Foldings of KLR algebras

Ying Ma, Toshiaki Shoji and Zhiping Zhou

Abstract. Let $U_q^-$ be the negative half of the quantum group associated to the Kac-Moody algebra $\mathfrak{g}$, and $\mathbf{U}_q^-$ the quantum group obtained by a folding of $\mathfrak{g}$. Let $A = \mathbb{Z}[q, q^{-1}]$. McNamara showed that $U_q^-$ is categorified over a certain extension $\widetilde{A}$ of $A$, by using the folding theory of KLR algebras. He posed a question whether $\widetilde{A}$ coincides with $A$ or not. In this paper, we give an affirmative answer for this problem.

Introduction

Let $\mathcal{X} = (I, ( \cdot, \cdot ))$ be a Cartan datum, and $U_q^-$ the negative half of the quantum group associated to $\mathcal{X}$. Let $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the root lattice of $\mathcal{X}$, and set $Q^+ = \sum_{i \in I} \mathbb{N} \alpha_i$. Set $A = \mathbb{Z}[q, q^{-1}]$, and let $A U_q^-$ be Lusztig’s integral form of $U_q^-$, which is an $A$-subalgebra of $U_q^-$. In the case where $\mathcal{X}$ is of symmetric type, Lusztig constructed in [L1] the canonical basis of $U_q^-$, by using the geometry of a quiver $\overrightarrow{Q}$ associated to $\mathcal{X}$. More precisely, he constructed, for each $\beta \in Q^+$, a category $Q_{V_\beta}$ consisting of certain semisimple $\overrightarrow{Q}$-complexes on the representation space $V_\beta$ of $\overrightarrow{Q}$ associated to $\beta$, and showed that the direct sum $K(Q) = \bigoplus_{\beta \in Q^+} K(Q_{V_\beta})$ of the Grothendieck group $K(Q_{V_\beta})$ of $Q_{V_\beta}$ has a structure of $A$-algebra, which is isomorphic to $A U_q^-$. The canonical basis of $U_q^-$ is obtained from a natural basis of $K(Q_{V_\beta})$ coming from simple perverse sheaves in $Q_{V_\beta}$.

Let $\sigma$ be an admissible diagram automorphism on $\mathcal{X}$ (see 1.5), which gives a bijection $\sigma : I \rightarrow I$. Let $J$ be the set of $\sigma$-orbits in $I$. Then $\overrightarrow{X} = (J, ( \cdot, \cdot ))$ gives a Cartan datum. We denote by $\overrightarrow{U}_q^-$ the corresponding quantum group. In [L2], Lusztig constructed the canonical basis of $\overrightarrow{U}_q^-$, by extending the discussion in the symmetric case. By a suitable choice of the orientation, $\sigma$ acts on the quiver $\overrightarrow{Q}$, and it induces a periodic functor $\sigma^*$ on $Q_{V_\beta}$ whenever $\beta$ is $\sigma$-stable. He defined a new category $\overrightarrow{Q}_{V_\beta}$ attached to $Q_{V_\beta}$ and $\sigma^*$, and a (modified) Grothendieck group $K(\overrightarrow{Q}_{V_\beta})$. Let $Q^+_\sigma$ be the set of $\sigma$-stable elements in $Q^+$. Then the direct sum $K(\overrightarrow{Q}) = \bigoplus_{\beta \in Q^+_\sigma} K(\overrightarrow{Q}_{V_\beta})$ has a structure of $\mathbb{Z}[\zeta_n, q, q^{-1}]$-algebra, where $n$ is the order of $\sigma$, and $\zeta_n$ is a primitive $n$-th root of unity in $\overrightarrow{Q}_{Q^+}$. He proved that a certain $A$-subalgebra $A K(\overrightarrow{Q})$ is isomorphic to $A \overrightarrow{U}_q^-$. In that case, a natural basis of $A K(\overrightarrow{Q})$ gives the canonical basis of $\overrightarrow{U}_q^-$, up to sign, i.e., gives the canonical signed basis. The canonical basis of $\overrightarrow{U}_q^-$ is obtained from it by suitably fixing the signs, which was done by an algebraic argument using Kashiwara’s tensor product rule.
$K(Q)$ (for symmetric $X$) gives a geometric categorification of $U_q^-$.

The other hand, in [KL], [R1], Khovanov-Lauda, and Rouquier introduced the Khovanov-Lauda-Rouquier algebra $R = \bigoplus_{\beta \in Q^+_+} R(\beta)$ (KLR algebra in short) over an algebraically closed field $k$, and gave a categorification of $U_q^-$ (for general $X$) in terms of KLR algebras. For each $\beta \in Q^+_+$, let $R(\beta)$ be the category of finitely generated graded projective $R(\beta)$-modules, and $K_{gp}(\beta)$ the Grothendieck group of $R(\beta)$-$gp$. They proved that $K_{gp} = \bigoplus_{\beta \in Q^+_+} K_{gp}(\beta)$ has a structure of $A$-algebra, and is isomorphic to $\tilde{\Lambda} U_q^-$. It is natural to expect a similar discussion as in Lusztig’s case will hold also for KLR algebras. McNamara [M] developed the theory of foldings for KLR algebras. Let $X$ and $\underline{X}$ be as above (for $X$: general type), and choose $\beta \in Q^+_+$. By a suitable choice of defining parameters of $R$, $\sigma$ acts on $R(\beta)$, thus we obtain a periodic functor $\sigma^*$ on $R(\beta)$-gp. One can define a new category $\tilde{R}(\beta)$-gp $= \tilde{\mathcal{P}}_{\beta}$, and a modified Grothendieck group $K(\mathcal{P}_{\beta})$. $K(\mathcal{P}) = \bigoplus_{\beta \in Q^+_+} K(\mathcal{P}_{\beta})$ has a structure of $Z[\zeta_n, q, q^{-1}]$-algebra. McNamara constructed a natural basis $\tilde{B}$ of $K(\mathcal{P})$ by using the theory of crystals. Define an extension $\tilde{\Lambda}$ of $\Lambda$ as the smallest subring $Z[\zeta_n, q, q^{-1}]$ containing the structure constants of $\tilde{B}$ with respect to the product in $K(\mathcal{P})$. Hence the $\tilde{\Lambda}$-span $\tilde{\Lambda} K(\mathcal{P})$ of $\tilde{B}$ gives an $\tilde{\Lambda}$-algebra. He proved that $\tilde{\Lambda} K(\mathcal{P})$ is isomorphic to $\tilde{\Lambda} \tilde{U}_q^-$, where $\tilde{\Lambda} \tilde{U}_q^-$ is the extension of $\Lambda \tilde{U}_q^-$ to $\tilde{\Lambda}$, thus obtained a categorification of $\tilde{U}_q^-$. He showed that $\Lambda \subset \tilde{\Lambda} \subset Z[\zeta_n + \zeta_n^{-1}, q, q^{-1}]$, and posed a question whether $\tilde{\Lambda}$ coincides with $\Lambda$ or not ([M, Question 11.2]).

The aim of this paper is to show that this certainly holds (Theorem 3.16). The main ingredient for the proof is the following result proved in [MSZ]. We consider the case where the order $n$ of $\sigma$ is a power of a prime number $\ell$. Let $F = Z/\ell Z$ be the finite field of $\ell$-elements. Let $A' = F[\zeta_n, q, q^{-1}] = A/\ell A$, and consider the $A'$-algebras $\Lambda A' \tilde{U}_q^-$ and $\tilde{\Lambda} A' \tilde{U}_q^-$, by changing the base ring from $\Lambda$ to $A'$. $\sigma$ acts naturally on $\Lambda A' \tilde{U}_q^-$, and let $\tilde{\Lambda} A' \tilde{U}_q^- - \sigma$ be the fixed point subalgebra by $\sigma$. In [MSZ, Thm. 3.4, Thm. 4.27], it is proved that $\tilde{\Lambda} A' \tilde{U}_q^-$ is realized as the quotient algebra of $\Lambda A' \tilde{U}_q^- - \sigma$, and that the canonical basis of $\tilde{U}_q^-$ is obtained from the $\sigma$-stable canonical basis of $\tilde{U}_q^-$, up to sign, through the quotient map. This gives an alternate approach for Lusztig’s geometric construction of canonical signed basis of $\tilde{U}_q^-$. A similar argument also works for the KLR algebras. In this scheme, the finite field $F$ enters in the theory, but $\zeta_n$ is not involved. By comparing this with McNamara’s isomorphism, we obtain the required result.

In the case where $X$ is symmetric, and $R$ is a symmetric type KLR algebra (see 5.1) over a field $k$ of characteristic zero, Varagnolo-Vasserot [VV] and Rouquier [R2] proved that the natural basis of $K_{gp}$ gives the canonical basis (which coincides with Kashihara’s lower global basis by [GL]) of $U_q^-$, by using the geometric interpretation of $R$ in terms of the Ext algebra coming from Lusztig’s category $Q_{\mathcal{V}_g}$. A similar result holds also for $K(\mathcal{P})$, namely, the basis $\tilde{B}$ corresponds, via McNamara’s isomorphism, to the canonical basis of $\tilde{U}_q^-$, which coincides with the lower global basis. This is stated in the last paragraph of [M], with a brief indication for the proof, which is based on the geometric argument. In Theorem 5.13 and Theorem 5.16, we
give a simple proof of this fact, without using the geometry (though assuming the results in [VV], [R2]).

1. Quantum Groups

1.1. Let \( X = (I, (\ , \ ) ) \) be a Cartan datum, where \( I \) is a finite set, and \((\ , \ ) \) is a symmetric bilinear form on the vector space \( \bigoplus_{i \in I} \mathbb{Q} \alpha_i \) with basis \( \alpha_i \), satisfying the properties that

\[ (i) \ (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \text{ for any } i \in I, \]
\[ (ii) \ \frac{2(\alpha_i, \alpha_{i'})}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\leq 0} \text{ for any } i \neq i' \in I. \]

For \( i, i' \in I \), set \( a_{i' i} = 2(\alpha_i, \alpha_{i'})/(\alpha_i, \alpha_i) \in \mathbb{Z} \). \( A = (a_{i i'}) \) is called the Cartan matrix associated to \( X \). The Cartan datum is called symmetric if \((\alpha_i, \alpha_i) = 2\) for any \( i \in I \).

Let \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) be the root lattice of \( X \). We set \( Q_+ = \sum_{i \in I} \mathbb{N} \alpha_i \), and \( Q_- = -Q_+ \).

1.2. Let \( q \) be an indeterminate, and for an integer \( n \), a positive integer \( m \), set

\[ [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]^! = [1][2] \cdots [m], \quad [0]^! = 1. \]

For each \( i \in I \), set \( d_i = (\alpha_i, \alpha_i)/2 \in \mathbb{N} \), and \( q_i = q^{d_i} \). We denote by \([n]_i \) the element obtained from \([n]\) by replacing \( q \) by \( q_i \).

For the Cartan datum \( X \), let \( U_q^- = U^-_q(X) \) be the negative half of the quantum group associated to \( X \). \( U_q^- \) is an associative algebra over \( \mathbb{Q}(q) \) generated by \( f_i \ (i \in I) \) satisfying the \( q \)-Serre relations

\[ \sum_{k + k' = 1 - a_{i i'}} (-1)^k f_i^{(k)} f_{i'}^{(k')} = 0, \quad \text{for } i \neq i' \in I, \]

where \( f_i^{(n)} = f_i^n/[n]_i^! \).

Set \( A = \mathbb{Z}[q, q^{-1}] \), and let \( A U_q^- \) be Lusztig’s integral form of \( U_q^- \), namely, the \( A \)-subalgebra of \( U_q^- \) generated by \( f_i^{(n)} \) for \( i \in I, n \in \mathbb{N} \).

We define a \( Q \)-algebra automorphism, called the bar-involution, \( - : U_q^- \to U_q^- \) by \( \overline{q} = q^{-1}, \overline{f_i} = f_i \) for \( i \in I \). We define an anti-algebra automorphism \( * : U_q^- \to U_q^- \) by \( f_i^* = f_i \) for any \( i \in I \).

1.3. \( U_q^- \) has a weight space decomposition \( U_q^- = \bigoplus_{\beta \in Q_-} (U_q^-)_\beta \), where \((U_q^-)_\beta \) is a subspace of \( U_q^- \) spanned by elements \( f_{i_1} \cdots f_{i_N} \) such that \( \alpha_{i_1} + \cdots + \alpha_{i_N} = -\beta \). \( x \in U_q^- \) is called homogeneous with \( \text{wt } x = \beta \) if \( x \in (U_q^-)_\beta \). We define a multiplication on \( U_q^- \otimes U_q^- \) by

\[ (x_1 \otimes x_2) \cdot (x'_1 \otimes x'_2) = q^{-(\text{wt } x_2, \text{wt } x'_1)} x_1 x'_1 \otimes x_2 x'_2 \]
where $x_1, x'_1, x_2, x'_2$ are homogeneous in $U_q^-$. Then $U_q^- \otimes U_q^-$ becomes an associative algebra with respect to this twisted product. One can define a homomorphism $r : U_q^- \to U_q^- \otimes U_q^-$ by $r(f_i) = f_i \otimes 1 + 1 \otimes f_i$ for each $i \in I$. It is known that there exists a unique bilinear form $(\ , \ )$ on $U_q^-$ satisfying the following properties; $(1, 1) = 1$ and

\begin{align}
(f_i, f_j) &= \delta_{ij}(1 - q_i^2)^{-1}, \\
(x, y'y'') &= (r(x), y' \otimes y''), \\
(x'x'', y) &= (x' \otimes x'', r(y)),
\end{align}

(1.3.2)

where the bilinear form on $U_q^- \otimes U_q^-$ is defined by $(x_1 \otimes x_2, x'_1 \otimes x'_2) = (x_1, x'_1)(x_2, x'_2)$. Thus defined bilinear form is symmetric and non-degenerate.

Following [L2, 1.2.13], for each $i$, we define a $\mathbb{Q}(q)$-linear map $i_r : U_q^- \to U_q^-$ by the condition that

\begin{equation}
(1.3.3) \quad r(x) = f_i \otimes i_r(x) + \sum y \otimes z,
\end{equation}

where $y$ runs over homogeneous elements such that $\text{wt} \ y \neq -\alpha_i$. Since $i_r$ coincides with the operator $c'_i$ defined in [K1, 3.4], (see [L2, Prop. 3.1.6] and [K1, Lemma 3.4.1]), hereafter we write $i_r$ as $c'_i$. Thus, if we define the action of $f_i$ on $U_q^-$ by the left multiplication, then $c'_i, f_i$ satisfies the $q$-boson relations, as operators on $U_q^-$,

\begin{equation}
(1.3.4) \quad c'_i f_{i'} = q^{-(\alpha_i, \alpha_{i'})} f_{i'} c'_i + \delta_{i'i'},
\end{equation}

1.4. Let $V$ be a $\mathbb{Q}(q)$-subspace of $U_q^-$. A basis $\mathcal{B}$ of $V$ is said to be almost orthonormal if

\begin{equation}
(1.4.1) \quad (b, b') \in \begin{cases} 
1 + q\mathbb{Z}[[q]] \cap \mathbb{Q}(q) & \text{if } b = b', \\
q\mathbb{Z}[[q]] \cap \mathbb{Q}(q) & \text{if } b \neq b'.
\end{cases}
\end{equation}

Recall that $A = \mathbb{Z}[q, q^{-1}]$. Let $A_0 = \mathbb{Q}[[q]] \cap \mathbb{Q}(q)$. Set

\begin{equation}
(1.4.2) \quad \mathcal{L}_\mathbb{Z}(\infty) = \{ x \in A U_q^- \mid (x, x) \in A_0 \}.
\end{equation}

Then $\mathcal{L}_\mathbb{Z}(\infty)$ is a $\mathbb{Z}[q]$-submodule of $A U_q^-$. It is known that if $\mathcal{B}$ is an $A$-basis of $A U_q^-$, which is almost orthonormal, then $\mathcal{B}$ gives a $\mathbb{Z}[q]$-basis of $\mathcal{L}_\mathbb{Z}(\infty)$ by [L2, Lemma 16.2.5].

We define a subset $\tilde{\mathcal{B}}$ of $U_q^-$ by

\begin{equation}
(1.4.3) \quad \tilde{\mathcal{B}} = \{ x \in A U_q^- \mid \overline{x} = x, (x, x) \in 1 + q\mathbb{Z}[[q]] \}.
\end{equation}

If there exists a basis $\mathcal{B}'$ of $U_q^-$ such that $\tilde{\mathcal{B}} = \mathcal{B}' \sqcup -\mathcal{B}'$, then $\tilde{\mathcal{B}}$ is called the canonical signed basis. In that case, the basis $\mathcal{B}'$ is determined uniquely, up to sign, by the
condition (1.4.3). Lusztig proved in [L2], by using the geometric theory of quivers, that $\mathcal{B}$ is the canonical signed basis. Kashiwara’s global crystal basis $\mathcal{B}$ in [K1] also satisfies the condition that $\mathcal{B} = \mathcal{B} \sqcup -\mathcal{B}$ (see [K1, Prop. 5.1.2]), but note that the inner products $(\ , \ )$ on $U^-_q$ defined by Lusztig and by Kashiwara differ by a scalar factor on each weight space.

1.5. A permutation $\sigma : I \to I$ is called an admissible automorphism on $X$ if it satisfies the property that $(\alpha_i, \alpha_{i'}) = (\alpha_{\sigma(i)}, \alpha_{\sigma(i')})$ for any $i, i' \in I$, and that $(\alpha_i, \alpha_{i'}) = 0$ if $i$ and $i'$ belong to the same $\sigma$-orbit in $I$. Assume that $\sigma$ is admissible. We denote by $n$ the order of $\sigma : I \to I$. Let $J$ be the set of $\sigma$-orbits in $I$. For each $j \in J$, set $\alpha_j = \sum \alpha_i$, and consider the subspace $\bigoplus_{j \in J} Q\alpha_j$ of $\bigoplus_{i \in I} Q\alpha_i$, with basis $\alpha_j$. We denote by $|j|$ the size of the orbit $j$ in $I$. The restriction of the form $(\ , \ )$ on $\bigoplus_{j \in J} Q\alpha_j$ is given by

$$ (\alpha_j, \alpha_{j'}) = \begin{cases} (\alpha_i, \alpha_i)|j| & (i \in j), \\ \sum_{i \in j, i' \in j}(\alpha_i, \alpha_{i'}) & (i \not\in j'). \end{cases} $$

Then $X = (J, (\ , \ ))$ turns out to be a Cartan datum, which is called the Cartan datum induced from $(X, \sigma)$. $\sigma$ acts naturally on $Q$, and the set $Q^\sigma$ of $\sigma$-fixed elements in $Q$ is identified with the root lattice of $X$. We have $Q^\sigma = \sum_{j \in J} N\alpha_j$.

Let $\sigma$ be an admissible automorphism of $X$. Then $\sigma$ induces an algebra automorphism on $U^-_q \to U^-_q$ by $f_i \mapsto f_{\sigma(i)}$, which we also denote by $\sigma$. The action of $\sigma$ leaves $A U^-_q$ invariant. We denote by $A U^-_q^{\sigma}$ the fixed point subalgebra of $A U^-_q$.

Let $X$ be the Cartan datum induced from $(X, \sigma)$. Let $U^-_q = U^-_q(X)$ the quantum group associated to $X$. Thus $U^-_q$ is an associative $Q(q)$-algebra, with generators $f_j$ ($j \in J$), and similar relations as in (1.2.1). The integral form $A U_q^{-}$ of $U_q^{-}$ is the $A$-subalgebra generated by $f_j^{(n)}$ ($j \in J, n \in \mathbb{Z}$).

1.6. Quotient algebras $V_q$. In general, let $V$ be a vector space on which $\sigma$ acts. For each $x \in V$, we denote by $O(x)$ the orbit sum of $x$, namely $O(x) = \sum_{0 \leq i < k} \sigma^i(x)$, where $k$ is the smallest integer such that $\sigma^k(x) = x$. Hence $O(x)$ is a $\sigma$-invariant element in $V$.

Let $\sigma : X \to X$ be as in 1.5, and $n$ the order of $\sigma : I \to I$. In the rest of this section, we assume that $n$ is a power of a prime number $\ell$. Let $F = \mathbb{Z}/\ell\mathbb{Z}$ be the finite field of $\ell$ elements, and set $A' = F[q, q^{-1}] = A/\ell A$. We consider the $A'$-algebra

$$ A' U_q^{-,\sigma} = A' \otimes_A A U_q^{-,\sigma} \simeq A U_q^{-,\sigma}/\ell(A U_q^{-,\sigma}). $$

Let $J$ be an $A'$-submodule of $A' U_q^{-,\sigma}$ generated by $O(x)$ for $x \in A' U_q^{-,\sigma}$ such that $\sigma(x) \not= x$. Then $J$ is a two-sided ideal of $A' U_q^{-,\sigma}$. We define an $A'$-algebra $V_q$ as the quotient algebra of $A' U_q^{-,\sigma}$,

$$ V_q = A' U_q^{-,\sigma}/J. \quad (1.6.1) $$

Let $\pi : A' U_q^{-,\sigma} \to V_q$ be the natural projection.
For each $j \in J$ and $a \in \mathbb{N}$, set $\overline{f}_j^{(a)} = \prod_{i \in j} f_i^{(a)}$. Since $\sigma$ is admissible, $f_i^{(a)}$ and $f_i^{(a)'}$ commute each other for $i, i' \in j$. Hence $\overline{f}_j^{(a)}$ does not depend on the order of the product, and $\overline{f}_j^{(a)} \in \mathbb{A} \mathbb{U}_q^{-\sigma}$. We denote its image in $\mathbb{A} \mathbb{U}_q^{-\sigma}$ also by $\overline{f}_j^{(a)}$. Thus we can define $g_j^{(a)} \in \mathbb{V}_q$ by

\begin{equation}
(1.6.2) \quad g_j^{(a)} = \pi(\overline{f}_j^{(a)}).
\end{equation}

In the case where $a = 1$, we set $\overline{f}_j^{(1)} = \overline{f}_j = \prod_{i \in j} f_i$ and $g_j^{(1)} = g_j$.

We define an $\mathbb{A}'$-algebra $\mathbb{A'} \mathbb{U}_q^{-}$ by $\mathbb{A'} \mathbb{U}_q^{-} = \mathbb{A'} \otimes \mathbb{A} \mathbb{U}_q^{-}$.

The following result was proved in [MSZ].

**Theorem 1.7** ([MSZ, Thm. 3.4]). The assignment $f_j^{(a)} \mapsto g_j^{(a)} (j \in J)$ gives an isomorphism $\Phi : \mathbb{A'} \mathbb{U}_q^{-} \cong \mathbb{V}_q$ of $\mathbb{A}'$-algebras.

**Remark 1.8** In Theorem 3.4 in [MSZ], the existence of the canonical signed basis of $\mathbb{U}_q^{-}$ is assumed. This certainly holds, by 1.4. In [MSZ, Thm. 4.27], the canonical signed basis for $\mathbb{U}_q^{-}$ was constructed from the canonical basis in the symmetric case, by making use of Theorem 1.7.

1.9. **The dual space** $(\mathbb{U}_q^{-})^*$. Let $\mathbb{U}_q^{-} = \bigoplus_{\beta \in \mathbb{Q}^-} (\mathbb{U}_q^{-})_\beta$ be the weight space decomposition of $\mathbb{U}_q^{-}$. We define the graded dual $(\mathbb{U}_q^{-})^*$ of $\mathbb{U}_q^{-}$ by

\[ (\mathbb{U}_q^{-})^* = \bigoplus_{\beta \in \mathbb{Q}^-} (\mathbb{U}_q^{-})^*_\beta, \]

where $(\mathbb{U}_q^{-})^*_\beta = \text{Hom}_{\mathbb{Q}(q)}((\mathbb{U}_q^{-})_\beta, \mathbb{Q}(q))$. $f_i^{(n)}$ and $e_i'$ act on $\mathbb{U}_q^{-}$, and we define the actions of $e_i'^{(n)}$ and $f_i$ on $(\mathbb{U}_q^{-})^*$ as the transpose of $f_i^{(n)}, e_i'$. We write $e_i' = e_i'^{(n)}$ for $n = 1$ (as operators on $(\mathbb{U}_q^{-})^*$).

$\mathbb{A} \mathbb{U}_q^{-}$ is decomposed as

\[ \mathbb{A} \mathbb{U}_q^{-} = \bigoplus_{\beta \in \mathbb{Q}^-} \mathbb{A} (\mathbb{U}_q^{-})_\beta, \]

where $\mathbb{A} (\mathbb{U}_q^{-})_\beta = \mathbb{A} \mathbb{U}_q^{-} \cap (\mathbb{U}_q^{-})_\beta$ is a free $\mathbb{A}$-module. We define $\mathbb{A} (\mathbb{U}_q^{-})^*$ by

\[ \mathbb{A} (\mathbb{U}_q^{-})^* = \bigoplus_{\beta \in \mathbb{Q}^-} \mathbb{A} (\mathbb{U}_q^{-})^*_\beta, \]

where $\mathbb{A} (\mathbb{U}_q^{-})^*_\beta = \text{Hom}_{\mathbb{A}}(\mathbb{A} (\mathbb{U}_q^{-})_\beta, \mathbb{A})$. Since $\mathbb{Q}(q) \otimes_{\mathbb{A}} \mathbb{A} (\mathbb{U}_q^{-})^* \simeq (\mathbb{U}_q^{-})^*$, $\mathbb{A} (\mathbb{U}_q^{-})^*$ is regarded as an $\mathbb{A}$-submodule of $(\mathbb{U}_q^{-})^*$, and $\mathbb{A} (\mathbb{U}_q^{-})^*$ gives an $\mathbb{A}$-lattice of $(\mathbb{U}_q^{-})^*$.

The actions of $f_i^{(n)}, e_i'$ preserve $\mathbb{A} \mathbb{U}_q^{-}$, and the actions of $e_i'^{(n)}, f_i$ preserve $\mathbb{A} (\mathbb{U}_q^{-})^*$.

Recall that $u \mapsto \overline{u}$ is the bar involution on $\mathbb{U}_q^{-}$. We define the bar involution $\varphi \mapsto \overline{\varphi}$ on $(\mathbb{U}_q^{-})^*$ by $\overline{\varphi(u)} = \overline{\varphi}(\overline{u})$ ($\varphi \in (\mathbb{U}_q^{-})^*, u \in \mathbb{U}_q^{-}$).
In [K2], Kashiwara introduced the upper global basis $B^*$ of $(U_q^-)^*$, which is characterized by the following properties.

**Theorem 1.10.** There exists a unique $A$-basis $B^*$ of $A(U_q^-)^*$ satisfying the following properties. For $b \in B^*$, set

$$
\varepsilon_i(b) = \max\{ k \geq 0 \mid e_i^k b \neq 0 \}.
$$

(i) $B^*$ has a weight space decomposition $B^* = \bigsqcup_{\beta \in Q^-} B^*_\beta$, where $B^*_\beta = B^* \cap A(U^-_q)^{\beta}$.

(ii) $1^* \in B^*$, where $\{1^*\} \subset A(U^-_q)_0$ is the dual basis of $\{1\} \subset A(U^-_q)_0$.

(iii) Assume that $e_i^k b \neq 0$. Then there exists a unique $\widetilde{e}_ib \in B^*$ satisfying the following.

$$
e_i' b = [\varepsilon_i(b)] \widetilde{e}_ib + \sum_{b' \in B^*, \varepsilon_i(b') < \varepsilon_i(b) - 1} E_{bb'}^{(i)} b' \quad (E_{bb'}^{(i)} \in q_i^{1-\varepsilon_i(b)} Z[q]).
$$

(iv) There exists a unique $\widetilde{f}_ib \in B^*$ satisfying the following.

$$
f_i b = q_i^{-\varepsilon_i(b)} \widetilde{f}_ib + \sum_{b' \in B^*, \varepsilon_i(b') < \varepsilon_i(b) + 1} F_{bb'}^{(i)} b' \quad (F_{bb'}^{(i)} \in q_i^{1-\varepsilon_i(b)} Z[q]).
$$

(v) $\widetilde{e}_i \widetilde{f}_ib = b$.

(vi) $\widetilde{f}_i \widetilde{e}_ib = b$ if $e_i'b \neq 0$.

(vii) $\widetilde{b} = b$.

The $A$-basis $B$ of $A(U_q^-)$ is defined as the dual basis of $B^*$. $B$ is called the lower global basis. $B$ coincides with the global crystal basis given in 1.4.

1.11. **The dual module $V_q^*$.** We define $A'(U_q^-)^*$ similarly to $A(U_q^-)$. Then $e_i^{[n]}$ act on $A'(U_q^-)^*$.

We define an action of $\sigma$ on $(U_q^-)^*$ by $(\sigma \varphi)(x) = \varphi(\sigma^{-1}(x))$ for $\varphi \in (U_q^-)^*$ and $x \in U_q^-$. Thus the natural pairing $\langle \ , \rangle : U_q^- \times (U_q^-)^* \to Q(q)$ satisfies the property $\langle \sigma(x), \sigma(\varphi) \rangle = \langle x, \varphi \rangle$. By the above discussion, the pairing $\langle \ , \rangle$ induces a perfect pairing $\langle \ , \rangle : A'(U_q^-) \times A'(U_q^-)^* \to A'$, which is compatible with the action of $\sigma$.

By Theorem 1.10, $\sigma$ acts on $B^*$ as a permutation, hence acts on $B$ similarly.

Let $A'(U_q^-)^{*,\sigma}$ be the submodule of $A'(U_q^-)^*$ of $\sigma$-fixed elements. Let $J^*$ be an $A'$-submodule of $A'(U_q^-)^{*,\sigma}$ consisting of $O(x)$ for $x \in A'(U_q^-)^*$ such that $O(x) \neq x$. We define $V_q^*$ as the quotient $A'$-module

$$(1.11.1) \quad V_q^* = A'(U_q^-)^{*,\sigma}/J^*.$$

Since $\langle A(U_q^-)^{*,\sigma}, J^* \rangle = \langle J, A'(U_q^-)^{*,\sigma} \rangle = 0$, the pairing $\langle \ , \rangle$ induces a pairing $\langle \ , \rangle : V_q \times V_q^* \to A'$. Let $B^\sigma$ be the set of $\sigma$-fixed elements in $B$, and $B^{*,\sigma}$ the set of $\sigma$-fixed elements in $B^*$. Then the image of $B^\sigma$ gives a basis of $V_q$, and the image
of $\mathcal{B}^{\ast,\sigma}$ gives a basis of $V_q^\ast$. Hence the pairing $\langle \ , \rangle : V_q \times V_q^\ast \to A'$ gives a perfect pairing on $A'$.

For each $j \in J$, we define an action of $\tilde{f}_j^{(n)}$ on $V_q$ by the left multiplication by $g_j^{(n)} = \pi(f_j^{(n)})$ (see (1.6.2)). We define an operator $\tilde{e}_j^{(n)}$ on $A'(U_q^-)^\ast$ by $\tilde{e}_j^{(n)} = \prod_{i \in j} e_i^{(n)}$. Then $\tilde{e}_j^{(n)}$ commutes with $\sigma$, and induces an action on $A'(U_q^-)^{\ast,\sigma}$. Since $\tilde{e}_j^{(n)}$ preserves $J^\ast$, $\tilde{e}_j^{(n)}$ acts on $V_q^\ast$, which we also denote by $\tilde{e}_j^{(n)}$. $\tilde{e}_j^{(n)}$ coincides with the transpose of the operation $\tilde{f}_j^{(n)}$ on $V_q$.

The operators $\tilde{e}_j^{(n)}$ on $A'(U_q^-)^\ast$ is defined as in the case of $A'(U_q^-)^\ast$. The following result is immediate from Theorem 1.7.

**Proposition 1.12.** Let $\Phi^\ast : V_q^\ast \cong A'(U_q^-)^\ast$ be the $A'$-module isomorphism obtained as the transpose of the map $\Phi : A'U_q^- \cong V_q$. Then $\Phi^\ast$ commutes with the actions $\tilde{e}_j^{(n)}$ on $V_q^\ast$ and $\tilde{e}_j^{(n)}$ on $A'(U_q^-)^\ast$.

### 2. KLR Algebras

**2.1.** Let $X = (I, (\ ))$ be a Cartan datum, and $\sigma$ an admissible automorphism on $X$. Assume that $k$ is an algebraically closed field. We define a KLR algebra associated to $(X, \sigma)$ as follows. For each $i, i' \in I$, choose a polynomial $Q_{i,i'}(u, v) \in k[u, v]$ satisfying the following properties;

(i) $Q_{i,i}(u, v) = 0$,

(ii) $Q_{i,i'}(u, v)$ is written as $Q_{i,i'}(u, v) = \sum_{p, q} t_{ii',pq}u^p v^q$, where the coefficients $t_{ii',pq} \in k$ satisfy the condition

\[(\alpha_i, \alpha_{i'})p + (\alpha_{i'}, \alpha_i)q = -2(\alpha_i, \alpha_{i'}) \quad \text{if } t_{ii',pq} \neq 0.\]

Moreover, $t_{ii',-a_{i},0} = t_{ii',0,-a_{i'}} = 0$.

(iii) $Q_{i,i'}(u, v) = Q_{i',i}(v, u)$,

(iv) $Q_{\sigma(i),\sigma(i')}(u, v) = Q_{i,i'}(u, v)$.

For $\beta = \sum_{i \in I} n_i \alpha_i \in Q_+$, we set $|\beta| = \sum n_i$. For $\beta \in Q_+$ such that $|\beta| = n$, define $I^\beta$ by

\[I^\beta = \{(i_1, \ldots, i_n) \in I^n \mid \sum_{1 \leq k \leq n} \alpha_i = \beta\}.\]
The KLR algebra $R(\beta)$ is an associative $k$-algebra defined by the generators, $e(\nu)$ ($\nu = (\nu_1, \ldots, \nu_n) \in I^\beta$), $x_k$ ($1 \leq k \leq n$), $\tau_k$ ($1 \leq k < n$), with relations

1. $e(\nu)e(\nu') = \delta_{\nu, \nu'}e(\nu)$, $\sum_{\nu \in I^\beta} e(\nu) = 1$,
2. $x_k x_l = x_l x_k$, $x_k e(\nu) = e(\nu)x_k$,
3. $\tau_k e(\nu) = e(s_k \nu) \tau_k$, $\tau_k \tau_l = \tau_l \tau_k$, if $|k - l| > 1$,
4. $\tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu)$,
5. $e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}
6. $(\tau_k x_l - x_{s_k} \tau_k)e(\nu) = \begin{cases} Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ x_k - x_{k+2} & \text{otherwise}. \end{cases}$

(Here $s_k$ ($1 \leq k < n$) is a transvection $(k, k+1)$ in the symmetric group $S_n$. $S_n$ acts on $I^n$ by $w : (\nu_1, \ldots, \nu_n) \mapsto (\nu_{w^{-1}(1)}, \ldots, \nu_{w^{-1}(n)})$.)

We define the algebra $R_n = \bigoplus_{[\beta] = n} R(\beta)$, and the algebra $R$ by

$R = \bigoplus_{n \geq 0} R_n = \bigoplus_{\beta \in Q_+} R(\beta)$. 

Note that the condition (iv) is used in considering the $\sigma$-setup for KLR algebras. The discussion until 2.9 is independent of $\sigma$, hence the condition (iv) is redundant.

The algebra $R(\beta)$ is a $\mathbb{Z}$-graded algebra, where the degree is defined, for $\nu = (\nu_1, \ldots, \nu_n) \in I^\beta$, by

$\deg e(\nu) = 0$, $\deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k})$, $\deg \tau_k e(\nu) = -(\alpha_{\nu_k}, \alpha_{\nu_{k+1}})$.

For a graded $R(\beta)$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, the grading shift by $-1$ is denoted by $qM$, namely $(qM)_i = M_{i-1}$. (Here $q$ is the indeterminate given in Section 1). There exists an anti-involution $\psi : R(\beta) \to R(\beta)$, which leaves all the generators $e(\nu), x_k$ and $\tau_k$ invariant.

2.2 For $m, n \in \mathbb{N}$, there exists a $k$-bilinear map $R_m \times R_n \to R_{m+n}$ defined by

$(x_k, 1) \mapsto x_k$, $(\tau_k, 1) \mapsto \tau_k$, $(1, x_k) \mapsto x_{m+k}$, $(1, \tau_k) \mapsto \tau_{m+k}$,
$(e(\nu), e(\nu')) \mapsto e(\nu, \nu')$, 

where for $\nu \in I^m, \nu' \in I^n$, we write their juxtaposition by $(\nu, \nu') \in I^{m+n}$. This defines an injective $k$-algebra homomorphism $R_m \otimes R_n \to R_{m+n}$. By this map, $R_{m+n}$ has a structure of a right $R_m \otimes R_n$-module.
For an \( R_m \)-module \( M \) and an \( R_n \)-module \( N \), we define an \( R_{m+n} \)-module \( M \circ N \) by

\[
M \circ N = R_{m+n} \otimes_{R_m \circ R_n} M \otimes N.
\]

\( M \circ N \) is called the convolution product of \( M \) and \( N \). Assume that \( M \) is an \( R(\beta) \) module, and \( N \) is an \( R(\gamma) \)-module with \(|\beta| = m, |\gamma| = n\). Then \( M \) is an \( R_m \)-module by the projection \( R_m \to R(\beta) \), and similarly, \( N \) is an \( R_n \)-module. The \( R_{m+n} \)-module \( M \circ N \) has the property that \( e(\beta + \gamma)(M \circ N) = M \circ N \), hence \( M \circ N \) is regarded as an \( R(\beta + \gamma) \)-module. The \( R(\beta + \gamma) \)-module \( M \circ N \) is defined directly as

\[
M \circ N = R(\beta + \gamma)e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} M \otimes N,
\]

where

\[
e(\beta, \gamma) = \sum_{\nu, \nu' \in \mathbb{Z}^n} e(\nu, \nu').
\]

Let \( \beta, \gamma \in Q_+ \). For an \( R(\beta + \gamma) \)-module \( M \), \( e(\beta, \gamma)M \) has a natural structure of \( R(\beta) \otimes R(\gamma) \)-module.

2.3. Let \( R(\beta) \)-Mod be the abelian category of graded \( R(\beta) \)-modules. Let \( R(\beta) \)-gm be the full subcategory of \( R(\beta) \)-Mod consisting of finite dimensional graded \( R(\beta) \)-modules, and \( R(\beta) \)-gp the full subcategory of \( R(\beta) \)-Mod consisting of finitely generated graded projective \( R(\beta) \)-modules. Thus \( R(\beta) \)-gm is an abelian category, and \( R(\beta) \)-gp is an additive category. Let \( K_{gm}(\beta) \) be the Grothendieck group of the category \( R(\beta) \)-gm, and \( K_{gp}(\beta) \) the Grothendieck group of the category \( R(\beta) \)-gp. Set

\[
(2.3.1) \quad K_{gm} = \bigoplus_{\beta \in Q_+} K_{gm}(\beta), \quad K_{gp} = \bigoplus_{\beta \in Q_+} K_{gp}(\beta).
\]

By the convolution product, \( K_{gm} \) and \( K_{gp} \) are equipped with structures of associative algebras over \( \mathbb{Z} \). The shift operation \( M \mapsto qM \) on the graded module \( M \) induces an action of \( A \) on the Grothendieck groups. Thus \( K_{gm} \) and \( K_{gp} \) have the structure of \( A \)-algebras.

2.4. Let \( P \) be a finitely generated projective \( R(\beta) \)-module. We define a dual module \( DP \) by

\[
(2.4.1) \quad DP = \text{Hom}_{R(\beta)}(P, R(\beta)) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{R(\beta)}(q^n P, R(\beta))_0,
\]

where \( \text{Hom}_{R(\beta)}(-, -)_0 \) is the space of degree preserving homomorphisms. Hence \( DP \) is a graded \( k \)-vector space. \( R(\beta) \) acts on \( DP \) by

\[
(2.4.2) \quad x(\lambda)(m) = \lambda(\psi(x)m)
\]

for \( x \in R(\beta), \lambda \in DP, m \in P \). Then \( DP \in R(\beta) \)-gp.
Let $M$ be a finite dimensional $R(\beta)$-module. Define a dual $DM$ by

\begin{equation}
DM = \text{Hom}_k(M, k) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_k(q^nM, k)_0.
\end{equation}

Similarly to (2.4.2), $DM$ has a structure of $R(\beta)$-module, and $DM \in R(\beta)\gm$.

Thus $D$ gives a contravariant functor $R(\beta)\gm \to R(\beta)\gm$ or $R(\beta)\gm \to R(\beta)\gm$ such that $D^2 \cong \text{Id}$.

The object $M$ in $R(\beta)\gm$ or in $R(\beta)\gm$ is said to be self-dual if $DM \cong M$.

For $\beta \in Q_+$, let $B_\beta$ be the set of $[P]$ in $K_{\gm}(\beta)$, where $P$ runs over self-dual finitely generated projective indecomposable $R(\beta)$-modules. Let $B_\beta^*$ be the set of $[L]$ in $K_{\gm}(\beta)$, where $L$ runs over self-dual finite dimensional simple $R(\beta)$-modules. Then $B_\beta$ gives an $A$-basis of $K_{\gp}(\beta)$, and $B_\beta^*$ gives an $A$-basis of $K_{\gm}(\beta)$. We set $B = \bigsqcup_{\beta \in Q_+} B_\beta$, and $B^* = \bigsqcup_{\beta \in Q_+} B_\beta^*$.

2.5. For any $P \in R(\beta)\gp$, $M \in R(\beta)\gm$, we define

\begin{equation}
\langle [P], [M] \rangle = q\dim_k(P^\vee \otimes_{R(\beta)} M) = \sum_{d \in \mathbb{Z}} \dim_k(P^\vee \otimes_{R(\beta)} M)_d q^d.
\end{equation}

Then this defines a $A$-bilinear map $K_{\gp}(\beta) \times K_{\gm}(\beta) \to A$. Since $k$ is algebraically closed, an irreducible module $L$ is absolutely irreducible. Then for a finitely generated projective indecomposable module $P$ and a finite dimensional simple module $L$, we have

\begin{equation}
\langle [P], [L] \rangle = \begin{cases} 
1 & \text{if } P \text{ is the projective cover of } DL, \\
0 & \text{otherwise.}
\end{cases}
\end{equation}

Thus $\langle , \rangle : K_{\gp}(\beta) \times K_{\gm}(\beta) \to A$ gives a perfect pairing on $A$, and $B_\beta$ and $B_\beta^*$ are dual to each other.

On the other hand, for any $P, Q \in R(\beta)\gp$, we define

\begin{equation}
\langle [P], [Q] \rangle = \sum_{d \in \mathbb{Z}} \dim(P^\vee \otimes_{R(\beta)} Q)_d q^d.
\end{equation}

This gives a well-defined symmetric bilinear form $\langle , \rangle : K_{\gp}(\beta) \times K_{\gp}(\beta) \to \mathbb{Z}((q))$.

2.6. For $i \in I$, let $L_i = k$ be the (graded) simple $R(\alpha_i)$-module, and $P_i$ the projective cover of $L_i$. Then $P_i = k[x_i]e(i) = R(\alpha_i)$. For any $n \geq 1$, set

\begin{equation}
L_i^{(n)} = q_i^{(n)} L_i^{(n)} , \quad P_i^{(n)} = q_i^{(n)} P_i^{(n)},
\end{equation}

where $L_i^{(n)} = L_i \circ \cdots \circ L_i$, and similarly for $P_i^{(n)}$. Then $L_i^{(n)}$ is a simple $R(n\alpha_i)$-module lying in $R(n\alpha_i)\gm$, and $P_i^{(n)} \in R(n\alpha_i)\gp$ is the projective cover of $L_i^{(n)}$. 
Assume that $\beta, \in Q_+$ and $n \in \mathbb{N}$. We further assume that $\beta - n\alpha_i \in Q_+$. We define a functor $E_i^{(n)} : R(\beta) \text{-gm} \to R(\beta - n\alpha_i) \text{-gm}$ by

$$M \mapsto \left(R(\beta - n\alpha_i) \circ P_i^{(n)}\right)_{\psi} \otimes_{R(\beta)} M.$$  

(2.6.2)

If $n = 1$, we write $E_i^{(n)}$ as $E_i$. $E_i^{(n)} M$ is also written as

$$E_i^{(n)} M = \left(R(\beta - n\alpha_i) \otimes_k (P_i^{(n)})_{\psi} \otimes_{R(\beta - n\alpha_i) \otimes R(n\alpha_i)} M\right)$$

$$= \left(R(\beta - n\alpha_i) \otimes_k (P_i^{(n)})_{\psi} \otimes_{R(\beta - n\alpha_i) \otimes R(n\alpha_i)} e(\beta - n\alpha_i, n\alpha_i) M\right).$$

The action of $R(\beta - n\alpha_i) \otimes R(n\alpha_i)$ on $M$ is equal to the $R(\beta - n\alpha_i) \otimes R(n\alpha_i)$-module $e(\beta - n\alpha_i, n\alpha_i) M$, and it acts as 0 outside. Since $P_i = R(\alpha_i)$, we have $R(\beta - \alpha_i) \circ P_i = R(\beta)e(\beta - \alpha_i, \alpha_i)$. Hence

$$E_i M = e(\beta - \alpha_i, \alpha_i)R(\beta) \otimes_{R(\beta)} M = e(\beta - \alpha_i, \alpha_i)M.$$  

(2.6.3)

$E_i^{(n)}$ is an exact functor, and it induces an $A$-homomorphism

$$e_i^{(n)} : K_{gm}(\beta) \to K_{gm}(\beta - n\alpha_i).$$

For $\beta \in Q_+, n \in \mathbb{N}$, define an additive functor $F_i^{(n)} : R(\beta) \text{-gp} \to R(\beta + n\alpha_i) \text{-gp}$ by

$$P \mapsto P \circ P_i^{(n)}.$$  

(2.6.4)

The functor $F_i^{(n)}$ induces an $A$-homomorphism on the Grothendieck groups,

$$f_i^{(n)} : K_{gp}(\beta) \to K_{gp}(\beta + n\alpha_i).$$

It is known that the operators $f_i^{(n)}$ on $K_{gp}$, and $e_i^{(n)}$ on $K_{gm}$ satisfy the following adjunction relation with respect to the pairing $\langle \ , \ \rangle$. (The proof is also given as the special case of Lemma 3.10).

$$\langle f_i^{(n)}[P], [M] \rangle = \langle [P], e_i^{(n)}[M] \rangle.$$  

(2.6.5)

The following categorification theorem was proved by Khovanov-Lauda [KL], and Rouquier [R1].

**Theorem 2.7** ([KL], [R1]). There exists an isomorphism of $A$-algebras

$$\tilde{\Theta}_0 : A U_q^- \cong K_{gp}.$$  

(2.7.1)

which maps $f_i^{(n)}$ to $[P_i^{(n)}]$. Moreover, $\tilde{\Theta}_0$ is an isometry.
2.8. Let \([k]\) be the isomorphism class of the trivial representation \(k\) of \(R(0)\), which gives a basis of \(K_{\text{gm}}(0)\) and of \(K_{\text{gp}}(0)\). Let \(1^* \in A(U_q^-)^*_0\) be the dual basis of \(1 \in A(U_q^-)\). As a corollary to Theorem 2.7, we have the following result.

Proposition 2.9. (i) There exists an isomorphism of \(A\)-modules (actually, an anti-algebra isomorphism),

\[ \tilde{\Theta} : A U_q^- \simeq K_{\text{gp}}, \]

which maps \(1 \in A(U_q^-)\) to \([k] \in K_{\text{gp}}(0)\). \(\tilde{\Theta}\) commutes with the actions of \(f_i^{(n)}\). \(\tilde{\Theta}\) is an isometry with respect to the inner product \((\cdot, \cdot)\) on \(A U_q^-\) and on \(K_{\text{gp}}\).

(ii) There exists an isomorphism of \(A\)-modules,

\[ \tilde{\Theta}^* : K_{\text{gm}} \simeq A(U_q^-)^* \]

which maps \([k]\) to \(1^*\). Moreover, \(\tilde{\Theta}^*\) commutes with the actions of \(e_i^{(n)}\).

Proof. We define \(\tilde{\Theta} : A U_q^- \to K_{\text{gp}}\) by \(\tilde{\Theta} = \tilde{\Theta}_0 \circ \ast\), where \(\ast : A U_q \to A U_q^-\) is the anti-involution on \(A U_q^-\) (see 1.2). Since the action of \(f_i^{(n)}\) on \(K_{\text{gp}}\) is defined by the functor \(F_i^{(n)} : R(\beta)\text{-gp} \to R(\beta + n\alpha_i)\text{-gp}\), \(\tilde{\Theta}\) commutes with \(f_i^{(n)}\). Since the inner product on \(U_q^-\) is invariant under the action of \(\ast\), \(\tilde{\Theta}\) is an isometry. Hence (i) holds. By using \(\langle \cdot , \cdot \rangle\), we have an isomorphism \(K_{\text{gm}}(\beta) \simeq \text{Hom}_A(K_{\text{gp}}(\beta), A)\). Thus one can define a map \(\tilde{\Theta}^*\) as the transpose of \(\tilde{\Theta}\). Then by (2.6.5), the map \(\tilde{\Theta}^*\) commutes with \(e_i^{(n)}\). Hence (ii) holds. \(\Box\)

2.10. From now on, we consider the diagram automorphism \(\sigma : I \to I\). \(\sigma\) acts on \(\nu = (\nu_1, \ldots, \nu_n) \in I^n\) by \(\sigma \nu = (\sigma(\nu_1), \ldots, \sigma(\nu_n))\). In view of the condition (iv) in 2.1, the assignment \(e(\nu) \mapsto e(\nu(\nu)), x_i \mapsto x_i, \tau_i \mapsto \tau_i\) induces an algebra isomorphism \(R(\beta) \simeq R(\sigma(\beta))\). Thus \(\sigma\) gives a functor \(\sigma^* : R(\sigma(\beta))\text{-Mod} \to R(\beta)\text{-Mod}\), where, for an \(R(\sigma(\beta))\)-module \(M, \sigma^* M\) is the pull-back of \(M\) by the homomorphism \(R(\beta) \to R(\sigma(\beta))\). We consider the functor \(\tau = (\sigma^{-1})^* : R(\beta)\text{-Mod} \to R(\sigma(\beta))\text{-Mod}\) for any \(\beta \in Q_+\). Then \(\tau\) gives functors \(R(\beta)\text{-gm} \to R(\sigma(\beta))\text{-gm}\), and \(R(\beta)\text{-gp} \to R(\sigma(\beta))\text{-gp}\). Thus \(\tau\) induces automorphisms on \(K_{\text{gp}}\) and on \(K_{\text{gm}}\), which we also denote by \(\tau\). Note that \(\tau\) permutes the bases \(B\) and \(B^*\).

Since \(\tau(L_i^{(n)}) = L_{\sigma(i)}^{(n)}, \tau(P_i^{(n)}) = P_{\sigma(i)}^{(n)}\), we see that

\[ (2.10.1) \quad \tau \circ f_i^{(n)} = f_{\sigma(i)}^{(n)} \circ \tau \quad \text{on} \quad K_{\text{gp}}, \quad \tau \circ e_i^{(n)} = e_{\sigma(i)}^{(n)} \circ \tau \quad \text{on} \quad K_{\text{gm}}. \]

It follows that the map \(\tilde{\Theta} : A U_q^- \simeq K_{\text{gp}}\) is compatible with the actions of \(\sigma\) and \(\tau\). Hence the map \(\tilde{\Theta}^* : K_{\text{gm}} \simeq A(U_q^-)^*\) is also compatible with the actions of \(\tau\) and \(\sigma\).

2.11. We follow the setup in 1.6. Hence we assume that \(n\) is a power of a prime number \(\ell\), and set \(A' = F[q, q^{-1}]\). We define \(A' K_{\text{gp}} = A' \otimes_A K_{\text{gp}}, A' K_{\text{gm}} = \)
\[ A' \otimes_{A} K_{gm}. \] Then \( B \) and \( B^* \) give the \( A' \)-bases of \( A'K_{gp} \), and \( A'K_{gm} \), respectively. The pairing \( \langle \langle , \rangle \rangle \) is extended to a perfect \( A' \)-pairing \( A'K_{gp} \times A'K_{gm} \to A' \).

Let \( A'K_{gp}^\tau \) be the set of \( \tau \)-fixed elements in \( A'K_{gp} \). Then \( A'K_{gp}^\tau \) is an \( A' \)-submodule of \( A'K_{gp} \), and \( \{O(b) \mid b \in B\} \) gives an \( A' \)-basis of \( A'K_{gp}^\tau \). Let \( J \) be the subset of \( A'K_{gp}^\tau \) consisting of \( O(x) \) for \( x \in A'K_{gm} \) such that \( O(x) \not= x \). Then \( J \) is an \( A' \)-submodule of \( A'K_{gp}^\tau \). We define the quotient module \( V_q \) by

\[(2.11.1) \quad V_q = A'K_{gp}^\tau / J.\]

Since the isomorphism \( \tilde{\Theta} : AU_q \cong K_{gp} \) is compatible with the actions of \( \sigma \) and \( \tau \) by 2.10, it induces an isomorphism \( A'U_q^{\sigma} \cong A'K_{gp}^\tau \), which maps \( J \) onto \( J \). Thus \( \tilde{\Theta} \) induces an isomorphism

\[(2.11.2) \quad \Theta : V_q \cong V_q.\]

For each \( j \in J \), let \( j = \{i_1, \ldots, i_t\} \). We define a functor \( \tilde{F}^{(n)}_j : R \cdot gp \to R \cdot gp \) by \( \tilde{F}^{(n)}_j = F^{(n)}_{i_1} \cdots F^{(n)}_{i_t} \). This induces an \( A' \)-homomorphism \( \tilde{f}^{(n)}_j = f^{(n)}_{i_1} \cdots f^{(n)}_{i_t} \) on \( A'K_{gp}^\tau \). Since \( f^{(n)}_j \) commute each other by Theorem 2.7, and \( \tau \) permutes \( f^{(n)}_j \), \( \tilde{f}^{(n)}_j \) commutes with \( \tau \), and acts on \( V_q \). The map \( \Theta : V_q \to V_q \) is compatible with the action of \( \tilde{f}^{(n)}_j \).

Next consider the \( A' \)-submodule \( A'K_{gm}^\tau \) consisting of \( \tau \)-fixed elements. We define \( J^* \) as the \( A' \)-submodule of \( A'K_{gm}^\tau \) consisting of \( O(x) \) for \( x \in A'K_{gm} \) such that \( O(x) \not= x \). We define a quotient module \( V_q^* \) by

\[(2.11.3) \quad V_q^* = A'K_{gm}^\tau / J^*.\]

Similarly to the case of \( \tilde{\Theta} \), the isomorphism \( \tilde{\Theta}^* : K_{gm} \cong A(\tilde{U}_q)^* \) induces an isomorphism \( A'K_{gm}^\tau \cong A'(\tilde{U}_q)^{*, \sigma} \), which maps \( J^* \) onto \( J^* \). Thus \( \tilde{\Theta}^* \) induces an isomorphism

\[(2.11.4) \quad \Theta^* : V_q^* \cong V_q^*.\]

For \( j \in J \), we define a functor \( \tilde{E}^{(n)}_j : R \cdot gm \to R \cdot gm \) by \( \tilde{E}^{(n)}_j = E^{(n)}_{i_1} \cdots E^{(n)}_{i_t} \). This induces an \( A' \)-homomorphism \( \tilde{e}^{(n)}_j = e^{(n)}_{i_1} \cdots e^{(n)}_{i_t} \), commuting with \( \tau \). Thus, it induces an action of \( \tilde{e}^{(n)}_j \) on \( V_q^* \). The map \( \Theta^* \) commutes with the action of \( \tilde{e}^{(n)}_j \).

Let \( B_n \) be the image of \( B^* \) onto \( V_q \), and \( B_n^* \) the image of \( B^{*, \tau} \) onto \( V_q^* \). Then \( B_n \) gives a basis of \( V_q \), and \( B_n^* \) gives a basis of \( V_q^* \), which are dual to each other.

Summing up the above discussion, we obtain the following.

**Proposition 2.12.** There exist isomorphisms of \( A' \)-modules,

\[ \Theta : V_q \cong V_q, \quad \Theta^* : V_q^* \cong V_q^*. \]
The map $\Theta$ commutes with the actions of $\tilde{f}_j^{(n)}$, and the map $\Theta^*$ commutes with the actions of $\tilde{e}_j^{(n)}$.

3. Periodic functors associated to KLR algebras

3.1. First we recall the general theory of periodic functors due to [L2, 11.1]. Let $\mathcal{C}$ be a linear category over a field $k$, namely, a category where the space of morphisms between any two objects has the structure of $k$-vector space such that the composition of morphisms is bilinear, and that finite direct sums exist. A functor between two linear categories is said to be linear if it preserves the $k$-vector space structure.

A linear functor $\sigma^* : \mathcal{C} \to \mathcal{C}$ is said to be periodic if there exists $n \geq 1$ such that $(\sigma^*)^n = \text{id}_\mathcal{C}$. Assume that $\sigma^*$ is a periodic functor on $\mathcal{C}$. We define a new category $\tilde{\mathcal{C}}$ as follows. The objects of $\tilde{\mathcal{C}}$ are the pairs $(A, \phi)$, where $A \in \mathcal{C}$, and $\phi : \sigma^* A \cong A$ is an isomorphism in $\mathcal{C}$ such that the composition satisfies the relation

$$\phi \circ \sigma^* \phi \circ \cdots \circ (\sigma^*)^{n-1} \phi = \text{id}_A.$$

(3.1.1)

A morphism from $(A, \phi)$ to $(A', \phi')$ in $\tilde{\mathcal{C}}$ is a morphism $f : A \to A'$ in $\mathcal{C}$ satisfying the following commutative diagram,

$$\begin{array}{ccc}
\sigma^* A & \xrightarrow{\sigma^* f} & \sigma^* A' \\
\phi & & \phi' \\
A & \xrightarrow{f} & A'.
\end{array}$$

(3.1.2)

The category $\tilde{\mathcal{C}}$ is a linear category, hence is an additive category. $\tilde{\mathcal{C}}$ is called the category associated to the periodic functor $\sigma^*$ on $\mathcal{C}$

3.2. Let $\tilde{\mathcal{C}}$ be as in 3.1. Let $n$ be the smallest integer such that $(\sigma^*)^n = \text{id}_\mathcal{C}$. Assume that $k$ is an algebraically closed field such that $\text{ch} k$ does not divide $n$. Let $\zeta_n$ be a primitive $n$-th root of unity in $\mathbb{C}$, and fix, once for all, a ring homomorphism $\mathbb{Z}[\zeta_n] \to k$, which maps $\zeta_n$ to a primitive $n$-th root of unity in $k$. If $(A, \phi) \in \tilde{\mathcal{C}}$, then $(A, \zeta_n \phi) \in \tilde{\mathcal{C}}$, which we denote by $\zeta_n(A, \phi)$.

An object $(A, \phi) \in \tilde{\mathcal{C}}$ is said to be traceless if there exists an object $B \in \mathcal{C}$, an integer $t \geq 2$ which is a divisor of $n$ such that $(\sigma^*)^t B \cong B$, and an isomorphism

$$A \simeq B \oplus \sigma^* B \oplus \cdots \oplus (\sigma^*)^{t-1} B,$$

where, $\phi : \sigma^* A \cong A$ is given by the identity maps $(\sigma^*)^k B \cong (\sigma^*)^k B$ $(1 \leq k \leq t - 1)$ and an isomorphism $(\sigma^*)^t B \cong B$ on each direct summand, under the above isomorphism.
For the category $\tilde{C}$, $K(\tilde{C})$ is defined as a $\mathbb{Z}[\zeta_n]$-module generated by symbols $[A, \phi]$ associated to the isomorphism class of the object $(A, \phi) \in \tilde{C}$, subject to the relations

(i) $[X] = [X'] + [X'']$ if $X \cong X' \oplus X''$,
(ii) $[A, \zeta_n \phi] = \zeta_n [A, \phi]$,
(iii) $[X] = 0$ if $X$ is traceless.

In the case where $\tilde{C}$ is an abelian category, the condition (i) is replaced by the condition

(i') $[X] = [X'] + [X'']$ if there exists a short exact sequence

$$0 \to X' \to X \to X'' \to 0.$$  

$K(\tilde{C})$ is an analogue of the Grothendieck group. In fact, if $n = 1$, $\tilde{C} = C$, then $K(\tilde{C})$ coincides with the Grothendieck group $K(C)$. We call $K(\tilde{C})$ the Grothendieck group of $\tilde{C}$.

### 3.3.

We consider the KLR algebra $R = \bigoplus_{\beta \in Q_+} R(\beta)$ associated to $(X, \sigma)$ as in 2.1. Let $n$ be the order of $\sigma$, and assume that $\text{ch} k$ does not divide $n$ (here we don’t give any restriction on $n$). Assume that $\beta$ is $\sigma$-stable. Then by 2.10, we obtain a functor $\sigma^* : R(\beta) \text{-Mod} \to R(\beta) \text{-Mod}$. We apply the discussion in 3.1, and let $\mathcal{C}_\beta$ be the category $\tilde{C}$ associated to the periodic functor $\sigma^*$ on $C = R(\beta) \text{-Mod}$. Then as remarked in [M, Lemma 2.2], the category $\mathcal{C}_\beta$ is equivalent to the category of graded representations of the algebra

$$R(\beta)^\#(\mathbb{Z}/n\mathbb{Z}) = R(\beta) \otimes_k k[\mathbb{Z}/n\mathbb{Z}],$$

where $k[\mathbb{Z}/n\mathbb{Z}]$ is the group algebra of $\mathbb{Z}/n\mathbb{Z}$, and $k[\mathbb{Z}/n\mathbb{Z}]$ acts on $R(\beta)$ through the action of $\sigma^*$. Hence $\mathcal{C}_\beta$ is an abelian category. We denote by $\mathcal{P}_\beta$ the full subcategory of $\mathcal{C}_\beta$ consisting of finitely generated projective objects in $\mathcal{C}_\beta$. Let $\mathcal{L}_\beta$ be the full subcategory of $\mathcal{C}_\beta$ consisting of $(M, \phi)$ such that $M$ is a finite dimensional $R(\beta)$-module. Then $\mathcal{P}_\beta$ is an additive category and $\mathcal{L}_\beta$ is an abelian category.

Let $K(\mathcal{P}_\beta)$ (resp. $K(\mathcal{L}_\beta)$) be the Grothendieck group of the category $\mathcal{P}_\beta$ (resp. $\mathcal{L}_\beta$) as defined in 3.2. $K(\mathcal{P}_\beta)$ and $K(\mathcal{L}_\beta)$ have structures of $\mathbb{Z}[\zeta_n, q, q^{-1}]$-modules, where $q$ acts as the grading shift as in the case of $K_{\text{gp}}(\beta)$ or $K_{\text{gm}}(\beta)$. We set

$$K(\mathcal{P}) = \bigoplus_{\beta \in Q_+} K(\mathcal{P}_\beta), \quad K(\mathcal{L}) = \bigoplus_{\beta \in Q_+} K(\mathcal{L}_\beta).$$

### 3.4.

The contravariant functor $D$ on $R(\beta) \text{-gm}$ or on $R(\beta) \text{-gp}$ given in 2.4 can be extended to the case of $\mathcal{L}_\beta$ or $\mathcal{P}_\beta$ as follows. For $(M, \phi)$ in $\mathcal{P}_\beta$ or in $\mathcal{L}_\beta$, we define the dual object by

$$D(M, \phi) = (DM, D(\phi)^{-1}),$$
where $D(\phi) : DM \cong D(\sigma^*M) = \sigma^*DM$. It is clear that $D$ gives a functor $\mathcal{L}_\beta \to \mathcal{L}_\beta$. It was shown in [M, 5] that $D$ gives a functor $\mathcal{P}_\beta \to \mathcal{P}_\beta$.

$(M, \phi)$ in $\mathcal{L}_\beta$ or in $\mathcal{P}_\beta$ is said to be self-dual if $D(M, \phi) \simeq (M, \phi)$ in that category. Note that if $(M, \phi)$ is a self-dual object, then $\phi$ is uniquely determined by $M$ in the case where $n$ is odd, and unique up to $\pm 1$ in the case where $n$ is even.

In particular, $(k, \text{id}) \in \mathcal{L}_0$ is a unique self-dual simple object if $n$ is odd, and there exist two self-dual simple object $(k, \pm \text{id}) \in \mathcal{L}_0$. We write $[\pm 1] = [k, \pm \text{id}] \in K(\mathcal{L}_0)$.

Let $\mathcal{B}_\beta$ be the set of elements in $K(\mathcal{L}_\beta)$ consisting of isomorphism classes of self-dual objects $(M, \phi)$ such that $M$ is a finite dimensional simple $R(\beta)$-module. It was proved in [M, Thm.10.8], by using the crystal structure of $\mathcal{B}_\beta^*$, that there exists a $Z[\zeta_n, q, q^{-1}]$-basis $\mathcal{B}_\beta^*$ of $K(\mathcal{L}_\beta)$ such that $\mathcal{B}_\beta^* \subset \mathcal{B}_\beta^*$ and that $[1] \in \mathcal{B}_0^*$. Also there exists a $Z[\zeta_n, q, q^{-1}]$-basis $\mathcal{B}_\beta$ of $K(\mathcal{P}_\beta)$ consisting of isomorphism classes of finitely generated projective indecomposable self-dual objects in $\mathcal{P}_\beta$ which are projective cover of elements in $\mathcal{B}_\beta^*$. We set $\mathcal{B} = \bigsqcup_{\beta \in Q_+} \mathcal{B}_\beta$, and $\mathcal{B}^* = \bigsqcup_{\beta \in Q_+} \mathcal{B}_\beta^*$.

### 3.5

We extend the pairing $\langle \ , \ \rangle : K_{\text{gp}}(\beta) \times K_{\text{gm}}(\beta) \to A$ given in 2.5 to our situation. We define a bilinear map $\langle \ , \ \rangle : K(\mathcal{P}_\beta) \times K(\mathcal{L}_\beta) \to Z[\zeta_n, q, q^{-1}]$ by

\begin{equation}
\langle [P, \phi], [M, \phi'] \rangle = \sum_{d \in Z} \text{Tr} (\phi \otimes \phi', (P^\psi \otimes_{R(\beta)} M)_d) q^d.
\end{equation}

Here for given $\phi : \sigma^*P \cong P, \phi' : \sigma^*M \cong M$, $\phi \otimes \phi'$ is the induced map $\sigma^*P^\psi \otimes \sigma^*M \to P^\psi \otimes M$. Since $\sigma^*P^\psi \otimes \sigma^*M$ is canonically isomorphic to $P^\psi \otimes M$, $\phi \otimes \phi'$ is regarded as a linear transformation on $P^\psi \otimes M$.

Note that in [M, Lemma 3.1], instead of $\langle \ , \ \rangle$, a sesqui-linear form $\langle \ , \ \rangle$ is used, which is defined by

\begin{equation}
\langle [P, \phi], [M, \phi'] \rangle = \sum_{d \in Z} \text{Tr}(\sigma_{\phi, \phi'}, \text{Hom}_{R(\beta)}(P, M)_d) q^d.
\end{equation}

The relation between these forms is given by the isomorphism of graded vector spaces,

\begin{equation}
\text{Hom}_{R(\beta)}(P, M) \simeq D P^\psi \otimes_{R(\beta)} M.
\end{equation}

By [M, Thm.11.1], we have the following.

\begin{equation}
\text{The pairing } \langle \ , \ \rangle : K(\mathcal{P}_\beta) \times K(\mathcal{L}_\beta) \to Z[\zeta_n, q, q^{-1}] \text{ gives a perfect pairing. } \mathcal{B} \text{ and } \mathcal{B}^* \text{ are dual to each other.}
\end{equation}

The symmetric bilinear form $(\ , \ ) : K_{\text{gp}}(\beta) \times K_{\text{gp}}(\beta) \to Z((q))$ is also extended to our situation as follows. For $(P, \phi), (Q, \phi') \in \mathcal{P}_\beta$, set

\begin{equation}
\langle [P, \phi], [Q, \phi'] \rangle = \sum_{d \in Z} \text{Tr}(\phi \otimes \phi', (P^\psi \otimes_{R(\beta)} Q)_d) q^d.
\end{equation}
Here $\phi \otimes \phi'$ gives an isomorphism $\sigma^*P^\psi \otimes_{R(\beta)} \sigma^*Q \simeq P^\psi \otimes_{R(\beta)} Q$. By using the canonical isomorphism $\sigma^*P^\psi \otimes_{R(\beta)} \sigma^*Q \simeq P^\psi \otimes_{R(\beta)} Q$, we regard $\phi \otimes \phi'$ as a linear transformation on $P^\psi \otimes_{R(\beta)} Q$. (3.5.3) induces a symmetric bilinear pairing $(\ , \ ) : K(\mathcal{P}_\beta) \times K(\mathcal{P}_\beta) \to \mathbb{Z}[[c]]((q))$.

3.6. We extend the convolution product defined in 2.2 to the case of $\mathcal{C}_\beta$. Take $(M, \phi) \in \mathcal{C}_\beta, (N, \phi') \in \mathcal{C}_\gamma$. Set

$$
(3.6.1) \quad (M, \phi) \circ (N, \phi') = (M \circ N, \phi \circ \phi'),
$$

where $\phi \circ \phi'$ is given by the composite of the map $\sigma^*M \circ \sigma^*N \to M \circ N$ and the canonical isomorphism $\sigma^*(M \circ N) \simeq \sigma^*M \circ \sigma^*N$.

For $\beta, \gamma \in Q^\alpha$, the automorphisms $\sigma$ on $R(\beta)$ and $R(\gamma)$ induces an automorphism $\sigma$ on $R(\beta) \otimes R(\gamma)$ by $\sigma(u \otimes w) = \sigma(u) \otimes \sigma(w)$. Thus one can consider the category of graded representations of $(R(\beta) \otimes R(\gamma))\sharp(\mathbb{Z}/n\mathbb{Z})$, which we denote by $\mathcal{C}_{\beta \gamma}$. We can define full subcategories $\mathcal{L}_{\beta \gamma}, \mathcal{P}_{\beta \gamma}$ of $\mathcal{C}_{\beta \gamma}$ similarly.

The convolution product (3.6.1) gives an induction functor $\text{Ind}_{\beta \gamma} : \mathcal{C}_{\beta \gamma} \to \mathcal{C}_{\beta + \gamma}$. On the other hand, for each $(M, \phi) \in \mathcal{C}_{\beta + \gamma}$, we define

$$
(3.6.2) \quad \text{Res}_{\beta \gamma}(M, \phi) = (e(\beta, \gamma)M, \phi'),
$$

where $\text{Res} M = e(\beta, \gamma)M$ is an $R(\beta) \otimes R(\gamma)$-module given in 2.2. We have the canonical isomorphism $\sigma^*(\text{Res} M) \simeq \text{Res}(\sigma^*M)$ since $e(\beta, \gamma)$ is $\sigma$-invariant, which induces the isomorphism $\phi' : \sigma^*(\text{Res} M) \simeq \text{Res} M$. Thus $\text{Res}_{\beta \gamma}(M, \phi)$ belongs to $\mathcal{C}_{\beta \gamma}$, and we obtain the restriction functor $\text{Res}_{\beta \gamma} : \mathcal{C}_{\beta + \gamma} \to \mathcal{C}_{\beta \gamma}$.

It is known by [M, Thm. 4.3] that the induction functor and the restriction functor form an adjoint pair of exact functors between $\mathcal{C}_{\beta \gamma}$ and $\mathcal{C}_{\beta + \gamma}$.

We have isomorphisms of $\mathbb{Z}[\zeta, q, q^{-1}]$-modules

$$
K(\mathcal{P}_\beta) \otimes_{\mathbb{Z}[\zeta, q^{\pm 1}]} K(\mathcal{P}_\gamma) \simeq K(\mathcal{P}_{\beta \gamma}), \quad K(\mathcal{L}_\beta) \otimes_{\mathbb{Z}[\zeta, q^{\pm 1}]} K(\mathcal{L}_\gamma) \simeq K(\mathcal{L}_{\beta \gamma}),
$$

by [M, Prop. 4.1], and the induction functors $\text{Ind}_{\beta \gamma}$ induces homomorphisms of Grothendieck groups,

$$
K(\mathcal{P}_\beta) \otimes_{\mathbb{Z}[\zeta, q^{\pm 1}]} K(\mathcal{P}_\gamma) \to K(\mathcal{P}_{\beta + \gamma}), \quad K(\mathcal{L}_\beta) \otimes_{\mathbb{Z}[\zeta, q^{\pm 1}]} K(\mathcal{L}_\gamma) \to K(\mathcal{L}_{\beta + \gamma}).
$$

It is shown that $K(\mathcal{P}_\gamma)$ and $K(\mathcal{L}_\gamma)$ turn out to be associative algebras over $\mathbb{Z}[\zeta, q^{\pm 1}]$ with respect to this product.

3.7. Let $j = \{i_1, \ldots, i_t\} \in J$, and set $\alpha_j = \alpha_{i_1} + \cdots + \alpha_{i_t}$, then $\alpha_j \in Q^\alpha$. We consider $R(\alpha_j)$. Let $\nu = (i_1, \ldots, i_t)$. Then $I^{\alpha_j} = \{w \nu \mid w \in S_j\}$, and in particular, for any $\nu' = (i_1', \ldots, i_t') \in I^{\alpha_j}, \nu_1', \ldots, \nu_t'$ are all distinct. Moreover, $Q_{i,j}(u, v) \in k^*$ since $(\alpha_j, \alpha_{i'}) = 0$ if $i \neq i' \in j$. Hence by the defining relations on the KLR algebra $R(\alpha_j)$ in 2.1, we have, for $1 \leq k < t$,

$$
(3.7.1) \quad \tau_k^2 \epsilon(\nu) \in k^*e(\nu), \quad \tau_k \epsilon \tau_k = x_{s_k(t)} \tau_k, \quad \tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}.
$$
In particular, $\tau_w$ is well-defined for $w \in S_t$. Let
\[
P[x_1, \ldots, x_t] = \bigoplus_{\nu' \in I^{r_j}} k[x_1, \ldots, x_t]e(\nu') = \bigoplus_{w \in S_t} k[x_1, \ldots, x_t]e(w\nu)
\]
be the polynomial ring of $R(\alpha_j)$. By the basis theorem, $R(\alpha_j)$ is a free $P[x_1, \ldots, x_t]$-module with basis $\{\tau_w \mid w \in S_t\}$, namely,
\[(3.7.2) \quad R(\alpha_j) = \bigoplus_{w \in S_t} P[x_1, \ldots, x_t]\tau_w.
\]
For any $f \in k[x_1, \ldots, x_t]$, we have $\tau_wf = r_w(f)\tau_w$, where $r_w \in k[S_t]$ is the element corresponding to $w \in S_t$. Thus $\tau_wP[x_1, \ldots, x_t] = P[x_1, \ldots, x_t]\tau_w$. The left multiplication of $\tau_w$ sends $P[x_1, \ldots, x_t]\tau_{w'}$ to $P[x_1, \ldots, x_t]\tau_{ww'}$.

Recall that $R(\alpha_{ij}) = P_{i_k} = k[x_{ij}]e(i_k)$ as $R(\alpha_{ik})$-modules. Then
\[
P_{i_1} \circ \cdots \circ P_{i_t} = R(\alpha_j) \otimes_{R(\alpha_{i_1}) \otimes \cdots \otimes R(\alpha_{i_t})} P_{i_1} \otimes \cdots \otimes P_{i_t}
\]
\[
\simeq \bigoplus_{w \in S_t} \tau_w k[x_1, \ldots, x_t]e(i_1, \ldots, i_t)
\]
\[
= \bigoplus_{w \in S_t} k[x_1, \ldots, x_t]\tau_we(i_1, \ldots, i_t).
\]

Set $P_j = P_{i_1} \circ \cdots \circ P_{i_t}$. Then $P_j$ is written as $P_j = \bigoplus_{w \in S_t} k[x_1, \ldots, x_t][w]$, where $[w] = \tau_w e(i_1, \ldots, i_t)$. Thus $P_j$ is a free $k[x_1, \ldots, x_t]$-module with basis $[w]$, where $e(\nu')$ acts on $[w]$ by 1 if $\nu' = w\nu$, and by 0 otherwise, $\tau_i$ acts on $[w]$ by $[s_i(w)]$.

We define a quotient module $L_j$ of $P_j$ by
\[
L_j = P_j/\langle x_k P_j \mid 1 \leq k \leq t \rangle.
\]
Then $L_j \simeq \bigoplus_{w \in S_t} k[w] \simeq L_{i_1} \circ \cdots \circ L_{i_t}$, where $x_k$ acts on $[w]$ as 0. Thus $P_j$ and $L_j$ are isomorphic to the ones defined in [M, 7]. $L_j$ is a simple $R(\alpha_j)$-module lying in $R(\alpha_j)$-$\mathbf{gm}$, and $P_j$ is the projective cover of $L_j$ lying in $R(\alpha_j)$-$\mathbf{gp}$.

We have $P_j = R(\alpha_j)e(\nu)$ with $\nu = (i_1, \ldots, i_t)$, and
\[(3.7.3) \quad R(\alpha_j) = \bigoplus_{\nu' \in I^{r_j}} R(\alpha_j)e(\nu') = \bigoplus_{\nu' \in I^{r_j}} P_{j, \nu'},
\]
where $P_{j, \nu'} = \bigoplus_{w \in S_t} k[x_1, \ldots, x_t]\tau_w e(\nu')$, which is isomorphic to $P_j$ as $R(\alpha_j)$-modules.

Since $\alpha_j$ is $\sigma$-stable, one can consider the category $\mathcal{C}_{\alpha_j}, \mathcal{L}_{\alpha_j}$ and $\mathcal{P}_{\alpha_j}$. We define $\phi_j : \sigma^* L_j \simeq L_j$ by $\phi_j[w] = [\sigma(w)]$ (here $[w]$ is identified with the permutation $(i'_1, \ldots, i'_t)$ of $(i_1, \ldots, i_t)$ on which $\sigma$ acts). We define $L(j) = (L_j, \phi_j)$, which is a simple object in $\mathcal{L}_{\alpha_j}$. We define $P(j)$ the projective cover of $L(j)$, which lies in $\mathcal{P}_{\alpha_j}$. $P(j)$ is given as $P(j) = (P_j, \phi_j)$, where $\phi_j : \sigma^* P_j \simeq P_j$ is the extension of $\phi_j : \sigma^* L_j \simeq L_j$. 
3.8. For $n \geq 1$, we define $L(j)^{(n)} \in \mathcal{L}_{\alpha j}$ by

$$L(j)^{(n)} = q_j^{(2)} L(j)^{on}. \quad (3.8.1)$$

By [M, Lemma 7.2], $L(j)^{(n)}$ is a self-dual simple object in $\mathcal{L}_{\alpha j}$. Note that $L(j)^{(n)} = (L_j^{(n)}, \phi_{nj})$, where $L_j^{(n)} = q_j^{(2)} L_j^{on}$, and $\phi_{nj} : \sigma^* L_j^{(n)} \cong L_j^{(n)}$ is induced from $\phi_j : \sigma^* L_j \cong L_j$. Let $P(j)^{(n)}$ be the projective cover of $L(j)^{(n)}$. Then

$$P(j)^{(n)} = q_j^{(2)} P(j)^{on}. \quad (3.8.2)$$

We have $P(j)^{(n)} = (P_j^{(n)}, \phi_{nj})$, where $P_j^{(n)} = q_j^{(2)} P_j^{on}$ is the projective cover of $L_j^{(n)}$, and $\phi_{nj} : \sigma^* P_j^{(n)} \cong P_j^{(n)}$ is the extension of $\phi_{nj} : \sigma^* L_j^{(n)} \cong L_j^{(n)}$.

3.9. Assume that $\beta \in Q^\alpha_+$ and $j \in J$. We define a functor $F_j^{(n)} : \mathcal{C}_\beta \to \mathcal{C}_{\beta + \alpha j}$ by

$$F_j^{(n)} : P \mapsto P \circ P(j)^{(n)}. \quad (3.9.1)$$

If we write $P = (P_0, \phi)$, then the functor is given by

$$F_j^{(n)} P = (R(\beta + n\alpha_j) \otimes_{R(\beta)} R(\alpha_j)) (P_0 \otimes P_j^{(n)}),$$

where $\phi'$ is obtained from $\phi \otimes \phi_{nj} : \sigma^* (P_0) \otimes \sigma^* (P_j^{(n)}) \to P_0 \otimes P_j^{(n)}$. By 3.6, $F_j^{(n)}$ gives a functor $\mathcal{P}_\beta \mapsto \mathcal{P}_{\beta + \alpha j}$. It induces an operator $f_j^{(n)} : K(\mathcal{P}_\beta) \to K(\mathcal{P}_{\beta + \alpha j})$.

On the other hand, assume that $\beta \in Q^\alpha_+$ such that $\beta - n\alpha_j \in Q^\alpha_+$ for $j \in J$. We define a functor $E_j^{(n)} : \mathcal{L}_\beta \to \mathcal{L}_{\beta - \alpha j}$ as follows; write $P(j)^{(n)} = (P_j^{(n)}, \phi_{nj})$ as in 3.8, $P_j^{(n)}$ is the projective $R(\alpha j)$-module. Let $M = (M_0, \phi) \in \mathcal{L}_\beta$, where $M_0 \in R(\beta)$-gm. Set

$$E_j^{(n)} : (M_0, \phi) \mapsto \left( (R(\beta - n\alpha_j) \circ P_j^{(n)})^\psi \otimes_{R(\beta)} M_0, \phi' \right) = (M_1, \phi'), \quad (3.9.3)$$

where $\phi'$ is induced naturally from $\phi$ and $\phi_{nj}$. As in 2.6, $M_0 \mapsto M_1$ is an exact functor. Hence $E_j^{(n)}$ is an exact functor. It induces an operator $e_j^{(n)} : K(\mathcal{L}_\beta) \to K(\mathcal{L}_{\beta - \alpha j})$.

In the case where $n = 1$, for $M = (M_0, \phi_0) \in \mathcal{C}_\beta$, $E_j M = (\tilde{E}_j M_0, \phi)$, where $\tilde{E}_j M_0$ is given by

$$\tilde{E}_j M_0 = (R(\beta - \alpha_j) \otimes_k P_j^{\phi} \otimes_{R(\beta - \alpha_j) \otimes R(\alpha_j)} e(\beta - \alpha_j, \alpha_j) M_0.$$
Note that $P_j = R(\alpha_j)e(\nu)$ with $\nu = (i_1, \ldots, i_t)$ by (3.7.3). Thus $P_j^\psi = e(\nu)R(\alpha_j)$, and so

$$\widetilde{E}_j M_0 = e(\beta - \alpha_j, \alpha_{i_1}, \ldots, \alpha_{i_t})M_0 = E_{i_1} \cdots E_{i_t} M_0,$$

where $e(\beta - \alpha_j, \alpha_{i_1}, \ldots, \alpha_{i_t})$ is defined similarly to $e(\beta, \gamma)$ in 2.2, and $E_{i_k}$ is the functor defined in (2.6.3).

The following result is a generalization of (2.6.5).

**Lemma 3.10.** Assume that $j \in J$, $\beta \in Q_+^\ell$. Then we have

$$\langle j f_j^{(n)}[P], [M] \rangle = \langle [P], e_j^{(n)}[M] \rangle, \quad (P \in \mathcal{P}_\beta, M \in \mathcal{L}_{\beta+n\alpha_j}).$$

**Proof.** We define functors

$$\widetilde{F}_j^{(n)} : R(\beta)-\mathfrak{g}p \to R(\beta + n\alpha_j)-\mathfrak{g}p, \quad P_0 \mapsto P_0 \circ P_j^{(n)};$$

$$\widetilde{E}_j^{(n)} : R(\beta + n\alpha_j)-\mathfrak{g}m \to R(\beta)-\mathfrak{g}m, \quad M_0 \mapsto (R(\beta) \circ P_j^{(n)})^{\psi} \otimes R(\beta + n\alpha_j) M_0.$$

Let $P = (P_0, \phi) \in \mathcal{P}_\beta$, $M = (M_0, \phi') \in \mathcal{L}_{\beta+n\alpha_j}$. We have

$$(\ref{3.10.1})$$

$$\left(\widetilde{F}_j^{(n)} P_0\right)^\psi \otimes R(\beta + n\alpha_j) M_0$$

$$\simeq \left(P_0^\psi \otimes R(\beta) \otimes (P_j^{(n)})^{\psi} \otimes R(\beta + n\alpha_j)\right) \otimes R(\beta + n\alpha_j) M_0$$

$$= P_0^\psi \otimes R(\beta) \widetilde{E}_j^{(n)} M_0.$$  

Let $Z$ be the left hand side of (3.10.1). The isomorphisms $\phi : \sigma^* P_0 \simeq P_0, \phi' : \sigma^* M_0 \simeq M_0$ induce the isomorphism $\sigma^* Z \simeq Z$, which coincides with the isomorphism $\sigma^* (P_0^\psi \otimes R(\beta) \widetilde{E}_j^{(n)} M_0) \simeq P_0^\psi \otimes R(\beta) \widetilde{E}_j^{(n)} M_0$ induced from $\phi$ and $\phi'$, through the isomorphisms in (3.10.1). Thus the lemma is proved.  

**3.11.** As discussed in 3.6, $K(\mathcal{P})$ is an associative algebra over $\mathbb{Z}[\zeta_n, q, q^{-1}]$ with the basis $\mathcal{B}$. We define $\mathcal{A}$ as the smallest subring of $\mathbb{Z}[\zeta_n, q, q^{-1}]$ containing $\mathcal{B}$ such that all the structure constants of the algebra $K(\mathcal{P})$ with respect to the basis $\mathcal{B}$ lie in $\mathcal{A}$.

Let $\bar{A} K(\mathcal{P})$ be the $\bar{A}$-submodule of $K(\mathcal{P})$ spanned by $\mathcal{B}$. Then $\bar{A} K(\mathcal{P})$ is the $\bar{A}$-subalgebra of $K(\mathcal{P})$. Set $\bar{A} \mathcal{U}_q = \bar{A} \otimes_\mathcal{A} \mathcal{U}_q$. The following result was proved by McNamara [M].

**Theorem 3.12 ([M, Thm. 6.1]).** There exists a unique $\bar{A}$-algebra isomorphism

$$(\ref{3.12.1})$$

$$\gamma_0 : \bar{A} \mathcal{U}_q \simeq \bar{A} K(\mathcal{P}).$$
such that $\gamma_0(f_j^{(n)}) = [P(j)^{(n)}]$. Moreover, $\gamma_0$ is an isometry.

The following result is a generalization of Proposition 2.9.

**Proposition 3.13.** (i) Consider the actions of $f_j^{(n)}$ on $K(\mathcal{P})$ defined in (3.9.1), and on $U_q^-$ by the left multiplication. Then there exists an isomorphism of $\tilde{A}$-modules (actually anti-algebra isomorphism)

$$(3.13.1) \quad \gamma : \tilde{A} U_q^- \cong \tilde{A} K(\mathcal{P})$$

such that $\gamma$ maps $1 \in \tilde{A} (U_q^-)_0$ to $[k, id] \in \tilde{A} K(\mathcal{P}_0)$, commuting with $f_j^{(n)}$. Moreover, $\gamma$ is an isometry.

(ii) Consider the actions of $e_j^{(n)}$ on $K(\mathcal{L})$ defined in (3.9.2), and on $(U_q^-)^*$ as in 1.9. Then there exists an isomorphism of $\tilde{A}$-modules,

$$(3.13.2) \quad \gamma^* : \tilde{A} K(\mathcal{L}) \cong \tilde{A} (U_q^-)^*$$

such that $\gamma^*$ maps $[k, id] \in \tilde{A} K(\mathcal{L}_0)$ to $1^* \in \tilde{A} (U_q^-)_0^*$, commuting with $e_j^{(n)}$.

**Proof.** The proof is similar to that of Proposition 2.9. We define $\gamma : \tilde{A} U_q^- \to \tilde{A} K(\mathcal{P})$ by $\gamma = \gamma_0 \circ \ast$, where $\ast : \tilde{A} U_q \to \tilde{A} U_q^-$ is the anti-involution. Since the action of $f_j^{(n)}$ on $K(\mathcal{P})$ is defined by (3.9.1), (i) follows from Theorem 3.12. If we define $\gamma^*$ as the transpose of $\gamma$, (ii) follows from (i) by Lemma 3.10.

**3.14.** $f_j^{(n)} \in \tilde{A} U_q^-$ satisfies the relation $[n]_j^m f_j^{(n)} = f_j^{n}$. Hence by (3.13.1), the operator $f_j^{(n)}$ on $\tilde{A} K(\mathcal{P})$ satisfies a similar relation. Then by using Lemma 3.10, we obtain the corresponding relation on the operator $e_j^{(n)}$ on $\tilde{A} K(\mathcal{L})$. Thus we have

$$(3.14.1) \quad [n]_j^m f_j^{(n)} = f_j^{n}, \quad [n]_j^m e_j^{(n)} = e_j^{m}$$

as operators on $\tilde{A} K(\mathcal{P})$ and on $\tilde{A} K(\mathcal{L})$.

**3.15.** In [M, Lemma 11.3], McNamara showed that $\tilde{A} \subset \mathbb{Z}[\zeta_n + \zeta_n^{-1}, q, q^{-1}]$, and paused a question whether $\tilde{A}$ coincides with $A$ or not ([M, Question 11.2]). In this paper, we show that this certainly holds, namely,

**Theorem 3.16.** The structure constants in $K(\mathcal{P})$ with respect to the basis $B$ are all contained in $A$. Thus the isomorphisms (3.12.1), (3.13.1) and (3.13.2) can be replaced by the isomorphisms

$$\gamma_0 : \tilde{A} U_q^- \cong A K(\mathcal{P}) \quad \text{and} \quad \gamma^* : \tilde{A} K(\mathcal{L}) \cong A (U_q^-)^*.$$

The theorem will be proved in the next section.
4. THE PROOF OF THEOREM 3.16

4.1. We keep the setup in Section 3. In particular, \( n \) is the order of \( \sigma : I \to I \), and \( \zeta_n \) is a primitive \( n \)-th root of unity in \( C \). If \( n = 1 \), Theorem 3.16 certainly holds by Theorem 2.7. We prove the theorem by induction on \( n \). Hence we assume that \( n > 1 \), and that the theorem holds for \( n' < n \). We write \( n = n_1n_2 \), where \( n_2 > 1 \) is a power of a prime number \( \ell \), and \( n_1 \) is prime to \( n_2 \). Choose \( a, b \in \mathbb{Z} \) such that \( 1 = n_1a + n_2b \), and set \( \sigma_1 = \sigma^{n_2b}, \sigma_2 = \sigma^{n_1a} \). Thus \( \sigma = \sigma_1\sigma_2 \), and the order of \( \sigma_1 \) (resp. \( \sigma_2 \)) is \( n_1 \) (resp. \( n_2 \)). We define \( \zeta_{n_1} = (\zeta_n)^{n_2b} \) and \( \zeta_{n_2} = (\zeta_n)^{n_1a} \). Then \( \zeta_{n_1} \) (resp. \( \zeta_{n_2} \)) is a primitive \( n_1 \)-th root (resp. \( n_2 \)-th root) of unity in \( C \). The ring homomorphism \( \mathbb{Z}\zeta_n \to k \) restricts to the ring homomorphism \( \mathbb{Z}\zeta_{n_1} \to k \), or to \( \mathbb{Z}[\zeta_{n_2}] \to k \).

4.2. Assume that \( \beta \in Q^*_\ell \). Let \( \mathcal{C}_\beta \) be the abelian category associated to \( \sigma^*: R(\beta) \cong R(\beta') \). For each \( (M, \phi) \in \mathcal{C}_\beta \), we define an object \( \sigma^*_2(M, \phi) = (M', \phi') \in \mathcal{C}'_\beta \) as follows; set \( M' = \sigma^*_2M \), then \( \sigma^*_2(\phi) : \sigma^*_2\sigma_1^*M \cong \sigma^*_2M \) induces an isomorphism \( \sigma_1^*M' = \sigma_1^*\sigma^*_2M = \sigma_2^*\sigma_1^*M \cong \sigma^*_2M = M' \), which we denote by \( \phi' : \sigma^*_2M' \cong M' \). Clearly \( \phi' = \sigma^*_2(\phi) \) satisfies the relation (3.1.1), and we obtain \( (M', \phi') \in \mathcal{C}'_\beta \). The functor \( \sigma^*_2 : \mathcal{C}_\beta \to \mathcal{C}'_\beta \) is a periodic functor on the linear category \( \mathcal{C}_\beta \). Thus we define \( \mathcal{C}''_\beta \) as the category associated to the periodic functor \( \sigma^*_2 \) (see 3.1).

The category \( \mathcal{C}''_\beta \) is described as follows; objects are triples \( (M, \phi_1, \phi_2) \) where \( (M, \phi_1) \in \mathcal{C}_\beta \), and \( \phi_2 : \sigma^*_2(M, \phi_1) \cong (M, \phi_1) \) in \( \mathcal{C}_\beta \) satisfies the condition that \( \phi_2 \circ \sigma^*_2 \phi_2 \cdots \circ (\sigma^*_2)^{n_2-1} \phi_2 = \text{id}_{\mathcal{C}_\beta} \). A morphism from \( (M, \phi_1, \phi_2) \) to \( (M', \phi_1', \phi_2') \) is defined by using a similar diagram as in (3.1.2) by replacing \( M \) by \( (M, \phi_1) \) and \( \sigma \) by \( \sigma_2 \).

By 3.3, \( \mathcal{C}_\beta \) is equivalent to the category of graded representations of the algebra \( R(\beta)[\mathbb{Z}/n\mathbb{Z}] \). Similarly, \( \mathcal{C}'_\beta \) is equivalent to the category of graded representations of the algebra \( R(\beta)[\mathbb{Z}/n_1\mathbb{Z}] \). By a similar argument, \( \mathcal{C}''_\beta \) is equivalent to the category of graded representations of the algebra \( (R(\beta)[\mathbb{Z}/n_1\mathbb{Z}]) \iso (\mathbb{Z}/n_2\mathbb{Z}) \). Since \( k[\mathbb{Z}/n\mathbb{Z}] \cong k[\mathbb{Z}/n_1\mathbb{Z}] \oplus_k k[\mathbb{Z}/n_2\mathbb{Z}] \), we see that

\[
(4.2.1) \quad (R(\beta)[\mathbb{Z}/n_1\mathbb{Z}]) \iso (\mathbb{Z}/n_2\mathbb{Z}) \cong R(\beta)[\mathbb{Z}/n\mathbb{Z}].
\]

Assume given \((M, \phi_1, \phi_2) \in \mathcal{C}''_\beta \). \( \phi_2 : \sigma^*_2(M, \phi_1) \cong (M, \phi_1) \) induces an isomorphism \( \sigma^*_2M \cong M \), which we also denote by \( \phi_2 \). Then \( \phi_2 \circ \sigma^*_2(\phi_1) : \sigma^*_2\sigma_1^*M \cong \sigma^*_2M \cong M \) gives an isomorphism \( \sigma^*M \cong M \), and the correspondence \((M, \phi_1, \phi_2) \mapsto (M, \phi_2 \circ \sigma^*_2(\phi_1))\) gives a functor \( \mathcal{C}''_\beta \to \mathcal{C}_\beta \). By (4.2.1), this gives an equivalence of the categories \( \mathcal{C}''_\beta \cong \mathcal{C}_\beta \).

Recall that \( \mathcal{P}_\beta \) is the full subcategory of finitely generated graded objects in \( \mathcal{C}_\beta \), which is equivalent to the subcategory of finitely generated projective \( R(\beta)[\mathbb{Z}/n\mathbb{Z}] \)-modules. Similarly, the full subcategory \( \mathcal{P}''_\beta \) of \( \mathcal{C}''_\beta \), and the full subcategory \( \mathcal{P}'_\beta \) of \( \mathcal{C}'_\beta \), are defined. Let \( \mathcal{L}_\beta \) be the full subcategory of finite dimensional graded objects in \( \mathcal{C}_\beta \), namely, the full subcategory of finite dimensional graded
where $\phi (4.4.1)$ (an isomorphism $R$ there exists an Grothendieck group $K$ ($C$ and $A$). \[ \text{Lemma 4.3.} \text{The category } C'' \text{ (resp. } \mathcal{P}''', \mathcal{L}'') \text{ is equivalent to the category } C' \text{ (resp. } \mathcal{P}_\beta, \mathcal{L}_\beta). \]

4.4. By the discussion in 3.2, if $(M, \phi) \in C'$, then $(M, \zeta_n \phi) \in C'$, which is denoted by $\zeta_n(M, \phi)$. Similarly, $\zeta_n(M, \phi_1)$ is defined for $(M, \phi_1) \in C'$. For a given $(M, \phi_1, \phi_2) \in C''$, we have $(M, \zeta_n \phi_1, \zeta_n \phi_2) \in C''$, which we denote by $\zeta_n(M, \phi_1, \phi_2)$. Thus under the category equivalence $C'' \simeq C'$, the action of $\zeta_n$ on $(M, \phi_1, \phi_2)$ corresponds to the action of that on $(M, \phi_2 \circ \sigma_2(\phi_1))$.

The notion of traceless elements in $C''$, and the Grothendieck groups $K(C')$ and $K(L'_\beta)$ were defined in 3.2 and 3.3. This definition works also for $C'$, and the Grothendieck group $K(P')$, $K(L'')$ are defined.

The notion of traceless elements for $C''$ is translated to our situation as follows. An object $(A, \phi_1, \phi_2) \in C''$ is said to be traceless if there exists an object $(M, \phi) \in C'$, an integer $t_2 \geq 1$ which is a divisor of $n_2$ such that $(\sigma_2^{t_2}(M, \phi) \supseteq (M, \phi)$, and an isomorphism

\begin{equation}
(A, \phi_1) \simeq (M, \phi) \oplus \sigma_2^*(M, \phi) \oplus \cdots \oplus (\sigma_2^{t_2-1}(M, \phi),
\end{equation}

where $\phi_2 : \sigma_2^*(A, \phi_1) \supseteq (A, \phi_1)$ is defined as in 3.2. Furthermore, we assume that there exists an $R(\beta)$-module $M_1$, an integer $t_1 \geq 1$ which is a divisor of $n_1$ such that $(\sigma_1^{t_1}M_1 \supseteq M_1$, and an isomorphism

\begin{equation}
M \simeq M_1 \oplus (\sigma_1^*)M_1 \oplus \cdots \oplus (\sigma_1^{t_1-1}M_1,
\end{equation}

where $\phi : \sigma_1^*M \supseteq M$ is defined as in 3.2. Finally, we assume that $t_1t_2 \geq 2$.

Under the equivalence $C'' \simeq C'$, the traceless elements in $C''$ correspond to the traceless elements in $C'$. We define the Grothendieck group $K(C'')$ in a similar way as in 3.2, by using the action of $\zeta_n$, and the traceless elements as above. Then $K(C'') \simeq K(C')$. The Grothendieck groups $K(P'')$, $K(L'')$ are defined similarly, and we have

\begin{equation}
K(P'') \simeq K(P'), \quad K(L'') \simeq K(L').
\end{equation}

4.5. Recall that the functor $D : C'' \rightarrow C'$ is given by $D(M, \phi) = (DM, D(\phi)^{-1})$ with respect to the contravariant functor $D : R(\beta)$-Mod $\rightarrow R(\beta)$-Mod. Under the category equivalence $C'' \simeq C'$, this is given by

\begin{equation}
D : (M, \phi_1, \phi_2) \mapsto (DM, D(\phi_1)^{-1}, D(\phi_2)^{-1}),
\end{equation}

where $D(\phi_2)$ is the dual of $\phi_2 : \sigma_2^*(M, \phi_1) \rightarrow (M, \phi_1)$ in $C'$, and it is also the dual of $\phi_2 : \sigma_2^*M \rightarrow M$ in $R(\beta)$-Mod. Similar results hold also for $P'' \simeq P$, and $L'' \simeq L$. 

$R(\beta)\mathbb{Z}/(\mathbb{Z}/n\mathbb{Z})$-modules. The full subcategory $\mathcal{L}_\beta$ of $C'$, and the full subcategory $\mathcal{L}''$ of $C''$ are defined similarly.

Summing up the above discussion, we obtain the following lemma.

\[ \text{Lemma 4.3.} \text{The category } C'' \text{ (resp. } \mathcal{P}''', \mathcal{L}'') \text{ is equivalent to the category } C' \text{ (resp. } \mathcal{P}_\beta, \mathcal{L}_\beta). \]
Here we note a lemma.

**Lemma 4.6.** Assume that $(P, \phi) \in \mathcal{P}_\beta$ is a projective indecomposable object, which is not traceless. Then $P$ is a projective indecomposable $R(\beta)$-module.

**Proof.** Let $(P, \phi) \in \mathcal{P}_\beta$ be a projective indecomposable object, where $\phi : \sigma^*P \cong P$. We assume that $(P, \phi)$ is not traceless. Then $P$ is a projective $R(\beta)$-module. Let $P_0$ be an indecomposable direct summand of $P$. Let $t$ be the smallest integer such that $(\sigma^*)^tP_0 \cong P_0$. Then $f \circ \sigma^*(\phi) \circ \cdots \circ (\sigma^*)^t \cdot \phi : (\sigma^*)^tP \cong P$ induces an isomorphism $(\sigma^*)^tP_0 \cong P_0$, which we denote by $\phi_0$. Set $M = P_0 \oplus \sigma^*P_0 \oplus \cdots \oplus (\sigma^*)^tP_0$. Then $\sigma^*M \cong M$, and the restriction of $\phi : \sigma^*P \cong P$ on $\sigma^*M$ gives an isomorphism $\phi_0 : \sigma^*M \cong M$, where $\phi_0$ is defined by $\phi_0 = (\sigma^*)^kP_0 \cong (\sigma^*)^kP_0$ for $k = 1, \ldots, t - 1$, and $\phi_0 : (\sigma^*)^tP_0 \cong P_0$. Hence $(M, \phi_0) \in \mathcal{P}_\beta$ is a direct summand of $(P, \phi)$. Since $(P, \phi)$ is indecomposable, we have $(P, \phi) = (M, \phi_0)$. If $t \geq 2$, then $(M, \phi_0)$ is traceless. Hence by assumption, $t = 1$, i.e., $P = P_0$ is indecomposable. The lemma holds. \( \square \)

**4.7.** For a given $\beta \in \mathcal{Q}_+^\ast$, let $\mathcal{B}_\beta$ (resp. $\mathcal{B}'_\beta$) be the $\mathbb{Z}[\zeta_n, q, q^{-1}]$-basis of $K(\mathcal{P}_\beta)$ (resp. $K(\mathcal{L}_\beta)$) given in 3.4. Similarly, for a given $\beta \in \mathcal{Q}_+^\ast$, $\mathbb{Z}[\zeta_n, q, q^{-1}]$-basis of $K(\mathcal{P}'_\beta)$ (resp. $K(\mathcal{L}_\beta)$) is defined, which we denote by $\mathcal{B}_\beta$ (resp. $\mathcal{B}'_\beta$).

Let $(P, \phi, \phi') \in \mathcal{P}'_\beta$ be a self-dual projective indecomposable object corresponding to an element in $\mathcal{B}_\beta$ under the equivalence $\mathcal{P}'_\beta \cong \mathcal{P}_\beta$. Then $P \in R(\beta)$-$\text{gp}$ is projective indecomposable by Lemma 4.6, and $(P, \phi) \in \mathcal{P}_\beta$ is self-dual projective indecomposable by (4.5.1). Thus we have

(4.7.1) The projection $(P, \phi_1, \phi_2) \mapsto (P, \phi_1)$ induces a map $\mathcal{B}_\beta \to \pm \mathcal{B}'_\beta$.

Let $\beta, \gamma \in \mathcal{Q}_+^\ast$. For $(M, \phi) \in \mathcal{C}_\beta, (N, \phi') \in \mathcal{C}_\gamma$, the convolution product is defined as $(M, \phi) \circ (N, \phi') = (M \circ N, \phi \circ \phi')$ as in (3.6.1), by using the convolution product $M \circ N \in R(\beta + \gamma)$-$\text{Mod}$. Under the equivalence $\mathcal{C}'_\beta \cong \mathcal{C}_\beta$, this convolution product is expressed as

(4.7.2) $(M, \phi_1, \phi_2) \circ (N, \phi'_1, \phi'_2) = (M \circ N, \phi_1 \circ \phi'_1, \phi_2 \circ \phi'_2),$

where $(M \circ N, \phi_1 \circ \phi'_1) = (M, \phi_1) \circ (N, \phi'_1)$ applied for the category $\mathcal{C}'$, and $\phi_2 \circ \phi'_2 : \sigma_2^t(M \circ N) \cong M \circ N$ is defined similarly to $\phi_1 \circ \phi'_1 : \sigma_1^t(M \circ N) \cong (M \circ N)$.

Similarly to $\mathcal{A}$ defined in 3.11, we define $\mathcal{A}_1$ as the smallest subring of $\mathbb{Z}[\zeta_{\mathbf{n}_1}, q, q^{-1}]$ containing $\mathcal{A}$ such that the structure constants with respect to $\mathcal{B}'$ are contained in $\mathcal{A}_1$. Since $\mathbf{n}_1 < \mathbf{n}$, by induction hypothesis, we have $\mathcal{A}_1 = \mathcal{A}$.

By (4.7.1) and (4.7.2), we have the following.

**Lemma 4.8.** The structure constants of $\mathcal{B}$ in $\mathcal{A} K(\mathcal{P})$ lie in $\mathbb{Z}[\zeta_{\mathbf{n}_2}, q, q^{-1}]$.

**4.9.** Recall that $\mathcal{X} = (J, ( , ))$ is the Cartan datum induced from $(X, \sigma)$, and $\mathcal{U}_q$ is the quantum group associated to $\mathcal{X}$. Here $\sigma = \sigma_1 \sigma_2$, with $\sigma_2$ : a power of $\ell$. Let $J_1$ be the set of $\sigma_1$-orbits in $I$, and $X_1 = (J_1, ( , ))$ the Cartan datum induced from $(X, \sigma_1)$. $\sigma_2$ acts on $J_1$, and induces an admissible diagram automorphism on $X_1$. The set of $\sigma_2$-orbits on $J_1$ coincides with $J$, and the Cartan datum induced from $(X_1, \sigma_2)$ is isomorphic to $\mathcal{X}$. We denote by $\mathcal{U}'_q$ the quantum group associated to $X_1$. $\sigma_2$ acts on $\mathcal{U}'_q$ as an automorphism.
Let $A K(\mathcal{P}')$ be the $A$-submodule of $K(\mathcal{P}')$ spanned by $B'$. Then by the induction hypothesis, $A K(\mathcal{P}')$ is the $A$-subalgebra of $K(\mathcal{P}')$. By applying Theorem 3.16 to $K(\mathcal{P}')$, we have an isomorphism of $A$-modules

\begin{equation}
\gamma_1 : A U_q^- \cong A K(\mathcal{P}'),
\end{equation}

which maps $1 \in A(U_q^-) \otimes k, id \in A K(\mathcal{P}_0')$, and commutes with the actions of $f_j^{(n)} (j \in J, n \in \mathbb{N})$.

Let $A K(\mathcal{L}')$ be the $A$-submodule of $K(\mathcal{L}')$ spanned by $B^*$. Then $A K(\mathcal{L}')$ is $A$-dual to $A K(\mathcal{P}')$, and $B'$ and $B^*$ are dual to each other. Let $(U_q^-)^*$ be the dual space of $U_q^-$. By taking the transpose of $\gamma_1$, we have an isomorphism of $A$-modules,

\begin{equation}
\gamma_i^* : A K(\mathcal{L}') \cong A(U_q^-)^*,
\end{equation}

which maps $[k, id] \in A K(\mathcal{L}_0')$ to $1^* \in A(U_q^-)^*_0$, and commutes with the actions of $e_j^{(n)} (j \in J, n \in \mathbb{N})$.

4.10. We consider the functor $\tau = (\sigma_2^{-1})^* : \mathcal{C}_q^I \rightarrow \mathcal{C}_q^{I'}$ for any $\beta \in Q^*_{+}$.

Note that the order of $\tau$ is equal to $n_2$, which is a power of $\ell$. Let $A' = F[q, q^{-1}]$ with $F = \mathbb{Z}/\ell \mathbb{Z}$. The discussion in Section 2 can be applied to our situation, by replacing $K_{gp}$ (resp. $K_{gm}$) by $A K(\mathcal{P}')$ (resp. $A K(\mathcal{L}')$) as follows. Set $A' K(\mathcal{P}') = A' \otimes A K(\mathcal{P}')$, and define $A' K(\mathcal{L}')$ similarly. $B'(\text{resp. } B'^*)$ gives an $A'$-basis of $A' K(\mathcal{P}')$ (resp. $A' K(\mathcal{L}')$). Let $A' K(\mathcal{P}')^\tau$ be the set of $\tau$-fixed elements in $A' K(\mathcal{P}')$. Then $\{O(b) | b \in B'\}$ gives an $A'$-basis of $A' K(\mathcal{P}')^\tau$. Let $\mathcal{J}'$ be the subset of $A' K(\mathcal{P}')^\tau$ consisting of $O(x)$ for $x \in A' K(\mathcal{P}')$ such that $O(x) \neq x$. Then $\mathcal{J}'$ is an $A'$-submodule of $A' K(\mathcal{P}')^\tau$. Similarly, $A'$-submodule $\mathcal{J}'^*$ of $A' K(\mathcal{L}')^\tau$ is defined. We define the quotient modules $\mathcal{V}'_q$ and $\mathcal{V}'_q^*$ by

\begin{equation}
\mathcal{V}'_q = A' K(\mathcal{P}')^\tau/\mathcal{J}', \quad \mathcal{V}'_q^* = A' K(\mathcal{L}')^\tau/\mathcal{J}'^*.
\end{equation}

A $\sigma$-orbit $j$ in $I$ is regarded as a $\sigma_2$-orbit in $J_1$. Thus for each $j \in J$, one can write as $j = \{j_1, \ldots, j_l\}$ with $j_k \in J_1$. We define a functor $\tilde{F}_j^{(n)} : \mathcal{P}' \rightarrow \mathcal{P}'$ by $\tilde{F}_j^{(n)} = F_{j_1}^{(n)} \cdots F_{j_l}^{(n)}$. This induces an $A'$-homomorphism $\tilde{f}_j^{(n)} = f_{j_1}^{(n)} \cdots f_{j_l}^{(n)}$ on $A' K(\mathcal{P}')$. As in 2.10, $\tilde{f}_j^{(n)}$ commutes with $\tau$, and acts on $\mathcal{V}'_q$.

Similarly, for each $j \in J$, we define a functor $\tilde{E}_j^{(n)} : \mathcal{L}' \rightarrow \mathcal{L}'$ by $\tilde{E}_j^{(n)} = E_{j_1}^{(n)} \cdots E_{j_l}^{(n)}$. This induces an $A'$-homomorphism $\tilde{e}_j^{(n)} = e_{j_1}^{(n)} \cdots e_{j_l}^{(n)}$ commuting with $\tau$. Thus we have an action of $\tilde{e}_j^{(n)}$ on $\mathcal{V}'_q^*$. The pairing $\langle , \rangle : A' K(\mathcal{P}') \times A' K(\mathcal{L}') \rightarrow A$ induces a perfect pairing $\langle , \rangle : \mathcal{V}'_q \times \mathcal{V}'_q^* \rightarrow A'$, and $\tilde{e}_j^{(n)}$ coincides with the transpose of $\tilde{f}_j^{(n)}$.

The quotient algebra $\mathcal{V}'_q = A' (U_q^-)_{\sigma_2} / \mathcal{J}'$ of $A' (U_q^-)_{\sigma_2}$ is defined as in (1.6.1) by replacing $\sigma : U_q^- \rightarrow U_q^-$ by $\sigma_2 : U_q^- \rightarrow U_q^-$. Also the quotient module $\mathcal{V}'_q^* = A' (U_q^-)_{\sigma_2} / \mathcal{J}'^*$ is defined as in (1.11.1). ($\mathcal{J}'$, $\mathcal{J}'^*$ are the objects corresponding
to $\mathbf{J}, \mathbf{J}^\ast$, respectively.) Note that the Cartan datum induced from $(X_1, \sigma_2)$ coincides with $\mathbf{X}$. Thus by Theorem 1.7 and Proposition 1.12, we see that there exist isomorphisms

$$
(4.10.2) \quad \Phi_1 : A(U_q^-) \cong V_q', \quad \Phi_1^* : V_q^* \cong A(U_q^-)^*,
$$

where under these isomorphisms, the actions of $j_{\beta}^{(n)}$ (resp. $e_{\beta}^{(n)}$) corresponds to the actions of $j_{\beta}^{(n)}$ (resp. $e_{\beta}^{(n)}$).

Since the isomorphism $\gamma_1 : A(U_q^-) \cong K(\mathcal{P})$ is compatible with the action of $\sigma_2$ and $\tau$, it induces an isomorphism $A(U_q^-)^{\sigma_2} \cong A.K(\mathcal{P})^\tau$, which maps $\mathbf{J}$ onto $\mathbf{J}'$. Similarly, the isomorphism $\gamma_1^* : K(\mathcal{P}') \cong A(U_q^-)^*$ induces an isomorphism $A.K(\mathcal{P}')^\tau \cong A(U_q^-)^{\sigma_2}$, which maps $\mathbf{J}'$ onto $\mathbf{J}'^*$. It follows that $\gamma_1$ and $\gamma_1^*$ induce isomorphisms

$$
(4.10.3) \quad \Theta_1 : V_q' \cong V_q', \quad \Theta_1^* : V_q^* \cong V_q^*,
$$

where $\Theta_1$ (resp. $\Theta_1^*$) commutes with the actions of $j_{\beta}^{(n)}$ (resp. $e_{\beta}^{(n)}$).

### 4.11

Let $\tilde{A}$ be as in 3.11. By Lemma 4.8, $A \subset \tilde{A} \subset \mathbb{Z}[\kappa_2, q, q^{-1}]$. We define

$$
\tilde{A}' = A' \otimes_A \tilde{A} = \tilde{A}/\ell\tilde{A}.
$$

Then $A'$ is a subalgebra of $\tilde{A}'$.

We compare the structure of $A.K(\mathcal{P}')$ (resp. $\tilde{A}.K(\mathcal{P})$) and $V_q^*$ (resp. $V_q'$) by changing the base ring to $\tilde{A}'$. Set $\tilde{A}.K(\mathcal{P}) = \tilde{A}' \otimes_{\tilde{A}} A.K(\mathcal{P})$, and $\tilde{A}.K(\mathcal{P}') = \tilde{A}' \otimes_{\tilde{A}} A'.K(\mathcal{P}')$. The definition of $V_q'$ and $V_q^*$ in (4.8.1) can be applied by replacing $A'$ by $\tilde{A}'$, and one can define $\tilde{A}.V_q'$ and $\tilde{A}.V_q^*$ as the quotients of $\tilde{A}.K(\mathcal{P})^\tau$ and $\tilde{A}.K(\mathcal{P}')^\tau$, respectively, where $\mathbf{J}' \subset \tilde{A}.K(\mathcal{P})^\tau, \mathbf{J}'^* \subset \tilde{A}.K(\mathcal{P}')^\tau$ are defined similarly.

Assume that $(M, \phi_1, \phi_2) \in \mathcal{L}'_\beta$ is a traceless element. Then by 4.4, $(M, \phi_1) \in \mathcal{L}'_\beta$ is traceless, or there exists $(M', \phi') \in \mathcal{L}'_\beta$, and an integer $t \geq 2$ which is a divisor of $n_2$, such that $(\sigma_2)^t(M', \phi') \simeq (M', \phi')$ and that

$$
(M, \phi_1) \simeq (M', \phi') \oplus (\sigma_2)^2(M', \phi') \oplus \cdots \oplus (\sigma_2)^{t-1}(M', \phi'),
$$

where $\phi_2 : (\sigma_2)^{t}(M, \phi_1) \cong (M, \phi_1)$ is defined as in (4.4.1). It follows that $[M, \phi_1] = 0$ in $K(\mathcal{L}'_\beta)$, or

$$
[M, \phi_1] = [M', \phi'] + \tau[M', \phi'] + \cdots + \tau^{t-1}[M', \phi'] \in \mathbf{J}'^*.
$$

In particular,

#### (4.11.1)

If $(M, \phi_1, \phi_2)$ is traceless, then $[M, \phi_1] \in \mathbf{J}'^*$.

Let $(L, \phi_1, \phi_2) \in \mathcal{L}'_\beta$ be a simple object in $\mathcal{L}'_\beta$, which is not traceless, such that $[(L, \phi_1, \phi_2)] \in \tilde{A}.K(\mathcal{P})$. By [M, Thm. 3.6], $(L, \phi_1)$ is a simple object in $\mathcal{L}'_\beta$.
such that \( \sigma_2^* (L, \phi_1) \cong (L, \phi_1) \). Here \( L \) is a simple \( R(\beta) \)-module, and is isomorphic to a self-dual simple module, up to degree shift. By using the category equivalence \( \mathcal{L}'' \simeq \mathcal{L} \), there exist self-dual simple objects \( (L, \phi_1^*, \phi_2^*) \) in \( \mathcal{L}_\beta'' \). Then \( (L, \phi_1^*) \) is self-dual in \( \mathcal{L}_\beta'' \), and there exists a power \( \zeta_1 \) of \( \zeta_{n_1} \), with degree shift \( q^a \), such that \( (L, \phi_1^*) = \zeta_1 q^a (L, \phi_1) \). By our assumption in 4.1, we may only consider the case where \( \phi_1^* = \pm \phi_1 \), up to degree shift. Now if \( n_2 \) is odd, \( \phi_2^* : \sigma_2^* (L, \phi_1^*) \cong (L, \phi_1^*) \) is unique, while if \( n_2 \) is even, there exist exactly two \( \pm \phi_2^* \) such that \( (L, \phi_1^*, \phi_2^*) \) is self-dual. Thus for \( (L, \phi_1, \phi_2) \) as above, there exist \( \zeta \) which is a power of \( \zeta_{n_2} \), with some degree shift \( q^a \), such that \( (L, \phi_1, \phi_2) = \zeta q^a (L, \phi_1^*, \phi_2^*) \). We attach \( \zeta q^a [L, \phi_1] \in \tilde{A} K(\mathcal{L}_\beta'') \) to \( (L, \phi_1, \phi_2) \in \mathcal{L}_\beta'' \). This gives a well-defined map since we are working on \( F = \mathbb{Z}/l \mathbb{Z} \).

In view of (4.11.1), we obtain a well-defined \( \tilde{A}' \)-module homomorphism

\[
\Psi^* : \tilde{A} K(\mathcal{L}'') \to \tilde{A}' V_q^*,
\]

where \( [(L, \phi_1, \phi_2)] \) goes to \( [\zeta q^a (L, \phi_1)] \) for a simple module \( L \in R(\beta) \)-gm such that \( \sigma^* L \cong L \). By considering the transpose of (4.11.2), one can define an \( \tilde{A}' \)-module homomorphism

\[
\Psi : \tilde{A}' V_q \to \tilde{A}' K(\mathcal{P}'').
\]

Under the isomorphisms \( K(\mathcal{L}) \simeq K(\mathcal{L}'') \), \( K(\mathcal{P}) \simeq L(\mathcal{P}'') \), \( e_j^{(n)} \) (resp. \( f_j^{(n)} \)) acts on \( \tilde{A}' K(\mathcal{L}'') \) (resp. on \( \tilde{A}' K(\mathcal{P}'') \)).

**Lemma 4.12.** (i) The map \( \Psi^* \) commutes with the action of \( e_j^{(n)} \) on \( \tilde{A}' K(\mathcal{L}'') \), and that of \( e_j^{(n)} \) on \( \tilde{A}' V_q^* \).

(ii) The map \( \Psi \) commutes with the action of \( f_j^{(n)} \) on \( \tilde{A}' K(\mathcal{P}'') \), and that of \( f_j^{(n)} \) on \( \tilde{A}' V_q^* \).

**Proof.** (ii) follows from (i) by the discussion in 4.10, and by Lemma 3.10.

We prove (i). First consider the case where \( n = 1 \). Assume that \( \beta \in Q^* \) such that \( \beta - \alpha_j \in Q^*_\mathbb{Z} \). Take \( (M, \phi) \in \mathcal{L}_\beta \), which corresponds to \( (M, \phi_1, \phi_2) \in \mathcal{L}_\beta'' \) under the equivalence \( \mathcal{L}_\beta'' \simeq \mathcal{L}_\beta \). By the discussion in 3.9, \( E_j (M, \phi) \) is written as \( E_j (M, \phi) = (\hat{E}_j M, \phi') \), where \( \hat{E}_j M \) is given as in (3.9.4), and \( \phi' \) is determined uniquely from \( \phi \). Thus \( E_j (M, \phi_1, \phi_2) \) is written as \( E_j (M, \phi_1, \phi_2) = (\hat{E}_j (M, \phi_1), \phi_2) \), where \( \hat{E}_j (M, \phi_1) \in \mathcal{L}_\beta' \) is defined as in 4.10, and \( \phi_2 \) is determined uniquely from \( \phi_1, \phi_2 \). The functor \( (M, \phi_1) \mapsto \hat{E}_j (M, \phi_1) \) induces the operation \( \tilde{e}_j : K(\mathcal{L}_\beta') \to K(\mathcal{L}_\beta'_{\alpha-j}) \), and \( \tilde{e}_j ([M, \phi_1]) \) is \( \tau \)-invariant. Hence it induces the operation \( \tilde{e}_j \) on \( \tilde{A}' V_q^* \). This shows that the map \( \Psi^* \) commutes with the action of \( e_j \) on \( \tilde{A}' K(\mathcal{L}'') \) and that of \( \tilde{e}_j \) on \( \tilde{A}' V_q^* \).
We have \([n]_j^\dagger e_j^{(n)} = e_j^n\) on \(\tilde{A}^\dagger K(\mathcal{L}^n)\) by (3.14.1). On the other hand, we have \(\tilde{e}_j^{(n)} = e_j^{(n)} \cdots e_j^{(n)}\) on \(\tilde{A}^\dagger K(\mathcal{L}^n)\). Note that \([n]_j^\dagger e_j^{(n)} = e_j^n\), where \(q_j = q^{(\alpha_j, \alpha_j)/2} = q_i^t\) for \(i = j_k\). Hence we have
\[
([n]_j^\dagger e_j^{(n)})^t = e_j^n.
\]
Since \([n]_j^\dagger = [n]_j\) in \(\tilde{A}\), and \(\tilde{A}V_q^\ast\) is a free \(\tilde{A}\)'-module, the assertion holds for any \(n\). Thus (i) is proved.

**Proposition 4.13.**

(i) The map \(\Psi : \tilde{A}\gamma^\dagger V_q^\prime \to \tilde{A}\gamma K(\mathcal{P}^n)\) gives an isomorphism of \(\tilde{A}\)'-modules. Furthermore the isomorphism \(\gamma : \tilde{A}\gamma U_q^\dagger \cong \tilde{A}\gamma K(\mathcal{P})\) factors through isomorphisms

\[
(4.13.1) \quad \gamma : \tilde{A}\gamma U_q^\dagger \xrightarrow{\Phi_1} \tilde{A}\gamma V_q^\prime \xrightarrow{\Theta_1} \tilde{A}\gamma V_q^\prime \xrightarrow{\Psi} \tilde{A}\gamma K(\mathcal{P}),
\]

where \(\Phi_1, \Theta_1, \Psi\) commute with the action of \(f_j^{(n)}\) or \(\tilde{f}_j^{(n)}\).

(ii) The map \(\Psi^* : \tilde{A}\gamma K(\mathcal{L}^n) \to \tilde{A}\gamma V_q^\ast\) gives an isomorphism of \(\tilde{A}\)'-modules. Furthermore the isomorphism \(\gamma^* : \tilde{A}\gamma K(\mathcal{L}) \cong \tilde{A}\gamma (U_q^\prime)^\ast\) factors through isomorphisms

\[
(4.13.2) \quad \gamma^* : \tilde{A}\gamma K(\mathcal{L}^n) \xrightarrow{\Psi^*} \tilde{A}\gamma V_q^\ast \xrightarrow{\Theta^*_1} \tilde{A}\gamma V_q^\ast \xrightarrow{\Phi^*_1} \tilde{A}\gamma (U_q^\prime)^\ast,
\]

where \(\Psi^*, \Theta^*_1, \Phi^*_1\) commute with the action of \(e_j^{(n)}\) or \(\tilde{e}_j^{(n)}\).

**Proof.** We define a map \(\tilde{\gamma} : \tilde{A}\gamma U_q^\dagger \to \tilde{A}\gamma K(\mathcal{P}^n)\) as the composite of \(\Phi_1, \Theta_1\) and \(\Psi\),
\[
\tilde{\gamma} : \tilde{A}\gamma U_q^\dagger \xrightarrow{\Phi_1} \tilde{A}\gamma V_q^\prime \xrightarrow{\Theta_1} \tilde{A}\gamma V_q^\prime \xrightarrow{\Psi} \tilde{A}\gamma K(\mathcal{P}^n).
\]

By (4.10.2), the action of \(f_j^{(n)}\) on \(\tilde{A}\gamma U_q^\dagger\) and that of \(\tilde{f}_j^{(n)}\) on \(\tilde{A}\gamma V_q^\prime\) are compatible with \(\Phi_1\). By (4.10.3), the action of \(f_j^{(n)}\) is compatible with \(\Theta_1\). By Lemma 4.12, the action of \(f_j^{(n)}\) and that of \(\tilde{f}_j^{(n)}\) are compatible with \(\Psi\). It follows that \(\tilde{\gamma}\) is compatible with the action of \(f_j^{(n)}\). Moreover, \(\tilde{\gamma}\) maps \(1 \in \tilde{A}\gamma U_q^\dagger\) to \([\langle k, \text{id}, \text{id}\rangle] \in \tilde{A}\gamma K(\mathcal{P}^n)\). On the other hand, by Proposition 3.13, we have an anti-algebra isomorphism \(\gamma : \tilde{A}\gamma U_q^\dagger \cong \tilde{A}\gamma K(\mathcal{P})\), which commutes with the action of \(f_j^{(n)}\), under the identification \(K(\mathcal{P}) \cong K(\mathcal{P}^n)\). Since \(\gamma\) is determined uniquely by the action of \(f_j^{(n)}\), we conclude that \(\gamma = \tilde{\gamma}\). In particular, \(\Psi\) is an isomorphism, and (i) is proved. (ii) follows from (i). The proposition is proved.

**4.14.** We are now ready to prove Theorem 3.16. Let \(\mathbf{B}\) be the basis of \(\tilde{A}\gamma K(\mathcal{P})\) given in 3.4, and \(\mathbf{B}^\ast\) be the dual basis of \(\tilde{A}\gamma K(\mathcal{L})\). Similarly, we obtain the basis \(\mathbf{B}'\)
of $K(\mathcal{P}^\prime)$, and its dual basis $B^\ast$ of $K(\mathcal{L}^\prime)$ over a suitable extension of $A$. But by our assumption in 4.1, we may assume that $B^\prime$ is a basis of $A^\prime K(\mathcal{P}^\prime)$, and $B^\ast$ is the dual basis of $A^\prime K(\mathcal{L}^\prime)$. Let $(B^\prime)^\tau$ (resp. $(B^\ast)^\tau$) be the set of $\tau$-fixed elements in $B^\prime$ (resp. in $B^\ast$). Then the image $\tilde{B}$ of $(B^\prime)^\tau$ in $V_q$ gives an $A'$-basis of $V_q^\ast$, and the image $\tilde{B}^\ast$ of $(B^\ast)^\tau$ gives an $A'$-basis of $V_q^\ast$. Now by the construction of the map $\psi^*: A^\prime K(\mathcal{L}^\prime) \to A^\prime V_q^\ast$, and by Proposition 4.13, $\psi^*(B^\ast)$ coincides with $\tilde{B}^\ast$. Hence $\psi^{-1}(\tilde{B})$ coincides with $B^\prime$. In the diagram (4.13.1), the maps $\Phi_1, \Theta_1$ are defined over $A'$. Thus $\gamma^{-1}(\tilde{B})$ gives an $A'$-basis of $\Lambda U_q^\ast$.

On the other hand, since originally $\gamma$ is an isomorphism $\Lambda U_q^\ast \cong A^\prime K(\mathcal{P}^\prime)$, we see that $\gamma^{-1}(\tilde{B})$ is an $A$-basis of $\Lambda U_q^\ast$. Fix $\beta \in Q_-$, and let $Y$ (resp. $Y'$) be the $A$-submodule (resp. $A'$-submodule) of $U_q^\ast$ spanned by $\gamma^{-1}(B_\beta)$. Then $Y/Y'$ is a finitely generated $Z$-module, and by the above discussion, $\ell(Y/Y') = 0$. Hence by Nakayama’s lemma, $Y$ coincides with $Y'$ if the base ring is extended from $A = \mathbb{Z}[q, q^{-1}]$ to $\mathbb{Z}_q[q, q^{-1}]$, where $\mathbb{Z}_q$ is the ring of $\ell$-adic integers. It follows that, for $b, b' \in \gamma^{-1}(\tilde{B})$, we have $bb' = \sum_{b'' \in \gamma^{-1}(\tilde{B})} a_{b'b''}b''$, where

$$a_{b'b''} \in \mathbb{Z}[\zeta_{2n}, q, q^{-1}] \cap \mathbb{Z}_q[q, q^{-1}] = \mathbb{Z}_q[q^{-1}].$$

Thus $\gamma^{-1}(\tilde{B})$ is an $A$-basis of $\Lambda U_q^\ast$. Since $\gamma$ is an anti-algebra isomorphism, we see that the structure constants of the basis $B$ all lie in $A$, namely we have $\Lambda = A$. Theorem 3.16 is proved.

As a corollary to Theorem 3.16, a refinement of Proposition 4.13 is obtained.

**Corollary 4.15.**

(i) There exist isomorphisms of $A$-modules,

$$\gamma: \Lambda U_q^\ast \cong A^\prime K(\mathcal{P}^\prime), \quad \gamma^*: A^\prime K(\mathcal{L}^\prime) \cong \Lambda U_q^\ast.$$

(ii) The isomorphisms $\gamma, \gamma^*$ can be factored, as $A'$-modules, through

$$\gamma: \Lambda U_q^\ast \xrightarrow{\Phi_1} A' V_q^\ast \xrightarrow{\Theta_1} A' V_q^\ast \xrightarrow{\psi} A' K(\mathcal{P}^\prime),$$

$$\gamma^*: A^\prime K(\mathcal{L}^\prime) \xrightarrow{\psi^*} A' V_q^\ast \xrightarrow{\Theta_1^*} A' V_q^\ast \xrightarrow{\Phi_1^*} A' U_q^\ast.$$

5. **KLR algebras and global bases**

5.1. In this section, we assume that $R = \bigoplus_{\beta \in Q_+} R(\beta)$ is symmetric type, namely, the Cartan matrix $A = (a_{ij})$ is symmetric, and $Q_{i,j}(u, v)$ is a polynomial in $u - v$. Furthermore, we assume that $k$ is an algebraically closed field of characteristic 0. We consider the isomorphisms $\gamma: \Lambda U_q^\ast \cong A^\prime K(\mathcal{P}^\prime)$ and $\gamma^*: A^\prime K(\mathcal{L}^\prime) \cong \Lambda U_q^\ast$ as in Corollary 4.15. In the last paragraph of [M], he states a result that $\gamma^{-1}(\tilde{B})$ coincides with the canonical basis, and also coincides with the lower global basis of $U_q^\ast$, with
a brief indication for the proof, which relies on the geometric argument. In this section we give a simple proof of this fact, without using the geometry.

Let $\tilde{\Theta}_0 : A U_q^{-} \rightarrow K_{\text{gp}}$ be the isomorphism of $A$-algebras as given in Theorem 2.7. Recall that $B$ is the basis of $K_{\text{gp}}$, and $B^*$ is the basis of $K_{\text{gn}}$ given in 2.4. Let $B^*$ be the upper global basis of $A (U_q^-)^*$ as given in Theorem 1.10, and $B$ the lower global basis of $A U_q^-$. Also let $B^2$ be the canonical basis of $A U_q^-$. In the case where $X$ is symmetric, it is known by Grojnowski-Lusztig [GL] that $X$ is also self-dual. Thus $\Theta$ is the basis of $A U_q^{-}$.

The following result was proved by Varagnolo-Vasserot [VV], and Rouquier [R2], by using the Ext algebra obtained from Lusztig’s category $\mathcal{Q}_V$ related to the geometry of quivers (see Introduction).

**Theorem 5.2** ([VV], [R2]). Under the setup in 5.1, the isomorphisms $\tilde{\Theta}_0 : A U_q^{-} \rightarrow K_{\text{gp}}$ sends $B^2$ to $B$.

**5.3.** Since $B^2$ is invariant under $*$, $B^2$ corresponds to $B$ under the isomorphism $\tilde{\Theta} : A U_q^{-} \rightarrow K_{\text{gp}}$ in Proposition 2.9. Since the basis $B^2$ is almost orthonormal, $B$ is also almost orthonormal in the sense of 1.4, i.e., for $b, b' \in B$,

$$(5.3.1) \quad (b, b') \in \begin{cases} 1 + qZ[[q]] & \text{if } b = b', \\ qZ[[q]] & \text{if } b \neq b'. \end{cases}$$

We want to show that $B$ has a similar property.

**Proposition 5.4.** Assume that $b \in B$. Then we have $Db = b$, and

$$(5.4.1) \quad (b, b') \in \begin{cases} 1 + qZ[[q]] & \text{if } b = b', \\ qZ[[q]] & \text{if } b \neq b'. \end{cases}$$

**Proof.** Take $b, b' \in B$, and write it as $b = [P], b' = [Q]$ for $P = (P_0, \phi), Q = (Q_0, \phi') \in \mathcal{P}_\beta$. The relation $Db = b$ is clear from the definition. We show (5.4.1). By Lemma 4.6, $P_0$ is an indecomposable projective $R(\beta)$-module. Since $(P_0, \phi)$ is self-dual, $P_0$ is also self-dual. Thus $[P_0]$ corresponds to some element in $B$. Hence by (5.3.1), $P_0$ satisfies the relation $([P_0], [P_0]) \in 1 + qZ[[q]]$. We compare the inner product (2.5.3) for $K_{\text{gp}}(\beta)$ and the inner product (3.5.4) for $K(\mathcal{P}_\beta)$. We show that

$$(5.4.2) \quad ([P], [Q]) \in \begin{cases} 1 + qZ[\zeta_n][[q]] & \text{if } [P] = [Q], \\ qZ[\zeta_n][[q]] & \text{if } [P] \neq [Q]. \end{cases}$$

Since $DP_0 \simeq P_0$, by using the relation (3.5.2) (it also holds if $M$ is projective), in the definition of the inner product on $K_{\text{gp}}(\beta)$ and on $K(\mathcal{P}_\beta)$, one can replace the tensor product $P_0^\psi \otimes_{R(\beta)} P_0$ by the Hom space Hom$_{R(\beta)}(P_0, P_0)$. Then (5.3.1) shows that $\dim_k \text{Hom}_{R(\beta)}(P_0, P_0)_0 = 1$, and $\dim_k \text{Hom}_{R(\beta)}(P_0, q^k P_0)_0 = 0$ for $k > 0$. In particular, the grading preserving homomorphisms $P_0 \rightarrow P_0$ are only the scalar multiplications. The action of $\phi \otimes \phi$ on $(P_0^\psi \otimes_{R(\beta)} P_0)_0$ corresponds to the action $f \mapsto \phi^{-1} \circ f \circ \phi$ for Hom$_R(P_0, P_0)_0$, hence it induces the identity map on
Corollary 5.5. Let $\mathcal{B}$ be the lower global basis of $\mathcal{A}(\mathcal{U}_q^{-})$, and $\mathcal{B}^\ast$ the upper global basis of $\mathcal{A}(\mathcal{U}_q^{-})\ast$. Then $\gamma^{-1}(\mathcal{B})$ coincides with $\mathcal{B}$ or with $\mathcal{B}^\ast$, up to sign, and $\gamma^\ast(\mathcal{B}^\ast)$ coincides with $\mathcal{B}^\ast$, up to sign.

Proof. By Proposition 5.4, $\gamma^{-1}(\mathcal{B})$ gives an almost orthonormal basis in $\mathcal{A}(\mathcal{U}_q^{-})$. Since $\gamma^{-1}(\mathcal{B}) \subset \mathcal{B}$, we have $\mathcal{B} = \gamma^{-1}(\mathcal{B}) \sqcup -\gamma^{-1}(\mathcal{B})$. Hence $\gamma^{-1}(\mathcal{B})$ coincides with $\mathcal{B}$ or with $\mathcal{B}^\ast$, up to sign. The second statement follows easily from this. \qed

5.6. Take $j \in J$. Consider the operator $E_j : \mathcal{L}_\beta \to \mathcal{L}_{\beta - \alpha_j}$, for $\beta \in Q_+^\ast$ such that $\beta - \alpha_j \in Q_+^\ast$, defined as in (3.9.3) ($E_j = E_j^{(n)}$ with $n = 1$). $E_j$ induces an operator $E'_j : K(\mathcal{L}_\beta) \to K(\mathcal{L}_{\beta - \alpha_j})$. For $\beta \in Q_+^\ast$, we define a functor $F'_j : \mathcal{L}_\beta \to \mathcal{L}_{\beta + \alpha_j}$ by

$$F'_j : M \mapsto M \circ L_j.$$ 

Since the convolution product is an exact functor, $F'_j$ is an exact functor. Thus it induces an operator $f_j : K(\mathcal{L}_\beta) \to K(\mathcal{L}_{\beta + \alpha_j})$.

For a simple object $M \in \mathcal{L}_\beta$, we set

$$\varepsilon_j(M) = \max\{k \geq 0 \mid E_j^k M \neq 0\}.$$ 

We define crystal operators $\tilde{E}_j M \in \mathcal{L}_{\beta - \alpha_j}$, $\tilde{F}_j M \in \mathcal{L}_{\beta + \alpha_j}$ by

$$\tilde{E}_j M = q_j^{1-\varepsilon_j(M)} \text{soc} E_j M,$$

$$\tilde{F}_j M = q_j^{\varepsilon_j(M)} \text{hd} F'_j M.$$ 

If we use the restriction functor defined in 3.6, $E_j M$ is written as

$$E_j M \simeq \text{Hom}_{R(\alpha_j)}(L(j), \text{Res}_{\beta - \alpha_j, \alpha_j} M).$$

Thus $\tilde{E}_j M$ (resp. $\tilde{F}_j M$) coincides with $\tilde{E}_j^* M$ (resp. $\tilde{F}_j^* M$) given in [M, 9], where

$$\tilde{E}_j^* M = q_j^{1-\varepsilon_j(M)} \text{soc} \text{Hom}_{R(\alpha_j)}(L(j), \text{Res}_{\beta - \alpha_j, \alpha_j} M),$$

$$\tilde{F}_j^* M = q_j^{\varepsilon_j(M)} \text{hd}(M \circ L(j)) \text{.}$$

(Note that we use $\varepsilon_j(M)$ as in (5.6.1). This corresponds to $\varepsilon_j^*(M)$ in the notation of [M].)

The following result is immediate from [M, Lemma 8.3], though this formula is not used in later discussions.
Lemma 5.7. Let \( M \in \mathcal{L}_\beta \) be a simple object. Then \( \widetilde{F}_j M \) is a simple object such that 
\[
\varepsilon_j(\widetilde{F}_j M) = \varepsilon_j(M) + 1,
\]
and \([F'_j M]\) is written in the Grothendieck group \( K(\mathcal{L}_{\beta + \alpha_j})\),
\[
[F'_j M] = q_j^{-\varepsilon_j(M)}[\widetilde{F}_j M] + \sum_k [L_k],
\]
where \( L_k \in \mathcal{L}_{\beta + \alpha_j} \) are simple objects such that 
\[
\varepsilon_j(L_k) < \varepsilon_j(M) + 1.
\]

In order to consider the action of \( E_j \), we need a lemma, which is obtained from Lemma 8.1 and Lemma 8.2 in \([M]\).

Lemma 5.8 (\([M]\)).

(i) Let \( N \) be a simple object in \( \mathcal{C}_\beta \) such that \( \varepsilon_j(N) = 0 \). Set \( M = N \circ L(j)\) for any \( m \geq 1 \). Then \( \text{hd} M \) is irreducible, and \( \varepsilon_j(M) = m \).

(ii) Let \( N \) be a simple object in \( \mathcal{C}_\beta \) such that \( \varepsilon_j(N) = m \). Then there exists a simple object \( X \) in \( \mathcal{C}_{\beta - \alpha_j} \) with \( \varepsilon_j(X) = 0 \) such that
\[
\text{Res}_{\beta - \alpha_{j}, m\alpha_{j}}(N) \simeq X \otimes L(j)^{(m)}.
\]

We show the following.

Proposition 5.9. Let \( M \in \mathcal{L}_\beta \) be a simple object such that \( \varepsilon_j(M) = m > 0 \). Then \( \widetilde{E}_j M \) is a self-dual simple object in \( \mathcal{L}_{\beta - \alpha_j} \), and in the Grothendieck group \( K(\mathcal{L})\), \([E_j M]\) is written as
\[
(E_j M) = G(q)[\widetilde{E}_j M] + \sum_{1 \leq k \leq p} [L_k],
\]
where \( G(q) \in \mathbb{N}[q, q^{-1}] \), \( \varepsilon_j(\widetilde{E}_j M) = m - 1 \), and \( L_k \in \mathcal{L}_{\beta - \alpha_j} \) are simple objects such that \( \varepsilon_j(L_k) < m - 1 \).

Proof. The proof is done by a similar argument as in the proof of \([M, \text{Lemma 9.4}]\).
We give an outline of the proof below. It is known by \([M, \text{Lemma 9.4}]\), \( \widetilde{E}_j M \) is a self-dual simple object in \( \mathcal{L}_{\beta - \alpha_j} \). We fix a composition series of \( E_j M \), and let \( N \) be a simple object such that \( \varepsilon_j(N) = m - 1 \), appearing as a composition factor. We show
\[
N \text{ is isomorphic to } \widetilde{E}_j M, \text{ up to degree shift}.
\]

Note that \( E_j M \simeq \text{Hom}_{\mathcal{R}(\alpha_j)}(L(j), \text{Res}_{\beta - \alpha_j, \alpha_j} M) \). Let \( \widetilde{N} \) be a subobject of \( E_j M \) such that \( N \) is a simple quotient of \( \widetilde{N} \). By the adjunction isomorphism, we have
\[
\text{Hom}_{\mathcal{L}_\beta}(\widetilde{N} \circ L(j), M) \simeq \text{Hom}_{\mathcal{L}_{\beta - \alpha_j}}(\widetilde{N}, \text{Hom}_{\mathcal{R}(\alpha_j)}(L(j), \text{Res}_{\beta - \alpha_j, \alpha_j} M)).
\]
Since the right hand side of this equality is non-zero, \( M \) is a simple quotient of \( \widetilde{N} \circ L(j) \). By Lemma 5.8 (ii), there exists a simple \( X \) such that
\[
\text{Res}_{\beta - m\alpha_j, m\alpha_j}(M) \simeq X \otimes L(j)^{(m)}.
\]
Since the restriction functor is exact, we have a surjective map

$$\text{Res}_{\beta-\text{ma}_j,\text{ma}_j}(\tilde{N} \circ L(j)) \to X \otimes L(j)^{(m)}.$$  

We consider the Mackey filtration (see [M, Thm. 4.5]) for

$$\text{Res}_{\beta-\text{ma}_j,\text{ma}_j}(\tilde{N} \circ L(j)) = \text{Res}_{\beta-\text{ma}_j,\text{ma}_j} \circ \text{Ind}_{\beta-\text{a}_j,\alpha}(\tilde{N} \otimes L(j)).$$

Since $\varepsilon_j(N) = m - 1$, this Mackey filtration contains a non-zero factor which involves $\text{Res}_{\beta-\text{ma}_j,(m-1)\alpha_j} N$, up to degree shift. By Lemma 5.8 (ii), there exists a simple object $Y$ such that

$$(5.9.4) \quad \text{Res}_{\beta-\text{ma}_j,(m-1)\alpha_j} N \cong Y \otimes L(j)^{(m-1)}.$$  

We have a surjective homomorphism $Y \otimes L(j)^{(m)} \to X \otimes L(j)^{(m)}$ with degree shift. Since $X, Y$ are simple, we have $X \cong Y$, up to degree shift.

By considering the adjunction isomorphism in (5.9.4), we have a non-zero homomorphism $Y \circ L(j)^{(m-1)} \to N$. Since $\varepsilon_j(N) = m - 1$, we have $\varepsilon_j(Y) = 0$. By Lemma 5.8 (i), $\text{hd}(Y \circ L(j)^{(m-1)})$ is irreducible. It follows that $N \cong \text{hd}(Y \circ L(j)^{(m-1)})$. By applying this argument to $\tilde{E}_j M$ (the original setup in [M]), we see that $\tilde{E}_j M \cong \text{hd}(X \circ L(j)^{(m-1)})$. Hence $N \cong \tilde{E}_j M$, up to degree shift, and (5.9.2) holds.

Since $\varepsilon_j(M) = m$, all the composition factors $L$ of $E_j M$ satisfy the relation $\varepsilon_j(L) \leq m - 1$. The proposition is proved.

$\Box$

5.10. Let $B = \bigcup_{\beta \in Q_+} B_\beta$ be the basis of $K(\mathcal{P})$, and $B^* = \bigcup_{\beta \in Q_+} B^*_\beta$ be the basis of $K(\mathcal{L})$ defined as before. The functors $\tilde{E}_j$ and $\tilde{F}_j$ induce operators on $K(\mathcal{L})$, which we denote by $\tilde{e}_j$ and $\tilde{f}_j$. The following result was proved by Lemma 9.1, Lemma 9.3 and Lemma 9.4, together with the definition of $B^*$ in [M].

Proposition 5.11. (i) $\tilde{e}_j$ sends $B^*_\beta$ to $B^*_j$, 

(ii) $\tilde{f}_j$ sends $B^*_\beta$ to $B^*_\beta \cup \{0\}$.

(iii) For $b, b' \in B^*$, $b = \tilde{f}_j b'$ if and only if $b' = \tilde{e}_j b$.

The following result is a generalization of Theorem 5.2.

Theorem 5.12. Let $\gamma : A \mathcal{U}^- \cong A K(\mathcal{P})$, $\gamma^* : A K(\mathcal{L}) \cong A(\mathcal{U}^-)^*$ be the isomorphisms as in Corollary 4.15.

(i) $\gamma^*$ sends $B^*$ to $\tilde{B}^*$.

(ii) $\gamma$ sends $\tilde{B}$ to $B$.

Proof. We show (i). We prove, by induction on $|\beta|$, that

$$(5.12.1) \quad \gamma^*(B_{\beta}) = \tilde{B}^-_{\beta}, \quad (\beta \in Q_+^*).$$

Since $\gamma^*$ maps $[1] = [k, \text{id}] \in A K(\mathcal{L}_0)$ to $1^* \in (\mathcal{U}^-)^*_0$, we have $\gamma^*(B_0^*) = B_0^*$. Hence (5.12.1) holds for $\beta = 0$. Take $0 \neq \beta \in Q_+^*$, and assume that (5.12.1) holds for
\[ \beta' \in \mathbb{Q}_+^* \text{ such that } |\beta'| < |\beta|. \text{ Let } b \in \mathbb{B}_q^*. \text{ There exists } j \in J \text{ such that } e_j' b \neq 0. \]

Then by Proposition 5.9 and by Proposition 5.11, \( e_j' b \) can be written as

\[ e_j' b = G(q)\tilde{e}_j b + \sum_{b'} a_{bb'} b', \quad (G(q) \in \mathbb{N}[q, q^{-1}], a_{bb'} \in A), \]

where \( \varepsilon_j(\tilde{e}_j b) = \varepsilon_j(b) - 1 \), and \( b' \) runs over the elements in \( \mathbb{B}_q^{*+\alpha_j} \) such that \( \varepsilon_j(b') < \varepsilon_j(b) - 1 \). Let \( b_0 = \gamma^*(b) \). By Corollary 5.5, \( b_0 \in \mathbb{B}_q^* \), up to sign. Since the map \( \gamma^* \) commutes with the action of \( e_j \), we have

\[ e_j'(b_0) = G(q)\gamma^*(\tilde{e}_j b) + \sum_{b'} a_{bb'} \gamma^*(b'). \]

Here by induction, \( \gamma^*(\tilde{e}_j b), \gamma^*(b') \in \mathbb{B}_q^{*-\beta - \alpha_j} \). Moreover, since \( \gamma^* \) commutes with \( e_j' \), \( \varepsilon_j(\gamma^*(\tilde{e}_j b)) = \varepsilon_j(b_0) - 1 \), and \( \varepsilon_j(\gamma^*(b')) < \varepsilon_j(b_0) - 1 \). Comparing (5.12.3) with Theorem 1.10 (iii) (applied for \( \mathbb{A}(U_q^{-}) \)), we see that \( b_0 = \gamma^*(b) \in \mathbb{B}_q^{*-\beta} \). This proves (5.12.1). Hence (i) holds. (ii) follows from (i).

\[ \Box \]

**Remarks 5.13.**

(i) The quantum group of arbitrary type is obtained as \( \mathbb{U}_q^{-} \) from some \( \mathbb{U}_q^{-} \) of symmetric type, through an admissible automorphism \( \sigma \). Hence our result shows that the global crystal basis of the quantum group of arbitrary type is obtained from the KLR algebras through foldings.

(ii) In general, the natural basis \( \mathbb{B} \) of \( K_{s,p} \) has the positivity property, i.e., for \( b, b' \in \mathbb{B}, bb' \) is a linear combination of the basis with coefficients in \( \mathbb{N}[q, q^{-1}] \). This property does not hold for the global crystal basis \( \mathcal{B} \) of non-symmetric type. Note that the basis \( \mathbb{B} \) of \( K(\mathcal{P}) \) not necessarily has the positivity property.

**5.14.** By Theorem 5.2, the map \( \tilde{\Theta}_0 \) sends the canonical basis \( \mathcal{B}^3 \) to \( \mathbb{B} \). By applying Theorem 5.12 for the case where \( \sigma = \text{id} \), we rediscover the result of [GL] that \( \mathcal{B} = \mathcal{B}^3 \). In the following, we show that this also holds in the non-symmetric case.

For \( b \in \mathbb{B}, b' \in \mathbb{B}^* \) and \( j \in J \), we set

\[ \varepsilon_j(b) = \max\{k \geq 0 \mid b \in f_j^k K(\mathcal{P})\}, \]

\[ \varepsilon_j(b') = \max\{k \geq 0 \mid e_j^k b' \neq 0\}. \]

Note that the definition \( \varepsilon_j(b') \) is compatible with the definition \( \varepsilon_j(M) \) in (5.6.1).

Take \( b \in \mathbb{B}_q^* \), and let \( b_* \in \mathbb{B}_q^* \) be the basis dual to \( b \). Then we have

\[ \varepsilon_j(b) = \varepsilon_j(b_*). \]

In fact, assume that \( b \in f_j^k K(\mathcal{P}) \). Then \( \langle f_j^n K(\mathcal{P}), b_* \rangle \neq 0 \), and there exists \( b' \in \mathbb{B}_q^* \) such that \( \langle f_j^n b, b_* \rangle = \langle b', e_j^n b_* \rangle \neq 0 \), hence \( e_j^n b_* \neq 0 \). On the contrary, assume that \( e_j^n b_* \neq 0 \). Then there exists \( b' \in \mathbb{B}_q^* \) such that \( \langle b', e_j^n b_* \rangle = \langle f_j^n b', b_* \rangle \neq 0 \). It is known ([L2, Thm. 14.3.2]) that a subset of \( \mathcal{B}^3 \) gives a basis of
Let \( f_j^n \mathbb{U}_q^- \) for each \( n \geq 0 \). Then by Corollary 5.5, a subset of \( \mathbb{B} \) gives a basis of \( f_j^n K(\mathcal{P}) \) for each \( n \geq 0 \). It follows that \( b \in f_j^n K(\mathcal{P}) \). Thus (5.14.1) holds.

**Theorem 5.15.** \( \gamma^{-1} \) sends \( \mathbb{B} \) to \( \mathbb{B}^* \). Hence the canonical basis \( \mathbb{B}^* \) and the global crystal basis \( \mathcal{B} \) of \( \mathbb{U}_q^- \) coincide with.

**Proof.** The proof is done just by applying some properties of \( K(\mathcal{P}) \) and \( K(\mathcal{L}) \), together with Theorem 5.12. Below, we give an outline of the proof.

By Proposition 5.11, Theorem 5.12, together with Theorem 1.10 (iii), we have, for \( b \in \mathbb{B}^* \) such that \( \varepsilon_j(b) = m > 0 \),

\[
(5.15.1) \quad e'_j b = [\varepsilon_j(b)]_j \tilde{e}_j b + \sum_{b' \in \mathbb{B}^*} a_{bb'} b', \quad (a_{bb'} \in A),
\]

where \( \varepsilon_j(\tilde{e}_j b) = \varepsilon_j(b) - 1 \), and \( a_{bb'} = 0 \) unless \( \varepsilon_j(b') < \varepsilon_j(b) - 1 \).

We define \( \tilde{f}_j : \mathbb{B}_{\beta-\alpha_j} \rightarrow \mathbb{B}_j \) as the transpose of the map \( \tilde{e}_j : \mathbb{B}_j^* \rightarrow \mathbb{B}_{\beta-\alpha_j}^* \), i.e., \( \tilde{f}_j(b) = b' \) if and only if \( \tilde{e}_j((b')_*) = b_*, \) where \( b_*, (b')_* \) are dual of \( b, b' \), respectively. \( \tilde{f}_j \) gives a bijection \( \mathbb{B}_{\beta-\alpha_j} \cong \mathbb{B}_j \). Since \( \mathbb{B} \) is a dual basis of \( \mathbb{B}^* \), (5.15.1) and (5.14.1) implies, for \( b \in \mathbb{B} \), that

\[
(5.15.2) \quad f_j b = [\varepsilon_j(b) + 1]_j \tilde{f}_j b + \sum_{b' \in \mathbb{B}} c_{bb'} b', \quad (c_{bb'} \in A),
\]

where \( \varepsilon_j(\tilde{f}_j b) = \varepsilon_j(b) + 1 \), and \( c_{bb'} = 0 \) unless \( \varepsilon_j(b') > \varepsilon_j(b) + 1 \). By (5.15.2), we have the following.

(5.15.3) Assume that \( b \in \mathbb{B} \) is such that \( \varepsilon_j(b) = 0 \). Then for any \( n \geq 1 \), there exists \( b_1 \in \mathbb{B} \) such that

\[
f_j^{(n)} b = b_1 + \sum_{b' \in \mathbb{B}} c_{bb'} b', \quad (c_{bb'} \in A),
\]

where \( \varepsilon_j(b_1) = n \), and \( c_{bb'} = 0 \) unless \( \varepsilon_j(b') > n \). Note that \( c_{bb'} \in A \) follows from Corollary 4.15.

For each \( j \in J, n \in \mathbb{N} \), set \( \mathbb{B}_{n;j} = \{ b \in \mathbb{B} \mid \varepsilon_j(b) = n \} \). We define a map \( \pi_{n;j} : \mathbb{B}_{n;j} \rightarrow \mathbb{B}_{n;j} \) by \( b \mapsto b_1 \) in (5.15.3). \( \pi_{n;j} \) is a bijection since \( \tilde{f}_j \) is a bijection.

By using Proposition 5.11, we have

\[
\bigcap_{j \in J} \{ b \in \mathbb{B}^* \mid \varepsilon_j(b) = 0 \} = [1] \in A K(\mathcal{L}_0)
\]

This implies, for each \( \beta \in Q_+^\sigma \), that

\[
(5.15.4) \quad \mathbb{B}_\beta = \bigcup_{\begin{subarray}{c} j \in J, n > 0 \\ \beta - n\alpha_j \in Q_+^\sigma \end{subarray}} \pi_{n;j}(\mathbb{B}_{n;j} \cap \mathbb{B}_{\beta-n\alpha_j}).
\]
We now consider the isomorphism \( \gamma^{-1} : \mathbb{A}K(\mathcal{P}) \xrightarrow{\sim} \mathbb{A}\mathbf{U}_q^{-} \), and act \( \gamma^{-1} \) on both sides of (5.15.4). Since \( \gamma \) commutes with the action of \( \mathfrak{f}_j^{(n)} \), the right hand side of (5.15.4) corresponds exactly an inductive construction of canonical basis (see [L2, 14.4.2, Thm. 14.4.3]). Thus \( \gamma^{-1}(\mathcal{B}) \) coincides with the canonical basis \( \mathcal{B}^{\natural} \) of \( \mathbb{A}\mathbf{U}_q^{-} \). The theorem now follows from Theorem 5.12.

\[ \square \]

**Remark 5.17.** We have deduced the coincidence of \( \mathcal{B} \) and \( \mathcal{B}^{\natural} \) by making use of the crystal structure of the basis \( \mathcal{B}^{\ast} \) of \( K(\mathcal{L}) \). However, if we assume the properties of canonical basis as in (5.15.4), then the coincidence \( \mathcal{B} = \mathcal{B}^{\natural} \) is shown, directly without appealing KLR algebras, by using a similar formula as (5.15.3) for \( \mathcal{B} \).

**References**

[GL] I. Grojnowski and G. Lusztig; A comparison of bases of quantized enveloping algebras, Linear algebraic groups and their representations, Contemp. Math. 153 (1993), pp. 11-19.

[K1] M. Kashiwara; On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J. 63 (1991), 465-516.

[K2] M. Kashiwara; Global crystal bases of quantum groups, Duke Math. J. 69 (1993), 455-485.

[KL] M. Khovanov and A. D. Lauda; A diagrammatic approach to categorification of quantum groups, I, Represent. Theory, 13 (2009), 309-347.

[L1] G. Lusztig; Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), 365-421.

[L2] G. Lusztig; Introduction to quantum groups, Progress in Math. Vol. 110 Birkhauser, Boston/Basel/Berlin, 1993.

[M] P. J. McNamara; Folding KLR algebras, J. London Math. Soc (2), 100 (2019), 447-469.

[MSZ] Y. Ma, T. Shoji and Z. Zhou; Diagram automorphisms and canonical bases for quantized enveloping algebras, to appear in J. Algebra.

[R1] R. Rouquier; 2-Kac-Moody algebras, Preprint, 2008, arXiv:0812.5023.

[R2] R. Rouquier; Quiver Hecke algebras and 2-Lie algebras, Algebra Colloq. 19 (2012), 359-410.

[VV] M. Varagnolo and E. Vasserot; Canonical bases and KLR-algebras, J. reine angew. Math. 659 (2011), 67-100.

Y. Ma
School of Mathematical Sciences, Tongji University
1239 Siping Road, Shanghai 200092, P.R. China
E-mail: 1631861@tongji.edu.cn

T. Shoji
School of Mathematics Sciences, Tongji University
1239 Siping Road, Shanghai 200092, P.R. China
E-mail: shoji@tongji.edu.cn
Z. Zhou
School of Mathematical Sciences, Tongji University
1239 Siping Road, Shanghai 200092, P.R. China
E-mail: forza2p2h0u@163.com