Optimizing a variable-rate diffusion to hit an infinitesimal target at a set time

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Abstract

I consider a stochastic optimization problem for a one-dimensional continuous martingale whose diffusion rate is constrained to be between two positive values \( r_1 < r_2 \). The problem is to find an optimal adapted strategy for the choice of diffusion rate in order to maximize the chance of hitting an infinitesimal region around the origin at a set time in the future. More precisely, the parameter associated with “the chance of hitting the origin” is the exponent for a singularity induced at the origin of the final time probability density. I show that the optimal exponent solves a transcendental equation depending on the ratio \( r_2/r_1 \).

1 Introduction

Pick \( y \in \mathbb{R} \) and positive numbers \( r_1, r_2, T \) with \( r_1 < r_2 \). For a Borel measurable function \( D : \mathbb{R} \times [0, T] \to [r_1, r_2] \), let \( X_t \in \mathbb{R} \) be the weak solution to the one-dimensional stochastic differential equation

\[
dX_t = \sqrt{D(X_t, t)} dB_t, \quad X_0 = y, \quad t \in [0, T]
\]  

(1.1)

for a standard Brownian motion \( B_t \). Weak solutions to diffusion equations of the form (1.1) exist and are unique by [4, Sect. 7.3]. In broad terms the question I address in this article is the following: What choice of diffusion coefficient maximizes the probability that \( X_t \) lands in an infinitesimal region around the origin at the final time \( T \) given the constraint \( r_1 \leq D(x, t) \leq r_2 \)? To form a more precise statement of the question, let us consider the probability densities \( P_{y,t}^{(D)}(x) \) satisfying the forward Kolmogorov equation corresponding to the stochastic differential equation above:

\[
\frac{d}{dt} P_{y,t}^{(D)}(x) = \frac{1}{2} \frac{d^2}{dx^2} \left( D(x, t) P_{y,t}^{(D)}(x) \right), \quad P_{y,0}^{(D)}(x) := \delta_y(x), \quad t \in [0, T].
\]  

(1.2)

Maximizing the chance of landing in “an infinitesimal region around the origin” at time \( T \) does not merely mean maximizing the density \( P_{y,T}^{(D)}(x) \) at \( x = 0 \), it turns out, because the density \( P_{y,T}^{(D)}(x) \) can have an infinite singularity at \( x = 0 \). Thus the problem shifts to maximizing the strength of the singularity for \( P_{y,T}^{(D)}(x) \) at \( x = 0 \), which I will characterize through the limit

\[
\mathcal{I}(D) = \liminf_{\epsilon \to 0} \left\{ 1 - \frac{\log \left( \int_{|x| \leq \epsilon} P_{y,T}^{(D)}(x) \right)}{\log(\epsilon)} \right\} \in [0, 1).
\]  

(1.3)
Notice that if $\mathcal{P}^{(D)}_{x,t}(x) \sim |x|^{-\eta}$ around $x = 0$ for $\eta > 0$, then $\mathcal{I}(D) = \eta$. In particular, if $D(x, t)$ is constant, then $\mathcal{P}^{(D)}_{x,t}(x)$ is bounded and $\mathcal{I}(D) = 0$.

If we think of $D(x, t)$ as the strategy of a random walker $X_t$ attempting to maximize his chance of arriving to the origin at time $T$, it is reasonable that he should rush with the maximum diffusion rate $r_2$ when he judges himself to be far given the time remaining, and he should choose to hide his time with the minimum diffusion rate $r_1$ when he judges himself to be close. Thus it natural to have $D(x, t) \nearrow r_2$ as $|x| \nearrow \infty$ for all $t \in [0, T)$. Since $X_t$ is a time-changed Brownian motion with diffusion rates restricted to $[r_1, r_2]$, our optimization problem inherits a Brownian scale invariance when viewed from the origin and the final time $T \in \mathbb{R}^+$; the random walker should make the same choice of diffusion rate at space-time points $(x, t)$ and $(x', t')$ in $\mathbb{R} \times [0, T)$ for which $\frac{x^2}{T-t} = \frac{x'^2}{T-t'}$. Any strategy $D(x, t)$ consistent with the above scale invariance should satisfy

$$D(x, t) = D\left( x \left( \frac{T-t'}{T-t} \right)^{\frac{1}{2}}, t' \right), \quad t', t' \in [0, T). \quad (1.4)$$

For the above reason, I will focus my analysis on diffusion coefficients of the form $D(x, t) = R\left( \frac{x}{\sqrt{T-t}} \right)$ for measurable functions $R : \mathbb{R} \to [r_1, r_2]$. It is also natural to assume that $R$ is symmetric about zero.

Theorem 1.1 is the main result of this article. To state the result we need to define positive numbers $\kappa$ and $\eta$ solving the pair of equations (1.5), which depend on the diffusion bounds $r_1, r_2$ through their ratio $V := (\frac{r_2}{r_1})^{\frac{1}{2}}$. For $\eta > 0$ define the function $Y_\eta : \mathbb{R} \to \mathbb{R}^+$ by $Y_\eta(x) := \int_0^\infty dz z^{\eta-1} e^{-\frac{1}{2} (z+x)^2}$. Given $V \in \mathbb{R}^+$ let the constants $\eta \equiv \eta(V), \kappa \equiv \kappa(V)$ be determined by

$$\frac{Y_{\eta+1}(\frac{\kappa}{\sqrt{V}})}{Y_\eta(\frac{\kappa}{\sqrt{V}})} = 1 - \eta - \frac{\kappa^2}{2V} \quad \text{and} \quad \frac{\kappa \int_0^\infty d a \left( Y_\eta(a) + Y_\eta(-a) \right) e^{\frac{\kappa^2}{2} - \frac{a^2}{2V}}}{Y_\eta(\kappa) + Y_\eta(-\kappa)} = 1. \quad (1.5)$$

I will denote the collection of symmetric, Borel measurable functions from $\mathbb{R}$ to $[r_1, r_2]$ such that $\lim_{z \to \infty} R(z) = r_2$ by $B(\mathbb{R}, [r_1, r_2])$.

**Theorem 1.1.** Fix $y \in \mathbb{R}$ and positive numbers $T, r_1, r_2$ with $r_2 > r_1$. Let $\mathcal{P}^{(R)}_{y,t} \in L^1(\mathbb{R})$ obey the Kolmogorov equation (1.2) with $D(x, t) = R\left( \frac{x}{\sqrt{T-t}} \right)$. For $V := (\frac{r_2}{r_1})^{\frac{1}{2}}$ the following equality holds:

$$\eta(V) = \max_{R \in B(\mathbb{R}, [r_1, r_2])} \lim_{\epsilon \nearrow 0} \epsilon^{-1} \left( \frac{1 - \log \left( \int_{|x| \leq \epsilon} \mathcal{P}^{(R)}_{y,t}(x) \right)}{\log(\epsilon)} \right). \quad (1.6)$$

The above maximum is attained uniquely for $R^* : \mathbb{R} \to [r_1, r_2]$ of the form

$$R^*(x) := r_1 + (r_2 - r_1) \chi(|x| \geq r_1 \kappa^2(V)). \quad (1.7)$$

It is instructive to examine the limiting behavior of the exponent $\eta(V)$ and the cut-off parameter $\kappa(V)$ characterizing the optimal solution (1.5) in the respective limits $V \searrow 1$ and $V \nearrow \infty$. One surprise is that for large $V$ the optimal cut-off $\kappa(V)$ approaches the finite value $\kappa \in \mathbb{R}^+$ solving the equation $1 = \kappa \int_0^\infty da e^{\frac{\kappa^2}{2} - \frac{a^2}{2V}}$.

- The values $\eta(V), \kappa(V)$ increase continuously with the parameter $V \in (1, \infty)$.
- As $V \searrow 1$, $\eta(V) \searrow 0$ and $\kappa(V) \searrow 1$.
- As $V \nearrow \infty$, $\eta(V) \nearrow 1$ and $\kappa \nearrow \kappa$. Moreover $1 - \eta(V) \approx \frac{\sqrt{2\pi} V^{\frac{1}{2}}}{\sqrt{\kappa}} V^{-1}$ for large $V$. 
The optimization problem described above focuses on maximizing the probability of certain vanishingly low chance events. In particular there is no penalty for landing far from the target region. It is a much different problem, for instance, to minimize a quantity of the form

\[ \bar{I}(D) = \int_{\mathbb{R}} dx \mathcal{P}^{(D)}_{y,T}(x) \varphi(x), \]  

where \( \mathcal{P}^{(D)}_{y,T} \) is defined as in (1.2) and \( \varphi : \mathbb{R} \to \mathbb{R}^+ \) is a convex function quantifying the penalty for landing away from the target point at the final time \( T \). When \( D(x,t) \) is restricted to the range \([r_1, r_2]\), the optimal strategy for the penalty problem is simply to always use the lowest available diffusion rate \( r_1 \). Discussion of optimization problems in stochastic settings can be found in [1, 2].

2 The stationary dynamics

The restriction of the diffusion coefficient \( D(x,t) \) to the parabolic form \( R(\sqrt{\frac{x}{T-t}}) \) for a measurable function \( R : \mathbb{R} \to [r_1, r_2] \) implies that a solution to the Kolmogorov equation (1.2) is equivalent under a time-space reparameterization to the solution of a stationary dynamics (2.2). For \( (x,t) \in \mathbb{R} \times [0,T) \) let \( (z,s) \in \mathbb{R} \times [-\log(T), \infty) \) be given by

\[ (x,t) \mapsto (z,s) = \left( \frac{x}{\sqrt{T-t}}, -\log(T-t) \right). \]  

Through the transformation (2.1), we can use \( \mathcal{P}^{(R)}_{y,T}(x) \) to define new probability densities \( \psi^{(R)}_{b,s}(z) := e^{-\frac{1}{2}s} \mathcal{P}^{(R)}_{b,T-e^{-s}}(e^{-\frac{1}{2}s}z) \), which satisfy the stationary forward Kolmogorov equation

\[ \frac{d}{ds} \psi^{(R)}_{b,s}(z) = -\frac{1}{2} \frac{d}{dz} \left( z \psi^{(R)}_{b,s}(z) \right) + \frac{1}{2} \frac{d^2}{dz^2} \left( R(z) \psi^{(R)}_{b,s}(z) \right), \]  

where \( b := \frac{y}{\sqrt{T}}, s \in [-\log(T), \infty) \), and \( \psi^{(R)}_{b,-\log(T)}(z) := \delta_b(z) \). The reparameterized process is thus a diffusion with a repulsive drift that grows proportionally to the distance from the origin:

\[ dZ_s = \frac{1}{2} Z_s ds + \sqrt{R(Z_s)} dB_s', \quad Z_0 = b, \]  

where \( B_s' \) is a copy of standard Brownian motion. The trajectories for the process \( Z_t \) will undergo an essentially exponential divergence to infinity after wandering near the origin for a finite time period. The state of the original process \( X_t \) at the final time \( T \) is recovered by the limit

\[ X_T = \lim_{s \to \infty} \frac{Z_s}{e^s} = Z_0 + \int_0^\infty e^{-\frac{s}{2}} \sqrt{R(Z_v)} dB_v'. \]  

In the next two sections I will study the stationary dynamics more closely.

3 Analysis of the generators for the stationary dynamics

Let \( B(\mathbb{R}, [r_1, r_2]) \) denote the collection of Borel measurable functions from \( \mathbb{R} \) to \([r_1, r_2]\). For a given element \( R \in B(\mathbb{R}, [r_1, r_2]) \), consider the backwards Kolmogorov generator

\[ \mathcal{L}^{(R)} f := \frac{1}{2} x \frac{d}{dx} f + \frac{1}{2} R(x) \frac{d^2}{dx^2} f. \]  

(3.1)
The operator $\mathcal{L}^{(R)}$ is self-adjoint when acting on the weighted $L^2$-space defined below. Let $L^2(\mathbb{R}, w(x)dx)$ be the Hilbert space with inner product

$$\langle f | g \rangle_R := \int_{\mathbb{R}} dx w(x) \overline{f(x)} g(x) \quad \text{for weight} \quad w(x) := \frac{e^{\int_0^x dv \frac{v^2}{R(v)}}}{R(x)}.$$  

The corresponding norm is denoted by $\|f\|_{2,R} := \sqrt{\langle f | f \rangle_R}$.

**Lemma 3.1.** Let $R \in B(\mathbb{R}, [r_1, r_2])$. The operator $\mathcal{L}^{(R)}$ is self-adjoint when assigned the domain

$$\mathcal{D} = \left\{ f \in L^2(\mathbb{R}, w(x)dx) \mid \left\| \frac{d^2}{dx^2} f \right\|_{2,R} < \infty \right\}.$$  

Moreover, $(\mathcal{L}^{(R)}, \mathcal{D})$ and $(\frac{d^2}{dx^2}, \mathcal{D})$ are mutually relatively bounded.

**Proof.** The first task is to verify that $\mathcal{L}^{(R)}$ maps $\mathcal{D}$ into $L^2(\mathbb{R}, w(x)dx)$. Using integration by parts, I have the equality below for all smooth functions $f \in L^2(\mathbb{R}, w(x)dx)$ with $\frac{d^2}{dx^2} f \in L^2(\mathbb{R}, w(x)dx)$.

$$\|\mathcal{L}^{(R)} f\|_{2,R}^2 = \frac{1}{4} \left\| R(x) \frac{d^2}{dx^2} f \right\|_{2,R}^2 - \frac{1}{4} \int_{\mathbb{R}} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2$$

$$\geq \frac{r_1^2}{4} \left\| \frac{d^2}{dx^2} f \right\|_{2,R}^2 - \frac{1}{4} \int_{\mathbb{R}} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2.$$  

The inequality (3.3) simply uses that $R(x) \geq r_1$. The equality (3.2) extends to all elements in $\mathcal{D}$ and implies that $\|\mathcal{L}^{(R)} f\|_{2,R} \leq r_2 \left\| \frac{d^2}{dx^2} f \right\|_{2,R}$ since $R(x) \leq r_2$. Hence $\mathcal{L}^{(R)}$ is well-defined on $\mathcal{D}$. The operator $(\mathcal{L}^{(R)}, \mathcal{D})$ is symmetric by integration by parts, and the domain $\mathcal{D}'$ for $(\mathcal{L}^{(R)})^*$ consists of all functions $f \in L^2(\mathbb{R}, w(x)dx)$ with $\|\mathcal{L}^{(R)} f\|_{2,R} < \infty$. To show that $(\mathcal{L}^{(R)}, \mathcal{D})$ is self-adjoint, it is sufficient to prove that $(\frac{d^2}{dx^2}, \mathcal{D})$ is relatively bounded to $(\mathcal{L}^{(R)}, \mathcal{D})$.

I focus on showing that $\frac{d^2}{dx^2}$ is also relatively bounded to $\mathcal{L}^{(R)}$. With the lower bound (3.3), it will be enough to demonstrate that there is a $C > 0$ such that

$$\int_{\mathbb{R}} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2 \leq C \|f\|_{2,R}^2 + \frac{r_1^2}{2} \left\| \frac{d^2}{dx^2} f \right\|_{2,R}^2.$$  

It is convenient to split the integration over $\mathbb{R}$ into the domains $|x| \leq L$ and $|x| > L$ for some $L \gg 1$ to get the bound

$$\int_{|x| \leq L} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2 \leq e^{\frac{L}{r_1}} \int_{\mathbb{R}} dx \left| \frac{df}{dx} (x) \right|^2 + \int_{|x| \geq L} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2.$$  

For the first term on the right side of (3.5), using integration by parts, Cauchy-Schwarz, and the inequality $2uv \leq u^2 + v^2$ yields the first inequality below for any $c > 0$:

$$\int_{\mathbb{R}} dx \left| \frac{df}{dx} (x) \right|^2 \leq c \int_{\mathbb{R}} dx f(x)^2 + \frac{1}{c} \int_{\mathbb{R}} dx \left| \frac{d^2 f}{dx^2} (x) \right|^2 \leq cr_2 \|f\|_{2,R}^2 + \frac{r_2^2}{c} \left\| \frac{d^2}{dx^2} f \right\|_{2,R}^2.$$  

The second inequality of (3.6) follows from the relation $w(x) \geq r_2^{-1}$. For the second term on the right side of (3.5), I have the inequalities

$$\int_{|x| \geq L} dx e^{\int_0^x dv \frac{v^2}{R(v)}} \left| \frac{df}{dx} (x) \right|^2 \leq \frac{r_2^2}{L^2} \left\| x \frac{d}{dx} f \right\|_{2,R}^2 \leq \frac{4r_2^3}{L^2} \left\| \frac{d^2}{dx^2} f \right\|_{2,R}^2.$$  

\]
The first inequality of (3.7) is Chebyshev’s, and the second inequality is discussed below. By writing $\mathcal{L}^{(R)} f = \frac{1}{2} x \frac{d}{dx} f + \frac{1}{2} R(x) \frac{d^2}{dx^2} f$ and expanding the left side of (3.2), I obtain the following inequality:

$$\| x \frac{d}{dx} f \|_{2,R}^2 \leq -2 \text{Re} \left( \langle x \frac{d}{dx} f | R(x) \frac{d^2}{dx^2} f \rangle_{2,R} \right) \leq 2r_2 \| x \frac{d}{dx} f \|_{2,R} \| \frac{d^2}{dx^2} f \|_{2,R}.$$  

The second inequality is by Cauchy-Schwarz and $R(x) \leq r_2$. Thus $\| x \frac{d}{dx} f \|_{2,R}$ is smaller than $2r_2 \| \frac{d^2}{dx^2} f \|_{2,R}$ as required to get the second inequality of (3.7).

By picking $L \in \mathbb{R}^+$ with $L^2 \geq \frac{16b^3}{r_2^2}$ and $c \in \mathbb{R}^+$ with $c \geq e^{\frac{4r_2^2}{16}}$, I obtain the inequality (3.4) for $C = cr_2 e^{\frac{4r_2^2}{16}}$. The mutual relative boundedness of $\mathcal{L}^{(R)}$ and $\frac{d^2}{dx^2}$ was shown in the analysis above.

In the statement of the proposition below, I denote the maximum element in the spectrum of $\mathcal{L}^{(R)}$ by $\Sigma(\mathcal{L}^{(R)})$.

**Proposition 3.2.** Let $R \in B(\mathbb{R}, [r_1, r_2])$ be symmetric about zero and $f, \frac{d^2}{dx^2} f \in L^2(\mathbb{R}, w(x)dx)$.

1. The operator $\mathcal{L}^{(R)}$ has compact resolvent.
2. The eigenvalues for $\mathcal{L}^{(R)}$ are negative. When $R$ is increasing over the interval $[0, \infty)$ and not strictly constant, then the largest eigenvalue $\Sigma(\mathcal{L}^{(R)})$ lies in the interval $(-\frac{1}{2}, 0)$.
3. The eigenvalue $\Sigma(\mathcal{L}^{(R)})$ is non-degenerate, and the phase of the corresponding eigenfunction can be chosen so that the following properties hold:
   - The values $\phi(x)$ are strictly positive for all $x \in \mathbb{R}$ and $\phi(x) = \phi(-x)$.
   - The second derivative of $\phi$ is contained in $L^2(\mathbb{R}, w(x)dx)$ and $R(x) \frac{d^2}{dx^2} (x)$ is continuous.
   - The function $\phi$ is strictly decreasing over the interval $(0, \infty)$.
   - The function $\phi$ has a unique inflection point $c > 0$ over the interval $(0, \infty)$.
4. The following equality holds for any $b \in \mathbb{R}$:

$$\lim_{s \to \infty} \log \left( \int_\mathbb{R} dx \psi_{b,s}^{(R)}(x)f(x) \right) = \Sigma(\mathcal{L}^{(R)})$$

**Proof.**

Part (1): Define the functions $\ell_+: \mathbb{R} \to \mathbb{R}$ such that $\ell_+(x) := \ell_-(x)$ and $\ell_-(x) := \int_{-\infty}^x dz e^{-\int_0^z \frac{d}{dy} R(y) dy}$. Notice that $\ell_+, \ell_-$ are the fundamental solutions to the differential equation

$$(\mathcal{L}^{(R)} g)(x) = \frac{1}{2} x \frac{dg}{dx}(x) + \frac{1}{2} R(x) \frac{d^2}{dx^2}(x) = 0.$$  

Also, define the functions $c_\pm : \mathbb{R} \to \mathbb{R}^+$ as $c_\pm(z) := \frac{2\ell_\pm(x) \gamma(z)}{\gamma(R(z))}$, where $\gamma := \int_{-\infty}^\infty dz e^{-\int_0^z \frac{d}{dy} R(y) dy}$. By the standard technique of pasting together the fundamental solutions, the Green function $G : \mathbb{R}^2 \to \mathbb{R}$ satisfying $-((\mathcal{L}^{(R)})^{-1} f)(x) = \int_\mathbb{R} dx G(x, z)f(z)$ can be written in the form

$$G(x, z) = c_-(z) \ell_-(x) \chi(x \leq z) + c_+(z) \ell_+(x) \chi(x > z).$$  

(3.8)

There is a canonical isometry from $L^2(\mathbb{R}, w(x)dx)$ to $L^2(\mathbb{R})$ given by the map sending $f(x)$ to $w^{-\frac{1}{2}}(x)f(x)$. Thus the kernel $\widehat{G}(x, z) := w\frac{1}{2}(x)G(x, z)w^{-\frac{1}{2}}(z)$ yields the Hilbert-Schmidt norm though
the standard formula \( \| (\mathcal{L}(R))^{-1} \|_{HS} = \int_{\mathbb{R}^2} dx dz |\hat{G}(x, z)|^2 \). However, the quantity \( \int_{\mathbb{R}^2} dx dz |\hat{G}(x, z)|^2 \) is finite given the form (3.8), and hence \( \mathcal{L}(R) \) has compact resolvent.

Part (2): The largest eigenvalue of \( \mathcal{L}(R) \) is the negative inverse of the largest eigenvalue for \(- (\mathcal{L}(R))^{-1}\). Since \(- (\mathcal{L}(R))^{-1}\) has a strictly positive integral kernel \(G(x, z)\), the eigenfunction \(\phi\) associated with the leading eigenvalue of \(- (\mathcal{L}(R))^{-1}\) is strictly positive-valued (for the correct choice of phase) and unique. The leading eigenvalue for \(- (\mathcal{L}(R))^{-1}\) is positive and given by the convex integral of values

\[
\int_{\mathbb{R}} dz \frac{\phi(z)}{||\phi||_1} \int_{\mathbb{R}} dx G(x, z). \tag{3.9}
\]

Note that \(\phi \in L^2(\mathbb{R}, w(x)dx)\) ensures that \(||\phi||_1\) is finite. However, I have the following equality:

\[
\int_{\mathbb{R}} dx G(x, z) = 2 + 2e^{-\int_0^z dv \frac{R(v)}{R(z)}} \int_z^\infty dR(x)e^{-\int_0^x dv \frac{R(v)}{R(x)}}, \tag{3.10}
\]

where the expression \(\int_0^\infty dR(z)e^{-\int_0^z dv \frac{R(v)}{R(z)}}\) happens to be non-negative for \(R\) symmetric and increasing over \(\mathbb{R}^+\). For \(R\) symmetric and increasing over \(\mathbb{R}^+\) and not constant, there must be some regions of \(z \in \mathbb{R}\) such that (3.10) is strictly greater than 2. Thus (3.9) must be > 2, and the largest eigenvalue for \(\mathcal{L}(R)\) is in the interval \((-\frac{1}{2}, 0)\).

Part (3): As remarked in Part (2), the eigenfunction \(\phi(x)\) with leading eigenvalue \(E := \Sigma(\mathcal{L}(R)) < 0\) must be strictly positive for all \(x \in \mathbb{R}^+\). The function \(\phi(x)\) is symmetric around the origin since \(R(x)\) is symmetric and otherwise \(\phi(-x)\) would be a second eigenfunction for the non-degenerate eigenvalue \(\Sigma(\mathcal{L}(R))\).

By Lem. 3.1, \(d^2 \phi dx^2\) is relatively bounded to \(\mathcal{L}(R)\), and thus the eigenfunctions of \(\mathcal{L}(R)\) lie in the domain of \(d^2 \phi dx^2\). The continuity of \(R(x)\) \(d^2 \phi dx^2\) follows from the equality

\[
-\frac{1}{2} \frac{d\phi}{dx} (x) = -E\phi(x) + \frac{1}{2} R(x) \frac{d^2 \phi}{dx^2} (x). \tag{3.11}
\]

Since \(R(x)\) \(\frac{d^2 \phi}{dx^2}\) is continuous and \(R(x) \geq r_1\) is bounded away from zero, an inflection point \(u\) for \(\phi\) must satisfy \(\frac{d^2 \phi}{dx^2}(u) = \lim_{x \to u} \frac{d^2 \phi}{dx^2}(x) = 0\).

From (3.11) we can see that the second derivative \(\frac{d^2 \phi}{dx^2}(x)\) is negative in a region around the origin \(|x| < c\), where \(c > 0\) denotes the inflection point closest to the origin over the interval \((0, \infty)\). An inflection point must exist since \(\phi(x)\) is positive, continuously differentiable, and decaying at infinity. By the symmetry of \(\phi\), \(\frac{d\phi}{dx}(x)\) is zero for \(x = 0\). Since the derivative of \(\frac{d\phi}{dx}(x)\) is negative over the interval \((0, c)\), we must have that \(\frac{d\phi}{dx}(x)\) is negative over the interval \((0, c)\). Suppose to reach a contradiction that there is some point \(u \in (c, \infty)\) such that either

\[
\text{(i). } \frac{d\phi}{dx}(u) = 0 \quad \text{or} \quad \text{(ii). } \frac{d^2 \phi}{dx^2}(u) = 0. \tag{3.12}
\]

I will let \(u\) denote the smallest such value. Notice that I can not have both \(\frac{d\phi}{dx}(u) = 0\) and \(\frac{d^2 \phi}{dx^2}(u) = 0\) since the term \(-E\phi(x)\) in (3.11) is strictly positive. For the cases (3.12), the following reasoning applies:

(i). If \(\frac{d\phi}{dx}(u) = 0\), then the continuous function \(\frac{1}{2} R(x) \frac{d^2 \phi}{dx^2}\) must be positive over the interval \([c, u]\). This, however, contradicts equation (3.11) for \(x = u\) since the terms on the right side of (3.11) are both positive.
(ii). If \( \frac{d^2 \phi}{dx^2}(u) = 0 \), then \( \frac{d \phi}{dx}(x) \) must be negative over the interval \([c, u]\). A linear approximation of equation (3.11) about the point \( u \) yields that

\[
\delta \frac{d \phi}{dx}(u) \left( -\frac{1}{2} + E \right) + O(\delta^2) = \frac{1}{2} R(u + \delta) \frac{d^2 \phi}{dx^2}(u + \delta), \quad |\delta| \ll 1. \tag{3.13}
\]

Since \( \frac{d \phi}{dx}(u) \) and \( E \) are negative, it follows from (3.13) that \( u \) must be an inflection point at which the concavity changes from down to up. However, by my definitions, \( \phi(x) \) is concave up over the interval \((c, u)\), which brings me to a contradiction.

It follows that \( \phi(x) \) is strictly decreasing over \((0, \infty)\) and has exactly one inflection point over that interval.

Part (4): Using the backward representation of the dynamics, I have the equality

\[
\int_{\mathbb{R}} da \psi_{b,s}^{(R)}(a) f(a) = (e^{s\mathcal{L}^{(R)}} f)(b),
\]

where by assumption \( f, \frac{d^2}{dx^2} f \in L^2(\mathbb{R}, w(x)dx) \). The function \( e^{s\mathcal{L}^{(R)}} f \) can be written as

\[
e^{s\mathcal{L}^{(R)}} f = e^{sE} \langle \phi | f \rangle_R \phi + e^{s\mathcal{L}^{(R)}} g \quad \text{for} \quad g := f - \langle \phi | f \rangle_R \phi,
\]

where \( \phi \) is the eigenfunction for \( \mathcal{L}^{(R)} \) corresponding to the leading eigenvalue \( E := \Sigma(\mathcal{L}^{(R)}) \), as before. Note that \( \frac{d^2}{dx^2} g \in L^2(\mathbb{R}, w(x)dx) \) by our assumption \( \frac{d^2}{dx^2} f \in L^2(\mathbb{R}, w(x)dx) \). Let \( E_1 \) be the largest eigenvalue following \( E \). I will show that \( e^{s\mathcal{L}^{(R)}} g \) decays uniformly with exponential rate \( -E_1 \) as \( s \to \infty \) over any compact interval \([-L, L]\). I have the following inequalities:

\[
\| e^{s\mathcal{L}^{(R)}} g \|_{2,R} \leq e^{sE_1} \| g \|_{2,R}, \tag{3.14}
\]

\[
\| \frac{d^2}{dx^2} e^{s\mathcal{L}^{(R)}} g \|_{2,R} \leq C e^{sE_1} \left( \| g \|_{2,R} + \| \frac{d^2}{dx^2} g \|_{2,R} \right), \tag{3.15}
\]

where the second inequality holds for some \( C > 0 \). The first inequality in (3.14) uses that \( g \) lies in the orthogonal space to \( \phi \). For the second inequality in (3.14), recall from Lem. 3.1 that \( \frac{d^2}{dx^2} \) and \( \mathcal{L}^{(R)} \) are mutually relative bounded so that I have the first and third inequalities below for some constants \( c, C > 0 \):

\[
\| \frac{d^2}{dx^2} e^{s\mathcal{L}^{(R)}} g \|_{2,R} \leq c \left( \| e^{s\mathcal{L}^{(R)}} g \|_{2,R} + \| \mathcal{L}^{(R)} e^{s\mathcal{L}^{(R)}} g \|_{2,R} \right) \\
\leq c e^{sE_1} \left( \| g \|_{2,R} + \| \mathcal{L}^{(R)} g \|_{2,R} \right) \\
\leq C e^{sE_1} \left( \| g \|_{2,R} + \| \frac{d^2}{dx^2} g \|_{2,R} \right). \tag{3.16}
\]

The second inequality above follows since \( \mathcal{L}^{(R)} \) and \( e^{s\mathcal{L}^{(R)}} \) commute and \( g, \mathcal{L}^{(R)} g \) are orthogonal to \( \phi \).

Next I use (3.14) and (3.15) to bound the supremum of \( e^{s\mathcal{L}^{(R)}} g \) over a finite interval \([-L, L]\). For \( L \geq 1 \) there must be a point \( x \in [-L, L] \) such that the first inequality below holds

\[
\| (e^{s\mathcal{L}^{(R)}} g)(x) \| \leq r \frac{1}{2} \| e^{s\mathcal{L}^{(R)}} g \|_{2,R} \leq r \frac{1}{2} e^{sE_1} \| g \|_{2,R}. \tag{3.17}
\]
For \( x \) satisfying (3.17), the fundamental theorem of calculus applied to the function \( e^{s\mathcal{L}(R)} g \) gives the first inequality below:

\[
\sup_{y \in [-L, L]} |(e^{s\mathcal{L}(R)} g)(y)| \leq |(e^{s\mathcal{L}(R)} g)(x)| + \int_{-L}^{L} \frac{d}{dz} (e^{s\mathcal{L}(R)} g)(z) \leq C r_{1}^{2} e^{sE_{1}} \| g \|_{2,R} + (2Lr_{2})^{1/2} \| \frac{d}{dx} e^{s\mathcal{L}(R)} g \|_{2,R} \leq C r_{1}^{2} e^{sE_{1}} \| g \|_{2,R} + (2Lr_{2})^{3/2} \| \frac{d^{2}}{dx^{2}} e^{s\mathcal{L}(R)} g \|_{2,R},
\]

(3.18)

The second inequality is by Jensen’s inequality and \( R(x) \leq r_{2} \). The last inequality in (3.18) follows from the relation \( \| \frac{d}{dx} e^{s\mathcal{L}(R)} g \|_{2,R} \leq \frac{r_{2}}{r_{1}} \| \frac{d^{2}}{dx^{2}} e^{s\mathcal{L}(R)} g \|_{2,R} \), which can be seen from the equality (3.2).

Finally, the last line of (3.18) decays on the order \( e^{sE_{1}} \) by (3.14) and (3.15).

\( \square \)

In the special case in which the function \( R \) has the form \( R(x) = r_{1} + (r_{2} - r_{1}) \chi(|x| > \text{c}) \) for some \( c \in \mathbb{R}^{+} \), I denote the corresponding generator by \( \mathcal{L}_{c} \). Note that any \( \mathcal{L}^{(R)} \) with \( R \) symmetric and increasing from \( r_{1} \) to \( r_{2} \) over the interval \( [0, \infty) \) can be formally written as a convex combination of the operators \( \mathcal{L}_{c} \) through the formula \( \mathcal{L}^{(R)} = \frac{1}{r_{2} - r_{1}} \int_{0}^{\infty} dR(c) \mathcal{L}_{c} \). Thus the operators \( \mathcal{L}_{c} \) are extremal in the class of operators \( \mathcal{L}^{(R)} \) corresponding to reasonable strategies \( R \in \mathcal{B}(\mathbb{R}, [r_{1}, r_{2}]) \).

**Lemma 3.3.** For any measurable and symmetric function \( R : \mathbb{R} \to [r_{1}, r_{2}] \), the following inequality holds:

\[
\Sigma(\mathcal{L}^{(R)}) \leq \sup_{c \in (0, \infty)} \Sigma(\mathcal{L}_{c}).
\]

Moreover, the above supremum is attained as a maximum for a value \( c > 0 \) satisfying the following property: The unique inflection point over the interval \( (0, \infty) \) for the eigenfunction \( \phi_{c} \) corresponding to the eigenvalue \( \Sigma(\mathcal{L}_{c}) \) (see Part (3) of Prop. 3.2) occurs at the value \( c \).

**Proof.** The argument will proceed by examining certain perturbations of the operator \( \mathcal{L}^{(R)} \). The perturbations that I consider will be of the form

\[
\mathcal{L}^{(R+hA)} = \mathcal{L}^{(R)} + \frac{h}{2} A(x) \frac{d^{2}}{dx^{2}}
\]

for \( h \ll 1 \) and a well-chosen bounded measurable function \( A : \mathbb{R} \to \mathbb{R} \). By Lem. 3.1, the operator \( \frac{d^{2}}{dx^{2}} \) is relatively bounded to \( \mathcal{L}^{(R)} \). It follows that operators of the form \( A(x) \frac{d^{2}}{dx^{2}} \) are also relatively bounded to \( \mathcal{L}^{(R)} \) when \( A \) is bounded, and we can use standard perturbation theory [3] to characterize the leading eigenvalue of \( \mathcal{L}^{(R+hA)} \) for small \( h \).

By Part (3) of Prop. 3.2, the eigenfunction \( \phi \) with eigenvalue \( E := \Sigma(\mathcal{L}^{(R)}) \) must satisfy that

\[
\frac{d^{2}\phi}{dx^{2}}(x) < 0 \quad \text{for} \quad |x| < c \quad \text{and} \quad \frac{d^{2}\phi}{dx^{2}}(x) > 0 \quad \text{for} \quad |x| > c
\]

(3.19)

for some \( c > 0 \). Define \( A : \mathbb{R} \to \mathbb{R} \) to be of the form

\[
A(x) := \begin{cases} r_{2} - R(x) & |x| \geq c, \\ r_{1} - R(x) & |x| \leq c. \end{cases}
\]

(3.20)

Notice that \( R(x) + hA(x) \) maps into the interval \([r_{1}, r_{2}]\) for every \( h \in [0, 1] \). For \( h \ll 1 \) perturbation theory yields that

\[
\Sigma(\mathcal{L}^{(R+hA)}) = \Sigma(\mathcal{L}^{(R)}) + \frac{h}{2} \left\langle \phi \bigg| A(x) \frac{d^{2}}{dx^{2}} \phi \right\rangle + o(h).
\]

(3.21)
Since the values $\phi(x)$ are strictly positive by Part (3) of Prop. 3.2, the property (3.19) implies that the expression $\langle \phi | A(x) \frac{d^2}{dx^2} \phi \rangle$ must be strictly positive unless $A(x) = 0$. However, $A(x) = 0$ implies that $\mathcal{L}^{(R)} = \mathcal{L}_c$.

The above analysis shows that any operator $\mathcal{L}^{(R)}$ admits a small perturbation $\mathcal{L}^{(R+hA)}$, $h \ll 1$ leading to a higher maximum eigenvalue unless $R(x)$ is of the form $r_1 + (r_2 - r_1)\chi(|x| \geq c)$.

\section{The extremal strategies}

By Part (4) of Prop. 3.2 it is sufficient to focus attention on the extremal generators $\mathcal{L}_c$.

\textbf{Lemma 4.1.} Let $r_1 < r_2$ and $V = \left(\frac{r_1}{r_2}\right)^2$. For $\eta(V)$, $\kappa(V)$ defined as in Thm. 1.1,

$$\max_{c \in \mathbb{R}^+} \overline{\Sigma}(\mathcal{L}_c) = \frac{\eta(V) - 1}{2}.$$ 

The maximizing value $c \in \mathbb{R}^+$ is unique and given by $c = \kappa(V)^{\frac{2}{3}}$.

\textbf{Proof.} Let $\phi_c$ denote the eigenfunction of $\mathcal{L}_c$ with leading eigenvalue $E_c := \overline{\Sigma}(\mathcal{L}_c)$. Recall from Part (2) of Prop. 3.2 that $E_c \in (-\frac{1}{2}, 0)$. In parts (i) and (ii) below, I discuss the equations determining the eigenvalue $E_c$ and the additional criterion determining $\max_{c \in \mathbb{R}^+} E_c$, respectively.

(i) By Part (3) of Prop. 3.3 the values $\phi_c(x) \in \mathbb{C}$ have a single phase for all $x \in \mathbb{R}$ that can be chosen to be positive. The function $\phi_c : \mathbb{R} \rightarrow \mathbb{R}^+$ satisfies the differential equation

$$0 = -E_c \phi_c(x) + \frac{1}{2} \frac{d^2 \phi_c}{dx^2}(x) + \frac{1}{2} r_1 \frac{d^2 \phi_c}{dx^2}(x) \quad |x| \leq c, \quad (4.1)$$

$$0 = -E_c \phi_c(x) + \frac{1}{2} \frac{d^2 \phi_c}{dx^2}(x) + \frac{1}{2} r_2 \frac{d^2 \phi_c}{dx^2}(x) \quad |x| > c. \quad (4.2)$$

The fundamental solutions $L_{r,E_c}^\pm$ to the differential equations (4.1) and (4.2) have the forms

$$L_{r,E_c}^-(x) := \int_0^\infty dy \, y^2 E_c e^{-\frac{1}{2}x^2} \quad \text{and} \quad L_{r,E_c}^+(x) := L_{r,E_c}^-(x)$$

for $r = r_1$ and $r = r_2$, respectively. Hence the function $\phi_c$ is a linear combination of $L_{r_1,E_c}^+$, $L_{r_1,E_c}^-$ over the domain $|x| \leq c$ and a linear combination of $L_{r_2,E_c}^+$, $L_{r_2,E_c}^-$ over the domain $|x| > c$. In order for the function $\phi_c$ to be positive, an element of $L^2(\mathbb{R}, w(x)dx)$, and symmetric about zero, I am left with the following unnormalized form:

$$\phi_c(x) = \begin{cases} 
L_{r_2,E_c}^-(x) & x < -c, \\
\gamma(L_{r_1,E_c}^+(x) + L_{r_1,E_c}^-(x)) & |x| \leq c, \\
L_{r_2,E_c}^+(x) & x > c, 
\end{cases} \quad (4.3)$$

for some constant $\gamma \in \mathbb{R}^+$. The values $\gamma$ and $E_c$ are fixed by the requirement that $\phi_c$ is continuously differentiable at $x = c$. Equivalently, $E_c$ can be determined first through the Wronskian identity $W(L_{r_1,E_c}^+ + L_{r_1,E_c}^-, L_{r_2,E_c}^+)(c) = 0$, and then $\gamma$ is given by $\gamma = \frac{L_{r_2,E_c}^-(c)}{L_{r_1,E_c}^+(c) + L_{r_1,E_c}^-(c)}$.

To see that $W(L_{r_1,E_c}^+ + L_{r_1,E_c}^-, L_{r_2,E_c}^+)(c) = 0$ has a solution for some $E_c \in (-\frac{1}{2}, 0)$, notice that the Wronskian equaling zero is equivalent to

$$J_{r_1,r_2}^{(c)}(E_c) = 1 \quad \text{for} \quad J_{r_1,r_2}^{(c)}(E) := \frac{dL_{r_1,E_c}^+(c)}{dx} \frac{dL_{r_2,E_c}^-(c)}{dx} + \frac{dL_{r_2,E_c}^+(c)}{dx} \frac{dL_{r_1,E_c}^+(c)}{dx} \left( L_{r_1,E_c}^+(c) + L_{r_1,E_c}^-(c) \right) - \frac{dL_{r_2,E_c}^+(c)}{dx} \left( L_{r_1,E_c}^+(c) + L_{r_1,E_c}^-(c) \right).$$
and notice that for any fixed \( c, r_1, \) and \( r_2 \)
\[
\lim_{E \to 0} J^{(c)}_{r_1, r_2}(E) = 0 \quad \text{and} \quad \lim_{E \to \frac{1}{2} \frac{r_2}{r_1}} J^{(c)}_{r_1, r_2}(E) = \frac{r_2}{r_1} > 1.
\]

The intermediate value theorem guarantees that there exists a solution \( J^{(c)}_{r_1, r_2}(E_c) = 1 \) for some \( E_c \in (-\frac{1}{2}, 0) \). The solution \( E_c \) must be unique since otherwise I can construct two positive-valued eigenfunctions \( \phi_{c,1}(x) \) and \( \phi_{c,2}(x) \) for \( L_c \) of the form \( \mathcal{L}_c \). However, \( \mathcal{L}_c^{(R)} \) is self-adjoint in the weighted Hilbert space \( L^2(\mathbb{R}, w(x)dx) \) by Lem. 3.3 so \( \phi_{c,1}(x) \) and \( \phi_{c,2}(x) \) must be orthogonal, which contradicts the possibility of both functions being strictly positive.

I remark that the continuous differentiability of \( \phi_c \) along with the equations (4.1) and (4.2) imply that the second derivative of \( \phi_c \) is discontinuous at \( x = c \) unless \( \lim_{x \to c} \frac{d^2 \phi_c}{dx^2}(x) = \frac{d^2 \phi_c}{dx^2}(c) = 0 \).

(ii). By Lem. 3.3 the parameter value \( c \in \mathbb{R}^+ \) at which \( E_c \) is maximized also has the property that the eigenfunction \( \phi_c \) has an inflection point at \( c \). Recall from the remark following (3.11) that the inflection point must be a continuity point for the second derivative: \( \lim_{x \to c} \frac{d^2 \phi_c}{dx^2}(x) = \frac{d^2 \phi_c}{dx^2}(c) = 0 \). These results can be alternatively found by combining the relations \( W(L_{r_1, E_c}^+ + L_{r_1, E_c}^-) \) for \( c = 0 \) and \( \frac{dE_c}{dc} = 0 \).

The constraint \( \lim_{x \to c} \frac{d^2 \phi_c}{dx^2}(x) = 0 \) and the form (4.3) yield that
\[
0 = \frac{d^2 L_{r_1, E_c}^+}{d^2 x}(c) + \frac{d^2 L_{r_1, E_c}^-}{d^2 x}(c) \quad \text{and} \quad 0 = \frac{d^2 L_{r_2, E_c}^+}{d^2 x}(c).
\]

Moreover, combining equations (4.1) and (4.2) with (4.3) implies that the values \( c, E_c \) satisfy the equations
\[
\frac{2E_c}{c} \left[ L_{r_1, E_c}^+(c) + L_{r_1, E_c}^-(c) \right] = \frac{dL_{r_1, E_c}^+}{dx}(c) - \frac{dL_{r_1, E_c}^-}{dx}(c) \quad \text{and} \quad \frac{2E_c}{c} L_{r_2, E_c}^+(c) = \frac{dL_{r_2, E_c}^+}{dx}(c).
\]

Denote \( \kappa := c r_1^{-\frac{1}{2}}, V := (\frac{2c}{\pi})^\frac{1}{2}, \eta := 1 + 2E_c \) and \( Y_\eta(\kappa) := \int_0^\infty dz \eta^{-1} e^{-\frac{1}{2}(z+\kappa)^2} \). By changing variables \( yr_1^{-\frac{1}{2}} \to y \) in the integrals defining \( L_{r_1, E_c}^\pm(c) \) and \( L_{r_2, E_c}^\pm(c) \), the above equations are equivalent to
\[
\frac{1 - \eta}{\kappa} \left( Y_\eta(\kappa) + Y_\eta(-\kappa) \right) = -\frac{dY_\eta}{dx}(\kappa) + \frac{dY_\eta}{dx}(-\kappa) \quad \text{and} \quad \frac{1 - \eta}{\kappa} \frac{Y_\eta(\kappa)}{V} = -\frac{dY_\eta}{dx}(\kappa).
\]

Finally, the above equations are equivalent to (1.5) since \( -\frac{dY_\eta}{dx}(x) = Y_{\eta+1}(x) + x Y_\eta(x) \) for \( x = \kappa \) and \( x = \frac{\kappa}{V} \) and since
\[
Y_{\eta+1}(\kappa) - Y_{\eta+1}(-\kappa) = -e^{-\frac{\kappa^2}{2}} \int_0^\infty dz \eta^{-1} e^{-\frac{z^2}{2}} \left( e^{\kappa z} - e^{-\kappa z} \right)
\]
\[
= -\kappa \left( Y_\eta(\kappa) + Y_\eta(-\kappa) \right) + (1 - \eta) e^{-\frac{\kappa^2}{2}} \int_0^\infty dz \eta^{-1} e^{-\frac{z^2}{2}} \left( e^{\kappa z} + e^{-\kappa z} \right)
\]
\[
= -\kappa \left( Y_\eta(\kappa) + Y_\eta(-\kappa) \right) + (1 - \eta) e^{-\frac{\kappa^2}{2}} \int_0^\infty da \eta(a) + Y_\eta(\kappa) + Y_\eta(-\kappa) e^{\frac{a^2}{2}}.
\]

where the second equality applies integration by parts.

The following proposition lists the basic properties for the solutions \( \kappa(V) \) and \( \eta(V) \) to (1.5).

**Proposition 4.2.** For \( V > 1 \) let \( \kappa(V) \) and \( \eta(V) \) be determined by the equations (4.3). Also let \( \pi > 0 \) be the solution to the equation \( 1 = \pi \int_0^\infty da e^{-\frac{a^2}{2}} \) and define \( \tilde{W}(\kappa) := \int_\mathbb{R} dx e^{-\frac{1}{2} \kappa^2} \log \left( |\kappa - x| \right) \).
1. $\eta(V)$ and $\kappa(V)$ increase monotonically with $V$ and are restricted to the intervals $\eta(V) \in (0, 1)$ and $\kappa(V) \in (1, \overline{\kappa})$. Moreover, $\kappa(V) \leq V \sqrt{1 - \eta}$.

2. In the limit $V \searrow 1$,
\[
\eta(V) = \sqrt{\frac{8}{\pi e}}(V - 1) + O((V - 1)^2) \\
\kappa(V) = 1 + \left(1 - \sqrt{\frac{2}{\pi e}} \int_0^\infty dz e^{-\frac{1}{2}z^2} - \sqrt{\frac{2}{\pi e}}\right)(V - 1) + O((V - 1)^2).
\]

3. In the limit $V \nearrow \infty$,
\[
\eta(V) = 1 - \frac{\sqrt{2\overline{\kappa}}}{\sqrt{\pi}} \frac{1}{V} + O\left(\frac{1}{V^2}\right) \\
\kappa(V) = \overline{\kappa} - \frac{\pi^2}{\overline{\kappa}} \left(\int_0^{\overline{\kappa}} da \widetilde{W}(a)e^{\frac{a^2}{2}} - \widetilde{W}(\overline{\kappa})\right) \frac{1}{V} + O\left(\frac{1}{V^2}\right).
\]

Proof of Prop. 4.2. The existence of solutions $\kappa \equiv \kappa(V), \eta \equiv \eta(V) > 0$ to the equations (1.5) follows from the analysis in the proof of (3.3). A perturbation argument similar to that used in the proof of Lem. 3.3 implies that for $V < V'$
\[
\eta(V) = \max_{R \in \bar{B}(\mathbb{R}, [r_1, Vr_1])} \left(1 - \Sigma(L(R))\right) < \max_{R \in \bar{B}(\mathbb{R}, [r_1, V^\prime r_1])} \left(1 - \Sigma(L(R))\right) = \eta(V').
\]

Thus $\eta(V)$ is an increasing function. For a given $\eta \in (0, 1)$ the value $\kappa > 0$ is the unique positive inflection point for the function
\[
Z_\eta(x) := Y_\eta(x) + Y_\eta(-x) = \int_\mathbb{R} dz e^{-\frac{1}{2}z^2} |z - x|^\eta - 1;
\]
see equation (4.4). Notice that $Z_\eta(x)$ is smooth, symmetric about zero, and decreases monotonically away from zero. From the form (1.5) it is clear that the inflection point $\kappa$ of $Z_\eta$ increases with $\eta$ and approaches 1 in the limit $\eta \searrow 0$. Hence $\kappa(V)$ is an increasing function with image contained in $(1, \infty)$.

Since $Y_{\eta + 1}$, $Y_\eta$, $\kappa$, $V$ are strictly positive, the left side of (1.5) immediately yields the inequality $V \sqrt{1 - \eta} \geq \kappa$. We can write the right equation in (1.5) in the form $1 = \frac{\kappa \int_0^\kappa da \widetilde{Z}(a)e^{\frac{a^2}{2}}}{Z_\eta(\kappa)}$. Since $Z_\eta(a)$ is decreasing over the interval $[0, \kappa]$,
\[
1 = \frac{\kappa \int_0^\kappa da \widetilde{Z}(a)e^{\frac{a^2}{2}}}{Z_\eta(\kappa)} \geq \kappa \int_0^\kappa da e^{\frac{a^2}{2} - \frac{a^2}{2}}.
\]
Thus $\kappa < \overline{\kappa}$ and $\kappa(V)$ has image contained in $(1, \overline{\kappa})$. As $\eta \nearrow 1$ the functions $Z_\eta(x)$ converge uniformly over compact sets to the constant $\sqrt{2\pi}$, and from the right side of (1.5) it is clear that $\kappa \nearrow \overline{\kappa}$.

Parts (2) and (3) follow from expanding (1.5) around $(\eta, \kappa) \approx (0, 1)$ and $(\eta, \kappa) \approx (1, \overline{\kappa})$ for $V - 1 \ll 1$ and $V \gg 1$, respectively. For the regime $(\eta, \kappa) \approx (0, 1)$ it is useful to use that the right equality in (1.5) is equivalent to $\frac{1 - \eta - \kappa^2}{\kappa} = \frac{Y_{\eta + 1}(\kappa) - Y_{\eta + 1}(-\kappa)}{Y_\eta(\kappa) + Y_\eta(-\kappa)}$.

5 Proof of Theorem 1.1

For a bounded measurable function $h : \mathbb{R} \to \mathbb{R}$, define $G_h^{(R)}(x) := \mathbb{E}x\left[h\left(\lim_{s \to \infty} e^{-\frac{x}{2}Z_s}\right)\right]$, where the limit exists by the remark (2.4).
Lemma 5.1. Let $h$ be smooth and have compact support and $R \in \widehat{B}(\mathbb{R}, [r_1, r_2])$. The functions $G^{(R)}_h$ and $\frac{d^2}{dx^2} G^{(R)}_h$ are elements in $L^2(\mathbb{R}, w(x) dx)$.

Proof. I can assume without losing generality that $h(x)$ is non-negative. Since $h$ has compact support, it is smaller than $c[l,-L]$ for some $c, L > 0$. To show that $G^{(R)}_h \in L^2(\mathbb{R}, w(x) dx)$, I can bound the values $G^{(R)}_h(x)$ through the relations below:

$$G^{(R)}_h(x) = \mathbb{E}_x \left[ h \left( x + \int_0^\infty e^{-\frac{x}{2}} \sqrt{R(Z_s)} dB_s \right) \right]$$

$$\leq c \mathbb{P}_x \left[ \left| \int_0^\infty e^{-\frac{x}{2}} \sqrt{R(Z_s)} dB_s \right| \geq \|x - L\| \right]$$

$$\leq c e^{-\frac{1}{2}(|x| - L)^2} \mathbb{E}_x \left[ e^{-\frac{x}{2}} \left( \int_0^\infty \sqrt{R(Z_s)} dB_s \right)^2 \right]$$

$$\leq c \delta^{-\frac{1}{2}} e^{-\frac{1}{2}(|x| - L)^2} \delta^{\frac{1}{2}}.$$ (5.1)

where $\delta$ is picked from $(0,1)$. The second equality above holds by (2.4), and the second inequality is Chebyshev’s. The third inequality in (5.1) holds because $e^{-\frac{1}{2}x^2}$ is a convex function and thus the expectation in the fourth line is larger when $R(Z_s)$ is replaced by $r_2$:

$$\mathbb{E}_x \left[ e^{-\frac{x}{2}} \left( \int_0^\infty \sqrt{R(Z_s)} dB_s \right)^2 \right] \leq \mathbb{E}_x \left[ e^{-\frac{x}{2}} \left( \int_0^\infty \sqrt{R(Y_s)} dB_s \right)^2 \right] = \delta^{-\frac{1}{2}}.$$ (5.1)

By (5.1) the norm of $\|G^{(R)}_h\|_{2,R}^2$ is bounded by $c r^{-\frac{1}{2}} \delta^{-\frac{1}{2}} \int_\mathbb{R} dx e^{-\frac{x}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(|x| - L)^2}$, and I can pick $\delta > 0$ small enough so that the right side below is finite since $R(y) \geq r_1$ and $R(y) \not\to r_2$ as $|y| \not\to \infty$.

Now I will show that $\frac{d^2}{dx^2} G^{(R)}_h \in L^2(\mathbb{R}, w(x) dx)$. Notice that by inserting a conditional expectation into the formula for $G^{(R)}_h(x)$

$$G^{(R)}_h(x) = \mathbb{E}_x \left[ h \left( \lim_{s \to \infty} e^{-\frac{x}{2}} Z_s \right) \right] = \mathbb{E}_x \left[ \mathbb{E} \left[ h \left( e^{-\frac{x}{2}} \lim_{s \to \infty} e^{-\frac{x}{2}} Z_s \right) \right] \mathbb{E}_r \right]$$ (5.2)

and considering a first-order expansion of (5.2) for small $r > 0$ leads to the equation

$$(\mathcal{L}^{(R)} G^{(R)}_h)(x) = G^{(R)}_h(x) \quad \text{for} \quad \hat{h}(x) := \frac{x}{2} \frac{dh}{dx}(x).$$ (5.3)

The function $\hat{h}(x)$ is smooth and has compact support by our assumptions on $h(x)$. By our analysis above, $G^{(R)}_h(x)$ is an element in $L^2(\mathbb{R}, w(x) dx)$. It follows from (5.3) that $G^{(R)}_h(x)$ is in the domain of $\mathcal{L}^{(R)}$. Finally, $\frac{d^2}{dx^2} G^{(R)}_h(x)$ is in $L^2(\mathbb{R}, w(x) dx)$ since the operator $\frac{d^2}{dx^2}$ is relatively bounded to $\mathcal{L}^{(R)}$ by Lem. 3.1. 

\[
\]

For $z \in \mathbb{R}$ and $r \in \mathbb{R}^+$, define $F^{(R)}_{\epsilon, t}: \mathbb{R} \to \mathbb{R}^+$ to be the solution to $\frac{d}{dt} F^{(R)}_{\epsilon, t}(x) = \frac{1}{2} R \left( \frac{x}{\sqrt{t} - t} \right) \frac{d^2}{dx^2} F^{(R)}_{\epsilon, t}(x)$ with $F^{(R)}_{\epsilon, 0}(x) = 1_{[-\epsilon, \epsilon]}(x)$. We can also express $F^{(R)}_{\epsilon, t}(z)$ as

$$F^{(R)}_{\epsilon, t}(x) = \mathbb{P} \left[ |X_T| \leq \epsilon \right] = \mathbb{E}_x \left[ \left| \lim_{s \to \infty} e^{-\frac{x}{2}} Z_s \right| \leq \epsilon \right],$$ (5.4)

where the second inequality follows from the remark (2.4). For $\lambda > 0$ there is the following scale invariance:

$$F^{(R)}_{\epsilon, t}(x) = F^{(R)}_{\lambda \epsilon, \lambda^2 t}(\lambda x).$$ (5.5)
Proof of Thm. 1. Let \( b = \frac{y}{\sqrt{T}} \) and \( s = -2 \log(\epsilon) \). By the Markov property, I have the equality
\[
\int_{[-\epsilon, \epsilon]} dx \mathcal{P}^{(R)}_{y,T}(x) = \int_{\mathbb{R}} dx \mathcal{P}^{(R)}_{y,T-\epsilon^2}(x) F^{(R)}_{\epsilon, \epsilon^2}(x).
\]
Moreover,
\[
\int_{\mathbb{R}} dx \mathcal{P}^{(R)}_{y,T-\epsilon^2}(x) F^{(R)}_{\epsilon, \epsilon^2}(x) = \int_{\mathbb{R}} dz \psi^{(R)}_{b,s}(z) F^{(R)}_{\epsilon, \epsilon^2}(e^{-s/2}z),
\]
where the second equality above uses the scale invariance (5.5). Notice that by (5.4) \( F^{(R)}_{1,1}(z) = G^{(R)}_{1[1][-1]}(z) \). Hence we can bound \( F^{(R)}_{1,1}(z) \) above and below by \( G^{(R)}_{h}(z) \) for compact, smooth functions \( h : \mathbb{R} \to \mathbb{R}^+ \). Moreover, from Lem. 5.1 and Part (4) of Prop. 3.2 I have the first equality below
\[
1 + 2 \Sigma(L_R) = \lim_{s \to \infty} \left\{ 1 + \frac{2 \log \left( \int_{\mathbb{R}} dz \psi^{(R)}_{b,s}(z) G^{(R)}_{h}(z) \right)}{s} \right\}.
\]
It follows from the above that \( 1 - \frac{\log(\int_{[-\epsilon, \epsilon]} dx \mathcal{P}^{(R)}_{y,T}(x))}{\log(\epsilon)} \) also converges to \( 1 + 2 \Sigma(L_R) \) as \( \epsilon \searrow 0 \). By Lems. 3.3 and 4.1 the value \( 1 + 2 \Sigma(L_R) \) attains the maximum \( \eta(V) \) over all \( R \in \hat{B}(\mathbb{R}, [r_1, r_2]) \) when \( R \) has the form \( R(x) = r_1 + (r_2 - r_1) \chi(|x| > r_1 \kappa(V)) \).

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