A note on radial solutions to the critical Lane-Emden equation with a variable coefficient

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Abstract

In this note, we consider the following problem

\[
\begin{aligned}
-\Delta u &= (1 + g(x))u^{\frac{N+2}{N-2}} , \quad u > 0 \text{ in } B, \\
u &= 0 \text{ on } \partial B,
\end{aligned}
\]

where \(N \geq 3\) and \(B \subset \mathbb{R}^N\) is a unit ball centered at the origin and \(g(x)\) is a radial Hölder continuous function such that \(g(0) = 0\). We prove the existence and nonexistence of radial solutions by the variational method with the concentration compactness analysis and the Pohozaev identity.

1 Introduction

We study the following problem

\[
\begin{aligned}
-\Delta u &= (1 + g(x))u^{\frac{N+2}{N-2}} , \quad u > 0 \text{ in } B, \\
u &= 0 \text{ on } \partial B,
\end{aligned}
\] (1.1)

where \(B \subset \mathbb{R}^N\) is a unit ball centered at the origin with \(N \geq 3\), \(g\) is a locally Hölder continuous function in \(B\) and radial, i.e., \(g(x) = g(|x|)\). We note that a typical case is given by \(g(x) = |x|^\beta\) with \(\beta \geq 0\). We will show some existence and nonexistence results on (1.1).

First let us consider the next basic problem which is extensively investigated by many authors;

\[
\begin{aligned}
-\Delta u &= u^{\frac{N+2}{N-2}} , \quad u > 0 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{aligned}
\] (1.2)

where \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^N\) with \(N \geq 3\). Since the nonlinearity \(u^{\frac{N+2}{N-2}}\) has the critical growth, as is well-known, due to the lack of the compactness of the associated Sobolev embedding \(H^1_0(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)\), the existence/nonexistence of solutions of (1.2) becomes a very delicate and interesting question. In fact, in contrast to the subcritical case, we can prove that (1.2) has no smooth solution if \(\Omega\) is a star-shaped domain by the Pohozaev identity [15].
(See also [6]). Hence in order to ensure the existence of solutions of (1.2), we need some “perturbation” to (1.2). A celebrated work in this direction is given by [6]. They add a lower order term \( \lambda u^q \) \((1 \leq q < (N+2)/(N-2))\) to the critical nonlinearity \( u^{\frac{N+2}{N-2}} \) (i.e., replace \( u^{\frac{N+2}{N-2}} \) by \( u^{\frac{N+2}{N-2}} + \lambda u^q \)) and successfully show the existence of solutions of (1.2). After that, [8], [5] and [4] prove that the topological perturbation to the domain can also induce solutions to (1.2). See also [9], [14] for the effect of the geometric perturbation to the domain. Furthermore, another perturbation is found by Ni [13]. He considers a variable coefficient \( |x|\alpha \) with \( \alpha > 0 \) on \( u^{\frac{N+2}{N-2}} \). More precisely he investigates

\[
\begin{align*}
-\Delta u &= |x|\alpha u^p, \quad u > 0 \text{ in } B, \\
u &= 0 \text{ on } \partial B,
\end{align*}
\]

(1.3)

where \( \alpha > 0 \) and \( p \in \left(1, \frac{N+2+2\alpha}{N-2}\right)\). The crucial role of the variable coefficient \( |x|\alpha \) appears in the following compactness lemma for radially symmetric functions in \( H^1_0(B) \). Here we define \( H^r_0(B) \) is a subspace of \( H^1_0(B) \) which consists of all radial functions.

**Lemma 1.1** ([13]). The map \( u \mapsto |x|^m u \) from \( H^r_0(B) \) to \( L^p(B) \) is compact, for \( p \in [1, \tilde{m}) \) where

\[
\tilde{m} = \begin{cases} 
\frac{2N}{N-2-2m} & \text{if } m < \frac{N-2}{2} \\
\infty & \text{otherwise}
\end{cases}
\]

Applying this, one successfully obtains the existence of a mountain pass solution of (1.3) for all \( p \in \left(1, \frac{N+2+2\alpha}{N-2}\right)\). The exponent \( p \) can be supercritical (i.e., \( p > \frac{N+2}{N-2} \)) if \( \alpha > 0 \). We here note that, for the critical case, the essential point seems that \( u^{\frac{N+2}{N-2}} \) has a variable coefficient which is radial and attains 0 at the origin (see Example 2.1 in [17]). In view of this it is an interesting question that whether it is possible to ensure the existence of solutions in the case where the coefficient does not attain 0 at the origin. Very recently, Ai-Cowan [2] study another problem including our problem (1.1). Applying their dynamical system approach, which is developed in [1], we can confirm the existence of radially symmetric solutions of (1.1) for the case \( g(x) = |x|^\beta \) with \( \beta \in (0, N-2) \). An interesting point in this case is that the coefficient \((1 + g(x))\) attains the local minimum at the origin but not 0. Hence we cannot apply Lemma 1.1 directly. Then it is an interesting question to investigate how the coefficient can exclude the non-compactness of their nonlinearity. Motivated by this, we investigate (1.1) via the variational method. Our aim is to give a variational interpretation on the results in [2] and further, to extend their results to a more general coefficient which has a local minimum at the origin.

Now in order to explain our main results, we give an observation to the results in [2]. In the variational point of view, it seems better to write the right hand side of the equation of (1.1) as \( u^{\frac{N+2}{N-2}} + g(x)u^{\frac{N+2}{N-2}} \). Then the first term is actually noncompact. On the other hand, the second one becomes compact by Lemma 1.1 if \( g(x) \) behaves like \( |x|^\beta \) with \( \beta > 0 \). Then we clearly expect that it would
play the role of the subcritical perturbation $\lambda u^q$ with $1 \leq q < (N + 2)/(N - 2)$ in (1) mentioned above.

Then, it is natural to consider the next more general problem. (See also the generalization in [2].)

$$
\begin{cases}
-\Delta u = u^{\frac{N+2}{N-2}} + \lambda k(x)f(u), \ u > 0 \quad \text{in } B, \\
u = 0 \quad \text{on } \partial B
\end{cases}
$$

(1.4)

where $\lambda > 0$ is a parameter and $k : \overline{B} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy some of the next assumptions.

(k1) $k(x) \not\equiv 0$ is a nonnegative Hölder continuous function on $\overline{B}$ and radial, i.e., $k(x) = k(|x|)$.

(k2) $k(x) = O(|x|^\beta)$ ($|x| \to 0$) for some $\beta > 0$.

(k3) There exist constants $\gamma \geq \beta > 0$ and $C, \delta > 0$ such that $k(|x|) \geq C|x|^{\gamma}$ for all $|x| \in (0, \delta)$.

(f1) $f(t)$ is locally Hölder continuous function on $[0, \infty]$ and $f(t) \geq 0$ for all $t > 0$ and $f(t) = 0$ for all $t \leq 0$.

(f2) $\lim_{t \to 0} \frac{f(t)}{t} = 0$ and $\lim_{t \to \infty} \frac{f(t)}{t^q} = 0$ for $q = (N + 2 + 2\beta)/(N - 2)$.

(f3) There exists a constant $\theta > 2$ such that $f(t) t^{1/p} \geq \theta F(t)$ for all $t \geq 0$ where $F(t) := \int_0^t f(s)ds$.

Now, we give our main results.

**Theorem 1.2.** We have the following.

(i) If $k, f$ satisfy (k1), (k2), (k3), (f1), (f2), (f3) and further,

(f4) $\lim_{t \to \infty} \frac{f(t)}{t^p} = \infty$ for $p = \max\left\{1, \frac{2\gamma + 6 - N}{N - 2}\right\}$,

then (1.4) admits a radially symmetric solution for all $\lambda > 0$.

(ii) If $k, f$ verify (k1), (k2), (f1), (f2), (f3) and further,

(k4) there exists a point $x_0 \in \overline{B}$ such that $k(x_0) > 0$ and,

(f5) there exists a constant $c > 0$ such that $f(t) > 0$ for all $t \in (0, c)$,

then, there exists a constant $\lambda^* > 0$ such that (1.4) has a radially symmetric solution for all $\lambda > \lambda^*$.

**Remark 1.3.** The hypothesis in Theorem 1.2 (i) permits the case where $k(x) = |x|^\beta$ for $\beta > 0$ and $f(u) = u_+^q$ with any $q \in \max\{1, (2\beta + 6 - N)/(N - 2)\}, (N + 2 + 2\beta)/(N - 2))$. The condition $q > \max\{1, (2\beta + 6 - N)/(N - 2)\}$ is assumed to lower the mountain pass energy down to the level for which the local compactness of the Palais-Smale sequences is valid. See Lemmas 2.3 and 2.4 for the detail. On the other hand, (ii) is valid for $f(u) = u_+^q$ with any $q \in (1, (N + 2 + 2\beta)/(N - 2))$. 

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Remark 1.4. A similar problem is considered in [7] and [10]. The existence and nonexistence for the linear perturbation case with $k(r) = r^\beta$ for $\beta > 0$ and $f(t) = t^q$ are completed by [4]. Furthermore, the superlinear perturbation case with $k(r) = r^\beta$ for $\beta > 0$ and $f(t) = t^q$, with $q \in (1, (N + 2 + 2\beta)/(N - 2))$ is treated in [17]. Our theorem gives a generalization of a part of their results.

A nonexistence result on (1.4) is given by the Pohozaev identity as follows.

Theorem 1.5. Let $\lambda \in \mathbb{R}$, $k(x) = |x|^\beta$ with $\beta \geq 0$, $f(u) = u^q$, and $q \geq 1$. Then (1.4) admits no solution if one of the following is true;

(i) $q \in [1, (2\beta + N + 2)/(N - 2)]$ and $\lambda \leq 0$, or
(ii) $q \geq \frac{2\beta + N + 2}{N - 2}$ and $\lambda \geq 0$, or,
(iii) $\beta = 0$ and $q = (N + 2)/(N - 2)$.

Remark 1.6. The same conclusion holds even if we replace the domain $B$ by any star-shaped domain. See the argument in Section 3.

Now we come back to our main question on (1.1). The desired existence results are given as a corollary of (i) of Theorem 1.2.

Corollary 1.7. We assume

(g1) $g(x)$ is H"older continuous and $g \geq -1$ on $\overline{B}$ and radial, i.e., $g(x) = g(|x|)$,
(g2) $g(0) = 0$, and
(g3) there exist constants $\gamma \in (0, N - 2)$, $\delta \in (0, 1]$ and $C > 0$ such that $g(|x|) \geq C|x|^\gamma$ for all $|x| \in (0, \delta)$.

Then (1.1) admits at least one radially symmetric solution.

Remark 1.8. This theorem generalizes Theorem 2 in [2] for the case $g(|x|, u) = g(|x|)$. To see this, note first that their condition (6) in [2] implies (g2) and (g3). Furthermore, since (g3) is a condition for the behavior of $g$ only near the origin, we can easily construct an example which satisfies (g2) and (g3), but not (6). In addition, they prove Theorem 2 in [2] by dynamical system approach while we shall prove it via the variational method with the concentration compactness analysis. Hence our proof can give a variational interpretation and a generalization of their theorem.

By Corollary 1.7, we have the existence of solution of (1.1) if $g(x) = \lambda |x|^\beta$ with $\beta \in (0, N - 2)$ and $\lambda > 0$. For the case including $\beta \geq N - 2$, we have the next corollary as a direct consequence of (ii) in Theorem 1.2.

Corollary 1.9. Let $\lambda > 0$, $g(x) = \lambda k(x)$ and $k(x)$ is a nonnegative H"older continuous function in $\overline{B}$ such that $k(0) = 0$ and $k(|x|) = k(|x|)$. Furthermore, assume there exists a point $x_0 \in \overline{B}$ such that $k(x_0) > 0$. Then there exists a constant $\lambda^* > 0$ such that (1.1) admits at least one radially symmetric solution for all $\lambda > \lambda^*$. 

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Remark 1.10. Corollary 1.9 implies that if $g(x) = \lambda|x|^\beta$ with $\beta > 0$, a radially symmetric solution exists for all sufficiently large $\lambda > 0$. Furthermore, we remark that this generalizes Theorem 1 of [3].

The existence results above are best possible in the following sense. We have the following nonexistence result.

Theorem 1.11. Let $g(x) = \lambda|x|^\beta$ with $\beta \geq 0$ and $\lambda \in \mathbb{R}$. Then (1.1) does not admit any radially symmetric solution if $\beta = 0$ and $\lambda \in \mathbb{R}$, or $\beta > 0$ and $\lambda \leq 0$. In addition if $\beta \geq N - 2$, there exists a constant $\lambda_\ast > 0$ which depends on $\beta$ and $N$ such that (1.1) has no radially symmetric solution for all $\lambda \in [0, \lambda_\ast]$.

Remark 1.12. In our computation, we can choose $\lambda_\ast = \begin{cases} \frac{2(N-1)}{N-2} & \text{if } \beta = N - 2, \\ \frac{2(N-1)}{N-2} \left( \frac{2N-2+\beta}{\beta-N+2} \right)^\frac{2-N+2}{\beta-N+2} & \text{if } \beta > N - 2. \end{cases}$

For the detail, see the proof of Theorem 1.11 in Section 3.

Organization of this paper

This paper consists of three sections with an appendix. In Section 2 we give the proof of the existence results. In Section 3 we show the nonexistence assertions by the Pohozaev identity. Lastly in Appendix A we give a remark on the proof for the reader’s convenience. Throughout this paper we define $H_r(B)$ as a subspace of $H^1_0(B)$ which consists of all the radial functions. Furthermore we put $2^* = 2N/(N - 2)$ and define the Sobolev constant $S > 0$ as usual by

$$S := \inf_{u \in H^1_0(B) \setminus \{0\}} \frac{\|u\|^2}{\int_B |\nabla u|^2 \,dx}$$

where $\|u\|^2 = \int_B |\nabla u|^2 \,dx$. Finally we define $B_s(0)$ as a $N$ dimensional ball centered at the origin with radius $s > 0$.

2 Existence results

In this section, we give a proof of the existence results of our main theorems and corollaries. In the following we always suppose (k1), (k2), (f1) and (f2). For the problem (1.4), we define the associated energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*} \int_B u^2^* \,dx - \int_B kF(u) \,dx \quad (u \in H_r(B)).$$

Then noting our assumptions and Lemma 1.4, it is standard to see that $I(u)$ is well-defined and continuously differentiable on $H_r(B)$. In addition, by (k1) and (f1), the usual elliptic theory and the strong maximum principle ensure that every critical point of $I$ is a solution of (1.4). Hence our aim becomes to look for critical points of $I$. We first prove the mountain pass geometry of $I$. 

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Lemma 2.1. We have

(a) \( \exists \rho, a > 0 \) such that \( I(u) \geq a \) for all \( u \in H_r(B) \) with \( \|u\| = \rho \), and

(b) for all \( u \in H_r(B) \setminus \{0\} \), \( I(tu) \to -\infty \) as \( t \to \infty \), for all \( \lambda > 0 \).

Proof. First note that by (f1) and (f2), we have that for any \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that 
\[
|f(t)| \leq \varepsilon t + C t^p
\]
for all \( t \geq 0 \) and some \( p \in (1, (N + 2 + 2\beta)/(N - 2)) \). Then Lemma 1.1 and the Sobolev inequality give
\[
I(u) \geq \left( \frac{1}{2} - \frac{\lambda \varepsilon}{\mu_1} \right) \|u\|^2 - \lambda C \|u\|^{p+1} - C \|u\|^2
\]
for all \( u \in H_r(B) \). Taking \( \varepsilon \in (0, \mu_1/(4\lambda)) \), we get (a) for all \( \lambda \in (0, \infty) \).

Next, since \( k(x)f(u) \geq 0 \) for all \( x \in B \) and \( u \in \mathbb{R} \), we have for all \( t > 0 \) and \( u \in H_r(B) \setminus \{0\} \) that
\[
I(tu) \leq \frac{t^2}{2} \|u\|^2 - \frac{t^{2^*}}{2^*} \int_B u^{2^*} dx.
\]
Since \( 2 < 2^* \), we obtain \( I(tu) \to -\infty \) as \( t \to \infty \), which shows (b). This finishes the proof.

Noting Lemma 2.1, we define
\[
\Gamma := \{ \gamma \in C([0, 1], H_r(B)) \mid \gamma(0) = 0, \gamma(1) = e \}
\]
with \( e \in H_r(B) \) satisfying \( \|e\| > \rho \) and \( I(e) \leq 0 \). Then we put
\[
c_\lambda := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0, 1])} I(u).
\]
We next show the local compactness property of the Palais-Smale sequences of \( I \). Here, as usual, we call \( (u_n) \subset H_r(B) \) is a (PS)\(_c\) sequence for \( I \) if \( I(u_n) \to c \) for some \( c \in \mathbb{R} \) and \( I'(u_n) \to 0 \) in \( H_{r^{-1}}(B) \) as \( n \to \infty \) where \( H_{r^{-1}}(B) \) is the dual space of \( H_r(B) \).

Lemma 2.2. Suppose \( f \) satisfies (f3) and \( \lambda > 0 \). If \( (u_n) \subset H_r(B) \) is a (PS)\(_c\) sequence for a value \( c < S^{N/2}/N \), then \( (u_n) \) has a subsequence which strongly converges in \( H_r(B) \) as \( n \to \infty \).

Proof. By (f3), we obtain that
\[
c + o(1) = I(u_n) - \frac{1}{\min\{2^*, \theta\}} \langle I'(u_n), u_n \rangle + o(1)\|u_n\|
\geq \left( \frac{1}{2} - \frac{1}{\min\{2^*, \theta\}} \right) \|u_n\|^2 + o(1)\|u_n\|
\]
This shows the claim. Hence noting (f1), (f2) and Lemma 1.1, we have that, up to a subsequence, there exists a nonnegative function \( u \in H^r(B) \) such that

\[
\begin{aligned}
&u_n \rightharpoonup u \text{ weakly in } H^1_0(B), \\
&\int_B kf(u_n) dx \to \int_B kf(u) dx, \\
&\int_B r^\beta |u_n|^{s+1} dx \to \int_B r^\beta |u|^{s+1} dx \text{ for any } s \in [1, (N + 2 + 2\beta)/(N - 2)), \\
&u_n \to u \text{ a.e. on } B,
\end{aligned}
\]

as \( n \to \infty \). Furthermore, since \( (u_n) \subset H^r(B) \), the concentration compactness lemma (Lemma I.1 in [12]) implies that there exist values \( \nu_0, \mu_0 \geq 0 \) such that

\[
|\nabla u_n|^2 \rightharpoonup d\mu \geq |\nabla u|^2 + \mu_0 \delta_0, \\
(u_n)^2^+ \rightharpoonup d\nu = u^{2^*} + \nu_0 \delta_0,
\]

in the measure sense where \( \delta_0 \) denotes the Dirac measure with mass 1 which concentrates at \( 0 \in \mathbb{R}^N \) and

\[
S_{\nu_0}^{\frac{2}{2^*}} \leq \mu_0. \tag{2.2}
\]

Let us show \( \nu_0 = 0 \). If not, we define a smooth test function \( \phi \) in \( \mathbb{R}^N \) such that \( \phi = 1 \) on \( B(0, \varepsilon) \), \( \phi = 0 \) on \( B(0, 2\varepsilon)^c \) and \( 0 \leq \phi \leq 1 \) otherwise. We also assume \( |\nabla \phi| \leq 2/\varepsilon \). Then noting (f1), (f2) and using (k1), (k2), (2.1) and Lemma 1.1, we get

\[
0 = \lim_{n \to \infty} (I'(u_n), u_n \phi)
\]

\[
= \lim_{n \to \infty} \left( \int_B \nabla u_n \cdot \nabla (u_n \phi) dx - \int_B (u_n)^2^+ \phi dx - \lambda \int_B kf(u_n) u_n \phi dx \right)
\]

\[
= \lim_{n \to \infty} \left( \int_B |\nabla u_n|^2 \phi dx - \int_B (u_n)^2^+ \phi dx - \lambda \int_B kf(u_n) u_n \phi dx + \int_B u_n \nabla u_n \cdot \nabla \phi dx \right)
\]

\[
= \int_B \phi d\mu - \int_B \phi d\nu + o(1)
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). It follows that

\[
0 \geq \mu_0 - \nu_0.
\]

Then by (2.2), we obtain

\[
\nu_0 \geq S_{\nu_0}^{\frac{2}{2^*}}.
\]

Using this estimate, we have by (f3) that

\[
c = \lim_{n \to \infty} \left( I(u_n) - \frac{1}{2} (I'(u_n), u_n) \right)
\]

\[
\geq \frac{1}{N} \lim_{n \to \infty} \int_B d\nu
\]

\[
\geq \frac{S_{\nu_0}^{\frac{2}{2^*}}}{N}
\]
which contradicts our assumption. It follows that
\[ \lim_{n \to \infty} \int_B (u_n)_2^* \, dx = \int_B u^2 \, dx. \]
Then the usual argument proves \( u_n \to u \) in \( H_r(B) \). We finish the proof. \( \Box \)

Next we estimate the mountain pass energy \( c_\lambda \). To do this, we use the Talenti function \( U_\varepsilon(x) := \frac{\varepsilon^{2-N} N - 2}{(\varepsilon^2 + |x|^2)^{N-2}} \). Moreover we define a cut off function \( \psi \in C_0^\infty(B) \) such that \( \psi(x) = \psi(|x|) \), supp\{\( \psi \)\} \( \subset B_{\delta}(0) \) and \( \psi = 1 \) on \( B_{\eta}(0) \) for some \( \eta \in (0, \delta) \). We set \( u_\varepsilon := \psi U_\varepsilon \) and \( v_\varepsilon := u_\varepsilon/\|u_\varepsilon\|_{L^2(B)} \in H_r(B) \). Then, if \( q > \max(2\gamma + 6 - N)/(N - 2) \), a similar calculation with that in [6] shows that
\[
\begin{aligned}
&\|v_\varepsilon\|^2 = S + O(\varepsilon^{N-2}) \\
&\|v_\varepsilon\|_{L^2(B)} = 1, \\
&\int_B kv_\varepsilon^{q+1} \, dx \geq C \int_B |x|^{\gamma} v_\varepsilon^{q+1} \, dx = C' \varepsilon^a + O(\varepsilon^{N-2})
\end{aligned}
\]
(2.3)
where \( a = \gamma + N - \frac{(N-2)(q+1)}{2} \) and \( C, C' > 0 \) are constants. Let us prove the next lemma (Cf. Lemma 2.1 in [6].)

**Lemma 2.3.** Assume that \( k \) verifies (k3). Then if
\[ \lim_{\varepsilon \to 0} \varepsilon^{\gamma+2} \int_0^{\varepsilon^{-1}} F \left( \frac{\varepsilon^{-1} r}{1 + r^2} \right)^{\frac{N-2}{2}} r^{\gamma+N-1} \, dr = \infty \]
holds, we have \( c_\lambda < S^{N/2}/N \) for all \( \lambda > 0 \).

**Proof.** Let \( v_\varepsilon \in H_r(B) \) as above. Then from Lemma 2.1, we find a constant \( t_\varepsilon > 0 \) such that \( I(t_\varepsilon v_\varepsilon) = \max_{t \geq 0} I(tv_\varepsilon) \). Since
\[ 0 = \frac{d}{dt} \big|_{t=t_\varepsilon} I(tv_\varepsilon) = t_\varepsilon \|v_\varepsilon\|^2 - t_\varepsilon^{q-1} - \int_B kf(t_\varepsilon v_\varepsilon) v_\varepsilon \, dx \]
and \( \int_B kf(v_\varepsilon) v_\varepsilon \, dx \geq 0 \) by (k1) and (f1), we have
\[ t_\varepsilon \leq \|v_\varepsilon\|_{\overline{W}} =: T_\varepsilon. \]

Since \( T_\varepsilon = \|v_\varepsilon\|_{\overline{W}}^2 \) is the maximum point of the map \( t \mapsto \frac{T_\varepsilon^2}{2} \|v_\varepsilon\|^2 - \frac{T_\varepsilon^q}{2} \), we get by (2.3) that for any \( t > 0 \)
\[ I(tv_\varepsilon) \leq I(t_\varepsilon v_\varepsilon) \leq \frac{T_\varepsilon^2}{2} \|v_\varepsilon\|^2 - \frac{T_\varepsilon^q}{2} - \int_B kf(t_\varepsilon v_\varepsilon) \, dx \leq \frac{S^N}{N} - \int_B kf(t_\varepsilon v_\varepsilon) \, dx + O(\varepsilon^{N-2}). \]
Therefore once we prove
\[ \lim_{\varepsilon \to 0} \varepsilon^{-(N-2)} \int_B k F(t\varepsilon v\varepsilon) dx = \infty, \] (2.5)
we conclude \( c_{\lambda} \leq I(t\varepsilon v\varepsilon) < S^{N/2}/N \) for all small \( \varepsilon > 0 \). This completes the proof. Lastly let us ensure (2.5). To do this, we first claim that \( \lim_{\varepsilon \to 0} t\varepsilon \rightarrow S^{(N-2)/4} \). Indeed, using (f2), for any \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that
\[ \int_B \frac{\varkappa f(t\varepsilon v\varepsilon)^\varepsilon}{t\varepsilon} dx \leq t\varepsilon^{\varepsilon-1}\delta \int_B |x|^{\varepsilon^2} v\varepsilon^{\varepsilon+1} dx + C_\delta \int_B |x|^{\varepsilon^2} v\varepsilon^{\varepsilon+2} dx. \]
Since \( t\varepsilon \leq T\varepsilon = O(1) \), \( \int_B |x|^{\varepsilon^2} v\varepsilon^{\varepsilon+1} dx = O(1) \) by \( q = (N+2+2\beta)/(N-2) \) and \( \int_B |x|^{\varepsilon^2} v\varepsilon^{\varepsilon+2} dx = o(1) \) as \( \varepsilon \to 0 \), we get
\[ \lim_{\varepsilon \to 0} \int_B \frac{\varkappa f(t\varepsilon v\varepsilon)^\varepsilon}{t\varepsilon} dx = 0. \]
Then since \( \langle f(t\varepsilon v\varepsilon), v\varepsilon \rangle = 0 \), we have
\[ t\varepsilon = \left( \|v\varepsilon\|^2 - \int_B \frac{\varkappa f(t\varepsilon v\varepsilon)^\varepsilon}{t\varepsilon} dx \right)^{-1}. \]
This with (2.3) proves the claim. In particular, \( t\varepsilon \) converges to a positive value as \( \varepsilon \to 0 \). Now we calculate by (k3) that
\[ \varepsilon^{-(N-2)} \int_B k F(t\varepsilon v\varepsilon) dx \geq C_1 \varepsilon^{-(N-2)} \int_0^\eta F \left[ t\varepsilon \left( \frac{\varepsilon}{\varepsilon^2 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr \]
\[ \geq C_2 \varepsilon^{\gamma + 2} \int_0^\eta F \left[ t\varepsilon \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr \]
\[ \geq C_3 \varepsilon^{\gamma + 2} \int_0^\eta F \left[ \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr \]
for some constant \( C_1, C_2, C_3, D > 0 \) where in the last inequality we replace \( \varepsilon / t\varepsilon^{(N-2)/2} \) by \( \varepsilon \) which does not change the conclusion below. If \( D \geq 1 \), we clearly get (2.5) by our assumption (2.4). If \( D < 1 \), we obtain
\[ \varepsilon^{\gamma + 2} \int_0^\eta F \left[ \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr = \varepsilon^{\gamma + 2} \int_0^\frac{1}{\beta} F \left[ \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr \]
\[ - \varepsilon^{\gamma + 2} \int_0^\frac{1}{2} F \left[ \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr. \]
Finally, note that (f2) shows
\[ \varepsilon^{\gamma + 2} \int_0^\frac{1}{2} F \left[ \left( \frac{\varepsilon^{-1}}{1 + r^2} \right)^{N-2} \right] r^{\gamma+N-1} dr = o(1) \]
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). This finishes the proof. \( \square \)
The next lemma confirms that under our assumptions, $f(t)$ satisfies (2.4).

**Lemma 2.4.** Assume (k3). Then, if $f$ satisfies (f4), then (2.4) holds true.

**Proof.** By (f4), for any $M > 0$, there exists a constant $R > 0$ such that $f(t) \geq Mt^p$ where $p = \max\{1, \frac{2z+6-N}{N-2}\}$. Furthermore, note that if $r \leq C\varepsilon^{-1/2}$ for $C = (2R)^{-(N-2)/2}$, we get

$$\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}} \geq R$$

for all small $\varepsilon > 0$. It follows that

$$\varepsilon^{\gamma+2} \int_0^{\varepsilon^{-1}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1} dr \geq \varepsilon^{\gamma+2} \int_0^{C\varepsilon^{-1/2}} F\left[\left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{N-2}{2}}\right] r^{\gamma+N-1} dr \geq \varepsilon^{\gamma+2} \frac{M}{p+1} \int_0^{C\varepsilon^{-1/2}} \left(\frac{\varepsilon^{-1}}{1+r^2}\right)^{\frac{(N-2)(p+1)}{2}} r^{\gamma+N-1} dr \to \infty$$

as $\varepsilon \to 0$. This completes the proof.

**Lemma 2.5.** If $k, f$ satisfy (k4) and (f5), we have a constant $\lambda^* > 0$ such that $c_\lambda \leq S^{N/2}/N$ for all $\lambda > \lambda^*$.

**Proof.** Since $k(x_0) > 0$ by (k4), there exist constants $0 < r_1 < |x_0| < r_2 < 1$ such that $k > 0$ on $B(0,r_2) \setminus B(0,r_1)$. Then we choose a radial function $u \in C^\infty_0(B) \setminus \{0\}$ such that $u \geq 0$ and supp$\{u\} \subset B(0,r_2) \setminus B(0,r_1)$. Then by Lemma 2.4 we have a constant $t_\lambda > 0$ such that $I(t_\lambda u) = \max_{t>0} I(tu)$. Since $\frac{d}{dt}|_{t=t_\lambda} I(tu) = 0$, we get

$$\|u\|^2 - t_\lambda^{2\gamma-2} \int_B u^{2\gamma} \, dx - \lambda \int_B \frac{k(t_\lambda u)u}{t_\lambda} \, dx = 0$$

It follows that $t_\lambda \to 0$ as $\lambda \to \infty$. If not, there exists a sequence $(\lambda_n) \subset (0,\infty)$ such that $\lambda_n \to \infty$ and $t_{\lambda_n} \to t_0 > 0$ for some value $t_0 > 0$ as $n \to \infty$. But this is impossible in view of the previous formula and (f5). Then it follows from (k1) and (f1) that

$$c_\lambda \leq I(t_\lambda u) \leq t_0^2 \|u\|^2 \to 0$$

as $\lambda \to \infty$. This finishes the proof.

Then we prove the existence assertions of main theorems.

**Proof of Theorem 1.2.** First note that under the assumption in Lemma 2.4 and the mountain pass theorem (3, see also Theorem 2.2 in [6]), there exists a $(PS)_{c_\lambda}$ sequence $(u_n) \subset H_r(B)$ of $I$. Hence our aim is to see that $(u_n)$ has a subsequence which strongly converges in $H_r(B)$. This fact follows from Lemmas 2.1, 2.2 and 2.3, which proves (i). The proof of (ii) is completed by Lemmas 2.1, 2.2 and 2.4. This completes the proof of Theorem 1.2.
Proof of Corollary 1.7. The proof is clear from (i) of Theorem 1.2. Here we remark on (g1) and (g2). We first note that non-negativity of $k$ in (k1) is needed only to apply the maximum principle. Hence it is clear that in the present case it can be weakened to $g \geq -1$ in (g1). Furthermore, by (g1), the associated energy functional

$$I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2^*} \int_B (1 + g)|u|^{2^*} dx$$

is always well-defined. Hence we can weaken (k2) in Theorem 1.2 to the condition $k(0) = 0$. Finally, in the present case, since we do not assume $k(|x|) = O(|x|^\beta)$ for $\beta > 0$, in principle, we cannot use Lemma 1.1 directly in the proof of Lemma 2.2. Although the modification is trivial, we will give the modified proof in Appendix A for the readers’ convenience.

Proof of Corollary 1.9. The proof is immediate by (ii) of Theorem 1.2.

3 Nonexistence results

In this section, we prove the nonexistence results by the Pohozaev identity. Since some results still hold true for the star-shaped domain, we first consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + g|u|^{q-1}u \text{ in } \Omega \\ u = 0 \text{ on } \partial \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ is a bounded smooth domain, $q \geq 1$ and $g$ is a $C^1$ function. Now, let us recall the formula

$$\int_{\Omega} \left\{ \frac{x \cdot \nabla g}{q + 1} + \left( \frac{N}{q + 1} - \frac{N - 2}{2} \right) g \right\} |u|^q du = \frac{1}{2} \int_{\partial \Omega} (x \cdot \nu)|\nabla u|^2 ds$$

holds for any solution $u \in C^1(\Omega)$. This is the Pohozaev identity for (3.1).

Theorem 3.1. Let $\lambda \in \mathbb{R}$ and $g(x) = \lambda|x|^\beta$ with $\beta \geq 0$. Then if $\Omega$ is a star-shaped domain, (3.1) has no $C^1$ solution if either one of the following holds:

(i) $\lambda \leq 0$ and $q \leq (N + 2 + 2\beta)/(N - 2)$ or,

(ii) $\lambda \geq 0$ and $q \geq (N + 2 + 2\beta)/(N - 2)$ or otherwise,

(iii) $\beta = 0$, $\lambda \in \mathbb{R}$ and $q = (N + 2)/(N - 2)$.

Proof. Let $u \in C^1(\Omega)$ be a solution of (3.1). Then under the assumption in the theorem, we get by (3.2) that

$$\lambda \int_{\Omega} \left( \frac{\beta + N}{q + 1} - \frac{N - 2}{2} \right) |x|^\beta |u|^{q+1} du = \frac{1}{2} \int_{\partial \Omega} (x \cdot \nu)|\nabla u|^2 ds.$$ 

Then if one of (i)-(iii) holds, the left hand side is nonpositive. On the other hand, since $x \cdot \nu \geq 0$ by our assumption, we have $|\nabla u| \equiv 0$ on $\partial \Omega$. Then from
the principle of unique continuation we must have \( u \equiv 0 \) in \( \Omega \). This shows the proof.

**Proof of Theorem 1.5.** The proof is a direct consequence of Theorem 3.1.

Lastly let us show the proof of Theorem 1.11. To do this, we assume \( q \geq 1 \) and \( u = u(r) \ (r \in [0, 1]) \) is a solution of

\[
\begin{cases}
-u'' - \frac{(N - 1)}{r}u' = |u|^{\frac{q}{q-1}}u + g|u|^{q-1}u & \text{in } (0, 1), \\
u'(0) = 0 = u(1).
\end{cases}
\]

(3.3)

with a \( C^1 \) function \( g(r) \) on \([0, 1]\). In addition, we suppose \( \psi(r) \ (r \in [0, 1]) \) is a smooth test function such that \( \psi(0) = 0 \). Then we have the following. (See [6] and also [11].)

**Theorem 3.2.** If \( u \) is a solution of (3.3), we get

\[
\psi(1)|u'(1)|^2 = \frac{1}{2} \int_0^1 u^2r^{N-4} \left\{ r^3\psi''' - (N - 1)(N - 3)r\psi' + (N - 1)(N - 3)\psi \right\} dr \\
+ \frac{2(N - 1)}{N} \int_0^1 |u|^2 \left\{ r^{N-1}\psi' - r^{N-2}\psi \right\} dr \\
+ \frac{1}{q + 1} \int_0^1 |u|^{q+1} \left\{ (q + 3)gr^{N-1}\psi' - (q - 1)(N - 1)gr^{N-2}\psi + 2gr^{N-1}\psi \right\} dr.
\]

(3.4)

**Proof.** Multiplying the equation in (3.3) by \( r^{N-1}\psi u' \) gives

\[
\psi(1)|u'(1)|^2 - \int_0^1 |u'|^2 \left\{ r^{N-1}\psi' - (N - 1)r^{N-2}\psi \right\} dr \\
= \frac{N - 2}{N} \int_0^1 |u|^2 \left\{ r^{N-1}\psi' + (N - 1)r^{N-2}\psi \right\} dr \\
+ \frac{\lambda(q + 1)}{2} \int_0^1 |u|^{q+1} \left\{ g'r^{N-1}\psi + r^{N-2}g\psi' + (N - 1)r^{N-2}g \right\} dr.
\]

(3.5)

On the other hand, we multiply the equation in (3.3) by \( (r^{N-1}\psi' - (N - 1)r^{N-2}\psi)u \) and compute

\[
\int_0^1 |u|^2 \left\{ r^{N-1}\psi' - (N - 1)r^{N-2}\psi \right\} dr \\
- \frac{1}{2} \int_0^1 u^2 \left\{ r^{N-1}\psi''' + (N - 1)(N - 3)r^{N-4}(\psi - rv') \right\} dr \\
= \int_0^1 |u|^2 \left\{ r^{N-1}\psi' - (N - 1)r^{N-2}\psi \right\} dr \\
+ \lambda \int_0^1 g(r)|u|^{q+1} \left\{ r^{N-1}\psi' - (N - 1)r^{N-2}\psi \right\} dr.
\]

(3.6)

Combining (3.5) and (3.6), we complete the proof.

\[
\square
\]
Proof of Theorem 1.11  The first assertion follows from Theorem 3.1. Let us prove the second assertion. To do this, assume \( \lambda \geq 0 \) and \( u = u(r) \) \( (r \in [0,1]) \) is a radially symmetric solution of \((1.1)\). Then it satisfies

\[
\begin{cases}
-uu'' - \frac{(N-1)}{r} u' = (1 + g)|u|^\frac{4}{N-2}u \text{ in } (0,1), \\
u'(0) = u(1) = 0,
\end{cases}
\]

(3.7)

where we put \( g(r) = \lambda r^\beta \). Again choose a smooth test function \( \psi \) such that \( \psi(0) = 0 \). Then by Theorem 3.2 we have

\[
\frac{1}{2} \int_0^1 u^2 r^{N-4} \{ r^3 \psi'' - (N-1)(N-3) r \psi' + (N-1)(N-3) \psi \} \, dr = \psi(1)|u'(1)|^2
\]

\[
+ \frac{1}{N} \int_0^1 |u|^2 \{ -(N-2) g' r^{N-1} \psi + 2(N-1)(1 + g(r))(r^{N-2} \psi - r^{N-1} \psi') \} \, dr.
\]

(3.8)

We fix \( \beta \geq N-2 \) and then select \( \psi(r) = ar^{N-1} + br \) so that \( r^3 \psi'' - (N-1)(N-3) r \psi' + (N-1)(N-3) \psi = 0 \) and \( \psi(0) = 0 \). This ODE has an explicit solution \( \psi(r) = ar^{N-1} + br + cr^{-(N-3)} \) where \( a, b, c \in \mathbb{R} \) are arbitrary constants. Since we assume \( \psi(0) = 0 \), we must have \( c = 0 \), i.e., \( \psi(r) = ar^{N-1} + br \). Then we get

\[
\psi(1)|u'(1)|^2
\]

\[
+ \frac{1}{N} \int_0^1 |u|^2 \{ -(N-2) g' r^{N-1} \psi + 2(N-1)(1 + g(r))(r^{N-2} \psi - r^{N-1} \psi') \} \, dr = 0.
\]

(3.9)

Substituting \( \psi(r) = ar^{N-1} + br \) into

\[
h(r) := -(N-2) k' r^{N-1} \psi + 2(N-1)(1 + k)(r^{N-2} \psi - r^{N-1} \psi'),
\]

we see

\[
h(r) = r^{2N-3} \times
\]

\[
\times \left[ -\lambda a(N-2) (2(N-1) + \beta) r^\beta - \lambda b \beta (N-2) r^{\beta-N+2} - 2a(N-1)(N-2) \right].
\]

Finally, we choose \( a < 0 \) and \( b = |a| > 0 \). In particular, we have \( \psi(1) = a + b = 0 \).

Then some elementary calculations show that if we set

\[
\lambda_* = \begin{cases}
\frac{2(N-1)}{N-2} \text{ if } \beta = N-2, \\
\frac{2(N-1)}{N-2} \left( \frac{2N-2+\beta}{\beta-N+2} \right)^{\frac{\beta-N+2}{N-2}} \text{ if } \beta > N-2,
\end{cases}
\]

we assure that \( h \neq 0 \) and \( h \geq 0 \) for all \( \lambda \in [0, \lambda_*] \). Therefore in view of \( (3.9) \), we reach to a contradiction if \( \lambda \in [0, \lambda_*] \). This finishes the proof. \( \square \)
A Critical case

In this appendix, we give a proof of Lemma 2.2 under the assumption in Corollary 1.7 for the readers’ convenience. Especially we will use only the condition (g2) which is weaker than (k2).

Lemma A.1. Assume (g1), (g2) and \((u_n) \subset H_r(B)\) is a \((PS)_c\) sequence of

\[
I(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2r} \int_B (1 + g) u^r \, dx.
\]

Then if \(c < S^{\frac{2N}{N-2}}/N\), \((u_n)\) has a subsequence which strongly converges in \(H_r(B)\).

Proof. From the definition we have

\[
c + o(1) = I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle + o(1) \|u_n\|
\geq \frac{1}{N} \|u_n\|^2 + o(1) \|u_n\|.
\]

This implies \((u_n)\) is bounded in \(H_r(B)\). Then we can asume that there exists a nonnegative function \(u \in H_r(B)\) such that

\[
\begin{aligned}
 & u_n \rightharpoonup u \text{ weakly in } H_r(B), \\
 & u_n \rightarrow u \text{ a.e. on } B,
\end{aligned}
\]

up to a subsequence. By the concentration compactness lemma, we can suppose that there exist values \(\mu_0, \nu_0 \geq 0\) such that

\[
\begin{cases}
|\nabla u_n|^2 \to d\mu \geq |\nabla u|^2 + \mu_0 \delta_k, \\
u_n \rightharpoonup u \text{ in } L^p(B) \text{ for all } p \in (1, 2N/(N-2)), \\
(u_n)^r \to d\nu = u^r + \nu_0 \delta_0,
\end{cases}
\]

in the measure sense, where \(\delta_0\) denotes the Dirac delta measure concentrated at the origin with mass 1 as before. Furthermore, we have

\[
S\nu_0^{\frac{2}{r}} \leq \mu_0. \tag{A.1}
\]

We show \(\nu_0 = 0\). To this end, we assume \(\nu_0 > 0\) on the contrary. Then, for small \(\varepsilon > 0\), we define a smooth test function \(\phi\) as in the proof of Lemma 2.2.

Since \(I'(u_n) \to 0\) in \(H^{-1}(B)\) and \((u_n)\) is bounded, we have

\[
0 = \lim_{n \to \infty} \langle I'(u_n), u_n \phi \rangle
= \lim_{n \to \infty} \left( \int_B \nabla u_n \cdot \nabla (u_n \phi) \, dx - \int_B (1 + g)(u_n)^r \phi \, dx \right)
= \lim_{n \to \infty} \left( \int_B |\nabla u_n|^2 \phi \, dx - \int_B (1 + g)(u_n)^r \phi \, dx + \int_B u_n \nabla u_n \cdot \nabla \phi \, dx \right)
= \int_B \phi \, d\mu - \int_B (1 + g) \phi \, d\nu + o(1)
\]
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). Taking \( \varepsilon \to 0 \) and noting \( g(0) = 0 \), we obtain
\[
0 \geq \mu_0 - \nu_0.
\]
Then using (A.1), we get
\[
\nu_0 \geq S^\frac{2}{N}.
\]
Finally, noting this estimate, we see
\[
c = \lim_{n \to \infty} \left( I(u_n) - \frac{1}{2} \langle J'(u_n), u_n \rangle \right)
= \frac{1}{N} \lim_{n \to \infty} \int_B (1 + g) d\nu
\geq \frac{S^\frac{2}{N}}{N}
\]
since \( g(0) = 0 \), which is a contradiction. It follows that
\[
\lim_{n \to \infty} \int_B (1 + g)(u_n)^{2^*}_+ dx = \int_B (1 + g)u^{2^*}_+ dx.
\]
Then a standard argument shows that \( u_n \to u \) in \( H_r(B) \). This completes the proof.

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