Consistent Estimation for PCA and Sparse Regression with Oblivious Outliers

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Abstract

We develop machinery to design efficiently computable and consistent estimators, achieving estimation error approaching zero as the number of observations grows, when facing an oblivious adversary that may corrupt responses in all but an \( \alpha \) fraction of the samples. As concrete examples, we investigate two problems: sparse regression and principal component analysis (PCA). For sparse regression, we achieve consistency for optimal sample size \( n \gtrsim (k \log d) / \alpha^2 \) and optimal error rate \( O(\sqrt{(k \log d) / (n \cdot \alpha^2)}) \) where \( n \) is the number of observations, \( d \) is the number of dimensions and \( k \) is the sparsity of the parameter vector, allowing the fraction of inliers to be inverse-polynomial in the number of samples. Prior to this work, no estimator was known to be consistent when the fraction of inliers \( \alpha \) is \( o(1/\log \log n) \), even for (non-spherical) Gaussian design matrices. Results holding under weak design assumptions and in the presence of such general noise have only been shown in dense setting (i.e., general linear regression) very recently by d’Orsi et al. \([dNS21]\). In the context of PCA, we attain optimal error guarantees under broad spikiness assumptions on the parameter matrix (usually used in matrix completion). Previous works could obtain non-trivial guarantees only under the assumptions that the measurement noise corresponding to the inliers is polynomially small in \( n \) (e.g., Gaussian with variance \( 1/n^2 \)).

To devise our estimators, we equip the Huber loss with non-smooth regularizers such as the \( \ell_1 \) norm or the nuclear norm, and extend d’Orsi et al.’s approach \([dNS21]\) in a novel way to analyze the loss function. Our machinery appears to be easily applicable to a wide range of estimation problems. We complement these algorithmic results with statistical lower bounds showing that the fraction of inliers that our PCA estimator can deal with is optimal up to a constant factor.

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1 Introduction

Estimating information from structured data is a central theme in statistics that by now has found applications in a wide array of disciplines. On a high level, a typical assumption in an estimation problem is the existence of a –known a priori– family of probability distributions $\mathcal{P} := \{ \mathbb{P}_\beta \mid \beta \in \Omega \}$ over some space $\mathcal{Z}$ that are each indexed by some parameter $\beta \in \Omega$. We then observe a collection of $n$ independent observations $Z = (Z_1, \ldots, Z_n)$ drawn from an unknown probability distribution $\mathbb{P}_{\beta^*} \in \mathcal{P}$. The goal is to (approximately) recover the hidden parameter $\beta^*$.

Oftentimes, real-world data may contain skewed, imprecise or corrupted measurements. Hence, a desirable property for an estimator is to be robust to significant, possibly malicious, noise perturbations on the given observations. Indeed, in the last two decades, a large body of work has been developed on designing robust algorithms (e.g., see [CLMW11, ZLW10, TJSO14, BJKK17, SZF18, SBRJ19, PJK21, dNS21]). However, proving strong guarantees often either demands strong assumptions on the noise model or requires that the fraction of perturbed observations is small. More concretely, when we allow the noise to be chosen adaptively, i.e., chosen dependently on the observations and hidden parameters, a common theme is that consistent estimators – estimators whose error tends to zero as the number of observations grows – can be attained only when the fraction of outliers is small.

In order to make vanishing error possible in the presence of large fractions of outliers, it is necessary to consider weaker adversary models that are oblivious to the underlying structured data. In recent years, a flurry of works have investigated oblivious noise models [CLMW11, ZLW10, TJSO14, BJKK17, SZF18, SBRJ19, PJK21, dNS21]. These results however are tailor-made to the specific models and problems. To overcome this limitation, in this paper we aim to provide a simple blueprint to design provably robust estimators under minimalistic noise assumptions for a large class of estimation problems. As a testbed for our blueprint, we investigate two well-studied problems:

Principal component analysis (PCA): Given a matrix $Y := L^* + N$ where $L^* \in \mathbb{R}^{n \times n}$ is an unknown parameter matrix and $N$ is an $n$-by-$n$ random noise matrix, the goal is to find an estimator $\hat{L}$ for $L^*$ which is as close as possible to $L^*$ in Frobenius norm.

Sparse regression: Given observations $(X_1, y_1), \ldots, (X_n, y_n)$ following the linear model $y_i = \langle X_i, \beta^* \rangle + \eta_i$ where $X_i \in \mathbb{R}^d$, $\beta^* \in \mathbb{R}^d$ is the $k$-sparse parameter vector of interest (by $k$-sparse we mean that it has at most $k$ nonzero entries) and $\eta_1, \ldots, \eta_n$ is noise, the goal is to find an estimator $\hat{\beta}$ for $\beta^*$ achieving small squared prediction error $\frac{1}{n} \| X(\hat{\beta} - \beta^*) \|^2$ where $X$ is the matrix whose rows are $X_1, \ldots, X_n$.

Principal component analysis. A natural way to describe principal component analysis under oblivious perturbations is that of assuming the noise matrix $N$ to be an $n$-by-$n$ matrix with a uniformly random set of $\alpha \cdot n^2$ entries bounded by some small $\zeta > 0$ in absolute value. In these settings, we may think of the $\alpha$ fraction of entries with small noise as the

1In this paper, we assume $\Omega \subseteq \mathbb{R}^d$ and $Z \subseteq \mathbb{R}^D$ for some $d, D$, denote random variables in bold face, and hide absolute constant factors with the notation $\mathcal{O}(\cdot), \Omega(\cdot), \gtrsim, \lesssim$ and logarithmic factors with $\tilde{\mathcal{O}}(\cdot), \tilde{\Omega}(\cdot)$.

2Our analysis also works for the parameter error $\| \beta^* - \hat{\beta} \|$. 

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set of uncorrupted observations. Moreover, for $\zeta > 0$, the fact that even for uncorrupted observations the noise is non-zero allows us to capture both gross sparse errors and small entry-wise noise in the measurements at the same time (for example if $\zeta = 1$ then the model captures settings with additional standard Gaussian noise). Remarkably, for $\zeta = 0$, Candès et al.’s seminal work [CLMW11] provided an algorithm that exactly recovers $L^*$ even for a vanishing fraction of inliers, under incoherence conditions on the signal $L^*$. The result was slightly extended in [ZLW+10] where the authors provided an algorithm recovering $L^*$ up to squared error $O(\zeta^2 \cdot n^4)$ thus allowing polynomially small measurement noise $\zeta$, but still failing to capture settings where standard Gaussian measurement noise is added to the sparse noise. Even for simple signal matrices $L^*$, prior to this work, it remained an open question whether consistent estimators could be designed in presence of both oblivious noise and more reasonable measurement noise (e.g., standard Gaussian).

**Linear regression.** Similarly to the context of principal component analysis, a convenient model for oblivious adversarial corruptions is that of assuming the noise vector $\eta$ to have a random set of $\alpha \cdot n$ entries bounded by $\zeta$ in absolute value (here all results of interest can be extended to any $\zeta > 0$ just by scaling, so we consider only the case $\zeta = 1$). This model trivially captures the classical settings with Gaussian noise (a Gaussian vector with variance $\sigma^2$ will have an $\alpha = \Theta(1/\sigma)$ fraction of inliers) and, again, allows us to think of the $\alpha$-fraction of entries with small noise as the set of uncorrupted observations. Early works on consistent regression focused on the regime with Gaussian design $X_1, \ldots, X_n \sim N(0, \Sigma)$ and deterministic noise $\eta$.[BJKK17] presented an estimator achieving error $\tilde{O}(d/(\alpha^2 \cdot n))$ for any $\alpha$ larger than a fixed constant. [SBRJ19] extended this result by achieving comparable error rates even for a vanishing fraction of inliers $\alpha \gtrsim 1/\log \log n$. Assuming $n \gtrsim d^2/\alpha^2$ [TJSO14] proved that the Huber-loss estimator [Hub64] achieves optimal error rate $O(d/\alpha^2 \cdot n)$ even for polynomially small fraction of inliers $\alpha \gtrsim \sqrt{d/n}$. This line of work culminated in [dNS21] which extended the result of [TJSO14] achieving the same guarantees with optimal sample complexity $n \gtrsim d/\alpha^2$. For Gaussian design, similar result as [dNS21] can be extracted from independent work [PF20]. Furthermore, the authors of [dNS21] extended these guarantees to deterministic design matrices satisfying only a sparseness condition (trivially satisfied by sub-Gaussian design matrices).

Huber loss was also analyzed in context of linear regression in [SZF18, DT19, SF20, PJL21]. These works studied models that are different from our model, and these results do not work in the case $\alpha = o(1)$. [SZF18] used assumptions on the moments of the noise, [DT19] and [SF20] studied the model with non-oblivious adversary, and the model from [PJL21] allows corruptions in covariates.

In the context of sparse regression, less is known. When the fraction of observations is constant $\alpha \geq \Omega(1)$, [NTN11] provided a first consistent estimator. Later [DT19] and [SF20] improved that result (assuming $\alpha \geq \Omega(1)$, but these results also work with non-oblivious

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3For Gaussian design $X$ one may also consider a deterministic noise model. This noise model is subsumed by the random noise model discussed here (if $X$ is Gaussian). As shown in [dNS21], roughly speaking the underlying reason is that the Gaussianity of $X$ allows one to obtain several other desirable properties “for free”. For example, one could ensure that the noise vector is symmetric by randomly flipping the sign of each observation $(y_i, X_i)$, as the design matrix will still be Gaussian. See Section 2 for a more in-depth discussion.
adversary). In the case $\alpha = o(1)$, the algorithm of [SBRJ19] (for Gaussian designs) achieves nearly optimal error convergence $\tilde{O}\left(\frac{k \log^2 d}{\alpha^2 n}\right)$, but requires $\alpha \geq 1/\log \log n$. More recently, [dNS21] presented an algorithm for standard Gaussian design $X \sim N(0, 1)^{n \times d}$, achieving the nearly optimal error convergence $\tilde{O}(k/(\alpha^2 \cdot n))$ for nearly optimal inliers fraction $\alpha \geq \sqrt{(k \cdot \log^2 d)/n}$. Both [SBRJ19] and [dNS21] use an iterative process, and require a bound $\|\beta^*\| \leq d^{O(1)}$ (for larger $\|\beta^*\|$ these algorithms also work, but the fraction of inliers or the error convergence is worse). The algorithm from [dNS21], however, heavily relies on the assumption that $X \sim N(0, 1)^{n \times d}$ and appears unlikely to be generalizable to more general families of matrices (including non-spherical Gaussian designs).

**Our contribution.** We propose new machinery to design efficiently computable consistent estimators achieving optimal error rates and sample complexity against oblivious outliers. In particular, we extend the approach of [dNS21] to structured estimation problems, by finding a way to exploit adequately the structure therein. While consistent estimators have already been designed under more benign noise assumptions (e.g. the LASSO estimator for sparse linear regression under Gaussian noise), it was previously unclear how to exploit this structure in the setting of oblivious noise. One key consequence of our work is hence to demonstrate what minimal assumptions on the noise are sufficient to make effective recovery (in the sense above) possible. Concretely, we show

**Oblivious PCA:** Under mild assumptions on the noise matrix $N$ and common assumption on the parameter matrix $L^*$ – traditionally applied in the context of matrix completion [NW12] – we provide an algorithm that achieves optimal error guarantees.

**Sparse regression:** Under mild assumptions on the design matrix and the noise vector – similar to the ones used in [dNS21] for dense parameter vectors $\beta^*$ – we provide an algorithm that achieves optimal error guarantees and sample complexity.

For both problems, our analysis improves over the state-of-the-art and recovers the classical optimal guarantees, not only for Gaussian noise, but also under much less restrictive noise assumptions. At a high-level, we achieve the above results by equipping the Huber loss estimator with appropriate regularizers. Our techniques closely follow standard analyses for $M$-estimators, but crucially depart from them when dealing with the observations with large perturbations. Furthermore, our analysis appears to be mechanical and thus easily applicable to many different estimation problems.

## 2 Results

Our estimators are based on regularized versions of the Huber loss. The regularizer we choose depends on the underlying structure of the estimation problem: We use $\ell_1$ regularization to enforce sparsity in linear regression and nuclear norm regularization to enforce a low-rank structure in the context of PCA. More formally, the *Huber penalty* is defined as the function $f_h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that
\[ f_h(t) := \begin{cases} \frac{1}{2}t^2 & \text{for } |t| \leq h, \\ h(|t| - \frac{h}{2}) & \text{otherwise}. \end{cases} \] (2.1)

where \( h > 0 \) is a penalty parameter. For \( X \in \mathbb{R}^D \), the Huber loss is defined as the function \( F_h(X) := \sum_{i \in [D]} f_h(X_i) \). We will define the regularized versions in the following sections. For a matrix \( A \), we use \( \|A\|, \|A\|_{\text{nuc}}, \|A\|_F, \|A\|_{\text{max}} \) to denote its spectral, nuclear, Frobenius, maximum\(^4\) norms, respectively. For a vector \( v \), we use \( \|v\| \) and \( \|v\|_1 \) to denote its \( \ell_2 \) and \( \ell_1 \) norms.

### 2.1 Oblivious principal component analysis

For oblivious PCA, we provide guarantees for the following estimator (\( \zeta \) will be defined shortly):

\[ \hat{L} := \underset{L \in \mathbb{R}^{n \times n}}{\text{argmin}} \left( F_h(Y - L) + 100\sqrt{n}(\zeta + \rho/n)\|L\|_{\text{nuc}} \right). \] (2.2)

**Theorem 2.1.** Let \( L^* \in \mathbb{R}^{n \times n} \) be an unknown deterministic matrix and let \( N \) be an \( n \)-by-\( n \) random matrix with independent, symmetrically distributed (about zero) entries and \( \alpha := \min_{i,j \in [n]} P\{|N_{ij}| \leq \zeta\} \) for some \( \zeta > 0 \). Suppose that \( \text{rank}(L^*) = r \) and \( \|L^*\|_{\text{max}} \leq \rho/n \).

Then, with probability at least \( 1 - 2^{-n} \) over \( N \), given \( Y = L^* + N \), \( \zeta \) and \( \rho \), the estimator Eq. (2.2) with Huber parameter \( h = \zeta + \rho/n \) satisfies

\[ \|\hat{L} - L^*\|_F \leq O\left( \frac{\sqrt{n}}{\alpha} \right) \cdot (\zeta + \rho/n). \]

We first compare the guarantees of Theorem 2.1 with the previous results on robust PCA [CLMW11, ZLW\(^+\)10].\(^5\) The first difference is that they require \( L^* \) to satisfy certain incoherent conditions.\(^6\) Concretely, they provide theoretical guarantees for \( r \leq O(\mu^{-1}n(\log n)^{-2}) \), where \( \mu \) is the incoherence parameter. In certain regimes, such a constraint strongly binds with the eigenvectors of \( L^* \), restricting the set of admissible signal matrices. Using the different assumption \( \|L^*\|_{\text{max}} \leq \rho/n \) (commonly used for matrix completion, see [NW12]), we can obtain nontrivial guarantees (i.e. \( \|\hat{L} - L^*\|_F/\|L^*\|_F \to 0 \) as \( n \to \infty \)) even when the \( \mu \)-incoherence conditions are not satisfied for any \( \mu \leq n/\log^2 n \), and hence the results [CLMW11, ZLW\(^+\)10] cannot be applied. We remark that, without assuming incoherence, the dependence of the error on the maximal entry of \( L^* \) is inherent (see Remark 3.2).

\(^4\)For \( n \times m \) matrix \( A, \|A\|_{\text{max}} = \max_{i,j \in [n], i \in [m]} |A_{ij}|. \)

\(^5\)We remark that in [CLMW11] the authors showed that they can consider non-symmetric noise when the fraction of inliers is large \( \alpha \geq 1/2 \). However for smaller fraction of inliers their analysis requires the entries of the noise to be symmetric and independent, so for \( \alpha < 1/2 \), their assumptions are captured by Theorem 2.1.

\(^6\)A rank-\( r \) \( n \times n \) dimensional matrix \( M \) is \( \mu \)-incoherent if its singular vector decomposition \( M := U \Sigma V^T \) satisfies \( \max_{i \in [n]} \|U^T e_i\|^2 \leq \frac{\mu}{\pi}, \max_{i \in [n]} \|V^T e_i\|^2 \leq \frac{\mu}{\pi} \) and \( \|UV^T\|_{\infty} \leq \frac{\sqrt{\mu}}{\pi}. \)
The second difference is that Theorem 2.1 provides a significantly better dependence on the magnitude \( \zeta \) of the entry-wise measurement error. Specifically, in the settings of Theorem 2.1, [ZLW+10] showed that if \( L^* \) satisfies the incoherence conditions, the error of their estimator is \( O(n^2 \zeta) \). If the entries of \( N \) are standard Gaussian with probability \( a \) (and hence \( \zeta \leq O(1) \)), and the entries of \( L^* \) are bounded by \( O(1) \), then the error of our estimator is \( O(\sqrt{n}/\alpha) \), which is considerably better than \( O(n^2) \) as in [ZLW+10]. On the other hand, their error does not depend on the magnitude \( \rho/n \) of the signal entries, so in the extreme regimes when the singular vectors of \( L^* \) satisfy the incoherence conditions but \( L^* \) has very large singular values (so that the magnitude of the entries of \( L \) is significantly larger than \( n \)), their analysis provides better guarantees than Theorem 2.1.

As another observation to understand Theorem 2.1, notice that our robust PCA model also captures the classical matrix completion settings. In fact, any instance of matrix completion can be easily transformed into an instance of our PCA model: for the entries \( (i,j) \) that we do not observe, we can set \( N_{i,j} \) to some arbitrarily large value \( \pm C(\rho, n) \gg \rho/n \), making the signal-to-noise ratio of the entry arbitrarily small. The observed (i.e. uncorrupted) \( \Theta(\alpha \cdot n^2) \) entries may additionally be perturbed by Gaussian noise with variance \( \Theta(\zeta^2) \). The error guarantees of the estimator in Theorem 2.1 is \( O((\rho/n + \zeta) \sqrt{n}/\alpha) \). Thus, the dependency on the parameters \( \rho, n, \zeta, \) and \( r \) is the same as in matrix completion and the error is within a factor of \( \Theta(\sqrt{1/\alpha}) \) from the optimum for matrix completion. However, this worse dependency on \( \alpha \) is intrinsic to the more general model considered and it turns out to be optimal (see Theorem 2.4). On a high level, the additional factor of \( \Theta(\sqrt{1/\alpha}) \) comes from the fact that in our PCA model we do not know which entries are corrupted. The main consequence of this phenomenon is that a condition of the form \( \alpha \geq \sqrt{1/n} \) appears inherent to achieve consistency. To get some intuition on why this condition is necessary, consider the Wigner model where we are given a matrix \( Y = xx^\top + \sigma W \) for a flat vector \( x \in \{ \pm 1 \}^n \) and a standard Gaussian matrix \( W \). Note, that the entries of \( W \) fit our noise model for \( \zeta = 1, \rho/n = 1, r = 1 \) and \( \alpha = \Theta(1/\sigma) \). The spectral norm of \( \sigma W \) concentrates around \( 2\sigma\sqrt{n} \) and thus it is information-theoretically impossible to approximately recover the vector \( x \) for \( \sigma = 1/\alpha = \omega(\sqrt{n}) \) (see [PWBM16]).

### 2.2 Sparse regression

Our regression model considers a fixed design matrix \( X \in \mathbb{R}^{n \times d} \) and observations \( y := X\beta^* + \eta \in \mathbb{R}^n \) where \( \beta^* \) is an unknown \( k \)-sparse parameter vector and \( \eta \) is random noise with \( P(|\eta_i| \leq 1) \geq \alpha \) for all \( i \in [n] \). Earlier works [BJKK17], [SBRJ19] focused on the setting that the design matrix consists of i.i.d. rows with Gaussian distribution \( N(0, \Sigma) \) and the noise is \( \eta = \zeta + w \) where \( \zeta \) is deterministic \( (\alpha \cdot n) \)-sparse vector and \( w \) is subgaussian. As in [dNS21], our results for a fixed design and random noise can, in fact, extend to yield the same guarantees for this early setting (see Theorem 2.3). Hence, a key advantage of our results is that the design \( X \) does not have to consist of Gaussian entries. Remarkably, we can handle arbitrary deterministic designs as long as they satisfy some mild conditions. Concretely, we make the following three assumptions, the first two of which are standard in the literature of sparse regression (e.g., see [Wai19], section 7.3):

1. For every column \( X^i \) of \( X, \|X^i\| \leq \sqrt{n} \).
2. Restricted eigenvalue property (RE-property): For every vector \( u \in \mathbb{R}^d \) such that\(^7\)
\[
\|u_{\text{supp}(\beta^*)}\|_1 \geq 0.1 \cdot \|u\|_1,
\]
we have \( \frac{1}{\lambda} \|Xu\|^2 \geq \lambda \cdot \|u\|^2 \) for some parameter \( \lambda > 0 \).

3. Well-spreadness property: For some (large enough) \( m \in [n] \) and for every vector \( u \in \mathbb{R}^d \)
such that \( \|u_{\text{supp}(\beta^*)}\|_1 \geq 0.1 \cdot \|u\|_1 \) and for every subset \( S \subseteq [n] \) with \( |S| \geq n - m \), it holds that \( \| (Xu)_S \| \geq \frac{1}{2} \|Xu\| \).

Denote \( F_2(\beta) := \sum_{i=1}^n f_2(y_i - \langle X_i, \beta \rangle) \), where \( X_i \) are the rows of \( X \), and \( f_2 \) is as in Eq. (2.1).

We devise our estimator for sparse regression and state its statistical guarantees below:

\[
\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^d} \left( F_2(\beta) + 100\sqrt{n \log d} \cdot \|\beta\|_1 \right) . \quad (2.3)
\]

**Theorem 2.2.** Let \( \beta^* \in \mathbb{R}^d \) be an unknown \( k \)-sparse vector and let \( X \in \mathbb{R}^{n \times d} \) be a deterministic matrix such that for each column \( X^i \) of \( X \), \( \|X^i\| \leq \sqrt{n} \), satisfying the RE-property with \( \lambda > 0 \) and

well-spreadness property with \( m \geq \frac{k \log d}{\lambda \alpha} \) (recall that \( n \geq m \)).

Further, let \( \eta \) be an \( n \)-dimensional random vector with independent, symmetrically distributed (about zero) entries and \( \alpha = \min_{i \in [n]} \mathbb{P}\{|\eta_i| \leq 1\} \).

Then with probability at least \( 1 - d^{-10} \) over \( \eta \), given \( X \) and \( y = X\beta^* + \eta \), the estimator Eq. (2.3) satisfies

\[
\frac{1}{n} \|X(\hat{\beta} - \beta^*)\|^2 \leq O\left( \frac{1}{\lambda} \cdot \frac{k \log d}{\alpha^2 \cdot n} \right)
\]

and

\[
\|\hat{\beta} - \beta^*\|^2 \leq O\left( \frac{1}{\lambda^2} \cdot \frac{k \log d}{\alpha^2 \cdot n} \right).
\]

There are important considerations when interpreting this theorem. The first is the special case \( \eta \sim N(0, \sigma^2)^n \), which satisfies our model for \( \alpha = \Theta(1/\sigma) \). For this case, it is well known (e.g., see [Wai19], section 7.3) that under the same RE assumption, the LASSO estimator achieves a prediction error rate of \( O(\frac{\sigma^2}{\lambda} \cdot \frac{k \log d}{n}) = O(\frac{k \log d}{\lambda \cdot \alpha^2 \cdot n}) \), matching our result. Moreover, this error rate is essentially optimal. Under a standard assumption in complexity theory (\( \text{NP} \not\subseteq \text{P/poly} \)), the RE assumption is necessary when considering polynomial-time estimators [ZWJ14]. Further, this also shows that the dependence on the RE constant seems unavoidable. Under mild conditions on the design matrix, trivially satisfied if the rows are i.i.d. Gaussian with covariance \( \Sigma \) whose condition number is constant, our guarantees are optimal up to constant factors for all estimators if \( k \leq d^{1-O(1)} \) (e.g. \( k \leq d^{0.99} \)), see [RWW11]. This optimality also shows that our bound on the number of samples is best possible since otherwise we would not be able to achieve vanishing error. The (non-sparse version of) well-spreadness property was first used in the context of regression in [dNS21]. In the same work the authors also showed that, under oblivious noise assumptions, some weak form of spreadness property is indeed necessary.

The second consideration is the optimal dependence on \( \alpha \): Theorem 2.2 achieves consistency as long as the fraction of inliers satisfies \( \alpha = \omega(\sqrt{k \log d/n}) \). To get an intuition, observe that lower bounds for standard sparse regression show that, already for

\(^7\)For a vector \( v \in \mathbb{R}^d \) and a set \( S \subseteq [d] \), we denote \( v_{\text{S}} \) the restriction of \( v \) to the coordinates in \( S \).
Theorem 2.2. Theorem 2.2 is a standard assumption in sparse regression and it is easier to consider \( \text{BJKK17} \) is the first result to achieve consistency under such minimalistic noise settings and deterministic designs. Previous results [BJKK17, SBRJ19, dNS21] focused on simpler settings of Gaussian design \( X \) and deterministic noise, and provide no guarantees for more general models. The techniques for Theorem 2.2 also extend to this case.

Theorem 2.3. Let \( \beta^* \in \mathbb{R}^d \) be an unknown \( k \)-sparse vector and let \( X \) be a \( n \times d \) random matrix with i.i.d. rows \( X_1, \ldots, X_n \sim N(0, \Sigma) \) for a positive definite matrix \( \Sigma \). Further, let \( \eta \in \mathbb{R}^n \) be a deterministic vector with \( \alpha \cdot n \) coordinates bounded by 1 in absolute value. Suppose that \( n \geq \frac{\nu(\Sigma) \cdot k \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2} \), where \( \nu(\Sigma) \) is the maximum diagonal entry of \( \Sigma \) and \( \sigma_{\min}(\Sigma) \) is its smallest eigenvalue.

Then, with probability at least \( 1 - d^{-10} \) over \( X \), given \( X \) and \( y = X \beta^* + \eta \), the estimator \( \text{Eq. (2.3)} \) satisfies

\[
\frac{1}{n} \left\| X (\hat{\beta} - \beta^*) \right\|^2 \leq O \left( \frac{\nu(\Sigma) \cdot k \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2 \cdot n} \right) \quad \text{and} \quad \left\| \hat{\beta} - \beta^* \right\|^2 \leq O \left( \frac{\nu(\Sigma) \cdot k \log d}{\sigma_{\min}^2(\Sigma) \cdot \alpha^2 \cdot n} \right).
\]

Even for standard Gaussian design \( X \sim N(0, 1)^{n \times d} \), the above theorem improves over previous results, which required sub-optimal sample complexity \( n \geq (k/\alpha^2) \cdot \log d \cdot \| \beta^* \|^2 \). For non-spherical Gaussian designs, the improvement over state of the art [SBRJ19] is more serious: their algorithm requires \( \alpha \geq \Omega(1/\log \log n) \), while our Theorem 2.3 doesn’t have such restrictions and works for all \( \alpha \geq \sqrt{\frac{\nu(\Sigma)}{\sigma_{\min}(\Sigma)} \cdot \frac{k \log d}{n}} \); in many interesting regimes \( \alpha \) is allowed to be smaller than \( n^{-\Omega(1)} \). The dependence on \( \alpha \) is nearly optimal: the estimator is consistent as long as \( \alpha \geq \omega \left( \sqrt{\frac{\nu(\Sigma)}{\sigma_{\min}(\Sigma)} \cdot \frac{k \log d}{n}} \right) \), and from the discussion after Theorem 2.3, if \( \alpha \leq O(\sqrt{(k \log d)/n}) \), no estimator is consistent.

Note that while we can deal with general covariance matrices, to compare Theorem 2.3 with Theorem 2.2 it is easier to consider \( \Sigma \) in a normalized form, when \( \nu(\Sigma) \leq 1 \). This can be easily achieved by scaling \( X \). Also note that Theorem 2.2 can be generalized to the case \( \| X \| \leq \sqrt{\nu \gamma} \) for arbitrary \( \nu > 0 \), and then the error bounds and the bound on \( m \) should be multiplied by \( \nu \).

The RE-property of Theorem 2.2 is a standard assumption in sparse regression and is satisfied by a large family of matrices. For example, with high probability a random matrix \( X \) with i.i.d. rows sampled from \( N(0, \Sigma) \), with positive definite \( \Sigma \in \mathbb{R}^{d \times d} \) whose diagonal entries are bounded by 1, satisfies the RE-property with parameter \( \Omega(\sigma_{\min}(\Sigma)) \) for all subsets of \([d]\) of size \( k \) (so for every possible support of \( \beta^* \)) as long as \( n \geq \frac{1}{\sigma_{\min}(\Sigma)} \cdot k \log d \) (see [Wai19], section 7.3.3). The well-spread assumption is satisfied for such \( X \) with for all sets \( S \subset [n] \) of size \( m \leq cn \) (for sufficiently small \( c \)) and for all subsets of \([d]\) of size \( k \) as long as \( n \geq \frac{1}{\sigma_{\min}(\Sigma)} \cdot k \log d \).

\( \eta \sim N(0, \sigma \cdot \text{Id}_n) \), it is possible to achieve consistency only for \( n \geq \omega(\sigma^2 k \log d) \) (if \( k \leq d^{-\Omega(1)} \)). As for this \( \eta \), the number of entries of magnitude at most 1 is \( O(n/\sigma) \) with high probability, it follows that for \( \alpha = \Theta(1/\sigma) \leq O\left(\sqrt{(k \log d)/n}\right) \), no estimator is consistent.

To the best of our knowledge, Theorem 2.2 is the first result to achieve consistency under such minimalistic noise settings and deterministic designs. Previous results [BJKK17, SBRJ19, dNS21] focused on simpler settings of Gaussian design \( X \) and deterministic noise, and provide no guarantees for more general models. The techniques for Theorem 2.2 also extend to this case.
2.3 Optimal fraction of inliers for principal component analysis under oblivious noise

We show here that the dependence on $\alpha$ we obtain in Theorem 2.1 is information therapeutically optimal up to constant factors. Concretely, let $L^*, N, Y, \alpha, \rho$ and $\zeta$ be as in Theorem 2.1, and let $0 < \varepsilon < 1$ and $0 < \delta < 1$. A successful $(\varepsilon, \delta)$-weak recovery algorithm for PCA is an algorithm that takes $Y$ as input and returns a matrix $\hat{L}$ such that $\|L - L^*\|_F \leq \varepsilon \cdot \rho$ with probability at least $1 - \delta$.

It can be easily seen that the Huber-loss estimator of Theorem 2.1 fails to be a successful weak-recovery algorithm if $\alpha = o(\sqrt{r/n})$ (for both cases $\zeta \leq \rho/n$ and $\rho/n \leq \zeta$, we need $\alpha = \Omega(\sqrt{r/n})$). A natural question to ask is whether the condition $\alpha = \Omega(\sqrt{r/n})$ is necessary in general. The following theorem shows that if $\alpha = o(\sqrt{r/n})$, then weak-recovery is information-theoretically impossible. This means that the (polynomially small) fraction of inliers that the Huber-loss estimator of Theorem 2.1 can deal with is optimal up to a constant factor.

**Theorem 2.4.** There exists a universal constant $C_0 > 0$ such that for every $0 < \varepsilon < 1$ and $0 < \delta < 1$, if $\alpha := \min_{i,j \in [n]} P[|N_{i,j}| \leq \zeta]$ satisfies $\alpha < C_0 \cdot (1 - \varepsilon^2)^2 \cdot (1 - \delta) \cdot \sqrt{r/n}$, and $n$ is large enough, then it is information-theoretically impossible to have a successful $(\varepsilon, \delta)$-weak recovery algorithm.

The problem remains information-theoretically impossible (for the same regime of parameters) even if we assume that $L^*$ is incoherent; more precisely, even if we know that $L^*$ has incoherence parameters that are as good as those of a random flat matrix of rank $r$, the theorem still holds.

**Notation and outline**

**Notation.** For $\beta \in \mathbb{R}^d$ we define the function $\|\beta\|_0 := \sum_{i \in [d]} 1_{[\beta_i \neq 0]}$. For a subspace $\Omega \subseteq \mathbb{R}^d$ we denote the projection of $\beta$ onto $\Omega$ by $\beta_{\Omega}$. We write $\Omega^\perp$ for the orthogonal complement of $\Omega$. For $N \in \mathbb{N}$ we denote $[N] := \{1, 2, \ldots, N\}$. We write log for the logarithm to the base $e$.

For a matrix $X \in \mathbb{R}^{d \times d}$ we denote by $\text{rspan}(X)$ and $\text{cspan}(X)$ respectively the rows and columns span of $X$, and we write $\|X\|$ for the spectral norm of $X$, $\|X\|_F$ for its Frobenius norm, $\|X\|_{\text{nuc}}$ for its nuclear norm and $\|X\|_{\text{max}} := \max_{i,j \in [n]} |X_{ij}|$. For a vector $v \in \mathbb{R}^n$ we write $\|v\|$ for its Euclidean norm, $\|v\|_1 = \sum_{i=1}^N |v_i|$ and $\|v\|_\infty = \max_{i \in [N]} |v_i|$. For a norm $\|\cdot\|$ we write $\|\cdot\|$ for its dual. We denote by $G \sim N(0, 1)^{n \times d}$ a random $n$-by-$d$ matrix $G$ with i.i.d. standard Gaussian entries. Similarly, we denote by $g \sim N(0, 1)^n$ an $n$-dimensional random vector $g$ with i.i.d. standard Gaussian entries.

For a set $S$ and a metric $\rho : S \times S \to [0, \infty)$, we denote an $\varepsilon$-net in $S$ by $N_{\varepsilon, \rho}(S)$. That is, $N_{\varepsilon, \rho}(S)$ is a subset of $S$ such that for any $u \in S$ there exists $v \in N_{\varepsilon, \rho}(S)$ satisfying $\rho(u, v) \leq \varepsilon$.

**Outline.** In Section 3, we present the main ideas behind our results. In Section 4, we introduce a general framework that allows us to prove our main theorems about consistent
estimation (Theorem 2.1, Theorem 2.2 and Theorem 2.3).

Sections 5 to 8 are devoted to the proofs of theorems from Section 2. Concretely, in Section 5 we prove Theorem 2.1, Section 6 contains a proof of Theorem 2.2 and in Section 7 we prove Theorem 2.3. Complementing these algorithmic results, we prove a lower bound (Theorem 2.4) in Section 8.

Finally, Appendix A contains the proofs of a few facts about the Huber loss and Appendix B contains a few facts from probability theory.

3 Techniques

To illustrate our techniques in proving statistical guarantees for the Huber-loss estimator, we first use sparse linear regression as a running example. Then, we discuss how the same ideas apply to principal component analysis. Finally, we also remark our techniques for lower bounds.

3.1 Sparse linear regression under oblivious noise

We consider the model of Theorem 2.2. Our starting point to attain the guarantees for our estimator Eq. (2.3), i.e., \( \hat{\beta} := \arg\min_{\beta \in \mathbb{R}^p} F_2(\beta) + 100 \sqrt{n \log d}\|\beta\|_1 \), is a classical approach for M-estimators (see e.g. [Wai19], chapter 9). For simplicity, we will refer to \( F_2(\beta) \) as the loss function and to \( \|\beta\|_1 \) as the regularizer. At a high level, it consists of the following two ingredients:

(I) an upper bound on some norm of the gradient of the loss function at the parameter \( \beta^* \),

(II) a lower bound on the curvature of the loss function (in form of a local strong convexity bound) within a structured neighborhood of \( \beta^* \). The structure of this neighborhood can roughly be controlled by choosing the appropriate regularizer.

The key aspect of this strategy is that the strength of the statistical guarantees of the estimator crucially depends on the directions and the radius in which we can establish lower bounds on the curvature of the function. Since these features are inherently dependent on the landscape of the loss function and the regularizer, they may differ significantly from problem to problem. This strategy has been applied successfully for many related problems such as compressed sensing or matrix completion albeit with standard noise assumptions.\(^8\)

Under oblivious noise, [dNS21] used a particular instantiation of this framework to prove optimal convergence of the Huber-loss – without any regularizer – for standard linear regression. Such estimator, however, doesn’t impose any structure on the neighborhood of \( \beta^* \) considered in (II) and thus, can only be used to obtain sub-optimal guarantees for sparse regression.

In the context of sparse regression, the above two conditions translate to: (I) an upper bound on the largest entry in absolute value of the gradient of the loss function at \( \beta^* \), and

\(^8\)The term "standard noise assumptions" is deliberately vague; as a concrete example, we will refer to (sub)-Gaussian noise distributions. See again chapter 9 of [Wai19] for a survey.
(II) a lower bound on the curvature of $F_2$ within the set of approximately $k$-sparse vectors\footnote{We will clarify this notion in the subsequent paragraphs.} close to $\beta^*$. We use this recipe to show that all approximate minimizers of $F$ are close to $\beta^*$. While the idea of restricting to only approximately sparse directions has also been applied for the LASSO estimator in sparse regression under standard (sub)-Gaussian noise, in the presence of oblivious noise, our analysis of the Huber-loss function requires a more careful approach.

More precisely, under the assumptions of Theorem 2.2, the error bound can be computed as
\[
O\left(\frac{s\|G\|_{\text{reg}}}{\kappa}\right),
\]
where $G$ is a gradient of Huber loss at $\beta^*$, $\|\cdot\|_{\text{reg}}$ is a norm dual to the regularization norm (which is equal to $\|\cdot\|_{\text{max}}$ for $\ell_1$ regularizer), $s$ is a structure parameter, which is equal to $\sqrt{k/\lambda}$, and $\kappa$ is a restricted strong convexity parameter. Note that by error here we mean $\frac{1}{\sqrt{n}}\|X(\hat{\beta} - \beta^*)\|$

Similarly, under assumptions of Theorem 2.1, we get the error bound Eq. (3.1), where $G$ is a gradient of Huber loss at $L^*$, $\|\cdot\|_{\text{reg}}$ is a norm dual to the nuclear norm (i.e. the spectral norm), structure parameter $s$ is $\sqrt{r}$, and $\kappa$ is a restricted strong convexity parameter. For more details on the conditions of the error bound, see the supplementary material. Below, we explore the bounds on the norm of the gradient and on the restricted strong convexity parameter.

**Bounding the gradient of the Huber loss.** The gradient of the Huber-loss $F_2(\cdot)$ at $\beta^*$ has the form $\nabla F_2(\beta^*) = \sum_{i=1}^n f'_2[\eta_i] \cdot X_i$, where $X_i$ is the $i$-th row of $X$. The random variables $f'_2[\eta_i], i \in [n]$, are independent, centered, symmetric and bounded by 2. Since we assume that each column of $X$ has norm at most $\sqrt{n}$, the entries of the row $X_i$ are easily bounded by $\sqrt{n}$. Thus, $\nabla F_2(\beta^*)$ is a vector with independent, symmetric entries with bounded variance, so its behavior can be easily studied through standard concentration bounds. In particular, by a simple application of Hoeffding’s inequality, we obtain, with high probability,
\[
\|\nabla F_2(\beta^*)\|_{\text{max}} = \max_{j \in [d]} \left| \sum_{i \in [n]} f'_2[\eta_i] \cdot X_{ij} \right| \leq O\left(\sqrt{n \log d}\right).
\]

**Local strong convexity of the Huber loss.** Proving local strong convexity presents additional challenges. Without the sparsity constraint, [dNS21] showed that under a slightly stronger spreadness assumptions than Theorem 2.2, the Huber loss is locally strongly convex within a constant radius $R$ ball centered at $\beta^*$ whenever $n \geq d/\alpha^2$. (This function is not globally strongly convex due to its linear parts.) Using their result as a black-box, one can obtain the error guarantees of Theorem 2.2, but with suboptimal sample complexity. The issue is that with the substantially smaller sample size of $n \geq \tilde{O}(k/\alpha^2)$ that resembles the usual considerations in the context of sparse regression, the Huber loss is not locally...
strongly convex around \( \beta^* \) uniformly across all directions, so we cannot hope to prove convergence with optimal sample complexity using this argument. To overcome this obstacle, we make use of the framework of M-estimators: Since we consider a regularized version of the Huber loss, it will be enough to show local strong convexity in a radius \( R \) uniformly across all directions which are approximately \( k \)-sparse. For this substantially weaker condition, \( \tilde{O}(k/\alpha^2) \) will be enough.

More in details, for observations \( y = X\beta^* + \eta \) and an arbitrary \( u \in \mathbb{R}^d \) of norm \( \|u\| \leq R \), it is possible to lower bound the Hessian\(^\text{10}\) of the Huber loss at \( \beta^* + u \) by:\(^\text{11}\)

\[
HF_2(\beta^* + u) = 2 \sum_{i=1}^{n} f_2''((Xu)_i - \eta_i) \cdot X_iX_i^T
\]

\[
= 2 \sum_{i=1}^{n} \mathbf{1}_{\| (Xu)_i - \eta_i \| < 2} \cdot X_iX_i^T
\]

\[
\geq M(u) := 2 \sum_{i=1}^{n} \mathbf{1}_{\| Xu \| \leq \lambda} \cdot \mathbf{1}_{\| \eta_i \| \leq \lambda} \cdot X_iX_i^T
\]

As can be observed, we do not attempt to exploit cancellations between \( Xu \) and \( \eta \). Let \( Q := \{ i \in [n] \mid \| \eta_i \| \leq \lambda \} \) be the set of uncorrupted entries of \( \eta \). Given that with high probability \( Q \) has size \( \Omega(\alpha \cdot n) \), the best outcome we can hope for is to provide a lower bound of the form \( \langle u, M(u)u \rangle \geq \alpha n \) in the direction \( \hat{\beta} - \beta^* \). In [dNS21], it was shown that if the span of the measurement matrix \( X \) is well spread, then \( \langle u, M(u)u \rangle \geq \Omega(\alpha \cdot n) \).

If the direction \( \hat{\beta} - \beta^* \) was fixed, it would suffice to show the curvature in that single direction through the above reasoning. However, \( \hat{\beta} \) depends on the unknown random noise vector \( \eta \). Without the regularizer in Eq. (2.3), this dependence indicates that the vector \( \hat{\beta} \) may take any possible direction, so one needs to ensure local strong convexity to hold in a constant-radius ball centered at \( \beta^* \). That is, \( \min_{\| u \| \leq R} \lambda_{\min}(M(u)) \geq \Omega(\alpha \cdot n) \). It can be shown through a covering argument (of the ball) that this bound holds true for \( n \geq d/\alpha^2 \). This is the approach of [dNS21].

The minimizer of the Huber loss follows a sparse direction. The main issue with the above approach is that no information concerning the direction \( \hat{\beta} - \beta^* \) is used. In the settings of sparse regression, however, our estimator contains the regularizer \( \|\beta\|_1 \). The main consequence of the regularizer is that the direction \( \hat{\beta} - \beta^* \) is approximately flat in the sense \( \| \hat{\beta} - \beta^* \|_1 \leq O(\sqrt{k}\| \hat{\beta} - \beta^* \|) \). The reason\(^\text{12}\) is that due to the structure of the objective function in Eq. (2.3) and concentration of the gradient Eq. (3.2), the penalty for dense vectors is larger than the absolute value of the inner product \( \langle \nabla F(\beta^*), \hat{\beta} - \beta^* \rangle \) (which, as previously argued, concentrates around its (zero) expectation).

\(^\text{10}\)The Hessian does not exist everywhere. Nevertheless, the second derivative of the penalty function \( f_2 \) exists as an \( L_1 \) function in the sense that \( f_2''(b) - f_2''(a) = 2 \int_{a}^{b} \mathbf{1}_{\| t \| \leq 2} dt \). This property is enough for our purposes.

\(^\text{11}\)A more extensive explanation of the first part of this analysis can be found in [dNS21].

\(^\text{12}\)This phenomenon is a consequence of the decomposability of the \( L_1 \) norm, see the supplementary material.
This specific structure of the minimizer implies that it suffices to prove local strong convexity only in approximately sparse directions. For these set of directions, we carefully construct a sufficiently small covering set so that \( n \geq \tilde{O}(k/\alpha^2) \) samples suffice to ensure local strong convexity over it.

**Remark 3.1 (Comparison with LASSO).** It is important to remark that while this approach of only considering approximately sparse directions has also been used in the context of sparse regression under Gaussian noise (e.g., the LASSO estimator), obtaining the desired lower bound is considerably easier in these settings as it directly follows from the restricted eigenvalue property of the design matrix. In our case, we require an additional careful probabilistic analysis which uses a covering argument for the set of approximately sparse vectors. As we see however, it turns out that we do not need any additional assumptions on the design matrix when compared with the LASSO estimator except for the well-spreadness property (recall that some weak version of well-spreadness is indeed necessary in robust settings, see [dNS21]).

### 3.2 Principal component analysis under oblivious noise

A convenient feature of the approach in Section 3.1 for sparse regression, is that it can be easily applied to additional problems. We briefly explain here how to apply it for principal component analysis. We consider the model defined in Theorem 2.1. We use an estimator based on the Huber loss equipped with the nuclear norm as a regularizer to enforce the low-rank structure in our estimator

\[
\hat{\mathbf{L}} := \arg\min_{\mathbf{L} \in \mathbb{R}^{m \times n}, \|\mathbf{L}\|_{\text{max}} \leq \rho/n} \left( F_{\zeta + \rho/n}(\mathbf{Y} - \mathbf{L}) + 100\sqrt{n}(\zeta + \rho/n)\|\mathbf{L}\|_{\text{nuc}} \right), \tag{3.3}
\]

In this setting, the gradient \( \nabla F_{\zeta + \rho/n}(\mathbf{Y} - \mathbf{L}^*) \) is a matrix with independent, symmetric entries which are bounded (by \( \zeta + \rho/n \)) and hence its spectral norm is \( O((\zeta + \rho/n)\sqrt{n}) \) with high probability. Local strong convexity can be obtained in a similar fashion as shown in Section 3.1: due to the choice of the Huber transition point all entries with small noise are in the quadratic part of \( F \). Moreover, the nuclear norm regularizer ensures that the minimizer is an approximately low-rank matrix in the sense that \( \|\mathbf{M}\|_{\text{nuc}} \leq O(\sqrt{n}\|\mathbf{M}\|_F) \).

So again, it suffices to provide curvature of the loss function only on these subset of structured directions.

**Remark 3.2 (Incoherence vs. spikiness).** Recall the discussion on incoherence in Section 2. If for every \( \mu \leq n/\log^2 n \) matrix \( \mathbf{L}^* \) doesn’t satisfy \( \mu \)-incoherence conditions, the results in [CLMW11, ZLW+10] cannot be applied. However, our estimator achieves error \( \|\hat{\mathbf{L}} - \mathbf{L}^*\|_F/\|\mathbf{L}^*\|_F \to 0 \) as \( n \to \infty \). Indeed, let \( \omega(1) \leq f(n) \leq o(\log^2 n) \) and assume \( \zeta = 0 \). Let \( \mathbf{u} \in \mathbb{R}^n \) be an \( f(n) \)-sparse unit vector whose nonzero entries are equal to \( 1/\sqrt{f(n)} \). Let \( \mathbf{v} \in \mathbb{R}^n \) be a vector with all entries equal to \( 1/\sqrt{n} \). Then, \( \mathbf{u}\mathbf{v}^\top \) does not satisfy incoherence with any \( \mu < n/f(n) \). We have \( \|\mathbf{u}\mathbf{v}^\top\|_F = 1 \), and the error of our estimator is \( O(1/(\alpha\sqrt{f(n)})) \), so it tends to zero for constant (or even some subconstant) \( \alpha \).

Furthermore notice that the dependence of the error of Theorem 2.1 on the maximal entry of \( \mathbf{L}^* \) is inherent if we do not require incoherence. Indeed, consider \( \mathbf{L}_1 = b \cdot \mathbf{e}_1\mathbf{e}_1^\top \) for...
large enough $b > 0$ and $L_2 = e_2 e_2^T$. For constant $\alpha$, let $|N_{ij}|$ be 1 with probability $\alpha/2$, 0 with probability $\alpha/2$ and $b$ with probability $1 - \alpha$. Then given $Y$ we cannot even distinguish between cases $L^* = L_1$ or $L^* = L_2$, and since $\|L_1 - L_2\|_F \geq b$, the error should also depend on $b$.

Remark 3.3 (\(\alpha\) vs \(\alpha^2\): what if one knows which entries are corrupted?). As was observed in Section 2, the error bound of our estimator is worse than the error for matrix completion by a factor $1/\sqrt{\alpha}$. We observe similar effect in linear regression: if, as in matrix completion, we are given a randomly chosen $\alpha$ fraction of observations \(\{(X_i, y_i = \langle X_i, \beta^* \rangle + \eta_i\}\}_{i=1}^n\) where $\eta \sim N(0, 1)^n$, and since for the remaining samples we may not assume any bound on the signal-to-noise ratio, then this problem is essentially the same as linear regression with $an$ observations. Thus the optimal prediction error rate is $\Theta(\sqrt{d/(an)})$. Now, if we have $y = X\beta^* + \eta$, where $\eta \sim N(0, 1/\alpha^2)$, then with probability $\Theta(\alpha)$, $|\eta_i| \leq 1$, but the optimal prediction error rate in this case is $\Theta(\sqrt{d/(\alpha^2 n)})$. So in both linear regression and robust PCA, prior knowledge of the set of corrupted entries makes the problem easier.

3.3 Optimal fraction of inliers for principal component analysis under oblivious noise

In order to prove Theorem 2.4, we will adopt a generative model for the hidden matrix $L^*$: We will generate $L^*$ randomly but assume that the distribution is known to the algorithm. This makes the problem easier. Therefore, any impossibility result for this generative model would imply impossibility for the more restrictive model for which $L^*$ is deterministic but unknown.

We generate a random flat matrix $L^*$ using $n \cdot r$ independent and uniform random bits in such a way that $L^*$ is of rank $r$ and incoherent with high probability. Then, for every constant $0 < \xi < 1$, we find a distribution for the random noise $N$ in such a way that the fraction of inliers satisfies $\alpha := \mathbb{P}[|N_{ij}| \leq \zeta] = \Theta\left(\xi \sqrt{r/n}\right)$, and such that the mutual information between $L^*$ and $Y = L^* + N$ can be upper bounded as $I(L^*; Y) \leq O(\xi \cdot n \cdot r)$. Roughly speaking the smaller $\xi$ gets, the more independent $L^*$ and $Y$ will be. Now using an inequality that is similar to the standard Fano-inequality but adapted to weak-recovery, we show that if there is a successful $(\varepsilon, \delta)$-weak recovery algorithm for $L^*$ and $N$, then $I(L^*; Y) \geq \Omega((1 - \varepsilon^2)^2 \cdot (1 - \delta) \cdot n \cdot r)$. By combining all these observations together, we can deduce that if $\xi$ is small enough, it is impossible to have a successful $(\varepsilon, \delta)$-weak recovery algorithm for $L^*$ and $N$.

4 Meta-Theorem

We present a high-level theorem which will be applied to prove Theorem 2.1, Theorem 2.2 and Theorem 2.3. Recall the general setting an estimation problem: we start with a family of probability distributions $\mathcal{P} := \{P_\theta \mid \theta \in \Omega\}$ over some space $\mathcal{Z}$ and indexed by some parameter $\theta \in \Omega$. We observe a collection of $n$ independent samples $Z = (Z_1, \ldots, Z_n)$ taking value in $\mathcal{Z}$, drawn from an unknown probability distribution $P_{\theta^*} \in \mathcal{P}$. We assume $\Omega \subseteq \mathbb{R}^d$. 

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and $\mathcal{Z} \subseteq \mathbb{R}^D$ for some integers $d$ and $D$. Our goal is then to recover $\theta^*$. That is, given $Z$, the goal is to find $\hat{\theta} \in \mathbb{R}^d$ such that for some suitable error function $E : \mathbb{R}^d \to [0, \infty)$, the value $E(\theta^* - \hat{\theta})$ is as small as possible. It is clear that this general setting also captures settings in which the observations are perturbed by oblivious adversarial noise.

On a high level, we will use the following scheme:

1. Let $\|\cdot\|_{\text{reg}} : \mathbb{R}^d \to [0, \infty)$ be a norm, and let $\gamma \in \mathbb{R}$ be a scalar. Design a cost function $F : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which depends on $Z$.

2. For a set $C \subseteq \mathbb{R}^d$, show that the target parameter (or some approximation of it)

$$
\hat{\theta} := \arg\min_{\theta \in C} (F(\theta) + \gamma \|\theta\|_{\text{reg}})
$$

satisfies $E(\theta^* - \hat{\theta}) \leq R$ for some acceptable $R \geq 0$ with high probability over the samples $Z$.

3. Argue that $\hat{\theta}$ can be computed efficiently.

The norm $\|\cdot\|_{\text{reg}}$ is often referred to as a regularizer. Its role is to enforce a certain structure on the target parameter. For example, in the context of sparse linear regression $y = X\beta^* + \eta$ with $\beta^* \in \mathbb{R}^d$ being a $k$-sparse vector, the LASSO estimator:

$$
\hat{\beta} := \arg\min_{\beta \in \mathbb{R}^d} \left( \|X\beta - y\|^2 + \gamma \|eta\|_1 \right)
$$

follows the description above. In this example, the cost function is the squared euclidean norm and the regularizer corresponds to a convex relaxation of the norm $\|\beta\|_0$.

If the cost function and the set $C$ are convex and satisfy mild assumptions, the estimator can be computed efficiently (in polynomial time). The estimators that we use for PCA and sparse linear regression can be computed in polynomial time. For more details on computational aspects of convex optimization, see [Vis18].

For convex cost functions the meta-theorem below (which appears in different forms in the literature, e.g. see [Wai19], section 9.4) can be used to mechanically bound the guarantees of the estimator. Before stating the theorem, let’s define the following set: for a norm $\|\cdot\|_{\text{reg}}$ and for a vector subspace $V \subseteq \mathbb{R}^d$ and $b \geq 1$, we denote

$$
S_b(V) = \{ u \in \mathbb{R}^d \mid \|u\|_{\text{reg}} \leq b\|u_V\|_{\text{reg}} \},
$$

where $u_V$ is the orthogonal projection of $u$ on $V$.

**Theorem 4.1.** Let $\gamma, \kappa, R, s$ be positive real numbers and let $C \subseteq \mathbb{R}^d$ be a convex set. Consider a vectors space $\Omega \subseteq \mathbb{R}^d$ and let $\theta^* \in \Omega \cap C$.

Let $\|\cdot\|_{\text{reg}} : \mathbb{R}^d \to [0, \infty)$ be a norm and consider a continuous error function $E : \mathbb{R}^d \to [0, \infty)$ such that $E(0) = 0$. Let $F : \mathbb{R}^d \to \mathbb{R}$ be a convex differentiable cost function.

Suppose that there exists a vector space $\overline{\Omega}$ such that $\Omega \subseteq \overline{\Omega} \subseteq \mathbb{R}^d$ and such that the following properties hold:
(Decomposability) For all \( u \in \Omega \) and \( v \in \Omega^\perp \),
\[
\|v + u\|_{\text{reg}} = \|v\|_{\text{reg}} + \|u\|_{\text{reg}}. 
\]
(4.1)

(Contraction) For all \( u \in \mathcal{S}_4(\Omega) \),
\[
\|u\|_{\text{reg}} \leq s \cdot \mathcal{E}(u) .
\]
(4.2)

(Gradient bound) The dual norm of \( \|\cdot\|_{\text{reg}} \) of gradient of \( F \) at \( \theta^* \) satisfies
\[
\|\nabla F(\theta^*)\|_{\text{reg}}^* \leq \gamma/2 .
\]
(4.3)

(Restricted local strong convexity) Let
\[
\mathcal{B}_R := \{ u \in \mathbb{R}^d \mid \mathcal{E}(u) = R, \theta^* + u \in C \} .
\]
Then
\[
\forall u \in \mathcal{B}_R \cap \mathcal{S}_4(\Omega) \quad F(\theta^* + u) \geq F(\theta^*) + \langle \nabla F(\theta^*), u \rangle + \frac{\kappa}{2} (\mathcal{E}(u))^2.
\]
(4.4)

(Bound on radius) Parameters \( \gamma, \kappa, R \) and \( s \) satisfy
\[
\frac{\gamma \cdot s}{\kappa} \leq R/4 .
\]
(4.5)

Then, for every \( \theta' \in C \) such that \( F(\theta^*) + \gamma \|\theta'\|_{\text{reg}} \leq F(\theta^*) + \gamma \|\theta^*\|_{\text{reg}} \),
\[
\mathcal{E}(\theta' - \theta^*) < R .
\]

For completeness, we include a proof of Theorem 4.1. We will need the following lemma.

**Lemma 4.2.** Consider the settings of Theorem 4.1. If \( \theta \in C \) satisfies
\[
F(\theta) + \gamma \|\theta\|_{\text{reg}} \leq F(\theta^*) + \gamma \|\theta^*\|_{\text{reg}} ,
\]
then \( \theta - \theta^* \in \mathcal{S}_4(\Omega) \).

**Proof.** Denote \( \Delta = \theta - \theta^* \). By the decomposability of the regularizer Eq. (4.1),
\[
\|\theta^* + \Delta\|_{\text{reg}} = \|\theta^* \|_{\text{reg}} + \|\Delta\|_{\text{reg}} \geq \|\theta^* \|_{\text{reg}} - \|\Delta\|_{\text{reg}} ,
\]
(Triangle Inequality)
\[
= \|\theta^* \|_{\text{reg}} + \|\Delta\|_{\text{reg}} - \|\Delta\|_{\text{reg}} .
\]
(Decomposability of \( \|\cdot\|_{\text{reg}} \).)
By convexity of the cost function and Hölder’s inequality,

\[ F(\theta^* + \Delta) - F(\theta^*) \geq -|\langle \nabla F(\theta^*), \Delta \rangle| \geq -\|\nabla F(\theta^*)\|_{\text{reg}}^2 \cdot \|\Delta\|_{\text{reg}}. \]

Hence by the gradient bound and the decomposability of the regularizer,

\[ F(\theta^* + \Delta) - F(\theta^*) \geq -\gamma^2 \cdot \|\Delta\|_{\text{reg}} = -\gamma^2 \left( \|\Delta\|_{\text{reg}}^2 + \|\Delta\|_{\text{reg}}^2 \right). \]

Recall that \( F(\theta^* + \Delta) + \gamma \|\theta^* + \Delta\|_{\text{reg}} \leq F(\theta^*) + \gamma \|\theta^*\|_{\text{reg}} \), hence

\[ 0 \geq \gamma \left( \|\theta^* + \Delta\|_{\text{reg}} - \|\theta^*\|_{\text{reg}} \right) + (F(\theta^* + \Delta) - F(\theta^*)) \]
\[ \geq \gamma \left( \|\theta^* + \Delta\|_{\text{reg}} - \|\theta^*\|_{\text{reg}} \right) - \gamma^2 \left( \|\Delta\|_{\text{reg}}^2 + \|\Delta\|_{\text{reg}}^2 \right) \]
\[ \geq \gamma \left( \|\Delta\|_{\text{reg}}^2 - \|\Delta\|_{\text{reg}}^2 \right) - \gamma^2 \left( \|\Delta\|_{\text{reg}}^2 + \|\Delta\|_{\text{reg}}^2 \right) \]
\[ = \gamma^2 \left( \|\Delta\|_{\text{reg}}^2 - 3 \|\Delta\|_{\text{reg}} \right). \]

Therefore, we have \( \|\Delta\|_{\text{reg}} \leq 3 \|\Delta\|_{\text{reg}} \), and thus

\[ \|\Delta\|_{\text{reg}} \leq \left\|\Delta\right\|_{\text{reg}} + \|\Delta\|_{\text{reg}} \leq 4 \|\Delta\|_{\text{reg}}. \]

\[ \square \]

We are now ready to prove the theorem.

**Proof of Theorem 4.1.** Denote \( G(\theta) = F(\theta) + \gamma \|\theta\|_{\text{reg}} \).

Assume by contradiction that there exists \( \theta' \in \mathcal{C} \) such that \( \mathcal{E}(\theta' - \theta^*) \geq R \) and \( G(\theta') \leq G(\theta^*) \). By continuity of \( \mathcal{E} \), there should exist a point \( \tilde{\theta} \) on the segment between \( \theta^* \) and \( \theta' \) such that \( \mathcal{E}(\tilde{\theta} - \theta^*) = R \). Since \( \mathcal{C} \) is convex, \( \tilde{\theta} \in \mathcal{C} \), so \( \tilde{\theta} - \theta^* \in \mathcal{B}_R \). By convexity of \( G \), \( G(\tilde{\theta}) \leq G(\theta^*) \). Denote \( \tilde{\Delta} = \tilde{\theta} - \theta^* \). We get

\[ F\left( \theta^* + \tilde{\Delta} \right) - F(\theta^*) \leq \gamma \left( \|\theta^*\|_{\text{reg}} - \|\tilde{\Delta} + \theta^*\|_{\text{reg}} \right) \]
\[ \leq \gamma \cdot \|\tilde{\Delta}\|_{\text{reg}} \quad \text{(Triangle Inequality)} \]
\[ \leq \gamma \cdot s \cdot \mathcal{E}(\tilde{\Delta}) \quad \text{(Lemma 4.2 & Eq. (4.2))} \]

By restricted local strong convexity (Eq. (4.4)) and the Gradient bound (Eq. (4.3)), we have

\[ \left( \mathcal{E}(\tilde{\Delta}) \right)^2 \leq \frac{2}{\kappa^2} \left( |\langle \nabla F(\theta^*), \Delta \rangle| + \left( F\left( \theta^* + \tilde{\Delta} \right) - F(\theta^*) \right) \right) \]
\[ \leq \frac{2}{\kappa^2} \left( |\langle \nabla F(\theta^*), \Delta \rangle| + \gamma \cdot s \cdot \mathcal{E}(\tilde{\Delta}) \right) \quad \text{(Eq. (4.4))} \]

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\[ \frac{2}{\kappa} \left( \|\nabla F(\theta^*)\|_{\text{reg}} \|\tilde{\Delta}\|_{\text{reg}} + \gamma \cdot s \cdot \mathcal{E}(\tilde{\Delta}) \right) \quad \text{(Hölder’s inequality)} \]

\[ < 4 \cdot \frac{\gamma \cdot s \cdot \mathcal{E}(\tilde{\Delta})}{\kappa} \quad \text{(Eq. (4.3) & Eq. (4.2))} \]

\[ \leq R \cdot \mathcal{E}(\tilde{\Delta}) \quad \text{(Eq. (4.5))} \]

So \( \mathcal{E}(\tilde{\Delta}) < R \), leading to a contradiction. Hence every \( \theta' \in C \) such that \( G(\theta') \leq G(\theta^*) \) satisfies \( \mathcal{E}(\theta' - \theta^*) < R \).

\[ \square \]

5 Principal component analysis with oblivious outliers (Theorem 2.1)

We will prove Theorem 2.1, that we restate in this section.

Recall that for \( L \in \mathbb{R}^{n \times n} \), \( F_h(L) = \sum_{i,j \in [n]} f_h(L_{ij}) \), where

\[ f_h(t) := \begin{cases} \frac{1}{2} t^2 & \text{for } |t| \leq h, \\ h(|t| - \frac{h}{2}) & \text{otherwise.} \end{cases} \]

**Theorem** (Restatement of Theorem 2.1). Let \( L^* \in \mathbb{R}^{n \times n} \) be an unknown deterministic matrix, let \( N^* \in \mathbb{R}^{n \times n} \) be a random matrix with independent, symmetrically distributed (about zero) entries and let \( \alpha := \min_{i,j \in [n]} \Pr\{|N_{ij}| \leq \zeta\} \) for some \( \zeta > 0 \). Suppose that \( \text{rank}(L^*) = r \) and \( \|L^*\|_{\text{max}} \leq \rho/n \).

Consider the following estimator:

\[ \hat{L} := \arg\min_{L \in \mathbb{R}^{n \times n}, \|L\|_{\text{max}} \leq \rho/n} (F_h(Y - L) + \gamma \|L\|_{\text{nuc}}) , \quad (5.1) \]

where \( h = \zeta + \rho/n \) and \( \gamma = 100 \sqrt{n} (\zeta + \rho/n) \).

Then, with probability at least \( 1 - 2^{-n} \) over \( N \), given \( Y = L^* + N \), \( \zeta \) and \( \rho \), the estimator \( \hat{L} \) satisfies

\[ \|\hat{L} - L^*\|_F \leq O \left( \frac{\sqrt{nh}}{\alpha} \right) \cdot (\zeta + \rho/n) . \]

In light of Theorem 4.1, we can prove Theorem 2.1 by showing that the estimator \( \hat{L} \) in Eq. (5.1) fulfills all the conditions of Theorem 4.1 with \( F(L) := F_h(Y - L) = F_{\zeta + \rho/n}(Y - L) \), \( \|\cdot\|_{\text{reg}} := \|\cdot\|_{\text{nuc}} \), \( \gamma = 100 \sqrt{n} (\zeta + \rho/n) \) and \( \mathcal{E}(\cdot) := \|\cdot\|_F \).

To this end, we define the two vector spaces in Theorem 2.1, \( \Omega \) and \( \overline{\Omega} \), as follows:

\[ \Omega := \left\{ L \in \mathbb{R}^{n \times n} \mid \text{rspan}(L) \subseteq \text{rspan}(L^*), \text{cspan}(L) \subseteq \text{cspan}(L^*) \right\} , \quad (5.2) \]

\[ \overline{\Omega} := \left\{ L \in \mathbb{R}^{n \times n} \mid \text{rspan}(L) \subseteq \text{rspan}(L^*)^\perp, \text{cspan}(L) \subseteq \text{cspan}(L^*)^\perp \right\} . \quad (5.3) \]

It is easy to see that \( \Omega \subseteq \overline{\Omega} \) and the nuclear norm is decomposable per Eq. (4.1) with respect to \( \Omega \) and \( \overline{\Omega} \). That is, for all \( L \in \Omega \) and \( L' \in \overline{\Omega} \), we have \( \|L + L'\|_{\text{nuc}} = \|L\|_{\text{nuc}} + \|L'\|_{\text{nuc}} \), satisfying condition Eq. (4.1).
Moreover, since $L^*$ has rank $r$, Eq. (5.3) implies that any matrix in $\Omega$ has rank at most $2r$. Hence, we immediately obtain that for all $L \in S_4(\Omega) = \{ L \in \mathbb{R}^{n \times n} \mid ||L||_{\text{nuc}} \leq 4||L||_{\text{nuc}} \},$ $||L||_{\text{nuc}} \leq 4\sqrt{2r}||L||_F$, satisfying condition Eq. (4.2) with $s = 4\sqrt{2r}$.

It remains to prove the gradient bound of the condition Eq. (4.3), i.e., a bound on the spectral norm of $\nabla F_h(Y - L^*)$ (since the dual norm of the nuclear norm is the spectral norm), and the local strong convexity of the condition Eq. (4.4).

We start with proving the gradient bound:

**Lemma 5.1 (Gradient bound of spectral norm).** Consider the settings of Theorem 2.1, and let $\delta \in (0, 1)$. Then with probability at least $1 - \delta/2$,

$$||\nabla F_h(Y - L^*)|| \leq 10h \sqrt{n + \log (2/\delta)}.$$ 

**Proof.** By definition of the Huber penalty for all $i, j \in [n]$

$$-h \leq \nabla f_h(Y_{ij} - L^*_{ij}) = \nabla f_h(N_{ij}) \leq h.$$ 

That is, entries are independent, symmetric and bounded by $h$ in absolute value. Hence by Fact B.7, with probability $1 - \delta/2$ the spectral norm of this matrix is bounded by $10h \sqrt{n + \log (2/\delta)}$. 

**Proof of local strong convexity.** We first bound the size of an $\varepsilon$-net for the set of approximately low-rank matrices (Lemma 5.2) and then apply this bound to derive a lower bound for the second-order integral of the Huber-loss function with penalty $h$ (Lemma 5.3).

**Lemma 5.2 ($\varepsilon$-Net for approximately low-rank matrices).** Let $0 < \varepsilon < 1$ and $s \geq 1$. Define

$$\mathcal{L}_s := \{ L \in \mathbb{R}^{n \times n} \mid ||L||_{\text{nuc}} \leq s||L||_F, ||L||_F \leq 1 \}.$$ 

Then $\mathcal{L}_s$ has an $\varepsilon$-net of size $\exp\left[\frac{16s^2 n}{\varepsilon^2}\right]$.

**Proof.** Let $W$ be a $n$-by-$n$ random matrix with i.i.d entries $W_{ij} \sim N(0, 1)$. By Sudakov’s minoration Fact B.9, we have

$$\sqrt{\log |\mathcal{N}_{L_s}||_F(L_s)|} \leq \frac{2}{\varepsilon} \mathbb{E} \sup_{L \in L_s} \langle W, L \rangle$$  

$$\leq \frac{2}{\varepsilon} \mathbb{E} \sup_{L \in L_s} ||W|| \cdot ||L||_{\text{nuc}}$$  

$$\leq \frac{2s}{\varepsilon} \mathbb{E} ||W||$$  

$$\leq \frac{4s \sqrt{n}}{\varepsilon}$$  

where in the last inequality we use a bound on the expected spectral norm of a Gaussian matrix Fact B.6. 

\(\square\)
Hence the intersection of the set $S_4(\Omega) = \{ L \in \mathbb{R}^{n \times n} \mid \|L\|_{\text{nuc}} \leq 4\|L\|_{\text{nuc}} \}$ with the ball 
$\{ L \in \mathbb{R}^{n \times n} \mid \|L\|_F \leq 1 \}$ has $\varepsilon$-net of size \( \exp \left[ \frac{16 \cdot 32 \cdot n}{\varepsilon^2} \right] \leq \exp \left[ \frac{600 \cdot n}{\varepsilon^2} \right] \).

Now we can prove the restricted local strong convexity:

**Lemma 5.3 (Restricted local strong convexity of Huber-loss).** Consider the settings of Theorem 2.1. Let $0 < \delta < 1$, $R > 0$ and $h \geq \rho / n + \zeta$.

Define

$$B_R := \{ \Delta \in \mathbb{R}^{n \times n} \mid \|\Delta\|_F = R, \|L^* + \Delta\|_{\text{max}} \leq \rho / n \}.$$ 

Suppose that

$$R \geq 200 \cdot \frac{\rho / n}{\alpha} \cdot \sqrt{rn + \log(2/\delta)}.$$ 

Then with probability at least $1 - \delta / 2$, for all $\Delta \in B_R \cap S_4(\Omega)$,

$$F_h(L^* + \Delta) \geq F_h(L^*) + \langle \nabla F_h(L^*), \Delta \rangle + 0.01 \cdot \alpha \cdot \|\Delta\|_F^2.$$ 

**Proof.** Denote $M = \rho / n$. Consider $L$ such that $\|L\|_{\text{max}} \leq M$. Since $h \geq \zeta + M$, by Lemma A.2,

$$F_h(L) - F_h(L^*) - \langle \nabla F_h(L^*), L - L^* \rangle \geq \frac{1}{2} \sum_{i,j \in [n]} (L_{ij} - L^*_{ij})^2 \mathbf{1}[|L_{ij}| < h - \zeta] \cdot \mathbf{1}[|L^*_{ij}| < \zeta].$$ (Lemma A.2)

$$= \frac{1}{2} \sum_{i,j \in [n]} (L_{ij} - L^*_{ij})^2 \mathbf{1}[|N_{ij}| < \zeta].$$ (\(\|L^*\|_{\text{max}} \leq M \leq h - \zeta\))

We will lower bound this quantity for every $L$ such that $L - L^* \in B_R \cap S_4(\Omega)$. Denote $C_R := B_R \cap S_4(\Omega)$ and let $\Delta := L - L^* \in C_R$. By Lemma 5.2, there exists $(\varepsilon \cdot R)$-net $N_{\varepsilon R, \|\cdot\|_F}(C_R)$ of size at most \( \exp \left[ \frac{16 \cdot 32 \cdot n}{\varepsilon^2} \right] \leq \exp \left[ \frac{600 \cdot n}{\varepsilon^2} \right] \) (recall that $s^2 = 32r$). Thus, we can write $\Delta \in C_R$ as a sum $A + B \in \mathbb{R}^{n \times n}$ where $A \in N_{\varepsilon R, \|\cdot\|_F}(C_R)$ and $\|B\|_F \leq \varepsilon R$. It follows that

$$\sum_{i,j \in [n]} \Delta_{ij}^2 \cdot \mathbf{1}[|N_{ij}| < \zeta] = \sum_{i,j \in [n]} (A_{ij} + B_{ij})^2 \cdot \mathbf{1}[|N_{ij}| < \zeta]$$

$$\geq \frac{1}{2} \sum_{i,j \in [n]} A_{ij}^2 \cdot \mathbf{1}[|N_{ij}| < \zeta] - \sum_{i,j \in [n]} B_{ij}^2 \cdot \mathbf{1}[|N_{ij}| < \zeta].$$ (5.4)

Let $\varepsilon = \sqrt{\alpha} / 4$. Then

$$\sum_{i,j \in [n]} B_{ij}^2 \cdot \mathbf{1}[|N_{ij}| < \zeta] \leq \|B\|_F^2 \leq \varepsilon^2 R^2 \leq \frac{\alpha \cdot R^2}{16}. $$ (5.5)

Denote $E := \mathbb{E} \sum_{i,j \in [n]} A_{ij}^2 \cdot \mathbf{1}[|N_{ij}| < \zeta]$. Since $A \in N_{\varepsilon R, \|\cdot\|_F}(C_R) \subset C_R$, we have $\|A\|_F = R$, hence

$$E \geq \alpha \|A\|_F^2 \geq \frac{\alpha \cdot R^2}{2}.$$ (5.6)
Moreover, since \(\|A\|_{\max} \leq \|-L^*\|_{\max} + \|A + L^*\|_{\max} \leq 2M\) (recall \(A \in \mathcal{B}_R\)) and \(\alpha_{ij} = \mathbb{P}(|N_{ij}| \leq \zeta)\), we have \(|A_{ij}^2 \left(1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right)| \leq 4M^2\), implying that

\[
\sum_{i,j \in [n]} \mathbb{E} A_{ij}^4 \left(1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right)^2 \leq 4M^2 \sum_{i,j \in [n]} \mathbb{E} A_{ij}^2 \left|1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right|
\]

\[
= 4M^2 \sum_{i,j \in [n]} \left(\alpha_{ij} \cdot A_{ij}^2 \cdot 0 + (1 - \alpha_{ij}) \cdot A_{ij}^2 \cdot \alpha_{ij}\right)
\]

\[
= 4M^2 \sum_{i,j \in [n]} A_{ij}^2 \cdot (\alpha_{ij} - \alpha_{ij}^2)
\]

\[
\leq 4M^2 E.
\]

Applying Bernstein’s inequality (Fact B.3) with \(t \geq 1\) we get

\[
\mathbb{P}\left(\left|\sum_{i,j \in [n]} A_{ij}^2 \left(1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right)\right| \geq t \cdot 2M \cdot \sqrt{E} + t^2 \cdot 4M^2\right) \leq 2 \exp\left(-t^2/4\right).
\]

Note that \(|\mathcal{N}_{\epsilon R, \|\cdot\|_p}(C_R)| \leq \exp\left[\frac{600rn}{\epsilon^2}\right] \leq \exp\left[\frac{10000rn}{\epsilon^2}\right]\). Therefore, if we set

\[
t = \sqrt{\frac{40000rn}{\alpha}} + 8 \log(2/\delta)
\]

and take a union bound over \(\mathcal{N}_{\epsilon R, \|\cdot\|_p}(C_R)\), we obtain that with probability at least \(1 - \delta/2\), we have

\[
\left|\sum_{i,j \in [n]} A_{ij}^2 \cdot \left(1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right)\right| \leq 400M \cdot \sqrt{E} \cdot \sqrt{\frac{rn}{\alpha} + \log(2/\delta)}
\]

\[
+ (400M)^2 \left(\frac{rn}{\alpha} + \log(2/\delta)\right).
\]

for all \(A \in \mathcal{N}_{\epsilon R, \|\cdot\|_p}(C_R)\). Now since \(E \geq \frac{\alpha R^2}{2}\) and \(R \geq \frac{1000M}{\sqrt{\alpha}} \sqrt{\frac{rn}{\alpha} + \log(2/\delta)}\), we have

\[
\sqrt{E} \geq \frac{\sqrt{\alpha} \cdot R}{\sqrt{2}} \geq 1400M \sqrt{\frac{rn}{\alpha} + \log(2/\delta)},
\]

hence, with probability at least \(1 - \delta/2\),

\[
\left|\sum_{i,j \in [n]} A_{ij}^2 \cdot \left(1_{[|N_{ij}| \leq \zeta]} - \alpha_{ij}\right)\right| \leq \frac{2}{7} \cdot E + \left(\frac{2}{7}\right)^2 \cdot E \leq 4E.
\]
By combining this with Eq. (5.4), Eq. (5.5) and Eq. (5.6), we obtain that with probability at least $1 - \delta$, we have
\[
\sum_{i,j \in [n]} \Delta_{ij}^2 \cdot 1[|N_{ij}| \leq 1] \geq \frac{1}{2}(E - 0.4E) - \frac{\alpha \cdot R^2}{16} \geq 0.15\alpha \cdot R^2 - 0.0625\alpha \cdot R^2 \geq 0.08\alpha R^2
\]
concluding the proof. \qed

**Putting everything together.** We can now combine the above results with Theorem 4.1 to prove Theorem 2.1.

**Proof of Theorem 2.1.** By Lemma 5.1 and Lemma 5.3, we can apply Theorem 4.1 with $\gamma = 100(\zeta + \frac{R}{n})\sqrt{n + \log(2/\delta)}$, $\kappa = 0.01\alpha$, and $s = 4\sqrt{2r}$.

It follows that for
\[
R \geq (\zeta + \rho/n)\sqrt{\frac{r(n + \log(2/\delta))}{\alpha^2}}
\]
the estimator $\hat{L}$ defined in Eq. (5.1) satisfies $\|\hat{L} - L^*\|_F < R$ with probability at least $1 - \delta$. With $\delta = 2^{-n}$ we get the desired bound. \qed

## 6 Sparse linear regression with oblivious outliers (Theorem 2.2)

We prove Theorem 2.2, which will be restated below. Before the restatement, for easier reference, we list the three assumptions in Section 2.2 for the design matrix $X \in \mathbb{R}^{n \times n}$:

1. For every column $X^i$ of $X$, $\|X^i\| \leq \sqrt{n}$.

2. **Restricted eigenvalue property (RE-property):** For every vector $u \in \mathbb{R}^d$ such that\(^{13}\)
\[
\|u_{\text{supp}(\beta)}\|_1 \geq 0.1 \cdot \|u\|_1, \quad \frac{1}{n} \|Xu\|^2 \geq \lambda \cdot \|u\|^2
\]
for some parameter $\lambda > 0$.

3. **Well-spreadness property:** For some $m \in [n]$ and for every vector $u \in \mathbb{R}^d$ such that $\|u_{\text{supp}(\beta)}\|_1 \geq 0.1 \cdot \|u\|_1$ and for every subset $S \subseteq [n]$ with $|S| \geq n - m$, it holds that $\|(Xu)_S\| \geq \frac{1}{2} \|Xu\|$.

Recall that $F_2(\beta) = \sum_{i=1}^n f_2(y_i - \langle X_i, \beta \rangle)$, where
\[
f_2(t) := \begin{cases} 
\frac{1}{2}t^2 & \text{for } |t| \leq 2, \\
2|t| - 2 & \text{otherwise}.
\end{cases}
\]
\(^{13}\)For a vector $v \in \mathbb{R}^d$ and a set $S \subseteq [d]$, we denote by $v_S$ the restriction of $v$ to the coordinates in $S$. 

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Theorem 6.1 (Restatement of Theorem 2.2). Let $\beta^* \in \mathbb{R}^d$ be an unknown $k$-sparse vector and let $X \in \mathbb{R}^{n \times d}$ be a deterministic matrix such that for each column $X'$ of $X$, $\|X'\| \leq \sqrt{vn}$, satisfying the RE-property with $\lambda > 0$ and well-spreadness property with $m \geq \frac{k \log d}{\lambda \cdot n}$ (recall that $n \geq m$). Further, let $\eta$ be an $n$-dimensional random vector with independent, symmetrically distributed (about zero) entries and $\alpha = \min_{i \in [n]} \mathbb{P}\{\|\eta_i\| \leq 1\}$. Consider the following estimator:

$$\hat{\beta} := \arg \min_{\beta \in \mathbb{R}^d} \left( F_2(\beta) + 100\sqrt{vn \log d} \cdot \|\beta\|_1 \right).$$

(6.1)

Then, with probability at least $1 - d^{-10}$ over $\eta$, given $X$ and $y = X\beta^* + \eta$, the estimator $\hat{\beta}$ satisfies

$$\frac{1}{n} \|X(\hat{\beta} - \beta^*)\|^2 \leq O \left( \frac{\nu}{\lambda} \cdot \frac{k \log d}{\alpha^2 \cdot n} \right) \quad \text{and} \quad \|\hat{\beta} - \beta^*\|^2 \leq O \left( \frac{\nu}{\lambda^2} \cdot \frac{k \log d}{\alpha^2 \cdot n} \right).$$

We assume $d \geq 2$ since for $d = 1$ Theorem 6.1 is trivially true (since the probability $1 - d^{-10} = 0$ in this case).

As for principal component analysis (Section 5), we will prove Theorem 6.1 by showing that the estimator Eq. (6.1) fulfills the conditions of Theorem 4.1 with $F(\beta) = F_2(\beta)$, $\|u\|_{\text{reg}} = \|u\|_1$, $\gamma = 100\sqrt{n \log d}$ and $\mathcal{E}(u) = \frac{1}{\sqrt{n}} \|Xu\|$.

Let $\Omega := \{\beta \in \mathbb{R}^d \mid \text{supp} \ \beta \subseteq \text{supp}(\beta^*)\}$ and $\overline{\Omega} := \Omega$. Clearly, for any $v \in \Omega$ and any $v' \in \overline{\Omega}$,

$$\|v + v'\|_1 = \|v\|_1 + \|v'\|_1.$$

That is, $\|\cdot\|_1$ is decomposable, satisfying condition Eq. (4.1).

The contraction condition Eq. (4.2) holds for $s = 4\sqrt{k/\lambda}$ since for all $v \in S_4(\overline{\Omega}) = \{v \mid \|v\|_1 \leq 4\|v\|_{\Omega}\}$, $\|v\|_1 \leq 4\sqrt{k/\lambda} \|v\| \leq 4\sqrt{k/\lambda} \cdot \sqrt{\frac{1}{n}} \|Xv\|$, where the last inequality comes from the RE-property.

It remains to provide a gradient bound of the form in Eq. (4.3) and local strong convexity in Eq. (4.4).

Lemma 6.2 (Gradient bound). Consider the settings of Theorem 6.1. Then, with probability at least $1 - \delta/2$,

$$\|\nabla F_2(\beta^*)\|_{\max} \leq 20\sqrt{v \cdot n \cdot (\log d + \log(2/\delta))}.$$

Proof. By definition of $f_2$,

$$\nabla \left( \sum_{i=1}^n f_2(y_i - \langle X_i, \beta^* \rangle) \right) = z^T X$$

where $z$ is an $n$-dimensional random vector with independent, symmetric entries $f_2'(\eta_i)$ bounded by $2$ in absolute value. By Hoeffding’s inequality (Fact B.2), for $t \geq 0$,

$$\mathbb{P}(|\langle z, X_i \rangle| \geq 10t \cdot 2 \cdot \|X_i\|) \leq \exp(-t^2).$$

Since $\|X_i\| \leq \sqrt{vn}$, taking a union bound over all $j \in [d]$ yields the statement. \(\square\)
**Proof of local strong convexity.** We first bound the size of an $\ell_0$-net for the set of approximately sparse vectors (Lemma 6.3) and then prove the required local strong convexity bound (Lemma 6.4).

**Lemma 6.3 ($\ell_0$-Net for approximately sparse vectors).** Let $0 < \varepsilon < 1$ and 

$$
\mathcal{U}_s := \left\{ \beta \in \mathbb{R}^d \left| \|\beta\|_1 \leq s \cdot \frac{1}{\sqrt{n}} \|X\beta\|, \frac{1}{\sqrt{n}} \|X\beta\| \leq 1 \right. \right\}.
$$

Then $\mathcal{U}_s$ has an $\varepsilon$-net of size $\exp \left[ \frac{16 s^2 \log d}{\varepsilon^2} \right]$ in terms of distance $\rho(\beta, \beta') := \frac{1}{\sqrt{n}} \|X(\beta - \beta')\|.$

**Proof.** Let $w$ be an $n$-dimensional random Gaussian vector $w \sim N(0, \mathbb{I}_n)$. By Sudakov’s minoration (Fact B.9), for

$$
\sqrt{\log |\mathcal{N}_{\varepsilon, \rho(\cdot, \cdot)}(\mathcal{U}_s)|} \leq \frac{2}{\varepsilon} \mathbb{E} \frac{1}{\sqrt{n}} \sup_{\beta \in \mathcal{U}_s} \langle w, X\beta \rangle \quad \text{(Fact B.9)}
$$

$$
= \frac{2}{\varepsilon} \mathbb{E} \frac{1}{\sqrt{n}} \sup_{\beta \in \mathcal{U}_s} \langle X^T w, \beta \rangle
$$

$$
\leq \frac{2}{\varepsilon} \mathbb{E} \frac{1}{\sqrt{n}} \sup_{\beta \in \mathcal{U}_s} \|X^T w\|_{\max} \|\beta\|_1 \quad \text{(Hölder’s inequality)}
$$

$$
\leq \frac{2s}{\varepsilon} \mathbb{E} \frac{1}{\sqrt{n}} \|X^T w\|_{\max} \quad \text{(Definition of $\mathcal{U}_s$)},
$$

where in the last inequality we use the bound on the expected maximal entry of a vector with $\nu$-subgaussian entries Fact B.4. □

**Lemma 6.4 (Restricted local strong convexity of Huber-loss).** Consider the settings of Theorem 6.1. Let $0 < \delta < 1, R > 0$. Define

$$
\mathcal{B}_R := \left\{ u \in \mathbb{R}^d \left| \frac{1}{\sqrt{n}} \|Xu\| = R \right. \right\}.
$$

Suppose that the set size $m$ from the well-spread property satisfies $m \geq 4R^2 n$ and

$$
R \geq 100 \cdot \sqrt{\frac{v \log d + \log(2/\delta)}{\lambda \cdot \alpha^2 \cdot n}}.
$$

Then with probability at least $1 - \delta/2$, for all $u \in \mathcal{B}_R \cap S_4(\Omega),$

$$
F_2(\beta) \geq F_2(\mathcal{y} - X\mathcal{b}^*) + \langle \nabla F_2(\mathcal{b}^*), u \rangle + 0.01 \cdot \alpha n \cdot \frac{1}{n} \|Xu\|^2.
$$
Proof. Denote \( C_R = \mathcal{B}_R \cap \mathcal{S}_4(\Omega) \) and let \( u \in C_R \). By Lemma A.2,
\[
F_2(\beta^* + u) - F_2(\beta^*) - \langle \nabla F_2(\beta^*), u \rangle \geq \frac{1}{2} \sum_{i \in [n]} \langle X_i, u \rangle^2 \mathbf{1}_{|[\eta_i] \leq 1} \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1}.
\]
Note that for any \( u \in \mathcal{B}_R \) there are at most \( 4R^2 n \) coordinates of \( Xu \) larger than \( 1/4 \) in absolute value, and since \( X \) is well-spread for sets of size \( m = 4R^2 n \),
\[
\sum_{i \in [n]} \langle X_i, u \rangle^2 \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \geq \frac{1}{4} \|Xu\|^2.
\]
Thus
\[
E := \mathbb{E} \sum_{i \in [n]} \langle X_i, u \rangle^2 \mathbf{1}_{|[\eta_i] \leq 1} \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \geq \frac{1}{4} \cdot \alpha \cdot \|Xu\|^2 = \frac{\alpha R^2 n}{4}
\]
We now bound the deviation. We have
\[
\text{for all } i \in [n], \quad \langle X_i, u \rangle^2 \mathbf{1}_{|[\eta_i] \leq 1} \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \leq 1
\]
and
\[
\mathbb{E} \sum_{i \in [n]} \left[ \langle X_i, u \rangle^2 \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \cdot \left( \mathbf{1}_{|[\eta_i] \leq 1} - \alpha_i \right) \right]^2 \\
\leq \mathbb{E} \sum_{i \in [n]} \langle X_i, u \rangle^2 \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \cdot \mathbf{1}_{|[\eta_i] \leq 1} \\
\leq E.
\]
Applying Bernstein’s inequality Fact B.3
\[
\mathbb{P}\left( \sum_{i \in [n]} \langle X_i, u \rangle^2 \cdot \mathbf{1}_{[(X_i, u)^\perp] \leq 1/4} \cdot \left( \mathbf{1}_{|[\eta_i] \leq 1} - \alpha_i \right) \geq t \cdot \sqrt{E} + t^2 \right) \leq \exp\left\{ -t^2/4 \right\}.
\]
It remains to extend uniformly this bound over all \( u \in C_R \). By Lemma 6.3 there exists an \((\varepsilon \cdot R)\)-net \( \mathcal{N}_{\varepsilon R}(C_R) \) of size \( \exp\left[ \frac{256 \nu k \log d}{\lambda \varepsilon^2} \right] \) (recall that \( s = 4\sqrt{k/\lambda} \)). Thus for any \( u \in C_R \) there exists \( u' \in \mathcal{N}_{\varepsilon R}(C_R) \) such that \( \frac{1}{\sqrt{n}} \|X(u - u')\| \leq \varepsilon R \) and consequently
\[
\sum_{i \in [n]} \langle X_i, u' \rangle^2 \cdot \mathbf{1}_{|[\eta_i] \leq 1, (X_i, u)^\perp \leq 1/4} \leq \sum_{i \in [n]} \langle X_i, u' \rangle^2 \cdot \mathbf{1}_{|[\eta_i] \leq 1, (X_i, u)^\perp \leq 1/4} + \varepsilon^2 R^2 n
\]
\[
\leq 2 \sum_{i \in [n]} \langle X_i, u \rangle^2 \cdot \mathbf{1}_{|[\eta_i] \leq 1, (X_i, u)^\perp \leq 1/4} + \varepsilon^2 R^2 n
\]
\[
+ 2 \sum_{i \in [n]} \langle X_i, u' - u \rangle^2 \cdot \mathbf{1}_{|[\eta_i] \leq 1, (X_i, u)^\perp \leq 1/4}
\]
\[
\leq 2 \sum_{i \in [n]} \langle X_i, u \rangle^2 \cdot \mathbf{1}_{|[\eta_i] \leq 1, (X_i, u)^\perp \leq 1} + 3\varepsilon^2 R^2 n.
\]
The first inequality holds since each term at the first sum that doesn’t appear in the second sum corresponds to the index $i \in [n]$ such that $\langle X_i, u - u' \rangle^2 \geq 1/4$, and since each term is bounded by $1/4$, their sum is bounded by $\sum_{i \in [n]} \langle X_i, u - u' \rangle^2 \leq \varepsilon^2 R^2 n$.

Setting $\varepsilon = \sqrt{n}/4$ and taking a union bound, with probability at least $1 - \delta/2$ for all unit vectors $u \in \mathcal{L}_{k,R}$ we get

$$\sum_{i \in [n]} \langle X_i, u \rangle^2 1[|\eta_i| \leq 1] 1[\langle X, u \rangle^2 \leq 4] \geq \frac{E}{2} - \frac{3\varepsilon^2 R^2 n}{2} - \sqrt{E} \sqrt{64vk \log d + 4 \log(\frac{2}{\delta})} \frac{\sqrt{\lambda \varepsilon}}{\lambda \varepsilon^2} - \frac{128vk \log d + 4 \log(\frac{2}{\delta})}{\lambda \varepsilon^2} \geq 0.01 \cdot \alpha \cdot R^2 n .$$

\[\square\]

**Putting things together.** We combine the above results with Theorem 4.1.

**Proof of Theorem 2.2.** By Lemma 6.2 and Lemma 6.4, we can apply Theorem 4.1 with $\gamma = 100 \sqrt{v \cdot n (\log d + \log(2/\delta))}$, $\kappa = 0.01 \cdot \alpha \cdot n$ and $s = 4 \sqrt{k/\lambda}$. It follows that for

$$R \geq \sqrt{\frac{v \cdot k \cdot (\log d + \log(2/\delta))}{\lambda \cdot \alpha^2 \cdot n}} ,$$

the estimator $\hat{\beta}$ defined in Eq. (6.1) with probability $1 - \delta$ satisfies $\frac{1}{n} \| X (\hat{\beta} - \beta^*) \| \leq R$. Taking $\delta = d^{-10}$, we get the desired bound. Since $\hat{\beta} - \beta \in S_4(\Omega)$, we also get the desired parameter error $\| \hat{\beta} - \beta \| \leq R/\sqrt{\lambda}$.

\[\square\]

### 7 Sparse linear regression with Gaussian design (Theorem 2.3)

In this section we will prove Theorem 2.3. As before, we will use Theorem 4.1. Recall that in this setting, our model looks as follows:

$$y = X \beta^* + \eta$$

where $X \in \mathbb{R}^{n \times d}$ is a random matrix whose rows $X_1, \ldots, X_n$ are i.i.d. $N(0, \Sigma)$ and $\eta \in \mathbb{R}^n$ is a deterministic vector such that an coordinates have absolute value bounded by 1. We restate Theorem 2.3 here for completeness:
Theorem 2.3 (Restatement of Theorem 2.3). Let $\beta^* \in \mathbb{R}^d$ be an unknown $k$-sparse vector and let $X$ be a $n$-by-$d$ random matrix with i.i.d. rows $X_1, \ldots, X_n \sim N(0, \Sigma)$ for a positive definite matrix $\Sigma$. Further, let $\eta \in \mathbb{R}^n$ be a deterministic vector with $\alpha \cdot n$ coordinates bounded by 1 in absolute value. Suppose that $n \geq \frac{\nu(\Sigma) \cdot k \cdot \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2}$, where $\nu(\Sigma)$ is the maximum diagonal entry of $\Sigma$ and $\sigma_{\min}(\Sigma)$ is its smallest eigenvalue. Then, with probability at least $1 - d^{-10}$ over $X$, given $X$ and $y = X\beta^* + \eta$, the estimator Eq. (2.3) satisfies

$$\frac{1}{n} \|X(\hat{\beta} - \beta^*)\|^2 \leq O\left(\frac{\nu(\Sigma) \cdot k \cdot \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2 \cdot n}\right) \quad \text{and} \quad \|\hat{\beta} - \beta^*\|^2 \leq O\left(\frac{\nu(\Sigma) \cdot k \cdot \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2 \cdot n}\right).$$

As in the previous section, we assume $d \geq 2$ since for $d = 1$ Theorem 2.3 is true (since the probability $1 - d^{-10} = 0$ in this case).

First, we bound the gradient of Huber loss. Then, to prove restricted local strong convexity of Huber loss, we show that the values of the empirical covariance (as a quadratic form) on approximately $k$-sparse vectors are well-concentrated near the values of the actual covariance. The proof first appeared in [RWY10], but they only stated the result in terms of a lower bound on the values of empirical covariance and did not discuss an upper bound, though the proof of the upper bound is very similar. Then we use this concentration to prove well-spreadness and restricted local strong convexity.

Recall that $F_2(\beta) = \sum_{i=1}^n f_2(y_i - \langle X_i, \beta \rangle)$, where

$$f_2(t) := \begin{cases} \frac{1}{2} t^2 & \text{for } |t| \leq 2, \\ 2|t| - 2 & \text{otherwise}. \end{cases}$$

Gradient bound for Gaussian design.

Lemma 7.2. Consider the settings of Theorem 2.3. Then with probability at least $1 - \delta/2$

$$\|\nabla F_2(\beta^*)\|_{\max} \leq 4\sqrt{\nu(\Sigma) \cdot n \cdot (\log d + \log(2/\delta))}. $$

Proof. By definition of the Huber loss and choice of the Huber penalty

$$\nabla F_2(\beta^*) = \nabla \left(\sum_{i=1}^n f_2(y_i - \langle X_i, \beta^* \rangle)\right) = z^T X$$

where $z$ is an $n$-dimensional vector whose entries $f_2'(\eta_i)$ are bounded by 2 in absolute value. Since $\frac{1}{\|z\|} \Sigma^{-1/2} X^T z = g \sim N(0, 1)^n$,

$$\|z^T X\| = \|z\| \cdot \|\Sigma^{1/2} g\| \leq 2\sqrt{n} \cdot \sqrt{\nu(\Sigma) \cdot (2 \log d + 4 \log(2/\delta))},$$

where we used the union bound over all $j \in [d]$ and the standard tail bounds for Gaussian variables $(\Sigma^{1/2} g)_j$ whose variance is $\Sigma_{jj}$. \qed
Concentration of empirical covariance on approximately $k$-sparse vectors. To prove well-spreadness and restricted local strong convexity in case of Gaussian design $X$, we will need the fact that for all approximately $k$-sparse vectors $u$, $\frac{1}{n}\|Xu\|^2 \approx \|\Sigma^{1/2}u\|^2$ as long as $n \gtrsim \frac{\nu(\Sigma) \log d}{\sigma_{\text{min}}(\Sigma)}$. Formally, we will use the following theorem:

**Theorem 7.3.** Let $X$ be a $n$-by-$d$ random matrix with i.i.d. rows $X_1, \ldots, X_n \sim N(0, \Sigma)$, where $\Sigma$ is a positive definite matrix. Suppose that for some $K \geq 1$, $n \geq 1000 \cdot \frac{\nu(\Sigma)}{\sigma_{\text{min}}(\Sigma)} \cdot K \log d$. Then with probability at least $1 - \exp(-n/100)$, for all $u \in \mathbb{R}^d$ such that $\|u\|_1 \leq \sqrt{n} \|u\|$, 

$$\frac{1}{2} \|\Sigma^{1/2}u\| \leq \frac{1}{\sqrt{n}} \|Xu\| \leq 2 \|\Sigma^{1/2}u\|. \quad (7.1)$$

The first inequality of Theorem 7.3 was shown in [RWY10] (see also [Wai19], section 7.3.3), and the second inequality can be proved in a very similar way. For completeness, we provide a proof of second inequality.

**Proof of the second inequality of Theorem 7.3.** Since the inequality is scale invariant, it is enough to show it for $u \in \mathbb{R}^d$ such that $\|\Sigma^{1/2}u\| = 1$. For $s > 0$ denote

$$\mathcal{U}_s := \{u \in \mathbb{R}^d \mid \|\Sigma^{1/2}u\| = 1, \|u\|_1 \leq s\} \quad \text{and} \quad \mathcal{M}_s(X) := \sup_{u \in \mathcal{U}_s} \frac{1}{\sqrt{n}} \|Xu\|.$$

First, we bound the expectation of $\mathcal{M}_s(X)$:

**Lemma 7.4.**

$$\mathbb{E} \mathcal{M}_s(X) \leq 1 + 2s \sqrt{\frac{\nu(\Sigma) \log d}{n}}.$$

**Proof.** Consider Gaussian process $W_{u,v} = v^T X u$ for $(u, v) \in \mathcal{U}_s \times S^{n-1}$, where $S^{n-1}$ is a unit sphere in $\mathbb{R}^n$. Denote $\mathcal{P} = \mathcal{U}_s \times S^{n-1}$. Our goal is to bound $\frac{1}{\sqrt{n}} \mathbb{E} \sup_{(u,v) \in \mathcal{P}} W_{u,v}$.

Denote $G = X \Sigma^{-1/2}$. For all $(u, v), (\tilde{u}, \tilde{v}) \in \mathcal{P}$,

$$\mathbb{E}(W_{u,v} - W_{\tilde{u},\tilde{v}})^2 = \mathbb{E}(X^T, uv^T - \tilde{u}\tilde{v}^T)^2$$

$$= \mathbb{E}(G^T, \Sigma^{1/2}uv^T - \Sigma^{1/2}\tilde{u}\tilde{v}^T)^2$$

$$= \|\Sigma^{1/2}u - \Sigma^{1/2}\tilde{u}\|_F^2.$$

Now consider another Gaussian process $Z_{u,v} = g^T \Sigma^{1/2}u + h^Tv$, where $g \sim N(0, \text{Id}_d)$ and $h \sim N(0, \text{Id}_n)$. For all $(v, u), (\tilde{v}, \tilde{u}) \in \mathcal{P}$,

$$\mathbb{E}(Z_{u,v} - Z_{\tilde{u},\tilde{v}})^2 = \mathbb{E}(g, \Sigma^{1/2}(u - \tilde{u}))^2 + \mathbb{E}(h, v - \tilde{v})^2$$

$$= \|\Sigma^{1/2}u - \Sigma^{1/2}\tilde{u}\|_F^2 + \|v - \tilde{v}\|^2.$$

Note that for every quadruple of unit vectors $x, \tilde{x} \in \mathbb{R}^d, y, \tilde{y} \in \mathbb{R}^n$,

$$\|xy^T - \tilde{x}\tilde{y}^T\|_F^2 = \|(x - \tilde{x})y^T + \tilde{x}(y^T - \tilde{y}^T)\|_F^2.$$
\[ = \|y\|^2 \|x - \bar{x}\|^2 + \|\bar{x}\|^2 \|y - \bar{y}\|^2 + 2 \text{Tr} y(x - \bar{x})^T (y\bar{y}^T) \]
\[ = \|x - \bar{x}\|^2 + \|y - \bar{y}\|^2 + 2(\langle x, \bar{x} \rangle - \|\bar{x}\|^2) \cdot (\|y\|^2 - \langle y, \bar{y} \rangle) \]
\[ \leq \|x - \bar{x}\|^2 + \|y - \bar{y}\|^2. \]

Hence for all \((u, v), (\bar{u}, \bar{v}) \in \mathcal{P}\), \(\mathbb{E}(W_{u, v} - W_{\bar{u}, \bar{v}})^2 \leq \mathbb{E}(Z_{u, v} - Z_{\bar{u}, \bar{v}})^2\), and by Sudakov–Fernique theorem Fact B.8,
\[
\mathbb{E} \sup_{(u, v) \in \mathcal{P}} W_{u, v} \leq \mathbb{E} \sup_{(u, v) \in \mathcal{P}} Z_{u, v}.
\]

Therefore, it is enough to bound \(\mathbb{E} \sup_{u \in \mathcal{U}_1} g^T \Sigma^{1/2} u + \mathbb{E} \sup_{\|v\| = 1} h^T v\). The second term is just an expectation of \(\chi\)-distributed variable, and can be bounded using Jensen’s inequality:
\[
\mathbb{E} \sup_{\|v\| = 1} h^T v = \mathbb{E} \|h\| \leq \sqrt{\mathbb{E} \|h\|^2} \leq \sqrt{n}.
\]

The first term can be bounded as follows:
\[
\mathbb{E} \sup_{u \in \mathcal{U}_1} g^T \Sigma^{1/2} u \leq \mathbb{E} \|u\|_1 \cdot \|\Sigma^{1/2} g\|_{\text{max}} \leq s \mathbb{E} \|\Sigma^{1/2} g\|_{\text{max}} \leq 2s \sqrt{\nu(\Sigma) \log d},
\]
where we used Fact B.4 to bound the max norm of a vector \(\Sigma^{1/2} g\) whose entries are \(\nu(\Sigma)\)-subgaussian. Dividing by \(\sqrt{n}\), we get the desired bound. \(\square\)

Now, we bound the deviation of \(\mathcal{M}_s(X)\):

**Lemma 7.5.** For all \(t \geq 0\),
\[
\mathbb{P}[|\mathcal{M}_s(X) - \mathbb{E} \mathcal{M}_s(X)| \geq t] \leq 2 \exp\left[-nt^2/2\right].
\]

**Proof.** For \(A \in \mathbb{R}^{n \times d}\) denote \(\mathcal{F}_s(A) = \sqrt{n} \cdot \mathcal{M}_s(A \Sigma^{1/2}) = \sup_{u \in \mathcal{U}_1} \|A \Sigma^{1/2} u\|\). Note that for all \(A, B \in \mathbb{R}^{n \times d}\),
\[
\mathcal{F}_s(A) - \mathcal{F}_s(B) \leq \sup_{u \in \mathcal{U}_1} \left(\|A \Sigma^{1/2} u\| - \|B \Sigma^{1/2} u\|\right)
\]
\[
\leq \sup_{u \in \mathcal{U}_1} \|(A - B) \Sigma^{1/2} u\|
\]
\[
\leq \|A - B\| \leq \|A - B\|_F.
\]

Hence \(\mathcal{F}_s\) is 1-Lipschitz, and by Fact B.5, for all \(t \geq 0\),
\[
\mathbb{P}[|\mathcal{F}_s(G) - \mathbb{E} \mathcal{F}_s(G)| \geq t] \leq 2 \exp\left[-t^2/2\right],
\]
where \(G = X \Sigma^{-1/2}\) is a matrix with i.i.d. standard Gaussian entries. Taking \(t = t \sqrt{n}\), we get the desired bound. \(\square\)
Taking $t = 0.2$, we conclude that with probability at least $1 - 2 \exp(-0.02n)$,

$$
M_s(X) \leq 1.2 + 2s\sqrt{v(\Sigma) \log d \over n}.
$$

For $s = \sqrt{K/\sigma_{\min}(\Sigma)}$ this bound implies that for all $u$ such that $\|\Sigma^{1/2}u\| = 1$ and $\|u\|_1 \leq \sqrt{K}\|u\|$, with probability at least $1 - 2 \exp(-0.02n)$,

$$
\frac{1}{\sqrt{n}} \|Xu\| \leq 1.2 + 2\sqrt{v(\Sigma) K \log d \over \sigma_{\min}(\Sigma) \cdot n} \leq 1.3,
$$

and we get the desired bound. \hfill \Box

**Well-spreadness of Gaussian matrices.** If $n \geq {v(\Sigma) \over \sigma_{\min}(\Sigma)} \cdot k \log d$, then an $n \times d$ random matrix $X$ with i.i.d. rows $X_1, \ldots, X_n \sim N(0, \Sigma)$ satisfies the RE-property with parameter $\sigma_{\min}(\Sigma)/4$ over all sets of size $k$ where $\sigma_{\min}(\Sigma)$ is the smallest eigenvalue of $\Sigma$ (it is a consequence of Theorem 7.3). Also, norms of columns of $X$ are bounded by $O(\sqrt{n})$ with high probability. Hence, $X$ satisfies Assumption 1 and 2 of Theorem 6.1, with high probability.

In the next lemma, we show that it also satisfies the last assumption, namely the well-spreadness assumption:

**Lemma 7.6.** Let $X$ be a $n$-by-$d$ random matrix with i.i.d. rows $X_1, \ldots, X_n \sim N(0, \Sigma)$, where $\Sigma$ is a positive definite matrix. Suppose that for some $K \geq 1$, $n \geq 10^6 \cdot {v(\Sigma) \over \sigma_{\min}(\Sigma)} \cdot K \log d$. Then with probability at least $1 - \exp(-n/1000)$, for all $u \in \mathbb{R}^d$ such that $\|u\|_1 \leq \sqrt{K}\|u\|$ and for all sets $S \subseteq [n]$ of size $\lceil 0.999n \rceil$,

$$
\|X_Su\| \geq \frac{1}{2}\|Xu\|. \tag{7.2}
$$

**Proof.** For a set $M \subseteq [n]$ of size at most $n/1000$ independent of $X$, Theorem 7.3 implies that $\|X_Mu\| \leq 0.1\sqrt{n} \cdot \|\Sigma^{1/2}u\|$ and $\|Xu\| \geq 0.5\sqrt{n} \cdot \|\Sigma^{1/2}u\|$ with probability at least $1 - 2 \exp(-n/100)$. Using a union bound over all sets $M$ of size $n - \lceil 0.999n \rceil$, with probability

$$
1 - 2 \exp(-n/100 + n \log(1000e)/100) \geq 1 - \exp(-n/1000),
$$

we get

$$
\|X_Mu\|^2 \leq 0.1\|Xu\|^2.
$$

Since for $S = [n] \setminus M$, $\|Xu\|^2 = \|X_Mu\|^2 + \|X_Su\|^2$, we get the desired bound. \hfill \Box

Now we can prove restricted strong convexity.
Restricted local strong convexity of Huber loss for Gaussian design.

**Lemma 7.7.** Consider the settings of Theorem 2.3. Let $0 < \delta < 1$, $R > 0$. Define

$$\mathcal{B}_R := \left\{ u \in \mathbb{R}^d \mid \frac{1}{\sqrt{n}}||X(u)|| = R \right\}.$$

Suppose that $R \leq \frac{1}{200}$. Then with probability at least $1 - 3 \exp(-\alpha n/1000)$, for all $u \in \mathcal{B}_R \cap S_4(\Omega)$,

$$F_2(\beta^* + u) \geq F_2(\beta^*) + \langle \nabla F_2(\beta^*), u \rangle + \frac{\alpha n}{200} \cdot \frac{1}{n}||Xu||^2.$$

**Proof.** Let $u \in \mathcal{B}_R \cap S_4(\Omega)$, where $\Omega$ is the support of $\beta^*$. By Lemma A.2,

$$F_2(\beta^* + u) - F_2(\beta^*) - \langle \nabla F_2(\beta^*), u \rangle \geq \frac{1}{2} \sum_{i \in [n]} \langle X_i, u \rangle^2 \mathbf{1}_{[|\eta_i| \leq 1]} \cdot \mathbf{1}_{[|\langle X_i, u \rangle| \leq 1]}.$$

Denote $A = \{ i \in [n] \mid |\eta_i| \leq 1 \}$. Matrix $X_A$ is an $an \times d$ random matrix with i.i.d. rows $X_j \sim N(0, \Sigma)$. By Theorem 7.3, with probability $1 - 2 \exp(-an/100)$,

$$16\alpha R^2 n = 16\alpha ||Xu||^2 \geq 4an||\Sigma^{1/2}u||^2 \geq ||X_Au||^2 \geq \frac{\alpha n}{4}||\Sigma^{1/2}u||^2 \geq \frac{\alpha}{16}||Xu||^2 = \frac{\alpha}{16}R^2 n.$$

By Lemma 7.6, with probability $1 - \exp(-an/1000)$, $X_A$ satisfies well-spread property for sets of size $an/1000$ and for all $u \in S_4(\mathcal{K})$. Since number of entries of $X_Au$ which are larger than 1 is at most $16\alpha R^2 n \leq an/1000$, we get

$$\sum_{i \in [n]} \langle X_i, u \rangle^2 \mathbf{1}_{[|\eta_i| \leq 1]} \cdot \mathbf{1}_{[|\langle X_i, u \rangle| \leq 1]} = \sum_{i \in A} \langle X_i, u \rangle^2 \mathbf{1}_{[|\langle X_i, u \rangle| \leq 1]} \geq \frac{1}{4}||X_Au||^2 \geq \frac{\alpha}{64}R^2 n.$$

Hence with probability at least $1 - 3 \exp(-an/1000)$ we get the desired bound. □

**Putting everything together.** Let’s check that the conditions of Theorem 4.1 are satisfied for $\Omega = \mathbb{O} = \text{supp}(\beta^*)$ and $\mathcal{E}(u) = \frac{1}{\sqrt{n}}||Xu||$. Decomposability is obvious. As a consequence of Theorem 7.3 $X$ satisfies the RE-property with $\lambda \geq \sigma_{\min}(\Sigma)/4$ with probability at least $1 - \exp(-n/100)$, so contraction is satisfied with $s = 8\sqrt{k/\sigma_{\min}(\Sigma)}$. By Lemma 7.2, gradient
is bounded by $15\sqrt{\nu(\Sigma) \cdot n \cdot (\log d)}$ with probability $1 - d^{-10}/2$. By Lemma 7.7, with probability at least $1 - 3 \exp(n/1000)$, Huber loss satisfies restricted local strong convexity with parameter $\kappa = 0.01\alpha n$. Hence for

$$n \gtrsim \frac{\nu(\Sigma) \cdot k \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2} \quad \text{and} \quad R \gtrsim \sqrt{\frac{\nu(\Sigma) \cdot k \log d}{\sigma_{\min}(\Sigma) \cdot \alpha^2 \cdot n}},$$

and since then we have $\hat{\beta} - \beta^* \in \mathbb{B}_R \cap S_4(\Omega)$ the estimator $\hat{\beta}$ defined in Eq. (6.1) satisfies $\frac{1}{\sqrt{n}} \| X(\hat{\beta} - \beta^*) \| \leq R$ with probability at least $1 - d^{-10}$. Since $\hat{\beta} - \beta \in S_4(\Omega)$, we also get the desired parameter error $\| \hat{\beta} - \beta \| \leq 2R/\sqrt{\sigma_{\min}(\Sigma)}$.

8 Optimal fraction of inliers for principal component analysis under oblivious noise (Theorem 2.4)

In this section we prove Theorem 2.4. Recall that a successful $(\epsilon, \delta)$-weak recovery algorithm (where $\epsilon, \delta \in (0, 1)$) for PCA is an algorithm that takes $Y$ as input and returns a matrix $\hat{L}$ such that $\|\hat{L} - L^*\|_F \leq \epsilon \cdot \rho$ with probability at least $1 - \delta$ (where $\rho$, $Y$ and $L^*$ are as in Theorem 2.1).

Let’s restate Theorem 2.4:

**Theorem 8.1** (Restatement of Theorem 2.4). Let $Y = L^* + N \in \mathbb{R}^{n \times n}$, where $\text{rank}(L^*) = r$, $\|L^*\|_{\text{max}} \leq \rho/n$ and the entries of $N$ are independent and symmetric about zero. Let $\zeta > 0$.

Then there exists a universal constant $C_0 > 0$ such that for every $0 < \epsilon < 1$ and $0 < \delta < 1$, if $\alpha := \min_{i,j \in [n]} P[|N_{i,j}| \leq \zeta]$ satisfies $\alpha < C_0 \cdot (1 - \epsilon^2)^2 \cdot (1 - \delta) \cdot \sqrt{r/n}$, and $n$ is large enough, then it is information-theoretically impossible to have a successful $(\epsilon, \delta)$-weak recovery algorithm. The problem remains information-theoretically impossible (for the same regime of parameters) even if we assume that $L^*$ is incoherent; more precisely, even if we know that $L^*$ has incoherence parameters that are as good as those of a random flat matrix of rank $r$, the theorem still holds.

More in detail, we construct distributions over $L^*$ and $N$ such that the assumptions of the theorem are satisfied and if $\alpha < C_0 \cdot (1 - \epsilon^2)^2 \cdot (1 - \delta) \cdot \sqrt{r/n}$, weak recovery is not possible.

We will assume without loss of generality that $0 \leq \zeta \leq \rho/n = 1$. Indeed, weak recovery property is scale invariant, so we can assume $\rho = n$. We can assume $\zeta \leq 1$ since if the theorem is true for $\zeta = 1$, then it is true for all $\zeta > 1$.

A generative model for the hidden matrix

In the following, we will denote the all-zeros vector of dimension $n$ as $\mathbf{0}_n$. Similarly, we will denote the all-ones vector of dimension $n$ as $\mathbf{1}_n$.

For the sake of simplicity, we will assume that $\frac{n}{r}$ is an integer. We will divide the the matrix $L^*$ into $r$ blocks of $\frac{n}{r} \times n$ sub-matrices.

---

\(^{14}\) All the subsequent proofs can be adapted for a general $r$ with minor modifications.
For every \(1 \leq k \leq r\), let \(u_k\) be an arbitrary but fixed and deterministic vector in the set \(\{0_{(k-1)}\,\vec{a}\} \times \{-1, +1\}^n \times \{0_{(r-k)}\,\vec{a}\}\), and let \(v_k\) be a random flat vector chosen uniformly from \(\{-1, +1\}^n\). We further assume that the random vectors \(\{v_k\}_{1 \leq k \leq r}\) are mutually independent. The hidden matrix \(L^*\) is constructed as follows:\(^{15}\)

\[
L^* = \sum_{k=1}^{r} u_k \cdot v_k^T.
\]

Note that \(L^*\) is a flat matrix, i.e., \(L^* \in \{-1, +1\}^{n \times n}\). Furthermore, the rank of \(L^*\) is at most \(r\), and with high probability, \(L^*\) is incoherent with parameter \(\mu \leq O(\log n)\).

### The noise distribution

Let \((N_{ij})_{i,j \in [n]}\) be i.i.d. random variables that are sampled according to the distribution

\[
\mathbb{P}[N_{ij} = \ell] = \begin{cases} 
\frac{\xi \sqrt{r}}{2\sqrt{n} - \xi \sqrt{r}} \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{|\ell|/2} & \text{if } \ell \text{ is even} \\
0 & \text{otherwise,}
\end{cases}
\]

where \(0 < \xi \leq 1/2\) is a constant. Furthermore, we assume that \(N\) is independent from \(L^*\).

The distribution of \(N\) is symmetric and satisfies

\[
\alpha := \mathbb{P}[|N_{ij}| \leq 1] = \mathbb{P}[N_{ij} = 0] = \frac{\xi \sqrt{r}}{2\sqrt{n} - \xi \sqrt{r}} = \Theta\left(\xi \sqrt{\frac{r}{n}}\right).
\]

Define

\[
Y = L^* + N.
\]

### Upper bound on the mutual information

**Lemma 8.2.** The mutual information \(I(L^*; Y)\) between \(L^*\) and \(Y\) can be upper bounded as follows:

\[
I(L^*; Y) \leq O(\xi \cdot n \cdot r).
\]

**Proof.** Notice that for every \(\ell \in 2\mathbb{Z}\), we have

\[
\mathbb{P}[N_{ij} = \ell + 2] = \mathbb{P}[N_{ij} = \ell] \cdot \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\text{sign}(\ell+1)},
\]

where

\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

---

\(^{15}\)For the general case in which \(\vec{a}\) may not be an integer, we divide \(L^*\) into \(r\) blocks of disjoint sub-matrices of dimensions \(\lceil \frac{n}{r} \rceil \times n\) and \(\lfloor \frac{n}{r} \rfloor \times n\).
For every $L^* \in \{-1, +1\}^{n \times n}$ and every $Y \in (2\mathbb{Z} + 1)^{n \times n}$, we have

$$
P[Y = Y|L^* = L^*] = P[N = Y - L^*]$$

$$= \prod_{i,j} P[N_{ij} = Y_{ij} - L^*_{ij}]$$

$$= \prod_{i,j} P[N_{ij} = Y_{ij} - 1 + 1 - L^*_{ij}]$$

$$\equiv \prod_{i,j} \left[ P[N_{ij} = Y_{ij} - 1] \cdot \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\frac{1}{2}(1 - L^*_{ij}) \cdot \text{sign}(Y_{ij} - 1 + 1)} \right]$$

$$= P[N = Y - 1_n 1_n^T] \cdot \prod_{i,j} \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\frac{1}{2}(1 - L^*_{ij}) \cdot \text{sign}(Y_{ij})},$$

where $\equiv$ follows from Eq. (8.2). Therefore, we can write

$$P[Y = Y|L^* = L^*] = P[N = Y - 1_n 1_n^T] \cdot f(L^*, Y), \quad (8.3)$$

where

$$f(L^*, Y) = \prod_{i,j} \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\frac{1}{2}(1 - L^*_{ij}) \cdot \text{sign}(Y_{ij})}$$

$$= \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\frac{1}{2} \sum_{i,j} (1 - L^*_{ij}) \cdot \text{sign}(Y_{ij})} \quad (8.4)$$

$$= \left(1 - \xi \sqrt{\frac{r}{n}}\right)^{\frac{1}{2} \langle 1_n 1_n^T - L^*, \text{sign}(Y) \rangle},$$

where $\text{sign}(Y)$ is the $n \times n$ matrix defined as $\text{sign}(Y)_{i,j} = \text{sign}(Y_{i,j})$. Furthermore, from Eq. (8.3) we can deduce that for every $Y \in (2\mathbb{Z} + 1)^{n \times n}$, we have

$$P[Y = Y] = P[N = Y - 1_n 1_n^T] \cdot g(Y), \quad (8.5)$$

where

$$g(Y) = \mathbb{E}_{L^*}[f(L^*, Y)]. \quad (8.6)$$

Now from Hölder’s inequality, we have

$$|\langle 1_n 1_n^T - L^*, \text{sign}(Y) \rangle| \leq \|1_n 1_n^T - L^*\|_{\text{nuc}} \cdot \|\text{sign}(Y)\|$$

$$\leq \left(\|1_n 1_n^T\|_{\text{nuc}} + \|L^*\|_{\text{nuc}}\right) \cdot \|\text{sign}(Y)\|$$

$$= (n + \|L^*\|_{\text{nuc}}) \cdot \|\text{sign}(Y)\|. $$

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If $L^*$ is in the support of the distribution of $L^*$, then there exist $r$ vectors $\{v_k\}_{1 \leq k \leq r}$ such that $v_k \in \{-1, +1\}^n$ for every $1 \leq k \leq r$, and

$$L^* = \sum_{k=1}^{r} u_k \cdot v_k^T.$$ 

Therefore,

$$\|L^*\|_{\text{nuc}} \leq \sum_{k=1}^{r} \|u_k \cdot v_k^T\|_{\text{nuc}} = \sum_{k=1}^{r} \sqrt{\frac{n}{r}} \cdot \sqrt{n} = r \cdot \sqrt{\frac{n}{r}} = n \sqrt{r}. \quad (8.7)$$

Hence,

$$\langle 1_n 1_n^T - L^*, \text{sign}(Y) \rangle \leq n \cdot (\sqrt{r} + 1) \cdot \|\text{sign}(Y)\| \leq 2n \sqrt{r} \cdot \|\text{sign}(Y)\|.$$ 

By combining this with Eq. (8.4), we get

$$\left(1 - \xi \sqrt{\frac{r}{n}} \|\text{sign}(Y)\| \right) \leq f(L^*, Y) \leq \left(1 - \xi \sqrt{\frac{r}{n}} \right)^{-n \sqrt{r}} \|\text{sign}(Y)\|. \quad (8.8)$$

Furthermore, from Eq. (8.6) and Eq. (8.8), we get

$$\left(1 - \xi \sqrt{\frac{r}{n}} \|\text{sign}(Y)\| \right) \leq g(Y) \leq \left(1 - \xi \sqrt{\frac{r}{n}} \right)^{-n \sqrt{r}} \|\text{sign}(Y)\|. \quad (8.9)$$

The mutual information between $L^*$ and $Y$ is given by

$$I(L^*; Y) = \sum_{L^*, Y} \mathbb{P}[L^* = L^*, Y = Y] \cdot \log_2 \frac{\mathbb{P}[Y = Y | L^* = L^*]}{\mathbb{P}[Y = Y]}$$

$$\overset{(t)}{=} \sum_{L^*, Y} \mathbb{P}[L^* = L^*, Y = Y] \cdot \log_2 \frac{\mathbb{P}[N = Y - 1_n 1_n^T] \cdot f(L^*, Y)}{\mathbb{P}[N = Y - 1_n 1_n^T] \cdot g(Y)}$$

$$= \sum_{L^*, Y} \mathbb{P}[L^* = L^*, Y = Y] \cdot \log_2 \frac{f(L^*, Y)}{g(Y)}$$

$$= \mathbb{E} \left[ \log_2 \frac{f(L^*, Y)}{g(Y)} \right],$$
where (‡) follows from Eq. (8.3) and Eq. (8.5). Now from Eq. (8.8) and Eq. (8.9), we get

\[
I(L^*; Y) \leq \mathbb{E} \left[ \log_2 \left( 1 - \frac{2n \sqrt{r}}{\log 2} \| \text{sign}(Y) \| \right) \right]
= -2n \sqrt{r} \cdot \log_2 \left( 1 - \frac{2n \sqrt{r}}{\log 2} \right) \cdot \mathbb{E} \left[ \| \text{sign}(Y) \| \right]
= -\frac{2}{\log 2} n \sqrt{r} \cdot \log \left( 1 - \frac{2n \sqrt{r}}{\log 2} \right) \cdot \mathbb{E} \left[ \| \text{sign}(Y) \| \right]
\]

(8.10)

where (‡) follows from the fact that \(- \log(1 - t) \leq 2t\) for every \(t \in [0, 1/2]\).

Now let \(S = \text{sign}(Y)\). In order to estimate \(\mathbb{E} \left[ \| \text{sign}(Y) \| \right] = \mathbb{E} \left[ \| S \| \right]\), we first condition on \(L^* = L^*\) for a fixed \(L^*\):

\[
\mathbb{E} \left[ \| S \| | L^* = L^* \right] = \mathbb{E} \left[ \| S - \mathbb{E}[S | L^* = L^*] + \mathbb{E}[S | L^* = L^*] \| | L^* = L^* \right]
\leq \mathbb{E} \left[ \| S - \mathbb{E}[S | L^* = L^*] \| | L^* = L^* \right] + \| \mathbb{E}[S | L^* = L^*] \|.
\]

Notice that

\[
\mathbb{E}[S | L^* = L^*] = \mathbb{E}[\text{sign}(L^* + N) | L^* = L^*]
= \mathbb{E}[\text{sign}(L^* + N)]
= \alpha \cdot L^*,
\]

where

\[
\alpha = \mathbb{P}[N_{ij} = 0] = \frac{\xi \sqrt{r}}{2 \sqrt{n} - \xi \sqrt{r}}.
\]

Therefore,

\[
\mathbb{E} \left[ \| S \| | L^* = L^* \right] \leq \mathbb{E} \left[ \| \hat{S} \| | L^* = L^* \right] + \alpha \cdot \| L^* \|,
\]

where

\[
\hat{S} = S - \mathbb{E}[S | L^* = L^*] = S - \alpha \cdot L^*.
\]

Now given \(L^* = L^*\), the entries of \(\hat{S}\) are centered and conditionally mutually independent. Furthermore, \(\| \hat{S} \|_{\text{max}} \leq \| S \|_{\text{max}} + \alpha \cdot |L^*|_{\text{max}} = 1 + \alpha \leq 2\). Therefore, by Fact B.7, there is a universal constant \(C \geq 2\) such that

\[
\mathbb{E} \left[ \| \hat{S} \| | L^* = L^* \right] \leq C \sqrt{n}.
\]
We conclude that
\[ \mathbb{E}[||\text{sign}(Y)||] = \mathbb{E}[||S||] \leq C \sqrt{n} + \alpha \cdot \mathbb{E}[||L^*||]. \quad (8.11) \]

Now notice that \( ||L^*|| = U \cdot V^T \), where \( U = [u_1 \ldots u_r] \) is the \( n \times r \) matrix whose columns are \( u_1, \ldots, u_r \), and \( V = [v_1 \ldots v_r] \) is the \( n \times r \) matrix whose columns are \( v_1, \ldots, v_r \). We have:
\[
\mathbb{E}[||L^*||] = \mathbb{E}[||U \cdot V^T||] \leq \mathbb{E}[||U|| \cdot ||V^T||] = ||U|| \cdot \mathbb{E}[||V^T||] = \sqrt{n} \cdot \mathbb{E}[||V^T||] \]
\[
= \frac{\xi \sqrt{r}}{2 \sqrt{n - \xi \sqrt{r}}} \cdot C \sqrt{n} = \frac{C \cdot n}{\sqrt{r}},
\]
where \((i)\) follows from the fact that \( V \) is an \( n \times r \) matrix with i.i.d. zero-mean entries and satisfying \( ||V||_{\text{max}} = 1 \) and Fact B.7. By inserting this in Eq. (8.11), we get
\[
\mathbb{E}[||\text{sign}(Y)||] \leq C \sqrt{n} + \frac{\xi \sqrt{r}}{2 \sqrt{n - \xi \sqrt{r}}} \cdot \frac{C \cdot n}{\sqrt{r}}
\]
\[
\leq C \sqrt{n} + \frac{\sqrt{r}}{\sqrt{n}} \cdot \frac{C \cdot n}{\sqrt{r}}
\]
\[
= 2C \sqrt{n},
\]
By combining this with Eq. (8.10), we get
\[
I(L^*; Y) \leq \frac{4 \xi \cdot \sqrt{n} \cdot r}{\log 2} \cdot 2C \sqrt{n}
\]
\[
\leq \frac{8C \xi}{\log 2} \cdot n \cdot r
\]
\[
= O(\xi \cdot n \cdot r).
\]

**Successful weak-recovery reduces entropy**

**Lemma 8.3.** If there exists a \((\delta, \varepsilon)\)-successful weak recovery algorithm that takes \( Y = L^* + N \) as input and returns a matrix \( \hat{L} \) as output in such a way that
\[
\mathbb{P}[||\hat{L} - L^*||_{\text{F}} \leq \varepsilon \cdot n] \geq 1 - \delta,
\]
then the mutual information between \( L^* \) and \( Y \) can be lower bounded as follows:
\[
I(L^*; Y) \geq \frac{(1 - \varepsilon^2)^2}{8 \log 2} \cdot (1 - \delta) \cdot n \cdot r - 1.
\]
Proof. Define the set

$$\Omega = \left\{ \sum_{k=1}^{r} u_k v_k^T : \forall k \in [r], v_k \in \mathbb{R}^n \text{ and } \|v_k\|_{\text{max}} \leq 1 \right\}.$$  

It is easy to see that $\Omega$ is a closed and convex set. Let $\hat{L}_\Omega$ be the orthogonal projection of $\hat{L}$ onto $\Omega$. Since $L^* \in \Omega$, we have $\|\hat{L}_\Omega - L^*\|_F \leq \|\hat{L} - L^*\|_F$. Therefore,

$$P[\|\hat{L}_\Omega - L^*\|_F \leq \varepsilon \cdot n] \geq P[\|\hat{L} - L^*\|_F \leq \varepsilon \cdot n] \geq 1 - \delta.$$

Using an inequality that is similar to the standard Fano-inequality, we will show that the existence of a successful weak-recovery algorithm implies a linear decrease in the entropy of the random vectors $(v_k)_{k \in [r]}$.

Define the random variable $Z$ as follows:

$$Z = 1[\|\hat{L}_\Omega - L^*\|_F \leq \varepsilon \cdot n].$$

Furthermore, for every $L \in \Omega$, define

$$B_{L, \varepsilon} = \left\{ (v_k)_{k \in [r]} \in \{-1, +1\}^{n \cdot r} : \|L - \sum_{k=1}^{r} u_k v_k^T\|_F \leq \varepsilon \cdot n \right\}.$$  

Clearly, if $Z = 1$, then $(v_k)_{k \in [r]} \in B_{L_\Omega, \varepsilon}$.

Let $H((v_k)_{k \in [r]}|\hat{L}_\Omega)$ be the conditional entropy of $(v_k)_{k \in [r]}$ given $\hat{L}_\Omega$. We have:

$$H((v_k)_{k \in [r]}|\hat{L}_\Omega) \leq H(Z, (v_k)_{k \in [r]}|\hat{L}_\Omega) = H(Z|\hat{L}_\Omega) + H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z) \leq H(Z) + H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 0) \cdot P[Z = 0] + H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 1) \cdot P[Z = 1] \leq 1 + n \cdot r \cdot P[Z = 0] + (1 - P[Z = 0]) \cdot H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 1),$$

where the last inequality follows from the fact that $Z$ is a binary random variable (and hence $H(Z) \leq \log_2(2) = 1$), and the fact that $(v_k)_{k \in [r]} \in \{-1, +1\}^{n \cdot r}$, which implies that $H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 0) \leq \log_2|\{-1, +1\}^{n \cdot r}| = n \cdot r$.

Since $P[Z = 0] \leq \delta$ and $H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 1) \leq \log_2|\{-1, +1\}^{n \cdot r}| = n \cdot r$, we have

$$H((v_k)_{k \in [r]}|\hat{L}_\Omega) \leq 1 + n \cdot r + (1 - \delta) \cdot H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 1).$$

Now notice that

$$H((v_k)_{k \in [r]}|\hat{L}_\Omega, Z = 1) \stackrel{(\ast)}{=} \log_2|B_{\hat{L}_\Omega, \varepsilon}| \leq \max_{L \in \Omega} \log_2|B_{L, \varepsilon}|,$$

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where (*) follows from the fact that given \( Z = 1 \), we have \((v_k)_{k \in [r]} \in B_{L,\epsilon}^\prime\). On the other hand, for every \( L \in \Omega \), we have

\[
\log_2 |B_{L,\epsilon}| = n \cdot r + \log_2 \left( \frac{|B_{L,\epsilon}|}{2^{n \cdot r}} \right) = n \cdot r + \log_2 P[(v_k)_{k \in [r]} \in B_{L,\epsilon}],
\]

where the last equality follows from the fact that \((v_k)_{k \in [r]}\) is uniformly distributed in \(\{-1, +1\}^{n \cdot r}\). Therefore,

\[
H\left((v_k)_{k \in [r]}; \tilde{L}_\Omega\right) \leq 1 + n \cdot r + (1 - \delta) \cdot \max_{L \in \Omega} \log_2 P[(v_k)_{k \in [r]} \in B_{L,\epsilon}]. \tag{8.12}
\]

Now fix \( L \in \Omega \) and let \((v_k)_{k \in [r]}\) be vectors in \(\mathbb{R}^n\) such that \(\|v_k\|_{\max} \leq 1\) and \(L = \sum_{k=1}^{r} u_k v_k^T\). We have \((v_k)_{k \in [r]} \in B_{L,\epsilon}\) if and only if \(\|L' - L\|_F \leq \epsilon \cdot n\). Notice that

\[
\|L' - L\|_F^2 = \left( \sum_{k=1}^{r} u_k \cdot (v_k - v_k')^T, \sum_{k'=1}^{r} u_{k'} \cdot (v_{k'} - v_{k'})^T \right)
\]

\[
= \text{Tr}\left( \left( \sum_{k=1}^{r} u_k \cdot (v_k - v_k') \right)^T \cdot \left( \sum_{k'=1}^{r} u_{k'} \cdot (v_{k'} - v_{k'}) \right) \right)
\]

\[
= \sum_{k=1}^{r} \sum_{k'=1}^{r} \text{Tr}\left( (v_k - v_k') \cdot u_k^T \cdot u_{k'} \cdot (v_{k'} - v_{k'}) \right)
\]

\[
\overset{(*)}{=} \frac{n}{r} \cdot \sum_{k=1}^{r} \text{Tr}\left( (v_k - v_k') \cdot (v_k - v_k')^T \right) = \frac{n}{r} \cdot \sum_{k=1}^{r} \|v_k - v_k\|^2
\]

\[
= \frac{n}{r} \cdot \sum_{k=1}^{r} \sum_{i=1}^{n} (v_{k,i} - v_{k,i})^2 = \frac{n}{r} \cdot \sum_{k=1}^{r} \sum_{i=1}^{n} \left( v_{k,i}^2 + v_{k,i}^2 - 2v_{k,i} \cdot v_{k,i} \right)
\]

\[
\overset{(**)}{=} \frac{n}{r} \cdot \left( n \cdot r - 2 \sum_{k=1}^{r} \sum_{i=1}^{n} v_{k,i} \cdot v_{k,i} \right),
\]

where (*) follows from the fact that \((u_k)_{k \in [r]}\) are orthogonal to each other, and \(\|u_k\|^2 = \frac{n}{r}\) for every \( k \in [r] \). Note that \((v_{k,i})_{1 \leq i \leq n}\) and \((v_{k,i})_{1 \leq i \leq n}\) are the entries of \(v_k\) and \(v_k\), respectively. (** follows from the fact that \(v_k \in \{-1, +1\}^n\) for every \( k \in [r] \). Therefore,

\[
\|L' - L\|_F^2 \geq n^2 - \frac{2n}{r} \cdot \sum_{k=1}^{r} \sum_{i=1}^{n} v_{k,i} \cdot v_{k,i},
\]

which implies that

\[
P[(v_k)_{k \in [r]} \in B_{L,\epsilon}] = P\left[\|L' - L\|_F^2 \leq \epsilon^2 \cdot n^2\right]
\]

\[
\leq P\left[ n^2 - \frac{2n}{r} \cdot \sum_{k=1}^{r} \sum_{i=1}^{n} v_{k,i} \cdot v_{k,i} \leq \epsilon^2 \cdot n^2\right]
\]

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\[
= \mathbb{P}\left[ \sum_{k=1}^{r} \sum_{i=1}^{n} v_{k,i} \cdot v_{i} \geq \frac{1}{2} \cdot (1 - \varepsilon^2) \cdot n \cdot r \right].
\]

Note that the random variables \((v_{k,i} \cdot v_{i})\) are independent. Moreover, for every \(1 \leq k \leq r\) and every \(1 \leq i \leq n\), we have

\[
\mathbb{E}[v_{k,i} \cdot v_{i}] = 0.
\]

Furthermore, since \(\|v_k\|_{\text{max}} \leq 1\) and \(v_k \in \{-1, +1\}^n\), the random variables \((v_{k,i} \cdot v_{i})\) can be uniformly bounded as

\[|v_{k,i} \cdot v_{i}| \leq |v_{k,i}| \leq 1.\]

It follows from Hoeffding’s inequality Fact B.2 that

\[
\mathbb{P}\left[ (v_k)_{k \in [r]} \in B_{L,\varepsilon} \right] \leq \exp\left( -\frac{(1 - \varepsilon^2)^2 \cdot n^2 \cdot r^2}{8 \cdot n \cdot r} \right)
= \exp\left( -\frac{(1 - \varepsilon^2)^2}{8 \cdot n \cdot r} \right).
\]

Since this is true for every \(L \in \Omega\), we get from Eq. (8.12) that

\[
H\left( (v_k)_{k \in [r]} \right| \hat{L}_{\Omega} \right) \leq 1 + n \cdot r - (1 - \delta) \cdot \frac{(1 - \varepsilon^2)^2}{8 \log 2} \cdot n \cdot r.
\]

Therefore, the mutual information between \((v_k)_{k \in [r]}\) and \(\hat{L}_{\Omega}\) satisfies:

\[
I\left( (v_k)_{k \in [r]} ; \hat{L}_{\Omega} \right) = H\left( (v_k)_{k \in [r]} \right) - H\left( (v_k)_{k \in [r]} \right| \hat{L}_{\Omega} \right)
\geq n \cdot r - 1 - n \cdot r + (1 - \delta) \cdot \frac{(1 - \varepsilon^2)^2}{8 \log 2} \cdot n \cdot r
= \frac{(1 - \varepsilon^2)^2}{8 \log 2} \cdot (1 - \delta) \cdot n \cdot r - 1.
\]

Now since \((v_k)_{k \in [r]} \rightarrow L^* \rightarrow Y \rightarrow \hat{L} \rightarrow \hat{L}_{\Omega}\) is a Markov chain, it follows from the data-processing inequality that

\[
I(L^*; Y) = I((v_k)_{k \in [r]} ; Y) \geq I\left( (v_k)_{k \in [r]} \right| \hat{L}_{\Omega} \right) \geq \frac{(1 - \varepsilon^2)^2}{8 \log 2} \cdot (1 - \delta) \cdot n \cdot r - 1.
\]

\(\Box\)

**Putting everything together**

Now we are ready to prove Theorem 2.4:
Proof of Theorem 2.4. From Lemma 8.2 and Lemma 8.3 we can deduce that if there exists a \((\delta, \epsilon)-\)successful weak recovery algorithm then we must have

\[
\frac{8C\xi}{\log 2} \cdot n \cdot r \geq I(L'; Y) \geq \frac{(1 - \epsilon^2)^2}{8\log 2} \cdot (1 - \delta) \cdot n \cdot r - 1.
\]

Therefore, if \(n\) is large enough and

\[
\xi < \frac{(1 - \epsilon^2)^2}{64C} \cdot (1 - \delta) - \frac{\log 2}{8C} \cdot \frac{1}{r \cdot n},
\]

it is impossible to have a \((\delta, \epsilon)-\)successful weak recovery algorithm. Now since \(\alpha = \Theta\left(\xi \sqrt{\frac{r}{n}}\right)\), we get the result. \(\square\)

References

[Adl90] Robert J. Adler, An introduction to continuity, extrema, and related topics for general gaussian processes, Lecture Notes-Monograph Series 12 (1990), i–155.

[BBC11] Dimitris Bertsimas, David B. Brown, and Constantine Caramanis, Theory and applications of robust optimization, SIAM Review 53 (2011), no. 3, 464–501.

[BGN09] Aharon Ben-Tal, Laurent El Ghaoui, and Arkadi Nemirovski, Robust optimization, Princeton Series in Applied Mathematics, vol. 28, Princeton University Press, 2009.

[BJKK17] Kush Bhatia, Prateek Jain, Parameswaran Kamalaruban, and Purushottam Kar, Consistent robust regression, NIPS, 2017, pp. 2107–2116.

[CLMW11] Emmanuel J. Candès, Xiaodong Li, Yi Ma, and John Wright, Robust principal component analysis?, J. ACM 58 (2011), no. 3, 11:1–11:37.

[dNS21] Tommaso d’Orsi, Gleb Novikov, and David Steurer, Consistent regression when oblivious outliers overwhelm, 38th International Conference on Machine Learning, ICML 2021, 18-24 July 2021, Virtual Event, Proceedings of Machine Learning Research, PMLR, 2021.

[DT19] Arnak Dalalyan and Philip Thompson, Outlier-robust estimation of a sparse linear model using \(l_1\)-penalized huber’s \(m\)-estimator, Advances in Neural Information Processing Systems, 2019, pp. 13188–13198.

[Hub64] Peter J. Huber, Robust estimation of a location parameter, Ann. Math. Statist. 35 (1964), no. 1, 73–101.
[MMYSB19] Ricardo A. Maronna, R. Douglas Martin, Victor J. Yohai, and Matías Salibián-Barrera, *Robust statistics: Theory and methods*, 2 ed., Wiley Series in Probability and Statistics, John Wiley & Sons, 2019.

[NTN11] Nasser Nasrabadi, Trac Tran, and Nam Nguyen, *Robust lasso with missing and grossly corrupted observations*, Advances in Neural Information Processing Systems (J. Shawe-Taylor, R. Zemel, P. Bartlett, F. Pereira, and K. Q. Weinberger, eds.), vol. 24, Curran Associates, Inc., 2011.

[NW12] Sahand Negahban and Martin J. Wainwright, *Restricted strong convexity and weighted matrix completion: Optimal bounds with noise*, Journal of Machine Learning Research 13 (2012), 1665–1697.

[PFW20] Scott Pesme and Nicolas Flammarion, *Online robust regression via sgd on the l1 loss*, Advances in Neural Information Processing Systems (H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, eds.), vol. 33, Curran Associates, Inc., 2020, pp. 2540–2552.

[PJL21] Ankit Pensia, Varun Jog, and Po-Ling Loh, *Robust regression with covariate filtering: Heavy tails and adversarial contamination*, 2021.

[PWB16] Amelia Perry, Alexander S. Wein, Afonso S. Bandeira, and Ankur Moitra, *Optimality and sub-optimality of PCA for spiked random matrices and synchronization*, CoRR abs/1609.05573 (2016).

[RV10] Mark Rudelson and Roman Vershynin, *Non-asymptotic theory of random matrices: Extreme singular values*, Proceedings of the International Congress of Mathematicians 2010, ICM 2010 (2010).

[RWY10] Garvesh Raskutti, Martin J. Wainwright, and Bin Yu, *Restricted eigenvalue properties for correlated gaussian designs*, Journal of Machine Learning Research 11 (2010), no. 78, 2241–2259.

[RWY11] Garvesh Raskutti, Martin J. Wainwright, and Bin Yu, *Minimax rates of estimation for high-dimensional linear regression over ℓq -balls*, IEEE Transactions on Information Theory 57 (2011), no. 10, 6976–6994.

[SBRJ19] Arun Sai Suggala, Kush Bhatia, Pradeep Ravikumar, and Prateek Jain, *Adaptive hard thresholding for near-optimal consistent robust regression*, Conference on Learning Theory, COLT 2019, 25-28 June 2019, Phoenix, AZ, USA, 2019, pp. 2892–2897.

[SF20] Takeyuki Sasai and Hironori Fujisawa, *Robust estimation with lasso when outputs are adversarially contaminated*, 2020.

[SZF18] Qiang Sun, Wenxin Zhou, and Jianqing Fan, *Adaptive huber regression*. 3, 4
[TJSO14] Efthymios Tsakonas, Joakim Jaldén, Nicholas D Sidiroopoulos, and Björn Ottersten, *Convergence of the huber regression m-estimate in the presence of dense outliers*, IEEE Signal Processing Letters 21 (2014), no. 10, 1211–1214. 3, 4

[Ver18] Roman Vershynin, *High-dimensional probability: An introduction with applications in data science*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2018. 46

[Vis18] Nisheeth K Vishnoi, *Algorithms for convex optimization*, Cambridge University Press, 2018. 16

[Wai19] Martin J. Wainwright, *High-dimensional statistics: A non-asymptotic viewpoint*, Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, 2019. 7, 8, 9, 11, 16, 29, 46, 47

[ZLW+10] Zihan Zhou, Xiaodong Li, John Wright, Emmanuel J. Candès, and Yi Ma, *Stable principal component pursuit*, ISIT, IEEE, 2010, pp. 1518–1522. 3, 4, 6, 7, 14

[ZWJ14] Yuchen Zhang, M. Wainwright, and Michael I. Jordan, *Lower bounds on the performance of polynomial-time algorithms for sparse linear regression*, COLT, 2014. 8
A Facts about Huber loss

Fact A.1 (Integration by parts for absolutely continuous functions). Let \( F, G : \mathbb{R} \rightarrow \mathbb{R} \) be absolutely continuous functions, i.e. there exist locally integrable functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( a, b \in \mathbb{R} \),
\[
\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{and} \quad \int_a^b g(t) \, dt = G(b) - G(a) .
\]
Then for all \( a, b \in \mathbb{R} \),
\[
\int_a^b f(t) G(t) \, dt = F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t) \, dt .
\]
Proof. Let
\[
\int_a^b f(t) G(t) \, dt = G(a) \cdot (F(b) - F(a)) + \int_a^b f(t) \int_a^b 1_{[\tau \in [a,b]]} g(\tau) \, d\tau \, dt
\]
\[
= G(a) \cdot (F(b) - F(a)) + \int_a^b g(\tau) \int_a^b f(t) 1_{[\tau \in [\tau, b]]} \, d\tau \, dt
\]
\[
= G(a) \cdot (F(b) - F(a)) + \int_a^b g(\tau) \cdot (F(b) - F(\tau)) \, d\tau
\]
\[
= G(a) \cdot (F(b) - F(a)) + F(b)(G(b) - G(a)) - \int_a^b g(\tau) F(\tau) \, d\tau
\]
\[
= F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t) \, dt .
\]
\[\square\]

Lemma A.2 (Second order behavior of Huber-loss function). Let \( h > 0 \). For all \( \eta, \delta \in \mathbb{R} \), and all \( 0 \leq \tau \leq h \),
\[
f_h(\eta + \delta) - f_h(\eta) - f'_h(\eta) \cdot \delta \geq \frac{\delta^2}{2} 1_{[|\eta| \leq h - \tau]} \cdot 1_{[|\delta| \leq \tau]} .
\]
Proof. Consider \( g : \mathbb{R} \rightarrow \mathbb{R} \) defined as \( g(t) = f_h(\eta + t \cdot \delta) \). Note that for all \( a, b \in \mathbb{R} \),
\[
f'_h(\eta + b \delta) - f'_h(\eta + a \delta) = \int_{\eta + a \delta}^{\eta + b \delta} 1_{[|x| \leq h]} \, dx .
\]
Changing the variable \( x = \eta + t \delta \), we get
\[
g'(b) - g'(a) = \delta^2 \int_a^b 1_{[|\eta + t \delta| \leq h]} \, dt .
\]
By Fact A.1,
\[
\delta^2 \int_0^1 1_{[|\eta + t \delta| \leq h]} \cdot (1 - t) \, dt = -g'(0) + g(1) - g(0) .
\]
Note that \( g(0) = f_h(\eta), g(1) = f_h(\eta + \delta) \) and \( g'(0) = \delta f'_h(\eta) \). Since for all \( 0 \leq \tau \leq h \),
\[
1_{[|\eta + t \delta| \leq h]} \geq 1_{[|\eta| \leq h - \tau]} \cdot 1_{[|\delta| \leq \tau]} \quad \text{and} \quad \int_0^1 (1 - t) \, dt = 1/2 ,
\]
we get the desired bound. \[\square\]
B Tools for Probabilistic Analysis

This section contains some technical results needed for the proofs in the main body of the paper.

Fact B.1 (Chernoff’s inequality, [Ver18]). Let $\zeta_1, \ldots, \zeta_n$ be independent Bernoulli random variables such that $\Pr(\zeta_i = 1) = \Pr(\zeta_i = 0) = p$. Then for every $\Delta > 0$,

$$
\Pr\left( \sum_{i=1}^{n} \zeta_i \geq pn(1 + \Delta) \right) \leq \left( \frac{e^{-\Delta}}{(1 + \Delta)^{1+\Delta}} \right)^{pn}.
$$

and for every $\Delta \in (0, 1)$,

$$
\Pr\left( \sum_{i=1}^{n} \zeta_i \leq pn(1 - \Delta) \right) \leq \left( \frac{e^{-\Delta}}{(1 - \Delta)^{1-\Delta}} \right)^{pn}.
$$

Fact B.2 (Hoeffding’s inequality, [Wai19]). Let $z_1, \ldots, z_n$ be mutually independent random variables such that for each $i \in [n]$, $z_i$ is supported on $[-c_i, c_i]$ for some $c_i \geq 0$. Then for all $t \geq 0$,

$$
\Pr\left( \sum_{i=1}^{n} (z_i - \mathbb{E} z_i) \geq t \right) \leq \exp\left( -\frac{t^2}{2 \sum_{i=1}^{n} c_i^2} \right),
$$

and

$$
\Pr\left( \left| \sum_{i=1}^{n} (z_i - \mathbb{E} z_i) \right| \geq t \right) \leq 2 \exp\left( -\frac{t^2}{2 \sum_{i=1}^{n} c_i^2} \right).
$$

Fact B.3 (Bernstein’s inequality, [Wai19]). Let $z_1, \ldots, z_n$ be mutually independent random variables such that for each $i \in [n]$, $z_i$ is supported on $[-B, B]$ for some $B \geq 0$. Then for all $t \geq 0$,

$$
\Pr\left( \sum_{i=1}^{n} (z_i - \mathbb{E} z_i) \geq t \right) \leq \exp\left( -\frac{t^2}{2 \sum_{i=1}^{n} \mathbb{E} z_i^2 + \frac{2Bt}{3}} \right).
$$

Fact B.4 (Subgaussian maxima, [Wai19]). Let $d \geq 2$ be an integer and let $z$ be a $d$-dimensional random vector with zero mean $\sigma$-subgaussian entries. Then

$$
\mathbb{E}\|z\|_{\text{max}} \leq 2\sigma \sqrt{\log d}.
$$

Fact B.5 (Lipschitz functions of Gaussian vectors, [Wai19]). Let $g \sim N(0, 1)^m$ for some $m \in \mathbb{N}$ and let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be $L$-Lipschitz with respect to Euclidean norm, where $L > 0$. Then for all $t \geq 0$,

$$
\Pr\left( |F(g) - \mathbb{E} F(g)| \geq t \right) \leq 2 \exp\left( -\frac{t^2}{2L^2} \right).
$$

Fact B.6 (Spectral norm of Gaussian matrices, [Wai19]). Let $W \sim N(0, 1)^{n\times d}$. Then

$$
\mathbb{E}\|W\| \leq \sqrt{n} + \sqrt{d}.
$$

Moreover, for all $t \geq 0$,

$$
\Pr\left( \|W\| \geq \sqrt{n} + \sqrt{d} + t \right) \leq 2 \exp(-t^2/2).
$$
Fact B.7 (Spectral norm of matrices with bounded independent zero-mean entries, [RV10]). Let $M$ be an $n$-by-$n$ random matrix with independent zero-mean entries $M_{ij}$ supported on $[-1, 1]$. Then

$$\mathbb{E}\|M\| \leq (2 + o(1)) \sqrt{n}$$

as $n \to \infty$. Moreover, for all $t \geq 0$,

$$\mathbb{P}\left(\|M\| \geq \mathbb{E}\|M\| + \sqrt{2\pi} + t\right) \leq 2 \exp(-t^2/2).$$

Fact B.8 (Sudakov–Fernique theorem, [Adl90]). Let $\Theta$ be a compact subset of $\mathbb{R}^m$, where $m \in \mathbb{N}$. Let $W_\theta$ and $Z_\theta$ be real-valued sample-continuous zero-mean Gaussian processes indexed by elements of $\Theta$. Suppose that $\forall \theta, \theta' \in \Theta$, $\mathbb{E}(W_\theta - W_{\theta'})^2 \leq \mathbb{E}(Z_\theta - Z_{\theta'})^2$. Then

$$\mathbb{E}\sup_{\theta \in \Theta} W_\theta \leq \mathbb{E}\sup_{\theta \in \Theta} Z_\theta.$$

Fact B.9 (Sudakov Minoration, [Wai19]). Let $\{g_\theta \mid \theta \in \Theta\}$ be a zero-mean Gaussian process indexed by elements of some non-empty set $\Theta$. Let $\rho : \Theta \times \Theta \to [0, \infty)$ be a (pseudo)metric $\rho(\theta, \theta') := \left(\mathbb{E}(g_\theta - g_{\theta'})^2\right)^{1/2}$. Then

$$\mathbb{E}\sup_{\theta \in \Theta} g_\theta \geq \sup_{\varepsilon > 0} \varepsilon \sqrt{2\log|\mathcal{N}_{\varepsilon, \rho}(\Theta)|},$$

where $|\mathcal{N}_{\varepsilon, \rho}(\Theta)|$ is the minimal size of $\varepsilon$-net in $\Theta$ with respect to $\rho$. 

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