Hadamard Multipliers on Weighted Dirichlet Spaces

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Abstract. The Hadamard product of two power series is obtained by multiplying them coefficientwise. In this paper we characterize those power series that act as Hadamard multipliers on all weighted Dirichlet spaces on the disk with superharmonic weights, and we obtain sharp estimates on the corresponding multiplier norms. Applications include an analogue of Fejér’s theorem in these spaces, and a new estimate for the weighted Dirichlet integrals of dilates.

Mathematics Subject Classification. Primary 41A10; Secondary 41A17, 40G05, 40G10.

Keywords. Dirichlet space, Superharmonic weight, Fejér theorem.

1. Introduction and Statement of Results

1.1. Weighted Dirichlet Spaces

Let $\mathbb{D}$ be the open unit disk and $\mathbb{T}$ the unit circle. We write $\text{Hol}(\mathbb{D})$ for the set of holomorphic functions on $\mathbb{D}$. Given a positive superharmonic function $\omega$ on $\mathbb{D}$ and $f \in \text{Hol}(\mathbb{D})$, we define

$$D_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \, dA(z),$$

(1.1)

where $dA$ denotes normalized area measure on $\mathbb{D}$. The weighted Dirichlet space $D_\omega$ is the set of $f \in \text{Hol}(\mathbb{D})$ with $D_\omega(f) < \infty$. Defining

$$\|f\|^2_{D_\omega} := |f(0)|^2 + D_\omega(f) \quad (f \in D_\omega),$$

(1.2)

makes $D_\omega$ into a Hilbert space.

The class of superharmonic weights was introduced and studied by Aleman [1]. It includes two important subclasses:

JM supported by an NSERC grant. TR supported by grants from NSERC and the Canada Research Chairs program.
• the power weights $\omega(z) := (1 - |z|^2)^\alpha$ ($0 \leq \alpha \leq 1$), which form a scale linking the classical Dirichlet space $\mathcal{D}$ (where $\alpha = 0$) to the Hardy space $H^2$ (where $\alpha = 1$);

• the harmonic weights, introduced earlier by Richter [6] in connection with his analysis of shift-invariant subspaces of the classical Dirichlet space.

1.2. Hadamard Multipliers

Given formal power series $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \sum_{k=0}^{\infty} b_k z^k$, we define their Hadamard product to be the formal power series given by the formula

$$(f \ast g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k.$$ 

Obviously, if one of $f$ or $g$ is a polynomial, then $f \ast g$ is a polynomial too. Also, if both $f, g \in \text{Hol}(\mathbb{D})$, then $f \ast g \in \text{Hol}(\mathbb{D})$ as well.

In this paper we study the Hadamard multipliers of $\mathcal{D}_\omega$, namely those $h$ with the property that $h \ast f \in \mathcal{D}_\omega$ whenever $f \in \mathcal{D}_\omega$. We also seek optimal estimates for $\mathcal{D}_\omega(h \ast f)$ in terms of $\mathcal{D}_\omega(f)$ and the Taylor coefficients of $h$.

1.3. Statement of Main Results

Given a sequence of complex numbers $(c_k)_{k \geq 1}$, we write $T_c$ for the infinite matrix

$$T_c := \begin{pmatrix}
  c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \ldots \\
  0 & c_2 & c_3 - c_2 & c_4 - c_3 & \ldots \\
  0 & 0 & c_3 & c_4 - c_3 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \quad (1.3)$$

This matrix may or may not act as a bounded operator on $\ell^2$. If it does, then we write $\|T_c\|$ for its operator norm. If not, then we write $\|T_c\| = \infty$.

If the $(c_k)$ are the coefficients of a formal power series $h(z) = \sum_{k=0}^{\infty} c_k z^k$, then we also write $T_h$ in place of $T_c$. Note that, in this situation, the coefficient $c_0$ plays no role. Our main result is the following theorem.

**Theorem 1.1.** Let $h(z)$ be a formal power series. The following statements are equivalent.

(i) $h$ is a Hadamard multiplier of $\mathcal{D}_\omega$ for every superharmonic weight $\omega$.

(ii) $T_h$ acts a bounded operator on $\ell^2$.

In this case, for all superharmonic weights $\omega$ on $\mathbb{D}$ and all $f \in \mathcal{D}_\omega$, we have

$$\mathcal{D}_\omega(h \ast f) \leq \|T_h\|^2 \mathcal{D}_\omega(f). \quad (1.4)$$

The constant $\|T_h\|^2$ is best possible.

**Remark.** Note that, for an individual weight $\omega$, the equivalence between (i) and (ii) may well fail to hold. For example, the Hadamard multipliers of the classical Dirichlet space $\mathcal{D}$ are precisely those $h(z)$ with bounded coefficients.
In particular, the function $h(z) = z + z^3 + z^5 + z^7 + \ldots$ is a Hadamard multiplier of $D$. However, in this case,

$$T_h := \begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & \ldots \\
0 & 0 & 1 & -1 & 1 & -1 & \ldots \\
0 & 0 & 1 & -1 & 1 & -1 & \ldots \\
0 & 0 & 0 & 0 & 1 & -1 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},$$

which does not act as a bounded operator on $\ell^2$.

To apply Theorem 1.1, it is helpful to have at our disposal some criteria for the boundedness of the matrix $T_c$, as well as quantitative estimates for its norm. The next theorem collects together some results of this kind.

**Theorem 1.2.** Let $c := (c_k)_{k \geq 1}$ be a sequence of complex numbers, and let $T_c$ be defined by (1.3).

(i) We have the pair of estimates:

$$\|T_c\| \leq \sup_{k \geq 1} |c_k| + 2 \sup_{k \geq 2} |c_k - c_{k-1}|,$$

$$\|T_c\| \geq \sup_{k \geq 1} \left\{ |c_k|^2 + (k-1)|c_k - c_{k-1}|^2 \right\}^{1/2}.$$

(ii) If there exists $n \geq 1$ such that $c_k = 0$ for all $k > n$, then

$$\|T_c\|^2 \leq (n + 1) \sum_{k=1}^n |c_{k+1} - c_k|^2.$$

(iii) If $c_k \to 0$ as $k \to \infty$, then

$$\|T_c\| \leq \sum_{k=1}^{\infty} \sqrt{k(k+1)}|c_{k+2} - 2c_{k+1} + c_k|.$$

Theorems 1.1 and 1.2 lead to the following corollary.

**Corollary 1.3.** For $h(z) := \sum_{k=0}^{\infty} c_k z^k$ to be a Hadamard multiplier of $D_\omega$ for all superharmonic weights $\omega$:

- a necessary condition is that $c_k = O(1)$ and $(c_k - c_{k-1}) = O(1/\sqrt{k})$,

- a sufficient condition is that $c_k = O(1)$ and $(c_k - c_{k-1}) = O(1/k)$.

As a further application of Theorems 1.1 and 1.2, we obtain sharp estimates for the Dirichlet, Fejér and de la Vallée Poussin kernels as Hadamard multipliers of $D_\omega$.

**Corollary 1.4.** Let $n \geq 0$ and let $\omega$ be a superharmonic weight.

(i) If $D_n(z) := \sum_{k=0}^{n} z^k$, then

$$D_\omega(D_n * f) \leq (n + 1)D_\omega(f) \quad (f \in D_\omega).$$
(ii) If $K_n(z) := \sum_{k=0}^{n}(1 - k/(n+1))z^k$, then
\[ D_\omega(K_n * f) \leq \frac{n}{n+1}D_\omega(f) \quad (f \in \mathcal{D}_\omega). \]

(iii) If $V_n(z) := \sum_{k=0}^{n-1}z^k + \sum_{k=n}^{2n-1}(2 - k/n)z^k$, then
\[ D_\omega(V_n * f) \leq 2D_\omega(f) \quad (f \in \mathcal{D}_\omega). \]

Moreover, there exists a superharmonic weight $\omega$ such that, for each $n \geq 0$, the constants are best possible in (i), (ii) and (iii).

Another application of Theorems 1.1 and 1.2 is to radial dilates. Given $f \in \text{Hol}(\mathbb{D})$ and $r \in [0,1)$, we define $f_r \in \text{Hol}(\mathbb{D})$ by $f_r(z) := f(rz)$. Notice that $f_r = P_r * f$, where $P_r$ is the Poisson kernel, given by $P_r(z) := \sum_{k=0}^{\infty} r^k z^k$.

We obtain the following corollary.

**Corollary 1.5.** Let $\omega$ be a superharmonic weight and let $f \in \mathcal{D}_\omega$. Then $f_r \in \mathcal{D}_\omega$ for all $r \in [0,1)$ and
\[ D_\omega(f_r) \leq r^2(2 - r)D_\omega(f) \quad (0 \leq r < 1). \] (1.5)

Inequalities of the type $D_\omega(f_r) \leq C D_\omega(f)$ have been studied by several authors, with the value of $C$ being improved over time. Here is a brief summary of the history:

- $C = 4$ for harmonic weights $\omega$ (Richter and Sundberg [7]).
- $C = 5/2$ for superharmonic weights $\omega$ (Aleman [1]).
- $C = 2r/(1 + r) \leq 1$ for harmonic $\omega$ (Sarason [8]).
- $C = 2r/(1 + r) \leq 1$ for superharmonic $\omega$ (El-Fallah et al [3]).

In the last two cases, the method used was to identify certain $\mathcal{D}_\omega$ with an appropriate de Branges–Rovnyak space, and prove the desired inequality in that space. Our proof of (1.5) is direct, and our constant $C$ is better.

One reason for studying inequalities of the type (1.5) is that they can be used to prove that polynomials are dense in $\mathcal{D}_\omega$. This fact was originally established by Richter [6] (for harmonic $\omega$) and Aleman [1] (for general superharmonic $\omega$) using a different technique, based on a certain type of wandering-subspace theorem.

We end this section by stating an analogue of Fejér’s theorem for $\mathcal{D}_\omega$, which yields another, direct proof of the density of polynomials in $\mathcal{D}_\omega$. Given a power series $f(z) := \sum_{k=0}^{\infty} a_k z^k$, we write
\[ s_n(f)(z) := \sum_{k=0}^{n} a_k z^k \quad \text{and} \quad \sigma_n(f)(z) := \frac{1}{n+1} \sum_{k=0}^{n} s_k(f)(z). \]

**Theorem 1.6.** (i) If $\omega$ is a superharmonic weight and if $f \in \mathcal{D}_\omega$, then $\|\sigma_n(f) - f\|_{\mathcal{D}_\omega} \to 0$ as $n \to \infty$.

(ii) There exist a superharmonic weight and a function $f \in \mathcal{D}_\omega$ such that $\|s_n(f) - f\|_{\mathcal{D}_\omega} \not\to 0$ as $n \to \infty$.

Part (ii) was already known (see e.g. [4, Exercise 7.3.2]). We include it to complement part (i), which we believe to be new.
The rest of the paper is structured as follows. In Sect. 2 we review
some basic properties of Dirichlet spaces with superharmonic weights. The
theorems and corollaries stated above are proved in Sects. 3–7. We conclude
in Sect. 8 with some remarks and questions.

2. Background on Superharmonic Weights

Let \( \omega \) be a positive superharmonic function on \( \mathbb{D} \). In this section we summa-
alyze some basic properties of the weighted Dirichlet space \( \mathcal{D}_\omega \). For detailed
proofs and further information, we refer to [1].

By standard results from potential theory, \( \omega \) is locally integrable on \( \mathbb{D} \), and
\[
\frac{1}{r^2} \int_{|z| \leq r} \omega \, dA \text{ is a decreasing function of } r \text{ for } 0 < r < 1 \quad \text{(see e.g. [5,}

Theorems 2.5.1 and 2.6.8]. It follows that \( \omega \in L^1(\mathbb{D}) \), and thus \( \mathcal{D}_\omega \) contains
the polynomials.

As \( \omega \) is a positive superharmonic function, there exists a unique positive
finite Borel measure \( \mu \) on \( \mathbb{D} \) such that, for all \( z \in \mathbb{D} \),
\[
\omega(z) = \int_{\mathbb{D}} \log \left| \frac{1 - \overline{\zeta}z}{\zeta - z} \right| \frac{2}{1 - |\zeta|^2} \, d\mu(\zeta) + \int_{T} \left| \frac{1 - |z|^2}{\zeta - z} \right|^2 \, d\mu(\zeta) \quad (2.1)
\]
(see e.g. [5, Theorem 4.5.1]). When \( \mu = \delta_\zeta \), we write \( \mathcal{D}_\zeta \) for \( \mathcal{D}_\omega \). T h u s , f o r
\( f \in \text{Hol}(\mathbb{D}) \),
\[
\mathcal{D}_\zeta(f) = \begin{cases} 
\int_{\mathbb{D}} \log \left| \frac{1 - \overline{\zeta}z}{\zeta - z} \right| \frac{2}{1 - |\zeta|^2} |f'(z)|^2 \, dA(z), & \zeta \in \mathbb{D}, \\
\int_{T} \left| \frac{1 - |z|^2}{\zeta - z} \right|^2 |f'(z)|^2 \, dA(z), & \zeta \in T.
\end{cases}
\]

For general \( \omega \), corresponding to a measure \( \mu \) on \( \overline{\mathbb{D}} \), we can recover \( \mathcal{D}_\omega(f) \)
from \( \mathcal{D}_\zeta(f) \) via Fubini’s theorem:
\[
\mathcal{D}_\omega(f) = \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(f) \, d\mu(\zeta). \quad (2.2)
\]

The spaces \( \mathcal{D}_\zeta \) are sometimes called local Dirichlet spaces. The following
result gives a very simple alternative description of them. Recall that \( H^2 \)
denotes the Hardy space.

**Theorem 2.1.** Let \( \zeta \in \overline{\mathbb{D}} \). Then \( f \in \mathcal{D}_\zeta \) if and only if
\( f(z) = a + (z - \zeta)g(z) \), where \( g \in H^2 \) and \( a \in \mathbb{C} \). In this case \( \mathcal{D}_\zeta(f) = \|g\|_{H^2}^2 \).

Thus, if \( \zeta \in \mathbb{D} \), then \( \mathcal{D}_\zeta \) is just \( H^2 \) with a different (but equivalent)
norm. But if \( \zeta \in T \), then \( \mathcal{D}_\zeta \) is a proper subspace of \( H^2 \). For a proof of this
result, see [1, Chapter IV, §1].

3. Proof of Theorem 1.1

**Proof that (ii)⇒(i).** Let \( h(z) := \sum_{k=0}^{\infty} c_k z^k \) be a power series such that
\( \|T_h\| < \infty \). It follows that the sequence \( (c_{k+1} - c_k)_{k \geq 1} \) belongs to \( \ell^2 \).

Let \( \zeta \in \overline{\mathbb{D}} \), and let \( f \in \mathcal{D}_\zeta \). By Theorem 2.1, there exist \( g \in H^2 \)
and \( a \in \mathbb{C} \) such that \( f(z) = a + (z - \zeta)g(z) \), and \( \mathcal{D}_\zeta(f) = \|g\|_{H^2}^2 \). Writing \( f(z) :=
\[ \sum_{k=0}^{\infty} a_k z^k \] and \( g(z) := \sum_{k=0}^{\infty} b_k z^k \), and equating coefficients of \( z^k \) in the resulting formula, we obtain the relations
\[
\begin{cases}
    a_0 = a - b_0 \zeta, \\
    a_k = b_{k-1} - b_k \zeta \quad (k \geq 1).
\end{cases} \tag{3.1}
\]

As both the sequences \((b_k)_{k \geq 1}\) and \((c_{k+1} - c_k)_{k \geq 1}\) belong to \( \ell^2 \), the series \( \sum_{k=1}^{\infty} (c_{k+1} - c_k) b_k \) converges absolutely, and it thus makes sense to define a formal power series \( F \) by
\[
F(z) := \sum_{j=0}^{\infty} \left( c_{j+1} b_j + \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k \zeta^{k-j} \right) z^j.
\]

If we multiply this power series by \((z - \zeta)\), then we obtain a new power series whose \( j \)th coefficient (for \( j \geq 1 \)) is given by
\[
c_j b_{j-1} + \sum_{k=j}^{\infty} (c_{k+1} - c_k) b_k \zeta^{k-j+1} - c_j b_j \zeta - \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k \zeta^{k-j+1}
= c_j (b_{j-1} - b_j \zeta).
\]

In view of the relations (3.1), this is exactly equal to \( c_j a_j \). In other words, we have \( (h \ast f)(z) = (z - \zeta) F(z) + A \), where \( A \) is a constant.

We claim that \( F \in H^2 \). Indeed,
\[
\|F\|_{H^2}^2 = \sum_{j=0}^{\infty} \left| c_{j+1} b_j + \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k \zeta^{k-j} \right|^2
\leq \sup_{|\eta|=1} \sum_{j=0}^{\infty} \left| c_{j+1} b_j + \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k \eta^{k-j} \right|^2 \quad \text{(max. principle)}
= \sup_{|\eta|=1} \sum_{j=0}^{\infty} \left| c_{j+1} b_j \eta^j + \sum_{k=j+1}^{\infty} (c_{k+1} - c_k) b_k \eta^k \right|^2,
\]
and so, writing \( v_\eta := (b_j \eta^j)_{j \geq 0} \), we have
\[
\|F\|_{H^2}^2 \leq \sup_{|\eta|=1} \|T_\hbar(v_\eta)\|_{\ell^2}^2 = \sup_{|\eta|=1} \|T_\hbar\|_{\ell^2}^2 \|v_\eta\|_{\ell^2}^2
= \|T_\hbar\|_{\ell^2}^2 \sum_{k=0}^{\infty} |b_k|^2 = \|T_\hbar\|^2 \|g\|^2_{H^2} = \|T_\hbar\|_{H^2}^2 \mathcal{D}_\zeta(f).
\]

In combination with Theorem 2.1, these observations imply that \( h \ast f \in \mathcal{D}_\zeta \) and that
\[
\mathcal{D}_\zeta(h \ast f) = \|F\|_{H^2}^2 \leq \|T_\hbar\|_{H^2}^2 \mathcal{D}_\zeta(f).
\]

Now let \( \omega \) be a superharmonic weight on \( \mathbb{D} \), and let \( f \in \mathcal{D}_\omega \). Let \( \mu \) be the associated measure on \( \overline{\mathbb{D}} \) so that (2.1) holds. Formula (2.2) then gives
\[
\mathcal{D}_\omega(f) = \int_{\overline{\mathbb{D}}} \mathcal{D}_\zeta(f) \, d\mu(\zeta).
\]
Thus $f \in D_\zeta$ for $\mu$-almost every $\zeta \in \mathbb{D}$, and, by what we have proved above, 
$D_\zeta(h \ast f) \leq \|T_h\|^2 D_\zeta(f)$ for all such $\zeta$. Consequently, 

$$D_\omega(h \ast f) = \int_{\mathbb{D}} D_\zeta(h \ast f) d\mu(\zeta) \leq \|T_h\|^2 \int_{\mathbb{D}} D_\zeta(f) d\mu(\zeta) \leq \|T_h\|^2 D_\omega(f).$$

Thus $h$ is a Hadamard multiplier of $D_\omega$, and (1.4) holds. \hfill \Box

Proof that (i) $\Rightarrow$ (ii). Consider the closed subspace of $D_1$ defined by 

$$D_1^0 := \{ f \in D_1 : f(0) = 0 \}.$$

Note that $\|f\|_{D_1}^2 = D_1(f)$ for all $f \in D_1^0$. Also, by Theorem 2.1, we have $f \in D_1^0$ if and only if there exists $g \in H^2$ such that $f(z) = g(0) + (z-1)g(z)$, and in this case $D_1(f) = \|g\|_{H^2}^2$. Thus, if we define $U : H^2 \to D_1^0$ by 

$$Ug(z) := g(0) + (z-1)g(z) \quad (g \in H^2),$$

then $U$ is an isometry of $H^2$ onto $D_1^0$, i.e., $U$ is a unitary operator.

Now let $h(z) := \sum_{k=0}^\infty c_k z^k$ be a Hadamard multiplier of $D_1$. Clearly it is also a Hadamard multiplier of $D_1^0$. Define $M_h : D_1^0 \to D_1^0$ by 

$$M_h(f) := h \ast f \quad (f \in D_1^0).$$

An application of the closed graph theorem shows that $M_h$ is a bounded linear map of $D_1^0$ into itself. Hence $U^* M_h U$ is a bounded linear map of $H^2$ into itself. We shall show that the matrix of this map with respect to the standard orthonormal basis $\{1, z, z^2, \ldots\}$ of $H^2$ is exactly $T_h$.

Fix $k \geq 0$. Then, identifying $H^2$ with $\ell^2$ in the standard way, we have 

$$UT_h(z^k) = U \left( (c_{k+1} - c_k)(1 + z + \cdots + z^{k-1}) + c_{k+1}z^k \right)$$

$$= (c_{k+1} - c_k) + (c_{k+1} - c_k)(z^k - 1) + c_{k+1}(z^{k+1} - z^k)$$

$$= c_{k+1}z^{k+1} - c_k z^k = M_h(z^{k+1} - z^k) = M_h U(z^k).$$

Thus $U^* M_h U(z^k) = T_h(z^k)$ for all $k \geq 0$, as claimed.

It follows that $T_h$ acts as a bounded linear operator on $\ell^2$. Also, we have 

$$\|T_h\|^2 = \|U^* M_h U\|^2 = \|M_h\|^2 = \sup\{ D_1(f) : f \in D_1^0, \ D_1(f) = 1 \},$$

which shows that the constant $\|T_h\|^2$ in (1.4) is best possible. \hfill \Box

We note in passing the following consequence of the proof above.

Corollary 3.1. A power series $h$ is a Hadamard multiplier of $D_1$ if and only if it is a Hadamard multiplier of $D_\omega$ for all superharmonic weights $\omega$.

4. Proofs of Theorem 1.2 (i) and Corollary 1.3

Proof of Theorem 1.2 (i). We can decompose $T_c$ as

$$T_c = \begin{pmatrix}
    c_1 & 0 & 0 & 0 & \cdots \\
    0 & c_1 & 0 & 0 & \cdots \\
    0 & 0 & c_2 & 0 & \cdots \\
    0 & 0 & 0 & c_3 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} + \begin{pmatrix}
    0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\
    0 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\
    0 & 0 & c_3 - c_2 & c_4 - c_3 & \cdots \\
    0 & 0 & 0 & c_4 - c_3 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$
The first matrix has norm at most $\sup_{k \geq 1} |c_k|$. As for the second matrix, the absolute value of each entry is at most $\sup_{k \geq 2} k |c_k - c_{k-1}|$ times the corresponding entry in the Cesàro matrix,

$$C := \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \ldots \\ 0 & 1/2 & 1/3 & 1/4 & \ldots \\ 0 & 0 & 1/3 & 1/4 & \ldots \\ 0 & 0 & 0 & 1/4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

It is well known that $\|C\| = 2$ (see e.g. [2, pp. 128–129]). It follows that

$$\|T_c\| \leq \sup_{k \geq 1} |c_k| + 2 \sup_{k \geq 2} k |c_k - c_{k-1}|.$$  

This is the desired upper bound for $\|T_c\|$. The lower bound is obtained by noting that $\|T_c\| \geq \sup_{k \geq 1} \|T_c(e_k)\|_{\ell^2}$, where $(e_k)$ is the standard unit vector basis of $\ell^2$. □

**Proof of Corollary 1.3.** This corollary follows directly from Theorems 1.1 and 1.2 (i). □

5. Proofs of Theorem 1.2 (ii) and Corollary 1.4

**Proof of Theorem 1.2 (ii).** Let $n \geq 1$, and suppose that $c_k = 0$ for all $k > n$. Let $\xi := (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2$. Then we have

$$\|T_c(\xi)\|_{\ell^2}^2 = \sum_{j=1}^n |c_j \xi_j + \sum_{k=j}^n (c_{k+1} - c_k) \xi_{k+1}|^2$$  

$$= \sum_{j=1}^n \left( \sum_{k=j}^n |c_{k+1} - c_k| |\xi_{k+1} - \xi_j|^2 \right)$$  

$$\leq \sum_{j=1}^n \left( \sum_{k=j}^n |c_{k+1} - c_k|^2 \right) \left( \sum_{k=j}^n |\xi_{k+1} - \xi_j|^2 \right)$$  

$$\leq \left( \sum_{k=1}^n |c_{k+1} - c_k|^2 \right) \sum_{j=1}^n \sum_{k=j}^n |\xi_{k+1} - \xi_j|^2.$$  

Now,

$$\sum_{j=1}^n \sum_{k=j}^n |\xi_{k+1} - \xi_j|^2 = \frac{1}{2} \sum_{p=1}^{n+1} \sum_{q=1}^{n+1} |\xi_p - \xi_q|^2 = (n+1) \sum_{k=1}^{n+1} |\xi_k|^2 - \sum_{k=1}^{n+1} |\xi_k|^2.$$  

The right-hand side is at most $(n+1) \sum_{k=1}^{n+1} |\xi_k|^2 \leq (n+1) \|\xi\|^2_{\ell^2}$. Hence

$$\|T_c(\xi)\|^2_{\ell^2} \leq (n+1) \sum_{k=1}^n |c_{k+1} - c_k|^2 \|\xi\|^2_{\ell^2}.$$  

This shows that $\|T_c\|^2 \leq (n+1) \sum_{k=1}^n |c_{k+1} - c_k|^2$, completing the proof. □
Proof of Corollary 1.4. The inequalities in (i), (ii) and (iii) all follow directly from Theorems 1.1 and 1.2 (ii). It remains to justify the sharpness of the constants.

(i) Let \( f(z) := nz^{n+1} - (n+1)z^n + 1 = (z-1)(nz^n - z^{n-1} - z^{n-2} - \ldots - 1). \)
Then \( (D_n \ast f)(z) = -(n+1)z^n + 1 = -n - (n+1)(z-1)(z^{n-1} + \ldots + z + 1). \) By Theorem 2.1, we have
\[
\mathcal{D}_1(f) = n(n+1) \quad \text{and} \quad \mathcal{D}_1(D_n \ast f) = (n+1)^2 n = (n+1)\mathcal{D}_1(f).
\]

(ii) Let \( f(z) := z^{n+1} - (n+1)z^n + n = (z-1)(z^n + z^{n-1} + \ldots + z - n). \) Then
\[
(K_n \ast f)(z) = n - nz = -n(z-1). \] By Theorem 2.1 again,
\[
\mathcal{D}_1(f) = n(n+1) \quad \text{and} \quad \mathcal{D}_1(K_n \ast f) = n^2 = \frac{n}{n+1} \mathcal{D}_1(f).
\]

(iii) Let \( f(z) := z^{2n} - 2z^n + 1 = (z-1)(z^{2n-1} + \ldots + z^n - z^{n-1} - \ldots - 1). \)
Then \( (V_n \ast f)(z) = -2z^n + 1 = -1 - 2(z-1)(z^{n-1} + \ldots + z + 1). \) Once more, using Theorem 2.1, we have
\[
\mathcal{D}_1(f) = 2n \quad \text{and} \quad \mathcal{D}_1(V_n \ast f) = 4n = 2\mathcal{D}_1(f).
\]
Thus the constants in (i), (ii) and (iii) are sharp for all \( n \geq 0. \)

6. Proofs of Theorem 1.2 (iii) and Corollary 1.5

We shall prove Theorem 1.2 (iii) using a superposition technique enshrined in the following lemma.

Lemma 6.1. Let \((h_n)_{n \geq 0}\) be a sequence of formal power series such that \( \sum_{n=0}^{\infty} |h_n(0)| < \infty \) and \( \sum_{n=0}^{\infty} \|T_{h_n}\| < \infty. \) Then the formal power series \( h := \sum_{n=0}^{\infty} h_n \) converges coefficientwise, and \( T_h \) acts as a bounded operator on \( \ell^2, \) with
\[
\|T_h\| \leq \sum_{n=0}^{\infty} \|T_{h_n}\|. \quad (6.1)
\]

Proof. For each pair of integers \( r \leq s, \) we have
\[
\left\| \sum_{n=r}^{s} T_{h_n} \right\| \leq \sum_{n=r}^{s} \|T_{h_n}\|. \quad (6.2)
\]
It follows that the series \( \sum_{n=0}^{\infty} T_{h_n} \) is Cauchy in the Banach space of bounded operators on \( \ell^2, \) hence convergent in that space, to \( T \) say. As convergence in the space of bounded operators on \( \ell^2 \) implies the entrywise convergence of the corresponding matrices, it follows that \( \sum_{n=0}^{\infty} h_n \) converges coefficientwise to \( h \) and that \( T = T_h. \) (Note that, since the constant coefficients of \( h_n \) do not appear in \( T_{h_n}, \) we need to assume \( \sum_{n} |h_n(0)| < \infty \) separately.) Thus \( T_h \)
acts as a bounded operator on \( \ell^2. \) Finally, letting \( r = 0 \) and \( s \to \infty \) in (6.2), we obtain (6.1).

Proof of Theorem 1.2 (iii). Let \((c_n)_{n \geq 1}\) be a sequence such that
\[
\lim_{n \to \infty} c_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \sqrt{n(n+1)}|c_{n+2} - 2c_{n+1} + c_n| < \infty.
\]
Set \( c_0 := 0 \). For each \( n \geq 0 \), let \( h_n := \lambda_n K_n \), where
\[
\lambda_n := (n + 1)(c_{n+2} - 2c_{n+1} + c_n) \quad \text{and} \quad K_n(z) := \sum_{k=0}^{n} (1 - k/(n + 1)) z^k.
\]
Then \( K_n(0) = 1 \) for each \( n \), and \( \|T_{K_n}\| = \sqrt{n/(n + 1)} \) by Corollary 1.4(ii).
Since \( \sum_{n=0}^{\infty} |\lambda_n| < \infty \), we may apply Lemma 6.1 to deduce that \( \sum_{n=0}^{\infty} \lambda_n K_n \) converges coefficientwise to a power series \( h \) such that
\[
\|T_h\| \leq \sum_{n=0}^{\infty} \|T_{K_n}\| = \sum_{n=0}^{\infty} \sqrt{n(n + 1)} |c_{n+2} - 2c_{n+1} + c_n|.
\]
We claim that \( h(z) = \sum_{k=0}^{\infty} c_k z^k \). If so, then
\[
\|T_h\| = \sum_{n=1}^{\infty} \sqrt{n(n + 1)} |c_{n+2} - 2c_{n+1} + c_n|,
\]
and the theorem is proved.

It remains to justify the claim. We have
\[
\sum_{n=0}^{\infty} \lambda_n K_n(z) = \sum_{n=0}^{\infty} \lambda_n \left( \sum_{k=0}^{n} (1 - k/(n + 1)) z^k \right) = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} \lambda_n \left( 1 - \frac{k}{n + 1} \right) \right) z^k.
\]
Thus \( h(z) = \sum_{k=0}^{\infty} d_k z^k \), where
\[
d_k := \sum_{n=k}^{\infty} \lambda_n \left( 1 - \frac{k}{n + 1} \right) \quad (k \geq 0).
\]
Now, for each \( k \geq 0 \), we have
\[
d_k - d_{k+1} = \sum_{n=k+1}^{\infty} \frac{\lambda_n}{n + 1},
\]
and so
\[
(d_k - d_{k+1}) - (d_{k+1} - d_{k+2}) = \frac{\lambda_k}{k + 1} = c_{k+2} - 2c_{k+1} + c_k.
\]
It follows that \( u_k := c_k - d_k \) satisfies the relation \( u_{k+2} - 2u_{k+1} + u_k = 0 \) for all \( k \geq 0 \). Therefore \( u_k = \alpha k + \beta \) for some constants \( \alpha, \beta \). Now \( c_k \to 0 \) by hypothesis, and \( d_k \to 0 \) by its very definition, so \( u_k \to 0 \), and thus necessarily \( \alpha = \beta = 0 \). Hence \( d_k = c_k \) for all \( k \), and so finally \( h(z) = \sum_{k=0}^{\infty} c_k z^k \), as claimed.

\textbf{Proof of Corollary 1.5.} As remarked in the introduction, we have \( f_r = P_r \ast f \), where \( P_r(z) := \sum_{k=0}^{\infty} r^k z^k \), the Poisson kernel. By Theorems 1.1 and 1.2(iii),
we have $D_\omega(P_r \ast f) \leq \|T_{P_r}\|D_\omega(f)$, where

$$
\|T_{P_r}\| \leq \sum_{k=1}^{\infty} \sqrt{k(k+1)}|r^{k+2} - 2r^{k+1} + r^k|
$$

$$
= (1 - r)^2 \sum_{k=1}^{\infty} \sqrt{k(k+1)} r^k
$$

$$
\leq (1 - r)^2 \left( \sum_{k=1}^{\infty} kr^k \right)^{1/2} \left( \sum_{k=1}^{\infty} (k+1)r^k \right)^{1/2}
$$

$$
= r(2 - r)^{1/2},
$$

the last inequality arising from the Cauchy–Schwarz inequality.

\[\square\]

7. Proof of Theorem 1.6

Note that $s_n(f)$ and $\sigma_n(f)$ can be written as Hadamard products, namely

$$
s_n(f) = D_n \ast f \quad \text{and} \quad \sigma_n(f) = K_n \ast f,
$$

where $D_n$ and $K_n$ are the Dirichlet and Fejér kernels already studied in Corollary 1.4. We make key use of this remark in what follows.

\textbf{Proof of Theorem 1.6.} (i) By the parallelogram identity, we have

$$
D_\omega(\sigma_n(f) - f) + D_\omega(\sigma_n(f) + f) = 2D_\omega(\sigma_n(f)) + 2D_\omega(f).
$$

Corollary 1.4 (ii) gives

$$
D_\omega(\sigma_n(f)) = D_\omega(K_n \ast f) \leq D_\omega(f) \quad (n \geq 0).
$$

Also, $\sigma_n(f) \to f$ locally uniformly on $\mathbb{D}$, so, by (1.1) and Fatou’s lemma,

$$
\liminf_{n \to \infty} D_\omega(\sigma_n(f) + f) \geq 4D_\omega(f).
$$

It follows that

$$
\limsup_{n \to \infty} D_\omega(\sigma_n(f) - f) \leq 2D_\omega(f) + 2D_\omega(f) - 4D_\omega(f) = 0.
$$

Finally, since $\sigma_n(f)(0) \to f(0)$, we conclude that

$$
\|\sigma_n(f) - f\|_{D_\omega}^2 = |\sigma_n(f)(0) - f(0)|^2 + D_\omega(\sigma_n(f) - f) \to 0.
$$

(ii) We consider $f \mapsto s_n(f)$ as a linear map : $\mathcal{D}_1 \to \mathcal{D}_1$. The calculations in the proof of Corollary 1.4 show that, if $f_n(z) := nz^{n+1} - (n + 1)z^n$,

$$
\|f_n\|_{\mathcal{D}_1}^2 = \mathcal{D}_1(f_n) = n(n + 1),
$$

$$
\|s_n(f_n)\|_{\mathcal{D}_1}^2 = \|D_n \ast f_n\|_{\mathcal{D}_1}^2 = (n + 1)^2 n.
$$

Thus the norm of $s_n$ as a linear operator satisfies $\|s_n\| \geq \sqrt{n + 1}$. Hence $\sup_n \|s_n\| = \infty$. By the Banach–Steinhaus theorem, there exists $f \in \mathcal{D}_1$ such that $\sup_n \|s_n(f)\|_{\mathcal{D}_1} = \infty$. In particular, $\|s_n(f) - f\|_{\mathcal{D}_1} \not\to 0$.

\[\square\]
8. Concluding Remarks and Questions

(1) Once it is known that polynomials are dense in $D\omega$, it is a routine matter to prove that the following statements are equivalent:

- $\lim_{n\to\infty} \|h_n * f - h * f\|_{D\omega} = 0$ for all superharmonic $\omega$ and all $f \in D\omega$.
- $h_n \to h$ coefficientwise and $\sup_n \|T_{h_n}\| < \infty$.

Evidently, this generalizes Theorem 1.6. However, the interest of Theorem 1.6 lies in the fact that, since its proof makes no appeal to the density of polynomials, it can be used to give an independent proof of this density.

(2) Although (1.5) provides an estimate of the constant $C$ such that $D\omega(f_r) \leq C D\omega(f)$ that is better than those known to date, it is clearly still not optimal. Indeed, we know that best constant is exactly $\|T_{P_r}\|^2$. Is there a ‘nice’ algebraic expression for this norm?

(3) Which sequences $(c_k)$ have the property that $\|T_{c}\| < \infty$? There are various equivalent ways to reformulate this question. For example, which real sequences $(a_k)$ have the property that the self-adjoint, L-shaped matrix

$$
\begin{pmatrix}
   a_1 & a_2 & a_3 & a_4 & \ldots \\
   a_2 & a_2 & a_3 & a_4 & \ldots \\
   a_3 & a_3 & a_3 & a_4 & \ldots \\
   a_4 & a_4 & a_4 & a_4 & \ldots \\
   \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

acts as a bounded operator on $\ell^2$?

(4) Can we characterize the Hadamard multipliers $h(z) := \sum_{k=0}^\infty c_k z^k$ of $D\omega$ for a fixed superharmonic weight $\omega$ on $\mathbb{D}$? We know that:

- if $\zeta \in \mathbb{D}$, then $h$ is a Hadamard multiplier of $D\zeta \iff \sup |c_k| < \infty$,
- if $\zeta \in \mathbb{T}$, then $h$ is a Hadamard multiplier of $D\zeta \iff \|T_{c}\| < \infty$.

Are there examples of superharmonic weights $\omega$ for which the characterization lies somewhere strictly between these two extremes?

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Received: July 2, 2019.
Revised: October 12, 2019.