SHOCK-LIKE FREEZE-OUT IN RELATIVISTIC HYDRODYNAMICS

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Abstract

We have formulated a self-consistent model of freeze-out on an arbitrary hyper-surface. It conserves energy and momentum across the discontinuity between ideal fluid and the gas of free particles. Energy and momentum of those free particles have non-equilibrium values that could be a signal for the formation of hot and dense matter in heavy ion collisions.

Key words: freeze-out, particle spectra, conservation laws

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1. Introduction

In this paper we present our model of freeze-out for relativistic hydrodynamics. It solves the problem where and how to stop solving the hydrodynamical equations in such a way that it is consistent with the conservation of energy and momentum. When one solves the hydrodynamical equations one has to terminate the solution on the boundary with the vacuum. Also, one has to stop solving hydrodynamics at some freeze-out temperature or baryonic density. However, in most cases one continues to solve the equations for such values of these quantities, for which hydrodynamics is no longer valid (see [1, 2]). There are several papers [3, 4, 5] on a shock-like treatment of the freeze-out problem. However, in our opinion these approaches are ad hoc, since the existence of such a shock is postulated and not obtained as a result of the equations of motion.

It should be noted that this is only part of the trouble. Another one comes from the problem of calculating particle spectra on a time-like hypersurface. For a space-like hypersurface of freeze-out the correct answer for the spectra of particles is given by the formula of Cooper and Frye [6]. However, one cannot use that formula for time-like hypersurfaces, since it leads to negative numbers of particles. This is due to the fact that it was obtained only for the space-like case, where the decay of one element of gas does not affect the decay of adjacent elements.

Here we present our model which is based on the conservation laws of energy and momentum between fluid and gas of free particles. The paper is organized as follows: the second part contains a short derivation of the momentum spectra for the gas of free particles on a time-like hypersurface, which was also obtained recently in [7]; the third part contains the derivation of the conservation laws; some useful formulae are derived there; and in the final part we discuss possible effects on observables.

We also hope that the model suggested here is not only of academic interest, but will be used by other researchers to solve the equations of relativistic hydrodynamics in practical calculations.

2. Decay of the gas of free particles

In order to obtain the particle spectra for the gas, we will use the method derived by Gorenstein and Sinyukov [8]. In this paper we shall deduce the equations for the case of one-dimensional hydrodynamical motion, but the final result will not depend on this assumption, since it will be written in covariant form. Suppose there is a boundary between fluid and gas. Let us consider the decay of a small element $\Delta x$ of the gas of free particles in its rest frame. The gas is supposed to be located in the left hemisphere...
and to have the freeze-out temperature $T = T^*$. (We suppose that the derivative to the freeze-out hypersurface $v_\sigma$ in the $t - x$ plane is positive). We note that this frame is the rest frame of the gas before decay. Hereafter we shall call it the reference frame of the gas. Suppose the particles in the gas have an equilibrium distribution function $\phi\left(\frac{p_0}{T^*}\right)$.

First we consider the contribution from particles with negative momenta that leave the element $\Delta x$ (see Fig.1)

$$\frac{dN_1}{d^2p_\perp \Delta S_\perp} = \phi\left(\frac{p_0}{T^*}\right) \Delta x \, dp \, \Theta(-p) ,$$

where $p_\perp$ is the transverse momentum of the particle, and $\Delta S_\perp$ the transverse size of the element. The second contribution is given by particles with negative momenta from the element $-\frac{p}{p_0} \Delta t$

$$\frac{dN_2}{d^2p_\perp \Delta S_\perp} = -\phi\left(\frac{p_0}{T^*}\right) \frac{p}{p_0} \Delta t \, dp \, \Theta(-p) .$$

Finally, the third contribution comes from particles with positive momenta from the element $\Delta x - \frac{p}{p_0} \Delta t$. However, those particles will cross the freeze-out hypersurface only if their velocity is smaller than the derivative to the hypersurface $v_\sigma$ in the $t - x$ plane. Thus, the third term reads as follows (see Fig.2):

$$\frac{dN_3}{d^2p_\perp \Delta S_\perp} = \phi\left(\frac{p_0}{T^*}\right) \left[ \Delta x - \frac{p}{p_0} \Delta t \right] \frac{p}{p_0} \Delta t \, \Theta(p) \Theta\left(v_\sigma - \frac{p}{p_0}\right) .$$

After some simple algebra one obtains the formula for the spectrum of the gas of free particles

$$\frac{dN_{\text{tot}}}{d^2p_\perp \, dp \Delta S_\perp} = \phi\left(\frac{p_0}{T^*}\right) \left[ \Delta x - \frac{p}{p_0} \Delta t \right] \Theta\left(v_\sigma - \frac{p}{p_0}\right) .$$

As will be shown below, the modification of the spectrum due to the $\Theta$-function will lead to an energy-momentum tensor that differs from the equilibrium case.

Our last step is to write the formula for the spectrum in a fully relativistic form. For that we need to change the energy of the particles appearing in the distribution function in the reference frame of the gas to the product of the four-vectors of momentum and hydrodynamical velocity, $p_\mu u^\mu$, and change the integration over the hypersurface of freeze-out to the product of the four-vectors of momentum and normal vector to the freeze-out hypersurface, $p_\mu d\sigma^\mu$. Finally, we have
\[ p_0 \frac{dN_{tot}}{d^3p dS} = \phi \left( \frac{p_\mu u_\mu}{T^*} \right) p_\nu d\sigma^\nu \Theta \left( p_\rho d\sigma^\rho \right), \]  

(5)

where the vector \( d\sigma_\mu = (v_\sigma, -1)dt \) is the normal vector to the freeze-out hypersurface in the left hemisphere. It is, however, easy to check that the above formula is valid for the right hemisphere as well.

The meaning of this nice result is that it is the formula of Cooper and Frye \(^[6]\), but without negative particle numbers!

It is easy to see that for a space-like hypersurface, where \( v_\sigma > 1 \), the above expression gives the result obtained by Cooper and Frye \(^[6]\).

The energy-momentum tensor of free particles in the reference frame of the gas has the following form:

\[ T_2^{\mu\nu}(v_\sigma) = \int \frac{d^3p}{p_0} p_\mu p_\nu \Theta \left( v_\sigma - \frac{p}{p_0} \right) \phi \left( \frac{p_0}{T} \right). \]  

(6)

It is easy to calculate this tensor for the case of noninteracting massless particles; it has the form:

\[ T_2^{00}(v_\sigma) = \epsilon (T^*) \frac{1 + v_\sigma}{2}, \]  

(7)

\[ T_2^{01}(v_\sigma) = \epsilon (T^*) \frac{v_\sigma^2 - 1}{4}, \]  

(8)

\[ T_2^{11}(v_\sigma) = \epsilon (T^*) \frac{v_\sigma^3 + 1}{6}, \]  

(9)

where \( \epsilon \) is the usual energy density. The above result is valid for the left hemisphere. The corresponding formulae for the right hemisphere can be obtained in the same way. Now one can see that for the case \( v_\sigma = 1 \) the above expressions give the usual formula for the ideal gas. We hope that this unusual behavior of energy and momentum can be found in experimental pion spectra.

In the paper by Sinyukov \(^[3]\) another model for freeze-out was suggested. He considered the decay from a box and did not account for the additional contributions from the intrinsic volume of the gas, namely from the element \(-\frac{p}{p_0} dt\). Due to that, the energy-momentum tensor obtained in his paper is not symmetric! It means that the orbital momentum of the considered system is not conserved!

On the other hand, he considered an \textit{ad hoc} model for deflagration from very hot pionic matter into gas of free particles. We do not think that such a simple picture corresponds
to the real situation in heavy ion collisions, since the solution of the hydrodynamical equations \[1, 2\] does not exhibit shock-like transitions in the expansion of hot and dense pionic matter.

Now we would like to clarify an important question: "What is the difference between an ideal fluid and the gas of free particles at freeze-out temperature?" It seems that the main difference is that they have different values of the cross-section. Due to that, there are collisions in the fluid which lead to thermodynamical equilibrium. In contrast, there are about no collisions in the gas of free particles (and we shall neglect them completely), because the cross-section for collisions is very small. Of course, fluid and gas have somewhat different values of temperature, but due to the fact that the mean free-path strongly depends on the temperature, this difference should be small. It was found [10] that for pions the mean free-path depends on the fifth power of the inverse temperature: \( \lambda \approx \frac{\text{const}}{T^5} \).

Thus, the difference between the temperatures of the fluid and the gas should be small, but due to the strong dependence of the collision cross-section on the temperature, their mean free-paths should be very different.

Now we would like to formulate a more physical concept for freeze-out, based on the conservation laws on the discontinuity between ideal fluid and the gas of free particles.

3. Conservation laws on the surface between fluid and gas

We would like to consider a simple model of freeze-out assuming that the system consists of the fluid and the gas of free particles. We do not take into account the transition region between them. Then, the total energy-momentum tensor of the system is as follows

\[
T^{\mu\nu} = T_1^{\mu\nu} \Theta(T - T_1^*) + T_2^{\mu\nu} \Theta(T_* - T),
\]

(10)

where index 1 corresponds to the fluid, and index 2 to the gas of free particles, \( T \) is the temperature, \( T_1^* \) and \( T_2^* \) are the freeze-out temperatures for fluid and gas, respectively. We assume that those temperatures satisfy \( T_1^* \geq T_2^* \).

The equations of motion have the form:

\[
\partial_\mu T^{\mu\nu} = 0.
\]

(11)

Taking derivatives and using the equations for the evolution of fluid and gas,

\[
\partial_\mu T_a^{\mu\nu} = 0, \quad a = 1, 2,
\]

(12)

one finds that the terms with the delta-functions must vanish:
\[ T^{\mu\nu}_1 \partial_\mu T^*_1 = T^{\mu\nu}_2 \partial_\mu T^*_2 . \] (13)

Thus, we have obtained two equations (for the 1+1-dimensional case, but the case of 3+1 dimensions gives the same result). However, we have to add to this equation the condition that the derivatives of the temperatures of fluid and gas are equal, since there is only one hypersurface of freeze-out:

\[ \partial_\mu T^*_1 = \partial_\mu T^*_2 . \] (14)

Using this condition and dividing the first equation above by the second one and canceling derivatives of \( T^* \), we get a new equation:

\[ \left( T^{11}_1 - T^{11}_2 \right) \left( T^{00}_1 - T^{00}_2 \right) = \left( T^{10}_1 - T^{10}_2 \right) \left( T^{01}_1 - T^{01}_2 \right) , \] (15)

or

\[ \text{Det} \left( T^{\mu\nu}_1 - T^{\mu\nu}_2 \right) = 0 . \] (16)

However, there is one problem. It seems that we know the energy-momentum tensor of the gas only in its reference frame, but the energy-momentum tensor of the fluid is known in the laboratory system. On the other hand, we do not know the velocity of the gas of free particles in the laboratory system. Fortunately, this problem can be easily solved.

Let us suppose that the fluid with temperature \( T = T^*_1 \) has a velocity \( v_1 \) in the laboratory system, and the gas has a velocity \( v_2 \) in the same system. Then, the velocity of the fluid in the reference frame of the gas is

\[ v_{\text{rel}} = \frac{v_1 - v_2}{1 - v_1 v_2} . \] (17)

Now the energy-momentum tensors of the fluid \( T^{\mu\nu}_1(v_{\text{rel}}, T^*_1) \) and the gas \( T^{\mu\nu}_2(v_\sigma, T^*_2) \) are known in the reference frame of the gas. And we have one equation that was obtained before:

\[ \text{Det} \left( T^{\mu\nu}_1(v_{\text{rel}}, T^*_1) - T^{\mu\nu}_2(v_\sigma, T^*_2) \right) = 0 . \] (18)

Fortunately, we have one more equation. Due to the definition of the freeze-out hypersurface:

\[ T(x,t) = T^*_2 \] (19)
and the fact that the velocity \( v_\sigma \) is the derivative of the implicit function \( T(x, t) = T_2^* \) one gets the following equation

\[
v_\sigma = \left( \frac{dx}{dt} \right)_{T = T_2^*} = -\frac{\partial_0 T_2^*}{\partial_1 T_2^*} = \frac{T_1^{10}(v_{\text{rel}}, T_1^*) - T_2^{10}(v_\sigma, T_2^*)}{T_1^{00}(v_{\text{rel}}, T_1^*) - T_2^{00}(v_\sigma, T_2^*)}.
\]  

(20)

Another way to obtain the above result is as follows. One has to remember that the derivatives of the freeze-out temperature \( T_2^* \) are the components of the normal line to the hypersurface \( T(x, t) = T_2^* \). Then, energy-momentum conservation reads

\[
T_1^{11}(v_{\text{rel}}, T_1^*) - T_1^{10}(v_{\text{rel}}, T_1^*) v_\sigma = T_2^{11}(v_\sigma, T_2^*) - T_2^{10}(v_\sigma, T_2^*) v_\sigma, \tag{21}
\]

\[
T_1^{01}(v_{\text{rel}}, T_1^*) - T_1^{00}(v_{\text{rel}}, T_1^*) v_\sigma = T_2^{01}(v_\sigma, T_2^*) - T_2^{00}(v_\sigma, T_2^*) v_\sigma. \tag{22}
\]

Thus, we have two equations for three unknowns \( T_1^*, v_{\text{rel}} \) and \( v_\sigma \), since we suppose that the freeze-out temperature for the gas of free particles is known.

From the above equations one finds the freeze-out hypersurface in the reference frame of the gas. However, this hypersurface is still unknown in the laboratory frame, because the above equations do not fix the velocity of the gas in the laboratory frame. In order to find the freeze-out hypersurface in the laboratory frame, one has to solve the hydrodynamical equations. Thus, the problem is solved.

We have to add the following. The equations

\[
T_1^{\mu \nu} \partial_\mu T_2^* = T_2^{\mu \nu} \partial_\mu T_2^*
\]  

(23)

are the conservation laws of energy and momentum on the discontinuity between the fluid and the gas of free particles, since the derivatives of the freeze-out temperature \( T_1^* \) give the normal vector to the hypersurface. However, one cannot obtain them from the solution of the equations of hydrodynamics for the fluid. Suppose one knows the freeze-out point \( T(x, t) = T_1^* \) at a fixed time. In order to find this point at the next moment of time, one has to solve the hydrodynamical equations for unphysical temperatures, namely, \( T < T_1^* \! \). Thus, from the above equations one has to find the velocities \( v_\sigma \) and \( v_{\text{rel}} \) in the reference frame of the gas. Since the velocity of the fluid in the laboratory system \( v_1 \) is known from the solution of the hydrodynamical equations, one makes a Lorenz transformation and finds the velocity of the gas in the laboratory system:

\[
v_2 = \frac{v_1 - v_{\text{rel}}}{1 - v_1 v_{\text{rel}}}. \tag{24}
\]
Using this velocity one can find the derivative of the freeze-out hypersurface in the laboratory frame by a Lorenz transformation.

In the case of baryon-rich matter, the freeze-out condition is as follows

\[ n(x, t) = n^*, \quad (25) \]

and the thermodynamical functions depend on temperature and chemical potential. In this case we have one more hydrodynamical equation which is the conservation law of baryonic charge, and one more equation for the baryonic densities between fluid and the gas of free particles.

4. Results and discussion

We have developed a freeze-out model for relativistic hydrodynamics. We have obtained the conservation laws on the discontinuity between ideal fluid and the gas of free particles. We have derived an expression for the energy-momentum tensor of the gas of free particles and its momentum distribution function. It is important to emphasize that this tensor differs from the equilibrium one. Thus, one can hope to find this unusual behavior in experiments, and this can give detailed information about the freeze-out process in heavy ion collisions.

One possible effect is related to measuring the ratio of the total (longitudinal) momentum density in the backward hemisphere to the zero component of the baryonic charge flux. In the reference frame of the gas, that component reads (for the left hemisphere)

\[ N^0 = n(T, \mu) \frac{1 + v_\sigma}{2}. \quad (26) \]

If it is possible to neglect the contribution of the space-like part of the freeze-out hypersurface, then that ratio will depend on the thermodynamical quantities and an average value of \( v_\sigma \) on the time-like part of the freeze-out hypersurface. Thus, measuring this ratio for the gas of free particles one can find the average value of \( v_\sigma \). We hope that it is possible to do so for asymmetric heavy ion collisions. For example, the S + Au reaction that was studied in [11] indicates that the space-like part of the freeze-out hypersurface is small and that the velocities on it are approximately constant. Therefore, finding the suggested ratio can give important information about the freeze-out process of hot and dense hadronic matter.
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Appendix A

In this Appendix we would like to discuss the question: How to solve the hydrodynamical equations together with the above equations on the freeze-out hypersurface. In fact we shall only show that the suggested system of equations for the shock is consistent with the hydrodynamical equations, so that they can be solved simultaneously.

A. Suppose our system consists of pions only. In this case thermodynamical quantities depend only on the temperature. Suppose we have two hydrodynamical equations for the fluid that can be written in the form

\[ F_a(\epsilon_{1i}, p_{1i}, v_{1i}, x_i) = 0, \quad a = 1, 2, \]

with unknown values of energy density \( \epsilon_{1i} \), pressure \( p_{1i} \), velocity \( v_{1i} \), and at the coordinate point \( x_i \). Now let us consider the evolution of the discontinuity between fluid and the gas of free particles. First, let us count the unknowns. They are \( T_1, v_{\text{ref}}^e, v_1, v_2 \) (instead of \( v_{\text{rel}} \)) and \( x_i^{\text{shock}} \). The last one we can change to the velocity of the shock in the laboratory system: \( v_{\sigma}^{\text{Lab}} \). Thus, we have 5 unknowns. On the other hand we have precisely 5 equations. They are: two hydrodynamical equations for the fluid, two conservation laws on the discontinuity between fluid and gas and the relation between the velocities in different frames:

\[ v_{\sigma}^{\text{Lab}} = \frac{v_{\sigma}^{\text{ref}} + v_2}{1 + v_{\sigma}^{\text{ref}} v_2}. \]

Thus, solving this system of equations one finds the trajectory of the freeze-out hypersurface, temperature of the fluid and velocities of fluid and gas.

B. Now let us consider the case of baryonic matter. Then one has more unknowns and more equations. Let us count them.

The unknown quantities are: three thermodynamical functions of the fluid - \( \epsilon_1, p_1, n_1 \), two thermodynamical functions for the gas - \( T_2, \mu_2 \), the derivative to the freeze-out hypersurface in the reference frame of the gas - \( v_{\sigma}^{\text{ref}} \), the velocities of the fluid and gas in the laboratory system \( v_1, v_2 \), and the derivative to the freeze-out hypersurface in the
laboratory system $v_{\sigma}^{Lab}$. Thus, one has 9 unknowns. At the same time there are also 9 equations: 3 hydrodynamical equations for the fluid (two for energy and momentum, one for the baryonic charge), 3 equations on the discontinuity between fluid and gas (two for energy and momentum, one for the baryonic charge), the equation of state for the fluid that relates its thermodynamical functions $\epsilon_1, p_1, n_1$, the relation between the velocity of the shock in the reference frame of the gas and in the laboratory frame, and the equation of freeze-out for the gas of free particles $n_2(x, t) = n^*$. Now one has to solve the above system of equations and find all unknowns.
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Fig.1. Decay of the element $\Delta x = AD$.

Particles with negative momenta are leaving elements $AD$ and $DE = -\frac{p}{p_0} \Delta t$.

The passage of particles is indicated by lines with arrows.
Fig. 2. Decay of the element  $\Delta x = AD$.  

Particles with positive momenta are leaving elements $AF = \Delta x - p/p_0 \Delta t$.  

The passage of particles is indicated by lines with arrows.