Quantum corrections to the Lorentz algebra due to mixed gravitational-$U(1)$-chiral anomalies.

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Abstract

We calculate the quantum corrections to the Lorentz algebra for chiral Weyl fermions interacting with an external $U(1)$ gauge field in a background Riemann-Cartan (RC) spacetime. This was achieved by setting up the equal-time commutation relations (ETCR) for the canonical spin current of chiral Weyl fermions. Furthermore, these quantum corrections lead to an order sensitive commutator, i.e., swapping the Lorentz generators in the commutator doesn’t merely lead to a sign change, but rather a completely different correction term to the Lorentz algebra. Thus, the algebra of Lorentz is altered due to anomalies associated with chiral particles.
NOTATIONS

Spacetime coordinates will be labeled with Latin indices \( i, j, k \ldots = 0, 1, 2, 3 \). Spatial coordinates will be denoted by \( a, b, \ldots = 1, 2, 3 \). The frame (tetrad) fields are denoted as \( e_\alpha \) with components \( e^i_\alpha \), where \( \alpha, \beta, \ldots = 0, 1, 2, 3 \) are the Lorentz indices. The coframe field is denoted as \( \vartheta^\beta \) with components (vierbeins) \( e^i_\beta \). \( \omega_\alpha = \ast \vartheta_\alpha \), \( \omega_{\alpha\beta} = \ast (\vartheta_\alpha \wedge \vartheta_\beta) \), \( \omega_{\alpha\beta\gamma} = \ast (\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma) \), \( e := \text{det}(e^j_\beta) = \sqrt{-g} \). Parentheses around the indices denote antisymmetrization \([ij] := \frac{1}{2}(ij - ji)\). The (Fock-Ivanenko) covariant exterior derivative components for spinors are \( D_\alpha = e^i_\alpha D_i \) with \( D_i = \partial_i + i\Gamma^\alpha_i \Sigma_{\alpha\beta} \), where \( \Gamma^\alpha_i \Sigma_{\alpha\beta} \) is the Lorentz (spin) connection and \( \Sigma_{\alpha\beta} \) are the representations of the Lorentz generators. The metric field will be denoted as \( g_{ij}(x) \) and the Minkowski metric as \( \eta_{ij} \).
I. INTRODUCTION

The Einstein-Cartan (EC) theory is a gauge theory of gravity, obtained by gauging the Poincaré group. The Poincaré group is a 10 parameter non-Abelian and non-compact Lie group which is the semi-direct product of 4 translations $T_4$ and 6 Lorentz transformations $SO(1,3)$, i.e., $P(1,3) = T_4 \rtimes SO(1,3)$. This approach is built over the concept of Wigner classification, where quantum particles are indexed by its invariants mass and spin which are linked to the set of translations and Lorentz transformations respectively. The gauge connection one-forms (potentials) are the spin connection $\Gamma^{\alpha\beta}(x)$ linked to the Lorentz group and the orthonormal coframe field (tetrads) $\vartheta^\alpha(x)$ linked to the translations. The translational field strength two-form $T^\alpha(x)$ is called torsion while the rotational field strength two-form $R^{\alpha\beta}(x)$ is called Lorentz curvature. Thus the underlying arena of the Einstein-Cartan theory is called a Riemann-Cartan (RC) spacetime. The matter current three-forms associated with these gauge potentials are the canonical energy-momentum current $\mathcal{T}_\alpha(x)$, which are the currents coupled to $\vartheta^\alpha(x)$ and sources the underlying spacetime curvature while the canonical spin current $\mathcal{S}_{\alpha\beta}(x)$ are currents coupled to the spin (Lorentz) connection and are sources of spacetime torsion [1]. In this paper, we concentrate specifically on the canonical spin current for spin-$\frac{1}{2}$ spinorial fields which are surprisingly dual to the chiral (axial) vector currents $J^i_\alpha(x)$.

In spinor electrodynamics, quantum anomalous non-conservation of the chiral (axial) vector current even for massless fermions leads to a topological term. This was firstly explained in the context of PCAC problem of a neutral meson decaying into two photons $\pi^0 \rightarrow 2\gamma$. This is the celebrated Adler-Bell–Jackiw (ABJ) anomaly or chiral anomaly problem [2].$^1$ In particle physics, the intrinsic angular momentum or spin plays a pivotal role. In [3] it was already claimed that the conservation law for spin current of hadrons would satisfy a PCAC (partially conserved axial vector current) like equation. It was also shown that the tensorial part of the spin current could source massive spin-2 mesons. Hence, we take this study further to set up an analogous PCAC and chiral anomaly like expression for the spin tensor.

$^1$ Definitely, there exists chiral anomalies for non-Abelian gauge fields too.
Recently, a major breakthrough happened in condensed matter systems, where the chiral anomaly analogue was experimentally confirmed in new topological phases of matter, namely the \textit{Weyl-semimetal}. Electrons in these material behave like a Weyl fermion which are massless relativistic particles. A Weyl semimetal carries non-trivial topological properties and the Weyl fermions with opposite chiralities are separated in momentum space and host a monopole and an antimonopole of Berry flux in momentum space respectively. In this situation, parallel magnetic and electric fields ($\vec{E} \cdot \vec{B}$) can pump electrons between Weyl cones of opposite chirality that are separated in momentum space. This process violates the chiral charge conservation and the number of particles of left and right chirality are not separately conserved. This arises because of the topological term ($\frac{1}{2} \varepsilon_{ijkl} F^{ij}(x) F^{kl}(x) \approx \vec{E} \cdot \vec{B}$)[4].

Recently, there have been attempts in the direction of mixed gravitational-axial anomalies in Weyl-semimetals [5]. But what is important to note is that mixed gravitational-axial anomalies are purely based on General relativity (GR). GR is built over the concept of a symmetric metric field $g_{ij}(x)$ whose second-derivatives lead to the Riemann curvature tensor $R_{ijkl}(x)$. This doesn’t offer a way to rather probe the interaction of spin of chiral particles with gravitational fields, which is only possible in the context of EC gravity.

Ideally, there have not been enough concrete experimental proofs of spin current arising from a gauge perspective. Although, one should not forget that there have been some experimental signatures related to the anomalous phases of $^3$He in the A phase at low temperatures having a net macroscopic spin. This also points strongly towards the existence of an asymmetric energy-momentum tensor. Also, the EC length scale $l_{EC} \approx (\lambda_{Compton} l_{Planck}^2)^{\frac{1}{3}} \approx 10^{-29}m$ is seven orders of magnitude larger than the reduced Planck scale $l_{Planck} \approx 10^{-36}m$. PLANCK data indicate that General Relativity can be verified to scales of $\approx 10^{-28} m$ [6]. Thus, signatures of EC gravity may well be detected in the near future.

Since, we have some hopes of spacetime being RC and also have access to such topological materials to look into anomalies associated with chiral Weyl fermions experimentally, we take up an interesting problem which deals with finding quantum non-conservation laws for the canonical spin current of fermions.
A. Motivation

The question that we want to address in this paper is if the conservation of spin angular momentum holds in the quantum level. If not, then the “anomalies” arising in QFT, would lead to the breakdown of Lorentz symmetry in the quantum regime. This would imply that the algebra of Lorentz suffers from anomalies due to quantum effects.

It is quite surprising that the canonical spin current (sources of torsion) of spinorial fields are related to the chiral (axial) currents. Hence, it is motivating enough to look for gravitational analogues of the chiral anomaly. This phenomena is not merely calculating the chiral anomaly in a curved background [7, 8], but rather a completely different anomaly where we compute the divergence of the spin current in a background Riemann-Cartan (RC) spacetime (curved + contorted) or interacting with a combination of external Abelian $U(1)$ gauge fields in a RC background.\textsuperscript{2}

Infact, such an anomaly has already been calculated in [9, 10] and is also famously known as the “Lorentz-Anomaly”. Thus computing these anomalies could always be useful since experimental tests of these anomalies could be possible using the topologically non-trivial “Weyl semimetal”.

It was shown in [9] that, purely gravitational anomalies do not exist in $4n$ dimensions as a consequence of charge conjugation properties of the gravitational interactions. On the contrary they are present only in $4n + 2$ dimensions [9, 11, 12]. Infact, the existence of such chiral Lorentz anomalies leads to a breakdown of Lorentz symmetry in $4n+ 2$ dimensions due to quantum effects.

Thus our main motivation is to calculate using the results obtained in [10], the quantum corrections to the algebra of Lorentz present for chiral particles interacting with a combined gravitational and $U(1)$ gauge field. Thus, by studying chiral gauge theories in curved space-time, one can understand the interactions between intrinsic angular momentum of fermions coupled to the underlying geometry of spacetime.

\textsuperscript{2} It would be further clarified why non-Abelian fields were not mentioned here.
II. PLAN OF THE PAPER

We firstly describe the canonical spin current of Dirac fields and its irreducible decomposition into different components. It turns out to be that only the totally *antisymmetric axial vector* part survives, i.e., \( AX \mathcal{S}_{\alpha \beta}^i(x) = \frac{i\hbar e}{8} \epsilon^{\gamma \delta} \epsilon_{\alpha \beta \gamma \delta} J_5^i(x) \). Then in subsection (III.C), we revisit the ETCR/ spin current algebra. Later, we briefly go through the similarities between the ABJ anomaly and the Lorentz anomaly by considering, (i) a purely gravitational (RC) background, (ii) spin of chiral Weyl fermions interacting with a combination of an external \( U(1) \) gauge field in a background RC spacetime.

Our main work can be found in section (IV), wherein, we compute the quantum modifications to the spin current algebra which shows up as “anomalies” due to \( U(1) \) gauge fields and Riemann-Cartan spacetime combined. Furthermore, using these results, we reduce these spin current commutators to the Lorentz algebra for chiral Weyl fermions. This leads to a very interesting consequence that, one obtains different quantum correction terms either proportional to \( \alpha \) ("Fine-structure constant") or rather a *Dirac quantized product of the electric and magnetic charges*, if we include magnetic monopoles in Maxwell’s theory. A very interesting result that we show here is that, merely swapping the Lorentz generators in the commutation relation not only leads to a sign change but rather a completely different correction terms. Thus, we show that “quantum anomalies” leads to the breakdown of local Lorentz symmetry and ultimately, alterations to the Lorentz algebra.

III. THE CANONICAL SPIN CURRENT OF A SPINORIAL FIELD.

A. Spin currents irreducible decomposition

The irreducible decomposition of the spin tensor \((6 \times 4 = 24 \text{ components})\) consists of a tensor part, a vector part and an axial vector part [13],

\[
\mathcal{S}_{ij}^k = TEN \mathcal{S}_{ij}^k + VEC \mathcal{S}_{ij}^k + AX \mathcal{S}_{ij}^k.
\]

The tensor piece, \(TEN \mathcal{S}_{ij}^k\) consists of 16 components. The vector piece \(VEC \mathcal{S}_{ij}^k := \frac{2}{3} \mathcal{S}_{[ij]}^k \delta_{k}^{ij}\) contains 4 components and the axial vector piece being completely antisymmetric \(AX \mathcal{S}_{ijk} = \mathcal{S}_{[ijk]}\) also contains 4 components.
The Dirac Lagrangian in a Riemann-Cartan spacetime reads,

\[ \mathcal{L}_D = \frac{\hbar c}{2} e \left( i (\bar{\psi} \gamma^\alpha D_\alpha \psi - D_\alpha \bar{\psi} \gamma^\alpha \psi) - 2 \frac{mc}{\hbar} \bar{\psi} \psi \right). \]  

(2)

Where \( \gamma^\alpha \) are the constant Dirac matrices and are the components of the Clifford-algebra valued exterior forms, i.e.,

\[ \gamma := \gamma_\alpha \theta^\alpha, \]

and the Fock-Ivanenko type covariant derivative acts on the spinors,

\[ D_i \psi(x) = \left( \partial_i + i \frac{e}{4} \Gamma_{i}^{\alpha\beta} [\gamma_\alpha, \gamma_\beta] \right) \psi(x) \]

Where, \( \Sigma_{\alpha\beta} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta] \) is the spinor representation of the Lorentz generators. The Dirac equation for the spinor field \( \psi \) and the conjugate field \( \bar{\psi} \) are,

\[ i e^k_a \gamma^\alpha \partial_k \psi = m \psi - i e^k_a \gamma^\alpha \Gamma_k \psi, \]  

(3a)

\[ -i \partial_k \bar{\psi} \gamma^a e^k_\alpha = \bar{\psi} m - i \bar{\psi} \Gamma_k \gamma^a e^k_\alpha, \]  

(3b)

where \( D_\alpha \psi = e^k_\alpha (\partial_k + \Gamma_k) \psi \), \( D_\alpha \bar{\psi} = \bar{\psi}(\partial_k - \Gamma_k) e^k_\alpha \).

B. Relation between spinorial spin current and chiral (axial) vector currents

The canonical spin current of the Dirac field are totally antisymmetric (axial part with 4-components) and is dual to the axial vector currents,

\[ A_X \mathcal{S}_{\alpha\beta}^i(x) := \frac{i}{4} \hbar c e^{i \gamma^\alpha} e_{\alpha\beta\gamma\delta} \bar{\psi}(x) \gamma_5 \gamma^\delta \psi(x) = \frac{i}{4} \hbar c e^{i \gamma^\alpha} e_{\alpha\beta\gamma\delta} J^\delta_5(x). \]  

(4)

The above Eq. (4), obeys the classical conservation equation for spin angular momentum, i.e., [13],

\[ D_i \mathcal{S}_{\alpha\beta}^i(x) - 2 \mathcal{T}_{[\alpha\beta]} = 0. \]  

(5)

Where \( \mathcal{T}_{[\alpha\beta]} \) is the antisymmetric part of the canonical energy-momentum current. Our goal is to study whether the above Eq. (5) is subjected to “quantum non-conservation” due to anomalies. Hence, we derive an analogue of the chiral anomaly. From this point on, we work with chiral Weyl fermions which are massless and carry a handedness (chirality).
C. Spin current algebra

The ETCRs for the spin current can be derived using the Schwinger quantum action principle by varying the matrix elements of Eq. (5), w.r.t to the spin (Lorentz) connection $\Gamma^i_{\alpha\beta}(x)$. Thus, the canonical spin current Lie algebra reads,\(^3\)

$$\left[\mathcal{S}_{\alpha\beta}^0(x), \mathcal{S}_{\gamma\delta}^j(x')\right]_{x_0=x'_0} = 2i\left(\eta_{\gamma\delta}^\alpha \mathcal{S}^j_{\beta\delta}(x) - \eta_{\delta\alpha}^\beta \mathcal{S}^j_{\gamma\beta}(x)\right)\delta^0(x-x')$$

$$+ i\left(\partial_i \left(\frac{\delta \mathcal{S}^i_{\alpha\beta}(x)}{\delta \Gamma^\gamma_j(x')} - 2\frac{\delta \Sigma^j_{[\alpha\beta]}(x)}{\delta \Gamma^\gamma_j(x')}\right)\right). \quad (6)$$

The second bracket contains the Schwinger terms. The first term in the second bracket of Eq. (6) is always zero for fermions since $\frac{\delta \mathcal{S}^i_{\alpha\beta}(x)}{\delta \Gamma^\gamma_j(x')} = 0$, i.e the current is independent of the connection. For the sake of ease, one could avoid the other term in the second bracket.

D. Anomalous divergence of spin currents

Using the Lagrangian for chiral particles (chiral Lagrangian) and local Lorentz invariance, we find that the chiral Weyl spinors also satisfy the classical conservation law,

$$D_i \mathcal{S}^i_{\alpha\beta}(x)\chi - 2\Sigma^j_{[\alpha\beta]}(x)\chi = 0, \quad (7)$$

where subscript $\chi$ denotes chiral Weyl fermions. The spin tensor and the antisymmetric canonical energy-momentum tensors are,

$$\mathcal{S}^i_{\alpha\beta}(x) = \frac{i\hbar c}{4} \epsilon^{\gamma\epsilon}_{\alpha\gamma\delta} \chi(x) \gamma^\epsilon(x) \gamma^\delta(x) \chi(x) = \frac{i\hbar c}{2} \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\epsilon\gamma} J^\epsilon_{\gamma\delta}(x), \quad (8a)$$

$$\Sigma^j_{[\alpha\beta]}(x) = \frac{i\hbar c}{2} \epsilon(\bar{\chi}\gamma_{[\alpha} D_{\beta]}\chi - D_{[\beta} \bar{\chi}\gamma_{\alpha]} \chi). \quad (8b)$$

These two currents Eqs. (8a, 8b) are related to Lorentz invariance and diffeomorphism invariance respectively. Hence, fermions interacting with purely gravitational field should possess both Lorentz invariance and diffeomorphism invariance.

In order to study the anomalies associated with the divergence of spin currents, we concentrate on the local Lorentz symmetry. In order to achieve this we construct a gauge

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\(^3\) The spin current Lie-algebra corresponds to $\mathfrak{so}(1,3) \otimes \mathfrak{so}(1,3)$. 

invariant axial vector current interacting with a background Riemann-Cartan spacetime,

\[ J_5^i(x|\epsilon) = \lim_{\epsilon \to 0} \frac{1}{2} \psi(x + \frac{\epsilon}{2}) \gamma^5 \gamma^i \exp \left\{ i \lambda \int_{x_0}^{x+\frac{\epsilon}{2}} dy' \Gamma_j^\alpha \gamma^\alpha \gamma^j \Sigma_{\alpha \beta} \right\} \psi(x - \frac{\epsilon}{2}). \]  (9)

Here, the Fujikawa point splitting on the fields in terms of the spin connection has been used since \( J_5^i(x) \) is singular. Here, \( \lambda \) corresponds to a coupling associated with gravity. Substituting Eqs. (9, 8a) into Eq. (7), we obtain the anomalous divergence of the spin tensor;

\[ D_\alpha \mathcal{S}_{\alpha \beta}^i(x) - 2 \Sigma_{\alpha \beta} \approx (\tilde{R}_{\alpha \gamma} R_{\beta \gamma ij} - \tilde{R}_{\beta \gamma} R_{\alpha ij}) = 0. \]  (10)

Where \( \tilde{R}_{\alpha \beta}^{ij}(x) = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} \tilde{R}^{\gamma \delta ij}(x) \) is the dual Lorentz curvature. The anomaly term \( \tilde{R} R \) in the r.h.s. of Eq. (10) is always zero for \( 4n \) dimensions because of charge conjugation properties of the gravitational interactions. Thus the quantum corrections to the spin current algebra is non-existent in 4-dimensional spacetime. Hence, the quantum conservation equation holds for the spin in a background RC spacetime. This is a marked difference between the above Eq. (10) and the ABJ anomaly, where the later contains anomalous topological terms in 4-dimensional spacetime even for the massless case.

IV. ANOMALY DUE TO GRAVITATIONAL + U(1) GAUGE FIELDS AND BREAKDOWN OF LORENTZ INVARIANCE

In order to better understand the interaction between spin and the fundamental interactions, consider the chiral Weyl spinors interacting with an external Abelian \( U(1) \) gauge field \( A_i(x) \) and other non-Abelian \( SU(N) \) gauge fields \( A^A_i(x) \) in a background Riemann-Cartan spacetime.

It is pretty surprising that anomalies are solely contributed by the Abelian \( U(1) \) gauge fields on a curved background. Surprisingly, the non-Abelian gauge fields don’t contribute to anomalies since there is no breakdown of Poincaré invariance, This leads to the breakdown of Lorentz invariance in physical spacetime. Thus the corresponding expression for the Lorentz-chiral anomaly is, \[ 10, \] 5

\[ D_\alpha \mathcal{S}_{\alpha \beta}^i x - 2 \Sigma_{\alpha \beta} x = -\frac{i q_e}{96\pi^2} \left( R_{\alpha \beta} F_{\alpha \beta}^i + \tilde{R}_{\alpha \beta kl} F_{kli}^i + 2 F_{\alpha \beta i}^i \right). \]  (11)

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4 We avoid the torsion tensor coupling to the chiral currents in Eq. (10). This term potentially leads to anomalies and would be investigated more in the future.

5 The covariant derivative \( ; \) in Eq. (11) contains both Lorentz-affine connections. we only vary w.r.t the spin connection and don’t bother about the affine part.
Here, $\mathcal{F}_{\alpha\beta}(x) = \frac{1}{\sqrt{g}}\gamma_{\alpha\beta\delta}F^{\gamma\delta}(x)$ is the dual of the electromagnetic field strength tensor. A major difference between Eq. (11) and the chiral anomaly is that the presence of a *Laplacian of the electromagnetic field strength* tensor, $F_{\alpha\beta\gamma}$. Also, the first two terms in the r.h.s of Eq.(11) shows coupling between the electromagnetic field strength and the rotational field strength (curvature) of the gravitational fields. On the contrary the chiral anomaly contains the topological term $\frac{q^2}{8\pi^2}\tilde{F}^{ij}F_{ij}(x)$ [2]. Thus, the anomaly due to spin leads to several new terms which are absent in chiral anomalies.

The spin current commutators are obtained by varying Eq. (11) w.r.t. the spin connection $\Gamma_j^{\gamma\delta}(x')$. We record directly the spin current commutators with anomalies included,

$$
[\mathcal{S}_{\alpha\beta}^{0j}(x), \mathcal{S}_{\gamma\delta}^{j}(x')] = 2i(\eta_{\gamma[a}\mathcal{S}_{\beta]d}^{j}(x) - \eta_{\delta[a}\mathcal{S}_{\beta]d}^{j}(x))\delta^3(x-x') - W_{\alpha\beta\gamma\delta}^{0j}(x, x') - \frac{iq_e}{24\pi^2}(\partial_k\delta^3(x-x'))
$$

The third term in r.h.s of Eq. (12) is the quantum modification or the anomaly term. By the same argument as above, the $W$ terms for spinorial fields completely vanish. Here, we explicitly write down the different components,

$$
[\mathcal{S}_{ab}^{0}(x), \mathcal{S}_{cd}^{e}(x')] = 2i(\eta_{[c[a}\mathcal{S}_{d]b}^{e}(x) - \eta_{d[a}\mathcal{S}_{b]c}^{e}(x))\delta^3(x-x') - \frac{iq_e}{24\pi^2}(\epsilon_{abg}F^g(x)e^{[e}[\epsilon_{d]}(x))\partial_f\delta^3(x-x')
$$

$$
[\mathcal{S}_{a0}^{0}(x), \mathcal{S}_{b0}^{e}(x')] = i\mathcal{S}_{ab}^{e}(x)\delta^3(x-x') - \frac{iq_e}{24\pi^2}(B_{a}(x)e^{[e}[\epsilon_{b]}(x))\partial_f\delta^3(x-x')
$$

$$
[\mathcal{S}_{ab}^{0}(x), \mathcal{S}_{0b}^{e}(x')] = 2i\eta_{[c[a}\mathcal{S}_{d]b}^{e}(x)\delta^3(x-x') - \frac{iq_e}{24\pi^2}(\tilde{F}_{ab}(x)e^{[e}[\epsilon_{d]}(x))\partial_f\delta^3(x-x') - \frac{1}{2}F^{e}\partial_f\delta^3(x-x'))
$$

$$
[\mathcal{S}_{a0}^{0}(x), \mathcal{S}_{bc}^{e}(x')] = 2i\eta_{[c[a}\mathcal{S}_{d]c}^{e}(x)\delta^3(x-x') - \frac{iq_e}{24\pi^2}(\tilde{F}_{a0}(x)e^{[e}[\epsilon_{d]}(x))\partial_f\delta^3(x-x') + \frac{1}{2}F^{e}\partial_f\delta^3(x-x'))
$$
Writing down the equations explicitly in terms of the electric and magnetic components, yields,

\[
[S_0^a(x), S_{bc}^e(x')] = 2i\eta_{d[a} \tilde{S}_{c]0}^e(x) \delta^3(x-x') \\
- \frac{iq_e c}{24\pi^2} \left( \frac{1}{2} (B \times \nabla)^e \delta^3(x-x') + B_d e_{[d} e^{f]} c(x) \partial_f \delta^3(x-x') \right),
\]

\[
[S_{ab}^0(x), S_{c0}^0(x')] = 2i\eta_{[a} \tilde{S}_{b]0}^0(x) \delta^3(x-x') \\
- \frac{iq_e c}{24\pi^2} \left( \frac{1}{2} (E \cdot \nabla)^0 \delta^3(x-x') + \epsilon_{a[b} E_{c]} e^f(x) \partial_f \delta^3(x-x') \right),
\]

\[
[S_{a0}^0(x), S_{e0}^0(x')] \approx -\frac{iq_e c}{24\pi^2} (B \cdot \nabla)^0 \delta^3(x-x'),
\]

\[
[S_{ab}^0(x), S_{a0}^e(x')] \approx -\frac{iq_e c}{24\pi^2} (E \times \nabla)^e \delta^3(x-x').
\]

The symbol \(\approx\) means there are some other terms which are not of our interest. In fact the Lorentz generators are nothing but the integral of the time component of the spin current over a space-like 3 dimensional hypersurface, i.e.,

\[
\Sigma_{bc} := \int d^3 x S_{bc}^0(x) \quad (21a)
\]

\[
\Sigma_{a0} = \int d^3 x S_{a0}^0(x). \quad (21b)
\]

Where \(\Sigma_{ab}\) are the Lorentz rotations while \(\Sigma_{a0}\) are the Lorentz boosts.

Integrating Eq. (18) w.r.t. \(x\ & x'\) and using the Gauss theorem yields,

\[
[S_{ab}, \Sigma_{c0}] = \left( \eta_{ac} \Sigma_{b0} - \eta_{bc} \Sigma_{a0} \right) - \frac{ihc}{12\pi} \alpha + \ldots. \quad (22)
\]

Where \(\alpha = \frac{q^2}{\hbar c}\) is the fine structure constant.

This is a remarkable result, since the second term in Eq. (22) is the alteration to the Lorentz algebra due to quantum corrections. Hence, the the existence quantum anomalies leads to the breakdown of local Lorentz symmetry.

Infact, integrating Eq. (19) w.r.t. \(x\ & x'\) and using the Gauss theorem yields a very surprising result,

\[
[S_{a0}, \Sigma_{0a}] = -\frac{ihc}{12\pi} n. \quad (23)
\]

Where \(n = \frac{2q_m q_e}{\hbar c} \in \mathbb{Z}\) is the Dirac quantization. \(q_m\) is the magnetic charge. Ideally, the algebra of two same boost generators is zero. If we don’t consider magnetic monopoles, this yields the standard result, which is zero.
V. DISCUSSION AND CONCLUSION

Firstly we infer that the purely gravitational anomaly term proportional to $\tilde{R}_\alpha^{\gamma ij} R_{\beta \gamma ij} - \tilde{R}_\beta^{\gamma ij} R_{\alpha \gamma ij}$, vanishes identically in four dimensions. This is related to the charge conjugation properties of the gravitational field. Hence a quantum conservation law holds for spin currents in four dimensions and the spin current algebra contains no anomalous terms.

We see that $[\mathcal{G}_r, \mathcal{G}_{r,b}]$ always contains correction terms related to the $\vec{E}$ fields. On the contrary, $[\mathcal{G}_b, \mathcal{G}_{r,b}]$, contains terms proportional to $\vec{B}$ fields as closure failure. Thus local active transformations performed on Weyl fermions in the presence of $U(1)$ gauge fields, produces, apart from a mere sign change, completely different fields and charges (electric or magnetic). Hence, anomalies are sensitive to the order of performing symmetry transformations.

A very interesting result is Eq. (23), where the rotation-boost commutation contains a correction term proportional to $\frac{q^2 e}{\hbar c}$ or the “fine-structure constant”.

The commutation between the rotation and boosts generators contains a quantum correction term proportional to the electric charge squared. While the same boost generators close on a non-trivial topological number $n \in \mathbb{Z}$ which is the Dirac quantized product of electric and magnetic charges.

If magnetic charges are not taken into consideration, Eqs. (17, 18) (rotation-boost or boost-rotation commutators) contain the Maxwell source equations. The source free Maxwell equations show up in - Eqs. (19, 20) (rotation-rotation or boost-boost commutators). This could have some important implications and would be investigated in future works.

Similar anomalous current commutation relations were found for the vector and axial vector currents (cf. [14]). Here too, the additional anomaly terms corresponded to electric and magnetic fields or charges and the current commutation were order sensitive.

\[\text{Here, } \mathcal{G}_{r,b} \text{ implies either } \mathcal{G}_r \text{ or } \mathcal{G}_b\]
Our further goal would be to rather come up with viable experimental approaches to look for chiral-Lorentz anomaly signatures using Weyl semimetals. This shall be investigated in the near future.

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