PRIVATE DISTRIBUTED MEAN ESTIMATION

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ABSTRACT

Ever since its proposal, differential privacy has become the golden standard for rigorous privacy protection. Output perturbation is the most widely used differentially private mechanism. It works by adding calibrated noise drawn from the real domain to the output. However, the finite computers can only represent real numbers to a given machine precision. To guarantee the strictness of differential privacy under fixed precision, several pioneering works (Dwork et al. (2006); Ghosh et al. (2012); Balcer & Vadhan (2017); Agarwal et al. (2018); Canonne et al. (2020)) proposed to use noise drawn from discrete for differential privacy. However, discrete differentially private mechanisms are not used in dense tasks such as deep learning because of the lack of tight composition theorem like moments accountant. In this paper, we prove Rényi differential privacy for discrete Gaussian mechanism. Thus, we can directly use analytical moments accountant (Wang et al. (2018)) to provide tight composition for distributed mean estimation.

1 INTRODUCTION

Federated learning (McMahan et al. (2016); Konečný et al. (2016a;b); Kairouz et al. (2019)) has drawn numerous attention in the past few years as an emerging distributed learning paradigm. In federated learning, the goal is to train a high-quality centralized model when the training data remains distributed across many clients. The communication cost is of great significance in federated learning because the clients, such as mobile phones, might have slow and unstable network connections. On the other hand, it is also of utmost importance to protect the privacy of the clients’ data.

There have been a line of researches (Geyer et al. (2017); Agarwal et al. (2018); Truex et al. (2019)) focusing on enhancing privacy guarantee of the federated learning systems. Among these works, we identify cpSGD (Agarwal et al. (2018)) as most comparable to our work because it also considers both communication cost and privacy guarantee. Concretely, cpSGD uses Binomial noise (Dwork et al. (2006)) to provide differential privacy for distributed mean estimation and leverages random rotation to maintain low communication cost. However, as pointed out in Canonne et al. (2020), the privacy protection provided by Binomial mechanism is quite limited because shifted Binomial distributions have different supports. As a result, it falls into the worst-case pitfall (Mironov (2017)) that completely exposes the secret with a small probability. As an alternative, Canonne et al. (2020) proposed to use discrete Gaussian for differential privacy and provided concentrated differential privacy proof for it.

In this paper, we manage to apply discrete Gaussian mechanism in the federated learning setting and further reduce the communication cost. Specifically, we make the following contributions.

• We provide Rényi differential privacy analysis for discrete Gaussian mechanism, a stronger privacy guarantee than Binomial mechanism.

• With analytical moments accountant for Rényi differential privacy (Wang et al. (2018)), we manage to further improve the total communication cost in distributed mean estimation under the same privacy guarantee.

• We reconcile discrete Gaussian mechanism with secure aggregation scheme by mapping the values to its quotient groups.
2 DISCRETE GAUSSIAN MECHANISM

In this section, we introduce discrete Gaussian mechanism and provide the Rényi differential privacy analysis.

Discrete Gaussian mechanism works by adding noise drawn from calibrated discrete Gaussian distribution (Definition 1).

**Definition 1 (Discrete Gaussian Distribution).** Discrete Gaussian is a probability distribution on a discrete additive subgroup \(L\) parameterized by \(\sigma\). For a discrete Gaussian distribution \(N_L(\sigma)\) and \(x \in L\), the probability mass on \(x\) is proportional to \(e^{-\pi x^2/(2\sigma^2)}\).

Variables following discrete Gaussian mechanism have several useful features in distributed mean estimation. For instance, the sum of two discrete Gaussian variables still follows discrete Gaussian distribution (Lemma 2). The probability mass on \(x\) obeys \(\alpha\)-RDP for all \(\alpha, \epsilon\) and \(\delta\).

**Lemma 1 (Sum of discrete Gaussian variables).** Let \(X \sim N_L(\sigma_x)\) and \(Y \sim N_L(\sigma_y)\). If a third variable \(Z\) is generated by the sum of \(X\) and \(Y\): \(Z = X + Y\), then \(Z \sim N_L(\sqrt{\sigma^2_x + \sigma^2_y})\).

**Lemma 2 (Dilation of discrete Gaussian variables).** Let \(X \sim N_L(\sigma)\). If \(Z = X/\alpha\), then \(Z \sim N_L(\sigma/\alpha)\).

**Canonne et al. (2020)** prove concentrated differential privacy for discrete Gaussian mechanism. However, there lacks tight composition theorem like moments accountant for concentrated differential privacy. Thus, we provide Rényi differential privacy analysis for discrete Gaussian mechanism and use analytical moments accountant (Wang et al. (2018)) for tight composition.

**Theorem 1 (Rényi differential privacy for discrete Gaussian mechanism).** If \(f\) has sensitivity 1, then the discrete Gaussian mechanism: \(f() + N_L(\sigma)\) satisfies \((\alpha, \alpha/(2\sigma^2))-\text{RDP}\).

**Proof.**

\[
D_\alpha(N(0, \sigma^2) || N(\mu, \sigma^2)) = \frac{1}{\alpha - 1} \log \sum_L \exp(-\frac{(x - \mu)^2}{2\sigma^2}) \cdot \exp(-\frac{\alpha x^2}{2\sigma^2}) \cdot \exp((-1 - \alpha)(x - \mu)^2/(2\sigma^2))
\]

\[
= \frac{1}{\alpha - 1} \log \sum_L \exp(-\frac{(x - \mu)^2}{2\sigma^2}) \sum_L \exp((\alpha^2 - \alpha)\mu x - (1 - \alpha)\mu^2)/(2\sigma^2))
\]

\[
= \frac{1}{\alpha - 1} \log \{\exp((\alpha^2 - \alpha)\mu^2/(2\sigma^2))\}
\]

\[
= \alpha \mu^2/(2\sigma^2)
\]

**Lemma 3 (Analytical moments accountant for Discrete Gaussian mechanism (Wang et al. (2018))).** Given a dataset of \(n\) points drawn from a domain \(X\) and a discrete Gaussian mechanism \(\mathcal{M}\) that takes an input from \(X^m\) for \(m \leq n\), let \(\mathcal{M} \circ \text{subsample}\) be the composition of subsample and \(\mathcal{M}\). For all integers \(\alpha > 2\), if \(\mathcal{M}\) obeys \((\alpha, \epsilon(\alpha))-\text{RDP}\), then the subsampled discrete Gaussian mechanism \(\mathcal{M} \circ \text{subsample}\) obeys \((\alpha, \epsilon'(\alpha))-\text{RDP}\) where,

\[
\epsilon'(\alpha) \leq \frac{1}{\alpha - 1} \log(1 + \gamma^2 \frac{\alpha}{2}) \min\{4e^{\epsilon(2)} - 1, 2e^{\epsilon(2)}\} + \sum_{j=3}^{\alpha} 2\gamma^{j-1}(\frac{\alpha}{j})e^{(j-1)\epsilon(j)}.
\]

**Lemma 4 (Adaptive composition of RDP (Mironov (2017))).** If (randomized) mechanism \(\mathcal{M}_1\) obeys \((\alpha, \epsilon_1)-\text{RDP}\) and \(\mathcal{M}_2\) obeys \((\alpha, \epsilon_2)-\text{RDP}\) then their composition obeys \((\alpha, \epsilon_1 + \epsilon_2)-\text{RDP}\).

**Lemma 5 (RDP-DP conversion (Mironov (2017))).** If \(\mathcal{M}\) obeys \((\alpha, \epsilon)-\text{RDP}\), then \(\mathcal{M}\) obeys \((\epsilon + \log(1/\delta)/(\alpha - 1), \delta)-\text{DP}\) for all \(0 < \delta < 1\).
Secure aggregation is another challenge when applying discrete Gaussian in federated learning setting. Because discrete Gaussian contains infinite possible values, it is impossible to encode the noise within finite coding length. One candidate solution is to use variable length encoding (McMahan et al. (2016)). However, it is unclear how to reconcile secure aggregation (Bonawitz et al. (2017)) with variable length encoding schemes. To address the challenge, we propose to project the involved values into the quotient group. According to the post-processing theorem of RDP (Mironov (2017)), the result should still follow rigorous Rényi differential privacy.

**Algorithm 1**: Distributed mean estimation with discrete Gaussian and secure aggregation.

**Input**: $\mathbb{L}$, $\sigma$, $\mathbb{L}_q$: quotient group of $\mathbb{L}$, $\phi$: mapping from $\mathbb{L}$ to $\mathbb{L}_q$

1. for each client $i \in [m]$ do
2.   for each client $j \in [m]$ and $j \neq i$ do
3.     $\mu_{ij} \sim \text{Unif}(\mathbb{L}_q)$
4.     Send $\mu_{ij}$ to $j$
5.   for each client $i \in [m]$ do
6.     $\nu_i \sim N_{\mathbb{L}_q}(\sigma)$
7.     $\hat{g}_i = \phi(g_i + \nu_i + \sum_{j \neq i} \mu_{ij} - \sum_{j \neq i} \mu_{ji})$
8.     Send $\hat{g}_i$ to the server
9. Server computes $\sum_{i \in [m]} \hat{g}_i$

**Theorem 2** (RDP for Algorithm 1). Algorithm 1 obeys the same RDP guarantee as vanilla global discrete Gaussian mechanism with the same parameters.

**Proof Sketch.** According to modular arithmetic,

\[
\sum_{i \in [m]} \hat{g}_i = \phi\left(\sum_{i \in [m]} \phi(g_i + \nu_i + \sum_{j \neq i} \mu_{ij} - \sum_{j \neq i} \mu_{ji})\right)
\]

\[
= \phi\left(\sum_{i \in [m]} (g_i + \nu_i + \sum_{j \neq i} \mu_{ij} - \sum_{j \neq i} \mu_{ji})\right)
\]

\[
= \phi\left(\sum_{i \in [m]} (g_i + \nu_i)\right)
\]

Algorithm 1 is a simplified version of the full secure aggregation protocol (Bonawitz et al. (2017)) and we only use it to clarify the idea behind the reconciliation. Indeed, discrete Gaussian can be used with the complete version of secure aggregation using the same trick.

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