TWO WEIGHT NORM INEQUALITIES FOR THE
BILINEAR FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, we give a characterization of the two weight
strong and weak type norm inequalities for the bilinear fractional inte-
grals. Namely, we give the characterization of the following inequalities,
\[ \|I_{\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q(w)} \leq N^2 \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)} \]
and
\[ \|I_{\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q, \infty(w)} \leq N_{\text{weak}}^2 \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}, \]
when \( q \geq p_1, p_2 > 1 \) and \( p_1 + p_2 \geq p_1 p_2 \).

1. Introduction and Main Results

By a weight we mean a positive locally finite Borel measure on \( \mathbb{R}^n \). We
begin with the definition of the bilinear fractional integral \( I_{\alpha}(\cdot | \sigma_1, \cdot | \sigma_2) \). For
suitable functions \( f_1 \) and \( f_2 \), define
\[ I_{\alpha}(f_1 \sigma_1, f_2 \sigma_2)(x) = \int_{\mathbb{R}^2n} \frac{f_1(y_1)f_2(y_2)}{(|x - y_1| + |x - y_2|)^{2n-\alpha}} d\sigma_1 d\sigma_2. \]
Observe that
\[ |x - y_1| + |x - y_2| \simeq |y_1 - x| + |y_1 - y_2| \simeq |y_2 - x| + |y_2 - y_1|. \]
We know that \( I_{\alpha} \) is equivalent to its duals \( I_{\alpha}^{1,*} \) and \( I_{\alpha}^{2,*} \).

In this paper, we concern the following strong type weighted norm in-
equality,
\[ (1.1) \quad \|I_{\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q(w)} \leq N \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}, \]
and the weak type weighted norm inequality,
\[ (1.2) \quad \|I_{\alpha}(f_1 \sigma_1, f_2 \sigma_2)\|_{L^q, \infty(w)} \leq N_{\text{weak}} \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}, \]
where $\mathcal{N}$ and $\mathcal{N}_{\text{weak}}$ are the best constants such that the above inequalities hold, respectively. We aim to give a characterization of (1.1) and (1.2) using Sawyer type test conditions.

In the linear case, the characterization of weighted norm inequalities have attracted many authors. For the maximal operators, we refer the readers to the works of Sawyer [25] and Moen [21]. For the fractional integrals, we refer the readers to [8, 26, 27, 30]. And for the Calderón-Zygmund operators, this problem is referred to as the Nazarov-Treil-Volberg conjecture [31]. This conjecture has been solved for the Hilbert transform, see the remarkable work of Lacey [10, 11] and the recent work of Hytönen [6]. For the vector Riesz transform, it was partially solved by Sawyer, Shen and Uriarte-Tuero in [28], where they gave a characterization under the assumption that at least one of the two weights is supported on a line. And in [15], Lacey and Wick gave a characterization under the hypotheses that the two weights separately are not concentrated on a set of codimension one, uniformly over locations and scales. There is also another approach on this topic, namely, finding a minimal sufficient condition of the weights such that the two weight inequality holds. We refer the readers to [1, 2, 4, 5, 9, 16] and references therein. For other related works, we refer the readers to [17, 18, 19, 20, 22, 23].

Now the story goes to the multilinear case. In [18], we studied the characterization of two weight norm inequalities for the multilinear fractional maximal operators using Sawyer type test conditions. Recall that the multilinear fractional maximal operators are defined by

$$
\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{1}{|Q|^{1-\alpha/mn}} \int_Q |f_i(y_i)| dy_i,
$$

where $0 \leq \alpha < mn$.

In this paper, we give a characterization of the two weight strong and weak type norm inequalities for the bilinear fractional integrals. Specifically, we prove the following.

**Theorem 1.1.** Let $\sigma_1, \sigma_2, w$ be positive locally finite Borel measures and $q \geq p_1, p_2 > 1$ with $p_1 + p_2 \geq p_1 p_2$. Then (1.1) holds if and only if the following test conditions hold

$$
\mathcal{T} := \sup_Q \left( \frac{\|I_\alpha(1_Q \sigma_1, 1_Q \sigma_2)^q dw\|^{1/q}}{\sigma_1(1_Q)^{1/p_1} \sigma_2(1_Q)^{1/p_2}} \right) < \infty;
$$

$$
\mathcal{T}_1^* := \sup_Q \left( \frac{\|I_\alpha(1_Q w, 1_Q \sigma_2)^{p_1'} d\sigma_1\|^{1/p_1'}}{w(1_Q)^{1/q} \sigma_2(1_Q)^{1/p_2}} \right) < \infty;
$$

$$
\mathcal{T}_2^* := \sup_Q \left( \frac{\|I_\alpha(1_Q \sigma_1, 1_Q w)^{p_2'} d\sigma_2\|^{1/p_2'}}{\sigma_1(1_Q)^{1/p_1} w(1_Q)^{1/q'}} \right) < \infty.
$$

Moreover, $\mathcal{N} \simeq \mathcal{T} + \mathcal{T}_1^* + \mathcal{T}_2^*$.

And (1.2) holds if and only if $\mathcal{T}_1^*, \mathcal{T}_2^* < \infty$. Moreover, $\mathcal{N}_{\text{weak}} \simeq \mathcal{T}_1^* + \mathcal{T}_2^*$. 

This paper is organized as follows. In Section 2, we reduce the problem to the dyadic bilinear fractional integral and give some preliminary estimates. In Section 3 and Section 4, we give a proof for Theorem 1.1.

2. Preliminaries

In this section, as in [3], we define two dyadic versions of the bilinear fractional integral. We show that they are equivalent with the bilinear fractional integral pointwise. Firstly, we introduce the following result, which can be found in [7, Proof of Theorem 1.7].

**Proposition 2.1.** There are $2^n$ dyadic grids $D_t$, $t \in \{0, 1/3\}^n$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_t \in D_t$ satisfying $Q \subset Q_t$ and $l(Q_t) \leq 6l(Q)$, where

$$D_t := \{2^{-k}([0,1]^n + m + (-1)^kt) : k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, 1/3\}^n.$$ 

Given a dyadic grid $D$, we define the dyadic bilinear fractional integral

$$I_D^\alpha(f_1 \sigma_1, f_2 \sigma_2)(x) := \sum \prod_{i=1}^2 \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i \cdot \chi_Q(x).$$

Analogue to the argument in [3], we have the following result.

**Proposition 2.2.** Given $0 < \alpha < 2n$, positive locally finite Borel measures $\sigma_1, \sigma_2$ and non-negative functions $f_1, f_2$, then for any dyadic grid $D$,

$$I_D^\alpha(f_1 \sigma_1, f_2 \sigma_2)(x) \lesssim I_\alpha(f_1 \sigma_1, f_2 \sigma_2)(x).$$

Conversely, we have

$$I_\alpha(f_1 \sigma_1, f_2 \sigma_2)(x) \lesssim \max_{t \in \{0, 1/3\}^n} I_D^{\alpha t}(f_1 \sigma_1, f_2 \sigma_2)(x).$$

Notice that with Proposition 2.2, we can get the following

$$I_\alpha(f_1 \sigma_1, f_2 \sigma_2)(x) \simeq \sum_{t \in \{0, 1/3\}^n} I_D^{\alpha t}(f_1 \sigma_1, f_2 \sigma_2)(x).$$

**Proof.** Fix some $x \in \mathbb{R}^n$. Denote by $\{Q_k\}_{k \in \mathbb{Z}}$ the unique sequence in $D$ such that $x \in Q_k$ and $l(Q_k) = 2^k$. Fix $N \geq 1$. We have

$$\sum_{Q \in D, 2^{-N} \leq l(Q) \leq 2^N} \prod_{i=1}^2 \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i \cdot \chi_Q(x)$$

$$= \sum_{k=-N}^N 2 \prod_{i=1}^2 \frac{1}{|Q_k|^{1-\alpha/2n}} \int_{Q_k} f_i d\sigma_i$$

$$= \sum_{k=-N}^N \frac{1}{|Q_k|^{2-\alpha}} \int\int_{(Q_k \times Q_k) \setminus (Q_{k-1} \times Q_{k-1})} f_1 f_2 d\sigma_1 d\sigma_2.$$
For the second inequality, we refer the readers to [17, 20]. This completes the proof for every 

\[ \alpha < 2n \]

Given \( \alpha < 2n \), by rearranging the terms and letting \( N \to \infty \), we get

\[ I_\alpha^\varnothing (f_1 \sigma_1, f_2 \sigma_2)(x) \lesssim I_\alpha (f_1 \sigma_1, f_2 \sigma_2)(x). \]

For the second inequality, we refer the readers to [17, 20]. This completes the proof. \( \square \)

Next, we define a sparse version of \( I_\alpha^\varnothing \). We call \( S \subset \mathcal{D} \) a sparse family if for every \( Q \in S \),

\[ \left| \bigcup_{Q' \subseteq Q, Q' \in S} Q' \right| \leq \frac{1}{2} |Q|. \]

Now we can define the sparse dyadic bilinear fractional integral by

\[ I_\alpha^S (f_1 \sigma_1, f_2 \sigma_2)(x) := \sum_{Q \in S} \prod_{i=1}^{2} \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i \cdot \chi_Q(x). \]

Next we show that \( I_\alpha^S \) and \( I_\alpha^\varnothing \) are equivalent in some sense.

**Proposition 2.3.** Given \( 0 < \alpha < 2n \), positive locally finite Borel measures \( \sigma_1, \sigma_2 \) and bounded, non-negative, compactly supported functions \( f_1, f_2 \), then for any dyadic grid \( \mathcal{D} \), there exists a sparse family \( S \subset \mathcal{D} \) such that

\[ I_\alpha^\varnothing (f_1 \sigma_1, f_2 \sigma_2)(x) \lesssim I_\alpha^S (f_1 \sigma_1, f_2 \sigma_2)(x). \]

Notice that \( S \) is a subfamily of \( \mathcal{D} \). So we have

\[ I_\alpha^\varnothing (f_1 \sigma_1, f_2 \sigma_2)(x) \simeq I_\alpha^S (f_1 \sigma_1, f_2 \sigma_2)(x). \]

The proof of Proposition 2.3 is not essentially different from the linear case, which can be found in [3, 12, 24, 29].

**Proof.** Let \( a = 2^{2(n+1)} \). We split \( \mathcal{D} \) to the following subfamilies,

\[ \mathcal{P}_k = \{ Q \in \mathcal{D} : a^k < \prod_{i=1}^{2} \frac{1}{|Q|} \int_Q f_i d\sigma_i \leq a^{k+1} \}. \]
Then for every $Q$ with $\prod_{i=1}^{2} \frac{1}{|Q|} \int_{Q} f_i d\sigma_i \neq 0$, there is a unique $k$ such that $Q \in \mathcal{P}_k$. Therefore, we can write

$$I_{\alpha}^\vartheta(f_1, f_2)(x) = \sum_{k} \sum_{Q \in \mathcal{P}_k} \prod_{i=1}^{2} \left| Q \right|^{-\frac{1}{|Q|}} \int_{Q} f_i d\sigma_i \cdot \chi_Q(x)$$

$$\leq \sum_{k} a^{k+1} \sum_{Q \in \mathcal{P}_k} \left| Q \right|^\frac{\vartheta}{n} \chi_Q(x).$$

Denote by $\mathcal{S}_k$ the collection of maximal dyadic cubes $P \in \mathcal{D}$ such that

$$\prod_{i=1}^{2} \frac{1}{|P|} \int_{P} f_i d\sigma_i > a^k.$$

Since $\sigma_1$ and $\sigma_2$ are locally finite and $f_1$ and $f_2$ are bounded and compactly supported, such a collection exists. Notice that the cubes in $\mathcal{S}_k$ are pairwise disjoint. Set $\mathcal{S} = \bigcup_k \mathcal{S}_k$. We have

$$I_{\alpha}^\vartheta(f_1, f_2)(x) \leq \sum_{k} a^{k+1} \sum_{P \in \mathcal{S}_k} \sum_{Q \subset P} \left| Q \right|^\frac{\vartheta}{n} \chi_Q(x)$$

$$\leq \sum_{k} \sum_{P \in \mathcal{S}_k} \left( \prod_{i=1}^{2} \frac{1}{|P|} \int_{P} f_i d\sigma_i \right) \sum_{r=0}^{\infty} \sum_{Q \in \mathcal{P}_k, Q \subset P} \left| Q \right|^\frac{\vartheta}{n} \chi_Q(x)$$

$$= \sum_{S \in \mathcal{S}} \left( \prod_{i=1}^{2} \frac{1}{|S|} \int_{S} f_i d\sigma_i \right) \sum_{r=0}^{\infty} \sum_{k; S \in \mathcal{S}_k} \sum_{Q \in \mathcal{P}_k, Q \subset S} \left| Q \right|^\frac{\vartheta}{n} \chi_Q(x)$$

$$\leq \sum_{S \in \mathcal{S}} \left( \prod_{i=1}^{2} \frac{1}{|S|} \int_{S} f_i d\sigma_i \right) \sum_{r=0}^{\infty} \sum_{Q \in \mathcal{S}, Q \subset S} \left| Q \right|^\frac{\vartheta}{n} \chi_Q(x)$$

$$\leq I_{\alpha}^\vartheta(f_1, f_2)(x).$$

It remains to demonstrate that $\mathcal{S}$ is sparse. In fact, fix some $P \in \mathcal{S}_k$. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the collection of the maximal dyadic cubes in $\mathcal{S}$ which are strictly contained in $P$. Then for any $\lambda \in \Lambda$,

$$\prod_{i=1}^{2} \frac{1}{|P_\lambda|} \int_{P_\lambda} f_i d\sigma_i > a^{k+1}.$$

It follows that

$$\sum_{\lambda} |P_\lambda| \leq a^{-(k+1)/2} \sum_{\lambda} \left( \prod_{i=1}^{2} \int_{P_\lambda} f_i d\sigma_i \right)^{1/2}$$

$$\leq a^{-(k+1)/2} \prod_{i=1}^{2} \left( \sum_{\lambda} \int_{P_\lambda} f_i d\sigma_i \right)^{1/2}$$
\[ a^{-(k+1)/2} \leq \left( \int_P f_i d\sigma_i \right)^{1/2}, \]
\[ \leq \frac{1}{2} |P|, \]
where in the last step we use the fact that
\[ \prod_{i=1}^2 \frac{1}{|P|} \int_P f_i d\sigma_i \leq 2^{2n} \prod_{i=1}^2 \frac{1}{|\hat{P}|} \int_{\hat{P}} f_i d\sigma_i \leq 2^{2n} |\hat{P}|, \]
thanks to the maximal property. Recall that \( \hat{P} \) denotes the father cube of \( P \).

Now we reduce the problem to show the following result.

**Theorem 2.4.** Let \( D \) be a dyadic grid and \( S \subset D \) be a sparse family. Suppose that \( \sigma_1, \sigma_2, w \) are positive Borel measures and \( q \geq p_1, p_2 > 1 \) with \( p_1 + p_2 \geq p_1 p_2 \). Then

\[ \| \mathcal{I}^S_\alpha (f_1 \sigma_1, f_2 \sigma_2) \|_{L^q(w)} \leq N \prod_{i=1}^2 \| f_i \|_{L^{p_i}(\sigma_i)} \]
holds if and only if the following test conditions hold

\[ T^S := \sup_{Q \in S} \frac{\left( \int_Q \mathcal{I}^S_\alpha (1_Q \sigma_1, 1_Q \sigma_2)^q d\sigma_1 \sigma_2(Q)^{1/p_2} \right)^{1/q}}{\sigma_1(Q)^{1/p_1} \sigma_2(Q)^{1/p_2}} < \infty; \]

\[ T^{S,*}_1 := \sup_{Q \in S} \frac{\left( \int_Q \mathcal{I}^S_\alpha (1_Q w, 1_Q \sigma_2)^{p'_1} d\sigma_1 \sigma_2(Q)^{1/p_2} \right)^{1/p'_1}}{w(Q)^{1/p'_1} \sigma_2(Q)^{1/p_2}} < \infty; \]

\[ T^{S,*}_2 := \sup_{Q \in S} \frac{\left( \int_Q \mathcal{I}^S_\alpha (1_Q \sigma_1, 1_Q w)^{p'_2} d\sigma_2 \sigma_1(Q)^{1/p_1} w(Q)^{1/p'_2} \right)^{1/p'_2}}{\sigma_1(Q)^{1/p_1} w(Q)^{1/p'_2}} < \infty. \]

Moreover, if \( N \) is the best constant such that (2.2) holds, then \( N \simeq T^S + T^{S,*}_1 + T^{S,*}_2 \). And

\[ \| \mathcal{I}^S_\alpha (f_1 \sigma_1, f_2 \sigma_2) \|_{L^{q,\infty}(w)} \leq N_{\text{weak}} \prod_{i=1}^2 \| f_i \|_{L^{p_i}(\sigma_i)} \]
holds if and only if \( T^{S,*}_1, T^{S,*}_2 < \infty \). Moreover, if \( N_{\text{weak}} \) is the best constant such that (2.3) holds, then \( N_{\text{weak}} \simeq T^{S,*}_1 + T^{S,*}_2 \).

In the following, we give some elementary estimates. Assume that \( f_1 \) and \( f_2 \) are non-negative. By the monotone convergence theorem, it suffices to consider

\[ \mathcal{I}^{S(R)}_\alpha (f_1 \sigma_1, f_2 \sigma_2)(x) = \sum_{Q \in S} \prod_{i=1}^2 \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i \cdot \chi_Q(x), \]
where $R$ is a cube. In fact, we can further assume that the side-length of any cube in $\mathcal{S}(R)$ is at least $2^{-m}$. To avoid miscellaneous subscripts, we omit the index $m$ in the rest of the paper. Let

$$\Omega_k = \{x : T_\alpha^{S(R)}(f_1\sigma_1, f_2\sigma_2)(x) > 2^k\} = \bigcup_j Q_j^k,$$

where $\{Q_j^k\}_j \subset \mathcal{S}(R)$ is the collection of maximal dyadic cubes in $\Omega_k$ and we denote this collection by $Q_k$. We have the following dyadic maximum principle:

$$\sum_{Q \in \mathcal{S}, Q \supset Q_j^k} 2 \prod_{i=1}^{2} \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i > 2^k$$

and

$$\sum_{Q \in \mathcal{S}, Q \supset Q_j^k} 2 \prod_{i=1}^{2} \frac{1}{|Q|^{1-\alpha/2n}} \int_Q f_i d\sigma_i \leq 2^k.$$

Let $E(Q_j^k) = Q_j^k \cap \Omega_{k+1} \setminus \Omega_{k+2}$. Then for any $x \in E(Q_j^k)$, we have

$$2^{k+1} < T_\alpha^{S(R)}(f_1\sigma_1, f_2\sigma_2)(x) \leq T_\alpha^{S(R)}(1_{Q_j^k} f_1\sigma_1, 1_{Q_j^k} f_2\sigma_2)(x) + 2^k.$$

Therefore, for any $x \in E(Q_j^k)$,

$$T_\alpha^{S(R)}(1_{Q_j^k} f_1\sigma_1, 1_{Q_j^k} f_2\sigma_2)(x) > 2^k. \quad (2.4)$$

Now we have

$$\|T_\alpha^{S(R)}(f_1\sigma_1, f_2\sigma_2)\|_{L^q(w)}^q \leq \sum_{k,j} 2^{kq} w(E(Q_j^k))$$

$$= \sum_{k,j : w(E(Q_j^k)) > \delta w(Q_j^k)} 2^{kq} w(E(Q_j^k)) + \sum_{k,j : w(E(Q_j^k)) \leq \delta w(Q_j^k)} 2^{kq} w(E(Q_j^k))$$

$$\leq \sum_{w(E(Q_j^k)) > \delta w(Q_j^k)} w(E(Q_j^k))^{1-q} \left( \int_{E(Q_j^k)} T_\alpha^{S(R)}(1_{Q_j^k} f_1\sigma_1, 1_{Q_j^k} f_2\sigma_2) dw \right)^q$$

$$+ \delta \|T_\alpha^{S(R)}(f_1\sigma_1, f_2\sigma_2)\|_{L^q(w)}^q.$$

By letting $\delta$ be sufficiently small, it suffices to estimate

$$\sum_{k,j : w(E(Q_j^k)) > \delta w(Q_j^k)} w(E(Q_j^k))^{1-q} \left( \int_{E(Q_j^k)} T_\alpha^{S(R)}(1_{Q_j^k} f_1\sigma_1, 1_{Q_j^k} f_2\sigma_2) dw \right)^q. \quad (2.5)$$

In the following, we assume that all $k$ are in the same parity. Without loss of generality, we further assume that all $k$ are even. Then $E(Q_j^k)$ will be
pairwise disjoint. Denote
\[ \mathbb{K} := \{ k : k \text{ is even and } w(E(Q_j^k)) > \delta w(Q_j^k) \}. \]
In the rest of this paper, all the sum on \( k \) will be understood as on \( k \in \mathbb{K} \).
Notice that for \( k \in \mathbb{K}, w(E(Q_j^k)) \neq 0 \), which means that \( Q_j^k \notin Q_{k+2} \).

3. Proof of Theorem 1.1: The Strong Type

3.1. The special case. First, we investigate the special case \( f_1 = 1_Q \) and \( \text{supp } f_2 \subset Q \), where \( Q \in \mathcal{S} \). We have the following result.

Lemma 3.1. Let \( \sigma_1, \sigma_2, w \) be positive locally finite Borel measures and \( q \geq p_2 \). Then for any sparse family \( S \subset \mathcal{D} \) and cube \( Q \in \mathcal{S} \),

\[
(3.1) \quad \int_Q T_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2)^q \, dw \lesssim (T^S + T_2^{S_1})^q \sigma_1(Q)^{q/p_1} \| f_2 \|_{L^{p_2}(\sigma_2)}^q.
\]

Proof. Without loss of generality, we can assume that \( f_2 \) is non-negative. First of all, notice that

\[
1_Q I_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2) = I_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2) + 1_Q \sum_{Q \in S} \frac{\sigma_1(Q) \int_Q f_2 \, d\sigma}{|Q|^{2-\alpha/n}}
\]

\[
\lesssim I_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2).
\]

Therefore, by the previous arguments, it suffices to estimate

\[
(3.2) \quad \sum_{k,j} \frac{w(E(Q_j^k))}{w(E(Q_j^k))} \sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{E(Q_j^k)} T_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2) \, dw \right)^q.
\]

We have

\[
\sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{E(Q_j^k)} T_{\alpha}^S(1_Q \sigma_1, 1_Q f_2 \sigma_2) \, dw \right)^q = \sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{Q_j^k} f_2 T_{\alpha}^S(1_Q \sigma_1, 1_{E(Q_j^k)} \, w) \, d\sigma_2 \right)^q
\]

\[
\lesssim \sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{Q_j^k} f_2 T_{\alpha}^S(1_Q \sigma_1, 1_{E(Q_j^k)} \, w) \, d\sigma_2 \right)^q
\]

\[
+ \sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{Q_j^k \setminus Q_{k+2}} f_2 T_{\alpha}^S(1_Q \sigma_1, 1_{E(Q_j^k)} \, w) \, d\sigma_2 \right)^q
\]

\[
:= J_1 + J_2.
\]
First, we estimate $J_1$. We have

\[
J_1 \leq \sum_{k,j} w(Q_j^k)^{1-q} \left( \int_{Q_j^k \setminus \Omega_{k+2}} T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)^{p_2}d\sigma_2 \right)^{q/p_2} \\
\times \left( \int_{Q_j^k \setminus \Omega_{k+2}} f_2^{p_2}d\sigma_2 \right)^{q/p_2} \\
\leq (T_2^{S,*})^q \sum_{k,j} w(Q_j^k)^{1-q} w(Q_j^k)^{q-1} \sigma_1(Q_j^k)^{q/p_1} \\
\times \left( \int_{Q_j^k \setminus \Omega_{k+2}} f_2^{p_2}d\sigma_2 \right)^{q/p_2} \\
\leq (T_2^{S,*})^q \sigma_1(Q)^{q/p_1} \left( \sum_{k,j} \int_{Q_j^k \setminus \Omega_{k+2}} f_2^{p_2}d\sigma_2 \right)^{q/p_2} \\
\leq (T_2^{S,*})^q \sigma_1(Q)^{q/p_1} \|f_2\|_{L^{p_2}(\sigma_2)}^q.
\]

Next we estimate $J_2$. We can write

\[
\int_{Q_j^k \setminus \Omega_{k+2}} f_2 T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)d\sigma_2 \\
= \sum_{R \in Q_k \setminus Q_j^k} \int_R f_2 T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)d\sigma_2.
\]

Notice that for $x \in R$, $T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)(x)$ is a constant. Define $E_Q^\mu f = \mu(Q)^{-1} \int_Q f d\mu$. Then,

\[
\int_{Q_j^k \setminus \Omega_{k+2}} f_2 T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)d\sigma_2 \\
= \sum_{R \in Q_k \setminus Q_j^k} E_Q^{\sigma_2} f_2 \int_R T_\alpha^S(Q)(1,1_{E(Q_j^k)}w)d\sigma_2.
\]

To estimate the above sum we need the tool of principal cubes. Since $Q_j^k \subset Q$, we denote by $G_0$ the collection of the maximal cubes in $\bigcup_{k \in \mathbb{Z}} Q_k$. We define $G_n$ inductively. That is,

\[
G_{n+1} = \bigcup_{G' \in G_n} \{ G : \text{maximal dyadic subcube of } G' \text{ such that } E_{G'}^{\sigma_2} f_2 > 4E_G^{\sigma_2} f_2 \},
\]

where the dyadic system in the above is $\bigcup_{k \in \mathbb{Z}} Q_k$. Then the collection of principal cubes is $G = \bigcup_{n \geq 0} G_n$. By the definition, we immediately have the
following

\( \sum_{G \in \mathcal{G}} (E_{G}^{\sigma_{2}} f_{2})^{p_{2}} \sigma_{2}(G) \lesssim \|M_{\widetilde{G}}^{p_{2}} f_{2}\|_{L^{p_{2}}(\sigma_{2})}^{p_{2}} \lesssim \|f_{2}\|_{L^{p_{2}}(\sigma_{2})}^{p_{2}}. \)

Denote by \( G(Q) \) the minimal principal cube contains \( Q \). We see from the definition that

\[ E_{G(Q)}^{\sigma_{2}} f_{2} \leq 4E_{G(Q)}^{\sigma_{2}} f_{2}. \]

Note that

\[
\sum_{R \in \mathcal{Q}_{k+2}^{j} \cap G_{j}} E_{R}^{\sigma_{2}} f_{2} \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w) d\sigma_{2}
\]

\[
= \sum_{R \in \mathcal{Q}_{k}^{j}, G(R) = G(Q_{j}^{k})} E_{R}^{\sigma_{2}} f_{2} \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w(x)) d\sigma_{2}
\]

\[
+ \sum_{R \in \mathcal{Q}_{k}^{j}, R \in \mathcal{Q}_{k+2}} E_{R}^{\sigma_{2}} f_{2} \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w(x)) d\sigma_{2}.
\]

We have

\[
\sum_{k, j} w(Q_{j}^{k})^{1-q} \left( \sum_{R \in \mathcal{Q}_{k+2}^{j} \cap G_{j}} E_{R}^{\sigma_{2}} f_{2} \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w) d\sigma_{2} \right)^{q}
\]

\[
\lesssim \sum_{k, j} w(E(Q_{j}^{k}))^{1-q} \left( \sum_{R \in \mathcal{Q}_{k+2}^{j}, G(R) = G(Q_{j}^{k})} E_{R}^{\sigma_{2}} f_{2}
\]

\[
\times \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w(x)) d\sigma_{2} \right)^{q}
\]

\[
+ \sum_{k, j} w(Q_{j}^{k})^{1-q} \left( \sum_{R \in \mathcal{Q}_{k}^{j}, R \in \mathcal{Q}_{k+2}} E_{R}^{\sigma_{2}} f_{2} \int_{R} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{E(Q_{j}^{k})} w(x)) d\sigma_{2} \right)^{q}
\]

\[
:= J_{21} + J_{22}.
\]

For \( J_{21} \), we have

\[
J_{21} \lesssim \sum_{G \in \mathcal{G}} \sum_{k, j, G(Q_{j}^{k}) = G} w(E(Q_{j}^{k}))^{1-q} \left( \int_{E(Q_{j}^{k})} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{Q_{j}^{k}}) dw \right)^{q}
\]

\[
\times \left( \int_{E(Q_{j}^{k})} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{Q_{j}^{k}}) dw \right)^{q}
\]

\[
\lesssim \sum_{G \in \mathcal{G}} \sum_{k, j, G(Q_{j}^{k}) = G} \left( E_{G}^{\sigma_{2}} f_{2} \right)^{q} \int_{E(Q_{j}^{k})} T_{\alpha}^{S(G)}(1_{Q_{j}^{k}} f_{1}, 1_{Q_{j}^{k}}) dw.
\]
\[
\sum_{G \in \mathcal{G}} (E_G^p f_2)^q \int_G \mathcal{T}_\alpha^S(G)(1_G \sigma_1, 1_G \sigma_2)^q dw
\leq (\mathcal{T}_\alpha^S)^q \sum_{G \in \mathcal{G}} (E_G^p f_2)^q \sigma_1(G)^{q/p_1} \sigma_2(G)^{q/p_2}
\]
\[
\leq (\mathcal{T}_\alpha^S)^q \sigma_1(Q)^{q/p_1} \|f_2\|_{L_p^2(\sigma)}^q.
\]

And for \( J_{22} \), we have
\[
J_{22} \leq \sum_{k,j} w(Q_j^k)^{1-\frac{q}{p}} \left( \sum_{R \in \mathcal{Q}_k, R \in \mathcal{G}_k} (E_R^p f_2)^{p_2} \sigma_2(R) \right)^{q/p_2}
\times \left( \sum_{R \in \mathcal{Q}_k, R \in \mathcal{G}_k} \sigma_2(R)^{-\frac{q}{p}} \left( \int_R \mathcal{T}_\alpha^S(Q)(1_{Q_j^k} \sigma_1, 1_{E(Q_j^k)} w)^{q/p_2} d\sigma_2 \right) \right)^{q/p_2}
\leq (\mathcal{T}_\alpha^S)^q \sigma_1(Q)^{q/p_1} \left( \sum_{k,j} \sum_{R \in \mathcal{Q}_k, R \in \mathcal{G}_k} (E_R^p f_2)^{p_2} \sigma_2(R) \right)^{q/p_2}
\leq (\mathcal{T}_\alpha^S)^q \sigma_1(Q)^{q/p_1} \left( \sum_{G \in \mathcal{G}_k} (E_G^p f_2)^{p_2} \sigma_2(G) \right)^{q/p_2}
\leq (\mathcal{T}_\alpha^S)^q \sigma_1(Q)^{q/p_1} \|f_2\|_{L_p^2(\sigma)}^q,
\]

where the fact that \( k \in K \) and therefore any \( R \) appears only once is used.

3.2. The general case. In this subsection, we investigate the general case. Again, we can assume that \( f_1 \) and \( f_2 \) are non-negative. By the arguments in Section 2, we only need to estimate the following
\[
\sum_{k,j} w(Q_j^k)^{1-\frac{q}{p}} \left( \int_{E(Q_j^k)} \mathcal{T}_\alpha^S(R)(1_{Q_j^k} f_1 \sigma_1, 1_{Q_j^k} f_2 \sigma_2) dw \right)^q.
\]

Since \( p_1 + p_2 \geq p_1 p_2 \), we have \( p_1' \geq p_2 \). Hence
\[
\int_{E(Q_j^k)} \mathcal{T}_\alpha^S(R)(1_{Q_j^k} f_1 \sigma_1, 1_{Q_j^k} f_2 \sigma_2) dw
\leq \int_{Q_j^k} f_1 \mathcal{T}_\alpha^S(R)(1_{E(Q_j^k)} w, 1_{Q_j^k} f_2 \sigma_2) d\sigma_1
\]
First, we estimate \( \sum_{k,j} w(Q_{j}^{k})^{-q} \left( \sum_{Q \subset Q_{j}^{k}} E_{Q_{j}}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{q} \frac{1}{q} \left( \int_{Q_{j}^{k} \setminus Q_{k+2}} f_{1}^{p_{1}} d\sigma_{1} \right) \frac{1}{p_{1}} \)

\[
\leq \left( \int_{Q_{j}^{k} \setminus Q_{k+2}} T_{\alpha}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{\frac{1}{p_{1}}} \sum_{Q \subset Q_{j}^{k}} \frac{1}{q} \left( \int_{Q_{j}^{k} \setminus Q_{k+2}} f_{1}^{p_{1}} d\sigma_{1} \right) \frac{1}{p_{1}}
\]

\[
\leq (T_{1}^{S,*} + T_{2}^{S,*})^{q} \| f_{2} \|_{L^{p_{2}}(\sigma_{2})} \left( \sum_{Q \subset Q_{j}^{k}} \frac{1}{q} \left( \int_{Q_{j}^{k} \setminus Q_{k+2}} f_{1}^{p_{1}} d\sigma_{1} \right) \frac{1}{p_{1}} \right)
\]

where we use Lemma 3.1 in the last step. The summation on the first term is easy to estimate. In fact,

\[
\sum_{k,j} w(Q_{j}^{k})^{-q} \left( \sum_{Q \subset Q_{j}^{k}} E_{Q_{j}}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{q} \frac{1}{q} \left( \int_{Q_{j}^{k} \setminus Q_{k+2}} f_{1}^{p_{1}} d\sigma_{1} \right)
\]

\[
\leq (T_{1}^{*} + T_{2}^{*})^{q} \| f_{2} \|_{L^{p_{2}}(\sigma_{2})}^{q} \left( \sum_{k,j} \int_{Q_{j}^{k} \setminus Q_{k+2}} f_{1}^{p_{1}} d\sigma_{1} \right)^{\frac{1}{p_{1}}}
\]

It remains to estimate the summation on the second term. Let \( \tilde{G} \) be the principal cubes associated to \( f_{1} \) and \( \sigma_{1} \). We have

\[
\sum_{k,j} w(Q_{j}^{k})^{-q} \left( \sum_{Q \subset Q_{j}^{k}} E_{Q_{j}}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{q}
\]

\[
\leq \sum_{k,j} w(Q_{j}^{k})^{-q} \left( \sum_{G \subset Q_{j}^{k}} E_{G}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{q}
\]

\[
+ \sum_{k,j} w(Q_{j}^{k})^{-q} \left( \sum_{Q \subset Q_{j}^{k}} E_{Q}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) d\sigma_{1} \right)^{q}
\]

\[
:= I_{1} + I_{2}.
\]

First, we estimate \( I_{1} \). We have

\[
I_{1} \leq \sum_{G \subset \tilde{G}} \sum_{Q_{j}^{k} \subset G} \left( E_{G}^{S}(1_{E(Q_{j})}w, 1_{Q_{j}} f_{2} \sigma_{2}) \right)^{q} \int_{E(Q_{j})} T_{\alpha}^{S}(1_{Q_{j}^{k}} \sigma_{1}, 1_{Q_{j}} f_{2} \sigma_{2})^{q} \, dw
\]
\[ \leq \sum_{\tilde{G} \in \tilde{G}} (E_{\tilde{G}}^{\sigma_1}(f_1))^{q} \int_{\tilde{G}} T_{\alpha}^{S(R)}(1_{\tilde{G}}^{\sigma_1}, 1_{\tilde{G}} f_2 \sigma_2)^{q} dw \]
\[ \lesssim (T^{S} + T^{S, *})^{q} \| f_2 \|_{L^p(\sigma_2)}^{q} \sum_{\tilde{G} \in \tilde{G}} (E_{\tilde{G}}^{\sigma_1}(f_1))^{q} (\tilde{G})^{q/p_1} \quad (\text{by Lemma 3.1}) \]
\[ \leq (T^{S} + T^{S, *})^{q} \| f_2 \|_{L^p(\sigma_2)}^{q} \left( \sum_{\tilde{G} \in \tilde{G}} (E_{\tilde{G}}^{\sigma_1}(f_1))^{p_1} (\tilde{G}) \right)^{q/p_1} \]
\[ \lesssim (T^{S} + T^{S, *})^{q} \| f_1 \|_{L^q(\sigma_1)}^{q} \| f_2 \|_{L^p(\sigma_2)}^{q}. \]

Next we estimate \( I_2 \). We have
\[ I_2 \leq \sum_{k,j} w(Q_j^{k-1-q}) \left( \sum_{Q \subset Q_j^k, Q \in \tilde{G}} \left( E_{Q_j^k}^{\sigma_1}(f_1) \right)^{p_1} (\tilde{Q}) \right)^{q/p_1} \]
\[ \times \left( \sum_{Q \subset Q_j^k, Q \in \tilde{G}} \sigma_1(\tilde{Q})^{-p_1'/p_1} \left( \int_{\tilde{Q}} T_{\alpha}^{S(R)}(1_{E(Q_j^k)} w, 1_{Q_j^k} f_2 \sigma_2)(x) d\sigma_1 \right)^{p_1'} \right)^{q/p_1'} \]
\[ \leq \sum_{k,j} w(Q_j^{k-1-q}) \left( \sum_{Q \subset Q_j^k, Q \in \tilde{G}} \left( E_{Q_j^k}^{\sigma_1}(f_1) \right)^{p_1} (\tilde{Q}) \right)^{q/p_1} \]
\[ \times \left( \int_{Q_j^k} T_{\alpha}^{S(R)}(1_{E(Q_j^k)} w, 1_{Q_j^k} f_2 \sigma_2)(x) d\sigma_1 \right)^{p_1'} \]
\[ \leq (T^{S, *}_1 + T^{S, *}_2)^{q} \| f_2 \|_{L^p(\sigma_2)}^{q} \sum_{k,j} \left( \sum_{Q \subset Q_j^k, Q \in \tilde{G}} \left( E_{Q_j^k}^{\sigma_1}(f_1) \right)^{p_1} (\tilde{Q}) \right)^{q/p_1} \]
\[ \quad (\text{by Lemma 3.1}) \]
\[ \leq (T^{S, *}_1 + T^{S, *}_2)^{q} \| f_2 \|_{L^p(\sigma_2)}^{q} \left( \sum_{\tilde{G} \in \tilde{G}} (E_{\tilde{G}}^{\sigma_1}(f_1))^{p_1} (\tilde{G}) \right)^{q/p_1} \]
\[ \leq (T^{S, *}_1 + T^{S, *}_2)^{q} \| f_1 \|_{L^q(\sigma_1)}^{q} \| f_2 \|_{L^p(\sigma_2)}^{q}. \]

4. Proof of Theorem 1.1: The Weak Type

In this section, we focus on the weak type inequality \( (2.3) \). Again, we only need to consider \( (2.3) \) and we assume that \( f_1 \) and \( f_2 \) are non-negative. Notice that \( q > 1 \). If \( (2.3) \) holds, we see from the Kolmogorov inequality that
\[ \frac{1}{w(Q)} \int_{Q} T_{\alpha}^{S}(1_{Q} f_1 \sigma_1, 1_{Q} f_2 \sigma_2) dw \]
gence theorem, we get that (2.3) holds, regardless a constant independent
of the weights.

Therefore, by letting \( \delta \) be sufficiently small and using the monotone convergence theorem, we get that (4.1) holds, regardless a constant independent of the weights.

Now we assume that (4.1) holds for any \( Q \in \mathcal{S} \). For any \( R \in \mathcal{Q} \), we have

\[
\| T_{\alpha}^{S(R)}(f_1, f_2) \|_{L^{q,\infty}(u)}^q \lesssim \sup_k 2^{(k+1)q} w(\Omega_{k+1}).
\]

Denote \( F_j^k = Q_j^k \cap \Omega_{k+1} \). By the discussion in Section 2, we have

\[
T_{\alpha}^{S(R)}(1_{Q_j^k} f_1, 1_{Q_j^k} f_2)(x) > 2^k, \quad x \in F_j^k.
\]

Then

\[
2^{(k+1)q} w(\Omega_{k+1}) \leq \sum_j 2^{(k+1)q} w(F_j^k),
\]

\[
= \sum_{j: w(F_j^k) \geq \delta w(Q_j^k)} 2^{(k+1)q} w(F_j^k) + \sum_{j: w(F_j^k) < \delta w(Q_j^k)} 2^{(k+1)q} w(F_j^k)
\]

\[
\lesssim \sum_{j: w(F_j^k) \geq \delta w(Q_j^k)} w(F_j^k)^{1-q} \left( \int_{F_j^k} T_{\alpha}^{S(R)}(1_{Q_j^k} f_1, 1_{Q_j^k} f_2)(x)dw \right)^q + \delta \| T_{\alpha}^{S(R)}(f_1, f_2) \|_{L^{q,\infty}(u)}^q
\]

\[
\lesssim \mathcal{N}_{\text{weak}}^q \prod_j \|1_{Q_j^k} f_1\|_{L^{p_1}(\sigma_i)}^q + \delta \| T_{\alpha}^{S(R)}(f_1, f_2) \|_{L^{q,\infty}(u)}^q
\]

\[
\leq \mathcal{N}_{\text{weak}}^q \prod_j \left( \sum_{j} \|1_{Q_j^k} f_1\|_{L^{p_1}(\sigma_i)}^{p_1} \right)^{q/p_1} + \delta \| T_{\alpha}^{S(R)}(f_1, f_2) \|_{L^{q,\infty}(u)}^q
\]

\[
\lesssim \mathcal{N}_{\text{weak}}^q \prod_i \|f_i\|_{L^{p_i}(\sigma_i)}^q + \delta \| T_{\alpha}^{S(R)}(f_1, f_2) \|_{L^{q,\infty}(u)}^q.
\]

Therefore, by letting \( \delta \) be sufficiently small and using the monotone convergence theorem, we get that (2.3) holds, regardless a constant independent of the weights.

We see from the above arguments that (2.3) and (4.1) are equivalent. So we only need to give a characterization for (4.1). By the duality argument,
it is easy to see that (4.1) is equivalent to the following,

\[
\begin{align*}
(4.2) & \left( \int_Q I_\alpha^S (1_{Q w}, 1_{Q f_2})^{p'_1} d\sigma_1 \right)^{1/p'_1} \lesssim N_{\text{weak}} w(Q)^{1/q'} \| 1_{Q f_2} \|_{L^p(\sigma_2)}; \\
& \left( \int_Q I_\alpha^S (1_{Q f_1}, 1_{Q w})^{p'_2} d\sigma_2 \right)^{1/p'_2} \lesssim N_{\text{weak}} w(Q)^{1/q'} \| 1_{Q f_1} \|_{L^p(\sigma_1)}.
\end{align*}
\]

Therefore, the necessity part follows immediately, i.e., \( \mathcal{T}_1^{S,*}, \mathcal{T}_2^{S,*} \lesssim N_{\text{weak}} < \infty \). For the sufficiency part, since \( p'_1 \geq p_2 \), we focus on (4.2). By Lemma 3.1 we know that \( N_{\text{weak}} < \infty \). Moreover,

\[ N_{\text{weak}} \simeq \mathcal{T}_1^{S,*} + \mathcal{T}_2^{S,*}. \]

REFERENCES

[1] T. Anderson, D. Cruz-Uribe, K. Moen, Logarithmic bump conditions for Calderón-Zygmund operators on spaces of homogeneous type, Publ. Mat., to appear.
[2] D. Cruz-Uribe, J.M. Martell, C. Pérez, Sharp weighted estimates for classical operators, Adv. Math., 229(2012), 408–441.
[3] D. Cruz-Uribe and K. Moen, One and two weight norm inequalities for Riesz potentials, Illinois J. Math., to appear.
[4] D. Cruz-Uribe and C. Pérez, Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators, Indiana Univ. Math. J., 49(2000), 697–721.
[5] D. Cruz-Uribe and C. Pérez, Sharp two-weight, weak-type norm inequalities for singular integral operators, Math. Res. Lett., 6(1999), 417–427.
[6] T. Hytönen, The two-weight inequality for the Hilbert transform with general measures, http://arxiv.org/abs/1312.0843.
[7] T. Hytönen and C. Pérez, Sharp weighted bounds involving \( A_\infty \), Analysis & PDE, 6(2013), 777–818.
[8] A. Kairema, Two-weight norm inequalities for potential type and maximal operators in a metric space, Publ. Mat., 57(2013), 3–56.
[9] M. Lacey, On the separated bumps conjecture for Calderón-Zygmund operators, http://arxiv.org/abs/1310.3507.
[10] M. Lacey, Two weight inequality for the Hilbert transform: A real variable characterization, II, http://arxiv.org/abs/1301.4663.
[11] M. Lacey, The two weight inequality for the Hilbert transform: A primer, http://arxiv.org/abs/1304.5004.
[12] M. Lacey, K. Moen, C. Pérez and R.H. Torres, Sharp weighted bounds for fractional integral operators, J. Funct. Anal., 259 (2010), 1073–1097.
[13] M. Lacey, E. Sawyer and I. Uriarte-Tuero, A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure, J. Anal. & P.D.E. 5(2012), 1–60.
[14] M. Lacey, E. Sawyer and I. Uriarte-Tuero, Two weight inequalities for discrete positive operators, http://arxiv.org/abs/0911.3437.
[15] M. Lacey and B. Wick, Two weight inequalities for Riesz transforms, available at http://arxiv.org/abs/1312.6163.
[16] A.K. Lerner, On an estimate of Calderón-Zygmund operators by dyadic positive operators, J. Anal. Math., (to appear).
[17] K. Li, K. Moen and W. Sun, Sharp weighted inequalities for multilinear fractional maximal operator and fractional integrals, http://arxiv.org/abs/1304.2973.
[18] K. Li and W. Sun, *Characterization of a two weight inequality for multilinear fractional maximal operators*, [http://arxiv.org/abs/1305.4267](http://arxiv.org/abs/1305.4267).

[19] T. Mei, Q. Xue, S. Lan, *Sharp weighted bounds for multilinear fractional maximal type operators with rough kernels*, [http://arxiv.org/abs/1305.1865](http://arxiv.org/abs/1305.1865).

[20] K. Moen, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math., 60 (2009), 213–238.

[21] K. Moen, *Sharp one-weight and two-weight bounds for maximal operators*, Studia Mathematica, 194 (2009), 163–180.

[22] F. Nazarov, S. Treil and A. Volberg, *The Bellman function and two weight inequalities for Haar multipliers*, J. Amer. Math. Soc., 12 (1999), 909–928.

[23] F. Nazarov, S. Treil and A. Volberg, *Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures*, preprint (2004) [http://arxiv.org/abs/1003.1596](http://arxiv.org/abs/1003.1596).

[24] C. Pérez, *Two weighted inequalities for potential and fractional type maximal operators*, Indiana Univ. Math. J., 43 (1994), 663–683.

[25] E. Sawyer, *A characterization of a two weight norm inequality for maximal operators*, Studia Math., 75 (1982), 1–11.

[26] E. Sawyer, *A two weight weak type inequality for fractional integrals*, Trans. Amer. Math. Soc., 281 (1984), 339–345.

[27] E. Sawyer, *A characterization of two weight norm inequalities for fractional and Poisson integrals*, Trans. Amer. Math. Soc., 308 (1988), 533–545.

[28] E. Sawyer, C. Shen and I. Uriarte-Tuero, *A geometric condition, necessity of energy, and two weight boundedness of fractional Riesz transforms*, preprint, available at [http://arxiv.org/abs/1310.4484](http://arxiv.org/abs/1310.4484).

[29] E. Sawyer, R. Wheeden, *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces*, Amer. J. Math., 114 (1992), 813–874.

[30] E. Sawyer, R. Wheeden and S. Zhao, *Weighted norm inequalities for operators of Potential type and fractional maximal functions*, Potential Anal., 5 (1996), 523–580.

[31] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces*, CBMS Regional Conference Series in Mathematics (2003).

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