Commutators of free random variables

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Abstract

Let $\mathcal{A}$ be a unital $C^*$-algebra, given together with a specified state $\varphi : \mathcal{A} \to \mathbb{C}$. Consider two selfadjoint elements $a, b$ of $\mathcal{A}$, which are free with respect to $\varphi$ (in the sense of the free probability theory of Voiculescu). Let us denote $c := i(ab - ba)$, where the $i$ in front of the commutator is introduced to make $c$ selfadjoint. In this paper we show how the spectral distribution of $c$ can be calculated from the spectral distributions of $a$ and $b$. Some properties of the corresponding operation on probability measures are also discussed. The methods we use are combinatorial, based on the description of freeness in terms of non-crossing partitions; an important ingredient is the notion of $R$-diagonal pair, introduced and studied in our previous paper [12].

1. Introduction and presentation of the results

In this paper we show how the combinatorial description of freeness can be used to determine the distribution of the commutator of two free random variables.

The concept of freeness was introduced by Voiculescu [14] as a tool for studying free products of operator algebras. It soon became clear that it is a promising point of view to consider freeness as a non-commutative analogue of the classical probabilistic notion of independence, and this led to the development of a free probability theory (see the monograph [18], or the recent survey in [17]).

Let $a$ and $b$ be free random variables. Two basic problems in free probability theory, both solved by Voiculescu, consist in the study of $a + b$ and $ab$. The distributions of these new random variables are given by the additive ($\boxplus$) and respectively multiplicative ($\boxtimes$) free convolution of the distributions of $a$ and $b$. Voiculescu [17, 18] provided with the $R$- and $S$-transform two efficient analytical tools for dealing with $\boxplus$ and $\boxtimes$.

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To be more precise, let us recall that if \((\mathcal{A}, \varphi)\) is a non-commutative probability space – i.e. \(\mathcal{A}\) is a unital algebra, endowed with a linear functional \(\varphi: \mathcal{A} \to \mathbb{C}\) such that \(\varphi(1) = 1\) – then the distribution \(\mu_a\) of an element \(a \in \mathcal{A}\) is the linear functional on \(\mathbb{C}[X]\) defined by \(\mu_a(f) = \varphi(f(a))\) for all \(f \in \mathbb{C}[X]\). For every such distribution \(\mu: \mathbb{C}[X] \to \mathbb{C}\) (\(\mu\) linear, \(\mu(1) = 1\)) one defines its \(R\)-transform \(R(\mu)\) and its \(S\)-transform \(S(\mu)\) as special formal power series in an indeterminate \(z\) in such a way that, for \(a\) and \(b\) free in some \((\mathcal{A}, \varphi)\), we have the formulas

\[
R(\mu_{a+b}) = R(\mu_a) + R(\mu_b) \tag{1.1}
\]
and

\[
S(\mu_{ab}) = S(\mu_a) \cdot S(\mu_b), \quad \text{if } \varphi(a) \neq 0 \neq \varphi(b). \tag{1.2}
\]

In the case of selfadjoint operators, where the distributions of the random variables can be viewed as probability measures, the \(R\)- and the \(S\)-transforms are related with the corresponding Cauchy transforms. (For the definition of freeness, see Definition 2.1.4 below – or, for more details, the Section 2.5 of [18]. For the definitions of \(R(\mu)\) and \(S(\mu)\), as given by Voiculescu, we refer the reader to [18], Sections 3.2, 3.6.)

Having solved the problems of free addition and free multiplication, the next ‘canonical’ open problem is the one of the free commutator, i.e. of describing the distribution \(\mu_{ab−ba}\) in terms of \(\mu_a\) and \(\mu_b\). The goal of this paper is to present a solution to the free commutator problem.

We shall use the alternative, combinatorial, approach to free random variables which was developed by Speicher [13] and Nica and Speicher [10, 11, 12]. The main observation of [13] was that the coefficients of the \(R\)-transform \(R(\mu)\) – which we call free cumulants of \(\mu\) – can be calculated via a precise combinatorial prescription using the lattice of non-crossing partitions. Furthermore, in [10, 11] it was also shown that the multiplicative free convolution can be described directly in terms of the \(R\)-transform by

\[
R(\mu_{ab}) = R(\mu_a) \star R(\mu_b); \tag{1.3}
\]

here \(\star\) is a combinatorial convolution of formal power series, defined again by a precise combinatorial prescription using the structure of non-crossing partitions. The connection with the \(S\)-transform is achieved (see [11]) by establishing a combinatorial Fourier transform \(\mathcal{F}\) which converts the \(R\)-transform into the \(S\)-transform,

\[
S(\mu) = \mathcal{F}(R(\mu)), \tag{1.4}
\]
and which converts \(\star\) into multiplication of power series,

\[
\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{1.5}
\]

One of the most important advantages of this combinatorial approach is that it can be generalized in a straightforward way to multi-dimensional situations [13, 11]. But this is exactly what is needed for the commutator problem, because the crucial point turns out to consist in understanding, for \(a\) and \(b\) free, the relation between \(ab\) and \(ba\), i.e. in describing the two-dimensional joint distribution of the pair \((ab, ba)\).
In the same spirit as in Eqns. (1.1), (1.3), we will approach the distribution of the free commutator in terms of its $R$-transform. The main result of the paper, Theorem 1.2, gives a description of this $R$-transform in terms of our combinatorial convolution $\star$; two other alternative descriptions, derived from the one in Theorem 1.2, will be presented in the Corollaries 1.4 and 1.6. We prefer to work with $i(ab - ba)$, since, in a $C^*$-framework, this element is selfadjoint if $a$ and $b$ are so.

1.1 Notations. Let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be a distribution (linear functional, normalized by $\mu(1) = 1$) and let $[R(\mu)](z) = \sum_{n=1}^{\infty} \alpha_n z^n$ be its $R$-transform. We shall denote by $R_E(\mu)$ the generating series of the even free cumulants $(\alpha_{2n})_{n=1}^{\infty}$, i.e.

$$[R_E(\mu)](z) := \sum_{n=1}^{\infty} \alpha_{2n} z^n. \quad (1.6)$$

In general, $R_E(\mu)$ contains less information than $R(\mu)$; however, in the case when $\mu$ is even – i.e. $\mu(X^{2n+1}) = 0$ for all $n$ and hence also $\alpha_{2n+1} = 0$ for all $n$ – then $R(\mu)$ can be recaptured from $R_E(\mu)$ by

$$[R(\mu)](z) = [R_E(\mu)](z^2). \quad (1.7)$$

1.2 Theorem. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, let $a, b$ be in $\mathcal{A}$ and consider the element $c := i(ab - ba)$. If $a$ is free from $b$ then
i) $\mu_c$ is even, and
ii) we have the equation

$$R_E(\mu_c) = 2(R_E(\mu_a) \star R_E(\mu_b) \star Zeta), \quad (1.8)$$

where $Zeta$ is the special series

$$Zeta(z) := \sum_{n=1}^{\infty} z^n.$$

Of course, parts i) and ii) of our theorem can be combined into only one formula:

$$[R(\mu_{i(ab - ba)})](z) = 2(R_E(\mu_a) \star R_E(\mu_b) \star Zeta)(z^2). \quad (1.9)$$

Note that Theorem 1.2 does not only state that the distribution of the commutator is always even, but also, even more strikingly, that the distribution of the commutator depends only on the even free cumulants of $a$ and on the even free cumulants of $b$.

It is not mandatory to keep the result of Theorem 1.2 formulated in terms of $\star$, since (modulo a technical detail) we can apply in (1.8) the combinatorial Fourier transform and turn $\star$ into usual multiplication. Before doing this let us first present a direct application of formula (1.8) itself.

1.3 Application (commutator with symmetric Bernoulli distribution). Let $a, b$ be free in some non-commutative probability space $(\mathcal{A}, \varphi)$, such that $b$ has distribution $\mu_b = 1/2(\delta_{-1} + \delta_{+1})$. Then we have:

$$\mu_{i(ab - ba)} = \mu_a \boxplus \mu_{-a}. \quad (1.10)$$
Indeed, one finds out in this case (for instance by plugging \( \mu_b \) into the Eqn. (5.2) of Proposition 5.2 below) that \( R_E(\mu_b) \) is equal to \( \text{Moeb} \), the inverse of \( \text{Zeta} \) under \( \ast \). Therefore (1.9) reduces to

\[
[R(\mu_i(ab-ba))](z) = 2[R_E(\mu_a)](z^2) = [R(\mu_a)](z) + [R(\mu_{-a})](z) = [R(\mu_a \boxplus \mu_{-a})](z),
\]

which implies (1.10).

We now give the ‘analytical’ reformulation of Theorem 1.2, which was announced prior to 1.3. Recall that if \( \mu : C[X] \to C \) is a linear functional with \( \mu(1) = 1 \), then the series

\[
[M(\mu)](z) := \sum_{n=1}^{\infty} \mu(X^n) z^n
\]

is called the moment series of \( \mu \). By applying the combinatorial Fourier transform in (1.8) and by taking into account the relations which exist between the series \( R(\mu) \), \( S(\mu) \), \( M(\mu) \), one obtains the following.

**1.4 Corollary.** Let \((A, \varphi)\) be a non-commutative probability space and let \( a, b \in A \) be free and with non-zero variances \( (\varphi(a^2) - (\varphi(a))^2) \neq 0 \neq (\varphi(b^2) - (\varphi(b))^2) \). Then the moment series of \( i(ab - ba) \) is given by

\[
[M(\mu_i(ab-ba))](z) = \left( \frac{2}{w(1 + \frac{w}{2})(1 + w)^2} \cdot [R_E(\mu_a)]^{-1>}(\frac{w}{2}) \cdot [R_E(\mu_b)]^{-1>}(\frac{w}{2}) \right)^{-1>}(z^2),
\]

where the notation ‘\( h^{-1>} \)’ is used for the inverse under composition of a formal power series \( h \) without constant term, and with non-zero linear term \( (h(0) = 0 \neq h'(0)) \).

This corollary can be thought of as an algorithm which tells how to obtain the moment series of \( i(ab - ba) \) from the ones of \( a \) and \( b \), via a succession of algebraic operations, simple substitutions, and inversions under composition. Strictly speaking, the Equation (1.11) only connects \( M(\mu_i(ab-ba)) \) to the sequences of even free cumulants of \( a \) and \( b \), and not to \( M(\mu_a) \) and \( M(\mu_b) \); but it is known that the free cumulants are obtained from the moments by using the same types of operations – see, e.g., [18], Section 3.3.

Let us now examine some concrete examples of how Corollary 1.4 can be applied. The first step of a calculation based on 1.4 is to find out the inverse \( R_E \)-transforms \( [R_E(\mu_a)]^{-1>} \) and \( [R_E(\mu_b)]^{-1>} \). The inverse \( R_E \)-transforms of a few distributions which occur frequently in free probability are listed in the following table.
Table 1

| Distribution $\mu$                                                                 | $[R_E(\mu)]^{<-1>}(w)$                                      |
|-----------------------------------------------------------------------------------|-------------------------------------------------------------|
| semicircular of radius $r > 0$                                                     | $\frac{4w}{r^2}$                                           |
| $d\mu(t) = 2\pi^{-1}r^{-2}(r^2 - t^2)^{1/2}dt$ on $[-r, r]$                       |                                                             |
| $[R(\mu)](z) = r^2z^2/4$ see [18], 3.4.2                                          |                                                             |
| free Poisson of parameters $\alpha, \beta > 0$                                    | $\frac{w}{\beta^2(\alpha + w)}$                           |
| $([R(\mu)](z) = \alpha\beta z/(1 - \beta z)$ see [18], 3.7                      |                                                             |
| arcsine law on $[-r, r]$, $r > 0$                                                 | $\frac{2w + w^2}{r^2}$                                    |
| $d\mu(t) = \pi^{-1}(r^2 - t^2)^{-1/2}dt$ on $[-r, r]$                            |                                                             |
| $[R(\mu)](z) = -1 + \sqrt{1 + r^2z^2}$ see [18], 3.4.5                           |                                                             |
| Bernoulli                                                                         |                                                             |
| $\mu = \lambda \delta_{t_0} + (1 - \lambda) \delta_{t_1}$ for some $t_0 < t_1$ in $\mathbb{R}, \lambda \in (0, 1)$ | $\frac{w(1 + w)(1 + 2w)^2}{(t_1 - t_0)^2(w^2 + w + (\lambda - \lambda^2))}$ |

By using Eqn. (1.11) one can in principle obtain the distribution of the commutator of two free elements $a, b$ whenever $\mu_a$ and $\mu_b$ are taken from the Table 1. A detail which cannot be ignored, though, is that the calculation of $M(\mu_{i(ab - ba)})$ requires one more inversion under composition; if we just restrict our attention to the situations in Table 1, then, due to the fact that all the $[R_E(\mu)]^{<-1>}'s in the table are rational functions, the remaining inversion under composition comes to solving a (possibly not too pleasant) algebraic equation.

Let us point out that, even though the Corollary 1.4 is formulated in a combinatorial framework, its applications take place in a $C^*$-context. If the non-commutative probability space $(\mathcal{A}, \varphi)$ is actually a $C^*$-probability space – i.e., $\mathcal{A}$ is a unital $C^*$-algebra, and $\varphi$ is a state on $\mathcal{A}$ – then the distribution $\mu_c$ of a selfadjoint element $c \in \mathcal{A}$ is actually a probability measure with compact support on $\mathbb{R}$. In such a case, the moment series $M(\mu_c)$ can still be viewed as a formal power series, but it is also an analytic function on a neighborhood of zero, and is related to the Cauchy transform $G$ of $\mu_c$ via the formula

$$1 + [M(\mu_c)](z) = z^{-1}G(z^{-1}), \quad 0 < |z| < \|c\|^{-1}. \hspace{1cm} (1.12)$$
The algebraic equation for $M(\mu_{i(ab-ba)})$ obtained by using (1.11) can thus be converted (with the help of (1.12), written for $c = i(ab - ba)$), into an algebraic equation satisfied by $G$. From this point one can, in nice cases, solve for $G$. The algebraic equation for $M$ is semicircular of radius $\beta$, and that $\mu$ is absolutely continuous with respect to Lebesgue measure. The support of $\mu$ is a projection with $\phi$ is a projection with $\psi$ is a projection with $\chi$. In all the examples discussed here, $\lambda$ is semicircular (say, for definiteness, that it has radius 2) and that $\mu$ is semicircular of radius 2. Then (1.11) gives us a cubic equation for $M(\mu_c)$; when converted into an equation for $G$, this becomes:

$$\zeta^2 G(\zeta)^2 - 2\zeta^3 G(\zeta) + (2\zeta^2 - 1 + 4\lambda(1 - \lambda)) = 0, \quad \zeta \in \mathbb{C}, \Im \zeta > 0. \quad (1.13)$$

By solving for $G$ and then by applying the Stieltjes inversion formula, we obtain:

$$\mu_c = \sqrt{1 - 4\lambda(1 - \lambda)} \delta_0 + \frac{1}{\pi|t|} \sqrt{4\lambda(1 - \lambda) - (t - 1)^2} \chi_{[-\beta, -\alpha] \cup [\alpha, \beta]} \, dt, \quad (1.14)$$

where $\alpha := \sqrt{1 - 2\sqrt{\lambda(1 - \lambda)}} \in (0, 1)$ and $\beta := \sqrt{1 + 2\sqrt{\lambda(1 - \lambda)}} \in (1, \sqrt{2}]$. Note that the spectrum of $c$ (which is $[-\beta, -\alpha] \cup \{0\} \cup [\alpha, \beta]$) is an interval only if $\lambda = 1/2$; in this case $\mu_c$ is semicircular of radius $\sqrt{2}$, as one could also infer directly from Application 1.3.

2) Assume that both $a$ and $b$ are semicirculars of radius 2. Then (1.11) gives us a cubic equation for $M(\mu_c)$; hence we also get via (1.12) a cubic equation for $G$, which turns out to be:

$$\zeta G(\zeta)^3 + G(\zeta)^2 - \zeta G(\zeta) + 1 = 0. \quad (1.15)$$

By solving for $G$ and then by using the Stieltjes inversion formula we obtain in this case that $\mu_c$ is absolutely continuous with respect to Lebesgue measure. The support of $\mu_c$ is the interval $[-r, r]$, with $r = \sqrt{(11 + 5\sqrt{5})/2}$; its density is

$$\frac{d\mu(t)}{dt} = \frac{\sqrt{3}}{2\pi|t|} \left( \frac{3t^2 + 1}{9h(t)} - h(t) \right), \quad |t| \leq \sqrt{(11 + 5\sqrt{5})/2}, \quad (1.16)$$

where

$$h(t) = \sqrt{\frac{18t^2 + 1}{27}} + \sqrt{\frac{t^2(1 + 11t^2 - t^4)}{27}}, \quad (1.17)$$

This example has an alternative direct derivation, as follows. One notes first that $c$ has the same distribution as the anti-commutator $ab + ba$ (see Proposition 1.10 below). Then one writes

$$ab + ba = \left( \frac{a + b}{\sqrt{2}} \right)^2 - \left( \frac{a - b}{\sqrt{2}} \right)^2, \quad (1.18)$$
and uses the fact that \((a + b)/\sqrt{2}\) and \((a - b)/\sqrt{2}\) are also free semicircular of radius 2 (see e.g. [18], Section 2.6). Since the square of a semicircular of radius 2 is Poisson of parameters 1,1 (with \(R\)-transform \(Zeta(z) = z/(1 - z)\), as listed in Table 1), one gets:

\[
[R(\mu_c)](z) = [R(\mu_{ab+ba})](z) = [R(\mu_{(a+b)^2/2})](z) + [R(\mu_{(a-b)^2/2})](z) = \frac{z}{1 - z} + \frac{-z}{1 + z} = \frac{2z^2}{1 - z^2}.
\]

The formulas (1.15-17) then follow from (1.19) by using the direct relation which exists between the \(R\)-transform and the Cauchy transform (see [18], Section 3.3).

In view of (1.19), the distribution \(\mu_c\) of this example could be called ‘symmetric Poisson’. We note that (1.19) is also obvious from the statement of Theorem 1.2. Indeed, in this case we have \([R(\mu_a)](z) = [R(\mu_b)](z) = z\), which is the unit for the operation \([\ ]\) thus Eqn. (1.9) becomes

\[
[R(\mu_c)](z) = 2 Zeta(z^2) = \frac{2z^2}{1 - z^2}.
\]

The nice simple trick of (1.18) does not seem to work in other examples, because \((a + b)/\sqrt{2}\) and \((a - b)/\sqrt{2}\) are not free in general (actually, the freeness of these elements implies that \(a\) and \(b\) are semicircular – see [8], Section 5).

3) Assume that both \(a\) and \(b\) are projections. Then the equations obtained for \(M(\mu_c)\) and \(G\) are of degree four, but can be solved without much difficulty, because they are bi-quadratic. We only present the formula for \(\mu_c\) in two particular cases where the calculations are nicer than in the generic one. For both the particular cases it is convenient to denote

\[
\xi := 4\varphi(a)(1 - \varphi(a)) \in (0, 1]
\]

(1.20)

(where it is assumed that \(\varphi(a) \neq 0, 1\)).

a) If \(\varphi(b) = 1/2\), then (with \(\varphi(a) \in (0, 1)\) arbitrary and \(\xi\) as in (1.20)) we have:

\[
\mu_c = \sqrt{1 - \xi} \delta_0 + \frac{1}{\pi |t|} \frac{\sqrt{4t^2 - 1 + \xi}}{1 - 4t^2} \chi_{[-1/2, -\sqrt{1 - \xi/2}]\cup[\sqrt{1 - \xi/2}, 1/2]} dt.
\]

(1.21)

Here too, the spectrum of \(c\) is an interval only if \(\varphi(a) = 1/2\); in this case \(\mu_c\) is the arcsine law on \([-1/2, 1/2]\).

b) If \(\varphi(a) = \varphi(b) =: \lambda \in (0, 1)\), then there are several subcases to consider. For \(0 < \lambda \leq \frac{1}{2} - \frac{1}{\sqrt{8}}\) we have (with \(\xi\) from (1.20)) that:

\[
\mu_c = \sqrt{1 - \xi} \delta_0 + \frac{2}{\pi \sqrt{(1 - 4t^2)(1 + \sqrt{1 - 4t^2})(1 - 2\xi + \sqrt{1 - 4t^2})}} \chi_{[-\sqrt{\xi(1-\xi)}\sqrt{\xi(1-\xi)}]} dt
\]

(1.22)

(which means in particular that the norm of \(c\) decreases with \(\lambda\)). For \(\frac{1}{2} - \frac{1}{\sqrt{8}} \leq \lambda \leq \frac{1}{2}\) we have:

\[
\mu_c = \sqrt{1 - \xi} \delta_0 + h_1(t) \chi_{[-\sqrt{\xi(1-\xi)}\sqrt{\xi(1-\xi)}]} dt + h_2(t) \chi_{[-1/2, -\sqrt{\xi(1-\xi)}]\cup[\sqrt{\xi(1-\xi)}, 1/2]} dt,
\]

(1.23)
where the densities \( h_1, h_2 \) are given by:

\[
h_1(t) = \frac{1}{\pi} \sqrt{\frac{2t - 1 + \sqrt{1 - 4t^2}}{(1 + \sqrt{1 - 4t^2})(1 - 4t^2)}}
\]

\( (1.24) \)

\[
h_2(t) = \frac{1}{\pi} \sqrt{\frac{2t^2 - 1 + \xi + 2t\sqrt{t^2 - \xi(1 - \xi)}}{t^2(1 - 4t^2)}}.
\]

The situations when \( \frac{1}{2} \leq \lambda < 1 \) are obtained from those when \( 0 < \lambda \leq \frac{1}{2} \), by replacing \( a \) and \( b \) with \( 1 - a \) and \( 1 - b \), respectively.

An element \( a \) in a non-commutative probability space \( (A, \varphi) \) is called even if its distribution \( \mu_a \) is so, i.e. if \( \varphi(a^n) = 0 \) for \( n \) odd. In the particular case when we look at the free commutator of two even elements, we have a third reformulation of Eqn. (1.8) in Theorem 1.2, in terms of S-transforms.

**1.6 Corollary.** Let \( (A, \varphi) \) be a non-commutative probability space, and let \( a, b \in A \) be even, free, and with non-zero variances \( \gamma_a = \varphi(a^2) - (\varphi(a))^2, \gamma_b = \varphi(b^2) - (\varphi(b))^2 \). Consider the element \( c := i(ab - ba) \in A \), and its variance \( \gamma_c = \varphi(c^2) - (\varphi(c))^2 \). Then \( \gamma_c = 2\gamma_a\gamma_b(\neq 0) \), and we have the equation in S-transforms:

\[
[S(\mu_{c^2/\gamma_c})](w) = \frac{1 + w}{1 + w} \cdot [S(\mu_{a^2/\gamma_a})](\frac{w}{2}) \cdot [S(\mu_{b^2/\gamma_b})](\frac{w}{2}).
\]

(1.25)

Note that the S-transform \( S(\mu_{c^2/\gamma_c}) \) determines the distribution of \( c^2 \), which in turn determines the one of \( c \) (because \( c \) is even, by Theorem 1.2.i). Hence Equation (1.25) can also be viewed as a solution to the free commutator problem, in the particular situation when we work with even elements. The proof of Corollary 1.6 goes in parallel with the one of 1.4 – see Section 5.5 below.

**1.7 Application (iterated free commutators)** Let \( (A, \varphi) \) be a non-commutative probability space, and let \( (a_m)_{m=1}^\infty \) be a sequence of free elements of \( A \), all having the same even distribution \( \mu \). We form the sequence of iterated commutators:

\[
c_1 = a_1, \quad c_m = i(c_{m-1}a_m - a_mc_{m-1}), \quad m \geq 2,
\]

and we consider the problem whether this sequence has a limit distribution for \( m \to \infty \).

Let \( \gamma \) be the variance of \( \mu \). From the formula for variances in Corollary 1.6 it follows that the variance of \( c_m \) is \( \frac{1}{2}(2\gamma)^m, \quad m \geq 1 \). Hence, in order for the limit distribution of the \( c_m \)'s to exist and be non-trivial, we need to make the additional assumption that \( \gamma = 1/2 \) (which implies that \( c_m \) has variance 1/2 for all \( m \)).

The distribution \( \mu_{2a_m^2} \) does not depend on \( m \) (but only on \( \mu \), the common distribution of the \( a_m \)'s); we denote \( S(\mu_{2a_m^2}) =: g \). Then the repeated application of Corollary 1.6 leads
to the formula:

\[
[S(\mu_{2c_m^2})](w) = \prod_{k=1}^{m-1} g\left(\frac{w}{2^k}\right) \cdot g\left(\frac{w}{2^{m-1}}\right) \cdot \frac{1 + \frac{w}{2^{m-1}}}{1 + w}, \quad m \geq 2. \tag{1.26}
\]

It is easy to check that the series \(\prod_{k=1}^{m-1} g\left(\frac{w}{2^k}\right)\) have, for \(m \to \infty\), a coefficient-wise limit distribution, which we will denote \(\prod_{k=1}^{\infty} g\left(\frac{w}{2^k}\right)\). From (1.26) it follows that \(S(\mu_{2c_m^2})\) converges coefficient-wise, for \(m \to \infty\), to \(\frac{1}{1+w} \cdot \prod_{k=1}^{\infty} g\left(\frac{w}{2^k}\right)\). Since the moments of a functional are recaptured from the coefficients of its \(S\)-transform via polynomial equations (see e.g. [10]), we can thus infer the moment-wise convergence of the functionals \((\mu_{2c_m^2})_{m=1}^{\infty}\). Finally, due to the fact that all the \(c_m\)'s are even, the convergence of the sequence \((\mu_{2c_m^2})_{m=1}^{\infty}\) is equivalent to the one of the sequence \((\mu_{c_m})_{m=1}^{\infty}\).

The conclusion is hence that: under the additional assumption that \(\mu\) has variance \(1/2\), the sequence of distributions \((\mu_{c_m})_{m=1}^{\infty}\) converges moment-wise, and moreover, if \(c_\infty\) is an element (in some non-commutative probability space) such that \(\mu_{c_\infty} = \lim_{m \to \infty} \mu_{c_m}\), then we have:

\[
[S(\mu_{2c_\infty^2})](w) = \frac{1}{1+w} \cdot \prod_{k=1}^{\infty} g\left(\frac{w}{2^k}\right). \tag{1.27}
\]

From (1.27) it is clear that the limit distribution \(\mu_\infty\) of the \(\mu_{c_m}\)'s effectively depends on the input distribution \(\mu\) we start with. Some examples that can be easily calculated are:

a) If \(\mu\) is the symmetric Bernoulli distribution \(\frac{1}{2}(\delta_{-1/2} + \delta_{1/2})\), then \(\mu_\infty\) is semicircular of radius \(\sqrt{2}\).

b) If \(\mu\) is the arcsine law \(2\pi^{-1}(1 - 4t^2)^{-1/2}\) on \([-1/2, 1/2]\), then \(\mu_\infty\) is the symmetric (free) Poisson distribution of Example 1.5.2, properly normalized with a dilation by 1/4. This example, b), can be in fact viewed as a consequence of the previous one, a) – see Section 1.17 below.

c) If \(\mu\) is semicircular of radius \(\sqrt{2}\), then the common distribution of the elements \(2c_{m_0}\) is free Poisson of parameters 1,1, and this implies that the function \(g\) defined prior to (1.26) is just \(g(w) = 1/(1 + w)\). The right-hand side of (1.27) becomes hence \(\prod_{k=0}^{\infty} 1/(1 + \frac{w}{2^k})\); one can transform this infinite product into a sum (see e.g. [3], Theorem 2.1), and rewrite it as \(\exp(\frac{w}{2})\), where the \(\frac{1}{2}\)-exponential series is

\[
\exp_{1/2}(t) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!_{1/2}},
\]

with \([n]_{1/2} = 1 + 1/2 + \cdots + 1/2^{n-1} = 2 - 1/2^{n-1} \cdot 2 - 1/2^{n-1}.\) It is not clear though how \(\mu_\infty\) explicitly looks in this case.

Up to now, we have restricted our attention to the level of random variables. But as it follows directly from general freeness considerations, taking the commutator of free random variables is really an operation on the level of distributions. The main result of the paper, Theorem 1.2, can be viewed as a description of this operation in terms of the combinatorial convolution [4] and the \(R_E\)-transform. In the rest of the Introduction we will switch to this alternative point of view and we will examine the structure of this operation more closely.
1.8 Notation. Let \( \nu_1, \nu_2 \) be two distributions (in the algebraic sense of Notations 1.1). We shall denote the free commutator of \( \nu_1 \) and \( \nu_2 \) by \( [\nu_1 \square \nu_2] \), and their free anti-commutator by \( \{\nu_1 \square \nu_2\} \). More explicitly, if \( a \) and \( b \) are free random variables in some \((A, \varphi)\) such that \( \nu_1 = \mu_a \) and \( \nu_2 = \mu_b \), then
\[
[\nu_1 \square \nu_2] := \mu_{i(ab - ba)}, \quad \text{and} \quad \{\nu_1 \square \nu_2\} := \mu_{ab + ba}.
\] (1.28)

1.9 Remark. In the \( C^* \)-context, we thus have the operations of taking the free commutator and of taking the free anti-commutator for probability measures with compact support. In [3] it is shown that one can extend these operations (or, more generally, operations given by non-commutative selfadjoint polynomials) from probability measures with compact support to all probability measures.

In general, the commutator and the anti-commutator are of course different. But one of the byproducts of our results of Section 4 is the fact that they coincide for even distributions.

1.10 Proposition. Let \( \nu_1 \) and \( \nu_2 \) be even distributions. Then the commutator and the anti-commutator of \( \nu_1 \) and \( \nu_2 \) are the same,
\[
[\nu_1 \square \nu_2] = \{\nu_1 \square \nu_2\}.
\]

The description of the free anti-commutator for general distributions is still an open problem. Our general result for the free commutator relies on the fact that due to cancellations we can reduce the general case to the even case.

Clearly, the operation of taking the free commutator is not associative. But, Theorem 1.2 shows that only the factor 2 in formula (1.8) prevents it from being so. We can get rid of this factor by taking the convolution square root. Let us recall that for a distribution \( \mu \) its convolution square root \( \mu \Box^{1/2} \) (in algebraic sense) is uniquely determined by
\[
\left( \mu \Box^{1/2} \right) \Box \left( \mu \Box^{1/2} \right) = \mu \quad \text{or equivalently} \quad R(\mu \Box^{1/2}) = \frac{1}{2} \cdot R(\mu).
\]
Thus we get the following striking corollary of Theorem 1.2.

1.11 Corollary. The operation
\[
(\nu_1, \nu_2) \mapsto [\nu_1 \square \nu_2] \Box^{1/2}
\]
on distributions is associative.

1.12 Remarks.
1) This corollary raises the question, whether the operation \([\nu_1 \square \nu_2] \Box^{1/2}\) also makes sense in a \( C^* \)-context, i.e. whether the convolution square root of the free commutator of two
probability measures is again a probability measure. The following counterexample shows that this is not the case in general.

Namely, let $p$ and $q$ be two free projections with $\varphi(q) = 1/2$ and $0 < \varphi(p) \leq 1/2$. We saw in Example 1.5.3.a that the distribution of $i(pq - qp)$ is the probability measure

$$\mu_\xi = \sqrt{1-\xi} \delta_0 + \frac{1}{\pi |t|} \sqrt{\frac{4t^2 - 1 + \xi}{1 - 4t^2}} \chi_{[-1/2,1/2]} |t|^{\sqrt{1-\xi/2,1/2}}dt,$$

where $\xi := 4\varphi(p)(1-\varphi(p)) \in (0,1]$. We claim now that $\mu_\xi^{1/2}$ is not positive for $\xi$ sufficiently close to 1. Let us assume, by contradiction, that there exist probability measures $\nu_\xi$ with

$$\mu_\xi = \nu_\xi \square \nu_\xi \quad (0 < \xi \leq 1).$$

For $\xi = 1$ (i.e. $\varphi(p) = 1/2$) the Application 1.3 gives us

$$\nu_1 = \frac{1}{2}(\delta_{-1/4} + \delta_{1/4}).$$

For $0 < \xi < 1$, $\mu_\xi$ has an atom at 0. By Theorem 7.4 of [1], this implies that there exist atoms $\alpha$ and $\beta$ for $\nu_\xi$ such that $\alpha + \beta = 0$ and $\nu_\xi(\{\alpha\}) + \nu_\xi(\{\beta\}) > 1$. Since $\mu_\xi$, and hence also $\nu_\xi$, is even we can conclude that $\alpha = \beta = 0$, i.e. $\nu_\xi$ has, for all $0 < \xi < 1$, an atom at 0 with $\nu_\xi(\{0\}) > 1/2$. Consider now the mapping $\xi \mapsto \nu_\xi$ for $\xi \in (0,1]$. Clearly, all moments of $\nu_\xi$ are continuous in $\xi$. Since we can find a compact interval which supports all $\nu_\xi$ (by the fact that the same is true for all $\mu_\xi$ and by invoking Lemma 3.1 of [13]), this implies that $\xi \mapsto \nu_\xi$ is weakly continuous. But this is in contradiction with the fact that for $\xi < 1$ we have mass at least 1/2 at 0 whereas for $\xi = 1$ all mass sits at $\pm 1/4$. Thus we can conclude that at least for $\xi$ sufficiently close to 1 there exist no probability measure $\nu_\xi$ with $\mu_\xi = \nu_\xi \square \nu_\xi$.

2) Theorem 1.2 shows that the main information needed for calculating the free commutator consists of just the even free cumulants. Thus it is natural to define a mapping $\mu \mapsto \text{Even}(\mu)$ from all distributions to even distributions by the requirement that $\text{Even}(\mu)$ is even (i.e. its odd free cumulants are zero) and that $\text{Even}(\mu)$ has the same even free cumulants as $\mu$:

$$[\text{R}(\text{Even}(\mu))](z) = [\text{R}_E(\mu)](z^2).$$

This raises the question whether this mapping also makes sense on the $C^*$-level, i.e. whether it preserves positivity. Using the same kind of arguing as before one can convince oneself that this is not true in general.

Namely, according to Application 1.3, $\mu_\xi$ appearing above is essentially (up to rescaling) $\mu_p \square \mu_{-p}$. But this implies that $\nu_\xi$ is up to rescaling equal to $\text{Even}(\mu_p)$. Since $\nu_\xi$ is not positive for $\xi$ sufficiently close to 1, positivity fails also for $\text{Even}(\mu_p)$ with $\varphi(p)$ sufficiently near to 1/2.

Whereas $[\nu_1 \square \nu_2]^{1/2}$ does not make sense for probability measures in general, it is again the case of even probability measures which has a special status in this context. As we have seen in Corollary 1.6, our results take for even random variables an especially simple form if
we formulate them in terms of the square of the variables. In this spirit we will now make a transition from even probability measures to probability measures which have their support on the positive real line, by just taking the square of the corresponding random variables. We will denote by \( Q(\mu) \) the image of \( \mu \) under the transformation \( Q : t \mapsto t^2 \), i.e. we have \( Q(\mu_a) = \mu_a^2 \).

1.13 Corollary.

1) Let \( \nu_1 \) and \( \nu_2 \) be even probability measures. Then \( \{\nu_1 \boxtimes \nu_2\}^{1/2} = [\nu_1 \boxtimes \nu_2]^{1/2} \) is an even probability measure, too.

2) The map \( Q : t \mapsto t^2 \) transports the operation \( (\nu_1, \nu_2) \mapsto \{\nu_1 \boxtimes \nu_2\}^{1/2} = [\nu_1 \boxtimes \nu_2]^{1/2} \) on even probability measures to the multiplicative free convolution \( (\nu_1, \nu_2) \mapsto \nu_1 \boxtimes \nu_2 \) on positively supported probability measures; i.e., we have:

\[
Q(\{\nu_1 \boxtimes \nu_2\}^{1/2}) = Q([\nu_1 \boxtimes \nu_2]^{1/2}) = Q(\nu_1) \boxtimes Q(\nu_2). \tag{1.29}
\]

Note that, although we will prove Corollary 1.13 only for probability measures with compact support, the continuity properties of freeness (in particular Theorem 4.11 of \([3]\)) ensure that it is also true for arbitrary even probability measures on \( \mathbb{R} \). We next present an example which uses this unbounded version of Corollary 1.13.

1.14 Example. Let \( a \) and \( b \) be semicircular variables of radius 2 which are free. Then the free commutator of \( a \) and \( 1/b \) has a Cauchy distribution of parameter \( \gamma = 2 \), i.e.

\[
\mu_{i(ab^{-1} - b^{-1}a)} = \frac{\gamma}{\pi(t^2 + \gamma^2)} dt. \tag{1.30}
\]

This comes out from the Corollary 1.13 combined with the results of \([3]\) on free stable distributions, as follows. The distribution of \( 1/b^2 \) is the free stable distribution \( \nu_\alpha \) of index \( \alpha = 1/2 \), and on the other hand the distribution of \( a^2 \) can be written as \( \nu_{1/2} \) where \( \nu \) denotes the image of \( \nu \) by the map \( t \mapsto 1/t \). The multiplicative free convolution \( \nu_\alpha \boxtimes \nu_{1/2} \) is given explicitly in Proposition 8 of \([3]\) (comp. also \([5]\)) in the form

\[
\nu_{1/2} \boxtimes \nu_{1/2} = \frac{1}{\pi t + 1} \chi_{(0,\infty)} dt,
\]

which turns out to be \( Q^{-1} \) of the Cauchy distribution of parameter \( \tilde{\gamma} = 1 \). Since the additive free convolution of a Cauchy distribution of parameter \( \tilde{\gamma} \) with itself is a Cauchy distribution of parameter \( \gamma = 2\tilde{\gamma} \), we obtain, by Corollary 1.13, that the distribution of \( i(ab^{-1} - b^{-1}a) \) is Cauchy of parameter \( \gamma = 2 \).

Let us now come back to the general structure of the free commutator \( [\nu_1 \boxtimes \nu_2] \). Since \( \boxtimes \) is commutative, Theorem 1.2 implies in particular that also the free commutator is commutative in its two variables. But the special form of our formula (1.8) in Theorem 1.2 implies that we even have a form of higher order commutativity.
1.15 Notations.

1) A commutator expression \( f(\nu_1, \ldots, \nu_n) \) of \( n \) arguments (for some \( n \geq 1 \)) is an expression in which we take in some iterated way \( n \) commutators of the distributions \( \nu_1, \ldots, \nu_n \). More formally, we define the class of commutator expressions in the following recursive way:
   i) \( f(\nu_1) = \nu_1 \) is a commutator expression of one argument;
   ii) \( f(\nu_1, \ldots, \nu_n) = [f_1(\nu_1, \ldots, \nu_k) \square f_2(\nu_{k+1}, \ldots, \nu_n)] \) is a commutator expression of \( n \) arguments if, with some \( 1 \leq k < n \), \( f_1 \) and \( f_2 \) are commutator expressions of \( k \) and \( n-k \) arguments, respectively.

2) To each commutator expression \( f \) of \( n \) arguments we assign a depth vector \((d_1, \ldots, d_n)\) in the following recursive way:
   i) \( f(\nu_1) = \nu_1 \) has depth \((0)\).
   ii) If \( f(\nu_1, \ldots, \nu_n) = [f_1(\nu_1, \ldots, \nu_k) \square f_2(\nu_{k+1}, \ldots, \nu_n)] \) and if \( f_1 \) has depth \((d_1, \ldots, d_k)\) and \( f_2 \) has depth \((d_{k+1}, \ldots, d_n)\), then \( f \) has depth \((d_1 + 1, \ldots, d_k + 1, d_{k+1} + 1, \ldots, d_n + 1)\).

Intuitively, the depth vector of a commutator expression counts how many brackets we have to cross in order to reach the arguments inside the expression. For example, the expression \([\nu_1 \square \nu_2] \square [\nu_3 \square \nu_4]\) has depth \((2, 2, 2)\), whereas \([[[\nu_1 \square \nu_2] \square [\nu_3 \square \nu_4]]\) has depth \((3, 3, 2, 1)\).

The announced higher order commutativity of the free commutator can now be stated in the form that iterated commutators are commutative in such variables which have the same depth, i.e. which have equal entries in the depth vector. More generally, we can compare two different commutator expressions if their depth vectors differ only by a permutation.

1.16 Corollary. Let \( f \) and \( \hat{f} \) be two commutator expressions of \( n \) arguments with depth \((d_1, \ldots, d_n)\) and \((\hat{d}_1, \ldots, \hat{d}_n)\), respectively. If there exists a permutation \( \tau \in S_n \) such that \((d_1, \ldots, d_n) = (\hat{d}_{\tau(1)}, \ldots, \hat{d}_{\tau(n)})\) then

\[
f(\nu_{\tau(1)}, \ldots, \nu_{\tau(n)}) = \hat{f}(\nu_1, \ldots, \nu_n)
\]

for arbitrary distributions \( \nu_1, \ldots, \nu_n \); (1.31)

in particular

\[
f(\nu, \ldots, \nu) = \hat{f}(\nu, \ldots, \nu)
\]

for all distributions \( \nu \). (1.32)

A concrete example of this corollary is, with \( f = \hat{f} = [[\cdot \square \cdot] \square [\cdot \square \cdot]] \), that

\[
[[\nu_1 \square \nu_2] \square [\nu_3 \square \nu_4]] = [[\nu_1 \square \nu_2] \square [\nu_2 \square \nu_4]]
\]

for any distributions \( \nu_1, \ldots, \nu_4 \).

1.17 Application (continuation of 1.7). Let us denote the canonical iterated commutator of \( m \) arguments by \( f_m \), i.e.

\[
f_1(\nu_1) := \nu_1, \quad f_m(\nu_1, \ldots, \nu_m) := [f_{m-1}(\nu_1, \ldots, \nu_{m-1}) \square \nu_m] \quad (m \geq 2).
\]

Moreover, for every distribution \( \nu \), let us denote

\[
\nu_m := f_m(\nu, \ldots, \nu), \quad (m \geq 1).
\]
Then we have

\[(\nu_m)_n = (\nu_n)_m \quad \text{for all } m, n \geq 1. \quad (1.33)\]

Indeed, given \( m, n \geq 1 \), let us denote by \((f_m)_n\) the commutator expression of \( m \cdot n \) arguments defined by

\[(f_m)_n(\nu_{11}, \ldots, \nu_{1m}, \ldots, \nu_{nm}) = f_n(f_m(\nu_{11}, \ldots, \nu_{1m}), \ldots, f_m(\nu_{1n}, \ldots, \nu_{nm})).\]

An easy depth-counting argument shows that \((f_m)_n\) and \((f_n)_m\) satisfy the hypothesis of Corollary 1.16; but then the Eqn. (1.32) applied to this situation gives exactly (1.33).

Assume now that, in addition, the variance of \( \nu \) is equal to \( \frac{1}{2} \). Then we know, by Application 1.7, that the moment-wise limit \( \nu_\infty := \lim_{m \to \infty} \nu_m \) exists. By fixing \( n \) and letting \( m \to \infty \) in (1.33) it follows that:

\[(\nu_n)_\infty = (\nu_\infty)_n \quad \text{for all } n \geq 1. \quad (1.34)\]

If we take for \( \nu \) the symmetric Bernoulli distribution \( \frac{1}{2}(\delta_{-1/2} + \delta_{1/2}) \), then \( \nu_2 \) is the arcsine distribution on \([-1/2, 1/2]\) (see, e.g., Example 1.5.3.a), hence we get the connection between the examples a) and b) in Application 1.7 via the Eqn. (1.34), considered for \( n = 2 \).

An important ingredient in our proof of Theorem 1.2 is the notion of an \( R \)-diagonal pair, which was introduced and studied in our previous paper [12]. The connection of this concept with our present problem comes from the fact that if \( a \) and \( b \) are free and both even, then \((ab, ba)\) is an \( R \)-diagonal pair – which means that the joint distribution \( \mu_{ab,ba} \) has a nice, computable (two-dimensional) \( R \)-transform. This gives a definite and treatable relation between the elements \( ab \) and \( ba \) (even though, of course, they are not free). The \( R \)-transforms of \( \mu_{i(ab−ba)} \) and \( \mu_{ab+ba} \) can be recovered from the one of \( \mu_{ab,ba} \) by formulas which have as immediate consequence that actually \( \mu_{i(ab−ba)} = \mu_{ab+ba} \) (as stated in Proposition 1.10). We take the occasion to note that another consequence of the \( R \)-diagonality of \((ab, ba)\) is the following polar decomposition result.

\[1.18 \text{ Proposition.} \]

Let \((\mathcal{A}, \varphi)\) be a non-commutative probability space and assume that \( \mathcal{A} \) is a von Neumann algebra and \( \varphi \) is a faithful normal state. Consider \( a, b \in \mathcal{A} \) such that \( a \) and \( b \) are both selfadjoint and even, \( a \) is free from \( b \), and \( \text{Ker} \ a = \text{Ker} \ b = \{0\} \). Then \( ab \) has a polar decomposition \( ab = up \) where \( u \) is a Haar unitary and where \( \{u, u^*\} \) is free from \( p \).

The concatenation of arguments proving 1.10 and 1.18 is presented at the end of Section 4 of the paper.

Finally, let us describe how the paper is organized. In the next section, we will collect the preliminaries on the combinatorial description of freeness which will be needed in the forthcoming sections. In Section 3, we will reduce the general case to the case where \( a \) and \( b \) have even distribution. In Section 4, we will recall the definition and the relevant results from [12] on \( R \)-diagonal pairs and show that if \( a \) and \( b \) are free and both even, then \((ab, ba)\).
is an $R$-diagonal pair. This result will be used in Section 5 to conclude the proof of Theorem 1.2 and of its Corollaries 1.4, 1.6, 1.13, 1.16.

2. Preliminaries

We start by collecting together a few basic free probabilistic terms (some of them already implicitly reviewed and used in the Introduction).

2.1 Definitions.

1) We will call non-commutative probability space a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional normalized by $\varphi(1) = 1$. If we require in addition that $\mathcal{A}$ is a $C^*$-algebra and $\varphi$ is positive, then $(\mathcal{A}, \varphi)$ is called a $C^*$-probability space.

2) The joint distribution of a family of elements $a_1, \ldots, a_m \in \mathcal{A}$, in the non-commutative probability space $(\mathcal{A}, \varphi)$, is the linear functional $\mu_{a_1,\ldots,a_m} : \mathbb{C}\langle X_1, \ldots, X_m \rangle \to \mathbb{C}$ given by:

\[
\begin{align*}
\mu_{a_1,\ldots,a_m}(1) &= 1, \\
\mu_{a_1,\ldots,a_m}(X_{i_1} \cdots X_{i_n}) &= \varphi(a_{i_1} \cdots a_{i_n}) \quad \text{for } n \geq 1 \text{ and } 1 \leq i_1, \ldots, i_n \leq m,
\end{align*}
\]

(2.1)

where $\mathbb{C}\langle X_1, \ldots, X_m \rangle$ is the algebra of polynomials in $m$ non-commuting indeterminates $X_1, \ldots, X_m$.

3) If we only have one element $a \in \mathcal{A}$, then its distribution $\mu_a : \mathbb{C}[X] \to \mathbb{C}$, and the element $a$ itself are called even if $\mu_a(X^n)$ (or equivalently, $\varphi(a^n)$) vanishes for all odd $n$.

4) A family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ is said to be free in the non-commutative probability space $(\mathcal{A}, \varphi)$ if for every reduced word $w = a_1 \cdots a_k$ made with elements from the $\mathcal{A}_i$’s (i.e., $a_1 \in \mathcal{A}_{i_1}, \ldots, a_k \in \mathcal{A}_{i_k}$ with $i_1 \neq i_2, \ldots, i_{k-1} \neq i_k$) we have the implication $\varphi(a_1) = \cdots = \varphi(a_k) = 0 \Rightarrow \varphi(w) = 0$. This definition extends to subsets of $\mathcal{A}$, by putting $X_1, \ldots, X_n \subseteq \mathcal{A}$ to be free in $(\mathcal{A}, \varphi)$ if and only if the unital subalgebras generated by them are so.

5) Let $(\mathcal{A}, \varphi)$ be a $C^*$-probability space. Then:

- an element $a \in \mathcal{A}$ is called (standard) semicircular in $(\mathcal{A}, \varphi)$ if $a = a^*$ and if $\mu_a$ is the semicircle law $(2\pi)^{-1}\sqrt{4-t^2}dt$ on $[-2,2]$;

- an element $a \in \mathcal{A}$ is called (standard) circular in $(\mathcal{A}, \varphi)$ if it is of the form $(a+ib)/\sqrt{2}$, with $a, b$ free and semicircular;

- an element $u \in \mathcal{A}$ is called a Haar unitary in $(\mathcal{A}, \varphi)$ if it is a unitary and if $\varphi(u^n) = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$.

We next review some combinatorial facts about non-crossing partitions.
2.2 Definitions.

1) If \( \pi = \{B_1, \ldots, B_r\} \) is a partition of \( \{1, \ldots, n\} \), then the equivalence relation on \( \{1, \ldots, n\} \) with equivalence classes \( B_1, \ldots, B_r \) will be denoted by \( \sim \); the sets \( B_1, \ldots, B_r \) are called the blocks of \( \pi \). The number of elements in each set \( B_k \) will be denoted by \( |B_k|\).

2) A partition \( \pi \) of \( \{1, \ldots, n\} \) is called non-crossing if for every \( 1 \leq i < j < i' < j' \leq n \) such that \( i \sim j \) and \( j \sim j' \), it necessarily follows that \( i \sim j \sim i' \sim j' \). The set of all non-crossing partitions of \( \{1, \ldots, n\} \) will be denoted by \( NC(n) \). By \( NCE(n) \) we will denote the set of non-crossing partitions \( \pi \in NC(n) \) such that every block of \( \pi \) has an even number of elements. The complement \( NC(n) \setminus NCE(n) \) will be denoted by \( NCO(n) \).

3) On \( NC(n) \) we will consider the partial order relation, called refinement order, which is defined by \( \pi \preceq \rho \) if and only if each block of \( \rho \) is a union of blocks of \( \pi \).

2.3 Definition (complementation maps on \( NC(n) \)). Let us now fix \( n \geq 1 \) and an \( n \)-tuple of 1’s and 2’s, \( \varepsilon = (l_1, \ldots, l_n) \in \{1, 2\}^n \). We describe here, following [12], the construction of a map \( C_\varepsilon : NC(n) \to NC(n) \) which we will call \( \varepsilon \)-complementation map, and which plays an important role in the sequel.

Let us agree to call \( \varepsilon \)-insertion bead a \( 2n \)-tuple \( (P_1, \ldots, P_n; Q_1, \ldots, Q_n) \) of points on a circle, constructed according to the following rules:

- first, \( P_1, \ldots, P_n \) are drawn around the circle, equidistant and in clockwise order;
- for every \( 1 \leq i \leq n \) such that \( l_i = 1 \), \( Q_i \) is placed on the arc of circle going from \( P_i \) to \( P_{i+1} \) (clockwisely), such that the length of the arc \( P_iQ_i \) is 1/3 of the length of the arc \( P_iP_{i+1} \);
- for every \( 1 \leq i \leq n \) such that \( l_i = 2 \), \( Q_i \) is placed on the arc of circle going from \( P_i \) to \( P_{i-1} \) (counterclockwisely), such that the length of the arc \( P_iQ_i \) is 1/3 of the length of the arc \( P_iP_{i-1} \).

In the description of these rules, \( P_{i+1} \) and \( P_{i-1} \) are considered modulo \( n \) (i.e. \( P_{n+1} := P_1 \) and \( P_0 := P_n \)).

Having fixed an \( \varepsilon \)-insertion bead \( (P_1, \ldots, P_n; Q_1, \ldots, Q_n) \), the \( \varepsilon \)-complement \( \rho = C_\varepsilon(\pi) \in NC(n) \) of a given partition \( \pi \in NC(n) \) is defined via the rule that for every \( 1 \leq i, j \leq n \) we have: \( i \sim j \) if and only if the line segment \( Q_iQ_j \) does not intersect any of the line segments \( P_hP_k \), with \( h \sim k \).

To give a concrete example: for \( n = 5 \), \( \varepsilon = (1, 1, 2, 2, 1) \) and \( \pi = \{\{1, 2\}, \{3, 4, 5\}\} \), we have \( C_\varepsilon(\pi) = \{\{1\}, \{2, 3, 5\}, \{4\}\} \), as shown by Figure 1.
Figure 1.

We shall be interested in the combinatorics of the $\varepsilon$-complementation map in two distinct situations. On one hand, we shall consider the case when the sequence $\varepsilon$ contains an equal number of 1’s and 2’s. Two properties of $C_\varepsilon$ for such $\varepsilon$ are stated next.

2.4 Definition and Proposition. Let $n \geq 1$ and $\varepsilon = (l_1, \ldots, l_n) \in \{1, 2\}^n$ be such that $n$ is even and $|\{1 \leq i \leq n \mid l_i = 1\}| = |\{1 \leq i \leq n \mid l_i = 2\}| = n/2$. We shall say that a partition $\pi \in NC(n)$ is $\varepsilon$-alternating if for every block $B = \{i_1 < i_2 < \ldots < i_k\}$ of $\pi$ we have that $l_{i_1} \neq l_{i_2}, \ldots, l_{i_{k-1}} \neq l_{i_k}, l_{i_k} \neq l_{i_1}$. (Obviously, any such $\pi$ must be in $NCE(n)$, i.e. all its blocks must have even cardinality.) Given $\pi \in NC(n)$, the following hold:

1) ([12], Prop.7.7) If $\pi$ is $\varepsilon$-alternating, then $C_\varepsilon(\pi)$ is $\varepsilon$-alternating, too.
2) ([12], Prop.8.11) If both $\pi$ and $C_\varepsilon(\pi)$ are in $NCE(n)$, then $\pi$ is $\varepsilon$-alternating.

At the other extreme, we shall be interested in the case when $\varepsilon = (1, 1, \ldots, 1)$.

2.5 Definition. If $\varepsilon = (1, 1, \ldots, 1) \in \{1, 2\}^n$, then $C_\varepsilon$ will be denoted by $K$, and is called the Kremeras complementation map on $NC(n)$.

The map $K : NC(n) \to NC(n)$ was introduced in [1]. It turns out to be a bijection, and in fact an anti-automorphism of $(NC(n), \leq)$ (i.e. $\pi \leq \rho \iff K(\pi) \geq K(\rho)$). The inverse
of $K$ is the $\varepsilon$-complementation corresponding to $\varepsilon = (2, 2, \ldots, 2)$.

Recall that in the presentation of results made in Section 1, an important role was played by the operation of combinatorial convolution $\star$ (see e.g. Eqn. (1.8) of Theorem 1.2). By using the Kreweras complementation map, this operation can be defined as follows.

2.6 Definitions.

1) Let $\Theta$ be the set of formal power series of the form $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, with $\alpha_1, \alpha_2, \alpha_3, \ldots \in \mathbb{C}$. For $f \in \Theta$ and $n \geq 1$, we shall use the notation $[\text{coef } (n)](f)$ for the coefficient of order $n$ of $f$. Moreover, for $f \in \Theta$, $n \geq 1$ and $\pi = \{B_1, \ldots, B_r\} \in NC(n)$, we shall use the notation

$$[\text{coef } (n); \pi](f) := \prod_{i=1}^{r} [\text{coef } (|B_i|)](f).$$

(2.2)

2) $\star$ is the binary operation on $\Theta$ which is defined by the prescription:

$$[\text{coef } (n)](f \star g) = \sum_{\pi \in NC(n)} [\text{coef } (n); \pi](f) \cdot [\text{coef } (n); K(\pi)](g),$$

(2.3)

for every $f, g \in \Theta$ and $n \geq 1$.

For the interpretation of $\star$ which justifies the name of ‘combinatorial convolution’ (or, to be more precise: ‘convolution of multiplicative functions on non-crossing partitions’) we refer to [13], [10].

The operation $\star$ defined in 2.6 is associative, commutative, and has the series $Id(z) = z$ as a unit (see e.g. [10], Section 1.4). An important role in the considerations related to $\star$ is played by the series

$$\text{Zeta}(z) := \sum_{n=1}^{\infty} z^n, \quad \text{Moeb}(z) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{(n-1)!n!} z^n,$$

(2.4)

called the Zeta and Moebius series, respectively. $\text{Zeta}$ and $\text{Moeb}$ are inverse to each other with respect to $\star$.

Although chronologically the $R$-transform preceded the $\star$-operation, it is convenient to define it here in the following way.

2.7 Definition. If $\mu$ is a distribution (in the algebraic sense of Notations 1.1), then its moment series $M(\mu) \in \Theta$ is

$$[M(\mu)](z) := \sum_{n=1}^{\infty} \mu(X^n) z^n,$$

(2.5)

and its $R$-transform $R(\mu) \in \Theta$ is

$$R(\mu) := M(\mu) \star \text{Moeb}.$$  

(2.6)

The coefficients of $R(\mu)$ are called the free cumulants of $\mu$. 

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2.8 Remark. The Equation (2.6) can also be stated in the equivalent form

\[ M(\mu) = R(\mu) \Box Zeta, \quad (2.7) \]

which we will call the moment-cumulant formula. By identifying the coefficient of order \( n \) in (2.7), and by recalling how \( \Box \) is defined (Eqn. (2.3)), we get the explicit reformulation:

\[ \mu(X^n) = \sum_{\pi = \{B_1, \ldots, B_r\} \in NC(n)} \alpha_{|B_1|} \cdots \alpha_{|B_r|} \quad (n \geq 1), \quad (2.8) \]

where \((\alpha_n)_{n=1}^\infty\) are the free cumulants of \( \mu \).

From (2.8) (or (2.6-7)) it is clear that \( \mu \to R(\mu) \) is a bijection from the set of distributions onto \( \Theta \). Another fact immediately implied by (2.8) is that a distribution \( \mu \) is even (in the sense of Definition 2.1.3) if and only if all its odd free cumulants are equal to zero.

2.9 Remark. The direct connection between the operation \( \Box \) and freeness is provided by the fact, already mentioned in the Introduction, that:

\[ R(\mu_{ab}) = R(\mu_a) \Box R(\mu_b) \quad (2.9) \]

whenever \( a, b \) are free in some non-commutative probability space \((A, \varphi)\). For the proof of (2.9), see [10], Section 3.5. This equation can be reformulated as follows: first, we \( \Box \)-operate with Zeta on both its sides and thus (by also taking (2.7) into account) we turn it into:

\[ M(\mu_{ab}) = R(\mu_a) \Box M(\mu_b); \quad (2.10) \]

then by taking the coefficient of order \( n \) in (2.10), we obtain the formula:

\[ \varphi((ab)^n) = \sum_{\pi \in NC(n)} [\text{coef}(n); \pi](R(\mu_a)) \cdot [\text{coef}(n); K(\pi)](M(\mu_b)), \quad (2.11) \]

valid for \( a, b \) free in \((A, \varphi)\), and \( n \geq 1 \).

In the Sections 3 and 4 below we will need the generalization of (2.11) to the situation when \( K \) is replaced by an arbitrary \( \varepsilon \)-complementation map \( C_\varepsilon \); this is given in the next proposition, and is a particular case of Proposition 7.3 in [12].

2.10 Proposition ([12]). Let \( a, b \) be free in the non-commutative probability space \((A, \varphi)\), and let us denote \( ab =: x_1, ba =: x_2 \). Then for every \( n \geq 1 \) and \( \varepsilon = (l_1, \ldots, l_n) \in \{1, 2\}^n \) we have

\[ \varphi(x_{l_1} \cdots x_{l_n}) = \sum_{\pi \in NC(n)} [\text{coef}(n); \pi](R(\mu_a)) \cdot [\text{coef}(n); C_\varepsilon(\pi)](M(\mu_b)). \quad (2.12) \]

Even though all the results in the Introduction are concerning 1-dimensional distributions, the key fact in their proofs will be that we will often work in 2 variables. We thus conclude this section by reviewing the facts that will be needed about the 2-dimensional \( R \)-transform; this goes in parallel with the path taken in the Definitions 2.6, 2.7.
2.11 Definitions.

1) Let \( \Theta_2 \) be the set of formal power series without constant coefficient, in 2 non-commuting variables \( z_1, z_2 \). An element of \( \Theta_2 \) is thus a series of the form

\[
f(z_1, z_2) = \sum_{n=1}^{\infty} \sum_{(i_1, \ldots, i_n) \in \{1,2\}^n} \alpha_{(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n},
\]

(2.13)

where the \( \alpha_{(i_1, \ldots, i_n)} \)'s are some complex coefficients. For \( f \in \Theta_2 \), \( n \geq 1 \) and \( (i_1, \ldots, i_n) \in \{1,2\}^n \) we shall use the notation \( \lbrack \text{coef} \ (i_1, \ldots, i_n) \rbrack(f) \) for the coefficient of \( z_{i_1} \cdots z_{i_n} \) in \( f \). Moreover, for \( f \in \Theta_2 \), \( n \geq 1 \), \( (i_1, \ldots, i_n) \in \{1,2\}^n \) and \( \pi = \{B_1, \ldots, B_r\} \in NC(n) \) we shall use the notation

\[
\lbrack \text{coef} \ (i_1, \ldots, i_n); \pi \rbrack(f) := \prod_{j=1}^{r} \lbrack \text{coef} \ (i_1, \ldots, i_n \mid B_j) \rbrack(f),
\]

(2.14)

where in the right-hand side of (2.14) we write

\[
(i_1, \ldots, i_n \mid B) := (i_{h_1}, \ldots, i_{h_m})
\]

(2.15)

whenever \( B = \{h_1 < \cdots < h_m\} \) is a non-void subset of \( \{1, \ldots, n\} \).

2) We denote by \( \boxplus \) the binary operation on \( \Theta_2 \) which is defined by the prescription:

\[
\lbrack \text{coef} \ (i_1, \ldots, i_n); \pi \rbrack(f \boxplus g) = \sum_{\pi \in NC(n)} \lbrack \text{coef} \ (i_1, \ldots, i_n); \pi \rbrack(f) \cdot \lbrack \text{coef} \ (i_1, \ldots, i_n); K(\pi) \rbrack(g),
\]

(2.16)

for every \( f, g \in \Theta_2 \), \( n \geq 1 \) and \( (i_1, \ldots, i_n) \in \{1,2\}^n \). Then \( \boxplus \) is associative (but not commutative), and has the series \( Sum(z_1, z_2) = z_1 + z_2 \) as a unit. The series

\[
Zeta_2(z_1, z_2) := \sum_{n=1}^{\infty} \sum_{(i_1, \ldots, i_n) \in \{1,2\}^n} z_{i_1} \cdots z_{i_n},
\]

(2.17)

\[
Moeb_2(z_1, z_2) := \sum_{n=1}^{\infty} \sum_{(i_1, \ldots, i_n) \in \{1,2\}^n} (-1)^{n+1} \frac{(2n-2)!}{(n-1)!n!} z_{i_1} \cdots z_{i_n},
\]

are called the 2-variable Zeta and Moebius series; they are inverse to each other with respect to \( \boxplus \), and they are central (i.e. \( f \boxplus Zeta_2 = Zeta_2 \boxplus f \) for every \( f \in \Theta_2 \), and similarly for \( Moeb \)). For the proofs of all these facts, see [1], Section 3.

3) If \( \mu : \mathbb{C}[X_1, X_2] \to \mathbb{C} \) is a linear functional normalized by \( \mu(1) = 1 \), then its moment series \( M(\mu) \in \Theta_2 \) is

\[
\lbrack M(\mu) \rbrack(z_1, z_2) := \sum_{n=1}^{\infty} \sum_{(i_1, \ldots, i_n) \in \{1,2\}^n} \mu(X_{i_1} \cdots X_{i_n}) z_{i_1} \cdots z_{i_n},
\]

(2.18)
and its $R$-transform $R(\mu) \in \Theta_2$ is

$$R(\mu) := M(\mu) \Box \text{Moeb}_2.$$  \hspace{1cm} (2.19)

The coefficients of $R(\mu)$ are called the free cumulants of $\mu$.

2.12 Remark. Same as in the 1-dimensional case, Eqn. (2.19) can be stated in the equivalent form

$$M(\mu) = R(\mu) \Box \text{Zeta}_2,$$  \hspace{1cm} (2.20)

called the moment-cumulant formula (in two variables). From (2.19–20) it follows that $\mu \rightarrow R(\mu)$ is a bijection from the set of normalized linear functionals on $C\langle X_1, X_2 \rangle$ onto $\Theta_2$. Also, by identifying the coefficient of $z_{i_1} \cdots z_{i_n}$ on the two sides of (2.20), one can easily prove the following fact: let $\mu : C\langle X_1, X_2 \rangle \rightarrow C$ be a normalized linear functional, with free cumulants $(\alpha_{i_1, \ldots, i_n})_{n \geq 1}$; then $'\mu(X_{i_1} \cdots X_{i_n}) = 0$ for every odd $n$ and every $(i_1, \ldots, i_n) \in \{1, 2\}^n'$ is equivalent to $'\alpha_{i_1, \ldots, i_n} = 0$ for every odd $n$ and every $(i_1, \ldots, i_n) \in \{1, 2\}^n$'.

We mention that the important Equation (2.9), relating $\Box$ with the multiplication of free random variables, extends to the multi-dimensional case; in 2 variables we have:

$$R(\mu_{a_1b_1,a_2b_2}) = R(\mu_{a_1,a_2}) \Box \text{R}(\mu_{b_1,b_2}),$$  \hspace{1cm} (2.21)

whenever $\{a_1, a_2\}$ is free from $\{b_1, b_2\}$ in a non-commutative probability space $(\mathcal{A}, \varphi)$. The multi-variable versions of (2.9) have some interesting applications to freeness, see [11].

We also state here, for future reference, a ‘partly 2-variable’ version of Proposition 2.10; this is still a particular case of Proposition 7.3 in [12].

2.13 Proposition ([12]). Let $a$ be free from $\{b_1, b_2\}$ in the non-commutative probability space $(\mathcal{A}, \varphi)$, and let us denote $ab_1 := y_1, b_2a := y_2$. Then for every $n \geq 1$ and $\varepsilon = (l_1, \ldots, l_n) \in \{1, 2\}^n$ we have

$$\varphi(y_{l_1} \cdots y_{l_n}) = \sum_{\pi \in NC(n)} [\text{coef } (n); \pi](R(\mu_a)) \cdot [\text{coef } (l_1, \ldots, l_n); C_\varepsilon(\pi)](M(\mu_{b_1,b_2})).$$  \hspace{1cm} (2.22)

The 2-dimensional $R$-transform comes into the free commutator problem via the following fact, which is an immediate application of how the $R$-transform behaves under linear transformations (see [8], Section 5).

2.14 Proposition. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a, b$ be in $\mathcal{A}$. Then we have the relations:

$$[R(\mu_{i(ab-ba)})](z) = [R(\mu_{ab,ba})](iz, -iz),$$  \hspace{1cm} (2.23)

and

$$[R(\mu_{ab+ba})](z) = [R(\mu_{ab,ba})](z, z).$$  \hspace{1cm} (2.24)
3. **Reduction to the case of even random variables**

The result which will enable us to make such a reduction is the following.

3.1 **Theorem.** Let \( (\mathcal{A}, \varphi) \) and \( (\tilde{\mathcal{A}}, \tilde{\varphi}) \) be non-commutative probability spaces, and let \( a, b \in \mathcal{A}, \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}} \) be such that:

i) \( a \) is free from \( b \) in \( (\mathcal{A}, \varphi) \), \( \tilde{a} \) is free from \( \tilde{b} \) in \( (\tilde{\mathcal{A}}, \tilde{\varphi}) \);

ii) \( R_E(\mu_a) = R_E(\mu_{\tilde{a}}) \) (with \( R_E \) as in Notations 1.1);

iii) \( R_E(\mu_b) = R_E(\mu_{\tilde{b}}) \).

Then the distribution of \( ab - ba \) in \( (\mathcal{A}, \varphi) \) coincides with the one of \( \tilde{a} \tilde{b} - \tilde{b} \tilde{a} \) in \( (\tilde{\mathcal{A}}, \tilde{\varphi}) \).

The way how Theorem 3.1 is used will be shown later on in the paper (Section 5.4); for the moment we only concentrate on its proof. We start by making a further reduction in the statement of the theorem.

3.2 **Lemma.** It suffices to prove Theorem 3.1 in the case when its hypothesis (iii) is replaced by the stronger one that: (iii') \( \mu_b = \mu_{\tilde{b}} \).

**Proof.** Let us assume the Theorem 3.1 to be true when the hypothesis (i)+(ii)+(iii') are holding, and let us consider \( (\mathcal{A}, \varphi), (\tilde{\mathcal{A}}, \tilde{\varphi}), a, b \in \mathcal{A}, \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}} \) which only satisfy (i)+(ii)+(iii'). By using a free product construction, we can produce a non-commutative probability space \( (\hat{\mathcal{A}}, \hat{\varphi}) \) and elements \( \hat{a}, \hat{b} \in \hat{\mathcal{A}} \) which are free with respect to \( \hat{\varphi} \) and such that \( \mu_{\hat{a}} = \mu_a \), \( \mu_{\hat{b}} = \mu_{\tilde{b}} \). Then on one hand, \( \alpha, \beta \in \mathcal{A} \) and \( \alpha, \beta \in \mathcal{A} \) satisfy (i)+(ii)+(iii'), hence \( \hat{a} \hat{b} - \hat{b} \hat{a} \) has the same distribution as \( \tilde{a} \tilde{b} - \tilde{b} \tilde{a} \); on the other hand, \( b, a \in \mathcal{A} \) and \( \tilde{b}, \tilde{a} \in \tilde{\mathcal{A}} \) also satisfy (i)+(ii)+(iii'), hence \( \tilde{b} \tilde{a} - \tilde{a} \tilde{b} \) has the same distribution as \( \tilde{b} \tilde{a} - \tilde{a} \tilde{b} \). From these two facts it is immediate that \( ab - ba \) and \( \tilde{a} \tilde{b} - \tilde{b} \tilde{a} \) have identical distributions. QED

We shall prove the Theorem 3.1 (in the reduced form of 3.2) by exhibiting an expression for the moments of \( ab - ba \), where only the even free cumulants of \( a \) are involved. The first step towards doing this is the following.

3.3 **Proposition.** Let \( (\mathcal{A}, \varphi) \) be a non-commutative probability space, and let \( a, b \in \mathcal{A} \) be free with respect to \( \varphi \). Then for every \( n \geq 1 \) we have:

\[
\varphi((ab - ba)^n) = \sum_{\pi \in NC(n)} (-1)^{d(\varepsilon)} [\text{coef } (n); \pi] R(\mu_a) \cdot [\text{coef } (n); C_{\varepsilon}(\pi)] M(\mu_b),
\]

where for an \( n \)-tuple \( \varepsilon \in \{1, 2\}^n \), \( d(\varepsilon) \) denotes the number of components of \( \varepsilon \) that are equal to 2 (and the notations \( C_{\varepsilon}, [\text{coef } (n); \pi] \) are as established in Section 2).
Proof. Let us make the notations $ab = x_1$, $ba = x_2$, and for every $\varepsilon := (l_1, \ldots, l_n) \in \{1, 2\}^n$ let us put $x_\varepsilon := x_{l_1}x_{l_2} \cdots x_{l_n} \in A$. It is then obvious that

$$\varphi((ab - ba)^n) = \varphi((x_1 - x_2)^n) = \sum_{\varepsilon \in \{1, 2\}^n} (-1)^{d(\varepsilon)} \varphi(x_\varepsilon). \quad (3.2)$$

Equation (3.1) follows from (3.2) and the expression for $\varphi(x_\varepsilon)$ as summation over non-crossing partitions which is provided by Eqn. (2.12) of Proposition 2.10. QED

The second step is to put in evidence some remarkable involutions on $NCO(n)$ and $NCO(n) \times \{1, 2\}^n$.

3.4 Notations.

1) Recall from Section 2.2 that we denote by $NCO(n)$ the set of non-crossing partitions $\pi$ of $\{1, \ldots, n\}$ which have at least one block with an odd number of elements. Given $\pi \in NCO(n)$, it is easy to show that there exist blocks $B$ of $\pi$ having $|B|$ odd and min $B$, max $B$ of the same parity (see e.g. [12], Lemma 4.7); among these blocks let us pick the one, call it $B_o$, which has the smallest value of min $B_o$. It is convenient that (for the purposes of this section only) we give a name to the interval $[\min B_o, \max B_o]$; we shall call it the twist interval of $\pi$.

2) For $\pi \in NCO(n)$ we shall denote by $tw(\pi)$ (with $tw$ for ‘twist’) the partition of $\{1, \ldots, n\}$ obtained in the following way. Let $[t_0, t_1]$ be the twist interval of $\pi$, as defined above, and let $\tau$ be the permutation of $\{1, \ldots, n\}$ given by: $\tau(i) = t_0 + t_1 - i$, if $t_0 \leq i \leq t_1$, and $\tau(i) = i$ otherwise. Then we put $tw(\pi)$ to be $'\tau(\pi)'$, i.e.:

$$tw(\pi) := \{\tau(B) \mid B \text{ block of } \pi\}. \quad (3.3)$$

It is immediate that $tw(\pi)$ is also in $NCO(n)$, and has the same twist interval $[t_0, t_1]$ as $\pi$ itself. The latter fact has as consequence that $tw(tw(\pi)) = \pi$; i.e., $tw: NCO(n) \to NCO(n)$ is an involution.

3) We need to adapt the twist $tw$ to the case when an $n$-tuple $\varepsilon \in \{1, 2\}^n$ is considered at the same time with the partition $\pi \in NCO(n)$. Thus, we shall denote by $\tilde{tw}$ the map from $NCO(n) \times \{1, 2\}^n$ into itself which is defined by the formula:

$$\tilde{tw}(\pi, (l_1, \ldots, l_n)) = (tw(\pi), (l'_1, \ldots, l'_n)), \quad \pi \in NCO(n), \; l_1, \ldots, l_n \in \{1, 2\}, \quad (3.4)$$

where

$$l'_i = \begin{cases} 3 - t_0 + t_1 - i & \text{if } t_0 \leq i \leq t_1 \\ l_i & \text{otherwise}, \end{cases} \quad (3.5)$$

and where in (3.5) $t_0$ and $t_1$ are the end-points of the twist interval of $\pi$.

If $\tilde{tw}(\pi, \varepsilon) = (tw(\pi), \varepsilon')$ and if $\tilde{tw}(tw(\pi), \varepsilon') = (\pi, \varepsilon'')$, then from (3.5) and the fact that $tw(\pi)$ has the same twist interval as $\pi$, we infer that $\varepsilon'' = \varepsilon$. Hence $\tilde{tw}$ is an involution on $NCO(n) \times \{1, 2\}^n$. 23
3.5 Lemma. If \((\pi, \varepsilon)\) is in \(NCO(n) \times \{1, 2\}^n\) for some \(n\), and if \(\tilde{tw}(\pi, \varepsilon) =: (\pi', \varepsilon')\), then the partitions \(C_\varepsilon(\pi)\) and \(C_{\varepsilon'}(\pi')\) have the same block structure (i.e., for every \(1 \leq m \leq n\), the two partitions have the same number of blocks with \(m\) elements).

Proof. Let \([t_0, t_1]\) be the twist interval of \(\pi\), and let \(\tau\) be the permutation of \(\{1, \ldots, n\}\) defined by \(\tau(i) = t_0 + t_1 - i\), if \(t_0 \leq i \leq t_1\), and \(\tau(i) = i\) otherwise. We also denote \(C_\varepsilon(\pi) =: \rho, C_{\varepsilon'}(\pi') =: \rho'\). We will show that \(\rho = \tau(\rho')\) (where \(\tau(\rho')\) is defined as \(\{\tau(B) \mid B\) block of \(\rho'\}\)); this will of course imply that \(\rho\) and \(\rho'\) have the same block structure.

It will actually suffice to show that
\[
\rho \geq \tau(\rho') \quad \text{(in the refinement order)}; \tag{3.6}
\]
since at the moment it is not verified that \(\tau(\rho')\) is non-crossing, we consider this inequality in the larger poset of all the partitions of \(\{1, \ldots, n\}\).

Indeed, let us assume that (3.6) is proved. Then by applying this inequality to \((\pi', \varepsilon')\) instead of \((\pi, \varepsilon)\), and by taking into account that \(\tilde{tw}(\pi', \varepsilon') = (\pi, \varepsilon)\) and that \(\pi'\) has the same twist interval as \(\pi\), we get: \(\rho' \geq \tau(\rho)\). But \(\tau^2 = id\), hence the application of \(\tau\) to the both sides of the latter inequality leads to \(\tau(\rho') \geq \rho\), and hence (3.6) must be an equality.

Due to the way how \(\rho = C_\varepsilon(\pi)\) and \(\rho' = C_{\varepsilon'}(\pi')\) are concretely defined, the proof of (3.6) is an exercise in elementary geometry. We consider an \(\varepsilon\)-insertion bead \((P_1, \ldots, P_n; Q_1, \ldots, Q_n)\) (see Definition 2.3), and on the same circle we mark the points \(X_0, X_1, Y_0, Y_1\) in the way shown in Figure 2; if the length of the arc of circle between two consecutive \(P_i\)'s is denoted by \(\lambda\), then the length of the arcs \(Y_0P_{t_0}\) and \(P_{t_1}Y_1\) is \(\lambda/3\), and the one of the arcs \(X_0Y_0\) and \(Y_1X_1\) is \(\lambda/6\).

Let us record the remark that:
\[
[P_h, P_k] \cap [Y_0, Y_1] = \emptyset = [P_h, P_k] \cap [X_0, X_1] \tag{3.7}
\]
for every \(1 \leq h, k \leq n\) such that \(h \not\sim k\), and where we use the notation \([A, B]\) for the line segment connecting the points \(A, B\) in the plane. The verification of (3.7) is easily made by using that \(t_0\) and \(t_1\) are the min and max of the same block of \(\pi\) (hence \(\{t_0, \ldots, t_1\}\) and \(\{1, \ldots, n\} \setminus \{t_0, \ldots, t_1\}\) are both unions of blocks of \(\pi\)).

Let us next denote by \(S\) the open-half plane determined by \(X_0\) and \(X_1\), which contains \(Y_0, Y_1, P_{t_0}, P_{t_1}\). We shall denote by \(T\) the bijective transformation of the plane which reflects the points of \(S\) in the mediator line of \([X_0, X_1]\), and leaves fixed the points in the complement of \(S\) (see Figure 2). It is obvious that
\[
T(P_i) = P_{\tau(i)}, \quad 1 \leq i \leq n, \tag{3.8}
\]
where \(\tau\) is the permutation mentioned in the first phrase of the proof. We do not have a similar relation for the points \(Q_1, \ldots, Q_n\); we therefore define the points \(Q'_1, \ldots, Q'_n\) on the circle by requiring that the analogue of (3.8) holds, i.e. by putting:
\[
Q'_i := T(Q_{\tau(i)}), \quad 1 \leq i \leq n. \tag{3.9}
\]
It is readily checked that \((P_1, \ldots, P_n; Q'_1, \ldots, Q'_n)\) is an \(\varepsilon'\)-insertion bead, with \(\varepsilon'\) as in the statement of the lemma.
With these notations, the inequality (3.6) that is to be proved has the following geometric interpretation:

\[
\begin{cases}
\text{If } [Q_i, Q_j] \text{ intersects at least one } [P_h, P_k] \text{ with } h \ncong k, \\
\text{then } [T(Q_i), T(Q_j)] \text{ intersects at least one } [P_h, P_k] \text{ with } h \ncong k.
\end{cases}
\]  

(3.10)

Indeed, the first line of (3.10) means \( i \ncong j \) (see 2.3), while the second line of (3.10) means \( \tau(i) \ncong \tau(j) \) (where we also took (3.9) into account). Hence the statement in (3.10) is that:

\( i \ncong j \Rightarrow \tau(i) \ncong \tau(j) \) ( \( \Leftrightarrow i \ncong \tau(j) \) ), or in other words: \( i \ncong j \Rightarrow i \ncong j \); but this is exactly (3.6).

So we are left to prove (3.10). We shall divide the argument in three cases.

a) If \( Q_i \) and \( Q_j \) are on opposite sides of the line \( P_{t_0}P_{t_1} \). Then, as is clear from the definition of \( T \), the points \( T(Q_i) \) and \( T(Q_j) \) are still on opposite sides of \( P_{t_0}P_{t_1} \); hence, \( [T(Q_i), T(Q_j)] \) intersects \( [P_{t_0}, P_{t_1}] \), with \( t_0 \ncong t_1 \).

\[ \text{Figure 2.} \]
b) If \( Q_i \) and \( Q_j \) are on the same side of the line \( X_0X_1 \). Consider \( 1 \leq h, k \leq n \) such that \( h \sim k \) and \( \{Q_i, Q_j\} \cap [P_h, P_k] \neq \emptyset \). From (3.7) it follows that \( P_h \) and \( P_k \) are also on the same side of \( X_0X_1 \). But then, since \( T \) is affine on any of the two open half-planes determined by \( X_0X_1 \), we get:

\[
[T(Q_i), T(Q_j)] \cap [P_{\tau(h)}, P_{\tau(k)}] = T([Q_i, Q_j]) \cap T([P_h, P_k]) = T([Q_i, Q_j] \cap [P_h, P_k]) \neq \emptyset;
\]

and in addition we have \( \tau(h) \sim \tau(k) \), because \( h \sim k \) and \( \pi = \tau(\pi) \).

(c) If \( Q_i \) and \( Q_j \) are on the same side of the line \( P_0P_1 \), but are on different sides of the line \( X_0X_1 \). This is only possible if one of \( i, j \) is in \( \{t_0, t_1\} \) (say, for definiteness, that \( i = t_0 \)), and the other one \( (j) \) is in \( \{1, \ldots, n\} \setminus \{t_0, \ldots, t_1\} \). In order to have \( Q_i \) and \( Q_j \) on the same side of \( P_0P_1 \), we must also assume that the \( t_0 \)-th component of \( \varepsilon \) is 1; this implies that \( Q_i = Y_0, T(Q_i) = Y_1 \).

Consider now \( 1 \leq h, k \leq n \) such that \( h \sim k \) and \( \{Q_i, Q_j\} \cap [P_h, P_k] \neq \emptyset \). By using (3.7) we see that \( h, k \) must necessarily be in \( \{1, \ldots, n\} \setminus \{t_0, \ldots, t_1\} \); note that we also have \( h \sim k \), since \( \pi' = \tau(\pi) \) and \( h, k \) are fixed by \( \tau \).

We know that \( [P_h, P_k] \cap [Y_0, Q_j] \neq \emptyset \) (previous paragraph), and that \( [P_h, P_k] \cap [Y_0, Y_1] = \emptyset \) (Eqn. (3.7)). The line \( P_hP_k \) cannot cross exactly one edge of the triangle \( Y_0Y_1Q_j \), therefore we must have \( [P_h, P_k] \cap [Y_1, Q_j] \neq \emptyset \); but this means exactly that \( [P_h, P_k] \cap [T(Q_i), T(Q_j)] \neq \emptyset \), and since \( h \sim k \), the verification of (3.10) is complete.

At some points during the presentation of this proof, it was implicitly assumed that \( t_0 \neq t_1 \) and that \( \{1, \ldots, n\} \setminus \{t_0, \ldots, t_1\} \neq \emptyset \); it is however easy to see that, with the appropriate minor modifications, the argument is also working for the two extreme cases when \( t_0 = t_1 \) or \( \{t_0, t_1\} = \{1, n\} \). QED

3.6 Lemma. If \( (\pi, \varepsilon) \in NCO(n) \times \{1, 2\}^n \) and if \( \tilde{w}(\pi, \varepsilon) =: (\pi', \varepsilon') \), then the numbers \( d(\varepsilon) \) and \( d(\varepsilon') \) (counting how many 2's are in the \( n \)-tuples \( \varepsilon \) and \( \varepsilon' \)) have different parities.

**Proof.** The sequence \( \varepsilon' \) is obtained from \( \varepsilon \) by changing its components with indices in the twist interval \( [t_0, t_1] \) of \( \pi \). Thus if \( \varepsilon := (l_1, \ldots, l_n) \) and if we put: \( d' = |\{m \mid t_0 \leq m \leq t_1, l_m = 2\}|, d'' = |\{m \mid m < t_0 \text{ or } m > t_1, \text{ and } l_m = 2\}| \), then we obtain:

\[
d(\varepsilon) = d' + d'', \quad d(\varepsilon') = (t_1 - t_0 + 1) - d' + d''.
\]

Hence \( d(\varepsilon) + d(\varepsilon') = (t_1 - t_0 + 1) + 2d'' \). Since \( t_0 \) and \( t_1 \) have the same parity (see Notations 3.4.1), we obtain that \( d(\varepsilon) + d(\varepsilon') \) is an odd number, and the conclusion follows. QED

3.7 Proposition. Let \((A, \varphi)\) be a non-commutative probability space, and let \( a, b \in A \) be free with respect to \( \varphi \). Then for every \( n \geq 1 \) we have:

\[
\sum_{\pi \in NCO(n)} \sum_{\varepsilon \in \{1, 2\}^n} (-1)^{d(\varepsilon)} \text{coef}(\pi) \cdot \text{coef}(\mu_a) \cdot \text{coef}(\varepsilon) \cdot C_{\varepsilon}(\mu_b) = 0. \tag{3.11}
\]
Proof. Let us denote the sum appearing in the left-hand side of (3.11) by \( L \). By performing (in the named sum) the change of variable provided by the bijection \( \tilde{tw} \) of Definition 3.4.3, we obtain that:

\[
L = \sum_{\pi \in NCO(n) \atop \varepsilon \in \{1,2\}^n \atop \tilde{tw}(\pi, \varepsilon) = (\pi', \varepsilon')} (-1)^{d(\varepsilon')} \left[ \text{coef } (n); \pi' \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon'}(\pi') \right] (M(\mu_b)). \tag{3.12}
\]

Hence we can write:

\[
2L = \sum_{\pi \in NCO(n) \atop \varepsilon \in \{1,2\}^n \atop \tilde{tw}(\pi, \varepsilon) = (\pi', \varepsilon')} \left\{ (-1)^{d(\varepsilon)} \left[ \text{coef } (n); \pi \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon}(\pi) \right] (M(\mu_b)) \right\}
\]

\[
+ (-1)^{d(\varepsilon')} \left[ \text{coef } (n); \pi' \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon'}(\pi') \right] (M(\mu_b)). \tag{3.13}
\]

But for every \((\pi, \varepsilon) \in NCO(n) \times \{1,2\}^n\), with \( \tilde{tw}(\pi, \varepsilon) = (\pi', \varepsilon') \), it happens that

\[
(-1)^{d(\varepsilon)} \left[ \text{coef } (n); \pi \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon}(\pi) \right] (M(\mu_b)) + \]

\[
+ (-1)^{d(\varepsilon')} \left[ \text{coef } (n); \pi' \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon'}(\pi') \right] (M(\mu_b)) = 0. \tag{3.14}
\]

Indeed, we have \([\text{coef } (n); \pi \] \( (R(\mu_a)) \) = \([\text{coef } (n); \pi'] \( (R(\mu_a)) \) because \( \pi \) and \( \pi' \) have the same block structure (obvious from (3.3)); we have \([\text{coef } (n); C_{\varepsilon}(\pi) \] \( (M(\mu_b)) \) = \([\text{coef } (n); C_{\varepsilon'}(\pi') \] \( (M(\mu_b)) \) because \( C_{\varepsilon}(\pi) \) and \( C_{\varepsilon'}(\pi') \) have the same block structure (Lemma 3.5); and \((-1)^{d(\varepsilon)} + (-1)^{d(\varepsilon')} = 0\), by Lemma 3.6.

\(~\}From (3.13) and (3.14) it is obvious that \( \text{L} = 0\). QED

3.8 Conclusion of the proof of Theorem 3.1. Let \((\mathcal{A}, \varphi)\) be a non-commutative probability space, and let \( a, b \in \mathcal{A} \) be free with respect to \( \varphi \). By combining the Propositions 3.3 and 3.7 we obtain that, for every \( n \geq 1 \):

\[
\varphi((ab - ba)^n) = \sum_{\pi \in NCE(n) \atop \varepsilon \in \{1,2\}^n} (-1)^{d(\varepsilon)} \left[ \text{coef } (n); \pi \right] (R(\mu_a)) \cdot \left[ \text{coef } (n); C_{\varepsilon}(\pi) \right] (M(\mu_b)). \tag{3.15}
\]

Note that on the right-hand side of (3.15) only the even free cumulants of \( a - b \) – i.e. the coefficients of \( R_{E}(\mu_a) - \) are appearing, because one takes \([\text{coef } (n); \pi \] \( (R(\mu_a)) \) for \( \pi \)'s having only blocks of even size.

Consider now the situation discussed in Lemma 3.2, where we also have a non-commutative probability space \((\tilde{\mathcal{A}}, \tilde{\varphi})\) and \( \tilde{a}, \tilde{b} \in \tilde{\mathcal{A}} \), such that the hypothesis (i)+(ii)+(iii') are fulfilled. By writing the counterpart of Eqn. (3.15) for \( \tilde{a} \) and \( \tilde{b} \), we clearly obtain that \( \varphi((\tilde{a} \tilde{b} - \tilde{b} \tilde{a})^n) = \varphi((\tilde{a} \tilde{b} - \tilde{b} \tilde{a})^n) \) for every \( n \geq 1 \), as desired. QED

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4. R-diagonal pairs and free commutator

If $a$ and $b$ are free in some non-commutative probability space, then $ab$ and $ba$ are of course not free; but if in addition we assume that $a$ and $b$ are both even, then there still exists a definite and treatable relation between $ab$ and $ba$ – namely that they form an R-diagonal pair (notion introduced in [12]). This fact has a key role in our approach to the free commutator, and the main goal of the present section is the presentation of its proof. Before arriving to this (in Theorem 4.5), we shall review from [12] some facts about R-diagonality.

4.1 Definition. Let $(\mathcal{A}, \varphi)$ be a tracial non-commutative probability space and $x_1, x_2 \in \mathcal{A}$. We call $(x_1, x_2)$ an R-diagonal pair if its R-transform has the special form

$$[R(\mu_{x_1, x_2})](z_1, z_2) = f(z_1z_2) + f(z_2z_1)$$

(4.1)

for some series $f$ of only one variable; this $f$ is called the determining series of the pair $(x_1, x_2)$.

Two of the most important examples of R-diagonal pairs are $(c, c^*)$ and $(u, u^*)$ for $c$ a circular variable and $u$ a Haar unitary. The determining series in these cases are given by $f = \text{Id}$ and by $f = \text{Moeb}$, respectively (with $\text{Id}$, $\text{Moeb}$ as in the comments following the Definitions 2.6).

The main results of [12] about R-diagonal pairs are collected in the next theorem.

4.2 Theorem ([12]). Let $(\mathcal{A}, \varphi)$ be a tracial non-commutative probability space.

1) Let $(x_1, x_2)$ be an R-diagonal pair in $(\mathcal{A}, \varphi)$. Then the determining series $f$ of $(x_1, x_2)$ is given by

$$f = R(\mu_{x_1, x_2}) \Box \text{Moeb}.$$  

(4.2)

2) Let $x_1, x_2, p_1, p_2 \in \mathcal{A}$ such that $(x_1, x_2)$ is an R-diagonal pair and such that $\{x_1, x_2\}$ is free from $\{p_1, p_2\}$. Then $(x_1p_1, p_2x_2)$ is also an R-diagonal pair.

3) Let, in the situation of part 2), $f$ and $g$ be the determining series of $(x_1, x_2)$ and $(x_1p_1, p_2x_2)$, respectively. Then these two series are related by

$$g = f \Box R(\mu_{p_1p_2}).$$

(4.3)

Our interest in R-diagonal pairs came from the fact that some important properties of the circular element have natural generalizations to this framework. Two situations of this kind that we found are concerning polar decompositions of certain R-diagonal pairs (see [12], Application 1.9), and the occurrence of R-diagonal pairs as free off-diagonal compressions (see [12]). We state the polar decomposition result precisely, because we will need to refer to it in Section 4.7.
4.3 Proposition (12). Let \((A, \varphi)\) be a tracial probability space and assume that \(A\) is a von Neumann algebra and \(\varphi\) a normal faithful state. Consider \(x \in A\) such that \((x, x^*)\) is an \(R\)-diagonal pair and such that \(\text{Ker} \ x = \{0\}\). Let \(x = up\) be the polar decomposition of \(x\). Then it follows that \(u\) is a Haar unitary and that \(\{u, u^*\}\) is free from \(p\).

The following characterization of \(R\)-diagonal pairs will be used in the proof of Theorem 4.5.

4.4 Proposition. Let \((A, \varphi)\) be a tracial non-commutative probability space and \(x_1, x_2 \in A\). Then the following two properties are equivalent:

a) The pair \((x_1, x_2)\) is \(R\)-diagonal.

b) If \(u \in A\) is a Haar unitary such that \(\{u, u^*\}\) is free from \(\{x_1, x_2\}\), then \(\mu_{x_1, x_2} = \mu_{x_1u, u^*x_2}\).

Proof. \(a) \Rightarrow b): By Theorem 4.2, the pair \((x_1u, u^*x_2)\) is \(R\)-diagonal and its determining series \(g\) is connected with the determining series \(f\) of \((x_1, x_2)\) by

\[
g = f \bigodot R(\mu_{uu^*}) = f \bigodot R(\mu_1) = f \bigodot \text{Id} = f. \tag{4.4}
\]

This implies that the \(R\)-transforms -- and hence also the distributions -- of the two pairs are the same.

\(b) \Rightarrow a): The pair \((x_1u, u^*x_2)\) is \(R\)-diagonal by Theorem 4.2.2 and the fact that \(\{u, u^*\}\) is \(R\)-diagonal. Hence \((x_1, x_2)\) is also \(R\)-diagonal (since it has the same distribution as \((x_1u, u^*x_2)\)). QED

We now arrive to the result announced at the beginning of the section.

4.5 Theorem. Let \((A, \varphi)\) be a non-commutative probability space and consider \(a, b \in A\) such that \(a\) and \(b\) are both even and such that \(a\) is free from \(b\). Then \((ab, ba)\) is an \(R\)-diagonal pair.

Proof. Note first that we can replace \((A, \varphi)\) by \((A', \varphi')\), where \(A'\) is the unital algebra generated by \(a\) and \(b\) and \(\varphi'\) is the restriction of \(\varphi\) to \(A'\). Since \((A', \varphi')\) is always tracial (by the fact that the reduced free product of traces gives again a trace, see e.g. Proposition 2.5.3 in [18]), we can assume without loss of generality that \((A, \varphi)\) is a tracial non-commutative probability space.

Another assumption we can make (by embedding \((A, \varphi)\) into a larger tracial non-commutative probability space) is that there exists a Haar unitary \(u \in A\) such that \(\{u, u^*\}\) is free from \(\{a, b\}\).

Put \(x_1 := ab\), \(x_2 := ba\), and \(y_1 := x_1u = abu\), \(y_2 := u^*x_2 = u^*ba\). For a sequence \(\varepsilon = (l_1, \ldots, l_n)\) of 1’s and 2’s we will use the abbreviations \(x_\varepsilon := x_{l_1} \ldots x_{l_n}\), \(y_\varepsilon := y_{l_1} \ldots y_{l_n}\). By Proposition 4.4, what we have to show is that \(\mu_{x_1, x_2} = \mu_{y_1, y_2}\), or equivalently that:

\[
\varphi(x_\varepsilon) = \varphi(y_\varepsilon) \tag{4.5}
\]

for all sequences \(\varepsilon\). For the rest of the proof we fix such a sequence \(\varepsilon = (l_1, \ldots, l_n)\) (with \(n \in \mathbb{N}\) and \(l_i \in \{1, 2\}\)), about which we will prove that (4.5) holds.
By Proposition 2.10, we have
\[
\varphi(x_\varepsilon) = \sum_{\pi \in NC(n)} [\text{coef}(n); \pi](R(\mu_a)) \cdot [\text{coef}(n); C_\varepsilon(\pi)](M(\mu_b)). \tag{4.6}
\]
Since \(a\) and \(b\) are even, the odd cumulants of \(a\) and the odd moments of \(b\) vanish; therefore, by Proposition 2.4.2, only \(\varepsilon\)-alternating \(\pi\)'s give a non-vanishing contribution to the sum.

On the other hand, by Proposition 2.13,
\[
\varphi(y_\varepsilon) = \sum_{\pi \in NC(n)} [\text{coef}(n); \pi](R(\mu_a)) \cdot [\text{coef}(\varepsilon); C_\varepsilon(\pi)](M(\mu_{b,u,u^*b})). \tag{4.7}
\]
The pair \((bu, u^*b)\) is \(R\)-diagonal (by Theorem 4.2), and in particular all the free cumulants of odd length of \(\mu_{b,u,u^*b}\) are equal to zero; as a consequence, the joint moments of odd length of \((bu, u^*b)\) are all equal to zero, too (see Remark 2.12). But then Proposition 2.4.2 implies again that we can restrict in our summation to \(\varepsilon\)-alternating \(\pi\)'s.

So let us fix such an \(\varepsilon\)-alternating \(\pi\); by comparing (4.6) and (4.7), it only remains to show that
\[
[\text{coef}(n); C_\varepsilon(\pi)](M(\mu_b)) = [\text{coef}(\varepsilon); C_\varepsilon(\pi)](M(\mu_{b,u,u^*b})). \tag{4.8}
\]
Finally, (4.8) is indeed true, because the partition \(C_\varepsilon(\pi)\) is also \(\varepsilon\)-alternating (by Proposition 2.4.1) – hence when we explicitly write the right-hand side of (4.8) as a product (comp. Eqn. (2.14)), all the appearing \(u\)'s are cancelled by corresponding \(u^*\)'s. **QED**

**4.6 Remarks.**

1) Both the \(R\)-diagonal pairs \((c, c^*)\) and \((u, u^*)\), with \(c\) circular and \(u\) Haar unitary, can arise in the form \((ab, ba)\) with \(a, b\) free and both even; in fact both examples can be obtained when we ask in addition that \(b^2 = 1\), i.e. \(\mu_b = 1/2(\delta_1 + \delta_1)\). With this assumption, the \(R\)-diagonal pair \((x_1, x_2) := (ab, ba)\) has determining series
\[
f = R(\mu_{aba}) \boxtimes \text{Moeb} = R(\mu_{a^2}) \boxtimes \text{Moeb}. \tag{4.9}
\]
If we also ask that \(a^2 = 1\), then \(f = \text{Moeb}\) and \((x_1, x_2)\) is a Haar unitary. If \(a\) is a semicircular element, then \(a^2\) is a free Poisson element with \(R(\mu_{a^2}) = \text{Zeta}\); hence \(f = \text{Zeta} \boxtimes \text{Moeb} = \text{Id}\), and \((x_1, x_2)\) is a circular element.

2) The ‘dual version’ of the statement of Theorem 4.5 is also true, that is: if \((\mathcal{A}, \varphi)\) is a tracial non-commutative probability space and if \(x_1, x_2 \in \mathcal{A}\) are such that \((x_1, x_2)\) is an \(R\)-diagonal pair, then \(x_1x_2\) and \(x_2x_1\) are free. The proof of this assertion can be made as follows. Without loss of generality we can assume that there exists a Haar unitary \(u \in \mathcal{A}\) such that \(\{u, u^*\}\) is free from \(\{x_1, x_2\}\). Then, by Proposition 4.4, \((x_1, x_2)\) has the same distribution as \((x_1u, u^*x_2)\); so instead of proving that \(x_1x_2\) is free from \(x_2x_1\), it suffices to prove that \((x_1u)(u^*x_2)\) is free from \((u^*x_2)(x_1u)\). But the latter assertion is easily verified, by just using the definition of freeness.
4.7 Proofs of Propositions 1.10, 1.18.
1) In 1.10 we have to show that if \(a, b\) are free and both even in the non-commutative probability space \((\mathcal{A}, \varphi)\), then \(\mu_{i(ab-ba)} = \mu_{ab+ba}\). We saw in Theorem 4.5 that the pair \((ab, ba)\) is \(R\)-diagonal, hence that

\[
[R(\mu_{ab,ba})](z_1, z_2) = f(z_1z_2) + f(z_2z_1),
\]

with \(f\) a series of one variable. But then the Eqns.(2.23-24) of Proposition 2.14 give us that

\[
[R(\mu_{i(ab-ba)})](z) = 2f(z^2) = [R(\mu_{ab+ba})](z),
\]

which entails (Remark 2.8) that \(i(ab - ba)\) and \(ab + ba\) have the same distribution.

2) Proposition 1.18 is obtained by putting together the results in 4.3 and 4.5. QED

5. Proof of the free commutator formula

We shall first prove the Theorem 1.2 under the extra assumption that the elements \(a, b\) in its statement are even, and then in full generality – in the Sections 5.3 and 5.4, respectively. Before going to the proof of 1.2, we need to put into evidence one more formula, concerning the \(R\)-transform of the square of an even element. This formula follows from a natural bijection between \(NCE(2n)\) and the set of intervals of the poset \(NC(n)\), which was observed in [12], Corollary 4.5:

5.1 Proposition ([12]). Recall that \(NCE(2n)\) denotes the set of non-crossing partitions \(\sigma\) of \(\{1, 2, \ldots, 2n\}\), with the property that every block of \(\sigma\) has an even number of elements. There exists a bijection \(\Psi: NCE(2n) \to \{((\pi, \rho) \mid \pi, \rho \in NC(n), \pi \leq \rho)\}\), such that if \(\Psi(\sigma) = (\pi, \rho)\), then: \(\sigma\) and \(\pi\) have the same number of blocks, say \(k\); and moreover, one can write \(\sigma = \{B_1, \ldots, B_k\}\), \(\pi = \{A_1, \ldots, A_k\}\), in such a way that \(|B_j| = 2|A_j|\) for every \(1 \leq j \leq k\).

5.2 Proposition. We have the formula

\[
[R(\mu_a)](z) = [R(\mu_{a^2})] \mathbb{E} \text{Moeb } (z^2),
\]

or equivalently

\[
R_E(\mu_a) = R(\mu_{a^2}) \mathbb{E} \text{Moeb},
\]

holding for every even element \(a\) in some non-commutative probability space \((\mathcal{A}, \varphi)\).

Proof. Let us denote \([R(\mu_a)](z) := \sum_{n=1}^{\infty} a_n z^n\). Due to the hypothesis that \(a\) is even, we have \(a_m = 0\) for every odd \(m\) (Remark 2.8). Then the moment-cumulant formula (2.8) gives us:

\[
\varphi(a^{2n}) = \sum_{\sigma \in NCE(2n)} \alpha_{|B_1|} \cdots \alpha_{|B_k|}, \quad n \geq 1.
\]

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We next use the bijection $\Psi$ of Proposition 5.1 to make a “change of variable” in the summation in (5.3), thus obtaining:

\[
\varphi(a^{2n}) = \sum_{\pi \leq \rho, \rho \in NC(n)} \alpha_{2|A_1|, \ldots, \alpha_{2|A_k|} = 1/n}
\]

\[
= \sum_{\pi \in NC(n), \pi \in \{A_1, \ldots, A_k\}} \alpha_{2|A_1|, \ldots, \alpha_{2|A_k|}, K(\pi)} \cdot \left|\{\rho \in NC(n) | \rho \geq \pi\}\right|.
\]

(5.4)

Since

\[
R_E(\mu_a)(z) = \sum_{n=1}^{\infty} \alpha_{2n} z^n,
\]

the product $\alpha_{2|A_1|, \ldots, \alpha_{2|A_k|}}$ appearing in (5.4) is just $[\text{coef}(n); \pi](R_E(\mu_a))$. The other factor showing up in the general term of the summation in (5.4), $\left|\{\rho \in NC(n) | \rho \geq \pi\}\right|$, can also be viewed as $\left|\{\rho' \in NC(n) | \rho' \leq K(\pi)\}\right|$ – because the Kreweras complementation map on $NC(n)$ sends $\left\{\rho \in NC(n) | \rho \geq \pi\right\}$ bijectively onto $\left\{\rho' \in NC(n) | \rho' \leq K(\pi)\right\}$. A direct inspection of how the operation $\star$ is defined (Equation (2.3)) gives us that the latter cardinality can be identified as $[\text{coef}(n); K(\pi)](Zeta \star Zeta)$.

Hence our formula for $\varphi(a^{2n})$ becomes:

\[
\varphi(a^{2n}) = \sum_{\pi \in NC(n)} [\text{coef}(n); \pi](R_E(\mu_a)) \cdot [\text{coef}(n); K(\pi)](Zeta \star Zeta), \quad n \geq 1. \quad (5.5)
\]

But the right-hand side of (5.5) is exactly the coefficient of order $n$ in $R_E(\mu_a) \star Zeta \star Zeta$, again by the formula (2.3). Since $\varphi(a^{2n})$ can be, on the other hand, viewed as the coefficient of order $n$ in the moment series $M(\mu_a^2)$, the conclusion we draw from this calculation is that:

\[
M(\mu_a^2) = R_E(\mu_a) \star Zeta \star Zeta. \quad (5.6)
\]

Finally, we $\star$-operate with $Moeb \star Moeb$ on both sides of (5.6). If we take into account that $Moeb$ is the inverse of $Zeta$ under $\star$, and that $M(\mu_a^2) \star Moeb = R(\mu_a^2)$ (by Eqn. (2.6)), we see that (5.6) is turned into (5.2), which is equivalent to (5.1). QED

5.3 Proof of Theorem 1.2 for even elements. If, in the notations of 1.2, we assume in addition that $a$ and $b$ are even, then the pair $(ab, ba)$ is $R$-diagonal (Theorem 4.5); or in other words, the joint $R$-transform of $ab$ and $ba$ has the form

\[
[R(\mu_{ab,ba})](z_1, z_2) = h(z_1 z_2) + h(z_2 z_1),
\]

(5.7)

with $h$ the determining series of the pair $(ab, ba)$. From (5.7) and and Eqn. (2.23) of 2.14 it follows that

\[
[R(\mu_{i(ab-ba)})](z) = h((iz)(-iz)) + h((-iz)(iz)) = 2h(z^2);
\]

(5.8)
so what we actually have to prove is the equality:

\[ h = R_E(\mu_a) \boxtimes R_E(\mu_b) \boxtimes \text{Zeta}. \]  

(5.9)

Now, Theorem 4.2 gives us the formula

\[ h = R(\mu_{(ab)(ba)}) \boxtimes \text{Moeb}. \]  

(5.10)

Without loss of generality we can assume that we are working in a tracial non-commutative probability space (compare beginning of proof of Theorem 4.5). Thus we have that \( \mu_{abba} = \mu_{a^2b^2} \), and by applying Equation (2.9) we get:

\[ R(\mu_{abba}) = R(\mu_{a^2b^2}) = R(\mu_{a^2}) \boxtimes R(\mu_{b^2}) \]  

(5.11)

(where we used, of course, that \( a^2 \) is free from \( b^2 \)). Equations (5.10) and (5.11) imply together:

\[ h = R(\mu_{a^2}) \boxtimes R(\mu_{b^2}) \boxtimes \text{Moeb} \]  

(5.12)

\[ = (R(\mu_{a^2}) \boxtimes \text{Moeb}) \boxtimes \text{Zeta} \boxtimes (R(\mu_{b^2}) \boxtimes \text{Moeb}). \]

\[ \equiv R_E(\mu_a) \boxtimes \text{Zeta} \boxtimes R_E(\mu_b), \]  

and thus (5.9) is obtained. QED

5.4 Proof of Theorem 1.2 (general case). Let \((A, \varphi)\) and \(a, b \in A\) be as in Theorem 1.2. By a standard free product construction we can produce a second non-commutative probability space \((\tilde{A}, \tilde{\varphi})\) and two elements \(\tilde{a}, \tilde{b} \in \tilde{A}\) which are free and both even, such that \(R_E(\mu_a) = R_E(\tilde{\mu}_a)\), \(R_E(\mu_b) = R_E(\tilde{\mu}_b)\). Theorem 3.1 gives us that \(\mu_{abba} = \mu_{\tilde{a}\tilde{b}\tilde{a}\tilde{b}}\) (calculated in \((\tilde{A}, \tilde{\varphi})\)) coincides with \(\mu_{ab-ba}\) (calculated in \((A, \varphi)\)). The result of the previous section applies to \(\tilde{a}\) and \(\tilde{b}\), hence we just have to write:

\[ R_E(\mu_{i(ab-ba)}) = R_E(\mu_{\tilde{i}(ab-ba)}) = 2(R_E(\mu_{a^-}) \boxtimes R_E(\mu_{b^-}) \boxtimes \text{Zeta}) \]

\[ = 2(R_E(\mu_a) \boxtimes R_E(\mu_b) \boxtimes \text{Zeta}). \quad \text{QED} \]

We now head towards the proofs of the corollaries of Theorem 1.2. As mentioned in the Introduction, the proofs of 1.4 and 1.6 are obtained by applying the combinatorial Fourier transform \(F\) in Equation (1.8) of the theorem. Let us recall from [10] that for a series \(f(z) = \sum_{n=1}^{\infty} \alpha_n z^n\) with \(\alpha_1 \neq 0\), the series \(F(f)\) is simply defined by

\[ [F(f)](w) = \frac{1}{w} f^{<1>}(w), \]  

(5.12)

where \(<'-1 '>\) denotes inversion under composition. Recall also that if \(\mu : \mathbb{C}[X] \to \mathbb{C}\) is a distribution with \(\mu(X) \neq 0\), then its transform \(S(\mu)\) is defined (10) as

\[ [S(\mu)](w) = \frac{1+w}{w} (M(\mu))^{<-1>}(w), \]  

(5.13)
with $M(\mu)$ the moment series of $\mu$, as in Definition 2.7. The combinatorial Fourier transform connects $R(\mu)$ with $S(\mu)$, and converts the $\boxtimes$-operation into multiplication, in the way shown in the Eqs.(1.4–5) of the Introduction.

In the next proof we shall use the notation ‘$\circ D_\lambda$’ for the dilation of a formal power series with a $\lambda \in \mathbb{C} \setminus \{0\}$; more precisely, if $f(z) = \sum_{n=1}^{\infty} a_n z^n$, then

$$ (f \circ D_\lambda)(z) = f(\lambda z) := \sum_{n=1}^{\infty} (\alpha_n \lambda^n)z^n. \quad (5.14) $$

The $R$-transform and the $\boxtimes$-operation behave nicely with respect to dilations, in the sense that we have the formulas:

$$ R(\mu_\lambda) = R(\mu) \circ D_\lambda \quad (5.15) $$

for every $a$ in some $(\mathcal{A}, \varphi)$ and for every $\lambda$ (see e.g. [18], Example 3.4.3); and

$$ (f \circ D_\lambda) \boxtimes g = f \boxtimes (g \circ D_\lambda) = (f \boxtimes g) \circ D_\lambda \quad (5.16) $$

for every series $f, g$ and every $\lambda$ (see [11], Section 4.1).

### 5.5 Proof of Corollaries 1.4 and 1.6

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a, b \in \mathcal{A}$ be free and with non-zero variances $\gamma_a = \varphi(a^2) - (\varphi(a))^2$, $\gamma_b = \varphi(b^2) - (\varphi(b))^2$. The fact that the variance of $c := i(ab - ba)$ satisfies $\gamma_c = 2\gamma_a \gamma_b$ comes out by equating the coefficients of degree 1 on the two sides of Eqn. (1.8); indeed, the linear coefficients of $R_E(\mu_c), R_E(\mu_b), R_E(\mu_c)$ are exactly $\gamma_a, \gamma_b, \gamma_c$, respectively, and at the level of linear coefficients $\boxtimes$ is just usual multiplication (see (2.3)).

It is convenient to rewrite (1.8) in the form:

$$ \begin{cases} R_E(\mu_c) \circ D_{1/\gamma_c} = 2h \circ D_{1/2}, \\ h := (R_E(\mu_a) \circ D_{1/\gamma_a}) \boxtimes (R_E(\mu_b) \circ D_{1/\gamma_b}) \boxtimes Zeta. \end{cases} \quad (5.17) $$

By applying $\mathcal{F}$ in the two Equations (5.17), we obtain:

$$ [\mathcal{F}(R_E(\mu_c) \circ D_{1/\gamma_c})](w) = [\mathcal{F}(h)](\frac{w}{2}) $$

(by just using the definition of $\mathcal{F}$ in (5.12)), and respectively

$$ \mathcal{F}(h) = \mathcal{F}(R_E(\mu_a) \circ D_{1/\gamma_a}) \cdot \mathcal{F}(R_E(\mu_b) \circ D_{1/\gamma_b}) \cdot \mathcal{F}(Zeta). $$

If we also take into account that $[\mathcal{F}(Zeta)](w) = 1/(1 + w)$, we thus arrive to:

$$ [\mathcal{F}(R_E(\mu_c) \circ D_{1/\gamma_c})](w) = \frac{[\mathcal{F}(R_E(\mu_a) \circ D_{1/\gamma_a})](\frac{w}{2}) \cdot [\mathcal{F}(R_E(\mu_b) \circ D_{1/\gamma_b})](\frac{w}{2})}{1 + \frac{w}{2}}. \quad (5.18) $$

Now let us establish the formula:

$$ [\mathcal{F}(R_E(\mu_c) \circ D_{1/\gamma_c})](w) = (1 + w) \cdot [S(\mu_{c^2/\gamma_c})](w). \quad (5.19) $$
Indeed, from Equation (5.2) we have, by also using (5.15–16),

\[ R_E(\mu_c) \circ D_{1/\gamma_c} = (R(\mu_{a^2}) \circ D_{1/\gamma_c}) \boxtimes \text{Moeb} = R(\mu_{a^2/\gamma_c}) \boxtimes \text{Moeb}; \]  

(5.20)

then by applying \( \mathcal{F} \) in (5.20) and by substituting \( \mathcal{F}(R(\mu_{a^2/\gamma_c})) = S(\mu_{a^2/\gamma_c}) \), \( [\mathcal{F}(\text{Moeb})](w) = 1 + w \), we obtain (5.19).

By putting (5.18) and (5.19) together, we obtain:

\[ [S(\mu_{c^2/\gamma_c})](w) = \frac{[\mathcal{F}(R_E(\mu_a) \circ D_{1/\gamma_a})](\frac{w}{2}) \cdot [\mathcal{F}(R_E(\mu_b) \circ D_{1/\gamma_b})](\frac{w}{2})}{(1 + \frac{w}{2})(1 + w)}. \]  

(5.21)

If \( a \) and \( b \) happen to be even, then we can write the counterparts of (5.19) for \( a \) and \( b \), and substitute them into (5.21); this leads exactly to the Equation (1.25) of Corollary 1.6.

If \( a \) and \( b \) are not assumed to be even, then we just replace the \( \mathcal{F} \)s on the right-hand side of (5.21) from their definition (Eqn. (5.12)); then (5.21) becomes:

\[ [S(\mu_{c^2/\gamma_c})](w) = \frac{4\gamma_a\gamma_b}{w^2(1 + \frac{w}{2})(1 + w)} \cdot [R_E(\mu_a)]^{<1>}(\frac{w}{2}) \cdot [R_E(\mu_b)]^{<1>}(\frac{w}{2}). \]  

(5.22)

In (5.22) we multiply both sides with \( w/(1 + w) \), and then take their inverse under composition; in this way the left-hand side of the equation becomes \( M(\mu_{c^2/\gamma_c}) \) (see (5.13)). More precisely, we get the formula:

\[ [M(\mu_{c^2/\gamma_c})](z) = \left( \frac{4\gamma_a\gamma_b}{w(1 + \frac{w}{2})(1 + w)^2} \cdot [R_E(\mu_a)]^{<1>}(\frac{w}{2}) \cdot [R_E(\mu_b)]^{<1>}(\frac{w}{2}) \right)^{<1>}(z). \]  

(5.23)

If, finally, we do a dilation with \( \gamma_c = 2\gamma_a\gamma_b \) in (5.23), and replace \( z \) by \( z^2 \), then Equation (1.11) of Corollary 1.4 is obtained. QED

\subsection*{5.6 Proof of Corollary 1.13.}

Let \( a, b \) be free selfadjoint elements in a \( C^* \)-probability space \((\mathcal{A}, \varphi)\), such that \( \mu_a = \nu_1 \) and \( \mu_b = \nu_2 \). Without loss of generality we can assume that \( \varphi \) is a trace and (by appropriately enlarging \((\mathcal{A}, \varphi)\)) that there exists \( d = d^* \in \mathcal{A} \) with \( \mu_d = \frac{1}{2} (\delta_1 + \delta_{-1}) \) and such that \( d \) is classically independent (not free!) from \( \{a, b\} \).

We know, by (5.8),(5.10), that

\[ [R(\mu_{(ab-ba)^2/2})](z) = h(z^2) \quad \text{with} \quad h = R(\mu_{aba}) \boxtimes \text{Moeb}. \]  

(5.24)

We define \( \hat{a} := \sqrt{aba} \cdot d \). Then \( \hat{a} \) is selfadjoint and even, and has \( \hat{a}^2 = abba \). Equations (5.1) and (5.24) imply together that

\[ [R(\mu_{\hat{a}})](z) = [R(\mu_{aba}) \boxtimes \text{Moeb}](z^2) = h(z^2) = [R(\mu_{(ab-ba)^2/2})](z), \]

hence that

\[ [\nu_1 \boxplus \nu_2]^{tr} = \mu_{(ab-ba)} = \mu_{\hat{a}}. \]  

(5.25)
Since \( \hat{a} \) is a selfadjoint element in a \( C^* \)-probability space, its distribution is a probability measure; thus (5.25) implies part 1) of Corollary 1.13. Moreover, formula (1.29) in the part 2) of 1.13 also follows from (5.25), via the calculation:

\[
R(Q([\nu_1 \square \nu_2]^{1/2})) = R(Q(\mu_0)) = R(\mu_{\square 2}) = R(\mu_{a^2 \square b^2}) = R(Q(\mu_a)Q(\mu_b)). \text{QED}
\]

We are only left to present the proof of Corollary 1.16. In order to do this, we introduce one more notation concerning commutator expressions:

**5.7 Notation.** To each commutator expression \( f \) of \( n \) arguments (in the sense of 1.15) we assign a \( \square \)-depth vector \((t_1, \ldots, t_{n-1})\), in the following recursive way.

i) \( f(\nu_1) = \nu_1 \) has \( \square \)-depth \( \emptyset \),

ii) If \( f(\nu_1, \ldots, \nu_n) = [f_1(\nu_1, \ldots, \nu_k \square f_2(\nu_{k+1}, \ldots, \nu_n)) \) and if \( f_1 \) has \( \square \)-depth \((t_1, \ldots, t_{k-1})\) and \( f_2 \) has \( \square \)-depth \((t_{k+1}, \ldots, t_{n-1})\), then \( f \) has \( \square \)-depth \((t_1 + 1, \ldots, t_{k-1} + 1, 1, t_{k+1} + 1, \ldots, t_{n-1} + 1)\).

The \( \square \)-depth vector records how many brackets we have to cross in order to reach the various ‘\( \square \)’ signs inside the commutator expression (in the same way as the depth vector of 1.15.2 was doing this for the arguments of the expression). We could afford not to mention the \( \square \)-depth in the statement of Corollary 1.16, due to the following fact.

**5.8 Lemma.** Let \( f \) and \( \hat{f} \) be two commutator expressions of \( n \) arguments. If the depth vector of \( f \) differs from the one of \( \hat{f} \) only by a permutation, then the same is true for the \( \square \)-depth vectors of \( f \) and \( \hat{f} \).

**Proof.** By induction on \( n \). For \( n = 1 \) and \( n = 2 \) the assertion is trivial. Let us assume we have proved it for \( n - 1 \) and consider the statement for \( n \). We denote the depth and \( \square \)-depth of \( f \) by \((d_1, \ldots, d_n)\) and \((t_1, \ldots, t_{n-1})\), and those of \( \hat{f} \) by \((\hat{d}_1, \ldots, \hat{d}_n)\) and \((\hat{t}_1, \ldots, \hat{t}_{n-1})\), respectively.

Let \( m \) be the biggest depth appearing in \( f \) and \( \hat{f} \), i.e.

\[
m := \max\{d_1, \ldots, d_n\} = \max\{\hat{d}_1, \ldots, \hat{d}_n\}.
\]

It is clear that there exist \( k \) and \( l \) such that \( d_k = d_{k+1} = m = \hat{d}_l = \hat{d}_{l+1} \) and that \( t_k = m = \hat{t}_l \). Furthermore, there are commutator expressions \( g \) and \( \hat{g} \) of \( n - 1 \) arguments such that \( f \) and \( \hat{f} \) are of the form

\[
f(\nu_1, \ldots, \nu_n) = g(\nu_1, \ldots, \nu_{k-1}, [\nu_k \square \nu_{k+1}], \nu_{k+2}, \ldots, \nu_n)
\]

and

\[
\hat{f}(\nu_1, \ldots, \nu_n) = \hat{g}(\nu_1, \ldots, \nu_{l-1}, [\nu_l \square \nu_{l+1}], \nu_{l+2}, \ldots, \nu_n).
\]

The depths of \( g \) and \( \hat{g} \) are determined by the depths of \( f \) and \( \hat{f} \), respectively, just by replacing \((\ldots, m, m, \ldots)\) by \((\ldots, m - 1, \ldots)\). Thus the depth vectors of \( g \) and \( \hat{g} \) differ also
only by a permutation; this implies, by the induction hypothesis, that the same is true for
the □-depth vectors of g and ˆg. But the □-depths of f and ˆf differ from those of g and ˆg
just by t_k and ˆt_l, respectively. Since t_k = ˆt_l, the assertion for f and ˆf follows. QED

5.9 Proof of Corollary 1.16. For f, g formal power series without constant coefficient
(in 1 variable), and for λ ∈ C \ {0}, we have the formula:

\[(λf) \boxdot (λg) = λ(⟨f⟩ \boxdot g) \circ D_λ,\]

where ‘⊙D_λ’ is as defined in Eqn. (5.14) (see [11], Lemma 4.4). By using (5.26) twice, we
can rewrite the Equation (1.8) of Theorem 1.2 in the alternative form:

\[R_E(ν_1 □ ν_2) = (2R_E(ν_1)) \boxdot (2R_E(ν_2)) \boxdot (2Zeta) \circ D_{1/4},\]

with ν_1, ν_2 arbitrary distributions. The recursive use of (5.27) is very convenient for calculating the R_E-transforms of higher order free commutator expressions. More precisely, as
easily checked by induction, we obtain the following general formula: let f be a commutator
expression of n arguments, with depth vector (d_1, . . . , d_n) and □-depth vector (t_1, . . . , t_n−1);
if ν_1, . . . , ν_n are arbitrary distributions, and if we denote ν := f(ν_1, . . . , ν_n), then:

\[R_E(ν) = [(2^{d_1} R_E(ν_1)) \boxdot \ldots \boxdot 2^{d_n} R_E(ν_n)) \boxdot (2^{t_1} Zeta) \boxdot \ldots \boxdot (2^{t_{n−1}} Zeta)] \circ D_{4^{−(n−1)}}.\]

But, if we also take the Lemma 5.8 into account, the assertion of Corollary 1.16 is immediately implied by Eqn. (5.28). QED

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