Abstract

We investigate multidimensional gravity with the Gauss–Bonnet term and with torsion on the space of extra dimensions chosen to be the group manifold of a simple Lie group. We take the Robertson-Walker ansatz for the 4-dimensional space-time and study the potential of a dilaton and torsion fields. It is shown that for certain values of the parameters of the initial theory the potential has classically stable minima, corresponding to the spontaneous compactification of the extra dimensions. However, these minima have zero torsion.

1 Introduction

It is a common belief that space-time may have more than four dimensions. Theories based on this assumption are often referred to as Kaluza-Klein theories, since the initial idea was put forward by T. Kaluza and O. Klein in 1921 [1]. For additional dimensions not to contradict the observable reality they must be small enough so that they cannot be probed in modern experiments. Usually, it is assumed that the multidimensional space-time has the structure $E = M^{(4)} \times S$, with $M^{(4)}$ being the four dimensional space-time and $S$ being the internal space of extra dimensions with a characteristic size of the order of the Plank scale $L_{pl} \approx 10^{-33}$ cm. In so doing, it is supposed that the direct product structure of the multidimensional space-time $E = M^{(4)} \times S$ appeared dynamically, as a result of some symmetry breaking at a very early stage of the cosmological evolution of the universe.

However, investigations within the framework of pure multidimensional Einstein gravity have shown that this theory is plagued with many serious difficulties, which leads to the fact that the effective 4-dimensional theory is hard to interpret. For example, there is no acceptable vacuum solution of the form $E = M^{(4)} \times K/H$, where $M^{(4)}$ is Minkowski space-time and $K/H$ is a homogeneous space with non-abelian isometry group $K$ (which is necessary to obtain non-abelian gauge fields on $M^{(4)}$ [2]). To solve this problem one can add Yang-Mills or matter fields to the initial theory from the very beginning (see, e.g. [3]), but this would be a rather strong deviation from the original Kaluza-Klein idea. To remain closer to Kaluza-Klein ansatz one can consider some generalized gravitational theories, for example, theories with torsion.

An attractive feature of Kaluza-Klein theories is that scalar fields, which are necessary, for example, to ensure the inflation in four dimensions or spontaneous symmetry
breaking mechanism, appear quite naturally in four dimensions within this framework. They emerge from extra components of the metric and can also be induced by torsion on internal space. The potential of these fields is determined by the geometry and symmetries of the multidimensional space-time.

Compactification to manifolds with torsion has been investigated in many papers (see, e.g. [10] and references therein). It is well known that torsion is not a dynamical variable in pure Einstein-Cartan gravity [5]. In the presence of matter, its dynamics is carried by the spin density of matter. This property is no longer preserved if one adds higher order curvature terms to the standard Einstein-Cartan Lagrangian. There are many reasons for choosing the Gauss-Bonnet term $\mathcal{R}^2 = R_{ABCD} R^{CDAB} - 4 R_{AB} R^{BA} + R^2$. For example, as it was shown in [6], [7], if one adds such term to the initial Lagrangian, the equations of motion include the derivatives of the metric not higher than of the second order therefore such theories are ghost-free. Such form of the curvature squared term is also motivated by quantum field theory limit of the heterotic string models [8]. Besides, the presence of second order curvature terms could explain the inflation of the observable Universe (see e.g. [9]).

In this paper we will investigate multidimensional gravity with torsion. The ground state is chosen to have the structure of the direct product of Robertson-Walker universe and a group manifold $S$ represented as a homogeneous space $S = K/H$ with $K = S \times S$ being the isometry group and $H = \text{diag}\{S \times S\}$ being the isotropy subgroup.

The paper is organized as follows. In the next section we will give a brief description of $K$-invariant metric connections with torsion on group manifolds. In section 3 we calculate the components of the multidimensional curvature tensor $R^A_{BCD}$ and carry out the dimensional reduction of the initial action. Section 4 is devoted to the effective scalar fields potential. We will analyze its minima and consider possible cosmological consequences of the model.

## 2 Invariant connections with torsion on group manifolds

In this section we briefly present some results from our previous paper [13] with the purpose to make the discussion complete.

It is well known (see, e.g. [12]) that the group manifold of a simple Lie group $S$ (dim $S = d$) can be represented as a reductive homogeneous space $S = K/H$ with $K = S \times S$ being the isometry group and $H = \text{diag}\{S \times S\}$ being the isotropy subgroup. We consider the class of $K$-invariant metrics $g$ on $S$ (that is metrics which are invariant under both right and left action of the group $S$) and metric connections with torsion $\omega$ on the principal fiber bundle $O(S)$ of the orthonormal frames over $S$. The group $K$ acts transitively on the base $S = K/H$ by left multiplication of cosets and induces a natural automorphism of the bundle $O(S)$. The lift of the $K$-action to the bundle automorphism is characterized by the homomorphism $\lambda : H \rightarrow SO(d)$. We would like to remind that if torsion is present then connection cannot be expressed only in terms of derivatives of metric. Let $\mathfrak{k}$ be the Lie algebra of the group $K$, so that $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{s}$ and $\mathfrak{h} = \{(X, X), X \in \mathfrak{s}\}$ (gothic letters stand for the corresponding Lie algebras). The Lie algebra $\mathfrak{k}$ admits three natural reductive decompositions: $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ with
\[ m = m_0, \ m_+, \text{ and } m_- \]

\[ m_0 = \{(X/2, -X/2), \ X \in S\}, \ (0) \text{ connection}, \]  

\[ m_+ = \{(0, -X), \ X \in S\}, \ (-) \text{ connection}, \]  

\[ m_- = \{(X, 0), \ X \in S\}, \ (+) \text{ connection}. \]

We denote by \( o \in K/H \) the origin in \( K/H \). It is easy to see that there is a natural isomorphism between the spaces \( T_o(K/H), \ m \) and \( R^d \), that is \( T_o(K/H) \cong m \cong R^d \). All \( K \)-invariant connections \( \omega \) on the bundle \( O(S) \) are given by Wang’s theorem [12]. It states that there is a 1-1 correspondence between the \( K \)-invariant connections on the bundle \( O(S) \) and linear mappings \( \Lambda \)

\[ \Lambda : m \to SO(d) \equiv \text{Lie}(SO(d)) \]

such that

\[ \Lambda(\text{ad} h(\tilde{X})) = \text{ad}(\Lambda(h))\Lambda(\tilde{X}), \quad \tilde{X} \in m, \quad h \in H. \]  

In terms of the mapping \( \Lambda \) the formulas for the invariant torsion \( T \) and curvature \( R \) on \( K/H \) at the point \( o \) take the form

\[ T_o(\tilde{X}, \tilde{Y}) = \Lambda(\tilde{X})\tilde{Y} - \Lambda(\tilde{Y})\tilde{X} - [\tilde{X}, \tilde{Y}]m, \]

\[ R_o(\tilde{X}, \tilde{Y}) = [\Lambda(\tilde{X}), \Lambda(\tilde{Y})] - \Lambda([\tilde{X}, \tilde{Y}])m - \lambda([\tilde{X}, \tilde{Y}], \mathfrak{h}), \quad \tilde{X}, \tilde{Y} \in m. \]

Since \( SO(d) \cong R^d \wedge R^d \) we can introduce the mapping \( \beta : m \otimes m \to m \)

\[ \beta(\tilde{X}, \tilde{Y}) \equiv \Lambda(\tilde{X})\tilde{Y}. \]

Then we decompose the connection form \( \omega \) into a sum of the Levi-Civita connection \( \tilde{\omega} \) and the so-called contorsion form \( \tilde{\omega} \). Consequently, for the mappings \( \Lambda \) and \( \beta \) we have \( \Lambda = \tilde{\Lambda} + \Lambda \) and \( \beta = \tilde{\beta} + \beta \). The expression for \( \tilde{\Lambda} \) was obtained by Nomizu [12]

\[ \tilde{\beta}(\tilde{X}, \tilde{Y}) \equiv \tilde{\alpha}(\tilde{X})\tilde{Y} = \frac{1}{2}[\tilde{X}, \tilde{Y}]m + \tilde{U}(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in m, \]

where \( \tilde{U}(\tilde{X}, \tilde{Y}) \) is some symmetric bilinear mapping \( m \otimes m \to m \). We will discuss it later.

It is also useful to decompose \( \tilde{\beta} \), which describes contorsion, into symmetric and antisymmetric parts \( \beta = \beta_s + \beta_as \).

Hence

\[ \beta_{as} = \frac{1}{2}[\tilde{X}, \tilde{Y}]m + \tilde{U}(\tilde{X}, \tilde{Y}), \]  

\[ \beta_s = \tilde{U}(\tilde{X}, \tilde{Y}) + \tilde{\beta}_s(\tilde{X}, \tilde{Y}). \]

It is well known [12] that the \( K \)-invariant metrics \( \gamma \) on \( K/H \) are in 1-1 correspondence with non-degenerated symmetric \( \text{ad} H \)-invariant bilinear forms \( B \) on \( m \), i.e.

\[ B(\tilde{X}, \tilde{Y}) = \gamma_o(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in m. \]  

The invariance of \( B \) with respect to \( \text{ad} H \) means that

\[ B\left([\tilde{A}, \tilde{X}], \tilde{Y}\right) + B\left(\tilde{X}, [\tilde{A}, \tilde{Y}]\right) = 0, \quad \tilde{X}, \tilde{Y} \in m, \quad \tilde{A} \in \mathfrak{h}. \]
In can be shown that in terms of $\beta$ the metricity condition reads

$$B \left( \beta(\tilde{X}, \tilde{Y}), \tilde{Z} \right) + B \left( \beta(\tilde{Z}, \tilde{X}), \tilde{Y} \right) = 0, \quad \tilde{X}, \tilde{Y} \in \mathfrak{m}. $$

Combining this condition with two other formulas obtained from it by the cyclic permutation of $\tilde{X}$, $\tilde{Y}$ and $\tilde{Y}$, it is easy to derive the following relation between the symmetric part $\beta_s$ and the antisymmetric part $\beta_{as}$ of the full mapping $\beta(\tilde{X}, \tilde{Y})$

$$B \left( \beta_s(\tilde{X}, \tilde{Y}), \tilde{Z} \right) = B \left( \beta_s(\tilde{Y}, \tilde{Z}), \tilde{X} \right) + B \left( \beta_{as}(\tilde{Z}, \tilde{X}), \tilde{Y} \right), \quad \tilde{X}, \tilde{Y} \in \mathfrak{m},$$

which enables us to calculate the symmetric part through the antisymmetric one. So, our next step will be to find the operator $\beta_{as}$.

Let us write down the invariance condition (2) in the infinitesimal form

$$\tilde{\beta}_{as} \left( (\text{Ad} \rho \wedge 1 + 1 \wedge \text{Ad} \rho) \xi \right) = \text{Ad} \rho \left( \tilde{\beta}_{as} (\xi) \right), \quad \xi \in \mathfrak{m} \wedge \mathfrak{m}, \quad \rho \in \mathfrak{h}. $$

Now we see that $\tilde{\beta}_{as}$ can be interpreted as an operator which intertwines the equivalent representations of the algebra $\mathfrak{h}$ in the linear spaces $\mathfrak{m} \wedge \mathfrak{m}$ and $\mathfrak{m}$. To construct this operator explicitly we use the general method applied to the dimensional reduction of the invariant connections in multidimensional Yang-Mills theories (see e.g. [14]).

It is evident that $\mathfrak{m} \cong \mathfrak{h} \cong \mathcal{S}$ and the linear spaces $\mathfrak{h}$ and $\mathfrak{m}$ carry the adjoint representation of the algebra $\mathcal{S}$. So, it is necessary to decompose the antisymmetrized tensor product $\text{ad} \mathcal{S} \wedge \text{ad} \mathcal{S}$ into irreducible representations (irreps) of the algebra $H$ and to check, whether there is the adjoint representation $\text{ad} \mathcal{S}$ among them. In [13] it has been demonstrated that for the simple classical Lie algebras the decomposition of $\text{ad} \mathcal{S} \wedge \text{ad} \mathcal{S}$ has the following form

$$\text{ad} \mathcal{S} \wedge \text{ad} \mathcal{S} = \text{ad} \mathcal{S} + \nu + \nu^*, \quad \text{for } S = A_n,$$

$$\text{ad} \mathcal{S} \wedge \text{ad} \mathcal{S} = \text{ad} \mathcal{S} + \mu \quad \text{otherwise}.$$  

(9)

Here $\mu$ and $\nu$ stand for irreps different from the adjoint. This decomposition is a generalization of the well known theorem which states that $\text{ad} \mathcal{S}$ always appears in $\text{ad} \mathcal{S} \wedge \text{ad} \mathcal{S}$ [17]. Before presenting the explicit formula for the antisymmetric contorsion form we will make two remarks. We note that the structure of the Lie algebra of the reductive spaces admits two natural intertwining operators

$$\phi(X \wedge Y) = [X, Y]_{\mathfrak{m}} \quad \text{and} \quad \psi(X \wedge Y) = [X, Y]_{\mathfrak{h}}, \quad X, Y \in \mathfrak{m},$$

and we introduce the mapping $j: \mathfrak{h} \to \mathfrak{m}$.

So, now it is clear that the contorsion form $\tilde{\beta}_{as}$ (which accordingly to Schur’s lemma must be proportional to the operator $\phi$ or $j \circ \psi$) can be written as

$$\tilde{\beta}_{as}(\tilde{X} \wedge \tilde{Y}) = \frac{f(x)}{2} [\tilde{X}, \tilde{Y}], \quad \text{for } (\pm) \text{ connection},$$

$$\tilde{\beta}_{as}(\tilde{X} \wedge \tilde{Y}) = 2f(x) \circ j ([\tilde{X}, \tilde{Y}]), \quad \text{for } (0) \text{ connection}.$$  

(10)

where $f(x)$ is an arbitrary scalar function on $M^{(4)}$.

Let us define $\text{ad} K$-invariant symmetric bilinear form $B(\cdot, \cdot)$ on $\mathfrak{k}$

$$B(\tilde{X}, \tilde{Y}) = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle, \quad \tilde{X} = (X_1, X_2), \quad \tilde{Y} = (Y_1, Y_2).$$
Here $\langle \cdot, \cdot \rangle$ denotes an $adS$-invariant symmetric bilinear form on $S$. Inserting the expression for $\beta_{as}$ into (7) we see that r.h.s. vanishes identically, therefore $\beta_s = 0$. Thus, the invariant connections with torsion on group manifolds form a one parametric family given by

$$ \Lambda(\tilde{X})\tilde{Y} = \frac{1+f}{2}[\tilde{X}, \tilde{Y}]m, \quad \text{for (±) connection}, $$

$$ \Lambda(\tilde{X})\tilde{Y} = 2f \circ j \left(|\tilde{X}, \tilde{Y}|_H\right), \quad \text{for (0) connection}. $$

notice that the result for (0) connection, when the group manifold is realized as a symmetric homogeneous space, differs from that obtained in [15] for simply connected irreducible symmetric spaces. In the latter case the only invariant connection is the Levi-Civita one.

Introducing the homomorphism $i : S \to m$, $i(X) = \tilde{X}$ and having in mind that $i(R(X_k, X_p)X_j) = \tilde{R}(\tilde{X}_k, \tilde{X}_p)\tilde{X}_j$ where $\tilde{R}$ is the curvature tensor on $K/H$, we get from formula (5)

$$ R_o(X_k, X_p)X_j = F(x) [[X_k, X_p], X_j], \quad X_i \in S, \quad F(x) = \frac{f^2(x) - 1}{2}, $$

which yields the following curvature tensor components

$$ R_{ijkp} = F(x) C^a_{kp} C^b_{aj} g(\tilde{X}_b, \tilde{X}_i). $$

Here $C^a_{kp}$ are the structure constants of the Lie algebra $S$.

Analogously, we obtain from formula (3) that

$$ T_o(X_i, X_j) = f(x) [X_i, X_j] $$

and, therefore we have for the torsion components

$$ T^k_{ij} = f(x) C^k_{ij}. $$

We will use expressions (13) and (14) in the next section.

### 3 Dimensional reduction

We are interested in a model of multidimensional gravity in $D = 4 + d$ dimensions, with curvature squared terms and with torsion on the space of extra dimension.

Let $\{\hat{\theta}^A\}, A = 1, \ldots, D$ be a basis of orthonormal 1-forms on $E$ and $\hat{\omega}_B^A$ a metric connection 1-form which is invariant under the action of the symmetry group $K$. The curvature 2-form is constructed from the connection form $\hat{\omega}_B^A$ by the formula

$$ \hat{\Omega}_B^A = d\hat{\omega}_B^A + \hat{\omega}_C^A \wedge \hat{\omega}_B^C $$

and is related to the Riemann curvature tensor

$$ \hat{\Omega}_B^A = \frac{1}{2} R_{BCD}^A \hat{\theta}^C \otimes \hat{\theta}^D. $$

The action we consider is of the form

$$ S = \int_E \left(\hat{\lambda}_0 \mathcal{L}_0 + \hat{\lambda}_1 \mathcal{L}_1 + \hat{\lambda}_2 \mathcal{L}_2\right), $$

where $\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2$ are the Lagrangian multipliers, and $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ are the Lagrangians of the reduced theories.
where $\hat{\lambda}_n$ are multidimensional coupling constants and 

$$L_n = \hat{\Omega}^{A_1B_1} \wedge \ldots \wedge \hat{\Omega}^{A_nB_n} \wedge \epsilon_{A_1B_1 \ldots A_nB_n}, \quad 2n \leq D.$$ 

$$\epsilon_{A_1 \ldots A_n} = \frac{1}{(D - n)!} \epsilon_{A_1 \ldots A_n A_{n+1} \ldots A_D} \hat{\theta}^{A_{n+1}} \wedge \ldots \wedge \hat{\theta}^{A_D}.$$

In usual tensor notations we have for $L_k$

$$L_0 = d^4x \sqrt{-\hat{g}} - \text{volume},$$

$$L_1 = d^4x \sqrt{-\hat{g}} R - \text{the Einstein Lagrangian},$$

$$L_2 = d^4x \sqrt{-\hat{g}} (R_{ABCD} R^{CDAB} - 4R_{AB} R^{BA} + R^2) - \text{the Gauss-Bonnet term}.$$ 

Here $R$ is the scalar curvature, $R_{AB}$ is Ricci tensor, the first four coordinates are labeled by $x$, and the remaining internal coordinates by $\xi$. We recall that in the case when $D = 4$ the Gauss-Bonnet term is proportional to the Euler form, therefore it does not contribute to the equations of motion.

We assume that the multidimensional space-time has the structure $E = M^{(4)} \times S$, with $M^{(4)}$ being the Robertson-Walker space-time and $S$ being the internal compact group manifold. We suppose that the field equations have a vacuum solution with a symmetry $P \times K$: $P$ is the symmetry group of the 4-dimensional space-time and $K$ is a group of transformation of the internal coordinates. In this case the metric of the vacuum solution can be taken in the form (we will use the greek subindices for the 4-dimensional space-time and the latin ones for the internal space)

$$\hat{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(x, \xi) \end{pmatrix}, \quad g_{mn}(x, \xi) = \frac{L^2(x)}{L_0^2} \theta^m_a(\xi) \theta^b_n(\xi) \gamma_{ab} \quad (21)$$

with $g_{\mu\nu}$ being a 4-dimensional metric on $M^{(4)}$ and $g_{mn}$ being the metric of the internal space. $L(x)$ is the radius of the internal space $S$ and $L_0$ is some constant (which will be discussed later). $\gamma_{ab}$ is a flat metric at the origin $o$, $\theta^m_a(\xi)$ are the vielbeins.

We are interested in the case when there is torsion on the internal space. So, we choose the following ansatz for the $K$-invariant connection form:

$$\hat{\omega}^A_B = \begin{pmatrix} \hat{\omega}_A^\alpha_b \\ \hat{\omega}_A^a_b \end{pmatrix}. \quad \hat{\omega}_A^\alpha_b = \hat{\omega}_A^\alpha_\beta \hat{\omega}_\beta^\alpha_b + \bar{\omega}_b^\alpha (22)$$

Here, as before, the small circle denotes the Levi-Civita connection and $\bar{\omega}_b^\alpha$ stands for the contorsion form already introduced above.

Our next step is to calculate the curvature tensor components (16). They can be found using the metricity condition

$$d\hat{g}_{AB} - \hat{\omega}^C_B g_{AC} - \hat{\omega}_A^C g_{CB} = 0 \quad (23)$$

and the structure equation

$$\hat{T}^A = d\hat{\theta}^A + \hat{\omega}_B^A \wedge \hat{\theta}^B, \quad \hat{T}^\alpha = 0, \quad T^\alpha = f C_{bc}^a \theta^b \wedge \theta^c. \quad (24)$$

where $T^\alpha = 0$. \quad $T^\alpha = f C_{bc}^a \theta^b \wedge \theta^c$. \quad 6
After some straightforward and rather tedious calculations we obtain the following expressions for the connection form
\[ \hat{\omega}^\alpha_\beta = \Gamma^\alpha_{\nu\beta} \theta^\nu, \quad \hat{\omega}^\alpha_b = -\eta_{bd} \left( \frac{L}{L_0} \right) \partial^a \left( \frac{L(x)}{L_0} \right) \theta^a, \]
\[ \hat{\omega}^\alpha = \partial_\beta \ln \left( \frac{L(x)}{L_0} \right) \theta^\alpha, \quad \hat{\omega}^\alpha_b = \delta^a_\beta d \ln \left( \frac{L}{L_0} \right), \quad \hat{\omega}^\alpha_b = \frac{f + 1}{2} C^\alpha_{db} \theta^d. \] (25)
where \( \Gamma^\alpha_{\nu\beta} \) is the Christoffel symbols for the Robertson-Walker metric.

Using formulas (15) and (16) we obtain the following values for the curvature tensor components \( R^A_{\beta\gamma\delta} \).

\[ R^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta} = R^\alpha_{\beta\gamma\delta} = 0, \]
\[ R^\alpha_{\beta\gamma\delta} = \frac{L(x)}{L_0} \nabla \left( \partial^\nu \left( \frac{L(x)}{L_0} \right) \eta_{bd} \right), \quad R^\alpha_{\beta\gamma\delta} = \delta^a_\beta \frac{L_0}{L(x)} \nabla \left( \partial_\beta \frac{L(x)}{L_0} \right), \]
\[ R^\alpha_{\beta\gamma\delta} = \frac{\partial_\gamma f(x)}{2} C^a_{gb}, \quad R^\alpha_{\beta\gamma\delta} = f(x) \partial_\beta \ln \frac{L(x)}{L_0} C^a_{fg}, \]
\[ R^\alpha_{\beta\gamma\delta} = -f(x) \frac{L(x)}{L_0} \partial^a \left( \frac{L(x)}{L_0} \right) C^d_{gb} \eta_{df}, \quad R^\alpha_{\beta\gamma\delta} = f(x) \partial_\beta \ln \frac{L(x)}{L_0} C^a_{fg}, \]
\[ R^\alpha_{\beta\gamma\delta} = F(x) C^a_{db} C^d_{fg} - \left( \partial_\gamma \frac{L(x)}{L_0} \right)^2 \left( \delta^a_f \eta_{bg} - \delta^a_g \eta_{bf} \right), \] (26)
where
\[ F(x) = \frac{f^2(x) - 1}{2}. \]

For the latter use it is convenient to introduce the so-called dilaton field \( \Psi \) defined by
\[ \Psi = \ln \left\{ \frac{L(x)}{L_0} \right\}. \] (27)

When calculating the curvature \( R \) and the Gauss-Bonnet term \( \mathcal{R}^2 \) we are faced with products of the structure constants. We can express them through the eigenvalues of the second order Casimir operator \( C_2 \). Using that \( C_2 = 1 \) for the adjoint representation [18], we get
\[ R = R^{(4)} - 2d \Box \Psi(x) - d (d + 1) (\partial \Psi(x))^2 - d F(x) L^{-2}(x), \] (28)
\[ \mathcal{R}^2 = \mathcal{R}^{(4)2} + 8d (\nabla_{\mu} \Psi + \partial_{\mu} \Psi \partial_{\nu} \Psi) R^{(4)\mu\nu} - 2d \left[ 2 \Box \Psi + (d + 1)(\partial \Psi)^2 + FL^{-2} \right] R^{(4)} - 4d (d - 1) \nabla_{\mu} \Psi \nabla^{\mu} \Psi - 8d (d - 1) \nabla_{\mu} \Psi \partial^{\mu} \Psi \partial^{\nu} \Psi + d (d - 1)(d - 2)(d + 1)(\partial \Psi)^4 + 4d (d - 1)(\Box \Psi)^2 + d (d - 3) F^2 L^{-4} + 4d (d - 2) F L^{-2} \Box \Psi + 2 F d (d - 1)(d - 2) L^{-2} (\partial \Psi)^2 + 4d^2 (d - 1)(\partial \Psi)^2 \Box \Psi + 2d f \partial_{\mu} f \partial^{\mu} \Psi L^{-2}. \] (28')

The invariance of the metric and the connection allows us to carry out the dimensional reduction of the action. We insert expressions (28) and (28') into (17) and omit total derivatives. Then, after having integrated the Lagrangian over the group manifold, we obtain the reduced action in the form
\[ S = v_d \int d^4 x e^{\Phi} \sqrt{-g^{(4)}} \left\{ \lambda_1 R^{(4)} + \lambda_0 e^{-\Psi} d + \lambda_1 \left( -\frac{1}{2} d(d+2)(\partial \Psi)^2 - e^{-(d+2)\Phi} \bar{F} d \right) \\
+ \lambda_2 \left[ e^{-2\Psi} \left( d(d^2 - 2d - 12) \bar{F} (\partial \Psi)^2 + 2d(d+6) \partial_\mu \bar{F} \partial^\mu \Psi - 2d \bar{F} R^{(4)} \right) \right] \\
+ e^{d\Psi} \left\{ R^{(4)} - d(d-1)(d+2)(\partial \Psi)^4 + d(d^2 - 4) \partial \Psi (\partial \Psi)^2 + 4d G_{\mu \nu} \partial^\mu \Psi \partial^\nu \Psi \right\} \right\}, \tag{29} \]

where \( v_d \) is the volume of the internal space with the scale factor \( L = L_0 \), \( g^{(4)} = \det g_{\mu \nu} \), and \( G_{\mu \nu} \) is the Einstein tensor.

Notice that in the action we obtained the term proportional to \( R^{(4)} \) should correspond to the Einstein gravity; therefore, to bring the action (29) to the correct form we will introduce the metric \( \eta_{\alpha \beta}(x) \) related to \( g_{\alpha \beta}(x) \) by the formula

\[ g_{\alpha \beta}(x) = \left( \frac{L(x)}{L_0} \right)^{-d} \eta_{\alpha \beta}(x). \]

In terms of the metric \( \eta_{\alpha \beta}(x) \) the action (29) reads

\[ S = \int d^4 x \sqrt{-\eta} \left\{ \lambda_1 R^{(4)} + \lambda_0 e^{-\Psi} d + \lambda_1 \left( -\frac{1}{2} d(d+2)(\partial \Psi)^2 - e^{-(d+2)\Phi} \bar{F} d \right) \\
+ \lambda_2 \left[ e^{-2\Psi} \left( d(d^2 - 2d - 12) \bar{F} (\partial \Psi)^2 + 2d(d+6) \partial_\mu \bar{F} \partial^\mu \Psi - 2d \bar{F} R^{(4)} \right) \right] \\
+ e^{d\Psi} \left\{ R^{(4)} - d(d-1)(d+2)(\partial \Psi)^4 + d(d^2 - 4) \partial \Psi (\partial \Psi)^2 + 4d G_{\mu \nu} \partial^\mu \Psi \partial^\nu \Psi \right\} \right\}, \tag{30} \]

where \( \lambda_i = v^d \lambda_i \) are the effective four-dimensional coupling constants; \( \bar{F} = F L_0^{-2} \); \( R^{(4)} \), \( R^{(4)} \) and \( G_{\mu \nu} \) have been calculated with respect to the metric \( \eta_{\mu \nu} \).

We consider the spatially homogeneous and isotropic Robertson-Walker universe. The interval for the metric \( \eta_{\alpha \beta} \) in the coordinate basis reads

\[ ds_{\text{RW}}^2 = -dt^2 + a^2(t) \left[ d\chi^2 + \sin^2 \chi (d\hat{\theta}^2 + \sin^2 \hat{\theta} d\phi^2) \right]. \tag{31} \]

In Robertson-Walker cosmology the function \( a(t) \) is usually interpreted as the radius of the universe. Notice that the flat Minkowski universe is the limiting case \( a \to \infty \).

For the latter use it is convenient to define the following dimensionless variable

\[ a(t)/L_0 = e^\alpha(t). \tag{32} \]

Straightforward calculations of the curvature \( R^{(4)} \) of the Robertson-Walker universe give the following result

\[ R^{(4)} = 6(\ddot{a} + 2\dot{a}^2) + \frac{e^{-2\alpha}}{L_0^2}. \tag{33} \]

For the scalar fields \( \Psi \) and \( f \) to be consistent with the homogeneous and isotropic Robertson-Walker ansatz, which has been chosen for the metric \( \eta_{\mu \nu} \), they can depend on time only. So, now the action is

\[ S = \int d^4 x \sqrt{-\eta} \left\{ \lambda_1 R^{(4)} + \lambda_1 \left( \frac{d}{2}(d+2)(\partial \Psi)^2 \right) \right\}, \]

8
\[
\begin{align*}
+ \bar{\lambda}_2 \left[ e^{-2\Psi} \left\{ -d\bar{F}((d^2 - 2d - 12)\bar{F}\dot{\Psi}^2 + 2\Psi\dot{\Psi}\dot{\bar{\alpha}} - 12\dot{\bar{\alpha}}^2) + 2d\dot{\bar{F}}(6\dot{\bar{\alpha}} - (d + 6)\dot{\Psi}) \right\} \\
+ e^{d\Psi} \left\{ -\frac{1}{3} d(d + 2)(d^2 + d - 3)\dot{\Psi}^4 + 2d(d^2 - 4)\dot{\Psi}\dot{\bar{\alpha}} + 4d\dot{\Psi}^2(3\alpha^2 + \frac{1}{2} e^{-2\alpha}) \right. \\
- 8d\dot{\Psi}\alpha^3 - 4dR^{(3)}e^{-2\alpha}\dot{\Psi} \left\} \right] - W(\Phi, f; \alpha; d, \bar{\lambda}_0, \bar{\lambda}_1, \bar{\lambda}_2) \right\},
\end{align*}
\]

with

\[ W = e^{-\Psi d}\left\{ -\bar{\lambda}_0 + \bar{\lambda}_1 d\bar{F}e^{-2\Psi} - \bar{\lambda}_2 d(d - 3)\bar{F}^2 e^{-4\Psi} \right\} + \frac{2d}{L_0^2} \bar{\lambda}_2 \bar{F} e^{-2\Psi} e^{-2\alpha}. \]

We notice that terms proportional to \(e^{-4\alpha}\), which one could expect to be present in the potential, in fact do not appear due to a special combinations of the coefficients in the Gauss-Bonnet term (20).

Thus, after the dimensional reduction we have obtained an effective 4-dimensional theory on \( M(4) \), which describes Einstein gravity coupled to the scalar fields \( \Phi(t) \) and \( f(t) \). We will devote the next section to a more detailed analysis of the potential (35).

### 4 Analysis of the potential

To investigate the behavior of the effective 4-dimensional system we need to derive equations of motion from the action (34) and to solve them. But the system of equations of motion which we obtain is so complicated, that we could not find any exact solution to it without additional assumptions. So, we restrict ourselves to the analysis of the static solutions. We think that even in this particular case we can see some typical

Fig. 1.

Fig. 2.
features in the evolution of the investigated system. Static solutions of the equations of motion correspond to the extrema of the potential (35). The analysis of the shape of the potential can trace qualitatively the dynamics of the internal and the 4-dimensional scale factors, determine the values of parameters for which the minima are separated from the decompactification area by a finite barrier and to examine the role of curvature and torsion in these processes. A simple analysis shows that the form of the potential strongly depends on the values of the parameters $\bar{\lambda}_i$. There are three typical shapes of the potential considered as a function of $\Psi$. They are shown in Fig.1 – Fig.3. The first two regimes are not of physical interest, because the absence of a local minimum corresponds to the absence of a compactifying solution with a finite size of the internal space. If we assume that the torsion field takes an equilibrium value $f = f_{\text{min}}$, so in the first regime (which is the case, for example, when $\bar{\lambda}_2 > 0$, $\bar{\lambda}_0 < 0$) the potential is unbounded from below. Therefore, the system either enters the quantum domain (see Fig.1, negative $\Psi$) and, therefore, the classical analysis will fail there, or $L(t) \to \infty$ as $t \to \infty$ (for large positive $\Psi$) and we have the decompactification of the extra dimensions. In the second regime (with $\bar{\lambda}_0 < 0$, $-\frac{\lambda_1^2}{4\lambda_0 \lambda_2} < \frac{(d+4)(d-3)}{(d+2)^2}$) we also see that the decompactification of the extra dimensions takes place (see Fig.2, large positive $\Psi$).

So, in the next two subsection we will study only those values of the parameters for which the potential has classically stable local minima, separated from the decompactification region by a finite barrier (Fig.3).

For the purpose of comparison we will use some results from our previous paper [13], where the 4-dimensional space-time was chosen to be the flat Minkowski space-time (that is the case $\alpha \to \infty$). The potential of our present model, which corresponds to the Robertson-Walker metric, differs from the Minkowski case by the presence of the last term in the expression (35). This term makes the analysis much more difficult, so we are forced to apply the method of perturbations with respect to a small parameter $\varepsilon$. Using the typical estimations for the coupling constants $\bar{\lambda}_i$ [4, 20], we found that it is convenient to choose $\varepsilon$ to be $\varepsilon \equiv e^{-2\alpha}$, where $\varepsilon$ satisfies the condition $\varepsilon \ll \bar{\lambda}_1 L_0^4$ and we also assume that $\varepsilon \ll \bar{\lambda}_0 L_0^4$, and $\varepsilon \ll \lambda_2$. In our calculations of the minima of the potential (35) it seems to be natural to choose the corresponding quantities of the flat Minkowski case as zero approximation.

We will study the potential as a function of two variables $W = W(\Psi, f)$ and neglect the fact that $W$ also depends on $\alpha$. We may assume that $\alpha$ changes little in comparison with $\Psi$ and $f$, so that this assumption allows us to consider $\alpha$ as a ‘constant background’.
Before investigating the minima of the potential we discuss the constant $L_0$ which still remains a free parameter in the model. Let us consider the flat Minkowski metric (that is the limiting case $\varepsilon = 0$). We can fix the parameter $L_0$ by the condition that the potential $W = W(\Psi)$ takes its minimum at $\Psi = \Psi_{\text{min}} = 0$. This allows us to interpret $L_0$ as the size of the internal space at the vacuum state, corresponding to the compactification of the extra dimensions. The explicit formula for $L_0$ will be presented in the next subsections.

We will investigate those values of the parameters $\bar{\lambda}_i$ for which there were minima of the potential in the Minkowski case, so that we will be able to compare the results. In what follows we will also use the notation $\Delta \equiv \frac{d \lambda_2^2}{4(d-3)\lambda_0 \lambda_2}$.

4.1 Case 1: $\bar{\lambda}_0 < 0, \quad \bar{\lambda}_2 < 0, \quad d \geq 4, \quad \Delta = 1$

Under these values of the parameters in the case of the flat Minkowski metric the minima of the corresponding potential are degenerated and located on the curve with nonzero torsion $|f| < 1$

$$\Psi_{\text{min}}(f) = \frac{1}{2} \ln \left\{ -(d-3) \frac{\bar{\lambda}_2}{\lambda_1 L_0^2} \frac{1-f^2}{2} \right\}. \quad (36)$$

This curve consists from two gutters, which join each other at the point with zero torsion.

Imposing the condition $\Psi_{\text{min}} = 0$, as it was explained earlier, we obtain the following value of the parameter $L_0$

$$L_0 = \sqrt{-\frac{\lambda_2 (d-3)}{2\lambda_1}}. \quad (37)$$

Looking at formula (34) we see that the value of the potential in the minimum generates an effective cosmological constant $\Lambda^{(4)} = W(\Psi_{\text{min}}, f_{\text{min}})$ in four dimensions. It can be checked that under this values of the parameters the potential vanishes identically on the curve (36), therefore the four-dimensional cosmological constant $\Lambda^{(4)}_{\text{Min}}$ is equal to zero, as it was expected.

When we pass to the Robertson-Walker metric we find that if the curvature is present the curve of the minima (36) with nonzero torsion reduces to the only point and we are left with the only minimum corresponding to zero torsion

$$\left( \Psi_{\text{min}} = -\frac{1}{2} \ln \left\{ 1 + 2 \frac{\bar{\lambda}_2}{\lambda_1 L_0^2} \varepsilon \right\}, \ f_{\text{min}} = 0 \right), \quad (38)$$

where we had to apply the second order of the theory of perturbation to see that the degeneracy is removed and the previous gutters ascend and no longer correspond to minima. Now we obtain a nonzero value for the effective four dimensional cosmological constant $\Lambda^{(4)}_{\text{RW}}$

$$\Lambda^{(4)}_{\text{RW}} \sim \frac{d \bar{\lambda}_1}{d-3 L_0^2} \varepsilon.$$

We see that $\Lambda^{(4)}_{\text{RW}}$ is given by the well defined quantities: $\varepsilon$ (related to the curvature), $\bar{\lambda}_1$ (which is the Newton constant) and $d$ (the dimension of the internal space). Thus, there are no free parameters which we could turne with the aim to set $\Lambda^{(4)}_{\text{RW}}$ equal to zero.
4.2 Case 2: $\tilde{\lambda}_0 < 0, \quad \tilde{\lambda}_2 < 0, \quad d \geq 4, \quad \frac{d(d+4)}{(d+2)^2} < \Delta < 1$

In the flat Minkowski case we found the there was a minimum with zero torsion for these values of the parameters. For $\varepsilon \neq 0$ our calculations within the perturbation theory shows that, as in the previous case, the potential (35) has a minimum only when torsion is zero. This minimum is located at the point

$$\Psi_{\min} = -\frac{1}{2} \ln \left[ 1 + \frac{4}{H(d+2)} \frac{\tilde{\lambda}_2}{\lambda_1 L_0^2} \varepsilon \right], \quad f_{\min} = 0 \quad (39)$$

where $H \equiv \sqrt{1 - \frac{D(d)}{\Delta}}, \quad D(d) \equiv \frac{d(d+4)}{(d+2)^2}$.

In this case we obtain the following expression for $L_0$

$$L_0 = \frac{\lambda_1}{\lambda_0} \frac{(d+2)}{8(H-1)}, \quad (40)$$

which turns to the form (37) for $\Delta = 1$.

As in the previous subsection, we also obtain a nonvanishing four dimensional cosmological constant, shifted by the presence of the curvature; but this expression is too lengthy, so we will not present it here.

5 Conclusions

The analysis we made shows that the curvature of the Robertson-Walker space-time plays an important role and essentially affects the model. In contrast to the flat Minkowski metric for non-vanishing curvature there are no longer minima with non-zero torsion. Let us suppose that the initial conditions for scalar fields $\Psi, f$ and for their derivatives $\dot{\Psi}, \dot{f}$ were such that the universe has reached the minimum and remained there without overcoming the barrier of the potential (see Fig.3). Such a situation corresponds to the compactification of the internal space with the characteristic size given by (37) or (40). So, in this case we see that if the curvature of the 4-dimensional space is nonzero, there is no solution of spontaneous compactification with nonzero torsion on the internal space. This is in contrast with the case of the Minkowski space-time, where minima of the potential with nonzero torsion do exist.

We found that the value of the potential for the spontaneous compactification minimum is non zero, which immediately implies that the vacuum state is stable only classically. So, in this case there is the possibility for decompactification of the internal space via quantum tunneling through the potential barrier. However, under some general assumptions about the values of the parameters of the model, the time of such a tunneling exceeds the life-time of the universe.

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