Kähler Calculus: Idempotents for Solutions with Symmetry of Exterior Systems in Metric Spaces

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To the memory of Erich Kähler

Abstract. Symmetry in differential equations constitutes a significant starting point for theoretical arguments. With his Clifford algebra of differential forms, Kähler addresses the overlooked manifestation of symmetry in the solutions themselves of such equations. In “Kähler’s algebra”, solutions with a given symmetry are members of left ideals generated by corresponding idempotents. These combine with phase factors to take care of all the dependence of the solution on $x^i$ and $dx^i$ for each $\partial/\partial x^i = 0$ symmetry. The maximum number of idempotents (each for a one-parameter group) that can, therefore, go into a solution is the dimensionality $n$ of the space. This number is further limited by non-commutativity of idempotents.

We consider the tensor product of Kähler algebra with tangent Clifford algebra, i.e. of valuedness. It has a commutative subalgebra of “mirror elements”, i.e. where the valuedness is dual to the differential form (like in $dx^i$, but not in $dx^j$). It removes the aforementioned limitation in the number of idempotent factors that can simultaneously go into a solution.

In the Kähler calculus, which is based on the Kähler algebra, total angular momentum is linear in its components, even it is not valued in tangent algebra. In particular, it is not a vector operator.

We carry Kähler’s treatment over to the aforementioned commutative algebra. Non-commutativity thus ceases to be a limiting factor in the formation of products of idempotents representing each a one-parameter symmetry. The interplay of factors that emerges in the expanded set of such products is inimical to operator theory. For practical reasons, we stop at products involving three one-parameter idempotents —henceforth call ternary— even though the dimensionality of the all important spacetime manifold allows for four of them.
1 Introduction

Erich Kähler proposed a treatment of symmetry of solutions of exterior systems of differential equations which goes beyond the traditional approach (Any system of differential equations, ordinary or partial, can be represented as an exterior system). Phase factors are just not enough to deal with this issue. Products of idempotents, each of them associated with a corresponding one-parameter group of symmetries, are required. That is a deeper treatment of symmetry than through the commutativity or not of operators that reflect those symmetries, translational and rotational ones in Euclidean spaces.

In section 4, we shall consider Clifford-valued Kähler differential forms. We keep the terminology “Kähler” for whenever the differential forms have a Clifford algebra structure, regardless of their valuedness. In other words, we mean “Kähler” whenever

\[ dx^i dx^j + dx^j dx^i = 2 \delta_{ij} \text{ or } 2 \eta_{ij}, \]

is satisfied.

We are interested in valuedness on the tangent Clifford algebra, which is defined by the relation

\[ a_i a_j + a_j a_i = 2 \delta_{ij} \text{ or } 2 \eta_{ij}, \]

where \( \eta_{ij} \) is constituted by ones and minus ones as dictated by the signature of the metric of the space considered. We later generalize Kähler’s work to apply to Clifford-valued Kähler differential forms, meaning Clifford-valued clifforms. We shall not, however, use the term clifform, since we shall no be interested in differential forms which are not so.

Of special interest because of its richness is Kähler’s study of angular momentum and of total angular momentum on idempotents with his algebra of scalar-valued differential forms. Kähler dealt at length also with tensor-valued differential forms, but it did not have consequence for his treatment of solutions of exterior systems.

2 Kähler’s treatment of translation and rotation symmetry

We have brought into this section pertinent results of Kähler’s calculus (KC) of scalar-valued differential forms [1], [2] [3]. The last of these publications
is a more comprehensive treatment of the first one. Nevertheless, some important results are exclusive to just one of those publications, like his treatment of Lie differentiation [1], his treatment of symmetry in solutions of differential systems [2], and his treatment of total angular momentum [3] (in the sense of linearly combining into one expression all three components of angular momentum). All of these three specific subjects will be reproduced in this section.

2.1 Kaehler’s monary and binary solutions with symmetry

Monary idempotents are those belonging to a one-parameter symmetry. By the term binary idempotents, we mean the product of two monary idempotents. A Kähler binary idempotent is constituted by the product of the idempotents for third component of angular momentum and for time translation symmetry. With a view to future physical applications, those idempotents will also be our starting point for the construction of ternary idempotents, i.e. constituted by the product of three monary idempotents. In the all too important dimension four, it will not make much of a difference in which order we approach the three factors. This will become retrospectively obvious.

There are two main types of symmetry idempotents, translational and rotational. Of special interest is the interplay between the two types and also the linear coexistence of different one-parameter rotational idempotents within a given equation. Such a coexistence happens in Kähler’s, where total angular momentum, though linear in its components (in a generalized sense of the term) is not a vector operator.

Except for notational changes, Kähler defines the idempotents

$$\varepsilon^\pm \equiv \frac{1}{2} (1 \mp i \, dt), \quad I_{xy}^{\pm} \equiv \frac{1}{2} (1 \pm i \, dx \, dy),$$

with $$(dx^i)^2 = 1 = -dt^2$$ and $c = 1$. The inversion of the signs in the definition of $\varepsilon^\pm$ has to do with his use of signature. The unit imaginary is there for the purpose of making idempotents. The squares of $i \, dt$ and $i \, dx \, dy$ are +1, as required. The squares of $i \, dt$ and $i \, dx \, dy$ are not.

The $\varepsilon^\pm$ commute with the $I_{xy}^{\pm}$. We also have

$$\varepsilon^+ + \varepsilon^- = 1 = I_{xy}^+ + I_{xy}^-, \quad \varepsilon^+ \varepsilon^\mp = I_{xy}^+ I_{xy}^- = 0.$$

From these equations follows that the statement

$$u = u^+ \varepsilon^+ + u^- \varepsilon^-$$
defines $dt$-independent spatial differentials $u^\pm$. Given a spatial differential, $u$, the statement
\[ u = +u \ I_{xy}^+ + -u \ I_{xy}^- \] (6)
similarly defines so called meridian differentials $\pm u$, i.e. that they do not depend on $d\phi$ when written in terms of the basis $(d\rho, d\phi, dz)$. The coefficients in (5)-(6) may still depend on all coordinates.

Combining these definitions and observations, we have
\[ 1 \equiv \epsilon^+ I_{xy}^+ + \epsilon^- I_{xy}^- + \epsilon^+ I_{xy}^- + \epsilon^- I_{xy}^+ \] (7)
and, therefore,
\[ u \equiv u^+ I_{xy}^+ + u^- I_{xy}^- + u^+ I_{xy}^- + u^- I_{xy}^+ \] (8)
The set of the four idempotents $\epsilon^\pm I_{xy}^\pm$ spans the same 4-dimensional module over the complex numbers as the set $(1, dt, dx dy, dt dx dy)$ does. The asterisk means that we may choose for $I_{xy}$ the superscripts $+$ and $-$ independently of the specific superscript in $\epsilon$.

2.2 Of Kaehler’s idempotents and commutativity

Idempotents like $\epsilon^\pm$ require only a 1-dimensional space. We also need just a one-dimensional space for space translation idempotents. $I_{xy}^\pm$ requires at least but not necessarily more than a 2-dimensional space, where we also have $(1/2)(1 \pm i dx)$ and $(1/2)(1 \pm i dy)$. But the last two ones, when not mutually annulling, do not commute, nor do they commute with the rotational ones (Commutativity will be key in the considerations that follow). In two dimensions, we could also have a $(t, x)$ space, the terminology speaking of the signature of the metric. We will not deal, however, with hyperbolic rotation symmetry, as it will loose relevance when we deal with Lorentzian spaces of dimensions three and four.

The idempotents in (3) would fit in 3-dimensional and 4-dimensional spaces, $(t, x, y)$ and $(t, x, y, z)$. Why Kähler did not consider translations in the $x$, $y$ and $z$ dimensions in his study of symmetry in spacetime? Once he had to consider $I_{xy}^\pm$ in order to deal with rotations for the problem of the fine structure of the hydrogen atom (and also of time translation because of rest mass) there was no room for $(1/2)(1 \pm i dx^i)$ since they do not commute with $\epsilon^\pm$. And there is no room for $I_{yz}^\pm$ and $I_{zx}^\pm$ (once $I_{xy}^\pm$ has been chosen) since they do not commute with $I_{xy}^\pm$.

We proceed to see what role commutativity and the mutually annulling of idempotents exhibited in the last equations (4) play. The impact of
commutativity is that it allows for the mutually annulling of idempotents such as, say, $I^+_{xy} e^-$ and $I^-_{xy} e^-$. Suppose we multiply (11) by $I^-_{xy} e^-$ on the right. We get $I^+_{xy} e^- I^-_{xy} e^- = I^+_{xy} I^-_{xy} e^- e^- = 0$. If $I^+_{xy}$ and $e^-$ did not commute, we could not have obtained this result. The impact of the vanishing of three of the four terms annulling when multiplying on the right of (7) by each of the four $e^\pm I^*_xy$ can be ascertained by trying to make the same considerations with

$$1 \equiv P^+_x I^+_{xy} + P^+_x I^-_{xy} + P^-_x I^+_{xy} + P^-_x I^-_{xy},$$

and, therefore,

$$u \equiv u P^+_x I^+_{xy} + u P^+_x I^-_{xy} + u P^-_x I^+_{xy} + u P^-_x I^-_{xy},$$

where $P^\pm_x \equiv \frac{1}{2}(1 \pm i dx)$. These are valid equations but the terms on the right of (9) and (10) do not have the same properties as the terms on the right of (7) and (8), which are disjoint, read orthogonal.

The $e^\pm I^*_xy$ are so call constant differentials [1], [3], meaning that $\partial(e^\pm I^*_xy) = 0$, where $\partial$ is the sum of the interior (read also divergence and co-derivative) and exterior derivatives. As a consequence of one of the properties of constant differentials, we have

$$\partial(u e^\pm I^*_xy) = (\partial u)e^\pm I^*_xy.$$ 

Hence all four terms on the right hand side of (8) are solutions of the same equation of the type $\partial u = au$ if $u$ is. The $u e^\pm I^*_xy$ are mutually orthogonal solutions of the same equation. We are thus justified in writing

$$u = + u^+ I^+_{xy} e^+ + u^- I^-_{xy} e^- + u^+ I^+_{xy} e^- + u^- I^-_{xy} e^-, $$

with uniquely defined meridian differentials $u^\pm$. 

One is thus led to seek spinorial solutions that take the form of meridian differentials times $I^\pm_{xy} e^*$. In a exterior system, the dependence on $t$ and $f$ of proper functions for time translations and rotational symmetry takes place through the standard phase factors. It is thus unavoidable (English translation of the term used by Kähler to make the present point) to consider spinorial solutions of the form

$$u = pe^{i m f - i E t} e^\pm I^*_xy$$

where $p$ is a strict meridian differential, i.e. meridian differential whose coefficients depend on $f$ and $z$, but not on $\phi$ and $t$.

With a view to future developments, it is worth pointing out at the nature of the different factors in the last expression. The $e^{i m f - i E t}$ and
$\epsilon^\pm I_{xy}$ are structural. On the other hand, $p$ will contain more fleeting info, not further restricted by those symmetry considerations but by the specifics of particular problems. Between $e^{im\phi-iEt}$ on the one hand and $\epsilon^\pm I_{xy}$ on the other hand, the idempotents contain the most rigid factor in (13); in the phase factor, one still has the freedom of having different values for $m$ and $E$. There is not a comparable freedom in $\epsilon^\pm I_{xy}$.

Let us finally deal with the relation of $\epsilon^\pm$ to charge, in addition to energy. In 1962, Kähler derived a conservation law

$$d(u, \eta \bar{u})_1 = 0, \quad (14)$$

from his equation that replaces Dirac’s [3]. It is quadratic, in the sense that it is bilinear in $u$ and its complex conjugate, $\bar{u}$. The operator $\eta$ reverses the sign of differential forms of odd grade, and $(\ldots)_1$ is a product of differential forms of grade $n-1$, where $n$ is the dimensionality of the space. We need not enter more details for present purposes. It is well known that equations like this represent conservation laws. What is conserved in this case?

Kähler was able to develop (14) to the point where it acquired the form

$$\frac{\partial \rho}{\partial t} + j + \frac{\partial \rho'}{\partial t} + j' = 0 \quad (15)$$

(Needless to say that $\rho$ and $\rho'$ stand here for charge densities, and $j$ and $j'$ stand for physical currents). Primed and unprimed quantities correspond to the two terms on the right of (5), which associates $\epsilon^\pm$ with the two signs of charge, not of energy, which is the same in both cases. The problem of association of antiparticles with the opposite sign of energy does not exist in Kähler’s relativistic quantum mechanics.

### 2.3 Kaehler’s approach to Lie differentiation

We have dealt with the form of solutions with the stated symmetries. We now deal with the related issue of operators for the same symmetries.

In KC, spin emerges through Lie differentiation as an integral part of angular momentum, not as a late attachment to the orbital one. One has to ask oneself, like Kähler did, what is the action of an operator

$$A \equiv \xi_i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \quad (16)$$

on differential forms. It is not what would appear to be at first sight. Suffice to consider the case of definite grade:

$$u \equiv \frac{1}{p!} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \quad (17)$$
The issue thus lies in extending the action of operator (16) from the ring of
differential $0$–forms to the ring of differential forms. Kähler ([3], section 16)
rightly credits Cartan with such an extension [4]. But the latter’s treatment
is too cryptic, as he was not interested in providing an expression for the
action of the operator (He instead directly derived with a very brief argument
the theorem that is sometimes referred to as the golden theorem). It amounts
to a different version of that action. Because of its clarity, we now reproduce
Kähler’s derivation of the formula for $A\mathbf{u}$ that he published in [1].

The right hand side of (16) could be viewed as the partial derivative with
respect to some coordinate in suitable coordinate systems ($y^i$). In order to
find them, Kähler considers the general solution of the system

$$\frac{dx^i}{dy^n} = \xi^i(x^1, ..., x^n), \quad (18)$$

which one may express as

$$x^i = x^i(y^1, ..., y^{n-1}), \quad (19)$$

where the $y^1, ..., y^{n-1}$ are first integrals of (18). The last of the first integrals
is additive to the variable $y^n$, with which we may thus identify it. In the
neighborhood of a point, (19) can be viewed as a coordinate transformation.
The pull-back of (17) to the $y$ coordinate system is

$$u \equiv \frac{1}{p!}a_{i_1...i_p} \frac{\partial x^{i_1}}{\partial y^{k_1}} ... \frac{\partial x^{i_p}}{\partial y^{k_p}} dy^{k_1} \wedge ... \wedge dy^{k_p}. \quad (20)$$

We now compute $\frac{\partial u}{\partial y^n}$ using that

$$\frac{\partial}{\partial y^n} = \frac{\partial x^i}{\partial y^n} \frac{\partial}{\partial x^i} = \xi^i \frac{\partial}{\partial x^i} = A. \quad (21)$$

Notice that, in (18), we had the $\xi^i$ as total derivatives since there was just
one independent variable. The same variable is viewed as the last one in a
different coordinate system, hence the partial differentiation notation. The
derivatives continue to be the same. The result of the computation is

$$Au = \xi^i \frac{\partial u}{\partial x^i} + d\xi^i \wedge e_i u. \quad (22)$$

Kähler writes $d\xi^i$ as $d(\xi^i)$ since he reserves the notation $d\xi^i$ for covariant
derivatives (See [1], end of his section 17, and second equation in section
34). The non-intuitive implication

$$Au \neq \frac{1}{p!} \xi^i \frac{\partial a_{i_1...i_p}}{\partial x^i} dx^{i_1} \wedge ... \wedge dx^{i_p} \quad (23)$$
is due to the fact that what remains constant in partial differentiations changes from term to term in (16). In order to avoid inadvertent errors, one should thus view $\partial / \partial \phi^i$ as the starting point for rotational symmetry with preference over $J_i = x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j}$, \hspace{1cm} (24)

Here $\phi^i$ represents the angle of rotation around three orthonormal axes and $(i, j, k)$ constitute cyclic permutations of $(1, 2, 3)$. Clearly each $\phi$ plays the role of $y^n$ in the previous argument. Coordinates $y^1, ..., y^{n-1}$ would then be the other coordinates in systems such as those of spherical and cylindrical coordinates, but different systems for different index $i$.

### 2.4 Angular momentum in Kaehler’s theory

In terms of Cartesian coordinates, the action of $\partial / \partial \phi^i$ on differential forms is given by

\[ J_i u = x^j \frac{\partial u}{\partial x^k} - x^k \frac{\partial u}{\partial x^j} + \frac{1}{2} w_i u - \frac{1}{2} u w_i, \]  \hspace{1cm} (25)

where

\[ w_i = dx^j dx^k \equiv dx^{jk}. \] \hspace{1cm} (26)

We shall refer to $\frac{1}{2} w_i u - \frac{1}{2} u w_i$ as the spin term(s), though chirality terms would also be appropriate. These two terms are intrinsically associated, like (we have seen) energy and charge also are.

Notice that, if $w_i$ and $u$ commute, the last two terms in (25) add up to zero. Such $u$’s are proper values of $\frac{1}{2} w_i - \frac{1}{2} w_i$ with proper value zero. The same two terms add up to $w_i u$ if they anticommute. Clearly, $w_k$ commutes with $dx^k$ and $(1/2)(1+w_k)$. It anticommutes with $dx^j$, $dx^k$, $w_j$ and $w_k$, and it neither commutes nor anticommutes with $(1/2)(1+w_i)$ and $(1/2)(1+w_j)$.

Consider now total angular momentum, not just components thereof. Kähler defines an operator $K$ by

\[ (K + 1) u \equiv \sum_{i=1}^{i=3} J_i u \vee w_i, \] \hspace{1cm} (27)

acting on scalar-valued differential forms $u$. The reason for Kähler’s defining $(K + 1) u$ where he could have simply written $K u$ has the unstated purpose of later ending with the same terminology as in the paradigm. He showed that

\[ (K + 1)^2 u = \sum_{i=1}^{i=3} J_i^2 u + (K + 1) u. \] \hspace{1cm} (28)
If we take the last term to the left side, we get the equation

\[(K + 1)Ku = \sum_{i=1}^{i=3} J_i^2 u. \tag{29}\]

It is a relation about operators which has a parallel in a proper values equations in the standard theory of angular momentum. Equation (27) implicitly speaks of \(J_i\) as its components, though not in the standard sense of components in a vector space or in a module. The operator

\[K + 1 \equiv \sum_{i=1}^{i=3} J_i \wedge w_i \tag{30}\]

is differential 2–form valued, in the sense that it is the contraction of the \(J_i\) with the elements of a basis of differential 2–forms.

The issue of the grade of the operator \(K + 1\) is a subtle one. We write its components as

\[J_i \equiv x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} + \frac{1}{2} w_i \wedge - \frac{1}{2} \wedge w_i, \tag{31}\]

where we have made explicit the symbol for Clifford product. The first two terms on the right of (31) are of grade zero; the last two appear to be of grade two. This is, however, deceiving, as we now explain.

Start by considering that the operator definition (31) follows (25), not the other way around. Given arbitrary differential forms \(u\) and \(v\), each of homogeneous grade, the grade of \(u \wedge v - v \wedge u\) is two units less than the sum of the respective grades (see [3], p. 478). By virtue of this, the grade of \(J_i u\) and \((K + 1)u\) is the same as the grade of \(u\) (if of definite grade \(r\)) in spite of the fact that each of \(\frac{1}{2} w_i u\) and \(\frac{1}{2} uw_i\) may consist of terms of grades \(r + 2\), \(r\) and \(r - 2\).

Notice that, in this treatment of angular momentum, there has not been any need for the unit imaginary. At a certain point, however, it becomes necessary in Kähler’s treatment in order to match operators with idempotents and thus with phase factors.

For completeness purposes, we call attention to his remark that the components of angular momentum should be \(\frac{\hbar}{2\pi i} J_i\), “and not as is usually the case

\[\frac{\hbar}{2\pi i} \left( J_1 + \frac{1}{2} \wedge w_1 \right), \quad \frac{\hbar}{2\pi i} \left( J_2 + \frac{1}{2} \wedge w_2 \right), \quad \frac{\hbar}{2\pi i} \left( J_3 + \frac{1}{2} \wedge w_3 \right) \tag{32}\]

[3] (section 27). (Compare with [5], chapter 7, Exercise 10).
2.5 Action of angular momentum on rotation idempotents

The action of \( J_l \) on \( w_k \) and on the idempotents \( \frac{1}{2}(1 \pm iw_k) \) will be the same as the action of \( \frac{1}{2}w_l - \frac{1}{2}w_l \). Obviously

\[
J_k I^\pm_{ij} = \frac{1}{2}w_k \frac{1}{2}(1 \pm iw_k) - \frac{1}{2}(1 \pm iw_k)w_k = 0. \tag{33}
\]

The idempotents \( I^\pm_{ij} \) are proper functions of \( J_k \) and \( \frac{1}{2}(w_k - w_k) \) with proper value zero. Nothing impedes that spin still be included in the factor \( e^{imb} \) in (13), through the first two terms in (25). The proper value zero becomes a pair of opposite proper values for solutions which have \( I^\pm_{ij} \) as factor and anticommute with the operator \( J_k \). The difference \( \frac{1}{2}w_k u - \frac{1}{2}iw_k \) becomes \( -i w_k \). Then \( w_k \) is absorbed into \( (1 \pm iw_k) \). The sign of the proper value will depend on the sign of the action of \( w_k \) on \( I^\pm_{ij} \).

For the other combinations of indices, we first get

\[
J_i w_k = w_j, \quad J_j w_k = -w_i \tag{34}
\]

Then, clearly,

\[
J_i I^\pm_{ij} = \pm \frac{1}{2}iw_j, \quad J_i I^\pm_{ij} = \mp \frac{1}{2}iw_i. \tag{35}
\]

The \( I^\pm_{ij} \) are not proper "functions" of \( J_i, J_j, \frac{1}{2}(w_i - w_i) \) and \( \frac{1}{2}(w_j - w_j) \).

We use (34) and (33) to obtain

\[
\sum_{i=1}^{3} J_l w_1 \vee w_l = 0 + w_3 w_2 - w_2 w_3 = 2w_3 w_2 = 2w_1. \tag{36}
\]

Thus, in general,

\[
(K + 1)w_k = 2w_k. \tag{37}
\]

Hence, \( w_k \) and \( iw_k \) are proper functions of \( K \) with proper value +1. Finally,

\[
(K + 1)\frac{1}{2}(1 \pm iw_k) = \pm(K + 1)iw_k = \pm iw_k. \tag{38}
\]

The rotational idempotents are not per se proper functions of \( K \), but can, of course, make part of expressions that are. Let us solve for \( K \frac{1}{2}(1 + iw_k) \).

We have, using (38),

\[
K \frac{1}{2}(1 \pm iw_i) = \pm iw_i - \frac{1}{2}(1 \pm iw_i) = -\frac{1}{2} \pm \frac{1}{2}iw_i = -\frac{1}{2}(1 \mp iw_i). \tag{39}
\]
It thus follows that $-K$ moves $\frac{1}{2}(1 \pm i w_i)$ down and up:

$$- K \frac{1}{2}(1 \pm i w_i) = \frac{1}{2}(1 \mp i w_i).$$

(40)

Notice that the unit imaginary has come into play through the idempotents. It had not done so through the angular momentum operators on their own.

### 3 Symmetry Related Transformations and the Unit Imaginary

In this section, we shall treat rotations through Clifford algebra, which is the canonical algebra of Euclidean and pseudo-Euclidean spaces. As a byproduct, we associate rotational symmetries with gauge transformations. This association can then be extended to other symmetries. All this is relevant for our later replacement of the unit imaginary in the next section. It usurps the role of geometric quantities of square $-1$.

#### 3.1 Rotation of vectors

The contents of this subsection is well known by every practitioner of Clifford algebra and can be found in one of our posted papers in arXiv [?]. We repeat it here for completeness purposes. In the plane, let $t$ be a unit vector, and let $u'$ be the reflection of $u$ with respect to the direction $t$. We clearly have $u' = tu$ because $(u', t)$ is indistinguishable from $(t, u)$ (picture this situation with $u$ of unit magnitude). Since $t^{-1} = t$, we have

$$u' = tu t^{-1} = tut.$$  

(41)

Consider next the reflection of $u$ with respect to a plane $\pi$ with normal unit vector $n$. Let $t$ now be a unit vector in the direction of the projection of $u$ on $\pi$. The three vectors ($n$, $t$ and $u$) are on the same plane, $n$ and $t$ being perpendicular. By a known elementary theorem, the reflections of $u$ with respect to two perpendicular vectors are opposite;

$$nnu = -ttu.$$  

(42)

In 3-D, a rotation of $u$ by an angle $\phi$ around an axis in the direction of the unit vector $n_3$ yields the vector $u''$ resulting from successive reflections
on two planes making an angle $\phi/2$, interacting along $\mathbf{N}$ and with normal unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. We thus have

$$u'' = \mathbf{n}_2\mathbf{n}_1u\mathbf{n}_1\mathbf{n}_2 = (\mathbf{n}_1\mathbf{n}_2)^{-1}u(\mathbf{n}_1\mathbf{n}_2). \quad (43)$$

We write the Clifford product $\mathbf{n}_1\mathbf{n}_2$ in terms of its scalar and exterior components:

$$\mathbf{n}_1\mathbf{n}_2 = \cos \frac{\phi}{2} + \lambda \mathbf{a}_1\mathbf{a}_2, \quad (44)$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are any two orthonormal vectors in the plane determined by $\mathbf{n}_1$ and $\mathbf{n}_2$. Clearly then

$$\mathbf{n}_1\mathbf{n}_2 = \cos \frac{\phi}{2} + \sin \frac{\phi}{2} \mathbf{a}_1\mathbf{a}_2 = e^{\frac{\phi}{2}\mathbf{a}_1\mathbf{a}_2}. \quad (45)$$

since $(\mathbf{a}_1\mathbf{a}_2)^2 = -1$. It thus follows that

$$u'' = e^{-\frac{\phi}{2}\mathbf{a}_1\mathbf{a}_2} u e^{\frac{\phi}{2}\mathbf{a}_1\mathbf{a}_2}. \quad (46)$$

### 3.2 (Mainly) rotational and related gauge symmetry

We now extend the action of rotations to elements of the tangent Clifford algebra. We apply it to monomials; the extension to polynomials is trivial. Define

$$\mathbf{R} \equiv e^{\frac{\phi}{2}\mathbf{a}_i\mathbf{a}_j}. \quad (47)$$

Under a rotation around $\mathbf{a}_k$, the Clifford product of vectors $\mathbf{A} \equiv \mathbf{ab}...\mathbf{k}$ becomes

$$\mathbf{A}' \equiv (\mathbf{R}^{-1}\mathbf{aR})(\mathbf{R}^{-1}\mathbf{bR})...(\mathbf{R}^{-1}\mathbf{kR}) = \mathbf{R}^{-1}\mathbf{ab}...\mathbf{kR} = \mathbf{R}^{-1}\mathbf{AR}. \quad (48)$$

Spinors are members of left ideals. Rotational spinors in the tangent algebra are the elements of ideals

$$\mathbf{A}^{\frac{1}{2}}(1 \pm i\mathbf{a}_i\mathbf{a}_j), \quad (49)$$

where $\mathbf{A}$ represents all elements in the algebra. The unit imaginary is there to build the idempotents since $(\mathbf{a}_i\mathbf{a}_j)^2$ equals $-1$, and not the required $+1$. Hence something is not quite right with Clifford algebra, as one should not need units imaginary to deal with idempotents in real vector spaces. It looks as if there must be “something Clifford” which is more fundamental than the Clifford algebra itself. We have the solution in the tensor product of two specific, related Clifford algebras. But that is not the only reason why
we shall have something more fundamental [1]. More on this in the next section.

Since (49) belongs to the algebra, its rotation in the plane determined by $a_i$ and $a_j$ is given by

$$R^{-1}A\frac{1}{2}(1 \pm ia_i a_j)R = R^{-1}AR\frac{1}{2}(1 \pm ia_i a_j) = A'\frac{1}{2}(1 \pm ia_i a_j), \quad (50)$$

which means that the ideal transforms into itself. But general members of this ideal transform to $R^{-1}A\frac{1}{2}(1 \pm ia_i a_j)R$, not to

$$R^{-1}A\frac{1}{2}(1 \pm ia_i a_j). \quad (51)$$

Transformation (51) —equivalently $(R^{-1}AR, R^{-1}\psi)$— is the gauge transformation associated with but not identical to a rotation, even if $R$ is given by (47).

By virtue of their structure, gauge transformations apply to equations of the type

$$(\partial - A)\psi = 0, \quad (52)$$

where, suffice to say, $A$ and $\psi$ are respective input and output and where $R$ might not be as for rotations, (47). The exponent in $R$ could be, for instance, a scalar times a vector of square +1, or times $i$ times a vector of square −1. This is so by virtue of the fact that

$$[R^{-1}(\partial - a)R] R^{-1}\psi = 0, \quad (53)$$

follows from (52) for any operator which has an inverse. Of course, the physical relevance of $R$ will depend on whether $R^{-1}(\partial - a)R$ is an invariant.

Let $\lambda$ be an arbitrary scalar and let $R$ be of the type $e^{\lambda \Sigma}$ with $\Sigma^2 = -1$. The idempotents

$$\frac{1}{2}(1 \pm i\Sigma) \quad (54)$$

define ideals

$$A\frac{1}{2}(1 \pm i\Sigma) \quad (55)$$

associated with the gauge transformation

$$(\partial - a) \rightarrow R^{-1}(\partial - a)R, \quad \psi \rightarrow R^{-1}\psi. \quad (56)$$

If the two idempotents (54) are associated with solutions of an equation possessing some one-parameter group of classical geometric symmetry, the
gauge transformation (56) is associated with that symmetry and we may say that it is a classical gauge symmetry, as opposed to gauge symmetries not directly related to the tangent bundle.

We have seen gauge transformations as invariance properties of equations, (56). If $R$ is as in (50), the second of these can then be seen as a gauge transformation of the associated ideal, in turn associated with a rotation. Hence the two groups in the respective equations (56) are different but they respond to the same symmetry. Hence, it is correct but not very illuminating to speak of rotation groups and spin groups, when not absolutely necessary. They reflect different applications of the same symmetry.

3.3 Of units imaginary in translational and related gauge symmetry

Assume that Kähler had been concerned with gauge symmetry under space translation, instead of the $\epsilon^\pm$ given in (3). He would have had idempotents like $\frac{1}{2}(1 \pm dz)$. Then $R$ would be $R \equiv e^{-\lambda a_3}$. It is not relevant here what $R^{-1}R$ would mean. We have mentioned gauge symmetry under space translation, not space translation symmetry. it would, however, be a relevant issue in quantum mechanics, or perhaps the beginning of a hint of why one has problems with position operators in standard quantum mechanics.

Also, if the signature of the metric were not positive definite, we would not have the unit imaginary in the phase factor for hyperbolic rotations.

We shall see in the next section that the right choice of algebra makes the unit imaginary unnecessary for building the idempotents or for gauge transformations. We may thus advance that the presence/absence of the standard unit imaginary is a function of the algebra where the equation with gauge symmetry lives. We shall say that a gauge symmetry is geometric in a given algebra if its idempotent and corresponding exponential factor belong to the tangent Clifford algebra. From this point on in this paper, we shall deal with tangent Clifford algebra valued differential forms. Hence, the gauge symmetries that we shall explicitly or implicitly consider will be of the geometric type.

The natural question arising from the considerations about to be made is whether a non-geometric gauge transformation actually is a geometric one that simply has gone unnoticed by an unfortunate choice of algebra. But that is a physical issue, which will thus not be considered in this paper. Let us summarize by saying that there is no need for the standard unit imaginary when one deals with the algebras that shall occupy us in the next section. The issue of the unit imaginary is far more subtle than usually
acknowledged.

4 Clifford-valued Clifforms

An algebraically minded mathematician who were exposed for the first time to the expression \( dx + dy + dz \) would see in there the tensor product of two modules, and/or the tensor product of the two exterior or Clifford algebras spanned by those modules. On the other hand, most of us have been introduced to the wrong concept of \((dx, dy, dz)\) so that \( dx + dy + dz \) is a vector with weird components. It was already obvious to É. Cartan that one should be dealing with tensor products of structures; he referred to the curvature of Euclidean connections as bivector-valued differential 2-forms \[6\] (See specially his sections on integral invariants in the chapter on Riemannian manifolds with metric connections of that reference). Both of the structures in his product were exterior algebras.

4.1 Tensor product of Clifford algebras

Exterior and Clifford algebras are quotient algebras of general tensor algebras. Being algebras, they are modules or vector spaces in the first place, of dimension \(2^n\) if constructed upon modules or vector spaces of dimension \(n\). Consider the tensor product of the two Clifford algebras respectively defined by the relations (1) and (2) in terms of orthonormal bases. Simple bases in those algebras are \((1, dx, dy, dz, dydz, dzdx, dxdy, ...)\) and \((1, i, j, k, ij, jk, ki, ...\)) respectively. The element \(dxj\) of the product structure has components \((0, j, 0, 0, 0, ...\) in one algebra and \((0, 0, dx, 0, 0, 0, ...)\) in the other. A basis in the product structure is constituted by the \(2^n \times 2^n\) elements resulting from product of all the basis elements in one algebra by all the basis elements in the other one.

Each Clifford algebra factor is isomorphic to the structure spanned by the degenerate product of \((1, dx, dy, dz, dydz, dzdx, dxdy, ...)\) with \((1, 0, 0, 0, ...0)\). The other Clifford algebra factor is isomorphic to the degenerate product of \((1, 0, 0, 0, ...0)\) with \((1, i, j, k, ij, jk, ki, ...\))

In the new structures just considered, we have Clifford-valued clifforms, the term clifform being used when the exterior product of differential forms is replaced with their Clifford product. Kähler developed calculus for tensor-valued differential (clifforms, which he called tensors whose coefficients are differential forms. And he considered physical applications for scalar-valued differential forms, but not for those of more general valuedness. In particul-
lar, he did not extend to them his study of symmetry in solutions of exterior systems.

In more recent times, tensor products of Clifford algebras have been considered by Oziewicz [7]. Interestingly, products of similar type have emerged in very deep algebraic work by Helmstetter [8]. Unlike Oziewicz'es and our own work, such tensor products are not per se the primary concepts, but the idea that “in matters Clifford”, there is something more fundamental than the Clifford algebra itself. The significance of work of such a nature may not be overlooked, specially (for us) if one were to find one day that his work and ours are related in some form or another. We devote subsection 4.3 to briefly introduce his work in simple terms.

### 4.2 Commutative, mirror algebra

The $2^n$ structure considered in the previous subsection has a subalgebra of mirror-symmetric elements. We proceed to explain the concept via some simple examples. $dx^1a_1$ is mirror symmetric and $dx^1a_2$ is not, since the superscript of one factor is not equal to the subscript of the other factor. But $dx^1y^1$ is mirror symmetric and $dx^1y^2$ is not. Certainly, $dr = dx^1dy^1 + dx^1y^2 + dx^1z^1$ is a non-monomial example of mirror symmetric element. The inhomogeneous grade element $dx^1 + dx^1y^1$ also is. As a member of the structure spanned by $(1, i, j, k, ik, ji, jk, ...)$, it has respective coefficients $(0, dx, 0, 0, 0, 0, dx^1y_0, 0, 0, ...).$ And as a member of the structure spanned by $(1, dx, dy, dz, dydz, dzdx, dx^1y, ...) it has coefficients $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, ...).$ This is to be compared with, say, $dx^1y^1$, whose components in one algebra are $((0,0, dx, 0, 0, 0, 0, ...)$ and $(0, 0, 0, 0, 0, 0, 0, 0, 0, ...)$ in the other (third place versus second place).

What is the motivation for considering the structure of mirror elements, either in itself or as a substructure of the aforementioned tensor product of structures? If the pervasiveness of $dx^1 + dy^1 + dz^1$ were not enough to answer this question, consider rotations and associated gauge transformations. The unit imaginary $i$ in the exponent in standard treatments usurps the role of real geometric units of square minus one in the tangent Clifford algebra of $E_3$. Thus “$i$” unduly replaces $a_1 \wedge a_2$ (= $a_1a_2$ by virtue of orthogonality) in the expressions for rotations around the $z$ axis when one uses Clifford algebra [9], [10], [11]. Consequently, the replacement of $i$ with $a_1a_2$ should also take place in the phase factor for third component of angular momentum and in the idempotents for solutions with rotational symmetry. Under such a replacement, elements like $dxdy^1a_1a_2$ (no sum) emerge in the rotational idempotents.
Let $\Omega^{lm}$ be differential 2-forms, say the components $R_{pq}^{nm} dx^p \wedge dx^q$ of the curvature of Euclidean connections as bivectors. The $\Omega^{lm}$ are not mirror-symmetric, but the $R_{pq}^{nm} dx^p \wedge dx^q e_p \wedge e_q$ are, both with and without summation over $p$ and $q$ (We use $e$ instead of $a$ to signify that the bases are different at different points).

The set of mirror symmetric elements is closed under mirror symmetric products, i.e. with the same symbol in both algebras. $(\wedge, \wedge)$ and $(\cdot, \cdot)$ are mirror products, but not $(\lor, \wedge)$, $(\lor, \cdot)$, $(\land, \lor)$, etc. The little computation

$$dy^i(\lor, \lor) dx^i = dy dx^i j = (-dx dy)(-ij) = dx dy ij$$

(57)

exhibits both the meaning of the symbol $(\lor, \lor)$ and the commutativity of this product. This commutativity has nothing to do with the fact that we multiplied monomials. Thus, for example, we have

$$(dx^i + dx dy ij)(\lor, \lor) dx dz^i k = dx dz i i k + dx dy dx dz ij ik =$$

$$= dz^i k + (-dy dz)(-jk) = dz^i k + dy dz j i k = dx dz i k (\lor, \lor) (dx^i + dx dy ij).$$

(58)

Coefficients that are not scalar functions do not change the mirror symmetry element property. Thus

$$(\mu dx^i + \nu dx dy ij)(\lor, \lor) \lambda dx dz^i k = \mu \lambda dx dz i i k + \nu \lambda dx dy dz ij ik =$$

$$= \mu \lambda dz^i k + \nu \lambda (-dy dz)(-jk) = \mu \lambda dz^i k + \nu \lambda dy dz j k.$$ 

(59)

We have just illustrated that, under mirror-symmetric products, mirror-symmetric elements constitute a commutative algebra, subalgebra of the aforementioned tensor product of Clifford algebras. The reason is that any minus signs emerging in a product of monomials in one algebra is cancelled by a minus sign from the product of mirror images of the factors. The non-symmetric product of mirror symmetric elements is not a mirror symmetric element.

We shall refer with the names mirror symmetric algebra, commutative algebra and commutative subalgebra to the restriction of that tensor product of Clifford algebras to the set of mirror symmetric elements endowed with mirror symmetric multiplication.

Given the element $dx + dx dy$ in the Kähler algebra, it uniquely determines the mirror element $dx^i + dx dy ij$. For comparison of meanings, we observe that $a_1 a_2$ represents any geometric object of unit area in the $xy$ plane of any given Euclidean space. $dx dy$ represents an integrand whose integration gives a number, namely the area of a figure contained in the $xy$ plane.
\( dxdy \mathbf{a}_1 \mathbf{a}_2 \) represents an integrand whose integration gives a bivector \( \lambda \mathbf{a}_1 \mathbf{a}_2 \), namely any geometric object of area \( \lambda \) in the \( xy \) plane of any given Euclidean space with a reference frame attached to it.

### 4.3 On the foundations of Clifford algebra.

We have developed Clifford-related structure that has not been exploited by the community of Clifford practitioners. We shall show that it evokes Helmstetter’s work on the foundations of Clifford algebra through the concept to which he refers as the Clifford monoid (also known as Lipschitz monoid, Lipschitz semigroup and Clifford-Lipschitz monoid). He finds it to be more important than the Clifford algebra itself \[8\].

The Clifford monoid is one that satisfies some property too sophisticated to be dealt with here. In addition, Helmstetter gives credit to Sato, Miwa and Jimbo \[12\] for having done work in a similar direction but with an approach different from, but equivalent to, the approach to infrastructure of Clifford algebra through the Clifford monoid (In his reference to those authors, Helmstetter points out that the physical title of their work does not let one guess that it is about algebras de Clifford). Our work is further removed from those works, than they are to each other.

In order to start illustrating his main point of “insufficiency” of the classical theory of Clifford algebra, Helmstetter states:

> “The essential object in each of these works is a topological vector space \( A \) over a field \( R \) or \( C \) of real or complex numbers, and where a dense subspace \( C \) is a Clifford algebra or looks like one such algebra;” (our emphasis).

Further down the text he continues with the related statement:

> "It is important to add that the multiplication in the algebra \( C \) is not continued for the ambient topology of \( A \), in such a way that this multiplication is prolonged only for certain privileged pairs \((x, y)\) of elements of \( A \)." (our emphasis)

With the emphasizing of terms in those two quotations, we have intimated what the answer is to the following question: what could be more fundamental than Clifford algebras, being as they are canonical Euclidean and pseudo-Euclidean structures? (Vector algebra is disfigured Clifford algebra, peculiar to 3-D). In order to capture the significance of the quoted remarks, we momentarily leave Helmstetter work to recall another work where
connection with the environment of topology and set theory is spoused as a very desirable feature in mathematics.

Let us provide context with the remark that a Euclidean space is a special case of affine space, which in turn is a particular case of projective space, in turn of ..., in turn of topological space, in turn of ... and in turn of set (Fill the blanks as per your understanding). Ideally, algebraic concepts at one of those levels (say Clifford algebra at the Euclidean and pseudo-Euclidean levels) should mimic algebraic concepts of more general levels. This point has been made very explicitly by Barnabei, Brini and the late Professor Rota [13] in connection with the work of Peano [14] on the algebra which is to projective space what Clifford algebra is for Euclidean space (This should not, however, obscure the fact that the mathematical philosophy of Professors Helmstetter and Rota is otherwise very different). Their key issue is that the basic products in projective algebra should mimic operations in set theory like those of reunion and intersection. Such conceptual prolongation is already present, though in less overreaching way, in É. Cartan’s view of the theory of Euclidean connections as theory of affine connections where affine tangent bundles are restricted to Euclidean tangent bundles [6].

Helmstetter’s work does not start with tensor products. Hence, the relation of our work to his would have to be documented far beyond the superficial considerations to follow. To get to the heart of the potential relevance of his work in connection with our product of tensor algebras, let us get directly to the point of closest approach. In his development of the concept of Clifford monoid, he gives the multiplication

\[(x \otimes y) \cdot (x' \otimes y') = \sigma(y, x')\sigma(y, y') \cdot xx' \otimes yy',\]

where the \(\sigma\)'s are just signs (±) that depend on the arguments (contents of the parentheses) and where \(x, y\) and \(x', y'\) are members of the tensor product of two copies of the same Clifford algebra. \(x\) and \(y\) belong to respective copies. But this is a by-product; the pairs \((x, y)\) and \((x', y')\) to which Eq. (60) refers start as privileged pairs of elements of \(A\). The tensor product on the right hand side of (60) is not an hoc multiplication, but implied by an abstract development whose underlying motivation will escape those who, like the present author, lack the required algebraic sophistication. Without need to understand the intricacies of this equation and of the argument that led to it, one may nevertheless see the similarities and differences with our tensor product of algebras. We mean the likes of the structural similarity of (60) with our tensor product of algebras. until we hit differences like Helmstetter’s \(xx' \otimes yy'\) versus our own \(xx' \otimes yy'\).
The fundamental character of our structures is due to the significance of elements like $dx_i + dy_j + dz_k$. It belongs to the tensor product of two algebras and contains the commutative algebra of mirror symmetric elements under the symmetric products restriction. However, approaching something as meaningful as a commutative algebra from a non-commutative one through a tensor product may lack a depth which would be achieved through a more topological approach, of the “Helmstetter kind”. In any case, we have to mention Helmstetter for at least stating his priority in doing deep, extensive work on product structures (See also references in [8]).

4.4 Action of angular momentum on the mirror algebra of 3-D Euclidean space

Of special significance is the action of angular momentum on rotational idempotents in the mirror algebra.

Define

$$ I_{ij}^\pm \equiv \pm \frac{1}{2} (1 \pm w_k a_{ij}). \quad (61) $$

Instead of (33) and (35) we now have

$$ I_{ij}^\pm = \pm \frac{1}{2} J_l w_k a_{ij}, \quad (62) $$

where $a_{ij}$ is compact notation for $a_i \wedge a_j$ (equal to $a_i a_j$ by virtue of orthonormality). Hence

$$ J_l I_{ij}^\pm = \pm \frac{1}{2} w_j a_{ij}, \quad J_j I_{ij}^\pm = \mp \frac{1}{2} w_i a_{ij}, \quad J_k I_{ij}^\pm = 0, \quad (63) $$

and, therefore,

$$ (K + 1) I_{ij}^\pm = \pm \frac{1}{2} (K + 1) w_k a_{ij}, \quad (64) $$

since the action of $(K + 1)$ on $1/2$ is zero.

We proceed to develop $(K + 1) I_{12}^\pm$ and generalize by cyclic permutations. Thus

$$ (K + 1) I_{12}^\pm = \pm \frac{1}{2} \sum_l (J_l w_3 a_{12}) w_l = \pm \frac{1}{2} (w_2 w_1 - w_1 w_2) a_{12} = \pm w_3 a_{12}, \quad (65) $$

and, therefore,

$$ (K + 1) I_{ij}^\pm = \pm w_k a_{ij}. \quad (66) $$
The action of $K$ itself on $I_{ij}^\pm$ is of interest:

$$K I_{ij}^\pm = \pm w_k a_{ij} - \frac{1}{2}(1 \pm w_k a_{ij}) = -\frac{1}{2} \pm \frac{1}{2} w_k a_{ij} = -I_{ij}^\mp,$$

(67)

Hence,

$$-K I_{ij}^\pm = I_{ij}^\mp, \quad K^2 I_{ij}^\pm = I_{ij}^\pm.$$

(68)

We see that the rotational idempotents, $I_{ij}^\pm$, are not proper functions of either $K$ or $K + 1$. The $w_k a_{ij}$ are:

$$(K + 1)w_k a_{ij} = 2w_k a_{ij}, \quad Kw_k a_{ij} = w_k a_{ij}.$$

(69)

For later purposes, we proceed to consider proper values of elements of different grades in a Cartesian basis of differential 1–forms and 2–forms. Let $dx^i$ denote $dx^i a_i$ (no sum). We have

$$J_i dx^1 = (0, \, dx^3, -dx^2)a_1$$

(70)

and

$$(K + 1)dx^1 = \sum (J_i dx^1) w^i = (dx^3 w_3 - dx^2 w_2) a_1 = 2dx^1 a_1 = 2dx^1,$$

(71)

and similarly for other values of the index. Hence

$$K dx^l = dx^l.$$

(72)

On the other hand

$$(K + 1)dx^{ij} a_{ij} = 2dx^{ij} a_{ij},$$

(73)

and, therefore,

$$K dx^{ij} a_{ij} = dx^{ij} a_{ij}.$$

(74)

It follows that the proper values of $(K + 1)$ are always +1 for all members in the module of mirror 1–forms and mirror 2–forms. They are zero for 0–forms and mirror 3–forms (Like scalars, $w$ commutes with the whole algebra). The action of $K + 1$ on it is, therefore, zero.

5 Idempotents in Commutative Algebra

5.1 Hierarchy of products of idempotents in commutative algebra

In the previous section, we introduced the three pairs of rotational idempotents $I_{ij}^\pm$ (i.e. $\pm \frac{1}{2}(1 \pm w_k a_{ij})$). Although commutativity is no longer a
problem, once we have one of those pairs there is no room for the other two in a product.

In a ways similar to our definition of rotational idempotents in the commutative algebra, we now define time and space translation symmetries. Instead of the first equations 3, we now have, for time translation symmetry,

\[ \varepsilon^\pm = \frac{1}{2}(1 \mp dt e_0) \equiv \frac{1}{2}(1 \mp dt), \tag{75} \]

with \( dt \equiv dt e_0 \).

Space-translational symmetry is different from time-translational symmetry in that the phase factor is not of the standard type, since the square of \( a_i \) is not minus one. For this reason, time translational symmetry has pre-eminence over space translation symmetry.

Commutativeness enters the concept of primitiveness. Since it is now present everywhere, it is no longer significant as a concept. In the commutative algebra, the replacement \( \frac{1}{2}(1 \pm dz a_3) \) for the idempotents \( \frac{1}{2}(1 \pm idz) \) will now have a role to play. As we did with the rotational idempotents, we shall consider all three directions,

\[ P^\pm_i \equiv \frac{1}{2}(1 \pm dx^i), \tag{76} \]

with \( dx^i \equiv dx^i a_i \).

All these idempotents will have to be accompanied by corresponding geometric phase factors,

\[ e^{m_{\alpha \beta \gamma} a_i I_{ij}^\pm}, \quad e^{-E_{\tau \theta a_0} \varepsilon^\pm} \quad e^{{\lambda}_{ij} x^i a_i P^\pm_i}, \quad \text{no sums!} \tag{77} \]

We might think of proceeding with \( d\rho \) as we have done with \( dz \). We do not see a reason at this point to do so in dealing with ternary idempotents. From a physical perspective, “\( d\rho \) translational symmetry” is unheard of. From a mathematical perspective, \( dz \) and the corresponding idempotent are constant differentials. \( d\rho \) and its associated idempotent are not. Also, the metric of Euclidean space depends on \( \rho \) when written in cylindrical coordinates (We have to use these coordinates if rotational symmetry is assumed in the first place).

Needless to say that \( I_{ij}^\pm P_{jk}^\pm \) may be expected to play a more significant role than \( I_{ij}^\pm P_{kj}^\pm \) and \( I_{ij}^\pm P_{ji}^\pm \), if, in the first place, these two play any role at all by themselves. Similarly, in dealing with applications, products involving \( I_{ij}^\pm \)'s and/or \( \varepsilon^\pm \)'s will be more significant than products involving \( P^\pm_i \)'s because of the remark made about phase factors.
Hence, we have been able to justify consideration of a set of idempotents for symmetry which goes far beyond the set of Kähler’s four binary idempotents for dealing with the electromagnetic interaction.

We have not yet considered boosts. There are physical reasons for why not, but this is not the place to deal with such reasons. Let us deal with the mathematical arguments. Recall that, in dimension four, we had at our disposal $\rho$ and $z$ — but not on $\phi$ and $t$ — once joined time translational and rotational symmetry had been assumed. There is no room to accommodate any boost under such restrictions. One might retort that there is no room either for translations along the $x$ and $y$ axes. But there is an important distinction between the boosts and the translations, the implications of this difference to be considered in a future paper. Although there is no room for solutions of field equations with translational symmetry in directions contained in the plane of rotations, there still is room for them in connection with curves. Indeed curves satisfy the “natural lifting conditions” $dx^i - u^i dt = 0$. Nothing similar takes place with the differentials $dx^idt$ that pertain to boosts, like the $dx^idx^j$‘s pertain to rotations.

We are thus led to view the ternary idempotents

$$\epsilon^{\pm}_{ij} P_1^\pm$$

as candidates for physical applications in the same way as Kähler considered $\epsilon^{\pm}_{xy}$ for applications in relativistic quantum mechanics. We shall refer to (78) as the canonical ternary idempotents of 4-D Lorentzian space.

5.2 Canonical ternary idempotents of 4-D Lorentzian space

In connection with (78), there are two issues to be considered. One of them is that some of those are repeated. The other one is that, after removing repetitions, there still are linear combinations of them that add to zero. We shall deal only with the first of these issues, since dealing with the second one would have too much of a physics flavor. That is purposefully avoided to avoid conflicts of classification in the arXiv.

Consider the following equalities:

$$4\Gamma^+_1 P^+_1 = (1 + dx dy)(1 \pm dx) = (1 + dx dy)(1 \pm dy) = 4\Gamma^+_2 P^+_2. \quad (79)$$

$$4\Gamma^-_1 P^-_1 = (1 - dx dy)(1 \pm dx) = (1 - dx dy)(1 \mp dy) = 4\Gamma^-_2 P^-_2. \quad (80)$$

Notice the change of sign in the exponent of (80), which is not the case in the exponent of (79). At the level of solutions of equations and not just idempotents, there would still be a difference in the phase factors.
Multiplication of $I_{12}$ with $P_3^\pm$ will not be in the same footing with multiplication with $P_1^\pm$ and $P_2^\pm$, since $z$ (thus $P_3^\pm$) relates to $I_{12}$ in a different way than to $P_1^\pm$ and $P_2^\pm$. In the case of $P_3$, we have a different symmetry if the plus and minus directions are indistinguishable. In that case, $I_{12}^+P_3^+ = I_{12}^-P_3^-$ and $I_{12}^+P_3^- = I_{12}^-P_3^+$. In view of these symmetries, we can organize the six different $I_{12}P$ binary idempotents in three columns and two rows respectively for the subscripts and superscripts of $P$. We left-multiply them by $\varepsilon^\pm$. We thus have the following table of twelve different canonical ternary idempotents

$$
\begin{bmatrix}
\varepsilon^\pm I_{12}\tilde{1}\P_1^+ & \varepsilon^\pm I_{12}\tilde{12}\P_2^+ & \varepsilon^\pm I_{12}\tilde{3}\P_3^+
\varepsilon^\pm I_{12}\tilde{1}\P_1^- & \varepsilon^\pm I_{12}\tilde{12}\P_2^- & \varepsilon^\pm I_{12}\tilde{3}\P_3^-
\end{bmatrix}
$$

(81)

We could also have organized the same group of idempotents with $I_{12}^-$ in columns 1 and 3, and $I_{12}^-\tilde{2}$ in column 2, everything else remaining the same.

What we have done for $I_{12}$ can be repeated for $I_{23}$ and $I_{31}$. In these cases, the role of $P_3$ in (81) would be played by $P_1$ and $P_2$ respectively. Putting these options together yields a set of 36 different ternary idempotents. They cannot be linearly independent since the dimension of the commutative algebra is 16. We shall not enter into that since the results would look very much like physics, and we intentionally want to let this paper be of a purely mathematical nature.

6 Concluding Remarks

Among the many results obtained in this paper we mention those that constitute its core. Kähler’s idempotents for solutions with symmetry contain the unit imaginary. We have reformulated them so that their now geometric form allows for the removal of that unit. It happens in a commutative algebra embedded in the tensor product of Kähler’s real algebra of scalar-valued differential forms with its dual Clifford algebra of valuedness (i.e. tangent Clifford algebra). More importantly, Kähler’s focus on solutions of equations rather than on the equations themselves allows one to extend the manifestation of symmetry through the new and extended set of idempotents in the new venue. The contribution of the concomitant exponential factors that accompany those idempotents has not been studied here.

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