Renormalization group equations and integrability in Hamiltonian systems

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Abstract

We investigate Hamiltonian systems with two degrees of freedom by using renormalization group method. We show that the original Hamiltonian systems and the renormalization group equations are integrable if the renormalization group equations are Hamiltonian systems up to the second leading order of a small parameter.

Many systems in the nature are described by Hamiltonian equations of motion, and we are interested in dynamical properties of the systems. Although behavior of systems is revealed by solving equations of motion, we cannot generally obtain exact solutions to Hamiltonian systems with more than one degree of freedom because of lack of integrals. We therefore solve equations of motion by using perturbation techniques. Naive perturbation often fails to give global solutions since it yields secular terms, and hence improved perturbation methods have been developed, for instance, averaging method, multiple scale method, matched asymptotic expansions and canonical perturbation theory[1, 2].

Recently, renormalization group method [3, 4] is proposed as one of the most powerful tools to construct approximate solutions since it unifies many of the perturbation techniques listed above. This method reduces equations of motion to amplitude equations called renormalization group equations (RGEs) by ignoring fast motion. The reduced equations, RGEs, must reflect features of original systems, and we expect that dynamical properties of the original systems are
obtained from RGEs. In this article, we consider two types of perturbed Hamiltonian systems with two degrees of freedom. We investigate symplectic properties, integrability and integrals of RGEs, which are related to properties in the original systems. We first present our main result:

**Theorem** Let Hamiltonian be represented as follows:

\[ H(q_1, q_2, p_1, p_2) = H_0(q_1, q_2, p_1, p_2) + \epsilon V_1(q_1, q_2), \]

where the potential \( V_1(q_1, q_2) \) is a homogeneous cubic or quartic function of \( q_1, q_2 \) and \( \epsilon \) is a small parameter, i.e. \( |\epsilon| << 1 \). If renormalization group equation of the system (3) is a Hamiltonian system up to the second leading order of \( \epsilon \), then the original system (1) and the renormalization group equation are integrable.

Let us briefly review renormalization group method by using a simple system with one degree of freedom:

\[ H(q, p) = H_0(q, p) + \epsilon V_1(q), \]

\[ H_0(q, p) = \frac{1}{2}(p^2 + q^2), \quad V_1(q) = \frac{1}{2}q^2. \]

The equation of motion is

\[ \frac{d^2q}{dt^2} + q = -\epsilon q, \]

and the exact solution is

\[ q = B_0 \cos(\sqrt{1+\epsilon}t) + C_0 \sin(\sqrt{1+\epsilon}t), \]

where \( B_0 \) and \( C_0 \) are constants of integration and determined by initial condition at the initial time \( t = t_0 \). We perturbatively solve Eq.(3) by expanding \( q \) as a series of positive powers of \( \epsilon \):

\[ q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots. \]

This naive expansion gives

\[ q(t; t_0, B_0, C_0) = B_0 \cos t + C_0 \sin t + \frac{\epsilon}{2}(t - t_0)(C_0 \cos t - B_0 \sin t) - \frac{\epsilon^2}{8} \left[(t - t_0)(C_0 \cos t - B_0 \sin t) + (t - t_0)^2(B_0 \cos t + C_0 \sin t)\right] + O(\epsilon^3), \]

and this expansion breaks when \( \epsilon(t - t_0) \geq 1 \) because of secular terms. Here we ignored homogeneous parts of \( q_j (j \geq 1) \) which is the kernel of the linear operator \( L = \frac{d^2}{dt^2} + 1 \), since they can be included in \( q_0 \). Renormalization group method
removes the secular terms by regarding $B_0$ and $C_0$ as functions of the initial time $t_0$, and the evolutions of $B(t_0)$ and $C(t_0)$ are determined by RGE [3, 4]:

$$\frac{\partial q}{\partial t_0} \bigg|_{t_0=t} = 0, \text{ for any } t. \quad (9)$$

In other words, the RGE is

$$\frac{d B(t)}{d t} = \left(\frac{\epsilon}{2} - \frac{\epsilon^2}{8}\right) C(t) + O(\epsilon^3), \quad (10a)$$

$$\frac{d C(t)}{d t} = -\left(\frac{\epsilon}{2} - \frac{\epsilon^2}{8}\right) B(t) + O(\epsilon^3), \quad (10b)$$

and it gives the renormalized solution $q^{\text{RG}}$:

$$q^{\text{RG}}(t) = q(t; t, B(t), C(t))$$

$$= B_0 \cos \left( \left(1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}\right) t \right) + C_0 \sin \left( \left(1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8}\right) t \right) + O(\epsilon^3). \quad (11)$$

where $B_0 = B(t_0)$ and $C_0 = C(t_0)$. This expression gives an approximate but global solution to Eq.(5) up to $O(\epsilon^2)$. In the RGE (10) $B(t)$ and $C(t)$ are canonical conjugate variables because the equation is yielded by Hamiltonian

$$H^{\text{RG}} = \left(\frac{\epsilon}{4} - \frac{\epsilon^2}{16}\right) (B^2 + C^2), \quad (12)$$

and

$$\frac{d B}{d t} = \frac{\partial H^{\text{RG}}}{\partial C} + O(\epsilon^3), \quad (13a)$$

$$\frac{d C}{d t} = -\frac{\partial H^{\text{RG}}}{\partial B} + O(\epsilon^3). \quad (13b)$$

Moreover, the system governed by the $H^{\text{RG}}$ is obviously integrable and $(B^2 + C^2)/2$, which corresponds to $H_0$, is an integral of $H^{\text{RG}}$. Do these statements hold even in systems where chaotic motion appears?

To answer this question, we calculate RGEs in system (1), whose equation of motion is

$$\frac{d^2 q_j}{d t^2} + q_j = -\epsilon \frac{\partial V_1}{\partial q_j}, \quad (j = 1, 2). \quad (14)$$

First let potential $V_1$ be homogeneous quartic functions of $q_1, q_2$:

$$V_1(q_1, q_2) = \alpha_1 q_1^4 + \alpha_2 q_1^3 q_2 + \alpha_3 q_1^2 q_2^2 + \alpha_4 q_1 q_2^3 + \alpha_5 q_2^4. \quad (15)$$
We expand $q_1$ and $q_2$ as
\[ q_j = q_{j,0} + \epsilon q_{j,1} + \epsilon^2 q_{j,2} + \cdots, \quad (j = 1, 2), \] (16)
and write the zero-th order solution to the equation (14) as
\[ q_{j,0} = B_j \cos(t) + C_j \sin(t), \quad (j = 1, 2), \] (17)
where we omit subscript 0 of $B_j$ and $C_j$ for simplicity of symbols.

Following the procedure to obtain RGE, we get
\[
\frac{dB_j}{dt} = f_j(B_1, C_1, B_2, C_2; \epsilon, \alpha_1, \cdots, \alpha_5),
\] (18a)
\[
\frac{dC_j}{dt} = g_j(B_1, C_1, B_2, C_2; \epsilon, \alpha_1, \cdots, \alpha_5),
\] (18b)
although the explicit forms of $f_j$ and $g_j$ are not shown here because of complexity.

To clarify the condition of $\alpha_1, \cdots, \alpha_5$ with which the RGEs (18) become Hamiltonian systems, we define $\Delta_j$ as
\[
\Delta_j \equiv \frac{\partial f_j}{\partial B_j} + \frac{\partial g_j}{\partial C_j}, \quad (j = 1, 2). \] (19)

For a RGE, both of the equations $\Delta_1 = 0$ and $\Delta_2 = 0$ are satisfied if and only if Hamiltonian exists and
\[
\frac{dB_j}{dt} = \frac{\partial H_{RG}}{\partial C_j},
\] (20a)
\[
\frac{dC_j}{dt} = -\frac{\partial H_{RG}}{\partial B_j}. \] (20b)

We show the $\Delta_j$ up to the second order of $\epsilon$:
\[
\Delta_1 = \frac{\epsilon^2}{8}(B_1C_2 - C_1B_2) \left( (9\alpha_2^2 + 4\alpha_3^2 - 24\alpha_1\alpha_3 - 9\alpha_2\alpha_4)(B_1B_2 + C_1C_2) \right.
\] + $3[(\alpha_2 + \alpha_4)\alpha_3 - 6(\alpha_1\alpha_4 + \alpha_2\alpha_5)](B_2^2 + C_2^2)),
\] (21a)
\[
\Delta_2 = -\frac{\epsilon^2}{8}(B_1C_2 - C_1B_2) \left( (9\alpha_4^2 + 4\alpha_3^2 - 24\alpha_3\alpha_5 - 9\alpha_2\alpha_4)(B_1B_2 + C_1C_2) \right.
\] + $3(\alpha_2 + \alpha_4)\alpha_3 - 6(\alpha_1\alpha_4 + \alpha_2\alpha_5)](B_1^2 + C_1^2)\right\}. \] (21b)

The $\Delta_1$ and $\Delta_2$ take zeros irrespective of values of $B_j$ and $C_j$ if and only if the three equations hold:
\[
9\alpha_2^2 + 4\alpha_3^2 - 24\alpha_1\alpha_3 - 9\alpha_2\alpha_4 = 0, \] (22a)
\[
9\alpha_4^2 + 4\alpha_3^2 - 24\alpha_3\alpha_5 - 9\alpha_2\alpha_4 = 0, \] (22b)
\[
(\alpha_2 + \alpha_4)\alpha_3 - 6(\alpha_1\alpha_4 + \alpha_2\alpha_5) = 0. \] (22c)
Consequently, RGEs are Hamiltonian systems when the coefficients of $V_1$, namely $\alpha_1, \cdots, \alpha_5$, satisfy the condition (22). We remark that $\Delta_j$ has no terms of order $\epsilon$, which is the leading order of Eq. (18), and hence the RGEs are Hamiltonian systems for any coefficients up to $O(\epsilon)$.

To understand the meaning of the condition (22), we use the Bertrand-Darboux theorem\[5\], with which we judge integrability of “natural” Hamiltonian systems with two degrees of freedom whose form is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y).$$  \hfill (23)

The theorem states that the following three conditions are equivalent:

1. There is an integral which is independent of Hamiltonian and quadratic with respect to the momenta.
2. For a set of constants, $(a, b, b', c_1, c_2) \neq (0, 0, 0, 0)$, the potential $V(x, y)$ satisfies the Darboux equation

$$\left( V_{yy} - V_{xx} \right)(-2axy - b' y - bx + c_1) + 2V_{xy}(ay^2 - ax^2 + by - b'x + c_2) + V_x(6ay + 3b) + V_y(-6ax - 3b') = 0,$$  \hfill (24)

where $V_x = \partial V/\partial x$, $V_{xx} = \partial^2 V/\partial x^2$ and so on.
3. The system is separable in Cartesian, polar, elliptic or parabolic coordinates.

We apply the Bertrand-Darboux theorem to the original Hamiltonian systems written by Eqs. (1) and (15) to judge whether the systems are integrable or not. The Darboux equation restricts the coefficients $\alpha_1, \cdots, \alpha_5$ in the potential function (15) (Table I), and we can find in the Table I the condition (22). Consequently, when RGEs become Hamiltonian systems, the original systems are integrable. Moreover, the RGEs are also integrable since they have another integral $I$,

$$I \equiv (B_1^2 + C_1^2 + B_2^2 + C_2^2)/2,$$  \hfill (25)

except for Hamiltonian $H^{RG}$ itself. Note the quantity $I$ corresponds to $H_0$ in Eq. (14).

For the potential of homogeneous cubic functions of $q_1$ and $q_2$,

$$V_1(q) = \alpha_1 q_1^3 + \alpha_2 q_1^2 q_2 + \alpha_3 q_1 q_2^2 + \alpha_4 q_2^3,$$  \hfill (26)

secular terms appear only in even orders of $\epsilon$ and the leading and the second leading orders of RGEs are $\epsilon^2$ and $\epsilon^4$, respectively. Then we calculate $\Delta_1$ and $\Delta_2$ up to the fourth order of $\epsilon$ and

$$\Delta_1 = \frac{\epsilon^4}{108}(B_1 C_2 - B_2 C_1)(3(\alpha_1 \alpha_3 + \alpha_2 \alpha_4) - \alpha_2^2 - \alpha_3^2)h_1,$$  \hfill (27a)

$$\Delta_2 = -\frac{\epsilon^4}{108}(B_1 C_2 - B_2 C_1)(3(\alpha_1 \alpha_3 + \alpha_2 \alpha_4) - \alpha_2^2 - \alpha_3^2)h_2,$$  \hfill (27b)
Potential function $V + 9\alpha n$ian systems up to the second leading order of a small parameter. The main result of this article is that the integrable part is harmonic oscillators and homogeneous cubic and quartic potential Hamiltonian systems with two degrees of freedom were considered, whose integrals are added as perturbation. Consequently, the original systems (1) with the cubic perturbation are integrable when RGEs are Hamiltonian systems. We therefore proved our Theorem. In summary, we investigated dynamical properties of renormalization group equation (RGE) which are symplectic properties, integrability and integrals. Hamiltonian systems with two degrees of freedom were considered, whose integrable part is harmonic oscillators and homogeneous cubic and quartic potential functions are added as perturbation. The main result of this article is that the original Hamiltonian systems and RGEs are integrable if the RGEs are Hamiltonian systems up to the second leading order. We therefore proved our Theorem.

In summary, we investigated dynamical properties of renormalization group equation (RGE) which are symplectic properties, integrability and integrals. Hamiltonian systems with two degrees of freedom were considered, whose integrable part is harmonic oscillators and homogeneous cubic and quartic potential functions are added as perturbation. The main result of this article is that the original Hamiltonian systems and RGEs are integrable if the RGEs are Hamiltonian systems up to the second leading order of a small parameter.

We have two future works. One is generalization of potential function. We can directly compare the Darboux equation with condition for RGE being Hamiltonian systems if we write RGEs for the general potential. The other is extension to systems with higher degrees of freedom. For the purpose, extended Bertrand–Darboux theorem is available. Taking the contraposition of our main result, we suppose that we can obtain information on chaotic behavior by considering how symplectic properties of RGEs break in non-integrable systems.

Table 1: Condition of $\alpha_1, \cdots, \alpha_5$, to satisfy the Darboux equation. Here $\alpha_1, \cdots, \alpha_5$ are the coefficients of homogeneous quartic potential functions. There are two cases I and II, and case II must simultaneously satisfy the three equations, which coincide with the condition that RGEs become Hamiltonian systems.

|  | Condition for $\alpha_j$ | Potential function |
|---|---|---|
| I | $\alpha_2 = \alpha_4 = 0, \alpha_3 = 2\alpha_4 = 2\alpha_5$ | $V_1 = \alpha_1(q_1^2 + q_2^2)^2$ |
| II | $9\alpha_2^2 + 4\alpha_3^2 - 24\alpha_1\alpha_3 - 9\alpha_2\alpha_4 = 0$ | $V_1 = \alpha_1q_1^4 + \alpha_2q_1^2q_2 + \alpha_3q_1^2q_2^2 + \alpha_4q_1^3 + \alpha_5q_2^4$ |
|   | $9\alpha_4^2 + 4\alpha_3^2 - 24\alpha_3\alpha_5 - 9\alpha_2\alpha_4 = 0$ |   |
|   | $(\alpha_2 + \alpha_4)\alpha_3 - 6(\alpha_1\alpha_4 + \alpha_2\alpha_5) = 0$ |   |

where

$$h_1 = 40\alpha_2(3\alpha_1 + \alpha_3)(B_1^2 + C_1^2) + (66\alpha_1\alpha_2 + 139\alpha_2\alpha_3 - 495\alpha_1\alpha_4 + 186\alpha_3\alpha_4)(B_2^2 + C_2^2) + 6(33\alpha_2^2 + 34\alpha_3^2 + 12\alpha_2^2 - 40\alpha_1\alpha_3 + 15\alpha_2\alpha_4)(B_1B_2 + C_1C_2), \quad (28a)$$

$$h_2 = 40\alpha_3(3\alpha_4 + \alpha_2)(B_2^2 + C_2^2) + (66\alpha_3\alpha_4 + 139\alpha_2\alpha_3 - 495\alpha_1\alpha_4 + 186\alpha_1\alpha_2)(B_1^2 + C_1^2) + 6(33\alpha_2^2 + 34\alpha_3^2 + 12\alpha_2^2 - 40\alpha_2\alpha_4 + 15\alpha_1\alpha_3)(B_1B_2 + C_1C_2). \quad (28b)$$

One condition for $\Delta_1 = \Delta_2 = 0$ is $h_1 = h_2 = 0$ whose unique solution is the trivial one, i.e. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, 0, 0)$, and hence we ignore this condition. The other condition is $3(\alpha_1\alpha_3 + \alpha_2\alpha_4) - \alpha_2^2 - \alpha_3^2 = 0$, and this condition is found in Table 2 which shows the condition for the Darboux equation being satisfied. Consequently, the original systems (II) with the cubic perturbation are integrable when RGEs are Hamiltonian systems up to the second leading order. We therefore proved our Theorem.

Theorem
Table 2: Condition of $\alpha_1, \ldots, \alpha_4$, to satisfy the Darboux equation. Here $\alpha_1, \ldots, \alpha_4$ are the coefficients of homogeneous cubic potential functions. Four cases I, \ldots, IV are obtained, and the condition in case II coincides to the one that RGEs become Hamiltonian systems.

| I | $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ | $V_1 = 0$ |
| II | $3(\alpha_1\alpha_3 + \alpha_2\alpha_4) - \alpha_2^2 - \alpha_3^2 = 0$ | $V_1 = \alpha_1q_1^3 + \alpha_2q_1^2q_2 + \alpha_3q_1q_2^2 + \alpha_4q_2^3$ |
| III | $\alpha_1 = 2\alpha_3$, $\alpha_2 = \alpha_4 = 0$ | $V_1 = \alpha_3(2q_1^3 + q_1q_2^2)$ |
| IV | $\alpha_1 = \alpha_3 = 0$, $2\alpha_2 = \alpha_4$ | $V_1 = \alpha_2(2q_1^3 + q_1^2q_2)$ |

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