Axiomatic Rejection for the Propositional Fragment of Leśniewski’s Ontology

Takao Inoué, Arata Ishimoto* and Mitsunori Kobayashi

August 15, 2021

Abstract

A Hilbert-type axiomatic rejection HAR for the propositional fragment $L_1$ of Leśniewski’s ontology is proposed. Also a Gentzen-type axiomatic rejection GAR of $L_1$ is proposed. Models for $L_1$ are introduced. By axiomatic rejection, Ishimoto’s embedding theorem will be proved. One of our main theorems is:

**Theorem (Main Theorem)**

\[ \vdash_T A \iff \vdash_H A \iff TA \text{ is valid in first-order predicate logic with equality} \iff \text{not } \vdash_H A. \]

where $\vdash_T A$ means that $A$ is provable in the tableau method of $L_1$, while $\vdash_H A$ means that $A$ is provable in the Hilbert-type $L_1$.

In the last section, as the characterization theorem, we shall show:

**Theorem (Characterization Theorem)** The following statements are equivalent:

1. The Cut elimination theorem for the Gentzen-type $L_1$ (i.e. tableau method) holds,

2. No Hintikka formula of the form $A_1 \lor A_2 \lor \cdots \lor A_n$ ($n \geq 1$) is provable in the Hilbert-type $L_1$, where $A_i$ ($1 \leq i \leq n$) is an atomic formula or a negated atomic one,

*Deceased. This paper is dedicated to the memory of Professor Emeritus Arata Ishimoto. This paper is cited in Ishimoto [27].
For any formula \( A \) of \( \mathbf{L}_1 \), if \( A \) is provable in the Hilbert-type \( \mathbf{L}_1 \), then it is provable in the Gentzen-type \( \mathbf{L}_1 \) (tableau method),

(4) Contradiction and Dichotomy theorems for the Hilbert-type \( \mathbf{L}_1 \) hold,

(5) No Hintikka formula is provable in the Hilbert-type \( \mathbf{L}_1 \).

(6) The Hilbert-type \( \mathbf{L}_1 \) is \( L \)-decidable with respect to \( \mathbf{HL}_1 \) (i.e. the set of all formula of \( \mathbf{L}_1 \) is the disjoint union of the set of all the theorem of the Hilbert-type \( \mathbf{L}_1 \) and that of all the theorem of \( \mathbf{HL}_1 \)), where \( \mathbf{HL}_1 \) is the axiomatic rejection with Hintikka formulas as axioms.

Then we shall show that \( \mathbf{HL}_1 \) has the same strength with \( \mathbf{HAR} \).

\textit{Keywords:} axiomatic rejection, the propositional fragment of Leśniewski’s ontology, tableau method, refutation calculus, Hintikka formula, positive part, negative part, Schütte-type formulation of logic, cut elimination theorem, model, embedding, Russellian-type definite description.

1 Introduction

In Ishimoto [21] and later Kobayashi-Ishimoto [38] a logical system called \( \mathbf{L}_1 \), was proposed as the propositional fragment of Leśniewski’s ontology designated as \( \mathbf{L} \). The fragment \( \mathbf{L}_1 \) of Leśniewski’s ontology is defined in its Hilbert-type version as the smallest class of formulas containing all the instances of tautology and the formulas of the form:

1.1 \( \vdash \epsilon ab \supset \epsilon aa \),
1.2 \( \vdash \epsilon ab \land \epsilon bc \supset \epsilon ac \),
1.3b \( \vdash \epsilon ab \land \epsilon bb \supset \epsilon ba \),

being closed under detachment as a rule (1.3 is due to Kanai [32]). Instead of 1.3b we may take, as an axiom,

1.3 (original) \( \vdash \epsilon ab \land \epsilon bc \supset \epsilon ba \),

which is the original one in Ishimoto [21].

Leśniewski’s (elementary) ontology \( \mathbf{L} \), on the other hand, is defined on the basis of the formula of the following form:

1.4 \( \vdash \epsilon ab \equiv .\exists x \epsilon xa \land \forall x \forall y (\epsilon xa \land \epsilon ya \supset \epsilon xy) \land \forall x (\epsilon xa \supset \epsilon xb) \), or more simply,
1.5 \( \vdash \epsilon ab \equiv .\exists x(\epsilon xa \land \epsilon xb) \land \forall x \forall y (\epsilon xa \land \epsilon ya \supset \epsilon xy) \),

with first-order predicate logic (without equality) as underlying logic.
The (well-formed) formulas of $L_1$ to be referred to by such meta-logical variables as $A$, $B$, \ldots are defined in the well-known way in terms of $\epsilon$ (Leśniewski’s epsilon) and a (countable) infinite list of name variables, $a$, $b$, \ldots as well as a number of logical symbols sufficient for developing classical propositional logic and some auxiliary symbols. The (well-formed) formulas of $L$ are defined analogously with quantifiers added.

In what follows, all these symbols and their combinations will be employed only meta-logically. Outermost parentheses are always suppressed if no ambiguity arises therefrom.

As seen above we shall use a Polish-style notation such as $\epsilon ab$ for Leśniewski’s epsilon as in Sobociński [71] (for other notations, see a useful table in Simons [61, p. 99]).

For Leśniewski’s ontology in general refer, among others, to Iwanuś [31], Lejewski [42], Luschei [44], Miéville [46], Rickey [50], Simons [58, 61], Slupecki [65], Szednicki-Rickey [72], Surma et al. [74], Stachniak [75], and Urbaniak [81].

Now the purpose of this paper is to prove, among others, that $L_1$ is embedded in first-order predicate logic with equality via the translation $T$ to be specified below, that is, embedding theorem:

$$\vdash_H A \iff T A$$

is valid in first-order predicate logic with equality, where $H$ is the Hilbert-type version of $L_1$.

The translation $T$ which transforms every formula of $L_1$ into a formula of first-order predicate logic with equality. The inductive definition $T$ is as follows. (This definition was first proposed by Ishimoto [21] on the basis of Prior [48].)

- $T\epsilon ab = F_b\langle x \rangle F_a x$,
- $TA \lor B = TA \lor TB$,
- $T \sim A = \sim TA$,

$F_a, F_b, \ldots$ are monadic predicate (variables) corresponding to name variables $a, b, \ldots$ not necessarily, exhausting all of them. $F_b\langle x \rangle F_a x$, on the other hand, is the Russellian-type definite description and stands for:

$$\exists x (F_a x \land F_b x) \land \forall x \forall y (F_a x \land F_a y \supset x = y),$$

with the scope of the description confining to $F_b$. This embedding theorem was already proved in Ishimoto [21] and later Kobayashi-Ishimoto [38] by
a different method. In the sequel, the theorem will be proved anew on the
basis of a more general setting, which is summarized as the following (meta-
equivalences:
\[ \vdash_T A \iff \vdash_H A \]
\[ \iff TA \text{ is valid in first-order predicate logic with equality} \]
\[ \iff \text{not } \dashv_H A, \]
where \( \vdash_T A \) (\( \vdash_H A \)) signifies that \( A \) is a thesis of the tableau method or the
Gentzen-type version (Hilbert-type version) of \( L_1 \), and \( \vdash_H A \), on the other
hand, means that \( A \) is axiomatically rejected in its Hilbert-type version: in
other words, \( A \) is an anti-thesis of it. \( TA \) is the result from \( A \) by applying
the translation \( T \) to \( A \). We will call it the \( T\)-transform of \( A \). As easily seen,
there are many formulas of predicate logic which are not a \( T\)-transform of a
formula of \( L_1 \). For other translation and embedding of \( L_1 \), refer to Blass [1],
Inoué [16, 17, 18], Smirnov [67, 69] and Takano [78].

The paper to follow consists of nine sections with this introduction in-
cluded with one appendix by the second author of this paper.

To begin with, the following second section will concern the tableau
method version of \( L_1 \) and the proof of :
\[ \vdash_T A \implies \vdash_H A, \]
along with a number of theorems which will turn out to be essential in the
later development. In the third section, an axiomatic rejection \( \text{HAR} \) for \( L_1 \),
i.e. a logical system to derive all anti-theses of \( L_1 \), will be introduced formally
in its Hilbert-type version, whereas a Gentzen-type axiomatic rejection \( \text{GAR} \)
for \( L_1 \) will be proposed in the section seven. In the sections 4, 5, and 6, we
shall prove on the basis of axiomatic rejection that:
\[ \vdash_H A \]
\[ \implies TA \text{ is valid in first-order predicate logic with equality} \]
\[ \implies \text{not } \dashv_H A. \]

For the first (meta-)implication, we can treat it as in Ishimoto [21]. For
the second one, our argument for the proof is based on the model construction
with assignments of finite subsets of the set of natural numbers to name vari-
ables as seen in Ishimoto [24]. (The original idea of the model construction
is due to the third author of the present paper.)

It is emphasized that such a model for \( L_1 \) is, without any change, one for
\( L \) by making use of the arguments in Kobayashi-Ishimoto [38] and that the
model constructed is finite. In the section six, the extension of a model for \( L_1 \) to one for \( L \) will be used to prove a version of Separation theorem (i.e. \( L \) is a conservative extension of \( L_1 \)), which was first proved in Ishimoto [21]. Our treatment about the model is a correction and a refinement of Ishimoto [24].

Combining all those (meta-)implications with:

\[
\neg \vdash_H A \implies \vdash_H A,
\]

which will be proved in the third section (from Dichotomy theorem (Theorem 3.2)), we shall finally obtain the looked-for equivalences. The Dichotomy theorem and Contradiction theorem (Theorem 6.4 to be proved in the section six) for the Hilbert-type version of \( L_1 \) provide us with the decidability of \( L_1 \) (cf. Slupecki [66]).

In the seventh section we shall introduce a Gentzen-type axiomatic rejection for \( L_1 \) and Contradiction theorem (Theorem 6.4) will be syntactically proved under the following postulate: no Hintikka formula of the form \( A_1 \lor A_2 \lor \cdots \lor A_n \) \((n \geq 1)\) is provable in the Hilbert-type version of \( L_1 \), where \( A_i \) \((1 \leq i \leq n)\) is an atomic formula or a negated atomic one. The syntactical treatment of the theorem leads us to a novel syntactical proof of the cut elimination theorem for the tableau method version of \( L_1 \), which will be carried out in the eighth section. The idea of such a proof would be applied to many logics. Our approach to prove cut elimination theorem was first explored in Inoué-Ishimoto [19] for classical propositional logic.

Here, we wish to take an opportunity of emphasizing that the proposed tableau method for \( L_1 \) is a system to be developed within the bounds of its Hilbert-type version up until the sixth section. Such was the insight that Slupecki and Lukasiewicz had when they were working with the Aristotelian syllogistic in its Hilbert-type version. In fact, they saw through the Hilbert-type syllogistic a Gentzen structure hidden under the surface, although they never developed the structure as a self contained Gentzen-type logic. (This was attempted by Ishimoto-Kani-Kagiwada [29], Kanai [33] for the Aristotelian syllogistic, and by Inoué-Ishimoto [20] for the Brentano-type syllogistic with Leśniewski’s epsilon \( \epsilon \).) Roughly speaking, the Gentzen structure thus discovered was made use of very skillfully for the benefit of the Hilbert-type version of the logic concerned. And, the theses referring to the tableau method should be understood only within the framework of \( L_1 \) in its Hilbert-type version. Thus, \( \vdash_T A \), for example, may be thought of, not only as a
thesis of the Gentzen-type $L_1$, but also as a theorem which belongs to the
Hilbert-type $L_1$ (see Theorem 2.2 in the following section).

In the last section nine, as the characterization theorem, we shall show:

**Theorem (Characterization Theorem)**
The following statements are equivalent:

1. The Cut elimination theorem for the Gentzen-type $L_1$ (tableau method) holds,
2. No Hintikka formula of the form $A_1 \lor A_2 \lor \cdots \lor A_n$ ($n \geq 1$) is provable in the Hilbert-type $L_1$, where $A_i$ ($1 \leq i \leq n$) is an atomic formula or a negated atomic one,
3. For any formula $A$ of $L_1$, if $A$ is provable in the Hilbert-type $L_1$, then it is provable in the the Gentzen-type $L_1$ (tableau method),
4. Contradiction and Dichotomy theorems for the Hilbert-type $L_1$ hold,
5. No Hintikka formula is provable in the Hilbert-type $L_1$,
6. The Hilbert-type $L_1$ is $L$-decidable with respect to $HL_1$ (i.e. the set of all formula of $L_1$ is the disjoint union of the set of all the theorem of the Hilbert-type $L_1$ and that of all the theorem of $HL_1$), where $HL_1$ is the axiomatic rejection with Hintikka formulas as axioms.

Then we shall show that the $HL_1$ has the same strength with $HAR$. $HL_1$ was proposed in Inoué [15].

This paper contains an philosophical appendix by the second author, Arata Ishimoto.

## 2 Tableau method

In this section, as in Kobayashi-Ishimoto [38], $L_1$ will be developed by means of the tableau method, which in spite of its appearance may be understood in terms of the Hilbert-type $L_1$ as remarked above. For this purpose, the notion of the positive and negative parts due to Schütte [52, 53, 54] will be introduced with a view to simplifying the subsequent development.

**Definition 2.1** The **positive** and **negative** parts of a formula $A$ are defined only as follows:

1. $A$ is a positive part of $A$,
2. If $B \lor C$ is a positive part of $A$, then $B$ and $C$ are positive parts of $A$,
3. If $\sim B$ is a positive part of $A$, then $B$ is negative parts of $A$,
2.14 If \( \sim B \) is a negative part of \( A \), then \( B \) is positive parts of \( A \).

As suggested in this definition, the logical symbols to be employed in the sequel are \( \lor \) (disjunction) and \( \sim \) (negation) with other logical symbols being defined, if necessary, in their terms.

The specified occurrence of a formula \( A \) as a positive (negative) part of another is indicated by \( F[A+] \) (\( G[A-] \)) as exemplified below:

\[
\begin{align*}
F[A+] &= A, \\
F[A+] &= \sim \sim A \lor B, \\
F[A+] &= \sim B \lor (A \lor C), \\
G[A-] &= \sim A, \\
G[A-] &= \sim \sim \sim A \lor B, \\
G[A-] &= \sim \sim (\sim A \lor C) \lor B,
\end{align*}
\]

where all the formulas involved are assumed to be different. Such expressions as \( F[A+, B-] \) and the like are understood analogously with the proviso that specified formulas do not overlap with each other.

On the basis of the above definition of the positive and negative parts of a formula, the tableaux for \( \mathbf{L}_1 \) are defined by the following four reduction rules to be applied to a formula of \( \mathbf{L}_1 \):

\[
\begin{align*}
\lor & \quad \frac{G[A \lor B-]}{G[A \lor B-] \lor \sim A \mid G[A \lor B-] \lor \sim B} \\
\epsilon_1 & \quad \frac{G[eab_-]}{G[eab_-] \lor \sim eaa} \\
\epsilon_2 & \quad \frac{G[eab_-, ebc_-]}{G[eab_-, ebc_-] \lor \sim eac} \\
\epsilon_{3b} & \quad \frac{G[eab_-, ebb_-]}{G[eab_-, ebb_-] \lor \sim eba},
\end{align*}
\]

where all these reduction rules, as will be seen presently, should be understood as derived rules put up-side down as far as we remain in the Hilbert-type \( \mathbf{L}_1 \). The rule \( \epsilon_{3b} \) can be replaced by the following rule \( \epsilon_3 \) for the tableau method:

\[
\epsilon_3 \quad \frac{G[eab_-, ebc_-]}{G[eab_-, ebc_-] \lor \sim eba}.
\]

By reducing a formula by way of these reduction rules, a tableau is obtained for the formula. A branch of a tableau is closed if it is ending with
a formula of the form $F[A_+, A_-]$. A tableau is said to be *closed* if every branch of it is closed. A tableau is *open* if it is not closed. A formula of $L_1$ is *provable* in the tableau method for $L_1$ if there exists a closed tableau of it. It is also known that $\lor_-$ is sufficient for tableaux yielding all the instances of tautology. (For the formal definition of tableaux, consult Fitting [5] and Smullyan [70].)

The principal formulas of these rules, such as $A \lor B$ in the case of $\lor_-$, are minimal negative parts of the formulas to be reduced. Here, the *minimal positive* or *negative parts* of a formula are the positive (negative) parts of a formula which does not contain properly any positive or negative parts of the formula. The presence of the formulas repeated in the results of a reduction will be justified when we come across Hintikka formulas to be defined below.

For the purpose of illustration, the axioms 1.1, 1.2 and 1.3b of the Hilbert-type $L_1$ will be proved by the proposed tableau method:

\[
\begin{align*}
\epsilon_1 & \quad \sim \epsilon_{ab} \lor \epsilon_{aa} \quad (= 1.1) \\
& \quad (\sim \epsilon_{ab} \lor \epsilon_{aa}) \lor \sim \epsilon_{aa} , \end{align*}
\]

\[
\begin{align*}
\epsilon_2 & \quad \sim\sim (\sim \epsilon_{ab} \lor \sim \epsilon_{bc}) \lor \epsilon_{ac} \quad (= 1.2) \\
& \quad (\sim\sim (\sim \epsilon_{ab} \lor \sim \epsilon_{bc}) \lor \epsilon_{ac}) \lor \sim \epsilon_{ac} , \end{align*}
\]

\[
\begin{align*}
\epsilon_{3b} & \quad \sim\sim (\sim \epsilon_{ab} \lor \sim \epsilon_{bb}) \lor \epsilon_{ba} \quad (= 1.3b) \\
& \quad (\sim\sim (\sim \epsilon_{ab} \lor \sim \epsilon_{bb}) \lor \epsilon_{ba}) \lor \sim \epsilon_{ba} , \end{align*}
\]

where all these tableaux are closed.

The following theorem (Theorem 2.3) fundamental in the subsequent development was stated and proved in Kobayashi-Ishimoto [38]. But, here we take the opportunity of repeating it with a simple proof of it.

Before proceeding to the theorem, the notion of Hintikka formulas (for $L_1$) is in order and it reads as follows:

**Definition 2.2** A *Hintikka formula* $A$ is a formula which satisfies the following conditions:

2.21 $A$ is not of the form $F[B_+, B_-]$,

2.22 If $A$ contains $B \lor C$ as a negative part of $A$, then it contains $B$ or $C$ as a negative part of it,

2.23 If $A$ contains $\epsilon_{ab}$ as a negative part of $A$, then it contains $\epsilon_{aa}$ as a negative part of it,
2.24 If \( A \) contains \( \epsilon_{ab} \) and \( \epsilon_{bc} \) as a negative part of \( A \), then it contains \( \epsilon_{ac} \) as a negative part of it,

2.25 If \( A \) contains \( \epsilon_{ab} \) and \( \epsilon_{bb} \) as a negative part of \( A \), then it contains \( \epsilon_{ba} \) as a negative part of it,

We shall show some examples of Hintikka formula:

\[
\sim \epsilon_{ab} \lor \epsilon_{ba} \lor \sim \epsilon_{aa},
\]

\[
\sim (\epsilon_{ab} \lor \epsilon_{bc}) \lor \sim \epsilon_{ab} \lor \sim \epsilon_{aa},
\]

\[
\sim \epsilon_{ab} \lor \sim \epsilon_{bc} \lor \sim \epsilon_{ac} \lor \sim \epsilon_{ba} \lor \sim \epsilon_{aa} \lor \sim \epsilon_{bb},
\]

where some of the variables could be identical with each other. Here and in what follows, disjuncts are assumed to be associated in any way.

**Theorem 2.1** (Fundamental Theorem) Given a formula (of \( L_1 \)), by reducing it by reduction rules there obtains a finite tableau, each branch of which ends either with a formula of the form \( F[A_+, A_-] \) or with a Hintikka formula, whereby a branch is extended by a reduction rule only if the formula to be reduced is not of the form \( F[A_+, A_-] \) and the reduction gives rise to a formula not occurring in the formula to be reduced as negative part thereof.

With a view to proving the Theorem, it is remarked in advance that there are only a finite number of subformulas of the given formula, and only some of them could be employed as a principal formula of a \( \lor_- \) application. The principal formula for \( \lor_- \) is \( A \lor B \), while that for \( \epsilon_1 \) (\( \epsilon_2 \) and \( \epsilon_{3b} \)) is \( \epsilon_{ab} \) (\( \epsilon_{ab}, \epsilon_{bc} \) and \( \epsilon_{ab}, \epsilon_{bb} \)). There are also a finite number of the possible pairings of name variables in the given formula, and only some of them could be combined by the applications of \( \epsilon_1, \epsilon_2 \) or \( \epsilon_{3b} \) to yield a fresh occurrence of a negative part which was not in occurrence as such in the formula to be reduced. This proves the first half of the Fundamental Theorem.

For proving the second half of the theorem, let us assume that extending a branch by way of reduction rules which is not ending with a formula of the form \( F[A_+, A_-] \), we come soon or later across a formula to which no rules are applicable any more without violating the requirement of the Theorem.

We wish to show that the formula already constitutes a Hintikka formula. If not, the formula would, for example, contain an \( \epsilon_{ab} \) as a negative part without containing another negative part \( \epsilon_{aa} \). We could, then, reduce the
formula by $\epsilon_1$ against the assumption. The other properties of Hintikka formulas are taken care of analogously.

From the requirement (in Theorem 2.1) for extending a branch, it immediately follows that any principal formula (or formulas) used as such before is (are) never employed again in the same status in the same branch.

A tableau, which is constructed in compliance with the requirement, is said to be normal. The tableaux obtained for the axioms of $L_1$ are all normal if the name variables involved are different from each other.

In the sequel, we shall need the operation (due to Schütte) of removing a formula from another which contains the former as its positive of negative part. If a given formula is $F[A_+]$ $(G[A_-])$, the formula or the empty expression resulting by removing $A$ from the formula is denoted by $F[\_+]$ $(G[\_])$.

**Definition 2.3** Given $F[A_+]$ $(G[A_-])$, $F[\_+]$ $(G[\_])$ is defined only as follows:

2.41 If $F[A_+]$ is $A$, then $F[\_+]$ is the empty expression,

2.42 If $F[A_+]$ is $F_1[A \lor B_+]$ or $F_1[B \lor A_+]$, then $F[\_+]$ is $F_1[B_+]$,

2.43 If $F[A_+]$ is $G_1[\sim A_-]$, then $F[\_+]$ is $G_1[\_+]$,

2.44 If $G[A_-]$ is $F_1[\sim A_+ ]$, then $G[\_+]$ is $F_1[\_+]$.

The removal of $A$ from $F[A_+]$ $(G[A_-])$ is defined by induction on the number of procedures 2.11-2.14 used for specifying $A$ as a positive (negative) part of $F[A_+]$ $(G[A_-])$. By the same induction, it follows that $F[\_+]$ $(G[\_])$ is a well-formed formula or the empty expression, given $F[A_+]$ $(G[A_-])$.

The operation thus defined will be exemplified as follows.

If $F[A_+] = F_1[A \lor B_+] = F_2[(A \lor B) \lor C_+] = (A \lor B) \lor C$, then $F[\_+] = F_1[B_+] = F_2[B \lor C_+] = B \lor C$.

If $F[A_+] = G_1[\sim A_-] = F_1[\sim \sim A_+] = F_2[\sim \sim A \lor B_+] = \sim \sim A \lor B$, then $F[\_+] = G_1[\_+] = F_1[\_+] = F_2[B_+] = B$.

If $G[A_-] = F_1[\sim A_+] = G_1[\sim \sim A_-] = F_2[\sim \sim \sim A_+] = \sim \sim \sim A$, then $G[\_+] = F_1[\_+] = G_1[\_+] = F_2[\_+] = \text{the empty expression}$.

If $G[A_-] = F_1[\sim A_+] = F_2[\sim A \lor B_+] = G_1[\sim (A \lor B)_+] = F_2[\sim \sim (A \lor B)_+] = \sim \sim (A \lor B)_+ = \sim \sim (A \lor B)$, then $G[\_+] = F_1[\_+] = F_2[B_+] = G_1[\sim B_-] = F_3[\sim \sim B_+] = \sim \sim B$.

**Lemma 2.1**

$\vdash_H F[A_+] \equiv .F[\_+] \lor A,$

$\vdash_H G[A_-] \equiv .G[\_+] \lor \sim A.$
The Lemma is proved simultaneously by induction on the number of procedures 2.11–2.14 applied for specifying \( A \) as a positive (negative) part of \( F[A_+] \) \((G[A_-])\).

2.51 The basis is forthcoming right away, since we have \( \vdash H F[A_+] \equiv .F[ +] \lor A \) with \( F[ +] \) being the empty expression, and the disjunction of the empty expression with any formula is identified with the formula, which is regarded as a stipulation.

2.52 If \( F[A_+] \) is \( F_1[A \lor B] \) (or \( F_1[A \lor A_+] \)), then \( \vdash H F[A_+] \equiv F_1[A \lor B] \) \((F_1[B \lor A_+]) \equiv .F_1[ +] \lor (A \lor B) \) (or \( F_1[ +] \lor (B \lor A) \)) (by induction hypothesis) \( \equiv .(F_1[ +] \lor A) \lor B. \equiv .F_1[B_+] \lor A \) (by induction hypothesis).

2.53 If \( F[A_+] = G[\sim A_-] \), then \( \vdash H F[A_+] \equiv G[\sim A_-] \equiv .G[ -] \lor \sim A \) (by induction hypothesis) \( \equiv .G[ -] \lor A. \equiv .F[ -] \lor A. \equiv G[ -] \lor A. \)

2.54 If \( G[A_-] = F[\sim A_+] \), then \( \vdash H G[A_-] \equiv F[\sim A_+] \equiv .F[ -] \lor \sim A \) (by induction hypothesis) \( \equiv .F[ -] \lor \sim A. \equiv G[A_-]. \)

Lemma 2.2

\[ \vdash H F[ +] \supset F[A_+], \]
\[ \vdash H G[ -] \supset G[A_-], \]

where \( F[ +] \) \((G[-]) \) is not empty.

These two implications are proved on the basis of Lemma 2.1, respectively as follows:

\[ \vdash H F[ +] \lor .F[ +] \lor A, \quad \vdash H F[ +] \lor A. \supset F[A_+], \]
\[ \vdash H G[ -] \lor .G[ -] \lor \sim A, \quad \vdash H G[ -] \lor \sim A. \supset G[A_-]. \]

The Lemma is to the effect of thinning in the sense of Gentzen [3]. Analogously, we have:

Lemma 2.3

\[ \vdash H A \supset F[A_+], \]
\[ \vdash H \sim A \supset G[A_-]. \]

The Lemma is again to the effect of thinning.

Theorem 2.2 For any formula \( A \) of \( L_1 \), we have

\[ \vdash_T A \implies \vdash_H A, \]

where \( \vdash_T A \) means (as already stated) that \( A \) is a thesis of the tableau method version of \( L_1 \) as interpreted in its Hilbert-type counterpart.
The theorem is proved by induction on the length of the tableau.

2.21 [Basis] The basis is taken care of by the following equivalence to be obtained on the basis of Lemma 2.1:

\[ \vdash_H F[A_+, A_-] \equiv F[+, -] \lor (A \lor \sim A), \]

the right-hand side of which is a tautology and, therefore, provable in the Hilbert-type version of \( L_1 \).

2.22 [Induction steps] Induction steps are dealt with by the following equivalences all reduction rules, namely, \( \lor, \sim, \epsilon_1, \epsilon_2 \) and \( \epsilon_{3b} \) (or \( \epsilon_3 \)), which we are resorting, among others, to Lemmas 2.1 and 2.2 as well as to 1.1, 1.2 and 1.3b.

From \( \vdash_H G(A \lor B_-) \equiv \ldots \) and
\[ \vdash_H G[\ldots] \lor \sim (A \lor B) \equiv \ldots \]
\[ \equiv \ldots (G[A \lor B_-] \lor \sim (A \lor B)). \quad \text{(for } \lor), \]
we have
\[ \vdash_H G[A \lor B_-] \equiv \ldots (G[A \lor B_-] \lor \sim A) \land (G[A \lor B_-] \lor \sim B). \]

From \( \vdash_H G[eab_-] \lor \sim eaa \supset G[eab_-] \lor \sim eab \) and
\[ \vdash_H G[eab_-] \lor \sim eab \equiv \ldots \]
\[ \equiv \ldots G[eab_-] \quad \text{(for } \epsilon_1), \]
it follows that \( \vdash_H G[eab_-] \lor \sim eaa \supset G[eab_-] \).

From
\[ \vdash_H G[eab_-, ebc_-] \lor \sim eac \supset G[eab_-, ebc_-] \lor \sim eab \lor \sim ebc \]
and
\[ \vdash_H G[eab_-, ebc_-] \lor \sim eab \lor \sim ebc \]
\[ \equiv \ldots G[\ldots] \lor \sim eab \lor \sim ebc \lor \sim ebc \]
\[ \equiv \ldots G[\ldots] \lor \sim eab \lor \sim ebc \quad \text{(for } \epsilon_2), \]
we obtain \( \vdash_H G[eab_-, ebc_-] \lor \sim eac \supset G[eab_-, ebc_-] \).

From \( \vdash_H G[eab_-, ebb_-] \lor \sim eba \supset G[eab_-, ebb_-] \lor \sim eab \lor \sim ebb \)
and
\[ \vdash_H G[eab_-, ebb_-] \lor \sim eab \lor \sim ebb \]
\[ \equiv \ldots G[\ldots] \lor \sim eab \lor \sim eab \lor \sim ebb \]
\[ \equiv \ldots G[\ldots] \lor \sim eab \lor \sim ebb \quad \text{(for } \epsilon_3), \]
we get \( \vdash_H G[eab_-, ebb_-] \lor \sim eba \supset G[eab_-, ebb_-] \).

In view of the Theorem just proved, a proof in the tableau-method-type \( L_1 \) is transformed into that of its correspondent in the Hilbert-type version of \( L_1 \). In other words, a proof having tableau-method-proof is now thought of
as a proof in the Hilbert-type $L_1$ with each reduction rule to be understood as a derived rule of $L_1$.

The Theorem, it is remarked, constitutes the first step in obtaining the looked-for meta-equivalences as announced in the introduction.

Lemma 2.4 Every Hintikka formula contains at least one occurrence of atomic formulas either as its positive or negative part.

Suppose, if possible, to the contrary. There would, then, be the shortest positive or negative part of the given Hintikka formula. If that be shorter positive parts against the assumption. If a formula were the shortest positive part of the form $A \lor B$, then $A$ and $B$ would the shorter positive parts against the assumption. If a formula having the shorter than $A \lor B$ again contrary to the hypothesis. If $\sim A$ were the shortest positive part, $A$ would be a shorter negative part against the assumption. Lastly, if $\sim A$ were the shortest negative part, $A$ would be a shorter positive part again contrary to hypothesis.

3 Axiomatic rejection - its Hilbert-type version HAR

We are now in a position to state the axioms and rules for axiomatic rejection. The axiomatic rejection HAR to developed hereunder, it is noticed, is for the Hilbert-type version of $L_1$. It constitutes a Hilbert-type axiomatic rejection in distinction to the Gentzen-type one to be introduced in what follows.

Axioms:

3.11 $\vdash_H \epsilon aa$,
3.12 $\vdash_H \sim \epsilon aa$,

where $a$ is a name variable specified for the purpose.

Rules:

3.13 $\vdash_H A \supset B$, $\vdash_H B \rightarrow \vdash_H A$,
3.14 $\vdash_H A \rightarrow \vdash_H B$,

in the second rule $A$ is obtained from $B$ by uniform substitution of a name variable for some occurring in $B$. (These two rules are due to Lukasiewicz [43] and Slupecki [63, 64].)

3.15 $\vdash_H A \rightarrow \vdash_H A \lor \epsilon ab$, (Kobayashi’s rule) 

$^1$Kobayashi is the third author of the present paper.
where $A$ is a Hintikka formula constituting a disjunction with all the disjuncts being either an atomic formula or a negatied atomic formula and $\epsilon ab$ does not occur in $A$ negated, i.e. as its negative part. Above $\not\vdash_H A$ means that $A$ is axiomatically rejected in the Hilbert-type $L_1$, i.e. $\text{HAR}$. Instead of it, we may denote it by $\vdash_{\text{HAR}} A$. As easily seen, $A \lor \epsilon ab$ also constitutes a Hintikka formula. (For axiomatic rejection, refer besides Slupecki [63, 64], Lukasiewicz [43] and Härting [8] also to Goranko-Pulcini-Skura [7], Inoué [10, 11], Ishimoto [22, 27], Iwanuś [31], Skura [62] and so on.)

We are now presenting an example of axiomatic rejection for the purpose of illustrating how our axioms and rules work in combination:

3.2 $\not\vdash_H \epsilon ab \lor \epsilon bc \supset \epsilon aa,$

where $a$, $b$ and $c$ are name variables different from each other.

(1) $\vdash_H \sim \epsilon bb \lor \sim \epsilon bb \supset \sim \epsilon bb$, tautology,

(2) $\not\vdash_H \sim \epsilon bb \lor \sim \epsilon bb$ (1), 3.12, 3.13, 3.14,

(3) $\not\vdash_H \sim \epsilon bc \lor \sim \epsilon bb$ (2), 3.14,

(4) $\not\vdash_H (\sim \epsilon bc \lor \sim \epsilon bb) \lor \epsilon aa$ (3), 3.15,

(5) $\vdash_H (\epsilon ab \lor \epsilon bc) \lor \sim \epsilon bc \lor \epsilon aa \supset \sim \epsilon bc \lor \sim \epsilon bb \lor \epsilon aa$ tautology,

(6) $\not\vdash_H (\epsilon ab \lor \epsilon bc) \lor \sim \epsilon bc \lor \epsilon aa$ (5), (4), 3.13,

(7) $\vdash_H (\epsilon ab \lor \epsilon bc) \lor \epsilon aa \supset \sim (\epsilon ab \lor \epsilon bc) \lor \sim \epsilon bc \lor \epsilon aa$ tautology,

(8) $\not\vdash_H (\epsilon ab \lor \epsilon bc) \lor \epsilon aa$ (7), (6), 3.13.

From (8), we obtain 3.2.

**Lemma 3.1** For any formula $A$ of $L_1$, we have:

$$\vdash_H A \equiv . A_1 \lor A_2 \lor \cdots \lor A_n \quad (n \geq 1),$$

where $A_1, A_2, \cdots, A_n$ exhaust all the formulas which occur in $A$ as its minimal positive or negative parts with minimal negative parts prefixed with negation. (The minimal positive (negative) parts of a formula, it is remembered, are those which contain neither positive nor negative parts of the formula except themselves.)

This is proved by induction on the number of the minimal positive and negative parts which can be brought out by way of Lemma 2.1.

The Lemma is exemplified as follows:

$$\vdash_H \sim \sim \sim \epsilon ab \lor \sim \sim \epsilon ba \lor \sim \epsilon aa. \equiv . \sim \epsilon ab \lor \epsilon ba \lor \sim \epsilon aa,$$
\[ \vdash_H \sim (\epsilon ab \vee \sim \epsilon bc) \vee \sim \sim \sim \epsilon ac \vee \sim \epsilon bc \vee \sim \epsilon aa. \]

\[ \equiv \sim (\epsilon ab \vee \sim \epsilon bc) \vee \sim \epsilon ac \vee \sim \epsilon bc \vee \sim \epsilon aa. \]

If \( A \) is a Hintikka formula as is the case with the examples, at least, one formula among \( A_1, A_2, \ldots, A_n \) is atomic or the negation of an atomic formula in view of Lemma 2.4.

We, next, wish to prove that every Hintikka formula of \( L_1 \) is axiomatically rejected in \( \text{HAR} \), that is, a thesis of \( \text{HAR} \). With this in view we are proving a number of preparatory lemmas.

**Lemma 3.3** Any formula of \( L_1 \) of the form:

\[ \sim B_1 \vee \sim B_2 \vee \cdots \vee \sim B_n \quad (n \geq 1), \]

is axiomatically rejected in \( \text{HAR} \), where \( B_1, B_2, \ldots, B_n \) are atomic formulas.

For proving the Lemma we substitute the name variable \( a \) as specified in 3.12 for all the name variables occurring in the given formula. There, then, obtains,

\[ \sim \epsilon aa \vee \sim \epsilon aa \vee \cdots \vee \sim \epsilon aa, \]

which in turn is equivalent to \( \sim \epsilon aa \) (by classical propositional logic). From this it follows that the given formula is axiomatically rejected by 3.12, 3.13, 3.14.

**Lemma 3.3** Any Hintikka formula of \( L_1 \) of the form:

\[ A_1 \vee A_2 \vee \cdots \vee A_n \vee \sim B_1 \vee \sim B_2 \vee \cdots \vee \sim B_m \quad (n \geq 1, m \geq 1), \]

is axiomatically rejected in \( \text{HAR} \), where \( A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m \) are atomic.

As shown by Lemma 3.2,

\[ 3.31 \quad \sim B_1 \vee \sim B_2 \vee \cdots \vee \sim B_m \quad (m \geq 1), \]

is axiomatically rejected. The rule 3.15 for axiomatic rejection is, then, rejection of the given formula, whereby 3.31 is a Hintikka formula to begin with and the result of the application of 3.15 to a Hintikka formula again gives rise to another as remarked earlier in connection with the statement of the rule 3.15.
Lemma 3.4 Any Hintikka formula of L₁ of the form:

\[ A₁ ∨ A₂ ∨ ⋯ ∨ A_n \quad (n \geq 1), \]

is axiomatically rejected in HAR, where \( A₁, A₂, ⋯, A_n \) are atomic formulas.

By substituting the variable \( a \) as specified in 3.11 for all the variables taking place in the given formula, there obtains,

\[ εaa ∨ εaa ∨ ⋯ ∨ εaa, \]

which is equivalent to \( εaa \) (by classical propositional logic). The given formula is again axiomatically rejected in view of 3.11, 3.13 and 3.14.

We are now in a position to treat the case of general Hintikka formulas. The next is one of our main results.

Theorem 3.1 (Basic Theorem)
Every Hintikka formula of L₁ is axiomatically rejected in HAR.

For its proof, we shall begin with a given Hintikka formula is of the form as stipulated in the right-hand side of Lemma 3.1, namely,

\[ \text{3.41} \quad A₁ ∨ A₂ ∨ ⋯ ∨ A_n \quad (n \geq 1), \]

where, at least, one \( A_i \) (\( 1 \leq i \leq n \)) is atomic or the negation of an atomic formula. A Hintikka formula is always transformed into a Hintikka formula of the form 3.41 by Lemma 3.1 in view of Lemma 2.4 and the definition of Hintikka formula.

The proof of Theorem 3.1 is carried out by induction on the number of the \( A_i \)'s having the form \( ∼ (B₁ ∨ B₂) \).

The basis is to the effect that in 3.41 there does not occur any formula of the form \( ∼ (B₁ ∨ B₂) \) and \( ∼ B_i \) (\( i = 1 \) or \( i = 2 \)). The basis holds from Lemmas 3.2, 3.3 and 3.4. Since 3.41 is a Hintikka formula, the given Hintikka formula 3.41 is equivalent to a formula of the form:

\[ ⋯ ∨ ∼ (B₁ ∨ B₂) ∨ ∼ B_i ∨ ⋯. \]

Now, by classical propositional logic, we have:

\[ \text{3.42} \quad \vdash_H \left( ⋯ ∨ ∼ (B₁ ∨ B₂) ∨ ∼ B_i ∨ ⋯ \right) ⊃ \left( ⋯ ∨ ∼ B_i ∨ ⋯ \right), \]

the consequence of which is again a Hintikka formula.

By induction hypothesis we have:
3.43 \( \vdash_H \cdots \lor \sim B_i \lor \cdots \).

which in turn gives rise to,

3.44 \( \vdash_H \cdots \lor (B_1 \lor B_2) \lor \sim B_i \lor \cdots \),

by 3.42, 3.43 and 3.13 as requested.

Since any Hintikka formula is equivalent to a formula of the form 3.41 by
Lemmas 2.4 and 3.1, this completes the proof of our Basic theorem.

Lemma 3.5 Given a branch of a tableau, which is ending with a Hintikka
formula, every constituent formula of the branch is a positive part of the
Hintikka formula, and such a formula implies the succedent one and is ax-
iomatically rejected in HAR.

The first part is proved by induction on the length of the branch. The
second and third parts are taken care of on the basis of the following these of
Hilbert-type version \( L_1 \) corresponding to reduction rules and the rule 3.13.

\[
\begin{align*}
\vdash_H G[A \lor B] & \supset G[A \lor B] \lor \sim A, \\
\vdash_H G[A \lor B] & \supset G[A \lor B] \lor \sim B, \\
\vdash_H G[\epsilon a b] & \supset G[\epsilon a b] \lor \sim \epsilon a a, \\
\vdash_H G[\epsilon a b, \epsilon b c] & \supset G[\epsilon a b, \epsilon b c] \lor \sim \epsilon a c, \\
\vdash_H G[\epsilon a b, \epsilon b b] & \supset G[\epsilon a b, \epsilon b b] \lor \sim \epsilon b a,
\end{align*}
\]

which were already mentioned in 2.22.

Corollary 3.1 Every positive (the negation of negative) part of a Hintikka
formula of \( L_1 \) is axiomatically rejected in HAR.

This follows from Lemma 2.3 and Lemma 3.5 (or Theorem 3.1) and the
rule 3.13.

By Theorem 2.1 (Fundamental Theorem) and Lemma 3.5, we have:

Corollary 3.2 Every formula of \( L_1 \), which is not provable by the tableau
method for \( L_1 \), is axiomatically rejected in HAR.

From the Corollary and Theorem 2.2 there follows immediately an important
theorem.

Theorem 3.2 (Dichotomy theorem) Every formula of \( L_1 \), which is not a
thesis of the Hilbert-type \( L_1 \), is axiomatically rejected in HAR (in its Hilbert-
type version).
In other words, every formula is provable or axiomatically rejected.

Suppose $A$ is not provable in the Hilbert-type version of $L_1$. By Theorem 2.2, $A$ is also not provable in the tableau method for the Hilbert-type $L_1$. By Corollary 3.2, $A$ is, then, axiomatically rejected in its Hilbert-type version.

Since the set of provable formulas as well as that of axiomatically rejected formulas in the Hilbert-type $L_1$ are both recursively enumerable, Theorem 3.2, namely, Dichotomy theorem and Contradiction theorem (i.e. Theorem 6.4) to be proved in the sequel provides us with the decidability for the Hilbert-type version of $L_1$. Corollary 3.2 gives a decision procedure for the Hilbert-type version of $L_1$. On the basis of a setting similar to ours, Slupecki [63, 64] and Łukasiewicz [43] gave a decision method for the Aristotelian syllogistic (cf Slupecki [66]). As will be seen in what follows, a Gentzen-type axiomatic rejection will give us a much more simpler decision method for $L_1$.

Now, we are in a position to give a normal form to each axiomatic rejection of the formula 3.2, which was already carried out by a more round about way.

With this in view, a (normal) tableau will be constructed for 3.2 as follows with all the name variables involved being different from each other:

$$
\begin{array}{c}
\sim (\varepsilon ab \lor \varepsilon bc) \lor \varepsilon aa \quad (= 3.2) \\
\sim (\varepsilon ab \lor \varepsilon bc) \lor \varepsilon aa \lor \sim \varepsilon bc \quad (= 3.23)
\end{array}
\frac{3.21}{3.22} (\varepsilon_1) \quad \frac{\sim (\varepsilon ab \lor \varepsilon bc) \lor \varepsilon aa \lor \sim \varepsilon bc \lor \varepsilon bb \quad (= 3.24)}{(\lor)}

\frac{3.22}{(\lor)}
\frac{3.22}{(\lor)}

where

\begin{align*}
3.21 &= \sim (\varepsilon ab \lor \varepsilon bc) \lor \varepsilon aa \lor \sim \varepsilon ab, \\
3.22 &= \sim (\varepsilon ab \lor \varepsilon bc) \lor \varepsilon aa \lor \sim \varepsilon ab \lor \sim \varepsilon aa.
\end{align*}

We have by Lemma 3.5:

$$
\vdash_H 3.2 \supset 3.23,
\vdash_H 3.23 \supset 3.24.
$$

By Theorem 3.2, 3.24 is axiomatically rejected, since it is a Hintikka formula. In view of the above two implications and the rule 3.13, we are given an axiomatic rejection of 3.2 in the normal form, which consists in first obtaining a Hintikka formula and reject the given formula on its basis.

The above example for the normal form theorem can be regarded as a prototype for the argument developed in Inoué [10, 11], in which axiomatic rejection with Hintikka formulas as axioms is proposed for classical propositional logic and its extensions. We will again touch on this in the section seven where a Gentzen-type axiomatic rejection is introduced for $L_1$. 

18
4 Translation and soundness

In this section, we shall first recall the translation $T$ explained in the introduction. $T$ transforms every formula of $L_1$ into a formula of first-order predicate logic with equality. The inductive definition of $T$ is as follows. (This definition was first proposed by Ishimoto [21] on the basis of Prior [48].)

\begin{align*}
4.11 & \quad T\epsilon a b = F_{b} \iota x F_{a} x, \\
4.12 & \quad TA \lor B = TA \lor TB, \\
4.13 & \quad T \sim A = \sim TA,
\end{align*}

$F_{a}, F_{b}, \ldots$ are monadic predicate (variables) corresponding to name variables $a, b, \ldots$ not necessarily, exhausting all of them. $F_{b} \iota x F_{a} x$, on the other hand, is the Russellian-type definite description and stands for

$$\exists y (F_{a} x \land F_{b} x) \land \forall x \forall y (F_{a} x \land F_{a} y \supset x = y)$$

with the scope of the description confining to $F_{b}$. As easily seen, there are some formulas of predicate logic which are not a $T$-transform of a formula of $L_1$.

**Theorem 4.1** (Soundness theorem) If $\vdash_{H} A$, then $TA$ is a thesis of first-order predicate logic with equality.

The proof is carried out by induction on the length of the proof in the Hilbert-type version of $L_1$.

The basis is taken care of on the basis of the following theses of predicate logic, respectively, corresponding to 1.11, 1.12 and 1.13:

\begin{align*}
\vdash & \quad F_{b} \iota x F_{a} x \supset F_{a} \iota x F_{a} x, \\
\vdash & \quad F_{b} \iota x F_{a} x \land F_{c} \iota x F_{b} x \supset F_{c} \iota x F_{a} x, \\
\vdash & \quad F_{b} \iota x F_{a} x \land F_{b} \iota x F_{b} x \supset F_{a} \iota x F_{b} x.
\end{align*}

For treating the induction steps, let us assume that $TA$ and $TA \supset B$ ($TA \supset TB$) are provable in predicate logic. $TB$ is, then, forthcoming as a thesis of the logic by detachment.

This complete the proof of the Soundness theorem for the Hilbert-type $L_1$. 

19
5 Models for $L_1$

For proving,

$$TA$$ is valid in first-order predicate logic with equality

$$\implies \ not \ |-_H A,$$

which will be demonstrated in the next section as Theorem 6.1, we need some preparatory lemmas and definitions concerning the construction of models for $L_1$, which are defined on the basis of the models for first-order predicate logic with equality.

Theorem 5.1 For every Hintikka formula $A$, there is a model for $L_1$ which falsifies $A$ and every positive (negative) part of it is false (true) there.

This is proved by defining a model for $L_1$, where every atomic positive (negative) part of the Hintikka formula is made false (true), the existence of which is guaranteed by Lemma 2.4. The second part is taken care of by Lemma 2.3. But, we are proposing a method, which, though seemingly more complicated, will be convenient for obtaining a model for predicate logic for falsifying $TA$. The model in turn gives rise to the one for $L_1$, as will be seen in what follows.

Definition 5.1 A chain a Hintikka formula $A$ is a (finite) collection of name variables $a_1, a_2, \ldots, a_n \ (n \geq 1)$ such that

5.21 Every pair $a_i$ and $a_j \ (1 \leq i \leq n, 1 \leq j \leq n)$ belonging to the collection are connected by the relation defined as $\epsilon a_i a_j$ and $\epsilon a_j a_i$ both of which constitute negative parts of $A$.

5.22 The collection is maximal with respect to this property 5.21.

As easily seen, the relation defined by 5.21 is reflexive, symmetric and transitive.

Definition 5.2 A tail of a chain (of a Hintikka formula) is a name variable $b$ such that $\epsilon ab$ is a negative part of the Hintikka formula with the $a$, but not the $b$, being a member of the chain.

For illustrative purposes, a number of Hintikka formulas will be presented with chains and tails associated thereto. The name variables involved, it is assumed, are different from each other.
5.31 \( \sim \epsilon ab \lor \sim \epsilon ba \sim \epsilon aa \lor \sim \epsilon bb, \)
where \( \{a, b\} \) is a chain without tails. (\( \{a, b\} \), for example, is a set consisting of \( a \) and \( b \).)

We shall here introduce a convenient notation for chains and tails. An expression \([x_1, x_2, \ldots, x_n] \) means a chain \( \{x_1, x_2, \ldots, x_n\} \) of a Hintikka formula, where \( x_1, x_2, \ldots, x_n \) are name variables. By

\[ [x_1, x_2, \ldots, x_n] \longrightarrow y, \]

we mean that \( y \) is a tail of a chain \([x_1, x_2, \ldots, x_n]\). With this notation, 5.31 is of the type \([a, b]\).

5.32 \( \sim \epsilon ab \lor \sim \epsilon bc \sim \epsilon ac \sim \epsilon ba \sim \epsilon aa \lor \sim \epsilon bb, \)
where \( \{a, b\} \) is a chain with \( c \) being its tail: 5.32 is of the type:

\[ [a, b] \longrightarrow c. \]

5.33 \( \sim \epsilon ab \lor \sim \epsilon bc \lor \epsilon ab \lor \sim \epsilon aa \lor \sim \epsilon bb, \)
where \( \{a\} \) and \( \{b\} \), respectively, are different chains with \( c \) being \( a \) tail common to these two chains:

\[
\begin{array}{ccc}
& [a] & \\
/ & & \\
& c. & \\
& & [b]
\end{array}
\]

5.34 \( \sim (\epsilon aa \lor \sim \epsilon bb) \lor \sim \epsilon ab \lor \sim \epsilon dc \lor \epsilon cb \lor \sim \epsilon bb \lor \sim \epsilon ba \lor \sim \epsilon aa \lor \sim \epsilon dd \lor \sim \epsilon ae \lor \sim \epsilon be, \)
where \( \{a, b\} \) constitutes a chain with \( e \) as its tail, while \( \{d\} \) is another chain whose tail is \( c \): 5.34 is of the type:

\[
\begin{array}{ccc}
[a, b] & \longrightarrow & e, \\
& & \\
[d] & \longrightarrow & c.
\end{array}
\]

5.35 \( \sim \epsilon ab \lor \sim \epsilon ac \lor \epsilon bc \lor \epsilon da \lor \epsilon aa, \)
where \{a\} is the only chain having both b and c as its tails: 5.35 is of the type:

\[
\begin{array}{c}
  [a] \\
  b \\
  c.
\end{array}
\]

We are now going to describe a method of defining a model for \(L_1\) which falsifies the given Hintikka formula. We shall first confine ourselves to the definition of the models specific to the Hintikka formulas as above presented. All these models for \(L_1\) falsifies the given Hintikka formula, since every atomic formula taking place there as a positive (negative) part is false (true) in these models, and this makes every positive (negative) part of the Hintikka formula false (true) as easily proved by induction on the length of positive and negative parts (cf. Schütte [54, p. 12 Theorem 1.6]).

5.41 To begin with, we wish to define a model \(M\) (for \(L_1\)) which falsifies the Hintikka formula 5.31.

The model \(M\) consists of two elements, namely, \{1\} assigned to the members a and b of the chain and \emptyset (the empty set) assigned to the infinite list of the remaining name variables. The truth value of atomic formula \(eab\) is that of \(Teab\), i.e. \(F_e F_a x\) to be defined on the basis of the model \(M'\) for first-order predicate logic with equality with the domain consisting only and equality standing for the identity between numbers. On the basis of this model, \(e\{1\}\{1\}\) is true since \(Te\{1\}\{1\} = \{1\}\{1\}x\) is true in \(M'\), while \(e\{1\}\emptyset\), \(e\emptyset\{1\}\) and \(e\emptyset\emptyset\) are all seen false in \(M\), because their \(T\)-transforms are all false in \(M'\). The truth values of other formulas are defined on the basis of those for atomic formulas. (Here, \{a\}, for example, denotes a (monadic) predicate which is true only fo a, and \emptyset the predicate constantly galse for any argument with respect to the model for the predicate logic. \{a, b\}, \{a, b, c\} and the like are understood analogously.)

5.42 For defining a model (for \(L_1\)) which falsifies the Hintikka formula 5.32, we assign \{1\} to the members of the chain, e.g. a and b, while the tail is given \{1, n\} as its vvalue with \(n\) being any natural number other that 1, say 2. The remaining name variables are assigned \emptyset as before.

A model \(M\) (for \(L_1\) is, then, defined with \emptyset, \{1\}, \{1, 2\} as the elements
of the domain (the universe), while the truth value of any atomic formula is identified with that of its $T$-transform to be defined on the basis of the model $\mathcal{M}'$ for first-order predicate logic with $\{1, 2\}$ as its domain.

For example, the truth value of $\epsilon\{1\}\{1, 2\}$ is that of the following:

$$\exists x(\{1\}x \land \{1, 2\}x) \land \forall x\forall y(\{1\}x \land \{1\}y. \supset x = y),$$

whose truth value is obtained by considering the following formula:

$$\{\{1\}1 \land \{1, 2\}1, \lor ,\{\{1\}2 \land \{1, 2\}2 \land (\{\{1\}1 \land \{1, 2\}1. \supset 1 = 1) \land (\{\{1\}1 \land \{1, 2\}2. \supset 1 = 2) \land (\{\{1\}2 \land \{1, 2\}1. \supset 2 = 1) \lor (\{\{1\}2 \land \{1, 2\}2. \supset 2 = 2).$$

The truth value of $\epsilon\{1\}\{1, 2\}$ is of course true.

5.43 The Hintikka formula 5.34 is taken care of by assigning $\{1\}$, $\{2\}$, $\{1\}$, $\{1, 2, 3\}$ and $\emptyset$, respectively, to $a$, $b$, $c$ and the remaining variables and defining a model $\mathcal{M}$ (for $L_1$ with its domain consisting of $\{1, 2, 3\}$ and $\emptyset$. The truth value of atomic formulas are again defined on the basis of a model $\mathcal{M}'$ for first-order predicate logic with equality constructed analogously to the preceding two case with the domain consisting of $a$, $2$, $3$. $\mathcal{M}$ is a model for $L_1$, and falsifies 5.33, since $T5.33$ is false in $\mathcal{M}'$.

5.44 The treatment of the Hintikka formula 5.34 proceeds by assigning $\{1\}$, $\{2\}$, $\{1\}$, $\{1, 3\}$ $\{2, 4\}$, respectively, to $a$, $b$, $d$, $e$, $c$ with the remaining variables assigned $\emptyset$. The $\mathcal{M}$ and $\mathcal{M}'$ are defined analogously to the preceding cases with the domain of $\mathcal{M}'$ consisting of $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ and 5.34 is false in $\mathcal{M}$, since $T5.34$ is false in $\mathcal{M}'$.

5.45 The Hintikka formula 5.35 is taken care of by assigning $\{1\}$, $\{1, 2\}$, $\{1, 3\}$ and $\emptyset$, respectively, to $a$, $b$, $c$ and the remaining variables. Everything goes as before, and $T5.35$ is false in $\mathcal{M}'$.

The definitions of models $\mathcal{M}$ and $\mathcal{M}'$ for $L_1$ and first-order predicate logic with equality is respectively generalized in the following way, given a Hintikka formula:

5.51 Every member of a chain is assigned one and the same unit set (of a natural number) with a different unit set assigned to a member of different chains.

5.52 To tail we assign a set of natural numbers $\{m_1, m_2, \ldots, m_k, N\}$, where $\{m_i\}$ $(1 \leq i \leq k)$ is the unit set associated with a member of a chain which is ending with the tail and $N$ is a number never employed so far in

\[23\]
defining the model. The natural number $k$ depending on the tail should be maximal. To make sure of the given situation, it is illustrated in the notation in 5.31 as follows:

Type

$$[a_1^1, a_2^1, \ldots]$$

$$\tilde{t} \leftarrow [a_1^2, a_2^2, \ldots]$$

$$\cdots$$

$$[a_1^k, a_2^k, \ldots],$$

After the assignment

$$\tilde{t} \leftarrow \tilde{m}_1$$

$$\tilde{m}_1 \leftarrow \tilde{m}_2$$

$$\cdots$$

$$\tilde{m}_k,$$

where

$$\tilde{t} = \{m_1, m_2, \ldots, m_k, N\},$$

$$\tilde{m}_1 = [\{m_1\}, \{m_1\}, \ldots],$$

$$\tilde{m}_2 = [\{m_2\}, \{m_2\}, \ldots],$$

$$\tilde{m}_k = [\{m_k\}, \{m_k\}, \ldots].$$

5.53 To all other names, we assign the empty set $\emptyset$.

5.54 A model $\mathcal{M}'$ is, then, defined for first-order predicate logic with equality with the domain consisting of the (fine) set of natural numbers so far introduced. In case $\emptyset$ be the only set assigned to name variables, the domain of $\mathcal{M}'$ is any non-empty set of natural numbers. The truth value of any atomic formula, say $\epsilon ab$, in $\mathcal{M}$ is, then, identified with that of its $T$-transform $T\epsilon ab$, namely $F_{\epsilon \mu x} F_{a} x$ in $\mathcal{M}'$, and that of other formulas is defined on the basis of the truth values of atomic formulas. To the domain of $\mathcal{M}'$, any non-empty set of natural numbers could be adjoined without effecting the truth value in the model $\mathcal{M}$ (for $L_1$).

As expected, the model thus defined in general is the one for $L_1$. With a view to proving this, all the entities obtained in the course of the model construction are classified into the following three categories, namely,

$\emptyset$ (the empty set as a predicate),

$\{a\}$ (a unit set as a predicate),
\{m_1, m_2, \ldots, m_k, N\} \ (k \geq 1) \quad \text{(a finite set as a predicate)},

which are, respectively, assigned to name variables occurring neither as a members of a chain nor as a tail, members of a chain and tails.

On the basis of such a classification, the \(T\)-transforms of 1.1, 1.2 and 1.3b turn out to be true in \(M'\), and the truth, then, gives rise to the satisfaction of axioms 1.1, 1.2 and 1.3b in \(M\) to be defined by way of \(M'\).

5.61 To begin with, \(T1.1\) is seen to be true in \(M'\) for all the assignments to the name variables \(a\) and \(b\) as shown below:

For \(a = \emptyset\) or \(\{m_1, m_2, \ldots, m_k, N\}\), \(Teab\) is always false in \(M'\) irrespectively of any assignment to \(b\) in view of the Russelian-type definite description. This makes \(T1.1\) and 1.1 true in \(M'\) and in \(M\), respectively.

For \(a = \{n\}\), \(Teaa\) is true again by the definite description. This makes \(T1.1\) and, consequently, 1.1 true in \(M'\) and in \(M\), respectively.

5.62 The axiom 1.2 is taken care of as follows:

For \(a = \emptyset\), \(eab\) is false in view of the definite description, and this gives rise to the truth of 1.2 in \(M\) through that of \(T1.2\).

For \(a = \{n\}\), and \(b = \emptyset\), \(T1.2\) is easily seen true by the definite description.

For \(a = \{n\}\), \(b = \{m\}\) and \(c = \emptyset\), \(T1.2\) is true in \(M'\) since \(Tebc\) is false there making 1.2 true in \(M\).

For \(a = \{n\}\), \(b = \{m\}\) and \(c = \{1\}\), \(Teab\), \(Tebc\) and \(Teab\) are all true if \(n = m = 1\). This then, makes 1.2 true in \(M\). If \(n = m\) and \(m \neq 1\), \(Tebc\) is false in \(M'\) making \(T1.2\) and 1.2 true in \(M'\) and in \(M\), respectively. For \(n \neq m\), \(Teab\) is false and \(T1.2\) and 1.2 are true in \(M'\) and in \(M\), respectively.

For \(a = \{n\}\), \(b = \{m\}\) and \(c = \{m_1, m_2, \ldots, m_k, N\}\), \(Teac\) is true if \(n = m\) and \(m\) is a member of \(c\), making \(T1.2\) and 1.2 true in \(M'\) and in \(M\), respectively. If \(n = m\) and \(m\) is not a member of \(c\), then \(Tebc\) is false, making \(T1.2\) and 1.2 true in \(M'\) and in \(M\), respectively. If \(n \neq m\), \(Tebc\) is false and \(T1.2\) and 1.2 true in \(M'\) and in \(M\), respectively.

For \(a = \{n\}\) and \(b = \{m_1, m_2, \ldots, m_k, N\}\), \(Tebc\) is false, making \(T1.2\) and 1.2 true in \(M'\) and in \(M\), respectively.

For \(a = \{m_1, m_2, \ldots, m_k, N\}\), \(Tebc\) is always false, for any value of \(b\), and \(T1.2\) and consequently, 1.2 is ture in \(M'\) and in \(M\), respectively.

5.63 The axiom 1.3 is seen to be true in \(M\) in the following way:

For \(b = \emptyset\) or \(b = \{m_1, m_2, \ldots, m_k, N\}\), \(Tebb\) is false in \(M'\) making \(T1.3\) and 1.3 true, respectively, in \(M'\) and in \(M\).

For \(b = \{n\}\), \(a = \emptyset\), \(Teab\) is false in \(M'\) making 1.3 true in \(M\).

For \(b = \{n\}\), \(a = \{m\}\) and \(n = m\), \(Teba\) is true in \(M'\) and this makes \(T1.3\) and 1.3 true respectively, in \(M'\) and in \(M\).
For $b = \{n\}$, $a = \{m\}$ and $n \neq m$, $Teab$ is false in $\mathcal{M}'$. This makes $T1.3$ and $1.3$ true respectively, in $\mathcal{M}'$ and in $\mathcal{M}$.

5.64 This completes the proof that the axioms 1.1–1.3 for $L_1$ are satisfied by the model $\mathcal{M}$ since we have $TA \supset B \equiv .TA \supset TB$. Therefore, this model constructed by 5.51–5.54 constitutes a finite model for $L_1$.

5.65 We next have to check whether such a model (for $L_1$) as constructed above actually falsifies the given Hintikka formula. By the definition of Hintikka formula, we may only consider atomic positive (negative) parts of it. In other words, if each atomic positive (negative) part of it is assigned falsity (truth) (via $\mathcal{M}'$), then the Hintikka formula is false in $\mathcal{M}$ (via $\mathcal{M}'$) (cf, for example Schütte [54, p. 12, Theorem 1.6]). (Note that such atomic positive (negative) parts of it are the minimal positive (negative) ones of it, but they do not always exhaust all the minimal ones in some cases.)

5.66 If the Hintikka formula has $Teab$ as its positive part, the $n$ we have the following possibilities as the result of our assignment:

$e\emptyset\emptyset$, $e\emptyset\{n\}$, $e\emptyset\{m_1, m_2, \ldots, m_k, N\}$, $e\{n\}\emptyset$, $e\{p\}\{q\}$, $e\{r\}\{m_1, m_2, \ldots, m_k, N\}$, $e\{m_1, m_2, \ldots, m_k, N\}\emptyset$, $e\{m_1, m_2, \ldots, m_k, N\}\{n\}$, $e\{m_1, m_2, \ldots, m_k, N\}\{m'_1, m'_2, \ldots, m'_j, L\}$,

which are all false in $\mathcal{M}$, where $p \neq q$ and $r$ is not a member of $\{m_1, m_2, \ldots, m_k, N\}$. The other possibilities do not happen because of the definitions 2.2, 5.2, 5.3, 5.51–5.53.

5.67 If the Hintikka formula has $eab$ as its negative part, then we have the only following possibilities as the result of our assignment: $e\{n\}\{n\}$ and $e\{m\}\{m_1, m_2, \ldots, m_k, N\}$, which are true in $\mathcal{M}$, where $m$ is an element of $\{m_1, m_2, \ldots, m_k, N\}$. By similar reasoning as in 5.66, the other possibilities do not happen. It is, however, remarked that the Hintikka formula does not contain formulas of the form $etb$ as its negative part, where $t$ is a tail and $a$ is an arbitrary name variable. If such a formula were a negative part of the Hintikka formula, then the Hintikka formula would be of the form $F[eta, ett, \ldots]$ by the definition of Hintikka formula. Thus $t$ is an element of some chain. This contradicts the following.

26
Proposition 5.1 Suppose that $A$ is a Hintikka formula of $L_1$. Then no tail of a chain of $A$ belongs to other chains of it.

For proving the Proposition, let $C_1$ and $C_2$ be chains of $A$ and $b$ a tail of $C_1$. Suppose that $b$ is a member of $C_2$. Then there is a name variable of $C_1$ such that $eab$ is a negative part of $A$. Since $b$ is an element of $C_2$, $A$ contains $ebb$ as its negative part. By 2.25, $A$ thus contains $eba$ as its negative part. In other words, $b$ is a member of $C_1$, which contradicts the definition of tail.

5.68 From 5.68–5.67, it follows that the given Hintikka formula is falsified in $M$ and that every positive (negative) part of it is false (true) in $M$ as understood by induction of the length of positive (negative) part. This is the second part of Theorem 5.1.

By 5.65–5.67, we complete the proof of Theorem 5.1. (The original idea of our model construction is due to the third author of the present paper.)

We are, now, taking the opportunity of demonstrating that the model $M$ as above defined for $L_1$ also constitutes the one for $L$ as well. This will play an important role, in particular, with reference to the proof of Separation theorem to be shown in the section six.

The model thus constructed for falsifying the given Hintikka formula is a model $M = \langle D, \epsilon \rangle$ such that $D$ is a finite set of subsets of the set of natural numbers, which are regarded as Leśniewskian names, and the truth values of the formulas are reduced to those of their $T$-transforms, which are in turn based upon a model $M' = \langle D', F_a, F_b, \ldots \rangle$ for first-order predicate logic with equality. Here, $D'$ is non-empty set of natural numbers, not necessary infinite. Secondly and more importantly $F_a, F_b, \ldots$ do not exhaust all the subsets of $D'$. Put it the other way round, a number of subsets of $D'$ are remaining anonymous with being named by any names.

Such a situation is sometimes responsible for the appearance of the so-called singular names which are taking place in the process of defining models for $L_1$. A singular name, which was introduced in Ishimoto [21], is a name which is not an atom, but contains only one name in the sense of $\epsilon$-relation, where an atom means a member $a$ of the domain of a model $M$ such that $eaa$ is true in $M$. Here, note that a name used in Ishimoto [21] is a subset of the set of natural numbers, i.e. an element of the domain of a model for $L_1$ in our present context. A singular name is occurring, for example, in the model as defined in 5.42, where $b$, which happens to be a tail, is singular without being an atom, but containing only one name, namely, $a$. As will be seen presently, such a model fails to be a model for $L$.27
The remedy is not difficult to think of. It is only requested to adjoin \{N\} with \(m_1, m_2, \ldots, m_k, N\) of each given singular name. In the case of 5.42, in addition to \(a\) and \(b\) there takes place another name, say, \(c\) which is the unit set \{2\}.

The remedy sketched above, which makes use of introducing some new names, is based on the argument in Ishimoto [21] and later in Kobayashi-Ishimoto [38], which will be discussed below in our context. The reader will see that the argument can be applied to our case without any change, because our models constructed above coincide with ones defined by the model construction in Kobayashi-Ishimoto [38], if we identify names in the just cited paper with the set of natural numbers: in other words, every atomic positive (negative) part of a given Hintikka formula is false (true) and the rest of all atomic formulas is falsified in both models. This is the truth concept for atomic formulas when we argue about the models for \(L\), while the counterpart of the models for \(L_1\) is on the definite description. This identification is the very trick of connecting the argument of ours to that of the paper cited above. (The idea is due to the second author of the present paper.) To continue the remedy, we shall show the following straightforward

**Lemma 5.1** Given a model \((L_1)\), another model is defined which does not involve singular names, and the truth value of the formula in the original model are remaining the same in the new model.

It is remarked in passing that any model thus constructed always contains atoms. This is because such a model involves names corresponding to unit sets, and names of unit sets are atoms.

For demonstrating that the models for \(L_1\) thus augmented, if necessary, are the models for \(L\) as well, we need in advance the following preparatory theorem.

**Theorem 5.2** Given a model for \(L_1\), it is also the one for \(L\) if and only if it does not involve any singular names.

Necessity: Given a model \(M =< \mathcal{D}, \epsilon >\) for \(L_1\), suppose it contains a singular name \(b\), of which the only element is a (in the sense of \(\epsilon\)-relation) and it is, of course, an atom.

Then,

\[
1 \quad \exists x (\epsilon x b \land \epsilon x b),
\]

is true in the model \(M\). This is because
(2) \( \epsilon ab \land \epsilon ab \),
is true in \( M \) with \( a \) and \( b \) belonging to \( D \).
We also have the truth of
(3) \( \forall x \forall y (\epsilon xb \land \epsilon yb \supset \epsilon xy) \),
in \( M \), since \( a \) is the only name (\( \in D \)) such that \( \epsilon ab \) is tre. \( \epsilon bb \), on the other hand, is false, since \( b \) is singular not being an atom. This makes the axiom schema 1.5 for \( L \) false for \( a = b \) in the model \( M \).

Sufficiency: Given a model \( M = \langle D, \epsilon \rangle \) for \( L_1 \). Suppose \( M \) is not a model for \( L \), although it is for \( L_1 \). Then, there are, at least, two names \( a \) and \( b \). This is to the effect that
(4) \( \epsilon ab \),
is false with
(5) \( \exists x (\epsilon xa \land \epsilon xb) \land \forall x \forall y (\epsilon xa \land \epsilon ya \supset \epsilon xy) \),
being true in the model. In this connection, it is remarked, the converse implication (4) \( \supset \) (5) is true in the model for \( L_1 \).
Since (5) is true in the model, there is an \( x \) (\( \in D \)) such that
(6) \( \epsilon xa \land \epsilon xb \),
is true there.
Further, suppose, if possible, that there is an \( y \) (\( \in D \)) such that
(7) \( \epsilon ya \land \sim \epsilon xb \).
In view of (6), (7) and (5), \( \epsilon yx \) is true in the model, which gives rise to the truth of \( \epsilon yb \) by 1.2. This, however, contradicts (7).
We, thus, have the truth in the model of
(8) \( \forall x (\epsilon xa \supset \epsilon ab) \),
from which follows by (4) that
(9) \( \epsilon aa \),
is not true in the model. This is to the effect that \( a \) is a singular name of the model in view of (5).
This completes the proof of Theorem 5.2.
From the Theorem there straightforwardly obtain:

**Theorem 5.3** A model constructed as above for \( L_1 \), if properly extended when necessary, constitutes at the same time one for \( L \).

In the above argument we followed the usual model-theoretic interpretation for Leśniewskian quantification, while an alternative interpretation, namely the substitutional one has been much discussed so far by some of the leading philosophers. In our setting we are assigning to each name variable
a certain element of a structure, i.e. a subset of the set of natural numbers, whereas the substitutional interpretation does not assign anything to the name variables and thus does not need any domain of our model or the like. Because of the reason the substitutional interpretation would appeal to some, although it could not be the only reason. We will, here, not go into the alternative interpretation further. But, we shall cite the literature about the interpretation, where we see a variety of arguments or it, as follows: Küng [39, 40], Küng-Canty [41], Quine [49], Rickey [51], Simons [59, 60] and so on.

As a concluding remark of the present section, it is emphasized that the domain of the model for $L_1$ and $L$ is a finite set whose elements are all finite sets and that our argument is treated within the bounds of first-order logic. In addition, we mention that the treatment for the remedy of the models with singular names is not the only one: we can, for example, make use of the result of Takano [77] for an alternative treatment. (The idea for the alternative treatment is due to the first author of the present paper. We decide to take the second author’s idea for the present paper.) Takano's paper contains a proof of the completeness theorem for Leśniewski’s ontology $L$ with respect to a natural truth concept,

$$
\epsilon ab \text{ is true (in a structure)} \iff \exists p(a = \{p\} \land p \in b),
$$

where the right $\in$ means the membership relation of the set theory, which is similar to ours. The embedding of Leśniewski’s ontology into the monadic second-order predicate logic in Smirnov [59] may be regarded as a syntactical version of such a natural interpretation of Leśniewski’s epsilon (cf. Takano [78]).

For a similar model construction for the Aristotelian syllogistic, one may consult Kanai [33].

In addition, we shall here mention some application of $L_1$ for natural language as follows: Ishimoto [23, 24, 26], Ishimoto-Shimidzu [30] and Shimidzu [55, 56].

6 Axiomatic rejection and embedding theorem

With a view to proving the theorem announced at the beginning of the preceding section, let us assume that $\models_H A$. We, then, wish to prove that $T_A$ is not valid in first-order predicate logic with equality.
The proof is carried out by induction on the number of rules applied for axiomatically rejecting $A$.

6.1 The basis does not present any difficulties, since both $\epsilon aa$ and $\sim \epsilon aa$ constitute Hintikka formulas, and their $T$-transforms are both falsified by a model for first order predicate logic with equality in view of Theorem 5.1. In this connection, it is remembered, the model for $L_1$, which falsifies the Hintikka formula, is defined on the basis of a model for predicate logic, in which the $T$-transform of the Hintikka formula is also false.

6.2 Induction steps:

6.21 The last applied rule for rejection is 3.13. By induction hypothesis, there is a model for first-order predicate logic which falsifies $T_B$. This is to te effect that $T_B$ is not a thesis of first-order predicate logic by the soundness theorem for the logic. Now, we have $\vdash_H A \supset B$. By Soundness theorem (Theorem 4.1), $T_A \supset B = T_A \supset T_B$ is a thesis of first-order predicate logic with equality. From this it follows that $T_A$ is not valid in the predicate logic along with $T_B$.

6.22 The last applied rule for rejection is 3.14. For taking care of this case, let us assume that $T$ is falsified by a model for first-order predicate logic with equality, and $A$ is obtained from $B$ by uniform substitution for some name variables occurring in $B$. As easily seen, $T_B$ is also falsified by the same model by identifying the value of $b$ with that of $a$, where $a$ is substituted for $b$ in $B$.

6.23 The last applied rule for rejection is 3.15. The case is taken care of without resorting to induction hypothesis. In fact, not only the given Hintikka formula, but $A \vee \epsilon ab$ is also a Hintikka formula, and its $T$-transform is made false by a model for first-order predicate logic with equality by Theorem 5.1.

This completes the proof of:

**Theorem 6.1**

$T_A$ is valid in first-order predicate logic with equality

$\implies$ not $\vdash_H A$.

Making use of the completeness theorem of first-order predicate logic, this Theorem together with Theorems 2.2 and 4.1 as well as with Corollary 3.2 gives rise to the looked-for equivalences:
Theorem 6.2 (Main Theorem)
\[ \vdash_T A \iff \vdash_H A \]
\[ \iff TA \text{ is valid in first-order predicate logic with equality} \]
\[ \iff \text{not } \vdash_H A. \]

which were announced in the first section.

In particular, from the second equivalence, there obtains a theorem to the effect that \( L_1 \) is embedded in first-order predicate logic with equality via the translation \( T \), namely,

**Theorem 6.3** \( \vdash_H A \text{ if and only if } TA \text{ is a thesis of first-order predicate logic with equality.} \)

Here, we are again making use of the completeness of the predicate logic.

In the third section, Dichotomy theorem (Theorem 3.2) was proved to the effect that every formula (of \( L_1 \) is either provable or axiomatically rejected in the Hilbert-type \( L_1 \). Now, another theorem to be coupled with this Theorem will be proved in this section as already mentioned in the section three. It will be called Contradiction theorem.

**Theorem 6.4** (Contradiction theorem) It is not the case that for any formula \( A \) (of \( L_1 \)), \( \vdash_H A \) and \( \vdash_H A \) at the same time.

The proof is straightforward in view of the second and third equivalences of Theorem 6.2. A syntactical proof is also possible. I will be presented in the section eight (under some postulate).

Availing ourselves of Contradiction theorem just proved, there is forthcoming:

**Corollary 6.1** For any formula \( A \) of \( L_1 \), we have:
\[ \vdash_H A \text{ if and only if } TA \text{ is not valid in first-order predicate logic with equality.} \]

Suppose \( \vdash_H A \), then we have not \( \vdash_H A \) by Contradiction theorem, which in turn gives rise to that \( TA \) is not valid in the predicate logic by Theorem 6.3. If not \( \vdash_H A \), then we have \( \vdash_H A \) by Dichotomy theorem (Theorem 3.2), which, then yields the negation of the right side of the Corollary by Theorem 4.1 (Soundness theorem).

The following theorem is a version of Separation theorem, which was first proved in Ishimoto [21, Theorem 3.4, p. 293].
**Theorem 6.5** (Separation theorem) *If a quantifier-free formula $A$ of $L$, i.e. a formula belonging to $L_1$ is valid, then $A$ is already a thesis of $L_1$.*

In other words, $L$ is a conservative extension of $L_1$. Suppose, if possible, $A$ is not provable in the Hilbert-type $L_1$. Then, by Theorem 2.2 it is not the case that $\vdash_T A$. Form this it follows that $A$ is a positive part of a Hintikka formula by lemma 3.5. In view of Theorem 5.1, there is a model for $L_1$ which falsifies the Hintikka formula as well as $A$. As shown in the preceding section, this model for $L_1$ could also be the model for $L$, and there $A$ is again false. But, this is contrary to the assumption.

This is a model-theoretic proof of Separation theorem. A syntactical proof of the original Separation theorem is given Takano [79].

With this we are coming to the end of the Hilbert-type axiomatic rejection for $L_1$. In the following sections, a more constructive Gentzen-type axiomatic rejection will be developed again for $L_1$.

Before concluding this section, we wish to make a supplementary remark to Fundamental theorem, i.e. Theorem 2.1.

According to the Theorem, there obtains in a finite number of steps either a closed tableau or the one, which is not closed, namely open, by reducing the given formula in compliance with the stipulation as stated in the Theorem. Nevertheless, there might be a possibility that some reductions are resulting in a closed tableau, while others do not produce any closed tableaux although starting with one and the same formula. We wish to show that this is not the case. In fact, if a tableau, which is open, were forthcoming by reducing the given formula in a way different from the successful reduction with a branch ending with a Hintikka formula, the Hintikka formula would be axiomatically rejected by Theorem 3.1. From this it follows that the $T$-transform of the formula would be not valid in first-order predicate logic with equality by Theorem 6.1. In view of the successful reduction, the given formula is a thesis of the Hilbert-type $L_1$ by Theorem 2.2, and from this obtains that the $T$-transform of the given formula is provable in first-order predicate logic with equality, i.e. valid there by Theorem 4.1, namely Soundness theorem. This is a contradiction. (For this, refer to Kleene [35, 36, 37].)

Summing up the above argument, we obtain the following.

**Theorem 6.6** (Permutability theorem) *Once we obtain a closed tableau by reducing a formula, there is no possibility of getting another tableau, which is open by way of a reduction different from the given one.*
7 Gentzen-type axiomatic rejection GAR

The axiomatic rejection for $L_1$ developed so far has been the version based upon the Hilbert-type $L_1$. Thus, $\vdash_T A$, for example, has been thought of as formalized within the bounds of the Hilbert-type version of $L_1$ notwithstanding its appearance. As mentioned earlier, this was also the policy adopted by Łukasiewicz for the decision method of the Aristotelian syllogistic.

In this and following section, we are returning to the purely Gentzen-type or tableau method version of $L_1$ availing ourselves of its Schütte-style formalization as above introduced, and wish to develop a Gentzen-type counterpart. All the syntactical preliminaries are also understood in the Gentzen-style.

In the Gentzen-type axiomatic rejection, we are again starting with Theorem 2.1, namely Fundamental theorem. For completeness, we repeating the Theorem hereunder:

Given a formula (of $L_1$), by reducing it by reduction rules there obtains a finite tableau, each branch of which ends either with a formula of th form $F[A_+, A_-]$ or with a Hintikka formula, whereby a branch is extended by a reduction rule only if the formula to be reduced is not of the form $F[A_+, A_-]$ and the reduction gives rise to a formula not occurring in the formula to be reduced as negative part thereof.

The Theorem may be understood in the sense of the Gentzen-style formulation of $L_1$. In other words, the reduction rules are the inference rules given outright in the sense of Gentzen only put up-side-down.

Before proceeding further, some well-known theorem will be cited of the Gentzen-type logic for subsequent reference:

**Theorem 7.1** (Thinning theorem)

\[
\begin{align*}
\vdash_T A & \implies \vdash_T F[A_+], \\
\vdash_T \sim A & \implies \vdash_T G[A_-].
\end{align*}
\]

**Theorem 7.2** (Interchange theorem)

\[
\begin{align*}
\vdash_T F[A_+, B_+] & \implies \vdash_T F[B_+, A_+], \\
\vdash_T G[A_-, B_-] & \implies \vdash_T G[B_-, A_-].
\end{align*}
\]

**Theorem 7.3** (Translation theorem)

\[
\begin{align*}
\vdash_T F[\_+, \_+] & \implies \vdash_T F[\_+, A_+], \\
\vdash_T G[\_-, \_] & \implies \vdash_T G[\_-, A_-].
\end{align*}
\]
Theorem 7.4 (Contraction theorem)

\[ \vdash T_{F[A_+, A_+]} \Rightarrow \vdash T_{F[A_-, A_-]} \]

\[ \vdash T_{G[A_-, A_-]} \Rightarrow \vdash T_{G[A_-, A_-]} \]

All these theorems are known as structural rules en bloc. Every one of them is easily proved by induction on the length of the proof (or tableau) of the assumption, except Contraction theorem. Hereby, it is noticed the length of the proof remains invariant or even gets shorter in the conclusion.

Definition 7.1 A formula \( A \) of \( L_1 \) is axiomatically rejected in \( GAR \) (in the Getzen-type axiomatic rejection) (denoted by \( \vdash_T A \)) if there exists a tableau of it, at least, one branch of which is ending with a Hintikka formula (i.e. an open tableau of it).

This is a Gentzen-type counterpart of the definition of axiomatic rejection.

Roughly speaking, any Hintikka formula is now playing the role of the axioms for the Gentzen-type axiomatic rejection for \( L_1 \), whereas its counterpart of the Hilbert-type \( L_1 \), was more complicated as described in detail in the third section. Nevertheless, it is remembered, we come nearer the Gentzen-type axiomatic rejection in the same section. The theorem is to the effect that any formula, if it is rejected at all, is rejected through a Hintikka formula and after the Hintikka formula only the rule 3.13 is made use of. (The use of Hintikka formulas as axiom for axiomatic rejection is dating from Inoué [10, 11] concerning classical propositional logic and its extensions.)

Therefore we may regard an axiomatization for axiomatic rejection with Hintikka formulas as axioms as a realization of the normal form theorem in the Hilbert-type logic.

The idea of such an axiomatization for axiomatic rejection can be applied to an axiomatization for the set of all satisfiable formulas of classical propositional logic (and its extensions). This was pointed out in Inoué [13].

Theorem 7.5 (Dichotomy theorem for the Gentzen-type \( L_1 \)) Every formula, which is not provable in the Gentzen-type \( L_1 \) (tableau method for \( L_1 \)), is axiomatically rejected in \( GAR \) (in the Gentzen-type axiomatic rejection for \( L_1 \)).

This is a Gentzen-type counterpart of Theorem 3.2 for the Hilbert-type \( L_1 \). The Dichotomy theorem is to be effect that every formula is either provable or axiomatically rejected providing us with a decision method for
the Gentzen-type $L_1$, while Theorem 3.2 itself is not yet enough to give a decision method for the Hilbert-type version of $L_1$. This fact is a very remarkable difference between the Hilbert- and Gentzen-type versions of $L_1$.

With a view to proving Dichotomy theorem, let us assume that $A$ is not provable in the Gentzen-type $L_1$. In view of Theorem 2.1, i.e. Fundamental theorem, there obtains by reducing $A$ a (finite) tableau, at least, one branch of which is ending with a Hintikka formula. $A$ is, thus, axiomatically rejected by Definition 7.1.

**Theorem 7.6 (Contradiction theorem for the Gentzen-type $L_1$)** It is not the case that for any formula $A$ (of $L_1$), $\vdash_T A$ and $\nabla_T A$ at the same time.

Here, $\nabla_T A$, it is remembered, signified that $A$ is axiomatically rejected in the Gentzen-type $L_1$.

For the Hilbert-type version of $L_1$, the Theorem was demonstrated as Theorem 6.4 in the preceding section. The proof there was not syntactical based upon Theorem 6.2.

Now, we are proceeding to a syntactical proof of Theorem 7.6, i.e. Contradiction theorem.

With this in view, let us assume, if possible, $\vdash_T A$ and $\nabla_T A$ simultaneously for a formula $A$.

In view of $\vdash_T A$, there obtains a closed (normal) tableau by reducing $A$. $\nabla_T A$, on the other hand, gives rise to a tableau, at least, one branch of which is ending with a Hintikka formula. This, however, is not possible on the basis of the theorem stated at the end of the sixth section, namely Theorem 6.4. According to the theorem, if a reduction is successful giving rise to a closed (normal) tableau, there is no possibility of any other reductions to fail in producing a closed (normal) tableau as far as they are under the proviso as stated in Theorem 2.1, namely Fundamental theorem. Thus, $\vdash_T A$ and $\nabla_T A$ are not compatible, and Contradiction theorem was proved.

The proof of the above cited theorem given in the preceding section, i.e. Theorem 6.4, it is remembered, was model-theoretic resorting to Theorem 6.2. A purely syntactical proof is also possible of this theorem, though laborious, and we have in mind the theorem syntactically proved with a view to making the Gentzen-type axiomatic rejection for $L_1$ purely syntactical.
8 Cut elimination theorem

In this section, we shall take up one of the highlight of this paper, namely the cut elimination theorem for \( L_1 \) to be proved on the basis of the Hilbert-and Gentzen-type axiomatic rejections for \( L_1 \).

As well-known, cut is a rule which is applied (in the Gentzen-Schütte-type formalism) is the following form (see Schütte [52]):

\[
\vdash T[F[A^+]] \land \vdash T[G[A^-]] \implies \vdash T[F[+]] \lor G[-],
\]

where \( A \) is called the cut formula of the cut-application.

**Theorem 8.1** (Cut elimination theorem) The cut rule 8.1 is a derived rule in the Gentzen-type version of \( L_1 \).

In other word, cut is a rule to be dispensed with.

Here, we shall prove cut elimination theorem model-theoretically.

With this in view, let us assume that \( \vdash T[F[A^+]] \) and \( \vdash T[G[A^-]] \). By Translation theorem (i.e. Theorem 7.3), they, respectively, give rise to \( \vdash T[F[+]] \land A \) and \( \vdash T[G[-]] \land \neg A \), from which we obtain,

\[
\vdash T (F[+]) \lor G[-]) \land A \quad \text{and} \quad \vdash T (F[+]) \lor G[-]) \lor \neg A
\]

by Thinning and Interchange theorems (i.e. Theorems 7.1 and 7.2). These two formulas, then, give rise to:

\[
\vdash T (F[+]) \lor G[-]) \lor (A \land \neg A)
\]

(i.e. \( \vdash T (F[+]) \lor G[-]) \lor (TA \land \neg TA) \)),

in view of its reduction by \( \lor_\neg \). By Theorem 6.2,

\[
T(F[+]) \lor G[-]) \lor (A \land \neg A)
\]

is valid in first-order predicate logic with equality. Since \( TA \land \neg TA \) is contradictory, we have \( T(F[+]) \lor G[-]) \) is valid in the logic. Again by Theorem 6.2, we have the looked-for \( \vdash T F[+] \lor G[-] \).

This is a semantical proof of the Cut elimination theorem for \( L_1 \). From the standpoint of Gentzen-type formalization, it is not so interesting. In the sequel, we will prove it purely syntactically under some assumption, making use of axiomatic rejection. (In a traditional way, the cut elimination theorem
was proved of Leśniewski’ elementary) ontology by Takano [79], and that for $L_1$ is forthcoming therefrom.

As well-known, with respect to other Gentzen-type logics, both the Hilbert- and Gentzen-type versions of $L_1$ are proved to be equivalent by way of the Cut elimination theorem. The equivalence, it is remembered, was already demonstrated in Theorem 6.2 availing ourselves of the embedding of $L_1$ in first-order predicate logic with equality.

Nevertheless, the equivalence thus established is confined to the provability in both versions of $L_1$, and its counterpart for axiomatic rejection has not been proved yet. In what follows, we wish to demonstrate this theorem, namely:

**Theorem 8.2** For any formula $A$ of $L_1$, we have

$$\vdash_H A \iff \vdash_T A.$$

The Theorem is demonstrated availing ourselves of the equivalence of the Hilbert- and Gentzen-type versions of $L_1$.

With this in view, let us assume $\vdash_H A$, but not $\vdash_T A$. By Dichotomy theorem for the Gentzen-type $L_1$, we have $\vdash_T A$, which gives rise to $\vdash_H A$ by Theorem 2.2 or the equivalence of the Hilbert-type $L_1$.

Conversely, suppose $\vdash_T A$, but not $\vdash_H A$. From this it follows in turn $\vdash_T A$ by Theorem 6.2. But, this is in contradiction to $\vdash_T A$ by Contradiction theorem for the Gentzen-type $L_1$.

This proof is evidently model-theoretic. But, we can present a syntactical proof of $\iff$ of Theorem 8.2.

That is carried out by induction on the length of the branch which leads to a Hintikka formula starting from $A$.

The basis is taken care of by Theorem 3.1 to the effect that every Hintikka formula is axiomatically rejected in the Hilbert-type $L_1$.

Induction steps are dealt with by the following theses of the Hilbert-type $L_1$ and induction hypothesis:

$$\vdash_H G[A \lor B] \supset (G[A \lor B] \lor A) \land (G[A \lor B] \lor B),$$

$$\vdash_H G[eab] \supset G[eab] \lor \sim ea,$$

$$\vdash_H G[eab, ebc] \supset G[eab, ebc] \lor \sim ec,$$

$$\vdash_H G[eab, ebb] \supset G[eab, ebb] \lor \sim eb,$$

$$\vdash_H G[A \lor B] \lor A \quad \text{or} \quad \vdash_H G[A \lor B] \lor B,$$

$$\vdash_H G[eab] \lor \sim ea.$$
\[-1_H \ G[\epsilon_{ab_-}, \epsilon_{bc_-}] \lor \sim \epsilon_{ac}, \]
\[-1_H \ G[\epsilon_{ab_-}, \epsilon_{bb_-}] \lor \sim \epsilon_{ba}. \]

Before concluding this section, we wish to present a novel syntactical proof hitherto unknown of the cut elimination theorem for the Gentzen-type $L_1$. (The proof is essentially due to the first author of the present paper.) Nevertheless, an additional postulate for the Hilbert-type $L_1$ is in order for demonstrating cut elimination theorem. It is:

8.4 No Hintikka formula of the form $A_1 \lor A_2 \lor \cdots \lor A_n$ ($n \geq 1$) is provable in the Hilbert-type version of $L_1$, where $A_i$ ($1 \leq i \leq n$) is an atomic formula or a negated atomic one.

The postulate appears to be intuitively plausible, since we can always construct a model for $L_1$, which falsified a given Hintikka formula. From 8.4 there forthcoming th consistency of $L_1$ in its Hilbert-type version. In fact, if the version is inconsistent, there obtains $\vdash_H \epsilon_{aa}$, which contradicts that not $\vdash_H \epsilon_{aa}$ in view of the postulate 8.4. Nevertheless, the consistency of $L_1$ does not give rise to the postulate 8.4. This is the situation different from what we have in the case of classical propositional logic where the consistency is equivalent to the analogue of 8.4. (For the details, refer to Inoué-Ishimoto [19].)

Contradiction theorem for the Hilbert-type $L_1$, Theorem 6.4 it is remembered, was proved in the sixth section by resorting to Theorem 6.2, which was model-theoretic.

A purely syntactic proof of the Contradiction theorem for the Hilbert-type version of $L_1$ is carried out by induction on the length of the axiomatic rejection of the given $A$, where the postulate 8.4 is playing an important role.

As easily seen, for proving Contradiction theorem it is sufficient to derive not $\vdash_H A$ from $\vdash_H A$.

The basis cases (3.11 and 3.12) are taken care of by 8.4. Indeed, $\vdash_H \epsilon_{aa}$ and $\vdash_H \epsilon_{aa}$ ($\vdash_H \sim \epsilon_{aa}$ and $\vdash_H \sim \epsilon_{aa}$) are not the case simultaneously in view of the postulate 8.4.

For taking care of induction steps, suppose $\vdash_H \epsilon_{aa}$ is obtained from $\vdash_H A \supset B$ and $\vdash_H B$ by the rule 3.13. Suppose, further, $\vdash_H A$, which gives rise to $\vdash_H B$ by detachment against induction hypothesis. This takes care of the rule 3.13 for axiomatic rejection.

With a view to dealing with the rule 3.14, suppose $\vdash_H B$ and $B$ is forthcoming from $A$ by a uniform substitution for some name variables occurring in $A$. Further, suppose, if possible, that $\vdash_H B$, which gives rise to $\vdash_H A$. 

39
against induction hypothesis.

Lastly, assume \( \not\vdash_H A \vee \alpha \alpha \) is obtained from \( \vdash_H A \) by means of the rule 3.15, \( A \vee \alpha \alpha \) is a Hintikka formula with the condition in 3.15, \( \not\vdash_H A \vee \alpha \alpha \) does not hold by the postulate 8.4.

This completes the syntactical proof of the Contradiction theorem for the Hilbert-type version of \( L_1 \) (under the postulate 8.4).

We are now in a position to demonstrate the Cut elimination theorem for the Gentzen-type \( L_1 \) on the basis of the Contradiction for the Hilbert-type \( L_1 \) just proved.

The demonstration of the Cut elimination theorem proceeds in the following way:

To start with, suppose \( \vdash_T F[A_+] \) and \( \vdash_T G[A_-] \), which, respectively, give rise to \( \vdash_H F[A_+] \) and \( \vdash_H G[A_-] \) by Theorem 2.2. They will be referred to as (*) and (**) below, respectively.

With a view to obtaining \( \vdash_T F[+] \vee G[-] \), let us assume, if possible, to the contrary, namely not \( \not\vdash_T F[+] \vee G[-] \), which in turn yields \( \not\vdash_T F[+] \vee G[-] \) by the Dichotomy theorem for Gentzen-type \( L_1 \), namely Theorem 7.5. In view of Definition 7.1, there obtains a Hintikka formula by reducing \( F[+] \vee G[-] \). By Theorem 3.1 and Lemma 3.5, we, then, have \( \not\vdash_H F[+] \vee G[-] \).

We, now, wish to derived \( \vdash_H F[+] \vee G[-] \) on the basis of (*) and (**), which is going on in the following way:

(1) \( \vdash_H F[A_+] \) (*)
(2) \( \vdash_H G[A_-] \) (**)
(3) \( \vdash_H F[+ \vee A] \) (1), Lemma 2.1,
(4) \( \vdash_H G[- \vee A] \) (2), Lemma 2.1,
(5) \( \vdash_H (F[+] \vee G[-]) \vee A \) (3), tautology,
(6) \( \vdash_H (F[+] \vee G[-]) \vee A \) (4), tautology,
(7) \( \vdash_H (F[+] \vee G[-]) \vee (A \wedge \sim A) \) (5), (6), tautology,
(8) \( \vdash_H F[+] \vee G[-] \vee (A \wedge \sim A) \) (7), (8), tautology,
(9) \( \vdash_H A \wedge \sim A \) (9), tautology,
(10) \( \vdash_H F[+] \vee G[-] \) (7), (8), (9), tautology.

The last formula (10), namely \( \vdash_H F[+] \vee G[-] \), however contradicts \( \not\vdash_H F[+] \vee G[-] \) as above obtained in view of the Contradiction theorem for the Hilbert-type \( L_1 \).

This completes the demonstration of Cut elimination theorem on the basis of the postulate 8.4.
Here, it is remarked in passing that Theorem 6.2, in virture of which we are allowed to obtain $\vdash_T A$ from $\vdash_H A$ is not employing Cut elimination theorem for its proof. Otherwise, it would be preposterous.

This kind of proof of Cut elimination theorem was first explored in Inoué-Ishimoto [19] for classical propositional logic and will be made use of in other syllogistic systems as shown in Inoué-Ishimoto [20].

By the Cut elimination theorem for $L_1$, we obtain, in a routine way, a syntactical proof of:

8.5 $\vdash_H A \implies \vdash_T A$,

which was model-theoretically proved in Theorem 6.2.

As a concluding remark of the present section, we shall show the following.

**Theorem 8.3** The following statements are equivalent:

1. The Cut elimination theorem for $L_1$ holds,
2. No Hintikka formula of the form $A_1 \lor A_2 \lor \cdots \lor A_n$ ($n \geq 1$) is provable in the Hilbert-type version of $L_1$, where $A_i$ ($1 \leq i \leq n$) is an atomic formula or a negated atomic one. (= the postulate 8.4),
3. $\vdash_H A \implies \vdash_T A$,
4. Contradiction and Dichotomy theorems for the Hilbert-type $L_1$ hold.

We have already demonstrated the proof of (2) $\Rightarrow$ (4) $\Rightarrow$ (1) above. And we mentioned (1) $\Rightarrow$ (3) in 8.5. We shall prove the implication (3) $\Rightarrow$ (2). The proof of it is not difficult to think of. Suppose

$$A_1 \lor A_2 \lor \cdots \lor A_n \quad (n \geq 1)$$

is a Hintikka formula, where $A_i$ ($1 \leq i \leq n$) is an atomic formula or a negated atomic one. Thus, not $\vdash_T A_1 \lor A_2 \lor \cdots \lor A_n$. By the contraposition of (3), the formula is not provable in the Hilbert-type $L_1$.

This completes the proof of Theorem 8.3. We wish to emphasize that the proof is purely syntactical.

9 Characterization theorem and axiomatic rejection with Hintikka formulas as axioms

The reader will find a similar equivalence as Theorem 8.3 in Inoué-Ishimoto [19] for classical propositional logic, on the basis of which the argument for
Theorem 8.3 was developed. Such equivalences would hold for a variety of logics, if (2) is appropriately changed for a given logic.

Inoué [15] obtained the following theorem.

**Theorem 9.1** (Theorem 1.1 in [15]) The following statements are equivalent:

1. No Hintikka formula is provable in the Hilbert-type \( \mathbf{L}_1 \),
2. The Hilbert-type \( \mathbf{L}_1 \) is \( \mathcal{L} \)-decidable with respect to \( \mathbf{HL}_1 \) (i.e. the set of all formula of \( \mathbf{L}_1 \) is the disjoint union of the set of all the theorem of the Hilbert-type \( \mathbf{L}_1 \) and that of all the theorem of \( \mathbf{HL}_1 \)),
3. The Cut elimination theorem for the Gentzen-type \( \mathbf{L}_1 \) holds,
4. For any formula \( A \) of \( \mathbf{L}_1 \), if \( A \) is provable in the Hilbert-type \( \mathbf{L}_1 \), then it is provable in the the Gentzen-type \( \mathbf{L}_1 \).

Theorem 9.1 was proved on the basis of the Hilbert-type axiomatic rejection \( \mathbf{HL}_1 \) for \( \mathbf{L}_1 \) which is defined with Hintikka formulas as axioms in [15] in the following.

Axioms:

9.1 For every Hintikka formula \( A \) of \( \mathbf{L}_1 \), \( \vdash \mathbf{HL}_1 A \),

Rule:

9.2 \( \vdash \mathbf{H} A \supset B, \vdash \mathbf{HL}_1 B \implies \vdash \mathbf{HL}_1 A \),

where \( \vdash \mathbf{HL}_1 A \) means that \( A \) is axiomatically rejected by \( \mathbf{HL}_1 \).

Combining with Theorem 8.3 with the above Theorem 9.1, we obtain our last principal result of this paper as follows.

**Theorem 9.2** (Characterization Theorem) The following statements are equivalent:

1. The Cut elimination theorem for the Gentzen-type \( \mathbf{L}_1 \) (tableau method) holds,
2. No Hintikka formula of the form \( A_1 \lor A_2 \lor \cdots \lor A_n \) \( (n \geq 1) \) is provable in the Hilbert-type version of \( \mathbf{L}_1 \), where \( A_i \) \( (1 \leq i \leq n) \) is an atomic formula or a negated atomic one,
3. For any formula \( A \) of \( \mathbf{L}_1 \), if \( A \) is provable in the Hilbert-type \( \mathbf{L}_1 \), then it is provable in the the Gentzen-type \( \mathbf{L}_1 \) (tableau method).

---

2Please let the first author of this paper give a correction of his paper. The original statement of Theorem 1.1 (i) in [15] is not correct. ‘\( \mathbf{HL}_1 \)’ of Theorem 1.1 (i) in [15] should be ‘the Hilbert-type \( \mathbf{L}_1 \)."
(4) Contradiction and Dichotomy theorems for the Hilbert-type $L_1$ hold.
(5) No Hintikka formula is provable in the Hilbert-type $L_1$,
(6) The Hilbert-type $L_1$ is $L$-decidable with respect to $HL_1$ (i.e. the set of all formula of the Hilbert-type $L_1$ is the disjoint union of the set of all the theorem of $L_1$ and that of all the theorem of $HL_1$).

From Theorem 9.2.(4), (6), Theorems 3.2 and 6.4, we have

**Theorem 9.3** For any formula $A$ of $L_1$,

$$
\vdash_{HL_1} A \iff \vdash_{HAR} A.
$$

Thus this means that the $HL_1$ has the same strength with $HAR$.

**Appendix by Arata Ishimoto, the second author**

Up to the last section, we have mainly been concerned with the technical matters of $L_1$, namely the propositional fragment of Leśniewski’s ontology ignoring its philosophical implications. But, Leśniewski’s ontology is a system intended to be a logic in the wider sense of the word, not a mere formalism as emphasized by Leśniewski himself. Roughly speaking, Leśniewski’s ontology is an ontology in the traditional sense of the word. This is to the effect that Leśniewski’s ontology is a science to inquire into the most general aspects of the entities existing in the world.

From such a point of view, we wish to scrutinize some philosophico-ontological problem underlying the logical techniques developed so far. More specifically, we are looking into the philosophico-ontological significance of $L_1$ to be seen under formalism. This, it is believed, is the very task of philosophical logic.

As well-known, Leśniewski’s ontology has traditionally been interpreted in the spirit a rather extreme nominalism, which has culminated in the so-called ‘reism’ as propounded by Kotarbiński. Reism is a philosophy which advocates an ontology that only material things are legitimate intitites in existence, and beyond them there is nothing. (For reism, refer to Wolenski [84] and also to Sinisi [57, p. 59] with respect to Leśniewski’s ontology.)

Nevertheless, we wish to oppose to such a nominalism another ontology diagonally different therefrom. To be more specific, a conceptula realism will

---

3In the original version of this paper, this appendix is the section 9.
be advanced here as an alternative ontology to underlie Leśniewski’s ontology or its propositional fragment $L_1$. The conceptual realism we are defending is a Platonist philosophy to the effect that only concepts or universals are in existence independently and they are remaining invariant through every interpretation. In reference to the $L_1$ we have developed in this paper, name variables are representing concepts, i.e. universals, and they are susceptible of a large number of different interpretations. This was shown in the concrete in the fifth and sixth sections with respect to the model construction for $L_1$. More specifically, Theorem 6.2 tells us that $A$ is a thesis of $L_1$ if and only if $TA$ is valid in first-order predicate logic with equality, where $T$ is a translation to transform a formula of $L_1$ into its correspondent in predicate logic. This is an embedding theorem of $L_1$ in first-order predicate logic. As shown in the fifth section, there are a variety of possibilities of defining models for $L_1$. Thus, for one and the same $A$, we have a large number of different interpretations, and every one of them makes $TA$ true. As indicated in the sixth section, $A$ is also made true in the models which are defined in terms of those for $TA$.

In defining a model for the given $A$, it is remarked, each name variable involved in $A$ is assigned a set of individuals (in the sense of predicate logic), and the set varies from one interpretation to another. Nevertheless, the name variable, on the basis of which we are defining sets of individuals, is transcending all these sets remaining the same concept or universal.

This is nothing but the conceptual realism we are defending availing ourselves of the technical apparatus as developed up to the preceding section. Put it the other way round, $L_1$ is a logic deprived of individuals, namely the entities belonging to the lowest type which are called out temporarily legitimate entities in the proposed ontology of conceptual realism. (For such a conceptual realism, refer also to Ishimoto [24].)

Acknowledgments. The first author of this paper, as the representative of us, would like to thank the late Professor V. A. Smirnov for inviting us to Institute of Philosophy, Russian Academy of Sciences in Moscow in order to present this work at the conference. This paper is an enlarged and refined version of the paper of it. For Professor Smirnov, see Karpenko [34], Bystrov [2] and Finn [4]. The first author would also like to thank the late Professor Emeritus Arata Ishimoto, my teacher, and Mr. Mitsunori Kobayashi, my research friend, for fruitful research and discussions.
References

[1] Blass, A. “A faithful modal interpretation of propositional ontology”, *Mathematica Japonica*, 40, 217–223 (1994).

[2] Bystrov, P., “In memory of Professor V. A. Smirnov (1931-1996)”, *Modern Logic*, 6, 198–200 (1996).

[3] Cirulis, Y., “Logic with inclusion”, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 21, 247–266 (1977) (in Russian). DOI https://doi.org/10.1002/malq.19750210132

[4] Finn, V. K., “Vladimir Alexandrovich Smirnov as a Founder of Research Schools in Logic and Methodology of Science in the USSR and Russia”, *Studia Logica*, 66, 205–213 (2000). DOI https://doi.org/10.1023/A:1005287928736

[5] Fitting, M., *Proof Methods for Modal and Intuitionistic Logics*, Dordrecht: D. Reidel, 1983.

[6] Gentzen, G., “Untersuchungen über das logische Schliessen”, *Mathematische Zeitschrift*, 39, 176–210 (1935). English translation in Szabo [76], pp. 68–131. DOI https://doi.org/10.1007/BF01201353

[7] Goranko, G., G. Pulcini and T. Skura, “Refutation Systems: An Overview and Some Applications to Philosophical Logics”, pp. 173–197 in F. Liu, H. Ono and J. Yu (eds.), *Knowledge, Proof and Dynamics*, Springer, Singapor: Springer, 2020. DOI https://doi.org/10.1007/978-981-15-2221-5

[8] Härtig, K., “Zur Axiomatisierung der Nicht-Identitäten des Aussagenkalkülus”, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 6, 240–247 (1960). DOI https://doi.org/10.1002/malq.19600061504

[9] Hintikka, K. J. J., “Form and content in quantification theory”, *Acta Philosophica Fennica*, 8, 7–55 (1955).

[10] Inoué, T., “On Ishimoto’s theorem in axiomatic rejection -the philosophy of unprovability-”, *Philosophy of Science*, 22, 77–93
(1989) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.22.77

[11] Inoué, T., “On rejected formulas -Hintikka formula and Ishimoto formula-“ (abstract), *The Journal of Symbolic Logic*, 56, 1129 (1991).

[12] Inoué, T., “Cut elimination theorem, tableau method, axiomatic rejections”, (abstract), *Abstracts of Papers Presented to the American Mathematical Society*, 14, 264 (1993).

[13] Inoué, T., “Some topological properties of some class of rejected formulas and satisfiable formulas”, (abstract), *Abstracts, vol.1: Logic, The Proceedings of the 9th International Congress of Logic, Methodology and Philosophy of Science* (Logic Colloquim ‘91), held in Upssala, Sweden (August 7-14, 1991), p. 128. *The Journal of Symbolic Logic*, 58, 760–761 (1993).

[14] Inoué, T., “Partial interpretation of Leśniewski’s epsilon in modal and intensional logics”, (abstract), *The Bulletin of Symbolic Logic*, 1, 95–96 (1995). (I decided not to publish the full paper of this abstract, because [1] has been published and the essence of it is contained in [16].) DOI http://dx.doi.org/10.2307/420948

[15] Inoué, T., “Hintikka formulas as axioms of refutation calculus, a case study”, *Bulletin of the Section of Logic*, 24, 105–114 (1995).

[16] Inoué, T., “Partial interpretations of Leśniewski’s epsilon in von Wright-type deontic logics and provability logics”, *Bulletin of the Section of Logic*, 24, 223–233 (1995).

[17] Inoué, T., “On Blass translation for Leśniewski’s propositional ontology and modal logics”, Forthcoming. (arXiv:2006.15421v2 [math.LO], 2020) To appear in Studia Logica.

[18] Inoué, T., “A sound interpretation of Leśniewski’s epsilon in modal logic KTB”, Forthcoming. (arXiv:2007.12006 [math.LO], 2020)

[19] Inoué, T., and A. Ishimoto, “Cut elimination theorem and Hilbert-and Gentzen-style axiomatic rejections”, (abstract), *Abstracts of Papers Presented to the American Mathematical Society*, 13, 499–500 (1992).
[20] Inoué, T. and A. Ishimoto, “The Brentano-type syllogistic with Leśniewski’s epsilon”. In preparation.

[21] Ishimoto, A., “A propositional fragment of Leśniewski’s ontology”, Studia Logica, 36, 285–299 (1977). DOI http://dx.doi.org/10.1007/BF02120666

[22] Ishimoto, A., “On the method of axiomatic rejection in classical propositional logic”, Philosophy of Science, 14, 45–60 (1981) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.14.45

[23] Ishimoto, A., “A Lesniewskian version of Montague grammar”, pp. 139–144 in J. Horecký (ed.) Colling 82, Amsterdam: North-Holland, 1982. DOI https://doi.org/10.3115/991813.991835

[24] Ishimoto, A., “An idealistic approach to situation semantics”, pp. 401–416 in M. Nagao (ed.) Language and Artificial intelligence, Amsterdam: North-Holland, 1986.

[25] Ishimoto, A., “The logical structure of natural language understanding -from the standpoint of Leśniewski’s ontology-”, Philosophy of Science, 21, 145–159 (1988) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.21.145

[26] Ishimoto, A. (ed.), The Logic of Natural Language and its Ontology, Tokyo: Taga Shyuppan, 1990.

[27] Ishimoto, A., “Axiomatic Rejection for Classical Propositional Logic”, pp. 257–270 in Philosophical Logic and Logical Philosophy, Essays in Honour of Vladimir A. Smirnov, edited by P. I. Bystrov and V. N. Sadosky, Springer-Science+Business Media, B.V., 1996. DOI https://doi.org/10.1007/978-94-015-8678-8

[28] Ishimoto, A., “Logicism revisited in the propositional fragment of Leśniewski’s ontology”, pp. 219–232 in Philosophy of Mathematics Today, (Episteme vol. 22), edited by E. Agazzi and G. Darvas, Kluwer Academic Publishers, 1997. DOI https://doi.org/10.1007/978-94-011-5690-52
[29] Ishimoto, A., N. Kanai and K. Kagiwada, “On the Gentzen-type Formulation of Aristotelian Syllogistic”, *Philosophy of Science*, 17, 117–132 (1984) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.17.117

[30] Ishimoto, A. and S. Shimidzu, “The structure of language understanding -dialogue between Montague and Lesniewskian grammarians-”, *Philosophy of Science*, 19, 61–74 (1986) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.19.61

[31] Iwanuś, B., “On Leśniewski’s elementary ontology”, *Studia Logica*, 31, 73–119 (1973). DOI http://dx.doi.org/10.1007/BF02120531 Reprinted in Srzednicki-Rickey [72], pp. 165–215.

[32] Kanai, N., “The propositional fragment of Leśniewski’s ontology and its simplified formulation”, *Philosophy of Science*, 21, 145–159 (1988) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.21.133

[33] Kanai, N., “A Gentzen-type formulation of the Aristotelian syllogistic and its completeness with respect to models of first-order predicate logic”, in Ishimoto[26], pp. 269–389. (in Japanese)

[34] Karpenko, A. S., “V. A. Smirnov (1931-1996): Work and Life”, *Studia Logica*, 66, 201–204 (2000). DOI https://doi.org/10.1023/A:1005238627828

[35] Kleene, S. C., “Permutability of inferences in Gentzen’s calculi LK and LJ”, *Memoires of the American Mathematical Society*, 10, 1–26 (1952).

[36] Kleene, S. C., *Introduction to Metamathematics*, Amsterdam: North-Holland, 1952.

[37] Kleene, S. C., *Mathematical Logic*, Reading: Addison-Wesley, 1967.

[38] Kobayashi, M. and A. Ishimoto, “A propositional fragment of Leśniewski’s ontology and its formulation by the tableau method”, *Studia Logica*, 41, 181–195 (1982). DOI https://doi.org/10.1007/BF00370344
[39] Künk, G., “Prologue-functors”, Journal of Philosophical logic, 3, 241–254 (1974). DOI https://doi.org/10.1007/BF00247225

[40] Künk, G., “The meaning of the quantifiers in the logic of Leśniewski”, Studia Logica, 36, 309–322 (1977). DOI https://doi.org/10.1007/BF02120668 Reprinted in E. Morscher, J. Czermak and P. Weingartner (eds.), Problems in Logic and Ontology, Graz: Akademische Druck-u. Verlangsanstalt, pp. 75–88, 1977.

[41] Künk, G. and J. T. Canty, “Substitutional quantification and Leśniewskian quantifiers”, Theoria, 36, 165–182 (1977). DOI https://doi.org/10.1111/j.1755-2567.1970.tb00418.x

[42] Lejewski, C., “On Leśniewski’s ontology”, Ratio, 1, 150–176 (1958).

[43] Łukasiewicz, J., Aristotle’s Syllogistic from the Standpoint of Modern Formal Logic, Oxford: Clarendon Press, 1951.

[44] Luschei, E. C., The Logical Systems of Leśniewski, Amsterdam, North-Holland, 1962.

[45] McCall, S., Polish Logic 1920-1939, Oxford, Clarendon Press, 1967.

[46] Miéville, D., Un développement des systèmes logiques de Stanislaw Leśniewski, Protothétique-Ontologie-Méréologie, Bern: Peter Lang, 1984.

[47] Prior, A. N., Formal Logic, (2nd. ed), Oxford: Clarendon Press, 1962.

[48] Prior, A. N., “Existence in Leśniewski and in Russell”, pp. 149–155, in J. N. Crossley and M. A. E. Dummett (eds.) Formal Systems and Recursive Functions, Amsterdam: North-Holland, 1965.

[49] Quine, W. V. O., Ontological Relativity and other Essays, New York: Columbia University Press, 1962.

[50] Rickey, V. F., “A survey of Leśniewski’s ontology”, Studia Logica, 36, 407–426 (1977). DOI https://doi.org/10.1007/BF02120674

[51] Rickey, V. F., “Interpretation of Leśniewski’s ontology”, Dialectica, 39, 189–192 (1985). DOI https://doi.org/10.1111/j.1746-8361.1985.tb01256.x
[52] Schütte, K., *Beweistheorie*, Springer-Verlag, 1960.

[53] Schütte, K., *Vollständige Systeme modaler und intuitionistischer Logik*, Springer-Verlag, 1968.

[54] Schütte, K., *Proof Theory*, Springer-Verlag, 1977.

[55] Shimidzu, S., “On Lesniewskian Logical Grammar”, *Philosophy of Science*, 22, 95–109 (1989) (Tokyo: Waseda University Press) (in Japanese). DOI https://doi.org/10.4216/jpssj.22.95

[56] Shimidzu, S., “The first-order predicate subsystem of Leśniewski’s ontology and its relation to logical grammar”, in Ishimoto[26], pp. 171–182. (in Japanese)

[57] Sinisi, V. F., “The development of ontology”, *Topoi*, 2, 53–61 (1983). DOI https://doi.org/10.1007/BF00139701

[58] Simons, P. M., “On understanding of Leśniewski”, *History and Philosophy of Logic*, 3, 165–191 (1982). DOI https://doi.org/10.1080/01445348208837038

[59] Simons, P. M., “Leśniewski’s logic and its relation to classical and free logics”, pp. 369–400 in G. Dorn and P. Weingartner (eds.), *Foundation of Logic and Linguistics*, New York: Plenum, 1985.

[60] Simons, P. M., “A semantic for ontology”, *Dialectica*, 39, 193–216 (1985). DOI https://doi.org/10.1111/j.1746-8361.1985.tb01257.x

[61] Simons, P. M., *Parts, A Study in Ontology*, Oxford: Clarendon Press, 1987.

[62] Skura, T., “On pure refutation formulations of sentential logics”, *Bulletin of the Section of Logic*, 19, 102–107 (1990).

[63] Slupecki, J., *Z badań nad syllogistyka Arystotelesa*, Wrocław: Travaux de la Société des Sciences et des Lettres de Wrocław, Serie B, no. 6, 1948.

[64] Slupecki, J., “On Aristotelian syllogistic”, *Studia Philosophica*, 4, 275–300 (1949–1950). (This is an English translation of Slupecki [63].)
[65] Słupecki, J., “S. Leśniewski’s calculus of names”, *Studia Logica*, 3, 7–71 (1955). DOI http://dx.doi.org/10.1007/BF02067245. Reprinted in Srzednicki-Rickey [72], pp. 59-122.

[66] Słupecki, J., “L-decidability and decidability”, *Bulletin of the Section of Logic*, 1, 38–43 (1972).

[67] Smirnov, V. A., “Embedding the elementary ontology of Stanisław Leśniewski into the monadic second-order calculus of predicates”, *Studia Logica* 42:197–207, 1983; ‘Correction’, *Studia Logica*, 45, 231 (1986). DOI https://doi.org/10.1007/BF01063840

[68] Smirnov, V. A., “Logical relations between theories”, *Synthese*, 66, 71–87 (1986). DOI https://doi.org/10.1007/BF00413580

[69] Smirnov, V. A., “Strict embedding of the elementary ontology into the monadic second-order calculus of predicates admitting the empty individual domain”, *Studia Logica*, 46, 1–15 (1987). DOI https://doi.org/10.1007/BF00396902

[70] Smullyan, R. M., *First-Order Logic*, Berlin: Springer-Verlag, 1968.

[71] Sobociński, B., “On the successive simplifications of the axiom-system of Prof. S. Leśniewski’s ontology”, pp. 188–200 in McCall [45].

[72] Srzednicki, J. T. J. and V. F. Rickey (eds.), *Leśniewski’s system*, The Hague: Martinus Nijhoff Publishers, 1984.

[73] Srzednicki, J. T. J. and V. Z. Stachniak (eds.), *S. Leśniewski’s Lecture Notes in Logic*, Dordrecht: Kluwer Academic Publishers, 1988.

[74] Surma, S. J., J. T. J. Srzednicki, D. I. Barnett and V. F. Rickey (eds.), *Stanisław Leśniewski’s Collected Works*, vol. I, II, Dordrecht: Kluwer Academic Publishers, 1992.

[75] Stachniak, Z., *Introduction to Model Theory for Leśniewski’s Ontology*, Wroclaw: Acta Universitatis Wratislaviensis No 586, Prace Filozoficzne XXXI, Logika 9, 1981.

[76] Szabo, M. E., *The Collected Papers of Gerhard Gentzen*, Amsterdam: North-Holland, 1969.
[77] Takano, M., “A semantical investigation into Leśniewski’s axiom of his ontology”, *Studia Logica*, 44, 71–77 (1985). DOI https://doi.org/10.1007/BF00370810

[78] Takano, M., “Embeddings between the elementary ontology with an atom and the monadic second-order predicate logic”, *Studia Logica*, 46, 248–253 (1987). DOI https://doi.org/10.1007/BF00372549

[79] Takano, M., “Syntactical proof of translation and separation theorems on subsystems of elementary ontology”, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 37, 129–138 (1991). DOI https://doi.org/10.1002/malq.1991037090

[80] Urbaniak, R., *Leśniewski’s Systems of Logic and Mereology; History and Re-evaluation*, PhD thesis, Department of Philosophy, University of Calgary, 2008.

[81] Urbaniak, R. *Leśniewski’s Systems of Logic and Foundations of Mathematics*, Cham: Springer, 2014.

[82] Woleński, J., *Logic and Philosophy in the Lvov-Warsaw School*, Dordrecht: Kluwer Academic Publishers, 1989.

[83] Woleński, J., “On comparison of theories by their contents”, *Studia Logica*, 48, 617–622 (1989). DOI https://doi.org/10.1007/BF00370211

[84] Woleński, J. (ed.), *Kotarbiński: Logic, Semantics and Ontology*, Dordrecht: Kluwer Academic Publishers, 1990.

[85] Woleński, J. and J. Zygmunt, “Jerzy Słupecki (1904–1984): life and works”, *Studia Logica*, 48, 401–411 (1989). DOI https://doi.org/10.1007/BF00370196

Takao Inoué
1. Meiji Pharmaceutical University
   Department of Medical Molecular Informatics
   Tokyo, Japan

2. Hosei University
   Graduate School of Science and Engineering
   Tokyo, Japan
3. Hosei University  
Faculty of Science and Engineering  
Department of Applied Informatics  
Tokyo, Japan  
ta-inoue@my-pharm.ac.jp  
takao.inoue.22@hosei.ac.jp  
takaoapple@gmail.com  

Arata Ishimoto  
Professor Emeritus of Tokyo Institute of Technology  
Tokyo, Japan  
Deceased  

Mitsunori Kobayashi  
Logician and Composer, Japan