K-MOTIVES AND KOSZUL DUALITY

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Abstract. We construct an ungraded version of Beilinson–Ginzburg–Soergel’s Koszul duality, inspired by Beilinson’s construction of rational motivic cohomology in terms of $K$-theory. For this, we introduce and study the category $DK(X)$ of constructible $K$-motives on varieties $X$ with an affine stratification. There is a natural and geometric functor from the category of mixed sheaves $D_{mix}(X)$ to $DK(X)$. We show that when $X$ is the flag variety, this functor is Koszul dual to the realisation functor from $D_{mix}(X^\vee)$ to $D(X^\vee)$, the constructible derived category.

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1. Introduction

Let $G \supset B$ be a split reductive group with a Borel subgroup and $X = G/B$ the flag variety. Denote the Langlands dual by $G^\vee \supset B^\vee$ and $X^\vee$. In this article we prove and make sense the following Theorem, providing an ungraded version of Beilinson–Ginzburg–Soergel’s Koszul duality.

Theorem. There is a commutative diagram of functors

$$
egin{array}{ccc}
D_{mix}(X) & \xrightarrow{\Kos} & D_{mix}(X^\vee) \\
\downarrow \iota & & \downarrow \iota^v \\
DK(X) & \xrightarrow{\Kos} & D(X^\vee).
\end{array}
$$

Let us explain the ingredients of this diagram. In [BG86], [Soe90] and [BGS96], Beilinson–Ginzburg–Soergel consider a category $D_{mix}(X)$ of mixed sheaves on $X$, which is a graded version of either the constructible derived category of sheaves $D(X^\vee) = D^b_{\text{c}}(X^\vee,\text{an}(\mathbb{C}),\mathbb{Q})$ or equivalently the derived BGG category $\mathcal{O}$ of representations of Lie($G^\vee(\mathbb{C})$). In particular, there is an autoequivalence (1) of...
$D_{\text{mix}}(X)$ called Tate twist, which behaves as a shift of grading functor, and a functor $v : D_{\text{mix}}(X^\vee) \to D(X^\vee)$ called Betti realisation, which behaves as a functor forgetting the grading. Most remarkably, Beilinson–Ginzburg–Soergel construct a triangulated equivalence called Koszul duality

$$\hat{Kos} : D_{\text{mix}}(X) \to D_{\text{mix}}(X^\vee)$$

mapping intersection complexes to projective perverse sheaves and intertwining the Tate twist $\mathbb{1}$ with the shift twist $\mathbb{1}[2]$.

The main idea of this article is to fill in the bottom left corner of the above diagram by taking a “$K$-theoretic” point of view. It is based on the simple observation that where Betti realisation “forgets” the shift by $\mathbb{1}$, passage to $K$-theory “forgets” the shift by $\mathbb{1}[2]$.

To make this observation more precise, we use Soergel–Wendt’s very satisfying construction of $D_{\text{mix}}(X)$ as a full subcategory of the category of Beilinson motives $\text{DM}_B(X/\mathbb{F}_p, \mathbb{Q})$, see [SW16]. We define the category $D_K(X)$ as a full subcategory of the category of $K$-motives $\text{DM}_K(X/\mathbb{F}_p, \mathbb{Q})$ analogously.

Following Cisinski–Dégile [CD12], who construct the categories of Beilinson motives and $K$-motives, there is a functor

$$\iota : \text{DM}_B(X/\mathbb{F}_p, \mathbb{Q}) \to \text{DM}_K(X/\mathbb{F}_p, \mathbb{Q})$$

compatible with all six operations, which we call Beilinson realization. The functor expresses Beilinson’s realization that rational motivic cohomology can be defined in terms of Adam’s eigenspaces of algebraic $K$-theory. There is a natural isomorphism, called Bott isomorphism, $\mathbb{Q} \cong \mathbb{Q}(1)[2]$ in $\text{DM}_K(X/\mathbb{F}_p, \mathbb{Q})$ and hence $\iota$ “forgets” the shift by $\mathbb{1}[2]$. The functor $\iota$ descends to a functor $\iota : D_{\text{mix}}(X) \to D_K(X)$.

The construction of the functor $\hat{Kos}$ is just a copy of Soergel’s construction of $\hat{Kos}$. Both $D_K(X)$ and $D(X^\vee)$ admit a combinatorial description in terms of the homotopy category of Soergel modules. The commutativity of the diagram is hence immediate.

We proceed as follows. In the second section we recall Cisinski–Dégile’s construction of Beilinson motives and $K$-motives. In the third section we consider the categories $D_{\text{mix}}(X)$ and $D_K(X)$ for general affinely stratified varieties. We study their weight structures and recall Soergel’s Erweiterungssatz. In the fourth and last section we return to the flag variety. We recall Soergel’s construction of the Koszul duality functor and prove the Theorem mentioned above. In the appendix we collect some useful facts about weight structures and $t$-structures.

Remark 1.1. (1) The case of modular coefficients is work in progress joint with Shane Kelly and building on [EK19].

(2) It would be desirable to have an equivariant (both in the sense of Borel and Bredon) version of $D_K(X)$. One should be able to construct the Borel equivariant version using Soergel–Virk–Wendt’s work [SVW18]. For the Bredon equivariant case one would need to generalize the work of Hoyois [Hoy17] on equivariant $K$-motives to varieties over finite fields. Then one could prove a motivic Springer correspondence involving the affine Hecke algebra, generalizing [Rid13, RR16, RR17] and [Eke18], and proving a derived version of Lusztig’s comparison between the graded affine Hecke algebra and the affine Hecke algebra.
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2. Motivic Sheaves après Cisinski–Dégilde

2.1. Construction of Motivic Categories. Let $S$ be a scheme. In [CD12, Chapter 13-15] Cisinski–Dégilde construct the motivic triangulated categories (see [CD12, Definition 2.4.45]) of $K$-motives and Beilinson motives over $S$ in the following fashion. First they consider the spectrum $KGL_{Q,S}$ representing rational homotopy invariant $K$-theory in the stable homotopy category $SH(S)$. They show that $KGL_{Q,S}$ can be represented by a homotopy cartesian commutative monoid. This allows them to consider the category of rational $K$-motives over $S$ $DM_K(S,Q) := KGL_{Q,S} \text{-mod}(S) := Ho(KGL_{Q,S} \text{-mod})$ as the homotopy category of modules over $KGL_{Q,S}$. They show that this system of categories for every scheme $S$ forms a motivic triangulated category. More or less by construction $DM_K(S,Q)$ “computes $K$-theory”, so for smooth $S$ there is a natural isomorphism

$$
\text{Hom}_{DM_K(S,Q)}(\mathbb{Q}, \mathbb{Q}(p)[q]) = K_{2p-q}(S) \otimes \mathbb{Q},
$$

where $K_n(S)$ denotes Quillen’s algebraic $K$-theory of $S$ and by $\mathbb{Q}$ we always denote the tensor unit in any monoidal $\mathbb{Q}$-linear category. Furthermore, there is an isomorphism, called Bott isomorphism, $\mathbb{Q} \cong \mathbb{Q}(1)[2]$ in $DM_K(S,Q)$.

Rational algebraic $K$-theory naturally decomposes in eigenspaces of Adam’s operations $K^n(S) \otimes \mathbb{Q} = \bigoplus_i K^{(i)}(S)$ which turn $K^*(S) \otimes \mathbb{Q}$ into a bigraded ring, see [Wei13, IV.5]. Following Beilinson, the groups $K^{(i)}(S)$ can be used as a definition of rational motivic cohomology.

As shown by Riou [Rio10], the spectrum $KGL_S$ also admits an Adam’s decomposition, compatible with the monoid structure,

$$
KGL_{Q,S} = \bigoplus_i KGL^{(i)}_S.
$$

Denoting $H_{B,S} := KGL^{(0)}_S$, Cisinski–Dégilde define the category of Beilinson motives over $S$ $DM_B(S,Q) := Ho(H_{B,S} \text{-mod})$ as the homotopy category of modules over $H_{B,S}$. They show that also $DM_B(S,Q)$ forms a motivic triangulated category. Again by construction, $DM_B(S,Q)$ “computes motivic cohomology”, so for regular $S$ there is a natural isomorphism

$$
\text{Hom}_{DM_B(S,Q)}(\mathbb{Q}, \mathbb{Q}(p)[q]) \cong K^{(p)}_{2p-q}(S).
$$

We observe, see [Blo86] and [Lev94], that the latter group can be identified with Bloch’s higher Chow groups $K^{(p)}_{2p-q}(S) \cong CH^q(S, \mathbb{Q}(p))$, which in the very particular cases we will consider just boil down to Borel–Moore homology of the according complex manifold.
2.2. The Functor $\iota$. For all $i$, the graded pieces $KGL_{Q,S}^{(i)}$ are naturally isomorphic to $H_{B,S}$. In fact, Cisinski–Dégilde show in [CD12, Corollary 14.2.17] that there is an isomorphism of monoids

$$KGL_{Q,S} \cong H_{B,S}[t, t^{-1}] := \bigoplus_{i \in \mathbb{Z}} H_{B,S}(i)[2i].$$

We can hence define the functor, which we call Beilinson realization functor,

$$\iota : DM_{B}(S, \mathbb{Q}) \to DM_{K}(S, \mathbb{Q}), \iota(F) = KGL_{Q,S} \otimes H_{B,S} F = \bigoplus_{i \in \mathbb{Z}} F(i)[2i]$$

and show

**Theorem 2.1.**

1. The functor $\iota$ is a functor of premotivic categories, that is, compatible with $\otimes$ and $f^{*}$, $f_{*}$, for arbitrary $f : S \to T$ as well as $f^{\#}$ for $f : S \to T$ smooth.
2. The functor $\iota$ is left adjoint to the forgetful functor, say $F$, such that their composition yields

$$F(\iota(M)) = \bigoplus_{i \in \mathbb{Z}} M(i)[2i], \text{ for all } M \in DM_{B}(S, \mathbb{Q}).$$

3. If we restrict $DM_{B}(S, \mathbb{Q})$ and $DM_{K}(S, \mathbb{Q})$ to schemes $S$ over some fixed field $k$, the functor $\iota$ is moreover compatible with the operations $f_{!}, f_{*}$ and $\mathcal{H}om$.

**Proof.** See also [BL16, Proposition 4.1.1] for a similar statements.

(1) and (2) follow from [CD12, Proposition 7.2.13] using that $KGL_{Q,S}$ is a monoid in $H_{B,S}$-mod.

(3) To check that $\iota$ also commutes with $f_{!}, f_{*}$ and $\mathcal{H}om$ we will apply [CD12, Theorem 4.4.25], which needs several assumptions on $DM_{B}(S, \mathbb{Q})$ and $DM_{K}(S, \mathbb{Q})$, namely that

1. $DM_{B}(S, \mathbb{Q})$ is dualizable with respect to Tate twists, $\mathbb{Q}$-linear and separated;
2. $DM_{K}(S, \mathbb{Q})$ is $\mathbb{Q}$-linear and separated;
3. the object $\mathbb{Q}(i)$ is rigid in $DM_{B}(S, \mathbb{Q})$ for any $i \in \mathbb{Z}$.

The categories $DM_{B}(S, \mathbb{Q})$ and $DM_{K}(S, \mathbb{Q})$ are $\mathbb{Q}$-linear by construction. The separatedness (conservativity of $f^{*}$ for surjective morphisms $f : S \to T$) of $DM_{B}(S, \mathbb{Q})$ is proven in [CD12, Theorem 14.3.3]. The proof relies on the trace-formula for $KGL$ shown in [CD12, Proposition 13.7.6] and hence translates word for word to $DM_{K}(S, \mathbb{Q})$. That $DM_{B}(S, \mathbb{Q})$ is dualizable with respect to Tate twists is a consequence of absolute purity and hence follows from [CD12, Theorem 14.4.1].

We can think of $\iota$ as a “degrading” functor with respect to the shift of grading $(1)[2]$ and get the following functorial version of the Adam’s decomposition in $K$-theory.

**Corollary 2.2.** Let $M, N \in DM_{B}(S, \mathbb{Q})$. There is a natural isomorphism

$$\text{Hom}_{DM_{K}(S, \mathbb{Q})}(\iota(M), \iota(N)) \cong \text{Hom}_{DM_{B}(S, \mathbb{Q})}(M, F(\iota(N)))$$

$$\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{DM_{B}(S, \mathbb{Q})}(M, N(i)[2i]).$$
2.3. Tate Motives over Affine Spaces of Finite Fields. Let $k = \mathbb{F}_q$ be a finite field. Since the rational higher $K$-theory of finite fields vanishes, the categories of Tate motives over $A^n_k$ become semisimple. Denote by
\[
  \text{DMT}_B(A^n_k, \mathbb{Q}) = \langle \mathbb{Q}(n) | n \in \mathbb{Z} \rangle \subset \text{DM}_B(A^n_k, \mathbb{Q})
\]
\[
  \text{DMT}_K(A^n_k, \mathbb{Q}) = \langle \mathbb{Q} \rangle \subset \text{DM}_B(A^n_k, \mathbb{Q})
\]
the full triangulated subcategories generated by Tate objects (observe that $\mathbb{Q}(n) \cong \mathbb{Q}[-2n]$ in $\text{DM}_K$ and is hence not needed as a generator). Since there are non-trivial homomorphisms between the generators and their shifts, one can easily show

**Theorem 2.3.** There are equivalences of triangulated monoidal categories
\[
  \text{DMT}_B(A^n_k, \mathbb{Q}) \cong \text{Der}^b(\mathbb{Q}\text{-mod}^2)
\]
and
\[
  \text{DMT}_K(A^n_k, \mathbb{Q}) \cong \text{Der}^b(\mathbb{Q}\text{-mod})
\]
with the bounded derived categories of (graded) finite dimensional vector spaces over $\mathbb{Q}$. Here we let $\mathbb{Q}(p) \in \text{DM}_B(A^n_k, \mathbb{Q})$ correspond to $\mathbb{Q}$ sitting in grading degree $-p$ by convention.

This equips the categories $\text{DMT}_B(A^n_k, \mathbb{Q})$ and $\text{DMT}_K(A^n_k, \mathbb{Q})$ with canonical $t$-structures, see Remark A.3(1), which we denote by
\[
  (\text{DMT}_B(A^n_k, \mathbb{Q})^{t \leq 0}, \text{DMT}_B(A^n_k, \mathbb{Q})^{t \geq 0})
\]
and
\[
  (\text{DMT}_K(A^n_k, \mathbb{Q})^{t \leq 0}, \text{DMT}_K(A^n_k, \mathbb{Q})^{t \geq 0}).
\]

For arbitrary schemes $S$ of finite type over fields, the categories $\text{DM}_B(S, \mathbb{Q})$ and $\text{DM}_K(S, \mathbb{Q})$ are equipped with a weight structure $w$, see [Hoe11] and [BL16]. This weight structure descends to Tate motives and assigns the weight $2p-q$ to $\mathbb{Q}(p)[q]$. So for $\delta \in \{B, K\}$ we have
\[
  \text{DM}_\delta(A^n_k, \mathbb{Q})^{w \leq 0} = \langle \mathbb{Q}(p)[q] | 2p - q \leq 0 \rangle_{\oplus, \cong} \subset \text{DM}_\delta(A^n_k, \mathbb{Q})
\]
and
\[
  \text{DM}_\delta(A^n_k, \mathbb{Q})^{w \geq 0} = \langle \mathbb{Q}(p)[q] | 2p - q \geq 0 \rangle_{\oplus, \cong} \subset \text{DM}_\delta(A^n_k, \mathbb{Q}),
\]
where by $\oplus, \cong$ we denote closure under finite direct sums and isomorphisms.

We observe that the $t$-structure and weight structure on $\text{DMT}_K(A^n_k, \mathbb{Q})$ coincide! As explained in Remark A.3(1), $t$-structures and weight structures usually behave very differently. Our case just happens to be quite degenerate, since $\text{DMT}_K(A^n_k, \mathbb{Q})$ is semisimple.

We observe that the induced functor
\[
  \iota : \text{DMT}_B(A^n_k, \mathbb{Q}) \to \text{DMT}_K(A^n_k, \mathbb{Q})
\]
is not compatible with the $t$-structures, since $\iota(\mathbb{Q}(1)[2]) = \mathbb{Q}$. But $\iota$ is compatible with the weight structures.

**Proposition 2.4.** Let $M$ be in $\text{DMT}_K(A^n_k, \mathbb{Q})$, then
\[
  M \in \text{DMT}_B(A^n_k, \mathbb{Q})^{w \leq 0} \iff \iota(M) \in \text{DMT}_K(A^n_k, \mathbb{Q})^{w \leq 0} = \text{DMT}_K(A^n_k, \mathbb{Q})^{t \leq 0}
\]
and
\[
  M \in \text{DMT}_B(A^n_k, \mathbb{Q})^{w \geq 0} \iff \iota(M) \in \text{DMT}_K(A^n_k, \mathbb{Q})^{w \geq 0} = \text{DMT}_K(A^n_k, \mathbb{Q})^{t \geq 0}.
\]

As explained in [SW16] Section 3.4, the interplay of the $t$-structure and weight structure on $\text{DMT}_B(A^n_k, \mathbb{Q})$ can be seen as toy case of Koszul duality. So the preceding Proposition gives us a subtle first hint that $\iota$ should be related Koszul duality!
3. Motives On Affinely Stratified Varieties

3.1. Constructible Motives. Let $k = \mathbb{F}_q$ be a finite field. Let $X/k$ be a variety with a cell decomposition (also called affine stratification), that is, $X = \bigcup_{s \in \mathcal{S}} X_s$ where $\mathcal{S}$ is some finite set and each $i_s : X_s \to X$ is a locally closed subvariety isomorphic to $\mathbb{A}^n_k$ for some $n \geq 0$. In this situation, Soergel–Wendt [SW16] define

**Definition 3.1.** The category of mixed stratified Tate motives on $X$ is $\text{MTDer}_{\mathcal{S}}(X, \mathbb{Q}) = \{ M \in \text{DM}_B(X, \mathbb{Q}) \mid i_s^*M \in \text{DMT}_B(X_s, \mathbb{Q}) \text{ for all } s \in \mathcal{S} \}$ the full subcategory of the category of Beilinson motives $\text{DM}_B(X, \mathbb{Q})$ of objects which restrict to Tate motives on the strata.

For this category to be well-behaved, so for example closed under Verdier duality, Soergel–Wendt impose the following technical condition on the stratification.

**Definition 3.2.** The stratification $\mathcal{S}$ on $X$ is called Whitney–Tate if $i_t^*i_s^*\mathbb{Q} \in \text{DMT}_B(X_t, \mathbb{Q})$ for all $s, t \in \mathcal{S}$.

We will abbreviate $D_{\text{mix}}(X) = \text{MTDer}_{\mathcal{S}}(X, \mathbb{Q})$ and speak of constructible mixed motives, and assume that $\mathcal{S}$ is Whitney–Tate from now on.

We can now copy their definition in the context of $K$-motives.

**Definition 3.3.** The category of constructible $K$-motives is $D_{K}(X) = \{ M \in \text{DM}_K(X, \mathbb{Q}) \mid i_s^*M \in \text{DMT}_K(X_s, \mathbb{Q}) \text{ for all } s \in \mathcal{S} \}$ the full subcategory of the category of $K$-motives $\text{DM}_K(X, \mathbb{Q})$ of objects which restrict to Tate motives on the strata.

Since the functor $\iota : \text{DM}_B(X, \mathbb{Q}) \to \text{DM}_K(X, \mathbb{Q})$ commutes with the six operations, we see that it descends to a functor $\iota : D_{\text{mix}}(X) \to D_{K}(X)$ and observe that the Whitney–Tate condition with respect to $\text{DM}_B(X, \mathbb{Q})$ implies the one for $\text{DM}_K(X, \mathbb{Q})$.

In order to be closed under the six functors, we need to restrict us to morphisms of varieties which are compatible with their affine stratification in the following sense.

**Definition 3.4.** Let $(X, \mathcal{S})$ and $(Y, \mathcal{S}')$ be varieties with affine stratifications. We call $f : X \to Y$ an affinely stratified map if

1. for all $s \in \mathcal{S}'$ the inverse image $f^{-1}(Y_s)$ is a union of strata;
2. for each $X_s$ mapping into $Y_{s'}$, the induced map $f : X_s \to Y_{s'}$ is a surjective linear map.

3.2. Weight Structures. Following the discussion in Appendix A we can obtain $t$-structures and weight structures on $D_{\text{mix}}(X)$ and $D_{K}(X)$ by glueing inductively. Beilinson Motives and $K$-motives naturally come with a weight structure, see [Heb11]. We will still define the weight structures on $D_{\text{mix}}(X)$ and $D_{K}(X)$ manually, but want to note that they coincide with the natural weight structures.
Theorem 3.5. Setting

\[ D_{mix}(X)^{w\leq 0} = \{ M \in D_{mix}(X) \mid i_{s}^{*}M \in DMT_{B}(X_{s})^{w\leq 0} \text{ for all } s \in \mathcal{S} \}, \]
\[ D_{mix}(X)^{w\geq 0} = \{ M \in D_{mix}(X) \mid i_{s}^{*}M \in DMT_{B}(X_{s})^{w\geq 0} \text{ for all } s \in \mathcal{S} \}, \]
\[ DK(X)^{w\leq 0} = \{ M \in DK(X) \mid i_{s}^{*}M \in DMT_{K}(X_{s})^{w\leq 0} \text{ for all } s \in \mathcal{S} \} \]
\[ DK(X)^{w\geq 0} = \{ M \in DK(X) \mid i_{s}^{*}M \in DMT_{K}(X_{s})^{w\geq 0} \text{ for all } s \in \mathcal{S} \} \]
defines weight structures on \( D_{mix}(X) \) and \( DK(X) \).

Proof. Use Theorem 3.7 inductively. \qed

The weight structures on \( D_{mix}(X) \) and \( DK(X) \) are closely related.

Proposition 3.6. Let \( M \in D_{mix}(X) \). Then

\[ M \in D_{mix}(X)^{w\leq 0} \text{ if and only if } i(M) \in DK(X)^{w\leq 0} \text{ and } \]
\[ M \in D_{mix}(X)^{w\geq 0} \text{ if and only if } i(M) \in DK(X)^{w\geq 0}. \]

Proof. The compatibility of \( i \) with \( i_{s}^{*} \) and \( i_{s}^{!} \) shown in Theorem 3.7 reduces the statement to Proposition 3.4. \qed

We prove some exactness properties of the six functors.

Theorem 3.7. Let \( (X/\mathbb{F}_{p}, \mathcal{S}) \) and \( (Y/\mathbb{F}_{p}, \mathcal{S}') \) be varieties with a Whitney–Tate affine stratification and \( f : X \to Y \) be an affinely stratified map. Then we get

1. \( f^{*} \), \( f_{!} \) and \( \otimes \) are right \( w \)-exact.
2. \( f^{!} \), \( f_{*} \) are left \( w \)-exact.

Proof. See [EK19, Proposition 3.2]. \qed

3.3. Pointwise Purity and the Weight Complex Functor.

Definition 3.8. Let \( M \in D_{mix}(X) \) (resp. \( DK(X) \)). We say that \( M \) is pointwise pure if \( i_{s}^{*}M \) and \( i_{s}^{!}M \) are in \( DMT_{B}(X_{s}, \mathbb{Q})^{w=0} \) (resp. \( DMT_{K}(X_{s}, \mathbb{Q})^{w=0} \)) for all \( s \in \mathcal{S} \).

Proposition 3.9. \( M \in D_{mix}(X) \) is pointwise pure if and only if \( i(M) \in DK(X) \) is pointwise pure.

One can construct pointwise pure objects by different methods, for example:

1. Using affinely stratified resolutions of singularities of closures of strata in \( X \), see [EK19, Theorem 4.5].
2. Using contracting \( \mathbb{G}_{m} \) actions, see [SW16, Proposition 7.3].
3. By an inductive process in the case of flag varieties, see [SW16, Lemma 6.6].

All of those methods can be used to show that all objects in \( D_{mix}(X)^{w=0} \) and \( DK(X)^{w=0} \) are pointwise pure in the case of flag varieties.

Pointwise pure objects in \( D_{mix}(X) \) are under some assumptions in fact sums of (appropriately shifted and twisted) intersection complexes, that is, simple perverse motives. See [SW16, Corollary 11.11].

Pointwise pure objects are very special since they have no non-trivial extensions amongst each other.
Proposition 3.10. Let $M, N$ be in $D_{\text{mix}}(X)$ (resp. $DK(X)$) be pointwise pure. Then for all $n \neq 0$ we have
\[
\text{Hom}_{D_{\text{mix}}(X)}(M, N[n]) = 0 \quad \text{(resp. Hom}_{DK(X)}(M, N[n]) = 0).
\]

Proof. The statement for $D_{\text{mix}}(X)$ follows from the one of $DK(X)$ using $\iota$. For $DK(X)$, we observe that the pointwise purity implies that $M, N$ live in $DK(X)^{w=0} \cap DK(X)^{t=0}$, where by $DK(X)^{t=0}$ we denote the heart of the bottom $p = 0$ perverse $t$-structure on $DK(X)$. Hence the statement for negative $n$ follows from the axioms of the $t$-structure and the statement for positive $n$ from the axioms of the weight structure.

Pointwise purity allows us to consider the category $D_{\text{mix}}(X)$ and $DK(X)$ as homotopy categories of their weight zero objects.

Theorem 3.11. Assume that all objects in $DK(X)^{w=0}$ are pointwise pointwise pure. Then the weight complex functor (see Theorem A.8) induces an equivalences of categories,
\[
D_{\text{mix}}(X) \cong \text{Hot}^b(D_{\text{mix}}(X)^{w=0}) \\
DK(X) \cong \text{Hot}^b(DK(X)^{w=0})
\]
compatible with the functor $\iota$.

Proof. We prove the statement for $D_{\text{mix}}(X)$, the case of $DK(X)$ is done in the same way. First, $D_{\text{mix}}(X)$ is generated by $D_{\text{mix}}(X)^{w=0}$ as a triangulated category. Hence the essential image of the weight complex functor is $\text{Hot}^b(D_{\text{mix}}(X)^{w=0}) \subset \text{Hot}(D_{\text{mix}}(X)^{w=0})$. The pointwise purity assumption and Proposition 3.10 shows that there are no non-trivial extensions in $D_{\text{mix}}(X)$ between objects in $D_{\text{mix}}(X)^{w=0}$. Trivially, the same holds true in $\text{Hot}(D_{\text{mix}}(X)^{w=0})$. Since the weight complex functor restricts to the inclusion $D_{\text{mix}}(X)^{w=0} \rightarrow \text{Hot}(D_{\text{mix}}(X))$, an inductive argument (“dévissage”) shows that the functor is indeed fully faithful, where we again use that $D_{\text{mix}}(X)$ is generated by $D_{\text{mix}}(X)^{w=0}$ as a triangulated category.

The compatibility with $\iota$ follows since $\iota$ is weight exact.

We note that there is a different way of proving the last theorem using a formalism called “tilting”, see [SW16] and [SVW18]. We prefer the weight complex functor, since it also exists without the pointwise purity assumption. The weight complex functor even exists for all Beilinson and $K$-motives, where the heart of the weight structure is the category Chow motives, see [Bon10].

3.4. Erweiterungssatz. The Erweiterungssatz as first stated in [Soe90] and re-proven in a more general setting in [Gin91] allows a combinatorial description of pointwise pure weight zero sheaves on $X$ in terms of certain modules over the cohomology ring of $X$. In the case of $X$ being the flag variety, these modules are called Soergel modules. In [SW16] a motivic version is considered, which easily extends to $K$-motives.

Definition 3.12. We denote by
\[
\mathbb{H} : D_{\text{mix}}(X) \rightarrow \mathbb{H}(X) - \text{mod}^{\mathbb{Z}} \quad M \mapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_{\text{mix}}(X)}(\mathbb{Q}, M(n)[2n])
\]
\[
\mathbb{K} : DK(X) \rightarrow \mathbb{K}(X) - \text{mod} \quad M \mapsto \text{Hom}_{DK(X)}(\mathbb{Q}, M)
\]
the hypercohomology functors. Here \( H(X) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_{\text{mix}}(X)}(\mathbb{Q}, \mathbb{Q}(n)[2n]) \), and \( K(X) = \text{Hom}_{D_K(X)}(\mathbb{Q}, \mathbb{Q}) \), and the former is interpreted as a graded ring.

The rings \( H(X) \) and \( K(X) \) are nothing else than the motivic cohomology and \( K \)-theory of \( X \) and we collect some of the important properties.

**Theorem 3.13.** (1) The map \( \iota : H(X) \to K(X) \) induced by \( \iota : D_{\text{mix}}(X) \to D_K(X) \) is an isomorphism. (2) The following diagram commutes up to natural transformation

\[
\begin{array}{ccc}
D_{\text{mix}}(X) & \xrightarrow{\iota} & H(X) - \text{mod}^\mathbb{Z} \\
\downarrow & & \downarrow \\
D_K(X) & \xrightarrow{K} & K(X) - \text{mod} \\
\end{array}
\]

where the right vertical arrow is forgetting the grading.

(3) The ring \( H(X) \) is the Chow ring of \( X \). Assume that \( X \) and the stratification is already defined over \( \mathbb{Z} \). Then \( H(X) \) coincides with the Borel–Moore singular homology of \( X^{an}(\mathbb{C}) \).

(4) Assume that \( X \) is smooth, then \( K(X) = K_0(X) \) is 0-th \( K \)-group of \( X \), that is, the Grothendieck group of the category of vector bundles on \( X \).

**Proof.** (1) follows from Corollary 2.2. (2) is clear. For (3) and (4) we refer to the discussion in Section 2.1. \( \square \)

We remark that motivic cohomology is bigraded (higher Chow groups) and \( K \)-theory graded (higher \( K \)-groups). In our particular setup (affine stratification, finite field base, rational coefficients) all the higher groups vanish. We hence see one grading less.

Under a certain technical assumption the functors \( H \) and \( K \) are fully faithful on pointwise pure objects.

**Theorem 3.14** (Erweiterungssatz). Assume that all objects in \( D_{\text{mix}}(X)^{w=0} \) are pointwise pure and for each stratum \( i : X_s \to X \) and \( M \in D_{\text{mix}}(X)^{w=0} \) the map \( H(M) \to H(i_*i^* M) \) is surjective and the map \( H(i_! i^! M) \to H(M) \) is injective. Then the functors

\[
\begin{align*}
H : D_{\text{mix}}(X)^{w=0} & \to H(X) - \text{mod}^\mathbb{Z} \\
K : D_K(X)^{w=0} & \to K(X) - \text{mod}
\end{align*}
\]

are fully faithful.

**Proof.** The statement for \( D_{\text{mix}}(X) \) is proven in [SW16, Section 8]. Then one for \( D_K(X)^{w=0} \) follows by applying \( \iota \) and Corollary 2.2. \( \square \)

The assumptions are fulfilled if there are contracting \( \mathbb{G}_m \) actions for the closure of strata, see [SW16] Proposition 8.8. The theorem in particular applies to flag varieties.

**Definition 3.15.** We denote the essential images of \( D_{\text{mix}}(X)^{w=0} \) and \( D_K(X)^{w=0} \) under \( H \) and \( K \) by \( H(X)-\text{Smod}^\mathbb{Z} \) and \( K(X)-\text{Smod}^\mathbb{Z} \) and call them Soergel modules.
Corollary 3.16. Under the assumptions of Theorem 3.14, there are equivalences of categories
\[ D_{\text{mix}}(X) \cong \text{Hot}^b(\mathbb{H}(X)^\text{-Sm}) \] and
\[ DK(X) \cong \text{Hot}^b(\mathbb{K}(X)^\text{-Sm}). \]

4. Flag Varieties and Koszul Duality

4.1. Flag Varieties. Let \( G \supset B \supset T \) be a split reductive group over \( \mathbb{F}_p \) with a Borel subgroup and maximal torus. Denote the Langlands dual by \( G^\vee \supset B^\vee \supset T^\vee \). Denote by \( X(T) = \text{Hom}(T, \mathbb{G}_m) \) the character lattice, by \( W = N_G(T)/T \subset S \) the Weyl group with the set of simple reflections corresponding to \( B \), and for \( w \in W \) by \( l(w) \in \mathbb{Z}_{\geq 0} \) the length of an element. The flag variety \( X = G/B \) has an affine stratification by its \( B \)-orbits, called the Bruhat stratification,

\[ X = \bigcup_{w \in W} X_w \]

where \( X_w = BwB/B \cong \mathbb{A}_k^{l(w)} \). By [SW16, Proposition 4.10] this stratification fulfills the Whitney–Tate condition. More generally, the partial flag varieties \( G/P \) for parabolics \( B \subset P \subset G \) with their stratification by \( B \)-orbits are Whitney–Tate. It hence make sense to consider the categories \( D_{\text{mix}}(X) \) and \( DK(X) \).

4.2. Translation Functors and Pointwise Purity. We recall the inductive construction of pointwise pure objects in \( D_{\text{mix}}(X) \), see [SW16, Section 6].

First of all, the object \( i_{e, !}\mathbb{Q} \) is pointwise pure, where \( e \in W \) denotes the identity. For a simple reflection \( s \in S \) we denote by \( P_s = B \cup BsB \) the minimal parabolic and the smooth proper morphism (in fact the map is a projective bundle)

\[ \pi_s : X \to G/P_s. \]

The functor \( \theta_s = \pi_s^* \pi_{s, *} \) is called translation functor. It clearly preserves pointwise pure objects. For an arbitrary \( w \in W \) with \( l = l(w) \) we choose a shortest expression \( w = s_1 \cdots s_l \). Then the object \( \theta_{s_1} \cdots \theta_{s_l} i_{e, !}\mathbb{Q} \) is called a Bott–Samelson motive. It is pointwise pure, has support \( \bigoplus X_w \) and a unique indecomposable direct summand, which we will denote by \( \mathcal{E}_w \), with support \( \bigoplus X_w \). In fact all pointwise objects are sums of shifts twits of the motives \( \mathcal{E}_w \) and the objects \( \mathcal{F}\mathcal{E}_w = \mathcal{E}_w[l(w)] \) are simple perverse motives (intersection complexes) by the decomposition theorem! Subsumed, we get

\[ D_{\text{mix}}(X)^{w=0} = \langle \theta_{s_1} \cdots \theta_{s_l} i_{e, !}\mathbb{Q}(n) [2n] | s_i \in S, l \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \rangle_{\mathbb{R}, \mathbb{Z}, \mathbb{C}} = \langle \mathcal{E}_w(n)[2n] | w \in W, n \in \mathbb{Z} \rangle_{\mathbb{R}, \mathbb{Z}, \mathbb{C}}. \]

where by \( \cong, \oplus, \in \) we denote closure under isomorphism, finite direct sum and direct summands. We see that all objects in \( D_{\text{mix}}(X)^{w=0} \) are pointwise pure.

We observe that exactly the same construction works for \( DK(X) \). We denote \( \mathcal{E}_w = i(\mathcal{E}_w) \) and obtain

\[ DK(X)^{w=0} = \langle \theta_{s_1} \cdots \theta_{s_l} i_{e, !}\mathbb{Q} | s_i \in S, l \in \mathbb{Z}_{\geq 0} \rangle_{\mathbb{R}, \mathbb{Z}, \mathbb{C}} = \langle \mathcal{E}_w | w \in W \rangle_{\mathbb{R}, \mathbb{Z}, \mathbb{C}}. \]
Furthermore, the weight complex functor induces equivalences of categories, see Theorem 3.11.

\[ D_{mix}(X) \cong \text{Hot}^b(D_{mix}(X)^{w=0}) \]

\[ DK(X) \cong \text{Hot}^b(DK(X)^{w=0}) \]

compatible with the functor \( \iota \) in the obvious way.

4.3. **Soergel Modules I.** The categories \( D_{mix}(X)^{w=0} \) and \( DK(X)^{w=0} \) can be described combinatorially in terms of Soergel modules, using the functors \( H \) and \( K \) and the Erweiterungssatz, see Section 3.4. We recall the explicit description of \( \mathbb{H}(X) \) and \( \mathbb{K}(X) \). Recall that \( X(T) = \text{Hom}(T, \mathbb{G}_m) \) denotes the character lattice. Then there are natural isomorphisms

\[ C = S(X(T) \otimes \mathbb{Q})/ S(X(T) \otimes \mathbb{Q})_+^W \cong \mathbb{H}(X) \cong \mathbb{K}(X) \]

where \( S(X(T) \otimes \mathbb{Q}) \) denotes the symmetric algebra, and \( S(X(T) \otimes \mathbb{Q})_+^W \) the ideal of invariants of positive degree under the action of \( W \). In [Soe90] it is shown that the Bott–Samelson motives

\[ \theta_{s_1} \cdots \theta_{s_l} \mathcal{Q} \]

are mapped to the Bott–Samelson modules

\[ C \otimes C_{s_1} \cdots \otimes C_{s_l} \mathcal{Q} \]

under \( \mathbb{H} \) and hence also under \( \mathbb{K} \). We denote \( D_w = \mathbb{H}(\mathcal{E}_w) = \mathbb{K}(\mathcal{F}_w) \). Then \( D_w \)

\[ \cong (D_w(n) \mid w \in W, n \in \mathbb{Z})_{\cong, \oplus} \]

\[ \cong C\text{-}\text{Smol} \]

of weight zero objects in \( D_{mix}(X) \) and the category of graded **Soergel modules.** Here we denote by \( \langle n \rangle \) the shift of grading in \( C\text{-}\text{smol} \). In the same way there is an equivalence

\[ DK(X)^{w=0} = \langle \mathcal{E}_w \mid w \in W \rangle_{\cong, \oplus} \]

\[ \cong \langle D_w \mid w \in W \rangle_{\cong, \oplus} \]

\[ = C\text{-}\text{smol} \]

between pointwise pure \( K \)-motives and ungraded **Soergel modules.** Both descriptions are compatible with the functor \( \iota \) in the obvious way.
4.4. Projective Perverse Sheaves. We describe the “Koszul dual” of the last sections. This is the “classical story”. First, there is a functor, called Betti realization functor,
\[ v : D_{\text{mix}}(X^\vee) \rightarrow D(X^\vee) \]
where \( D(X^\vee) = D_{(B)}^b(X^{\vee,an}(\mathbb{C}), \mathbb{Q}) \) is the constructible derived category of sheaves. The functor \( v \) is a degrading functor with respect to the Tate twist \((n)\). We have \( v(-(n)) \cong v(-) \) and for \( M,N \in D_{\text{mix}}(X^\vee) \) the functor \( v \) induces an isomorphism
\[ \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_{\text{mix}}(X^\vee)}(M,N(n)) = \text{Hom}_{D(X^\vee)}(v(M),v(N)). \]

The functor \( v \) is furthermore clearly exact for the perverse \( t \)-structures on \( D_{\text{mix}}(X^\vee) \) and \( D(X^\vee) \). We denote the categories of projective perverse sheaves by
\[ \text{Proj} D_{\text{mix}}(X^\vee)^t=0 \text{ and } \text{Proj} D(X^\vee)^t=0. \]

One can show, using Theorem [A.3] that there are equivalences of categories
\[ \text{Hot}^b(\text{Proj} D_{\text{mix}}(X^\vee)^t=0) \cong \text{Der}^b(D_{\text{mix}}(X^\vee)^t=0) \cong D_{\text{mix}}(X^\vee) \text{ and } \]
\[ \text{Hot}^b(\text{Proj} D(X^\vee)^t=0) \cong \text{Der}^b(D(X^\vee)^t=0) \cong D(X^\vee) \]
which are all compatible with \( v \). Denote by \( w_0 \in W \) the longest element. Let \( \mathcal{P}_w \in \text{Proj} D_{\text{mix}}(X^\vee)^t=0 \) be the projective cover of \( j_{w_0}! Q[l(w)] \). Then \( \mathcal{P}_w = v(\mathcal{P}_w) \in \text{Proj} D(X^\vee)^t=0 \) the projective cover of \( j_{w_0}! Q[l(w)] \in D(X)^t=0. \)

4.5. Soergel Modules II. Soergel shows in [Soe90] that categories of projective perverse objects \( \text{Proj} D_{\text{mix}}(X^\vee)^t=0 \) and \( \text{Proj} D(X^\vee)^t=0 \) can be described in terms of Soergel modules as well.

First, Soergel’s Endomorphismensatz states that there is an isomorphism of graded algebras
\[ C \cong \text{Hom}_{D(X^\vee)}(\mathcal{P}_{w_0}, \mathcal{P}_{w_0}) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_{\text{mix}}(X^\vee)}(\mathcal{P}_{w_0}, \mathcal{P}_{w_0}(n)). \]

In [Soe90] this statement is originally proven representation-theoretically for category \( O \). There is also a topological proof, due to Bezrukavnikov–Riche, see [BR18].

Then, Soergel’s Struktursatz shows that the functors
\[ \hat{\mathcal{V}} : D_{\text{mix}}(X^\vee) \rightarrow C \cdot \text{mod}^Z, M \mapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D_{\text{mix}}(X^\vee)}(\mathcal{P}_{w_0}, M(n)) \]
\[ \mathcal{V} : D(X) \rightarrow C \cdot \text{mod}, M \mapsto \text{Hom}_{D(X^\vee)}(\mathcal{P}_{w_0}, M) \]
are fully faithful on projective perverse objects. In fact there are isomorphisms
\[ \hat{\mathcal{V}}(\mathcal{P}_w) \cong \mathcal{V}(\mathcal{P}_w) \cong D_w. \]

Hence there are equivalences of categories
\[ \text{Proj} D_{\text{mix}}(X)^t=0 = C \cdot \text{Sm}^Z \text{ and } \]
\[ \text{Proj} D(X)^t=0 = C \cdot \text{Sm}. \]
4.6. **Koszul duality.** The existence of the following Koszul duality functor $\hat{\text{Kos}}$ for $D_{\text{mix}}(X)$ was first conjectured by Beilinson–Ginzburg in [BG86] and proven by Soergel in [Soe90] using the combinatorial descriptions in terms of Soergel modules from above. The very elegant formulation using motivic sheaves is due to Soergel–Wendt, [SW16]. The functor $\hat{\text{Kos}}$ can be constructed as the composition

$$
\hat{\text{Kos}} : D_{\text{mix}}(X) \cong \text{Hot}^b(\text{C-Smod}) \\
\cong \text{Hot}^b(\text{Proj}

\text{D}_{\text{mix}}(X^\vee)_{t=0}) \cong D_{\text{mix}}(X^\vee)
$$

Under this equivalence the intersection complex $\hat{\mathcal{E}}_w$ is sent to the projective perverse motive $\hat{\mathcal{P}}_w$. It also intertwines the grading shifts $(n)[2n]$ and $(n)$. For further properties we refer to [BGS96].

We can now consider the ungraded version of Koszul duality in exactly the same way, namely, we have equivalences

$$
\text{Kos} : DK(X) \cong \text{Hot}^b(DK(X)^{w=0}) \\
\cong \text{Hot}^b(\text{C-Smod}) \\
\cong \text{Hot}^b(\text{Proj}

\text{D}(X^\vee)_{t=0}) \cong D(X^\vee)
$$

Under this equivalence the $K$-motive $\mathcal{E}_w$ is sent to the projective perverse sheaf $\mathcal{P}_w$. The functor $\text{Kos}$ inherits all the nice properties of $\hat{\text{Kos}}$.

Combining everything, we hence obtain the quite satisfying commutative diagram

$$
\begin{array}{ccc}
D_{\text{mix}}(X) & \xrightarrow{\hat{\text{Kos}}} & D_{\text{mix}}(X^\vee) \\
\downarrow & & \downarrow \\
DK(X) & \xrightarrow{\text{Kos}} & D(X^\vee).
\end{array}
$$

**Appendix A. Weight Structures and t-Structures**

For the convenience of the reader, we briefly recall the definitions and gluing of $t$-structures and weight structures of triangulated categories.

**A.1. Definitions.**

**Definition A.1.** [BBDS82, Definition 1.3.1] Let $\mathcal{C}$ be a triangulated category. A $t$-structure $t$ on $\mathcal{C}$ is a pair $t = (\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of full subcategories of $\mathcal{C}$ such that with $\mathcal{C}^{t n} := \mathcal{C}^{\leq 0}[-n]$ and $\mathcal{C}^{t n} := \mathcal{C}^{\geq 0}[-n]$ the following conditions are satisfied:

1. $\mathcal{C}^{t 0} \subseteq \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{t 1} \subseteq \mathcal{C}^{\geq 0}$;
2. for all $X \in \mathcal{C}^{t 0}$ and $Y \in \mathcal{C}^{t 1}$, we have $\text{Hom}_\mathcal{C}(X, Y) = 0$;
3. for any $X \in \mathcal{C}$ there is a distinguished triangle

$$
A \rightarrow X \rightarrow B \rightarrow +1
$$

with $A \in \mathcal{C}^{t 0}$ and $B \in \mathcal{C}^{t 1}$.

The full subcategory $\mathcal{C}^{t 0} = \mathcal{C}^{\leq 0} \cap \mathcal{C}^{t 0}$ is called the heart of the $t$-structure.

**Definition A.2.** [Bon10, Definition 1.1.1] Let $\mathcal{C}$ be a triangulated category. A weight structure $t$ on $\mathcal{C}$ is a pair $t = (\mathcal{C}^{w \leq 0}, \mathcal{C}^{w \geq 0})$ of full subcategories of $\mathcal{C}$,
which are closed under direct summands, such that with $C^{\leq n} := C^{\leq 0}[-n]$ and $C^{\geq n} := C^{\geq 0}[-n]$ the following conditions are satisfied:

1. $C^{\leq 0} \subseteq C^{\leq 1}$ and $C^{\geq 1} \subseteq C^{\geq 0};$
2. for all $X \in C^{\geq 0}$ and $Y \in C^{\leq -1}$, we have $\text{Hom}_C(X, Y) = 0;$
3. for any $X \in C$ there is a distinguished triangle

$$A \rightarrow X \rightarrow B \rightarrow +1$$

with $A \in C^{\geq 1}$ and $B \in C^{\leq 0}.$

The full subcategory $C^{w=0} = C^{\leq 0} \cap C^{\geq 0}$ is called the heart of the weight structure.

**Remark A.3.** (1) The standard example of a t-structure is of course the derived category $\text{Der}(\mathcal{A})$ of an abelian category $\mathcal{A}$, where we set

$$\text{Der}(\mathcal{A})^{t \leq 0} = \{ X \in \text{Der}(\mathcal{A}) | \mathcal{H}^i X = 0 \text{ for all } i > 0 \} \text{ and } \text{Der}(\mathcal{A})^{t \geq 0} = \{ X \in \text{Der}(\mathcal{A}) | \mathcal{H}^i X = 0 \text{ for all } i < 0 \}.$$  

The standard example of a weight structure is the homotopy category of chain complexes $\text{Hot}(\mathcal{A})$ of an additive category $\mathcal{A}$, where we set

$$\text{Hot}(\mathcal{A})^{w \leq 0} = \langle X \in \text{Hot}(\mathcal{A}) | X^i = 0 \text{ for all } i > 0 \rangle,$$

$$\text{Hot}(\mathcal{A})^{w \geq 0} = \langle X \in \text{Hot}(\mathcal{A}) | X^i = 0 \text{ for all } i < 0 \rangle,$$

and by $\cong$ we denote closure under isomorphism. This already showcases an important distinction between t-structures and weight structures. While the heart of a t-structure is abelian, the heart of a weight structure is only additive in general, and behaves more like the subcategory of projectives or injectives in an abelian category.

(2) We use the cohomological convention for weight and t-structures. One can easily translate to the homological convention, by setting $C^{\leq 0} = C^{\geq 0}$ and $C^{\geq 0} = C^{\leq 0}.$

**Proposition A.4.** Let $\mathcal{C}$ be a triangulated category with a t-structure or weight structure. The categories $\mathcal{D} = \mathcal{C}^{t \leq 0}, \mathcal{C}^{t \geq 0}, \mathcal{C}^{w \leq 0}, \mathcal{C}^{w \geq 0}$ are extension stable. That is, for any distinguished triangle in $\mathcal{C}$

$$A \rightarrow B \rightarrow C \rightarrow +1$$

with $A, C \in \mathcal{D}$, also $B \in \mathcal{D}.$

We will use standard terminology for exactness of functors.

**Definition A.5.** Let $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two triangulated categories with t-structures (weight structures). We say that $F$ is left t-exact (or left w-exact) if $F(C^{t \leq 0}_1) \subseteq C^{t \leq 0}_2$ (or $F(C^{w \leq 0}_1) \subseteq C^{w \leq 0}_2$) and $F$ is right t-exact (or right w-exact) if $F(C^{t \geq 0}_1) \subseteq C^{t \geq 0}_2$ (or $F(C^{w \geq 0}_1) \subseteq C^{w \geq 0}_2$). We say that $F$ is t-exact (w-exact) if $F$ is both left and right t-exact (or w-exact).

**A.2. Glueing.** As explained in [BBDS2], t-structures can be glued together. In fact the axiomatic setup required to perform such a glueing also works for weight structures. But there is subtle and essential difference in the definition of the glueing of t-structures and weight structures, exchanging * and ! functors.

**Definition A.6.** [BBDS2] Section 1.4.3] We call sequence of triangulated functors and categories

$$\mathcal{C}_2 \xrightarrow{i_* = h_!} \mathcal{C} \xrightarrow{j^* = j^!} \mathcal{C}_U$$
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(1) The functor $i_* = i!$ admits triangulated left and right adjoints, denoted by $i^*$ and $i^!$.
(2) The functor $j^* = j!$ admits triangulated left and right adjoints, denoted by $j_!$ and $j_*$.
(3) One has $j^* i_* = 0$.
(4) For all $K \in \mathcal{C}$ the units and counits of the adjunctions can be completed to distinguished triangles

\[ j_! j^! K \to K \to i_* i^* K \to \quad \text{and} \quad \]
\[ i_! i^! K \to K \to j_* j^* K \to \]

(5) The functors $i_* = i!$, $j_!$ and $j^* = j!$ are fully faithful.

**Theorem A.7.** Assume that $\mathcal{C}_Z \xrightarrow{i_* \sim i!} \mathcal{C}_{U} \to \mathcal{C}_Z$ is a glueing datum.

(1) If $(\mathcal{C}_U^{\leq 0}, \mathcal{C}_U^{\geq 0})$ and $(\mathcal{C}_Z^{\leq 0}, \mathcal{C}_Z^{\geq 0})$ are $t$-structures on $\mathcal{C}_U$ and $\mathcal{C}_Z$, then

\[ \mathcal{C}_{t}^{\leq 0} := \{ X \in \mathcal{C} | j_! K \in \mathcal{C}_U^{\leq 0} \land i^* K \in \mathcal{C}_Z^{\leq 0} \} \]
\[ \mathcal{C}_{t}^{\geq 0} := \{ X \in \mathcal{C} | j_* K \in \mathcal{C}_U^{\geq 0} \land i^! K \in \mathcal{C}_Z^{\geq 0} \} \]

defines a $t$-structure on $\mathcal{C}$.

(2) If $(\mathcal{C}_U^{w\leq 0}, \mathcal{C}_U^{w\geq 0})$ and $(\mathcal{C}_Z^{w\leq 0}, \mathcal{C}_Z^{w\geq 0})$ are weight structures on $\mathcal{C}_U$ and $\mathcal{C}_Z$, then

\[ \mathcal{C}_{w}^{\leq 0} := \{ X \in \mathcal{C} | j^* K \in \mathcal{C}_U^{w\leq 0} \land i_! K \in \mathcal{C}_Z^{w\leq 0} \} \]
\[ \mathcal{C}_{w}^{\geq 0} := \{ X \in \mathcal{C} | j^! K \in \mathcal{C}_U^{w\geq 0} \land i_* K \in \mathcal{C}_Z^{w\geq 0} \} \]

defines a weight structure on $\mathcal{C}$.

**Proof.** The statement for $t$-structures is [BBDS2 Theorem 1.4.10]. The statement for weight structures is [Bon10 Theorem 8.2.3]. \(\square\)

**A.3. Weight complex and Realization functors.** It is often possible to realize a triangulated category with $t$-structure as the derived category of its heart. Similarly, one can often realize a triangulated category with a weight structure as the homotopy category of chain complexes of its heart. We recall some statements from the literature.

**Theorem A.8.** Let $\mathcal{C}$ be an “enhanced” triangulated category, meaning that either

(1) (Derivator) $\mathcal{C} = \mathbb{D}(pt)$, where $\mathbb{D}$ is a strong stable derivator.
(2) ($\infty$-category) $\mathcal{C} = \mathcal{H}o(\mathcal{C}')$, where $\mathcal{C}'$ is a stable $\infty$-category.
(3) (f-category) There is an f-category $\mathcal{D}F$ over $\mathcal{C}$.

Assume that $\mathcal{C}$ is equipped with a $t$-structure. Then there is a triangulated functor called realization functor

\[ \mathbb{D}er^b(\mathcal{C}_{\leq 0}) \to \mathcal{C} \]

restricting to the inclusion of the heart $\mathcal{C}_{\leq 0} \to \mathcal{C}$.

Assume that $\mathcal{C}$ is equipped with a weight structure. Then there is a triangulated functor called weight complex functor

\[ \mathcal{C} \to \mathcal{H}ot(\mathcal{C}_{w=0}) \]
restricting to the inclusion of the heart $\mathcal{C}^{w=0} \to \text{Hot}(\mathcal{C}^{w=0})$.

Proof. For the statement about $t$-structures, we refer to [Vir18] for derivators, [Lur17] for $\infty$-categories and [Bei87] for $f$-categories. For the statement about weight structures, we refer to [Bon10] for $f$-categories and [Sos17], [Aok19] for $\infty$-categories. In fact, the derivator assumption implies the $f$-category assumption by [Mod19]. □

There are different assumptions under which the above functors can be shown to be fully faithful. We refer to the references in the proof above. Furthermore, it can be shown that realization and weight complex functors are compatible with “enhanced” exact triangulated functors between “enhanced” triangulated categories. We note that the categories of motives and the six operations between them are all “enhanced”.

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