Thick planar domain wall: its thin wall limit and dynamics

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Abstract

We consider a planar gravitating thick domain wall of the $\lambda\phi^4$ theory as a spacetime with finite thickness glued to two vacuum spacetimes on each side of it. Darmois junction conditions written on the boundaries of the thick wall with the embedding spacetimes reproduce the Israel junction condition across the wall in the limit of infinitesimal thickness. The thick planar domain wall located at a fixed position is then transformed to a new coordinate system in which its dynamics can be formulated. It is shown that the wall’s core expands as if it were a thin wall. The thickness in the new coordinates is not constant anymore and its time dependence is given.

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1 Introduction

Domain walls are solutions to the coupled Einstein-scalar field equations with a potential having a spontaneously broken discrete symmetry and a discrete set of degenerate minima. In the simplest case of two minima, a domain wall having a non-vanishing energy density appears in the separation layer, with the scalar field interpolating between these two values. Domain walls in the cosmological context have a long history [1]. It was realized very early that the formation of domain walls with a typical energy scale of $\geq 1\,\text{MeV}$ must be ruled out [2], because a network of such objects would dominate the energy of the universe, violating the observed isotropy and homogeneity. Domain walls were reconsidered in a possible late time phase transition scenario at the scale of $\leq 100\,\text{MeV}$. Such walls were supposed to be thick because of the low temperature of the phase transition [3]. The suggestion that Planck size topological defects could be regarded as triggers of inflation, revived the discussion of thin and thick domain walls [4]. The realization of our universe as a $(3 + 1)$-dimensional domain wall immersed in a higher dimensional spacetime has led to the recent numerous studies [5].

First attempts to investigate the gravitational properties of domain walls were based on the so-called thin wall approximation. In this approach one forgets about the underlying field theory and simply treats the domain wall as a zero thickness $(2+1)$-dimensional timelike hypersurface embedded in a four-dimensional spacetime. The Darmois-Israel thin wall formalism [6] is then used to match the solutions of the Einstein equations on both sides of the wall in the embedding spacetime across the thin wall. However, such spacetimes have delta function-like distributional curvature and energy-momentum tensor supported on the hypersurface. Using this procedure, the first vacuum solutions for a spacetime containing an infinitely thin planar domain wall was found by Vilenkin [7], and Ipser and Sikivie [8]. The very interesting feature of such domain walls is that they are not static, but have a de Sitter-like expansion in the wall’s plane. External observers experience a repulsion from the wall, and there is an event horizon at finite proper distance from the wall’s core. These results were initially obtained within the framework of the Darmois-Israel thin wall formalism which has been used as an approximative description of real wall of finite thickness. Typically, a self-gravitating domain wall has two length scales, its thickness $w$ and its distance to the event horizon which can be compared to $w$. Since these lengths are expressed in terms of the coupling constants of the theory, thin walls turn out to be an artificial construction in terms of these underlying parameters as mentioned in [9]. The first exact dynamical solution to thick planar domain walls was obtained by Goetz [10] and later a static solution was recovered with the price of sacrificing reflection symmetry [11]. Silveira [12] studied the dynamics of a spherical thick domain wall by appropriately defining an average radius $< R >$, and then used the well-known plane wall scalar field solution as the first approximation to derive a formula relating $< \dot{R} >$, $< \ddot{R} >$, and $< R >$ as the equation of motion for the thick wall. Widrow [13] used the Einstein-scalar equations for a static thick domain wall with planar symmetry. He then took the zero-thickness limit of his solution and showed that the orthogonal components of the energy-momentum tensor vanish in that limit. Garfinkle and Gregory [14] presented a modification of the Israel thin shell equations to treat the evolution of thick domain walls in vacuum. They used an expansion of the coupled Einstein-scalar field equations describing the thick gravitating wall in powers of the thickness of the domain wall around the well-known solution of the hyperbolic tangent kink for a $\lambda\phi^4$ wall and concluded that the effect of the thickness at the first approximation was to effectively reduce the energy
density of the wall compared to the thin case, leading to a faster collapse of a spherical wall as well as a slower expansion of a planer wall in vacuum. Barra`bes et al. [15] applied the expansion in the wall action and integrated it out perpendicular to the wall to show that the effective action for a thick domain wall in vacuum, apart from the usual Nambu term, consists of a contribution proportional to the induced Ricci curvature scalar. Arod´z et al. [16] applied a cubic polynomial ansatz for the scalar field inside the wall while the field gets the vacuum values outside the wall. Then the field equations and boundary conditions were used to determine the coefficients of polynomial. They obtained a Nambu-Goto type for the core of the wall together with a nonlinear equation for its thickness. Within the context of a fully nonlinear treatment of a scalar field coupled to gravity, Bonjour, Charmousis and Gregory (BCG) found an approximate but analytic description of the spacetime of a thick planar domain wall of the $\lambda\phi^4$ model by examining the field equations perturbed in a parameter characterizing the gravitation interaction of the scalar field [17], but they did not analyze the motion of the thick wall. The thin wall limit of Goetz's solution was studied in [18] and it was shown that this solution has a well-defined limit.

A completely different method based on the gluing of a thick wall, considered as a regular manifold, to two different manifolds on each side of it was developed by Khakshournia-Mansouri (KM) in [19] and recently in a more generalized context in [20]. In this paper, we will first investigate the thin wall limit of the thick BCG-like domain wall adapted to the KM formalism. We then introduce a coordinate transformation from the BCG solution to the one in which one can observe evolution of the thick wall.

The organization of this paper is as follows. In section 2 we give a brief introduction to BCG thick wall solution and summarize all the useful equations we need in this paper. Section 3 is devoted to the thick wall formalism developed in [19] and its application to the thick planar domain wall. The thin wall limit is then established in section 4. The dynamics of this thick domain wall in the new coordinates is studied in section 5. We then summarize our results in section 6.

2 Planar domain wall solution

Domain walls as regions of varying scalar field separating two vacua with different field values are usually described by the following matter Lagrangian:

$$L = \nabla_\mu \phi \nabla^\mu \phi - V(\phi),$$

where $\phi$ is a real scalar field and $V(\phi)$ is a symmetry breaking potential which we take it to be $V(\phi) = \lambda(\phi^2 - \eta^2)^2$, where $\lambda$ is a coupling constant and $\eta$ is the symmetry breaking scale. Looking for a static solution of the equation of motion derived from this Lagrangian in flat space-time, one gets

$$X = \tanh \frac{z}{w},$$

where $X = \frac{\phi}{\eta}$, $w = \frac{1}{\sqrt{\lambda\eta}}$, and $z$ is the coordinate normal to the wall. This particular solution represents an infinite planar domain wall centered at $z = 0$. From the stress-energy tensor of the wall, one can easily observe that the wall energy density is peaked around $z = 0$ and falls down effectively at $z = w$. We may therefore call $w$, being a characteristic length scale of the
domain wall, the effective thickness of the wall. We now look at the planar gravitating domain wall solutions. The line element of a plane symmetric spacetime may be written in the general form

\[ ds^2 = A^2(z)dt^2 - B^2(z,t)(dx^2 + dy^2) - dz^2, \] (3)

which displays reflection symmetry around the wall’s core assuming to be located at \( z=0 \), where \( z \) is the proper length along the geodesics orthogonal to the wall. In order to obtain a thick gravitating wall solution one should solve the coupled system of the Einstein and scalar field equations which are as follows

\[ R_{\mu\nu} = \epsilon \left( 2X_{,\mu} X_{,\nu} - \frac{1}{w^2} g_{\mu\nu} (X^2 - 1)^2 \right), \] (4)

\[ \Box X + \frac{2}{w^2} X(X^2 - 1) = 0, \] (5)

where \( R_{\mu\nu} \) is the Ricci tensor. For a static field \( X(z) \), the Einstein equations (4) leads to the constraint \( B(z,t) = A(z) \exp(kt) \).

BCG investigated the spacetime of a thick gravitating planar domain wall for a \( \lambda \phi^4 \) potential [17]. In this context, the dimensionless parameter \( \epsilon \) in front of the equation (4) can be taken to characterize the coupling of gravity to the scalar field:

\[ \epsilon \equiv 8\pi G\eta^2. \] (6)

Supposing that gravity is weakly coupled to the scalar field, \( A(z) \) and \( X(z) \) may be expanded in the powers of \( \epsilon \): 

\[ A_i(z) = A_0(z) + \epsilon A_1(z) + O(\epsilon^2), \] (7)

\[ X(z) = X_0(z) + \epsilon X_1(z) + O(\epsilon^2), \] (8)

where we have introduced the index \( i \) to indicate the quantities inside the wall.

In the limit \( \epsilon \to 0 \), one should obtain the same results as that of the non-gravitating planar wall, i.e. \( A_i(z) = 1 \) and \( X(z) = \tanh\left( \frac{z}{w} \right) \). Using these expansions, BCG solved the coupled Einstein and scalar field equations to first order in \( \epsilon \) and obtained the following results:

\[ A_i(z) = 1 - \frac{\epsilon}{3} \left[ 2 \ln(\cosh \frac{z}{w}) + \frac{1}{2} \tanh^2 \frac{z}{w} \right] + O(\epsilon^2), \] (9)

\[ k = \frac{2}{3} \epsilon \] (10)

\[ X(z) = \tanh \frac{z}{w} - \frac{\epsilon}{2} \text{sech}^2 \frac{z}{w} \left( \frac{z}{w} + \frac{1}{3} \tanh \frac{z}{w} \right) + O(\epsilon^2). \] (11)

Eqs. (9-11) represent the perturbative solution to the spacetime of the thick wall described by Eq. (3). In the presence of gravity, there is an event horizon at the proper distance \( z = \frac{1}{k} \) from the core of the wall where \( A_i(z) \) goes to zero and the coordinate system used in (3) breaks down as discussed by BCG.
3 The thick wall formalism

In this section we first make a short review of the thick wall formalism developed by KM in [19]. Then we apply it to the thick planar domain wall described by the metric (3). In KM formalism a thick wall is modelled with two boundaries $\Sigma_1$ and $\Sigma_2$ dividing a spacetime $\mathcal{M}$ into three regions. Two regions $\mathcal{M}_+$ and $\mathcal{M}_-$ on either side of the wall and region $\mathcal{M}_0$ within the wall itself. Treating the two surface boundaries $\Sigma_1$ and $\Sigma_2$ separating the manifold $\mathcal{M}_0$ from two distinct manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively, as nonsingular timelike hypersurfaces, we do expect the intrinsic metric $h_{\mu\nu}$ and extrinsic curvature tensor $K_{\mu\nu}$ of $\Sigma_j$ ($j=1,2$) to be continuous across the corresponding hypersurfaces. These requirements named the Darmois conditions are formulated as

$$[h_{ab}]_{\Sigma_j} = 0 \quad j = 1, 2,$$

(12)

$$[K_{ab}]_{\Sigma_j} = 0 \quad j = 1, 2,$$

(13)

where the square bracket denotes the jump of any quantity that is discontinuous across $\Sigma_j$, Latin indices range over the intrinsic coordinates of $\Sigma_j$, and Greek indices over the coordinates of the 4-manifolds. To apply the Darmois conditions on two surface boundaries of a given thick wall one needs to know the metrics in three distinct spacetimes $\mathcal{M}_+, \mathcal{M}_-$ and $\mathcal{M}_0$ being jointed at $\Sigma_j$. While the metrics in $\mathcal{M}_+$ and $\mathcal{M}_-$ are usually given in advance, knowing the metric within the wall spacetime $\mathcal{M}_0$ requires a nontrivial work.

Let us now impose these junction conditions for the self gravitating thick planar domain wall described in the previous section. Recalling $\epsilon \ll 1$, corresponding to weak self gravity, we first introduce a dimensionless parameter $\frac{1}{\epsilon} > \Delta \gg 1$ to assume the scalar field takes its vacuum values on the wall boundaries $\Sigma_1$ and $\Sigma_2$ being located at the proper distances $z = \pm \Delta w/2$ far from the wall’s core surface while the coordinate system in (3) is still valid. We may then think of $\Delta w$ as the proper thickness of the planar domain wall. Now, in the coordinate frame of the metric (3), in which the wall is stationary, the nonvanishing components of the intrinsic metric $h_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$ of $\Sigma_j$ take the following simple forms:

$$h_{\mu\nu} = g_{\mu\nu}, \quad \mu, \nu \neq z,$$

(14)

$$K_{\mu\nu} = -\frac{1}{2}g_{\mu\nu, z}.$$  

(15)

In order to find the spacetime metric on both sides of $\Sigma_j$, we first note that within the vacuum region $\mathcal{M}_+$ ($\mathcal{M}_-$) in which $\phi = \eta(-\eta)$, the spacetime metric can be easily determined by solving the Einstein equations (4) yielding

$$A_o(z) = -k_o |z| + C_o,$$

(16)

where the index $o$ denotes the quantities in the vacuum region, and $k_o$ and $C_o$ are the relevant integration constants. Using Eqs. (14) and (15) we write down the junction conditions (12) and (13) as

$$k = k_o,$$

(17)

$$A_i(z)|_{z=\Delta w/2} = A_o(z)|_{z=\Delta w/2},$$

(18)

$$\frac{\partial A_i(z)}{\partial z}|_{z=\Delta w/2} = \frac{\partial A_o(z)}{\partial z}|_{z=\Delta w/2}.$$  

(19)
We now use the solutions (9), (10) and (16) due to BCG for the wall metric in the region $M_0$, where the scalar field varies according to (11), and for the metric in the vacuum regions, respectively. Then the junction conditions (18) and (19) lead to the following constraints on the vacuum metric constants $C_o$ and $k_o$

$$C_o = 1 + k_o \frac{\Delta w}{2} - \frac{\epsilon}{3} \left(2 \ln (\cosh \frac{\Delta}{2}) + \frac{1}{2} \tanh^2 \frac{\Delta}{2}\right),$$ \hspace{1cm} (20)

$$k_o = \frac{\epsilon}{w} \left(\tanh \frac{\Delta}{2} - \frac{1}{3} \tanh^3 \frac{\Delta}{2}\right).$$ \hspace{1cm} (21)

Note that within the context of BCG work it is supposed that the boundaries of the wall, where the scalar field takes its vacuum values, are at infinity. In contrast, we have modelled the thick planar wall in such a way that the wall boundaries $\Sigma_j$ are situated at the finite proper distances $\pm \Delta w/2$ from the core of the wall. Hence, we choose $\Delta$ to be sufficiently large in order to simulate BCG solution within our thick wall model.

4 From the thick to thin domain walls

We now turn our attention to the thin wall limit of our thick wall model. First let us define the process of passing from a thick gravitating domain wall to a thin one by letting $\epsilon$ and $w$ go to zero while keeping their ratio $\frac{\epsilon}{w}$ fixed. This has the effect that the distance of the event horizon to the domain wall remains finite. The continuity of the extrinsic curvature tensor $K_{\mu\nu}$ across the thick wall boundary, say $\Sigma_1$, located at the proper distance $z = \Delta w/2$ as a consequence of the Darmois junction conditions (13) yields:

$$K^o_{\mu\nu}|_{z=\Delta w/2} = K^i_{\mu\nu}|_{z=\Delta w/2}. \hspace{1cm} (22)$$

Considering $(xx)$ component of the condition (22) and using (15) we can evaluate the right hand side of Eq. (22) for the BCG wall metric solution (9). This yields

$$K^{o}_{xx}|_{z=\Delta w/2} = \frac{\epsilon}{w} \left(\tanh \frac{\Delta}{2} - \frac{1}{3} \tanh^3 \frac{\Delta}{2}\right) \exp(2k_o t),$$ \hspace{1cm} (23)

where we have also used the condition (17). Imposing the above thin wall limit prescription, Eq. (23) reduces to

$$K^{o}_{xx}|_{z=0} = \frac{\epsilon}{w} \left(\tanh \frac{\Delta}{2} - \frac{1}{3} \tanh^3 \frac{\Delta}{2}\right) \exp(2k_o t).$$ \hspace{1cm} (24)

To identify the right hand side of the equation (24) we recall the definition of the surface energy density $\sigma$ of an infinitely thin wall. Within our thick wall model it takes the form

$$\sigma = \lim_{(w \to 0, \epsilon \to 0)} \int_{\Delta w/2}^{\Delta w/2} \rho dz,$$ \hspace{1cm} (25)

where $\rho = \rho(z)$ is the energy density of the scalar field determined by the BCG scalar field solution (11). Using Eq. (25) we obtained the following expression for $\sigma$:

$$\sigma = \frac{1}{2\pi G w} \frac{\epsilon}{w} \left(\tanh \frac{\Delta}{2} - \frac{1}{3} \tanh^3 \frac{\Delta}{2}\right),$$ \hspace{1cm} (26)
where we used the definition of $\epsilon$ given by (23). Comparing the results (24) and (26) one immediately obtains

$$K_{xx}^0|_{z=0} = 2\pi G \sigma \exp(2k_o t).$$

(27)

Now, this is just the ($xx$) component of the Israel junction condition $K_{\mu\nu}^0 = 2\pi G \sigma h_{\mu\nu}$ for a planar thin domain wall placed at $z = 0$. The Israel thin wall approximation treats the wall as a singular hypersurface with the surface energy $\sigma$ separating the two plane symmetric vacuum spacetimes $M_+$ and $M_-$ from each other. We see, therefore, within our thick wall formulation, in the thin wall limit the Darmois junction conditions for the extrinsic curvature tensor at the wall boundaries generate the well-known Israel jump condition. In the process of passing from a thick planar domain wall to the thin one all the information about the internal structure of the wall is squeezed in the parameter $\sigma$ characterizing the wall surface energy density according to (26).

5 Evolution of the thick planar domain wall

In the coordinate system (3) used to describe the thick wall, the wall is locally static and its core is situated at the fixed position $z = 0$. To study the evolution of the thick wall we will look for a reference frame in which the wall is moving. In order to do this, let us introduce a coordinate system $(v, r, X, Y)$ in which the plane symmetric metric (3) with the solutions (9) and (10) for $A$ and $k$, respectively is transformed to the form

$$ds^2 = f(z, t)dv^2 + 2H(z, t)dvdv - r^2(dX^2 + dY^2),$$

(28)

by applying the following appropriate coordinate transformations:

$$X = kx,$$

$$Y = ky,$$

$$r = \frac{A(z)}{k}e^{kt},$$

$$v = v(z, t),$$

(29)

where $v(z, t)$ is an unknown function to be determined. Identification of the metrics (3) and (28) leads to

$$f(z, t) \left( \frac{\partial v}{\partial t} \right)^2 + 2H(z, t) \frac{\partial v}{\partial t} \frac{\partial r}{\partial t} = A^2(z),$$

$$f(z, t) \left( \frac{\partial v}{\partial z} \right)^2 + 2H(z, t) \frac{\partial v}{\partial z} \frac{\partial r}{\partial z} = -1,$$

$$f(z, t) \frac{\partial v}{\partial t} \frac{\partial v}{\partial t} + 2H(z, t) \left( \frac{\partial v}{\partial t} \frac{\partial r}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial r}{\partial t} \right) = 0.$$ 

(30)

Solving the above equations we find

$$f(z, t) = \frac{H^2 \left( A^2 \left( \frac{\partial v}{\partial t} \right)^2 - \left( \frac{\partial v}{\partial z} \right)^2 \right)}{A^2(z)},$$

(31)
\[ \frac{\partial v}{\partial t} = A^2 \left( \frac{s_1 A \frac{\partial r}{\partial z} - \frac{\partial r}{\partial t}}{A^2 \left( \frac{\partial r}{\partial z} \right)^2 - \left( \frac{\partial r}{\partial t} \right)^2} \right), \]

\[ \frac{\partial v}{\partial z} = -A \frac{\partial r}{\partial z} \left( \frac{s_2}{A^2 \left( \frac{\partial r}{\partial z} \right)^2 - \left( \frac{\partial r}{\partial t} \right)^2} \right), \]

where the sign parameters \( s_1 \) and \( s_2 \) can be taken independently to be \( \pm 1 \). Focusing our attention on the \( z > 0 \) side of the wall, we infer that the continuity of the extrinsic curvature on the wall boundary placed at \( z = \Delta w/2 \), however, requires \( s_1 = -1 \) and \( s_2 = -1 \).

In addition, for the coordinate transformations to be integrable we require:

\[ \frac{\partial^2 v}{\partial t \partial z} = \frac{\partial^2 v}{\partial z \partial t}. \]  

Imposing the condition (33) on the solutions (32) we arrive at the following:

\[ \frac{1}{H(z,t)} \frac{\partial H(z,t)}{\partial t} - \frac{A(z)}{H(z,t)} \frac{\partial H(z,t)}{\partial z} = g(z), \]

\[ g(z) \equiv -A'(z) - k + \frac{A(z)A''(z)}{A'(z) - k}. \]

We have now all the prerequisites to find the unknown functions defining the transformation and to understand the dynamics of the thick wall. Assuming a factorizable solution in the form \( H(z,t) = h_z(z)h_t(t) \), then Eq. (34) leads to the two following equations:

\[ \frac{1}{h_t(t)} \frac{\partial h_t(t)}{\partial t} = C, \]

\[ \frac{A(z)}{h_z(z)} \frac{\partial h_z(z)}{\partial z} + g(z) = C, \]

where \( C \) is an arbitrary constant. Integrating Eqs. (36) and (37) over \( t \) and \( z \), respectively, we find

\[ H(z,t) = Dh_z(z) \exp(Ct), \]

with

\[ h_z(z) = \frac{2e^{(C+k)z - \frac{k}{2} \tanh^2 \frac{z}{w}}}{(2 - \tanh \frac{z}{w})(1 + \tanh \frac{z}{w})^2 (\cosh \frac{z}{w})^2}. \]

where \( D \) is an integration constant. The transformation (29) is now fully determined, except for the explicit functional dependence of \( v \) which will be considered later. The wall in the new coordinates is defined by \( R(\tau) = r(\tau, \tau) \), with \( \tau \) being its proper time as measured by observers at rest on the \( z = \text{const} \) surface. It is then easily seen that dynamics of the thick wall in the new coordinates is given by

\[ R(\tau) = \frac{A(z)}{k} \left. \frac{e^{\frac{kr}{A(z)}}}{z = \text{const}} \right|_{z = \text{const}}. \]
This dynamical equation takes a simple form for the core of the wall defined by \( R(\tau_0) = \tau(t(\tau_0), z = 0) \). In this case we have
\[
R(\tau_0) = \frac{1}{k} e^{k\tau_0}.
\]
(41)

Comparing Eq. (41) with the thin wall dynamics of \([8]\), it is obvious that the core of our thick wall evolves as if it were a thin wall with the effective surface tension \( \tilde{\sigma} = \frac{k}{2\pi G} \). In addition, taking into account Eqs. (17), (21), and (26) it is seen that in the thin wall limit Eq. (41) is reduced to the Ipser-Sikivie solution for the evolution of a thin planar wall \([8]\).

We now intend to express the equation of motion (41) as a function of the new coordinate \( v \).

Note that from the 3-metric \( h_{ab} \) induced on each \( z = \text{const} \) surface of the wall we have:
\[
f(z, \tau) \dot{v}^2(z, \tau) + 2 \dot{R}(\tau)H(z, \tau)\dot{v}(z, \tau)\big|_{z=\text{const}} = 1,
\]
(42)

where dot denotes the derivative with respect to \( \tau \). Solving this for \( \dot{v} \) gives
\[
\dot{v} = -\dot{R}H + \sqrt{\dot{R}^2H^2 + f}\big|_{z=\text{const}},
\]
(43)

where we have made the sign choice to make sure of having a finite solution on each \( z = \text{const} \) surface specifically on the wall boundaries where the continuity of the extrinsic curvature requires \( f \) given by the expression (31) to be zero. To proceed further, we focus our attention on the core of the thick wall and first write down explicitly the solution (43) there
\[
\dot{v} = \frac{e^{-(C+k)\tau_0}}{D},
\]
(44)

where we have used Eqs. (31), (33), (39), and (41). Now, requiring that \( \tau_0 \) and \( v \) point to the same direction, i.e. \( \dot{v} > 0 \), leads to the constraint \( D > 0 \). Integrating this equation over \( \tau_0 \) we get the following results
\[
v(\tau_0) = \frac{1}{D(C + k)} \left( 1 - e^{-(C+k)\tau_0} \right), \quad C > -k,
\]
(45)
\[
v(\tau_0) = \frac{1}{D|C + k|} \left( e^{|C+k|\tau_0} - 1 \right), \quad C < -k,
\]
(46)

where we have put \( v = 0 \) at \( \tau_0 = 0 \). Inserting the solutions (45) and (46) into Eq. (41) yields the functional dependence of the equation of motion of the wall’s core on the time coordinate \( v \) as follows:
\[
R(v) = \frac{1}{k} \left( 1 - D(C + k)v \right)^{\frac{k}{C+k}}, \quad C > -k,
\]
(47)
\[
R(v) = \frac{1}{k} \left( 1 + D|C + k|v \right)^{\frac{k}{|C+k|}}, \quad C < -k.
\]
(48)

It follows that the core of the wall expands when looked upon from the bulk frame \( v \) regardless of whether \( C > k \) or \( C < k \).
To learn more about the evolution of the wall, we write down the derivatives of Eqs. (47) and (48) with respect to the coordinate time $v$

\[
\frac{dR}{dv} = D \left( \frac{1}{1 - D(C + k)v} \right)^{\frac{2k + C}{C + k}}, \quad C > -k, \quad (49)
\]

\[
\frac{dR}{dv} = D (1 + D|C + k|v)^{-\frac{k}{|C + k| - 1}}, \quad C < -k. \quad (50)
\]

Consequently, from Eqs. (49) and (50) we see that the expansion of the wall’s core slows down provided $C < -2k$, otherwise it speeds up.

From the coordinate transformation we have chosen to investigate the dynamics of the wall, it is clear that the wall thickness is no longer a constant. To see the time dependence of the thickness we first note that the spacetime parts ($z > 0$) and ($z < 0$) are transformed separately to spheres. Therefore, it is more appropriate to look at the half-thickness ($w^*$) defined by

\[
w^* = R(v)|_{z=0} - R(v)|_{z=\frac{\Delta w}{2}}. \quad (51)
\]

Following the above procedure leading to the equations of motion for the core of the wall one can find the corresponding equation for the boundary of the wall located at $z = \frac{\Delta w}{2}$. We get

\[
R(v)|_{z=\frac{\Delta w}{2}} = A \left( 1 - \frac{2D(C + k)h_z(z)}{A}(v - v_0) \right)^{-\frac{k}{C + k}} |_{z=\frac{\Delta w}{2}}, \quad (52)
\]

where we have considered the case $C > -k$, $v_0$ denotes the initial value of $v$ when the proper time on the boundary of the wall is taken to be zero. Plugging Eqs. (47) and (52) into the definition (51) we end up with

\[
w^* = \frac{1}{k} (1 - D(C + k)v)^{-\frac{k}{C + k}} - A \left( 1 - \frac{2D(C + k)h_z(z)}{A}(v - v_0) \right)^{-\frac{k}{C + k}} |_{z=\frac{\Delta w}{2}}, \quad (53)
\]

where $A(z)$ and $h_z(z)$ are given by the solutions (9) and (39), respectively. This gives the time dependence of the half-width of the wall in terms of the new coordinate $v$. Noting that $\frac{\partial r}{\partial z} < 0$, we can see that the half-width $w^*$ is an increasing function of $v$.

6 Conclusion

We have studied the thick planar domain wall as a spacetime having two boundaries at the same proper distance from the wall’s core, as first formulated in [19]. It has then been shown that in the thin wall limit the Darmois junction conditions generate the familiar Israel’s jump condition at the boundary of the corresponding thin wall with the embedding spacetimes. It is realized that in the limiting process to the thick planar domain wall all the information about the internal structure of the wall is squeezed in the surface energy density $\sigma$ introduced in (26).

We have also given explicitly a coordinate transformation from the plane symmetric metric (3) describing a thick planar domain wall at rest to a reference frame in which the thick wall evolves.
Starting from an ansatz for the transformed metric of the form \[28\] this aim was achieved by analytically solving the transformation equations. It is then seen that the core of the thick wall is moving with respect to the transformed reference frame as if it were a thin wall. In contrast, according to Eq. \[40\] the other layers of the wall evolve differently. We have also studied the thin wall limit of the moving thick wall and showed that in this limit the thick wall’s solution becomes the well-known thin wall solution of \[8\]. In this dynamical picture, the wall thickness shows a time variation, which has been given explicitly.

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