A quantified Tauberian theorem for Laplace-Stieltjes transform

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Abstract. We prove a quantified Tauberian theorem involving Laplace-Stieltjes transform which is motivated by the work of Ingham and Karamata. For this, we consider functions which are locally of bounded variation and, therefore, get a generalisation of some results of Batty and Duyckaerts. We show that our theorem can be applied to special Dirichlet series.

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1. Introduction

Considering Tauberian theorems which involve Laplace-Stieltjes transform is a business with a history of over 100 years. In 1916 Riesz generalised his observations about Dirichlet series to Laplace-Stieltjes transforms of functions which are locally of bounded variation \cite{10}. Some years later his work was refined independently by Ingham \cite{5} and Karamata \cite{6} who stated the following Tauberian theorem.

**Theorem 1.1.** Let $X$ be a Banach space, $A : [0, \infty) \to X$ locally of bounded variation, $A(0) = 0$ and assume that there are $C' > 0$ and $x_0 > 0$ so that

$$\limsup_{t \to \infty} \left| e^{-x_0 t} \int_0^t e^{x_0 s} dA(s) \right| \leq C'.$$

Then $f(z) = \int_0^\infty e^{-zs} dA(s)$ is convergent for every $z \in \mathbb{C}$ with Re($z$) > 0. Suppose further, that for some $A_\infty \in X$ the function $z \mapsto \frac{f(z) - A_\infty}{z}$ admits a continuous extension to the closed half-plane $\{ z \in \mathbb{C} \mid \text{Re}(z) \geq 0 \}$. Then

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\[ \limsup_{t \to \infty} \| A(t) - A_\infty \| \leq 2C'. \]

Actually, Ingham and Karamata showed this only for scalar-valued \( A \) but there is no difficulty to adapt the proof for the vector-valued case.

In his approach to the Prime Number Theorem, Newman gave a new proof of Riesz’ Tauberian theorem for Dirichlet series \[9\]. This proof was adapted to Laplace transforms of bounded measurable functions by Korevaar \[7\] and Zagier \[11\] who obtained special cases of the Ingham-Karamata-theorem. For an overview of the development of Tauberian theory see \[8\].

The Newman-Korevaar-Zagier technique helped to state new kinds of results in the theory of stability of operator semigroups, see for example \[1\]. Recent results in stability theory gave not only conditions for stability but stated convergence rates for semigroups \((T(t))_{t \geq 0}\) and, analogously, for bounded measurable functions \( f : [0, \infty) \to X, \) where \( X \) is a Banach space, for large times, see \[2,3\].

In this paper we combine the ideas of Ingham and Karamata with those of Batty and Duyckaerts. Therefore, we get both a quantitative version of Theorem 1.1 and a generalization of \[2, \text{Theorem 4.1.}\] in the case \( k = 1 \). In fact, we show the following.

**Theorem 1.2.** Take the same assumptions as in Theorem 1.1. In addition, let \( M : [0, \infty) \to [1, \infty) \) be a continuous, increasing function and \( R : [0, \infty) \to [1, \infty] \) an increasing function. Assume that there exist \( C > 0, T \geq 0 \) so that

\[
\sup_{t > T} \sup_{x_0 \leq x \leq R(t)} \left| xe^{-xt} \int_0^t e^{xs} dA(s) \right| \leq C. \tag{1.1}
\]

Suppose further that

\[ f \text{ has an analytic extension into the region } Q := \left\{ z \in \mathbb{C} \mid 0 \geq x > -\frac{1}{M(|y|)} \right\} \tag{1.2} \]

and that

\[ ||f(z)|| \leq M(|y|) \text{ holds throughout } Q, \text{ where } z = x + iy. \tag{1.3} \]

Then there exist \( K > 0, T' \geq 0 \) so that

\[ ||A(t) - f(0)|| \leq K \max \left\{ \frac{1}{M_{\log}^{-1} \left( \frac{t}{T'} \right)}, \frac{1}{R(t)} \right\} \]

for every \( t > T' \), where \( M_{\log}^{-1} \) is the inverse of the function \( M_{\log} \) defined by \( M_{\log}(a) = M(a)(\log a + \log M(a) - \frac{1}{2}\log(5C)) \) for \( a \geq 1 \).

If \( R(t) \), for increasing \( t \), is growing quickly enough then we get exactly the same rate as in \[2, 3\] – see Remark 4.1 below – but for a wider class of
functions, namely functions which are locally of bounded variation. Regarding
the assumptions we remark the following.

(i) In addition to Ingham and Karamata, we assume the Tauberian condi-
tion \( \{1.1\} \). There is a function \( A \) such that this condition is not true for
\( T = 0 \); see Remark 2.4

(ii) The continuation property (1.2) and the growth condition (1.3) are as
in [2]. They ensure that we get a quantitative result.

In the following section we give three useful lemmas for the proof of Theorem
1.2 which we present in Section 3. Subsequently, we show that Theorem 1.2
includes the result from [2] and can be applied to Dirichlet series
\( f(z) = \sum_{n=1}^{\infty} b_n n^{-z} \) with a bounded sequence of coefficients \( (b_n)_{n \in \mathbb{N}} \). For the rest of
the article, we define \( H := \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \) and \( \mathbb{R}_+ := [0, \infty) \).

2. Preliminaries

In this section we prove three lemmas which will be applied in the proof of
Theorem 1.2. In all of them we deduce different conclusions from the same
condition, namely a condition similar to (1.1).

Lemma 2.1. Let \( X \) be a Banach space, \( A : \mathbb{R}_+ \to X \) locally of bounded
variation, \( A(0) = 0 \) and suppose that

\[
\sup_{t \in \mathbb{R}_+} \left\| e^{-xt} \int_0^t e^{xs} \, dA(s) \right\| \leq C
\]

for some \( x > 0 \) and some \( C > 0 \). Let \( z = x + iy, y \in \mathbb{R} \). Then

\[
\sup_{t \in \mathbb{R}_+} \left\| e^{-xt} \int_0^t e^{ys} \, dA(s) \right\| \leq C \left( 1 + \frac{|y|}{x} \right).
\]

Proof. First, define \( G : \mathbb{R}_+ \to X \) by \( G(s) = \int_0^s e^{xr} \, dA(r), s \in \mathbb{R}_+ \). Then we
have by properties of the Riemann-Stieltjes integral

\[
\left\| e^{-xt} \int_0^t e^{ys} \, dA(s) \right\| = \left\| e^{-xt} \int_0^t e^{xs} e^{ys} \, dA(s) \right\| = \left\| e^{-xt} \int_0^t e^{ys} \, dG(s) \right\|.
\]

Integration by parts (cf. [4] p.63) and suitable estimates yield

\[
\left\| e^{-xt} \int_0^t e^{ys} \, dG(s) \right\| = \left\| e^{-xt} \left( [e^{ys} G(s)]_0^t - iy \int_0^t e^{ys} G(s) \, ds \right) \right\|
\leq \left\| e^{-xt} e^{yt} \int_0^t e^{xr} \, dA(r) \right\|
+ |y| \left\| e^{-xt} \int_0^t e^{ys} e^{xs} \left( \int_0^s e^{xr} \, dA(r) \right) \, ds \right\|
\leq C + |y| e^{-xt} \int_0^t |e^{ys}| |e^{xs}| \left\| e^{-xs} \int_0^s e^{xr} \, dA(r) \right\| \, ds
\]
\[ \leq C + C|y|e^{-xt} \int_0^t e^{xs} \, ds \]
\[ = C \left( 1 + \frac{|y|}{x} (1 - e^{-xt}) \right) \leq C \left( 1 + \frac{|y|}{x} \right). \]

As this is true for every \( t \in \mathbb{R}_+ \), we proved the claim. \( \square \)

**Lemma 2.2.** Let \( X \) be a Banach space, \( A : \mathbb{R}_+ \to X \) locally of bounded variation, \( A(0) = 0 \) and suppose that

\[ \sup_{t \in \mathbb{R}_+} \left| \int_0^t e^{xs} \, dA(s) \right| \leq C \]

for some \( x > 0 \) and some \( C > 0 \). Let \( z = x + iy \), \( y \in \mathbb{R} \). Then

\[ \sup_{t \in \mathbb{R}_+} \left| \int_t^\infty e^{-zs} \, dA(s) \right| \leq C \left( 3 + \frac{|y|}{x} \right). \quad (2.1) \]

**Proof.** Again we consider the function \( G \) given by \( G(s) = \int_0^s e^{yr} \, dA(r) \), \( s \in \mathbb{R}_+ \). First, we show that the integral in (2.1) exists. We have

\[ \left| \int_t^v e^{-zs} \, dA(s) \right| = \left| \int_t^v e^{-isy} e^{-xs} e^{xs} \, dA(s) \right| \]
\[ = \left| \int_t^v e^{-isy} e^{-xs} \, dG(s) \right|. \]

Integration by parts and suitable estimates yield

\[ \left| \int_t^v e^{-isy} e^{-xs} \, dG(s) \right| = \left| \left[ e^{-isy} e^{-xs} G(s) \right]_t^v + (2x + iy) \int_t^v e^{-isy} e^{-xs} G(s) \, ds \right| \]
\[ \leq \left| e^{-isy} e^{-xs} \int_t^v e^{yr} \, dA(r) - e^{-isy} e^{-xs} \int_0^t e^{yr} \, dA(r) \right| + 2x \int_t^v e^{-xs} \left| e^{-xs} \int_0^s e^{yr} \, dA(r) \right| \, ds \]
\[ + |y| \int_t^v e^{-xs} \left| e^{-xs} \int_0^s e^{yr} \, dA(r) \right| \, ds \]
\[ \leq C e^{-vx} + Ce^{-xt} + 2x C \int_t^v e^{-xs} \, ds + C|y| \int_t^v e^{-xs} \, ds \]
\[ = C(e^{-vx} + e^{-xt}) + 2C(e^{-vx} - e^{-xt}) \]
\[ - C \frac{|y|}{x} (e^{-vx} - e^{-xt}) \]

which converges to 0 for \( v, t \to \infty \) and, therefore, the improper integral exists. For the estimate (2.1) we write

\[ \left| \int_t^\infty e^{-zs} \, dA(s) \right| = \lim_{v \to \infty} \left| \int_t^v e^{-isy} e^{-2xs} \, dG(s) \right|. \]
The above estimate gives, for every \( t \in \mathbb{R}_+ \),

\[
\lim_{v \to \infty} \left| e^{xt} \int_t^v e^{-ys} e^{-2xs} dG(s) \right| \leq \lim_{v \to \infty} \left[ C(e^{x(t-v)} + 1) - 2C(e^{x(t-v)} - 1) - C\frac{|y|}{x}(e^{x(t-v)} - 1) \right] \\
= C \left( 3 + \frac{|y|}{x} \right).
\]

\[\square\]

**Lemma 2.3.** Let \( X \) be a Banach space, \( A : \mathbb{R}_+ \to X \) locally of bounded variation, \( A(0) = 0 \) and suppose that

\[
\sup_{t \in \mathbb{R}_+} \left| e^{-xt} \int_0^t e^{xs} dA(s) \right| \leq C
\]

for some \( x_0 > 0 \) and some \( C > 0 \). Then

\[
\sup_{t \in \mathbb{R}_+} \left| e^{-xt} \int_0^t e^{xs} dA(s) \right| \leq \frac{Cx_0}{x}
\]

for every \( x \) with \( 0 < x \leq x_0 \).

**Proof.** By using \( G(s) = \int_0^s e^{x_0r} dA(r), s \in \mathbb{R}_+ \) we have

\[
\left| \int_0^t e^{xs} dA(s) \right| = \left| \int_0^t e^{xs} e^{-x_0s} e^{x_0s} dA(s) \right| \\
= \left| \int_0^t e^{s(x-x_0)} dG(s) \right|.
\]

Remember that \( ||G(s)|| \leq Ce^{x_0s} \) according to the assumptions. We integrate by parts and estimate:

\[
\left| \int_0^t e^{s(x-x_0)} dG(s) \right| = \left| \left[ e^{s(x-x_0)} G(s) \right]_0^t - (x-x_0) \int_0^t e^{s(x-x_0)} G(s) ds \right| \\
\leq \left| e^{t(x-x_0)} G(t) \right| + |x-x_0| \int_0^t e^{xs} e^{-x_0s} ||G(s)|| ds \\
\leq Ce^{xt} + C|x-x_0| \int_0^t e^{xs} ds \\
= Ce^{xt} + C\frac{|x-x_0|}{x} (e^{xt} - 1) \\
\leq \left( 1 + \frac{|x-x_0|}{x} \right) Ce^{xt}. \tag{2.2}
\]

The coefficient can be simplified to
and we get the result. □

**Remark 2.4.** For proving Theorem 1.2 it would be nice to extend the result of Lemma 2.3 to all \( x \in (0, R(t)) \). In fact, this is not possible for all \( t \in \mathbb{R}_+ \).

For example, fix \( T > 0 \) and consider

\[
A : \mathbb{R}_+ \to \{0, 1\} \quad \text{with} \quad A(t) = \begin{cases} 0, & 0 \leq t \leq T, \\ 1, & t > T. \end{cases}
\]

Then \( A \) is of bounded variation and \( A(0) = 0 \). Let \( x > 0 \). Define \( g_x : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
g_x(t) := e^{-xt} \int_0^t e^{xt} dA(t) = \begin{cases} 0, & 0 \leq t \leq T, \\ e^{x(T-t)}, & t > T. \end{cases}
\]

Now choose \( x_0 = 1 \) and set \( C := 1 = \sup_{t \in \mathbb{R}_+} |g_{x_0}(t)| \). Because \( \sup_{t \in \mathbb{R}_+} |g_x(t)| = 1 \) holds for every \( x > 0 \) we have

\[
\sup_{t \in \mathbb{R}_+} |g_x(t)| > \frac{1}{x} = \frac{Cx_0}{x}
\]

for every \( x > x_0 \). But notice that there exists \( T' > T \) so that

\[
\sup_{t > T'} |g_x(t)| < \frac{1}{x} = \frac{Cx_0}{x}
\]

is true for all \( x > 0 \) and, in particular, for all \( x \in (0, R(t)) \).

**3. Proof of Theorem 1.2**

First, we show that the condition

\[
\sup_{t \in \mathbb{R}_+} \left\| e^{-x_0 t} \int_0^t e^{x_0 s} dA(s) \right\| \leq C
\]

for some \( x_0 > 0 \) and some \( C > 0 \) is sufficient for the existence of the Laplace-Stieltjes transform \( f(z) = \int_0^\infty e^{-zs}dA(s) \) of \( A \) for every \( z \in H \). Using inequality (2.2) in the proof of Lemma 2.3 we conclude that (3.1) is valid for every \( x > 0 \). By Lemma 2.2 we get that \( \int_0^\infty e^{-zs}dA(s) \) exists for every \( z = x + iy \) with \( x > 0 \) and \( y \in \mathbb{R} \). Therefore, we proved the above claim.

For proving the quantitative statement we use the notation

\[
f_t(z) = \int_0^t e^{-zs}dA(s)
\]

so that

\[
f_t(0) = \int_0^t dA(s) = A(t)
\]
and consider now the behaviour of $||f_t(0) - f(0)||$ for large $t$. Fix $t > T$ and consider $R \in [1, R(t)]$. We form a contour $\Gamma$ which consists of two parts: $\Gamma_1$ is the arc $\{ z \in \mathbb{C} ||z| = R, \text{Re}(z) \geq 0 \}$ in the closed right half-plane. $\Gamma_2$ consists of the three segments $[iR, -\frac{1}{2M(R)} + iR], [-\frac{1}{2M(R)} + iR, -\frac{1}{2M(R)} - iR]$ and $[-\frac{1}{2M(R)} - iR, -iR]$. Therefore, $\Gamma_2$ is contained in $Q$. By Cauchy’s integral formula we get

$$||f_t(0) - f(0)|| = \left| \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_t(z) - f(z)}{z} e^{tz} \left( 1 + \frac{z^2}{R^2} \right)^2 \, dz \right|. \quad (3.2)$$

As $f_t$ is an entire function we can replace the integral $\int_{\Gamma_2} f_t(z) e^{tz} \left( 1 + \frac{z^2}{R^2} \right)^2 \, dz$ by an integral over $\tilde{\Gamma}_1 = \{ z \in \mathbb{C} ||z| = R, \text{Re}(z) < 0 \}$, which is the reflection of $\Gamma_1$ through the origin. Let us split the integral in (3.2) into three parts:

$$||f_t(0) - f(0)|| \leq \left| \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f_t(z) - f(z)}{z} e^{tz} \left( 1 + \frac{z^2}{R^2} \right)^2 \, dz \right| + \left| \frac{1}{2\pi i} \int_{\tilde{\Gamma}_1} \frac{f_t(z)}{z} e^{tz} \left( 1 + \frac{z^2}{R^2} \right)^2 \, dz \right| + \left| \frac{1}{2\pi i} \int_{\Gamma_2} f(z) e^{tz} \left( 1 + \frac{z^2}{R^2} \right)^2 \, dz \right| =: I + II + III$$

Now, we estimate every single integral. For that we use

$$\left| 1 + \frac{z^2}{R^2} \right| = \frac{2|x|}{R}, \quad \frac{1}{|z|} = \frac{1}{R}, \quad |y| \leq R$$

on the circle $|z| = R$, with $z = x + iy$. By Lemma 2.3 we know that

$$\sup_{t>T} \left| e^{-xt} \int_0^t e^{xs} dA(s) \right| \leq \frac{C}{x_0} \frac{x_0}{x} = \frac{C}{x}$$

for every $x$ with $0 < x < x_0$. So all estimations do not depend on whether $x$ is smaller than $x_0$ or fulfills $x_0 \leq x \leq R$. Then $I$ can be estimated by (remember $z = x + iy$ and notice that $x \geq 0$ on $\Gamma_1$)

$$I \leq \frac{1}{2\pi} \int_{\Gamma_1} \left| \int_0^t (f_t(z) - f(z)) e^{tx} \frac{4x^2}{R^2} \, dz \right| \leq \frac{1}{2\pi} \int_{\Gamma_1} \left| \int_0^\infty e^{tx} \int_t^\infty e^{-zs} dA(s) \right| \frac{4x^2}{R^2} \, dz \leq \frac{1}{2\pi} \int_{\Gamma_1} \frac{C}{x} \left( 3 + \frac{|y|}{x} \right) \frac{4x^2}{R^2} \, dz$$
\[
\begin{align*}
\leq & \frac{6C}{\pi R^3} \int_{\Gamma_1} x \, |dz| + \frac{2C}{\pi R^2} \int_{\Gamma_1} |dz| \\
= & \frac{12C}{\pi R} + \frac{2C}{R} \leq \frac{6C}{R},
\end{align*}
\]
where we used Lemma 2.2. For the estimation of \(I\) we assume \(x < 0\) and define \(\tilde{z} := -z = -x - iy\), with \(y \in \mathbb{R}\) and \(\tilde{z}\) lies in the right half-plane. Now, similar considerations give

\[
\begin{align*}
II \leq & \frac{1}{2\pi} \int_{\tilde{\Gamma}_1} ||f_t(z)||e^{tx} \cdot \frac{4(-x)^2}{R^3} |dz| \\
= & \frac{1}{2\pi} \int_{\tilde{\Gamma}_1} \left| e^{tx} \int_0^t e^{\tilde{z}s} dA(s) \right| \frac{4(-x)^2}{R^3} |dz| \\
\leq & \frac{1}{2\pi} \int_{\tilde{\Gamma}_1} C \left( 1 + \frac{|y|}{-x} \right) \frac{4(-x)^2}{R^3} |dz| \\
\leq & \frac{2C}{\pi R^3} \int_{\tilde{\Gamma}_1} (-x) |dz| + \frac{2C}{\pi R^2} \int_{\tilde{\Gamma}_1} |dz| \\
= & \frac{4C}{\pi R} + \frac{2C}{R} \leq \frac{4C}{R},
\end{align*}
\]
where we used Lemma 2.1. Finally, we consider \(III\). By assumption, \(||f(z)||\) is less than or equal to \(M(|y|)\) for every \(z = x + iy\) on the path of integration. Along the segments \([iR, -\frac{1}{2M(R)} + iR]\) and \([-\frac{1}{2M(R)} - iR, -iR]\) we have (remember \(R \geq 1, M(R) \geq 1\))

\[
\left| 1 + \frac{z^2}{R^2} \right| \leq \frac{\sqrt{2}}{R} \quad \text{and} \quad \frac{1}{|z|} \leq \frac{1}{R}.
\]
For the segment from \(-\frac{1}{2M(R)} + iR\) to \(-\frac{1}{2M(R)} - iR\) we can estimate

\[
\left| 1 + \frac{z^2}{R^2} \right| \leq \sqrt{2} \quad \text{and} \quad \frac{1}{|z|} \leq 2M(R).
\]
Therefore, we get

\[
\begin{align*}
III \leq & \frac{1}{2\pi} \left( 2 \int_{-\frac{1}{2M(R)}}^{0} M(R) e^{tx} \cdot \frac{1}{R} \cdot \frac{2}{R^2} \, dx \right) \\
+ & \frac{1}{2\pi} \int_{-R}^{0} M(R) e^{-\frac{t}{2M(R)}} \cdot 2M(R) \cdot 2 \, dy.
\end{align*}
\]
Since

\[
\int_{-\frac{1}{2M(R)}}^{0} e^{tx} \, dx = \frac{1}{t} - \frac{1}{t} e^{-\frac{t}{2M(R)}} \leq \frac{1}{t},
\]
we conclude
III \leq \frac{M(R)}{tR^3} + 2R(M(R))^2e^{-\frac{t}{2M(R)}}.

If we summarize all estimates we have

\[ ||A(t) - f(0)|| \leq \frac{10C}{R} + \frac{M(R)}{tR^3} + 2R(M(R))^2e^{-\frac{t}{2M(R)}}. \quad (3.3) \]

Now, we optimize this estimate over \( R \) by equating the first and the third term of the right-hand side:

\[
\frac{10C}{R_{\text{opt}}} = 2R_{\text{opt}}(M(R_{\text{opt}}))^2e^{-\frac{t}{2M(R_{\text{opt}})}}.
\]

Therefore, we get

\[ t = 4M(R_{\text{opt}}) \left( \log R_{\text{opt}} + \log M(R_{\text{opt}}) - \frac{1}{2} \log(5C) \right), \quad (3.4) \]

that is

\[ R_{\text{opt}} = M^{-1}_\log \left( \frac{t}{4} \right), \]

where \( M^{-1}_\log \) is the inverse of the function on the right-hand side of (3.4), i.e. \( M_{\log}(\cdot) = M(\cdot)(\log \cdot + \log M(\cdot) - \frac{1}{2} \log(5C)) \).

Since \( R \geq 1 \) by assumption, we have \( t \geq 4M(1) \left( \log M(1) - \frac{1}{2} \log(5C) \right) \). So we define

\[ T' := \max \left\{ T, 4M(1) \left( \log M(1) - \frac{1}{2} \log(5C) \right) \right\}. \]

If we insert \( t \) into the middle term of the sum in (3.3) it follows

\[
\frac{M(R_{\text{opt}})}{tR^3_{\text{opt}}} \leq \frac{1}{\log M(1) - \log(\sqrt{5C})} \cdot \frac{1}{R^3_{\text{opt}}} \leq \frac{K'}{R_{\text{opt}}} = \frac{K'}{M^{-1}_\log \left( \frac{t}{4} \right)},
\]

where \( K' := (\log M(1) - \log(\sqrt{5C}))^{-1} \). Finally, we check if \( R_{\text{opt}} \in [1, R(t)] \) is true. If \( R_{\text{opt}} \in [1, R(t)] \), then

\[ ||A(t) - f(0)|| \leq \frac{20C}{R_{\text{opt}}} + \frac{K'}{R_{\text{opt}}} \leq \frac{K}{M^{-1}_\log \left( \frac{t}{4} \right)} \quad (3.5) \]

for every \( t > T' \), with a suitable \( K > 0 \). Otherwise, we have

\[ \frac{1}{M^{-1}_\log \left( \frac{t}{4} \right)} = \frac{1}{R_{\text{opt}}} \leq \frac{1}{R(t)}, \]

so that

\[ ||A(t) - f(0)|| \leq \frac{20C}{R_{\text{opt}}} + \frac{K'}{R_{\text{opt}}} \leq \frac{K}{R(t)} \quad (3.6) \]
for every $t > T'$, with a suitable $K > 0$. Combining (3.5) and (3.6) gives the result of Theorem 1.2.

**Remark 3.1.** The integrand in (3.2) is multiplied by the terms $\left(1 + \frac{z^2}{R^2}\right)^2$, which is the so-called ”fudge factor”, and $e^{tx}$. Both do not change the value of the contour integral but help to estimate the integral, see [7]. This idea is due to Newman [9].

### 4. Different Remarks

In this section we show that Theorem 1.2 includes the results from [2] and can be applied to Dirichlet series with bounded coefficients.

**Remark 4.1.** Let $X$ be a Banach space and $f : \mathbb{R}_+ \to X$ a bounded measurable function. We know that $\hat{f}(z) = \int_0^\infty e^{zs} f(s) ds$ exists for every $z \in H$ and that $f \in L^1_{loc}(\mathbb{R}_+; X)$. Furthermore, the function $A : \mathbb{R}_+ \to X$ with $A(t) := \int_0^t f(s) ds$ is locally of bounded variation and differentiable almost everywhere. So $A$ is an antiderivative of $f$ with $A(0) = 0$. Denote by $C$ the bound of $\|f(s)\|$. Then

$$\left\| e^{-xt} \int_0^t e^{zs} dA(s) \right\| = \left\| e^{-xt} \int_0^t e^{zs} f(s) ds \right\|$$

$$\leq e^{-xt} \int_0^t e^{zs} \|f(s)\| ds$$

$$\leq C e^{-xt} \cdot \frac{1}{x} (e^{xt} - 1)$$

$$\leq \frac{C}{x},$$

for every $t \in \mathbb{R}_+$ and every $x > 0$. Choosing $R(t) = \infty$ for every $t > 0$ we see that condition (4.1) is satisfied. For this choice of $R(t)$, the condition $R_{opt} \in [1, R(t)]$ is always true.

Further, if we make the same assumptions as in Section 4 of [2], all conditions of our Theorem 1.2 are fulfilled. So we can conclude that there exist $T' \geq 0, K > 0$ so that

$$\left\| \int_0^t f(s) ds - f(0) \right\| \leq K \max \left\{ \frac{1}{M_{\log}^{-1} \left( \frac{4}{7} \right)}, \frac{1}{R(t)} \right\}$$

(4.1)

for every $t > T'$, where $M_{\log}^{-1}$ is the inverse of the function $M_{\log}$ defined by $M_{\log}(a) = M(a)(\log a + \log M(a) - \frac{1}{7} \log(5C))$ for $a \geq 1$.

Note that by the above choice of $R(t)$ the maximum in (4.1) is equal to $(M_{\log}^{-1} \left( \frac{4}{7} \right))^{-1}$ for every $t > T'$. In this way we recover [2, Theorem 4.1.] in the case $k = 1$. 
Remark 4.2. We show that condition (1.1) of Theorem 1.2 is automatically true for special Dirichlet series.

Let $X$ be a Banach space and $(b_n)_{n \in \mathbb{N}} \in l^\infty(X)$. Set $D := \max\{\|b\|, 1\}$. Define a sequence $(a_n)_{n \in \mathbb{N}}$ by $a_n := \frac{b_n}{n}$ for every $n \in \mathbb{N}$. Consider the Dirichlet series

$$f(z) = \sum_{n=1}^\infty \frac{a_n}{n^z} = \sum_{n=1}^\infty \frac{b_n}{n^{z+1}},$$

which is analytic in $H$. Furthermore, we define $A : \mathbb{R}_+ \to X$ by $A(s) := \sum_{\log n < s} a_n$ so that $A$ is locally of bounded variation, continuous from the left and $A(0) = 0$. For $t > 0$ and $x > 0$ we get

$$\left\| e^{-xt} \int_0^t e^{xs} dA(s) \right\| = \left\| e^{-xt} \sum_{\log n < t} e^{x \log n} a_n \right\|$$

$$= \left\| e^{-xt} \sum_{\log n < t} n^x a_n \right\|$$

$$\leq D e^{-xt} \sum_{\log n < t} n^{x-1}$$

$$\leq D e^{-xt} \sum_{n=1}^{[e^t]} n^{x-1}.$$

Further estimates yield

$$e^{-xt} \sum_{n=1}^{[e^t]} n^{x-1} \leq e^{-xt} \int_0^{e^t} (s + 1)^{x-1} ds$$

$$\leq \frac{1}{x} e^{-xt} (e^t + 1)^x$$

$$= \frac{1}{x} (1 + e^{-t})^x,$$

which is bounded by $e \cdot x^{-1}$ for every $t \in \mathbb{R}_+$ and every $x$ with $0 < x \leq e^t$. Define $R(t) := e^t$ for every $t > 0$. It follows that $R(t) > 1$. Defining $C := D \cdot e$ we conclude

$$\left\| e^{-xt} \int_0^t e^{xs} dA(s) \right\| \leq \frac{C}{x}$$

for every $t > 0$ and $x \in (0, e^t]$. Therefore, we state the following corollary of Theorem 1.2.
Corollary 4.3. Let $X$ be a Banach space and $(b_n)_{n \in \mathbb{N}} \in l^\infty(X)$. Define $a_n := \frac{b_n}{n}$ for every $n \in \mathbb{N}$. Then the Dirichlet series

$$f(z) = \sum_{n=1}^\infty \frac{a_n}{n^z}$$

is analytic in $\{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$. Let $M : [0, \infty) \to [1, \infty)$ be a continuous, increasing function and define $R : [0, \infty) \to [1, \infty)$ by $R(t) := e^t$. Assume that $f$ has an analytic extension into the region

$$Q := \left\{ z \in \mathbb{C} \mid 0 \geq x > -\frac{1}{M(|y|)} \right\}$$

so that $\|f(z)\| \leq M(|y|)$ holds throughout $Q$, where $z = x + iy$. Then there exist $K > 0, T' \geq 0$ so that

$$\left\| \sum_{\log n < t} a_n - f(0) \right\| \leq K \max \left\{ \frac{1}{M_{\log^{-1}(\frac{t}{4})}}, \frac{1}{R(t)} \right\}$$

for every $t > T'$, where $M_{\log}^{-1}$ is the inverse of the function $M_{\log}$ defined by $M_{\log}(a) = M(a)(\log a + \log M(a) - \frac{1}{2}\log(5C))$ for $a \geq 1$.

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