Affine extensions of loops
Ágota Figula and Karl Strambach

1 Introduction

Most of the known examples of loops $L$ with strong relations to geometry have classical groups as the groups generated by their left translations ([7], [10], [9], [6], [8], Chapter 9, [12], Chapters 22 and 25, [4], [5]). These groups $G$ may be seen as subgroups of the stabilizer of 0 in the group of affinities of suitable affine spaces $A_n$, and as the elements of the loops $L$ one can often take certain projective subspaces of the hyperplane at infinity of $A_n$. The semidirect products $T \rtimes G$, where $T$ is the translation group of the affine space $A_n$, have in many cases a geometric interpretation as motion groups of affine metric geometries. In the papers [4], [5] three dimensional connected differentiable loops are constructed which have the connected component of the motion group of the 3-dimensional hyperbolic or pseudo-euclidean geometry as the group topologically generated by the left translations and which are Bol, Bruck or left A-loops. The set of the left translations of these loops induces on the plane at infinity the set of left translations of a loop isotopic to the hyperbolic plane loop (cf. [12], Chapter 22, p. 280, [9], p. 189). This and the fact that, up to our knowledge, there are only few known examples of sharply transitive sections in affine metric motion groups, motivated us to seek a simple geometric procedure for an extension of a loop realized as the image $\Sigma^*$ of a sharply transitive section in a subgroup $G^*$ of the projective linear group $PGL(n - 1, K)$ to a loop realized as the image of a sharply transitive section in a group $\Delta = T' \rtimes C$ of affinities of the $n$-dimensional projective plane $A_n = \mathbb{K}^n$ over a commutative field $\mathbb{K}$. Moreover, we desire that $T'$ is a large subgroup of affine translations and that $\alpha(C) = G^*$ holds for the canonical homomorphism $\alpha : GL(n, \mathbb{K}) \to PGL(n, \mathbb{K})$. We show that this goal can be achieved if in the $(n - 1)$-dimensional projective

\footnotesize
\begin{flushleft}
\textsuperscript{1}2000 Mathematics Subject Classification: 20N05, 51F25, 12D15, 51N30, 51M05. \\
\textsuperscript{2}Key words and phrases: loop, Bol loops, unitary and orthogonal geometries with positive index.
\end{flushleft}
hyperplane \( E \) of infinity of \( \mathcal{A}_n \) for \( G^* \) there exists an orbit \( \mathcal{O} \) of \( m \)-dimensional subspaces such that \( \Sigma^* \) acts sharply transitively on \( \mathcal{O} \), if there is a subspace of dimension \((n - 1 - m)\) having empty intersection with any element of \( \mathcal{O} \) and if the restriction of \( \alpha^{-1} \) to \( \Sigma^* \) defines a bijection from \( \alpha^{-1}(\Sigma^*) \) onto \( \Sigma^* \).

In the third section we demonstrate that our construction successfully can be applied to sharply transitive sections in unitary and orthogonal groups \( SU_{p_2}(n, F) \) of positive index \( p_2 \) over ordered pythagorean \( n \)-real fields \( F \). In this way we obtain many non-isotopic topological loops. The groups generated by the left translations of these loops are semidirect products \( T \rtimes C \), where \( T \) is the full translation group of \( \mathcal{A}_n \) and where \( \alpha(C) \) is a non-solvable normal subgroup of \( \alpha(SU_{p_2}(n, F)) \).

In the last section we take for the groups \( G \) unitary or orthogonal Lie groups of any positive index in order to obtain differentiable loops \( L \) such that the group topologically generated by the left translations of \( L \) is a pseudo-unitary motion group or the connected component of a pseudo-euclidean motion group.

## 2 Some basic notations of loop theory

A set \( L \) with a binary operation \( (x, y) \mapsto x \cdot y \) is called a loop if there exists an element \( e \in L \) such that \( x = e \cdot x = x \cdot e \) holds for all \( x \in L \) and the equations \( a \cdot y = b \) and \( x \cdot a = b \) have precisely one solution which we denote by \( y = a \setminus b \) and \( x = b / a \). The left translation \( \lambda_a : y \mapsto a \cdot y : L \rightarrow L \) is a bijection of \( L \) for any \( a \in L \). Two loops \((L_1, \cdot)\) and \((L_2, \ast)\) are isotopic if there are three bijections \( \alpha, \beta, \gamma : L_1 \rightarrow L_2 \) such that \( \alpha(x) \ast \beta(y) = \gamma(x \cdot y) \) holds for any \( x, y \in L_1 \). A loop \((L, \cdot)\) is called topological if \( L \) is a topological space and the mappings \((x, y) \mapsto x \cdot y, (x, y) \mapsto x \setminus y, (x, y) \mapsto y / x : L^2 \rightarrow L \) are continuous. A loop \((L, \cdot)\) is called differentiable if \( L \) is a \( C^\infty \)-differentiable manifold and the mappings \((x, y) \mapsto x \cdot y, (x, y) \mapsto x \setminus y, (x, y) \mapsto y / x : L^2 \rightarrow L \) are differentiable.

A loop \( L \) is a Bol loop if the identity \( x(y \cdot xz) = (x \cdot yx)z \) holds. A Bruck loop is a Bol loop \((L, \cdot)\) satisfying the automorphic inverse property, i.e. the identity \((x \cdot y)^{-1} = x^{-1} \cdot y^{-1} \) for all \( x, y \in L \). A loop \( L \) is a left A-loop if each \( \lambda_{x,y} = \lambda_{y,x}^{-1} \lambda_x \lambda_y : L \rightarrow L \) is an automorphism of \( L \).

Let \( G \) be the group generated by the left translations of \( L \) and let \( H \) be the stabilizer of \( e \in L \) in the group \( G \). The left translations of \( L \) form a subset of \( G \) acting on the cosets \( \{xH; x \in G\} \) such that for any given cosets \( aH \) and \( bH \) there exists precisely one left translation \( \lambda_x \) with \( \lambda_x aH = bH \).

Conversely let \( G \) be a group, let \( H \) be a subgroup of \( G \) and let \( \sigma : G/H \rightarrow G \) be a section with \( \sigma(H) = 1 \in G \) such that the subset \( \sigma(G/H) \) generates \( G \).
and acts sharply transitively on the space $G/H$ of the left cosets $\{xH, x \in G\}$ (cf. [12], p. 18). We call such a section sharply transitive. Then the multiplication defined by $xH \ast yH = \sigma(xH)yH$ on the factor space $G/H$ or by $x \ast y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. If $N$ is the largest normal subgroup of $G$ contained in $H$ then the factor group $G/N$ is isomorphic to the group generated by the left translations of $L(\sigma)$.

Two loops $L_1$ and $L_2$ having the same group $G$ as the group generated by the left translations and the same stabilizer $H$ of $e \in L_1, L_2$ are isomorphic if there is an automorphism of $G$ leaving $H$ invariant and mapping $\sigma_1(G/H)$ onto $\sigma_2(G/H)$. The automorphisms of a loop $L$ corresponding to a sharply transitive section $\sigma : G/H \to G$ are given by the automorphisms of $G$ leaving $H$ and $\sigma(G/H)$ invariant. If two loops are isotopic then the groups generated by their left translations are isomorphic ([13], Theorem III.2.7, p. 65). Loops $L$ and $L'$ having the same group $G$ generated by their left translations are isotopic if and only if there is a loop $L''$ isomorphic to $L'$ having $G$ again as the group generated by its left translations and there exists an inner automorphism $\tau$ of $G$ mapping $\sigma''(G/H)$ belonging to $L''$ onto the set $\sigma(G/H)$ corresponding to $L$ (cf. [12], Theorem 1.11. pp. 21-22).

3 Affine extensions

Let $G$ be a subgroup of the general linear group $GL(n, \mathbb{K})$ over a commutative field $\mathbb{K}$. Denote by $\alpha$ the canonical epimorphism from $GL(n, \mathbb{K})$ onto $PGL(n, \mathbb{K})$. The kernel $Z$ of $\alpha$ is the centre of $GL(n, \mathbb{K})$. Let $\tilde{H}$ be a subgroup of $G$ with $Z \cap G \leq \tilde{H}$ such that for the pair $G^* = \alpha(G)$ and $H^* = \alpha(\tilde{H})$ there exists a sharply transitive section $\sigma^* : G^*/H^* \to G^*$ determining a loop $L^*$. Moreover, we assume that $\Sigma^* := \sigma^*(G^*/H^*)$ generates $G^*$ and that for the preimage $(\alpha|G)^{-1}(\Sigma^*) = \Sigma \subseteq G$ one has $\tilde{H} \cap \Sigma = \{1\}$. Then the mapping $\alpha$ induces a bijection from $\Sigma$ onto $\Sigma^*$.

We denote by $\mathcal{A}_n$ the $n$-dimensional affine space $\mathbb{K}^n$ and by $E$ the projective hyperplane of dimension $(n-1)$ at infinity of $\mathcal{A}_n$. Let $U^*$ be an $m$-dimensional subspace of $E$ having $H^*$ as the stabilizer of $U^*$ in $G^*$. Let $\mathcal{X}$ be the set

$$\mathcal{X} = \{\gamma U^*; \gamma \in \Sigma^*\}.$$  

The elements of $\mathcal{X}$ may be seen as the elements of $L^*$ such that $U^*$ is the identity of $L^*$ and the multiplication is given by $X^* \circ Y^* = \tau_{U^*,X^*}(Y^*)$ for all $X^*, Y^* \in \mathcal{X}$, where $\tau_{U^*,X^*}$ is the unique element of the sharply transitive set $\Sigma^*$ of the linear transformations of $E$ mapping $U^*$ onto $X^*$.

Let $A = T \rtimes S$ be the semidirect product consisting of affinities of $\mathcal{A}_n = \mathbb{K}^n$, where $T$ is the translation group of $\mathcal{A}_n$ and $S$ is the stabilizer of $0 \in \mathcal{A}_n$. 

3
isomorphic to the group $GL(n, \mathbb{K})$. We consider the group $G$ as a subgroup of $S$ in the group $\Theta = \mathbb{K}^* \rtimes G$ of affinities of $\mathcal{A}_n$. The subgroup $\hat{H}$ of $S$ fixes the point $0 \in \mathcal{A}_n$ and the subspace $U^*$ of the hyperplane $E$. Let $U$ be the $(m + 1)$-dimensional affine subspace containing 0 and intersecting $E$ in $U^*$. If $H$ is the stabilizer of $U$ in the group $\Theta$, then one has $\hat{H} = H \cap \Theta_0$, where $\Theta_0$ is the stabilizer of the point 0 in $\Theta$.

Let $W$ be a subspace of $\mathcal{A}_n$ such that $W$ contains 0, has affine dimension $(n - m - 1)$ and intersects any subspace of the set $Z := \{ \rho(U); \rho \in \Sigma \}$ only in 0. Let $T_W$ be the group of affine translations $x \mapsto x + w : \mathcal{A}_n \to \mathcal{A}_n$ with $w \in W$. Then $W$ intersects any subspace $\delta(Y)$, where $\delta \in T_W$ and $Y \in Z$, in precisely one point. Moreover, the stabilizer of $\delta(Y)$ in $T_W$ consists only of the identity.

**Theorem 1.** The subset $\Xi = T_W \Sigma = \{ \tau \rho; \tau \in T_W, \rho \in \Sigma \}$ of the group $\Theta = T \rtimes G$ acts sharply transitively on the set

$$U = \{ \psi(U); \psi \in \Xi \} = \{ \psi(U); \psi \in \Theta \}.$$

The elements of $U$ can be taken as the elements of a loop $L_\Xi$ which has $U$ as the identity and for which the multiplication is defined by

$$X \circ Y = \tau_{U,X}(Y) \quad \text{for all } X, Y \in U,$$

where $\tau_{U,X}$ is the unique element of $\Xi$ mapping $U$ onto $X$.

The set $\Xi$ is the set of the left translations of $L_\Xi$ and generates a group $\Delta$ which is a semidirect product $\Delta = T' \rtimes C$, where the normal subgroup $T'$ consists of translations of the affine space $\mathcal{A}_n$ and $C$ is a subgroup of $G$ with $\alpha(C) = G^*$. There is a sharply transitive section $\sigma : \Delta/\hat{H} \to \Delta$ such that $\sigma(\Delta/\hat{H}) = \Xi$, the group $\hat{H}$ is the stabilizer of $U$ in $\Delta$ and the subgroup $T' \cap \hat{H}$ consists of all translations $x \mapsto x + u : \mathcal{A}_n \to \mathcal{A}_n$ with $u \in U$.

**Proof.** Let $D_1$ and $D_2$ be elements belonging to $U$. We show that there is precisely one element $\beta \in \Xi$ with $\beta(D_1) = D_2$. Let $D_1^* = D_1 \cap E$ and $D_2^* = D_2 \cap E$, where $E$ is the hyperplane at infinity of $\mathcal{A}_n$. Thus there exists precisely one element $\rho^* \in \Sigma^*$ and hence there exists precisely one element $\rho \in \Sigma$ with $\alpha(\rho) = \rho^*$ such that $\rho^*(D_1^*) = D_2^*$. The subspaces $\rho(D_1)$ and $D_2$ intersect $E$ in $D_2^*$. In the group $T_W$ there exists precisely one translation $\tau$ mapping the point $\rho(D_1) \cap W$ onto the point $D_2 \cap W$. Hence the element $\beta = \tau \rho$ is the only element in $\Xi$ mapping $D_1$ onto $D_2$ and the set $\Xi$ is a sharply transitive set on $U$. It follows that the subspaces in $U$ can be taken as the elements of a loop $L_\Xi$ having $U$ as the identity, such that the multiplication is defined as in the assertion of the theorem.
The group \( \Delta \) generated by the left translations of \( L \) is a subgroup of \( \Theta = T \rtimes G \). Let \( \hat{H} \) be the stabilizer of \( U \) in \( \Delta \). Since \( \Xi \) is the image of a sharply transitive section \( \sigma : \Delta / \hat{H} \to \Delta \) we have \( \Delta(U) = \Xi \hat{H}(U) = \Xi(U) \). Let \( \hat{H} \) be the stabilizer of \( U \) in \( \Delta \). Since \( \Xi \) is the image of a sharply transitive section \( \sigma : \Delta / \hat{H} \to \Delta \) we have \( \Delta(U) = \Xi \hat{H}(U) = \Xi(U) \).

Let \( T_U \) be the group of affine translations \( x \mapsto x + u : \mathbb{A}^n \to \mathbb{A}^n \) with \( u \in U \). Since \( W \oplus U = \mathbb{K}^n \) we have that \( T = T_W \times T_U \). Thus one has \( \Delta T(U) = \Delta T_W T_U(U) = \Delta T_W(U) = \Delta(U) \) since \( T_W \leq \Delta \). For the group \( \Lambda \) of dilatations \( x \mapsto ax : \mathbb{A}^n \to \mathbb{A}^n \) with \( a \in \mathbb{K} \setminus \{0\} \) we have that \( T \Lambda \) is a normal subgroup of \( \Theta \Lambda \) and \( \Lambda(U) = U \). Moreover \( \Theta(U) = \Delta T \Lambda(U) \) since the kernel of the restriction of \( \alpha : GL(n, \mathbb{K}) \to PGL(n, \mathbb{K}) \) to \( G \) consists only of dilatations.

The group \( \Delta \) contains a normal subgroup \( N \) fixing the hyperplane \( E \) at infinity pointwise. Since \( \Sigma^* \) generates \( G^* \) we see that \( \Delta/N \) is isomorphic to \( G^* \).

Let \( T' = T \cap \Delta \). Then \( \Delta \) is the semidirect product of \( \Delta = T' \rtimes C \), where \( C \) is the stabilizer of 0 in \( \Delta \) and \( CN/N \) is isomorphic to \( G^* \).

4 Applications

Let \( R \) be an ordered pythagorean field and let \( K = R(i) \) be the algebraic extension of \( R \) such that \( i^2 = -1 \). Let \( F \in \{ R, K \} \) and let \( V = F^n \) be an \( n \)-dimensional \( F \)-vector space for a fixed \( n \geq 3 \). The automorphism \( a \mapsto \bar{a} : F \to F \) is the identity if \( F = R \) or the involutory automorphism fixing \( R \) elementwise and mapping \( i \) onto \( -i \) if \( F = K \). Denote by \( M_n(F) \) the set of the \((n \times n)\)-matrices over \( F \). If \( A = (a_{i,j}) \) is a matrix in \( M_n(F) \) then \( \bar{A}^t = (\bar{a}_{j,i}) \). Let \( \mathcal{H}(n, F) \) be the set of positive definite hermitian \((n \times n)\)-matrices, i.e. the set

\[
\mathcal{H}(n, F) = \{ A \in M_n(F); A = \bar{A}^t \text{ with } \bar{v}^t Av > 0 \text{ for all } v \in V \setminus \{0\} \}.
\]

We assume that the field \( R \) is \( n \)-real which means that the characteristic polynomial of every matrix in \( \mathcal{H}(n, F) \) splits over \( K \) into linear factors. Thus this polynomial splits into linear factors already over \( R \) (cf. [8], p. 14). The class of \( n \)-real fields contains the class of totally real fields (cf. [8], p. 13), which is larger than the class of real closed fields and the class of hereditary euclidean fields. A hereditary euclidean field \( k \) is an ordered field such that every formally real algebraic extension of \( k \) has odd degree over \( k \) (cf. [15], Satz 1.2 (3), p. 197).

The group

\[
U(n, F) = \{ B \in GL(n, F); B \bar{B}^t = I_n \},
\]

where \( I_n \) is the identity in \( GL(n, F) \), is called the orthogonal group for \( F = R \).
and the unitary group for \( F = K \). Let
\[
J_{(p_1,p_2)} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)
\]
be the diagonal \((n \times n)\)-matrix such that the first \( p_1 \) entries are 1 and the remaining \( p_2 \) entries are -1. We have \( p_1 + p_2 = n \). The matrix \( J_{(p_1,p_2)} \) defines a hermitian form on \( F^n \) for \( F = K \) and an orthogonal form for \( F = R \) by
\[
\bar{v}^t J v = \sum_{i=1}^{p_1} \bar{v}_i v_i - \sum_{j=p_1+1}^{n} \bar{v}_j v_j.
\]

Let \( p_2 > 0 \). The unitary (orthogonal) group of index \( p_2 \) is the set
\[
U_{p_2}(n, F) = \{ A \in GL_n(F); \bar{A}^t J_{(p_1,p_2)} A = J_{(p_1,p_2)} \}.
\]

Since the group \( U_{p_2}(n, F) \) is isomorphic to the group \( U_{(n-p_2)}(n, F) \) (cf. [14], Proposition 9.11, p. 153) we may assume that \( p_1 \geq p_2 \). Let
\[
\Omega_{(p_1,p_2)}(F) = U_{p_2}(n, F) \cap U(n, F) \quad \text{and} \quad \Sigma_{(p_1,p_2)}(F) = U_{p_2}(n, F) \cap H(n, F).
\]
The group \( \Omega_{(p_1,p_2)}(F) \) is the direct product \( \Omega_{(p_1,p_2)}(F) = U(p_1, F) \times U(p_2, F) \), where \( U(p_1, F) \) may be identified with the group \( \left( \begin{array}{cc} U(p_1, F) & 0 \\ 0 & I_{p_2} \end{array} \right) \) and \( U(p_2, F) \) may be identified with the group \( \left( \begin{array}{cc} I_{p_1} & 0 \\ 0 & U(p_2, F) \end{array} \right) \); here \( I_{p_i} \) is the identity in \( GL(p_i, F) \) (cf. [5], Theorem 9.13, p. 123).

According to [5] (Theorem 9.11, p. 121) the set \( \Sigma_{(p_1,p_2)}(F) \) is the image of a sharply transitive section \( \sigma' : U_{p_2}(n, F)/\Omega_{(p_1,p_2)}(F) \to U_{p_2}(n, F) \) such that the corresponding loop \( L_{(p_1,p_2)} \) is a Bruck loop.

The group \( G_{(p_1,p_2)} \) generated by the set \( \Sigma_{(p_1,p_2)}(F) \) of the left translations of \( L_{(p_1,p_2)} \) is contained in the group \( SU_{p_2}(n, F) := \{ A \in U_{p_2}(n, F); \det A = 1 \} \) (cf. [5], 9.14, p. 124). Thus the loop \( \bar{L}_{(p_1,p_2)} \) corresponds also to the section
\[
\sigma : SU_{p_2}(n, F)/\Phi \to SU_{p_2}(n, F),
\]
where \( \Phi := (U(p_1, F) \times U(p_2, F)) \cap SU_{p_2}(n, F) \).

The kernel of the restriction of \( \alpha : GL(n, F) \to PGL(n, F) \) to the group \( SU_{p_2}(n, F) \) consists of the matrices \( D_a = \text{diag}(a, \ldots, a), \ a \in F \setminus \{0\} \) and \( a^2 = 1 \). Moreover one has \( a\bar{a} = 1 \) since any matrix \( D_a \) satisfies \( \bar{D}_a^t J_{(p_1,p_2)} D_a = J_{(p_1,p_2)} \). Thus any matrix \( D_a \) is contained in \( \Phi \) and \( \alpha \) induces a bijection from \( \Sigma_{(p_1,p_2)}(F) \) onto \( \alpha(\Sigma_{(p_1,p_2)}(F)) \). The set \( \alpha(\Sigma_{(p_1,p_2)}(F)) \) is the image of a sharply transitive section
\[
\sigma^* : \alpha(SU_{p_2}(n, F))/\alpha(\Phi) \to \alpha(SU_{p_2}(n, F))
\]
which corresponds to a Bruck loop \( L^*_{(p_1,p_2)} \).
The elements of \( \Sigma_{(p_1,p_2)}(F) \) are matrices \( A \in SU_{p_2}(n, F) \) satisfying the relations \( A = \tilde{A}^t \) and \( \tilde{v}^t Av > 0 \) for all \( v \in V \setminus \{0\} \). With \( A \) also \( A^{-1} \) is contained in \( \Sigma_{(p_1,p_2)}(F) \) ([8] 1.11, p. 16). Because of \( B^{-1} = B^t \) for all \( B \in \Phi \) and \( B^t AB \in \Sigma_{(p_1,p_2)}(F) \) ([8] 1.11, p. 16) one has

\[
B^{-1} AB \in \Sigma_{(p_1,p_2)}(F) \quad \text{for all } B \in \Phi \text{ and } A \in \Sigma_{(p_1,p_2)}(F). \tag{1}
\]

Since \( \sigma \) is a section every element \( S \) of \( SU_{p_2}(n, F) \) can be written in a unique way as \( S = S_1 C \) with \( S_1 \in \Sigma_{(p_1,p_2)}(F) \) and \( C \in \Phi \). The set

\[
\Sigma_{(p_1,p_2)}(F)^G_{(p_1,p_2)} = \{ Y^{-1}XY; \ X \in \Sigma_{(p_1,p_2)}(F), Y \in G_{(p_1,p_2)} \}
\]

is invariant with respect to the conjugation by the elements \( S \in SU_{p_2}(n, F) \):

\[
S^{-1}Y^{-1}XYS = C^{-1}S_1^{-1}Y^{-1}XYS_1C = [(C^{-1}S_1^{-1}C)(C^{-1}Y^{-1}C)][(C^{-1}YC)(C^{-1}S_1C)] \in \Sigma_{(p_1,p_2)}(F)^G_{(p_1,p_2)}. \]

Hence the group \( G_{(p_1,p_2)} \), which is generated also by \( \Sigma_{(p_1,p_2)}(F)^G_{(p_1,p_2)} \), is a normal non-central subgroup of \( SU_{p_2}(n, F) \). Then according to Théorème 5 in [2] p. 70 the group \( G_{(p_1,p_2)} \) coincides with \( SU_{p_2}(n, F) \) if \( F = K \). If \( F = R \) and \( (n, p_2) \neq (4, 2) \) then the group \( G_{(p_1,p_2)} \) contains the commutator subgroup \( [SU_{p_2}(n, F)]' =: K_{(n,p_2)} \) of \( SU_{p_2}(n, F) \) ([3], p. 63 and pp. 58-59). If \( F = R \) and \( (n, p_2) = (4, 2) \) then the commutator subgroup \( K_{(4,2)} \) is isomorphic to the direct product \( PSL_2(R) \times PSL_2(R) \) ([3], p. 59). Since the hermitian matrices in the set \( \Sigma_{(2,2)}(F) \) depend on 3 free parameters ([8], 9.12, p. 122) the group \( G_{(2,2)} \) contains \( K_{(4,2)} \). Therefore in any case the group \( G_{(p_1,p_2)} \) is a normal subgroup of \( SU_{p_2}(n, F) \) containing \( K_{(n,p_2)} \).

The group \( G_{(p_1,p_2)} \) leaves the value \( \tilde{v}^tJ_{(p_1,p_2)}v \) invariant since

\[
\tilde{v}^t(A^tJ_{(p_1,p_2)}A)v = \tilde{v}^tJ_{(p_1,p_2)}v \quad \text{for } A \in SU_{p_2}(n, F).
\]

We see the group \( G_{(p_1,p_2)} \) as a subgroup of the stabilizer of the element 0 in the group \( A \) of affinities of \( A_n = F^n \), and the group \( \alpha(G_{(p_1,p_2)}) := G^*_{(p_1,p_2)} \) as a subgroup of the group \( PGL(n, F) \) which acts on the \( (n-1) \)-dimensional projective hyperplane \( E \) at infinity of \( A_n \).

We embed the affine space \( A_n \) into the \( n \)-dimensional projective space \( P_n(F) \) such that \( (x_1, \ldots, x_n) \mapsto F^*(1, x_1, \ldots, x_n) \), \( x_i \in F \) for all \( 1 \leq i \leq n \) and \( F^* = F \setminus \{0\} \). With respect to this embedding the hyperplane \( E \) consists of the points \( \{ F^*(0, x_1, \ldots, x_n), x_i \in F, \text{not all } x_i = 0 \} \). The cone in \( A_n \) which is described by the equation

\[
(*) \quad \sum_{i=1}^{p_1} \bar{x}_i x_i - \sum_{j=p_1+1}^{n} \bar{x}_j x_j = 0
\]
intersects $E$ in a hyperquadric $C$; the points $\{F^*(0, x_1, \cdots, x_n)\}$ of $C$ satisfy the equation $(\ast)$. The hypersurface $C$ of $E$ divides the points of $E \setminus C$ into two regions $R_1$ and $R_2$. A point $F^*(0, x_1, \cdots, x_n)$ belongs to $R_1$ if and only if
\[ \sum_{i=1}^{p_1} \bar{x}_i x_i > \sum_{j=p_1+1}^{n} \bar{x}_j x_j. \]
It belongs to $R_2$ if and only if
\[ \sum_{i=1}^{p_1} \bar{x}_i x_i < \sum_{j=p_1+1}^{n} \bar{x}_j x_j. \]

The group $\alpha(SU_{p_2}(n, F)) = SU_{p_2}(n, F)/\Lambda'$, where $\Lambda'$ is the group of dilations contained in $SU_{p_2}(n, F)$, leaves $R_1, R_2$ as well as $C$ invariant since for any $f \in F$ and $v \in V = F^n$ one has $(\bar{f} \bar{v}^t) J_{(p_1, p_2)}(fv) = (\bar{f} f) (\bar{v}^t J_{(p_1, p_2)} v)$ and $\bar{f} f > 0$. The group $\alpha(\Phi) = \Phi/(\Phi \cap \Lambda')$ leaves the subspace
\[ W_1^* = \{(0, x_1, \ldots, x_{p_1}, 0, \ldots, 0); x_i \in F\} \subseteq E \]
as well as the subspace
\[ W_2^* = \{(0, \ldots, 0, x_{p_1+1}, \ldots, x_n); x_i \in F\} \subseteq E \]
invariant. The intersection $W_1^* \cap W_2^*$ is empty since $W_i^* \subseteq R_i$, $i = 1, 2$.

Let $W_i, i = 1, 2$, be the $p_i$-dimensional affine subspace of $A_n$ containing $0$ such that $W_i \cap E = W_i^*$. Thus $W_1 \cap W_2 = \{0\}$. Let $W_j$ be a $p_j$-dimensional affine subspace of $A_n$ such that $p_j = n - p_i$ and $W_j$ intersects $W_i$ only in the point $0$. Thus $W_j$ intersects any subspace of the set
\[ Z_i = \{\rho(W_i), \rho \in G_{(p_1, p_2)}\} = \{\lambda(W_i), \lambda \in \Sigma_{(p_1, p_2)}(F)\}, \]
where $i \neq j$, only in $0$. Affine subspaces $\tilde{W}_j$ with these properties exist, one can take for instance $\tilde{W}_j = \rho(W_j) \in Z_j$.

Let $\Theta$ be the semidirect product $\Theta = T \rtimes G_{(p_1, p_2)}$, where $T$ is the translation group of $A_n$. By Theorem 1 the set $\Xi_{(p, \tilde{W}_j)} = \{T_{\tilde{W}_j} \Sigma_{(p_1, p_2)}(F)\}, i \neq j$, acts sharply transitively on the set
\[ U_i = \{\psi(W_i); \psi \in \Xi_{(p, \tilde{W}_j)}\}. \]

Thus a loop $L_{(p, \tilde{W}_i)}$ is realized on $U_i$.

The group $SU_{p_2}(n, K)$ acts irreducibly on the vector space $V = K^n$ and the commutator subgroup $K_{(n, p_2)}$ of $SO_{p_2}(n, R)$ acts irreducibly on $V = R^n$ (cf. [1], Theorem 3.24, p. 136). Hence the group $\Delta$ generated by the left translations $\Xi_{(p, \tilde{W}_j)}$ of the loop $L_{(p, \tilde{W}_j)}$ contains all translations of the affine space $A_n$. It follows that $\Delta$ is the semidirect product $\Delta = T \rtimes C$ of the translation group $T$ by a subgroup $C$ of the stabilizer of $0 \in A_n$ in the group $A$ of affinities. If $F = K$ then $C$ is isomorphic to $SU_{p_2}(n, K)$ and the stabilizer $\tilde{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{\tilde{W}_i} \rtimes \Phi$ since any
element $g \in G_{(p_1, p_2)} = SU_{p_2}(n, K)$ has a unique representation as $g = g_1 g_2$ with $g_1 \in \Sigma_{(p_1, p_2)}(K)$ and $g_2 \in \Phi$. If $F = R$ then $C$ is a normal subgroup of $SO_{p_2}(n, R)$ containing $\mathcal{K}_{(n, p_2)}$ and the stabilizer $\mathcal{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{W_i} \rtimes \Gamma$, where $\Gamma = \Phi \cap C$.

For $p_1 > p_2$ the loop $L_{(p_1, W_2)}$ is never isotopic to a loop $L_{(p_2, W_1)}$. This follows from the fact that the stabilizer $H_k$, $k = 1, 2$, of the identity of $L_{(p_k, W_k)}$ with $l \neq k$ in $\Delta$ contains the group $T_{W_k}$ as the largest normal subgroup consisting of affine translations. Since $T_{W_1}$ is not isomorphic to $T_{W_2}$ one has that $H_1$ is not isomorphic to $H_2$. (cf. [13], Theorem III.2.7, p. 65)

Now we consider the loops $L_{(p_1, W_j)}$ and $L_{(p_i, W_j)}$ for $W_j \neq W_j$. According to (1) the subspaces $W_1$ and $W_2$ are invariant under the subgroup $\Phi$ of the stabilizer of $0 \in A_n$ in the group $A$ of affinities. Hence if $g \in \Phi$ then one has $g \Sigma_{(p_1, p_2)}(F) g^{-1} = \Sigma_{(p_1, p_2)}(F)$ and $g T_{W_k} g^{-1} = T_{W_k}$, $k = 1, 2$, for the group $T_{W_k} = \{ x \mapsto x + w_k; w_k \in W_k \}$. This yields $g \Xi_{(p_1, W_j)} g^{-1} = \Xi_{(p_i, W_j)}$. For $W_j \neq W_j$ the group $\Phi$ does not normalize the translation group $T_{W_j}$. Therefore

$$g T_{W_j} \Sigma_{(p_1, p_2)}(F) g^{-1} = (g T_{W_j} g^{-1})(g \Sigma_{(p_1, p_2)}(F) g^{-1}) =$$

$$(g T_{W_j} g^{-1}) \Sigma_{(p_1, p_2)}(F) \neq \Xi_{(p_i, W_j)}$$

for suitable elements $g \in \Phi$. This means that not all elements of $\Phi$ induce automorphisms of $L_{(p_1, W_j)}$. Therefore the loops $L_{(p_1, W_j)}$ and $L_{(p_i, W_j)}$ are not isomorphic if $W_j \neq W_j$.

**Proposition 2.** Any loop $L_{(p_1, W_j)}$ is a topological loop with respect to the topology induced on the set $\mathcal{U}$ by the topology on the set of the $p_i$-dimensional subspaces of $A_n$ which is derived from the topology of the topological field $F$.

**Proof.** Since $R$ is an ordered field, $R$ as well as $K = R(i)$ are topological fields with respect to the topology given by the ordering of $R$. Then the ring $\mathcal{M}_n(F)$ of $(n \times n)$-matrices over $F$ is a topological ring such that the open $\varepsilon$-neighbourhoods of $0 \in \mathcal{M}_n(F)$ consist of matrices $(c_{ij})$ with $|c_{ij}| < \varepsilon$. The group $GL(n, F) \leq \mathcal{M}_n(F)$ is a topological group. Since the set $Z = \{ \text{diag}(a, \ldots, a), a \in F \setminus \{0\} \}$ is a closed subgroup of $GL(n, F)$ the group $PGL(n, F) = GL(n, F)/Z$ is a topological group. The subgroups $SU_{p_2}(n, F)$ and $\Phi = (U(p_1, F) \times U(p_2, F)) \cap SU_{p_2}(n, F)$ are closed subgroups of $GL(n, F)$. Moreover $SU_{p_2}(n, F)/Z$ as well as $\Phi Z/Z$ are closed subgroups of $PGL(n, F)$.

The affine space $A_n = F^n$ and the $(n - 1)$-dimensional projective hyperplane $E$ carry topologies derived from the topology of the field $F$ (cf. [13].
Chapter XI). The semidirect product $A = T \rtimes GL(n, F)$ is a topological group consisting of continuous affinities; it induces on the hyperplane $E$ a continuous group of projective collineations. Any subset of $A$ is a topological space with respect to the topology induced from $A$ and any subgroup of $A$ becomes a topological group in this manner.

Let $Q_1$ be a fixed $p_1$-dimensional subspace of $A_n$ and let $Q$ be the set of the affine $(n - p_1)$-dimensional affine subspaces with $|Q_1 \cap Q| = 1$ for $Q \in Q$. The set $Q$ also carries a topology determined by the topology of $F$. The set $Q^*$ of intersections $Q_1$ of the affine subspaces $Q$ of $Q$ with $E$ inherits the topology of the Grassmann manifold of the $(n - p_1 - 1)$-dimensional subspaces of the hyperplane $E$. The geometric operation $(Q, Q_1) \mapsto Q \cap Q_1 : Q \rightarrow Q_1$ is continuous.

On the topological space $\Sigma_{(p_1, p_2)}(F)$ a topological Bruck loop $L_{(p_1, p_2)}$ is realized by the multiplication

$$A \circ B = \sqrt{AB^2A} \text{ for all } A, B \in \Sigma_{(p_1, p_2)}(F),$$

where $X \mapsto \sqrt{X}$ is the inverse map of the bijection $X \mapsto X^2 : \Sigma_{(p_1, p_2)}(F) \rightarrow \Sigma_{(p_1, p_2)}(F)$ (cf. [8] (1.14), p. 17 and (9.1) Theorem (4), p. 108, [12], p. 121). We denote by $[\rho(W_i)]^*$ with $\rho \in \Sigma_{(p_1, p_2)}(F)$ the intersection of the subspace $\rho(W_i)$ with the hyperplane $E$ and by $Z_i^*$ the set $\{[\rho(W_i)]^* ; \rho \in \Sigma_{(p_1, p_2)}(F)\}$. For the elements of the loop $L_{(p_1, W_j)}$ one can take the elements of the set

$$U_{(p_1, W_j)} = \{w(W_i); \psi \in \Xi_{(p_1, W_j)}\} = \{\tau \rho(W_i) ; \tau \in T_{\tilde{W}_j}, \rho \in \Sigma_{(p_1, p_2)}(F)\}.$$

The subspace $\tilde{W}_j$ is homeomorphic to the group $T_{\tilde{W}_j}$, and the set $Z_i$ is homeomorphic to $\Sigma_{(p_1, p_2)}(F)$. Any element $\tau \rho(W_i) \in U_{(p_1, W_j)}$ is uniquely determined by $[\rho(W_i)]^*$ and $(\tau \rho(W_i)) \cap \tilde{W}_j$. The mapping

$$\omega : \tau \rho(W_i) \mapsto ((\tau \rho(W_i)) \cap \tilde{W}_j, [\rho(W_i)]^*)$$

from $U_{(p_1, W_j)}$ onto the topological product $\tilde{W}_j \times Z_i^*$ is a bijection such that

$$\omega^{-1} : (w, Z^*) \mapsto w \lor Z^*,$$

where $w \lor Z^*$ is the $p_1$-dimensional affine subspace containing $w \in \tilde{W}_j$ and intersecting $E$ in $Z^* \in Z_i^*$. Since the geometric operations of joining and of intersecting of distinct subspaces are continuous maps, $\omega$ is a homeomorphism.

Let $(w_k, Z_k^*) \in \tilde{W}_j \times Z_i^*$ with $k = 1, 2$ and let $\tau_k \rho_k(W_i)$ be the subspaces of $U_{(p_1, W_j)}$ such that $\omega(\tau_k \rho_k(W_i)) = (w_k, Z_k^*)$. The multiplication given by

$$(w_1, Z_1^*) \circ (w_2, Z_2^*) = (w_3, Z_3^*),$$

(3)
where $Z^*_3 = [\rho_1 \rho_2(W_i)]^*$ and

\[ w_3 = \tau_1[\rho_1(\tau_2 \rho_2(W_i)) \cap \tilde{W}_j] = \tau_1[(\rho_1(\tau_2 \rho_2(W_i)) \cap \tilde{W}_j) \vee [\rho_1 \rho_2(W_i)]^*] \cap \tilde{W}_j \]

yields a topological loop. This loop is homeomorphic to $L_{(p_1, \tilde{W}_j)}$ since

$[\rho_1 \tau_2 \rho_2(W_i)]^* = [\rho_1 \rho_2(W_i)]^*$ and $[\tau_1 \rho_1 \tau_2 \rho_2(W_i)]^* = [\rho_1 \rho_2(W_i)]^*$.

\[ \square \]

5 Special cases: $\mathbb{R}$ and $\mathbb{C}$

**Proposition 3.** The loop $L_{(p_1, \tilde{W}_j)}$ is a differentiable loop diffeomorphic to $\mathbb{R}^d$, where $d = \varepsilon(p_2 + p_1)$, with $\varepsilon = 1$ if $F = \mathbb{R}$ and $\varepsilon = 2$ if $F = \mathbb{C}$.

If $F = \mathbb{C}$ then the group $\Delta$ generated by the left translations of $L_{(p_1, \tilde{W}_j)}$ is the semidirect product $\mathbb{C}^n \rtimes SU_{p_2}(n, \mathbb{C})$ and the stabilizer of $W_i$ in $\Delta$ is the semidirect product $\mathbb{C}^n \rtimes \Pi$, where $\Pi$ is an epimorphic image of the direct product $SU_{p_1}(n, \mathbb{C}) \times SU_{p_2}(n, \mathbb{C}) \times SO_2(\mathbb{R})$.

If $F = \mathbb{R}$ then $\Delta$ is the semidirect product $\mathbb{R}^n \rtimes SO_{p_2}(n, \mathbb{R})^\circ$, where $SO_{p_2}(n, \mathbb{R})^\circ$ is the connected component of $SO_{p_2}(n, \mathbb{R})$, and the stabilizer of $W_i$ in $\Delta$ is the semidirect product $\mathbb{R}^n \rtimes (SO(p_1, \mathbb{R}) \times SO(p_2, \mathbb{R}))$.

**Proof.** Clearly the topological manifold $L_{(p_1, \tilde{W}_j)}$ carries the differentiable structure of the real differentiable manifold $\Xi_{(p_1, \tilde{W}_j)}$ which is the topological product of $T_{\tilde{W}_j}$ and $\Sigma_{(p_1, p_2)}(F)$.

According to Section 4 the group $\Delta$ topologically generated by the left translations $\Xi_{(p_1, \tilde{W}_j)}$ is the semidirect product $\Delta = F^n \rtimes C$, where $C$ contains the commutator subgroup of $SU_{p_2}(n, F)$.

If $F = \mathbb{C}$ then $C = SU_{p_2}(n, \mathbb{C})$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $T_{\tilde{W}_j} \rtimes \Phi$ with $\Phi = [U_{p_1}(n, \mathbb{C}) \times U_{p_2}(n, \mathbb{C})] \cap SU_{p_2}(n, \mathbb{C})$ which is a maximal compact subgroup of $SU_{p_2}(n, \mathbb{C})$ ([12], p. 28). The groups $SU_{p_2}(n, \mathbb{C})$ and $\Phi$ are connected therefore the groups $\Delta$ and $\hat{H}$ are connected. Since $\Delta$ is the topological product $\Xi_{(p_1, \tilde{W}_j)} \times \hat{H} = \Xi_{(p_1, \tilde{W}_j)} \times T_{\tilde{W}_j} \rtimes \Phi$ it follows that the manifold $\Xi_{(p_1, \tilde{W}_j)}$ and hence the loop $L_{(p_1, \tilde{W}_j)}$ are diffeomorphic to an affine space.

If $F = \mathbb{R}$ then $C$ is a subgroup of $SO_{p_2}(n, \mathbb{R})$ containing the commutator subgroup $K_{(n, p_2)}$. According to [3] p. 57 the factor group $SO_{p_2}(n, \mathbb{R})/K_{(n, p_2)}$ has order 2. Hence $K_{(n, p_2)}$ is the connected component of $SO_{p_2}(n, \mathbb{R})$. The group $\Phi = [O(p_1, \mathbb{R}) \times O(p_2, \mathbb{R})] \cap SO_{p_2}(n, \mathbb{R})$ is not connected since the factor group $O(p_i, \mathbb{R})/SO(p_i, \mathbb{R})$ has order 2 ([13], Corollary 9.37, p. 158) and the product $\alpha_1 \alpha_2$ with $\alpha_i \in O(p_i, \mathbb{R})$, but $\alpha_i \notin SO(p_i, \mathbb{R})$ for $i = 1, 2$, is an element of $SO_{p_2}(n, \mathbb{R})$. The group $SO_{p_2}(n, \mathbb{R})$ is homeomorphic to the topological product $\Sigma_{(p_1, p_2)}(\mathbb{R}) \times \Phi$. Since $SO_{p_2}(n, \mathbb{R})$ has two connected
components and $\Phi$ is not connected the manifold $\Sigma_{(p_1,p_2)}(\mathbb{R})$ is connected. It follows that the group $C$ generated by $\Sigma_{(p_1,p_2)}(\mathbb{R})$ is connected and hence isomorphic to the connected component $SO_{p_2}(n,\mathbb{R})^o = k_{(n,p_2)}$ of $SO_{p_2}(n,\mathbb{R})$. Thus the group $\Delta = T \ltimes C$ is connected. Moreover $\Delta$ is the topological product $\Xi_{(p_1,W_j)} \times \hat{H} = \Xi_{(p_1,W_j)} \times T_{W_i} \times (\Phi \cap \hat{H})$. Since $\Delta$, $\Xi_{(p_1,W_j)}$ and $T_{W_i}$ are connected, the group $\Phi \cap \hat{H}$ is connected and hence a maximal compact subgroup of $SO_{p_2}(n,\mathbb{R})$. This yields that $\Xi_{(p_1,W_j)}$ and $L_{(p_1,W_j)}$ are diffeomorphic to an affine space.

The group $\Delta$ is the topological product $\Xi_{(p_1,W_j)} \times \hat{H}$. Thus for the real dimension of $L_{(p_1,W_j)}$ one has

$$\dim L_{(p_1,W_j)} = \dim \Xi_{(p_1,W_j)} = \dim \Delta - \dim \hat{H}$$

where $\dim \Delta = \dim \Xi_{(p_1,W_j)} = \dim \Phi \cap \hat{H}$.

If $F = \mathbb{C}$ then the group $\Phi = C \cap \hat{H}$ is an epimorphic image of the direct product $SU(p_1,\mathbb{C}) \times SU(p_2,\mathbb{C}) \times SO_2(\mathbb{R})$ (cf. [16], p. 28). This yields

$$\dim L_{(p_1,W_j)} = [(p_1 + p_2)^2 - 1] + 2p_j - (p_j^2 - 1) - (p_j^2 - 1) = 2p_j + 2p_1p_2 - 2(2(m - 1)^2 + 2(m - 1))$$

for $0 \leq k \leq m$ (16, p. 26 and p. 28). It follows that $L_{(p_1,W_j)}$ is diffeomorphic to $\mathbb{R}^{2(p_1+p_1p_2)}$.

The group $\Delta$ is the semidirect product $\Delta = \mathbb{C}^n \rtimes C$, where $C$ is the group $SU_{p_2}(n,\mathbb{C})$ and the stabilizer $\hat{H}$ is the semidirect product $T_{W_i} \rtimes \Phi$, where $\Phi$ is an epimorphic image of $SU(p_1,\mathbb{C}) \times SU(p_2,\mathbb{C}) \times SO_2(\mathbb{R})$.

If $F = \mathbb{R}$ then $C \cap \hat{H} = SO(p_1,\mathbb{R}) \times SO(p_2,\mathbb{R})$ ([16], p. 31 and p. 38). It follows that

$$\dim L_{(p_1,W_j)} = \frac{1}{2}((p_1 + p_2)(p_1 + p_2 - 1) + p_j - \frac{1}{2}p_1(p_1 - 1) - \frac{1}{2}p_2(p_2 - 1) = p_j + p_1p_2.$$ 

Hence the loop $L_{(p_1,W_j)}$ is diffeomorphic to $\mathbb{R}^{p_1+p_1p_2}$.

The group $\Delta$ is the semidirect product $\Delta = \mathbb{R}^n \rtimes C$, where $C$ is the group $SO_{p_2}(n,\mathbb{R})^o$ and the stabilizer $\hat{H}$ of $W_i$ in $\Delta$ is the semidirect product $\mathbb{R}^n \times (SO(p_1,\mathbb{R}) \times SO(p_2,\mathbb{R})).$

The loop $L_{(p_1,W_j)}$ is diffeomorphic to the manifold $\tilde{W}_j \times Z_i$ since $Z_i$ is diffeomorphic to $\Sigma_{(p_1,p_2)}(\mathbb{R})$. The mapping $(x, D^*) \mapsto x \vee D^*$ assigning to a point $x \in \mathcal{A}_n = F^n, F \in \{\mathbb{R}, \mathbb{C}\}$ and to an element $D^*$ of the Graßmannian manifold of the $(p_1 - 1)$-dimensional $F$-subspaces of the hyperplane $E$ the affine subspace $D$ containing $x$ and intersecting $E$ in $D^*$ is differentiable. Also the mapping $D \mapsto D \cap \tilde{W}_j$ assigning to a $p_1$-dimensional affine $F$-subspace $D$ of $\mathcal{A}_n$ the point $D \cap \tilde{W}_j$ is differentiable. Since the loop realized on $\Sigma_{(p_1,p_2)}(F)$ by the multiplication (2) is differentiable, the representation of $L_{(p_1,W_j)}$ on the manifold $\tilde{W}_j \times Z_i$ by the multiplication (3) yields that $L_{(p_1,W_j)}$ is differentiable. 

\[\square\]
References

[1] E. Artin, *Geometric Algebra*, Interscience Publishers, New York London, 1957.

[2] J. A. Dieudonné, *Sur les groupes classiques*, Hermann, Paris, 1958.

[3] J. A. Dieudonné, *La géométrie des groupes classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer Verlag, Berlin Heidelberg New York, 1971.

[4] Á. Figula, *3-dimensional Bol loops as sections in non-solvable Lie groups*, accepted for publication in Forum Math.

[5] Á. Figula, *3-dimensional loops on non-solvable reductive spaces*, accepted for publication in Adv. Geometry.

[6] E. Gabrieli, H. Karzel, *Point reflection geometries, geometric K-loops and unitary geometries*, Resultate Math. 32 (1997), 66-72.

[7] B. Im, *K-loops in the Minkowski world over an ordered field*, Resultate Math. 25 (1994), 60-63.

[8] H. Kiechle, *Theory of K-Loops*, Lecture Notes in Mathematics. 1778. Springer-Verlag, Berlin, 2002.

[9] E. Kolb, A. Kreuzer, *Geometry of kinematic K-loops*, Abh. Math. Sem. Univ. Hamburg 65 (1995), 189-197.

[10] A. Konrad, *Hyperbolische loops über Oktaven und K-loops*, Resultate Math. 25 (1994), 331-338.

[11] H. Lenz, *Vorlesungen über projektive Geometrie*, Akademische Verlagsgesellschaft Geest and Portig, Leipzig, 1965.

[12] P. T. Nagy and K. Strambach, *Loops in Groups Theory and Lie Theory*, de Gruyter Expositions in Mathematics. 35. Berlin, New York, 2002.

[13] H. O. Pflugfelder, *Quasigroups and Loops: Introduction*, Heldermann-Verlag, Berlin, 1990.

[14] I. R. Porteous, *Topological Geometry*, Van Nostrand Reinhold Company, London, 1969.

[15] A. Prestel and M. Ziegler, *Erblich euklidische Körper*, J. Reine Angew. Math. 274-275 (1975), 196-205.
[16] J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, Lecture notes in mathematics 40. Springer Verlag, Berlin, 1967.

Ágota Figula  
Mathematisches Institut  
der Universität Erlangen-Nürnberg  
Bismarckstr. 1 1/2  
D-91054 Erlangen, Germany,  
figula@mi.uni-erlangen.de  
and  
Institute of Mathematics,  
University of Debrecen  
P.O.B. 12, H-4010 Debrecen,  
Hungary, figula@math.klte.hu

Karl Strambach  
Mathematisches Institut  
der Universität Erlangen-Nürnberg  
Bismarckstr. 1 1/2  
D-91054 Erlangen, Germany,  
strambach@mi.uni-erlangen.de