SEMIQUASITRIANGULAR HOPF ALGEBRAS

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Abstract. We say that a Hopf algebra $H$ is semicocommutative if the right adjoint coaction factorizes through $H \otimes Z(H)$, where $Z(H)$ denotes the centre of $H$. For instance the commutative and the cocommutative Hopf algebras are semicocommutative. The quasitriangular Hopf algebras generalize the cocommutative Hopf algebras. In this paper we introduce and begin the study of a similar generalization for the semicocommutative ones. These algebras, which we call semiquasitriangular Hopf algebras have many of the basic properties of the quasitriangular ones. In particular, they have associated braided categories of representations in a natural way.

Introduction

Let $k$ be a field. All the algebras and vector spaces considered in this paper are over $k$, all the maps are $k$-linear maps and the unadorned tensor product will denote the tensor product over $k$. As usual, given a Hopf algebra $H$, we write $\mu$, $\eta$, $\Delta$, $\epsilon$ and $S$, adorned with a subscript if necessary, to denote the multiplication, the unit, the comultiplication, the counit and the antipode of $H$, respectively. For the comultiplication we use the Sweedler notation $\Delta(h) = h_1 \otimes h_2$, without summation symbol. Moreover, given a right $H$-comodule $M$ with coaction $\nu_M$, we write $\nu_M(m) = m_0 \otimes m_1$ (also without any summation symbol). If $M$ is a left $H$-module, then the notation that we will use is $\nu_M(m) = m_{-1} \otimes m_0$.

Recall that a Hopf algebra $H$ is called cocommutative if $\Delta = \tau \Delta$, where $\Delta$ denotes the comultiplication of $H$ and $\tau: H \otimes H \rightarrow H \otimes H$ is the flip $\tau(h \otimes l) = l \otimes h$. Let $S$ be the antipode of $H$. It is easy to check that $H$ is cocommutative if and only if $\operatorname{Ad}(h) := h_2 \otimes S(h_1)h_3 = h \otimes 1$ for all $h \in H$. That is, if the right adjoint coaction is trivial. In fact, in this case, $S^2(h) = S^2(h)S(h_1)h_3 = h$ for all $h \in H$, and then $h_2 \otimes h_1 = h_2 \otimes S^2(h_1) = h_3 \otimes S^2(h_2)S(h_1)h_4 = h_1 \otimes h_2$. The converse assertion is trivial.

Let $H$ be a Hopf algebra and let $Z(H)$ the centre of $H$. We say that a Hopf algebra $H$ is semicocommutative if $\operatorname{Ad}(h) \in H \otimes Z(H)$ for all $h \in H$. That is, if the right adjoint coaction factorizes through $H \otimes Z(H)$.

For instance, the commutative and cocommutative Hopf algebras are semicommutative Hopf algebras. Moreover, the class of these algebras is closed under the operations of taking tensor products, subHopfalgebras and quotients.

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Let $H$ be a semicommutative Hopf algebra. A normal $H$-module is a vector space $M$, endowed with a left action $\rho: H \otimes M \to M$ and a right coaction $\nu: M \to M \otimes H$, such that for all $h \in H$ and $m \in M$,

1. $\nu(m) \in M \otimes Z(H)$,
2. $\nu(h \cdot m) = h_2 \cdot m_0 \otimes S(h_1) h_3 m_1$.

A morphism $f: M \to N$ of normal $H$-modules is a map which is a morphism of $H$-modules and $H$-comodules.

For instance, $H$, endowed with the left regular action and the right adjoint coaction, is a normal $H$-module, and, if $H$ is a cocommutative Hopf algebra, then each left $H$-module, endowed with the trivial coaction, is a normal $H$-module.

Note that the definition of normal $H$-module is similar to the definition of Yetter-Drinfeld module. In fact, when $H$ is commutative both notions coincide.

Next, we mention without proof some results about semicommutative Hopf algebras.

**Theorem.** For each semicommutative Hopf algebra $H$, the category $_H \mathcal{M}^H(n)$, of normal $H$-modules, is a braided category. The unit object is $k$, endowed with the trivial action and the trivial coaction, and the tensor product is the usual tensor product over $k$, endowed with the diagonal action and the diagonal coaction. The associative and unit constraints are the usual ones and the braid $c$ is given by $c_{MN}(m \otimes n) = n_0 \otimes n_1 \cdot m$.

**Example.** Let $G$ be a group and let $Z(G)$ be the center of $G$. A normal $k[G]$-module is a direct sum $M = \bigoplus_{g \in Z(G)} M_g$, of left $G$-modules $M_g$. A map $f: M \to N$ is a morphism of normal $k[G]$-modules if it is $G$-linear and $f(M_g) \subseteq N_g$ for all $g \in Z(G)$. The braid $c_{MN}: M \otimes N \to N \otimes M$ is given by $c_{MN}(m \otimes n) = n \otimes g \cdot m$, for $m \in M$ and $n \in N_g$.

**Example.** Let $G$ be a finite group. Using that a right $k[G]^*$-comodule is the same that a left $k[G]$-module and that a $k[G]^*$-module is the same that a $G$-graduate $k$-module, it is easy to check that a normal $k[G]^*$-module is a left $G$-module $M$, endowed with a decomposition $M = \bigoplus_{g \in G} M_g$, such that $m \in M_g \Rightarrow y \cdot m \in M_{gy^{-1}}$, for all $g, y \in G$.

Moreover, a map $f: M \to N$ is a morphism of normal $k[G]^*$-modules if it is $G$-linear and $f(M_g) \subseteq N_g$ for all $g \in G$. Finally, the braid $c_{MN}: M \otimes N \to N \otimes M$ is given by $c_{MN}(m \otimes n) = g \cdot n \otimes m$, for $m \in M_g$ and $n \in N$.

**Theorem.** For each semicommutative Hopf algebra $H$, the full braided subcategory of $_H \mathcal{M}^H(n)$, consisting of all finite dimensional normal $H$-modules, is rigid.

**Proposition.** If $H$ is an semicommutative Hopf algebra, then

$$S^2(h) = h_2 S(h_1) h_3, \quad \text{for all } h \in H.$$

One can think the quasitriangular Hopf algebras [D] as a generalization of the cocommutative ones. The aim of this paper is to introduce and begin the study of a similar generalization for the semicommutative ones, that we call semiquasitriangular Hopf algebras. Such an algebra is a pair $(H, R)$, consisting of a Hopf
algebra $H$ and an invertible element $R$ of $H \otimes H$, satisfying suitable conditions. From the definition it follows that $(H, 1 \otimes 1)$ is semiquasitriangular if and only if $H$ is semicocommutative. Moreover, the quasitriangular Hopf algebras are semiquasitriangular.

Our main results are Theorem 2.5, Corollary 2.10, Theorem 2.11 and Proposition 3.2. In particular we get generalizations of the semicocommutative Hopf algebras results mentioned above.

We want to note that one can arrive to the notion of semiquasitriangular Hopf algebra in a different way to the considered in this paper. In [G-G1] (see also [G-G2]) we define a notion of Hopf crossed products, that generalize the classical one introduced in [B-C-M] and [D-T]. In [M1] was proved that if $H$ admits a quasitriangular structure, then the Drinfeld double $D(H)$ of $H$ is isomorphic to a classical Hopf crossed product $A\#H$. In [D-G-G] was proved that for $D(H)$ be isomorphic to a Hopf crossed product $A\#H$ in the sense of [G-G1], it suffices that $H$ admits a semiquasitriangular structure. This gives a version for this setting of the Majid result.

1. Semiquasitriangular Hopf algebras

In this section we introduce the notion semiquasitriangular Hopf algebras and study its basic properties. Moreover, we show that this concept includes the ones of quasitriangular and semicocommutative Hopf algebras. All the results of this section are immediate in the last case.

Before beginning we establish some notations. Let $H$ be a Hopf algebra, $R = \sum_i R^{(1)}_i \otimes R^{(2)}_i$ an invertible element of $H \otimes H$ and $\tau : H \otimes H \to H \otimes H$ the flip $\tau(h \otimes l) = l \otimes h$.

1. We will write $R = R^{(1)} \otimes R^{(2)}$, understanding the summation symbol and the index $i$. Similarly $R^{-(1)} \otimes R^{-(2)}$ denotes $R^{-1}$. When it is necessary we let $R^{(1)}_{\prime} \otimes R^{(2)}_{\prime}$, etcetera denote copies of $R$.

2. Given $n \geq 1$ let $H^\otimes n$ be the tensor products of $n$ copies of $H$. For any 2-tuple $(k_1, k_2)$ of distinct elements of $\{1, \ldots, n\}$, we let $R_{k_1, k_2}$ denote the element of $H^\otimes n$, given by

$$R_{k_1, k_2} = \sum_i y_{i, (1)}^{(1)} \otimes \cdots \otimes y_{i, (n)}^{(n)},$$

where $y_{i, (k_j)}^{(j)} = R_{i, (j)}$ for any $j = 1, 2$ and $y_{i, (k)}^{(j)} = 1$, otherwise.

3. Given $1 \leq i < n$ we let $\tau_{\otimes n} : H^\otimes n \to H^\otimes n$ denote the map $\tau_{\otimes n} := H^\otimes i-1 \otimes \tau \otimes H^\otimes n-i-1$. Here we extend the notation of $H^\otimes n$, taking $H^\otimes 0 = k$.

**Definition 1.1.** A semiquasitriangular Hopf algebra is a pair $(H, R)$, where $H$ is a Hopf algebra with bijective antipode and $R \in H \otimes H$ is an invertible element satisfying

1. $R_{1,2} = R_{1,1} \otimes R_{2,2}$,
2. $R^{(1)}_{1,2} \otimes R^{(2)}_{1,2} = R^{(1)}_{1,1} \otimes R^{(2)}_{1,2} \otimes R^{(2)}_{1,1}$,
3. $R^{(1)}_{1,2} \otimes R^{(2)}_{1,2} R^{(1)}_{1,2} R^{(2)}_{1,2} = R^{(1)}_{1,1} \otimes R^{(1)}_{1,2} R^{(2)}_{1,1} \otimes R^{(2)}_{1,2} R^{(2)}_{1,2}$,
Let $H$ be a semiquasitriangular Hopf algebra and let $n \geq 1$. The following assertions are valid:

1. $(\Delta^n \otimes H)(R) = R_{1,n+2}R_{2,n+2} \cdots R_{n+1,n+2}$,
2. $(H \otimes \Delta^n)(R) = R_{1,n+2}R_{1,n+1} \cdots R_{12}$,
3. $(H \otimes \tau_n \Delta^n)(R)R_{i+1,i+2} = R_{i+1,i+2}(H \otimes \Delta^n)(R)$ for all $1 \leq i \leq n$,
4. $(\tau_n \Delta^n \otimes H)(R)R_{i,i+1} = R_{i,i+1}(\Delta^n \otimes H)(R)$ for all $1 \leq i \leq n$.

**Proof.** Items (1) and (2) follow easily by induction on $n$. We prove item (3) and leave the last one to the reader. Assume by induction the formula is true for $n$. Since,

$$(H \otimes \Delta^{n-1})(R) = R_{1,n+1}R_{1n} \cdots R_{12},$$

we have

$$(H \otimes \tau_n \Delta^n)(R)R_{i+1,i+2} = (H^i \otimes \Delta^\text{cop} \otimes H^{n-i})(R_{1,n+1}R_{1n} \cdots R_{12})R_{i+1,i+2} = R_{i,n+2}R_{i,n+1} \cdots R_{i+1,i+3}(H^i \otimes \Delta^\text{cop})(R_{1,i+1})R_{i+1,i+2}R_{i,i-1} \cdots R_{12}.$$

Hence, by item (3) of Definition 1.1,

$$(H \otimes \tau_n \Delta^n)(R)R_{i+1,i+2} = R_{i,n+2}R_{i,n+1} \cdots R_{i+1,i+3}R_{i+1,i+2}(H^i \otimes \Delta)(R_{1,i+1})R_{i,i-1} \cdots R_{12} = R_{i+1,i+2}(H \otimes \Delta^n)(R),$$

where the last equality follows by (2). 

**Proposition 1.3.** If $(H, R)$ is a semiquasitriangular Hopf algebra, then

$$(\epsilon \otimes H)(R) = (H \otimes \epsilon)(R) = 1, \quad (S \otimes H)(R) = R^{-1} \quad \text{and} \quad (H \otimes S)(R^{-1}) = R.$$

Hence, $(S \otimes S)(R) = R$.

**Proof.** The proof given in [M2, Lemma 2.1.2] for quasitriangular Hopf algebras only use items (1) and (2) of Definition 1.1. 

**Proposition 1.4.** If $(H, R)$ is an semiquasitriangular Hopf algebra, then

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$ 

**Proof.** The proof given in [M2, Lemma 2.1.4] for quasitriangular Hopf algebras only use items (2) and (3) of Definition 1.1.
Example 1.5. A Hopf algebra $H$ is semicommutative if and only if $(H, 1_{H \otimes H})$ is a semiquasitriangular Hopf algebra.

Example 1.6. Each quasitriangular Hopf algebra is semiquasitriangular. In fact, if $(H, R)$ is a quasitriangular Hopf algebra, then it is well known that $(H, R)$ satisfies conditions (1)–(4) of Definition 1.1. Moreover, for all $h \in H$,

$$
\nu(h) = R^{(2)} h_2 R^{(2)} \otimes S(h_1) S(R^{(1)}) h_3 R^{(1)} \\
= R^{(2)} h_2 R^{(2)} \otimes S(R^{(1)} h_1) h_3 R^{(1)} \\
= h_1 R^{(2)} R^{(2)} \otimes S(h_2 R^{(1)}) h_3 R^{(1)} \\
= h R^{(2)} R^{(2)} \otimes S(R^{(1)}) R^{(1)} \\
= h \otimes 1,
$$

which clearly belongs to $H \otimes Z(H)$, and

$$
R^{(1)} h_2 R^{(1)} \otimes S(R^{(2)}) S(h_1) R^{(2)} h_3 = h_3 R^{(1)} R^{(1)} \otimes S(R^{(2)}) S(h_1) h_2 R^{(2)} \\
= h R^{(1)} R^{(1)} \otimes S(R^{(2)}) R^{(2)} \\
= h R^{(1)} R^{(1)} \otimes S(S^{-1}(R^{(2)}) R^{(2)}) \\
= h \otimes 1.
$$

Example 1.7. If $(H, R_H)$ and $(L, R_L)$ are semiquasitriangular Hopf algebras, then $(H \otimes L, R_H \otimes R_L)$, where $R_H \otimes R_L := (H \otimes \tau \otimes L)(R_H \otimes R_L)$, also is.

Example 1.8. The class of semiquasitriangular Hopf algebra is closed under the operations of taking quotients. Moreover, if $(H, R)$ is a quasitriangular Hopf algebra, $L \subseteq H$ is a subHopf algebra of $H$ and $R \in L \otimes L$, then $(L, R)$ is a semiquasitriangular Hopf algebra.

Proposition 1.9. If $(H, R)$ is a semiquasitriangular Hopf algebra, then

$$
\nu(h) = R^{(2)} h_2 R^{(2)} \otimes S^2(R^{(1)}) S(h_1) S(R^{(1)}) h_3 \\
= R^{(1)} h_2 S(R^{(1)}) \otimes S(h_1) R^{(2)} h_3 R^{(2)},
$$

for all $h \in H$.

Proof. Since, by Proposition 1.3, $R^{(2)} R^{(2)} \otimes S^2(R^{(1)}) S(R^{(1)}) = 1 \otimes 1$, we have

$$
\nu(h) = R^{(2)} h_2 R^{(2)} \otimes S(h_1) S(R^{(1)}) h_3 R^{(1)} \\
= R^{(2)} h_2 R^{(2)} R^{(2)} R^{(2)} \otimes S(h_1) S(R^{(1)}) h_3 R^{(1)} S^2(R^{(1)}) S(R^{(1)}) \\
= R^{(2)} h_2 R^{(2)} R^{(2)} R^{(2)} \otimes S^2(R^{(1)}) S(h_1) S(R^{(1)}) h_3 R^{(1)} S(R^{(1)}) \\
= R^{(2)} h_2 R^{(2)} \otimes S^2(R^{(1)}) S(h_1) S(R^{(1)}) h_3.
$$

So, the first equality is true. The second one follows in a similar way. □

Let $H$ be a braided Hopf algebra. Since $S(Z(H)) = Z(H)$, it is true that

$$(S \otimes S) \nu S^{-1}(h) \in H \otimes Z(H) \quad \text{for all } h \in H.$$
Using the expressions for $\nu$ given in Definition 1.1 and Proposition 1.9 to compute this map, we obtain that for all $h \in H$:

\[
S \otimes S \nu S^{-1}(h) = R^{(2)}h_2R^{(2)} \otimes R^{(1)}h_1S(R^{(1)})S(h_3)
\]

\[
= R^{(1)}h_2R^{(1)} \otimes h_1R^{(2)}S(h_3)S(R^{(2)})
\]

\[
= R^{(2)}h_2R^{(2)} \otimes h_1S(R^{(1)})S(h_3)S^2(R^{(1)})
\]

\[
= S(R^{(1)})h_2R^{(1)} \otimes R^{(2)}h_1R^{(2)}S(h_3).
\]

**Proposition 1.10.** If $R$ is a semiquasitriangular structure for a Hopf algebra $H$, then so is $\tau(R^{-1})$.

**Proof.** Conditions (1)–(4) of Definition 1.1 follow from standard arguments for quasitriangular Hopf algebras [M2, Exercise 2.1.3]. Hence, we only check conditions (5) and (6). By Proposition 1.3 and the fact that $(H, R)$ satisfies Proposition 1.9,

\[
\nu_{(H, \tau(R^{-1}))}(h) = S(R^{(1)})h_2S(R^{(1)}) \otimes S(h_1)S(R^{(2)})h_3R^{(2)}
\]

\[
= R^{(1)}h_2S(R^{(1)}) \otimes S(h_1)R^{(2)}h_3R^{(2)}
\]

\[
= \nu_{(H, R)}(h).
\]

Consequently, $(H, \tau(R^{-1}))$ satisfies condition (5). Moreover, by Proposition 1.3, Proposition 1.9 and condition (5) of Definition 1.1 for $R$, we have

\[
\nu_{(H, \tau(R^{-1}))}(h) = \nu_{(H, R)}(h)
\]

\[
= R^{(2)}h_2R^{(2)} \otimes S^2(R^{(1)})S(h_1)S(R^{(1)})h_3
\]

\[
= R^{(-2)}h_2R^{(-2)} \otimes S(R^{(-1)})S(h_1)R^{(-1)}h_3.
\]

So, $(H, \tau(R^{-1}))$ also satisfies condition (6). \qed

**Proposition 1.11.** If $R$ is a semiquasitriangular structure for a Hopf algebra $H$, then $\tau(R)$ and $R^{-1}$ are semiquasitriangular structures for $H^{\text{op}}$ and $H^{\text{cop}}$.

**Proof.** By Proposition 1.10 it suffices to prove it for $\tau(R)$. As before, we only check conditions (5) and (6) of Definition 1.1, since conditions (1)–(4) follow from standard arguments for quasitriangular Hopf algebras. In the rest of the proof all the multiplications are in $H$. By Proposition 1.3 and the fact that $(H, R)$ satisfies condition (5) of Definition 1.1, we have

\[
(S^{-1} \otimes H) \nu_{(H^{\text{op}}, \tau(R))}(h) = S^{-1}(R^{(1)})h_2R^{(1)} \otimes R^{(2)}h_3S^{-1}(R^{(2)})S^{-1}(h_1)
\]

\[
= S^{-1}(R^{(1)})S^{-1}(h_2)S^{-1}(R^{(3)}) \otimes R^{(2)}S(S^{-1}(h_1))S^{-1}(R^{(2)})S^{-1}(h_3)
\]

\[
= R^{(1)}S^{-1}(h_2)R^{(1)} \otimes S(R^{(2)})S(S^{-1}(h_1))R^{(2)}S^{-1}(h_3)
\]

\[
= \nu_{(H, R)}(S^{-1}(h)),
\]

for all $h \in H$. From this it follows immediately that $\nu_{(H^{\text{op}}, \tau(R))}(h) \in H^{\text{op}} \otimes Z(H^{\text{op}})$. Now, let $h \in H$. Using Propositions 1.3 and 1.9, we obtain

\[
\nu_{(H, R)}(S^{-1}(h)) = R^{(2)}S^{-1}(h_2)R^{(2)} \otimes S(S^{-1}(h_1))S(R^{(1)})S^{-1}(h_3)R^{(1)}
\]

\[
= S^{-1}(R^{(2)})S^{-1}(h_2)S^{-1}(R^{(2)}) \otimes S(h_3)R^{(1)}S^{-1}(h_1)S^{-1}(R^{(1)})
\]

\[
= S^{-1}(R^{(2)})h_2R^{(2)} \otimes h_3R^{(1)}S^{-1}(h_1)S^{-1}(R^{(1)}).
\]

From these facts it follows easily that $(H^{\text{op}}, \tau(R))$ satisfies condition (6) of Definition 1.1. It remains to prove that $\tau(R)$ is a semiquasitriangular structure for $H^{\text{cop}}$. We leave this task to the reader. \qed
2. Normal modules

Let \((H, R)\) be a semiquasitriangular Hopf algebra. In this section we introduce the category \((H, R)\mathcal{M}^{(H, R)}(n)\) of left-right normal \((H, R)\)-modules and begin the study of its properties. We assert that this is the suitable category of representations of \((H, R)\). Evidence that this assertion is “right” is given by the facts (proved in this section) that

- \(H\) is a left-right normal \((H, R)\)-module in a natural sense,
- \((H, R)\mathcal{M}^{(H, R)}(n)\) is a braided category,
- The full braided subcategory of \((H, R)\mathcal{M}^{(H, R)}(n)\) made out of the finite dimensional modules is rigid,
- if \((H, R)\) is quasitriangular, then the category of left \(H\)-modules is, in a natural way, a braided subcategory of \((H, R)\mathcal{M}^{(H, R)}(n)\).

For an exposition of the theory of braided Hopf algebras, we remit to [J-S], [Ka], [Ch-P] and [M2].

**Definition 2.1.** Let \((H, R)\) be an semiquasitriangular Hopf algebra.

A left-right normal \((H, R)\)-module is a vector space \(M\), endowed with a left action \(\rho_M : H \otimes M \to M\) and a right coaction \(\nu_M : M \to M \otimes H\), such that for all \(h \in H\) and \(m \in M\),

1. \(\nu_M(m) \in M \otimes Z(H)\),
2. \(\nu_M(h \cdot m) = R^{(2)}h_2R^{(2)} \cdot m_0 \otimes S(h_1)S(R^{(1)})h_3R^{(1)}m_1\).

A left normal \((H, R)\)-module is a vector space \(M\), endowed with a left action \(\rho_M : H \otimes M \to M\) and a left coaction \(\nu_M : M \to H \otimes M\), such that for all \(h \in H\) and \(m \in M\),

1. \(\nu_M(m) \in Z(H) \otimes M\),
2. \(\nu_M(h \cdot m) = S^{-1}(h_3)S^{-1}(R^{(2)})h_1R^{(2)}m_{-1} \otimes R^{(1)}h_2R^{(1)} \cdot m_0\).

A right normal \((H, R)\)-module is a vector space \(M\), endowed with a right action \(\rho_M : M \otimes H \to M\) and a right coaction \(\nu_M : M \to M \otimes H\), such that for all \(h \in H\) and \(m \in M\),

1. \(\nu_M(m) \in M \otimes Z(H)\),
2. \(\nu_M(m \cdot h) = m_0 \cdot R^{(1)}h_2R^{(1)} \otimes m_1R^{(2)}h_3S^{-1}(R^{(2)})S^{-1}(h_1)\).

A right-left normal \((H, R)\)-module is a vector space \(M\), endowed with a right action \(\rho_M : M \otimes H \to M\) and a left coaction \(\nu_M : M \to H \otimes M\), such that for all \(h \in H\) and \(m \in M\),

1. \(\nu_M(m) \in Z(H) \otimes M\),
2. \(\nu_M(m \cdot h) = m_{-1}R^{(1)}h_1S(R^{(1)})S(h_3) \otimes m_0 \cdot R^{(2)}h_2R^{(2)}\).

In all the cases a morphism \(f : M \to N\) of normal \(H\)-modules is a map which is a morphism of \(H\)-modules and \(H\)-comodules.

For instance, if \(H\) is a semicommutative Hopf algebra, then each normal \(H\)-module is a left-right normal \((H, 1_{H \otimes H})\)-module, and if \((H, R)\) is a quasitriangular Hopf algebra, then each left \(H\)-module can be think as a left-right normal \((H, R)\)-module with trivial coaction.

From now on by a normal \((H, R)\)-module we understand a left-right normal \((H, R)\)-module.
Lemma 2.2. Let \((H, R)\) be an semiqasitriangular Hopf algebra. For each \(h \in H\),
\[(H \otimes \Delta) \nu(h) \in H \otimes \mathbb{Z}(H) \otimes H.\]

Proof. By item (1) of Definition 1.1, we have:
\[(H \otimes \Delta) \nu(h) = (H \otimes \Delta)(R(2)h_2 R(2) \otimes S(h_1)S(R(1))h_3 R(1)h_1)\]
\[= R(2)h_3 R(2) \otimes S(h_2)S(R(1)h_4 R(1)) \otimes S(h_1)S(R(1))h_5 R(1)h_1\]
\[= R(2) \tilde{R}(2)h_3 R(2) \tilde{R}(2) \otimes S(h_2)S(\tilde{R}(1))h_4 R(1) \otimes S(h_1)S(R(1))h_5 \tilde{R}(1).\]

Now, the assertion follows immediately from item (5) of Definition 1.1. \(\square\)

Proposition 2.3. If \((H, R)\) is an semiqasitriangular Hopf algebra, then \(H\), endowed with the left regular action and the coaction \(\nu\) introduced in Definition 1.1, is a normal \(H\)-module.

Proof. It is immediate that \(\nu\) is a counitary map that satisfies condition (1) of Definition 2.1. Next, we check that it is coassociative. By item (2) of Proposition 1.2, we have:
\[(\nu \otimes H) \nu(h) = (\nu \otimes H)(R(2)h_2 R(2) \otimes S(h_1)S(R(1))h_3 R(1)h_1)\]
\[= \tilde{R}(2)R(2)h_3 \tilde{R}(2) \tilde{R}(2) \otimes S(R(2))h_2 \tilde{R}(2) \tilde{R}(2) \otimes S(R(1)h_4 R(1)) \otimes S(h_1)S(R(1))h_5 R(1)\]
\[= \tilde{R}(2) \tilde{R}(2)h_3 \tilde{R}(2) \tilde{R}(2) \otimes S(R(2))h_2 \tilde{R}(2) \tilde{R}(2) \otimes S(R(1)h_4 R(1)) \otimes S(h_1)S(R(1))h_5 \tilde{R}(1).\]

Hence, by conditions (1) and (4) of Definition 1.1 and Proposition 1.3, we have:
\[(\nu \otimes H) \nu(h) = \tilde{R}(2)h_3 \tilde{R}(2) \tilde{R}(2) \otimes S(R(2))h_2 \tilde{R}(2) \tilde{R}(2) \otimes S(R(1)h_4 R(1)) \otimes S(h_1)S(R(1))h_5 \tilde{R}(1)\]
On the other hand, by Proposition 1.3 and Lemma 2.2,

\[
(H \otimes \Delta) \nu(h) = \tilde{R}^{(2)} h_{3} \tilde{R}^{(2)} \otimes S(h_{2}) S(\tilde{R}^{(1)}_{2}) h_{4} \tilde{R}^{(1)}_{1} \otimes S(h_{1}) S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \\
= \tilde{R}^{(2)} h_{3} \tilde{R}^{(2)} \otimes S(h_{2}) S(\tilde{R}^{(1)}_{2}) h_{4} \tilde{R}^{(1)}_{1} \otimes S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \\
\otimes S(h_{1}) S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \\
= \tilde{R}^{(2)} h_{3} \tilde{R}^{(2)} \otimes S(h_{2}) S(\tilde{R}^{(1)}_{2}) h_{4} \tilde{R}^{(1)}_{1} \otimes S(\tilde{R}^{(1)}_{1}) \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \\
\otimes S(h_{1}) S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \\
= \tilde{R}^{(2)} h_{3} \tilde{R}^{(2)} \otimes S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \\
\otimes S(h_{1}) S(\tilde{R}^{(1)}_{1}) h_{5} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \tilde{R}^{(1)}_{2} \\
= \nu(h) \nu(l),
\]

as desired. \(\square\)

**Corollary 2.4.** If \((H, R)\) be a semiquasitriangular Hopf algebra, then \(H\) is a left normal \((H, R)\)-module via the left regular action and the left coaction

\[
\nu_{1}(h) := S^{-1}(h_{3}) S^{-1}(R^{(2)}) h_{1} R^{(1)} h_{2} R^{(1)},
\]

it is a right normal \((H, R)\)-module via the right regular action and the right coaction

\[
\nu_{2}(h) := R^{(1)} h_{2} R^{(1)} \otimes R^{(2)} h_{3} S^{-1}(R^{(2)}) S^{-1}(h_{1}),
\]

and it is a right-left \((H, R)\)-module via the right regular action and the left coaction

\[
\nu_{3}(h) := R^{(1)} h_{1} S(R^{(1)}) S(h_{3}) \otimes R^{(2)} h_{2} R^{(2)}.
\]

**Proof.** By Proposition 2.3, it suffices to note that a left normal \((H, R)\)-module is the same that a normal \((H^{\text{cop}}, R)\)-module, a right normal \((H, R)\)-module is the same that a normal \((H^{\text{cop}}, R)\)-module and a right-left \((H, R)\)-module is the same that a normal \((H^{\text{cop}, R})\)-module. \(\square\)

**Theorem 2.5.** Let \((H, R)\) be a semiquasitriangular Hopf algebra. The category \((H, R) \mathcal{M}^{(H, R)}(n)\), of normal \((H, R)\)-modules, is a braided category. The unit object is \(k\), endowed with the trivial action and the trivial coaction, and the tensor product is the usual tensor product over \(k\), endowed with the diagonal action and the diagonal
coaction. The associative and unit constraints are the usual ones and the braid \( c \) is given by \( c_{MN}(m \otimes n) = R^{(2)} \cdot n_0 \otimes R^{(1)} n_1 \cdot m \).

**Proof.** Let \( M \) and \( N \) be normal \( H \)-modules. It is obvious that \( m_0 \otimes n_0 \otimes m_1 n_1 \in M \otimes N \otimes Z(H) \) for all \( h \in H, n \in N \) and \( m \in M \). Moreover, by item (2) of Definition 2.1 and items (2) and (5) of Definition 1.1, we have

\[
\nu(h \cdot (m \otimes n)) = (h_1 \cdot m)_0 \otimes (h_2 \cdot n)_0 \otimes (h_1 \cdot m)_1 (h_2 \cdot n)_1
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot m_0 \otimes \overline{R}^{(2)} h_5 \overline{R}^{(2)} \cdot n_0
\]

\[
\otimes S(h_1) S(R^{(1)} h_3 R^{(1)} m_1 S(h_4) S(\overline{R}^{(1)} h_6 \overline{R}^{(1)} n_1)
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot m_0 \otimes \overline{R}^{(2)} h_5 \overline{R}^{(2)} \cdot n_0
\]

\[
\otimes S(h_1) S(R^{(1)} h_3 S(h_4) S(\overline{R}^{(1)} h_6 \overline{R}^{(1)} R^{(1)} m_1 n_1)
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot m_0 \otimes \overline{R}^{(2)} h_5 \overline{R}^{(2)} \cdot n_0
\]

\[
\otimes S(h_1) S(\overline{R}^{(1)} R^{(1)} h_7 \overline{R}^{(1)} R^{(1)} m_1 n_1)
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot m_0 \otimes R^{(2)} h_3 R^{(2)} \cdot n_0 \otimes S(h_1) S(R^{(1)} h_4 R^{(1)} m_1 n_1)
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot (m_0 \otimes n_0) \otimes S(h_1) S(R^{(1)} h_3 R^{(1)} m_1 n_1)
\]

for each \( h \in H \). Hence, the tensor product \( M \otimes N \) is a normal \( H \)-module via the diagonal action and the diagonal coaction. Moreover, it is immediate that \( k \) is a normal \( H \)-module, and it is clear that the usual associative and unit constraints are \( H \)-linear and \( H \)-colinear maps. So, \( (H, R) M^{(H, R)}(n) \) is a monoidal category. To prove that it is a braided category with braid \( c \), we must show that \( c \) is a natural isomorphism of normal \((H, R)\)-modules, and that

\[
c_{M \otimes N, P} = (c_{MP} \otimes N) (M \otimes c_{NP}) \quad \text{and} \quad c_{MN \otimes P} = (N \otimes c_{MP}) (c_{MN} \otimes P)
\]

for all \( M, N, P \in (H, R) M^{(H, R)}(n) \). We do this in several steps.

\( c_{MN} \) is \( H \)-linear: For \( h \in H, m \in M \) and \( n \in N \), we have:

\[
c_{MN}(h \cdot (m \otimes n)) = c_{MN}(h_1 \cdot m \otimes h_2 \cdot n)
\]

\[
= R^{(2)} h_2 R^{(2)} \cdot (h_2 \cdot n)_0 \otimes R^{(1)} (h_2 \cdot n)_1 \cdot (h_1 \cdot m)
\]

\[
= R^{(2)} R^{(2)} h_3 R^{(2)} \cdot n_0 \otimes R^{(1)} S(h_2) S(\overline{R}^{(1)} h_4 \overline{R}^{(1)} n_1 h_1 \cdot m
\]

\[
= R^{(2)} R^{(2)} h_3 R^{(2)} \cdot n_0 \otimes R^{(1)} h_1 S(h_2) S(\overline{R}^{(1)} h_4 \overline{R}^{(1)} n_1 \cdot m
\]

\[
= R^{(2)} R^{(2)} h_3 R^{(2)} \cdot n_0 \otimes R^{(1)} S(\overline{R}^{(1)} h_2 R^{(1)} n_1 \cdot m
\]

\[
= h_1 R^{(2)} \cdot n_0 \otimes h_2 R^{(2)} \cdot n_1 \cdot m
\]

\[
= h \cdot (R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot n_1 \cdot m)
\]

\[
= h \cdot c_{MN}(m \otimes n),
\]

where the third equality follows from item (2) of Definition 2.1, the fourth one follows from item (1) of Definition 2.1 and item (5) of Definition 1.1 and the sixth one follows from Proposition 1.3.
$c_{MN}$ is $H$-colinear: For $m \in M$ and $n \in N$, we have:

\[
\nu(c_{MN}(m \otimes n)) = \nu(R^{(2)} \cdot n_0 \otimes R^{(1)}n_1 \cdot m)
\]
\[
= (R^{(2)} \cdot n_0)_0 \otimes (R^{(1)}n_1 \cdot m)_0 \otimes (R^{(2)} \cdot n_0)_1 (R^{(1)}n_1 \cdot m)_1
\]
\[
= R^{(2)}R_2^{(2)}R^{(2)} \cdot n_0 \otimes R^{(2)}R_2^{(1)}n_3 R^{(2)} \cdot m_0 \otimes
\]
\[
S(R_1^{(1)}R_3^{(1)}R_4^{(1)}) n_1 S(n_2) S(R_1^{(1)}) S(R_3^{(1)}) n_4 R^{(1)}m_1
\]
\[
= R^{(2)}R_2^{(2)}R^{(2)} \cdot n_0 \otimes R^{(2)}R_2^{(1)}n_1 R^{(2)} \cdot m_0 \otimes
\]
\[
S(R_1^{(1)}R_3^{(2)}R_4^{(1)}R_5^{(1)}S(R_1^{(1)}R_3^{(1)}) n_2 R^{(1)}m_1
\]
\[
= R^{(2)}R_2^{(2)}R^{(2)} \cdot n_0 \otimes R^{(2)}R_2^{(1)}R^{(2)} \cdot n_1 \cdot m_0 \otimes S(R_1^{(1)}) R^{(1)}S(R_3^{(1)}) R^{(1)}m_1n_2
\]
\[
= R^{(2)} \cdot n_0 \otimes R^{(1)}n_1 \cdot m_0 \otimes m_1n_2
\]
\[
= (c_{MN} \otimes H) \nu(m \otimes n),
\]

where the third equality follows from item (2) of Definition 2.1, the fifth one follows from items (3) and (4) of Definition 1.1 and the seventh one follows from Proposition 1.3.

$c$ is a natural isomorphism: Let $f: M \to M'$ and $g: N \to N'$ morphisms of normal modules. For each $m \in M$ and $n \in N$, we have:

\[
(g \otimes f)c_{MN}(m \otimes n) = g(R^{(2)} \cdot n_0) \otimes f(R^{(1)}n_1 \cdot m)
\]
\[
= R^{(2)} \cdot g(n_0) \otimes R^{(1)}n_1 \cdot f(m)
\]
\[
= R^{(2)} \cdot g(n_0) \otimes R^{(1)}g(n_1) \cdot f(m)
\]
\[
= c_{M'N'}(f(m) \otimes g(n)).
\]

This shows that $c$ is a natural transformation. In order to prove that $c_{MN}$ is a bijective map it suffices to note that $c = l_{\tau(R)} \overline{\tau}$, where $\overline{\tau}: M \otimes N \to N \otimes M$ and $l_{\tau(R)}: N \otimes M \to N \otimes M$ are the maps defined by

\[
\overline{\tau}(m \otimes n) := n_0 \otimes n_1 \cdot m \quad \text{and} \quad l_{\tau(R)}(n \otimes m) := R^{(2)} \cdot n \otimes R^{(1)} \cdot m,
\]

which clearly are bijective.

It is true that $(c_{MP} \otimes N)(M \otimes c_{NP}) = c_{M \otimes N,P}$: by condition (2) of Definition 2.1, conditions (1) and (3) of Definition 1.1 and Proposition 1.3,

\[
(c_{MP} \otimes N)(M \otimes c_{NP})(m \otimes n \otimes p) = (c_{MP} \otimes N)(m \otimes R^{(2)} \cdot p_0 \otimes R^{(1)}p_1 \cdot n)
\]
\[
= R^{(2)} \cdot (R^{(2)} \cdot p_0)_0 \otimes R^{(1)}(R^{(2)} \cdot p_0)_1 \cdot m \otimes R^{(1)}p_1 \cdot n
\]
\[
= R^{(2)}R_2^{(2)}R^{(2)} \cdot p_0 \otimes R^{(1)}S(R_1^{(2)}) S(R_3^{(2)}) R^{(1)}p_1 \cdot m \otimes R^{(1)}p_2 \cdot n
\]
\[
= R^{(2)}R_1^{(2)}R_2^{(2)}R^{(2)} \cdot p_0 \otimes R^{(1)}S(R_1^{(2)}) S(R_3^{(2)}) R^{(1)}p_1 \cdot m \otimes R^{(1)}p_2 \cdot n
\]
It is true that \((N \otimes c_{MP})(c_{MN} \otimes P) = c_{M,N \otimes P}\): by item (1) of Definition 2.1 and item (2) of Definition 1.1,

\[
(N \otimes c_{MP})(c_{MN} \otimes P)(m \otimes n \otimes p) = (N \otimes c_{MP})(R^{(2)} \cdot n_0 \otimes R^{(1)}n_1 \cdot m \otimes p)
= R^{(2)} \cdot n_0 \otimes \overline{R}^{(2)} \cdot p_0 \otimes \overline{R}^{(1)}p_1 \cdot n_1 \cdot m
= R^{(2)} \cdot n_0 \otimes \overline{T}^{(2)} \cdot p_0 \otimes \overline{T}^{(1)}R_1p_1 \cdot n_1 \cdot m
= R^{(2)}_1 \cdot n_0 \otimes R^{(2)}_2 \cdot p_0 \otimes R^{(1)}n_1p_1 \cdot m
= c_{M,N \otimes P}(m \otimes n \otimes p).
\]

This finish the proof. \(\square\)

**Remark 2.6.** The inverse of the braid \(c\) introduced in Theorem 2.5 is given by \(c^{-1}_{NM}(n \otimes m) = S(n_1)S(R^{(1)}) \cdot m \otimes R^{(2)} \cdot n_0\). In fact, let \(\overline{\tau}: M \otimes N \to N \otimes M\) and \(l_{\tau(R)} : N \otimes M \to N \otimes M\) be as in the proof of Theorem 2.5. It is easy to see that \(\overline{\tau}^{-1}(n \otimes m) = S(n_1) \cdot m \otimes n_0\) and \(l_{\tau(R)}^{-1}(n \otimes m) = R^{(2)} \cdot n \otimes S(R^{(1)}) \cdot m\). Hence, by item (5) of Definition 1.1, item (3) of Proposition 1.2 and Proposition 1.3, we have

\[
c^{-1}_{NM}(n \otimes m) = \overline{\tau}^{-1}(l^{-1}_{\tau(R)}(n \otimes m))
= S((R^{(2)} \cdot n_1)S(R^{(1)}) \cdot m \otimes (R^{(2)} \cdot n_0)
= S(S(R^{(1)} \cdot n_1)S(R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m \otimes (R^{(2)} \cdot n_0)
= S(S(R^{(1)} \cdot n_1)S(R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m \otimes (R^{(2)} \cdot n_0)
= S(n_1)S(R^{(1)}) \cdot m \otimes R^{(2)} \cdot n_0,
\]

as we assert.

**Corollary 2.7.** Let \((H,R)\) be a semiquasitriangular Hopf algebra. The categories \((H,R)_{(H,R)}M(n)\), of left normal \((H,R)\)-modules, \(M_{(H,R)}(H,R)(n)\), of right normal \((H,R)\)-modules, and \((H,R)_{(H,R)}M_{(H,R)}(n)\), of right-left normal \((H,R)\)-modules, are braided categories. In all the cases the unit object is \(k\), endowed with the trivial action and the trivial coaction; the tensor product is the usual tensor product over \(k\), endowed with the diagonal action and the diagonal coaction, and the associative and unit constraints are the usual ones. The braids are given respectively by \(c_{1MN}(m \otimes n) = R^{(1)} \cdot n_0 \otimes R^{(2)}n_{-1} \cdot m\), \(c_{2MN}(m \otimes n) = n_0 \cdot R^{(1)} \otimes m \cdot n_1R^{(2)}\) and \(c_{3MN}(m \otimes n) = n_0 \cdot R^{(2)} \otimes m \cdot n_1R^{(1)}\).

**Proof.** Proceed as in the proof of Corollary 2.4. \(\square\)
Recall that an object $V$ of a braided category $\mathcal{C}$ is rigid if there exists an object $V^*$, endowed with arrows $\text{ev}_V : V^* \otimes V \to 1_\mathcal{C}$ and $\text{coev}_V : 1_\mathcal{C} \to V \otimes V^*$, where $1_\mathcal{C}$ is the unit object of $\mathcal{C}$, satisfying

$$\text{id}_V = (V \otimes \text{ev}_V)(\text{coev}_V \otimes V) \quad \text{and} \quad \text{id}_{V^*} = (\text{ev}_V \otimes V^*)(V^* \otimes \text{coev}_V).$$

The object $V^*$, which is unique unless a canonical isomorphism, is called the left dual of $V$, and the morphisms $\text{ev}_V$ and $\text{coev}_V$ are called the evaluation and the coevaluation maps of $V$, respectively. Let $U, V$ rigid objects of $\mathcal{C}$ and let $f : U \to V$ be a map of $\mathcal{C}$. The transpose map $f^* : V^* \to U^*$ of $f$ is defined by

$$f^* := (\text{ev}_V \otimes U^*)(V^* \otimes f \otimes U^*)(V^* \otimes \text{coev}_V).$$

A braided category is said to be rigid if each object has a left dual. Let $(H, R)$ be a semi-quasitriangular Hopf algebra. We are going to prove that the category of finite dimensional left-right normal $(H, R)$-modules is a rigid braided category.

Let $M$ be a finite dimensional left-right normal $(H, R)$-module. Given $f \in M^*$ and $h \in H$ we define $h \cdot f$ by $(h \cdot f)(m) = f(S(h) \cdot m)$ and we define $\nu_M(f) \in M^* \otimes \mathbb{Z}(H)$ by $\nu(f) = \sum_i f(m_i) m_i^* \otimes S^{-1}(m_i)$, where $\{m_i, m_i^*\}_{i \in I}$ are dual bases of $M$.

**Theorem 2.8.** $M^*$ is a left-right normal $(H, R)$-module.

**Proof.** It is immediate that $h \otimes f \mapsto h \cdot f$ is an action and that $(M^* \otimes \iota) \nu(f) = f$. Let us see that $(M^* \otimes \Delta) \nu = (\nu \otimes H) \nu$. By definition

$$(M^* \otimes \Delta) \nu_M (f) = \sum_{i \in I} f(m_i) m_i^* \otimes S^{-1}(m_i) \otimes S^{-1}(m_i),$$

and

$$(\nu \otimes H) \nu_M (f) = \sum_{i,j \in I} f(m_i) m_i^* (m_j) m_j^* \otimes S^{-1}(m_j) \otimes S^{-1}(m_i).$$

Evaluating in the first factor of these expressions in $m_k$ for $1 \leq k \leq n$, we reduce to prove that

$$f(m_k) \otimes S^{-1}(m_k^2) \otimes S^{-1}(m_k) = \sum_{i \in I} f(m_i) m_i^* (m_k) \otimes S^{-1}(m_k) \otimes S^{-1}(m_i).$$

This follows to applying $(f \otimes S^{-1} \otimes S^{-1}) (N \otimes \tau) (\nu_M \otimes H)$ to the equality

$$m_k^0 \otimes m_k^1 = \sum_{i \in I} m_i m_i^* (m_k) \otimes m_k^0.$$ 

It remains to prove that condition (2) of Definition 2.1 is satisfied. We must see that

$$(h \cdot f)_0 (m) h^* ((h \cdot f)_1) = (R^{(2)} h R^{(2)} \cdot f_0)(m) h^* (S(h_1) S(R^{(1)}) h R^{(1)} f_1)$$

where $R = R^{(1)} \otimes R^{(2)}$. The proof will be presented elsewhere.
for all $m \in M$ and $h^* \in H^*$. On one hand we have
\[
(h \cdot f)_0(m)h^*((h \cdot f)_1) = \sum_{i \in I} f(S(h)m_i_0)m_i^*(m)m^*(S^{-1}(m_{i_1}))
= \sum_{i \in I} f(S(h)m_i_0)m_i^*(m)h(S^{-1}(m_{i_1}))
= f(S(h)m_0)h^*(S^{-1}(m_1)),
\]
where the last equality follows from the fact that $\sum_{i \in I} m_i m_i^*(m) = m$. On the other hand,
\[
(R^{(2)} h_2 R^{(2)} \cdot f_0)(m)h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} f_1))
= \sum_{i \in I} f(m_i_0)m_i^*(S(R^{(2)} h_2 R^{(2)} \cdot m))h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} S^{-1}(m_{i_1}))
= \sum_{i \in I} f(m_i_0)m_i^*(S(R^{(2)} h_2 R^{(2)} \cdot m))h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} S^{-1}(m_{i_1}))
= f((S(R^{(2)} h_2 R^{(2)} \cdot m)_0)h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} S^{-1}(((S(R^{(2)} h_2 R^{(2)} \cdot m)_1))
= f(R^{(2)} S(R^{(2)} h_2 R^{(2)} \cdot m)_0)h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} S^{-1}(S(R^{(2)} h_2 R^{(2)} \cdot m)_1)))
\]
where the third equality follows from the fact that $\sum_{i \in I} m_i m_i^*(m) = m$, the fourth one follows from the fact that $\nu_M$ satisfies condition (2) of Definition 2.1, the sixth one follows from Proposition 1.3 and the seventh and eighth ones follow from item (3) of Proposition 1.2. Since, by the discussion that follows Proposition 1.9,
\[
h_1 \otimes S(R^{(2)} S(h_3) S(R^{(2)} \otimes R^{(1)} h_2 S(R^{(1)} h_4) S(h_4) \in H \otimes H \otimes Z(H),
\]
we obtain
\[
(R^{(2)} h_2 R^{(2)} \cdot f_0)(m)h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} f_1))
= f(S(R^{(2)} R^{(2)} S(h_3) S(R^{(2)} R^{(2)} \cdot m_0)
\]
\[
h^*(S(h_1)S(R^{(1)} h_3 R^{(1)} S^{-1}(m_1) R^{(1)} h_2 S(R^{(1)} h_4) S(h_4))}
Then, the left-right normal \( (H,R) \)-module \( M^* \) is a left dual of \( M \).

**Proof.** It is well known that ev\(_M\) and coev\(_M\) are \( H \)-linear maps and that

\[
\text{id}_M = (M \otimes \text{ev}_M) (\text{coev}_M \otimes M) \quad \text{and} \quad \text{id}_{M^*} = (\text{ev}_M \otimes M^*) (M^* \otimes \text{coev}_M).
\]

It remains to prove that ev\(_M\) and coev\(_M\) are \( H \)-colinear maps. Let \( \{m_i, m_i^*\}_{i \in I} \) be dual bases of \( M \) and let \( m \in M \). Since \( \sum_{i \in I} m_i m_i^*(m_0) \otimes m_1 = m_0 \otimes m_1 \) and \( m_0 \otimes m_1 \in H \otimes \mathbb{Z}(H) \), we have

\[
(\text{ev}_M \otimes H) \nu(f) = (\text{ev}_M \otimes H)(f(m_0) m_i^* m_0 \otimes m_1 S^{-1}(m_{i1}))
\]

\[
= f(m_0) m_i^* m_0 \otimes m_1 S^{-1}(m_{i1})
\]

\[
= f(m_0) \otimes m_2 S^{-1}(m_1)
\]

\[
= f(m) \otimes 1
\]

for each \( f \in M^* \). So, ev\(_M\) is a morphism of comodules. To check that coev\(_M\) is also, it suffices to note that since

\[
\sum_{i \in I} m_{i0} \otimes m_{i1} \otimes m_i^* = \sum_{ij \in I} m_j \otimes m_{i1} \otimes m_j^*(m_{i0}) m_i^*,
\]

we have

\[
\nu \text{ coev}(1) = \sum_{j \in I} \nu(m_j \otimes m_j^*)
\]

\[
= \sum_{ij \in I} m_j^* (m_{i0}) m_i^* S^{-1}(m_{i1}) m_j
\]

\[
= \sum_{i \in I} m_{i0} m_i^* S^{-1}(m_i) m_{i1}
\]

\[
= \sum_{i \in I} m_1 m_i^* \otimes 1
\]

\[
= (\text{coev} \otimes H) \nu(1),
\]

as desired. \( \square \)
Corollary 2.10. The category $(H,R)\mathcal{M}^{(H,R)}_\text{fin}(n)$, of left-right finite dimensional normal $(H,R)$-modules, is rigid.

Let $(H,R)$ be a finite dimensional semiquasitriangular Hopf algebra and let $\{h_i,h_i^*\}_{i \in I}$ be dual bases of $H$ and $H^*$. Let $H \bowtie H^*$ denote the tensor product $H \otimes H^*$, endowed with the multiplication $(h \bowtie \psi)(l \bowtie \phi) = hR(2)l_2R(2)^\bowtie (\psi \leftarrow SL(S(R(1))l_3R(1))\phi$ and the coadiagonal comultiplication. We write $h \bowtie \psi$ to denote the element $h \otimes \psi$ of $H \bowtie H^*$. Let $T \in (H \bowtie H^*) \otimes (H \bowtie H^*)$ be the element $T := \sum_{i \in I}(R(1)h_i \bowtie \epsilon) \otimes (R(2) \bowtie h_i^*)$.

Theorem 2.11. $H \bowtie H^*$ is a Hopf algebra with unit $1 \bowtie \epsilon$, counit $\epsilon_H \otimes \epsilon_{H^*}$ and antipode $S_{H \bowtie H^*}(h \bowtie \psi) := (1 \bowtie \phi S)(S(h) \bowtie \epsilon)$. Moreover the category of left representations of $H \bowtie H^*$ coincide with the category of normal $(H,R)$-modules and $(H \bowtie H^*, T)$ is a quasitriangular Hopf Algebra.

Proof. This result can be proved using [Ch-P, Theorem 5.1.11], but here we prefer to give a direct proof. Let $\chi : H^* \otimes H \rightarrow H \otimes H^*$ be the map given by $\chi(\psi \otimes h) = (1 \bowtie h)(\psi \bowtie 1)$. In order to prove that $H \bowtie H^*$ is a associative algebra with unit $1 \bowtie \epsilon$, it suffices to check that $\chi$ is a twisted map in the sense of [C-V-S]. That is,

$$
\begin{align*}
\chi(\mu \otimes \mu) & = (\mu \otimes H^*)(H \otimes \chi)(\chi \otimes H), \\
\chi(\mu\mu_H^* \otimes H) & = (H \otimes \mu_{H^*})(\chi \otimes H^*)(H^* \otimes \chi), \\
\chi(\epsilon \otimes h) & = h \bowtie \epsilon \\
\chi(\psi \otimes 1) & = 1 \bowtie \psi.
\end{align*}
$$

We leave this to the reader. Now, it is immediate that the category of left representations of $H \bowtie H^*$ coincide with the category of normal $(H,R)$-modules. In fact, if $M$ is a normal $(H,R)$-module, then

$$(h \bowtie \psi) \cdot m = h \cdot m_\psi(m_1) \quad \text{for} \quad m \in M.$$

Since the category of finite dimensional normal $(H,R)$-modules is rigid monoidal, it is true that $H \bowtie H^*$ is a Hopf algebra with comultiplication and antipode given by

$$
\Delta_{H \bowtie H^*}(h \bowtie \psi) = (h \bowtie \psi) \cdot [(1 \bowtie \epsilon) \otimes (1 \bowtie \epsilon)] = (h_1 \bowtie \psi_1) \otimes (h_2 \bowtie \psi_2)
$$

and

$$
S_{H \bowtie H^*}(h \bowtie \psi) = (1 \bowtie S(\phi))(S(h) \bowtie \epsilon).
$$

Finally, since the category of normal $(H,R)$-modules is braided,

$$
\tau \epsilon((1 \bowtie \epsilon) \otimes (1 \bowtie \epsilon)) = \tau \left( \sum_i R(2)_i \cdot (1 \bowtie h_i^*) \otimes R(1)_i h \cdot (1 \bowtie \epsilon) \right)
$$

$$
= \sum_i (R(1)_i h \bowtie \epsilon) \otimes (R(2)_i \bowtie h_i^*),
$$

is an $R$-structure of $H \bowtie H^*$. \hfill \Box

Example 2.12. When $(H,R)$ is quasitriangular, then $H \bowtie H^* = H \otimes H^*$.

Example 2.13. Let $G$ be a finite group, $H = k[G]^*$ and $R = 1 \otimes 1$. For each $x \in G$ let $\delta_x : G \rightarrow k$ be the map $\delta_x(y) = \delta_{x,y}$, where $\delta_{x,y}$ is the Kronecker symbol. Then $H \bowtie H^*$ is the tensor product of $k[G]^*$ with $k[G]$, endowed with the multiplication given by

$$(\delta_x \bowtie y)(\delta_{x'} \bowtie y') = \begin{cases}
\delta_x \bowtie yy' & \text{if } x' = yxy^{-1}, \\
0 & \text{in other case}.
\end{cases}$$

In this case, the $R$-matrix $T$ is $\sum_{x,y \in G}(\delta_x \bowtie 1) \otimes (\delta_y \bowtie x)$. 


3. The Drinfeld Element of a Semiquasitriangular Hopf Algebra

In this section we show that the properties of the Drinfeld element of a quasitriangular Hopf algebra remain valid in the semiquasitriangular setting. However, in this last case some formulas are more involved (see for instance Proposition 3.2).

**Definition 3.1.** Let \((H, R)\) be a semiquasitriangular Hopf algebra. The Drinfeld element of \((H, R)\) is the element \(u := S(R^{(2)})R^{(1)}\) of \(H\).

**Proposition 3.2.** Assume that \((H, R)\) is a semiquasitriangular Hopf algebra. Let \(T: H \to H\) be the map defined by

\[
T(h) := R^{(2)}h_2R^{(2)}S(h_1)S(R^{(1)})h_3R^{(1)}.
\]

The Drinfeld element \(u\) is invertible with inverse \(R^{(2)}S^2(R^{(1)})\). Moreover, \(S^2(h) = uT(h)u^{-1}\) for all \(h \in H\).

**Proof.** By conditions (5) and (6) of Definition 1.1 and Proposition 1.3, we have

\[
S(h_2)uT(h_1) = S(h_1)uR^{(1)}h_2R^{(1)}S(R^{(2)})S(h_1)R^{(2)}h_3
= S(R^{(2)})S(h_1)R^{(1)}uR^{(1)}h_2R^{(1)}
= S(R^{(2)})S(h_1)h_2R^{(1)}
= u\epsilon(h).
\]

Hence, \(S^2(h)u = S^2(h_2)\epsilon(h_1)u = S^2(h_3)S(h_2)uT(h_1) = uT(h)\). It remain to check that \(u\) is invertible and \(u^{-1} = R^{(2)}S^2(R^{(1)})\). By the formula proved above, condition (4) of Definition 1.1 and Proposition 1.3, we have

\[
\tilde{R}^{(2)}S^2(\tilde{R}^{(1)})u = \tilde{R}^{(2)}u\tilde{R}^{(1)}R^{(2)}S(\tilde{R}^{(1)})S(R^{(1)})\tilde{R}^{(1)}R^{(1)}
= \tilde{R}^{(2)}u\tilde{R}^{(1)}R^{(2)}S(R^{(1)})S(\tilde{R}^{(1)})\tilde{R}^{(1)}R^{(1)}
= \tilde{R}^{(2)}u\tilde{R}^{(1)}
= \tilde{R}^{(2)}S(\tilde{R}^{(2)})\tilde{R}^{(1)}\tilde{R}^{(1)}
= 1.
\]

Hence, \(\tilde{R}^{(2)}S^2(\tilde{R}^{(1)})\) is a left inverse of \(u\). To prove that it is also a right inverse, we note that by condition (3) of Definition 1.1 and Proposition 1.3,

\[
S^2(\tilde{R}^{(2)})uS^2(\tilde{R}^{(1)}) = uR^{(1)}R^{(2)}S(\tilde{R}^{(2)})S(R^{(1)})\tilde{R}^{(1)}R^{(2)}S^{(2)}(\tilde{R}^{(1)})
= uR^{(1)}R^{(1)}\tilde{R}^{(2)}S(\tilde{R}^{(2)})S(R^{(1)})\tilde{R}^{(1)}R^{(2)}S^{(2)}(\tilde{R}^{(1)})
= uR^{(1)}R^{(1)}S(R^{(2)})\tilde{R}^{(2)}R^{(2)}S^{(2)}(\tilde{R}^{(1)})
= u\tilde{R}^{(2)}S^2(\tilde{R}^{(1)}).
\]

So, \(u\tilde{R}^{(2)}S^2(\tilde{R}^{(1)}) = S^2(\tilde{R}^{(2)})uS^2(\tilde{R}^{(1)}) = \tilde{R}^{(2)}u\tilde{R}^{(1)} = 1\), as we want. \(\square\)
Proposition 3.3. Let \((H, R)\) be a semiquasitriangular Hopf algebra. The Drinfeld element \(u\) satisfies

\[
\epsilon(u) = 1, \quad \Delta(u) = (R_{21}R)^{-1}(u \otimes u) = (u \otimes u)(R_{21}R)^{-1}, \\
\Delta(S(u)) = (R_{21}R)^{-1}(S(u) \otimes S(u)) = (S(u) \otimes S(u))(R_{21}R)^{-1}
\]

and

\[
\Delta(uS(u)) = (R_{21}R)^{-2}(uS(u) \otimes uS(u)) = (uS(u) \otimes uS(u))(R_{21}R)^{-2}.
\]

Proof. From Proposition 1.3 it is immediate that \(\epsilon(u) = 1\). By Proposition 1.3 and item (3) of Definition 1.1, we have

\[
R_{32}R_{23}(R^{(1)} \otimes \Delta(S(R^{(2)}))) = R_{32}R_{23}(R^{(1)} \otimes (S \otimes S) \Delta^\text{cop}(R^{(2)})) \\
= R_{32}(H \otimes S \otimes S)(R_{23})(H \otimes S \otimes S)(R^{(1)} \otimes \Delta^\text{cop}(R^{(2)})) \\
= R_{32}(H \otimes S \otimes S)((R^{(1)} \otimes \Delta^\text{cop}(R^{(2)}))R_{23}) \\
= R_{32}(H \otimes S \otimes S)(R_{23}(R^{(1)} \otimes \Delta(R^{(2)}))) \\
= R_{32}(H \otimes S \otimes S)(R^{(1)} \otimes \Delta(R^{(2)}))R_{23} \\
= (H \otimes S \otimes S)((R^{(1)} \otimes \Delta(R^{(2)}))R_{32})R_{23} \\
= (H \otimes S \otimes S)(R_{32}(R^{(1)} \otimes \Delta^\text{cop}(R^{(2)})))R_{23} \\
= (R^{(1)} \otimes \Delta(S(R^{(2)})))R_{32}R_{23}.
\]

Hence, \(R_{21}R\Delta(u) = R_{21}R\Delta(S(R^{(2)}))S(R^{(1)}) = \Delta(S(R^{(2)}))R_{21}RS(R^{(1)})\). Similarly, \(\Delta(u)R_{21}R = \Delta(S(R^{(2)}))R_{21}RS(R^{(1)})\). On the other hand arguing as in [Ka, Proposition VIII.4.5], can be check that this last expression equals \(u \otimes u\). This gives the formula for \(\Delta(u)\). Now it is easy to check the formulas for \(\Delta(S(u))\) and \(\Delta(uS(u))\). \(\square\)

Proposition 3.4. Let \((H, R)\) be a semiquasitriangular Hopf algebra. The elements \(u, S(u)\) and \(uS(u)\) are coinvariants for the coaction \(\nu\).

Proof. Since, by Proposition 2.3, \(\nu\) is multiplicative, it suffices to prove the assertion for \(u\) and \(S(u)\). We consider the first case we leave the second one to the reader. By items (3) and (4) of Proposition 1.2 and Proposition 1.3,

\[
\nu(u) = R^{(2)}S(R_{21}^{(2)})R_{2}^{(1)}R^{(2)}S(R_{1}^{(1)})S^{2}(R_{3}^{(2)})S(R^{(1)})S(R_{1}^{(2)})R_{3}^{(1)}R^{(1)} \\
= R^{(2)}S(R_{21}^{(2)})R^{(2)}R_{3}^{(1)} \otimes S(R_{1}^{(1)})S^{2}(R_{3}^{(2)})S(R^{(1)})S(R_{1}^{(2)})R_{3}^{(1)}R_{2}^{(1)} \\
= R^{(2)}S(R_{21}^{(2)})S(R_{21}^{(2)})R_{3}^{(1)} \otimes S(R_{1}^{(1)})S^{2}(R_{3}^{(2)})S(R^{(1)})S(R^{(1)})S(R_{2}^{(2)})R_{3}^{(1)}R_{2}^{(1)} \\
= S(R_{21}^{(2)})R_{3}^{(1)} \otimes S(R_{1}^{(1)})S^{2}(R_{3}^{(2)})S(R^{(1)})S(R^{(1)})S(R_{2}^{(2)})R_{2}^{(1)} \\
= S(R_{21}^{(2)})R_{3}^{(1)} \otimes S(R_{1}^{(1)})R_{2}^{(1)} \\
= S(R_{21}^{(2)})R_{1}^{(1)} \otimes 1 \\
= u \otimes 1,
\]

as we want. \(\square\)
Corollary 3.5. For each semiquasitriangular Hopf algebra $(H,R)$, it is true that $S^2(u) = u$, $S^2(u^{-1}) = u^{-1}$ and $uS(u) = S(u)u$.

Proof. By Proposition 3.2 it suffices to prove that $T(u) = u$ and $T(S(u)) = S(u)$. These assertions follow immediately from Proposition 3.4, since $T = \mu \nu$. □

Proposition 3.6. Let $(H,R)$ be a semiquasitriangular Hopf algebra, and let $\tilde{\nu}$ be the left coaction of $H$ defined by $\tilde{\nu} := (S \otimes S)\nu S^{-1}$. The following facts are equivalents:

1. $S(u)u \in Z(H)$,
2. $ST = TS$,
3. $\mu \nu = \mu \tilde{\nu}$.

Proof. Let $h \in H$. By Proposition 3.2, we have

$$S(u)^{-1}S(T(h))S(u)S(S^2(h)) = S^2(S(h)) = uT(S(h))u^{-1}.$$

That is $S(T(h))S(u)u = S(u)uT(S(h))$. Since $S$ and $T$ are bijective maps and $S(u)u$ is invertible, this implies that the items (1) and (2) are equivalents. It remains to prove that (2) ⇔ (3). To do this it suffices to note that $T = \mu \nu$ and that

$$ST S^{-1} = S \mu \nu S^{-1} = S \mu \mu S^{-1} = \mu (S \otimes S)\nu S^{-1} = \mu \tilde{\nu},$$

where the second equality follows from the fact that $\text{Im} \nu \subseteq H \otimes Z(H)$. □

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