On Quasi-Hopf superalgebras

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Abstract

In this work we investigate several important aspects of the structure theory of the recently introduced quasi-Hopf superalgebras (QHSAs), which play a fundamental role in knot theory and integrable systems. In particular we introduce the opposite structure and prove in detail (for the graded case) Drinfeld’s result that the coproduct $\Delta' \equiv (S \otimes S) \cdot T \cdot \Delta \cdot S^{-1}$ induced on a QHSA is obtained from the coproduct $\Delta$ by twisting. The corresponding “Drinfeld twist” $F_D$ is explicitly constructed, as well as its inverse, and we investigate the complete QHSA associated with $\Delta'$. We give a universal proof that the coassociator $\Phi' = (S \otimes S \otimes S)\Phi_{321}$ and canonical elements $\alpha' = S(\beta), \beta' = S(\alpha)$ correspond to twisting the original coassociator $\Phi = \Phi_{123}$ and canonical elements $\alpha, \beta$ with the Drinfeld twist $F_D$. Moreover in the quasi-triangular case, it is shown algebraically that the R-matrix $R' = (S \otimes S)R$ corresponds to twisting the original R-matrix $R$ with $F_D$. This has important consequences in knot theory, which will be investigated elsewhere.
1 Introduction

The main aim of this paper, in conjunction with [1], is to continue the work introduced in [2] which defines $\mathbb{Z}_2$ graded versions of Drinfeld’s quasi-Hopf algebras [3], called quasi-Hopf superalgebras (QHSAs). In particular, we show that the special QHSA structure obtained by application of the antipode (see proposition [4]) actually coincides with the quasi-Hopf superalgebra structure induced by twisting with $F_D$, the “Drinfeld twist” (see equation (4.10)). In the quasi-triangular case, our results in this direction are new, even in the non-graded case.

The potential for application of these new structures is enormous. They give rise to new (non-standard) representations of the braid group and corresponding link polynomials which will be investigated elsewhere. Moreover, it has already been shown in [4–8] and [2] that QHSAs are directly relevant to elliptic quantum (super)groups [9,10], which are useful in obtaining elliptic solutions [11–16] to the (graded) quantum Yang-Baxter equation.

The importance of QHSAs in supersymmetric integrable models and the theory of knots and links [17] should become evident as the theory is developed further, which is the aim of this paper. In particular, the opposite structure is introduced and several aspects of their structure theory are investigated.

2 Quasi-Hopf Superalgebras and Twistings

This section is mostly a summary of the definitions and results given in [2]. They are important and worth restating here since they will be used frequently.

Definition 1 A $\mathbb{Z}_2$ graded quasi-bialgebra $A$ over $\mathbb{C}$ is a unital associative algebra equipped with algebra homomorphisms $\epsilon : A \to \mathbb{C}$ (counit), $\Delta : A \to A \otimes A$ (coproduct) together with an invertible homogeneous $\Phi \in A \otimes A \otimes A$ (coassociator) satisfying

\begin{align*}
(1 \otimes \Delta)\Delta(a) &= \Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi, \forall a \in A \quad (2.1) \\
(\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi &= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi) \quad (2.2) \\
(\epsilon \otimes 1)\Delta &= 1 = (1 \otimes \epsilon)\Delta \quad (2.3) \\
(1 \otimes \epsilon \otimes 1)\Phi &= 1 \quad (2.4)
\end{align*}

Properties (2.2), (2.3) and (2.4) imply that

\[(\epsilon \otimes 1 \otimes 1)\Phi = 1 = (1 \otimes 1 \otimes \epsilon)\Phi.\]
In this case, multiplication of tensor products is \( \mathbb{Z}_2 \) graded and defined as

\[
(a \otimes b)(c \otimes d) = ac \otimes bd \times (-1)^{[b][c]}
\]

for homogeneous \( a, b, c, d \in H \) and where \( [a] \in \mathbb{Z}_2 \) denotes the grading of \( a \), so that we have the following important result which will be used frequently:

\[
[a] = 1 \Rightarrow \epsilon(a) = 0.
\]

Also, the twist map \( T : H \otimes H \to H \otimes H \) is defined by

\[
T(a \otimes b) = (-1)^{[a][b]} b \otimes a.
\]

Since \( \Phi \) is homogeneous, the counit properties imply that \( \Phi \) is even (\( [\Phi] = 0 \)).

**Definition 2** A QHSA \( H \) is a \( \mathbb{Z}_2 \) graded quasi-bialgebra equipped with a \( \mathbb{Z}_2 \) graded antiautomorphism \( S : H \to H \) (antipode) and homogeneous canonical elements \( \alpha, \beta \in H \) such that for all \( a \in H \)

\[
m \cdot (1 \otimes \alpha)(S \otimes 1)\Delta(a) = \epsilon(a)\alpha,
\]

\[
m \cdot (1 \otimes \beta)(1 \otimes S)\Delta(a) = \epsilon(a)\beta,
\]

\[
m(m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha \otimes \beta)(1 \otimes 1 \otimes S)\Phi = 1,
\]

\[
m(m \otimes 1) \cdot (1 \otimes \beta \otimes \alpha)(1 \otimes S \otimes 1)\Phi^{-1} = 1.
\]

Here \( m : H \otimes H \to H \) is the multiplication map, \( m(a \otimes b) = ab, \forall a, b \in H \), and \( S \) is defined by \( S(ab) = (-1)^{[a][b]} S(b)S(a) \) for homogeneous \( a, b \). This can be extended to inhomogeneous elements by linearity. Also, since \( H \) is associative, \( m(m \otimes 1) = m(1 \otimes m) \).

If we apply \( \epsilon \) to (2.7) and (2.8) we obtain, in view of equation (2.4),

\[
\epsilon(\alpha)\epsilon(\beta) = \epsilon(\alpha\beta) = 1,
\]

so that \([\alpha] = [\beta] = 0\). It then follows by applying \( \epsilon \) to (2.5) and (2.6) that

\[
\epsilon(S(a)) = \epsilon(a), \forall a \in H.
\]

If we write

\[
\Phi = \sum_{\nu} X_\nu \otimes Y_\nu \otimes Z_\nu,
\]

and using the standard coproduct notation of Sweedler [18]

\[
\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)},
\]

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(2.5), (2.6), (2.7) and (2.8) may be expressed

\[ \sum_{(a)} S(a(1)) a a(2) = \epsilon(a) \alpha, \]

\[ \sum_{(a)} a(1) \beta S(a(2)) = \epsilon(a) \beta, \]

\[ 1 = \sum_{\nu} S(X_{\nu}) \alpha Y_{\nu} \beta S(Z_{\nu}) = \sum_{\nu} \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) \alpha \bar{Z}_{\nu}. \]

The definition of a QHSA is designed to ensure that its finite dimensional representations constitute a monoidal category.

For example, a Hopf superalgebra is a QHSA with \( \alpha = \beta = 1 \) and \( \Phi = 1^{\otimes 3} \). In fact, the relation between QHSAs and Hopf superalgebras is analogous to that between quasi-triangular Hopf superalgebras and cocommutative ones. In the latter case cocommutativity is weakened while in the former case coassociativity is weakened (in the same sense).

Before proceeding, it is important to establish some notation. For the coassociator and its inverse, we set

\[ \Phi_{123} \equiv \Phi = \sum_{\nu} X_{\nu} \otimes Y_{\nu} \otimes Z_{\nu}, \]

\[ \Phi_{-1}^{123} \equiv \Phi^{-1} = \sum_{\nu} \bar{X}_{\nu} \otimes \bar{Y}_{\nu} \otimes \bar{Z}_{\nu}. \]

We may then define the elements \( \Phi_{132} \) and \( \Phi_{312} \) (for example) by applying appropriate twists to the positions so that

\[ \Phi_{132} = (1 \otimes T) \Phi_{123} = \sum_{\nu} X_{\nu} \otimes Z_{\nu} \otimes Y_{\nu} \times (-1)^{[Y_{\nu}][Z_{\nu}]}, \]

\[ \Phi_{312} = (T \otimes 1) \Phi_{132} = \sum_{\nu} Z_{\nu} \otimes X_{\nu} \otimes Y_{\nu} \times (-1)^{[Y_{\nu}][Z_{\nu}]+[X_{\nu}][Z_{\nu}]}, \]

and similarly for \( \Phi^{-1} \), so that, for example,

\[ \Phi_{-1}^{231} = (1 \otimes T) \Phi_{-1}^{123} = (1 \otimes T)(T \otimes 1) \Phi_{-1}^{123} = \sum_{\nu} \bar{Y}_{\nu} \otimes \bar{Z}_{\nu} \otimes \bar{X}_{\nu} \times (-1)^{[X_{\nu}][Y_{\nu}]+[X_{\nu}][Z_{\nu}]}.
Note that our convention differs from the usual one (see [3] for example) which employs the inverse permutations on the positions. However, this is simply notation and is not important below.

We now have the following definition, which once again appears in [2], and which we include here for convenience.

**Definition 3** A QHSA $H$ is called quasi-triangular if there exists an invertible homogeneous $R \in H \otimes H$ such that

\[
\Delta^T(a)R = R \Delta(a), \quad \forall a \in H \tag{2.9}
\]

\[
(\Delta \otimes 1)R = \Phi^{-1}_{231} R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} \tag{2.10}
\]

\[
(1 \otimes \Delta)R = \Phi_{312} R_{13} \Phi^{-1}_{213} R_{12} \Phi_{123}, \tag{2.11}
\]

where $\Delta^T \equiv T \cdot \Delta$. Moreover, if $R$ satisfies $R^{-1} = T \cdot R \equiv R^T$, then $H$ is called triangular.

Note that this definition of quasi-triangular QHSAs ensures that the family of finite dimensional $H$-modules constitutes a quasi-tensor category.

Equations (2.10) and (2.11) immediately imply

\[
(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1,
\]

and hence $[R] = 0$. It can be shown that $R$ also satisfies the graded quasi-quantum Yang-Baxter equation (graded QQYBE)

\[
R_{12} \Phi^{-1}_{231} R_{13} \Phi_{132} R_{23} \Phi^{-1}_{123} = \Phi^{-1}_{321} R_{23} \Phi_{312} R_{13} \Phi^{-1}_{213} R_{12}. \tag{2.12}
\]

Now we come to twistings. Here we point out that the category of quasi-triangular QHSAs is invariant under a kind of gauge-transformation. Let $F \in H \otimes H$ be an invertible homogeneous element satisfying the property

\[
(1 \otimes \epsilon)F = (\epsilon \otimes 1)F = 1, \tag{2.13}
\]

(so that $[F] = 0$) with $H$ a (quasi-triangular) QHSA. Set

\[
\Delta_F(a) = F \Delta(a) F^{-1}, \quad \forall a \in H,
\]

\[
\Phi_F = (F \otimes 1) \cdot (\Delta \otimes 1)F \cdot \Phi \cdot (1 \otimes \Delta)F^{-1} \cdot (1 \otimes F^{-1}), \tag{2.14}
\]

and

\[
\alpha_F = m \cdot (1 \otimes \alpha)(S \otimes 1)F^{-1},
\]

\[
\beta_F = m \cdot (1 \otimes \beta)(1 \otimes S)F. \tag{2.15}
\]
Also put
\[ R_F = F^T R F^{-1}, \]  
(2.16)
where \( F^T \equiv T \cdot F \equiv F_{21} \). The following theorem summarises results proven in [4]. Let \((H, \Delta, \epsilon, \Phi, S, \alpha, \beta)\) denote the entire QHSA structure. Given this structure, we have

**Theorem 1** \((H, \Delta_F, \epsilon, \Phi_F, S, \alpha_F, \beta_F)\) is also a QHSA. Moreover, if \( H \) is quasi-triangular with R-matrix \( R \), then \((H, \Delta_F, \epsilon, \Phi_F, S, \alpha_F, \beta_F)\) is also quasi-triangular with R-matrix \( R_F \).

We refer to \( F \) as a twistor. \((H, \Delta_F, \epsilon, \Phi_F, S, \alpha_F, \beta_F)\) is said to be the structure of \( H \) twisted under \( F \).

It is possible to impose on \( F \) the cocycle condition
\[(F \otimes 1)(\Delta \otimes 1)F = (1 \otimes F)(1 \otimes \Delta)F. \]  
(2.17)

It is worth pointing out that if we have a quasi-triangular Hopf superalgebra \((\Phi = 1 \otimes 3, \alpha = \beta = 1)\) with structure \((H, \Delta, \epsilon, S)\) and R-matrix \( R \), and then applying a twist \( F \) that satisfies (2.17), we would obtain a Hopf superalgebra \((H, \Delta_F, \epsilon, S)\) with new R-matrix \( R_F \).

### 3 Opposite structure

Let
\[ \Delta^T = T \cdot \Delta \]
be the opposite coproduct on a QHSA \( H \). Also set
\[ \Phi^T = \Phi^{-1}_{321} = \sum Z_\nu \otimes Y_\nu \otimes X_\nu \times (-1)^{|X_\nu||Y_\nu| + |X_\nu||Z_\nu| + |Y_\nu||Z_\nu|}, \]
\[ \alpha^T = S^{-1}(\alpha), \]
and
\[ \beta^T = S^{-1}(\beta). \]

Our aim here is to prove the following.

**Proposition 1** \((H, \Delta^T, \epsilon, \Phi^T, S^{-1}, \alpha^T, \beta^T)\) is a QHSA. This is called the opposite structure on \( H \).
**proof** Firstly we prove that we indeed have a \( \mathbb{Z}_2 \) graded quasi-bialgebra structure. We note that (2.3) and (2.4) are obvious. For \( a \in A \), (2.1) may be written (in Sweedler's notation [18])

\[
a(1) \otimes \Delta(a(2)) = \Phi_{123}^{-1}(\Delta(a(1)) \otimes a(2))\Phi_{123}.
\]

Below we set

\[
\Delta(a(1)) = \sum_i a(1)_i \otimes a^i_{(1)},
\]

\[
\Delta(a(2)) = \sum_i a(2)_i \otimes a^i_{(2)}
\]

so that (2.1) becomes

\[
\sum a(1) \otimes a(2)_i \otimes a^i_{(2)} = \sum \Phi_{123}^{-1}(a(1)_i \otimes a^i_{(1)} \otimes a(2))\Phi_{123}. \quad (3.1)
\]

If we then apply the algebra homomorphism \((1 \otimes T)(T \otimes 1)(1 \otimes T)\) to (3.1) we obtain

\[
(\Delta^T \otimes 1)\Delta^T(a) = \Phi_{321}^{-1}(1 \otimes \Delta^T)\Delta^T(a)\Phi_{321}
\]

which can be written

\[
(1 \otimes \Delta^T)\Delta^T(a) = (\Phi^T)^{-1}(\Delta^T \otimes 1)\Delta^T(a)\Phi^T
\]

with \( \Phi^T \) as stated. Taking the inverse of (2.2) and applying the algebra homomorphism \((T \otimes T)(1 \otimes T \otimes 1)(T \otimes T)(1 \otimes T \otimes 1)\) to both sides, we have

\[
(\Delta^T \otimes 1 \otimes 1)\Phi^T \cdot (1 \otimes 1 \otimes \Delta^T)\Phi^T = (\Phi^T \otimes 1) \cdot (1 \otimes \Delta^T \otimes 1)\Phi^T \cdot (1 \otimes \Phi^T)
\]

which is (2.2) for the opposite structure. Hence we have proved the \( \mathbb{Z}_2 \) graded quasi-bialgebra properties.

As to the remaining properties, we use (2.3) to obtain the following:

\[
m \cdot (1 \otimes \alpha^T)(S^{-1} \otimes 1)\Delta^T(a) = \sum S^{-1}(a(2))S^{-1}(\alpha)a(1) \times (-1)^{[a(1)]}[a(2)]
\]

\[
= \sum S^{-1}(S(a(1))\alpha a(2))
\]

\[
= S^{-1}(\epsilon(a)\alpha)
\]

\[
= \epsilon(a)\alpha^T
\]

and similarly, we can use (2.4) to obtain

\[
m \cdot (1 \otimes \beta^T)(1 \otimes S^{-1})\Delta^T(a) = \epsilon(a)\beta^T.
\]
As to the opposite of (2.7), we have
\[ m(m \otimes 1) \cdot (S^{-1} \otimes 1 \otimes 1)(1 \otimes \alpha^T \otimes \beta^T)(1 \otimes 1 \otimes S^{-1})\Phi^T \]
\[ = \sum S^{-1}(\bar{Z}_\nu)S^{-1}(\beta)S^{-1}(\bar{X}_\nu) \times (-1)[\bar{X}_\nu][\bar{Y}_\nu]+[\bar{X}_\nu][\bar{Z}_\nu]+[\bar{Y}_\nu][\bar{Z}_\nu] \]
\[ = \sum S^{-1}(\bar{X}_\nu)\beta S(\bar{Y}_\nu)\alpha \bar{Z}_\nu \]
\[ = 1. \]

In a similar way, we can show the opposite of (2.8) is
\[ m(m \otimes 1) \cdot (1 \otimes \beta^T \otimes \alpha^T)(1 \otimes S^{-1} \otimes 1)\Phi^T = 1. \]

This completes the proof. \( \Box \)

Now consider (2.9). This immediately shows that the opposite R-matrix \( R^T \equiv T \cdot R \) satisfies the intertwining property under the opposite coproduct \( \Delta^T \). We now investigate (2.10) and (2.11) for this opposite structure.

Set
\[ R = \sum_i e_i \otimes e^i. \]

Applying the homomorphism \((1 \otimes T)(T \otimes 1)(1 \otimes T)\) to (2.10) gives
\[ (1 \otimes \Delta^T)R^T = \sum (X_\nu \otimes Z_\nu \otimes Y_\nu)(e^j \otimes 1 \otimes e^j)(Y_\rho \otimes Z_\rho \otimes X_\rho)(e^k \otimes e_k \otimes 1) \]
\[ \cdot (Z_\mu \otimes \bar{Y}_\mu \otimes \bar{X}_\mu) \times (-1)[\bar{Y}_\nu][\bar{Z}_\nu]+[\bar{X}_\nu][\bar{Y}_\nu][\bar{Z}_\nu]+[\bar{X}_\nu][\bar{Y}_\nu][\bar{Z}_\nu] \times (-1)[\bar{X}_\mu][\bar{Y}_\mu]+[\bar{X}_\nu][\bar{Y}_\nu][\bar{Z}_\nu] \]
\[ = \Phi_{132}^{-1} \Phi_{13}^{T} \Phi_{231} R_{123}^T \Phi_{321}^{-1}. \]

Since
\[ \Phi_{321}^{-1} = \Phi_{123}^T, \]
\[ \Phi_{231} = (\Phi^T)^{-1}_{213}, \]
\[ \Phi_{132}^{-1} = \Phi_{312}^T, \]

we have
\[ (1 \otimes \Delta^T)R^T = \Phi_{312}^T R_{13}^T (\Phi^T)^{-1}_{213} R_{12}^T \Phi_{123}^T \]
which proves (2.11) for the opposite structure.

Now applying the homomorphism \((T \otimes 1)(1 \otimes T)(T \otimes 1)\) to (2.11), we can obtain equation (2.10) for the opposite structure in a similar way:
\[ (\Delta^T \otimes 1)R^T = (\Phi^T)^{-1}_{231} R_{13}^T \Phi_{132}^T R_{23}^T (\Phi^T)^{-1}_{123}. \]
Thus we have proved

**Proposition 2** \((H, \Delta^T, \epsilon, \Phi^T, S^{-1}, \alpha^T, \beta^T)\) is a quasi-triangular QHSA with R-matrix \(R^T \equiv T \cdot R\).

It is worth noting that if \(H\) is a quasi-triangular QHSA, then its R-matrix \(R\) satisfies (2.13), so we may consider twisting \(H\) with its own R-matrix. Obviously the coproduct now reduces to the opposite one:

\[
\Delta_R(a) = R\Delta(a)R^{-1} = \Delta^T(a)
\]

for every \(a \in H\). In this case, in view of the graded QQYBE (2.12), the coassociator induced by \(R\) coincides with the opposite coassociator:

\[
\Phi_R \equiv R_{12} \cdot (\Delta \otimes 1)R \cdot \Phi \cdot (1 \otimes \Delta)R^{-1} \cdot R_{23}^{-1} \\
R_{12} \cdot \Phi_{231}^{-1} \Phi_{132} \Phi_{23} \Phi_{123}^{-1} \cdot R_{12}^{-1} \Phi_{213} \Phi_{312}^{-1} \Phi_{321} R_{23}
\]

\[
= \Phi^T
\]

The corresponding canonical elements are given, from (2.15), by

\[
\alpha_R = m \cdot (1 \otimes \alpha)(S \otimes 1)R^{-1}, \\
\beta_R = m \cdot (1 \otimes \beta)(1 \otimes S)R,
\]

while the R-matrix induced by twisting with \(R\) is, from (2.16),

\[
R^T \cdot R \cdot R^{-1} = R^T
\]

which is simply the opposite R-matrix. It thus appears that the structure induced by twisting with \(R\) corresponds to the opposite quasi-triangular QHSA structure. Note however that \(\alpha_R\) and \(\beta_R\) are defined with respect to the antipode \(S\) rather than the opposite antipode \(S^{-1}\).

So now we come to consider the opposite structure of the twisted quasi-triangular QHSA \((H, \Delta_F, \Phi_F, \epsilon, S, \alpha_F, \beta_F)\) with R-matrix \(R_F\). The opposite coproduct is clearly given by

\[
(\Delta_F)^T(a) = F^T \Delta^T(a)(F^T)^{-1}
\]
which obviously corresponds to twisting the opposite coproduct on $H$ with $F^T$. That is, 
$$(\Delta_F)^T(a) = (\Delta^T)_{F^T}(a).$$ 
To see this is in fact the case for the remaining structure, we note that the opposite coassociator to $\Phi_F$ is

$$(\Phi_F)^T = (\Phi_F^{-1})_{321}$$

$$= (T \otimes 1)(1 \otimes T)(T \otimes 1)(\Phi_F^{-1})_{123}$$

$$= (T \otimes 1)(1 \otimes T)(T \otimes 1) \cdot \{ F_{12} \cdot (1 \otimes \Delta)F \cdot \Phi_{123}^{-1} \cdot (\Delta \otimes 1)F^{-1} \cdot F_{12}^{-1} \}$$

$$= F_{23}^T \cdot (\Delta^T \otimes 1)F^T \cdot \Phi_{321}^{-1} \cdot (1 \otimes \Delta^T)(F^T)^{-1} \cdot (F_{23}^T)^{-1}$$

$$= F_{23}^T \cdot (\Delta^T \otimes 1)F^T \cdot \Phi_{123}^T \cdot (1 \otimes \Delta^T)(F^T)^{-1} \cdot (F^T)^{-1}_{23}$$

Similarly for the opposite R-matrix we have

$$(R_F)^T = FR^T(F^T)^{-1} = (R^T)_{F^T}.$$

It remains to consider the canonical elements (2.13). To this end,

$$(\alpha_F)^T = S^{-1}(\alpha_F)$$

$$= \sum S^{-1}(S(f_i)\alpha f_i)$$

$$= \sum S^{-1}(\bar{f}^{\bar{i}})S^{-1}(\alpha)f_i \times (-1)^{[\bar{i}][\bar{f}]}$$

$$= \sum m \cdot (1 \otimes S^{-1}(\alpha))(S^{-1} \otimes 1)(\bar{f}^{\bar{i}} \otimes \bar{f}_i) \times (-1)^{[\bar{i}][\bar{f}]}$$

$$= m \cdot (1 \otimes \alpha^T)(S^{-1} \otimes 1)(F^T)^{-1}$$

$$= (\alpha^T)_{F^T}$$

and similarly $(\beta_F)^T = (\beta^T)_{F^T}$. Here we have used proposition 3 and the fact that $S^{-1}$ is the antipode under the opposite structure. Thus we have proved

**Proposition 3**

$$(H, (\Delta_F)^T, \epsilon, (\Phi_F)^T, S^{-1}, (\alpha_F)^T, (\beta_F)^T) = (H, (\Delta^T)_{F^T}, \epsilon, (\Phi^T)_{F^T}, S^{-1}, (\alpha^T)_{F^T}, (\beta^T)_{F^T}).$$

Moreover, if $H$ is quasi-triangular with R-matrix $R$, then $(R_F)^T = (R^T)_{F^T}$.

Now take $H$ to be a normal quasi-triangular Hopf superalgebra and consider a twistor $F(\lambda) \in H \otimes H$ which depends on $\lambda \in H$, where we assume $\lambda$ depends on one or possibly several parameters. Here we assume that $F(\lambda)$ satisfies the shifted cocycle condition (cf. equation (2.17))

$$F_{12}(\lambda) \cdot (\Delta \otimes 1)F(\lambda) = F_{23}(\lambda + h^{(1)}) \cdot (1 \otimes \Delta)F(\lambda) \quad (3.2)$$
where \( h^{(1)} = h \otimes 1 \otimes 1 \) and \( h \in H \) fixed. We then have the following QHSA structure induced by twisting with \( F(\lambda) \):

\[
\Phi(\lambda) \equiv \Phi_{F(\lambda)} = F_{23}(\lambda + h^{(1)})F_{23}(\lambda)^{-1}
\]
\[
\Delta_{\lambda}(a) = F(\lambda)\Delta(a)F(\lambda)^{-1}, \quad \forall a \in H
\]
\[
\alpha_{\lambda} = m \cdot (S \otimes 1)F(\lambda)^{-1}
\]
\[
\beta_{\lambda} = m \cdot (1 \otimes S)F(\lambda)
\]
\[
R(\lambda) = F(\lambda)^T R F(\lambda)^{-1}
\]

\( (3.3) \)

It is straightforward to show that equations \((2.10, 2.11)\) in this case reduce to

\[
(\Delta_{\lambda} \otimes 1)R(\lambda) = \Phi_{231}^{-1}(\lambda)R_{13}(\lambda)R_{23}(\lambda + h^{(1)}),
\]
\[
(1 \otimes \Delta_{\lambda})R(\lambda) = R_{13}(\lambda + h^{(2)})R_{12}(\lambda)\Phi_{123}(\lambda),
\]

\( (3.4) \)

while the QQYBE \((2.12)\) becomes

\[
R_{12}(\lambda + h^{(3)})R_{13}(\lambda)R_{23}(\lambda + h^{(1)}) = R_{23}(\lambda)R_{13}(\lambda + h^{(2)})R_{12}(\lambda).
\]

This is the graded dynamical QYBE, of interest in obtaining elliptic solutions to the QYBE.

We can also determine the opposite structure of the above. Recall that \( H \) is also a QHSA with the opposite coproduct \( \Delta_{\lambda}^T \) and with the opposite coassociator

\[
\Phi(\lambda)^T = \Phi(\lambda)_{321}^{-1} F_{12}^T(\lambda)F_{12}^T(\lambda + h^{(3)})^{-1}.
\]

It is worth noting, in view of proposition 3, that this coincides with the QHSA structure induced on the opposite QHSA structure of \( H \) by twisting with \( F^T(\lambda) \). By applying \((1 \otimes T)(T \otimes 1)(1 \otimes T)\) to the shifted cocycle condition \((3.2)\), it can be shown that \( F^T(\lambda) \) satisfies the opposite shifted cocycle condition

\[
F_{23}(\lambda)(1 \otimes \Delta^T)F^T(\lambda) = F_{12}^T(\lambda + h^{(3)})(\Delta^T \otimes 1)F^T(\lambda).
\]

To complete the opposite QHSA structure the antipode is \( S^{-1} \), while the canonical elements are now given by

\[
\alpha_{\lambda}^T = S^{-1}(\alpha_{\lambda}),
\]
\[
\beta_{\lambda}^T = S^{-1}(\beta_{\lambda}).
\]
Applying \((T \otimes 1)(1 \otimes T)(T \otimes 1)\) to (3.4) gives the coproduct properties

\[
(\Delta^T \otimes 1)R^T(\lambda) = R^T_{13}(\lambda + h^{(2)})R^T_{23}(\lambda)\Phi_{321}(\lambda),
\]

\[
(1 \otimes \Delta^T)R^T(\lambda) = \Phi_{132}(\lambda)R^T_{13}(\lambda)R^T_{12}(\lambda + h^{(3)}),
\]

which are special cases of (2.10) and (2.11), for the coassociator concerned. Finally, the graded QQYBE satisfied by \(R^T(\lambda)\) reduces to

\[
R^T_{12}(\lambda)R^T_{13}(\lambda + h^{(2)})R^T_{23}(\lambda) = R^T_{23}(\lambda + h^{(1)})R^T_{13}(\lambda)R^T_{12}(\lambda + h^{(3)})
\]

which we refer to as the opposite graded dynamical QQYBE.

4 Drinfeld twist

This section is concerned with the QHSA structure induced by the Drinfeld twist \([3]\), and gives details of some remarkable results relating to this construction.

First it is worth establishing some useful notation. Set

\[
(1 \otimes \Delta)\Delta(a) = \sum a_{(1)} \otimes \Delta(a_{(2)}) = \sum a^R_{(1)} \otimes a^R_{(2)} \otimes a^R_{(3)},
\]

\[
(\Delta \otimes 1)\Delta(a) = \sum \Delta(a_{(1)}) \otimes a_{(2)} = \sum a^L_{(1)} \otimes a^L_{(2)} \otimes a^L_{(3)}.
\]

The following result will be used later.

**Lemma 1** \(\forall a \in H, \) we have

\[
\sum X_{\nu} a \otimes Y_{\nu} \beta S(Z_{\nu})(-1)^{[a][X_{\nu}]} = \sum a_{(1)}^L X_{\nu} \otimes a_{(2)}^L Y_{\nu} \beta S(Z_{\nu})S(a_{(3)}^L) x (-1)^{[X_{\nu}][a_{(2)}^L]}, \quad (4.1)
\]

\[
\sum S(X_{\nu}) a Y_{\nu} \otimes a Z_{\nu} (-1)^{[a][Z_{\nu}]} = \sum S(a_{(1)}^R) S(X_{\nu}) a Y_{\nu} a_{(2)}^R \otimes Z_{\nu} a_{(3)}^R x (-1)^{[Z_{\nu}][a_{(2)}^R]}, \quad (4.2)
\]

\[
\sum a \bar{X}_{\nu} \otimes S(\bar{Y}_{\nu}) a \bar{Z}_{\nu} = \sum a_{(1)} a^L_{(1)} \otimes S(a_{(2)}^L) S(\bar{Y}_{\nu}) a \bar{Z}_{\nu} a_{(3)}^L x (-1)^{[X_{\nu}][a_{(2)}^L] + [a_{(1)}^L]}, \quad (4.3)
\]

\[
\sum \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) \otimes \bar{Z}_{\nu} a = \sum a_{(1)}^R \bar{X}_{\nu} \beta S(\bar{Y}_{\nu}) S(a_{(2)}^R) \otimes a_{(3)}^R \bar{Z}_{\nu} x (-1)^{[Z_{\nu}][a_{(2)}^R] + [a_{(3)}^R]}.
\]

(4.4)
proof. For (4.1), $\Phi(1 \otimes \Delta)\Delta(a) = (\Delta \otimes 1)\Delta(a)\Phi$ can be rewritten as

$$
\sum X_\nu a_{(1)}^R \otimes Y_\nu a_{(2)}^R \otimes Z_\nu a_{(3)}^R (-1)^{[Z_\nu][a_{(1)}^R] + [a_{(2)}^R] + [Y_\nu][a_{(3)}^R]}
= \sum a_{(1)}^L X_\nu \otimes a_{(2)}^L Y_\nu \otimes a_{(3)}^L Z_\nu (-1)^{[X_\nu][a_{(1)}^L] + [a_{(2)}^L] + [Y_\nu][a_{(3)}^L]}.
$$

Then applying $(1 \otimes m)(1 \otimes 1 \otimes \beta S)$ to both sides we obtain

\begin{align*}
\text{l.h.s.} &= \sum X_\nu a_{(1)}^R \otimes Y_\nu a_{(2)}^R \beta S(a_{(3)}^R) S(Z_\nu)(-1)^{[Z_\nu][a_{(2)}^R] + [a_{(3)}^R] + [Y_\nu][a_{(1)}^R]} \\
&= \sum X_\nu a_{(1)}^R \otimes Y_\nu \beta S(Z_\nu)(-1)^{[a_{(1)}^R][Y_\nu]} \\
&= \sum a_{(1)}^L X_\nu \otimes a_{(2)}^L Y_\nu \beta S(Z_\nu)(-1)^{[a][Y_\nu]} \\
&= \text{r.h.s.} \sum a_{(1)}^L X_\nu \otimes a_{(2)}^L Y_\nu \beta S(Z_\nu)(-1)^{[a_{(2)}^L][Y_\nu]}.
\end{align*}

This proves (4.1). Parts (4.2),(4.3) and (4.4) are proved similarly and we shall only outline how they are obtained. We can arrive at (4.2) by applying $(m \otimes 1)(S \otimes \alpha \otimes 1)$ to $(\Delta \otimes 1)\Delta(a)\Phi = \Phi(1 \otimes \Delta)\Delta(a)$. Equation (4.3) can be obtained by applying $(1 \otimes m)(1 \otimes S \otimes \alpha)$ to $(1 \otimes \Delta)\Delta(a)\Phi^{-1} = \Phi^{-1}(\Delta \otimes 1)\Delta(a)$. Finally, if we apply $(m \otimes 1)(1 \otimes \beta S \otimes 1)$ to $\Phi^{-1}(\Delta \otimes 1)\Delta(a) = (1 \otimes \Delta)(\Delta(a)\Phi^{-1}$ we arrive at (4.4). This completes the proof. \[ \square \]

Also, the following equations, which arise from equation (2.2), will prove useful throughout:

\begin{align}
\Phi \otimes 1 &= (\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi \cdot (1 \otimes \Phi^{-1}) \cdot (1 \otimes \Delta \otimes 1)\Phi^{-1} \\
&= \sum (X_{(1)} \otimes X_{(2)} \otimes X_{(3)}) Y_{(1)} \otimes Y_{(2)} Z_{(1)} \otimes Z_{(2)} Z_{(3)} \otimes \tilde{Z}_{(4)} \otimes \tilde{Z}_{(5)} \\
&\quad \times (-1)^{[Z_{(1)}][a_{(1)}^R] + [a_{(2)}^R] + [Y_{(2)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R]} \\
&\quad \times (-1)^{[Z_{(3)}][a_{(1)}^R] + [a_{(2)}^R] + [Y_{(2)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R]} \\
&\quad \times (-1)^{[Y_{(2)}][a_{(2)}^R] + [a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R]}. (4.5)
\end{align}

\begin{align}
1 \otimes \Phi &= (1 \otimes \Delta \otimes 1)\Phi^{-1} \cdot (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1)\Phi \cdot (1 \otimes 1 \otimes \Delta)\Phi \\
&= \sum (X_{(1)} \otimes X_{(2)} \otimes X_{(3)}) Y_{(1)} \otimes Y_{(2)} Z_{(1)} \otimes Z_{(2)} Z_{(3)} \otimes \tilde{Z}_{(4)} \otimes \tilde{Z}_{(5)} \\
&\quad \times (-1)^{[X_{(1)}][a_{(2)}^R] + [a_{(3)}^R] + [X_{(2)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R]} \\
&\quad \times (-1)^{[X_{(2)}][a_{(2)}^R] + [a_{(3)}^R] + [X_{(3)}][a_{(3)}^R] + [Z_{(3)}][a_{(3)}^R]} \\
&\quad \times (-1)^{[X_{(3)}][a_{(2)}^R] + [a_{(3)}^R] + [X_{(3)}][a_{(3)}^R] + [Z_{(3)}][a_{(3)}^R]}. (4.6)
\end{align}

\begin{align}
\Phi^{-1} \otimes 1 &= (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes 1) \cdot (1 \otimes 1 \otimes \Delta)\Phi^{-1} \\
&= \sum (X_{(2)} \otimes X_{(3)} \otimes Y_{(2)} \otimes Z_{(2)} \otimes \tilde{Z}_{(4)} \otimes \tilde{Z}_{(5)} \\
&\quad \times (-1)^{[X_{(2)}][a_{(2)}^R] + [a_{(3)}^R] + [X_{(3)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R] + [Z_{(2)}][a_{(3)}^R]}. (4.6)
\end{align}
homomorphism and thus a new coproduct on $H$. It follows that $\Delta'$ is obtained from $\Delta$ by twisting. Given a QHSA $H$, we note that $(S \otimes S)\Delta^T$ and $\Delta^T \cdot S^{-1}$ both determine $\mathbb{Z}_2$ graded algebra antihomomorphisms. It follows that $\Delta' \equiv (S \otimes S)\Delta^T \cdot S^{-1}$ determines an algebra homomorphism and thus a new coproduct on $H$. That is, 

$$\Delta'(a) = (S \otimes S)\Delta^T(S^{-1}(a)), \ \forall a \in H.$$

**Remarks:** In the case $H$ is a normal Hopf superalgebra, $\Delta' = \Delta$ (cf. Sweedler [18]).

In what follows, we work towards showing that $\Delta'$ is obtained from $\Delta$ by twisting. Apply $(S \otimes S)\Delta^T \otimes 1$ to lemma [1], (4.1), to give

$$1 \otimes \Phi^{-1} = (1 \otimes 1 \otimes \Delta)\Phi^{-1} \cdot (\Delta \otimes 1 \otimes 1)\Phi^{-1} \cdot (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi$$

$$= \sum (X_\nu Y_\mu X_\sigma \otimes Y_\nu Y_\mu Y_\rho Y_\gamma \otimes Z_\mu Z_\nu Z_\rho)
\times (-1)^{([X_\nu]+[Y_\mu]+[X_\sigma]+[Y_\nu]+[Y_\mu]+[Y_\gamma])}
\times (-1)^{([Y_\nu]+[Y_\gamma]+[Z_\mu]+[Z_\nu]+[Z_\rho])}
\times (-1)^{([X_\sigma]+[X_\mu]+[X_\nu])}
\times (-1)^{([X_\mu]+[X_\nu]+[X_\sigma]+[Y_\nu]+[Y_\gamma]+[Z_\mu]+[Z_\nu]+[Z_\rho])}
\times (-1)^{([X_\nu]+[X_\mu]+[X_\sigma])}.$$

Now let $\gamma \in H \otimes H$ be an even element (ie. $[\gamma] = 0$). If we apply $(1^\otimes 2 \otimes \gamma)(1^\otimes 2 \otimes \Delta)$ to the above equation, we obtain

$$\sum (S \otimes S)\Delta^T(a) \cdot (S \otimes S)\Delta^T(X_\nu) \otimes Y_\nu \beta S(Z_\nu)
\Delta(Y_\nu \beta S(Z_\nu))
\Delta(S(a_{(3)}))
\times (-1)^{([X_\nu]+[a_{(1)}]+[a_{(2)}])}.$$

Then applying $(m \otimes m)(1 \otimes T \otimes 1)$ gives

$$\sum (S \otimes S)\Delta^T(a) \cdot (S \otimes S)\Delta^T(X_\nu) \cdot \gamma \cdot \Delta(Y_\nu \beta S(Z_\nu))
\Delta(Y_\nu \beta S(Z_\nu))
\Delta(S(a_{(3)}))
\times (-1)^{([X_\nu]+[a_{(1)}]+[a_{(2)}])}.$$

so that if $\gamma$ satisfies

$$\sum (S \otimes S)\Delta^T(a_{(1)}) \cdot \gamma \cdot \Delta(a_{(2)}) = \epsilon(a)\gamma,$$

(4.9)
then
\[
(S \otimes S)\Delta^T(a) \sum (S \otimes S)\Delta^T(X) \cdot \gamma \cdot \Delta(Y) \beta Z
= \sum (S \otimes S)\Delta^T(X) \cdot \epsilon(a_1) \gamma \cdot \Delta(Y) \beta Z \Delta(S(a_2))(-1)^{|a_1||X|}
= \sum (S \otimes S)\Delta^T(X) \cdot \gamma \cdot \Delta(Y) \beta Z \Delta(S(a)).
\]

This can be rewritten
\[
(S \otimes S)\Delta^T(a) F_D = F_D \Delta(S(a)), \quad \forall a \in H
\]
where
\[
F_D = \sum (S \otimes S)\Delta^T(X) \cdot \gamma \cdot \Delta(Y) \beta Z.
\] (4.10)

To find \( \gamma \in H \otimes H \) satisfying (4.9), we first note, \( \forall a \in H \),
\[
(\Delta \otimes \Delta)\Delta(a) = (\Delta \otimes 1 \otimes 1)(1 \otimes \Delta)\Delta(a)
= (\Delta \otimes 1 \otimes 1)(\Phi^{-1}(\Delta \otimes 1)\Delta(a)\Phi)
= (\Delta \otimes 1 \otimes 1)(\Phi^{-1} \cdot ((\Delta \otimes 1 \otimes 1)\Delta(a) \cdot (\Delta \otimes 1 \otimes 1)) \Phi
= (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \cdot (1 \otimes \Delta) \Delta \otimes 1)\Delta(a) \cdot (\Phi^{-1} \otimes 1)
\cdot (\Delta \otimes 1 \otimes 1) \Phi.
\]

We thus arrive at
\[
(\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi \cdot (\Delta \otimes \Delta)\Delta(a) = (1 \otimes \Delta \otimes 1)(\Delta \otimes 1)\Delta(a) \cdot (\Phi^{-1} \otimes 1) \cdot (\Delta \otimes 1 \otimes 1) \Phi. \quad (4.11)
\]

Now write
\[
(\Delta \otimes \Delta)\Delta(a) = \sum \Delta(a_1) \otimes \Delta(a_2)
= \sum a_{(1)}^L \otimes a_{(2)}^L \otimes a_{(1)}^R \otimes a_{(2)}^R.
\]
\[
(1 \otimes \Delta \otimes 1)(\Delta \otimes 1)\Delta(a) = \sum (1 \otimes \Delta)(a_{(1)}^L \otimes a_{(2)}^L \otimes a_{(3)}^L)
= \sum a_{(1)}^L \otimes a_{(2)}^L \otimes a_{(3)}^L.
\]

Lemma 2
\[
\gamma = (m \otimes m) \cdot (1 \otimes \alpha \otimes 1 \otimes \alpha)(S \otimes 1 \otimes S \otimes 1)
\cdot (1 \otimes T \otimes 1)(T \otimes 1 \otimes 1)(\Phi^{-1} \otimes 1)(\Delta \otimes 1 \otimes 1) \Phi \quad (4.12)
satisfies (4.7). Moreover
\[
\gamma = (m \otimes m) \cdot (1 \otimes \alpha \otimes 1 \otimes \alpha)(S \otimes 1 \otimes S \otimes 1) \\
\cdot (1 \otimes T \otimes 1)(T \otimes 1 \otimes 1)(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}.
\]

**Proof** First we set
\[
\sum_i A_i \otimes B_i \otimes C_i \otimes D_i \equiv \sum \bar{\chi}_\nu X^{(1)}_\mu \otimes \bar{Y}_\nu X^{(2)}_\mu \otimes \bar{Z}_\nu Y_\mu \otimes Z_\mu (-1)^{[X^{(1)}_\mu][X^{(2)}_\mu]+[Z_\mu]}
\]
\[
= (\Phi^{-1} \otimes 1)(\Delta \otimes 1 \otimes 1)\Phi.
\]

Note that \([A_i] + [B_i] + [C_i] + [D_i] = 0 \pmod{2}\). Now we have, from (1.11),
\[
\sum A_i a^{(1)}_i \otimes B_i a^{(2)}_i \otimes C_i a^{R}_i \otimes D_i a^{R}_i (-1)^{[a^{(1)}_i][[A_i] + [a^{(2)}_i][[B_i]] + [a^{R}_i][[C_i]]] + [a^{R}_i][[D_i]]}
\]
\[
= \sum a^{(1)}_i A_i \otimes a^{(2)}_i B_i \otimes a^{R}_i C_i \otimes a^{R}_i D_i
\]
\[
\times (-1)^{[A_i][[B_i]] + [a^{(2)}_i][[C_i]] + [a^{R}_i][[B_i]] + [a^{R}_i][[D_i]]}.
\]

Applying \((m \otimes m)(S \otimes \alpha \otimes S \otimes \alpha)(1 \otimes T \otimes 1)(T \otimes 1 \otimes 1)\) to the above we obtain
\[
\text{l.h.s.} = \sum S(a^{(2)}_i) S(B_i) \alpha C_i a^{R}_i \otimes S(a^{(1)}_i) S(A_i) \alpha D_i a^{R}_i
\]
\[
\times (-1)^{[a^{(1)}_i][[A_i] + [a^{(2)}_i][[B_i]] + [a^{R}_i][[C_i]] + [a^{R}_i][[D_i]]}
\]
\[
= \sum (S \otimes S)(a^{(2)}_i \otimes a^{R}_i) (S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i) (a^{R}_i \otimes a^{(2)}_i)
\]
\[
\times (-1)^{[A_i][[B_i]] + [a^{(2)}_i][[C_i]] + [a^{R}_i][[B_i]] + [a^{R}_i][[D_i]]}
\]
\[
= \sum (S \otimes S) \Delta^T(a^{(1)}_i) (S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i) \Delta(a^{(2)}_i)(-1)^{[A_i][[B_i]] + [a^{R}_i][[C_i]]}
\]
\[
= \sum (S \otimes S) \Delta^T(a^{(1)}_i) \cdot \gamma \cdot \Delta(a^{(2)}_i)
\]
\[
= \text{r.h.s.} = \sum S(B_i) \epsilon(a^{(2)}_i) \alpha C_i \otimes S(A_i) S(a^{(1)}_i) \alpha a^{R}_i D_i (-1)^{[D_i][[a^{(1)}_i] + [a^{R}_i]] + [A_i][[B_i]] + [C_i]]
\]
\[
= \sum S(B_i) \alpha C_i \otimes S(A_i) S(a^{(1)}_i) \alpha a^{R}_i D_i (-1)^{[D_i][[a^{(1)}_i] + [a^{R}_i]] + [A_i][[B_i]] + [C_i]]
\]
\[
= \epsilon(a) \sum S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i (-1)^{[A_i][[B_i]] + [C_i]]
\]
\[
= \epsilon(a) \gamma
\]

with \(\gamma\) given by (1.12). As to the second part, note that
\[
\gamma = \sum S(\bar{Y}_\nu X^{(2)}_\mu) \alpha \bar{Z}_\nu Y_\mu \otimes S(\bar{X}_\nu X^{(1)}_\mu) \alpha Z_\mu (-1)^{[X^{(1)}_\mu][X^{(2)}_\mu]+[X^{(2)}_\nu]Z_\mu+([X^{(1)}_\nu][X^{(2)}_\mu]+[X^{(1)}_\nu][X^{(2)}_\nu]+[Y^{(2)}_\nu]+[Y^{(2)}_\mu])}
\]
\[
= \sum (S \otimes S) \Delta^T(X^{(2)}_\mu)(S(\bar{Y}_\nu) \alpha Z_\nu Y_\mu \otimes S(\bar{X}_\nu) \alpha Z_\nu (-1)^{[X^{(1)}_\mu][1+X^{(2)}_\nu]}).
\]

From (2.2),
\[
(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1} = (1 \otimes \Delta \otimes 1)\Phi^{-1} \cdot (\Phi^{-1} \otimes 1)(\Delta \otimes 1 \otimes 1)\Phi
\]
\[
= \sum \bar{X}_\sigma \bar{X}_\nu X^{(1)}_\mu \otimes \bar{Y}_\sigma X^{(2)}_\mu \otimes \bar{Z}_\sigma Y_\mu \otimes \bar{Z}_\sigma Z_\mu
\]
\[
\times (-1)^{[X^{(1)}_\mu][X^{(2)}_\mu]+[X^{(2)}_\nu][Z_\mu]+([X^{(1)}_\nu][X^{(2)}_\mu]+[X^{(1)}_\nu][X^{(2)}_\nu]+[Y^{(2)}_\nu]+[Y^{(2)}_\mu])}([X^{(1)}_\mu][X^{(2)}_\nu]+[X^{(1)}_\nu][X^{(2)}_\mu]+[X^{(1)}_\nu][X^{(2)}_\nu]+[Y^{(2)}_\nu]+[Y^{(2)}_\mu])\bar{Z}_\nu].
\]
If we then apply \((m \otimes m)(S \otimes \alpha \otimes S \otimes \alpha)(1 \otimes T \otimes 1)(T \otimes 1 \otimes 1)\) to this equation, straightforward calculation reveals

\[
\begin{align*}
(m \otimes m)(S \otimes \alpha \otimes S \otimes \alpha)(1 \otimes T \otimes 1)(T \otimes 1 \otimes 1)(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1} & = \sum (S \otimes S)\Delta^T(X_\mu)(S(Y_\nu)\alpha\bar{Z}_\nu \otimes S(\bar{X}_\nu)\alpha\bar{Z}_\nu)(-1)^{[X_\nu][1+[Y_\mu]]} \\
& = \gamma.
\end{align*}
\]

□

Thus we have shown that \(F_D\) defined by (4.10) satisfies

\[
\Delta'(a)F_D = F_D\Delta(a), \quad \forall a \in H. \tag{4.13}
\]

It remains to show that \(F_D\) is invertible and thus qualifies as a twist. We proceed by constructing \(F_D^{-1}\) explicitly.

**Note:** From the definition of \(\gamma\), it is easily seen that

\[
(1 \otimes \epsilon)\gamma = \alpha \otimes \epsilon(\alpha), \quad (\epsilon \otimes 1)\gamma = \epsilon(\alpha) \otimes \alpha
\]

so that

\[
(1 \otimes \epsilon)F_D = (\epsilon \otimes 1)F_D = \sum \epsilon(\alpha)S(X_\nu)\alpha Y_\nu \beta S(Z_\nu) = \epsilon(\alpha).
\]

It then becomes clear, since \(\epsilon(\alpha)\epsilon(\beta) = 1\), that strictly speaking \(\epsilon(\beta)F_D\) qualifies as a twist. This corresponds to a non-zero scalar multiple of \(F_D\) which is not important below.

Now let \(\tilde{\gamma} \in H \otimes H\) be an even element. Apply \((1 \otimes \tilde{\gamma})(\Delta \otimes \Delta')\) to lemma \([4], (4.3)\), to give

\[
\begin{align*}
l.h.s. & = \sum \Delta(a)\Delta(\bar{X}_\nu) \otimes \tilde{\gamma}\Delta'(S(\bar{Y}_\nu)\alpha\bar{Z}_\nu) \\
= r.h.s. & = \sum \Delta(\bar{X}_\nu)\Delta(a_{(1)}^T) \otimes \tilde{\gamma}\Delta'(S(a_{(2)}^L)\Delta^T(a_{(3)}^L)\alpha\bar{Z}_\nu)\Delta'(S(\bar{Y}_\nu)\alpha\bar{Z}_\nu)(-1)^{[\bar{X}_\nu][\mu ]+[\nu ]}. \\
& = \sum \Delta(\bar{X}_\nu)\Delta(a_{(1)}^T) \otimes \tilde{\gamma}(S \otimes S)\Delta^T(a_{(2)}^L)\Delta'(S(\bar{Y}_\nu)\alpha\bar{Z}_\nu)\Delta'(a_{(3)}^L)(-1)^{[\bar{X}_\nu][\mu ]+[\nu ]}.
\end{align*}
\]

On applying \((m \otimes m)(1 \otimes T \otimes 1)\), we obtain

\[
\sum \Delta(a)\Delta(\bar{X}_\nu) \cdot \tilde{\gamma} \cdot \Delta'(S(\bar{Y}_\nu)\alpha\bar{Z}_\nu) = \sum \Delta(\bar{X}_\nu)\Delta(a_{(1)}^T) \cdot \tilde{\gamma}(S \otimes S)\Delta^T(a_{(2)}^L)\Delta'(S(\bar{Y}_\nu)\alpha\bar{Z}_\nu)\Delta'(a_{(3)}^L)(-1)^{[\bar{X}_\nu][\mu ]+[\nu ]}.
\]

If \(\tilde{\gamma}\) satisfies

\[
\sum \Delta(a_{(1)}^T) \cdot \tilde{\gamma}(S \otimes S)\Delta^T(a_{(2)}^L) = \epsilon(a)\tilde{\gamma}, \quad \forall a \in H, \tag{4.14}
\]

\]

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then
\[ F_D^{-1} \Delta'(a) = \Delta(a) F_D^{-1}, \quad \forall a \in H, \]
where
\[ F_D^{-1} = \sum \Delta(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(\bar{Y}_\nu) \alpha \bar{Z}_\nu). \]

To explicitly construct \( \bar{\gamma} \in H \otimes H \) satisfying (4.14), we note
\[
(\Delta \otimes \Delta) \Delta(a) \cdot (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \\
= (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Delta \otimes 1) \Delta(a). \tag{4.17}
\]

**Lemma 3**
\[
\bar{\gamma} = (m \otimes m) \cdot (1 \otimes \beta S \otimes 1 \otimes \beta S) \\
\cdot (1 \otimes T \otimes 1)(1 \otimes 1 \otimes T)(\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1)
\]
satisfies (4.14). Moreover,
\[
\bar{\gamma} = (m \otimes m) \cdot (1 \otimes \beta S \otimes 1 \otimes \beta S) \\
\cdot (1 \otimes T \otimes 1)(1 \otimes 1 \otimes T)(1 \otimes 1 \otimes \Delta) \Phi \cdot (1 \otimes \Phi^{-1})
\]

**Proof** The proof is very similar to that of lemma 2. We obtain the first part by applying
\[(m \otimes m)(1 \otimes \beta S \otimes 1 \otimes \beta S)(1 \otimes T \otimes 1)(1 \otimes 1 \otimes T) \text{ to } (4.17).\]
The second part is obtained by noting that \( \bar{\gamma} \) can be written as
\[
\bar{\gamma} = \sum \Delta(\bar{X}_\nu) \cdot (X_\mu \beta S(\bar{Z}_\nu) \otimes Y_\mu \beta S(\bar{Y}_\nu Z_\mu))(-1)^{[\bar{Z}_\nu][(Y_\mu)+[\bar{Y}_\nu]]+[X_\mu][Z_\mu]},
\]
then applying \((m \otimes m)(1 \otimes \beta S \otimes 1 \otimes \beta S)(1 \otimes T \otimes 1)(1 \otimes 1 \otimes T) \text{ to } \)
\[(1 \otimes 1 \otimes \Delta) \Phi \cdot (1 \otimes \Phi^{-1}) = (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1) \Phi,
\]
which is a restatement of (2.2). This proves the second part. \(\square\)

It remains to show that \( F_D^{-1} \) is indeed the inverse of \( F_D \). To this end, the following result is useful.

**Lemma 4**
\[
F_D \Delta(\alpha) = \gamma
\]
\[
\Delta(\beta) F_D^{-1} = \bar{\gamma}.
\]
proof Note that

\[ F_D \otimes 1 = (m(1 \otimes m) \otimes 1) \cdot ((S \otimes S)\Delta^T \otimes \gamma \Delta \otimes \Delta \otimes 1) \cdot (1 \otimes 1 \otimes \beta S \otimes 1) \cdot (\Phi \otimes 1) \]

\[
\sum (S \otimes S) \{ \Delta^T(X_\mu)\Delta^T(X_\rho) \} \epsilon(Y_\rho) \cdot \gamma \cdot \Delta(Y_\mu X_\sigma \beta S(Z^{(1)}_\mu Y_\sigma)) \\
\Delta(S(Y_\mu)) \epsilon(X_\mu) \otimes Z_\nu Z^{(2)}_\mu \bar{Z}_\sigma \bar{Z}_\rho (-1)^{[\bar{X}_\nu][X_\mu]+[\bar{X}_\sigma][Z_\nu]+[\bar{Y}_\tau][Z^{(2)}_\mu]+[X_\nu][\bar{Z}_\sigma]+[\bar{X}_\nu]+[Z^{(1)}_\mu]) .
\]

Now applying \( 1 \otimes 1 \otimes S \) to both sides, this reduces to

\[ F_D \otimes 1 = \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu X_\sigma \beta S(Z^{(1)}_\mu Y_\sigma)) \otimes S(Z^{(2)}_\mu \bar{Z}_\sigma)(-1)^{[\bar{X}_\nu][Z_\mu]+[\bar{Y}_\tau][Z^{(2)}_\mu]} .
\]

Further, applying \((1 \otimes 1 \otimes \Delta)(1 \otimes 1 \otimes S^{-1})\) to both sides gives

\[ F_D \otimes 1 \otimes 1 = \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu X_\sigma \beta S(Z^{(1)}_\mu Y_\sigma)) \otimes \Delta(Z^{(2)}_\mu \bar{Z}_\sigma)(-1)^{[\bar{X}_\nu][Z_\mu]+[\bar{Y}_\tau][Z^{(2)}_\mu]} .
\]

Now multiply by \( \Delta(\alpha) \otimes 1 \otimes 1 \) from the right and apply \((m \otimes m)(1 \otimes T \otimes 1)\) so that

\[ F_D \Delta(\alpha) = \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu X_\sigma \beta S(Y^{(1)}_\sigma) S(Z^{(1)}_\mu \alpha Z^{(2)}_\mu \bar{Z}_\sigma)(-1)^{[\bar{Y}_\tau][Z^{(1)}_\mu]+[\bar{Y}_\tau][Z^{(2)}_\mu]+[X_\nu][Z_\mu]} \\
= \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu X_\sigma \beta S(Y^{(1)}_\sigma) \epsilon(Z^{(2)}_\mu) \alpha \bar{Z}_\sigma) \\
= \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu \epsilon(Z^{(2)}_\mu)) \Delta(X_\sigma \beta S(Y^{(1)}_\sigma) \alpha \bar{Z}_\sigma) \\
= \sum (S \otimes S) \Delta^T(X_\mu) \cdot \gamma \cdot \Delta(Y_\mu \epsilon(Z^{(2)}_\mu)) \\
= \gamma.
\]

The second part \( \Delta(\beta)F_D^{-1} = \bar{\gamma} \) is proved similarly with the help of (1.7) and (1.13). \( \Box \)

Now set

\[
\sum \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i \equiv \sum \bar{X}^{(1)}_\nu X_\mu \otimes X^{(2)}_\nu Y_\mu \otimes \bar{Y}_\nu Z_\mu (-1)^{[\bar{X}_\nu][\bar{Y}_\tau]+[X_\nu][\bar{X}_\nu]}+\bar{X}^{(2)}_\nu[X_\nu] \\
= (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1).
\]

We compute \( F_D^{-1} \cdot F_D \):

\[
F_D^{-1} \cdot F_D = \sum \Delta(\bar{X}_{\sigma} \beta S(\bar{Y}_{\sigma}) \alpha \bar{Z}_{\sigma}) F_D^{-1} \cdot F_D \\
\sum \Delta(\bar{X}_{\sigma} \beta S(\bar{Y}_{\sigma})) F_D^{-1} \Delta' (\alpha \bar{Z}_{\sigma}) F_D \\
\sum \Delta(\bar{X}_{\sigma}) \Delta(\beta) \Delta(S(\bar{Y}_{\sigma})) F_D^{-1} \cdot F_D \Delta(\alpha) \Delta(\bar{Z}_{\sigma}) \\
\sum \Delta(\bar{X}_{\sigma}) \Delta(\beta) F_D^{-1} \Delta' (S(\bar{Y}_{\sigma})) \cdot F_D \Delta(\alpha) \Delta(\bar{Z}_{\sigma}).
\]

Using lemma 4 this reduces to

\[
F_D^{-1} \cdot F_D = \sum (\bar{X}^{(1)}_\sigma \bar{A}_i \otimes \bar{X}^{(2)}_\sigma \bar{B}_i) (\beta \otimes \beta)(S \otimes S) \cdot T(A_j \bar{Y}^{(1)}_\sigma \bar{C}_i \otimes B_j \bar{Y}^{(2)}_\sigma \bar{D}_i) \\
\cdot (\alpha \otimes \alpha)(C_j Z^{(1)}_\sigma \otimes D_j \bar{Z}^{(2)}_\sigma) (+1)^{\xi},
\]

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where
\[
\xi = [B_j([\bar{D}_i] + [Y_\sigma])] + [Y_\sigma][A_j] + [C_i] + [\bar{D}_i] + [A_j][C_i] + [\bar{D}_i] + [A_i][\bar{X}_\sigma^{(2)}]
\]
+ [C_i][\bar{Y}_\sigma^{(2)}] + [D_j][\bar{Z}_\sigma^{(1)}] + [B_j][Y_\sigma^{(1)}].

On the other hand, setting
\[
r \equiv \sum (1^{\otimes 2} \otimes A_j \otimes B_j \otimes C_j \otimes D_j) \cdot (\Delta \otimes \Delta \otimes \Delta) \Phi^{-1} \cdot (\bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i \otimes 1^{\otimes 2})
\]
\[
= \sum \bar{X}_\sigma^{(1)} \bar{A}_i \otimes \bar{X}_\sigma^{(2)} \bar{B}_i \otimes A_j \bar{Y}_\sigma^{(1)} \bar{C}_i \otimes B_j \bar{Y}_\sigma^{(2)} \bar{D}_i \otimes C_j \bar{Z}_\sigma^{(1)} \otimes D_j \bar{Z}_\sigma^{(2)} (-1)^\xi,
\]
implies
\[
F_D^{-1} \cdot F_D = \varphi(r)
\]
with \( \varphi : H^{\otimes 6} \to H^{\otimes 2} \) defined by
\[
\varphi(a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \otimes a_6) = (a_1 \otimes a_2)(\beta \otimes \beta)(S \otimes S) \cdot T(a_3 \otimes a_4) \cdot (\alpha \otimes \alpha)(a_5 \otimes a_6).
\]

**Remark:** The two equivalent expressions of \( \bar{\gamma} \) (\( \bar{\gamma} \)) implies that we can choose either
\[
\sum \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i = \begin{cases} (\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1), & \text{or} \\ (1 \otimes 1 \otimes \Delta) \Phi \cdot (1 \otimes \Phi^{-1}), & \end{cases}
\]
\[
\sum A_j \otimes B_j \otimes C_j \otimes D_j = \begin{cases} (1 \otimes \Phi) \cdot (1 \otimes 1 \otimes \Delta) \Phi^{-1}, & \text{or} \\ (\Phi^{-1} \otimes 1) \cdot (1 \otimes 1 \otimes \Delta) \Phi. & \end{cases}
\]
Similarly, we can show
\[
F_D^{-1} \cdot F_D = \bar{\varphi}(\bar{r}),
\]
where
\[
\bar{r} = \sum (A_j \otimes B_j \otimes C_j \otimes D_j \otimes 1^{\otimes 2}) \cdot (\Delta \otimes \Delta \otimes \Delta) \Phi \cdot (1^{\otimes 2} \otimes \bar{A}_i \otimes \bar{B}_i \otimes \bar{C}_i \otimes \bar{D}_i)
\]
with \( \bar{\varphi} : H^{\otimes 6} \to H^{\otimes 2} \) defined by
\[
\bar{\varphi}(a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \otimes a_6) = (S \otimes S) \cdot T(a_1 \otimes a_2) \cdot (\alpha \otimes \alpha)(a_3 \otimes a_4)
\]
\[
\cdot (\beta \otimes \beta)(S \otimes S) \cdot T(a_5 \otimes a_6).
\]

Before proceeding, it is worth noting the following properties of \( \varphi \) and \( \bar{\varphi} \) which follow immediately from their definition:
\[
\varphi(h\Delta_{23}(a)) = \epsilon(a)\varphi(h) = \varphi(\Delta_{45}(a)h), \quad (4.18)
\]
\[
\varphi(h\Delta_{14}(a)) = \epsilon(a)\varphi(h) = \varphi(\Delta_{36}(a)h), \quad (4.19)
\]
\[
\bar{\varphi}(\Delta_{23}(a)h) = \epsilon(a)\bar{\varphi}(h) = \bar{\varphi}(h\Delta_{45}(a)), \quad (4.20)
\]
\[
\bar{\varphi}(\Delta_{14}(a)h) = \epsilon(a)\bar{\varphi}(h) = \bar{\varphi}(h\Delta_{36}(a)), \quad (4.21)
\]
\( \forall a \in H, h \in H^6 \) and where we have used the notation \( \Delta_{14}(a) = \sum a_{(1)} \otimes 1 \otimes a_{(2)} \otimes 1 \otimes 1 \) (ie. \( \Delta(a) \) acting in the first and fourth components of the tensor product) etc.

Now we choose the following expressions for \( r \) and \( \bar{r} \):

\[
\begin{align*}
\bar{r} &= ((\Phi^{-1} \otimes 1)(1 \otimes 1 \otimes \Delta) \Phi \otimes 1^{\otimes 2}) \cdot (\Delta \otimes \Delta \otimes \Delta) \Phi^{-1} \cdot ((\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \otimes 1^{\otimes 2}), \\
r &= (1^{\otimes 2} \otimes (1 \otimes \Phi)(1 \otimes 1 \otimes \Delta) \Phi^{-1}) \cdot (\Delta \otimes \Delta \otimes \Delta) \Phi^{-1} \cdot ((\Delta \otimes 1 \otimes 1) \Phi^{-1} \cdot (\Phi \otimes 1) \otimes 1^{\otimes 2}),
\end{align*}
\]

which implies

\[
\begin{align*}
\bar{r} &= \sum \Delta_{45}(\bar{Z}_{\nu}^{(1)})(\Delta(\bar{X}_{\nu}) \otimes \bar{Y}_{\nu} \otimes 1^{\otimes 2} \otimes \bar{Z}_{\nu}^{(2)})(\Phi \otimes 1^{\otimes 3}) \\
&\quad \cdot (1^{\otimes 3} \otimes \Phi)(\bar{X}_{\mu}^{(1)} \otimes 1^{\otimes 2} \otimes \bar{Y}_{\mu} \otimes \Delta(\bar{Z}_{\mu}))\Delta_{23}(\bar{X}_{\mu}^{(2)})(-1)[\bar{Z}_{\nu}^{(1)}][\bar{Z}_{\nu}]+[\bar{X}_{\nu}^{(2)}][\bar{X}_{\nu}],
\end{align*}
\]

Equation (4.18) implies

\[
\varphi(r) = \varphi(s)
\]

where

\[
s = \sum (\Delta(\bar{X}_{\nu}) \otimes \bar{Y}_{\nu} \otimes 1^{\otimes 2} \otimes \bar{Z}_{\nu})(\Phi \otimes 1^{\otimes 3})(1^{\otimes 3} \otimes \Phi)(\bar{X}_{\mu} \otimes 1^{\otimes 2} \otimes \bar{Y}_{\mu} \otimes \Delta(\bar{Z}_{\mu})).
\]

Using (2.2), and noting that

\[
\begin{align*}
\Phi^{-1}_{236} &= (1^{\otimes 3} \otimes (1 \otimes T)(T \otimes 1))(1 \otimes \Phi^{-1} \otimes 1^{\otimes 2}), \\
\Phi^{-1}_{145} &= ((T \otimes 1)(1 \otimes T) \otimes 1^{\otimes 3})(1^{\otimes 2} \otimes \Phi^{-1} \otimes 1),
\end{align*}
\]

the expression for \( s \) reduces to

\[
\begin{align*}
s &= \sum \Delta_{36}(Z_{\mu})\Delta_{45}(Y_{\sigma}) \cdot (X_{\mu} \otimes Y_{\mu} \otimes 1^{\otimes 4}) \cdot \Phi^{-1}_{236} \cdot (\bar{X}_{\nu} \otimes 1^{\otimes 4} \otimes \bar{Z}_{\nu})(\bar{X}_{\sigma} \otimes 1^{\otimes 4} \otimes \bar{Z}_{\sigma}) \\
&\quad \cdot \Phi^{-1}_{145} \cdot (1^{\otimes 4} \otimes Y_{\rho} \otimes Z_{\rho})\Delta_{23}(\bar{Y}_{\nu})\Delta_{14}(X_{\rho})(-1)[Y_{\sigma}][Z_{\rho}]+[Y_{\sigma}][X_{\rho}]+[Y_{\sigma}][Z_{\rho}]+[Z_{\rho}]+[X_{\rho}].
\end{align*}
\]

Equations (4.18) and (4.19) then imply

\[
\varphi(s) = \varphi(t),
\]

where

\[
\begin{align*}
t &= \Phi^{-1}_{236} \cdot \Phi^{-1}_{145} \\
&= \sum \bar{X}_{\mu} \otimes X_{\nu} \otimes \bar{Y}_{\mu} \otimes Y_{\nu} \otimes Z_{\mu} \otimes Z_{\nu}(-1)[\bar{Z}_{\nu}][\bar{Z}_{\nu}],
\end{align*}
\]
which then implies

$$\varphi(r) = \varphi(t)$$

and

$$\varphi(r) = \varphi(t) - \sum (X_\mu \otimes X_\nu)(\beta \otimes \beta)(S(Y_\mu) \otimes S(Y_\nu))(\alpha \otimes \alpha)(\tilde{Z}_\mu \otimes \tilde{Z}_\nu)(-1)^{|\tilde{Z}_\nu|Z_\mu}+|Y_\nu||Y_\mu]$$

which reduces to

$$\varphi(r) = \varphi(t) - \sum X_\mu \beta S(Y_\mu)\alpha \tilde{Z}_\mu \otimes X_\nu \beta S(Y_\nu)\alpha \tilde{Z}_\nu = 1 \otimes 1.$$ 

Similarly, with the following choice of $\bar{r}$,

$$\bar{r} = (\Phi^{-1} \otimes 1^{\otimes 3}) \cdot (\Delta \otimes 1^{\otimes 2} \otimes \Delta)(\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)\Phi \cdot (1 \otimes \Phi)) \cdot (1^{\otimes 3} \otimes \Phi^{-1}),$$

and using (2.2) and (2.1), we obtain

$$\tilde{\varphi}(\bar{r}) = \tilde{\varphi}(\bar{s}),$$

with $\bar{s}$ defined by

$$\bar{s} = \sum (X_\mu \otimes 1^{\otimes 2} \otimes Y_\mu \otimes \Delta(Z_\mu))(1^{\otimes 3} \otimes \Phi^{-1}) \cdot (\Phi^{-1} \otimes 1^{\otimes 3})(\Delta(X_\mu) \otimes Y_\rho \otimes 1^{\otimes 2} \otimes Z_\nu)$$

which reduces to

$$\bar{s} = \sum \Delta_{14}(X_\nu)\Delta_{23}(Y_\rho)(1^{\otimes 4} \otimes Y_\nu \otimes \tilde{Z}_\nu) \cdot \Phi_{145} \cdot (X_\mu \otimes 1^{\otimes 4} \otimes Z_\mu)(X_\rho \otimes 1^{\otimes 4} \otimes Z_\rho)$$

$$\cdot \Phi_{236} \cdot (X_\sigma \otimes \tilde{Y}_\sigma \otimes 1^{\otimes 4})\Delta_{45}(Y_\mu)\Delta_{36}(\tilde{Z}_\sigma)(-1)^{|X_\nu|Z_\mu}+|Y_\nu||X_\mu|+|X_\mu|+|\tilde{Z}_\sigma|.$$

This implies that

$$\tilde{\varphi}(\bar{r}) = \tilde{\varphi}(\bar{s}) = \tilde{\varphi}(\bar{t}),$$

where

$$\bar{t} = \Phi_{145} \cdot \Phi_{236} = t^{-1},$$

from which it follows that

$$\tilde{\varphi}(\bar{r}) = \tilde{\varphi}(\bar{t}) = 1 \otimes 1,$$

so that $F_D^{-1}$ is indeed the inverse of $F_D$.

Summarising the above results, we have proved

**Theorem 2** $\Delta'$ is obtained from $\Delta$ by twisting with $F_D$. That is,

$$\Delta'(a) = F_D \Delta(a) F_D^{-1}, \ \forall a \in H$$

with $F_D$ as in (4.10) and $\gamma$ as in lemma 3. Moreover $F_D^{-1}$ is given explicitly by (4.10) with $\bar{\gamma}$ as in lemma 3.
Remark: It is actually $\bar{F}_D = \epsilon(\beta)F_D$ which qualifies as a twist. Thus we have

$$\Delta'(a) = \bar{F}_D \Delta(a) \bar{F}_D^{-1}, \forall a \in H$$

with $F_D^{-1} = \epsilon(\alpha)F_D^{-1}$. Thus $H$ is a QHSA with coproduct $\Delta'$ under the twisted structure induced by $\bar{F}_D$.

The following gives alternative expressions for $F_D$ and $F_D^{-1}$ (the proof is straightforward).

Lemma 5

$$F_D = \sum \Delta'(\bar{X}_\nu \beta S(\bar{Y}_\nu)) \cdot \gamma \cdot \Delta(\bar{Z}_\nu)$$

$$F_D^{-1} = \sum \Delta(S(X_\nu)\alpha Y_\nu) \cdot \bar{\gamma} \cdot (S \otimes S)\Delta^T(Z_\nu).$$

5 QHSA structure induced by $\Delta'$

In this section we give the full QHSA induced by $\Delta'$.

Proposition 4 $H$ is a QHSA with coproduct, coassociator and canonical elements given respectively by

$$\Delta', \Phi' \equiv (S \otimes S \otimes S)\Phi_{321}, \alpha' = S(\beta), \beta' = S(\alpha).$$

proof First we note that $\Phi' = (S \otimes S \otimes S)(\Phi^T)^{-1}$, $\Phi^T = \Phi_{321}^{-1}$. $\Phi^T$ is the coassociator associated with the opposite QHSA structure, and obeys

$$(1 \otimes \Delta^T)\Delta^T(a)(\Phi^T)^{-1} = (\Phi^T)^{-1}(\Delta^T \otimes 1)\Delta^T(a).$$

Applying $S \otimes S \otimes S$ to both sides of this expression yields

$$\Phi' \sum S(a_{(2)}) \otimes (S \otimes S)\Delta^T(a_{(1)})(-1)^{[a_{(1)}][a_{(2)}]}$$

$$= (\sum (S \otimes S)\Delta^T(a_{(2)}) \otimes S(a_{(1)})(-1)^{[a_{(1)}][a_{(2)}]} \cdot \Phi'$$

which reduces to

$$\Phi' \cdot (1 \otimes \Delta')(S \otimes S)\Delta^T(a) = \Delta'(1)(S \otimes S)\Delta^T(a) \cdot \Phi'$$

or

$$(1 \otimes \Delta')\Delta'(a) = (\Phi')^{-1}(\Delta' \otimes 1)\Delta'(a)\Phi', \forall a \in H.$$
Next, from
\[(\Delta^T \otimes 1 \otimes 1)\Phi^T \cdot (1 \otimes 1 \otimes \Delta^T)\Phi^T = (\Phi^T \otimes 1) \cdot (1 \otimes \Delta^T \otimes 1)\Phi^T \cdot (1 \otimes \Phi^T)\]
we take the inverse
\[\frac{1}{(1 \otimes 1 \otimes \Delta^T)}(\Phi^T)^{-1} \cdot (\Delta^T \otimes 1 \otimes 1)(\Phi^T)^{-1}\]
\[= \frac{1}{(1 \otimes (\Phi^T)^{-1})} \cdot (1 \otimes \Delta^T \otimes 1)(\Phi^T)^{-1} \cdot ((\Phi^T)^{-1} \otimes 1)\]
and then apply \(S \otimes S \otimes S \otimes S\) to both sides:
\[\text{l.h.s.} = ((S \otimes S)\Delta^T \cdot S^{-1} \otimes 1 \otimes 1)(S \otimes S \otimes S)(\Phi^T)^{-1}
\cdot (1 \otimes 1 \otimes (S \otimes S)\Delta^T \cdot S^{-1})(S \otimes S \otimes S)(\Phi^T)^{-1}
= (\Delta' \otimes 1 \otimes 1)\Phi' \cdot (1 \otimes 1 \otimes \Delta')\Phi'\]
\[\text{r.h.s.} = (\Phi' \otimes 1)(1 \otimes (S \otimes S)\Delta^T \cdot S^{-1} \otimes 1)(S \otimes S \otimes S)(\Phi^T)^{-1} \cdot (1 \otimes \Phi')
= (\Phi' \otimes 1) \cdot (1 \otimes \Delta' \otimes 1)\Phi' \cdot (1 \otimes \Phi').\]

Thirdly, from
\[(1 \otimes \epsilon \otimes 1)\Phi^T = 1,\]
and applying \(S \otimes S \otimes S\) to both sides gives
\[(1 \otimes \epsilon \otimes 1)\Phi' = 1.\]

As to the canonical elements \(\alpha'\) and \(\beta'\),
\[m \cdot (1 \otimes \alpha')(S \otimes 1)\Delta'(a) = m \cdot (1 \otimes S(\beta))(S \otimes 1)(S \otimes S)\Delta^T(S^{-1}(a))\]
\[= m \cdot (1 \otimes S(\beta))(S \otimes 1)(S \otimes S)\sum \bar{a}_{(2)} \otimes \bar{a}_{(1)}(-1)^{[\bar{a}_{(2)}][\bar{a}_{(1)}]}\]
\[= \sum S^2(\bar{a}_{(2)})S(\beta)S(\bar{a}_{(1)})(-1)^{[\bar{a}_{(2)}][\bar{a}_{(1)}]}\]
\[= S(\sum \bar{a}_{(1)} \beta S(\bar{a}_{(2)}))\]
\[= \epsilon(\bar{a})S(\beta)\]
\[= \epsilon(S^{-1}(a))S(\beta)\]
\[= \epsilon(a)\alpha'\]

and similarly
\[m \cdot (1 \otimes \beta')(1 \otimes S)\Delta'(a) = \epsilon(a)\beta'.\]
Finally,

\[ m(m \otimes 1) \cdot (1 \otimes \beta' \otimes \alpha')(1 \otimes S \otimes 1)(\Phi')^{-1} \]

\[ = m(m \otimes 1) \cdot (1 \otimes S(\alpha) \otimes S(\beta))(1 \otimes S \otimes 1)(S \otimes S \otimes S)\Phi_{321}^{-1} \]

\[ = \sum S(\bar{Z}_\nu)S(\alpha)S^2(\bar{Y}_\nu)S(\beta)S(\bar{X}_\nu)(-1)^{[\bar{Z}_\nu]+[\bar{X}_\nu][\bar{Y}_\nu]} \]

\[ = S(\sum \bar{X}_\nu \beta S(\bar{Y}_\nu)\alpha \bar{Z}_\nu) \]

\[ = S(1) \]

\[ = 1 \]

and similarly

\[ m(m \otimes 1) \cdot (S \otimes 1 \otimes 1)(1 \otimes \alpha' \otimes \beta')(1 \otimes 1 \otimes S)\Phi' = 1. \]

This proves that \( H \) is a QHSA with the structure given. \( \square \)

### 5.1 Connection with the Drinfeld twist

Our aim is to show that the twisted structure induced by \( F_D \) coincides precisely with the QHSA structure of proposition 4. We have already shown in theorem 2 that \( \Delta' = \Delta_{FD} \), so it remains to show that \( \Phi' = \Phi_{FD} \), while \( \alpha' \) and \( \beta' \) are equivalent to \( \alpha_{FD} \) and \( \beta_{FD} \) respectively.

For the coassociator, it remains to prove

\[ \Phi' = (S \otimes S \otimes S)\Phi_{321} \]

\[ = \Phi_{FD} \]

\[ = (F_D \otimes 1)(\Delta \otimes 1)F_D \cdot \Phi \cdot (1 \otimes \Delta)F_D^{-1} \cdot (1 \otimes F_D^{-1}), \]

or

\[ \Phi' \cdot (1 \otimes F_D)(1 \otimes \Delta)F_D = (F_D \otimes 1)(\Delta \otimes 1)F_D \cdot \Phi. \] (5.1)

To this end,

\[ (1 \otimes F_D)(1 \otimes \Delta)F_D \]

\[ \overset{1.13}{=} (1 \otimes \Delta'')F_D \cdot (1 \otimes F_D) \]

\[ \overset{1.10}{=} \sum (1 \otimes \Delta')(S(X_\nu)) \cdot (1 \otimes F_D)(1 \otimes F_D^{-1}) \]

\[ \cdot (1 \otimes \Delta')\gamma \cdot (1 \otimes F_D)(1 \otimes F_D^{-1}) \cdot (1 \otimes \Delta'\Delta(Y_\nu \beta S(Z_\nu))(1 \otimes F_D) \]

\[ \overset{\parallel}{=} \sum (1 \otimes \Delta')(S(X_\nu)) \cdot (1 \otimes F_D) \cdot (1 \otimes \Delta) \gamma \]

\[ \cdot \Phi^{-1}(\Delta \otimes 1)\Delta(Y_\nu \beta S(Z_\nu)) \cdot \Phi. \]
Now multiplying both sides by $\Phi'$ on the left gives

$$\Phi' \cdot (1 \otimes F_D)(1 \otimes \Delta)F_D$$

$$= \sum (\Delta' \otimes 1) \Delta'(S(X_\nu)) \cdot \Phi' \cdot (1 \otimes F_D) \cdot (1 \otimes \Delta) \gamma \cdot \Phi^{-1}(\Delta \otimes 1) \Delta(Y_\nu \beta S(Z_\nu)) \cdot \Phi,$$

while we can likewise show

$$(F_D \otimes 1)(\Delta \otimes 1)F_D \cdot \Phi = (\Delta' \otimes 1)F_D \cdot (F_D \otimes 1) \cdot \Phi$$

$$= \sum (\Delta' \otimes 1) \Delta'(S(X_\nu)) \Phi' \cdot (\Phi')^{-1} \cdot (F_D \otimes 1)(\Delta \otimes 1) \gamma \cdot \Phi^{-1} \cdot (\Delta \otimes 1) \Delta(Y_\nu \beta S(Z_\nu)) \cdot \Phi.$$

So to prove (5.1), it suffices to prove

$$(1 \otimes F_D)(1 \otimes \Delta) \gamma = (\Phi')^{-1} \cdot (F_D \otimes 1)(\Delta \otimes 1) \gamma \cdot \Phi,$$

or

**Lemma 6**

$$(\Phi')^{-1} \cdot (F_D \otimes 1)(\Delta \otimes 1) \gamma = (1 \otimes F_D)(1 \otimes \Delta) \gamma \cdot \Phi^{-1}. \quad (5.2)$$

**proof** Since

$$\gamma = \sum S(B_i) \alpha C_i \otimes S(A_i) \alpha D_i (-1)^{[A_i][B_i]+[C_i]},$$

we have

$$(F_D \otimes 1)(\Delta \otimes 1) \gamma = \sum F_D \Delta(S(B_i)) \Delta(\alpha) \Delta(C_i) \otimes S(A_i) \alpha D_i (-1)^{[A_i][B_i]+[C_i]}$$

$$= \sum (S \otimes S) \Delta^T(B_i) F_D \Delta(\alpha) \Delta(C_i) \otimes S(A_i) \alpha D_i (-1)^{[A_i][B_i]+[C_i]}$$

$$= \sum (S \otimes S) \Delta^T(B_i) \cdot \gamma \cdot \Delta(C_i) \otimes S(A_i) \alpha D_i (-1)^{[A_i][B_i]+[C_i]}$$

$$= \sum (S \otimes S) \Delta^T(B_i) \cdot (S \otimes S) T(A_j \otimes B_j) \cdot (\alpha \otimes \alpha) \cdot (C_j \otimes D_j)$$

$$\cdot \Delta(C_i) \otimes S(A_i) \alpha D_i (-1)^{[A_i][B_i]+[C_i]}$$

where in the penultimate equation we have used theorem 13. Set

$$(\Phi')^{-1} = \sum (S \otimes S \otimes S)(Z_\nu \otimes Y_\nu \otimes X_\nu) (-1)^{[Z_\nu]+[X_\nu][Y_\nu]}$$

which implies

$$(\Phi')^{-1}(F_D \otimes 1)(\Delta \otimes 1) \gamma = \sum (S \otimes S) T(Y_\nu \otimes Z_\nu) \cdot (S \otimes S) T(\Delta(B_i) \cdot (S \otimes S) T(A_j \otimes B_j)$$

$$\cdot (\alpha \otimes \alpha) \cdot (C_j \otimes D_j) \cdot \Delta(C_i) \otimes S(X_\nu) S(A_i) \alpha D_i$$
\[(S \otimes S)T\{(A_j \otimes B_j)\Delta(B_i)(\tilde{Y}_\nu \otimes \tilde{Z}_\nu)\} \cdot (\alpha \otimes \alpha) \cdot (C_j \otimes D_j) \cdot \Delta(C_i) \otimes S(A_i \tilde{X}_\nu) \alpha D_i \cdot (-1)^{([A_j]+[B_j])([B_i]+[X_\nu])+[B_i][X_\nu]}\]

where

\[p = \sum A_i \tilde{X}_\nu \otimes (A_j \otimes B_j) \cdot \Delta(B_i) \cdot (\tilde{Y}_\nu \otimes \tilde{Z}_\nu) \otimes (C_j \otimes D_j) \cdot \Delta(C_i) \otimes D_i \cdot (-1)^{([A_j]+[B_j])([B_i]+[X_\nu])+[B_i][X_\nu]}\]

and with \(\zeta : H^{\otimes 6} \to H^{\otimes 3}\) defined by

\[\zeta(a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \otimes a_6) = S(a_3)\alpha a_4 \otimes S(a_2)\alpha a_5 \otimes S(a_1)\alpha a_6 \cdot (-1)^{[a_1][a_2][a_3][a_4][a_5][a_6]}\]

Also, \(p\) can be reduced to

\[p = \sum (1 \otimes A_j \otimes B_j \otimes C_j \otimes D_j \otimes 1) \cdot (1 \otimes \Delta \otimes \Delta \otimes 1) (A_i \otimes B_i \otimes C_i \otimes D_i) \cdot (\Phi^{-1} \otimes 1^{\otimes 3}) \cdot \Phi^{-1} \otimes 1^{\otimes 3}\]

Now we compute the right hand side of (5.2):

\[(1 \otimes F_D)(1 \otimes \Delta_F)\gamma \cdot \Phi^{-1}\]

\[= \sum S(B_i)\alpha C_i \otimes F_D \Delta(S(A_i)\alpha D_i) \cdot \Phi^{-1}(-1)^{[A_i][B_i]+[C_i]}\]

\[= \sum S(B_i)\alpha C_i \otimes (S \otimes S)\Delta^T(A_i)F_D \Delta(\alpha) \Delta(D_i) \cdot \Phi^{-1}(-1)^{[A_i][B_i]+[C_i]}\]

\[= \sum S(B_i)\alpha C_i \otimes (S \otimes S)\Delta^T(A_i) \cdot \gamma \cdot \Delta(D_i) \cdot \Phi^{-1}(-1)^{[A_i][B_i]+[C_i]}\]

\[= \sum S(B_i)\alpha C_i \tilde{X}_\nu \otimes (S \otimes S)T\{(A_j \otimes B_j)\Delta(A_i)\} \cdot (\alpha \otimes \alpha) \cdot \Delta(D_i) \cdot (\tilde{Y}_\nu \otimes \tilde{Z}_\nu)(-1)^{[X_\nu][A_i]+[D_i]}\]

where, in the third equality we used theorem 2. Here

\[\bar{p} = \sum (A_j \otimes B_j)\Delta(A_i) \otimes B_i \otimes C_i \tilde{X}_\nu \otimes (C_j \otimes D_j) \cdot \Delta(D_i) \cdot (\tilde{Y}_\nu \otimes \tilde{Z}_\nu) \cdot (-1)^{[X_\nu][A_i]+[D_i]}\]

\[= \sum (A_j \otimes B_j \otimes 1^{\otimes 2} \otimes C_j \otimes D_j) \cdot (\Delta \otimes 1^{\otimes 2} \otimes \Delta)(A_i \otimes B_i \otimes C_i \otimes D_i) \cdot (1^{\otimes 3} \otimes \Phi^{-1}).\]
Therefore, to prove (5.2), it suffices to show that
\[ \zeta(p) = \zeta(\tilde{p}). \] (5.3)

We first note that \( \forall h \in H^{\otimes 6} \) and \( \forall a \in H \) (notation as in equations (4.18-4.21))
\[ \zeta(\Delta_{34}(a)h) = \epsilon(a)\zeta(h) = \zeta(\Delta_{25}(a)h) = \zeta(\Delta_{16}(a)h). \] (5.4)

We can also write
\[ \tilde{p} = \{(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}\}_{1256} \cdot \tilde{p}, \]
where
\[ \tilde{p} = (\Delta \otimes 1^\otimes \otimes \Delta)\{(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}\} \cdot (1^\otimes \otimes \Phi^{-1}). \]

In the following we use \( \sim \) to denote equivalence under the map \( \zeta \):
\[ \tilde{p} \rightarrow (4.2) \rightarrow (4.3) \rightarrow \]
\[ = \sum (1 \otimes X_\mu \otimes 1^\otimes \otimes Y_\mu \otimes Z_\mu)(\check{X}_\nu \otimes \check{Y}_\nu \otimes 1^\otimes \otimes \Delta(\check{Z}_\nu)) \cdot \{(1^\otimes \otimes (\Delta \otimes 1 \otimes 1)\Phi) \cdot \tilde{p} \]
\[ \rightarrow \sum (1 \otimes \Phi)_{1256} \cdot (\check{X}_\nu \otimes \check{Y}_\nu \otimes \Delta(\check{Z}_\nu^{(1)}) \otimes \check{Z}_\nu^{(2)} \otimes \check{Z}_\nu^{(3)}) \cdot \{(1^\otimes \otimes (\Delta \otimes 1 \otimes 1)\Phi) \cdot \tilde{p} \]
\[ \rightarrow \sum (1 \otimes X_\mu \otimes \Delta(Y_\mu^{(1)}) \otimes Y_\mu^{(2)} \otimes Z_\mu)\{(1^\otimes \otimes (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1)\Delta)\Phi^{-1} \]
\[ \rightarrow \sum (1 \otimes (1 \otimes (\Delta \otimes 1)\Delta \otimes 1)\Phi) \sim \{(1^\otimes \otimes ((\Delta \otimes 1)\Delta \otimes 1)\Delta)\Phi^{-1} \]
\[ = \{(1 \otimes (\Delta \otimes 1 \otimes \Delta)\otimes 1)\Phi\} \sim \{(1^\otimes \otimes (\Delta \otimes 1 \otimes \Delta)\otimes 1)\Phi\} \cdot \tilde{p}. \]

That is,
\[ \zeta(\tilde{p}) = \zeta(u) \]
where
\[ u = \{(1 \otimes (\Delta \otimes 1)(\Delta \otimes 1)\Phi) \sim \{(1^\otimes \otimes ((\Delta \otimes 1)\Delta \otimes 1)(\Delta \otimes 1)\Delta)\Phi^{-1} \sim \{(1^\otimes \otimes (\Delta \otimes 1 \otimes \Delta)\otimes 1)\Phi\} \sim \tilde{p}. \]

We now compute \( p \). Using equation (2.2) we obtain
\[ p = (1^\otimes \otimes \Phi \otimes 1) \sim \{(1 \otimes (\Delta \otimes 1)(\Delta \otimes 1)\Phi^{-1} \otimes 1) \sim \{(1 \otimes (\Delta \otimes \Delta \otimes 1)\Phi \}
\[ \sim (1^\otimes \otimes \Phi \otimes 1) \sim \{(1 \otimes (\Delta \otimes 1)(\Delta \otimes 1)\Phi^{-1} \otimes 1) \sim \{(1 \otimes (\Delta \otimes \Delta \otimes 1)\Phi \}
\]
Thus we have proved (5.3), i.e
\[ \zeta(p) = \zeta(u) = \zeta(\bar{p}). \]
This proves lemma 4, so that
\[ \Phi' = \Phi_{FD}, \]
as required. \(\Box\)

For the canonical elements, we begin with the following useful result:

**Lemma 7** For any \( \eta \in H \otimes H \),
\[
m \cdot (1 \otimes \alpha)(S \otimes 1) \{ \Delta(a) \eta \} = \epsilon(a)m \cdot (1 \otimes \alpha)(S \otimes 1)\eta, \tag{5.5}
\]
\[
m \cdot (1 \otimes \beta)(1 \otimes S) \{ \eta \Delta(a) \} = \epsilon(a)m \cdot (1 \otimes \beta)(1 \otimes S)\eta. \tag{5.6}
\]

**proof** For (5.5),
\[
l.h.s. = m \cdot (1 \otimes \alpha)(S \otimes 1) \{ \sum (a_{(1)} \otimes a_{(2)}) (\eta_i \otimes \eta^i) \} \]
\[
= \sum S(\eta_i) S(a_{(1)}) \alpha a_{(2)} \eta^i (-1)^{|\eta_i|(|a_{(1)}|+|a_{(2)}|)} \]
\[
= \epsilon(a)S(\eta_i) \alpha \eta^i \]
\[
= \epsilon(a)m \cdot (1 \otimes \alpha)(S \otimes 1)\eta \]
\[
= r.h.s.
\]
The proof of (5.6) is similar. □

For \( \alpha_{FD} \), we have

\[
\alpha_{FD} = m \cdot (1 \otimes \alpha)(S \otimes 1) F_D^{-1}
\]

\[
= m \cdot (1 \otimes \alpha)(S \otimes 1) \sum \Delta(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(\bar{Y}_\nu)\alpha \bar{Z}_\nu)
\]

\[
\overset{(5.6)}{=} m \cdot (1 \otimes \alpha)(S \otimes 1) \sum \epsilon(\bar{X}_\nu) \cdot \bar{\gamma} \cdot \Delta'(S(\bar{Y}_\nu)\alpha \bar{Z}_\nu)
\]

\[
= m \cdot (1 \otimes \alpha)(S \otimes 1) \{ \bar{\gamma} \cdot \Delta'(\alpha) \}
\]

\[
= m \cdot (1 \otimes \alpha)(S \otimes 1) \sum \Delta(\bar{X}_\nu)(X_\mu \beta S(\bar{Z}_\nu) \otimes \bar{Y}_\mu \beta S(\bar{Y}_\nu) Z_\mu) \cdot \Delta'\alpha\}
\]

\[
\times (-1)^{[Z_\nu][\{Y_\mu\}]+[\bar{Y}_\mu][Z_\mu]}
\]

\[
\overset{(5.6)}{=} m \cdot (1 \otimes \alpha)(S \otimes 1) \sum \epsilon(\bar{X}_\nu)(X_\mu \beta S(\bar{Z}_\nu) \otimes \bar{Y}_\mu \beta S(\bar{Y}_\nu) S(\bar{Z}_\nu)) \cdot \Delta'\alpha\}
\]

\[
\times (-1)^{[Z_\nu][\{Y_\mu\}]+[\bar{Y}_\mu][Z_\mu]}
\]

\[
= \sum S(\beta \bar{\alpha}_{(1)}) S(X_\mu) \alpha Y_\mu \beta S(Z_\mu) \bar{\alpha}_{(2)}
\]

\[
= \sum S(\beta \bar{\alpha}_{(1)} ) \bar{\alpha}_{(2)}
\]

\[
= S(\sum S^{-1}(\bar{\alpha}_{(2)}) \beta \bar{\alpha}_{(1)})
\]

where we have used the notation

\[
\Delta'(\alpha) = \sum \bar{\alpha}_{(1)} \otimes \bar{\alpha}_{(2)}.
\]

Now observe

\[
\sum S^{-1}(\bar{\alpha}_{(2)}) \beta \bar{\alpha}_{(1)} = m \cdot (1 \otimes \beta)(S^{-1} \otimes 1) \Delta'^T(\alpha)
\]

\[
= m \cdot (1 \otimes \beta)(S^{-1} \otimes 1)(S \otimes S) \Delta(S^{-1}(\alpha))
\]

\[
= m \cdot (1 \otimes \beta)(1 \otimes S) \Delta(S^{-1}(\alpha))
\]

\[
= \epsilon(S^{-1}(\alpha)) \beta
\]

\[
= \epsilon(\alpha) \beta
\]

which implies

\[
\alpha_{FD} = m \cdot (1 \otimes \alpha)(S \otimes 1) F_D^{-1}
\]

\[
= S(\epsilon(\alpha) \beta)
\]

\[
= \epsilon(\alpha) S(\beta)
\]

\[
= \epsilon(\alpha) \alpha'.
\]

The result for \( \beta_{FD} \), namely

\[
\beta_{FD} = m \cdot (1 \otimes \beta)(1 \otimes S) F_D = \epsilon(\beta) \beta'
\]
is proved similarly.

We have therefore proved the following:

**Theorem 3**  The QHSA structure defined on $H$ by proposition 4 is precisely equivalent to that induced by the Drinfeld twist $F_D$.

### 5.2 Drinfeld twisting on quasi-triangular QHSAs

Our aim here is to extend theorem 3 to the important case of quasi-triangular QHSAs. We begin with

**Proposition 5**  With the full QHSA structure of proposition 4, $H$ is quasi-triangular with $R$-matrix

$$ R' = (S \otimes S)R. $$

**proof**  Applying $S \otimes S$ to (2.9) gives, $\forall a \in H$

$$ R'(S \otimes S)\Delta^T(a) = (S \otimes S)\Delta(a)R', $$

so that

$$ R'\Delta'(a) = (\Delta')^T R'. $$

Applying $T \otimes 1$ to (2.10) gives

$$ (\Delta^T \otimes 1)R = \Phi_{321}^{-1}R_{23}\Phi_{312}R_{13}\Phi_{213}^{-1}. $$

Then applying $S \otimes S \otimes S$ we obtain

$$ \text{l.h.s.} = ((S \otimes S)\Delta^T \cdot S^{-1} \otimes 1)(S \otimes S)R $$

$$ = (\Delta' \otimes 1)R' $$

$$ = \text{r.h.s.} = (S \otimes S \otimes S)\Phi_{213}^{-1} \cdot (S \otimes S \otimes S)R_{13} \cdot (S \otimes S \otimes S)\Phi_{312} $$

$$ \cdot (S \otimes S \otimes S)R_{23} \cdot (S \otimes S \otimes S)\Phi_{321}^{-1}. $$

Since

$$ \Phi'_{123} = (S \otimes S \otimes S)\Phi_{321}, $$

$$ (\Phi')^{-1}_{123} = (S \otimes S \otimes S)\Phi_{321}^{-1}, $$

$$ (\Phi')^{-1}_{231} = (S \otimes S \otimes S)\Phi_{213}^{-1}, $$

$$ \Phi'_{132} = (S \otimes S \otimes S)\Phi_{312}, $$

$$ (\Phi')^{-1}_{132} = (S \otimes S \otimes S)\Phi_{312}^{-1}, $$

$$ (\Phi')^{-1}_{321} = (S \otimes S \otimes S)\Phi_{213}^{-1}. $$
we have
\[(\Delta' \otimes 1)R' = (\Phi'_{13})_{231}^{-1} R'_{132} (\Phi')_{23}^{-1} \cdot (R')_{23} (\Phi')_{123}^{-1}.\]

Similarly, applying \((S \otimes S \otimes S)(1 \otimes T)\) to (2.11) we arrive at
\[(1 \otimes \Delta')R' = \Phi'_{312} (R')_{13} (\Phi')_{213}^{-1} (R')_{12} \Phi'_{123}.\]

This completes the proof. □

We now show that the R-matrix \(R'\) coincides with the R-matrix \(R_{FD}\) induced from \(R\) by the Drinfeld twist \(F_D\). Our main result is

**Theorem 4** The quasi-triangular QHSA structure on \(H\), defined by propositions 4, 5 is precisely equivalent to the quasi-triangular QHSA structure induced on \(H\) by the Drinfeld twist \(F_D\). Namely,
\[R' = F_D^T R F_D^{-1} = R_{FD}.\]

**proof** To prove this, it suffices to show
\[R' F_D = F_D^T R\]
where
\[F_D^T = \sum (S \otimes S) \Delta(X_\nu) \cdot \gamma^T \cdot \Delta^T(Y_\nu \beta S(Z_\nu)) \cdot T \cdot F_D,\]
and \(\gamma^T = T \cdot \gamma\). To this end,
\[R' F_D = R' \sum (S \otimes S) \Delta^T(X_\nu) \cdot \gamma \cdot \Delta(Y_\nu \beta S(Z_\nu)) \cdot \sum (S \otimes S) \Delta(X_\nu) \cdot \gamma \cdot \Delta(Y_\nu \beta S(Z_\nu)) \]
and similarly
\[F_D^T R = \sum (S \otimes S) \Delta(X_\nu) \cdot \gamma^T \cdot \Delta(Y_\nu \beta S(Z_\nu)).\]
It therefore suffices to show

**Lemma 8**
\[R' \gamma = \gamma^T R.\]
proof Write \( R = \sum a_t \otimes a^t \) and note that \( R \) is even. We then have for the left hand side

\[
R' \gamma = \sum (S(a_t) \otimes S(a^t))(S(B_i)\alpha C_i \otimes S(A_i)\alpha D_i)(-1)^{[A_i][B_i]+[C_i][D_i]} \\
= \sum (S \otimes S)T\{(A_i \otimes B_i)(a^t \otimes a_t)\} \cdot (\alpha \otimes \alpha) \cdot (C_i \otimes D_i) \\
\times (-1)^{[B_i][a^t]+([A_i]+[a^t])([B_i]+[a_t])+[A_i]([B_i]+[a^t])} \\
= \sum (S \otimes S)T\{(A_i \otimes B_i)R^T\} \cdot (\alpha \otimes \alpha) \cdot (C_i \otimes D_i) \\
= \psi(v),
\]

where

\[
v = \sum (A_i \otimes B_i \otimes C_i \otimes D_i)(R^T \otimes 1^{\otimes 2})
\]

and \( \psi : H^{\otimes 4} \rightarrow H^{\otimes 2} \) is defined by

\[
\psi(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = (S \otimes S)T(a_1 \otimes a_2) \cdot (\alpha \otimes \alpha) \cdot (a_3 \otimes a_4).
\]

For the right hand side (using obvious notation), we have

\[
\gamma^T R = T\sum S(B_i')\alpha C_i' \otimes S(A_i')\alpha D_i' \cdot (e_t \otimes e^t)(-1)^{[A_i'][B_i']+[C_i'][D_i']} \\
= \sum (S \otimes S)(A_i' \otimes B_i') \cdot (\alpha \otimes \alpha)(D_i' \otimes C_i') (e_t \otimes e^t)(-1)^{[D_i'][C_i']} \\
= \sum (S \otimes S)T\{T(A_i' \otimes B_i')\} \cdot (\alpha \otimes \alpha) \cdot T\{(C_i' \otimes D_i')R^T\} \\
= \psi(\tilde{v}),
\]

where

\[
\tilde{v} = (T \otimes T) \sum (A_i' \otimes B_i' \otimes C_i' \otimes D_i')(1^{\otimes 2} \otimes R^T),
\]

so it suffices to show \( \psi(v) = \psi(\tilde{v}) \). Above we have used lemma 3, so that

\[
\sum A_i \otimes B_i \otimes C_i \otimes D_i = (\Phi^{-1} \otimes 1)(\Delta \otimes 1 \otimes 1)\Phi,
\]

\[
\sum A_i' \otimes B_i' \otimes C_i' \otimes D_i' = (1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}.
\]

In view of equation (2.9), \( v \) immediately reduces to

\[
v = (\Phi^{-1}_{123}(R^T)_{12} \otimes 1)(\Delta^T \otimes 1 \otimes 1)\Phi.
\]

With the help of the equation

\[
(1 \otimes \Delta)R^T = (T \otimes 1)(1 \otimes T)(\Delta \otimes 1)R \\
= \Phi^{-1}_{123}(R^T)_{12} \Phi_{213}(R^T)_{13} \Phi^{-1}_{312},
\]
\( v \) can be written

\[
v = \{(1 \otimes \Delta)R^T \cdot \Phi_{312}(R^T)_{13}^{-1} \Phi_{213}^{-1} \otimes 1\}(\Delta \otimes 1 \otimes 1)\Phi
\]

\[
= \sum \Delta_{23}(a_t)(a_t' \otimes 1 \otimes 1)^3\{(\Phi_{312}(R^T)_{13}^{-1} \Phi_{213}^{-1} \otimes 1\}(\Delta \otimes 1 \otimes 1)\Phi.
\]

Now observe

\[
\psi(\Delta_{23}(a)h) = \epsilon(a)\psi(h) = \psi(\Delta_{41}(a)h), \quad (5.7)
\]

which holds \( \forall a \in H, h \in H^{\otimes 4} \). In what follows, we use \( \sim \) to denote equivalence under \( \psi \).

We then have

\[
v \sim \sum \epsilon(\alpha_t)(\alpha_t' \otimes 1 \otimes 1)^3\{(\Phi_{312}(R^T)_{13}^{-1} \Phi_{213}^{-1} \otimes 1\}} \cdot (\Delta \otimes 1 \otimes 1)\Phi
\]

\[
= (T \otimes 1 \otimes 1)^2\{(\Phi_{312} \otimes 1)((R^T)_{23}^{-1} \otimes 1)\Phi^{-1} \otimes 1\}(\Delta \otimes 1 \otimes 1)\Phi
\]

\[
= (T \otimes 1 \otimes 1)^3\{(\Phi_{312} \otimes 1)(1 \otimes (R^T)^{-1} \otimes 1)\Phi \cdot (1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}\}
\]

By straightforward application of equation (5.7) we obtain

\[
v \sim \sum \epsilon(\alpha_t)(\alpha_t' \otimes 1 \otimes 1)^3\{(\Phi_{312} \otimes 1)((R^T)_{23}^{-1} \otimes 1)\Phi \cdot (1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}\}
\]

\[
= (T \otimes 1 \otimes 1)^3\{(1 \otimes \Phi_{213}^{-1})(\Phi_{312}^{-1} \otimes 1))(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1}\}
\]

As to \( \tilde{v} \) we note that

\[
(\Delta \otimes 1)R^T = (1 \otimes T)(T \otimes 1)\Phi \otimes 1 \otimes \Delta)R
\]

\[
= \Phi_{123}(R^T)_{23} \Phi_{132}^{-1}(R^T)_{13} \Phi_{231}^{-1}.
\]

Paying particular attention to equations (2.9) and (5.4), we have

\[
\tilde{v} = (T \otimes T) \cdot \{(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1} \cdot (1 \otimes 1 \otimes \Phi^T)\}
\]

\[
\sim (T \otimes T) \cdot \{(1 \otimes \Phi)(1 \otimes 1 \otimes \Phi^T)\}
\]

\[
= (T \otimes T) \{(1 \otimes \Phi_{123} R^T_{23} \otimes 1 \otimes \Delta)\Phi^{-1}\}
\]

\[
= \sum \Delta_{14}(a_t)(1 \otimes \Phi_{123} R^T_{23} \otimes 1 \otimes \Delta)(T \otimes T) \{(1 \otimes \Phi_{231}^{-1} (R^T)_{13} \otimes \Phi_{132})
\]

\[
\cdot (T \otimes T)(1 \otimes 1 \otimes \Delta)\Phi^{-1}(-1)^{|\alpha_t| |\alpha_t'|}.
\]
\[ \sum \epsilon(t^2 \otimes a_t \otimes 1)(T \otimes T) \{(1 \otimes (\Phi^{-1}_{231}(R^T)^{-1}_{13})(1 \otimes \Phi_{132})) \} \cdot (T \otimes 1^{\otimes 2})(1 \otimes 1 \otimes \Delta)\Phi^{-1} \]

\[ = \ (T \otimes 1^{\otimes 2})\{(1 \otimes 2 \otimes T)\{(1 \otimes \Phi^{-1}_{231})(1 \otimes (R^T)^{-1}_{13})(1 \otimes \Phi_{132})\} \} \cdot (1 \otimes 1 \otimes \Delta)\Phi^{-1} \}. \]

We therefore have

\[ \tilde{v} \sim (T \otimes 1 \otimes 1)\{(1 \otimes \Phi^{-1}_{231})(1 \otimes (R^T)^{-1} \otimes 1)(1 \otimes \Phi)(1 \otimes 1 \otimes \Delta)\Phi^{-1} \}. \]

Thus \( \psi(v) = \psi(\tilde{v}) \) from which the lemma follows. \( \square \) This is sufficient to prove theorem 4.

6 Concluding remarks

As noted in the introduction, the potential for applications of QHSAs is enormous, particularly in knot theory and supersymmetric integrable models, and these applications will be investigated elsewhere. In applications such as these, it is important to have a well developed and accessible structure theory, which has been the main focus of this paper. It is worth noting, even in the non-graded case, that the structure induced by the Drinfeld twist (4.10) has only been investigated for quasi-bialgebras [3]. Thus our results on the complete (graded) quasi-Hopf algebra structure, and in particular the purely algebraic and universal proof of theorem 4, are new even in the non-graded case.

Note added: After this paper was posted to the math.QA bulletin board, we were informed by F. Hausser of their paper [19], in which the result of theorem 4 was proved (in the non-graded case only) using graphical techniques on the category of finite dimensional modules of \( H \). However, as we have mentioned above, our proof is purely algebraic and universal.

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