Homogeneous Finsler spaces and the flag-wise positively curved condition*

Ming Xu¹, Shaoqiang Deng²†

¹College of Mathematics
Tianjin Normal University
Tianjin 300387, P. R. China
Email: mgmgmgxu@163.com.

²School of Mathematical Sciences and LPMC
Nankai University
Tianjin 300071, P. R. China
E-mail: dengsq@nankai.edu.cn

Abstract

In this paper, we introduce the flag-wise positively curved condition for Finsler spaces (the (FP) Condition), which means that in each tangent plane, we can find a flag pole in this plane such that the corresponding flag has positive flag curvature. Applying the Killing navigation technique, we find a list of compact coset spaces admitting non-negatively curved homogeneous Finsler metrics satisfying the (FP) Condition. Using a crucial technique we developed previously, we prove that most of these coset spaces cannot be endowed with positively curved homogeneous Finsler metrics. We also prove that any Lie group whose Lie algebra is a rank 2 non-Abelian compact Lie algebra admits a left invariant Finsler metric satisfying the (FP) condition. As by-products, we find the first example of non-compact coset space $S^3 \times \mathbb{R}$ which admits homogeneous flag-wise positively curved Finsler metrics. Moreover, we find some non-negatively curved Finsler metrics on $S^2 \times S^3$ and $S^6 \times S^7$ which satisfy the (FP) condition, as well as some flag-wise positively curved Finsler metrics on $S^3 \times S^3$, shedding some light on the long standing general Hopf conjecture.

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Key words: Finsler metric; homogeneous Finsler space; flag curvature; flag-wise positively curved condition.

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†Corresponding author
1 Introduction

Many curvature concepts in Riemannian geometry can be naturally generalized to Finsler geometry. For example, flag curvature is the generalization of sectional curvature, which measures how the space curves along a tangent plane. The study of compact Finsler spaces of positive flag curvature (we call these spaces positively curved for simplicity) and the classification of positively curved homogeneous Finsler spaces are important problems in Finsler geometry. Recently we have made some big progress in the study of these problems; see [15], [16] and [17].

It is well known that the flag curvature $K^F(x, y, P)$ depends not only on the tangent plane $P \subset T_xM$, but also on the nonzero base vector $y \in P$ (the flag pole). This implies that flag curvature is much more localized than sectional curvature in Riemannian geometry. This feather leaves us more options in studying flag curvature. For example, concerning the positively curved property mentioned above, we can define the following flag-wise positively curved condition, or (FP) condition for simplicity.

**Condition 1.1** We say that a Finsler space $(M, F)$ satisfies the (FP) condition if for any $x \in M$ and tangent plane $P \subset T_xM$, there exists a nonzero vector $y \in T_xM$ such that the flag curvature $K^F(x, y, P) > 0$.

The flag-wise positively curved condition is equivalent to the positively curved condition in Riemannian geometry, but they seem essentially different in Finsler geometry. Nevertheless, up to now no example has been constructed to give an interpretation of this phenomenon.

The first purpose of this work is to give some examples of non-negatively curved homogeneous Finsler spaces satisfying the (FP) condition. We first prove the following theorem.

**Theorem 1.2** The compact coset spaces

$$SU(p + q)/SU(p)SU(q), \ SO(n)/SO(n - 2), \ Sp(n)/SU(n), $$
$$SO(2n)/SU(n), \ E_6/\text{SO}(10) \text{ and } E_7/E_6$$

admit non-negatively curved homogeneous Randers metrics satisfying the (FP) condition.

The coset spaces in (1.1) can be viewed as canonical circle bundles over compact irreducible Hermitian symmetric spaces. In Riemannian geometry they are weakly symmetric spaces [5], while in Finsler geometry they may not be. The metrics indicated by the theorem are constructed from Riemannian normal homogeneous metrics and invariant Killing vector fields through the navigation process. The method is originated from [9], where Z. Hu and S. Deng classified homogeneous Randers spaces with positive flag curvature and vanishing S-curvature.

It is not hard to observe that, except for a few homogeneous spheres, the coset spaces in (1.1) do not admit positively curved homogeneous Riemannian metrics [11]. In fact, we have also proven that they do not admit positively curved reversible homogeneous Finsler metrics [19]. The second purpose of this work is to prove that the
same statement holds for some spaces in (1.1) without the reversibility condition. This will be based on the following crucial criterion for the existence of positively curved homogeneous metrics on an odd dimensional compact coset space.

**Theorem 1.3** Let \((G/H, F)\) be an odd dimensional positively curved homogeneous Finsler space such that \(\mathfrak{g} = \text{Lie}(G)\) is a compact Lie algebra with a bi-invariant inner product \(\langle \cdot, \cdot \rangle_{\text{bi}}\), and a bi-invariant orthogonal decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\). Let \(t\) be a fundamental Cartan subalgebra of \(\mathfrak{g}\), i.e., \(t \cap \mathfrak{h}\) is a Cartan subalgebra of \(\mathfrak{h}\). Then there do not exist a pair of linearly independent roots \(\alpha\) and \(\beta\) of \(\mathfrak{g}\) satisfying the following conditions:

1. Neither \(\alpha\) nor \(\beta\) is a root of \(\mathfrak{h}\);
2. \(\pm \alpha\) are the only roots of \(\mathfrak{g}\) contained in \(\mathbb{R}\alpha + t \cap \mathfrak{m}\);
3. \(\beta\) is the only root of \(\mathfrak{g}\) contained in \(\beta + \mathbb{R}\alpha + t \cap \mathfrak{g}\).

Applying Theorem 1.3 to the coset spaces in (1.1), we get the following corollary.

**Corollary 1.4** None of the coset spaces

\[
\begin{align*}
\text{SU}(p+q)/\text{SU}(p)\text{SU}(q), & \quad \text{with } p > q \geq 2 \text{ or } p = q > 3, \\
\text{Sp}(n)/\text{SU}(n), & \quad \text{with } n > 4, \\
\text{SO}(2n)/\text{SU}(n), & \quad \text{with } n = 5 \text{ or } n > 6, \\
\text{E}_6/\text{SO}(10) \text{ and } \text{E}_7/\text{E}_6 & \quad \text{(1.2)}
\end{align*}
\]

admits a positively curved homogeneous Finsler metrics.

Theorem 1.3 can also be applied to other odd dimensional compact coset spaces, and this excludes many candidates from the list of positively curved homogeneous Finsler spaces. This has been partially performed in [16] and [19], where we assume that the homogeneous metric is reversible. It is a very surprising fact since in a rather long period we believed that the homogeneous flag curvature formula (see Theorem 2.2 below) of L. Huang can only be effectively applied to the homogeneous positive flag curvature problem (i.e. to conduct a meaningful calculation for the flag curvature \(K^F(o, u, u \wedge v)\) with \(o = eH \in G/H\) and \(u, v \in T_o(G/H) = \mathfrak{m} \subset \mathfrak{g}\), when a pair of tangent vectors \(u, v\) satisfy the following condition:

\[
[u, v] = 0 \text{ and } \langle u, [u, \mathfrak{m}] \rangle^F_u = 0. \quad (1.3)
\]

This is in fact the reason why the reversibility assumption in [16] and [19] is needed, since under these conditions, the homogeneous flag curvature formula can be reduced to be a much simpler form (see [17]). But in this paper, we will show that the formula can even be applied in the cases where none of the conditions in (1.3) is satisfied.

To summarize, we have found an infinite family of compact coset spaces which can not be homogeneously positively curved, but admit non-negatively and flag-wise positively curved homogeneous Finsler metrics. We have noticed that using a similar argument one can find many more homogeneous examples which are circle bundles over
the compact Hermitian symmetric spaces, among which SU(3) × SU(3)/S^1 is the most simple and typical example.

The metrics we have constructed on these examples are in fact normal homogeneous. This raises the problem to classify the normal homogeneous Finsler spaces which satisfy the (FP) condition. We will take up this problem in the future. Here we will only pose some problems related to the above examples, and conjecture that the answer to these problem will be positive.

Concerning the (FP) condition alone, without the non-negatively curved condition, we prove

**Theorem 1.5** Let G be a Lie group whose Lie algebra g is compact and non-Abelian with rk g = 2. Then G admits left invariant Finsler metrics satisfying the (FP) condition.

According to [8], the group in Theorem 1.5 does not admit any left invariant positively curved Finsler metrics. Note that the Lie groups in this theorem can even be non-compact, e.g., SU(2) × R = S^3 × R, which does not admit any positively curved homogeneous Finsler metric by the Bonnet-Myers Theorem in Finsler geometry [3].

There are some interesting by-products of the main results of this paper. Applying Theorem 1.2 to SO(4)/SO(2) (which is diffeomorphic to S^2 × S^3) and SO(8)/SO(6) (which is diffeomorphic to S^6 × S^7), and applying Theorem 1.5 to SU(2) × SU(2) (which is diffeomorphic to S^3 × S^3), we obtain

**Corollary 1.6** (1) Both S^2 × S^3 and S^6 × S^7 admit non-negatively curved Finsler metrics satisfying the (FP) condition.

(2) S^3 × S^3 admits Finsler metrics satisfying the (FP) condition.

Note that the existence of a positively curved Riemannian metric on a product manifold, which is a part of the general Hopf conjecture, is a very hard long standing open problem. The interest of the above corollary lies in the fact that at least there exist non-negatively curved Finsler metrics satisfying the (FP) condition on certain product manifolds, and it sheds some light on such an involved and significant problem.

In Section 2, we review some basic concepts in Finsler geometry which will be needed for this work. We also introduce the Killing navigation process and a key flag curvature formula as well as the homogeneous flag curvature formulas. In Section 3, we use the Killing navigation method to construct flag-wise positively and non-negatively curved homogeneous Randers metrics, and prove Theorem 1.2. In Section 4, we prove Theorem 1.5 and raise more questions concerning the (FP) condition. In Section 5, we prove Theorem 1.3 and Corollary 1.4.

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2 Preliminaries

2.1 Finsler metric and Minkowski norm

A Finsler metric on a smooth manifold $M$ is a continuous function $F : TM \to [0, +\infty)$ satisfying the following conditions:

1. $F$ is a smooth positive function on the slit tangent bundle $TM \setminus 0$;
2. $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_x M$, and $\lambda \geq 0$;
3. On any standard local coordinates $x = (x^i)$ and $y = y^i \partial_{x^i}$ for $TM$, the Hessian matrix

$$
(g^{ij}(x, y)) = \frac{1}{2}[F^2(x, y)]_{y^i y^j}
$$

is positive definite for any nonzero $y \in T_x M$, i.e., it defines an inner product (sometimes denoted as $g^F_y$)

$$
\langle u, v \rangle^F_y = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(y + su + tv) |_{s=t=0}, \ u, v \in T_x M,
$$
on $T_x M$.

We call $(M, F)$ a Finsler space or a Finsler manifold. The restriction of the Finsler metric to a tangent space is called a Minkowski norm. Minkowski norm can also be defined on any real vector space by similar conditions as (1)-(3), see [3] and [6]. A Finsler metric (or a Minkowski norm) $F$ is called reversible if $F(x, y) = F(x, -y)$ for all $x \in M$ and $y \in T_x M$ (or $F(y) = F(-y)$ for all vector $y$, respectively).

Riemannian metrics are the special Finsler metrics such that for any standard local coordinates the Hessian matrix $(g_{ij}(x, y))$ is independent of $y$. In this case, $g_{ij}(x, y)dx^i dx^j$ is a well defined global section of $\text{Sym}^2(T^*M)$, which is often referred to as the Riemannian metric. The most important and simple non-Riemannian Finsler metrics are Randers metrics [13], which are of the form $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric, and $\beta$ is a 1-form. The condition for (the Hessian of) $F$ to be positive definite is that the $\alpha$-norm of $\beta$ is smaller than 1 at each point, see [3]. $(\alpha, \beta)$-metrics are generalizations of Randers metrics. They are of the form $F = \alpha \phi(\beta/\alpha)$, where $\alpha$ and $\beta$ are the same as in Randers metrics, and $\phi$ is a smooth function. The condition for $F$ to be positive definite can be found in [6]. Recently, there are many research works on $(\alpha, \beta)$-spaces, see for example [2].

The differences between Riemannian metrics and non-Riemannian Finsler metrics can be detected by the Cartan tensor $C^F_y(u, v, w)$, where $y$, $u$, $v$ and $w$ are tangent vectors in $T_x M$, and $y \neq 0$. The Cartan tensor is defined by

$$
C^F_y(u, v, w) = \frac{1}{4} \frac{\partial^3}{\partial r \partial s \partial t} F^2(x, y + ru + sv + tw) |_{r=s=t=0},
$$
or equivalently,

$$
C^F_y(u, v, w) = \frac{1}{2} \frac{d}{dt} \langle u, v \rangle^F_y |_{t=0}.
$$

A Finsler metric is Riemannian if and only if its Cartan tensor vanishes everywhere. The Cartan tensor can also be defined for a Minkowski norm which can also be used to determine whether the norm is induced by an inner product.
2.2 Geodesic spray and flag curvature

On a Finsler space \((M, F)\), the geodesic spray is a globally defined vector field \(G(x, y)\) which can be expressed as \(G = y^i \partial_{x^i} - 2G^i_\ell \partial_{y^\ell}\) on a standard local coordinate system, where \(G^i_\ell = \frac{1}{2}g^{i\ell}([F^2]_{x^i}y^\ell - [F^2]_{y^\ell})\). A curve \(c(t)\) on \(M\) is called a geodesic, if \((c(t), \dot{c}(t))\) is an integration curve of \(G\), i.e., if on any standard local coordinate, we have \(\ddot{c}(t) + 2G^i_\ell(c(t), \dot{c}(t)) = 0\). It is well known that the speed function \(F(c(t), \dot{c}(t))\) of a geodesic is a constant function.

The geodesic spray are very useful for the presentation of curvatures in Finsler geometry. For example, on a standard local coordinate system, we can write the Riemann curvature formula for Killing navigation is crucial in this work.

\[ R^i_k(y) = 2\partial_{x^i}G^i - y^i\partial^2_{x^j}y^\ell G^\ell + 2G^i_\ell \partial_{y^\ell} G^j - \partial_{y^i}G^i_\ell \partial_{y^\ell}G^j. \]

Flag curvature is a natural generalization of sectional curvature in Riemannian geometry. Now let us recall the definition. Given a nonzero vector \(y \in T_x M\) (the flag pole), and a tangent plane \(P \subset T_x M\) spanned by \(y\) and \(w\), the flag curvature is defined as

\[ K^F(x, y, P) = K^F(x, y, y \wedge w) = \frac{\langle R^F_y, w \rangle_y F}{\langle w, w \rangle_y F(y, y) F - \langle (y, w) F \rangle^2}. \] (2.4)

Notice that the flag curvature depends only on \(y\) and \(P\) rather than \(w\). When \(F\) is a Riemannian metric, it is just the sectional curvature and it is irrelevant to the choice of the flag pole.

2.3 Navigation process and the flag curvature formula for Killing navigation

The navigation process is an important technique in studying Randers spaces and flag curvature [4]. Let \(V\) be a vector field on the Finsler space \((M, F)\) with \(F(V(x)) < 1\), for any \(x \in M\). Given \(y \in T_x M\), denote \(\tilde{y} = y + F(x, y)V(x)\). Then \(\tilde{F}(x, \tilde{y}) = F(x, y)\) defines a new Finsler metric on \(M\). We call it the metric defined by the navigation datum \((F, V)\). When \(V\) is a Killing vector field of \((M, F)\), i.e., \(L_V F = 0\), we call this process a Killing navigation, and \((F, V)\) a Killing navigation datum. The following flag curvature formula for Killing navigation is crucial in this work.

\textbf{Theorem 2.1} Let \(\tilde{F}\) be the metric defined by the Killing navigation datum \((F, V)\) on the smooth manifold \(M\) with \(\text{dim} M > 1\). Then for any \(x \in M\), and any nonzero vectors \(w\) and \(y \in T_x M\) such that \(\langle w, y \rangle \tilde{F} = 0\), we have \(K^F(x, y, y \wedge w) = K^F(x, \tilde{y}, \tilde{y} \wedge w)\).

The proof can be found in [11] or [12], where some more general situations are also considered.

2.4 Homogeneous flag curvature formulas

Using local coordinates to study the flag curvature in homogenous Finsler geometry is not very convenient. In [10], L. Huang provided a homogeneous flag curvature which is expressed by some algebraic data of the homogeneous Finsler space.
To introduce his formula, we first define some notations.

Let \((G/H, F)\) be a homogeneous Finsler space, where \(H\) is the compact isotropy subgroup at \(o = eH\), \(\text{Lie}(G) = \mathfrak{g}\) and \(\text{Lie}(H) = \mathfrak{h}\). Then we have an \(\text{Ad}(H)\)-invariant decomposition \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\). Denote the projection to the \(\mathfrak{h}\)-factor (or \(\mathfrak{m}\)-factor) in this decomposition as \(\text{pr}_\mathfrak{h}\) (or \(\text{pr}_\mathfrak{m}\)), and define \([\cdot, \cdot]_\mathfrak{h} = \text{pr}_\mathfrak{h} \circ [\cdot, \cdot]\) and \([\cdot, \cdot]_\mathfrak{m} = \text{pr}_\mathfrak{m} \circ [\cdot, \cdot]\), respectively. Notice that \(\mathfrak{m}\) can be canonically identified with the tangent space \(T_o(G/H)\).

For any \(u \in \mathfrak{m}\setminus \{0\}\), the spray vector field \(\eta(u)\) is defined by
\[
\langle \eta(u), w \rangle^F_u = \langle u, [w, u]_\mathfrak{m} \rangle^F_u, \quad \forall w \in \mathfrak{m},
\]
and the connection operator \(\mathcal{N}(u, \cdot)\) is a linear operator on \(\mathfrak{m}\) determined by
\[
2\mathcal{N}(u, w_1), w_2)_u^F = \langle [w_2, w_1]_\mathfrak{m}, u \rangle^F_u + \langle [w_2, u]_\mathfrak{m}, w_1 \rangle^F_u + \langle [w_1, u]_\mathfrak{m}, w_2 \rangle^F_u
- 2\mathcal{C}_u^F(w_1, w_2, \eta(u)), \quad \forall w_1, w_2 \in \mathfrak{m}.
\]

L. Huang has proven the following theorem.

**Theorem 2.2** Let \((G/H, F)\) be a homogeneous Finsler space, and \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\) an \(\text{Ad}(H)\)-invariant decomposition, where \(\mathfrak{g} = \text{Lie}(G)\), \(\mathfrak{h} = \text{Lie}(H)\), and \(\mathfrak{m} = T_o(G/H)\). Then for any nonzero vector \(u \in T_o(G/H)\), the Riemann curvature \(R_u^F : T_o(G/H) \to T_o(G/H)\) satisfies
\[
\langle R_u^F v, w \rangle^F_u = \langle [v, [u]_\mathfrak{h}, v]_\mathfrak{m}, u \rangle^F_u + \langle \bar{R}(u) w, w \rangle^F_u, \quad \forall w \in \mathfrak{m},
\]
where the linear operator \(\bar{R}(u) : \mathfrak{m} \to \mathfrak{m}\) is given by
\[
\bar{R}(u)v = D_{\eta(u)} \mathcal{N}(u, v) - \mathcal{N}(u, \mathcal{N}(u, v)) + \mathcal{N}(u, [u, v]_\mathfrak{m}) - [u, \mathcal{N}(u, v)]_\mathfrak{m},
\]
and \(D_{\eta(u)} \mathcal{N}(u, v)\) is the derivative of \(\mathcal{N}(\cdot, v)\) at \(u \in \mathfrak{m}\setminus \{0\}\) in the direction of \(\eta(u)\).

This provides an explicit presentation for the nominator in (2.4), which is the only crucial term for studying the flag curvature. So we call (2.5) the homogeneous flag curvature formula.

In particular, when \(\eta(u) = 0\) (i.e. \(\langle u, [\mathfrak{m}, u]_\mathfrak{m} \rangle_u^F = 0\)), and \(\text{span}\{u, v\}\) is a two dimensional commutative subalgebra of \(\mathfrak{g}\), we get a very simple and useful homogeneous flag curvature.

**Theorem 2.3** Let \((G/H, F)\) be a connected homogeneous Finsler space, and \(\mathfrak{g} = \mathfrak{h} + \mathfrak{m}\) be an \(\text{Ad}(H)\)-invariant decomposition for \(G/H\). Then for any linearly independent commutative pair \(u\) and \(v\) in \(\mathfrak{m}\) satisfying \(\langle [u, \mathfrak{m}], u \rangle^F_u = 0\), we have
\[
K^F(o, u, u \wedge v) = \frac{\langle U(u, v), U(u, v) \rangle^F_u}{\langle u, u \rangle^F_u \langle v, v \rangle^F_u - \langle [u, v]_\mathfrak{m}, u \rangle^F_u [\langle [u, v]_\mathfrak{m}, u \rangle^F_u]^2},
\]
where \(U\) is the bilinear map from \(\mathfrak{m} \times \mathfrak{m}\) to \(\mathfrak{m}\) defined by
\[
\langle U(u, v), w \rangle^F_u = \frac{1}{2} ([[w, u]_\mathfrak{m}, v]_\mathfrak{m}^F_u + [[w, v]_\mathfrak{m}, u]_\mathfrak{m}^F_u), \quad \forall w \in \mathfrak{m}.
\]

Theorem 2.3 can be proven either by using Theorem 2.2 or directly using the Finslerian submersion technique. See [17] for the details.
3 Non-negatively curved homogeneous Finsler spaces satisfying the (FP) Condition

In this section, we will construct some examples of compact non-negatively curved homogeneous Finsler spaces satisfying the (FP) condition. Our main tool is the following lemma.

Lemma 3.1 Let $G$ be a compact Lie group, $H$ a closed connected subgroup, Lie($G$) = $\mathfrak{g}$ and Lie($H$) = $\mathfrak{h}$. Fix a bi-invariant inner product on $\mathfrak{g}$ = Lie($G$), denoted as $\langle \cdot, \cdot \rangle_{\text{bi}}$, and an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume that there exists a vector $v \in \mathfrak{m}$ satisfying the following conditions:

1. There exists a one-dimensional subspace $\mathfrak{m}_0$ in $\mathfrak{m}$ such that the action of Ad($H$) on $\mathfrak{m}_0$ is trivial;
2. Any 2-dimensional commutative subalgebra $t' \subset \mathfrak{m}$ satisfies $\langle \mathfrak{m}_0, t' \rangle_{\text{bi}} = 0$.

Then $G/H$ admits non-negatively curved homogeneous Randers metrics satisfying the (FP) condition.

Proof. First we endow $G/H$ with a Riemannian normal homogeneous metric $F$ defined by a bi-invariant inner product $\langle \cdot, \cdot \rangle_{\text{bi}}$ on $\mathfrak{m}$. By (1) of the lemma, we can find a vector $v$ satisfying $[v, \mathfrak{h}] = 0$ and $\langle v, v \rangle_{\text{bi}} < 1$. The left invariant vector field $v \in \mathfrak{m} \subset \mathfrak{g}$ induced on $G$ also defines a $G$-invariant vector field $V$ on $G/H$. Furthermore, since for any $u \in \mathfrak{m}$, we have

$$\langle [v, u]_{\mathfrak{m}}, u \rangle_{\text{bi}} = \langle [v, u], u \rangle_{\text{bi}} = 0,$$

and

$$\langle [v, u]_{\mathfrak{m}}, v \rangle_{\text{bi}} = \langle [v, u], v \rangle_{\text{bi}} = 0,$$

$V$ is a Killing vector field of $(G/H, F)$. Therefore $(F, V)$ is a Killing navigation datum. Thus it defines a Finsler metric $\tilde{F}$ on $G/H$ by the navigation process. Since both $F$ and $V$ are $G$-invariant, so is $\tilde{F}$.

Note that the Riemannian normal homogeneous space $(G/H, F)$ is non-negatively curved. So by Theorem 2.1, $(G/H, \tilde{F})$ is also non-negatively curved. To show that $\tilde{F}$ satisfies the (FP) condition, we only need to prove the assertion at $o = eH$. The tangent space $T_o(G/H)$ can be identified with $\mathfrak{m}$. Now let $P \subset \mathfrak{m}$ be a tangent plane at $e$. Then there are only the following two cases.

Case 1 $P$ is a commutative subalgebra. By (2) of the lemma, $\langle P, v \rangle_{\text{bi}} = 0$. Fixing an orthogonal basis \{\(v_1, v_2\)\} of $P$, we can find a nonzero vector $v_1'$ such that

$$\tilde{v}_1' = v_1' + F(v_1')v = v_1.$$

Since $\langle v_1', v_2 \rangle_{\tilde{F}} = \langle v_1', v_2 \rangle_{\text{bi}} = 0$, by Theorem 2.1, we have

$$K^{\tilde{F}}(o, v_1, v_1 \wedge v_2) = K^F(o, v_1', v_1' \wedge v_2).$$

On the other hand, by (2) of the lemma, we have $[v_1, v_2] \neq 0$. Therefore

$$[v_1', v_2] = -F(v_1')[v_1, v_2] \neq 0.$$
Then by the curvature formula of the Riemannian normal homogeneous spaces, we have
\[ K^F(o, v_1, P) = K^F(o, v'_1, v'_2) > 0. \]

**Case 2** \( P \) is not a commutative subalgebra in \( m \). Fix an orthogonal basis \( \{v_1, v_2\} \) of \( P \) such that \( \langle v, v_2 \rangle_{\mathfrak{h}_0} = 0 \). Select the nonzero vectors \( v'_1 \) and \( v''_1 \) such that
\[ v'_1 = v'_1 + F(v'_1)v = v_1 \text{ and } v''_1 = v''_1 + F(v''_1)v = -v_1. \]
If \( [v'_1, v_2] \neq 0 \) or \( [v''_1, v_2] \neq 0 \), then by a similar argument as above, we can get \( K^F(o, v_1, P) \neq 0 \) or \( K^F(o, -v_1, P) \neq 0 \). If \( [v'_1, v_2] = [v''_1, v_2] = 0 \), we then have \( [v, v_2] = [v_1, v_2] = 0 \), which is a contradiction.

Therefore, the homogeneous metric \( F \) on \( G/H \) is non-negatively curved and satisfies the (FP) condition. This completes the proof of the lemma. \( \blacksquare \)

The assumptions in Lemma 3.1 in fact implies that \( \text{rk } \mathfrak{g} = \text{rk } \mathfrak{h} + 1 \). Moreover, we assert that \( H \) is a regular subgroup of \( G \). In fact, any Cartan subalgebra of \( \mathfrak{h} \) can be extended to a Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \) by adding \( m_0 \). For simplicity we call such \( \mathfrak{t} \) a fundamental Cartan subalgebra, since in this case \( \mathfrak{t} \cap \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{h} \). By (1) of Lemma 3.1 we have \( [m_0, \mathfrak{h}] = 0 \). Hence any root plane of \( \mathfrak{h} \) with respect to \( \mathfrak{t} \cap \mathfrak{h} \) is also a root plane of \( \mathfrak{g} \) with respect to \( \mathfrak{t} \). Thus \( H \) is a regular subgroup of \( G \).

If we further decompose \( m \) orthogonally as \( m = m_0 + m_1 \), then \( m_1 \) is the sum of some root planes with respect to the Cartan subalgebra \( \mathfrak{t} \) mentioned above.

Now we are ready to prove the following theorem which indicates that the canonical circle bundles over compact irreducible Hermitian symmetric spaces admit non-negatively curved homogeneous Randers metrics satisfying the (FP) condition.

**Theorem 3.2** Let \( H \subset K \subset G \) be a triple of compact connected Lie groups and \( \mathfrak{h} \subset \mathfrak{t} \subset \mathfrak{g} \) the corresponding triple of Lie algebras, such that \( G \) is a compact simple Lie group, \( G/K \) is an irreducible Hermitian symmetric space, and \( \mathfrak{h} \) is the semisimple factor of \( \mathfrak{t} \). Then \( G/H \) admits non-negatively curved homogeneous Randers metrics satisfying the (FP) condition.

**Proof.** Given an irreducible Hermitian symmetric pair \( (\mathfrak{g}, \mathfrak{t}) \), we have orthogonal decompositions \( \mathfrak{g} = \mathfrak{h} + m \), \( m = m_0 + m_1 \), and \( \mathfrak{t} = \mathfrak{h} + m_0 \), which are all invariant under \( \text{Ad}(H) \). In particular, the \( \text{Ad}(H) \)-actions on \( m_0 \) is trivial. Fix a fundamental Cartan subalgebra \( \mathfrak{t} \) containing \( m_0 \), and a nonzero vector \( v \in m_0 \). It is well known that for any root plane \( \mathfrak{g}_{\pm \alpha} \subset m_1 \), we have \( \langle v, \alpha \rangle_{\mathfrak{h}} \neq 0 \). Therefore for any nonzero \( v' \in m_1 \), the Lie bracket \( [v, v'] \) is a nonzero vector in \( m_1 \).

Now we will apply Lemma 3.3 to prove the theorem. Condition (1) of Lemma 3.3 is naturally satisfied. Consider Condition (2) of Lemma 3.3. Given any 2-dimensional commutative subalgebra \( \mathfrak{t}' \) of \( \mathfrak{g} \) contained in \( m \), there exists a bi-invariant basis of \( \mathfrak{t}' \), consisting of \( v_1 = av + v'_1 \) and \( v_2 \), where \( a \in \mathbb{R} \) and \( v'_1, v_2 \in m_1 \). It is easily seen that \( [v_1, v_2] = a[v, v_2] + [v'_1, v_2] = 0 \). By the symmetric condition, we have \( [v'_1, v_2] \subset \mathfrak{h} + m_0 \). Moreover, it is easy to check that \( [v, v_2] \neq 0 \) is a nonzero vector in \( m_1 \). Therefore we have \( a = 0 \), that is, \( \langle v, v' \rangle_{\mathfrak{h}} = 0 \). This implies that Condition (2) of Lemma 3.3 is also satisfied. So by Lemma 3.3 \( G/H \) admits a non-negatively curved homogeneous Finsler metric satisfying the (FP) condition. \( \blacksquare \)
The compact irreducible Hermitian symmetric pairs \((g, \mathfrak{k})\) can be listed as the following:

\[(A_{p+q-1}, A_{p-1} \oplus A_{q-1} \oplus \mathbb{R}), (B_n, B_{n-1} \oplus \mathbb{R}), (C_n, A_n \oplus \mathbb{R}), (D_n, D_{n-1} \oplus \mathbb{R}), (E_6, D_5 \oplus \mathbb{R}) \text{ and } (E_7, E_6 \oplus \mathbb{R}).\]

Correspondingly, by Theorem 3.2 the list of coset spaces in (1.1) admits non-negatively curved homogeneous Randers metrics satisfying the (FP) condition, which completes the proof of Theorem 1.2.

4 Left invariant metrics on quasi-compact Lie groups satisfying the (FP) condition

All the examples of non-negatively curved homogeneous Finsler spaces satisfying the (FP) condition provided by Theorem 1.2 are compact. Furthermore, they all satisfy the rank inequality \(\text{rk}_g \leq \text{rk}_h + 1\) which is satisfied by all the positively curved homogeneous spaces. These observations lead to the following problems

**Problem 4.1** Is a non-negatively curved homogeneous Finsler space satisfying the (FP) condition compact?

**Problem 4.2** Does any non-negatively curved and flag-wise positively curved homogeneous Finsler spaces \((G/H, F)\) with compact \(G\) satisfy the condition \(\text{rk}_g \leq \text{rk}_h + 1\) for their Lie algebras?

In Riemannian geometry, the (FP) Condition is equivalent to the positively curved condition, hence the answers to both the above problems are positive. In Finsler geometry, however, the non-negatively curved condition seems to be more subtle, since Theorem 1.5 provides counter examples for both the problems with the non-negatively curved condition dropped.

**Proof of Theorem 1.5** We start with the bi-invariant Riemannian metric \(F\) on \(G\), which is determined by the bi-invariant inner product \(\langle \cdot, \cdot \rangle_{\text{bi}}\). Let \(t_1\) be a Cartan subalgebra in \(g\) and \(v_1\) a generic vector in \(t_1\), i.e., \(t_1 = c_g(v_1)\). Then \(v_1\) defines a left invariant vector field \(V_1\) on \(G\), which is also a Killing vector field of constant length for \((G, F)\). For any sufficiently small \(\epsilon > 0\), the navigation datum \((F, \epsilon V_1)\) defines a Finsler metric \(\tilde{F}_{1,\epsilon}\). Since both \(F\) and \(V_1\) are left invariant, so is \(\tilde{F}_{1,\epsilon}\). As we have argued before, \((G, \tilde{F}_{1,\epsilon})\) is non-negatively curved. The following lemma indicates that \(t_1\) is the only 2-dimensional subspace \(P \subset g = T_eG\) for which the (FP) condition may not be satisfied for \((G, \tilde{F}_{1,\epsilon})\).

**Lemma 4.3** Keep all the above assumptions and notations. Given any sufficiently small \(\epsilon > 0\), if the 2-dimensional subspace \(P \subset g\) satisfies \(K_{\tilde{F}_{1,\epsilon}}(e, y, P) \leq 0\) for any nonzero \(y \in P\), then \(P = t_1\).

**Proof.** Given any \(P \subset g\) as in the lemma, we can find a nonzero vector \(w_2 \in P\) with \(\langle w_2, v_1 \rangle_{\text{bi}} = 0\). Then there exists a nonzero vector \(w_1 \in P\) such that \(\langle w_1, w_2 \rangle_{\text{bi}} = 0\).
One can also find some suitable positive numbers $a$ and $b$, such that $w'_1 = w_1 - av_1$ and $w'_2 = -w_1 - bv_1$ satisfying the conditions

\[ w'_1 = w'_1 + F(w'_1)v_1 = w_1 \text{ and } w'_2 = w'_2 + F(w'_2)v_1 = -w_1. \]

Moreover, we also have

\[ \langle w'_1, w_2 \rangle_{bi} = \langle w'_2, w_2 \rangle_{bi} = 0. \]

By Theorem 2.1 we have

\[ K^F(e, w'_1 \wedge w_2) = K^\tilde{F}_{1, \epsilon}(e, w_1, P) \leq 0, \quad (4.6) \]

and

\[ K^F(e, w'_2 \wedge w_2) = K^\tilde{F}_{1, \epsilon}(e, -w_1, P) \leq 0. \quad (4.7) \]

Since $(G, F)$ is non-negatively curved, the equality holds for both (4.6) and (4.7), that is, we have

\[ [w'_1, w_2] = [w_1, w_2] - a[v_1, w_2] = 0, \]

and

\[ [w'_2, w_2] = -[w_1, w_2] - b[v_1, w_2] = 0, \]

from which we conclude that $[w_1, w_2] = [v, w_2] = 0$. Since $v_1$ is a generic vector in $t_1$, we have $w_2 \in t_1$. Now if we change the flag pole to another generic $w_3 = w_1 + cw_2 \in P$, then there is a nonzero number $d$ such that the vector $w_4 = w_2 + dw_1$ satisfies the condition $\langle w_3, w_4 \rangle_{\tilde{F}_{1, \epsilon}} = 0$. Let $w'_3$ be the nonzero vector such that $w'_3 = w'_3 + F(w'_3)v = w_3$. Then by Theorem 2.1 we have

\[ K^F(e, w'_3 \wedge w_4) = K^\tilde{F}_{1, \epsilon}(e, w_3, w_3 \wedge w_4) \leq 0. \]

So we have $K^F(e, w'_3 \wedge w_4) = 0$, and

\[ [w'_3, w_4] = [w_1 + cw_2 - F(w'_3)v, w_2 + dw_1] = F(w'_3)d[v, w_1] = 0, \]

i.e., $[v, w_1] = 0$. Thus $P \subset t_1$. Now it follows from the assumption $\text{rk}_g = 2$ that $P = t_1$. This completes the proof of the lemma. ■

Now we fix the sufficiently small $\epsilon > 0$. The left invariant metric $\tilde{F}_{1, \epsilon}$ on $G$ is real analytic. So for any 2-dimensional subspace $P \neq t_1$, the inequality $K^\tilde{F}_{1, \epsilon}(e, y, P) > 0$ is satisfied for all nonzero $y$ in a dense open subset of $P$.

Let $t_2 \neq t_1$ be another Cartan subalgebra of $g$, and $v_2 \in t_2$ a nonzero generic vector. Then for any sufficiently small $\epsilon > 0$, We can similarly define the left invariant metric $\tilde{F}_{2, \epsilon}$ from $F$ and $v_2$.

Now we can construct the left invariant metric indicated in the theorem.

Let $v$ be a bi-invariant unit vector in $t_1$. Then we can find two disjoint closed subsets $D_1$ and $D_2$ in $S = \{u \in g | (u, u)_{bi} = 1\}$ such that there are two small open neighborhood $U_1$ and $U_2$ of $v$ in $S$ satisfying $U_1 \subset D_2 \subset S \backslash D_1 \subset U_2$. Also we have two closed cones in $g \backslash 0$, defined by

\[ C_1 = \{ty | \forall t > 0 \text{ and } y \in D_1\}, \quad \text{and}\]
\[ C_2 = \{ty | \forall t > 0 \text{ and } y \in D_2\}. \]
Viewing $\tilde{F}_{1,\epsilon}$ and $\tilde{F}_{2,\epsilon}$ as Minkowski norms on $\mathfrak{g}$, we now glue $\tilde{F}_{1,\epsilon}|_{C_1}$ and $\tilde{F}_{2,\epsilon}|_{C_2}$ to produce a new Minkowski norm. Let $\mu : \mathfrak{g}\backslash 0 \to [0,1]$ be a smooth positively homogeneous function of degree 0 (i.e. $\mu(\lambda y) = \lambda \mu(y)$ whenever $\lambda > 0$ and $y \neq 0$) such that $\mu|_{C_1} \equiv 1$ and $\mu|_{C_2} \equiv 0$. Define $F_\epsilon = \mu \tilde{F}_{1,\epsilon} + (1 - \mu) \tilde{F}_{2,\epsilon}$. Obviously $F_\epsilon$ coincides with $\tilde{F}_{1,\epsilon}$ on $C_1$, and with $\tilde{F}_{2,\epsilon}$ on $C_2$. When $\epsilon > 0$ is small enough, $F_\epsilon$ can be sufficiently $C^4$-close to $F_0 = F$, when restricted to a closed neighborhood of $S$. It implies that the strong convexity condition (3) in the definition of Minkowski norms is satisfied for $F_\epsilon$. The other conditions are easy to check. So $F_\epsilon$ is a Minkowski norm. Using the left actions of $G$, we get a family of left invariant Finsler metrics $F_\epsilon$ on $G$, where $\epsilon > 0$ is sufficiently small.

Let us check the (FP) condition for $F_\epsilon$ for a fixed $\epsilon > 0$. We only need to check at $e$. Given any 2-dimensional subspace $P \subset \mathfrak{g}$ different from $t_1$, we can find a generic vector $w_1 \in P$ such that a neighborhood of $w_1$ is contained in $C_1$, i.e., $F_\epsilon$ and $\tilde{F}_{1,\epsilon}$ coincides on a neighborhood of $w_1$, so we have $K^{F_\epsilon}(e, w_1, P) = K^{\tilde{F}_{1,\epsilon}}(e, w_1, P) > 0$. For $P = t_1$, we can find a generic vector $w_1 \in P$ from $U_1$. Then similarly we can show that $K^{F_\epsilon}(e, w_1, P) = K^{\tilde{F}_{1,\epsilon}}(e, w_1, P) > 0$. Thus $F_\epsilon$ satisfies the (FP) condition.

This completes the proof of Theorem 1.5.

Finally, we must remark that the above argument can not be easily generalized to $G$ with higher ranks. On the other hand, if $\dim \mathfrak{c}(\mathfrak{g}) > 1$, then Theorem 2.3 implies that the flag curvature vanishes for any flag contained in the center, regardless of the choice of the flag pole. So the (FP) condition is not satisfied by any left invariant Finsler metric on $G$.

We end this section with the following

**Problem 4.4** Let $G$ be a Lie group such that $\text{Lie}(G) = \mathfrak{g}$ is a compact non-Abelian Lie algebra with $\dim \mathfrak{c}(\mathfrak{g}) \leq 1$. Does $G$ admit a flag-wise positively curved left invariant metric?

## 5 A key technique for the homogeneous positive flag curvature problem

In this section, we complete the proof of Theorem 1.3. We first recall some result on odd dimensional positively curved homogeneous Finsler spaces.

### 5.1 General theories for odd dimensional positively curved homogeneous Finsler spaces

Let $(G/H, F)$ be an odd dimensional positively curved homogeneous Finsler space such that $\mathfrak{g} = \text{Lie}(G)$ and $\mathfrak{h} = \text{Lie}(H)$ are compact Lie algebras. We fix a bi-invariant inner product $\langle . , . \rangle_{\text{bi}}$ of $\mathfrak{g}$, and an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with respect to $\langle . , . \rangle_{\text{bi}}$.

The rank inequality in [17] implies that $\text{rk}\mathfrak{g} = \text{rk}\mathfrak{h} + 1$. Fix a fundamental Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, i.e., $\mathfrak{t} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{t}$. Then we have $\mathfrak{t} = \mathfrak{t} \cap \mathfrak{h} + \mathfrak{t} \cap \mathfrak{m}$ and $\dim \mathfrak{t} \cap \mathfrak{m} = 1$. In later discussions, roots and root planes of $\mathfrak{g}$ and $\mathfrak{h}$ are meant to be with respect to $\mathfrak{t}$ and $\mathfrak{t} \cap \mathfrak{h}$, respectively. By the restriction of $\langle . , . \rangle_{\text{bi}}$ to $\mathfrak{t}$ and $\mathfrak{t} \cap \mathfrak{h}$, roots of $\mathfrak{g}$ and $\mathfrak{h}$ will also be viewed as vectors in $\mathfrak{t}$ and $\mathfrak{t} \cap \mathfrak{h}$, respectively.
The most fundamental decomposition for $g$ is

$$g = t + \sum_{\alpha \in \Delta_g} g_{\pm \alpha},$$

where $\Delta_g \subset t$ is the root system of $g$ and $g_{\pm \alpha}$ is the root plane of $g$ for the roots $\pm \alpha$.

Another decomposition for $g$ is

$$g = \sum_{\alpha' \in t \cap h} \hat{g}_{\pm \alpha'},$$

(5.8)

where $\hat{g}_{\pm \alpha'} = \sum_{pr_{\alpha'}(\alpha') = \alpha'} g_{\pm \alpha}$ when $\alpha' \neq 0$ and $\hat{g}_0 = c_{\hat{g}}(t \cap h)$. Notice that $\hat{g}_0 = t + g_{\pm \alpha}$ if there are roots $\pm \alpha \in t \cap m$, and otherwise $\hat{g}_0 = t$. The following lemma indicates that this decomposition is compatible with the orthogonal decomposition $g = h + m$, and the decomposition

$$h = t \cap h + \sum_{\alpha' \in \Delta_h} h_{\pm \alpha'},$$

where $\Delta_h \subset t \cap h$ is the root system of $h$ and $h_{\pm \alpha'}$ is the root plane of $h$ for the roots $\pm \alpha'$.

**Lemma 5.1** For any $\alpha' \in t \cap h$, we have the following.

1. $\hat{g}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap h + \hat{g}_{\pm \alpha'} \cap m$.
2. If $\hat{g}_{\pm \alpha'} \cap h \neq 0$, then we have $\pm \alpha' \in \Delta_h$ and $h_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap h$.
3. Denote $\hat{m}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap m$. Then $\hat{m}_{\pm \alpha'} = \hat{g}_{\pm \alpha'}$ if and only if $\alpha' \notin \Delta_h$. In particular, $\hat{m}_0 = \hat{g}_0$ is either one dimensional or three dimensional.

By this lemma, the decomposition (5.9) implies that

$$m = \sum_{\alpha' \in t \cap h} \hat{m}_{\pm \alpha'}.$$  

(5.9)

The proof of Lemma 5.1 is easy and will be omitted.

Let $\alpha'$ be a nonzero vector in $t \cap h$. Denote by $t'$ the orthogonal complement of $\alpha'$ in $t \cap h$ with respect to $\langle \cdot, \cdot \rangle_{bi}$, and $pr_{t'}$ the orthogonal projection to $t'$. Then we have a decomposition

$$m = \sum_{\alpha'' \in t'} \hat{m}_{\pm \alpha''},$$

(5.10)

where $\hat{m}_{\pm \alpha''} = \sum_{\alpha' \in t \cap h, pr_{t'}(\alpha') = \alpha''} \hat{m}_{\pm \alpha'}$.

The importance of (5.9) and (5.10) lies in the fact that they are orthogonal with respect to certain inner product $g^F_{bi}$ defined by the homogeneous Finsler metric $F$. Thus we have the following orthogonality lemma.

**Lemma 5.2** Keep all above assumptions and notations. Then we have:

1. For any nonzero vector $u \in \hat{m}_0$, the decomposition (5.9) is $g^F_{bi}$-orthogonal.
2. For any nonzero vector \( u \in \hat{\mathfrak{m}}_{\pm \alpha^\prime} \) with nonzero \( \alpha^\prime \in t \cap \mathfrak{h} \), the decomposition (5.10) is \( g^F_u \)-orthogonal.

3. When \( \dim \hat{\mathfrak{m}}_0 = 3 \), we can choose a suitable fundamental Cartan subalgebra \( t \) such that there is a nonzero vector \( u \in t \cap \mathfrak{m} \) satisfying \( \langle t \cap \mathfrak{m}, g_{\pm \alpha} \rangle_u^F = 0 \), where \( g_{\pm \alpha} = [t \cap \mathfrak{m}, \hat{\mathfrak{m}}_0] \) is the root plane contained in \( \hat{\mathfrak{m}}_0 \).

4. For any nonzero vector \( u \in \hat{\mathfrak{m}}_{\pm \alpha^\prime} \) with \( \alpha^\prime \neq 0 \), we have \( \langle u, [u, t \cap \mathfrak{h}] \rangle_u^F = 0 \).

Proof. This lemma is a summarization of the techniques we have used in [16]. Here we only present a very sketchy proof of it.

1. Let \( T_H \) be the maximal torus in \( H \) generated by \( t \cap \mathfrak{h} \). Then the inner product \( g_u^F \) is \( \text{Ad}(T_H) \)-invariant when the nonzero vector \( u \) belongs to \( \hat{\mathfrak{m}}_0 \). Since the summands in (5.9) correspond to different irreducible representations of \( T_H \), (5.9) is \( g^F_u \)-orthogonal.

2. Let \( T' \) be the connected subgroup in \( T_H \) generated by \( t' \), the orthogonal complement of \( \alpha^\prime \) in \( t \cap \mathfrak{h} \). Notice that \( T' = \exp t' \) is in fact a closed sub-torus. Thus for any nonzero vector \( u \in \hat{\mathfrak{m}}_{\pm \alpha^\prime} \), the inner product \( g_u^F \) is \( \text{Ad}(T') \)-invariant, and the summands in the decomposition (5.10) correspond to different irreducible representations of \( T' \). This proves the \( g^F_u \)-orthogonality of (5.10).

3. Choose the fundamental Cartan subalgebra \( t \) which contains the \( F \)-unit vector \( u \in \hat{\mathfrak{m}}_0 \) whose length reaches the maximum or minimum among all \( F \)-unit vectors in \( \hat{\mathfrak{m}}_0 \). Then \( t \) and \( u \) satisfy the requirement in the lemma.

4. By the \( \text{Ad}(T_H) \)-invariance of \( F \), the restriction of \( F \) to the 2-dimensional subspace \( \mathbb{R} u + [t \cap \mathfrak{h}, u] \) coincides with the bi-invariant inner product up to a scalar change. Hence the assertion (4) follows immediately.

5.2 Proof of Theorem 1.3

Assume conversely that for the odd dimensional positively curved homogeneous Finsler space, with a bi-invariant orthogonal decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \), we can find roots \( \alpha \) and \( \beta \) satisfying (1)-(4) in the theorem. We will deduce assertions by the following steps:

1. We first assert that neither \( \alpha \) nor \( \beta \) is contained in \( t \cap (\mathfrak{h} \cup \mathfrak{m}) \).

If one of \( \alpha \) and \( \beta \) belongs to \( t \cap \mathfrak{h} \), then there will be a contradiction will the following lemma of [10].

Lemma 5.3 Let \( (G/H, F) \) be an odd-dimensional homogeneous Finsler space with positive flag curvature and a bi-invariant orthogonal decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \) for the compact Lie algebra \( \mathfrak{g} \). Assume \( \alpha \in t \cap \mathfrak{h} \) is the only root of \( \mathfrak{g} \) contained in \( \alpha + t \cap \mathfrak{h} \). Then \( \alpha \) is a root of \( \mathfrak{h} \) and \( \mathfrak{g}_{\pm \alpha} = \hat{\mathfrak{g}}_{\pm \alpha} = \mathfrak{h}_{\pm \alpha} \).

Proof. Assume conversely that \( \alpha \) is not a root of \( \mathfrak{h} \). Then \( \mathfrak{g}_{\pm \alpha} = \hat{\mathfrak{g}}_{\pm \alpha} \subset \mathfrak{m} \). Let the Cartan subalgebra \( t \) and the nonzero vector \( u \in t \cap \mathfrak{m} \) be chosen as in (3) of Lemma 5.2. Let \( v \) be any nonzero vector in \( \mathfrak{g}_{\pm \alpha} \). Then they are a linearly independent commutative pair in \( \mathfrak{m} \). By (1) and (3) of Lemma 5.2, we get

\[
\langle u, [u, \mathfrak{m}] \rangle_u^F = \langle u, [v, \mathfrak{m}] \rangle_u^F = \langle v, [u, \mathfrak{m}] \rangle_u^F = 0.
\]

Then Theorem 2.3 implies that \( K^F(u, u, u \wedge v) = 0 \). This is a contradiction.
If \( \alpha \) or \( \beta \) belongs to \( t \cap m \), then by the orthogonality, the other belongs to \( t \cap h \), which is also a contradiction.

To summarize, \( pr_h(\alpha) \), \( pr_h(\beta) \), \( pr_m(\alpha) \) and \( pr_m(\beta) \) are all nonzero vectors.

2. Define an orthogonal decomposition for \( m \), with respect to some inner product \( g^F_u \).

Denote \( \alpha' = pr_h(\alpha) \), \( \beta' = pr_h(\beta) \), and \( t' \) the bi-invariant orthogonal complement of \( \alpha' \) in \( t \cap h \). Let \( \beta'' = pr_v(\beta') \). Then by (3) of the theorem, we have \( \beta'' \neq 0 \), \( \hat{m}_0 = t \cap m + g_{\pm \alpha} \) and \( \hat{m}_{\pm \beta''} = g_{\pm \beta} \).

For simplicity, denote \( m'_0 = t \cap m \), \( m'_1 = g_{\pm \alpha} \), \( m' = m'_0 + m'_1 \), \( m'' = g_{\pm \beta} \), and the sum of all other \( \hat{m}_{\pm \gamma''} \) as \( m''' \). Then (3) of the theorem also indicates that

\[
[m' + m'', m''']_m \subset m''', \quad \text{and} \quad [g_{\pm \alpha}, g_{\pm \beta}] = 0. \tag{5.11}
\]

By (2) of Lemma 5.2, the decomposition

\[
m = m' + m'' + m'''
\]

is \( g^F_u \)-orthogonal for any nonzero vector \( u \in m' \).

3. Now we fix a bi-invariant unit vector \( u_0 \in t \cap m \), and prove that \( \langle u_0, u_0 \rangle^F_{u + tu_0} \) and \( \langle v, v \rangle^F_{u + tu_0} \) only depend on \( t \) for all bi-invariant unit vectors \( u \in g_{\pm \alpha} \) and \( v \in g_{\pm \beta} \).

Let \( \{u(s), \tilde{u}(s)\} \) be a smooth family of bi-invariant orthonormal basis of \( g_{\pm \alpha} \), where \( u(s) \) exhaust all bi-invariant unit vectors in \( g_{\pm \alpha} \). Similarly define \( \{v(r), \tilde{v}(r)\} \) for \( g_{\pm \beta} \). Denote \( u(s, t) = u(s) + tu_0 \). Then there exist two smooth families of group elements \( g'_s \) and \( g''_s \) from the maximal torus \( T_H \) in \( H \) generated by \( t \cap h \) such that

\[
\text{Ad}(g'_s)u(0) = u(s), \quad \text{Ad}(g'_s)v(r) = v(r),
\]

\[
\text{Ad}(g''_s)u(s) = u(s), \quad \text{and} \quad \text{Ad}(g''_s)v(0) = v(r).
\]

Then we have

\[
\langle u_0, u_0 \rangle^F_{u(s, t)} = \langle \text{Ad}(g'_s)u_0, \text{Ad}(g'_s)u_0 \rangle^F_{\text{Ad}(g'_s)u(0, t)} = \langle u_0, u_0 \rangle^F_{u(0, t)},
\]

and

\[
\langle v(r), v(r) \rangle^F_{u(s, t)} = \langle \text{Ad}(g'_s g''_s) v(0), \text{Ad}(g'_s g''_s) v(0) \rangle^F_{\text{Ad}(g'_s g''_s) u(0, t)} = \langle v(0), v(0) \rangle^F_{u(0, t)},
\]

which proves the assertion.

Notice that the statement that \( \langle v(r), v(r) \rangle^F_{u(s, t)} \) only depends on \( t \) implies that

\[
\langle v(r), \tilde{v}(r) \rangle^F_{u(s, t)} = 0. \tag{5.13}
\]

In the following, we calculate the components in (2.5) one by one.

(1) \( \eta(u(s, t)) \).

Restricted to \( m' \), \( F \) is an \((\alpha, \beta)\)-norm corresponding to the decomposition \( m' = m'_0 + m'_1 \) by its \( \text{Ad}(T_H) \)-invariance. So (4) of Lemma 5.2 implies that \( \langle u(s, t), \tilde{u}(s) \rangle^F_{u(s, t)} = 0 \). Since

\[
\langle \eta(u(s, t)), [m'_0 + \mathbb{R} u(s, t) + m'' + m''']_m \rangle^F_{u(s, t)} = \langle u(s, t), [m'_0 + \mathbb{R} u(s, t) + m'' + m''']_m \rangle^F_{u(s, t)} = 0,
\]

\[
\langle \eta(u(s, t)), m'_0 + \mathbb{R} u(s, t) + m'' + m''']_m \rangle^F_{u(s, t)} = \langle u(s, t), \mathbb{R} u(s) + m'' + m''']_m \rangle^F_{u(s, t)} = 0,
\]

\[
\langle \eta(u(s, t)), m'_1 + \mathbb{R} u(s, t) + m'' + m''']_m \rangle^F_{u(s, t)} = \langle u(s, t), \mathbb{R} u(s) + m'' + m''']_m \rangle^F_{u(s, t)} = 0,
\]

\[
\langle \eta(u(s, t)), m'' + m''']_m \rangle^F_{u(s, t)} = \langle u(s, t), m'' + m''']_m \rangle^F_{u(s, t)} = 0,
\]

\[
\langle \eta(u(s, t)), m''' \rangle^F_{u(s, t)} = \langle u(s, t), m''' \rangle^F_{u(s, t)} = 0.
\]
we get $\eta(u(s, t)) = c_1(s, t)\tilde{u}(s)$, where the real function $c_1(s, t)$ depends smoothly on $s$ and $t$.

We now prove that $c_1(s, t) = c_1(t)$ only depends on $t$. Applying $g'_s$ defined above, we get

\[
\langle \text{Ad}(g'_s)\eta(u(0, t)), \text{Ad}(g'_s)w\rangle_{u(s, t)} = \langle \eta(u(0, t)), w\rangle_{u(0, t)} = \langle u(0, t), [w, u(0, t)]m\rangle_{u(0, t)} = \langle \text{Ad}(g'_s)u(0, t), \text{Ad}(g'_s)[w, u(0, t)]m\rangle_{\text{Ad}(g'_s)u(0, t)} = \langle u(s, t), [\text{Ad}(g'_s)w, u(s, t)]m\rangle_{u(s, t)}.
\]

Thus $\eta(u(s, t)) = \text{Ad}(g'_s)\eta(u(0, t)) = c_1(0, t)\text{Ad}(g'_s)\tilde{u}(0) = c_1(0, t)\tilde{u}(s)$.

(2) The Cartan tensor $C^F_{u(s, t)}(w_1, w_2, \tilde{u}(s))$.

If $w_1 \in m''$ and $w_2 \in m''$, then there exist a vector $h \in t \cap h$ such that $\langle h, \alpha \rangle_{bi} \neq \langle \alpha, h \rangle_{bi} = 0$. By Theorem 1.3 in [7], we have

\[
\langle [h, w_1], w_2 \rangle_{u(s, t)} + \langle w_1, [h, w_2] \rangle_{u(s, t)} + 2C^F_{u(s, t)}(w_1, w_2, [h, u(s, t)]) = 0. \tag{5.14}
\]

Since $[h, w_1] = [h, w_2] = 0$ and $[h, u(s, t)]$ is a nonzero multiple of $\tilde{u}(s)$, we have $C^F_{u(s, t)}(w_1, w_2, \tilde{u}(s)) = 0$.

In the case that $w_1 \in m''$ and $w_2 \in m' + m''$, we can similarly prove that

$C^F_{u(s, t)}(w_1, w_2, \tilde{u}(s)) = 0$,

using (5.14) and the $g'^F_{u(s, t)}$-orthogonality (by (2) of Lemma 5.2).

To summarize, we have

\[
C^F_{u(s, t)}(m'', \cdot, \eta(u(s, t))) \equiv 0. \tag{5.15}
\]

(3) $N(u(s, t), v(r))$.

If $w_1 \in m''$ and $w_2 \in m' + m''$, then by (5.11), (5.15) and the $g^F_u$-orthogonality of (5.12), we have

\[
\langle N(u(s, t), m''), m' + m'' \rangle_{u(s, t)} = 0, \tag{5.16}
\]

by the definition (2.5).

On the other hand, there exists $h \in t \cap h$ such that $\langle u_0 - h, \beta \rangle_{bi} = 0$. So by (5.15), we have

\[
\langle N(u(s, t), v(r)), v(r) \rangle_{u(s, t)} = \langle [v(r), u(s, t)]m, v(r) \rangle_{u(s, t)} + C^F_{u(s, t)}(v(r), v(r), \eta(u(s, t))) = \langle [v(r), u(s, t)]m, v(r) \rangle_{u(s, t)} = t\langle [v(r), h], v(r) \rangle_{u(s, t)} = tC^F_{u(s, t)}(v(r), v(r), [h, u(s, t)]) = 0, \tag{5.17}
\]
where we have applied Theorem 1.3 in [7].

By (5.16), (5.17) and (5.13), we have

\[ N(u(s,t), v(r)) = c_2(r, s, t)v(r). \] (5.18)

Considering the vector \( h \in \mathfrak{t} \cap \mathfrak{h} \), which satisfies \( \langle u_0 - h, \beta \rangle_{\mathfrak{h}_0} = 0 \), we get

\[
2\langle \langle N(u(s,t), v(r)), \tilde{v}(r) \rangle_{u(s,t)} \rangle_{u(s,t)} = \langle \langle v(r), u(s,t) \rangle_{\mathfrak{m}} \rangle_{u(s,t)} + \langle \langle \tilde{v}(r), u(s,t) \rangle_{\mathfrak{m}} \rangle_{v(r)} \rangle_{u(s,t)} = 2ct\langle u_0, u_0 \rangle_{u(0,t)} + 2c\langle u_0, u(0) \rangle_{u(0,t)} \]

where in the last step, we have used (5.15) and Theorem 1.3 in [7]. This leads to a more important assertion, namely,

\[
\langle N(u(s,t), v(r)), \tilde{v}(r) \rangle_{u(s,t)} = ct\langle u_0, u_0 \rangle_{u(0,t)} + c\langle u_0, u(0) \rangle_{u(0,t)},
\]

where \( c \) is the nonzero constant determined by \( \langle \tilde{v}(r), v(r) \rangle_{\mathfrak{m}} = \langle \tilde{v}(0), v(0) \rangle_{\mathfrak{m}} = 2cu_0 \).

Note that the smooth function \( c_2(r, s, t) \) in (5.18) depends on \( t \) only, since

\[
c_2(r,s,t)\langle v(0), v(0) \rangle_{u(0,t)}^F = c_2(r,s,t)\langle \tilde{v}(r), \tilde{v}(r) \rangle_{u(s,t)}^F = \langle N(u(s,t), v(r)), \tilde{v}(r) \rangle_{u(s,t)}^F = c\langle u_0, u(0) \rangle_{u(0,t)}^F + ct\langle u_0, u_0 \rangle_{u(0,t)}^F,
\] (5.19)

so we can denote \( c_2(t) = c_2(r, s, t) \).

To summarize, we have

\[
D_{\eta(u(0,t))}N(u(0,t), v) = 0,
\]

\[
N(u(0,t), N(u(0,t), v(0))) = -c_2(t^2)v(0),
\]

\[
[u(0,t), N(u(0,t), v(0))] = N(u(0,t), [u(0,t), v(0)]) = c'c_2(t)v(0),
\]

where \( c' \) is the nonzero constant determined by \( [u_0, \tilde{v}(0)] = c'v(0) \). Moreover, it is obvious that

\[
[u(0,t), v(0)]_h = 0.
\]

Now applying the homogeneous flag curvature formula (2.5) in Theorem 2.2, we get

\[
\langle R_{u(0,t)}v(0), v(0) \rangle_{u(0,t)}^F = c_2(t^2)\langle v(0), v(0) \rangle_{u(0,t)}^F.
\] (5.20)
Finally, it is easily seen that there are positive constants \(0 < C_1 < C_2\), such that for all \(t \in (-\infty, \infty)\), we have

\[
C_1 < \langle v(0), v(0) \rangle_{u(0,t)} < C_2,
\]

\[
C_1 < \langle u_0, u_0 \rangle_{u(0,t)} < C_2,
\]

and

\[
|\langle u_0, u(0) \rangle_{u(0,t)}| < C_2.
\]

So by the continuity of \(c_2(t)\), there must be a \(t_0\) such that \(c_2(t_0) = 0\). Then by (5.20), we get \(K^F(o, u(0,t_0), u(0,t_0) \wedge v(0)) = 0\), which is a contradiction.

The proof of Theorem 1.3 is now completed.

**Remark 5.4** The main technique in the proof of Theorem 1.3 can be summarized as the following lemma, which may be applied to more general situations.

**Lemma 5.5** Let \((G/H, F)\) be a homogeneous Finsler space with compact Lie algebra \(g = \text{Lie}(G)\), and compact subalgebra \(h = \text{Lie}(H)\) with \(rk h = rk g - 1\). Fix a bi-invariant inner product \(\langle \cdot, \cdot \rangle_{bi}\) on \(g\), an orthogonal decomposition \(g = h + m\) accordingly, a fundamental Cartan subalgebra \(t\), and a bi-invariant unit vector \(u_0 \in t \cap m\). Assume that \(\alpha\) and \(\beta\) are roots of \(g\) satisfying the following conditions:

1. \([g \pm \alpha, g \pm \beta] = 0\).
2. Neither \(\alpha\) nor \(\beta\) is contained in \(t \cap h\) or \(t \cap m\).
3. \(m\) can be decomposed as the direct sum of three subspaces \(m' = t \cap m + g_{\pm \alpha}\), \(m'' = g_{\pm \beta}\), and \(m'''\), such that this decomposition is \(g^F_u\)-orthogonal for any nonzero \(u \in m'\) and \([m' + m'', m''']_m \subset m''\).
4. \(\langle u_0, u_0 \rangle_{u+tu_0}^F\) and \(\langle v, v \rangle_{u+tu_0}^F\) only depend on \(t\) for any bi-invariant unit vectors \(u \in m'\) and \(v \in g_{\pm \beta}\).

Then \((G/H, F)\) can not be positively curved.

### 5.3 Proof of Corollary 1.4

We conclude this paper with a case by case discussion about the coset spaces in (1.2), which gives a proof of Corollary 1.3. In the following, we will use some notations and convention for root systems of compact simple Lie algebras, which can be found in [18].

**Case 1** \(G/H = \text{SU}(p+q)/\text{SU}(p)\text{SU}(q)\) with \(p > q = 2\) or \(p = q > 4\). The root system of \(g = \text{su}(p+q)\) can be isometrically identified with

\[
\{e_i - e_j, \forall 1 \leq i < j \leq p + q\},
\]

and the fundamental Cartan subalgebra \(t\) can be chosen so that

\[
t \cap m = \mathbb{R}[q(e_1 + \cdots + e_p) - p(e_{p+1} + \cdots + e_{p+q})].
\]

Now consider \(\alpha = e_1 - e_{p+1}\) and \(\beta = e_2 - e_{p+2}\). Then they satisfies (1)-(4) in Theorem 1.3. So \(\text{SU}(p+q)/\text{SU}(p)\text{SU}(q)\) does not admit any positively curved homogeneous Finsler metric.
Case 2 \( G/H = \text{Sp}(n)/\text{SU}(n) \) with \( n > 4 \). The root system of \( \mathfrak{g} = \text{sp}(n) \) can be isometrically identified with

\[
\{ \pm e_i, \forall 1 \leq i \leq n; \pm e_i \pm e_j, \forall 1 < j \leq n \},
\]

and the fundamental Cartan subalgebra \( \mathfrak{t} \) can be chosen so that

\[
\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + \cdots + e_n).
\]

Now consider \( \alpha = 2e_1 \) and \( \beta = 2e_2 \). Then it is easily seen that they satisfy the conditions (1)-(4) in Theorem 1.3. Hence \( \text{Sp}(n)/\text{SU}(n) \) do not admit any positively curved homogeneous Finsler metric.

Case 3 \( G/H = \text{SO}(2n)/\text{SU}(n) \) with \( n = 5 \) or \( n > 6 \). The root system of \( \mathfrak{g} = \text{so}(2n) \) can be isometrically identified with

\[
\{ \pm e_i \pm e_j, \forall 1 \leq i < j \leq n \},
\]

and the fundamental Cartan subalgebra \( \mathfrak{t} \) can be chosen so that

\[
\mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(e_1 + \cdots + e_n).
\]

Now consider \( \alpha = e_1 + e_2 \) and \( \beta = e_3 + e_4 \). It is easy to check that they satisfy the conditions (1)-(4) in Theorem 1.3. Thus \( \text{SO}(2n)/\text{SU}(n) \) do not admit any positively curved homogeneous Finsler metric.

Case 4 \( G/H = E_6/\text{SO}(10) \). The root system of \( \mathfrak{g} = e_6 \) can be isometrically identified with

\[
\{ \pm e_i \pm e_j, \forall 1 \leq i < j \leq 5; \pm \frac{1}{2}e_1 \pm \cdots \pm \frac{1}{2}e_5 \pm \frac{\sqrt{3}}{2}e_6 \text{ with an odd number of } + \text{’s} \},
\]

and the fundamental Cartan subalgebra \( \mathfrak{t} \) can be chosen so that \( \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}e_6 \). Now set \( \alpha = e_1 + e_2 \) and \( \beta = e_3 + e_4 \). Then \( \alpha, \beta \) satisfy the conditions (1)-(4) in Theorem 1.3. Thus \( E_6/\text{SO}(10) \) do not admit any positively curved homogeneous Finsler metric.

Case 5 \( G/H = E_7/E_6 \). The root system of \( \mathfrak{g} = e_7 \) can be isometrically identified with

\[
\{ \pm e_i \pm e_j, \forall 1 \leq i < j \leq 7; \pm \sqrt{2}e_7; \frac{1}{2}(\pm e_1 \pm \cdots \pm e_6 \pm \sqrt{2}e_7)
\]

with an odd number of plus signs among the first six coefficients,

and the fundamental Cartan subalgebra \( \mathfrak{t} \) can be chosen so that \( \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(\sqrt{2}e_6 + e_7) \). Now set \( \alpha = e_1 + e_2 \) and \( \beta = e_3 + e_4 \). Then \( \alpha, \beta \) satisfy the conditions (1)-(4) in Theorem 1.3. Thus \( E_7/E_6 \) do not admit any positively curved homogeneous Finsler metric.

Now the proof of Corollary 1.4 is completed.
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