if two groups, one generated by $l_1$ and $g$, and the second one generated by $r_1$ and $h$, are finite. The two groups are finite if and only if $g$ and $h$ are any of the following quaternions (see (6)):

$$
\frac{1}{\sqrt{2}}(\pm 1, 0, \pm 1, 0), \quad \frac{1}{\sqrt{2}}(0, \pm 1, 0, \pm 1), \quad \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1).
$$

Recall that a twist is represented by the matrix (7). If $s = \pi k$ for integer $k$, then the homoclinic cycle belongs to a three-dimensional subspace. Since changing $t \to -t$ or $s \to \pi - s$ yields twists linked with different cycles that belong to the same homoclinic network, only $t > 0$ and just one of the two values, $s$ or $\pi - s$, are considered. We also restrict the angle $	heta$ to take the smallest possible value (for the given group $\Gamma$), because otherwise an one-dimensional invariant subspace in $P_2$ separates $\xi$ and $\gamma'\xi$, making impossible the connection $\xi \to \gamma'\xi$. Calculating the matrices of the mappings $q \to gq$ for $g$ and $h$ found above reveals the existence of two twists $\gamma_1$ satisfying these conditions. For the first one,

$$
(g, h) = \frac{1}{2}((0, 1, 0, 1), (1, 0, 1, 0))
$$

and the group generated by $s_1$ and $\gamma_1$ is $\Gamma_1 = \mathbb{D}_8 \rtimes \mathbb{Z}_2 \cong (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_4$ comprised of 32 elements. For the second one,

$$
(g, h) = \frac{1}{2\sqrt{2}}((0, 1, 0, -1), (1, 1, -1, 1))
$$

and the group generated by $s_1$ and $\gamma_2$ is $\Gamma_2 = \mathbb{Z} \rtimes \mathbb{Z}_4$ is comprised of 192 elements.

QED
Suppose now either of the two groups \( \Gamma_1 \) is a proper subgroup of the group of symmetries, \( G \), of a dynamical system. Then the system cannot possess a simple homoclinic cycle linked with the respective twist \( \gamma \) for the following reasons. If \( G \supset \Gamma_1 \), then either \( G = \mathbb{D}_{2k} \times \mathbb{Z}_{2k} \) is generated by \( s_1 \) and \(((0, \cos \beta, 0, \sin \beta), (\cos \beta, 0, \sin \beta, 0))\) for \( \beta = \pi/4k \), or \( G = \mathbb{D} \times \mathbb{Z}_4 \).

In the first case, the reflection \(((0, \cos 2\beta, 0, \sin 2\beta), (0, \cos 2\beta, 0, \sin 2\beta))\) is an element of \( G \), and by lemma 3 the plane, invariant with respect to the reflection, intersects with \( P_2 \). In the second case, the reflection about the plane intersecting with \( P_2 \) along the line \( x_2 = x_3 \) is an element of \( G \). Intersection of two invariant planes implies the existence of an one-dimensional invariant subspace in \( P_2 \) that prohibits the connection \( \xi \rightarrow \gamma \xi \). If \( G \supset \Gamma_2 \), then \( G = \mathbb{D} \times \mathbb{T} \) has the sixth-order element \( \frac{1}{3}((1, 1, 1, 1), (1, 1, 1, -1)) \) for which \( e_2 \) is invariant, and by lemma 2 \((0, a, 0, 0)\) is not an equilibrium in any homoclinic cycle.

Second, suppose \( \gamma \notin \mathfrak{so}(4) \). The preimage, under \( \Phi \), of the reflection \( s_3 \) about the plane \( P_3 \) is

\[ \Phi^{-1}s_3 = (l_3, r_3) = ((0, \cos(t + s), \sin(t + s), 0), (0, \cos((t - s + \pi), \sin(t - s + \pi)), 0)). \]

The reflections \( s_1 \) and \( s_3 \) are conjugate by \( \gamma^2 \). Denote by \((g, h)\) a preimage of the square of the twist; \( \gamma^2 \) belongs to \( \mathfrak{so}(4) \) and satisfies

\[ g_{l_3}^{-1} = l_3, \quad hr_{r_3}^{-1} = r_3. \tag{8} \]

If \( t + s \neq k_1 \pi/2 \) and \( t - s \neq k_2 \pi/2 \), then the only possibilities are

\[ g = (\cos((t + s)/2), 0, 0, \sin((t + s)/2)) \]

or \((0, \cos((t + s)/2), \sin((t + s)/2), 0)\), \( h = (\cos((t - s + \pi)/2), 0, 0, \sin((t - s + \pi)/2)) \]

or \((0, \cos((t - s + \pi)/2), \sin((t - s + \pi)/2), 0)\). \( \tag{9} \)

For \( A \) given by (7) and \( \alpha = -1 \),

\[ A^2 = \begin{pmatrix}
\cos s \cos t & \cos s \sin t & -\sin^2 s & -\cos s \sin s \\
\cos t \sin s & \cos t \cos s & -\cos s \sin t & \sin s \sin t \\
-\sin^2 t & \cos t \sin s & \cos t \cos s & -\sin s \cos s \\
\cos t \cos s & \sin t \sin s & \sin s \cos s & \cos^2 s
\end{pmatrix}, \tag{11} \]

It easy to check that no \( g (9) \) and \( h (10) \) yield a linear transformation \( q \rightarrow gqh^{-1} : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) whose matrix has this form.

Now suppose \( t - s = k_2 \pi \). A preimage of the reflection about \( s_3 \) is

\[ (l_3, r_3) = ((0, \pm \cos(t + s), \pm \sin(t + s), 0), (0, 1, 0, 0)). \]

For \( k_2 = 0 \), (8) holds for any element \( h \in \mathfrak{so}(4) \) and for \( g \) satisfying (9). Under the condition that the matrix of \( \Phi(g, h) \) is \( A^2 \) (11), there are two possibilities for the square of the twist:

\[ (g, h) = ((\cos((t + s)/2), 0, 0, \sin((t + s)/2), \cos(\pi - (t + s)/2), 0, 0, \sin(\pi - (t + s)/2)) \]

or \( \tag{12} \)

\[ (g, h) = ((0, \cos((t + s)/2), \sin((t + s)/2), 0), (0, \cos(\pi - (t + s)/2), \sin(\pi - (t + s)/2), 0)). \]

The group generated by \((l_1, r_1)\) and \((g, h)\) is finite if \( t + s = 4\pi/k \). In fact, the group generated by (12) and \( s_1 \), and the group generated by (13) and \( s_1 \), are identical. Two cycles linked with the square of the twist (12) belong to the same homoclinic network as the cycle linked with (13). Note that the order of (12) is \( k \) and the order of (13) is two. The group generated by (13)
and $s_1$ is $D_{2k} \times Z_4$. The group $\Gamma$ is a product of $D_{2k} \times Z_k$ with $Z_2$, where $Z_2$ is generated by an orientation-reversing symmetry.

We checked directly that for the pair $t + s \neq k_4 \pi/2$ and $t - s \neq k_2 \pi/2$, as well as for the pair $t + s = k_4 \pi/2$ and $t - s = k_2 \pi/2$, no twists of the form (11) are possible.

Suppose now a dynamical system has a group of symmetries $G_0 \rtimes Z_k$, where $D_{2k} \rtimes Z_k$ is a proper subgroup of $G_0$. $D_{2k} \rtimes Z_k$ is a subgroup of $D_{2k} \rtimes D_{2k}$ and of $D_{2k} \rtimes Z_k$ for any $r \geq 2$; for $k > 2$ it is not a subgroup of any other finite subgroup of $SO(4)$. The groups $D_{2k} \rtimes D_{2k}$ and $D_{2k} \rtimes Z_{2k}$ contain reflections about the planes that intersect with the plane $P_2$, prohibiting connections $\xi \rightarrow \gamma \xi$. For $k = 2$, the two remaining possibilities are $G_0 = D_6 \rtimes Z_2 \cong (Z_2)^3 \times Z_4$ or $G_0 = O \rtimes Z_4$. In the first case, a system with the symmetry $G_0 \rtimes Z_2 \cong (Z_2)^4 \times Z_4$ can possess a homoclinic cycle linked with the twist $\gamma$ (such a system is an instance of example 2 in section 3.1). The reflection about the plane intersecting with $P_2$ along the line $x_2 = x_3$ belongs to $O \rtimes Z_4$. Thus the line $x_2 = x_3$ is an invariant subspace of the dynamical system, that prohibits the connection $\xi \rightarrow \gamma \xi$ in the second case.

Table 1 summarizes the results of this section.

4. Homoclinic cycles in $\mathbb{R}^5$

In this section we list all symmetry groups $\Gamma$ such that a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^5$ can possess a simple homoclinic cycle. We rely on theorem 2 proven in [22], which presents a complete list of the finite subgroups of $SO(5)$. Examining each of the subgroups individually, we prove in lemmas 4–6 that the only $\Gamma$’s acting on $\mathbb{R}^5$ irreducibly are $(Z_2)^3 \rtimes Z_5$ and $(Z_2)^3 \times Z_5$, both already listed in table 1 (lines 2 and 3). If the action of $\Gamma$ is reducible, the homoclinic cycle belongs to a subspace $N$ of $\mathbb{R}^5$, that is isomorphic to $\mathbb{R}^3$ or $\mathbb{R}^4$. Hence, $\Gamma = \tilde{\Gamma} \times K$, where $\Gamma$ is one of the groups listed in table 1, and $K \subset O(2)$ if $N$ isomorphic to $\mathbb{R}^3$, or $K \subset O(1)$ if $N$ isomorphic to $\mathbb{R}^4$. By inspecting various subgroups $K$ and various actions of $\tilde{\Gamma}$ of $\mathbb{R}^5 \ominus N$ and noting that homoclinic cycles with dim $P = 2, 3$ or 4 are possible, we compile a complete list of simple homoclinic cycles that can exist in $\mathbb{R}^5$. The results are summarized in table 2.

**Theorem 2 (Corollary 2 in the arXiv version of [22]).** Let $\Gamma$ be a finite subgroup of the orthogonal group $SO(5)$ or $O(5)$. Then at least one of the following statements is true:

(i) $\Gamma$ is conjugate to a subgroup of $W = (Z_2)^4 \rtimes S_5$ or $\tilde{W} = (Z_2)^5 \rtimes S_5$;

(ii) $\Gamma$ is isomorphic to $A_5$, $S_5$, $A_6$ or $S_6$, or to the product of one of these groups with $Z_2 = \langle -1 \rangle$;

(iii) $\Gamma$ is conjugate to a subgroup of $O(4) \times O(1)$ or $O(3) \times O(2)$.

Note that these possibilities are not mutually exclusive. Here the semidirect product $\tilde{W} = (Z_2)^2 \rtimes S_4$ acts on $\mathbb{R}^5$ by inversion and permutation of coordinates, and $W = (Z_2)^4 \rtimes S_5$.
Classification and stability of simple homoclinic cycles in $\mathbb{R}^5$.

| $G_{Es}$ | $\dim V_{Es}$ | $N$ | $\Gamma$ | $\dim P$ | $d(\Gamma')$ | $d(G)$ | Class |
|----------|---------------|-----|----------|----------|--------------|--------|-------|
| $(Z_2)^3 \times Z_2$ | 5 | 5.10 | $G_{Es}$ | 2 | 0 | 0 | 3-3-[2][3][1] |
| $(Z_2)^3 \times Z_2$ | 5 | 5.10 | $G_{Es}$ | 3 | 0 | 0 | 2-2-[2][1] |
| $(Z_2)^3 \times Z_2$ | 5 | 5.10 | $G_{Es}$ | 4 | 0 | 0 | 1-1 |
| $(Z_2)^3 \times Z_3$ | 5 | 10 | $G_{Es}$ | 2 | 0 | 0 | 3-3-[2][3][1] |
| $(Z_2)^3 \times Z_3$ | 5 | 10 | $G_{Es}$ | 3 | 0 | 0 | 2-2-[2][1] |
| $(Z_2)^3 \times Z_4$ | 5 | 10 | $G_{Es}$ | 4 | 0 | 0 | 1-1 |
| $(Z_2)^4 \times Z_4$ | 4 | 4.8 | $G_{Es}$ | 3 | 1 | 1 | 2-2-[2][1] |
| $(Z_2)^4 \times Z_4$ | 4 | 4.8 | $G_{Es}$ | 3 | 0 | 0 | 2-2-[2][1] |
| $(Z_2)^4 \times Z_4$ | 4 | 4.8 | $G_{Es} \times Z_2$ | 2 | 0 | 0.1 | 3-3-[2][3][1] |
| $(Z_2)^4 \times Z_4$ | 4 | 8 | $G_{Es}$ | 3 | 1 | 1 | 2-1-[12] |
| $(Z_2)^4 \times Z_4$ | 4 | 8 | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $(Z_2)^4 \times Z_4$ | 4 | 8 | $G_{Es} \times Z_2$ | 2 | 0 | 0.1 | 3-2-[12][3] |
| $O \times Z_4$ | 4 | 12.24 | $G_{Es}$ | 3 | 1 | 1 | 2-1-[12] |
| $O \times Z_4$ | 4 | 12.24 | $G_{Es}$ | 2 | 0 | 0 | 3-1-[123] |
| $O \times Z_4$ | 4 | 12.24 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-1-[123] |
| $O \times Z_4$ | 4 | 12.24 | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $D_4 \times Z_2 \times Z_2$ | 4 | 4.2k | $G_{Es}$ | 3 | 1 | 1 | 2-1-[12] |
| $D_4 \times Z_2 \times Z_2$ | 4 | 4.2k | $G_{Es}$ | 2 | 0 | 0 | 3-1-[123] |
| $D_4 \times Z_2 \times Z_2$ | 4 | 4.2k | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $D_4 \times Z_2 \times Z_2$ | 4 | 4.2k | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-2-[12][3] |
| $(Z_2)^3 \times Z_4$ | 4 | 4.8 | $G_{Es}$ | 4 | 1 | 1 | 1-1 |
| $(Z_2)^3 \times Z_4$ | 4 | 4.8 | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $(Z_2)^3 \times Z_4$ | 4 | 4.8 | $G_{Es}$ | 4 | 0 | 0 | 1-1 |
| $(Z_2)^3 \times Z_4$ | 4 | 4.8 | $G_{Es} \times Z_2$ | 3 | 0 | 0.1 | 2-1-[1][3][2] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es}$ | 4 | 2 | 2 | 1-1 |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es}$ | 3 | 1 | 1 | 2-1-[12] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 3 | 1 | 1 | 2-1-[1][2] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es}$ | 2 | 0 | 0 | 3-1-[123] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-2-[12][3] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 3 | 0 | 1 | 2-1-[1][2] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 3 | 0 | 1 | 2-1-[1][2] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 3 | 0 | 0 | 3-2-[1][3] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 2-1-[1][2] |
| $(Z_2)^3 \times Z_4$ | 3 | 3.6 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-1-[1][3][2] |
| $(Z_2)^3 \times Z_4$ | 4 | 4.8 | $G_{Es} \times K$ | 2 | 0 | 0.1,2 | 3-2-[1][23]m’ |
| $D_4$ | 2 | 2 | $G_{Es}$ | 4 | 2 | 2 | 1-1 |
| $D_4$ | 2 | 2 | $G_{Es}$ | 3 | 1 | 1 | 2-1-[12] |
| $D_4$ | 2 | 2 | $G_{Es}$ | 4 | 2 | 2 | 1-1 |
| $D_4$ | 2 | 2 | $G_{Es} \times Z_2$ | 3 | 1 | 1.2 | 2-2-[1][2] |
| $D_4$ | 2 | 2 | $G_{Es}$ | 2 | 0 | 0 | 3-1-[123] |
| $D_4$ | 2 | 2 | $G_{Es}$ | 4 | 0 | 0 | 1-1 |
| $D_4$ | 2 | 2 | $G_{Es}$ | 3 | 0 | 0 | 2-1-[12] |
| $D_4$ | 2 | 2 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-2-[1][3] |
| $D_4$ | 2 | 2 | $G_{Es} \times Z_2$ | 3 | 0 | 1 | 2-2-[1][2] |
| $D_4$ | 2 | 2 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-3-[1][3][2] |
| $D_4$ | 2 | 2 | $G_{Es} \times Z_2$ | 2 | 0 | 0 | 3-2-[1][23]m’ |

Note: $N$ denotes the number of equilibria involved in a cycle. $d(\Gamma')$ stands for $\dim\text{Fix}(\Gamma')$ and $d(G)$ for $\dim\text{Fix}(G_{Es})$. Label 3-2-[1][23]m’ indicates cycles which are either of class 3-2-[1][23] or 3-2-[1][23]m.
is its subgroup, of index two, consisting of orientation-preserving elements. The symmetric group $S_n$ acts on $\mathbb{R}^n$ by permutation of coordinates, and also on its subspace isomorphic $\mathbb{R}^{n-1}$ in which the sum of all coordinates is zero.

**Lemma 4.** Consider a $\Gamma$-equivariant system (1)–(2) in the subspace of $\mathbb{R}^6$ in which the sum of all coordinates is zero (isomorphic to $\mathbb{R}^5$), where $\Gamma$ is one of the following: $S_6$, $A_6$, $S_6 \times \mathbb{Z}_2$ or $\mathbb{Z}_6 \times \mathbb{Z}_2$ for $Z_2 = \{-1\}$. Such a system cannot possess a simple homoclinic cycle.

**Proof.** We begin by showing that if $\Gamma = S_6$ or $\Gamma = A_6$, then the system can only possess a homoclinic cycle that is not simple. Suppose the system has an equilibrium $\xi_1 = (a,a,a,-a,-a,-a)$ with an unstable eigenspace $V_1 = (b,c,d,0,0,0)$ such that $b+c+d = 0$ and a stable eigenspace $V_2 = (0,0,0,e,f,g)$, $e + f + g = 0$. The symmetry $s : x \mapsto (x_1,x_5,x_6,x_1,x_2,x_3)$ maps $\xi_1$ to $\xi_2 = (-a,-a,-a,a,a,a)$. In the hyperplane $(b,c,d,a,a,a)$, where $b+c+d+3a = 0$, the steady state $\xi_1$ is unstable and $\xi_2$ is stable; therefore, the heteroclinic connection $\kappa_1 = W^u(\xi_1) \cap W^s(\xi_2)$ is structurally stable. The symmetry $s$ maps $V_1$ to $V_2$, where the connection $s\kappa_1 : \xi_2 \to \xi_1$ exists. Note that both $\kappa_1$ and $\kappa_2$ are two-dimensional manifolds, and therefore the cycle is not simple. This connection does not exist if $-I \in \Gamma$, because the symmetry $-I$ maps $\xi_1$ to $\xi_2$ and the hyperplane $(b,c,d,a,a,a)$ is $-I$-invariant. Hence the equilibria are simultaneously stable or unstable in this subspace.

To show that no other homoclinic cycles exist, we consider a fixed-point subspace of dimension not exceeding three for a group listed in the condition of the theorem and prove that no simple homoclinic cycles exist that connect equilibria of a given symmetry type. First note that two equilibria, $(a,-a,0,0,0,0)$ and $(0,0,a,-a,0,0)$, might be connected within the subspace $(b,-b,c,-c,0,0)$, but the existence of the invariant subspace $(b,-b,b,-b,0,0)$ separating the two equilibria prohibits this. Second, steady states $(a,a,b,b,b,b)$ and $(b,b,a,a,b,b)$, for $a + 2b = 0$, belong to the plane $(c,c,d,d,e,e)$, where $c + d + e = 0$. The plane is invariant under the transformation $x \mapsto (x_3,x_4,x_1,x_2,x_5,x_6)$ that interchanges the two steady states; hence within the plane they are simultaneously stable and unstable and no robust homoclinic connection from one steady state to another exists. Proofs on the non-existence of homoclinic connections between any other pair of steady states reduce to the two above arguments.

QED

**Definition 15.** Consider a system (1)–(2) possessing a robust $\gamma$-symmetric heteroclinic cycle $X = \{\xi_1, \ldots, \xi_m\}$ (i.e. $\gamma(\xi_j) = \xi_{j+k}$ holds for all $j$, $1 \leq j \leq m$, and some $k$ independent of $j$; for a homoclinic cycle, $k = 1$). The symmetry subgroup of $X$, $G(X)$, is the subgroup of $\Gamma$ generated by $\Sigma_j$, $1 \leq j \leq m$, and $\gamma$.

**Definition 16.** Consider a system (1)–(2) possessing a robust heteroclinic cycle $X = \{\xi_1, \ldots, \xi_m\}$. The essential subspace of $X$, $V_{\text{Ess}}(X)$, is the smallest $G(X)$-invariant subspace of $\mathbb{R}^n$ which contains all contracting and expanding eigenvectors of the equilibria involved in the cycle.

**Definition 17.** Consider a $\Gamma$-equivariant system (1)–(2) possessing a robust $\gamma$-symmetric heteroclinic cycle $X = \{\xi_1, \ldots, \xi_m\}$. The essential subgroup of $X$, $G_{\text{Ess}}(X)$, is the group $G(X)/\Sigma_j\gamma(\xi)$.

**Lemma 5.** Suppose a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^5$ possesses a simple homoclinic cycle $X$ with an equilibrium $\xi$ and $T$ is the isotropy group of $\xi$. Consider the isotypic decomposition of $\mathbb{R}^5$ under the action of $T$:

$$\mathbb{R}^5 = W_1 \oplus \cdots \oplus W_K.$$
Suppose $W_1$ is the isotypic component in which $T$ acts trivially, $W_2$ contains the contracting eigenvector and $W_3$ contains the expanding eigenvector. Then $\dim W_2 = \dim W_3$ and $\dim W_i \leq 2$ for any $2 \leq k \leq K$.

The proof is similar to that of lemma 2 and is not presented. It is based on the fact that for an equilibrium involved in a homoclinic network the numbers of incoming and outgoing trajectories are equal.

**Lemma 6.** Suppose a $\Gamma$-equivariant system (1)–(2) in $\mathbb{R}^5$ possesses a simple homoclinic cycle which does not belong to any proper subspace of $\mathbb{R}^5$. Suppose $\Gamma \subset \tilde{W}$, $\Gamma \not\subset O(4) \times O(1)$ and $\Gamma \not\subset O(3) \times O(2)$. Then

- $\Gamma = (\mathbb{Z}_2)^5 \rtimes \mathbb{Z}_5$ or $\Gamma = (\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_5$;
- $\dim P = 2, 3$ or $4$ and $\Sigma = (\mathbb{Z}_2)^3$, $(\mathbb{Z}_2)^5$ or $\mathbb{Z}_2$.

**Proof.** Since $\Gamma$ is a subgroup of $\tilde{W} = S_5 \rtimes (\mathbb{Z}_2)^5$, we have $\Gamma = G \rtimes \mathcal{R}$, where $R \subset (\mathbb{Z}_2)^5$ is the subgroup of $\Gamma$ comprised of reflections and $G = \Gamma/R$. For the sake of simplicity we assume that all symmetries in $G$ are just coordinate permutations. If this is not the case, the proof is similar and is not presented.

Since the group $\Gamma$ is not a subgroup of $O(4) \times O(1)$ or $O(3) \times O(2)$, it has a fifth-order element which is a cyclic permutation of coordinates. Without any loss of generality we assume that it is the permutation $x \mapsto (x_2, x_3, x_4, x_5, x_1)$.

Suppose $\dim P = 2$. We begin by showing that the equilibria involved in the cycle are located on coordinate axes. Let us assume the converse. An equilibrium belongs to an one-dimensional subspace of fixed points for a subgroup of $\Gamma$. Such subspaces are

$$(a, 0, 0, 0, 0), \ (a, a, 0, 0, 0), \ (a, a, a, 0, 0) \text{ or } (a, a, a, a, a).$$

We need to find $\Gamma = G \rtimes R \subset S_5 \rtimes (\mathbb{Z}_2)^5$ for which such a subspace (not of the first type) is a fixed-point subspace of a subgroup of $\Gamma$. The permutation $x \mapsto (x_2, x_3, x_4, x_5, x_1)$ generates $\mathbb{Z}_5 = \langle (1, 2, 3, 4, 5) \rangle$. By our assumption, the permutation belongs to $\Gamma$; hence $G$, a subgroup of $S_5$, contains $\mathbb{Z}_5$. Only five such subgroups exist:

$$\mathbb{Z}_5, \ D_5 = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4) \rangle, \ GA(1,5) = \langle (1, 2, 3, 4, 5), (1, 2, 3, 4) \rangle, \ A_5 = \langle (1, 2, 3, 4, 5), (1, 2, 3) \rangle \text{ and } S_5.$$

The subspace $(a, a, 0, 0, 0)$ is invariant, if $G = D_5$ or $G = S_5$. If $G = D_5$, then the system does not have an invariant plane $P$ that contains $(a, a, 0, 0, 0)$ and its symmetric copy. If $G = S_5$, then by lemma 5 an equilibrium $(a, a, 0, 0, 0)$ is not involved in a homoclinic cycle. Similarly, lemma 5 prohibits homoclinic cycles involving $(a, a, a, 0, 0), (a, a, a, a, 0)$ or $(a, a, a, a, a)$.

Thus we have established that the equilibria involved in the cycle are located on coordinate axes. The twist $\gamma$ is a cyclic permutation of the coordinates by the assumption that the cycle does not belong to any proper subspace of $\mathbb{R}^5$. The group $\Sigma$ acts trivially on the coordinate plane $P$ and non-trivially on $P^\perp \cong \mathbb{R}^5$. If $\Sigma$ contains a symmetry $r$ that acts on $P^\perp$ as $-I$, then $R = (\mathbb{Z}_2)^5$, because $\gamma^k r \gamma^{-k}$ for $1 \leq k \leq 5$ generate all reflections in $\mathbb{R}^5$. In this case $\Sigma = (\mathbb{Z}_2)^5$. If $\Sigma$ does not contain $-I$, then $\Sigma = (\mathbb{Z}_2)^3$ is generated by symmetries changing signs of two coordinates, and $R = (\mathbb{Z}_2)^4$ is comprised of symmetries changing signs of an even number of coordinates.

Thus it remains to show that $G = \mathbb{Z}_5$. Assume the converse, i.e. that $\mathbb{Z}_5$ is a proper subgroup of $G$. Only four subgroups of $S_5$ contain $\mathbb{Z}_5$, they are $D_5$, GA(1,5), $A_5$ and $S_5$.

By lemma 5, the group $G$ is not GA(1,5), $A_5$ or $S_5$, because in these cases $T$, the isotypy subgroup of $\xi$, contains an element permuting three or four coordinates. The group $G$ is not $D_5$,
because the line $x_3 = x_4$ in the coordinate plane $(0, 0, x_3, x_4, 0)$ is invariant for the permutation \((2,5)(3,4)\); this prohibits a connection from \((0, 0, a, 0, 0)\) to \((0, 0, a, 0, 0)\). Therefore, $G = \mathbb{Z}_5$ and for dim $P = 2$ the lemma is proved.

For $\Gamma = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_4$ or $\Gamma = (\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_5$, a $\Gamma$-equivariant system can possess homoclinic cycles with dim $P = 3$ or dim $P = 4$. A cycle with dim $P = 3$ involves the equilibria on coordinate planes, e.g. $\xi = (a, b, 0, 0, 0)$ and its $\gamma$-symmetric copies, the connection from $\xi$ to $\gamma \xi$ belongs to the hyperplane $P = (d, e, f, 0, 0)$ and $\Sigma$ is either $\mathbb{Z}_2$ or $(\mathbb{Z}_2)^2$. A cycle with dim $P = 4$ involves the equilibria on coordinate hyperplanes, e.g. $\xi = (a, b, c, 0, 0)$ and its $\gamma$-symmetric copies, the connection from $\xi$ to $\gamma \xi$ belongs to the hyperplane $P = (d, e, f, g, 0)$ and $\Sigma = \mathbb{Z}_2$. The proof that $\Gamma$ is not any other subgroup of $W$ is similar to the one presented for dim $P = 2$ and is omitted. QED

Theorem 2 and lemmas 4 and 6 enable us to find a complete list of subgroups $\Gamma \subset O(5)$ such that a $\Gamma$-equivariant system (1)–(2) can possess a simple homoclinic cycle $X$. Theorem 2 states that any finite subgroup of $O(5)$ is either one of those considered in lemmas 4 and 6, or it is a subgroup of $O(4) \times O(1)$ or $O(3) \times O(2)$.

In the latter cases the homoclinic cycle $X$ in $\mathbb{R}^5$ belongs to a subspace $N \cong \mathbb{R}^3$ or $\mathbb{R}^4$. We denote by $\hat{X}$ this cycle in the restriction of the dynamical system into the respective subspace, by $\hat{\Gamma}$ the symmetry group of the restricted system, by $\hat{P}$ the fixed point subspace containing the homoclinic connection in $\hat{X}$ and by $\hat{\Sigma}$ the isotropy subgroup of the subspace $\hat{P}$. By definition 17 of the essential subgroup, $G_{\text{Ess}}(X) = \hat{\Gamma}$. Type $Z$ and type $A'$ cycles are defined in terms of the isotypic decomposition of $\hat{P}^\perp$ under $\hat{\Sigma}$. For cycles in $\mathbb{R}^3$ and $\mathbb{R}^4$, the decomposition involves one or two isotypic components,

$$\hat{P}^\perp = \hat{U}_1 \quad \text{or} \quad \hat{P}^\perp = \hat{U}_1 \oplus \hat{U}_2.$$  

If the decomposition has a single component, then the cycle is of type $A'$, otherwise it is of type $Z$. If the auxiliary subspace $\mathbb{R}^5 \odot N$ is $\mathbb{R}$, then either it increases the dimension of $\hat{P}$ or of an existing $\hat{U}_r$, or it constitutes a new isotypic component in the decomposition of $\hat{P}^\perp$. If $\mathbb{R}^5 \odot N = \mathbb{R} \oplus \mathbb{R}$, then for any $\mathbb{R}$ in the direct sum the possibilities are the same.

For the cycles listed in table 1, dim $\hat{P} = 2$. Note that a system in $\mathbb{R}^4$, whose symmetry group is $\Gamma = (\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_4$, can also possess a homoclinic cycle for which dim $\hat{P} = 3$ and $\hat{\Sigma} = \mathbb{Z}_2$. No cycles exist with dim $\hat{P} = 3$ in a dynamical system that has any other symmetry group listed in this table.

Properties of homoclinic cycles are summarized in table 2. We introduce classes of cycles (see the last column of table 2) in the next section in order to derive conditions for the asymptotic stability of the cycles (see table 4). The first six lines in table 2 describe genuinely 5-dimensional homoclinic cycles in dynamical systems, whose groups of symmetries are not subgroups of $O(4) \times O(1)$ and $O(3) \times O(2)$; they are categorized by lemma 6.

Suppose now $\Gamma \subset O(4) \times O(1)$, but $\Gamma \nsubseteq O(3) \times O(2)$. Then $X$ belongs to a subspace $N$ of $\mathbb{R}^5$ isomorphic to $\mathbb{R}^4$, as discussed above. The group $\Gamma$ can be either $\hat{\Gamma}$ or $\hat{\Gamma} \times \mathbb{Z}_2$, where $\hat{\Gamma}$ is a symmetry group of a dynamical system in $\mathbb{R}^4$ and $\mathbb{Z}_2$ acts on $\mathbb{R}$ as $-I$. The number of equilibria involved into $X$ is the same as in the original cycle $\hat{X}$, dim Fix($V_{\text{Ess}}(X)$) = dim Fix($V_{\text{Ess}}(\hat{X})$) and $G_{\text{Ess}}(X) = \hat{\Gamma}$. Other quantities shown in the table depend on how $\hat{\Gamma}$ acts on $\mathbb{R} = \mathbb{R}^4 \odot N$.

Table 1 shows four $\hat{\Gamma}$’s acting on $\mathbb{R}^4$ for which homoclinic cycles can exist. For three of them, $\hat{\Sigma} = \mathbb{Z}_2 = \langle \sigma \rangle$, and for one $\hat{\Sigma} = (\mathbb{Z}_2)^2$. If $\Gamma = \hat{\Gamma}$ and it acts trivially on $\mathbb{R}$, then dim Fix($\Gamma$) = 1, dim $P = \dim \hat{P} + 1$, $\Sigma = \hat{\Sigma}$ and $X$ belongs to the same class as $\hat{X}$. If $\Gamma = \hat{\Gamma}$ and it acts non-trivially on $\mathbb{R}$, or if $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$, then dim Fix($\Gamma$) = 0 and we have:

(i) if $\Gamma = \hat{\Gamma}$ and $\sigma$ acts on $\mathbb{R}$ as $-I$ (possible if $\hat{\Sigma} = \mathbb{Z}_2$), then dim $P = \dim \hat{P}$ and the cycle remains of type $A'$;
(ii) if $\Gamma = \hat{\Gamma}$, $\Sigma$ acts on $\mathbb{R}$ as $I$, $\gamma$ as $-I$ (possible if $\gamma$ is of an even order), then $
abla \Gamma = \dim \nabla \Gamma + 1$ and the cycle $X$ belongs to the same class as $\hat{X}$;

(iii) if $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$, then $\dim \nabla \Gamma = \dim \nabla \Gamma$, $\Sigma = \hat{\Sigma} \times \mathbb{Z}_2$ and the isotypic decomposition of $\nabla \Gamma$ involves one component more, than the one of $\hat{\nabla} \Gamma$ (e.g. if $\hat{X}$ belongs to the class 2-1-[12], then $X$ belongs to 3-2-[12](3)).

Suppose now $\Gamma \subset \Omega(3) \times \Omega(2)$. Then $X$ belongs to a subspace of $\mathbb{R}^3$, $N$, that is isomorphic to $\mathbb{R}^3$. The group $\Gamma$ can be either $\hat{\Gamma}$, $\hat{\Gamma} \times \mathbb{Z}_2$ or $\hat{\Gamma} \times \K$, where $\hat{\Gamma}$ is a symmetry group of a dynamical system in $\mathbb{R}^3$, $\mathbb{Z}_2 \subset \Omega(1)$ and $\K \subset \Omega(2)$, $\K \not\subset \Omega(1)$. There are two distinct homoclinic cycles in $\mathbb{R}^3$, for both of them $\hat{\Sigma} = \mathbb{Z}_2 = \langle \sigma \rangle$ (see table 1).

If $\Gamma = \hat{\Gamma}$ and it acts trivially on $\mathbb{R}^2 = \mathbb{R}^2 \oplus N$, then $\dim \text{Fix}(\Gamma) = 2$, $\dim \nabla \Gamma = \dim \nabla \Gamma + 2$, $\Sigma = \hat{\Sigma}$ and $X$ belongs to the 1-1 class. If $\Gamma = \hat{\Gamma}$ or $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$ acting trivially on one of $\mathbb{R}$ comprising $\mathbb{R}^2$, then depending on how $\gamma$ acts on the remaining $\mathbb{R}$, one of the cases (i)–(iii) takes place (however, in the cases (i) and (iii) $\dim \nabla \Gamma = \dim \nabla \Gamma + 1$ and in the case (ii) $\dim \nabla \Gamma = \dim \nabla \Gamma + 2$).

If $\Gamma$ acts non-trivially in $\mathbb{R}^2$ or $\Gamma = \hat{\Gamma} \times \K$ (the two conditions are not mutually exclusive), then $\dim \text{Fix}(\Gamma) = 0$ and exactly one of the following statements holds true:

(iv) if $\Gamma = \hat{\Gamma}$ and $\sigma$ acts on $\mathbb{R}^2$ as $-I$, then $\dim \nabla \Gamma = 2$ and the cycle $X$ belongs to the class 3-1-[123];

(v) if $\Gamma = \hat{\Gamma}$, $\Sigma$ acts on $\mathbb{R}^2$ as $I$, $\gamma$ as $-I$ (possible if $\gamma$ is of an even order), then $\dim \nabla \Gamma = 4$ and the cycle $X$ belongs to the class 2-1-

(vi) if $\Gamma = \hat{\Gamma}$ and $\Gamma$ acts of $\mathbb{R}^2$ as $\mathbb{D}_k$ (only $k = 2, 3, 4$ are possible), then $\dim \nabla \Gamma = 3$ and the cycle $X$ belongs to the class 2-1-[12];

(vii) if $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$, $\mathbb{Z}_2$ acts on one $\mathbb{R}$ as $I$ and on the other $\mathbb{R}$ as $-I$, and $\sigma$ acts on $\mathbb{R}^2$ as $-I$, then $\dim \nabla \Gamma = 2$ and the cycle $X$ belongs to the class 3-2-[12](3);

(viii) if $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$, $\mathbb{Z}_2$ acts on one $\mathbb{R}$ as $I$ and on the other $\mathbb{R}$ as $-I$, and $\sigma$ acts on one $\mathbb{R}$ as $I$ and on the other as $-I$, then $\dim \nabla \Gamma = 3$ and the cycle $X$ belongs to the class 2-1-[12][2];

(ix) if $\Gamma = \hat{\Gamma} \times \mathbb{Z}_2$, $\mathbb{Z}_2$ acts on one $\mathbb{R}$ as $I$ and on the other $\mathbb{R}$ as $-I$, and $\gamma$ permutes the two $\mathbb{R}$ (possible if $\gamma$ is of the second order), then $\dim \nabla \Gamma = 2$ and the cycle belongs to the class 3-3-[1][2][3][2];

(x) if $\Gamma = \hat{\Gamma} \times \K$, $\K = \mathbb{Z}_r$ or $\mathbb{D}_r$ with $r > 2$, then $\dim \nabla \Gamma = 2$ and the cycle belongs to a class 3-2-[1][23] or 3-2-[1][23][m], which are discussed in appendix A. Note that the conditions for asymptotic stability of these cycles are identical.

5. Stability

In this section we derive necessary and sufficient conditions for the asymptotic stability of homoclinic cycles in $\mathbb{R}^3$. The classification of homoclinic cycles introduced in section 5.1 is applied in section 5.2 to determine the general form of the Poincaré map for each class in $\mathbb{R}^3$.

Lemma 7 proven in section 5.1 is employed to find all classes of homoclinic cycles in $\mathbb{R}^3$. The conditions for stability are presented in section 5.4. Some of them rely on the known results that are stated in section 5.3, others are derived in appendices B and C.

5.1. Classification of homoclinic cycles in $\mathbb{R}^n$

By definition 8, a homoclinic cycle is structurally stable, if there exists a subgroup $\Sigma \subset \Gamma$ such that the connection from $\xi$ to $\gamma \xi$ belongs to a fixed-point subspace $P = \text{Fix}(\Sigma)$. We introduce a classification of simple homoclinic cycles in $\mathbb{R}^n$ that is based on the actions of $\Sigma$ and $\gamma$ (note that $\gamma \notin \Sigma$) on $P^\perp$. The classification suffices to determine conditions for stability.
of homoclinic cycles in $\mathbb{R}^5$—for each class, they have the form of inequalities for eigenvalues of the linearization $df(\xi)$ (see table 4).

Let $e_k$, $1 \leq k \leq K$, be a basis in $P^\perp$ comprised of eigenvectors of $df(\xi)$, and let $h_k$, $1 \leq k \leq K$, be a basis in $P^+$ comprised of eigenvectors of $df(\gamma \xi)$. We assume that $e_1$ is the contracting eigenvector of $df(\xi)$, $h_1$ is the expanding eigenvector of $df(\gamma \xi)$ and the remaining transverse eigenvectors are related: $h_k = \gamma e_k$, $2 \leq k \leq K$. Suppose

$$P^\perp = U_1 \oplus \ldots \oplus U_J$$

is the isotypic decomposition of $P^\perp$ under the action of $\Sigma$. Let the eigenvectors in the basis be ordered in such a way that the $h_k$, belonging to one isotypic component, $U_j$, (whose dimension we denote by $l_j$), have consecutive indices, $k = s + 1, \ldots, s + l_j$; namely, $U_1$ is spanned by $h_1, \ldots, h_{l_1}$, $U_2$ by $h_{l_1+1}, \ldots, h_{l_1+l_2}$, etc. Each isotypic component $U_j$ is also spanned by the eigenvectors $e_k$ for some $k = i_{s_{j-1}+1}, i_{s_{j-1}+2}, \ldots, i_s$, where we have denoted $s_j = l_1 + \ldots + l_j$. We label such a homoclinic cycle by a sequence of numbers, where subsequences associated with individual isotypic components are enclosed in square brackets:

$$[i_1, i_2, \ldots, i_s][i_{t_1+1}, i_{t_2+2}, \ldots, i_{t_2}][i_{s_{j-1}+1}, i_{s_{j-1}+2}, \ldots, i_{s_j}].$$

However, not all possible permutations and combinations of brackets are encountered in a homoclinic cycle, as shown in the following lemma.

**Lemma 7.** Let $X$ be a simple robust homoclinic cycle in a system (1)–(2). Consider the isotypic decomposition of $P^\perp$ under the action of $\Sigma$,

$$P^\perp = U_1 \oplus \cdots \oplus U_J.$$  \hspace{1cm} (16)

Recall that for our ordering of eigenvectors $h_1 \in U_1$. Denote by $U$ the isotypic component in (16) that contains $e_1$. Then

(i) $\dim U_1 = \dim U$.

(ii) $\gamma e_j \notin U_1$ for any $e_j \notin U$.

**Proof.**

(i) Write $S = \Sigma \cap \gamma \Sigma^{-1}$ and $Q = P^\perp \cap \gamma P^\perp$. Consider the isotypic decomposition of the subspace $Q$ under the action of $S$,

$$Q = V_1 \oplus \cdots \oplus V_{l_1},$$

where $V_1$ is the isotypic component in which $S$ acts trivially. The isotypic decomposition of $\gamma P^\perp$ under the action of $\gamma \Sigma \gamma^{-1}$ is

$$\gamma P^\perp = \gamma U_1 \oplus \cdots \oplus \gamma U_J.$$  \hspace{1cm} (15)

By the definition of simple homoclinic cycles, $P^\perp = Q \oplus \langle h_1 \rangle$ and $\gamma P^\perp = Q \oplus \langle \gamma e_1 \rangle$. The group $S$ acts trivially on $\langle h_1 \rangle$ and $\langle \gamma e_1 \rangle$. Therefore, $U_1 = V_1 \oplus \langle h_1 \rangle$ and $\gamma U = V_1 \oplus \langle \gamma e_1 \rangle$; this implies $\dim U_1 = \dim V_1 + 1 = \dim U$.

(ii) As we have found, $U = \gamma^{-1} V_1 \oplus \langle e_1 \rangle$. Therefore, the condition of the lemma implies $e_j \notin \gamma^{-1} V_1$, and hence $\gamma e_j \notin V_1$. Since $\gamma e_j \in Q$ and $Q \perp \langle h_1 \rangle$, $\gamma e_j \notin V_1 \oplus \langle h_1 \rangle = U_1$. QED

All classes of homoclinic cycles in $\mathbb{R}^5$ are listed in table 3. The sequences defined above are supplemented by two numbers: the dimension of $P^\perp$ and the number of isotypic components (e.g. 3-1-[123] labels a cycle with a three-dimensional $P^\perp$ comprised of a single isotypic component).
5.2. Poincaré maps for homoclinic cycles in $\mathbb{R}^5$

Following [15, 19, 26], in order to examine the stability of a homoclinic cycle we consider a Poincaré map near the cycle. In section 2.2 we have defined radial, contracting, expanding and transverse eigenvalues of the linearization $df(\xi)$. Let $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ be coordinates in the coordinate system with the origin at $\xi$ and the basis comprised of the associated eigenvectors in the following order: radial, contracting, expanding and transverse.

If $\delta$ is small, in a $\delta$-neighbourhood of $\xi$ the system (1) can be approximated by the linear system

\begin{align}
\dot{u}_l &= -r_l u_l, & 1 \leq l \leq n_r, \\
\dot{v} &= -c v, \\
\dot{w} &= e w, \\
\dot{z}_l &= t_l z_l, & 1 \leq l \leq n_t.
\end{align}

Here, $(u, v, w, z)$ denote the scaled coordinates $(u, v, w, z) = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})/\delta$.

Let $(u_0, v_0, w_0, z_0)$ be the point in $\gamma^{-1}P$ where the trajectory $\gamma^{-1}\kappa$ intersects with the sphere $|u|^2 + v^2 = 1$, and $q$ be local coordinates in the hyperplane tangent to the sphere at the point $(u_0, v_0)$. We consider two crossections:

\[{\tilde{H}}^{\text{out}} = \{(u, v, w, z) : |u|, |v|, |z| \leq 1, w = 1\}\]

and

\[{\tilde{H}}^{\text{in}} = \{(q, w, z) : |q|, |w|, |z| \leq 1\}\].

Near $\xi$, trajectories of system (1) can be approximated by a local map (called the first return map) $\tilde{\phi} : {\tilde{H}}^{\text{in}} \to {\tilde{H}}^{\text{out}}$ that associates a point, where a trajectory crosses $\gamma^{-1}\kappa$ with the point, where the trajectory crossed $\gamma^{-1}\kappa$. The global map $\gamma : \gamma^{-1}\kappa \to \gamma^{-1}\kappa$ associates a point where a trajectory crosses $\gamma^{-1}\kappa$ with the point where its previously crossed $\gamma^{-1}\kappa$. The Poincaré map is the superposition $\tilde{\psi} = \tilde{\psi}\tilde{\phi}$. The $w$- and $z$-components of the map $\tilde{\psi}$ are independent of $q$: this was shown in [25, 26] for slightly different systems, but the proof can be trivially modified to serve the case considered here. Thus, one can define the map $g(w, z)$ that is the restriction of the map $\tilde{\psi}$ into the $(w, z)$ subspace. The stability properties of fixed points of the maps $\tilde{\psi}$ and $g$ are identical; hence, the stability of a cycle is determined by the stability of the fixed point $(w, z) = 0$ of the map $g$.

Denote by $\phi$ and $\psi$ the restrictions of $\tilde{\psi}$ and $\tilde{\phi}$ into $P^\perp$. In the leading order, the map $\phi$ is

\[\phi(w, \{z_i\}) = (v_0 w^{e/i}, \{z_i|w|^{1-\epsilon/i}\})\]

(we use the coordinates $(u, z)$ in $H^{\text{in}}$ and $(u, z)$ in $H^{\text{out}}$). Note that the local map is expressed by (18) for any homoclinic cycle of whichever class.

Expressions for global maps are different for different classes of homoclinic cycles. The conditions of lemma 8 (see appendix A) are satisfied for all simple homoclinic cycles in $\mathbb{R}^5$

\[2\text{ We assume here that all eigenvalues are real. The system under consideration can have a pair of complex conjugate radial eigenvalues, if the dimension of the radial eigenspace is larger than one, or it can have a pair of complex conjugate transverse eigenvalues, if the homoclinic cycle is of the classes 3-1-[123] or 3-2-[1][23]. The radial eigenvalues are not relevant in the study of stability. If transverse eigenvalues are complex, the estimates}

\[k_1(|z_1| + |z_2|)|w|^{-\epsilon/t} \leq |z_1| \leq K_1(|z_1| + |z_2|)|w|^{-\epsilon/t}, \quad k_2(|z_1| + |z_2|)|w|^{-\epsilon/t} \leq |z_2| \leq K_2(|z_1| + |z_2|)|w|^{-\epsilon/t},\]

where $t = \text{Re}(t_1) = \text{Re}(t_2)$, can be employed in the proofs of stability and instability, respectively, instead of the exact expressions

\[z_1 = z_1|w|^{-\epsilon/t}, \quad z_2 = z_2|w|^{-\epsilon/t}.\]

Since only the values of exponents are important in the proofs, in the conditions for stability (see table 4) $t$ is replaced by $\text{Re}(t_1)$, and no other modifications are required.
but for the 3-2-[1][23] cycles. We do not consider the 3-2-[1][23] cycles henceforth in this section; the global map for them is derived in appendix A. By the lemma, each isotypic component of \( P^\perp \) is an invariant subspace of the map \( \psi \).

Generically, in the leading order the global map \( \psi \) is linear in each isotypic component. The matrix, \( C \), of the linear map

\[
(w^\xi, z^\xi) = \psi(v^\xi, z^\xi) = C \begin{pmatrix} v^\xi \\ z^\xi \end{pmatrix},
\]

(here superscripts indicate whether the components are in the basis of eigenvectors of \( df(\xi) \) or of \( df(y^\xi) \)) is the product

\[
C = BA,
\]

where \( A \) is the matrix of the map \( \psi \) in the basis of eigenvectors of \( df(\xi) \) and \( B \) is the matrix of the transformation of \( (v^\xi, z^\xi) \) into \( (w^\xi, z^\xi) \).

In the study of stability, we focus on the location of blocks in \( C = \{c_{ij}\} \) that vanish because the map \( \psi \) has invariant subspaces. Generically, \( c_{ij} \neq 0 \), if \( e_i \) and \( h_j \) belong to the same isotypic component in the decomposition \((15)\). Hence, the location of zero entries of \( C \) can be determined applying the classification presented in section 5.1. Below we list exhaustively the possible forms of matrices \( C \) (where non-zero entries are shown by *), for various types of homoclinic cycles in \( \mathbb{R}^5 \), and determine the general forms of Poincaré maps (see table 3).

If the subspace \( P^\perp \) is one-dimensional, then \( C \) is an \( 1 \times 1 \) matrix and the Poincaré map is just \( g(w) = c_{11} w^{\xi/e} \).

If the subspace \( P^\perp \) is two-dimensional, then the classification of simple homoclinic cycles in \( \mathbb{R}^5 \) is applicable, since for such cycles \( P^\perp \) is two-dimensional. Alternatively, note that such cycles are either of type \( A' \) (if the decomposition of \( P^\perp \) under \( \Sigma \) has only one isotypic component) or \( Z \) (if there are two components). In the former case the cycle is classified as 2-1-[12], generically none of the entries of its matrix \( C \) vanish, and thus the Poincaré map is

\[
g(w, z) = (c_{11} w^{\xi/e} + c_{12} z|w|^{-\xi/e}, c_{21} w^{\xi/e} + c_{22} z|w|^{-\xi/e}).
\]

In the latter case the cycles are of the 2-2-[1][2] or 2-2-[2][1] classes, the matrices of the global map \( \psi \) are

\[
\begin{pmatrix}
* & 0 \\
0 & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & * \\
* & 0
\end{pmatrix}
\]

\[(20)\]

and the Poincaré maps are \( g(w, z) = (c_{11} w^{\xi/e}, c_{22} z|w|^{-\xi/e}) \) (for the 2-2-[1][2] cycle) and \( g(w, z) = (c_{12} z|w|^{-\xi/e}, c_{21} w^{\xi/e}) \) (for the 2-2-[2][1] cycle).

If the subspace \( P^\perp \) is three-dimensional, then homoclinic cycles are of types \( A' \) (if the decomposition \((15)\) involves just one isotypic component), \( Z \) (if the decomposition involves three components), or of other types not studied so far. For the type \( A' \) cycle listed in tables 1 and 2 as 3-1-[123], generically all entries of \( C \) are non-zero. For type \( Z \) cycles, \( A \) is a diagonal matrix and \( B \) is a permutation matrix (provided vectors \( e_j \) and \( h_j \) in the bases are normalised) and hence \( C \) has one of the following forms:

\[
\begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
* & 0 & 0 \\
0 & 0 & * \\
0 & * & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & * & 0 \\
0 & 0 & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & * \\
0 & * & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & * & 0 \\
0 & 0 & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & * & 0 \\
0 & 0 & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & *
\end{pmatrix}
\]

\[
(21)\]

Now suppose decomposition \((15)\) involves two isotypic components. Then the cycles are either 3-2-[-][·][·], or 3-2-[-][·][·], where dots stand for a permutation of indices \{1, 2, 3\}. Five permutations are possible: \{1\}[23], \{2\}[13], \{12\}[3], \{13\}[2] and \{23\}[1]. However, by lemma 7 only two of them are realized in homoclinic cycles. The matrices of global maps for these
classification and stability of simple homoclinic cycles in $\mathbb{R}^5$

Table 3. Poincaré maps for different classes of homoclinic cycles in $\mathbb{R}^5$. The final column indicates cycles of types A or Z. $a_j$ are the ratios (23) of eigenvalues of the linearization.

| Class | Poincaré map | Type |
|-------|--------------|------|
| 1-1-[1] | $g(w) = (c_{11} w^{a_1})$ | \(A', Z\) |
| 2-1-[12] | $g(w, z) = (c_{11} w^{a_1} + c_{12} z^{a_2}, c_{21} w^{a_1} + c_{22} z^{a_2})$ | \(A'\) |
| 2-1-[1][2] | $g(w, z) = (c_{11} w^{a_1}, c_{22} z^{a_2})$ | \(Z\) |
| 2-2-[2][1] | $g(w, z) = (c_{12} w^{a_2}, c_{21} w^{a_1})$ | \(Z\) |
| 3-1-[123] | $g(w, z, \xi) = (c_{11} w^{a_1} + c_{12} z^{a_2} + c_{13} \xi^{a_3}, c_{21} w^{a_1} + c_{31} z^{a_2} + c_{32} \xi^{a_3})$ | \(A'\) |
| 3-2-[12][3] | $g(w, z, \xi) = (c_{11} w^{a_1} + c_{12} z^{a_2} + c_{13} \xi^{a_3}, c_{21} w^{a_1} + c_{31} z^{a_2} + c_{32} \xi^{a_3})$ | \(Z\) |
| 3-2-[1][2][3] | $g(w, z, \xi) = (c_{11} w^{a_1} + c_{22} z^{a_2} + c_{31} \xi^{a_3}, c_{32} z^{a_2} + c_{33} \xi^{a_3})$ | \(Z\) |
| 3-2-[1][3][2] | $g(w, z, \xi) = (c_{11} w^{a_1}, c_{22} z^{a_2} + c_{31} \xi^{a_3}, c_{32} z^{a_2} + c_{33} \xi^{a_3})$ | \(Z\) |
| 3-2-[2][1][3] | $g(w, z, \xi) = (c_{12} z^{a_2}, c_{21} w^{a_1} + c_{32} \xi^{a_3}, c_{33} \xi^{a_3})$ | \(Z\) |
| 3-2-[3][1][2] | $g(w, z, \xi) = (c_{12} z^{a_2}, c_{23} z^{a_2} + c_{33} \xi^{a_3}, c_{13} \xi^{a_3})$ | \(Z\) |

Homoclinic cycles have the following forms:

\[
\begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & * & 0
\end{pmatrix}
\] (22)

Poincaré maps for these cycles are listed in table 3. The exponents $a_j$, $j = 1, 2, 3$, can be expressed in terms of eigenvalues of the linearization $df(\xi)$ by the relations

\[ a_1 = c/e, \; a_2 = -t_1/e \; (\text{or } -t_1/e \; \text{if } \dim P^{\perp} = 1), \; a_3 = -t_2/e. \] (23)

5.3. Stability of cycles of types A and Z

In this section we review conditions for the asymptotic stability of cycles of types A [17, 18] and Z [25]. Following the long-established tradition, we use the expression ‘asymptotic stability of homoclinic cycles’, when in fact we discuss the asymptotic stability of a homoclinic network. Note that a homoclinic cycle can never be asymptotically stable, as discussed in section 2.5 of [25], while the respective homoclinic network can.

Theorem 3 ([17, 18], adapted for homoclinic cycles). Let $-c$, $e$ and $t_j$, $1 \leq j \leq J$, be the contracting, expanding and transverse eigenvalues of $df(\xi)$ for the steady states $\xi$ involved in a homoclinic cycle of type A.

(a) If $c > e$ and $t_j < 0$ for all $1 \leq j \leq J$, then the cycle is asymptotically stable.
(b) If $c < e$ or $t_j > 0$ for some $j$ then the cycle is completely unstable.

Stability of the fixed point $(w, z) = 0$ of the map $g$ associated with a type $Z$ homoclinic cycle was studied in [25] by considering the map in the coordinates

\[ \eta = (\ln |w|, \ln |z_1|, \ldots, \ln |z_n|), \] (24)

in which the map is linear:

\[ g\eta = M\eta + F. \] (25)
Here

\[
M = B \begin{pmatrix}
a_1 & 0 & 0 & \ldots & 0 \\
a_2 & 1 & 0 & \ldots & 0 \\
a_3 & 0 & 1 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
a_N & 0 & 0 & \ldots & 1
\end{pmatrix}
\]  

(26)

is the transition matrix of the map \( g \), \( B \) is a permutation matrix (see section 5.2), and the entries \( a_j \) in the matrix of the local map are

\[
a_1 = c/e \quad \text{and} \quad a_{j+1} = -t_j/e, \quad 1 \leq j \leq J.
\]  

(27)

Any permutation is a superposition of cyclic permutations. We assume that vectors \( e_j \) in the basis are ordered in such a way that the first \( n_s \) vectors (the subscript \( s \) stands for significant) are involved in a single cyclic permutation and \( e_1 \) is the contracting eigenvector of \( df(\xi) \). Matrix \( B \) permutes the vectors: \( e_1 \to e_2 \to \cdots \to e_{n_s} \to e_1 \). Any eigenvalue of the upper left \( n_s \times n_s \) submatrix of \( M \) is also an eigenvalue of \( M \), because the upper right \( (J - n_s) \times n_s \) submatrix of \( M \) vanishes. These eigenvalues are called significant; generically they differ from one in absolute value. All other eigenvalues of \( M \) are one in absolute value. Denote by \( \lambda_{\max} \) the largest in absolute value significant eigenvalue of \( M \) and by \( v_{\max} \) the associated eigenvector. Conditions for the asymptotic and fragmentary asymptotic stability of type Z cycles in terms of \( \lambda_{\max} \) and components of \( v_{\max} \) are stated in theorems 4 and 5:

**Theorem 4 ([25]).** Let \( M \) be the transition matrix of a homoclinic cycle of type Z. Suppose all transverse eigenvalues of \( df(\xi) \) are negative.

(a) If \( |\lambda_{\max}| > 1 \), then the cycle is asymptotically stable.

(b) If \( |\lambda_{\max}| < 1 \), then the cycle is completely unstable.

**Theorem 5 ([25]).** Let \( M \) be the transition matrix of a homoclinic cycle of type Z. The cycle is fragmentarily asymptotically stable if and only if the following conditions are satisfied:

(i) \( \lambda_{\max} \) is real;

(ii) \( \lambda_{\max} > 1 \);

(iii) \( v_{j}^{\max} v_{q}^{\max} > 0 \) for all \( l \) and \( q, 1 \leq l, q \leq N \).

5.4. Stability of homoclinic cycles in \( \mathbb{R}^5 \)

Conditions for asymptotic stability and fragmentary asymptotic stability for various classes of homoclinic cycles are presented in table 4. For type A' cycles the conditions follow from theorem 3.

For type Z cycles the conditions are determined from theorems 4 and 5 by calculating eigenvalues and eigenvectors of transition matrices. For the 2-2-[1][2] cycles the transition matrices are

\[
\begin{pmatrix}
a_1 & 0 \\
a_2 & 1 \\
a_1 & 0
\end{pmatrix}
\]  

\[
2-2-[1][2] \quad 2-2-[2][1]
\]  

(28)

The first and second matrices have a one- and two-dimensional significant subspace, respectively. Calculating the eigenvectors and eigenvalues, we determine the conditions for asymptotic stability listed in table 4 (previously found in [19, 25, 26]). When such a cycle is not asymptotically stable, it is completely unstable.
Table 4. Conditions for asymptotic stability and fragmentary asymptotic stability of different classes of homoclinic cycles in \( \mathbb{R}^5 \) in terms of eigenvalues of the linearization \( df(\xi) \).

| Class | Conditions for stability |
|-------|--------------------------|
| 1-1-[1] | A. s.: \( c > e \) |
| 2-1-[12] | A. s.: \( c > e, t < 0 \) |
| 2-2-[1] | A. s.: \( c - t > e, t < 0 \) |
| 3-1-[123] | A. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-2-[1] | A. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[1]/[23] | A. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[1][2] | F. a. s.: \( c > e, ct_1 + et_2 < 0, ct_2 + et_1 < 0 \) |
| 3-3-[1][3][2] | A. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[1][3][2] | F. a. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[1][2][3] | F. a. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[1][2][3] | F. a. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[2][1][3] | A. s.: \( c > e, t_1 < 0, t_2 < 0 \) |
| 3-3-[2][3][1] | F. a. s.: \( c > e, t_1 < 0, t_2 < 0 \) |

The transition matrices of the 3-3-[1][2][3] cycles are

\[
\begin{pmatrix}
  a_1 & 0 & 0 \\
  a_2 & 1 & 0 \\
  a_3 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
  a_1 & 0 & 0 \\
  a_3 & 0 & 1 \\
  a_2 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  a_2 & 1 & 0 \\
  a_3 & 0 & 1 \\
  a_1 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  a_2 & 1 & 0 \\
  a_3 & 0 & 1 \\
  a_1 & 0 & 0 \\
\end{pmatrix}
\]

The dimension of the significant subspace of the first, second, third and fourth matrix is one, one, two and three, respectively. For the matrices with one- and two-dimensional significant subspaces, the conditions of theorems 4 and 5 can be expressed in terms of \( a_i \) by explicitly calculating the eigenvectors; the respective maps are either asymptotically stable or completely unstable. For the fourth matrix, the relations between the entries \( a_i \), that are equivalent to the conditions of the theorems, are derived in appendix B. Substituting (27), we obtain the conditions listed in table 4.

The Poincaré map for the 3-2-[1][2][3] cycle is

\[
g(w, z_1, z_2) = (c_{11} w^{\alpha_1} + c_{12} w^{\alpha_2}, c_{21} w^{\alpha_1} + c_{22} z_1 w^{\alpha_2}, c_{33} z_2 w^{\alpha_3}, c_{32} z_1 w^{\alpha_2} + c_{33} z_2 w^{\alpha_3})
\]

implying that the condition \( a_3 > 0 \) is necessary for fragmentary stability of the cycle. The first two components of \( g \) do not depend on \( z_2 \), and for them we use the conditions for the asymptotic stability of the 2-1-[12] cycle.

The Poincaré map for the 3-2-[1][2][3] cycle is

\[
g(w, z_1, z_2) = (c_{11} w^{\alpha_1} + c_{22} z_1 w^{\alpha_2} + c_{33} z_2 w^{\alpha_3}, c_{32} z_1 w^{\alpha_2} + c_{33} z_2 w^{\alpha_3})
\]

Using this expression one can easily derive the necessary and sufficient conditions for asymptotic stability: \( a_1 > 1, a_2 > 0 \) and \( a_3 > 0 \). The asymptotic stability of the 3-2-[1][2][3] cycle is studied in appendix C.

6. Discussion

We have found in the present paper all simple homoclinic cycles in \( \mathbb{R}^5 \) and the respective conditions for asymptotic stability. Perhaps, the most fascinating finding is that no new kinds
of homoclinic cycles in $\mathbb{R}^5$ are revealed. They are either of type Z studied in [25], or belong to a subspace of $\mathbb{R}^5$ isomorphic to $\mathbb{R}^3$ or $\mathbb{R}^4$. A question arises, whether homoclinic cycles of other types exist in $\mathbb{R}^n$ for $n > 5$.

The conditions for stability of type Z are derived in [25]. In $\mathbb{R}^5$, only cycles of type Z can be fragmentarily asymptotically stable; other cycles can be asymptotically stable or completely unstable. A cycle that is not of type Z is asymptotically stable if and only if the contracting eigenvalue is larger than the expanding and all transverse eigenvalues are negative. Whether this simple criterion for the asymptotic stability of cycles that are not of type Z remains valid in $\mathbb{R}^n$ for $n > 5$ is an open question.

A natural continuation of the present work is an investigation of resonance bifurcations of simple homoclinic cycles in $\mathbb{R}^5$, similar to the study [8] for homoclinic cycles in $\mathbb{R}^4$. Another possible continuation is an investigation of simple heteroclinic cycles in $\mathbb{R}^4$ using the homomorphism $\mathbb{Q} \times \mathbb{Q} \to SO(4)$ or of simple heteroclinic cycles in $\mathbb{R}^5$ using theorem 2.

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Appendix A. Global map for the 3-2-[1][23] homoclinic cycles

Consider a map $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ equivariant under a symmetry group $\Sigma \subset O(3)$. Suppose

(a) $\Sigma$ decomposes $\mathbb{R}^3$ into two isotypic components

$$\mathbb{R}^3 = U_1 \oplus U_2,$$

where the dimension of $U_1$ is one and of $U_2$ is two;

(b) for any $x \in \mathbb{R}^3$ there exists $\sigma \in \Sigma$ such that $\sigma x \neq x$.

Consider the $(x, z)$ coordinates in $\mathbb{R}^3$, where $x$ is the coordinate in $U_1$ and $z$ in $U_2$. In this appendix we determine the leading terms of the expansion of $\psi$ in small $x$ and $z$. We use the following lemma:

Lemma 8. Let a group $\Sigma$ act on a linear space $V$. Consider the isotypic decomposition of $V$ under the action of $\Sigma$:

$$V = U_0 \oplus U_1 \oplus \cdots \oplus U_K.$$

Suppose

- the action of $\Sigma$ on $U_0$ is trivial;
- any $\sigma \in \Sigma$ acts on a $U_k$, $1 \leq k \leq K$, either as $I$ or as $-I$.

Then for any subscript $k$, $1 \leq k \leq K$, there exists a subgroup $G_k \subset \Sigma$ such that the subspace $V_k = U_0 \oplus U_k$ is a fixed-point subspace of the group $G_k$.

Proof. We define a subgroup of $\Sigma$

$$N_k = \{ \sigma \in \Sigma : \sigma u_k = u_k \text{ for any } u_k \in U_k\}$$

and consider $W_k = Fix(N_k)$. Evidently, $V_k \subset W_k$. If $V_k = W_k$, then choosing $G_k = N_k$ proves the lemma. Now suppose $V_k \neq W_k$. Then, for some $j \neq 0$ and $k$, there exists $u_j \in U_j$ such that $u_j \in W_k$ and $\sigma u_j = u_j$ for any $\sigma \in N_k$. Consequently, there exist $\kappa_1, \kappa_2 \in (\Sigma \setminus N_k)$ such
that $\kappa_i u_j = - u_j$ (because $j \neq 0$) and $\kappa_2 u_j = u_j$ (because $j \neq k$). The product $\sigma = \kappa_1 \kappa_2$ satisfies $\sigma u_k = u_k$ and therefore $\sigma \in N_k$. However, $\sigma u_j = - u_j$, which implies $u_j \notin W_k$. Hence, the case $V_k \neq W_k$ is impossible.

By (a) and (b), there exists a symmetry $\sigma_1 \in \Sigma$ such that $\sigma_1(x, 0) = (-x, 0)$. $\Sigma$ can act on $U_2$ in three ways:

(i) there exists $\sigma_2 \in \Sigma$ such that $\sigma_2 \neq \sigma_1^k$ for any $k$ and $\sigma_2(0, z) = (0, -z)$;
(ii) no $\sigma_2$ satisfying (i) exists; there exists $\sigma_3 \in \Sigma$ such that $\sigma_3(0, z) = (0, e^{2\pi i/k} z)$, where $k > 1$ is odd, and $\sigma_1(x, z) = (-x, z)$;
(iii) no $\sigma_2$ satisfying (i) exists; there exist $\sigma_3 \in \Sigma$ such that $\sigma_3(0, z) = (0, e^{2\pi i/k} z)$, where $k > 1$ is odd, and $\sigma_1(x, z) = (-x, -z)$.

Lemma 8 implies that in case (i) the $x$- and $z$-components of $\psi$ are

$$\psi^x = x F(x, z, \bar{z}), \quad \psi^z = z G(x, z, \bar{z}) + \bar{z} H(x, z, \bar{z}),$$

where $F$ is real, generically $F(0, 0, 0) \neq 0$ and $G(0, 0, 0) \neq 0$, $s > 0$ is odd (it is determined by $\Sigma$). In cases (ii) and (iii) the components of $\psi$ can be calculated by simple algebra:

(ii) $\psi^x = x F(x, z, \bar{z}), \quad \psi^z = z G(x, z, \bar{z}) + \bar{z}^{k-1} H(x, z, \bar{z})$

(iii) $\psi^x = x F(x, z, \bar{z}) + \bar{z}^k J(x, z, \bar{z}) + \bar{z}^k \bar{F}(x, z, \bar{z}), \quad \psi^z = z G(x, z, \bar{z})$

where $F$ is real, generically $F(0, 0, 0) \neq 0, G(0, 0, 0) \neq 0$ and $J(0, 0, 0) \neq 0$. Thus in cases (i) and (ii), for small $x$ and $z$ the asymptotically largest terms of $\psi$ are

$$\psi^x = ax, \quad \psi^z = bz + d\bar{z}^s, \quad (30)$$

where $a$ is real, $b$ and $d$ are complex, and generically $a \neq 0$ and $b \neq 0$. (Here $b$ and $d$ are further restricted by the action of other symmetries from $\Sigma$, but these restrictions are insignificant for the study of stability of the Poincaré map.) In case (iii) the asymptotically largest terms of the map $\psi$ are

$$\psi^x = ax + b\bar{z}^k + \bar{b}\bar{z}^k, \quad \psi^z = dz, \quad (31)$$

where $a$ is real, $b$ and $d$ are complex, $d^k$ is real, and generically $a \neq 0, b \neq 0$ and $d \neq 0$.

We refer to a cycle with the global map (30) in cases (i) or (ii) as a 3-2-[1][23] cycle, and to a cycle with the global map (31) in case (iii) as a 3-2-[1][23]m cycle.

**Appendix B. Stability of the 3-3-[2][1][3] homoclinic cycle**

In this appendix we derive necessary and sufficient conditions for the asymptotic stability and fragmentary asymptotic stability of the 3-3-[2][1][3] homoclinic cycle in terms of eigenvalues of the linearization near the equilibrium. As noted in section 5.2, such a cycle is of type Z. Conditions for stability of type Z cycles in terms of eigenvalues and eigenvectors of their transition matrices are given by theorems 4 and 5. The transition matrix of the cycle is

$$M = \begin{pmatrix} a_2 & 1 & 0 \\ a_3 & 0 & 1 \\ a_1 & 0 & 0 \end{pmatrix} \quad (32)$$

(see (29)), where $a_i$ are related to eigenvalues of the linearization by (27), $a_1 > 0$ ($a_2$ and $a_3$ can have arbitrary signs).

Let $\lambda_1, \lambda_2$ and $\lambda_3$ be the eigenvalues of (32); $\lambda_1$ denotes the largest eigenvalue if all eigenvalues are real, or the real eigenvalue if the matrix has complex ones. Let $w$ denote
the eigenvector associated with $\lambda_1$. By theorem 4, the necessary and sufficient conditions for asymptotic stability are

$$a_2 > 0, \ a_3 > 0, \ \max_j |\lambda_j| > 1.$$  \hfill (33)

By theorem 5, the cycle is fragmentarily asymptotically stable if and only if

$$\lambda_1 > 1;$$  \hfill (34)

$$\lambda_1 > \max(|\lambda_2|, |\lambda_3|);$$  \hfill (35)

$$w_i w_j > 0 \text{ for any } 1 \leq i, j \leq 3.$$  \hfill (36)

By applying the following lemmas, one can avoid calculating the eigenvalues $\lambda_i$ by Cardano’s formulae for the roots of a cubic polynomial.

**Lemma 9.** Let all $a_i > 0$ in matrix (32). Then $\max_j |\lambda_j| > 1$ if and only if

$$a_1 + a_2 + a_3 > 1.$$  \hfill (37)

**Proof.** Eigenvalues of matrix (32) are roots of its characteristic polynomial

$$p_M(\lambda) = -\lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_1.$$  \hfill (38)

Suppose the inequality (37) is satisfied. This implies $p_M(1) > 0$. Since $p_M(\infty) < 0$, the polynomial $p_M$ has a root larger than one, and thus $\max_j |\lambda_j| > 1$.

We prove now the converse. Denote by $\lambda_{\text{max}}$ the maximal in absolute value root of $p_M(\lambda)$. Since all $a_i > 0$ and $|\lambda_{\text{max}}| > 1$,

$$a_1 + a_2 + a_3 > a_2 + \frac{a_3}{|\lambda_{\text{max}}|} + \frac{a_1}{|\lambda_{\text{max}}|^2} > \left| a_2 + \frac{a_3}{\lambda_{\text{max}}} + \frac{a_1}{\lambda_{\text{max}}^2} \right| = |\lambda_{\text{max}}| > 1.$$  \hfill QED

Components of the eigenvector $w$ associated with the eigenvalue $\lambda_1$ satisfy the equations

$$a_2 w_1 + w_2 = \lambda_1 w_1,$$  \hfill (39)

$$a_3 w_1 + w_3 = \lambda_1 w_2,$$  \hfill (40)

$$a_1 w_1 = \lambda_1 w_3.$$  \hfill (41)

By the Viète formulae for the roots of the characteristic polynomial $p_M$,

$$\lambda_1 + \lambda_2 + \lambda_3 = a_2,$$  \hfill (42)

$$-\lambda_1 \lambda_2 - \lambda_2 \lambda_3 - \lambda_1 \lambda_3 = a_3,$$  \hfill (43)

$$\lambda_1 \lambda_2 \lambda_3 = a_1.$$  \hfill (44)

**Lemma 10.** Eigenvalues and eigenvectors of matrix (32) satisfy conditions (34)–(36) if and only if the following four inequalities hold true:

$$a_1 > 0,$$

$$a_1 + a_2 + a_3 > 1,$$

$$a_2 a_3 + a_1 > 0,$$

$$a_1 a_2^2 + a_3^3 > 0.$$  \hfill (45)
Proof. The cubic polynomial $p_M$ has either three real roots or one real root and two complex conjugate ones. We consider the two cases separately.

Suppose all eigenvalues $\lambda_i$ are real.

Assume that (34)–(36) hold true. By virtue of (41) and (34), and since $w_1$ and $w_3$ have the same signs (36), we have $a_1 > 0$; hence by (44) $\lambda_2$ and $\lambda_3$ have the same signs. Equations (42) and (39) yield

\[-w_1(\lambda_2 + \lambda_3) = w_2,\]  

(46)

whereby $\lambda_2 + \lambda_3 < 0$ (see (36)). Therefore, $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus, (35) and the identity

\[(\lambda_1^2 - \lambda_2\lambda_3)(\lambda_2^2 - \lambda_3\lambda_1)(\lambda_3^2 - \lambda_1\lambda_2) = a_1a_2^3 + a_3^3\]  

(47)

(which follows from the Viète formulae) yield

\[a_1a_2^3 + a_3^3 > 0.\]

The characteristic polynomial $p_M$ (38) has only one positive root that is larger than one. Since $p_M(\infty) < 0$, this implies $p_M(1) = -1 + a_1 + a_2 + a_3 > 0$. The identity

\[(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) = -a_2a_3 - a_1,\]  

(48)

and (35) yield $a_2a_3 + a_1 > 0$. Thus, all inequalities in (45) are proven.

We prove now the converse assuming that (45) is satisfied. Since $a_1 > 0$, by virtue of (44) $\lambda_1 > 0$ for our ordering of $\lambda_i$, and $\lambda_2\lambda_3 > 0$. The inequality $a_2a_3 + a_1 > 0$ and (48) imply that $\lambda_2 < 0$ and $\lambda_3 < 0$. From (41) we deduce that $w_1$ and $w_3$ have the same signs; by virtue of (46) $w_1$ and $w_2$ also have the same signs, which proves (36). Since $p_M(1) > 0$ and $p_M(\infty) < 0$, (34) holds true. Due to (47) and (48), the inequalities $a_2a_3 + a_1 > 0$ and $a_1a_2^3 + a_3^3 > 0$ imply that the condition (35) holds true.

Suppose now the polynomial $p_M$ has one real root $\lambda_1$ and two complex conjugate roots $\lambda_{2,3} = \alpha \pm i\beta$. Identities (48) and (47) yield, respectively,

\[a_2a_3 + a_1 = -2\alpha((\lambda_1 + \alpha)^2 + \beta^2)\]  

(49)

and

\[a_1a_2^3 + a_3^3 = \Theta(\lambda_1^2 - \lambda_2\lambda_3)(\alpha^2 + \beta^2)^{-1}\]  

(50)

where

\[\Theta \equiv (\lambda_1(\alpha^2 + \beta^2) - \alpha^2 + 3\alpha\beta)^2 + (3\alpha^2\beta - \beta^3)^2 > 0,\]  

(51)

unless $a_1 + a_2 + a_3 = 0$ and $3\alpha^2 = \beta^2$.

By (50) and (51), condition (35) is equivalent to the inequality $a_1a_2^3 + a_3^3 > 0$. Since $\lambda_1\lambda_3 = |\lambda_2|^2$, (41) and (44) imply that $w_1$ and $w_3$ have same signs. By (49) the inequalities $a_2a_3 + a_1 > 0$ and $\alpha < 0$ are equivalent. By virtue of (42), (39) reduces to

\[-2\alpha w_1 = w_2,\]

and hence $a_2a_3 + a_1 > 0$ is equivalent to (36).

For the characteristic polynomial $p_M$ (38) with only one real eigenvalue the conditions (34) and $a_1 + a_2 + a_3 > 1$ are equivalent. Finally, (44) imply that $\alpha$ and $\lambda_1$ have same signs.

\[\text{QED}\]

Substituting expressions (27) into the inequalities in the statements of lemmas 9 and 10, we establish conditions for stability of the cycle presented in table 2.
Appendix C. Stability of the 3-2-1[23]m homoclinic cycle

In this appendix we derive necessary and sufficient conditions for asymptotic stability of the 3-2-1[23]m homoclinic cycle. The Poincaré map near the cycle is

\[ g(w, z) = (Aw^{a_1} + |w|^{a_2} k \Re(b z^k), dz|w|^{a_2}) \]  

(52)

where \( z = z_1 + iz_2 \), A is real, B and d are complex and \( d^k \) is real (see table 3 and the expression for the global map (31) in appendix A). Recall that \( a_1 = -c/e \), and therefore by definition of expanding and contracting eigenvalues (see section 2.2) \( a_1 \) is always positive. Upon the coordinate transformation \( b z^k = w + iv \), (52) becomes

\[ g(w, u, v) = (Aw^{a_1} + |w|^{a_2} k u, Cu|w|^{a_2}, Cv|w|^{a_2}) \]  

(53)

where \( C = d^k \) is real.

For \( a_1 > a_2 k \) and \( a_1 < 1 \), we partition \( \mathbb{R}^3 \) into two regions (if \( a_1 < a_2 k \) or \( a_1 > 1 \), the partitioning is not needed, see the proof of lemma 12):

\[ \Omega_i = \{(w, u, v) : |Aw^{a_1} + |w|^{a_2} k u| < |w|^\alpha \} \]

(54)

where \( \alpha \) satisfies the inequality \( a_1 < \alpha < 1 \).

Denote by \( B_{\epsilon} \) the \( \epsilon \)-neighbourhood of the point \((w, u, v) = 0 \), and \( \Omega_j(\epsilon) = \Omega_j \cap B_{\epsilon} \) for \( J = I, II \). Also,

\[ f_1(x) \approx f_2(x) \text{ denotes that } f_1(x) - f_2(x) = o(f_1(x)) \text{ when } x \to 0 \]

and

\[ f_1(x) \sim f_2(x) \text{ denotes that } f_1(x) \approx F f_2(x) \text{ for a constant } F \neq 0. \]

Finally, write

\[ Q(f_1, f_2) = \{(w, u, v) : f_1(w) < u < f_2(w) \} \text{ and } \mathcal{H}(f) = \{(w, u, v) : u = f(w)\}. \]

(55)

Let \( f_0(w) \) be the solution to \( Aw^{a_1} + |w|^{a_2} k f_0(w) = 0, l_0 = H(f_0) \) and \( l_{j+1} = g^{-1} l_j \) be the preimage of \( l_j \) under \( g \).

Lemma 11. Suppose \( Q(d_1, d_2) = \mathcal{G}(f_1, f_2) \), where

\[ f_j = f_0 + h_j, h_j \approx D w^{a_1/(a_1-a_2 k)} a_2 k \text{ for } j = 1, 2, \]

\[ h_1 - h_2 \sim w^\beta, \beta > a_1/(a_1-a_2 k) - a_2 k. \]

Then

\[ d_1 - d_2 \sim w^{\beta+a_1/(a_1-a_2 k)}. \]

Proof. By the condition of the lemma,

\[ g(w, f_1(w), u) \approx (w^{a_2} k h_1, E_1 w^{a_1}, \tilde{u}) \approx (E_2 w^{a_1/(a_1-a_2 k)}, E_1 w^{a_1}, \tilde{u}), \]  

(56)

\[ g(w, f_2(w), u) \approx (w^{a_2} k (h_2 - h_1), E_1 w^{a_1}, \tilde{u}) \approx ((w^{a_2} k h_1)^{(a_1-a_2 k)/a_1} + E_3 w^{a_1/(a_1-a_2 k)}(a_1-a_2 k) a_1/(a_1-a_2 k), E_1 w^{a_1}, \tilde{u}). \]  

(57)

(Note that \( w^{a_2} k (h_1)^{(a_1-a_2 k)/a_1} \sim w \).) For small \( |w_1 - w_2| \),

\[ d_1(w_1) - d_2(w_2) = d_1(w_1) - d_2(w_1) + d_2(w_1) - d_2(w_2) \approx d_1(w_1) - d_2(w_1) + d_2(w_1)(w_1 - w_2). \]

Set \( w_1 = (w^{a_2} k h_1)^{(a_1-a_2 k)/a_1} \) and \( w_2 = w_1 + E_3 w^{a_1/(a_1-a_2 k)}. \) By (56), \( d_2(w_1) \sim w^{a_1/(a_1-a_2 k)} - 1 \) and \( d_1(w_1) = d_2(w_2). \) Therefore, \( d_1(w) - d_2(w) \sim w^{\delta}, \) where \( \delta = \beta+a_1/(a_1-a_2 k) \). QED
Lemma 12. Consider the map
\[ g(w, u, v) = (A\,w^{a_1} + |w|^{a_2}u, C\,w^{a_2}u, C\,v|w|^{a_2}). \]
where \(a_1 > 0\).

(i) If
\[ a_1 > 1 \text{ and } a_2 > 0, \]
then the fixed point \((w, u, v) = 0\) of the map \(g\) is asymptotically stable.

(ii) If
\[ a_1 < 1 \text{ or } a_2 < 0, \]
then the fixed point \((w, u, v) = 0\) of the map \(g\) is completely unstable.

Proof.

(i) Suppose \(a_1 > 1 \text{ and } a_2 > 0\). Then the iterates \((w_{j+1}, u_{j+1}, v_{j+1}) = g(w_j, u_j, v_j)\) satisfy the inequalities
\[ |w_{j+1}| < \max(2|A\,w^{a_1}|, 2|w_j|^{a_2}u_j) \quad \text{and} \quad |u_{j+1}| < |w_j|^{a_2}u_j, \]
Therefore, if \(|w_j, u_j, v_j| < \epsilon\), we have
\[ |w_{j+1}| < \frac{2\epsilon}{1-\epsilon} |w_j|^{a_2}u_j. \]
Thus, \(|w_j| < \frac{2\epsilon}{1-\epsilon} \|w_0\| \quad \text{for} \quad j \geq 1, \epsilon \to 0\).

(ii) Suppose \(a_2 < 0\). Then the iterates \((w_j, u_j, v_j)\) escape from \(B_\epsilon\), unless \((w_0, u_0, v_0) \in I_k^\alpha\) for some \(k > 0\). The measure of the union of the sets \(I_k^\alpha\) is zero. This implies the statement of the lemma.

Suppose now \(a_2 > 0 \text{ and } a_1 < 1\). Consider the sets
\[ \mathcal{H}_j = \bigcup_{\epsilon \leq \epsilon \leq \infty} S^{-j} \Omega_\epsilon, \quad \mathcal{H}_j(\epsilon) = \mathcal{H}_j \cap B_\epsilon. \]
Let \(\mu\) denote the Lebesgue measure in \(\mathbb{R}^3\). Since \(\Omega_\epsilon = Q(f_1, f_2)\) where \(f_1 - f_2 \sim w^{a_1-a_2k}\), we have \(\mu(\Omega_\epsilon) \sim \epsilon^{2(a_1-a_2k)/a_2}\). By lemma 11, \(g^{-1}(\Omega_j) = Q(d_1, d_2)\) where \(d_1 - d_2 \sim w^{a_1-a_2k}t\), and \(s = a_1/(a_1 - a_2k) - a_1 > 0\). Therefore, \(\mu(\mathcal{H}_j(\epsilon)) \sim \epsilon^{2(a_1-a_2k+a_1t)/(a_1-a_2k)}\).

Thus, \(\lim_{j \to \infty} \mu(\mathcal{H}_j(\epsilon)) = 0\).

If \((w_0, u_0, v_0) \notin \mathcal{H}_j \text{ and } (w_0, u_0, v_0) \notin \bigcup_{\epsilon \leq \epsilon \leq \infty} I_k^\alpha\), then \(w_j \neq 0\) and \(|w_{j+1}| = |w_j| < \epsilon\) for \(l \geq j\). Since \(\alpha < 1\), this indicates that almost all \((w_0, u_0, v_0)\) (except for a set of zero measure) escape from \(B_\epsilon\) for a sufficiently small \(\epsilon > 0\).

\[ \text{QED} \]

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