The Simulated Greedy Algorithm for Several Submodular Matroid Secretary Problems

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\textbf{Abstract}

We study the matroid secretary problems with submodular valuation functions. In these problems, the elements arrive in random order. When one element arrives, we have to make an immediate and irrevocable decision on whether to accept it or not. The set of accepted elements must form an independent set in a predefined matroid. Our objective is to maximize the value of the accepted elements. In this paper, we focus on the case that the valuation function is a non-negative and monotonically non-decreasing submodular function.

We introduce a general algorithm for such submodular matroid secretary problems. In particular, we obtain constant competitive algorithms for the cases of laminar matroids and transversal matroids. Our algorithms can be further applied to any independent set system defined by the intersection of a constant number of laminar matroids, while still achieving constant competitive ratios. Notice that laminar matroids generalize uniform matroids and partition matroids.

On the other hand, when the underlying valuation function is linear, our algorithm achieves a competitive ratio of 9.6 for laminar matroids, which significantly improves the previous result.

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1 Introduction

In the classical secretary problem [8, 12, 13], one interviewer is interviewing $n$ candidates for a secretary position. The candidates arrive in an online fashion and the interviewer has to decide whether or not to hire the current candidate when he/she arrives. The goal is to hire the best secretary. It has been shown that when the candidates are arriving in random order, there exists an algorithm that hires the best candidate with probability $1/e$, where $e$ is the base of the natural logarithm.

Recently, Babaioff et al. [3] formulated the matroid secretary problem. Instead of hiring one candidate (element), in the matroid secretary problem, we seek to select a set of elements which form an independent set in a matroid. Again, the elements arrive in random order and the weights of the elements are revealed when they arrive. When one element arrives, we have to make an immediate and irrevocable decision on whether to accept this element or not. The important constraint is that the set of accepted elements must form an independent set in the predefined matroid. The objective is to maximize the total weights of the selected elements. Notice that the decision on accepting a particular element will impact our ability in accepting future elements.

In the matroid secretary problem, the value of a set of elements is the summation of the weights on these elements, i.e., the valuation function is linear. In some applications, however, it is more natural to measure the quality of a set by a valuation function, which is not necessarily linear. One set of functions widely used in the optimization community are the submodular functions. Such functions are characterized as functions with diminishing returns. We give the formal definition in Section 2.

For example, consider the following scenario. An advertiser is targeting a few platforms to reach a good coverage of audience. However, the coverage from different platforms may overlap with each other. In this case, the performance of a particular set of platforms can only be modelled as a submodular function. Assume the advertiser has to negotiate with the platforms one by one in an online fashion and has a hard budget limit on targeting at most $k$ platforms. This is exactly the matroid secretary problem with a submodular valuation function on a uniform matroid.

We can also consider multiple arriving advertisers, while assuming platforms are available offline. One can impose constraints both on the advertisers and platforms, e.g., each advertiser can afford $k$ platforms, and each platform can support at most $\ell$ advertisers. This scenario can be modelled as an intersection of two partition matroids, with a submodular valuation function, where the objective is to maximize the value of an overall online assignment.

In this paper, we extend the matroid secretary problem to the case with submodular valuation functions. In other words, the weights are not directly associated with elements. Instead, there exists an oracle to query the value of any subset of the elements we have seen. Our objective is to accept a set of elements which are independent in a given matroid with maximum value with respect to a submodular valuation function. We refer such problems as submodular matroid secretary problems. We refer the original matroid secretary problems, i.e., those with linear valuation functions, as linear matroid secretary problems.

We use the competitive analysis to measure the performance of our algorithms following the matroid secretary problem literature. More formally, let $U$ be the set of elements and $\mathcal{M}$ be a matroid defined on $U$. Before the process starts, an adversary assigns a submodular valuation function $f(\cdot) : 2^{|U|} \rightarrow \mathbb{R}^+ \cup \{0\}$, which maps any subset of $U$ to a non-negative real number. After that, there is a random permutation applied to the elements to decide their
arriving order to our online algorithm. Our algorithm can only query $f(\cdot)$ using elements that have been seen. In other words, the algorithm does not know $f(\cdot)$ before any element arrives.

Let $OPT_f(M) = \max_{S \in M} f(S)$ be the value of the optimal independent set. The objective of the submodular matroid secretary problem is to find an algorithm $Alg$ which maximizes the following ratio:

$$\inf_{f} \frac{\mathbb{E}_{P, A}[f(Alg_f(P, A))]}{OPT_f(M)},$$

where $Alg_f(P, A)$ is the solution generated by the algorithm given permutation $P$ and the internal randomness $A$ of the algorithm with valuation function $f(\cdot)$. The expectation is taken over all permutations and the internal randomness of the algorithm. We call the algorithm is $C$-competitive, i.e., with competitive ratio $C$, if the ratio in Eqn. (1) is at least $1/C$.

Our contributions. In this paper, we study the submodular matroid secretary problem with submodular valuation functions that are non-negative and monotonically non-decreasing. Our contribution is two-fold. First, we develop a general simulated greedy algorithm, which is inspired by the algorithm for the linear matroid secretary problem with transversal matroids in [6, 18]. Our algorithm is constant competitive for the submodular matroid secretary problem with laminar matroids and transversal matroids. Our analysis can be extended to the case that the independent set is defined as the intersection of a constant number of laminar matroids. Notice that laminar matroids generalize uniform matroids and partition matroids. When applying to the linear matroid secretary problem on laminar matroids, our algorithm improves the competitive ratio from $\frac{16000}{3}$ [16] to 9.6. Our algorithm is also much simpler than the one in [16].

Second, our technique in analyzing submodular functions could be of independent interest. Consider our simulated greedy algorithm for the uniform matroid case with cardinality $\mu$. We maintain two sets $M$ and $N$, which are initially empty. In each time, we will select an element $e \in U \setminus (M \cup N)$ such that $f_M(e)$ is maximized until $|M| = \mu$, where $f(\cdot)$ is the valuation function. With probability $p$, $e$ is placed into $M$. Otherwise, i.e., with probability $1 - p$, $e$ is placed into $N$. We develop machineries to show that $\mathbb{E}[f(N)] = \Theta(\mathbb{E}[f(M)])$, despite the fact that the elements are greedily selected with optimal marginal values against $M$. This fact is not intuitive though very important in our analysis. See our result in Section 4 for more details.

Related work. The secretary problem has been studied decades ago. It is first published in [13] and has been folklore even earlier [10]. Several results have appeared to generalize the classical secretary problem, while assuming that the elements arrive in random order. For example, Kleinberg [17] gave a $1 + O(1/\sqrt{k})$-competitive algorithm for selecting at most $k$ elements to maximize the sum of the weights. Babaioff et al. [2] provided a constant competitive algorithm for the Knapsack secretary problem, in which each element has a weight and a size, and the objective is to accept a set of elements whose total size is at most a given integer such that the total weight is maximized.

Babaioff et al. [3] systematically introduced the matroid secretary problem. The objective is to maximize the total weight of the selected elements $S$, which form an independent set in a given matroid. They gave an $O(\log r)$-competitive algorithm for a general matroid, i.e., the expected total weight of the elements in $S$ is $O(1/\log r)$ of the optimal solution, where $r$ is the rank of the matroid. The competitive ratio has been recently improved to $O(\sqrt{\log r})$ by Chakraborty et al. [5]. However, the conjecture that the matroid secretary problem with a general matroid allows a constant competitive algorithm is still widely open, while constant
competitive algorithms have been found for various matroids: uniform/partition matroids [2, 17], truncated partition matroids [3], graphical matroids [1, 18], transversal matroids [6, 18], laminar matroids [16], and regular and decomposable matroids [4]. For general matroids, Soto [19] developed a constant-competitive algorithm in random assignment model, i.e., the weights of the elements are assigned uniformly at random. This result can be extended to the case where the elements arrive in an adversarial order [14].

Gupta et al. [15] studied the non-monotone submodular matroid maximization problem for both offline and online (secretary) versions. For the online (secretary) version, they provided a $O(\log r)$-competitive algorithm for general matroids and a constant competitive algorithm for uniform matroids (algorithms achieving constant competitive ratios are obtained independently by Bateni et al. [4] and partition matroids. Feldman et al. [9] developed a simpler algorithm with a better competitive ratio for partition matroids for monotonically non-decreasing submodular functions.

Structure. In Section 2, we present some preliminaries and our algorithm. We then analyze a simple stochastic process in Section 3, which serves as a building block for later analysis. In Section 4, we analyze the algorithm for the cases of laminar matroids and the intersection of constant number of laminar matroids. We discuss the transversal matroid case in Section 5. We conclude with Section 6.

2 Preliminaries

2.1 Matroids

In the matroid secretary problem, the set of accepted elements must form an independent set defined by a given matroid.

Definition 1 (Matroids). Let $U \neq \emptyset$ be the ground set and $\mathcal{I}$ be a set of subsets of $U$. The system $\mathcal{M} = (U, \mathcal{I})$ is a matroid with independent sets $\mathcal{I}$ if:

1. If $A \subseteq B \subseteq U$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.
2. If $A, B \in \mathcal{I}$ and $|A| < |B|$, there exists an element $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{I}$.

In this paper, we work with the following two matroids.

Definition 2 (Laminar matroids). Let $U \neq \emptyset$ be the ground set. Let $\mathcal{F} = \{B_1, \ldots, B_\ell\}$ be a family of subsets over $U$. $\mathcal{F}$ is a laminar family, if for any $B_i, B_j$ such that $|B_i| \leq |B_j|$, either $B_i \cap B_j = \emptyset$ or $B_i \subseteq B_j$. Each set $B_i \in \mathcal{F}$ is associated with capacity $\mu(B_i)$. The laminar family $\mathcal{F}$ and $\mu(\cdot)$ define a matroid $\mathcal{M} = (U, \mathcal{I})$, such that any set $T \subseteq U$ is independent if for all $1 \leq i \leq \ell$, $|T \cap B_i| \leq \mu(B_i)$.

In particular, each $B_i$ defines a capacity constraint on the independent sets and a set is independent if it satisfies all such constraints. For simplicity, we assume all $B_i$s are distinct and $\mu(B_i) < \mu(B_j)$ if $B_i \subset B_j$. Otherwise, the capacity constraint in $B_i$ is redundant.

Definition 3 (Transversal matroids). Let $G = (L, R, E)$ be an undirected bipartite graph with left nodes $L$, right nodes $R$ and edges $E$. In the transversal matroid defined by $G$, the ground set is $L$ and a set of left nodes $S \subseteq L$ is independent if there exists a matching in $G$ such that the set of left nodes in the matching is $S$.

2.2 Submodular functions

In this paper, we assume the quality of the solution is measured by a submodular function. Notice that throughout this paper, we only work with non-negative and monotonically non-decreasing submodular functions.
Definition 4. Let $U$ be the ground set. Let $f(\cdot) : 2^{|U|} \rightarrow \mathbb{R}$ be a function mapping any subset of $U$ to a real number. $f(\cdot)$ is a submodular function if:

\[ \forall S, T \subseteq U, f(S) + f(T) \geq f(S \cup T) + f(S \cap T). \]

For simplicity, for any set $S \subseteq U$, we define its marginal function value $f_S(\cdot)$ as follows. For any $T \subseteq U$, $f_S(T) = f(S \cup T) - f(S)$. For singletons, we also write $f_S(e) = f_S(\{e\})$. It is not difficult to see that $f_S(\cdot)$ is submodular if $f(\cdot)$ is submodular.

2.3 The simulated greedy algorithm

Our general algorithm is based on the greedy algorithm, as in Algorithm 1.

| Input: Set $H \subseteq U$ of matroid $(U, \mathcal{I})$ and function $f(\cdot)$ |
| Output: A set of elements $T \subseteq H$ and $T \in \mathcal{I}$ |
| $T \leftarrow \emptyset$; |
| while $\exists e^* = \arg\max_{e \in H} \{f_T(e) \mid T \cup \{e\} \in \mathcal{I}\}$ do |
| $T \leftarrow T \cup \{e^*\}$; $H \leftarrow H \setminus \{e^*\}$; |
| end |
| return $T$; |

Algorithm 1: GREEDY

| Input: Matroid $(U, \mathcal{I})$ and function $f(\cdot)$ |
| Output: Selected elements ALG |
| $M, N, ALG \leftarrow \emptyset$; |
| $m \leftarrow \text{Binom}(|U|, p)$; |
| Observe the first $m$ elements $H$; |
| $M \leftarrow \text{GREEDY}(H)$; |
| for any subsequent element $e$ do |
| if $\text{GREEDY}(H \cup \{e\}) \neq \text{GREEDY}(H)$ then |
| $N \leftarrow N \cup \{e\}$; |
| if $ALG \cup \{e\} \in \mathcal{I}$ then |
| Accept $e$ and $ALG \leftarrow ALG \cup \{e\}$; |
| end |
| end |

Algorithm 2: ONLINE

Our simulated greedy algorithm ONLINE works as follows. (We will discuss the name of simulated greedy in a minute.) We observe the first $m$ elements $H$ without any selection, where $m$ is sampled from Binomial distribution $\text{Binom}(|U|, p)$ for some chosen probability $p$. Then we compute the greedy solution $\text{GREEDY}(H)$. After that, for any subsequent element $e$, we test that whether the greedy solution will change if $e$ is added to $H$ hypothetically. If so, we mark $e$ as a candidate and place it in $N$. Furthermore, if $ALG \cup \{e\} \in \mathcal{I}$ for candidate $e$ and current $ALG$, we accept $e$ into $ALG$. (Both $N$ and $ALG$ are initially empty.) The final $ALG$ will be the output of our algorithm. Observe that maintaining set $N$ is not necessary because $N$ only collects elements that has passed the greedy check and might be accepted
**Input:** Matroid \((U, I)\) and function \(f(\cdot)\)

**Output:** Selected elements \(S\)

\[
H, M, N, S \leftarrow \emptyset; \\
\text{for each element } e \text{ do} \\
\quad \text{Flip a coin with prob. } p \text{ of head;} \\
\quad \text{if head, } H \leftarrow H \cup \{e\}; \\
\end{aligned}
\]

\[
\text{while } \exists e^* = \arg\max_{e \in U \setminus (M \cup N)} \{f_M(e) \mid M \cup \{e\} \in I\} \text{ do} \\
\quad \text{if } e \in H \text{ then } M \leftarrow M \cup \{e\}; \\
\quad \text{else } N \leftarrow N \cup \{e\}; \\
\end{aligned}
\]

Prune \(N\) to produce a set of elements \(S \in I\);

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**Algorithm 3:** SIMULATE

potentially. However, we keep the notation in the algorithm because it corresponds to the same \(N\) in SIMULATE, which is heavily used throughout the analysis.

As we mentioned earlier, ONLINE is a generalization of the algorithms in [6, 18]. In particular, it has been observed that a simulated random algorithm in Algorithm 3 can be used in analyzing the performance of ONLINE. (We name ONLINE as a simulated greedy algorithm because of the corresponding greedy algorithm which simulates the online version.)

More specifically, SIMULATE works as follows. We maintain two sets \(M\) and \(N\) which are initially empty. In each step, we select an element \(e \in U \setminus (M \cup N)\) such that \(f_M(e)\) is maximized and \(M \cup \{e\} \in I\). (If no such element exists, SIMULATE terminates.) Then we toss a biased random coin with probability \(p\) to be head, which is the same probability in sampling \(m\) in ONLINE. If the coin is head, \(e\) is placed into \(M\). Otherwise, \(e\) is placed into \(N\). Since \(N\) may not be an independent set in \(I\) after SIMULATE terminates, we prune \(N\) to produce \(S \subset N\) such that \(S \in I\). The actual pruning step might be different in different application settings.

SIMULATE is useful in analyzing the performance of ONLINE with random arriving elements, because, as the naming suggests, both \(M\) and \(N\) have the same joint distribution in the two algorithms. This connection is extensively discussed in [6, 18]. For completeness, we provide a proof in Appendix [A]. We will guarantee that \(S\) in SIMULATE is stochastically dominated by ALG in ONLINE. Since we assume \(f(\cdot)\) is non-decreasing, in analyzing the performance of ONLINE, we can focus on \(S\) in SIMULATE.

**Lemma 5.** The sets of elements of \(H, M\) and \(N\) by SIMULATE have the same joint distribution as the \(H, M\) and \(N\) generated by ONLINE with a random permutation of the elements in \(U\).

### 3 A simple stochastic process

In this section, we study a simple stochastic process which serves as a building block of our analysis. We will apply this process to either the entire ground set \(U\) or some subsets of the elements in \(U\). Therefore, although we use the same notation for \(M\) and \(N\) in this section, they can be viewed as the intersections between the set of elements that are under consideration and the actual global \(M\) and \(N\) generated by the algorithm.

The simple stochastic process is defined by an underlying Bernoulli process, with an infinite sequence of independent and identical random variables \(X_t \in \{0, 1\}\) for \(t \geq 1\). Each
variable $X_t$ is a Bernoulli random variable with probability $p$ to be 1.

Our stochastic process is parametrized by a constant $\mu \geq 1$. We maintain two sets $M$ and $N$, which are initially empty, as follows. Starting from $t = 1$, if $X_t = 1$, we place $t$ into $M$; otherwise, $t$ is placed in $N$. The process immediately terminates after $|M| = \mu$.

We associate a non-negative weight $w_t$ to every time stamp $t$. In particular $w_t$ is a mapping from the previous $t - 1$ random variables $\{X_1, X_2, \ldots, X_{t-1}\}$ to a non-negative real number. ($w_t$ is constant by definition. If the process has been terminated before time $t$, we set $w_t = 0$.) For any set $T \subseteq N$, we define the weight as,

$$w(T) = \sum_{t \in T} w_t(X_1, X_2, \ldots, X_{t-1}).$$

(2)

Define $w(\emptyset) = 0$. The following proposition shows that the total weights of $M$ and $N$ are close to each other.

$\triangleright$ Proposition 6. $E[w(M)] = \frac{p}{1-p}E[w(N)]$.

$\triangleright$ Proof. Due to linearity of expectation, it is sufficient to consider the weights of $M$ and $N$ on a particular time stamp $t$. Let $F_t$ be the $\sigma$-algebra encoding all the randomness up to the time $t$. Notice that $w_t$ is $F_t$-measurable. Let $w_t^M = w_t$ if $X_t = 1$ and 0 otherwise. Similarly, we define $w_t^N = w_t$ if $X_t = 0$ and 0 otherwise. We immediately have $E[w_t^M | F_t] = \frac{p}{1-p}E[w_t^N | F_t]$. Therefore,

$$E[w(M)] = \sum_{t \geq 1} E_{F_t} [E[w_t^M | F_t]] = \frac{p}{1-p} \sum_{t \geq 1} E_{F_t} [E[w_t^N | F_t]] = \frac{p}{1-p} E[w(N)].$$

Notice that after the process terminates, we have $|M| = \mu$. On the other hand, the size of $N$ might be very large. Our analysis will be based on $N$s that are with size at most $\mu$. We produce an independent set $S$ from $N$ by a pruning process as follows.

$\triangleright$ Pruning. More formally, to address the issue of too large $N$s, we define $S = N$ if $|N| \leq \mu$ and $S = \emptyset$ otherwise. Clearly, we have $S \subseteq N$ and $w(S) \leq w(N)$.

We want to show that $w(S)$ is close to $w(N)$ in expectation. However, it is not possible for arbitrary set of $\{w_t\}$. In what follows, we impose a “decreasing weight” condition on $\{w_t\}$, which always holds in our applications. This condition is crucial in building the connection between $w(S)$ and $w(N)$.

$\triangleright$ Definition 7 (Decreasing weight mappings). The set of mappings $\{w_t\}$ forms a sequence of decreasing weight mappings if for any $i < j$ and $x_1, x_2, \ldots, x_{i-1}, x_i, \ldots, x_{j-1}$ we have:

$$w_i(x_1, \ldots, x_{i-1}) \geq w_j(x_1, \ldots, x_{i-1}, \ldots, x_{j-1}).$$

Proposition 6 makes a connection between $w(S)$ and $w(N)$. We briefly discuss the intuition behind this statement. Our objective is to show that the weight pruned from $N$ to $S$ is small. The random process indicates that the probability for having a large $N$ is exponentially decreasing on its size, e.g., by the Chernoff bound. Therefore, the probability mass of $N$ that is pruned is small. In terms of weight, on the other hand, those larger $N$s do have greater weights.

The condition of the decreasing weight mappings comes to rescue. In particular, in this case, the weight of $N$ grows roughly “linear” to its size. As the probability decreases exponentially with the size of $N$, the total weight pruned can still be bounded as the summation of a geometric sequence for those large $N$s. We concretely implement this proof as follows.
Proposition 8. Let $\beta = 2e(1-p)$. If $\{w_k\}$ forms a sequence of decreasing weight mappings, we have

$$\mathbb{E}[w(N)] - \mathbb{E}[w(S)] \leq \frac{(\mu + 1 - \mu \beta)\beta^\mu}{(1-\beta)^2} \cdot \mathbb{E}[w(S)] \leq \frac{(\mu + 1 - \mu \beta)\beta^\mu}{(1-\beta)^2} \cdot \mathbb{E}[w(N)]$$

If $\mu = 1$, it can be improved to

$$\mathbb{E}[w(N)] - \mathbb{E}[w(S)] \leq \frac{1 - p^2}{p^2} \cdot \mathbb{E}[w(S)] \leq \frac{1 - p^2}{p^2} \cdot \mathbb{E}[w(N)].$$

Proof. To simplify the notation, let $h_N = |N|$. By definition:

$$\mathbb{E}[w(N)] = \sum_{k=1}^{\infty} \mathbb{E}[w(N) \mid h_N = k] \mathbb{P}[h_N = k]; \quad \mathbb{E}[w(S)] = \sum_{k=1}^{\mu} \mathbb{E}[w(N) \mid h_N = k] \mathbb{P}[h_N = k]$$

Let $N_i$ be the set of the first $i$ elements of $N$ in our stochastic process. Let $A_k$ be all possible outcomes of $N_k$. Now for any fixed $1 \leq k \leq q < \mu$, by definition, we have

$$\mathbb{E}[w(N_k) \mid h_N = q] \mathbb{P}[h_N = q \mid N_k = A] = \sum_{A \in A_k} w(A) \mathbb{P}[h_N = q \mid N_k = A \land h_N = q]. \quad (3)$$

For a fixed $A$, let $\ell(A)$ be the number of 1s in $X_k$ when we pick the last element in $A$. (The last one must be 0 as it goes into $N$.) Since $A \in A_k$, $\ell(A) < \mu$. We have

$$\frac{\mathbb{P}[h_N = q \mid N_k = A \land h_N = k]}{\mathbb{P}[N_k = A \land h_N = k]} = \frac{\left(\begin{array}{c} k + \ell(A) - 1 \\ \ell(A) \end{array}\right)p^{\ell(A)}(1-p)^k \cdot (q-k+\mu-\ell(A)-1)q^\mu-\ell(A)(1-p)^{q-k}}{\left(\begin{array}{c} k + \ell(A) - 1 \\ \ell(A) \end{array}\right)p^{\ell(A)}(1-p)^k \cdot p^\mu-\ell(A)}$$

$$= \left(\begin{array}{c} k + \ell(A) - 1 \\ \ell(A) \end{array}\right)\frac{p^\mu-\ell(A)}{p^\mu-\ell(A)} (1-p)^{q-k}$$

$$\leq \left(\begin{array}{c} k + \ell(A) - 1 \\ \ell(A) \end{array}\right)\frac{p^\mu-\ell(A)}{p^\mu-\ell(A)} (1-p)^{q-k} \leq \left(\begin{array}{c} k + \ell(A) - 1 \\ \ell(A) \end{array}\right)\frac{p^\mu-\ell(A)}{p^\mu-\ell(A)} (1-p)^{q-k}$$

The first inequality comes from the fact that $\left(\begin{array}{c} n-t \\ k \end{array}\right) \leq \left(\begin{array}{c} n-t \\ k \end{array}\right)$ and $\left(\begin{array}{c} n-t \\ k \end{array}\right) \leq \left(\begin{array}{c} n-t \\ k \end{array}\right)$ when $t \geq 0$. The last inequality is due to $\left(\begin{array}{c} n-t \\ k \end{array}\right) \leq \left(\begin{array}{c} n-t \\ k \end{array}\right)$ and $\left(\begin{array}{c} n-t \\ k \end{array}\right) \leq \left(\begin{array}{c} n-t \\ k \end{array}\right)$. So we have, with Eqn. 3,

$$\mathbb{E}[w(N_k) \mid h_N = q] \mathbb{P}[h_N = q \mid N_k = A] \leq (2e)^{q-1}(1-p)^{q-k} \mathbb{E}[w(N_k) \mid h_N = k] \mathbb{P}[h_N = k]$$

(4)

On the other hand, by the decreasing order of $w_i$, we have that

$$\mathbb{E}[w(N_k) \mid h_N = q] \leq \frac{q}{k} \mathbb{E}[w(N_k) \mid h_N = q].$$

Therefore, for any $q > \mu$,

$$\mathbb{E}[w(S)] = \sum_{k=1}^{\mu} \mathbb{E}[w(N_k) \mid h_N = k] \mathbb{P}[h_N = k]$$

$$\geq \sum_{k=1}^{\mu} (2e)^{q-1}(1-p)^{k-q} \mathbb{E}[w(N_k) \mid h_N = q] \mathbb{P}[h_N = q] \quad \text{by Eqn. 3}$$

$$\geq \sum_{k=1}^{\mu} \frac{k}{q} (2e)^{q-1}(1-p)^{k-q} \mathbb{E}[w(N) \mid h_N = q] \mathbb{P}[h_N = q] \quad \text{by Eqn. 3}$$

$$= \frac{(1-p)^{-q}}{(2e)^{q-1}q} \cdot \mathbb{E}[w(N) \mid h_N = q] \mathbb{P}[h_N = q] \sum_{k=1}^{\mu} k(1-p)^k$$

$$\geq \frac{(1-p)^{-q}}{(2e)^{q-1}q} \cdot \mathbb{E}[w(N) \mid h_N = q] \mathbb{P}[h_N = q] \cdot (1-p)$$. 


Finally, recall that $\beta = 2e(1 - p)$, we have

$$\mathbb{E}[w(N)] - \mathbb{E}[w(S)] = \sum_{q=\mu+1}^{\infty} \mathbb{E}[w(N) \mid h_N = q] \Pr[h_N = q]$$

$$\leq \mathbb{E}[w(S)] \sum_{q=\mu+1}^{\infty} q(1 - p)^{q-1}(2e)^{q-1}$$

$$= \mathbb{E}[w(S)] \frac{(\mu + 1 - \mu \beta) \beta \mu}{(1 - \beta)^2}$$

The last equality come from the fact that, for any $\alpha < 1$, $\sum_{i=k}^{\infty} \frac{i \cdot \alpha^i}{(1 - \alpha)^2} = \frac{\alpha^k}{(1 - \alpha)^2}$.

Now consider the case that $\mu = 1$. By the stochastic process, $w(N_1)$ is either 0 or $w_1$.

Eqn. (5) still holds.

$$\mathbb{E}[w(S)] = \mathbb{E}[w(N_1) \mid h_N = 1] \Pr[h_N = 1]$$

$$= \mathbb{E}[w(N_1) \mid h_N = q] \Pr[h_N = q](1 - p)^{1-q}$$

$$\geq \frac{1}{q}(1 - p)^{1-q} \mathbb{E}[w(N) \mid h_N = q] \Pr[h_N = q]$$

$$\mathbb{E}[w(N)] - \mathbb{E}[w(S)] = \sum_{q=2}^{\infty} \mathbb{E}[w(N) \mid h_N = q] \Pr[h_N = q]$$

$$\leq \mathbb{E}[w(S)] \sum_{q=2}^{\infty} q(1 - p)^{q-1}$$

$$= \frac{1 - p^2}{p^2} \mathbb{E}[w(S)].$$

4 Laminar Matroid

In this section, we study the performance of our simulated greedy algorithm SIMULATE for the submodular matroid secretary problem with a laminar matroid. We first show that the entire process of SIMULATE can be casted as a simple stochastic process as discussed in the previous section. After that, we inspect the pruning stage in details. In particular, for each $B_i$ in the laminar matroid, we study a simple stochastic process restricted on the elements in $B_i$. The loss on the entire pruning steps can be divided into losses on the $B_i$s, which can be bounded by Proposition 8.

Let $\mu$ be the rank of the laminar matroid. Essentially, SIMULATE will select (at most) $\mu$ elements. We cast the SIMULATE process to the simple stochastic process with $\mu$ as follows.

In the $t$-th round, when the first $t - 1$ random coins are tossed, the current element $e$ in the greedy order is uniquely defined, as well as the current $M$ and $N$. We define the weight $w_t = f_{M_t}(e)$ where $M_t$ is the current elements in $M$.

Remark. We make two remarks regarding the connection between the two stochastic processes. First, the original simple stochastic process terminates when $|M| = \mu$. SIMULATE might terminate earlier because of the limit on the number of elements. In such cases, we assume the availability of an infinite number of dummy elements, with zero weights, which
Lemma 9.

The stochastic process consists of time stamps, while in all processes we study later we of elements in $M$ by definition. Furthermore, each element in the offline optimal solution has probability $H$ the optimal solution in $S$ independent. Furthermore, since ALG will be the greedy independent set of $H$, i.e., a head coin is associated with it. By submodularity of $H$, for $E$ obtain one constraint $E$ where $\triangleright$.

We extend the $w(\cdot)$ to elements besides those in $M$. In particular, $w(e) = f_{M_e}(e)$ for $e \in M \cap N$, i.e., $e$ appears in the greedy order of SIMULATE, where $M_e$ is the current set of elements in $M$ when $e$ appears. If $e \in M \cup N$, set $w(e) = 0$. Notice that $w(M) = f(M)$ by definition. Furthermore, each element in the offline optimal solution has probability $p$ in $H$, i.e., a head coin is associated with it. By submodularity of $f(\cdot)$, the expected value of the optimal solution in $H$ is at least $p \cdot OPT$. On the other hand, the greedy algorithm is a 2-approximation with a matroid constraint when the valuation function is monotone and submodular. Together with Proposition 6, we have

\[ \mathbb{E}[f(M)] = \mathbb{E}[w(M)] = \frac{p}{1 - p} \mathbb{E}[w(N)] \geq \frac{p}{2} \cdot OPT \]

Pruning. Notice that although $M$ is independent, $N$ might not be independent. We obtain $S$ by pruning $N$ as follows.

\[ S = N \setminus \left( \bigcup_{B \in F} 1_{|N \cap B| > \mu(B)} \cdot (N \cap B) \right), \]

where $1_{\text{cond}} \cdot (N \cap B) = N \cap B$ if $\text{cond}$ is true and empty otherwise. In other words, if one constraint $B_i$ is violated in $N$, we remove all elements in $B_i$ from $N$. Clearly, $S$ is independent. Furthermore, since ALG will be the greedy independent set of $N$ for a random order, it is straightforward to show that $S \subseteq ALG$.

Therefore, it is sufficient to bound $\mathbb{E}[f(S)]$. To do that, we first provide a lower bound for $\mathbb{E}[w(S)]$. After that, we bound $\mathbb{E}[f(S)]$ in terms of $\mathbb{E}[w(S)]$.

Roadmap. Here we briefly outline our strategy in getting the two pieces of results. To measure $\mathbb{E}[w(S)]$, we estimate the weight loss due to the pruning in Eqn. 5. For each constraint $B_i$, we cast the stochastic process in SIMULATE in processing elements in $B_i$ into a simple stochastic process with $\mu(B_i)$. By invoking Proposition 5 the weight loss $w(N \cap B_i) - w(S \cap B_i)$ is $2^{O(\mu(B_i))} \cdot w(N \cap B_i)$, which is charged to all elements in $B_i$, proportionally to $1_{e \in N} w(e)$ for all $e \in B_i$. The catch here is, for each element $e \in U$, the set of $\{B_i\}$ containing $e$ has a strictly increasing $\{\mu(B_i)\}$ sequence. Therefore, the charges on $e$ form a geometric sequence which in total will not exceed a constant fraction of $1_{e \in N} \cdot w(e)$. Since $w(N) = \sum_{e \in N} w(e)$, the total weight loss is a constant fraction.

The second piece of ingredient is to make a connection between $\mathbb{E}[f(S)]$ and $\mathbb{E}[w(S)]$. For simplicity, let us consider $\mathbb{E}[f(N)]$ and $\mathbb{E}[f(M)]$ instead to convey the idea. Recall that $w(N) = \sum_{e \in N} f_{M_e}(e)$, where $M_e$ is the set of elements in $M$ when $e$ arrives. Therefore, it is not intuitive why $\mathbb{E}[f(N)]$ should be large in the first place. To elaborate, we consider function $F = f(M) + f(N) - f(M \cup N)$ during the execution of the algorithm, which is a lower bound of $2f(N)$. We can view $f(M) + f(N) - f(M \cup N)$ as the intersection between $M$ and $N$, e.g., if $f(\cdot)$ is modeling a set cover. During the execution of the algorithm,
Lemma 10. Let $\beta = 2e(1-p)$. We have

$$\mathbb{E}[w(S)] \geq (1 - \frac{2\beta}{(1-\beta)^2})\mathbb{E}[w(N)].$$

Proof. Since for a fixed set of random outcomes, $w(\cdot)$ is a linear function. By Eqn. (8), we have that

$$\mathbb{E}[w(N)] \leq \mathbb{E}[w(S)] + \sum_{B \in \mathcal{F}} \mathbb{E}[w(1_{N \cap B} > \mu(B)) \cdot (N \cap B)].$$

Now we focus on the term $\mathbb{E}[w(1_{N \cap B} > \mu(B)) \cdot (N \cap B)]$ and the simulated greedy algorithm on elements in $B$, i.e., a particular constraint in $\mathcal{F}$. We isolate $B$ in the process by rearranging the randomness as follows. First, for each element in $U \setminus B$, we assign an independent random coin to it, i.e., if this element appears in the algorithm, its random coin will be tossed. For a fixed outcome of all random coins outside of $B$, the simulated greedy algorithm is a simple stochastic process for the elements in $B$. The only difference, however, is the process may terminate before $|M \cap B| = \mu(B)$.

This can be easily remedied by appending dummy elements as before. Recall that $\beta = 2e(1-p)$. By Proposition 8, we have:

$$\mathbb{E}[1_{N \cap B} > \mu(B) \cdot w(N \cap B)] \leq \frac{(\mu(B) + 1 - \mu(B)\beta^3\mu(B))}{(1-\beta)^2} \cdot \mathbb{E}[w(N \cap B)].$$

It follows that

$$\mathbb{E}[w(N)] \leq \mathbb{E}[w(S)] + \sum_{B \in \mathcal{F}} \frac{(\mu(B) + 1 - \mu(B)\beta^3\mu(B))}{(1-\beta)^2} \cdot \mathbb{E}[w(N \cap B)]$$

$$= \mathbb{E}[w(S)] + \frac{1}{(1-\beta)^2} \sum_{B \in \mathcal{F}} \sum_{e \in U} \mathbb{E}[w(e)1_{e \in N} \left( \sum_{B \in \mathcal{F}} (\mu(B) + 1 - \mu(B)\beta^3\mu(B)) \cdot 1_{e \in B} \right)]$$

$$\leq \mathbb{E}[w(S)] + \frac{1}{(1-\beta)^2} \sum_{e \in U} \mathbb{E}[w(e)1_{e \in N} \left( \sum_{i \geq 1} (i + 1 - i\beta)^2 \right)]$$

$$= \mathbb{E}[w(S)] + \frac{2\beta}{(1-\beta)^2} \mathbb{E}[w(N)]$$

Eqn. (8) uses the fact that the set of constrains $\{B_i\}$ containing an element $e$ has a strictly increasing sequence of $\{\mu(B_i)\}$. \hfill $\blacksquare$
Lemma 11. For any $t > 0$, let $\theta = 1 + \frac{(1-\beta)^3}{p}$. We have

$$\mathbb{E}[f(S)] \geq \left(1 - \frac{(1-\beta)^3}{t((1-\beta)^3 - 2\beta)}\right)\mathbb{E}[w(S)]$$

Proof. Let $g(S) = \sum_{e \in S} g(e)$. Since $S \subseteq N$, we have $f(S) \geq g(S)$ by the submodularity of $f(\cdot)$. We inspect the function $F(S, M, N) = t \cdot g(S) + f(M) - f(M \cup N)$. By the monotonicity of $f$, $f(S) \geq g(S) \geq F(S, M, N)/t$.

Define $\Delta_e = F(S', M'_e, N'_e) - F(S_e, M_e, N_e)$ where $M'_e$ (resp. $N'_e$ and $S'_e$) is the set $M$ (resp. $N$ and $S$) after we process element $e$. If $e \notin M \cup N$, define $\Delta_e = 0$. Therefore, $F(S, M, N) = \sum_{e \in U} \Delta_e$. Let $\mathcal{R}_e$ be the sub-$\sigma$-algebra encoding all randomness up to the time $e$ is picked in SIMULATE. Notice that $M_e$ and $N_e$ are $\mathcal{R}_e$ measurable. We have $\Pr[e \in M \mid \mathcal{R}_e] = p$ and $\Pr[e \in N \mid \mathcal{R}_e] = 1 - p$.

When $e \in S$,

$$\mathbb{E}[\Delta_e \mid \mathcal{R}_e] = t \cdot (\mathbb{E}[g(S') - g(S) \mid \mathcal{R}_e]) + (\mathbb{E}[f(M') - f(M) \mid \mathcal{R}_e])$$

$$= (t \cdot \mathbb{E}[f(M' \mid N') - f(M \cup N) \mid \mathcal{R}_e])$$

Then we bound $\mathbb{E}[\Delta_e \mid \mathcal{R}_e]$ by case analysis. Notice that $\Pr[e \in M \mid \mathcal{R}_e] + \Pr[e \in N \mid \mathcal{R}_e] = 1$ and $\Pr[e \in N \mid \mathcal{R}_e] \geq \Pr[e \in S \mid \mathcal{R}_e]$.

Case 1: $f(M_e) \geq \theta \cdot f(N_e)$.

$$\mathbb{E}[\Delta_e \mid \mathcal{R}_e] \geq \Pr[e \in M \mid \mathcal{R}_e] f(M_e) - \Pr[e \in N \mid \mathcal{R}_e] f(M_e)$$

$$\geq \frac{p}{1-p} \left(1 - \frac{1}{\theta}\right) \Pr[e \in S \mid \mathcal{R}_e] f(M_e) - \Pr[e \in N \mid \mathcal{R}_e] f(M_e)$$

Case 2: $f(M_e) < \theta \cdot f(N_e)$.

$$\mathbb{E}[\Delta_e \mid \mathcal{R}_e] \geq \frac{t}{\theta} \Pr[e \in S \mid \mathcal{R}_e] f(M_e) - \Pr[e \in N \mid \mathcal{R}_e] f(M_e)$$

By definition of $\theta$, we have $\frac{1}{\theta} - \frac{1}{1-\beta} = t/\theta$. So

$$\mathbb{E}[\Delta_e \mid \mathcal{R}_e] \geq \frac{t}{\theta} \Pr[e \in S \mid \mathcal{R}_e] f(M_e) - \Pr[e \in N \mid \mathcal{R}_e] f(M_e)$$

Therefore

$$t \cdot \mathbb{E}[f(S)] \geq \frac{t}{\theta} \sum_{e \in U} \mathbb{E}[\Delta_e \mid \mathcal{R}_e]$$

$$\geq \frac{t}{\theta} \sum_{e \in U} \left[\frac{t}{\theta} \Pr[e \in S \mid \mathcal{R}_e] f(M_e) - \Pr[e \in N \mid \mathcal{R}_e] f(M_e)\right]$$

$$= \frac{t}{\theta} \mathbb{E}[w(S)] - \mathbb{E}[w(N)]$$

By Lemma 10, we have $\mathbb{E}[f(S)] \geq \frac{1}{\theta} \mathbb{E}[w(S)] - \frac{(1-\beta)^3}{(1-\beta)^3 - 2\beta} \mathbb{E}[w(S)]$.

The last inequality is by Lemma 10. So $\mathbb{E}[f(S)] \geq \frac{1}{\theta} \mathbb{E}[w(S)] - \frac{(1-\beta)^3}{(1-\beta)^3 - 2\beta} \mathbb{E}[w(S)]$. 

---

1 We define $g(e)$ based on $N_e$ instead of $S_e$, i.e., the current set of elements in $S$, because $S_e$ is still a random set even all the randomness before $e$’s arrival is fixed.
Combining all the results together, we have an algorithm with competitive ratio at most 211 with \( p = 0.9794 \) and \( t = 10.1415 \).

**Theorem 12.** There is an online algorithm with competitive ratio at most 211 for the submodular matroid secretary problem with laminar matroids.

### 4.1 The intersection of constant number of laminar matroids

**Theorem 13.** For any constant \( k \), there is an online algorithm with competitive ratio at most \( \frac{1000k^2}{k+1} \) for the submodular matroid secretary problem with the intersection of \( k \) laminar matroids.

**Proof.** The independent set we considered is the intersection of \( k \) matroids. Therefore, the greedy algorithm is a \( \frac{1}{k+1} \)-approximation. Together with Proposition 6, we have

\[
E[f(M)] = \frac{p}{1-p} E[w(N)] \geq \frac{p}{k+1} \cdot \text{OPT.} \quad (10)
\]

Following the proof of Lemma 10, we have

\[
E[w(S)] \geq \left(1 - k \cdot \frac{2\beta}{(1-\beta)^2}\right) E[w(N)], \quad (11)
\]

where \( \beta = 2e(1-p) \). The additional \( k \) terms come from the fact that we have to sum up \( k \) geometric sequences instead of one in Eqn.(8).

Let \( a = \frac{1-p}{p} \). Recall that \( \theta = 1 + \frac{1-p}{p} \cdot t = 1 + at \). By taking \( t = \sqrt{\frac{1}{a(\sqrt{1-a})}} \). We have that

\[
E[f(S)] \geq \left(\sqrt{\gamma} - \sqrt{a}\right)^2 \frac{1-p}{k+1} \cdot \text{OPT} \quad (12)
\]

Thus overall we have that

\[
E[f(S)] \geq \left(\sqrt{\gamma} - \sqrt{a}\right)^2 \frac{1-p}{k+1} \cdot \text{OPT} \quad \text{by Eqn.(11) and Eqn.(12) *}/
\]

\[
\geq \left(\sqrt{1-a} - \sqrt{a}\right)^2 \frac{1-p}{k+1} \cdot \text{OPT} \quad \text{by Eqn.(10) *}/
\]

Now we analyze this ratio. Set \( p = 1 - \frac{c}{k} \) for some sufficiently small constant \( c \). Then \( \beta = 2ec/k \leq 2ec < 1 \) and \( \gamma = 1 - \frac{4ec}{(1-2ec)(1-\beta)} \geq 1 - \frac{4ec}{(1-2ec)^2} \). By enforcing \( c < 0.04 \), we have \( \gamma > a \). Then

\[
\left(\sqrt{1-a} - \sqrt{a}\right)^2 \frac{1-p}{k+1} \geq \frac{1}{k(k+1)} \cdot c \left(1 - \frac{4ec}{(1-2ec)^2} - \frac{4ec}{(1-2ec)^2} - \frac{c}{1-c}\right)^2
\]

By taking \( c = 0.02 \). We have that

\[
\left(\sqrt{1-a} - \sqrt{a}\right)^2 \frac{1-p}{k+1} \geq 0.009 \frac{1}{k(k+1)}
\]

\[\blacksquare\]
4.2 The linear case

In this section, we analyze the algorithm ONLINE for the laminar matroid secretary problem with linear functions. For this special case, we improve the competitive ratio to 9.6.

**Theorem 14.** Algorithm \( \text{SIMULATE} \) is a 9.6-competitive algorithm for the linear matroid secretary problem with laminar matroids.

For linear functions, our main idea is to prove that each element in the optimal solution has a good probability of staying in our solution set \( S \) in SIMULATE. Before proving the theorem directly, we first define some useful random variables and analyze the random process used in SIMULATE more precisely.

**Definition 15.** Let \( X_1, X_2, \ldots, X_n \) be independent Bernoulli trials such that \( \Pr[X_i = 1] = p \) and \( \Pr[X_i = 0] = 1 - p \). Define \( i^X_c(k) \) to be the random variable indicating the index of the \( k^{th} \) appearance of 0’s in the sequence, \( i^X_1(k) \) that of \( k^{th} \) appearance of 1’s. We define \( i^X_c(0) = i^X_c(0) = 0 \). Define \( G_p(m, n) \) for any positive integer \( m, n \) as follows.

\[
G_p(m, n) = \Pr[i^X_1(m) > i^X_c(n)]
\]

Intuitively, in SIMULATE, we flip a coin for each element and add it \( M \) if and only if the coin is head. We couple SIMULATE with \( \{X_i\} \) as follows. If \( X_i = 1 \), the \( i \)-th element in the greedy order of SIMULATE will be placed into \( M \). Otherwise, it is placed into \( N \). Consider the order of elements greedily selected in SIMULATE. Then \( i^X_1(k) \) (resp. \( i^X_c(k) \)) can be viewed as the index of the \( k^{th} \) element added to \( M \) (resp. \( N \)) in this greedy order. Since all elements considered in SIMULATE are ordered by weights, \( G_p(m, n) \) denotes the probability that the weight of the \( m \)-th element in \( M \) is smaller than the weight of the \( n \)-th element in \( N \).

Consider any element \( e \) in the offline optimal solution. In SIMULATE, \( e \) will be in \( M \) if the random coin comes with head when it is processed. Otherwise \( e \) will be placed in to \( N \). (Since the valuation function is linear, \( e \) will always show up in the greedy order in SIMULATE.) Therefore, the probability that \( e \in N \) is \( 1 - p \). The difficult part is to argue that \( e \) will survive the pruning with good probability.

We will use the same pruning process as in Eqn (6). In the following, we show that for any \( B \) that contains \( e \), the probability that \( B \) is violated, i.e., \( \mu(B) < N \cap B \), is at most \( G_p(\mu(B), \mu(B)) \). In particular, we have the following lemma.

**Lemma 16.**

\[
\forall e \in U, \Pr[e \in S \mid e \in N] \geq 1 - \sum_{n \geq 1} G_p(n, n)
\]

**Proof.** For any fixed \( B \) with \( e \in B \in F \), consider the sequence of coins that are tossed in the SIMULATE when the elements in \( B \setminus \{e\} \) arrive as \( \{X_1, X_2, \ldots\} \). (Note that conditioned on \( e \in N \), we know that the coin toss for \( e \) is 0.)

Conditioned on the event that \( e \in N \), the event \( |N \cap B| > \mu(B) \) implies \( i^X_1(\mu(B)) > i^X_c(\mu(B)) \). Otherwise, \( M \cap B \) will have cardinality \( \mu(B) \) before \( N \cap B \) has cardinality more than \( \mu(B) \), and will prevent any element in \( B \) being added to either \( M \) or \( N \). (Including \( e \), it means that \( N \) must reach size \( \mu(B) + 1 \) before \( M \) reaches size \( \mu(B) \).) That is, \( \Pr[|N \cap B| > \mu(B) \mid e \in N] \leq \Pr[i^X_1(\mu(B)) > i^X_c(\mu(B))] = G_p(\mu(B), \mu(B)) \). If for each \( B \in F \) that
contains e, we have \(|N \cap B| \leq \mu(B)|, then e must be in S. Thus by union bound,

\[
\Pr[e \in S \mid e \in N] \geq 1 - \sum_{B \in F \mid e \in B} \Pr[|N \cap B| > \mu(B) \mid e \in N]
\]

\[
\geq 1 - \sum_{B \in F \mid e \in B} G_p(\mu(B), \mu(B)) \geq 1 - \sum_{n \geq 1} G_p(n, n)
\]

To lower bound the term \(\Pr[e \in S \mid e \in N]\), it suffices to upper bound \(G_p(n, n)\) as shown in the following lemma.

\[\textbf{Lemma 17.}\]

\[G_p(m, n) = (1-p)^n \sum_{i=0}^{m-1} \binom{n-1+i}{i} p^i \leq (1-p)^n (1+p)^{n+m-2}\]

\[\textbf{Proof.}\] We prove the first equality by a counting argument. Notice that \(G_p(m, n)\) is the probability that when the number of 0s reaches \(n\), the number of 1s is still smaller than \(m\).

Let the number of 1s be \(i\) before the number of 0s reaches \(n\). Then we are interested in \(0 \leq i \leq m-1\). Consider the first \(n+i\) random variables in \(X\). Clearly \(X_{n+i} = 0\) because this is the time the number of 0s reaches \(n\). Therefore, the number of such configurations is \(\binom{n-1+i}{i}\), each appears with probability \((1-p)^n p^i\). The equality comes by summing over all such is.

Finally,

\[G_p(m, n) = (1-p)^n \sum_{i=0}^{m-1} \binom{n-1+i}{i} p^i \leq (1-p)^n \sum_{i=0}^{m-1} \binom{m+n-2}{i} p^i = (1-p)^n (1+p)^{m+n-2}\]

\[\textbf{Lemma 18.}\] For any element \(e \in \text{OPT}\), \(\Pr[e \in S] \geq (1-p) \left(1 - \frac{(1-p)}{1-(1-p)(1+p)^2}\right) \geq 1/9.6\) by taking \(p = 0.842\).

\[\textbf{Proof.}\] Since \(e \in \text{OPT}\), it is straightforward that \(e \in M \cup N\), and \(\Pr[e \in N] = 1 - p\).

\[
\Pr[e \in S] = \Pr[e \in S \mid e \in N] \cdot \Pr[e \in N]
\]

\[
= (1-p) \left(1 - \sum_{n \geq 1} G_p(n, n)\right)
\]

\[
\geq (1-p) \left(1 - \sum_{n \geq 1} (1-p)^n (1+p)^{2n-2}\right) \text{ by Lemma 17}
\]

\[
= (1-p) \left(1 - \frac{1-p}{1-(1-p)(1+p)^2}\right)
\]
5 Transversal matroid

In this section, we apply our simulated greedy algorithm to the submodular matroid secretary problem with transversal matroids. More specifically, we study the following submodular bipartite vertex-a-time matching problem.

5.1 Submodular Bipartite Vertex-a-time Matching Problem

Korula and Pál [18] generalized the transversal matroid secretary problem to an online bipartite graph matching problem, motivated by [6]. We further generalize to submodular valuation functions. In particular, we introduce the Submodular Bipartite Vertex-at-a-time Matching (SBVM) problem.

In the SBVM problem, there is an underlying bipartite graph $G(L \cup R, E)$. We are given the set of right nodes $R$. The nodes in $L$ are arriving sequentially in random order. When a vertex $\ell \in L$ arrives, all edges incident to $\ell$ are revealed. We assume the availability of an oracle for the submodular valuation function, which we can query the value of any subset of the edges we have seen. We must immediately decide to accept an edge to match $\ell$ with an unmatched vertex of $R$ or drop all edges incident to $\ell$.

We claim that the matroid secretary problem under a transversal matroid is a special case of the SBVM problem, when the valuation function is submodular. In particular, the valuation on $L$ in the transversal matroid can be extended to the valuation on the edges $E$.

Let $f'(\cdot)$ be the submodular function defined on the subsets of $L$. We define a function $f(\cdot)$ on the subsets of $E$ as follows: for $E' \subseteq E$, $f(E') = f'(L \cap E')$, where $L \cap E'$ is the set of left nodes incident to $E'$.

Lemma 19. If $f'(\cdot)$ is a monotonically non-decreasing submodular function, $f(\cdot)$ is a monotonically non-decreasing submodular function.

Proof. Clearly, if $f'(\cdot)$ is monotonically non-decreasing, $f(\cdot)$ must be monotonically non-decreasing as well. Let $E'' \subseteq E' \subseteq E$. We have $E'' \cap L \subseteq E' \cap L$. Therefore, for any edge $e \in E$, we want to show that

$$f(E'' \cup \{e\}) - f(E'') \geq f(E' \cap \{e\}) - f(E').$$

(13)

If $e$ is sharing the left node with $E'$, by monotonicity, the left term in Eqn. (13) is non-negative while the right term is zero. So the statement is true. On the other hand, when $e$ is not sharing the left node with $E'$, $e$ is not sharing the left node with $E''$ either. Eqn. (13) in this case comes directly from the submodularity of $f(\cdot)$.

Hence, for the submodular matroid secretary problem with a transversal matroid, we can extend the valuation function on $L$ to the set of edges in the underlying bipartite graph.

The optimal solutions for both problems are the same. In fact, if we find a matching, which is a good approximation of the SBVM problem, the left nodes of the matching are a good approximation of the matroid secretary problem with the same approximation ratio.

Now we are ready to show that our general online algorithm (Algorithm 2) with slightly modification gives a constant competitive ratio for the SBVM problem. We first prove that the greedy algorithm has a good approximation for the offline version of this submodular maximization problem.

\footnote{The ties in the valuation function have to be broken in a consistent way.}
Lemma 20. For a bipartite graph \( G(L, R, E) \) and a monotonically increasing submodular function \( f(\cdot) \geq 0 \) defined on all subsets of \( E \), GREEDY is a 3-approximate algorithm.

Proof. First we show that all matchings of \( G = (L, R, E) \) can be represented by independent sets, which are the intersection of two partition matroids. Both ground sets of these two partition matroids are \( E \). In the matroid \( M_1(E, I_1) \) (resp. \( M_2(E, I_2) \)), a set of edges is independent if no two edges in it have the same left (resp. right) node. It is easy to see that the set of matchings in \( G \) is exactly \( I_1 \cap I_2 \).

The Theorem 2.1 in [11] shows that the greedy algorithm is a \( k+1 \) approximation for the submodular function maximization problem under the intersection of \( k \) matroids. Therefore, our algorithm has approximation ratio 3.

Lemma 21.

\[
\mathbb{E}[f(M)] \geq \frac{p}{3} \cdot \text{OPT}
\]

Algorithm 4: SIMULATE2

5.2 Analysis

We cast the stochastic process in SIMULATE2 to our simple stochastic process as follows. In particular, at each time \( i \) an edge \( e \) is selected in SIMULATE2, we define \( w_i = f_M(e) \) where \( M \) is the current set of elements in \( M \). Clearly, \( w_i \) is a mapping from previous \( i-1 \) Bernoulli random variables. Our process will terminate after \( n \) edges are selected. So \( \mu = n \). In case the process terminates before \( |M| = n \), we can further append dummy edges in the process.

Notice that \( w(M) = f(M) \). By Proposition [10] we have

\[
\mathbb{E}[w(N)] = \frac{1-p}{p} \mathbb{E}[w(M)] = \frac{1-p}{p} \mathbb{E}[f(M)].
\]
Pruning. Since $N$ may not be a matching, we remove all edges in $N$ that share the same node in $R$ with other edges in $N$. Notice that no two edges in $N$ share the same left node. Let $S$ be the set of edges left. Define $E_r$ be the set of edges incident to $r \in R$. Then $|S \cap E_r| \leq 1$. We have

$$N = S \cup \left( \bigcup_{r \in R} 1_{|N \cap E_r| > 1} \cdot (N \cap E_r) \right),$$

and

$$\mathbb{E}[w(N)] \leq \mathbb{E}[w(S)] + \sum_{r \in R} \mathbb{E}[w(1_{|N \cap E_r| > 1} \cdot (N \cap E_r))].$$

Equation (15)

Now we focus on the term $\mathbb{E}[w(1_{|N \cap E_r| > 1} \cdot (N \cap E_r))$ for a particular node $r \in R$. We isolate our stochastic process on edges in $E_r$, by rearranging randomness as follows. For each edge in $e \in E \setminus E_r$, we associate a biased random coin. When $e$ arrives in the process, the random coin associated with it will be tossed. (Since all edges incident to the same left node will be processed only once on the first arriving edge, we will not toss two random coins for the same left node.)

For a fixed set of outcomes of random coins associated with edges in $E \setminus E_r$, the process on edges in $E_r$ is a simple stochastic process with $\mu = 1$. Therefore, by Proposition 8 we have

$$\mathbb{E}[w(1_{|N \cap E_r| > 1} \cdot (N \cap E_r))] \leq \frac{1 - p^2}{p^2} \mathbb{E}[w(S \cap E_r)].$$

Equation (16)

Since $E_r$-s are disjoint, and $w(\cdot)$ is linear for a fixed set of random outcomes, we have

$$\sum_{r \in R} w(S \cap E_r) = w(S)$$

Equation (17)

Combining Eqn. (15), Eqn. (16) and Eqn. (17), we immediately have:

$$\mathbb{E}[w(S)] \geq p^2 \cdot \mathbb{E}[w(N)].$$

Equation (18)

Finally, we bound $f(S)$ based on $w(S)$ following an approach similar to the laminar matroid case. Again, we define $g(e) = f_{\mathcal{N}_e}(e)$, i.e., if $e$ appears in the greedy order, $\mathcal{N}_e$ is the current set of elements in $N$; otherwise, $g(e) = 0$.

Lemma 22. For any $t > 0$, let $\alpha = \frac{p}{1 - p}$ and $\theta = \frac{t + \theta}{\alpha}$, we have

$$\mathbb{E}[f(S)] \geq \left( \frac{1}{\theta} - \frac{1}{1 - p^2} \right) \cdot \mathbb{E}[w(S)]$$

Proof. Let $g(S)$ be the function $\sum_{e \in S} g(e)$. Since $S \subset N$, we have $f(S) \geq g(S)$. We inspect the function $F(S, M, N) = t \cdot g(S) + f(M) - f(M \cup N) \leq t \cdot g(S)$.

Following exactly the same argument in the proof of Lemma 11 we have the same conclusion as in Eqn. (9):

$$t \cdot \mathbb{E}[f(S)] \geq \frac{t}{\theta} \mathbb{E}[w(S)] - \mathbb{E}[w(N)].$$

The lemma immediately follows from Eqn. (18).

Lemma 22. For any $t > 0$, let $\alpha = \frac{p}{1 - p}$ and $\theta = \frac{t + \theta}{\alpha}$, we have

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The lemma immediately follows from Eqn. (18).

Combine Lemma 21, Eqn. (13), Eqn. (18) and Lemma 22

$$\mathbb{E}[f(S)] \geq \left( \frac{1}{\theta} - \frac{1}{1 - p^2} \right) \cdot p^2 \cdot \frac{1 - p}{3} \cdot \text{OPT} \geq \frac{1}{95} \text{OPT}$$

The inequality comes from taking $p = 0.9$ and $t = 5.29$. We have the main result of this section.
Theorem 23. There is an online algorithm with competitive ratio at most $9.5$ for the submodular matroid secretary problem with transversal matroids.

6 Conclusion

In this paper, we develop a general algorithm for the submodular matroid secretary problems. In particular, we obtain constant competitive algorithms for laminar matroids and transversal matroids. Our algorithm can also handle the intersection of a constant number of laminar matroids, which makes our algorithm more applicable.

Our algorithm does not work on general matroids. Consider the following simple example on graphical matroids. There is a single heavy edge $(u, v)$ in the graph. There is a large number of nodes $K = \{u_1, u_2, \ldots, u_n\}$ and edges $\{(u, u_i), (u_i, v) \mid u_i \in K\}$. The weight on each such edge is very small. It is easy to verify that the probability that our algorithm will accept $(u, v)$ is exponentially small on $n$. Nevertheless, our algorithm can handle graphical matroids using the same decomposition technique [1], i.e., by reducing the problem to a partition matroid, which is randomly selected from two constructed partition matroids. On the other hand, it would be interesting to characterize the independent set constraints for which our algorithm framework is constant competitive.

In the distinction between the submodular case and linear case in matroid secretary problem, we still cannot adapt the recent $O(\sqrt{\log r})$ competitive algorithm in [5] as well as the constant competitive algorithm for the random assignment model in [19] previously on the linear case. It would be interesting to close this gap. Finally, it is still widely open whether the matroid secretary problem permits constant competitive algorithms for general matroids.

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Lemma 5 (restated). The sets of elements of $H$, $M$ and $N$ by SIMULATE have the same joint distribution as the $H$, $M$ and $N$ generated by ONLINE with a random permutation of the elements in $U$.

Proof. We couple the randomness in SIMULATE and ONLINE as follows. In ONLINE, the following randomness is used: (a) the random permutation $\pi_O$ of $U$; (b) a random number $k = \text{Binom}(|U|, p)$. Let $H_O$ be the first $k$ elements in $\pi_O$. For a permutation $\pi$ of $U$, let $H_\pi$ be a (non-ordered) prefix of $\pi$. Then for any fixed permutation $\pi$ and $H_\pi$,

$$\Pr[\pi_O = \pi \land H_O = H_\pi] = (n!)^{-1} \binom{n}{|H_\pi|} p^{|H_\pi|} (1 - p)^{n - |H_\pi|}.$$

In SIMULATE, we can associate each element in $U$ with a biased coin with head probability $p$. Let $H_S$ be the set of nodes in $U$ whose coin is head and $T_S = U \setminus H_S$. We append the randomness of SIMULATE by applying random permutations on $H_S$ and $T_S$. A permutation of $U$ is the concatenation of $H_S$ and $T_S$ denoted as $\pi_S$.

$$\Pr[\pi_S = \pi \land H_S = H_\pi] = p^{|H_\pi|} (1 - p)^{n - |H_\pi|} (|H_\pi|)!^{-1} ((n - |H_\pi|))!^{-1}.$$

Therefore, the probabilities of having a particular permutation and $H$ are the same in the two algorithms. It is then sufficient to show that both algorithms generate the same $M$ and $N$, given a fixed permutation $\pi$ of $U$ and $H$.

Notice that $M = \text{GREEDY}(H)$ in both algorithms, which must be identical. Now we prove for $N$. Let $N_O$ and $N_S$ be the $N$ in ONLINE and SIMULATE, respectively.

Consider element $e \in N_O$. By ONLINE, $e \in M' = \text{GREEDY}(H \cup \{e\})$. Assume $e$ is the $i$-th element added in $M'$. Let $M_{i-1}$ be the first $i - 1$ elements placed into $M$ in ONLINE. Notice that the first $i - 1$ elements placed into $M$ in SIMULATE is exactly $M_{i-1}$. Let $e_i$ be the $i$-th element placed into $M$ in SIMULATE. Since $e \in M'$, $f_{M_{i-1}}(e_i) > f_{M_{i-1}}(e_i)$. Therefore, in SIMULATE, $e$ must be processed before $e_i$ is placed into $M$. As the coin associated with $e$ is tail, we conclude that $e \in N_S$.

For the other direction, consider $e \in N_S$. Let $M_e \subseteq M$ be the set of elements in $M$ when $e$ is processed in SIMULATE. By the greedy nature of SIMULATE, $f_{M_e}(e)$ is larger than any other elements in $H \setminus \{M_e\}$. Therefore, $e \in \text{GREEDY}(H \cup \{e\})$, i.e., $e \in N_O$. ◼