Abstract

We generalize the Galileon symmetry and its relativistic extension to a de Sitter background. This is made possible by studying a probe-brane in a flat five-dimensional bulk using a de Sitter slicing. The generalized Lovelock invariants induced on the probe brane enjoy the induced Poincaré symmetry inherited from the bulk, while living on a de Sitter geometry. The non-relativistic limit of these invariants naturally maintain a generalized Galileon symmetry around de Sitter while being free of ghost-like pathologies. We comment briefly on the cosmology of these models and the extension to the AdS symmetry as well as generic FRW backgrounds.

1 Introduction

Over 44 orders of magnitudes, from millimeter length scales (TeV energy scales) all the way up to Cosmological scales ($\sim 10^{-32}$–$33$ eV), the theory of General Relativity is easily the force which has successfully passed the largest variety of tests. In view of these successes, modifying the theory of General Relativity may seem a purely academic endeavor. However, the unification of gravity with the rest of the standard model will undoubtedly result in a modification of the theory in the UV. Furthermore, at the other end of the energy spectrum, the Cosmological Constant and the late-time acceleration of the Universe might be seen as a sign of the breakdown of gravity on today’s cosmological scales.

Following this philosophy, there has been a return of interest in theories that modify gravity in the IR. Theories of massive gravity in particular, could play a crucial role within the framework of degravitation, [1, 2, 3], where the weakening of gravity on large distances could address the cosmological constant problem.

Among these models, the Dvali-Gabadadze-Porrati (DGP) scenario has played a crucial role, as the first theory of gravity mediated by an effectively softly massive gravity.
spin-2 field (resonance), without leading to any ghost like pathologies (when working around the stable branch), [4]. While the weakening of gravity in DGP is not sufficient to degravitate the vacuum energy, small departures from DGP are expected to provide a useful framework for degravitation, [3], in particular when extending DGP to higher dimensions, [5, 6]. Furthermore, theories of hard mass gravity with no ghosts in the decoupling limit, can potentially degravitate small amounts of vacuum energy [7].

Another (opposite) strategy in understanding the late-time acceleration of the Universe relies on the idea of self-acceleration, whereby the acceleration of the Universe is driven by the graviton’s own degrees of freedom (namely its helicity-0 mode) rather than a cosmological constant. While the DGP scenario allows for self-accelerating solutions, [8], that branch of solutions has been proven to be pathological, [9]. However, once again, small modifications of the model can also provide stable self-accelerating solutions, [10]. This has been explored more recently in the generalization of the decoupling limit of DGP, proposed by Nicolis et. al., [11]. The decoupling limit of DGP is obtained by taking the Planck scale to infinity, while sending the graviton mass to zero in such a way that the helicity-0 and -2 modes of the graviton decouple, [12, 13]. In that limit, the resulting effective theory for the helicity-0 mode satisfies two fundamental properties: First, its action is invariant under the Galilean symmetry $\pi \to \pi + c + v^\mu x^\mu$. Second, although the decoupling limit of DGP involves higher derivative terms in $\pi$ it does so in such a way that the equations of motion remain second order in derivative, hence avoiding the standard Ostrogradski instability. The idea behind the “Galileon” model proposed by Nicolis et. al., is to generalize the decoupling limit of DGP to all the possible interactions satisfying the two previous properties. In four dimensions, there are five possible interactions that satisfy the symmetry, while maintaining a well posed Cauchy problem. As shown in [11, 14], for a certain class of parameters, the model allows stable self-accelerating solutions. Interestingly, the Galileon family of interactions, has been shown to arise generically in other theories where gravity is mediated by a massive spin-2 field (hard mass graviton) which show no signs of the Boulware-Deser ghost in the decoupling limit [15, 16, 17, 18].

Whether the Galileon can tackle the cosmological constant problem or provide a viable phenomenology for self-acceleration, is still an open question, but it does provide a unique framework for the study of a scalar field that could play a role on cosmological scale while exhibiting the Vainshtein mechanism on small distance scales near sources, [19]. As such, there has been a large amount of interests in the cosmology and the phenomenology of such models, [20, 21, 22, 23, 24, 25]. However, in its original derivation from a decoupling limit where interactions with gravity are suppressed, the “Galileon” should really be though as a scalar field living around flat space-time. To bring the Galileon a step forward and understand its behaviour around generic geometries, in particular around cosmological backgrounds, it is essential to understand its potential covariantization.

When promoting the Galileon to curved backgrounds, Deffayet et. al. have en-
sured that the absence of ghosts was preserved, [26]. However this comes at the price of breaking the Galileon symmetry, which is only valid on a flat geometry. The “Co-
variant Galileon” can also be derived in a natural way by embedding a probe brane in a five-dimensional space-time and inducing the Lovelock invariants on the brane, [27]. The Galileon symmetry around Minkowski is then a simple consequence of five-dimensional Poincaré in the bulk, but it is clear that for an arbitrary bulk geometry the amount of symmetry is reduced. The symmetry enjoyed by the Galileon on an arbitrary background, should then in principle be expressed after deriving the exact bulk geometry.

In this work, we follow the same philosophy as in [27] and consider the specific case where the bulk geometry remains Ricci flat, while the brane geometry is (anti-) de Sitter. This is made possible by working in the de Sitter slicing of five-dimensional Minkowski, where the geometry on a brane at fixed position along the extra dimension is (anti-) de Sitter. Looking at the five-dimensional Poincaré symmetry in the de Sitter slicing, we can easily generalize the “flat” Galileon symmetry to a “de Sitter” equivalent. Not only is this framework most useful for cosmology, and in particular for inflation and late-time acceleration, but equally importantly, this framework can be used as a playground where corrections from the geometry to the Galileon symmetry and interactions can be understood. As such the de Sitter Galileon provides an important generalization of the Galileon symmetry around a curved background.

The rest of the paper is organized as follows: We begin in section 2 by presenting the full Poincaré symmetry when working in the de Sitter slicing of Minkowski. We also discuss the implications for the non-renormalization theorem which is essential for the viability of the effective field theory of this model. We then move onto the probe-brane picture in section 3 and describe the Lovelock invariants induced on the brane. By taking the “non-relativistic” limit, which corresponds here to a weak field limit, we recover the generalization of the Galileon, now valid around de Sitter. We also present the anti-de Sitter (AdS) equivalent of the Galileon and discuss its possible generalization on FRW. We then comment on the consequences for cosmology and specifically for inflation and late-time acceleration, before exploring further directions in section 4.

2 Symmetries and Non-Renormalization theorem

2.1 de Sitter Slicing of Minkowski

Following the same philosophy as in [27], we study the brane position modulus of a relativistic probe brane embedded in a five-dimensional bulk. We start with describing the five-dimensional Minkowski bulk in a de Sitter slicing and expressing the full Poincaré symmetry in this gauge,

\[ ds^2_5 = e^{-2Hy} (dy^2 + q_{\mu\nu}dx^\mu dx^\nu) \]  \hspace{1cm} (1)

3
where \( q_{\mu \nu} \) is the de Sitter metric,
\[
ds_{dS}^2 = q_{\mu \nu} dx^\mu dx^\nu = -n^2(t) dt^2 + a^2(t) dx^2
\]
with \( \frac{a}{a} = \text{const} = H \).

The equivalent of the translation (parameter \( c \)), rotations (\( v_i \)) and boost (\( v_0 \)) involving the \( y \) direction are given by \( x^a \rightarrow x^a + \delta x^a \) with

\[
\begin{align*}
\delta y &= e^{Hy} \left( a \left( c + v_i x^i + \frac{v_0}{2H} + \frac{1}{2} v_0 H \delta_{ij} x^i x^j \right) - \frac{v_0}{2aH} \right) \\
\delta t &= \frac{e^{Hy}}{n} \left( a \left( c + v_i x^i + \frac{v_0}{2H} + \frac{1}{2} v_0 H \delta_{ij} x^i x^j \right) + \frac{v_0}{2aH} \right) - \frac{1}{n} \left( c + \frac{v_0}{H} + v_i x^i \right) \\
\delta x^i &= \left( c H x^i + v_0 x^i + \frac{v_i}{2H} + v_j x^j x^i - \frac{1}{2} v^i x^j x_j - \frac{v^i}{2Ha^2} - \frac{e^{Hy}}{a} \left( \frac{v^i}{H} + v_0 x^i \right) \right).
\end{align*}
\]

We could have expressed the equivalent of translations along \( y \) in the simpler form \( \delta y = n \delta t = c e^{Hy} a(t) \) and \( \delta x^i = 0 \), but have chosen the specific form of the transformation which reproduces the standard translation and rotations in flat space in the limit when \( H \rightarrow 0 \) with the lapse \( n = 1 \)

\[
\begin{align*}
y &\rightarrow y + c + v_\mu x^\mu \\
x^\mu &\rightarrow x^\mu - v^\mu y.
\end{align*}
\]

For a brane localized at \( y = \pi(x^\mu) \), the 5d dimensional Poincaré symmetry implies the following symmetry under shifting and boosting of \( \pi \):

\[
\pi \rightarrow \pi + (\delta y - \delta x^\mu \partial_\mu \pi) \bigg|_{y=\pi(x^\mu)}.
\]

Provided that all the brane quantities transform as scalar, the induced action should then be invariant under the transformation (8).

### 2.2 Non-renormalization theorem

Let us first review the non-renormalization theorem as presented in Ref. [13]. Starting with the now standard form of the Galileon,

\[
\mathcal{L} = - (\partial \pi)^2 + \frac{1}{\Lambda^3} (\partial \pi)^2 \Box \pi,
\]

and expanding around a classical solution, \( \pi = \pi_{\text{cl}} + \varphi \), the leading order Lagrangian for the perturbations around that background is

\[
\mathcal{L}_\varphi = Z_{\mu \nu} (\pi_{\text{cl}}) \partial^\mu \varphi \partial^\nu \varphi,
\]

with

\[
Z_{\mu \nu} = -(1 + 2 \frac{\Box \pi_{\text{cl}}}{\Lambda^3}) \eta_{\mu \nu} - \frac{2}{\Lambda^3} \partial_\mu \partial_\nu \pi_{\text{cl}}.
\]
To simplify the argument, let us assume a simpler kinetic term of the form $Z_{\mu \nu} = -Z \eta_{\mu \nu}$, with $Z = 1 + 2 \Box \pi_{\text{cl}} / \Lambda^3$, so that the canonically normalized field $\hat{\phi}$ can easily be expressed in terms of $\phi$, $\hat{\phi} = \sqrt{Z} \phi$, and the Lagrangian around a non-trivial configuration is

$$L_{\phi} = - (\partial \hat{\phi})^2 + \frac{1}{2} \left[ \frac{(\partial Z)^2}{2Z^2} - \frac{\partial^2 Z}{Z} \right] \hat{\phi}^2,$$

(12)

giving rise to an effective mass term for $\phi$. The Coleman-Weinberg 1-loop effective action is then of the form

$$L_{1-\text{loop}} = (V''(\hat{\phi}))^2 \log \frac{V''}{M_*^2} \supset \left\{ \frac{(\partial Z)^4}{Z^4}, \frac{(\partial Z)^2 \partial^2 Z}{Z^2}, \frac{\partial^2 Z^2}{Z^2} \right\} \log \frac{V''}{M_*^2},$$

(13)

where $M_*$ is the cutoff scale, with $M_* \gg \Lambda$. Within the strong coupling region the higher interactions dominate, $\partial^2 \pi_{\text{cl}} \gg \Lambda^3$, and one might worry that since $Z$ is large, the effective field theory goes out of control. However as is explicit in (13), the counter terms do not depend on the absolute value of $Z$ but only on the ratio with at least one of its derivatives. Quantum corrections can thus be under control as long as $\partial Z \ll Z$ even though the classical solution has $Z \gg 1$. In other words, one can have large operators, $\partial^2 \pi_{\text{cl}} \gg \Lambda^3$ as long as the effective theory can be reorganized so as to define new irrelevant operators that carry extra gradients, $\partial^3 \pi_{\text{cl}} \ll \Lambda^4$, i.e. $\partial / \Lambda \ll 1$. Furthermore a key ingredient of the non-renormalization theorem is the nature of the new counter terms. It is clear from the form of the one-loop effective action, that any counter term, will arise with extra gradients as compared to the original interactions, $\partial Z \sim \partial^3 \pi$. This implies that the Galileon interactions themselves will not be renormalized, and the one-loop effective action will only affect higher derivative terms.

Having summarized the argument in the usual Galileon scenario, it is straightforward to apply the same philosophy when working around de Sitter. In this case, the Galileon interaction, as we shall see later, is dressed with corrections coming from the Hubble parameter, schematically $\partial \pi \rightarrow \partial \pi + \beta H \pi$, with a given constant $\beta$ so that for instance, $(\partial \pi)^2 \Box \pi \rightarrow (\partial \pi)^2 \Box \pi + 6H^2 \pi (\partial \pi)^2 - 8H^4 \pi^3$. In this framework, the Hubble parameter plays a similar role to the gradient, $H \sim \partial$, so that the previous counting remains identical. As a consequence, the de Sitter Galileon interactions are again not renormalized, and the effective field theory remains under control as long as $\partial / \Lambda \sim H / \Lambda \ll 1$, i.e. $\partial^3 \pi_{\text{cl}} / \Lambda^4 \sim H^3 \pi_{\text{cl}} / \Lambda^4 \ll 1$ even around classical configurations with $\partial^2 \pi_{\text{cl}} / \Lambda^3 \sim H^2 \pi_{\text{cl}} / \Lambda^3 \sim 1$.

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4Sufficiently close to a source all the non-zero elements of $Z_{\mu \nu}$ will indeed become of the same order.


\section{de Sitter Galileon}

We use the same notation as [11], with \( \Pi_{\mu\nu} = D_\mu D_\nu \pi \) where the covariant derivative is taken w.r.t. the 4D metric \( q_{\mu\nu} \) and square brackets \([\ldots]\) represent the trace (w.r.t. \( q_{\mu\nu} \)) of a tensor. Furthermore we also introduce the following notation

\[ [\phi^n] \equiv \partial^n \Pi \cdot \partial^n \pi, \tag{14} \]

so in particular \( [\phi] = D_\mu D_\nu \pi \partial^\mu \pi \partial^\nu \pi = q^{\mu\alpha} q^{\nu\beta} D_\mu D_\nu \pi \partial_\alpha \pi \partial_\beta \pi. \) We also express the Lorentz factor as

\[ \gamma = \frac{1}{\sqrt{1 + (\partial \pi)^2}} \tag{15} \]

with \((\partial \pi)^2 = q^{\mu\nu} \partial_\mu \pi \partial_\nu \pi.\)

\subsection{Probe Brane in de Sitter Slicing}

We consider a probe brane embedded in a flat, 5-D Minkowski bulk, which we take to be in a de Sitter slicing \((1)\). The brane is positioned at \( y = \pi(x_\mu) \). From the five-dimensional theory, five invariant quantities can be induced on the brane [27]: The tadpole \( \mathcal{L}_1 \), the DBI equivalent \( \mathcal{L}_2 \), the extrinsic curvature \( \mathcal{L}_3 \), the induced Ricci tensor \( \mathcal{L}_4 \) and finally the boundary term from the Gauss-Bonnet curvature in the bulk \( \mathcal{L}_5 \). These are constructed out of the induced metric at \( y = \pi(x_\mu) \)

\[ g_{\mu\nu} = e^{-2H\pi} (q_{\mu\nu} + \partial_\mu \pi \partial_\nu \pi) \tag{16} \]

It is convenient to remind ourselves first of their expression around flat space-time as derived in [27],

\[ \mathcal{L}_2^f = \gamma^{-1} \tag{17} \]

\[ \mathcal{L}_3^f = ([\Pi] - \gamma^2[\phi]) \tag{18} \]

\[ \mathcal{L}_4^f = \gamma (([\Pi]^2 - [\Pi^2]) + 2\gamma^2 ([\phi^2] - [\Pi][\phi])) \tag{19} \]

\[ \mathcal{L}_5^f = \gamma^2 \left( ([\Pi]^3 + 2[\Pi^3] - 3[\Pi][\Pi^2]) + 6\gamma^2([\Pi][\phi^2] - [\phi^3]) \right. \]

\[ \left. - 3\gamma^2([\Pi]^2 - [\Pi^2])[\phi] \right). \tag{20} \]
When working around a de Sitter background, the previous Lagrangians do not respect the full Poincaré symmetry. For a de Sitter geometry the invariants are

\[
\begin{align*}
\sqrt{-q} \mathcal{L}_{1}^{dS} &= \frac{1}{5H} \sqrt{-q} \left( e^{-5H\pi} - 1 \right) \quad (21) \\
\sqrt{-q} \mathcal{L}_{2}^{dS} &= \sqrt{-q} = e^{-4H\pi} \sqrt{-q} \mathcal{L}_{2}^{f} \quad (22) \\
\sqrt{-q} \mathcal{L}_{3}^{dS} &= -\sqrt{-q} K = e^{-3H\pi} \sqrt{-q} \left( \mathcal{L}_{3}^{f} + 4H\gamma\mathcal{L}_{2}^{f} \right) \quad (23) \\
\sqrt{-q} \mathcal{L}_{4}^{dS} &= \sqrt{-q} R = e^{-2H\pi} \sqrt{-q} \left( \mathcal{L}_{4}^{f} + 6H\gamma\mathcal{L}_{3}^{f} + 12H^{2}\gamma^{2} \mathcal{L}_{2}^{f} \right) \quad (24) \\
\sqrt{-q} \mathcal{L}_{5}^{dS} &= -\frac{3}{2} \sqrt{-q} \mathcal{K}_{GB} \\
&= e^{-H\pi} \sqrt{-q} \left( \mathcal{L}_{5}^{f} + 6H\gamma\mathcal{L}_{4}^{f} + 18H^{2}\gamma^{2} \mathcal{L}_{3}^{f} + 24H^{3}\gamma^{3} \mathcal{L}_{2}^{f} \right) \quad (25)
\end{align*}
\]

where the \( \mathcal{L}_{i}^{f} \) are the flat space invariants (17-20) and \( \mathcal{K}_{GB} \) is the Gibbons-Hawking-York boundary term associated with a bulk Gauss-Bonnet term \( \mathcal{R}_{GB} \) [28, 27],

\[
\begin{align*}
\mathcal{R}_{GB} &= R^{2} - 4R_{\mu\nu}R^{\mu\nu} \quad (26) \\
\mathcal{K}_{GB} &= -\frac{2}{3} K_{\mu\nu}^{3} + K K_{\mu\nu}^{2} - \frac{1}{3} K^{3} - 2G_{\mu\nu}K^{\mu\nu} \quad (27)
\end{align*}
\]

We have used the following expressions for the extrinsic and intrinsic curvature on the brane,

\[
\begin{align*}
K_{\mu\nu} &= \gamma e^{-H\pi} \left( \Pi_{\mu\nu} + H\gamma_{\mu\nu} + H \partial_{\mu}\pi \partial_{\nu}\pi \right) \\
R_{\mu\nu} &= 2H^{2}\Pi_{\mu\nu} + \gamma^{2} \left( \left[ \Pi \right] \Pi_{\mu\nu} - \Pi_{\mu\nu}^{2} \right) + \gamma^{4} \left( \phi_{\mu\nu}^{2} - \left[ \phi \right][\Pi] \right) \\
&\quad + H\gamma^{2} \left( 3H + \gamma^{2} \left[ \Pi \right] + \gamma^{2} \left( \left[ \Pi \right] \left( \partial\pi \right)^{2} - \left[ \phi \right] \right) \right) \left( q_{\mu\nu} + \partial_{\mu}\pi \partial_{\nu}\pi \right) \\
R &= e^{2H\pi} \left( 12H^{2}\gamma^{2} + 6H^{2}\gamma^{2} \left( \left[ \Pi \right] - \gamma^{2} \left[ \phi \right] \right) + \gamma^{2} \left( \left[ \Pi \right]^{2} - \left[ \Pi \right] \right) + 2\gamma^{4} \left( \phi^{2} - \left[ \phi \right][\pi] \right) \right)
\end{align*}
\]

One can check explicitly that the Lagrangians (21-25) defined in this way are symmetric under the transformation (3-5). Furthermore they also satisfy the recursive relations related to the ‘Universal Field Equations’, first introduced by Fairlie et al. [29], and explained more recently within the context of the Galileon [30, 27]

\[
\mathcal{L}_{n+1}^{dS} = -e^{H\pi} \gamma^{-1} \frac{\delta \mathcal{L}_{n}^{dS}}{\delta \pi} \quad \text{for} \quad n \geq 1 ,
\]

which generalize the flat space-time relations. It is straightforward to check that there cannot be any further invariant beyond \( n = 5 \) because \( \delta_{\pi} \mathcal{L}_{5}^{dS} \) is a total derivative.

### 3.2 Non-relativistic Limit

When building the action for this Galileon field the Lagrangian \( \mathcal{L}_{2}^{dS} \) will enter with a coefficient \( \lambda \) related to the tension on the brane, \( S \supset \int d^{4}x \lambda \mathcal{L}_{2}^{dS} \). We can build
the non-relativistic limit, \((\partial \pi)^2 \ll 1\), of this theory by canonically normalizing the field \(\pi = \hat{\pi}/\sqrt{\lambda}\) and sending \(\lambda \to \infty\). In this limit the Galileon symmetry becomes (after setting the lapse to \(n = 1\)),

\[
\hat{\pi} \longrightarrow \hat{\pi} + e^{Ht} \left( c + v_i x^i + \frac{1}{2} v_0 H x^i x_i \right) + \frac{v_0}{H} \sinh(Ht), \tag{31}
\]

which in the flat space limit \(H \to 0\) reduces to the Galileon shift symmetry \(\hat{\pi} \to \hat{\pi} + (c + v_\mu x^\mu)\) of Nicolis et al. \[11\]. Keeping the lapse arbitrary to retain the gauge freedom, the Galilean transformation becomes,

\[
\hat{\pi} \longrightarrow \hat{\pi} + a(t) \left( c + v_i x^i + v_0 H x^i x_i - v_0 \left( \frac{n(t)}{a(t)} \right)^2 \right), \tag{32}
\]

with \(\frac{\dot{a}}{am} = H = \text{const.}\).

The specific combination of \(\mathcal{L}^{\text{dS}}_1\) and \(\mathcal{L}^{\text{dS}}_2\) which remains finite in the limit \(\lambda \to \infty\) is

\[
\mathcal{L}^{\text{NR}}_2 = \lambda (\mathcal{L}^{\text{dS}}_2 + 4 \mathcal{L}^{\text{dS}}_1 - 1) \quad \xrightarrow{\lambda \to \infty} \quad \frac{1}{2} ((\partial \hat{\pi})^2 - 4H^2 \hat{\pi}^2), \tag{33}
\]

which is indeed invariant under the non-relativistic transformation (31), and which gives back the usual first Galileon kinetic term, \(1/2(\partial \pi)^2\), in the limit \(H \to 0\).

We can now go further and consider the non-relativistic limit of the higher order invariants (extrinsic curvature term, scalar curvature, etc...) which are also invariant under (31). The “de Sitter” generalization of the Galileon derivative interactions are then:

\[
\begin{align*}
\mathcal{L}^{\text{NR}}_2 &= (\partial \hat{\pi})^2 - 4H^2 \hat{\pi}^2 \tag{34} \\
\mathcal{L}^{\text{NR}}_3 &= (\partial \hat{\pi})^2 \Box \hat{\pi} + 6H^2 \hat{\pi} (\partial \hat{\pi})^2 - 8H^4 \hat{\pi}^3 \tag{35} \\
\mathcal{L}^{\text{NR}}_4 &= (\partial \hat{\pi})^2 \left( [\hat{\Pi}]^2 - [\hat{\Pi}]^2 \right) - 6H^2 \hat{\pi} (\partial \hat{\pi})^2 \Box \hat{\pi} \\
&\quad - \frac{1}{2} H^2 (\partial \hat{\pi})^4 - 18H^4 \hat{\pi}^2 (\partial \hat{\pi})^2 + 12H^6 \hat{\pi}^4 \tag{36} \\
\mathcal{L}^{\text{NR}}_5 &= (\partial \hat{\pi})^2 \left( [\hat{\Pi}]^3 - 3[\hat{\Pi}]^2 [\hat{\Pi}] + 2[\hat{\Pi}]^3 \right) - 4H^2 \hat{\pi} (\partial \hat{\pi})^2 \left( [\hat{\Pi}]^2 - [\hat{\Pi}]^2 \right) \\
&\quad - \frac{1}{2} H^2 (\partial \hat{\pi})^4 \Box \hat{\pi} + 12H^4 \hat{\pi}^2 (\partial \hat{\pi})^2 \Box \hat{\pi} + 2H^4 \pi (\partial \hat{\pi})^4 \\
&\quad + 24H^6 \hat{\pi}^3 (\partial \hat{\pi})^2 - \frac{48}{5} H^8 \hat{\pi}^5. \tag{37}
\end{align*}
\]

One can then check that the actions \(S^{\text{NR}}_i = \int a^3(t)n(t)\mathcal{L}^{\text{NR}}_i\) are indeed invariant under the transformation (32) up to a total derivative. In flat space-time, in addition to these invariants, there is also a tadpole \(\mathcal{L}^{\text{NR}}_1 = \pi\) which enjoys the accidental Galileon symmetry. More precisely, under a Galileon transformation, the tadpole is
invariant up to a constant term which is irrelevant around flat space-time. In de Sitter, however this constant would gravitate, and affect the equations of motion, so strictly speaking around a curved background, there is no equivalent of the tadpole contribution. However it is worth pointing out that, in its fully relativistic form, $L_{1}^{(dS)}$ in (21) does satisfy the relativistic symmetry (8) up to a total derivative, with no constant left over, so in the full picture, the tadpole remains an invariant at the same level as the other interactions.

### 3.3 Anti-de Sitter

To recover the symmetries for the Galileon living in AdS, we can embed a 4d AdS brane in an extra dimension of time rather than space as performed previously. We emphasize that this is only a standard “trick” to recover the AdS symmetry in 4d, and by no means do we consider the higher dimensional scenario physical. This is equivalent as to perform the following Wick-rotation, $y \rightarrow iy$ which implies,

$$
\begin{align*}
\text{dS variables} & \quad \rightarrow \quad \text{AdS variables} \\
\{y, t, x^3\} & \quad \rightarrow \quad i\{y, -z, t\} \\
\{\pi, H, K, K_{GB}\} & \quad \rightarrow \quad i\{\pi, -H, K, -K_{GB}\} \\
L_n^{dS} & \quad \rightarrow \quad (-i)^n L_n^{AdS}
\end{align*}
$$

(38)

More specifically, we start with the AdS slicing of five-dimensional Minkowski,

$$
ds^2 = e^{-2Hy} \left(-dy^2 + dz^2 + e^{-2Hz} \eta_{\alpha\beta} dx^\alpha dx^\beta\right),
$$

(39)

where in this section the indices $\alpha, \beta$ run over time and two of the space directions, $\alpha, \beta = 0, 1, 2$, with coordinates $x^\alpha = \{t, x^1, x^2\}$. These indices are raised and lowered using the 1 + 2-dimensional Minkowski metric $\eta_{\alpha\beta}$. In this gauge, the equivalent of the translations and boosts involving the extra dimensions are $x^\alpha \rightarrow x^\alpha + \delta x^\alpha$ with

$$
\begin{align*}
\delta y &= (c + v_\alpha x^\alpha)e^{H(y-z)} - \frac{v_z}{H} \partial_z g(y, z) \\
\delta z &= (c + v_\alpha x^\alpha)(1 - e^{H(y-z)}) - v_z g(y, z) \\
\delta x^\alpha &= c H x^\alpha + v_\alpha \left(\frac{e^{Hz}}{H}(e^{Hy} - \sinh Hz) - \frac{H}{2} x^\alpha x^\alpha\right) \\
&\quad + x^\alpha H v_\beta x^\beta - v_z x^\alpha (1 - e^{H(y+z)}),
\end{align*}
$$

(40-42)

with $g(y, z) = \frac{1}{H}(1 - e^{Hy} \cosh Hz) + \frac{H}{2} e^{Hy} x_\alpha x^\alpha$. Once again, we then consider a brane localized at $y = \pi(x^\mu)$, and the resulting symmetry on the brane is given by (8). The invariants on the brane, are then easily expressed as in (21-25). In particular,

$$
L_1^{AdS} = \frac{1}{5H} \left(e^{-5H\pi} - 1\right)
$$

(43)
and the other invariants are expressed in terms of $\mathcal{L}^{\text{AdS}}_1$ using a similar ‘Universal Field Equations’ recursive relation as (30),

$$
\mathcal{L}^{\text{AdS}}_{n+1} = -e^{H\pi}\tilde{\gamma}^{-1}\frac{\delta\mathcal{L}^{\text{AdS}}_n}{\delta\pi} \quad \text{for} \quad n \geq 1,
$$

with

$$
\tilde{\gamma} = \frac{1}{\sqrt{1 - (\partial\pi)^2}},
$$

where now all contractions are performed with respect to the background AdS metric.

### 3.4 FRW

As such, the symmetry we have inferred is only preserved on a pure (anti)-de Sitter background. However, for any cosmological scenario, we would need to include at slow-roll departures from de Sitter. One could in principle covariantize this model, similarly to that performed in [26], however we would then loose the Galileon symmetry. Instead, another way forward is to generalize the description to a generic Friedmann-Robertson-Walker (FRW) background. In principle, this extension is straightforward when considering the FRW-slicing of five-dimensional Minkowski

$$
ds^2 = e^{-2Hy}\left[a^2(t)dx^2 + dy^2 + 2(1 + Hy - e^{Hy})\frac{\dot{H}}{nH^2}dydt\right.
- \left(1 - \frac{\dot{H}}{nH}\right) \left(1 + 2(1 - e^{Hy})\frac{\dot{H}}{nH} + y\frac{\dot{H}}{nH}\right)dt^2],
$$

however the realization of the Poicaré symmetry in this gauge is highly non-trivial, and the its non-relativistic limit appears non-local. This is however not surprising, as there is no reason why FRW which is not maximally symmetric should enjoy similar amount of symmetry. In the five-dimensional picture, the coordinate transformation to transfer from flat slicing of Minkowski to an FRW slicing is non-local, and so the Poincaré symmetry expressed in the five-dimensional FRW slicing does not have the similar close form as in de Sitter.

### 4 Outlooks

The Galileon theory, as formulated in flat space, was particularly interesting for cosmology because the Galileon symmetry generalises the shift symmetry required (at least in a softly broken form) for a successful realisation of the simplest models of inflation [24]. However, by definition, inflation does not occur in flat space and so the flat space Galileon symmetry will be broken during inflation. While inflation
is occurring the Universe will look like a realisation of a de Sitter geometry, (with some corrections so as to gracefully exit and let the Universe reheat). The de Sitter geometry is then a very good approximation whilst inflation is taking place, and therefore we can ask whether the generalisation of the Galileon symmetry to a de Sitter background gives a theory that is suitable for inflation.

From the presence of the tachyonic mass term in (34) it is clear that the Galileon will suffer from an instability on time scales proportional to the Hubble time and that there will be problems with attempting to build models of inflation satisfying the slow roll conditions \( \pi \ll 3H|\dot{\pi}| \ll 3M_P H^2 \). It might seem possible at first to combine several of the Galileon interactions, so as to construct a suitable potential with a stable minimum for \( \pi \), however the existence of the shift symmetry indicates that any configuration is subject to decay. In flat space, the Galileon respects a shift symmetry \( \pi \rightarrow \pi + c \) which is precisely what is required for inflation. On de Sitter, however, that symmetry is promoted to \( \pi \rightarrow \pi + a(t)c \), which corresponds to an unstable mode. Therefore even if a slow roll solution, \( \pi_{\text{infl}}(t) \), can be found it will be unstable to decay into \( \pi_{\text{infl}} + a(t)c \). This cannot obey the slow roll conditions for more than a few e-foldings unless \( c \) is tuned to be very small. Since all values of \( c \) will be generated by the symmetry this is not a natural scenario.

This does not prove, however, that there are large corrections to the inflationary models built around the flat space Galileon theory. What we consider here is a disjoint scenario, although the motivations are similar. In this article we consider a scalar field scenario for which the full de Sitter Galileon symmetry is realized. For the flat space Galileon inflationary model [24], the Galileon symmetry is softly broken allowing slow roll inflation. A coupling to the non-flat background geometry will induce, via quantum effects, small corrections of order \( H/\Lambda \) but these will not act to drive the theory towards one in which the de Sitter Galileon symmetry is realized. A straightforward way to see that these two theories are qualitatively different is to consider the five-dimensional brane world parent scenarios. The orientation of the branes in the two scenarios is totally different; corresponding to arranging the brane to lie along the fifth dimension of either a flat or de Sitter slicing of the bulk geometry. Therefore one scenario is not a small perturbation of the other unless \( H \) is vanishingly small, which is certainly not the case during inflation.

In some scalar field models, such as DBI, inflation is still possible even though the scalar mode is not slow rolling. In the non-relativistic (Galileon) version of this theory this is not possible, as the unstable mode will push the model outside the inflationary region. In the relativistic realization, however, the possibility might still be open. Another alternative, is to realize inflation in a de Sitter Galileon model by breaking the symmetry. For example by introducing a positive mass term, which would flip the sign of the tachyonic mass \(-4H^2 \pi^2\) in (34), or by considering the tadpole term \( L_1^{(NR)} = \pi \). Then it would be possible to obtain a slowly rolling-scalar field, although the size of the symmetry breaking term would have to be fine tuned. As shown in [24], although such terms would break the symmetry, they would not lead to any counter-terms and would themselves not be renormalized from the other
Galileon interactions.

One of the initial motivations for studying Galileon theories was the possibility to explain the late time accelerated expansion of the Universe through a self accelerated solution. A slow roll solution in the de Sitter Galileon theory may also explain the late time accelerated expansion as a form of dark energy. The instability present in the de Sitter Galileon, although disastrous during inflation, would actually be acceptable in the late-time de Sitter expansion. In that case the instability would only manifest itself over periods of time comparable to the age of the Universe. In such a model, we would have a small time variation of dark energy, due to a very light scalar field on cosmological scale. On solar system scales, however, the usual Vainshtein mechanism would take over, evading the usual fifth force constraints, [19].

The flat space Galileon interactions contain only derivatives of $\pi$. In contrast the de Sitter Galileon interactions (34)-(37) contain additional, potential-like, terms which are functions of $\pi$ and terms which a mixtures of $\pi$ and gradients of $\pi$. One way to ensure that no worse instability exist, and to have solutions suitable for dark energy is to work near extrema of the potential-like contributions. In that case, we have to limit ourselves to a subset of possible choices of coefficients for the $L_i^{(\text{NR})}$ which have extrema of the potential within the regime of validity of the effective field theory, $\pi \ll \Lambda^4/H^3$, with the positive vacuum energy measured today.

Finally, it is worth commenting on the possibility of covariantizing the de Sitter Galileon beyond its fixed background. Similarly as in the case of the standard Galileon, the covariantization breaks the fundamental symmetry of the theory, and nothing else is expected here. However, one should be able to generalize this theory beyond any background without introducing new (ghost) degrees of freedom. When doing so, the first Galileon interaction could for instance be promoted to

\[
L_2^{(\text{NR})} \rightarrow (\partial \pi)^2 - \frac{1}{3} R \pi \tag{47}
\]

\[
L_3^{(\text{NR})} \rightarrow (\partial \pi)^2 \Box \pi - 2\pi G_{\mu\nu} \partial^{\mu} \pi \partial^{\nu} \pi - \frac{1}{3} R_{\text{GB}} \pi^3 \tag{48}
\]

where $R$, $G_{\mu\nu}$ and $R_{\text{GB}}$ are respectively the scalar curvature, the Einstein tensor and the Gauss-Bonnet term in the arbitrary geometry\(^5\). These non-minimal couplings to gravity could have interesting consequences of their own. First of all, that theory could be viable without the need of a separate Einstein-Hilbert. Furthermore, these couplings could play a relevant role in the study of superluminal modes that have been know to appear around non-trivial spherically symmetric or cosmological backgrounds, [11, 22], if we took that couplings of the form $\pi R$ seriously beyond de Sitter.

In the very latest stage of this work we became aware that very similar ideas have been explored by Goon et. al., [31].

\(^5\)We point out however that no covariantization is unique. The most natural way to generalize this theory would be to start with an arbitrary five-dimensional metric and derive the induced Lovelock invariants on the brane.
acknowledgments

We would like to thank Andrew Tolley for useful conversations. This work is supported by the SNF.

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