A finite dimensional approximation to pinned Wiener measure on some symmetric spaces

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Abstract

Let $M$ be a Riemannian manifold, $o \in M$ be a fixed base point, $W_o(M)$ be the space of continuous paths from $[0,1]$ to $M$ starting at $o \in M$, and let $\nu_x$ denote Wiener measure on $W_o(M)$ conditioned to end at $x \in M$. The goal of this paper is to give a rigorous interpretation of the informal path integral expression for $\nu_x$:

$$d\nu_x(\sigma) = \delta_x(\sigma(1)) \frac{1}{Z} e^{-\frac{1}{2}E(\sigma)}D\sigma, \quad \sigma \in W_o(M).$$

In this expression $E(\sigma)$ is the “energy” of the path $\sigma$, $\delta_x$ is the $\delta$–function based at $x$, $D\sigma$ is interpreted as an infinite dimensional volume “measure” and $Z$ is a certain “normalization” constant. We will interpret the above path integral expression as a limit of measures, $\nu^P_{1,P,x}$, indexed by partitions, $P$ of $[0,1]$. The measures $\nu^P_{1,P,x}$ are constructed by restricting the above path integral expression to the finite dimensional manifolds, $H^P_{P,x}(M)$, of piecewise geodesics in $W_o(M)$ which are allowed to have jumps in their derivatives at the partition points and end at $x$. The informal volume measure, $D\sigma$, is then taken to be a certain Riemannian volume measure on $H^P_{P,x}(M)$. When $M$ is a symmetric space of non–compact type, we show how to naturally interpret the pinning condition, i.e. the $\delta$–function term, in such a way that $\nu^P_{1,P,x}$ are in fact well defined finite measures on $H^P_{P,x}(M)$. The main theorem of this paper then asserts that $\nu^P_{1,P,x} \rightarrow \nu_x$ (in a weak sense) as the mesh size of $P$ tends to zero.

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1 Introduction

Let (\nabla) be the Levi-Civita covariant derivative on \(M\) which we add to the default set up \((M^d, g, \nabla, o)\).

The path space
\[
W_o(M) := \{\sigma \in C([0,1] \to M) \mid \sigma(0) = o\}
\]
is known as the Wiener space on \(M\) and let \(\nu\) be the Wiener measure on \(W_o(M)\)—i.e., the law of \(M\)-valued Brownian motion which starts at \(o \in M\).

Consider the heat equation of the following form:
\[
\frac{\partial}{\partial t} \phi = -H \phi, \quad \phi(x,0) = f(x),
\]
where \(H = -\frac{1}{2} \Delta_g + V\) is the Schrödinger operator, \(\Delta_g\) is the Laplace-Beltrami operator on \((M, g, o)\) and \(V : M \to \mathbb{R}\) is an external potential. Let \(e^{-tH}\) be the solution operator to (1.1). Under modest regularity conditions, this operator admits an integral kernel \(p_t^H(\cdot, \cdot)\). In the physics literature one frequently finds Feynman type informal identities of the form,
\[
p_{1}^H(o,x) = \frac{1}{Z} \int_{W_o(M)} \delta_x(\sigma(1)) e^{-\int_{0}^{1} (\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))) d\tau} D\sigma,
\]
and
\[
(e^{-H} f)(o) = \frac{1}{Z} \int_{W_o(M)} f(\sigma(1)) e^{-\int_{0}^{1} (\frac{1}{2}|\dot{\sigma}(\tau)|^2 + V(\sigma(\tau))) d\tau} D\sigma.
\]

Variants of these informal path integrals are often used as the basis for “defining” and making computations in quantum-field theories. From a mathematical perspective, making sense of such path integrals is thought to be a necessary step to developing a rigorous definition of interacting quantum field theories, (see for example; Glimm and Jaffe [13], Barry Simon [21], the Clay Mathematics Institute’s Millennium problem involving Yang-Mills and Mass Gap). In this paper we give an interpretation of the formal identity using a finite dimensional approximation scheme when the manifold is a symmetric space of non—compact type, for example, hyperbolic space \(\mathbb{H}^d\).

\[
\int_{W_o(M)} \delta_x(\sigma(1)) \frac{1}{Z} e^{-\int_{0}^{1} \frac{1}{2} |\dot{\sigma}(\tau)|^2 d\tau} D\sigma := p_t(o,x)
\]
where \(p_t(x,y)\) is the heat kernel associated to \(\frac{1}{2} \Delta_g\) on \(M\).
1.1 Finite Dimensional Approximation Scheme for Path Integrals

The central idea behind finite dimensional approximation scheme is to define a path integral as a limit of the same integrands restricted to “natural” approximate path spaces, for example, piecewise linear paths, broken lines, polygonal paths and so on. The ill–defined expression under these finite dimensional approximations usually becomes well–defined or has better interpretations, see (12, 15). For example, when \( M = \mathbb{R}^d \), it is known that Wiener measure on \( W_0 (\mathbb{R}^d) \) may be approximated by Gaussian measures on piecewise linear path spaces. More specifically, Eq. (1.3) with \( V = 0 \) and restricted to a finite dimensional subspace of piecewise linear paths based on a partition of \([0,1]\) has a natural interpretation as Gaussian probability measure. The interpretation results from the canonical isometry between the piecewise linear path space and \( \mathbb{R}^{dn} \), where \( n \) is the number of partition points. By combining Wiener’s theorem on the existence of Wiener measure with the dominated convergence theorem, one can see that these Gaussian measures converge weakly to \( \nu \) as the mesh of partition tends to zero, (see for example [9] Proposition 6.17 for details). An analogous theory on general manifolds was also developed, see for example Pinsky [20], Atiyah [2], Bismut [3], Andersson and Driver [1] and references therein. In [1], followed by [19] and [18], the finite dimensional approximation problem is viewed in its full geometric form by restricting the expression in Eq. (1.3) to finite dimensional sub-manifolds of piecewise geodesic paths on \( M \). Unlike the flat case \( (M = \mathbb{R}^d) \) where the choice of translation invariant Riemannian metric on path spaces is irrelevant, various Riemannian metrics on approximate path spaces are explored. Based on these metrics, different approximate measures are constructed which lead to different limiting measures on \( W_o(M) \), see [1], [18], and [19]. In this paper we adopt a so–called \( G^1_\sigma \) metric on the piecewise geodesic space.

In the remainder of this section, we establish some necessary notations.

**Definition 1.1 (Cameron-Martin space on \((M,g,o))** Let

\[
H(M) := \left\{ \sigma \in C([0,1] \to M) : \sigma(0) = o, \sigma \text{ is a.c. and } \int_0^1 |\sigma'(s)|_g^2 ds < \infty \right\}
\]

be the **Cameron–Martin space** on \((M,g,o)\). (Here a.c. means absolutely continuous.)

**Notation 1.2** Let \( \Gamma(TM) \) be differentiable sections of \( TM \) and \( \Gamma_\sigma(TM) \) be differentiable sections of \( TM \) along \( \sigma \in H(M) \).

The space, \( H(M) \), is an infinite dimensional Hilbert manifold which is a central object in problems related to calculus of variations on \( W_o(M) \). Klingenberg [16] contains a good exposition of the manifold of paths. In particular, Theorem 1.2.9 in [16] presents its differentiable structure in terms of atlases. We will be interested in certain Riemannian metrics on \( H(M) \) and on certain finite dimensional submanifolds where the formal path integrals make sense.

**Definition 1.3** For any \( \sigma \in H(M) \) and \( X,Y \in \Gamma_{\sigma-\text{c.}}(TM) \), We define a metric \( G^1 \) as follows:

\[
\langle X,Y \rangle_{G^1} = \int_0^1 \left\langle \nabla_X \frac{d\sigma}{ds}(s), \nabla_Y \frac{d\sigma}{ds}(s) \right\rangle_g ds
\]

where \( \Gamma_{\sigma-\text{c.}}(TM) \) is the set of absolutely continuous vector fields along \( \sigma \) with finite energy, i.e. \( \int_0^1 \left\langle \frac{\nabla_X}{ds}(s), \frac{\nabla_X}{ds}(s) \right\rangle_g ds < \infty \).

**Remark 1.4** To see that \( G^1 \) is a metric on \( H(M) \), we identify the tangent space \( T_\sigma H(M) \) with \( \Gamma_{\sigma}^{a-c.}(TM) \). To motivate this identification, consider a differentiable one-parameter family of curves \( \sigma_t \) in \( H(M) \) such that \( \sigma_0 = \sigma \). By definition of tangent vector, \( \frac{d}{dt} \bigg|_{t=0} \sigma_t(s) \) should be viewed as a tangent vector at \( \sigma \). This is actually the case, for detailed proof, see Theorem 1.3.1 in [17].

**Definition 1.5 (Piecewise geodesic space)** Given a partition \( \mathcal{P} := \{0 = s_0 < \cdots < s_n = 1\} \) of \([0,1]\), define:

\[
H_{\mathcal{P}}(M) := \{ \sigma \in H(M) \cap C^2([0,1] \setminus \mathcal{P}) : \nabla \sigma'(s) / ds = 0 \text{ for } s \notin \mathcal{P} \}.
\]
The piecewise geodesic space $H_P(M)$ is a finite dimensional embedded submanifold of $H(M)$. As for its tangent space, following the argument of Theorem 1.3.1 in [10], for any $\sigma \in H_P(M)$, the tangent space $T_\sigma H_P(M)$ may be identified with vector-fields along $\sigma$ of the form $X(s) \in T_{\sigma(s)}M$ where $s \to X(s)$ is piecewise $C^2$ and satisfies Jacobi equation for $s \notin \mathcal{P}$, i.e.

$$\frac{\nabla^2 X}{ds^2}(s) = R(\dot{\sigma}(s),X(s))\dot{\sigma}(s),$$

where $R$ is the curvature tensor. (See Theorem 2.29 below for a more detailed description of $TH_P(M)$). After specifying the tangent space of $H_P(M)$, we can define the $G_P^1$ metric as follows.

**Definition 1.6** For any $\sigma \in H_P(M)$ and $X,Y \in T_\sigma H_P(M)$, let

$$\langle X,Y \rangle_{G_P^1} := \sum_{j=1}^n \left\langle \frac{\nabla X}{ds}(s_{j-1}+),\frac{\nabla Y}{ds}(s_{j-1}+) \right\rangle_g \Delta_j$$

where $\Delta_j = s_j - s_{j-1}$ and $\frac{\nabla Y}{ds}(s_{j-1}+) = \lim_{s \to s_{j-1}+} \frac{\nabla Y}{ds}(s)$.

Endowed with the Riemannian metric $G_P^1$, $H_P(M)$ becomes a finite dimensional Riemannian manifold and the right hand side of (1.3) is now well-defined on $H_P(M)$ if $d\sigma$ is interpreted as the volume measure induced from this Riemannian metric. This motivates the following approximate measure definition.

**Definition 1.7** (Approximate measure on $H_P(M)$) Let $\nu_P^1$ be the probability measure on $H_P(M)$ defined by:

$$d\nu_P^1(\sigma) = \frac{1}{Z_P^1}e^{-\frac{1}{2} \int_0^1 \langle \sigma'(s),\sigma'(s) \rangle ds}dvol_{G_P^1}(\sigma),$$

where $dvol_{G_P^1}$ is the volume measure on $H_P(M)$ induced from the metric $G_P^1$ and $Z_P^1$ is the normalization constant.

**1.2 Main Theorems**

In this section we state the main results of this paper while avoiding many technical details.

**Definition 1.8** (Pinned piecewise geodesic space) For any $x \in M$,

$$H_{P,x}(M) := \{\sigma \in H_P(M) : \sigma(1) = x\}.$$ 

We prove below in Proposition 3.4 that when $M$ has non-positive sectional curvature, $H_{P,x}(M)$ is an embedded submanifold of $H_P(M)$.

**Theorem 1.9** If $M$ is a Hadamard manifold with bounded sectional curvature and $\mathcal{P} = \{k/n\}_{k=0}^n$ are equally-spaced partitions, then there exists a finite measure $\nu_{P,x}^1$ supported on $H_{P,x}(M)$, such that for any bounded continuous function $f$ on $H_P(M)$,

$$\lim_{m \to \infty} \int_{H_P(M)} \delta_x^{(m)}(\sigma(1))f(\sigma)d\nu_{P}^1(\sigma) = \int_{H_P(M)} f(\sigma)d\nu_{P,x}^1(\sigma),$$

where $\delta_x^{(m)}$ is an approximate sequence of $\delta_x$ in $C^\infty_0(M)$.

**Remark 1.10** The formula for $d\nu_{P,x}^1$ is explicitly given, see Definition 2.21.

The next theorem asserts, under additional geometric restrictions, that the measure $\nu_{P,x}^1$ we obtained from Theorem 1.9 serves as a good approximation to pinned Wiener measure $\nu_x$.

**Theorem 1.11** If $M$ is a symmetric space of non-compact type, i.e. it is a Hadamard manifold with parallel curvature tensor, then for any cylinder function $f \in \mathcal{F}_{C_0}^1$, see Definition 2.26

$$\lim_{|\mathcal{P}| \to 0} \int_{H_P(M)} f(\sigma)d\nu_{P,x}^1(\sigma) = \int_{W_x(M)} f(\sigma)d\nu_x(\sigma)$$

where $\nu_x$ is pinned Wiener measure, see Theorem 2.13 below.
1.3 Structure of the Paper

For the guidance to the reader, we give a brief summary of the contents of this paper.

In Section 2 we set up some notations and preliminaries in probability and geometry. In particular we present the Eells-Elworthy-Malliavin construction of Brownian motion on manifolds.

In Section 3 we define explicitly the pinned approximate measure $\nu^\gamma_{x,\pi}$ and study its properties. In Theorem 3.13 we prove that $\nu^\gamma_{x,\pi}$ is a finite measure and that $x \rightarrow \int_{\pi^{-1}(x)} f d\nu^\gamma_{x,\pi}$ is a continuous function on $M$ provided $f$ is bounded and continuous. This property is the key ingredient in proving Theorem 1.9. The proof of Theorem 1.9 is also given in this section.

In Section 4 we develop the so-called orthogonal lift of a vector field $X$ on $M$ to vector fields $\hat{X}_\pi$ on $H_P(M)$ and $\hat{X}$ on $W_\pi(M)$. Integration by parts formulae for these two operators are presented which will serve as an important tool in the proof of Theorem 1.11.

In Section 5 (using the development maps introduced in Section 2), we view $\hat{X}_\pi$ as defined on all of $W_\pi(M)$ and show that for any bounded cylinder function $f$ (also introduced in Section 2), $\bigl\| \hat{X}_\pi f - \hat{X} f \bigr\|_{L^1(W_\pi(M))} \rightarrow 0$ as $|P| \rightarrow 0$ for any $q \geq 1$ and more challengingly, we show the same result for $\hat{X}^{tr,v}$ $f - \hat{X}_\pi^{tr,v} f$, where $\hat{X}^{tr,v}$ is the adjoint of $\hat{X}$ with respect to $\nu$ and $\hat{X}_\pi^{tr,v}$ is the adjoint of $\hat{X}_\pi$ with respect to $\nu^1$.

In Section 6 we combine all the tools that are developed from previous sections to prove the main Theorem 1.11 of this paper.

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2 Preliminaries in Geometry and Probability

For the remainder of this paper, let $u_0 : \mathbb{R}^d \rightarrow T_o M$ be a fixed linear isometry which we add to the standard setup $(M,g,o,u_0,\nabla)$. We use $u_0$ to identify $T_o M$ with $\mathbb{R}^d$. Suggested references for this section are Chapter 2 of [14] and Sections 2, 3 of [7]. Some other references are [1, 10, 4] and [8] to name just a few.

Definition 2.1 (Orthonormal Frame Bundle $(\mathcal{O}(M),\pi)$) For any $x \in M$, denote by $\mathcal{O}(M)_x$ the space of orthonormal frames on $T_x M$, i.e. the space of linear isometries from $\mathbb{R}^d$ to $T_x M$. Denote $\mathcal{O}(M) := \cup_{x \in M} \mathcal{O}(M)_x$ and let $\pi : \mathcal{O}(M) \rightarrow M$ be the (fiber) projection map, i.e. for each $u \in \mathcal{O}(M)_x$, $\pi(u) = x$. The pair $(\mathcal{O}(M),\pi)$ is the orthonormal frame bundle over $M$.

In this paper we use the connection on $\mathcal{O}(M)$ that are specified by the following connection form.

Definition 2.2 (Connection Form on $\mathcal{O}(M)$) We define a $\mathfrak{so}(d)$-valued connection form $\omega^\gamma$ on $\mathcal{O}(M)$ in the following way: for any $u \in \mathcal{O}(M)$ and $X \in T_u \mathcal{O}(M)$,

$$\omega^\gamma_u (X) := u^{-1} \frac{\partial u(s)}{\partial s} \bigg|_{s=0}$$

where $u(\cdot)$ is a differentiable curve on $\mathcal{O}(M)$ such that $u(0) = u$ and $\frac{du(s)}{ds} \bigg|_{s=0} = X$. For any $\xi \in \mathbb{R}^d$, $\frac{\partial u(s)}{\partial s} \bigg|_{s=0} \xi := \frac{\partial u(s) \xi}{\partial s} \bigg|_{s=0}$ is the covariant derivative of $u(\cdot) \xi$ along $\pi(u(\cdot))$ at $\pi(u)$.

Definition 2.3 (Horizontal Bundle $\mathcal{H}$) Given a connection form $\omega^\gamma$, the horizontal bundle $\mathcal{H} \subset T\mathcal{O}(M)$ is defined to be the kernel of $\omega^\gamma$.

Definition 2.4 For any $a \in \mathbb{R}^d$, define the horizontal lift $B_a \in \Gamma(\mathcal{H})$ in the following way: for any $u \in \mathcal{O}(M)$, $B_a(u) \in \mathcal{H}_u \subset T_u \mathcal{O}(M)$ is uniquely determined by

$$\omega^\gamma_u (B_a(u)) = 0 \text{ and } \pi_*(B_a(u)) = ua.$$

Definition 2.5 (Horizontal Lift of a Path) For any $\sigma \in H(M)$, a curve $u : [0,1] \rightarrow \mathcal{O}(M)$ is said to be a horizontal lift of $\sigma$ if $\pi \circ u = \sigma$ and $u'(s) \in \mathcal{H}_{u(s)} \forall s \in [0,1]$.
Remark 2.6 In this paper we only consider horizontal lift with fixed start point $u_0 \in \pi^{-1}(\sigma(0))$. Under this assumption, given $\sigma \in H(M)$, its horizontal lift $u(\sigma, \cdot)$ is unique.

We denote $u$ by $\psi(\sigma)$ and call $\psi$ the horizontal lift map.

Definition 2.7 (Development Map) Given $w \in H(\mathbb{R}^d)$, the solution to the ordinary differential equation

$$du(s) = \sum_{i=1}^{d} B_{e_i}(u(s)) \, dw^i(s), \quad u(0) = u_0$$

is defined to be the development of $w$ and we will denote this map $w \to u$ by $\eta$, i.e. $\eta(w) = u$.

Here $\{e_i\}_{i=1}^{d}$ is the standard basis of $\mathbb{R}^d$.

Definition 2.8 (Rolling Map) $\phi = \pi \circ \eta : H(\mathbb{R}^d) \to H(M)$ is said to be the rolling map to $H(M)$.

Definition 2.9 (Anti-rolling Map) Given $\sigma \in H(M)$ with $u = \psi(\sigma)$. The anti-rolling of $\sigma$ is a curve $w \in H(\mathbb{R}^d)$ defined by:

$$w_t = \int_0^t u_s^{-1} \sigma_s' ds$$

Remark 2.10 It is not hard to see $w = \phi^{-1}(\sigma)$ and $u(\sigma, s) u_0^{-1}$ is the parallel translation along $\sigma \in H(M)$.

The Eells-Elworthy-Malliavin construction of Brownian motion depends in essence on a stochastic version of the maps defined above. Since the development maps on the smooth category are defined through ordinary differential equations, a natural way to introduce probability is to replace ODEs by (Stratonovich) stochastic differential equations.

First we set up some measure theoretic notations and conventions. Suppose that $(\Omega, \{ \mathcal{G}_s \}, \mathcal{G}, P)$ is a filtered measurable space with a finite measure $P$. For any $\mathcal{G}$—measureable function $f$, we use $P(f)$ and $\mathbb{E}_P[f]$ (if $P$ is a probability measure) to denote the integral $\int f \, dP$. Given two filtered measurable spaces $(\Omega, \{ \mathcal{G}_s \}, \mathcal{G}, P)$ and $(\Omega', \{ \mathcal{G}'_s \}, \mathcal{G}', P')$ and a $\mathcal{G}/\mathcal{G}'$ measurable map $f : \Omega \to \Omega'$, the law of $f \circ \eta$ under $P$ is the push-forward measure $f_*P(\cdot) := P(f^{-1}(\cdot))$. We are mostly interested in the path spaces $W_0(M)$, $W_0(\mathbb{R}^d)$ and $W_{u_0}(\mathcal{O}(M))$, where the following notation is being used.

Notation 2.11 If $(Y, y)$ is a pointed manifold, let $W(Y) := C([0, 1], Y)$ be the space of all continuous paths in $Y$ equipped with the uniform topology, $W_y(Y) := \{ w \in W(Y) | w(0) = y \}$ be the subset of continuous paths that start at $y$.

Definition 2.12 For any $s \in [0, 1]$ let $\Sigma_s : W_y(Y) \to Y$ be the coordinate functions given by $\Sigma_s(\sigma) = \sigma(s)$.

We will often view $\Sigma$ as a map from $W_y(Y)$ to $W_y(Y)$ in the following way: for any $\sigma \in W_y(Y)$ and $s \in [0, 1]$, $\Sigma(\sigma)(s) = \Sigma_s(\sigma)$. Let $\mathcal{F}_s^\sigma$ be the $\sigma$—algebra generated by $\{ \Sigma_\tau : \tau \leq s \}$. We use $\mathcal{F}_s^\sigma$ as the raw $\sigma$—algebra and $\{ \mathcal{F}_s^\sigma \}_{0 \leq s \leq 1}$ as the filtration on $W_y(Y)$. The next theorem defines the Wiener measure $\nu$ and pinned Wiener measure $\nu_x$ on $(W_y(Y), \mathcal{F}_s^\sigma)$.

Theorem 2.13 Assume $Y$ is a stochastically complete Riemannian manifold, then there exist two finite measures $\nu$ and $\nu_x$ on $(W_y(Y), \mathcal{F}_s^\sigma)$ which are uniquely determined by their finite dimensional distributions as follows. For any partition $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = 1$ of $[0, 1]$ and bounded functions $f : Y^n \to \mathbb{R}$;

$$\nu(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^n} f(x_1, \ldots, x_n) \prod_{i=1}^{n} p_{\Delta_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_n \tag{2.1}$$

and

$$\nu_x(f(\Sigma_{s_1}, \ldots, \Sigma_{s_n})) = \int_{Y^{n-1}} f(x_1, \ldots, x_n) \prod_{i=1}^{n} p_{\Delta_i}(x_{i-1}, x_i) \, dx_1 \cdots dx_{n-1} \tag{2.2}$$

where $p_{\cdot}(\cdot, \cdot)$ is the heat kernel on $Y$ associated to $\frac{1}{2} \Delta_y$, $\Delta_i = s_i - s_{i-1}$, $x_0 = y$ and $s_n = y$ in (2.2).
**Definition 2.14 (Brownian motion)** A stochastic process $X : (\Omega, \mathcal{F}, \{\mathcal{G}_t\}, P) \to (W_y(Y), \nu)$ is said to be a Brownian motion on $Y$ if the law of $X$ is $\nu$ i.e. $X_\pi P := P \circ X^{-1} = \nu$.

**Remark 2.15** From Theorem 2.17 it is clear that the law of the adapted process $\Sigma : W_y(Y) \to W_y(Y)$ is $\nu$ and $\Sigma$ is a Brownian motion. We will call $\Sigma$ the canonical Brownian motion on $Y$.

**Remark 2.16** Using Theorem 2.13, we can construct Wiener measure and pinned Wiener measure on $W_0(\mathbb{R}^d)$, $W_\omega(M)$ and $W_{u_0}(O(M))$ respectively. In order to avoid ambiguity from moving between $W_0(\mathbb{R}^d)$ and $W_\omega(M)$, we fix the symbol $\mu (\mu_\Sigma)$ as the Wiener (pinned Wiener) measure on $W_0(\mathbb{R}^d)$ and reserve the symbol $\nu(\nu_\Sigma)$ as the Wiener (pinned Wiener) measure on $W_\omega(M)$. Meanwhile we reserve $\Sigma$ as the canonical Brownian motion on $M$.

**Theorem 2.17 (Stochastic Horizontal Lift of Brownian Motion)** If $\Sigma$ is the canonical Brownian motion on $M$, then there exists a unique (up to $\nu$-equivalence) $\tilde{u} \in W_{u_0}(O(M))$ such that

$$\pi (\tilde{u}) = \Sigma.$$  \hspace{1cm} (2.3)

**Proof.** See Theorem 2.3.5 in [14] □

**Definition 2.18 (Stochastic Anti–rolling Map)** If $\Sigma$ is the canonical Brownian motion on $M$, then the stochastic anti–rolling $\beta$ of $\Sigma$ is defined by,

$$d\beta_s = \tilde{u}_s^{-1}d\Sigma_s, \beta_0 = 0.$$  \hspace{1cm} (2.4)

$\tilde{u}$ and $\beta$ defined above are linked through the (stochastic) development map.

**Definition 2.19 (Stochastic Development Map)** Let $\tilde{u}$ and $\beta$ be as defined in Theorem 2.17 and Definition 2.18, then $\tilde{u}$ satisfies the following SDE driven by $\beta$,

$$d\tilde{u}_s = \sum_{i=1}^{d}B_{e_i}(\tilde{u}_s)\delta\beta_s, \tilde{u}(0) = u_0,$$

and $\tilde{u}$ is said to be the development of $\beta$.

**Fact 2.20** The following facts are well known, the proofs may be found in the references listed at the beginning of this section, for example, Theorem 3.3 in [7].

- $\phi$ is a diffeomorphism from $H(\mathbb{R}^d)$ to $H(M)$,
- $\beta$ is a Brownian motion on $(W_\omega(\mathbb{R}^d), \mu)$.

From now on some notations are fixed for the convenience of consistency.

**Notation 2.21** For any $\sigma \in H(M)$, $u_\beta(\sigma) \in H_{u_0}(O(M))$ is its horizontal lift and $b_\beta(\sigma) \in H(\mathbb{R}^d)$ is its anti-rolling. Recall that $\{\Sigma_s\}_{0 \leq s \leq 1}$ is fixed to be the canonical Brownian motion on $(W_\omega(M), \nu)$. We also fix $\beta(\cdot)$ to be the stochastic anti-rolling of $\Sigma$, (which is a Brownian motion on $\mathbb{R}^d$) and $\tilde{u}(\cdot)$ to be the stochastic horizontal lift of $\Sigma$.

**Notation 2.22** Given a partition $P$ of $[0, 1]$, $\beta_P$ is the piecewise linear approximation to the Brownian motion $\beta$ on $\mathbb{R}^d$ given by:

$$\beta_P(s) := \beta(s_{i-1}) + \frac{\Delta_i \beta}{\Delta_i}(s - s_{i-1}) \text{ if } s \in [s_{i-1}, s_i]$$

where $\Delta_i \beta = \beta(s_i) - \beta(s_{i-1})$ and $\Delta_i = s_i - s_{i-1}$.

**Notation 2.23 (Geometric Notation)**
• curvature tensor For any \( X,Y,Z \in \Gamma(TM) \), define the (Riemann) curvature tensor \( R : \Gamma(TM) \times \Gamma(TM) \to \Gamma(End(TM)) \) to be:

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z
\]

• For any \( \sigma \in H(M) \), define \( R_u(\sigma,s) (\cdot,\cdot) \) to be a map from \( \mathbb{R}^d \otimes \mathbb{R}^d \) to \( \text{End}(\mathbb{R}^d) \) given by:

\[
R_u(\sigma,s) (a,b) \cdot = u(\sigma,s)^{-1} R(u(\sigma,s)a,u(\sigma,s)b) u(\sigma,s) \forall a,b \in \mathbb{R}^d
\]

\[(2.5)\]

where \( R \) is the curvature tensor of \( M \). Similarly we define \( R_{\tilde{u}}(\sigma,s) (\cdot,\cdot) \) to be a random map (up to \( \nu \)-equivalence) from \( \mathbb{R}^d \otimes \mathbb{R}^d \) to \( \mathbb{R}^d \) as follows:

\[
R_{\tilde{u}}(\sigma,s) (\cdot,\cdot) = \tilde{u}(\sigma,s)^{-1} R(\tilde{u}(\sigma,s)\cdot,\tilde{u}(\sigma,s)\cdot) \tilde{u}(\sigma,s)
\]

\[(2.6)\]

• \( \text{Ric}(\cdot) := \sum_{i=1}^{d} R(v_i,\cdot)v_i \) is the Ricci curvature tensor on \( M \). Here \( \{v_i\}_{i=1}^{d} \) is an orthonormal basis of proper tangent space. Using \( u(\sigma,s) \) or \( \tilde{u}(\sigma,s) \) to pull back \( R \) as in \[(2.5)\] and \[(2.6)\], we can define \( \text{Ric}_{u(\sigma,s)} \) and \( \text{Ric}_{\tilde{u}(\sigma,s)} \) to be maps (random maps) from \( \mathbb{R}^d \) to \( \mathbb{R}^d \).

**Convention 2.24** Since most of our results require a curvature bound, it would be convenient to fix a symbol \( N \) for it, i.e. \( ||R|| \leq N \) when it is viewed as a tensor of order 4. Following this manner, we have \( ||\text{Ric}|| \leq (d-1)N \). A generic constant will be denoted by \( C \), it can vary from line to line. Sometimes \( C(\cdot) \) or \( C(\cdot) \) are used to specify its dependence on some parameters.

**Definition 2.25** \( f : W_\sigma(M) \to \mathbb{R} \) is a cylinder function if there exists a partition

\[\mathcal{P} := \{0 < s_1 < \cdots < s_n \leq 1\}\]

of \([0,1]\) and a function \( F : C^m(M^n,\mathbb{R}) \) such that

\[f = F(\Sigma_{s_1}, \Sigma_{s_2}, \ldots, \Sigma_{s_n})\]

We denote this space by \( \mathcal{F}C^m \).

**Notation 2.26** Denote

\[\mathcal{F}C^1_{\mathcal{P}} := \{ f := F(\Sigma) \in \mathcal{F}C^1, F \text{ and all its partial differentials } \text{grad}_s F \text{ are bounded} \}\].

**Remark 2.27** In this paper the partition \( \mathcal{P} \) is always equally spaced, so \( \mathcal{P} \equiv \mathcal{P}_i \equiv \frac{1}{n} \) for \( i = 1, \ldots, n \).

**Definition 2.28** (Jacobi equation) For \( \sigma \in H(M) \), \( Y \in \Gamma_\sigma(TM) \), we say \( Y(s) \in T_\sigma(s)M \) satisfies Jacobi equation if:

\[
\frac{\nabla^2}{ds^2} Y(s) = R(\sigma'(s),Y(s))\sigma'(s).
\]

Further if the horizontal lift \( u(s) \) of \( \sigma \) is used, we let \( y(s) := u^{-1}(s)Y(s) \). It then follows that \( y(s) \) satisfies the pulled back Jacobi equation,

\[
y''(s) = R_u(s)(b'(s),y(s))b'(s),
\]

\[(2.7)\]

where \( b'(s) = u(s)^{-1}\sigma'(s) \). Once we have Jacobi equation, we can describe the tangent space \( TH_\mathcal{P}(M) \) of \( H_\mathcal{P}(M) \).

We formalize the tangent space of \( H_\mathcal{P}(M) \) mentioned in **Definition 1**.

**Theorem 2.29** (Tangent space to \( H_\mathcal{P}(M) \)) For all \( \sigma \in H_\mathcal{P}(M) \),

\[
T_\sigma H_\mathcal{P}(M) = \{ s \to u(s)J(s) \mid J \in C([0,1],\mathbb{R}^d), J \in H_{\mathcal{P},\sigma} \text{ with } J(0) = 0 \}\.
\]

\[(2.8)\]

where \( J \in H_{\mathcal{P},\sigma} \) iff

\[
J''(s) = R_{u(s)}(b'(s),J(s))b'(s) \text{ for } s \in [s_i-1,s_i) \text{ for } i = 1, \ldots, n.
\]
Proof. See Theorem 1.3.1 in [10]. □

Notation 2.30 Given \( h (\cdot) \in H (\mathbb{R}^d) \), denote \( X^h (\sigma, s) := u (\sigma, s) h (s) \).

Notation 2.31 \( \{ (C_{P,i}^j (\sigma, s))_{i=1}^n \} \) and \( \{ (S_{P,i}^j (\sigma, s))_{i=1}^n \} \)

Let

\[ P := \{ 0 = s_0 < s_1 < \cdots < s_n = 1 \} \]

be a partition of \([0, 1]\), \( K_i := [s_{i-1}, s_i] \) and \( \Delta_i := s_i - s_{i-1} \) for \( 1 \leq i \leq n \), and say that \( f (s) \) satisfies the \( i \)-Jacobi’s equation if

\[ f'' (s) = R_{i,s} (u^{-1} \sigma' (s_{i-1} +), f (s)) u^{-1} \sigma' (s_{i-1} +) \text{ for } s \in K_i. \]

where \( u^{-1} \sigma' (s) := u (\sigma, s)^{-1} \sigma' (s) \in \mathbb{R}^d \).

We now let \( C_{P,i}^j (\sigma, s) \) and \( S_{P,i}^j (\sigma, s) \in \text{End}(\mathbb{R}^d) \) denote the solution to Eq. (3.1) with initial conditions,

\[ C_{P,i}^j (s_{i-1}) = I, \quad C_{P,i}^j (s_{i-1}) = 0, \quad S_{P,i}^j (s_{i-1}) = 0 \quad \text{and} \quad S_{P,i}^j (s_{i-1}) = I \]

and we further let

\[ C_{P,i} (\sigma) := C_{P,i} (\sigma, s_i) \quad \text{and} \quad S_{P,i} (\sigma) := S_{P,i} (\sigma, s_i). \]

Here we view \( C_{P,i} (s) \) and \( S_{P,i} (s) \) as maps from \( H_P (M) \) to \( \text{End}(\mathbb{R}^d) \).

Definition 2.32 Define for all \( i = 1, \cdots, n \),

\[ f_{P,i} (\sigma, s) = \begin{cases} 0 & s \in [0, s_{i-1}] \\ \frac{S_{P,i} (\sigma, s)}{\Delta_i} & s \in [s_{i-1}, s_i] \\ \frac{C_{P,i} (\sigma, j) \cdots C_{P,i} (\sigma, j) S_{P,i} (\sigma)}{\Delta_i} & s \in [s_{j-1}, s_j] \text{ for } j = i + 1, \cdots, n \end{cases} \]

with the convention that \( S_{P,0} = |P| I \) and \( f_{P,0} = I \).

Remark 2.33 \( S_{P,j} (s), C_{P,j} (s) \) may be expressed in terms of \( \{ f_{P,i,j} \}_{i=0}^n \) by

\[ S_{P,j} (s) = \Delta_j f_{P,j} (s) \]

\[ C_{P,j} (s) = f_{P,j-1} (s) f_{P,j-1}^{-1} (s). \]

3 Approximate Pinned Measures

3.1 Representation of \( \delta \) – function

Let \( Y \) be a smooth Riemannian manifold, we will denote the distribution on \( Y \) by \( \mathcal{D}' (Y) \) and, compactly supported distribution by \( \mathcal{E}' (Y) \). For a matrix \( A \), let \( \text{eig} (A) \) denote the set of eigenvalues of \( A \). For each \( x \in Y \), let \( \delta_x \in \mathcal{E}' (Y) \) be the \( \delta \)-function at \( x \) defined by

\[ \delta_x (f) = f (x) \quad \forall f \in C(\infty) (Y). \]

Lemma 3.1 (Representation of \( \delta_0 \) on flat space) There exist functions \( \{ g_i \}_{i=0}^d \) with \( g_0 \in C_0^\infty (\mathbb{R}^d) \), \( \{ g_j \}_{j=1}^d \subset C^\infty (\mathbb{R}^d / \{ 0 \}) \) with supports contained in a compact subset \( K \subset \mathbb{R}^d \) and satisfying

\[ |g_j (x)| \leq c |x|^{1-d} \quad \text{for } j = 1, \cdots, d, \quad (3.1) \]

such that

\[ \delta_0 = g_0 + \sum_{j=1}^d \frac{\partial g_j}{\partial x_j} \text{in} \mathcal{E}' (\mathbb{R}^d). \quad (3.2) \]

In more detail, for any \( f \in C_0^\infty (\mathbb{R}^d) \),

\[ f (0) = \int_{\mathbb{R}^d} \left( g_0 + \sum_{j=1}^d \frac{\partial g_j}{\partial x_j} \right) f dx = \int_{\mathbb{R}^d} \left( g_0 f - \sum_{j=1}^d \frac{\partial f}{\partial x_j} g_j \right) dx. \quad (3.3) \]
Proof}. This lemma can be derived from Lemma 10.10 in [22]. ■

Based on this representation we can get a representation of \( \delta_p \) for any \( p \in M \).

**Theorem 3.2 (Representation of \( \delta \)-function on manifold)** For any \( p \in M \), there exist functions \( \{g_j\}_{j=0}^d \subset C^\infty (M/\{p\}) \cap L^\infty (M) \) with supports in a compact subset \( K \) of \( M \) and smooth vector fields \( \{X_j\}_{j=1}^d \subset \Gamma^\infty (TM) \) with compact support such that

\[
\delta_p = g_0 + \sum_{j=1}^d X_j g_j \quad \text{in} \quad \mathcal{E}'(M).
\] (3.4)

**Proof.** Pick a chart \( \{U, x\} \) near \( p \in M \) such that \( x(p) = 0 \). Since \( x(U) = \mathbb{R}^d \), one can apply Lemma 3.1 on \( x(U) \cong \mathbb{R}^d \) and get:

\[
\delta_0 = \tilde{g}_0 - \sum_{j=1}^d \frac{\partial \tilde{g}_j}{\partial x_j}
\]

where \( \delta_0 \) is the delta mass on \( x(U) \) supported at the origin. So for any \( h \in C^\infty (U) \)

\[
h(p) = h \circ x^{-1}(0) = \int_{\mathbb{R}^d} \left( \tilde{g}_0 - \sum_{j=1}^d \frac{\partial \tilde{g}_j}{\partial x_j} \right) h \circ x^{-1} \, d\lambda = \int_{\mathbb{R}^d} \left( \tilde{g}_0 + \sum_{j=1}^d \tilde{g}_j \frac{\partial}{\partial x_j} \right) h \circ x^{-1} \, d\lambda
\]

where \( d\lambda \) is the Lebesgue measure on \( \mathbb{R}^d \). Consider \( \left( \frac{\tilde{g}_i}{\sqrt{\det g}} \circ x \right)_{j=0}^d \) where \( g = (g_{ij})_{1 \leq i,j \leq d} \) is the metric matrix, i.e. \( g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_g \). From Lemma 3.1 we know that \( \frac{\tilde{g}_i}{\sqrt{\det g}} \circ x \) has compact support in \( U \) and therefore \( K := \bigcup_{j=1}^d \text{supp} \left( \frac{\tilde{g}_i}{\sqrt{\det g}} \circ x \right) \) is compact in \( U \). Using the smooth Urysohn lemma we can construct a smooth function \( \phi \in C^\infty (M \to [0,1]) \) such that \( \phi^{-1}(\{0\}) = M/U \) and \( \phi^{-1}(\{1\}) = K \). Define

\[
\tilde{g}_0 = \phi \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x
\]

and

\[
\tilde{g}_j = \phi \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x, \quad X_j = \phi \cdot (x^{-1})_* \frac{\partial}{\partial x_j} \quad \text{for} \quad j = 1, \ldots, d.
\]

Then for any \( f \in C^\infty (M) \),

\[
\int_M \left( \tilde{g}_0 + \sum_{j=1}^d \tilde{g}_j X_j \right) f \, d\nuol = \int_U \left( \tilde{g}_0 + \sum_{j=1}^d \tilde{g}_j X_j \right) f \, d\nuol = \int_U \left( \frac{\tilde{g}_0}{\sqrt{\det g}} \circ x \cdot \phi f \right) d\nuol + \sum_{j=1}^d \int_U \phi^2 \frac{\tilde{g}_j}{\sqrt{\det g}} \circ x \left( (x^{-1})_* \frac{\partial (\phi f)}{\partial x_j} - (x^{-1})_* \frac{\partial \phi}{\partial x_j} f \right) d\nuol
\]

where \( d\nuol \) is the volume measure on \( M \).
Since \( \phi \cdot (x^{-1}) \cdot \frac{\partial \phi}{\partial x_j} \equiv 0 \) and \( \phi \equiv 1 \) on \( K \), we have:
\[
\int_M \left( \tilde{g}_0 + \sum_{j=1}^d \tilde{g}_j X_j \right) f \text{dvol} = \int_U \left( \frac{\tilde{g}_0}{\sqrt{\text{det } g}} \circ x + \sum_{j=1}^d \frac{\tilde{g}_j}{\sqrt{\text{det } g}} \circ x (x^{-1}) \cdot \frac{\partial}{\partial x_j} \right) f \text{dvol} = \int_{\mathbb{R}^d} \left( \frac{\tilde{g}_0}{\sqrt{\text{det } g}} + \sum_{j=1}^d \frac{\tilde{g}_j}{\sqrt{\text{det } g}} \frac{\partial}{\partial x_j} \right) f \circ x^{-1} \sqrt{\text{det } g} \text{d} \lambda = f \circ x^{-1} (0) = f (p).
\]

Therefore, by the Divergence Theorem, we can write down \( \delta_p \) in distributional sense as
\[
\delta_p = g_0 + \sum_{j=1}^d X_j g_j
\]
where \( g_0 = \tilde{g}_0 - \sum_{j=1}^d \tilde{g}_j \cdot \text{div } X_j \) and for \( j = 1, \ldots, n, \) \( g_j = -\tilde{g}_j \).

From the construction one can see that \( X_j \in \Gamma^\infty (TM) \) with compact support and \( \{ g_j \}_{j=0}^d \subset C^\infty (M/\{p\}) \cap L^\infty (M) \) with supports being a compact subset of \( M \). \( \blacksquare \)

**Remark 3.3** Since \( C^\infty_0 (M) \) is dense in \( L^q (M) \), \( \forall q \geq 1 \), for any \( g_j, j = 1, \ldots, d \), we can find a sequence \( \{ g_j^{(m)} \}_m \subset C^\infty_0 (M) \) such that
\[
g_j^{(m)} \to g_j \text{ in } L^\infty (M)
\]
In particular, we can make \( \cup_m \text{supp } (g_j^{(m)}) \) to be compact.

**Corollary 3.4** Define
\[
\delta_x^{(m)} := g_0^{(m)} + \sum_{j=1}^d X_j g_j^{(m)} \in C^\infty_0 (M).
\]
Then \( \{ \delta_x^{(m)} \}_m \) is an approximating sequence of delta mass \( \delta_x \), i.e. \( \delta_x^{(m)} \to \delta_x \) in \( \mathcal{D}' (M) \).

**Proof.** Using integration by parts, we have for any \( f \in C^\infty (M) \),
\[
\int_M f \delta_x^{(m)} \text{d} \lambda = \int_M \left( g_0^{(m)} + \sum_{j=1}^d X_j g_j^{(m)} \right) f \text{d} \lambda = \int_M \left( g_0^{(m)} f + \sum_{j=1}^d g_j^{(m)} \cdot X_j f \right) \text{d} \lambda
\]
Since \( K := \cup_m \text{supp } (g_j^{(m)}) \) is compact, \( f \cdot 1_K \) and \( X_j f \cdot 1_K \in L^\infty (M) \), then Corollary 3.4 easily follows from Holder’s inequality. \( \blacksquare \)

### 3.2 Definition of \( \nu_{p,x}^1 \)

In this section we will give an explicit definition of \( \nu_{p,x}^1 \) proposed in Theorem 1.9.

**Definition 3.5 (End point map)** Define \( E_1 : H (M) \to M \) to be \( E_1 (\sigma) = \sigma (1) \) and let \( E_1^p \) denote \( E_1 |_{H^p (M)} \).
Recall from Definition 3.7 that
\[ H_{P,x} (M) := \{ \sigma \in H_P (M) \mid \sigma (1) = x \} = (E_1^P)^{-1} (\{x\}). \]
In general, it is not guaranteed that \( E_1^P \) is a submersion, which would guarantee that \( H_{P,x} (M) \) is an embedded submanifold of \( H_P (M) \). The following is an easy, yet illuminating, example showing what can go wrong:

**Example 3.6** If \( M = S^2 \), \( o \) is the north pole and \( P := \{0,1\} \), then \( \dim H_P (M) = 2 \). Consider
\[ X (\sigma, s) := (0, \pi \sin ss \pi, 0) \in T_{\sigma} H_P (M) \]
where
\[ \sigma (s) = (\sin ss \pi, 0, \cos ss \pi). \]
An one parameter family realizing \( X (\sigma, s) \) would be
\[ \sigma_t (s) = (\sin ss \pi \cos st \pi, \sin ss \pi \sin st \pi, \cos ss \pi), \]
from which one can easily see that:
\[ E_{1,s}^P (X) = \frac{d}{dt} \bigg|_0 E_1^P (\sigma_t) = \frac{d}{dt} \bigg|_0 \sigma_t (1) = X (\sigma, 1) = 0. \]
So by Rank-Nullity theorem, \( E_{1,s}^P \) is not surjective.

The problem comes from the conjugate points on \( M \). Two points \( p \) and \( q \) are conjugate points along a geodesic \( \sigma \) if there exists non-zero Jacobi field (smooth vector field along \( \sigma \) satisfying Jacobi equation) vanishing at \( p \) and \( q \). This fact will allow the kernel of \( E_1^P \) to be “overly large” (more accurately dimension exceeds \((n - 1) d\)), so by Rank-nullity theorem, \( E_1^P \) can not be surjective. In this paper we consider manifolds with non-positive sectional curvature. These manifolds do not have conjugate points. From the next proposition we will see that \( E_1^P \) is a submersion on these manifolds.

**Notation 3.7** We construct a \( G_1^P \)-orthonormal frame
\[ \{ X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n \} \]
of \( H_P (M) \) as follows: for any \( \sigma \in H_P (M) \), \( X^{h_{\alpha,i}} (\sigma, \cdot) = u_\alpha (\sigma) h_{\alpha,i} (\sigma, \cdot) \), where
\[ h_{\alpha,i} \in H_{P,\sigma} \text{ and } h'_{\alpha,i} (s_j +) = \frac{\delta_{i=1,j} e_\alpha}{\sqrt{\Delta_j + 1}} \text{ for } j = 0, \ldots, n - 1 \] (3.5)
and the definition of \( H_{P,\sigma} \) can be found in Eq. (2.8).

**Remark 3.8** Using Proposition 3.1 it is not hard to see that
\[ h_{\alpha,i} (s) = \frac{1}{\sqrt{n}} f_{P,i} (s) e_\alpha \] (3.6)
where \( \{ f_{P,i} (s) \} \) is given in Definition 2.5E.

**Proposition 3.9** If \( M \) is complete with non-positive sectional curvature, then for any \( x \in M \), \( H_{P,x} (M) := (E_1^P)^{-1} (\{x\}) \) is an embedded submanifold of \( H_P (M) \).

**Proof.** It suffices to show \( E_1^P \) is a submersion. Since \( M \) is complete, for any \( y \in M \), there exists a geodesic \( \sigma \) parametrized on \([0,1]\) and connecting \( o \) and \( y \). So \( E_1^P \) is surjective. To show \( E_1^P \) is surjective, we use a class of vector fields \( \{ X^{h_{\alpha,n}} \}_{\alpha=1}^d \) in Notation 3.7. Since
\[ E_{1,s}^P (X^{h_{\alpha,n}}) = X^{h_{\alpha,n}} = \sqrt{n} u (1) S_{P,n} e_\alpha, \]
where \( u (\cdot) = u (\sigma, \cdot) \) is the horizontal lift of \( \sigma \in H_P (M) \). From Proposition A.3 we know \( S_{P,n} \) is invertible, therefore \( \{ E_{1,s}^P (X^{h_{\alpha,n}}) \}_{\alpha=1}^d \) spans \( T_{E_1^P (\sigma)} M \). So \( E_{1,s}^P \) is surjective. \( \blacksquare \)

Since \( H_{P,x} (M) \) is an embedded submanifold of \( H_P (M) \), we can restrict the Riemannian metric \( G_1^P \) on \( TH_P (M) \) in Eq. (1.6) to a Riemannian metric on \( TH_{P,x} (M) \).

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Notation 3.10 Assuming $M$ has non-positive sectional curvature, for any $x \in M$, let $G^{1}_{p,x}$ be the restriction of $G^{1}_{p}$ to $T_{x}H_{p} (M) \subset T_{x}H_{p} (M)$. Further, let $\text{vol}_{G^{1}_{p,x}}$ be the associated volume measure on $H_{p,x} (M)$.

Based on the volume measure $\text{vol}_{G^{1}_{p,x}}$ on $H_{p,x} (M)$, we can construct the pinned approximated measure $\nu_{p,x}^{1}$:

**Definition 3.11** Let $\nu_{p,x}^{1}$ be the measure on $H_{p,x} (M)$ defined by

$$
dv_{p,x}^{1} (\sigma) = \frac{1}{J_{p}(\sigma)} \frac{1}{Z_{p}^{1}} e^{E(\sigma)} \text{dvol}_{G_{p,x}}^{1} (\sigma) 
$$

where $J_{p}(\sigma) := \sqrt{\text{det} (E_{1}^{p} E_{1}^{p\prime})}$, $Z_{p} := (2\pi)^{1/2}$ and $E(\sigma) = \int_{0}^{1} \langle \sigma'(s), \sigma'(s) \rangle_{g} \text{d}s$ is the energy of $\sigma$.

### 3.3 Continuous Dependence of $\nu_{p,x}^{1}$ on $x$

Throughout this section we further assume $M$ is simply connected, i.e. $M$ is a Hadamard manifold, and the sectional curvature of $M$ is bounded from below by $-N$. The following theorem illustrates that the measures, $\nu_{p,x}^{1}$, are finite and “continuously varying” with respect to $x$.

**Notation 3.12** We will denote by $C_{b}(Y)$ bounded continuous functions on a topological space $Y$.

**Theorem 3.13** For any $x \in M$, $\nu_{p,x}^{1}$ is a finite measure. Moreover, for any $f \in C_{b} (H_{p,x} (M))$, define

$$
h_{p}^{f} (x) := \int_{H_{p,x} (M)} f (\sigma) \text{d}nu_{p,x}^{1} (\sigma) . 
$$

If the mesh size $|P| := \frac{1}{n}$ of the partition $P$ is small enough, i.e. $n \geq 3dN$, then $h_{p}^{f} (x) \in C (M)$.

Before proving this theorem, we need to set up some notations and auxiliary results.

**Notation 3.14** We fix $n \in \mathbb{N}$ and let $s_{i} := \frac{i}{n}$ and $\tau := 1 - \frac{1}{n} = s_{n-1}$. We further define $K := H_{p} ([0, \tau], M)$ be the space of piecewise geodesic paths, $\sigma : [0, \tau] \rightarrow M$ such that $\sigma (0) = o \in M$.

**Lemma 3.15** For $x, y \in M$, we can choose an unique element $\log_{x} (y) \in T_{x}M$ so that

$$
\gamma_{y,x} (t) := \exp_{x} \left( (t - \tau) \frac{1}{n} \log_{x} (y) \right), \quad t \leq t \leq 1 .
$$

is the unique minimal-length-geodesic connecting $x$ to $y$ such that $\gamma_{y,x} (\tau) = x$ and $\gamma_{y,x} (1) = y$.

**Proof.** Since $M$ is a Hadamard manifold, by the Theorem of Hadamard (See Theorem 3.1 in [5], $\exp_{x} : T_{x}M \rightarrow M$ is a diffeomorphism. Therefore we can see that $\log_{x} (y) = \exp_{x}^{-1} (y)$ is unique and it follows that the geodesic $\gamma_{y,x}$ is unique. $\blacksquare$

**Definition 3.16** For any given $y \in M$, let $\psi_{y} : K \rightarrow H_{p,y} (M) := (E_{1}^{p})^{-1} (\{y\})$ be defined by

$$
\psi_{y} (\sigma) := \gamma_{y,\sigma(\tau)} \ast \sigma
$$

where

$$
(\gamma_{y,\sigma(\tau)} \ast \sigma) (t) = \begin{cases} 
\sigma (t) & \text{if } 0 \leq t \leq \tau \\
\gamma_{y,\sigma(\tau)} (t) & \text{if } \tau \leq t \leq 1 .
\end{cases}
$$

**Notation 3.17** For any $\sigma \in H_{p,y} (M)$, let $\xi_{y,\sigma} := u (\sigma, \tau)^{-1} \log_{\sigma(\tau)} (y) \in \mathbb{R}^{d}$ and

$$
A_{\xi_{y}} \xi_{y}(\sigma, s) = R_{u(\sigma,1-s)} (\xi_{y}, \sigma, \sigma) \xi_{y} \xi_{y} \text{ and } 0 \leq s \leq \frac{1}{n} .
$$

Denote by $C_{y} (\sigma, s), S_{y} (\sigma, s)$ the solutions to $y''(\sigma, s) = A_{\xi_{y}} (\sigma, s) y(\sigma, s)$ with initial values $C_{y}(\sigma, 0) = I, C'_{y} (\sigma, 0) = 0, S_{y}(\sigma, 0) = 0, S'_{y} (\sigma, 0) = I$. 

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The next lemma characterizes the differential of \( \psi_y \):

**Lemma 3.18** For any \( \sigma \in K \) and \( X^h(\sigma, \cdot) := u(\sigma, \cdot)h(\sigma, \cdot) \in T_\sigma K \),

\[
\psi_y \left( X^h(\sigma, \cdot) \right) = X^h(\psi_y(\sigma), \cdot) := u(\psi_y(\sigma), \cdot) \hat{h}(\psi_y(\sigma), \cdot)
\]

where

\[
\hat{h}(\psi_y(\sigma), s) = \begin{cases} \hat{h}(\psi_y(\sigma), s) & s \in [0, \tau] \\ S_y(\psi_y(\sigma), 1 - s) S_y(\psi_y(\sigma), \frac{1}{n})^{-1} \hat{h}(\sigma, \tau) & s \in [\tau, 1]. \end{cases} (3.9)
\]

**Proof.** From now on we will suppress the path argument \( \psi_y(\sigma) \) in \( \hat{h} \). Suppose that \( t \to \sigma_t \in K \) is an one-parameter family of curves in \( K \) such that \( \sigma_0 = \sigma \) and \( \frac{d}{dt} |_{0} \sigma_t = X^h(\sigma) \). Then we have

\[
\psi_y \left( X^h(\sigma) \right) = \frac{d}{dt} |_{0} \psi_y(\sigma_t) = \frac{d}{dt} |_{0} \gamma_{y, \sigma_t} \star \sigma_t.
\]

If \( s \in [0, \tau] \), then

\[
\frac{d}{dt} |_{0} \left( \gamma_{y, \sigma_t(\tau)} \star \sigma_t \right)(s) = \frac{d}{dt} |_{0} \sigma_t(s) = X^h_s(\sigma).
\]

While if \( s \in [\tau, 1] \) we have

\[
\frac{d}{dt} |_{0} \left( \gamma_{y, \sigma_t(\tau)} \star \sigma_t \right)(s) = \frac{d}{dt} |_{0} \gamma_{y, \sigma_t(\tau)}(t) =: X^h_s(\psi_y(\sigma)),
\]

where \( X^h_s \) is uniquely determined by,

1. \( \hat{h} \) satisfies Jacobi’s equation,
2. \( \hat{h}(\tau) = h(\tau) \) and \( \hat{h}(1) = 0. \)

Denote \( \hat{h}(s) \) by \( g(1 - s) \) for \( s \in [\tau, 1] \), the above conditions are equivalent to \( g \) being the solution to the following boundary value problem:

\[
\begin{cases}
g''(s) = A_{\xi_y}(s) g(s) \\
g(0) = 0 \\
g \left( \frac{1}{n} \right) = \hat{h}(\tau) \end{cases}.
\]

Then we use \( S_y(\cdot) \) to express the solution. Here we have used Proposition A.3 to see that \( S_y(s) \) is invertible for \( s \in [0, \frac{1}{n}] \), therefore

\[
g(s) = S_y(s) S_y \left( \frac{1}{n} \right)^{-1} \hat{h}(\tau) \text{ for } s \in [0, \tau]
\]

and thus

\[
\hat{h}(s) = g(1 - s) = S_y(1 - s) S_y \left( \frac{1}{n} \right)^{-1} \hat{h}(\tau) \text{ for } s \in [\tau, 1].
\]

**Corollary 3.19** For any \( y \in M \), \( \psi_y \) is a diffeomorphism.

**Proof.** From Lemma 3.18 it is easy to see that the push forward \( (\psi_y)_* \) of \( \psi_y \) is one to one and thus an isomorphism since \( \dim(K) = \dim(HP_\nu(M)) \). Therefore the inverse function theorem implies that \( \psi_y \) is a local diffeomorphism. Furthermore, \( M \) being a Hadamard manifold implies that \( \psi_y \) is bijective, so \( \psi_y \) is actually a diffeomorphism.
**Definition 3.21** An orthonormal frame \( \{ X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n-1 \} \) of \( \mathcal{K} \) can be constructed similarly to Notation 3.7
\[
h_{\alpha,i} \in H_{P,\sigma} \text{ and } h'_{\alpha,i}(s_j) = \frac{\delta_{1-j} e_\alpha}{\sqrt{\Delta_{j+1}}} \text{ for } j = 0, \ldots, n-2.
\]

In this section we will use the same notation for both these two sets of orthonormal frames as the meaning should be clear from the context.

**Definition 3.22** \( f : M \to N \) is a differentiable map between two Riemannian manifolds \( M, N \). The **Normal Jacobian** of \( f \) is defined to be \( \sqrt{\det(f_1^\gamma) \rho} \).

We will use the orthonormal frame \( \{ X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n-1 \} \) of \( \mathcal{K} \) to estimate the Normal Jacobian \( J_P \) of \( E_1 \) in Lemma 3.22 and the “volume change” \( V_x \) (See precise definition in Lemma 3.24) brought by the diffeomorphism \( \psi_x \) in Lemma 3.24 and 3.25.

**Lemma 3.22** Let \( J_P := \sqrt{\det(E_1^P, (E_1^P)_r)^{tr}} \) be the Normal Jacobian of \( E_1^P \), then
\[
J_P(\sigma) = \sqrt{\det \left( \frac{1}{n} \sum_{i=1}^{d} f_{P,i}(\sigma,1) f_{P,i}^{tr}(\sigma,1) \right)} \forall \sigma \in H_P(M).
\]

**Proof.** Note that
\[
E_1^P, \sigma X^h(\sigma) = X^h(\sigma,1),
\]
so if \( v \in T_{E_1^P(\sigma)} M \), then
\[
\left\langle (E_1^P)_{tr} v, X^h \right\rangle_{G_\rho} = \langle v, E_1^P X^h \rangle_{T_{E_1^P(\sigma)} M} = \langle u(1)^{-1} v, h(1) \rangle_{\mathbb{R}^d}.
\]
Therefore, using the orthonormal frame of \( TH_P(M) \) given by
\[
\{ X^{h_{\alpha,i}} : 1 \leq \alpha \leq d, 1 \leq i \leq n \},
\]
we find
\[
(E_1^P)_{tr} v = \sum_{i, \alpha} \left\langle (E_1^P)_{tr} v, X^{h_{\alpha,i}} \right\rangle_{G_\rho} X^{h_{\alpha,i}} = \sum_{i, \alpha} \left\langle u(1)^{-1} v, h_{\alpha,i}(1) \right\rangle \mathbb{R}^d X^{h_{\alpha,i}}.
\]
Let \( \{ e_\alpha \}_{\alpha=1}^d \) be the standard basis of \( \mathbb{R}^d \), since \( u(1) \) is an isometry, \( \{ u(1) e_\alpha \}_{\alpha=1}^d \) is an O.N. basis of \( T_{E_1^P(\sigma)} M \). Using
\[
h_{k,i}(1) = \frac{1}{\sqrt{n}} f_{P,i}(1) e_k \text{ for } 1 \leq k \leq d,
\]
we can compute:
\[
\det \left( E_1^P, (E_1^P)_{tr} \right) = \det \left\{ \left\langle (E_1^P)_{tr} u(1) e_\alpha, (E_1^P)_{tr} u(1) e_\beta \right\rangle_{G_\rho} \right\}_{\alpha, \beta}
\]
\[
= \det \left\{ \sum_{i=1}^{d} \sum_{\gamma=1}^{d} \left\langle h_{\gamma,i}(1), e_\alpha \right\rangle \left\langle h_{\gamma,i}(1), e_\beta \right\rangle \right\}_{\alpha, \beta}
\]
\[
= \det \left\{ \sum_{i=1}^{d} \frac{1}{n} \left\langle e_\gamma, f_{P,i}(1) e_\alpha \right\rangle \left\langle e_\gamma, f_{P,i}(1) e_\beta \right\rangle \right\}_{\alpha, \beta}
\]
\[
= \det \left\{ \frac{1}{n} \sum_{i=1}^{d} f_{P,i}(1) e_\alpha \left\langle f_{P,i}(1) e_\beta \right\rangle \right\}_{\alpha, \beta}
\]
\[
= \det \left( \frac{1}{n} \sum_{i=1}^{d} f_{P,i}(1) f_{P,i}^{tr}(1) \right).
\]

Using the expression of \( J_P \) in Lemma 3.22 we can easily derive the following estimate.
Corollary 3.23 Let $J_P$ be defined as above, then for any $\sigma \in H_P(M)$, $J_P(\sigma) \geq 1$.

Proof. For any $v \in \mathbb{C}^d$, using Proposition 4.1 we have:

$$\left\langle \frac{1}{n} \sum_{i=1}^{n} f_{P,i}(\sigma, 1) f_{P,i}^{tr}(\sigma, 1) v, v \right\rangle = \frac{1}{n} \sum_{i=1}^{n} \|f_{P,i}(\sigma, 1) v\|^2$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} \|v\|^2$$

$$= \|v\|^2.$$

So by Min-max theorem, $\text{eig} \left( \frac{1}{n} \sum_{i=1}^{n} f_{P,i}(\sigma, 1) f_{P,i}^{tr}(\sigma, 1) \right) \subset [1, +\infty)$ and therefore:

$$J_P(\sigma) = \sqrt{\det \left( \frac{1}{n} \sum_{i=1}^{n} f_{P,i}(\sigma, 1) f_{P,i}^{tr}(\sigma, 1) \right)} \geq 1.$$

Lemma 3.24 For any $\sigma \in K$, let $V_x: K \to \mathbb{R}_+^+$ be the normal Jacobian of $\psi_x: K \to H_P(x)(M)$, i.e. $V_x := \sqrt{\det \left( (\psi_x)^{tr} \psi_x \right)}$, then

$$V_x(\sigma) = \sqrt{\det \left( I + L_x(\sigma) F_P(\sigma) L_x(\sigma)^{tr} \right)} \forall \sigma \in K,$$

where

$$L_x(\sigma) := C_x \left( \sigma, \frac{1}{n} \right) S_x \left( \sigma, \frac{1}{n} \right)^{-1} \text{ and } F_P(\sigma) := \frac{1}{n} \sum_{i=0}^{n-2} f_{P,i}(\sigma, \tau) f_{P,i}(\sigma, \tau)^{tr}.$$

Proof. Using (3.9) and differentiating $\hat{h}$ with respect to $s$, we get:

$$\hat{h}'(\sigma, \tau) = -C_x \left( \sigma, \frac{1}{n} \right) S_x \left( \sigma, \frac{1}{n} \right)^{-1} h(\sigma, \tau) := -L_x(\sigma) h(\sigma, \tau). \tag{3.10}$$

Also from Proposition 4.1 we have

$$h(\sigma, \tau) = \frac{1}{n} \sum_{i=0}^{n-1} f_{P,i+1}(\sigma, \tau) h'(\sigma, s_i+),$$

so

$$\hat{h}'(\sigma, \tau+) = -L_x(\sigma) \frac{1}{n} \sum_{i=0}^{n-1} f_{P,i+1}(\sigma, \tau) h'(\sigma, s_i+). \tag{3.11}$$

For any $\alpha, \beta \in \{1, ..., d\}$ and $i, j \in \{1, ..., n-1\},$

$$\left\langle \psi_x(\sigma), \psi_x(\sigma) \right\rangle_{T_{\psi_x(\sigma)}H_P(x)(M)} = \frac{1}{n} \sum_{k=0}^{n-2} \langle h'_{\alpha,i}(s_k+), h'_{\beta,j}(s_k+) \rangle + \frac{1}{n} \langle h'_{\alpha,i}(\tau+), h'_{\beta,j}(\tau+) \rangle$$

$$= \delta_{(\alpha,i)} \frac{1}{n} \left( L_x(\sigma) \frac{1}{n} f_{P,i}(\tau) e_\alpha, L_x(\sigma) \frac{1}{n} f_{P,j}(\tau) e_\beta \right)$$

$$= \delta_{(\alpha,i)} + \left( L_x(\sigma) \frac{1}{n} f_{P,i}(\tau) e_\alpha, L_x(\sigma) \frac{1}{n} f_{P,j}(\tau) e_\beta \right),$$

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where
\[ \delta^{(\beta,j)}_{(\alpha,i)} = \begin{cases} 1 & \alpha = \beta, i = j \\ 0 & \text{otherwise}. \end{cases} \]

It follows that the volume change
\[ V_x (\sigma) = \sqrt{\det \left( I_{(\mathbb{R}^d)^{n-1}} + \hat{T}_x (\sigma) \right)} \tag{3.12} \]

where \( \hat{T}_x (\sigma) \in \text{End} \left( (\mathbb{R}^d)^{n-1} \right) \) is defined by
\[ \left( \hat{T}_x (\sigma) \right)_{d(i-1)+\alpha,d(j-1)+\beta} = \left( L_x (\sigma) \frac{1}{n} f_{P,i} (\sigma,\tau) e_\alpha, L_x (\sigma) \frac{1}{n} f_{P,j} (\sigma,\tau) e_\beta \right), \]

where \( 1 \leq i, j \leq n-1, 1 \leq \alpha, \beta \leq d \). If
\[ S_\sigma = \left( \begin{array}{c} I_{(\mathbb{R}^d)^{n-1}} \\ A_x (\sigma) \end{array} \right) \in M_{nd \times (n-1)d} \]

and
\[ A_x (\sigma) = \left( \frac{1}{n} L_x (\sigma) f_{P,0} (\sigma,\tau) e_1, \ldots, \frac{1}{n} L_x (\sigma) f_{P,n-2} (\sigma,\tau) e_d \right) \in M_{d \times (n-1)d}, \]

then
\[ I_{(\mathbb{R}^d)^{n-1}} + \hat{T}_x (\sigma) = S_\sigma^t S_\sigma. \]

Applying Lemma 3.1 we get:
\[ \det \left( I_{(\mathbb{R}^d)^{n-1}} + \hat{T}_x (\sigma) \right) = \det \left( I_{(\mathbb{R}^d)^{n-1}} + A_x (\sigma) (A_x (\sigma))^t \right) \]
\[ = \det \left( I + \frac{1}{n^2} \sum_{i=0}^{n-2} \sum_{\alpha=1}^{d} L_x f_{P,i} (\tau) e_\alpha e_\alpha^t f_{P,i} (\tau)^t L_x^t \right) \]
\[ = \det \left( I + L_x F_p L_x^t \right) \]

where \( F_p (\sigma) \) is as in Eq. 3.2.4.

**Lemma 3.25** For any \( \sigma \in K \),
\[ V_x (\sigma) \leq \sum_{k=0}^{d} \binom{d}{k} n^k e^{\frac{k}{2} E^2 (\sigma, x) \Pi_{j=0}^{n-2} e^{k \lambda_{(s_j, s_{j+1})}}} \tag{3.13} \]

**Proof.** From Lemma 3.24 and Appendix B one can see, after suppressing \( \sigma \),
\[ \det \left( I_{(\mathbb{R}^d)^{n-1}} + \hat{T}_x \right) = \det \left( I + L_x F_p L_x^t \right) = \prod_{i=1}^{d} \left( 1 + \lambda_{i,x} \right) \leq \left( 1 + \max_{1 \leq i \leq d} \lambda_{i,x} \right)^d, \]

where \{\lambda_{i,x}\}_{i=1}^{d} = \text{eig} \left( L_x F_p L_x^t \right).

Note that
\[ \max_{1 \leq i \leq d} \lambda_{i,x} = \left\| L_x (\sigma) F_p L_x (\sigma)^t \right\| \leq \left\| L_x (\sigma) \right\|^2 \left\| F_p \right\| \leq \frac{1}{n} \left\| L_x (\sigma) \right\|^2 \sup_{0 \leq i \leq n-2} \left\| f_{P,i} (\tau) \right\|^2. \]

Using Proposition A.5 we get:
\[ \left\| C_x \left( \sigma, \frac{1}{n} \right) \right\| \leq e^{\frac{1}{2} E^2 (\sigma, x)}, \]

where for any \( x, y \in M \), \( d (x, y) \) is the geodesic distance between \( x \) and \( y \), and
\[ \left\| S_x^{-1} \left( \sigma, \frac{1}{n} \right) \right\| \leq n, \]

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so

\[ \| L_x (\sigma) \|^2 \leq n^2 e^{N d^2 (\sigma(\tau), x)} \]

and

\[ \max_{1 \leq i \leq d} \lambda_i, x \leq n e^{N d^2 (\sigma(\tau), x)} \sup_{0 \leq i \leq n-2} \| f_{P,i} (\sigma, \tau) \|^2. \]

Therefore

\[
V_x (\sigma) = \left( 1 + \max_{1 \leq i \leq d} \lambda_i, x \right)^{\frac{d}{2}} \leq \left( 1 + n e^{N d^2 (\sigma(\tau), x)} \sup_{0 \leq i \leq n-2} \| f_{P,i} (\sigma, \tau) \|^2 \right)^{\frac{d}{2}}
\]

\[ \leq \left( 1 + n^2 e^{\frac{d}{2} d (\sigma(\tau), x)} \sup_{0 \leq i \leq n-2} \| f_{P,i} (\sigma, \tau) \| \right)^{d}
\]

\[ = \sum_{k=0}^{d} \binom{d}{k} n^{\frac{k}{2}} e^{\frac{k}{2} d (\sigma(\tau), x)} \sup_{0 \leq i \leq n-2} \| f_{P,i} (\sigma, \tau) \|^k. \quad (3.14)\]

Applying Proposition A.5 to \( f_{P,i} (\sigma, \tau) \) shows

\[
\| f_{P,i} (\tau) \| \leq \| G_{P,n-1} \| \cdots \| G_{P,i+1} \| \| \frac{S_i}{\Delta_i} \|
\]

\[ \leq e^{\frac{1}{2} N d^2 (\sigma(s_{n-2}), \sigma(s_{n-1}))} \cdots e^{\frac{1}{2} N d^2 (\sigma(s_{i}), \sigma(s_i))} \left( 1 + \frac{N d^2 (\sigma(s_{i-1}), \sigma(s_i))}{6} \right)
\]

\[ \leq \Pi_{j=i-1}^{n-2} e^{\frac{1}{2} N d^2 (\sigma(s_j), \sigma(s_{j+1}))} \cdot e^{\frac{N d^2 (\sigma(s_{i-1}), \sigma(s_i))}{6}}
\]

\[ \leq \Pi_{j=i-1}^{n-2} e^{N d^2 (\sigma(s_j), \sigma(s_{j+1}))}
\]

\[ \leq \Pi_{j=0}^{n-2} e^{N d^2 (\sigma(s_j), \sigma(s_{j+1}))}
\]

Taking supremum over \( i \), we get

\[ \sup_{0 \leq i \leq n-2} \| f_{P,i} (\sigma, \tau) \| \leq \Pi_{j=0}^{n-2} e^{N d^2 (\sigma(s_j), \sigma(s_{j+1}))}. \quad (3.15)\]

and Eq. (3.13) follows. \( \blacksquare \)

**Definition 3.26** For any \( X, Y \in \mathcal{T} K \) (the tangent bundle of \( K \)), define two metrics \( G^0_{P,\tau}, G^1_{P,\tau} \) to be:

\[ \langle X, Y \rangle_{G^0_{P,\tau}} = \sum_{i=1}^{n-1} \langle X (s_i), Y (s_i) \rangle \Delta_i \]

and

\[ \langle X, Y \rangle_{G^1_{P,\tau}} = \sum_{i=1}^{n-1} \left( \frac{\nabla X}{ds} (s_{i-1}), \frac{\nabla Y}{ds} (s_{i-1}) \right) \Delta_i. \]

**Lemma 3.27** \( G^0_{P,\tau} \) is a metric on \( K \).

**Proof.** The only non–trivial part is to check \( \langle X, X \rangle_{G^0_{P,\tau}} = 0 \implies X = 0 \). Since \( M \) has non–positive curvature, there are no conjugate points. For each \( 0 \leq i \leq n-1 \), there is a unique piece of Jacobi field \( X \) along \( \sigma \) \( |s_i, s_{i+1}| \) with specified boundary values \( X (s_i) \) and \( X (s_{i+1}) \). \( \langle X, X \rangle_{G^0_{P,\tau}} = 0 \implies X (s_i) = 0 \) for any \( 1 \leq i \leq n \). By the uniqueness of Jacobi field, \( X \equiv 0 \). \( \blacksquare \)

Based on the metric \( G^0_{P,\tau} \) and \( G^1_{P,\tau} \), we define measures \( \nu^0_{P,\tau} \) and \( \nu^1_{P,\tau} \) on \( K \) as follows.

**Definition 3.28** Let

\[ d\nu^0_{P,\tau} := \frac{n^{(n-1)d}}{(2\pi)^{(n-1)d/2}} e^{-\frac{1}{2} E} d\nu^0_{G^0_{P,\tau}} \]

and

\[ d\nu^1_{P,\tau} = \frac{1}{(2\pi)^{(n-1)d/2}} e^{-\frac{1}{2} E} d\nu^1_{G^1_{P,\tau}}. \]
Lemma 3.29 If
\[ \rho_P (\sigma) := \Pi_{i=1}^{n-1} \det \left( \frac{S_{P,i} (\sigma)}{n} \right) \quad \forall \sigma \in \mathcal{K}, \]
then \( d\nu_P^{\rho} = \rho_P d\nu_{P,\tau} \), and moreover, \( \rho_P (\sigma) \geq 1 \, \forall \sigma \in \mathcal{K} \). Proof. The argument to show \( \rho_P \) is the density of \( \nu_P^{\rho} \) with respect to \( \nu_{P,\tau} \) is almost exactly the same as Theorem 5.9 in [1] with a slight change of ending point from 1 to \( \tau \). Here we focus on the lower bound estimate of \( \rho_P (\sigma) \). Since for any \( v \in \mathbb{C}^d \),
\[ \left\| \frac{S_{P,i}}{n} v \right\| \geq \| v \|, \]
we know from Proposition A.3 that for any \( \lambda \in \text{eig} \left( \frac{S_{P,i}}{n} \right) \),
\[ |\lambda| \geq 1, \]
and from which we know:
\[ \rho_P (\sigma) = \Pi_{i=1}^{n-1} \det \left( \frac{S_{P,i} (\sigma)}{n} \right) \geq 1. \]

Proof of Theorem 3.13. Since \( \psi_x \) is a diffeomorphism,
\[ h_P^f (x) = \int_{H_P,\tau (M)} \frac{1}{Z_P J_P} \psi_x (\sigma) e^{-\frac{1}{2} E(\sigma)} d\text{vol}_{G_{b,r}} (\sigma) \]
\[ = \int_{K} \frac{1}{Z_P J_P} \psi_x (\sigma) e^{-\frac{1}{2} E(\sigma)} d\text{vol}_{G_{b,r}} (\sigma). \]
Notice that
\[ \frac{1}{Z_P} e^{-\frac{1}{2} E(\sigma)} = \frac{1}{(2\pi)^{n/2}} \frac{1}{2^{n/2}} e^{-\frac{1}{2} E(\sigma)} e^{-\frac{1}{2} d^2(\sigma(\tau),x)}, \]
so
\[ h_P^f (x) = \frac{1}{(2\pi)^{n/2}} \int_{K} \frac{f}{J_P} \psi_x (\sigma) e^{-\frac{1}{2} d^2(\sigma(\tau),x)} d\nu_{G_{b,r}} (\sigma). \]
Combining (3.14), (3.15) we know that:
\[ e^{-\frac{1}{2} d^2(\sigma(\tau),x)} V_x (\sigma) \leq \sum_{k=0}^{d} \binom{d}{k} n^k e^{-\frac{Nk-n}{2} d^2(\sigma(\tau),x)} \Pi_{j=0}^{n-2} e^{N^2 d^2(\sigma(s_j),\sigma(s_{j+1})).} \]
So
\[ \sup_{x \in M} e^{-\frac{1}{2} d^2(\sigma(\tau),x)} V_x (\sigma) \leq \sup_{x \in M} e^{-\frac{Nk-n}{2} d^2(\sigma(\tau),x)} \sum_{k=0}^{d} \binom{d}{k} n^k \Pi_{j=0}^{n-2} e^{N^2 d^2(\sigma(s_j),\sigma(s_{j+1})).} \]
When \( n \) is large enough, i.e. \( n > Nk, e^{-\frac{Nk-n}{2} d^2(\sigma(\tau),x)} \leq 1 \) and thus it suffices to show
\[ \mathbb{E}_{\nu_{G_{b,r}}} \left[ \sum_{k=0}^{d} \binom{d}{k} n^k \Pi_{j=0}^{n-2} e^{N^2 k d^2(\sigma(s_j),\sigma(s_{j+1})).} \right] < \infty. \]
For each \( k \leq d \) we have:
\[ \mathbb{E}_{\nu_{G_{b,r}}} \left[ \binom{d}{k} n^k \Pi_{j=0}^{n-2} e^{N^2 k d^2(\sigma(s_j),\sigma(s_{j+1})).} \right] = C_n \mathbb{E}_{\nu} \left[ \Pi_{j=0}^{n-2} e^{N^2 k |\Delta_{j+1}\beta|^2} \right] \]
Using Lemma A.2 in Appendix A we obtain a bound of the right–hand side of Eq. (3.18) (the bound here depends on \( n \)). Since for any \( \sigma \in \mathcal{K} \), \( \int_{P} \psi_x (\sigma) e^{-\frac{1}{2} d^2(\sigma(\tau),x)} V_x (\sigma) \) is continuous with respect to \( x \in M \), so by dominated convergence theorem, \( h_P^f (x) \in \mathcal{C} (M) \).
Proposition 3.30 \( \sup_{\mathcal{P}} \sup_{x \in M} \nu_{\mathcal{P}, x}^{-1} (H_{\mathcal{P}, x} (M)) < \infty. \)

**Proof.** Based on Eq. (3.17) and Corollary 3.23

\[
\nu_{\mathcal{P}, x}^{-1} (H_{\mathcal{P}, x} (M)) \leq C_d \int_{\mathcal{K}} e^{-\frac{N}{2} d^2(\sigma, x)} V_x (\sigma) \, d\nu_{\mathcal{P}, x} (\sigma)
\]

Combining Eq. (3.14) and Eq. (3.15) we know that:

\[
e^{-\frac{N}{2} d^2(\sigma, x)} V_x (\sigma) \leq \sum_{k=0}^{d} \binom{d}{k} \left( \frac{Nk}{n} \right)^{2d(\sigma, x)} \prod_{j=0}^{n-2} e^{Nk^2 d^2(\sigma_j, \sigma_{(s_j, s_{(s_j+1))})}}
\]

For each \( k \leq d \), applying Lemma 3.29 we have:

\[
\mathbb{E}_{\nu_{\mathcal{P}, x}} \left[ e^{-\frac{N}{2} d^2(\sigma, x)} \right] \left( \frac{d}{k} \right) \int_{\mathcal{K}} e^{-\frac{N}{2} d^2(\sigma, x)} \prod_{j=0}^{n-2} e^{Nk^2 d^2(\sigma_j, \sigma_{(s_j, s_{(s_j+1))})}} d\nu_{\mathcal{P}, x} (\sigma)
\]

\[
= \left( \frac{d}{k} \right) \prod_{j=0}^{n-2} e^{-\frac{N}{2} d^2(\sigma_j, \sigma_{(s_j, s_{(s_j+1))})}} d\nu_{\mathcal{P}, x} (\sigma)
\]

Now define the projection map \( \pi_P : \mathcal{K} \rightarrow M^{n-1} \), for any \( \sigma \in \mathcal{K} \),

\[
\pi_P (\sigma) := (\sigma (s_1), \ldots, \sigma (s_{n-1}))
\]

Since \( M \) is a Hadamard manifold, \( \pi_P \) is a diffeomorphism. From there one can get:

\[
\left( \frac{d}{k} \right) \prod_{j=0}^{n-2} e^{-\frac{N}{2} d^2(\sigma_j, \sigma_{(s_j, s_{(s_j+1))})}} d\nu_{\mathcal{P}, x} (\sigma)
\]

\[
= \left( \frac{d}{k} \right) \prod_{j=0}^{n-2} e^{-\frac{N}{2} d^2(\sigma_j, \sigma_{(s_j, s_{(s_j+1))})}} d\nu_{\mathcal{P}, x} (\sigma)
\]

Corollary 4.2 in [23] gives a lower bound of the heat kernel of manifold \( M \), provided \( \text{Ric} \geq (1 - d) N \):

\[
p_t (x, y) \geq (2\pi t)^{-\frac{n}{2}} e^{-\frac{N}{2} \left( \frac{\sinh \sqrt{N} \rho}{\sqrt{N} \rho} \right)^{1/d} e^{-Ct}}
\]

where \( N \) is the curvature bound and \( C \) is some constant depending only on \( d \) and \( N \) and \( \rho = d (x, y) \).

Using the fact that

\[
\frac{\sinh \sqrt{N} \rho}{\sqrt{N} \rho} \leq e^{\frac{\rho^2}{2}}
\]

it follows that

\[
p_t (x, y) \geq (2\pi t)^{-\frac{n}{2}} e^{-\frac{1}{2} \left( 1 + \frac{N(n-1)}{2} \right) \rho^2} e^{-Ct}
\]

Let \( t = \frac{1}{n-N_1} \), where \( N_1 = 2Nd + \frac{N(n-1)}{2} \), we have, for any \( j \in \{0, \ldots, n-1\} \):

\[
e^{-\frac{1}{2} (n-2N_1) d^2(x_j, x_{j+1})} \leq e^{Ct} p_t (x_j, x_{j+1}) (2\pi t)^{\frac{d}{2}}.
\]

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So

\[
\left(\frac{d}{k}\right) n^{\frac{k+(n-1)d}{2}} (2\pi)^{\frac{(n-1)d}{2}} \int_{M^{n-1}} e^{-\frac{n-2N_{1}}{2} d^2(x_{n-1},x)} \prod_{j=0}^{n-2} e^{-\frac{1}{2} (n-2Nd)d^2(x_{j},x_{j+1})} dx_1 \cdots dx_n
\]

\[
\leq \left(\frac{d}{k}\right) n^{\frac{k+(n-1)d}{2}} (n - N_{1})^{\frac{d}{2}} e^{C\frac{n-2N_{1}}{n}} \int_{M^{n-1}} p_{n-1}^{\frac{1}{N_{1}}} (x_{n-1},x) \prod_{j=0}^{n-2} p_{n-1}^{\frac{1}{N_{1}}} (x_{j},x_{j+1}) dx_{1} \cdots dx_{n-1}
\]

\[
= \left(\frac{d}{k}\right) n^{\frac{d-k}{2}} (1 - \frac{N_{1}}{n})^{\frac{d}{2}} \int_{M} p_{n-1}^{\frac{1}{N_{1}}} (x_{n-1},x) p_{n-1}^{\frac{1}{N_{1}}} (o,x_{n-1}) dx_{n-1}
\]

\[
= \left(\frac{d}{k}\right) n^{\frac{d-k}{2}} (1 - \frac{N_{1}}{n})^{\frac{d}{2}} p_{n-1}^{\frac{1}{N_{1}}} (0,x)
\] (3.21)

Since the heat kernel is continuous with respect to time t, combining (3.19), (3.20) and (3.21) we get

\[
\left(\frac{d}{k}\right) e^{C\frac{n-2N_{1}}{n}} \int_{0}^{\infty} e^{-\frac{t}{2} N_{1}} p_{n-1}^{\frac{1}{N_{1}}} (0,x) \leq C.
\]

and hence

\[\nu_{p,x}^{1} (H_{p,x} (M)) \leq C.\]

where C is a constant depending only on d and N.

Theorem 3.31 shows that the class of approximate pinned measures \(\{\nu_{p,x}^{1}\}\) are finite measures and using the continuity result for \(h_{\nu_{p}} (x)\), one can see that \(\nu_{p,x}^{1}\) is deserved to be formally expressed as \(\delta_{x} (\sigma (1)) \nu_{p}^{1}\) and it should be interpreted in the sense of Corollary 3.32 below. First we state a co–area formula.

**Theorem 3.31 (Theorem 2.3 in Federer [11])** Let H and M be two Riemannian manifolds with volume measures \(dvol_{H}\) and \(dvol_{M}\) respectively. If \(p : H \to M\) is a smooth submersion, \(g : H \to [0,\infty)\) is a density function, for each \(x \in M\), let \(dvol_{H_{x}}\) be the volume measure on \(H_{x} := p^{-1} \{x\}\) and \(J (y) := \sqrt{\det(p_{xy} p_{xy}^{\tau})}\) on \(y \in H_{x}\), then for any non–negative measurable function \(f : M \to [0,\infty)\),

\[
\int_{H} (f \circ p) g dvol_{H} = \int_{M} dvol_{M} (x) f (x) \int_{H_{x}} \frac{1}{J (y)} g (y) dvol_{H_{x}} (y).
\] (3.22)

**Corollary 3.32** Denote by \(\delta_{x} \in \mathcal{E}' (M)\) the delta–function at \(x \in M\) and \(\{\delta_{x}^{(m)}\}_{m} \subset C_{0}^\infty (M)\) is an approximate sequence to \(\delta_{x} (\sigma (1)) \delta_{x}^{(m)}\) in \(\mathcal{E}' (M)\) i.e. for any \(h \in C_{0}^\infty (M)\), we have:

\[
\lim_{m \to \infty} \int_{M} h (y) \delta_{x}^{(m)} (y) dy = \int_{M} h (y) \delta_{x} (y) dy =: h (x)
\]

where \(dy\) is the volume measure on \(M\).

Then for any \(f \in C_{0}^\infty (H_{p} (M))\),

\[
\lim_{m \to \infty} \int_{H_{p} (M)} f (\sigma) \delta_{x}^{(m)} (\sigma (1)) d\nu_{p}^{1} (\sigma) = \int_{H_{p,x} (M)} f (\sigma) d\nu_{p,x}^{1} (\sigma).
\]

**Proof.** Using the co–area formula (3.22) with

\[(H, M, p, g, f) = \left( H_{p} (M), M, E_{1}^{p}, \frac{1}{Z_{p}} e^{-\frac{d}{2}}, \delta_{x}^{(m)} (\sigma (1)) f (\sigma) \right),\]
we have
\[ \int_{H^p(M)} \delta_x^{(m)} (\sigma \, (1)) \, f (\sigma) \, d\nu^p_M (\sigma) = \int_M dy \delta_x^{(m)} (y) \int_{H^p_y(M)} f (\sigma) 
\]
\[ = \int_M h^f_p (y) \delta_x^{(m)} (y) \, dy. \]

From Theorem 3.13 we know \( h^f_p (x) \in C (M) \), therefore:

\[ \lim_{m \to \infty} \int_{H^p(M)} \delta_x^{(m)} (\sigma \, (1)) \, f (\sigma) \, d\nu^p_M (\sigma) = \lim_{m \to \infty} \int_M h^f_p (y) \delta_x^{(m)} (y) \, dy = h^f_p (x) \]
\[ = \int_{H^p_y(M)} f (\sigma) 
\]

\[ = \int_{H^p_y(M)} f (\sigma) \, d\nu^1_{p,x} (\sigma). \]

\[ \blacksquare \]

4 Orthogonal Lifting Technique

4.1 The Orthogonal Lift \( \hat{X}_p \) on \( H^p (M) \)

As a remainder, unless mentioned separately, \( M \) is a complete Riemannian manifold without positive sectional curvature bounded below by \( -N \). In this subsection we focus on the unpinned piecewise geodesic space \( H^p (M) \).

4.1.1 A Parametrization of \( T_p H^p (M) \)

Recall from Theorem 2.29 that \( Y \in \Gamma (TH^p (M)) \) iff for each \( \sigma \in H^p (M) \), \( J (\sigma, s) := u (\sigma, s)^{-1} \, Y (\sigma, s) \) satisfies (in the following equation we suppress \( \sigma \))

\[ J'' (s) = R_{u(s)} (b' (s_{i-1} +) \, J (s)) b' (s_{i-1} +) \text{ for } s \in [s_{i-1}, s_i) \, i = 1, \ldots, n. \quad (4.1) \]

where \( b = \phi (\sigma) \in H (\mathbb{R}^d) \) is the anti–rolling of \( \sigma \).

From above we observe that \( J \) can be parametrized by

\[ \{ J' (s_i +) = k_i \}_{i=0}^{n-1} \quad (4.2) \]

where \( (k_0, k_1, \ldots, k_{n-1}) \) is an arbitrary element of \( (\mathbb{R}^d)^n \). Proposition 4.1 explains this parametrization in more detail.

**Proposition 4.1** If \( (k_0, k_1, \ldots, k_{n-1}) \in (\mathbb{R}^d)^n \), then the unique \( J (\cdot) \in C ([0, 1], \mathbb{R}^d) \) satisfying (4.1) and (4.2) above is given by

\[ J (s) = \frac{1}{n} \sum_{i=0}^{l-1} f_{\mathcal{P},i+1} (s) \, k_i \text{ for } s \in [s_{i-1}, s_i] , \quad 1 \leq l \leq n. \quad (4.3) \]

**Proof.** From the definition of \( f_{\mathcal{P},i+1} \) (see Definition 2.32), \( J \) in Eq. (4.3) may be written as

\[ J (s) = C_{\mathcal{P},l} (s) \left[ \sum_{i=0}^{l-2} C_{\mathcal{P},l-2} \ldots C_{\mathcal{P},i+2} S_{\mathcal{P},i+1} k_i \right] + S_{\mathcal{P},l} (s) \, k_{l-1} \text{ when } s \in [s_{l-1}, s_l]. \]

To finish the proof, we need only verify that \( J \) is continuous, \( J' (s_{i+}) = k_i \) for \( 0 \leq i \leq n - 1 \) and \( J \) solves the Jacobi equation (4.1). Since \( C_{\mathcal{P},l} (s) \) and \( S_{\mathcal{P},l} (s) \) satisfies Jacobi equation for
\( s \in [s_{l-1}, s_l] \), \( J \) satisfies (4.11) and is continuous at \( s \notin \mathcal{P} \). For each \( s_l, 1 \leq l \leq n-1 \), since \( C_{P_{l+1}}(s_l) = I, s_{P_{l+1}}(s_l) = 0 \) and \( J \) is right continuous on \([0,1]\),

\[
J(s_l-) = \lim_{s \uparrow s_l} J(s) = C_{P_{l+1}} \left[ \sum_{i=0}^{l-2} C_{P_{l-1}} \ldots C_{P_{i+2}} s_{P_{i+1}} k_i \right] + s_{P_{l+1}} k_{l-1}
\]

\[= C_{P_{l+1}}(s_l) \left[ \sum_{i=0}^{l-1} C_{P_{l-1}} \ldots C_{P_{i+2}} s_{P_{i+1}} k_i \right] + s_{P_{l+1}}(s_l) k_l \]

\[= J(s_l) = J(s_l^+) . \]

So \( J \) is also continuous at partition points. Then since

\[
J'(s_{l-1}+) = C'_{P_{l+1}}(s_{l-1}+) \left[ \sum_{i=0}^{l-2} C_{P_{l-1}} \ldots C_{P_{i+2}} s_{P_{i+1}} k_i \right] + s'_{P_{l+1}}(s_{l-1}+) k_{l-1}
\]

\[= 0 + I \cdot k_{l-1} = k_{l-1}, \]

\( J \) satisfies (4.2). The uniqueness of \( J \) is easily seen from the uniqueness of solutions to linear ODE with initial values.

**Definition 4.2** For each \( s \in [0,1] \), define \( L_s : (\mathbb{R}^d)^n \to \mathbb{R}^d \) as follows: for \( s \in [s_{l-1}, s_l] \),

\[
L_s(k_0, \ldots, k_{n-1}) = \frac{1}{n} \sum_{i=0}^{l-1} f_{P_{i+1}}(s) k_i. \tag{4.4}
\]

We now compute the adjoint of \( L_1 \).

**Lemma 4.3** For any \( v \in \mathbb{R}^d \), let \( L_1^* : \mathbb{R}^d \to (\mathbb{R}^d)^n \) be the adjoint of \( L_1 \), then

\[
L_1^* v = \frac{1}{n} \left( f_{P_{1}}^* (1) v, f_{P_{2}}^* (1) v, \ldots, f_{P_{n}}^* (1) v \right), \tag{4.5}
\]

where \( f_{P_{i}}^* (1) \) is the matrix adjoint of \( f_{P_{i}} (1) \).

**Proof.** Equation (4.5) immediately follows from the identity;

\[
\langle L_1 (k_0, \ldots, k_{n-1}) , v \rangle = \sum_{i=0}^{n-1} \left( \frac{1}{n} f_{P_{i+1}} (1) k_i , v \right) = \sum_{i=0}^{n-1} \left( k_i , \frac{1}{n} f_{P_{i+1}} (1) v \right) . \tag{4.6}
\]

**Definition 4.4** We now define

\[
K_P(s) v := nL_s (L_1^* v). \tag{4.7}
\]

In particular,

\[
K_P(1) v = \frac{1}{n} \sum_{i=0}^{n-1} f_{P_{i+1}} (1) f_{P_{i+1}}^* (1) v. \tag{4.8}
\]

Recall that given a matrix \( A \), \( \text{eig}(A) \) denotes the eigenvalues of \( A \).

**Lemma 4.5 (Invertibility of \( K_P(1) \))** If \( M \) has non-positive sectional curvature, then

\[
\text{eig}(K_P(1)) \subset [1, \infty) \tag{4.9}
\]

and thus \( K_P(1) \) is invertible.
Proof. Denote \( R_{u_i}(b'(s_{i-1}+), \cdot) b'(s_{i-1}+) \) by \( A_{P,i}(s) : H_P(M) \to \text{End} (\mathbb{R}^d) \). Notice that \( M \) having non-positive sectional curvature guarantees \( A_{P,i}(s) \) is positive semi-definite. Then apply Proposition 4.5 to find, for any \( i = 1, \ldots, n \) and \( v \in \mathbb{C}^d \),

\[
\|C_{P,i} v\| \geq \|v\| \quad \text{and} \quad \|S_{P,i} v\| \geq \frac{1}{n} \|v\|.
\]

From these inequalities it follows that

\[
\|f_{P,i}(1)v\| = n \|C_{P,n}C_{P,n-1} \cdots C_{P,i+1}S_{P,i}v\| \geq n \cdot \frac{1}{n} \|v\| = \|v\|.
\]

So \( f_{P,i}(1) \) is invertible and \( \|f_{P,i}^*(1)^{-1}\| = \|f_{P,i}(1)^{-1}\| \leq 1 \). Therefore for any \( v \in \mathbb{C}^d \),

\[
\|f_{P,i}^*(1)^{-1}v\| \leq \|v\|
\]

now replace \( v \) by \( f_{P,i}^*(1)v \), we get \( \|f_{P,i}^*(1)v\| \geq \|v\| \) and thus

\[
\langle K_P(1)v, v \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle f_{P,i+1}(1)f_{P,i+1}^*(1)v, v \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \|f_{P,i+1}^*(1)v\|^2 \geq \frac{1}{n} \cdot n \|v\|^2 = \|v\|^2 \quad \forall v \in \mathbb{C}^d.
\]

This implies that

\[
eig(K_P(1)) \subset [1, \infty).
\]

In particular, \( K_P(1) \) is invertible. \( \blacksquare \)

4.1.2 Orthogonal Lift on \( H_P(M) \)

In this subsection we use the least square method to lift a vector field \( X \in \Gamma(TM) \) to a vector field \( \tilde{X}_P \in \Gamma(TH_P(M)) \), here lift means \( \tilde{X}_P \) satisfies Eq. (4.10).

**Theorem 4.6 (Orthogonal lift)** For all \( X \in \Gamma(TM) \), there exists a unique vector field \( \tilde{X}_P \in \Gamma(TH_P(M)) \) satisfying:

1. For all \( h \in C^1(M) \),

\[
\tilde{X}_P(h \circ E_1) (\sigma) = (Xh)(E_1(\sigma)), \quad \text{i.e.} \quad E_{1*}\tilde{X}_P(\sigma) = X(\sigma(1)). \tag{4.10}
\]

2. For all \( \sigma \in H_P(M) \),

\[
\left\| \tilde{X}_P(\sigma) \right\|_{G_P} = \inf \{ \|Y(\sigma)\|_{G_P} : Y \in \Gamma(TH_P(M)), Y \text{ satisfies } (4.10) \}. \tag{4.11}
\]

In this paper \( \tilde{X}_P \) is referred to as the **orthogonal lift** of \( X \) to \( (H_P(M), G_P^1) \).

First we use the parametrization in Subsection 4.1.1 to characterize \( \{ \text{Nul} (E_{1*}\sigma) \}^\perp \).

**Lemma 4.7** Suppose \( Y \in \Gamma(TH_P(M)) \) with \( k(\cdot) := u(\cdot)^{-1}Y(\cdot) : H_P(M) \to H(\mathbb{R}^d) \). Then \( Y \in \{ \text{Nul} (E_{1*}\sigma) \}^\perp \) iff

\[
(k'(s_0+), \ldots, k'(s_{n-1}+)) \in (\text{Nul} \, L_1)^\perp = \text{Ran} (L_1^*).
\]

**Proof.** Given \( Y(\cdot) := u(\cdot)k(\cdot) \) and \( Z(\cdot) := u(\cdot)J(\cdot) \in \Gamma(TH_P(M)) \), then

\[
\langle Y(\sigma), Z(\sigma) \rangle_{G_P} = 0 \iff \sum_{i=0}^{n-1} \langle J'(\sigma, s_i+), k'(\sigma, s_i+) \rangle_{\Delta_{i+1}} = 0 \iff \sum_{i=0}^{n-1} \langle J'(\sigma, s_i+), k'(\sigma, s_i+) \rangle = 0,
\]

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and
\[ Z(\sigma) \in \text{Nul} \left( E_{1*}^\perp \right) \iff E_{1*}^\perp (Z) = u_1(\sigma) J(\sigma, 1) = 0 \iff J(\sigma, 1) = 0. \]

Recall from Proposition 4.1 and Definition 4.2 that (here we suppress \( \sigma \))
\[ J(1) = L_1 (J'(s_0^+), \ldots, J'(s_{n-1}^+)), \]
so
\[ J(1) = 0 \iff (J'(s_0^+), \ldots, J'(s_{n-1}^+)) \in \text{Nul} (L_1). \tag{4.12} \]

Since
\[ \sum_{i=0}^{n-1} \langle J'(s_i^+), K'(s_i^+) \rangle = \langle (J'(s_0^+), \ldots, J'(s_{n-1}^+)), (K'(s_0^+), \ldots, K'(s_{n-1}^+)) \rangle, \]
so \( Y \in \{ \text{Nul} (E_{1*}) \}^\perp \) iff
\[ (K'(s_0^+), \ldots, K'(s_{n-1}^+)) \in \{ \text{Nul} (L_1) \}^\perp = \text{Ran} (L_1^*). \]

**Remark 4.8** According to [4.3], it is immediate that
\[ \text{Ran} (L_1^*) = \left\{ \left( \frac{1}{n} f_{P*1}(1) v, \frac{1}{n} f_{P*2}(1) v, \ldots, \frac{1}{n} f_{P*n}(1) v \right) : \forall v \in \mathbb{R}^d \right\}. \]

**Definition 4.9** Given \( X \in \Gamma (TM) \), define \( \tilde{X}_P \in \Gamma (TH_P (M)) \) to be \( \tilde{X}_P (\cdot) = u(\cdot) J_P (\cdot) \) where
\[ J_P (s) := K_P (s) K_P (1)^{-1} u_1^{-1} X \circ E_1. \]

**Proof of Theorem 4.6** We will show \( \tilde{X}_P \) is the unique orthogonal lift of \( X \). Since \( T_\sigma H_P (M) = \text{Nul} (E_{1*}^\perp) \oplus_{G^*_P} \{ \text{Nul} (E_{1*}^\perp) \}^\perp \), given a lift \( Z \in \Gamma (TH_P (M)) \) of \( X \in \Gamma (TM) \), its orthogonal projection to \( \{ \text{Nul} (E_{1*}^\perp) \}^\perp \) is also a lift but with smaller \( G^*_P \) norm. So if \( Z \) is an orthogonal lift, then \( Z \in \{ \text{Nul} (E_{1*}^\perp) \}^\perp \). From Lemma 4.7 and Remark 4.8 it follows that if \( k (\cdot) := u^{-1} (\cdot) Z (\cdot), \) then
\[ (k'(s_0), \ldots, k'(s_{n-1})) = \left( \frac{1}{n} f_{P*1}(1) v, \frac{1}{n} f_{P*2}(1) v, \ldots, \frac{1}{n} f_{P*n}(1) v \right) = L_1^* v \]
for some \( v \in \mathbb{R}^d \). Then using Definition 4.4 and Proposition 4.11 \( k \) must have the following form,
\[ k_s = K_P (s) v \]
for some \( v \in \mathbb{R}^d \) to be determined. To specify \( v \), we use condition 4.10
\[ \tilde{X}_P (\sigma, 1) = X (\sigma (1)). \]

This implies \( K_P (1) v = u_1^{-1} X \circ E_1 \). Since \( K_P (1) \) is invertible, we can just pick \( v \) to be \( K_P (1)^{-1} u_1^{-1} X \circ E_1 \).

**Definition 4.10** We will view \( \tilde{X}_P \) as a differential operator with domain,
\[ \mathcal{D} \left( \tilde{X}_P \right) := C^1_b (H_P (M)). \]

Here
\[ C^1_b (H_P (M)) := \left\{ f \in C^1 (H_P (M)) : \exists C \text{ s.t. } \| (df)_\sigma X \| \leq C \langle X, X \rangle_{G^*_P} \forall \sigma \in H_P (M), X \in T_\sigma H_P (M) \right\}. \]

Since \( C^1_b (H_P (M)) \) is dense in \( L^2 (H_P (M), \nu_P^2) \), we can also view \( \tilde{X}_P \) as a densely defined operator on \( L^2 (H_P (M), \nu_P^2) \). ■
### 4.1.3 Restricted Adjoint $\tilde{\mathcal{X}}_{P}^{tr,v_P}$

In this subsection we study $\tilde{\mathcal{X}}_{P}^{tr,v_P}$—the adjoint of $\tilde{\mathcal{X}}_{P}$ with respect to $\nu_P^1$ restricted to $D(\tilde{\mathcal{X}}_{P})$, i.e.
we require $D(\tilde{\mathcal{X}}_{P}^{tr,v_P}) = D(\tilde{\mathcal{X}}_{P})$.

**Lemma 4.11** Given $X \in \Gamma(TM)$, if $\tilde{\mathcal{X}}_{P}$ is the orthogonal lift of $X$, then

$$
\tilde{\mathcal{X}}_{P}^{tr,v_P} = -\tilde{\mathcal{X}}_{P} + M_{\tilde{\mathcal{X}}_{P}} (\phi (s+),b(s+)) ds - M_{\text{div}} \tilde{\mathcal{X}}_{P} 
$$

(4.13)

where $M$ is the multiplication operator, $b$ is the anti-rolling of $\sigma$ and $\text{div} \tilde{\mathcal{X}}_{P}$ is the divergence of $\tilde{\mathcal{X}}_{P}$ with respect to $\text{vol}_{G_{b}}$.

**Proof.** In this proof we identify the measure $\nu_P^1$ with the associated $nd$—form. So by “Cartan’s magic formula”, first assume $f \in C^1_b(H_P(M))$ with compact support,

$$
\mathcal{L}_{\tilde{\mathcal{X}}_{P}} (f \nu_P^1) = d (i_{\tilde{\mathcal{X}}_{P}} (f \nu_P^1)) + i_{\tilde{\mathcal{X}}_{P}} (d (f \nu_P^1)).
$$

Since $f \nu_P^1$ is a top degree form, $d (f \nu_P^1) = 0$. By Stokes’ theorem,

$$
\int_{H_P(M)} d (i_{\tilde{\mathcal{X}}_{P}} (f \nu_P^1)) = 0.
$$

Therefore we have:

$$
\int_{H_P(M)} \mathcal{L}_{\tilde{\mathcal{X}}_{P}} (f \nu_P^1) = 0
$$

and thus

$$
\int_{H_P(M)} (\tilde{\mathcal{X}}_{P} f) d\nu_P^1 = \int_{H_P(M)} \mathcal{L}_{\tilde{\mathcal{X}}_{P}} (f \nu_P^1) - \int_{H_P(M)} f \mathcal{L}_{\tilde{\mathcal{X}}_{P}} (\nu_P^1) = - \int_{H_P(M)} f \mathcal{L}_{\tilde{\mathcal{X}}_{P}} (\nu_P^1). 
$$

(4.14)

Recall that $\nu_P^1 = \frac{1}{Z_P} e^{-\frac{4}{E}} \text{vol}_{G_{b}}$, so

$$
\mathcal{L}_{\tilde{\mathcal{X}}_{P}} (\nu_P^1) = \left[ \tilde{\mathcal{X}}_{P} \left( \frac{1}{Z_P} e^{-\frac{4}{E}} \right) \right] \text{vol}_{G_{b}} + \left( \text{div} \tilde{\mathcal{X}}_{P} \right) \nu_P^1.
$$

(4.15)

In (4.15)

$$
\tilde{\mathcal{X}}_{P} \left( \frac{1}{Z_P} e^{-\frac{4}{E}} \right) = - \frac{1}{2} \tilde{\mathcal{X}}_{P} (E) \frac{1}{Z_P} e^{-\frac{4}{E}}
$$

$$
= - \int_0^1 \left\langle \sigma' (s+), \frac{\nabla \tilde{\mathcal{X}}_{P}}{ds} (s+) \right\rangle ds \frac{1}{Z_P} e^{-\frac{4}{E}}
$$

$$
= - \int_0^1 \left\langle b' (s+), J_{P} (s+) \right\rangle ds \frac{1}{Z_P} e^{-\frac{4}{E}}.
$$

(4.16)

Combining (4.14), (4.15) and (4.16) we get, if $f \in C^1_b(H_P(M))$ with compact support, then

$$
\int_{H_P(M)} \tilde{\mathcal{X}}_{P} f d\nu_P^1 = \int_{H_P(M)} f \cdot (\tilde{\mathcal{X}}_{P}^{tr,v_P},1) d\nu_P^1,
$$

(4.17)

where $\tilde{\mathcal{X}}_{P}^{tr,v_P}$ is defined in Eq. (4.13). For the general case choose a cut–off function $\phi \in C^\infty_c (\mathbb{R}, [0, 1])$ such that $\phi \equiv 1$ on $[-1, 1]$ and $\phi \equiv 0$ on $\mathbb{R} / [-2, 2]$. Let $f_n := f \cdot \phi (\frac{E}{n})$, observe, using product rule, that

$$
\tilde{\mathcal{X}}_{P} f_n = \phi \left( \frac{E}{n} \right) \cdot \tilde{\mathcal{X}}_{P} f + \frac{2}{n} f \cdot \phi' \left( \frac{E}{n} \right) \int_0^1 \left( J_{P} (s+), b' (s+) \right) ds.
$$

(4.18)

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so $\dot{X}_P f_n \to \dot{X}_P f$ as $n \to \infty$ $\nu^1_P$ a.s.

Using Proposition 4.20 and Lemma 6.1 we have for any $q \geq 1$, there exists $M = M(q) > 0$ such that $\forall \mathcal{P}$ with $|\mathcal{P}| \leq \frac{1}{M},$

$$
\int_0^1 \langle J'_P (s+), b' (s+) \rangle \, ds \in L^q \left( H_P (M), d\nu^1_P \right).
$$

Since $f$ has bounded differential, from Definition 4.9 and 4.4 we have

$$
\left| \dot{X}_P f \right| \leq C \left\langle \dot{X}_P, \dot{X}_P \right \rangle_{G^1_P}
$$

$$
= C \sum_{i=1}^{n} \langle J'_P (s_i-1+), J'_P (s_i-1+) \rangle \Delta_i
$$

$$
= \frac{C}{n} \left\| f^{P, i-1}_P (1) K^1_P (1)^{-1} u^{-1}_1 X \circ E_1 \right\|
$$

$$
\leq C \max_{1 \leq i \leq n} \left\| f^{P, i-1}_P \right\|^2 \left\| K^1_P (1)^{-1} u^{-1}_1 X \circ E_1 \right\|^2
$$

Lemma 4.5 states $K^1_P (1)^{-1}$ is bounded. Then utilizing Lemma 5.6 we have for any $q \geq 1$, there exists $M = M(q) > 0$ such that $\forall \mathcal{P}$ with $|\mathcal{P}| \leq \frac{1}{M},$

$$
\dot{X}_P f \in L^q \left( H_P (M), d\nu^1_P \right).
$$

Lemma 6.2 shows that $\dot{X}^{\text{tr}, v^1}_P \in L^q \left( H_P (M), d\nu^1_P \right)$ provided $|\mathcal{P}| \leq \frac{1}{M}$ for some $M = M(q)$, so applying DCT to both sides of Eq. (4.17) with $f_n \to f$ gives Eq. (4.18). \(\blacksquare\)

4.1.4 Computing $\text{div} \dot{X}_P$

Recall from Notation 5.7 and Remark 5.8 that

$$
X^{b_{\alpha, i}} (\sigma, s) = u (\sigma, s) \frac{1}{\sqrt{n}} f_{P, i} (s) e_{\alpha}, \ 1 \leq \alpha \leq d, \ 1 \leq i \leq n
$$

is an orthonormal frame on $(TH_P (M), G^1_P)$. Using this orthonormal frame, one can get an expression of $\text{div} \dot{X}_P$.

Proposition 4.12 Let $\dot{X}_P$ be the orthogonal lift of $X \in \Gamma (TM)$, then

$$
\text{div} \dot{X}_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle X^{b_{\alpha, j}}, J'_P (s_{j-1}+), e_{\alpha} \rangle \sqrt{\Delta_j}
$$

(4.19)

Proof. By definition

$$
\text{div} \dot{X}_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle \left[ X^{b_{\alpha, j}}, \dot{X}_P \right], X^{b_{\alpha, j}} \rangle_{G^1_P},
$$

(4.20)

where $[\cdot, \cdot]$ is the Lie bracket of vector fields.

Now fix $j$ and $\alpha$, notice that $\dot{X}_P = X^{J_P}$, apply Theorem 3.5 in [1] to find

$$
\left[ X^{b_{\alpha, j}}, \dot{X}_P \right] = X^{f (b_{\alpha, j}, J_P)},
$$

where

$$
f_s (h_{\alpha, j}, J_P) = (X^{b_{\alpha, j}} J_P (s) - (X^{J_P} h_{\alpha, j}) (s) + q_s (X^{b_{\alpha, j}}) J_P (s) - q_s (X^{J_P}) h_{\alpha, j} (s)
$$

and

$$
q_s (X^f) = \int_0^s R_{u_r} (b' (r+), f (r)) \, dr.
$$
Therefore
\begin{equation}
\left\langle \left[ X^{h_{a,j}}, \tilde{X}_P \right], X^{h_{a,j}} \right\rangle_{G^p} = \sum_{i=1}^{n} \left\langle f', h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} \tag{4.21}
\end{equation}

\begin{equation}
= \sum_{i=1}^{n} \left\langle \left( X^{h_{a,j}, J_p} \right)' - \left( X^{J_p h_{a,j}} \right)' , h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} + \sum_{i=1}^{n} \left\langle \left( q_s \left( X^{h_{a,j}} \right) J_p \left( s \right) \right)' - \left( q_s \left( X^{J_p} h_{a,j} \left( s \right) \right) \right)' \right\rangle_{s_{i-1} + \Delta_i} \tag{4.22}
\end{equation}

Here \( \prime \) is the derivative with respect to (time) \( s \).

Since \( h'_{a,j} \left( s_{i-1} + \right) \) is independent of \( \sigma \), so
\( \left( X^{J_p h_{a,j}} \right)' \left( s_{i-1} + \right) = X^{J_p} \left( \sigma \rightarrow h'_{a,j} \left( \sigma, s_{i-1} + \right) \right) = 0, \)
and thus
\begin{equation}
\sum_{i=1}^{n} \left\langle \left( X^{J_p h_{a,j}} \right)' , h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} = 0. \tag{4.23}
\end{equation}

We now claim that
\begin{equation}
\left( q_s \left( X^{h_{a,j}} \right) J_p \left( s \right) \right)' = q'_s \left( X^{h_{a,j}} \right) J_p \left( s \right) + q_s \left( X^{h_{a,j}} \right) J'_p \left( s \right) = 0 \text{ for } s \in \mathcal{P}. \tag{4.24}
\end{equation}

Since
\( h'_{a,j} \left( s_{i-1} + \right) \neq 0 \text{ iff } i = j, \)
and when \( i = j, \)
\( h_{a,j} \left( s \right) = 0 \text{ for } s \leq s_{i-1}, \)
so both \( q'_{s_{i-1}} \left( X^{h_{a,j}} \right) = 0 \) and \( q_{s_{i-1}} \left( X^{h_{a,j}} \right) = 0. \) It then follows that the claim is true and
\begin{equation}
\sum_{i=1}^{n} \left\langle q_s \left( X^{h_{a,j}} \right) J_p \left( s \right) , h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} = 0, \tag{4.25}
\end{equation}

Lastly because \( q_s \left( X^{J_p} \right) \) is skew-symmetric,
\begin{equation}
\sum_{i=1}^{n} \left\langle q_s \left( X^{J_p} \right) h'_{a,j} , h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} = 0. \tag{4.26}
\end{equation}

Combining Eq. (4.22), (4.23), (4.24) and (4.25) shows,
\begin{equation}
\left\langle \left[ X^{h_{a,j}}, \tilde{X}_P \right], X^{h_{a,j}} \right\rangle_{G^p} = \sum_{i=1}^{n} \left\langle X^{h_{a,j}, J_p}, h'_{a,j} \right\rangle_{s_{i-1} + \Delta_i} = \left\langle X^{h_{a,j}, J_p} \left( s_{j-1} + \right) \right\rangle_{\pi \left( X \right)} \sqrt{\Delta_j}. \tag{4.26}
\end{equation}

Summing Eq. (4.26) on \( a' \) and \( j \) while making use of (4.19) gives (4.10). \( \blacksquare \)

4.2 The Orthogonal Lift \( \tilde{X} \) on \( W_0 \left( M \right) \)

Definition 4.13 (Cameron-Martin vector field) A Cameron-Martin process, \( h, \) is an \( \mathbb{R}^d \)-valued process on \( W_0 \left( M \right) \) such that \( s \rightarrow h(s) \) is in \( H \left( \mathbb{R}^d \right) \) \( \nu \)-a.s. and a \( TM \)-valued process \( X^h \) on \( \left( W_0 \left( M \right), \nu \right) \) is called a Cameron-Martin vector field (denote this space by \( X \)) if \( \nu(X_s) = \Sigma_s \nu \text{ -a.s.}, h(s) := u_s^{-1} X_s^{h} \text{ is a Cameron-Martin process and} \)
\begin{equation}
\left\langle X^h, X^h \right\rangle_X := \mathbb{E} \left[ \left\| h \right\|^2_{H \left( \mathbb{R}^d \right)} \right] < \infty. \tag{4.27}
\end{equation}
Cameron-Martin vector field is the key concept in path space analysis. In this section we are going to introduce a non-adapted Cameron-Martin vector field (see Definition 4.21) which “lift” a vector field on a manifold \( M \) to a “vector field” on the corresponding path space \( W_o(M) \).

**Definition 4.14** Define \( \hat{T}:(0,1] \times W_o(M) \rightarrow \text{End}(\mathbb{R}^d) \) to be the solution to the following initial value problem:

\[
\begin{align*}
\frac{d}{ds} \hat{T}_s + \frac{1}{2} \text{Ric}_{\hat{\alpha}_s} \hat{T}_s &= 0 \\
\hat{T}_0 &= I.
\end{align*}
\]

**Definition 4.15** Using \( \hat{T}_s \), we define \( \tilde{K}: [0,1] \times W_o(M) \rightarrow \text{End}(\mathbb{R}^d) \):

\[
\tilde{K}_s := \hat{T}_s \left[ \int_0^s \hat{T}_{r}^{-1} \left( \hat{T}_{r}^{-1} \right)^* dr \right] \hat{T}_1^*.
\]

**Remark 4.16** Both \( \hat{T} \) and \( \tilde{K} \) are defined up to \( \nu \)-equivalence. We can pick a version at first place in order to avoid stating \( \nu \)-a.s. in the following results.

**Lemma 4.17** For all \( s \in [0,1] \), \( \hat{T}_s \) is invertible. Further both \( \sup_{0 \leq s \leq 1} \| \hat{T}_s \| \) and \( \sup_{0 \leq s \leq 1} \| \hat{T}_s^{-1} \| \) are bounded by \( e^{\frac{1}{2}(d-1)N} \), where \( (d-1)N \) is a bound of \( \| \text{Ric} \| \).

**Proof.** Denote by \( U_s \in \text{End}(\mathbb{R}^d) \) the solution to the following initial value problem:

\[
\frac{d}{ds} U_s = -\frac{1}{2} U_s \text{Ric}_{\hat{\alpha}_s}, \quad U_0 = I,
\]

then direct computation shows that \( Y_s := \hat{T}_s U_s \in \text{End}(\mathbb{R}^d) \) satisfies

\[
\frac{d}{ds} Y_s = \frac{1}{2} \left( \text{Ric}_{\hat{\alpha}_s} Y_s - Y_s \text{Ric}_{\hat{\alpha}_s} \right), \quad Y_0 = I.
\]

By the uniqueness of solutions for linear ODE, we get \( Y_s = I \), and this shows that \( U_s \) is a left inverse to \( \hat{T}_s \). As we are in finite dimensions it follows that \( \hat{T}_s^{-1} \) exists and is equal to \( U_s \). The stated bounds now follow by Gronwall’s inequality. ■

**Lemma 4.18** \( \tilde{K}_1 \) is invertible and \( \| \tilde{K}_1^{-1} \| \leq e^{(d-1)N} \), provided \( \| \text{Ric} \| \leq (d-1)N \).

**Proof.** Since

\[
\tilde{K}_1 := \int_0^1 \left( \hat{T}_1 \hat{T}_r^{-1} \right) \left( \hat{T}_1 \hat{T}_r^{-1} \right)^* dr
\]

is a symmetric positive semi-definite operator such that

\[
\left\langle \tilde{K}_1 v, v \right\rangle = \int_0^1 \left\| \left( \hat{T}_1 \hat{T}_r^{-1} \right)^* v \right\|^2 dr \quad \forall v \in \mathbb{C}^d.
\]

Apply Lemma 4.17 to the expression given;

\[
\left\langle \tilde{K}_1 v, v \right\rangle \geq \int_0^1 e^{-(d-1)N} \left\| \left( \hat{T}_r^{-1} \right)^* v \right\|^2 dr \geq \int_0^1 e^{-(d-1)N} \| v \|^2 dr = e^{-(d-1)N} \| v \|^2
\]

from which it follows that \( \text{eig}(\tilde{K}_1) \subset [e^{-(d-1)N}, \infty) \) and \( \| \tilde{K}_1^{-1} \| = \frac{1}{\min \{ \lambda : \lambda \in \text{eig}(\tilde{K}_1) \}} \leq e^{(d-1)N} \). ■

**Definition 4.19** For each \( X \in \Gamma \left( \hat{T}M \right) \) define two \( \nu \)-equivalent maps \( \hat{H}: W_o(M) \rightarrow \mathbb{R}^d \) and \( \hat{J}: W_o(M) \rightarrow H(\mathbb{R}^d) \) by

\[
\hat{H} = \tilde{u}_1^{-1} X \circ E_1
\]

and

\[
\hat{J}_s := \tilde{K}_s \tilde{K}_1^{-1} \hat{H} \text{ for } s \in [0,1].
\]
Notation 4.20 Given a measurable function \( h : W_o(M) \to H(\mathbb{R}^d) \), let \( Z_h : W_o(M) \to H(\mathbb{R}^d) \) be the solution to the following initial value problem:

\[
\begin{align*}
Z_h'(s) &= -\frac{1}{2}Ric_u Z_h(s) + h'(s) \\
Z_h(0) &= 0.
\end{align*}
\]

Definition 4.21 (Orthogonal Lift on \( W_o(M) \)) For any \( X \in \Gamma(TM) \), define \( \tilde{X} \in \mathcal{X} \) as follows.

\[
\tilde{X}_s = X^{Z_{\Phi}}_s := \tilde{u}_s Z_{\Phi}(s) \quad \text{for} \quad 0 \leq s \leq 1
\]

where

\[
\Phi_s = \int_0^s \left( \tilde{T}_{_s}^{-1} \right)^* \left[ \int_0^1 \left( \tilde{T}_t \tilde{T}_{_s} \right)^{-1} \tilde{T}^{-1}_s \tilde{H} \right] dt.
\]

Given \( f \in \mathcal{F}^1 \), define the gradient operator \( Df \) as follows,

\[
D_s f := \tilde{u}_s \sum_{i=1}^n (s \wedge s_i) \tilde{u}_s^{-1} \text{grad}_i F
\]  \hspace{1cm} (4.31)

where \( F(\Sigma_s, \cdots, \Sigma_{s_i}) \) is a representation of \( f \) and \( \text{grad}_i F \) is the differential of \( F \) with respect to the \( i \)th variable.

Then we define \( \tilde{X} f := \langle Df, \tilde{X} \rangle_{G^t} \).

Since \( \mathcal{F}^1 \) is dense in \( L^2(W_o(M), \nu) \), \( \tilde{X} \) can be viewed as a densely defined operator on \( L^2(W_o(M), \nu) \) which admits an integration by parts formula as below.

Theorem 4.22 For any \( f, g \in \mathcal{F}^1 \), we have

\[
\mathbb{E}_\nu \left[ \tilde{X} f \cdot g \right] = \mathbb{E}_\nu \left[ f \cdot \tilde{X}^{tr,\nu} g \right],
\]  \hspace{1cm} (4.32)

where

\[
\tilde{X}^{tr,\nu} = -\tilde{X} + \sum_{a=1}^d \left( \tilde{C} H, e_a \right) \int_0^1 \left( \tilde{T}_s^{-1} \right)^* e_a d\beta_s + \sum_{a=1}^d \left( -X^{Z_{\Phi}} e_a \right) \tilde{C} H, e_a \right)
\]

and

\[
\tilde{C} = \left[ \int_0^1 \left( \tilde{T}_r \tilde{T}_{_s} \right)^{-1} \tilde{T}_s^{-1} \tilde{T}^{-1}_s \right] dr.
\]

Proof. See Lemma 4.23 of IVP. \( \blacksquare \)

Remark 4.23 The orthogonal lift \( \tilde{X} \) on \( W_o(M) \) can be viewed as a stochastic extension of the orthogonal lift in the sense of Theorem 4.4 where the path space is the curved Cameron-Martin space \( H(M) \) and the Riemannian metric is a damped metric related to Ricci curvature. Interested readers may refer to IVP for more details in this topic.

5 Convergence Result

In this section \( M \) is a complete Riemannian manifold with non-positive and bounded sectional curvature. Other conditions will be mentioned specifically in theorems if needed. First we modify and abuse a few notations we have defined before in order to avoid messy arguments.

Notation 5.1 Recall that \( \beta : W_o(M) \to W_0(\mathbb{R}^d) \) is the Brownian motion on \( \mathbb{R}^d \) defined in Definition 2.18. We have also defined \( \beta_P : W_o(M) \to H_P(\mathbb{R}^d) \) to be the linear approximation to Brownian motion on \( \mathbb{R}^d \) as in Notation 2.23. Now denote by \( \beta_P := \eta \circ \beta_P \) the development map of \( \beta_P \). Notice that \( \phi \circ \beta_P \in H_P(M) \) a.s. here \( \phi \) is the rolling map onto \( H(M) \). So after identifying \( C_{\beta_P}, S_{\beta_P} \) and hence \( \phi_{\beta_P} \) with \( C_{\beta_P}, \phi \circ \beta_P, S_{\beta_P} \circ \phi \circ \beta_P \) and \( \phi_{\beta_P} \circ \phi \circ \beta_P \), we can view them as maps from \( W_o(M) \) to \( \text{End}(\mathbb{R}^d) \). The point here is to make the notations short and it should not cause confusions after this explanation.
Lemma 5.6
Recall from the beginning of this section that

Lemma 5.4
For all

Remark 5.5
Convention 5.2
We use

5.1 Convergence of $\tilde{X}_P$ to $\tilde{X}$

5.1.1 Some Useful Estimates for $\{C_P,i\}_{i=1}^n$ and $\{S_P,i\}_{i=1}^n$
We apply Proposition A.5 to get some commonly used estimates listed as Lemmas 5.3.

Lemma 5.3
For any $i \in \{1, \ldots, n\}$ and $s \in [s_{i-1}, s_i]$, we have

\[ |C_P,i(s)| \leq \cosh \left( \sqrt{N} |\Delta_i \beta| \right) \leq e^{\frac{N}{2} |\Delta_i \beta|^2}, \tag{5.1} \]

\[ |S_P,i(s)| \leq \sqrt{N} |\Delta_i \beta| e^{\frac{N}{2} |\Delta_i \beta|^2}, \tag{5.2} \]

\[ |S_P,i - \Delta_i I| \leq \frac{N |\Delta_i \beta|^2 \Delta_i}{6} e^{\frac{N}{2} |\Delta_i \beta|^2}, \tag{5.3} \]

\[ |C_P,i - I| \leq \frac{N |\Delta_i \beta|^2}{2} e^{\frac{N}{2} |\Delta_i \beta|^2}. \tag{5.4} \]

Lemma 5.4
For all $\gamma \in \left( 0, \frac{1}{4} \right)$, define $K_\gamma := \sup_{s,t \in [0,1], s \neq t} \left\{ \frac{|\beta_r - \beta_s|}{|t-s|^\gamma} \right\}$, then there exists an $\epsilon_\gamma > 0$ such that $\mathbb{E} \left[ e^{\epsilon K_\gamma^2} \right] < \infty$.

Proof. See Fernique’s Theorem (Theorem 3.2) in [17].

Remark 5.5
From Lemma 5.4, it is easy to see any polynomial of $\epsilon K_\gamma$ has finite moments of all orders.

5.1.2 Size Estimates of $f_P,i(s)$
Recall from Definition 2.32 that $f_P,i : W_0(M) \times [0,1] \to \text{End}(\mathbb{R}^d)$ $0 \leq i \leq n$ is given by

\[ f_P,i(s) = \begin{cases} 0 & s \in [0, s_{i-1}] \\ \frac{S_P,i(s)}{C_P,i(s) C_{P,i-1} \cdots C_{P,i+1} S_{P,i}} & s \in [s_{i-1}, s_i] \\ \Delta_i & s \in [s_{j-1}, s_j] \text{ for } j = i+1, \ldots, n \end{cases} \]

with the convention that $S_{P,0} \equiv |P| I$ and $f_{P,0} \equiv I$.

Using the estimates in Subsection 5.1.1 it is easy to get an estimate of $f_P,i(s)$.

Lemma 5.6
Recall from the beginning of this section that $n := \frac{1}{|P|}$ and $N$ is the sectional curvature bound. For each $q \geq 1$, we have

\[ \sup_{n \geq 2qN} \mathbb{E} \left[ \sup_{i,j \in \{0, \ldots, n\}} |f_P,i(s_j)|^q \right] < \infty. \tag{5.5} \]

Proof. For all $i, j \in \{0, \ldots, n\}$, if $j < i$, $f_P,i(s_j) \equiv 0$. So we only need to consider the case when $j \geq i$. Since

\[ f_P,i(s_j) = \frac{C_{P,j} C_{P,j-1} \cdots C_{P,i+1} S_{P,i}}{\Delta_i}, \]

so

\[ |f_P,i(s_j)|^q \leq |C_{P,j}|^q |C_{P,j-1}|^q \cdots |C_{P,i+1}|^q \left| \frac{S_{P,i}}{\Delta_i} \right|^q. \]
Applying Eq. (5.1) and (5.3) to find
\[ |f_{P,i}(s_j)|^q \leq e^{\frac{q}{2}N \sum_{k=1}^n |\Delta_k \beta|^2} \left( e^{-\frac{q}{2} |\Delta_i \beta|^2} + \frac{N |\Delta_i \beta|^2}{6} \right)^q \]
\[ \leq e^{\frac{q}{2}N \sum_{k=1}^n |\Delta_k \beta|^2} \left( 1 + \frac{N |\Delta_i \beta|^2}{6} \right)^q \]
\[ \leq e^{\frac{q}{2}N \sum_{k=1}^n |\Delta_k \beta|^2} \left( e^{\frac{N |\Delta_i \beta|^2}{6}} \right) \]
\[ \leq e^{qN \sum_{k=1}^n |\Delta_k \beta|^2}. \]  

Since \( e^{qN \sum_{k=1}^n |\Delta_k \beta|^2} \) is independent of \( i \) and \( j \), we have
\[ \sup_{i,j \in \{1, \ldots, n\}} |f_{P,i}(s_j)|^q \leq e^{qN \sum_{k=1}^n |\Delta_k \beta|^2}. \tag{5.6} \]

Then Eq. (5.5) follows from Lemma A.2 in Appendix A. ■

**Notation 5.7** Given \( n \in \mathbb{N} \) and \( s \in [0,1] \), let \( \bar{s} = s_{k-1} \) when \( s \in (s_{k-1}, s_k) \), \(|P| = \frac{1}{n}\) is the mesh size of the partition \( P \) and also let
\[ A_{P,k}(s) := R_{uv}(s) (\beta_P(s_{k-1}), \cdot, \beta_P(s_{k-1})). \]

**Lemma 5.8** For each \( q \geq 1 \), \( \gamma \in (0, \frac{1}{2}) \) there exists a constant \( C \) such that for all \( n > 5qN \),
\[ \mathbb{E} \left[ \sup_{i \in \{0, \ldots, n\}, s \in [0,1]} |f_{P,i}(s) - f_{P,i}(\bar{s})|^q \right] \leq C |P|^{2q\gamma}. \tag{5.7} \]

**Proof.** For \( s \in (s_{k-1}, s_k) \), Taylor’s expansion gives
\[ f_{P,i}(s) - f_{P,i}(\bar{s}) = \int_{\bar{s}}^s A_{P,k}(r) f_{P,i}(r) (s-r) \, dr \]
\[ = \int_{\bar{s}}^s A_{P,k}(r) (f_{P,i}(r) - f_{P,i}(\bar{s})) (s-r) \, dr + \int_{\bar{s}}^s A_{P,k}(r) f_{P,i}(\bar{s}) (s-r) \, dr. \]

Since \( |A_{P,k}(s)| \leq N |\Delta_k \beta|^2 \), we have
\[ |f_{P,i}(s) - f_{P,i}(\bar{s})| \leq \frac{N}{\Delta_k} |\Delta_k \beta|^2 \int_{\bar{s}}^s |f_{P,i}(r) - f_{P,i}(\bar{s})| \, dr + \frac{1}{2} N |\Delta_k \beta|^2 \sup_{j:j \geq i} |f_{P,i}(s_j)|. \]

By Gronwall’s inequality, we have:
\[ |f_{P,i}(s) - f_{P,i}(\bar{s})| \leq \frac{1}{2} N |\Delta_k \beta|^2 \sup_{j:j \geq i} |f_{P,i}(s_j)| e^{N|\Delta_k \beta|^2} \]

Using estimate (5.6) gives
\[ |f_{P,i}(s) - f_{P,i}(\bar{s})|^q \leq \frac{N^q}{2^q} |\Delta_k \beta|^{2q} e^{N|\Delta_k \beta|^2} e^{qN \sum_{k=1}^n |\Delta_k \beta|^2} \]
\[ \leq C |P|^{2q\gamma} e^{2qN \sum_{k=1}^n |\Delta_k \beta|^2} K^{2q}. \tag{5.8} \]

Then Eq. (5.7) follows from Lemma A.2, 5.4 and Holder’s inequality. ■

**Theorem 5.9** Let \( \tilde{T}(\cdot) \) be as in Definition 4.14 then for each \( q \geq 1 \), \( \gamma \in (0, \frac{1}{2}) \), there exists a constant \( C \) such that for all \( n > 5q\gamma \),
\[ \mathbb{E} \left[ \sup_{i \in \{0, \ldots, n\}, s \in [s_i,1]} \left| f_{P,i}(s) - \tilde{T}_{s_i} \tilde{T}_{s_i}^{-1} \right|^q \right] \leq C |P|^q. \]
In order to prove Theorem 5.9 we need the following result.

**Lemma 5.10** For each \( q \geq 1, \gamma \in (0, \frac{1}{q}) \), there exists a constant \( C \) such that for all \( n > 5q\gamma \),

\[
E \left[ \sup_{i \in \{1, \ldots, n\}} \sup_{j \geq i} \left| f_{P,i}(s_j) - \left( f_{P,i}(s_i) - \int_{s_i}^{s_j} Ric_{u_P(q)} f_{P,i}(\mathbf{L}) \, dr \right) \right|^q \right] \leq C |P|^q. \tag{5.9}
\]

**Proof.** For each \( s_j \in P \) with \( j \geq i+1 \) and for \( k = i, \ldots, j-1 \), we have

\[
f_{P,i}(s_{k+1}) = f_{P,i}(s_k) + \int_{s_k}^{s_{k+1}} A_{P,k+1}(r) f_{P,i}(r) (s_{k+1} - r) \, dr \tag{5.10}
\]

where

\[
e_{i,k} = \int_{s_k}^{s_{k+1}} A_{P,k+1}(r) f_{P,i}(r) (s_{k+1} - r) \, dr - \int_{s_k}^{s_{k+1}} A_{P,k+1}(s_k) f_{P,i}(s_k) (s_{k+1} - r) \, dr.
\]

Since \( \{f_{P,i}(s_j)\}_{j} \) is adapted, by Itô’s lemma

\[
\frac{\Delta^2_{k+1}}{2} A_{P,k+1}(s_k) f_{P,i}(s_k) = \frac{1}{2} R_{u_P(s_k)}(\Delta_{k+1} \beta, f_{P,i}(s_k)) \Delta_{k+1} \beta
\]

\[
= \frac{1}{2} \int_{s_k}^{s_{k+1}} R_{u_P(s_k)}(\beta_r - \beta_{s_k}, f_{P,i}(s_k)) \, d\beta_r
\]

\[
+ \frac{1}{2} \int_{s_k}^{s_{k+1}} R_{u_P(s_k)}(d\beta_r, f_{P,i}(s_k)) (\beta_r - \beta_{s_k})
\]

\[
- \frac{1}{2} Ric_{u_P(s_k)} f_{P,i}(s_k) \Delta_k.
\]

Summing (5.10) over \( k \) from \( i \) to \( j-1 \), we have

\[
f_{P,i}(s_j) = f_{P,i}(s_i) - \frac{1}{2} \int_{s_i}^{s_j} Ric_{u_P(q)} f_{P,i}(\mathbf{L}) \, dr + M_{P,s_j} + \sum_{k=i}^{j-1} e_{i,k} \tag{5.11}
\]

where

\[
M_{P,s} := \frac{1}{2} \int_{s_i}^{s} R_{u_P(q)}(\beta_r - \beta_{s_i}, f_{P,i}(\mathbf{L})) \, d\beta_r + \frac{1}{2} \int_{s_i}^{s} R_{u_P(q)}(d\beta_r, f_{P,i}(\mathbf{L})) (\beta_r - \beta_{s_i})
\]

is a \( \mathbb{R}^d \)-valued martingale starting from \( s_i \). By Burkholder-Davis-Gundy inequality, for \( q \geq 1 \),

\[
E \left[ \sup_{s \in [s_i, s_j]} \left| M_{P,s} \right|^q \right] \leq CE \left[ \langle M_P \rangle^\frac{q}{2} \right]. \tag{5.12}
\]

where \( \langle M_P \rangle \) is the quadratic variation process of \( M_P \). An estimate of \( \langle M_P \rangle \) gives

\[
\langle M_P \rangle_1 \leq dN^2 \int_{s_i}^{s} \left| \beta_r - \beta_{s_i} \right|^2 |f_{P,i}(\mathbf{L})|^2 \, dr \leq dN^2 \int_{0}^{1} \left| \beta_r - \beta_{s_i} \right|^2 |f_{P,i}(\mathbf{L})|^2 \, dr,
\]

and by Jensen’s inequality,

\[
\langle M_P \rangle^\frac{q}{2} \leq d^\frac{q}{2} N^q \int_{0}^{1} \left| \beta_r - \beta_{s_i} \right|^q |f_{P,i}(\mathbf{L})|^q \, dr.
\]
Since \(\{f_{P,i}(r)\}_{r \in [0,1]}\) is adapted to the filtration generated by \(\beta\), using the independence of \(|\beta_r - \beta_u|^q\) and \(f_{P,i}(r)\) we have:

\[
\mathbb{E}\left[ (M_P)_{j}^{q} \right] \leq d^2 N^{q} \int_{0}^{1} \mathbb{E}\left[ |\beta_r - \beta_u|^q \right] \mathbb{E}\left[ |f_{P,i}(r)|^q \right] dr = C \sup_{j \in \{0, \ldots, n\}} \mathbb{E}\left[ |f_{P,i}(s_j)|^q \right] |P|^\frac{q}{2}.
\]

By Lemma 5.3, we know

\[
\mathbb{E}\left[ (M_P)_{j}^{q} \right] \leq C |P|^\frac{q}{2}.
\]

So to finish the proof of Lemma 5.10, it suffices to show:

\[
\mathbb{E}\left[ \sup_{i \in \{0, \ldots, n\}, j \in \{i+1, \ldots, n\}} \left| \sum_{k=i}^{j-1} e_{i,k} \right|^q \right] \leq C |P|^\gamma q.
\]

Since \(|e_{i,k}| \leq I_P(i, k) + II_P(i, k)\), where

\[
I_P(i, k) = \left| \int_{s_k}^{s_{k+1}} A_{P,k}(r) (f_{P,i}(r) - f_{P,i}(s_k)) (s_{k+1} - r) dr \right|
\]

and

\[
II_P(i, k) = \left| \int_{s_k}^{s_{k+1}} (A_{P,k}(r) - A_{P,k}(s_k)) f_{P,i}(s_k) (s_{k+1} - r) dr \right|
\]

using 5.8, we know

\[
I_P(i, k) \leq \frac{N}{2} \sup_{i \in \{1, \ldots, n\}, r \in [0,1]} |f_{P,i}(r) - f_{P,i}(r)| |\Delta_{k+1} \beta|^2 \leq CK_3^4 |P|^4 \gamma e^{2N \sum_{k=1}^{n} |\Delta_{k} \beta|^2}.
\]

Since

\[
|R_{u,P}(s_k) - R_{u,P}(r)| \leq C \int_{s_k}^{s_{k+1}} |\beta_r^\gamma (s)| ds = C |\Delta_{k+1} \beta| \leq CK_3 \gamma |P|^\gamma,
\]

using 3.6, we have

\[
II_P(i, k) \leq C \sup_{i, j \in \{1, \ldots, n\}, r \in [0,1]} |f_{P,i}(s_j)| |\Delta_{k+1} \beta|^2 \sup_{r \in [s_k, s_{k+1}]} |R_{u,P}(s_k) - R_{u,P}(r)| \leq CK_3 \gamma |P|^3 \gamma e^{2N \sum_{k=1}^{n} |\Delta_{k} \beta|^2}.
\]

So

\[
\sum_{k=i}^{j-1} e_{i,k} \leq \frac{1}{|P|} (I_P(i, k) + II_P(i, k)) \leq C \left( K_3^4 |P|^4 \gamma - 1 + K_3^3 |P|^3 \gamma - 1 \right) e^{2N \sum_{k=1}^{n} |\Delta_{k} \beta|^2}.
\]

For any \(\gamma' \in (0, \frac{1}{2})\), we can choose \(\lambda \in \left(\frac{1}{3}, \frac{1}{2}\right)\) such that \(\gamma' = 3\gamma - 1\) and thus using Lemma 5.4, we get

\[
\mathbb{E}\left[ \sup_{i \in \{0, \ldots, n\}, j \in \{i+1, \ldots, n\}} \left| \sum_{k=i}^{j-1} e_{i,k} \right|^q \right] \leq C |P|^{\gamma' q}.
\]

Combining Eq. 5.11, 5.13 and 5.14, we obtain 5.9. 

**Proof of Theorem 5.9** For \(s \geq s_i\), define

\[
\hat{f}_{P,i}(s) := f_{P,i}(s) - \frac{1}{2} \int_{s_i}^{s} Ric_{u,P}(r) f_{P,i}(r) dr.
\]

Then

\[
|\hat{f}_{P,i}(s_j) - f_{P,i}(s_j)| \leq \left| \frac{1}{2} \int_{s_i}^{s_j} (Ric_{u,P}(r) - Ric_{u,P}(s_i)) f_{P,i}(r) dr \right| + \left| \frac{1}{2} \int_{s_i}^{s_j} Ric_{u,P}(r) (f_{P,i}(r) - f_{P,i}(s_i)) dr \right| + |M_{P,s_j}| + \sum_{k=i}^{j-1} e_{i,k}.
\]

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By Gronwall’s inequality we have
\[ |Ric_{u,p}(r) - Ric_{u,p}(\xi)| = |\nabla_{(r-\xi)}\beta_p(\xi)Ric| \leq C \sup_i |\Delta_i\beta| \leq CK_\gamma |\mathcal{P}|, \]
using Lemma 5.6 and Holder’s inequality we know
\[ E \left[ \int_{s_i}^{s_j} (Ric_{u,p}(r) - Ric_{u,p}(\xi)) f_{p,i}(\xi) \, dr \right]^q \leq C |\mathcal{P}|^q. \tag{5.16} \]
Then we consider
\[ \int_{s_i}^{s_j} Ric_{u,p}(r) (f_{p,i}(r) - f_{p,i}(\xi)) \, dr. \]
By Lemma 5.8 one can easily see
\[ E \left[ \sup_{i \in \{0, \ldots, n\}} \int_{s_i}^{s_j} Ric_{u,p}(r) (f_{p,i}(r) - f_{p,i}(\xi)) \, dr \right]^q \leq C |\mathcal{P}|^{2\gamma}. \tag{5.17} \]
Combining Eq. (5.16) and (5.17) and Lemma 5.10 we get
\[ E \left[ \sup_{i \in \{0, \ldots, n\}, j \geq i} \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right|^q \right] \leq C |\mathcal{P}|^{\gamma}. \tag{5.18} \]
Then using Lemma 5.3 and notice that
\[ \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right| \leq C (s - \xi) \sup_{0 \leq s \leq 1} \left| f_{p,i}(s) \right|, \]
we have
\[ E \left[ \sup_{i \in \{0, \ldots, n\}, s \geq s_i} \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right|^q \right] \leq C |\mathcal{P}|^{\gamma}. \tag{5.19} \]
Then for \( s \geq s_i \), define \( \hat{f}_{p,i}(s) \) to be the solution to the following ODE
\[ \begin{aligned}
\frac{d}{ds} \hat{f}_{p,i}(s) + \frac{1}{2} Ric_{u,p}(s) \hat{f}_{p,i}(s) &= 0, \\
\hat{f}_{p,i}(s_i) &= I.
\end{aligned} \]
Therefore
\[ \hat{f}_{p,i}(s) = I - \frac{1}{2} \int_{s_i}^{s} Ric_{u,p}(r) \hat{f}_{p,i}(r) \, dr \]
and
\[ \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right| \leq |f_{p,i}(s_i) - I| + C \int_{s_i}^{s} \left| \hat{f}_{p,i}(r) - f_{p,i}(r) \right| \, dr + C \sup_{s \geq s_i} \left| f_{p,i}(s) - \hat{f}_{p,i}(s) \right|. \]
By Gronwall’s inequality we have
\[ \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right| \leq \left( |f_{p,i}(s_i) - I| + C \sup_{s \geq s_i} \left| f_{p,i}(s) - \hat{f}_{p,i}(s) \right| \right) e^{\frac{1}{2}}. \]
Thus by Lemma 5.3 and Eq. (5.18) it follows that
\[ E \left[ \sup_{i \in \{0, \ldots, n\}, s \geq s_i} \left| \hat{f}_{p,i}(s) - f_{p,i}(s) \right|^q \right] \leq C |\mathcal{P}|^{\gamma}. \tag{5.19} \]
Lastly, we look at \( \hat{T}_{s_i} \hat{T}_{s_i}^{-1} \) where \( s \geq s_i \). Note that \( \hat{T}_{s_i} \hat{T}_{s_i}^{-1} \) satisfies the following ODE,
\[ \begin{aligned}
\frac{\left( \hat{T}_{s_i} \hat{T}_{s_i}^{-1} \right)'}{2} + \frac{1}{2} Ric_{\alpha} \left( \hat{T}_{s_i} \hat{T}_{s_i}^{-1} \right) &= 0, \\
\left( \hat{T}_{s_i} \hat{T}_{s_i}^{-1} \right) &= I.
\end{aligned} \]
Proof. For all \( q \), so for all \( n > 5 \).

The proof is completed by combining Lemma 5.10 and (5.17), (5.18), (5.19), and (5.20).

5.1.3 Convergence of \( K_P (s) \) to \( \hat{K} \)

Recall from Definition 4.4 that \( K_P (s) \) satisfies the piecewise Jacobi equation:

\[
\begin{align*}
K_P^{(0)} (s) &= A_{P,i} (s) K_P (s) \text{ for } s \in [s_{i-1}, s_i] \\
K_P^{(i-1)} (s) &= f_{P,i}^{(i-1)} (1) \text{ and } K_P (0) = 0, \text{ for } i = 1, \ldots, n
\end{align*}
\]

where \( f_{P,i}^{(1)} \) is given in Definition 2.32.

Before we state the main theorem in this section, we need some supplementary lemmas.

Lemma 5.11 Recall that \( n := |P| \) and \( N \) is the curvature bound. For each \( q \geq 1 \),

\[
\sup_{n > 2q N} \mathbb{E} \left[ \sup_{s \in \{0, \ldots, n\}} |K_P (s)|^q \right] < \infty.
\]

Proof. For all \( i \in \{1, \ldots, n\} \), recall from (4.7) that

\[
K_P (s_i) = \frac{1}{n} \sum_{j=0}^{i-1} f_{P,j+1} (s) f_{P,j+1}^{*} (1).
\]

So for all \( q \geq 1 \), we have

\[
|K_P (s_i)|^q \leq i^{q-1} \frac{1}{n^q} \sum_{j=0}^{i-1} |f_{P,j+1} (s_i)|^q |f_{P,j+1}^{*} (1)|^q.
\]

Using (5.6) we have

\[
|K_P (s_i)|^q \leq e^{2q N} \sum_{k=1}^{N} |\Delta_k|^2.
\]

Then taking expectations as in Lemma 5.6 gives (5.22).

Lemma 5.12 For each \( q \geq 1 \) and \( \gamma \in \left(0, \frac{1}{2}\right) \), there exists a constant \( C > 0 \) such that for all \( n > 5q N \),

\[
\mathbb{E} \left[ \sup_{i \in \{1, \ldots, n\}, r \in [0,1]} |K_P (s) - K_P (r, \tilde{r})|^q \right] \leq C |P|^{2q \gamma}
\]

Proof. For \( s \in [s_{i-1}, s_i] \),

\[
K_P (s) = K_P (s_{i-1}) + f_{P,i}^{*} (1) (s - s_{i-1}) + \int_{s_{i-1}}^s A_{P,i} (r) K_P (r) (s - r) \, dr.
\]
Therefore
\[
|K_P(s) - K_P(s_{i-1})| \\
\leq |f_{P,i}(1)|(s - s_{i-1}) + \int_{s_{i-1}}^{s} A_{P,i}(r) (K_P(r) - K_P(s_{i-1}) + K_P(s_{i-1})) (s - r) \, dr \\
\leq |f_{P,i}(1)|(s - s_{i-1}) + N \frac{|\Delta_i \beta|^2}{\Delta_i^2} \int_{s_{i-1}}^{s} |K_P(r) - K_P(s_{i-1})| (s - r) \, dr + \frac{1}{2} N |\Delta_i \beta|^2 |K_P(s_{i-1})|. 
\]
(5.24)

We use the shorthand
\[
f(s) := |f_{P,i}(1)|(s - s_{i-1}) + N \frac{|\Delta_i \beta|^2}{\Delta_i^2} \int_{s_{i-1}}^{s} |K_P(r) - K_P(s_{i-1})| (s - r) \, dr \\
+ \frac{1}{2} N |\Delta_i \beta|^2 |K_P(s_{i-1})|. 
\]

Then it is easily seen that
\[
f'(s) = |f_{P,i}(1)| + N \frac{|\Delta_i \beta|^2}{\Delta_i^2} \int_{s_{i-1}}^{s} |K_P(r) - K_P(s_{i-1})| \, dr, \\
f''(s) = N \frac{|\Delta_i \beta|^2}{\Delta_i^2} |K_P(s) - K_P(s_{i-1})| \leq N \frac{|\Delta_i \beta|^2}{\Delta_i^2} f(s), 
\]
and \( f(s) \) satisfies the following ODE
\[
\begin{cases}
  f''(s) = N \frac{|\Delta_i \beta|^2}{\Delta_i^2} f(s) + \delta(s) \\
  f'(s_{i-1}) = |f_{P,i}(1)| \\
  f(s_{i-1}) = \frac{1}{2} N |\Delta_i \beta|^2 |K_P(s_{i-1})|.
\end{cases}
\]
(5.25)

where
\[
\delta(s) = f''(s) - N \frac{|\Delta_i \beta|^2}{\Delta_i^2} f(s) \leq 0.
\]

This ODE can be solved exactly to obtain
\[
f(s) = C_{s_{i-1}}(s) \frac{1}{2} N |\Delta_i \beta|^2 |K_P(s_{i-1})| + S_{s_{i-1}}(s) |f_{P,i}(1)| + \int_{s_{i-1}}^{s} C_r(s) \delta(r) \, dr 
\]
where
\[
C_r(s) := \cosh \left( \sqrt{N} |\beta_P'(s_{i-1}+)| (s - r) \right) 
\]
and
\[
S_r(s) := \frac{\sinh \left( \sqrt{N} |\beta_P'(s_{i-1}+)| (s - r) \right)}{\sqrt{N} |\beta_P'(s_{i-1}+)|}. 
\]

Since \( \delta(r) \leq 0 \) and \( C_r(s) \geq 0 \), we have
\[
f(s) \leq C_{s_{i-1}}(s) \frac{1}{2} N |\Delta_i \beta|^2 |K_P(s_{i-1})| + S_{s_{i-1}}(s) |f_{P,i}(1)|. 
\]

Then using the following estimate
\[
\frac{S_{s_{i-1}}(s)}{\Delta_i} \leq C_{s_{i-1}}(s) \frac{s - s_{i-1}}{\Delta_i} \leq e^{N |\Delta_i \beta|^2}, 
\]
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we obtain

\[
 f(s) \leq e^{N|\Delta_i|_\beta^2} \left( \frac{1}{2} N |\Delta_i|_\beta^2 |K_P(s_{i-1})| + |\mathcal{P}| |f_{P,i}(1)| \right)
\]

(5.26)

\[
 \leq e^{NK^2|\mathcal{P}|^2\gamma} \left( \frac{1}{2} NK^2 |\mathcal{P}|^{2\gamma} \sup_{i \in \{1, \ldots, n\}} |K_P(s_{i-1})| + |\mathcal{P}| \sup_{i \in \{1, \ldots, n\}} |f_{P,i}(1)| \right).
\]

Note that \( f \geq 0 \), using (5.6) and (5.23) we have for \( q \geq 1 \),

\[
 f^q(s) \leq U_q |\mathcal{P}|^{2q\gamma},
\]

where

\[
 U_q = e^{qNK^2|\mathcal{P}|^2\gamma} \left( \frac{1}{2} NK^2 + |\mathcal{P}|^{1-2\gamma} \right)^qe^{qN \sum_{i=1}^{n} |\Delta_i\beta|_2^2}
\]

is a random variable with finite first moment which can be bounded uniformly for \( n > 5qN \). Therefore,

\[
 E \left[ \sup_{i \in \{1, \ldots, n\}, r \in [0,1]} |K_P(r) - K_P(s)|^q \right] \leq C |\mathcal{P}|^{2q\gamma}
\]

(5.27)

**Proposition 5.13** Let \( K_P \) and \( \tilde{K} \) be defined as in Definition 4.4 and 4.15. Then for each \( q \geq 1 \) and \( \gamma \in (0, \frac{1}{2}) \), there exists a constant \( C > 0 \) such that for all \( n > 5qN \),

\[
 E \left[ \sup_{i \in \{0, \ldots, n\}} \left| K_P(s_i) - \tilde{K}_{s_i} \right|^q \right] \leq C |\mathcal{P}|^q.
\]

(5.28)

**Proof.** For all \( i \in \{1, \ldots, n\} \), \( K_P(s_i) \) and \( \tilde{K}_{s_i} \) can be rewritten as

\[
 K_P(s_i) = f_{P,i-1}(s_i) f_{P,i-1}(1)^{-1} \left( \sum_{j=0}^{i-1} f_{P,j+1}(1) f_{P,j+1}^*(1) \right) \Delta_{j+1}
\]

(5.29)

and

\[
 \tilde{K}_{s_i} = \tilde{T}_{s_i} \tilde{T}^{-1} \int_0^{s_i} \left( \tilde{T}_{s_i} \tilde{T}^{-1} \right) \left( \tilde{T}_{s_i} \tilde{T}^{-1} \right)^* \, dr.
\]

First define

\[
 \tilde{K}_P(s_i) := \tilde{T}_{s_i} \tilde{T}^{-1} \int_0^{s_i} \left( \tilde{T}_{s_i} \tilde{T}^{-1} \right) \left( \tilde{T}_{s_i} \tilde{T}^{-1} \right)^* \, dr,
\]

where \( \bar{s} = s_i \) if \( s \in [s_{i-1}, s_i] \). We will show, for each \( q \geq 1 \),

\[
 \sup_{i \in \{0, \ldots, n\}} \left| \tilde{K}_{s_i} - K_P(s_i) \right|^q \leq C |\mathcal{P}|^q.
\]

(5.30)

Recall from (4.2) that \( \tilde{T}_r \tilde{T}_r^{-1} \) satisfies the following ODE,

\[
 \frac{d}{dr} \left( \tilde{T}_r \tilde{T}_r^{-1} \right) = \frac{1}{2} \left( \tilde{T}_r \tilde{T}_r^{-1} \right) \text{Ric}_{\bar{s}}.
\]

So by Lemma 4.17

\[
 \left| \frac{d}{dr} \left( \tilde{T}_r \tilde{T}_r^{-1} \right) \right| \leq N \left| \tilde{T}_r \tilde{T}_r^{-1} \right| \leq N.
\]

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Therefore
\[
\left| \left( \tilde{T}_1 \tilde{T}_r^{-1} \right) \left( \tilde{T}_1 \tilde{T}_r^{-1} \right)^* - \left( \tilde{T}_1 \tilde{T}_r^{-1} \right) \left( \tilde{T}_1 \tilde{T}_r^{-1} \right)^* \right| \leq \int_r^T \left| \frac{d}{ds} \left[ \left( \tilde{T}_1 \tilde{T}_s^{-1} \right) \left( \tilde{T}_1 \tilde{T}_s^{-1} \right)^* \right] \right| ds
\]
\[
\leq 2 \int_r^T \left| \frac{d}{ds} \left( \tilde{T}_1 \tilde{T}_s^{-1} \right) \right| \left( \left( \tilde{T}_1 \tilde{T}_s^{-1} \right)^* \right| ds
\]
\[
\leq C \left( \tau - r \right) \leq C |\mathcal{P}|,
\]
and
\[
\left| K_{s_i} - K_{\mathcal{P}} (s_i) \right| \leq \left| T_{s_i} \tilde{T}_1^{-1} \right| \int_0^{s_i} \left| \left( \tilde{T}_1 \tilde{T}_r^{-1} \right) \left( \tilde{T}_1 \tilde{T}_r^{-1} \right)^* - \left( \tilde{T}_1 \tilde{T}_r^{-1} \right) \left( \tilde{T}_1 \tilde{T}_r^{-1} \right)^* \right| dr \leq C |\mathcal{P}|.
\]
Since the right-hand side is independent of \( i \), we proved (5.30). Secondly, define
\[
\dot{K}_{\mathcal{P}} (s_i) := \tilde{T}_s \tilde{T}_1^{-1} \left( \sum_{j=0}^{i-1} f_{\mathcal{P}, j+1} (1) f_{\mathcal{P}, j+1}^* (1) \right) \Delta_{j+1}.
\]
We will show, for each \( q \geq 1, \gamma \in (0, \frac{1}{2}) \), there exists a constant \( C > 0 \) such that for all \( n > 5qN \), we have
\[
E \left[ \sup_{s \in \mathcal{P}} \left| \dot{K}_{\mathcal{P}} (s) - \dot{K}_{\mathcal{P}} (s) \right|^q \right] \leq C |\mathcal{P}|^{q\gamma}.
\]
(5.31)
For each \( j \in \{1, \ldots, n\} \),
\[
\left| f_{\mathcal{P}, j+1} (1) f_{\mathcal{P}, j+1}^* (1) - \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* \right|
\]
\[
\leq \left| f_{\mathcal{P}, j+1} (1) f_{\mathcal{P}, j+1}^* (1) - f_{\mathcal{P}, j+1} (1) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* \right|
\]
\[
+ \left| f_{\mathcal{P}, j+1} (1) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* - \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* \right|
\]
\[
\leq \left( \left| f_{\mathcal{P}, j+1} (1) \right| + \left| \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right| \right) \left| f_{\mathcal{P}, j+1} (1) - \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right|.
\]
Since \( |f_{\mathcal{P}, j+1} (1)| \leq e^{N \sum_{k=1}^n |\Delta_k|^2} \) by (5.10), ans also \( \left| \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right| \leq 1 \), we have
\[
\left| f_{\mathcal{P}, j+1} (1) f_{\mathcal{P}, j+1}^* (1) - \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* \right|
\]
\[
\leq \left( e^{N \sum_{k=1}^n |\Delta_k|^2} + 1 \right) \sup_{j \in \{1, \ldots, n\} \setminus \{ j \}} \left| f_{\mathcal{P}, j+1} (1) - \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right|.
\]
Thus for all \( i \in \{1, \ldots, n\} \),
\[
\left| K_{\mathcal{P}} (s_i) - \dot{K}_{\mathcal{P}} (s_i) \right|^q \leq |\mathcal{P}|^{q-i} \sum_{j=0}^{i-1} \left| f_{\mathcal{P}, j+1} (1) f_{\mathcal{P}, j+1}^* (1) - \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right) \left( \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right)^* \right|^q
\]
\[
\leq \left( e^{N \sum_{k=1}^n |\Delta_k|^2} + 1 \right)^q \sup_{j \in \{1, \ldots, n\} \setminus \{ j \}} \left| f_{\mathcal{P}, j+1} (1) - \tilde{T}_1 \tilde{T}_{s_{j+1}}^{-1} \right|^q.
\]
Since \( \left( e^{N \sum_{k=1}^n |\Delta_k|^2} + 1 \right)^q \leq e^{N q \sum_{k=1}^n |\Delta_k|^2} \), using Holder’s inequality and Theorem 5.9 we get
\[
E \left[ \sup_{s \in \mathcal{P}} \left| K_{\mathcal{P}} (s) - \dot{K}_{\mathcal{P}} (s) \right|^q \right] \leq C |\mathcal{P}|^{q\gamma}.
\]
Lastly, we estimate $\hat{K}_P(s_i) - K_P(s_i)$. Using (5.29) we have

$$\left| \hat{K}_P(s_i) - K_P(s_i) \right| \leq \left| f_{P,i-1}(s_i) f_{P,i-1}^{-1}(1) - T_s T_1^{-1} \right| \left( \sum_{j=0}^{i-1} f_{P,j+1}(1) f_{P,j+1}^{-1}(1) \right) \Delta_{j+1} \leq \left| f_{P,i-1}(s_i) f_{P,i-1}^{-1}(1) - T_s T_1^{-1} \right| \sup_{j \in \{1, \ldots, n\}} \left| f_{P,j+1}(1) \right|^2.$$  

Since

$$\left| f_{P,i-1}(s_i) f_{P,i-1}^{-1}(1) - T_s T_1^{-1} \right| = \left| f_{P,i-1}(s_i) - T_s T_{s,i}^{-1} \right| \left| f_{P,i-1}^{-1}(1) \right| + \left| T_s T_{s,i}^{-1} \right| \left| (T_1 T_{s,i}^{-1})^{-1} - f_{P,i-1}^{-1}(1) \right|,$$

and from Proposition A.3 we know $\left| f_{P,i-1}(1)^{-1} \right| \leq 1$, and

$$\left| (T_1 T_{s,i}^{-1})^{-1} - f_{P,i-1}(1)^{-1} \right| \leq \left| T_1 T_{s,i}^{-1} - f_{P,i-1}(1) \right| \left| f_{P,i-1}(1)^{-1} \right| \leq \left| T_1 T_{s,i}^{-1} - f_{P,i-1}(1) \right|.$$

So

$$\left| f_{P,i-1}(s_i) f_{P,i-1}^{-1}(1)^{-1} - T_s T_1^{-1} \right| \leq 2 \sup_{1 \leq i,j \leq n} \left| T_s T_{s,i}^{-1} - f_{P,i}(s_j) \right|.$$  

Then using Theorem 5.9 Lemma 5.6 and Holder’s inequality we have

$$E \left[ \sup_{s \in \mathcal{P}} \left| \hat{K}_P(s) - K_P(s) \right|^{q} \right] \leq C |\mathcal{P}|^{q/q}$$  

(5.32)

Finally Lemma 5.13 is proved by combining (5.30), (5.31) and (5.32).  

**Lemma 5.14** For each $q \geq 1$, there exists a constant $C > 0$ such that

$$\sup_{s \in [0,1]} \left| \tilde{K}_s - \tilde{K}_s \right|^q \leq C |\mathcal{P}|^q$$

**Proof.** By the fundamental theorem of calculus, we have

$$\tilde{K}_s = -\frac{1}{2} \int_0^s \operatorname{Ric}_{\tilde{s}} \tilde{K}_r dr + \int_0^s \left( T_1 T_r^{-1} \right)^* dr.$$  

Using Lemma 1.17 note that $\operatorname{Ric}$ is bounded by $(d - 1) N$, we have

$$\left| \tilde{K}_s \right| \leq (d - 1) N \int_0^s \left| \tilde{K}_r \right| dr + C$$  

where $C$ and $(d - 1) N$ are two constants independent of $s$. Then using Gronwall’s inequality we get

$$\left| \tilde{K}_s \right| \leq Ce^{Ns} \leq Ce^N$$  

(5.33)

so $\sup_{s \in [0,1]} \left| \tilde{K}_s \right|$ is bounded. Then using the fundamental theorem of calculus again from $\tilde{s}$ to $s$ we have

$$\tilde{K}_s - \tilde{K}_s = -\frac{1}{2} \int_{\tilde{s}}^s \operatorname{Ric}_{\tilde{s}} \tilde{K}_r dr + \int_{\tilde{s}}^s \left( T_1 T_r^{-1} \right)^* dr$$

$$= -\frac{1}{2} \int_{\tilde{s}}^s \operatorname{Ric}_{\tilde{s}} \left( \tilde{K}_r - \tilde{K}_s \right) dr + \int_{\tilde{s}}^s \left( T_1 T_r^{-1} \right)^* dr + \frac{1}{2} \int_{\tilde{s}}^s \operatorname{Ric}_{\tilde{s}} \tilde{K}_r dr.$$
Therefore
\[ |\bar{K}_s - \bar{K}_\omega| \leq \frac{N}{2} \int_\omega^s \left| \tilde{K}_r - \bar{K}_\omega \right| \, dr + C |\mathcal{P}|. \]

By Gronwall’s inequality again we have
\[ |\bar{K}_s - \bar{K}_\omega| \leq C |\mathcal{P}| e^{\frac{N}{2}} \]
and thus
\[ \sup_{s \in [0,1]} |\bar{K}_s - \bar{K}_\omega|^q \leq C |\mathcal{P}|^q \]

The next theorem is a generalization to Proposition 5.13 in the sense that \( s \) now can be taken to be arbitrary between 0 and 1.

**Theorem 5.15** For each \( q \geq 1 \) and \( \gamma \in (0, \frac{1}{2}) \), there exists a constant \( C > 0 \) such that for all \( n > 5qN \),
\[ \mathbb{E} \left[ \sup_{s \in [0,1]} \left| \bar{K}_s - \mathbb{K}_\mathbb{P} (s) \right|^q \right] \leq C |\mathcal{P}|^q \gamma^q \]

**Proof.** For any \( s \in [0,1] \), \( s \in [s_{i-1}, s_i] \) for some \( i \in \{1, \ldots, n\} \). So
\[ \left| \mathbb{K}_\mathbb{P} (s) - \bar{K}_s \right| \leq \left| \mathbb{K}_\mathbb{P} (s) - \mathbb{K}_\mathbb{P} (1) \right| + \left| \mathbb{K}_\mathbb{P} (1) - \bar{K}_{s_i} \right| + \left| \bar{K}_{s_{i-1}} - \bar{K}_s \right|. \]

Then using Lemma 5.12, Proposition 5.13 and 5.14 we prove this theorem. ■

### 5.1.4 Convergence of \( \mathbb{P}_\mathbb{P} (s) \) to \( \bar{J}_s \)

Recall from Definition 4.9 that \( \mathbb{P}_\mathbb{P} (s) := \mathbb{K}_\mathbb{P} (s) \mathbb{K}_\mathbb{P} (1)^{-1} \mathbb{H}_\mathbb{P} \), where \( \mathbb{H}_\mathbb{P} : \mathbb{W}_\mathbb{P} (\mathbb{M}) \rightarrow \mathbb{R}^d \) is given by \( \mathbb{H}_\mathbb{P} = u_\mathbb{P} (1)^{-1} \mathbb{X} (\pi \circ u_\mathbb{P} (1)) \) and \( u_\mathbb{P} \) is interpreted in Notation 5.1.

**Proposition 5.16** Let \( \bar{J}_s \) be as in Definition 4.19 and \( \mathbb{X} \in \Gamma (\mathbb{T} \mathbb{M}) \) with compact support, then for any \( q \geq 1 \),
\[ \lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \sup_{s \in [0,1]} \left| \mathbb{P}_\mathbb{P} (s) - \bar{J}_s \right|^q \right] = 0. \]

**Proof.**
\[ \left| \mathbb{P}_\mathbb{P} (s) - \bar{J}_s \right| \leq I_{\mathbb{P}} (s) + II_{\mathbb{P}} (s) + III_{\mathbb{P}} (s), \]
where
\[ I_{\mathbb{P}} (s) = \left| \bar{K}_s - \mathbb{K}_\mathbb{P} (s) \right| \left| \mathbb{K}_\mathbb{P} (1)^{-1} \right| |\mathbb{H}_\mathbb{P}| \]
\[ II_{\mathbb{P}} (s) = \left| \bar{K}_s \right| \left| \mathbb{K}_\mathbb{P} (1)^{-1} \mathbb{K}_1^{-1} \right| |\mathbb{H}_\mathbb{P}| \]
\[ III_{\mathbb{P}} (s) = \left| \bar{K}_s \right| \left| \mathbb{K}_1^{-1} \right| |\mathbb{H}_\mathbb{P} - \bar{H}|. \]

For \( I_{\mathbb{P}} (s) \), since \( \mathbb{X} \) has compact support, \( |\mathbb{H}_\mathbb{P} (\sigma)| \) is bounded. By Lemma 4.5, \( \left| \mathbb{K}_\mathbb{P} (1)^{-1} \right| \leq 1 \). Then using Theorem 5.13 we have
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq 1} I_{\mathbb{P}} (s) \right] \leq C |\mathcal{P}|^{q+1} \text{ for } n > 5qN. \]

For \( II_{\mathbb{P}} (s) \) : since
\[ \mathbb{K}_\mathbb{P} (1)^{-1} - \mathbb{K}_1^{-1} = \mathbb{K}_\mathbb{P} (1)^{-1} \left( \bar{K}_1 - \mathbb{K}_\mathbb{P} (1) \right) \mathbb{K}_1^{-1}, \]

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Recall from (5.38) that $\sup_{s \in [0,1]} |\tilde{K}_s|$ is bounded (the bound is deterministic), using Theorem 5.16, again we have

$$\mathbb{E} \left[ \sup_{0 \leq s \leq 1} I_{p_{(s)}} \right] \leq C |\mathcal{P}|^q \text{ for } n > 5qN. \quad (5.37)$$

For $III_{p_{(s)}}$ (5.17): Since $F : \mathcal{O}(M) \to \mathbb{R}^d$ given by $F(y) = y^{-1}X \circ \pi(y)$ is bounded and continuous, and by the Wong–Zakai approximation Theorem (for example, see Theorem 10 in [10]), $u_{p_{(1)}} \to \tilde{u}_1$ in probability as $|\mathcal{P}| \to 0$, by DCT,

$$\tilde{H}_p \to \tilde{H} \text{ in } L^\infty^-(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \quad (5.38)$$

Also since $\sup_{s \in [0,1]} |\tilde{K}_s|$ and $|\tilde{K}_1^{-1}|$ are bounded, we have

$$\sup_{0 \leq s \leq 1} III_{p_{(s)}} \to 0 \text{ in } L^\infty^-(W_o(M)) \text{ as } |\mathcal{P}| \to 0. \quad (5.39)$$

Combining Eq. (5.36), (5.37) and (5.39) we prove this proposition. ■

### 5.2 Convergence of $\tilde{X}^{tr,\nu}_{p_{(s)}}$ to $\left(\tilde{X}\right)^{tr,\nu}$

Recall from Lemma 4.11 and Theorem 4.22 that

$$\tilde{X}^{tr,\nu}_{p_{(s)}} = -\tilde{X}_p + \int_0^1 \langle J_{p_{(s^+)}}(s), d\beta_{p_{(s)}} \rangle - div\tilde{X}_p \quad (5.40)$$

and

$$\tilde{X}^{tr,\nu} = -\tilde{X} + \sum_{a=1}^d \left\langle \tilde{C}\tilde{H}, e_a \right\rangle \int_0^1 \langle \tilde{T}_s^{-1} \rangle e_a, d\beta_s \rangle - \sum_{a=1}^d \left\langle XZ_a \left(\tilde{C}\tilde{H}\right), e_a \right\rangle. \quad (5.41)$$

**Theorem 5.17** If $M$ has parallel curvature tensor, i.e. $\nabla R \equiv 0$, then for any $f \in FC^1_b$ and $q \geq 1$,

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \left| \tilde{X}^{tr,\nu}_{p_{(s)}} f - \tilde{X}^{tr,\nu} f \right|^q \right] = 0,$$

where according to Notation 5.7, $\tilde{X}^{tr,\nu}_{p_{(s)}} f$ is interpreted as $(\tilde{X}^{tr,\nu}_{p_{(s)}} (f \mid_{H_p(M)})) \circ \phi \circ \beta_p$.

**Proof.** In correspondence with the three–term formulae (5.40) and (5.41), this theorem is decomposed as three propositions: Proposition 5.19 states that

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \left| \tilde{X}_p f - \tilde{X} f \right|^q \right] = 0,$$

Proposition 5.20 states that

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \left| \int_0^1 \langle J_{p_{(s^+)}}(s), d\beta_{p_{(s)}} \rangle - \sum_{a=1}^d \left\langle \tilde{C}\tilde{H}, e_a \right\rangle \int_0^1 \langle \tilde{T}_s^{-1} \rangle e_a, d\beta_s \rangle \right|^q \right] = 0$$

and Proposition 5.21 states that

$$\lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \left| div\tilde{X}_p - \sum_{a=1}^d \left\langle XZ_a \left(\tilde{C}\tilde{H}\right), e_a \right\rangle \right|^q \right] = 0.$$

Thus the proof will be complete once the stated propositions are proved. ■

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Remark 5.18 For Proposition 5.19 and 5.20, we assume the assumption of bounded sectional curvature as is mentioned in the beginning of this section. For Proposition 7.21, we further require the curvature tensor to be covariantly constant.

Proposition 5.19 If \( X \in \Gamma(TM) \) with compact support and \( f \in \mathcal{F}C^1 \), then for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} \mathbb{E} \left[ |\hat{X}f| \right] = 0.
\]

Proposition 5.20 Keeping the notation above, we have for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} \mathbb{E} \left[ \left| \int_0^1 \langle J'_{P}(s^+), d\beta_P(s) \rangle - \sum_{\alpha=1}^d \langle \tilde{\nabla}_s \tilde{C}H_e, e_{\alpha} \rangle \int_0^1 \langle \tilde{P}^{-1} e_{\alpha} \rangle^q \right| \right] = 0.
\]

Proposition 5.21 Continuing the notation above, if we further assume \( \nabla R \equiv 0 \), then for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} \mathbb{E} \left[ \left| \text{div} \hat{X}_P - \sum_{\alpha=1}^d \left( X Z_{\alpha} \langle \tilde{C}H_e, e_{\alpha} \rangle \right)^q \right| \right] = 0.
\]

Proof of Proposition 5.19 Using Eq. (1.31) and the fact that \( \pi \circ u_P = \phi \circ \beta_P \), we have

\[
\hat{X}_P f = \sum_{i=1}^n \langle \text{grad}_i F(\pi \circ u_P), u_P(s_i) J_P(s_i) \rangle = \sum_{i=1}^n \langle u^{-1}_P(s_i), \text{grad}_i F(\pi \circ u_P), J_P(s_i) \rangle,
\]

and

\[
\hat{X} f = \sum_{i=1}^n \langle \text{grad}_i F(\pi \circ \hat{u}), \tilde{u}_s, \tilde{J}_s \rangle = \sum_{i=1}^n \langle \tilde{u}^{-1}_s, \text{grad}_i F(\pi \circ \hat{u}), \tilde{J}_s \rangle.
\]

where \( F \) is a representation of \( f \) as in Definition 2.25.

Since \( W(\mathcal{O}(M)) \ni y \to u^{-1}_s(\text{grad}_i F(\pi \circ y)) \in \mathbb{R}^d \) is continuous and bounded, using Theorem 10 in [10] and DCT, we know

\[
u^{-1}_P(s_i) \langle \text{grad}_i F(\pi \circ u_P) \rangle \to \tilde{u}^{-1}_s(\text{grad}_i F(\pi \circ \hat{u})) \text{ in } L^\infty -(W_o(M)) \text{ as } |P| \to 0.
\]

The proof is then completed by making use of (5.42) and Proposition 5.10.

Proof of Proposition 5.20

\[
\int_0^1 \langle J'_{P}(s^+), d\beta_P(s) \rangle = \sum_{i=1}^n \left( \frac{J_P(s_i) - J_P(s_{i-1})}{\Delta_i}, \Delta_i \beta \right) = \sum_{i=1}^n \left( J_P(s_{i-1}), \Delta_i \beta \right) + \sum_{i=1}^n \frac{1}{\Delta_i} \left( \int_{s_{i-1}}^{s_i} J''_P(s) (s - s_{i-1}) ds, \Delta_i \beta \right) = I_P + II_P,
\]

where

\[
I_P = \sum_{i=1}^n \left( J_P(s_{i-1}), \Delta_i \beta \right)
\]

and

\[
II_P = \sum_{i=1}^n \frac{1}{\Delta_i} \left( \int_{s_{i-1}}^{s_i} J''_P(s) (s - s_{i-1}) ds, \Delta_i \beta \right).
\]
Using the fact that $J_P$ satisfies Jacobi equation, we further have

\[
II_P = \sum_{i=1}^{n} \left\langle \frac{1}{\Delta_i^3} \int_{s_{i-1}}^{s_i} R_{\nu_P(s)} (\Delta_i \beta, J_P(s)) \Delta_i \beta (s - s_{i-1}) ds, \Delta_i \beta \right\rangle \\
= \sum_{i=1}^{n} \frac{1}{\Delta_i^3} \int_{s_{i-1}}^{s_i} \left\langle R_{\nu_P(s)} (\Delta_i \beta, J_P(s)) \Delta_i \beta, \Delta_i \beta \right\rangle (s - s_{i-1}) ds.
\]

Since the curvature tensor is anti-symmetric,

\[
\left\langle R_{\nu_P(s)} (\Delta_i \beta, J_P(s)) \Delta_i \beta, \Delta_i \beta \right\rangle \equiv 0 \nu-a.s,
\]

so $II_P \equiv 0 \nu-a.s.$

\[
I_P = \sum_{i=1}^{n} \left\langle f_{P,i}^* (1) K_P (1)^{-1} H_P, \Delta_i \beta \right\rangle \\
= \sum_{i=1}^{n} \left\langle K_P (1)^{-1} H_P, f_{P,i} (1) \Delta_i \beta \right\rangle = \left\langle K_P (1)^{-1} H_P, \sum_{i=1}^{n} f_{P,i} (1) \Delta_i \beta \right\rangle.
\]

For each $i \geq 1, s \in [s_{i-1}, s_i],$ define $g_i(s) := S_{P,i}(s) - C_{P,i}(s) S_{P,i-1}.$ Then Taylor’s expansion of $g_i$ at $s_{i-1}$ gives

\[
g_i(s) = -S_{P,i-1} + (s - s_{i-1}) I + \int_{s_{i-1}}^{s} A_{P,i}(r) g_i(r)(s - r) dr.
\]

So

\[
|g_i(s)| \leq |S_{P,i-1} - (s - s_{i-1}) I| + N |\beta_P'(s_{i-1})|^2 \int_{s_{i-1}}^{s} |g_i(r)|(s - r) dr.
\]

By Gronwall’s inequality and Eq. (5.3), we have

\[
|g_i(s_i)| \leq \frac{N}{6} K^2 \gamma |P|^{2\gamma + 1} e^{\frac{N}{2} |\Delta_i \beta|^2}.
\]

Note that $g_i(s_i) = S_{P,i} - C_{P,i} S_{P,i-1},$ so by Eq. (5.1),

\[
|f_{P,i}(1) - f_{P,i-1}(1)| \leq \frac{1}{|P|} |C_{P,n}| \cdots |C_{P,i+1}| \cdot |S_{P,i} - C_{P,i} S_{P,i-1}| \\
\leq C K^2 \gamma |P|^{2\gamma} e^{\sum_{i=1}^{n} N|\Delta_i \beta|^2}
\]

and thus

\[
\left| \sum_{i=1}^{n} f_{P,i}(1) \Delta_i \beta - \sum_{i=1}^{n} f_{P,i-1}(1) \Delta_i \beta \right|^q \leq |P|^{1-q} \left[ \sum_{i=1}^{n} |f_{P,i}(1) - f_{P,i-1}(1)|^q |\Delta_i \beta|^q \right] \\
\leq C K^2 e^{\sum_{i=1}^{n} qN|\Delta_i \beta|^2}.
\]

Picking $\gamma \in \left( \frac{1}{2}, \frac{4}{3} \right)$ we know for any $q \geq 1,$

\[
\mathbb{E} \left[ \left| \sum_{i=1}^{n} f_{P,i}(1) \Delta_i \beta - \sum_{i=1}^{n} f_{P,i-1}(1) \Delta_i \beta \right|^q \right] \to 0 \text{ as } |P| \to 0. \quad (5.43)
\]

Since $f_{P,i-1}(1) = f_{P,0}(1) f_{P,0}^{-1}(s_{i-1}) S_{P,i-1} \Delta_{i-1}^{-1},$ so

\[
\left\langle K_P (1)^{-1} H_P, \sum_{i=1}^{n} f_{P,i-1}(1) \Delta_i \beta \right\rangle = \left\langle f_{P,0}(1) K_P (1)^{-1} H_P, \sum_{i=1}^{n} f_{P,0}(s_{i-1}) S_{P,i-1} \Delta_{i-1}^{-1} \Delta_i \beta \right\rangle. \quad (5.44)
\]
Using Proposition \[\text{Proposition} \, \Delta \] we have \( \left| f_{p_0}^{-1} (s_{i-1}) \right| \leq 1 \). Then using Eq. \( \text{Eq.} \, 5.3 \) we obtain

\[
\left| f_{p_0}^{-1} (s_{i-1}) \frac{S_{p_0, i-1}}{\Delta_{i-1}} - f_{p_0}^{-1} (s_{i-1}) \right| \Delta_i \leq \left| \frac{S_{p_0, i-1}}{\Delta_{i-1}} - I \right| \Delta_i \leq \frac{N K \gamma |\mathcal{P}|^{\gamma}}{1 - \gamma} e^{\frac{1}{s} |\Delta_{i-1}|^2}.
\]

Therefore for each \( q \geq 1 \),

\[
\sum_{i=1}^{n} \left| f_{p_0}^{-1} (s_{i-1}) \frac{S_{p_0, i-1}}{\Delta_{i-1}} \Delta_i \beta - \sum_{i=1}^{n} f_{p_0}^{-1} (s_{i-1}) \Delta_i \beta \right|^q \leq |\mathcal{P}|^{1-q} \sum_{i=1}^{n} \left| N \gamma K \gamma |\mathcal{P}|^{\gamma} e^{\frac{1}{s} |\Delta_{i-1}|^2} \right|^q
\]

\[
\leq C |\mathcal{P}|^{(3\gamma-1)q} K^{\gamma q} e^{\sum_{i=1}^{n} \frac{N \gamma}{|\sum_{i}^{n} |\Delta_{i-1}|^2|}}. \quad (5.45)
\]

Picking \( \gamma \in (\frac{1}{q}, \frac{1}{2}) \), we have

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} \left| f_{p_0}^{-1} (s_{i-1}) \frac{S_{p_0, i-1}}{\Delta_{i-1}} \Delta_i \beta - \sum_{i=1}^{n} f_{p_0}^{-1} (s_{i-1}) \Delta_i \beta \right|^q \right) \right] \rightarrow 0 \text{ as } |\mathcal{P}| \rightarrow 0.
\]

Rewrite \( \sum_{i=1}^{n} \left| f_{p_0}^{-1} (s_{i-1}) \Delta_i \beta \right| \) as \( \int_{0}^{1} f_{p} (s) ds \beta_s \), where \( f_{p} (s) := \sum_{i=1}^{n} f_{p_0}^{-1} (s_{i-1}) 1_{[s_{i-1}, s_i)} (s) \). Define a martingale \( M_r := \int_{0}^{r} f_{p} (s) ds \beta_s - \int_{0}^{r} \tilde{T}_s^{-1} ds \beta_s \). Then by Burkholder-Davis-Gundy inequality, for each \( q \geq 1 \),

\[
\mathbb{E} \left[ \sup_{r \in [0,1]} |M_r|^q \right] \leq C \mathbb{E} \left[ \langle M \rangle_1^{\frac{q}{2}} \right],
\]

where

\[
\langle M \rangle_1 \leq \int_{0}^{1} \left| f_{p} (s) - \tilde{T}_s^{-1} \right|^2 ds \leq 2 \int_{0}^{1} \left| f_{p} (s) - \tilde{T}_s^{-1} \right| ds + 2 \int_{0}^{1} \left| \tilde{T}_s^{-1} - \tilde{T}_s^{-1} \right| ds.
\]

Since

\[
\int_{0}^{1} \left| f_{p} (s) - \tilde{T}_s^{-1} \right|^2 ds = \sum_{i=1}^{n} \left| f_{p_0}^{-1} (s_{i-1}) - \tilde{T}_{s_{i-1}}^{-1} \right|^2 \Delta_i
\]

\[
\leq \sum_{i=1}^{n} \left| f_{p_0}^{-1} (s_{i-1}) \right|^2 \left| f_{p_0}^{-1} (s_{i-1}) - \tilde{T}_{s_{i-1}}^{-1} \right|^2 \Delta_i
\]

\[
\leq \sup_{i \in \{0, \ldots, n\}} \left| f_{p_0}^{-1} (s_i) - \tilde{T}_s^{-1} \right|^2 \quad (5.46)
\]

and

\[
\int_{0}^{1} \left| \tilde{T}_s^{-1} - \tilde{T}_s^{-1} \right|^2 ds = \int_{0}^{1} \left| \int_{0}^{r} (\tilde{T}_r^{-1})' ds \right|^2 ds \leq \int_{0}^{1} N |s - 2| ds \leq N |\mathcal{P}|^2,
\]

so

\[
\langle M \rangle_1^{\frac{q}{2}} \leq C \left( \int_{0}^{1} \left| f_{p} (s) - \tilde{T}_s^{-1} \right|^2 ds \right)^{\frac{q}{2}} + C \left( \int_{0}^{1} \left| \tilde{T}_s^{-1} - \tilde{T}_s^{-1} \right|^2 ds \right)^{\frac{q}{2}}
\]

\[
\leq C \left( \sup_{i \in \{0, \ldots, n\}} \left| f_{p_0}^{-1} (s_i) - \tilde{T}_s^{-1} \right|^2 + |\mathcal{P}|^q \right).
\]

Then using Theorem \( \text{Thm.} \, 5.3 \) we have

\[
\mathbb{E} \left[ \langle M \rangle_1^{\frac{q}{2}} \right] \leq C |\mathcal{P}|^{\theta q}. \quad (5.47)
\]

Then it follows that for each \( q \geq 1 \),

\[
\int_{0}^{1} f_{p} (s) ds \beta_s - \int_{0}^{1} \tilde{T}_s^{-1} ds \beta_s \rightarrow 0 \text{ in } L^q (W_n (M)) \text{ as } |\mathcal{P}| \rightarrow 0.
\]
Then using Eq. (5.36), Eq. (5.38) and Theorem 5.15 we have
\[
\mathbf{K}_P (1)^{-1} H_P \to \tilde{\mathbf{K}}_1^{-1} \tilde{H} \text{ in } L^\infty (W_o (M)) \text{ as } |P| \to 0
\]
and
\[
f^*_P (1) \to \tilde{T}_1^* \text{ in } L^\infty (W_o (M)) \text{ as } |P| \to 0,
\]
therefore
\[
I_P \to \left< T_1^* \tilde{\mathbf{K}}_1^{-1} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right> \text{ in } L^\infty (W_o (M)) \text{ as } |P| \to 0. \tag{5.48}
\]
Lastly, notice that
\[
\tilde{\mathbf{K}}_1 = \tilde{T}_1 \int_0^1 \left( T_1^* \tilde{T}_1^* \right)^{-1} \tilde{d}T_1^* ,
\]
so
\[
\tilde{\mathbf{K}}_1^{-1} = \left( \tilde{T}_1^{-1} \right)^* \tilde{C}
\]
where \( \tilde{C} \) is defined in Theorem 5.22 and
\[
\left< T_1^* \tilde{\mathbf{K}}_1^{-1} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right> = \left< \tilde{C} \tilde{H}, \int_0^1 \tilde{T}_s^{-1} d\beta_s \right> = \sum_{a=1}^d \left< \tilde{C} \tilde{H}, e\alpha \right> \int_0^1 \left< \left( \tilde{T}_s^{-1} \right)^* e\alpha , d\beta_s \right>. \tag{5.49}
\]
The proof is completed by combining Eq. (5.48) and (5.49).

We state two supplementary lemmas.

**Lemma 5.22** If the curvature tensor is parallel, i.e. \( \nabla R \equiv 0 \), then
\[
- \sum_{a=1}^d \left< X \mathbf{Z}_a \left( \tilde{C} \tilde{H} \right), e\alpha \right> = \text{div} X \circ E_1 - \sum_{a=1}^d \left< \tilde{C} \mathbf{A}_\gamma (Z\alpha) \tilde{H}, e\alpha \right>. \tag{5.50}
\]
where \( \mathbf{A}_\gamma (Z\alpha) = \int_0^\gamma R_{\alpha\gamma} (Z\alpha (r), \delta r) \).

**Proof.** See Lemma 4.25 in IVP.

**Lemma 5.23** Fix \( s \in [0, 1] \), consider an one parameter family of paths \( \{ \sigma_t \} \subset H_P (M) \) and denote by \( u_t (\cdot) \) the horizontal lift of \( \sigma_t \). For simplicity, we will denote \( u_t (s) \) by \( u_t \), \( \sigma_0 \) by \( \sigma \), the derivative with respect to \( t \) by \( \dot{\cdot} \) and the derivative with respect to \( s \) by \( \tau \). For any \( X \in \Gamma (TM) \), define \( f_X : \mathcal{O} (M) \to \mathbb{R}^d \cong T_o M \) by
\[
f_X (u) = u^{-1} (X \circ \pi) (u)
\]
Then:
\[
\frac{d}{dt} \mid_{0} f_X (u_t) = \left( \frac{d}{dt} \mid_{0} u_t \right) f_X = u_0^{-1} \nabla_{\dot{\sigma}(s)} X - \int_0^\gamma R_{\alpha(r)} \left( u_0 (r)^{-1} \dot{\sigma} (r^+) , u_0 (r)^{-1} \dot{\sigma} (r) \right) d f_X (u_0).
\]

**Proof.** The connection on \( \mathcal{O} (M) \) defined in Definition 4.22 gives the following decomposition:
\[
\dot{u}_0 = B_a (u_0) + \ddot{A} (u_0)
\]
where \( a = u_0^{-1} \frac{d}{dt} \mid_{0} \sigma_t (s) = u_0^{-1} \dot{\sigma} (s) \in T_o M \) and \( \ddot{A} (u_0) = \frac{d}{dt} \mid_{0} u_0 e t A \) for some \( A = u_0^{-1} \frac{d}{dt} \mid_{0} \gamma (0) \in so(d) \) and \( B_a (u_0) = \frac{d}{dt} \mid_{0} / /_t (\gamma) u_0 \) where \( \gamma \) satisfies \( \dot{\gamma} (0) = u_0 a \) and \( \gamma (0) = \sigma (s) \). In this example, we can choose \( \gamma (\cdot) \) to be \( \sigma (\cdot) \). So
\[
B_a (u_0) f_X = \frac{d}{dt} \mid_{0} u_0^{-1} / /_t (\gamma) (X \circ \pi) (/ /_t (\gamma) u_0) = u_0^{-1} \nabla_{\dot{\sigma}(s)} X
\]
and
\[
\ddot{A} (u) f_X = \frac{d}{dt} \mid_{0} u e^{-t A} u^{-1} (X \circ \pi) (ue^{A}) = - A u_0^{-1} X (\sigma (1)) = - Af_x (u_0)
\]

Following the computation in Theorem 3.3 in [1], we know that
\[ A = \int_0^s R_{u_0(r)} \left( u_0(r)^{-1} \sigma'(r^+), u_0(r)^{-1} \dot{\sigma}(r) \right) dr. \]

**Proof of Proposition 5.21.** Because of Lemma 5.22 it suffices to prove
\[
\lim_{|P| \to 0} \mathbb{E} \left[ \left| \operatorname{div} \tilde{X}_P - \left( \operatorname{div} X \circ E_1 - \sum_{\alpha=1}^d \left\langle \tilde{C} \mathbf{A}_1 (Z_\alpha) \tilde{H}, e_\alpha \right\rangle \right) \right|^q \right] = 0.
\]

From Definition 4.9 we get, for each \( \alpha \in \{1, \ldots, d\} \) and \( j \in \{1, \ldots, n\} \), that
\[
J^\prime_P (s_{j-1}+) = K^\prime_P (s_{j-1}+) \cdot K^\prime_P (1) - 1 \cdot H^P = f^*_P (1) K^\prime_P (1) - 1 \cdot H^P.
\]
Therefore
\[
X^{h_{\alpha,j}} J^\prime_P (s_{j-1}+) = I_P (\alpha, j) + II_P (\alpha, j) + III_P (\alpha, j),
\]
where
\[
I_P (\alpha, j) = \left( X^{h_{\alpha,j}} f^*_P (1) \right) K^\prime_P (1) - 1 \cdot H^P
\]
\[
II_P (\alpha, j) = f^*_P (1) \left( X^{h_{\alpha,j}} K^\prime_P (1) \right) H^P
\]
\[
III_P (\alpha, j) = f^*_P (1) K^\prime_P (1) \left( X^{h_{\alpha,j}} H^P \right).
\]
Using Proposition 4.12 we have
\[
\operatorname{div} \tilde{X}_P = \sum_{\alpha=1}^d \sum_{j=1}^n \left( \left( I_P + II_P + III_P \right) (\alpha, j), e_\alpha \right) \sqrt{\Delta_j}.
\]

Based on the expression above, Proposition 5.21 will be proved as a corollary of Lemma 5.24 to Lemma 5.27. In Lemma 5.24 and Lemma 5.25 we show that
\[
\sum_{\alpha=1}^d \sum_{j=1}^n \left( III_P (\alpha, j), e_\alpha \right) \sqrt{\Delta_j} \to 0 \quad \text{as} \quad |P| \to 0.
\]

In Lemma 5.26 we show that
\[
\sum_{\alpha=1}^d \sum_{j=1}^n \left( II_P (\alpha, j), e_\alpha \right) \sqrt{\Delta_j} \to 0 \quad \text{as} \quad |P| \to 0.
\]

In Lemma 5.27 we show that
\[
\sum_{\alpha=1}^d \sum_{j=1}^n \left( I_P (\alpha, j), e_\alpha \right) \sqrt{\Delta_j} \to 0 \quad \text{as} \quad |P| \to 0.
\]

**Lemma 5.24** If \( \nabla R \equiv 0 \), then for any \( q \geq 1 \),
\[
\lim_{|P| \to 0} \mathbb{E} \left[ \left| \sum_{\alpha=1}^d \sum_{j=1}^n \left( III_P (\alpha, j), e_\alpha \right) \sqrt{\Delta_j} - \left( \operatorname{div} X \circ E_1 - \sum_{\alpha=1}^d \left\langle \tilde{C} \mathbf{A}_1 (Z_\alpha) \tilde{H}, e_\alpha \right\rangle \right) \right|^q \right] = 0. \quad (5.52)
\]
Proof. Applying Lemma 5.24 to $X^{\omega, r} H P$ gives

$$
\sum_{\alpha=1}^{d} \sum_{j=1}^{n} \langle III_\alpha (\alpha, j), e_\alpha \rangle \sqrt{\Delta j} = IV_P - V_P,
$$

where

$$
IV_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} u_P (1)^{-1} \nabla u_P (1)^{-1} \Delta f_{P,j} (1) e_\alpha, e_\alpha \right\rangle \sqrt{\Delta j}
$$

and

$$
V_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} \int_0^1 R_{u_P} (1) (\beta_P (r^+), h_{\alpha,j} (r)) dr H_P, e_\alpha \right\rangle \sqrt{\Delta j}.
$$

We first compute $IV_P$. After viewing $L (\cdot) = u_P (1)^{-1} \nabla u_P (1) X$ as a linear functional on $\mathbb{R}^d$ we have

$$
IV_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} L (f_{P,j} (1) e_\alpha), e_\alpha \right\rangle \Delta j
$$

$$
= \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} L f_{P,j} (1) \right\rangle \Delta j
$$

$$
= \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} L \right\rangle \Delta j
$$

$$
= \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j} (1) K_P (1)^{-1} L \right\rangle \Delta j
$$

$$
= \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j} (1) K_P (1)^{-1} L \right\rangle \Delta j
$$

(5.53)

where in Eq. (5.53) we use identity (1.8) $\sum_{j=1}^{n} \Delta_j f_{P,j} (1) f_{P,j}^*(1) = K_P (1)$ and given $A \in M_{d \times d}$, $Tr (A) := \sum_{\alpha=1}^{d} (\langle A e_\alpha, e_\alpha \rangle)$ is the trace of the matrix $A$.

The proof of the lemma will be completed by Lemma 5.25 below which shows term $V_P$ converges to the right side of Eq. (5.53). ■

Lemma 5.25 Let $V_P$ be defined as in Lemma 5.24 and $\nabla R \equiv 0$, then for any $q \geq 1$,

$$
\lim_{q \to 0} \mathbb{E} \left[ V_P - \sum_{\alpha=1}^{d} \left\langle \tilde{C} A_1 (Z_\alpha) \tilde{H}, e_\alpha \right\rangle \right]^q = 0.
$$

(5.54)

Proof. Recall that

$$
V_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left\langle f_{P,j}^* (1) K_P (1)^{-1} \int_0^1 R_{u_P} (1) (\beta_P (r^+), h_{\alpha,j} (r)) dr H_P, e_\alpha \right\rangle \sqrt{\Delta j}.
$$

For each $\alpha \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$, since $h_{\alpha,j} (r) = \sqrt{\Delta j} f_{P,j} (r) e_\alpha$, we have

$$
\int_0^1 R_{u_P} (1) (\beta_P (r^+), \frac{1}{\sqrt{\Delta j}} h_{\alpha,j} (r)) dr = \int_0^1 R_{u_P} (1) (\beta_P (r^+), f_{P,j} (r) e_\alpha) dr
$$

$$
= \int_0^1 R_{u_P} (1) (\beta_P (r^+), f_{P,j} (r) e_\alpha) dr + e_0
$$

(5.55)
where \( e_0 := e_{0,1} + e_{0,2} \)

\[
e_{0,1} = \int_0^1 R_{\omega_p(r)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr - \int_0^1 R_{\omega_p(2)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr
\]

and

\[
e_{0,2} = \int_0^1 R_{\omega_p(2)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr - \int_0^1 R_{\omega_p(2)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr.
\]

Since \( \nabla R \equiv 0 \), \( dR_u = \nabla d_b R \equiv 0 \). So \( R_u \equiv R_{\omega_p} \) is independent of \( u \), therefore \( e_{0,1} = 0 \). As for \( e_{0,2} \), since

\[
|e_{0,2}|^q \leq N \sup_{r \in [0,1]} |\beta_p(r^+)|^q \sup_{r \in [0,1], j \in \{1, \ldots, n\}} |f_{\beta, j}(r) - f_{\beta, j}(r)|^q,
\]

using \( \boxplus \) we have

\[
|e_{0,2}|^q \leq C K_\gamma^3 q |P| q^{(3 \gamma - 1)} e^{2qN \sum_{k=1}^n |\Delta_k e|}^2,
\]

and from which it follows

\[
E [|e_{0,2}|^q] \leq C |P| q^{(3 \gamma - 1)} \forall n \geq 5qN.
\] (5.56)

Picking \( \gamma > \frac{1}{3} \) such that \( q (3 \gamma - 1) > 0 \) for any \( q \geq 1 \), so \( E [|e_{0,2}|^q] \to 0 \) as \( |P| \to 0 \).

Next we analyze

\[
\int_0^1 R_{\omega_p(2)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr = \sum_{k=1}^n R_{\omega_p(s_{k-1})} (\Delta_k e, f_{\beta, j} (s_{k-1}) e_\alpha) = \int_0^1 g_1 (s) \, d\beta
\]

where

\[
g_1 (s) = \sum_{k=1}^n R_{\omega_p(s_{k-1})} (\cdot, f_{\beta, j} (s_{k-1}) e_\alpha) 1_{[s_{k-1}, s_k]} (s).
\]

Define

\[
g_2 (s) = \sum_{k=1}^n R_{\tilde{\alpha}_{s_k-1}} (\cdot, f_{\beta, j} (s_{k-1}) e_\alpha) 1_{[s_{k-1}, s_k]} (s)
\]

\[
g_3 (s) = \sum_{k=j+1}^n R_{\tilde{\alpha}_{s_k-1}} (\cdot, \tilde{T}_{s_k-1} e_\alpha) 1_{[s_{k-1}, s_k]} (s)
\]

\[
g_4 (s) = R_{\tilde{\alpha}_s} (\cdot, \tilde{T}_{s_{j-1}} e_\alpha) 1_{[s_{j-1}, 1]} (s)
\]

\[
g_5 (s) = R_{\tilde{\alpha}_s} (\cdot, \tilde{T}_{s_{j-1}} e_\alpha) 1_{[s_{j-1}, 1]} (s).
\]

For each \( i = 1, 2, 3, 4 \), denote \( e_{\beta, i} (r) = \int_0^r g_i (s) \, d\beta - \int_0^r g_{i+1} (s) \, d\beta \), then

\[
\int_0^1 R_{\omega_p(2)} (\beta_p^r (r^+) , f_{\beta, j} (r) e_\alpha) \, dr - \int_{s_j}^1 R_{\tilde{\alpha}_r} (d\beta, \tilde{T}_{s_{j-1}} e_\alpha) = \sum_{i=1}^4 e_{\beta, i} (1). \] (5.57)

Notice that \( \{g_i\}_{i=1}^5 \) are all adapted, so based on the same computation as in Lemma \( \boxplus \) (mainly Burkholder-Davis-Gundy inequality), we can show for each \( i \in \{1, 2, 3, 4\} \) and for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} E_{\nu} [|e_{\beta, i} (1)|^q] = 0 \] (5.58)

Using Eq. (5.55), (5.56), (5.57) and (5.58) we have for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} E_{\nu} \left[ |V_{\beta} - \tilde{V}_{\beta}|^q \right] = 0 \] (5.59)
where

\[
\dot{V}_P = \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \left( (\tilde{T}_{s_j}^{-1})^* \tilde{T}_s^{-1} \int_{s_j}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r \tilde{T}_{s_j}^{-1}) \dot{H}, e_{\alpha} \right) \Delta_j.
\]

For each \( j \in \{1, \ldots, n\} \), denote by \( B_j \) the bilinear form on \( \mathbb{R}^d \):

\[
B_j(u, v) = \left( (\tilde{T}_{s_j}^{-1})^* \tilde{T}_s^{-1} \int_{s_j}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r \tilde{T}_{s_j}^{-1}) \dot{H}, v \right) \Delta_j.
\]

Then

\[
\dot{V}_P = \sum_{j=1}^{n} \sum_{\alpha=1}^{d} B(e_{\alpha}, e_{\alpha}) = \sum_{j=1}^{n} \sum_{\alpha=1}^{d} B(\tilde{T}_{s_j} e_{\alpha}, (\tilde{T}_{s_j}^{-1})^* e_{\alpha})
\]

\[
= \sum_{\alpha=1}^{d} \int_{0}^{1} \left( \tilde{T}_s^{-1} \tilde{T}_s^{-1} \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \dot{H}, (\tilde{T}_{s_j}^{-1} (\tilde{T}_{s_j}^{-1})^* e_{\alpha}) \right) ds.
\]

Define

\[
\dot{\hat{V}}_P := \sum_{\alpha=1}^{d} \int_{0}^{1} \left( \tilde{T}_s^{-1} \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \dot{H}, (\tilde{T}_{s_j}^{-1} (\tilde{T}_{s_j}^{-1})^* e_{\alpha}) \right) ds.
\]

Then we are about to show for any \( q \geq 1 \),

\[
\lim_{|\mathcal{P}| \to 0} \mathbb{E} \left[ \left| \dot{V}_P - \dot{\hat{V}}_P \right|^q \right] = 0.
\]

Using Eq. (5.60) we know

\[
\left| \dot{V}_P - \dot{\hat{V}}_P \right| \leq \sum_{\alpha=1}^{d} \int_{0}^{1} (I_P (s) + II_P (s)) ds,
\]

where

\[
I_P (s) = \left( \tilde{T}_s^{-1} \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \dot{H}, (\tilde{T}_{s_j}^{-1} (\tilde{T}_{s_j}^{-1})^* e_{\alpha}) \right)
\]

and

\[
II_P (s) = \left( \tilde{T}_s^{-1} \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \dot{H}, (\tilde{T}_{s_j}^{-1} (\tilde{T}_{s_j}^{-1})^* - \tilde{T}_{s_j}^{-1} (\tilde{T}_{s_j}^{-1})^* e_{\alpha}) \right).
\]

Since

\[
|I_P (s)|^q \leq C \left( \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \right)^q,
\]

\[
|II_P (s)|^q \leq C |\mathcal{P}|^q \left( \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \right)^q
\]

and by Burkholder-Davies-Gundy inequality, \( \mathbb{E} \left[ \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \right]^q \leq C |\mathcal{P}|^{\frac{q}{2}} \), we obtain

\[
\mathbb{E} \left[ \left| \dot{V}_P - \dot{\hat{V}}_P \right|^q \right] \leq C \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \int_{s_j}^{s_{j-1}} \mathbb{E} \left[ |I_P (s)|^q + |II_P (s)|^q \right]
\]

\[
\leq C \sum_{\alpha=1}^{d} \sum_{j=1}^{n} \int_{s_j}^{s_{j-1}} (C + |\mathcal{P}|^q) \mathbb{E} \left[ \left( \int_{s}^{1} R_{\tilde{\alpha}r} (d\beta_r, \tilde{T}_r e_{\alpha}) \right)^q \right]
\]

\[
= C |\mathcal{P}|^{\frac{q}{2}}
\]
and from which Eq. (5.61) follows.

Then we show a change of integration order, i.e.

\[ \hat{V}_P = \sum_{a=1}^{d} B_1(e_a, e_a) \]  

(5.63)

, where \( B_1(u, v) := \left< \tilde{T}_1^s K_1^{-1} \int_0^t \tilde{R}_a \left( d \beta_r, \tilde{T}_r u \right) \tilde{H}, \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* v ds \right> \). Define

\[ f(t) = \sum_{a=1}^{d} \int_0^t \tilde{T}_1^s K_1^{-1} \int_0^t \tilde{R}_a \left( d \beta_r, \tilde{T}_r e_a \right) \tilde{H}, \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* e_a \right> ds \]

and

\[ g(t) = \sum_{a=1}^{d} \int_0^t \tilde{T}_1^s K_1^{-1} \int_0^t \tilde{R}_a \left( d \beta_r, \tilde{T}_r e_a \right) \tilde{H}, \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds e_a \right> . \]

Then

\[ df = \sum_{a=1}^{d} \left( \tilde{T}_1^s K_1^{-1} \tilde{R}_a \left( d \beta_t, \tilde{T}_t e_a \right) \tilde{H}, \int_0^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds e_a \right) \]

Since

\[ dg = \sum_{a=1}^{d} \left( \tilde{T}_1^s K_1^{-1} \tilde{R}_a \left( d \beta_t, \tilde{T}_t e_a \right) \tilde{H}, \int_0^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds e_a \right) = df \]

and \( g(0) = 0 = f(0), \) Eq. (5.63) is proved by observing that \( \hat{V}_P = f_1 = g_1 = \sum_{a=1}^{d} B_1(e_a, e_a). \)

Lastly, notice that

\[ \sum_{a=1}^{d} B_1(e_a, e_a) = Tr(\sum_{a=1}^{d} B_1) = \sum_{a=1}^{d} B_1 \left( \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds e_a, \left[ \left( \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds \right)^{-1} \right] e_a \right) \]

and \( \tilde{T}_r \int_r^t \tilde{T}_s^{-1} (\tilde{T}_s^{-1})^* ds e_a = Z_\alpha(r), \) we combine Eq. (5.59), (5.61) and (5.68) to prove Eq. (5.64).

\[ \text{Lemma 5.26 If } \nabla R \equiv 0, \text{ then for any } q \geq 1, \]

\[ \lim_{|\alpha| \to 0} \mathbb{E} \left[ \left| \sum_{a=1}^{n} \sum_{j=1}^{d} \left< I_{\alpha} (\alpha, j), e_a \right> \sqrt{\Delta_j} \right|^q \right] = 0. \]

\[ \text{Proof. Define } \tilde{g}_j(s) := X^{h_{\alpha,j}} f_{P,j} (s) \text{ and } g_j(s) := \tilde{g}_j(s) - \tilde{g}_j(\tilde{s}). \text{ Then we know that } g_j(s) \text{ satisfies the following ODE: for } k = j, \cdots, n \]

\[ \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} (\tilde{g}_j(s), \tilde{A}_{P,k} (s) (f_{P,j} (s) - f_{P,j}(\tilde{s}))) & s \in [s_{k-1}, s_k] \\ g_j(s) & = 0 \\ g_j(\tilde{s}) & = 0 \end{cases} \]

where \( \tilde{A}_{P,k} (s) = \frac{d}{dt} \left( R_{\tilde{u}_P(t,s)} (\beta_{P}(t,s), \cdot) \beta_{P}(t,s) \right) \), \( \beta(t, \cdot) := \tilde{\phi}^{-1} (E(tX^{h_{\alpha,j}})) \) is the stochastic anti-rolling of the approximate flow \( E(tX^{h_{\alpha,j}}) \) (See Corollary 4.6 in [6]), \( \beta_P(t, \cdot) \) is the piecewise linear approximation of \( \beta(t, \cdot) \) and \( \tilde{u}_P(t, \cdot) = \eta \circ \beta_P(t, \cdot) \) is the horizontal lift of \( \beta_P(t, \cdot). \)

Since \( \nabla R \equiv 0, \) \( \tilde{A}_{P,k}(s) \) is a constant operator for \( s \in [s_{k-1}, s_k], \) therefore by DuHammel’s principle,

\[ g_j(s) = \int_{s_{k-1}}^{s} S_{P,k} (s - r) \tilde{A}_{P,k} (f_{P,j} (r) - f_{P,j}(s_{k-1})) dr. \]
Using Eq. 5.3 and 5.8 we obtain the following estimate,

\[
|g_j(s)| \leq \int_{s_{k-1}}^{s} |S_{P,k}(s-r)| |\dot{A}_{P,k}| |f_{P,j}(r) - f_{P,j}(s_{k-1})| \, dr \\
\leq C \sup_{1 \leq k \leq n} |\dot{A}_{P,k}| |P|^{2\gamma + 2} K_\gamma^2 e^{3N \sum_{k=1}^{n} |\Delta_k|}.
\]  

(5.64)

Therefore

\[
|\tilde{g}_j(1)| \leq \sum_{k=j}^{n} |g_j(s_k)| \leq C \sup_{1 \leq k \leq n} |\dot{A}_{P,k}| |P|^{2\gamma} K_\gamma^2 e^{3N \sum_{k=1}^{n} |\Delta_k|}. 
\]  

(5.65)

Then we analyze \( \sup_{k \in \{1, \ldots, n\}} |\dot{A}_{P,k}| \). Since \( \nabla R \equiv 0 \),

\[
\dot{A}_{P,k} = 2R_{w_P(s)} \left( \frac{d}{dt} |\beta_P'(t,s)|, \right) \beta_P'(s).
\]

Notice that \( \beta_P'(t,s) = u_P^{-1}(s) \sigma_P(t,s) \), where \( \sigma_P(t,:) = \phi \circ \beta_P(t,:) \) is the rolling of \( \beta_P(t,:) \), so we can use Lemma 5.23 to get

\[
X^{\beta_j,0} \beta_P'(s_{k-1}+) = \delta^i e_\alpha \sqrt{|\Delta_j|} - \int_{0}^{s_{k-1}} R_{w_P(\tau)} (\beta_P' (\tau+), h_{\alpha,j} (\tau)) \, d\tau \beta_P'(s_{k-1}+). 
\]  

(5.66)

Therefore

\[
|\dot{A}_P'(r)| \leq N \left| X^{\beta_j,0} \beta_P'(s_{k-1}+) \right| |\beta_P'(s_{k-1})| \\
\leq N \left( \frac{1}{\sqrt{|P|}} + N \sup_{j,s} |\alpha_{\beta_j}(s)| \sup_{s \in [0,1]} |\beta_P'(s)|^2 \right) |\beta_P'(s_{k-1})| \\
\leq N \left( \frac{1}{\sqrt{|P|}} + Nf(K_\gamma) \sqrt{|P|} |P|^{2(\gamma-1)} \right) K_\gamma |P|^{\gamma-1} \\
\leq f(K_\gamma) |P|^{\beta \gamma - 3/2}
\]

where \( f(K_\gamma) \) is some random variable in \( L^1 \left( W_o (M) \right) \), so

\[
|\tilde{g}_j(1)| \leq Cf(K_\gamma) |P|^{\beta \gamma - 3/2}. 
\]  

(5.67)

From above one can see

\[
\sum_{\alpha,j=1}^{d,n} \langle I, e_\alpha \rangle \sqrt{|\Delta_j|} = \sum_{\alpha,j=1}^{d,n} \langle X^{\beta_j,0} T_j^* \rangle K_\gamma^{-1} (1) H_{P}, e_\alpha \rangle \sqrt{|\Delta_j|} \\
= \sum_{\alpha=1}^{d} \left( \sum_{j=1}^{n} \left( \tilde{g}_j(1) \sqrt{|P|} \right) K_\gamma^{-1} H_{P}, e_\alpha \right).
\]

From (5.67) we know that \( \sum_{j=1}^{n} \left( \tilde{g}_j(1) \sqrt{|P|} \right) \to 0 \) in \( L^\infty \left( W_\cdot \right) \), also notice that for any \( q \geq 1 \),

\[
\lim_{|P| \to 0} E_\nu \left[ \left| K_\gamma^{-1} H_{P} - K_\gamma^{-1} \hat{H} \right|^q \right] = 0.
\]

So

\[
\lim_{|P| \to 0} E_\nu \left[ \left| \sum_{\alpha=1}^{d} \left( \sum_{j=1}^{n} \left( \tilde{g}_j(1) \sqrt{|P|} \right) K_\gamma^{-1} H_{P}, e_\alpha \right) \right|^q \right] = 0 \forall q \geq 1.
\]
Lemma 5.27 If $\nabla R \equiv 0$, then for any $q \geq 1$,
\[
\lim_{|P| \to 0} E \left[ \sum_{\alpha=1}^{d} \sum_{j=1}^{n} (II_P(\alpha, j), e_\alpha) \sqrt{\Delta_j} \right] |P|^q = 0.
\]
Proof. Since $X^{\alpha,j} = -K_P(1)^{-1} X^{\alpha,j} (K_P(1)) K_P(1)^{-1}$,
\[
\left| X^{\alpha,j} (K_P(1)^{-1}) \right| \leq \left| X^{\alpha,j} (K_P(1)) \right|.
\]
Then let $\tilde{g}_j(s) := X^{\alpha,j} (K_P(s))$ and this lemma follows from a Lemma 6.24 type argument. \]

6 Proof of Theorem 1.11
First we collect a list of supplementary results.

Lemma 6.1 For any $f \in F_2^1$, $\tilde{X}^{tr,v} f \in L^\infty(-W_o(M), \nu)$.
Proof. See Lemma 4.24 in IVP.

Lemma 6.2 For any $f \in F_2^1$ and $q \geq 1$, there exists a constant $M = M(q)$ such that for all partition $P$ with $|P| < \frac{1}{M}$, $\tilde{X}^{tr,v}_P f \in L^q (H_P(M), \nu^1_P)$.
Proof. From Theorem 5.17 we know that
\[
\lim_{|P| \to 0} E \left[ \int_{W_o(M)} \left| \tilde{X}^{tr,v}_P f (\phi(P)) - \tilde{X}^{tr,v}_P f \right|^q d\nu \right] = 0.
\]
By Lemma 6.1 $\tilde{X}^{tr,v} f \in L^\infty(-W_o(M))$. Therefore Lemma 6.2 follows from triangle inequality and the fact that the law of $\phi(\beta_P)$ under $\nu$ is $\nu^1$.

Notation 6.3 Denote by $g$ any one of $\{g_i\}_{i=0}^d$ as in Theorem 7.1 and $\{g^{(m)}\}_{m} \subset C^\infty_0(M)$ be the approximate sequence in $L^{d-\epsilon}(M)$ as defined in Remark 3.5.

Lemma 6.4 Define $\tilde{g}(\sigma) = g(\sigma(1))$ and $\tilde{g}^{(m)}(\sigma) = g^{(m)}(\sigma(1))$, then for any $f \in F_2^1$,
\[
\int_{W_o(M)} \tilde{g}(\tilde{X}^{tr,v}_P f)(\sigma) d\nu(\sigma) < \infty
\]
and
\[
\lim_{m \to \infty} \int_{W_o(M)} \tilde{g}^{(m)}(\tilde{X}^{tr,v} f)(\sigma) d\nu(\sigma) = \int_{W_o(M)} \tilde{g}(\tilde{X}^{tr,v} f)(\sigma) d\nu(\sigma).
\]
Proof. Since $\nu \{ \sigma : \sigma(1) = x \} = 0$, so $\tilde{g}$ is $\nu - a.s.$ well-defined. In particular, for any $p > 0$,
\[
\int_{W_o(M)} |g(\sigma)|^p d\nu(\sigma) = \int_{M} |g(x)|^p p_1(0,x) d\lambda(x),
\]
where $\lambda$ is the volume measure on $M$.

Since $g$ has compact support and $p_1(0,\cdot) \in C^\infty(M)$,
\[
\int_{M} |g(x)|^p p_1(0,x) d\lambda(x) \leq C \|g\|^p_{L^p(M)}.
\]
Combining Eq. (3.3) and (3.4) and letting $p = \frac{d}{\alpha - 1}$ we get
\[
\tilde{g} \in L^{1+\frac{\alpha}{d-1}}(W_o(M)).
\]
Since $\tilde{X}^{tr,v} f \in L^\infty(-W_o(M))$ by Lemma 6.1, using Holder’s inequality we prove Eq. (6.1).

Since $\cup_m \text{supp}(g^{(m)})$ is contained in a compact set, Eq. (6.2) can be proved similarly with $g$ replaced by $g^{(m)} - g$. \]

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Lemma 6.5 Define \( \tilde{g} : H_P(M) \to \mathbb{R} \) to be \( \tilde{g}(\sigma) = g(\sigma(1)) \), then \( \tilde{g} \in L^{\frac{d}{d-1}}(H_P(M), \nu_P^+) \).

Proof. Set
\[
V_P := \{ \sigma \in H_P(M) : E_1^P(\sigma) = x \}.
\]
Applying the co-area formula Eq. (3.22) to \( f(y) = 1_{\{y=x\}} \), we have
\[
\nu_P^+(V_P) = \int_{H_P(M)} f(\sigma(1)) d\nu_P^+(\sigma) = \int_M f(y) h_P(y) dy = 0.
\]
So \( \tilde{g} \) is \( \nu_P^+ - a.s. \) well-defined. Then applying the co-area formula Eq. (3.22) again to \( |\tilde{g}|^{\frac{d}{d-1}} \), we have
\[
\int_{H_P(M)} |\tilde{g}(\sigma)|^{\frac{d}{d-1}} d\nu_P^+(\sigma) = \int_M |g(x)|^{\frac{d}{d-1}} h_P(x) dx,
\]
where \( h_P(x) \in C(M) \) is defined in Theorem 8.13 with \( f \equiv 1 \). Since \( g \) has compact support,
\[
\int_M |g(x)|^{\frac{d}{d-1}} h_P(x) dx \leq C \int_M |g(x)|^{\frac{d}{d-1}} dx.
\]
Therefore \( \tilde{g} \in L^{\frac{d}{d-1}}(H_P(M), \nu_P^+) \).

Lemma 6.6 Define \( \tilde{g}(\sigma) = g(\sigma(1)) \) and \( \tilde{g}^{(m)}(\sigma) = g^{(m)}(\sigma(1)) \), then there exists a constant \( M \) such that for any \( f \in FC_b \) and \( P \) with \( |P| < \frac{1}{M} \),
\[
\int_{H_P(M)} |\tilde{g} \cdot \left( X_{tr,\nu} f \right)(\sigma) d\nu_P^+(\sigma) < \infty
\]
and
\[
\lim_{m \to \infty} \int_{H_P(M)} \tilde{g}^{(m)}(\sigma) \left( X_{tr,\nu} f \right)(\sigma) d\nu_P^+(\sigma) = \int_{H_P(M)} \tilde{g}(\sigma) \left( X_{tr,\nu} f \right)(\sigma) d\nu_P^+(\sigma).
\]

Proof. Using Lemma 6.4, Lemma 6.5 and Holder’s inequality, we can easily see Eq. (6.7). Then applying the co-area formula (3.22) with
\[
(H, M, p, g, f) = \left( H_P(M), M, E_1^P, \frac{1}{Z_P} e^{-\frac{d}{2}}, \left( |\tilde{g}^{(m)} - \tilde{g}|(\sigma) \right)^{\frac{d}{d-1}} \right),
\]
we have:
\[
\int_{H_P(M)} \left| \left( \tilde{g}^{(m)} - \tilde{g} \right)(\sigma) \right|^{\frac{d}{d-1}} d\nu_P^+(\sigma) = \int_M |(g^m - g)(x)|^{\frac{d}{d-1}} h_P(x) dx.
\]
Since \( \cup_m \text{supp} (g^{(m)}) \) is contained in a compact set, Eq. (6.8) can be proved exactly the same argument as in Lemma 6.6 with \( g \) replaced by \( g^{(m)} - g \) and letting \( m \to \infty \).

Lemma 6.7 For any \( p \leq \frac{d}{d-1} \), \( \sup_P \mathbb{E}[|\tilde{g}(\phi \circ \beta_P)|^p] < \infty \).

Proof. Since the law of \( \phi \circ \beta_P \) under \( \nu \) is \( \nu_P^+ \), we have
\[
\mathbb{E}[|\tilde{g}(\phi \circ \beta_P)|^p] = \int_{H_P(M)} |\tilde{g}|^p(\sigma) d\nu_P^+(\sigma).
\]
Then applying co-area formula (3.22) exactly as Eq. (6.6) we get
\[
\int_{H_P(M)} |\tilde{g}|^p(\sigma) d\nu_P^+(\sigma) = \int_M |g(x)|^p \nu_{P,x}^+(H_P,M) dx.
\]
Using Proposition 3.30 note that \( g \) has compact support, we have
\[
\sup_P \int_M |g(x)|^p \nu_{P,x}^+(H_P,M) dx \leq \|g\|_{L^{\frac{d}{d-1}}(M)} \|\sup_P \nu_{P,x}^+(H_P,M) \cdot 1_{\text{supp}(g)}(x)\|_{L^{\frac{d}{d-1}}(\supp(g))} \leq C \|g\|_{L^{\frac{d}{d-1}}(M)}.
\]
Theorem 6.8 For any $f \in \mathcal{F}^1_{\nu}$,
\[
\lim_{|p| \to 0} \int_{H_p(M)} \tilde{g}(\sigma) \hat{X}_{P}^{tr,v_P} f(\sigma) d\nu_P(\sigma) = \int_{W_o(M)} \tilde{g}(\sigma) \hat{X}_{\nu}^{tr,v} f(\sigma) d\nu(\sigma).
\]

Proof. Since
\[
\int_{H_p(M)} \tilde{g}(\sigma) \left( \hat{X}_{P}^{tr,v_P} f(\sigma) \right) d\nu_P(\sigma) = E_{\nu} \left[ \tilde{g} \cdot \left( \hat{X}_{P}^{tr,v_P} f(\sigma) \right) (\phi \circ \beta_P) \right]
\]
and
\[
\int_{W_o(M)} \tilde{g}(\sigma) \left( \hat{X}_{\nu}^{tr,v} f(\sigma) \right) d\nu(\sigma) = E_{\nu} \left[ \tilde{g} \cdot \hat{X}_{\nu}^{tr,v} f \right],
\]
we have
\[
\left| \int_{H_p(M)} \tilde{g}(\sigma) \hat{X}_{P}^{tr,v_P} f(\sigma) d\nu_P(\sigma) - \int_{W_o(M)} \tilde{g}(\sigma) \hat{X}_{\nu}^{tr,v} f(\sigma) d\nu(\sigma) \right|
\]
\[
\leq E \left[ \tilde{g} \cdot \hat{X}_{P}^{tr,v_P} f (\phi \circ \beta_P) - \tilde{g} \cdot \hat{X}_{\nu}^{tr,v} f \right] + E \left[ \tilde{g} \cdot \hat{X}_{\nu}^{tr,v} (\phi \circ \beta_P) - \hat{X}_{\nu}^{tr,v} f \right] + E \left[ \tilde{g} (\phi \circ \beta_P) - \tilde{g} \right] \left| \hat{X}_{\nu}^{tr,v} f \right|.
\]

Choosing $p < \frac{d}{d-1}$ and using Holder’s inequality, we have
\[
E \left[ \tilde{g} (\phi \circ \beta_P) \right] \left| \hat{X}_{P}^{tr,v_P} f (\phi \circ \beta_P) - \hat{X}_{\nu}^{tr,v} f \right|
\]
\[
\leq \| \tilde{g} (\phi \circ \beta_P) \|_{L^P(W_o(M))} \cdot \left| \hat{X}_{P}^{tr,v_P} f (\phi \circ \beta_P) - \hat{X}_{\nu}^{tr,v} f \right|_{L^1(W_o(M))}
\]
\[
\leq \sup_p \| \tilde{g} (\phi \circ \beta_P) \|_{L^P(W_o(M))} \cdot \left| \hat{X}_{P}^{tr,v_P} f (\phi \circ \beta_P) - \hat{X}_{\nu}^{tr,v} f \right|_{L^1(W_o(M))}.
\]

Then using Theorem 5.17 and Lemma 6.1 we have
\[
\lim_{|p| \to 0} E \left[ \tilde{g} (\phi \circ \beta_P) \right] \left| \hat{X}_{P}^{tr,v_P} f (\phi \circ \beta_P) - \hat{X}_{\nu}^{tr,v} f \right| = 0.
\]

Then because of Lemma 6.1, it suffices to find a $p < \frac{d}{d-1}$ such that
\[
\lim_{|p| \to 0} E_{\nu} \left[ \| \tilde{g} (\phi \circ \beta_P) - \tilde{g} \|^{p} \right] = 0. \tag{6.12}
\]

Since for any $\epsilon > 0$, there exists a constant $C_{p,\epsilon}$ such that
\[
\| \tilde{g} (\phi \circ \beta_P) - \tilde{g} \|^{p(1+\epsilon)} \leq C_{p,\epsilon} \left( \| \tilde{g} (\phi \circ \beta_P) \|^{p(1+\epsilon)} + \| \tilde{g} \|^{p(1+\epsilon)} \right),
\]
We choose $p$ and $\epsilon$ such that $p(1+\epsilon) < \frac{d}{d-1}$. From Eq. 6.5 we know $E \left[ \| \tilde{g} \|^{p(1+\epsilon)} \right] < \infty$. Then using Lemma 6.7, we have
\[
\sup_p E_{\nu} \left[ \| \tilde{g} (\phi \circ \beta_P) \|^{p(1+\epsilon)} \right] < \infty.
\]
So $\sup_p E_{\nu} \left[ \| \tilde{g} (\phi \circ \beta_P) - \tilde{g} \|^{p(1+\epsilon)} \right] < \infty$ and thus
\[
\{ \| \tilde{g} (\phi \circ \beta_P) - \tilde{g} \|^{p} \} \text{ is uniformly integrable under } \nu.
\]

Then we want to show $\tilde{g} (\phi \circ \beta_P) - \tilde{g} \|^{p} \to 0$ in probability.

Let $U_P := \{ \sigma \in W_o(M) : E_{\nu} (\phi^{-1} \circ \beta_P (\sigma)) = x \}$, since the law of $\Phi^{-1} \circ \beta_P$ under $\nu$ is $\nu_P$, recall from Lemma 6.3 that $V_P := \{ \sigma \in H_P(M) : E^{1}_{\nu} (\sigma) = x \}$ and $\nu_P (V_P) = 0$, so
\[
\nu (U_P) = \nu_P (V_P) = 0.
\]
From there we can construct a \( \nu \)-Null set;

\[
N := \cup P U_P \cup \{ \sigma \in W_\alpha (M) : E_1 (\sigma) = x \}.
\]

Recall from Theorem 10 in \([1]\) that

\[
|u_P (1) - \tilde{u}(1)| \to 0 \text{ in probability.}
\]

Notice that \( g \in C^\infty (M/ \{ x \}) \) and \( \pi : O (M) \to M \) is smooth, so excluding \( N \), we have

\[
|\hat{g} (\phi \circ \beta_P) - \hat{g}| = |g \circ \pi (u_P (1)) - g \circ \pi (\tilde{u}(1))| \to 0 \text{ in probability.} \tag{6.13}
\]

Combining this with uniformly integrability we proved Eq. (6.12).

**Proposition 6.9** Let \( f \in F_C^1_b \), then

\[
\lim_{m \to \infty} \int_{W_\alpha (M)} \delta_x^{(m)} (\Sigma_1) \, f \, d\nu = \int_{W_\alpha (M)} f \, d\nu_x,
\]

where \( \Sigma_\nu (\sigma) = \sigma (r) \) is the canonical process on \( W_\alpha (M) \).

**Proof.** Since \( f = F (\Sigma_{a_1}, \ldots, \Sigma_{a_d}) \), we have by Markov property,

\[
\int_{W_\alpha (M)} \delta_x^{(m)} (\Sigma_1) \, f \, d\nu = \int_M \delta_x^{(m)} (x_n) \int_{M^n} F (x_1, \ldots, x_n) \Pi_{j=1}^n p_{x_{j-1}, x_j} (x_{j-1}, x_j) \, dx_1 \cdots dx_n.
\]

Viewing \( \int_{M^n} F (x_1, \ldots, x_n) \Pi_{j=1}^n p_{x_{j-1}, x_j} (x_{j-1}, x_j) \, dx_1 \cdots dx_n \) as a function of \( x_n \), observe that it is uniformly integrable with respect to \( x_n \), therefore it is a continuous function of \( x_n \). Thus

\[
\lim_{m \to \infty} \int_{W_\alpha (M)} \delta_x^{(m)} (\Sigma_1) \, f \, d\nu = \lim_{m \to \infty} \int_M \delta_x^{(m)} (x_n) \int_{M^{n-1}} F (x_1, \ldots, x_n) \Pi_{j=1}^{n-1} p_{x_{j-1}, x_j} (x_{j-1}, x_j) \, dx_1 \cdots dx_{n-1}
\]

\[
= \int_{M^{n-1}} F (x_1, \ldots, x_{n-1}, x) \Pi_{j=1}^{n-1} p_{x_{j-1}, x_j} (x_{j-1}, x_j) \cdot p_{x_{n-1}, x} (x_{n-1}, x) \, dx_1 \cdots dx_{n-1}
\]

\[
= \int_{W_\alpha (M)} f \, d\nu_x.
\]

**Proof of Theorem 1.11** Recall from Remark 3.3 that we can construct an approximate sequence of the delta mass \( \delta_x \) on \( M \):

\[
\delta_x^{(m)} := \hat{g}_0^{(m)} + \sum_{j=1}^d X_j \hat{g}_j^{(m)} \in C_0^\infty (M)
\]

where \( \left\{ \hat{g}_j^{(m)} : 0 \leq j \leq d, m \geq 1 \right\} \subseteq C_0^\infty (M) \) and \( \{ X_j : 1 \leq j \leq d \} \subseteq \Gamma (TM) \) with compact supports. Consider their orthogonal lift on \( H_P (M) \) (referring to Theorem 4.0) as follows

\[
\tilde{\delta_x}^{(m)} := \hat{g}_0^{(m)} + \sum_{j=1}^d X_{P,j} \hat{g}_j^{(m)} \in C^\infty (H_P (M))
\]

where \( \hat{g} (\sigma) = g \circ E_1^P (\sigma) \) for any \( g \in C (M) \) and \( X_{P,i} \) is the orthogonal lift of \( X_i \) into \( \Gamma (TH_P (M)) \). For any \( 0 \leq j \leq d \) (with the convention that \( X_{P,0} = I \)), using integration by parts, we get:

\[
\int_{H_P (M)} \left( \hat{g}_0^{(m)} + \sum_{j=1}^d X_{P,j} \hat{g}_j^{(m)} \right) \, f \, d\nu_P = \int_{H_P (M)} \left( \hat{g}_0^{(m)} \cdot f + \sum_{j=1}^d X_{P,j} \hat{g}_j^{(m)} \cdot f \right) \, d\nu_P. \tag{6.14}
\]
Now let $m \to \infty$, from Corollary 3.32 we have:

$$\text{the left-hand side of (6.14)} = \int_{H_{P,x}(M)} f d\nu^1_{P,x}.$$  

Applying Lemma 6.6 to each pair $(\tilde{g}^{(m)}_j, X_{P,j})$, we have:

$$\text{the right-hand side of (6.14)} = \int_{H_{P}(M)} \left( \tilde{g}_0 \cdot f + \sum_{j=1}^{d} X_{P,j}^{tr,\nu} f \cdot \tilde{g}_j \right) d\nu. \quad (6.15)$$

Then let $|P| \to 0$, from Theorem 6.8 we have:

$$\lim_{|P| \to 0} \int_{H_{P,x}(M)} f d\nu^1_{P,x} = \int_{W_o(M)} \left( \tilde{g}_0 \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j \right) d\nu. \quad (6.16)$$

According to Lemma 6.4

$$\int_{W_o(M)} \left( \tilde{g}_0 \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j \right) d\nu = \lim_{m \to \infty} \int_{W_o(M)} \left( \tilde{g}_0^{(m)} \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j^{(m)} \right) d\nu. \quad (6.17)$$

Then using integration by parts formula developed in Theorem 4.19 we have:

$$\int_{W_o(M)} \left( \tilde{g}_0^{(m)} \cdot f + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} f \cdot \tilde{g}_j^{(m)} \right) d\nu = \int_{W_o(M)} \left( \tilde{g}_0^{(m)} + \sum_{j=1}^{d} \tilde{X}_j^{tr,\nu} \tilde{g}_j^{(m)} \right) \cdot f d\nu$$

$$= \int_{W_o(M)} \tilde{g}_x^{(m)} f d\nu. \quad (6.18)$$

Lastly, using Proposition 6.9 we have

$$\lim_{m \to \infty} \int_{W_o(M)} \tilde{g}_x^{(m)} f d\nu = \int_{W_o(M)} f d\nu_x. \quad (6.19)$$

Combining Eq. (6.16), (6.17) and (6.18) we have

$$\lim_{|P| \to 0} \int_{H_{P,x}(M)} f d\nu^1_{P,x} = \int_{W_o(M)} f d\nu_x.$$  

**A. ODE estimates**

**Lemma A.1** If $X$ is a normal random variable with mean 0 and variance $t$, then

$$\mathbb{E} \left[ e^{k|X|^2} \right] = \begin{cases} \infty & \text{if } k \geq \frac{1}{2t} \\ (1 - 2kt)^{-1/2} & \text{if } k < \frac{1}{2t} \end{cases}$$

**Proof.** The result follows from the direct computation below.

$$\mathbb{E} \left[ e^{k|X|^2} \right] = \int_{-\infty}^{\infty} e^{kx^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{k(x - \frac{1}{2t}x^2)} dx.$$  

If $k \geq \frac{1}{2t}$, then

$$\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{(k-\frac{1}{2t})x^2} dx \geq \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} dx = \infty.$$  

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If \( k < \frac{1}{2\pi} \), then
\[
\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{(k - \frac{x^2}{2})} dx = \frac{1}{\sqrt{2\pi (\frac{1}{2\pi} - k)}} \int_{-\infty}^{\infty} e^y dy = (1 - 2kt)^{-\frac{1}{2}}.
\]

Lemma A.2 Let \( \beta \) be a standard Brownian motion on \( \mathbb{R}^d \), \( \{s_i = \frac{i}{n}\}_{i=0}^n \) be an equally spaced partition of \([0, 1]\) and \( \Delta_i \beta \) be \( \beta_{s_i} - \beta_{s_{i-1}} \), then for any \( q > 0 \), we have
\[
\sup_{n:n > 2q} E \left[ e^{q \sum_{j=1}^n |\Delta_j \beta|^2} \right] < \infty. \tag{A.1}
\]

Proof. Since for each \( j \), \( |\Delta_j \beta|^2 = \sum_{l=1}^d |(\Delta_j \beta)_l|^2 \), where \( \{(\Delta_j \beta)_l\}_{l=1}^d \) are coordinates of \( \Delta_j \beta \), i.e. \( \Delta_j \beta = ((\Delta_j \beta)_1, \ldots, (\Delta_j \beta)_d) \). Since \( \beta \) is a Brownian motion on \( \mathbb{R}^d \), \( \{(\Delta_j \beta)_l\}_{l=1}^d \) are i.i.d with Gaussian distribution of mean 0 and variance \( \frac{1}{n} \). Using Lemma A.2 we have
\[
E \left[ e^{q \sum_{j=1}^n |\Delta_j \beta|^2} \right] = \prod_{j=1}^n \prod_{l=1}^d E \left[ e^{q |(\Delta_j \beta)_l|^2} \right] = \left( 1 - \frac{2q}{n} \right)^{-\frac{nd}{2}}.
\]
Then Eq. (A.1) follows since \( (1 - \frac{2q}{n})^{-\frac{nd}{2}} \) converges as \( n \to \infty \).

Proposition A.3 Consider an ODE:
\[
Y''(s) = A(s) Y(s)
\]
where \( Y(s), A(s) \in M_{n \times n}(\mathbb{R}) \) are real \( n \times n \) matrices and \( A(s) \) is positive semi-definite.
Denote by \( \{C(s), S(s)\} \) the solutions to this ODE with initial values:
\[
C(0) = I, C'(0) = 0 \text{ and } S(0) = 0, S'(0) = I
\]
Recall that in this paper we use \( \text{eig}(X) \) to denote the set of eigenvalues of matrix \( X \). Then
- If \( \lambda \in \text{eig}(C(s)) \), then \( |\lambda| \geq 1 \).
- If \( \lambda \in \text{eig}(S(s)) \), then \( |\lambda| \geq s \).

Proof. For all \( v \in \mathbb{C}^d \), define \( v(s) := C(s) v \), then:
\[
\langle v''(s), v(s) \rangle = \langle A(s) v(s), v(s) \rangle \geq 0.
\]
Therefore,
\[
\frac{d}{ds} \langle v'(s), v(s) \rangle = \langle v''(s), v(s) \rangle + \|v'(s)\|^2 \geq 0.
\]
Since \( \langle v'(0), v(0) \rangle = 0 \), so \( \langle v'(s), v(s) \rangle \geq 0 \). Therefore
\[
\frac{d}{ds} \|v(s)\|^2 = 2 \text{Re} \langle v'(s), v(s) \rangle \geq 0.
\]
Notice that \( \|v(0)\|^2 = \|v\|^2 \), so
\[
\|v(s)\|^2 \geq \|v\|^2.
\]
Therefore if \( \lambda \in \text{eig}(C(s)) \), choose \( v \in \mathbb{C}^d \) to be an eigenvector associated to \( \lambda \), then
\[
\|\lambda v\|^2 = \|C(s)v\|^2 \geq \|v\|^2.
\]
and $S$ the solutions to

From there it follows

Therefore

A lower bound result for $\|S(s)v\|$ can be found in Appendix E:

$$\|S(s)v\| \geq s \|v\|.$$  

From there it follows

If $\lambda \in \text{eig}(S(s))$, then $|\lambda| \geq s$

and $S(s)$ is invertible with

$$\|S(s)\| = \max_{\lambda \in \text{eig}(S(s))} |\lambda| \geq s.$$

Definition A.4 Fix $\xi \in \mathbb{R}^d$, $\sigma \in H(M)$, denote $R_u(s) (\xi, \cdot) \xi$ by $A_\xi(s)$, and let $C_\xi(s), S_\xi(s) \in M_{d \times d}$ be the solutions to $V''(s) = A_\xi(s)V(s)$ with initial values $C_\xi(0) = I, C'_\xi(0) = 0$ and $S_\xi(0) = 0, S'_\xi(0) = I$.

Proposition A.5 If $R$ is bounded by a constant $N$, i.e. $|R(\xi, \cdot)\xi| \leq N |\xi|^2$, then

$$|C_\xi(s)| \leq \cosh\left(\sqrt{N} |\xi| s\right) \leq e^{\frac{1}{2}N|\xi|^2s^2},$$  \hspace{1cm} (A.2)

$$|S_\xi(s)| \leq \sqrt{N} |\xi| s \frac{\sinh\left(\sqrt{N} |\xi| s\right)}{\sqrt{N} |\xi| s} \leq \cosh\left(\sqrt{N} |\xi| s\right) \sqrt{N} |\xi| s \leq \sqrt{N} |\xi| se^{\frac{1}{2}N|\xi|^2s^2},$$  \hspace{1cm} (A.3)

$$|S_\xi(s) - sI| \leq \frac{N |\xi|^2 s^3}{6} e^{\frac{1}{2}N|\xi|^2s^2},$$  \hspace{1cm} (A.4)

and

$$|C_\xi(s) - I| \leq \frac{N |\xi|^2 s^2}{2} e^{\frac{1}{2}N|\xi|^2s^2}.$$  \hspace{1cm} (A.5)

Proof. A.2 and A.3 are quite elementary, so here we only present the proof of A.4 and A.5.

By Taylor’s expansion,

$$S_\xi(s) = sI + \int_0^s R_u(\xi, S_\xi(r)) \xi (s - r) \, dr.$$  

$$|S_\xi(s) - sI| \leq N |\xi|^2 \int_0^s |S_\xi(r)| (s - r) \, dr \leq N |\xi|^2 \int_0^s ||S_\xi(r) - rI|| + r \, (s - r) \, dr$$

Define $f(s) := |S_\xi(s) - sI|$, then we have:

$$f(s) \leq \int_0^s N |\xi|^2 (s - r) f(r) \, dr + N |\xi|^2 s^3 \frac{3}{6}$$

By Gronwall’s inequality:

$$f(s) \leq N |\xi|^2 \frac{s^3}{6} e^{\frac{1}{2}N|\xi|^2s^2}$$

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Then we consider $C_\xi(s)$:

$$C_\xi(s) = I + \int_0^s R_{\alpha_\varepsilon}(\xi, C_\xi(r)) \xi(s - r) \, dr.$$  

So

$$|C_\xi(s) - I| \leq N|\xi|^2 \int_0^s |C_\xi(r)| (s - r) \, dr \leq N|\xi|^2 \int_0^s \|C_\xi(r) - I\| (s - r) \, dr.$$  

Define $f(s) := |C_\xi(s) - I|$, then we have:

$$f(s) \leq \int_0^s N|\xi|^2 (s - r) f(r) \, dr + N|\xi|^2 \frac{s^2}{2}.$$  

By Gronwall’s inequality:

$$f(s) \leq N|\xi|^2 \frac{s^2}{2} e^{N|\xi|^2 s^2}.$$  

B Matrix Analysis

**Theorem B.1** Suppose that $V$ is a finite dimensional inner product space, $A : V^n \to V$ is a linear map, and

$$S := \left[ \begin{array}{c} I_{V^n \times V^n} \\ A \end{array} \right] : V^n \to V^{n+1}.$$  

Then

$$\det(S^\text{tr}S) = \det[I_V + AA^\text{tr}].$$  

**Proof.** First observe that

$$S^\text{tr}S = \left[ \begin{array}{cc} I & A^\text{tr} \\ A & I \end{array} \right] = I + A^\text{tr}A.$$  

We denote $\dim(V) = d$ and let $\{u_j\}_{j=1}^d \subset V$ be an orthonormal basis of eigenvectors for $AA^\text{tr} : V \to V$ so that $AA^\text{tr}u_j = \lambda_j u_j$ and then let $v_j := A^\text{tr}u_j$. Then it follows that

$$A^\text{tr}Av_j = A^\text{tr}AA^\text{tr}u_j = A^\text{tr}\lambda_j u_j = \lambda_j A^\text{tr}u_j = \lambda_j v_j.$$  

Now extend $\{v_j\}_{j=1}^d$ to a basis for all $V^n$. From this we will find that $S^\text{tr}S$ has eigenvalues $\{1\} \cup \{1 + \lambda_j\}_{j=1}^d$ and therefore

$$\det(S^\text{tr}S) = \prod_{j=1}^d (1 + \lambda_j) = \det(I + AA^\text{tr}).$$  

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