The principle of fatigue model test for spur gears based on similarity theory

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Abstract. Spur gear fatigue bench test is an expensive and time-consuming process. Especially for the big spur gears, the installations and measurements of them are usually very difficult. In order to solve this problem, this paper analysed the similarity principles of geometry, contact stress, bending stress, number of cycles and other factors for spur gears. The principle of fatigue model test for spur gears is proposed based on similarity theory. It is the basis of spur gear fatigue model test.

1. Introduction
Spur gear is the widely used many kinds of transmission systems. It is one of the most important parts of the mechanical productions [1]. Due to the complexity of their structure, loading, and working environment, most of the developed spur gears must be tested and verified before they are used in mechanical products [2]. While gear fatigue bench test is an expensive and time-consuming process. And the full-scale gears usually are too big and the installations and measurements of them are very difficult in the bench test process [3]. Fatigue model test is a good way to solve this problem. Fatigue similarity analysis is the foundation to do model test [4].

Similarity refers to the similarity relationship in nature and function between things or physical process. It is a way to study objective laws by using the similarity features between things or physical processes. The basis of similarity applications are the three similarity theorems, which is the theoretical basis of model test [5]. In the process of studying the similarity of physical processes, it is usually depend on the dimensional analysis method.

The physical quantities describing a physical process are composed of some fundamental quantities, which can express other physical quantities. These fundamental quantities are called dimensions. The aim of dimensional analysis method is to analyze the relationship of the quantities of a physical process. Through dimensional analysis, the dimensional variables can be converted to no dimensional quantities, and the no dimensional quantity equations of a physical process can be got. The similar physical process can be expressed by the same equation with no dimensional quantities. After that, the similar principle between two similar physical processes can be established [6].

The three dimension independent physical quantities can be selected in geometry, kinematics and dynamics. This is helpful to meet the requirement of dimension independent [7]. In the process of determining the dimensionless Pi items, the founder mental dimension independent quantities should be selected from the quantities which can describe the physical process. If the dimensions of the other physical quantities can be expressed by the products of the powers of the fundamental dimensions, the
ratios of the quantities can be considered the dimensionless Pi terms. After that, according to the second theorem of similarity, the results and the data of model test can be applied to the prototype [8].

In this paper, the similarity principles of geometry, contact stress, bending stress, number of cycles, and other factors are analyzed with dimensional analysis method for spur gears. Then, the fatigue similarity principles for spur gears are proposed. It provides the theoretical basis of fatigue model test for spur gears.

2. Geometric similarity of spur gears

According to the geometry similar theory, two helical gears can be considered similar in geometry if they have the same ratio of characteristic length dimension. Actually, cylindrical spur gear has six basic geometric parameters. They are tooth number \( z \), modulus \( m \), pressure Angle \( \alpha \), tooth width \( b \), addendum coefficient \( h^*a \), clearance coefficient \( c^* \). The pressure angles at different points on the involute tooth profile have different values. In order to make it convenient for design and manufacture of spur gears, the pressure angles on the reference circle are standardized in China. Usually, the value of it is \( 20^\circ \). The parameter addendum is the product of modulus multiplied by addendum coefficient. And the parameter dedendum is the product of modulus multiplied by the sum of addendum coefficient and clearance coefficient. In China, as for normal spur gears, when \( m \geq 1 \) mm, \( h^*a = 1 \), \( c^* = 0.25 \); When \( m < 1 \) mm, \( h^*a = 1 \), \( c^* = 0.35 \). And the other geometrical parameters of spur gears can be expressed by the basic parameters. The tooth number and pressure angle are dimensionless parameters. The similar ratio of them is 1. The other parameters have the dimensions of the lengths.

The calculation formulas, dimensions, and similarity ratio of spur gear parameters are shown in Table 1.

| Physical quantity | Symbol | Calculation formula | Dimension | Similarity ratio |
|-------------------|--------|---------------------|-----------|-----------------|
| Modulus           | \( m \) | —                   | \( L \)    | \( L_r \)       |
| Pressure angle    | \( \alpha \) | —                   | —         | 1               |
| Tooth number      | \( z \)   | —                   | —         | 1               |
| Reference diameter| \( d \)   | \( d_i = mz_i \)    | \( L \)    | \( L_r \)       |
| Base diameter     | \( d_b \) | \( d \cos \alpha \) | \( L \)    | \( L_r \)       |
| Addendum          | \( h_a \) | \( h_a = h^*a \)    | \( L \)    | \( L_r \)       |
| Dedendum          | \( h_f \) | \( h_f = (h^*a + c^*)m \) | \( L \)    | \( L_r \)       |
| Tooth thickness   | \( s \)   | \( s_t = \pi m / 2 \) | \( L \)    | \( L_r \)       |
| Pitch             | \( p_t \) | \( p_t = \pi m \)   | \( L \)    | \( L_r \)       |
| Facewidth         | \( b \)   | —                   | \( L \)    | \( L_r \)       |

It can be seen from the geometric similar analysis for spur gears that the requirements in geometric similar are: the same similar ratios of module, reference diameter, face width and the other parameters which have dimensions of length.

As for a gear pair, in addition to the geometric features of involved individual gears, another important geometry feature is the contact ratio. It is another important geometrical factor involved in the gear fatigue life. For a spur gear pair, the calculation formula of contact ratio \( \varepsilon_a \) is [9, 10]:

\[
\varepsilon_a = \frac{1}{2\pi} \left[ z_1 \left( \tan \alpha_{a1} - \tan \alpha' \right) + z_2 \left( \tan \alpha_{a2} - \tan \alpha' \right) \right]
\]  

(1)

Where, \( z_1 \) is the tooth number pinion; \( z_2 \) is the tooth number of wheel; \( \alpha' \) is the actual engaging angle, for standard gear and standard installation \( \alpha' = \alpha = 20^\circ \). \( \alpha_{a1} \) and \( \alpha_{a2} \) can be expressed by the following formulas:
$\alpha_{a1} = \arccos \frac{z_1 \cos \alpha}{z_1 + 2h^*}$

(2)

$\alpha_{a2} = \arccos \frac{z_2 \cos \alpha}{z_2 + 2h^*}$

(3)

If the wheels and pinions of two spur gear pairs have the same tooth number, pressure angle, module and tooth width, they must have the same contact ratio. According to the definition of the geometric similarity, spur gear pairs, which are similar in geometry, must have the same contact ratio value.

3. Fatigue similarity model for spur gears

Gear drive system transmits torque under a center time, space and load environment. It is a typical engineering mechanical system. There are many factors affect the fatigue life of spur gears. Such as contact stress, tooth root stress, number of stress cycles, materials, processing quality, lubrication and the working environment etc.[11, 12]. Among them, contact stress and tooth root stress, number of stress cycles, and materials are the governing factors. In order to get the fatigue similarity model for the spur gears. It needs to analyze the similar principles of these governing factors.

3.1. Tooth contact stress similarity analysis

For the spur gears, the tooth contact stress generally refers to the contact stress at pitch point. It can be expressed by the following formulas [13]:

$$\sigma_H = \sqrt{\frac{2KT}{bd^2} \frac{u \pm 1}{u} \cdot Z_H Z_E}$$

(4)

where, the “+” is for the external gear, and the “-” is for the internal gear; $K$ is the load coefficient, which is related to the gear working environment, the prime mover, and the load characteristics; $d$ is the reference circle diameter; $T$ is the transmission torque; $b$ is the tooth width; $\mu$ is the ratio of tooth number; $Z_H$ is the pitch point area coefficient, when the pressure angle $\alpha = 20^\circ$, $Z_H = 2.5$; $Z_E$ is the elastic coefficient of the gear material, which represents the influence of the material elastic module $E$ and Poisson's ratio $\nu$, and it can be denoted as:

$$Z_E = \sqrt{\frac{1}{\pi \left( \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right)}}$$

(5)

Where, $E_1$ is the elastic module of the driving gear material; $E_2$ is the elastic module of the driven gear material; $\nu_1$ is the Poisson's ratio of the driving gear material; $\nu_2$ is the Poisson's ratio of the driven gear material.

There are eight parameters involved in the contact stress of spur gears. They are the input torque $T$, the pitch circle diameter $d$, the ratio of tooth number $\mu$, the pressure angle $\alpha$, he poisson's ratio $\nu$ and elastic module $E$ of the material. The contact stress of the helical gear can be expressed as the following function:
\[ \sigma_H = f(T, d, b, u, \varepsilon_\alpha, \alpha, \nu, E) \] (6)

In the equation (6) \( \mu, \alpha \) and \( \nu \) are dimensionless variables. The other variables can be expressed by power law formulas in terms of dimensions of the fundamental quantities length (L), mass (M) and time (T). For example, the dimension of torque \( T \) can be denoted as: \([T] = [L]^2[M]^{2}[T]^{-2}\). The variables involved in the spur gear contact stress and their corresponding exponents of fundamental quantities are list in Table 2.

In Table 2, there are two dimensional in depended variables. Therefore, the number of dimensionless Pi terms is \(8-2=6\). If \( E \) and \( d \) are selected as the founder mental dimensional independent variables. The dimensionless Pi terms can be written as:

\[ \pi_1 = \frac{\sigma_H}{Td^{-3}}, \pi_2 = \frac{E}{Td^{-3}}, \pi_3 = \frac{b}{d}, \pi_4 = u, \pi_5 = \alpha, \pi_6 = \nu \] (7)

### Table 2. Dimensions of the variables about contact stress of spur gears

|       | \([\sigma_H]\) | \([T]\) | \([d]\) | \([b]\) | \([\mu]\) | \([\alpha]\) | \([\nu]\) | \([E]\) |
|-------|-------------|------|------|------|-------|------|------|------|
| \([\text{L}]\) | -1 | 2 | 1 | 1 | 0 | 0 | 0 | -1 |
| \([\text{M}]\) | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 1 |
| \([\text{T}]\) | -2 | -2 | 0 | 0 | 0 | 0 | 0 | -2 |

If two spur gears are similar in geometry and they have the same material performance \( \nu \) and \( E \), there are: \( \pi_3 = L_r = \text{const}, \pi_4 = 1 = \text{const}, \pi_5 = \text{const}, \pi_6 = \text{const} \). According to the similarity theorem, \( \pi_1 \) and \( \pi_2 \) also should be const. They can be expressed by the following equations.

\[ \pi_1 = \frac{\sigma_H}{Td^{-3}} = \text{const} \] (8)

\[ \pi_2 = \frac{E}{Td^{-3}} = \text{const} \] (9)

From the equations (8) and (9), it can be known that the requirement of two geometric similar spur gears have the same contact stress is the ratio of torque transmitted by them should have the following relations:

\[ \frac{T_m}{T_p} = \left( \frac{d_m}{d_p} \right) = L_r^3 \] (10)

Where \( T_m \) is the torque transmitted by the scale-down or scale-up similar gear model, \( T_p \) is the torque transmitted by the full-scale gear.

From the analysis of the contact stress similarity for spur gears, it can be known that the geometric similar gears have the same value of contact stress if they have the same material’s elastic module and Possion’s ratio, and the ratio of the torque transmitted by them is equal to the cube of their geometric similarity ratio.
3.2. Tooth root bending stress similarity analysis

Because the bending moment of the spur gear has the largest value at the root of the tooth and there is stress concentration at this area. The bending fatigue strength at the area of root radius is weakest. Therefore, the tooth bending fatigue stress of the external spur gear is calculated by the following equation [14]:

\[
\sigma_F = \frac{2KT}{bdm} Y_{fa} Y_{sa} Y_{\epsilon}
\]  

(11)

Where, \(K\) is the load coefficient; \(Y_{fa}\) is the tooth shape coefficient; \(Y_{sa}\) is the stress correction coefficient; \(Y_{\epsilon}\) is the contact ratio coefficient.

It can be known from equation (11) that the tooth root bending stress of spur gear is mainly determined by the physical quantities of torque, tooth width, reference circle diameter, modulus, tooth width and tooth number. The tooth root stress of spur gear can be expressed by the following function:

\[
\sigma_F = f(T,d,b,m,z)
\]  

(13)

The variables involved in the spur gear tooth root bending stress and their corresponding exponents of fundamental quantities are shown in Table 3.

In Table 3, tooth number \(z\) is a dimensionless variable. There are two dimension independent variables. Therefore, the number of dimensionless Pi terms is: 6-2=4. If \(T\) and \(d\) are selected as dimensional independent founder mental variables, the dimensionless Pi terms can be written as:

\[
\pi_1' = \frac{\sigma_F d^3}{T}, \quad \pi_2' = \frac{b}{d}, \quad \pi_3' = \frac{m}{d}, \quad \pi_4' = z
\]  

(14)

| Variables | [\(\sigma_F\)] | [\(T\)] | [\(d\)] | [\(b\)] | [\(m\)] | [\(z\)] |
|-----------|----------------|--------|---------|--------|--------|--------|
| \([\text{L}]\) | -1              | 2      | 1       | 1      | 1      | 0      |
| \([\text{M}]\) | 1               | 2      | 0       | 0      | 0      | 0      |
| \([\text{T}]\) | -2              | -2     | 0       | 0      | 0      | 0      |

For the geometric similar spur gears, \(\pi_3' = \text{const}, \pi_4' = \text{const}\). According to the similarity theorem, the tooth bending stress similarity requires all of the involved Pi terms must be const, so \(\pi_1' = \text{const}\), which can be denoted as:

\[
\pi_1' = \frac{\sigma_F d^3}{T} = \text{const}
\]  

(15)

From the equation (15) it can be known that the tooth root bending stress of the geometry similar spur gears will have the same value, if the proportional ratio of the torque transmitted by them is equal to the cube of the geometry ratio. It can be expressed by the following equation:

\[
\frac{T_m}{T_p} = L_r^3
\]  

(16)
Where $T_m$ is the torque transmitted by the scale-down or scale-up similar gear model, $T_p$ is the torque transmitted by the full-scale gear.

From the analysis of the tooth root stress for spur gears, it can be concluded that the geometric similar spur gears would have the same value of tooth bending stress, if the torque proportional ratio of them is equal to the cube of the geometric similarity ratio.

### 3.3. Number of stress cycles analysis for spur gears

During the gear working, the gear teeth go into meshing and exit from meshing for one time in a rotating circle. The tooth bending stress and the contact stress is changed for one time. The number of stress cycles $N$ can be denoted as [9]:

$$N = 60njL_n$$

where, $n$ is the speed of gear, it’s unit is r/min; $j$ is the engagement times of the same tooth surface for one rotation; $L_n$ is the working time of the gear, it’s unit is hour.

As for the geometric similar spur gears, they have the same value of ‘$j$’. If the product of their speed and working time has the same value, they should have the same number of contact stress cycles and the same number of tooth root bending stress cycles.

### 3.4. Other factors affecting fatigue similarity of spur gears

Gear transmission is very complex. Except for the stress amplitude, number of stress cycles and material performance, there are many other factors affect the fatigue performance of spur gears, such as the machining and assembly precision, heat treatment, linear velocity and working environments etc. In order to make the fatigue similarity of spur gears as far as possible, these factors should be taken into account.

Firstly, in order to ensure the materials of the similar spur gears have the same performance, the material’s type and grade of the gear should be as same as possible. If it is possible, the similar spur gears should be made by the same furnace smelting materials. Secondly, the machining process and the heat treatment of them, such as annealing, rolling, cutting, casting, etc. should be the same. In addition, their metallographic structure and mechanical properties should be kept consistent. In addition, the working environments, such as lubrication and rotation speed also have a great influence on the gear fatigue life. Therefore, the similar spur gears should use the same viscosity lubricant and the same lubrication method. Moreover, they should use the similar load spectrum, and the linear velocity of them should be kept as consistent as possible [15].

In the gear design process, the machining accuracy of spur gears is up to their linear velocity. While, the linear velocities of the similar spur gears with different geometrical size are different, if they have the same rotation speeds. This will affect the additional dynamic load of the gears, and then affect the fatigue life of them. Therefore, the linear velocity of the similar spur gears should be kept the same value. The linear velocity of any point on the spur gear can be denoted as:

$$V_n = r_n \omega$$

Where $r_n$ is the distance from point $n$ to the center of rotation $O$, and $\omega$ is the rotation speed.

From equation (18) it can be known that making two similar spur gears have the same linear velocity needs their rotate speed proportional ratio is equal to $1/L_n$. Therefore, the scale-down similar spur gears have higher rotate speed and shorter fatigue life time with the same number of stress cycles.

### 4. Conclusion

In this paper, we analyzed the similar principles of geometry, contact stress, tooth root stress, number of stress cycles, material performance and other factors that affect the fatigue life of spur gears. It can
be concluded that the spur gears will have the same fatigue performance if they meet the following similar requirements:

1. The spur gears should be similar in geometry. That means they should have the same geometric ratio, such as the similar ratios of module, reference circle diameter and tooth width. In addition, they should have the same tooth number, pressure angle, addendum coefficient and clearance coefficient.

2. The geometrical similar spur gears should have the same tooth contact stress and tooth root stress. The spur gear scale-up or scale-down similar models must have the same value of tooth contact stress and the tooth bending stress, if they are similarity in geometry, their materials have the same value of poisson's ratio and elastic module, and their torque ratio is equal to the cube of their geometric similar ratio.

3. The number of stress circles of the similar spur gears should be the same. If two spur gears have the same product value of the speed and the working time, their number of stress circles have the same value.

4. The other factors affecting fatigue performance should be kept consistent. For the similar spur gears, the machining and assembly precision, heat treatment, linear velocity and working environments of them should be kept as same as possible.

The fatigue similar model of spur gears is the basis of fatigue model test for spur gears. While, the gear fatigue damage process is very complex, which are affected by many kinds of factors. In our spur gear fatigue similar model, the material and lubrication factors were simplified. The fatigue life relationship between the similar spur gears with different materials under different lubrication conditions should be further studied in the following investigations.

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