HOMOLOGY OPERATIONS AND COSIMPLICIAL ITERATED LOOP SPACES

PHILIP HACKNEY

Abstract. The mod 2 homology spectral sequence associated to a cosimplicial $E_{n+1}$-space admits homology operations. We prove this by constructing, for any cosimplicial space $X$, external operations (including a Browder operation) landing in the spectral sequence associated to $S^n \times_{S^2} (X \times X)$. When $X$ is a cosimplicial $E_{n+1}$-space we couple the external operations with the levelwise structure maps to produce internal operations in the spectral sequence.

Bousfield identified $H^*(\text{Tot } X)$ as the target of this spectral sequence. Noting that $\text{Tot } X$ is an $E_{n+1}$-space when $X$ is a cosimplicial $E_{n+1}$-space, we show that the operations constructed above agree with the usual Araki-Kudo and Browder operations in the target.

1. Introduction

For each $n \geq 2$, let $C_{n+1}$ be a fixed $E_{n+1}$ operad. In this paper we extend the results of [6] to the setting of cosimplicial $C_{n+1}$-spaces. In particular, using the indexing convention on the homology spectral sequence so that $E^2_{-s,t} = \pi^s H_t(X; \mathbb{Z}/2)$, we prove the following:

Theorem 1.1. Suppose that $X$ is a cosimplicial object in the category of $C_{n+1}$-spaces. Then there are operations in the mod-2 homology spectral sequence associated to $X$

$$Q^{m}: E^r_{-s,t} \to E^r_{-s,m+t} \quad m \in [t, t - s + n] \text{ and } n \geq s$$

$$Q^{m}: E^r_{-s,t} \to E^r_{m-s-t,2t} \quad m \in [t - s, \min(t, t - s + n)]$$

where

$$w = \begin{cases} r & m = t - s \\ 2r - 2 & m \in [t - s + 1, t - r + 2] \\ r + t - m & m \in [t - r + 3, t] \end{cases}$$

(notating that some of these intervals may be empty). These are homomorphisms unless $s = 0$ and $m = t - s + n$. There is also a Browder operation

$$\lambda_n: E^r_{-s,t} \otimes E^r_{-s',t'} \to E^r_{-s-s',t+t'+n}$$

and, if $s = 0$, then

$$Q^{t+n}(x + y) = Q^{t+n}(x) + Q^{t+n}(y) + \lambda_n(x, y).$$
This Theorem is the $E_{n+1}$ analogue of a theorem of Turner (see [11] and also our paper [3]). Prior to Turner’s work, Dyer-Lashof operations were constructed in Eilenberg-Moore spectral sequences in [1] and [8], though in both cases only the ‘vertical’ operations were given. The theory of Steenrod operations, on the other hand, has a rich literature. Appropriate starting points are [5] and [10, Chapter 7].

As expected, the operations of Theorem 1.1 are consistent for various choices of $n < \infty$. Namely, it will be evident from our construction that, given $C_{n+1} \to C_{n+2}$, the operations are compatible with the forgetful functor

$$(C_{n+2}\text{-spaces})^\Delta \to (C_{n+1}\text{-spaces})^\Delta.$$ 

In addition, we show that the operations converge to the usual Dyer-Lashof or Araki-Kudo operations in the homology of $\text{Tot}(X)$. The precise statement is given in Theorem 7.1. We also show convergence of the multiplication (Theorem 7.3) and of the Browder operation (Theorem 7.4). These convergence results are similar to those proved in the case $n = \infty$ in [7].

Much in this paper relies on its companion [6], and the overall scheme is similar. We work with simple Bousfield-Kan universal examples, which we think of as cosimplicial spheres. We construct, for each $n$, external operations for these examples and use the universal property to transport these operations into the spectral sequence for an arbitrary cosimplicial space. When that cosimplicial space is actually a cosimplicial $C_{n+1}$-space, we then obtain internal operations by combining the external operations with the $C_{n+1}(2)$-structure.

1.1. Notation and Conventions. We generally work over the field $k = \mathbb{Z}/2$, and all modules should be interpreted as being $k$-vector spaces. For a chain complex $C$, we write $(\Sigma C)_{q+1} = C_q$ for the suspension, $\text{sk}_t(C)$ for the brutal truncation with

$$\text{sk}_t(C)_k = \begin{cases} C_k & k \leq t \\ 0 & k > t, \end{cases}$$

and $C^v$ for the bicomplex with

$$(C^v)_{a,b} = \begin{cases} C_b & a = 0 \\ 0 & a \neq 0. \end{cases}$$

If $B$ is a bicomplex, we will always consider the suspension as operating vertically, so that $(\Sigma B)_{a,b+1} = B_{a,b}$. If $A$ is an abelian category, we write $N : A^{\Delta^{op}} \to \text{Ch}_{\geq 0}(A)$ for normalization and $C : A^\Delta \to \text{coCh}^{\geq 0}(A)$ for conormalization, with the convention that

$$CY^p = \text{coker} \left( \bigoplus_{k=1}^p d^k : \bigoplus Y_{p-1} \to Y^p \right).$$

If $A$ happens to be the category $\text{Ch}$ of chain complexes over $k$, then we interpret $C$ as landing in the category of (left-plane) bicomplexes. If $Y$ is a cosimplicial chain complex, the indexing is given by

$$C(Y)_{-p,q} = C(Y_q)^p.$$ 

Given a bicomplex $B$, we will let $TB$ denote the product total complex

$$TB_m = \prod_j B_{j,m-j}.$$
together with the filtration by columns:

\[ F^k_m = \prod_{j \leq k} B_{j,m-j}. \]

The spectral sequences used in this paper are derived from this filtration. In particular, the spectral sequence associated to a cosimplicial chain complex \( Y \) is, by definition, the spectral sequence associated to the above filtration for the bicomplex \( B = CY \).

The Bousfield-Kan universal examples (see [3] for the universal property) are cosimplicial simplicial pointed sets defined by

\[ D_{(r,s,t)} := \Sigma^{t-s} \operatorname{coker} (\sk_{s-1} \Delta_+ \to \sk_k \Delta_+). \]

where \( \Sigma \) is the Kan suspension and \( \Delta_+ \) is obtained by adding a disjoint basepoint to each cosimplicial degree of the standard cosimplicial simplicial set \( \Delta \). We also define a cosimplicial chain complex (see [6] for the universal property)

\[ D_{rst} := \Sigma^{t-s} \operatorname{coker} (\sk_{s-1} \Delta_* \to \sk_k \Delta_*), \]

where \( \Delta_* \) is the normalization of the cosimplicial simplicial \( k \)-module \( k \Delta \). From now on, when we write \( \Delta^p \) we will always mean the normalized chains on the standard simplicial \( p \)-simplex.

Except in section 7, we are not concerned about what type of spaces we use (simplicial sets or topological spaces), and consequently the mod-2 chains functor \( S_* : \text{Spaces} \to \text{Ch} \) will mean either the normalized simplicial chains functor \( \operatorname{Nk} \) or this functor composed with the singular functor \( \text{Top} \to \text{Set}^{op} \). The spectral sequence associated to a cosimplicial space \( X \) is defined to be the spectral sequence associated to the cosimplicial chain complex \( S_* X \).

1.2. External Operations. Since we work mod 2, the basic strategy for understanding operations follows that from [9], rather than the more complicated picture which occurs at odd primes as in [4]. We notice that \( C_{n+1}(2) \simeq S^n \) in the category of \( \pi \)-spaces (where \( \pi := \Sigma_2 = \{e, \sigma\} \)). Recall the definition of \( W \):

\[ W_i = \begin{cases} e_i \cdot k\pi & i \geq 0 \\ 0 & i < 0 \end{cases} \]

\[ d(e_i) = (1 + \sigma)e_{i-1} \]

The chain complex \( W \) is a model for \( S_* (C_{\infty}(2)) \simeq S_* (E\pi) \simeq S_* (S^\infty) \), and its brutal truncation \( \sk_n W \) with

\[ (\sk_n W)_i = \begin{cases} W_i & i \in [0, n] \\ 0 & \text{else} \end{cases} \]

is a model for \( S_* (S^n) \simeq S_* (C_{n+1}(2)) \) (where all equivalences are as \( \pi \)-modules). We define, for a chain complex \( C \),

\[ E^n(C) := \sk_n W \otimes_{\pi} (C \otimes C). \]

If \( Z \) is a \( C_{n+1} \)-space, then, as in [9], there is a map

\[ E^n(S_* (Z)) \to S_* (Z) \]

induced from the structure map

\[ C_{n+1}(2) \times (Z \times Z) \to Z \]
and the $\pi$-homotopy equivalence $S^n \simeq C_{n+1}(2)$. The usual Araki-Kudo operations are obtained by defining chain maps (of degree $m$)

$$q^m : C \to \mathcal{E}^n(C)$$

and then post-composing with the map $\mathcal{E}^n(C) \to C$, provided it exists. We will call the maps $H(q^m) : H_*(C) \to H_{*+m}(\mathcal{E}^n(C))$ external operations.

Suppose $Y$ is a cosimplicial chain complex, and consider $\mathcal{E}^n(Y)$:

$$\Delta \overset{Y}{\to} \text{Ch} \overset{\mathcal{E}^n}{\to} \text{Ch}.$$

We regard the (homology) spectral sequence associated to $\mathcal{E}^n(Y)$ as the correct target for external operations. Indeed, if $X$ is a cosimplicial $C_{n+1}$-space and $Y = S^* (X)$, then the levelwise structure maps

$$C_{n+1}(2) \times (X^p \times X^p) \to X^p$$

induce a map of cosimplicial chain complexes

$$\mathcal{E}^n(Y) \to Y.$$

Thus, for this application, it is enough to construct external operations with the indicated target.

As in [6], we construct external operations for the Bousfield-Kan universal examples $D_{rst}$. Sections 2–4 are devoted to computations in the spectral sequence of $\mathcal{E}^n(Y)$. In sections 5 and 6 we use these calculations to define external operations for an arbitrary cosimplicial chain complex $Y$. Finally, in section 7 we extend the results of [7] to give compatibility of our operations with those in the target.

2. Calculation of $E^1$ and $E^2$

In this section we begin the calculation of the spectral sequence for

$$\mathcal{E}^n(D_{rst}) = (\text{sk}_n W) \otimes_\pi (D_{rst} \otimes D_{rst}),$$

following closely the calculation for $n = \infty$ in [9]. We assume throughout that $r \geq 2$, $n \geq 2$, and $s, t \geq 0$.

Let us first give a basis for $E^1$. If $Y$ is a cosimplicial chain complex, then we have an isomorphism (of complexes of graded modules) $E^1(Y) = H_*(C(Y) \cong CH_*(Y)$ by [3, 3.1], where $H_*(-)$ refers to taking homology in the ‘chain complex direction.’ The first step towards computing $E^1(\mathcal{E}^n(Y))$ is thus to compute $H_*(\mathcal{E}^n(Y))$. By [9, Lemma 1.1], we have an isomorphism of functors out of chain complexes

$$H_*(-) \cong H_*(\mathcal{E}^n(H_*(-))),$$

so we have an isomorphism of cosimplicial graded modules

$$H_*(\mathcal{E}^n(Y)) \cong H_*(\mathcal{E}^n(H_*(Y))).$$

We now give a brief indication of why [9, Lemma 1.3], which computes the homology of $\text{sk}_n W \otimes_\pi (C \otimes C)$, is true. For a $\mathbb{k}\pi$-module $M$ (such as $H_*(D^p) \otimes H_*(D^p)$), the complex $\text{sk}_n W \otimes_\pi M$ is just

$$0 \to M \overset{1+\sigma}{\to} \cdots \overset{1+\sigma}{\to} M \overset{1+\sigma}{\to} M \overset{1+\sigma}{\to} M \to 0.$$
Thus the homology is
\[
H_i(\text{sk}_n W \otimes M) = \begin{cases} 
\frac{M/(1 + \sigma)}{\ker(1 + \sigma)/\text{im}(1 + \sigma)} & i = 0 \\
\ker(1 + \sigma) & 0 < i < n \\
\ker(1 + \sigma) & i = n 
\end{cases}
\]
and zero otherwise.

Recall that in [6] we let
\[
\Lambda^p_r = \{ \varepsilon \mid \varepsilon : [r] \rightarrow [p], \varepsilon(0) = 0 \}
\]
which was used to give a basis for \(H_*(D_{r,s})\) in [6] Proposition 2.3.

**Theorem 2.1.** The \(E^1\) page of the spectral sequence associated to \(\mathcal{E}^n(D_{r,s})\) has a basis consisting of bigraded sets (column, bottom, top, middle)
\[
\mathcal{C}, \mathcal{C}^d, \mathcal{B}, \mathcal{B}^d, \Theta, \Theta^d, \mathcal{M}, \text{ and } \mathcal{M}^2.
\]
We give an exhaustive list of their elements. For \(m \in [0, n]\), we have
\[
e_m \otimes \text{id}_{[s]} \otimes \text{id}_{[s]} \in \mathcal{C}_{-s,2t+m}
\]
\[
e_m \otimes \text{id}_{[s+r]} \otimes \text{id}_{[s+r]} \in (\mathcal{C}^d)_{-s-r,2t+2r+m-2}.
\]
For each pair \(\varepsilon < \varepsilon' \in \Lambda^p_s\) and \([p] = \text{im} \varepsilon \cup \text{im} \varepsilon'\) we have
\[
e_0 \otimes \varepsilon \otimes \varepsilon' \in \mathcal{B}_{-p,2t}
\]
\[
(1 + \sigma)_{e_0} \otimes \varepsilon \otimes \varepsilon' \in \Theta_{-p,2t+n}.
\]
Here \(-p\) is between \(-s-1\) and \(-2s\).

For \(\varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r}\) and \([p] = \text{im} \varepsilon \cup \text{im} \gamma\), we have
\[
e_0 \otimes \varepsilon \otimes \gamma \in (\mathcal{M}^1)_{-p,2t+r-1}
\]
\[
(1 + \sigma)_{e_0} \otimes \varepsilon \otimes \gamma \in (\mathcal{M}^2)_{-p,2t+r+n-1}
\]
which live in degrees with \(-p\) between \(-s-r\) and \(-2s-r\).

For \(\gamma < \gamma' \in \Lambda^p_{s+r}\) and \([p] = \text{im} \gamma \cup \text{im} \gamma'\) we have
\[
e_0 \otimes \gamma \otimes \gamma' \in (\mathcal{B}^d)_{-p,2t+2r-2}
\]
\[
(1 + \sigma)e_0 \otimes \gamma \otimes \gamma' \in (\Theta^d)_{-p,2t+2r+n-2}
\]
with \(-p\) between \(-s-r-1\) and \(-2s-2r\). If \(r = \infty\) then the basis only consists of \(\mathcal{C}, \mathcal{B}, \text{ and } \Theta\).

**Proof.** Applying [6] Lemma 1.3, the homology of \(\mathcal{E}^n(H_*(D_{r,s}))\) has a basis given by the disjoint union of the following sets:
\[
\{ e_m \otimes \varepsilon \otimes \varepsilon \mid \varepsilon \in \Lambda^p_s, m \in [0, n] \}
\]
\[
\{ e_m \otimes \gamma \otimes \gamma \mid \gamma \in \Lambda^p_{s+r}, m \in [0, n] \}
\]
\[
\{ e_0 \otimes \varepsilon \otimes \varepsilon' \mid \varepsilon, \varepsilon' \in \Lambda^p_s, \varepsilon < \varepsilon' \}
\]
\[
\{ e_0 \otimes \varepsilon \otimes \gamma \mid \varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r} \}
\]
\[
\{ e_0 \otimes \gamma \otimes \gamma' \mid \gamma, \gamma' \in \Lambda^p_{s+r}, \gamma < \gamma' \}
\]
\[
\{ (1 + \sigma)e_0 \otimes \varepsilon \otimes \varepsilon' \mid \varepsilon, \varepsilon' \in \Lambda^p_s, \varepsilon < \varepsilon' \}
\]
\[
\{ (1 + \sigma)e_0 \otimes \varepsilon \otimes \gamma \mid \varepsilon \in \Lambda^p_s, \gamma \in \Lambda^p_{s+r} \}
\]
\[
\{ (1 + \sigma)e_0 \otimes \gamma \otimes \gamma' \mid \gamma, \gamma' \in \Lambda^p_{s+r}, \gamma < \gamma' \}
\]
where the total order on $\Lambda_p^s$ is given, as in [6], by the reverse dictionary order on $\text{im} \varepsilon$. Now conormalize $H_*(\mathcal{E}^n(D_{rst}))$ as in [6].

It is helpful to visualize this page. There are five potential pictures for $E^1$ that can be drawn, depending on the relationship between $n$ and $r$. One such picture (for the case $n > 2r - 2$) is given in Figure 1 where we have indicated modules with rank greater than zero by snaky lines and modules of rank equal to one by straight lines.

![Figure 1. $E^1(\mathcal{E}^n(D_{rst}))$](image-url)

The basis elements with $e_0$ and $e_n$ give us up to six interesting rows to talk about. There may be overlaps of snaky segments when $n = r - 1$ or $2r - 2$, but this does not affect the calculation of the homology of $\delta^1$.

It turns out that we have already computed the homology of each of the six snaky line segments in [6], and we now recall three families of complexes from that paper. First, we have

$$\Omega_{s,s'} = C(H_s(D_{\infty,s}) \otimes H_{s'}(D_{\infty,s'})).$$

When $s = s'$ there is an action of $\pi$, and we then have the associated complexes

$$\bar{\Omega}_s = \Omega_{s,s}/\pi$$

and

$$\Upsilon_s = \ker(1 + \sigma : \Omega_{s,s} \rightarrow \Omega_{s,s}).$$

The interpretation of [6, Proposition 5.1] in our current notation is that we have isomorphisms of complexes

$$\bar{\Omega}_s \rightarrow (k(\mathcal{B} \sqcup \mathcal{C}_{s,2t}), \delta^1)$$

$$\Omega_{s,s+r} \rightarrow (k(\mathcal{M}^1), \delta^1)$$

$$\bar{\Omega}_{s+r} \rightarrow (k(\mathcal{B}^d \sqcup (\mathcal{C}^d)_{s-r,2t+2r-2}), \delta^1).$$

This takes care of the identification of three of the six snaky line segments. For the other three, we have
Proposition 2.2. There are isomorphisms of complexes

\[ Υ_s \to (k(\mathcal{F} \sqcup \mathcal{E}_{-s,2t+n}), δ^1) \]
\[ Ω_{s,s+r} \to (k(\mathcal{M}^2), δ^1) \]
\[ Υ_{s+r} \to (k(\mathcal{F}^d \sqcup (\mathcal{E}_{d})_{-s-r,2t+2r+n-2}), δ^1) \].

Proof. The map

\[ H_s(D_{rss}) \otimes H_{s+r-1}(D_{rss}) \to H_s(\mathcal{E}^n(H_s(D_{rss}))) \]
\[ ε \otimes γ' \mapsto (1 + σ)e_n \otimes ε \otimes γ' \]
induces the middle isomorphism, as in the proof of \[6\, Proposition 5.1\]. Furthermore, if \( M^* \) is one of the cosimplicial modules

\[ H_s(D_{rss}) \otimes H_s(D_{rss}) \text{ or } H_{s+r-1}(D_{rss}) \otimes H_{s+r-1}(D_{rss}) \],

then we have a cosimplicial map

\[ \ker(1 + σ : M \to M) \to H_s(\mathcal{E}^n(H_s(D_{rss}))) \]
\[ ζ \otimes ζ' \mapsto e_n \otimes ζ \otimes ζ' \].

This is an inclusion by \[13\, Lemma 1.3\], and remains so after conormalizing. Finally, it is easy to see that the conormalized map

\[ Υ \to E^1(\mathcal{E}^n(H_s(D_{rss}))) \]

has the appropriate image. □

All of the work to compute \( E^2 \) has now been completed. Figure 1 becomes Figure 2.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2}
\caption{\( E^2(\mathcal{E}^n(D_{rst})) \) for \( n > 2r - 2 \)}
\end{figure}

Theorem 2.3. Suppose \( s > 0 \). Then a basis for \( E^2(\mathcal{E}^n(D_{rst})) \) is given by the disjoint union of bigraded sets \( ε, ε^d, b, b^d, t, t^d, m^1, \) and \( m^2 \), which consist of a single
element in each of the indicated bidegrees:

\[
\begin{align*}
c : \{-s\} \times [2t, 2t + n - 1] & \quad c^d : \{-s - r\} \times [2t + 2r - 2, 2t + 2r + n - 3] \\
b : [-2s, -s - 1] \times \{2t\} & \quad b^d : [-2s - 2r, -s - r - 1] \times \{2t + 2r - 2\} \\
t : [-2s, -s - 2] \times \{2t + n\} & \quad t^d : [-2s - 2r, -s - r - 2] \times \{2t + 2r + n - 2\} \\
m^1 : \{(-2s - r, 2t + r - 1)\} & \quad m^2 : \{(-2s - r, 2t + n + r - 1)\}
\end{align*}
\]

If \(s = 0\) then the statement is the same except that \(c\) is also nonempty in bidegree \((-s, 2t + n)\) (and of course \(b = \emptyset = t\)). If \(r = \infty\) then the basis is given by \(c\), \(b\), and \(t\).

**Proof.** Ignoring the vertical grading, \((E^1, \delta^1)\) is the direct sum of the complexes

\[
\begin{align*}
(k(\mathfrak{B} \sqcup \mathcal{C}_{-s,2t}), \delta^1) & \quad (k(\mathfrak{B}^d \sqcup (\mathcal{C}^d)_{-s-r,2t+2r-2}), \delta^1) \\
(k(\mathfrak{C} \sqcup \mathcal{C}_{-s,2t+n}), \delta^1) & \quad (k(\mathfrak{C}^d \sqcup (\mathcal{C}^d)_{-s-r,2t+2r+n-2}), \delta^1) \\
\bigoplus_{i=1}^{n-1} (k\mathcal{C}_{-s,2t+i}, 0) & \quad \bigoplus_{i=1}^{n-1} (k\mathcal{C}^d_{-s-r,2t+2r-2+i}, 0) \\
(k\mathfrak{M}^1, \delta^1) & \quad (k\mathcal{M}^2, \delta^1).
\end{align*}
\]

The result now follows from Proposition 2.22 in the present paper, and, from \cite{6}, Theorems 6.2 and 5.2 and Propositions 6.1 and 5.1. \(\Box\)

A variation of the proof of \cite{6} Theorem 8.3] gives that

**Proposition 2.4.** If \(r\) is finite then \(TE^\infty(\mathcal{E}^n(D_{rst})) = HTC(\mathcal{E}^n(D_{rst})) = 0\).

In \cite{6} (the case \(n = \infty\)), the structure of \(E^2\), combined with the vanishing of \(E^\infty\), determined the higher differentials in the spectral sequence. This is not the case here. Considering naturality would allow us to compute the higher differentials, but even a statement along the lines of \cite{6} Proposition 9.1] is horrendously complicated with many cases. Luckily, we will only need partial information about the differentials in this spectral sequence. We make some computations in section 4 but before that we restrict to the case \(r = \infty\) where we are able to give a complete answer. Section 3 is also an important component of the convergence results in section 7.

3. **Special Case:** \(r = \infty\)

In Figure 3 we visualize the statement of Theorem 2.3] when \(r = \infty\). Notice that \(E^2 \neq E^\infty\) when \(n < \infty\) and \(s > 0\). The reason for this is that there are \(n + 1\) nonzero terms on \(E^\infty\), but \(E^2\) has \(2s + n - 2\) nonzero classes. To justify the first part of this sentence, notice that an easy variant of the proof of \cite{6} Theorem 8.3] gives

**Theorem 3.1.** Let \(2 \leq r \leq \infty\) and \(0 \leq n \leq \infty\). Then

\[H_rTC(\mathcal{E}^n(D_{rst})) \cong TE^\infty(\mathcal{E}^n(D_{rst})).\]

Let \(I = b \cup c\) denote the generating set for the lower right portion of \(E^2(\mathcal{E}^n(D_{(\infty st)}))\). Since there is some overlap in total degrees, we need to check that the terms that survive to \(E^\infty\) are all in \(I\).
HOMOLOGY OPERATIONS AND COSIMPLICIAL ITERATED LOOP SPACES

Theorem 3.2. Consider the spectral sequence for $E^n(D_{\infty st})$. The classes from $E^2$ which survive and are nonzero at $E^\infty$ are exactly those of $l$ in total degrees $[2t - 2s, 2t - 2s + n]$.

Proof. We wish (according to Theorem 3.1 and [6, Proposition 8.4]) to compute the cohomology of

$$T((sk_n W)^v \otimes \pi C(D \otimes D)),$$

where $D = D_{\infty ss}$. If we set

$$b^n_m = \dim H_m T((sk_n W)^v \otimes \pi C(D \otimes D)),$$

we already know from Theorem 3.1 that

$$b^\infty_m = \begin{cases} 1 & m \geq 0 \\ 0 & m < 0 \end{cases}$$

and from Theorem 3.1 and [6, Proposition 4.2] that

$$b^0_m = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0. \end{cases}$$

We use comparison and induction to interpolate between these two extremes and show that

$$b^n_m = \begin{cases} 1 & m \in [0, n] \\ 0 & \text{otherwise.} \end{cases}$$

We have two exact sequences of complexes

(1) \hspace{1cm} 0 \to sk_n W \to W \to \Sigma^{n+1} W \to 0

and

(2) \hspace{1cm} 0 \to sk_0 W \to sk_n W \to \Sigma sk_{n-1} W \to 0

where the first map is the obvious inclusion in both cases. Freeness of $W_i$ over $k\pi$ tells us that if we apply $(-)^v \otimes \pi C(D \otimes D)$ to either of these exact sequences we will still have a short exact sequence of bicomplexes. Furthermore, taking products is exact, so applying $T$ we again get short exact sequences of complexes – write

$$B^n = T((sk_n W)^v \otimes \pi C(D_{\infty ss} \otimes D_{\infty ss})).$$
We first apply $T((-) \otimes_{\pi} C(D \otimes D))$ to $\text{SES}(2)$. The short exact sequence of complexes

$$0 \to B^0 \to B^n \to \Sigma B^{n-1} \to 0$$

gives us a long exact sequence in homology, so we have

\[
\begin{array}{c}
H_m B^0 \longrightarrow H_m B^n \longrightarrow H_m \Sigma B^{n-1} \longrightarrow H_{m-1} B^0 \\
0 \hspace{2cm} 0 \\
H_{m-1} B^{n-1} \hspace{2cm} 0
\end{array}
\]

for $m - 1 > 0$. If we assume inductively that

$$b_{m-1}^n = \begin{cases} 
1 & m \in [0, n-1] \\
0 & \text{otherwise}
\end{cases}$$

then we see that $b_m^n$ is zero for $m \in [n+1, \infty)$ and one for $m \in [2, n]$.

Next we apply $T((-) \otimes_{\pi} C(D \otimes D))$ to $\text{SES}(1)$, so we have

$$0 \to B^n \to B^\infty \to \Sigma B^{n+1} \to 0$$

and in the long exact sequence in homology we get

\[
\begin{array}{c}
H_{m+1} \Sigma B^{n+1} \longrightarrow H_m B^n \longrightarrow H_m B^\infty \longrightarrow H_m \Sigma B^n \longrightarrow H_{m-1} B^\infty \\
H_{m-n} B^\infty \hspace{2cm} H_{m-n-1} B^\infty
\end{array}
\]

This tells us that $b_m^n = b_m^\infty$ for $m < n$, since there $b_{m-n}^\infty = 0 = b_{m-n-1}^\infty$.

We thus know $b_m^n$ for $m \in [2, \infty) \cup [0, n-1]$, so for $n \geq 2$ we know it for all $m$.

The only thing we are missing is $b_1^1$, but this follows from the exact sequence

\[
\begin{array}{c}
0 \longrightarrow H_1 B^1 \longrightarrow H_1 \Sigma B^0 \longrightarrow H_0 B^0 \longrightarrow H_0 B^1 \longrightarrow 0 \\
k \hspace{2cm} k \hspace{2cm} k \hspace{2cm} k
\end{array}
\]

The following Corollary says that there is only one way for something in $t$ to be hit by something in $l$. Recall that $l = b \sqcup c$ and $t$ consist of at most one element in each bidegree.

**Corollary 3.3.** Consider the spectral sequence for $E^n(D_{\infty st})$. If $a \in [\max(n+1-s, 0), n-1]$, then examination of bidegrees tells us that

$$c_{-s, 2t+a} = \{c\} \quad \text{and} \quad t_{a-s-n-1, 2t+n} = \{v\}$$

are nonempty. The element $c$ survives to $E^{n+1-a}$ and

$$\delta^{n+1-a}[c] = [v] \neq 0.$$

If $a \in [n+1, s-1]$, then

$$b_{a-2s, 2t} = \{c\} \quad \text{and} \quad t_{a-2s-n-1, 2t+n} = \{v\}$$

are nonempty. The element $c$ survives to $E^{n+1}$ and

$$\delta^{n+1}[c] = [v] \neq 0.$$
**Proof.** When \( s = 0 \) this Corollary says that there are no nontrivial differentials, which is obvious since \( E_2 \) consists of a single column.

Assume \( s > 0 \) and \( t = 0 \). First note that \( l \) lives in total degrees \([-2s, -s + n - 1]\) and \( t \) lives in total degrees \([-2s + n, -s + n - 2]\). All differentials out of \( t \) are zero, so those elements in total degrees \([-2s + n, -s + n - 2]\) must be hit by something (Theorem 3.2). Thus elements of \( l \) in total degrees \([-2s + n + 1, -s + n - 1]\) have some nonzero differential. Collectively, these classes hit the classes of elements of \( t \) in total degrees \([-2s + n, -s + n - 2]\]. The classes that are unaccounted for are the elements of \( l \) in total degrees \([-2s, -2s + n + 1]\) and the element of \( t \) in total degree \(-2s + n\). In particular, the element \( c \) which lives in total degree \(-2s + n + 1\) doesn’t survive to \( E_\infty \), and since it lives in the second page it cannot map to something in \( l \). Hence there is a nontrivial differential

\[
0 \neq \delta^r([c]) = [v]
\]

where \( v \in t \) is in total degree \(-2s + n\). The statement then follows by passing from total degree to bidegree. \( \square \)

4. **Remaining pages, case \( r < \infty \)**

We now turn our attention to partial external operations. For a cosimplicial chain complex \( Y \), these are operations whose target is the spectral sequence for

\[
E^n(Y) = \text{sk}_n W \otimes_\pi (Y \otimes Y).
\]

They are of particular interest when we have a map

\[
E^n(Y) \rightarrow Y,
\]

such as when \( Y = S_* X \), where \( X \) is a cosimplicial \( E_{n+1} \)-space.

Parallel to what we did in [6], we first define operations on \( r \)-cycles:

**Definition.** Let

\[
\tilde{Q}^m : Z^r_{s,t}(Y) \rightarrow E^r(E^n(Y))
\]

be the functions defined, for \( y \in Z^r_{s,t}(Y) \), by

\[
\tilde{Q}_v^m(y) = E^r(E^n(\Theta_y))(q_{s,m+1}) \quad m \in [t, t - s + n] \\
\tilde{Q}_h^m(y) = E^r(E^n(\Theta_y))(q_{m,s-t,2t}) \quad m \in [t - s, \min(t, t - s + n)]
\]

where \( q_{p,q} \in E^2_{p,q}(E^n(D_{rst})) \) is nonzero.

Unfortunately, this definition does not make any sense yet. Why should the indicated \( q \) survive to page \( r \)? How do we know that there is only one generator in the indicated bidegrees? This section is devoted to these questions.

We expect that the top operation will not be additive, so we cannot immediately carry out the program given in section 12 of [6] to induce operations on the spectral sequence. We will define the Browder operation in Section 5 in order to study the deviation from additivity of \( \tilde{Q}^{s+n} \). For now, we perform the necessary spectral sequence computations in order to make the above definition a good one. In particular, we compute enough of the differentials of the spectral sequence associated to \( E^n(D_{rst}) \) to give a partial analogue to [6, Corollary 9.2].

The main tool is the comparison

\[
\phi : E^n(D_{rst}) \rightarrow E(D_{rst})
\]
induced by the inclusion

$$sk_n W \hookrightarrow W.$$ 

**Proposition 4.1.** Let \( n \geq 1 \) and \( \infty > r \geq 2 \). The kernel of \( \phi \) on the second page is

$$\ker(E^2(\phi)) = \ker(t \sqcup t^d \sqcup m^2).$$

**Proof.** Note that the map \( C(\phi) \) is just an inclusion. At \( E^1 \) the representatives of \( \mathcal{T}, \mathcal{T}^d \), and \( \mathcal{M}^2 \) are all vertical boundaries, so \( \ker(t \sqcup t^d \sqcup m^2) \subset \ker(E^2(\phi)) \). Comparing representatives in [6, Theorem 3.1] with the representatives in Theorem 2.1 gives that this inclusion is equality.

Define a set of integral lattice points \( L = L_{rst}^n \) by

\[
[-2s - 2r, -2s - 2r + n] \times \{2t + r - 2\}
\]

if \( n \leq s + r - 1 \) and by

\[
(-2s - 2r, -s - r - 1) \times \{2t + r - 2\}
\]

\[
\cup (-s - r) \times [2t + r - 2, 2t + r - s - 2 + n]
\]

if \( n \geq r + s \).

**Proposition 4.2.** If \((p, q) \in L\), then

$$E^2_{p,q}(E^n(D_{rst})) = \ker.$$ 

**Proof.** We know that \( 1 \) is a lower bound for dimension since at each of these lattice points there is an element of \( t^d \). The only classes at \( E^2 \) which might share a bidegree with \( t^d \) (and hence with \( L \)) are \( t \) and \( m^2 \). Notice that the lattice points of \( L \) cover the following range of total degrees:

\([2t - 2s - 2, 2t - 2s + n - 2],\)

while by Theorem 2.3 we know that \( m^2 \) lives in total degree \( 2t - 2s + n - 1 \) and \( t \) lives in total degree \( 2t - 2s + n \) and above.

**Lemma 4.3.** Consider \( E^2(E^n(D_{rst})) \) and let

\[
\{c\} = m_{-2s-r,2t+r-1}^1
\]

\[
\{v\} = b_{-2s-2r,2t+2r-2}^d
\]

Then

$$\delta^* [c] = [v] \neq 0.$$ 

**Proof.** Assume that \( t = 0 \). We list the ranges of total degrees of each of the various subsets which constitute a basis for \( E^2 \) (when \( s > 0 \):

\[
t : [-2s, -s + n - 1]
\]

\[
t^d : [-2s - 2, -s + r + n - 3]
\]

\[
t : [-2s + n, -s + n - 2]
\]

\[
t^d : [-2s + n - 2, -s + r + n - 4]
\]

\[
m^1 : \{-2s - 1\}
\]

\[
m^2 : \{-2s + n - 1\}
\]

Then element \( v \) is in the smallest possible total degree \(-2s - 2\) so must be hit by something in total degree \(-2s - 1\) since \( E^\infty = 0 \) (Proposition 2.4). The only elements which are in total degree \(-2s - 1\) are \( c \) and, if \( n = 1 \), the element in \( t^d_{-2s-2r,2r-1} \) (here we use that \( n \geq 1 \)). Examination of bidegrees indicates that the
second of these could only hit \( v \) via \( \delta^1 \), so we have the stated result for \( s > 0 \). The proof for \( s = 0 \) is similar. \( \square \)

**Proposition 4.4.** Suppose that \( v \in I \) has total degree in

\[
[2t - 2s, 2t - 2s - 1 + n]
\]

and \( j \) is such that \( \delta^j[\phi v]_j \neq 0 \). Then

\[
0 \neq \delta^j[v]_j \in I^d.
\]

**Proof.** Assume \( t = 0 \). If \([v]_j \) makes sense (that is \( \delta^k[v]_k = 0 \) for \( k < j \)), then

\[
\phi \delta^j[v]_j = \delta^j \phi[v]_j = \delta^j[\phi v]_j \neq 0
\]

so \( \delta^j[v]_j \neq 0 \). Proposition 4.2 coupled with \( [6, \text{Proposition 9.1}] \) then tell us that it must land in the stated place.

We now show that \( \delta^k[v]_k = 0 \) for \( 2 \leq k < j \). By Lemma 4.3 we know that \( v \) does not hit \( m^1 \) nontrivially. The differential of \( v \) is in the following range of total degrees,

\[
[-2s - 1, -2s + n - 2]
\]

so we see (as in the proof of Prop. 4.2) that \( v \) cannot hit any of the bidegrees spanned by \( t \) or \( m^2 \). On the other hand, \( t^d \) lives in the following range of total degrees

\[
[-2s + n - 2, r - s + n - 4],
\]

so it’s possible that \( v \) hits something in \( t^d \) if it has total degree \( -2s + r + n \). But \( t^d \) is so far away that this must happen on a page bigger than \( j \) (see Figure 2 on page 7). To be precise, the differential would be one of (writing \( c \in t_{-2s-2r, 2r+n-2} \))

\[
\delta^{2r+s}[v] = [c] \quad \text{if } v \in c_{-s-1+n}
\]

\[
\delta^{2r+n-1}[v] = [c] \quad \text{if } v \in b_{-2s-1+n, 0}
\]

whereas \( j \leq 2r - 1 \) by \([6, \text{Proposition 9.1}] \). \( \square \)

This Proposition tells us that the spectral sequence for \( E(E^{D_{rst}}) \) vanishes in the bidegrees of \( L \) at the same time as in \( E(D_{rst}) \). This is precisely what we will need to help us show that the \( Q^m \) vanish on appropriate boundaries.

**5. Additivity and the Browder Operation**

We would like to mimic Section 12 of \([6]\), but on first glance it appears that \([6, \text{Proposition 11.1}] \) fails when \( m = t - s + n \) because of the classical formula \([9, \text{Proposition 6.5}] \)

\[
\xi_n(x + y) = \xi_n(x) + \xi_n(y) + \lambda_n(x, y)
\]

where \( \lambda_n \) is the Browder operation and \( \xi_n(x_q) = Q^{n+n}(x) \). Surprisingly, additivity holds for the top operation as long as \( s > 0 \). We will see in a moment that this happens because the Browder operation lands in a lower filtration degree, but first we prove the additivity statement.

**Proposition 5.1 (Additivity).** Let \( r \geq 2 \) and

\[
b = \begin{cases} 
  t - s + n & s > 0 \\
  t + n - 1 & s = 0.
\end{cases}
\]
The functions
\[ \tilde{Q}^m : Z^r_{-s,t}(Y) \to E^r_{-s,m+t}(\mathcal{E}^n(Y)) \quad m \in [t,b] \]
\[ \tilde{Q}^m_n : Z^r_{-s,t}(Y) \to E^r_{m-s-t,2t}(\mathcal{E}^n(Y)) \quad m \in [t-s, \min(t, t-s+n)] \]
are homomorphisms.

Proof. As in the proof of \[6\] Proposition 11.1, we have
\[ E^1(\text{sk}_n W \otimes D_{rst} \otimes D_{rst}) \cong H_*(\text{sk}_n W) \otimes E^1(D_{rst}) \otimes E^1(D_{rst}) \]
which is nonzero only in the following list of bidegrees:
\[ (-2s, 2t), (-2s - r, 2t + r - 1), (-2s - 2r, 2t + 2r - 2) \]
\[ (-2s, 2t + n), (-2s - r, 2t + r - 1 + n), (-2s - 2r, 2t + 2r - 2 + n) \]
since \( H_*(\text{sk}_n W) = k e_0 \oplus k(1 + \sigma)e_n \). The only possible overlap with bidegrees of the operations are \((-2s, 2t)\), which is the external square, and, if \( s = 0, (0, 2t + n) \). But this last bidegree corresponds to the operation \( \tilde{Q}^{t+n}_v \) (when \( s = 0 \)), which is the one operation that is excluded from the statement of the Proposition. \( \square \)

Definition (Browder Operation). Let \( Y \) be a cosimplicial chain complex. Consider \( k \) as a chain complex in degree 0. Using the map \( k \to \Sigma^{-n} \text{sk}_n W \) which sends 1 to \((1 + \sigma)e_n\) for the middle arrow below, we consider the map of bicomplexes
\[ C(Y) \otimes C(Y) \xrightarrow{AW} C(Y \otimes Y) \xrightarrow{\Sigma^{-n}C(\text{sk}_n W \otimes Y \otimes Y)} \Sigma^{-n}C(\mathcal{E}^n(Y)) \]
Then we have a map
\[ \lambda_n : E^r_{-s,t}(Y) \otimes E^r_{-s',t'}(Y) \to E^r_{-s-s',t+t'}(\mathcal{E}^n(Y)) \]
which we call the external Browder operation.

There is a discrepancy in bidegrees. Our top operation for an element in \( Z^r_{-s,t} \) is in bidegree \((-s, 2t + n - s)\) or \((-2s + n, 2t)\), whereas the Browder operation of two elements in \( E^r_{-s,t} \) is in bidegree \((-2s, 2t + n)\). When \( s = 0 \) the Browder operation still measures the deviation from additivity of the top operation.

Proposition 5.2. Suppose that \( x, y \in Z^r_{0,t}(Y) \). Then
\[ \tilde{Q}^{t+n}(x + y) = \tilde{Q}^{t+n}_v(x) + \tilde{Q}^{t+n}_v(y) + \lambda_n([x], [y]). \]

Before beginning the proof, we make note of the following variation of \[6\] Lemma 11.2):

Lemma 5.3. Let \( X \) and \( Y \) be cosimplicial chain complexes. Then
\[ \mathcal{E}^n(X \oplus Y) \cong \mathcal{E}^n(X) \oplus \mathcal{E}^n(Y) \oplus (\text{sk}_n W \otimes X \otimes Y) \]
via
\[ e_i \otimes (x + y) \otimes (x' + y') \mapsto e_i \otimes x \otimes x' + e_i \otimes y \otimes y' + e_i \otimes x' \otimes y + \sigma e_i \otimes x' \otimes y. \]
\( \square \)
Proof of Proposition 5.2. Examine the diagram from [6, Proposition 11.1]

\[
\begin{array}{ccc}
D_{r0t} & \rightarrow & D_{r0t} \oplus D_{r0t} \\
\Theta_x \oplus \Theta_y & \rightarrow & Y
\end{array}
\]

where the top map is the diagonal. Lemma 5.3 still works when we replace \( W \) by \( sk_n W \), and we again get the decomposition

\[
E_n(D) \rightarrow E_n(D \oplus D) \oplus (sk_n W \otimes D \otimes D) \rightarrow E_n(Y)
\]

where \( D = D_{r0t} \). The image of \( q_{0,2t+n} \) under \( E^n(\Theta_x) \) and \( E^n(\Theta_y) \) give \( \tilde{Q}^{t+n}(x) \) and \( \tilde{Q}^{t+n}(y) \). We now seek to identify the image of \( q_{0,2t+n} \) under the composite

\[
E^n(D) \rightarrow sk_n W \otimes D \otimes D \rightarrow E^n Y.
\]

For maps \( f : A \rightarrow C \) and \( g : B \rightarrow C \), the following commutes

\[
\begin{array}{ccc}
sk_n W \otimes A \otimes B & \rightarrow & sk_n W \otimes C \otimes C \\
E^n(A \oplus B) & \rightarrow & E^n(C)
\end{array}
\]

where the left vertical arrow is the inclusion from Lemma 5.3. Replacing \( A = B = D_{r0t} \) and \( C = Y \), we extend this to the diagram

\[
\begin{array}{ccc}
C(D) \otimes C(D) & \rightarrow & C(Y) \otimes C(Y) \\
\rightarrow & \rightarrow \\
C(D \otimes D) & \rightarrow & C(Y \otimes Y) \\
\rightarrow & \rightarrow \\
\Sigma^{-n}C(sk_n W \otimes D \otimes D) & \rightarrow & \Sigma^{-n}C(sk_n W \otimes Y \otimes Y) \\
\rightarrow & \rightarrow \\
\Sigma^{-n}C(E^n(D \oplus D)) & \rightarrow & \Sigma^{-n}C(E^n(Y))
\end{array}
\]

The composite of the vertical maps on the right is what was used to define the external Browder operation, so

\[
E^r(D) \otimes E^r(D) \rightarrow E^r(Y) \otimes E^r(Y)
\]

takes \( \iota \otimes \iota \) to \( \lambda_n([x]_r, [y]_r) \). Furthermore, the Alexander-Whitney map is particularly simple on elements in cosimplicial degree 0: \( AW(id_0 \otimes id_0) = id_0 \otimes id_0 \). So the
vertical maps on the left give
\[ C(D) \otimes C(D) \rightarrow C(D \otimes D) \rightarrow \Sigma^{-n}C(\text{id}_n \otimes D \otimes D) \]
\[ \text{id}_0 \otimes \text{id}_0 \rightarrow \text{id}_0 \otimes \text{id}_0 \rightarrow (1 + \sigma)e_n \otimes \text{id}_0 \otimes \text{id}_0 . \]

At \( E^1 \) this coincides with the image of \( q_{0,2t+n} \) by Lemma 5.4.

Lemma 5.4. Let \( C \) be a chain complex. Consider the composite
\[ E^n(C) \xrightarrow{\sigma} E^n(C \oplus C) \rightarrow \text{sk}_n W \otimes C \otimes C \]
where the projection map is the one given by Lemma 5.3:
\[ E^n(X \oplus Y) \rightarrow \text{sk}_n W \otimes X \otimes Y \]
\[ e_m \otimes (x + y) \otimes (x' + y') \rightarrow e_m \otimes x \otimes y' + \sigma e_m \otimes x' \otimes y. \]

Then the homology of the composite sends \( e_n \otimes [c] \otimes [c] \rightarrow (1 + \sigma)e_n \otimes [c] \otimes [c]. \)

Proof. Fix a quasi-isomorphism \( C \rightarrow HC \). The following commutes,
\[ H_*E^n(C) \xrightarrow{\cong} H_*(E_n(C \oplus C)) \xrightarrow{\cong} H_*(\text{sk}_n W \otimes C \otimes C) \]
so it is enough to prove that for a module \( M \),
\[ H_*(E^n(M)) \rightarrow H_*(\text{sk}_n W \otimes M \otimes M) \]
sends \( e_n \otimes m \otimes m \) to \( (1 + \sigma)e_n \otimes m \otimes m \). This is an easy computation.

Remark. The formula given in Proposition 5.2 says that if \( y \) happens to be in \( B^r_{0,t} \) then
\[ \tilde{Q}_{v}^{t+n}(x + y) = \tilde{Q}_{v}^{t+n}(x) + \tilde{Q}_{v}^{t+n}(y) \]
since \( [y]_r = 0 \). If so we show that \( \tilde{Q}_{v}^{t+n}(y) = 0 \) for \( y \in B^r_{0,t} \) then we will know that \( \tilde{Q}_{v}^{t+n} \) induces a function
\[ E^r_{0,1}(Y) \rightarrow E^r_{0,2t+n}(E^n(Y)). \]

6. Definition of Operations

The proofs of nearly everything in section 12 of [10] now go through, with perhaps the only subtle point that the analogue of [6] Lemma 12.4 relies on the vanishing statement Proposition 5.4.

Lemma 6.1. The homomorphisms \( \tilde{Q}^m \) vanish on \( \tilde{Z}_{s-1,t+1}^r \) for \( r \geq 2 \).

Proof. Write \( r' = r - 1, s' = s + 1, t' = t + 1 \) and let \( y \in \tilde{Z}_{s',t'}^r(Y) \subset \tilde{Z}_{s,t}^r(Y). \)
Then the following commutes
\[ D_{rst} \xrightarrow{\Theta} D_{s't'} \]
\[ \Theta^r \]
\[ X \]
\[ \Theta^r \]

[10]
If \( r' \geq 2 \), then Theorem 2.3 says that \( E^r(\mathcal{E}^n(D_{r's't'})) \) is zero on the ranges \( \{ -s \} \times [2t, 2t + n - s] \) and \( [-2s, \min(-s - 1, 2t + n - s)] \times \{ 2t \} \) we are interested in. The diagram

\[
\begin{array}{ccc}
E^r(\mathcal{E}^n(D_{rst})) & \xrightarrow{\Theta_{p,q}} & E^r(\mathcal{E}^n(D_{r's't'})) \\
\downarrow & & \downarrow \\
E^r(\mathcal{E}^n(Y)) & & E^r(\mathcal{E}^n(Y))
\end{array}
\]

commutes and the rightmost composition takes \( q_{p,q} \) to zero for \( (p, q) \) in the appropriate range, so all of the \( Q \) must vanish on \( x \). If \( r = 2 \) then \( E^2(\mathcal{E}^n(D_{1s't'})) = 0 \). \( \square \)

In particular, this shows that the \( \hat{Q}^m \) vanish on \( \partial F^{-s-1} \), and the proof of the following is a minor variation of that of the corresponding Proposition in Section 12 of [6].

**Proposition 6.2.** The homomorphisms \( \hat{Q}^m \) vanish on \( \partial F^{-s} \).

**Lemma 6.3.** The vertical maps \( \hat{Q}_v \) vanish on \( \partial Z^{-s+1}_{-s+r-1,t-r+2} \) for \( r > 2 \).

**Proof.** Notice that we may assume that \( n \geq s \), otherwise we have not defined the vertical maps and the statement is vacuously true. Let \( r' = r - 1, s' = s + 1 - r, t' = t - r + 2 \). We may assume that \( y \in Z^{-s+1}_{-s+r-1,t-r+2} \) has the form

\[
y = \sum_{j=s-r+1}^{s-1} y^j.
\]

The following diagram commutes

\[
\begin{array}{ccc}
D_{rst} & \xrightarrow{\Theta_{p,q}} & D_{r's't'} \\
\downarrow \Theta_{p,q} & & \downarrow \Theta_{p,q} \\
Y & & Y
\end{array}
\]

Applying Proposition 4.2 to \( (r', s', t') \), we see that the vector space \( E^2_{p,q}(\mathcal{E}^n(D_{r's't'})) \) is one-dimensional for \( p = -s \) and \( q \in [2t, 2t - s + n] \). Furthermore, Proposition 4.2 tells us that all of these classes vanish at page \( r' + 1 = r \). These are exactly the bidegrees where we have defined vertical operations, so applying \( E^r(\mathcal{E}^n(\cdot)) \) to the above diagram we see that \( \hat{Q}_v(\partial y) = 0 \) on \( E^r \). \( \square \)

**Lemma 6.4.** If \( r = 2 \) then the homomorphisms \( \hat{Q} \) vanish on \( \partial Z^{-s+1}_{-s+1,t} \).

**Proof.** As in [6]. \( \square \)

**Theorem 6.5.** The maps above define functions

\[
\begin{align*}
Q_{r_1}^m : E^r_{-s,t}(Y) & \to E^r_{-s,m+t}(\mathcal{E}^n(Y)) \quad m \in [t, t - s + n] \\
Q_{r_1}^m : E^r_{-s,t}(Y) & \to E^r_{m-s,t+2}(\mathcal{E}^n(Y)) \quad m \in [t - s, \min(t, t - s + n)]
\end{align*}
\]

where

\[
w = \begin{cases} 
  r & m = t - s \\
  2r - 2 & m \in [t - s + 1, t - r + 2] \\
  r + t - m & m \in [t - r + 3, t].
\end{cases}
\]

They are homomorphisms unless \( s = 0 \) and \( m = t - s + n \), in which case there is an error term given by Proposition 5.2.
Proof. The only missing ingredient is the vanishing of $\tilde{Q}_n^m$ on an element $\partial y$ where $y \in Z_{r-s+r-1,t-r+2}^{-1}$ is of the form

$$y = \sum_{j=s-r+1}^{s-1} y^j.$$  

This is an extension of the proof of Lemma 6.3. According to Propositions 4.2 and 4.3 applied to $(r', s', t')$, part of [11 Corollary 9.2] applies in the spectral sequence for $E^n(D_{r's't'})$ to give appropriate vanishing in the range of bidegrees $[-2s+1, \min(-2s+n, -s-1)] \times \{2t\}$. In particular,

$$E^r_{p,2t}(E^n(D_{r's't'})) = 0$$

for $p \in [-2s+1, \min(-2s+n, -s-1)] \cap I$ and

$$E^r_{p,2t}(E^n(D_{r's't'})) = 0$$

for $p \in [\min(-r-s+3, -s)] \cap I$, where $I = [-2s+1, \min(-2s+n, -s-1)]$. Furthermore, Lemma 4.3 tells us that

$$E^{2r-2}_{-2s,2t}(E^n(D_{r's't'})) = 0.$$  

The statement then follows by going from $p$ to $m = t + s + p$.  

7. CONVERGENCE OF OPERATIONS

In this section, we take “space” to mean “simplicial set,” as we did in [11]. If $X$ is a cosimplicial space, then the target for the homology spectral sequence is $H_*(\text{Tot } X)$, using a filtration given by Bousfield in [2]. The arguments of [11] may be modified to give

**Theorem 7.1.** Suppose that $X$ is a cosimplicial space and $m \leq t - s + n$. Then

$$Q^n[F^{-s}H_{t-s}(\text{Tot } X)] \subset F^{-v}H_{t-s+m}(\text{Tot } (S^n \times X))$$

where

$$v(t, s, m) = \begin{cases} 
  s & m \geq t \\
  t + s - m & t - s \leq m \leq t. 
\end{cases}$$

Furthermore, for each $m$ the following diagram commutes:

\[
\begin{array}{ccc}
(F^{-s}/F^{-s-1})H_{t-s}(\text{Tot } X) & \longrightarrow & E^\infty_{s,t}(S_*(X)) \\
\downarrow \quad \text{Q}^m & & \quad \uparrow \text{Q}^m \\
(F^{-s}/F^{-s-1})H_{t-s+m}(\text{Tot } (S^n \times X)) & \longrightarrow & E^\infty_{v, v-s+m+t}(S_*(S^n \times X)) \\
\end{array}
\]

The obvious necessary change is to replace, throughout [11], $E_\pi$ and $B\pi$ by $S^n = \text{sk}_n E\pi$ and $\mathbb{R}P^n = \text{sk}_n B\pi$, respectively. The one other change we must make in comparison with the proof of [11, Theorem 2.1] is that we must use the results of Section 8 to get a handle on the $E^\infty$ term for the spectral sequence of $E^n(D_{\infty, st})$.

In particular, Theorem 7.1 implies convergence of the internal operations when $X$ is a cosimplicial $C_{n+1}$-space:
**Theorem 7.2.** Suppose that $X$ is a cosimplicial $C_{n+1}$-space. Then the Araki-Kudo operation $Q^m$ has the following effect on the filtration:

$$Q^m [F^{-s}H_{t-s}(\text{Tot}(X))] \subset F^{-v}H_{t-s+m}(\text{Tot}(X)),$$

where $v$ is given above. Furthermore, for each $m \in [t - s, t - s + n]$, the following diagram commutes:

$$
\begin{array}{ccc}
(F^{-s}/F^{-s-1})H_{t-s}(\text{Tot}(X)) & \longrightarrow & E_{s,t}^\infty(X) \\
\downarrow^{Q^m} & & \downarrow^{Q^m} \\
(F^{-v}/F^{-v-1})H_{t-s+m}(\text{Tot}(X)) & \longrightarrow & E_{v,v-s+m+1}^\infty(X)
\end{array}
$$

We also note that the results of [7] which are related to multiplication, namely Theorem 4.4 and Corollary 4.5, may be extended to the present situation with only minor changes to the proofs that appear in [7].

**Theorem 7.3.** Fix $n \geq 2$, and let $X$ be a cosimplicial space. Then the external multiplication $\mu: H^*\text{Tot}X \otimes H^*\text{Tot}X \rightarrow H^*(S^n \times_\pi X \times 2)$ is compatible with the filtration in the sense that

$$\mu(F^{-s} \otimes F^{-s'}) \subset F^{-s-s'}.$$

Furthermore, we have that the following diagram commutes.

$$
\begin{array}{ccc}
(F^{-s}/F^{-s-1})H^*\text{Tot}X \otimes (F^{-s'}/F^{-s'-1})H^*\text{Tot}X & \longrightarrow & E_{s,s}^\infty(S_*(X)) \otimes E_{s,s}^\infty(S_*(X)) \\
\downarrow & & \downarrow \\
(F^{-s-s'}/F^{-s-s'-1})H^*\text{Tot}(S^n \times_\pi X \times 2) & \longrightarrow & E_{s,s}^\infty(S_*(S^n \times_\pi X \times 2))
\end{array}
$$

If $X$ is a cosimplicial $C_{n+1}$-space, then we have the corresponding internal statement that the diagram

$$
\begin{array}{ccc}
(F^{-s}/F^{-s-1})H^*\text{Tot}X \otimes (F^{-s'}/F^{-s'-1})H^*\text{Tot}X & \longrightarrow & E_{s,s}^\infty(S_*(X)) \otimes E_{s,s}^\infty(S_*(X)) \\
\downarrow & & \downarrow \\
(F^{-s-s'}/F^{-s-s'-1})H^*\text{Tot}X & \longrightarrow & E_{s,s}^\infty(S_*(X))
\end{array}
$$

commutes. 

We now turn to the Browder operation, which is fundamentally an algebraic operation. Regarding $kS^n$ also as a constant cosimplicial simplicial $k$-module, we can iterate [7] Theorem 4.2 to get that bottom of the diagram

$$
\begin{array}{ccc}
H_*(\text{Tot}_\ell U) \otimes H_*(\text{Tot}_\ell V) & \longrightarrow & H_*(T_\ell CNU) \otimes H_*(T_\ell CNV) \\
\downarrow^{(1+\sigma)e_{n} \otimes - \otimes -} & & \downarrow^{(1+\sigma)e_{n} \otimes - \otimes -} \\
H_*(S^n) \otimes H_*(\text{Tot}_\ell U) \otimes H_*(\text{Tot}_\ell V) & \longrightarrow & H_*(S^n) \otimes H_*(T_\ell CNU) \otimes H_*(T_\ell CNV) \\
\downarrow & & \downarrow \\
H_*(\text{Tot}_\ell (kS^n \otimes U \otimes V)) & \longrightarrow & H_*(T_\ell CN(kS^n \otimes U \otimes V))
\end{array}
$$
commutes. The left hand composite

\[ \N \Tot U \otimes \N \Tot V \to \sk_n W \otimes \N \Tot U \otimes \N \Tot V \]

\[ \nabla : \N[k \text{S}^n \otimes \Tot U \otimes \Tot V] \to \N[\Tot(k \text{S}^n \otimes U \otimes V)] \]

can be seen a cosimplicial extension of the usual external Browder operation (see [9, p.184]). The right hand composite is given by

\[ T_l CNU \otimes T_l CNV \to \sk_n W \otimes T_l CNU \otimes T_l CNV \]

\[ \to T_l((\sk_n W)^{\nu} \otimes CNU \otimes CNV) \]

\[ \xrightarrow{AW} T_l C(\sk_n W \otimes NU \otimes NV) \]

\[ \nabla : T_l C(k \text{S}^n \otimes U \otimes V). \]

The composite

\[ T_l CNU \otimes T_l CNV \to \sk_n W \otimes T_l CNU \otimes T_l CNV \]

\[ \to T_l((\sk_n W)^{\nu} \otimes CNU \otimes CNV) \xrightarrow{AW} T_l C(\sk_n W \otimes NU \otimes NV) \]

is equal to

\[ T_l CNU \otimes T_l CNV \to T_l(CNU \otimes CNV) \]

\[ \to T_l[(\sk_n W)^{\nu} \otimes CNU \otimes CNV] \to T_l C(\sk_n W \otimes NU \otimes NV). \]

We see this show up as the the bottom left portion of the commutative diagram

\[ C(NU) \otimes C(NV) \xrightarrow{AW} C(NU \otimes NV) \]

\[ (\sk_n W)^{\nu} \otimes C(NU) \otimes C(NV) \xrightarrow{AW} C(\sk_n W \otimes NU \otimes NV) \]

Our definition of the Browder operation from page [14] is essentially the upper right portion of this diagram. Thus we see that our definition agrees with the classical one.

**Theorem 7.4.** We consider the map

\[ H_k(\Tot X) \otimes H_{k'}(\Tot X) \to H_{k+k'+n}(S^n \times \Tot X \times \Tot X) \to H_{k+k'+n}(\Tot(S^n \times \pi X \times 2)) \]

as the classical external Browder operation. This map descends to filtration quotients and, furthermore, our spectral sequence version \( \lambda_n \) from page [12] makes the following diagram commute.

\[ (F^{-s}/F^{-s-1})H \Tot X \otimes (F^{-s'}/F^{-s'-1})H \Tot X \xrightarrow{E_{\infty}^s(S_*(X)) \otimes E_{\infty}^{s'}(S_*(X))} \]

\[ (F^{-s-s'}/F^{-s-s'-1})H \Tot(S^n \times \pi X \times 2) \xrightarrow{E_{-s-s'}^{-\infty}(S_*(S^n \times \pi X \times 2))} \]

**Proof.** Most of the argument precedes the Theorem statement, and the rest is as in the proof of [7] Theorem 4.4. \( \square \)

**Acknowledgements.** Parts of this work are adapted from the author’s Ph.D. thesis, and the author thanks his advisor, Jim McClure, for careful readings and numerous clarifying suggestions.
HOMOLOGY OPERATIONS AND COSIMPLICIAL ITERATED LOOP SPACES

References

[1] Anthony P. Bahri. Operations in the second quadrant Eilenberg-Moore spectral sequence. J. Pure Appl. Algebra, 27(3):207–222, 1983.
[2] A. K. Bousfield. On the homology spectral sequence of a cosimplicial space. Amer. J. Math., 109(2):361–394, 1987.
[3] A. K. Bousfield and D. M. Kan. A second quadrant homotopy spectral sequence. Trans. Amer. Math. Soc., 177:305–318, 1973.
[4] Frederick R. Cohen. The homology of $C_{n+1}$-spaces, $n \geq 0$. In The Homology of Iterated Loop Spaces, Lecture Notes in Mathematics, Vol. 533, pages vii+490. Springer-Verlag, Berlin, 1976.
[5] W. G. Dwyer. Higher divided squares in second-quadrant spectral sequences. Trans. Amer. Math. Soc., 260(2):437–447, 1980.
[6] Philip Hackney. Homology operations and cosimplicial infinite loop spaces. I. Preprint, arXiv:1101.3798 [math.AT].
[7] Philip Hackney. Homology operations and cosimplicial infinite loop spaces. II. Preprint, arXiv:1101.5395 [math.AT].
[8] Hans Ligaard and Ib Madsen. Homology operations in the Eilenberg-Moore spectral sequence. Math. Z., 143:45–54, 1975.
[9] J. Peter May. A general algebraic approach to Steenrod operations. In The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod’s Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, pages 153–231. Springer, Berlin, 1970.
[10] William M. Singer. Steenrod Squares in Spectral Sequences, volume 129 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2006.
[11] James M. Turner. Operations and spectral sequences. I. Trans. Amer. Math. Soc., 350(9):3815–3835, 1998.

Department of Mathematics, University of California, Riverside
E-mail address: hackney@math.ucr.edu