THE PRESCRIBED CURVATURE PROBLEM FOR ENTIRE HYPERSURFACES IN MINKOWSKI SPACE

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Abstract. We prove three results in this paper. First, we prove for a wide class of functions $\varphi \in C^2(S^{n-1})$ and $\psi(X, \nu) \in C^2(\mathbb{R}^{n+1} \times \mathbb{H}^n)$, there exists a unique, entire, strictly convex, spacelike hypersurface $M_u$ satisfying $\sigma_k(\kappa[M_u]) = \psi(X, \nu)$ and $u(x) \to |x| + \varphi \left( \frac{x}{|x|} \right)$ as $|x| \to \infty$. Second, when $k = n-1, n-2$, we show the existence and uniqueness of entire, $k$-convex, spacelike hypersurface $M_u$ satisfying $\sigma_k(\kappa[M_u]) = \psi(x, u(x))$ and $u(x) \to |x| + \varphi \left( \frac{x}{|x|} \right)$ as $|x| \to \infty$. Last, we obtain the existence and uniqueness of entire, strictly convex, downward translating solitons $M_u$ with prescribed asymptotic behavior at infinity for $\sigma_k$ curvature flow equations. Moreover, we prove that the downward translating solitons $M_u$ have bounded principal curvatures.

1. Introduction

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$ 

In this paper, we will devote ourselves to the study of spacelike hypersurfaces with prescribed $\sigma_k$ curvature in Minkowski space $\mathbb{R}^{n,1}$. Here, $\sigma_k$ is the $k$-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$ 

Any such hypersurface $M$ can be written locally as a graph of a function $x_{n+1} = u(x), x \in \mathbb{R}^n$, satisfying the spacelike condition

$$|Du| < 1.$$ 

More precisely, we will focus on the following equation:

$$\sigma_k(\kappa[M_u]) = \psi(X, \nu),$$

where $X = (x, u(x))$ is the position vector of $M_u = \{(x, u(x)) | x \in \mathbb{R}^n\}$, $\nu = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}}$ is the upward unit normal lying on the hyperboloid $\mathbb{H}^n$, and $\kappa[M_u] =$

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(\kappa_1, \ldots, \kappa_n) are the principal curvatures of \( M_u \). Thus equation (1.2) can be rewritten as

\[
\sigma_k(\kappa[M_u]) = \psi(x, u(x), Du).
\]

Notice that the right hand side functions \( \psi \) of (1.2) and (1.3) are different. Slightly extending the notation, we use the same symbol here.

The classical Minkowski problem asks for the construction of a strictly convex compact surface \( \Sigma \) whose Gaussian curvature is a given positive function \( f(\nu(X)) \), where \( \nu(X) \) denotes the normal to \( \Sigma \) at \( X \). This problem has been discussed by Nirenberg [24], Pogorelov [27], and Cheng-Yau [11]. The general problem of finding strictly convex hypersurfaces with prescribed surface area measures is called the Christoffel–Minkowski problem. This type of problems can be deduced to a fully nonlinear equation of the form (1.2). It may be traced back to Alexandrov [1] who established the problem of prescribing zeroth curvature measure. Later on, the prescribed curvature measure problem in convex geometry has been extensively studied by Alexandrov [2], Pogorelov [26], Guan-Lin-Ma [18], and Guan-Li-Li [17]. A more general form of the prescribed curvature measure problem can be expressed as (1.3). In particular, Guan-Ren-Wang [19] solved this problem in Euclidean space for convex hypersurfaces. Other related studies and references may be found in [3, 9, 10, 15, 25, 33].

Our goal here is to construct entire, spacelike hypersurfaces satisfying equation (1.2) in Minkowski space. The main results of this paper are the following.

**Theorem 1.** Suppose \( \varphi \) is a \( C^2 \) function defined on \( S^{n-1} \), i.e., \( \varphi \in C^2(S^{n-1}) \), \( \psi(X, \nu) \in C^2(\mathbb{R}^{n+1} \times \mathbb{H}^n) \) is a positive function, and \( c_1 \geq \psi(X, \nu) \geq c_2 \) for some positive constants \( c_1, c_2 \). We further assume that \( \psi_{x_{n+1}} \geq 0 \) (or \( \psi_u \geq 0 \)). If either \( \psi^{-1/k}(X, \nu) \) is locally strictly convex with respect to \( X \) for any \( \nu \) or \( \psi \) only depends on \( \nu \), then there exits a unique, entire, strictly convex, spacelike hypersurface \( M_u = \{(x, u(x))| x \in \mathbb{R}^n \} \) satisfying (1.2). Moreover, as \( |x| \to \infty \),

\[
u(x) \to |x| + \varphi \left( \frac{x}{|x|} \right).
\]

**Remark 2.** Indeed, from the proof of the \( C^2 \) global estimate Lemma 4, we can see that, the assumption \( \psi(X, \nu) \) does not depend on \( X \) can be replaced by a weaker assumption, that is, \( \psi^{-1/k}(X, \nu) \) is convex with respect to \( X \) and the corresponding form \( \psi(x, u, Du) \) does not depend on \( |x| \).

**Remark 3.** In the proof, we only can see that the hypersurface \( M_u \) we constructed is convex. In order to say its strictly convex, we need to apply the Constant Rank Theorem (see Theorem 1.2 in [16] and Theorem 27 in [35]) and the Splitting Theorem (see Theorem 28 in [35]) to obtain that if \( M_u \) has a degenerate point in the interior, then \( M_u = M^l \times \mathbb{R}^{n-l} \), where \( M^l \subset \mathbb{R}^l \) is a strictly convex, space like hypersurface. This contradicts (1.4).

Before stating our second result, we need the following definition:
Definition 4. A $C^2$ regular hypersurface $M \subset \mathbb{R}^{n+1}$ is $k$-convex, if the principal curvatures of $M$ at $X \in M$ satisfy $\kappa[X] \in \Gamma_k$ for all $X \in M$, where $\Gamma_k$ is the Gårding cone
\[
\Gamma_k = \{ \kappa \in \mathbb{R}^n | \sigma_m(\kappa) > 0, m = 1, \ldots, k \}.
\]

Using the newly developed methods in [28] and [29], we are able to generalize results in [5]. We prove

Theorem 5. Suppose $\varphi$ is some $C^2$ function defined on $\mathbb{S}^{n-1}$ and $\psi(x,u(x)) \in C^2(\mathbb{R}^{n+1})$ is a positive function satisfying $c_1 \geq \psi(x,u(x)) \geq c_2$ for $c_1, c_2 > 0$. We further assume that $k = n-1, n-2$, and $\psi_u \geq 0$. Then there exists a unique, $k$-convex, spacelike hypersurface $M_u = \{(x,u(x))| x \in \mathbb{R}^n\}$ satisfying
\[
\sigma_k(\kappa[M_u]) = \psi(x,u(x)).
\]
Moreover, as $|x| \to \infty$,
\[
u(1.5) \quad u(x) \to |x| + \varphi \left( x \left/ |x| \right. \right).
\]

Now, let’s consider the $\sigma_k$ curvature flow with forcing term in Minkowski space:
\[
u(1.7) \quad \frac{dX}{dt} = -\left( C - \frac{\sigma_{1/k}^k(\kappa[M_u])}{(n/k)^{1/k}} \right) \nu,
\]
where $\kappa[M_u] \in \Gamma_k$. This can be rewritten as the equation for the height function $u$,
\[
u(1.8) \quad \frac{u_t}{\sqrt{1-|Du|^2}} = \frac{\sigma_{1/k}^k(\kappa[M_u])}{(n/k)^{1/k}} - C.
\]
The downward translating soliton to (1.8) is of the form
\[
u(1.9) \quad u(x,t) = u(x) - t,
\]
where $u(x)$ satisfies
\[
u(1.10) \quad \left( \frac{\sigma_k}{(n/k)^{1/k}} \right)^{1/k}(\kappa[M_u]) = C - \frac{1}{\sqrt{1-|Du|^2}}.
\]
The above equation (1.10) can be viewed as the “degenerate” type of (1.2). In this case, we prove the following theorem:

Theorem 6. Suppose $\varphi$ is a $C^2$ function defined on $\mathbb{S}^{n-1}$ := \{ $x \in \mathbb{R}^n | |x| = \tilde{C}$ \}, where $\tilde{C} = \sqrt{1 - (\frac{1}{C})^2}$ and $C > 1$ is a constant. There exists a unique, strictly convex solution $u : \mathbb{R}^n \to \mathbb{R}$ of (1.10) such that as $|x| \to \infty$,
\[
u(1.11) \quad u(x) \to \tilde{C}|x| - \frac{1}{C^2} \sqrt{\frac{n-k}{n}} \log |x| + \varphi \left( \tilde{C} x \left/ |x| \right. \right).
\]
Moreover, $M_u = \{(x,u(x))| x \in \mathbb{R}^n\}$ has bounded principal curvatures.

When $k = 1$, (1.10) has been studied in [20] and [31]; when $k = 2$, (1.10) has been studied in [6].
Remark 7. Under our assumptions on $\psi$, we can see that the linearized operators of equations (1.2), (1.5), and (1.10) satisfy the maximum principle. Therefore, the uniqueness properties in Theorem 1, 5, and 6 follow from the maximum principle directly.

The rest of this paper is organized as follows. In Section 2, we introduce some basic formulas and notations. The solvability of equations (1.2) and (1.5) on bounded domain (Dirichlet problem) is discussed in Section 3. We prove the local $C^1$ and $C^2$ estimates for solutions of equations (1.2) and (1.5) in Section 4. This leads to the completion of the proof of our first two main results, Theorem 1 and Theorem 5 in Section 6. Section 6 and Section 7 are devoted to Theorem 6. In particular, in Section 6, we study the radially symmetric solution to equation (1.10), this solution will be used to construct barrier functions in Section 7. We finish the proof of Theorem 6 in Section 7.

2. Preliminaries

In this paper, we will follow notations in [35]. For readers convenience, we will include some basic notations and formulas in this section. Readers who are already familiar with calculations in Minkowski space can skip this section.

We first recall that the Minkowski space $\mathbb{R}^{n,1}$ is $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

\[ ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2. \]

Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n,1}$.

2.1. Vertical graphs in $\mathbb{R}^{n,1}$. A spacelike hypersurface $M$ in $\mathbb{R}^{n,1}$ is a codimension one submanifold whose induced metric is Riemannian. Locally $M$ can be written as a graph

\[ M_u = \{ X = (x, u(x)) | x \in \mathbb{R}^n \} \]

satisfying the spacelike condition (1.1). Let $E = (0, \cdots, 0, 1)$, then the height function of $M$ is $u(x) = -\langle X, E \rangle$. It's easy to see that the induced metric and second fundamental form of $M$ are given by

\[ g_{ij} = \delta_{ij} - D_x u D_{x_j} u, \quad 1 \leq i, j \leq n, \]

and

\[ h_{ij} = \frac{u_{x_i x_j}}{\sqrt{1 - |Du|^2}}, \]

while the timelike unit normal vector field to $M$ is

\[ \nu = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}}, \]

where $Du = (u_{x_1}, \cdots, u_{x_n})$ and $D^2 u = (u_{x_i x_j})$ denote the ordinary gradient and Hessian of $u$, respectively. By a straightforward calculation, we have the principle curvatures of $M$ are eigenvalues of the symmetric matrix $A = (a_{ij})$:

\[ a_{ij} = \frac{1}{w} \gamma_{ik} \gamma_{lj}. \]
where \( \gamma_{ik} = \delta_{ik} + \frac{u_i u_k}{w(1+w)} \) and \( w = \sqrt{1 - |Du|^2} \). Note that \((\gamma_{ij})\) is invertible with inverse \( \gamma_{ij} = \delta_{ij} - \frac{u_i u_j}{1+w} \), which is the square root of \((g_{ij})\).

Let \( S \) be the vector of \( n \times n \) symmetric matrices and

\[
S_k = \{ A \in S : \lambda(A) \in \Gamma_k \},
\]

where \( \lambda(A) = (\lambda_1, \cdots, \lambda_n) \) denotes the eigenvalues of \( A \). Define a function \( F \) by

\[
F(A) = \sigma_k(\lambda(A)), \quad A \in S_k,
\]

then (1.3) can be written as

\[
(2.1) \quad F\left( \frac{1}{w} \gamma_{ik} u_{kl} \gamma_{lj} \right) = \psi(x, u(x), Du).
\]

Throughout this paper we denote

\[
F^{ij}_{kl}(A) = \frac{\partial F}{\partial a^{ij}_{kl}}(A), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}.
\]

Now, let \( \{ \tau_1, \tau_2, \cdots, \tau_n \} \) be a local orthonormal frame on \( TM \). We will use \( \nabla \) to denote the induced Levi-Civita connection on \( M \). For a function \( v \) on \( M \), we denote \( v_i = \nabla_{\tau_i} v, \quad v_{ij} = \nabla_{\tau_i} \nabla_{\tau_j} v \), etc. In particular, we have

\[
|\nabla u| = \sqrt{g_{ij} u_x^i u_x^j} = \frac{|Du|}{\sqrt{1 - |Du|^2}}.
\]

Using normal coordinates, we also need the following well known fundamental equations for a hypersurface \( M \) in \( \mathbb{R}^{n,1} \):

\[
\begin{align*}
X_{ij} &= h_{ij} \nu \quad \text{(Gauss formula)} \\
(\nu)_i &= h_{ij} \tau_j \quad \text{(Weigarten formula)} \\
h_{ijk} &= h_{ikj} \quad \text{(Codazzi equation)} \\
R_{ijkl} &= -(h_{ik} h_{jl} - h_{il} h_{jk}) \quad \text{(Gauss equation)},
\end{align*}
\]

and the Ricci identity,

\[
R_{ijkl} = h_{ijkl} + h_{mj} R_{imlk} + h_{im} R_{jmkl} = h_{kl} h_{ij} - (h_{mj} h_{il} - h_{ml} h_{ij}) h_{mk} - (h_{mj} h_{kl} - h_{ml} h_{kj}) h_{mi}.
\]

2.2. The Gauss map. Let \( M \) be an entire, strictly convex, spacelike hypersurface, \( \nu(X) \) be the timelike unit normal vector to \( M \) at \( X \). It’s well known that the hyperbolic space \( \mathbb{H}^n(-1) \) is canonically embedded in \( \mathbb{R}^{n,1} \) as the hypersurface

\[
\langle X, X \rangle = -1, \quad x_{n+1} > 0.
\]

By parallel translating to the origin we can regard \( \nu(X) \) as a point in \( \mathbb{H}^n(-1) \). In this way, we define the Gauss map:

\[
G : M \to \mathbb{H}^n(-1); \quad X \mapsto \nu(X).
\]

Next, let’s consider the support function of \( M \). We denote

\[
v := \langle X, \nu \rangle = \frac{1}{\sqrt{1 - |Du|^2}} \left( \sum_i x_i \frac{\partial u}{\partial x_i} - u \right).
\]
Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$. We will also denote $\{e^*_1, \cdots, e^*_n\}$ the pull-back of $e_i$ by the Gauss map $G$. Similar to the convex geometry case, we denote

$$\Lambda_{ij} = v_{ij} - v\delta_{ij}$$

the hyperbolic Hessian. Here $v_{ij}$ denote the covariant derivatives with respect to the hyperbolic metric.

Let $\nabla$ be the connection of the ambient space. Then, we have

$$X = \sum_i v_i e_i - v \nu$$

and

$$\nabla_{e^*_j} X = \sum_k (e_j(v_k)e_k + v_k \nabla_{e_j} e_k) - v_j \nu - v \nabla_{e_j} \nu = \sum_k \Lambda_{kj} e_k.$$ 

Note also that,

$$g_{ij} = \langle \nabla_{e^*_i} X, \nabla_{e^*_j} X \rangle = \sum_k \Lambda_{ik} \Lambda_{kj},$$

(2.4)

$$h_{ij} = \langle \nabla_{e^*_i} X, \nabla_{e^*_j} \nu \rangle = \Lambda_{ij},$$

(2.5)

This implies that the eigenvalues of the hyperbolic Hessian are the curvature radius of $M$. Therefore, equation (1.2) can be written as

$$F(v_{ij} - v\delta_{ij}) = \frac{1}{\psi(X, \nu)},$$

(2.6)

where $F(A) = \frac{\sigma_n}{\sigma_{n-k}}(\lambda(A))$. Moreover, it is clear that

$$(\nabla_{e_j} \nabla_{e_i} \nu)^\perp = \delta_{ij} \nu,$$

(2.7)

which yields, for $k = 1, 2, \cdots, n + 1$,

$$\nabla_{e_j} \nabla_{e_i} x_k = x_k \delta_{ij},$$

(2.8)

where $x_k$ is the coordinate function.

2.3. **Legendre transform.** Suppose $M$ is an entire, strictly convex, spacelike hypersurface. Then $M$ is the graph of a convex function

$$x_{n+1} = -\langle X, E \rangle = u(x_1, \cdots, x_n),$$

where $E = (0, \cdots, 0, 1)$. Introduce the Legendre transform

$$\xi_i = \frac{\partial u}{\partial x_i}, \quad u^* = \sum x_i \xi_i - u.$$

Next, we calculate the first and the second fundamental forms in terms of $\xi_i$. Since it is well known that,

$$\left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right) = \left( \frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j} \right)^{-1},$$

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We have, the first and the second fundamental forms can be rewritten as:

\[ g_{ij} = \delta_{ij} - \xi_i \xi_j, \quad h_{ij} = \frac{u^{*ij}}{\sqrt{1 - |\xi|^2}}, \]

where \((u^{*ij})\) denotes the inverse matrix of \((u^*_{ij})\) and \(|\xi|^2 = \sum_i \xi_i^2\). Now, let \(W\) be the Weingarten matrix of \(M\), then

\[(W^{-1})_{ij} = \sqrt{1 - |\xi|^2} g_{ik} u^*_{kj}.\]

From the discussion above, we can see that if \(M_u = \{(x, u(x)) | x \in \mathbb{R}^n\}\) is an entire, strictly convex, spacelike hypersurface satisfying \(\sigma_k(\kappa[M]) = \psi\), then the Legendre transform of \(u\) denoted by \(u^*\), satisfies

\[
F(w^* \gamma^*_{ik} u^*_{kj} \gamma^*_{lj}) = \frac{\sigma_n}{\sigma_{n-k}} (\kappa^*[w^* \gamma^*_{ik} u^*_{kj} \gamma^*_{lj}]) = \frac{1}{\psi}.
\]

Here, \(w^* = \sqrt{1 - |\xi|^2}\) and \(\gamma^*_{ij} = \delta_{ij} - \frac{\xi_i \xi_j}{1 + w}\) is the square root of the matrix \(g_{ij}\).

3. The Dirichlet Problem

We will divide this section into two subsections. In the first subsection, we only consider the convex solution to (1.2). In the second subsection, we restrict ourselves to the case when \(k = n - 1 (n \geq 3), n - 2 (n \geq 5)\), and we will consider the \(k\)-convex, spacelike solution to (1.5). When \(k = 2\), this problem has been studied by [4] and [34].

3.1. Dirichlet Problem for \(1 \leq k \leq n\). Recall that in [35] we proved the following Lemma.

**Lemma 8.** Let \(F \subset S^{n-1}\), \(\tilde{F} = \text{Conv}(F)\), and \(u^*\) be a solution of

\[
\begin{cases}
\tilde{F}(w^* \gamma^*_{ik} u^*_{kj} \gamma^*_{lj}) = \frac{1}{(\frac{n}{k})^\frac{1}{k}} & \text{in } \tilde{F} \\
u^* = \varphi & \text{on } \partial \tilde{F},
\end{cases}
\]

where \(\tilde{F}(w^* \gamma^*_{ik} u^*_{kj} \gamma^*_{lj}) = \left(\frac{\sigma_n}{\sigma_{n-k}}\right)^\frac{1}{k} (\kappa^*[w^* \gamma^*_{ik} u^*_{kj} \gamma^*_{lj}])\). Then, the Legendre transform of \(u^*\) denoted by \(u\) satisfies, when \(\frac{x}{|x|} \in F\)

\[
u(x) - |x| \to -\varphi \left(\frac{x}{|x|}\right) \quad \text{as } |x| \to \infty, \text{ uniformly.}
\]

Notice that the proof of the above Lemma is independent of the equation that the function \(u^*\) satisfies. Therefore, adapting the above Lemma to the settings in this paper, this Lemma tells us that if a strictly convex function \(u^* : B_1 \to \mathbb{R}\) satisfies \(u^*(\xi) = -\varphi(\xi)\) for \(\xi \in \partial B_1\), then the Legendre transform of \(u^*\) denoted by \(u\), satisfies \(u(x) \to |x| + \varphi \left(\frac{x}{|x|}\right)\) as \(|x| \to \infty\). Moreover, by Theorem 4 in [35], there exists two solutions \(u, \bar{u}\) such that

\[
\sigma_k(\kappa[M_y]) = c_1,
\]
\[ \sigma_k(\kappa[M_u]) = c_2, \]

and as \( |x| \to \infty \)

\[ y(x) - |x|, \bar{u}(x) - |x| \to \varphi \left( \frac{x}{|x|} \right). \]

Here, the constants \( c_1, c_2 \) are the same as the ones in Theorem 1. Throughout this paper, we will denote the Legendre transforms of \( y, \bar{u} \) by \( y^*, \bar{u}^* \) respectively. It’s easy to see that \( \bar{u}^* \) and \( \bar{u}^* \) are the super- and sub- solutions of (2.9).

Combining the discussions above with Section 2, we conclude that in order to find an entire, strictly convex solution \( u \) of (1.3), we only need to solve the following equation:

(3.3)

\[
\begin{cases}
F(w^*\gamma_{ik}u^*_kl^*_l) = \psi^* \text{ in } B_1, \\
u^* = -\varphi \text{ on } \partial B_1,
\end{cases}
\]

where

\[ \psi^*(\xi, u^*, Du^*) = \frac{1}{\psi(x, u, Du)} = \frac{1}{\psi(Du^*, \xi \cdot Du^* - u^*, \xi)}, \]

and

\[ F(w^*\gamma_{ik}u^*_kl^*_l) = \frac{\sigma_n}{\sigma_{n-k}}(\kappa^*[w^*\gamma_{ik}u^*_kl^*_l]). \]

Note that by our assumption in Theorem 1 we have,

(3.4)

\[ \psi_{u^*} = \frac{\psi_u}{\psi^2} \geq 0. \]

Thus, equation (3.3) possesses the maximum principle.

Notice that equation (3.3) is degenerate on \( \partial B_1 \). Therefore, we will consider the approximate equation:

(3.5)

\[
\begin{cases}
F(w^*\gamma_{ik}u^*_kl^*_l) = \psi^* \text{ in } B_r, \\
u^* = u^* \text{ on } \partial B_r
\end{cases}
\]

where \( 0 < r < 1 \).

By continuity method we know that, if we can obtain a prior estimates up to the second order, then we can show (3.5) has a unique, strictly convex solution \( u^* \). In view of the super- and sub- solutions \( y^*, \bar{u}^* \), the \( C^0 \) estimates are easy to obtain. The \( C^1 \) estimates can be derived by following the argument in Subsection 9.2 of [30]. The \( C^2 \) estimate on the boundary can be derived from Lemma 27 in [30] and the argument of Bo Guan [14]. In the following, we only need to consider the global \( C^2 \) estimate.

Let \( M_u = \{(x, u(x))|x \in \mathbb{R}^n\} \) be a strictly convex, spacelike hypersurface, \( v = \langle X, \nu \rangle \) be the support function of \( M_u \), and \( u^* \) be the Legendre transform of \( u \). From Subsection 2.2 and 2.3 we know that \( \lambda[v_{ij} - v\delta_{ij}] = \kappa^*[w^*\gamma_{ik}u^*_kl^*_l] \). Therefore, to study the global \( C^2 \) estimate of (3.5) is equivalent to study the global \( C^2 \) estimate of (2.6).
For our convenience, we will consider the equation

\[
\hat{F}(\Lambda) = \left( \frac{\sigma_n}{\sigma_{n-k}} \right) \frac{1}{k} (\Lambda) = \tilde{\psi},
\]

where \( \Lambda = (\Lambda_{ij}) = (v_{ij} - v\delta_{ij}) \), \( \tilde{\psi} = \psi^{-1/k}(X, \nu) \), and \( v_{ij} \) is the covariant derivatives with respect to the hyperbolic metric.

We will use \( \lambda[\Lambda] = (\lambda_1, \lambda_2, \cdots, \lambda_n) \) to denote the eigenvalues of the matrix \( \Lambda \).

We define the Riemann curvature tensor:

\[
R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.
\]

Let \( \{e_1, e_2, \cdots, e_n\} \) be an orthonormal frame on \( \mathbb{H}^n \), we use the notation

\[
R_{ijkl} = R(e_i, e_j)e_k \cdot e_l, \quad R_{ijkl} = g^{lp}R_{ijkp}.
\]

Then the commutation formulae are

\[
v_{ijk} - v_{ikj} = R_{ljk}v_{il}, \quad v_{ijkl} - v_{ijlk} = R_{mkl}v_{jm} + R_{mlk}v_{im}.
\]

Note that in hyperbolic space we have,

\[
R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}.
\]

Therefore, given an orthonormal frame on \( \mathbb{H}^n \), we obtain the following geometric formulae:

\[
\Lambda_{ijk} = \Lambda_{ikj}, \quad \Lambda_{ikji} - \Lambda_{ikij} = v_{ikji} - v_{ikij} = -v_{ij}\delta_{ik} + v_{li}\delta_{jk} - v_{jk}\delta_{il} + v_{ik}\delta_{jl}.
\]

We will prove

**Lemma 9.** Let \( v \) be the solution of \((3.6)\) in a bounded domain \( U \subset \mathbb{H}^n \). Denote the eigenvalues of \((v_{ij} - v\delta_{ij})\) by \( \lambda[v_{ij} - v\delta_{ij}] = (\lambda_1, \cdots, \lambda_n) \). Then

\[
\lambda_{\text{max}} \leq \max\{C, \lambda|_{\partial U}\},
\]

where \( \lambda_{\text{max}} = \max\{\lambda_1, \cdots, \lambda_n\} \), and \( C \) is a positive constant only depending on \( U \) and \( \tilde{\psi} \).

**Proof.** Set

\[
M = \max_{P \in \mathcal{U}} \max_{|\xi| = 1, \xi \in T_p\mathbb{H}^n} (\log \Lambda_{\xi\xi} + Nx_{n+1}),
\]

where \( x_{n+1} \) is the coordinate function. Without loss of generality, we assume \( M \) is achieved at an interior point \( P_0 \in U \) for some direction \( \xi_0 \). Chose an orthonormal frame \( \{e_1, \cdots, e_n\} \) around \( P_0 \) such that \( e_1(P_0) = \xi_0 \) and \( \Lambda_{ij}(P_0) = \lambda_i\delta_{ij} \).

Now, let’s consider the test function

\[
\phi = \log \Lambda_{11} + Nx_{n+1}.
\]
At its maximum point $P_0$, we have

\begin{align}
0 &= \phi_i = \frac{\Lambda_{1ii}}{\Lambda_{11}} + N(x_{n+1})_i \\
0 &> \phi_{ii} = \frac{\Lambda_{1ii}}{\Lambda_{11}} - \frac{\Lambda_{11}^2}{\Lambda_{11}^2} + N(x_{n+1})_{ii}.
\end{align}

Note that $(x_{n+1})_{ij} = x_{n+1}\delta_{ij}$, thus

\begin{equation}
\hat{F}^{ii}\phi_{ii} = \frac{\hat{F}^{ii}\Lambda_{11}^{ii}}{\Lambda_{11}} - \frac{\hat{F}^{ii}\Lambda_{11}^2}{\Lambda_{11}^2} + N x_{n+1} \sum_i \hat{F}^{ii}.
\end{equation}

In view of (3.7), we get

$$\Lambda_{11ii} = \Lambda_{i1ii} = v_{ii} - v_{11} = \Lambda_{ii11} + \Lambda_{ii} - \Lambda_{11}.$$ 

This yields,

\begin{equation}
\hat{F}^{ii}\Lambda_{11ii} = \hat{F}^{ii}\Lambda_{i1ii} + \hat{F}^{ii} \Lambda_{ii} - \Lambda_{11} \sum_i \hat{F}^{ii}.
\end{equation}

Differentiating equation (3.6) twice we obtain,

\begin{equation}
\hat{F}^{ii}\Lambda_{ii1} = -\hat{F}^{pq,rs} \Lambda_{pq1} \Lambda_{rs1} + \tilde{\psi}_{11}
\end{equation}

\begin{equation}
= -\hat{F}^{pq,qq} \Lambda_{pq1} \Lambda_{qq1} - \sum_{p \neq q} \frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q} \Lambda_{pq1}^2 + \tilde{\psi}_{11}.
\end{equation}

By the concavity of $(\sigma_n/\sigma_{n-k})^{1/k}$ we can see that the first term on the right hand side is nonnegative. Combining (3.10)-(3.12) we have,

\begin{equation}
\hat{F}^{ii}\phi_{ii} \geq \frac{\tilde{\psi}_{11}}{\Lambda_{11}} - \frac{1}{\Lambda_{11}} \sum_{p \neq q} \frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q} \Lambda_{pq1}^2 - \frac{\hat{F}^{ii}\Lambda_{11}^2}{\Lambda_{11}^2} + (N x_{n+1} - 1) \sum_i \hat{F}^{ii}\Lambda_{ii1} + \tilde{\psi}_{11}
\end{equation}

\begin{equation}
\geq \frac{\tilde{\psi}_{11}}{\Lambda_{11}} + \frac{1}{\Lambda_{11}} \sum_{i \neq 1} \frac{\hat{F}^{ii} - \hat{F}^{ii1}}{\lambda_i - \lambda_1} \Lambda_{11i}^2 - \frac{\hat{F}^{ii}\Lambda_{111}}{\Lambda_{11}^2} + (N x_{n+1} - 1) \sum_i \hat{F}^{ii}.
\end{equation}

We need an explicit expression of $\hat{F}^{ii}$. A straightforward calculation gives

\begin{equation}
k\hat{F}^{k-1} \hat{F}^{ii} = \frac{\sigma_n^2 \sigma_{n-k} - \sigma_n^2 \sigma_{n-k}}{\sigma_{n-k}^2},
\end{equation}

where for $1 \leq l \leq n$, $\sigma_n^l = \frac{\partial \sigma_n}{\partial \lambda_l}$. Since

$$\sigma_n^i \sigma_{n-k} - \sigma_n^i \sigma_{n-k} = \sigma_{n-1}(\lambda|i)(\lambda_{n-k-1}(\lambda|i) + (\lambda_{n-k}(\lambda|i) - \lambda_1 \sigma_{n-1}(\lambda|i) \sigma_{n-k-1}(\lambda|i) = \sigma_{n-1}(\lambda|i) \sigma_{n-k}(\lambda|i).$$
Here and in the following, $\sigma_l(\lambda | a)$ and $\sigma_l(\lambda | ab)$ are the $l$-th elementary symmetric polynomials of $\lambda_1, \cdots, \lambda_n$ with $\lambda_a = 0$ and $\lambda_a = \lambda_b = 0$, respectively. It follows

$$k \hat{F}^{k-1} = \frac{\sigma_{n-1}(\lambda | i) \sigma_{n-k}(\lambda | i)}{\sigma^2_{n-k}}. \tag{3.15}$$

Therefore, we get

$$k \hat{F}^{k-1} (\hat{F}^i - \hat{F}^1) \tag{3.16}$$

$$= \frac{1}{\sigma^2_{n-k}} [\sigma_{n-1}(\lambda | i) \sigma_{n-k}(\lambda | i) - \sigma_{n-1}(\lambda | 1) \sigma_{n-k}(\lambda | 1)]$$

$$= \frac{\sigma_{n-2}(\lambda | 1) \sigma_{n-k}(\lambda | i) - \sigma_{n-1}(\lambda | 1) \sigma_{n-k} - \lambda_i] \sigma_{n-k}}{\sigma^2_{n-k}}$$

$$= \frac{\sigma_{n-2}(\lambda | 1) (\lambda_1 + \lambda_i) \sigma_{n-k-1}(\lambda | 1) + \sigma_{n-k}(\lambda | 1) i}. \tag{3.17}$$

When $i \geq 2$, we can see that

$$k \hat{F}^{k-1} \left( \frac{\hat{F}^i - \hat{F}^1}{\lambda_1 - \lambda_i - \hat{F}^i} \right) \tag{3.19}$$

$$= \frac{\sigma_{n-2}(\lambda | 1) \sigma_{n-k-1}(\lambda | 1) \sigma_{n-k-1}(\lambda | 1)}{\sigma^2_{n-k}}$$

$$> 0.$$

Plugging (3.17) into (3.13), we obtain

$$\hat{F}^{ii} \phi_{ii} \geq \frac{\psi_{11}}{\Lambda_{11}} - \frac{\hat{F}^{11} \Lambda^2_{11}}{\Lambda^2_{11}} + (N x_{n+1} - 1) \sum_i \hat{F}^{ii} \tag{3.18}$$

$$= \frac{\psi_{11}}{\Lambda_{11}} - \hat{F}^{11} N^2 (y_{n+1})^2_1 + (N x_{n+1} - 1) \sum_i \hat{F}^{ii}.$$

Here, in the last equality, we have used (3.8).

Now, let’s calculate $\hat{F}_{11}$. We denote the connection of the ambient space by $\nabla$, and $\{e^*_1, e^*_2, \cdots, e^*_n\}$ denotes the pull back of $\{e_1, e_2, \cdots, e_n\}$ via the Gauss map. Differentiating $\hat{F}$ with respect to $e_1$ twice we get,

$$\hat{F}_{11} = d_X \psi^{-1/k}(\nabla e^*_1 X) + d_y \psi^{-1/k} (e_1), \tag{3.19}$$
where the first inequality comes from the locally strict convexity assumption on \( \psi^{-1/k} \), i.e., for any spacelike vector \( \xi \in \mathbb{R}^{n,1} \),

\[
d_X d_X \psi^{-1/k}(\xi, \xi) \geq c_0 |\xi|_E^2 \geq c_0 |\xi|_M^2.
\]

Here \( c_0 > 0 \) is some constant depending on the defining domain, and \( |\cdot|_E, |\cdot|_M \) are the Euclidean norm and Minkowski norm respectively. At the point \( P_0 \), in view of (3.8) and the assumption that \( \psi_{x_{n+1}} > 0 \), we derive

\[
\begin{align*}
\tilde{\psi}_{11} & \geq c_0 \lambda_1 - N \sum_k (x_{n+1})_k d_X \psi^{-1/k}(e_k) - C - \frac{C}{\lambda_1} \\
& = c_0 \lambda_1 + N \frac{k}{\psi^{-1/k}} d_X \psi(\nabla x_{n+1}) - C - \frac{C}{\lambda_1} \\
& = c_0 \lambda_1 + N \frac{k}{\psi^{-1/k}} d_X \psi \left( -\frac{\partial}{\partial x_{n+1}} + x_{n+1} \nu \right) - C - \frac{C}{\lambda_1} \\
& = c_0 \lambda_1 + N \frac{k}{\psi^{-1/k}} d_X \psi \left( |x|^2 \frac{\partial}{\partial x_{n+1}} + x_{n+1} \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right) - C - \frac{C}{\lambda_1} \\
& = c_0 \lambda_1 + N \frac{k}{\psi^{-1/k}} \psi^{-1/k} - \frac{C}{\lambda_1} \\
& \geq c_0 \lambda_1 + N \frac{k}{\psi^{-1/k}} x_{n+1} \sum_{i=1}^n x_i \frac{\partial \psi}{\partial x_i} - C - \frac{C}{\lambda_1} \\
& \geq -C - \frac{C}{\lambda_1}.
\end{align*}
\]

Here, in the last inequality we have assumed \( \lambda_1 = \lambda_1(|\psi|_{E^2}) > 0 \) is large at \( P_0 \). On the other hand, note that the functional \( \tilde{F} \) is concave and homogenous of degree
one. Therefore,
\[
\sum_i \hat{F}^{ii} = \hat{F}(\lambda) + \sum_i \hat{F}^{ii}(1 - \lambda_i)
\]
(3.22)
\[
\geq \hat{F}(1) = \left(\frac{n}{k}\right)^{-1/k}.
\]
Combining (3.18)-(3.22), we obtain
\[
0 \geq \hat{F}^{ii} \phi_{ii} \geq -C - \frac{C}{\lambda_1} N^2 (x_{n+1})_i^2 + (N x_{n+1} - 1) \left(\frac{n}{k}\right)^{-1/k}.
\]
Let $N, \lambda_1$ be sufficiently large, then we obtain a contradiction. This completes the proof of Lemma 9.

Notice that this is the only place we need to use the locally strict convexity assumption of $\psi^{-1/k}$ in Theorem 1. It’s also clear that the above proof can be easily modified to the case when $\psi^{-1/k}$ is convex with respect to $X$ and the corresponding $\psi(x, u(x), Du(x))$ does not depend on $|x|$ (see the second inequality in (3.21)), as stated in the Remark 2. Therefore, (3.21) is solvable when either $\psi^{-1/k}$ is locally strictly convex with respect to $X$ or $\psi^{-1/k}$ is convex with respect to $X$ and $\psi(x, u(x), Du(x))$ does not depend on $|x|$.

3.2. Dirichlet problem for $k = n - 1, n - 2$. Let $n \in \mathbb{N}$ and $\Omega_n := \{x \in \mathbb{R}^n | u(x) = n\}$, we will consider the following Dirichlet problem:
\[
(3.23)
\left\{
\begin{array}{l}
\sigma_k(\kappa[M_u]) = \psi(x, u(x)) \text{ in } \Omega_n, \\
u = n \text{ on } \partial \Omega_n.
\end{array}
\right.
\]
Note that since $u$ is strictly convex, $\Omega_n$ is strictly convex. It’s easy to see that if $u$ is a solution of (3.23), then $u \leq u \leq \bar{u}$. Therefore, in order to find a $k$-convex solution $u$ for (3.23), we only need to study the $C^1$ and $C^2$ estimates of $u$.

3.2.1. $C^1$ estimate for equation (3.23).

Lemma 10. Let $u$ be a solution of (3.23), then $|Du| < C < 1$. Here $C$ is a constant depending on $|Du|_{\Omega_n}$ and $\psi$.

Proof. Let $V = -\langle \nu, E \rangle = \frac{1}{\sqrt{1-|Du|^2}}$, and consider the test function $\phi = \ln V + Ku$, where $K > 0$ to be determined. If $\phi$ achieves its maximum at an interior point $P_0 \in M_u$, then at this point, we may choose a normal coordinate $\{\tau_1, \cdots, \tau_n\}$ such that $h_{ij} = \kappa_i \delta_{ij}$. Since at $P_0$ we have
\[
\phi_i = \frac{V_i}{V} + Ku_i = 0
\]
and
\[
0 \geq \phi_{ii} = \frac{V_{ii}}{V} - \frac{V^2}{V^2} + Ku_{ii}.
\]
A straightforward calculation yields
\[
0 \geq -\frac{\langle \nabla \sigma_k, E \rangle}{V} - \frac{\sigma_k^{ij} \kappa^2_i u_{ij}^2}{V^2} + Kk\psi V + \sigma_k^{ij} \kappa_i^2.
\]
Note that $|\langle \nabla \sigma_k, E \rangle| \leq CV^2$, where $C$ only depends on $|\psi|_{C^1}$. Choose $K > C + 1$ we have
\[-\frac{\langle \nabla \sigma_k, E \rangle}{V} - \frac{\sigma_k^{ij} \kappa_i^2 u_i^2}{V^2} + K k \psi V + \frac{\sigma_k^{ij} \kappa_i^2}{V^2} > 0.\]
This leads to a contradiction. \(\square\)

3.2.2. $C^2$ boundary estimates for equation (3.23). Now, we will establish the $C^2$ boundary estimate. For our convenience, we will consider the solvability of the following Dirichlet problem:
\[
\begin{aligned}
G(Du, D^2u) &= F \left( \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj} \right) = \psi(x, u(x)) \quad \text{in } \Omega, \\
\text{u} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $\Omega$ is strictly convex. We will follow the idea of [10].

**Infinitesimal stretching.** If $u$ is a solution of (3.24), let $v(x) = \frac{1}{t} u(tx)$, where $t > 0$. Then the principal curvatures of $M_v$ satisfies $\kappa[M_v(x)] = t \kappa[M_u(tx)]$. Therefore
\[
G(Dv, D^2v) = t^k \psi(tx, u(tx))
= t^k \psi(tx, tv(x)).
\]
We denote $\dot{v} = \frac{dv}{dt} = -\frac{1}{t^2} u(tx) + x \cdot Du(tx)$, when $t = 1$
\[
\dot{v} = x \cdot Du(x) - u(x).
\]
Differentiating equation (3.25) with respect to $t$, then evaluate it at $t = 1$ we obtain
\[
G^{ij} \partial_{ij} \dot{v} + G^s \partial_s \dot{v}
= k \psi + \psi_x (v + \dot{v}) + x \psi_x.
\]
Denote $L := G^{ij} \partial_{ij} + G^s \partial_s$, we have
\[
L(x \cdot Du - u) = k \psi + \psi_x (u + x \cdot Du - u) + x \psi_x
= k \psi + x \psi_x + \psi_x x \cdot Du.
\]

**Infinitesimal rotation in Minkowski space.** Keeping the coordinates $x' = (x_1, \ldots, x_{n-1})$ fixed, we rotate in the $(x_n, u)$ variables,
\[
\begin{bmatrix}
cosh \theta & \sinh \theta \\
\sinh \theta & \cosh \theta
\end{bmatrix}
\begin{bmatrix}
x_n \\
u
\end{bmatrix}
= \begin{bmatrix}
cosh \theta x_n + \sinh \theta u \\
cosh \theta u + \sinh \theta x_n
\end{bmatrix}.
\]
To the first order in $\theta$ the image of $(x, u(x))$ under such rotation is
\[(x', x_n + u(x) \theta, u(x) + x_n \theta).
\]
Therefore, to the first order in $\theta$ the image of
\[(x', x_n - u(x) \theta, u(x', x_n - u(x) \theta))
\]
is $(x', x_n, u(x', x_n - u(x) \theta) + x_n \theta)$. Denote this image as a graph function
\[v(x) = u(x', x_n - u(x) \theta) + x_n \theta + \text{higher order in } \theta,
\]
then we have
\[ G(Dv, D^2v) = \psi(x', x_n - u(x)\theta, u(x'), x_n - u(x)\theta) + \text{higher order in } \theta \]
\[ = \psi(x', x_n - u(x)\theta, v(x) - x_n\theta) + \text{higher order in } \theta. \]

Notice that \( \frac{\partial w}{\partial \theta} |_{\theta=0} = x_n - u_n u \), we obtain
\[ G^{ij} \partial_{ij}(x_n - u_n u) + G^s \partial_s (x_n - u_n u) \]
\[ \quad = \psi_n(-u(x)) + \psi_z(x_n - u_n u - x_n). \]

Thus, we conclude that
\[ L(x_n - uu_n) = -w\psi_n - u_n w\psi_z. \]

**Lemma 11.** Let \( u \) be a solution of \( \text{(3.24)} \), then \( |D^2u| < C \) on \( \partial \Omega \). Here \( C \) is a constant depending on \( \Omega \) and \( \psi \).

**Proof.** For any \( p \in \partial \Omega \), we suppose \( p \) is the origin and that the \( x_n \)-axis is the interior normal of \( \partial \Omega \) at \( p \). We may also assume the boundary near the origin \( p \) is represented by
\[ x_n = \frac{1}{2} \sum_{\alpha=1}^{n-1} \lambda_\alpha x_\alpha^2 + O(|x'|^3), \ x' = (x_1, \cdots, x_{n-1}), \]
where \( \lambda_\alpha > 0 \), \( 1 \leq \alpha \leq n - 1 \) are the principal curvatures of \( \partial \Omega \) at the origin. Let \( T_\alpha = \partial_\alpha + \lambda_\alpha (x_\alpha \partial_n - x_n \partial_\alpha) \). Note that \( G^{ij} u_{ij\alpha} + G^s u_{s\alpha} = \psi_\alpha + \psi_z u_\alpha \). In view of the fact that \( \text{(3.24)} \) is invariant under rotation (see equation (3.1) in [10]), we get
\[ |LT_\alpha u| \leq C. \]

Moreover, it’s easy to see we have \( |T_\alpha u| \leq C|x'|^2 \) on \( \partial \Omega \) near the origin. In the following, we denote \( \Omega_\beta := \Omega \cap \{x_n < \beta\} \). Set
\[ h = (x \cdot Du - u) - \frac{\delta}{\beta}(x_n - uu_n). \]

On \( \partial \Omega \cap \partial \Omega_\beta \), note that \( u = 0 \), we have \( x \cdot Du \leq C_1|x'|^2 \). This implies on \( \partial \Omega \cap \partial \Omega_\beta \),
\[ h = x \cdot Du - \frac{\delta}{\beta} x_n \leq \left(C_1 - \frac{\delta}{\beta}\right)|x'|^2, \]
where \( a > 0 \) depends on the principal curvatures of \( \partial \Omega \). Notice that \( u \) is a spacelike function, we suppose \( |Du| \leq \theta_0 \) in \( \Omega \) for some \( \theta_0 \in (0, 1) \). Then we have \( 0 \leq -u \leq \theta_0 \beta \) in \( \Omega_\beta \). Therefore, on \( \{x_n = \beta\} \) we obtain
\[ h = \beta uu_n + \sum_{\alpha=1}^{n-1} x_\alpha u_\alpha - u + \frac{\delta}{\beta} uu_n - \delta \]
\[ \leq \beta \theta_0 + C\beta^{1/2} + \theta_0 \beta + \theta_0^2 \delta - \delta \]
\[ \leq C\beta^{1/2} + \delta(\theta_0 - 1) \]
with $C$ being independent of $\beta$ and $\delta$. Moreover,

$$Lh = k\psi + x\psi_x + \psi_x \cdot Du - \frac{\delta}{\beta}(-u\psi_n - u_n u\psi_x)$$

(3.32)

$$\geq k\psi - C\beta^{1/2} - C\delta$$

$$\geq \frac{k}{2}\psi,$$

where $\delta$ and $\beta$ are small positive constants.

Now choose $A = A(\delta) > 0$ large such that

$$Ah \leq -|T_\alpha u| \text{ on } \partial\Omega_\beta,$$

and $LAh > |L T_\alpha u|$ in $\Omega_\beta$. By the maximum principle we conclude that

$$Ah \pm T_\alpha u \leq 0 \text{ in } \bar{\Omega}_\beta.$$

On the other hand we have $h(0) = T_\alpha u(0) = 0$. Therefore,

$$|\partial_n T_\alpha u(0)| \leq -Ah_n(0) \leq \frac{A\delta}{\beta},$$

which yields

$$|u_{n\alpha}(0)| \leq C.$$  

Since $p \in \partial\Omega$ is arbitrary, we get

$$|u_{n\alpha}(x)| \leq C \text{ for any } x \in \partial\Omega.$$  

Applying Lemma 1.2 in [8] we obtain

$$|u_{n\alpha}(x)| \leq C \text{ for any } x \in \partial\Omega.$$  

This completes the proof of this Lemma. \(\square\)

3.2.3. $C^2$ global estimate for equation (3.23). Finally, we will prove the $C^2$ global estimate. In this subsubsection, for the greater generality, we will assume $\psi = \psi(X, \nu)$.

**Lemma 12.** Let $u$ be a solution of (3.24) with $\psi = \psi(X, \nu)$, then

$$|D^2 u| < \max\{C, \max_{\partial\Omega} |D^2 u|\}$$

on $\Omega$. Here $C$ is a constant depending on $|Du|_\Omega$ and $\psi$.

**Proof.** we consider the following test function whose form first appeared in [19],

$$\phi = \log \log P - N \langle \nu, E \rangle.$$  

Here, the function $P$ is defined by $P = \sum_i e^{\kappa_i}$ and $N$ is a sufficiently large constant to be determined later.

We may assume that the maximum of $\phi$ is achieved at some point $P_0 \in M_u$, where $u$ is the solution of (3.24). Suppose $\{\tau_1, \tau_2, \cdots, \tau_n\}$ is a normal coordinate near $P_0$ such that at $P_0$, $h_{ij} = \kappa_i\delta_{ij}$ and $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$. 

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Differentiating the function \( \phi \) twice at \( P_0 \), we have

\[
(3.33) \quad \phi_i = \frac{P_i}{P \log P} + Nh_{ii}u_i = 0,
\]

and

\[
\phi_{ii} = \frac{P_{ii}}{P \log P} - \frac{P_i^2}{P^2 \log P} - \frac{P_{ii}^2}{(P \log P)^2} - Nh_{ii}^2 \langle \nu, E \rangle + \sum_s N u_s h_{iss}.
\]

Contracting with \( \sigma_{ii}^k \), we get

\[
(3.34) \quad \sigma_{ii}^k \phi_{ii} = \sum_l e^{\kappa_l} h_{lii} + \sum_l e^{\kappa_l} h_{lii}^2 + \sum_{l \neq q} e^{\kappa_p} - e^{\kappa_q} h_{pqi} - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2
\]

\[-Nh_{ii}^2 \langle \nu, E \rangle + \sum_s N u_s \sigma_{ii}^k h_{iss}.\]

At \( P_0 \), differentiating the equation (1.2) twice yields,

\[
(3.35) \quad \sigma_{ii}^k h_{ii} = d_X \psi(\tau_l) + \kappa_l d_\nu \psi(\tau_l),
\]

and

\[
(3.36) \quad \sigma_{ii}^k h_{ill} + \sigma_{ii}^{pq,rs} h_{pq} h_{rsl} \geq -C - Ch_{11}^2 + \sum_s h_{sll} d_\nu \psi(\tau_s),
\]

where \( C \) is some uniform constant only depending on \( \psi \). Note that

\[
(3.37) \quad h_{llii} = h_{ill} - h_{ii} h_{ll}^2 + h_{ii}^2 h_{ll}.
\]

Inserting (3.36) and (3.37) into (3.34), we obtain

\[
(3.38) \quad \sigma_{ii}^k \phi_{ii} \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \left( -C - C \kappa_l^2 - \sigma_{ii}^{pq,rs} h_{pq} h_{rsl} + \sum_s h_{sll} d_\nu \psi(\tau_s) \right) \right]
\]

\[+ \sum_l \sigma_{ii}^k e^{\kappa_l} h_{lii}^2 + \sum_{l \neq q} \sigma_{ii}^k e^{\kappa_p} - e^{\kappa_q} h_{pqi} - \left( \frac{1}{P} + \frac{1}{P \log P} \right) \sigma_{ii}^k P_i^2 \]

\[-Nh_{ii}^2 \langle \nu, E \rangle + \sum_s N u_s \sigma_{ii}^k h_{sii} - \sigma_{ii}^k \kappa_l^2.\]

By (3.33) and (3.35), we have

\[
\frac{1}{P \log P} \sum_s \sum_l e^{\kappa_l} h_{sll} d_\nu \psi(\tau_s) + \sum_s N u_s \sigma_{ii}^k h_{sii} \geq -C.
\]
Now, for any constant $K > 1$, we denote
\[
A_i = e^{\kappa_i} \left[ K (\sigma_k)_{i_1}^2 - \sum_{p \neq q} \sigma_k^{pp,qq} h_{ppi} h_{qqi} \right], \quad B_i = 2 \sum_{l \neq i} \sigma_k^{ii,li} e^{\kappa_i} h_{lii}^2, \\
C_i = \sigma_k^{ii} \sum_l e^{\kappa_i} h_{lii}^2, \quad D_i = 2 \sum_{l \neq i} \sigma_k^{ii,l} \frac{e^{\kappa_i} - e^{\kappa_i}}{\kappa_i - \kappa_i} h_{lii}^2, \quad E_i = \frac{1 + \log P}{P \log P} \sigma_k^{ii} P_i^2.
\]

Combining
\[
- \sum_l \sigma_k^{pq,rs} h_{pql} h_{rsi} = \sum_{p \neq q} \sigma_k^{pp,qq} h_{pql}^2 - \sum_{p \neq q} \sigma_k^{pp,qq} h_{ppi} h_{qqi},
\]
with (3.38), we get
\[
(3.39) \quad \sigma_k^{ii} \phi_{ii} \geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i) + (-N \langle \nu, E \rangle - 1) \sigma_k^{ii} \kappa_i^2 - C \kappa_1.
\]

**Claim 1.** For any given $0 < \varepsilon < \frac{1}{2}$, we let $\alpha = \frac{1 - 2 \varepsilon}{1 + \varepsilon}$. There exists a positive constant $\delta < \frac{1}{2}$ such that, for any $|\kappa_i| \leq \delta \kappa_1, 1 \leq i \leq n$, if the constant $K$ and the maximum principal curvature $\kappa_1$ both are sufficiently large, we have
\[
A_i + B_i + C_i + D_i - E_i - \frac{\alpha}{P \log P} \sigma_k^{ii} P_i^2 \geq 0.
\]

Applying Lemma 6 in [28], we can see that when $K$ is chosen to be sufficiently large, then $A_i \geq 0$. By the Cauchy-Schwarz inequality, we have
\[
(3.40) \quad P_i^2 = e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iiil} h_{lli} + \left( \sum_{l \neq i} e^{\kappa_l} h_{lli}^2 \right)^2 
\leq e^{2\kappa_i} h_{iii}^2 + 2 \sum_{l \neq i} e^{\kappa_i + \kappa_l} h_{iiil} h_{lli} + (P - e^{\kappa_i}) \sum_{l \neq i} e^{\kappa_l} h_{lli}^2.
\]

Thus,
\[
(3.41) \quad B_i + C_i + D_i - E_i - \frac{\alpha}{P \log P} \sigma_k^{ii} P_i^2 \geq 2 \sum_{l \neq i} e^{\kappa_i} \sigma_k^{ii,l} h_{lli}^2 + 2 \sum_{l \neq i} \frac{e^{\kappa_i} - e^{\kappa_l}}{\kappa_i - \kappa_l} \sigma_k^{ii,l} h_{lli}^2 - \frac{1 + \alpha \log P}{P \log P} \sum_{l \neq i} \sigma_k^{ii,l} h_{lli}^2 
+ \frac{1 + \alpha \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_k^{ii,l} h_{lli}^2 + e^{\kappa_i} \sigma_k^{ii} h_{lli}^2 
- \frac{1 + \alpha \log P}{P \log P} e^{2\kappa_i} \sigma_k^{ii} h_{lli}^2 - 2 \frac{1 + \alpha \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_l} \sigma_k^{ii,l} h_{ili} h_{lli}.
\]
Let $\varepsilon$ be equal to the $\varepsilon_T$ in Lemma 12 of [28]. Then we know there exists a positive constant $\delta < \varepsilon$ such that, when $|\kappa_i| < \delta \kappa_1$

\[(3.42) \quad (2 - \varepsilon) \sum_{l \neq i} e^{\kappa_i} \sigma_k^{li} h_{li}^2 + (2 - \varepsilon) \sum_{l \neq i} \frac{e^{\kappa_i} - e^{\kappa_i}}{\kappa_l - \kappa_i} \sigma_k^{li} h_{li}^2 - \frac{1 + \alpha}{\log P} \sum_{l \neq i} e^{\kappa_i} \sigma_k^{li} h_{li}^2 \geq 0.
\]

On the other hand, we have

\[(3.43) \quad \sum_{l \neq i, 1} e^{\kappa_i + \kappa_i} \sigma_k^{li} h_{li}^2 - 2 \sum_{l \neq i, 1} e^{\kappa_i + \kappa_i} \sigma_k^{li} h_{li}^2 \geq - \sum_{l \neq i, 1} e^{\kappa_i + \kappa_i} \sigma_k^{li} h_{li}^2.
\]

It follows

\[(3.44) \quad B_i + C_i + D_i - E_i - \frac{\alpha}{P \log P} \sigma_k^{ii} P^2 \geq \frac{1 + \alpha + \log P}{P \log P} e^{\kappa_1 + \kappa_i} \sigma_k^{ii} h_{li}^2 + e^{\kappa_i} \sigma_k^{ii} h_{li}^2 - \frac{1 + \alpha + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_i} \sigma_k^{ii} h_{li}^2 - 2 \frac{1 + \alpha + \log P}{P \log P} e^{\kappa_i + \kappa_i} \sigma_k^{ii} h_{li} h_{1l} - \varepsilon e^{\kappa_i} \sigma_k^{11, ii} h_{1l}^2 + \varepsilon \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_k^{11, ii} h_{li}^2.
\]

A straightforward calculation shows that when $\kappa_1$ is very large the following inequalities hold:

\[
e^{\kappa_i} \sigma_k^{ii} h_{li}^2 - \frac{1 + \alpha + \log P}{P \log P} \sum_{l \neq i} e^{\kappa_i + \kappa_i} \sigma_k^{ii} h_{li}^2 \geq \left(\frac{e^{\kappa_1}}{P} - \frac{1 + \alpha}{\log P}\right) e^{\kappa_i} \sigma_k^{ii} h_{li}^2 \geq \frac{1}{n + 1} e^{\kappa_i} \sigma_k^{ii} h_{li}^2,
\]

and

\[
-2 \frac{1 + \alpha + \log P}{P \log P} e^{\kappa_i + \kappa_i} \sigma_k^{ii} h_{li}^2 h_{1l}^2 \geq - \frac{3}{P} e^{\kappa_i + \kappa_i} \sigma_k^{ii} h_{li}^2 h_{1l}^2 \geq -3 e^{\kappa_i} \sigma_k^{ii} h_{li}^2 h_{1l}^2.
\]

Moreover, it is easy to see that

\[(3.45) \quad e^{\kappa_i} \sigma_k^{11, ii} h_{li}^2 + \frac{e^{\kappa_i} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_k^{11, ii} h_{li}^2 = e^{\kappa_i} \sigma_k^{11, ii} h_{li}^2 + \frac{e^{\kappa_1} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_k^{11, ii} h_{li}^2.
\]

By the Taylor expansion, we have

\[(3.46) \quad \frac{e^{\kappa_i} - e^{\kappa_i}}{\kappa_1 - \kappa_i} \sigma_k^{ii} h_{li}^2 = e^{\kappa_i} \sum_{m \geq 1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} \sigma_k^{ii} h_{li}^2.
\]
Combining the previous four formulae with (3.44), we obtain when $\kappa_1$ is sufficiently large and $|\kappa_i| < \delta \kappa_1$,
\[
B_i + C_i + D_i - E_i - \frac{\alpha}{P \log P} \sigma_k^i p_i^2 \geq e^{\kappa_1} \sigma_k^i \left[ \frac{1}{n+1} h_{ii}^2 - 3 |h_{iii} h_{11i}| + \varepsilon \sum_{m=1} \frac{(\kappa_1 - \kappa_i)^{m-1}}{m!} h_{11i}^2 \right] 
\geq 0.
\]
Therefore, Claim 1 is proved.

Now, recall Section 4 of [28] and the proof of Theorem 14 in [29], we know the following claim is true.

**Claim 2.** Suppose $k = n - 1$ ($n \geq 3$) and $k = n - 2$ ($n \geq 5$). For any index $1 \leq i \leq n$, if the positive constant $K$ and the maximum principal curvature $\kappa_1$ both are sufficiently large, we have
\[
A_i + B_i + C_i + D_i - E_i \geq 0.
\]

By Claim 1 and Claim 2, (3.39) becomes
\[
(3.47) \quad 0 \geq \sum_{|\kappa_i| < \delta \kappa_1} \frac{\alpha}{(P \log P)^2} \sigma_k^i p_i^2 + (-N \langle \nu, E \rangle - 1) \sigma_k^i \kappa_i^2 - C \kappa_1.
\]

Here, the constant $\delta$ is the constant chosen in Claim 1. Choose $N > 0$ such that $\sigma_k^{11} \kappa_1^2 (-N \langle \nu, E \rangle - 1) - C \kappa_1 > 0$, we get a contradiction. Therefore, our desired estimate follows immediately.

By Lemma 10, Lemma 11, and Lemma 12, we conclude that when $k = n - 1, n - 2$, the Dirichlet problem (3.23) admits a $k$-convex solution.

### 4. The Local Estimates

We will devote this section to establishing the local $C^1$ and $C^2$ estimates for the solution $u$ of (1.3).

#### 4.1. Local $C^1$ Estimates

In this subsection, we will prove the local $C^1$ estimate. We will split it into two cases. In the first case, we will assume $u$ is a convex solution of (1.2); in the second case, we will assume $u$ is a $k$-convex solution of (1.5). Note that in both cases our results hold for $1 \leq k \leq n$.

For strictly convex, spacelike hypersurfaces, Bayard-Schnürer [7] proved the following local gradient estimate lemma.

**Lemma 13.** *(Lemma 5.1 in [7]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that $u$ is strictly convex and $u < \bar{u}$ in $\Omega$. Also assume that near $\partial \Omega$, we have $\Psi > \bar{u}$. Consider the set, where $u > \Psi$. For every $x$ in this set, we have the following gradient estimate for $u$:
\[
\frac{1}{\sqrt{1 - |Du|^2}} \leq \frac{1}{u(x) - \Psi(x)} \sup_{\{u > \Psi\}} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.
\]
For $k$-convex, spacelike hypersurfaces, Bayard [5] proved a similar result when $k=2$. In the following, we will extend it to all $k$. Our argument is a modification of Bayards’ in [5]. We would also like to mention that the basic idea of this argument had appeared in Chow-Wang [12].

Lemma 14. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that $M_u = \{(x, u(x))| x \in \Omega\}$ is a $k$-convex hypersurface satisfying

$$\sigma_k(\kappa[M_u]) = \psi(x, u(x))$$

and $u \leq \bar{u}$ in $\Omega$. Also assume that near $\partial \Omega$, we have $\Psi > \bar{u}$. Consider the set, where $u > \Psi$.

For every $x$ in this set, we have the following gradient estimate for $u$:

$$\frac{1}{\sqrt{1-|Du|^2}} \leq \left[ \frac{1}{u(x) - \Psi(x)} \cdot \sup_{\{u > \Psi\}} \left( \bar{u} - \Psi \right) \right]^N C.$$

Here, $N = N(n, k)$ is a uniform constant only depending on $n, k$, and $C = C(\bar{u} - \Psi, |\Psi|_{C^1}, |\psi|_{C^1})$ is a uniform constant depending on the upper bound of $\bar{u} - \Psi$, $\frac{1}{\sqrt{1-|D\Psi|^2}}$, $D^2\Psi$, and $|\psi|_{C^1}$.

Proof. Consider the test function:

$$\phi = (u - \Psi)^N(-\langle \nu, E \rangle),$$

where $N$ is a large undetermined constant. Assume the function $\phi$ achieves its maximum at $P$. We may choose a local normal coordinate $\{\tau_1, \ldots, \tau_n\}$ such that at $P$, $h_{ij} = \kappa_i \delta_{ij}$. Differentiating $\phi$ twice at $P$, we have,

$$0 = \frac{\phi_i}{\phi} = N \frac{u_i - \Psi_i}{u - \Psi} + \frac{h_{im}u_m}{-\langle \nu, E \rangle},$$

$$0 \geq \frac{\phi_{ii}}{\phi} - \frac{\phi_i^2}{\phi^2} = N \frac{u_{ii} - \Psi_{ii}}{u - \Psi} - N \frac{(u_i - \Psi_i)^2}{(u - \Psi)^2}$$

$$+ \sum_m \frac{h_{im}^2(-\langle \nu, E \rangle) + \sum_m \frac{h_{imi}u_m}{-\langle \nu, E \rangle}}{(-\langle \nu, E \rangle)^2}$$

Contracting with $\sigma_{k_i}^i$, we get

$$0 \geq \frac{\sigma_{k_i}^{ij} \phi_{ij}}{\phi} = N \frac{\sigma_{k_i}^{ij} u_{ij} - \sigma_{k_i}^{ij} \Psi_{ij}}{u - \Psi} - N \frac{\sigma_{k_i}^{ij} (u_i - \Psi_i)^2}{(u - \Psi)^2}$$

$$+ \sigma_{k_i}^{ij} \kappa_i^2 + \frac{\sigma_{k_i}^{ij} \sum_m h_{imi}u_m}{-\langle \nu, E \rangle} - \frac{\sigma_{k_i}^{ii} \kappa_i^2 u_i^2}{(-\langle \nu, E \rangle)^2}$$

Without loss of generality, we may assume that at $P$

$$u_i^2 \geq \frac{\nabla u_i^2}{n},$$

where $\nabla$ is the Levi-Civita connection on $M_u$. By (4.1), we have

$$\kappa_1 = N \frac{\langle \nu, E \rangle}{u - \Psi} \left( 1 - \frac{\Psi_1}{u_1} \right).$$
We may also assume $|\nabla u(P)|$ is so large that $\frac{|\Psi u_1|}{u_1} < \frac{1}{2}$. Then at $P$ we can see,

\[(4.3) \quad \kappa_1 \leq \frac{N \langle \nu, E \rangle}{2 u - \Psi}.
\]

Thus, if $N$ is sufficiently large, $\kappa_1$ is negative and its norm is large. Using the inequality (26) in Lin-Trudinger [23], we obtain

\[
\sum_{i \geq 2} \sigma^i_k \kappa^2_i \geq \eta \sigma_{11}^1 \kappa^2_1,
\]

where $\eta$ is a uniform constant only depending on $n, k$. Therefore,

\[
\sigma^i_k \kappa^2_i - \frac{\sigma^i_k \kappa^2_i u^2}{(-\langle \nu, E \rangle)^2} \geq \sum_{i \geq 2} \sigma^i_k \kappa^2_i - \left(1 - \frac{1}{n}\right) \sigma^i_k \kappa^2_i \geq \eta \sigma_{11}^1 \kappa^2_1 := \eta_0 \sigma_{11}^1 \kappa^2_1.
\]

By (4.3), we get

\[(4.4) \quad \sigma^i_k \kappa^2_i - \frac{\sigma^i_k \kappa^2_i u^2}{(-\langle \nu, E \rangle)^2} \geq \frac{\eta_0 N^2}{4} (-\langle \nu, E \rangle)^2 (u - \Psi)^2.
\]

Inserting (1.2) and (4.4) into (4.2) yields,

\[(4.5) \quad 0 \geq N(u - \Psi) \sigma^i_k \kappa_i (-\langle \nu, E \rangle) - \sigma^i_k \Psi_i - N \sigma^i_k (u_i - \Psi_i)^2 + (u - \Psi)^2 \sum_m \psi^m u^m - \langle \nu, E \rangle + \frac{\eta_0 N^2}{4} \sigma_{11} (-\langle \nu, E \rangle)^2.
\]

Notice that

\[
\psi_m = \sum_{l=1}^n \psi x_l \left\langle \tau_m, \frac{\partial}{\partial x_l} \right\rangle + \psi u \left\langle -\tau_m, E \right\rangle,
\]

we calculate,

\[(4.6) \quad \sum_m \psi^m u^m - \langle \nu, E \rangle \geq -C \left(1 + \langle -\nu, E \rangle \right).
\]

Combing (4.5) with (4.6), we get

\[(4.7) \quad 0 \geq -(n - k + 1) N(u - \Psi) \sigma_{k-1} |\nabla^2 \Psi| - 2(n - k + 1) N \sigma_{k-1} (|\nabla u|^2 + |\nabla \Psi|^2) - C(u - \Psi)^2 \left(1 + \langle -\nu, E \rangle \right) + \frac{\eta_0 N^2}{4} \sigma_{11} (-\langle \nu, E \rangle)^2.
\]

Notice that when $\kappa_1 < 0$, we have

\[
\sigma_{k-1} = \kappa_1 \sigma_{k-2} (\kappa[1] + \sigma_{k-1} (\kappa[1]) \leq \sigma_{11}^1.
\]

Moreover, $-\langle \nu, E \rangle = \sqrt{1 + |\nabla u|^2}$. Let $N$ be sufficiently large in (4.7), we obtain the desired estimate. □
4.2. The Pogorelov type local $C^2$ estimates. Recall that in [35] (see Lemma 24), we proved the Pogorelov type local $C^2$ estimate for strictly convex, spacelike, constant $\sigma_k$ curvature hypersurfaces. With small modifications, we can show

**Lemma 15.** Let $u^*$ be the solution of (3.5) and $u^r$ be the Legendre transform of $u^*$. For any given $s > 2C_0 + 1$, where $C_0 > \min \bar{u}$ is an arbitrary constant, let $r_s > 0$ be a positive number such that when $r > r_s$, $u^r|_{\partial \Omega_r} > s$, where $\Omega_r = Du^* (B_r)$. Let $\kappa_{\max} (x)$ be the largest principal curvature of $\mathcal{M}_{u^r}$ at $x$, where $\mathcal{M}_{u^r} = \{(x, u^r (x)) | x \in \Omega_r\}$. Then, for $r > r_s$ we have

\[
\max_{\mathcal{M}_{u^r}} (s - u^r) \kappa_{\max} \leq C.
\]

Here, $C$ depends on the local $C^1$ estimates of $u^r$ and $s$.

In the rest of this subsection, we will establish the Pogorelov type local $C^2$ estimates for the $k$-convex solution of equation (1.2), where $k = n - 1 (n \geq 3), n - 2 (n \geq 5)$.

**Lemma 16.** Let $u^m$ be the $k$-convex solution of (3.23) with $\psi = \psi (X, \nu)$, where $k = n - 1 (n \geq 3), n - 2 (n \geq 5)$. For any given $s > 1$, let $m > s$, then $u^m|_{\partial \Omega_m} = m > s$. Let $\kappa_{\max} (x)$ be the largest principal curvature of $\mathcal{M}_{u^m}$ at $x$, where $\mathcal{M}_{u^m} = \{(x, u^m (x)) | x \in \Omega_m\}$. Then, for $m > s$ we have

\[
\max_{\mathcal{M}_{u^m}} (s - u^m) \kappa_{\max} \leq C.
\]

Here, $C$ depends on the local $C^1$ estimates of $u^m$ and $s$.

**Proof.** In this proof, for our convenience when there is no confusion, we will drop the superscript on $u^m$. Now, on $\Omega_m$, we consider the following test function whose form first appeared in [19],

\[
\phi = \beta \log (s - u) + \log \log P - N \langle \nu, E \rangle.
\]

Here the function $P$ is defined by

\[
P = \sum_l e^{\kappa_l},
\]

and $\beta, N$ are constants to be determined later.

Let $U_s = \{x \in \mathbb{R}^n | u(x) < s\}$, we may assume that the maximum of $\phi$ is achieved at $P_0 \in U_s$. Choose a local normal coordinate $\{\tau_1, \tau_2, \ldots, \tau_n\}$ such that at $P_0$, $h_{ij} = \kappa_j \delta_{ij}$ and $\kappa_1 \geq \kappa_2 \cdots \geq \kappa_n$.

Differentiating the function $\phi$ twice at $P_0$, we get

\[
\phi_i = -\frac{\beta u_i}{s - u} + \frac{P_i}{P \log P} + Nh_{ii} u_i = 0,
\]
and 

\[ 0 \geq \phi_{ii} \]

\[ = \frac{P_{ii}}{P \log P} - \frac{P^2}{P^2 \log P} - \frac{P^2}{(P \log P)^2} + \frac{\beta h_{ii} \langle \nu, E \rangle}{s - u} - \frac{\beta u_i^2}{(s - u)^2} \]

\[-Nh_{ii}^2 \langle \nu, E \rangle + \sum_s N u_s h_{isi} \]

\[ = \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lilii}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \]

\[ + \frac{\beta h_{ii} \langle \nu, E \rangle}{s - u} - \frac{\beta u_i^2}{(s - u)^2} - Nh_{ii}^2 \langle \nu, E \rangle + \sum_s N u_s h_{isi} \]

Contracting with \( \sigma^i_{ki} \), we have

\[ \sigma^i_{ki} \phi_{ii} \]

\[ = \frac{\sigma^i_{ki}}{P \log P} \left[ \sum_l e^{\kappa_l} h_{llii} + \sum_l e^{\kappa_l} h_{lilii}^2 + \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 - \left( \frac{1}{P} + \frac{1}{P \log P} \right) P_i^2 \right] \]

\[ + \frac{\beta \sigma^i_{ki} \kappa_i \langle \nu, E \rangle}{s - u} - \frac{\beta u_i^2}{(s - u)^2} - \frac{\kappa_i^2}{2} N u_s h_{isi} \]

At \( P_0 \), differentiating the equation (1.2) twice yields,

\[ \sigma^{ii}_{ki} h_{ii} = d_X \psi(\tau_i) + \kappa_i d_\nu \psi(\tau_i), \]

and

\[ \sigma^{ii}_{ki} h_{iii} + \sigma^{pq,rs}_{ki} h_{pqr} h_{rs} \geq -C - Ch_{11}^2 + \sum_s h_{slid} d_\nu \psi(\tau_s), \]

where \( C \) is some uniform constant. Note that

\[ h_{lij} = h_{iii} - h_{ii} h_{il}^2 + h_{li}^2. \]

Inserting (4.12) and (4.13) into (4.10), we obtain

\[ \sigma^i_{ki} \phi_{ii} \]

\[ \geq \frac{1}{P \log P} \left[ \sum_l e^{\kappa_l} \left( -C - C \kappa_i^2 - \sigma^{pq,rs}_{ki} h_{pqr} h_{rs} + \sum_s h_{slid} d_\nu \psi(\partial_s) \right) \right] \]

\[ + \sum_l \sigma^i_{ki} e^{\kappa_l} h_{lilii}^2 + \sigma^i_{ki} \sum_{p \neq q} \frac{e^{\kappa_p} - e^{\kappa_q}}{\kappa_p - \kappa_q} h_{pqi}^2 \right) \]

\[ + \frac{\beta k \sigma^l \langle \nu, E \rangle}{s - u} - \frac{\beta \sigma^i_{ki} u_i^2}{(s - u)^2} - \frac{\kappa_i^2}{2} N u_s \sigma^{ii}_{ki} h_{isi} + \sum_s N u_s \sigma^{ii}_{ki} h_{isi} \]

From (1.9) and (4.11), we deduce

\[ \frac{1}{P \log P} \sum_j \sum_l e^{\kappa_l} h_{ijkl} d_\nu \psi(\tau_j) + \sum_j N u_j \sigma^{ii}_{ki} h_{sii} \geq \sum_l d_\nu \psi(\tau_l) \frac{\beta u_i}{s - u} - C. \]
For any constant $K > 1$, denote

$$A_i = e^{\kappa_i} \left[ K (\sigma_k)^2 - \sum_{p \neq q} \sigma_k^{pq} h_{pp} h_{qq} \right], \quad B_i = 2 \sum_{i \neq i} \sigma_k^{ii} e^{\kappa_i} h_{lii},$$

$$C_i = \sigma_k^{ii} \sum_l e^{\kappa_l} h_{lii}^2, \quad D_i = 2 \sum_{i \neq i} \sigma_k^{li} e^{\kappa_i - \kappa_l} h_{lii}^2, \quad E_i = \frac{1 + \log P}{P \log P} \sigma_k^{ii} p_i^2.$$

Note that

$$- \sum_l \sigma_k^{pq,rs} h_{pq} h_{rs} = \sum_{p \neq q} \sigma_k^{pq} h_{pp}^2 - \sum_{p \neq q} \sigma_k^{pq} h_{pq} h_{qq}.$$

Therefore, (4.14) becomes

$$\sigma_k^{ii} \phi_{ii}$$

$$\geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i)$$

$$+ \frac{\beta_k \sigma_k \langle \nu, E \rangle}{s - u} - \frac{\beta \sigma_k^{ii} u_i^2}{(s - u)^2} + \left( -N \langle \nu, E \rangle - 1 \right) \sigma_k^{ii} \kappa_i^2 + \sum_l d_{\nu} \psi(\tau_l) \frac{\beta u_l}{s - u} - C\kappa_1.$$

Following the same argument as the one in the proof of Lemma 12, from (4.15) we obtain,

$$\sigma_k^{ii} \phi_{ii}$$

$$\geq \frac{1}{P \log P} \sum_i (A_i + B_i + C_i + D_i - E_i)$$

$$+ \frac{\beta_k \sigma_k \langle \nu, E \rangle}{s - u} - \frac{\beta \sigma_k^{ii} u_i^2}{(s - u)^2} + \left( -N \langle \nu, E \rangle - 1 \right) \sigma_k^{ii} \kappa_i^2 + \sum_l d_{\nu} \psi(\tau_l) \frac{\beta u_l}{s - u} - C\kappa_1.$$

Here, the constant $\delta$ is the same constant as the one chosen in Claim 1 of Lemma 12. Moreover, by (4.9), we have

$$- \frac{\beta \sigma_k^{ii} u_i^2}{(s - u)^2} \geq - \frac{\sigma_k^{ii}}{\beta} \left[ 2 \left( \frac{P_i}{P \log P} \right)^2 + 2N^2 u_i^2 \kappa_i^2 \right].$$

Choose $\beta > 0$ such that $\alpha \beta > 2$, then (4.16) implies

$$0 \geq \beta k \sigma_k \langle \nu, E \rangle \frac{\sigma_k^{ii} u_i^2}{s - u}$$

$$+ \left( -N \langle \nu, E \rangle - 1 \right) \sigma_k^{ii} \kappa_i^2 + \sum_l d_{\nu} \psi(\tau_l) \frac{\beta u_l}{s - u} - C\kappa_1 - \sum_{|\kappa_i| \leq \delta \kappa_1} \sigma_k^{ii} 2N^2 u_i^2 \kappa_i^2.$$

Now, first choose $N > 0$ such that $\frac{1}{2} \sum_{|\kappa_i| \leq \delta \kappa_1} \sigma_k^{ii} \kappa_i^2 (-N \langle \nu, E \rangle - 1) - C\kappa_1 \geq 0$, then choose $\beta = \beta(N)$ sufficiently large such that $\sum_{|\kappa_i| \leq \delta \kappa_1} \left( \sigma_k^{ii} \kappa_i^2 (-N \langle \nu, E \rangle - 1) - C\kappa_1 \right)$
\[ \frac{\sigma_{ii}}{\beta} - 2N^2u_i^2 \kappa_i^2 \geq 0. \] We deduce
\[
\frac{\beta C}{s - u} + \sum_{|\kappa_i| \geq \delta \kappa_1} \frac{2\beta \sigma_{ii}^\gamma u_i^2}{(s - u)^2} \geq \sum_{|\kappa_i| \geq \delta \kappa_1} \sigma_{ii}^\gamma \kappa_i^2 (-N \langle \nu, E \rangle - 1).
\]
If \( \frac{C}{s - u} \geq \sum_{|\kappa_i| \geq \delta \kappa_1} \frac{2\beta \sigma_{ii}^\gamma u_i^2}{(s - u)^2} \), we get
\[
\frac{2C\beta}{s - u} \geq \sigma_{11}^\gamma \kappa_1^2 (-N \langle \nu, E \rangle - 1) \geq c_0 (N - 1) \kappa_1,
\]
which implies the desired estimate. If \( \frac{C}{s - u} \leq \sum_{|\kappa_i| \geq \delta \kappa_1} \frac{2\beta \sigma_{ii}^\gamma u_i^2}{(s - u)^2} \), we let \( i_0 \) denote the index of the maximum value element of the set
\[
\left\{ \frac{2\beta \sigma_{ii}^\gamma u_i^2}{(s - u)^2} ; |\kappa_i| \geq \delta \kappa_1 \right\}.
\]
Then, we obtain
\[
\frac{4n \beta \sigma_{ii}^\gamma u_{i_0}^2}{(s - u)^2} \geq \sigma_{i_0 i_0}^\gamma \kappa_{i_0}^2 (-N \langle \nu, E \rangle - 1) \geq C (N - 1) \sigma_{i_0 i_0}^\gamma \delta^2 \kappa_{i_0},
\]
which also implies our desired estimate. \( \square \)

5. The prescribed curvature problem

We will prove Theorem 1 and 5 in this section.

Let’s consider the proof of Theorem 1 first. Recall that in Subsection 3.1, we have solved the approximate Dirichlet problem (3.5) on \( B_r \), for \( r < 1 \). We will denote the strictly convex solution of (3.5) by \( u_r^* \). We further denote the Legendre transform of \((B_r, u_r^*)\) to be \((\Omega_r, u_r^*)\), where \( \Omega_r = Du_r^*(B_r) \) is the domain of \( u_r^* \). By Lemma 19 and 20 in [35] we have
\[
\bar{u} \leq u_r^* \leq \bar{u},
\]
in \( \Omega_r \).

In the following, we will denote \( \bar{\Omega}_r = Du_r^*(B_r) \) to be the domain of \( u_r := u_r^{|\bar{\Omega}} \). It is not difficult to see that these domains are increasing, namely,
\[
\bar{\Omega}_r \subset \bar{\Omega}_{s}, \text{ for } r < s.
\]
Moreover, by the choice of \( u \) in Subsection 3.1 we have
\[
u_{|\partial \bar{\Omega}_r} \to +\infty, \text{ as } r \to 1.
\]
Thus, by the comparison principle, we have
\[
u_{|\partial \bar{\Omega}_r} = [\xi \cdot Du_r^*(\xi) - u_r^* (\xi)]|_{\partial B_r}
\geq [\xi \cdot Du_r^*(\xi) - u_r^* (\xi)]|_{\partial B_r}
= u|_{\partial \bar{\Omega}_r}.
\]
From this we can see that, as \( r \to 1 \), \( u_r|_{\partial \Omega_r} \to +\infty \). This in turn implies, for any compact set \( K \subset \mathbb{R}^n \), there exists a constant \( c_K = c(K) < 1 \) such that when \( r > c_K \),
Therefore, for any compact set $K \subset \mathbb{R}^n$, we can apply Lemma 13 and Lemma 15 to obtain uniform $C^1$ and $C^2$ bounds for $u_r$ in $K$.

More precisely, in order to obtain the local $C^1$ estimate, we introduce a new subsolution $u_1$ of (1.2), where $u_1$ satisfies

$$\sigma_k(\kappa_1, \ldots, \kappa_n) = c_1 + 100,$$

and as $|x| \to \infty$

$$u_1 \to |x| + \varphi \left( \frac{x}{|x|} \right).$$

By the strong maximum principle we have, when $x \in \mathbb{R}^n$

$$u_1(x) < u(x).$$

Thus, for any compact convex domain $K$, let $2\delta = \min_K (\bar{u} - u_1)$.

We define a strict spacelike function $\Psi = u_1 + \delta$. Denote $K' = \{x \in \mathbb{R}^n; \Psi \leq \bar{u}\}$. Since as $|x| \to \infty$, $u_1 - \bar{u} \to 0$, we know that $K'$ is a compact set only depending on $K$. Applying Lemma 13 for any $(\Omega_r, u_r)$, if $K' \subset \Omega_r$, we have the gradient estimate:

$$\sup_{K'} \frac{1}{\sqrt{1 - |Du_r|^2}} \leq \frac{1}{\delta} \sup_{K'} \frac{\bar{u} - \Psi}{\sqrt{1 - |D\Psi|^2}}.$$ 

Next, we want to show that for any given compact set $K \subset \mathbb{R}^n$, $\{|D^2u_r|\}$ is uniformly bounded in $K$. Without loss of generality, let’s consider any $B_R \subset \mathbb{R}^n$. Let $C_0 = \max_{B_R} \bar{u}$ and $s = 2C_0 + 1$ in Lemma 15 Denote $U_s = \{x \in \mathbb{R}^n; u(x) < s\}$, then by earlier discussion, it’s easy to see that there exists $r_s > 0$ such that when $r > r_s$, $\Omega_r \supset U_s$. Applying Lemma 15 we obtain when $r > r_s$

$$\sup_{B_R} \kappa_{\text{max}}(M_{u_r}) \leq C.$$

Here $C$ depends on the upper bound of $\frac{1}{\sqrt{1 - |Du|^2}}$ on $\bar{U}_s$, which is independent of $r$.

Using the classical regularity theorem and convergence theorem, we conclude that $(\Omega_r, u_r)$ converges locally smoothly to an entire, smooth convex function $u$ satisfying (1.2). In view of (5.1) and the asymptotic behavior of $u, \bar{u}$, we know that as $|x| \to \infty$, $u \to |x| + \varphi \left( \frac{x}{|x|} \right)$. Moreover, by Remark 2 we also know that $u$ is strictly convex. Therefore, its Gauss map image is $B_1$, i.e., $Du(\mathbb{R}^n) = B_1$.

Theorem 5 follows by replacing Lemma 13 and Lemma 15 in the proof of Theorem 4 with Lemma 14 and Lemma 16.

6. The radial downward translating soliton

In this section, we will study the radially symmetric downward translating soliton. Recall that we say $\mathcal{M}_u$ is a downward translating soliton when its principal curvatures satisfy

$$(6.1) \quad \sigma_k(\kappa[\mathcal{M}_u]) = \binom{n}{k} \left( C - \frac{1}{\sqrt{1 - |Du|^2}} \right)^k,$$
where $C > 1$ is a constant. We want to point out that in this section and the next section, $C$ is the fixed constant in (6.1). We also denote 
$$\tilde{C} = \sqrt{1 - \frac{1}{C^2}}$$

as in Theorem 6. The following theorem is a generalization of Theorem 1 in [6].

**Theorem 17.** Let $C > 1$ be a positive constant. Then there exists a strictly convex radial solution $u : \mathbb{R}^n \to \mathbb{R}$ of (6.1), satisfying 
\[|Du| \to \tilde{C}, \quad \text{as } |x| \to +\infty.\]

Moreover, as $|x| \to \infty$, $u(x)$ has the following asymptotic expansion
\[
(6.2) \quad u(x) = \tilde{C}|x| - \frac{1}{C^2} k \sqrt{\frac{n-k}{n}} \log |x| + c_0 + o(1)
\]
for some constant $c_0 \in \mathbb{R}$. In particular, the radial solution $u$ is unique up to the addition of a constant.

For radial solutions, we will reduce the equation (6.1) to an ODE. Let $u = u(r)$ and $y = \frac{\partial u}{\partial r}$, then a straightforward calculation yields,
\[
D_i u = y \frac{x_i}{|x|}, \quad D_{ij}^2 u = y \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) + y' \frac{x_i x_j}{|x|^2}.
\]

Therefore,
\[
\kappa[\mathcal{M}_u] = \frac{1}{\sqrt{1 - y^2}} \left( \frac{y'}{1 - y^2}, \frac{y}{r}, \ldots, \frac{y}{r} \right),
\]
and (6.1) becomes
\[
(6.3) \quad \frac{1}{(1 - y^2)^{k/2}} r^{k-1} \left( \frac{k}{n} \frac{y'}{1 - y^2} + \frac{n-k}{n} \frac{y}{r} \right) = \left( C - \frac{1}{\sqrt{1 - y^2}} \right)^k.
\]

By a small modification of the proof of Proposition 2.1 in [6], we obtain

**Proposition 18.** Under the hypotheses of Theorem 17, there exists a solution $y$ of (6.3), which is defined on $[0, +\infty)$ and smooth on $(0, +\infty)$, such that
\[
y(0) = 0, \quad 0 \leq y < \tilde{C}, \quad y'(0) = C - 1, \quad \text{and } y' > 0 \text{ on } (0, +\infty).
\]

Moreover, as $r \to 0^+$, we have
\[
\kappa[\mathcal{M}_u(r)] \to (C - 1)(1, 1, \ldots, 1).
\]

Since the proof is a small modification of the proof of Proposition 2.1 in [6], we skip it here. Now, let’s study the asymptotic behavior of $y$.

**Proposition 19.** Let $y$ be the solution of (6.3). Then as $r \to \infty$, $y$ has the following asymptotic expansion
\[
y(r) = \tilde{C} \left( 1 - \frac{1}{C^2} \sqrt{\frac{n-k}{n}} \frac{1}{r^2} + O \left( \frac{1}{r^2} \right) \right).
\]
Proof. By Proposition 18 we may assume
\[ y(r) = \bar{C} - \frac{z}{r}. \]
Then we have,
\[ \sqrt{1 - y^2} - \frac{1}{C} = \frac{1 - \frac{1}{C} y^2}{\sqrt{1 - y^2 + \frac{1}{C}}} = \frac{z}{r} A(r), \quad \text{where} \quad A(r) = \frac{\sqrt{1 - \frac{1}{C} y^2} + y}{\sqrt{1 - y^2 + \frac{1}{C}}}. \]
Differentiating (6.4), then substituting it into (6.3), we get
\[ k_n y^{k-1} \left( \frac{-z'}{r^k} + \frac{z}{r^{k+1}} \right) + \frac{n - k}{n} y^k = c^k \left( \sqrt{1 - y^2 + \frac{1}{C}} \right)^k. \]
By (6.5), (6.6) can be simplified as
\[ k_n y^{k-1} \left( -z' + \frac{z}{r} \right) + \frac{n - k}{n} y^k = c^k z^k A^k(r). \]
Thus, we obtain
\[ z' = -B(r) z^k + C(r), \]
where
\[ B(r) = c^k n \frac{1 - y^2}{k} y^{k-1} A^k(r) \quad \text{and} \quad C(r) = \frac{z}{r} + \frac{n - k}{k} y(1 - y^2). \]
Applying Proposition 18 we can see that
\[ \lim_{r \to +\infty} B(r) = \frac{n}{k} c^{2k-2} \bar{C} \quad \text{and} \quad \lim_{r \to +\infty} C(r) = \frac{n - k}{k} \frac{1}{C^2} \bar{C}. \]
Here, we have used \( \lim_{r \to +\infty} \frac{z}{r} = 0 \), which is a direct consequence of Proposition 18.
Next Lemma is a generalization of Proposition A.2 in [6].

**Lemma 20.** Assume \( z : (0, +\infty) \to \mathbb{R} \) is a positive solution of the equation
\[ z' = -A(r) z^k + B(r), \]
where \( A, B : (0, \infty) \to \mathbb{R} \) are continuous functions such that
\[ \lim_{r \to +\infty} A(r) = A_0 > 0, \quad \lim_{r \to +\infty} B(r) = B_0 > 0. \]
Then
\[ \lim_{r \to +\infty} z(r) = \sqrt[1/k]{\frac{B_0}{A_0}}. \]

**Proof.** In order to prove this Lemma, we only need to prove

**Claim 3.** Assume \( z : (0, +\infty) \to \mathbb{R} \) is a positive solution of the equation
\[ z' = A_0 z^k + B_0, \]
with \( A_0 < 0, B_0 > 0 \) being constants. Then
\[ \lim_{r \to +\infty} z(r) = \left( -\frac{B_0}{A_0} \right)^{1/k}. \]
If this claim is true, following the same argument as Proposition A.2 in [6], we can prove Lemma 20. We will prove this claim below.

Without loss of generality, let’s consider the positive solution of equation

\[ z' = B - z^k \]

instead. We will show that

\[ \lim_{r \to \infty} z(r) = B^{1/k}. \]

First, since \( z \) is a positive solution of (6.9), let’s assume \( 0 < z(r_0) = z_0 < B^{1/k} \) then we have \( z_0 < z(r) < B^{1/k} \) on \((r_0, \infty)\). Denote \( z_1 = B^{1/k} \) we get

\[ z^k - B = (z - z_1)(z^{k-1} + z^{k-2}z_1 + \cdots + z_1^{k-1}). \]

Therefore (6.9) can be written as

\[ -(r) = \left[ \frac{A_1}{z - z_1} + \frac{Q_{k-2}(z)}{z^{k-1} + z^{k-2}z_1 + \cdots + z_1^{k-1}} \right] dz, \]

where \( A_1 = \frac{1}{k}z_1^{-k} \) and \( Q_{k-2}(z) \) is a polynomial of degree \( k - 2 \). It’s easy to see that \( Q_{k-2}(z) = -A_1 z^{k-2} + Q(k-3)(z) \) and \( Q_{k-3}(z) \) is a polynomial of degree \( k - 3 \).

Integrating (6.11) from \( r_0 \) to \( r \) yields

\[ -(r) - r + r_0 = A_1 \ln \left[ \frac{z(r) - z_1}{z_0 - z_1} \right] - \int_{z_0}^{z(r)} \frac{A_1 z^{k-2}}{z^{k-1} + z^{k-2}z_1 + \cdots + z_1^{k-1}} dz \]

\[ + \int_{z_0}^{z(r)} \frac{Q_{k-2}(z)}{z^{k-1} + z^{k-2}z_1 + \cdots + z_1^{k-1}} dz. \]

Notice that as \( r \to \infty \) the left hand side of (6.12) goes to \(-\infty\), while

\[ -(r) \Rightarrow -A_1 \ln \left( \frac{z_1}{z_0} \right), \]

and

\[ \int_{z_0}^{z(r)} \frac{Q_{k-2}(z)}{z^{k-1} + z^{k-2}z_1 + \cdots + z_1^{k-1}} dz \]

is bounded. Therefore, \( \lim_{r \to \infty} z(r) = z_1 = B^{1/k} \). Similarly, we can prove the case when \( z(r_0) = z_0 > z_1 \). □

From Lemma 20 and equation (6.7), we conclude

\[ \lim_{r \to +\infty} z(r) = \frac{1}{C^2} \sqrt{n - k} \frac{1}{n}. \]

We further assume

\[ z(r) = \frac{1}{C^2} \sqrt{n - k} \frac{w(r)}{r} + \frac{w(r)}{r}. \]

Inserting it into (6.7), we get

\[ w' = -D(r)w + F(r), \]
where
\[ D(r) = B(r) \sum_{i=1}^{k} \binom{k}{i} \left( \frac{1}{C^2} \sqrt{\frac{n-k}{n}} \right)^{k-i} \left( \frac{w}{r} \right)^{i-1}, \quad F(r) = r \left( C(r) - \frac{B(r) n - k}{C^{2k} n} \right) + \frac{w}{r} \]

Notice that \( \lim_{r \to +\infty} \frac{w}{r} = 0 \) and \( D(r) \) has a uniform positive lower bound. In the following, we want to find a positive upper bound for \( F(r) \). Using the expressions (6.8) of \( B(r), C(r) \), we obtain
\[
F(r) = \frac{w}{r} + z + \frac{n-k}{k} y^{k-1} r \left[ y^k - \left( \frac{A(r)}{C} \right)^k \right]
\]
Therefore, we only need to show \( r \left( y - \frac{A(r)}{C} \right) \) is bounded as \( r \to \infty \). By (6.5), we have
\[
r \left( y - \frac{A(r)}{C} \right) = r \left( y - \frac{1}{C} \sqrt{\frac{1 - \frac{1}{C^2}}{1 - y^2} + \frac{1}{C}} \right) = r \left( \frac{y \sqrt{1 - y^2} - \frac{1}{C} \sqrt{1 - \frac{1}{C^2}}}{\sqrt{1 - y^2 + \frac{1}{C}}} \right).
\]
Combining (6.14) with the expression of \( y \) and (6.5), we can derive
\[
y \sqrt{1 - y^2} - \frac{1}{C} \sqrt{1 - \frac{1}{C^2}} = \left( \sqrt{1 - \frac{1}{C^2}} - \frac{z}{r} \right) \left( \frac{1}{C} + \frac{z A(r)}{r} \right) - \frac{1}{C} \sqrt{1 - \frac{1}{C^2}}
\]
\[
= \frac{z}{r} \left( -\frac{1}{C} + A(r) \sqrt{1 - \frac{1}{C^2}} \right) - \frac{z^2 A(r)}{r^2}.
\]
From (6.14), (6.15), and Lemma 20 we conclude that \( r \left( y - \frac{A(r)}{C} \right) \) is uniformly bounded from above. Thus, \( F(r) \) has an uniform upper bound. Applying Proposition A.3 in [6], we obtain a uniform upper bound for \( w \). This completes the proof. \( \square \)

It’s not hard to see that Theorem 17 follows from Proposition 18 and Proposition 19.

7. The existence results

In this section we will prove Theorem 6. First, we want to prove the following existence Theorem.

**Proposition 21.** Suppose \( \varphi \) is a \( C^2 \) function defined on \( S_{\tilde{C}}^{n-1} := \{ x \in \mathbb{R}^n | |x| = \tilde{C} \} \), where \( \tilde{C} = \sqrt{1 - \left( \frac{1}{\tilde{C}} \right)^2} \). There exists a unique, strictly convex solution \( u : \mathbb{R}^n \to \mathbb{R} \) of (1.10) such that as \( |x| \to \infty \),
\[
u(x) \to \tilde{C}|x| - \frac{1}{C^2} \sqrt{\frac{n-k}{n}} \log |x| + \varphi \left( \frac{\tilde{C}}{|x|} \right).
\]
7.1. Constructing barriers. We first construct the barrier functions of equation (1.10). Following the ideas of [31, 32], we denote the radial solution of (1.10) by \( z_0^k(|x|) \), whose asymptotic expansion satisfies (6.2) with \( c_0 = 0 \). Let
\[
p_i(\tilde{C}y) = D\varphi(\tilde{C}y) + (-1)^{i+1}2M\tilde{C}y, \quad i = 1, 2
\]
for any \( y \in S^{n-1} \). Set,
\[
z_i^k(x, y) = \varphi(\tilde{C}y) - p_i(\tilde{C}y) \cdot \tilde{C}y + z_0^k(|x + p_i(\tilde{C}y)|), \quad \forall x \in \mathbb{R}^n, y \in S^{n-1}.
\]
Then,
\[
q_1^k(x) = \sup_{y \in S^{n-1}} z_1^k(x, y)
\]
is a subsolution of (1.10) and
\[
q_2^k = \inf_{y \in S^{n-1}} z_2^k(x, y)
\]
is a supersolution of (1.10). Moreover, \( q_1^k(x) \leq q_2^k(x) \) and when \( |x| \to +\infty \), we have
\[
q_i^k(x) \to \tilde{C}|x| - \frac{1}{C^2} \left( \frac{n-k}{n} \log |x| + \varphi \left( \frac{x}{|x|} \right) \right), \quad i = 1, 2.
\]

7.2. The Dirichlet problem. First, let’s solve equation (1.10) for the case when \( k = n \). For any \( t > \min_{\mathbb{R}^n} q_2^k \), we let
\[
\partial \Omega_t = \{ x \in \mathbb{R}^n | q_1^k(x) < t < q_2^k(x) \},
\]
and \( \Omega_t \) be a smooth, strictly convex domain in \( \mathbb{R}^n \). Consider the following Dirichlet problem:
\[
\begin{cases}
\sigma^{\frac{1}{n}}(\kappa(Mu_t)) = C + \langle \nu, E \rangle & \text{in } \Omega_t \\
u_t = t & \text{on } \partial \Omega_t.
\end{cases}
\]
By a small modification of [13], we know that there exists a unique solution \( u_t \) of (7.2). Then, applying the local \( C^1 \), \( C^2 \) estimates obtained in [7] we conclude that, there exists a subsequence \( \{u_{t_i}\}_{i=1}^{\infty} \) \( (t_i \to \infty \text{ as } i \to \infty) \), that converges to an entire, strictly convex solution \( u \) of (1.10) for \( k = n \). Moreover, it’s easy to see that \( u(x) \) satisfies the desired asymptotic behavior as \( |x| \to \infty \). From now on, we will denote this solution by \( u^n \). We will also denote the Legendre transform of \( u^n \) by \( u^{n*} \).

Next, we consider the case when \( k < n \). We denote the legendre transform of \( z_0^k \) by \((z_0^k)^*\), that is,
\[
(z_0^k)^*(\tau) = r \cdot \frac{\partial z_0^k}{\partial r} - z_0^k(r), \quad \text{where } \tau = \frac{\partial z_0^k}{\partial r}.
\]
Using the asymptotic expansion of \( z_0 \) derived in Section 6 we know
\[
(z_0^k)^*(\tau) = \frac{1}{C^2} \left( \frac{n-k}{n} \log r - 1 \right) + O \left( \frac{1}{r} \right).
\]
We denote its principal part:
\[
(\tilde{z}_0^k)^*(\tau) = \frac{1}{C^2} \left( \frac{n-k}{n} \log r - 1 \right),
\]
it is clear that $(z_{0}^{k})^{*}$ is unbounded in $B_{\tilde{C}}$.

To make sure our solution is convex, we consider the dual Dirichlet problem on $B_{\tau}$ for any $\tau < \tilde{C}$,

$$
(7.3) \quad \begin{cases} 
\hat{F}(w^{*}\gamma_{ik}^{*}u_{kl}^{*}\gamma_{ij}^{*}) = \frac{\binom{n}{k}^{-1/k}}{C - \sqrt{1-|\xi|^2}} & \text{in } B_{\tau}, \\
\hat{u}^{*} = u_{n}^{*} + (z_{0}^{k})^{*} & \text{on } \partial B_{\tau}.
\end{cases}
$$

Here, we have $w^{*} = \sqrt{1-|\xi|^2}$, $\gamma^{*}_{ij} = \delta_{ij} - \frac{\xi \xi_{j}}{1+w}$, $u_{kl}^{*} = \frac{\partial^{2}u}{\partial x_{k}\partial x_{l}}$, $\hat{F}(w^{*}\gamma_{ik}^{*}u_{kl}^{*}\gamma_{ij}^{*}) = \left(\frac{\sigma_{n-k}}{\sigma_{n-k}}(\kappa^{*}[w^{*}\gamma_{ik}^{*}u_{kl}^{*}\gamma_{ij}^{*}])\right)^{1/k}$, and $\kappa^{*}[w^{*}\gamma_{ik}^{*}u_{kl}^{*}\gamma_{ij}^{*}] = (\kappa_{1}^{*}, \ldots, \kappa_{n}^{*})$ are the eigenvalues of the matrix $(w^{*}\gamma_{ik}^{*}u_{kl}^{*}\gamma_{ij}^{*})$. The solvability of (7.3) has been established in Section 3. Therefore, by standard PDE theorems, in order to prove Proposition 21 we only need to obtain local $C^{1}$ and local $C^{2}$ estimates for the translating soliton equation (1.10). In order to do so, we will need the following Lemma.

**Lemma 22.** Let $u^{*}$ be a solution to equation (7.3) and $u^{\tau}$ be the Legendre transform of $u$. Then, for any $x \in Du^{*}(B_{\tau})$, we have $q_{1}^{*}(x) \leq u^{\tau}(x) \leq q_{2}^{*}(x)$.

**Proof.** Without causing confusion we shall drop the superscript $\tau$ in the proof. We only need to prove that

$$
(z_{1}^{k})^{*}(x,y) \leq u(x) \leq (z_{2}^{k})^{*}(x,y),
$$

for any $x \in Du^{*}(B_{\tau})$ and $y \in S^{n-1}$. This is equivalent to prove

$$
(z_{1}^{k})^{*}(\xi,y) \leq u^{*}(\xi) \leq (z_{2}^{k})^{*}(\xi,y),
$$

for any $\xi \in B_{\tau}$ and $y \in S^{n-1}$. Since we have

$$
(7.4) \quad (z_{1}^{k})^{*}(\xi,y) = (z_{1}^{k})^{*}(|\xi|) - p_{i}(\hat{C}y) \cdot \xi - \varphi(\hat{C}y) + p_{i}(\hat{C}y) \cdot \hat{C}y
$$

$$
= (z_{1}^{k})^{*}(|\xi|) - (z_{0}^{n})^{*}(\xi) + (z_{0}^{n})^{*}(\xi,y),
$$

and

$$
(z_{2}^{n})^{*}(\xi,y) < u_{n}^{*}(\xi) < (z_{0}^{n})^{*}(\xi,y),
$$

we obtain on $\partial B_{\tau}$,

$$
(z_{2}^{n})^{*}(\xi,y) < u_{n}^{*}(\xi) < (z_{1}^{k})^{*}(\xi,y),
$$

By comparison principle, we finish the proof. $\square$

### 7.3. Local $C^{1}$ and $C^{2}$ estimates.

Similar to Lemma 13, we have the following local $C^{1}$ estimate Lemma for translating solitons.

**Lemma 23.** Let $\Omega \subset \mathbb{R}^{n}$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \rightarrow \mathbb{R}^{n}$ be strictly $C$-spacelike, i.e.

$$
|Du|, |D\bar{u}|, |D\Psi| < \tilde{C}.
$$

Assume that $u$ is strictly convex and $u \leq \bar{u}$ in $\Omega$. Also assume that near $\partial \Omega$, we have $\Psi > \bar{u}$. Consider the set, where $u > \Psi$. For every $x$ in that set, we have the following gradient estimate for $u$:

$$
\frac{1}{\sqrt{\tilde{C}^{2} - |Du|^{2}}} \leq \frac{1}{u(x) - \Psi(x)} \cdot \sup_{\{u > \Psi\}} \frac{\bar{u} - \Psi}{\sqrt{\tilde{C}^{2} - |D\Psi|^{2}}}.
$$
Since the proof is the same as the proof of Lemma 5.1 in [7], we skip it here.

We now construct $\Psi$. Following the argument in Section 4 of [6], let

$$\Psi(x) = -A_0 + \tilde{C} \sqrt{1 + |x|^2}.$$ 

It is clear that when $|x|$ sufficiently large we have $\Psi(x) > q_2(x)$. On the other hand, for any compact set $\mathcal{K} \subset \mathbb{R}^n$, we can always choose $A_0$ sufficiently large such that $\Psi(x) < q_1(x)$ in $\mathcal{K}$. Applying Lemma 23 we obtain that for any $\mathcal{K} \subset \mathbb{R}^n$ and any strictly convex function $q_1(x) < u(x) < q_2(x)$ satisfying (1.10), whose domain of definition contains $\mathcal{K}$, there exists a local $C^1$ bound $C_{\mathcal{K}}$ for $u(x)$ in $\mathcal{K}$ that is only depending on $\mathcal{K}$.

Using the idea of [35], we can prove the following Pogorelov type local $C^2$ estimate for translating solitons.

**Lemma 24.** Let $u$ be the solution of (1.10) defined on $\Omega$. For any given $s > \min_{\mathbb{R}^n} u + 1$, suppose $u|_{\partial \Omega} > s$. Let $\kappa_{\text{max}}(x)$ be the largest principal curvature of $\mathcal{M}_u = \{(x, u(x)) | x \in \Omega \}$ at $x$. Then, we have

$$\max_{\mathcal{M}_u} (s - u) \kappa_{\text{max}} \leq C_1.$$ 

Here, $C_1$ only depends on the local $C^1$ estimate of $u$. More specifically, $C_1$ depends on the lower bound of $C + \langle \nu, E \rangle$.

Following the argument in Section 5, we complete the proof of Proposition 21.

### 7.4. Proof of Theorem 6

In this subsection, we will prove that the hypersurface $\mathcal{M}_u$ constructed in Proposition 21 has bounded principal curvatures. This completes the proof of Theorem 6. For our convenience, in the following, we will drop the superscript $k$, and the updated configuration $z_k^0$ now becomes $z_0$.

Suppose $u$ is a strictly convex solution of (1.10) and $u^*$ is the Legendre transform of $u$. Then $u^*$ satisfies

$$\hat{F}(w^* \gamma_{ik}^* u_{kl}^* \gamma_{lj}^*) = \frac{n}{C - \frac{1}{\sqrt{1 - |\xi|^2}}} \text{ in } B_{\tilde{C}}.$$ 

We also denote the Legendre transform of $z_0$ by $z_0^*$, that is,

$$z_0^*(\tau) = r \cdot \frac{\partial z_0}{\partial r} - z_0(r), \text{ where } \tau = \frac{\partial z_0}{\partial r}.$$ 

Using the asymptotic expansion of $z_0$ derived in Section 6 we know

$$z_0^*(\tau) = \frac{1}{C^2} \sqrt{n - \frac{k}{n}} (\log r - 1) + O \left( \frac{1}{r} \right).$$ 

We denote its principal part as

$$z_0^* (\tau) = \frac{1}{C^2} \sqrt{n - \frac{k}{n}} (\log r - 1),$$

it is clear that $z_0^*(\tau)$ is unbounded in $B_{\tilde{C}}$. 

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Lemma 25. Let $u^*$ and $\tilde{z}_0^*$ be defined as above. Then we have,

$\lim_{{\xi \to \xi_0}} (u^*(\xi) - \tilde{z}_0^*(|\xi|)) = -\varphi(\xi_0)$, for any $\xi_0 \in \partial B_{\tilde{C}}, \xi \in B_{\tilde{C}}$.

Proof. We will use the auxiliary functions $z_i(x, y)$, $i = 1, 2$, constructed in Subsection 7.1. It’s easy to see that

$z_1(x, y) < u(x) < z_2(x, y)$, for any $x \in \mathbb{R}^n, y \in S^{n-1}$.

By the strictly convexity of $z_i(x, y)$ we have

$\varphi(\xi) = z_i^*(|\xi|) - p_i(\tilde{C} y) \cdot \xi - \varphi(\tilde{C} y) + p_i(\tilde{C} y) \cdot \tilde{C} y$.

Notice that

$\varphi(\xi) = z_i^*(|\xi|) - p_i(\tilde{C} y) \cdot \xi - \varphi(\tilde{C} y) + p_i(\tilde{C} y) \cdot \tilde{C} y$.

Therefore, let $\tilde{C} y = \xi_0$ and $\xi \to \xi_0$, we get

$z_i(\xi, \tilde{C}^{-1} \xi_0) - z_i^*(|\xi|) \to -\varphi(\xi_0)$.

This together with (7.7) yields (7.6). \qed

Now we let

$\partial = \xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}$

be the angular derivative. Similar to Section 10 in [30], we obtain following Lemmas.

Lemma 26. Let $u^*$ be the solution of equation (7.5). Then, $|\partial u^*|$ are bounded above by a constant depends on $|\varphi|_{C^1}$ and $\partial^2 u^*$ are bounded above by a constant depends on $|\varphi|_{C^2}$.

Proof. Notice that $\partial |\xi|^2 = 0$, we have the angular derivative of the right hand side of equation (7.5) is zero. Therefore, following the proof of Lemma 29 and 30 in [30], we have

$F^{ij} w^* \gamma_{ik} (\partial(u^* - \tilde{z}_0^*))_{ij} \gamma_{lj}^* = 0, F^{ij} w^* \gamma_{ik} (\partial^2(u^* - \tilde{z}_0^*))_{ij} \gamma_{lj}^* \geq 0$.

In view of (7.6) and the maximum principle, we obtain the desired estimates. \qed

We further have

Lemma 27. Let $u^*$ be the solution of equation (7.5). There is a positive constant $b$ such that

$\sqrt{\tilde{C}^2 - |\xi|^2} |\partial^2 u^*| < b$.

Proof. We consider $u^* - \tilde{z}_0^*$, which has $C^0$ bound on $B_{\tilde{C}}$. Since $\partial^2 u^* = \partial^2(u^* - \tilde{z}_0^*)$, the rest of the proof is same as the one of Lemma 5.3 in [21]. \qed

Lemma 28. Suppose $a_0 < r < \tilde{C}$ for some $a_0 \in (0, \tilde{C})$, and $S^{n-1}(r) = \{\xi \in \mathbb{R}^n| \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in S^{n-1}(r)$, there is a function

$\tilde{u}_0^* = z_0^* + b_1 \xi_1 + \cdots + b_n \xi_n + b$

such that

$\tilde{u}_0^*(\hat{\xi}) = u^*(\hat{\xi})$. 

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and
\[ \tilde{u}_0^*(\hat{\xi}) > u^*(\xi), \text{ for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}. \]

Here \( b_1, \ldots, b_n \) are constants depending on \( \hat{\xi} \), and \( b \) is a positive constant independent of \( \hat{\xi} \) and \( r \).

Proof. The proof is almost the same as the proof of Lemma 5.4 in [21]. We only need to replace \( u, u^*, -k\sqrt{1 - |x|^2} \) by \( u^* - \bar{\hat{z}}_0^*, \tilde{u}_0^* - \bar{\hat{z}}_0^* \) and \( z_0^* - \bar{\hat{z}}_0^* \) in Li's proof. \( \square \)

Similarly, we can prove the following Lemma analogous to Lemma 5.5 in [21].

**Lemma 29.** Suppose \( a_0 < r < \hat{C} \) for some \( a_0 \in (0, \hat{C}) \), and \( \mathbb{S}^{n-1}(r) = \{ \xi \in \mathbb{R}^n \mid \sum_{i=1}^n \xi_i^2 = r^2 \} \). For any point \( \hat{\xi} \in \mathbb{S}^{n-1}(r) \), there is a function
\[ u_0^* = z_0^* + a_1 \xi_1 + \cdots + a_n \xi_n - a \]
such that
\[ u_0^*(\hat{\xi}) = u^*(\hat{\xi}), \]
and
\[ u_0^*(\hat{\xi}) < u^*(\xi), \text{ for any } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}. \]

Here \( a_1, \ldots, a_n, a \) are constants depending on \( \hat{\xi} \), and \( a > 0, a\sqrt{\hat{C}^2 - |\hat{\xi}|^2} < C_1 \), where \( C_1 \) is a positive constant only depending on \( |\varphi|_{C^2} \).

Using Lemma 28 and Lemma 29 we can show

**Lemma 30.** Let \( u \) be the solution of equation (1.10) and \( u^* \) be the Legendre transform of \( u \). There are positive constants \( d_2 > d_1 \) such that
\[ 0 < d_1 \leq u(\hat{C}^2 - |Du|^2) \leq d_2. \]

Here \( d_2 \) depends on the \( |u|_{C^0(\Omega)} \) and \( \Omega = \{ x \in \mathbb{R}^n \mid |Du| \leq a_0 \} \).

Proof. We modify the proof of Li [21]. We first consider the lower bound. For any \( \hat{\xi} \in \mathbb{S}^{n-1}(r) \), using Lemma 28, we have
\[ u^*(\hat{\xi}) = \tilde{u}_0^*(\hat{\xi}), \text{ and } u^*(\xi) < \tilde{u}_0^*(\xi) \text{ for } \xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}. \]

Thus, using \( \tilde{u}_0^* \) is a supersolution, we get \( u^*(\xi) < \tilde{u}_0^*(\xi) \) in \( B_r \). Therefore, at \( \hat{\xi} \), we get
\[ u(\hat{x}) = \hat{\xi} \cdot Du^* - u^* > \hat{\xi} \cdot \tilde{u}_0^* - \tilde{u}_0^* = z_0(\hat{r}) - b, \]

where we assume \( \hat{x} = Du^*(\hat{\xi}) \) and \( z_0(\hat{r}) := \frac{\partial z_0}{\partial r}(\hat{r}) = |\hat{\xi}| \). Thus, at \( \hat{x} \), we have
\[ u(\hat{C}^2 - |Du|^2) > z_0(\hat{r})(\hat{C}^2 - |z_0(\hat{r})|^2) - b(\hat{C}^2 - |\hat{\xi}|^2). \]

Using the asymptotic behavior of \( z_0 \), we have
\[ z_0(\hat{C}^2 - |z_0'|^2) = \left[ \hat{C}r - \frac{1}{\hat{C}^2} \sqrt{\frac{n-k}{n}} \log r + O\left(\frac{1}{r}\right) \right] \left[ \hat{C}^2 - \left( \hat{C} - \frac{1}{\hat{C}^2} \sqrt{\frac{n-k}{n}} \frac{1}{r} + O\left(\frac{1}{r^2}\right) \right)^2 \right] \]
\[ = \frac{2\hat{C}^2}{\hat{C}^2} \sqrt{\frac{n-k}{n}} + o(1) \]
We denote
\[ 2c_0 = 2\tilde{C}\frac{n-k}{C^2\sqrt{n}}. \]

Therefore, by (7.9), we obtain
\[ u(\tilde{C}^2 - |Du|^2) > \frac{c_0}{2}, \]
for \( r \) being sufficiently close to \( \tilde{C} \), which we may assume \( r > a_0 \). For \( r < a_0 \), without loss of generality, we can assume \( u \geq 1 \). Therefore
\[ u(\tilde{C}^2 - |\tilde{\xi}|^2) \geq \tilde{C}^2 - a_0^2. \]

Thus, we obtain the uniform lower bound. For the upper bound. Applying a similar argument, for \( r \) being sufficiently close to \( \tilde{C} \), which we will still assume \( r \geq a_0 \), we have
\[ u(\tilde{C}^2 - |Du|^2) < z_0(\hat{r})(\tilde{C}^2 - |\hat{\xi}'|^2) + a(\tilde{C}^2 - |\hat{\xi}|^2) \leq 3c_0 + C_1\tilde{C}. \]
We obtain a uniform upper bound. \( \square \)

Finally, we are ready to adapt the ideas in [30, 21] to estimate the principal curvatures of \( M_u \).

**Proposition 31.** Let \( u \) be the solution of equation (1.10). Then the hypersurface \( M_u = \{(x, u(x)) | x \in \mathbb{R}^n\} \) has bounded principal curvatures.

**Proof.** We will establish a Pogorelov type interior estimate. For any \( s > 0 \), consider
\[ \phi = e^{\frac{s}{s-u}[u(\mathcal{C} + \langle \nu, E \rangle)] - N P_m^{1/m}}, \]
where \( P_m = \sum_j \kappa_j^m \) and \( m, N > 0 \) are constants to be determined later. Without loss of generality, we also assume \( u \geq 1 \) in \( \mathbb{R}^n \). It's easy to see that \( \phi \) achieves its local maximum at an interior point of \( U_s = \{x \in \mathbb{R}^n | u(x) < s\} \), we will assume this point is \( x_0 \). We can choose a local normal coordinate \( \{\tau_1, \cdots, \tau_n\} \) such that at \( x_0 \),
\[ h_{ij} = \kappa_i \delta_{ij} \text{ and } \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n. \]

Differentiating \( \log \phi \) at \( x_0 \) we get,
\[ (7.10) \quad \frac{\phi_i}{\phi} = \sum_j \frac{\kappa_j^{m-1} h_{jjj}}{P_m} - N \frac{h_{ii} \langle \tau_i, E \rangle}{\mathcal{C} + \langle \nu, E \rangle} - N \frac{u_i}{u} - \frac{s u_i}{(s-u)^2} = 0, \]
\[
\frac{\phi_{ii}}{\phi^2} - \frac{\phi_{i}^2}{\phi^2} = \frac{1}{P_m} \left[ \sum_j \kappa_j^{m-1} h_{jii} + (m-1) \sum_j \kappa_j^{m-2} h_{jii}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{pqi}^2 \right] \\
- \frac{m}{P_m^2} \left( \sum_j \kappa_j^{m-1} h_{jii} \right)^2 - N \sum_i h_{ii} \frac{\langle \tau_i, E \rangle}{C + \langle \nu, E \rangle} + N h_{ii}^2 \frac{-\langle \nu, E \rangle}{C + \langle \nu, E \rangle} \\
+ N h_{ii}^2 \frac{u_{i}^2}{(C + \langle \nu, E \rangle)^2} + N \frac{h_{ii} \langle \nu, E \rangle}{u} + N \frac{u_{i}^2}{u^2} + s \frac{h_{ii} \langle \nu, E \rangle}{(s-u)^2} - 2s \frac{u_{i}^2}{(s-u)^3} \leq 0.
\]

By equation (1.10), we derive

\[
\sigma_{ii}^k = \left( \frac{n}{k} \right)^k (C + \langle \nu, E \rangle)^{k-1} (-h_{jj} u_j),
\]

and

\[
\sigma_{iijj}^k = -\sigma_{i}^{pq,r} h_{pqj}^k h_{rsj} + \left( \frac{n}{k} \right)^k (C + \langle \nu, E \rangle)^{k-2} h_{jj}^2 u_j^2 \\
+ \left( \frac{n}{k} \right)^k (C + \langle \nu, E \rangle)^{k-1} \left( -\sum_i h_{jj} u_i + h_{jj}^2 \langle \nu, E \rangle \right) \\
\geq -\sigma_{i}^{pq,r} h_{pqj}^k h_{rsj} + \left( \frac{n}{k} \right)^k (C + \langle \nu, E \rangle)^{k-1} \left( -\sum_i h_{jj} u_i \right) \\
-K_0 (C + \langle \nu, E \rangle)^{k-1} \kappa_1^2,
\]

where \( K_0 = K_0(n, k, C) > 0 \) is a constant depending on \( n, k \) and \( C \). Recall that in Minkowski space we have

\[
h_{jjj} = h_{iijj} + h_{ii}^2 h_{jj} - h_{ii} h_{jj}^2.
\]

Thus,

\[
\sigma_{iijj}^k = \sigma_{i}^{ii} h_{jjj} + \sigma_{i}^{ii} h_{jj}^2 - \sigma_{i}^{ii} h_{jj} h_{jj}^2 \geq \sigma_{i}^{ii} h_{jjj} - k \left( \frac{n}{k} \right) (C + \langle \nu, E \rangle)^k h_{jj}^2.
\]
Combining (7.13) with (7.11) we obtain

\(0 \geq \sigma_{k_i}^2 \phi_{ii} = \frac{\sigma_{k_i}^2}{P_m} \left[ \sum_j \kappa_j^{m-1} h_{jjj} + (m - 1) \sum_j \kappa_j^{m-2} h_{jjj}^2 + \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{ppq} \right] \)

\[-m \sigma_{k_i}^2 \left( \sum_j \kappa_j^{m-1} h_{jjj} \right)^2 - N \sigma_{k_i}^2 \kappa_{ii} \sum_l h_{lli} \left( \frac{\tau_l, E}{C + \langle \nu, E \rangle} \right) \]

\[+ N \sigma_{k_i}^2 \frac{h_{ii} \langle \nu, E \rangle}{C + \langle \nu, E \rangle} + N \sigma_{k_i}^2 \frac{u_i^2}{C + \langle \nu, E \rangle} \]

\[+ N \sigma_{k_i}^2 \frac{h_{ii} \langle \nu, E \rangle}{u} + N \sigma_{k_i}^2 \frac{u_i^2}{u^2} + \frac{\sigma_{k_i}^2 h_{ii} \langle \nu, E \rangle}{u} - 2s \frac{\sigma_{ii}^2 u_i^2}{(s - u)^2} \]

\[\geq -K_0 (C + \langle \nu, E \rangle)^{k-1} \kappa_1 + \sum_i (A_i + B_i + \kappa_{ii}^2 - E_i) \]

\[+ \binom{n}{k} (C + \langle \nu, E \rangle)^{k-1} - \sum_{j,l} h_{jjj} \kappa_j^{m-1} u_l \]

\[-N k \binom{n}{k} (C + \langle \nu, E \rangle)^{k-2} \sum_{x} \kappa_{ii}^2 u_i^2 \]

\[+ N \sigma_{k_i}^2 \kappa_{ii} \frac{h_{ii} \langle \nu, E \rangle}{C + \langle \nu, E \rangle} + N \sigma_{k_i}^2 \frac{u_i^2}{C + \langle \nu, E \rangle} \]

\[+ N \sigma_{k_i}^2 \frac{h_{ii} \langle \nu, E \rangle}{u} + N \sigma_{k_i}^2 \frac{u_i^2}{u^2} + \frac{\sigma_{k_i}^2 h_{ii} \langle \nu, E \rangle}{u} - 2s \frac{\sigma_{ii}^2 u_i^2}{(s - u)^2} \].

Here

\(A_i = \frac{\kappa_i^{m-1}}{P_m} \left[ K (\kappa_i)_{ii}^2 - \sum_{p,q} \sigma_{p,q}^2 h_{ppq} h_{qqi} \right] \), for some constant \(K > 1\),

\(B_i = \frac{2\kappa_i^{m-1}}{P_m} \sum_j \sigma_{k_j}^{jjj} h_{jjj}^2 \),

\(C_i = \frac{m - 1}{P_m} \sigma_{k_i}^2 \sum_j \kappa_j^{m-2} h_{jjj}^2 \),

\(D_i = \frac{2\sigma_{k_i}^{jjj}}{P_m} \sum_{j \neq i} \frac{\kappa_j^{m-1} - \kappa_i^{m-1}}{\kappa_j - \kappa_i} h_{jjj} \),

and

\(E_i = \frac{m \sigma_{k_i}^2}{P_m^2} \left( \sum_j \kappa_j^{m-1} h_{jjj} \right)^2 \).

By Lemma 8, Lemma 9, and Corollary 10 in [22] we can assume the following claim holds.
Claim 4. There exists two small positive constants \( \delta \) and \( \eta < 1 \). If \( \kappa_k \leq \delta \kappa_1 \), we have

\[
\sum_i A_i + B_i + C_i + D_i - \left( 1 + \frac{\eta}{m} \right) E_i \geq 0,
\]

where \( m > 0 \) is sufficiently large.

If (7.15) doesn’t hold, we would have \( \kappa_k > \delta \kappa_1 \). Since \( \sigma_k \leq \binom{n}{k} c^k \), we get

\[
\delta^{k-1} \kappa_1^k \leq \kappa_1 \kappa_2 \cdots \kappa_k \leq \sigma_k \leq \binom{n}{k} c^k.
\]

This gives an upper bound for \( \kappa_1 \) at \( x_0 \) directly, then we would be done. Therefore, we assume (7.15) holds. Plugging (7.15) into (7.14) yields,

\[
0 \geq -K_0 (C + \langle \nu, E \rangle)^{k-1} \kappa_1 + \eta \frac{\sigma_k^{ij}}{P_m} \left( \sum_j \kappa_j^{m-1} h_{jj} \right)^2 - k \left( \frac{n}{k} \right) (C + \langle \nu, E \rangle)^{k-1} |\nabla u| u^2 \left( \frac{N}{u} + \frac{s}{(s-u)^2} \right) + N \sigma_k^{ij} \frac{h_{ii} \langle \nu, E \rangle}{u} + N \sigma_k^{ii} \frac{u_i^2}{u^2} + s \frac{\sigma_k^{ii} h_{ii} \langle \nu, E \rangle}{u^2} (s-u)^2 - 2s \frac{\sigma_k^{ii} u_i^2}{(s-u)^3}.
\]

From equation (7.10) we obtain

\[
\left( \sum_j \kappa_j^{m-1} h_{jj} \right)^2 = N^2 \frac{\kappa_i u_i^2}{(C + \langle \nu, E \rangle)^2} + N^2 \frac{u_i^2}{u^2} + \frac{s^2 u_i^2}{(s-u)^4} - 2N^2 \frac{\kappa_i u_i^2}{u(C + \langle \nu, E \rangle)} - 2Ns \frac{\kappa_i u_i^2}{(C + \langle \nu, E \rangle)(s-u)^2} + 2Ns \frac{u_i^2}{u(s-u)^2}.
\]

Inserting (7.17) into (7.16), we derive

\[
0 \geq -K_0 (C + \langle \nu, E \rangle)^{k-1} \kappa_1 + \eta \frac{s^2 \sigma_k^{ii} u_i^2}{(s-u)^4} + N(N\eta + 1) \sigma_k^{ii} \frac{u_i^2}{(C + \langle \nu, E \rangle)^2} - 2N^2 \eta \frac{\sigma_k^{ii} \kappa_i u_i^2}{u(C + \langle \nu, E \rangle)} - 2Ns \eta \frac{\sigma_k^{ii} u_i^2}{u(s-u)^2} + 2Ns \frac{\sigma_k^{ii} u_i^2}{(s-u)^2} + N \sigma_k^{ij} \frac{h_{ii} \langle \nu, E \rangle}{u} + N(\eta N + 1) \sigma_k^{ii} \frac{u_i^2}{u^2} + s \frac{\sigma_k^{ii} h_{ii} \langle \nu, E \rangle}{(s-u)^2} - 2s \frac{\sigma_k^{ii} u_i^2}{(s-u)^3} - k \left( \frac{n}{k} \right) (C + \langle \nu, E \rangle)^{k-1} |\nabla u| u^2 \left( \frac{N}{u} + \frac{s}{(s-u)^2} \right) + N \sigma_k^{ii} \kappa_i^2 \frac{\langle \nu, E \rangle}{C + \langle \nu, E \rangle}.
\]
It’s clear that
\[(7.19)\]
\[|\nabla u| = \frac{|Du|}{\sqrt{1 - |Du|^2}} < -\langle \nu, E \rangle \leq C.\]

We also notice that for any \(1 \leq i \leq n, \sigma_k^{ii} \kappa_i \leq \binom{n}{k} C^k \) (no summation). By a simple calculation we get, when \(N > \frac{1}{\eta^2}\)
\[(7.20)\]
\[\eta s^2 \sigma_k^{ii} u_i^2 + 2Ns\eta \frac{\sigma_k^{ii} u_i^2}{u(s-u)^2} - 2s \frac{\sigma_k^{ii} u_i^2}{(s-u)^3} \geq 0.\]

Moreover, applying Lemma \[30\] we know there exists two positive constants \(\tilde{d}_2 > \tilde{d}_1 > 0\) such that
\[(7.21)\]
\[\tilde{d}_1 \leq u(C + \langle \nu, E \rangle) \leq \tilde{d}_2.\]

Therefore, for \(N > \frac{1}{\eta^2}\) being sufficiently large, combining \[(7.19)-(7.21)\] with \[(7.18)\] we have,
\[
0 \geq -K_0(C + \langle \nu, E \rangle)^{k-1} \kappa - \frac{2N}{d_1} |\nabla u|^2 \sigma_k^{ii} \kappa_i - 2Ns \frac{|\nabla u|^2 \sigma_k^{ii} \kappa_i}{(C + \langle \nu, E \rangle)(s-u)^2} \]
\[- NC \sigma_k^{ii} \kappa_i - C \sigma_k^{ii} \kappa_i \frac{s}{(s-u)^2} - kC^2 \binom{n}{k} (C + \langle \nu, E \rangle)^{k-1} \frac{s}{(s-u)^2} \]
\[- k \binom{n}{k} C^2 (C + \langle \nu, E \rangle)^{k-1} N + N \frac{c_0 \sigma_k^{ii} \kappa_i}{C + \langle \nu, E \rangle}.\]

It’s easy to see that the above inequality yields, at \(x_0\)
\[\kappa_1 \leq K(N, C, \tilde{d}_1) \frac{s^2}{(s-u)^2}.\]

Therefore, in \(U_s\), by \[(7.21)\], we have
\[\phi \leq K(N, C, \tilde{d}_1) e^{-\frac{s}{(s-u)^2}} \frac{s^2}{(s-u)^2}.\]

Note that for any \(t \in [0, s]\),
\[\varphi(t) = e^{-\frac{s}{(s-t)^2}} \frac{s^2}{(s-t)^2} \leq 4e^{-2}.\]

We obtain that at any point \(x \in U_s\),
\[(7.22)\]
\[\phi \leq K(N, C, \tilde{d}_1).\]

Now, for any \(x \in \mathbb{R}^n\), we can choose \(s > 0\) large such that \(x \in U_{s/2}\). Then by \[(7.22)\] and \[(7.21)\], we conclude
\[\kappa_1(x) \leq K(N, C, \tilde{d}_1, \tilde{d}_2).\]

Since \(x\) is arbitrary, we finish proving Proposition \[31\].

Theorem \[6\] follows from Proposition \[21\] and Proposition \[31\] immediately.
References

[1] A.D. Alexandrov, *Existence and uniqueness of a convex surface with a given integral curvature*. Doklady Akad. Nauk Kazah SSSR, 36 (1942), 131–134.
[2] A.D. Alexandrov, *Uniqueness theorems for surfaces in the large. I (Russian)* Vestnik Leningrad. Univ., 11, (1956), 5–17. English translation: AMS Translations, series 2, 21, (1962), 341–354.
[3] I. Bakelman and B. Kantor, *Existence of spherically homeomorphic hypersurfaces in Euclidean space with prescribed mean curvature*. Geometry and Topology, Leningrad, 1, (1974), 3–10.
[4] P. Bayard, *Dirichlet problem for space-like hypersurfaces with proescribed scalar curvature in R^n*. Calc. Var. 18 (2003), 1–30.
[5] P. Bayard, *Entire spacelike hypersurfaces of prescribed scalar curvature in Minkowski space*. Cal. Var. 26(2), 2006, 245–264.
[6] P. Bayard, *Entire downward solitons to the scalar curvature flow in Minkowski space*. arXiv:2002.07685.
[7] P. Bayard and O.C. Schnürer, *Entire spacelike hypersurfaces of constant Gauss curvature in Minkowski space*, J. Reine Angew. Math. 627 (2009), 1–29.
[8] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian*. Acta Math. 155 (1985), no. 3-4, 261–301.
[9] L. A. Caffarelli, L. Nirenberg and J. Spruck, *Nonlinear second order elliptic equations IV: Starshaped compact Weingarten hypersurfaces*. Current topics in partial differential equations, Y.Ohyu, K.Kasahara and N.Shimakura (eds), Kinokunize, Tokyo, 1985, 1–26.
[10] Caffarelli, Luis; Nirenberg, Louis; Spruck, Joel *Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces*. Comm. Pure Appl. Math. 41 (1988), no. 1, 47–70.
[11] S.Y. Cheng and S.T. Yau, *On the regularity of the solution of the n-dimensional Minkowski problem*. Comm. Pure Appl. Math., 29, (1976), 495–516.
[12] K.S. Chou and X.J. Wang, *A variational theory of the Hessian equation*. Comm. Pure Appl. Math., 54, (2001), 1029–1064.
[13] Delanoë, F. *The Dirichlet problem for an equation of given Lorentz-Gaussian curvature*. Ukrain. Mat. Zh. 42 (1990), no. 12, 1704–1710; translation in Ukrainian Math. J. 42 (1990), no. 12, 1538–1545 (1991).
[14] B. Guan, *The Dirichlet problem for Hessian equations on Riemannian manifolds*, Calc. Var. Partial Differential Equations 8 (1999), no. 1, 45–69.
[15] B. Guan and P. Guan, *Convex Hypersurfaces of Prescribed Curvature*. Ann. of Math., 156, (2002), 655–674.
[16] Guan, Pengfei; Lin, Changshou; Ma, Xi’nan *The Christoffel-Minkowski problem. II. Weingarten curvature equations*. Chinese Ann. Math. Ser. B 27 (2006), no. 6, 595–614.
[17] P. Guan, J. Li and Y.Y. Li, *Hypersurfaces of Prescribed Curvature Measure*. Duke Math. J., 161, (2012), 1927–1942.
[18] P. Guan, C.S. Lin and X. Ma, *The Existence of Convex Body with Prescribed Curvature Measures*. Int. Math. Res. Not., (2009) 1947–1975.
[19] P. Guan, C. Ren and Z. Wang, *Global C^2 estimates for curvature equation of convex solution*, Comm. Pure Appl. Math. LXVIII(2015) 1287–1325.
[20] Jian, Huaiyu; Ju, Hongjie; Lu, Jian; *Translating solutions to mean curvature flow with a forcing term in Minkowski space*, Commun. Pure Appl. Anal. 9 (2010), no. 4, 963–973.
[21] A.-M. Li, *Spacelike hypersurfaces with constant Gauss-Kronecker curvature in the Minkowski space*, Arch. Math., Vol.64, 534–551 (1995).
[22] M. Li, C. Ren and Z. Wang, *An interior estimate for convex solutions and a rigidity theorem*, J. Funct. Anal. 270 (2016), no. 7, 2691–2714.
[23] M. Lin and N.S. Trudinger, *On some inequalities for elementary symmetric functions*. Bull. Austral. Math. Soc. 50 (1994), 317–326.
[24] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*. Comm. Pure Appl. Math., 6, (1953), 337–394.

[25] V.I. Oliker, *Hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed Gaussian curvature and related equations of Monge-Ampère type*. Commun. Partial Differ. Equ. 9(8), 807–838 (1984).

[26] A.V. Pogorelov, *On the question of the existence of a convex surface with a given sum principal radii of curvature (in Russian)*. Uspekhi Mat. Nauk., 8, (1953), 127–130.

[27] A.V. Pogorelov, *The Minkowski Multidimensional Problem*. John Wiley, 1978.

[28] C. Ren and Z. Wang, *On the curvature estimates for Hessian equations*, American Journal of Mathematic, Vol.141, No.5, 1281–1315, 2019.

[29] C. Ren and Z. Wang, *The global curvature estimate for the $n - 2$ Hessian equation*. arXiv: 2002.08702.

[30] C. Ren, Z. Wang, and L. Xiao, *Entire spacelike hypersurfaces with constant $\sigma_{n-1}$ curvature in Minkowski space*. [arXiv:2005.06109]

[31] Spruck, Joel; Xiao, Ling *Entire downward translating solitons to the mean curvature flow in Minkowski space*. Proc. Amer. Math. Soc. 144 (2016), no. 8, 3517–3526.

[32] A. E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math. 66, 39–56 (1982).

[33] A. Treibergs and S.W. Wei, *Embedded hypersurfaces with prescribed mean curvature*. J. Diff. Geom. , 18, (1983), 513–521.

[34] J. Urbas, *The Dirichlet problem for the equation of prescribed scalar curvature in Minkowski space*. Calc. Var. 18 (2003) 307–316.

[35] Z. Wang and L. Xiao, *Entire spacelike hypersurfaces with constant $\sigma_k$ curvature in Minkowski space*. [arXiv:2007.01495]

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