Abstract

A simple algorithm is described to target any desired operation point for simple one-dimensional and two-dimensional dynamical systems. What makes the algorithm unique is the fact that it targets any desired point, not merely a stable/unstable fixed point of the dynamics. It is shown how the method may be applied to the logistic map and the Hénon map. Generalisations to the case of n-dimensional dynamical systems are discussed.

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1 Introduction

The idea of chaos control using only periodic orbits was introduced by Ott,Grebogi and Yorke in their seminal paper[1]. Ever since, the ideas presented in that paper and subsequent modifications of the algorithm [2-3]have been used to control chaotic dynamics in a variety of experimental systems[4-6]. There have also been advances in using noise to control the dynamics of systems, in particular by Parmananda and co-workers[7], and by Ditto et al[8].

This paper seeks to answer the following question : Is it possible, using a targeting algorithm, to (perturbatively) achieve any desired operation point for a dynamical system? In particular, the focus is on points that are not necessarily stable/unstable fixed points of the underlying dynamics. It will
be shown that the method to be described works very well for the logistic and the Hénon map.

The paper is organized as follows: In section 2, we introduce the basic algorithm to be employed to control the dynamics. We illustrate what a working model of the algorithm might look like, and try to motivate the reasons for its success. We consider a general m-dimensional dynamical system and show how the algorithm works in this case. Section 3 seeks to apply the algorithm to the logistic map. It is conclusively shown (both theoretically and using numerical simulations) that the method works very well for the logistic map. In section 4 the algorithm is applied to the Hénon map – it is again demonstrated that the algorithm is very effective and rather fast in converging to the desired target point. In Section 5, we show how the algorithm can be modified easily to include a stepwise control, thus ensuring a gradual rate of convergence to the target point. We end the paper with some conclusions.

### 2 The Algorithm

In this section, we describe in detail the targeting algorithm discussed in the Introduction. For systems whose control parameter can be varied in time, similar control mechanisms have been investigated by a number of authors [9-10]. The principal assumption throughout this paper is that we are dealing with (a) systems where the control parameter cannot be varied at all, (b) systems where it is necessary to sit at a fixed value of the control parameter and yet be able to work at a desired operation point. The algorithm therefore assumes that the control parameter is fixed throughout the duration of the dynamics of the system.

Consider a general m-dimensional dynamical system described by m coupled difference equations:

\[ x_i^{(n+1)} = f_i(x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)}; \mu_1, \mu_2, \ldots, \mu_m) \]  

(1)

Here, \( n \) is the discrete time index and the variable \( i \) runs from 1 to \( m \). The control parameters are given by \( \mu_1, \ldots, \mu_m \) and \( x_i^{(n)} \) are the values of the dynamical variables that describe the system \( (x_i) \) at time \( n \). It is assumed that the dynamics of the system at time \( n + 1 \) depend only on the values of the various dynamical variables, \( x_i \) at time \( n \).

For the purposes of this discussion, as alluded to earlier, the \( \mu_i \)'s are fixed. Let us say that a desired target state of the dynamical system (not necessar-
ily a stable/unstable fixed point of the dynamics) is \( \vec{T} \equiv (T_1, T_2, \ldots, T_m) \). Given this desired target, the targeting algorithm works as follows. We define a "distance" of the \( i^{th} \) dynamical variable, \( x^{(n)}_i \) from its target value \( T_i \) by:

\[
d^{(n)}_i = |x^{(n)}_i - T_i|
\]

(2)

The new value of the dynamical variables describing the system are then given by the algorithm as:

\[
X^{(n+1)}_j = x^{(n+1)}_j \prod_{i=1}^{m} d^{(n)}_i + T_j, \forall j = 1, \ldots, m
\]

(3)

We make the assignment \( x^{(n+1)}_j = X^{(n+1)}_j \) firstly. Then the map is iterated according to these modified values of the dynamical variables and the procedure in Eqs. (2)-(3) is repeated. Notice how the algorithm works now. The product term in (3) is zero when ANY \( d^{(n)}_i \) vanishes. Since this happens when \( x^{(n)}_i = T_j \), it is clearly seen that the dynamics of the modified system is such that it rapidly converges to the target point once any one dynamical variable has reached its target value. In fact, it can be seen from the above steps that it takes the system only one further iteration to reach the desired target point, \( \vec{T} \). Though the algorithm appears "uncontrolled" here, it will be demonstrated in Section 5 that this is not so. A stepwise control is implemented for both the logistic and the Hénon map. In sections 3 and 4, we also show that the target point is stable. We use numerical simulations to show the robustness of the algorithm to noise.

3 The Logistic Map

In this section, we apply the targeting algorithm introduced in the previous section to the logistic map. In this section, since the system is a one-dimensional discrete dynamical system, the time index appears as a subscript only. The logistic map is defined by the difference equation:

\[
x_{n+1} = \mu x_n (1 - x_n)
\]

(4)

Here, \( 0 \leq \mu \leq 4 \) and \( 0 \leq x_n \leq 1 \). Let us call the target point, \( t \). Applying the algorithm to the logistic map then yields the "modified logistic map",

\[
x_{n+1} = \mu x_n (1 - x_n) | x_n - t | + t
\]

(5)
As is seen, $t$ is a fixed point of the map. Let us analyse the stability of this point. To do this, we compute the derivative, $D = \frac{dx_{n+1}}{dx_n}$ at $t$. We obtain $D = D_1 + D_2 + D_3$, where $D_1, D_2,$ and $D_3$ are given by:

$$D_1 = \mu(1 - x_n) \mid x_n - t \mid$$

$$D_2 = -\mu x_n \mid x_n - t \mid$$

$$D_3 = \pm \mu x_n(1 - x_n)$$

Combining all these expressions and evaluating the derivative at $t$, we see that $D(t) = \pm \mu t(1 - t)$. From the properties of the logistic map, $\mid D(t) \mid \leq 1$ (the equality being reached only when $t = 0.5$ and $\mu = 4$. Therefore, we see that the new desired target point is indeed stable. The algorithm thus ensures stability of the fixed point. A couple of points are in order here – it is seen that applying the algorithm to the system does not always ensure that the iterates are in the range $[0, 1]$. This, therefore, has to be enforced during the simulations.

Having analysed the stability of the target point, we now show some examples where the algorithm has been applied successfully. Figure 1 shows the plot of the iterates versus the discrete time index, $n$, for the uncontrolled logistic map for two values of the control parameter, $\mu$. The corresponding plots in Figure 2 show the logistic map with the targeting algorithm applied. Notice how quick the convergence to the target point is in both cases. We have observed that the convergence is typically that good. Though only two representative values of $\mu$ have been chosen here, the algorithm has been tested over a large number of control parameter values and is found to work just as well in all cases.

To look at how noise affects the ability of the algorithm to target, we kicked the logistic map iterates randomly for the above two values of the control parameter. The noise used to kick the iterates is uniform in $[0, 1]$. Shown in Figure 3 are plots of the iterates versus the discrete time index, $n$. We see that the noise only momentarily disturbs the targeting ability, and that the algorithm is indeed robust against noise. This is a good feature, since noise is almost unavoidable in experiments, and the algorithm would not be applicable to any real experiments if it were sensitive to noise.
4 The Hénon map

In this section, we apply the algorithm to the Hénon map and demonstrate that targeting works very well here as well. The robustness of the algorithm to noise is also investigated and it is shown that a conclusion similar to that found for the logistic map holds.

The Hénon map is the simplest extension of the logistic map to the case of 2 dimensions. In this section as in the previous one, the time index appears as a subscript throughout. The Hénon map is defined by the following set of coupled difference equations:

\[
\begin{align*}
x_{n+1} &= 1 - \mu x_n^2 + \alpha y_n \\
y_{n+1} &= x_n 
\end{align*}
\]

(9)

The map is invertible as long as \( \alpha \neq 0 \). Let the desired target point in this case be \((x_t, y_t)\). Applying the algorithm to the map yields the following "modified Hénon map" :

\[
\begin{align*}
x_{n+1} &= (1 - \mu x_n^2 + \alpha x_n)M + x_t \\
y_{n+1} &= x_nM + y_t 
\end{align*}
\]

(10)

(11)

where \( M = |x_n - x_t||y_n - y_t| \).

To look at the stability of the target point \((x_t, y_t)\), we need to examine the Jacobian of the system at that point. To evaluate the derivatives of \(x_{n+1}\) and \(y_{n+1}\) at \((x_t, y_t)\), we use the fact that \( |z| = z\theta(z) - z\theta(-z) \), where \( z = x_n - x_t \) or \( y_n - y_t \), and then take derivatives. Here we find something remarkable. At the target point, the Jacobian of the system is 0. Every term in the Jacobian is 0, and therefore the eigenvalues of the Jacobian matrix are 0 too. From the theory of fixed points, this implies a superstable fixed point. Therefore, the targeting algorithm does indeed ensure that the desired target point is stable in this case as well.

Figures 4 shows the iterates of the uncontrolled map for two sets of parameters, while Figure 5 shows the map for the same two sets of parameters when the targeting algorithm is applied. We again see that the number of iterations taken for the system to settle down at the target point is very small. Similar experiments have been performed over a fairly wide range of parameter values with similar results. Figure 6 shows the effect of noise on the algorithm. It is again seen clearly that the algorithm is robust to noise and that the system settles down from any random perturbations very quickly to attain the desired target point.
It is important that the initial point be in the basin of attraction for the given parameter set. Numerical experiments were done where this was not the case, and the dynamical system is quickly attracted to the point at $\infty$ for almost any targeting point.

5 Stepwise Control Using The Algorithm

In the previous sections as seen in the figures the approach to the target point is not gradual. However, it is not hard to envisage applications where it might be crucial to have a stepwise control whereby the target point could be approached in a smooth fashion. In this section, it is shown that the algorithm can be implemented in a way as to allow for a gradual approach to the target point.

To achieve a stepwise control, we break up the ”distance” of the first iterate from the target point into a large number of steps (for the simulations shown, the number of steps is 1000). After that, the targeting algorithm is just run so that each step is regarded as a temporary target point, so that the system moves from one temporary target point to another until it finally reaches the desired target point. By choosing the steps to be small, one can therefore approach the target in a controlled fashion.

Figures 7 and 8 demonstrate this idea for the logistic and the Hénon map, respectively.

6 Conclusions

In conclusion, we have demonstrated a simple algorithm to target any desired operation point for simple one and two dimensional systems. In fact, as mentioned in Section 2, the algorithm can be easily generalised to a m-dimensional dynamical system. The advantages of the algorithm are the fact that it can target any point (not necessarily a fixed point of the system), it is very fast, and that it is fairly robust against noise and random perturbations.

The disadvantages are that one does need to know the functional form of the equations describing the dynamical system, and that unlike some other control algorithms, the control has to be applied during the full duration of the dynamics of the system. Work is ongoing to see how this limitation can be relaxed. We are also working on trying to implement (atleast theoretically) a simple electrical circuit that would achieve the objective. The results of this investigation will be published at a later date.
7 Acknowledgements

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8 Captions

Figure 1a: The uncontrolled logistic map for $\mu = 1.05$. The iterates are plotted against the discrete time index $n$.
Figure 1b: The uncontrolled logistic map for $\mu = 3.7$. The iterates are plotted against the discrete time index $n$.
Figure 2a: The controlled logistic map for $\mu = 1.05$, the target points being at $t = 0.75$ and $t = 0.42$.
Figure 2b: The controlled logistic map for $\mu = 3.7$, the target points being at $t = 0.005$ and $t = 0.95$.
Figure 3a: The robustness of the algorithm to noise - Random kicks applied once every 100 iterations to the logistic map for $\mu = 1.05$. The
dotted line denotes iterations for the target point $t = 0.42$, and the solid line denotes iterations for the target point $t = 0.75$.

**Figure 3b**: Random kicks applied once every 100 iterations to the logistic map for $\mu = 3.7$. The dotted line denotes iterations for the target point $t = 0.75$ and the solid line denotes iterations for the target point $t = 0.42$.

**Figure 4a**: The uncontrolled Hénon map for parameter values, $\mu = 1.1$ and $\alpha = 0.3$, with $y_n$ plotted against $x_n$.

**Figure 4b**: The uncontrolled Hénon map for parameter values, $\mu = 1.1$ and $\alpha = -0.3$, with $y_n$ plotted against $x_n$.

**Figure 5a**: The controlled Hénon map for $\mu = 1.1$ and $\alpha = 0.3$ with $y_n$ plotted against $x_n$ - the target. The solid line denotes initial point at $(−0.005, −0.5)$ and target point at $(−1, −1)$. The dotted line denotes initial point at $(−0.1, −0.1)$ and target point at $(0.9, 0.9)$. The small dashed line denotes initial point at $(0.7, −0.4)$ and target point at $(0, 0.5)$. The large dashed line denotes initial point at $(0.95, 0.5)$ and target point at $(−1, −1)$.

**Figure 5b**: The controlled Hénon map for $\mu = 1.1$ and $\alpha = -0.3$ with $y_n$ plotted against $x_n$. The lines, initial and target points are exactly the same as for Figure 5a.

**Figure 6a**: Robustness of the algorithm to noise - demonstrated for the Hénon map for parameter values $\mu = 1.1$ and $\alpha = 0.3$. The initial point is $(-0.5, 0.5)$, the target point is $(0.2, 1.0)$ and the kicks are applied once every 100 iterations.

**Figure 6b**: Robustness of the algorithm to noise - demonstrated for the Hénon map for parameter values $\mu = 1.1$ and $\alpha = -0.3$. The initial point is $(−0.5, 0.5)$, the target point is $(0.2, 1.0)$ and the kicks are applied once every 100 iterations.

**Figure 7**: Stepwise control of the logistic map for $\mu = 1.05$ - the initial point is 0.005, the target point is $t = 0.95$ and the number of steps is 1000. The achieved target point is plotted versus the iteration index, $n$.

**Figure 8**: Stepwise control of the Hénon map for parameter values $\mu = 1.1$ and $\alpha = 0.3$ - the initial point is $(0.5, 0.5)$, the target point is $(−1, −1)$ and the number of steps is 1000. The achieved target point is plotted versus iteration index, $n$. 

