The Lawson homology and Deligne-Beilinson cohomology for Fulton-MacPherson configuration spaces

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Abstract

In this paper, we compute the Lawson homology groups and Deligne-Beilinson cohomology groups for Fulton-MacPherson configuration spaces. The explicit formulas are given.

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1 Introduction

In this paper, all varieties are defined over \( \mathbb{C} \). Let \( X \) be an \( d \)-dimensional projective variety. Let \( Z_p(X) \) be the space of algebraic \( p \)-cycles on \( X \).

The \textbf{Chow group} \( \text{Ch}_p(X) \) of \( p \)-cycles is defined by \( Z_p(X) \) modulo the rational equivalence. For general background on Chow groups, the reader is referred to Fulton’s book.
The Lawson homology $L_pH_k(X)$ of $p$-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(Z_p(X)) \text{ for } k \geq 2p \geq 0,$$

where $Z_p(X)$ is provided with a natural topology (cf. [F], [L1]). For general background on Lawson homology, the reader is referred to [L2].

It is convenient to extend the definition of Lawson homology by setting

$$L_pH_k(X) = L_0H_k(X), \text{ if } p < 0.$$

It was proved in [H] that, for any smooth projective variety $X$, the formula on Lawson homology for a blowup holds:

**Theorem 1.1 (H)** Let $X$ be smooth projective variety and $Y \subset X$ be a smooth subvariety of codimension $r \geq 2$. Let $\sigma : \tilde{X}_Y \to X$ be the blowup of $X$ along $Y$, $\pi : D := \sigma^{-1}(Y) \to Y$ the natural map, and $i : D \to \tilde{X}_Y$ the exceptional divisor of the blowup. Then for integers $p, k$ with $k \geq 2p \geq 0$, there is an isomorphism

$$I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \cong L_pH_k(\tilde{X}_Y).$$

Now we give minimal notations for the Fulton-MacPherson configuration spaces enough for stating the main theorem (see section 2.2 for a construction of the Fulton-MacPherson configuration spaces by a sequence of blowups.)

Let $X$ be a smooth projective variety of dimension $d$ and let $n \geq 1$ be an integer. Consider the cartesian product $X^n := X \times \cdots \times X$ of $n$ copies of $X$. Denote by $\Delta_I$ the diagonal in $X^n$ where $x_i = x_j$ if $i, j \in I$.

The configuration space $F(X, n)$ is the complement of all diagonals in $X^n$, i.e.,

$$F(X, n) = X^n \setminus \bigcup_{|I| \geq 2} \Delta_I = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j, \forall i \neq j\}.$$ 

For each subset $I \in [n] := \{1, \ldots, n\}$ with at least two elements, denote by $\text{Bl}_{\Delta}(X^I)$ the blowup of the corresponding cartesian product $X^I$ along its small diagonal. In [FuM], Fulton and MacPherson have given the definition of their compactification $X[n]$ as follows.

**Theorem 1.2 (Fulton-MacPherson)** The closure of the natural locally closed embedding

$$i : F(X, n) \hookrightarrow X^n \times \prod_{|I| \geq 2} \text{Bl}_{\Delta}(X^I)$$

is smooth, and the boundary is a simple normal crossing divisor. The closure is called the Fulton-MacPherson configuration space, denoted by $X[n]$. 

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We call two subsets \( I, J \subseteq [n] := \{1, 2, \ldots, n\} \) are \textit{overlapped} if \( I \cap J \) is a nonempty proper subset of \( I \) and of \( J \).

A \textit{nest} \( S \) is a set of subsets of \([n]\) such that any two elements \( I \neq J \in S \) are not overlapped, and all singletons \( \{1\}, \ldots, \{n\} \) are in \( S \). Notice that the nest defined here, unlike the one defined in \([FM]\), contains singletons.

Given a nest \( S \), define \( S^\circ = S \setminus \{\{1\}, \ldots, \{n\}\} \). In the description of nests by forests below, \( S^\circ \) corresponds to the forest \( S \) cutting of all leaves.

A nest \( S \) naturally corresponds to a not necessarily connected tree (which is also called a \textit{forest} or a \textit{grove}), each node of which is labeled by an element in \( S \). For example, the following forest corresponds to a nest \( S = \{1, 2, 3, 23, 123\} \).

\[
\begin{array}{c}
\begin{array}{c}
\bullet & \bullet & \bullet \\
\{1\} & \{2\} & \{3\}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow & \downarrow & \downarrow \\
123 & 23 & 1
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\downarrow & \downarrow & \downarrow \\
2 & 3 & \\
\end{array}
\end{array}
\]

Denote by \( c(S) \) the number of connected components of the forest, i.e., the number of maximal elements of \( S \). Denote by \( c_I(S) \) (or \( c_I \) if no ambiguity arise) the number of maximal elements of the set \( \{J \in S | J \subset I\} \), i.e. the number of sons of the node \( I \). In the above example, \( c(S) = 1, c_{123} = c_{23} = 2 \).

For a nest \( S \neq \{\{1\}, \ldots, \{n\}\} \) (i.e. \( S^\circ \neq \emptyset \)), define a set \( M_S \) of lattice points in the integer lattice \( \mathbb{Z}^{S^\circ} \) as follows

\[
M_S := \{ \mu = \{ \mu_I \} \in \mathbb{Z}^{S^\circ} : 1 \leq \mu_I \leq d(c_I - 1) - 1 \}.
\]

(Recall that \( d = \dim X, c_I = c_I(S) \)) and define \( \|\mu\| := \sum_{I \in S^\circ} \mu_I, \forall \mu \in M_S \).

For \( S = \{\{1\}, \ldots, \{n\}\} \), assume \( M_S = \{\mu\} \) with \( \|\mu\| = 0 \).

It was proved in \([Li]\) that, for any smooth projective variety \( X \) the following holds:

\textbf{Theorem 1.3 (\([Li]\))} \ Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \). Then for each \( p \geq 0 \), there is an isomorphism of Chow groups:

\[
\text{Ch}_p(X[n]) \cong \bigoplus_S \bigoplus_{\mu \in M_S} \text{Ch}_{p-\|\mu\|}(X^{c(S)}).
\]

where \( S \) runs through all nests of \([n]\).

The first main result in this paper is the following

\textbf{Theorem 1.4} \ Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \). Then for each pair of integers \( p, k \), \( k \geq 2p \geq 0 \), there is an isomorphism of Lawson homology groups:

\[
L_pH_k(X[n]) \cong \bigoplus_S \bigoplus_{\mu \in M_S} L_{p-\|\mu\|}H_{k-2\|\mu\|}(X^{c(S)}).
\]

where \( S \) runs through all nests of \([n]\).
Remark 1.1 When $p = 0$, Theorem 1.4 reduces to the formula of singular homology groups with integer coefficient for $X[n]$. In particular, the integer singular homology of $X[n]$ depends only the integer singular homology of $X$.

As a corollary, we have the following more explicit formula:

Corollary 1.1 Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers $p, k$, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:

$$L_pH_k(X[n]) \cong \bigoplus_{1 \leq m \leq n} L_{p-i}H_{k-2i}(X^m) \oplus \left[\frac{z^m n}{m!}\right]_{m!},$$

where $N$ and $\oplus [\frac{z^m n}{m!}]_{m!}$ are defined in (7).

Let $X$ be a complex manifold of complex dimension $d$. Let $\Omega^k_X$ the sheaf of holomorphic $k$-form on $X$. The **Deligne complex of level $p$** is the complex of sheaves:

$$\mathbb{Z}_D(p) : 0 \to \mathbb{Z} \to \Omega^0_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{p-1}_X \to 0$$

The **Deligne-Beilinson cohomology** of $X$ in level $p$ we mean the hypercohomology of this complex:

$$H^*_D(X, \mathbb{Z}(p)) := H^*(X, \mathbb{Z}_D(p))$$

For Deligne-Beilinson cohomology $H^*_D(\cdot, \mathbb{Z}(p))$, we obtain the following result:

Theorem 1.5 Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers $p, k$, there is an isomorphism of Deligne-Beilinson cohomology groups:

$$H^k_D(X[n], \mathbb{Z}(p)) \cong \bigoplus_{S} \bigoplus_{\mu \in M_S} H^{k-2\|\mu\|}_D(X^{c(S)}, \mathbb{Z}(p - \|\mu\|)).$$

The main tools used to prove the main result are: The formula on the Lawson homology for a blowup proved in [1] and the method in computing the Chow groups of the Fulton-MacPherson configuration space in [3].

2 Some fundamental materials

2.1 Lawson homology

Recall that for a morphism $f : U \to V$ between projective varieties, there exist induced homomorphism $f_* : L_pH_k(U) \to L_pH_k(V)$ for all $k \geq 2p \geq 0$. Furthermore, it has been
shown by C. Peters [Pe] that if $U$ and $V$ are smooth and projective, there are Gysin “wrong way” homomorphism $f^*: L_pH_k(V) \to L_{p-c}H_{k-2c}(U)$, where $c = \dim(V) - \dim(U)$.

Let $X$ be a smooth projective variety and $i_0: Y \hookrightarrow X$ a smooth subvariety of codimension $r$. Let $\sigma: \tilde{X}_Y \to X$ be the blowup of $X$ along $Y$, $\pi: D = \sigma^{-1}(Y) \to Y$ the natural map, and $i: D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowing up. Set $U \equiv X - Y \cong \tilde{X}_Y - D$. Denote by $j_0$ the inclusion $U \subset X$ and $j$ the inclusion $U \subset \tilde{X}_Y$. Note that $\pi: D = \sigma^{-1}(Y) \to Y$ makes $D$ into a projective bundle of rank $r - 1$, given precisely by $D = P(N_{Y/X})$ and we have (cf. [V2], pg.271)

$$O_{\tilde{X}_Y}(D)|_D = O_{P(N_{Y/X})}(-1).$$

Denote by $h$ the class of $O_{P(N_{Y/X})}(-1)$ in Pic(D). We have $h = -D|_D$ and $-h = i^*i_*: L_qH_m(D) \to L_{q-1}H_{m-2}(D)$ for $0 \leq 2q \leq m$ ([FG, Theorem 2.4], [Pe, Lemma 11]). The last equality can be equivalently regarded as a Lefschetz operator

$$-h = i^*i_*: L_qH_m(D) \to L_{q-1}H_{m-2}(D), \quad 0 \leq 2q \leq m.$$

The proof of the main result are based on the following Theorems:

**Theorem 2.1 (Lawson homology for a blowup)** Let $X$ be smooth projective manifold and $Y \subset X$ be a smooth subvariety of codimension $r$. Let $\sigma: \tilde{X}_Y \to X$ be the blowup of $X$ along $Y$, $\pi: D = \sigma^{-1}(Y) \to Y$ the natural map, and $i: D = \sigma^{-1}(Y) \to \tilde{X}_Y$ the exceptional divisor of the blowing up. Then for each $p, k$ with $k \geq 2p \geq 0$, we have the following isomorphism

$$I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} L_{p-j}H_{k-2j}(Y) \right\} \oplus L_pH_k(X) \xrightarrow{\cong} L_pH_k(\tilde{X}_Y)$$

given by

$$I_{p,k}(u_1, \ldots, u_{r-1}, u) = \sum_{j=1}^{r-1} i_* (h^j \cdot \pi^*u_j) + \sigma^*u.$$

**2.2 The Fulton-MacPherson configuration spaces**

Fulton and MacPherson have constructed in [FuM] a compactification of the configuration space of $n$ distinct labeled points in a non-singular algebraic variety $X$. It is related to several areas of mathematics. In their original paper, Fulton and MacPherson use it to construct a differential graded algebra which is a model for $F(X, n)$ in the sense of Sullivan [FuM]. Axelrod-Singer constructed the compactification in the setting of smooth manifolds. $\mathbb{P}^1[n]$ is related to the Deligne-Mumford compactification $\overline{M}_{0,n}$ of the moduli space of nonsingular genus-0 projective curves.
Now we explain an explicit inductive construction of this compactification given in [FuM]. \(X[2]\) is the blowup of \(X^2\) along the diagonal \(\Delta_{12}\). \(X[3]\) is a sequence of blowups of \(X[2] \times X\) along non-singular subvarieties corresponding to \(\{\Delta_{123}; \Delta_{13}, \Delta_{23}\}\). More specifically, denote by \(\pi\) the blowup \(\pi: X[2] \times X \to X^3\), we blow up first along \(\pi^{-1}(\Delta_{123})\), then along the strict transforms of \(\Delta_{13}\) and \(\Delta_{23}\) (the two strict transforms are disjoint, so they can be blown up in any order). In general, \(X[n + 1]\) is a sequence of blowups of \(X[n] \times X\) along smooth subvarieties corresponding to all diagonals \(\Delta_I\) where \(|I| \geq 2\) and \((n + 1) \in I\).

Later, a symmetric construction of \(X[n]\) has been given by several people: De Concini and Procesi [DP], MacPherson and Procesi [MP], and Thurston [Th]. To construct \(X[n]\) we can blow up along diagonals by the order of ascending dimension, which is different from the non-symmetric order of the original construction. For example, \(X[4]\) is the blowup of \(X^4\) along diagonals corresponding to:

\[
1234; 123, 124, 134, 234; 12, 13, \ldots, 34.
\]

Compare it with the order in [FuM]:

\[
12; 123; 13, 23; 124, 134, 234; 14, 24, 34.
\]

It is proved in [Li] that, for any smooth projective variety \(X\) the following holds:

**Theorem 2.2 ([Li])** Let \(X\) be a smooth projective variety defined over \(\mathbb{C}\). Then for each \(p \geq 0\), there is an isomorphism of Chow groups:

\[
\text{Ch}^p(X[n]) \cong \bigoplus_S \bigoplus_{\mu \in M_S} \text{Ch}^{p-\|\mu\|}(X^{c(S)}).
\]

where \(S\) runs through all nests of \([n]\).

Notice that we use upper indices for the Chow groups in the above theorem. By changing variable \(\mu_I\) to \(d(c_I - 1) - \mu_I\), we get exactly Theorem 1.3 appeared in the introduction.

**Remark 2.1** The above theorem proved in [Li] holds for non-singular projective varieties \(X\) over any algebraic closed field.

Equivalently, but more explicitly, the Chow groups \(X[n]\) of can be calculated by using exponential generating functions. Here we adopt R. Stanley’s notation \([x^i]F(x)\) as the coefficient of \(x^i\) in the power series \(F(x)\), which is generalized in an obvious way to the following situation [St]:

\[
[x^i t^n n!]\sum_{j,q} a_{jq} x^j t^q q! = a_{in}.
\]
Corollary 2.1 If $h_i(x)$ are polynomials whose exponential generating function $N(x, t) = \sum_{i \geq 1} h_i(x) \frac{t^i}{i!}$ satisfies the identity

$$(1 - x)x^d + (1 - x^{d+1}) = \exp (x^d N) - x^{d+1}\exp (N),$$

then we have

$$Ch_p(X[n]) = \bigoplus_{1 \leq m \leq n} \bigoplus_{0 \leq i \leq p} Ch_{p-i}(X^m)^{\otimes [x^i m^p]} \Delta^m \otimes m'. $$

\[\square\]

2.3 Deligne-Beilinson cohomology

Let $X$ be a complex manifold of complex dimension $d$. Let $\Omega^k_X$ the sheaf of holomorphic $k$-form on $X$. The Deligne complex of level $p$ is the complex of sheaves

$$Z_D(p) : 0 \to \Omega^{(2\pi P)^p} X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{p-1}_X \to 0$$

The Deligne-Beilinson cohomology of $X$ in level $p$ we mean the hypercohomology of this complex:

$$H^*_D(X, \mathbb{Z}(p)) := H^*(X, \mathbb{Z}(p))$$

There is a multiplication of complexes

$$\nu : \mathbb{Z}(p)_D \otimes \mathbb{Z}(q)_D \to \mathbb{Z}(p + q)_D$$

defined as follows

$$\nu(x \bullet y) = \begin{cases} 
  x \cdot y, & \text{if } \deg x = 0 \\
  x \wedge dy, & \text{if } \deg x > 0 \text{ and } \deg y = q > 0 \\
  0, & \text{otherwise}
\end{cases}$$

This gives a product structure on the Deligne–Beilinson cohomology as follows

$$\cup : H^k_D(X, \mathbb{Z}(p)) \otimes \mathbb{Z} H^{k'}_D(X, \mathbb{Z}(q)) \to H^{k+k'}_D(X, \mathbb{Z}(p + q)).$$

(2)

For details, the reader is referred to [EV].

Let $X$ be an $d$-dimensional compact Kähler manifold. The Hodge filtration

$$\cdots \subseteq F^p H^k(X, \mathbb{C}) \subseteq F^{p-1} H^k(X, \mathbb{C}) \subseteq \cdots \subseteq F^0 H^k(X, \mathbb{C}) = H^k(X, \mathbb{C})$$
is defined by

\[ F^p H^k(X, \mathbb{C}) = \bigoplus_{i \geq p} H^{i,k-i}(X). \]

We denote by \( p^k_X \) the natural quotient map \( p^k_X : H^k(X, \mathbb{C}) \to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}). \)

It was proved (cf. [EV], Corollary 2.4; [V1], Proposition 12.26) that

\[
\cdots \to H^{k-1}(X, \mathbb{C})/F^p H^{k-1}(X, \mathbb{C}) \to H^k_D(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \to \cdots \tag{3}
\]

Now let \( X \) be an \( d \)-dimensional projective variety over \( \mathbb{C} \) and \( i_0 : Y \hookrightarrow X \) a smooth subvariety of codimension \( r \geq 2 \). Let \( \sigma : \tilde{X}_Y \to X \) be the blowup of \( X \) along \( Y \), \( \pi : D = \sigma^{-1}(Y) \to Y \) the natural map, and \( i : D = \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y \) the exceptional divisor of the blowup. Set \( U := X - Y \cong \tilde{X}_Y - D \). Denote by \( j_0 \) the inclusion \( U \subset X \) and \( j \) the inclusion \( U \subset \tilde{X}_Y \). Note that \( \pi : D = \sigma^{-1}(Y) \to Y \) makes \( D \) into a projective bundle of rank \( r - 1 \), given precisely by \( D = \mathbb{P}(N_{Y/X}) \) and we have (cf. [V2], pg. 271]

\[ \mathcal{O}_{\tilde{X}_Y}(D)|_D = \mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1). \]

Denote by \( h \) the class of \( \mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1) \) under the first Chern class \( c_1 : H^1(D, \mathcal{O}_D) \to H^2_D(D, \mathbb{Z}(1)) \) (cf. [EV, p. 88]).

The following proposition was proved in [EV].

**Proposition 2.1 ([EV], Prop. 8.5)** The Deligne-Beilinson cohomology \( H^k_D(D, \mathbb{C}(p)) \) of the projective bundle \( \pi : D \to Y \) is given by the following isomorphism:

\[
\bigoplus_{0 \leq j \leq r-1} H^{k-2j}_D(Y, \mathbb{C}(p-j)) \xrightarrow{\pi_*} H^k_D(D, \mathbb{Z}(p))
\]

**Remark 2.2** We omit the cup product of elements in \( H^{k-2j}_D(Y, \mathbb{Z}(p-j)) \) with \( h^j \).

Moreover, Barbieri-Viale proved the following blowup formula for Deligne-Beilinson cohomology:

**Theorem 2.3 ([Bv])** Let \( X, Y, D, \tilde{X}_Y, Y \) be as above. Then for each \( p, k \) with \( p \geq r \geq 0 \), we have the following isomorphism

\[
I_{p,k} : \left\{ \bigoplus_{1 \leq j \leq r-1} H^{k-2j}_D(Y, \mathbb{C}(p-j)) \right\} \oplus H^k(D, \mathbb{C}(p)) \xrightarrow{\sim} H^k_D(\tilde{X}_Y, \mathbb{Z}(p)). \tag{4}
\]

**Remark 2.3** Barbieri-Viale proved a general result, including the blowup formula for \( \acute{e} \text{tale} \) cohomology, to Theorem 2.3.
3 Lawson homology for Fulton-MacPherson configuration spaces

In this section, we give a proof of Theorem 1.4. According to the construction, the Fulton-MacPherson configuration space $X[n]$ is obtained by a sequence of blowups along all diagonals $\Delta_I$ in a suitable order. Each of them is a blowup of a smooth projective variety along a smooth projective subvariety. Therefore, we can calculate the Lawson homology groups of $X[n]$ by successively applying the blowup formula for Lawson homology (Theorem 2.1).

We have the following

Theorem 3.1 Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers $p, k$, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:

$$L_p H_k(X[n]) \cong \bigoplus_{S, \mu \in M_S} \bigoplus_{c(S)} \prod_{i=1}^{p-1} H_{k-2i}(|\mu|)(X^{c(S)}).$$

Proof. This follows essentially from the construction of the Fulton-MacPherson configuration space $X[n]$ and the blowup formula for Lawson homology groups. The detailed computation for explicit formulas will be given in the corollary below.

More explicitly, we have the following

Corollary 3.1 Let $X$ be a smooth projective variety defined over $\mathbb{C}$. Then for each pair of integers $p, k$, $k \geq 2p \geq 0$, there is an isomorphism of Lawson homology groups:

$$L_p H_k(X[n]) \cong \bigoplus_{1 \leq m \leq n} \bigoplus_{0 \leq i \leq p} L_{p-i} H_{k-2i}(X^m) \oplus \left[ \frac{x^n}{n!} \right] N^m_m.$$

where $N$ and $\oplus \left[ \frac{x^n}{n!} \right] N^m_m$ are the same as those in Corollary 2.1.

Proof. Let $h_n(x)$ be the polynomial

$$\sum_{\{S, \mu \in M_S \mid c(S) = 1} x^{||\mu||}. $$

Given a fixed nest $S$ with $n$ leaves and $c(S) = 1$, its contribution to $h_n(x)$ is the product of $\sigma_{c_I-1}$, where $I$ goes through all non-leaves of $S$ (if $S$ has no non-leaves, i.e., it contains only singletons, then the contribution is 1). Therefore we have the following recurrence formula

$$h_n(x) = \sum_{\{I_1, \ldots, I_k \mid \text{partition of } [n]} h_{|I_1|} h_{|I_2|} \cdots h_{|I_k|} \sigma_{k-1}. $$
where \( \sigma_k = \sum_{i=1}^{dk-1} x^i \) for \( k > 0 \), and \( \sigma_0 = 0. \)

By the Compositional Formula of exponential generating functions (cf. [St], Theorem 5.1.4), the generating function \( N(t) := \sum_{i \geq 1} h_i \frac{t^i}{i!} \) of \( h_n \) satisfies the identity
\[
N - t + 1 = E_g(N),
\]
where \( E_g(N) = 1 + \sum_{i > 0} \sigma_i - 1 N_i. \)

Since \( \sigma_j = (x^j - x)/(x - 1) \), calculation shows
\[
E_g(N) = 1 + N + \frac{1}{x - 1} \left[ \frac{1}{x^d} (e^{xdN-1} - 1) - xe^N + x \right].
\]

Put it in the above identity, we have
\[
(1 - x)x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1} \exp(N).
\]

For a partition \( \Pi = \{I_1, \ldots, I_k\} \) of \( [n] \), the number of times of \( h_\Delta(\Pi)(i) \) appear in the decomposition of \( h(X[n]) \) is equal to \( [x^k](h_{|I_1}(x) \cdots h_{|I_k}(x)) \). Add up this number for all partitions with \( k \) blocks, we will get the number of times of \( h(X^k)(i) \) appear in the decomposition, denoted by \( a_{k,i}. \)

Denote
\[
F_n(y) = \sum_{\text{partition of } [n]} h_{|I_1} h_{|I_2} \cdots h_{|I_k} y^k.
\]

Then the coefficient \( [y^k]F_n(y) = \sum a_{k,i} x^i. \) Use the Compositional Formula again,
\[
F_n = \left[ \frac{t^n}{n!} \right] \exp(yN).
\]

Therefore
\[
[y^k]F_n(y) = \left[ \frac{y^k}{n!} \right] \frac{t^n}{n!} \exp(yN) = \left[ \frac{y^k}{n!} \right] \frac{t^n}{n!} \exp(yN) = \left[ \frac{y^k}{n!} \right] \frac{t^n}{n!} \frac{n^k}{x^k}.
\]

Now the results follow from Theorem [St] \( \Box \).

Similarly, we compute the Deligne-Beilinson cohomology for Fulton-MacPherson configuration spaces.

**Theorem 3.2** Let \( X \) be a smooth projective variety defined over \( \mathbb{C} \). Then for each pair of integers \( p, k \), there is an isomorphism of Deligne-Beilinson cohomology groups:
\[
H^k_{\text{D}}(X[n], \mathbb{Z}(p)) \cong \bigoplus_{S} \bigoplus_{\mu \in \mathcal{M}_S} H^{k-2||\mu||}_{\text{D}}(X^{e(S)}, \mathbb{Z}(p - ||\mu||)).
\]
Proof. The method of the proof is the same as that in Theorem 3.1. We get the result by using the explicit construction of Fulton-MacPherson configuration spaces and Theorem 2.3.

Remark 3.1 By using the same method, we can compute the étale cohomology for Fulton-MacPherson configuration spaces.

Remark 3.2 The decomposition of Lawson homology (Theorem 3.1) and Deligne-Beilinson cohomology (Theorem 3.2) of the Fulton-MacPherson configuration spaces can be generalized without any difficulty to the wonderful compactifications of arrangements of subvarieties, since the latter compactifications can also be constructed by a sequence of blowups along smooth centers (for definition and construction of these compactifications, see [Li]).

4 Examples

1. The Lawson homology group of $X[2]$.

The morphism $\pi : X[2] \to X^2$ is a blowup along the diagonal $\Delta_{12}$. Corollary 3.1 asserts

$$L_pH_k(X[2]) \cong L_pH_k(X^2) \oplus \bigoplus_{j=1}^{d-1} L_{p-j}H_{k-2j}(X).$$

2. The Lawson homology group of $X[3]$.

Note that $X[3]$ is the blowup of $X^3$ first along small diagonal $\Delta_{123}$, then along three disjoint proper transforms of diagonals $\Delta_{12}$, $\Delta_{13}$ and $\Delta_{23}$.

Apply again Corollary 3.1 we have

$$L_pH_k(X[3]) \cong L_pH_k(X^3) \oplus \bigoplus_{j=1}^{d-1} \bigoplus_{i=1}^{2d-1} \left( L_{p-j}H_{k-2j}(X) \right)^{\oplus \min\{3i-2, 6d-3i-2\}}$$

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