Potential-capacity and some applications

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Abstract

We introduce a new capacity associated to a non negative function \( V \). We apply this notion to the study of a necessary and sufficient condition to ensure the existence and uniqueness of a Schrödinger type equation with measure data and with an operator whose coefficients are discontinuous. Namely, for a potential \( V \), \( f \) a bounded Radon measure on \( \Omega \), then the equation \( \mathcal{L}_V u = -\Delta u + U \cdot \nabla u + Vu = f \) has a solution in \( L^1(V) \cap L^1_0(\Omega) = \{ g \text{ measurable, } \int_\Omega |g|V dx \text{ is finite and } \lim_{\varepsilon \to 0} \int_{\{x : \text{dist}(x, \partial \Omega) \leq \varepsilon\}} |g|dx = 0 \} \) if and only if \( f \) does not charge "irregular points" of \( V \), provided that the set of "irregular points" have a zero potential capacity. As a byproduct of our results, we have the non existence of a Green operator for some \( \mathcal{L}_V \).

Our method is also based on a new topology and density of \( C^2_c(\Omega \setminus K) \) in \( C^2_0(\Omega) \) whenever \( K \) has a zero potential-capacity.

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1 Introduction

In recent works (see [4], [5]), we have studied the equation

\[ -\Delta \omega + U \cdot \nabla \omega + V \omega = f \]
in a smooth open bounded domain $\Omega$ whenever the potential $V$ is locally integrable on the domain, under the Dirichlet condition $u = 0$ on $\partial \Omega$.

The first natural question is: what happens if we remove this integrability condition on $V$?

When examining the prototype of $V$ say $V(x) = |x - a|^{-m}$, with $m > 0$, $a \in \Omega$, we observe that there is an interaction between the point $a$, the power $m$ and the right hand side $f$.

To describe the growth of $V$ and such interaction, we introduce here a new capacity associated to the potential $V$. Roughly speaking, the more $V$ contains "irregular points" the more its capacity will be small. In particular, we will focus on potential whose "irregular points" are of capacity zero.

This new capacity is slightly different to the usual one considered by many authors (see [12] [17] for a complete review). Indeed, we recall that, if $K$ is a compact subset of an open set $\Omega$ of $\mathbb{R}^n$, then, for $1 \leq k < +\infty$, $1 \leq q < +\infty$ the $W^{k,q}$ capacity of $K$ is usually defined as

$$\mathrm{Cap}_{W^{k,q}}(K) = \inf \left\{ \| \varphi \|_{W^{k,q}(\Omega)} : \varphi \in B^k_K \right\}$$

where

$$B^k_K = \left\{ \varphi \in C^k_c(\Omega), \ 0 \leq \varphi \leq 1, \ \varphi = 1 \text{ in a neighborhood of } K \right\}.$$ (2)

Here, we shall consider a potential $V \geq 0$ on $\Omega$, $V \neq 0$ for $\varphi \in C^2_c(\Omega)$, we define

$$\| \varphi \|_{V,\infty} = \| \varphi \|_{L^1(\Omega)} + \left\| \frac{\nabla \varphi}{\sqrt{V}} \right\|_{\infty} + \left\| \frac{\Delta \varphi}{V} \right\|_{\infty},$$

and we shall associate, the following capacity function, for a compact $K$ included in $\Omega$

$$\mathrm{Cap}_{V,\infty}(K) = \inf \left\{ \| \varphi \|_{V,\infty}, \ \varphi \in B^2_K \right\}$$ (4)

Such capacity possesses common properties as for the above classical capacities (see Section 2 below), namely, we will show in particular that

$$\text{if } \mathrm{Cap}_{V,\infty}(K_i) = 0, \ \text{for } i \in J(\text{finite}) \text{ then } \mathrm{Cap}_{V,\infty}\left( \bigcup_{i \in J} K_i \right) = 0.$$ 

Roughly speaking, such capacity will measure how singular is the potential $V$? And how "large" is this singularity. For instance, if $a \in \Omega$ and $V$ behaves like $|x - a|^{-m}$ near $a$, then
\[ \text{Cap}_{V,\infty}(\{a\}) = 0 \text{ if } m \geq 2 \text{ and } \text{Cap}_{V,\infty}(\{a\}) > 0 \text{ if } m < 2. \] But one of the most important properties that we need for the applications are:

**Theorem 1.**

Let \( K \) be compact included in \( \Omega \). Assume that \( \text{Cap}_{V,\infty}(K) = 0 \). Then there exists a sequence \((\psi_j)_j, \psi_j \in C_c^2(\Omega)\) such that

1. \( \psi_j(x) \xrightarrow{j \to +\infty} 0 \) for a.e in \( \Omega \) and strongly in \( L^1(\Omega) \), \( \psi_j = 1 \) on \( K \).

\[ \frac{|\nabla \psi_j|}{\sqrt{V}} \xrightarrow{j \to +\infty} 0 \text{ and } \frac{\Delta \psi_j}{V} \xrightarrow{j \to +\infty} 0 \text{ strongly in } L^\infty(\Omega). \]

2. If furthermore \( V \in L^{\frac{n}{n-1}}_{\text{loc}}(\Omega_K) \) with \( \Omega_K = \{x \in \Omega, \ dist \ (x; K) > 0\} \), then, for all \( x \in \Omega - K \)

\[ \psi_j(x) \xrightarrow{j \to +\infty} 0, \]

more precisely, if \( \Omega_{K,0} \subset \subset \Omega_K \), then

\[ \text{Max}_{\Omega_{K,0}} |\psi_j(x)| \xrightarrow{j \to +\infty} 0, \ |||\nabla \psi_j|||_{L^{n-1}(\Omega_{K,0})} \xrightarrow{j \to +\infty} 0. \]

3. If \( a \in \Omega, V(x) = |x - a|^{-m} \) then \( V \) is in \( L^{\frac{n}{n-1}}(\Omega) \) if and only if \( m < 2 \).

The natural question is then, can we give sufficient conditions to ensure that

\[ \text{Cap}_{V,\infty}(K) = 0? \quad (5) \]

The answer to that question is naturally linked with the motivations of our study. One of them is the following:

Let \( \mu_0 \) be the Dirac mass at the origin, \( m \) a positive parameter, then we observe the following phenomena:

If \( m \geq 2 \) then there is no solution of

\[ (M_1) \begin{cases} -\Delta u(x) + \frac{u(x)}{|x|^m} = \mu_0 \quad \text{in } B(0; 1) \subset \mathbb{R}^n, \ n \geq 2, \\ u(x) = 0 \quad \text{if } |x| = 1. \end{cases} \]

But if \( m < 2 \), the above problem \((M_1)\) possesses at least one solution \( u \). The same phenomena
were also given in [1].

Let us notice that \((M_1)\) has a solution if \(n = 1\).

Another motivation that we shall prove in this note is the following removable type singularities result:

**Proposition (removable singularities with potential)**

Assume (for simplicity) that \(V(x) = \sum_{i=1}^{n} \frac{b_i}{|x - a_i|^{m_i}}, m_i \geq 0, b_i > 0, a_i \in \Omega\) and consider \(K = \{a_i, m_i \geq 2\}, w \in L^1(\Omega; V) \cap L^1_{\text{loc}}(\Omega)\) such that \(\forall \varphi \in C^2_c(\Omega \setminus K)\) we have

\[
\int_{\Omega} w(-\Delta \varphi + V \varphi) dx = 0.
\]

Then

\(w \equiv 0.\)

*Here \(\delta(x) = \text{distance}(x; \partial \Omega)\).*

So, the natural question is that if we consider an arbitrary potential \(V \geq 0\), how can we replace the set of singularities \(K = \{a_i, m_i \geq 2\}\)?

The question seems to be linked with some density problem (with an adequate topology).

The tough problem linked with that question is the construction of an appropriate sequence smooth function vanishing over \(K\) and disappearing when we pass to the limit for an adequate topology. These are the purpose of our main results stated in the next section. Namely a generalization of the above proposition for a large class of potential \(V\) and applications to some existence and non existence result for weak or very weak solution. We shall provide few examples of compact \(K\) whose \((V, \infty)\)-capacity is zero.

### 2 Notations Definitions - Primary definitions and results

We shall keep the notation we used to employ. We set

\[
L^0(\Omega) = \{v : \Omega \to \mathbb{R} \text{ Lebesgue measurable}\}
\]
and we denote by $L^p(\Omega)$ the usual Lebesgue space $1 \leq p \leq +\infty$. Although it is not too often used, we shall use the notation

$$W^{1,p}(\Omega) = W^1 L^p(\Omega)$$

for the associated Sobolev space. We need the following definitions:

**Definition 1. of the distribution function and monotone rearrangement**

Let $u \in L^0(\Omega)$. The distribution function of $u$ is the decreasing function

$$m = m_u : \mathbb{R} \rightarrow [0, |\Omega|]$$

$$t \mapsto \text{measure } \{x : u(x) > t\} = |\{u > t\}|.$$ 

The generalized inverse $u_*$ of $m$ is defined by, for $s \in [0, |\Omega|[$,

$$u_*(s) = \inf \left\{ t : |\{u > t\}| \leq s \right\},$$

and is called the decreasing rearrangement of $u$. We shall set $\Omega_* = ]0, |\Omega|[.$

**Definition 2.**

Let $1 \leq p \leq +\infty$, $0 < q \leq +\infty$:

- If $q < +\infty$, one defines the following norm for $u \in L^0(\Omega)$

$$\|u\|_{p,q} = \|u\|_{L^{p,q}} := \left[ \frac{1}{|\Omega|} \int_{\Omega_*} \left( t^\frac{p}{q} |u|_*^{\frac{q}{p}}(t) \right)^q \frac{dt}{t} \right]^\frac{1}{q} \text{ where } |u|_*^{\frac{q}{p}}(t) = \frac{1}{t} \int_0^t |u|_*(\sigma)d\sigma.$$ 

- If $q = +\infty$,

$$\|u\|_{p,\infty} = \sup_{0 < t \leq |\Omega|} t^\frac{p}{q} |u|_*(t).$$

The space

$$L^{p,q}(\Omega) = \left\{ u \in L^0(\Omega) : \|u\|_{p,q} < +\infty \right\}$$

is called a Lorentz space.

- If $p = q = +\infty$, $L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$.

- The dual of $L^{1,1}(\Omega)$ is called $L_{\exp}(\Omega)$

**Remark 1.**

We recall that $L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega)$ for any $p > 1$, $q \geq 1$. 

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Definition 3.

If $X$ is a Banach space in $L^0(\Omega)$, we shall denote the Sobolev space associated to $X$ by

$$W^1X = \{ \varphi \in L^1(\Omega) : \nabla \varphi \in X \}$$

or more generally for $m \geq 1$,

$$W^mX = \{ \varphi \in W^1X, \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n \leq m, \ D^{[\alpha]} \varphi \in X \}.$$

We also set

$$W^1_0X = W^1X \cap W^{1,1}_0(\Omega).$$

We also need to recall the Hardy’s inequality in $L^{n',\infty}$ saying that if $\Omega$ is a bounded Lipschitz domain:

$$\int_\Omega \left| \frac{u}{\delta} \right|^q \leq c \| \nabla u \|_{L^{n',\infty}}^q \quad \forall u \in W^1_0 L^{n',\infty}(\Omega),$$

(7)

with $n' = \frac{n}{n-1}$, $1 < q < n'$. This inequality can be obtained from the results of [15] (see also [6]) since $W^1_0 L^{n',\infty}(\Omega) \subset W^1_0(\Omega; 1 + |\log \delta|)$.

We need the following Lemma whose proof is given in [17, 9, 16]

Lemma 2.1.

Let $A \subset \mathbb{R}^n$ be closed and for $x \in \mathbb{R}^n$ let $d(x) = d(x;A)$ denote the distance from $x$ to $A$. Let

$$U = \{ x : d(x) < 1 \}.$$

Then there is a function $\rho \in C^\infty(U - A)$ and a positive number $M = M(n)$ such that

$$M^{-1}d(x) \leq \rho(x) \leq M \ d(x), \ x \in U - A$$

$$|D^\alpha \rho(x)| \leq c(\alpha) \ d(x)^{1-|\alpha|}, \ x \in U - A, \ |\alpha| = \alpha_1 + \ldots + \alpha_n.$$
In particular, the result holds if \( A = \partial \Omega \) boundary of an open bounded set \( \Omega \), in this case

\[
\rho \in C^\infty(\Omega) \text{ and } d(x) = \delta(x) = \text{dist}(x; \partial \Omega).
\]

**Definition 4. of \((V, \infty)\)-capacity or potential-capacity**

Let \( V \geq 0 \) be a measurable function on \( \Omega \), \( V \) non identically zero, \( V \) is called a potential function.

The \((V, \infty)\)-capacity of a compact \( K \) included in \( \Omega \) (or potential-capacity of \( K \)) is given by relation (4).

We will denote by \( c \) different constant, sometimes we will specify the dependence with respect to the data.

**Property 1. of \((V, \infty)\)-capacity**

For any compact \( K \) in \( \Omega \), we have

1. measure(\( K \)) \( \leq \) \text{Cap}_{V, \infty}(\( K \)).

2. If \( K_1 \) is another compact included in \( K \) then

\[
\text{Cap}_{V, \infty}(K_1) \leq \text{Cap}_{V, \infty}(K).
\]

3. For all \( \epsilon > 0 \), there exists an open set \( \omega \) containing \( K \) such that for all compact \( K' \) satisfying \( K \subset K' \subset \omega \), on has

\[
\text{Cap}_{V, \infty}(K') \leq \text{Cap}_{V, \infty}(K) + \epsilon.
\]

4. If \( V_1, V_2 \) are two nonnegative potential \( V_1 \leq V_2 \) then

\[
\text{Cap}_{V_2, \infty}(K) \leq \text{Cap}_{V_1, \infty}(K).
\]

**Proof :**

1. For \( \psi \in B_\infty^r \) we have measure(\( K \)) \( \leq \int_\Omega \psi(x)dx \leq ||\psi||_{V, \infty} \) which gives the result.
2. If \( K_1 \subset K \) then \( B^2_K \subset B^2_{K_1} \). Therefore

\[
\text{Cap}_{V, \infty}(K_1) \subset \text{Cap}_{V, \infty}(K).
\]

3. Let \( \varepsilon > 0 \), then there exists \( \psi \in B^2_K \) such that

\[
||| \psi_e |||_{V, \infty} - \varepsilon \leq \text{Cap}_{V, \infty}(K) \leq ||| \psi_e |||_{V, \infty}.
\]

Since \( \psi_e = 1 \) in a neighborhood of \( K \), thus there exists an open set of \( \omega \) on which \( \psi_e = 1 \). Then for all compact \( K' \) with \( K \subset K' \subset \omega \) one has \( \psi_e = 1 \) on \( K' \) and then \( \psi_e \in B^2_{K'} \).

Thus,

\[
\text{Cap}_{V, \infty}(K') \leq ||| \psi_e |||_{V, \infty} \leq \text{Cap}_{V, \infty}(K) + \varepsilon.
\]

4. If \( 0 \leq V_1 \leq V_2 \) then \( \frac{1}{V_2^\alpha} \leq \frac{1}{V_1^\alpha} \) if \( \alpha = \frac{1}{2}, \alpha = 1 \) from which we get the result. \( \diamond \)

Remark 2.

- *In the definition of \((V, \infty)\)-capacity, we can add a different power on the potential \( V \) but the choice of the power is linked with the applications.*

- *The property (3) is the so-called continuity from the right in Choquet’s capacity theory.*

Proof of Theorem 1

1. As for relation (8) considering \( \varepsilon = \frac{1}{j}, j \geq 1 \) we have a sequence \( (\psi_j)_j \):

\[
\psi_j = 1 \text{ on } K, \quad 0 \leq ||| \psi_j |||_{V, \infty} - \text{Cap}_{V, \infty}(K) \leq \frac{1}{j}.
\]

If \( \text{Cap}_{V, \infty}(K) = 0 \) then

\[
||| \psi_j |||_{V, \infty} \xrightarrow{j \to +\infty} 0
\]

which implies

\[
\frac{\nabla \psi_j}{V} \xrightarrow{j \to +\infty} 0, \quad \frac{\Delta \psi_j}{V} \xrightarrow{j \to +\infty} 0, \text{ and } ||\psi_j||_{L^1} \xrightarrow{j \to +\infty} 0.
\]
This last convergence implies that for a subsequence still denoted by \( \psi_j \) that
\[
\psi_j(x) \xrightarrow[j \to +\infty]{} 0 \text{ for a.e.}
\]

2. Let \( x \in \Omega_K \), there exists \( r > 0 \) so that \( B(x; r) \subset \Omega_K \). From Poincaré-Sobolev’s inequality or P.D.E. regularity (see [7])
\[
\|\nabla \psi_j\|_{L^{n,1}(B(x;r))} \leq c \left[ \|\psi_j\|_{L^1(B(x;r))} + \|\Delta \psi_j\|_{L^{1,1}(B(x;r))} \right] \leq c \left[ \|\psi_j\|_{L^1(\Omega)} + \frac{\|\Delta \psi_j\|_V}{V} \right] \xrightarrow[j \to +\infty]{} 0.
\]

\[
\max_{y \in B(x;r)} |\psi_j(y)| \leq c \|\nabla \psi_j\|_{L^{n,1}(B(x;r))} + c \|\psi_j\|_{L^1(\Omega)} \xrightarrow[j \to +\infty]{} 0.
\]

If \( \Omega_{K,0} \) is open set relatively compact in \( \Omega \) by recovering \( \Omega_{K,0} \) and using the same argument as the above result we deduce
\[
\max_{y \in \Omega_{K,0}} |\psi_j(y)| \xrightarrow[j \to +\infty]{} 0, \quad \|\nabla \psi_j\|_{L^{n,1}(\Omega_{K,0})} \xrightarrow[j \to +\infty]{} 0.
\]

3. A direct and simple computation shows that
\[
\| |x - a|^m\|_{L^{1,1}} < +\infty \text{ if and only if } m < 2.
\]

\( \Diamond \)

3 Few examples of compact \( K \) having a \((V, \infty)\)-capacity zero

**Theorem 2. (Comparison near a compact)**

Let \( K \) be a compact in \( \Omega \), \( V_1 \) and \( V_2 \) two nonnegative potentials satisfying
1. \( \exists \eta > 0 \) such that \( V_1 \leq V_2 \) on a compact set

\[
K_\eta = \left\{ x \in \Omega, \ d(x; K) \geq \text{dist} (x; K) \leq 2\eta \right\} \subset \Omega.
\]

2. \( V_1 \) is bounded from below and above on

\[
O_\eta = \left\{ x \in \Omega : \frac{1}{2} \eta < d(x; K) < 2\eta \right\} \Rightarrow 0 \leq \inf_{\partial \eta} \text{ess } V_1 \leq \sup_{\partial \eta} \text{ess } V_1 < +\infty.
\]

Then there exists a constant \( c_\eta > 0 \) such that

\[
\text{Cap}_{V_2, \infty}(K) \leq c_\eta \text{Cap}_{V_1, \infty}(K). \tag{10}
\]

In particular,

if \( \text{Cap}_{V_1, \infty}(K) = 0 \) then \( \text{Cap}_{V_2, \infty}(K) = 0. \tag{11} \)

**Proof:**

Let \( \theta \) be in \( C^\infty_0(\Omega) \) such that \( 0 \leq \theta \leq 1 \) and \( \theta = 1 \) on \( \left\{ x \in \Omega : d(x, K) \leq \eta \right\} \) and \( \text{support}(\theta) \cap \left\{ x \in \Omega : d(x; K) < \frac{3}{2} \eta \right\} \).

Let us show that there exists a constant \( c_\theta > 0 \) such that for all \( \psi \in B^2_K \),

\[
\| \theta \psi \|_{V_2, \infty} \leq c_\theta \| \psi \|_{V_1, \infty} \tag{12}
\]

We need the following

**Lemma 3.1.**

There exists a constant \( c_{0\eta} > 0 \) such that for all \( \psi \in B^2_K \) we have

\[
|\psi(x)| \leq c_{0\eta} \left[ \| \psi \|_{L^1(\Omega)} + \| \nabla \psi \|_{L^\infty(O_\eta)} \right] \quad \forall x \in \text{supp}(\theta) \cap \left\{ \eta \leq d(\cdot; K) \leq \frac{3}{2} \eta \right\}. \tag{13}
\]

**Proof:**

By the compactness of the set

\[
H_0 = \text{supp}(\theta) \cap \left\{ \eta \leq d(\cdot; K) \leq \frac{3}{2} \eta \right\}
\]
we have a family \((B(x_i; r_i))_{i=1,...,p}\) such that

\[ H_0 \subset \bigcup_{i=1}^{p} B(x_i; r_i) = \mathcal{O} \subset \mathcal{O}_\eta. \]

Applying the Sobolev embedding, we have a constant \(c_{\theta \eta} > 0\)

\[ \|\psi\|_{L^\infty(\mathcal{O})} \leq c_{\theta \eta} \left( \|\psi\|_{L^1(\Omega)} + \|\nabla \psi\|_{L^\infty(\Omega)} \right), \quad \psi \in \mathbb{B}_K^2. \]

This gives the result

\[ \Box \]

Let \(\psi \in \mathbb{B}_K^2\) then \(\|\theta \psi\|_{L^1} \leq \|\psi\|_{L^1}\). For \(x \in \Omega\)

\[ \nabla (\theta \psi)(x) = \nabla \theta \psi(x) + \theta(x) \nabla \psi(x). \]

We distinguish 3 cases

1. If \(x \notin \text{supp}(\theta)\) then \(\nabla (\theta \psi)(x) = 0\), therefore we have

\[ \frac{\nabla (\theta \psi)}{\sqrt{V_2}}(x) = 0 \leq \|\psi\|_{V_1, \infty}. \]

2. If \(x \in \text{supp} \theta, \ x \notin \mathcal{O}_\eta\) then \(V_1(x) \leq V_2(x)\) and

\[ |\nabla (\theta \psi)(x)| \leq |\nabla \psi(x)| \quad \text{so that} \quad \frac{\nabla (\theta \psi)}{\sqrt{V_2}}(x) \leq \frac{\nabla \psi}{\sqrt{V_1}}(x) \leq \|\psi\|_{V_1, \infty}. \]

3. If \(x \in \text{supp} \theta, \ x \in \mathcal{O}_\eta\) we still have \(V_1(x) \leq V_2(x)\) but \(V_1(x) \geq \text{ess inf}_{\mathcal{O}_\eta} V_1 > 0\) so that

(a) if \(d(x; K) \leq \eta\), \(\nabla (\theta \psi)(x) = \nabla \psi(x)\) so we still have

\[ \frac{\nabla (\theta \psi)}{\sqrt{V_2}}(x) \leq \frac{\nabla \psi}{\sqrt{V_1}}(x) \leq \|\psi\|_{V_1, \infty}. \]

(b) if \(\eta < d(x; K) \leq \frac{3}{2}\eta\), we use Lemma 3.1 to

\[ \frac{\nabla (\theta \psi)}{\sqrt{V_2}}(x) \leq c_\theta \left[ \psi(x) + \frac{\nabla \psi}{\sqrt{V_1}}(x) \right] \leq c_{\theta \eta} \|\psi\|_{V_1, \infty}. \]

Since \(\Delta (\theta \psi)(x) = \Delta \theta(x) \psi(x) + 2 \nabla \theta(x) \nabla \psi(x) + \theta(x) \Delta \psi(x)\), we can argue as before to deduce
that
\[ \left| \frac{\Delta (\theta \psi)}{\sqrt{V_2}} (x) \right| \leq c_\theta \eta \| \psi \|_{V_1, \infty}, \ \forall x \in \Omega. \]  
(14)

We have shown
\[ \left\| \frac{\nabla (\theta \psi)}{\sqrt{V_2}} \right\|_{L^\infty(\Omega)} + \left\| \frac{\Delta (\theta \psi)}{V_2} \right\|_{L^\infty(\Omega)} \leq c_\theta \eta \| \psi \|_{V_1, \infty}. \]  
(15)

Thus we deduce relation (12) \( \forall \psi \in \mathbb{B}_K^2 \). We then have
\[ \text{Cap}_{V_2, \infty}(K) \leq c_\theta \text{Cap}_{V_1, \infty}(K). \]

◊

Here are few examples of Theorem 2.

**Corollary 2.1. of Theorem 2**

*Let \( A \), be a closed set included in \( \Omega \) whose measure is zero, \( m > 2, \ m \in \mathbb{R}, \ V \) a potential such that*

*there exists \( \eta > 0, \ c > 0 \) with \( V(x) \geq \frac{c}{d(x; A)^m} \) for \( x \in \{ y : d(y; A) \leq 2\eta \} \). Then*

\[ \text{Cap}_{V, \infty}(A) = 0. \]

**Proof :**

Let us set \( V_1(x) = \frac{c}{d(x; A)^m}, \ x \in \Omega. \) According to Theorem 2 it is sufficient to show that
\[ \text{Cap}_{V_1, \infty}(A) = 0. \]

Let \( H \in C^\infty(\mathbb{R}) \) such that
\[ H \in C^\infty(\mathbb{R}) \text{ such } H(t) = \begin{cases} 1 & \text{if } t \geq 2, \\ 0 & \text{if } t \leq 1. \end{cases} \]  
(16)

and denote by \( \delta(x) = \text{dist}(x; \partial \Omega). \)

According to Lemma 2.1 that we have a function \( \rho \in C^\infty(\Omega), \) two constants \( c_1 > 0, \ c_2 > 0 \) such that
1. \( c_1 \delta(x) \leq \rho \leq c_2 \delta(x), \forall x \in \Omega \)

2. \( \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \exists c_\alpha > 0 \) such that

\[
|D^\alpha \rho(x)| \leq c_\alpha \rho(x)^{1-|\alpha|} \quad \text{with} \quad |\alpha| = \alpha_1 + \ldots + \alpha_n, \quad D^\alpha = \frac{D^{\alpha_1}}{\partial x_1} \ldots \frac{D^{\alpha_n}}{\partial x_n}.
\]

More, we have \( \rho_A \in C^\infty(\Omega \setminus A), M = M(n) > 0, c(\alpha), |\alpha| \leq 2 \) such that for all \( x \in \Omega \setminus A, \)

3. \( M^{-1} d(x; A) \leq \rho_A(x) \leq M d(x; A), \)

4. \( |D^\alpha \rho_A(x)| \leq c(\alpha) d(x; A)^{1-|\alpha|}. \)

Consider the sequence \( \psi_j(x) = (1 - H(j \rho_A(x))) H(j \rho(x)). \) Then \( \psi_j \in C^\infty(\Omega) \) and \( j \geq j_a, \) large enough so that \( \{ x \in \Omega : \rho_A(x) < \frac{1}{j_a} \} \subset \{ x \in \Omega : \text{dist}(x; A) < \text{dist}(A; \partial \Omega) \}. \)

\[
\psi_j(x) = \begin{cases} 
1 & \text{if } \rho_A(x) < \frac{1}{j}, \\
0 & \text{if } \rho_A(x) \geq \frac{2}{j}, \\
1 - H(j \rho_A(x)) & \text{if } \frac{1}{j} < \rho_A(x) < \frac{2}{j}.
\end{cases}
\]

On the set \( D_j = \left\{ \frac{1}{j} < \rho_A(x) < \frac{2}{j} \right\} \) one has

\[
\nabla \psi_j(x) = -j H'(j \rho_A(x)) \nabla \rho_A(x),
\]

so that

\[
\rho_A(x) |\nabla \psi_j(x)| \leq c_1 ||H'||_\infty j \rho_A(x) \leq c_1 H
\]

and

\[
\Delta \psi_j(x) = -H''(j \rho_A(x)) j^2 |\nabla \rho_A(x)|^2 - H'(j \rho_A(x)) j \Delta \rho_A(x).
\]

From which we have

\[
\rho_A(x)^2 |\nabla \psi_j(x)| \leq c_3 ||H'||_\infty (j \rho_A(x))^2 + ||H'||_\infty c_4 j \rho_A(x) \leq c_2 H.
\]

Since the measure of \( A \) is zero and \( \psi_j(x) \xrightarrow[j \to +\infty]{} 0 \quad \forall x \in \Omega \setminus A, \)
we deduce by the Lebesgue dominated theorem that
\[ \| \psi_j \|_{L^1} \xrightarrow{j \to +\infty} 0. \]

Since \( \Delta \psi_j(x) = \nabla \psi_j(x) = 0 \) outside of \( D_j \), we deduce from the above estimates
\[ \left\| \frac{\nabla \psi_j}{\sqrt{V}} \right\|_{L^\infty(\Omega)} \leq c'_1 H_j \frac{1}{j^{\frac{m}{2}-1}} \xrightarrow{j \to +\infty} 0 \] (19)
and
\[ \left\| \frac{\Delta \psi_j}{V_1} \right\|_{L^\infty(\Omega)} \leq c'_2 H_j \frac{1}{j^{m-2}} \xrightarrow{j \to +\infty} 0. \] (20)

Since \( \psi_j \in B^2_A \), we deduce
\[ \text{Cap}_{V,\infty}(A) \leq \| \psi_j \|_{V_i,\infty} \xrightarrow{j \to +\infty} 0. \]

\[ \diamond \]

**Corollary 2.2. of Theorem 2**

Let \( S_1 = \{ x \in \mathbb{R}^n : |x| = 1 \} \) the unit sphere of \( \mathbb{R}^n \), \( m > 2 \) assume that \( S_1 \subset \Omega \) and let \( V \) a nonnegative potential such that there exists \( \eta > 0 \), \( c > 0 \) such that \( V(x) \geq c |x| - 1 |^m \) for all \( x \in \{ y \in \Omega, d(y; S_1) \leq 2\eta \} \). Then
\[ \text{Cap}_{V,\infty}(S_1) = 0. \]

One important property concerns the potential-capacity of a finite union of compact \( \bigcup_{i \in J} K_i \) such that \( \text{Cap}_{V,\infty}(K_i) = 0 \) we are not able to prove the subadditivity, but we also have:

**Theorem 3.**

Let \( V \) be a nonnegative potential, \( K_i, i \in J \) be a finite number of compact sets included in \( \Omega \).
Assume that \( \text{Cap}_{V,\infty}(K_i) = 0 \ \forall \ i \in J \). Then
\[ \text{Cap}_{V,\infty}(\bigcup_{i \in J} K_i) = 0. \]

**Proof:**

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Since \( \text{Cap}_{V,\infty}(K_i) = 0 \) there exists a sequence \( \psi_{ij} \in C^2_c(\Omega) \) such that for a.e \( x \),

\[
\psi_{ij}(x) \xrightarrow{j \to +\infty} 0, \quad ||\psi_{ij}||_{V,\infty} \xrightarrow{j \to +\infty} 0,
\]

\( \psi_{ij}(x) = 1 \) in a neighborhood of \( K_i \), \( 0 \leq \psi_{ij} \leq 1 \). Let us consider \( H \in C^\infty(\mathbb{R}) \), \( 0 \leq H \leq 1 \) as in relation (16), \( \rho \in C^\infty(\Omega) \) equivalent to the distance function \( \delta(x) = \text{dist}(x;\partial\Omega) \).

Since \( \psi_{ij} \in C^2_c(\Omega) \) then we have a set \( \Delta_{ij} \subset \{ x \in \overline{\Omega} : \psi_{ij}(x) = 0 \} \) which is an open neighborhood of the boundary. Therefore, we can consider the open set \( \Delta_j = \bigcap_{i \in J} \Delta_{ij} \) neighborhood of \( \partial\Omega \).

Since \( \alpha_j = \text{dist}(\Omega \setminus \Delta_j;\partial\Omega) > 0 \), we can consider a sequence \( \mu_j > 0 \), such that \( \mu_j < \alpha_j \) and \( \mu_j \to 0 \) as \( j \to +\infty \). One has, in this case, the set

\[
\left\{ x \in \Omega : \rho(x) \leq \mu_j \right\} \subset \Delta_j,
\]

otherwise, we will have a point \( x \) such \( \rho(x) \leq \mu_j \) and \( x \in \Omega \setminus \Delta_j \) so that

\[
\text{dist}(\Omega \setminus \Delta_j;\partial\Omega) \leq \rho(x).
\]

The function

\[
\Phi_j(x) = \left( 1 - \prod_{i \in J} (1 - \psi_{ij}(x)) \right) H\left( \frac{2}{\mu_j} \rho(x) \right) \quad \text{with} \quad 3\mu_j < \text{dist} \left( \bigcup_{i \in J} K_i;\partial\Omega \right)
\]

satisfies

1. \( \Phi_j(x) = 1, \ x \in \bigcup_{i \in J} K_i \),
2. \( \Phi_j \in C^2_c(\Omega) \),
3. \( 0 \leq \Phi_j(x) \leq 1, \ \Phi_j(x) = 1 - \prod_{i \in J} (1 - \psi_{ij}(x)) \) if \( \rho(x) > \mu_j \), \( \Phi_j(x) = 0 \) if \( \rho(x) \leq \mu_j \).

We shall set for simplicity \( \Phi_{ij}(x) = 1 - \psi_{ij}(x) \).

For \( x \in \Omega \) such that \( \rho(x) > \mu_j \), we have \( H\left( \frac{2}{\mu_j} \rho \right) = 1 \) and

\[
\nabla\Phi_j(x) = -\sum_{k \in J, i \in J, k \neq i} \Phi_{ij}(x) \nabla\Phi_{kj}(x)
\]

(21)
\[
\frac{\nabla \Phi_j}{\sqrt{V}}(x) \leq \sum_{k \in J} \frac{\nabla \psi_{kj}}{\sqrt{V}}(x) = \sum_{k \in J} \frac{\nabla \psi_{kj}}{\sqrt{V}}(x) \leq \sum_{k \in J} \|\psi_{kj}\|_{V,\infty}. \tag{22}
\]

We also have
\[
|\Delta \Phi_j(x)| \leq \sum_{k \in J} |\Delta \psi_{kj}(x)| + \sum_{k \in J} \sum_{\ell \in J} |\nabla \psi_{kj}(x)||\nabla \psi_{\ell j}(x)|
\]
\[
\frac{|\Delta \Phi_j(x)|}{\sqrt{V}} \leq \sum_{k \in J} \frac{|\Delta \psi_{kj}(x)|}{\sqrt{V}} + \sum_{(k, \ell) \in J^2} \frac{\nabla \psi_{kj}(x)}{\sqrt{V}} \frac{\nabla \psi_{\ell j}(x)}{\sqrt{V}}
\]
\[
\leq \sum_{k \in J} \|\psi_{kj}\|_{V,\infty} + \sum_{(k, \ell) \in J^2} \|\psi_{kj}\|_{V,\infty} \|\psi_{\ell j}\|_{V,\infty}. \tag{23}
\]

If \(\rho(x) \leq \mu_j\) then \(x \in \Delta_j\) and
\[
1 - \prod_{i \in J}(1 - \psi_{ij}(x)) = 0 : \Phi_j(x) = 0.
\]

We conclude that relations (22) and (23) hold true. Therefore, we always have
\[
\left\| \frac{\nabla \Phi_j}{\sqrt{V}} \right\|_{L^\infty(\Omega)} \leq \sum_{k \in J} \|\psi_{kj}\|_{V,\infty}, \tag{24}
\]
\[
\left\| \frac{\Delta \Phi_j}{V} \right\|_{L^\infty(\Omega)} \leq \sum_{k \in J} \|\psi_{kj}\|_{V,\infty} + \left( \sum_{k \in J} \|\psi_{kj}\|_{V,\infty} \right)^2. \tag{25}
\]

On other hand, by the Lebesgue dominated convergence theorem, we have
\[
\|\Phi_j\|_{L^1(\Omega)} \xrightarrow{j \to +\infty} 0. \tag{26}
\]

Relations (24) to (26) yield that
\[
\|\Phi_j\|_{V,\infty} \xrightarrow{j \to +\infty} 0.
\]

Since
\[
\text{Cap}_{V,\infty} \left( \bigcup_{i \in J} K_i \right) \leq \|\Phi_j\|_{V,\infty},
\]
we derive the result.

As a consequence of Theorem 3, we have
Corollary 3.1. of Theorem 3

For \( i \in \{1, \ldots, m\} \), let \( a_i \in \Omega \), \( r_i \geq 0 \), \( m_i > 2 \), \( c_i > 0 \) real numbers.

Define \( S_i = \{ x \in \mathbb{R}^n : |x - a_i| = r_i \} \), \( K = \bigcup_{i=1}^{m} S_i \) assumed to be included in \( \Omega \). Let \( V \) be a nonnegative potential such that there exists \( \eta > 0 \) such that

\[
V(x) \geq \sum_{i=1}^{m} \frac{c_i}{(|x - a_i| - r_i)^{m_i}} \text{ on } \{ y : \text{dist}(y; K) \leq \eta \}.
\]

Then

\[
\text{Cap}_{V,\infty}(K) = 0.
\]

Proof:

We have seen in Corollary 2.1 of Theorem 2 that \( \text{Cap}_{V,\infty}(S_i) = 0 \) whenever \( S_i \subset \Omega \). Applying Theorem 3, we deduce the result.

\[\square\]

In the above Corollary 1 and 2 of Theorem 2 we may replace \( S_1 \) by any compact included in \( \Omega \) whose measure is zero. Concrete examples for application are given in [3, 13].

As we have announced in the introduction, we have \( \text{Cap}_{|x| - m,\infty}(\{0\}) > 0 \) if \( m < 2 \). Here is the proof

Theorem 4.

Let \( V \) be a nonnegative potential, \( a \in \Omega \) be such that there exist \( \eta > 0 \), \( c > 0 \)

\[
V(x) \leq \frac{c}{|x - a|^{m}}, \quad x \in B(a; 2\eta) \text{ for some } m < 2.
\]

Then

\[
\text{Cap}_{V,\infty}(\{a\}) > 0.
\]

Proof:

Let us set \( V_1(x) = \frac{c}{|x - a|^{m}}, \quad x \in \Omega \setminus \{a\} \).

Following Theorem 2, \( \text{Cap}_{V,\infty}(\{a\}) \geq c_0 \text{Cap}_{V_1,\infty}(\{a\}), \ c_0 > 0 \).
We have for \( \varphi \in B^2_{\{a\}}, \)

\[
\|\nabla \varphi\|_{L^{n,1}(B(a,\eta))} \leq c \|\varphi\|_{V_{1,\infty}} \left[ 1 + \|V_i\|_{L^{n,1}(B(a,\eta))} \right]
\]

\[
\|\varphi\|_{V_{1,\infty}} \leq c_1 \|\varphi\|_{V_{1,\infty}} < +\infty.
\]

Applying the Sobolev-Lorentz embedding

\[
\|\varphi\|_{L^{\infty}(B(a,\eta))} \leq c_2 \left[ \|\varphi\|_{L^{n,1}(B(a,\eta))} + \|\nabla \varphi\|_{L^{n,1}(B(a,\eta))} \right] \leq c_3 \|\varphi\|_{V_{1,\infty}}.
\]

Since \( \varphi(x) = 1 \) in a neighborhood of \( a \), this last inequality implies \( 1 \leq c_3 \text{Cap}_{V_{1,\infty}}(\{a\}) \), this implies the result. 

\[\Diamond\]

Remark 3.

1. As we state before, the choice of the power \( \frac{1}{2} \) and 1 in the definition is linked with the application, it is clear we can use other power as \( \|\nabla \psi\|_{V^\alpha} \) and \( \|\Delta \psi\|_{V^\beta} \), \( \alpha > 0, \beta > 0 \) (see [14]).

2. In Corollary 2.1 of Theorem 2, we may take \( m = 2 \), but the proof to show that \( \text{Cap}_{V_{1,\infty}}(A) = 0 \) uses a different argument ( [14] work in progress).

We define

\[
C^2_0(\Omega) = \{ \varphi \in C^2(\Omega), \varphi = 0 \text{ on } \partial \Omega \}.
\]

4 Approximation of functions in \( C^2_0(\Omega) \)

We shall introduce the following sets :

\[
L^1(\Omega; V) = \left\{ g : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |g(x)|V(x)dx < +\infty \right\}
\]

\[
L^1_0(\Omega) = \left\{ g \in L^1(\Omega) \text{ such that } \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{x : \delta(x) \leq \varepsilon\}} |g(x)|dx = 0 \right\}
\]
Remark 4.

One has

$$L^1(\Omega; \delta) = \left\{ g \in L^1(\Omega) : \int_{\Omega} |g(x)| \frac{dx}{\delta(x)} < +\infty \right\}$$

is strictly included in $L^1_{\infty}(\Omega)$.

Indeed, it was shown in [15] that

$$f \geq 0 \quad f \in L^1(\Omega; \delta) \setminus L^1(\Omega; \delta(1 + |\log \delta|))$$

then the unique solution $g \in L^1_{\infty}(\Omega)$ of

$$- \int_{\Omega} g \Delta \varphi \, dx = \int_{\Omega} \varphi f \, dx \quad \forall \varphi \in C^2_0(\Omega)$$

verifies

$$\int_{\Omega} \frac{g}{\delta}(x) \, dx = +\infty.$$ 

But A. Ponce ([12], chap. 20) shows that we have $g \in L^1_{\infty}(\Omega)$.

Definition 5.

Let $\phi$ be in $C^2_0(\Omega)$. We will say that a sequence $(\varphi_j)_j$ of $C^2_0(\Omega)$ converges weakly in the sense of the potential $V$ to $\phi$ if for all $g \in L^1(V) \cap L^1_{\infty}(\Omega)$:

1. $\int_{\Omega} g \varphi_j \, V \, dx \xrightarrow{j \to +\infty} \int_{\Omega} g \phi \, V \, dx$
2. $\int_{\Omega} g \partial_k \varphi_j \, dx \xrightarrow{j \to +\infty} \int_{\Omega} g \partial_k \phi \, dx, \quad k = 1, \ldots, n$
3. $\int_{\Omega} g \Delta \varphi_j \, dx \xrightarrow{j \to +\infty} \int_{\Omega} g \Delta \phi \, dx$.

Here $\partial_k$ is the partial derivative with respect to the $k^{th}$ derivate.

Definition 6.

Let $\phi$ be in $C^2_0(\Omega)$. We will say that a sequence $(\varphi_j)_j$ of $C^2_0(\Omega)$ converges weakly-strongly in the sense of $V$ to $\phi$ if for all $g \in L^1(V) \cap L^1_{\infty}(\Omega)$

$$0 = \lim_{j} \int_{\Omega} |g| |\varphi_j - \phi| \, V \, dx = \lim_{j} \int_{\Omega} |g| |\nabla \varphi_j - \nabla \phi| \, dx = \lim_{j} \int_{\Omega} |g| |\Delta \varphi_j - \Delta \phi| \, dx.$$
We have the

**Theorem 5.**

Let $K$ be a compact in $\Omega$ and $V$ a nonnegative potential. Assume that $\text{Cap}_{V,\infty}(K) = 0$. Then

the set $C^2_c(\Omega \setminus K)$ is **weakly-strongly dense** in $C^2_0(\bar{\Omega})$ in the sense of the potential $V$.

**Proof:**

Let $\Phi$ be in $C^2_0(\bar{\Omega})$. Since $\text{Cap}_{V,\infty}(K) = 0$ one has a sequence $(\varphi_j)_j$, $\varphi_j \in C^2_c(\Omega)$ such that $0 \leq \varphi_j \leq 1$, $\varphi_j = 1$ in a neighborhood of $K$ and $||\varphi_j||_{V,\infty} \xrightarrow[j \to +\infty]{} 0$, $\varphi_j \xrightarrow[j \to +\infty]{} 0$ a.e in $\Omega$.

As before, we then have a sequence $(\mu_j)_j$ tending to zero $\mu_j > 0$ such that

the set $\{x : \delta(x) \leq \mu_j\}$ is included in $\{x : \varphi_j(x) = 0\}$.

Let $H$ be the function given in (16).

Then the sequence $\Phi_j = (1 - \varphi_j)H \left( \frac{2}{\mu_j} \rho \right) \Phi$ where $\rho$ is the smooth function equivalent to the distance function $\delta$ with the conditions

\[ ||\nabla \rho||_\infty < +\infty, \quad ||\Delta \rho||_\infty < +\infty \text{ (see Lemma 2.1)} \]

$\Phi_j$ belongs to $C^2_c(\Omega \setminus K)$. We have the following pointwise relations

\[ \Phi_j(x) \xrightarrow[j \to +\infty]{} \Phi(x) \text{ a.e in } \Omega. \] (27)

If $\rho(x) > \mu_j$, we have:

\[
\begin{align*}
\Phi_j(x) &= (1 - \varphi_j(x))\Phi(x) \\
\nabla \Phi_j(x) &= -\nabla \varphi_j(x)\Phi(x) + (1 - \varphi_j(x))\nabla \Phi(x) \\
\Delta \Phi_j(x) &= -\Delta \varphi_j(x)\Phi(x) - 2\nabla \varphi_j(x)\nabla \Phi(x) + (1 - \varphi_j(x))\Delta \Phi(x).
\end{align*}
\]

(28)

If $\rho(x) \leq \mu_j$, we know that $\varphi_j(x) = 0$ so that

\[ \Phi_j(x) = H \left( \frac{2}{\mu_j} \rho(x) \right) \Phi(x) \text{ if } \rho(x) \geq \frac{1}{2}\mu_j \text{ and } \Phi_j(x) = 0 \text{ otherwise.} \]

(29)
Then on $\left\{ \frac{1}{2} \mu_j \leq \rho \leq \mu_j \right\}$, one has

$$\nabla \Phi_j(x) = \frac{2}{\mu_j} \nabla \rho H' \left( \frac{2}{\mu_j} \rho(x) \right) \Phi(x) + H \left( \frac{2}{\mu_j} \rho(x) \right) \nabla \Phi(x)$$

and

$$\Delta \Phi_j(x) = \frac{2}{\mu_j} \Delta \rho(x) \Phi(x) H' \left( \frac{2}{\mu_j} \rho(x) \right) + \left( \frac{2}{\mu_j} \right)^2 \Phi(x) |\nabla \rho(x)|^2 H'' \left( \frac{2}{\mu_j} \rho(x) \right)$$

$$+ \frac{1}{\mu_j} H' \left( \frac{2}{\mu_j} \rho(x) \right) \nabla \rho(x) \cdot \nabla \Phi(x) + H \left( \frac{2}{\mu_j} \rho(x) \right) \Delta \Phi(x).$$

(30)

Let $g \in L^1(V) \cap L^1_0(\Omega)$. By the Lebesgue dominated theorem, we have

$$\lim_\Omega |g(x)| |\Phi_j(x) - \Phi(x)| V(x)dx = 0$$

(31)

From relation (28), we derive

$$\int_{\left\{ \rho > \mu_j \right\}} |\nabla \Phi_j(x) - \nabla \Phi(x)| |g(x)| dx \leq ||\nabla \Phi||_\infty \int_\Omega |\varphi_j(x)| |g(x)| dx$$

$$+ ||\varphi_j||_{V,\infty} ||\Phi||_\infty \left( \int_\Omega |g(x)| V(x)dx \right) \left( \int_\Omega |g|dx \right).$$

(32)

(We have used the Cauchy Schwarz inequality: $\int_\Omega |g|\sqrt{V} dx \leq \left( \int_\Omega |g|Vdx \right) \left( \int_\Omega |g|dx \right)$)

Setting $A_j = \left\{ \frac{1}{2} \mu_j \leq \rho \leq \mu_j \right\}$, one has

$$\int_{\left\{ \rho \leq \mu_j \right\}} |g(x)| |\nabla \Phi_j(x) - \nabla \Phi(x)| dx \leq \int_{\left\{ \rho \leq \mu_j \right\}} |g(x)| |\nabla \Phi(x)| dx + \int_{A_j} |g(x)| |\nabla \Phi_j(x)| dx.$$  

(33)

The first integral tends to zero using Lebesgue dominated theorem, while the second integral can be bound as

$$\int_{A_j} |g(x)| |\nabla \Phi_j(x)| dx \leq c \left( \frac{1}{\mu_j} \int_{A_j} |g(x)| dx + \int_{A_j} |g(x)| dx \right) \xrightarrow{j \to +\infty} 0 \text{ since } g \in L^1_0(\Omega).$$

(34)

From relations (32) to (34) we derive

$$\lim_\Omega |g(x)| |\nabla \Phi_j(x) - \nabla \Phi(x)| dx = 0.$$  

(35)
Using the same argument as above, we have

\[
\lim \int_{\Omega} |g(x)| |\Delta \Phi_j(x) - \Delta \Phi(x)| dx = 0. \quad (36)
\]

Indeed, we have

\[
\int_{\{\rho \geq \mu_j\}} |g(x)| |\Delta \Phi_j(x) - \Delta \Phi(x)| dx \leq \int_{\{\rho \geq \mu_j\}} |\Delta \varphi_j| |\Phi| |g| dx + \int_{\{\rho \geq \mu_j\}} |\varphi_j| |\Delta \Phi| |g| dx
\]
\[
+ 2 \int_{\{\rho \geq \mu_j\}} |\nabla \varphi_j| |\nabla \Phi| |g| dx
\]
\[
\leq c_\Phi |\varphi_j|_{V, \infty} \left[ \int_{\Omega} |g| \frac{V dx}{\rho} \right] \left[ 1 + \int_{\Omega} |g| dx \right]
\]
\[
+ c_\Phi \int_{\Omega} |\varphi_j| |g| dx \xrightarrow{j \to +\infty} 0. \quad (37)
\]

On \( \{\rho \leq \mu_j\} \), we have :

\[
\int_{\{\rho \leq \mu_j\}} |g| |\Delta \Phi_j - \Delta \Phi| dx \leq \int_{\{\rho \leq \mu_j\}} |g| |\Delta \Phi| + \int_{A_j} |g| |\Delta \Phi_j| dx. \quad (38)
\]

The first term tends to zero, while for the last term we replace \( \Delta \Phi_j \) by its expression :

\[
\int_{A_j} |g| |\Delta \Phi_j| dx \leq I_{1j} + I_{2j} + I_{3j} + I_{4j}. \quad (39)
\]

Using the fact that \( \frac{\Phi(x)}{\rho(x)} \leq c |\nabla \Phi|_{\infty} \), and \( |\rho \Delta \rho(x)| \leq c_2 \) for all \( x \in \Omega \), we have :

\[
I_{1j} \leq c \frac{1}{\mu_j} \int_{A_j} \frac{\Phi(x)}{\rho(x)} |\Delta \rho(x) | \frac{\rho(x)}{\rho(x)} |g(x)| dx \leq c \frac{1}{\mu_j} \int_{A_j} |g(x)| dx, \quad (40)
\]

\[
I_{2j} \leq c \left( \frac{1}{\mu_j} \right)^2 \int_{A_j} |g(x)| \rho(x) dx \leq c \frac{1}{\mu_j} \int_{A_j} |g(x)| dx, \quad (41)
\]

\[
I_{3j} \leq c \frac{1}{\mu_j} \int_{A_j} |g(x)| dx, \quad I_{4j} \leq c \int_{A_j} |g(x)| dx \quad (42)
\]

Thus

\[
\int_{A_j} |g| |\Delta \phi_j| dx \leq c \left[ \frac{1}{\mu_j} \int_{A_j} |g(x)| dx + \int_{A_j} |g(x)| dx \right], \quad (43)
\]

the constant \( c \) is independent of \( j \) and \( g \). From relations (37) to (43), we derive the result. ∆
One may also give sufficient conditions to ensure that a sequence converges weakly in the sense of $V$.

Here is an example of such result:

**Theorem 6.**

Let $(\varphi_j)_j$ be a sequence of $C^2_0(\Omega)$, $K$ a compact in $\Omega$, $V$ a nonnegative potential such that $V$ is upper semi-continuous, that is for all real $t$, the set $\{V \geq t\}$ is closed in $\Omega$, and assume also that the set $\{x : V(x) = +\infty\}$ is of measure zero, and:

1. $||\varphi_j||_{V,\infty}$ remains bounded in $\mathbb{R}_+$,

2. there exists $\varphi \in C^2_0(\Omega)$ such that the sequence $(\varphi_j)_j$ converges to $\varphi$ in $L^\infty(\Omega)$-weak-star.

Then, $(\varphi_j)_j$ converges weakly to $\varphi$ in the sense of the potential $V$.

**Sketch of the proof**

Let $1 > \eta > 0$, the, \( \{x \in \Omega : V(x) \geq \frac{1}{\eta}\}\) is closed in $\Omega$ thus $\Omega_\eta = \{x \in \Omega : \frac{1}{V}(x) > \eta\}$ is open and we have a constant $M$ such that \( \forall j, \forall \eta \in [0,1], ||\nabla \varphi_j||_{L^\infty(\Omega_\eta)} + ||\Delta \varphi_j||_{L^\infty(\Omega_\eta)} \leq \eta^{-1}||\varphi_j||_{V,\infty} \leq M\eta^{-1}. \) (44)

Then we deduce that for all $\eta \in [0,1]$, $||\nabla \varphi_j - \varphi||_{C^1(\Omega_\eta), j \to +\infty} \to 0$.

Since $\Omega \setminus \bigcup_{\eta>0} \Omega_\eta$ is of measure zero, therefore,

\[ \frac{\nabla \varphi_j}{\sqrt{V}} \to \frac{\nabla \varphi}{\sqrt{V}} \quad \text{and} \quad \frac{\Delta \varphi_j}{\sqrt{V}} \to \frac{\Delta \varphi}{V} \quad \text{in} \quad L^\infty\text{-weak-star when} \quad j \to +\infty. \]

From those convergences, we derive the result.

\[ \Box \]

**An example of sequence satisfying Theorem 6**

As example, we can take $A$ as in Corollary 2.1 of Theorem 2, $V(x) = d(x; A)^{-2}$ and $\psi_j(x) = \left(1 - H(j\rho_A(x))\right)H(j\rho(x))$, as in the proof of Corollary 2.1 of Theorem 2. Then, for $\varphi \in C^2_0(\Omega)$ the sequence $\varphi_j(x) = (1 - \psi_j(x))\varphi(x)$ satisfies conditions 1. and 2. .

Indeed, since $\varphi_j(x) \to \varphi(x) \quad \forall x \in \Omega \setminus A$ and $||\varphi_j||_{\infty} \leq ||\varphi||_{\infty}$, we deduce that $(\varphi_j)_j$ converges to $\varphi$ in $L^\infty(\Omega)$-weakly-star.
The set \( \{ x : V(x) = +\infty \} = A \) is of measure zero and \( V \) is upper semi continuous.

More, \( \varphi_j(x) = \varphi(x) \) if \( \rho_A(x) \geq \frac{2}{j} \), and \( \nabla \varphi_j(x) = 0 \) if \( \rho_A(x) < \frac{1}{j} \),

on \( D = \{ \frac{1}{j} < \rho_A < \frac{2}{j} \} \), we have

\[
\rho_A(x) |\nabla \varphi_j(x)| \leq c_1 \varphi \text{ and } \rho_A(x)^2 |\Delta \varphi_j(x)| \leq c_2 \varphi.
\]

This implies \( \| \varphi_j \|_{V, \infty} \leq M < +\infty \).

Corollary 6.1. of Theorem 6

Let \( V(x) = d(x; A)^{-2} \), \( A \) a compact set of measure zero in \( \Omega \). Then

\( C^2_c(\Omega \setminus A) \) is weakly dense in \( C^2_0(\Omega) \) in the sense of the potential \( V \).

Proof :

Let \( \varphi \in C^2_0(\Omega) \) then \( \varphi_j H(j \rho(x)) \) is in \( C^2_c(\Omega \setminus A) \) and the above arguments imply the statement.

5 Applications of the potential-capacity and the approximation of \( C^2_0(\Omega) \)

As a first application of the above results, we shall prove a removable type problem.

Theorem 7.

Let \( K \) be compact included in \( \Omega \). Assume that \( C^2_c(\Omega \setminus K) = C^2_c(\Omega_K) \) is weakly dense in \( C^2_0(\Omega) \) in the sense of potential \( V \) and let \( w \in L^1(\Omega; V) \cap L^1_0(\Omega) \) be such that for all \( \varphi \in C^2_c(\Omega_K) \) we have

\[
\int_{\Omega} w(-\Delta \varphi + V \varphi) dx = 0.
\]

Then \( w \) satisfies the same equation (46) with \( \varphi \in C^2_0(\Omega) \).

Proof :

Let \( \varphi \) be in \( C^2_0(\Omega) \). Then, we have a sequence \( (\varphi_j)_j, \varphi_j \in C^2_c(\Omega \setminus K) \) such that
\[
\lim_{j \to \infty} \int_{\Omega} w \varphi_j V \, dx = \int_{\Omega} w \varphi V \, dx.
\] (47)

\[
\lim_{j \to \infty} \int_{\Omega} w \Delta \varphi_j \, dx = \int_{\Omega} w \Delta \varphi \, dx.
\] (48)

Since \(0 = \int_{\Omega} w(-\Delta \varphi_j + V \varphi_j) \, dx\) thus we have the result by passing to the limit. \(\diamondsuit\)

Next, we recall the following Kato’s inequality (see [12, 10, 8, 2]).

**Lemma 5.1. Kato’s inequality and weak maximum principle**

Assume that \(w\) and \(f\) are in \(L^1(\Omega)\) such \(-\Delta w = f\ in \mathcal{D}'(\Omega)\). Then

1. \(-\Delta |w| \leq f \text{ sign}(w) in \mathcal{D}'(\Omega)\), (49)

2. \(-\Delta w_+ \leq f \text{ sign}_+(w) in \mathcal{D}'(\Omega)\),

3. if \(-\Delta w \leq 0 in C^2_0(\Omega)' (dual space) then w \leq 0.\)

\[
\text{sign}_+(\sigma) = \begin{cases} 
1 & \text{if } \sigma > 0, \\
0 & \text{otherwise,}
\end{cases}
\]

and \(\text{sign}(\sigma) = \begin{cases} 
1 & \text{if } \sigma > 0, \\
0 & \text{if } \sigma = 0, \\
-1 & \text{if } \sigma < 0.
\end{cases}\)

**Corollary 7.1. of Theorem 7 and Lemma 5.1**

Under the same assumption as for Theorem 7, the function \(w\) verifying relation (46) satisfies

\[w \equiv 0.\]

**Proof :**

Since \(\mathcal{D}(\Omega) = C_0^\infty(\Omega) \subset C^2_0(\Omega)\), then following Theorem 7, we have

\[-\Delta w = -Vw \in L^1(\Omega) in \mathcal{D}'(\Omega).\]
From Kato’s inequality, one has:

\[-\Delta(|w|) \leq -V|w| \leq 0.\]

Therefore, using the same arguments as for Theorem 7, the inequality holds in the dual space $C^2_0(\Omega)'$, we conclude that $|w| \leq 0 : w = 0$.

Let $V$ be a nonnegative potential and define the subset of $\Omega$ by

\[\Omega_V = \left\{ x \in \Omega, \exists r_x > 0 \text{ such that } ||V||_{L^{\frac{2}{m-1}}(B(x,r_x))} < +\infty \right\} .\]

One can show that $\Omega_V$ is an open set in $\Omega$.

Thus its complement $K_V = \Omega - \Omega_V$ is a compact included in $\Omega$.

**Definition 7.**

The points $K_V$ are called the irregular points of $V$.

**Remark 5.**

The choice of $K_V$ can be modified according to the application that one wants to do.

If $V(x) = |x - a|^{-m}$, $a \in \Omega$ and applying the first theorem, then

\[K_V = \begin{cases} \{a\} & \text{if } m \geq 2, \\ \emptyset & \text{otherwise}. \end{cases}\]

And as consequence of the above result, if $m > 2$, $A$ compact subset of $\Omega$

\[V(x) = \text{dist}(x;A)^{-m} \text{ then } K_V = A.\]

**Corollary 7.2. of Theorem 1 and Theorem 5**

Under the same assumptions as for Theorem 5 and Theorem 1, with $K = K_V$, then

for $\Phi \in C^2_0(\Omega)$ the sequence $(\Phi_j)_j$ given in the proof of Theorem 5 say

\[\Phi_j = (1 - \varphi_j)H \left( \frac{2}{\mu_j} \right) \Phi\]

satisfies: For all open set $\Omega_{V,\delta}$ relatively compact in $\Omega_V$ one has:
1. \[ \lim_{j \to +\infty} \max_{\Omega_{V,0}} |\Phi_j(x) - \Phi(x)| = 0 \]
2. \[ \lim_{j \to +\infty} |\nabla (\Phi_j - \Phi)|_{L^{n-1}(\Omega_{V,0})} = 0. \]

**Proof:**

Let \( \Omega_{V,0} \subset \subset \Omega_V \). Then according to Theorem 1

\[ \max_{\Omega_{V,0}} |\varphi_j(x)| \longrightarrow 0 \quad (50) \]

\[ ||\nabla \varphi_j||_{L^{n-1}(\Omega_{V,0})} \longrightarrow 0. \quad (51) \]

On the other hand for \( j \geq j_0 \), we have \( \Omega_{V,0} \subset \{ \rho > \mu_j \} \). Therefore we have

\[ \max_{\Omega_{V,0}} |\Phi_j(x) - \Phi(x)| \leq ||\Phi||_{L^{\infty}} \max_{\Omega_{V,0}} |\varphi_j(x)| \quad (52) \]

\[ ||\nabla (\Phi_j - \Phi)||_{L^{n-1}(\Omega_{V,0})} \leq ||\nabla \Phi||_{L^{\infty}} \max_{\Omega_{V,0}} |\varphi_j(x)| + ||\Phi||_{L^{\infty}} ||\nabla \varphi_j||_{L^{n-1}(\Omega_{V,0})} \quad (53) \]

Relations (50) to (53) give the result. \( \diamond \)

As in [4], we may add a transport term \( U \cdot \nabla \varphi \) in the above equation (46).

**Lemma 5.2.**

Let \( V \) a nonnegative potential \( K \) be a compact in \( \Omega \) with \( \text{Cap}_{V,\infty}(K) = 0 \).

Consider \( U \in L^{n,1}(\Omega)^n \), \( p > n \), \( w \in L^1(V) \cap L^q(\delta^{-1}) \), \( q < p' \). Assume that \( w \in L^{\infty}(\delta^{-1}) \) if \( n \geq 3 \) and \( w \in L^{\exp}(\Omega) \) if \( n = 2 \).

Then, for all \( \Phi \in C_0^2(\overline{\Omega}) \), the sequence given in Theorem 5, \( \Phi_j = (1 - \varphi_j) H \left( \frac{2}{\mu_j} \rho \right) \Phi \) satisfies

1. \[ \lim_{j \to \infty} \int_{\Omega} w \Delta \Phi_j \, dx = \int_{\Omega} w \Delta \Phi \, dx \]
2. \[ \lim_{j \to \infty} \int_{\Omega} w \Phi_j \, V \, dx = \int_{\Omega} \Phi \, w \, V \, dx, \]
3. \[ \lim_{j \to \infty} \int_{\Omega} w U \cdot \nabla \Phi_j \, dx = \int_{\Omega} w U \cdot \nabla \Phi \, dx. \]

**Proof:**

The two first statements are the consequence of the fact \( w \in L^1(V) \cap L^1(\delta^{-1}) \), \( L^1(\delta^{-1}) \subset L^1(\Omega) \) and the fact that \( \Phi_j \) converges weakly-strongly to \( \Phi \) in the sense of the potential \( V \). Moreover,
we have a constant $c_H > 0$

$$\int_{\Omega} \left| wU \cdot (\nabla \Phi_j - \nabla \Phi) \right| dx \leq c_{H\Phi} \left[ \left\| \frac{\nabla \Phi_j}{\sqrt{V}} \right\|_{L^\infty} \int_{\Omega} \left| w \right| U \sqrt{V} dx + \int_{\Omega} \left| \varphi_j \right| \left| w \right| |U| dx \right]. \tag{54}$$

By Hölder, $|w|U$ and $|w|U\sqrt{V}$ are in $L^1(\Omega)$ since

$$\int_{\Omega} \left| w \right| |U| dx \leq \left\| \frac{w}{\delta} \right\|_{L^q} \left\| U \right\|_{L^{q'}} < +\infty, \quad \frac{1}{q} + \frac{1}{q'} = 1$$

and

$$\int_{\Omega} \left| w \right| U \sqrt{V} dx \leq c \left\| w \right\|_{L^{1}(V)} \left\| w \right\|_{L^{\frac{n}{n-2},\infty}} \left\| U \right\|_{L^{n,1}} \text{ if } n \geq 3.$$ 

Idem for $n = 2$. Therefore, relation (54) leads to statement 3. knowing

$$\lim_{j \to +\infty} \varphi_j(x) = 0, \quad 0 \leq \varphi_j \leq 1 \quad \text{and} \quad \left\| \frac{\nabla \varphi_j}{\sqrt{V}} \right\|_{\infty} \to 0.$$

\[\square\]

**Theorem 8.**

*Under the same assumption as for Lemma 5.2, if furthermore $w$ satisfies*

$$\int_{\Omega} w(-\Delta \Phi - U \cdot \nabla \Phi + V \Phi) dx = 0 \quad \forall \Phi \in C_0^2(\Omega \setminus K) \tag{55}$$

*then, (55) holds for all $\Phi \in C_0^2(\Omega)$, and if $\partial \Omega \in C^{1,1}$ and $\text{div} \left( \overrightarrow{U} \right) = 0$ in $\mathcal{D}'(\Omega)$ with $\overrightarrow{U} \cdot \overrightarrow{\nu} = 0$ on $\partial \Omega$ (\nu exterior normal to $\partial \Omega$) then*

$$w \equiv 0.$$

**Proof:**

If $\Phi \in C_0^2(\overline{\Omega})$ we have a sequence $\Phi_j$ in $C_0^2(\Omega \setminus K)$ such that, $\Phi_j$ satisfies the conclusion of Lemma 5.2. Thus, we have (55) with $\Phi \in C_0^2(\Omega)$ as test function. To prove that $w \equiv 0$ we need to employ the following variant of Kato’s inequality (see [4]).

**Theorem 9.** Variant of Kato’s inequality

*Let $\overline{u}$ be in $W^{1,1}_{loc}(\Omega) \cap L^{n',\infty}(\Omega)$ with $\overline{u} / \delta \in L^1(\Omega)$ and $\overline{U} \in L^{n-1}(\Omega)^n$ with $\text{div} \left( \overrightarrow{U} \right)$ in $\mathcal{D}'(\Omega)$, $\overrightarrow{U} \cdot \overrightarrow{\nu} = 0$ on $\partial \Omega$.*

*Assume that $\overline{\Pi} = -\Delta \overline{u} + \text{div} \left( \overrightarrow{U} \overline{u} \right) \in L^1(\Omega; \delta)$. Then for all $\phi \in C_0^2(\overline{\Omega})$, $\phi \geq 0$ one has*
\[ \int_{\Omega} [\nabla u + L^* \phi] dx \leq \int_{\Omega} \phi \text{sign}(\nabla u) L^* \phi dx \]

\[ \int_{\Omega} |u| L^* \phi dx \leq \int_{\Omega} \phi |w| V dx, \]

where \( L^* \phi = -\Delta \phi - \nabla \cdot \nabla \phi = -\Delta - \text{div} (U \phi) \).

According to equation (55), \( Lw = -Vw \in L^1(\Omega) \). Thus the above Kato’s type inequality holds and

\[ \forall \phi \in C^2_0(\Omega), \quad \phi \geq 0 : \int_{\Omega} |w| L^* \phi \leq -\int_{\Omega} \phi |w| V dx \leq 0. \]

Thus one has

\[ \int_{\Omega} |w| L^* \phi = 0 \quad \forall \phi \in C^2_0(\Omega). \]

By density result the same equation holds

\[ \forall \phi \in H^1_0(\Omega) \cap W^{2,1}(\Omega), \quad \phi \geq 0. \]

Resolving \( L^* \phi = 1 \) we derive that \( w \equiv 0 \).

Next, we want to discuss some existence problem related to equation (55).

We always assume that \( U \in L^p(\Omega)^n, \quad p > n, \quad \text{div}(U) = 0 \) in \( \mathcal{D}'(\Omega), \quad U \cdot \nu = 0 \) on \( \partial \Omega \) and \( \partial \Omega \in C^{1,1} \).

**Theorem 10.**

Let \( f \) be a bounded Radon measure in \( \Omega \). Assume that \( \text{Cap}_{V,\infty}(K_V) = 0 \).

If \( |f|(K_V) = 0 \) (\( f \) does not charge the compact set \( K_V \)) then there exists an unique solution \( u \in L^1(V) \cap L^1(\Omega; \delta^{-1}) \) such that

\[ \int_{\Omega} u(-\Delta \varphi - U \cdot \nabla \varphi + V \varphi) dx = \int_{\Omega} \varphi df, \quad \forall \varphi \in C^2_0(\Omega). \]  

**Proof:**

The uniqueness is a consequence of Theorem 8.

For the existence, we first notice that the problem is linear, we may assume that \( f \geq 0 \). We shall set as usual

\[ M^1(\Omega) = \left\{ f : \text{ bounded Radon measure on } \Omega \right\}, \]

\[ M^1(\Omega) = C_0(\Omega)'/, \quad C_0(\Omega) = \left\{ \varphi : \Omega \to \mathbb{R} \text{ continuous, } \varphi = 0 \text{ on } \partial \Omega \right\} \]

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Let us introduce \( V_j = \min(j; V) \) we have proved the following result in [4, 5].

**Lemma 5.3.**

There exists \( u_j \geq 0, \, u_j \in W^1_0 L^{n', \infty}(\Omega) \) such that

1. \( \forall \varphi \in H^1_0(\Omega) \cap W^{2,1}(\Omega) \)
   \[
   \int_\Omega u_j[-\Delta \varphi - U \cdot \nabla \varphi + V_j \varphi] \, dx = \int_\Omega \varphi df. \tag{57}
   \]

2. There exists a constant \( c_0 \) independent of \( j \) such that
   \[
   \| u_j \|_{W^1_0 L^{n', \infty}(\Omega)} + \int_\Omega V_j u_j \, dx \leq c_0 \| f \|_{M^1(\Omega)}. \tag{58}
   \]

3. In particular, there exist a function \( u \geq 0 \) and a subsequence \( u_j \) such that
   
   (a) \( \lim_{j \to +\infty} u_j(x) \) a.e in \( \Omega \), strongly in \( L^1(\Omega) \) and weakly in \( W^1_0 L^{n', \infty}(\Omega) \).
   
   (b)
   \[
   \| u \|_{W^1_0 L^{n', \infty}(\Omega)} + \int_\Omega V u \, dx \leq c_0 \| f \|_{M^1(\Omega)}. \tag{59}
   \]

**Proof of Lemma 5.3**

Since \( f \in M^1(\Omega) \), there is a sequence \( f_k \in L^\infty(\Omega) \) such that

\[
\| f_k \|_{L^1(\Omega)} \leq \| f \|_{M^1(\Omega)} \text{ and } f_k \text{ converges to } f \text{ weakly in } C_c(\Omega)'.
\]

(i.e \( \forall \varphi \in C_c(\Omega) \quad < f_k, \varphi > \to < f, \varphi > \).

According to [4, 5], one has a function \( u_{jk} \in W^1_0 L^{n', \infty}(\Omega) \) satisfying, \( \forall \varphi \in C^2_0(\bar{\Omega}) \)

\[
\int_\Omega u_{jk}[-\Delta \varphi - U \cdot \nabla \varphi + V_j \varphi] \, dx = \int_\Omega f_k \varphi \, dx \tag{60}
\]

and

\[
\| u_{jk} \|_{W^1_0 L^{n', \infty}(\Omega)} + \int_\Omega V_j u_{jk} \, dx \leq c_0 \| f_k \|_{L^1(\Omega)} \leq c_0 \| f \|_{M^1(\Omega)} \tag{61}
\]

where \( c_0 \) is independent of \( j \) and \( k \) (in fact \( c_0 \) depends on \( \Omega \) and \( \| U \|_{L^{n,1}(\Omega)} \)). More \( u_{jk} \geq 0 \).

Thus we have a subsequence still denoted \( (u_{jk})_k \) and a function \( u_j \in W^1_0 L^{n', \infty}(\Omega) \), \( u_j \geq 0 \) such

\[
u_{jk} \to u_j \text{ weakly in } W^1_0 L^{n', \infty}(\Omega), \text{ strongly in } L^1(\Omega) \text{ and almost everywhere in } \Omega.
\]

Thus, we can pass easily to the limit in relations (60) and (61) to derive the part 1.) and 2.)
of Lemma 5.3. By the same reason as above, we have a subsequence still denoted \( u_j \) and a function \( u \geq 0 \) such that \( u_j \rightharpoonup u \) weakly in \( W^1_0 L^{n'} \infty(\Omega) \) strongly in \( L^1(\Omega) \), almost everywhere in \( \Omega \). From relation (61) using among other Fatou’s lemma, we have relation (59).

\[ \Box \]

**Lemma 5.4.**

Let \( \varphi \in W^1_0 L^{n,1}(\Omega) \) with support(\( \varphi \)) \( \cap K_V = \emptyset \).

Then

\[ \lim_{j \to +\infty} \int_\Omega |u_j V_j - u V| \varphi dx = 0. \]

**Proof:**

Let \( \varphi \) be in \( W^1_0 L^{n,1}(\Omega) \) with support(\( \varphi \)) \( \cap K_V = \emptyset \).

Thus \( V \varphi \in L^{\frac{n}{n-1}}(\Omega) \) and support(\( V \varphi \)) \( \subset \subset \Omega \setminus K_V = \Omega_V \),

since support(\( \varphi \)) \( \subset \left\{ x : \text{dist (} x; K \text{)} > \eta \right\} \) for some \( \eta > 0 \). We have :

\[
\begin{align*}
||u_j - u||_{L_{exp}(\Omega)} &\leq c_N ||u_j - u||_{W^1_0 L^{2,\infty}(\Omega)} \text{ if } n = 2 \\
||u_j - u||_{L^{\frac{n}{n-2}}(\Omega)} &\leq c_N ||u_j - u||_{W^1_0 L^{n',\infty}(\Omega)} \text{ if } n \geq 3.
\end{align*}
\]

(62)

Therefore, applying Hölder’s inequality, we have a constant \( c_0 > 0 \) (independent of \( u_j, u, V_j \)) such that for any measurable subset \( E \subset \Omega \)

\[ \int_E V |\varphi| |u_j - u| dx \leq c_0 ||V \varphi||_{L^{\frac{n}{n-1}}(E)} \xrightarrow{|E| \to 0} 0. \]

(63)

Therefore, using the Egoroff’s theorem or Vitali’s theorem one has

\[ \lim_{j \to \infty} \int_\Omega V |\varphi| |u_j - u| dx = 0. \]

(64)

Since we have

\[ \lim_{j \to \infty} \int_\Omega |u| |V_j - V| \varphi dx = 0. \]

(65)

Finally we have

\[ \lim_{j \to \infty} \int_\Omega |u_j V_j - u V| \varphi dx = 0. \]

(66)

\[ \Box \]
Lemma 5.5.

The function $u$ found in the preceding Lemma 5.3, satisfies, $\forall \varphi \in C^2_c(\Omega_V)$

$$\int_{\Omega} u(-\Delta \varphi - U \cdot \nabla \varphi + V \varphi)dx = \int_{\Omega} \varphi df.$$  

Proof:

Let $\varphi$ be in $C^2_c(\Omega_V)$ then,

$$\int_{\Omega} u_j[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi_j]dx = \int_{\Omega} \varphi_j df$$  

and

$$\lim_{j \to +\infty} \int_{\Omega} |u_j V_j - u V| \varphi dx = 0 \text{ (since support}(\varphi) \cap K_V = \emptyset).$$

Thus we may pass to the limit in relation (67).  

Lemma 5.6.

If $\text{Cap}_{V,\infty}(K_V) = 0$ and $|f|(K_V) = 0$ then,

the function $u$ given in Lemma 5.5 satisfies relation (56)

and

$$u \in L^1(V) \cap W^1_0 L^{n',\infty}(\Omega) \subset L^1(V) \cap L^q \left(\Omega; \frac{1}{\delta} \right) \quad q < n'.$$

Proof:

Let $\Phi$ be in $C^2_c(\Omega)$. From our assumption we have a sequence $\Phi_j \in C^2_c(\Omega_V)$ such that :

1. $\Phi_j$ converges weakly to $\Phi$ in the sense of potential $V$,

2. $\Phi_j(x) \longrightarrow \Phi(x)$ for all $x \in \Omega - K_V = \Omega_V$ (see Corollary 7.2 of Theorem 5 and Theorem 1), $\Phi_j = 0$ in the neighborhood of $K_V$

Since

$$\int_{\Omega} u(-\Delta \Phi_j - U \cdot \nabla \varphi_j + V \Phi_j)dx = \int_{\Omega} \Phi_j df,$$  

We pass to the limit since $u \in L^1(V) \cap W^1_0 L^{n',\infty}\Omega \subset L^1(V) \cap L^1_0(\Omega)$ in the first integral and in
the second integral using Lebesgue dominated theorem to derive

\begin{align}
\lim_{j \to \infty} \int_{\Omega} u(-\Delta \Phi_j + V \Phi_j) dx &= \int_{\Omega} u(-\Delta \Phi + V \Phi) dx \\
\lim_{j \to \infty} \int_{\Omega} \Phi_j df &= \int_{\Omega \backslash K_V} \Phi df.
\end{align}

(69) \hspace{2cm} (70)

Applying Lemma 5.3, we have

\begin{align}
\lim_{j \to \infty} \int_{\Omega} u U \cdot \nabla \Phi_j dx &= \int_{\Omega} u U \cdot \nabla \Phi dx.
\end{align}

(71)

But \(|f|_{(K_V)} = 0\) so we have

\begin{align}
\int_{\Omega \backslash K_V} \Phi df &= \int_{\Omega} \Phi df.
\end{align}

(72)

From relations (68) to (72), we derive

\begin{align}
\int_{\Omega} u(-\Delta \Phi - U \cdot \nabla \Phi + V \Phi) dx &= \int_{\Omega} \Phi df.
\end{align}

(73)
For the converse, we will first prove

**Theorem 11.**

Assume that $\text{Cap}_{V,\infty}(K_V) = 0$, $f = \mu_a$, the Dirac measure at $a \in \Omega$.

If $a \in K_V$ then there is no solution $u \in L^1(V) \cap L^1_0(\Omega)$ of

$$\int_{\Omega} u[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi] dx = \varphi(a) \quad \forall \varphi \in C^2_{0}(\Omega).$$

(74)

**Proof:**

If there was a solution, then, $\forall \varphi \in C^2_c(\Omega \setminus K_V)$ we have

$$\int_{\Omega} u[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi] dx = 0.$$

But $\text{Cap}_{V,\infty}(K_V) = 0$ thus the same equation holds for all $\varphi \in C^2_{0}(\Omega)$ which implies that $\forall \varphi \in C^2_{0}(\Omega)$, $\varphi(a) = 0$. This is impossible.

One can generalize Theorem 11 as follow

**Theorem 12.**

Assume that $\text{Cap}_{V,\infty}(K_V) = 0$. Let $f$ be a bounded Radon measure such that $G = \text{support}(f) \cap K_V$ is an isolate subset of support$(f)$, i.e. there exists an open set $\omega$ such that $\omega \cap \text{support}(f) = G$.

$$\text{if } |f|(K_V) > 0 \text{ then there is no solution of (73).}$$

(75)

**Proof:**

If $|f|(K_V) > 0$, then $G$ is an isolate subset of support of $f$, therefore, we can consider $\theta \in C^\infty_c(\Omega)$ such that $\theta = 1$ on $G$, support$\theta \subset \omega$.

We write $f = \theta f + (1 - \theta)f$ so that measure $f_1 = (1 - \theta)f$ does not charge $K_V$. By the preceding result, we have $u_1 \in L^1(V) \cap L^1_0(\Omega)$ such that

$$\int_{\Omega} u_1[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi] dx = \int_{\Omega} \varphi df_1 \quad \forall \varphi \in C^2_{0}(\Omega).$$
Assume that we have a solution \( u \) of (73) so, \( w = u - u_1 \) is a solution of

\[
\int_{\Omega} w[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi]dx = \langle \theta f, \varphi \rangle, \quad \forall \varphi \in C^2_0(\Omega).
\]

In particular

\[
\int_{\Omega} w[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi] = 0 \quad \forall \varphi \in C^2(\Omega \setminus K_V).
\]

Applying Theorem 8, we deduce that \( w = 0 \) say \( u = u_1 \) which mean \( f = (1 - \theta)f : \theta f \equiv 0 \) this is a contradiction with the fact that \( |f|(K_V) > 0 \).

\[\Diamond\]

**Theorem 13.**

Assume that \( \text{Cap}_{V,\infty}(K_V) = 0 \), \( f \in M^1(\Omega) \) such that \( G = \text{support}(f) \cap K \) is an isolate subset of \( \text{support}(f) \).

Then one has a solution \( u \in L^1(V) \cap L^1_0(\Omega) \) of

\[
\int_{\Omega} u[-\Delta \varphi - U \cdot \nabla \varphi + V \varphi] = \int_{\Omega} \varphi df \quad \forall \varphi \in C^2_0(\Omega) \text{ if and only if } |f|(K_V) = 0.
\]

After submitting this work, we have received the paper [11] where a similar result as for this last theorem is given but only for solution in \( W^{1,1}_0(\Omega) \cap L^1(V) \) which is strictly included in \( L^1_0(\Omega) \cap L^1(V) \).

More, our proofs are totally different.

**References**

[1] Ph. Benilan, H. Brezis, *Nonlinear problems related to Thomas-Fermi equation* J. Evol. Equ. 3 (2004) 673-770.

[2] H. Brezis, A.C. Ponce, *Kato’s inequality when \( \Delta u \) is a measure*, C.R.A.S. Paris 338 8 (2004) 599-604.
[3] F. Cooper, A. Khare, U. Sukhatme, *Supersymmetry and quantum mechanics*, Physics Reports 251 (5-6) (1995) 267-385.

[4] J. I. Díaz, D. Gómez-Castro, J.M. Rakotoson and R. Temam, *Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach*, Discrete and Continuous Dynamical Systems, 38, 2 (2018) 509-546.

[5] J. I. Díaz, D. Gómez-Castro, J.M. Rakotoson, *Existence and uniqueness of solutions of Schrödinger type stationary equations with very singular potentials without prescribing boundary conditions and some applications*, D.E.A. 10 1 (2018) 47-74.

[6] J.I. Díaz, J.M. Rakotoson, *Elliptic Problems on the Space of Weighted With the Distance To the Boundary Integrable Functions Revisited*, Electron. J. Differ. Equations Conf. (2012) 21, 45-59.

[7] A. Fiorenza, M.R. Formica, J.M. Rakotoson, *Pointwise estimates for $\mathcal{G}_\Gamma$-functions and applications*, Differential and Integral Equations 30 11-12 (2017) 809-824.

[8] T. Kato, *Schrödinger operators with singular potential*, Israel J. Math 13 (1972) 135-148.

[9] A. Kufner, 'Weighted Sobolev spaces,' Teuber Verlagsgesellschaft Prague, 1980.

[10] M. Marcus, L. Veron, 'Nonlinear second order elliptic equations involving measures,' de Gruyter, Berlin 2013.

[11] L. Orsina, C. Ponce, *On The nonexistence of Grenn's function and failure of the strong maximum principle*, (22 August 2018) www.arXiv:1808.07267v1 [math.AP]

[12] A. Ponce, 'Elliptic PDEs, Measures and Capacities From the poisson Equation to Non-linear Thomas-Fermi Problems,' (2016) www.ems-ph.org.

[13] G. Pöschl, E. Teller, *Bemerkungen zur Quantenmechanik des anharmonischen Oszillators*, Zeitschrift für Physik, 83 (3-4) (1933) 143-151.

[14] J.M. Rakotoson, 'Linear equations with variable coefficients and Banach function spaces,' work in progress.

[15] J.M. Rakotoson, *New Hardy inequalities and behaviour of linear elliptic equations*, Journal of Functional Analysis 263 (2012) 2893-2920.
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