The suggestion that power-law like distributions frequently reported in experiments may be the effect of threshold dynamics and metastability was made by Bak, Tang and Wiesenfeld in 1987 and was called Self-Organized Criticality (SOC). Avalanche dynamics in granular piles was from the onset used as a metaphor and laboratory for SOC behavior and has inspired many models and experiments [1]. One of the most celebrated of these efforts is the experimental study of avalanches in one dimensional rice piles by Frette et al. and the theoretical Oslo model inspired by the rice pile experiment. The general interest and relevance of such studies rely on the assumption, guided by equilibrium critical phenomena, that the critical behavior of scale invariant systems falls into universality classes determined solely by a few general characteristics of the system, such as symmetry and dimension. So-called “relevant” parameters can decide which of the symmetries the system is asymptotically dominated by.

The rôle of anisotropy in SOC has been highlighted very early by Hwa and Kardar and Grinstein et al., who used anisotropic Langevin equations to describe sandpiles. On the cellular automata level, Kadanoff who used anisotropic Langevin equations to describe (crossover) is of great importance for the interpretation of experimental results and consistent relevant dependence on anisotropy. This is termed by the introduction of stochasticity nor by multiple topplings.

Moreover, contrary to suggestions in former studies, the switch between different universality classes (crossover) is not triggered by the introduction of stochasticity nor by multiple topplings.

Similar to the system size at which the crossover occurs, $L_x$, depends on the strength of the anisotropy $v$. We exemplify two possible mechanisms causing anisotropy in experiments, one of which vanishes with system size $L$ fast enough to keep $vL$ constant. This represents a marginal case and consequently makes a unique identification of the critical exponents impossible.

The original Oslo model (oOM) consists, in one dimension, of a lattice of sites $i = 1, \ldots, L$. Two coupled dynamical variables are associated with each lattice site: the primary variable $z_i \in \{0, 1, 2, \ldots\}$ and the threshold variable $z^c_i \in \{1, 2\}$. The initial configuration consists of $z_i = 0 \forall i$ and a random configuration of the $z^c_i$. The system is driven by increasing $z_i$ by one (a “grain”) followed by a relaxation of all sites $1 \leq i \leq L$ for which $z_i > z^c_i$ (“over-critical” sites). In case a site $i$ is over-critical, the following updates are performed (“toppling” or “relaxation”): $z_i \rightarrow z_i - 2$ and $z_{i+1} \rightarrow z_{i+1} + 1$ and, importantly, the existing value of the threshold $z^c_i$ is afterwards replaced by $1$ with probability $p$ and by $2$ with probability $1 - p$. The boundaries are updated the same way except that for $i = 1$ ($i = L$) addition on site $0$ ($L + 1$) is omitted.

We now introduce a tunable degree of anisotropy into the dynamics. An over-critical site $i$ is relaxed in the following way. Only left movement: with probability $p_l(1 - p_r)$ perform the updates: $z_i \rightarrow z_i - 1$ and $z_{i-1} \rightarrow z_{i-1} + 1$. Only right movement: with probability $p_r(1 - p_l)$ perform the updates $z_i \rightarrow z_i - 1$ and $z_{i+1} \rightarrow z_{i+1} + 1$. Both left and right movement: with probability $p_l p_r$ perform the updates $z_{i+1} \rightarrow z_{i+1} + 1$ and $z_i \rightarrow z_i - 1$. A new threshold $z^c_i$ is chosen, at random, after every successful update, i.e. when at least one grain has been redistributed.

We call this version of the model the anisotropic Oslo model (aOM). The strength of the anisotropy is described by the drift velocity $v = (p_r - p_l)/(p_r + p_l)$ which is the net flux of grains through the system. Clearly it is only sensible to study $p_r + p_l > 0$. The case $p_r = p_l = 1$ corresponds exactly to the oOM, while $p_r = p_l = 0$ represents a stochastic variant of the oOM. The avalanche exponents for the extreme, totally asymmetric case $p_l = 0$ and $p_r = 1$ can be obtained exactly and describe, as we shall see below, the scaling behavior for all $v > 0$. We are interested in the statistics of the sizes, $s$, of the avalanches of relaxation induced by the driving $z_1 \rightarrow z_1 + 1$. The size of an avalanche is in both versions of the model defined as the number of times the relaxation rule was successfully applied after the drive $z_1 \rightarrow z_1 + 1$ in order to make $z_1 < z^c_1 \forall i$ yet again. Thus $s \geq 0$. From the definition

**Anisotropy and universality: the Oslo model, the rice pile experiment and the quenched Edwards-Wilkinson equation.**

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(Dated: 16.07.03)

We show that any amount of anisotropy moves the Oslo model to another known universality class, the exponents of which can be derived exactly. This amounts to an exact solution of the quenched Edwards-Wilkinson equation with a drift term. We argue that anisotropy is likely to be experimentally relevant and may explain why consistent exponents have not been extracted in the rice pile experiments.

PACS numbers: 45.70.Ht, 05.65.+b, 64.60.Ht, 68.35.Fx
above it is clear that the model is equally well defined in terms of reduced probabilities, omitting the case where no grain is redistributed.

In the totally anisotropic or asymmetric limit [14] the model resembles some features of other exactly solved, directed models [10, 13, 15, 16, 14]. In two dimensions a very similar model has been studied numerically [14]. However, we stress that contrary to some other “exact” solutions, the model is solvable directly on the lattice and without assuming any scaling behavior [18]. Also the amplitudes of the moments can be calculated exactly.

It is a tedious, but straightforward task to show that the aOM is “Abelian”, i.e. the order of updates is irrelevant for its statistical properties. Since the microdynamics which prescribes the order of updates is irrelevant, there is no unique way to define a microscopic timescale. Presumably universal exponents of the duration of avalanches are therefore mainly a property of the arbitrary choice of the microdynamics. According to Hughes and Paczuski [20], a non-Abelian variant of an Abelian model may or may not remain in the same universality class.

Moreover, one notes that the oOM as well as the aOM contains multiple toppings, i.e. a single site can relax several times during a single avalanche.

We now describe the avalanche statistics of the aOM. In Fig. 1 we demonstrate that the avalanche size distribution $P(s; L, p_r, p_l)$ follows simple (finite size) scaling

$$P(s; L, p_r, p_l) = a(p_r, p_l) s^{-\tau} G \left( \frac{s}{b(p_r, p_l) L^D} \right) \quad \text{for } s > s_l$$  

(1)

where $G$ is the universal scaling function, $a(p_r, p_l)$ and $b(p_r, p_l)$ are two anisotropy and system dependent parameters, and $s_l$ is the lower cutoff independent of $L$. The values for the scaling exponents are $\tau_0 = 4/3$ and $D_n = 3/2$, which can be derived exactly in the asymmetric limit $v = 1$ [18] and represent the known universality class of the stochastic, directed sandpile in two dimensions [12, 13, 17], which is in turn closely related to the directed sandpile [10]. Numerically, these exponents have been found for all $v > 0$ studied at sufficiently large system sizes $L \gg L_X$. The crossover that occurs around $L_X$ is discussed in detail below. The two exponents of the aOM are to be compared with the exponents for the oOM, of $\tau_0 = 1.556(4)$ and $D_n = 2.25(2)$. Since the average avalanche size scales linearly with the system size, the exponents are related by $D (2 - \tau) = 1$ [4, 21].

The easiest way to derive the exponents from numerical data is by analysis of the moments [22], which scale according to $n > \tau - 1$ in leading order like

$$\langle s^n \rangle = \int_0^\infty ds s^n P(s; L, p_r, p_l) = a (b L^D)^{1+n-\tau} g_n + \ldots$$  

(2)

where $g_n$ is discussed below and \ldots denotes subleading terms, especially Wegner’s corrections to scaling [23]. In
the following, the crossover is studied by means of the
terms of the rescaled second moment, \( \langle s^2 \rangle / L^{5/2} \), which is shown in Fig. 2. For non-vanishing anisotropy, \( v > 0 \), it approaches a constant as \( L \to \infty \). For very small but finite values of \( v \) and \( L \) the rescaled moment increases with \( L \) like \( L^{0.75} \), corresponding to the oOM behavior, but at \( L \approx L_X(v) \) it crosses over and eventually converges to a finite constant. Below, we shall relate the behavior of \( L_X(v) \) to the effective anisotropy relevant to an experiment of a given size. Here we emphasize that Fig. 2 clearly demonstrates that the universality class of the extremely anisotropic case, \( p_r = 1, p_l = 0 \), contains all systems with non-vanishing anisotropy \( v > 0 \). That renders the oOM with \( v = 0 \) a special case; however, it is remarkable that even for \( p_r = p_l \neq 1 \) the model still shows oOM behavior. Thus, it is not the stochasticity itself which induces the change in critical behavior.

Eq. 1 allows the definition of universal amplitude ratios

\[
  g_n = \frac{\langle s^n \rangle}{\langle s^2 \rangle^{n-2}}
\]

which can easily be proven to be asymptotically independent of \( a, b \) and \( L \). The two constraints on \( G \), which fix \( a \) and \( b \) in 1, can be chosen such that \( g_n \) are the moments of \( x^{-\gamma}G(x) \) as used in 2, namely by imposing that \( \int_0^\infty x^{-1-\gamma}G(x) = \int_0^\infty x^{2-\gamma}G(x) = 1 \). The universal amplitude ratio \( g_3 \) as shown in Fig. 3 indicates not only the same crossover behavior as observed in Fig. 2 but also the universality of \( G \).

The importance of the above result is highlighted when we recall that the oOM in the continuum limit is described by the quenched Edwards-Wilkinson (EW) equation 3. A similar derivation shows that the aOM is a quenched EW equation with an additional drift term

\[
  \partial_t h(x, t) = D \partial_x^2 h(x, t) - v \partial_x h(x, t) + \eta(x, h(x, t)),
\]

where \( D \) is the surface tension and \( v \) the anisotropy or drift velocity as defined above. In the aOM, the quenched noise \( \eta(x, h(x, t)) \) represents the randomly chosen \( x_i \) and \( h(x, t) \) is the number of charges received by site \( x \) at time \( t \). The quenched nature of the noise makes it difficult to solve 4 directly. Together with the boundary conditions, it prevents the drift term \( v \partial_x h \) from being absorbed by a Galilean transformation. However, the above results determines the roughness exponent via \( D = 1 + \chi \) to be \( \chi = 1/2 \) for \( v > 0 \), which has already been suggested in another case of anisotropic depinning 24. For \( v = 0 \) numerics for the oOM suggest correspondingly that \( \chi = 1.25(2) \).

The crossover for small \( v > 0 \) with increasing \( L \) can also be illuminated by a study of the individual grains in the system, which behave like biased random walkers. This leads again to a diffusion equation with drift term and two absorbing boundaries 25. The average avalanche size is the average time the particles spend in the system divided by the average number of grains redistributed per toppling. The crossover is expected as soon as the ballistic motion dominates over the diffusion, \( L^2/D > L/v \), thus \( L_X(v) = D/v \). This has also been tested numerically, based on heuristic estimation of \( L_X \), as shown by the marks in Fig. 2.

FIG. 3: Scaling of \( g_3 \) (see Eq. 3) for different anisotropies. The filled circles are results for \( p_r = p_l \) in the same vertical order as shown in Fig. 2 and \( p_r = 1 \). Open circles show other parameters \( (p_r, p_l) \) as indicated. The dashed lines are the two extreme cases oOM \( (p_r = p_l = 1) \) and the solvable model \( (p_r = 1, p_l = 0) \). The open arrows mark the crossover points in Fig. 3.

FIG. 4: A stylized toppling of a single grain: If an elongated grain of width \( b \) and height \( a \) topples from site \( i \) to site \( i + 1 \), it reduces the height at \( i \) by \( a \) and increases the height at \( i + 1 \) by \( b \), thereby increasing the slope at site \( i - 1 \) by \( b \), decreasing the slope at \( i \) by \( a + b \) and increasing the slope at \( i + 1 \) by \( a \). The net flux of slope is \( a - b \) to the right.

We now discuss the relevance of anisotropy to real experimental granular systems. We stress that the anisotropy is in the amount of slope transported between sites involved in a relaxation event. Any net flux of the slope is eventually compensated by the toppling of the last site and the slope is therefore asymptotically stationary.
One process leading to an anisotropic redistribution of slope arises when the toppling grain is elongated, see Fig. 4. Most remarkably, in the original experiment it was noted that only the elongated rice samples showed scale invariant behavior. A reorientation of a single grain, as shown in Fig. 4, leads to a net flux of slope to the right. It can happen only once, and in fact it depends on how and whether the rice enters and/or leaves the system with a typical orientation. If that is the case, then an average and whether the rice enters and/or leaves the system with $L \gg L_x(v)$ or not. The dependence of, say, the ratio $s^2/v^{2.5}$ on $L$ would in this case be given by an (unknown) trajectory through the diagram in Fig. 2 since a change in $L$ also leads to a change in $v$, which allows non-universal quantities to enter.

We do not know if the experiment by Frette et al. involves this complication, but the exponents extracted from the experiment by the authors are not consistent.

Local rearrangements, such as expansions (on the site that looses a grain) and compressions (on the site that receives the grain) lead to an anisotropy which does not vanish with $L$: Say column $i-1$ from which the grain leaves expands by $\epsilon(h_i)$ and column $i+1$ which receives the grain is compressed by an amount $\epsilon(h_{i+1})$. Then the changes of the slopes toppling are $\Delta z_{i-1} = 1 - \epsilon(h_i)$, $\Delta z_i = -2 + \epsilon(h_i) + \epsilon(h_{i+1})$ and $\Delta z_{i+1} = 1 - \epsilon(h_{i+1})$. Assuming that the columns behave elastically, $\epsilon$ would be an increasing function of $h$ which results in a net flow to the right, i.e. $v > 0$. However, it remains unclear whether any of these effects can be seen in experimental systems. Moreover, having shown that anisotropy is a relevant field, it would not be surprising to find other relevant fields which lead to yet another universality class.

We have demonstrated that for any amount of anisotropy the exponents of the original Oslo model change and are given by simple rational numbers which can all be obtained exactly. The crossover has been studied numerically using a moment analysis, Eq. 3, and universal amplitude ratios, Eq. 3, and the crossover length has been determined. The generalized model described above continuously connects the established original Oslo model and an exactly solvable, directed variant. This variant has, compared to the original Oslo model, an enormous basin of attraction, so that the latter may be regarded a special case of the former.

Moreover, we find a change in critical behavior of an SOC model, genuinely due to anisotropy, rather than stochasticity or the presence of multiple topplings.

The results are theoretically interesting especially because of their relation to the Edwards-Wilkinson equation, the roughness exponent of which has been obtained in case of the presence of a drift term in one dimension to be $\chi = 1/2$. Moreover, according to our study, experiments are seriously complicated due to a coupling between system size and effective anisotropy. That might provide a clue as to the apparent difficulties to find theoretically predicted exponents in the real world.

It is a pleasure to thank Kim Christensen for very helpful discussions and Paul Anderson for proofreading. The authors gratefully acknowledge the support of EPSRC. G.P. would like to thank Quincy Thoeren for hospitality.

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[26] There are some problems in the derivation: The scaling ansatz is unphysical, as higher moments diverge even in finite systems. The result $\beta = \nu$ is due to a neglect of the lower cutoff. Moreover, the possible interpretation of $\alpha \approx 2.02$ as $\tau$ leads to system size independent finite first moment, which contradicts $\langle s \rangle \propto L$. In this light it would be very interesting to repeat the data analysis.