Isometries of absolute order unit spaces

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Received: 13 June 2019 / Accepted: 12 December 2019 / Published online: 23 December 2019
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Abstract
We prove that a unital, bijective linear map between absolute order unit spaces is an isometry if and only if it is absolute value preserving. We deduce that, on (unital) JB-algebras, such maps are precisely Jordan isomorphisms. Next, we introduce the notions of absolutely matrix ordered spaces and absolute matrix order unit spaces and prove that a unital, bijective ∗-linear map between absolute matrix order unit spaces is a complete isometry if, and only if, it is completely absolute value preserving. We obtain that on (unital) C*-algebras such maps are precisely C*-algebra isomorphisms.

Keywords Absolutely ordered space · Absolute order unit space · Isometry · Absolute value preserving maps · Absolute matrix order unit space

Mathematics Subject Classification Primary 46B40; Secondary 46L05 · 46L30

1 Introduction

In [12], Kakutani proved that an abstract M-space is precisely a concrete C(K, R) space for a suitable compact and Hausdorff space K. In [6], Gelfand and Naimark proved that an abstract (unital) commutative C*-algebra is precisely a concrete C(K, C) space for a suitable compact and Hausdorff space K. Thus Gelfand-Naimark theorem for commutative C*-algebras, in the light of Kakutani theorem, yields that the self-adjoint part of a commutative C*-algebra is, in particular, a vector lattice.

The Amit Kumar was financially supported by the Senior Research Fellowship of the University Grants Commission of India.

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However, Kadison’s anti-lattice theorem suggest that the self-adjoint part of a general C*-algebra may not be a vector lattice [10].

Following Kadison’s functional representation of the self-adjoint part of a unital C*-algebra as the space of continuous affine functions on the state space S(A) of A, it becomes evident that the order structure of a C*-algebra is rich with many properties. The works of Kadison, Effros, Størmer and Pedersen, besides many others, highlight various aspects of order structure of a C*-algebra and encourages us to expect a ‘non-commutative vector lattice’ or a ‘near lattice’ structure in it. The monograph [19] (and references therein), for example, is a good source of information for this purpose.

Keeping this point of view, the first author also worked in this direction [13–15]. In [16], he introduced the notion of absolutely ordered spaces and that of absolute order unit spaces. The self-adjoint parts of unital C*-algebras and (unital) M-spaces are examples of absolute order unit spaces. It was shown that under an additional condition (see [15, Theorem 4.12]) an absolutely ordered space turns out to be a vector lattice. Now one can easily show that under the same condition, an absolute order unit space becomes an M-space. Therefore, an absolutely ordered space may be termed as a ‘non-commutative vector lattice’.

For an element a in a C*-algebra A, we define the ‘absolute value’ of a as |a| := (a*a)½ and for an element v in a vector lattice V, we define the ‘absolute value’ of v as |v| := v ∨ (−v). The absolute values defined in two different contexts have a connection. We recall that for a pair of positive elements a and b in A, we have ab = 0 if, and only if, |a − b| = a + b. Also, for a pair of positive elements u and v in V, we have u ∧ v = 0 if, and only if, |u − v| = u + v. Thus in both the cases, we can say that a ⊥ b if, and only if, |a − b| = a + b. In other words, the two kinds of orthogonality relate to the same kind of relation in terms of absolute value. The definition of an absolutely ordered space is influenced by some of the basic properties of the orthogonality which hold in the both kinds of the above-mentioned (ordered) spaces.

In [11], Kadison characterized bijective linear isometries between unital C*-algebras. Since then, many generalizations and extensions of this result have been studied. Surjective isometries of C*-algebras have been characterized as Jordan triple preserving maps. Further, these results have been extended to JB*-triples. In another direction, Jordan isomorphisms have been characterized in terms of absolute value preserving maps together with some or the other conditions. (See, for example, [5,7,17,18,20].) A matricial version of the results of this type were studied by Blecher et al. in [2,3].

In this paper, we study absolute value preserving maps between absolute order unit spaces. We prove that a unital, bijective linear map between absolute order unit spaces is an isometry if, and only if, it is absolute value preserving (Theorem 3.3). We deduce that on (unital) JB*-algebras such maps are precisely Jordan isomorphisms (Corollary 3.4). Besides this, we study some elementary properties of absolute value preserving maps. Next, we introduce the notions of absolutely matrix ordered spaces and absolute matrix order unit spaces in the context of matrix ordered spaces and present a matricial version of these results. We prove that a unital, bijective *-linear map between absolute matrix order unit spaces is a complete isometry if, and only if,
it is completely absolute value preserving (Theorem 4.6). From here, we prove that on (unital) C*-algebras such maps are precisely C*-algebra isomorphisms (Corollary 4.7). This result was proved in [2, Corollary 3.2]. (Also see, [3].) We give a simple, order-theoretic proof using a trick which is apparently new.

The purpose of this paper is to establish absolute matrix order unit spaces as an order theoretic prototype of unital C*-algebras. The results in this paper, we believe, are baby-steps in this direction [14-16] where complete absolute value preserving maps may be connoted as morphisms.

2 Absolute value preservers on absolutely ordered spaces

We begin by recalling some basic order theoretic notions. Let V be a real vector space. A non-empty subset V+ of V is called a cone if V+ is closed under vector addition as well as scalar multiplication with non-negative real numbers. In this case, (V, V+) is called a real ordered vector space. Also, then (V, ≤) is a partially ordered space with the partial order u ≤ v if v − u ∈ V+ in a unique way, in the sense that (i) u ≤ u for all u ∈ V, (ii) u ≤ w whenever u ≤ v and v ≤ w and, (iii) u + w ≤ v + w and ku ≤ kv whenever u ≤ v, w ∈ V and k is a positive real number. The cone V+ is said to be proper, if V+ ∩ −V+ = {0}. It is said to be generating, if V = V+ − V+. Recall that V+ is proper if and only if ≤ is anti-symmetric.

A positive element e ∈ V+ is said to be an order unit for V if for each v ∈ V, there is a positive real number k such that ke ± v ∈ V+. The cone V+ is said to be Archimedean, if for any v ∈ V with ku + v ∈ V+ for a fixed u ∈ V+ and all positive real numbers k, we have v ∈ V+.

Let W be a vector subspace of V. Then W is said to be an order ideal of (V, V+) if, for v ∈ V+, w ∈ W with v ≤ w, we have v ∈ W.

Let (V, V+) be a real ordered vector space with an order unit e such that V+ is proper and Archimedean. Then e determines a norm on V given by

\[ \|v\| := \inf \{k > 0 : ke ± v ∈ V+\} \]

in such a way that V+ is norm-closed and for each v ∈ V, we have \|v\|e ± v ∈ V+. In this case, we say that V is an order unit space and denote it by (V, e).

Now, we recall the notion of absolutely ordered spaces which was recently introduced by the first author as a possible non-commutative model for vector lattices.

**Definition 2.1** [16, Definition 3.4] Let (V, V+) be a real ordered vector space and let | · | : V → V+ be a mapping satisfying the following conditions:

(a) |v| = v if v ∈ V+;
(b) |v| ± v ∈ V+ for all v ∈ V;
(c) |k · v| = |k| · |v| for all v ∈ V and k ∈ \mathbb{R};
(d) If u, v and w ∈ V with |u − v| = u + v and 0 ≤ w ≤ v, then |u − w| = u + w;
(e) If u, v and w ∈ V with |u − v| = u + v and |u − w| = u + w, then |u − v| ± w | = u + |v ± w|.

Then (V, V+, | · |) is said to be an absolutely ordered space.
It was shown in [16] that a vector lattice and the self-adjoint part of a (unital) C*-algebra are examples of absolutely ordered spaces.

**Definition 2.2** Let \((V, V^+, |·|)\) be an absolutely ordered space. Let \(W\) be a vector subspace of \(V\) and put \(W^+ := \text{int } V^+\). Then \(W\) is said to be an **absolutely ordered subspace** of \((V, V^+, |·|)\) if \(|w| \in W^+\) for all \(w \in W\). A vector subspace \(W\) of \(V\) which is an order ideal of \((V, V^+, |·|)\) and an absolutely ordered subspace of \((V, V^+, |·|)\) is called an **absolute order ideal** of \((V, V^+, |·|)\).

**Remark 2.3** Let \((V, V^+, |·|)\) be an absolutely ordered space.

1. The cone \(V^+\) is proper and generating. In fact, if \(±v \in V^+\), then by (a) and (c), we get \(v = |v| = |−v| = −v\) so that \(v = 0\). Next, by (b), for any \(v \in V\), we have
   \[v = \frac{1}{2} ((|v| + v) − (|v| − v)) \in V^+ − V^+\].

2. Let \(u, v \in V\) be such that \(|u − v| = u + v\). Then \(u, v \in V^+\). For such a pair \(u, v \in V^+\), we shall say that \(u\) is **orthogonal** to \(v\) and denote it by \(u \perp v\).

3. We write, \(v^+ := \frac{1}{2}(|v| + v)\) and \(v^- := \frac{1}{2}(|v| − v)\). Then \(v^+ \perp v^−, v = v^+ − v^−\) and \(|v| = v^+ + v^−\). This decomposition is unique in the following sense: If \(v = v_1 − v_2\) with \(v_1 \perp v_2\), then \(v_1 = v^+\) and \(v_2 = v^−\). In other words, every element in \(V\) has a unique orthogonal decomposition in \(V^+\).

**Definition 2.4** Let \(V\) and \(W\) be absolutely ordered spaces. A linear map \(φ : V \to W\) is said to be an **absolute value preserving map** (\(|·|\)-preserving map, in short), if
\[φ(|v|) = |φ(v)|\] for all \(v \in V\).

**Remark 2.5** Let \(φ : V \to W\) be a bijective, linear and \(|·|\)-preserving map. Then \(φ^{-1}\) is also a bijective, linear and \(|·|\)-preserving map. To see this, let \(w \in W\). Since \(φ\) is bijective, there exists a unique \(v \in V\) such that \(φ(v) = w\). Now
\[φ(|φ^{-1}(w)|) = φ(|v|) = |φ(v)| = |w|\] so that \(|φ^{-1}(w)| = φ^{-1}(|w|)\). Hence \(φ^{-1}\) is \(|·|\)-preserving.

The next result is an elementary characterization of \(|·|\)-preserving maps.

**Proposition 2.6** Let \(V\) and \(W\) be absolutely ordered spaces and let \(φ : V \to W\) be a linear map. Then the following statements are equivalent:

1. \(φ\) is \(|·|\)-preserving;
2. \(φ ≥ 0\) and \(φ(v_1) \perp φ(v_2)\) for all \(v_1, v_2 \in V^+\) with \(v_1 \perp v_2\);
3. \(φ(v^+) = φ(v)^+\) for all \(v \in V\);
4. \(φ(v^-) = φ(v)^-\) for all \(v \in V\).
Proof (1) \(\implies\) (2): Let \(v \in V\) with \(v \geq 0\). Now, as \(\phi\) is an \(|\cdot|\)-preserving and by 2.1(a), we have \(\phi(v) = \phi(|v|) = |\phi(v)| \geq 0\), therefore \(\phi \geq 0\). Let \(v_1, v_2 \in V^+\). Put \(w = v_1 - v_2\). Then \(|w| = v_1 + v_2\). Since \(\phi\) is an additive \(|\cdot|\)-map (by (1)), we get \(\phi(v_1) + \phi(v_2) = \phi(|v|) = |\phi(v)| = |\phi(v_1) - \phi(v_2)|\). Thus \(\phi(v_1), \phi(v_2) \in W^+\) with \(\phi(v_1) \perp \phi(v_2)\).

(2) \(\implies\) (3): Let \(v \in V\). Then \(v^+ \perp v^-\) so that by (2), \(\phi(v^+) \perp \phi(v^-)\). As \(\phi(v) = \phi(v^+) - \phi(v^-)\), we get \(\phi(v)^+ = \phi(v^+), \phi(v)^- = \phi(v^-)\).

(3) \(\implies\) (4): If we use the fact, \(v^- = (v^-)^+\).

(4) \(\implies\) (1): Let \(v \in V\). Then \(|v| = v^+ + v^- = (v^-)^+ + v^-\). Thus by (4), we get

\[
\phi(|v|) = \phi(v^-)^+ + \phi(v^-) = \phi(v^+)^+ + \phi(v^-) = |\phi(v)|.
\]

\(\blacksquare\)

Theorem 2.7 Let \(V\) and \(W\) be absolutely ordered spaces and let \(\phi : V \rightarrow W\) be a linear \(|\cdot|\)-preserving map. Then

1. \(\ker(\phi)\) is an absolute order ideal of \(V\).
2. \(\phi(V)\) is an absolutely ordered subspace of \(W\). In particular, \(\phi(V)^+ = \phi(V^+)\).
3. For each \(v \in V\), we define \(|v + \ker(\phi)| = |v| + \ker(\phi)\). Then

\[
(V / \ker(\phi), (V / \ker(\phi))^+, |\cdot|)
\]

is also an absolutely ordered space, where

\[
(V / \ker(\phi))^+ := \{v + \ker(\phi) : v \in V^+\}.
\]

Proof (1) Let \(v \in \ker(\phi)\). Then \(\phi(v) = 0\) so that \(0 = |\phi(v)| = |\phi(|v|)|\). Thus \(|v| \in \ker(\phi)\) and consequently, \(\ker(\phi)\) is an absolutely ordered subspace of \(V\). Now, as \(\phi\) is positive, \(\ker(\phi)\) is an order ideal.

(2) Let \(w \in \phi(V)\), say \(w = \phi(v)\) for some \(v \in V\). Then \(\phi(v^+) = \phi(v)^+ = w^+\) and \(\phi(v^-) = \phi(v)^- = w^-\in \phi(V)\). Thus \(\phi(V)\) is an absolutely ordered subspace of \(W\). Next, if \(w \in \phi(V)^+\), then \(\phi(v^-) = w^- = 0\) so that \(w = \phi(v) = \phi(v^+)\). Thus \(\phi(V)^+ \subset \phi(V^+)\). Now, being \(|\cdot|\)-preserving, \(\phi \geq 0\) so that \(\phi(V^+) \subset \phi(V^+)\). Hence \(\phi(V^+) = \phi(V^+)\).

(3) By [1, Proposition II.1.1], we note that \((V / \ker(\phi))^+\) is a proper cone of \(V / \ker(\phi)\). We show that the absolute value is well defined in \(V / \ker(\phi)\). To see this, let \(u, v \in V\) such that \(u + \ker(\phi) = v + \ker(\phi)\). Then \(\phi(u) = \phi(v)\). Now, \(\phi\) is \(|\cdot|\)-preserving so that \(\phi(|u|) = |\phi(u)| = |\phi(v)| = |\phi(|v|)|\) and hence \(|u| + \ker(\phi) = |v| + \ker(\phi)\).

(a) Let \(v \in V\) with \(v + \ker(\phi) \in (V / \ker(\phi))^+\). There exists \(v_0 \in V^+\) such that \(v + \ker(\phi) = v_0 + \ker(\phi)\). Thus

\[
|v + \ker(\phi)| = |v_0 + \ker(\phi)|
\]

\[
= |v_0| + \ker(\phi)
\]

\[
= v_0 + \ker(\phi)
\]

\[
= v + \ker(\phi).
\]
(b) Let \( v \in V \). Then \(|v + \ker(\phi)| \pm (v + \ker(\phi)) = (|v| \pm v) + \ker(\phi) \in (V/\ker(\phi))^+\).

(c) Let \( k \in \mathbb{R} \). Then

\[
|k(v + \ker(\phi))| = |(kv + \ker(\phi))|
= |kv| + \ker(\phi)
= |k||v| + \ker(\phi)
= |k|(|v| + \ker(\phi))
= |k||v + \ker(\phi)|.
\]

(d) Let \( u, v, w \in V \) such that \( u + \ker(\phi), v + \ker(\phi), w + \ker(\phi) \in (V/\ker(\phi))^+ \) with \(|u - v| + \ker(\phi) = u + v + \ker(\phi)\) and \( w + \ker(\phi) \leq v + \ker(\phi)\). Then \(|\phi(u - v)| = \phi(|u - v|) = \phi(u + v)\) and \(0 \leq \phi(w) \leq \phi(v)\). Since \( \phi(V) \) is an absolutely ordered space, we may conclude that

\[\phi(|u - w|) = |\phi(u) - \phi(w)| = \phi(u) + \phi(w).\]

Thus \(|u - w| + \ker(\phi) = u + w + \ker(\phi)\).

(e) Let \(|u - v| + \ker(\phi) = u + v + \ker(\phi)\) and \(|u - w| + \ker(\phi) = u + w + \ker(\phi)\). Then \(|\phi(u) - \phi(v)| = \phi(u) + \phi(v)\) and \(|\phi(u) - \phi(w)| = \phi(u) + \phi(w)\). Since \( \phi(V) \) is an absolutely ordered space, we may conclude that \(|\phi(u) - |\phi(v) + \phi(w)|| = |\phi(u) + |\phi(v) + \phi(w)||\). Thus, it follows that \(|u - |v \pm w|| + \ker(\phi) = u + |v \pm w| + \ker(\phi)\).

Hence \((V/\ker(\phi), (V/\ker(\phi))^+, |\cdot|)\) is an absolutely ordered space. \(\square\)

**Corollary 2.8** Let \( V \) and \( W \) be absolutely ordered spaces and let \( \phi : V \to W \) be a linear \(|\cdot|\)-preserving map. Put \( \ker^+(\phi) := \{v \in V^+ : \phi(v) = 0\}\), then

1. \( \phi \) is injective if, and only if, \( \ker^+(\phi) = \{0\}\).
2. \( \phi \) is surjective if, and only if, \( \phi(V^+) = W^+\).
3. The quotient map \( \tilde{\phi} : V/\ker(\phi) \to \phi(V) \) is a bijective \(|\cdot|\)-preserving map.

**Proof** (1) By Theorem 2.7(1), \( \ker(\phi) \) is an absolutely ordered space. Thus we have \( \ker(\phi) = \ker^+(\phi) - \ker^-(\phi) \). Now, the proof of (1) is immediate.

(2) If \( \phi \) is surjective, it follows from Theorem 2.7(2) that \( \phi(V^+) = W^+\). Conversely, assume that \( \phi(V^+) = W^+\). If \( w \in W \), by assumption there exist \( v_1, v_2 \in V^+ \) such that \( \phi(v_1) = w^+, \phi(v_2) = w^-\). Put \( v = v_1 - v_2 \) so that \( \phi(v) = w \). Hence \( \phi \) is surjective.

(3) It is an immediate consequence of Theorem 2.7(3). \(\square\)

### 3 Absolute value preservers on absolute order unit spaces

We begin this section by recalling the notion of an absolute order unit space. First, we recall three types of orthogonality in an absolutely ordered space.
Theorem 4. If we combine this result with Theorem 3.3, we may deduce the following:

Definition 2.8(2),

**Theorem 3.3**

Let \( \phi \) be a unital, bijective linear map. Then \( |\cdot| \) is an isometry, we get that

\[
\|\phi(v)\| = \|\phi(v)\|
\]

for all \( v \in V \) so that \( \phi \) is an isometry.

Conversely, let \( \phi \) be an isometry. We show that \( \phi \) preserves \( |\cdot| \). Let \( v \in V^+ \) with \( \|v\| \leq 1 \). Then \( 0 \leq v \leq e_v \) so that \( 0 \leq e_v - v \leq e_v \). Thus \( \|e_v - v\| \leq 1 \). Since \( \phi \) is an isometry, we get that \( \|e_w - \phi(v)\| \leq 1 \) so that \( e_w - \phi(v) \leq e_w \). Then \( \phi(v) \in W^+ \) and hence \( \phi \geq 0 \). Now, as \( \phi^{-1} \) is also an isometry, we have \( \phi^{-1} \geq 0 \).

Let \( v_1, v_2 \in V^+ \) with \( v_1 \perp v_2 \). If \( v_1 = 0 \) or \( v_2 = 0 \), then \( \phi(v_1) \perp \phi(v_2) \). Now, assume that \( v_1 \neq 0, v_2 \neq 0 \). Then \( w_i = \phi(v_i) \in W^+ \setminus \{0\} \) for \( i = 1, 2 \). Let \( 0 \leq u_i \leq w_i, i = 1, 2 \). Then \( 0 \leq \phi^{-1}(u_i) \leq v_i, i = 1, 2 \). Since \( v_1 \perp v_2 \), we have \( v_1 \perp^a v_2 \) and consequently, \( \phi^{-1}(u_1) \perp^a \phi^{-1}(u_2) \). Thus, by [13, Theorem 3.3], we have

\[
1 = \|\phi^{-1}(u_1)\|^{-1} \phi^{-1}(u_1) + \|\phi^{-1}(u_2)\|^{-1} \phi^{-1}(u_2) = \|u_1\|^{-1}u_1 + \|u_2\|^{-1}u_2
\]

as \( \phi^{-1} \) is an isometry. Again, applying [13, Theorem 3.3], we get that \( u_1 \perp^a u_2 \) so that \( w_1 \perp^a w_2 \). Now, by the definition of an absolute order unit space, we get that \( w_1 \perp w_2 \). Hence, by Proposition 2.6, \( \phi \) is \( |\cdot| \)-preserving.

Maitland Wright and Youngson proved that any unital, surjective, linear isometry \( \phi : A \to B \) between unital JB-algebras \( A \) and \( B \) is a Jordan isomorphism [18, Theorem 4]. If we combine this result with Theorem 3.3, we may deduce the following:
Corollary 3.4 Let $A$ and $B$ be unital $JB$-algebras and let $\phi : A \to B$ be a bijective linear map. Then the following statements are equivalent:

1. $\phi$ is a unital isometry;
2. $\phi$ is a unital $\|\cdot\|$-preserving map;
3. $\phi$ is a Jordan isomorphism.

Proof Let $\phi$ be a Jordan isomorphism. Let $\phi(1_A) = p \in B$ and let $\phi^{-1}(1_B) = q \in A$. Then

$$1_B = \phi(q) = \phi(1_A o q) = \phi(1_A) o \phi(q) = p o 1_B = p$$

so that $\phi$ is unital. Also, $\phi$ is positive. In fact, if $a \in A^+$, then $a = (a^2)^{\frac{1}{2}}$ so that $\phi(a) = \phi(a^{\frac{1}{2}})^2 \in B^+$. Now, for any $x \in A$, we have

$$|\phi(x)|^2 = \phi(x)^2 = \phi(x^2) = \phi(|x|^2) = \phi(|x|)^2$$

so that $\phi(|x|) = |\phi(x)|$ for all $x \in A$. Thus (3) implies (2). Now, by Theorem 3.3, the proof is complete. \qed

Corollary 3.5 Let $(V, e_V)$ and $(W, e_W)$ be absolute order unit spaces and let $\phi : V \to W$ be a bijective linear map. Consider the three statements:

1. $\phi$ is unital;
2. $\phi$ is an isometry; and
3. $\phi$ is $\|\cdot\|$-preserving.

Then any two of these statements imply the third.

Proof By Theorem 3.3, we have that (1) and (2) imply (3) and that (1) and (3) imply (2). Now, assume that (2) and (3) hold. Then by Remark 2.5, we note that $\phi^{-1}$ is also $\|\cdot\|$-preserving. Thus $\phi$ and $\phi^{-1}$ are positive isometries. Put $\phi(e_V) = w_0$. Then $w_0 \in W^+$ with $\|w_0\| = 1$ so that $w_0 \leq e_W$. Thus $e_V = \phi^{-1}(w_0) \leq \phi^{-1}(e_W)$. Since $\phi^{-1}$ is an isometry, we get that $\phi^{-1}(e_W) \leq e_V$ so that $\phi^{-1}(e_W) = e_V$. Hence $\phi$ is unital. \qed

3.1 Absolute compatibility

Orthogonality in $C^*$-algebras or more generally, absolute $\infty$-orthogonality in absolute order unit space has a curious by-product. Let $(V, e)$ be an absolute order unit space and let $0 \leq u, v \leq e$. Then $u \perp \infty v$ if and only if $u + v \leq e$ and $|u - v| + |e - u - v| = e$ [16, Proposition 4.1]. We segregate the later part of this observation as a property. Let $u, v \in V^+$. We say that $u$ is absolutely compatible with $v$ (we write, $u \triangle v$) if $|u - v| + |e - u - v| = e$.

Absolute compatibility is an useful notion. It was applied to derive spectral decomposition in an absolute order unit space [16]. This property has been separately studied in the context of operator algebras [8,9]. Here, we study absolute compatibility preservers for absolute order unit spaces and their relation with absolute value preservers.
We also need to recall the notion of order projection of $V$ as given in [16, Definition 5.2]: Let $0 \leq p \leq e$. We say that $p$ is an order projection, if $p \perp e - p$. We write $\mathcal{OP}(V)$ for the set of all order projections in $V$. Recall that in a unital $C^*$-algebra, an order projection is precisely a projection [16, Theorem 5.3].

**Remark 3.6** Let $(V, e_V)$ and $(W, e_W)$ be absolute order unit spaces. Then a unital $|\cdot|$-preserving map $\phi : V \rightarrow W$ preserves order projections. To see this, let $p \in \mathcal{OP}(V)$. Then $p \perp e_V - p$. As $\phi(e_V) = e_W$ and $\phi(|v|) = |\phi(v)|$ for all $v \in V$, by Proposition 2.6, we get $\phi(p) \perp e_W - \phi(p)$. Thus $\phi(p) \in \mathcal{OP}(W)$.

Now, in the next Theorem 3.7(1), we generalize Remark 3.6.

**Theorem 3.7** Let $V$ and $W$ be absolute order unit spaces and let $\phi : V \rightarrow W$ be an $|\cdot|$-preserving map such that $\phi(e_V) \in \mathcal{OP}(W)$. Then

1. $\phi(\mathcal{OP}(V)) \subset \mathcal{OP}(W)$.
2. For $u, v \in V^+$ with $u \triangle v$, we have $\phi(u) \triangle \phi(v)$.

**Proof** (1) Put $\phi(e_V) = q \in \mathcal{OP}(W)$ and let $p \in \mathcal{OP}(V)$. Then

$$|e_V - 2p| = |(e_V - p) - p| = e_V.$$  

Since $\phi$ is $|\cdot|$-preserving, we have

$$q = \phi(e_V) = |\phi(e_V) - 2\phi(p)| = |q - 2\phi(p)|.$$  

Also, as $p \leq e_V$, we get that $\phi(p) \leq q$.

Now $(e_W - q) \perp q$ and $0 \leq \phi(p), q - \phi(p) \leq q$ so that, by 2.1(d), we get that $(e_W - q) \perp \phi(p)$ and $(e_W - q) \perp q - \phi(p)$. Using [16, Remark 3.5], we have

$$|e_W - 2\phi(p)| = |(e_W - q) + ((q - \phi(p)) - \phi(p))|$$  

$$= |(e_W - q)| + |q - \phi(p) - \phi(p)|$$  

$$= (e_W - q) + |q - 2\phi(p)|$$  

$$= e_W.$$  

Hence $\phi(p) \in \mathcal{OP}(W)$.

(2) Let $u, v \in V^+$ such that $u \triangle v$. Then $|u - v| + |e_V - u - v| = e_V$. By [16, Proposition 4.2], we conclude that $0 \leq u, v \leq e_V$. Since $\phi \geq 0$, we have $0 \leq \phi(u), \phi(v) \leq \phi(e_V)$. As $\phi$ is $|\cdot|$-preserving, we further get that

$$|\phi(u) - \phi(v)| + |\phi(e_V) - \phi(u) - \phi(v)| = \phi(e_V).$$

Now $\phi(e_V) \in \mathcal{OP}(W)$ so that $\phi(e_V) \perp e_W - \phi(e_V)$. Thus as $0 \leq \phi(u), \phi(v) \leq \phi(e_V)$ and $\phi(e_V) \perp e_W - \phi(e_V)$, by Definition 2.1(d), we have $\phi(u) \perp e_W - \phi(e_V)$.
\[\phi(e_V) \text{ and } \phi(v) \perp e_W - \phi(e_V).\] Now, by Definition 2.1(e), we get \[\phi(u) + \phi(v) \perp e_W - \phi(e_V).\] Thus applying [16, Remark 3.5], we obtain that

\[|e_W - \phi(u) - \phi(v)| = |(e_W - \phi(e_V)) + (\phi(e_V) - \phi(u) - \phi(v))| = (e_W - \phi(e_V)) + |\phi(e_V) - \phi(u) - \phi(v)|.
\]

Therefore, we get

\[|\phi(u) - \phi(v)| + |e_W - \phi(u) - \phi(v)| = |\phi(u) - \phi(v)| + (e_W - \phi(e_V)) + |\phi(e_V) - \phi(u) - \phi(v)| = e_W,
\]

so that \(\phi(u) \Delta \phi(v).\)

\[\square\]

**Theorem 3.8** Let \(V\) and \(W\) be absolute order unit spaces and let \(\phi : V \to W\) be a linear map such that \(\phi \geq 0\). If \(\phi\) is \(\Delta\)-preserving, then

1. \(\phi\) is a contraction.
2. \(\phi\) is \(|\cdot|\)-preserving.

**Proof** (1) First, we show that \(\phi\) is contractive on \(V^+\). To see this, let \(v \in V^+\). Without loss of generality, we may assume that \(\|v\| \leq 1\). Since \(v \bot 0\), by [16, Proposition 4.1], we have \(v \Delta 0\) so that \(\phi(v) \Delta 0\). Now, by [16, Proposition 4.2], we get that \(0 \leq \phi(v) \leq e_W\). Thus \(\phi\) is contractive on \(V^+\). Now let \(v \in V\) be an arbitrary element with \(\|v\| \leq 1\). Consider the orthogonal decomposition \(v = v^+ - v^-\). Then \(v^+ \perp v^-\) so that \(\max\{\|v^+\|, \|v^-\|\} = \|v\| \leq 1\). Also then \(-v^- \leq v \leq v^+\). Since \(\phi\) is positive, we have \(-\phi(v^-) \leq \phi(v) \leq \phi(v^+)\). Thus \(\|\phi(v)\| \leq \max\{\|\phi(v^-)\|, \|\phi(v^+)\|\} \leq 1\) as \(\phi\) is contractive on \(V^+\). Hence \(\phi\) is a contraction on \(V\).

(2) Let \(u, v \in V^+\) with \(u \perp v\) and assume that \(\|u\| \leq 1, \|v\| \leq 1\). Then by [16, Proposition 4.1], we have \(u + v \leq e_V\) and \(u \Delta v\). Since \(\phi \geq 0\) and \(\Delta\)-preserving, we get \(\phi(u) + \phi(v) \leq \phi(e_V)\) and \(\phi(u) \Delta \phi(v)\). Also, by (1), \(\phi(e_V) \leq e_W\). Thus again applying [16, Proposition 4.1], we may conclude that \(\phi(u) \perp \phi(v)\). Hence, by Proposition 2.6, \(\phi\) is \(|\cdot|\)-preserving.

\[\square\]

**Remark 3.9** Let \(V\) and \(W\) be absolute order unit spaces and let \(\phi : V \to W\) be a linear map. Then

1. If \(\phi \geq 0\) and \(\phi(e_V) \in \mathcal{OP}(W)\), then \(\phi\) is \(\Delta\)-preserving if, and only if, it is \(|\cdot|\)-preserving.
2. Let \(V\) and \(W\) be absolute order unit spaces and let \(\phi : V \to W\) be a unital surjective isometry. Then for all \(u\) and \(v \in V^+\), \(\phi(u) \Delta \phi(v)\) if, and only if, \(u \Delta v\).
4 A matricial version of absolute value preserving maps

A matrix ordered space is a $*$-vector space $V$ together with a sequence $\{M_n(V)^+\}$ with $M_n(V)^+ \subset M_n(V)_{sa} := \{v \in M_n(V) : v = v^*\}$ for each $n \in \mathbb{N}$ satisfying the following conditions:

(a) $(M_n(V)_{sa}, M_n(V)^+)$ is a real ordered vector space, for each $n \in \mathbb{N}$; and
(b) $\alpha^* v \alpha \in M_m(V)^+$ for all $v \in M_n(V)^+$, $\alpha \in M_{n,m}$ and $n, m \in \mathbb{N}$.

It is denoted by $(V, \{M_n(V)^+\})$. If, in addition, $e \in V^+$ is an order unit in $V_{sa}$ such that $V^+$ is proper and $M_n(V)^+$ is Archimedean for all $n \in \mathbb{N}$, then $V$ is called a matrix absolute order unit space and is denoted by $(V, \{M_n(V)^+\}, e)$ [4].

In this section, we introduce the matricial version of absolutely ordered spaces and that of absolute order unit spaces.

**Definition 4.1** Let $(V, \{M_n(V)^+\})$ be a matrix ordered space and assume that $|\cdot|_{m,n} : M_{m,n}(V) \to M_n(V)^+$ for $m, n \in \mathbb{N}$. Let us write $|\cdot|_{n,n} = |\cdot|_n$ for every $n \in \mathbb{N}$. Then $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ is called an absolutely matrix ordered space, if it satisfies the following conditions:

1. For all $n \in \mathbb{N}$, $(M_n(V)_{sa}, M_n(V)^+, |\cdot|_n)$ is an absolutely ordered space;
2. For $v \in M_{m,n}(V)$, $\alpha \in M_{r,m}$ and $\beta \in M_{n,s}$, we have

$$|\alpha v \beta|_{r,s} \leq \|\alpha\||v|_{m,n}\beta|_{n,s};$$

3. For $v \in M_{m,n}(V)$ and $w \in M_{r,s}(V)$, we have

$$|v \oplus w|_{m+r,n+s} = |v|_{m,n} \oplus |w|_{r,s}.$$

Here $v \oplus w := \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$.

**Proposition 4.2** Let $(V, \{M_n(V)^+\}, \{|\cdot|_{m,n}\})$ be an absolutely matrix ordered space. Then

1. If $\alpha \in M_{r,m}$ is an isometry i.e. $\alpha^* \alpha = I_m$, then $|\alpha v|_{r,n} = |v|_{m,n}$ for any $v \in M_{m,n}(V)$.
2. If $v \in M_{m,n}(V)$, then $\begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix}|_{m+n} = |v^*|_{m,n} \oplus |v|_{m,n}$.
3. $\begin{bmatrix} v^*|_{m,n} \\ v \end{bmatrix} \in M_{m+n}(V)^+$ for any $v \in M_{m,n}(V)$.
4. $|v|_{m,n} = \begin{bmatrix} v \\ 0 \end{bmatrix}|_{m+r,n}$ for any $v \in M_{m,n}(V)$ and $r \in \mathbb{N}$.
5. $|v|_{m,n} \oplus 0_s = \begin{bmatrix} v \\ 0 \end{bmatrix}|_{m,n+s}$ for any $v \in M_{m,n}(V)$ and $s \in \mathbb{N}$.

**Proof** (1) Let $\alpha \in M_{r,m}$ be an isometry. Then, using 4.1(2), we get that

$$|\alpha v|_{r,n} \leq \|\alpha\||v|_{m,n} = |\alpha^* \alpha v|_{m,n} \leq \|\alpha^*\||\alpha v|_{r,n} = |\alpha v|_{r,n}.$$

Thus $|\alpha v|_{r,n} = |v|_{m,n}$.
(2) Put \( \alpha = \begin{bmatrix} 0_{n,m} & I_n \\ I_m & 0_{m,n} \end{bmatrix} \in M_{n+m} \). Then \( \alpha \) is an isometry with
\[
\alpha \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} = \begin{bmatrix} v^* & 0_n \\ 0_m & v \end{bmatrix}.
\]

Now, by (1) and 4.1(3), it follows that
\[
\begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix}_{m+n} = \begin{bmatrix} v^* & 0_n \\ 0_m & v \end{bmatrix}_{m+n} = |v|_{n,m} \oplus |v|_{m,n}.
\]

(3) As \( \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \in M_{m+n}(V)_{sa} \), by 2.1(b), we have
\[
\begin{bmatrix} |v^*|_{n,m} & v \\ v^* & |v|_{m,n} \end{bmatrix} = \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix}_{m+n} + \begin{bmatrix} 0_m & v \\ v^* & 0_n \end{bmatrix} \in M_{m+n}(V)^+.
\]

(4) \( \alpha v = \begin{bmatrix} v \\ 0 \end{bmatrix} \in M_{m+r,n} \) for \( \alpha = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in M_{m+r.m} \). Since \( \alpha^* \alpha = I_m \), by (1), we conclude that \( |v|_{m,n} = \begin{bmatrix} v \\ 0 \end{bmatrix}_{m+r,n} \) if, \( v \in M_{m,n}(V) \) and \( r \in \mathbb{N} \).

(5) For \( \alpha = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \in M_{m+r.m} \), we get that \( \alpha \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \in M_{m+r,n+s} \). Since \( \alpha^* \alpha = I_m \), again using (1)
\[
\begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}_{m,n+s} = \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}_{m+r,n+s} = \begin{bmatrix} |v|_{m,n} \\ 0 \\ 0 \end{bmatrix},
\]
if \( v \in M_{m,n}(V) \) and \( r, s \in \mathbb{N} \).

\( \square \)

**Definition 4.3** Let \((V, \{M_n(V)^+\}, e)\) be a matrix order unit space such that

(a) \( \langle V, \{M_n(V)^+\}, \| \cdot \|_{m,n} \rangle \) is an absolutely matrix ordered space; and
(b) \( \perp = \perp^a_{\infty} \) on \( M_n(V)^+ \) for all \( n \in \mathbb{N} \).

Then \((V, \{M_n(V)^+\}, \| \cdot \|_{m,n}, e)\) is called an absolute matrix order unit space.

**Example 4.4** A unital \( C^* \)-algebra is an absolute matrix order unit space. Let \( A \) be a
unital \( C^* \)-algebra with unity \( 1_A \). Then, for each \( n \in \mathbb{N} \), \( M_n(A) \) is a \( C^* \)-algebra with unity \( I_A^n \) \( (\text{where } I_A^n := 1_A \oplus \cdots \oplus 1_A \in M_n(A) \). If \( M_n(A)^+ \) denotes the set of all the positive elements in \( M_n(A) \), then \( (A, \{M_n(A)^+\}_{n\in\mathbb{N}}, 1_A) \) is a matrix order unit space.

For \( m, n \in \mathbb{N} \), define \( \| \cdot \|_{m,n} : M_{m,n}(A) \rightarrow M_n(A)^+ \) given by \( \|a\|_{m,n} = (a^*a)^{\frac{1}{2}} \) for all \( a \in M_{m,n}(A) \). We show that \((A, \{M_n(A)^+\}, \| \cdot \|_{m,n}, 1_A)\) is an absolute matrix order unit space.
Theorem 4.6

Let \( (\alpha \alpha \beta)^{\#} (\alpha \alpha \beta) \)
\( = \beta^* a^* (\alpha a^*) \alpha a \beta \)
\( \leq \|\alpha\|^2 (\beta^* (a^* a) \beta) \)
\( = \|\alpha\|^2 (\beta^* |a|^2_{m,n} \beta) \)
\( = \|\alpha\|^2 ((|a|_{m,n} \beta)^* (|a|_{m,n} \beta)) \)
so that \( |\alpha \alpha \beta|^2_{r,s} \leq (\|\alpha\|||a|_{m,n} \beta|_{n,s})^2 \). Thus \( |\alpha \alpha \beta|_{r,s} \leq \|\alpha\|||a|_{m,n} \beta|_{n,s} \).

Next, let \( a \in M_{m,n}(A) \) and \( b \in M_{r,s}(A) \). Then

\[
|a \oplus b|^2_{m+r,n+s} = \left[ \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \left[ \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right] \\
= \left[ \begin{array}{cc} a^* a & 0 \\ 0 & b^* b \end{array} \right] \\
= \left[ \begin{array}{cc} |a|^2_{m,n} & 0 \\ 0 & |b|^2_{r,s} \end{array} \right] \\
= \left[ \begin{array}{cc} |a|_{m,n} & 0 \\ 0 & |b|_{r,s} \end{array} \right]^2.
\]

Thus \( |a \oplus b|_{m+r,n+s} = |a|_{m,n} \oplus |b|_{r,s} \).

Definition 4.5

Let \( (V, e_V) \) and \( (W, e_W) \) be absolute matrix order unit spaces and let \( \phi : V \to W \) be a *-linear map. We say that \( \phi \) is completely \( |\cdot|\)-preserving if, \( \phi_n : M_n(V) \to M_n(W) \) is an \( |\cdot|\)-preserving map for each \( n \in \mathbb{N} \).

Now we present the matricial version of Theorem 3.3.

Theorem 4.6

Let \( (V, e_V) \) and \( (W, e_W) \) be absolute matrix order unit spaces and let \( \phi : V \to W \) be a unital *-linear surjective isomorphism. Then \( \phi \) is a complete isometry if, and only if, it is completely \( |\cdot|\)-preserving.

Proof

First, let \( \phi \) be a complete isometry. Fix \( n \in \mathbb{N} \). Then \( \phi_n : M_n(V)_{sa} \to M_n(W)_{sa} \) is a surjective isometry. Since \( (M_n(V)_{sa}, M_n(V)^+, |\cdot|_n, e_V^n) \) is an absolute order unit space and \( \phi_n : M_n(V)_{sa} \to M_n(W)_{sa} \) is a unital, bijective linear isometry, thus by Theorem 3.3, we get that \( \phi_n(|v|_n) = |\phi_n(v)|_n \) for all \( v \in M_n(V)_{sa} \). Let \( v \in M_n(V) \). Then
\[
\begin{bmatrix} 0 & v^* \\ v & 0 \end{bmatrix} \in M_{2n}(V)_{sa}
\]
so that
\[
\phi_{2n} \left( \begin{bmatrix} 0 & v^* \\ v & 0 \end{bmatrix} \right)_{2n} = \phi_{2n} \left( \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right)_{2n}.
\]
Thus, by Proposition 4.2(2), we get
\[
\begin{bmatrix} \phi_n(|v^*|_n) & 0 \\ 0 & \phi_n(|v|_n) \end{bmatrix} = \begin{bmatrix} |\phi_n(v)^*|_n & 0 \\ 0 & |\phi_n(v)|_n \end{bmatrix}.
\]
Therefore, \( \phi_n |v|_n = |\phi_n(v)|_n \) for each \( n \in \mathbb{N} \).

Conversely, assume that \( \phi \) is a completely \( | \cdot | \)-preserving map. Fix \( n \in \mathbb{N} \). Then \( \phi_n : M_n(V)_{sa} \to M_n(W)_{sa} \) is an \( | \cdot | \)-preserving map. Since \( \phi_n : M_n(V)_{sa} \to M_n(W)_{sa} \) is a unital bijective \( | \cdot | \)-preserving map, again applying Theorem 3.3, we have that \( \phi_n : M_n(V)_{sa} \to M_n(W)_{sa} \) is an isometry. Let \( v \in M_n(V) \), then
\[
\begin{bmatrix}
0 & v \\
0 & 0
\end{bmatrix} \in M_{2n}(V)_{sa}.
\]
Since \( \phi_{2n} \) is an isometry on \( M_{2n}(V)_{sa} \), we get that
\[
\left\| v \right\|_n = \left\| \begin{bmatrix}
0 & v \\
0 & 0
\end{bmatrix} \right\|_{2n} = \left\| \phi_{2n} \left( \begin{bmatrix}
0 & v \\
0 & 0
\end{bmatrix} \right) \right\|_{2n} = \left\| \begin{bmatrix}
0 & \phi_n(v) \\
\phi_n(v)^* & 0
\end{bmatrix} \right\|_{2n} = \left\| \phi_n(v) \right\|_n.
\]
Hence \( \phi : V \to W \) is a complete isometry.

\[\Box\]

**Corollary 4.7** Let \( A \) and \( B \) be any two unital \( C^* \)-algebras and let \( \phi : A \to B \) be a \( * \)-linear bijective map. Then the following facts are equivalent:

1. \( \phi \) is a unital complete isometry;
2. \( \phi \) is a unital completely \( | \cdot | \)-preserving map;
3. \( \phi \) is a \( C^* \)-algebra isomorphism.

**Proof** Following Theorem 4.6, it suffices to show that (2) (or equivalently (1)) implies (3). Let \( \phi \) be a unital completely \( | \cdot | \)-preserving map. Then \( \phi_n : M_n(A)_{sa} \to M_n(B)_{sa} \) is a unital \( | \cdot | \)-preserving map for each \( n \in \mathbb{N} \). Thus, by Corollary 3.4, \( \phi_n : M_n(A)_{sa} \to M_n(B)_{sa} \) is a Jordan isomorphism for each \( n \in \mathbb{N} \). In particular, \( \phi_3(a^2) = \phi_3(a)^2 \) for any \( a \in M_3(A)_{sa} \). Let \( x, y \in A \) and consider \( a = \begin{bmatrix}
0 & x & 0 \\
x^* & 0 & y \\
0 & y^* & 0
\end{bmatrix} \in M_3(A)_{sa} \). Then
\[
\phi_3(a^2) = \phi_3(a)^2 \text{ yields that } \phi(xy) = \phi(x)\phi(y).
\]
Thus \( \phi \) is a \( C^* \)-algebra isomorphism.

\[\Box\]

**Remark 4.8** It follows, from Corollary 4.7, that a unital surjective \( * \)-linear map between unital \( C^* \)-algebras is complete isometry if, it is a 3-isometry.

**Acknowledgements** The authors are grateful to the referee(s) for their valuable suggestions.

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