TOPOLOGICAL EQUISINGULARITY OF FUNCTION GERMS WITH 1-DIMENSIONAL CRITICAL SET

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Abstract. We focus on topological equisingularity of families of holomorphic function germs with 1-dimensional critical set. We introduce the notion of equisingularity at the critical set and prove that any family which is equisingular at the critical set is topologically equisingular. We show that if a family of germs with 1-dimensional critical set has constant generic Lê numbers then it is equisingular at the critical set, and hence topologically equisingular (answering a question of D. Massey). We use this to modify the definition of singularity stem present in the literature, introducing and characterising topological stems (being this concept closely related with Arnold’s series of singularities). We provide another sufficient condition for topological equisingularity for families whose reduced critical set is deformed flatly. Finally we study how the critical set can be deformed in a topologically equisingular family and provide examples of topologically equisingular families whose critical set is a non-flat deformation with singular special fibre and smooth generic fibre.

1. Introduction

In this paper we answer several topological equisingularity questions for holomorphic function germs with 1-dimensional critical set.

The topological equisingularity notions that we will handle are the following. Denote by $B_\epsilon$ and $S_\epsilon$ the ball and the sphere of radius $\epsilon$ centered at the origin of $\mathbb{C}^n$. Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a holomorphic germ. The embedded link of $f$ is the topological type of the pair $(S_\epsilon, f^{-1}(0) \cap S_\epsilon)$ for $\epsilon$ sufficiently small; the abstract link is the topological type of $f^{-1}(0) \cap S_\epsilon$. Two germs $f$ and $g$ are topologically $R$-equivalent if there is a germ of self-homeomorphism $\varphi$ of $(\mathbb{C}^n, 0)$ such that $g = f \circ \varphi$; if we only have that $\varphi(f^{-1}(0)) = g^{-1}(0)$ then we say that $f$ and $g$ have the same topological type. By the conical structure of singularities having the same embedded link implies having the same topological type. A family $f_t : (\mathbb{C}^n, 0) \to \mathbb{C}$ of holomorphic germs depending continuously on a parameter $t$ varying in a manifold $T$ is topologically $R$-equisingular if there is a family of self-homeomorphisms $\varphi_t$ depending continuously in $t$ and parametrised over $T$ such that $f_t \circ \varphi_t = f_{t_0}$ for a fixed $t_0 \in T$ and any $t \in T$.

Topological equisingularity has proved to be a very subtle subject, with several long standing open questions. For instance Zariski’s Multiplicity Question asks whether two germs with the same topological type must have the same multiplicity (Question A, [21]).

Suppose that $f_t$ has an isolated singularity at the origin and that the Milnor number of $f_t$ is independent on $t$. Assume $n \neq 3$. Lê and Ramanujam [11] proved
that the diffeomorphism type of the Milnor fibration of $f$ and of the embedded link are independent on $t$. Later King \cite{King} and Timourian \cite{Timourian} proved that for any $t \in T$ there is a neighborhood $U$ of $t$ in $T$ such that the restriction of $f_t$ over $U$ is topologically $R$-equisingular. As the Milnor number is a topological invariant, if $n \neq 3$, it is an invariant characterising topological equisingularity for germs with isolated singularities. Answering whether families with constant Milnor number are topologically equisingular for $n = 3$ is a major open problem.

Topological equisingularity for germs having not necessarily isolated singularities turns out to be much more difficult, and the theory is less developed than in the case of germs with isolated singularities. As an illustration of this, and also to motivate the results of this paper, let us mention some of the latest developments. In \cite{Massey1}, \cite{Massey2} Massey introduced the Lê numbers, a set of polar invariants attached to a germ and to a coordinate system. Suppose that the family $f_t$ depends holomorphically on a complex parameter $t$ and that the critical set of $f_t$ has dimension $s$ (with $n \geq s + 4$) for any $t \in T$, Massey \cite{Massey1} proved that if the Lê numbers of $f_t$ with respect to a coordinate system (satisfying a certain genericity property) are constant then the diffeomorphism type of the Milnor fibration of $f_t$ is independent on $t$ (it was proved by the author in \cite{Bobadilla} that the homotopy type of the abstract links remains constant if $n \geq 4$, without the condition that $n \geq s + 4$). It remained open whether the embedded topological type of $f_t$ is independent on $t$. This was answered in the negative in \cite{Bobadilla}, where counterexamples with critical set of dimension 3 were provided (a counterexample of Zariski’s Question B, of \cite{Zariski} was constructed as an application). Probably, the main reason of the failure of the Lê numbers in controlling the embedded topological type is the fact that they control primarily the Milnor fibration, but, as it is shown in \cite{Bobadilla} in the case of non-isolated singularities the relation of the Milnor fibration with the embedded topological type is, in general, weak (in the case of isolated singularities the Milnor fibration determines the embedded topological type). On the other hand, in \cite{Bobadilla1}, it was shown that the generic Lê numbers are not topological invariants (even in the case of critical set of dimension 1). A different approach for topological equisingularity of families of functions satisfying certain conditions on their Newton polytope can be found in \cite{Bobadilla2}.

In this paper we adopt a new viewpoint towards topological equisingularity. Although a general formulation of it is possible (and the reader is encouraged to figure it out) we introduce it for the case of functions with 1-dimensional critical set (which are the object of interest along this paper). Functions with 1-dimensional critical set occupy a distinguished position among those with non-isolated singularities, since they are at the limit of series of singularities; we will be more explicit about this below. The critical set of such a function $f$ (the support of the sheaf of vanishing cycles) is a curve which is stratified by the Milnor number of the singularities of the generic transverse hyperplane sections. Roughly speaking, a family $f_t$ is equisingular at the critical set if there is a family of self-homomorphisms of $(\mathbb{C}^n, O)$ topologically trivialising the family of critical sets and respecting the stratification by transverse Milnor number outside the origin (see Definition \cite{Bobadilla3}). We may view equisingularity at the critical set as an intermediate notion between topological equisingularity itself and the constancy of any candidate set of invariants intended to imply topological equisingularity. The strategy is to prove that equisingularity
at the critical set implies (any notion of) topological equisingularity and to find numerical invariants implying equisingularity at the critical set.

Now we summarise the main results of this paper. In the families that one finds in practice, the dependence on the parameter is always at least differentiable. We will work in this paper with families \( f_t \) of holomorphic germs with 1-dimensional critical set depending smoothly on a parameter \( t \) varying on a manifold \( T \). We will gain, in case that the family is topologically equisingular, some additional regularity on the trivialising family of homeomorphisms. Often in this paper the space of parameters \( T \) is assumed to be (diffeomorphic to) a cube \([0, 1]^n\). This is due to the fact that we prove topological equisingularity theorems which are \( \text{global over the parameter space} \), and in order to achieve it is necessary that the parameter space has trivial topology. Anyway, if the objective was to obtain equisingularity results \( \text{local over the parameter space} \), the assumption that it is a cube does not decrease generality.

**Theorem A.** (See Theorems 34 and 40 for more details.) If \( f_t \) is equisingular at the critical set then

(i) The diffeomorphism type of the Milnor fibration of \( f_t \) is independent on \( t \).
(ii) The topological type of the embedded link and the topological right-equivalence type of \( f_t \) are independent on \( t \).
(iii) If \( T \) is a cube then \( f_t \) is topologically \( R \)-equisingular over \( T \). The family of homeomorphisms \( \varphi_t \) trivialising \( f_t \) satisfies that the restrictions of \( \varphi_t \) to the complement of the critical set, and to the critical set minus the origin is a diffeomorphism, and the dependence on \( t \) of \( \varphi_t \) restricted to these strata is differentiable.

**Theorem B.** (See Theorem 42.) If \( f_t \) depends holomorphically on a parameter and any of the following conditions hold:

1. there is a coordinate system \( Z \) such that the Lê numbers at the origin of \( f_t \) with respect to \( Z \) are defined and independent of \( t \),
2. the generic Lê numbers at the origin of \( f_t \) are independent of \( t \),

then \( f_t \) is equisingular at the critical set.

We immediately deduce

**Corollary C.** (See Theorem 41) Suppose that \( f_t \) depends holomorphically on a parameter, if one of the following conditions hold

1. there is a coordinate system \( Z \) such that the Lê numbers at the origin of \( f_t \) with respect to \( Z \) are defined and independent of \( t \),
2. the generic Lê numbers at the origin of \( f_t \) are independent of \( t \),

then the embedded topological type of \( f_t \) at the origin and the the diffeomorphism type of the Milnor fibration is independent of \( t \in T \). Moreover, if \( T \) is (diffeomorphic to) a cube, the restriction of the family over \( T \) is topologically \( R \)-equisingular.
This recovers Massey’s Theorem for 1-dimensional critical set and answers positively Massey’s question about the embedded topological type, in contrast with what happens for higher dimensional critical set.

Isolated hypersurface singularities are related to non-isolated ones through the idea of “series of singularities”. In his classification of low dimensional \( \mu \)-classes of isolated singularities, Arnol’d observed that the list of singularities splits into series. Although it was impossible to give a precise definition of what a series of singularities is, it became clear that series are associated with singularities of infinite codimension (non-isolated singularities). Inspired by Arnol’d remark C.T.C Wall formulated a conjecture (see the Conjecture at the introduction of [20]), or rather, a guiding principle for classification of singularities. Later D. Mond introduced the idea of singularity stem, as a first step to understand the notion of series of singularities in the context of mappings from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \) (the idea is that a singularity stem is a non-isolated singularity which lies in the limit of a series of singularities). Following a suggestion of Montaldi, Pellikaan defined inductively the notion of stem of degree \( d \), characterised stems of degree 1 as functions with irreducible 1-dimensional critical set and transversal type \( A_1 \), and proved that any stem of finite degree is a function with 1-dimensional critical set. He was also able to give bounds on the degree of the stem depending on the number of irreducible components of the critical set of the function and the transversal Milnor number. Here we modify the concept of stem and define topological stem (see Definition 50).

With this notion we prove

**Theorem D.** (See Theorem 51) A holomorphic function germ \( f : (\mathbb{C}^n, O) \to \mathbb{C} \) is a topological stem of positive finite degree if and only if its critical set is 1-dimensional at the origin. Moreover the degree of the stem is bounded above by the generic first Lê number at the origin of \( f \).

Our modification of the definition of stem consists essentially in replacing differentiable \( R \)-equivalence by topological \( R \)-equivalence. This is natural since series in Arnol’d classification of singularities are in fact topological series (series of \( \mu \)-classes), and thus it is therefore reasonable that the object to find at the limit of the series admits a topological definition as well. It is interesting to notice that Theorem D is an application of our sufficient conditions for topological \( R \)-equisingularity, which is not at reach using any of the previously known sufficient conditions for equisingularity.

A consequence of our theory and of the results of [3] is the following theorem, giving an alternative sufficient condition of topological equisingularity, for the case that the reduced critical set is deformed flatly:

**Theorem E.** (See Theorem 53) Let \( f : \mathbb{C} \times (\mathbb{C}^n, O) \to (\mathbb{C}, 0) \) be a family of holomorphic germs at the origin, holomorphically depending on a parameter \( t \), and having 1-dimensional critical set at the origin. Let \( \Sigma := V(\partial f/\partial z_1, \ldots, \partial f/\partial z_n) \). Suppose that all the irreducible components of \( \Sigma \) at \((0, O)\) are of dimension 2 and that the restriction

\[
\pi : \Sigma_{\text{red}} \to \mathbb{C}
\]

is a flat morphism with reduced fibres (where \( \Sigma_{\text{red}} \) is \( \Sigma \) with reduced structure and \( \pi \) is the restriction of the projection of \( \mathbb{C} \times \mathbb{C}^n \) to the first factor) such that the
Milnor number at the origin $\mu((\Sigma_{\text{red}}), O)$ (in the sense of [3]) is independent of $t$. If, in a neighborhood of $(0, O)$ in $\mathbb{C} \times \mathbb{C}^n$, the generic transversal Milnor number of $f_t$ at any point $(t, x)$ of $\Sigma_{\text{red}} \setminus \mathbb{C} \times \{O\}$ only depends on the irreducible component of $\Sigma_{\text{red}}$ to which $(t, x)$ belongs then the family $f$ is topologically $R$-equisingular.

Finally we study how the reduced critical set may vary in a family which is topologically $R$-equisingular. It seems that there are no further restrictions apart from the obvious topological ones. We observe a great amount of flexibility in the critical set of a topologically equisingular family. We exhibit a family $f_t$ (see Example 54) which is topologically $R$-equisingular and the reduced critical set is singular for $t = 0$ and smooth for $t \neq 0$ (thus the deformation of reduced critical sets is non flat). To the author’s knowledge it is the first time that this phenomenon is observed. Example 54 is a family for which the sufficient conditions for topological equisingularity given in Theorem E fail: we really need the finer Theorem A to establish the equisingularity. In Problem 56 we propose to construct more families which are equisingular at the critical set starting out of deformations of parametrisations (this is the way how Example 54 was constructed). The motivation for it is based on the following observation: the change at multiplicity of the critical set of $f_t$ implies, using Lê-Iomdine formulas (see [14]), that adding a high power of a linear function to $f_t$ will not yield a $\mu$-constant family. This suggests that, in the realm of function germs with 1-dimensional critical set, we have more space to find a possible counterexample of Zariski’s multiplicity question than in the realm of isolated singularities.

For the proof of Theorem A we need a improved version of the Theorems of King and Timourian as follows:

**Preliminary Theorem.** Let $f_t$ be a family of germs with isolated singularities at the origin, with constant Milnor number, and depending smoothly on a parameter $t$ which varies on a cube $T$. Then the family is topologically $R$-equisingular over $T$, and the restriction of the trivialising family of homeomorphisms $\varphi_t$, to $\mathbb{C}^n \setminus \{O\}$ is a family of diffeomorphisms depending smoothly on the parameter $t$.

The methods appearing in [7], [8] and [19] are certainly local in the base. We need to introduce a new device, which we call a cut, which allow us to prove the result above, which is global in the base, and also enables to prove the differentiability outside the origin. Moreover the method with which we prove the Preliminary Theorem provides tools that generalise well to prove Theorem A.

A natural question is to determine the relation between equisingularity at the critical set and topological equisingularity. We conjecture that they coincide:

**Conjecture A.** A family $f_t$ of function germs with 1-dimensional critical set depending smoothly on $t$ is equisingular at the critical set if and only if $f_t$ is topologically equisingular to $f_{t'}$, for any parameters $t, t'$.

It is easy to reduce the above conjecture to the following one:
Conjecture B. Let \( f \) be a function germ with smooth 1-dimensional critical set such that the transversal Milnor number of \( f \) at any point of its critical set is independent on the point. Then any function germ \( g \) topologically equisingular to \( f \) has smooth critical set and constant transversal Milnor number.

Actually I expect the following stronger statement to be true:

Conjecture C. Let \( g \) be a function germ with irreducible 1-dimensional critical set. Let \( \mu \) denote the transversal Milnor number of \( f \) at a generic point of its critical set. If the reduced Euler characteristic of the Milnor fibre of \( f \) is equal to \((-1)^n \mu\), then \( f \) is a function germ with smooth 1-dimensional critical set such that the transversal Milnor number of \( f \) at any point of its critical set is equal to \( \mu \).

In fact, by a recent result of Lé-Massey [11] and Tibar [18], in conjectures \( B \) and \( C \) it suffices to prove that the critical set of \( g \) is smooth. Conjecture \( C \) is a numerical characterisation of function germs with 1-dimensional critical set which are topologically a product.

The material is distributed as follows.

In Section 2 we introduce cuts and prove the Preliminary Theorem. In Section 3 we define a particular kind of neighborhoods of the origin in \( \mathbb{C}^n \) which are essential in the proof of Theorem A. In Section 4 we introduce equisingularity at the critical set. In Section 5 we prove parts (i) and (ii) of Theorem A. In Section 6 we prove part (iii) of Theorem A. In Section 7 we prove Theorem B. In Section 8 we define topological stems and prove Theorem D. Finally, in Section 9 we prove Theorem E, study how the reduce critical set deforms in a topologically equisingular family, and construct Example 54.

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2. The case of isolated singularities revisited

2.1. Cobordisms. A subspace \( X \subset \mathbb{R}^N \) is smoothly stratified if it admits an stratification whose strata are smooth submanifolds. A stratified subspace \( Y \subset X \) is a subspace which is a union of strata. Given any subset \( X \) of \( \mathbb{R}^N \) we denote by \( \bar{X} \) the subset of interior points, and by \( \partial X \) its frontier \( X \setminus \bar{X} \). Observe that if \( X \) is a stratified space, then both \( \bar{X} \) and \( \partial X \) are forcedly stratified subspaces. A stratified cobordism between \( X_0 \) and \( X_1 \) is, by definition, a triple \((X, X_0, X_1)\) given by a stratified space \( X \) and two stratified subspaces \( X_0, X_1 \) of \( \partial X \), satisfying the following condition: for any stratum \( A \subset X \) the "cobordism closure"

\[
\hat{A} := A \cup (\partial A \cap (X_1 \cup X_2))
\]

has a structure of manifold with boundary (being the boundary \( \partial A \cap (X_1 \cup X_2) \)). A stratified cobordism is homotopically (homologically) trivial if the inclusions of \( X_0 \) and \( X_1 \) in \( X \) are homotopy (homology) equivalences. Given a stratified space \( Y \), the product \( Y \times [0, 1] \) inherits a natural stratification which makes the triple \((Y \times [0, 1], Y \times \{0\}, Y \times \{1\})\) a stratified cobordism. A stratified cobordisms \((Y, Y_0, Y_1)\) is trivial if there is a homeomorphism

\[
\varphi : (X, X_0, X_1) \to (Y \times [0, 1], Y \times \{0\}, Y \times \{1\})
\]
such that for any stratum \( A \subset X \) there is a stratum \( B \subset Y \) such that
\[
\varphi|_{\hat{A}} : \hat{A} \to B \times [0,1]
\]
is a diffeomorphims between manifolds with boundary. Two special cases of strati-fied cobordisms will be of special interest. The first is the case in which \( X \) is a compact manifold with two boundary components \( X_1 \) and \( X_2 \). Then our definitions coincide with the usual ones. We will say that in this case we have a (non-stratified) cobordism between \( X_1 \) and \( X_2 \). The second is when \( X \) is a manifold with corners and the boundary \( \partial X \) admits a decomposition \( \partial X = X_0 \cup X_1 \cup Y \), where \( X_0, X_1 \) and \( Y \) are manifolds with boundary and we have \( \partial X_0 = X_0 \cap Y \), \( \partial X_1 = X_1 \cap Y \). Then \( \partial Y = \partial X_0 \cup \partial X_1 \) and \( (Y, Y \cap X_0, Y \cap X_1) \) is a cobordism. We say that \( (X, X_0, X_1) \) is a cobordism with boundary between \( X_0 \) and \( X_1 \), and that \( (Y, Y \cap X_0, Y \cap X_1) \) is the boundary cobordism.

**Setting and notation.** Throughout the rest of this section we denote by
\[
\pi : E \to U
\]
be a smooth complex vector bundle of rank \( n \) over a manifold \( U \). Let
\[
f : E \to \mathbb{C}
\]
be a smooth germ at \( \{O\} \times U \) whose restrictions to the fibres of \( \pi \) are holomorphic. Assume that the Milnor number \( \mu(f|_{E_p}) \) is independent of \( p \in U \). Denote by
\[
\phi : E \to U \times \mathbb{C}
\]
the mapping \( \phi := (\pi, f) \). Fix a hermitian metric for the vector bundle \( E \) and let \( \rho \) be the associated (fibrewise) "distance to the origin" function. Consider \( K \subset U \) and let \( \theta : K \to \mathbb{R}_+ \) be any positive continuous function. We define the \( \theta \)-neighborhood of \( K \) as
\[
B(K, \theta) := \{ x \in E : \pi(x) \in K \text{ and } \rho(x) \leq \theta(\pi(x)) \}.
\]
Given subsets \( A \subset U \) and \( B \subset E \), we will denote the intersection \( B \cap \pi^{-1}(A) \) by \( B_A \) or by \( B|_A \).

**2.2. Cuts.**

**Definition 1.** A cut for \( f \) over a submanifold (possibly non-closed and with corners) \( V \subset U \) with amplitude \( \delta \) is a closed smooth hypersurface \( H \) of \( \phi^{-1}(V \times D_\delta) \) with the following properties:

1. For any \( (t,s) \in V \times D_\delta \) the level set \( \phi^{-1}(t,s) \) is smooth outside the zero section of \( \pi \) and transverse to \( H \) at any of their intersection points.
2. There is a unique connected component of \( \phi^{-1}(V \times D_\delta) \setminus H \) containing the zero-section of \( E \). Its closure \( Y_{\text{int}}(H,V,\delta) \) is a smooth manifold with corners, called the interior component, and the restriction
\[
\pi : Y_{\text{int}}(H,V,\delta) \to V
\]
is a smooth locally trivial fibration with contractible fibres.
3. For any \( t \in V \) and any radius \( \epsilon_t \) for the Milnor ball of \( f_t \) the space
\[
\phi^{-1}(t,0) \cap Y_{\text{int}}(H,V,\delta) \setminus \hat{B}((O,t),\epsilon_t)
\]
is a smooth trivial cobordism.
Usually $V$ will be either an open subset of $U$ or the closure of an open subset. If $H$ is a cut over $V$ of amplitude $\delta$ then for any $V' \subset V$ submanifold and $\delta' < \delta$, the hypersurface $H \cap \delta^{-1}(V' \times D_\delta)$ is a cut over $V'$ with amplitude $\delta'$, we will abuse notation and denote by $Y_{\text{int}}(H, V', \delta')$ the interior component of the new cut. We need the following relation in the set of cuts: let $H$ and $H'$ be cuts over open subsets $V$ and $V'$ with amplitudes $\delta$ and $\delta'$ respectively. We say that $H \preceq H'$ if

$$Y_{\text{int}}(H, V \cap V', \min\{\delta, \delta'\}) \subset Y_{\text{int}}(H', V \cap V', \min\{\delta, \delta'\}).$$

The strict version of the relation is: we say that $H \prec H'$ if

$$Y_{\text{int}}(H, V \cap V', \min\{\delta, \delta'\}) \subset Y_{\text{int}}(H', V \cap V', \min\{\delta, \delta'\}).$$

Observe that $H_V \preceq H_W$ if $V \cap W = \emptyset$. Beware that the relation is not an ordering, and that given two cuts related in both directions, they are not necessarily equal.

The following Lemma shows that cuts appear naturally

**Lemma 2.** Suppose $n \neq 3$. For a fixed $t \in U$ we consider a pair $(\epsilon, \delta)$ of radii for the Milnor fibration of $f_1$. There exists an open neighborhood $V$ of $t \in U$ such that $\varphi^{-1}(V \times D_\delta) \cap \partial B(V, \epsilon)$ is a cut over $V$ of amplitude $\delta$.

*Proof.* Conditions (1) and (2) are easy. Condition (3) is contained in the proof of Lé-Ramanujam Theorem [10].

**Lemma 3.** Suppose $n \neq 3$. If we have an open subset $V \subset U$, a positive $\delta$ and a closed smooth hypersurface $H_V$ of $\varphi^{-1}(V \times D_\delta)$, and $H_V$ satisfies the conditions of a cut at a point $t \in V$, then there is an open neighborhood $V'$ of $p \in V$ such that $H_V \cap \varphi^{-1}(V' \times D_\delta)$ is a cut over $V'$ of amplitude $\delta$.

*Proof.* The proof is an application of h-cobordism Theorem as in the proof of Lé-Ramanujam Theorem [10].

### 2.3. Extension of cuts.

**Definition 4.** Let $V_1 \subset V_2$ be submanifolds of $U$ and $H_1$, $H_2$ cuts of the same amplitude $\delta$ over $V_1$ and $V_2$ respectively. We say that $H_2$ is an extension of $H_1$, if we have $H_2 \cap \varphi^{-1}(V_1 \cap D_\delta) = H_1$.

We consider the following situation: define $I := [0, 1]$. Denote by $C$ the $d$-dimensional cube $I^d$, with its natural structure of manifold with corners, and $U := (-\epsilon, 1 + \epsilon)^d$ for a certain $\epsilon > 0$.

All the cuts below are chosen with the same amplitude $\delta$. Consider two cuts $H_+$ and $H_-$ defined over $U$, with $H_- \prec H_+$. Let $A$ be a contractible union of faces of $C$. We consider a cut $H_0$ over a neighborhood $V$ of $A$, which satisfies $H_- \prec H_0 \prec H_+$. We consider also a number of pairs

$$(B_1, \mu_1), \ldots, (B_{k_1}, \mu_{k_1}) \quad (C_1, \nu_1), \ldots, (C_{k_2}, \nu_{k_2}),$$

where the $B_i$'s and the $C_i$'s are faces of $C$, each $\mu_i$ denotes a cut over a contractible neighborhood $B_i'$ of $B_i$ in $U$ and each $\nu_i$ denotes a cut over a contractible neighborhood $C_i'$ of $C_i$ in $U$. Suppose that the following relations hold: $H_- \prec \mu_i$ for any $i$, $H_- \prec \nu_i$ for any $i$, $\mu_i \prec \mu_j$ if $i < j$, $\mu_i \prec H_0$ for any $i$, $\mu_i \prec \nu_j$ for any $i, j$, $\mu_i \prec H_+$ for any $i$, $H_0 \prec \nu_i$ for any $i$, $\nu_i \prec \nu_j$ if $i > j$, and $\nu_i \prec H_+$ for any $i$.

We will need the following extension Lemma:
Lemma 5. In the setting above, after possibly shrinking $U$ and $V$ to smaller neighborhoods of the corresponding sets, and possibly decreasing $\delta$, there exists a cut $H'_0$ over $U$ with amplitude $\delta$ which extends $H_0$ and satisfies $H_- \prec H_0$, $\mu_i \prec H_0$ for any $i$, $H_0 \prec \nu_i$ for any $i$, and $H_0 \prec H_+$.  

Proof. Define

$$Y := Y_{int}(H_+, U, \delta) \setminus Y_{int}(H_-, U, \delta).$$

By Ehresmann Fibration Theorem and the contractibility of $U \times D_\delta$ the mapping

$$\phi : Y \to U \times D_\delta$$

is a trivial fibration. From the defining properties of a cut and an easy manipulation with cobordisms we deduce that each fibre of $\phi$ is a trivial cobordism.

We will use the following fact:

**Fact 1**: let $A$ be a contractible closed union of faces of $C$ and $V$ be a neighborhood of $A$ in $U$. There exist a compact neighborhood $D_1$ of $A$ in $V$, whose boundary $\partial D_1$ is a smooth hypersurface in $U$, an open neighborhood $D_2$ of $C$ in $U$ and a smooth mapping

(1) $$\xi : D_2 \setminus \hat{D}_1 \to \partial D_1$$

which is a trivial fibration over its image, which is contractible, with fibre diffeomorphic to the half-open interval $[0, 1)$ (the closed extreme of the fibres corresponding with the intersection of the fibre with $\partial D_2$). Let

(2) $$\Psi : D_2 \setminus \hat{D}_1 \to [0, 1) \times \xi(D_2 \setminus \hat{D}_1)$$

be a diffeomorphism giving a trivialisation.

This fact follows from easy geometrical considerations. As an example we do in detail the case in which $A = [0, 1]^{d_1} \times \{0\}^{d_2}$, with $d_1 + d_2 = d$. Let $\sigma$ denote the usual distance function in $\mathbb{R}^d$. We may consider

$$D_1 := \{p \in \mathbb{R}^d : \sigma(p, A) \leq \epsilon_1\},$$

$$D_2 := \{p \in \mathbb{R}^d : \sigma(p, C) < \epsilon_2\},$$

for $0 << \epsilon_2 << \epsilon_1 << 1$, and define $\xi(x_1, ..., x_d)$ to be the only intersection point of $\partial D_1$ with the segment joining $(x_1, ..., x_d)$ with $(x_1, ..., x_{d_1}, 0, ..., 0)$. We encourage the reader to draw a picture of this construction.

We shall proceed in two steps.

**Step 1**: we shall inductively reduce to the case in which $k_1 + k_2 = 0$. Suppose that $k_1 + k_2 > 0$. Assume that $k_2 > 0$.

Choose any point $t \in C_1$. From the defining properties of a cut, an easy manipulation with cobordisms, Ehresmann Fibration Theorem and the fact that $C'_1$ is contractible, it follows that the restrictions

$$\phi : Y_{int}(H_+, C'_1, \delta) \setminus Y_{int}(\nu_1, C'_1, \delta) \to C'_1 \times D_\delta,$$

$$\phi : Y_{int}(\nu_1, C'_1, \delta) \setminus Y_{int}(H_-, C'_1, \delta) \to C'_1 \times D_\delta,$$

are smooth trivial fibrations with fibre a trivial smooth cobordism. Hence, it is easy to construct a smooth vector field $\mathcal{Y}$ in $Y|_{C'_1 \times D_\delta}$ which is tangent to the fibres of $\phi$ such that its associated flow defines a diffeomorphism $\psi : (H_-)|_{C'_1 \times D_\delta} \times [0, 1] \to Y|_{C'_1}$ satisfying the equality

(3) $$Y_{int}(\nu_1, C'_1, \delta) \setminus Y_{int}(H_-, C'_1, \delta) = \psi((H_-)|_{C'_1 \times D_\delta} \times [0, 1/2]).$$
We are going to extend \( Y \) to a vector field \( \mathcal{X} \) defined in \( Y \). As \( C_1 \) is a face, we may choose \( D_1 \) to be the neighborhood of \( C_1 \) in \( C'_1 \), and \( D_2 \) the neighborhood of \( C \) in \( U \) so that the mapping \( \Phi \) exists as predicted by Fact 1. Using that \( \phi : Y \to U \times D_\delta \) is a trivial fibration, it is easy to prove that there exists a smooth diffeomorphism

\[
\Phi : Y_{D_2 \setminus \partial_1} \to [0,1) \times Y_{\xi(D_2 \setminus \partial_1)}
\]

such that, if \( q_i \) denotes the projection of \([0,1) \times Y_{\xi(D_2 \setminus \partial_1)}\) to the \( i \)-th factor \( (i = 1,2) \), then

(i) the mappings

\[
(\Psi, \text{Id}_{D_1}) \circ \phi : Y_{D_2 \setminus \partial} \to [0,1) \times \xi(D_2 \setminus \partial_1) \times D_\delta
\]

and

\[
(q_1, \phi|_{Y_{\xi(D_2 \setminus \partial_1)}} \circ \Phi : Y_{D_2 \setminus \partial_1} \to [0,1) \times \xi(D_2 \setminus \partial_1) \times D_\delta
\]

coincide,

(ii) the restriction \( q_2 \circ \Phi|_{Y_\xi(D_2)} \) is the identity; moreover, the mappings

\[
(0, \text{Id}_{Y_{D_1}}) : Y_{D_1} \to [0,1) \times Y
\]

and \( \Phi \) glue to a smooth mapping.

We define a smooth vector field \( \mathcal{X} \) in \( Y_{D_2} \) piecewise as follows: the restriction of \( \mathcal{X} \) to \( Y_{D_1} \) is equal to the restriction of \( \mathcal{X}|_{Y_{D_1}} \); the restriction of \( \mathcal{X} \) to \( Y(Y_{D_2 \setminus \partial_1}) \) is the pullback by \( \Psi \) of the vector field in \([0,1) \times Y_{\xi(D_2 \setminus \partial_1)}\) whose \([0,1)\)-component is zero and whose \( \xi(D_2 \setminus \partial_1) \)-component is equal to the restriction \( \mathcal{X}|_{Y_{\xi(D_2 \setminus \partial_1)}} \). Observe that \( \mathcal{X} \) is smooth, tangent to the fibres of \( \phi \), extends \( Y \), and its flow trivialises the cobordism given by each fibre of \( \phi \).

We redefine \( U := D_2 \). Then the flow of \( \mathcal{X} \) induces a diffeomorphism

\[
\varphi : H_- \times [0,1] \to Y.
\]

Let \( \{\theta_1, \theta_2\} \) be a partition of unity subordinated to \( \{C'_1 \times D_\delta, (U \setminus C_1) \times D_\delta\} \). Define \( \beta := 1/2\theta_1 + \theta_2 \). Let \( \sigma : Y \to [0,1] \) be the smooth function \( \sigma := p_2 \circ \varphi^{-1} \), with \( p_2 \) the projection of \( H_- \times [0,1] \) to the second factor. We claim that if \( C'_1 \) is a sufficiently small neighborhood of \( C_1 \) in \( U \), then:

(a) for any \( 1 \leq i \leq k_1 \) and any \( x \in \mu_i|_{B_1} \), we have \( \sigma(x) < \beta(\pi(x)) \),

(b) for any \( 2 \leq j \leq k_2 \) and any \( x \in \nu_j|_{C_2} \), we have \( \sigma(x) < \beta(\pi(x)) \), and

(c) for any \( x \in H_0 \) we have \( \sigma(x) < \beta(\pi(x)) \).

Consider \( x \in \mu_i|_{B_1} \cap D_1 \). Observe that we have Equality (8) together with the fact that \( \mu_i \preceq \nu_1 \) implies that \( \sigma(x) < 1/2 \). By continuity this inequality occurs in a neighborhood of \( x \) in \( \mu_i \). On the other hand \( \sigma(x) < 1 \) for any \( x \in \mu_i \). Hence, using that \( B_1 \times D_\delta \) is compact, it is easy to see that \( \beta \), by shrinking \( C'_1 \), we have \( \sigma(x) < \beta(\pi(x)) \) for any \( x \in \mu_i \). This shows property (a). The proofs of properties (b) and (c) are analogous.

Now, by continuity, shrinking \( B'_1 \), \( C'_1 \) and \( V \), we have

(a) for any \( 1 \leq i \leq k_1 \) and any \( x \in \mu_i|_{B'_i} \) we have \( \sigma(x) < \beta(\pi(x)) \),

(b) for any \( 2 \leq j \leq k_2 \) and any \( x \in \nu_j|_{C'_2} \) we have \( \sigma(x) < \beta(\pi(x)) \), and

(c) for any \( x \in H_0 \) we have \( \sigma(x) < \beta(\pi(x)) \).

By construction the subspace \( \nu'_i := \{\varphi(x, \beta(\pi(x))), x \in H_-\} \) is a closed smooth hypersurface of \( \pi^{-1}(U) \cap f^{-1}(D_\delta) \) which constitutes a cut over \( U \) and satisfies \( H_- \prec \nu'_i \), \( \mu_i \prec \nu'_i \) for any \( i \), \( H_0 \prec \nu'_i \) and \( \nu_i \prec \nu'_i \) for any \( i \neq 1 \).
Then we can redefine $H_+: = \nu'_t$ and we have $k_1 + k_2$ strictly smaller. We have worked out the case in which $k_2 > 0$. The case in which $k_1 > 0$ is identical. Inductively we are reduced to the case $k_1 + k_2 = 0$.

**Step 2:** We assume $k_1 = k_2 = 0$.

As $A$ is a contractible closed union of faces of $C$, a simpler version of the argument of Step 1 allows to construct, after eventually shrinking $U$ and $V$, a cut $H_0^\prime$ over $U$ which extends $H_0$ and satisfies

$$H_- \prec H_0^\prime \times H_+.$$

\[\square\]

2.4. Existence of cuts. In this section we let $L$ denote either a point or the circle $S^1$. Again $C$ denotes the cube $I^d \subset \mathbb{R}^d$. Let $U := L \times (-\epsilon, 1 + \epsilon)^d$ for a certain $\epsilon > 0$. We shall make the following assumption:

**Assumption A:** there exists $\zeta > 0$ such that $\rho^{-1}(e)$ meets $\phi^{-1}(t, 0)$ transversely for any $t \in L \times \{(0, ..., 0)\}$ and any positive $\epsilon \leq \zeta$.

**Remark 6.** Observe that, as the restriction of $\rho$ to each fibre of $\pi$ is real analytic, the assumption is satisfied automatically if $L$ is a point (see [11]), pages 21-25.

**Lemma 7.** Let $\theta : U \rightarrow \mathbb{R}_+$ be a positive continuous function. There exists a neighborhood $V$ of $L \times \{(0, ..., 0)\}$ in $U$, and positive $\epsilon$ and $\delta$ such that $H_V := \rho^{-1}(\epsilon) \cap \phi^{-1}(V \times D_\delta)$ is a cut over $V$ of amplitude $\delta$ such that $Y_{int}(H_V, V, \delta)$ is contained in $B(U, \theta)$.

**Proof.** The proof is easy after Lemmas 2 and 3. \[\square\]

**Proposition 8.** Let $\theta : U \rightarrow \mathbb{R}_+$ be any continuous function. After possibly shrinking $U$ to a smaller neighborhood of $L \times C$ in $L \times \mathbb{R}^d$, there exists positive $\epsilon$, $\delta$, a neighborhood $V$ of $L \times \{(0, ..., 0)\}$ in $U$ and a cut $H$ over $U$ of amplitude $\delta$ such that

1. the space $Y_{int}(H, U, \delta)$ is contained in $B(U, \theta)$,
2. the cut $H$ extends the cut $H_V := \rho^{-1}(\epsilon) \cap \phi^{-1}(V \times D_\delta)$ obtained in Lemma 4.

**Proof.** We will assume $L = S^1$. The proof for the case in which $L$ is a point is a simplification of the case presented here.

For any $t \in L \times C$ we choose a positive $\epsilon_t < \theta(t)$ such that $\rho(\phi^{-1}(t, 0) \cap \phi^{-1}(0, \alpha_t))$ is a submersion. By Lemma 2 there exists a neighborhood $U_t$ of $t$ in $U$ and a positive $\delta_t$ such that $H_+(U_t) := \phi^{-1}(U_t \times D_{\delta_t}) \cap \rho^{-1}(\epsilon_t)$ is a cut over $U_t$ with amplitude $\delta_t$ and $Y_{int}(H_+(U_t), U_t, \delta_t)$ is contained in $B(U, \theta)$.

By compactness of $L$ and $C$ there exists a finite cover $\{U_t\}_{t \in A}$ of $L \times C$ which contains a collection $\{U_t\}_{t \in A_0}$, with $A_0 \subset C \subset U$, which covers $L \times \{(0, ..., 0)\}$ and satisfies that $A_0 \subset L \times \{(0, ..., 0)\}$.

Let $\gamma : I \rightarrow S^1$ be the parametrisation defined by $\gamma(t) = e^{2\pi it}$. Given a partition

$$0 = \alpha_0 < \alpha_1 < ... < \alpha_r = 1$$

we define $I_j := [\alpha_{j-1}, \alpha_j]$ and also, for any multi-index $J = (j_0, ..., j_d)$, the cube

$$C_J := \gamma(I_{j_0}) \times \prod_{r=1}^d I_{j_r}.$$
which is contained in \( L \times C \). Choose the partition so fine that any cube \( C_j \) is contained at least in an open set of the cover \( \{ U_t \}_{t \in A} \), and moreover, if \( C_j \) meets \( L \times \{(0, ..., 0)\} \), then it is contained in an open set of the cover \( \{ U_t \}_{t \in A_0} \). We assign to each \( C_j \) a fixed set \( U_{t(j)} \) of the cover containing it, taking care that if \( C_j \) meets \( L \times \{(0, ..., 0)\} \) then \( t(J) \) belongs to \( A_0 \). We denote \( U_{t(j)} \) by \( U_J \). Define \( \epsilon_J := \epsilon_{t(j)} \). Define \( \delta_J := \delta_{t(j)} \). Choose \( \delta_{\min} \) and \( \epsilon_{\min} \) strictly smaller than any \( \delta_J \) and \( \epsilon_J \) respectively.

By Lemma 7 there exists a neighborhood \( V \) of \( L \times \{(0, ..., 0)\} \) in \( U \), positive \( \epsilon < \epsilon_{\min} \), \( \delta < \delta_{\min} \) such that \( H_V := \rho^{-1}(e) \cap \phi^{-1}(V \times D) \) is a cut over \( V \) of amplitude \( \delta \).

Choose a positive \( \eta \) strictly smaller than \( \epsilon \). For any \( t \in L \times C \) we choose a positive \( \xi_t < \eta \) such that \( \rho|_{\phi^{-1}((0,0) \cap \rho^{-1}((0,0)))} \) is a submersion. By Lemma 2 for any \( t \in L \times C \), there exists a neighborhood \( V_t \) of \( t \) in \( U \) and a positive \( \delta_t < \delta_{\min} \) such that \( H_{\rho^{-1}}(V_t) := \phi^{-1}(U_t \times D \delta_t) \cap \rho^{-1}(\xi_t) \) is a cut over \( V_t \) with amplitude \( \delta_t \). We pick \( V_t \) so small that it is contained in any \( U_{t(J)} \) which contains \( t \).

By compactness we can choose a finite cover \( \{ V_t \}_{t \in B} \) of \( L \times C \).

For each \( j \), given any subdivision

\[
\alpha_j = \alpha_{j,0} < \alpha_{j,1} < \ldots < \alpha_{j,s} = \alpha_{j+1},
\]

we define the intervals \( I_{j,k} := [\alpha_{j,k-1}, \alpha_{j,k}] \subset I \) and, for multi-indexes \( J = (j_0, ..., j_d) \) and \( K = (k_0, ..., k_d) \) we define the cubes

\[
C^{K}_J := \gamma(I_{j_0,k_0}) \times \prod_{r=1}^{d} I_{j_r,k_r}.
\]

Each cube \( C_J \) splits in an union of the cubes \( C^K_J \), where \( K \) vary.

We choose the subdivisions so fine that each cube \( C^K_J \) is contained in an open subset of the cover \( \{ V_t \}_{t \in B} \). We assign to each cube \( C^K_J \) a fixed open subset \( V_{t(J)} \) containing it. We denote \( V_{t(J)} \) by \( V^K_J \), and define \( \xi^K_J := \xi_{t(J)} \), \( \epsilon^K_J := \epsilon_J \) and \( \delta^K_J := \delta_{t(J)} \).

Given \( \zeta > 0 \) we define the intervals \( \tilde{I}_{j,k} := (\alpha_{i,j-1} - \zeta, \alpha_{i,j} + \zeta) \) and the open cubes

\[
\tilde{C}^{K}_J := \gamma(\tilde{I}_{j_0,k_0}) \times \prod_{r=1}^{d} \tilde{I}_{j_r,k_r}.
\]

Choose \( \zeta \) small enough so that

(i) the closure of \( \tilde{C}^{K}_J \) is contained in \( V^K_J \), and hence in \( U_J \),

(ii) for any choice of indexes, the sets \( \tilde{C}^{K}_J \) and \( \tilde{C}^{K'}_{J'} \) meet if and only if \( C^K_J \) and \( C^{K'}_{J'} \) meet.

Condition (i) and the compactness of the closure of \( \tilde{C}^{K}_{J'} \) imply that we can shrink \( \delta^K_J \) such that

\[
H_{+}(\tilde{C}^{K}_J) := \phi^{-1}(\tilde{C}^{K}_J \times D_{\delta^K_J}) \cap \rho^{-1}(\epsilon^K_J)
\]

and

\[
H_{-}(\tilde{C}^{K}_J) := \phi^{-1}(\tilde{C}^{K}_J \times D_{\delta^K_J}) \cap \rho^{-1}(\xi^K_J)
\]

are cuts over \( \tilde{C}^{K}_J \) with amplitude \( \delta^K_J \).

We choose \( \delta > 0 \) smaller than \( \delta^K_J \) for any choice of multi-indexes \( (J, K) \).

Given a fixed choice of multi-indexes \( (J, K) \) we consider the following cuts:
• for any indexes $(J', K')$ different from the fixed ones, and such that $\epsilon_{J'}^{K'} < \epsilon_j^K$ we consider the cut  
\[ \nu_j^{K'}(\tilde{C}_j^K) := \phi^{-1} ((\tilde{C}_j^K \cap \tilde{C}_j^{K'}) \times D_\delta) \cap \rho^{-1}(\epsilon_{J'}^{K'}) \]
over $\tilde{C}_j^K \cap \tilde{C}_j^{K'}$ with amplitude $\delta$,
• for any indexes $(J', K')$ different from the fixed ones, and such that $\xi_{J'}^{K'} > \xi_j^K$ we consider the cut  
\[ \mu_j^{K'}(\tilde{C}_j^K) := \phi^{-1} ((\tilde{C}_j^K \cap \tilde{C}_j^{K'}) \times D_\delta) \cap \rho^{-1}(\xi_{J'}^{K'}) \]
over $\tilde{C}_j^K \cap \tilde{C}_j^{K'}$ with amplitude $\delta$.

Because of Condition (i) and the compactness of the closure of $\tilde{C}_j^K$ it is clear that a small modification of $\epsilon_j^K$ and $\xi_j^K$ keeps all the properties obtained above. An adequate perturbation making the $\epsilon_j^K$’s and the $\xi_j^K$’s pairwise different gives rise to the following properties:

The cuts $\nu_j^{K'}(\tilde{C}_j^K)$ obtained by letting $(J', K')$ vary are linearly related by $\succeq$ and, moreover, if two multi-indexes $(J', K')$ and $(J'', K'')$ are such that $\tilde{C}_j^K$ and $\tilde{C}_j^{K''}$ meet then, if the cuts $\nu_j^{K'}(\tilde{C}_j^K)$ and $\nu_j^{K''}(\tilde{C}_j^K)$ are defined, they are necessarily related by $\prec$ in one of the directions. The same happens for the cuts $\mu_j^{K'}(\tilde{C}_j^K)$ obtained by letting $(J', K')$ vary. We have moreover

\[ \mu_j^{K'}(\tilde{C}_j^K) \prec \nu_j^{K''}(\tilde{C}_j^K), \]
\[ H_-(\tilde{C}_j^K) \prec \mu_j^{K'}(\tilde{C}_j^K), \]
\[ \mu_j^{K'}(\tilde{C}_j^K) \prec H_+(\tilde{C}_j^K), \]
\[ H_-(\tilde{C}_j^K) \prec \nu_j^{K'}(\tilde{C}_j^K), \]
\[ \nu_j^{K'}(\tilde{C}_j^K) \prec H_+(\tilde{C}_j^K) \]
for any choice of multi-indexes for which the expression makes sense.

Shrink $V$ to a neighborhood of $L \times \{(0, ..., 0)\}$ in $U$ which is small enough that it only meets $\tilde{C}_j^K$ if $\tilde{C}_j^K$ meets $L \times \{(0, ..., 0)\}$. If $C_j^K$ meets $L \times \{(0, ..., 0)\}$ we consider the cut  
\[ H_{\tilde{C}_j^K \cap V} := \phi^{-1} ((\tilde{C}_j^K \cap V) \times D_\delta) \cap H_V \]
obtained by restriction over $\tilde{C}_j^K \cap V$ of the previously constructed cut $H_V$.

By our constructions (choosing the perturbation of the $\epsilon_j^K$’s and the $\xi_j^K$’s small enough) we have:

(4) \[ H_-(\tilde{C}_j^K) \prec H_{\tilde{C}_j^K \cap V}, \]
(5) \[ H_{\tilde{C}_j^K \cap V} \prec H_+(\tilde{C}_j^K), \]
(6) \[ \mu_j^{K'}(\tilde{C}_j^K) \prec H_{\tilde{C}_j^K \cap V}, \]
(7) \[ H_{\tilde{C}_j^K \cap V} \prec \nu_j^{K'}(\tilde{C}_j^K). \]

We shall construct the cut $H$ extending $H_V$ step by step over the cubes $C_j^K$ using Lemma 6. Order the cubes $C_j^K$ lexicographically in $(j_0, ..., j_d, k_0, ..., k_d)$.
We start with the first cube, that is \((J, K) = (0, \ldots, 0)\). Observe that
\[
A := (S^1 \times \{(0, \ldots, 0)\}) \cap U_{(0, \ldots, 0)} = \gamma(I_0) \times \{0\}^d
\]
is a contractible union of faces of \(C_{0, \ldots, 0}^0\). We apply Lemma 9 and obtain a cut \(H(C_{(0, \ldots, 0)}^0)\) which extends \(H_{C_{(0, \ldots, 0)}^0} \cap V\) and satisfies the relations analogous to \((4)-(7)\) with respect to the cuts \(H_-(C_{(0, \ldots, 0)}^0), H_+(C_{(0, \ldots, 0)}^0), \mu_J^K(C_{(0, \ldots, 0)}^0)\) and \(v_J^K(C_{(0, \ldots, 0)}^0)\), for \((J', K')\) varying. We redefine \(V := V \cup C_{0, \ldots, 0}^0\) and the cut \(H_V\) as the union of the previous \(H_V\) and \(H(C_{(0, \ldots, 0)}^0)\). By construction, the new cut \(H_V\) satisfies the following property: for any \((J, K)\) the cut \(H_{C_{(0, \ldots, 0)}^0} \cap V\) given by the restriction of \(H_V\) over \(C_{(0, \ldots, 0)}^0\) satisfies relations \((4)-(7)\).

The inductive step runs similarly. 

\[\square\]

Remark 9. Choose any point \(p \in C\). The above proposition is valid (with the same proof, up to some notational complication) if we let \(L \times \{p\}\) play the role of \(L \times \{(0, \ldots, 0)\}\) both in Assumption A and in the statement of the Proposition.

2.5. Topological equisingularity of \(\mu\)-constant families. Let \(\theta_1 : U \to \mathbb{R}_+\) be any continuous function. We choose a positive \(\delta_1\) and a cut \(H_1\) as predicted by Proposition 8 In order to simplify the notation we denote \(Y_{\text{int}}(H_1, U, \delta)\) simply by \(Y_1\). We view \(U\) embedded in \(Y_1\) as the zero section of the vector bundle \(\pi : E \to U\). Define \(Y_1^* := Y_1 \setminus f^{-1}(0)\).

Lemma 10. There exists a smooth vector field \(\mathcal{X}\) in \(Y|_{L \times C} \setminus (L \times C)\) with the following properties:

1. It is tangent to the fibres of \(\pi\).
2. There exists a vector field \(W\) in \(D_{1,1}^*\), which is radial, pointing to the origin and of modulus \(|W(z)| = |z|^2\), such that \(df(X)(x) = W(f(x))\) for any \(x \in Y_1^*\).
3. The vector field \(\mathcal{X}\) is tangent to \(f^{-1}(0)\).
4. Any integral curve converges to the origin of the fibre of \(\pi\) in which it lies in infinite time.

Proof. We will construct \(\mathcal{X}\) as the amalgamation of two vector fields \(Y\) and \(Z\). We define each of them separately. Denote by \(V\) the vector field in \(D_{1,1}^*\) which is radial, pointing at the origin and of modulus \(|V(z)| = |z|^2\). Consider in \(U \times D_{1,1}^*\) the vector field \(V'\) characterised by being tangent to the fibres of the projection of \(U \times D_{1,1}^*\) to the first factor, and a lift by the projection of \(U \times D_{1,1}^*\) to the second factor of the vector field \(V\). As \(\phi : Y_1^* \to U \times D_{1,1}^*\) is submersive we can define the vector field \(Y'\) in \(Y_1^*\) to be a lifting of \(V'\) by the mapping \(\phi\).

For the definition of \(Z\) some auxiliary constructions are needed. We will shrink \(U\) when necessary without explicitly mentioning it. There is a continuous function \(\theta_2 : U \to \mathbb{R}\) such that the inclusion \(B(U, \theta_2) \subset Y_1\) is satisfied. Applying Proposition 5 to the function \(1/2\theta_2\) we obtain new positive \(\delta_2\) (which we choose to be smaller than \(\delta_1/2\)) and a cut \(H_2\) of amplitude \(\delta_2\) such that \(Y_2 := Y_{\text{int}}(H_2, U, \delta_2)\) is contained in \(B(U, 1/2\theta_2)\).

We iterate this procedure to obtain an infinite sequence of cuts \(H_i\) over \(L \times C\) with amplitude \(\delta_i\), which gives rise to an infinite, nested, sequence
\[
Y_1 \supset Y_2 \supset \ldots \supset Y_i \supset \ldots
\]
of closed neighborhoods of \( L \times C \), satisfying that \( \cap_{i=1}^{\infty} Y_i \) is equal to the zero section.

Define \( Z_i := Y_i \cap f^{-1}(D_{\delta_{i+1}}) \setminus \hat{Y}_{i+1} \). By Property (1) of Definition \( \ref{def:restricted} \) the restriction
\[
\phi_i := \phi|_{Z_i} : Z_i \to L \times C \times D_{\delta_{i+1}}
\]
is a locally trivial fibration. By Property (3) of Definition \( \ref{def:restricted} \) and an easy manipulation with cobordisms, its generic fibre is a trivial cobordism.

By the second property that Proposition \( \ref{prop:fields} \) predicts for the cut \( H_i \) there is a neighborhood \( V_i \) of \( L \times \{ (0, \ldots, 0) \} \) in \( U \) and a positive \( \epsilon_i \) such that
\[
H_i \cap \pi^{-1}(V_i) = \phi^{-1}(V_i \times D_{\delta_i}) \cap \rho^{-1}(\epsilon_i).
\]
We have that \( \{ \epsilon_i \}_{i \in \mathbb{N}} \) is a decreasing sequence converging to 0.

Observe that we have
\[
Z'_i := Z_i \cap \pi^{-1}(V_{i+1}) = \phi^{-1}(V_{i+1} \times D_{\delta_{i+1}}) \cap \rho^{-1}([\epsilon_i, 1]).
\]
Using Assumption A, and possibly shrinking \( V_{i+1} \) and \( \delta_{i+1} \) we obtain that the restriction of \( \rho \) to each fibre of \( \phi_i|_{Z'_i} \) is a submersion. Therefore there exists a vector field \( Z'_i \) on \( Z'_i \) which is tangent to the fibres of \( \phi_i|_{Z'_i} \), non-zero at any point, and such that \( d\rho(Z'_i) = -1 \). Consequently, its flow takes \( (Y_i \cap H_i \cap f^{-1}(D_{\delta_{i+1}}))_{V_i} \) to \( (Y_{i+1} \cap H_{i+1})_{V_i} \) in finite time \( T_i = \epsilon_i - \epsilon_{i+1} \).

We may suppose that \( V_i \) is of the form \( L \times B_i \), with \( B_i \) a small ball centered around \( (0, \ldots, 0) \) in \( \mathbb{R}^d \). Using that \( C \) is a cube and the fact that \( \phi_i \) is a locally trivial fibration, we may (in the same way that we extended \( \gamma \) to \( \chi \) in Step 1 of the proof of Lemma \( \ref{lem:extension} \) extend \( Z'_i \) to a vector field \( Z_i \) on \( Z_i \) which is tangent to the fibres of \( \phi_i \), non-zero at any point, and whose flow takes \( Y_i \cap H_i \cap f^{-1}(D_{\delta_{i+1}}) \) to \( Y_{i+1} \cap H_{i+1} \) in time \( T_i \). We rescale the vector field in order to ensure that \( T_i = 1 \).

As, for any \( i \in \mathbb{N} \), both \( Z_i \) and \( Z_{i+1} \) are transverse to \( H_{i+1} \), and both of them point into \( Y_{i+1} \). Using an adequate partition of unity, we may glue the vector fields \( \{ Z_i \}_{i \in \mathbb{N}} \) to a vector field \( Z \) defined in \( \cup_{i \in \mathbb{N}} Z_i \), nowhere vanishing and tangent to the fibres of \( \phi \). Up to a rescaling of \( Z \) we may assume that its flow takes \( Y_i \cap H_i \cap f^{-1}(D_{\delta_{i+1}}) \) to \( Y_{i+1} \cap H_{i+1} \) in time \( T_i = 1 \).

Let \( \rho_1 : D_\delta \to [0, 1] \) be a smooth function vanishing at 0 and positive in \( D_\delta^* \). Let \( \rho_2 : U \to \mathbb{R} \) be an smooth function with support contained in \( \cup_{i \in \mathbb{N}} Z_i \) and which is identically 1 in a neighborhood of \( f^{-1}(0) \cap (\cup_{i \in \mathbb{N}} Z_i) \) in \( \cup_{i \in \mathbb{N}} Z_i \). Define the vector field \( \chi := (\rho_1 \circ f') \gamma + \rho_2 Z \) on \( Y \). The first three properties that \( Z \) must satisfy are clear by construction (for the second one we take \( \mathcal{W} = \rho_1 V' \)).

For the fourth observe that as \( \sum_{i=1}^{\infty} T_i = \infty \) the convergence to the origin in infinite time is clear for any curve contained in \( f^{-1}(0) \). Let \( \gamma(t) \) be an integral curve lying off \( f^{-1}(0) \). By property (2) and the fact that \( \chi \) is tangent to the fibres of \( \pi \) (which are compact) it must accumulate in infinite time to a point in \( f^{-1}(0) \), but since \( \chi \) is defined and non-zero at \( f^{-1}(0) \setminus E_0 \), the only possibility is that \( \gamma \) satisfies property (4). \( \square \)

**Notation.** Let \( H \), \( \theta_1 \), \( Y_1 \), \( \delta_1 \) as in the beginning of the section. Denote the cut \( H_1 \) by \( H \), the space \( Y_1 \) by \( Y \), and \( \delta_1 \) by \( \delta \) for simplicity of notation. Consider the natural projections \( \sigma_1 : L \times C \to L \), \( \sigma_2 : L \times C \to C \). We view \( L \) as a subset of \( L \times C \) via its natural identification with \( L \times \{ (0, \ldots, 0) \} \). View \( L \times C \) as a subset of \( Y \) identifying it with the zero section of \( E|_{L \times C} \). Given a subset \( A \subset E \), we define \( \partial_A := \cup_{t \in U} \partial A_t \), that is, the union of the frontiers of the fibres over \( U \).
The globalisation of Timourian’s [19] and King’s [8] result that we will need is the following:

**Theorem 11.** There exist a homeomorphism

\[ \Psi : Y \to Y_L \times C \]

such that

1. We have the equality \( \pi = (\sigma_1 \circ \pi|_{Y_L}, \sigma_1 \circ \partial_1, \sigma_2 \circ \Psi) \), where \( p_i \) is the projection of \( Y_L \times C \) to the \( i \)-th factor \((i = 1,2)\).
2. We have the equality \( f|_{Y_L} \circ \Psi = f \).
3. The restriction of \( \Psi \) to \( Y \setminus (L \times C) \) is smooth.

**Proof.** By Ehresmann fibration Theorem the mappings

\[ \phi|_{Y \cap f^{-1}(\partial D_L)} : Y \cap f^{-1}(\partial D_L) \to U \times \partial D_L, \]

\[ \phi|_{H \cap Y} : H \cap Y \to U \times D_L \]

are locally trivial fibrations, (whose fibre are diffeomorphic respectively to the Milnor fibre of \( f_t \), and the abstract link of \( f_t \) for any \( t \in U \)).

Observe that we have

\[ \partial_\pi Y = (Y \cap f^{-1}(\partial D_L)) \cup (H \cap Y), \]
\[ \partial_\pi (Y \cap f^{-1}(\partial D_L)) = \partial_\pi (H \cap Y) = (Y \cap f^{-1}(\partial D_L)) \cap (H \cap Y) = Y \cap f^{-1}(\partial D_L) \cap H. \]

Using the fibrations above and the fact that \( C \) is a cube (and hence contractible) we will construct a diffeomorphism

\[ \psi : \partial_\pi Y \to C \times \partial_\pi Y_L \]

such that, if let \( \psi \) play the role of \( \Psi \), properties (1) and (2) are true at the points where \( \psi \) is defined. We define first the restriction of \( \psi \) to \( Y \cap f^{-1}(\partial D_L) \), and after extend it to \( H \cap Y \).

Lift a radial vector field \( Z_\lambda \) stemming from \((0,...,0)\) in \( C \), first to a vector field \( Z_2 \) in \( L \times C \) whose projection to \( L \) vanishes, and then to a vector field \( Z_3 \) in \( Y \cap f^{-1}(\partial D_L) \) which is tangent to the fibres of \( f \). The existence of the lifting \( Z_2 \) is obvious, and the existence of the lifting \( Z_3 \) follows because the mapping \( \mathbf{10} \) is locally trivial. The inverse of the restriction of \( \psi \) to \( Y_1 \cap f^{-1}(\partial D_L) \) can be obtained easily from the flow of \( Z_3 \).

Now we extend \( \psi \) to \( H \cap Y \). We consider a lift \( Z_4 \) of \( Z_2 \) in \( H \cap Y \) which is tangent to the fibres of \( f \), and which coincides with \( Z_3 \) at their common domain \( Y \cap f^{-1}(\partial D_L) \cap H \) (this lift exists by the local triviality of the mapping \( \mathbf{11} \)). The inverse of the desired extension is constructed easily from the flow of \( Z_3 \).

In order to obtain \( \Psi \) we have to extend \( \psi \) to the interior of \( Y \).

The set \( Z := \partial_\pi Y \times [0, \infty) \) is a manifold with corners The properties of the vector field \( X \) constructed in Lemma \( \mathbf{12} \) easily imply that, if \( \varphi \) denotes its flow, then the restriction

\[ \varphi|_Z : Z \to Y \setminus L \times C \]

is a diffeomorphism.

Define the restriction \( \Psi|_{Y \setminus L \times C} \) as the composition of the following sequence of mappings

\[ Y \setminus L \times C \xrightarrow{\varphi|_Z^{-1}} \partial_\pi Y \times [0, \infty) \xrightarrow{(\psi, id_{[0,\infty)})} C \times \partial_\pi Y_L \times [0, \infty) \xrightarrow{(id_C, \varphi|_Z)} C \times Y_L, \]
and the restriction $\Psi|_{L \times C}$ as
\[ \Psi|_{L \times C} := Id_{L \times C}. \]

By construction each of the restrictions $\Psi|_{Y \setminus L \times C}$ and $\Psi|_{L \times C} := Id_{L \times C}$ is smooth. The mapping $\Psi$ is continuous at $L \times C$ because $\psi$ satisfies Property (1) of the statement. Property (1) of the statement is also satisfied by $\Psi$ because $\psi$ satisfies it and $X$ satisfies Property (1) of Lemma 10. Property (2) of the statement is satisfied by $\Psi$ because $\psi$ satisfies it and $X$ satisfies Property (2) of Lemma 10. □

**Corollary 12.** Let $f : C \times (C^n, O) \to C$ be a smooth family of holomorphic germs defining isolated singularities at the origin with constant Milnor number. Let $p$ be any base point in $C$. Define the space $Y$ as in Theorem 11. There exists a homeomorphism $\Psi : Y \to C \times Y|_p$ satisfying properties (1 - 3) of Theorem 11.

**Proof.** We only have to notice that Assumption A is satisfied (Remark 6). □

**Remark 13.** The above result improves the results of King and Timourian ([8], [19]) in the sense that it is global in the base (we do not need to pick a small neighborhood of $p \in C$ in order to find the trivialisation), and also because the trivialisation $\Psi$ is smooth at all the strata where it can possibly be (smoothness at the origin can not be expected since isolated singularities have non-trivial moduli).

2.6. **Families over Riemann surfaces.** Now we prove Assumption A in an important situation in which $L = S^1$.

Let $S$ be a compact surface diffeomorphic to $S^1 \times I$ embedded as a smooth submanifold in a Riemann surface $\tilde{S}$. As we are only interested in a neighborhood of $S$ we can assume that $\tilde{S}$ is diffeomorphic to $S^1 \times (-\epsilon, 1 + \epsilon)$ for a certain positive $\epsilon$; we fix a product decomposition and denote by
\[ \alpha : \tilde{C} \to (-\epsilon, 1 + \epsilon) \]
the projection to the second factor. We suppose that $U = \tilde{C}$ and that the vector bundle $\pi : E \to U$ and the function $f$ are holomorphic. Recall that we have defined $\phi := (\pi, f)$.

**Proposition 14.** There is a finite subset $J \subset I$ such that, for any $p \in I \setminus J$, there exists $\zeta > 0$ satisfying that $\rho^{-1}(\zeta)$ meets $\varphi^{-1}(t, 0)$ transversely for any positive $\epsilon \leq \zeta$ and any $t \in \alpha^{-1}(p)$. In other words, up to a finite number of exceptional $p \in I$, assumption A is satisfied over $\alpha^{-1}(p) = S^1 \times \{p\}$.

**Proof.** Recall that the critical set of the function $f$ coincides with the zero section of $\pi$ and the restrictions of $f$ to the fibres of the bundle $\pi$ have an isolated singularity at the origin with Milnor number not depending on the particular fibre.

By the holomorphicity of $\pi$ and $f$, the locus $Z$ in which the stratification
\[ \{f^{-1}(0) \setminus \tilde{C}, \tilde{C}\} \]
fails to satisfy Whitney conditions is the complex analytic subset where the $\mu^*$ sequence jumps. The set $Z$ is of dimension 0, and hence discrete. Thus $Z \cap S$ is finite. Define $J := \alpha(Z) \cap I$.

Suppose $p \in I \setminus J$. If the statement of the proposition fails there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $E_{\alpha^{-1}(p)}$, converging to a point $x$ in the zero section over $\alpha^{-1}(p)$ such
that $x_n$ is a critical point for the restriction of $\rho$ to $\varphi^{-1}(0) \cap E_{\varphi(x_n)}$. Then it is easy to check that the tangent space $T_{x_n}\varphi^{-1}(0)$ of $\varphi^{-1}(0)$ at $x_n$ contains the orthogonal complement (by the hermitian inner product) of the line $\varphi(x_n), x_n$ in $E_{\varphi(x_n)}$. Choosing a subsequence we may assume that $T_{x_n}\varphi^{-1}(0)$ converges to a complex hyperplane $H$ of $T_xE$ and that $\varphi(x_n), x_n$ converges to a line $l$ contained in $\varphi^{-1}(\varphi(x))$, which satisfies that $H$ contains the hermitian orthogonal complement of $l$ in $\varphi^{-1}(\varphi(x))$. As $p$ does not belong to $J$, the stratification satisfies Whitney conditions at $x$, and therefore we have that $l$ is also included in $H$. Then $H$ is in fact equal (by dimensional reasons) to $\varphi^{-1}(\varphi(x))$.

Now we choose a sequence $\{y_n\}_{n \in \mathbb{N}}$, with $f(y_n) \neq 0$ and $y_n$ so close to $x_n$ that the sequence of hyperplanes $T_{y_n}\varphi^{-1}(f(y_n))$ has limit $H$. Such sequence shows that Thom’s $A_f$-condition is not satisfied at $x$ by the open stratum of the stratification

\begin{equation}
\{ E \setminus \varphi^{-1}(0), \varphi^{-1}(0) \setminus \tilde{C}, \tilde{C} \},
\end{equation}

which is in contradiction with the fact that the transversal Milnor number of $f$ is constant along $\tilde{C}$.

We recall our original setting: consider a smooth manifold $U = S^1 \times (-\epsilon, \epsilon)^d$, a smooth complex vector bundle $\varphi : E \to U$ and a complex smooth function $f : E \to \mathbb{C}$ whose restriction to the fibres of $\varphi$ is holomorphic. Suppose that $f$ has an isolated singularity at the origin of each fibre of $\varphi$, with Milnor number independent of the fibre.

The next corollary allows to apply the topological equisingularity results obtained in this section in many situations.

**Corollary 15.** Suppose $L = S^1$. Assume that there is a smoothly embedded path $\gamma : (-\zeta, \zeta) \hookrightarrow [0, 1]^n$ (with image $B := \gamma(-\zeta, \zeta)$) such that $\tilde{S} := S^1 \times B$ has a structure of Riemann surface compatible with the smooth structure, and that $E_{\tilde{S}}$ has a structure of holomorphic vector bundle which makes the restriction of $f|_{E_{\tilde{S}}}$ holomorphic. Then, for a generic point $p \in B$, Assumption A is satisfied if we let $S^1 \times \{p\}$ play the role of $S^1 \times \{(0, \ldots, 0)\}$. Therefore the statement of Proposition 8 holds letting $S^1 \times \{p\}$ play the role of $S^1 \times \{(0, \ldots, 0)\}$ (Remark 9). Hence also all the topological equisingularity results stated above hold for the family $f$.

### 3. Adapted Neighborhoods

Let $f : (\mathbb{C}^n, O) \to \mathbb{C}$ be any holomorphic germ. For the homotopical study of the Milnor fibration of $f$, different classes of systems of neighborhoods of the origin are possible (see the definition of system of Milnor neighborhoods in [14] page 105).

For the topological study of the Milnor fibration and of the embedded link of a function one has to be more careful. In [11] pages 21–25 it is proved that if $\rho : \mathbb{C}^n \to \mathbb{R}$ is non-negative continuous, real analytic except at the origin, and such that $\rho^{-1}(0) = \{O\}$ (we will refer to it as an analytic distance function), then, for $\epsilon$ small enough, the system of $\rho$-balls $\{\rho^{-1}([0, \epsilon])\}$ is a system of Milnor neighborhoods for $f$. Moreover it is shown that the homeomorphism type of the Milnor fibration and the embedded link of $f$ is independent of the chosen analytic distance function. We take as definition up to homeomorphism of Milnor fibration and embedded link of $f$ the ones obtained with an analytic distance function. This choice coincides with Milnor’s original definition (see [15]).
We will need a kind of systems of neighborhoods that can not be defined with analytic distance functions. We will have to check that the Milnor fibration and the embedded link defined with this kind of neighborhoods are, up to homeomorphisms, the same than the ones defined above.

**Definition 16.** Let $M$ be a manifold and $\Sigma \subset M \times \mathbb{C}^n$ be a smooth submanifold such that the restriction of the projection $\pi : M \times \mathbb{C}^n \to M$ to $\Sigma$ is a submersion and for any $m \in M$ the fibre $\Sigma_m \subset \{m\} \times \mathbb{C}^n$ is a complex submanifold of dimension 1 (a Riemann surface). Denote by $T(M \times \mathbb{C}^n, M) \to M \times \mathbb{C}^n$ the relative tangent bundle of $\pi$. Denote by $\mathcal{T}(\Sigma, M) \to \Sigma$ the relative tangent bundle of the submersion $\pi|_\Sigma$. Let $S \subset \Sigma$ be a submanifold. A normal bundle to $S$ over $\Sigma$ is a complex vector subbundle $p : N \to S$ of rank $n-1$ of $T(M \times \mathbb{C}^n, M)|_S$ such that $N + \mathcal{T}(\Sigma, M)|_S = \mathcal{T}(M \times \mathbb{C}^n, M)|_S$, (that is, its fibre $N_p$ is transverse to $\mathcal{T}(\Sigma, M)_p$ for any $p \in S$).

We endow $N$ with the hermitian metric coming from a hermitian product in $M \times \mathbb{C}^n$. If $S$ is compact there exists a positive $\eta$ such that the tubular neighborhood $T_N(\eta) := B(S, \eta) \subset N$ of the zero section of $N$ is naturally embedded in $M \times \mathbb{C}^n$.

**Assumption.** Assume from this point that the critical set $\Sigma_f$ of $f$ has dimension at most 1. Denote by $\Sigma_f^m$ the smooth part of $\Sigma_f$.

Observe that $f^{-1}(0)$ is a stratified space with the stratification $(f^{-1}(0) \backslash \Sigma_f, \Sigma_f^m, \{O\})$.

**Definition 17.** A system of neighborhoods of the origin adapted to $f$ is a system \(\{U_\alpha\}_{\alpha \in A}\) of neighborhoods of the origin such that for any $\alpha \in A$ there exists $\delta_\alpha > 0$ (which is called a Milnor disk radius for $U_\alpha$), satisfying

1. the neighborhood $U_\alpha$ is a compact manifold with boundary $\partial U_\alpha$.
2. There is a tubular neighborhood $W$ of $\Sigma_f \cap \partial U_\alpha$ in $\partial U_\alpha$ such that $W$ coincides with $T_N(\eta)$ for a certain normal bundle to $\Sigma_f^m$ called
   \[ p := N \to \Sigma_f \cap \partial U_\alpha, \]
   and a certain $\eta > 0$.
3. The function $f$ has no critical points in $U_\alpha \cap f^{-1}(D_{\delta_\alpha}^1)$, and for any $s \in D_{\delta_\alpha}$ the boundary $\partial U_\alpha$ meets $f^{-1}(s)$ transversely (in the stratified sense for $s = 0$).
4. For any $\beta$ such that $U_\beta \subset U_\alpha$, and any $\delta < \min\{\delta_\alpha, \delta_\beta\}$ there is a homeomorphism
   \[ \kappa : (U_\alpha \setminus \hat{U}_\beta) \cap f^{-1}(D_\delta) \to \partial U_\beta \cap f^{-1}(D_\delta) \times [0, 1] \]
   whose restrictions to \(((U_\alpha \setminus \hat{U}_\beta) \cap f^{-1}(D_\delta)) \setminus \Sigma_f)\) and to \((U_\alpha \setminus \hat{U}_\beta) \cap \Sigma_f\) are smooth and which satisfies $f = f \circ p_1 \circ \kappa$, where $p_1$ is the projection of $\partial U_\beta \cap f^{-1}(D_\delta) \times [0, 1]$ to the first factor.
5. For $\delta > 0$ small enough the space $U_\alpha \cap f^{-1}(D_\delta)$ is contractible.

**Remark 18.** The fourth condition of the last definition implies that the space

\[ f^{-1}(s) \cap U_\alpha \setminus \hat{U}_\beta \]

is a trivial (stratified when $s = 0$) cobordism for any $s \in D_\delta$. 
Let \( \{ U_\alpha \}_{\alpha \in A} \) be a system of neighborhoods of the origin adapted to \( f \). By Ehresmann Fibration Theorem, for any smooth neighborhood \( U_\alpha \) of the origin adapted to \( f \), the mapping
\[
(17) \quad f : U_\alpha \cap f^{-1}(D_{\delta_\alpha}^s) \to D_{\delta_\alpha}^t
\]
is locally trivial. Its restriction over \( \partial D_{\delta_\alpha} \) is called the \textit{Milnor Fibration} of \( f \) for \( U_\alpha \).

We define the \textit{embedded link} of \( f \) in \( U_\alpha \) as the pair
\[
(18) \quad (\partial(U_\alpha \cap f^{-1}(D_{\delta})) \cap f^{-1}(0)) \cap \partial(U_\alpha \cap f^{-1}(D_{\delta}))
\]
The \textit{abstract link} is, by definition, the space \( f^{-1}(0) \cap \partial(U_\alpha \cap f^{-1}(D_{\delta})) \).

Property (4) of Definition 17 implies that the topological type of the embedded link, and the diffeomorphism type of the Milnor fibration are independent of \( \alpha \in A \). Hence the the Milnor fibration and the embedded link associated to a system of neighborhoods adapted to \( f \) are well defined objects.

**Lemma 19.** The homotopy type of the abstract link and of the Milnor fibre is independent on the system of neighborhoods adapted to \( f \) used to define it. Any system of neighborhoods adapted to \( f \) is a system of Milnor neighborhoods for \( f \) (14, Definition/Proposition A.1).

**Proof.** Suppose that we have two systems of neighborhoods \( \{ U_\alpha \}_{\alpha \in A} \) and \( \{ V_\beta \}_{\beta \in B} \) adapted to \( f \). Consider sequences \( \{ \alpha_n \}_{n \in \mathbb{N}}, \{ \beta_m \}_{m \in \mathbb{N}} \) such that \( U_{\alpha_n} \subset V_{\beta_m} \) and \( V_{\beta_{m+1}} \subset \dot{U}_{\alpha_n} \). By definition of adapted system of neighborhoods, for any \( n \) there exists \( \delta \) such that the cobordisms \( f^{-1}(s) \cap (U_{\alpha_n} \setminus \dot{U}_{\alpha_{n+1}}) \) and \( f^{-1}(s) \cap (V_{\beta_m} \setminus \dot{U}_{\beta_{m+1}}) \) are trivial for any \( s \in D_{\delta} \). It is easy to deduce that the cobordisms
\[
(19) \quad f^{-1}(s) \cap (V_{\beta_m} \setminus \dot{U}_{\alpha_n})
\]
are homotopically trivial for any sufficiently small \( s \). This implies the lemma. \( \square \)

We will prove:
- the Milnor fibration of \( f \) for \( \{ U_\alpha \}_{\alpha \in A} \) is \( C^\infty \)-equivalent with the classical Milnor fibration of \( f \) defined by Milnor.
- the topological type of the embedded link associated to any neighborhood adapted to \( f \) coincides with the classical embedded link defined by Milnor.

In the remaining part of the section we exhibit a particular system of neighborhoods of the origin adapted to \( f \) for which these assertions are true. After this, it is enough to show that the Milnor fibration and the embedded link are independent of the neighborhood of the origin adapted to \( f \). This will be achieved in the next sections.

We will use the following Lemmas several times in the sequel.

**Lemma 20.** Let \( f : (\mathbb{C}^n, x) \to \mathbb{C} \) be a holomorphic germ with smooth 1-dimensional critical locus \( \Sigma_f \). Suppose that the generic transversal Milnor number of \( f \) at any point \( y \in \Sigma_f \) is equal to a constant \( \mu \). If \( H \) is a hyperplane transverse to \( \Sigma_f \) through \( x \) then \( \mu(f|_H, x) = \mu \).

**Proof.** The Lê numbers of \( f \) with respect to a generic coordinate system \( Z_0 \) are \( \lambda_{f, Z_0}(y) = \mu \) and \( \lambda_{f, Z_0}^1(y) = 0 \) for any \( y \in \Sigma_f \). As \( H \) is transverse to \( \Sigma_f \) there is another prepolar coordinate system, \( Z_1 := \{ z_1, ..., z_n \} \), with \( V(z_1) = H \). The first Lê number \( \lambda_{f, Z_1}(x) \) is equal to the Milnor number \( \mu' \) of \( f|_{z_1^{-1}(z_1(y))} \) at \( y \), for \( y \in \Sigma_f \) generic. We claim that \( \mu' = \mu \).
As \( \mu \) is the generic transversal Milnor number we have \( \mu' \geq \mu \). By definition of the first Lê number we have the equality \( \lambda^1_{f,z_i}(x) = \lambda^1_{f,z_i}(y) \) for any \( y \in \Sigma_f \). Hence as the alternating sum of the Lê numbers equals the reduced Euler characteristic of the Milnor fibre, which is equal to \( \mu \), if \( \mu' > \mu \) then \( \lambda^0_{f,z_i}(y) \neq 0 \). On the other hand, the locus where \( \lambda^0_{f,z_i}(y) \) is different from zero is the intersection locus of the relative polar curve \( \Gamma^1_{f,z_i} \) with \( \Sigma_f \). As, by definition of \( \Gamma^1_{f,z_i} \), it can not contain \( \Sigma_f \) we have that \( \lambda^1_{f,z_i}(y) = 0 \) at most points. Hence the claim is true.

As \( \mu \) is the generic transversal Milnor number we have \( \mu(f|_H, x) \geq \mu \). If the inequality is strict then the relative polar curve \( \Gamma^1_{f,z_i} \) meets \( \Sigma_f \) at \( x \), and hence \( \lambda^0_{f,z_i}(x) > 0 \). Then, as \( \lambda^1_{f,z_i}(x) = \mu \), the alternating sum of the Lê numbers cannot be equal to the reduced Euler characteristic \( \mu \). We get a contradiction which forces the equality \( \mu(f|_H, x) = \mu \). \( \square \)

**Remark 21.** Here we sketch an alternate proof of the previous Lemma suggested by T. Gaffney: let \( H_t \) be a family of generic hyperplanes, so \( f|_{H_t} \) is a \( \mu \)-constant family. Then Thom’s \( A_f \) condition holds, and therefore no hyperplane transverse to the critical line can be a limit of tangent hyperplanes. A result of Lê-Henry states that if \( H \) is not a limit of tangent hyperplanes then \( f|_H \) has the generic transversal Milnor number.

**Lemma 22.** Let \( M \) be a manifold and \( \Sigma \subset M \times \mathbb{C}^n \) be a smooth submanifold such that \( \pi|_{\Sigma} \) is a submersion and, for any \( m \in M \), the fibre \( \Sigma_m \subset \{m\} \times \mathbb{C}^n \) is a complex submanifold of dimension 1. Consider a compact submanifold \( S \subset \Sigma \) and a normal bundle \( p : N \to S \) to \( \Sigma \) over \( S \). There exists a normal bundle \( p' : N' \to \Sigma \) to \( \Sigma \) whose restriction to \( S \) coincides with \( p : N \to S \).

**Proof.** Let \( \sigma : \mathbb{P} \to M \times \mathbb{C}^n \) be the projectivised dual of \( T(M \times \mathbb{C}^n, M) \to M \times \mathbb{C}^n \). An element of \( \mathbb{P} \) is a pair \((H, x)\) where \( x \in M \times \mathbb{C}^n \) and \( H \) is a hyperplane of \( T(M \times \mathbb{C}^n, M)_x \). Define \( Z \subset \mathbb{P}|_{\Sigma} \) as

\[
Z := \{(H, x) \in \mathbb{P}|_{\Sigma} : H \cap T(\Sigma, M)_x\}.
\]

As \( T(\Sigma, M)_x \) is a line in \( T(M \times \mathbb{C}^n, M)_x \) the fibre \( Z_x \) is a hyperplane of the projective space \( \mathbb{P}_x \). Hence the restriction

\[
\sigma|_{\mathbb{P}|_{\Sigma}} \setminus Z : \mathbb{P}|_{\Sigma} \setminus Z \to \Sigma
\]

is an affine bundle. Any affine bundle admits a differentiable global section, and hence can be given a structure of vector bundle.

Observe that giving a normal bundle to \( \Sigma \) over a submanifold \( S \) of \( \Sigma \) is the same that giving a smooth section of the vector bundle \( \mu \) over \( S \). Using a partition of unity it is possible to extend the section smoothly to a global one. \( \square \)

**3.1. Construction of adapted neighbourhoods.** Consider any hermitian product in \( \mathbb{C}^n \) and let \( \rho \) be the distance function to the origin induced by it. Denote by \( B_\epsilon \) the \( \rho \)-ball centered at the origin of radius \( \epsilon \). Choose \( \epsilon_0 \) sufficiently small so that

1. the stratification

\[
\{B_\epsilon \setminus f^{-1}(0), B_\epsilon \cap f^{-1}(0) \setminus \Sigma_f, B_\epsilon \setminus \Sigma_f \setminus \{O\}, \{O\}\}
\]

satisfies Whitney conditions,

2. the sphere \( \mathbb{S}_\epsilon \) meets \( f^{-1}(0) \) transversely (in a stratified sense) for any \( \epsilon < \epsilon_0 \),
It is well known that the system of neighborhoods \( \{B_x\}_{0 < r < \epsilon_0} \) satisfies all the properties of a system of neighborhoods of the origin adapted to \( f \) except the second one. We are going to modify the neighborhoods so that this property is satisfied as well.

Fix \( 0 < \epsilon < \epsilon_0 \). The intersection \( \Sigma_f \cap S_\epsilon \) is a disjoint union of circles. We will modify \( S_\epsilon \) locally near each circle. We assume for simplicity of notation that there is only one circle \( S \). For \( \xi > 0 \) small enough \( A := \Sigma_f \cap B_{+\epsilon} \setminus B_{-\xi} \) is diffeomorphic to \( S \times [-\xi, \xi] \) by a diffeomorphism whose second component in the product \( S \times [-\xi, \xi] \) is the function \( \rho - \epsilon \). We identify \( A \) and \( S \times [-\xi, \xi] \) via this diffeomorphism.

I claim that there is a normal bundle \( p : N \to A \) such that

(i) for any \( s \in S \times [-\xi/3, \xi/3] \) the fibre \( N_s \) is contained in the tangent space of the level set of \( \rho \) at \( s \),

(ii) it is holomorphic over \( S \times [2\xi/3, \xi] \).

There is a normal bundle \( p_1 : N_1 \to S \times [-\xi/3, \xi/3] \) to \( T \) satisfying the first property: as \( T \) meets the level sets of \( \rho \) transversely, if \( \xi \) is small enough, for any \( s \in S \times [-\xi/3, \xi/3] \) the hermitian orthogonal complement to the vector \( s \in \mathbb{C}^n \) is transverse to \( T_s A \) (we are using the natural trivialisation of the tangent bundle \( T \mathbb{C}^n \)) and contained in \( T S_{\rho(s)} \). The fibres of \( N_1 \) are, by definition, those hermitian orthogonal complements.

Choose a coordinate system \( \{z_1, ..., z_n\} \) of \( \mathbb{C}^n \) such that, for any \( x \) in a punctured neighborhood of the origin, the hyperplane \( V(z_1 - z_1(x)) \) is transverse to \( T_x \Sigma_f \). Then the subbundle

\[
(20) \quad p : N_{z_1} \to \Sigma_f \setminus \{O\}
\]

whose fibre over \( x \) is \( V(z_1 - z_1(x)) \) is a normal bundle to \( \Sigma_f \) in the same punctured neighborhood. The bundle \( p_2 : N_2 \to S \times [2\xi/3, \xi] \) is defined as the restriction of the bundle \( 20 \) over \( S \times [\xi/2, \xi] \).

Any normal bundle \( p : N \to A \) to \( A \) expending \( N_1 \) and \( N_2 \) (which exists by Lemma 24) satisfies requirements (i) and (ii). The claim is proved.

Pick \( \eta > 0 \) so small that \( T_N(\eta) \) is embedded in \( \mathbb{C}^n \) and \( T_{N_\epsilon}(\eta) \) is contained in \( B_{+\xi} \). The restriction \( f|_N \) is a smooth function, whose restriction to \( N_3 \setminus [2\xi/3, \xi] \) and to the fibres of \( p \) is holomorphic. As a consequence of condition (1) above, the generic transversal Milnor number of \( f \) at any point \( t \in A \) is independent on the point \( t \). Hence, by Lemma 20 the restriction of \( f \) to the fibres of \( p \) has an isolated singularity at the origin with Milnor number independent of the fibre. As we are in the situation of Corollary 14 we can use Theorem 11 in order to obtain (after possibly shrinking \( \eta \)) a continuous flow

\[
(21) \quad \Phi : T_{N_\epsilon}(\eta) \times [-\xi, \xi] \to \mathbb{C}^n
\]

whose restrictions to the zero section \( S \times [-\xi, \xi] \) and its complement are smooth, such that

(a) for any \( s \in S \) we have \( \rho \circ \Phi(s, t) = \epsilon + t \),

(b) the flow lines are contained in the fibres of \( f \) and are compatible with the bundle projection \( p \) (the projection by \( p \) of a flow line is a flow line); the flow takes \( S \times [-\xi, \xi] \) into \( A \).

Conditions (i) and (a) above, and the compatibility of the flow lines with \( p \), imply that, if \( \eta \) is chosen small enough, there exists a positive small enough \( \xi' \) such
that any flow line of the restriction
\[ \Phi' := \Phi|_{T_{N_\eta}(\eta) \times [-\xi', \xi']} : T_{N_\eta}(\eta) \times [-\xi', \xi'] \to \mathbb{C}^n \]
is transverse to the level spheres of \( \rho \) and points outwards. Hence for any \( x \in T_{N_\eta}(\eta) \) there exists at most one value \( \alpha(x) \in (-\xi', \xi') \) such that \( \Phi'(x, \alpha(x)) \) belongs to \( S_\epsilon \).

Since \( S \) is contained in \( S_\epsilon \), we have that \( \alpha(x) \) is zero for any \( x \in S \). We have defined a function
\[ \alpha : T_{N_\eta}(\eta) \to \mathbb{R} \]
which is continuous, constant and equal to 0 in \( S \), and smooth outside \( S \).

The mapping \( \Psi : T_{N_\eta}(\eta) \to \mathbb{C}^n \) defined by the formula \( \Psi(x) = \Phi'(x, \alpha(x)) \) is a homeomorphism of \( T_{N_\eta}(\eta) \) onto its image \( W \), a neighborhood of \( S \) in \( S_\epsilon \). Consider a smooth function \( \mu : S_\epsilon \to [0, 1] \), with support \( K \) contained in \( W \), and identically 1 in a neighborhood of \( S \). We deform \( S_\epsilon \) as follows: we define the homotopy
\[ \varphi : S_\epsilon \times [0, 1] \to \mathbb{C}^n \]
pieces by \( \varphi(x, \lambda) := \Phi'(x, -\lambda \mu(x) \alpha((\Psi^{-1}(x)))) \) if \( x \in W \) and \( \varphi(x, \lambda) := x \) if \( x \)
is not contained in \( K \). For fixed \( \lambda \) the mapping \( \varphi(\cdot, \lambda) \) is a homomorphism onto its image and coincides with \( \text{Id}_{S_\epsilon} \) if \( \lambda = 0 \). We denote by \( U_{\epsilon, \lambda} \) the region of \( \mathbb{C}^n \)
bounded by \( \varphi(S_\epsilon, \lambda) \).

For any \( \lambda \in \{0, 1\} \) the neighborhood \( U_{\epsilon, \lambda} \) satisfies Properties (1) and (3) of Definition 17 and Property (2) is satisfied for \( \lambda = 1 \).

Using Property (b) of the flow \( (21) \) and the fact that Property (4) of Definition 17 is satisfied by the system \( \{B_\lambda\}_{0 < \epsilon < \epsilon_0} \) it is easy to show that for any \( \epsilon' < \epsilon \) there exists \( \delta > 0 \) such that there is a homeomorphism
\[ \kappa : (U_{\epsilon, 1} \setminus \mathring{B}_{\epsilon'}) \cap f^{-1}(D_\delta) \to (S_{\epsilon'} \cap f^{-1}(D_\delta)) \times [0, 1] \]
whose restriction to \( (U_{\epsilon, 1} \setminus \mathring{B}_{\epsilon'}) \cap f^{-1}(D_\delta) \setminus \Sigma_f \) and to \( (U_{\epsilon, 1} \setminus \mathring{B}_{\epsilon'}) \cap \Sigma_f \) is smooth and satisfies \( f = f \circ p_1 \circ \kappa \), where \( p_1 \) is the projection of \( (S_{\epsilon'} \cap f^{-1}(D_\delta)) \times [0, 1] \) to the first factor. In particular

- for any \( \epsilon' < \epsilon \), there exists \( \delta > 0 \) such that for any \( s \in D_\delta \) the space
\[ f^{-1}(s) \cap U_{\epsilon, 1} \setminus \mathring{B}_{\epsilon'} \]
is a trivial (stratified when \( s = 0 \)) cobordism,
- we have a homeomorphism
\[ \partial(B_{\epsilon'} \cap f^{-1}(D_\delta)), f^{-1}(0) \cap \partial(B_{\epsilon'} \cap f^{-1}(D_\delta)) \cong \partial(U_{\epsilon, 1} \cap f^{-1}(D_\delta)), f^{-1}(0) \cap \partial(U_{\epsilon, 1} \cap f^{-1}(D_\delta))). \]

Property (5) of Definition 17 for \( U_{\epsilon, \lambda} \) is a consequence of property (b) of the flow \( (21) \) and the fact that for any \( \epsilon' < \epsilon \) the ball \( B_{\epsilon'} \) satisfies it.

We construct a system of neighborhoods as follows. Give a decreasing sequence of positive numbers \( \{\epsilon_n\}_{n \in \mathbb{N}} \) converging to the 0. For any \( n \) we produce a deformation \( U_{\epsilon_n, \lambda} \) of \( B_{\epsilon_n} \) as above. The choices can be made so that \( U_{\epsilon_n, \lambda} \) is contained in \( \mathring{B}_{\epsilon_{n-1}} \) for any \( n \in \mathbb{N} \).

**Proposition 23.** The family \( \{U_{\epsilon_n, \lambda}\}_{n \in \mathbb{N}} \) is a system of neighborhoods adapted to \( f \). Moreover the topological type of the embedded link associated to it, and the diffeomorphism type of the Milnor fibration associated with it coincides with the classical ones.
Proof. Property (4) of Definition 17 (the only one that remains to be checked) follows easily using that it is satisfied for the system \( \{B_{n}\} \) and that we have the homeomorphism (23).

The triviality of cobordism (24) implies also the diffeomorphisms of Milnor fibrations. The fact that the embedded link associated to \( \{U_{n,1}\}_{n \in \mathbb{N}} \) is homeomorphic to the pair 

\[
(\partial B_{n} \cap f^{-1}(D_{b}), f^{-1}(0) \cap (\partial B_{n} \cap f^{-1}(D_{b})),
\]

which, by a Theorem of [15] is homeomorphic to \((S_{r}, f^{-1}(0) \cap S_{r})\) (the classical embedded link), is essentially due to homeomorphism (25).

4. EQUISINGULARITY AT THE CRITICAL SET

Setting. Let \( T \) be a connected smooth manifold. A smooth family of holomorphic germs \( f_{t} : (\mathbb{C}^{n}, O) \to \mathbb{C} \) depending on a parameter \( t \in T \), is given by a smooth function \( f : U \to \mathbb{C} \) defined at an open neighborhood of \( \{0\} \times T \) in \( T \times \mathbb{C}^{n} \) satisfying that the restriction \( f_{t} := F|_{\mathbb{C}^{n} \times \{t\}} \) is holomorphic for any \( t \in T \). We view \( T \times \mathbb{C}^{n} \) as a trivial vector bundle over \( T \) and consider the natural projection \( \tau : T \times \mathbb{C}^{n} \to T \). Let \( \Sigma_{t} \) denote the critical set of \( f_{t} \). Define \( \Sigma := \cup_{t \in T} \Sigma_{t} \). Consider a hermitian product in \( T \times \mathbb{C}^{n} \) and denote by \( \rho \) the distance to the origin in each fibre.

Now we define equisingularity at the critical set. Roughly speaking, the family \( f_{t} \) is equisingular at the critical set if the family of critical sets \( \Sigma_{t} \) is topologically equisingular by a family of homeomorphisms preserving the transversal Milnor number (outside the origin). This is formalised as follows.

Given any \( A \subset T, B \subset T \times \mathbb{C}^{n} \) and a positive function \( \theta : A \to \mathbb{R}_{+} \) we define \( B_{A} = B|_{A} \) and \( B(A, \theta) \) as in the previous section. Choose \( t \in T \), there exists a positive \( \epsilon_{t} \) such that \( \partial B((t,O), \epsilon) \) meets \( \Sigma_{t} \) transversely for any \( \epsilon \leq \epsilon_{t} \). Then \( \Sigma_{t} \cap B((t,O), \epsilon_{t}) \) is homeomorphic to the cone over \( \Sigma_{t} \cap \partial B((t,O), \epsilon_{t}) \), and there is an irreducible component of the germ \( (\Sigma_{t}, (t,O)) \) for each connected component of \( \Sigma_{t} \cap \partial B((t,O), \epsilon_{t}) \). Let \( \Sigma_{t} = \cup_{i=1}^{r} \Sigma_{t,i} \) be the decomposition of \( (\Sigma_{t}, (t,O)) \) in irreducible components. The number \( \epsilon_{t} \) can be chosen small enough that, for any \( i \leq r \), the generic transversal Milnor number at any point of \( \Sigma_{t,i} \cap B((t,O), \epsilon_{t}) \setminus \{(t,O)\} \) is equal to a constant \( \mu_{t,i} \).

We will identify \( T \) with \( T \times \{O\} \).

Definition 24. Let \( f_{t} \) be a smooth family of holomorphic germs. Suppose that \( \Sigma_{t} \) is 1-dimensional at the origin for any \( t \). We say that \( f_{t} \) is equisingular at the critical set if the following conditions are satisfied:

1. The space \( \Sigma \setminus T \) is smooth of real dimension \( 2 + \dim(T) \) at any of its points, and such that the restriction \( \tau|_{\Sigma \setminus T} \) is a submersion.

2. The previous property implies that for any \( t \in T \), there exists a neighborhood \( W_{t} \) of \( t \) in \( T \) so small that \( \Sigma_{t} \cap \partial B((t,O), \epsilon_{t}) \) transversely for any \( t' \in W_{t} \). For any \( t' \in W_{t} \) the space \( \Sigma_{t'} \cap B((t,O), \epsilon_{t}) \) is homeomorphic to \( \Sigma_{t} \cap B((t,O), \epsilon_{t}) \), (notice that if the homeomorphism exists it can be chosen sending \( (t,O) \) to \( (t,O) \)).

3. The previous property implies that for any \( t, t' \in W_{t} \), there is a bijective correspondence (induced by inclusion) between the connected components of \( \Sigma_{t} \cap B((t,O), \epsilon_{t}) \setminus \{(t,O)\} \),
\[ \Sigma_{W_t} \cap B(W_t, \epsilon_t) \backslash \{(O) \times W_t \} \]
and
\[ \Sigma_{t'} \cap B((t,O), \epsilon_t) \backslash \{(t,O)\}. \]
For any \(t' \in W_t\) let \(\Sigma_{t',i}\) be the connected component of \(\Sigma_{t} \cap B((t,O), \epsilon_t) \backslash \{(t,O)\}\) corresponding to \(\Sigma_{t,i}\). The generic transversal Milnor number of \(f_{t'}\) at any point of \(\Sigma_{t',i} \backslash \{(t',O)\}\) is \(\mu_{t,i}\).

**Remark 25.** Condition (2) of the previous Definition can be phrased informally saying that the underlying deformation of critical sets is topologically equisingular. It is important to notice that it implies that for any \(t'\) close to \(t\) the function \(f_{t'}\) has no isolated singularities in \(B((t,O), \epsilon_t)\). As \(\Sigma_{t'}\) is of real dimension \(2\) the condition can be rephrased in any of the following formulations: the cobordism
\[ \Sigma_{t'} \cap B((t,O), \epsilon_t) \backslash \dot{B}((t,O), \epsilon_{t'}) \]
- is trivial in the smooth category,
- is homologically trivial.

Let \(V\) be any neighborhood of the origin adapted to \(f_t\). A consequence is that \(\partial V\) meets \(\Sigma_{t}\) transversely for any \(\alpha\). Choose \(W_t\) so small that \(\Sigma_{t'}\) meets \(\partial V\) transversely for any \(t' \in W_t\). Let \(V'\) be any neighborhood of the origin adapted to \(f_{t'}\) such that \(V' \subset V\).

**Remark 26.** By Property (4) of Definition 14 Property (2) of Definition 24 and an easy cobordism manipulation, the cobordism
\[ \Sigma_{t'} \cap V \backslash \dot{V}' \]
is trivial.

Here it is a conormality Lemma (in the sense of 14, Definition A.7)

**Lemma 27.** Let \(f_t\) be a smooth family of holomorphic germs such that \(\Sigma_t\) is 1-dimensional at the origin for any \(t\). Suppose that \(f_t\) is equisingular at the critical set. Given \(t \in T\) and \(V\), a smooth neighborhood of the origin adapted to \(f_t\), there exists a neighborhood \(W_t\) of \(t\) in \(T\), and a positive \(\delta\) such that

1. there is a normal bundle \(\pi : N \to \Sigma \cap (W_t \times \partial V)\) to \(\Sigma\) such that for a certain positive \(\eta\) the space \(T_{\Sigma} \cap (W_t \times \partial V)\) is a tubular neighborhood of \(\Sigma \cap (W_t \times \partial V)\) in \(W_t \times \partial V\),
2. the fibre \(f_{t'}^{-1}(s)\) meets transversely (in the stratified sense when \(s = 0\)) the boundary \(\partial V\) for any \(t' \in W_t\) and \(s \in D_\delta^*\).

**Proof.** There is a normal bundle to \(\Sigma_t\) called \(p : N \to \Sigma_t \cap \partial V\), and a positive \(\eta\) such that \(T_N(\eta)\) is naturally embedded in \(\partial V\). There is a neighborhood \(W_t\) of \(t \in T\) such that \(\Sigma \cap (W_t \times \partial V)\) is included in \(W_t \times T_N(\eta)\), and the restriction
\[ \tau|_{\Sigma \cap (W_t \times \partial V)} : \Sigma \cap (W_t \times \partial V) \to W_t \]
is submersive (hence a locally trivial fibration).

For any \(t' \in W_t\) the fibre \(\Sigma_{t'} \cap \partial V\) is a disjoint union of circles, contained in \(\{t'\} \times T_N(\eta)\). In each connected component of \(\{t'\} \times T_N(\eta)\) we find exactly one circle. Define
\[ \Xi : W_t \times C^n \to \{t\} \times C^n \]
by \(\Xi(t',x) := (t,x)\). Fix any \(t' \in W_t\). The inclusion
\[ t : T_N(\eta) \to N \]
induces a fibration
\[ p^\circ|_{\{t^1\} \times T_{\eta}(n)} : \{t^1\} \times T_{\eta}(n) \rightarrow \Sigma_t \cap \partial V. \]
If \( W_t \) is small enough and \( t^1 \in W_t \), each circle of \( \Sigma_t' \cap \partial V \) meets transversely the fibres of \( p^\circ|_{\{t^1\} \times T_{\eta}(n)} \). Observe that the fibre of \( p^\circ|_{\{t^1\} \times T_{\eta}(n)} \) at any point \((t,y) \in \Sigma_t \cap \partial V\) is a \( n-1 \)-dimensional complex analytic submanifold of \( \{t^1\} \times \mathbb{C}^n \).

Given \((t^1,x) \in \Sigma_t' \cap \partial V\) we denote by \( H_{(t^1,x)} \) the complex tangent space at \((t^1,x)\) of the unique fibre of \( p^\circ|_{\{t^1\} \times T_{\eta}(n)} \) to which \((t^1,x)\) belongs. Define
\[ \overline{p} : \overline{N} \rightarrow \Sigma \cap (W_t \times \partial V) \]
as the bundle whose fibre over any point \((t^1,x) \in \Sigma \cap (W_t \times \partial V)\) is \( H_{(t^1,x)} \). Clearly \( \overline{p} \) is a normal bundle extending \( p \), and, if \( \eta \) is small enough then \( T_{\overline{p}(\eta)} \) is naturally embedded in \( W_t \times \partial V \) and satisfies Condition (1).

The space \( T_{\overline{p}(\eta)} \) has exactly one connected component \( C_i \) for each connected component of \( \Sigma_t \cap \partial V \). Let \( i \) be the transversal Milnor number corresponding to the \( i \)-th connected component of \( \Sigma_t \cap \partial V \). As \( \overline{p} \) is a normal bundle, by Property (3) of Definition 24 and Lemma 20 we deduce the following: for any connected component \( C_i \), the restriction of \( f \) to any fibre of \( \overline{p}_{|C_i} \), is an analytic function which has an isolated singularity at the origin with Milnor number \( \overline{\mu}_i \). By the constancy of the Milnor number we get, perhaps shrinking \( W_t \) and diminishing \( \eta \), that the restriction of \( f_{t^1} \) the the fibres of \( C_i \) have no critical points outside the origin. This implies the existence of a positive \( \delta \) such that \( f_{t^1}^{-1}(s) \) is transverse (in the stratified sense if \( s = 0 \)) to \( \partial V \) in \( T_{\overline{p}(\eta)} \), for any \( t^1 \in W_t \) and \( s \in D_\delta \).

If \( W_t \) and \( \delta \) are sufficiently small and \( t^1 \in W_t \) and \( s \in D_\delta \), the transversality of \( f_{t^1}^{-1}(s) \) to \( \partial V \) outside \( T_{\overline{p}(\eta)} \) follows because \( f_{t^1}^{-1}(0) \) is transverse to \( \partial V \) outside \( T_{\overline{p}(\eta)} \) and the transversality is an open condition. \( \square \)

5. THE EMBEDDED TOPOLOGICAL TYPE AND THE MILNOR FIBRATION

Assumption. Throughout the rest of the paper we assume \( n \geq 5 \).

Along this section we let \( f_t : (\mathbb{C}^n, O) \rightarrow \mathbb{C} \) be a smooth family of holomorphic germs depending on a parameter \( t \) varying in a smooth manifold \( T \). Let \( f : T \times \mathbb{C}^n \rightarrow \mathbb{C} \) be the smooth function defining the family. We assume that the critical set of \( f_t \) at the origin is of dimension 1 for any \( t \), and that the family is topologically equisingular at the critical set. Denote by
\[ \psi : T \times \mathbb{C}^n \rightarrow T \times \mathbb{C} \]
the mapping \((\tau,f)\).

Notation. Given \( A \subset T \times \mathbb{C}^n \times \) and \( B \subset T \) we define \( A_B = A \cap \tau^{-1}(B) \).

5.1. Cuts.

Definition 28. A cut for \( f \) with amplitude \( \delta \) over a submanifold \( V \subset T \) is a closed smooth hypersurface \( H \) of \( \psi^{-1}(V \times D_\delta) \) with the following properties:

1. There is a tubular neighborhood of \( \Sigma \cap H \) in \( H \) which coincides with \( T_{\eta}(n) \) for a certain normal bundle to \( \Sigma \) called \( \pi : N \rightarrow \Sigma \cap H \), and a certain \( \eta > 0 \).

2. For any \((t,s) \in V \times D_\delta \) the hypersurface \( H \) meets the (open) set of smooth points of \( \psi^{-1}(t,s) \) transversely.
(3) There is a unique connected component $X_{\text{int}}(H, V, \delta)$ (the interior component) of $\psi^{-1}(V \times D_\delta) \setminus H$, which contains the zero section. Moreover the restriction

$$\tau|_{X_{\text{int}}(H, V, \delta)} : X_{\text{int}}(H, V, \delta) \to V$$

is a smooth locally trivial fibration, with fibre a contractible compact manifold with corners.

(4) For any $t \in V$ and any neighborhood $W$ of the origin adapted to $f_t$ and contained in $X_{\text{int}}(Y, V, H)_t$, for any Milnor disk radius $\delta'$ for $(f_t, W)$ satisfying $\delta' < \delta$, and any $s \in D_\delta^*$, we have that $f_t^{-1}(s) \cap (X_{\text{int}}(Y, V, H)_t \setminus \tilde{W})$ is a homotopically trivial cobordism.

(5) For any $t \in V$ and any neighborhood $W$ of the origin adapted to $f_t$ and contained in $X_{\text{int}}(Y, V, H)_t$ we have that $\Sigma_t \cap (X_{\text{int}}(Y, V, H)_t \setminus \tilde{W})$ is a trivial cobordism.

(6) The previous property implies that, for any $t \in V$, there is a bijective correspondence (induced by inclusion) between the connected components of $\Sigma_t \cap X_{\text{int}}(Y, V, H)_t \setminus \{(t, 0)\}$ and $\Sigma_t \cap X_{\text{int}}(Y, V, H)_t \setminus \tilde{W}$. For any connected component $\Gamma$ of $\Sigma_t \cap X_{\text{int}}(Y, V, H)_t \setminus \tilde{W}$ and any $(t, p) \in \Gamma$, the transversal Milnor number of $f_t$ at $p$ is independent on the point $(t, p)$.

Remark 29. By the proof of Lemma 19 it is sufficient to check Property (4) for a single neighborhood adapted to $f$.

Remark 30. The definition of system of neighborhoods adapted to $f_1$ implies that for any neighborhood $W$ of the origin adapted to $f_t$, there exists $\delta$ such that $\psi^{-1}(\{t\} \times D_\delta) \cap \partial W$ is a cut over $\{t\}$ of amplitude $\delta$.

The relations $\preceq$ and $\prec$ are defined as in Section 2.2.

5.2. Construction. Let $C = [0, 1]^{d}$ be a cube embedded in $T$ and $V$ an open neighborhood of $C$ in $T$ diffeomorphic to the open cube $(-\epsilon, 1 + \epsilon)^{d}$ (for $\epsilon > 0$). Suppose that we have cuts $H_+$ and $H_-$ for $f$ over $V$ of the same amplitude $\delta$, such that $H_- \prec H_+$. Along this section we will shrink $\delta$ when it is necessary without explicitly mentioning it. Define $X := X_{\text{int}}(H_+, V, \delta) \setminus X_{\text{int}}(H_-, V, \delta)$.

By Definition 29, an easy argument with cobordisms, and Ehresmann Fibration Theorem, the restriction

$$\tau|_{\Sigma \cap X} : \Sigma \cap X \to V$$

is a locally trivial fibration (trivial in fact, since the base is contractible), with fibre a trivial cobordism diffeomorphic to a disjoint union of cylinders, one for each connected component of $\Sigma \setminus V$. We will assume for notational simplicity that $\Sigma \setminus V$ is connected, being the treatment of the general case completely analogous. We consider a diffeomorphism

$$\nu : \Sigma \cap X \to S^1 \times [0, 1] \times V$$

such that $\tau|_{\Sigma \cap X} = p_3 \circ \nu$ (being $p_i$ the projection of $S^1 \times [0, 1] \times V$ to the $i$-th factor for $i = 1, 2, 3$), $\nu(\Sigma \cap H_-) = S^1 \times \{0\} \times V$ and $\nu(\Sigma \cap H_+) = S^1 \times \{1\} \times V$.

By Definition 29 there exist a normal bundle $\pi_+ : N_+ \to \Sigma \cap H_+$ and a neighborhood of $\Sigma \cap H_+$ in $H_+$ which is equal to $T_{N_+}(\eta)$ for a certain positive $\eta$. Similarly there exist a normal bundle $\pi_- : N_- \to \Sigma \cap H_-$ and a neighborhood of $\Sigma \cap H_-$ in $H_-$ coinciding with $T_{N_-}(\eta)$. Continued...
For any \( t \in V \), if \( z_1 \) is a coordinate of \( \mathbb{C}^n \) not vanishing at \( \Sigma_t \), the \( z_1 \)-section (that is, the hyperplane \( V(z_1 - z_1(x)) \)) is transverse to \( \Sigma_t \) at any point \( x \in \Sigma_t \), with finitely many exceptions. This allows to find an annulus \( A \) embedded in \( \Sigma \cap X_t \) not meeting the boundary, whose embedding in \( \Sigma \cap X_t \) is a homotopy equivalence, such that the \( z_1 \)-section is transverse to \( \Sigma \) at any point of \( A \). Consequently the vector subbundle \( \pi_A : N_A \to A \) of \( T(T \times \mathbb{C}^n, T)_A \) whose fibre over \( a \in A \) is the \( z_1 \)-section of the fibre \( T(T \times \mathbb{C}^n, T)_a \) is a constant (and hence holomorphic) normal bundle to \( \Sigma \) over \( A \).

Using Lemma 22 we extend the bundles \( \pi_+, \pi_- \) and \( \pi_A \) to a normal bundle
\[
\pi : N \to \Sigma \cap X.
\]
We choose \( \eta \) small enough so that \( T_N(\eta) \) embeds as neighborhood of \( \Sigma \cap X \) in \( X \). Observe that we have automatically the compatibility
\[
\tau \circ \pi|_{T_N(\eta)} = \tau|_{T_N(\eta)}.
\]
By the last condition of Definition 28 and by Lemma 20 the Milnor number at the origin of the restriction of \( f \) to the fibre of \( \pi \) over \( x \in \Sigma \cap X \) is independent of \( x \). Observe that \( \Sigma \cap X_C \) is diffeomorphic to the product of \( C \) and a cylinder, which is the same than the product of a circle and a \((d + 1)\)-dimensional cube. Since the restriction \( \pi_A \) is holomorphic, an application of Corollary 15 enables us to use the results of Section 2 to the restriction of the function \( f \) to \( T_N(\eta) \), viewed as a \( \mu \)-constant family parametrised over \( \Sigma \cap X \).

Thus, by Lemma \( \ref{lem:extension} \) and Proposition \( \ref{prop:embedding} \) there exists a cut \( S \) for \( f \) over \( \Sigma \cap X \) with amplitude \( \delta \) such that \( Y := Y_{\text{int}}(S, \Sigma \cap X, \delta) \) is contained in \( T_N(\eta) \). We consider the composition \( \nu \circ \pi : Y \to S^1 \times [0, 1] \times C \). Given any \( t \in C \), consider the circle
\[
K_t := S^1 \times \{0\} \times \{t\}
\]
inside \( S^1 \times [0, 1] \times C \), and define
\[
Y_{K_t} := (\nu \circ \pi)^{-1}(K_t) = \pi^{-1}(\Sigma_t \cap H_-) = Y_t \cap H_-
\]
By Theorem 11 there exist a homeomorphism
\[
(30) \quad \Psi : Y \to Y_{K_t} \times [0, 1] \times C
\]
such that
\begin{enumerate}
\item We have the equality \( \nu \circ \pi = (p_1 \circ \nu \circ \pi|_{Y_{K_t}}, q_2, q_3) \circ \Psi \), where \( q_i \) is the projection of \( Y_{K_t} \times [0, 1] \times C \) to the \( i \)-th factor.
\item We have \( f|_{Y_{K_t}} \circ q_1 \circ \Psi = f \).
\item \( \) the restriction of \( \Psi \) to \( \Sigma \cap X \) coincides with \( \nu \), and the restriction of \( \Psi \) to \( Y \setminus (\Sigma \cap X) \) is smooth.
\end{enumerate}

We will use the notation \( Z = \overline{X \setminus Y} \). The subspaces \( Y \) and \( Z \) meet at a common boundary which coincides with the cut \( S \). As for any \( t \in C \) and \( s \in D_\delta \) the fibre \( f_t^{-1}(s) \) meets \( S \) transversely, by Ehresmann Fibration Theorem and the contractibility of the base the restriction
\[
(31) \quad \psi|_Z : Z \to C \times D_\delta
\]
is a trivial fibration, which we call the outer fibration.
5.3. **Topological trivialisation of the space between two cuts.** We start studying the outer fibration using cobordism theory.

**Lemma 31.** For any \( s \in D^*_t \), the cobordisms

\[
(f_t^{-1}(s) \cap X, f_t^{-1}(s) \cap H_+) \cap H_-
\]

\[
(f_t^{-1}(s) \cap Z, f_t^{-1}(s) \cap Z \cap H_+) \cap Z \cap H_-
\]

are simply connected \( h \)-cobordisms, the second one with boundary.

**Proof.** By Property (4) of Definition 28, we may find a neighborhood \( W \) of the origin adapted to \( f_t \) such that \( W \) is contained in \( X_{int}(H_+, V, \delta), X_{int}(H_-, V, \delta) \) and

\[
(f_t^{-1}(s) \cap X_{int}(H_+, V, \delta) \setminus W, f_t^{-1}(s) \cap H_+, f_t^{-1}(s) \cap \partial W)
\]

\[
(f_t^{-1}(s) \cap X_{int}(H_-, V, \delta) \setminus W, f_t^{-1}(s) \cap H_-, f_t^{-1}(s) \cap \partial W)
\]

are homotopically trivial cobordisms for \( s \) non-zero and sufficiently small. This easily implies that the cobordism \( \mathcal{C} \) is also homotopically trivial, and that the three spaces involved in the cobordism have the homotopy type of \( f_t^{-1}(s) \cap \partial W \), which is the boundary of the Milnor fibre of \( f_t \) for the adapted neighborhood \( W \).

Consider \( \pi : Y \to \Sigma \cap X \) as in Construction 5.2. Define \( \phi : Y \to (\Sigma \cap X) \times D_\delta \) by \( \phi := (\pi, f) \). Consider the space

\[
B = \phi^{-1}((\Sigma_t \cap H_-) \times D_\delta).
\]

Observe that \( B \) is included in \( H_- \) and that we have the equality

\[
f_t^{-1}(s) \cap B = \phi^{-1}((\Sigma_t \cap H_-) \times \{s\}).
\]

The mappings

\[
\pi|_B : B \to \Sigma_t \cap H_-
\]

\[
\pi|_{f_t^{-1}(s) \cap B} : f_t^{-1}(s) \cap B \to \Sigma_t \cap H_-
\]

are locally trivial fibrations over a circle with simply connected fibres: the fibres of the first mapping are contractible, and the fibres of the second are homeomorphic to the Milnor fibre of the transversal singularity at any point of \( \Sigma_t \), which is simply connected by the Kato-Matsumoto bound (recall that we have assumed \( n \geq 5 \)). Hence the inclusion \( f_t^{-1}(s) \cap B \subset B \) induces an isomorphism of (infinite cyclic) fundamental groups. Using this, an easy application of Seifert-van Kampen Theorem shows that the fundamental group of \( f_t^{-1}(s) \cap H_- \) isomorphic to the fundamental group of \( (f_t^{-1}(s) \cap H_-) \cup B \).

The restriction of the mapping \( \psi \) to \( (Z \cap H_-)_t \) yields a trivial fibration

\[
\psi|_{(Z \cap H_-)_t} : (Z \cap H_-)_t \to \{t\} \times D_\delta.
\]

Using it we deduce that the space \( (f_t^{-1}(s) \cap H_-) \cup B \) is homotopy equivalent to \( f_t^{-1}(D_\delta) \cap H_- \). Using the fibration \( \mathcal{C} \) we show that \( f_t^{-1}(D_\delta) \cap H_- \) admits \( f_t^{-1}(0) \cap H_- \) as a deformation retract. Working with the fibration \( \mathcal{C} \) we prove that, in turn, the space \( (f_t^{-1}(0) \cap H_-) \cup B \) admits \( f_t^{-1}(0) \cap H_- \) as a deformation retract.

Thus \( f_t^{-1}(s) \cap H_- \cup B \) and \( f_t^{-1}(0) \cap H_- \) have isomorphic fundamental groups.
Consequently $f_t^{-1}(s) \cap H_-$ and $f_t^{-1}(0) \cap H_-$ have isomorphic fundamental groups for any $(s, t) \in C \times D_\delta$.

We can show in the same way that the spaces $f_t^{-1}(s) \cap H_+$ and $f_t^{-1}(0) \cap H_+$ (respectively $f_t^{-1}(s) \cap X$ and $f_t^{-1}(0) \cap X$) have isomorphic fundamental groups for any $(t, s) \in C \times D_\delta$.

With a similar construction we can show that $f_t^{-1}(s) \cap \partial W$ has the same fundamental group than $f_t^{-1}(0) \cap \partial W$, which is homotopy equivalent to the classical link of $f_t$. The later space is simply connected by \cite{13}, Theorem 5.2 (use that $n \geq 5$).

We have shown that $\ref{32}$ is a simply connected $h$-cobordism.

Consider the decomposition

$$f_t^{-1}(s) \cap H_- = (f_t^{-1}(s) \cap H_- \cap Y) \bigcup (f_t^{-1}(s) \cap H_- \cap Z)$$

The mapping $\pi$ fibres the first piece and the intersection of the two pieces over the circle $\Sigma_t \cap H_-$, with fibres the Milnor fibre and the link of the transversal singularities respectively. The link of the transversal singularity is simply connected due to Theorem 5.2 of \cite{13} (we have assumed $n \geq 5$). Seifert-van Kampen Theorem implies now that $f_t^{-1}(s) \cap H_-$ and $f_t^{-1}(s) \cap H_- \cap Z$ have isomorphic fundamental groups. Therefore $f_t^{-1}(s) \cap H_- \cap Z$ is simply connected. Similarly we show that $f_t^{-1}(s) \cap H_+ \cap Z$ and $f_t^{-1}(s) \cap Z$ are simply connected.

It remains to be shown the following vanishing of relative homology groups:

$$H_*(f_t^{-1}(s) \cap Z, f_t^{-1}(s) \cap Z \cap H_-; Z) = 0$$
$$H_*(f_t^{-1}(s) \cap Z, f_t^{-1}(s) \cap Z \cap H_+; Z) = 0.$$

Using the ladder of long exact sequences formed by the Mayer-Vietoris sequences associated to the decomposition $\ref{39}$ and the decomposition

$$f_t^{-1}(s) = (f_t^{-1}(s) \cap Y) \bigcup (f_t^{-1}(s) \cap Z),$$

and the fact that the inclusions

$$f_t^{-1}(s) \cap H_- \cap Y \subset f_t^{-1}(s) \cap Y,$$
$$f_t^{-1}(s) \cap H_- \cap Y \cap Z \subset f_t^{-1}(s) \cap Y \cap Z$$

are homotopy equivalences (which is true due to the homeomorphism $\ref{30}$), we show that the first required vanishing follows from the vanishing

$$H_*(f_t^{-1}(s) \cap X, f_t^{-1}(s) \cap H_-; Z) = 0,$$

which is true since the cobordism $\ref{32}$ is an homotopically trivial. The second vanishing is proved similarly. \hfill \square

Lemma 32. If $H$ is a cut for $f$ over $V$ with amplitude $\delta$ then $f_t^{-1}(s) \cap H$ is simply connected for any $(t, s) \in V \times D_\delta$.

Proof. We may assume that $H$ is equal to the cut $H_-$ of the previous Lemma. We have shown that $f_t^{-1}(s) \cap H_-$ is simply connected for $s \neq 0$, and that the fundamental groups of $f_t^{-1}(s) \cap H_-$ and $f_t^{-1}(0) \cap H_-$ are the same. \hfill \square

We have the ingredients for the proof one of our main technical propositions.

Proposition 33. For any $t \in C$ there exists a homeomorphism

$$\Theta : X \to (H_-)_t \times [0, 1] \times C$$

such that
(1) We have $\tau = p_3 \circ \Theta$, where $p_i$ is the projection of $(H_-)_t \times [0, 1] \times C$ to the $i$-th factor $(i = 1, 2, 3)$.

(2) We have $f_t|_{(H_-)_t}\circ p_3 \circ \Theta = f$.

(3) The restriction of $\Theta$ to $\Sigma \cap X$ is smooth with target $((H_-)_t \cap \Sigma) \times [0, 1] \times C$ and the restriction of $\Theta$ to $X \setminus \Sigma$ is smooth.

(4) The restriction $\Theta|_Y$ coincides with the homeomorphism $\Theta|_Y$.

Proof. We have the equalities

$$S_{K_i} := S \cap (\nu \circ \pi)^{-1}(K_i) = S \cap \pi^{-1}(\Sigma \cap H_-) = (S \cap H_-)_t.$$  

The restriction of the homeomorphism $\Psi|_S$ to the boundary $S$ induces a diffeomorphism

$$\Psi|_S : S \to (S \cap H_-)_t \times [0, 1] \times C$$

preserving the fibres of $f$.

The restrictions

$$\psi|_S : S \to C \times D_\delta$$

of the trivial fibration (31) are trivial fibrations as well. Choose a smooth trivialisation

$$\varphi_1 := (\gamma, t, f_t) : (S \cap H_-)_t \to \psi|_{(S \cap H_-)_t}(t, 0) \times \{t\} \times D_\delta$$

of $\psi|_{(S \cap H_-)_t}$.

Denote by $q_i$ the projection of $(S \cap H_-)_t \times [0, 1] \times C$ to the $i$-th factor $(i = 1, 2, 3)$. Since $\Psi|_S$ is a restriction of the homeomorphism $\Theta$, the mapping

$$\Upsilon : S \to \psi|_{(S \cap H_-)_t}(t, 0) \times [0, 1] \times C \times D_\delta$$

defined by $\Upsilon := (\gamma \circ q_1, q_2, q_3, f_t \circ q_1) \circ \Psi|_S$ clearly satisfies $\psi|_S = (p_3, p_4) \circ \Upsilon$, where $p_i$ is the projection of $\psi|_{(S \cap H_-)_t}(t, 0) \times [0, 1] \times C \times D_\delta$ to the $i$-th factor.

Define

$$\Xi : \psi|_{(S \cap H_-)_t}(t, 0) \times [0, 1] \times C \times D_\delta \to \psi|_S^{-1}(t, 0) \times C \times D_\delta$$

by $\Xi(y, u, t', s) := (\Psi|_S^{-1}(y, u, t), t', s)$. Then the composition

$$\varphi_2 := \Xi \circ \Upsilon : S \to \psi|_S^{-1}(t, 0) \times C \times D_\delta$$

is a trivialisation of the fibration $\psi|_S$.

Consider an extension

$$\varphi_3 : Z \to \psi|_Z^{-1}(t, 0) \times C \times D_\delta$$

of $\varphi_2$ to a trivialisation of $\psi|_Z$.

By Lemma 33, the triple

$$\psi|_Z^{-1}(t, 0), \psi|_Z^{-1}(t, 0) \cap H_-, \psi|_Z^{-1}(t, 0) \cap H_+$$

is a simply connected $h$-cobordism with boundary. The diffeomorphism (35) induces a trivialisation of the boundary cobordism

$$\psi|_S^{-1}(t, 0), \psi|_S^{-1}(t, 0) \cap H_-, \psi|_S^{-1}(t, 0) \cap H_+.$$

By $h$-cobordism Theorem the cobordism (40) admits a trivialisation

$$\kappa : \psi|_Z^{-1}(t, 0) \to (\psi|_Z^{-1}(t, 0) \cap H_-) \times [0, 1]$$
which extends the given trivialisation at the boundary.

Consider the diffeomorphism

\[ \Upsilon' : Z \to (\psi^{-1}|_{Z(t, 0) \cap H_\perp} \times [0, 1] \times C \times D_\delta \]

defined by \( \Upsilon' := (\kappa \circ \sigma_1, \sigma_2, \sigma_3) \circ \varphi_3 \), where \( \sigma_i \) is the projection of \( \psi|_Z^{-1}(t, 0) \times C \times D_\delta \) to the \( i \)-th factor. By definition \( \Upsilon' \) extends \( \Upsilon \).

The mapping \( \varphi_3 \) restricts to a trivialisation

\[ \varphi_3|_{(Z \cap H_\perp)_t} : (Z \cap H_\perp)_t \to \psi|_{Z(t, 0) \cap H_\perp}^{-1}(t, 0) \cap H_\perp \times \{t\} \times D_\delta \]

de the fibration \( \psi|_{(Z \cap H_\perp)_t} : (Z \cap H_\perp)_t \to \{t\} \times D_\delta \). Define

\[ \Psi' : Z \to (Z \cap H_\perp)_t \times [0, 1] \times C \]

by \( \Psi'(z) = (\varphi_3|_{(Z \cap H_\perp)_t}^{-1}(x, s), u, t') \), where \( (x, u, t', s) := \Upsilon'(z) \).

As \( \Upsilon \) extends \( \Upsilon' \) we obtain that \( \Psi \) and \( \Psi' \) coincide at the intersection of their domains. Hence the desired homeomorphism \( \Theta \) can be defined piecewise over \( Y \) and \( Z \) gluing \( \Psi \) and \( \Psi' \).

In order to fulfill the differentiability of \( \Theta \) at \( S \) it is sufficient take a collar of \( S \) and modify \( \Psi \) and \( \Psi' \) in each of the sides so that the gluing is smooth. \( \square \)

**Corollary 34.** Let \( f : (\mathbb{C}^n, O) \to \mathbb{C} \) be a holomorphic germ with critical set of dimension 1 at the origin. Given a system of neighborhoods \( \{U_\alpha\}_{\alpha \in A} \) of the origin adapted to \( f \), the homeomorphism type of the embedded link and the diffeomorphism type of the Milnor fibration of \( f \) associated to \( \{U_\alpha\}_{\alpha \in A} \) coincide with the classical ones.

**Proof.** In Proposition 23 we exhibit a system of neighborhoods adapted to \( f \) for which the statement is true. Suppose that we have two neighborhoods \( W_1, W_2 \) of the origin adapted to \( f \) such that \( W_1 \subset W_2 \). By Remark 30 there is \( \delta > 0 \) such that \( \partial W_i \cap f^{-1}(D_\delta) \) is a cut for \( f \) of amplitude \( \delta \) for \( i = 1, 2 \). A straightforward application of Proposition 23 gives the result. \( \square \)

### 5.4. The embedded topological type and the Milnor fibration

We show that being a cut is an open property on the base:

**Lemma 35.** Let \( f_t : (\mathbb{C}^n, O) \to \mathbb{C} \) be a smooth family of holomorphic germs depending on a parameter \( t \) varying in a smooth manifold \( T \), and \( V \) be a compact submanifold of \( T \). Assume that \( f \) is equisingular at the critical set. Suppose that we have a positive \( \delta > 0 \) and a closed smooth hypersurface \( H \) of \( \psi^{-1}(T \times D_\delta) \) which satisfies Property (1) of Definition 25 and which is such that the restriction \( H|_V \) is a cut over \( V \) of amplitude \( \delta \). Then there is an open neighborhood \( U \) of \( V \) in \( T \) such that \( H|_U \) is a cut over \( U \) of amplitude \( \delta \).

**Proof.** We have to check that \( H|_T \) satisfies Properties (2)-(6) of Definition 25 for a certain open neighborhood \( U \) of \( V \) in \( T \). The second property follows from the fact that Property (1) holds and an argument like in the proof of Lemma 27. Let \( U \) be an open neighborhood over which Properties (1) and (2) hold. It is easy to check using Ehresmann fibration Theorem that Property (3) holds as well. The fact that Property (5) holds (after possibly shrinking \( U \)) is an easy argument involving manipulations with cobordisms, Ehresmann fibration Theorem, and the fact that cobordism 26 is trivial. To show that Property (6) holds (after possibly shrinking \( U \)) we only have to use that it holds over \( V \), that \( V \) is compact and that the family is equisingular at the critical set.
Proving Property (4) is slightly more involved. Let $U$ be an open neighborhood of $V$ in $T$ over which Properties (1)-(3), (5) and (6) are satisfied, and such that $V$ meets all the connected components of $U$. Then $X := X_{int}(H,U,\delta)$ is defined. The mapping
\begin{equation}
\tau : X \to U
\end{equation}
is a locally trivial fibration with contractible fibres, and the mapping
\begin{equation}
\psi = (\tau, f) : X \setminus f^{-1}(0) \to U \times D^*_\delta
\end{equation}
is a locally trivial fibration whose fibre is diffeomorphic to the Milnor fibre of $f_t$ for any $t \in V$ (notice that $H$ is a cut over $V$).

Choose any $t \in U$. Let $W$ be a neighborhood of the origin adapted to $f_t$ satisfying that $\{t\} \times W$ is contained in $\hat{X}_t$. We will prove that the cobordism
\begin{equation}
(f_t^{-1}(s) \cap X_t \setminus \hat{W}, f_t^{-1}(s) \cap H, f_t^{-1}(s) \cap \partial W)
\end{equation}
is a simply connected $h$-cobordism for any $s \in D^*_\delta$.

Using the fibration (48) and Lemma 32 we deduce that $f_t^{-1}(s) \cap H$ is simply connected for any $(t,s) \in U \times D^*_\delta$. The space $f_t^{-1}(s) \cap \partial W$ is simply connected by Lemma 32 since $\partial W$ is a cut over $t$. By fibration (48) and the fact that $H$ is a cut over $V$ we deduce that $f_t^{-1}(s) \cap X_t$ has the same homotopy type than the Milnor fibre of $f_t$, which is simply connected by Kato-Matsumoto bound. As $W$ is a neighborhood adapted to $f$ the space $f_t^{-1}(s) \cap W$ is also homotopic to the Milnor fibre of $f_t$. Seifert-van Kampen Theorem applied to the decomposition

$$f_t^{-1}(s) \cap X_t = f_t^{-1}(s) \cap X_t \setminus \hat{W} \cup f_t^{-1}(s) \cap W$$

gives that $f_t^{-1}(s) \cap X_t \setminus \hat{W}$ is simply connected.

To finish the proof it is enough to show that all the relative homology groups

$$H_*(f_t^{-1}(s) \cap X_t \setminus \hat{W}, f_t^{-1}(s) \cap \partial W; \mathbb{Z})$$

vanish. By excision it is equivalent to show the vanishing

$$H_*(f_t^{-1}(s) \cap X_t, f_t^{-1}(s) \cap W; \mathbb{Z}) = 0.$$
Theorem 36. Assume $n \geq 5$. Let $f_t$ be a smooth family of holomorphic germs parametrised over a connected family $T$, such that $\Sigma_t$ is 1-dimensional at the origin for any $t$. If $f_t$ is equisingular at the critical set then the diffeomorphism type of the Milnor fibration and the homeomorphism type of the embedded link of $f_t$ is independent of $t$.

Proof. As $T$ is connected we are reduced to prove a local statement in the base. Given any $t \in T$ we consider an adapted neighborhood $W$ to $f_t$. By Corollary 33 for a certain positive $\delta$, the Milnor fibration of $f_t$ is $C^\infty$-equivalent to

$$f_t : W \cap f_t^{-1}(\partial D_\delta) \to \partial D_\delta$$

and the embedded link is homeomorphic to

$$(\partial(W \cap f_t^{-1}(D_\delta)), f_t^{-1}(0) \cap \partial W).$$

Consider another neighborhood of the origin $W'$ adapted to $f_t$, such that we have the inclusion $W' \subset W$. By Remark 30 there exists $\delta > 0$ such that

$$\{\{t\} \times \partial W' \cap f_t^{-1}(D_\delta)$$

and

$$\{\{t\} \times \partial W' \cap f_t^{-1}(D_\delta)$$

are cuts over $t$ of amplitude $\delta$. By Lemma 27 (1) and Lemma 35 there is a neighborhood $C$ of $t$ in $T$ (which can be taken take cubical) such that

$$H_+ := (C \times \partial W) \cap \psi^{-1}(C \times D_\delta)$$

and

$$H_- := (C \times \partial W') \cap \psi^{-1}(C \times D_\delta)$$

are cuts over $C$ of amplitude $\delta$.

Then the fibration

$$\psi : X_{int}(H_+, C, \delta) \setminus f_t^{-1}(0) \to C \times D_\delta^*$$

is locally trivial. Consequently the diffeomorphism type of the fibration

$$(50) \quad f_{t'} : \{t'\} \times W \cap f_{t'}^{-1}(\partial D_\delta) \to \partial D_\delta$$

is independent of $t' \in C$. For $t' = t$ we obtain the diffeomorphism type of the Milnor fibration of $t$.

An easy argument using Proposition 33 applied to $H_-$ and $H_+$ easily implies that the homeomorphism type of the pair

$$(51) \quad (\partial X_{int}(H_+, C, \delta)|_{t'}, \partial X_{int}(H_+, C, \delta)|_{t} \cap f_{t'}^{-1}(0))$$

is independent of $t' \in C$. For $t' = t$ we obtain the homeomorphism type of the embedded link of $f_t$.

Fix any $t' \in C$. Consider any neighborhood $W''$ adapted to $f_{t'}$ such that $W'' \subset W$. By Corollary 34 the Milnor fibration of $f_{t'}$ is diffeomorphic to

$$(52) \quad f_{t'} : W'' \cap f_{t'}^{-1}(\partial D_\delta) \to \partial D_\delta$$

and the embedded link of $f_{t'}$ is homeomorphic to

$$(53) \quad (\partial(W'' \cap f_{t'}^{-1}(D_\delta)), f_{t'}^{-1}(0) \cap \partial W)$$

for $\delta$ small enough.

By Remark 30 if we shrink $\delta$ enough we have that $\{\{t'\} \times \partial W''(t') \cap f_{t'}^{-1}(D_\delta)$ is a cut over $t'$ of amplitude $\delta$. Another application of Proposition 33 now for
\[ C = \{t'\}, \ H_+ := \{(t') \times \partial W\} \cap f_{\nu}^{-1}(D_\delta) \text{ and } H_- := \{(t') \times \partial W''\} \cap f_{\nu}^{-1}(D_\delta) \] implies that the diffeomorphism types of the fibrations \[\text{(51)}\] and \[\text{(52)}\] are the same and that the pairs \[\text{(51)}\] and \[\text{(52)}\] are homeomorphic. \(\square\)

6. Topological R-equivalence

Our aim is to prove that any family \( f_t \) of analytic germs, parametrised over a cube \( C \), with critical set of dimension at most 1, which is topologically equisingular at the critical set, is, in fact, topologically equisingular with respect to \( R \)-equivalence. Once we have the results of the previous section we can follow closely the strategy that we used for \( \mu \)-constant families in Section \ref{mu-equisingularity}. We do it now pointing specially the aspects in which the proofs are different.

6.1. Extension of cuts: the non-isolated case. We define extension of cuts as in Definition \[\ref{extension}]. Consider the closed and open cubes \( C \) and \( U \) as in \[\ref{cubes}\]. Consider two cuts \( H_+ \) and \( H_- \) for \( f \) defined over \( U \) of the same amplitude \( \delta \), pairs \((B_1, \mu_1), \ldots, (B_{k_f}, \mu_{k_f})\) \((C_1, \nu_1), \ldots, (C_{k_\nu}, \nu_{k_\nu})\), as in \[\ref{cubes}\] and a cut \( H_0 \) of amplitude \( \delta \) over a neighbourhood \( V \) of a contractible union of faces \( A \).

We show that Lemma \[\ref{extension} \] still holds in the new setting:

**Lemma 37.** In the setting above, after possibly shrinking \( U \) and \( V \) to smaller neighborhoods of the corresponding sets, there exists a cut \( H'_0 \) over \( U \) which extends \( H_0 \) and satisfies \( H_\neq H_0 \), \( \mu_i < H_0 \) for any \( i \), \( H_0 < \nu_i \) for any \( i \), and \( H_0 < H_+ \).

**Proof.** The structure of the proof is the same than the one of Lemma \[\ref{extension} \]. However we need to adapt several arguments.

Define \[ X := X_{\text{int}}(H_+, U, \delta) \setminus X_{\text{int}}(H_-, U, \delta). \]

**Step 1:** we shall inductively reduce to the case in which \( k_1 + k_2 = 0 \). Suppose that \( k_1 + k_2 > 0 \). Assume that \( k_2 > 0 \).

Observe that by the definition of a cut the projections
\[ \tau : \Sigma \cap X \to U, \]
\[ \tau : \Sigma \cap (X_{\text{int}}(H_+, C_1', \delta)) \setminus (X_{\text{int}}(\nu_1, U, \delta)) \to C_1', \]
\[ \tau : \Sigma \cap (X_{\text{int}}(\nu_1, U, \delta)) \setminus (X_{\text{int}}(H_-, C_1', \delta)) \to C_1', \]
are trivial fibrations with fibres trivial cobordisms diffeomorphic to a disjoint union of cylinders, one for each connected component of \( \Sigma \cap H_+ \). We will assume for simplicity that there is a unique connected component, being the general case analogous. We denote \( \Sigma \cap X \) by \( \Sigma' \). Taking into account that the "\(<\)" relations between the cuts are preserved by intersecting with \( \Sigma \), an argument like Step 1 of Lemma \[\ref{extension} \] shows that (after possibly shrinking \( U \) and \( C_1' \) to smaller neighborhoods) there exists a smooth closed hypersurface \( K \subset \Sigma' \) with the following properties:

1. (extension) we have \( K_{C_1'} = \nu_1 \cap \Sigma_{C_1'} \).
2. The space \( \Sigma' \setminus K \) has two connected components \( A_+ \) and \( A_- \) each of them containing respectively \( H_+ \cap \Sigma \) and \( H_- \cap \Sigma \). The component \( A_- \) contains moreover the intersections \( H_0 \cap \Sigma, \mu_i \cap \Sigma \) and \( \nu_j \cap \Sigma \) for any \( i \) and \( j \neq 1 \).
3. The restrictions \( \tau : A_- \to U \) and \( \pi : A_+ \to U \) are locally trivial fibrations with fibre a trivial cobordism (a cylinder).
In this situation there exists a diffeomorphism
\[ \varphi : \Sigma' \to (\Sigma' \cap H_-) \times [0, 1] \]
such that
\[ \varphi(H_- \cap \Sigma') = (H_- \cap \Sigma') \times \{0\}, \]
\[ \varphi(H_+ \cap \Sigma') = (H_- \cap \Sigma') \times \{1\}, \]
\[ \varphi(K \cap \Sigma') = (H_- \cap \Sigma') \times \{1/2\} \]
and
\[ \tau = \tau \circ q_1 \circ \varphi, \]
with \( q_i \) the projection of \((\Sigma' \cap H_-) \times [0, 1]\) to the \(i\)-th factor \(i = 1, 2\).

Let \( \pi_+ : M_+ \to \Sigma' \cap H_+ \), \( \pi_- : M_- \to \Sigma' \cap H_+ \) and \( \pi_{\nu_1} : M_{\nu_1} \to \nu_1 \cap \Sigma'|c_1 \) be normal bundles such that a neighborhood of \( \Sigma' \cap H_+ \), \( \Sigma' \cap H_- \) and \( \Sigma' \cap \nu_1 \) in \( H_+ \), \( H_- \) and \( \nu_1 \) coincide with a tubular neighborhood of the zero section of \( \pi_+, \pi_- \), and \( \pi_{\nu_1} \) respectively. As in Construction 5.2 we construct a normal bundle \( \nu : \Sigma' \to X \) which extends \( \pi_+, \pi_- \) and \( \pi_{\nu_1} \) and is holomorphic over an annulus contained in \( \Sigma' \) for a certain \( t \in C \), whose inclusion into \( \Sigma' \) is a homotopy equivalence. Consider \( \eta \) such that \( T_N(\eta) \) is naturally embedded in \( T \times \mathbb{C}^n \) and observe that the Milnor number of the restriction of \( f \) to the fibres of \( \pi \) at the origin of the fibres is constant.

We have a \( \mu \)-constant family over \( \Sigma' \) for which Assumption A is satisfied (see Corollary 15).

Define
\[ \phi := (\tau, f) : T_N(\eta) \to \Sigma' \times \mathbb{C}. \]
Consider the restriction
\[ \psi|_X = (\tau, f)|_X : X \to U \times D_\delta. \]

As in Construction 5.2, we may find a smooth hypersurface \( S \subset X \), which splits \( X \) in two submanifolds with boundary, whose closures are denoted by \( Y \) and \( Z \), (being \( Y \) the one containing \( \Sigma' \)) and a homeomorphism
\[ \Psi : Y \to (Y \cap H_-) \times [0, 1] \]
whose restrictions to \( \Sigma' \) and \( Y \setminus \Sigma' \) are smooth, and satisfying \( \psi|_Y = \psi|_{Y \cap H_-} \circ q_1 \circ \Psi \), and \( \varphi|_Y = ((\varphi|_{Y \cap H_-}) \circ \Psi, \) being \( q_i \) the projection of \((Y \cap H_-) \times [0, 1]\) to the \(i\)-th factor \(i = 1, 2\). The inverse \( \Psi^{-1} \) may be seen as a flow that integrates a (not necessarily continuous) vector field \( \mathcal{Y} \) defined over \( Y \), which is smooth over \( \Sigma' \) and \( Y \setminus \Sigma' \), is tangent to the fibres of \( \psi \), and whose integral \( \Psi^{-1} \) extends \( \varphi^{-1} \) and takes fibres of \( \pi \) to fibres of \( \pi \).

Our aim now is to extend \( \mathcal{Y} \) to a vector field defined over \( X \). We work first in \( X|_{C_1'} \). Define
\[ B_+ := Z \cap X_{int}(H_+, C_1', \delta) \setminus X_{int}(\nu_1, C_1', \delta), \]
\[ B_- := Z \cap X_{int}(\nu_1, C_1', \delta) \setminus X_{int}(H_+, C_1', \delta). \]
By Ehresmann fibration Theorem and contractibility of the base the restrictions \( \psi : B_+ \to C_1' \times D_\delta \) and \( \psi : B_- \to C_1' \times D_\delta \) are trivial fibrations, with fibres cobordisms with boundary, being the boundary cobordisms the fibres of the restrictions \( \psi|_{B_+ \cap S} \) and \( \psi|_{B_- \cap S} \). All the boundary cobordisms are simultaneously trivialised by the restrictions \( \Psi|_{B_+ \cap S} \) and \( \Psi|_{B_- \cap S} \).
Applying the procedures of the proof of Proposition 33 to $B_-$ and $B_+$, we obtain that there is a diffeomorphism

$$\Theta : Z_{C'_1} \to (Z \cap H_-)_{C'_1} \times [0, 1]$$

satisfying

(i) $\Theta|_{S_{C'_1}} = \Psi|_{S_{C'_1}}$,

(ii) $\psi|_{Z_{C'_1}'} = \psi|(Z \cap H_-)_{C'_1} \circ p_1 \circ \Theta$ (being $p_1$ the projection of $(Z \cap H_-)_{C'_1} \times [0, 1]$ to the first factor),

(iii) $\Theta(\nu_1|_{C'_1} \cap Z) = (H_-|_{C'_1}) \cap Z \times \{1/2\}$.

The first compatibility implies that $\Theta$ and $\Psi$ glue to a homeomorphism $\Xi$ from the union of their domains $(Y \cup Z_{C'_1})$ to the union of the images. Using a collaring we may assume that $\Xi$ is smooth at $S$. Observe that the cut $\nu_1|_{C'_1}$ is contained in the domain of definition of $\Xi$ and that

$$\Xi(\nu_1|_{C'_1}) = H_-|_{C'_1} \times \{1/2\}.$$

The inverse $\Xi^{-1}$ may be seen as a flow of a vector field $Z$ defined on $Y \cup Z_{C'_1}$.

Observe that the restriction

$$\psi : Z \to U \times D_\delta$$

is a trivial fibration with fibre a trivial cobordism with boundary. Following the procedure used in the proof of Lemma 5 to extend the vector field $\mathcal{Y}$ to $\mathcal{X}$, we can construct a vector field $Z'$ in $(Z \setminus \hat{Z})_{C'_1}$ which coincides with $Z$ at the intersection $S \cup Z_{0C'_1}$ of the domains of $Z$ and $Z'$, and whose flow induces a diffeomorphism

$$\Xi' : (Z \setminus \hat{Z}|_{C'_1}) \to (H_- \cap ((Z \setminus \hat{Z})_{C'_1}) \times [0, 1].$$

satisfying the analog of property (ii) above. A careful construction of $Z'$ (using a collaring) yields that $Z$ and $Z'$ glue to a vector field $\mathcal{X}$ defined over $X$, which is the desired extension of $\mathcal{Y}$ to $X$.

Step 1 finishes using the flow of $\mathcal{X}$ as we use the flow $\varphi$ in the proof of Lemma 5.

Step 2 is also as in Lemma 5.

\[\square\]

6.2. Existence of cuts: the non-isolated case. Let $f : U \times \mathbb{C}^n \to \mathbb{C}$ be a smooth family of holomorphic functions (with $U = (-\epsilon, 1 + \epsilon)^d$). Consider $C = [0, 1]^d \subset U$. Let $\rho$ be a distance function associated to a hermitian metric in the trivial vector bundle $\tau : U \times \mathbb{C}^n \to U$. Recall that $\psi = (\tau, f)$. Given any subset $B \subset U \times \mathbb{C}^n$ we denote by $\partial_r B$ the union $\partial_r B := \cup_{u \in U} \partial B_u$.

**Proposition 38.** Let $\theta : C \to (0, \infty)$ be any continuous function. There exist a positive $\delta$ and a cut $H$ over $C$ with amplitude $\delta$ such that $X_{int}(H, C, \delta)$ is contained in $B(C, \theta)$.

**Proof.** The proof is completely analogous to the proof of Proposition 33 using the fact that for any neighborhood $W$ of the origin adapted to $f$, there is a neighborhood $V$ of $t$ in $U$ and a positive $\delta$ such that $\partial W \cap \psi^{-1}(V \times D_\delta)$ is a cut for $f$ over $V$ of amplitude $\delta$. \[\square\]
6.3. Topological equisingularity. Let $C$, $\tau : E \to U$, $f$ and $\psi$ as in Section 3.2. Let $\theta_1 : U \to \mathbb{R}_+$ be a positive continuous function. By Proposition \ref{prop:existence}, there exists $\delta_1 > 0$ and a cut $H_1$ for $f$ over $U$ of amplitude $\delta_1$ such that $X := X_{\text{int}}(H_1, C, \delta_1)$ is contained in $B(C, \theta_1)$. Define $X^* := X \setminus f^{-1}(0)$.

Construction (†). Let $H_2$ be a cut for $f$ over $C$ of amplitude $\delta_2 < \delta_1$ such that $H_2 < H_1$. Consider $Z_1 := X_{\text{int}}(H_1, C, \delta_2) \setminus X_{\text{int}}(H_2, C, \delta_2)$. As in Construction \ref{construction:topological} we obtain a normal bundle $\pi_1 : M_1 \to Z_1 \cap \Sigma$ such that $T_\eta(M_1)$ embeds in $Z_1$ as a tubular neighborhood of $\Sigma \cap Z_1$ for a positive $\eta$. Using Proposition \ref{prop:existence} it is easy to construct a vector field $Z_1$ in $Z_1$ which is smooth outside $\Sigma$, smooth and tangent to $\Sigma$ at $\Sigma \cap Z$, and such that its flow induces a homeomorphism

$$\Theta : Z_1 \to (Z_1 \cap H_1) \times [0, 1]$$

satisfying

1. we have $\psi = \psi|_{Z_1 \cap H_1} \circ p_1 \circ \Theta$, where $p_i$ is the projection of $(Z_1 \cap H_1) \times [0, 1]$ to the $i$-th factor,
2. we have the equalities

$$\Theta(Z_1 \cap H_1) = (Z_1 \cap H_1) \times \{0\}$$

$$\Theta(Z_1 \cap H_2) = (Z_1 \cap H_1) \times \{1\},$$

3. for $\eta$ sufficiently small, the diffeomorphism given by the restriction of the flow to $T_\eta(M_1)$ for a fixed time $s$ takes the fibres of $\pi_1$ to fibres of $\pi_1$.

Lemma 39. There exists a (not necessarily continuous) vector field $X$ in $X \setminus C$ with the following properties:

1. its restrictions to $X \setminus \Sigma$ and to $\Sigma \setminus C$ are smooth
2. it is tangent to the fibres of $\tau$,
3. there exists a vector field $W$ in $D_0^\ast$, which is radial, pointing to the origin and of modulus $|W(z)| \leq |z|^2$, such that $df(X)(x) = W(f(x))$ for any $x \in X \setminus f^{-1}(0)$,
4. the vector field $X$ is tangent to $f^{-1}(0)$ outside $\Sigma$ and it is tangent to $\Sigma$ in $\Sigma$,
5. any integral curve converges to the origin of the fibre of $\tau$ in which it lies in positive infinite time,
6. the flow of $X$ is continuous.

Proof. As in the proof of Lemma \ref{lemma:topological}, we construct $X$ as the amalgamation two vector fields $Y$ and $Z$.

There is a continuous function $\theta_2 : C \to \mathbb{R}$ such that $B(C, \theta_2)$ is included in $X_{\text{int}}(H_1, C, \delta_1)$. By Proposition \ref{prop:existence} there exists $\delta_2$ satisfying $\delta_1 > \delta_2 > 0$ and a cut $H_2$ for $f$ over $C$ of amplitude $\delta_2$ such that $X_2 := X_{\text{int}}(H_1, C, \delta_2)$ is contained in $B(C, 1/2\theta_2)$. We iterate this procedure to obtain an infinite sequence of constants $\delta_i$ and cuts $H_i$ over $C$ such that the sets $X_i := X_{\text{int}}(H_i, C, \delta_i)$ form a nested sequence

\begin{equation}
X = X_1 \supset X_2 \supset ... \supset X_i \supset ...
\end{equation}

of closed neighborhoods of $C$ such that $\cap_{i=1}^\infty X_i$ is equal to the zero section $C$.

Let $Z_i := (X_i \setminus X_{i+1}) \cap f^{-1}(D_{\theta_{i+1}})$. Let $\pi_i : M_i \to \Sigma \cap Z_i$ be the normal bundle appearing in Construction (†). If the normal bundles $\pi_i$ are chosen carefully enough they glue to a normal bundle $\pi : M \to \Sigma \setminus C$. Let $\xi : \Sigma \setminus C \to \mathbb{R}_+$ be a smooth
function such that $T_M(\xi)$ embeds as a neighborhood of $\Sigma \setminus C$ in $X \setminus C$. Construction (\dagger) produces a vector field $Z_i$ over $Z_i$. A partition of unity argument glues the vector fields $Z_i$ to a vector field $Z$ defined on $Z := \cup_{i=1}^{\infty} Z_i$, smooth in $Z \setminus \Sigma$, tangent to the fibres of $\psi$ at their smooth points, and such that $Z|_{\Sigma}$ is smooth and tangent to $\Sigma$. Observe that we have $df(Z) = 0$. Let

$$\Psi : W \to T_M(\xi)$$

be the flow of $Z|_{T_M(\xi)}$, where $W \subset T_M(\xi) \times \mathbb{R}$ is the maximal domain where it is defined. By the property (c) of the flow of $Z_i$ (see Construction (\dagger)), for any $(x,u) \in W$ the commutation relation

$$\pi(\Psi(x,u)) = \Psi(\pi(x),u) \tag{55}$$

holds.

Consider in $C \times D_0^+$ the vector field $V'$ characterised by being tangent to the fibres of the projection from $C \times D_0^+$ to the first factor, and a lift by the projection from $C \times D_0^+$ to the second factor of the vector field $V$ in $D_0^+$ which is radial, pointing at the origin and of modulus $||V(z)|| = ||z||^2$. As $\psi : X^* \to U \times D_0^+$ is submersive we can define the vector field $Y$ in $X^*$ to be a lifting of $V'$ by the mapping $\psi$. Moreover, as the restriction of $f$ to the fibres of $p : T_M(\xi) \to \Sigma$ only has critical points at $\Sigma$, (perhaps having to shrink $\xi$) we may construct $\Psi$ such that its restriction to $T_M(\xi)$ is tangent to the fibres of $\pi : T_M(\xi) \to \Sigma$.

Let $\rho_1 : D_0 \to [0,1]$ be a smooth function vanishing at 0 and positive in $D_0^+$. We choose $\rho_1$ small enough that the modulus $||\rho_1(f(z))V(z)||$ converges to 0 as $||f(z)||$ approaches 0. Let $\rho_2 : U \to \mathbb{R}$ be an smooth function with support contained in the interior of $Z$ and which is identically 1 in a neighborhood of $f^{-1}(0) \cap Z$ in $Z$. Define the vector field $\mathcal{X} := (\rho_1 \circ f)V + \rho_2 Z$ on $X$.

As $\mathcal{X}|_{f^{-1}(0)}$ coincides with $Z|_{f^{-1}(0)}$ it is clear that any integral curve in $f^{-1}(0)$ converges to the origin in positive infinite time. It is also clear that the restrictions of $\mathcal{X}$ to $Y \setminus \Sigma$ and to $\Sigma \setminus C$ are smooth, and that the later is tangent to $\Sigma \setminus C$. All the properties required to $\mathcal{X}$ are clear except (3), (5) and (6).

Property (3) is true taking $W := \rho_1 V$.

Observe that the restriction $Y|_{H_1 \cap \mathcal{Y}}$ is tangent to $H_1$, and that $Z$ points into $X$ at any $y \in Z \cap H_1$. On the other hand $df(\mathcal{X})$ is radial and pointing to the origin. This shows that no integral curve of $\mathcal{X}$ can go out of the domain $X$ in positive time. As $df(\mathcal{X}) = \rho_1 V$ and $V$ is radial, pointing to the origin, and of modulus small enough that any of its integral curves converges to the origin in positive infinite time, we deduce that any integral curve of $\mathcal{X}$ not lying in $f^{-1}(0)$ is defined in positive infinite time. As $\mathcal{X}$ coincides with $Z$ in $f^{-1}(0)$ it is also clear that the integral curves contained in $f^{-1}(0)$ are defined in positive infinite time.

The continuity of the flow of $\mathcal{X}$

$$\Phi : (X \setminus C) \times [0, \infty) \to X$$

is clear outside $(\Sigma \setminus C) \times [0, \infty)$, since $\mathcal{X}$ is smooth outside $\Sigma$. Observe that from the Relation (55) and from the tangency of $Y|_{T_M(\xi)}$ to the fibres of $\pi : T_M(\xi) \to \Sigma$ we obtain that if $\Phi : W' \to T_M(\xi)$ is the flow of $\mathcal{X}|_{T_M(\xi)}$ (where $W'$ is its maximal domain of definition), then we have the commutation relation

$$\pi(\Phi(x,u)) = \Phi(\pi(x),u) \tag{56}$$
for any \((x,u) \in W\). The continuity of \(\Phi\) at any point of \((\Sigma \setminus C) \times [0, \infty)\) follows easily using this relation and the fact that the modulus \(||\rho_1(f(z))Y(z)||\) converges to 0 as \(||f(z)||\) approaches 0. This shows (6).

As the vector field \(\mathcal{X}\) is smooth outside \(\Sigma\) the only accumulation points of an integral curve of \(\mathcal{X}\) as time tends to \(+\infty\) can be in \(\Sigma\). As any integral curve of \(\mathcal{X}\) inside \(\Sigma\) converges to the origin in positive infinite time, from the continuity of the flow of \(\mathcal{X}\) we deduce that the only accumulation point of an integral curve of \(\mathcal{X}\) can be the origin of the fibre of \(\tau\) where it is contained. This proves Property (5). □

Here is our main topological equisingularity theorem

**Theorem 40.** Let \(f: C \times \mathbb{C}^n \to \mathbb{C}\) be a family of holomorphic germs at the origin, with 1-dimensional critical set and smoothly parametrised over a cube \(C\). Suppose that it is equisingular at the critical set. Let \(t\) be any point of \(C\). Use the notations considered in this section. There exists a homeomorphism

\[
(57) \quad \Psi: X \to C \times X_t
\]

such that

1. we have \(\tau = p_1 \circ \Psi\), where \(p_i\) is the projection of \(C \times X_t\) to the \(i\)-th factor \((i = 1, 2)\),
2. we have \(f|_{X \setminus \Sigma} \circ p_2 \circ \Psi = f\),
3. the restriction of \(\Psi\) to \(X \setminus \Sigma\), and to \(\Sigma \setminus C\) is smooth.

**Proof.** The first step is to construct the restriction of \(\Psi\) to \(\partial X\). We have the decomposition \(\partial X = (X \cap f^{-1}(\partial D_\delta)) \cup (H_1 \cap f^{-1}(D_\delta))\). The restriction of \(\Psi\) to \(H_1 \cap f^{-1}(D_\delta)\) can be constructed applying Proposition 33 to the cuts \(H_2 \prec H_1\) (see Construction (†)). The extension to \(X \cap f^{-1}(\partial D_\delta)\) is easy to obtain using that

\[
\psi: X \cap f^{-1}(\partial D_\delta) \to C \times \partial D_\delta
\]

is a locally trivial fibration.

After this the proof is completely analogous to the proof of Theorem 2 replacing the reference to Lemma 10 by a reference to Lemma 39. □

7. **Families with constant Lê numbers**

Let \(f: T \times \mathbb{C}^n \to \mathbb{C}\) a holomorphic function, where \(T\) is a connected complex manifold. Unless we state the contrary we suppose that the critical set of \(f_t\) has dimension 1 at the origin for any \(t \in T\). Let \(Z := \{z_1, ..., z_n\}\) be a coordinate system of \(\mathbb{C}^n\). The symbol \(\lambda_{f_t}^i Z(x)\) denotes the \(i\)-th Lê number of a holomorphic function \(f\) at the point \(x\) with respect to \(Z\). Our aim in this section is to prove the following theorem:

**Theorem 41.** If any of the following conditions hold:

1. there is a coordinate system \(Z\) such that the Lê numbers at the origin of \(f_t\) with respect to \(Z\) are defined and independent of \(t\),
2. the generic Lê numbers at the origin of \(f_t\) are independent of \(t\),

then the embedded topological type of \(f_t\) at the origin and the the diffeomorphism type of the Milnor fibration is independent of \(t \in T\). Moreover, if \(T\) is (diffeomorphic to) a cube, the restriction of the family over \(T\) is topologically \(R\)-equisingular.
Because of the results of the previous sections to prove the Theorem it is enough to show that the family is equisingular at the singular set. Thus we are reduced to prove:

**Theorem 42.** If any of the conditions of the previous theorem hold then \( f_t \) is equisingular at the critical set.

Since equisingularity at the critical set is local in the base, we can assume that \( T \) is an open neighborhood of the origin in an affine space, and it is enough to prove that \( f \) is equisingular at the critical set in a small neighborhood \( V \) of the origin in \( T \).

A generic coordinate system \( Z \) for \( f_{t_0} \) satisfies that the Lê numbers at the origin of \( f_t \) with respect to it are defined for any \( t \) close enough to \( t_0 \). Moreover, by the lexicographical upper semicontinuity of the Lê numbers ([14] Corollary 4.16) the constancy of the generic Lê numbers implies the constancy of the Lê numbers of \( f_t \) with respect to \( Z \), for \( t \) close enough to \( t_0 \). Therefore Condition (2) implies Condition (1), at least locally in the base.

By the definition of equisingularity at the critical set it is clear that (by slicing), it is enough to assume that \( T \) has complex dimension 1. Thus we make the assumption \( T = \mathbb{C} \). To any coordinate system \( Z = \{z_1, \ldots, z_n\} \) for \( (\mathbb{C}^n, O) \) we associate the coordinate system \( Z' := \{t, z_1, \ldots, z_n\} \) of \( \mathbb{C} \times \mathbb{C}^n \).

Given \( A \subset \mathbb{C} \times \mathbb{C}^n \) we consider the notation \( A_t := A \cap (\{t\} \times \mathbb{C}^n) \).

Let \( \Sigma_f := \{\partial f/\partial t, \partial f/\partial z_1, \ldots, \partial f/\partial z_n\} \) denote the critical set of \( f \) and let \( \Sigma := \{\partial f/\partial z_1, \ldots, \partial f/\partial z_n\} \) be the union of the critical sets of the \( f_t \)’s.

**Lemma 43.** Let \( f_t \) be a family with critical set not necessarily 1-dimensional: let \( s := \dim_O(\Sigma_0) \). Suppose that for small \( t \) the Lê numbers \( \lambda_{f_t, Z}(O) \) are defined and independent on \( t \) for any \( 0 \leq i \leq s \). Then the sets \( \Sigma_f \) and \( \Sigma \) are equal in a neighborhood of the origin.

**Proof.** We only have to show \( \Sigma \subset \Sigma_f \). For this we show that for any \( t \) small enough the set \( (\Sigma_f)_t \) contains \( \Sigma_t \). If the inclusion does not hold there exists a sequence \( y_n = (t_n, x_n) \) converging to \( (O, 0) \) such that \( x_n \) belongs to \( \Sigma_{t_n} \setminus \Sigma_f \). Hence \( V(f - f(y_n)) \) is smooth at \( y_n \) and the tangent space \( T_{y_n} V(f) \) is equal to \( V(t - t_n) \). Consequently the limit of the tangent spaces is \( V(t) \).

On the other hand, by [14], Theorem 6.5, \( \mathbb{C} \times \{O\} \) satisfies Thom’s \( A_t \) condition with respect to the open stratum at the origin, which contradicts the fact that the limit of tangent spaces is \( V(t) \).

A consequence is that the first polar variety of \( f \) is empty, since, it is defined by

\[
\Gamma^1_{f, Z} = \overline{V(\partial f/\partial z_1, \ldots, \partial f/\partial z_n) \setminus \Sigma_f}.
\]

We deduce the following vanishings for \( t \) small enough (the Lê varieties and Lê cycles computations that follow are done using [14], Chapter 1):

\[
\gamma^1_{f, Z}(t, O) = 0 \quad \Lambda^0_{f, Z} = 0 \quad \lambda^0_{f, Z}(t, O) = 0.
\]

Let \( \Sigma = \cup_{i=1}^{k_1} \Sigma_i \) and \( \Sigma_0 = \cup_{i=1}^{k_2} \Sigma_{0,i} \) be decompositions of \( \Sigma \) and \( \Sigma_0 \) in irreducible components. Reordering conveniently we can find numbers \( 1 \leq k_1 \leq k_2 \leq k \) such that

1. The component \( \Sigma_i \) is 2-dimensional for \( 1 \leq i \leq k_1 \).
(2) The component $\Sigma_i$ is 1-dimensional and contained in $V(t)$ for $k_1 + 1 \leq i \leq k_2$.

(3) The component $\Sigma_i$ is 1-dimensional and not contained in $V(t)$ for $k_2 + 1 \leq i \leq k$.

(4) There exist numbers $0 = r_0 \leq r_1 \leq \ldots \leq r_{k_2} = r$ such that

$$\Sigma_j \cap V(t) = \bigcup_{i=r_{j-1}+1}^{r_j} \Sigma_{0,i}$$

for any $1 \leq j \leq k_2$.

Now we assume that all the Lé numbers of $f_i$ at the origin with respect to $Z$ are defined for any $t$.

In the following discussion we draw further consequences from the fact that the Lé numbers $\lambda_{f_i,Z}^j(O)$ are independent on $t$ for any $0 \leq i \leq s = 1$. Let $X,Y \subset U$ be closed analytic subspaces of an open subset $U$ of $\mathbb{C}^n$. We denote by $X \setminus Y$ the scheme theoretical closure of $X \setminus Y$ in $U$, and by $[X]$ the cycle associated with the scheme $X$ (see [14], Chapter 1).

Since the first Lé number at the origin of $f_i$ with respect to $Z$ is defined, and $\Sigma_i$ is 1-dimensional, all the irreducible components of $\Gamma_{f_i,Z}^2$ are of the expected dimension 2 for any $t$ (see [14], pages 11-14). As $\Sigma_{f_i}$ is of dimension 1 and, by definition,

$$\Gamma_{f_i,Z}^2 = V(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n) \setminus \Sigma_{f_i},$$

we conclude that all the irreducible components of $V(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n)$ are 2-dimensional. Hence $(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n)$ is a regular sequence and the analytic subscheme $V(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n)$ has no embedded components. Thus we have

$$\Gamma_{f_i,Z}^2 = V(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n).$$

The third polar variety of $f$ is

$$\Gamma_{f_i,Z}^3 = V(\partial f/\partial z_3, \ldots, \partial f/\partial z_n) \setminus \Sigma_f = V(\partial f/\partial z_3, \ldots, \partial f/\partial z_n).$$

The first equality is by definition. From the fact that all the irreducible components of $V(\partial f_i/\partial z_3, \ldots, \partial f_i/\partial z_n)$ are 2-dimensional and the Theorem on the dimension of the fibres of a morphism we deduce that all the irreducible components of $V(\partial f/\partial z_3, \ldots, \partial f/\partial z_n)$ have at most dimension 3, which is on the other hand its minimal possible dimension given the number of equations defining the set. Hence $(\partial f/\partial z_3, \ldots, \partial f/\partial z_n)$ is a regular sequence and the analytic subscheme $V(\partial f/\partial z_3, \ldots, \partial f/\partial z_n)$ has no embedded components. This, together with the fact that $\Sigma$ is 2-dimensional implies the second equality.

Consider a decomposition in irreducible components of $\Gamma_{f,Z}^3 \cap V(\partial f/\partial z_2) = V(\partial f/\partial z_2, \ldots, \partial f/\partial z_n)$:

$$V(\partial f/\partial z_2, \ldots, \partial f/\partial z_n) = \bigcup_{i=1}^{k_1} X^i \bigcup_{i=1}^{d} Y^i,$$

where $X^i$ coincides as a set with the component $\Sigma_i$ of $\Sigma$. As any $Y_i$ is (at least) 2-dimensional and the components $\Sigma_i$ are 1-dimensional for $i > k_1$, no $Y^i$ is a component of $\Sigma$. We have

$$\Gamma_{f,Z}^2 = \Gamma_{f,Z}^2 \cap V(\partial f/\partial z_2) \setminus \Sigma = \bigcup_{i=1}^{d} Y^i,$$
\[ \Lambda_{f,Z'}^2 = [\Gamma_{f,Z'}^3 \cap V(\partial f / \partial z_2)] - [\Gamma_{f,Z'}^2] = \sum_{i=1}^{k_1} [X^i]. \]

Now we want to compute the first Lê cycle of \( f_t \) for a certain value of the parameter \( t \). By equation (62) we have
\[ \Gamma_{f_t,Z}^2 \cap V(\partial f_t / \partial z_2) = V(\partial f_t / \partial z_2, ..., \partial f_t / \partial z_n). \]
Hence, taking into account Equation (64), we have
\[ [\Gamma_{f_t,Z}^2 \cap V(\partial f_t / \partial x_2)] = (\sum_{i=1}^{k_1} [X^i_t]) + (\sum_{i=1}^{d} [Y^i_t]). \]

Clearly \( (\bigcup_{i=1}^{k_1} X^i_t) \) is included in \( \Sigma_{f_t} \), and therefore
\[ \Lambda_{f_0,Z}^1 \geq \sum_{i=1}^{k_1} [X^i_{t_0}] = (\sum_{i=1}^{k_1} [X^i]) \cdot [V(t - t_0)]. \]

**Lemma 44.** The previous inequality is actually an equality for any \( t_0 \in T \):
\[ \Lambda_{f_0,Z}^1 = \sum_{i=1}^{k_1} [X^i_{t_0}] = (\sum_{i=1}^{k_1} [X^i]) \cdot [V(t - t_0)]. \]
Moreover, the components \( \Sigma_{k_1+1}, ..., \Sigma_{k_2} \) do not appear. In other words, we have \( k_1 = k_2 \).

**Proof.** We claim that no component of \( Y^i_{t_0} \) is contained in \( \Sigma_{f_{t_0}} \) for any \( i \leq d \) and any \( t_0 \). The claim implies easily both assertions of the Lemma.

By Lemma 43 it is enough to show that no component of \( Y^i_{t_0} \) is contained in \( \Sigma \). This is clear (after possibly shrinking the base of the family) for \( t_0 \neq 0 \). Thus equality (70) holds in this case.

Inequality (69) implies that the first Lê number of \( f_{t_0} \) satisfies
\[ \lambda_{f_{t_0},Z}^1 \geq \operatorname{mult}_{\{0\}}(\sum_{i=1}^{k_1} [X^i]) \cdot [V(t - t_0)] \cap [V(z_1)] = \operatorname{mult}_{\{0\}}(\sum_{i=1}^{k_1} [X^i \cap V(z_1)]) \cdot [V(t - t_0)]. \]

This inequality becomes an equality if and only if our claim is true. This is the case if \( t_0 \neq 0 \).

As \( \operatorname{mult}_{\{0\}}(\sum_{i=1}^{k_1} [X^i \cap V(z_1)]) \cdot [V(t)] \) is upper semicontinuous in \( t \) and the Lê number \( \lambda_{f_t,Z}^1 \) is independent on \( t \) we conclude that Inequality (71) must be an equality also for \( t_0 = 0 \). \( \square \)

**Lemma 45.** The Lê numbers \( \lambda_{f,Z}(t,O) \) are defined for \( i \leq 2 \) and any \( t \).

**Proof.** We work for \( t = 0 \). The case \( t \neq 0 \) is analogous. We have already checked that \( \lambda_{f,Z}^0(O) \) is defined (and zero).

Two consequences of Lemma 44 are the equalities
\[ \Lambda_{f_0,Z}^1 = \sum_{i=1}^{k_1} [X^i \cap V(t)], \]
\[ \Gamma_{f_0,Z}^1 = \sum_{i=1}^{d} [Y^i \cap V(t)]. \]

The definedness of \( \lambda_{f_0,Z}^1 (O) \) implies that \( X^i \cap V(t,x_1) \) is 0-dimensional at the origin for any \( i \). Hence, taking into account (66), we obtain that \( \lambda_{f,Z'}^2 (0,O) \) is defined.

It only remains to be shown that \( \lambda_{f,Z'}^1 (O) \) is defined. The first L\'e cycle is
\[ \Lambda_{f,Z'}^1 = [\Gamma_{f,Z'}^2 \cap V(\partial f/\partial z_1)] - [\Gamma_{f,Z'}^1]. \]

We only need to prove that \( Y^i \cap V(\partial f/\partial z_1) \cap V(t) \) is 0-dimensional at the origin. By Identity (73) the last set is equal to \( \Gamma_{f_0,Z}^1 \cap V(\partial f_0/\partial z_1) \). We have to show that \( \Gamma_{f_0,Z}^1 \cap V(\partial f_0/\partial z_1) \) is 0-dimensional. If this is not the case then \( \Gamma_{f_0,z}^1 \) is contained in \( V(\partial f_0/\partial z_1, ..., \partial f_0/\partial z_n) \) and meets the origin. Hence it is contained in \( V(f_0) \), but this is impossible by definition of the relative polar varieties.

\textbf{Lemma 46.} The components \( \Sigma_{k_2+1}, ..., \Sigma_k \) do not appear. In other words, we have \( k_2 = k \).

\textbf{Proof.} Applying Proposition 1.21 of [14], the vanishing (59) and the constancy of the L\'e numbers of \( f_t \) we obtain that for any \( t \) small enough \( \lambda_{f,Z'}^1 (t,O) \) is constant.

Suppose \( k_2 \neq k \). As \( \Sigma_{k_2+1} \) is a 1-dimensional irreducible component of the singular locus the cycle \( [\Sigma_{k_2+1}] \) is a positive summand of \( \Lambda_{f,Z'}^1 \). As it goes through \( (0,O) \) but not through \( (t,O) \) for \( t \) non-zero, we have that \( \lambda_{f,Z'}^1 (0,O) \) is strictly bigger than \( \lambda_{f,Z'}^1 (t,O) \), which is a contradiction. \( \square \)

Let \( \Sigma_{\text{red}} \) denote the analytic subset \( \Sigma \) with its reduced structure. We have

\textbf{Corollary 47.} The analytic subset \( \Sigma \) admits a decomposition in irreducible components at the origin of the form \( \Sigma := \bigcup_{i=1}^{k} \Sigma_i \) such that each component is 2-dimensional at the origin. Moreover the restriction
\[ \pi|_{\Sigma_{\text{red}}} : \Sigma_{\text{red}} \to \mathbb{C} \]

is flat.

\textbf{Proof.} We have proved everything except flatness. Each of the components of the primary decomposition of \( \Sigma_{\text{red}} \) is an irreducible component, and all of them map dominant by \( \pi|_{\Sigma_{\text{red}}} \) to the 1-dimensional target \( \mathbb{C} \). It is well known that this implies that \( \pi|_{\Sigma_{\text{red}}} \) is flat. \( \square \)

Now we study the geometry of the fibres of the mapping (74) in a neighborhood of the origin. To obtain a lighter notation we denote \( \Sigma_{\text{red}} \) by \( S \).

Consider the mapping
\[ \alpha := (z_1,t) : S \to \mathbb{C}^2. \]
As the origin $(0, O)$ is an isolated point of $\alpha^{-1}(O)$, there is a neighborhood $U$ of $(0, O)$ in $S$ such that, for any $\epsilon$, $\eta$ positive and small enough, the restriction

$$\alpha|_{E^}\colon E := \alpha^{-1}(D_\epsilon \times D_\eta) \cap U \rightarrow D_\epsilon \times D_\eta$$

is a finite mapping, the space $E$ is a neighborhood of the origin in $S$ and $\alpha|_{E^}^{-1}(0, 0) = \{(0, O)\}$. We will denote $\alpha|_E$ simply by $\alpha$. In the sequel $\epsilon$ and $\eta$ are chosen always so that $\alpha$ is finite.

We choose $\epsilon$ and $\eta$ so small that the decompositions $E = \cup_{i=1}^k E_i$ and $E_0 = \cup_{i=1}^r E_{0,i}$ in irreducible components coincide at the origin with the previously given decompositions; in other words, as germs of reduced sets at the origin we have $E_i = \Sigma_i$ and $E_{0,i} = \Sigma_{0,i}$. We will always choose $\epsilon$ small enough that any two different components of $E_0$ only intersect at the origin.

It is well known that for $\epsilon$ and $\eta$ sufficiently small the mapping

$$\pi|_E : E \rightarrow D_\eta$$

is a topologically trivial fibration over $D^n_\eta$.

We consider a decomposition of $E_t$ in irreducible components $E_{t,1}, \ldots, E_{t,r_t}$ for any $t$. The topological local triviality of $\pi|_E$ over $D^n_\eta$ implies that the number $r_gen := r_t$ is independent on $t$ as long as $t \in D^n_\eta$. For any $t \in D^n_\eta$ we consider a numbering $E_{t,1}, \ldots, E_{t,r_{t,gen}}$ of the irreducible components of $E_t$. There is a mapping $\beta_t : \{1, \ldots, r_{gen}\} \rightarrow \{1, \ldots, k\}$ defined by the relation $E_{t,i} \subset E_{\beta_t(i)}$. We also define a mapping $\beta_0 : \{1, \ldots, r\} \rightarrow \{1, \ldots, k\}$ by the relation $E_{0,j} \subset E_{\beta_0(j)}$.

Let $\mu_{t,j}$ denote the generic transversal Milnor number of $f_t$ at a generic point of $E_{t,j} \setminus \{O\}$. Observe that there is a finite set $I_t \subset D_t$ such that, for any $s \in D_t \setminus I_t$, the hyperplane $V(z_1 - s)$ meets $E_t$ transversely at points with transversal Milnor number $\mu_{t,j}$. By Lemma 20, for any $s \in D_t \setminus I_t$, the Milnor number of any of the isolated singularities of $f_t|_{V(z_1 - s)}$ contained in $E_{t,j}$ is equal to $\mu_{t,j}$. Moreover, for any $x \in V(z_1 - s) \cap E_t$, we have the equality

$$I_x(L^{1}_{f_t,x}, V(z_1 - s)) = \mu_{t,i};$$

indeed, since $f_t$ has smooth critical set at $s$ and $V(z_1 - s)$ meets $E_t$ transversely in $x$, the first Lé number $\lambda^{1}_{f_t,x}$ is equal to the generic transversal Milnor number. On the other hand, by definition, the Lé number is equal to the intersection multiplicity $I_x(L^{1}_{f_t,x}, V(z_1 - s))$.

Given any component $E_i$ we define a family of germs with isolated singularities parametrised over it, by imposing that, for any $x \in E_i$, we assign the singularity at $x$ defined by the restriction $f_{\pi(x)}|_{V(z_1 - z_1(x))}$. Let $\mu_i$ be the generic Milnor number of the family. Observe that, by the upper semicontinuity of the Milnor number, if $E_{t,j}$ is contained in $\Sigma_i$ then $\mu_i \leq \mu_{t,j}$. As $\mu_i$ and $\mu_{t,j}$ are the generic values of the Milnor number of the hyperplane sections $V(z_1 - s)$ at $E_t$ and $E_{t,j}$ respectively, we obtain that for any generic $t$ we have the equality $\mu_i = \mu_{t,j}$: the subset of $E_t$ in which the hyperplane section has Milnor number strictly bigger than $\mu_i$ is closed in the Zariski topology. We may express the last equality as

$$\mu_{t,j} = \mu_{\beta_t(j)}$$

for $t$ generic.

By Sard’s Theorem the mapping $\pi|_E$ is generically submersive at the regular locus of $E$.

**Proposition 48.** The following assertions hold:
(1) If $\epsilon$ is small enough, the analytic set $E_0 \setminus \{O\}$ is smooth and $\pi|_E$ is submersive at $E_0 \setminus \{O\}$.

(2) For any $j \in \{1, \ldots, r_0\}$ we have $\mu_{0,j} = \mu_{\beta_0(j)}$.

Proof. Clearly, if $\epsilon$ is small enough, the only singularity of $E_0$ is at the origin.

The first Lê number $\lambda^1_{f_0, z}(O)$ is $I_O(\Lambda^1_{f_0, z}, V(z_1))$ by definition. In Lemma 11 we have proved that $\Lambda^1_{f_0, z}$ is equal to $\sum_{i=1}^{k_1} [X^i] = \sum_{t=1}^{k_2} [X^i] \cdot [V(t - t_0)]$ for any $t_0$. By conservation of number in intersection theory we have that for a certain $\epsilon$ small enough, there exist positive $\xi$ and $\eta$ such that if $s \in D_\xi$ and $t \in D_\eta$, then

$$\lambda^1_{f_0, z}(O) = \sum_{x \in X_t \cap V(z_1 - s)} I_x([X_i], V(z_1 - s)).$$

By Equality (78), if $s \in D_\xi \setminus I_t$, this quantity is equal to the sum

$$\sum_{x \in V(f_t, z_1 - s)} \mu(f_t|V(z_1 - s), x)$$

of the Milnor numbers of the singularities of the restriction of $f_t$ to $V(z_1 - s)$ which are contained in the zero set $V(f_t, z_1 - s)$.

If either a component of $E_0$ is contained in the singular locus of $E$, or $\pi|_E$ is not generically submersive at it, then $E_0$ is contained in the ramification locus of $\alpha$, and therefore $D_\eta \times \{0\}$ is a component of the branching locus of $\alpha$. Hence, if $t \in D^*_\eta$ and $\eta$ is small enough the set $D_\xi \times \{t\}$ is not included in the branching locus of $\alpha$. Then, if $s$ does not belong to the finite set $I_0 \cup I_1$, the cardinality of $X_0 \cap V(z_1 - s)$ is strictly smaller than the cardinality of $X_t \cap V(z_1 - s)$ for $t \neq 0$. As the cardinality of $X_t \cap V(z_1 - s)$ is the number of singularities of $f_t|V(z_1 - s)$ contained in $f_t|V(z_1 - s)(0)$, by the non-splitting result of Lê-Lazzery we have the strict inequality

$$\sum_{x \in V(f_t, z_1 - s)} \mu(f_t|V(z_1 - s), x) < \sum_{x \in V(f_0, z_1 - s)} \mu(f_0|V(z_1 - s), x),$$

which is impossible since both quantities are equal to $\lambda^1_{f_0, z}(O)$ by Equalities (78) and (80). This proves that $\pi$ is generically submersive at any component of $E_0$. Shrinking $\epsilon$ we obtain assertion (1).

Fix $s_0 \neq 0$ small enough so that $V(z_1 - s_0)$ meets $E_0$ transversely in finitely many points. Then there exits $\xi$ small enough that $V(z_1 - s_0)$ meets $E_\xi$ transversely for any $t \in D_\xi$, and hence

$$\alpha|_{\alpha^{-1}(\{s_0\} \times D_\xi)} : \alpha^{-1}(\{s_0\} \times D_\xi) \to \{s_0\} \times D_\xi$$

is finite and etale. Thus, the space $\alpha^{-1}(\{s_0\} \times D_\xi)$ splits in a finite number of disks and we have a natural bijection between $\Sigma_{f_t} \cap V(z_1 - s_0)$ and $\Sigma_{f_0} \cap V(z_1 - s_0)$ induced by incision in $\alpha^{-1}(\{s_0\} \times D_\xi)$.

Choose $x \in \Sigma_{f_t} \cap V(z_1 - s_0)$ for a certain $t \in D^*_\xi$, and let $y$ be the corresponding point in $\Sigma_{f_0} \cap V(z_1 - s_0)$ by the bijection. Suppose that $x$ belongs to $E_{t,j_1}$ and that $y$ belongs to $E_{0,j_2}$. Clearly $E_{t,j_1}$ and $E_{0,j_2}$ belong to the same irreducible component of $E$, in other words $\beta_t(j_1) = \beta_0(j_2)$. The summand of the left hand side of Inequality 61 corresponding to $x$ is equal to $\mu_{t,j_1}$ and the summand of the right hand side of Inequality 61 corresponding to $y$ is equal to $\mu_{0,j_2}$. Observe that $\mu_{t,j_1}$ is equal to $\mu_{\beta_t(j_1)}$ for $t \in D^*_\eta$ generic, and that, in general, we have the inequality $\mu_{0,j_2} \geq \mu_{\beta_0(j_2)} = \mu_{\beta_0(j_1)}$. Hence the contribution of $x$ to the LHS of
Inequality (81) is smaller or equal than the contribution of the corresponding point y to the RHS, and the contribution is strictly smaller if and only if $\mu_{0,j} < \mu_{\beta_0(j)}$. Therefore, if there exists $j \in \{1, \ldots, r_0\}$ for which the inequality $\mu_{0,j} \leq \mu_{\beta_0(j)}$ is strict, then the Inequality (81) is strict and we get a contradiction again. \qed

**Proposition 49.** There exist positive $\epsilon$ and $\eta$ such that

1. the ramification locus of $\alpha|_E$ is $\mathbb{C} \times \{O\}$,
2. for any $t \in D_\eta$ and any $x \in E_{t,j} \setminus \{O\}$ we have $\mu(f_t|_{V(z_1 - z_1(x))}, x) = \mu_{\beta_0(j)}$ if $t \neq 0$ and $\mu(f_t|_{V(z_1 - z_1(x))}, x) = \mu_{\beta_0(j)}$ if $t = 0$.

**Proof.** We show that if any of the conclusions is false then the Lê number $\lambda_{f, z}^1(t, O)$ is not independent on $t$, which is a contradiction, as we have seen in the proof of Lemma 46.

Since the first polar variety of $f$ is empty, the first Lê cycle $\Lambda_{f, z}^1$ is the cycle associated to the scheme

$$W := V(\partial f/\partial z_2, ..., \partial f/\partial z_n) \setminus \Sigma \cap V(\partial f/\partial z_1).$$

The first Lê number of $f$ at $(t_0, O)$ is the intersection multiplicity

$$\lambda_{f, z}^1(t_0, O) = I_{(t_0, O)}([W], [V(t - t_0)]).$$

By conservation of number in intersection theory, if $t_0$ is small enough, we have

$$I_{(0, O)}([W], [V(t - t_0)]) = \sum_{(t_0, x) \in W \cap V(t - t_0)} I_{(t_0, x)}([W], [V(t - t_0)]).$$

Hence, if there is an irreducible component of $W$ different from $\mathbb{C} \times \{O\}$ and passing through $(0, O)$ then $\lambda_{f, z}^1(0, O)$ is strictly bigger than $\lambda_{f, z}^1(t, O)$ for $t \neq 0$.

Suppose that we have $(t, x) \in E$ belonging to the ramification locus of $\alpha|_E$. We may assume (if $\eta$ is small enough) that in the discriminant of $\alpha$ there are no components of the form $\mathbb{C} \times \{t\}$ for $t \neq 0$. Hence for any $s$ close enough to $z_1(x)$ the singularity that $f_t|_{V(z_1 - z_1(x))}$ has at $x$ splits in several critical points of $f_t|_{V(z_1 - s)}$. By the Lê-Lazaarzy non-splitting result we deduce that there is at least one of them, called $x_s$, such that $f_t(x_s)$ is not zero. As $\Sigma$ is contained in $f^{-1}(0)$ we have that $(t, x_s)$ belongs to $V(\partial f/\partial z_2, ..., \partial f/\partial z_n) \setminus \Sigma$. As $x_s$ converges to $x$ as $s$ converges to $z_1(x)$ we conclude that $(t, x)$ belongs to $V(\partial f/\partial z_2, ..., \partial f/\partial z_n) \setminus \Sigma$. As $\partial f/\partial z_1(t, x) = 0$ for being $(t, x) \in \Sigma$ we deduce that $(t, x)$ belongs to $W$. Therefore, if the first assertion is false there is an irreducible component of $W$ which is different from $\mathbb{C} \times \{O\}$ and we get a contradiction.

If $t = 0$, Assertion (2) follows easily (by shrinking $\epsilon$) from the second assertion of Proposition 49. Suppose now that we have $(t, x) \in E$, with $t \neq 0$, such that $x$ belongs to $E_{t,j}$ and

$$\mu(f_t|_{V(z_1 - z_1(x))}, x) > \mu_{\beta_0(j)}.$$ (82)

Choose a neighborhood $U$ of $x$ in $\mathbb{C}^n$ such that $x$ is the only critical point of $f_t|_{V(z_1 - z_1(x))}$ in $U \cap V(z_1 - z_1(x))$. As, for generic $s$, the Milnor number of any of the singularities of $f_t|_{V(z_1 - s)}$ at $\Sigma_{t,j}$ is equal to $\mu_{\beta_0(j)}$ (use previous Proposition applied to $E_t$ instead of $E_0$), by the strict inequality (82), we can approximate $z_1(x)$ by a sequence $\{s_n\}_{n \in \mathbb{N}}$ such that $f_t|_{V(z_1 - s_n)}$ has several critical points in $U$. By the Lê-Lazaarzy non-splitting result we deduce that there is at least one of them, called $x_n$, such that $f_t(x_n)$ is not zero. From this point the proof proceeds like the proof for the first assertion. \qed
**Proof of Theorem 44.** By the remarks after the statement of the theorem we know that it is sufficient to prove that the first condition implies the conclusion.

The dimensional requirement of the first condition for equisingularity at the critical set follows from Lemma 43, Lemma 44 and Lemma 46. The first assertion of Proposition 49 implies that $E_t$ is smooth outside the origin, and that

$$\pi|_{E_t \setminus (O \times \{O\})} : E \setminus (O \times \{O\}) \to D_\eta$$

is a submersion. This gives the first condition for equisingularity at the critical set.

The first assertion of Proposition 49 also implies that for any $t \in D_\eta$ the mapping $z_1 : E_t \setminus \{O\} \to \mathbb{C}$ has no critical points. Therefore the mapping $|z_1|^2 : E_t \to \mathbb{R}$ has the origin as its only minimum and has no critical points in $E_t \setminus \{O\}$. Thus the restriction $|z_1|^2 : E_t \setminus \{O\} \to (0, \epsilon]$ is a proper function without critical points for any $t \in D_\eta$. This function trivialises the desired cobordism and implies the second condition for equisingularity at the critical set.

The third condition of equisingularity at the critical set follows from the second assertion of Proposition 49. \qed

### 8. Topological Stems

Here we modify Pellikaan’s inductive definition of stem of degree $d$ in order to define topological stems, and prove Theorem D. We refer the reader to the introduction and to the papers cited there to find motivation for the definition and applications of stems, and for an explanation of why our modification of the definition is a reasonable one.

Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}(\mathbb{C}^n, O)$.

**Definition 50.** Let $d$ be a positive integer. We define topological stems of degree $d$ inductively as follows:

- A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$ is a topological stem of degree 1 if there exists a positive integer $N$ such that for any $g \in \mathfrak{m}^N$, and $t \in \mathbb{C}$ sufficiently small, either $f + tg$ has an isolated singularity at the origin, or it is topologically $R$-equisingular to $f$.

- A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$ is a topological stem of degree $d$ if there exists a positive integer $N$ such that for any $g \in \mathfrak{m}^N$, and $t \in \mathbb{C}$ sufficiently small, either $f + tg$ is a stem of degree strictly smaller than $d$, or it is topologically $R$-equisingular to $f$.

**Theorem 51.** A holomorphic function germ $f : (\mathbb{C}^n, O) \to \mathbb{C}$ is a topological stem of positive finite degree if and only if its critical set is 1-dimensional at the origin.

**Proof.** Suppose that the critical set of $f$ is 1-dimensional at the origin. We will show that $f$ is a topological stem of degree bounded by its first Lê number with respect to a generic coordinate system. We work by induction on the first Lê number. Let $Z$ be generic coordinate system. Then $f|_{V(z_1)}$ has an isolated singularity at the origin, with Milnor number $\mu$. As $f|_{V(z_1)}$ is $\mu + 1$-determined, for any $g \in \mathfrak{m}^{\mu + 2}$ and any $t \in \mathbb{C}$ the restriction of the function $h_t := f + tg$ to $V(z_1)$ has an isolated singularity at the origin (with the same Milnor number). In this situation the Lê numbers of $h_t$ at the origin with respect to $Z$ are well defined for any $t \in \mathbb{C}$.

Suppose that $\lambda^b_{h_t, Z}(O)$ is constant for $t$ sufficiently small. In view of Definition 4.1 of [13], it is easy to show that any polar ratio of $h_t$ is bounded above by $M_t := \lambda^b_{h_t, Z}(O) + 1$. By the lexicographical upper semicontinuity of the Lê numbers
we have that $M := M_0 \geq M_t$ for $t$ small. The function $h_0 + z_1^M$ has an isolated singularity at the origin (see Lemma 4.3, [14]). Denote by $\mu_0$ its Milnor number. Consider $N := \mu_0 + 2$. Notice that, as $h_0 + z_1^M$ is $N - 1$-determined, if $g$ belongs to $m^N$ we have the equality of Milnor numbers

\begin{equation}
\mu(h_0 + z_1^M) = \mu(h_t + z_1^M)
\end{equation}

for any $t$. By the Lê-Iomdine formula, if the first Lê number of $h_t$ is constant then the 0-th Lê number is constant as well, and applying Theorem 11 we find that $h_t$ is topologically $R$-equisingular for $t$ small enough.

Otherwise $\lambda^1_{h_t, Z}(O) (t \neq 0$ sufficiently small$)$ is strictly smaller than $\lambda^1_{h_0, Z}(O)$. In this case, by induction hypothesis, the germ $h_t$ is a topological stem of finite degree bounded by its generic first Lê number at the origin, which is in turn bounded by $\lambda^1_{h_t, Z}(O)$. We conclude that $f$ is a topological stem of degree bounded by $\lambda^1_{f, Z}(O)$.

\section{The underlying deformation of the critical set}

Notice that the condition of equisingularity at the critical set imposes no condition at the origin. This observation is probably the explanation of the phenomena shown in this section. Below we will show how to produce examples of families which are equisingular at the critical set, but such that the critical set experiences drastic changes from the analytic viewpoint. We also prove a new topological $R$-equisingularity condition for families for which the reduced critical set undergoes a flat $\mu$-constant deformation of reduced curves in the sense of [3].

We have shown that, if in a family of germs with 1-dimensional critical set the Lê numbers with respect to a generic coordinate system are constant, then the family is topologically $R$-equisingular. However topological equisingularity does not imply the constancy of the Lê numbers, even if the dimension of the critical set is 1 (in particular the Lê numbers are not topological invariants). In [6] the following counterexample was constructed:

**Example 52.** Define germs $f, g_t : (\mathbb{C}^3, 0) \to \mathbb{C}$ by

\begin{align*}
 f(x, y, z) &:= x^{15} + y^{10} + z^6 \\
 g_t(x, y, z) &:= xy + tz.
\end{align*}

The family $F_t$ has critical set of dimension 1, it is topologically $R$-equisingular, but the Lê numbers with respect to a generic coordinate system are not independent on $t$.

Let us mention that $(f, g_t) : (\mathbb{C}^3, 0) \to \mathbb{C}$ is an example due to Henry (appearing in [3]) of a family of i.c.i.s. with constant Milnor number and non-constant multiplicity. In [6] we present a proof of the topological $R$-equisingularity of this example based on the results of [3]. An alternate proof of this can be obtained proving that $F_t$ is equisingular at the critical set. To do this we first show that, after fixing a certain radius $\epsilon > 0$, the critical set of $F_t$ with reduced structure is given by $V(f, g_t)$ (this can be done in the same way that we treat the next Example 54). After we observe that, as $V(f, g_t)$ defines a $\mu$-constant deformation of the reduced curve singularity $V(f, g_0)$, we have that $V(f, g_t)$ is topologically equisingular (see [3]). This implies the first two conditions for equisingularity at the critical set. The third
condition, which deals with transversal Milnor numbers can be established by an easy inspection of the equation of $F_t$.

The following theorem shows that the previous example illustrates a general phenomenon.

**Theorem 53.** Let $f : \mathbb{C} \times (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ be a family of holomorphic germs at the origin, holomorphically depending on a parameter $t$, and having 1-dimensional critical set at the origin. Let $\Sigma := V(\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$. Suppose that all the irreducible components of $\Sigma$ at $(0, O)$ are of dimension 2 and that the restriction

$$\pi : \Sigma_{\text{red}} \to \mathbb{C}$$

is a flat morphism with reduced fibres (where $\Sigma_{\text{red}}$ is $\Sigma$ with reduced structure and $\pi$ is the restriction of the projection of $\mathbb{C} \times \mathbb{C}^n$ to the first factor) such that the Milnor number at the origin $\mu((\Sigma_{\text{red}})_t, O)$ (in the sense of [3]) is independent of $t$. If, in a neighborhood of $(0, O)$ in $\mathbb{C} \times \mathbb{C}^n$, the generic transversal Milnor number of $f_t$ at any point $(t, x)$ of $\Sigma_{\text{red}} \setminus \mathbb{C} \times \{O\}$ only depends on the irreducible component of $\Sigma_{\text{red}}$ to which $(t, x)$ belongs then the family $f$ is topologically $R$-equisingular.

**Proof.** Using [3], Section 5 one proves easily the first two conditions of equisingularity at the critical set. After this the third condition follows now easily from the hypothesis. The result follows applying Theorem 40. □

However, the condition given in the previous theorem is again not a characterisation of topological $R$-equisingularity. The point is that the family of reduced critical sets does not undergo, in general, a flat deformation. If a holomorphic family $f : \mathbb{C} \times (\mathbb{C}^n, O) \to (\mathbb{C}, 0)$ of function germs $f_t$ with 1-dimensional critical set at the origin, is equisingular with respect to the critical set then it is easy to see that the restriction $\pi : \Sigma_{\text{red}} \to \mathbb{C}$ is flat at $(0, O)$. However, what is not true in general is that the fibre $(\Sigma_{\text{red}})_0$ is reduced. In fact it may have embedded components at the origin.

Consider the following deformation of a parametrisation in $\mathbb{C}^3$ (with deformation parameter $t$):

$$x = s^3 \quad y = s^4 \quad z = ts.$$ (84)

The following equations in $\mathbb{C}\{t, x, y, z\}$ define the image $Z \subset \mathbb{C} \times \mathbb{C}^3$ of the family as a set:

$$ty - xz = 0 \quad tx^3 - y^2z = 0 \quad y^3 - x^4 = 0 \quad t^3x - z^3 = 0.$$ (85)

**Example 54.** The family

$$f_t := (ty - xz)^9 + (tx^3 - y^2z)^4 + (y^3 - x^4)^3 + (t^3x - z^3)^{12}$$ (86)

is equisingular at the critical set. For any $t$ its critical set is the image of the parametrisation (84). Hence, the family of reduced critical sets does not undergo, in general, a flat deformation (the fibre of $\pi : \Sigma_{\text{res}} \to \mathbb{C}$ at 0 has an embedded component at the origin). Observe that the critical set $(\Sigma_t)_{\text{red}}$ is smooth for $t \neq 0$ and singular (of multiplicity 3) for $t = 0$. The transversal Milnor number is 6 and, hence the first Lê number with respect to a generic coordinate system is 6 for $t \neq 0$ and 18 for $t = 0$. Clearly any stabilisation of the form

$$g_t := f_t + u^a + v^b$$ (87)

has all the above properties and, by Theorem 40, is topologically $R$-equisingular.
Before checking the assertions of the example, we observe the following:

**Remark 55.** We have obtained two topologically \( R \)-equivalent functions with critical locus of dimension 1, such that the critical set is smooth for one of them and singular at the origin for the other. To the author’s knowledge it is the first time that this behavior is observed.

The equation \( f_t \) is quasihomogeneous of degree 30 if we give weights \((3, 4, 1)\) to the variables \((x, y, z)\). This shows that for any \( t \) the critical set of \( f_t \) is contained in the central fibre. In the weighted homogeneous plane \( P(3, 4, 1) \) it is easy to see that the curve defined by \( f_t = 0 \) is precisely the curve \( C_t \). To show the equisingularity at the critical set of the family \( f_t \) we only have to check the condition on the transversal Milnor number, but this follows from an easy inspection of the equations (the reader may check that the generic transversal Milnor number at any point of \( C_t \setminus \{O\} \) is controlled by the following terms of the equation: \((tx^3 - y^2z)^4 + (y^3 - x^4)^3\). After this the equisingularity at the critical set of \( g_t \) follows easily. As \( g_t : \mathbb{C}^5 \to \mathbb{C} \) is defined in the good range of dimensions \((n \geq 5)\) we can apply Theorem 40 and conclude that the family \( g_t \) is topologically \( R \)-equisingular.

The fact that the transversal Milnor number in \( C_t \setminus \{O\} \) is controlled by the terms \((tx^3 - y^2z)^4 + (y^3 - x^4)^3\) suggests to play the following game. Observe that the multiplicity of \( f_t \) is 9 and that the monomials of order 9 appear in the terms \((y^3 - x^4)^3\) (for any \( t \)) and \((ty - xz)^9\) (only for \( t \neq 0 \)). Replacing the term \((ty - xz)^9\) by \((ty - xz)^8\) we expect not to change the generic transversal Milnor number at any point of \( C_t \setminus \{O\} \), since it is controlled by two different terms. In fact one may check that

\[
(88) \quad h_t := (ty - xz)^8 + (tx^3 - y^2z)^4 + (y^3 - x^4)^3 + (t^3x - z^3)^{12}
\]

is a family such that the only component of the critical set of \( f_t \) meeting the origin is precisely \( C_t \), with generic transversal Milnor number equal to 6 at any point of \( C_t \setminus \{O\} \). Unfortunately the family is not equisingular at the origin since for \( t \neq 0 \) the critical locus contains precisely 1 Morse point outside \( C_t \), and this Morse point converges to the origin as \( t \) approaches 0. As the family has not constant multiplicity, if this Morse point would have not appeared, then we would have obtained a counterexample of Zariski’s multiplicity conjecture (after adding high powers of new variables in order to meet the dimensional restrictions of our equisingularity results).

The last examples shows that we have a great deal of flexibility in deforming the critical set in a family which is equisingular at the critical set (and, in the correct range of dimensions, topologically \( R \)-equisingular). We propose the following problem:

**Problem 56.** Construct examples of non-equimultiple, but topologically equisingular, families of parametrised curves (like \( f_t \)), with one or several components, and families of functions whose critical set consists exactly with the parametrised curve and has a prescribed transversal Milnor number outside the origin (like \( f_t \)). Observe that in these conditions the constructed family is equisingular at the critical set.
Specially in the case in which $C_t$ is not an i.c.i.s., and thus we need more equations than its codimension to define it, we could have enough space to construct examples of equisingular at the critical set families which are not equimultiple.

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