RENORMALON CHAINS CONTRIBUTIONS TO THE NON-SINGLET EVOLUTION KERNELS IN $|\varphi^3|^6$ AND QCD.

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Abstract

The contributions to non-singlet evolution kernels $P(z)$, for the DGLAP equation, and $V(x, y)$, for the Brodsky–Lepage equation, are calculated for certain classes of diagrams including the renormalon chains. Closed expressions are obtained for the sums of contributions associated with these diagram classes. Calculations are performed in the $|\varphi^3|^6$ model and QCD in the $\overline{\text{MS}}$ scheme. The contribution of one of the classes of diagrams dominates for a large number of flavours $N_f \gg 1$. For the latter case, a simple solution to the Brodsky–Lepage evolution equation is obtained.

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1 Introduction

Evolution equations play an important role in both inclusive and exclusive hard processes. They describe the dependence of parton distribution functions of DIS and of the parton wave functions on the renormalization parameter $\mu^2$. The main ingredients of these equations are the perturbatively calculable evolution kernels. The two-loop DGLAP evolution kernels $P(z)$ were obtained in, and a more complicated two-loop kernel $V(x, y)$ of the Brodsky–Lepage (BL) evolution equation for meson wave functions was calculated in. A three-loop calculation of these kernels looks like being a tremendous problem. Recently, the results of very complicated calculations of the first few elements of the three-loop anomalous dimension of composite operators in DIS $- \gamma_{(2)}(N = 2, 4, 6, 8)$ have been presented in. Lately, these results were applied to improve the QCD analysis of DIS experimental data. Despite this important technical and phenomenological progress one may feel insufficiency because of the absence of the whole kernel in this order or, even at least, some parts of it. Of course, the kernel admitting a physical probability interpretation is a more general and convenient object from the theoretical point of view.

A method of estimating anomalous dimensions of composite operators in the limit of a large number of flavours, $N_f$, has been suggested by J. Gracey. It is based on conformal properties of the theory at the critical point $g = g_c$ corresponding to the non-trivial zero $g_c$ of the $D$-dimensional $\beta$-function $\beta(g_c) = 0$. The generating function has been constructed to obtain the leading large-$N_f$ asymptotics of the anomalous dimension $\gamma_{(n)}(N)$ to any order $n$ of perturbation theory (PT).

In this letter, I propose a method of calculating certain classes of multiloop diagrams directly for the DGLAP kernel $P(z)$ as well as for the BL kernel $V(x, y)$ in the $\overline{\text{MS}}$ scheme rather than their moments as in. These diagrams contain insertion of chains of one-loop self-energy parts (renormalon chains) into the one-loop diagrams (see Fig.1 a,b) for kernels. The kernels $P(n)$ are obtained for any order $n$ of PT by analysing of these “dressed by chain” diagrams. Then, the kernel $P(z; g)$ ($V$) for diagrams with the totally dressed propagator (i.e., in the “all loops” approximation) is calculated; this kernel appears to

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be a generating function to obtain the partial kernels $P_{(n)}$. Moreover, this kernel $P(z; g)$ is the analytic function in variable of a coupling constant $g$. The insertions of the renormalon chains lead to converged PT series, and the usual renormalon singularities don’t appear in the renormgroup function calculations. This was found first for $\beta$- and $\gamma$-functions in [11], the following development of this approach is presented in [11]. It is to be stressed that the method of obtaining $P$ or $V$ does not depend on the nature of self-energy insertions and does not appeal to the value of parameters $N_f T_R$, $C_A/2$ or $C_F$ (for QCD case) associated with different loops. Another distinctive feature is that the PT-improved evolution kernels are calculated in the direct and standard way by using rather elementary methods. This is possible due to a simple algebraic structure of the counterterms of the diagrams considered. In this letter, we mainly present technical results on resummation of the above mentioned counterterms within the framework of the $[\varphi^3]_6$ model and, in part, in QCD. The solution to the BL evolution equation for a large number of flavors $N_f$ is also briefly discussed.

To develop the diagrammatic analysis of multiloop evolution kernels, let us first consider a toy model describing $\psi_i^s$, $\psi_i$, the charged “quark” fields, and $\varphi$, the “gluon” one, in 6 space-time dimensions based on the Lagrangian $L_{int} = g \sum_i (\bar{\psi}_i^s \psi_i \varphi)(6)$. The number of “quark” flavors, $N_f$, will be considered as an arbitrary free parameter associated with the “quark” loops. This theory has much in common with QCD: it is renormalizable and its $\beta$-function has a structure similar to that of $\beta_{QCD}$,

$$\beta(a) = -a^2 b_0 + \ldots, \quad b_0 = \left( \frac{2}{3} - \frac{N_f}{6} \right), \quad a = \frac{g^2}{(4\pi)^3}.$$ 

Moreover, the one-loop evolution kernels $P_0$ and $V_0$, following from the simplest triangular diagram in Figs.1a and 1b, turn out to be proportional to the corresponding expressions for the same diagrams in QCD

$$a P_0(z) = a(1 - z), \quad a V_0(x,y) = a \left( \theta(y > x) \frac{x}{y} + \theta(y > \bar{x}) \frac{\bar{x}}{\bar{y}} \right),$$

where $\bar{y} = 1 - y$, $\bar{x} = 1 - x, \ldots$. The similarity of the kernels in both the theories extends to the two-loop level [13]. The advantage of the $[\varphi^3]_6$ model is that the relevant Feynman integrals are simple, and it is much easier to study the structure of multiloop expressions.

In general, calculation of the evolution kernels $P(z; \alpha)$ [3] and $V(x, y; \alpha)$ [3] is quite analogous to that of the anomalous dimensions. The major modification is the change $(kn)^N \to \delta(z - kn)$ of the vertex factor corresponding to the composite operator, $k$ being the relevant momentum related to the incoming “quark” line, see Figs. 1a,b. In these Figs., $p$ ( $\bar{q} p$ or $p$) is the external momentum of the diagrams; $n$ is a light-like vector ($n^2 = 0$) introduced to pick out the symmetric traceless composite operator; $pn = 1$. Detailed examples of similar calculations can be found in [12].

## 2 First triangular diagrams for DGLAP evolution kernel

In this section, the method will be demonstrated by using well-understood [4, 14, 13] quark-loop insertions as an example. Let us consider the one-loop triangular diagram $\Gamma$ for kernels in Fig.1c with the chain of self-energy part insertions $g_i$ into the gluon line. Expressions for $P(z)$ or $V(x, y)$ in terms of the renormalization constant $Z_{\Gamma}$ for the diagram $\Gamma$ in the $\overline{\text{MS}}$ scheme (see [4] and [3]) are given by

$$Z_{\Gamma} = 1 - \hat{K} R'(\Gamma), \quad P = -a \partial_{\alpha} \left( Z_{\Gamma}^{(1)} \right) = a \partial_{\alpha} \left( \hat{K} R'(\Gamma) \right).$$

Here, $R'$ is the incomplete BPHZ $R$-operation; $\varepsilon = (6 - D)/2$ and $D$ is the space-time dimension; $\hat{K}$ picks out poles in $\varepsilon$, where $\hat{K}$ projects a simple pole, and $Z^{(1)}$ is the coefficient of the simple pole in
the expansion of $Z_\Gamma$. For $g_i$, being the one-loop insertions, expression (2) can be simplified following the definition of the $R'$-operation as

$$\hat{K}_1 R'(\Gamma) \equiv \hat{K}_1 R'(G \otimes \prod_i g_i) = \hat{K}_1 \left[ G \otimes \prod_i (1 - \hat{K}_1) g_i \right] \Rightarrow P_{(n)} = (n + 1) a \hat{K}_1 \left[ G \otimes \prod_i (1 - \hat{K}_1) g_i \right], \quad (3)$$

where $G$ denotes the “outer triangle” part of the diagram $\Gamma$ without the chain of bubbles, and

$$R g_i(k^2) = (1 - \hat{K}_1) g_i(k^2) = -\frac{a N f}{\varepsilon} \left( \gamma_\varphi(\varepsilon) \left( \frac{\mu^2}{k^2} \right)^\varepsilon - \gamma_\varphi(0) \right) \quad (4)$$

$$\gamma_\varphi(\varepsilon) = C(\varepsilon) B(2 - \varepsilon, 2 - \varepsilon). \quad (5)$$

Here $B(a, b)$ is the Euler B-function, $\gamma_\varphi = \gamma_\varphi(0)$ is the one-loop anomalous dimension of the gluon field (at $N_f = 1$), and the constant $C(\varepsilon) = \Gamma(1 - \varepsilon) \Gamma(1 + \varepsilon)$ reflects the concrete choice of $\overline{\text{MS}}$ scheme where every loop integral is multiplied by the scheme factor $\Gamma(D/2 - 1)(\mu^2/4\pi)^\varepsilon$. Note, the function $\gamma_\varphi(\varepsilon)$ in (3) will play an important role in our consideration. Substituting (3) into (4) and performing a direct calculation (see, e.g., [12]), one arrives at the expression

$$P_{(n)}^{(1)}(z) = (n + 1) a P_0(z)(-a N f \gamma_\varphi)^n \hat{K}_1 \left[ \frac{C(\varepsilon)(1 - \varepsilon)}{\varepsilon^{n+1}} z^{-\varepsilon} \sum_{j=0}^n \frac{\gamma_\varphi(\varepsilon)}{\gamma_\varphi(0)} \frac{\Gamma(1 + (j + 1) \varepsilon)}{\Gamma(1 + j \varepsilon) \Gamma(1 + \varepsilon)} \left( \frac{n}{j} \right) \right] \quad (6)$$

It is instructive to consider the properties of the first few kernels $P_{(n)}^{(1)}(z)$. Their expressions are presented in Table 1 for $k = 1, \ldots, 8$. To obtain them, the FORM 2.0 program [15] has been used. The lessons are: i) the perturbation theory for kernels $P_{(n)}^{(1)}(z)$ looks like being improved in comparison with the corresponding anomalous dimensions $\gamma_{(n)}(N)$. Indeed, the term in $P_{(n)}^{(1)}(z)$ leading in $z$ is $\sim \frac{\ln^n(z)}{n!}$, i.e., it is factorial suppressed in $n$, while for the corresponding contribution to $\gamma_{(n)}(N) \sim (n)^0$ is not suppressed; ii) one can see factorial suppression in $n$ of all other logarithmic terms in $P_{(n)}^{(1)}(z)$; iii) the $n$-bubble chain generates $\zeta(n)$ (the Riemann zeta-function) in the non-logarithmic term in $P_{(n)}^{(1)}(z)$; this may be traced back to the expansion of the Euler B-function as their possible origin. These properties give hint about a possible resummation of the $P_{(n)}^{(1)}(z)$-series.

Indeed, the closed expression for the sum of partial kernels $P^{(1)}(z; A) = a \sum_{n=0}^\infty P_{(n)}^{(1)}(z)$ exist,

$$P^{(1)}(z; A) = a \sum_{n=0}^\infty P_{(n)}^{(1)}(z) = a P_0(z) z^{-A} (1 - A) \left( \frac{\gamma_\varphi(0)}{\gamma_\varphi(A)} \right), \quad \text{where } A = a N f \gamma_\varphi \quad (7)$$

and the kernel $P^{(1)}(z; A)$ is the generating function for $P_{(n)}^{(1)}(z)$. The proof of this assertion is presented in Appendix A. This result possesses several remarkable properties:

- the $P^{(1)}(z; A)$ becomes the dominant part of the total perturbative kernel $P(z)$ when $N_f \gg 1$. Below, the result for the kernel $P^{(1)}(z; A)$ will be completed by taking into account similar corrections to the “quark leg” (see Fig.1d) whose contribution is proportional to $\delta(1 - z)$.

- the analytic properties of $P^{(1)}(z; A)$ in the variable $A$ are completely determined by the one-loop “anomalous dimension” $\gamma_\varphi(A)$ in $D$ dimensions (see Eqs.(3)). The singularities of $P^{(1)}(z; A)$ correspond to zeros of the function $B(2 - A, 2 - A)$ in $\gamma_\varphi(A)$. Of course, $\gamma_\varphi(A)$ provides the scheme dependence in $P^{(1)}(z; A)$ due to the factor $C$ in the definition (3).

\footnote{I am greatly indebted to Dr. L. Avdeev who provided me with his brilliant FORM-based program for expansion in $\varepsilon$, see [13]. Note that the contents of Table 1 is limited here only by place.}
the kernel \( P^{(1)}(z; A) \) is an analytic function in variable \( A \), except for singularities at the points \( A = 2 + k + 1/2, \ k = 0, 1, ..., \) where the function has simple poles. The nearest singularity appears at \( A = 5/2 \), i.e., at \( aN_f = 15 \), determining the range of convergence of the PT series.

In a similar way, one can derive an expression for the sum of diagrams in Fig. 1d connected with the anomalous dimension of the quark field \( \gamma^{(1)}_{\psi}(A) \) obtained in the “all quark-loop” approximation. Collecting the results of resummation in the main approximation in \( A \), which correspond to the diagrams in Figs. 1c and d, we arrive at the final expression

\[
P^{(1)}(z; A) - \delta(1 - z)\gamma^{(1)}_{\psi}(A) = a \left( \frac{\gamma_{\psi}(0)}{\gamma_{\psi}(A)} \right) \left[ \bar{z}z^{-A}(1 - A) - \delta(1 - z) \frac{(1 - A)(3 - A)(2 - A)}{(3 - A)(2 - A)} \right].
\]

Integration produces the following expression for the anomalous dimension \( \gamma(N, A) \) of the composite operator

\[
\gamma(N, A) = \int_{0}^{1} z^N \left( P^{(1)}(z; A) - \delta(1 - z)\gamma_{\psi}(A) \right) dz = a\gamma_{\psi} \frac{\Gamma(4 - 2A)}{\Gamma(2 - A)\Gamma(1 - A)^2\Gamma(1 + A)} \left[ \frac{\Gamma(N + 1 - A)}{\Gamma(N + 3 - A)} - \frac{\Gamma(2 - A)}{\Gamma(4 - A)} \right].
\]

Formula (8) can be obtained by another method applied in [8] to the QCD case (see Eq. [27] below). Note that the anomalous dimension \( \gamma(1, A) \) corresponds to parton momentum. The equality \( \gamma(1, A) = 0 \), following from the r.h.s. of (8), is expected; it signals that the parton momentum is conserved.

3 Other triangular diagrams contributions

Consider now the class of diagrams in Fig. 2. There are closed expressions for sums of partial kernels

\[
P^{(2a)}(z; B) = a \sum_{m=0}^{\infty} P^{(2)}_{(m)}(z) = aP_{0}(z) \left( \bar{z}z^{-B} \frac{\gamma_{\psi}(0)}{\gamma_{\psi}(B)} \right), \text{ diag. in Fig.2a,}
\]

\[
\tilde{P}^{(2a)}(z; B) = a \sum_{m=0}^{\infty} (2 - \delta_{0,n}) P^{(2)}_{(m)}(z) = aP_{0}(z) \left( \frac{(2 - \bar{z}z^{-B} \frac{\gamma_{\psi}(0)}{\gamma_{\psi}(B)} - 1}{2} \right), \text{ diag. in Fig.2a + MC,}
\]

\[
P^{(2b)}(z; B) = a \sum_{m=0}^{\infty} (m + 1) P^{(2)}_{(m)}(z) = aP_{0}(z) \left( 1 + B d dB \right) \left( \frac{(2 - \bar{z}z^{-B} \frac{\gamma_{\psi}(0)}{\gamma_{\psi}(B)} \right), \text{ diag. in Fig.2b,}
\]

\[
\delta(1 - z)\gamma^{(2)}_{\psi} = a\delta(1 - z) \left( \frac{\gamma_{\psi}(0)}{\gamma_{\psi}(A)} \right) \frac{(1 - A)(3 - A)(2 - A)}{(3 - A)(2 - A)}, \text{ diagr. in Fig.2c ;}
\]

the functions \( P(z; B) \) appear as generating functions for the corresponding partial kernels. Here \( P^{(2)}_{(m)}(z) \) is the partial kernel with \( m \) insertions into one of the quark lines (see Fig.2a); \( \gamma_{\psi}(\bar{z}) \) is the one-loop anomalous dimension of the quark field in \( D \)-dimension; for this model \( \gamma_{\psi}(\bar{z}) = \gamma_{\psi}(z) \); \( B = a\gamma_{\psi} \); MC denotes a mirror-conjugate diagram. The first Eq. (10) corresponds to the diagrams in Fig. 2a where the chain of quark self-energy parts is substituted only into the left quark line of the triangle. To prove it, one has to repeat the way similar to proof in Appendix A. Equation (11) corresponds to the sum of diagrams in Fig.2a and its MC diagrams. This result will be used to restore the corresponding kernel \( V(x, y) \) in the next section. The analytic properties of the functions \( P^{(2a)}(z; B) \) and \( \tilde{P}^{(2a)}(z; B) \) in the parameter \( B \) are the same as for the kernel \( P^{(1)}(z; B) \); they are determined by the function \( \gamma_{\psi}(B) \). Equation (12) corresponds to substitutions of the chains into the both quark lines of the triangle in Fig.2b. At least, Eq. (13) corresponds to contributions to the anomalous dimension of the quark field from the diagram in Fig.2c.
The contribution $P^{(2)}(z; B)$ will be suppressed in the parameter $B$ in comparison with $P^{(1)}(z; A)$, if $N_f \gg 1$, i.e., $A \gg B$. Note, however, that the $N$-moments of the kernels $P^{(2a,b)} - \gamma^{(2)}(N, B)$ decrease in $N$ slower than $\gamma^{(1)}(N, A)$ corresponding to the kernel $P^{(1)}$. Therefore, at sufficiently large $N$ $\gamma^{(2)}(N, B) > \gamma^{(1)}(N, A)$ for any fixed parameters $A$ and $B$.

The expression for the kernel $P_{(n,m)}(z; A, B)$, corresponding to insertions of different one-loop parts in both gluon ($n$–bubble insertions) and quark ($m$–self-energy part insertions) lines of the triangular diagram (see Fig. 2d), is obtained as well. This formula is similar to Eq. (6) for $P_{(n)}$, but looks more cumbersome and is not shown here. The partial kernels $P_{(n,m)}(z; A, B)$ can be obtained by using the FORM program, in general, for any given $n$ and $m$. For illustration, we demonstrate here the first nontrivial kernel $aP_{(1,1)}$

$$P_{(1,1)} = P_0(z)AB\left(\frac{1}{2}\left[\ln(z) + \frac{8}{3}\right]^2 - \frac{20}{9} + \frac{1}{2}\left[\ln(z) + \frac{5}{3}\right]^2 - \frac{31}{18} - 2\left[\ln(z) + \frac{8}{3}\right]\left[\ln(z) + \frac{5}{3}\right] + \frac{80}{9} - 2\zeta(2)\right)$$

in comparison with partial kernels of the same order in $a$, $P_{(2,0)}$ ($\equiv P_{(2)}^{(1)}$) following from the expression for $P^{(1)}$ and $P_{(0,2)}$ ($\equiv 3P_{(2)}^{(2)}$), from $P^{(2b)}$

$$P_{(2,0)} = P_0(z)A^2\left(\frac{1}{2!}\left[\ln(z) + \frac{8}{3}\right]^2 - \frac{20}{9}\right), \quad P_{(0,2)} = 3P_0(z)B^2\left(\frac{1}{2!}\left[\ln(z) + \frac{5}{3}\right]^2 - \frac{31}{18}\right).$$

To complete the section, we conclude that contributions to kernel $P$ from any one-loop insertions into the lines of the triangular diagram are available now.

## 4 Triangular diagrams for the Brodsky–Lepage evolution kernel

Here, some partial results of the bubble resummation for the BL kernels $V$ are presented. We obtain them as a "byproduct" of our previous results for the kernel DGLAP $P$. The results obtained may be used, in particular, to check the regular calculation of the kernel $V$ in high orders of PT.

Note, the diagrams for the BL kernel differ from the DGLAP diagrams only by the "exclusive" kinematics of the input momentum, compare the diagrams in Fig.1a and 1b. So, one can repeat the proof similar to that in Appendix A for this case. There is another far more elegant way – to use exact relations between $V$ and $P$ kernels for triangular diagrams which were established for any order of perturbation theory in [5]. These relations work for both the $[\varphi^3]_0$ model and QCD. I quote these propositions without proofs.

Let the diagram in Fig.1c have a contribution to the DGLAP kernel in the form $P(z) = p(z) + \delta(1 - z) \cdot C$; then its contribution to the BL kernel is

$$V(x, y) = C\theta(y > x) \int_0^x \frac{p(z)}{z} dz + \delta(y - x) \cdot C,$$  \quad (14)

where $C \equiv 1 + (x \to \bar{x}, y \to \bar{y})$. Substituting Eq. (8) for $P^{(1)}(z)$ into relation (14) we immediately derive the expression for $V^{(1)}(x, y)$

$$V^{(1)}(x, y; A) = C a \left[\frac{\gamma_{\varphi}(0)}{\gamma_{\varphi}(A)}\right] \left[\theta(y > x) \left(\frac{x}{y}\right)^{1-A} - \frac{1}{2}\delta(y - x) \left(\frac{1 - A}{3 - A}(2 - A)\right)\right].$$  \quad (15)

The later expression may independently be verified by other relations by reducing any $V$ to $P$ [5, 17] ($V \to P$ reduction)

$$V^{(1)}(x, y; A) = C\theta(y > x)F\left(\frac{x}{y}; A\right) - \delta(y - x) \cdot C(A) \to P^{(1)}(z; A)$$

where $P^{(1)}(z; A) = \theta(1 - z)\bar{z} \frac{d}{dz} F(z; A) - \delta(1 - z) \cdot C$,  \quad (16)
Indeed, substituting \( (14) \) into \( (13) \) we return to the same Eq.\( (8) \) for \( P^{(1)}(z) \). Moreover, the first term of the Taylor expansion of \( V^{(1)}(x, y; A) \) in \( A \) coincides with the results of the two-loop calculation in \( [12] \). The contribution \( V^{(1)} \) should dominate for \( N_f \gg 1 \) in the whole kernel \( V \), and moreover, the function \( V^{(1)}(x, y) \) possesses an important symmetry. Really, the function \( V(x, y; A) = V^{(1)}(x, y; A) \cdot (\bar{y}y)^{1-A} \) is symmetric under the change \( x \leftrightarrow y, \ V(x, y) = V(y, x) \). This symmetry allows us to obtain the eigenfunctions \( \psi_n(x) \) of the equation

\[
\int_0^1 V^{(1)}(x, y; A) \psi_n(y) dy = \gamma(n; A) \psi_n(x),
\]

\[
\psi_n(y) \sim (\bar{y}y)^{d_\psi(A)-1} C_n^{(\alpha)(A)-\frac{1}{2}} (y - \bar{y}), \quad \text{here} \quad d_\psi(A) = D_A/2 - 1, \quad D_A = 6 - 2A,
\]

\( d_\psi(A) \) is the effective dimension of the scalar field when the anomalous dimension is taken into account, and \( C_n^{(\alpha)}(z) \) are the Gegenbauer polynomials of an order of \( \alpha \). This form of eigenfunctions \( \{\psi_n(y)\} \) as well as the \( x \leftrightarrow y \) symmetry of the function \( V(x, y) \) are the consequences of conformal symmetry conservation (see \( [13] \) for details) for the sum of diagrams under consideration. Equation \( (17) \) is tightly connected with the BL evolution equation in a special case when the \( \beta \)-function \( \beta = 0 \). In this case, the BL equation is simplified, the variables are separated, and the evolution equation may be reduced to \( (17) \). The partial solution to the BL equation turns out to be proportional to \( \psi_n(y) \) in \( (18) \).

For the diagrams in Fig. 2a plus its MC diagrams, the relation between \( \tilde{V}^{(2a)}(x, y) \) and \( \tilde{P}^{(2a)}(x) \) is (see \[3\])

\[
\tilde{V}^{(2a)}(x, y) = C\theta(y > x) \frac{1}{2} \left[ \frac{1}{y} \tilde{P}^{(2a)}(x) + \frac{1}{y} \tilde{P}^{(2a)}(\bar{x}) - \frac{1}{y} \tilde{P}^{(2a)} \left( \frac{x}{y} \right) \right],
\]

Substituting \( (14) \) into \( (13) \) one arrives at the expression for \( \tilde{V}^{(2a)}(x, y) \)

\[
\tilde{V}^{(2a)}(x, y) = C a \theta(y > x) \left\{ \left[ \frac{x^{1-B}}{y^{1-B}} + \frac{x^{1-B}}{y} - \frac{1}{y} \left( 1 - \frac{x}{y} \right)^{1-B} \right] \left( \frac{1}{y} \tilde{\gamma}_0(B) \right) - \frac{x}{y} \right\},
\]

which satisfies the same check as in the previous case. Of course, this formula does not cover the diagrams in Fig. 2b, where both the quark lines are dressed.

### 5 The results for QCD evolution kernels

The assertions about diagram contributions to \( P \) and \( V \), similar to the presented above, are also valid for the QCD diagrams. Their proof does not contain essentially new elements, but looks rather cumbersome. The complete results for the QCD evolution kernels will be presented in a subsequent paper. Here, at first, the QCD results for triangular diagrams in Fig. 1 c,d in the Feynman gauge are discussed. Based on the proof in Appendix A in the QCD version, one can derive the result for the sum of diagrams, in Fig. 1c, in QCD

\[
P^{(1c)}(z; A) = a_s C_F 2\bar{\varepsilon} \cdot (1 - A)^2 \frac{\Pi(0)}{\Pi(A)} z^{-A} - a_s C_F \cdot \delta(1 - z) \left( \frac{\Pi(0)}{\Pi(A)} \frac{1}{(1 - A)} - 1 \right).
\]

Here, \( a_s = \frac{\alpha_s}{4\pi} \); \( \Pi(\varepsilon) \) is twice the contribution to a one-loop \( D \)-dimensional anomalous dimension of the gluon field and \( \varepsilon = (4 - D)/2 \); \( \Pi(0) \) is the contribution to the standard anomalous dimension; \( A = -a_s \Pi(0) \). It should be emphasized again that the form of Exp.\( (21) \) does not depend on the nature of self-energy insertions into the gluon line. One can use it for the resummation of both the quark (\( \sim N_f T_R \))

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\(^3\) When this letter was considered by the Editorial Board, the work \[10\] appeared, where a similar solution has been presented for the case of QCD evolution kernel.
and gluon (∼ $C_A/2$) loops. Substituting into the general formula (21), the well-known expressions for Π(ε) from the quark or gluon (the ghost loop is also added) loops

$$
\Pi_q(ε) = -8N_f T_R B(D/2, D/2) C(ε),
$$
(22)

$$
\Pi_g(ε) = \frac{C_A}{2} B(D/2 - 1, D/2 - 1) \left( \frac{3D - 2}{D - 1} \right) C(ε),
$$
(23)

one obtains the expression $P_q^{(1)}(z; δ)$ for the quark–loop insertions

$$
P_q^{(1c)}(z; δ) = a_s C_F 2 z\bar{z}^{-\varepsilon} \left( \frac{(D_q/2 - 1)^2 B(2, 2)}{B(D_q/2, D_q/2) C(δ)} \right)
- a_s C_F δ(1 - z) \left( \frac{B(2, 2)}{B(D_q/2, D_q/2) C(δ)} \frac{1}{(D_q/2 - 1)} \right),
$$
(24)

where $D_q = 4 - 2δ$, $δ = -a_s Π_q(0) = -a_s N_f T_R 4/3$, $C(δ)$ is the scheme factor,

and the expression $P_g^{(1c)}(z; ε)$ for the gluon–loop insertions

$$
P_g^{(1c)}(z; ε) = a_s C_F 2 z\bar{z}^{-\varepsilon} \left( \frac{(D_g/2 - 1)(D_g - 1)}{3(3D_g - 2)B(D_g/2 - 1, D_g/2 - 1)C(ε)} \right)
- a_s C_F δ(1 - z) \left( \frac{10}{3(3D_g - 2)B(D_g/2 - 1, D_g/2 - 1)C(ε)} \frac{1}{(D_g/2 - 1)} \right),
$$
(25)

where $D_g = 4 - 2ε$, $ε = -a_s Π_g(0) = -a_s C_A 5/3$. The expression (25) for $P_g^{(1c)}(z; ε)$ has no any singularity in the parameter $ε < 0$ due to the asymptotic freedom.

Consider Eq.(24) for $P_q^{(1c)}(z; δ)$ in detail. By adding the contributions from the diagrams in Fig.1d to (24), one can find that the second term cancels in part and the final expression turns out to be the expected “plus form” (see, e.g. [3])

$$
P_q^{(1c,d)}(z; δ) = a_s C_F 2 \left( \bar{z}^\varepsilon \frac{(D_q/2 - 1)^2 Π_q(0)}{Π_q(δ)} \right), \text{ i.e., } \int_0^1 P_q^{(1c,d)}(z; δ) d\bar{z} = γ_q^{(1c,d)}(N = 0, δ) = 0
$$
(26)

due to the current conservation. The analytic properties of $P_q^{(1c,d)}(z; δ)$ in $δ$ are the same as for its scalar analogy $P^{(1)}(z; A)$ (see [3] in sect.2); they are determined by the behavior of the function $Π_q(δ)$ in $δ$, see Eq.(22). The nearest singularity of $P_q^{(1)}(z; δ)$ in $δ$ appears at $a_s N_f = 15/4$. The moments of Eq. (24) agree with the corresponding part of the generating function in [3]

$$
γ_q^{(1c,d)}(N, δ) = -a_s C_F \left( \frac{2}{3} \frac{N(D_q + N - 1)}{(D_q/2 + N)(D_q/2 - 1 + N)} \frac{(D_q/2 - 1)}{B(D_q/2, D_q/2)C(δ)D_q} \right),
$$
(27)

see the first term in Eq.(14) in [3], and ref. [3] (note that our moments differ in sign from the definition of the anomalous dimension of composite operators used there).

To complete the QCD calculations of the kernel $P^{(1)}$, we need the contribution from the last diagram in Fig.1e with the chain in the gluon line that is inserted into the composite operator. It can be obtained in a similar way, as in the previous QCD–calculations for Eq.(21)

$$
P^{(1e)}(z; A) = a_s C_F 2 \left( \frac{2 z^{1-A} Π(0)}{1 - z Π(A)} \right)_+, \quad (28)
$$
and the expression automatically has the “plus form”, see [4] for details. Collecting Eq. (24) and Eq. (28), one easily arrives at the complete QCD expression for \( P_{q}^{(1)}(z; \delta) \)

\[
P_{q}^{(1)}(z; \delta) = a_s C_F 2 \cdot \left[ \hat{z} z^{-\delta}(1 - \delta)^2 + \frac{2z^{1-\delta}}{1 - z} \right] + \frac{\Pi_q(0)}{\Pi_q(\delta)}. \tag{29}
\]

The contribution \( P_{q}^{(1)}(z; \delta) \) is gauge invariant. The moments of \( P_{q}^{(1)}(z; \delta) \) again agree with the complete generating function obtained in \([\mathfrak{S}]\) (see Eq.(14) there). To obtain the contribution from both kinds of insertions into the gluon line one would take certain substitutions in (24): \( \Pi_q(\varepsilon) + \Pi_g(\varepsilon) \rightarrow \Pi_q(\delta) \) and \( \varepsilon \rightarrow \delta \), here \( \varepsilon = -a(\Pi_q(0) + \Pi_g(0)) \). Of course, for this case a final result must be gauge dependent.

6 Conclusion

The method for calculating certain classes of multiloop diagrams for the kernel \( P(z) \) of the non-singlet DGLAP evolution equation is presented. These multiloop diagrams appear due to the insertion of chains of one-loop self-energy parts (renormalon chains) into the lines of the one-loop diagrams for the kernel. Closed expressions \( P^{(1,2)}(z, a_{1,2}) \) are found for sums of all the diagrams which belong to two of the diagram classes, see sect. 2-3. The corresponding proofs are based on a simple algebraic structure of the counterterms for the diagrams under consideration; this is outlined in Appendix A. Moreover, the kernels \( P^{(i)}(z, a_i) \) are generating functions for the partial kernels \( P_{(i)}^{(1)}(z) \) which correspond to the \( n \)-bubble insertion. The contribution \( P^{(1)}(z; a_1) \) (corresponding to one of the diagram classes) would dominate in the kernel for \( N_f \gg 1 \). The analytic properties of the function \( P^{(i)}(z, a_i) \) in the variable \( a_i \) are briefly discussed. The expressions for partial kernels \( P_{(n,m)}^{(1)}(z; a_1, a_2) \) for the diagrams of a “mixed class”, in any order of perturbation theory can also be obtained by using the FORM program.

The contributions \( V^{(i)}(x, y; a_i) \) to the Brodsky–Lepage kernel are obtained in sect. 4 for the same classes of diagrams as a “byproduct” of the previous technique. When \( N_f \gg 1 \), a special solution to the Brodsky–Lepage equation is derived. We emphasize that the method of calculating the evolution kernels \( P^{(i)} \) or \( V^{(i)} \) does not depend on the nature of self-energy insertions and does not appeal to the value of parameters \( N_f T_R, C_A/2 \) or \( C_F \) (for QCD case) associated with different loops.

The method and results are exemplified with a simple \([\mathfrak{φ}^3]_6\) model. It is clear that all the results obtained above in the framework of the scalar model have a wider meaning and can be applied to the QCD case. Here, in sect. 5, some QCD results are presented too; in particular, the kernel \( P_{q}^{(1)}(z; \delta) \), that corresponds to the diagram dressed by the main quark-loop chain, is derived. The anomalous dimension \( \gamma_{q}^{(1)}(N, \delta) \) corresponding to this kernel agrees with the generating function obtained earlier by Gracey \([\mathfrak{S}], [\mathfrak{B}]\). The contribution \( P_{g}^{(1)}(z; \epsilon) \) from the diagrams with the gluon-loop chain is derived in the same way. The same can be obtained for the mixed case when both kinds of bubbles (quark and gluon) are involved. The completed QCD case will be considered in detail in a subsequent paper.

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A Appendix

Here, formula (\( \mathfrak{H} \)) is proved; all others resummation formulæ in the text above can be obtained in a similar manner.
The elements (A.6) that can be constructed as superpositions of Γ-functions depending on similar trick with the identity (A.7) had been used in [10]. The proof implies an obvious generalization of the arguments like 1 + (1 + j)\varepsilon and the partial kernels \( S_\varepsilon \) of the l.h.s. of (A.1), and the last term in the r.h.s. of (A.1) may be associated with the S\( n+1 \) in the r.h.s. of (A.1) does not contribute to the pole part, by virtue of Lemma: \( S_{n+1}(\varepsilon) \leq O(\varepsilon^{n+1}) \). The expression for \( P^{(1)}(n)(z) \) can be derived by substituting decomposition (A.1) into (A.8) and using the above mentioned Lemma

\[
P^{(1)}(n)(z) = aP_0(z)(-A)^n\tilde{K}_1 \left[ \frac{1}{\varepsilon^{n+1}} z^{-\varepsilon(1-\varepsilon)} \left( C(\varepsilon)S_{n+1}(\varepsilon) + \frac{(-n)^n}{F(\varepsilon)} \right) \right] = aP_0(z)(A)^n \frac{\gamma\varepsilon(0)}{n!} \left\{ \frac{d^n}{d\varepsilon^n} \left( z^{-\varepsilon(1-\varepsilon)} \right) \right\}_{\varepsilon=0}.
\]

Now, let us take a sum of \( P^{(1)}(n)(z) \) over all \( n \) using Exp.(A.3); the differentials are naturally collected into the “shift argument operator” \( \exp(A\partial\varepsilon) \)

\[
P^{(1)}(z; A) = a \sum_n P^{(1)}(n)(z) = aP_0(z) \left\{ \exp(A\partial\varepsilon) \left[ z^{-\varepsilon(1-\varepsilon)} \right] \right\}_{\varepsilon=0}.
\]

Finally, performing the shift of argument in the square bracket in (A.4), we arrive at the expression

\[
P^{(1)}(z; A) = a \sum_{n=0}^{\infty} P^{(1)}(n)(z) = aP_0(z) z^{-A(1-A)} \left( \gamma\varepsilon(0) \right) \left( \gamma\varepsilon(A) \right),
\]

and the partial kernels \( P^{(1)}(n)(z) \) appear in the Taylor expansion of \( P^{(1)}(z; A) \) in \( A \) by construction.

Lemma: \( S_{n+1}(\varepsilon) \leq O(\varepsilon^{n+1}) \) for any analytic function \( F(\varepsilon) \) at point \( \varepsilon = 0 \).

Let us consider the expansion of every element of the sum in \( S_{n+1} \)

\[
\frac{F(\varepsilon)^j}{F(\varepsilon)} \frac{\Gamma(1+j\varepsilon)}{\Gamma(1+(j-1)\varepsilon)\Gamma(1+\varepsilon)}
\]

in powers of \( \varepsilon \) up to \( \varepsilon^n \). Any power \( \varepsilon^m \) of this expansion is accompanied by powers \( j^l \), where \( l \leq m \), and a coefficient that does not depend on \( j \). Therefore, the expansion of \( S_{n+1}(\varepsilon) \) as a whole in the power series \( \varepsilon^m \) will generate coefficients of the powers which are composed only of the elements proportional to \( \sum_{j=0}^{n+1} j^l(-)^{n+1-j} \binom{n+1}{j} \). All these elements are equal to 0 for \( m \leq n \) in virtue of the identity

\[
\sum_{j=0}^{n+1} j^l(-)^{n+1-j} \binom{n+1}{j} = 0, \text{ if } l < n + 1.
\]

Therefore, the power expansion of \( S_{n+1}(\varepsilon) \) can be started with the power higher than \( \varepsilon^n \). Note, the similar trick with the identity (A.7) had been used in [11]. The proof implies an obvious generalization of the elements (A.6) that can be constructed as superpositions of Γ-functions depending on \( j \) only through the arguments like \( 1 + j\varepsilon, 1 + (j-1)\varepsilon, \ldots \).
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| $n$ | The partial kernels $P_{(n)}^{(1)}(z;A)$ (the common factor $aP_0(z)A^n$ is dropped) |
|-----|-------------------------------------------------------------------------------------|
| 1   | $\left[\frac{\ln(z) + \frac{8}{3}}{3!}\right]_{1!}$ |
| 2   | $\left[\ln(z) + \frac{8}{3}\right]^{2}_{2!} - \frac{20}{9}$ |
| 3   | $\left[\ln(z) + \frac{8}{3}\right]^{3}_{3!} - \frac{20 \ln(z)}{9} - \frac{1!}{1!} - \left(\frac{256}{81} + 2\zeta(3)\right)$ |
| 4   | $\left[\ln(z) + \frac{8}{3}\right]^{4}_{4!} - \frac{20 \ln^2(z)}{9} - \frac{2!}{2!} - \left(\frac{256}{81} + 2\zeta(3)\right)\frac{\ln^1(z)}{1!} - \frac{512}{243} - 3\zeta(4) - \frac{16}{3}\zeta(3)$ |
| 5   | $\left[\ln(z) + \frac{8}{3}\right]^{5}_{5!} - \frac{20 \ln^3(z)}{9} - \frac{3!}{3!} - \left(\frac{256}{81} + 2\zeta(3)\right)\frac{\ln^2(z)}{2!} - \left(\frac{512}{243} + 3\zeta(4) - \frac{16}{3}\zeta(3)\right)\frac{\ln^1(z)}{1!} - \frac{4096}{3645}$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
| 6   | $\left[\ln(z) + \frac{8}{3}\right]^{6}_{6!} - \frac{20 \ln^4(z)}{9} - \frac{4!}{4!} - \left(\frac{256}{81} + 2\zeta(3)\right)\frac{\ln^3(z)}{3!} - \left(\frac{512}{243} + 3\zeta(4) - \frac{16}{3}\zeta(3)\right)\frac{\ln^2(z)}{2!} + \left(-\frac{4096}{3645}\right)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\left(\zeta(6) - 6\zeta(3)\zeta(4) + \frac{16}{3}\zeta^2(3)\right)$ |
| 7   | $\left[\ln(z) + \frac{8}{3}\right]^{7}_{7!} - \frac{20 \ln^5(z)}{9} - \frac{5!}{5!} - \left(\frac{256}{81} + 2\zeta(3)\right)\frac{\ln^4(z)}{4!} - \left(\frac{512}{243} + 3\zeta(4) - \frac{16}{3}\zeta(3)\right)\frac{\ln^3(z)}{3!} + \left(-\frac{4096}{3645}\right)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\left(\zeta(6) - 6\zeta(3)\zeta(4) + \frac{16}{3}\zeta^2(3)\right)$ |
| 8   | $\left[\ln(z) + \frac{8}{3}\right]^{8}_{8!} - \frac{20 \ln^6(z)}{9} - \frac{6!}{6!} - \left(\frac{256}{81} + 2\zeta(3)\right)\frac{\ln^5(z)}{5!} - \left(\frac{512}{243} + 3\zeta(4) - \frac{16}{3}\zeta(3)\right)\frac{\ln^4(z)}{4!} + \left(-\frac{4096}{3645}\right)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\left(\zeta(6) - 6\zeta(3)\zeta(4) + \frac{16}{3}\zeta^2(3)\right)$ |
|     | $\ln(z) + \frac{63}{2}\zeta(8) - 48\zeta(7) + \frac{40}{3}\zeta(6) + \frac{9}{2}\zeta^2(4)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\ln(z) + \frac{63}{2}\zeta(8) - 48\zeta(7) + \frac{40}{3}\zeta(6) + \frac{9}{2}\zeta^2(4)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\ln(z) + \frac{63}{2}\zeta(8) - 48\zeta(7) + \frac{40}{3}\zeta(6) + \frac{9}{2}\zeta^2(4)$ |
|     | $\left(\zeta(4) - 6\zeta(5) - \frac{8}{3}\right)$ |
|     | $\ln(z) + \frac{63}{2}\zeta(8) - 48\zeta(7) + \frac{40}{3}\zeta(6) + \frac{9}{2}\zeta^2(4)$ |

**Table 1.** The results of the $P_{(n)}^{(1)}(z;A)$ calculations, the $\zeta(n)$ is the Riemann zeta–function; note that $\zeta(2)$ and the Euler constant $\gamma_E$ do not appear in this expansion.
\[ \delta(z - kn) \]

(a) \hspace{2cm} (b)

\[ k \]
\[ p \]
\[ p \]
\[ yp \]
\[ \bar{yp} \]

\[ k \]
\[ \frac{1}{2} \]

(c) \hspace{2cm} (d) \hspace{2cm} (e)

\[ +MC \]

\[ +MC \]

\[ +\ldots \]
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Fig.2
Captions for Figures

Fig. 1
The diagrams in Figs. 1a - 1d are for the scalar model (and QCD); dashed line – for “gluons”, solid line – for “quarks”; the slash on the line denotes the delta function $\delta(z - kn)$, see Fig. 1a; black circle denotes the sum of all one-loop chains; MC denotes the mirror-conjugate diagram. The diagram in fig. 1e is for QCD; solid line – for quarks, dotted line – for gluons.

Fig. 2
The dashed circle in Figs. 2 denotes the sum of chains of one-loop self-energy “quark” parts.