Sorted Concave Penalized Regression

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Abstract

The Lasso is biased. Concave penalized least squares estimation (PLSE) takes advantage of signal strength to reduce this bias, leading to sharper error bounds in prediction, coefficient estimation and variable selection. For prediction and estimation, the bias of the Lasso can be also reduced by taking a smaller penalty level than what selection consistency requires, but such smaller penalty level depends on the sparsity of the true coefficient vector. The sorted ℓ1 penalized estimation (Slope) was proposed for adaptation to such smaller penalty levels. However, the advantages of concave PLSE and Slope do not subsume each other. We propose sorted concave penalized estimation to combine the advantages of concave and sorted penalizations. We prove that sorted concave penalties adaptively choose the smaller penalty level and at the same time benefits from signal strength, especially when a significant proportion of signals are stronger than the corresponding adaptively selected penalty levels. A local convex approximation, which extends the local linear and quadratic approximations to sorted concave penalties, is developed to facilitate the computation of sorted concave PLSE and proven to possess desired prediction and estimation error bounds. We carry out a unified treatment of penalty functions in a general optimization setting, including the penalty levels and concavity of the above mentioned sorted penalties and mixed penalties motivated by Bayesian considerations. Our analysis of prediction and estimation errors requires the restricted eigenvalue condition on the design, not beyond, and provides selection consistency under a required minimum signal strength condition in addition. Thus, our results also sharpens existing

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results on concave PLSE by removing the upper sparse eigenvalue component of the sparse Riesz condition.

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1 Introduction

The purpose of this paper is twofold. First, we provide a unified treatment of prediction, coefficient estimation, and variable selection properties of concave penalized least squares estimation (PLSE) in high-dimensional linear regression under the restrictive eigenvalue (RE) condition on the design matrix. Second, we propose sorted concave PLSE to combine the advantages of concave and sorted penalties, and to prove its superior theoretical properties and computational feasibility under the RE condition. Along the way, we study penalty level and concavity of multivariate penalty functions, including mixed penalties motivated by Bayesian considerations as well as sorted and separable penalties. Local convex approximation (LCA) is proposed and studied as a solution for the computation of sorted concave PLSE.

Consider the linear model

\[ y = X\beta^* + \varepsilon, \]  

(1.1)

where \( X = (x_1, ..., x_p) \in \mathbb{R}^{n \times p} \) is a design matrix, \( y \in \mathbb{R}^n \) is a response vector, \( \varepsilon \in \mathbb{R}^n \) is a noise vector, and \( \beta^* \in \mathbb{R}^p \) is an unknown coefficient vector. For simplicity, we assume throughout the paper that the design matrix is column normalized with \( \|x_j\|_2 = n \).

Our study focuses on local and approximate solutions for the minimization of penalized loss functions of the form

\[ \|y - X\beta\|_2^2/(2n) + \text{Pen}(\beta) \]  

(1.2)

with a penalty function \( \text{Pen}(\cdot) \) satisfying certain minimum penalty level and maximum concavity conditions as described in Section 2. The PLSE can be viewed as a statistical choice among local minimizers of the penalized loss.

Among PLSE methods, the Lasso [23] with the \( \ell_1 \) penalty \( \text{Pen}(\beta) = \lambda\|\beta\|_1 \) is the most widely used and extensively studied. The Lasso is relatively easy to compute as it is a convex minimization problem, but it is well known that the Lasso is biased. A consequence of this bias is the
requirement of a neighborhood stability/strong irrepresentable condition on
the design matrix $X$ for the selection consistency of the Lasso [13, 33, 24, 26].
Fan and Li [9] proposed a concave penalty to remove the bias of the
Lasso and proved an oracle property for one of the local minimizers of
the resulting penalized loss. Zhang [29] proposed a path finding algorithm
PLUS for concave PLSE and proved the selection consistency of the PLUS-
computed local minimizer under a rate optimal signal strength condition on
the coefficients and the sparse Riesz condition (SRC) [30] on the design. The
SRC, which requires bounds on both the lower and upper sparse eigenvalues
of the Gram matrix and is closely related to the restricted isometry property
(RIP) [7], is substantially weaker than the strong irrepresentable condition.
This advantage of concave PLSE over the Lasso has since become well
understood.

For prediction and coefficient estimation, the existing literature somehow
presents an opposite story. Consider hard sparse coefficient vectors satisfying
$|\text{supp}(\beta^*)| \leq s$ with $\log(p/s) \asymp \log p$ and small $(s/n) \log p$. Although rate
minimax error bounds were proved under the RIP and SRC respectively for
the Dantzig selector and Lasso in [6] and [30], Bickel et al. [4] sharpened their
results by weakening the RIP and SRC to the RE condition, and van de Geer
and Bühlmann [25] proved comparable prediction and $\ell_1$ estimation error
bounds under an even weaker compatibility or $\ell_1$ RE condition. Meanwhile,
rate minimax error bounds for concave PLSE still require two-sided sparse
eigenvalue conditions like the SRC [29, 32, 27, 10] or a proper known upper
bound for the $\ell_1$ norm of the true coefficient vector [12]. It turns out that
the difference between the SRC and RE conditions are quite significant as
Rudelson and Zhou [20] proved that the RE condition is a consequence of a
lower sparse eigenvalue condition alone. This seems to suggest a theoretical
advantage of the Lasso, in addition to its relative computational simplicity,
compared with concave PLSE.

Emerging from the above discussion, an interesting question is whether
the RE condition alone on the design matrix is also sufficient for the above
discussed results for concave penalized prediction, coefficient estimation and
variable selection, provided proper conditions on the true coefficient vector
and the noise. An affirmative answer of this question, which we provide in
this paper, amounts to the removal of the upper sparse eigenvalue condition
on the design matrix and actually also a relaxation of the lower sparse
eigenvalue condition or the restricted strong convexity (RSC) condition
[14] imposed in [12]; or equivalently, to the removal of the remaining
analytical advantage of the Lasso as far as error bounds for the afore
mentioned aims are concerned. Specifically, we prove that when the true
\(\beta\) is sparse, concave PLSE achieves rate minimaxity in prediction and coefficient estimation under the RE condition on the design. Furthermore, the selection consistency of concave PLSE is also guaranteed under the same RE condition and an additional uniform signal strength condition on the nonzero coefficients, and these results also cover non-separable multivariate penalties imposed on the vector \(\beta\) as a whole, including sorted and mixed penalties such as the spike-and-slab Lasso [19].

In addition to the above conservative prediction and estimation error bounds for the concave PLSE that are comparable with those for the Lasso in both rates and regularity conditions on the design, we also prove faster rates for concave PLSE when the signal is partially strong. For example, instead of the prediction error rate \((s/n)\log p\) in the worst case scenario, the prediction rate for concave PLSE is actually \(\sigma^2(s + s_1 \log p)/n\) where \(s_1\) is the number of small nonzero signals under the same RE condition on the design. Thus, concave PLSE adaptively benefits from signal strength with no harm to the performance in the worst case scenario where all signals are just below the radar screen. This advantage of concave PLSE is known under the sparse Riesz and comparable conditions, but not under the RE condition as presented in this paper.

The bias of the Lasso can be also reduced by taking a smaller penalty level than those required for variable selection consistency, regardless of signal strength. In the literature, PLSE is typically studied in a standard setting at penalty level \(\lambda \geq \lambda_* = (\sigma/\eta)\sqrt{(2/n)\log p}\). This lower bound has been referred to as the universal penalty level. However, as the bias of the Lasso is proportional to its penalty level, rate minimaxity in prediction and coefficient estimation requires smaller \(\lambda = \sigma\sqrt{(2/n)\log(p/s)}\) [22, 3]. Unfortunately, this smaller penalty level depends on \(s = \|\beta^*\|_0\), which is typically unknown. For the \(\ell_1\) penalty, a remedy for this issue is to apply the Slope or a Lepski type procedure [21, 3]. However, it is unclear from the literature whether the same can be done with concave penalties.

We propose a class of sorted concave penalties to combine the advantages of concave and sorted penalties. This extends the Slope beyond \(\ell_1\). Under an RE condition, we prove that the sorted concave PLSE inherits the benefits of both concave and sorted PLSE, namely bias reduction through signal strength and adaptation to the smaller penalty level. This provides prediction and \(\ell_2\) estimation error bounds of the order \(\sigma^2(s + s_1 \log(p/s))/n\) and comparable \(\ell_1\) estimation error bounds. Moreover, our results apply to approximate local solutions which can be viewed as output of computational algorithms for sorted concave PLSE.

To prove the computational feasibility of our theoretical results in
polynomial time, we develop an LCA algorithm for a large class of
multivariate concave PLSE to produce approximate local solutions to which
our theoretical results apply. The LCA is a majorization-minimization
(MM) algorithm and is closely related to the local quadratic approximation
(LQA) [9] and the local linear approximation (LLA) [34] algorithms. The
development of the LCA is needed as the LLA does not majorize sorted
concave penalties in general. Our analysis of the LCA can be viewed as
extension of the results in [32, 11, 14, 1, 27, 12, 10] where separable penalties
are considered, typically at larger penalty levels.

The rest of this paper is organized as follows. In Section 2 we study
penalty level and concavity of general multivariate penalties in a general
optimization setting, including separable, multivariate mixed and sorted
penalties, and also introduce the LCA for sorted penalties. In Section 3, we
develop a unified treatment of prediction, coefficient estimation and variable
selection properties of concave PLSE under the RE condition at penalty
levels required for variable selection consistency. In Section 4 we provide
error bounds for approximate solutions at smaller and sorted penalty levels
and output of LCA algorithms.

Notation: We denote by \( \beta^* \) the true regression coefficient vector, \( \Sigma = X^TX/n \) the sample Gram matrix, \( S = \text{supp}(\beta^*) \) the support set of the
coefficient vector, \( s = |S| \) the size of the support, and \( \Phi(\cdot) \) the standard
Gaussian cumulative distribution function. For vectors \( v = (v_1, \ldots, v_p) \), we
denote by \( \|v\|_q = \sum_j (|v_j|^q)^{1/q} \) the \( \ell_q \) norm, with \( \|v\|_\infty = \max_j |v_j| \) and
\( \|v\|_0 = \# \{ j : v_j \neq 0 \} \). Moreover, \( x_+ = \max(x, 0) \).

2 Penalty functions

We consider minimization of penalized loss

\[
L(\beta) + \text{Pen}(\beta), \quad \beta \in \mathbb{R}^p,
\]

with a general Fréchet differentiable loss function \( L(\beta) \) and a general
multivariate penalty function \( \text{Pen}(\beta) \) satisfying certain minimum penalty
level and maximum concavity conditions.

Penalty level and concavity of univariate penalty functions are well
understood as we will briefly describe in our discussion of separable penalties
in Subsection 2.2 below. However, for multivariate penalties, we need to
carefully define their penalty level and concavity in terms of sub-differential.
This is done in Subsection 2.1. We then study in separate subsections three
types of penalty functions, namely separable penalties, multivariate mixed
penalties, and sorted penalties. Moreover, we develop the LCA for sorted penalties in Subsection 2.5.

2.1 Sub-differential, penalty level and concavity

The sub-differential of a penalty $\text{Pen}(\cdot)$ at a point $b \in \mathbb{R}^p$, denoted by $\partial \text{Pen}(b)$ as a subset of $\mathbb{R}^p$, can be defined as follows. A vector $g \in \mathbb{R}^p$ belongs $\partial \text{Pen}(b)$ iff

$$
\liminf_{t \to 0^+} t^{-1}\{\text{Pen}(b + tu) - \text{Pen}(b)\} \geq g^T u, \quad \forall u \in \mathbb{R}^p.
$$

(2.2)

As $g^T u$ is continuous in $g$, $\partial \text{Pen}(b)$ is always a closed convex set.

Suppose $\dot{L}(b)$ is everywhere Fréchet differentiable with derivative $\dot{L} \partial \text{Pen}(b)$. It follows immediately from the definition of the sub-differential in (2.2) that

$$
\liminf_{t \to 0^+} \frac{1}{t}\left\{\{\dot{L}(\hat{\beta} + tu) + \text{Pen}(\hat{\beta} + tu)\} - \{\dot{L}(\hat{\beta}) + \text{Pen}(\hat{\beta})\}\right\} \geq 0
$$

for all $u \in \mathbb{R}^p$ iff $\dot{L}(\hat{\beta}) \in \partial \text{Pen}(\hat{\beta})$. This includes all local minimizers. Let $\hat{\text{Pen}}(b)$ denote a member of $\partial \text{Pen}(b)$. We say that $\hat{\beta}$ is a local solution for minimizing (2.1) iff the following estimating equation is feasible:

$$
-\dot{L}(\hat{\beta}) = \hat{\text{Pen}}(\hat{\beta}).
$$

(2.3)

As (2.3) characterizes all minimizers of the penalized loss when the penalized loss is convex, it can be viewed as a Karush-Kuhn-Tucker (KKT) condition.

We define the penalty level of $\text{Pen}(\cdot)$ at a point $b \in \mathbb{R}^p$ as

$$
\lambda(b) = \sup \left\{ \lambda : \{g_{S_b} : g \in \partial \text{Pen}(b)\} \supseteq [-\lambda, \lambda]|_{S_b}\right\}.
$$

(2.4)

This definition is designed to achieve sparsity for solutions of (2.3). Although $\lambda(b)$ is a function of $b$ in general, it depends solely on $\text{Pen}(\cdot)$ for many commonly used penalty functions. Thus, we may denote $\lambda(b)$ by $\lambda$ for notational simplicity. For example, in the case of the $\ell_1$ penalty $\text{Pen}(b) = \lambda\|b\|_1$, (2.4) holds with $\lambda(b) = \lambda$ for all $b$ with $|S_b| < p$. We consider a somewhat weaker penalty level for the sorted penalty in Subsection 2.4.

We define the concavity of $\text{Pen}(\cdot)$ at $b$, relative to an oracle/target coefficient vector $\beta^o$, as

$$
\kappa(b) = \kappa(b, \beta^o) = \sup \{ (\beta^o - b)^T (\hat{\text{Pen}}(b) - \hat{\text{Pen}}(\beta^o)) / \|b - \beta^o\|_2^2 \}
$$

with the convention $0/0 = 0$, where the supreme is taken over all choices $\hat{\text{Pen}}(b) \in \partial \text{Pen}(b)$ and $\hat{\text{Pen}}(\beta^o) \in \partial \text{Pen}(\beta^o)$. We use $\kappa = \kappa(\text{Pen}) = \kappa(b, \beta^o)$ for all $b$.
to denote the maximum concavity of Pen(·). For convex penalties, $-\pi(b, \beta^0)\|b - \beta^0\|^2_2$ is the symmetric Bregman divergence. A penalty function Pen(b) is convex if and only if $\pi \leq 0$. Given $s \geq \|\beta^0\|_0$ and $\xi > 0$, we may consider a relaxed concavity of Pen(·) at $b$ as
\[
\pi_{1,2}(b; \xi) = \inf \{ \pi_2(b) + (1 + \xi)^2 s \pi_1(b) \},
\] (2.6)
where infimum is taken over all nonnegative $\pi_1(b)$ and $\pi_2(b)$ satisfying
\[
h^T (\hat{\text{Pen}}(\beta^0) - \hat{\text{Pen}}(b)) \leq \pi_1(b)\|h\|_1^2 + \pi_2(b)\|h\|_2^2
\] (2.7)
with $h = b - \beta^0$ for all $\hat{\text{Pen}}(b) \in \partial\text{Pen}(b)$ and $\hat{\text{Pen}}(b + h) \in \partial\text{Pen}(b + h)$. This notion of concavity is more relaxed than the $\ell_2$ one in (2.5) because $\pi_{1,2}(b; \xi) \leq \pi(\cdot)$ always holds due to the option of picking $\pi_1(b) = 0$. The relaxed concavity is quite useful in our study of multivariate mixed penalties in Subsection 2.3. To include more solutions for (2.3) and also to avoid sometimes tedious task of fully characterizing the sub-differential, we allow $\hat{\text{Pen}}(b)$ to be a member of the following “completion” of the sub-differential,
\[
\partial\text{Pen}(b) = \text{convex.hull}\left\{ \lim_{t \to 0^+} \text{closure}\left( \bigcup_{\|v - b\|_2 \leq t} \partial\text{Pen}(v) \right) \right\}
\] (2.8)
in the estimating equation (2.3), as long as $\partial\text{Pen}(b)$ is replaced by the same subset in (2.3), (2.4), (2.5), (2.6) and (2.7). However, for notational simplicity, we may still use $\partial\text{Pen}(b)$ to denote $\partial\text{Pen}(b)$. We may also impose an upper bound condition on the penalty level of Pen(·):
\[
\sup\left\{ \|\hat{\text{Pen}}(b)\|_\infty / \lambda(b) : b \in \mathbb{R}^p, \hat{\text{Pen}}(b) \in \partial\text{Pen}(b) \right\} \leq \eta_*.
\] (2.9)
It is common to have $\eta_* = 1$ although $\eta_* \geq 1$ by (2.4). Without loss of generality, we impose the condition $\text{Pen}(0) = 0$.

2.2 Separable penalties

In general, separable penalty functions can be written as a sum of penalties on individual variables, $\text{Pen}(b) = \sum_{j=1}^p \rho_j(b_j)$. We shall focus on separable penalties of the form
\[
\rho(b; \lambda) = \sum_{j=1}^p \rho_j(b_j; \lambda),
\] (2.10)
where $\rho(t; \lambda)$ is a parametric family of penalties with the following properties:

(i) $\rho(t; \lambda)$ is symmetric, $\rho(t; \lambda) = \rho(-t; \lambda)$ with $\rho(0; \lambda) = 0$;
(ii) $\rho(t; \lambda)$ is monotone, $\rho(t_1; \lambda) \leq \rho(t_2; \lambda)$ for all $0 \leq t_1 < t_2$;

(iii) $\rho(t; \lambda)$ is left- and right-differentiable in $t$ for all $t$;

(iv) $\rho(t; \lambda)$ has selection property, $\dot{\rho}(0^+; \lambda) = \lambda \geq 0$;

(v) $|\dot{\rho}(t^-; \lambda)| \vee |\dot{\rho}(t^+; \lambda)| \leq \lambda$ for all real $t$,

where $\dot{\rho}(t \pm ; \lambda)$ denote the one-sided derivatives. Condition (iv) guarantees that the index $\lambda$ equals to the penalty level defined in (2.4), and condition (v) bounds the maximum penalty level with $\eta_n = 1$ in (2.9). We write $\dot{\rho}(t; \lambda) = x$ when $x$ is between the left- and right-derivative of $\rho(t; \lambda)$ at $t$, including $t = 0$ where $\dot{\rho}(0; \lambda) = x$ means $|x| \leq \lambda$, so that $\dot{\rho}(t; \lambda)$ is defined in the sense of (2.8). By (2.5), the concavity of $\rho(t; \lambda)$ is defined as

$$\pi(t; \rho, \lambda) = \sup_{t' > t} \{ \dot{\rho}(t'; \lambda) - \ddot{\rho}(t; \lambda) \}/(t' - t),$$

(2.11)

where the supreme is taken over all possible choices of $\dot{\rho}(t; \lambda)$ and $\dot{\rho}(t'; \lambda)$ between the left- and right-derivatives. Further, define the overall maximum concavity of $\rho(t; \lambda)$ as

$$\pi(\rho) = \max_{t > 0, \lambda > 0} \pi(t; \rho, \lambda).$$

(2.12)

Because $\rho(b; \lambda)$ is a sum in (2.10), the closure in (2.8) is the set of all vectors $\dot{\rho}(b; \lambda) = (\dot{\rho}(b_1; \lambda), \ldots, \dot{\rho}(b_p; \lambda))^T$, so that $\lambda$ gives the penalty level (2.4), and

$$\pi(b) = \pi(b; \rho, \lambda) \leq \max_{j \leq p} \pi(b_j; \rho, \lambda)$$

(2.13)

gives the concavity (2.5) of the multivariate penalty $\rho(b; \lambda)$.

Many popular penalty functions satisfy conditions (i)–(v) above, including the $\ell_1$ penalty $\rho(t; \lambda) = \lambda |t|$ for the Lasso with $\pi(\rho) = 0$, the SCAD (smoothly clipped absolute deviation) penalty [9] with

$$\rho(t; \lambda) = \int_0^{[t]} \{ \lambda - \pi(x - \lambda)_+ \}_+ \, dx$$

(2.14)

and $\pi(\rho) = \pi$, and the MCP (minimax concave penalty) [29] with

$$\rho(t; \lambda) = \int_0^{[t]} (\lambda - \pi x)_+ \, dx$$

(2.15)

and $\pi(\rho) = \pi$. An interesting way of constructing penalty functions is to mix penalties $\rho(t; \lambda)$ with a distribution $G(d\lambda)$ and a real $r_n$ as follows,

$$\rho_G(t) = -r_n^{-1} \log \left[ \int \exp \{ -r_n \rho(t; \lambda) \} G(d\lambda) \right].$$

(2.16)
This class of mixed penalties has a Bayesian interpretation as we discuss in Subsection 2.3. If we treat \( \exp \{-r_n \rho(t; \lambda)\} G(d\lambda) / \int \exp \{-r_n \rho(t; x)\} G(dx) \) as conditional density of \( \lambda \) under a joint probability \( \mathbb{P}_G \), we have

\[
\hat{\rho}_G(t) = \mathbb{E}_G[\hat{\rho}(t; \lambda)|t], \\
\lambda_G = \hat{\rho}_G(0+) = \int \lambda G(d\lambda) \quad \text{(penalty level),} \tag{2.17}
\]

\[
\mathcal{R}(\rho_G) \leq \mathcal{R}(\rho) + r_n \sup_t \text{Var}_G[\hat{\rho}(t; \lambda)|t],
\]

due to \( \hat{\rho}(0+; \lambda) = \lambda \) and \( \{\hat{\rho}(t_1; \lambda) - \hat{\rho}(t_2; \lambda)\}/(t_2 - t_1) \leq \mathcal{R}(\rho) \) for all \( t_1 \neq t_2 \) and \( \lambda > 0 \). For example, if \( G \) puts the entire mass in a two-point set \( \{\lambda', \lambda''\} \),

\[
\mathcal{R}(\rho_G) \leq \mathcal{R}(\rho) + r_n (\lambda' - \lambda'')^2/4.
\]

In particular, for \( \rho(t; \lambda) = |t|/\lambda \), \( r_n = n \) and two-point distributions \( G \), (2.16) gives the spike-and-slab Lasso penalty as in [19].

### 2.3 Multivariate mixed penalties

Let \( \pi(b|\theta) \) be a parametric family of prior density functions for \( \beta \). When \( \varepsilon \sim N(0, \sigma^2 I_{n\times n}) \) with known \( \sigma \) and \( \theta \) is given, the posterior mode can be written as the minimizer of

\[
\|y - Xb\|^2_2/(2n) + \text{Pen}_\theta(b)
\]

when \( \text{Pen}_\theta(b) = -(\sigma^2/n) \log(\pi(b|\theta)/\pi(0|\theta)) \). In a hierarchical Bayes model where \( \theta \) has a prior distribution \( \pi(d\theta) \), the posterior mode corresponds to

\[
\text{Pen}(b) = -r_n^{-1} \log \int \exp \{-r_n \text{Pen}_\theta(b)\} \nu(d\theta) \tag{2.18}
\]

with \( r_n = n/\sigma^2 \) and \( \nu(d\theta) = \pi(0|\theta)\pi(d\theta)/\int \pi(0|\theta)\pi(d\theta) \). This gives rise to (2.18) as a general way of mixing penalties \( \text{Pen}_\theta(\cdot) \) with suitable \( r_n \). When \( r_n = n/\sigma^2 \), it corresponds to the posterior for a proper hierarchical prior if the integration \( \int \exp \{-r_n \text{Pen}_\theta(b)\} \nu(d\theta) db \) is finite, and an improper one otherwise. When \( 0 < r_n \neq n/\sigma^2 \), it still has a Bayesian interpretation with respect to mis-specified noise level \( \sqrt{n/r_n} \) or sample size \( \sigma^2 r_n \). While \( r_n = 0 \) leads to \( \text{Pen}(b) = \int \text{Pen}(b) \nu(d\theta) \) as the limit at \( r_n = 0+ \), the formulation does not prohibit \( r_n < 0 \).

For \( \lambda = (\lambda_1, \ldots, \lambda_p)^T \in [0, \infty)^p \), let \( \rho(b; \lambda) = \sum_{j=1}^{p} \rho(b_j; \lambda_j) \) be a separable penalty function with different penalty levels for different
coefficients $b_j$, where $\rho(t; \lambda)$ is a family of penalties indexed by penalty level $\lambda$ as discussed in Subsection 2.2. As in (2.18),
\[
\rho_\nu(b) = -r_n^{-1} \log \left\{ \exp \left\{ -r_n \rho(b; \lambda) \right\} \nu(d\lambda) \right\},
\]
with the convention $\rho_\nu(b) = \int \rho(b; \lambda) \nu(d\lambda)$ for $r_n = 0$, is a mixed penalty for any probability measure $\nu(d\lambda)$. We study below the sub-differential, penalty level and concavity of such mixed penalties.

By definition, the sub-differential of (2.19) can be written as
\[
\partial \rho_\nu(b) = \left\{ \frac{\hat{\rho}(b; \lambda) \exp \left\{ -r_n \rho(b; \lambda) \right\} \nu(d\lambda)}{\int \exp \left\{ -r_n \rho(b; \lambda) \right\} \nu(d\lambda)} : \hat{\rho}(b; \lambda) \in \partial \rho(b; \lambda) \right\},
\]
with $\partial \rho(b; \lambda)$ being the set of all vectors $\hat{\rho}(b; \lambda) = (\hat{\rho}(b_1; \lambda_1), \ldots, \hat{\rho}(b_p; \lambda_p))^T$, provided that the lim inf can be taken under the integration over $\nu(d\lambda)$. This is allowed when $\|\hat{\rho}(t; \lambda)\|_\infty < \infty$. As in (2.17), we may write (2.20) as
\[
\partial \rho_\nu(b) = \left\{ \mathbb{E}_\nu[\hat{\rho}(b; \lambda) \mid b] : \hat{\rho}_j(b; \lambda) = (\hat{\rho}(b_1; \lambda_1), \ldots, \hat{\rho}(b_p; \lambda_p))^T \right\}.
\]
where the conditional $\mathbb{P}_\nu(d\lambda \mid b)$ is proportional to $\exp \left\{ -r_n \rho(b; \lambda) \right\} \nu(d\lambda)$. We recall that $\hat{\rho}(0; \lambda_j)$ may take any value in $[-\lambda_j, \lambda_j]$.

**Proposition 1.** Let $\rho_\nu(b)$ be a mixed penalty in (2.19) generated from a family of penalties $\rho(t; \lambda)$ satisfying conditions (i)–(v) in Subsection 2.2. Let $S_b = \text{supp}(b)$ with $s_b = |S_b| < p$. Then, the concavity of $\rho_\nu(b)$ satisfies
\[
\pi(b) \leq \pi(\rho) + \sup_u \phi_{\max}(r_n \text{Cov}_\nu(\hat{\rho}(u; \lambda), \hat{\rho}(u; \lambda) \mid u))
\]
with $\phi_{\max}$ being the largest eigenvalue, and (2.7) holds with
\[
\pi_2(b) \leq \pi(\rho) + (r_n \vee 0) \sup_{u} \max_{1 \leq j \leq p} \text{Var}_\nu(\hat{\rho}(u_j; \lambda_j) \mid u).
\]
If the components of $\lambda$ are independent given $\theta$, then (2.7) holds with
\[
\pi_2(b) \leq \pi(\rho) + (r_n \vee 0) \sup_u \max_{1 \leq j \leq p} \mathbb{E}_\nu[\text{Var}(\hat{\rho}(u_j; \lambda_j) \mid u, \theta)] \mid u],
\]
\[
\pi_1(b) \leq (r_n \vee 0) \sup_u \max_{1 \leq j \leq p} \text{Var}_\nu(\mathbb{E}_\nu[\hat{\rho}(u_j; \lambda_j) \mid u, \theta]) \mid u] .
\]
If in addition $\lambda$ is exchangeable under $\nu(d\lambda)$, the penalty level of (2.19) is
\[
\lambda(b) = \mathbb{E}[\lambda_j \mid b], \quad \forall j \notin S_b.
\]

Interestingly, (2.22) indicates that mixing $\rho(b; \lambda)$ with $r_n < 0$ makes the penalty more convex.
The penalty can be written as
\[ \lambda \ell_1(\theta) \] are bounded by
\[ \lambda \] where
\[ \beta \]
are univariate penalty functions
\[ \rho \]
with
\[ b \] is the j-th largest value among \(|b_1|, \ldots, |b_p|\).

Here we extend the sorted penalty beyond \( \ell_1 \). Given a family of univariate penalty functions \( \rho(t; \lambda) \) and a vector \( \lambda = (\lambda_1, \ldots, \lambda_p)^T \) with non-increasing nonnegative elements, we define the associated sorted penalty as
\[ \rho_{\#}(b; \lambda) = \sum_{j=1}^p \rho(b^{\#}_j; \lambda_j). \]

Although (2.25) seems to be a superficial extension of (2.24), it brings upon potentially significant benefits and its properties are nontrivial. We say that the sorted penalty is concave if \( \rho(t; \lambda) \) is concave in \( t \) in \([0, \infty)\). In Section 4, we prove that under an RE condition, the sorted concave penalty inherits the benefits of both the concave and sorted penalties, namely bias reduction for strong signal components and adaptation to the penalty level to the unknown sparsity of \( \beta \).
The following proposition gives penalty level and an upper bound for the maximum concavity for a broad class sorted concave penalties, including the sorted SCAD penalty and MCP. In particular, the construction of the sorted penalty does not increase the maximum concavity in the class.

**Proposition 2.** Let $\rho_\#(\mathbf{b}; \lambda)$ be as in (2.25) with $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$. Suppose $\rho(t; \lambda) = \int_0^{|t|} \hat{\rho}(x; \lambda)dx$ with a certain $\hat{\rho}(x; \lambda)$ non-decreasing in $\lambda$ almost everywhere in positive $x$. Let $\mathcal{S}_b = \text{supp}(b)$ and $s_b = |\mathcal{S}_b|$. Then, the sub-differential of $\rho_\#(\mathbf{b}; \lambda)$ includes all vectors $\mathbf{g} = (g_{k_j})_{j \in [p]}$ satisfying

$$g_{k_j} = \hat{\rho}(b_{k_j}; \lambda_j), \quad |b_{k_1}| \geq \cdots \geq |b_{k_s}| > 0, \quad j \leq s_b, \quad |g_{k_j}| \leq \lambda_j, \quad \{k_{s+1}, \ldots, k_p\} = \mathcal{S}_b^c, \quad j > s_b.$$  

Moreover, the maximum concavity of $\rho_\#(\mathbf{b}; \lambda)$ is no greater than that of the penalty family $\rho(t; \lambda)$:

$$h^T \left\{ \hat{\rho}_\#(\mathbf{b}; \lambda) - \hat{\rho}_\#(\mathbf{b} + h; \lambda) \right\} \leq \pi(h)\|h\|_2^2, \quad \forall \; \mathbf{b}, \; h.$$  

Moreover, the monotonicity condition on $\rho_\#(\mathbf{b}; \lambda)$ holds for the $\ell_1$, SCAD and MCP.

It follows from (2.26) that the maximum penalty level at each index $j \in \mathcal{S}_b^c$ is $\lambda_{s+1}$. Although the penalty level does not reach $\lambda_{s+1}$ simultaneously for all $j \in \mathcal{S}_b^c$ as in (2.4), we still take $\lambda_{s+1}$ as the penalty level for the sorted penalty $\rho_\#(\mathbf{b}; \lambda)$. This is especially reasonable when $\lambda_j$ decreases slowly in $j$. In Subsection 4.6, we show that this weaker version of the penalty level is adequate for Gaussian errors provided that for certain $A_0 > 1 > \alpha$,

$$\lambda_j \geq \lambda_{s,j} = A_0 \sigma \sqrt{(2/n) \log(p/(\alpha j))}, \quad j = 1, \ldots, p.$$  

More important, (2.26) shows sorted penalties automatically pick penalty level $\lambda_{s+1}$ from the sequence $\{\lambda_j\}$ without requiring the knowledge of $s$.

A key element of the proof of Proposition 2 is to write (2.25) as

$$\rho_\#(\mathbf{b}; \lambda) = \max \left\{ \sum_{j=1}^p \rho(b_j; \lambda_{k_j}) : (k_1, \ldots, k_p)^T \in \text{perm}(p) \right\},$$  

where $\text{perm}(p)$ is the set of all vectors generated by permuting $(1, \ldots, p)^T$.

### 2.5 Local convex approximation

We develop here LCA for penalized optimization (2.1), especially for sorted penalties. As a majorization-minimization (MM) algorithm, it is closely related to and in fact very much inspired by the LQA [9] and LLA [34, 32, 11].
Suppose for a certain continuously differentiable convex function $\Pen_-(b)$,

$$\Pen_+(b) = \Pen(b) + \Pen_-(b)$$

(2.29)

is convex. The LCA algorithm can be written as

$$b^{(new)} = \arg \min_b \left\{ L(b) + \Pen_+(b) - b^T \hat{\Pen}_-(b^{(old)}) \right\}. \quad (2.30)$$

This LCA is clearly an MM-algorithm: As

$$\Pen^{(new)}(b) = \Pen_+(b) - \Pen_-(b^{(old)}) - (b - b^{(old)})^T \hat{\Pen}_-(b^{(old)})$$

is a convex majorization of $\Pen(b)$ with $\Pen^{(new)}(b^{(old)}) = \Pen(b^{(old)})$,

$$L(b^{(new)}) + \Pen(b^{(new)}) \leq L(b^{(new)}) + \Pen^{(new)}(b^{(new)}) \leq L(b^{(old)}) + \Pen^{(new)}(b^{(old)}) \quad (2.31)$$

$$= L(b^{(old)}) + \Pen(b^{(old)}).$$

Let $\rho_{\#}(b; \lambda)$ be the sorted concave penalty in (2.25) with a penalty family $\rho(t; \lambda)$ and a vector of sorted penalty levels $\lambda = (\lambda_1, \ldots, \lambda_p)^T$. Suppose $\dot{\rho}(x; \lambda) = (\partial \rho(x; \lambda))_{\lambda}^{\partial x}$ is non-decreasing in $\lambda$ almost everywhere in positive $x$, so that Proposition 2 applies. Suppose for a certain continuously differentiable convex function $\rho_-(t)$

$$\rho_+(t; \lambda_j) = \rho(t; \lambda_j) + \rho_-(t) \text{ is convex in } t \text{ for } j = 1, \ldots, p. \quad (2.32)$$

By (2.28), $\rho_{+,\#}(b; \lambda) = \rho_{\#}(b; \lambda) + \rho_-(b)$, the sorted penalty with $\rho_+(t; \lambda)$, is convex in $b$, so that the LCA algorithm for $\rho_{\#}(b; \lambda)$ can be written as

$$b^{(new)} = \arg \min_b \left\{ L(b) + \rho_{+,\#}(b; \lambda) - b^T \dot{\rho}_-(b^{(old)}) \right\}, \quad (2.33)$$

where $\dot{\rho}_-(b)$ is the gradient of $\rho_-(b) = \sum_{j=1}^p \rho_-(b_j)$. The simplest version of LCA takes $\rho_-(t) = t^2 \beta / 2$ with the maximum concavity defined in (2.12), but this is not necessary as (2.32) is only required to hold for the given $\lambda$.

Figure 1 demonstrates that for $p = 1$, the LCA with $\rho_-(t) = t^2 \beta / 2$ also majorizes the LLA with $\Pen^{(new)}(b) = \rho(|b^{(old)}|; \lambda) + \dot{\rho}(|b^{(old)}|; \lambda)(|b| - |b^{(old)}|)$. With $\rho_-(t) = \lambda |t| - \rho(t; \lambda)$ in (2.32), the LCA is identical to an unfolded LLA with $\Pen^{(new)}(b) = \lambda |b| + \{\dot{\rho}(b^{(old)}; \lambda) - \lambda \text{sgn}(b^{(old)})\} (b - b^{(old)})$. The situation is the same for separable penalties, i.e. $\lambda_1 = \lambda_p$. However, the LLA is not feasible for truly sorted concave penalties with $\lambda_1 > \lambda_p$. 

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Figure 1: Local convex approximation (red dashed), local linear approximation (blue mixed) and original penalty (black solid) for MCP with $\lambda = 1$ and $\kappa = 1/3$ at $b^{(old)} = 1.5$. Left: penalty function and its approximations; Right: $\arg \min_b \{(x - b)^2/2 + \text{Pen}(b)\}$.

As the LCA also majorized the LLA, it imposes larger penalty on solutions with larger step size compared with the LLA, but this has little effect in our theoretical analysis in Subsections 4.5 and 4.6.

The LCA (2.33) can be computed by proximal gradient algorithms [15, 2, 18], which approximate $\mathcal{L}(b)$ by $\mathcal{L}(x) + (b - x)^T \nabla \mathcal{L}(x) + \|b - x\|^2_2/(2t)$ around $x$. For example, the ISTA [2] for LCA can be written as follows.

**Algorithm 1: ISTA for LCA**

**Initialization:** $b^0 = b^{(old)}$

**Iteration:**

$$b^{k+1} = \text{prox}(b^k - t \nabla \mathcal{L}(b^k) + t \rho_-(b^{(old)}); t \rho_+,(; \lambda))$$

where $\rho_+,(b; \lambda)) = \sum_{j=1}^p \rho_+(b^j; \lambda_j)$, $t \rho_+$ is the reciprocal of a Lipschitz constant for $\nabla \mathcal{L}$ or determined in the iteration by backtracking, and

$$\text{prox}(x; \text{Pen}) = \arg \min_b \{(b - x)^2/2 + \text{Pen}(b)\}$$  \hspace{1cm} (2.34)

is the so called proximal mapping for convex Pen, e.g. $\text{Pen}(b) = t \rho_+,(b; \lambda)$. We may also apply FISTA [2] as an accelerated version of Algorithm 1.

**Algorithm 2: FISTA for LCA**

**Initialization:** $x^1 = b^0 = b^{(old)}$, $t_1 = 1$

**Iteration:**

$$b^k = \text{prox}(x^k - t \nabla \mathcal{L}(x^k) + t \rho_-(b^{(old)}); t \rho_+,(; \lambda))$$

$$t_{k+1} = \{1 + (1 + 4t_k^2)^{1/2}\}/2$$

$$x^{k+1} = b^k + \{(t_k - 1)/t_{k+1}\}(b^k - b^{k-1})$$

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For sorted penalties $\rho_\#(b; \lambda)$, the proximal mapping is not separable but still preserves the sign and ordering in absolute value of the input. Thus, after removing the sign and sorting the input and output simultaneously, it can be solved with the isotonic proximal mapping,

$$\text{iso.prox}(x; \text{Pen}) = \arg\min_b \left\{ \|b - x\|_2^2/2 + \text{Pen}(b) : b_j \downarrow \text{ in } j \right\} \tag{2.35}$$

with $\text{Pen}(b) = \sum_{j=1}^p \rho(b_j; \lambda_j)$. Moreover, similar to the computation of the proximal mapping for the Slope in [5], this isotonic proximal mapping can be computed by the following algorithm.

**Algorithm 3:** iso.prox$(x; \rho(\cdot; \lambda))$

**Input:** $\lambda \downarrow$, $x \downarrow$

**Compute** $b_j = \arg\min_b \{(x_j - b)^2/2 + \rho(b; \lambda_j)\}$

**While** $b$ is not nonincreasing **do**

- Identify blocks of violators of the monotonicity constraint,
  - $b_{j'-1} > b_{j'} \leq b_{j'+1} \leq \cdots \leq b_{j''} > b_{j'+1}$, $b_{j'} < b_{j''}$
- Replace $b_j$, $j' \leq j \leq j''$, with the solution of
  $$\arg\min_b \sum_{j'=j}^{j''} \{(x_j - b)^2/2 + \rho(b; \lambda_j)\}$$

We formally state the above discussion in the following proposition.

**Proposition 3.** For $v = (v_1, \ldots, v_p)^T$, let $v^\# = (v_1^\#, \ldots, v_p^\#)^T$ with $v_j^\#$ being the $j$-th largest among $\{|v_1|, \ldots, |v_p|\}$. For $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$, let $\rho_\#(b; \lambda) = \sum_{j=1}^p \rho(b_j^\#; \lambda_j)$ as in (2.25) and $\rho(b; \lambda) = \sum_{j=1}^p \rho(b_j; \lambda_j)$. Then, $\text{sgn}(b_j) = \text{sgn}(x_j)$, $|b_j| \geq |b_k|$ whenever $|x_j| > |x_k|$, and

$$\left\{\text{prox}(x; \rho_\#(\cdot; \lambda))\right\}^\# = \text{iso.prox}(x^\#; \rho(\cdot; \lambda)). \tag{2.36}$$

Moreover, when $x_1 \geq \ldots \geq x_p \geq 0$ and $(x_j - b)^2/2 + \rho(b; \lambda_j)$ are convex in $b \geq 0$ for all $j$, iso.prox$(x; \rho(\cdot; \lambda))$ is solved by Algorithm 3.

For the MCP with $\rho_+(t; \lambda) = \rho(t; \lambda) + \pi t^2/2$, the univariate solution is

$$\text{prox}(x; t\rho_+ (\cdot; \lambda)) = \text{sgn}(x) \min \{(|x| - t\lambda)_+, |x|/(1 + t\pi)\},$$

which is a combination of the soft threshold and shrinkage estimators. Figure 2 plots this univariate proximal mapping for a specific $(\lambda, t)$. Algorithm 3 is more explicitly given as follows.
Figure 2: $\rho_+(\cdot; \lambda) = \rho(t; \lambda) + \pi t^2/2$ for LCA (red dashed), Lasso (blue mixed) and MCP (black solid) with $\lambda = 1$ and $\pi = 1/3$. Left: penalties; Right: proximal mappings

Algorithm 4: iso.prox($x; t \rho_+(\cdot; \lambda)$) for the LCA with sorted MCP

| Input: | $\lambda \downarrow$, $x \downarrow$, $t$, $\pi$ |
|-------|----------------------------------------|
| Compute | $b_j = \min \{ (x_j - t\lambda_j)_+, x_j/(1 + t\pi) \}$ |
| While $b$ is not nonincreasing do |
| Identify blocks of violators of the monotonicity constraint, $b_{j'-1} > b_{j'} \leq b_{j'+1} \leq \cdots \leq b_{j''} > b_{j''+1}$, $b_{j'} < b_{j''}$ |
| Replace $b_j$, $j' \leq j \leq j''$, with common value $b$ satisfying |
| $b = \frac{\sum_{j'=j'}^{j''} (x_j - t\lambda_j I\{\lambda_j > \pi b\})}{\sum_{j'=j'}^{j''} (1 + t\pi I\{\lambda_j \leq \pi b\})}$ |

The computation of the isotonic proximal mapping for SCAD can be carried out in a similar but more complicated fashion, as the region for shrinkage is broken by an interval for soft thresholding in each coordinate in the domain.

3 Properties of Concave PLSE

In this section, we present our results for concave PLSE at a sufficiently high penalty level to allow selection consistency. Smaller penalty levels are considered in Section 4. We divide the section into three subsections to describe conditions on the design matrix, the collection of PLSE under consideration, and error bounds for prediction, coefficient estimation and variable selection.
3.1 The restricted eigenvalue condition

We now consider conditions on the design matrix. The RE condition, proposed in [4], is arguably the weakest available on the design to guarantee rate minimax performance in prediction and coefficient estimation for the Lasso. The RE coefficient for the $\ell_2$ estimation loss can be defined as follows:

For $S \subset \{1, \ldots, p\}$ and $\xi > 0,$

$$\text{RE}_2(S; \xi) = \inf \left\{ \frac{(u^T \Sigma u)^{1/2}}{\|u\|_2} : \|u_{S^c}\|_1 < \xi \|u_S\|_1 \right\} \tag{3.1}$$

with $\inf \emptyset = 0$ for $S = \emptyset$. The RE condition refers to the property that $\text{RE}_2(S; \xi)$ is no smaller than a certain positive constant for all design matrices under consideration. For prediction and $\ell_1$ estimation, it suffices to impose a somewhat weaker compatibility condition [25]. The compatibility coefficient, also called $\ell_1$-RE [25], is defined as

$$\text{RE}_1(S; \xi) = \inf \left\{ \frac{(u^T \Sigma u)^{1/2}}{\|u_S\|_1/|S|^{1/2}} : \|u_{S^c}\|_1 < \xi \|u_S\|_1 \right\}. \tag{3.2}$$

In addition to the RE coefficients above, we define a relaxed cone invertibility factor (RCIF) for prediction as

$$\text{RCIF}_{\text{pred}}(S; \eta, w) = \inf \left\{ \frac{\|\Sigma u\|_\infty |S|}{u^T \Sigma u} : (1 - \eta)\|u_{S^c}\|_1 < -w_S^T u_S \right\}, \tag{3.3}$$

with $\eta \in [0, 1)$ and a vector $w \in \mathbb{R}^p$, and a RCIF for the $\ell_q$ estimation as

$$\text{RCIF}_{\text{est},q}(S; \eta, w) = \inf \left\{ \frac{\|\Sigma u\|_\infty |S|^{1/q}}{u^T \Sigma u} : (1 - \eta)\|u_{S^c}\|_1 < -w_S^T u_S \right\}. \tag{3.4}$$

The RCIF is a relaxation of the cone invertibility coefficient [28] for which the constraint $\|u_{S^c}\|_1 < \xi \|u_S\|_1$ is imposed.

The choices of $\xi$, $\eta$ and $w$ depend on the problem under consideration in the analysis, but typically we have $\|w\|_\infty \leq (1 - \eta)\xi$ so that the minimization in (3.3) and (3.4) is taken over a smaller cone. However, $\|w\|_2$ can be much smaller than $|S|^{1/2}(1 - \eta)\xi$ with partial signal strength. Moreover, it is feasible to have $w_S = 0$ under a beta-min condition for selection consistency. In our analysis, we use an RE condition to prove cone membership of the estimation error of the concave PLSE and the RCIF to bound the prediction and coefficient estimation errors. The following proposition, which follows from the analysis in Section 3.2 of [28], shows that the RCIF may provide sharper bounds than the RE does.
Proposition 4. If \( \|w_S\|_\infty \leq (1 - \eta)\xi \), then

\[
\begin{align*}
\text{RCIF}_{\text{pred}}(S; \eta, w) &\geq \text{RE}_1^2(S; \xi)/(1 + \xi)^2, \\
\text{RCIF}_{\text{est}, 1}(S; \eta, w) &\geq \text{RE}_1^2(S; \xi)/(1 + \xi)^2, \\
\text{RCIF}_{\text{est}, 2}(S; \eta, w) &\geq \text{RE}_1(S; \xi)\text{RE}_2(S; \xi)/(1 + \xi).
\end{align*}
\]

(3.5)

3.2 Concave PLSE

As discussed in Subsection 2.1, local minimizers of the penalized loss (1.2) must satisfy the KKT condition

\[
X^T(y - X\hat{\beta})/n = \hat{\text{Pen}}(\hat{\beta})
\]

(3.6)

for a certain member \( \hat{\text{Pen}}(\hat{\beta}) \) of the sub-differential \( \partial \text{Pen}(\beta) \). We shall treat (3.6) as an estimating equation to allow somewhat more general solutions, including solutions with \( \hat{\text{Pen}}(\hat{\beta}) \) in the completion of the sub-differential \( \partial \text{Pen}(b) \) at \( b = \hat{\beta} \) as defined in (2.8). Local solutions of form (3.6) include all estimators \( \hat{\beta} \) satisfying

\[
\begin{align*}
\left( \lambda - \kappa_\ell(\hat{\beta}_\ell) \right)_+ &\leq \text{sgn}(\hat{\beta}_\ell)X^T_j(y - X\hat{\beta})/n \leq \lambda, \\
|X^T_j(y - X\hat{\beta})/n| &\leq \lambda, \\
\hat{\beta}_\ell &\neq 0, \\
\hat{\beta}_\ell &= 0,
\end{align*}
\]

(3.7)

as solutions of (3.7) can be constructed with separable penalties \( \text{Pen}(b) = \sum_{j=1}^p \rho_j(b_j; \lambda) \) with a common penalty level \( \lambda \) and potentially different concavity satisfying \( \kappa(\rho_j) \leq \kappa_* \). Unfortunately, the more explicit (3.7) does not cover solutions with sorted penalties and some mixed penalties.

We study solutions of (3.6) by comparing them with an oracle coefficient vector \( \beta^o \). We assume that for a certain sparse subset \( S \) of \( \{1, \ldots, p\} \), \( \lambda_* > 0 \) and \( \eta \in [0, 1] \), the oracle \( \beta^o \) satisfy the following,

\[
\text{supp}(\beta^o) \subset S, \quad \|X^T(y - X^T\beta^o)/n\|_\infty < \eta \lambda_*.
\]

(3.8)

We may take \( \beta^o \in \mathbb{R}^p \) as the true coefficient vector \( \beta^* \) with \( S = \text{supp}(\beta^*) \), or the oracle LSE \( \hat{\beta}^o \) given by

\[
\hat{\beta}^o_S = (X^T_SX_S)^{-1}X^T_Sy, \quad \hat{\beta}^o_{S^c} = 0.
\]

(3.9)

When \( \varepsilon = y - X\beta^* \sim N(0, V) \) with \( \phi_{\text{max}}(V) \vee \max_{j \in I} x_j^TVx_j/n \leq \sigma^2 \),

\[
y - X\beta^o \sim N(0, V^o) \text{ with } \phi_{\text{max}}(V^o) \vee \max_{j \in I} x_j^TV^ox_j/n \in \mathbb{Z}(3d\hat{\beta})
\]

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for such \( \beta^o \), so that (3.8) holds with at least probability \( 1 - \sqrt{2/(\pi \log p)} \) for
\[
\lambda_* = (\sigma/\eta)\sqrt{(2/n) \log p}.
\]

Our analysis also applies to approximate local solutions satisfying
\[
X^T(y - X\hat{\beta})/n = \tilde{\text{Pen}}(\hat{\beta}) + \nu_{\text{approx}},
\]
including approximate solutions of (3.7), with \( \nu_{\text{approx}} \) an approximation error satisfying \( \hat{\beta}_j(\nu_{\text{approx}}) \geq 0 \) and a proper upper bound for \( \|\nu_{\text{approx}}\|_\infty \).

For computational efficiency, many algorithms only provide approximate solutions. In Subsection 4.1, we consider solutions with less restrictive approximation errors and provide a brief discussion with some references.

Given an oracle \( \beta^o \) satisfying (3.8), we consider solutions of (3.11) with penalties satisfying the following penalty level and concavity conditions:
\[
\lambda(\beta^o) \geq \lambda_*, \quad \bar{\rho}_{1,2}(\hat{\beta}; \xi) \leq \kappa_*, \quad \|w_S\|_\infty \leq (1 - \eta)\xi,
\]
where \( \lambda_* \) is as in (3.8), \( \kappa_* \) and \( \xi \) are positive constants, and
\[
w = \{\tilde{\text{Pen}}(\beta^o) + \nu_{\text{approx}} - X^T(y - X\beta^o)/n\}/\lambda(\beta^o).
\]

We recall that for any \( b \), \( \lambda(b) \) and \( \bar{\pi}_{1,2}(b; \xi) \) are the penalty level and relaxed concavity of \( \text{Pen}(\cdot) \) as defined in (2.4) and (2.6) respectively, and \( \tilde{\text{Pen}}(\beta^o) \) is a member of the sub-differential \( \partial \text{Pen}(\beta^o) \) in (2.2) or its completion in (2.8).

The third inequality in (3.12), assumed to hold for all choices of \( \tilde{\text{Pen}}(\beta^o) \in \partial \text{Pen}(\beta^o) \) in (3.13), follows from (3.8) and the penalty level condition (2.9) when \( \xi \) satisfies \( \eta_* \lambda(\beta^o) + \|\nu_{\text{approx}}\|_S/\lambda + \eta \lambda_* \leq (1 - \eta)\xi \lambda(\beta^o) \), but this may not be sharp.

For separable penalties \( \rho(b; \lambda) \) in (2.10), \( \lambda(b) = \lambda, \bar{\pi}_{1,2}(b; \xi) \leq \bar{\pi}(b) \leq \bar{\pi}(\rho) \) and \( \|w_S\|_\infty \leq 1 + \eta + \|\nu_{\text{approx}}\|_S/\lambda \) under conditions (i)–(v) and (3.8). The penalty level and upper bounds for the concavity are given in (2.17) for univariate mixed penalties like the spike-and-slab Lasso, and in multivariate mixed penalties in Proposition 1. These facts lead to more explicit versions of (3.12).

Let \( \mathcal{B}(\lambda_*, \kappa_*) \) be the set of all approximate local solutions satisfying (3.11) with different types of penalty functions satisfying (3.12); Given \( \beta^o \),
\[
\mathcal{B}(\lambda_*, \kappa_*) = \{\hat{\beta} : (3.11) \text{ holds with some Pen}(\cdot) \text{ satisfying (3.12)}\}.
\]

Given a penalty satisfying (3.12), \( \mathcal{B}(\lambda_*, \kappa_*) \) contains all local minimizer of the penalized loss (1.2). Our theory is applicable to the subclass
\[
\mathcal{B}_0(\lambda_*, \kappa_*) = \{\hat{\beta} : \hat{\beta} \text{ and } 0 \text{ are connected in } \mathcal{B}(\lambda_*, \kappa_*)\}.
\]
Here $\hat{\beta}$ and $0$ are not connected iff there exist disjoint closed sets $\mathcal{B}_0$ and $\mathcal{B}_1$ in $\mathbb{R}^p$ such that $0 \in \mathcal{B}_0$, $\hat{\beta} \in \mathcal{B}_1$ and $\mathcal{B}(\lambda_*, \kappa_*) \subseteq \mathcal{B}_0 \cup \mathcal{B}_1$. However, this condition will be relaxed in Proposition 5 below.

By definition, $\mathcal{B}_0(\lambda_*, \kappa_*)$ contains the set of all local solutions (3.6) computable by path following algorithms starting from the origin, with constraints on the penalty and concavity levels respectively. This is a large class of statistical solutions as it includes all local solutions connected to the origin regardless of the specific algorithms used to compute the solution and different types of penalties can be used in a single solution path. For example, the Lasso estimator belongs to the class as it is connected to the origin through the LARS algorithm \[16, 17, 8\]. The SCAD and MCP solutions with $\lambda \geq \lambda_*$ and $\kappa \leq \kappa_*$ belong to the class if they are computed by the PLUS algorithm \[29\] or by a continuous path following algorithm from the Lasso solution. As $\hat{\beta} = 0$ is the sparsest solution, $\mathcal{B}_0(\lambda_*, \kappa_*)$ can be viewed as the sparse branch of the solution space $\mathcal{B}(\lambda_*, \kappa_*)$. In Proposition 5, we prove that our theory is also applicable to approximate local solutions (3.11) computable through a discrete solution path in $\mathcal{B}(\lambda_*, \kappa_*)$ from $0$ with $\ell_1$ step size of order $\lambda$.

### 3.3 Error bounds

Let $\mathcal{B}(\lambda_*, \kappa_*)$ be as in (3.14), $\mathcal{B}_0(\lambda_*, \kappa_*)$ as in (3.15), $\mathcal{S} \supseteq \text{supp}(\beta^o)$, $\xi > 0$, $\mathcal{C}(\mathcal{S}; \xi) = \{ u : \| u_{\mathcal{S}^c} \|_1 \leq \xi \| u_{\mathcal{S}} \|_1 \}$ and

$$
\mathcal{B}_0^*(\lambda_*, \kappa_*) = \mathcal{B}_0(\lambda_*, \kappa_*) \cup \{ \hat{\beta} \in \mathcal{B}(\lambda_*, \kappa_*) : \hat{\beta} - \beta^o \in \mathcal{C}(\mathcal{S}; \xi) \}\, \text{(3.16)}
$$

Here under an RE condition on the design matrix, we provide prediction and coefficient estimation error bounds for solutions of (3.11) and conditions for variable selection consistency for solutions of (3.6) in the set (3.16) above.

**Theorem 1.** Suppose (3.8) holds for certain $\beta^o \in \mathbb{R}^p$ and $\text{RE}^2_2(\mathcal{S}; \xi) \geq \kappa_*$. Let $\hat{\beta}$ be a solution of (3.11) in $\mathcal{B}_0^*(\lambda_*, \kappa_*)$ with $\| \text{Pen}(\hat{\beta}) \|_\infty \leq \lambda$. Then,

$$
\| X\hat{\beta} - X\beta^o \|_2^2 / n \leq \frac{X^2 |\mathcal{S}|}{\text{RCIF}_{\text{pred}}(\mathcal{S}; \eta, \mathcal{W})} \leq \frac{\lambda^2 |\mathcal{S}|}{\text{RE}_1^2(\mathcal{S}; \xi)} \, , \, (3.17)
$$

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with \( \overline{\lambda} = \lambda + \eta \lambda_s + \| \nu_{\text{approx}} \|_\infty \) and the \( w \) in (3.13), and

\[
\begin{align*}
\| \hat{\beta} - \beta^o \|_1 & \leq \frac{\overline{\lambda}|S|}{\text{RCIF}_{1}(S; \eta, w)} \leq \frac{(1 + \xi)^2 \overline{\lambda}|S|}{\text{RE}_2^2(S; \xi)}, \\
\| \hat{\beta} - \beta^o \|_2 & \leq \frac{\overline{\lambda}|S|^{1/2}}{\text{RCIF}_{2}(S; \eta, w)} \leq \frac{(1 + \xi)|S|^{1/2}}{\text{RE}_1(S; \xi) \text{RE}_2(S; \xi)}, \\
\| \hat{\beta} - \beta^o \|_q & \leq \frac{\overline{\lambda}|S|^{1/q}}{\text{RCIF}_{q}(S; \eta, w)}, \quad \forall q \geq 1.
\end{align*}
\]

(3.18)

Moreover, if \( \hat{\beta} \) is a solution satisfying (3.7), then

\[
(\| X \hat{\beta} - X \beta^o \|_2^2 / n) \vee (\overline{\lambda} \| \hat{\beta} - \beta^o \|_1) \leq \overline{\lambda}^2 |S| / (\text{RE}_1(S; \xi) \text{RE}_2(S; \xi)).
\]

and \( \| \hat{\beta} - \beta^o \|_2^2 \leq \overline{\lambda}^2 |S| / (\text{RE}_1(S; \xi) \text{RE}_2(S; \xi)) \).

Compared with Theorem 1, existing results on statistical solutions of concave PLSE [29, 27, 10] cover only separable penalties of form (2.10), require stronger conditions on the design (such as upper sparse eigenvalue) and offer less explicit error bounds. While the error bounds for concave penalty in Theorem 1 match existing ones for the Lasso [4, 25], they also hold when \( \text{RE}_1(S; \xi) \) and \( \text{RE}_2(S; \xi) \) in (3.17) and (3.18) are replaced by the larger version with the constraint \( \| u_S \|_1 \leq \xi \| u_s \|_1 \) replaced by the more stringent \( (1 - \eta) \| u_S \|_1 \leq -w^T_S u_S \) in their definition, as \( -w^T_S u_S \) could be much smaller than \( (1 - \eta) \xi \| u_S \|_1 \) when significant proportion of the signals are strong. We describe the benefit of concave PLSE over the Lasso in such scenarios in the following two theorems.

**Theorem 2.** Suppose (3.8) holds for an oracle solution \( \beta^o \) of (3.6) and \( \text{RE}_2^2(S; \xi) \geq \kappa_s \). Let \( \hat{\beta} \) be a solution of (3.6) in \( B^*_0(\lambda_s, \kappa_s) \). Then,

\[
\phi_{\min}(\Sigma_{S,S}) > \overline{\pi}_{1,2}(\beta; \xi) \Rightarrow \hat{\beta} = \beta^o.
\]

(3.19)

where \( \phi_{\min}(\cdot) \) denotes the minimum eigenvalue for symmetric matrices. If a separable penalty of the form \( \rho(\beta; \lambda) \) is taken as in (2.10), then

\[
\hat{\beta}_{S_c} = 0 \quad \text{and} \quad \text{sgn}(\hat{\beta}_j) \text{sgn}(\beta^o_j) \geq 0 \quad \forall j \in S,
\]

(3.20)

and under the additional condition \( \overline{\pi}(0; \rho, \lambda) < \phi_{\min}(\Sigma_{S,S}) \),

\[
\text{sgn}(\hat{\beta}) = \text{sgn}(\beta^o).
\]

(3.21)
Theorem 2 provides selection consistency of concave PLSE under the restricted eigenvalue condition, compared with the required irrepresentable condition for the Lasso [13, 24, 33]. This is also new as existing results require the stronger sparse Riesz condition [29] or other combination of lower and upper sparse eigenvalue conditions on the design [31, 27, 10] for selection consistency. Theorem 2 also extends the selection consistency theory to non-separable multivariate concave penalties, e.g. non-separable spike-and-slab consistency. Theorem 2 provides selection consistency of concave PLSE under the restricted eigenvalue condition, compared with the required irrepresentable condition for the Lasso [13, 24, 33]. This is also new as existing results require the stronger sparse Riesz condition [29] or other combination of lower and upper sparse eigenvalue conditions on the design [31, 27, 10] for selection consistency. Theorem 2 also extends the selection consistency theory to non-separable multivariate concave penalties, e.g. non-separable spike-and-slab consistency. Theorem 2 unifies the analysis for prediction, estimation and variable selection, as the proof of selection consistency is simply done by inspecting the case of \( w_S = 0 \) in the prediction and estimation error bounds.

The condition \( w_S = 0 \), a consequence of the feasibility of \( \beta^o \) as an oracle solution of (3.6) in general, holds for the oracle LSE (3.9) under the beta-min condition \( \min_{j \in S} |\beta^o_j| \geq \gamma \lambda \) when the separable penalty (2.10) is used with \( \hat{\rho}(t; \lambda) = 0 \) for all \( |t| \geq \gamma \lambda \). As \( \text{RE}_2^2(S; \xi) \leq \phi_{\min}(\Sigma S, S) \) always holds and \( \pi_{1,2}(\hat{\beta}; \xi) \leq \kappa \leq \text{RE}_2^2(S; \xi) \) by (3.12) and the RE condition, the condition in (3.19) holds when \( \Sigma_{S, S'} \neq 0 \) as it implies \( \text{RE}_2^2(S; \xi) < \phi_{\min}(\Sigma S, S) \).

**Theorem 3.** Suppose (3.8) holds for \( \beta^o \in \mathbb{R}^p \) and \( \kappa \leq \text{RE}_2^2(S; \xi) \). Let \( \hat{\beta} \) and \( \lambda \) be as in Theorem 1 with \( \pi_{1,2}(\hat{\beta}; \xi) \leq (1 - 1/C_0)\text{RE}_2^2(S; \xi) \). Then,

\[
\|X\hat{\beta} - X\beta^o\|^2_2/n \leq (C_0\lambda)^2 \sup_{u \neq 0} \frac{\left[w_S^T u_S - (1 - \eta)\|u_S\|_1\right]^2}{u^T \Sigma u}
\]

(3.22)

with the \( w \) in (3.13), and for any seminorm \( \| \cdot \| \) as a loss function

\[
\|\hat{\beta} - \beta^o\| \leq C_0\lambda \sup_{u \neq 0} \frac{\|u_S^T u_S - (1 - \eta)\|u_S\|_1\}}{u^T \Sigma u}.
\]

(3.23)

**Corollary 1.** Suppose conditions of Theorem 3 with \( \lambda = \lambda_\ast = (\sigma/\eta)\sqrt{2/n}\log p, C_0^2/\text{RE}_2^2(S; \xi) = O(1) \), and \( \beta^o = \hat{\beta}^o \) being the oracle estimator in (3.9). Let \( s = |S| \) and \( s_1 = ||(\text{Pen}(\beta^o) + \nu_{\text{approx}})S||_2/\lambda \). Then,

\[
\|X\hat{\beta} - X\beta^o\|^2_2/n + \|\hat{\beta} - \beta^o\|^2_2 + \|\hat{\beta} - \beta^o\|_1^2/s = O_p(\sigma^2/n)s_1 \log p,
\]

implying \( \hat{\beta} = \hat{\beta}^o \) when \( s_1 = 0 \), and for the true \( \beta^* \)

\[
\|X\hat{\beta} - X\beta^*\|^2_2/n + \|\hat{\beta} - \beta^*\|^2_2 + \|\hat{\beta} - \beta^*\|_1^2/s = O_p(\sigma^2/n)(s_1 \log p + s).
\]

(3.24)

For \( \nu_{\text{approx}} = 0 \) and penalties (2.10) with \( \text{supp}(\hat{\rho}(t; \lambda)) \subseteq [-\gamma \lambda, \gamma \lambda] \),

\[
s_1 \leq \#\{j \in S : |\beta^o_j| < \lambda \gamma\},
\]

(3.25)
Theorem 3 and Corollary 1 demonstrate the benefits of concave PLSE, as \( s_1 = s = |S| \) in (3.24) for the Lasso but \( s_1 \) could be much smaller than \( s \) for concave penalties. For separable concave penalties, (3.25) holds with \( \gamma = 1 + 1/\pi \) for the SCAD (2.14) and \( \gamma = 1/\pi \) for the MCP (2.15).

For the exact Lasso solution with \( \pi(b) = 0 \), \( C_0 = 1 \) and \( \|w_S\|_\infty \leq 1 + \eta \), Theorem 3 yields the sharpest possible prediction and estimation error bounds based on the basic inequality \( u^T \Sigma u + (1 - \eta)\|u_{S^c}\|_1 \leq (1 + \eta)\|u_S\|_1 \) with \( u = (\beta - \beta^o)/\lambda \), as stated in the following corollary.

**Corollary 2.** Let \( \hat{\beta} \) be the Lasso estimator with penalty level \( \lambda \geq \lambda_* \). If (3.8) holds for a coefficient vector \( \beta^o \in \mathbb{R}^p \), then

\[
\frac{\|X\hat{\beta} - X\beta^o\|_2^2}{n(1 + \eta)^2 \lambda^2} \leq \sup_{u \neq 0} \frac{\psi^2(u)}{u^T \Sigma u} = \max_{0 < t < 1} \frac{|S|(1 - t)^2}{\text{RE}^2_1(S; t\xi)}
\]

with \( \psi(u) = [\|u_S\|_1 - \|u_{S^c}\|_1/\xi]_+ \) and \( \xi = (1 + \eta)/(1 - \eta) \),

\[
\frac{\|\hat{\beta} - \beta^o\|_2}{(1 + \eta)\lambda} \leq \sup_{u \neq 0} \frac{\|u\|_2 \psi(u)}{u^T \Sigma u} = \max_{0 < t < 1} \frac{|S|^{1/2}(1 - t)}{\text{RE}^2_{1,2}(S; t\xi)}
\]

with \( \text{RE}_{1,2}(S; \xi) = \inf_{u_{S^c} < \xi \|u_{S^c}\|_1} \frac{u^T \Sigma u}{(\|u\|_2^2\|u_S\|_1/|S|^{1/2})} \), and

\[
\frac{\|\hat{\beta} - \beta^o\|_1}{(1 + \eta)\lambda} \leq \sup_{u \neq 0} \frac{\|u\|_1 \psi(u)}{u^T \Sigma u} = \max_{0 < t < 1} \frac{|S|^{1/2}(1 + t\xi)(1 - t)}{\text{RE}^2_1(S; t\xi)}.
\]

As Theorems 1–3 deal with the same estimator under the same RE conditions on the design, they give a unified treatment of the prediction, coefficient estimation and variable selection performance of the PLSE, including the \( \ell_1 \) and concave penalties. For prediction and coefficient estimation, (3.17) and (3.18) match those of state-of-art for the Lasso in both the convergence rate and the regularity condition on the design, while (3.22), (3.23) and Corollary 1 demonstrate the advantages of concave penalization when \( s_1 \) is much smaller than \( s \). Meanwhile, for selection consistency, Theorem 2 weakens existing conditions on the design to the same RE condition as required for \( \ell_2 \) estimation with the Lasso. These RE-based results are significant as the existing theory for concave penalization, which requires substantially stronger conditions on the design, leaves a false impression that the Lasso has a technical advantage in prediction and parameter estimation by requiring much weaker conditions on the design than the concave PLSE.

The following lemma, which can be viewed as a basic inequality for analyzing concave PLSE, is the beginning point of our analysis.
Lemma 1. Let \( \lambda = \lambda(\beta^o) \) be as in (2.4), \( \hat{\beta} \) a solution of (3.11), \( h = \hat{\beta} - \beta^o \), and \( w = \{ \text{Pen}(\beta^o) + \nu_{\text{approx}} - X^T(y - X\beta^o)/n \} / \lambda \). Then,

\[
h^T \Sigma h \leq -\lambda h^T w + \alpha_1(\hat{\beta}) \| h \|_1^2 + \alpha_2(\hat{\beta}) \| h \|_2^2
\]

for all choices of \( \{ \pi_1(b), \pi_2(b) \} \) satisfying (2.7) at \( b = \hat{\beta} \). Moreover, for \( \lambda \in \text{supp}(\beta^o) \) and a proper choice of \( \text{Pen}(\beta^o) \) in \( \partial \text{Pen}(\beta^o) \),

\[
h^T \Sigma h + \left\{ \lambda - \| X^T_{\text{opt}}(y - X\beta^o)/n \|_{\infty} \right\} \| h \|_{\infty} \leq -\lambda h^T w + \alpha(\hat{\beta}) \| h \|_1^2 + \beta(\hat{\beta}) \| h \|_2^2.
\]

Next, we prove that the solutions in \( \mathcal{B}_0(\lambda_*, \kappa_*) \) in (3.16) and other approximate local solutions (3.11) in \( \mathcal{B}(\lambda_*, \kappa_*) \) are separated by a gap of size \( a_0 \lambda_0 \) in the \( \ell_1 \) distance for some \( a_0 > 0 \). Consequently, \( \hat{\beta} \in \mathcal{B}(\lambda_*, \kappa_*) \) implies \( \beta - \beta^o \in \mathcal{C}(S; \xi) \) for the \( \mathcal{B}_0(\lambda_*, \kappa_*) \) in (3.15) and \( \mathcal{C}(S; \xi) \) in (3.16).

Proposition 5. Let \( \pi_j(b) \) be as in (2.7), \( \mathcal{B}(\lambda_*, \kappa_*) \) as in (3.14) and \( \mathcal{C}(S; \xi) \) as in (3.16). Suppose \( \text{RE}_2(S; \xi) \geq \kappa_* \). Let \( a_1 = \eta - \| z_S \|_{\infty}/\lambda_*, \ a_2 = a_1 \xi/[2(\xi + 1)(\pi_1(\hat{\beta}) + \pi_2(\hat{\beta}))] \) and \( a_3 = a_1(1 - \eta)\xi/(1 - \eta)(\xi + 1) + a_1 \). Let \( \beta^o \) be a solution of (3.11) with penalty level \( \lambda \), and \( \hat{\beta} \in \mathcal{B}(\lambda_*, \kappa_*) \) with \( \hat{\beta} - \beta^o \in \mathcal{C}(S; \xi) \),

\[
\hat{\beta} - \beta^o \in \mathcal{C}(S; \xi) \text{ and } \| \hat{\beta} - \beta^o \|_1 \leq a_0 \lambda_0 \text{ imply } \beta - \beta^o \in \mathcal{C}(S; \xi),
\]

with \( a_0 = \min \{ a_2, a_2 a_3/(1 - \eta)(\xi + 1) \} \). Consequently,

\[
\mathcal{B}_0^*(\lambda_*, \kappa_*) = \{ \hat{\beta} \in \mathcal{B}(\lambda_*, \kappa_*) : \beta - \beta^o \in \mathcal{C}(S; \xi) \}.
\]

It follows from Proposition 5 that our theoretical results are applicable to statistical choices of approximate local solutions (3.11) computable through a discrete path of solutions \( \beta^{(i)} \) satisfying \( \| \beta^{(i)} - \beta^{(i-1)} \|_1 \leq a_0 \lambda^{(i)} \) and beginning from \( 0 \) or the Lasso solution, as \( \{ 0 \} \) and the Lasso solution both satisfy the condition \( \beta - \beta^o \in \mathcal{C}(S; \xi) \).

4 Smaller penalty levels, sorted penalties and LCA

We have studied in Section 3 exact solutions (3.6) and approximate solutions (3.11) for no smaller penalty level than \( \lambda_* \) in the event where \( \lambda_* \) is a strict upper bound of the supreme norm of \( z = X^T(y - X\beta^o)/n \) as in (3.8). Such penalty or threshold levels are commonly used in the literature
to study regularized methods in high-dimensional regression, but this is conservative and may yield poor numerical results. Under the normality assumption (3.10), (3.8) requires $\lambda \geq \lambda_s = (\sigma/\eta)\sqrt{(2/n)\log p}$, but the rate optimal penalty level is the smaller $\lambda \geq \sigma\sqrt{(2/n)\log(p/s)}$ for prediction and coefficient estimation with $s = |S|$. It is known that for $\log(p/s) \ll \log p$, rate optimal performance in prediction and coefficient estimation can be guaranteed by the Lasso with the nonadaptive smaller penalty level depending on $s$ [22, 3] or the Slope to achieve adaptation in the smaller penalty level [21, 3]. However, it is unclear from the literature whether the same can be done with concave penalties to also take advantage of noise strength, and under what conditions on the design. Moreover, for computational considerations, it is desirable to relax the condition imposed in Section 3 on the supreme norm of the approximation error $\nu_{\text{approx}}$ in (3.11). In this section, we consider approximate solutions for general penalties with nonadaptive penalty levels which are allowed to be smaller, concave slopes, and solutions produced by the LCA algorithm. Our analysis imposes a somewhat stronger RE condition, but that is the only condition required on the design matrix.

4.1 Approximate local solutions

As in (3.11), we write

$$X^T(y - X\hat{\beta})/n = \hat{\text{Pen}}(\hat{\beta}) + \nu_{\text{approx}}$$

(4.1)

as approximate solution of (3.6), including (3.7) for separable penalties. However, instead of imposing $\ell_\infty$ bounds on the approximation error as in Section 3, we will impose in this section a more practical $\ell_\infty$-$\ell_2$ split bound as in (4.10) below. Solutions of form (4.1) are called approximate local solutions in [31] where their uniqueness, variable selection properties and relationship to the global solution were studied. Computational algorithms for approximate solutions and statistical properties of the resulting estimators have been considered in [14, 1, 27, 12, 10] among others. However, these studies of approximate solutions all focus on separable penalties with penalty level (3.8) or higher.

Let $\beta^o$ be the oracle coefficient vector as in Section 3 and $S \subset \text{supp}(\beta^o)$. Let $\nu_{\text{div}} = \hat{\text{Pen}}(\hat{\beta}) - \text{Pen}(\beta^o)$. It follows from (4.1) that

$$X^T(y - X\hat{\beta})/n = \hat{\text{Pen}}(\beta^o) + \nu_{\text{div}} + \nu_{\text{approx}}.$$

(4.2)

With $z = X^T(y - X\beta^o)/n$, (4.2) leads to the identity

$$h^T\Sigma h = h^T\left\{z - \hat{\text{Pen}}(\beta^o)\right\} - h^T\nu_{\text{div}} - h^T\nu_{\text{approx}}.$$
with \( h = \hat{\beta} - \beta^0 \). While upper bounds for \(-h^T \nu_{\text{div}}\) and \(-h^T \nu_{\text{approx}}\) can be obtained via the maximum concavity of the penalty and the size restriction on \( \nu_{\text{approx}} \), a favorable choice of the sub-derivative \( \hat{\text{Pen}}(\beta^0) \) must be used to control the noise \( z \). When the penalty function is endowed with a constant penalty level \( \lambda = \lambda(\beta^0) \) on \( S^c \) as in (2.4) and concavity level as in (2.5),

\[
\begin{align*}
& h^T \Sigma h + (1 - \eta) \lambda \| h_{S^c} \|_1 - \bar{\pi}(\hat{\beta}) \| h \|_2^2 \\
\leq & \ (h_{S^c}^T z_{S^c} - \eta \lambda \| h_{S^c} \|_1) - \lambda h_{S^c}^T w_S - h^T \nu_{\text{approx}}.
\end{align*}
\]

This can be viewed as a basic inequality in our analysis of (4.1). We may use the relaxed concavity (2.7) as in Lemma 1, but (4.3) provides notational simplicity for a unified treatment with sorted concave penalties.

### 4.2 Sorted concave penalized estimation

Approximate solutions for sorted concave penalties are still defined by (4.1), so that (4.2) also holds. However, as sorted penalties are defined in (2.25) with a sequence of penalty levels \( \lambda_1 \geq \cdots \geq \lambda_p \geq 0 \), they do not provide \( \ell_1 \) control of the noise as in (4.3). Instead, sorted concave penalties control the noise through a sorted \( \ell_1 \) norm. More precisely, they regularize the correlation of the noise and design vectors in \( S^c \) with the norm

\[
\| b \|_{\#, s} = \sum_{j=1}^{p-s} (\lambda_{s+j}/\lambda_{s+1}) b^j, \quad \forall \ b \in \mathbb{R}^{p-s}, \ s = |S|,
\]

a standardized dual norm induced by the set of \( g_{S^c} \) given in (2.26). Here \( b^j \) is the \( j \)-th largest value among \([b_1], \ldots, [b_{p-s}]\).

As (4.2) holds, taking the most favorable \( \hat{\text{Pen}}(\beta^0) \) and the concavity bound in Proposition 2 in the derivation of (4.3), we find that

\[
\begin{align*}
& h^T \Sigma h + (1 - \eta) \lambda \| h_{S^c} \|_{\#, s} - \bar{\pi}(\hat{\beta}) \| h \|_2^2 \\
\leq & \ (h_{S^c}^T z_{S^c} - \eta \lambda \| h_{S^c} \|_{\#, s}) - \lambda h_{S^c}^T w_S - h^T \nu
\end{align*}
\]

with \( S \supseteq \text{supp}(\beta^0), \ z = X^T (y - X \beta^0)/n, \ w = \{ \hat{\text{Pen}}(\beta^0) - z \}/\lambda, \ \nu = \nu_{\text{approx}}, \ \lambda = \lambda_{s+1}, \ \text{and} \ \bar{\pi}(\hat{\beta}) = \bar{\pi}(\rho). \) This includes (4.3) as a special case, because \( \| b_{S^c} \|_{\#, s} = \| b_{S^c} \|_1 \) when \( \lambda_{s+1} = \cdots = \lambda_p \). More important, (4.5) is adaptive to the penalty level \( \lambda_{s+1} \) without the knowledge of \( s \).

We outline our analysis of (4.5) as follows. For \( r > 0 \) and \( \gamma > 0 \), define

\[
\Delta(r, w, \nu) = \sup_{u \neq 0} \frac{u^T z_{S^c}/\lambda - \eta \| u_{S^c} \|_{\#, s} - u_{S^c}^T w_S - u^T \nu/\lambda}{r \max \{ \| u \|_2, \| X u \|_2 \sqrt{\gamma}/n \}}.
\]

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We assume that for a certain constant $\xi > 0$, 
\[ \Delta((1-\eta)\xi|S|^{1/2}, w, \nu) < 1 \] (4.7)
for all approximate solutions (4.1) under consideration. This and (4.5) yield
\[ h^T \Sigma h + (1-\eta)\lambda \|h_{sc}\|_{#,s} - \pi(\beta)\|h\|^2_2 \leq (1-\eta)\xi|S|^{1/2} \max\{\|h\|_2, \|Xh\|_2 \sqrt{n/\gamma}\}. \] (4.8)
When $\pi(\beta)\|h\|^2_2 \leq h^T \Sigma h$, (4.8) provides the membership of $h$ in the cone $C_{#}(S; \xi, \gamma) = \{u : \|u_{sc}\|_{#,s} \leq \xi|S|^{1/2} \max(\|u\|_2, (\gamma u^T \Sigma u)^{1/2})\}$ (4.9)

On the other hand, when $h \in C_{#}(S; \xi, \gamma)$, a suitable RE condition provides $\pi(\beta)\|h\|^2_2 \leq h^T \Sigma h$. A key step in our analysis is to break this vicious circle, which will be done in Subsection 4.4.

The seemingly complicated (4.6), which summarizes conditions in our analysis on the noise vector $\varepsilon$, the penalty level $\lambda$ and approximation error $\nu$, is actually not hard to decipher. Consider $\nu$ satisfying
\[ u^T \nu \leq \eta_1 \lambda_{s+1} \|u_{sc}\|_{#,s} + r_2 \lambda_{s+1} \|u\|_2 \forall \ u \in \mathbb{R}^p \] (4.10)
with $\eta_1 \in (0, \eta)$ and $r_2 > 0$. The above condition on $\nu$ is fulfilled when
\[ \nu = \nu_1 + \nu_2 \text{ with } \nu_{1,j}^# \leq \eta_1 \lambda_j \text{ and } \|\nu_2\|_2 + \eta_1 \|\nu_1\|_{1:s} \leq \lambda_{s+1} r_2, \]
as this implies $u^T \nu \leq \eta_1 \sum_{j=1}^{p-s} u_{1,j}^# \lambda_j + \|u\|_2 \|\nu_2\|_2 \leq \eta_1 \lambda_{s+1} \|u_{sc}\|_{#,s} + (\eta_1 \|\nu_1\|_{1:s} + r_2 \|\nu_2\|_2) \|u\|_2$. Suppose $\lambda_j$ are bounded from below, $\lambda_j \geq \lambda_{s,j}$, $1 \leq j \leq p$, such as (2.27). Let
\[ \Delta(r, \eta, w, r_2) = \left\{ \sum_{j=1}^{p-s} \frac{(z_j^# - \eta \lambda_{s,j})^2}{r^2 \lambda_{s,j}^{#}} + \frac{w^2}{r^2} \right\}^{1/2} + \frac{r_2}{r}. \] (4.11)

It follows from the monotonicity of (4.11) in $\lambda_{s,s+j}$ and some algebra that
\[ \Delta(r, w, \nu)I_{|w_{sc}|_2 \leq w} \leq \Delta(r, \eta - \eta_1, w, r_2) \]
when (4.10) holds. In Subsection 4.6, we derive upper bound for the median of $\Delta$ by combing the arguments in [3, 22], and then apply concentration inequality to $\Delta$. In the simpler case with unsorted smaller penalty, $\lambda = \sigma L/(\eta^{1/2}(\eta - \eta_1))$ with $L = \sqrt{2 \log(p/s)}$,
\[ \mathbb{E} \Delta(r, \eta - \eta_1, w, r_2) \leq r^{-1} \left\{ \sqrt{4s(\eta - \eta_1)^2/(L^4 + 2L^2)} + w^2 + r_2 \right\} \]
\[ = o(1) + (w + r_2)/r \]

by Proposition 10 in [22] under the normality assumption (3.10). Thus, (4.7) holds with high probability when \( \|w_S\|_\infty \leq (1 - \eta)\xi' \) with \( \xi' < \xi, \ s = |S| \) and \( r_2/s^{1/2} \) is sufficiently small, in view of the third condition in (3.12).

The idea of including \( \|Xu\|_2 \), by setting \( \gamma > 0 \) in (4.6), comes from [3]. It provides upper bound \( 1/(\tau_\lambda, \sqrt{n}) \) for the Lipschitz norm of \( \Delta(r, w, \nu) \) in the noise \( y - X\beta^o \), and thus large deviation bounds for (4.6).

### 4.3 Approximate solutions of the LCA

As in (4.1), we consider approximate solutions of the LCA (2.30) of the form

\[ 0 = \hat{L}(b^{(new)}) + \hat{\text{Pen}}_+(b^{(new)}) - \hat{\text{Pen}}_-(b^{(old)}) + \nu_{\text{approx}}. \tag{4.12} \]

Such approximate solutions can be viewed as output of iterative algorithms such as those discussed in Subsection 2.5. The following lemma provides the LCA version of the basic inequality (4.5).

**Lemma 2.** Let \( h = b^{(new)} - \beta^o \) and \( z = -\hat{L}(\beta^o) \). Then,

\[
\begin{align*}
D_L(b^{(new)}, \beta^o) + D_+(b^{(new)}, \beta^o) &= h^T\{z - \hat{\text{Pen}}_+(\beta^o) + \hat{\text{Pen}}_-(b^{(old)}) - \nu_{\text{approx}}\} \\
&= h^T\{z - \hat{\text{Pen}}(\beta^o) + \nu_{\text{carry}} - \nu_{\text{approx}}\},
\end{align*}
\]

where \( D_L(b, \beta) = (b - \beta)^T\{\hat{L}(b) - \hat{L}(\beta)\} \) and \( D_+(b, \beta) = (b - \beta)^T\{\hat{\text{Pen}}_+(b) - \hat{\text{Pen}}_+(\beta)\} \) are respectively the symmetric Bregman divergence for the loss \( L(b) \) and the majorization penalty \( \text{Pen}(b) \) in (2.31), and \( \nu_{\text{carry}} = \hat{\text{Pen}}_-(b^{(old)}) - \hat{\text{Pen}}_+(\beta^o) \) is the carryover error in gradient.

In linear regression, \( L(b) = \|y - Xb\|_2^2/(2n) \) and (4.12) can be written as

\[ X^T(y - Xb^{(new)})/n = \hat{\text{Pen}}_+(b^{(new)}) - \hat{\text{Pen}}_-(b^{(old)}) + \nu_{\text{approx}} \tag{4.13} \]

as an approximate solution to the convex minimization problem of LCA penalized LSE. As in the derivation of (4.2), (4.3) and (4.5), this leads to

\[ X^T(y - Xb^{(new)})/n = \hat{\text{Pen}}(\beta^o) + \nu_{\text{div}} - \nu_{\text{carry}} + \nu_{\text{approx}}, \tag{4.14} \]

with \( \nu_{\text{div}} = \hat{\text{Pen}}_+(b^{(new)}) - \hat{\text{Pen}}_+(\beta^o) \) and \( \nu_{\text{carry}} \) as in Lemma 2, and then

\[
\begin{align*}
h^T\Sigma h + (1 - \eta)\lambda\|h_S^c\|_{., s}^2 &
\leq h^T\Sigma h + (1 - \eta)\lambda\|h_S^c\|_{., s}^2 + D_+(b^{(new)}, \beta^o) \\
&\leq (h_S^Tz_S - \eta\lambda\|h_S^c\|_{., s}) - \lambda h_S^T w_S - h^T\nu
\end{align*}
\]

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when the favorable $\tilde{\text{Pen}}(\beta^o)$ is taken as in (2.26), where $h = b^{(\text{new})} - \beta^o$, $D_+(b^{(\text{new})}, \beta^o) = h^T\nu_{\text{div}}$ is the symmetric Bregman divergence as in Lemma 2 and $\nu = \nu_{\text{approx}} - \nu_{\text{carry}}$. Thus, (4.5) holds with $\pi(\tilde{\beta}) = 0$ and an extra carryover term $h^T\nu_{\text{carry}}$ on the right-hand side.

For the sorted penalty $p_#(b; \lambda)$ in (2.25), we can always take $\rho_-(t)$ with $|(\partial/\partial t)^2 p_-(t)| \leq \pi(\rho)$, so that the carryover error is bounded by

$$\|\nu_{\text{carry}}\|_2 \leq \|\rho_-(b^{(\text{old})}) - \hat{\rho}_-(\beta^o)\|_2 \leq \pi(\rho)\|h^{(\text{old})}\|_2$$

with $h^{(\text{old})} = b^{(\text{old})} - \beta^o$ in the analysis of (4.13) through (4.5).

### 4.4 Good solutions

We define here a set of “good solutions” to which our error bounds in the following two subsections apply.

First, as in (3.12), we need to impose in addition to (4.7) a condition on the concavity of the penalty in (4.1). This leads to the solution set

$$\mathcal{B}(\lambda_*, \kappa_*, \gamma) = \left\{ \tilde{\beta} : (4.1) \text{ and } (4.7) \text{ hold, } \pi(\tilde{\beta}) \leq \kappa_* \right\}.$$  \hspace{1cm} (4.17)

Here $\lambda_*$ is a minimum penalty level requirement implicit in (4.7), e.g.

$$\lambda \geq \lambda_* = (\eta - \eta_t)^{-1} \sigma L/n^{1/2}$$

with $\Phi(-L) \leq s/p$, e.g. $L = \sqrt{2 \log(p/s)}$ for fixed penalty levels or $\lambda_* = \lambda_{s+1}$ for sorted ones satisfying (2.27). We note that $\mathcal{B}(\lambda_*, \kappa_*, \gamma)$ contains all solutions in the set $\mathcal{B}(\lambda_*, \kappa_*)$ in (3.14) with $\pi_1(\tilde{\beta}) = 0$ although (4.17) allows approximate solutions with smaller minimum penalty level $\lambda_*$. Let $\|u\|_{\#, \gamma} = \|u_S\|_{\#, \gamma} + |S|^{1/2} \max \{2\|u\|_2, \|Xu\|_2 \sqrt{\gamma/n} \}$. We say that two solutions $\tilde{\beta}$ and $\hat{\beta}$ in $\mathcal{B}_#(\lambda_*, \kappa_*, \gamma)$ are connected by an $a_0$-chain if there exist $\tilde{\beta}^{(k)}(\#) \in \mathcal{B}_#(\lambda_*, \kappa_*, \gamma)$ with sorted penalty levels $\{\lambda_1^{(k)}, \ldots, \lambda_p^{(k)}\}$ such that

$$\tilde{\beta}^{(0)} = \tilde{\beta}, \hat{\beta}^{(k^*)} = \hat{\beta}, \|\tilde{\beta}^{(k)} - \hat{\beta}^{(k-1)}\|_{\#, \gamma} \leq a_0|S|\lambda_{s+1}^{(k)}.$$  \hspace{1cm} (4.19)

$k = 1, \ldots, k^*$ with the $a_0$ specified in Proposition 6 below. This condition holds if $\tilde{\beta}$ and $\hat{\beta}$ are connected through a continuous path in $\mathcal{B}(\lambda_*, \kappa_*, \gamma)$. Similar to (3.15), we define

$$\mathcal{B}_0(\lambda_*, \kappa_*, \gamma) = \left\{ \tilde{\beta} \in \mathcal{B}(\lambda_*, \kappa_*, \gamma) : (4.19) \text{ holds with } \tilde{\beta} = 0 \right\}.$$ \hspace{1cm} (4.20)

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As the solutions are connected to 0 though a chain, \( \mathscr{B}_0(\lambda_s, \kappa_s, \gamma) \) can be viewed as the sparse branch of the solution set \( \mathscr{B}(\lambda_s, \kappa_s, \gamma) \).

With the cone (4.9), we define a set of “good solutions” as follows,

\[
\mathscr{B}^*_0(\lambda_s, \kappa_s, \gamma) = \left\{ \hat{\beta} \in \mathscr{B}(\lambda_s, \kappa_s, \gamma) : \hat{\beta} - \beta^o \in \mathcal{C}_#(S; \xi, \gamma) \right\} \cup \mathscr{B}_0(\lambda_s, \kappa_s, \gamma)
\]

\[
\cup \{ \hat{\beta} = b^{(new)} ; \ (4.13) \text{ and } (4.7) \text{ hold with the } \nu \text{ in } (4.15) \}. \tag{4.21}
\]

This is the set of approximate solutions (4.1) with estimation error inside the cone or connected to the origin through a chain, or approximate LCA solutions (4.13) with a reasonably small carryover component \( h^T \nu_{\text{carry}} \) in the \( h^T \nu \) in (4.15). Iterative applications of the LCA do provide a chain of good solutions to the final output without having to specify the step size. We note that for sorted penalties, (4.16) can be used to bound the solutions (4.13) with a reasonably small carryover component.

We prove below that good solutions all belong to the cone (4.9) when the following restricted eigenvalue is no smaller than the \( \kappa_s \) in (4.17),

\[
\text{RE}_#(S; \xi, \gamma) = \inf \left\{ \left( \frac{u^T \Sigma u}{\| u \|_2} \right)^{1/2} : 0 \neq u \in \mathcal{C}_#(S; \xi, \gamma) \right\}. \tag{4.22}
\]

As the cone \( \mathcal{C}_#(S; \xi, \gamma) \) in (4.9) depends on the sorted \( \lambda = (\lambda_1, \ldots, \lambda_p)^T \), the infimum in (4.22) is taken over all \( \lambda \) under consideration. We note that

\[
\text{RE}_#(S; \xi, \gamma) \geq \left[ \inf \left\{ \left( \frac{u^T \Sigma u}{\| u \|_2} \right)^{1/2} : \| u_{\mathcal{S}_c} \|_{\#, s} < \xi |S|^{1/2} \| u \|_2 \right\} \right] \wedge \frac{1}{\gamma}.
\]

When the cone is confined to \( \lambda_1 = \lambda_p \), i.e. \( \| u_{\mathcal{S}_c} \|_{\#, s} = \| u_{\mathcal{S}_c} \|_1 \), the RE condition on \( \text{RE}_#^2(S; \xi, \gamma) \) is equivalent to the restricted strong convexity condition [14] as \( \| u_{\mathcal{S}_c} \|_1 < \xi |S|^{1/2} \| u \|_2 \) implies \( \| u \|_1 < (\xi + 1)|S|^{1/2} \| u \|_2 \).

Compared with (3.1), the RE in (4.22) is smaller due to the use of a larger cone. However, this is hard to avoid because the smaller penalty does not control the \( \ell_\infty \) measure of the noise as in (3.8) and we do not wish to impose uniform bound on the approximation error \( \nu \).

**Proposition 6.** Let \( \bar{\pi}(b) \) be as in (2.5), \( \mathcal{B}(\lambda_s, \kappa_s, \gamma) \) as in (4.17), and \( \mathcal{C}_#(S; \xi, \gamma) \) as in (4.9). Suppose \( \text{RE}_#^2(S; \xi, \gamma) \geq \kappa_s \) and

\[
\Delta((1 - \eta - a_1)\xi |S|^{1/2}, w, \nu) \leq 1, \quad 0 < a_1 < 1 - \eta,
\]

as in (4.7) for all \( \{\lambda, w, \nu\} \) associated with solutions in \( \mathcal{B}(\lambda_s, \kappa_s, \gamma) \). Let

\[
a_2 = a_1 \xi^2 / (2 \bar{\pi}(\hat{\beta})), \quad a_3 = a_1 (1 - \eta) \xi / [(1 - \eta - a_1)(\xi + 1) + a_1] \quad \text{and} \quad a_0 = \min \left\{ a_2, a_2 a_3 / \{(1 - \eta)(\xi + 1)\} \right\}.
\]

Then,

\[
\mathscr{B}^*_0(\lambda_s, \kappa_s, \gamma) = \left\{ \hat{\beta} \in \mathcal{B}(\lambda_s, \kappa_s, \gamma) : \hat{\beta} - \beta^o \in \mathcal{C}_#(S; \xi, \gamma) \right\}.
\]

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4.5 Analytic error bounds

In this subsection we provide prediction and estimation error bounds for the set of approximate solutions in (4.21). While (4.7) is imposed on the entire $\mathcal{B}_0(\lambda, \kappa, \gamma)$, it can be sharpened to
\[
\Delta(r_1, w, \nu) \leq 1
\]
for a given $\hat{\beta}$ with $r_1 \leq (1 - \eta)\xi|S|^{1/2}$, in view of our discussion below (4.10).

**Theorem 4.** Let $\mathcal{B}_0(\lambda, \kappa, \gamma)$ be the solution set given through (4.17), (4.20) and (4.21), including approximate solutions for concave and sorted penalties and the LCA, with $S \supseteq \text{supp}(\beta^0)$. Suppose $\kappa \leq \text{RE}_2(S; \xi, \gamma)$. Let $\hat{\beta} \in \mathcal{B}_0(\lambda, \kappa, \gamma)$ satisfying $\pi(\hat{\beta}) \leq (1 - 1/C_0)\text{RE}_2(S; \xi, \gamma)$ and (4.24). Let $F(u) = \max \{|u|_2, (\gamma u^T \Sigma u)^{1/2}\}$. Then, for any semi-norm $\| \cdot \|$,  
\[
\|h\| \leq C_0 \lambda \sup_{u \neq 0} \frac{\|u\|_2 F(u) - (1 - \eta)\|u_{S^c}\|_s^s}{u^T \Sigma u}
\]  
with $h = \hat{\beta} - \beta^0$ for (4.1) and $h = b^{(new)} - \beta^0$ for (4.13). In particular, \[
\|h_S\|_1 + \|h_{S^c}\|_s^s \leq F(h) \leq (h^T \Sigma h)^{1/2} \leq \text{RE}_2(S; \xi, \gamma) \leq C_0 r_1 \lambda \text{RE}_2(S; \xi, \gamma).
\]

As an extension of Theorem 3, Theorem 4 provides prediction and $\ell_2$ estimation error bounds in the same form along with a comparable sorted $\ell_1$ error bound. It demonstrate the benefit of sorted concave penalization as $r_1^2 = s_1 + s/\log^2(p/s) + r_2^2$ in standard settings as described in Theorem 6 and Corollary 3, where $s = |S|$, $s_1$ can be viewed as the number of small nonzero coefficients, and $r_2$ is the $\ell_2$ component of the approximation error as in (4.10).

Theorem 4 applies to approximate solutions at smaller penalty levels, for penalties with fixed penalty level $(\lambda_1 = \cdots = \lambda_p)$, for sorted penalties adaptive choice of penalty level $\lambda_{s+1}$ from assigned sorted sequence $\lambda_1 \geq \cdots \geq \lambda_p$, and for general LCA with possibly sorted penalties. However, selection consistency of (4.1) is not guaanteed as false positive cannot be ruled out at the smaller penalty level or with general approximation errors. On the other hand, as properties of the Lasso at smaller penalty levels and the Slope have been studied in [22, 21, 3] among others, Theorem 4 can be viewed as an extension of their results to concave and/or sorted penalties discussed in Section 2 and to the LCA.
It is also possible to derive error bounds in the case of $C_0 = \infty$ as in Theorem 1 if the $\ell_\infty$ norm in the definition of the RCIF is replaced by the norm $\| \cdot \|_{\#, \ast}$ in (4.19). We omit details for the sake of space.

Next we apply Theorem 4 to iterative application of the LCA:

$$b^{(t)} \leftarrow \text{LCA}(b^{(t-1)}, \text{Pen}^{(t)}, \nu^{(t)}_{\text{approx}})$$

(4.26)

where $b^{(\text{new})} \leftarrow \text{LCA}(b^{(\text{old})}, \text{Pen}, \nu)$ is the one-step LCA as in (4.13) with a decomposition $\text{Pen}^{(t)}(b) = \text{Pen}^{(t)}_+(b) - \text{Pen}^{(t)}_-(b)$ as in (2.29).

**Theorem 5.** Let $b^{(t)}$ generated in (4.26) with penalty levels $\lambda^{(t)} = \lambda^{(t)}_s$ for $\text{Pen}^{(t)}$. Suppose $\| \hat{\text{Pen}}^{(t)}_-(b) - \hat{\text{Pen}}^{(t)}_-(\beta^0) \|_2 \leq \kappa_0 \| b - \beta^0 \|_2$ for all $b$ and

$$\Delta \left( r^{(t)}_1, w^{(t)}, \nu^{(t)}_{\text{approx}} \right) \leq 1, \ t = 1, \ldots, t_{\text{fin}},$$

(4.27)

with the $\Delta(r, w, \nu)$ in (4.6) and certain $r^{(t)}_1 > 0$. Let $h^{(t)} = b^{(t)} - \beta^0$ and $\nu_0 = \kappa_0 \| h^{(0)} \|_2$ be the initial carryover error. Suppose the RE condition

$$\text{RE}_\#(S; \xi, \gamma) \geq \kappa_0 \{ \lambda^{(t)}/\lambda^{(t+1)} \} \{ r^{(t)}_1 \lambda^{(t)}/\nu_0 + 1 \}$$

with $(1 - \eta)\xi_\ast < \nu_0/\lambda^{(1)} + r^{(t)}_1$, $t = 1, \ldots, t_{\text{fin}}$. Then, for $t \leq t_{\text{fin}}$

$$F(h^{(t)}) \leq \frac{r^{(t)}_1}{\text{RE}_\#(S; \xi, \gamma)} \lambda^{(t)} - \theta_0 \| h^{(t-1)} \|_2 \leq \sum_{k=1}^t \frac{\theta_0^k}{\text{RE}_\#(S; \xi, \gamma)} \| h^{(0)} \|_2$$

(4.27)

with $F(u) = \max \{ \| u \|_2, (\gamma u^T \Sigma u)^{1/2} \}$ and $\theta_0 = \kappa_0 / \text{RE}_\#(S; \xi, \gamma)$.

To find an approximate solution of PLSE with sorted penalty $\rho_\#(b; \lambda_\ast)$, we may implement (4.26) with a fixed penalty family $\rho(x; \lambda)$ and $\lambda^{(t)} \rightarrow \lambda_\ast$,

$$X^T (y - X b^{(t)})/\nu = \hat{\rho}_+\#(b^{(t)}; \lambda^{(t)}) - \hat{\rho}_-(b^{(t-1)}) + \nu^{(t)}_{\text{approx}},$$

(4.28)

$t = 1, \ldots, t^\ast$. For example, we may change the penalty levels proportionally by taking $\lambda^{(t)} = A^{(t)} \lambda_\ast$ with sufficiently large $A^{(0)}$ to ensure $b^{(0)} = 0$ and decreasing $A^{(t)}$.

Alternatively, we may move gradually from the Lasso to $\rho_\#(b; \lambda_\ast)$,

$$\lambda^{(t)}_j = \max (\lambda_\ast, \theta^t \lambda_\ast, 1), \ j \leq p,$$

(4.29)

$$\text{Pen}^{(t)}(b) = \theta^t \lambda_\ast \| b \|_1 + (1 - \theta^t) \rho_\#(b; \lambda^{(t)}),$$

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with suitable $\theta < 1$. The prediction and squared $\ell_2$ estimation error bounds are of the order $(\sigma^2/n)s \log p$ for the Lasso and $(\sigma^2/n)\{s + s_1 \log(p/s)\}$ for sorted concave penalties, where $s_1$ can be understood as the number of small nonzero coefficients. Thus, if we implement (4.29) for a total of $t^*$ steps, we need $\theta_0^{t^*} \leq \{1 + (s_1/s) \log(p/s)\}/\log p$ to achieve the better rate. This and parallel calculation for the weight for the Lasso and penalty level in (4.29) lead to the following requirement on $t^*$:

$$\max(\theta_0^{t^*}, \theta^{2t^*}) \leq \{1 + (s_1/s) \log(p/s)\}/\log p,$$

as $\lambda_{s,1}^2/\lambda_{s,s+1}^2 \approx \log(p/s)/\log p$ is of no smaller order.

### 4.6 Probabilistic error bounds

Here we prove conditions of Theorems 4 and 5 holds with high probability under the normality assumption (3.10) and the minimum penalty level condition (4.18) and (2.27) respectively for fixed and sorted penalty levels.

In addition to (3.10) we assume (4.10) holds with $0 < \eta_1 < \eta$ and $r_2 > 0$ and (4.18) holds for fixed penalty level. For sorted penalty levels we assume (2.27) holds with $\alpha \in (0, 1/4)$ and $A_0 = A/(\eta - \eta_1)$ for some $A > 1$. For nonnegative integer $s$ and the above $\{\alpha, A\}$, define $p_{\alpha,A} = 2\alpha \sum_k \alpha^{(A-1)A^k}$, $q_{\alpha,A} = (1 - \sqrt{2p_{\alpha,A}})_+$, $x_1 = s/q_{\alpha,A}$, $L_x = \sqrt{2 \log(p/(\alpha x))}$ and

$$\mu_{\#,s} = \left\{ \frac{2s(x_1/p)^{A^2-1}(2/q_{\alpha,A})^2}{A^2L_{s+1}^2(A^2L_{s+1}^2 + 2)} \right\}^{1/2} I_{\{s=0\}}. \tag{4.30}$$

We assume here that $x_1 \leq p$. This is reasonable when $p/s$ is large.

**Theorem 6.** Suppose (2.9), (3.10) and (4.10) with $\eta_1 \in (0, \eta)$ and $r_2 > 0$. Let $\beta^0 = \hat{\beta}^0$ be the oracle LSE as in (3.9) with $S = \text{supp}(\beta^0)$ and $s = |S|$. Let $\mathcal{B}_0^*(\lambda_s, \kappa_s, \gamma)$ be the solution set in (4.21), $\xi = (1-1/\alpha)(1-\eta)^{-1}\{(\eta-\eta_1)^2\mu_{\#,s}^2 + \eta_s^{2s^{1/2}} + r_2/s^{1/2}\}$ with the $\mu_{\#,s}$ in (4.30) and $a \in (0, 1)$. Let $F(u) = \max \{\|u\|_2, (\gamma u^T\Sigma u)^{1/2}\}$. Suppose $\kappa_s \leq (1 - 1/C^*)\text{RE}_\#$($S; \xi, \gamma$). Then, there exists an event $\Omega$ such that $\mathbb{P}\{\Omega\} \geq 1 - e^{-\xi * s \log(p/(\alpha(s+1)))}$ with $\xi = (1-\eta)^2 \xi^2 \eta/(\alpha(\eta - \eta_1))^2$, and that in the event $\Omega$

$$\|\hat{\beta} - \beta^0\| \leq (1 - \eta)C^* \lambda \sup_{u \neq 0} \frac{\|u\|\xi^{1/2}F(u) - \|u_{S^c}\|_{\#,s}}{u^T \Sigma u} \tag{4.31}$$

for all approximate solutions $\hat{\beta} \in \mathcal{B}_0^*(\lambda_s, \kappa_s, \gamma)$ and seminorms $\|\cdot\|$, where $\lambda = \lambda_{s+1}$ for sorted penalties and $\lambda = \lambda(\beta^0)$ as in (2.4) otherwise. In
addition, for all sorted or unsorted penalties satisfying \( \|\hat{\text{Pen}}(\beta^o)/\lambda\|_2 \leq s_1 \),

\[
\|\hat{\beta} - \hat{\beta}^o\| \leq C^* \lambda \sup_{u \neq 0} \frac{\|u\| r_1 F(u) - (1 - \eta)\|u_{S^c}\|_{\#}}{u^T \Sigma u}
\]

(4.32)

with \( r_1 = (1 - 1/a)^{-1}[\{(\eta - \eta_1)^2 \mu_{\#}^2 + s_1^2\}^{1/2} + r_2] \) and at least probability \( P\{\Omega\} - e^{-\xi r_1 \log(p/(\alpha(s+1)))} \) with \( \xi = \gamma/\{a(\eta - \eta_1)\}^2 \).

**Corollary 3.** Suppose \( \lambda = \lambda_{s+1} = \sigma \sqrt{(2/n) \log(p/s)} \) in (4.32) and \( \hat{\pi}(\hat{\beta}) \leq (1 - 1/C_0) \text{RE}_2^2(\mathcal{S}; \xi, \gamma) \) with \( C_0^2 \text{RE}_2^2(\mathcal{S}; \xi) = O_P(1) \). Then,

\[
\|X\hat{\beta} - X\hat{\beta}^o\|^2_n/\|X\|_2^2 + \|\hat{\beta} - \hat{\beta}^o\|^2_{\#} + \|\hat{\beta} - \hat{\beta}^o\|^2_{\#_s}/s = O_P(\sigma^2/n)\lambda_{s+1}^2 \log(p/s)
\]

with \( r_1^2 = s_1 + s/\log^2(p/s) + r_2^2 \), where \( s_1 = \|\hat{\text{Pen}}(\beta^o)/\lambda\|_2^2 \), and

\[
\|X\hat{\beta} - X\beta^o\|^2_n/\|X\|_2^2 + \|\hat{\beta} - \beta^o\|^2 + \|\hat{\beta} - \beta^o\|^2_{\#_s}/s
\]

= \( O_P(\sigma^2/n)\{(s_1 + r_2^2) \log(p/s) + s\} \).

(4.33)

For sorted concave penalty (2.25) with \( \sup(p(\cdot; \lambda)) \subseteq [-\gamma \lambda, \gamma \lambda] \),

\[
s_1 \leq \#\{j \leq |\mathcal{S}| : (\beta^o)^\#_{j} \leq \gamma \lambda_j\}.
\]

Corollary 3 extends Corollary 1 to smaller penalty levels \( \lambda = \lambda_{s+1} \geq A_0 \sigma \sqrt{(2/n) \log(p/\alpha(s+1))} \), sorted penalties, one-step application of the LCA, and their approximate solutions. In the worst case scenario where \( s_1 \approx s \), the error bounds in Corollary 3 attain the minimax rate [28, 3].

Theorem 6 and Corollary 3 provide sufficient conditions to guarantee simultaneous adaptation of sorted concave PLSE: (a) picking level \( \lambda_{s+1} \) automatically from \( \{\lambda_1, \ldots, \lambda_p\} \), and (b) partially removing the bias of the Slope [21, 3] when \( s_1 \ll s \), without requiring the knowledge of \( s \) or \( s_1 \).

Theorem 6 is a direct consequence of Theorem 4 and the following proposition.

**Proposition 7.** Suppose the normality assumption (3.10) holds. Let \( \eta > \eta_1 \) and \( \{\lambda_j\} \) be as in (2.27) for all sorted penalties. Let \( s = |\mathcal{S}| \) and \( \{q_{o,A}, \mu_{\#_s}\} \) be as in (4.30) with \( q_{o,A} > 0 \). Let \( a > 0, w > 0 \) and \( L = \sqrt{2 \log(p/(\alpha(s+1)))} \). Suppose \( \sup(p(\cdot)) \subseteq \mathcal{S}^c \). Then,

\[
P\{r_1 L \sqrt{\gamma}/a(\eta - \eta_1) \}
\]

(4.34)
when \((1 - 1/a)r_1 \geq \{(\eta - \eta_1)^2 \mu_{\#, s}^2 + w^2\}^{1/2} + r_2\). In particular, when \((4.1)\) is also confined to penalties satisfying \((2.9)\),

\[
P\left\{ (4.10) \text{ and } \|w\|_2 \leq \eta_2 s^{1/2} \implies (4.7) \right\} \geq \Phi \left( \frac{(1 - \eta) \xi s^{1/2} L}{a(\eta - \eta_1) \gamma^{-1/2}} \right)
\]

with \(\xi \geq \{(1 - 1/a)(1 - \eta)\}^{-1}\left[\{(\eta - \eta_1)^2 \mu_{\#, s}^2/s + \eta_2^2\}^{1/2} + r_2/s^{1/2}\right]\), where \(\mu_{\#, s}\) is as in \((4.30)\). Moreover, when \((4.1)\) is confined to penalties with the minimum fixed penalty level \(\lambda(\beta^o) = \lambda \geq \lambda_\ast\) as in \((2.4)\) and \((4.18)\), \((4.34)\) and \((4.35)\) hold with the \(L\) in \((4.18)\) and \(\mu_{\#, s} = \sqrt{4s/(L^4 + 2L^2)}\).

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A Appendices

We provide proofs in the following order: Proposition 1, Proposition 2, Proposition 3, Lemma 1, Proposition 5, Theorem 1–3, Proposition 6, Theorem 4, Theorem 5, and Proposition 7 along with an additional lemma. We omit proofs of Proposition 4, Lemma 2 and Theorem 6 as explained above or below their statements.

Proof of Proposition 1. For \( t_0 = 0 < t_1 < \ldots < t_K = 1 \), (2.20) yields

\[
\begin{align*}
    h^T \{ \mathbb{E}_\nu [\dot{\rho}(b; \lambda) | b] - \mathbb{E}_\nu [\dot{\rho}(b + h; \lambda) | b + h] \} \\
    = \sum_{k=1}^K h^T \{ \mathbb{E}_\nu [\dot{\rho}(b_{tk-1}; \lambda) | b_{tk-1}] - \mathbb{E}_\nu [\dot{\rho}(b_{tk}; \lambda) | b_{tk}] \} \\
    \leq \bar{\pi}(\rho) \| h \|_2^2 + \sum_{k=1}^K h^T \{ \mathbb{E}_\nu [\dot{\rho}(b_{tk-1}; \lambda) | b_{tk-1}] - \mathbb{E}_\nu [\dot{\rho}(b_{tk}; \lambda) | b_{tk}] \} \\
    \to \bar{\pi}(\rho) \| h \|_2^2 + \nu \int_0^1 \text{Var}_\nu(h^T \dot{\rho}(b_t; \lambda) | b_t) dt
\end{align*}
\]

with \( b_t = b + th_t \), where the limit is taken as \( \max_{k \leq K} |t_k - t_{k-1}| \to 0 \). Note that \( h^T \{ \dot{\rho}(b_{tk-1}; \lambda) - \dot{\rho}(b_{tk}; \lambda) \} \leq \| M \|_{\text{max}} \) for all matrices \( M \in \mathbb{R}^{p \times p} \). We note that \( \text{Cov}_\nu(\dot{\rho}(u; \lambda), \dot{\rho}(u; \lambda)|u, \theta) \) is a diagonal matrix when the components of \( \lambda \) are independent given \( \theta \).

Let \( \delta \in \{-1, 1\}^p \) and take \( \dot{\rho}(b_j; \lambda_j) = \delta_j \lambda_j \) for \( j \in S^c_b \). When \( \lambda \) is an exchangeable random vector under \( \nu(d\lambda) \),

\[
\left( \mathbb{E}_\nu [\dot{\rho}(b; \lambda) | b] \right)_{S^c_b} = \delta_{S^c_b} \mathbb{E}[\lambda_j | b], \; \forall \; j \in S^c_b
\]

As \( \varrho \text{Pen}(b) \) is convex, this gives \( \lambda(b) = \mathbb{E}[\lambda_j | b] \) for \( \rho_\nu(b), \; j \in S^c_b \).

Proof of Proposition 2. We omit the proof of (2.26) as it follows directly from the definition in (2.2). For \( \lambda \geq \lambda' \) and \( t \geq t' > 0 \),

\[
\rho(t; \lambda) + \rho(t'; \lambda') = \rho(t; \lambda') - \rho(t'; \lambda) = \int_{t'}^t \{ \dot{\rho}(x; \lambda) - \dot{\rho}(x; \lambda') \} dx \geq 0.
\]

Thus, (2.28) follows. By the convexity of \( \rho(t; \lambda) + \bar{\pi}(\rho)t^2/2 \) in \( t \),

\[
\rho_\#(b; \lambda) + \bar{\pi}(\rho) \| b \|_2^2/2 = \max \left\{ \sum_{j=1}^p \{ \rho(b_j; \lambda_{k_j}) + \bar{\pi}(\rho)|b_j|^2/2 : k \in \text{perm}(p) \} \right\}
\]

is convex, as the maximum of convex functions is convex. Hence, the maximum concavity of \( \rho_\#(b; \lambda) \) is no greater than \( \bar{\pi}(\rho) \).
**Proof of Proposition 3.** Because $\rho_{\#}(b; \lambda)$ depends only on $b^\#$, we have $\text{sgn}(b_j) = \text{sgn}(x_j)$, and the problem is to minimize

$$
\sum_{\ell=1}^p \frac{(|x_\ell| - |b_\ell|)^2}{2} + \rho_{\#}(b; \lambda) = \sum_{j=1}^p \frac{(|x_j| - b_j^\#)^2}{2} + \rho_{\#}(b; \lambda).
$$

By algebra, $(|x_j| - b_j^\#)^2 + (|x_k| - b_k^\#)^2 - (|x_j| - b_j^\#)^2 - (|x_k| - b_k^\#)^2 = (|x_j| - |x_k|)(b_k^\# - b_j^\#)$. Thus, when $(|x_j| - |x_k|)(b_k^\# - b_j^\#) > 0$, the solution can be improved by switching $x_j$ and $x_k$. This implies the monotonicity of $|b_j|$ in $|x_j|$ and (2.36).

Now consider $x_1 \geq \ldots \geq x_p \geq 0$. Let $\tilde{b}_j = \arg \min \{ (x_j - b)^2/2 + \rho(b; \lambda_j) \}$ be the univariate solution and $b^\# = \text{iso.prox}(x; \rho(:; \lambda))$ the multivariate isotonic solution. If $b_j > b_{j+1}$ and $b_j > \tilde{b}_j$, then we can improve $b$ by decreasing $b_j$. If $b_j > b_{j+1}$ and $b_j+1 < \tilde{b}_j$, then we can improve $b$ by increasing $b_{j+1}$. Thus, we must have $b_j > \tilde{b}_j$. Moreover, when

$$
\tilde{b}_{j-1} < \tilde{b}_j \leq \tilde{b}_{j+1} \leq \ldots \leq \tilde{b}_{j'} > \tilde{b}_{j'+1}, \quad \tilde{b}_j < \tilde{b}_{j'},
$$

we must have $b_j = b_{j+1} = \ldots = b_{j'}$. This groups the optimization problem in the first round of Algorithm 3. The argument also applies to the grouped optimization problem in the second round, so on and so forth. □

**Proof of Lemma 1.** Let $z = X^T(y - X\beta^o)/n$. Recall that $h = \beta - \beta^o$ and $w = \{\text{Pen}(\beta^o) + \nu_{\text{approx}} - z\}/\lambda$. By (3.11), $\text{Pen}(\beta) + \nu_{\text{approx}} = z - \Sigma h$, so that

$$
h^T \Sigma h = h^T \{ -\lambda w + \text{Pen}(\beta^o) - \text{Pen}(\bar{\beta}) \} \leq -\lambda h^T w + \bar{\gamma}_1(\bar{\beta})\|h\|^2 + \bar{\gamma}_2(\bar{\beta})\|h\|^2
$$

by the definition of $\bar{\gamma}_1(b)$ and $\bar{\gamma}_2(b)$ in (2.7). As $h_{\text{Sc}}^T(\nu_{\text{approx}})_{\text{Sc}} \geq 0$, this gives (3.26) with $\text{Pen}(\beta^o) \in \partial\text{Pen}(\beta^o)$ satisfying $\text{Pen}_j(\beta^o) = \text{sgn}(h_j)\lambda$, $j \in \text{Sc}$. □

**Proof of Proposition 5.** Let $h = \beta - \beta^o$ and $\tilde{h} = \beta - \beta^o$. We want to prove that

$$
\|h - \tilde{h}\|_1 \leq a_0\tilde{\lambda} \quad \text{and} \quad h \in \mathcal{C}(\mathcal{S}; \xi) \text{ imply } \tilde{h} \in \mathcal{C}(\mathcal{S}; \xi).
$$

(A.1)

As $\|z_{\text{Sc}}\|_\infty \leq (\eta - a_1)\lambda$ $\leq (\eta - a_1)\tilde{\lambda}$, Lemma 1 and (3.12) imply that

$$
\tilde{h}^T \Sigma \tilde{h} + (1 - \eta + a_1)\tilde{\lambda}\|h_{\text{Sc}}\|_1 \\
\leq (1 - \eta)\xi\tilde{\lambda}\|\tilde{h}_{\text{Sc}}\|_1 + \bar{\gamma}^1(\bar{\beta})\|h\|^2 + \bar{\gamma}^2(\bar{\beta})\|h\|^2
$$

(A.2)
Recall $a_2 = a_1\xi/[2(\xi + 1)]\{\pi_1(b) + \pi_2(b)\}$. When $\|h\|_1 \vee \|\tilde{h} - h\|_1 \leq a_2\bar{\lambda}$,  
$\pi_1(\tilde{\beta})\|\tilde{h}\|^2 + \pi_2(\tilde{\beta})\|\tilde{h}\|^2_2 \leq 2a_2\bar{\lambda}\|\tilde{h}\|_1\{\pi_1(\tilde{\beta}) + \pi_2(\tilde{\beta})\} \leq a_1\xi\bar{\lambda}\|\tilde{h}\|_1/(\xi + 1)$,  
so that (A.2) implies  
$$(1 - \eta + a_1)\|\tilde{h}_{S^c}\|_1 \leq (1 - \eta)\xi\|\tilde{h}_S\|_1 + \{a_1\xi/(\xi + 1)\}(\|\tilde{h}_S\|_1 + \|\tilde{h}_{S^c}\|_1),$$
which is equivalent to $\|\tilde{h}_{S^c}\|_1 \leq \xi\|\tilde{h}_S\|_1$ by algebra.  
Because $\pi_{1,2}(\tilde{\beta}; \xi) \leq \kappa_*$ and $\|h\|_1 \leq (1 + \xi)\|h_S\|_1 \leq (1 + \xi)|S|^{1/2}\|h\|_2$, the  
$\{h, \lambda\}$ version of (A.2) implies  
$$h^T\Sigma h + (1 - \eta + a_1)\lambda\|h_{S^c}\|_1 \leq (1 - \eta)\xi\lambda\|h_S\|_1 + \kappa_*\|h\|^2_2.$$  
By the RE condition, we have $\kappa_*\|h\|^2_2 \leq h^T\Sigma h$, so that  
$$(1 - \eta + a_1)\|h_{S^c}\|_1 \leq (1 - \eta)\xi\|h_S\|_1.$$  
(A.3)
Recall that $a_3 = a_1(1 - \eta)\xi/((1 - \eta)(\xi + 1) + a_1)$. If $\|h\|_1 > a_2\bar{\lambda}$ and  
$\|\tilde{h} - h\|_1 \leq \bar{\lambda}a_2a_3/(1 - \eta)(\xi \vee 1)$, (A.3) implies  
$$(1 - \eta)\|\tilde{h}_{S^c}\|_1 - (1 - \eta)\xi\|\tilde{h}_S\|_1 \leq (1 - \eta)\|h_{S^c}\|_1 - (1 - \eta)\xi\|h_S\|_1 + \{a_3\xi - \xi\|h_S\|_1\} = (1 - \eta + a_3)\|h_{S^c}\|_1 - \{a_1 + a_1\xi\}(\|h_S\|_1 - 1) \leq 0$$
by algebra. Thus, (A.1) holds in either cases. \hfill \Box

**Proof of Theorem 1–3.** Let $h = \tilde{\beta} - \beta^\circ$. By Proposition 5, $h \in \mathcal{C}(S; \xi)$. It follows that $\|h\|^2 \leq (1 + \xi)^2|S|\|h_S\|^2_2$, so that by Lemma 1 and (3.8)  
$$h^T\Sigma h + (1 - \eta)\lambda\|h_{S^c}\|_1 + \lambda w_S^T h_S \leq \pi_{1,2}(\tilde{\beta}; \xi)\|h\|^2_2 \leq \kappa_*\|h\|^2_2.$$  
Let $C_0$ be as in Theorem 3. As $h^T\Sigma h \geq \text{RE}^2(S; \xi)\|h\|^2_2$, Lemma 1 implies  
$$C_0^{-1}h^T\Sigma h + (1 - \eta)\lambda\|h_{S^c}\|_1 \leq -\lambda w_S^T h_S \leq \lambda\|w_S\|_2\|h\|_2.$$  
(A.4)
This immediately implies (3.22) and (3.23). For (3.17) and (3.18), we set $C_0 = \infty$. However, by (A.4) and the definition of RCIF,  
$$\text{RCIF}_{\text{pred}}(S; \eta, w)h^T\Sigma h \leq \|\Sigma h\|^2_\infty|S|.$$  
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Consequently, the first inequality in (3.17) follows from the fact that
\[
\|\Sigma h\|_\infty \leq \|X^T(y - X\beta)/n\|_\infty + \|X^T(y - X\beta^o)/n\|_\infty \\
= \lambda + \|\nu_{approx}\|_\infty + \eta\lambda_s = \lambda,
\]
and the second inequality follows from the first inequality in Proposition 4. Similarly, the first inequality in (3.18) follows from
\[
\text{RCIF}_{\text{est}, q}(S; \eta, w)\|h\|_q \leq \|\Sigma h\|_\infty |S|^{1/q} \leq \bar{X}|S|^{1/q},
\]
and the second from the second and third inequalities in Proposition 4.

Finally, we consider selection consistency for exact local solutions \(\hat{\beta}\) of (3.6) in \(B_0(\lambda_s, \kappa_s)\) under the assumption \(w_S = 0\). In this case, \(\nu_{approx} = 0\), \(\beta^o\) is a solution of (3.6), and \(\|h_S\|_1 = 0\) by (A.4). Moreover, because both \(\beta^o\) and \(\hat{\beta}\) are solutions of (3.6) with support in \(S\),
\[
h_S^T\Sigma_S h_S = h^T\{\text{Pen}(\beta^o) - \text{Pen}(\hat{\beta})\} \leq \pi_{1,2}(\hat{\beta}; \xi)\|h_S\|_2^2.
\]
Thus, \(\phi_{\min}(\Sigma_S, S) > \pi_{1,2}(\hat{\beta}; \xi)\) implies \(h = 0\).

For separable penalties of the form \(\rho(\beta; \lambda) = \sum_{j=1}^p \rho(b_j; \lambda)\) in (2.10),
\[
\kappa_s\|h_S\|_2^2 \leq h^T\{\text{Pen}(\beta^o) - \text{Pen}(\hat{\beta})\} = \sum_{j \in S}(\hat{\beta}_j - \beta^o_j)\{\hat{\beta}_j \rho(\beta^o; \lambda) - \hat{\beta}_j \rho(\hat{\beta}; \lambda)\}.
\]
As \((\hat{\beta}_j - \beta^o_j)\{\hat{\beta}_j \rho(\beta^o; \lambda) - \hat{\beta}_j \rho(\hat{\beta}; \lambda)\} \leq \min\{\pi(\beta^o_j; \rho, \lambda), \pi(\hat{\beta}_j; \rho, \lambda)\} h_j^2 \leq \kappa_s h_j^2\), equality must attain for every \(j\). This is possible only when \(sgn(\hat{\beta}_j)sgn(\beta^o_j) \geq 0\) for all \(j \in S\). Furthermore, \(sgn(\hat{\beta}_j)sgn(\beta^o_j) > 0\) for all \(j \in S\) when \(\pi(0; \rho, \lambda) < \phi_{\min}(\Sigma_S, S)\).

**Proof of Proposition 6.** Let \(\tilde{\beta} \in B(\lambda_s, \kappa_s, \gamma)\) with penalty level \(\lambda = \lambda_{s+1}\) and \(\tilde{\beta} - \beta^o \in C_\#(S; \xi, \gamma)\), and \(\tilde{\beta} \in B(\lambda_s, \kappa_s, \gamma)\) with \(\tilde{\lambda} = \tilde{\lambda}_{s+1}\). We first prove that \(\tilde{\beta} \in B(0; \lambda_s, \kappa_s, \gamma)\) implies \(\tilde{\beta} - \beta^o \in C_\#(S; \xi, \gamma)\), or equivalently
\[
\|\tilde{\beta} - \beta\|_{\#, S} \leq \tilde{\lambda}|S|a_0 \Rightarrow \tilde{\beta} - \beta^o \in C_\#(S; \xi, \gamma).
\] (A.5)

Let \(F(u) = \max\{\|u\|_2, \|Xu\|_2 \sqrt{n}\}\), \(h = \tilde{\beta} - \beta^o\) and \(\tilde{h} = \tilde{\beta} - \beta^o\). Assume without loss of generality \(\tilde{h}_{S^c} \|_{\#, S} \geq \xi S^{1/2} \tilde{h}_2\). By (4.5), (4.6) and (4.23)
\[
\tilde{h}^T \Sigma \tilde{h} + (1 - \eta)\tilde{\lambda}\|h_{S^c}\|_{\#, S} \geq \bar{\pi}(\tilde{\beta})\|\tilde{h}\|_2^2 \leq (1 - \eta - a_1)\tilde{\lambda}\xi |S|^{1/2} F(\tilde{h})\delta.
\]
Recall that \(a_2 = a\xi^2/(2\pi(\tilde{\beta}))\). When \(\|h\|_{\#, S} \geq \eta S^{1/2} \tilde{h}_2\), \(\bar{\pi}(\tilde{\beta})\|\tilde{h}\|_2^2 \leq \bar{\pi}(\tilde{\beta})\|h_{S^c}\|_{\#, S}/(\xi^2 |S|)\) \(\leq a_1\|h_{S^c}\|_{\#, S}\), so that \(\|h_{S^c}\|_{\#, S} \leq \xi |S|^{1/2} F(\tilde{h})\) by (A.6).
Because \( \pi(\hat{\beta}) \leq \kappa_a \), the \( \{h, \lambda\} \) version of (A.6) implies

\[
h^T \Sigma h + (1 - \eta) \lambda \|h_{Sc}\|_{\#, s} \leq (1 - \eta - a_1) \xi \|S\|^{1/2} F(h) + \kappa_a F^2(h).
\]

By the RE condition in \( \mathcal{C}_\#(S; \xi, \gamma) \), we have \( \kappa_a F^2(h) \leq h^T \Sigma h \), so that

\[
(1 - \eta) \|h_{Sc}\|_{\#, s} \leq (1 - \eta - a_1) \xi \|S\|^{1/2} F(h).
\]

Recall that \( a_3 = a_1(1 - \eta) \xi / \{(1 - \eta - a_1)(\xi + 1) + a_1\} \). If \( \|h\|_{\#, s} > a_2 \lambda \) and \( \|\tilde{h} - h\|_{\#, s} \leq \lambda a_2 a_3 / \{(1 - \eta)(\xi + 1)\} \), we have

\[
\begin{align*}
(1 - \eta) \|h_{Sc}\|_{\#, s} - (1 - \eta) \xi \|S\|^{1/2} F(\tilde{h}) & \leq (1 - \eta) \|h_{Sc}\|_{\#, s} - (1 - \eta) \xi \|S\|^{1/2} F(h) + \{(1 - \eta)(\xi + 1)\}\|\tilde{h} - h\|_{\#, s} \\
& = (1 - \eta + a_3) \|h_{Sc}\|_{\#, s} - (1 - \eta - a_3 / \xi) \xi \|S\|^{1/2} F(h) \\
& = \frac{(1 - \eta)(\xi + 1) + a_1}{\{(1 - \eta)(\xi + 1)\}} \leq 0
\end{align*}
\]

due to \( (1 - \eta + a_3)/(1 - \eta - a_3 / \xi) = (1 - \eta)/(1 - \eta - a_1) \). Hence, (A.5) holds in either cases. The proof for the approximate LCA solution is straightforward as (4.13) corresponds to (4.5) with \( \tilde{\pi}(\hat{\beta}) = 0 \).

\[\begin{proof}
\text{PROOF OF THEOREM 4. By (4.5), (4.6) and (4.24) that}
\end{proof}\]

\[
h^T \Sigma h + (1 - \eta) \lambda \|h_{Sc}\|_{\#, s} - \pi(\hat{\beta}) \|h\|_2^2 \leq r_1 \lambda F(h). \tag{A.7}
\]

with \( F(u) = \max \{\|u\|_2, \| Xu \|_{2\sqrt{\gamma/n}}\} \). By Proposition 6, \( h \in \mathcal{C}_\#(S; \xi, \gamma) \), so that \( \text{RE}_\#^2(S; \xi, \gamma) F^2(h) \leq h^T \Sigma h \). As \( \pi(\hat{\beta}) \leq (1 - 1/C_0) \text{RE}_\#^2(S; \xi, \gamma) \),

\[
C_0^{-1} h^T \Sigma h + (1 - \eta) \lambda \|h_{Sc}\|_{\#, s} \leq r_1 \lambda F(h),
\]

which implies (4.25).

\[\begin{proof}
\text{PROOF OF THEOREM 5. We begin induction by assuming}
\end{proof}\]

\[
\kappa_0 \|h^{(t-1)}\|_2 / \lambda^{(t)} \leq \nu_0 / \lambda^{(1)},
\]

which holds for \( t = 1 \). By (4.6), (4.27) and the condition on \( \xi \),

\[
1 \geq \Delta \left( r_1^{(t)} + \|\nu_{\text{carry}}^{(t-1)}\|_2 / \lambda^{(t)}, w^{(t)}, \nu_{\text{approx}}^{(t)} - \nu_{\text{carry}}^{(t-1)} \right) \\
\geq \Delta \left( r_1^{(t)} + \kappa_0 \|h^{(t-1)}\|_2 / \lambda^{(t)}, w^{(t)}, \nu_{\text{approx}}^{(t)} - \nu_{\text{carry}}^{(t-1)} \right)
\]

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Recall that Theorem 4 applies. As LCA is convex minimization, \( \pi(\hat{\beta}) = 0 \) in this application of Theorem 4. Thus,

\[
F(h^{(t)}) \leq \frac{r_1^{(t)}\lambda^{(t)} + \kappa_0 \|h^{(t-1)}\|_2}{\text{RE}^2_\#(S; \xi, \gamma)} \leq \frac{\lambda^{(t)} r_1^{(t)} + \nu_0 / \lambda^{(t)}}{\text{RE}^2_\#(S; \xi, \gamma)}.
\]

As \( \text{RE}^2_\#(S; \xi, \gamma) \geq \kappa_0 \{\lambda^{(t)}/\lambda^{(t+1)}\} \{r_1^{(t)}/\nu_0 + 1\} \),

\[
\frac{\kappa_0 F(h^{(t)})}{\lambda^{(t+1)}} \leq \frac{\lambda^{(t)} r_1^{(t)} + \nu_0 / \lambda^{(t)}}{\text{RE}^2_\#(S; \xi, \gamma)} \leq \nu_0 / \lambda^{(t)}.
\]

This completes the induction as \( \|\nu^{(t)}_{\text{carry}}\|_2 \leq \kappa_0 \|h^{(t)}\|_2 \leq \kappa_0 F(h^{(t)}) \).

We need the following lemma in the proof of Proposition 7.

**Lemma 3.** Let \( z_j \) be normal variables with \( \mathbb{E} z_j = 0 \) and \( \text{Var}(z_j) \leq \sigma_n^2 \). Let \( s, A, L_x, q_{\alpha,A}, x_1 \) and \( \mu_{\#,s} \) be as in (4.30). Then, for \( s > 0 \) and \( q_{\alpha,A} > 0 \)

\[
\mathbb{P}\left\{ \max_{x_1 \leq j \leq p} \frac{z_{j-s}^{\#}}{A \sigma_n L_j} < 1, \sum_{s < j < x_1} \frac{(z_{j-s}^{\#} - A \sigma_n L_j)^2}{(A \sigma_n L_{s+1})^2} < \mu_{\#,s}^2 \right\} > 1/2,
\]

and the right-hand side is greater than \((1 + q_{\alpha,A})/2 \) for \( s = 0 \).

**Proof of Lemma 3.** Set \( \sigma_n = 1 \) without loss of generality. As in [3],

\[
\mathbb{P}\{z_{j}^{\#} > t\} \leq \frac{(2p/j)e^{-t^2/2}}{1/2}.
\]

Recall that \( p_{\alpha,A} = 2\alpha \sum_{k=0}^{\infty} a^{(A-1)A^k}, q_{\alpha,A} = 1 - \sqrt{2p_{\alpha,A}} \) and \( x_1 = s/q_{\alpha,A} \leq p \). Define \( x_k \) by \( L_{x_k}^2 = A^{1-k} L_1^2, 2 \leq k \leq k^*, \) with \( AL_{p}^2 > L_{k^*}^2 \geq L_{p}^2 \). Let \( j_k \) be the solution of \( j_k - 1 < x_k \leq j_k \). Let \( y = 1-s/x_1 = \sqrt{2p_{\alpha,A}} \). Because \( z_{j}^{\#} \geq z_{j+1}^{\#} \) and \( x_k/(x_k-s) \leq 1/(1-s/x_1) = 1/y = y/(2p_{\alpha,A}) \),

\[
\mathbb{P}\left\{ \max_{x_1 \leq j \leq p} \frac{z_{j-s}^{\#}}{L_j} \geq A \right\} \leq \sum_{j_k < j_{k+1}} \mathbb{P}\left\{ z_{j_k-s}^{\#} \geq AL_{x_{k+1}} = A^{1/2} L_{x_k} \right\}
\]

[44]
\[ \sum_{j_k < j_{k+1}} \frac{2p}{j_k - s} \exp \left( -AL^2_{x_k}/2 \right) \tag{A.8} \]
\[ \leq \sum_{j_k < j_{k+1}} \frac{2p}{x_k - s} \left( \frac{\alpha x_k}{p} \right) \exp \left( -(A - 1)A^{k^*-k}L_p^2/2 \right) \]
\[ \leq \frac{1}{y} \sum_{k \leq k^*} 2^\alpha (A - 1)A^{k^*-k} = p_{\alpha,A}/y = y/2. \]

This completes the proof for \( s = 0 \), which gives \( x_1 = 0 \). For \( s > 0 \), Proposition 10 in [22] gives
\[ \mathbb{E} \sum_{s < j < x_1} \frac{(z_{j-s} - AL_j)^2}{A^2 L^2_{s+1}} \leq \mathbb{E} \left[ \|z - AL_{x_1}\|_2^2 \right] \leq \frac{4p(x_1/p)^{A^2}}{A^2 L^2_{s+1}(A^2 L_{x_1}^2 + 2)}. \]

As the right-hand side above equals to \( \mu_{\#s}^2 p_{\alpha,A}/2 \), by Markov’s inequality
\[ \mathbb{P} \left\{ \sum_{s < j < x_1} (z_{j-s} - AL_j)^2 > A^2 L^2_{s+1} \mu_{\#s}^2 \right\} \leq q_{\alpha,A}/2 = 1/2 - y/2. \]

This and (A.8) completes the proof with the union bound. \( \square \)

**Proof of Proposition 7.** Let \( \eta' = \eta - \eta_1 \), \( \lambda_{s,j} \) be as in (2.27) and
\[ \Delta(r, \eta, w, r_2) = \sup_{\|u\|_2 = 1} \frac{\mathbb{E}[\mathcal{S}^c \mathcal{S}^c \rho / \lambda_{s,s+1} - \eta \|u\|_2 + w \|u\|_2 + r_2]}{r \max \left\{ \|u\|_2/a, \|Xu\|_2 \sqrt{n} \right\}} \]
where \( \| \cdot \|_{\#s} \) is defined as in (4.4) with \( \lambda_j = \lambda_{s,j} \). By Lemma 3,
\[ \text{median}(\Delta(r, \eta, w, r_2)) \leq \left\{ (\eta^2 \mu_{\#s}^2 + w^2)^{1/2} + r_2 \right\} (a/r). \]

As \( \text{supp}(z) \subseteq \mathcal{S}^c \), \( \text{to} \Delta(r, \eta, w, r_2) \) is convex and unit Lipschitz in \( y - X\beta^o \), so that by the Gaussian concentration inequality
\[ \mathbb{P} \left\{ \Delta(r, \eta, w, r_2) \geq \text{median}(\Delta(r, \eta, w, r_2)) + 1 \right\} \leq \Phi(-t). \]

We first prove (4.34) and (4.35). Let \( r = r_1 \). Although \( \Delta(r, \eta, w, r_2) \) in (4.11) and \( \Delta(r, \eta, w, r_2) \) are defined with \( \lambda_{s,j} \) and \( \Delta(r_1, w, \nu) \) is defined in (4.6) with \( \lambda_j \geq \lambda_{s,j} \), (4.10) and the monotonicity of \( (z/\lambda - 1)_+ \) in \( \lambda \) still provide
\[ \Delta(r_1, w, \nu) I_{\{\|w\|_2 \leq w\}} \leq \Delta(r_1, \eta', w, r_2) / \text{median}(\Delta(r, \eta', w, r_2) + 1) \]
when

\[
\frac{1}{\text{median}(\tilde{\Delta}(r, \eta', w, r_2)) + 1} \geq \max \left( \frac{1}{a}, \frac{t \sigma}{r_1 \lambda \sqrt{\gamma n}} \right) = \max \left( \frac{1}{a}, \frac{\eta' t}{r_1 L \sqrt{\gamma}} \right),
\]

Setting \( t = r_1 L \sqrt{\gamma} / (a \eta') \) and \( \{(\eta')^2 \mu_{\#, s}^2 + w^2\}^{1/2} / r_1 + 1/a = 1 \), we have (4.34). Moreover, (4.35) follows with \( r_1 = (1 - \eta) \xi s^{1/2} \). For constant \( \lambda \) with \( \|u_S\|_{\#, s} = \|u_S\|_1 \), we replace the median with

\[
\mathbb{E} \tilde{\Delta}(r, \eta, w, r_2) \leq \left\{ (\eta^2 4s/(L^4 + 2L^2) + w^2)^{1/2} + r_2 \right\}(a/r),
\]

so that \( \mu_{\#, s}^2 \) can be replaced by \( 4s/(L^4 + 2L^2) \). \( \square \)