FOAM DIAGRAM SUMMATION
AT FINITE TEMPERATURE

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Abstract

We show that large-$N \phi^4$ theory is not trivial if one accepts the presence of a tachyon with a truly huge mass, and that it allows exact calculation. We use it to illustrate how to calculate the exact resummed pressure at finite temperature and verify that it is infrared and ultraviolet finite even in the zero-mass case. In 3 dimensions a residual effect of the resummed infrared divergences is that at low temperature or strong coupling the leading term in the interaction pressure becomes independent of the coupling and is $4/5$ of the free-field pressure. In 4 dimensions the pressure is well defined provided that the temperature is below the tachyon mass. We examine how rapidly this expansion converges and use our analysis to suggest how one might reorganise perturbation theory to improve the calculation of the pressure for the QCD plasma.

1 Introduction

In traditional calculations of the pressure in massless quantum field theories at finite temperature, there is a breakdown of perturbation theory because of infrared problems [1] [2]. This is true, in particular, of gauge theories at finite temperature. Formally the same types of problem crop up also in scalar $\lambda\phi^4$ theory. The traditional procedure [1] for summing the ring diagrams (figure 1a) produces a series expansion for the pressure $P$ in powers (and logarithms) of $\lambda^{1/2}$. This takes care of infinitely many otherwise problematic higher-loop diagrams, but it does not cover diagrams that include a single one-particle-irreducible self-energy insertion and so leaves arbitrarily high-loop orders to be considered. In nonabelian gauge theories, where some propagators remain massless perturbatively, the two-particle-irreducible higher-loop diagrams are still potentially infrared divergent. In an earlier paper [3] we have proposed a different and less direct method which absorbs all of the diagrams into a single resummed one-loop quantity which enjoys manifest infrared regularity in four dimensions. This introduces a variable mass $m$ for the field as a parameter, and gives the pressure in terms of an integral over the thermal propagator corresponding to mass $m$.

In this paper, we apply our summation method to sum all the extended ring diagrams of figure 1b, which we call foam diagrams. Other authors [4] [5] have called these super-daisy or Hartree-Fock diagrams. We began this analysis as a warm-up exercise for the gauge-theory problem, but have found

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Moreover, it allows us to investigate the convergence properties of thermal perturbation theory in a solvable model, from which we derive suggestions on how to optimize the perturbative treatment of the pressure also in more complicated theories such as QCD.

If we consider an $O(N)$ scalar field theory with unrenormalised Lagrangian density

$$
L(x) = \frac{1}{2} \left( (\partial \phi(x))^2 - m_0^2 (\phi(x))^2 \right) - \frac{\lambda_0}{4!} \frac{3}{N+2} \left( (\phi(x))^2 \right)^2
$$

then in the limit $N \to \infty$ the pressure per scalar particle coincides with the one obtained from the infinite sum of foam diagrams in the $N = 1$ theory. Hence our calculation in this paper of the foam diagrams may be regarded either as the leading-$N$ term in the pressure in the large-$N$ theory, or as an approximation (we do not know how good) to the pressure in the $N = 1$ theory. We obtain an expression for this infinite sum that is derived from a path integral and so has validity beyond perturbation theory. The renormalisation of the mass, and in 4 dimensions also of the coupling, is an essential part of the analysis; this is achieved through Dyson equations which can be solved exactly, and so again goes beyond perturbation theory.

We describe the renormalisation in section 2. As is well known [6], and as we find, in 4 dimensions there are problems with $\phi^4$ theory. For reasonably small coupling, these problems turn out to arise only at huge mass scales and so are not really important for physics, but in order to handle them we write our equations for $n$ dimensions. In section 3 we review our formula [3] for the pressure; as before, we choose to use the real-time formalism of thermal field theory. The formula involves the thermal addition $\delta m^2$ to the renormalised squared mass $m^2$. A key simplification is that, for the foam diagrams, $\delta m^2$ is both independent of momentum and real, and we show how this leads to rather...
simpler versions of our general formula for the pressure. We need to solve an integral equation for \( \delta m^2 \), which we do in section 4 for the case of 3 dimensions. We show that the infrared divergences that occur in the zero-mass case are rendered harmless by our resummation and that in the strong-coupling or low-temperature limit the leading term in the interaction pressure becomes independent of the coupling and is 4/5 of the free-field pressure.

In section 5 we investigate what happens in 4 dimensions if we ignore the triviality difficulties of the theory. We find that the integral equation for \( \delta m^2 \) has no solution above some critical temperature \( T_{\text{max}} \), so that it seems that then the pressure does not exist. However, for sufficiently weak coupling, \( T_{\text{max}} \) is exponentially huge, so that the 4-dimensional theory may be accepted as a highly accurate effective theory. In the particularly interesting massless case we make a detailed comparison of our nonperturbative results with those of resummed perturbation theory. We find that the latter converges quickly only for rather small coupling and that the rate of convergence depends critically on the renormalization scale. We end with some speculation on the case of QCD, where explicit calculation of the pressure up to and including order \( g^5 \) has revealed rather bad convergence properties of resummed perturbation theory there. We suggest that it may well be fruitful to reorganise the expansion so that \( \delta m/T \), rather than the coupling \( g \), is the expansion parameter.

2 Renormalisation

Zero temperature

We choose a zero-temperature renormalisation scheme that makes the formula for the pressure \( P(T) \) as simple as possible. In lowest-order perturbation theory the renormalised mass is defined to be

\[
m^2 = m_0^2 + \lambda_0 M(m_0^2)
\]  

(2.1a)

where \( M \) corresponds to the single-loop Feynman graph:

\[
M(m^2) = \frac{1}{2} \int \frac{dnq}{(2\pi)^d} \frac{i}{q^2 - m^2 + i\varepsilon}
\]  

(2.2a)

We extend this to all orders of perturbation theory by replacing (2.1a) with the Dyson equation

\[
m^2 = m_0^2 + \lambda_0 M(m^2)
\]  

(2.1b)

The integration over \( q^0 \) in the integral in (2.2a) may be done by closing the contour in one or other half plane and taking the residue at the pole. The result is that instead we may write

\[
M(m^2) = \frac{1}{2} \int \frac{dnq}{(2\pi)^d} 2\pi\delta^+(q^2 - m^2)
\]  

(2.2b)

The integration is simple:

\[
M(m^2) = \frac{\Gamma(1 - \frac{d}{2}n)}{2(4\pi)^{n/2}m^{n-2}}
\]  

(2.2c)

When we go to 4 dimensions we need also to renormalise the coupling. Initially we keep \( n \neq 4 \) as a regulator. We define the renormalised coupling \( \lambda \) to be the value of the \( ii \rightarrow jj \) scattering amplitude at \( s = 0 \), where \( i \) and \( j \) denote “colour” labels. The equation for \( \lambda \) is shown diagrammatically in figure 2. If we work to leading order in \( N \) in the large-\( N \) theory, or choose to include only foam diagrams in the \( N = 1 \) case, we must omit the last two terms in figure 2, so that we have

\[
\lambda = \lambda_0 + \lambda_0 \lambda L(m^2)
\]
Figure 2: coupling renormalisation

\[ L(m^2) = \frac{1}{\pi} \int \frac{d^n q}{(2\pi)^n} \frac{i}{(q^2 - m^2 + i\epsilon)^2} = M'(m^2) \]  

(2.3a)

where the prime denotes differentiation with respect to \( m^2 \). Both \( M \) and \( M' \) are ultraviolet divergent when \( n \to 4 \). The coupling has dimension \( m^{4-n} \). From (2.3a) and (2.2c),

\[ \lambda = \frac{\lambda_0}{1 + C_n \lambda_0 m^{n-4}} \]  

(2.4a)

so that if we want both the bare and the renormalised coupling to be non-negative

\[ 0 \leq \lambda \leq \frac{m^{1-n}}{C_n} \]  

(2.5)

Hence, when \( n \to 4 \), \( \lambda \) vanishes for all values of \( m \), because \( C_n \) diverges. This is the well-known triviality of \( \phi^4 \) theory in 4 dimensions [6].

One might perhaps believe that it does not matter whether the bare coupling is positive, because only the renormalised theory is relevant. However, if for \( n = 4 \) we choose \( \lambda \) to be greater than 0, the renormalised theory has a tachyon. To see this, we write the equation corresponding to figure 2 (with the last two terms omitted because they are negligible in the large-\( N \) limit) for the \( ii \to jj \) scattering amplitude for a general value of \( s \):

\[ T(s) = \lambda_0 + \lambda_0 M'(m^2, s) T(s) \]  

\[ M'(m^2, s) = -\frac{1}{2} \int \frac{d^n q}{(2\pi)^n} \frac{i}{((q + \frac{1}{2}p)^2 - m^2 + i\epsilon)((q - \frac{1}{2}p)^2 - m^2 + i\epsilon)} \]

\[ = -C_n \int_0^1 dx (m^2 - sx(1-x))^{n/2-2} \]  

(2.6)

From (2.4a) and (2.6),

\[ T(s) = \frac{\lambda}{1 + C_n \lambda \int_0^1 dx \{ (m^2 - sx(1-x))^{n/2-2} - m^{n-4} \}} \]  

(2.7a)

which becomes in 4 dimensions when \( |s| \) is large

\[ T(s) \approx \frac{\lambda}{1 - \frac{\lambda}{32 \pi^2} \log(-s/m^2)} \]  

(2.7b)
It is evident that, in 4 dimensions, \( T(s) \) has a pole at

\[ s = s_{\text{tachyon}} \approx -m^2 \exp \left( \frac{32\pi^2}{\lambda} \right) \]  

(2.8a)

Our definition (2.3a) of the renormalised coupling \( \lambda \) in terms of the value of the scattering amplitude \( T(s) \) at \( s = 0 \) makes \( \lambda \) vary with the renormalised mass \( m \), and the appearance of a tachyon pole in \( T(s) \) is related to the occurrence of a Landau pole in \( \lambda(m^2) \). Although this tachyon pole is in principle unacceptable, in practice it occurs at a very large negative value of \( s \). For example, if \( \lambda = 1 \), it is at \(-10^{137}m^2\). One could avoid the presence of this tachyon by introducing an UV cut-off \( \Lambda \), since then

\[ \lambda = \lambda_0 \left( 1 + \frac{\lambda_0}{32\pi^2} \log(\Lambda^2/m^2) \right) \]  

(2.4b)

which makes it possible to have both \( \lambda \) and \( \lambda_0 \) positive, with \( \lambda < \frac{32\pi^2}{\log(\Lambda^2/m^2)} \) and therefore \( |s_{\text{tachyon}}| > \Lambda^2 \).

Or alternatively we simply say that, for reasonably small coupling, the tachyon is so far away that it can be ignored. Either way, the \( n = 4 \) theory seems to be perfectly acceptable as an effective theory, for energies much below either the tachyon mass or the cut-off.

In what follows we shall be particularly interested in the case of massless theories, because there one encounters the infrared problems of thermal perturbation theory. Because of the tachyon (2.8a), this case may seem to be excluded, for then the tachyon cannot be kept far away, unless we send \( \lambda \to 0 \) as \( m \to 0 \). Contrary to appearances, this does not signal a trivial theory, however. In the massless case, there are infrared divergences in the defining equation for the renormalised coupling (2.3a). Switching to an alternative scheme defined by

\[ \bar{\lambda} = \lambda_0 + \lambda_0 \bar{\lambda} L(\bar{\mu}^2) \]  

(2.3b)

with some arbitrary scale \( \bar{\mu} \), shows that then, when \( \bar{\lambda} \neq 0 \),

\[ T(s) = \frac{\bar{\lambda}}{1 - \frac{\lambda}{32\pi^2} [\log(-s/\bar{\mu}^2) - 2]} \]  

(2.7c)

vanishes at \( s = 0 \), but not otherwise. So whenever we are going to inspect the massless limit, we shall switch over to using \( \bar{\lambda} \) rather than \( \lambda \). In terms of \( \bar{\lambda} \) and \( \bar{\mu} \), the tachyon is at

\[ s_{\text{tachyon}} = -\bar{\mu}^2 \exp \left( \frac{32\pi^2}{\bar{\lambda}} + 2 \right) \]  

(2.8b)

so for given choice of \( \bar{\mu} \) we can always make it as far away as we wish by taking \( \bar{\lambda} \) small enough.

In minimally subtracted massless theories, a renormalization scale is usually introduced through \( \lambda \to \mu^{4-n}\lambda \). Because

\[ \mu^{4-n}L(\bar{\mu}^2) \sim -\frac{1}{32\pi^2} \left( \frac{2}{4-n} - \gamma - \log \frac{\bar{\mu}^2}{4\pi\bar{\mu}^2} \right) \]

the scale \( \bar{\mu}^2 \) introduced as in (2.3b) is seen to coincide with the one of the modified minimal subtraction scheme \( (\overline{\text{MS}}) \), \( \bar{\mu}^2 = 4\pi e^{-\gamma} \bar{\mu}^2 \). Hence our notation.

A more physical renormalization scheme would be to define the renormalized coupling through the scattering amplitude at some scale \( \bar{\mu}^2 \) so that \( \bar{\lambda} = T(-\bar{\mu}^2) \). (2.7c) shows that \( \bar{\mu}^2 \equiv e^2 \bar{\mu}^2 \). However, in conformity with most of the literature on thermal field theory, we shall continue to use the \( \overline{\text{MS}} \) scheme.
**Nonzero temperature**

At nonzero temperature, in addition to the renormalised zero-temperature mass there is a thermal contribution, because (2.1) becomes replaced with

\[
m^2 + \delta m^2 = m_0^2 + \lambda_0 M_T(m^2 + \delta m^2)
\]

\[
M_T(m^2) = M(m^2) + N_T(m^2)
\]

\[
N_T(m^2) = \int \frac{d^nq}{(2\pi)^n} 2\pi \delta^+(q^2 - m^2) \frac{1}{e^{q^0/T} - 1}
\]

(2.9)

Eliminating the unrenormalised mass \(m_0^2\) with (2.1b) gives

\[
\delta m^2(m^2, T) = \lambda_0 [M_T(m^2 + \delta m^2) - M(m^2)]
\]

(2.10)

For less than 4 dimensions both \(\lambda_0\) and the expression in square brackets are finite, but when \(n \to 4\) this is no longer true and we must introduce also the renormalised coupling. If we define the function

\[
\hat{M}(m^2, \delta m^2) = M(m^2 + \delta m^2) - M(m^2)
\]

\[
= C_n \left\{ \left( (m^2 + \delta m^2)^{n/2-1} - (m^2)^{n/2-2}(m^2 + (n/2 - 1)\delta m^2) \right) \right\}
\]

(2.11a)

we find that

\[
\delta m^2(m^2, T) = \lambda \hat{M}(m^2, \delta m^2) + N_T(m^2 + \delta m^2)]
\]

(2.12)

This is an integral equation for \(\delta m^2(m^2, T)\) in which everything is finite when \(n \to 4\), with

\[
\hat{M}(m^2, \delta m^2) \to \frac{1}{32\pi^2} \left\{ (m^2 + \delta m^2) \log \left( 1 + \frac{\delta m^2}{m^2} \right) - \delta m^2 \right\}
\]

(2.11b)

The expansion of this in powers of \(\delta m^2/m^2\) begins with a term proportional to \((\delta m^2)^2\). In perturbation theory \(\delta m^2 = O(\lambda)\), so that \(\lambda \hat{M} = O(\lambda^3)\) and the first term on the right-hand side of (2.12) represents a contribution that first appears at three loop order. Since frequently calculations up to two loops have been taken as a clue to exact results, this particular contribution has been repeatedly missed in the literature [4] [7]. It owes its existence to the interplay between zero-temperature contributions plus their renormalisation with the thermal effects, and it will turn out to be of great importance to the existence and behaviour of the solutions of the equation for \(\delta m^2\).

In the massless limit \(m \to 0\), we are again confronted with infrared divergences, which can be avoided by switching to the alternative renormalisation scheme (2.3b) This amounts to substituting

\[
\lambda^{-1} = \bar{\lambda}^{-1} + \frac{1}{32\pi^2} \log \frac{\bar{\mu}^2}{m^2}
\]

(2.13)

which, in the limit \(m \to 0\), leads to

\[
\delta m^2(0, T) = \bar{\lambda} [M(\bar{\mu}^2, \delta m^2) + N_T(\delta m^2)]
\]

(2.14)

with

\[
M(\bar{\mu}^2, \delta m^2) = \frac{1}{32\pi^2} \delta m^2 \left( \log \frac{\delta m^2}{\bar{\mu}^2} - 1 \right)
\]

(2.15)

It is easy to see that \(\delta m^2(0, T)\) is, as it should be, independent of \(\bar{\mu}\) once the \(\bar{\mu}\)-dependence of \(\bar{\lambda}\) is taken into account.
3 The pressure

Our formula [3] for the pressure is derived from the grand partition function

$$Z(T) = \int d\vec{\phi} \exp(iS[\vec{\phi}, T])$$

$$S[\vec{\phi}, T]) = \int_C d^n x \mathcal{L}(x)$$ (3.1)

In the variant of the real-time thermal field theory we use*, the integration is over all $x$ and over the time contour $C$ which, in the complex $t$-plane, runs along the real axis from $-\infty$ to $+\infty$, back to $-\infty$, and then down to $-\infty - i/T$. (This is known as the Keldysh contour [9].) Differentiate with respect to $m_0^2$ keeping $\lambda_0$ fixed:

$$\frac{\partial}{\partial m_0^2} \log Z = -\frac{1}{2} i Z^{-1} \int d\vec{\phi} \left( \int_C d^n x \vec{\phi}^2(x) \right) \exp(iS[\vec{\phi}, T])$$

$$= -\frac{1}{2} i \left( \int_C d^n x \vec{\phi}^2(x) \right)$$ (3.2)

Here, $\langle \ldots \rangle$ denotes a thermal average. Space-time translation invariance tells us that the thermal average of $\vec{\phi}^2(x)$ is independent of $x$, and so the $x$ integration is trivial:

$$\frac{\partial}{\partial m_0^2} \log Z = -\frac{V}{2T} \langle \vec{\phi}^2(0) \rangle$$

$$= -\frac{NV}{2T} \int \frac{d^n q}{(2\pi)^n} D^{12}(q, T)$$ (3.3)

Here,

$$D^{12}(q, T) = \left\langle \int d^n x e^{i[q,x] \phi_i(0) \phi_i(x)} \right\rangle$$ (3.4)

where $\phi_i$ denotes any component of the $O(N)$-symmetric field $\vec{\phi}$. $D^{12}(q, T)$ is an element of the familiar $2 \times 2$ matrix propagator $\mathbf{D}(q, T)$ of Keldysh-contour real-time thermal field theory [2] [8]. Inserting its known form [10] [11] into (3.3) and using the relation $P(T) = (T/V) \log Z$, we obtain

$$\frac{\partial P(T)}{\partial m_0^2} = \int \frac{d^n q}{(2\pi)^n} \frac{1}{e^{q^0/T} - 1} \frac{1}{q^2 - m_0^2 - \Pi(q, T, T)}$$ (3.5a)

When, as in the application to the foam diagrams, the self energy $\Pi$ is real, we must apply the usual $i\epsilon$ prescription to $m_0^2$ and so

$$-\frac{\partial P(T)}{\partial m_0^2} = M_T(m_0^2 + \Pi(q, T, T))$$

$$= M_T(m^2 + \delta m^2)$$ (3.5b)

where $M_T$ is defined in (2.9). Subtracting off the zero-temperature pressure and using (2.10), we have

$$\frac{\partial}{\partial m_0^2} (P(T) - P(0)) = -\frac{\delta m^2(m^2, T)}{\lambda_0}$$ (3.6)

* For a review of the essentials of real-time thermal field theory, see reference [8]
Using (2.1a) and (2.3a) to express everything in terms of renormalised quantities and imposing the obvious boundary condition that the pressure vanishes for infinite mass, we find the remarkable formula

\[ P(T) - P(0) = \int_{m^2}^{\infty} dm'^2 \frac{\delta m^2(m'^2, T)}{\lambda(m'^2)} \]  \hspace{1cm} (3.7a)

Here,

\[ \lambda(m'^2) = \frac{\lambda}{1 + \lambda[L(m'^2) - L(m^2)]} \]  \hspace{1cm} (3.7b)

so that \( \lambda(m'^2) \) is a running coupling that is equal to \( \lambda \) when \( m'^2 = m^2 \); it is calculated from (2.3a) by varying the mass and keeping fixed \( \lambda_0 \).

In fact, the mass integration in (3.7a) can be carried out. Rewriting the right-hand side of (3.5b) as

\[ M_T(m^2 + \Pi(q, T, m)) = M_T(m^2 + \Pi(1 + \frac{\partial \Pi}{\partial m^2}) - \frac{1}{\lambda_0} \frac{\partial \Pi}{\partial m^2} \frac{\partial}{\partial m_0} \right) \]

\[ = M_T(m^2 + \delta m^2) \frac{\partial}{\partial m_0} (m^2 + \delta m^2) - \frac{1}{2\lambda_0} \frac{\partial}{\partial m_0} \Pi^2 \]  \hspace{1cm} (3.8a)

and introducing

\[ M(m^2) = \int_{m^2}^{\infty} dm'^2 M(m'^2) = \frac{1}{2} \int \frac{d^ndq}{(2\pi)^{n-1}} \theta(q^0) \theta(q^2 - m^2) \]

\[ N_T(m^2) = \int_{m^2}^{\infty} dm'^2 N_T(m'^2) = \int \frac{d^ndq}{(2\pi)^{n-1}} \theta(q^0) \theta(q^2 - m^2) \frac{1}{e^{q/T} - 1} \]  \hspace{1cm} (3.8b)

we have

\[ M_T(m^2 + \Pi(q, T, m)) = -\frac{\partial}{\partial m_0} \left[ M(m^2 + \pi) + \frac{1}{2\lambda_0} \Pi^2(T, m) \right] \]

so that with the boundary condition that the pressure vanishes for infinite mass

\[ P(T) - P(0) = M_T(m^2 + \pi) - M(m^2) + \frac{1}{2\lambda_0} (\Pi^2(T, m) - \Pi^2(0, m)) \]

\[ = N_T(m^2 + \pi) + M(m^2 + \delta m^2) - 1/2 \delta m^2 (M_T(m^2 + \delta m^2) + M(m^2)) \]  \hspace{1cm} (3.9)

A formula similar to (3.9) has been derived previously by Amelino-Camelia and Pi [5] using the CJT formalism [12], though their formula does not satisfy the physically-important constraint that the pressure vanishes when the mass is infinite.*

In order to highlight the interplay between thermal and quantum contributions, let us also give the following alternative version of the result (3.9)

\[ P(T) - P(0) = N_T(m^2 + \delta m^2) + \frac{1}{2} \left( \frac{\delta m^2}{\lambda} \right)^2 - \frac{1}{\lambda^2} \sum_{n=3}^{\infty} \frac{1}{m^2} M^{(n-1)}(m^2)(\delta m^2)^n \]  \hspace{1cm} (3.10)

This expression makes manifest the UV finiteness** of our result, and it exhibits three different kinds of contribution: \( N_T \), the classical expression for the pressure of a bosonic gas of particles with mass squared \( m^2 + \delta m^2 \); \( \frac{1}{2} \left( \frac{\delta m^2}{\lambda} \right)^2 \), which is \( O(\lambda) \) in perturbation theory, essentially a thermal interaction contribution; and the rest, which starts at three loop order, coming from the thermal mass shift in zero-temperature integrals.

* In reference [12], the effective potential is calculated, from which the pressure of our model follows by restricting to vanishing field expectation value and positive bare mass squared. We intend considering the symmetry-breaking sector of the theory in a future paper.

** Its IR finiteness is however better seen from the original version (3.9).
4 Three dimensions

When \( n = 3 \), (2.2c) and (2.9) become

\[
M(m^2) = -\frac{m}{8\pi} \\
N_T(m^2) = -\frac{T}{4\pi} \log(1 - e^{-m/T})
\]

(4.1)

Hence the integral equation (2.10) for \( \delta m^2 \) reduces to

\[
\frac{8\pi}{\lambda_0} \delta m^2 = m - \sqrt{m^2 + \delta m^2} - 2T \log \left(1 - \exp\left(-\sqrt{m^2 + \delta m^2}/T\right)\right)
\]

(4.2)

This must be solved numerically and the result plugged into (3.9). (We recall that, in 3 dimensions, \( \lambda_0 \) has the dimension of mass.)

When \( m = 0 \),

\[
\frac{8\pi T}{\lambda_0} \xi^2 + \xi = -2 \log(1 - \exp(-\xi))
\]

(4.3)

where \( \delta m^2 = \xi^2 T^2 \). The solution for \( \xi \) becomes small at high temperature, when \( 8\pi T \gg \lambda_0 \), but it goes to 0 quite slowly: it is about 0.4 for \( 8\pi T/\lambda_0 = 10 \) and 0.07 for \( 8\pi T/\lambda_0 = 1000 \).

In terms of \( \xi \), the formula for the pressure can be written as

\[
P(T) - P(0) = \frac{T^3}{2\pi} \left\{ \text{Li}_3(e^{-\xi}) + \xi \text{Li}_2(e^{-\xi}) - \frac{\xi^2}{4} \log(1 - e^{-\xi}) + \frac{\xi^3}{24} \right\}
\]

(4.4)

where \( \text{Li}_3 \) and \( \text{Li}_2 \) are the tri- and di-logarithmic functions, respectively [13]. In the high-temperature limit, where \( \xi \to 0 \), this approaches the free-field value \( \zeta(3)T^3/(2\pi) \).

Notice that a perturbation expansion of the pressure in the \( m = 0 \) case would be fraught with infrared problems. These would be encountered already at order \( \lambda_0^2 \), in the two-loop graph. The formula (3.9) has eliminated them through resummation. The right-hand side of (3.9) is finite when \( m \to 0 \), yielding (4.4), but it does not have a power-series expansion in powers of \( \delta m^2 \), because the derivative of \( M_T \) diverges at the origin. This infrared sensitivity leads to the interesting result that at low temperature or strong coupling the interaction pressure becomes a constant multiple of the free-field pressure, independent of the coupling. For \( 8\pi T \ll \lambda_0 \) the first term in (4.3) becomes negligible compared with the other two, and the solution for \( \xi \) approaches

\[
\xi \equiv \delta m/T \to \log \frac{3 + \sqrt{5}}{2} \quad \text{for } T/\lambda_0 \to 0
\]

(4.5)

that is \( \delta m \approx 0.96 T \).

Remarkably, for this value of \( \xi \) the polylogarithms in (4.4) can be evaluated (see equations (1.20) and (6.13) of reference [13]), yielding

\[
P(T) - P(0) \to \frac{2\zeta(3)}{5\pi} T^3 = \frac{4}{5}[P(T) - P(0)]_{\text{free}} \quad \text{for } T/\lambda_0 \to 0
\]

(4.6)

We have not found any physical explanation of this surprisingly simple result nor have we been able to derive it without recourse to the peculiar properties of polylogarithms.
5 Four dimensions

The massive case

When \( n \to 4 \) we must use the integral equation (2.12) for \( \delta m^2 \), which is written in terms of the renormalised coupling \( \lambda \). With \( x = \delta m^2/m^2 \) and \( z = T/m \), it reads

\[
\lambda = [F(x, z)]^{-1}
\]

\[
F(x, z) = \frac{1}{32\pi^2 x} \left\{ (1 + x) \log(1 + x) - x + 8(1 + x) \int_1^\infty d\omega \frac{\sqrt{\omega^2 - 1}}{e^{\omega \sqrt{1 + x/z}} - 1} \right\} \tag{5.1a}
\]

The function \([F(x, z)]^{-1}\) is plotted against \( x \) in figure 3, for various values of \( T/m \). Because of the factor \( 1/x \) in \( F \), all the curves go to 0 at small \( x \). Because of the logarithm term in \( F \), which arises from the (frequently neglected) contribution \( \hat{M} \) in (2.11b), they go to 0 again at large \( x \). So for each, there is a maximum choice of \( \lambda \) beyond which the equation has no real solution; for example, for \( T/m = 3 \) the critical value of \( \lambda \) is about 200. For values of \( \lambda \) below the critical value there are two solutions, as was found previously by Bardeen and Moshe [14]. However, only the smaller one is relevant, since by definition \( \delta m^2 \to 0 \) as \( T \to 0 \); the larger solution corresponds rather to having a second, nonperturbative solution for the renormalized mass \( m \) as a function of the bare mass \( m_0 \). For reasonable coupling the larger solution is in fact exponentially huge so that it would not matter anyway. For overcritical \( \lambda \), the two solutions become complex and have to be dismissed because we have started with the assumption of a real self-energy \( \Pi \) in (3.5b).

\[\text{Figure 3: the function } F^{-1} \text{ in (5.1a) plotted against } x \text{ for various values of } T/m\]

Alternatively, for a given choice of \( \lambda \), there is a critical value of \( T/m \) beyond which there is no solution. For \( \lambda < 10 \) this critical value is large and we may find it from (5.1a) approximately. The stationary value of the right-hand side of (5.1a) occurs when

\[
\frac{z}{4\sqrt{x}} = \int_1^\infty dk \frac{\sqrt{k^2 - 1}}{(e^{k\sqrt{x/z}} - 1)(1 - e^{-k\sqrt{x/z}})} \tag{5.2}
\]
The integral is approximately 0.8 when $\sqrt{\pi}/z = 1$ and it decreases rapidly as $\sqrt{\pi}/z = 1$ increases. Hence the stationary value is at $x$ just greater than $z^2$, and so the critical value is given by

$$T \approx m \exp \left( \frac{16\pi^2}{\lambda} \right)$$

That is, the critical temperature is of the order of the tachyon mass, which for $\lambda = 1$ is about $10^{68}m$.

The massless case

As we have seen in section 2, the massless case requires that $\lambda \to 0$ as $m \to 0$. This does not mean that we are driven to a trivial theory, but only that the definition of the coupling constant through the scattering at zero energy becomes inappropriate. Instead we have to introduce a renormalization scale $\bar{\mu}$ and a finite coupling constant $\bar{\lambda}$ as given by (2.3b) or (2.13). The integral equation to be solved for $\delta m^2$ is now (2.14). In place of (5.1a) we have $\bar{\lambda} = [\bar{F}(\delta m^2, \bar{\mu})]^{-1}$ and the situation is similar to the one of the massive case with $\bar{\mu}$ replacing $m$. If for example we choose $\bar{\mu}$ such that $\bar{\lambda}(\bar{\mu}) = 1$, the critical temperature above which there are no solutions is of the order of $\bar{\mu} \exp(16\pi^2)$.

In the remainder of this paper, we shall compare the exact results that we have found against a perturbative evaluation. At a given nonzero temperature it will turn out to be useful to define the coupling at $\bar{\mu} = T$ through

$$\bar{\lambda}(T) = \frac{\bar{\lambda}(\bar{\mu})}{1 + \bar{\lambda}(\bar{\mu})[L(\bar{\mu}) - L(T)]}$$

With this choice of the renormalization scheme the solutions to $\bar{\lambda} = [\bar{F}(\delta m^2, \bar{\mu} = T)]^{-1}$ are plotted in figure 4. The critical value of $\bar{\lambda}(T)$ beyond which no solutions exist turns out to be approximately 325.5. Below this, there are two solutions for the thermal mass, but the higher solution is found in the region close to the tachyon scale which is marked by the heavy line in figure 4. For $\bar{\lambda}(T) \ll 10^{2}$, the latter is exponentially far away and this loosely defines the range of coupling where we can accept the theory as an effective one.

By the way, had we chosen $\bar{\mu}/T > 1.62\ldots$ we would have found a seemingly different picture: for those values of $\bar{\mu}$ the critical value $\bar{\lambda}(\bar{\mu})$ has moved past infinity so that there always exist two solutions for positive coupling; but this is only because this change of the renormalization scheme maps sufficiently large values of $\bar{\lambda}(\bar{\mu} = T)$ onto negative $\bar{\lambda}(\bar{\mu})$. However, all this occurs in the region close to the tachyon mass scale, which we can ignore by avoiding too-large couplings.
We now compare the exact results one gets by a numerical evaluation of the above formulae against a perturbative one. This is particularly interesting in the case of a massless theory, for there ordinary perturbation theory runs into infrared singularities that need to be cured by resummation of the thermal mass.

In perturbation theory \( \delta m^2/T^2 \sim \lambda \), but a naive expansion of the functions \( N_T \) and \( N_T \) as a power series in their argument is bound to fail. However we may write the function \( N_T(\delta m^2) \) that appears in the integral equation (2.14) for \( \delta m^2 \) as

\[
N_T(\delta m^2) = \frac{T^2}{4\pi^2} \sum_{k=1}^{\infty} \frac{\delta m}{kT} K_1(k \delta m/T)
\]

where \( K_1 \) is the modified Bessel function of the second kind. Using a Mellin transform [15] one finds

\[
\sum_{k=1}^{\infty} \frac{\delta m}{kT} K_1 \left( \frac{k \delta m}{T} \right) = \pi^2/6 - \pi \frac{\delta m}{2T} \left[ \log \frac{\delta m}{4\pi T} + \gamma - 1/2 \right] - \frac{1}{4} \left( \frac{\delta m}{T} \right)^2 \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n+1)!n!} \zeta(2n+1) \left( \frac{\delta m}{4\pi T} \right)^{2n}
\]  \( (5.4) \)

With (5.4) we can solve the equation for the thermal mass perturbatively to any desired accuracy in \( \bar{\lambda} \) and insert the result into the pressure (3.9), which in the massless case is given by

\[
P(T) - P(0) = N_T(\delta m^2) + \frac{1}{2} \delta m^2 N_T(\delta m^2) + \frac{1}{128\pi^2} \delta m^4
\]  \( (5.5) \)

For this, we need also

\[
\frac{4\pi^2}{T^4} N_T(\delta m^2) = \frac{2\pi^4}{45} - \frac{\pi^2}{6} \left( \frac{\delta m}{T} \right)^2 + \frac{\pi}{3} \left( \frac{\delta m}{T} \right)^3 + \frac{1}{8} \left( \frac{\delta m}{T} \right)^4 \left[ \log \frac{\delta m}{4\pi T} + \gamma - \frac{3}{4} \right] + \frac{1}{4} \left( \frac{\delta m}{T} \right)^4 \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n+2)!n!} \zeta(2n+1) \left( \frac{\delta m}{4\pi T} \right)^{2n}
\]  \( (5.6) \)

The infrared divergences that appear in conventional perturbation theory without resummation of thermal masses have their origin in the fact that the series (5.4) and (5.6) are nonanalytic in \( \delta m^2 \) at \( \delta m = 0 \).

The first few terms of the series expansion in \( \bar{\lambda} \) are

\[
\frac{\delta m^2}{T^2} = \frac{\bar{\lambda}}{24} - \frac{\bar{\lambda}^{3/2}}{16\pi \sqrt{6}} + (3 - \gamma - \log \frac{\bar{\mu}}{4\pi T}) \frac{\bar{\lambda}^2}{384\pi^2} - (1 - 2\gamma - 2\log \frac{\bar{\mu}}{4\pi T}) \frac{\bar{\lambda}^{5/2}}{1024\pi^3 \sqrt{2/3}} + O(\bar{\lambda}^3)
\]  \( (5.7) \)

\[
\frac{P(T) - P(0)}{T^4} = \frac{\pi^2}{90} - \frac{\bar{\lambda}}{1152} + \frac{\bar{\lambda}^{3/2}}{576\pi \sqrt{6}} - (6 - \gamma - \log \frac{\bar{\mu}}{4\pi T}) \frac{\bar{\lambda}^2}{18432\pi^2} + (3 - 2\gamma - 2\log \frac{\bar{\mu}}{4\pi T}) \frac{\bar{\lambda}^{5/2}}{12288\pi^3 \sqrt{6}} - \left( 6 - \gamma - \log \frac{\bar{\mu}}{4\pi T} \right)^2 - 30 + \frac{\zeta(3)}{36} \frac{\bar{\lambda}^3}{294912\pi^4} + O(\bar{\lambda}^{7/2})
\]  \( (5.8) \)
It is simple to verify that (5.8) is independent of \( \tilde{\mu} \): differentiate it with respect to \( \log \tilde{\mu} \) and use \( d\lambda/d\log \tilde{\mu} = \lambda^2/16\pi^2 \) from (2.13).

In the case of the \( N = 1 \) theory, the pressure has been calculated up to and including order \( \tilde{\lambda}^{5/2} \) using hard-thermal-loop resummed perturbation theory \([16]\). Up to and including order \( \tilde{\lambda}^{3/2} \), there is no difference between the subset of foam diagrams and the complete set of diagrams, and the results indeed agree. Beyond this order, there are differences, in particular there are no terms involving \( \log \tilde{\lambda} \) in the foam-diagram subset, which do occur in the full set starting at order \( \tilde{\lambda}^{5/2} \). They come from the logarithmic terms in the expansion (5.4) and (5.6), which in the case of foam diagrams happen to combine with the log in (2.15) such that the thermal mass drops out from the arguments of the logarithms.

Despite this simplification, a comparison of the above perturbative result with the exact one might give some hints about the convergence properties of thermal perturbation series in general. It turns out that these depend strongly on the ratio of \( \tilde{\mu}/T \).

In figures 5-7 we juxtapose the exact and the perturbative results for the thermal mass \( \delta m/T \) and the ratio of the pressure \( |P(T) - P(0)| \) to its ideal-gas value \( \pi^2 T^4/90 \), including in (5.7) and (5.8) up to 10 terms beyond the leading one. We choose various values of the renormalization scale \( \tilde{\mu} \), but for ease of comparison in each case we plot against \( \tilde{\lambda} \) evaluated for \( \tilde{\mu} = T \) through the relation (5.3). (The actual expansion parameter \( \lambda(\tilde{\mu}) \) is larger (smaller) when \( \tilde{\mu} \) is larger (smaller) than \( T \).) The resulting 11 approximants are put on top of each other in order to give a visual impression of the rate (or failure) of convergence of the perturbative expansions in \( \lambda \); the exact results are indicated by dashed lines.

When \( \tilde{\mu} \) is very different from \( T \), the convergence of the series deteriorates markedly. In figures 5 and 6 the results for the thermal mass and the pressure are seen to become oscillatory for larger coupling when \( \tilde{\mu} = 100T \) (the vertical lines in figures 5a and 6a are part of these oscillating results), whereas with \( \tilde{\mu} = \frac{1}{100} T \) the perturbative results fail to improve with higher orders at roughly the same place, although in a less violent manner.

With the choice \( \tilde{\mu} = 2\pi T \), which has been advocated in reference \([17]\) on the grounds that this is the mass of the first nonzero Matsubara mode, the behaviour of the series expansion can be significantly improved, although for the first few approximants the results are slightly worse than at \( \tilde{\mu} = T \), see figure 7.

Attempting to improve perturbation theory by putting to zero one of the \( \tilde{\lambda}^2 \)-terms in (5.7) or (5.8) gives much larger \( \tilde{\mu} \)'s and rather bad convergence properties. The optimal choice of \( \tilde{\mu} \) seems to be around \( \tilde{\mu} = 4\pi \exp(-\gamma)T \), which absorbs all \( \gamma \)'s and \( \log(4\pi) \)'s. This is in fact rather close to the choice \( \tilde{\mu} = 2\pi T \) of reference \([17]\). Notice that the origin of the \( \gamma \)'s in (5.7) and (5.8) is entirely from the high-temperature expansions (5.4) and (5.6); those appearing in dimensional regularization have already been absorbed in \( \tilde{\mu} \). However, without the restriction to foam diagrams, the \( \gamma \)'s would not have the same coefficients as the logs, so it is not clear whether this result for the optimal renormalization point could be a general one. But it does confirm the expectation \([17]\) that the optimal renormalization scheme is to be found around \( 2\pi T \) rather than \( T \).

Another noteworthy observation is that the rate of convergence is markedly slower for the perturbation series of the pressure than it is for the thermal mass. Since this loss of accuracy comes from having inserted a perturbative result for \( \delta m/T \) into the series (5.6) and truncated at a given order in the coupling, a certain improvement would be simply to refrain from doing a high-temperature expansion of the integrals that appear in the expression for the pressure. The quality of the perturbation series for the pressure is then the same as that of the thermal mass. It would be interesting to see whether this could be implemented in QCD to ameliorate the frustratingly bad apparent convergence of resummed perturbation theory for the QCD pressure, which has been calculated up to order \( g^5 \) recently \([18]\).
Figure 5: A comparison of the perturbative results for $\delta m/T$ as a function of $\tilde{\lambda}^{1/2}(T)$ up to 10th order for different choices of the renormalization scale: a) $\bar{\mu} = 100T$, b) $\bar{\mu} = T$, c) $\bar{\mu} = \frac{1}{100}T$. In a) the “vertical” lines are part of two of the curves on the left.
Figure 6: A comparison of the perturbative results for $ar{P} \equiv [P(T) - P(0)]/\pi^2 T^4/90$ as a function of $\lambda^{1/2}(T)$ up to 12th order for different choices of the renormalization scale: a) $\bar{\mu} = 100T$, b) $\bar{\mu} = T$, c) $\bar{\mu} = \frac{1}{100}T$. 

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Figure 7: The perturbative results for $\delta m/T$ and $\bar{P} = [P(T) - P(0)]/\frac{2T^4}{90}$ for $\bar{\mu} = 2\pi T$. Compared to figures 5b and 6b, the rate of convergence becomes much more rapid after the first few approximations, for which $\bar{\mu} = T$ is slightly favoured.

A rather simple reorganization of perturbation theory is brought about by replacing truncated power series expansions by perturbatively equivalent Padé approximants [19], which amounts to replacing

$$F_n = c_0 + c_1 g + \ldots c_n g^n \to F_{[p,n]} = \frac{c_0 + a_1 g^1 + \ldots + a_p g^p + O(g^{n+1})}{1 + a_{p+1} g^1 + \ldots + a_n g^{n-p}}$$

In reference [20], this possibility has been studied in the context of thermal perturbation theory by testing for an unphysical dependence on the renormalization scale. Our results allow us to investigate the quality of Padé approximants by comparing directly with truncated power series and the exact result.

In figure 8 we have replaced the various $n$th-order results that gave rise to the curves displayed in figure 6b by $[n/2,n/2]$ Padé approximants up to $n = 8$ (for odd $n$ we rounded off the second entry at the expense of the first) and using $\bar{\mu} = T$. It turns out that we find a spectacular improvement of convergence up to really high values of the coupling which in our theory is bounded by the requirement to be sufficiently below the critical value, $\lambda^{1/2}(T) \ll 18$. The lower line in figure 8 is the Padé approximant $[0, 2]$ ($[1, 1]$ does not exist), which is only marginally better than its perturbative analogue.
[2,0], but already [1,2] (given by the next line upwards) is a very good approximation. The higher approximants approach the exact result from above and are extremely accurate with the exception of [3,3] which has a pole at \( \bar{\lambda}^{1/2} \approx 9.5 \). For \( \bar{\lambda}^{1/2} \) sufficiently smaller than that, this approximant is in fact quite good, but after the pole has been encountered it seems to be off by a constant\(^*\) and its quality is inferior to most of the lower-order approximants.

Curiously enough, with the choice \( \bar{\mu} = 2\pi T \) which has led to more rapid convergence in the case of truncated power series, the Padé approximants are not as good as in figure 8, although they are still a great improvement for \( \bar{\lambda}^{1/2} < 8 \). What happens is that all of the approximants run into poles in the range of \( \bar{\lambda} \) considered. The same holds true for larger \( \bar{\mu} \), whereas for \( \bar{\mu} \ll T \) the quality of the Padé approximations decreases, too, but not as rapidly.

All in all it appears that Padé approximants can give vast improvements of a truncated power series expansion unless the rational functions used as approximants develop poles. While there is no real theoretical explanation for the superiority of Padé approximants, it seems that their main advantage is that (in the absence of poles) the latter do not blow up at larger coupling as quickly as the corresponding truncated power series inevitably do. Since the exact result behaves rather unspectacularly, the odds seem to be in favour of Padé approximants.

Unfortunately, in QCD Padé approximants turn out to lead to less impressive improvements [20]. In figure 9a the perturbative result for QCD with 3 flavours is given, which shows that (resummed) perturbation theory is useful only up to \( g(T) \approx 1/2 \). But a real quark-gluon plasma as one hopes to produce in heavy-ion collisions has rather \( g(T) \approx 2 \), where the perturbative results are completely inconclusive. The corresponding Padé approximants are rendered in figure 9b. They seem to give some improvement of convergence, extending the allowed range of coupling to perhaps \( g(T) \approx 1 \), but this appears to break down before reaching \( g(T) = 2 \).

So ultimately one would have to find a different expansion scheme that does not involve truncated series in the coupling if one wants to cover more-strongly-coupled theories. Recently an interesting attempt towards an alternative perturbative scheme has been made in the example of a scalar theory in reference [21] using the numerical solution of an approximation to the gap equation in a loop

\* In reference [20] poles in Padé approximants were simply subtracted. Our observations imply that this works nicely only when the coupling is such that one is below the point where a pole arises.
In this context we observe that while the perturbative results are satisfactory only for small coupling, the high-temperature series (5.4) and (5.6) have excellent convergence properties, with convergence radius $\delta m/T = 2\pi$. In our model, the exact result for the thermal mass remains sufficiently smaller than $2\pi T$ even for extremely large coupling $\bar{\lambda}$. Only when we re-expanded in $\bar{\lambda}$ did the convergence become bad. In order to highlight this, figure 10a shows the solutions to the thermal mass equation when it is truncated at the leading or next-to-leading term in the expansion in $\delta m/T$, rather than being expanded in $\lambda$. Including the next power of $\delta m^2/T^2$, together with the logarithm, turns out to produce a result which is virtually indistinguishable from the exact one for the whole range of $\lambda$ bounded by the requirement that the tachyon mass remains larger than anything else. The same holds true for the pressure (figure 10b), with even smaller deviations from the exact result.

In the particularly interesting case of gauge theories it is crucial to have a consistent expansion scheme with a well-defined expansion parameter in order to be able to retain gauge fixing independence. If in QCD it would be possible to reorganize perturbation theory as a series in $\delta m/T$ rather than $g$, expansion of the pressure, which however required an ad hoc treatment of uncannisted ultraviolet divergences.
Figure 10: a) Solutions of the thermal mass equation when the latter is truncated at leading order in $\delta m/T$ (top curve) and next-to-leading order (bottom curve). All higher approximations are virtually indistinguishable from the exact solution, and they collectively form the middle line, which broadens only at the highest values of $\tilde{\lambda}$. b) Analogously for the pressure.

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