Meta-Stable Brane Configurations, Multiple NS5-Branes, and Rotated D6-Branes

Changhyun Ahn

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

We construct the type IIA nonsupersymmetric meta-stable brane configurations corresponding to the various $\mathcal{N} = 1$ supersymmetric gauge theories. The D6-branes are both displaced and rotated where these deformations are described as the mass term and quartic term for the fundamental flavors respectively. The multiplicity of the NS5-branes occurs in the superpotential order for adjoint, symmetric, or bifundamental matters. A rich pattern of nonsupersymmetric meta-stable states as well as the supersymmetric stable states is found.
1 Introduction

The dynamical supersymmetry breaking in meta-stable vacua [1, 2] occurs in the $\mathcal{N} = 1$ supersymmetric gauge theory with massive fundamental flavors and the corresponding type IIA meta-stable brane constructions have been studied in [3, 4, 5]. Other type IIA nonsupersymmetric meta-stable brane configuration has been constructed by considering the additional quartic term for the quarks in the electric superpotential besides the mass term [6, 7]. This extra deformation in the $\mathcal{N} = 1$ supersymmetric gauge theory corresponds to the rotation of D6-branes along the (45)-(89) directions in type IIA string theory. The nonsupersymmetric ground states arise only after the gravitational attraction of NS5-brane [8] is considered.

Branes can be used to describe the dynamics of a wide variety of $\mathcal{N} = 1$ supersymmetric gauge theories with different matter contents and superpotential [9]. It is very important to control the gauge singlet meson fields in appropriate way in order to find out the new metastable states. In general, one can consider the electric brane configuration of $k$ coincident left NS5-branes(012345) connected by $N_c$ D4-branes to $k'$ coincident right NS5'-branes(012389), with $N_f$ D6-branes(0123789) located between the left NS5- and right NS5'-branes [10]. The creation of D4-branes during the Seiberg dual is related to the gauge singlets in the magnetic brane configuration. Then there exist $k$ magnetic meson fields because each NS5-brane in the above brane configuration produces each meson field when it is crossing the D6-branes.

One can try to find out the new meta-stable brane configuration for a single meson field by taking a single left NS5-brane but multiple $k'$ right NS5'-branes rather than the general case with $k$ left NS5-branes and $k'$ right NS5'-branes where the corresponding dual gauge theory is not known so far. This question is also raised in [11] where the meta-stable brane configuration when $k = 2$ and $k' = 1$ was studied. In this paper, we find the various meta-stable brane configurations for the different gauge theories with matters and superpotential, along the lines of [12, 13, 14, 15]. All of these examples contain a set of a single NS5-brane and multiple NS5'-branes in their type IIA brane configurations. The single magnetic meson field is realized by the product of quark and anti-quark in the electric gauge theory in the presence of D6-branes with quartic superpotential for the fundamentals. On the other hand, when there are no D6-branes, it is realized by the product of bifundamentals with higher order superpotential.

In section 2, we review the type IIA brane configuration corresponding to the $\mathcal{N} = 1$ $SU(N_c)$ gauge theory with two adjoints and fundamentals and deform this theory by adding both the mass term and the quartic term for the fundamentals. Then we describe the dual $\mathcal{N} = 1$ $SU(\tilde{N}_c)$ gauge theory with corresponding dual matter as well as a gauge singlet. We
discuss the nonsupersymmetric meta-stable minimum and present the corresponding intersecting brane configuration of type IIA string theory. In section 3, we apply the method of section 2 to the $\mathcal{N} = 1 \text{Sp}(N_c)$ gauge theory with two adjoints and fundamentals.

In section 4, we review the type IIA brane configuration corresponding to the $\mathcal{N} = 1 \text{SU}(N_c)$ gauge theory with symmetric tensor and fundamentals and deform this theory by adding both the mass term and the quartic term for the fundamentals. Then we study the dual $\mathcal{N} = 1 \text{SU}(\tilde{N}_c)$ gauge theory with corresponding dual matters. We present the nonsupersymmetric meta-stable minimum and the corresponding intersecting brane configuration of type IIA string theory.

In section 5, we review the type IIA brane configuration corresponding to the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(N'_c)$ gauge theory with bifundamentals and fundamentals and deform this theory by adding both the mass terms and the quartic terms for each fundamentals. Then we describe the dual $\mathcal{N} = 1 \text{SU}(\tilde{N}_c) \times \text{SU}(N'_c)$ gauge theory with corresponding dual matters. We discuss the nonsupersymmetric meta-stable minimum and the corresponding intersecting brane configuration. In section 6, we use the method of section 5 to the $\mathcal{N} = 1 \text{Sp}(N_c) \times \text{SO}(2N'_c)$ gauge theory with bifundamentals, vectors and fundamentals. In section 7, we apply the method of section 5 to the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SO}(2N'_c)$ gauge theory with bifundamentals, vectors and fundamentals.

In section 8, we review the type IIA brane configuration corresponding to the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(N'_c)$ gauge theory with symmetric tensor and bifundamentals and deform this theory by adding both the mass term and the higher order term for bifundamentals. Then we describe the dual $\mathcal{N} = 1 \text{SU}(\tilde{N}_c) \times \text{SU}(N'_c)$ gauge theory with corresponding dual matters. We discuss the nonsupersymmetric meta-stable minimum and the corresponding intersecting brane configuration. In section 9, we use the method of section 8 to the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(N'_c)$ gauge theory with antisymmetric tensor, eight fundamentals and bifundamentals.

In section 10, we review the type IIA brane configuration corresponding to the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(N'_c)$ gauge theory with symmetric tensor, bifundamentals and fundamentals and deform this theory by adding both the mass terms and the quartic terms for each fundamentals. Then we describe the dual $\mathcal{N} = 1 \text{SU}(\tilde{N}_c) \times \text{SU}(N'_c)$ gauge theory or $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(\tilde{N}_c')$ gauge theory with corresponding dual matters. We study the nonsupersymmetric meta-stable minimum and the corresponding intersecting brane configurations. In section 11, we also describe the $\mathcal{N} = 1 \text{SU}(N_c) \times \text{SU}(N'_c)$ gauge theory with antisymmetric tensor, bifundamentals and fundamentals.

In section 12, we make some comments for the future directions.
2 $SU(N_c)$ with two adj. and $N_f$-fund.

2.1 Electric theory

The type IIA supersymmetric electric brane configuration [10] corresponding to $\mathcal{N} = 1$ $SU(N_c)$ gauge theory with two adjoint fields $\Phi$, $\Phi'$ and $N_f$-fundamental flavors $Q, \tilde{Q}$ can be described as follows: one NS5-brane(012345), $k'$ NS5'-branes(012389), $N_c$ D4-branes(01236), and $N_f$ D6-branes(0123789). Let us introduce two complex coordinates [9]

$$v \equiv x^4 + ix^5, \quad w \equiv x^8 + ix^9.$$ 

Since we consider a single NS5-brane, the adjoint field $\Phi$ is massive and can be integrated out. For large coupling in front of the quadratic $\Phi$ term, there is no $\Phi$-dependence in the superpotential. The $N_c$-color D4-branes are suspended between the left NS5-brane and the right NS5’-branes and similarly the $N_f$ D6-branes are located between the left NS5-brane and the right NS5’-branes. The $N_f$-fundamental flavors $Q, \tilde{Q}$ are strings stretching between $N_f$ D6-branes and $N_c$-color D4-branes while the adjoint field $\Phi'$ is related to the fluctuations of $N_c$-color D4-branes in $v$ direction.

Let us deform this theory by adding the mass term and the quartic term for fundamental quarks $Q, \tilde{Q}$ in order to find the new nonsupersymmetric meta-stable states. The former can be achieved by displacing the D6-branes along +$v$ direction leading to their coordinates $v = +v_{D6}$ [9] while the latter can be obtained by rotating the D6-branes [6] by an angle $-\theta$ in $(w, v)$-plane and we denote those rotated D6-branes by $D6_{-\theta}$-branes. Then, in the electric gauge theory, the deformed superpotential is given by

$$W_{elec} = \left[ \frac{g_{W}}{2} \text{tr} \Phi'^{k'+1} + Q\Phi'\tilde{Q} \right] - m \text{tr} Q\tilde{Q} + \frac{\alpha}{2} \text{tr}(Q\tilde{Q})^2$$

$$\alpha = \frac{\tan \theta}{\Lambda}, \quad m = \frac{v_{D6_{-\theta}}}{2\pi \ell_s^2} (2.1)$$

where $\Lambda$ is related to the scales of the electric and magnetic theories and $+v_{D6_{-\theta}}$ is the $v$ coordinate of the center of coincident rotated $D6_{-\theta}$-branes. When $k' = 1$, this theory reduces to the one in [6] [7] because the first two terms of (2.1) contribute to the additional quartic term for the quarks $Q, \tilde{Q}$ and we are left with the last two terms in (2.1). Therefore, we focus on the nontrivial case with the number of NS5’-branes $k' \geq 2$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration [10] with superpotential (2.1) in type IIA string theory as follows and draw this in Figure 1:

\footnote{Without the quartic term ($\alpha = 0$), the theory with $k' = N_c - 1$ obtained by resolving the superpotential(i.e., there exist also the lower order terms by splitting the $k'$ NS5'-branes in $v$ direction) was studied in [16] some time ago in the supersymmetric brane configuration and this theory without D6-branes when $k' \geq N_c + 1$ (multitrace interactions) was also studied in the context of supersymmetry breaking vacua from M5-branes recently [17].}
• One left NS5-brane in (012345) directions with $w = 0$
• $k'$ right NS5'-branes in (012389) directions with $v = 0$
• $N_c$-color D4-branes in (01236) directions with $v = 0 = w$
• $N_f$ $D6_{-\theta}$-branes in (01237) directions and two other directions in $(v, w)$-plane

Figure 1: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $SU(N_c)$ with two adjoints and fundamentals $Q, \tilde{Q}$. Note that there are $k'$ NS5'-branes and the fluctuations of $N_c$ color D4-branes in $v$ direction correspond to the adjoint field. A “rotation” of $N_f$ D6-branes in $(w, v)$-plane, which become $D6_{-\theta}$-branes, corresponds to a quartic term for the fundamentals while a “displacement” of $N_f$ D6-branes in $+v$ direction corresponds to a mass term for the fundamentals.

2.2 Magnetic theory

Let us move the left NS5-brane to the right all the way past the right NS5'-branes and one arrives at the Figure 2A. Note that there exists a creation of $N_f$-flavor D4-branes connecting $N_f$ $D6_{-\theta}$-branes and $k'$ NS5'-branes. Recall that the $N_f$ $D6_{-\theta}$-branes are not parallel to the NS5-brane in Figure 1 unless $\theta = \frac{\pi}{2}$. The linking number \[ \text{18} \] of the NS5-brane from Figure 2A is $l_m = \frac{N_f}{2} - \tilde{N}_c$. On the other hand, the linking number of the NS5-brane from Figure 1 is $l_e = -\frac{N_f}{2} + N_c$. From the equality between these two relations, one obtains the number of colors of dual magnetic theory \[ \text{10} \]

\[ \tilde{N}_c = N_f - N_c. \]

Note that the multiplicity $k'$ of NS5'-branes does not arise in this computation because the creation of flavor D4-branes appears via the dual process between the NS5-brane and $N_f$ $D6_{-\theta}$-branes which do not depend on $k'$.

The low energy theory on the dual color D4-branes has $SU(\tilde{N}_c)$ gauge group and an adjoint field $\phi'$ coming from 4-4 strings connecting the dual color D4-branes and $N_f$-fundamental dual quarks $q, \tilde{q}$ coming from 4-4 strings connecting between the color D4-branes and flavor D4-branes. Moreover, a single magnetic meson field $M \equiv Q\tilde{Q}$ is $N_f \times N_f$ matrix and comes from
Figure 2: The $\mathcal{N}=1$ supersymmetric magnetic brane configuration corresponding to Figure 1 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored (2A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered (2B). The $N_f$ flavor D4-branes connecting between $D6_\theta$-branes and NS5'-branes are splitting into $\tilde N_c, (N_c - l)$- and $l$- D4-branes (2A). Further $n$- D4-branes among $(N_c - l)$- D4-branes are moved to the NS5-brane (2B).

4-4 strings of flavor D4-branes. Then the dual magnetic superpotential is given by

$$W_{\text{mag}} = \left[ \frac{g_{\phi'}}{2} \text{tr} \phi'^{k' + 1} + q\phi'\tilde{q} + \frac{1}{\Lambda}Mq\tilde{q} \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \quad (2.2)$$

The undeformed expression, the first three terms, is already found in [10]. The case with $k' = 1$ leads to the one in [6]. In order to obtain the supersymmetric vacua, one computes the F-term equations for the superpotential (2.2):

$$\frac{1}{\Lambda}Mq + q\phi' = 0, \quad \phi'\tilde{q} + \frac{1}{\Lambda}\tilde{q}M = 0,$$

$$\frac{1}{2}g_{\phi'}(k' + 1)\phi'^{k'} + \tilde{q}q = 0, \quad \frac{1}{\Lambda}\tilde{q}q = m - \alpha M. \quad (2.3)$$

We choose the adjoint field $\phi'$ to be diagonal, i.e., $\phi' = \text{diag}(\phi'_1, \cdots, \phi'_{\tilde N_c})$. Then the first two equations of (2.3) imply that the upper left $\tilde N_c \times \tilde N_c$ block of $\frac{1}{\Lambda}M$ is given by diag($-\phi'_1, \cdots, -\phi'_{\tilde N_c}$). By substituting this into the last equation of (2.3), one obtains the upper left $\tilde N_c \times \tilde N_c$ block of $\frac{1}{\Lambda}q\tilde{q}$ is given by diag($m + \alpha \Lambda \phi'_1, \cdots, m + \alpha \Lambda \phi'_{\tilde N_c}$). Moreover, the lower right $(N_f - \tilde N_c) \times (N_f - \tilde N_c)$ block of $M$ is given by $l$'s zero eigenvalues and $(N_f - \tilde N_c - l)$'s eigenvalues $\frac{m}{\alpha}$. Then finally, the third equation of (2.3) gives rise to the expectation value for the adjoint field $\phi'$ satisfying that

$$\frac{1}{2}g_{\phi'}(k' + 1)\phi'^{k'} = -\Lambda(m + \alpha \Lambda \phi'_j) \quad (2.4)$$

for $j = 1, 2, \cdots, \tilde N_c$. For nonzero quark mass $m$, the expectation value $\phi'_j$ is not vanishing and $-\Lambda \phi'_j > \frac{m}{\alpha}$ for the positiveness of left hand side of (2.4) when $g_{\phi'}$ and $\phi'_j$ are real.
Let us first describe the nonsupersymmetric meta-stable states and supersymmetric ones when all the $N_f$ $D6_{−θ}$-branes and $k'$ NS5′-branes are coincident with each other.

- Coincident $N_f$ $D6_{−θ}$-branes and $k'$ NS5′-branes (equal massive flavors)

One writes $N_f \times N_f$ matrix $M$ with $\tilde{N}_c (= N_f - N_c)$ eigenvalues by the diagonal elements for $\phi'$, $l$’s eigenvalues by the zeros and $(N_c - l)$ eigenvalues $\frac{m}{\alpha} \tilde{N}_c$ as follows:

$$M = \begin{pmatrix} -\Lambda \phi' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{m}{\alpha} \tilde{N}_c \end{pmatrix}, \quad \text{with} \quad \phi' = \text{diag}(\phi'_1, \cdots, \phi'_{\tilde{N}_c}). \quad (2.5)$$

In the brane configuration of Figure 2A, the $l$ of the $N_f$-flavor D4-branes are connected with $l$-color D4-branes and the resulting $l$ D4-branes stretch from the $D6_{−θ}$-branes to the NS5-brane directly and the intersection point between the $l$ D4-branes and the $D6_{−θ}$-branes is given by $(v, w) = (+v_{D6_{−θ}}, 0)$. This corresponds to exactly the $l$’s eigenvalues from zeros of $M$ in (2.5). Now the remaining $(N_c - l)$-flavor D4-branes between the $D6_{−θ}$-branes and the NS5′-branes correspond to the eigenvalues $-\Lambda \phi'$ in (2.5) with (2.4) providing the exact nonzero $w$ coordinates for these flavor D4-branes.

One also represents the vacuum expectation value for the quarks as follows:

$$\tilde{q}q = \begin{pmatrix} \Lambda (m\tilde{N}_c + \alpha \Lambda \phi') & 0 & 0 \\ 0 & \Lambda m \tilde{N}_c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \phi' = \text{diag}(\phi'_1, \cdots, \phi'_{\tilde{N}_c}). \quad (2.6)$$

In the $l$-th vacuum the gauge symmetry is broken to $SU(\tilde{N}_c - l)$ and when the supersymmetric vacuum is drawn in Figure 2A with $l = 0$, the gauge group $SU(\tilde{N}_c)$ is unbroken. Then the supersymmetric ground state corresponds to the vacuum expectation values $M$ by $-\Lambda \phi'$ which has $\tilde{N}_c$’s eigenvalues and $\frac{m}{\alpha}$ with degeneracy $N_c$.

The theory has nonsupersymmetric meta-stable ground states since there exists an attractive gravitational interaction between the flavor D4-branes and the NS5-brane from the DBI action [8]. Let us rescale the meson field as $M = h\Lambda \Phi$. Then the magnetic superpotential (2.2) can be rewritten in terms of $\Phi, q, \tilde{q}$ and $\phi'$

$$W_{\text{dual}} = h\Phi \tilde{q}q + \frac{\mu_\phi}{2} h^2 \text{tr} \Phi^2 - h \mu^2 \text{tr} \Phi + \frac{g}{2} \text{tr} \phi^{k' + 1} + q \phi' \tilde{q} \quad (2.7)$$

with $\mu^2 = m\Lambda$ and $\mu_\phi = \alpha \Lambda^2$. 
Now one splits the \((N_c - l) \times (N_c - l)\) block at the lower right corner of \(M\) and \(q\tilde{q}\) into blocks of size \(n\) and \((N_c - l - n)\) and then (2.5) and (2.6) are rewritten as follows [7]:

\[
\begin{align*}
    h\Phi &= 
            \begin{pmatrix}
            -\phi' & 0 & 0 & 0 \\
            0 & 0 & 0 & 0 \\
            0 & 0 & \mu^2 \mathbb{1}_{N_c-1-n} & 0 \\
            0 & 0 & 0 & \mu^2 \mathbb{1}_{N_c-1-n}
            \end{pmatrix}, \\
    q\tilde{q} &= 
            \begin{pmatrix}
            \mu^2 \mathbb{1}_{N_c} + \alpha \Lambda^2 \phi' & 0 & 0 & 0 \\
            0 & \mu^2 \mathbb{1}_l & 0 & 0 \\
            0 & 0 & \varphi \tilde{\varphi} & 0 \\
            0 & 0 & 0 & \mathbb{1}_{N_c-1-n}
            \end{pmatrix}
    \end{align*}
\]

with \(\phi' = \text{diag}(\phi'_1, \cdots, \phi'_{N_c})\). Here \(\varphi\) and \(\tilde{\varphi}\) are \(n \times (N_c - l)\) matrices and correspond to \(n\)-flavors of fundamentals of the gauge group \(SU(N_c - l)\) which is unbroken. One can move \(n\) D4-branes from \((N_c - l)\) flavor D4-branes stretched between the \(D6_{-\theta}\)-branes and the NS5'-branes at \(w = +v_{D6_{-\theta}}\cot\theta\), to the local minimum of the potential and the end points of these \(n\) D4-branes are at a nonzero \(w\) [6]. In the brane configuration from Figure 2B, \(\varphi\) and \(\tilde{\varphi}\) correspond to fundamental strings connecting between the \(n\)-flavor D4-branes and \((N_c - l)\)-color D4-branes. Moreover, the \(h\Phi_n\) and \(\varphi \tilde{\varphi}\) are \(n \times n\) matrices. The supersymmetric ground state corresponds to the vacuum expectation values by \(h\Phi_n = \frac{\mu^2}{\mu_n} \mathbb{1}_n\) and \(\varphi \tilde{\varphi} = 0\).

The full one loop potential from (2.7) and (2.8) takes the form

\[
\frac{V}{|h|^2} = |\Phi_n \varphi + \varphi \phi'|^2 + |\tilde{\varphi} \Phi_n + \phi' \tilde{\varphi}|^2 + |\varphi \tilde{\varphi} - \mu^2 \mathbb{1}_n + h\mu_n \Phi_n|^2 + b|h\mu|^2 \text{tr} \Phi_n^\dagger \Phi_n
\]

(2.9)

where \(b = \frac{(\ln 4 - 1)}{8\pi^2} N_c\) and we did not include \(\Phi_{n-}\) or \(\Phi_n^\dagger\)-independent term and after differentiating this (2.9) with respect to \(\Phi_n^\dagger\) one obtains the local nonzero stable point given by

\[
h\Phi_n \simeq \frac{\mu_n}{b} \mathbb{1}_n \quad \text{or} \quad M_n \simeq \frac{\alpha \Lambda^3}{N_c} \mathbb{1}_n.
\]

This corresponds to the location of \(w\) coordinate, which is less than \(\frac{m_n}{\alpha \Lambda} = \frac{\mu^2}{\mu_n}\), of \(n\) flavor D4-branes between the \(D6_{-\theta}\)-branes and the NS5'-branes.

- Non-coincident \(N_f\) \(D6_{-\theta}\)-branes and \(k'\) NS5'-branes(different massive flavors)

Let us suppose that the numbers of \(D6_{-\theta}\)-branes and the NS5'-branes are equal to each other: \(N_f = k'\). Let us displace the \(k'\) \(D6_{-\theta}\)-branes and NS5'-branes given in Figure 2A in the \(v\) direction respectively to two \(k'\) different points denoted by \(v_{D6_{-\theta,j}}\) and \(v_{NS5_{j}}\) where \(j = 1, 2, \cdots, k'\). The color and flavor D4-branes attached to them are displaced also. The number of color D4-branes stretched between \(j\)-th \(NS5_{j}\)-brane and the NS5-brane is denoted by \(\tilde{N}_{c,j}\) while the number of flavor D4-branes stretched between the \(j\)-th \(D6_{-\theta,j}\)-brane and
the \( j \)-th \( NS5_j \)-brane is denoted by \( N_{f,j} \). Then it is obvious that there are relations between the color and flavor D4-branes \( \sum_{j=1}^{k'} \tilde{N}_{c,j} = \tilde{N}_c \) and \( \sum_{j=1}^{k'} N_{f,j} = N_f \). When all the \( NS5_j \)-branes and \( D6_{-\theta,j} \)-branes are distinct, the low energy physics corresponds to \( k' \)-decoupled supersymmetric gauge theories with gauge group \( \prod_{j=1}^{k'} SU(\tilde{N}_{c,j}) \).

One deforms the Figure 2B by displacing the multiple \( D6_{-\theta} \)-branes and \( NS5' \)-branes along \( v \) direction, as in \[15\]. Then the \( n \) curved flavor D4-branes attached to them (as well as other D4-branes) are displaced also as \( k' \) different \( n_j \)'s connecting between \( D6_{-\theta,j} \)-brane and \( NS5_j \)-brane. When we rescale the submeson field as \( M_j = h\Lambda \Phi_j \[15\], then the Kahler potential for \( \Phi_j \) is canonical and the magnetic quarks \( q_j \) and \( \tilde{q}_j \) are canonical near the origin of field space \[1\]. Then the magnetic superpotential \((2.7)\) can be rewritten in terms of \( \Phi_j, q_j, \tilde{q}_j \) and \( \phi'_{\tilde{N}_{c,j}} \), which is coming from 4-4 strings connecting between the \( j \)-th \( \tilde{N}_{c,j} \) D4-branes

\[
W_{\text{dual}} = \sum_{j=1}^{k'} \left[ h\Phi_j q_j \tilde{q}_j + \frac{\mu\phi}{2} h^2 \text{tr} \Phi_j^2 - h \text{tr} \mu^2 \Phi_j + \frac{g\phi'}{2} \text{tr} \phi'^{k'+1} \Phi_j + q_j \phi'_{\tilde{N}_{c,j}} \tilde{q}_j \right] \quad (2.10)
\]

with \( \mu_j^2 = m_j \Lambda_j \) and \( \mu\phi = \alpha \Lambda^2 \) as before.

One splits the \((N_{c,j} - l_j) \times (N_{c,j} - l_j)\) block at the lower right corner of \( h\Phi_j \) and \( q_j \tilde{q}_j \) into blocks of size \( n_j \) and \((N_{c,j} - l_j) - n_j\) for all \( j \) as follows \[15\]:

\[
h\Phi = \begin{pmatrix}
0_{\tilde{N}_c+l+n} & 0 & 0 & \cdots & 0 \\
0 & \mu^2_{\Phi,1} & 0_{N_{c,1}-l_1-n_1} & 0 & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \mu^2_{\Phi, k'} & 0 & \cdots \\
0 & 0 & 0 & 0 & \mu_{\Phi, k'} & 0 \\
0 & 0 & 0 & 0 & 0 & 0_{N_{c}-l}
\end{pmatrix}
- \text{diag}(\phi'_{\tilde{N}_{c,1}}, \cdots, \phi'_{\tilde{N}_{c,k'}, 0_{N_{c}}}) + \text{diag}(0_{\tilde{N}_c+l}, h\Phi_{n_1}, \cdots, h\Phi_{n_{k'}}, 0_{N_{c}-l-n}) \quad (2.11)
\]

and

\[
q\tilde{q} = \begin{pmatrix}
0_{\tilde{N}_c} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu^2_1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & \mu^2_{k'} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \mu_{k'} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0_{N_{c}-l}
\end{pmatrix}
+ \text{diag}(\mu^2_1 \tilde{N}_{c,1} + \alpha \Lambda^2 \phi'_{\tilde{N}_{c,1}}, \cdots, \mu^2_{k'} \tilde{N}_{c,k'} + \alpha \Lambda^2 \phi'_{\tilde{N}_{c,k'}}, 0_{N_{c}})
+ \text{diag}(0_{\tilde{N}_c+l}, \varphi_{n_1}, \varphi_{n_{k'}}, \cdots, \varphi_{n_{k'}}, 0_{N_{c}-l-n}) \quad (2.12)
\]

where \( l = \sum_{j=1}^{k'} l_j \) and \( n = \sum_{j=1}^{k'} n_j \). Here \( \varphi_{n_j} \) and \( \varphi_{n_j} \) are \( n_j \times (\tilde{N}_{c,j} - l_j) \) matrices and correspond to \( n_j \)-flavors of fundamentals of the gauge group \( SU(\tilde{N}_{c,j} - l_j) \) which is unbroken. They correspond to fundamental strings connecting between the \( n_j \)-flavor D4-branes and \((\tilde{N}_{c,j} - l_j)\)-color D4-branes. Moreover, the \( \Phi_{n_j} \) and \( \varphi_{n_j} \) are \( n_j \times n_j \) matrices. The
supersymmetric ground state corresponds to the vacuum expectation values by \( h \Phi_{n_j} = \frac{\mu_\phi}{\mu_\phi} 1_{n_j} \) and \( \varphi_{n_j} \tilde{\varphi}_{n_j} = 0 \). The full one loop potential from (2.10), (2.11) and (2.12) can be written similarly and the local nonzero stable point arises as

\[
h \Phi_{n_j} \simeq \frac{\mu_\phi}{b_j} 1_{n_j} \quad \text{or} \quad M_{n_j} \simeq \frac{\alpha \Lambda^3}{N_{c,j}} 1_{n_j}
\]

corresponding to the nonzero \( w \) coordinates of \( n_j \) flavor D4-branes between the \( D6_{-\theta,j} \)-brane and the \( NS5'_j \)-brane.

Therefore, the meta-stable states, for fixed \( k' \) which is related to the order of the adjoint field in the superpotential and \( \theta \) which is a deformation parameter by rotation angle of \( D6_{-\theta} \)-branes, are classified by the number of various D4-branes and the positions of multiple \( D6_{-\theta} \)-branes and \( NS5' \)-branes: \( (N_{c,j}, N_{f,j}, l_j, n_j) \) and \( (v_{D6_{-\theta,j}}, v_{NS5'_j}) \) where \( j = 1, 2, \ldots, k' \).

When \( N_f \) is not equal to \( k' \) (for example, when \( N_f > k' \)), then some coincident \( D6_{-\theta} \)-branes among \( N_f \) \( D6_{-\theta} \)-branes should be connected to the \( NS5' \)-branes in order to connect all the flavor D4-branes between the \( D6_{-\theta} \)-branes and the \( NS5' \)-branes. These coincident \( D6_{-\theta} \)-branes can be obtained by taking those quark masses equal. Then all the previous descriptions for the meta-stable states for the case with \( N_f = k' \) can be applied in this case also without any difficulty. On the other hand, when \( N_f < k' \), then some coincident \( NS5' \)-branes among \( k' \) \( NS5' \)-branes should be connected to the \( D6_{-\theta} \)-branes.

3 \( Sp(N_c) \) with two adj. and \( N_f \)-fund.

3.1 Electric theory

The type IIA supersymmetric electric brane configuration [10, 19, 20] corresponding to \( N = 1 \) \( Sp(N_c) \) gauge theory with two adjoint fields \( \Phi, \Phi' \) (which are symmetric tensors in the symplectic gauge group) and \( N_f \)-fundamental flavors \( Q \) can be described as follows: one NS5-brane, \( (2k' + 1) \) \( NS5' \)-branes, \( 2N_c \) D4-branes, \( 2N_f \) D6-branes and the \( O4^\pm \)-planes(01236). The \( 2N_c \)-color D4-branes are suspended between the left NS5-brane and the right NS5'-branes and the \( 2N_f \) D6-branes are located between the left NS5-brane and the right NS5'-branes. The \( 2N_f \)-fields \( Q \) are strings stretching between \( 2N_f \) D6-branes and \( 2N_c \)-color D4-branes. The adjoint field \( \Phi' \) is related to the fluctuations of color D4-branes in \( v \) direction.

Let us deform this theory by adding the mass and the quartic terms for fundamental quarks. The former can be achieved by displacing the D6-branes along \( \pm v \) direction leading to their coordinates \( v = \pm v_{D6} \) [9] while the latter can be obtained by rotating the D6-branes [6] by an angle \( -\theta \) in \( (w, v) \)-plane and we denote them by \( D6_{-\theta} \)-branes. That is, each \( N_f \)
D6-branes is moving to the $\pm v$ directions respectively and then is rotating by an angle $-\theta$ in $(w, v)$-plane. Therefore, in the electric gauge theory, the deformed superpotential is given by

$$W_{\text{elec}} = \left[ \frac{g \Phi'}{2} \text{tr} \Phi' \Phi'^2 + Q \Phi' Q \right] - m \text{tr} QQ + \frac{\alpha}{2} \text{tr}(QQ)^2, \quad \alpha = \frac{\tan \theta}{\Lambda}, \quad m = \frac{v_{D6-\theta}}{2 \pi \ell_s^2}.$$  \hspace{1cm} (3.1)

When $k' = 0$, this theory reduces to the one similar to the unitary case [6, 7] because the first two terms of (3.1) contribute to the additional quartic term for the quarks. Therefore, we focus on the nontrivial case with $k' \geq 1$. The order of the first term of (3.1) can be determined by replacing the $k'$ of (2.1) with $(2k' + 1)$ in the sense that when the additional $k'$ NS5'-branes are added into a single NS5'-brane, their mirrors also should be present, leading to the total number of NS5'-branes being $(2k' + 1)$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (3.1) in type IIA string theory as follows and draw this in Figure 3 which is nothing but the Figure 1 together with the addition of $O4^\pm$-planes:

- One left NS5-brane in (012345) directions with $w = 0$
- $(2k' + 1)$ right NS5'-branes in (012389) directions with $v = 0$
- $2N_c$-color D4-branes in (01236) directions with $v = 0 = w$
- $2N_f$ D6$_{-\theta}$-branes in (01237) directions and two other directions in $(v, w)$-plane
- $O4^\pm$-planes in (01236) directions with $v = 0 = w$

Figure 3: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $Sp(N_c)$ with two adjoints and fundamentals $Q$. Note that there are $(2k' + 1)$ NS5'-branes and the fluctuations of $2N_c$ color D4-branes in $v$ direction correspond to the adjoint field. A rotation of $N_f$ D6-branes in $(w, v)$-plane corresponds to a quartic term for the fundamentals while a displacement of $N_f$ D6-branes in $\pm v$ direction corresponds to a mass term for the fundamentals.

### 3.2 Magnetic theory

Let us move the left NS5-brane to the right past the right NS5'-branes as in previous section and we arrive at the Figure 4A. Note that there exists a creation of $N_f$-flavor D4-branes
connecting \( N_f \) \( D_{6-\theta} \)-branes and \((2k'+1)\) NS5'-branes (and their mirrors). The linking number of the NS5-brane from Figure 4A is \( l_m = \frac{2N_f}{2} - 2 - 2\tilde{N}_c \). The linking number of the NS5-brane from Figure 3 is \( l_c = \frac{-2N_f}{2} + 2 + 2N_c \). From the equality between these, one obtains the number of colors of dual magnetic theory \([10, 1, 21]\) \( \tilde{N}_c = N_f - N_c - 2 \).

Figure 4: The \( \mathcal{N} = 1 \) supersymmetric magnetic brane configuration corresponding to Figure 3 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored (4A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered (4B). The \( N_f \) upper flavor D4-branes connecting between upper \( D_{6-\theta} \)-branes and NS5'-branes are splitting into \( \tilde{N}_c \)-, \((N_f - \tilde{N}_c - l)\)- and \( l \)- D4-branes (4A). Further \( n \)- D4-branes among upper \((N_f - \tilde{N}_c - l)\)- D4-branes are moved to the NS5-brane (4B).

The low energy theory on the color D4-branes has \( Sp(\tilde{N}_c) \) gauge group and an adjoint field \( \phi' \) coming from 4-4 strings connecting the color D4-branes and \( N_f \)-fundamental quarks \( q \) coming from 4-4 strings connecting between the color D4-branes and flavor D4-branes. Moreover, a single magnetic meson field \( M \equiv QQ \) is \( 2N_f \times 2N_f \) matrix and comes from 4-4 strings of flavor D4-branes. Then the magnetic superpotential is given by

\[
W_{mag} = \left[ \frac{g_{\phi'}}{2} \text{tr} \phi'^{2k'+2} + q\phi'q + \frac{1}{\Lambda} Mqq \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \tag{3.2}
\]

When \( k' = 0 \) and there is no mass term for \( M \), this superpotential reduces to the one found in \([1, 21]\).

In order to obtain the supersymmetric vacua, one computes the F-term equations for the superpotential (3.2):

\[
\frac{1}{\Lambda} Mq + q\phi' = 0, \quad g_{\phi'}(k' + 1)\phi'^{2k'+1} + qq = 0, \quad \frac{1}{\Lambda} qq = m - \alpha M. \tag{3.3}
\]

We choose the adjoint field \( \phi' \) to be diagonal, i.e., \( \phi' = \text{diag}(\phi'_1, \cdots, \phi'_{\tilde{N}_c}) \otimes \sigma_3 \) \([19]\). Then the first equation of (3.3) implies that the upper left \( 2\tilde{N}_c \times 2\tilde{N}_c \) block of \( \frac{M}{\Lambda} \) is given by
diagonal of $D$ between the $v, w$ branes and the NS5-brane is given by $(\lambda, \phi)$.

Moreover, the lower right $2(N_f - \bar{N}_c) \times 2(N_f - \bar{N}_c)$ block of $M$ is given by $2l$'s zero eigenvalues and $2(N_f - \bar{N}_c - l)$'s eigenvalues $\pm \frac{m}{\alpha}$. Then finally, the second equation of (3.3) gives rise to the expectation value for the adjoint field $\phi'$ satisfying

$$g_{\phi'}(k' + 1)\phi_j^{2k' + 1} = -\Lambda(m + \alpha \Lambda \phi_j')$$

(3.4)

for $j = 1, 2, \ldots, \bar{N}_c$. For nonzero quark mass $m$, the expectation value $\phi_j'$ is not vanishing and $-\Lambda \phi_j' > \frac{m}{\alpha}$ for the positiveness of left hand side of (3.4) when $g_{\phi'}$ and $\phi_j'$ are real.

Let us first describe the nonsupersymmetric meta-stable states and supersymmetric ones when all the $2N_f$ $D6_{-\theta}$-branes and $(2k' + 1)$ NS5'-branes are coincident with each other.

- Two coincident $N_f$ $D6_{-\theta}$-branes and coincident $(2k' + 1)$ NS5'-branes

One writes $2N_f \times 2N_f$ matrix $M$ with $2\bar{N}_c$ eigenvalues by the diagonal elements for $\pm \varphi'$, $2l$'s eigenvalues by the zeros and $2(N_f - \bar{N}_c - l)$ eigenvalues $\pm \frac{m}{\alpha}$ as follows:

$$M = \begin{pmatrix}
-\Lambda \varphi' \otimes i\sigma_2 & 0 & 0 \\
0 & 0_{2l} & 0 \\
0 & 0 & \frac{m}{\alpha} 1_{N_f - \bar{N}_c - l} \otimes i\sigma_2
\end{pmatrix}, \quad \varphi' = \text{diag}(\varphi_1', \ldots, \varphi_{\bar{N}_c}')$$

(3.5)

Therefore, in the brane configuration of Figure 4A, the $l$ of the upper $N_f$ flavor D4-branes are connected with $l$ of $\tilde{N}_c$ color D4-branes and the resulting D4-branes stretch from the upper $D6_{-\theta}$-branes to the NS5-brane directly and the intersection point between the $l$ upper D4-branes and the NS5-brane is given by $(v, w) = (+v_{D6_{-\theta}}, 0)$. Now the $(N_f - \bar{N}_c - l)$ upper flavor D4-branes between the $D6_{-\theta}$-branes and the NS5'-branes are related to the corresponding half eigenvalues of $M: \frac{m}{\alpha} 1_{N_f - \bar{N}_c - l}$. The intersection point between the $(N_f - \bar{N}_c - l)$ upper D4-branes and the NS5'-branes is given by $(v, w) = (0, +v_{D6_{-\theta}} \cot \theta)$ corresponding to positive eigenvalues of $M$. Finally, the remnant $2\bar{N}_c$-flavor D4-branes between the $D6_{-\theta}$-branes and the NS5'-branes correspond to the eigenvalues $\pm \varphi'$ in (3.5) with (3.4) which shows the exact coordinate of $w$ for these flavor D4-branes.

One represents the vacuum expectation value for the quarks as follows:

$$qq = \begin{pmatrix}
\Lambda(m 1_{\bar{N}_c} + \alpha \Lambda \varphi') \otimes i\sigma_2 & 0 & 0 \\
0 & \Lambda m 1_{l} \otimes i\sigma_2 & 0 \\
0 & 0 & 0_{2(N_f - \bar{N}_c - l)}
\end{pmatrix}, \varphi' = \text{diag}(\varphi_1', \ldots, \varphi_{\bar{N}_c}')$$

(3.6)

In the $l$-th vacuum the gauge symmetry is broken to $Sp(\bar{N}_c - l)$ and when the supersymmetric vacuum is drawn in Figure 4A with $l = 0$, the gauge group $Sp(\bar{N}_c)$ is unbroken. Then the
supersymmetric ground state corresponds to the vacuum expectation values $M$ by $\pm \varphi'$ which has $2\tilde{N}_c$'s eigenvalues and $\pm \frac{m}{\alpha}$, with degeneracy $2(N_f - \tilde{N}_c)$.

Let us rescale the meson field as $M = h\Lambda \Phi$. Then the magnetic superpotential (3.2) can be rewritten in terms of $\Phi, q$ and $\varphi'$

$$W_{dual} = h\Phi q q + \frac{\mu_\varphi}{2} h^2 \text{tr} \Phi^2 - h\mu^2 \text{tr} \Phi + \frac{g_{\varphi'}}{2} \text{tr} \varphi'^2 + q \varphi' q$$

(3.7)

where $\mu^2 = m\Lambda$ and $\mu_\varphi = \alpha\Lambda^2$.

Now one splits the $2(N_f - \tilde{N}_c - l) \times 2(N_f - \tilde{N}_c - l - n)$ block at the lower right corner of $M$ and $qq$ into blocks of size $2n$ and $2(N_f - \tilde{N}_c - l - n)$ and then (3.5) and (3.6) are rewritten as follows:

$$h\Phi = \begin{pmatrix}
-\varphi' \otimes i\sigma_2 & 0 & 0 & 0 \\
0 & 0 & l & 0 \\
0 & 0 & h\Phi_{2n} & 0 \\
0 & 0 & 0 & \mu_\varphi^2 1_{(N_f - \tilde{N}_c - l - n)} \otimes i\sigma_2
\end{pmatrix},$$

$$qq = \begin{pmatrix}
(\mu^2 1_{\tilde{N}_c} + \alpha\Lambda^2 \varphi') \otimes i\sigma_2 & 0 & 0 & 0 \\
0 & 0 & \mu^2 1_l \otimes i\sigma_2 & 0 \\
0 & 0 & 0 & \varphi \varphi \\
0 & 0 & 0 & 0_{2(N_f - \tilde{N}_c - l - n)}
\end{pmatrix}.$$

Here $\varphi$ is $2n \times 2(\tilde{N}_c - l)$ dimensional matrix and corresponds to $2n$ flavors of fundamentals of the gauge group $Sp(\tilde{N}_c - l)$ which is unbroken by the nonzero expectation value of $q$. The $\Phi_{2n}$ and $\varphi \varphi$ are $2n \times 2n$ matrices. The supersymmetric ground state corresponds to the vacuum expectation values by $h\Phi_{2n} = \frac{\mu^2}{\mu_\varphi} 1_n \otimes i\sigma_2$ and $\varphi = 0$.

The full one loop potential takes the form

$$\frac{V}{|h|^2} = |\Phi_{2n} \varphi + \varphi \varphi'|^2 + |\varphi \varphi - \mu^2 1_{2n} + h\mu_\varphi \Phi_{2n}|^2 + b|h\mu|^2 \text{tr} \Phi_{2n} \Phi_{2n}$$

where $b = \frac{(n+4-1)\tilde{N}_c}{8\pi^2}$ and we did not include $\Phi_{2n}$-independent term and after differentiating this with respect to $\Phi_{2n}$ one obtains the local nonzero stable point given by

$$h\Phi_{2n} \simeq \frac{\mu_\varphi}{b} 1_n \otimes i\sigma_2 \quad \text{or} \quad M_{2n} \simeq \frac{\alpha\Lambda^3}{\tilde{N}_c} 1_n \otimes i\sigma_2$$

corresponding to the location of $w$ coordinates for $n$ flavor D4-branes, very close to the NS5-brane, between the $D6_{-\theta}$-branes and the NS5'-branes.

- Non-coincident $D6_{-\theta}$-branes and NS5'-branes

Let us consider when the numbers of $D6_{-\theta}$-branes and the number of NS5'-branes minus one are equal: $N_f = k'$. We displace the $k'$ upper $D6_{-\theta}$-branes and upper NS5'-branes, given
in Figure 4A, in the $+v$ direction respectively to two $k'$ different points denoted by $v_{D6_{-\theta,j}}$ and $v_{NS5_j'}$ where $j = 1, 2, \cdots, k'$ (and their mirrors in the $-v$ direction). A single NS5$'$-brane is located at $v = 0$. Then there are $\sum_{j=1}^{k'} N_{c,j} (= N_{f,j} - N_{c,j} - 2) = \tilde{N}_c$ and $\sum_{j=1}^{k'} N_{f,j} = N_f$. When all the upper NS5$'$-branes and $D6_{-\theta,j}$-branes ($j = 1, 2, \cdots, k'$) are distinct, the low energy physics corresponds to $k'$ decoupled supersymmetric gauge theories with gauge groups $\prod_{j=1}^{k'} Sp(\tilde{N}_{c,j})$. One deforms the Figure 4B by displacing the multiple $D6_{-\theta}$-branes and NS5$'$-branes along $v$ direction [13]. Then the $n$ curved flavor $D4$-branes attached to them as well as other $D4$-branes are displaced also as $k'$ different $n_j$'s connecting between $D6_{-\theta,j}$-brane and NS5$'$-brane. Then the magnetic superpotential (3.7) can be rewritten in terms of $\Phi_j, q_j$ and $\phi_{\tilde{N}_{c,j}}'$ which is coming from 4-4 strings connecting between the $j$-th $\tilde{N}_{c,j} D4$-branes

$$W_{\text{dual}} = \sum_{j=1}^{k'} \left[ h \Phi_j q_j \phi_{\tilde{N}_{c,j}}' + \mu_{\phi} \frac{h^2}{2} \text{tr} \Phi_j^2 - h \text{tr} \mu_{\phi}^2 \Phi_j + \frac{g_{\phi}}{2} \text{tr} \phi_{\tilde{N}_{c,j}}'^{2k'+2} + q_j \phi_{\tilde{N}_{c,j}}' q_j \right]$$

(3.8)

with $\mu_j^2 = m_j \Lambda_j$ and $\mu_{\phi} = \alpha \Lambda^2$ as before.

One splits the $2(N_{f,j} - \tilde{N}_{c,j} - l_j) \times 2(N_{f,j} - \tilde{N}_{c,j} - l_j)$ block at the lower right corner of $h \Phi_j$ and $q_j q_j$ into blocks of size $2n_j$ and $2(N_{f,j} - \tilde{N}_{c,j} - l_j - n_j)$ for all $j$ as follows:

$$h \Phi = \begin{pmatrix} 0_{2(\tilde{N}_c+l+n)} & \mu_{\phi}^2 1_{N_{f,1}-\tilde{N}_{c,1}-l_1-n_1} \otimes i \sigma_2 & 0 & \cdots & 0 \\ 0 & \mu_{\phi}^2 1_{N_{f,1}-\tilde{N}_{c,1}-l_1-n_1} \otimes i \sigma_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \mu_{\phi}^2 1_{N_{f,k'}-\tilde{N}_{c,k'}-l_{k'}-n_{k'}} \otimes i \sigma_2 \end{pmatrix}$$

$$- \text{diag}(\phi_{\tilde{N}_{c,1}}', \cdots, \phi_{\tilde{N}_{c,k'}}', 0_{N_{f,\tilde{N}_{c}}}) \otimes i \sigma_2$$

$$+ \text{diag}(0_{2(\tilde{N}_c+l)}), h \Phi_{2n_1}, \cdots, h \Phi_{2n_{k'}}, 0_{2(N_{f,\tilde{N}_{c}}-l-n)}$$

(3.9)

and

$$qq = \begin{pmatrix} 0_{2\tilde{N}_c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_{\phi}^2 1_{2l_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \mu_{\phi}^2 1_{2l_{k'}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0_{2(N_{f,\tilde{N}_{c}}-l)} \end{pmatrix}$$

$$+ \text{diag}(\mu_{\phi}^2 1_{\tilde{N}_{c,1}} + \alpha \Lambda^2 \phi_{\tilde{N}_{c,1}}', \cdots, \mu_{\phi}^2 1_{\tilde{N}_{c,k'}} + \alpha \Lambda^2 \phi_{\tilde{N}_{c,k'}}', 0_{N_{f,\tilde{N}_{c}}}) \otimes i \sigma_2$$

$$+ \text{diag}(0_{2(\tilde{N}_c+l)}, \varphi_{2n_1}, \varphi_{2n_1}, \cdots, \varphi_{2n_{k'}}, \varphi_{2n_{k'}}, 0_{2(N_{f,\tilde{N}_{c}}-l-n)})$$

(3.10)

where $l = \sum_{j=1}^{k'} l_j$ and $n = \sum_{j=1}^{k'} n_j$. Here $\varphi_{2n_j}$ is $2n_j \times 2(\tilde{N}_{c,j} - l_j)$ matrices and correspond to $2n_j$-flavors of fundamentals of the gauge group $Sp(\tilde{N}_{c,j} - l_j)$ which is unbroken. The supersymmetric ground state corresponds to the vacuum expectation values by $h \Phi_{2n_j} = \frac{\mu_{\phi}^2}{\mu_{\phi}} 1_{n_j} \otimes i \sigma_2$
and $\varphi_{2n_j} = 0$. The full one loop potential from (3.8), (3.9) and (3.10) can be written similarly and the local nonzero stable point arises as

$$h\Phi_{2n_j} \simeq \frac{\mu_0}{b_j} 1_{n_j} \otimes \sigma_2 \quad \text{or} \quad M_{2n_j} \simeq \frac{\alpha \Lambda^3}{N_{c,j}} 1_{n_j} \otimes \sigma_2$$

corresponding to the $w$ coordinates of $n_j$ flavor D4-branes between the $D6_{-\theta,j}$-brane and the $NS5'_j$-brane(and their mirrors).

Therefore, the meta-stable states, for fixed $k'$ and $\theta$, are classified by the number of various D4-branes and the positions of multiple $D6_{-\theta}$-branes and $NS5'$-branes: When $N_f$ is not equal to $k'$ (for example, when $N_f > k'$), then some coincident $D6_{-\theta}$-branes among $N_f$ $D6_{-\theta}$-branes should be connected to the $NS5'$-branes in order to connect all the flavor D4-branes between the $D6_{-\theta}$-branes and the $NS5'$-branes.

4 \hspace{1em} SU(N_c) with $N_f$-fund. and a symm.

4.1 Electric theory

The type IIA supersymmetric electric brane configuration [22, 23, 9] corresponding to $\mathcal{N} = 1$ SU($N_c$) gauge theory with a symmetric tensor $S$, a conjugate symmetric tensor $\tilde{S}$, $N_f$-fundamental flavors $Q, \tilde{Q}$ can be described as follows: one middle NS5-brane, $2k'$ $NS5'$-branes, $N_c$-color D4-branes, $2N_f$ D6-branes and the $O6^+$-plane(0123789). The $N_c$-color D4-branes are suspended between the left $k'$ $NS5'$-branes and the right $k'$ $NS5'$-branes and similarly each $N_f$ D6-branes is located between the middle NS5-brane and the $NS5'$-branes.

Let us deform this theory [24] by adding the mass and the quartic terms for fundamental quarks. The former can be achieved by displacing the D6-branes along $\pm v$ direction leading to their coordinates $v = \pm v_{D6}$ [9] while the latter can be obtained by rotating the D6-branes [6] by the angles $\mp \theta$ in $(w, v)$-plane and we denote them by $D6_{\mp \theta}$-branes. Therefore, in the electric gauge theory, the deformed superpotential is given by

$$W_{\text{elec}} = \frac{\alpha}{2} \text{tr}(Q \tilde{Q})^2 - m \text{tr} Q \tilde{Q} + \left[ -\frac{\beta}{2} \text{tr}(S \tilde{S})^{k'+1} + m_S \text{tr} S \tilde{S} \right]. \quad (4.1)$$

Here the $\alpha = \tan \frac{\theta}{\Lambda}$ and $m = \frac{v_{D6_{-\theta}}}{2\pi n_z}$ correspond to the rotation and displacement of D6-branes and they are the same as the ones in [12]. The last two terms in (4.1) are due to the rotation of NS5'-branes where $\beta = \tan \omega$ and $\omega$ is a rotation angle of NS5'-branes in the $(w, v)$-plane and the relative displacement of D4-branes where $m_S = v_{NS5'}$ is the distance of D4-branes in $v$ direction. When $k' = 1$ with small $\beta$ limit and $m_S = 0$, this theory reduces to the one [12].
Therefore, we focus on the case with \( k' \geq 2 \) and \( \beta, m_S \to 0 \), i.e., we consider the multiple NS5'-branes with the electric superpotential (4.1) with this particular limit.

Let us summarize the \( \mathcal{N} = 1 \) supersymmetric electric brane configuration with superpotential (4.1) in type IIA string theory as follows and draw this in Figure 5:

- One NS5-brane in (012345) directions with \( w = 0 = x^6 \)
- \( 2k' \) NS5'-branes in (012389) directions with \( v = 0 \)
- \( N_c \) color D4-branes in (01236) directions with \( v = 0 = w \)
- \( N_f \) D6\(_{\pm\theta}\)-branes in (01237) directions and two other directions in \((v, w)\)-plane
- \( O6^+\)-plane in (0123789) directions with \( x^6 = 0 = v \)

Figure 5: The \( \mathcal{N} = 1 \) supersymmetric electric brane configuration for the gauge group \( SU(N_c) \) with a symmetric tensor and fundamentals \( Q, \tilde{Q} \). Note that there are multiple \( k' \) NS5'-branes.

A rotation of \( N_f \) D6-branes in \((w, v)\)-plane, denoted by \( D6\_{\pm\theta}\)-branes, corresponds to a quartic term for the fundamentals while a displacement of \( N_f \) D6-branes in \( \pm v \) direction corresponds to a mass term for the fundamentals.

### 4.2 Magnetic theory

We move the \( D6_{\pm\theta}\)-branes and the left and right NS5'-branes through each other and use the linking numbers for the computation of creation of D4-branes as in [23]. Let us take the “extra” left- and right-(\( k' - 1 \)) NS5'-branes, compared with the single NS5'-brane case [23, 12], move them to the origin \( x^6 = 0 \), and rotate them by an angle \( \frac{\pi}{2} \) in \((w, v)\)-plane, coinciding with the middle NS5-brane. Temporarily, there exist \( 2k' - 1 \) middle NS5-branes at \( x^6 = 0 \). Let us move the left \( D6_{\theta}\)-branes to the right all the way(and their mirrors, right \( D6_{-\theta}\)-branes to the left) past the NS5-branes and the right single NS5'-brane. Then the linking number \( l_m \) of a \( D6_{\theta}\)-brane becomes \( l_m = \frac{1}{2} - n_{4L} \) while the one of the same \( D6_{\theta}\)-brane \( l_e = -\frac{1}{2} \) in the electric theory. Then the number of D4-branes to the left of this \( D6_{\theta}\)-brane \( n_{4L} \) in the magnetic theory becomes 1 and we must add \( N_f \) D4-branes to the left side of all \( N_f \) \( D6_{\theta}\)-branes(and their mirrors). Note that at the \( x^6 = 0 \), the \( D6_{\pm\theta}\)-branes become the \( D6_{\pm\frac{\theta}{2}}\)-branes instantaneously [23, 12].
Then the extra $2(k'-1)$ middle NS5-branes, which were left- and right-$(k'-1)$ NS5'-branes in the electric theory, are moving to $\pm x^6$ direction by performing the remaining dual process and rotating by an angle $\pi/2$ in $(w, v)$-plane. This leads to the left $k'$ NS5'-branes and the right $k'$ NS5'-branes which look similar to the brane configuration of electric theory but we need to further take Seiberg dual for the remaining single NS5'-brane here. In this process there is a creation of D4-branes because the O6-plane, which has 4 D6-brane charge, is not parallel to these $2(k'-1)$ NS5-branes at $x^6 = 0$. Finally, after moving the left NS5'-brane, which does not participate in the dual process so far, to the right all the way past O6-plane(and its mirror, the right NS5'-brane to the left), the linking number of the NS5'-brane can be computed as $l_m = \frac{N_f}{2k'} + \frac{4(k'-1)}{2k'} - \frac{N_c}{k'} + \frac{N_f}{k'}$. Note that the fractional contribution $\frac{k'-1}{k'}$ in the second term is due to the fact that only the left $(k'-1)$ NS5'-branes among $k'$ NS5'-branes in the electric theory are crossing the O6-plane with an angle. Of course, the remaining left single NS5'-brane among $k'$ NS5'-branes in the electric theory is crossing the $N_f$ D6-θ-branes with an angle. Originally, the linking number was given by $l_c = -\frac{N_f}{2k'} - \frac{4(k'-1)}{2k'} + \frac{N_c}{k'}$. This leads to the fact that the number of D4-branes becomes $\tilde{N}_c = 2N_f - N_c + 4(k'-1)$. Here the last constant term which depends on the number of NS5'-branes is due to the fact that the “extra” NS5'-branes are crossing the O6-plane. Of course, $k' = 1$ case reduces to the known number of D4-branes $\tilde{N}_c = 2N_f - N_c$ [23]. We present the Figure 6A.

Figure 6: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 5 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(6A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(6B). The $N_f$ flavor D4-branes connecting between $D6_{-\theta}$-branes and NS5'-branes are splitting into $(N_f - l)$- and $l$- D4-branes(6A). Further $n$- D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane(6B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c)$ gauge group and $N_f$-fundamental dual quarks $q, \bar{q}$ coming from 4-4 strings connecting between the color D4-branes and flavor D4-branes and a symmetric tensor and a conjugate symmetric tensor $s, \bar{s}$ coming from 4-4 strings connecting between the color D4-branes with $x^6 < 0$ and the color D4-branes with
Moreover, a single magnetic meson field $M \equiv Q \tilde{Q}$ is $N_f \times N_f$ matrix and comes from 4-4 strings of flavor D4-branes. Then the magnetic superpotential with the limit $\beta, m_S \to 0$ is given by

$$W_{mag} = \frac{1}{\Lambda} M q(\tilde{s}s)^{k'} \tilde{q} + \frac{\alpha}{2} \text{tr} M^2 - m \text{tr} M.$$  (4.2)

The case where $k' = 1$ and $\alpha = 0$ leads to the previous result in [23]. Note that the number of color $\tilde{N}_c$ depends on $k'$. In general, there are also different kinds of meson fields $M_j = Q(\tilde{S}S)^{i} \tilde{Q}, P_r = Q(\tilde{S}S)^{\gamma} \tilde{S}Q$ and $\tilde{P}_r = \tilde{Q}S(\tilde{S}S)^{r} \tilde{Q}$ where $j = 1, 2, \cdots, k'$ and $r = 0, 1, \cdots, k' - 1$ for general rotation angles of multiple NS5'-branes. The magnetic superpotential contains the interaction between these meson fields with $q, \tilde{q}, s$ and $\tilde{s}$ [24] as well as $(s\tilde{s})^{k+1}$ which will vanish for small $\beta$ limit. However, the particular route we take above from an electric theory to the magnetic theory does not produce these meson fields because the $N_f$ $D6_{\pm\theta}$-branes and the extra $2(k' - 1)$ NS5'-branes do not create the D4-branes corresponding to those meson fields.

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (4.2) and the expectation values for $M$ and $q\tilde{s}\tilde{q}$ are obtained since this superpotential is the same as the one in [12]. Let us consider two different cases.

- Coincident $N_f$ $D6_{\pm\theta}$-branes and $k'$ NS5'-branes (and their mirrors)

In this case, all the discussions given in [12] are satisfied with one exception that the number of colors here is different from the one in [12]. The theory has many nonsupersymmetric meta-stable ground states and when we rescale the meson field as $M = h\Lambda \Phi$, then the Kahler potential for $\Phi$ is canonical and the magnetic quarks are canonical near the origin of field space [1]. Then the magnetic superpotential can be written in terms of $\Phi$ or $M$

$$W_{mag} = h \Phi q(\tilde{s}s)^{k'} \tilde{q} + \frac{h^2 \mu_{s}}{2} \text{tr} \Phi^2 - h \mu_{s}^2 \text{tr} \Phi.$$

Now one splits the $(N_f - l) \times (N_f - l)$ block at the lower right corner of $h \Phi$ and $q\tilde{s}\tilde{q}$ into blocks of size $n$ and $(N_f - l - n)$ as follows [12]:

$$h \Phi = \begin{pmatrix} 0_l & 0 & 0 \\ 0 & h \Phi_n & 0 \\ 0 & 0 & \mu_{s}^2 \frac{1}{n_{\phi}} \mathbf{1}_{N_f - l - n} \end{pmatrix}, \quad q(\tilde{s}s)^{k'} \tilde{q} = \begin{pmatrix} \mu_{s}^2 \mathbf{1}_l & 0 & 0 \\ 0 & \varphi \beta \tilde{\varphi} & 0 \\ 0 & 0 & 0_{N_f - l - n} \end{pmatrix}.$$  

Here $\varphi$ and $\tilde{\varphi}$ are $n \times (\tilde{N}_c - l)$ dimensional matrices and correspond to $n$ flavors of fundamentals of the gauge group $SU(\tilde{N}_c - l)$ which is unbroken. In the brane configuration shown in Figure 6B, they correspond to fundamental strings connecting the $n$ flavor D4-branes and $(\tilde{N}_c - l)$ color D4-branes [12]. The $\Phi_n$ and $\varphi \beta \tilde{\varphi}$ are $n \times n$ matrices. The supersymmetric ground state corresponds to $h \Phi_n = \mu_{s}^2 \mathbf{1}_n$ and $\varphi \tilde{\varphi} = 0 = \beta \tilde{\varphi}$.
Now the full one loop potential for $\Phi_n, \hat{\varphi} \equiv \varphi \beta$ and $\hat{\varphi} \equiv \beta \tilde{\varphi}$ [23] takes the form

$$
\frac{V}{|h|^2} = |\Phi_n \hat{\varphi}|^2 + |\Phi_n \hat{\varphi}'|^2 + \mu^2 |\mu_n|^2 + b|\mu|^2 \text{tr} \Phi_n^\dagger \Phi_n,
$$

where $b = \frac{(\ln n - 4)}{8\pi^2} \tilde{N}_c$. Differentiating this potential with respect to $\Phi_n^\dagger$ and putting $\hat{\varphi} = 0 = \tilde{\varphi}$, one obtains

$$
h\Phi_n \simeq \frac{\mu_n}{b} \cdot 1_n \quad \text{or} \quad M_n \simeq \frac{\alpha \Lambda^3}{N_c} 1_n
$$

corresponding to the $w$ coordinates of $n$ curved flavor D4-branes between the $D6_{-\theta}$-branes and the NS5'-branes (and their mirrors).

- Non-coincident $D6_{-\theta}$-branes and NS5'-branes

Let us consider the case where the numbers of $D6_{-\theta}$-branes and the NS5'-branes are equal to each other: $N_f = N_{f'}$. We displace the $k'$ $D6_{-\theta}$-branes and NS5'-branes given in Figure 6A in the $v$ direction respectively to two $k'$ different points denoted by $v_{D6_{-\theta}, j}$ and $v_{NS5'}$ where $j = 1, 2, \cdots, k'$, as in section 2. There are relations between the color and flavor D4-branes as follows: $\sum_{j=1}^{k'} \tilde{N}_{c,j} = \tilde{N}_c$ and $\sum_{j=1}^{k'} N_{f,j} = N_f$.

One deforms the Figure 6B by displacing the multiple $D6_{-\theta}$-branes and NS5'-branes along $v$ direction. Then the $n$ curved flavor D4-branes attached to them are displaced also as $k'$ different $n_j$'s connecting between $D6_{-\theta}, j$-brane and NS5'$j$-brane. Let us rescale the submeson field as $M_j = h\Lambda \Phi_j$ [15] and the Kahler potential for $\Phi_j$ is canonical and the magnetic quarks $q_j$ and $\tilde{q}_j$ as well as $s_j$ and $\tilde{s}_j$ are canonical near the origin of field space [1]. Then the magnetic superpotential can be rewritten in terms of $\Phi_j, q_j, \tilde{q}_j, s_j$ and $\tilde{s}_j$

$$
W_{mag} = \sum_{j=1}^{k'} \left[ h\Phi_j q_j (\tilde{s}_j s_j)^{k'} \tilde{q}_j + \frac{h^2 \mu^2}{2 \text{tr} \Phi_j^2 - h \mu_j^2 \text{tr} \Phi_j} \right].
$$

with $\mu_j^2 = m_j \Lambda^2$ and $\mu \Phi = \alpha \Lambda^2$ as before.

One splits the $(N_{f,j} - l_j) \times (N_{f,j} - l_j)$ block at the lower right corner of $h\Phi_j$ and $q_j \tilde{s}_j s_j \tilde{q}_j$ into blocks of size $n_j$ and $(N_{f,j} - l_j - n_j)$ for all $j$ as follows [15]:

$$
h\Phi_n = \left( \begin{array}{cccccc} 0_{l+n} & 0 & 0 & \cdots & 0 \\ 0 & \mu_n \frac{1}{\mu} 1_{N_{f,1} - l_1 - n_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \mu_n \frac{1}{\mu} 1_{N_{f,f'} - l_{f'} - n_{f'}} \end{array} \right) + \text{diag}(0_l, h\Phi_{n_1}, \cdots, h\Phi_{n_{f'}}, 0_{N_f - l_n})
$$
and

\[ q(\tilde{s}s)^{k'} \tilde{q} = \begin{pmatrix}
\mu_2^2 l_1 & 0 & 0 & \cdots & 0 & 0 \\
. & . & . & . & . & . \\
0 & 0 & 0 & \cdots & \mu_2^2 l_{k'} & 0 \\
0 & 0 & 0 & 0 & 0 & 0_{N_f-l}
\end{pmatrix}
+ \text{diag}(0_{l}, \varphi_{n_1}, \tilde{\beta}_{n_1} \beta_{n_1}, \cdots, \varphi_{n_{k'}}, \tilde{\beta}_{n_{k'}} \beta_{n_{k'}}, 0_{N_f-l-n})\]

where \( l = \sum_{j=1}^{k'} l_j \) and \( n = \sum_{j=1}^{k'} n_j \), as before. Here \( \varphi_{n_j} \) and \( \tilde{\varphi}_{n_j} \) are \( n_j \times (\tilde{N}_{c,j} - l_j) \) matrices and correspond to \( n_j \)-flavors of fundamentals of the gauge group \( SU(\tilde{N}_{c,j} - l_j) \) which is unbroken. The supersymmetric ground state corresponds to the vacuum expectation values by \( h \Phi_{n_j} = \mu_2^2 l_{n_j} \) and \( \varphi_{n_j} \tilde{\beta}_{n_j} = \beta_{n_j} \tilde{\varphi}_{n_j} = 0 \). The full one loop potential can be written similarly and the local nonzero stable point arises as

\[ h \Phi_{n_j} \simeq \frac{\mu_2}{b_j} 1_{n_j} \quad \text{or} \quad M_{n_j} \simeq \frac{\alpha \Lambda^3}{\tilde{N}_{c,j}} 1_{n_j} \]

corresponding to the nonzero \( w \) coordinates of \( n_j \) flavor D4-branes between the \( D6_{-\theta,j} \)-brane and the \( NS5'_{j} \)-brane.

One can think of the brane configuration consisting of multiple outer NS5-branes as well as a single NS5'-brane, \( O6^{\pm} \)-planes and eight D6-branes. So we focus on the case where there is only a single outer NS5-brane [26, 25, 27, 28] when we discuss the other brane configurations containing this sub-brane configuration with NS5-brane, NS5'-brane, \( O6^{\pm} \)-planes and eight D6-branes later.

5. \( SU(N_c) \times SU(N'_c) \) with \( N_f \)- and \( N'_f \)-fund. and bifund.

5.1 Electric theory

The type IIA supersymmetric electric brane configuration [29, 30, 31, 32, 33] corresponding to \( \mathcal{N} = 1 \) \( SU(N_c) \times SU(N'_c) \) gauge theory with \( N_f \)-fundamental flavors \( Q, \bar{Q}, N'_f \)-fundamental flavors \( Q', \bar{Q}' \) and bifundamentals \( X, \bar{X} \) can be described as follows: one middle NS5-brane, 2\( k' \) NS5'-branes, \( N_c \)- and \( N'_c \)-D4-branes, and \( N_f \)- and \( N'_f \)-D6-branes. The \( N_c \)-color D4-branes are suspended between the middle NS5-brane and the right NS5'-branes, the \( N'_c \)-color D4-branes are suspended between the left NS5'-branes and the middle NS5-brane, the \( N_f \) D6-branes are located between the middle NS5-brane and the right NS5'-branes and the \( N'_f \) D6-branes are located between the left NS5'-branes and the middle NS5-brane.

Let us deform this theory by adding the mass term and the quartic term for fundamental quarks. The former can be achieved by displacing the D6-branes along +\( v \) direction leading
to their coordinates $v = +v_{D6}$ while the latter can be obtained by rotating the D6-branes by an angle $-\theta$ in $(w, v)$-plane and we denote them by $D6_{-\theta}$-branes. Then, in the electric gauge theory, the general deformed superpotential is given by

$$W_{elec} = \frac{\alpha}{2} \text{tr}(Q\tilde{Q})^2 - m \text{tr} Q\tilde{Q} + \frac{\alpha'}{2} \text{tr}(Q'\tilde{Q}')^2 - m' \text{tr} Q'\tilde{Q}' + \left[-\frac{\beta}{2} \text{tr}(X\tilde{X})^{k'+1} + m_X \text{tr} X\tilde{X}\right].$$

The last two terms are due to the rotation of NS5'-branes where $\beta = (\tan \omega_L + \tan \omega_R)$ and the relative displacement of D4-branes where the mass $m_X = v_{NS5'}$ is the distance of D4-branes in $v$ direction. When there are no D6-branes, this theory reduces to the one and the case where there exists further restriction on the number of NS5'-branes $k' = 1$ has been studied in [14]. We focus on the case with $k' \geq 2$ and the limit $\beta, m_X \to 0$. When we take the Seiberg dual for the gauge group $SU(N_c)$, we put $\alpha' = 0$ and $m' = 0$ (no displacement and no rotation of $N_f'$ D6-branes) while for the Seiberg dual for the gauge group $SU(N'_c)$, we take $\alpha = 0$ and $m = 0$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (5.1) in type IIA string theory as follows and draw this in Figure 7:

- One middle NS5-brane in $(012345)$ directions with $w = 0$
- $2k'$ NS5'-branes in $(012389)$ directions $v = 0$
- $N_f$ $D6_{-\theta}$-branes in $(01237)$ directions and two other directions in $(v, w)$-plane
- $N'_f$ D6-branes in $(0123789)$ directions
- $N_c$- and $N'_c$-color D4-branes in $(01236)$ directions with $v = 0 = w$

Figure 7: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $SU(N_c) \times SU(N'_c)$ with bifundamentals $X, \tilde{X}$ and fundamentals $Q, \tilde{Q}, Q', \tilde{Q}'$. Note that there exist multiple $2k'$ NS5'-branes. A rotation of $N_f$ D6-branes in $(w, v)$-plane corresponds to a quartic term for the fundamentals $Q, \tilde{Q}$ while a displacement of $N_f$ D6-branes in $+v$ direction corresponds to a mass term for the fundamentals $Q, \tilde{Q}$.
5.2 Magnetic theory for $SU(N_c)$

After we move a middle NS5-brane to the right all the way past the right NS5'-branes, we arrive at the Figure 8A. Note that there exists a creation of $N_f$ D4-branes connecting the $N_f$ $D6_{-\theta}$-branes and the $k'$ right NS5'-branes. The linking number of NS5-brane from Figure 8A is $l_m = \frac{N_f}{2} - \tilde{N}_c$. On the other hand, the linking number of NS5-brane from Figure 7 is $l_e = -\frac{N_f}{2} + N_c - N'_c$. From these two relations, one obtains the number of colors of dual magnetic theory as follows [30]: $\tilde{N}_c = N_f + N'_c - N_c$.

![Figure 8](image)

Figure 8: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 7 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(8A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(8B). The $N_f$ flavor D4-branes connecting between $D6_{-\theta}$-branes and NS5'-branes are splitting into $(N_f - l)$- and $l$- D4-branes(8A). Further $n$- D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane(8B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SU(N'_c)$ gauge group and $N_f$-fundamental dual quarks $q, \tilde{q}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N_f$ flavor D4-branes as well as $Q', \tilde{Q}', Y$ and $\tilde{Y}$ and gauge singlets. Moreover, a magnetic meson field $M \equiv Q\tilde{Q}$ is $N_f \times N_f$ matrix and comes from 4-4 strings of $N_f$ flavor D4-branes. Then the magnetic superpotential with the limit $\beta, m_X \to 0$ is given by

$$W_{dual} = \left[ \frac{1}{\Lambda} M q\tilde{q} + Y \tilde{F}'q + \tilde{Y} qF' + \Phi'Y\tilde{Y} \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \quad (5.2)$$

The case with $k' = 1$ and $\alpha = 0$ was studied in [30]. Although the superpotential (5.2) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [30] appears as two things: nonzero $\alpha$ in (5.2) and multiple NS5'-branes in Figure 8. Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F' \equiv \tilde{X}Q$ and $\tilde{F}' \equiv X\tilde{Q}$ [30].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (5.2) and the expectation values for $M$ and $q\tilde{q}$ are obtained. The F-term equations
are almost the same as the one in [30] and the derivative of (5.2) with respect to the meson field \( M \) has \( \alpha \) dependent term. The vacuum expectation values for \( Y, \tilde{Y}, F' \) and \( \tilde{F}' \) vanish as in [30].

- Coincident \( N_f \) \( D_{6-\theta} \)-branes and \( k' \) NS5'-branes

The theory has many nonsupersymmetric meta-stable ground states and when we rescale the meson field as \( M = h\Lambda \Phi \) as before, then the Kahler potential for \( \Phi \) is canonical and the magnetic quarks are canonical near the origin of field space [1]. Then the magnetic superpotential can be written as

\[
W_{\text{mag}} = h\Phi \tilde{q}q + \frac{h^2 \mu_\phi}{2} \text{tr} \Phi^2 - h\mu^2 \text{tr} \Phi + Y \tilde{F}' \tilde{q}q + \tilde{Y} q F' + \Phi Y \tilde{Y}
\]

where \( \mu^2 = m\Lambda \) and \( \mu_\phi = \alpha \Lambda^2 \). Now one splits the \((N_f - l) \times (N_f - l)\) block at the lower right corner of \( h\Phi \) and \( q\tilde{q} \) into blocks of size \( n \) and \((N_f - l - n)\) as follows [7]:

\[
h\Phi = \begin{pmatrix}
0_l & 0 & 0 \\
0 & h\Phi_n & 0 \\
0 & 0 & \frac{\mu^2}{\mu_\phi}1_{N_f - l - n}
\end{pmatrix}, \quad q\tilde{q} = \begin{pmatrix}
\mu^2 1_l & 0 & 0 \\
0 & \varphi \tilde{\varphi} & 0 \\
0 & 0 & 0_{N_f - l - n}\end{pmatrix}.
\]

Here \( \varphi \) and \( \tilde{\varphi} \) are \( n \times (\tilde{N}_c - l) \) dimensional matrices and correspond to \( n \) flavors of fundamentals of the gauge group \( SU(\tilde{N}_c - l) \) which is unbroken. In the brane configuration shown in Figure 8B, they correspond to fundamental strings connecting the \( n \) flavor D4-branes and \((\tilde{N}_c - l)\) color D4-branes [6]. The \( \Phi_n \) and \( \varphi \tilde{\varphi} \) are \( n \times n \) matrices. The supersymmetric ground state corresponds to \( h\Phi_n = \frac{\mu^2}{\mu_\phi}1_n \) and \( \varphi = 0 = \tilde{\varphi} \).

Now the full one loop potential takes the form

\[
\frac{V}{|h|^2} = |\Phi_n \varphi + Y \tilde{F}'|^2 + |\Phi_n \tilde{\varphi} + F'\tilde{Y}|^2 + |\varphi \tilde{\varphi} - \mu_\phi 1_n + h\mu \Phi_n|^2 + b|h|\mu|^2 \text{tr} \Phi_n^\dagger \Phi_n,
\]

where \( b = \frac{(ln 4 - 1)}{8\pi^2} \tilde{N}_c \) and we do not write down \( \Phi_n \) or \( \Phi_n^\dagger \)-independent terms. Differentiating this potential with respect to \( \Phi_n^\dagger \) and putting \( \varphi = 0 = \tilde{\varphi} \), one obtains

\[
h\Phi_n \simeq \frac{\mu_\phi}{b}1_n \quad \text{or} \quad M_n \simeq \frac{\alpha \Lambda^3}{\tilde{N}_c}1_n \quad (5.3)
\]

corresponding to the \( w \) coordinates of \( n \) curved flavor D4-branes between the \( D_{6-\theta} \)-branes and the NS5'-branes.

\footnote{Compared to a single gauge group, there exist extra \( k' \) left NS5'-branes, \( N_f' \) D6-branes and \( N_c' \) D4-branes. In Figure 8B, we consider the case where \( k' \) left NS5'-branes are far from \( k' \) right NS5'-branes. From the result of [34], in general, there exists a repulsive force between \( N_c' \) D4-branes and \( n \) D4-branes depending on their distance in \( w \) direction and an rotation angle between those D4-branes. In order to weaken this effect, we need to take the limit where their distance in \( w \) direction should be large and an rotation angle between them should be small. Furthermore, the distance beteen \( n \) D4-branes and the NS5-brane should be small also.}
• Non-coincident $N_f$ D6-$\theta$-branes and $k'$ NS5$'$-branes

When the numbers of D6-$\theta$-branes and the NS5$'$-branes are equal to each other $N_f = k'$, we displace the $k'$ D6-$\theta$-branes and NS5$'$-branes given in Figure 8A in the $v$ direction respectively to two $k'$ different points denoted by $v_{D6,\theta,j}$ and $v_{NS5'}$ where $j = 1, 2, \cdots, k'$, as in section 2. There are $\sum_{j=1}^{k'} N_{c,j} = \tilde{N}_c$ and $\sum_{j=1}^{k'} N_{f,j} = N_f$. Then the $n$ curved flavor D4-branes attached to them are displaced also as $k'$ different $n_j$'s connecting between D6-$\theta,j$-brane and NS5$'$-brane. Let us rescale the submeson field as $M_j = h \Lambda \Phi_j$ [15] and the Kahler potential for $\Phi_j$ is canonical and the magnetic quarks $q_j$ and $\tilde{q}_j$ are canonical near the origin of field space [1]. Then the magnetic superpotential can be rewritten in terms of $\Phi_j, q_j$ and $\tilde{q}_j$

$$W_{mag} = \sum_{j=1}^{k'} \left[ h \Phi_j q_j \tilde{q}_j + \frac{h^2 \mu_\phi}{2} \text{tr} \Phi_j^2 - h \mu_j^2 \text{tr} \Phi_j \right] + \cdots$$

with $\mu_j^2 = m_j \Lambda_j$ and $\mu_\phi = \alpha \Lambda^2$ as before.

One splits the $(N_{f,j} - l_j) \times (N_{f,j} - l_j)$ block at the lower right corner of $h \Phi_j$ and $q_j \tilde{q}_j$ into blocks of size $n_j$ and $(N_{f,j} - l_j - n_j)$ for all $j$ as follows [15]:

$$h \Phi = \begin{pmatrix}
0_{l+n} & 0 & 0 & \cdots & 0 \\
0 & \mu_j^2 \frac{1}{\mu_\phi} 1_{N_{f,j} - l_j - n_j} & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & 0 & \mu_\phi^2 1_{N_{f,j}'}
\end{pmatrix}
+ \text{diag}(0, h \Phi_{n_1}, \cdots, h \Phi_{n_{k'}}, 0_{N_f - l - n})$$

and

$$q \bar{q} = \begin{pmatrix}
\mu_j^2 1_{l_j} & 0 & 0 & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & 0 \\
0 & 0 & 0 & \cdots & \mu_\phi^2 1_{l_{k'}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0_{N_f - l}
\end{pmatrix}
+ \text{diag}(0, \varphi_{n_1} \bar{\varphi}_{n_1}, \cdots, \varphi_{n_{k'}} \bar{\varphi}_{n_{k'}}, 0_{N_f - l - n})$$

where $l = \sum_{j=1}^{k'} l_j$ and $n = \sum_{j=1}^{k'} n_j$. Here $\varphi_{n_j}$ and $\bar{\varphi}_{n_j}$ are $n_j \times (\tilde{N}_{c,j} - l_j)$ matrices and correspond to $n_j$-flavors of fundamentals of the gauge group $SU(\tilde{N}_{c,j} - l_j)$ which is unbroken. The supersymmetric ground state corresponds to the vacuum expectation values by $h \Phi_{n_j} = \mu_j^2 \frac{1}{\mu_\phi} 1_{n_j}$ and $\varphi_{n_j} \bar{\varphi}_{n_j} = 0$. The full one loop potential can be written similarly and the local nonzero stable point arises as

$$h \Phi_{n_j} \simeq \frac{\mu_\phi}{b_j} 1_{n_j} \quad \text{or} \quad M_{n_j} \simeq \frac{\alpha \Lambda^3}{\tilde{N}_{c,j}} 1_{n_j} \quad (5.4)$$
corresponding to the nonzero $w$ coordinates of $n_j$ flavor D4-branes between the $D6_{-\theta,j}$-brane and the $NS5'_j$-brane. Then, the meta-stable states, for fixed $k'$ and $\theta$, are specified by the number of various D4-branes and the positions of multiple $D6_{-\theta}$-branes and NS5'-branes.

6 $Sp(N_c) \times SO(2N'_c)$ with $N_f$-fund., $N'_f$-vectors. and bi-fund.

6.1 Electric theory

The type IIA supersymmetric electric brane configuration $[35, 36, 37]$ corresponding to $\mathcal{N} = 1$ $Sp(N_c) \times SO(2N'_c)$ gauge theory with $N'_f$-vectors $Q'$, $N_f$-fundamental flavors $Q$ and bifundamental $X$ can be described as follows: one middle NS5-brane, $2(2k' + 1)$ NS5'-branes, $2N_c$- and $2N'_c$-D4-branes, and $2N_f$- and $2N'_f$-D6-branes as well as $O4^\pm$-planes. The $2N_c$-color D4-branes are suspended between the middle NS5-brane and the right NS5'-branes, the $2N'_c$-color D4-branes are suspended between the left NS5'-branes and the middle NS5-brane, the $2N_f$-D6-branes are located between the middle NS5-brane and the right NS5'-branes and the $2N'_f$-D6-branes are located between the left NS5'-branes and the middle NS5-brane.

Let us deform this theory by adding the mass term and the quartic term for fundamental quarks. The former can be achieved by displacing the D6-branes along $\pm v$ direction leading to their coordinates $v = \pm v_{D6}$ [9] while the latter can be obtained by rotating the D6-branes [6] by an angle $-\theta$ in $(w, v)$-plane and we denote them by $D6_{-\theta}$-branes. Then, in the electric gauge theory, the general deformed superpotential is given by

$$W_{elec} = \frac{\alpha}{2} \text{tr}(QQ)^2 - m \text{tr} QQ + \frac{\alpha'}{2} \text{tr}(Q'Q')^2 - m' \text{tr} Q'Q'$$

$$+ \left[ -\frac{\beta}{2} \text{tr}(XX)^{2k'+2} + m_X \text{tr} XX \right]. \quad (6.1)$$

The last two terms are due to the rotation of NS5'-branes where $\beta = \tan \omega$ and the relative displacement of D4-branes where the mass $m_X = v_{NS5'}$ is the distance of D4-branes in $v$ direction. When there are no D6-branes, this theory reduces to the one [15] and the case where there exists a further restriction on the number of NS5'-branes $k' = 0$ has been studied in [14]. We focus on the case with $k' \geq 1$ and $\beta, m_X \rightarrow 0$. When we take the Seiberg dual for the gauge group $Sp(N_c)$, we put $\alpha' = 0$ and $m' = 0$ while for the Seiberg dual for the gauge group $SO(2N'_c)$, we take $\alpha = 0$ and $m = 0$ instead.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (6.1) in type IIA string theory as follows and draw this in Figure 9:

• One middle NS5-brane in (012345) directions with $w = 0$
- $2(2k' + 1)$ NS5'-branes in (012389) directions $v = 0$
- $2N_f$ $D6_{-\theta}$-branes in (01237) directions and two other directions in $(v, w)$-plane
- $2N'_f$ D6-branes in (012389) directions
- $2N_c$- and $2N'_c$-color $D4$-branes in (01236) directions with $v = 0 = w$
- $O4^\pm$-planes in (01236) directions with $w = 0 = v$

Figure 9: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $Sp(N_c) \times SO(2N'_c)$ with bifundamental $X$ and fundamentals and vectors $Q, Q'$. Note that there are multiple $2(2k' + 1)$ NS5'-branes. A rotation of $N_f$ D6-branes in $(w, v)$-plane corresponds to a quartic term for the fundamentals $Q$ while a displacement of $N_f$ D6-branes in $\pm v$ direction corresponds to a mass term for the fundamentals $Q$.

6.2 Magnetic theory for $Sp(N_c)$

After we move the middle NS5-brane to the right all the way past the right NS5'-branes, the linking number of NS5-brane from Figure 10A is given by $l_m = \frac{(2N_f)}{2} - 1 - (1) - 2\tilde{N}_c$. Originally, it was given by $l_e = -\frac{(2N_f)}{2} + 1 - (-1) + 2N_c - 2N'_c$ from Figure 9. Therefore, by the linking number conservation and equating these two $l$’s each other, we are left with the number of colors in the magnetic theory [35] $\tilde{N}_c = N_f + N'_c - N_c - 2$.

The low energy theory on the color $D4$-branes has $Sp(\tilde{N}_c) \times SO(2N'_c)$ gauge group and $N_f$-fundamental dual quarks $q$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ $D4$-branes and $N_f$ flavor $D4$-branes as well as $Q', Y$ and gauge singlets. Moreover, a magnetic meson field $M \equiv QQ$ is $2N_f \times 2N_f$ matrix and comes from 4-4 strings of $2N_f$ flavor $D4$-branes. Then the magnetic superpotential with the limit $\beta, m_X \rightarrow 0$ is given by

$$W_{\text{dual}} = \left[ \frac{1}{\Lambda} M qq + Y \Phi' Y + qNY \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \quad (6.2)$$

The case with $k' = 0$ and $\alpha = 0$ was studied in [35]. Although the superpotential (6.2) does not depend on the multiplicity ($2k' + 1$) of NS5'-branes in our particular limit, the difference from the previous result of [35] appears as 1) nonzero $\alpha$ in (6.2) and 2) multiple NS5'-branes in Figure 10A. Here other meson fields are given by $\Phi' \equiv XX$ and $N \equiv QX$. 
Figure 10: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 9 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored (10A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered (10B). The $N_f$ flavor D4-branes connecting between $D_{6-\theta}$-branes and NS5'-branes are splitting into $(N_f - l)$- and $l$- D4-branes (10A). Further $n$- D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane (10B).

One can compute the F-term equations for this superpotential (6.2) and the expectation values for $M$ and $qq$ are obtained. The F-term equations are almost the same as the one in [35] and the derivative of (6.2) with respect to the meson field $M$ has $\alpha$-dependent term. The vacuum expectation values for $Y$ and $N$ vanish as in [35].

• Coincident $N_f$ $D_{6-\theta}$-branes and $(k' + \frac{1}{2})$ NS5'-branes

The theory has many nonsupersymmetric meta-stable ground states and when we rescale the meson field as $M = h\Lambda\Phi$ as before. Then the magnetic superpotential can be written as

$$W_{mag} = h\Phi qq + \frac{h^2 \mu_\phi}{2} \text{tr} \Phi^2 - \hbar \mu^2 \text{tr} \Phi + Y\Phi^2 Y + qNY$$

where $\mu^2 = m\Lambda$ and $\mu_\phi = \alpha\Lambda^2$. Now one splits the $2(N_f - l) \times 2(N_f - l)$ block at the lower right corner of $h\Phi$ and $qq$ into blocks of size $2n$ and $2(N_f - l - n)$ as follows [7]:

$$h\Phi = \begin{pmatrix} 0_{2l} & 0 & 0 \\ 0 & h\Phi_{2n} & 0 \\ 0 & 0 & \frac{\mu^2}{\mu_\phi} 1_{N_f - l - n} \otimes i\sigma_2 \end{pmatrix}, \quad qq = \begin{pmatrix} \mu^2 1_{2l} & 0 & 0 \\ 0 & \varphi\varphi & 0 \\ 0 & 0 & 0_{2(N_f - l - n)} \end{pmatrix}.$$

Here $\varphi$ is $2n \times 2(\tilde{N}_c - l)$ dimensional matrices and correspond to $2n$ flavors of fundamentals of the gauge group $Sp(\tilde{N}_c - l)$ which is unbroken. The $\Phi_{2n}$ and $\varphi\varphi$ are $2n \times 2n$ matrices. The supersymmetric ground state corresponds to $h\Phi_{2n} = \frac{\mu^2}{\mu_\phi} 1_n \otimes i\sigma_2$ and $\varphi = 0$.

Now the full one loop potential takes the form

$$\frac{V}{|h|^2} = |\Phi_{2n}\varphi + NY|^2 + |\varphi\varphi - \mu^2 1_{2n} + h\mu_\phi \Phi_{2n}|^2 + b|h\mu|^2 \text{tr} \Phi_{2n}^2 \Phi_{2n},$$

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where \( b = \frac{(n+1)}{8\pi^2} \tilde{N}_c \) and we do not write down \( \Phi_{2n} \)-independent terms. Differentiating this potential with respect to \( \Phi_{2n} \) and putting \( \varphi = 0 \), one obtains

\[
h\Phi_{2n} \approx \frac{\mu_\phi}{b} 1_n \otimes i\sigma_2 \quad \text{or} \quad M_{2n} \approx \frac{\alpha \Lambda^3}{\tilde{N}_c} 1_n \otimes i\sigma_2
\]
corresponding to the \( w \) coordinates of \( n \) curved flavor D4-branes between the \( D6_{-\theta} \)-branes and the \( NS5' \)-branes.

- Non-coincident \( N_f \) \( D6_{-\theta} \)-branes and \((k' + \frac{1}{2}) \) \( NS5' \)-branes

When the numbers of \( D6_{-\theta} \)-branes and the number of \( NS5' \)-branes minus one are equal \( N_f = k' \), we displace the \( k' \) upper \( D6_{-\theta} \)-branes and upper \( NS5' \)-branes, given in Figure 10A, in the \( +v \) direction respectively to two \( k' \) different points denoted by \( v_{D6_{-\theta},j} \) and \( v_{NS5',j} \), where \( j = 1, 2, \ldots, k' \) (and their mirrors in the \(-v \) direction). A single \( NS5' \)-brane is located at \( v = 0 \). Then there exist \( \sum_{j=1}^{k'} \tilde{N}_{c,j} (\equiv N_{f,j} - N_{c,j} - 2) = \tilde{N}_c \) and \( \sum_{j=1}^{k'} N_{f,j} = N_f \).

When all the upper \( NS5'_j \)-branes and \( D6_{-\theta,j} \)-branes \((j = 1, 2, \ldots, k') \) are distinct, the low energy physics corresponds to \( k' \) decoupled supersymmetric gauge theories with gauge groups

\[
\prod_{j=1}^{k'} Sp(\tilde{N}_{c,j}) \times SO(2N'_c).
\]

One deforms the Figure 10B by displacing the multiple \( D6_{-\theta} \)-branes and \( NS5' \)-branes along \( v \) direction [15]. Then the \( n \) curved flavor D4-branes attached to them are displaced also as \( k' \) different \( n_j \)'s connecting between \( D6_{-\theta,j} \)-brane and \( NS5'_j \)-brane. Then the magnetic superpotential can be rewritten in terms of \( \Phi_j, q_j \)

\[
W_{mag} = \sum_{j=1}^{k'} \left[ h\Phi_j q_j q_j + \frac{h^2 \mu_\phi}{2} \text{tr} \Phi_j^2 - h\mu_\phi \text{tr} \Phi_j \right] + \cdots
\]

with \( \mu_j^2 = m_j \Lambda_j \) and \( \mu_\phi = \alpha \Lambda^2 \) as before.

One splits the \( 2(N_{f,j} - l_j) \times 2(N_{f,j} - l_j) \) block at the lower right corner of \( h\Phi_j \) and \( q_j q_j \) into blocks of size \( 2n_j \) and \( 2(N_{f,j} - l_j - n_j) \) for all \( j \) as follows:

\[
h\Phi = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \frac{\mu_\phi^2}{\mu_\phi} & 1_{N_{f,j} - l_j - n_j} \otimes i\sigma_2 & 0 & \cdots & 0 \\
. & . & . & . & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu_\phi^2}{\mu_\phi} & 1_{N_{f,j} - l_j - n_j} \otimes i\sigma_2
\end{pmatrix}
\]

and

\[
qq = \begin{pmatrix}
\mu_1^2 & 0 & \cdots & 0 & 0 \\
. & . & . & . & 0 & \cdots & 0 \\
0 & 0 & 0 & \mu_2^2 & 1_{2l_{k'}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
+ \text{diag}(0_{2l}, h\Phi_{2n_1}, \cdots, h\Phi_{2n_{k'}}, 0_{2(N_{f,j} - l_j - n_j)})
\]

and

\[
+ \text{diag}(0_{2l}, \varphi_{2n_1}, \cdots, \varphi_{2n_{k'}}, 0_{2(N_{f,j} - l_j - n_j)})
\]
where \( l = \sum_{j=1}^{k'} l_j \) and \( n = \sum_{j=1}^{k'} n_j \). Here \( \varphi_{2n_j} \) is \( 2n_j \times 2(\tilde{N}_{c,j} - l_j) \) matrices and correspond to \( 2n_j \)-flavors of fundamentals of the gauge group \( Sp(\tilde{N}_{c,j} - l_j) \) which is unbroken. Moreover, the \( \Phi_{2n_j} \) and \( \varphi_{2n_j} \) are \( 2n_j \times 2n_j \) matrices. The supersymmetric ground state corresponds to the vacuum expectation values by 

\[
\hbar \Phi_{2n_j} \simeq \mu \phi_{2n_j} \otimes i\sigma_2 \quad \text{and} \quad \Phi_{2n_j} \simeq 0.
\]

The full one loop potential can be written similarly and the local nonzero stable point arises as

\[
h \Phi_{2n_j} \simeq \mu \phi_{2n_j} \otimes i\sigma_2 \quad \text{or} \quad M_{2n_j} \simeq \frac{\alpha \Lambda^3}{N_{c,j}} 1_{n_j} \otimes i\sigma_2
\]

corresponding to the \( w \) coordinates of \( n_j \) flavor D4-branes between the \( D6_{-\theta,j} \)-brane and the \( NS5_j' \)-brane(and their mirrors). Then, the meta-stable states, for fixed \( k' \) and \( \theta \), are specified by the number of various D4-branes and the positions of multiple \( D6_{-\theta} \)-branes and \( NS5' \)-branes.

7 \( SU(N_c) \times SO(2N'_c) \) with \( N_f \)-fund., \( 2N'_f \)-vectors and bifund.

7.1 Electric theory

The type IIA supersymmetric electric brane configuration [38, 30] corresponding to \( N = 1 \) \( SU(N_c) \times SO(2N'_c) \) gauge theory with \( N_f \)-fundamental flavors \( Q, \tilde{Q}, 2N'_f \)-vectors \( Q' \) and bifundamentals \( X, \tilde{X} \) can be described as follows: two NS5-branes, \( 2k' \) NS5' -branes, \( N_c \)- and \( 2N'_c \)-D4-branes, \( 2N_f \)- and \( 2N'_f \)-D6-branes and \( O6^+ \)-plane. The \( 2N'_c \)-color D4-branes are suspended between the two NS5-branes and the \( N_c \)-color D4-branes are suspended between the NS5-brane and the NS5' -branes(and their mirrors).

Let us deform this theory by adding the mass term and the quartic term for the fundamentals. The former can be achieved by displacing the D6-branes along \(+v\) direction leading to their coordinates \( v = +v_{D6} \), while the latter can be obtained by rotating the D6-branes \([6]\) by an angle \(-\theta\) in \((w, v)\)-plane and we denote them by \( D6_{-\theta} \)-branes. Then, in the electric gauge theory, the general deformed superpotential is given by

\[
W_{\text{elec}} = \frac{\alpha}{2} \text{tr}(Q\tilde{Q})^2 - m \text{tr} Q\tilde{Q} + \frac{\alpha'}{2} \text{tr}(Q'Q')^2 - m' \text{tr} Q'Q' + \left[ -\frac{\beta}{2} \text{tr}(X\tilde{X})^{k'+1} + m_X \text{tr} X\tilde{X} \right].
\]

The last two terms are due to the rotation of NS5' -branes where \( \beta = \tan \omega \) and the relative displacement of D4-branes where the mass \( m_X = v_{NS5'} \) is the distance in \( v \) direction. This reduces to the one [30] when \( k' = 1 \) and \( \alpha = 0 = \alpha' = m' \). We focus on the case with \( k' \geq 2 \).
and $\beta, m_X \to 0$. When we take the Seiberg dual for the gauge group $SU(N_c)$, we put $\alpha' = 0$ and $m' = 0$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (7.1) in type IIA string theory as follows and draw this in Figure 11:

- Two NS5-branes in (012345) directions with $w = 0$
- $2k'$ NS5'-branes in (012389) directions $v = 0$
- $N_f$ $D6_{±θ}$-branes in (01237) directions and two other directions in $(v, w)$-plane
- $2N'_f$ D6-branes in (0123789) directions
- $N_c$- and $2N'_c$-color D4-branes in (01236) directions with $v = 0 = w$
- $O6^+$-plane in (0123789) directions with $x^6 = 0 = v$

Figure 11: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $SU(N_c) \times SO(2N'_c)$ with bifundamental $X, \tilde{X}$, fundamentals $Q, \tilde{Q}$ and vectors $Q'$. Note that there are multiple $k'$ NS5'-branes (and its mirrors). A rotation of $N_f$ D6-branes in $(w, v)$-plane corresponds to a quartic term for the fundamentals while a displacement of $N_f$ D6-branes in $±v$ direction corresponds to a mass term for the fundamentals.

7.2 Magnetic theory for $SU(N_c)$

After we move the right NS5-brane to the right all the way past the right NS5'-branes, we arrive at the Figure 12A. Note that there exists a creation of $N_f$ D4-branes connecting the $N_f$ $D6_{±θ}$-branes and the $k'$ right NS5'-branes. The linking number of NS5-brane from Figure 12A is $l_m = \frac{N_f}{2} - \tilde{N}_c$. On the other hand, the linking number of NS5-brane from Figure 11 is $l_e = -\frac{N_f}{2} + N_c - N'_c$. From these two relations, one obtains the number of colors of dual magnetic theory as follows [30]: $\tilde{N}_c = N_f + N'_c - N_c$.

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SO(2N'_c)$ gauge group and $N_f$-fundamental dual quarks $q, \tilde{q}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N_f$ flavor D4-branes as well as $Q', Y, \tilde{Y}$ and gauge singlets. Moreover, a magnetic meson field $M \equiv Q\tilde{Q}$ is $N_f \times N_f$ matrix and comes from 4-4 strings of $N_f$ flavor D4-branes.
Figure 12: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 11 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored (12A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered (12B). The $N_f$ flavor D4-branes connecting between $D_6-\theta$-branes and NS5'-branes are splitting into $(N_f - l)$- and $l$- D4-branes (12A). Further $n$- D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane (12B).

Then the magnetic superpotential with the limit $\beta, m_X \to 0$ is given by

$$W_{dual} = \left[ \frac{1}{\Lambda} M q\bar{q} + Q'\Phi'Q' + Y \tilde{F'}\tilde{q} + \tilde{Y} qF' + \Phi' Y\tilde{Y} \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \quad (7.2)$$

The case with $k' = 1$ and $\alpha = 0$ was studied in [30]. Although the superpotential (7.2) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [30] appears as two things: nonzero $\alpha$ in (7.2) and multiple NS5'-branes in Figure 12. Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F' \equiv \tilde{X}Q$ and $\tilde{F}' \equiv X\tilde{Q}$ [30].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (7.2) and the expectation values for $M$ and $q\bar{q}$ are obtained. The F-term equations are almost the same as the one in [30] and the derivative of (7.2) with respect to the meson field $M$ has $\alpha$ dependent term. The vacuum expectation values for $Y, \tilde{Y}, F', \tilde{F}'$ and $Q'$ vanish as in [30].

- Coincident $N_f$ $D_6-\theta$-branes and $k'$ NS5'-branes

By following the description of subsection 5.2, one obtains the local nonzero stable point given by (5.3) corresponding to the $w$ coordinates of $n$ curved flavor D4-branes between the $D_6-\theta$-branes and the NS5'-branes in Figure 12B.

- Non-coincident $N_f$ $D_6-\theta$-branes and $k'$ NS5'-branes

The local nonzero stable point arises as (5.4) corresponding to the nonzero $w$ coordinates of $n_j$ flavor D4-branes between the $D_6_{-\theta,j}$-brane and the $NS5'_j$-brane.
8 $SU(N_c) \times SU(N'_c)$ with a symm. and bifund.

8.1 Electric theory

The type IIA supersymmetric electric brane configuration [39, 40, 41, 42] corresponding to $\mathcal{N} = 1$ $SU(N_c) \times SU(N'_c)$ gauge theory with a symmetric tensor $S$, a conjugate symmetric tensor $\tilde{S}$ and bifundamentals $X, \tilde{X}$ can be described as follows: $(2k+1)$ NS5-brane, $2k'$ NS5'-branes, $N_c$- and $N'_c$-D4-branes and $O6^+$-plane. The $N_c$-color D4-branes are suspended between the left NS5'-branes and the right NS5'-branes and the $N'_c$-color D4-branes are suspended between the right NS5'-branes and the right NS5-branes (and their mirrors).

Let us deform this theory by adding the mass term and the higher order term for the bifundamentals. The former can be achieved by displacing the NS5-branes along $+v$ direction leading to their coordinates $v = +v_{NS5}$ [9] while the latter can be obtained by rotating the NS5-branes [6] by an angle $-\theta$ in $(w, v)$-plane and we denote them by $NS5_{-\theta}$-branes. Then, in the electric gauge theory, the general deformed superpotential by including the mass term and higher order term for symmetric tensor matter is given by

$$W_{elec} = -\frac{\alpha}{2} \text{tr}(X\tilde{X})^{k+1} + m \text{tr}X\tilde{X} + \left[-\frac{\beta}{2} \text{tr}(S\tilde{S})^{k'+1} + m_S \text{tr}S\tilde{S}\right].$$

The last two terms are due to the rotation of NS5'-branes where $\beta = \tan \omega$ and the relative displacement of D4-branes where the mass $m_S = v_{NS5'}$ is the distance in $v$ direction. This reduces to the one [14] when $k = k' = 1$ and $\beta = m_S = 0$. We focus on the case with $k, k' \geq 2$ and $\beta, m_S \to 0$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (8.1) in type IIA string theory as follows and draw this in Figure 13:

- One NS5-brane in $(012345)$ directions with $w = 0 = x^6$
- $2k'$ NS5'-branes in $(012389)$ directions with $v = 0$
- $k$ NS5$_{+\theta}$-branes in $(012389)$ directions with $v = 0$
- $N_c$- and $N'_c$-color D4-branes in $(01236)$ directions with $v = 0 = w$
- $O6^+$-plane in $(0123789)$ directions with $x^6 = 0 = v$

8.2 Magnetic theory for $SU(N_c)$

We apply the Seiberg dual to the $SU(N_c)$ factor and the two $k'$ NS5'-branes are interchanged each other. The linking number of the right NS5'-brane in Figure 14A is $l_m = -\frac{N_c}{k'} + \frac{N'_c}{k'}$ and the linking number of the left NS5'-brane in Figure 13 is given by $l_e = \frac{N_c}{k} - \frac{N'_c}{k}$. This leads to the fact that the number of D4-branes becomes $\tilde{N}_e = 2N'_c - N_c$. 

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Figure 13: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $SU(N_c) \times SU(N'_c)$ with a symmetric tensor $S, \tilde{S}$ and bifundamentals $X, \tilde{X}$. Note that there are multiple $k NS5_{\pm \theta}$-branes and $2k'$ NS5'-branes. A rotation of $k$ NS5-branes in $(w, v)$-plane corresponds to a higher term for the bifundamentals while a displacement of $k$ NS5-branes in $\pm v$ direction corresponds to a mass term for the bifundamentals.

Figure 14: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 13 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(13A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(13B). The $N'_c$ flavor D4-branes connecting between $NS5_{-\theta}$-branes and NS5'-branes are splitting into $(N'_c - l)$- and $l$- D4-branes(14A). Further $n$- D4-branes among $(N'_c - l)$- D4-branes are moved to the NS5-brane(14B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SU(N'_c)$ gauge group and $N'_c$-fundamental dual “quarks” $Y, \tilde{Y}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N'_c$ flavor D4-branes as well as $s$ and $\tilde{s}$. Moreover, a single magnetic meson field $M \equiv X\tilde{X}$ is $N'_c \times N'_c$ matrix and comes from 4-4 strings of $N'_c$ flavor D4-branes. Then the magnetic superpotential with the limit $\beta, m_S \rightarrow 0$ is given by

\[ W_{dual} = \frac{1}{\Lambda} M Y \tilde{Y} - \frac{\alpha}{2} M^{k+1} + mM. \]  

The case with $k = 1 = k'$ and $\alpha = 0$ was studied in [39]. Although the superpotential (8.2) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [39] appears as two things: nonzero $\alpha$ in (8.2) which has explicit $k$ dependent term and multiple $NS5_{-\theta}$-branes and NS5'-branes in Figure 14A. For the su-
persymmetric vacua, one can compute the F-term equations for this superpotential \( (8.2) \) and the expectation values for \( M \) and \( Y \tilde{Y} \) are obtained. The F-term equations are the same as the one in \([15]\).

- Coincident \( k \) NS5\(_{-\theta}\)-branes and \( k' \) NS5'-branes

The theory has many nonsupersymmetric meta-stable ground states due to the fact that there exists an attractive gravitational interaction between the flavor D4-branes and the NS5-brane from the DBI action. When we rescale the meson field as \( M = h \Lambda \Phi \), then the Kahler potential for \( \Phi \) is canonical and the magnetic quarks \( Y \) and \( \tilde{Y} \) are canonical near the origin of field space. Then the magnetic superpotential \((8.2)\) can be rewritten as

\[
W_{\text{dual}} = h \Phi Y \tilde{Y} + \frac{\mu_\phi}{2} h^{k+1} \text{tr} \Phi^{k+1} - h^2 \mu \text{tr} \Phi
\]  

(8.3)

with the new couplings \( \mu^2 = -m \Lambda \) and \( \mu_\phi = -\alpha \Lambda^{k+1} \).

Now one splits the \((N'_c - l) \times (N'_c - l)\) block at the lower right corner of \( M \) and \( Y \tilde{Y} \) into blocks of size \( n \) and \((N'_c - l - n)\) and then they are rewritten as follows \([7]\):

\[
h \Phi = \begin{pmatrix}
0_l & 0 & 0 \\
0 & h \Phi_n & 0 \\
0 & 0 & \left[ \frac{2 \mu^2}{\mu_\phi (k+1)} \right]^\frac{1}{k} 1_{N'_c - l - n}
\end{pmatrix}, \quad Y \tilde{Y} = \begin{pmatrix}
\mu^2 1_l & 0 & 0 \\
0 & \varphi \bar{\varphi} & 0 \\
0 & 0 & 0_{N'_c - l - n}
\end{pmatrix}.
\]  

(8.4)

Here \( \varphi \) and \( \bar{\varphi} \) are \( n \times (N'_c - l) \) matrices and correspond to \( n \)-flavors of fundamentals of the gauge group \( SU(N'_c - l) \) which is unbroken. The supersymmetric ground state corresponds to the vacuum expectation values by \( h \Phi_n = \left[ \frac{2 \mu^2}{\mu_\phi (k+1)} \right]^\frac{1}{k} 1_n \) and \( \varphi \bar{\varphi} = 0 \).

The full one loop potential for \( \Phi_n, \varphi, \) and \( \bar{\varphi} \) from \((8.3)\) and \((8.4)\) including the one loop result \([1]\) takes the form

\[
\frac{V}{|h|^2} = |\Phi_n \varphi|^2 + |\Phi_n \bar{\varphi}|^2 + |\varphi \bar{\varphi} - \mu^2 1_n + \frac{(k+1)h}{2} \mu_\phi \Phi_n^k|^2 + b |h \mu|^2 \text{tr} \Phi_n^\dagger \Phi_n,
\]  

(8.5)

where the positive numerical constant \( b \) is given by \( b = \frac{(\ln 4 - 1)}{8 \pi^2} \tilde{N}_c \). Differentiating this potential \((8.5)\) with respect to \( \Phi_n^\dagger \) and putting \( \varphi = 0 = \bar{\varphi} \), one obtains

\[
h \Phi_n \simeq \left[ \frac{2 b}{k (k+1) \mu_\phi} \right]^\frac{1}{k} 1_n \quad \text{or} \quad M_n \simeq \left[ \frac{\tilde{N}_c}{\alpha k (k+1) \Lambda^3} \right]^\frac{1}{k} 1_n.
\]  

(8.6)

It is evident that the \((N'_c - l - n)\) flavor D4-branes between the NS5\(_{-\theta}\)-branes and the NS5'-branes are related to the corresponding eigenvalues of \( h \Phi \) \((8.4)\), i.e., \( \left[ \frac{2 \mu^2}{\mu_\phi (k+1)} \right]^\frac{1}{k} 1_{N'_c - l - n} \) and the intersection point between the \((N'_c - l - n)\) D4-branes and the NS5'-branes is also given by \((v, w) = (0, +v_{NS5_{-\theta}} \cot \theta)\).
Non-coincident $k$ NS5-$\sigma$-branes and $k'$ NS5'-branes

When the numbers of NS5-$\sigma$-branes and the NS5'-branes are equal to each other $k = k'$, we displace the $k'$ NS5-$\sigma$-branes and NS5'-branes given in Figure 14A in the $v$ direction respectively to two $k'$ different points denoted by $v_{NS5-\sigma,j}$ and $v_{NS5',j}$ where $j = 1, 2, \ldots, k'$, as in section 2. There are $\sum_{j=1}^{k'} \tilde{N}_{c,j} = \tilde{N}_c$ and $\sum_{j=1}^{k'} N'_{c,j} = N'_c$.

One deforms the Figure 14B by displacing the multiple NS5-$\sigma$-branes and NS5'-branes along $v$ direction. Then the $n$ curved flavor D4-branes attached to them are displaced also as $k'$ different $n_j$’s connecting between NS5-$\theta$-brane and NS5'-brane. Let us rescale the submeson field as $M_j = h \Lambda \Phi_j$ [15] and the Kahler potential for $\Phi_j$ is canonical and the magnetic quarks $Y_j$ and $\tilde{Y}_j$ are canonical near the origin of field space [1]. Then the magnetic superpotential can be rewritten in terms of $\Phi_j, Y_j$ and $\tilde{Y}_j$

\[
W_{mag} = \sum_{j=1}^{k} \left[ h \Phi_j Y_j \tilde{Y}_j + \frac{h^{k+1} \mu_{\phi}}{2} \text{tr} \Phi_j^{k+1} - h \mu_{\phi}^2 \text{tr} \Phi_j \right]
\]

with $\mu_{\phi}^2 = -m_j \Lambda_j$ and $\mu_{\phi} = -\alpha \Lambda^{k+1}$ as before.

One splits the $(N'_{c,j} - l_j) \times (N'_{c,j} - l_j)$ block at the lower right corner of $h \Phi_j$ and $Y_j \tilde{Y}_j$ into blocks of size $n_j$ and $(N'_{c,j} - l_j - n_j)$ for all $j$ as follows [15]:

\[
h \Phi = \left[ \frac{2}{\mu_{\phi}(k+1)} \right]^k \begin{pmatrix}
0_{l+n} & 0 & 0 & \cdots & 0 \\
0 & \mu_{\phi}^2 \mathbf{1}_{N'_{c,j}-l_j-n_1} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \mu_{\phi}^2 \mathbf{1}_{N'_{c,j}-l_k-n_k} \\
\end{pmatrix}
+ \text{diag}(0_l, h \Phi_{n_1}, \ldots, h \Phi_{n_k}, 0_{N'_{c,j}-l-n})
\]

and

\[
Y \tilde{Y} = \left( \begin{array}{ccccc}
\mu_{\phi}^2 \mathbf{1}_{l_1} & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \mu_{\phi}^2 \mathbf{1}_{l_k} \\
\end{array} \right) + \text{diag}(0_l, \varphi_{n_1} \tilde{\varphi}_{n_1}, \ldots, \varphi_{n_k} \tilde{\varphi}_{n_k}, 0_{N'_{c,j}-l-n})
\]

where $l = \sum_{j=1}^{k'} l_j$ and $n = \sum_{j=1}^{k'} n_j$. Here $\varphi_{n_j}$ and $\tilde{\varphi}_{n_j}$ are $n_j \times (\tilde{N}_{c,j} - l_j)$ matrices and correspond to $n_j$-flavors of fundamentals of the gauge group $SU(\tilde{N}_{c,j} - l_j)$ which is unbroken. The supersymmetric ground state corresponds to the vacuum expectation values by $h \Phi_{n_j} = \left[ \frac{2n_j^2}{\mu_{\phi}(k+1)} \right]^k \mathbf{1}_{n_j}$ and $\varphi_{n_j} \tilde{\varphi}_{n_j} = 0$. The full one loop potential can be written similarly and the local nonzero stable point arises as

\[
h \Phi_{n_j} \simeq \left[ \frac{2b_j}{k(k+1)\mu_{\phi}} \right]^{\frac{1}{k-2}} \mathbf{1}_{n_j} \quad \text{or} \quad M_{n_j} \simeq \left[ \frac{\tilde{N}_{c,j}}{\alpha k(k+1)\Lambda_j^2} \right]^{\frac{1}{k-2}} \mathbf{1}_{n_j}
\]
corresponding to the nonzero $w$ coordinates of $n_j$ flavor D4-branes between the $NS5_{-\theta,j}$-brane and the $NS5_j$-brane. Then, the meta-stable states, for fixed $k, k'$ and $\theta$, are specified by the number of various D4-branes and the positions of multiple $NS5_{-\theta}$-branes and $NS5'$-branes.

9 $SU(N_c) \times SU(N'_c)$ with an antisym., eight-fund. and bifund.

9.1 Electric theory

The type IIA supersymmetric electric brane configuration [39] corresponding to $\mathcal{N} = 1$ $SU(N_c) \times SU(N'_c)$ gauge theory with an antisymmetric tensor $A$, a conjugate symmetric tensor $\tilde{S}$, eight fundamentals $\hat{Q}$ and bifundamentals $X, \tilde{X}$ can be described as follows: Two NS5-branes, $(2k' + 1)$ NS5'-branes, $N_c$- and $N'_c$-D4-branes, eight D6-branes and $O6^\pm$-planes. The $N_c$-color D4-branes are suspended between the left NS5-brane and the right NS5-brane and the $N'_c$-color D4-branes are suspended between the right NS5-brane and the right NS5'-branes (and their mirrors).

Let us deform this theory by adding the mass term and the higher order term for the bifundamentals. The former can be achieved by displacing the NS5'-branes along $+v$ direction leading to their coordinates $v = v_{NS5'}$ [9] while the latter can be obtained by rotating the NS5'-branes [6] by an angle $-\theta$ in $(w, v)$-plane and we denote them by $NS5_{-\theta}$-branes. Then, in the electric gauge theory, the general deformed superpotential is given by

$$W_{elec} = -\frac{\alpha}{2} \text{tr}(X\tilde{X})^{k'+1} + m \text{tr} X\tilde{X} + \left[ -\frac{\beta}{2} \text{tr}(A\tilde{S})^2 + \hat{Q}\tilde{S}\hat{Q} \right].$$

(9.1)

The third term is due to the rotation of NS5-branes where $\beta = \tan \omega$. This theory reduces to the one [14] when $k' = 1$ and $\beta = 0$. We focus on the case with $k' \geq 2$ and $\beta \to 0$.

Let us summarize the $\mathcal{N} = 1$ supersymmetric electric brane configuration with superpotential (9.1) in type IIA string theory as follows and draw this in Figure 15:

- Two NS5-branes in (012345) directions with $w = 0$
- One NS5'-brane in (012389) directions with $v = 0 = x^6$
- $k'$ $NS5_{\pm\theta}$-branes in (012345) directions with $w = 0$
- $N_c$- and $N'_c$-color D4-branes in (01236) directions with $v = 0 = w$
- Eight half D6-branes in (0123789) directions with $x^6 = 0 = v$
- $O6^\pm$-planes in (0123789) directions with $x^6 = 0 = v$
Figure 15: The $\mathcal{N} = 1$ supersymmetric electric brane configuration for the gauge group $SU(N_c) \times SU(N'_c)$ with an antisymmetric tensor $A, \tilde{S}$, eight fundamentals $\tilde{Q}$, and bifundamentals $X, \tilde{X}$. Note that there are $k'$ NS5-\pm\theta-branes. A rotation of $k'$ NS5'-branes in $(w, v)$-plane corresponds to a higher term for the bifundamentals while a displacement of $k'$ NS5-branes in $\pm v$ direction corresponds to a mass term for the bifundamentals.

### 9.2 Magnetic theory for $SU(N_c)$

We apply the Seiberg dual to the $SU(N_c)$ factor and the two NS5-branes are interchanged each other. The linking number of the right NS5-brane in Figure 16A is $l_m = \frac{4}{2} - \tilde{N}_c + N'_c$ and the linking number of the left NS5-brane in Figure 15 is given by $l_e = -\frac{4}{2} + N'_c - N'_c$. This leads to the fact that the number of D4-branes becomes $\tilde{N}_c = 2N'_c - N_c + 4$.

Figure 16: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 15 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(16A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(16B). The $N'_c$ flavor D4-branes connecting between NS5-\pm\theta-branes and NS5'-brane are splitting into $(N'_c - l)$- and $l$- D4-branes(16A). Further $n$- D4-branes among $(N'_c - l)$- D4-branes are moved to the NS5-brane(16B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SU(N'_c)$ gauge group and $N'_c$-fundamental dual “quarks” $Y, \tilde{Y}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N'_c$ flavor D4-branes as well as $a, \tilde{s}$ and $\tilde{q}$. Moreover, a single magnetic meson field $M \equiv X\tilde{X}$ is $N'_c \times N'_c$ matrix and comes from 4-4 strings of $N'_c$ flavor D4-branes. Then
the magnetic superpotential with the limit $\beta \to 0$ is given by

$$W_{\text{dual}} = \frac{1}{\Lambda} M Y \tilde{Y} - \frac{\alpha}{2} M^{k'+1} + m M + \tilde{q} s \tilde{q}. \quad (9.2)$$

The case with $k' = 1$ and $\alpha = 0$ was studied in [39]. The difference from the previous result of [39] appears as two things: nonzero $\alpha$ in (9.2) which has explicit $k'$ dependent term and multiple $N_{S5,\theta}$-branes in Figure 16A. For the supersymmetric vacua, one can compute the F-term equations for this superpotential (9.2) and the expectation values for $M$ and $Y \tilde{Y}$ are obtained.

- Coincident $k'$ $N_{S5,\theta}$-branes

One obtains the vacuum expectation value (8.6) with a replacement $k$ by $k'$ by applying the prescription of subsection 8.2. This provides the $w$ coordinates of $n$ flavor D4-branes between the $N_{S5,\theta}$-branes and the NS5'-brane in Figure 16B.

- Non-coincident $k'$ $N_{S5,\theta}$-branes

These non-coincident $N_{S5,\theta}$-branes can be obtained by taking those quark masses being unequal. Then all the previous descriptions for the meta-stable states can be applied in this case also without any difficulty.

10 $SU(N_c) \times SU(N'_c)$ with $N_f$- and $N'_f$-fund., a symm. and bifund.

10.1 Electric theory

The type IIA supersymmetric electric brane configuration [43] corresponding to $\mathcal{N} = 1$ $SU(N_c) \times SU(N'_c)$ gauge theory with $N_f$-fundamental flavors $Q, \tilde{Q}$, $N'_f$-fundamental flavors $Q', \tilde{Q}'$, a symmetric tensor $S$, a conjugate symmetric tensor $\tilde{S}$ and bifundamentals $X, \tilde{X}$ can be described as follows: Three NS5-branes, $2k'$ NS5'-branes, $N_c$- and $N'_c$-D4-branes, and $2N_f$- and $2N'_f$-D6-branes and $O6^+$-plane. The $N_c$-color D4-branes are suspended between the left NS5'-branes and the right NS5'-branes, the $N'_c$-color D4-branes are suspended between the right NS5'-branes and the right NS5-brane(and their mirrors), the $N_f$ D6-branes are located between the middle NS5-brane and the right NS5'-branes and the $N'_f$ D6-branes are located between the right NS5'-branes and the right NS5-brane(and their mirrors).

Let us deform this theory by adding the mass term and the quartic term for fundamental quarks. The former can be achieved by displacing the D6-branes along $\pm v$ direction leading to their coordinates $v = \pm v_D$. The latter can be obtained by rotating the D6-branes by an angle $\pm \theta$ in $(w, v)$-plane and we denote them by $D6_{\pm \theta}$-branes for $D6_{\pm \theta}$-branes for
the second gauge group). Then, in the electric gauge theory, the deformed superpotential is given by

\begin{equation}
W_{elec} = \frac{\alpha}{2} \text{tr}(Q\tilde{Q})^2 - m \text{tr}Q\tilde{Q} + \frac{\alpha'}{2} \text{tr}(Q'\tilde{Q}')^2 - m' \text{tr}Q'\tilde{Q}'
+ \left[-\frac{\beta}{2} \text{tr}(S\tilde{S})^{k'+1} + m_S \text{tr}S\tilde{S}\right] + \left[-\frac{\gamma}{2} \text{tr}(X\tilde{X})^2 + m_X \text{tr}X\tilde{X}\right].
\end{equation}

The last four terms are due to the rotation of NS5'-branes and NS5-branes where \(\beta = \tan \omega'\) and \(\gamma = \tan \omega\) and the relative displacement of D4-branes where the mass \(m_S = v_{NS5'}\) and the mass \(m_X = v_{NS5}\) are the distance in \(v\) direction. The case of \(k' = 1\) and \(\alpha = 0\) with \(\beta = \gamma = m_S = m_X = 0\) was studied in [43]. We focus on the case with \(k' \geq 2\) and \(\beta, \gamma, m_S, m_X \to 0\). When we take the Seiberg dual for the gauge group \(SU(N_c)\), we put \(\alpha' = 0\) and \(m' = 0\) and for the Seiberg dual on the gauge group \(SU(N_c')\) we take \(\alpha = 0\) and \(m = 0\).

Let us summarize the \(\mathcal{N} = 1\) supersymmetric electric brane configuration with superpotential (10.1) in type IIA string theory as follows and draw this in Figure 17:

- \(2k'\) NS5'-branes in \((012389)\) directions \(v = 0\)
- Three NS5-branes in \((012345)\) directions \(w = 0\)
- \(N_f\) D6±θ-branes in \((01237)\) directions and two other directions in \((v, w)\)-plane
- \(N_f'\) D6±θ'-branes in \((01237)\) directions and two other directions in \((v, w)\)-plane
- \(N_c\) and \(N_c'\)-color D4-branes in \((01236)\) directions with \(v = 0 = w\)
- O6+-plane in \((0123789)\) directions with \(x^6 = 0 = v\)

Figure 17: The \(\mathcal{N} = 1\) supersymmetric electric brane configuration for the gauge group \(SU(N_c) \times SU(N_c')\) with a symmetric tensor \(S, \tilde{S}\), bifundamentals \(X, \tilde{X}\) and fundamentals \(Q, \tilde{Q}, Q', \tilde{Q}'\). Note that there are \(2k'\) NS5'-branes. A rotation of \(N_f(N_f')\) D6-branes in \((w, v)\)-plane corresponds to a quartic term for the fundamentals \(Q, \tilde{Q}(Q', \tilde{Q}')\) while a displacement of \(N_f(N_f')\) D6-branes in \(\pm v\) direction corresponds to a mass term for the fundamentals \(Q, \tilde{Q}(Q', \tilde{Q}')\).
10.2 Magnetic theory for $SU(N_c)$

After the left NS5'-branes and $D6_\theta$-branes and the right NS5'-branes and $D6_{-\theta}$-branes are exchanged each other, we arrive at the Figure 18A. One reads off the number of colors of dual magnetic theory from section 4 by noticing the new $N'_c$ dependence: 

$$\tilde{N}_c = 2(N_f + N'_f) - N_c + 4(k' - 1).$$

Figure 18: The $\mathcal{N} = 1$ supersymmetric magnetic brane configuration corresponding to Figure 17 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(18A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(18B). The $N_f$ flavor D4-branes connecting between $D6_{-\theta}$-branes and NS5'-branes are splitting into $(N_f - l)$- and $l$- D4-branes(18A). Further $n$- D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane(18B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SU(N'_c)$ gauge group and $N_f$-fundamental dual quarks $q, \tilde{q}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N_f$ flavor D4-branes as well as $Q', \tilde{Q}', s, \tilde{s}, Y$ and $\tilde{Y}$ and gauge singlets. Moreover, a magnetic meson field $M \equiv Q\tilde{Q}$ is $N_f \times N_f$ matrix and comes from 4-4 strings of $N_f$ flavor D4-branes. Then the magnetic superpotential with the limit $\beta, \gamma, m_S, m_X \to 0$ is given by

$$W_{\text{dual}} = \left[ \frac{1}{\Lambda} M q\tilde{s}s\tilde{q} + Y \tilde{F}'\tilde{q} + \tilde{Y} q F' + \Phi' Y \tilde{Y} \right] + \frac{\alpha}{2} \text{tr} M^2 - mM.$$  \hspace{1cm} (10.2)

The case with $k' = 1$ and $\alpha = 0$ was studied in [43] as mentioned before. Although the superpotential (10.2) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [43] appears as 1) nonzero $\alpha$ in (10.2) and 2) multiple NS5'-branes in Figure 18A. Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F' \equiv \tilde{X}Q$ and $\tilde{F}' \equiv X\tilde{Q}$ [43].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (10.2) and the expectation values for $M$ and $q\tilde{s}\tilde{s}q$ are obtained. The F-term equations are almost the same as the one in [43] and the derivative of (10.2) with respect to the meson field $M$ has $\alpha$ dependent term. The vacuum expectation values for $Y, \tilde{Y}, F'$ and $\tilde{F}'$ vanish as in [43].

40
• Coincident \( N_f \) \( D_{6-\theta} \)-branes and \( k' \) \( NS5' \)-branes

One gets the local stable point (4.3) corresponding to the \( w \) coordinates of \( n \) curved flavor D4-branes between the \( D_{6-\theta} \)-branes and the \( NS5' \)-branes in Figure 18B by following the prescription of subsection 4.2.

• Non-coincident \( N_f \) \( D_{6-\theta} \)-branes and \( k' \) \( NS5' \)-branes

The local nonzero stable point arises similarly.

10.3 Magnetic theory for \( SU(N'_c) \)

After we move the right \( NS5' \)-branes to the right all the way past the right \( NS5 \)-brane (and its mirrors to the left), we arrive at the Figure 19A. Note that there exists a creation of \( k'N'_f \) D4-branes connecting the \( N'_f \) \( D_{6-\theta} \)-branes and the \( NS5' \)-brane. The linking number of \( NS5' \)-brane from Figure 19A is \( l_m = N'_f - \frac{N'_c}{k'} \). On the other hand, the linking number of \( NS5' \)-brane from Figure 17 is \( l_e = -\frac{N'_f}{2} + \frac{N'_c}{k'} - \frac{N'_f}{k'} \). From these two relations, one obtains the number of colors of dual magnetic theory \( \tilde{N}'_c = k'N'_f + N_c - N'_c \). In this subsection we consider only the single number of \( NS5' \)-brane \( k' = 1 \) in order to deal with a single meson field.

The low energy theory on the color D4-branes has \( SU(N_c) \times SU(\tilde{N}'_c) \) gauge group and \( N'_f \)-fundamental dual quarks \( q', \bar{q}' \) coming from 4-4 strings connecting between the color \( \tilde{N}'_c \) D4-branes and \( N'_f \)-flavor D4-branes as well as \( Q, \bar{Q}, S, \tilde{S}, Y, \tilde{Y} \) and gauge singlets. Moreover, a magnetic meson field \( M \equiv Q'\tilde{Q}' \) is \( N'_f \times N'_f \) matrix and comes from 4-4 strings of \( N'_f \)-flavor D4-branes. Then the magnetic superpotential with the limit \( \beta, \gamma, m_S, m_X \to 0 \) is given by

\[
W_{\text{dual}} = \left[ \frac{1}{\Lambda} M q' \bar{q}' + Y F q' + \tilde{Y} \bar{q}' \tilde{F} + \Phi Y \tilde{Y} \right] + \frac{\alpha}{2} \text{tr} M^2 - m'M. \tag{10.3}
\]
The case with $k' = 1$ and $\alpha = 0$ was studied in [43]. Although the superpotential (10.3) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [43] appears as nonzero $\alpha$ in (10.3). Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F \equiv XQ'$ and $\tilde{F} \equiv \tilde{X}\tilde{Q}'$ [43].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (10.3) and the expectation values for $M$ and $q'\tilde{q}'$ are obtained. The F-term equations are almost the same as the one in [43] and the derivative of (10.3) with respect to the meson field $M$ has $\alpha$ dependent term. The vacuum expectation values for $Y, \tilde{Y}, F$ and $\tilde{F}$ vanish as in [43].

- Coincident $N'_f$ D6-\(\theta\)'-branes

One obtains the local stable point as (5.3) by changing the role of $Q$ and $\tilde{Q}$ into the one of $Q'$ and $\tilde{Q}'$. This gives the $w$ coordinates of $n$ flavor D4-branes between the $D6_{-\theta}$-branes and the NS5'-'brane in Figure 19B.

- Non-coincident $N'_f$ D6-\(\theta\)'-branes

These non-coincident $D6_{-\theta}$-branes can be obtained by taking those quark masses being unequal. Then all the previous descriptions for the meta-stable states can be applied in this case also without any difficulty.

11 $SU(N_c) \times SU(N'_c)$ with $N_f$- and $N'_f$-fund., an antisymmm., eight-fund. and bifund.

11.1 Electric theory

The type IIA supersymmetric electric brane configuration [43] corresponding to $N = 1$ $SU(N_c) \times SU(N'_c)$ gauge theory with $N_f$-fundamental flavors $Q, \tilde{Q}$, $N'_f$-fundamental flavors $Q', \tilde{Q}'$, an antisymmetric tensor $A$, a conjugate symmetric tensor $\tilde{S}$, eight fundamentals $\tilde{Q}$ and bifundamentals $X, \tilde{X}$ can be described as follows: Two NS5-branes, $(2k' + 1)$ NS5'-'branes, $N_c$- and $N'_c$-D4-branes, and $2N_f$- and $2N'_f$-D6-branes, eight D6-branes and $O6^{\pm}$-planes. The $N_c$-color D4-branes are suspended between the two NS5-branes, the $N'_c$-color D4-branes are suspended between the right NS5-brane and the right NS5'-'brane(and their mirrors), the $N_f$ D6-branes are located between the middle NS5-'brane and the right NS5-brane and the $N'_f$ D6-branes are located between the right NS5-brane and the right NS5'-'brane(and their mirrors).

Let us deform this theory by adding the mass term and the quartic term for fundamental quarks. The former can be achieved by displacing the D6-branes along $\pm v$ direction leading to their coordinates $v = \pm v_{D6}$ [9] while the latter can be obtained by rotating the D6-branes
by an angle \( \mp \theta \) in \((w, v)\)-plane and we denote them by \( D_6_{\mp \theta} \)-branes. Then, in the electric gauge theory, the general deformed superpotential is given by

\[
W_{\text{elec}} = \frac{\alpha}{2} \text{tr}(Q\tilde{Q})^2 - m \text{tr} Q\tilde{Q} + \frac{\alpha'}{2} \text{tr}(Q'\tilde{Q}')^2 - m' \text{tr} Q'\tilde{Q}' - \frac{\beta}{2} \text{tr}(A\tilde{S})^2 + \left[ -\frac{\gamma}{2} \text{tr}(X\tilde{X})^{k'+1} + m_X \text{tr} X\tilde{X} \right].
\] (11.1)

The last three terms are due to the rotation of NS5-branes and NS5'-branes where \( \beta = \tan \omega \) and \( \gamma = \tan \omega' \) and the relative displacement of D4-branes where the mass \( m_X = v_{NS5'} \) is the distance in \( v \) direction. The case of \( k' = 1 \) and \( \alpha = 0 \) with \( \beta = \gamma = m_X = 0 \) was studied in [43]. We focus on the case with \( k' \geq 2 \) and \( \beta, \gamma, m_X \to 0 \). When we take the Seiberg dual for the gauge group \( SU(N_c) \), we put \( \alpha' = 0 \) and \( m = 0 \) and for the Seiberg dual on the gauge group \( SU(N'_c) \) we take \( \alpha = 0 \) and \( m = 0 \).

Let us summarize the \( \mathcal{N} = 1 \) supersymmetric electric brane configuration with superpotential (11.1) in type IIA string theory as follows and draw this in Figure 20:

- Two NS5-branes in \((012345)\) directions with \( w = 0 \)
- \((2k' + 1)\) NS5'-branes in \((012389)\) directions \( v = 0 \)
- \( N_f \) \( D_6_{\mp \theta} \)-branes in \((01237)\) directions and two other directions in \((v,w)\)-plane
- \( N'_f \) \( D_6_{\mp \theta} \)-branes in \((01237)\) directions and two other directions in \((v,w)\)-plane
- Eight \( D_6 \)-branes in \((0123789)\) directions
- \( N_c \)- and \( N'_c \)-color \( D_4 \)-branes in \((01236)\) directions with \( v = 0 = w \)
- \( O6^\pm \)-planes in \((0123789)\) directions with \( x^6 = 0 = v \)

Figure 20: The \( \mathcal{N} = 1 \) supersymmetric electric brane configuration for the gauge group \( SU(N_c) \times SU(N'_c) \) with an antisymmetric tensor \( A, \tilde{S}, X, \tilde{X}, Q, \tilde{Q}, Q', \tilde{Q}' \) and fundamentals \( \hat{Q} \). Note that there are multiple \( 2k' \) \( NS5' \)-branes. A rotation of \( N_f \) \( D6 \)-branes in \((w,v)\)-plane corresponds to a quartic term for the fundamentals \( Q, \tilde{Q} \) while a displacement of \( N_f \) \( D6 \)-branes in \( \pm v \) direction corresponds to a mass term for the fundamentals \( Q, \tilde{Q} \).
11.2 Magnetic theory for $SU(N_c)$

After the left NS5-branes and $D6_θ$-branes and the right NS5-branes and $D6_−θ$-branes are exchanged each other, we arrive at the Figure 21A. The number of colors of dual magnetic theory is given by

$$\tilde{N}_c = 2(N_f + N'_c) - N_c + 4.$$ 

Figure 21: The $\mathcal{N}=1$ supersymmetric magnetic brane configuration corresponding to Figure 20 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored(21A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered(21B). The $N_f$ flavor D4-branes connecting between $D6_−θ$-branes and NS5'-brane are splitting into $(N_f - l)$- and $l$- D4-branes(21A). Further $n$-D4-branes among $(N_f - l)$- D4-branes are moved to the NS5-brane(22B).

The low energy theory on the color D4-branes has $SU(\tilde{N}_c) \times SU(N'_c)$ gauge group and $N_f$-fundamental dual quarks $q, \tilde{q}$ coming from 4-4 strings connecting between the color $\tilde{N}_c$ D4-branes and $N_f$ flavor D4-branes as well as $Q', \tilde{Q}', a, \tilde{s}, \tilde{q}, Y, \tilde{Y}$ and gauge singlets. Moreover, a magnetic meson field $M \equiv Q\tilde{Q}$ is $N_f \times N_f$ matrix and comes from 4-4 strings of $N_f$ flavor D4-branes. Then the magnetic superpotential with the limit $\beta, \gamma, m_X \to 0$ is given by

$$W_{\text{dual}} = \left[ \frac{1}{\Lambda} M q\tilde{s}a\tilde{q} + \tilde{q}\tilde{s}\tilde{q} + YF'\tilde{q} + \tilde{Y}qF' + \Phi Y\tilde{Y} + \tilde{M}\tilde{q}\tilde{q} \right] + \frac{\alpha}{2} \text{tr} M^2 - mM. \quad (11.2)$$

The case with $k' = 1$ and $\alpha = 0$ was studied in [43] as mentioned before. Although the superpotential (11.2) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result appears as 1) nonzero $\alpha$ in (11.2) and 2) multiple NS5'-branes in Figure 21A. Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F' \equiv XQ, \tilde{M} \equiv Q\tilde{Q}$ and $\tilde{F}' \equiv X\tilde{Q}$ [43].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (11.2) and the expectation values for $M$ and $q\tilde{s}a\tilde{q}$ are obtained. The F-term equations are almost the same as the one in [43] and the derivative of (11.2) with respect to the meson field $M$ has $\alpha$ dependent term. The vacuum expectation values for $Y, \tilde{Y}, \tilde{q}, \tilde{M}, F'$ and $\tilde{F}'$ vanish as in [43].
• Coincident \( N_f \) \( D_{6-\theta} \)-branes

The theory has many nonsupersymmetric meta-stable ground states and when we rescale the meson field as \( M = h\Lambda\Phi \) as before, then the Kahler potential for \( \Phi \) is canonical and the magnetic quarks are canonical near the origin of field space \([1]\). Then the magnetic superpotential can be written as

\[
W_{\text{mag}} = h\Phi q\bar{s}\tilde{a}q + \frac{h^2\mu_\phi}{2} \mathrm{tr} \Phi^2 - h\mu^2 \mathrm{tr} \Phi + q\bar{s}\tilde{q} + YFq' + \tilde{Y}q'\tilde{F}' + \Phi'Y\tilde{Y} + \tilde{M}q\tilde{q}
\]

where \( \mu^2 = m\Lambda \) and \( \mu_\phi = \alpha\Lambda^2 \). Now one splits the \((N_f - l) \times (N_f - l)\) block at the lower right corner of \( h\Phi \) and \( q\bar{s}\tilde{a}q \) into blocks of size \( n \) and \((N_f - l - n)\) as follows \([7]\):

\[
h\Phi = \begin{pmatrix}
0_l & 0 & 0 \\
n_{N_f - l - n} & h\Phi_n & 0 \\
0 & 0 & \frac{\mu^2}{\mu_\phi}1_{N_f - l - n}
\end{pmatrix},
q\bar{s}\tilde{a}q = \begin{pmatrix}
\mu^21_l & 0 & 0 \\
0 & \varphi\bar{\beta}\gamma\bar{\varphi} & 0 \\
0 & 0 & 0_{N_f - l - n}
\end{pmatrix}.
\]

Here \( \varphi \) and \( \bar{\varphi} \) are \( n \times (\tilde{N}_c - l) \) dimensional matrices and correspond to \( n \) flavors of fundamentals of the gauge group \( SU(\tilde{N}_c - l) \) which is unbroken. The \( \Phi_n \) and \( \varphi\bar{\beta}\gamma\bar{\varphi} \) are \( n \times n \) matrices. The supersymmetric ground state corresponds to \( h\Phi_n = \frac{\mu^2}{\mu_\phi}1_n \) and \( \varphi\bar{\beta} = 0 = \gamma\bar{\varphi} \).

Now the full one loop potential takes the form

\[
\frac{V}{|h|^2} = |\Phi_n:\varphi\bar{\beta} + \tilde{Y}\tilde{F}|^2 + |\Phi_n:Y\bar{\varphi} + FY|^2 + |\varphi\bar{\beta}\gamma\bar{\varphi} - \mu^21_n + h\mu_\phi\Phi_n|^2 + b|h\mu|^2 \mathrm{tr} \Phi_n^\dagger\Phi_n,
\]

where \( b = \frac{(\ln 4 - 1)}{8\pi^2}\tilde{N}_c \) and we do not write down \( \Phi_n \) or \( \Phi_n^\dagger \)-independent terms. Differentiating this potential with respect to \( \Phi_n^\dagger \) and putting \( \varphi\bar{\beta} = 0 = \gamma\bar{\varphi} \), one obtains

\[
h\Phi_n \simeq \frac{\mu_\phi}{b}1_n \quad \text{or} \quad M_n \simeq \frac{\alpha\Lambda^3}{\tilde{N}_c}1_n
\]

corresponding to the \( w \) coordinates of \( n \) curved flavor D4-branes between the \( D_{6-\theta} \)-branes and the NS5'-branes.

• Non-coincident \( N_f \) \( D_{6-\theta} \)-branes

These non-coincident \( D_{6-\theta} \)-branes can be obtained by taking those quark masses being unequal.

### 11.3 Magnetic theory for \( SU(N'_c) \)

After we move the right NS5-brane to the right all the way past the right NS5'-branes (and its mirrors to the left), we arrive at the Figure 22A. The linking number of NS5-brane from Figure 22A is \( l_m = \frac{N'_f}{2} - \tilde{N}_c \). On the other hand, the linking number of NS5-brane from Figure...
Figure 22: The $N = 1$ supersymmetric magnetic brane configuration corresponding to Figure 20 with a misalignment between D4-branes when the gravitational potential of the NS5-brane is ignored (22A) and nonsupersymmetric brane configuration when the gravitational potential of the NS5-brane is considered (22B). Note that there are multiple $N'$ flavor D4-branes connecting between $D6_{-\theta^\prime}$-branes and NS5'-brane are splitting into $(N'_f - l)$- and $l$- D4-branes (22A). Further $n$- D4-branes among $(N'_f - l)$- D4-branes are moved to the NS5-brane (22B).

21 is $l = -\frac{N'_f}{2} + N_c - N'_c$. From these two relations, one obtains the number of colors of dual magnetic theory $\tilde{N}'_c = N'_f + N_c - N'_c$.

The low energy theory on the color D4-branes has $SU(N_c) \times SU(\tilde{N}'_c)$ gauge group and $N'_f$-fundamental dual quarks $q', \tilde{q}'$ coming from 4-4 strings connecting between the color $\tilde{N}'_c$ D4-branes and $N'_f$ flavor D4-branes as well as $Q, \tilde{Q}, A, \tilde{S}, \tilde{Q}, Y, \tilde{Y}$ and gauge singlets. Moreover, a single magnetic meson field $M \equiv Q'\tilde{Q}'$ is $N'_f \times N'_f$ matrix and comes from 4-4 strings of $N'_f$ flavor D4-branes. Then the magnetic superpotential with the limit $\beta, \gamma, m_X \to 0$ is given by

$$W_{\text{dual}} = \left[ \frac{1}{\Lambda} M q'\tilde{q}' + Y F q' + \tilde{Y} q' \tilde{F} + \Phi Y \tilde{Y} \right] + \frac{\alpha}{2} \text{tr} M^2 - m'M. \quad (11.3)$$

The case with $k' = 1$ and $\alpha = 0$ was studied in [43]. Although the superpotential (11.3) does not depend on the multiplicity $k'$ of NS5'-branes in our particular limit, the difference from the previous result of [43] appears as two things: nonzero $\alpha$ in (11.3) and multiple NS5'-branes in Figure 22A. Here other meson fields are given by $\Phi' \equiv X\tilde{X}, F \equiv XQ'$ and $\tilde{F} \equiv \tilde{X}\tilde{Q}'$ [43].

For the supersymmetric vacua, one can compute the F-term equations for this superpotential (11.3) and the expectation values for $M$ and $q'\tilde{q}'$ are obtained. The F-term equations are almost the same as the one in [43] and the derivative of (11.3) with respect to the meson field $M$ has $\alpha$ dependent term. The vacuum expectation values for $Y, \tilde{Y}, F$ and $\tilde{F}$ vanish as in [43].

- Coincident $N'_f D6_{-\theta^\prime}$-branes

One obtains the stable point (5.3) with appropriate replacement of $N_f D6_{-\theta}$-branes by the corresponding $N'_f D6_{-\theta^\prime}$-branes. This gives the $w$ coordinates of $n$ curved flavor D4-branes
between the $D6_{-\theta'}$-branes and the NS5'-branes in Figure 22B.

- Non-coincident $N'_f$ $D6_{-\theta'}$-branes and $k'$ NS5'-branes

The local nonzero stable point arises as (5.4).

12 Conclusions and outlook

As mentioned in the abstract, let us summarize the new features we have obtained from the meta-stable brane configurations.

Compared with the ones of [7, 6], the Figure 2 contains the multiple $k'$ NS5'-branes and the presence of $\tilde{N}_c$ flavor D4-branes connecting $D6_{-\theta}$-branes and NS5'-branes. Similarly, its symplectic version, characterized by the Figure 4, contains the multiple $(2k' + 1)$ NS5'-branes and the presence of $\tilde{N}_c$ flavor D4-branes connecting $D6_{-\theta}$-branes and NS5'-branes (and its mirrors). Compared with the one of [12], the Figure 6 has the multiple $k'$ NS5'-branes and the different value of $\tilde{N}_c$ which has $k'$-dependence. The Figures 8, 10, 12 have multiple NS5'-branes and rotated D6-branes, compared with [30].

Compared with [39], the Figure 14 has multiple rotated NS5-branes and multiple NS5'-branes while the Figure 16 has multiple rotated NS5'-branes. The Figure 18 contains the multiple $k'$ NS5'-branes and the different value of $\tilde{N}_c$ which has $k'$-dependence while the Figure 19 has only rotated D6-branes, compared with the previous result in [43]. Finally, the Figure 21 has only rotated D6-branes and the Figure 22 multiple NS5'-branes and rotated D6-branes, compared with [43].

We make some comments for the future directions along the line of meta-stable brane construction.

- As we mentioned, the construction for possible multiple outer NS5-branes on the gauge theory [24] with the antisymm. and conj. symm. as well as fundamentals is an open problem in the context of the analysis of supersymmetric ground states and nonsupersymmetric ground states also.

- Are there any variants of sections 2 and 3 by $k$ NS5-branes and a single NS5'-brane? When $k = 2$, the gauge theory analysis for unitary group was given in [44]. It would be interesting to study whether there exist nonsupersymmetric meta-stable vacua and if not, how one can think of the possible deformations in the superpotential to make them stabilize at one-loop?

- When there are multiple $k'$ middle NS5-branes, which is the same number as the one of NS5'-branes, in section 4, how one can analyze the meta-stable vacuum? For the supersymmetric brane configuration corresponding to the gauge theory [45 46 47], some of the analysis
was done in [25] where there are the higher order term for an adjoint field in the superpotential and the interaction terms between an adjoint and quarks and symmetric tensor.

- Similarly, when there are multiple $k'$ middle NS5-branes in section 5 (there are $(2k' + 1)$ middle NS5-branes in section 6), how one can analyze the meta-stable vacuum? For the supersymmetric brane configuration [29] corresponding to this gauge theory [46] which has extra two adjoint fields, the superpotential has more general form.

- It is possible to consider the case where there are multiple $k$ NS5-branes and a single NS5'-brane (and their mirrors) in section 7. It would be interesting to study whether there exist nonsupersymmetric meta-stable vacua.

- As mentioned before, when there are multiple $k'$ middle NS5-branes in section 8, the analysis for the supersymmetric brane configuration [25] explains the gauge theory result [46]. It is an open problem to find whether there exist nonsupersymmetric meta-stable vacua.

- Also one can think of the case when there are multiple $k$ NS5-branes in section 9. It would be interesting to find out the supersymmetric ground states and nonsupersymmetric ground states.

- When there are multiple $k'$ middle NS5-branes in section 10, how one can analyze the meta-stable vacuum? It is known that the supersymmetric brane configuration [25] describes the gauge theory [46]. On the other hand, one can consider the case where there exist multiple $k$ NS5-branes with a single NS5'-brane or combination of both multiple $k'$ middle NS5-branes and multiple $k$ NS5-branes in section 10. For the dual of second gauge group $SU(N'_c)$, we treated only $k' = 1$ NS5'-brane but it is also possible to consider the general $k'$ case where there are many meson fields.

- Also when there are multiple $k$ NS5-branes with a single NS5'-brane, or when there are multiple $k$ NS5-branes and multiple $k'$ NS5'-branes in section 11, how the meta-stable vacua as well as supersymmetric ones appear?

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