A Study of Function Spaces through a Functor

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Abstract: Let X be a locally compact Hausdorff space and let $F_{n}(X)$ = limit of the function spaces of maps of X into certain spaces of type $K(\pi,n)$ implies each of the spaces of sequences $SP_{n}^{\infty}$, $\sum SP_{n}^{\infty}$, $\sum SP_{n}^{\infty}$,..., $\sum SP_{n}^{\infty}$,... is a space of type $K(\pi,n)$.

For any space X, we define the space $F_{n,m}(X) = (\sum SP_{n}^{\infty})^{m}$ topologized by the compact-open topology. The aim of this paper is i) to investigate the properties of $F_{n,m}(X)$; ii) to study the object $F_{n,m}$. 

Keywords: Eilenberg-MacLane space, function spaces, $\Sigma$-homotopy classes, contravariant functor, compact open topology

1. Introduction
Throughout this paper we assume that all spaces are locally compact Hausdorff space, also all spaces are of type $K(\pi,n)$.

Now we recall the following definitions and statements:

Definition 1.1: Let $\pi$ be a discrete group. A based topological space X is called an Eilenberg-MacLane space of type $K(\pi,n)$, where $n \geq 1$; if all the homotopy groups $\pi_{n}(X)$ are trivial except for $\pi_{n}(X)$, which is isomorphic to $\pi$.

A pointed CW complex X is a $K(\pi,n)$ (Eilenberg-MacLane space) if $\pi_{n}(X) = \{n, k = n \}$ and $\pi_{n}(X) \neq \emptyset$.

Definition 1.2: Let $f: X \to Y$ be a continuous map, define $\Sigma f: \Sigma X \to \Sigma Y$ by $\Sigma f(x,t) = (f(x),t)$, then $\Sigma$ is a covariant functor. This implies that $\Sigma$ induces homotopic maps into homotopic maps i.e. $\Sigma$ induces a map $\Sigma : [X,Y] \to [\Sigma X, \Sigma Y]$.

Define $\Sigma^{n+1}(X) = \Sigma^{n}(\Sigma^{n}X)$ and $\Sigma^{n}(X) \to \Sigma^{n+1}(X)$ is an isomorphism.

In [5] we define $S$-category same as $\Sigma$-category is the category whose objects are topological spaces with base points and whose maps are from X to Y are the elements of $\{X, Y\}$.

For any space X we define the space $F_{n,m}(X) = (\sum SP_{n}^{\infty})^{m}$ topologized by the compact-open topology, then we have the following:

Lemma 1.3: Let X be a polyhedron, the map $F_{n,m}(X) \to F_{n,m+1}(X)$ is a weak homotopy equivalence for each $m \geq 0$.

Proof: Since $\pi_{n}(F_{n,m}(X)) = \pi_{n}(F_{n,m+1}(X))$ and $\pi_{n}(F_{n,m+1}(X)) \cong \pi_{n}(F_{n,m}(X))$. It follows that for each $m \geq 0$, $\pi_{n}(F_{n,m}(X)) \cong \pi_{n}(F_{n,m+1}(X))$.

Lemma 1.4: Each inclusion map $F_{n,m}(X) \subseteq F_{n,m+1}(X)$ is a weak homotopy equivalence.

Proof: Since $F_{n,m}(X)$ has the weak topology relative to the subsets $F_{n,m+1}(X)$, it follows that every subset of $F_{n,m}(X)$ is contained in $F_{n,m+1}(X)$ for some $m \geq 0$ (all the function spaces are easily seen to be Hausdorff). Therefore the inclusion maps $F_{n,m}(X) \subseteq F_{n,m+1}(X)$ induce the isomorphism $\pi_{n}(F_{n,m}(X)) \cong \pi_{n}(F_{n,m+1}(X))$, it follows from Lemma 1.3 that for any $m \geq 0$, $\pi_{n}(F_{n,m}(X)) \cong \pi_{n}(F_{n,m+1}(X))$.

Lemma 1.5: Let $\lambda: F_{n+1}(\Sigma X) \to F_{n}(X)$ be defined by $\lambda(\alpha)(x) = (\alpha(x),\alpha(t))$, then $\lambda$ is an isomorphism and if $f: X \to X$, commutativity holds in the diagram

\[
\begin{array}{c}
\frac{F_{n+1}(\Sigma X)}{F_{n}} \to F_{n+1}(X) \\
\downarrow \quad \downarrow \quad \downarrow \\
F_{n}(X)
\end{array}
\]

Proof: Since $\lambda : F_{n+1}(\Sigma X) \to F_{n}(X)$ is induced by the natural isomorphism $\lambda' : F_{n+1,m}(\Sigma X) \to F_{n,m}(X)$, for every $m \geq 1$ and so $\lambda$ is an isomorphism. Again since the diagram

\[
\begin{array}{c}
\frac{F_{n+1,m-1}(\Sigma X)}{F_{n,m-1}} \to F_{n+1,m-1}(X) \\
\downarrow \quad \downarrow \quad \downarrow \\
F_{n,m}(X)
\end{array}
\]

is commutative and so $\lambda$ is commutativity.

Let $\lambda: [F_{n+1}(\Sigma X), F_{n}(X)] \to [F_{n}(X), F_{n}(X)]$ be the isomorphism defined by $\lambda[h] = [\lambda D^{-1}h]$. Using the above Lemma 1.3, it follows that $\lambda: [F_{n+1}(\Sigma X), F_{n}(X)]$ is a homomorphism.

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Therefore we can extend the functor $F_n$ to a functor $F_n^f : [X,X'] \rightarrow [F_n(X)']$ such that the following diagram holds:

\[
\begin{array}{ccc}
[S^m X, S^m X'] & \to & [X,X'] \\
\lambda^m & \Rightarrow & \lambda^0 \\
[F_n(X), F_n(X)'] & \Rightarrow & \Delta_n
\end{array}
\]

Lemma 1.6: $F_n^f$ is a homomorphism.

Proof: We prove that $F_{n,m} : [S^m X, S^m X'] \rightarrow [F_n+m(X), F_n+m(X)]$ is a homomorphism for $m \geq 2$. Let $f, g : S^m X \rightarrow S^m X'$ such that $x_0 \in A \cap B$, $S^m X = A \cup B$, $f[B] = g[A] = x_0$ and $f \sim f'$, $g \sim g'$. Then $f' + g' : S^m X \rightarrow S^m X'$ is defined by $f' + g'[A]$. $\lambda \in F_{n,m}(X)$ be a constant map at $x \in X$. Then $F_n+m(\lambda)[A] = \lambda[A] = f(A)$ and $F_n+m(\lambda)[B] = \lambda[B] = f'[B]$. If $\lambda' \in F_{n,m}(X)$, then $F_n+m(\lambda')[A] = \lambda'[A] = f(A)$ and $F_n+m(\lambda')[B] = \lambda'[B] = f'[B]$. Since $(F_n+m(\lambda') + \lambda'[B]) = F_n+m(\lambda'[A])$, $\lambda'[A] = f(A)$ and $\lambda'[B] = f'[B]$. Then we have:

\[\begin{align*}
\lambda'[A] & = f(A) \\
\lambda'[B] & = f'[B]
\end{align*}\]

Therefore we can extend the functor $F_n$ to a functor $F_n^f : [X,X'] \rightarrow [F_n(X)']$ such that the following diagram holds:

\[
\begin{array}{ccc}
[S^m X, S^m X'] & \to & [X,X'] \\
\lambda^m & \Rightarrow & \lambda^0 \\
[F_n(X), F_n(X)'] & \Rightarrow & \Delta_n
\end{array}
\]

Lemma 1.8:

Let $f : X \to X'$, then the diagram holds:

\[
\begin{array}{ccc}
H^n(X) & \to & H^n(X') \\
\Delta_n & \Rightarrow & \Delta_n \\
\pi_n[F_n(X)] & \Rightarrow & \pi_n[F_n(X')]
\end{array}
\]

Lemma 1.9:

Let $f : X \to Y$ and $g_i : Y \to Z$, then $f \circ g_i = g_j$ for $i, j \in \{1, 2\}$ continuous. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1 \circ f_1 \sim g_2 \circ f_2$. The following diagram is commutative:

\[
\begin{array}{ccc}
\pi_n[F_n(X)] & \Rightarrow & \pi_n[F_n(X')]
\end{array}
\]

In section 2 we construct and investigate functor $F_{n,m}$.

Theorem 2.1

Let $f : X \to X'$, then $F_{n,m}(f) : F_{n,m}(X) \to F_{n,m}(X')$ is a continuous homomorphism. Proof: We define $F_{n,m}(f) = F_{n,m}(f)$ by $F_{n,m}(f)(x) = f(X)$, for $x \in F_{n,m}(X)$. Since for every $m$, $F_{n,m}(f) = F_{n,m}(f)$ is a continuous homomorphism and $F_{n,m}(f) = F_{n,m}(f)$. Therefore we can extend the functor $F_n$ to a functor $F_n^f : [X,X'] \to [F_n(X)']$ such that the following diagram holds:

\[
\begin{array}{ccc}
[S^m X, S^m X'] & \to & [X,X'] \\
\lambda^m & \Rightarrow & \lambda^0 \\
[F_n(X), F_n(X)'] & \Rightarrow & \Delta_n
\end{array}
\]

Lemma 1.8:

Let $f : X \to X'$, then the diagram holds:

\[
\begin{array}{ccc}
H^n(X) & \to & H^n(X') \\
\Delta_n & \Rightarrow & \Delta_n \\
\pi_n[F_n(X)] & \Rightarrow & \pi_n[F_n(X')]
\end{array}
\]

Lemma 1.9:

Let $f : X \to Y$ and $g_i : Y \to Z$, then $f \circ g_i = g_j$ for $i, j \in \{1, 2\}$ continuous. If $f_1 \sim f_2$ and $g_1 \sim g_2$, then $g_1 \circ f_1 \sim g_2 \circ f_2$. That is $\{g_1, f_1\} = \{g_2, f_2\}$.

In section 2, we construct and investigate functor $F_{n,m}$.

Theorem 2.1

If $f : X \to X'$, then $F_{n,m}(f) : F_{n,m}(X) \to F_{n,m}(X')$ is a continuous homomorphism. Proof: We define $F_{n,m}(f) : F_{n,m}(X') \to F_{n,m}(X)$ by $F_{n,m}(f)(x) = f'(x)$, for $x \in F_{n,m}(X')$, $m \geq 0$. Since for every $m$, $F_{n,m}(f)$ is a continuous homomorphism and $F_{n,m}(f) = F_{n,m}(f)$ is continuous.

Theorem 2.2

Let $[X,X']$ be the set of homotopy classes from $X$ to $X'$ and $[F_n(X), F_n(X)]_H$ denote the monoid of homotopy classes of homomorphisms, homotopic through homomorphisms, of one abelian monoid $F_n(X)$ into another $F_n(X)$, then we have a homomorphism $F_n : [X,X'] \to [F_n(X), F_n(X)]_H$ such that $F_n(f) = [F_n(f)]_H$.

Proof: Let $h : X \times I \to X'$ be a homotopy from $f_0$ to $f_1$. Then for each $m$ we have a continuous homomorphism $F_{n,m}(h) : (\Omega^m SP^{n+m}X') \to (\Omega^m SP^{n+m}X')$ which corresponds to a continuous map $h_m : (\Omega^m SP^{n+m}X') \times I \to (\Omega^m SP^{n+m}X')$ which is a continuous homomorphism for every $t \in I$.
Since commutativity holds in the diagram
\[
\begin{align*}
(G \circ \sum_{m+n=m} \prod_{i=1}^m X_i) 	imes 1 & \rightarrow \sum_{m+n=m} \prod_{i=1}^m X_i \\
(G \circ \sum_{m+n=m} \prod_{i=1}^m X_i) & \rightarrow \sum_{m+n=m} \prod_{i=1}^m X_i \\
(\sum_{m+n=m} \prod_{i=1}^m X_i) & \rightarrow \sum_{m+n=m} \prod_{i=1}^m X_i \\
\Rightarrow & \text{ the maps } h_n \text{ define a continuous map } h^*_n \text{.}
\end{align*}
\]

Therefore, the set of all monoid of homotopy classes of continuous homomorphisms, homotopic through continuous homomorphisms forms a category, then there exists a contravariant \( n \)-homotopy functor \( F_n : \mathcal{HC} \rightarrow \mathcal{FNHC} \).

**Theorem 2.6** Let \( \mathcal{HC} \) be the category of homotopy classes of homomorphisms and \( \mathcal{FNHC} \) be the category of homotopy classes of continuous homomorphisms, then there exists a contravariant \( (n,m) \) functor \( F_{n,m} : \mathcal{HC} \rightarrow \mathcal{FNHC} \).

**Proof:** Using the Theorem 2.1, Theorem 2.2 and Theorem 2.5, it follows

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