Multilevel Picard approximations for high-dimensional decoupled forward-backward stochastic differential equations

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Abstract

Backward stochastic differential equations (BSDEs) appear in numerous applications. Classical approximation methods suffer from the curse of dimensionality and deep learning-based approximation methods are not known to converge to the BSDE solution. Recently, Hutzenthaler et al. [33] introduced a new approximation method for BSDEs whose forward diffusion is Brownian motion and proved that this method converges with essentially optimal rate without suffering from the curse of dimensionality. The central object of this article is to extend this result to general forward diffusions. The main challenge is that we need to establish convergence in temporal-spatial Hölder norms since the forward diffusion cannot be sampled exactly in general.

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1 Introduction

Backward differential equations (BSDEs) belong to the most frequently studied equations in stochastic analysis and computational stochastics. BSDEs arise in the solution of stochastic optimal control problems (see, e.g., [52, 49, 54]), BSDEs appear in the approximative valuation of financial products such as financial derivative contracts (see, e.g., [22, 16, 18]), and BSDEs are strongly linked to nonlinear partial differential equations (PDEs) which themselves arise naturally in many applications (see, e.g., [46, 48, 45, 47]). BSDEs in applications typically do not have a solution in closed form and are often high-dimensional (where dimension refers to the underlying noise).

Key words and phrases: stochastic differential equation, strong convergence, Euler–Maruyama approximation, Lipschitz condition, Lyapunov function, curse of dimensionality, high-dimensional SDEs, high-dimensional PDEs, high-dimensional BSDEs, multilevel Picard approximations, multilevel Monte Carlo method.

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In view of the importance of BSDEs, it has been an active research field to construct efficient approximation methods for the last decades. We refer, for example, to [1, 2, 9, 10, 11, 12, 13, 17, 23, 25, 26, 27, 42, 50, 53, 55, 56]. These methods, however, suffer from the curse of dimensionality (cf., e.g., Bellman [8], Novak & Wozniakowski [44, Chapter 1], and Novak & Ritter [43]) in the sense that the number of computational operations to approximatively compute one sample path of the BSDE solution grows at least exponentially in the reciprocal $1/\varepsilon$ of the prescribed approximation accuracy $\varepsilon \in (0, \infty)$ or the dimension $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$. Moreover, we refer, for example, to [14, 19, 24, 30] and the references mentioned in the overview articles [6, 20] for deep learning-based approximation methods for BSDEs. These deep learning-based approximation methods, however, are not known to converge to the BSDE solution.

Recently, [21, 34] introduced full history recursive multilevel Picard (MLP) approximations. This nonlinear Monte Carlo method indeed overcomes the curse of dimensionality in the numerical approximation of a large class of semilinear partial differential equations (PDEs); see, e.g., [34, 36, 32, 37, 31, 4, 3, 35, 40] and, e.g., [7] for simulations. Based on the MLP method and on the multilevel approach of [28, 29], [33] then introduced an approximation method for BSDEs and proved that this new method converges essentially with rate $1/2$ to the BSDE solution without suffering from the curse of dimensionality. The main assumptions of [33, Theorem 5.1] are that the terminal condition is a Lipschitz function of a forward diffusion which can be sampled exactly (e.g., Brownian motion or the Ornstein-Uhlenbeck process) and that the nonlinearity is $\varepsilon$-independent and has a bounded and globally Lipschitz continuous derivative.

In this article we extend the results of [33] to decoupled forward-backward stochastic differential equations (FBSDEs). More precisely the contribution of this article is as follows:

(i) We extend the approximation method of [33] to more general forward diffusions (note for this that the Euler approximations in (1) agree with the solution of (4) if $\mu_d = 0$ and $\sigma_d$ is constant).

(ii) The improved (compared to [32]) efficiency has the drawback that the arguments on the right-hand side of (2) are not identical so that we need to analyze Hölder regularity of the MLP approximations. We prove for the first time that MLP approximations of semilinear PDEs converge in Hölder norms; see Theorem 4.2 below. The proof of Corollary 5.2 also uses that Euler approximations of SDEs converge in Hölder norms under suitable assumptions which was recently established in [39]; cf. also (177) below. We note that we assume higher regularity of all coefficient functions for our analysis with temporal-spatial Hölder norms.

(iii) Our error criterion is the $L^2$-norm of the path distance where supremum over time is inside expectation; cf. Theorem 1.1. This improves the error criterion of [33] where the $L^2$-distance at single time points was estimated. To achieve these improved error estimates, we analyze $L^p$-distances for MLP approximations (this was first done in [35]) for arbitrary $p \in [2, \infty)$ and apply an argument of [15] of Kolmogorov-Chentsov-type.

The following theorem is the main result of this article.

**Theorem 1.1.** Let $T, c, \delta \in (0, \infty)$, $\Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n$, for every $d \in \mathbb{N} = \{1, 2, \ldots\}$ let $f_d \in C^2(\mathbb{R}, \mathbb{R})$, $g_d \in C^2(\mathbb{R}^d, \mathbb{R})$, $\mu_d \in C^2(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma_d \in C^2(\mathbb{R}^d, \mathbb{R}^{d \times d})$, for every $d \in \mathbb{N}$ let $\| \cdot \|$ denote the standard norm on $\mathbb{R}^d$ and the Frobenius norm on $\mathbb{R}^{d \times d}$, assume for all $d \in \mathbb{N}$, $w \in \mathbb{R}$, $x, y, z \in \mathbb{R}^d$ that $|T f_d(0)| + \|\mu_d(0)\| + \|\sigma_d(0)\| \leq c$, $|f_d'(w)| \leq c$, $|f_d''(w)| \leq c$, $|D g_d(x)(y)| \leq c \|y\|$, $\|D \mu_d(x)(y)\| \leq c d \|y\|$, $\|D \sigma_d(x)(y)\| \leq c d \|y\|$, $|D^2 g_d(x)(y, z)| \leq c d^2 \|y\| \|z\|$, $\|D^2 \mu_d(x)(y, z)\| \leq c d^2 \|y\| \|z\|$, and $\|D^2 \sigma_d(x)(y, z)\| \leq c d^2 \|y\| \|z\|$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})$ be a filtered probability space which satisfies the usual conditions, let $\mathbb{P}^0(t \leq t) = t$, let $W^d, \theta : [0, T] \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be independent standard $(\mathbb{F}_t)_{t \in [0, T]}$-Brownian motions, assume that $(\mathbb{F}^d, \theta \in \Theta)$ and $(W^d, \theta)_{d \in \mathbb{N}, \theta \in \Theta}$ are independent, for every $d, n \in \mathbb{N}$, $\theta \in \Theta$, $s \in [0, T]$, $x \in \mathbb{R}^d$, let $(X^d_{s, n, \theta, x})_{t \in [s, T]} : [s, T] \times \mathbb{R}^d \times \Theta \to \mathbb{R}^{d^2}$ satisfy for all $k \in [0, n-1] \cap \mathbb{Z}$, $t \in \left[\max\{s, kT\}, \max\{s, (k+1)T\}\right]$ that $X^d_{s, s, \theta, x} = x$ and

\[ X^d_{s, t, \theta, x} = X^d_{s, \max\{s, \frac{kT}{n}\}, \theta, x} + \mu_d(X^d_{s, \max\{s, \frac{kT}{n}\}, \theta, x})(t - \max\{s, \frac{kT}{n}\}) + \sigma_d(X^d_{s, \max\{s, \frac{kT}{n}\}, \theta, x}) \left(\int_{s}^{t} \mu_d(X^d_{s, \max\{s, \frac{kT}{n}\}, \theta, x})(s - \max\{s, \frac{kT}{n}\}) ds \right)^2 - \int_{s}^{t} \sigma_d(X^d_{s, \max\{s, \frac{kT}{n}\}, \theta, x})(s - \max\{s, \frac{kT}{n}\}) dW^d_{s, \theta, x} \right), \]
let \( U_{n,m}^d : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R} \), \( d \in \mathbb{N}, n, m \in \mathbb{Z}, \theta \in \Theta \), satisfy for all \( d, n, m \in \mathbb{N}, \theta \in \Theta, t \in [0, T], x \in \mathbb{R}^d \) that \( U_{-1,m}^d(t, x) = U_{0,m}^d(t, x) = 0 \) and

\[
U_{n,m}^d(t, x) = \sum_{\ell=0}^{n-1} \frac{1}{m_n-\ell} \sum_{i=1}^{m_n-\ell} \left[ g_d(\mathcal{X}_{\ell,T}^{d,m',(\theta,\ell,i),x}) \right. \\
+ (T - t) \left( f_d \circ U_{\ell,m}^{d,(\theta,\ell,i)} \right) \left( t + (T-t) \mathcal{X}_{\ell,T}^{d,m',(\theta,\ell,i),x} \right) \\
\left. \right) - \mathbb{1}_N(\ell) g_d(\mathcal{X}_{\ell,T}^{d,m',(\theta,\ell,i),x}) \\
+ (T - t) \left( f_d \circ U_{\ell,m}^{d,(\theta,\ell,i)} \right) \left( t + (T-t) \mathcal{X}_{\ell,T}^{d,m',(\theta,\ell,i),x} \right) - \mathbb{1}_N(\ell) \right](T - t) \left( f_d \circ U_{\ell,m}^{d,(\theta,\ell,i)} \right) \left( t + (T-t) \mathcal{X}_{\ell,T}^{d,m',(\theta,\ell,i),x} \right) \\
\right],
\]

(2)

for every \( d, n, m \in \mathbb{N} \) let \( \mathcal{E}_{d,n,m} \subseteq \mathbb{N}_0 \) be the number of realizations of scalar random variables and the number of function evaluations of \( \{f_d, g_d, \mu_d, \sigma_d\} \) which are used to compute one realization of \( \left( \gamma_{d,n,m} \right)_{d \in \mathbb{N}, n \in \mathbb{N}_0} \) (cf. (158)–(159) for a precise definition), let \( M : \mathbb{N} \to \mathbb{N} \) satisfy for all \( n \in \mathbb{N} \)

\[
M(n) = \log(n) \in [0, 1],
\]

and for every \( d \in \mathbb{N} \) let \( X^d, Z^d : [0, T] \times \Omega \to \mathbb{R}^d \) and \( Y^d : [0, T] \times \Omega \to \mathbb{R}^d \) be \( \mathcal{F}_t \)\( (\mathbb{F}_t)_{t \in [0,T]} \)-predictable stochastic processes such that

\[
\int_0^T \mathbb{E} \left[ \|X^d_s\|^2 + \|Y^d_s\|^2 + \|Z^d_s\|^2 \right] ds < \infty \quad \text{and such that for all } t \in [0, T] \text{ it holds a.s. that}
\]

\[
X^d_t = \int_0^t \mu_d(X^d_s) \, dr + \int_0^t \sigma_d(X^d_s) \, dW^d,0_s,
\]

\[
Y^d_t = g_d(X^d_T) + \int_t^T f_d(Y^d_s) \, ds - \int_t^T (Z^d_s) T \, dW^d,0_s.
\]

(4)

(5)

Then there exist \( \gamma \in (0, \infty), n : \mathbb{N} \times (0, 1) \to \mathbb{N} \) such that for all \( d \in \mathbb{N}, \varepsilon \in (0, 1), n \in [n(d, \varepsilon), \infty) \cap \mathbb{N} \) it holds that

\[
\left( \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \gamma_{d,n,M(n)} - Y^d_t \right|^2 \right] \right)^{1/2} < \varepsilon \quad \text{and } \varepsilon^{2+\delta} \mathcal{E}_{d,n,M(n)} \leq \gamma d^\delta.
\]

Theorem 1.1 follows directly from Corollary 5.2.

Now we heuristically motivate our approximation method (3) for fixed \( d \in \mathbb{N} \). By the nonlinear Feynman-Kac formula of [46] the FBSDE solution \( Y^d \) satisfies a.s. that \( Y^d = (u(t, X^d_t))_{t \in [0, T]} \) where \( u \) is the viscosity solution of the associated PDE. This PDE solution also solves the stochastic fixed-point equation

\[
\forall (t, x) \in [0, T] \times \mathbb{R}^d : u(t, x) = (\Phi(u))(t, x) := \mathbb{E} \left[ g(X^t,x) + \int_t^T f_d(u(s, X^t,x)) \, ds \right]
\]

(6)

where \( X^t,x \) is the solution of SDE (4) starting in point \( x \in \mathbb{R}^d \) at time \( t \in [0, T] \); see [5]. The Picard approximation theorem suggests that the Picard iterates \( u_n := \Phi^n(0), n \in \mathbb{N}, \) converge to \( u \). Now in the telescope expansion

\[
u \approx u_n = \sum_{l=1}^{n-1} \left( u_{l+1} - \mathbb{1}_N(l) u_l \right) = \sum_{l=0}^{n-1} \left( \Phi(u_l - \mathbb{1}_N(l) \Phi(u_{l-1})) \right)
\]

(7)

we approximate expectations and temporal integrals by Monte Carlo averages where for large \( l \) we need fewer Monte Carlo samples (say \( m^{n-l} \)) since \( u_l - u_{l-1} \) is then already small. This roughly motivates
(2). To motivate (3) let \( \mathcal{L}_n \) be the piecewise affine-linear interpolation of its argument restricted to the uniform subgrid of \([0, T]\) with \( m^n \) subintervals. Now in the telescope expansion

\[
Y \approx \mathcal{L}_n(Y) = \sum_{l=0}^{n-1} (\mathcal{L}_{l+1}(Y) - \mathbb{I}_{N(l)}\mathcal{L}_l(Y))
\]

(8)

it suffices to approximate \( Y \) in the \( l \)-th summand by \( U_{n-l,m}(\cdot, X) \) since \( (\mathcal{L}_{l+1} - \mathcal{L}_l)(Y - U_{n-l,m}(\cdot, X)) \) is already of the desired order \( m^{-1} m^{l-n} = m^{-n} \) as the MLP approximations converge to \( u \) in suitable Hölder norms.

The remainder of this article is organized as follows. Section 2 establishes temporal-spatial Hölder-type regularity of solutions of the stochastic fixed-point equation (6). We derive error estimates for MLP approximations of PDEs in \( L^p \)-norms in Section 3 and in temporal-spatial Hölder norms in Section 4. Finally, in Section 5 we prove Theorem 1.1.

## 2 Hölder regularity of solutions to stochastic fixed-point equations

In this section we derive the Hölder-type regularity estimate (17) for \( f \circ u \). This could in a second step be used to derive Hölder regularity of \( u \). Later we only need the regularity of \( f \circ u \) and the regularity of \( f \circ u \) is slightly easier to derive from the fixed-point equations (6). For (15) below we assume a general forward diffusion which satisfies a flow property, a Lyapunov-type estimate, and certain strong continuity in the starting point.

**Setting 2.1.** Let \( d \in \mathbb{N}, p_1, p_2 \in [1, \infty), c, T \in (0, \infty), f \in C(\mathbb{R}, \mathbb{R}), g \in C(\mathbb{R}^d, \mathbb{R}), V \in C([0, T] \times \mathbb{R}^d, [1, \infty)) \) satisfy that \( \frac{d}{p_1} + \frac{d}{p_2} \leq 1 \), let \( \| \cdot \| : \mathbb{R}^d \to [0, \infty) \) be a norm, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, for every random variable \( X : \Omega \to \mathbb{R} \) let \( \| X \|_r \in [0, \infty), r \in [1, \infty) \), satisfy for all \( r \in [1, \infty) \) that \( \| X \|_r = \mathbb{E}[|X|^r] \), \( \mathbb{P} \left( X_{s,t} = X_{s,r} \right) = 1 \), \( \mathbb{P} \left( X_{s,t} \in A, X_{s,r} \in B \right) = \mathbb{P} \left( X_{s,t} \in A \right) \mathbb{P} \left( X_{s,r} \in B \right) \),

\[
\mathbb{P} \left( X_{s,t}^x = X_{s,r}^x \right) = 1, \quad \mathbb{P} \left( X_{s,t} \in A, X_{s,i} \in B \right) = \mathbb{P} \left( X_{s,t} \in A \right) \mathbb{P} \left( X_{s,i} \in B \right),
\]

\[
\left\| \left( X_{s,t}^x - X_{s,r}^x \right) - \left( X_{s,t}^y - X_{s,r}^y \right) \right\|_{p_2} \leq \frac{V(s,x)+V(s,y)+V(s,x)+V(s,y)}{4} \left\| \left( x - y \right) - \left( \tilde{x} - \tilde{y} \right) \right\| + \frac{\left\| \left( x - y \right) + \left( \tilde{x} - \tilde{y} \right) \right\| \left\| x - \tilde{x} \right\|}{2}\right),
\]

(13)

and

\[
\left\| \left( X_{s,t}^x - X_{s,t}^y \right) - \left( x - y \right) \right\|_{p_2} \leq \frac{V(s,x)+V(s,y)}{2} \left\| x - y \right\| \frac{\left| t-s \right|^{1/2}}{\sqrt{T}}.
\]

(14)

**Lemma 2.2.** Assume Setting 2.1 and assume for all \( s \in [0, T], t \in [s, T], x \in \mathbb{R}^d \) that

\[
\left\| \left( X_{s,t}^x - x \right) \right\|_{p_2} \leq V(s,x)\left| t-s \right|^{1/2} \quad \text{and} \quad \left\| V(t, X_{s,t}^x) \right\|_{p_1} \leq V(s,x).
\]

Then
(i) there exists a unique measurable \( u: [0, T] \times \mathbb{R}^d \to \mathbb{R} \) which satisfies for all \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that
\[
\mathbb{E}\left[|g(X_{t,T}^x)| \right] + \frac{1}{T} \mathbb{E}\left[|f(u(r, X_{t,r}^x))|\right] dr + \sup_{r \in [0,T], \xi \in \mathbb{R}^d} \frac{|u(r, \xi)|}{1 + |r|} \leq \infty \text{ and } u(t, x) = \mathbb{E}\left[g(X_{t,T}^x)\right] + \int_t^T \mathbb{E}\left[f(u(r, X_{t,r}^x))\right] dr,
\]
(ii) it holds for all \( s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d \) that \( |u(t, x)| \leq 2e^{c(T-t)} V(t, x) \) and
\[
|u(s, x) - u(t, y)| \leq 4e^{2cT} \left( \frac{V(s,x)+V(t,y)}{2} \right)^2 V(s,x)||t-s|^{2} + ||x-y|| \quad \text{(16)}
\]
and
(iii) it holds for all \( s \in [0, T], t \in [s, T], x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d \) that
\[
\left| \left| f(u(s, x)) - f(u(s, y)) \right| \right| \leq \left( \left| \left| f(v) - f(w) \right| \right| + \left| \left| f(v) - f(v) \right| \right| \right) < c|v - w|.
\]

Proof of Lemma 2.2. First, observe that (11), (10), and (13) show for all \( s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d, v, w \in \mathbb{R} \) that
\[
|f(v) - f(w)| = |(f(v) - f(w)) - (f(v) - f(v))| \leq c|v - w|,
\]
\[
g(x) - g(y) \leq \frac{1}{2} \left( \frac{3V(T,x)+V(T,y)}{V(T,x)} \right)^2 |x-y|^2 + \frac{1}{2} \left( \frac{3V(T,y)+V(T,x)}{V(T,y)} \right)^2 |y-x|^2
\]
and
\[
\left| \left| X_{s,t}^x - X_{s,t}^y \right| \right|_p \leq \frac{1}{2} \left( \frac{3V(s,x)+V(s,y)}{V(s,x)} \right)^2 |x-y|^2 + \frac{1}{2} \left( \frac{3V(s,y)+V(s,x)}{V(s,y)} \right)^2 |y-x|^2
\]
This, \( [33, \text{Lemma } 3.1] \) (applied with \( L \subset c, p_3 \subset p_1, f \subset ([0, T] \times \mathbb{R}^d \times \mathbb{R} \ni (t, x, w) \mapsto f(w) \in \mathbb{R} \), \( \phi \subset V, \psi \subset V \) in the notation of \( [33, \text{Lemma } 3.1] \)), the fact that \( \frac{1}{2} + \frac{1}{p_2} \leq 1 \), the fact that \( 1 \leq V \), the assumptions on measurability, (9), (12), and (15) prove (i) and (ii).

Next, (10), Hölder’s inequality, the fact that \( \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \), the triangle inequality, (15), (13), (20), and the fact that \( 1 \leq V \) show for all \( s \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that
\[
\left| \left| g(X_{s,t}^{x_1}) - g(X_{s,t}^{y_1}) \right| \right| - \left| \left| g(X_{s,t}^{x_2}) - g(X_{s,t}^{y_2}) \right| \right| \leq \frac{1}{4} \left( V(T) \left( \frac{X_{s,t}^{x_1}-X_{s,t}^{x_2}}{V(T)} \right)^2 + V(T) \left( \frac{X_{s,t}^{y_1}-X_{s,t}^{y_2}}{V(T)} \right)^2 \right)
\]
\[
\left| \left| X_{s,t}^{x_1} - X_{s,t}^{x_2} \right| \right|_p \leq \frac{1}{2} \left( \frac{3V(s,x)+V(s,y)}{V(s,x)} \right)^2 |x-y|^2 + \frac{1}{2} \left( \frac{3V(s,y)+V(s,x)}{V(s,y)} \right)^2 |y-x|^2
\]
\[
\leq 5 \left( \frac{V(s,x_1)+V(s,y_1)+V(s,x_2)+V(s,y_2)}{4} \right)^3 \left[ \frac{\|x_1-y_1\|}{\sqrt{T}} - \frac{\|x_2-y_2\|}{\sqrt{T}} \right] + \left[ \frac{\|x_1-y_1\|}{\sqrt{T}} + \frac{\|x_2-y_2\|}{\sqrt{T}} \right] \frac{\|x_1-x_2\|}{\sqrt{T}}.
\]

(21)

Next, (ii), Hölder’s inequality, the fact that \( \frac{2}{p_1} + \frac{1}{p_2} \leq \frac{1}{2} \), the triangle inequality, (15), and (20) show for all \( s \in [0, T] \), \( t \in [s, T] \), \( x, y \in \mathbb{R}^d \) that

\[
\begin{align*}
\|u(t, X_{s,t}^x) - u(t, X_{s,t}^y)\|_2 &\leq 4e^{2cT} \left( \frac{V(t, X_{s,t}^x)+V(t, X_{s,t}^y)}{2} \right)^2 \frac{\|X_{s,t}^x - X_{s,t}^y\|}{\sqrt{T}} \\
&\leq 4e^{2cT} \left( \frac{V(t, X_{s,t}^x)+V(t, X_{s,t}^y)}{2} \right)^2 \frac{\|X_{s,t}^x - X_{s,t}^y\|}{\sqrt{T}} \leq 4e^{2cT} \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 \frac{\|x-y\|}{\sqrt{T}}.
\end{align*}
\]

(22)

This, (11), Hölder’s inequality, the triangle inequality, and the fact that \( \frac{2}{4} \left( 4e^{2cT} \right)^2 6^0 = 2^0 \) imply for all \( s \in [0, T] \), \( t \in [s, T] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that

\[
\begin{align*}
&\| (f(u(t, X_{s,t}^x)) - f(u(t, X_{s,t}^y))) - (f(u(t, X_{s,t}^y)) - f(u(t, X_{s,t}^y))) \|_1 \\
&\quad - c \| (u(t, X_{s,t}^x) - u(t, X_{s,t}^y)) - (u(t, X_{s,t}^y) - u(t, X_{s,t}^y)) \|_1 \\
&\leq \frac{c}{2} \left\| (u(t, X_{s,t}^x) - u(t, X_{s,t}^y)) + (u(t, X_{s,t}^x) - u(t, X_{s,t}^x)) \right\|_1 \\
&\leq \frac{c}{2} 4e^{2cT} \left( \frac{V(s,x)+V(s,y)+V(s,x)+V(s,y)}{4} \right)^2 \frac{\|x_1-y_1\|}{\sqrt{T}} + \frac{\|x_2-y_2\|}{\sqrt{T}} \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 \frac{\|x_1-x_2\|}{\sqrt{T}}
\end{align*}
\]

(23)

This, (1), (12), the disintegration theorem (see, e.g., [34, Lemma 2.3]), and the triangle inequality imply for all \( s \in [0, T] \), \( t \in [s, T] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that

\[
\begin{align*}
&\| (f(u(t, X_{s,t}^x)) - f(u(t, X_{s,t}^x))) - (f(u(t, X_{s,t}^y)) - f(u(t, X_{s,t}^y))) \|_1 \\
&\quad - 2^{10} \frac{c}{c} 4e^{4cT} \left( \frac{V(s,x)+V(s,y)+V(s,x)+V(s,y)}{4} \right)^6 \frac{\|x_1-y_1\|}{\sqrt{T}} + \frac{\|x_2-y_2\|}{\sqrt{T}} \|x_1-x_2\| \\
&\leq c \left\| \left( u(t, \tilde{x}_1) - u(t, \tilde{y}_1) \right) - \left( u(t, \tilde{x}_2) - u(t, \tilde{y}_2) \right) \right\|_1 \\
&\leq c \left\| \left( g(X_{s,t}^{x_1}) - g(X_{s,t}^{y_1}) \right) - \left( g(X_{s,t}^{x_2}) - g(X_{s,t}^{y_2}) \right) + \int_t^T \left( f(u(r, X_{s,t}^{x_1})) - f(u(r, X_{s,t}^{y_1})) \right) \right\|
\end{align*}
\]

\[
\begin{align*}
&\quad - \left( f(u(r, X_{s,t}^{x_2})) - f(u(r, X_{s,t}^{y_2})) \right) \right\|_1 \\
&\quad + \int_t^T \left( f(u(r, X_{s,t}^{x_1})) - f(u(r, X_{s,t}^{y_1})) \right) - \left( f(u(r, X_{s,t}^{x_2})) - f(u(r, X_{s,t}^{y_2})) \right) \right\|_1 \\
&\leq c \left\| \left( g(X_{s,t}^{x_1}) - g(X_{s,t}^{y_1}) \right) - \left( g(X_{s,t}^{x_2}) - g(X_{s,t}^{y_2}) \right) \right\|_1 \\
&\quad + \int_t^T \left( f(u(r, X_{s,t}^{x_1})) - f(u(r, X_{s,t}^{y_1})) \right) - \left( f(u(r, X_{s,t}^{x_2})) - f(u(r, X_{s,t}^{y_2})) \right) \right\|_1 dr.
\end{align*}
\]

(24)

This, (21), the fact that \( 1 \leq V \), and the fact that \( 2^{10} \frac{c}{c} 4e^{4cT} + 5c \leq 1029 ce^{4cT} \) imply for all \( s \in [0, T] \), \( t \in [s, T] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that

\[
\begin{align*}
&\| (f(u(t, X_{s,t}^x)) - f(u(t, X_{s,t}^y))) - (f(u(t, X_{s,t}^x)) - f(u(t, X_{s,t}^y))) \|_1 \\
&\leq 1029 ce^{4cT} \left( \frac{V(s,x)+V(s,y)+V(s,x)+V(s,y)}{4} \right)^6 \left[ \|x_1-y_1\| - \frac{\|x_2-y_2\|}{\sqrt{T}} \right] + \left[ \|x_1-y_1\| + \frac{\|x_2-y_2\|}{\sqrt{T}} \right] \frac{\|x_1-x_2\|}{\sqrt{T}} \\
&\quad + \int_t^T c \left( f(u(r, X_{s,t}^{x_1})) - f(u(r, X_{s,t}^{y_1})) \right) - \left( f(u(r, X_{s,t}^{x_2})) - f(u(r, X_{s,t}^{y_2})) \right) \right\|_1 dr.
\end{align*}
\]

(25)
Moreover, (i) shows for all \( s \in [0, T] \), \( x \in \mathbb{R}^d \) that \( \int_s^T \left\| f(u(t, X^x_{s,t})) \right\|_1 \, dt < \infty \). This, (25), and the reverse Gronwall lemma (see, e.g., [36, Lemma 3.2]) show for all \( s \in [0, T] \), \( t \in [s, T] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that

\[
\left\| (f(u(t, X^x_{s,t})) - f(u(t, X^{y_1}_{s,t}))) - (f(u(t, X^x_{s,t})) - f(u(t, X^{y_2}_{s,t}))) \right\|_1 \\
\leq 1029 e^{5cT} \left( \frac{V(s,x_1)+V(s,y_1)+V(x_2)+V(y_2)}{4} \right)^6 \left[ \left\| (x_1-x_2)\right\| + \left\| (||x_1-x_2||+||y_2-y_1||)\right\| \right].
\]

(26)

This, (21), the fact that \( \forall x \in [0, \infty) : cx \leq e^x \), and the fact that \( 1029e^{-1} \leq 379 \) show for all \( t \in [0, T] \), \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) that \( 5 + 1029ce^{5cT} \leq 5 + 1029e^{-1}ce^{5cT} \leq 384e^{6cT} \) and

\[
\left\| (g(X^x_{s,t}) - g(X^{y_1}_{s,t})) - (g(X^x_{s,t}) - g(X^{y_2}_{s,t})) \right\|_1 \\
+ T \sup_{t \in [0,T]} \left\| f(u(r, X^x_{s,t})) - f(u(r, X^{y_1}_{s,t})) - \left( f(u(r, X^x_{s,t})) - f(u(r, X^{y_2}_{s,t})) \right) \right\|_1 \\
\leq 384e^{6cT} \left( \frac{V(t,x_1)+V(t,y_1)+V(t,x_2)+V(t,y_2)}{4} \right)^6 \left[ \left\| (x_1-x_2)\right\| + \left\| (||x_1-x_2||+||y_2-y_1||)\right\| \right].
\]

(27)

Next, the triangle inequality, (14), (20), and (15) show that for all \( s \in [0, T] \), \( t \in [s, T] \), \( x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d \) with \( V(s,x) \leq V(s,y) \) that

\[
\frac{1}{\sqrt{T}} \left\| (X^x_{s,t} - X^y_{s,t}) - (\tilde{x} - \tilde{y}) \right\|_p + \frac{1}{\sqrt{T}} \left\| (X^x_{s,t} - X^y_{s,t}) \right\|_p + \left\| \tilde{x} - \tilde{y} \right\|_p \\
\leq \left\| (X^x_{s,t} - X^y_{s,t}) - (x - y) \right\|_p + \left\| (X^x_{s,t} - X^y_{s,t}) \right\|_p + \left\| \tilde{x} - \tilde{y} \right\|_p \\
\leq (V(s,x)+V(s,y)) \left\| (x - y) \right\| \frac{1}{\sqrt{T}} + \left\| (x - y) \right\| \frac{1}{\sqrt{T}} + \frac{V(s,x)+V(s,y)}{2} \left\| (x - y) \right\| \frac{1}{\sqrt{T}} \\
\leq \frac{1}{\sqrt{T}} \left\| (x - y) \right\| + \frac{V(s,x)+V(s,y)}{2} \left\| (x - y) \right\| \frac{1}{\sqrt{T}} \\
\leq \frac{1}{\sqrt{T}} \left\| (x - y) \right\| + \frac{V(s,x)+V(s,y)}{2} \left\| (x - y) \right\| \frac{1}{\sqrt{T}}.
\]

(28)

Moreover, (i), (12), the distributional and independence properties, and the disintegration theorem (see, e.g., [34, Lemma 2.4]) show for all \( s \in [0, T] \), \( t \in [s, T] \), \( x \in \mathbb{R}^d \) that

\[
\mathbb{E}[u(t, X^x_{s,t})] = \mathbb{E}[u(t, \tilde{x})|_{\tilde{x} = X^x_{s,t}}] \\
= \mathbb{E}[\mathbb{E}[g(X_{s,t}^x) + \int_t^T f(u(r, X^x_{s,t})) \, dr |_{\tilde{x} = X^x_{s,t}}]] = \mathbb{E}[g(X_{s,t}^x) + \int_t^T f(u(r, X^x_{s,t})) \, dr].
\]

(29)

This, (i), Tonelli’s theorem, Hölder’s inequality, the fact that \( \frac{6}{p_1} + \frac{2}{p_2} \leq 1 \), the triangle inequality, (27), (15), and (28) show for all \( s \in [0, T] \), \( t \in [s, T] \), \( x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d \) that

\[
\mathbb{E}\left[\left( u(t, X^x_{s,t}) - u(t, X^{y_1}_{s,t}) \right) - (u(t, \tilde{x}) - u(t, \tilde{y})) \right] \\
\leq \left\|[g(X_{s,t}^x) - g(X_{s,t}^{y_1})] - (g(X_{s,t}^x) - g(X_{s,t}^{y_2})) \right\|_1 \\
\leq T \sup_{t \in [0,T]} \left\| f(u(r, X^x_{s,t})) - f(u(r, X^{y_1}_{s,t})) - \left( f(u(r, X^x_{s,t})) - f(u(r, X^{y_2}_{s,t})) \right) \right\|_1 \\
\leq 384e^{6cT} \left( \frac{V(t,x_1)+V(t,y_1)+V(t,x_2)+V(t,y_2)}{4} \right)^6 \left[ \left\| (x_1-x_2)\right\| + \left\| (||x_1-x_2||+||y_2-y_1||)\right\| \right].
\]

(30)

Moreover, (i), (12), the distributional and independence properties, and the disintegration theorem (see, e.g., [34, Lemma 2.4]) show for all \( s \in [0, T] \), \( t \in [s, T] \), \( x \in \mathbb{R}^d \) that

\[
\mathbb{E}\left[\left( u(t, X^x_{s,t}) - u(t, X^{y_1}_{s,t}) \right) - (u(t, \tilde{x}) - u(t, \tilde{y})) \right] \\
\leq \left\|[g(X_{s,t}^x) - g(X_{s,t}^{y_1})] - (g(X_{s,t}^x) - g(X_{s,t}^{y_2})) \right\|_1 \\
\leq T \sup_{t \in [0,T]} \left\| f(u(r, X^x_{s,t})) - f(u(r, X^{y_1}_{s,t})) - \left( f(u(r, X^x_{s,t})) - f(u(r, X^{y_2}_{s,t})) \right) \right\|_1 \\
\leq 384e^{6cT} \left( \frac{V(t,x_1)+V(t,y_1)+V(t,x_2)+V(t,y_2)}{4} \right)^6 \left[ \left\| (x_1-x_2)\right\| + \left\| (||x_1-x_2||+||y_2-y_1||)\right\| \right].
\]

(31)
\[ \leq 768e^{6cT} \left( \frac{V(s,x)+V(s,y)+V(t,x)+V(t,y)}{4} \right)^7 \]

\[ \times \left( \frac{|x-y-(\tilde{x}-\tilde{y})|}{\sqrt{T}} + \frac{|x-y|+|\tilde{x}-\tilde{y}|}{\sqrt{T}} \right) \left( \frac{3 V(s,x)+V(s,y)}{2} \frac{|t-s|^{1/2}}{\sqrt{T}} + \frac{|x-\tilde{x}|}{\sqrt{T}} \right). \]

(30)

Next, (29), (18), and (22) show for all \( s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d \) that

\[ |E[(u(s,x) - u(s,y)) - (u(t,X_{s,t}^x) - u(t,X_{s,t}^y))]| = \int_s^t E[f(u(r,X_{s,r}^x)) - f(u(r,X_{s,r}^y))] \, dr \]

\[ \leq \int_s^t \| f(u(r,X_{s,r}^x)) - f(u(r,X_{s,r}^y)) \|_1 \, dr \leq c|t-s| \sup_{r \in [s,t]} \| u(r,X_{s,r}^x) - u(r,X_{s,r}^y) \|_1 \]

\[ \leq cT^{1/2}e^{4cT} \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 \frac{|x-y|}{\sqrt{T}} = 4cTe^{2cT} \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 \frac{|x-y|}{\sqrt{T}}. \]

(31)

Next, (ii) shows for all \( s \in [0, T], t \in [s, T], x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d \) that

\[ \frac{c}{2} \left( |u(s,x) - u(s,y)| + |u(t,\tilde{x}) - u(t,\tilde{y})| \right) \leq c \left( \frac{4c^2T^{2/3}}{2} \right) \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 \frac{|x-y|}{\sqrt{T}} \left( \frac{V(s,x)+V(s,y)}{2} \right)^2 \frac{|\tilde{x}-\tilde{y}|}{\sqrt{T}} \]

\[ \leq 2^7ce^{4cT} \left( \frac{V(s,x)+V(s,y)+V(t,\tilde{x})+V(t,\tilde{y})}{4} \right)^4 \frac{|x-y|+|\tilde{x}-\tilde{y}|}{\sqrt{T}} \left( \frac{V(s,x)+V(s,y)}{2} \right)^3 + \frac{|x-\tilde{x}|}{\sqrt{T}}. \]

(32)

This, (11), the triangle inequality, (31), (30), (ii), the fact that \( 1 \leq V \), and the fact that \( 4e^{2T}e^{2cT}2^4 + 768ce^{6cT} + 2^7ce^{4cT} \leq 768ce^{6cT} + 2^7ce^{4cT}1 + cT \leq 896ce^{6cT} \) show for all \( s \in [0, T], t \in [s, T], x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d \) that

\[ |f(u(s,x)) - f(u(s,y))| \leq c \left( |u(s,x) - u(s,y)| + |u(t,\tilde{x}) - u(t,\tilde{y})| \right) \]

\[ \leq c \left[ |u(s,x) - u(s,y)| - |u(t,\tilde{x}) - u(t,\tilde{y})| \right] + c \left[ |u(s,x) - u(s,y)| + |u(t,\tilde{x}) - u(t,\tilde{y})| \right] \]

\[ \leq c \left[ |u(s,x) - u(s,y)| - |u(t,\tilde{x}) - u(t,\tilde{y})| \right] + c \left[ |u(s,x) - u(s,y)| + |u(t,\tilde{x}) - u(t,\tilde{y})| \right] \]

\[ \leq 4c^2Te^{2cT} \left( \frac{V(s,x)+V(s,y)+V(t,\tilde{x})+V(t,\tilde{y})}{2} \right)^3 \frac{|x-y|}{\sqrt{T}} \left( \frac{V(s,x)+V(s,y)}{2} \right)^2 \frac{|\tilde{x}-\tilde{y}|}{\sqrt{T}} \]

\[ \left( \frac{3 V(s,x)+V(s,y)}{2} \frac{|t-s|^{1/2}}{\sqrt{T}} + \frac{|x-\tilde{x}|}{\sqrt{T}} \right) \]

\[ \leq 896ce^{6cT} \left( \frac{V(s,x)+V(s,y)+V(t,\tilde{x})+V(t,\tilde{y})}{4} \right)^4 \frac{|x-y|+|\tilde{x}-\tilde{y}|}{\sqrt{T}} \left( \frac{2V(s,x)+V(s,y)}{2} \right) \frac{|t-s|^{1/2}}{\sqrt{T}} + \frac{|x-\tilde{x}|}{\sqrt{T}}. \]

(33)

This proves (iii). The proof of Lemma 2.2 is thus completed.

\[ \square \]

3 Error estimates for MLP approximations in \( L^p \)-norms

In this section we prove strong convergence rates of MLP approximations in \( L^p \)-norms; see Proposition 3.3 below. We note that we rescale the \( L^p \)-distance with a suitable Lyapunov-type function and take then the supremum over time and space; see the definition (43) of our semi-norms. The assumptions of Proposition 3.3 are collected in the following Setting 3.1 and include global Lipschitz continuity of the nonlinearity \( f \), local Lipschitz continuity of the terminal condition \( g \) whose local Lipschitz constant grows at most like the Lyapunov-type function \( V \), and strong regularity estimates (37) and (38) for the forward diffusion. Proposition 3.3 extends the analysis of [35] where the forward diffusion is Brownian motion. Moreover, Corollary 3.4 shows that MLP method approximates semilinear PDEs with a computational effort which is of order 2+ in the reciprocal accuracy \( 1/\varepsilon \) and at most of polynomial order in the dimension. Corollary 3.4 improves [32] which derived computational order 4+ in the reciprocal accuracy under slightly weaker assumptions on the coefficient functions.
Lemma 3.2 (Independence and distributional properties). Assume Setting 3.1. Then

i) it holds for all $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\theta \in \Theta$ that $U_{n,m}^\theta$ is measurable,

ii) it holds for all $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\theta \in \Theta$ that

$$
\sigma\left(\left\{ U_{n,m}^\theta(t,x) : t \in [0,T], x \in \mathbb{R}^d \right\}\right) 
\subseteq \sigma\left(\left\{ \tau^{(\theta,\nu)}_{s,t}, \sigma_{s,t}^{\nu}(x) : \nu \in \Theta, t \in [0,T], x \in \mathbb{R}^d \right\}\right),
$$

(iii) it holds for all $\theta \in \Theta$, $m \in \mathbb{N}$ that $U_{\ell,m}^{(\theta,\ell),i}(t,x) \in \mathbb{R}^d$, $(U_{\ell,\ell}^{(\theta,\ell),i}(t,x))_{t \in [0,T], x \in \mathbb{R}^d}$, $(U_{\ell,\ell-1,m}^{(\theta,\ell),i}(t,x))_{t \in [0,T], x \in \mathbb{R}^d}$, $(\sigma_{s,t}^{\nu}(x))_{s,t \in [0,T], \nu \in \Theta}$ are independent,

(iv) it holds for all $n \in \mathbb{N}_0$, $\theta \in \Theta$ that $(U_{n,m}^\theta(t,x))_{t \in [0,T], x \in \mathbb{R}^d}$, $\theta \in \Theta$, are identically distributed, and

v) it holds for all $\theta \in \Theta$, $\ell \in \mathbb{N}_0$, $m \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$
\begin{align*}
(T-t) & \left( f \circ U_{\ell,m}^{(\theta,\ell),i}(t) \right) 
\left( t + (T-t)\rho^{(\theta,\ell),i}, \lambda_{t,t+(T-t)\rho^{(\theta,\ell),i}}^{\rho^{(\theta,\ell),i}} \right) 
\left( f \circ U_{\ell,\ell-1,m}^{(\theta,\ell),i}(t) \right) 
\left( t + (T-t)\rho^{(\theta,\ell),i}, \lambda_{t,t+(T-t)\rho^{(\theta,\ell),i}}^{\rho^{(\theta,\ell),i}} \right),
\end{align*}
$$

where $\sigma\left(\left\{ U_{n,m}^\theta(t,x) : t \in [0,T], x \in \mathbb{R}^d \right\}\right)$ is the smallest sigma-algebra containing $\sigma\left(\left\{ \tau_{s,t}^{(\theta,\nu)}(x) : \nu \in \Theta, t \in [0,T], x \in \mathbb{R}^d \right\}\right)$.
are i.i.d. and have the same distribution as
\[
(T-t)(f \circ U_{t,m}^0)\left(t + (T-t)\theta_t^0, \mathcal{X}_{t,t+(T-t)\theta_t}^{m^0,0,x}\right)
- \mathbb{1}_n(t)(T-t)(f \circ U_{t-1,m}^1)\left(t + (T-t)\theta_t^0, \mathcal{X}_{t,t+(T-t)\theta_t}^{m^0,0,x}\right).
\] (42)

**Proof of Lemma 3.2.** The assumptions on measurability and distributions, basic properties of measurable functions, and induction prove (i) and (ii). Next, (ii) and the assumptions on independence prove (iii). Next, (iii), the fact that \(\forall \theta \in \Theta, m \in \mathbb{N}: U_{t,m}^0 = 0\), (39), the disintegration theorem (see, e.g., [34, Lemma 2.2]), the assumptions on distributions, and induction show (iv) and (v). ∎

**Proposition 3.3** (Error analysis by semi-norms). Assume Setting 3.1, let \(p \in [2, \infty), q_1 \in [3, \infty)\), assume that \(\frac{1}{p_1} + \frac{1}{p_2} \leq 1, p q_1 \leq p_1\) and \(\frac{1}{p_1} + \frac{1}{p_2} \leq 1\), and for every random field \(H: [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}\) let \(\|H\|_{1,a} \in [0, \infty], s \in [0, T]\), satisfy for all \(s \in [0, T]\) that
\[
\|H\|_{1,a} = \sup_{t \in [s, T]} \sup_{x \in \mathbb{R}^d} \|H(t, x)\|_p^{q_1}.
\] (43)

Then

(i) there exists a unique measurable \(u: [0, T] \times \mathbb{R}^d \to \mathbb{R}\) which satisfies for all \(t \in [0, T], x \in \mathbb{R}^d\) that
\[
\mathbb{E}\left[|g(X_{t,T}^x)| + \int_t^T \mathbb{E}\left[|f(u(r, X_{t,r}^x))|\right] dr + \sup_{r \in [0, t], \xi \in \mathbb{R}^d} \mathbb{P}(u(r, \xi)) < \infty\right.
\]
and \(u(t, x) = \mathbb{E}\left[g(X_{t,T}^x)\right] + \int_t^T \mathbb{E}\left[f(u(r, X_{t,r}^x))\right] dr\),

(ii) it holds for all \(n, m \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d, \theta \in \Theta\) that \(U_{n,m}^\theta(t, x), g(X_{t,T}^{m^0-n,0,x})\) and \((f \circ U_{n,m}^\theta(t, x))\) are integrable,

(iii) it holds for all \(n, m \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d\) that
\[
\mathbb{E}\left[U_{n,m}^\theta(t, x)\right] = \mathbb{E}\left[g(X_{t,T}^{m^0-n,0,x})\right] + (T-t)\mathbb{E}\left[(f \circ U_{n,m}^\theta(t, x))\right]
\]
and \(u(t, x) = \mathbb{E}\left[g(X_{t,T}^x)\right] + (T-t)\mathbb{E}\left[(f \circ u)(t + (T-t)\theta_t^0, X_{t,t+(T-t)\theta_t}^{m^0,0,x})\right]\),

(iv) it holds for all \(n, m \in \mathbb{N}, s \in [0, T]\) that
\[
\|U_{n,m}^\theta - u\|_{1,s} \leq \frac{16mn^2e^{C_T}}{\sqrt{m}} \sup_{x \in \mathbb{R}^d} \sum_{\ell=0}^{n-1} \frac{4(T-t)^{\frac{1}{2}}e^{C_T}}{\sqrt{m}} \left[\int_s^T \|U_{s,\ell,T}^\theta - u\|_{1,s}^{p_1} d\ell\right]^{1/p_1},
\] (44)

and

(v) it holds for all \(n, m \in \mathbb{N}, s \in [0, T]\) that
\[
\|U_{n,m}^\theta - u\|_{1,s} \leq 16mn(p - 1)^{n/2}e^{4ce(T-s)n}m^{p/2}p m^{-n/2}.
\] (45)

**Proof of Proposition 3.3.** Throughout this proof for every random variable \(X: \Omega \to \mathbb{R}\) with \(\mathbb{E}[|X|] < \infty\) let \(\mathbb{V}_p(X) \in [0, \infty]\) satisfy that \(\mathbb{V}_p(X) = \|X - \mathbb{E}[X]\|_p^2\). First, (38) and Fatou’s lemma yield for all \(s \in [0, T]\), \(t \in [s, T]\), \(x \in \mathbb{R}^d\) that \((X_{s,t}^{m^0,0,x})_{n \in \mathbb{N}}\) converges to \(X_{s,t}^x\) in probability and
\[
\|V(t, X_{s,t}^x)\|_{p_1} = \left\|\mathbb{P}\cdot \lim_{n \to \infty} V(t, X_{s,t}^{n,0,0,x})\right\|_{p_1} \leq \lim_{n \to \infty} \left\|V(t, X_{s,t}^{n,0,0,x})\right\|_{p_1} \leq V(s, x).
\] (46)

This, [33, Lemma 3.1] (applied with \(L \cap c, p_3 \cap p_1, f \cap (\{0, T]\times \mathbb{R}^d \times \mathbb{R} \ni (x, t, w) \mapsto f(w) \in \mathbb{R}\), \(\phi \cap V, \psi \cap V\) in the notation of [33, Lemma 3.1]), the fact that \(\frac{1}{p_1} + \frac{1}{p_2} \leq 1\), the assumptions on measurability and distributional properties, (34), (35), (36), (37), (43), and the fact that \(1 \leq V \leq V^p\) prove (i) and show for all \(t \in [0, T], x, y \in \mathbb{R}^d\) that
\[
|u(t, x)| \leq 2e^{C_T(T-t)}V(t, x), \quad \|u\|_{1,t} \leq 2e^{C_T(T-t)},
\]
and
\[
|u(t, x) - u(t, y)| \leq 4e^{C_T(T-t)} \left(\frac{V(t,x) + V(t,y)}{2}\right)^2 \|x-y\|_{\sqrt{T}}.
\] (47)
Next, (34), Jensen’ inequality, the fact that \( p \leq p_1 \), (38), and the fact that \( V \leq V^n \) show for all \( n \in \mathbb{N} \), \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that
\[
\left\| g(\mathcal{X}_{t,T}^n,0,x) \right\|_p \leq \left\| V(T, \mathcal{X}_{t,T}^n,0,x) \right\|_p \leq \left\| V(T, \mathcal{X}_{t,T}^n,0,x) \right\|_{p_1} \leq V(t, x) \leq (V(t, x))^{q_1}.
\] (48)

In addition, (35), Hölder’s inequality, the fact that \( \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p} \), (38), (46), and the fact that \( V^2 \leq V^n \) prove for all \( t \in [0, T] \), \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that
\[
\left\| g(\mathcal{X}_{t,T}^n,0,x) - g(X_{t,T}^x) \right\|_p \\
\leq \left\| \frac{V(T, \mathcal{X}_{t,T}^n,0,x) + V(X_{t,T}^x)}{2} \right\|_{\mathcal{F}^p} \leq \left\| V(T, \mathcal{X}_{t,T}^n,0,x) \right\|_{p_1} \left\| V(X_{t,T}^x) \right\|_{p_2} \\
\leq V(t, x) \cdot \frac{V(t, x)}{\sqrt{n}} \leq \frac{(V(t, x))^{q_1}}{\sqrt{n}}.
\] (49)

This and the triangle inequality show for all \( t \in [0, T] \), \( \ell, m \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that
\[
\left\| g(\mathcal{X}_{t,T}^{\ell,m,0,x}) - g(\mathcal{X}_{t,T}^{\ell,m-1,0,x}) \right\|_p \leq \sum_{j=0}^{\ell-1} \left\| g(\mathcal{X}_{t,T}^{\ell,m-1,0,x}) - g(X_{t,T}^x) \right\|_p \leq \sum_{j=0}^{\ell-1} \left\| g(\mathcal{X}_{t,T}^{\ell,m,0,x}) - g(X_{t,T}^x) \right\|_p \leq \sum_{j=0}^{\ell-1} \frac{(V(t, x))^{q_1}}{\sqrt{m}}.
\] (50)

Next, the disintegration theorem (see, e.g., [34, Lemma 2.2]), the assumptions on measurability and independence, Jensen’s inequality, the fact that \( pq_1 \leq p_1 \), and (38) prove for all \( t \in [0, T] \), \( \ell \in \mathbb{N}_0 \), \( m, n \in \mathbb{N} \), \( x \in \mathbb{R}^d \), \( \nu \in \mathbb{Z} \), \( H \in \text{span}_\mathbb{R}(\{f \circ U^0_{\ell,m}, f \circ u\}) \) that
\[
\left\| (T-t)H\left(t + (T-t)\nu, \mathcal{X}_{t,T}^{\ell,m,0,x}\right) \right\|_p = (T-t)\left\| \left\| H(r, y) \right\|_{q_1} \left\| g(\mathcal{X}_{t,T}^{\ell,m,0,x}) \right\|_{r=t+(T-t)e} \right\|_p \\
\leq (T-t)\left\| \left\| H\right\|_{1,T} \left\| \left\| \left( V(r, \mathcal{X}_{t,T}^{\ell,m,0,x})\right) \right\|_{q_1} \right\|_{r=t+(T-t)e} \right\|_p \\
\leq (T-t)c\left\| \left\| U^0_{\ell,m} - u \right\|_{1,T} \right\|_{e} \right\|_{p}.
\] (51)

Moreover, (35) and (43) show for all \( t \in [0, T] \) and all random fields \( H, K : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R} \) that \( \left\| f \circ H - f \circ K \right\|_{1,T} \leq c \left\| H - K \right\|_{1,T} \). This, (51), and the independence and distributional properties show for all \( t \in [0, T] \), \( \nu, \ell \in \mathbb{N}_0 \), \( m, n \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that
\[
\frac{1}{V(t, x)^{q_1}} \left\| (T-t)\left((f \circ U^\nu_{\ell,m}) - (f \circ u)\right)\left(t + (T-t)\nu, \mathcal{X}_{t,T}^{\ell,m,0,x}\right) \right\|_p \\
\leq (T-t)c\left\| \left\| U^\nu_{\ell,m} - u \right\|_{1,T} \right\|_{e} \right\|_{p}.
\] (52)

Next, (35), (47), Hölder’s inequality, the fact that \( \frac{2}{p_1} + \frac{1}{p_2} \leq \frac{2}{p} \), (38), (46), and the fact that \( V^3 \leq V^n \) prove for all \( t \in [0, T] \), \( r \in [t, T] \), \( x, y \in \mathbb{R}^d \), \( n \in \mathbb{N} \) that
\[
\left\| (f \circ u)(r, \mathcal{X}_{t,r}^{\ell,m,0,x}) - (f \circ u)(r, X_{t,r}^x) \right\|_p \leq c\left\| u(r, \mathcal{X}_{t,r}^{\ell,m,0,x}) - u(r, X_{t,r}^x) \right\|_p \\
\leq 4ce^{2T} \left\| \left( \frac{V(r, \mathcal{X}_{t,r}^{\ell,m,0,x}) + V(X_{t,r}^x)}{2} \right) \right\|_{q_1} \right\|_{p} \\
\leq 4ce^{2T} \left\| \left( \frac{V(r, \mathcal{X}_{t,r}^{\ell,m,0,x}) + V(X_{t,r}^x)}{2} \right) \right\|_{p_1} \left\| \left( \frac{\mathcal{X}_{t,r}^{\ell,m,0,x} - X_{t,r}^x}{\sqrt{T}} \right) \right\|_{p_2} \\
\leq 4ce^{2T}(V(t, x))^2 \cdot \frac{V(t, x)}{\sqrt{n}} \leq 4ce^{2T}(V(t, x)^{q_1}) \cdot \frac{V(t, x)}{\sqrt{n}}.
\] (53)

This, the triangle inequality, and (52) show for all \( \ell, m \in \mathbb{N} \), \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that
\[
\left\| (T-t)\left((f \circ U^\nu_{\ell,m})\left(t + (T-t)\nu, \mathcal{X}_{t,T}^{\ell,m,0,x}\right) - (f \circ U^\nu_{t-1,m})\left(t + (T-t)\nu, \mathcal{X}_{t,T}^{\ell,m-1,0,x}\right) \right) \right\|_p
\]
\[
\begin{align*}
&\leq \left( T-t \right) \left\| (f \circ U^0_{\ell,m}) - (f \circ u) \left( (T-t)X^0_{t,t+(T-t)\varnothing} \right) \right\|_p \\
&\quad + (T-t) \left\| (f \circ U^1_{\ell-1,m}) - (f \circ u) \left( (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) \right\|_p \\
&\quad + (T-t) \left\| (f \circ u) \left( t + (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) - (f \circ u) \left( (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) \right\|_p \\
&\quad + (T-t) \left\| (f \circ u) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) - (f \circ u) \left( (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right\|_p \\
&\leq \sum_{j=t-1}^t \left[ (T-t)c \left\| U^0_{j,m} - u \right\|_{1,t+(T-t)\varnothing} + \frac{4e(T-t)e^{2cT}}{\sqrt{m}} \right] \left( V(t,x) \right)^{d_1}. \tag{54}
\end{align*}
\]

This, (39), the triangle inequality, the fact that \( \forall m \in \mathbb{N}: U^0_{0,m} = 0 \), the independence and distributional properties, (48), (50), (47), and induction prove for all \( n \in \mathbb{N}_0 \), \( m \in \mathbb{N} \), \( x \in \mathbb{R}^d \), \( t \in [0,T] \), \( \theta \in \Theta \) that \( \left\| U^0_{n,m} \right\|_{1,t} + \left\| (T-t) (f \circ U^0_{n,m}) (t + (T-t)^0, X^{m,0,x}_{t,t+(T-t)\varnothing}) \right\|_p < \infty \). This, (43), and (48) establish (ii).

Next, (39), linearity, the independence and distributional properties, and a telescoping sum argument prove for all \( n,m \in \mathbb{N}, t \in [0,T], x \in \mathbb{R}^d \) that

\[
\mathbb{E} [U^0_{n,m}(t,x)] = \sum_{\ell=0}^{n-1} \frac{1}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \left( \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] - \mathbb{1}_n(\ell) \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] \right) \\
\quad + (T-t) \mathbb{E} \left[ (f \circ U^0_{\ell,m}) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right] \\
\quad - \mathbb{1}_n(\ell)(T-t) \mathbb{E} \left[ (f \circ U^0_{\ell-1,m}) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right] \\
= \sum_{\ell=0}^{n-1} \frac{1}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \left( \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] - \mathbb{1}_n(\ell) \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] \right) \\
\quad + (T-t) \mathbb{E} \left[ (f \circ U^0_{\ell,m}) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right] \\
\quad - \mathbb{1}_n(\ell)(T-t) \mathbb{E} \left[ (f \circ U^0_{\ell-1,m}) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right] \\
= \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] + (T-t) \mathbb{E} \left[ (f \circ u) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right]. \tag{55}
\]

Moreover, (i), the disintegration theorem (see, e.g., [34, Lemma 2.2]), and the independence and distributional properties show for all \( t \in [0,T], x \in \mathbb{R}^d \) that

\[
u(t,x) = \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) \right] + (T-t) \mathbb{E} \left[ (f \circ u) \left( t + (T-t)X^{0}_{t,t+(T-t)\varnothing} \right) \right]. \tag{56}
\]

This and (55) show (iii).

Next, (55) and (56) prove for all \( n,m \in \mathbb{N}, t \in [0,T], x \in \mathbb{R}^d \) that

\[
\begin{align*}
\mathbb{E} [U^0_{n,m}(t,x)] - u(t,x) &= \mathbb{E} \left[ g(X^{m-1,0,x}_{t,T}) - g(X^{m-1,0,x}_{t,T}) \right] \\
&\quad + (T-t) \mathbb{E} \left[ (f \circ u) \left( t + (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) \right] \\
&\quad + (T-t) \mathbb{E} \left[ (f \circ u) \left( t + (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) - (f \circ u) \left( t + (T-t)X^{m-1,0,x}_{t,t+(T-t)\varnothing} \right) \right]. \tag{57}
\end{align*}
\]

This, the triangle inequality, Jensen’s inequality, (49), (52), (53), and the fact that \( \forall t \in [0,T]: 4c(T-t)e^{2cT} + 1 \leq 4e^{2cT}(1 + c(T-t)) \leq 4e^{3cT} \) prove for all \( n,m \in \mathbb{N}, t \in [0,T], x \in \mathbb{R}^d \) that

\[
\left| \frac{\mathbb{E} [U^0_{n,m}(t,x)] - u(t,x)}{(V(t,x))^{d_1}} \right| \leq (T-t)e \left\| U^0_{n-1,m} - u \right\|_{1,t+(T-t)\varnothing} + \frac{4e^{3cT}}{\sqrt{m}}. \tag{58}
\]
Moreover, the Marcinkiewicz-Zygmund inequality (see [51, Theorem 2.1]), the fact that \( p \in [2, \infty) \),
the triangle inequality, and Jensen’s inequality show that for all \( n \in \mathbb{N} \) and all i.i.d. random variables \( X_k, k \in [1, n] \cap \mathbb{Z} \), with \( \mathbb{E}[|X_1|] < \infty \) it holds that \( (\text{Var}_p(\sum_{k=1}^n X_k))^{1/2} = \|\sum_{k=1}^n X_k - \mathbb{E}[X_k]\|_p \leq (p - 1)^{1/2} (\sum_{k=1}^n |X_k - \mathbb{E}[X_k]|^2)^{1/2} \leq 2(p - 1)^{1/2} |\mathbb{E}[X_1]|/\sqrt{n} \). This, (39), the triangle inequality, the properties on independence and distributions, (48), (50), (54), and the fact that \( \forall t \in [0, T] \): \( 4e(T-t)e^{2cT} + 1 \leq 4e^{2cT}(1 + c(T-t)) \leq 4e^{3cT} \) prove for all \( n, m \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d \) that
\[
\left\| \left( U_{n,m}^0(t, x) - \mathbb{E}[U_{n,m}^0(t, x)] \right) \right\|_p \leq \frac{\beta}{(T-t)^{1/2}} \left[ \left( \sum_{j=1}^{a_0} \sup_{t \in [s, T]} |(t - T)|^{1/2} \right) \left\| U_{n,m}^0(t, x) - u(t, x) \right\|_p \right]^{1/2} + \frac{4c(T-t)e^{2cT}}{\sqrt{m}^3}.
\]

In addition, the fact that \( \nu^0 \) is uniformly distributed on \([0, 1]\) and the substitution rule imply for all \( s \in [0, T], t \in [0, T], \) and all measurable \( h : [0, T] \to \mathbb{R} \) that
\[
(T-t) \left\| h(t + (T-t) \nu^0) \right\|_p = (T-t)^{1/2} \left[ f_1^T(h(\zeta)^p) \, d\zeta \right]^{1/2} \leq (T-s)^{1/2} \left[ f_1^T(h(\zeta)^p) \, d\zeta \right]^{1/2}.
\]
Since the exact computational effort is difficult to define, we instead assume that the computational effort is upper bounded by

\[ (T - t)c \left\| U_{n-1,m}^0 - u \right\|_{1,t+(T-t)^0} + \frac{4c_{3T}}{\sqrt{m^3}} \]

\[ \leq \sup_{t \in [s,T]} \left\{ \frac{2}{\sqrt{m^3}} + \sum_{t=0}^{n-1} \left( \frac{4}{\sqrt{m^{n-t}}} \left( (T - t)c \left\| U_{t,m}^0 - u \right\|_{1,t+(T-t)^0} + \frac{4c_{3T}}{\sqrt{m^3}} \right) \right) \right\} \]

\[ \leq \frac{2}{\sqrt{m^3}} + \sum_{t=0}^{n-1} \left( (T - s)^{T-1} \frac{1}{p} c \left[ \int_s^T \left\| U_{t,m}^0 - u \right\|_{1,s,T} d\zeta \right]^{1/p} + \frac{4c_{3T}}{\sqrt{m^3}} \right) \]

\[ \leq \frac{16mnc_{3T}}{\sqrt{m^3}} + \sum_{t=0}^{n-1} \left[ \frac{(T - s)^{T-1}}{\sqrt{m^{n-t}}} \left( \int_s^T \left\| U_{t,m}^0 - u \right\|_{1,s,T} d\zeta \right]^{1/p} \right). \tag{61} \]

This shows (iv).

Next, [32, Lemma 3.11] (applied for every \( s \in [0, T] \), \( n, m \in \mathbb{N} \) with \( M \cap m, N \cap n, \tau \cap s, a \cap 16mnc_{3T} \sqrt{p-n} \), \( b \cap 4(T - s)^{T-1} \frac{1}{p} c \sqrt{p-n} - \frac{(f_j) \cap (s, T) \cap \mathbb{N}_0}{} \), in the notation of [32, Lemma 3.11]), (iv), (47), the fact that \( \forall m \in \mathbb{N} : U_{0,m}^0 = 0 \), and the fact that \( \forall s \in [0, T] : 16mnc_{3T} - 2c(T - s) \leq 16mnc_{3T}(1 + c(T - s)) \leq 16mnc_{3T} \) for all \( n, m, n, s \in [0, T] \) that

\[ \left\| U_{n,m}^0 - u \right\|_{1,s} \leq \left( 16mnc_{3T} \sqrt{p-n} \right) + \left( 4(T - s)^{T-1} \frac{1}{p} c \sqrt{p-n} - \frac{(T - s)^{T-1}}{p} \right)^{n-1} \]

\[ \leq \sqrt{p-n} \left( 16mnc_{3T} \right) + \left( 4(T - s)^{T-1} \frac{1}{p} c \sqrt{p-n} - \frac{(T - s)^{T-1}}{p} \right)^{n-1} \]

\[ \leq 16mnc_{3T} \cdot \left( 4(T - s)^{T-1} \frac{1}{p} c \sqrt{p-n} - \frac{(T - s)^{T-1}}{p} \right)^{n-1} \]

This proves (v). The proof of Proposition 3.3 is thus completed.

The following Corollary 3.4 estimates the computational effort of the MLP approximations in (68). Since the exact computational effort is difficult to define, we instead assume that the computational effort satisfies the recursive inequality (69), which we now motivate. We think of the parameter \( a_1 \) as an upper bound for the effort to compute \( f(w) \) or \( g(x) \) for any \( w \in \mathbb{R}, x \in \mathbb{R}^d \). We think of \( a_2m^\ell \) as an upper bound for the effort to compute one realisation of \( \mathcal{X}_{s,T,\delta}^0, \mathcal{X}_{s,T,\delta}^\ell \), and we think of \( \mathcal{C}_{t,m} \) as an upper bound for the effort to compute one realisation \( U_{t,m}^0(t, x) \). We note that we approximate the forward diffusion by the Euler approximations in (67).

**Corollary 3.4** (Analysis of the computational effort). Let \( \|\cdot\| : \bigcup_{k,\ell \in \mathbb{N}} \mathbb{R}^{k \times \ell} \to [0, \infty) \) satisfy for all \( k, \ell \in \mathbb{N}, s = (s_{i,j})_{i \in [1,k]} \cap [1,\ell] \cap [1,\ell] \in \mathbb{R}^{k \times \ell} \) that \( \|s\|^2 = \sum_{i=1}^{k} \sum_{j=1}^{\ell} |s_{i,j}|^2 \), let \( T, \delta \in (0, \infty), c, \beta \in [1, \infty), p \in [2, \infty), d, a_1, a_2 \in \mathbb{N}, \Theta = \bigcup_{n \in \mathbb{N}} \mathbb{Z}^n, \phi \in C([0, \infty], \mathcal{C}_{t,m}^0) \), \( f \in C(\Theta, \mathbb{R}), g \in C(\mathbb{R}^d, \mathbb{R}) \), \( u \in C^{1,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}) \), \( \mu = (\mu_i)_{i \in [1,d]} \in \mathcal{C}^{2}(\mathbb{R}^d, \mathbb{R}^d), \sigma = (\sigma_{i,j})_{i,j \in [1,d]} \in \mathcal{C}^{2}(\mathbb{R}^d, \mathbb{R}^{d \times d}) \), assume for all \( x, y \in \mathbb{R}^d, w_1, w_2 \in \mathbb{R}, t \in [0, T] \) that

\[ \|\mu(0)\| + \|\sigma(0)\| \leq b, \quad \max\{\|\mu(x) - \mu(y)\|, \|\sigma(x) - \sigma(y)\|\} \leq c|x - y|, \tag{63} \]

\[ |Tf(0)| + |u(t, x)| + |g(x)| \leq \left( b^2 + c^2 \|x\|^2 \right)\beta, \tag{64} \]

\[ |g(x) - g(y)| \leq \max\{b^2, \|x\|^{2\beta}, \|y\|^{2\beta}\} \|x - y\|, \quad |f(t, w)| - |f(t, w)| \leq c|w_1 - w_2|, \tag{65} \]

\[ \left( \frac{\partial}{\partial t} u \right)(t, x) + \sum_{i=1}^{d} \left( \frac{\partial}{\partial x_i} u \right)(t, x) + \frac{1}{2} \sum_{i,j,k=1}^{d} \left( \frac{\partial^2}{\partial x_i \partial x_j} u \right)(t, x) \sigma_{i,k}(t, x) \sigma_{j,k}(t, x) \right) \]

\[ = -f(u(t, x)), \quad u(T, x) = g(x), \tag{66} \]
let \((\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0, T]})\) be a filtered probability space which satisfies the usual conditions, for every random variable \(X: \Omega \to \mathbb{R} \cup \{-\infty, \infty\}\) let \(||X||_r = \mathbb{E}[|X|^r]\), let \(r^0: \Omega \to [0, 1], \theta \in \Theta\), be i.i.d. random variables which satisfy for all \(t \in [0, 1]\) that \(\mathbb{P}(r^0 \leq t) = t\), let \(W^\theta: [0, T] \to \mathbb{R}^d\), \(\theta \in \Theta\), be independent \((\mathbb{F}_t)_{t \in [0, T]}\)-Brownian motions, assume that \((r^0)_{\theta \in \Theta}\) and \((W^\theta)_{\theta \in \Theta}\) are independent, for every \(n \in \mathbb{N}, \theta \in \Theta, s \in [0, T], x \in \mathbb{R}^d\) let \((\lambda_{n,s}^{\theta,x})_{t \in [s,T]}: [s, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d\) satisfy for all \(k \in [n, n-1] \cap \mathbb{Z}\), \(t \in [\max\{s, kT\}, \max\{s, (k+1)T\}]\) that \(\lambda_{n,s}^{\theta,x} = x\) and

\[\lambda_{n,s}^{\theta,x} = \lambda_{n,s}^{\theta,x} + \mu\left(\lambda_{n,s}^{\theta,x}, \max\{s, kT\}\right)\left(t - \max\{s, kT\}\right) + \sigma\left(\lambda_{n,s}^{\theta,x}, \max\{s, kT\}\right)\left(W^\theta_t - W^\theta_{\max\{s, kT\}}\right),\]

(67)

let \(U_{n,m}^\theta: [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}, n, m \in \Theta, s \in \Theta\), satisfy for all \(n, m \in \mathbb{N}, \theta \in \Theta, t \in [0, T], x \in \mathbb{R}^d\) that \(U_{n,m}^\theta(t,x) = U_{0,m}^\theta(t,x) = 0\) and

\[
U_{n,m}^\theta(t,x) = \sum_{\ell=0}^{n-1} \left[ \frac{1}{m-\ell} \sum_{i=1}^{m-\ell} \left( g(\lambda_{\ell,T}^{m\ell,\theta,i},x) - 1_N(\ell)g(\lambda_{\ell,T}^{m^\ell-1,\theta,i},x) \right) \right. \\
\left. + (T-t)\left( f \circ U_{\ell,m}^{\theta,i} \right) \left( t + (T-t)e^{\theta,i} \right) - 1_N(\ell)(T-t)\left( f \circ U_{\ell-1,m}^{\theta,\ell-1} \right) \left( t + (T-t)e^{\theta,i} \right) \right].
\]

(68)

let \((C_{n,m})_{n,m \in \mathbb{N}} \subseteq \mathbb{N}_0\) satisfy for all \(n, m \in \mathbb{N}\) that

\[C_{0,m} = 0 \quad \text{and} \quad C_{n,m} \leq \sum_{\ell=0}^{n-1} \left[ m-\ell \left[ 4a_1 + a_2 m^{\ell} + C_{\ell,m} + 1_N(\ell) C_{\ell-1,m} \right] \right],\]

(69)

and let \(M: \mathbb{N} \to \mathbb{C}, C \in [0, \infty]\) satisfy that

\[
\lim_{n \to \infty} \frac{M(n)}{n} = 0, \quad \sup_{n \in \mathbb{N}} \left[ \frac{M(n+1)}{M(n)} + \frac{(M(n))^{p/2}}{n} \right] < \infty,
\]

(70)

and

\[
C = \sup_{n \in \mathbb{N}} \left( \frac{6(M(n))^2 + 4 \left( \frac{7p^4 \pi^4 (M(k+1))^{(n+1)(2+k)/2}}{(M(n))^{n+2}} \right)^{2 \delta}}{(M(n))^{n+2}} \right),(71)
\]

Then

i) it holds that \(C < \infty\),

ii) there exists an up to indistinguishability unique continuous random field \((X_{s,t}^{x})_{s \leq t \leq T, x \in \mathbb{R}^d}: \Theta \times [0, T]^2 : s \leq t \in [0, T]^2 \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d\) such that for all \(s \in [0, T], \)

\(x \in \mathbb{R}^d\) it holds that \((X_{s,t}^{x})_{t \in [s,T]}\) is \((\mathbb{F}_t)_{t \in [s,T]}\)-adapted and such that for all \(s \in [0, T], t \in [s,T], \)

\(x \in \mathbb{R}^d\) it holds a.s. that

\[
X_{s,t}^{x} = x + \int_s^t \mu(X_{s,r}^{x}) \, dr + \int_s^t \sigma(X_{s,r}^{x}) \, dW^r_0;
\]

(72)

iii) for all \(t \in [0, T], x \in \mathbb{R}^d\) it holds that \(E\left[ \left| g(X_{s,t}^{x}) \right| \right] + \int_s^T E\left[ \left| f(u(r, X_{s,t}^{x})) \right| \right] \, dr < \infty\) and \(u(t,x) = E\left[ g(X_{s,t}^{x}) \right] + \int_s^T E\left[ f(u(r, X_{s,t}^{x})) \right] \, dr\),

iv) it holds for all \(n, m \in \mathbb{N}\) that \(C_{n,m} \leq \max\{4a_1, a_2 n\}(5m)^n\), and

v) there exists \(n: (0, 1) \to \mathbb{N}\) such that for all \(\varepsilon, \delta \in (0, 1)\) it holds that

\[
\sup_{t \in [0,T], x \in [-1,1]^d} \left\| U_{0,m}^{\varepsilon n(\varepsilon), M(n(\varepsilon))}(t,x) - u(t,x) \right\|_p \leq \varepsilon
\]

(73)

and

\[
\varepsilon^{2 \delta} C_{n(\varepsilon), M(n(\varepsilon))} \leq (a_1 + a_2) C \left( 5 \varepsilon e^2 \left( \sqrt{T + 16 \delta} \right)^2 \left( \sqrt{T + 16 \delta} \right)^3 e^{288 \delta p c T A^q (b^2 + c^2 d)^3} \right)^{\frac{3(\delta + 2)}{2}}.
\]

(74)
Proof of Corollary 3.4. First, (71) and (70) prove (i). Next, a standard result on stochastic differential equations with Lipschitz continuous coefficients (see, e.g., [41, Theorem 4.5.1]) and (63) show (ii).

Throughout the rest of this proof let \( q, c \in \mathbb{R} \) satisfy that \( q = 8\beta p \) and \( c = 16q^2c^2 \), let \( \varphi: \mathbb{R}^d \to [1, \infty) \), \( V: [0, T] \times \mathbb{R}^d \to [1, \infty) \) satisfy for all \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that

\[
\varphi(x) = 2^q \left( b^2 + c^2 \|x\|^2 \right) ^q
\]  

(75)

and

\[
V(t, x) = \left[ 5c e^{c^2[\sqrt{T+2q}]^2T} \left( \sqrt{T} + 2q \right) ^\frac{3}{4} e^{\frac{1}{2qT} (q(x))^{\frac{1}{4}}} \right] e^{\frac{1}{2qT} (q(x))^{\frac{1}{4}}}. \]

(76)

Observe that (64), (75), and (76) show for all \( t \in [0, T] \), \( x \in \mathbb{R}^d \) that

\[
|TF(0)| + |u(t, x)| + |g(x)| \leq \left[ b^2 + c^2 \|x\|^2 \right] ^\beta \leq (\varphi(x))^{\frac{\beta}{4}} \leq V(t, x). \]

(77)

Next, (65), the fact that \( \beta, c \in [1, \infty) \), (75), and (76) show for all \( x \in \mathbb{R}^d \), \( y \in \mathbb{R}^d \setminus \{x\} \) that

\[
\frac{\|g(x) - g(y)\|}{\|x - y\|} \leq \max\{b^2, \|x\|^2, \|y\|^2\}
\]

\[
\leq 2\left( b^2 + c^2 \|x\|^2 \right)^\beta + 2\left( b^2 + c^2 \|y\|^2 \right)^\beta \leq \left( \varphi(x) \right)^{\frac{\beta}{4}} + \left( \varphi(y) \right)^{\frac{\beta}{4}} \leq \frac{V(T,x) + V(T,y)}{2\sqrt{T}}. \]

(78)

Next, (63), the fact that \( \forall A, B \in [0, \infty) \) \( : A + B \leq 2\sqrt{A^2 + B^2} \), and (76) show for all \( x \in \mathbb{R}^d \) that

\[
\|\mu(0)\| + \|\sigma(0)\| + c\|x\| \leq b + c\|x\| \leq 2 \left[ b^2 + c^2 \|x\|^2 \right] ^\frac{1}{2} = (\varphi(x))^{\frac{1}{2}}. \]

(79)

Moreover, the fact that \( \forall x \in \mathbb{R}^d \) \( : \varphi(x) = (4b^2 + 4c^2 \|x\|^2)^q \) (see (76)), the fact that \( q \geq 3 \), [39, Lemma 3.1] (applied with \( p \equiv q \), \( a \equiv 4b^2 \), \( c \equiv 2c \), \( V \equiv \varphi \) in the notation of [39, Lemma 3.1]), and the fact that \( \bar{c} = 16q^2c^2 \) show for all \( x, y \in \mathbb{R}^d \) that

\[
\left( \frac{(D\varphi)(x))(y)}{\|y\|} \right) \leq 4qc(\varphi(x)) \frac{2q-1}{2q} \|y\| \leq \bar{c}(\varphi(x)) \frac{2q-1}{2q} \|y\| \quad \text{and} \quad \left( \frac{(D^2\varphi)(x))(y,y)}{\|y\|^2} \right) \leq 16q^2c^2(\varphi(x)) \frac{2q-1}{2q} \|y\|^2 = \bar{c}(\varphi(x)) \frac{2q-1}{2q} \|y\|^2. \]

This, (79), (63), (i), and [39, Theorem 3.2] (applied with \( m \equiv d \), \( b \equiv c \), \( p \equiv 2q \), \( V \equiv \varphi \) in the notation of [39, Theorem 3.2]), the fact that \( q \geq 4 \), Jensen’s inequality, the fact that \( 1 \leq q/\beta \leq q \), and the fact that \( \forall t, s \in [0, T] \) : \( |t - s|^{1/2} \leq \sqrt{T} \) show that

(1) it holds for all \( n \in \mathbb{N} \), \( s \in [0, T] \), \( t \in [s,T] \), \( x \in \mathbb{R}^d \) that

\[
\mathbb{E} \left[ \varphi(X^{n,0,x}_{s,t}) \right] \leq e^{1.5\bar{c}|t-s|} \varphi(x), \]

(80)

(II) it holds for all \( n \in \mathbb{N} \), \( s \in [0, T] \), \( t \in [s,T] \), \( x \in \mathbb{R}^d \) that

\[
\left\| X^{n,0,x}_{s,t} - X^{x}_{s,t} \right\|_{\mathbb{F}} \leq \sqrt{2}c \left[ \sqrt{T} + 2q \right] ^\frac{3}{4} e^{c^2[\sqrt{T+2q}]^2T} \left( e^{1.5\bar{c}T} \varphi(x) \right) ^{\frac{1}{4}} \sqrt{T}, \]

(81)

and

(III) it holds for all \( n \in \mathbb{N} \), \( s, \tilde{s} \in [0, T] \), \( t \in [s, \tilde{s}, T] \), \( x, \tilde{x} \in \mathbb{R}^d \) that

\[
\left\| X^{x}_{s,t} - X^{\tilde{x}}_{\tilde{s},\tilde{t}} \right\|_{\mathbb{F}} \leq \sqrt{2}c \frac{|x - \tilde{x}|}{\|x - \tilde{x}\|} e^{c^2[\sqrt{T+2q}]^2T}
\]

\[
+ 5e^{c^2[\sqrt{T+2q}]^2T} \left( \sqrt{T} + 2q \right) ^\frac{1}{2} e^{\frac{1.5\bar{c}T}{2q} (\varphi(x))^{\frac{1}{4}} + (\varphi(\tilde{x}))^{\frac{1}{4}}} \left[ |s - \tilde{s}|^{1/2} + |t - \tilde{t}|^{1/2} \right]. \]

(82)

This, the fact that \( \forall s \in [0, T] \), \( x \in \mathbb{R}^d \) \( : \mathbb{P}(X^{x}_{s,s} = x) = 1 \), (75), and (76) show for all \( s \in [0, T] \), \( t \in [s,T] \), \( x, y \in \mathbb{R}^d \) that

\[
\left\| X^{x}_{s,t} - x \right\|_{\mathbb{F}} \leq \left\| X^{x}_{s,t} - X^{x}_{s,s} \right\|_{\mathbb{F}} \leq 5e^{c^2[\sqrt{T+2q}]^2T} \left( \sqrt{T} + 2q \right) ^\frac{1}{2} e^{\frac{1.5\bar{c}T}{2q} (\varphi(x))^{\frac{1}{4}} |t - s|^{1/2} \leq V(s,x)|t - s|^{1/2}, \]

(83)
\[ \|X_{s,t}^x - X_{s,t}^y\|_B \leq \sqrt{2}\|x - y\| + T = \frac{V(s,x)+V(s,y)}{2}\|x - y\|, \]  

(84)

and

\[ \left\| \left| \chi_{s,t}^{n,0} - X_{s,t}^x \right| \right\|_{B} \leq V(s, x) \frac{\sqrt{T}}{\sqrt{n}}. \]  

(85)

Next, (80) shows for all \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \), \( s \in [0, T] \), \( t \in [s, T] \) that

\[ \left\| \frac{\left(\frac{1}{\beta} \left(\chi_{s,t}^{n,0} \right)^2\right)}{\| \varphi(x) \|} - \frac{\left(\frac{1}{\beta} \left(\chi_{s,t}^{n,0} \right)^2\right)}{\| \varphi(x) \|} \right\|_{B} \leq e^{\frac{1}{\beta} \left(\chi_{s,t}^{n,0} \right)^2}. \]  

(86)

This, (75), and (76) show for all \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \), \( s \in [0, T] \), \( t \in [s, T] \) that \( \|V(t, \chi_{s,t}^{n,0})\| \leq V(s, x) \). This, (81), continuity of \( V \), and Fatou’s lemma show for all \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \), \( s \in [0, T] \), \( t \in [s, T] \) that \( X_{s,t}^x = \mathbb{P}\lim_{n \to \infty} \chi_{s,t}^{n,0} \) and \( V(t, X_{s,t}^x) \), and \( \mathbb{P}\lim_{n \to \infty} V(t, \chi_{s,t}^{n,0}) \) show for all \( n \in \mathbb{N} \), \( x \in \mathbb{R}^d \) that

\[ \left\| \int_{V(t,x)}^{p(t,x)} \frac{r_{l,m}^0(u(t,x))}{\sqrt{V(t,x)}} du \right\| \leq \frac{16m(p-1)^{1/2}e^{2T}e^{m^{1/2}/p}}{m^{1/2}}. \]  

(87)

Next, (69) shows for all \( n, m \in \mathbb{N} \)

\[ C_{n,m} \leq \sum_{\ell=0}^{n-1} \left[ m^{n-\ell} \left[ 4a_1 + a_2 m^\ell + C_{\ell,m} + \mathbb{I}_{N}(\ell)C_{\ell-1,m} \right] \right] \]

\[ = a_2 mn^m + \sum_{\ell=0}^{n-1} \left[ m^{n-\ell} \left[ 4a_1 + C_{\ell,m} + \mathbb{I}_{N}(\ell)C_{\ell-1,m} \right] \right] \]

\[ \leq \max\{4a_1, a_2\} mn^m + \sum_{\ell=0}^{n-1} \left[ m^{n-\ell} \left[ \max\{4a_1, a_2\} + 1 + C_{\ell,m} + \mathbb{I}_{N}(\ell)C_{\ell-1,m} \right] \right]. \]  

(88)

This and [34, Lemma 3.6] (applied for every \( n \in \mathbb{N} \) with \( d \in \mathbb{R}_{d} \cap \mathbb{M}_{\mu} \in \mathbb{Z} \cap \mathbb{C}_{\mu} \in \mathbb{Z} \) in the notation of [34, Lemma 3.6]) show for all \( n, m \in \mathbb{N} \) that \( C_{n,m} \leq \max\{4a_1, a_2\} (5m)^n \). This shows (iv).

Throughout the rest of this proof let \( (n(\varepsilon))_{\varepsilon \in (0,1)} \subseteq [0, \infty] \) satisfy for all \( \varepsilon \in (0,1) \) that

\[ n(\varepsilon) = \inf\left\{ n \in \mathbb{N} : \sup_{k \in [n(\varepsilon), \infty]} \sup_{t \in [\varepsilon, \infty]} \left\| T_{k,M}(t, x) - u(t, x) \right\|_p \leq \varepsilon \right\} \cup \{\infty\}. \]  

(89)

Observe that (87), (70), and continuity of \( V \) prove for all \( \varepsilon \in (0,1) \) that

\[ \lim_{n \to \infty} \sup_{t \in [0, T], x \in [-1,1]^d} \left\| T_{n,M}(t, x) - u(t, x) \right\|_p \]

\[ \leq \lim_{n \to \infty} \sup_{t \in [0, T], x \in [-1,1]^d} \frac{4(M(n))^{(2p-1)/p} e^{(M(n))^{1/2}/p} (V(t,x))^3}{(M(n))^{n/2}} = 0. \]  

(90)
This and (89) show that \( n(\varepsilon) \in \mathbb{N} \). Next, the definitions of \( q, \bar{c}, (75) \), and (76) show that 
\[
\frac{e^{\text{4.5}T/2}}{q_2} = e^{\text{4.5}16\beta e^{2c}T} = e^{36\beta e^{2c}T} = e^{288\beta e^{2c}T} = \sup_{x \in [-1,1]^d} \varphi(x) = 2^{2q} \left( b^2 + \varepsilon^2 \right)^q = 4^q (b^2 + c^2 d)^q, \sup_{x \in [-1,1]^d} (\varphi(x))^{\beta} = 4\beta (b + c^2 d)^\beta,
\]
and
\[
\sup_{t \in [0,T], x \in [-1,1]^d} V(t,x) = 5\frac{e^{2\sqrt{T}+2q} T}{e^{\frac{1.5\sqrt{T}}{q} - 1}} e^{\frac{1.5\sqrt{T}}{q} - 1} (\varphi(x))^{\beta} = 5 ce^{2(\sqrt{T}+16\beta p)T} \left[ \sqrt{T}+16\beta p \right]^3 e^{288\beta e^{2c}T} 4\beta (b^2 + c^2 d)^\beta.
\]
Moreover, (iv), (70), and the fact that \( \forall n \in \mathbb{N}: n + 1 \leq 2^n \) prove for all \( n \in \mathbb{N} \) that
\[
C_{n+1,M(n+1)} \leq \max \{4a_1, a_2(n+1)\} (5M(n+1))^{n+1} 
\]
\[
\leq 4(a_1 + a_2) 2^n \left[ \left( \frac{M(k+1)}{M(k)} \right) M(n) \right]^{n+1}
\]
\[
\leq 4(a_1 + a_2) 2^n \left[ \frac{M(k+1)}{M(k)} \right] M(n) \leq (a_1 + a_2) C.
\]
This and (iv) show for all \( \varepsilon \in (0,1) \) that in the case \( n(\varepsilon) = 1 \) it holds that 
\[ \varepsilon^{2+\delta} C_n(\varepsilon, M(n)) \leq C_{1,1} \leq 20(a_1 + a_2) M(1) \]
and in the case \( n(\varepsilon) \in \mathbb{N} \cap [2, \infty) \) it holds that
\[
\varepsilon^{2+\delta} C_n(\varepsilon, M(n)) \leq \sup_{t \in [0,T], x \in [-1,1]^d} \left[ \frac{\sum_{k \in \mathbb{N}} \frac{M(k+1)}{M(k)} (n+1) \varepsilon^{2+\delta} e^{(2+4)n \frac{m^2/2}{p}}}{m^{n/2}} \right] \leq (a_1 + a_2) C (V(t,x))^{3(\delta+2)}.
\]
This, the fact that \( 20M(1) \leq C \), and the fact that \( 1 \leq V \) prove for all \( \varepsilon \in (0,1) \), \( x \in \mathbb{R}^d \) that 
\[ \varepsilon^{2+\delta} C_n(\varepsilon, M(n)) \leq (a_1 + a_2) C \sup_{t \in [0,T], x \in [-1,1]^d} (V(t,x))^{3(\delta+2)}. \]
This, (89), the fact that \( \forall \varepsilon \in (0,1): n(\varepsilon) \in \mathbb{N} \), and (91) imply (v). The proof of Corollary 3.4 is thus completed. □

4 Error estimates for MLP approximations in temporal-spatial Hölder-norms

In this section we prove strong convergence rates of MLP approximations in temporal-spatial Hölder
norms; see Theorem 4.2 below. Our Hölder-norms are defined in (99) below. It turned out that it
is advantageous to weight differences of time points and differences of space points with different
monomials of the Lyapunov-type function; see the denominator of (99). With this choice of Hölder-
norm we succeeded to derive the closed recursion (132).

**Setting 4.1.** Assume Setting 2.1, let \( \Theta = \cup_{n \in \mathbb{N}} \mathbb{Z}^n \), let \( v^0: \Omega \to [0,1] \), \( \theta \in \Theta \), be i.i.d. random
variables which satisfy for all \( t \in [0,1] \) that \( P(v^0(t) \leq t) = t \), let \( (\lambda_{s,t}^{\theta,x})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d} \) \{ (s,t) \in [0,T]^2; \ s \leq t \} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \), \( \theta \in \Theta \), \( n \in \mathbb{N} \), be measurable, assume that
(\chi_{s,t}^{n,\theta,x}, \chi_{s,t}^{n,\theta,\hat{x}})_{s,\hat{s} \in [0,T], t \in [s,T], \hat{t} \in [\hat{s},T], n, \hat{n} \in \mathbb{N}, x, \hat{x} \in \mathbb{R}^d, \theta \in \Theta}$, are i.i.d. random fields, assume for all $n \in \mathbb{N}$, $s \in [0,T]$, $\hat{s}, \hat{t} \in [s,T]$, $t \in [\hat{s},T]$, $x, \hat{x} \in \mathbb{R}^d$ that $(\chi_{s,t}^{n,\theta,x}, \chi_{s,t}^{n,\theta,\hat{x}})_{\theta \in \Theta}$ and $(\chi^\theta)_{\theta \in \Theta}$ are independent and that
\[
\left\| \chi_{s,t}^{n,\theta,x} - \chi_{s,t}^{n,\theta,\hat{x}} \right\|_{p_2} \leq \frac{V(s,x) + V(\hat{s},\hat{x})}{2} \left( |s-\hat{s}|^{1/2} + |t-\hat{t}|^{1/2} \right) + c_T \|x - \hat{x}\|,
\]
and let $U_{n,m}^\theta: [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $n, m \in \mathbb{Z}$, $\theta \in \Theta$, satisfy for all $n, m \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $U_{n,m}^\theta(t, x) = U_{n,m}^{\theta,0}(t, x) = 0$ and
\[
U_{n,m}^{\theta,0}(t, x) = \sum_{\ell=0}^{n-1} \frac{1}{n^{m-\ell}} \sum_{i=1}^{m^{n-\ell}} \left[ g(\chi_{t,T}^{m,\theta,(\ell,0),i}) - \mathbb{1}_N(\ell) g(\chi_{t,T}^{m,\theta,(\ell,0),i}) \right] + \left( T - t \right) \left( t + (T-t) \chi_{t,T}^{m,\theta,(\ell,0),i} \right) \left[ \chi_{t,T}^{m,\theta,(\ell,0),i} - \mathbb{1}_N(\ell) \right].
\]

**Theorem 4.2.** Assume Setting 4.1, let $p \in [2, \infty)$, $q_1 \in [3, \infty)$, $q_2 \in [9, \infty)$ satisfy that $\frac{q_1 + 1}{p} + 2 \leq \frac{1}{p}$ and $q_1 + 2 \leq q_2$, and for every $s \in [0,T]$ and every random field $H: [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ let $\|H\|_s, \|H\|_{1,s}, \|H\|_{2,s} \in [0, \infty)$ satisfy that
\[
\|H\|_s = \max \left\{ \|H\|_{1,s}, \|H\|_{2,s} \right\}, \quad \|H\|_{1,s} = \sup_{t \in [s,T], x \in \mathbb{R}^d} \frac{\|H(t,x)\|_p}{(t-x)^n}, \quad \text{and} \quad \|H\|_{2,s} = \sup_{t_1, t_2 \in [s,T], x_1, x_2 \in \mathbb{R}^d: t_1 \neq t_2} \left( \frac{(t_1-x_1)^{n/2} + (t_2-x_2)^{n/2}}{2} \right),
\]
Then
(i) there exists a unique measurable $u: [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that
\[
\mathbb{E}[g(X_{t,T}^x)] + \int_0^T \mathbb{E}[f(u(r, X_{r,T}^x))] dr + \sup_{r \in [0,T], \xi \in \mathbb{R}^d} \mathbb{E}\left[ |u(r, \xi)| \right] < \infty \quad \text{and} \quad \mathbb{E}[g(X_{t,T}^x)] + \int_0^T \mathbb{E}[f(u(r, X_{r,T}^x))] dr,
\]
(ii) it holds for all $s \in [0,T]$ that $\|u\|_{1,s} \leq \|u\|_{2,s} \leq 8 \mathbb{C}^{T}$, and
(iii) it holds for all $n, m \in \mathbb{N}$, $t \in [0,T]$ that
\[
\|U_{n,m}^\theta - u\|_t \leq 1776mnne^{6T}e^{mp^{1/2}/p}n^{-2}7e^{4ncT}(p-1)^{1/2}.
\]

**Proof of Theorem 4.2.** Throughout this proof for every random variable $X: \Omega \to \mathbb{R}$ with $\mathbb{E}[|X|] < \infty$ let $\text{Var}_p(X) \in [0, \infty]$ satisfy that $\text{Var}_p(X) = \mathbb{E}[|X - \mathbb{E}[X]|^p]$. First, (10), (11), (96), and (97) show for all $s \in [0,T]$, $t \in [s,T]$, $x, y \in \mathbb{R}^d$, $v, w \in \mathbb{R}$, $n \in \mathbb{N}$ that
\[
|g(x) - g(y)| \leq \frac{3V(T,x) + V(T,y) - |x-y|^2}{4} + \frac{3V(T,y) + V(T,x) - |y-x|^2}{4} = \frac{V(T,x) + V(T,y) - |x-y|^2}{2},
\]
and
\[
|f(v) - f(w)| = |(f(v) - f(w)) - (f(v) - f(v))| \leq |v-w|,
\]
and

$$\left\| X_{s,t}^{n,0,x} - X_{s,t}^{x} \right\|_{p_2} \leq V(s, x) \left(\frac{|t-s|^{1/2}}{\sqrt{n}} + \frac{\sqrt{TV(s,x)}}{2\sqrt{n}}\right).$$

This, the triangle inequality, (95), and the fact that $e^{CT} \leq V$ (see (97)) prove for all $s_1, s_2 \in [0, T]$, $t_1 \in [s_1, T]$, $t_2 \in [s_2, T]$, $x_1, x_2 \in \mathbb{R}^d$ that

$$\left\| X_{s_1,t_1}^{x_1} - X_{s_2,t_2}^{x_2} \right\|_{p_2} \leq \lim_{n \to \infty} \left[ \left\| X_{s_1,t_1}^{n,0,x_1} - X_{s_1,t_1}^{x_1} \right\|_{p_2} + \left\| X_{s_2,t_2}^{n,0,x_2} - X_{s_2,t_2}^{x_2} \right\|_{p_2} + \left\| X_{s_1,t_1}^{n,0,x_1} - X_{s_2,t_2}^{n,0,x_2} \right\|_{p_2} \right] \leq V(s, x) \left(\frac{|t_1-s_1|^{1/2}|t_2-s_2|^{1/2}}{\sqrt{n}} + e^{CT} \left\| x_1 - x_2 \right\| + \frac{\sqrt{TV(s_1,x_1) \cdot TV(s_2,x_2)}}{2\sqrt{n}} \right).$$

This and (97) show for all $s \in [0, T]$, $t \in [s, T]$, $x \in \mathbb{R}^d$ that

$$\left\| X_{s,t}^x - x \right\|_{p_2} = \left\| X_{s,t}^x - X_{s,t}^{x,s} \right\|_{p_2} \leq V(s, x) |t-s|^{1/2}.$$  

Next, (103), continuity of $V$, Fatou’s lemma, and (97) yield for all $s \in [0, T]$, $t \in [s, T]$, $x \in \mathbb{R}^d$ that $\mathbb{P}\lim_{n \to \infty} X_{s,t}^{n,0,x} = X_{s,t}^x$ and

$$\left\| V(t, X_{s,t}^x) \right\|_{p_1} = \left\| \mathbb{P}\lim_{n \to \infty} V(t, X_{s,t}^{n,0,x}) \right\|_{p_1} \leq \lim_{n \to \infty} \left\| V(t, X_{s,t}^{n,0,x}) \right\|_{p_1} \leq V(s, x).$$

This, (105), the assumptions of Theorem 4.2, and Lemma 2.2 prove that

(a) there exists a unique measurable $u: [0, T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that

$$\mathbb{E}\left[g(X_{T,t}^x)\right] + \int_0^T \mathbb{E}\left[|f(u(t, X_{s,t}^x))|\right] dt + \sup_{r \in [0,T], x \in \mathbb{R}^d} \frac{|u(r,x)|}{\mathbb{E}[g(X_{r,t}^x) + \int_0^T \mathbb{E}[|f(u(t, X_{s,t}^x))|] dt]} < \infty$$

and $u(t, x) = \mathbb{E}[g(X_{t,t}^x) + \int_0^T \mathbb{E}[|f(u(t, X_{s,t}^x))|] dt]$, $t \in [0, T]$.

(b) it holds for all $s \in [0, T]$, $t \in [s, T]$, $x, y \in \mathbb{R}^d$ that

$$|u(s, x) - u(t, y)| \leq 4e^{CT} \left(\frac{V(s, x) + V(t, y)}{2}\right)^{1/2} \sqrt{|t-s|^{1/2} + \|x-y\|}$$

and

$$|u(t, x)| \leq 2e^{CT} V(t, x),$$

and

(c) it holds for all $s \in [0, T]$, $t \in [s, T]$, $x, y, \tilde{x}, \tilde{y} \in \mathbb{R}^d$ that

$$\left| \left| f(u(s, x)) - f(u(t, y)) \right| - \left| f(u(t, \tilde{x})) - f(u(t, \tilde{y})) \right| \right| \leq 896ce^{CT} \left(\frac{V(s, x) + V(t, y) + V(t, \tilde{x}) + V(t, \tilde{y})}{4}\right)^7 \left(\frac{|x-y-(\tilde{x}-\tilde{y})|}{\sqrt{T}} + \frac{|x-y|}{\sqrt{T}} + \frac{\|x-y\|}{\sqrt{T}} \right).$$

This implies (i).

Next, (107) show for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $|u(t, x)| \leq 2e^{CT} V(t, x) \leq 2e^{CT} (V(t, x))^{\alpha_1}$. This and (99) imply for all $t \in [0, T]$ that $\|u\|_{1,1} \leq 2e^{CT}$. Next, (107), Jensen’s inequality, and the fact that $V^2 \leq V_{q_2}$ imply for all $s \in [0, T]$, $t_1 \in [s, T]$, $t_2 \in [t_1, T]$, $x_1, x_2 \in \mathbb{R}^d$ that

$$|u(t_1, x_1) - u(t_2, x_2)| \leq 4e^{2CT} \left(\frac{V(t_1, x_1) + V(t_2, x_2)}{2}\right)^{1/2} \left(\frac{|t_1-t_2|^{1/2}}{\sqrt{T}} + \frac{\|x_1-x_2\|}{\sqrt{T}} \right) \leq 8e^{2CT} \left(\frac{V(t_1, x_1) + V(t_2, x_2)}{2}\right)^{1/2} \left(\frac{|t_1-t_2|^{1/2}}{\sqrt{T}} + \frac{\|x_1-x_2\|}{\sqrt{T}} \right).$$

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This and (99) show that \( \sup_{t \in [0, T]} \| u \|_{2, t} \leq 8e^{2cT} \). This the fact that \( \sup_{t \in [0, T]} \| u \|_{1, t} \leq 2e^{cT} \) imply that \( \sup_{t \in [0, T]} \| u \|_{t} \leq 8e^{2cT} \). This shows (ii).

Next, (102), the fact that \( \forall s, t \in [0, T]: |s - t| \leq |s - t|^{1/2} \sqrt{T} \), (109) and prove for all \( s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d \) that

\[
| (t - s) \left[ f(u(x_s)) - f(u(s, y)) \right] | \\leq c \|t - s\|^{1/2} \sqrt{T} \left( \frac{V(s, x) + V(s, y)}{2} \right)^2 \frac{\|x - y\|}{\sqrt{T}} = 4cT e^{2cT} \left( \frac{V(s, x) + V(s, y)}{2} \right)^2 \frac{\|x - y\|}{\sqrt{T}} (110)
\]

This, the triangle inequality, (108), and the fact that \( 1 \leq V \) show for all \( s \in [0, T], t \in [s, T], x, y \in \mathbb{R}^d \) that

\[
| (T - s) \left[ f(u(s, x)) - f(u(s, y)) \right] - (T - t) \left[ f(u(t, x)) - f(u(t, y)) \right] | \\leq | (T - t) \left[ (f(u(s, x)) - f(u(s, y))) - (f(u(t, x)) - f(u(t, y))) \right] | \\leq (t - s) | f(u(x_s)) - f(u(s, y)) | + | (t - s) f(u(s, x)) - f(u(s, y)) | \\leq T \cdot 896c e^{6cT} \left( \frac{V(s, x) + V(s, y) + V(t, x) + V(t, y)}{4} \right)^7 \\cdot \left( \frac{\|x - y\|^2}{\sqrt{T}} + \frac{\|x - y\|^2}{\sqrt{T}} + \frac{\|x - y\|^2}{\sqrt{T}} \right)^2 \frac{(V(s, x) + V(s, y)) | t - s |^{1/2}}{\sqrt{T}} (111)
\]

This, the triangle inequality, Hölder’s inequality, the fact that \( \max \{ \frac{1}{p_1}, \frac{2}{p_2}, \frac{8}{p_1} + \frac{1}{p_2} \} \leq \frac{8}{p_1} + \frac{2}{p_2} \leq 1, \) (97), (106), (96), (103), (104), the fact that \( 1 \leq V \), the fact that \( 9 \leq q_2 \), and Jensen’s inequality show that for all \( t_1 \in [0, T], t_2, r_1 \in [t_1, T], r_2 \in [t_2, T] \cap [r_1, T], x_1, x_2 \in \mathbb{R}^d, n \in \mathbb{N} \) with \( |r_1 - r_2| \leq |t_1 - t_2| \) it holds that \( 900cT e^{6cT} \). 6 \leq 5400e^{-1} e^{cT} e^{6cT} \leq 1987e^{7cT} \) and

\[
\left\| (T - t_1) \left[ (f \circ u)(r_1, X_{t_1, t_1}^{r_1, x_1}) - (f \circ u)(r_1, X_{t_1, t_1}^{r_1, x_1}) \right] - (T - t_2) \left[ (f \circ u)(r_2, X_{t_2, t_2}^{r_2, x_2}) - (f \circ u)(r_2, X_{t_2, t_2}^{r_2, x_2}) \right] \right\|_p \\
\leq 900cT e^{6cT} \left( \sum_{i=1}^{\infty} \left| V(r_1, X_{t_1, t_1}^{r_1, x_1}) + V(r_1, X_{t_1, t_1}^{r_1, x_1}) \right| \right)^7 \left( \left| \left( X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right) \right| \right) \left\| (f \circ u)(r_1, X_{t_1, t_1}^{r_1, x_1}) - (f \circ u)(r_2, X_{t_2, t_2}^{r_2, x_2}) \right\|_p \\
+ \sum_{i=1}^{\infty} \left| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right| \left| V(r_1, X_{t_1, t_1}^{r_1, x_1}) + V(r_1, X_{t_1, t_1}^{r_1, x_1}) \right| \left| r_1 - r_2 \right|^{1/2} \left\| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right\|_p \\
\leq 900cT e^{6cT} \left( \sum_{i=1}^{\infty} \left| V(r_1, X_{t_1, t_1}^{r_1, x_1}) + V(r_1, X_{t_1, t_1}^{r_1, x_1}) \right| \right)^7 \left( \left| \left( X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right) \right| \right) \left\| (f \circ u)(r_1, X_{t_1, t_1}^{r_1, x_1}) - (f \circ u)(r_2, X_{t_2, t_2}^{r_2, x_2}) \right\|_p \\
+ \sum_{i=1}^{\infty} \left| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right| \left| V(r_1, X_{t_1, t_1}^{r_1, x_1}) + V(r_1, X_{t_1, t_1}^{r_1, x_1}) \right| \left| r_1 - r_2 \right|^{1/2} \left\| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right\|_p \\
\leq 900cT e^{6cT} \left( \frac{V(t_1, x_1) + V(t_2, x_2)}{2} \right)^7 \left[ \left( V(t_1, x_1) + V(t_2, x_2) \right) \left| t_1 - t_2 \right|^{1/2} + \left| x_1 - x_2 \right| \right] \left| r_1 - r_2 \right|^{1/2} \left\| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right\|_p \\
+ \frac{V(t_1, x_1) + V(t_2, x_2)}{2V_n} \left| t_1 - t_2 \right|^{1/2} + \frac{V(t_1, x_1) + V(t_2, x_2)}{2V_n} \left| t_1 - t_2 \right|^{1/2} \left\| x_1 - x_2 \right\|_p \\
\leq 900cT e^{6cT} \left( \frac{V(t_1, x_1) + V(t_2, x_2)}{2} \right)^8 \left[ \left| t_1 - t_2 \right|^{1/2} + \left| x_1 - x_2 \right| \right] \left| r_1 - r_2 \right|^{1/2} \left\| X_{t_1, t_1}^{r_1, x_1} - X_{t_1, t_1}^{r_1, x_1} \right\|_p \\
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Next, (11), the triangle inequality, (99), (109), Hölder’s inequality, the fact that \( q_1 + 2 \leq q_2 \), and the fact that \( 1 \leq V \) prove for all \( s \in [0, T] \), \( t_1 \in [s, T] \), \( t_2 \in [t_1, T] \), \( x_1, x_2 \in \mathbb{R}^d \), \( U \in \{ U_{n,m}^\theta : n, m \in \mathbb{Z}, \theta \in \Theta \} \) that \( \| (f \circ U) - (f \circ u) \|_{1,s} \leq c \| U - u \|_{1,s} \) and

\[
\begin{align*}
&\| (f(u(t_1, x_1)) - f(U(t_1, x_1))) - (f(u(t_2, x_2)) - f(U(t_2, x_2))) \|_2 \\
&\leq c \| (u(t_1, x_1) - U(t_1, x_1)) - (u(t_2, x_2) - U(t_2, x_2)) \|_2 \\
&\leq c \| (V(t_1, x_1))^{q_1} + (V(t_2, x_2))^{q_1} \|_{2, t_1,t_2}^{2} \| U - u \|_{1,s} \| V(t_1, x_1) + V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| V(t_1, x_1) - V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| \| x_1 - x_2 \|_{T}^{2} \\
&\leq 8c^{2eT} (V(t_1, x_1))^{q_1} + (V(t_2, x_2))^{q_1} \| U - u \|_{1,s} \| V(t_1, x_1) + V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| V(t_1, x_1) - V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| \| x_1 - x_2 \|_{T}^{2} \\
&\leq 8c^{2eT} (V(t_1, x_1))^{q_1} + (V(t_2, x_2))^{q_1} \| U - u \|_{1,s} \| V(t_1, x_1) + V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| V(t_1, x_1) - V(t_2, x_2) \|_{2, t_1,t_2}^{2} \| \| x_1 - x_2 \|_{T}^{2}.
\end{align*}
\]

Moreover, the fact that \( \forall x \in [0, \infty) : x \leq e^{-1}e^{x} \) shows that \( c + 8c^{2eT} \leq 9c^{2eT} \leq \frac{1}{4}c^{2eT} \cdot e^{2cT} \leq \frac{3c^{3eT}}{4c^{2eT}}. \) This, (113), (99), and symmetry prove for all \( s \in [0, T] \), \( U \in \{ U_{n,m}^\theta : n, m \in \mathbb{Z}, \theta \in \Theta \} \) that \( \| (f \circ U) - (f \circ u) \|_{2,s} \leq c \| U - u \|_{2,s} + 8c^{2eT} \| u - U \|_{1,s} \) and

\[
\| (f \circ U) - (f \circ u) \|_{s} \leq \frac{3c^{2eT}}{4c^{3eT}} \| U - u \|_{s}.
\]

Next, the triangle inequality, (99), and the fact that \( 1 \leq V^{q_2} \) show that for all \( s \in [0, T] \), \( t_1, t_2 \in [s, T] \), \( r_1 \in [t_1, T] \), \( r_2 \in [t_2, T] \), \( r \in [0, \min\{r_1, r_2\}] \), \( \xi_1, \xi_2 \in \mathbb{R}^d \), random fields \( H : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) with \( |r_1 - r_2| \leq |t_1 - t_2| \) and \( V(r_1, \xi_1) \leq V(r_2, \xi_2) \) it holds that

\[
|t_2 - t_1| \leq T \left( \frac{r - t_1}{T} \right)^{\frac{1}{p'}} \| |t_2 - t_1|^{1/2} \right) = T \left( \frac{r - t_1}{T} \right)^{\frac{1}{p'}} \| |t_2 - t_1|^{1/2}
\]

and

\[
\begin{align*}
&\| (T - t_1) H(r_1, \xi_1) - (T - t_2) H(r_2, \xi_2) \|_p \\
&\leq \| (T - t_2) H(r_1, \xi_1) - H(r_2, \xi_2) \|_p + \| (T - t_2) H(r_1, \xi_1) \|_p \\
&\leq T^{-\frac{1}{p'}} (T - s)^{\frac{1}{p'}} \left( |V(r_1, \xi_1)|^{q_2} + |V(r_2, \xi_2)|^{q_2} \right) \left[ \| |r_2 - r_1|^{1/2} \right] + \| |x_1 - x_2| \|_T \| \| H \|_{1,r} \\
&\leq T^{-\frac{1}{p'}} (T - s)^{\frac{1}{p'}} \| |r_2 - r_1|^{1/2} V(r_1, \xi_1) \|_{1,r} + \| |x_1 - x_2| \|_T \| \| H \|_{1,r} \\
&\leq T^{-\frac{1}{p'}} (T - s)^{\frac{1}{p'}} \| |r_2 - r_1|^{1/2} V(r_1, \xi_1) \|_{1,r} + \| |x_1 - x_2| \|_T \| \| H \|_{1,r}.
\end{align*}
\]

This, symmetry, the disintegration theorem (see, e.g., [34, Lemma 2.2]), the assumption on measurability and independence, the triangle inequality, Hölder’s inequality, the fact that \( \max\{ \frac{1}{p'}, \frac{1}{p'}, \frac{1}{p'} + \frac{1}{p'} \} \leq \frac{1}{p'} (97) \), and (95) show that for all \( s \in [0, T] \), \( t_1, t_2 \in [s, T] \), \( x_1, x_2 \in \mathbb{R}^d \), \( r_1 \in [t_1, T] \), \( r_2 \in [t_2, T] \), \( r \in [0, \min\{r_1, r_2\}] \), \( m, n \in \mathbb{N} \), \( \ell, \nu \in \mathbb{Z} \), \( H \in \text{span}_{\mathbb{N}}\{ (f \circ U_{\ell,m}^\nu, f \circ u) \} \) with \( |r_1 - r_2| \leq |t_1 - t_2| \) it holds that

\[
\begin{align*}
&\| (T - t_1) H(r_1, \xi_1, x_1, r_1, x_1) - (T - t_2) H(r_2, \xi_2, x_2) \|_p \\
&= \| (T - t_1) H(r_1, \xi_1, x_1, r_1, x_1) - (T - t_2) H(r_2, \xi_2, x_2) \|_p.
\end{align*}
\]
\[
\begin{align*}
&\leq T^{-\frac{1}{p}} (T-s)^{\frac{1}{p}} \left\| \left[ \| H \|_p, \frac{(V(r_1, \xi_1))^2 + (V(r_2, \xi_2))^2}{2} \left( 9 \frac{V(r_1, \xi_1) + V(r_2, \xi_2)}{2 \sqrt{T}} \| t_1 - t_2 \|^{1/2} + \| \xi_1 - \xi_2 \| \right) \right]_{\xi_1 = \nu_1, \xi_2 = \nu_2, r_1, r_2} \right\|_r \\
&\leq T^{-\frac{1}{p}} (T-s)^{\frac{1}{p}} \left\| H \|_p \left( \frac{1}{2} \sum_{i=1}^2 \left\| V(r_i, \mathcal{X}^{n,0,x_i}_i) \right\|_p \right) \right. \\
&\quad \cdot \left[ 2 \left( \frac{1}{2} \sum_{i=1}^2 \left\| V(r_i, \mathcal{X}^{n,0,x_i}_i) \right\|_p \right) \left\| t_1 - t_2 \|^{1/2} + \left\| \mathcal{X}^{n,0,x_1}_1 - \mathcal{X}^{n,0,x_2}_2 \right\|_r \right. \\
&\leq T^{-\frac{1}{p}} (T-s)^{\frac{1}{p}} \left\| H \|_p \left( \frac{2}{V(t_1,x_1) + V(t_2,x_2)} \left\| t_1 - t_2 \|^{1/2} + \frac{\| V(t_1,x_1) + V(t_2,x_2) \|_p \| t_1 - t_2 \|^{1/2} + \| r_1 - r_2 \|^{1/2}}{2 \sqrt{T}} + \frac{\| x_1 - x_2 \|}{\sqrt{T}} \right) \\
&\leq 3e^{CT} T^{-\frac{1}{p}} \left\| (T-s)^{\frac{1}{p}} \| H \|_p \left( \frac{2}{V(t_1,x_1) + V(t_2,x_2)} \left\| t_1 - t_2 \|^{1/2} + \frac{\| V(t_1,x_1) + V(t_2,x_2) \|_p \| t_1 - t_2 \|^{1/2} + \| x_1 - x_2 \|}{\sqrt{T}} \right) \right. \\
\end{align*}
\]

Next, a telescoping sum argument shows that for all \( \ell, m \in \mathbb{N}, t_1, t_2 \in [0, T], r_1 \in [t_1, T], r_2 \in [t_2, T], x_1, x_2 \in \mathbb{R}^d \) that

\[
\begin{align*}
(T-t_1) \left[ \left( f \circ U_{t_\ell}^0 \right)(r_1, \mathcal{X}^{m,f,0,x_1}_{t_\ell, r_1}) - \left( f \circ U_{t_\ell-1, m}^1 \right)(r_1, \mathcal{X}^{m,f,-1,0,x_1}_{t_\ell-1, r_1}) \right] \\
- (T-t_2) \left[ \left( f \circ U_{t_\ell}^0 \right)(r_2, \mathcal{X}^{m,f,0,x_2}_{t_\ell, r_1}) - \left( f \circ U_{t_\ell-1, m}^1 \right)(r_2, \mathcal{X}^{m,f,-1,0,x_2}_{t_\ell-1, r_1}) \right] \\
= \left\{ (T-t_1) \left[ \left( f \circ U_{t_\ell}^0 \right)(r_1, \mathcal{X}^{m,f,0,x_1}_{t_\ell, r_1}) - \left( f \circ U_{t_\ell}^0 \right)(r_2, \mathcal{X}^{m,f,0,x_2}_{t_\ell, r_1}) \right] \\
- (T-t_2) \left( f \circ U_{t_\ell}^1 \right)(r_1, \mathcal{X}^{m,f,-1,0,x_1}_{t_\ell-1, r_1}) - \left( f \circ U_{t_\ell}^1 \right)(r_2, \mathcal{X}^{m,f,-1,0,x_2}_{t_\ell-1, r_1}) \right\} \\
+ \left\{ (T-t_1) \left[ \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \\
- (T-t_2) \left[ \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \right\} \\
- \left\{ (T-t_1) \left[ \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) - \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) \right] \\
- (T-t_2) \left[ \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \right\} \\
= \left\{ (T-t_1) \left[ \left( f \circ U_{t_\ell}^0 \right)(r_1, \mathcal{X}^{m,f,0,x_1}_{t_\ell, r_1}) - \left( f \circ U_{t_\ell}^0 \right)(r_2, \mathcal{X}^{m,f,0,x_2}_{t_\ell, r_1}) \right] \\
- (T-t_2) \left( f \circ U_{t_\ell}^1 \right)(r_1, \mathcal{X}^{m,f,-1,0,x_1}_{t_\ell-1, r_1}) - \left( f \circ U_{t_\ell}^1 \right)(r_2, \mathcal{X}^{m,f,-1,0,x_2}_{t_\ell-1, r_1}) \right\} \\
+ \left\{ (T-t_1) \left[ \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \\
- (T-t_2) \left[ \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \right\} \\
- \left\{ (T-t_1) \left[ \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) - \left( f \circ u \right)(r_1, X_{t_\ell, x_1}) \right] \\
- (T-t_2) \left[ \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) - \left( f \circ u \right)(r_2, X_{t_\ell, x_2}) \right] \right\}.
\end{align*}
\]
\[
= \sum_{\nu=0}^{1} \left[ 9e^{4cT} T^{\nu-1/p} \left\| (T-s)^{1/p} \left[ U_{\nu,T,m} - u \right] \right\|_{
u} + \frac{1987e^{7cT}}{\sqrt{m+1}} \right].
\]

Next, \((10)\), the triangle inequality, Hölder’s inequality, the fact that \(\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p}\), \((97)\), \((106)\), \((96)\), \((103)\), \((104)\), the fact that \(1 \leq V\), the fact that \(3 \leq q_2\), and Jensen’s inequality show for all \(n \in \mathbb{N}\), \(t_1, t_2 \in [0, T]\), \(x_1, x_2 \in \mathbb{R}^d\) that

\[
\left\| g(\lambda^{n,0,x_1}_{t_1,T}) - g(X^{x_1}_{t_1,T}) \right\|_p \leq \left\| \sqrt{V(t_1)} \left( \lambda^{n,0,x_1}_{t_1,T} + V(t_1) X^{x_1}_{t_1,T} + V(T) X^{x_1}_{t_2,T} \right) \right\|_p + \left\| \lambda^{n,0,x_1}_{t_1,T} - X^{x_1}_{t_2,T} \right\|_p
\]

\[
\leq \left\| \sqrt{V(t_1)} \left( \lambda^{n,0,x_1}_{t_1,T} + V(T) X^{x_1}_{t_2,T} \right) \right\|_p + \left\| \lambda^{n,0,x_1}_{t_1,T} - X^{x_1}_{t_2,T} \right\|_p
\]

\[
\leq \frac{V(t_1,x_1)}{2} \left[ \left\| V(t_1) \left( \lambda^{n,0,x_1}_{t_1,T} \right) \right\|_p + 1 \right] \leq \left( \frac{V(t_1,x_1)}{2} \right)^{1/2} \left[ \left\| \lambda^{n,0,x_1}_{t_1,T} \right\|_p + 1 \right]
\]

\[
\leq \left( \frac{V(t_1,x_1)}{2} \right)^{1/2} \left[ \left\| \lambda^{n,0,x_1}_{t_1,T} \right\|_p + 1 \right] = \left( \frac{V(t_1,x_1)}{2} \right)^{1/2} \left[ \left\| \lambda^{n,0,x_1}_{t_1,T} \right\|_p + 1 \right]
\]

\[
= \sum_{\nu=0}^{1} \left[ 9e^{4cT} T^{\nu-1/p} \left\| (T-s)^{1/p} \left[ U_{\nu,T,m} - u \right] \right\|_{
u} + \frac{1987e^{7cT}}{\sqrt{m+1}} \right].
\]

This and the triangle inequality show for all \(\ell, m, n \in \mathbb{N}\), \(t_1, t_2 \in [0, T]\), \(x_1, x_2 \in \mathbb{R}^d\) that

\[
\left\| \left( g(\lambda^{n,0,x_1}_{t_1,T}) - g(X^{x_1}_{t_1,T}) \right) - \left( g(\lambda^{n,0,x_2}_{t_2,T}) - g(X^{x_2}_{t_2,T}) \right) \right\|_p \leq \sum_{\nu=0}^{1} \left[ 9e^{4cT} T^{\nu-1/p} \left\| (T-s)^{1/p} \left[ U_{\nu,T,m} - u \right] \right\|_{
u} + \frac{1987e^{7cT}}{\sqrt{m+1}} \right]
\]

\[
= \sum_{\nu=0}^{1} \left[ 9e^{4cT} T^{\nu-1/p} \left\| (T-s)^{1/p} \left[ U_{\nu,T,m} - u \right] \right\|_{
u} + \frac{1987e^{7cT}}{\sqrt{m+1}} \right].
\]

Next, \((101)\), Hölder’s inequality, the fact that \(\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p}\), \((97)\), \((95)\), the fact that \(2 \leq q_2\), and the fact that \(e^{cT} \leq V\) show for all \(n \in \mathbb{N}\), \(t_1, t_2 \in [0, T]\), \(x_1, x_2 \in \mathbb{R}^d\) that

\[
\left\| g(\lambda^{1,0,x_1}_{t_1,T}) - g(\lambda^{1,0,x_2}_{t_2,T}) \right\|_p \leq \left\| \sqrt{V(t_1)} \left( \lambda^{1,0,x_1}_{t_1,T} + V(t_1) X^{1,0,x_2}_{t_2,T} \right) \right\|_p + \left\| \lambda^{1,0,x_1}_{t_1,T} - X^{1,0,x_2}_{t_2,T} \right\|_p
\]

\[
= \left( \sqrt{V(t_1)} \right)^{1/2} \left[ \left\| \lambda^{1,0,x_1}_{t_1,T} \right\|_p + 1 \right] \leq \left( \sqrt{V(t_1)} \right)^{1/2} \left[ \left\| \lambda^{1,0,x_1}_{t_1,T} \right\|_p + 1 \right]
\]

\[
= \sum_{\nu=0}^{1} \left[ 9e^{4cT} T^{\nu-1/p} \left\| (T-s)^{1/p} \left[ U_{\nu,T,m} - u \right] \right\|_{
u} + \frac{1987e^{7cT}}{\sqrt{m+1}} \right].
\]

Next, Proposition 3.3, \((9)\), \((102)\), \((101)\), \((12)\), \((105)\), \((104)\), \((103)\), \((97)\), and the assumptions of Theorem 4.2 prove for all \(n, m \in \mathbb{N}\), \(t \in [0, T]\), \(x \in \mathbb{R}^d\) that \(U_{n,m}(t, x)\), \(g(\lambda^{m-1,0,x}_{t,T})\), and
\[(f \circ U_{n,m}^0) (t + (T-t)v^0, \mathcal{X}_{t,T}^{m^{n-1},0,x}) \text{ are integrable,}
\]

\[
\mathbb{E} \left[ U_{n,m}^0 (t, x) \right] = \mathbb{E} \left[ g(\mathcal{X}_{t,T}^{m^{n-1},0,x}) \right] + (T-t) \mathbb{E} \left[ (f \circ U_{n,m}^0) (t + (T-t)v^0, \mathcal{X}_{t,T}^{m^{n-1},0,x}) \right],
\]

\[
u(t, x) = \mathbb{E} \left[ g(X_{t,T}^x) \right] + (T-t) \mathbb{E} \left[ (f \circ u) (t + (T-t)v^0, X_{t,T}^x) \right].
\]

and

\[
\|U_{n,m}^0 - u\|_{1,t} \leq \frac{16mn^{3}\sqrt{p}}{\sqrt{T}} + \frac{n-1}{2} \left[ \frac{4(T-t)^{-3/2} \sqrt{p}}{\sqrt{T}} \left[ \int_1^T \|U_{r,m}^0 - u\|_{1,\xi}^p d\xi \right]^{1/p} \right].
\]

This, the disintegration theorem (see, e.g., [34, Lemma 2.2]), and the independence assumptions show for all \( n, m \in \mathbb{N}, t_1, t_2 \in [0, T], x_1, x_2 \in \mathbb{R}^d \) that

\[
\left| \mathbb{E} \left[ (U_{n,m}^0(t_1, x_1) - U_{n,m}^0(t_2, x_2) - u(t_1, x_1) + u(t_2, x_2)) \right] \right|
\]

\[= \mathbb{E} \left[ \left( g(\mathcal{X}_{t_1,T}^{m^{n-1},0,x_1}) - g(X_{t_1,T}^{x_1}) \right) - \left( g(\mathcal{X}_{t_2,T}^{m^{n-1},0,x_2}) - g(X_{t_2,T}^{x_2}) \right) \right]
\]

\[
+ \mathbb{E} \left[ \left( T-t_1 \right) \left( f \circ U_{n,m}^0(t, \mathcal{X}_{t,T}^{m^{n-1},0,x_1}) - f \circ u \right) \right] \left( r_1, \mathcal{X}_{t_1,T}^{m^{n-1},0,x_1} \right)
\]

\[
- \left( T-t_2 \right) \left( f \circ U_{n,m}^0(t, \mathcal{X}_{t,T}^{m^{n-1},0,x_2}) - f \circ u \right) \right] \left( r_2, \mathcal{X}_{t_2,T}^{m^{n-1},0,x_2} \right)
\]

\[
+ \left( T-t_1 \right) \left( f \circ u \right) \left( r_1, \mathcal{X}_{t_1,T}^{m^{n-1},0,x_1} \right)
\]

\[
- \left( T-t_2 \right) \left( f \circ u \right) \left( r_2, \mathcal{X}_{t_2,T}^{m^{n-1},0,x_2} \right) \right] \left| \left| r_1 = t_1 + (T-t_1)x_1 \right| \right| \left| r_2 = t_2 + (T-t_2)x_2 \right| \lambda = \nu \right). \quad (126)
\]

This, the triangle inequality, Jensen’s inequality, (120), (117), the fact that \( \forall \lambda \in [0, 1], t_1, t_2 \in [0, T]: |(t_1 + (T-t_1)\lambda) - (t_2 + (T-t_2)\lambda)| \leq |t_1 - t_2| \), (112), (114), the fact that \( 2 + 1987e^{7T} \leq 1989e^{7T} \), and the fact that \( 3e^{7T} \frac{3e^{3T}}{p} = 9e^{4T}T^{-1/p} \) show for all \( n, m \in \mathbb{N}, s \in [0, T], t_1 \in [s, T], t_2 \in [t_1, T], x_1, x_2 \in \mathbb{R}^d \) that

\[
\left| \mathbb{E} \left[ (U_{n,m}^0(t_1, x_1) - U_{n,m}^0(t_2, x_2) - u(t_1, x_1) + u(t_2, x_2)) \right] \right|
\]

\[
\leq \frac{2}{\sqrt{mn}} + 3e^{7T} \frac{3e^{3T}}{p} \left( T-s \right)^{1/p} \left| \left| U_{n,m}^0 \right| \right| _{1,\xi} \left| \left| u \right| \right| _{1,\xi} + \frac{1987e^{7T}}{\sqrt{mn}}
\]

\[
\leq 3e^{7T} \frac{3e^{3T}}{p} \left( T-s \right)^{1/p} \left| \left| U_{n,m}^0 \right| \right| _{1,\xi} \left| \left| u \right| \right| _{1,\xi} + \frac{1987e^{7T}}{\sqrt{mn}}
\]

\[
= 9e^{4T}T^{-1/p} \left( T-s \right)^{1/p} \left| \left| U_{n,m}^0 \right| \right| _{1,\xi} \left| \left| u \right| \right| _{1,\xi} + \frac{1989e^{7T}}{\sqrt{mn}}
\]

(127)

Next, the triangle inequality, and the independence and distributional properties show for all \( n, m \in \mathbb{N}, t_1, t_2 \in [0, T], x_1, x_2 \in \mathbb{R}^d \) that

\[
\left( \text{Var}_p \left( U_{n,m}^0(t_1, x_1) - U_{n,m}^0(t_2, x_2) \right) \right)^{1/p}
\]

\[
\leq \frac{1}{2} \sum_{\ell=0}^{n-1} \left( \text{Var}_p \left[ \frac{1}{m-n} \sum_{i=1}^{m-n} \left[ \left( g(\mathcal{X}_{t_1,T}^{m^{\ell-1},0,x_1}) - 1_{N}(f) g(\mathcal{X}_{t_1,T}^{m^{\ell-1},0,x_1}) \right) \right. \right.
\]

\[
- \left. \left. \left( g(\mathcal{X}_{t_2,T}^{m^{\ell-1},0,x_2}) - 1_{N}(f) g(\mathcal{X}_{t_2,T}^{m^{\ell-1},0,x_2}) \right) \right] \right) \right)^{1/p}
\]

\[
+ \sum_{\ell=0}^{n-1} \left( \text{Var}_p \left[ \frac{1}{m-n} \sum_{i=1}^{m-n} \left[ \left( T-t_1 \right) \left( f \circ U_{t_1,m}^{0,0,i} \right) \right. \right.
\]

\[
\left. \left. \left( t_1 + (T-t_1)v^0, \mathcal{X}_{t_1,T}^{m^{\ell-1},0,x_1} \right) \right] \right) \right) \right)^{1/p}
\]

\[
+ \sum_{\ell=0}^{n-1} \left( \text{Var}_p \left[ \frac{1}{m-n} \sum_{i=1}^{m-n} \left[ \left( T-t_2 \right) \left( f \circ U_{t_2,m}^{0,0,i} \right) \right. \right.
\]

\[
\left. \left. \left( t_2 + (T-t_2)v^0, \mathcal{X}_{t_2,T}^{m^{\ell-1},0,x_2} \right) \right] \right) \right)^{1/p}
\]

\[
+ \sum_{\ell=0}^{n-1} \left( \text{Var}_p \left[ \frac{1}{m-n} \sum_{i=1}^{m-n} \left[ \left( T-t \right) \left( f \circ U_{t,m}^{0,0,i} \right) \right. \right.
\]

\[
\left. \left. \left( t + (T-t)v^0, \mathcal{X}_{t,T}^{m^{\ell-1},0,x} \right) \right] \right) \right)^{1/p}
\]

\[
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\]
That fact that

\[
\sum_{t_1=t_2}^{n} \left| \left| (T - t_1) \left( f \circ U_{t_1}^{(0, t_1)} \right) \left( t_1 + (T - t_1) \mathbf{1}_{X_1}^{m^j, (0, t_1), x_1} \right) \right| \right|_p = 0, \quad \sum_{t_2}^{n} \left( T - t_2 \right) \left( f \circ U_{t_2}^{(0, t_2)} \right) \left( t_2 + (T - t_2) \mathbf{1}_{X_2}^{m^j, (0, t_2), x_2} \right) \right) \right|_p \]

\[
\left\| \mathbf{1}_{X_1}^{m^j, (0, t_1), x_1} \right\|_p = \sqrt[1/p]{ \mathbb{E} \left[ (T - t_1)^{1/p} \left( T - s \right)^{1/p} \left\| T_0^{j, m} - u \right\|_{s + (T - s)^{1/p}} \right] + 1989e^{T^{1/p}}} \right).}

(129)

This, the triangle inequality, (127), and the fact that \( \forall n, m \in \mathbb{N}, a_0, a_1, \ldots, a_{n-1} \in [0, \infty) : a_{n-1} + \sum_{j=1}^{n-1} \frac{a_j}{\sqrt{m^{n-j}}} \leq \sum_{j=1}^{n-1} \frac{2a_j}{\sqrt{m^{n-j}}}, \) prove for all \( n, m \in \mathbb{N}, s \in [0, T], t_1, t_2 \in [s, T], x_1, x_2 \in \mathbb{R}^d \) that

\[
\left\| \left( U_{n,m}^0 \left( t_1, x_1 \right) - u(t_1, x_1) \right) \right\|_p \leq \left( \mathbb{E} \left[ \left( U_{n,m}^0 \left( t_1, x_1 \right) - u(t_1, x_1) \right) \left( U_{n,m}^0 \left( t_2, x_2 \right) - u(t_2, x_2) \right) \right) \right] \right) \leq \left\| \mathbf{1}_{X_1}^{m^j, (0, t_1), x_1} \right\|_p \]
\[ 2\sqrt{p-1} + 4n\sqrt{p-1}\sqrt{m}1989e^{7cT} \leq 7958\sqrt{p-1} \cdot 9ne^{4cT} \text{ show for all } n, m \in \mathbb{N}, s \in [0, T] \text{ that} \]

\[ \|U_{n,m}^0 - u\|_{2,s} \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT}}{\sqrt{m}} + \sum_{\ell=0}^{n-1} \frac{4\sqrt{p-1}}{\sqrt{m}^{\ell+1}} 9e^{4cT} T^{-1/p} \left( f_s^T \|U_{\ell,m}^0 - u\|_{p,\zeta}^p \right)^{\frac{1}{p}}. \tag{131} \]

This, (125), and (99) prove for all \( n, m \in \mathbb{N}, s \in [0, T] \) that

\[ \|T_{n,m}^0 - u\|_s \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT}}{\sqrt{m}} + \sum_{\ell=0}^{n-1} \frac{4\sqrt{p-1}}{\sqrt{m}^{\ell+1}} 9e^{4cT} T^{-1/p} \left( \int_s^T \|T_{\ell,m}^0 - u\|_{p,\zeta}^p \right)^{\frac{1}{p}}. \tag{132} \]

This, (109), and [32, Lemma 3.11] (applied for every \( s \in [0, T] \) with \( M \succ m, N \succ n, \tau \succ s, a \succ 7958\sqrt{p-1} \cdot 9ne^{4cT}, b \succ 4\sqrt{p-1} \cdot 9e^{4cT} T^{-1/p}, (f_t)_{t \in \mathbb{N}_0} \cap ([s, T] \ni t \mapsto \|U_{n,m}^0 - u\|_t \in [0, \infty])_{t \in \mathbb{N}_0} \) in the notation of [32, Lemma 3.11]) show for all \( n, m \in \mathbb{N}, s \in [0, T] \) that

\[ \|U_{n,m}^0 - u\|_s \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT} + 4\sqrt{p-1} \cdot 9e^{4cT} T^{-1/p} (T - s)^{1/s}}{\sqrt{m}} \]

\[ \cdot \left( \sup_{t \in [s,T]} \left\| U_{0,m}^0 - u \right\|_t \right) e^{mp/2/p,m-n/2} \left[ 1 + 4\sqrt{p-1} \cdot 9e^{4cT} T^{-1/p} (T - s)^{1/p} \right]^{-n-1} \]

\[ \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT} + 36\sqrt{p-1} \cdot e^{4cT}}{\sqrt{m}} \left( 1 + \frac{36\sqrt{p-1} \cdot e^{4cT}}{2} \right)^{n-1} \]

\[ \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT} + 36\sqrt{p-1} \cdot e^{4cT}}{\sqrt{m}} \left( 1 + \frac{36\sqrt{p-1} \cdot e^{4cT}}{2} \right)^{n-1} (p - 1)^{\frac{n}{2}} \]

\[ \leq \frac{7958\sqrt{p-1} \cdot 9ne^{4cT} + 36\sqrt{p-1} \cdot e^{4cT}}{\sqrt{m}} \left( 1 + \frac{36\sqrt{p-1} \cdot e^{4cT}}{2} \right)^{n-1} (p - 1)^{\frac{n}{2}}. \tag{133} \]

This shows (iii). The proof of Theorem 4.2 is thus completed. \( \square \)

## 5 Error estimates for multigrid approximations of FBSDEs

In item (i) below we prove that our multigrid approximations (157) have a computational effort which grows essentially quadratically in the reciprocal accuracy \( 1/s \) and polynomially in the dimension. We note that we measure the error by \( L^p \)-norms of path distances. First, we apply in Theorem 5.1 below the Hölder-norm estimate (100) for MLP approximations of semilinear PDEs to obtain the error estimate (135) for our multigrid approximations of FBSDEs.

**Theorem 5.1 (Error estimate for FBSDEs).** Assume Setting 4.1, let \( \delta \in (0, \infty), p \in [2, \infty), q_1 \in [3, \infty), q_2 \in [9, \infty) \) satisfy that \( \frac{q_2+1}{p_1} + \frac{2}{p_2} \leq \frac{1}{p} \) and \( q_1 + 2 \leq q_2 \), let \( (\theta_n)_{n \in \mathbb{N}_0} \subseteq \Theta \), let \( \lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{R}, m \in \mathbb{N}, \text{ and } \lceil \cdot \rceil : \mathbb{N} \rightarrow \mathbb{N}, m \in \mathbb{N}, \text{ satisfy for all } n \in \mathbb{N}, t \in [0, T] \text{ that } \lceil t \rceil_m = \max(\{0, T \setminus \{T\} \cap \{0, \frac{T}{m}, 2\frac{T}{m}, \ldots\} \text{ and } \lceil t \rceil_m = \min(\{t, \infty\} \cup (T) \cap \{0, \frac{T}{m}, 2\frac{T}{m}, \ldots\}) \}, \text{ let } \mathcal{Y}^m_{n,m} = (\mathcal{Y}^m_{n,m})_{n \in [0,T]: [0, T]} \times \Omega \rightarrow \mathbb{R}, n, m \in \mathbb{N}, \text{ satisfy for all } n, m \in \mathbb{N}, t \in [0, T] \text{ that} \]

\[ \mathcal{Y}^m_{n,m} = \sum_{\ell=0}^{n-1} \left( \frac{\lceil t \rceil_m - \ell}{(T/m)^{\ell+1}} \right)^{U_{n-\ell,m}(\lceil t \rceil_{m+1}, \alpha_{0,\lceil t \rceil_{m+1}}^{m,0,0}) + \left( \frac{\lceil t \rceil_m - \ell}{(T/m)^{\ell+1}} \right)^{U_{n-\ell,m}(\lceil t \rceil_{m+1}, \alpha_{0,\lceil t \rceil_{m+1}}^{m,0,0})} - \mathbb{1}_N(t) \left( \frac{\lceil t \rceil_m - \ell}{(T/m)^{\ell+1}} \right)^{U_{n-\ell,m}(\lceil t \rceil_{m+1}, \alpha_{0,\lceil t \rceil_{m+1}}^{m,0,0}) + \left( \frac{\lceil t \rceil_m - \ell}{(T/m)^{\ell+1}} \right)^{U_{n-\ell,m}(\lceil t \rceil_{m+1}, \alpha_{0,\lceil t \rceil_{m+1}}^{m,0,0})} \right). \tag{134} \]

Then

i) there exists a unique measurable \( u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \) which satisfies for all \( t \in [0, T], x \in \mathbb{R}^d \text{ that} \]

\[ \mathbb{E}[g(X_T(t))] + \int_t^T \mathbb{E}[f(u(r, X_T(r)))] dr + \sup_{r \in [0,T], x \in \mathbb{R}^d} \frac{|u(r, x)|}{(r-x)^2} < \infty \text{ and } u(t, x) = \mathbb{E}[g(X_T(t))] + \int_t^T \mathbb{E}[f(u(r, X_T(r)))] dr, \]

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ii) it holds for all $t \in [0, T]$, $s \in [0, T] \setminus \{t\}$ that $\|u(t, X_{0,t}^n)\|_p \leq 2e^{2cT} (V(0, 0))^{q_1}$ and $\frac{\sqrt{T}\|u(t, X_{0,t}^n) - u(s, X_{0,s}^n)\|_p}{|t-s|^{1/2}} \leq 12e^{2cT} (V(0, 0))^{q_2+1}$, and

iii) it holds for all $n, m \in \mathbb{N}$, $t \in [0, T]$ that

$$\|\mathcal{P}^{n,m}_t - u(t, X_{0,t}^n)\|_p \leq 959mc^{6T}e^{mp/2}/n_m - n/2T^4n^{\epsilon T} (p - 1)^{2/7} (V(0, 0))^{q_2+1}. \quad (135)$$

Proof of Theorem 5.1. Throughout this proof for every $s \in [0, T]$ and every random field $H : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ let $\|H\|_s, \|H\|_{1,s}, \|H\|_{2,s} \in [0, \infty]$ satisfy that

$$\|H\|_s = \max \left\{ \|H\|_{1,s}, \|H\|_{2,s} \right\}, \quad \|H\|_{1,s} = \sup_{t \leq s} \sup_{x \in \mathbb{R}^d} \|H(t, x)\|_p, \quad \text{and} \quad \|H\|_{2,s} = \sup_{t_1, t_2 \in [s, T], x_1, x_2 \in \mathbb{R}^d : (t_1, x_1) \neq (t_2, x_2)} \frac{(V(t_1, x_1))^q + (V(t_2, x_2))^q}{2} \sqrt{\frac{[V(t_1, x_1) + V(t_2, x_2)]^{1/2}}{T} + \frac{|x_1 - x_2|}{\sqrt{T}}} \]. \quad (136)$$

Then Theorem 4.2, the assumptions of Theorem 5.1, the fact that $\forall m \in \mathbb{N} : U_{0,m}^0 = 0$, and the fact that $\forall n \in \mathbb{N}_0 : 2n \leq 2^n$ show that

a) there exists a unique measurable $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfies for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $E[g(X_{t,T}^x)] + \int_t^T E[\|f(u(r, X_{r,T}^x))\|_d + \sup_{r \in [0, T]} \|u(r, X_{r,T}^x)\|_d] \leq \infty$ and $u(t, x) = E[g(X_{t,T}^x)] + \int_t^T E[f(u(r, X_{r,T}^x))] \, dr$,

b) for all $s \in [0, T]$ it holds that

$$\|u\|_{1,s} \leq 2e^{2cT}, \quad \|u\|_{2,s} \leq 8e^{2cT}, \quad \text{and} \quad \|u\|_s \leq 8e^{2cT}, \quad (137)$$

and

c) for all $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $t \in [0, T]$ it holds that

$$\|U_{0,m}^0 - u\|_t \leq 88m e^{6cT} e^{mp/2}/n_m - n/2T^4n^{\epsilon T} (p - 1)^{2/7}. \quad (138)$$

This shows (i).

Next, (97) and (96) show for all $t \in [0, T]$, $n \in \mathbb{N}$ that

$$\left\|X_{0,t}^n - X_{0,t}^0\right\|_{p_2} = \left\|\left(X_{0,t}^n - X_{0,t}^0\right) - (X_{0,t}^n - X_{0,t}^0)\right\|_{p_2} \leq \frac{\sqrt{T}V(0, 0)}{2\sqrt{n}}. \quad (139)$$

This, the triangle inequality, and (95) show for all $s, t \in [0, T]$ that

$$\left\|X_{0,t}^0 - X_{0,s}^0\right\|_{p_2} \leq \lim_{n \to \infty} \left\|X_{0,t}^0 - X_{0,s}^n\right\|_{p_2} + \left\|X_{0,t}^n - X_{0,s}^n\right\|_{p_2} + \left\|X_{0,s}^n - X_{0,s}^0\right\|_{p_2} \leq V(0, 0) \frac{|t-s|^{1/2}}{2}. \quad (140)$$

In addition, (139), continuity of $V$, and (97) show for all $t \in [0, T]$ that

$$\left\|V(t, X_{0,t}^0)\right\|_{p_1} = \lim_{n \to \infty} \left\|V(t, X_{0,t}^n)\right\|_{p_1} \leq V(0, 0). \quad (141)$$

This, (136), and (137) show for all $t \in [0, T]$ that

$$\left\|u(t, X_{0,t}^0)\right\|_p \leq \left\|u\right\|_{1,t} \left\|V(t, X_{0,t}^0)\right\|_{p_1} \leq 2e^{2cT} (V(0, 0))^{q_1}. \quad (142)$$

Furthermore, (136), the triangle inequality, Hölder’s inequality, the fact that $\max\left\{\frac{q_2}{p_1} + \frac{1}{p_2}, \frac{q_2}{p_1} + \frac{1}{p_2}\right\} \leq \frac{1}{p}$, (137), (141), and (140) show for all $t, s \in [0, T]$ that

$$\left\|u(t, X_{0,t}^0) - u(s, X_{0,s}^0)\right\|_p \leq \left\|u\right\|_t \left\|V(t, X_{0,t}^0)\right\|_{p_1}^q + \left(V(s, X_{0,s}^0)\right)^q \frac{V(t, X_{0,t}^0) + V(t, X_{0,t}^0)}{2} \frac{|t-s|^{1/2}}{\sqrt{T} + \frac{|X_{0,t}^0 - X_{0,s}^0|}{\sqrt{T}}} \right\|_p \leq 2e^{2cT} (V(0, 0))^{q_1}. \quad (143)$$

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This shows for all \( t \in [0, T], s \in [0, T] \setminus \{ t \} \) that
\[
\frac{\sqrt{T} \| u(t, X_{0,t}) - u(s, X_{0,s}) \|_p}{|t-s|^{1/2}} \leq \frac{\sqrt{T}}{|t-s|^{1/2}} 12e^{2cT} (V(0,0))^{q_2+1} |t-s|^{1/2} = 12e^{2cT} (V(0,0))^{q_2+1}.
\] (143)

This and (142) prove (ii).

Moreover, (136), (137), Hölder’s inequality, the fact that \( \frac{q_2}{p_1} + \frac{1}{p_2} \leq \frac{1}{p} \), the triangle inequality, (97), (141), and (139) show for all \( t \in [0, T], m, n \in \mathbb{N} \) that
\[
\left\| (U_{n,t}^{m,n} - u)(t, X_{0,t}) - (U_{n,t}^{m,n} - u)(s, X_{0,s}) \right\|_p \leq \left\| U_{n,t}^{m,n} - u \right\|_p \leq \left\| \frac{2}{p_2} \left( (V(t, X_{0,t}^{m,n,0,0}))^{q_2} + (V(t, X_{0,t}^{0,0}))^{q_2} \right) \right\|_p \leq 8e^{2cT} (V(0,0))^{q_2} \leq 8e^{2cT} m^{-n/2} (V(0,0))^{q_2+1}.
\] (145)

Next, (136), the triangle inequality, Hölder’s inequality, the fact that \( \frac{q_2}{p_1} + \frac{1}{p_2} \leq \frac{1}{p} \), (138), (97), and (95) show for all \( n \in \mathbb{N}_0, \ell \in \{0, 1, \ldots, n\}, s \in [0, T], t \in [s, T] \) that
\[
\left\| (U_{n,t}^{\ell,s,m} - u)(t, X_{0,t}) - (U_{n,t}^{\ell,s,m} - u)(s, X_{0,s}) \right\|_p \leq \left\| U_{n,t}^{\ell,s,m} - u \right\|_p \leq \left\| \frac{2}{p_2} \left( (V(t, X_{0,t}^{\ell,s,m,0,0}))^{q_2} + (V(s, X_{0,s}^{\ell,s,m,0,0}))^{q_2} \right) \right\|_p \leq 888me^{6cT} e^{m^{p/2}/p m^{-(n-\ell)/2} (V(0,0))^{q_2+1}} (V(0,0))^{q_2} \leq 1332me^{6cT} e^{m^{p/2}/p m^{-(n-\ell)/2} (V(0,0))^{q_2+1}} (V(0,0))^{q_2+1}.
\] (146)
\[ + T^{1/2}2^{-1/2}m^{-n/2} \left[ \sup_{t,s \in [0,T]: t \not= s} \left\| u(t, X_{0,t}^{m,n,0,0} - u(s, X_{0,t}^{m,n,0,0}) \right\|_p \right] \]
\[ + \sum_{\ell=1}^{n-1} \left[ T^{1/2}2^{-1/2}m^{-\ell/2} \sup_{t,s \in [0,T]: t \not= s} \left\| (U^m_{n-\ell,m} - u(t, X_{0,t}^{m,n,0,0})) - (U^m_{n-\ell,m} - u(s, X_{0,t}^{m,n,0,0})) \right\|_p \right] \]
\[ = \sup_{t \in [0,T]} \left\| U^m_{n,m} - u(t, X_{0,t}^{m,n,0,0}) \right\|_p \]
\[ + \sum_{\ell=1}^{n} \left[ T^{1/2}2^{-1/2}m^{-\ell/2} \sup_{t,s \in [0,T]: t \not= s} \left\| (U^m_{n-\ell,m} - u(t, X_{0,t}^{m,n,0,0})) - (U^m_{n-\ell,m} - u(s, X_{0,t}^{m,n,0,0})) \right\|_p \right] \]
\[ \leq 888me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_1} \]
\[ + \sum_{\ell=1}^{n} \left[ T^{1/2}2^{-1/2}m^{-\ell/2}1332me^{6Te^{m/2}/p}m^{-(n-\ell)/2}4^{n-\ell} e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1} \right] \]
\[ \leq \sum_{\ell=0}^{n} \left[ \frac{1332}{\sqrt{e}} me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1} \right] \]
\[ \leq me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1} \]
\[ \leq 955me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1}. \tag{148} \]

This, the triangle inequality, and (145) show for all \( n, m \in \mathbb{N}, t \in [0,T] \) that
\[ \left\| g_t^{n,m} - u(t, X_{0,t}^{0,0}) \right\|_p \leq \left\| g_t^{n,m} - u(t, X_{0,t}^{0,0,0}) \right\|_p + \left\| u(t, X_{0,t}^{0,0}) \right\|_p \]
\[ \leq 955me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1} + 4e^{2Te^{m/2}(0,0)}^{q_2+1} \]
\[ \leq 955me^{6Te^{m/2}/p}m^{-n/2}4^n e^{A n c T} (p - 1)^2 (V(0,0))^{q_2+1}. \tag{149} \]

This shows (iii). The proof of Theorem 5.1 is thus completed. □

We discuss the computational effort of the approximation method in Corollary 5.2 below; see (158)–(159). First, (158) was explained in the paragraph above Corollary 3.4. Next, (159) is explained as follows. For every \( d, m, n \in \mathbb{N} \) we denote by \( C_{d,m,n} \) the upper bound for the computational effort to have one realization of the process \( (g_t^{d,n,m})_{k \in \{0,1, \ldots, m^n\}} \). For every \( d, m, n \in \mathbb{N} \) we think of \( a_3(d)(m^n + 1) \) as the upper bound for the computational effort to have one realization of the process \( (X_{0,kT/m^n})_{k \in \{0,1, \ldots, m^n\}} \). Here, we use Euler–Maruyama approximations; see (155). Furthermore, in (157) for every \( d, m, n \in \mathbb{N}, \ell \in \{0,1, \ldots, n - 1\} \) we need to evaluate \( U_{d,n-\ell,m}^{d,\ell} \) at \( m^{\ell + 1} \) space-time points.

**Corollary 5.2 (Complexity analysis).** Let \( ||: \cup_{k, \ell \in \mathbb{N}} \mathbb{R}^{k \times \ell} \to \{0, \infty\} \) satisfy for all \( k, \ell \in \mathbb{N}, s = (s_{ij})_{i \in \{1, \ldots, k\}, j \in \{1, \ldots, \ell\} \in \mathbb{N} \in \mathbb{R}^{k \times \ell} \) that \( ||s||^2 = \sum_{i=1}^{k} \sum_{j=1}^{\ell} |s_{ij}|^2 \), let \( T, \delta \in (0, \infty), \beta, c, p \in [2, \infty), \Theta = \cup_{n \in \mathbb{Z}^n} \mathbb{N}, b, a_1, a_2, a_3: \mathbb{N} \to \mathbb{N} \) satisfy that \( p > 2/\delta \) and \( \inf_{\gamma \in (0,\infty)} \sup_{u \in \mathbb{N}} (b(d) + \sum_{\gamma=1}^{3} a_1(d) \leq 1, \) for every \( d \in \mathbb{N} \) and \( f_{d} \in C^{2}(\mathbb{R}, \mathbb{R}), g_{d} \in C^{2}(\mathbb{R}, \mathbb{R}), u_{d} \in C^{1,2}(\{0,T\} \times \mathbb{R}^d, \mathbb{R}), \mu_{d} = (\mu_{d,i})_{i \in \{1, \ldots, m^n\} \in C^{2}(\mathbb{R}, \mathbb{R}), \mu_{d} = (\mu_{d,i})_{i \in \{1, \ldots, m^n\} \in C^{2}(\mathbb{R}, \mathbb{R}),)\) for all \( d \in \mathbb{N}, x, y, z \in \mathbb{R}^d, w \in \mathbb{R}, t \in [0,T] \) that
\[ |Tf_{d}(0)| + ||f_{d}(0)|| + |||f_{d}(0)||| \leq b(d), \quad |u_{d}(t, x)| + |g_{d}(x)| \leq b(d) + c^2 ||x||^2, \tag{150} \]
\[ |(Dg_{d}(x))(y)| \leq b(d)||y||, \quad |(D^2g_{d}(x))(y, z)| \leq b(d)||y||||z||, \quad |f_{d}(w)| + |f_{d}'(w)| \leq c, \tag{151} \]
\[ ||(D\mu_{d}(x))(y)|| \leq c||y||, \quad ||(D\mu_{d}(x))(y)|| \leq c||y||, \tag{152} \]
\[ ||(D^2\mu_{d}(x))(y, z)|| \leq b(d)||y||||z||, \quad ||(D^2\mu_{d}(x))(y, z)|| \leq b(d)||y||||z||, \tag{153} \]
\[
\frac{d}{dt}u_d(t,x) + (\mu_d(x), (\nabla_x u_d)(t,x)) + \frac{1}{2} \text{trace} (\sigma_d(x)(\sigma_d(x))^* (\text{Hess}_x u_d)(t,x)) = -f_d(u_d(t,x)),
\]
and \(u_d(T, x) = g_d(x)\), let \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})\) be a filtered probability space which satisfies the usual conditions, for every \(s \in [1, \infty)\), \(k, \ell \in \mathbb{N}\) and every random variable \(X: \Omega \to \mathbb{R}^{k \times \ell}\) let \(\|X\|_s = \mathbb{E}[\|X\|_s^s]\), let \(\mathcal{C}^d: \Omega \to [0, 1, \theta] \cap \mathbb{N}\) be i.i.d. random variables which satisfy for all \(t \in [0, 1]\) that \(\mathbb{P}(\mathcal{C}^d < t) = t\), let \(W^{d,\theta}: [0,T] \to \mathbb{R}^d\), \(d \in \mathbb{N}\), \(\theta \in \Theta\), be independent standard \((\mathcal{F}^d)_{t \in [0,T]}\)-Brownian motions with continuous sample paths, assume that \((\mathcal{F}^d)_{t \in [0,T]}\) and \((W^{d,\theta})_{d \in \mathbb{N}, \theta \in \Theta}\) are independent, for every \(d, n \in \mathbb{N}\), \(\theta \in \Theta\), \(s \in [0,T]\), \(x \in \mathbb{R}^d\) let \(X^{d,n,\theta,x}_{s,t} \in [s,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d\) satisfy for all \(k \in [0, n-1] \cap \mathbb{Z}\), \(t \in [\mathbb{N}(\epsilon/s), \mathbb{N}(\epsilon/s + \epsilon)]\) that \(X^{d,n,\theta,x}_{s,t} = x\) and
\[
X^{d,n,\theta,x}_{s,t} = X^{d,n,\theta,x}_{s,\mathbb{N}(\epsilon/s)} + \mu_d(X^{d,n,\theta,x}_{s,\mathbb{N}(\epsilon/s)})(t - \mathbb{N}(\epsilon/s)) + \sigma_d(X^{d,n,\theta,x}_{s,\mathbb{N}(\epsilon/s)})(W^{d,\theta}_t - W^{d,\theta}_{\mathbb{N}(\epsilon/s)}),
\]
let \(U^{d,\theta}_{d,n,m}(t, x) = U^{d,\theta}_{d,n,m}(t, x) = 0\) and
\[
U^{d,\theta}_{d,n,m}(t, x) = \frac{1}{m^{n-\ell}} \sum_{i=1}^{m^{n-\ell}} \left( g_d(X^{d,m,\theta,x}_{t,T}) - \mathbb{1}_T g_d(X^{d,m-1,\theta,x}_{t,T}) 
+ (T-t)(f_d \circ U^{d,\theta}_{d,n,m}(t, x) + \mathcal{A}^{d,n,\theta,x}_{s,t}(t, x) + \mathcal{A}^{d,n,\theta,x}_{s,t}(t, x) \right),
\]
let \(\mathcal{C}_{d,n,m} \in \mathbb{N}_0\), \(\mathcal{C}_{d,n,m} \in \mathbb{N}_0\) satisfy for all \(d, n, m \in \mathbb{N}\) that \(C_{d,n,m} = 0\),
\[
C_{d,n,m} \leq \sum_{\ell=0}^{n-1} \left( m^{n-\ell} \left( 4a_1(d) + a_2(d) m^{\ell} + C_{d,\ell,m} + \mathbb{1}_T g_d(X^{d,m,\theta,x}_{t,T}) \right) \right),
\]
and let \(M: \mathbb{N} \to \mathbb{N}\) satisfy that
\[
\lim_{n \to \infty} \frac{1}{M(n)} = 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \left[ \frac{M(n+1)}{M(n)} + \frac{(M(n))^{p/2}}{n} \right] < \infty.
\]
Then
(i) for all \(d \in \mathbb{N}\) there exists an up to indistinguishability unique continuous random field \((X^{d,x}_{s,t})_{s,t \in [0,T], s \in [s,T], x \in \mathbb{R}^d}\) such that for all \(s \in [0,T]\), \(x \in \mathbb{R}^d\) it holds that \((X^{d,x}_{s,t})_{s,T} \in [s,T] - \mathcal{F}^d_t\) adapted and such that for all \(s \in [0,T]\), \(t \in [s,T]\), \(x \in \mathbb{R}^d\) it holds a.s. that
\[
X^{d,x}_{s,t} = x + \int_s^t \mu_d(X^{d,x}_{s,r}) \, dr + \int_s^t \sigma_d(X^{d,x}_{s,r}) \, dW^d_{s,r},
\]
(ii) for all $d \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d$ it holds that $\mathbb{E}[|g_d(X_{t,T}^d,x})|] + \int_t^T \mathbb{E}[|f_d(u_d(r, X_{t,T}^d,x))|] \, dr < \infty$ and $u_d(t, x) = \mathbb{E}[g_d(X_{t,T}^d,x})] + \int_t^T \mathbb{E}[f_d(u_d(r, X_{t,T}^d,x))] \, dr$.

(iii) for all $d \in \mathbb{N}$ there exists a unique ($\mathbb{F}_t$)_{t \in [0,T]}-predictable stochastic process $Y_t^d = (Y_t^d, Z_{t}^{d,1}, Z_{t}^{d,2}, \ldots, Z_{t}^{d,d})$ on $[0, T] \times \Omega \to \mathbb{R}^{d+1}$ such that $\int_0^T \mathbb{E}[|Y_s^d| + \sum_{j=1}^d |Z_{s}^{d,j}|^2] \, ds < \infty$ and such that for all $t \in [0, T]$ it holds a.s. that

$$Y_t^d = g_d(X_{0,t}^d,x} + \int_t^T f_d(Y_s^d) \, ds - \sum_{j=1}^d \int_t^T Z_{s}^{d,j} \, dW_{s}^{d,j},$$

(162)

(iv) for all $d \in \mathbb{N}, t \in [0, T]$ it holds a.s. that $u_d(t, X_{0,0}^d) = Y_t^d$, and

(v) there exist $\gamma \in (0, \infty)$, $n: \mathbb{N} \times (0, 1) \to \mathbb{N}$ such that for all $d \in \mathbb{N}, \varepsilon \in (0, 1)$, $n \in [n(d, \varepsilon), \infty) \cap \mathbb{N}$ it holds that $(\mathbb{E}[\sup_{t \in [0,T]} |\varphi_d^n(t,\varepsilon) - Y_t^d|])^{1/p} < \varepsilon$ and $\varepsilon^{2+\delta} \mathbb{E}_{d,n,M(n)} \leq \gamma d^\ell$.

Proof Corollary 5.2. First, a standard result on stochastic differential equations with Lipschitz continuous coefficients (see, e.g., [41, Theorem 4.5.1]) and (152) show (i).

Throughout the rest of this proof let $q, \bar{c} \in \mathbb{R}$ satisfy that $q = 12\bar{c}p$ and $\bar{c} = 16q^2c^2$ and for every $d \in \mathbb{N}$ let $\varphi_d: \mathbb{R}^d \to [1, \infty)$, $V_d: [0, T] \times \mathbb{R}^d \to [1, \infty)$ satisfy for all $t \in [0, T], x \in \mathbb{R}^d$ that

$$\varphi_d(x) = 2^{\frac{2q}{q}} \left( (b(d))^2 + c^2 \|x\|^2 \right)^\frac{q}{2},$$

(163)

and

$$V_d(t, x) = \left[ 62(1 + b(d) + c)(c + 1) \left( \sqrt{T} + 2q \right)^\frac{q}{2} e^{5c^2[\sqrt{T} + 2q]^2} e^{\frac{4.5c^2}{\bar{c}}} e^{\frac{\gamma(4)\frac{\gamma}{2}}{\bar{c}}} \right] \left( \varphi_d(x) \right)^\frac{\gamma}{2q}.$$ 

(164)

First, (150), (163), and (164) show for all $d \in \mathbb{N}, t \in [0, T], x \in \mathbb{R}^d$ that

$$\max \left\{ |T f_d(0)|, |u_d(t, x)| + |g_d(x)| \right\} \leq \left( b(d) + c^2 \|x\|^2 \right)^{\frac{q}{\bar{c}}} \leq \left( \varphi_d(x) \right)^{\frac{q}{2}} \leq V_d(t, x).$$

(165)

Next, (151), [39, Lemma 3.3], and the fact that $\forall d \in \mathbb{N}: b(d) \leq \min \{ V_d, V_d \}$ show that for all $d \in \mathbb{N}, x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$, $v, w, \bar{v}, \bar{w} \in \mathbb{R}$ it holds that

$$\begin{align*}
\left| (g_d(x) - g_d(y)) - (g_d(\bar{x}) - g_d(\bar{y})) \right| &\leq b(d) \left( \|x - y\| + \|\bar{x} - \bar{y}\| + \frac{\|x - y\| + \|\bar{x} - \bar{y}\|}{2} \right) \\
&\leq \frac{V_d(T,x) + V_d(T,y) + V_d(T,\bar{x}) + V_d(T,\bar{y})}{4} \left( \|x - y\| + \|\bar{x} - \bar{y}\| \right) \\
&\leq \frac{V_d(T,x) + V_d(T,y)}{4} \left( \|x - y\| + \|\bar{x} - \bar{y}\| \right) \leq \frac{V_d(T,x) + V_d(T,y)}{2} \left( \|x - y\| \right) \\
&\leq \frac{V_d(T,x) + V_d(T,y)}{2} \left( \|x - y\| \right).
\end{align*}$$

(166)

and

$$\begin{align*}
\left| (f_d(v) - f_d(w)) - (f_d(\bar{v}) - f_d(\bar{w})) \right| &\leq c \left( |v - w| + \|\bar{v} - \bar{w}\| \right).
\end{align*}$$

(167)

This implies for all $d \in \mathbb{N}, x, y \in \mathbb{R}^d, v, w \in \mathbb{R}$ that

$$\begin{align*}
|g_d(x) - g_d(y)| &\leq \frac{|(g_d(x) - g_d(y)) - (g_d(\bar{x}) - g_d(\bar{y}))|}{2} + \frac{|(g_d(y) - g_d(x)) - (g_d(y) - g_d(\bar{x}))|}{2} + \frac{|(g_d(y) - g_d(x)) - (g_d(y) - g_d(\bar{y}))|}{2} \\
&\leq \frac{1}{2} \frac{3V_d(T,x) + V_d(T,y)}{2} \frac{|x - y|}{\sqrt{T}} + \frac{1}{2} \frac{3V_d(T,y) + V_d(T,x)}{2} \frac{|x - y|}{\sqrt{T}} + \frac{1}{2} V_d(T,x) + V_d(T,y) \frac{|x - y|}{\sqrt{T}} \\
&= \frac{1}{2} \left( \frac{3V_d(T,x) + V_d(T,y)}{2} + V_d(T,x) + V_d(T,y) \right) \frac{|x - y|}{\sqrt{T}} \\
&\leq \frac{1}{2} \left( \frac{3V_d(T,x) + V_d(T,y)}{2} + V_d(T,x) + V_d(T,y) \right) \frac{|x - y|}{\sqrt{T}}.
\end{align*}$$

(168)

and

$$\begin{align*}
|f_d(v) - f_d(w)| &\leq \frac{|(f_d(v) - f_d(w)) - (f_d(\bar{v}) - f_d(\bar{w}))|}{2} \leq c |v - w|.
\end{align*}$$

(169)

Next, (150), the fact that $\forall A, B \in [0, \infty): A + B \leq 2\sqrt{A^2 + B^2}$, and (163) show for all $d \in \mathbb{N}, x \in \mathbb{R}^d$ that

$$\|u_d(0)\| + \|\sigma_d(0)\| + c \|x\| \leq \|b(d)\| + c \|x\| \leq 2 \left( \|b(d)\|^2 + c^2 \|x\|^2 \right)^{\frac{1}{2}} = \left( \varphi_d(x) \right)^{\frac{1}{2q}}.$$}

(170)
Moreover, the fact that $\forall d \in \mathbb{N}, x \in \mathbb{R}^d$: $\varphi_d(x) = (4|b(d)|^2 + 4c^2\|x\|^2)^q$ (see (163)), the fact that $q \geq 3$, and [39, Lemma 3.1] (applied for every $d \in \mathbb{N}$ with $p \cap q, a \cap 4|b(d)|^2$, $c \cap 2c$, $V \cap \varphi_d$ in the notation of [39, Lemma 3.1]) show for all $d \in \mathbb{N}$, $x, y \in \mathbb{R}^d$ that $\|((D\varphi_d)(x))(y)\| \leq 4q(c\varphi_d(x))^{\frac{2q-1}{2q}} \|y\|$ and $\|((D^2\varphi_d)(x))(y, y)\| \leq 16q^2c^2(\varphi_d(x))^{\frac{2q-2}{2q}} \|y\|^2$. This and the fact that $\bar{c} = \max\{4qc, 16q^2c^2\}$ show for all $d \in \mathbb{N}, x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ that

$$\|((D\varphi_d)(x))(y)\| \leq \bar{c}(\varphi_d(x))^{\frac{2q-1}{2q}} \|y\| \quad \text{and} \quad \|((D^2\varphi_d)(x))(y, y)\| \leq \bar{c}(\varphi_d(x))^{\frac{2q-2}{2q}} \|y\|^2.$$  
(171)

Next, (151), (152), and [39, Lemma 3.3] show for all $d \in \mathbb{N}$, $\zeta \in \{\mu_d, \sigma_d\}$, $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ that

$$\|((\zeta(x) - \zeta(y)) - (\zeta(\bar{x}) - \zeta(\bar{y}))\| \leq \epsilon \|(x - y) - (\bar{x} - \bar{y})\| + b(d) \frac{\|x - y\| + \|\bar{x} - \bar{y}\|}{2} \|x - \bar{x}\|.$$  
(172)

This, (170), (i), (155), and [39, Theorem 3.2] (applied for every $d, n \in \mathbb{N}$ with $m \cap d, b \cap b(d), p \cap 2q, \mu \cap \mu_d, \sigma \cap \sigma_d, V \cap \varphi_d$ in the notation of [39, Theorem 3.2]), the fact that $q \geq 4$, Jensen’s inequality, the fact that $1 \leq q/\beta \leq q$, and the fact that $\forall t, s \in [0, T]: |t - s|^{1/2} \leq \sqrt{T} \leq \sqrt{T} + p$ show that

(I) it holds for all $d, n \in \mathbb{N}, s \in [0, T], t \in [s, T], x \in \mathbb{R}^d$ that

$$\mathbb{E}\left[\varphi_d(X_{s,t}^{d,n,0,x})\right] \leq e^{1.5\epsilon|t-s|}\varphi_d(x),$$  
(173)

(II) it holds for all $d, n \in \mathbb{N}, s \in [0, T], t \in [s, T], x \in \mathbb{R}^d$ that

$$\left\|X_{s,t}^{d,n,0,x} - X_{s,t}^{d,n,0,\bar{x}}\right\| \leq \sqrt{2}\epsilon \left[\sqrt{T} + 2q\right] e^{2\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}}.$$  
(174)

(III) it holds for all $d, n \in \mathbb{N}, s, \bar{s} \in [0, T], t \in [s, \bar{s}], \bar{t} \in [s, T], x, \bar{x} \in \mathbb{R}^d$ that

$$\left\|X_{s,\bar{t}}^{d,n,0,x} - X_{\bar{s},\bar{t}}^{d,n,0,\bar{x}}\right\| \leq \sqrt{2}\epsilon \left[\sqrt{T} + 2q\right] e^{2\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[\sqrt{T} + 2q\right] e^{1.5\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}}$$  
(175)

+ $5e^{2\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[\sqrt{T} + 2q\right] e^{1.5\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[|s - \bar{s}|^{1/2} + |t - \bar{t}|^{1/2}\right]$,

(IV) it holds for all $d, n \in \mathbb{N}, s \in [0, T], t \in [s, T], x \in \mathbb{R}^d$ that

$$\left\|\left(X_{s,t}^{d,x} - X_{s,t}^{d,\bar{x}}\right) - \left(X_{s,t}^{d,\bar{x}} - X_{s,t}^{d,\bar{y}}\right)\right\| \leq \epsilon \left[\sqrt{T} + 2q\right] e^{2\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[\sqrt{T} + 2q\right] e^{1.5\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[|s - \bar{s}|^{1/2} + |t - \bar{t}|^{1/2}\right]$.$$

(V) it holds for all $d, n \in \mathbb{N}, s \in [0, T], t \in [s, T], x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ that

$$\left\|\left(X_{s,t}^{d,x} - X_{s,t}^{d,\bar{x}}\right) - \left(X_{s,t}^{d,\bar{x}} - X_{s,t}^{d,\bar{y}}\right)\right\| \leq \sqrt{2}\epsilon \left[\sqrt{T} + 2q\right] e^{2\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[\sqrt{T} + 2q\right] e^{1.5\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[|s - \bar{s}|^{1/2} + |t - \bar{t}|^{1/2}\right]$.$$

and

(VI) it holds for all $d, n \in \mathbb{N}, s, \bar{s} \in [0, T], t \in [s, T], \bar{t} \in [s, T], x, \bar{x} \in \mathbb{R}^d$ that

$$\left\|\left(X_{s,\bar{t}}^{d,x} - X_{s,\bar{t}}^{d,\bar{x}}\right) - \left(X_{s,\bar{t}}^{d,\bar{x}} - X_{s,\bar{t}}^{d,\bar{y}}\right)\right\| \leq 62(b(d) + c) \left[\sqrt{T} + 2q\right] e^{3\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[\sqrt{T} + 2q\right] e^{1.5\epsilon\frac{|t-s|^{1/2} + |t-\bar{t}|^{1/2}}{\sqrt{T}}} \left[|s - \bar{s}|^{1/2} + |t - \bar{t}|^{1/2}\right]$.$$

(178)
This, the Markov property, (i), and (155) show for all \(d \in \mathbb{N}, s, \tilde{s} \in [0, T], t \in [s, T], \tilde{t} \in [\tilde{s}, T], r \in [t, T], x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d, \ A \in (B(\mathbb{R}^d))^\otimes \mathbb{R}^d, B \in (B(\mathbb{R}^d))^\otimes ([t, T] \times \mathbb{R}^d)\) that

\[
P\left( X_{s,t,r}^{d,x} = X_{s,t}^{d,x} \right) = 1, \quad P\left( X_{s,t}^{d,(\cdot)} \in A, X_{t,\tilde{t}}^{d,(\cdot)} \in B \right) = P\left( X_{s,t}^{d,(\cdot)} \in A \right) P\left( X_{t,\tilde{t}}^{d,(\cdot)} \in B \right). \tag{179} \]

\[
\left\| (X_{s,t}^{d,x} - X_{s,t}^{d,y}) - (X_{s,t}^{d,\tilde{x}} - X_{s,t}^{d,\tilde{y}}) \right\|_2 \leq V_d(s,x) + V_d(s,y) + V_d(s,\tilde{x}) + V_d(s,\tilde{y}) \left\| (x - y) - (\tilde{x} - \tilde{y}) \right\|_2 + \frac{\|x - y\| + \|\tilde{x} - \tilde{y}\|}{2\sqrt{T}} \tag{180} \]

\[
\left\| (X_{s,t}^{d,x} - X_{s,t}^{d,y}) - (x - y) \right\|_2 \leq \frac{V_d(s,x) + V_d(s,y)}{2} \frac{|x - y|}{\sqrt{T}} + e \left( \frac{\ln(2)}{2} + c^2 [\sqrt{T} + 2q]^2 \right) T \frac{|x - \tilde{x}|}{\sqrt{n}} \tag{181} \]

\[
P(X_{s,s}^{d,x} = x) = P\left( \lambda_{s,s}^{d,n,0,x} = x \right) = 1, \quad \sqrt{2} e \leq \sqrt{T} + 2q \leq (V_d(s,x)) \tag{184} \]

Next, (173) shows for all \(d, n \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that

\[
\left\| e \left( \frac{1.55(T-1)}{q} \right) \phi_d(\lambda_{s,t}^{d,n,0,x}) \right\|_q \leq e \left( \frac{1.55(T-1)}{q} \right) (\mathbb{E} \left( \phi_d(\lambda_{s,t}^{d,n,0,x}) \right)) \tag{185} \]

This, (163), and (164) show for all \(d \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that

\[
\left\| V_d(t, \lambda_{s,t}^{d,n,0,x}) \right\|_q \leq V_d(s, x). \tag{186} \]

This, (174), continuity of \(V_d, d \in \mathbb{N}\), and Fatou’s lemma show for all \(d \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that \(\lambda_{s,t}^{d,x} = \mathbb{P}\text{-}\lim_{n \to \infty} \lambda_{s,t}^{d,n,0,x}\) and \(\left\| V_d(t, \lambda_{s,t}^{d,x}) \right\|_q = \mathbb{P}\text{-}\lim_{n \to \infty} V_d(t, \lambda_{s,t}^{d,n,0,x}) \leq \lim_{n \to \infty} \left\| V_d(t, \lambda_{s,t}^{d,n,0,x}) \right\|_q \leq V_d(s, x). \) This, (i), the Feynman-Kac formula, the fact that \(f_d, d \in \mathbb{N}\), are Lipschitz continuous (see (151)), (165), and (154) show (ii).

Next, (174), (184), and (182) show for all \(d \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that

\[
\left\| \lambda_{s,t}^{d,n,0,x} - x \right\|_q = \mathbb{P}\text{-}\lim_{n \to \infty} \left\| \lambda_{s,t}^{d,n,0,x} - x \right\|_q \leq \frac{V_d(s,x) + V_d(s,x)}{2} \frac{|t - s|}{2}. \tag{187} \]

Moreover, (174) and (184) show for all \(d \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that

\[
\left\| \lambda_{s,t}^{d,x} - x \right\|_q = \mathbb{P}\text{-}\lim_{n \to \infty} \left\| \lambda_{s,t}^{d,n,0,x} - x \right\|_q \leq \frac{V_d(s,x) + V_d(s,x)}{2} \frac{|t - s|}{2}. \tag{188} \]

In addition, (184) and (183) show for all \(d, n \in \mathbb{N}, x \in \mathbb{R}^d, s \in [0, T], t \in [s, T]\) that

\[
\left\| \lambda_{s,t}^{d,n,0,x} - \lambda_{s,t}^{d,x} \right\|_q = \left\| (\lambda_{s,t}^{d,n,0,x} - \lambda_{s,t}^{d,x}) - (\lambda_{s,t}^{d,n,0,x} - \lambda_{s,t}^{d,n,0,x}) \right\|_q \leq V_d(s,x) \frac{|t - s|}{2}. \tag{189} \]
Next, for every \(d \in \mathbb{N}\) let \(\tilde{Y}_d = (\tilde{Y}_d, \tilde{Z}_{d,1}^d, \tilde{Z}_{d,2}^d, \ldots, \tilde{Z}_{d,d}^d) : [0, T] \times \Omega \to \mathbb{R}^{d+1}\) satisfy for all \(i \in [1, d] \cap \mathbb{Z}\), \(t \in [0, T]\) that \(Y^d_t = u_d(t, X^{d,0}_t)\) and \(\tilde{Z}_{d,i}^d = \left(\frac{\partial}{\partial x_i}\right) u_d(t, X^{d,0}_t, \sigma_d, \gamma_d, \delta_d, \gamma_d, \eta_d)\). Then Itô’s formula, the regularity assumptions of \(u_d\), \(d \in \mathbb{N}\), and the fact that \(\forall d \in \mathbb{N}, x \in \mathbb{R}^d\) \(g_d(x) = u_d(T, x)\) show that for all \(t \in [0, T]\) it holds a.s. that \(\tilde{Y}_d = g_d(X^{d,0}_T) + \int_0^T f_d(\tilde{Y}_s^d)\, ds - \sum_{j=1}^d \int_0^T \tilde{Z}^d_{s,j} \, dW^d_{s,j}\). This and a result on uniqueness of solutions to BSDEs (see, e.g., [38, Theorem 1.1]) imply that \(\tilde{Y}_d = Y_d\). This implies (iii) and (iv).

For the rest of this proof let \(\Delta_{d,n,m}, \mathcal{E}_{d,n,m} \in [0, \infty]\), \(d, m, n \in \mathbb{N}\), satisfy for all \(d, m, n \in \mathbb{N}\) that

\[
\Delta_{d,n,m} = m^{\min(\delta, 1)n/2} \left\{ \sup_{t \in [0,T]} \left\| \mathcal{Y}^{d,n,m}_t - Y^d_t \right\|_p \right. \\
+ m^{-n/2} \left[ \sup_{t \in [0,T]} \left\| \mathcal{Y}^d_t \right\|_p \right] \left( \sup_{t,s \in [0,T], t \neq s} \frac{\sqrt{T} \left| Y^d_t - Y^d_s \right|}{\left| t-s \right|^{1/p}} \right)^{1/p} \right\},
\]

and

\[
\mathcal{E}_{d,n,m} = \left( \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \mathcal{Y}^{d,n,m}_t - Y^d_t \right\|_p \right] \right)^{1/p},
\]

let \(K \in [0, \infty]\) satisfy that

\[
K = \inf \{ \gamma \in [2, \infty) : \forall d, n, m \in \mathbb{N} : \mathcal{E}_{d,n,m} \leq \gamma \Delta_{d,n,m} \cup \{ \infty \} \},
\]

and let \(n : \mathbb{N} \times (0, \infty) \to [0, \infty]\) satisfy for all \(d \in \mathbb{N}, \varepsilon \in (0, \infty)\) that

\[
n(d, \varepsilon) = \inf \{ n \in \mathbb{N} : \sup_{k \in \mathbb{N}} [n(\infty)] \sup_{t \in [0,T]} \Delta_{d,k,n,k} < \frac{1}{K} \cup \{ \infty \} \}.
\]

Now [15, Theorem 1.1] (applied with \(E \subset \mathbb{R}\), \(\varepsilon \subset \min(\delta, 1)/2\), \(\alpha \subset 0\), \(\beta \subset 1/2\) in the notation of [15, Theorem 1.1]) implies that \(K < \infty\). In addition, Theorem 5.1 (applied for every \(d \in \mathbb{N}\) with \(c \subset \frac{\ln(2) + c^2(\sqrt{T} + 2q)^2}{f \sup_{t \in [0,T]} g_d} \subset V \subset V_d\), \((X^{d,x}_{s,t})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d} \subset \Omega \subset (\mathcal{Y}^{d,n,m}_{s,t})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d} \subset \mathcal{A}^{n,m} \subset (U^{d,n}_{s,t})_{s \in [0,T], t \in [s,T], x \in \mathbb{R}^d} \subset \mathcal{C}_{n,m,n} \subset \mathcal{C}_{d,n,m} \subset \mathcal{C}_{d,n,m} \subset \mathcal{C}_{d,n,m}\) in the notation of Theorem 5.1, (165)–(167), (179)–(184), (186), the fact that \(\frac{\ln(2)}{f} + \frac{2q}{g} \leq \frac{\ln(2)}{f} \leq \frac{1}{q}\), the assumptions of Corollary 5.2, (ii), and (iv) show that there exists \(\gamma \in [1, \infty)\) such that \(\forall d \in \mathbb{N}\) it holds that \(\Delta_{d,n,M} \leq \gamma^n (M(n))^{n/2} (V_d(0, 0))^{10}\). This, (193), and the fact that \(K < \infty\) show for all \(d \in \mathbb{N}, \varepsilon \in (0, 1)\) that \(n(d, \varepsilon) < \infty\). This, (192), (193), and the fact that \(K < \infty\) show for all \(d \in \mathbb{N}, \varepsilon \in (0, 1), n \in \{n(d, \varepsilon), \infty\}\), that

\[
\mathcal{E}_{d,n,M} \leq K \Delta_{d,n,M} < \varepsilon.
\]

Next, Corollary 3.4 and (158) show for all \(d, n, m \in \mathbb{N}\) that \(\mathcal{C}_{d,n,m} \leq \max\{4a_1(d), a_2(d)n\}(5m)^n\). This and (159) imply that there exists \(\gamma \in [1, \infty)\) such that for all \(d, m, n \in \mathbb{N}\) it holds that

\[
\mathcal{E}_{d,n,m} \leq a_3(d)(m^n + 1) + \sum_{\ell=0}^{n-1} \left( m^{\ell+1} + 1 \right) \mathcal{E}_{d,n-\ell,m} \\
\leq 2a_3(d)m^n + \sum_{\ell=0}^{n-1} \left[ 2m^{\ell+1} \max\{4a_1(d), a_2(d)n\}(5m)^{n-\ell} \right] \leq \max\{a_1(d), a_2(d), a_3(d)\} \gamma^n m^{n+1}.
\]

This shows that there exists \(\gamma \in [1, \infty)\) such that for all \(d, n \in \mathbb{N}\) it holds that \(\mathcal{E}_{d,n+1,M(n+1)} \leq \max\{a_1(d), a_2(d), a_3(d)\} \gamma^{n+1} (M(n + 1))^{n+2}\). This and (160) imply that there exists \(\gamma \in [1, \infty)\)}
that $\mathcal{C}_{d,n+1,M(n+1)} \leq \max\{a_1(d), a_2(d), a_3(d)\}(\gamma M(n))^{n+2}$. This, (192), and the fact that $K < \infty$ show that there exist $\gamma, C \in [1, \infty)$ such that for all $d, n \in \mathbb{N}$ it holds that

$$
\mathcal{C}_{d,n+1,M(n+1)} |\mathcal{E}_{d,n+1,M(n)}|^{2+\delta} \leq \mathcal{C}_{d,n+1,M(n+1)} |K\Delta_{d,n,M(n)}|^{2+\delta} \\
\leq \max\{a_1(d), a_2(d), a_3(d)\}(\gamma M(n))^{n+2} \left(\frac{K\gamma^n(M(n))^{-n/2}}{V_d(0,0)}\right)^{\delta/2} \\
\leq \max\{a_1(d), a_2(d), a_3(d)\} \gamma^{(\delta+2)/2} (M(n))^{2}(M(n))^{-0.5\delta} (V_d(0,0))^{10(\delta+2)}.
$$

(196)

This and (195) show that there exists $C \in (0, \infty)$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$ it holds that

$$
\mathcal{C}_{d,n(\varepsilon),M(n(\varepsilon))} \varepsilon^{2+\delta} \leq \mathcal{C}_{d,1,M(1)} + 1\{2,\infty\}(n(d,\varepsilon)) \mathcal{C}_{d,n(\varepsilon),M(n(\varepsilon)))} \varepsilon^{2+\delta} \\
\leq \mathcal{C}_{d,1,M(1)} + 1\{2,\infty\}(n(d,\varepsilon)) \left[\mathcal{C}_{d,n+1,M(n+1)} |\mathcal{E}_{d,n,M(n)}|^{2+\delta}\right]_{n=n(d,\varepsilon)-1} \\
\leq \max\{a_1(d), a_2(d), a_3(d)\} C(V_d(0,0))^{10(\delta+2)}.
$$

(197)

This, (194), and the fact that $\exists \gamma \in (0, \infty) \forall d \in \mathbb{N}$: $V_d(0,0) + \sum_{i=1}^3 |a_i(d)| \leq \gamma d^{\gamma}$ (see (163) and (164) and recall the fact that $\exists \gamma \in (0, \infty) \forall d \in \mathbb{N}$: $b(d) + \sum_{i=1}^3 a_i(d) \leq \gamma d^{\gamma}$) imply (v). The proof of Corollary 5.2 is thus completed. \qed

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