THE UNTRUNCATED MARINARI-PARISI SUPERSTRING

G. Ferretti†

Institute of Theoretical Physics
Chalmers University of Technology
S-41296 Göteborg, Sweden

Abstract

It is shown that the bosonic angular degrees of freedom in the one dimensional Marinari-Parisi superstring can be integrated out exactly in the Hamiltonian formulation without having to perform the Dabholkar truncation. The resulting Hamiltonian is that of a supersymmetric Calogero system plus a four fermions interaction. This extra interaction vanishes for all physical states with fermion number zero or one where supersymmetry is manifest. We confirm that supersymmetry is nonperturbatively broken by instanton effects.

† E-mail: ferretti@fy.chalmers.se
a) The Marinari-Parisi superstring

The Marinari-Parisi model [1] describes a string moving in one dimensional superspace. This is achieved by extending the well known surface discretization techniques [2], [3], [4] to a supersurface. There are many reasons why such generalization is desirable. For one thing, one is interested in the non critical superstring just as much as the bosonic one. Also, the supersymmetric formulation allows one to study the $D = 0$, pure gravity in a way that avoids the instabilities present in the bosonic case. Finally, the model provides a good example of supersymmetry breaking.

The action generating the dual triangulation of the world-sheet embedded in one dimensional superspace is just the Wess-Zumino model of a $N \times N$ supermatrix $\Phi$ with cubic interaction. The components of the superfield are

$$\Phi(t, \theta, \bar{\theta}) = X(t) + \Psi(t)\theta + \bar{\Psi}(t)\bar{\theta} + A(t)\bar{\theta}\theta.$$ (1)

The bosonic fields $X$ and $A$ and the fermionic fields $\Psi$ and $\bar{\Psi}$ are $N \times N$ matrices and $\theta$, $\bar{\theta}$ are Grassmann numbers. If we introduce the covariant derivative

$$D = \bar{\theta} \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}, \quad D = \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{\theta}},$$ (2)

and the potential

$$W(z) = \frac{1}{2}z^2 - \frac{\lambda}{3}z^3,$$ (3)

the desired action is

$$S = \int dt d\bar{\theta} d\theta \text{ tr } \left( -\frac{1}{2} \Phi D \Phi + W(\Phi) \right).$$ (4)

It is straightforward to integrate out the auxiliary field $A$ and to perform the integration over the Grassmann variables $\theta$ and $\bar{\theta}$ to obtain the expression for the supersymmetric action in ordinary $D = 1$ space-time:

$$S = \frac{1}{2} \int dt \text{ tr } \left( \dot{X}^2 + \dot{\Psi}\bar{\Psi} + \Psi\dot{\Psi} + W'(X)^2 - \bar{\Psi}W''(X)\Psi - \bar{\Psi}\Psi W''(X) \right),$$ (5)

where, in writing down the last two terms under the trace, we used the fact that the potential $W$ is only a cubic polynomial.

The Hamiltonian that follows from quantizing this one dimensional system is the Hamiltonian of a supersymmetrical quantum mechanical system [5]. It can be written explicitly in terms of the supercharges $Q$ and $Q^\dagger$ as

$$H = \frac{1}{2} \{Q, Q^\dagger\},$$ (6)
where

\[ Q = i \text{ tr } \left( \left( \frac{\partial}{\partial X} - W'(X) \right) a \right), \quad Q^\dagger = i \text{ tr } \left( \left( \frac{\partial}{\partial X} + W'(X) \right) a^\dagger \right). \] (7)

We denote the fermionic creation and annihilation operator valued matrices by \( a^\dagger \) and \( a \) and impose the following CAR on their components \( \{ a^\dagger_i a^k \} = \delta^i_k \). Evaluating the anticommutator, one finds that the Hamiltonian consists of a bosonic and a fermionic piece, both involving \( N^2 \) degrees of freedom:

\[ H = H_B + H_F, \] (8)

where

\[ H_B = \frac{1}{2} \text{ tr } \left( -\frac{\partial^2}{\partial X^2} + W'^2(X) - W''(X) \right), \] (9)

and

\[ H_F = \frac{1}{2} \text{ tr } \left( a^\dagger W''(X) a + a^\dagger a W''(X) \right) = \text{ tr } \left( a^\dagger a - \lambda (a^\dagger X a + a^\dagger a X) \right). \] (10)

Marinari and Parisi have analyzed the bosonic sector of the model [1]. Neglecting the fermions allows one to integrate out the \( N^2 - N \) angular degrees of freedom and to rewrite the Schrödinger equation in terms on a gas of \( N \) noninteracting fermions [6]. Away from the continuum limit, using a WKB approximation, one finds that the vacuum energy is zero at weak coupling and scales like \( N^2 (\lambda - \lambda_c)^{5/2} \) at strong coupling.

As \( \lambda \) approaches its critical value the WKB energy vanishes perturbatively. Formally, there is a wave function of zero energy given by \( \exp(-\text{ tr } W(X))|0\rangle \). However, since the potential \( W \) is cubic, this is not a normalizable wave function, suggesting that there are nonperturbative corrections to the ground state energy. These corrections can be evaluated by instanton calculus [7] and found to be nonzero. Supersymmetry is therefore broken by nonperturbative effects.

b) The Dabholkar truncation

To estimate the nonperturbative correction to the energy in a manifestly supersymmetric way, one has to deal with states of fermion number different from zero. (For example, one has to compute the matrix element of the supercharge \( Q \) between the bosonic vacuum and its fermionic partner.)

The main problem when dealing with the full set of states is that the supermatrix \( \Phi \) can no longer be diagonalized by a unitary transformation. Dabholkar [8] has shown that there is a consistent way of truncating the theory that allows one to include some (but not all) states with non zero fermion number while preserving manifest supersymmetry and reducing the number of degrees of freedom from \( 2 \times N^2 \) to \( 2 \times N \).
This is done by considering the unitary matrix $\Omega$ that diagonalizes the bosonic matrix $X$ ($X = \Omega x \Omega^\dagger$) and by defining the new fermionic operator matrix $\hat{a} = \Omega^\dagger a \Omega$. (A short remark: throughout the paper, we will make a slight abuse of notation and denote both the diagonalized matrix and the ordered set of eigenvalues by $x$; what is meant is always clear from the contest.) One then considers only those fermionic states of the type created by the diagonal elements of $\hat{a}$

$$\hat{a}_{i_1}^\dagger \hat{a}_{i_2}^\dagger \hat{a}_{i_3}^\dagger \cdots |0>, \quad (11)$$

while excluding all non diagonal states. On the subspace defined by this truncation, the supercharges can be written in a way that is independent on the angular variables by introducing an effective potential

$$W_{\text{eff}}(x_1 \cdots x_N) = \sum_{i=1}^{N} W(x_i) + \sum_{i \neq j} \log |x_i - x_j|, \quad (12)$$

The logarithmic factor in the effective potential comes from the expression for the metric on the space of matrices when going from “Cartesian” to “polar” coordinates.

The explicit expression for the supercharge $Q$ in terms of the effective potential is

$$Q = i \sum_{k} \left( \frac{\partial}{\partial x_k} - \frac{\partial W_{\text{eff}}}{\partial x_k} \right) \hat{a}_k^\dagger. \quad (13)$$

The supersymmetric Hamiltonian $H = (1/2)\{Q, Q^\dagger\}$ is the Hamiltonian of a supersymmetric Calogero system. The consistency condition for this truncation to be physically meaningful is that the subspace defined in this way is invariant under the action of the supercharges $Q$ and $Q^\dagger$. This is indeed the the case for this model. One can then carry on the double scaling limit of the charges and evaluate the lifting of the vacuum energy. Dabholkar finds

$$E_{\text{g.s.}} \approx \kappa \epsilon^{-2S_{\text{inst.}}}/\kappa, \quad (14)$$

$\kappa$ being the string coupling constant. Supersymmetry is therefore broken. One can also give a collective field description of the theory [9].

c) The Untruncated theory

As long as one is only interested in the ground state energy or in correlation functions of operator that do not depend on the bosonic angular variables (such as $\text{tr} (a X^3)$ or $\text{tr} (aX^4 a^\dagger X^6)$), the truncated theory gives all the informations that are needed. On the other hand, the analysis of correlations functions for operators that are not in the singlet sector is precluded from the very beginning. Such operators are of interest since they create non trivial excitations and not much
is known about them [10]. Also, the above truncation, though consistent and supersymmetric, is basis dependent and it must be possible to write the theory in a way that does not depend on the basis in the representation of $U(N)$.

Once again, one is facing the problem that it is impossible to diagonalize simultaneously the bosonic and the fermionic degrees of freedom in the Lagrangian. We will show however, that, if one is willing to give up manifest supersymmetry, it is possible to integrate out the angular degrees of freedom from the Hamiltonian! The procedure used here is an adaptation of a method already used in the contest of Rajeev’s “Universal Yang–Mills Theory” [11], where one is also confronted with a matrix model with fermions [12]. One difference is that now, because of supersymmetry, the fermions are in the adjoint representation of $U(N)$ whereas in [12] they were in the fundamental one.

Let us begin our calculation by analyzing the symmetries of the most general state vector of the Marinari-Parisi model. The most general state of fermion number $n$ can be written as

$$|\Psi(X)\rangle = \Psi^{i_1\cdots i_n}j_1^{j_1}\cdots j_n^{j_n}a^{i_1}_{j_1}\cdots a^{i_n}_{j_n}|0\rangle \equiv \Psi(X) \cdot (\otimes a^\dagger)|0\rangle .$$

(15)

where $\Psi(X)$ is a tensor antisymmetric under the exchange of the pairs of indices $(i_k,j_k)$ and $(i_l,j_l)$ and $|0\rangle$ is the fermionic vacuum. (Do not confuse this $\Psi$ with the fermionic component of the superfield.) From now on, we will use tensor notation. The superfield $\Phi$ transforms under the adjoint representation of $U(N)$, $\Phi \rightarrow U\Phi U^\dagger = \text{Ad}_U\Phi$ and therefore, all its components $X$, $\Psi$, $\bar{\Psi}$, $A$ and the fermionic operators $a$ and $a^\dagger$ will also transform under the adjoint representation. Any state $|\Psi(X)\rangle$ must be invariant under the action of $U(N)$, i.e., in terms of the tensor $\Psi(X)$,

$$\Psi(UXU^\dagger) = (\otimes \text{Ad}_U\Psi)(X)$$

(16)

Note that in this particularly simple case we can assume the above relation to be imposed on the vectors themselves, not just on the rays. Now recall that any hermitian matrix $X$ can be diagonalized by a unitary matrix $\Omega$ as $X = \Omega x \Omega^\dagger$ and therefore we can factor out the dependence of the tensor $\Psi$ from the angular variables as

$$|\Psi(X)\rangle = (\otimes \text{Ad}_\Omega\Psi)(x) \cdot (\otimes a^\dagger)|0\rangle .$$

(17)

This is the most convenient form for further manipulations. From now on, we think of the tensor $\Psi$ as depending, not on the matrix $X$, but only on the $N$ eigenvalues $x$ and express the dependence on the angular variables through the action $\otimes \text{Ad}_\Omega$. 

5
Before we carry on, let us recall that the splitting \( X = \Omega x \Omega^\dagger \) does not fix the representative of the state in a unique way. We can always perform a subgroup of \( U(N) \) transformations that leave \( X \) invariant and \( x \) diagonal. This subgroup is clearly \( U(1)^N \times S_N \). The subgroup \( U(1)^N \) consists of all the diagonal matrices in \( U(N) \) and it acts on \( \Omega \) and \( x \) in the following way:

\[
H \in U(1)^N : \quad x \to x, \quad \text{and} \quad \Omega \to \Omega H. \tag{18}
\]

The subgroup \( S_N \) is the discrete group of permutations of \( N \) elements and it acts by permuting the eigenvalues of \( X \). \( S_N \) can always be embedded in \( U(N) \) by considering \( N \times N \) matrices \( P \) with \( N - 1 \) “zeroes” and 1 “one” in each row and column.

\[
P \in S_N : \quad x \to P^\dagger x P, \quad \text{and} \quad \Omega \to \Omega P. \tag{19}
\]

This invariance must be imposed as a constraint on the tensor wave function. Therefore, only those tensors that satisfy

\[
(\otimes \text{Ad}_H \Psi)(x) = \Psi(x) \quad \text{and} \quad (\otimes \text{Ad}_P \Psi)(x) = \Psi(P x P^\dagger) \tag{20}
\]

are allowed as physical states. We will return to this point at a later stage, for now let us proceed with the elimination of the angular variables.

Let us start by writing the expression of the metric on the space of hermitian matrices in polar coordinates. In Cartesian coordinates, the invariant length is just \( ds^2 = \text{tr} (dX dX) \). Let us denote the \( N \) radial variables as \( x^i \) (as usual) and the remaining \( N^2 - N \) angular variables by \( \theta^a \). By setting \( X = \Omega x \Omega^\dagger \) we obtain

\[
ds^2 = \sum_{i=1}^N dx_i \otimes dx_i + \sum_{a,b=1}^{N^2-N} g_{ab} d\theta^a \otimes d\theta^b, \tag{21}\]

where the angular part of the metric is given by

\[
g_{ab} = \sum_{k \neq l} (x_k - x_l)^2 \omega_{i\alpha}^k(x, \theta) \omega_{j\beta}^k(x, \theta), \tag{22}\]

and the complex functions \( \omega_{i\alpha}^j(x, \theta), \ i, j = 1 \cdots N, \ a = 1 \cdots N^2 - N \) are defined implicitly through the relation

\[
(\Omega^\dagger d\Omega)_j^i = \omega_{j\alpha}^i(x, \theta) d\theta^a \tag{23}\]

and satisfy \( \omega_{j\alpha}^i = -\omega_{i\alpha}^j \). The metric \( g_{ab} \) has an inverse \( g^{ab} \) satisfying the relation \( g_{ab} g^{ab} = N^2 - N \). This fact, together with \( S_N \) invariance implies

\[
g^{ab} \omega_{i\alpha}^k(x, \theta) \omega_{j\beta}^k(x, \theta) = \frac{1}{(x_k - x_l)^2} \delta_i^j \delta_l^k. \tag{24}\]
Also, it is well known that, setting $g = \det(g_{ab})$,

$$\int d\theta \sqrt{g} = \prod_{i<j}(x_i - x_j)^2 \equiv \Delta(x)^2. \quad (25)$$

Note that formula (25) also follows, up to an overall constant, from symmetry arguments and dimensional analysis.

Formulas (24) and (25) will allow us to integrate out the bosonic angular variables from the energy.

The energy of a normalized state is given, in Cartesian coordinates, by

$$E = \int dX <\Psi(X)|H_B + H_F|\Psi(X)> \equiv E_B + E_F, \quad (26)$$

where the two Hamiltonians $H_B$ and $H_F$ are defined in (9) and (10). The removal of the bosonic degrees of freedom from the fermionic energy does not present any problem. One simply writes

$$E_F = \int dX <\Psi(X)|H_F|\Psi(X)>$$

$$= \int dx \Delta(x)^2 <0|(\otimes a) \cdot (\otimes Ad_\Omega \Psi(x))\dagger tr\left(a\dagger a - \lambda(a\dagger Xa + a^\dagger aX)\right)(\otimes Ad_\Omega \Psi)(x) \cdot (\otimes a\dagger)|0>$$

$$= \int dx \Delta(x)^2 <0|(\otimes a) \cdot \Psi\dagger(x) \sum_{i,j=1}^N \left(1 - \lambda(x_i + x_j)a_j^\dagger a_i\right)\Psi(x) \cdot (\otimes a\dagger)|0>.$$

(27)

Now, just define a new tensor $\Xi(x) = \Delta(x)\Psi(x)$ satisfying the invariance conditions that follow from the constraints (20):

$$(\otimes Ad_H \Xi)(x) = \Xi(x) \quad \text{and} \quad (\otimes Ad_P \Xi)(x) = (-1)^{\sigma(P)}\Xi(P_xP^\dagger), \quad (28)$$

where $H$ and $P$ are the same as in (20) and $\sigma(P)$ is the parity of $P$. The expression of the fermionic Hamiltonian on the states $\Xi(x) \cdot (\otimes a\dagger)|0>$, satisfying the constraints (28) is therefore given by

$$H_F = \sum_{i,j=1}^N \left(1 - \lambda(x_i + x_j)a_j^\dagger a_i\right). \quad (29)$$

The bosonic energy requires a more careful analysis due to the dependence of the Laplacian on the angles $\theta^a$. We start by splitting the bosonic energy $E_B$ in two part, a radial part, containing the radial part of the Laplacian and the supersymmetric potential $W'^2 - W''$ and an angular part, containing the angular part of the Laplacian.

$$E_B = E_{\text{rad.}} + E_{\text{ang.}}. \quad (30)$$
The radial part does not present any difficulty and it can be rewritten at once as

\[
E_{\text{rad.}} = \int dx \Delta(x)^2 < 0|\otimes a\rangle \cdot \Psi^\dagger(x) \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + W'^2(x_i) - W''(x_i) \right) \Psi(x) \cdot (\otimes a^\dagger)|0 > . \tag{31}
\]

The angular part, on the other hand, is

\[
E_{\text{ang.}} = \frac{1}{2} \int dxd\theta \sqrt{gg}^{ab} < \Psi(X) \frac{\overleftarrow{\partial} \partial}{\partial \theta^a \partial \theta^b} |\Psi(X) >
= \frac{1}{2} \int dxd\theta \sqrt{gg}^{ab} < 0|\otimes a\rangle \cdot (\otimes Ad_\Omega \Psi)^\dagger(x) \frac{\overleftarrow{\partial} \partial}{\partial \theta^a \partial \theta^b} (\otimes Ad_\Omega \Psi)(x) \cdot (\otimes a^\dagger)|0 > . \tag{32}
\]

Using the fact that

\[
\frac{\partial}{\partial \theta^a} Ad_\Omega = Ad_\Omega \frac{\partial}{\partial \theta^a} \Omega, \tag{33}
\]

and that the generators of the adjoint action on the state can be expressed in terms of fermionic bilinears

\[
ad_\Omega \frac{\partial}{\partial \theta^a} \Omega = \omega^l_{ak}(a^l_p a^k_p - a^l_p a^k_p),
\]

we rewrite (32) as

\[
E_{\text{ang.}} = \frac{1}{2} \int dxd\theta \sqrt{gg}^{ab}\omega^l_{am}\omega^l_{bj} < 0|\otimes a\rangle \cdot \Psi^\dagger(x)(a^l_p a^m_p - a^l_p a^m_p)(a^l_q a^m_q - a^l_q a^m_q)\Psi(x) \cdot (\otimes a^\dagger)|0 > . \tag{35}
\]

Using (24) and (25), we can now integrate out the angular dependence altogether:

\[
E_{\text{ang.}} = \frac{1}{2} \int dx \Delta(x)^2 \sum_{l \neq m} \frac{1}{(x_l - x_m)^2} < 0|\otimes a\rangle \cdot \Psi^\dagger(x)(a^l_p a^m_p - a^l_p a^m_p)(a^l_q a^m_q - a^l_q a^m_q)\Psi(x) \cdot (\otimes a^\dagger)|0 > . \tag{36}
\]

The effect of the integration over the angular variables is the introduction of a (non manifestly supersymmetrical) four fermi interaction. We stress that this is an exact equivalence and it is valid on all the states without any truncation. It is not surprising that this expression is not manifestly supersymmetric because we have integrated out some of the bosonic degrees of freedom.

We can now carry out the same substitution \(\Psi(x) \to \Xi(x)\) and it is well known [6] that no extra terms are generated. The bosonic Hamiltonian acting on the states \(\Xi(x) \cdot (\otimes a^\dagger)|0 > \) is therefore obtained by adding the radial and the angular contributions;

\[
H_B = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + W'^2(x_i) - W''(x_i) \right) + \frac{1}{2} \sum_{l \neq m} \frac{1}{(x_l - x_m)^2} (a^l_p a^m_p - a^l_p a^m_p)(a^l_q a^m_q - a^l_q a^m_q). \tag{37}
\]

We conclude this section by rewriting the physical constraints (28) in a more explicit form and specializing, as an illustration, to the fermion numbers zero and one. For an arbitrary element \(T\) of
the Lie algebra of $SU(N)$, we have seen how to write the generators of $\otimes ad_T$ in terms of fermionic bilinears. Specializing to the Cartan subalgebra, we can rewrite the first constraint in (28) as
\[
\sum_{p, \text{ not } i} (a_i^\dagger p a_p^i - a_p^i a_i^\dagger p) \Xi(x) \cdot (\otimes a^\dagger)|0>, \quad \text{for all } i = 1 \cdots N. \tag{38}
\]
The second constraint implies the impossibility of factorizing the state vector as $f(x)|\text{fermi} >$ (where $f$ is a scalar) except when $|\text{fermi} >= |0 >$. In particular, the zero fermions wave function is given as $\chi(x)|0 >$ in terms of a totally antisymmetric scalar function $\chi(x)$, whereas the one fermion wave function is given as
\[
\sum_m \hat{\chi}(x_m, x_2, \cdots, x_{m-1}, x_1, x_{m+1} \cdots, x_N) a_{m}^\dagger a_{m}^i |0>, \tag{39}
\]
$\hat{\chi}$ being antisymmetric in the last $N-1$ variables. Note that all the zero and one fermion states belong to the Dabholkar truncated sector. This is obvious if one realizes that all the states in the truncated sector trivially satisfy the first of (28). However, not all the states in the truncated sector satisfy the second of (28); only those of the form (39) are allowed as true physical states. To find physical states that do not belong to the truncated sector, one has to go to fermion occupancy equal to two. There, one can construct states like $\Xi(x)_{kl}^i j^\dagger a_i^\dagger k a_j^\dagger|0 >$ that satisfy (28) but do not belong to the truncated sector.

As far as the calculation of the vacuum energy goes, the same results described in [8] apply here, mainly because the instanton calculation only involves states with fermion number zero and one. Hence, we see that supersymmetry is broken nonperturbatively.

The obvious (but important) thing to notice though is that the part of the Hamiltonian that is quartic in the fermionic operator vanishes on all states of fermion number zero or one. This means that supersymmetry is manifest on those states where the Hamiltonian reduces to that of a supersymmetric Calogero system with the \textit{same} potential $W$ as in the original action and not the effective potential described in [8] which is ultimately arising from the truncation. The calculation of the nonperturbative corrections to the vacuum energy proceeds just as in [8], without even having to worry about the effective potential. One finds once again that $E_{g.s.} \approx \kappa \exp(-2S_{\text{inst.}}/\kappa)$.

It would be interesting to investigate the effect of the quartic Hamiltonian on the states with higher fermion number. The explicit form of the Hamiltonian and of the states is simple enough to allow some investigations on the non singlet sector. This should be easier to do here than in the case of the bosonic string on the circle, let alone strings in higher dimensions.
The model described in this paper also suggests a possible generalization of the Calogero systems [13] in a direction that has not yet been explored. It might still be possible to solve the theory exactly even with this quartic term.

Acknowledgements

Most of the group theory techniques used here were explained to me by S.G. Rajeev while we were working on a similar problem. This work was initiated while visiting S.I.S.S.A., whose hospitality is gratefully acknowledged. Finally, I want to thank L. Brink and B. Nilsson for discussions.
References

1. E. Marinari and G. Parisi, Phys. Lett. B 240 (1990) 375.
2. F. David, Nucl. Phys. B 257 (1985) 45;
   F. David, Nucl. Phys. B 257 (1985) 543;
   V.A. Kazakov, Phys. Lett. B 150 (1985) 28;
   J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B 257 (1985) 433.
3. E. Brezin and V. Kazakov, Phys. Lett. B 236 (1990) 144;
   M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;
   D. Gross and A. Midgal, Phys. Rev. Lett. 64 (1990) 127;
4. D.J. Gross and N. Miljkovic, Phys. Lett. B 238 (1990) 217;
   E. Brezin, V.A. Kazakov and A.B. Zamolodchikov, Nucl. Phys. B 338 (1990) 673;
   P. Ginsparg and J. Zinn-Justin, Phys. Lett. B 240 (1990) 333.
5. E. Witten, Nucl. Phys. B 185 (1981) 513
   E. Witten, Nucl. Phys. B 202 (1983) 253.
6. E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Comm. Math. Phys. 59 (1978) 35.
7. G. Parisi, in Workshop on Random surfaces, quantum
   gravity and strings, Cargèse, France 1990;
   S. Shenker, in Workshop on Random surfaces, quantum
   gravity and strings, Cargèse, France 1990.
8. A. Dabholkar, Nucl. Phys. B 368 (1992) 293.
9. A. Jevicki and B. Sakita, Nucl. Phys. B 165 (1980) 511;
   S.R. Das and A. Jevicki, Mod. Phys. Lett., A 5 (1990) 1639;
   A. Jevicki and J.P. Rodrigues, Phys. Lett. B268 (1991) 53.
   J.D. Cohn and H. Dykstra, Mod. Phys. Lett. A 7 (1992) 1163;
   J.P. Rodrigues and A.J. Van Tonder, Int. J. Mod. Phys. A 8 (1993) 2517.
10. D. Gross and I.R. Klebanov, Nucl. Phys. B344 (1990) 475;
   P. Marchesini and E. Onofri, J. Math. Phys. 21 (1980) 1103.
11. S.G. Rajeev, Phys. Rev. D 42 (1990) 2779;
   S.G. Rajeev, Phys. Rev. D 44 (1991) 1836.
12. G. Ferretti and S.G. Rajeev, Phys. Lett. B 244 (1990) 265;
   G. Ferretti, Phys. Lett. B 284 (1992) 325.
13. F. Calogero, J. Math. Phys., 10 (1969) 2191;
F. Calogero, J. Math. Phys., 10 (1969) 2197;
F. Calogero, J. Math. Phys., 12 (1971) 419;
L. Brink, T.H. Hansson and M.A. Vasiliev, Phys. Lett., B 286 (1992) 109.