THE RATIO OF $\theta$-CONGRUENT NUMBERS

YAN LI AND SU HU

Abstract. Let $0 < \theta < \pi$ such that $\cos \theta \in \mathbb{Q}$. In this paper, we prove that for given positive square-free coprime integers $k, l$, there exist infinitely many pairs $(M, N)$ of $\theta$-congruent numbers such that $lN = kM$. This generalize the previous result of Rajan and Ramaroson [14] on the ratio of congruent numbers from congruent numbers (i.e. $\theta = \pi/2$) to arbitrary $\theta$-congruent numbers.

1. Introduction

A congruent number is a square-free integer which is the area of a right triangle with rational sides. The interesting problem to decide integers which are congruent numbers was systematically studied by Arab scholars in the tenth century. The congruent number problem are closely related to the arithmetic theory of elliptic curves. It is well known that a square-free integer $n$ is a congruent number if and only if the elliptic curve $ny^2 = x(x^2 - 1)$ has positive rank (see [7], for instance).

Fujiwara [11] extended the concept of congruent numbers by considering general (not necessarily right) triangles with rational sides. Let $\theta$ be a real number with $0 < \theta < \pi$. A triangle with an angle $\theta$ and rational sides is called a rational $\theta$-triangle. Notice that, for such a triangle, $\cos \theta$ is necessarily rational. In the sequel, we always assume $\cos \theta \in \mathbb{Q}$ and denote

\[
\cos \theta = \frac{s}{r}, \ r, s \in \mathbb{Z}, \ r > 0, \ \gcd(s, r) = 1.
\]

Definition 1.1. A natural number $n$ is $\theta$-congruent if $n$ is square-free and $nr \sin \theta$ is the area of a rational $\theta$-triangle.

For $\theta = \frac{\pi}{2}$, $\theta$-congruent numbers are just the usual congruent numbers. Like the congruent numbers, the $\theta$-congruent numbers are also closely related to the arithmetic of elliptic curves. Let $E_{n, \theta}$ be the elliptic curve defined by

\[
E_{n, \theta} : ny^2 = \frac{1}{r}x(x + \cos \theta - 1)(x + \cos \theta + 1).
\]

In a slight different form, Fujiwara [11] showed that

Theorem 1.2. (Fujiwara) Let $n$ be any square-free natural number. Then
(1) $n$ is $\theta$-congruent if and only if $E_{n,\theta}$ has a rational point of order greater than 2.

(2) For $n \neq 1, 2, 3, 6$, $n$ is $\theta$-congruent if and only if $E_{n,\theta}(\mathbb{Q})$ has positive rank.

For more properties of $\theta$-congruent numbers, see [6], [12].

For congruent numbers, Chahal [5] has proved that there exist infinitely many congruent numbers in each residue class modulo 8. Bennett [13] extended Chahal’s result to any integer $m > 1$. Johnstone and Spearman [16] made further improvements on Bennett’s result.

Recently, Rajan and Ramaroson [14] got the following interesting result on the ratio of congruent numbers.

**Theorem 1.3.** (Rajan and Ramaroson) If $k$ and $l$ are positive, square-free coprime integers, then there exist infinitely many pairs $(M, N)$ of congruent numbers such that $lN = kM$.

In this paper, following the method of Rajan and Ramaroson [14], we will generalize their results to arbitrary $\theta$-congruent numbers. Our main results are the following:

**Theorem 1.4.** Let $0 < \theta < \pi$ such that $\cos \theta \in \mathbb{Q}$. If $k$ and $l$ are positive, square-free coprime integers, then there exist infinitely many pairs $(M, N)$ of $\theta$-congruent numbers such that $lN = kM$.

**Corollary 1.5.** Assumption as above, there exist infinitely many square-free integers $N$ such that both $kN$ and $lN$ are $\theta$-congruent numbers.

2. **Generalized Holm’s Curve and its Jacobian**

Let $\beta$ be a rational number such that $\beta \neq \pm 1$. Let $k, l$ be coprime positive integers such that $k \neq l$. We call the curve

$$H_\beta : lx(x + \beta - 1)(x + \beta + 1) = ky(y + \beta - 1)(y + \beta + 1)$$

the generalized Holm’s curve since for $\beta = 0$, $H_0$ was considered by Holm [11] in a slight different form.

In this section, we will study $H_\beta$ and its Jacobian $E_\beta$. We will show that $H_\beta$ is a smooth irreducible curve of genus one with infinitely many rational points.

**Proposition 2.1.** $H_\beta$ is a smooth irreducible curve of genus one.

*Proof.* It is well-known that a smooth cubic is automatically irreducible of genus one. So we only need to check the smoothness. To do this, we will use the Jacobian criterion.
Let $G(x, y) = lx(x + \beta - 1)(x + \beta + 1) - ky(y + \beta - 1)(y + \beta + 1)$. The equation

$$
\begin{cases}
\frac{\partial G}{\partial x} = l(x + \beta - 1) + x(x + \beta + 1) + (x + \beta - 1)(x + \beta + 1) = 0 \\
\frac{\partial G}{\partial y} = -k(y + \beta - 1) + y(y + \beta + 1) + (y + \beta - 1)(y + \beta + 1) = 0
\end{cases}
$$

(2.1)

has four solutions $(\alpha_1, \alpha_1), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\alpha_2, \alpha_2)$, where we assume $\alpha_1 < \alpha_2$. Since $\alpha_1$ and $\alpha_2$ are the extreme points of cubic function $u = \nu(\nu + \beta - 1)(\nu + \beta + 1)$, from the graph, one can see $\alpha_1(\alpha_1 + \beta - 1)(\alpha_1 + \beta + 1) > 0$ and $\alpha_2(\alpha_2 + \beta - 1)(\alpha_2 + \beta + 1) < 0$. Since $k \neq l$ are positive integers, the four points $(\alpha_1, \alpha_1), (\alpha_1, \alpha_2), (\alpha_2, \alpha_1), (\alpha_2, \alpha_2)$ do not satisfy the equation $G(x, y) = 0$. Thus $H_\beta$ is smooth on the affine part. It is easily checked that $H_\beta$ is also smooth at the infinite points $(\sqrt[k]{l}, 1, 0), (\sqrt[k]{l}, 1, 0), (\sqrt[k]{l}, 1, 0), (\sqrt[k]{l}, 1, 0)$, where $\rho$ is a primitive cubic root of unity.

Using the method of [10] (p.23), one can change $H_\beta$ to its jacobian $E_\beta$:

$$
Y^2 = X^3 - \frac{1}{3}k^2l^2(3 + \beta^2)^6 + (2kl\beta^2(-9 + \beta^2)^2 + 27k^2(-1 + \beta^2)^2 + 27l^2(-1 + \beta^2)^2)
$$

by identifying $(0, 0)$ to the zero element. The computation is done by Mathematica 7.0, see the file “Eq.nb” in the supplementary materials.

**Proposition 2.2.** (i) The discriminant of $E_\beta$ is

$$
-\frac{16}{27}k^4l^4(-4k^2l^2(3 + \beta^2)^6 + (2kl\beta^2(-9 + \beta^2)^2 + 27k^2(-1 + \beta^2)^2 + 27l^2(-1 + \beta^2)^2).
$$

(ii) The $j$-invariant of $E_\beta$ is

$$
\frac{6912k^2l^2(3 + \beta^2)^6}{-4k^2l^2(3 + \beta^2)^6 + (2kl\beta^2(-9 + \beta^2)^2 + 27k^2(-1 + \beta^2)^2 + 27l^2(-1 + \beta^2)^2)}
$$

(iii) The rational transformation relating $H_\beta$ and $E_\beta$ are

$$
X = \frac{k(l(-3ly + k(3x + 6\beta - (3x + 4y)\beta^2 - 6\beta^3) + l\beta(-6 + 4x\beta + 3\beta(y + 2\beta))}}{3lx - 3ky},
$$

$$
Y = \frac{-k(k - l)(l(-1 + \beta^2))(kl(x - y)(1 + 2(x + y)\beta + 3\beta^2) - l^2x(-1 + \beta(x + \beta))(1 + \beta(y + \beta)))}{(lx - ky)^2},
$$

$$
x = \frac{3k(-1 + \beta^2)(l(2k^2\beta^2(-1 + \beta^2) + 3l^2(-1 + \beta^2)) + kl(3 + 16\beta^2 - 3\beta^4) + 6\beta Y - 3(k + l - (3k - 3l)\beta^2)X}{9(k + l)(-1 + \beta^2)Y + \beta(kl(3k^2(-1 + \beta^2) + 9l^2(-1 + \beta^2) - 2kl(9 + 5\beta^2(-6 + \beta^2)) - 6kl(3 + 5\beta^2)X + 18X^2)},
$$

$$
y = \frac{-3l(-1 + \beta^2)(l(2k^2\beta^2(-1 + \beta^2) + 3l^2(-1 + \beta^2)) + kl(3 + 16\beta^2 - 3\beta^4) + 6\beta Y - 3(k + l + (3k - l)\beta^2)X}{9(k + l)(-1 + \beta^2)Y + \beta(kl(3k^2(-1 + \beta^2) + 9l^2(-1 + \beta^2) - 2kl(9 + 5\beta^2(-6 + \beta^2)) - 6kl(3 + 5\beta^2)X + 18X^2)},
$$

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(iv): Under the above transformation, the nine rational points of $H_\beta$ correspond to the nine rational points of $E_\beta$ as follows:

$P_1: (-\beta - 1, -\beta + 1) \leftrightarrow \left(\frac{1}{3} kl(3 + \beta^2), k(k - l)(l + 1 + \beta^2)\right)$,

$P_2: (0, -\beta + 1) \leftrightarrow \left(\frac{1}{3} l(3l + \beta^2) - 2k\beta(3 + \beta), (k - l)(1 + \beta)(k(-1 + \beta) - l(1 + \beta^2))\right)$,

$P_3: (-\beta + 1, -\beta + 1) \leftrightarrow \left(\frac{1}{3} kl(-3 + (-6 + \beta)\beta), kl(k + l)(-1 + \beta^2)\right)$,

$P_4: (-\beta - 1, 0) \leftrightarrow \left(\frac{1}{3} k(3k(-1 + \beta)^2 - 2l(-3 + \beta)\beta), k(k - l)(-1 + \beta)(l + k(-1 + \beta^2 + l\beta))\right)$,

$P_5: (0, 0) \leftrightarrow O$,

$P_6: (-\beta + 1, 0) \leftrightarrow \left(\frac{1}{3} k(3k(1 + \beta)^2 - 2l\beta(3 + \beta)), k(k - l)(1 + \beta)(l - l\beta + k(1 + \beta^2))\right)$,

$P_7: (-\beta - 1, -\beta - 1) \leftrightarrow \left(\frac{1}{3} kl(-3 + \beta(6 + \beta)), -kl(k + l)(-1 + \beta^2)\right)$,

$P_8: (0, -\beta - 1) \leftrightarrow \left(\frac{1}{3} l(3l(-1 + \beta)^2 - 2k(-3 + \beta)\beta), -(k - l)(-1 + \beta)(l(-1 + \beta)^2 + k(1 + \beta))\right)$,

$P_9: (-\beta + 1, -\beta - 1) \leftrightarrow \left(\frac{1}{3} kl(3 + \beta^2), -k(k - l)(-1 + \beta^2)\right)$.

(v) $E_\beta(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z}$.

Proof. The (i), (ii), (iii) (iv) can be checked by Mathematica 7.0, see files “invariants.nb”, “trans.nb”, “coordinates.nb” in the supplementary materials. We only give the proof of (v). Let $E_\beta : Y^2 = f(X)$. Since

$$f'(\pm kl(3 + \beta^2)/3) = 0$$

and

$$f\left(\frac{1}{3} kl(3 + \beta^2)\right) = k^2(k - l)^2 l^2 (-1 + \beta^2)^2 > 0,$$

$f(X) = 0$ has only one real root. Therefore, $E_\beta(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z}$. □

The expression of addition of two points of $E_\beta$ is quite complicated since it has three parameters $k, l, \beta$. So it is hard to check a point on $E_\beta$ has infinite order by computer. Instead, we will give the following geometric and intuitive proof. The proof is based on Mazur’s famous result on the torsion group of rational points of elliptic curves over $\mathbb{Q}$ (see [2], [3]).

**Proposition 2.3.** $E_\beta(\mathbb{Q})$ has positive rank. Equivalently, $H_\beta$ has infinitely many rational points.

Proof. By (v) of Proposition [2,2] and Mazur’s result, if the rank of $E_\beta(\mathbb{Q})$ is 0, then $|E_\beta(\mathbb{Q})| \leq 12$. So in order to prove $E_\beta(\mathbb{Q})$ has positive rank, we will show it has at least 13 rational points. Since $E_\beta$ and $H_\beta$ are birationally equivalent over
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\( \mathbb{Q} \), it suffices to check it for \( H_\beta \). Changing the coordinate \((x + \beta, y + \beta)\) to \((x, y)\), we get a new equation

\[
H_\beta : l(x - \beta)(x - 1)(x + 1) = k(y - \beta)(y - 1)(y + 1).
\]

From this point to the end of the proof, we will use the above equation to show \(|H_\beta(\mathbb{Q})| \geq 13\). To do this, we can assume \(0 < \beta < 1\) and \(l > k\). The reason is as follows. We omitted the case of \(\beta = 0\) since it was already done in [14]. For \(\beta < 0\), change coordinate \((x, y)\) to \((-x, -y)\) and for \(\beta > 1\), change coordinate \((x, y)\) to \((\frac{1}{2}(\beta + 1)(x + 1) - 1, \frac{1}{2}(\beta + 1)(y + 1) - 1)\).

Letting \((x - \beta)(x - 1)(x + 1) = (y - \beta)(y - 1)(y + 1) = 0\), we get nine distinct rational points of \(H_\beta\):

\[
\begin{align*}
P_1 & : (-1, -1), & P_2 & : (-1, \beta), & P_3 & : (-1, +1), \\
P_4 & : (\beta, -1), & P_5 & : (\beta, \beta), & P_6 & : (\beta, +1), \\
P_7 & : (+1, -1), & P_8 & : (+1, \beta), & P_9 & : (+1, +1).
\end{align*}
\]

The tangent line of \(H_\beta\) at \((\beta, \beta)\): \(l(x - \beta) - k(y - \beta) = 0\) meets \(H_\beta\) at the new point:

\[
P_0 : \left(\frac{l}{l + k} \beta, \frac{l - k}{l + k}\right).
\]

From the group law of cubic curve (p.18-22 of [10]), the negative \(-P\) is the third intersection of cubic curve and the line passing through \(P\) and \(P_0\), where \(P\) is an arbitrary point of \(H_\beta\). So we have \(P_1 + P_9 = O\). From the figure above, one can see \(P_6 + P_7\) might be \(O\). By (iv) of Proposition [2.2] it is easy to see the \(Y\)-coordinates of \(P_2, P_3, P_4, P_8\) do not equal to zero under the assumption \(0 < \beta < 1\) and \(l > k > 0\). So they are not two torsion points, i.e., \(-P_i \neq P_i\) \((i = 2, 3, 4, 8)\). Drawing the lines connecting \(P_i\) \((i = 2, 3, 4, 8)\) and \(P_0\), one can easily see the set
\{-P_2, -P_3, -P_4, -P_8\} is disjoint with the set \( \{P_i|1 \leq i \leq 9\} \) from the figure above. Hence \(|H_\beta(\mathbb{Q})| \geq 13\), which concludes the proof.\( \square\)

In what following, we will apply the above results to the ratio of \(\theta\)-congruent numbers. From this point on, we will fixed an angle \(\theta\) with \(0 < \theta < \pi\) and let \(\beta = \cos \theta\). We will use notations \(H_\theta, E_\theta\) instead of \(H_\beta, E_\beta\), respectively. Also we will fixed coprime, unequal positive integers \(k, l\). In addition, we require \(k, l\) are square-free. Recall that \(r\) is the denominator of \(\cos \theta\).

Let \(A_x = x((x + \beta)^2 - 1)/r\) and \(A_y = y((y + \beta)^2 - 1)/r\). Then every rational point \((x, y)\) on \(H_\theta\) with \(A_x > 0\) gives rise two rational \(\theta\)-triangles whose areas are in the ratio

\[
\frac{A_x}{A_y} = \frac{k}{l}.
\]

Indeed, if \(A_x\) is positive, the rational \(\theta\)-triangle

\[
\{(x + \beta)^2 - 1, 2x, 1 + (x + \beta)^2 - 2(x + \beta)\beta\} \quad \text{(for } x > 0)\]

or

\[
\{1 - (x + \beta)^2, -2x, 1 + (x + \beta)^2 - 2(x + \beta)\beta\} \quad \text{(for } x < 0)\]

has area \(A_x r \sin \theta\), and similar for \(A_y\) since it is also positive. Therefore, by the definition of \(\theta\)-congruent numbers, every rational point \((x, y)\) on \(H_\theta\) with \(A_x > 0\) produces a pair of congruent numbers, \((N_x, N_y)\) when we take the square-free parts \(N_x\) of \(A_x\) and \(N_y\) of \(A_y\) respectively.

In the sequel, demonstrating the transformations in (iii) of proposition [2.2] we will show that there are infinitely many points \((X, Y)\) on \(E_\theta(\mathbb{Q})\) such that the corresponding point \((x, y)\) on \(H_\theta\) satisfies

\[
A_x > 0, \ A_y > 0, \ (l, N_x) = 1, \ (k, N_y) = 1.
\]

Taking the square-free part of both sides of \(lA_x = kA_y\), we will get

\[
ln_x = kN_y.
\]

To show the infinity of such points on \(E_\theta(\mathbb{Q})\), we use the valuation properties of elliptic curves over local fields.

3. THE VALUATION PROPERTIES OF GLOBAL POINTS ON ELLIPTIC CURVES

Let \(E\) be an elliptic curve over \(\mathbb{Q}\) defined by the Weierstrass equation:

\[
Y^2 = X^3 + aX + b, \ a, b \in \mathbb{Q}.
\]

Let \(S = \{p_1, p_2, ..., p_t\}\) be a set of prime numbers.

Given positive integers \(m_1, m_2, ..., m_t\), let

\[
U_{m_1, m_2, ..., m_t}(E) = \{P \in E(\mathbb{Q})|\text{ord}_{p_i}(X(P)) = -2m_i, \ \text{ord}_{p_i}(Y(P)) = -3m_i, \ \text{where } 1 \leq i \leq t\}.
\]

Improving the Proposition 3.3 of [14] a bit more, we get
Proposition 3.1. If \( E(\mathbb{Q}) \) has positive rank, then there exists an integer \( N \) such that \( U_{m_1, m_2, \ldots, m_t}(E) \neq \emptyset \), for all \( m_1, m_2, \ldots, m_t \geq N \).

Proof. Let \( E' \) with coordinates \((X', Y')\) be the global minimal Weierstrass Equation of \( E \) over \( \mathbb{Q} \). It is well-known that such equation exists (see [8] and [9], for instance). The coordinates \((X, Y)\) and \((X', Y')\) are related by

\[
X = u^2 X' + f \quad \text{and} \quad Y = u^3 Y' + gu^2 X' + h, \quad u, f, g, h \in \mathbb{Q}, \quad u \neq 0.
\]

Hence, there exists an integer \( M_1 \) such that

\[
U_{m_1-s_1, m_2-s_2, \ldots, m_t-s_t}(E) = U_{m_1, m_2, \ldots, m_t}(E').
\]

for all \( m_1, m_2, \ldots, m_t \geq M_1 \), where \( s_1 = \text{ord}_{p_1}(u), \ldots, s_t = \text{ord}_{p_t}(u) \).

By Proposition 3.1 and 3.3 of [14], there exists an integer \( M_2 \) such that

\[
U_{m_1, m_2, \ldots, m_t}(E') \neq \emptyset, \quad \text{for all} \quad m_1, m_2, \ldots, m_t \geq M_2.
\]

Let \( M = \max\{M_1, M_2\} \). Let \( N = \max\{M - s_1, M - s_2, \ldots, M - s_t\} \). Then

\[
U_{m_1, m_2, \ldots, m_t}(E) \neq \emptyset, \quad \text{for all} \quad m_1, m_2, \ldots, m_t \geq N.
\]

\[\square\]

Proposition 3.2. Assume \( E(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z} \) and \( E(\mathbb{Q}) \) has positive rank. Let \( Q_1, Q_2 \in E(\mathbb{Q}) \) such that \( Q_1 \) has infinite order. Then for any \( M > 0 \), there are infinitely many points \( P \) belonging to the set \( \{\pm([n]Q_1 + Q_2) | n \in \mathbb{Z}\} \) such that \( X(P) > M \) and \( y(P) > 0 \).

Proof. Since \( E(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z} \), \( E(\mathbb{R}) \) is a 1-dimensional commutative connected compact lie group. Hence, \( E(\mathbb{R}) \) is isomorphic to the unit circle group \( \{z \in \mathbb{C} | |z| = 1\} \) as lie groups (see p.7 of [15] or p.42 of [10], for instance). The image of the set \( \{[n]Q_1 + Q_2 | n \in \mathbb{Z}\} \) are everywhere dense in the circle group since \( Q_1 \) has infinite order. Pick up a sequence \( z_n \) from the image set such that \( \lim z_n = 1 \) as \( n \) goes to infinity. The corresponding sequence \( R_n \) on \( E(\mathbb{R}) \) goes to the zero element as \( n \) goes to infinity. This is equivalent to \( \lim X(R_n) = +\infty \). Taking negative if necessary, we can assume \( Y(R_n) > 0 \) for all \( n \). This concludes the proof. \[\square\]

4. Proof of the main theorem

We now apply Proposition 3.1 and 3.2 to the curve \( E_\theta \). From the results in section 2, we know that \( E_\theta(\mathbb{Q}) \) has positive rank and \( E_\theta(\mathbb{R})[2] = \mathbb{Z}/2\mathbb{Z} \). Recall that \( k \neq l \) are coprime square-free positive integers. So \( E_\theta \) satisfies the conditions of Proposition 3.1 and 3.2. Let \( S \) be the set of prime divisors of \( k \) and \( l \).

In what following, we will find an infinite set \( \mathcal{P} \) such that every point \((X, Y)\) of \( \mathcal{P} \) satisfies equation 2.2.
By Proposition \[3.1\]

\[(4.1) \quad U_{m_1, m_2, ..., m_t}(E_\theta) \neq \emptyset \]

for \(m_1, m_2, ..., m_t\) being sufficiently large.

Let \(P = (X, Y)\) be an arbitrary element of \(U_{m_1, m_2, ..., m_t}(E_\theta)\). Look carefully at the transformation formula in (iii) of Proposition \[2.2\] For \(1 \leq i \leq t\), we have

\[
\begin{align*}
\text{ord}_P(x) &= \text{ord}_P(6\beta Y) - \text{ord}_P(18\beta X^2) = \text{ord}_P(3) - m_i, \\
\text{ord}_P(y) &= \text{ord}_P(6\beta Y) - \text{ord}_P(18\beta X^2) = \text{ord}_P(3) - m_i, \\
\text{ord}_P(A_x) &= 3\text{ord}_P(x) - \text{ord}_P(r) = 3\text{ord}_P(3) - 3m_i - \text{ord}_P(r), \\
\text{ord}_P(A_y) &= 3\text{ord}_P(y) - \text{ord}_P(r) = 3\text{ord}_P(3) - 3m_i - \text{ord}_P(r),
\end{align*}
\]

if \(m_1, m_2, ..., m_t\) are sufficiently large.

Therefore, we can fix suitable, large \(m_1, m_2, ..., m_t\) such that equations \[4.1\] and \[4.2\] hold and \(\text{ord}_P(A_x), \text{ord}_P(A_y)\) are both even for all points \(P = (X, Y) \in U_{m_1, m_2, ..., m_t}(E_\theta)\), where \(1 \leq i \leq t\).

From the transformation formula in (iii) of Proposition \[2.2\] again, we have

\[
x \approx -\frac{6\beta Y}{18\beta X^2} \approx -\frac{1}{3\sqrt{X}} \to 0^- \quad \text{and} \quad y \approx -\frac{6\beta Y}{18\beta X^2} \approx -\frac{1}{3\sqrt{X}} \to 0^-
\]
as \(X\) goes to \(+\infty\) \((Y > 0)\). So there exists \(M > 0\) such that if \(X > M\) and \(Y > 0\), then \(-1 - \beta < x, y < 0\) which implies \(A_x, A_y > 0\).

Fix an element \(Q \in U_{m_1, m_2, ..., m_t}(E_\theta)\). Let \(h\) be a positive integer coprime to \(p_1p_2...p_t\). Set \(Q_1 = [p_1p_2...p_t]Q\) and \(Q_2 = [h]Q\). Applying Proposition \[3.2\] to such \(Q_1, Q_2, M\), we find an infinite set \(\mathcal{P}\) such that \(\mathcal{P} \subset \{\pm([n]Q_1 + Q_2)\mid n \in \mathbb{Z}\}\) and every element \((X, Y)\) of \(\mathcal{P}\) satisfies \(X > M\) and \(Y > 0\). So every element \((X, Y)\) of \(\mathcal{P}\) satisfies \(A_x > 0, A_y > 0\).

Since \([n]Q_1 + Q_2 = [np_1p_2...p_t + h]Q\) and \((h, p_1p_2...p_t) = 1\), by Proposition 3.1 of \[14\], we have \(\mathcal{P} \subset \{\pm([n]Q_1 + Q_2)\mid n \in \mathbb{Z}\} \subset U_{m_1, m_2, ..., m_t}(E_\theta)\). So every element \((X, Y)\) of \(\mathcal{P}\) satisfies that \(\text{ord}_P(A_x), \text{ord}_P(A_y)\) are both even for all \(p \in S\).

Recall that \(N_x\) (resp. \(N_y\)) is the square-free part of \(A_x\) (resp. \(A_y\)). Summing up, we have

\[
A_x > 0, \quad A_y > 0, \quad (l, N_x) = 1, \quad (k, N_y) = 1
\]

for every point \((X, Y)\) of \(\mathcal{P}\). Therefore

**Theorem 4.1.** For each \((X, Y) \in \mathcal{P}\), we have \(lN_x = kN_y\).

Following \[14\], we get

**Theorem 4.2.** Associated with the infinite set of points \((X, Y)\) in \(\mathcal{P}\), there are infinitely many pairs of square-free integers \((N_x, N_y)\).
Proof. Assume that there are only finitely many such pairs. Then there must exist a pair \((N, M)\) of square-free integers associated with infinitely many rational points \((X, Y)\) in \(\mathcal{P}\). Using \((x, y)\) instead of \((X, Y)\), we conclude that the curve \(C:\)

\[
\begin{align*}
x(x + \beta - 1)(x + \beta + 1)/r &= Nx^2 \\
x(x + \beta - 1)(x + \beta + 1) &= ky(y + \beta - 1)(y + \beta + 1)
\end{align*}
\]

has infinitely many rational points. Consider the rational map of curves:

\[C \to E_\beta, \quad (x, y, z) \mapsto (x, y).\]

This map is of degree 2 and ramified at the point \((x, y) = (0, 0)\). The Riemann-Hurwitz formula implies that the genus of \(C\) is greater than 1. By Faltings [4]'s theorem, \(C\) only has a finite number of rational points. Thus, we get a contradiction. So there are infinitely many such pairs.

Putting Theorem 4.1 and 4.2 together, we get our main theorem:

**Theorem 4.3.** Let \(0 < \theta < \pi\). If \(k\) and \(l\) are positive, square-free coprime integers, then there exist infinitely many pairs \((M, N)\) of \(\theta\)-congruent numbers such that \(lN = kM\).

Letting \(l = 1\), we get the following corollary.

**Corollary 4.4.** Let \(0 < \theta < \pi\). Given a positive, square-free integer \(k\), there exist infinitely many pairs \((M, N)\) of \(\theta\)-congruent numbers such that \(N = kM\).

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Supplementary Material: The files “Eq.nb”, “invariants.nb”, “trans.nb”, “coordinates.nb” will appear online.

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E-mail address: liyan.00@mails.tsinghua.edu.cn

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China (Su Hu)
E-mail address: hus04@mails.tsinghua.edu.cn