NOTES ON THE SZEGŐ MINIMUM PROBLEM. II. SINGULAR MEASURES

BY

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ABSTRACT

In this note, we prove several quantitative results concerning the Szegő minimum problem for classes of measures on the unit circle concentrated on small subsets. As a by-product, we refute a long-standing conjecture of Nevai.

This note can be read independently from the first one.

1. Introduction

In this note we will demonstrate several simple estimates of the quantity

\[ e_n(\rho)^2 = \min_{q_0, \ldots, q_{n-1}} \int_{\mathbb{T}} \left| t^n + q_{n-1} t^{n-1} + \cdots + q_1 t + q_0 \right|^2 d\rho(t) \]

for measures \( \rho \) supported by small subsets of the unit circle \( \mathbb{T} \).

We start with a straightforward lower bound for \( e_n(\rho) \) for measures \( \rho \) of the form

\[ \rho = \sum_{k \geq 1} a_k \rho_k, \]

where \( a_k \geq 0, \sum_k a_k = 1 \), and \( \rho_k \) are probability measures, \( \rho_k \) is invariant w.r.t. rotation of the circle by \( 2\pi/2^k \) radians. This lower bound yields a simple counter-example to the Nevai conjecture raised in [11] and then discussed by Rakhmanov in [13] and by Simon in [15, Sections 2.9, 9.4, 9.10].

Our second result (Theorem 6) deals with discrete probability measures

\[ \rho = \sum_j a_j \delta_{\lambda_j}, \quad \sum_j a_j = 1, \quad (\lambda_j) \subset \mathbb{T}. \]

Given a sequence \((a_j)\), we estimate the quantity \( \sup_{(a_j) \subset \mathbb{T}} e_n(\rho) \). Our proof relies on ideas from Denisov’s work [3].

Then we bring two results (Theorems 8 and 9) which provide conditions for super-exponential decay of \( e_n \). Note that [16, Chapter 4] contains a number of delicate conditions for sub-exponential decay of the sequence \( e_n(\rho) \) obtained by Erdős–Turán, Widom, Ullman, and Stahl–Totik.

We conclude this note with a discussion of the singular continuous Riesz products for which \( e_n(\rho) \) can be estimated in a simple and straightforward manner.
As in the first note, we use here the following notation: for positive $A$ and $B$, $A \lesssim B$ means that there is a positive numerical constant $C$ such that $A \leq CB$, while $A \gtrsim B$ means that $B \lesssim A$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$.

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2. Limit-invariant measures and the Nevai conjecture

2.1. Limit-invariant measures. We say that a measure $\rho$ on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is $\alpha$-invariant if it is invariant under the rotation $\theta \mapsto \theta + 2\pi\alpha \mod 2\pi$.

**Lemma 1:** Let $\rho$ be a $\frac{1}{k}$-invariant measure with $k \in \mathbb{N}$. Then $e_s(\rho)^2 = \rho(\mathbb{T})$, $s < k$.

**Proof of Lemma 1.** Suppose that $k > 1$ (for $k = 1$ the statement is obvious). By the $\frac{1}{k}$-invariance of the measure $\rho$, its moments of order $1 \leq |\ell| \leq k - 1$ vanish. Thus, the measures $\rho$ and $\rho(\mathbb{T})m$ (here and elsewhere, $m$ is the normalized Lebesgue measure on $\mathbb{T}$) have the same moments of order $0 \leq |\ell| \leq k - 1$, and therefore

$$e_s(\rho)^2 = e_s(\rho(\mathbb{T})m)^2 = \rho(\mathbb{T})e_s(m)^2 = \rho(\mathbb{T}), \quad s < k,$$

completing the proof. □

**Lemma 2:** Suppose that $\rho$ is a probability measure on $\mathbb{T}$ of the form

$$\rho = \sum_{k \geq 1} a_k \rho_k,$$

where $(\rho_k)$ is a sequence of probability measures such that $\rho_k$ is $2^{-k}$-invariant, and $(a_k)$ is a sequence of non-negative numbers such that $\sum_k a_k = 1$. Then

$$e_{2^n}(\rho)^2 \geq \sum_{k \geq n+1} a_k.$$

**Proof of Lemma 2.** The tail $v_n = \sum_{k \geq n+1} a_k \rho_k$ is a $2^{-(n+1)}$-invariant measure, so that

$$e_{2^n}(\rho)^2 \geq e_{2^n}(v_n)^2 = \sum_{k \geq n+1} a_k,$$

proving the lemma. □
It is curious to observe that, generally speaking, the lower bound from Lemma 2 cannot be significantly improved:

**Lemma 3**: Let $\Lambda_{2^k} = \{\lambda: \lambda^{2^k} = 1\}$, let the sequence $(a_k)$ be as in Lemma 2, let

$$\rho_k = \frac{1}{2^k} \sum_{\lambda \in \Lambda_{2^{k+1}} \setminus \Lambda_{2^k}} \delta_\lambda, \quad k \geq 0,$$

and let $\rho = \sum_{k \geq 0} a_k \rho_k$. Then

$$\sum_{k \geq n+1} a_k \leq e_2^n(\rho)^2 \leq 4 \sum_{k \geq n} a_k.$$

**Proof of Lemma 3.** The measure $\rho_k$ is $2^{-k}$-invariant, hence the lower bound follows from Lemma 2.

To prove the upper bound, we put $Q_{2^n}(z) = z^{2^n} - 1$. Since $Q_{2^n}$ vanishes at $\Lambda_{2^k}$ with $k \leq n$ and $|Q_{2^n}| \leq 2$ everywhere on $\mathbb{T}$, we have

$$e_2^n(\rho) \leq \|Q_{2^n}\|^2_{L^2(\rho)} \leq 4 \sum_{k \geq n} a_k \rho_k(\mathbb{T}) = 4 \sum_{k \geq n} a_k,$$

proving the upper bound.

2.2. Does the relative Szegő asymptotics always exist? Note that Lemma 2 yields the existence of singular measures $\rho$ with an arbitrary slow decay of the sequence $e_n(\rho)$ (as we will see later in Theorem 13, the Riesz products provide another construction of singular measures with such a property). Thus, taking an arbitrary measure $\mu$ with divergent logarithmic integral

$$(1) \quad \int_{\mathbb{T}} \log \mu' \, dm = -\infty, \quad \mu' = \frac{d\mu}{dm} > 0,$$

and adding to $\mu$ a singular measure $\rho$ as in Lemma 2, one can make the sequence $e_n(\mu + \rho)$ decaying incomparably slower than the sequence $e_n(\mu)$. It is not too difficult to achieve the same effect choosing an absolutely continuous $\rho$ such that $\mu + \rho = w\mu$ with $\log w \in L^1(m)$, or even with $\log w \in L^p(m)$ with any $p < \infty$.

**Theorem 4**: Suppose that $\mu$ is an absolutely continuous measure on $\mathbb{T}$ with $\mu' > 0$ $m$-a.e., and with divergent logarithmic integral (1). Then, for any sequence $\varepsilon_n \to 0$, there exists a positive function $w$ such that, for any $p < \infty$, $\log w \in L^p(m)$, while $e_n(w\mu)/\varepsilon_n \to \infty$ as $n \to \infty$. 
This theorem answers negatively a question raised by Nevai in [11], where he conjectured that for any measure $\mu$ with $\mu' > 0$ m.a.e. and for any positive function $w$ with $\log w \in L^1(m)$, one has

\begin{equation}
\lim_{n \to \infty} \frac{e_n(w\mu)}{e_n(\mu)} = \exp \left( \frac{1}{2} \int_T \log w \, dm \right).
\end{equation}

Note that when $\mu = m$ this becomes Szegő’s theorem. Nevai proved that this conjecture is correct when $w$ satisfies additional regularity assumptions. Further results in that direction were obtained by Rakhmanov [13] and Máté–Nevai–Totik [9]. In [13] (see the very end of Section 3) Rakhmanov discusses a similar question, and guesses that it may have a positive answer at least when $\mu$ has a smooth density and $\log w \in L^p(m)$ with some $p > 2$ (this is also refuted by Theorem 4). One can find a thorough discussion of the Nevai conjecture and related topics in the Simon treatise [15, Sections 2.9, 9.4, 9.10].

In the situation described in Theorem 4, relation (2) fails because for some unbounded $w$ with convergent logarithmic integral, we can have

$$
\frac{e_n(w\mu)}{e_n(\mu)} \to \infty, \quad n \to \infty.
$$

It turns out that for bounded $w$ with convergent logarithmic integral and for some $\mu$ with bounded density, we can have $e_n(w\mu)/e_n(\mu) \to 0$, $n \to \infty$, which gives a different example of failure of (2).

**Theorem 5:** There exist an absolutely continuous measure $\mu$ and a function $w$ on $T$ such that $0 < \mu' < 1$, $0 < w \leq 1$ m.a.e.,

$$
\int_T \log w \, dm > -\infty,
$$

and

$$
\lim_{n \to \infty} \frac{e_n(w\mu)}{e_n(\mu)} = 0.
$$

**Proof of Theorem 4.** Let $\mu = e^{-H} m$ be a measure satisfying the assumptions of Theorem 4, and set $\mu_0 = e^{-H_+} m \leq \mu$; here and later on,

$$
H_+ = \max(H, 0), \quad H_- = \max(-H, 0).
$$

Then $\mu_0$ is an absolutely continuous measure on $T$ with $\mu_0' > 0$ m.a.e., and with divergent logarithmic integral (1).
The idea of the proof is straightforward: we start with the same discrete measure \( \rho \) as above, i.e.,

\[
\rho = \sum_{k \geq 1} a_k \rho_k, \quad \rho_k = 2^{-k} \sum_{\lambda \in \Lambda_{2k+1} \setminus \Lambda_2} \delta_\lambda,
\]

and spread slightly each of the measures \( \rho_k \) retaining the \( 2^{-k} \)-invariance. First, using that \( H_+ < \infty \) a.e. on \( \mathbb{T} \), we fix \( A_k \) so that

\[
m\{ t \in \mathbb{T} : |\arg(t)| < 2^{-k} \pi, \max_{\lambda \in \Lambda_{2k+1} \setminus \Lambda_2} H_+(\lambda t) > A_k \} < 2^{-k-1},
\]

and then choose a measurable set \( X_k \subset \{ t \in \mathbb{T} : |\arg(t)| < 2^{-k} \pi \} \) of measure \( m(X_k) = \eta_k > 0 \) so that

\[
\sup_{t \in X_k} \max_{\lambda \in \Lambda_{2k+1} \setminus \Lambda_2} H_+(\lambda t) \leq A_k.
\]

We choose \( \eta_k \) in such a way that the sequence \( (\eta_k) \) is decreasing.

Note that, given \( k \), the sets \( \lambda X_k, \lambda \in \Lambda_{2k+1} \setminus \Lambda_{2k} \), are disjoint. Then we set

\[
E_k = \bigcup_{\lambda \in \Lambda_{2k+1} \setminus \Lambda_2} \lambda X_k, \quad E = \bigcup_{k \geq 1} E_k,
\]

and

\[
\tilde{\rho} = \sum_{k \geq 1} a_k \tilde{\rho}_k, \quad \tilde{\rho}_k = \frac{1}{2^k \eta_k} 1_{E_k} \cdot m
\]

for some sequence \( (a_n) \) of positive numbers to be chosen later on, of sum 1 (and observe that the measures \( \tilde{\rho}_k \) are \( 2^{-k} \)-invariant probability measures). Then we define a function \( w_0 \) by

\[
\mu_0 + \tilde{\rho} = e^{-H_+} w_0 \cdot m = w_0 \cdot \mu_0,
\]

i.e.,

\[
w_0 = 1 + (e^{H_+} 1_E) \cdot \sum_{k \geq 1} \frac{a_k}{2^k \eta_k} 1_{E_k}.
\]

Put \( w = \max(1, w_0 e^{-H_-}) \). Then

\[
w \cdot \mu = \max(1, w_0 e^{-H_-}) e^{-H_+} w_+ \cdot m \geq w_0 e^{-H_+} \cdot m = w_0 \cdot \mu_0,
\]

and

\[
0 \leq \log w \leq \log w_0 \leq H_+ 1_E + \log \left( \sum_{k \geq 1} \frac{a_k}{2^k \eta_k} 1_{E_k} \right) + \log 2.
\]
We need to choose the parameters $\eta_k$ to guarantee that both terms on the RHS are integrable in any power $p < \infty$. Furthermore, putting

$$\tilde{v}_n = \sum_{k \geq n+1} a_k \tilde{\rho}_k,$$

recalling that the measures $\tilde{\rho}_k$ are $2^{-k}$-invariant, and applying Lemma 1, we get

$$e_2^n (w \mu)^2 \geq e_2^n (w_0 \mu_0)^2 = e_2^n (\mu_0 + \tilde{\rho})^2 \geq e_2^n (\tilde{\rho})^2 \geq e_2^n (\tilde{v}_n)^2 = \tilde{v}_n (T) = \sum_{k \geq n+1} a_k.$$

To complete the proof of Theorem 4, we choose the sequence $a_k$ so that

$$\varepsilon_n = o\left( \sum_{k \geq \log_2 n+1} a_k \right), \quad n \to \infty.$$

It remains to show that the functions $(H_+ 1_E)^p$ and $\log^p_+ (\sum_{k \geq 1} \frac{a_k}{2^k \eta_k} \mathbb{I}_{E_k})$ are integrable for any $p < \infty$.

We have

$$\int_E H_+^p \, dm \leq \sum_{k \geq 1} \int_{E_k} H_+^p \, dm \leq \sum_{k \geq 1} A_k^p m(E_k) = \sum_{k \geq 1} A_k^p 2^k \eta_k < \infty,$$

provided that $\eta_k$ were chosen sufficiently small with respect to $A_k$.

The second estimate is also not difficult:

$$\int_E \log^p_+ \left( \sum_{k \geq 1} \frac{a_k}{2^k \eta_k} \mathbb{I}_{E_k} \right) \, dm = \sum_{r \geq 1} \int_{E_r \setminus \bigcup_{s > r} E_s} \log^p_+ \left( \sum_{k \geq 1} \frac{a_k}{2^k \eta_k} \mathbb{I}_{E_k} \right) \, dm$$

$$= \sum_{r \geq 1} \int_{E_r \setminus \bigcup_{s > r} E_s} \log^p_+ \left( \sum_{k=1}^r \frac{a_k}{2^k \eta_k} \mathbb{I}_{E_k} \right) \, dm$$

$$\leq \sum_{r \geq 1} \int_{E_r} \log^p \left( \frac{1}{\eta_r} \right) \, dm$$

$$\leq \sum_{r \geq 1} 2^r \eta_r \log^p \frac{1}{\eta_r} < \infty,$$

provided that $\eta_r$ tend to zero sufficiently fast. This finishes off the proof of Theorem 4. \qed
Proof of Theorem 5. Given $0 < \beta < \alpha < 1/2$, we set
\[
h_{\alpha, \beta}(e^{2\pi i \theta}) = \alpha \mathbb{1}_{[0,\alpha]}(\theta) + \beta \mathbb{1}_{(\alpha,1/2]}(\theta), \quad g_{\alpha}(e^{2\pi i \theta}) = \mathbb{1}_{[0,\alpha]}(\theta).
\]
Choose $N_k = 2^k$ (so that $N_{k+1} = N_k^4$). Next, choose $\alpha_k = e^{-N_k^2}$, $\beta_k = e^{-N_k^2}$, and define
\[
\mu = \left( \sum_{k \geq 2} h_{\alpha_k, \beta_k}(e^{2\pi i N_k \theta}) \right) \cdot m.
\]
(a) Clearly, $0 < \mu' < 1$ m-a.e.
(b) For every $k \geq 1$,
\[
\mu \geq \nu_k = \alpha_k \mathbb{1}_{[0,\alpha_k]}(e^{2\pi i N_k \theta}) m.
\]
Since the measure $\nu_k$ is $1/N_k$-invariant, by Lemma 1, we have
\[
e^2_s(\mu) \geq \nu_k(T) = \alpha_k^2, \quad 0 \leq s < N_k.
\]
(c) Set
\[
w(e^{2\pi i \theta}) = \exp \left( \sum_{k \geq 2} \log \frac{\beta_k}{\alpha_k} \cdot g_{\alpha_k}(e^{2\pi i N_k \theta}) \right).
\]
Then $0 < w \leq 1$ m-a.e. and
\[
\int_{T} \log(1/w) \, dm = \sum_{k \geq 2} \alpha_k \log \frac{\alpha_k}{\beta_k} = \sum_{k \geq 2} e^{-N_k^2} (N_{k+1} - N_k)
\]
\[
= \sum_{k \geq 2} N_k^{256} e^{-N_k^2} < \infty.
\]
(d) Given $k \geq 3$, by construction, we have $w\mu' < 2\beta_k$ on the arc
\[
J = \left\{ e^{2\pi i \theta} : 1 - \frac{1}{2N_k} < \theta < 1 \right\}
\]
of length $\pi/N_{k-1}$ (and, in fact, on $N_{k-1}2^{-k+3} - 1$ other arcs of the same length; we will not use this fact). Then, by [2, Lemma 11], there exists a monic polynomials $T_k$ of degree $N_k$ such that
\[
|T_k(e^{2\pi i \theta})| \leq 2 \cos^{N_k} \left( \frac{\pi}{2N_{k-1}} \right) < e^{-cN_k/N_{k-1}^2}, \quad e^{2\pi i \theta} \in T \setminus J.
\]
Furthermore, say, by the Remez inequality, we have
\[
|T_k(e^{2\pi i \theta})| \leq e^{CN_k m(J)} = e^{C N_k / N_{k-1}}, \quad e^{2\pi i \theta} \in J.
Let $N_k \leq n < N_{k+1}$. Then
\[
e_n(w\mu) \leq e_{N_k}(w\mu) \leq \int_T |T_k|^2 w\mu' \, dm \\
\leq 2\beta_k e^{CN_k/N_{k-1}} m(J) + e^{-cN_k/N_{k-1}} \\
= \frac{1}{N_{k-1}} e^{-N_{k+2}} e^{CN_k/N_{k-1}} + e^{-cN_k/N_{k-1}} \leq e^{-cN_k^{1/2}}.
\]

On the other hand,
\[
e_n(\mu) \geq e_{N_{k+1}-1}(\mu) \geq \alpha_{k+1} = e^{-N_{k-1}} = e^{-N_k^{1/4}}.
\]

We conclude that
\[
\lim_{n \to \infty} \frac{e_n(w\mu)}{e_n(\mu)} = 0,
\]
which completes the proof of Theorem 5.

3. Discrete measures on $T$

Given a sequence of positive numbers $a = (a_j)$ with $\sum_j a_j = 1$, and a sequence $(\lambda_j) \subset T$, consider the discrete measure
\[
\rho = \sum_{j \geq 1} a_j \delta_{\lambda_j}.
\]

Let
\[
e_n^*(a) = \sup_{(\lambda_j) \subset T} e_n(\rho),
\]
and $s_k = \sum_{j > k} a_j$.

**Theorem 6:**

(i) Suppose that the sequence $a$ is monotonic, i.e., $a_1 \geq a_2 \geq \cdots$. Then
\[
e_n^*(a)^2 \geq (n+1) \sum_{j \geq 1} a_j(n+1).
\]

In particular, $e_n^*(a)^2 \geq (n+1)a_{n+1}$.

(ii) Given $\gamma \in (0, 1)$, suppose that
\[
k|\log s_k|^{1+\frac{1}{\gamma}} \lesssim n.
\]

Then $e_n^*(a)^2 \leq C(\gamma)s_k$. 

(iii) Given $\sigma \in (0, \frac{1}{2}]$, suppose that

$$k^2|\log s_k|^{-1} \leq \frac{1}{8} \sigma n.$$ 

Then $e_n^*(a)^2 \leq s_k^{1-\sigma}$.

As we have already mentioned, the proofs of parts (ii) and (iii) follow ideas from Denisov’s paper [3].

3.1. EXAMPLES TO THEOREM 6. The following examples show that a combination of estimates from Theorem 6 provides relatively tight bounds.

Example A: Let $a = (2^{-j})_{j \geq 1}$. Then

$$2^{-n} \leq e_n^*(a)^2 \leq 2^{-cn}, \quad n \in \mathbb{N}.$$ 

**Proof.** The lower bound is a straightforward consequence of (i). To get the upper bound, we note that in this case $s_k = 2^{-k}$ so we can apply estimate (iii) with $\sigma = \frac{1}{2}$ and $k \geq cn$. 

Example B: Let $a = (c(p)j^{-p})_{j \geq 1}$ with $p > 1$. Then

$$c(p)\frac{1}{n^{p-1}} \leq e_n^*(a)^2 \leq C(p)\left(\frac{\log^3 n}{n}\right)^{p-1}.$$ 

**Proof.** The lower bound is again a straightforward consequence of (i). To prove the upper bound, first, we note that $s_k \simeq c(p)k^{1-p}$, so we can apply estimate (ii) with $\gamma = \frac{1}{2}$, and $k = C(p)n(\log n)^{-\frac{3}{2}}$.

**Remark:** Taking $\gamma$ closer to 1, one can improve $\log^3 n$ on the RHS to $\log^b n$ with any $b > 2$. On the other hand, it is not clear whether the logarithmic factor is needed at all.

Example C: Let $a = (c(p)j^{-1}\log^{-p}(j+1))_{j \geq 1}$ with $p > 1$. Then

$$\frac{c(p)}{(\log n)^{p-1}} \leq e_n^*(a)^2 \leq \frac{C(p)}{(\log n)^{p-1}}.$$ 

**Proof.** To prove the lower bound we note that

$$\sum_{j \geq 1} \frac{1}{j(n+1)\log^p(j(n+1)+1)} \gtrsim \frac{1}{n \log^p n} \sum_{1 \leq j \leq n} \frac{1}{j} \gtrsim \frac{1}{n \log^{p-1} n}.$$ 

To prove the upper bound, first, we note that $s_k \geq c(p)(\log k)^{1-p}$. This allows us to apply estimate (ii) with $\gamma = \frac{1}{2}$, $k = C(p)n(\log \log n)^{-3}$, for which $s_k = C(p)(\log n)^{1-p}$. 

Proof of estimate (i). Consider the measure
\[ \rho = \sum_{k=1}^{n+1} \left( \sum_{j \geq 0} a_{k+j(n+1)} \right) \delta_{e^{2\pi ik/(n+1)}}. \]
By the monotonicity of the sequence \( a \),
\[ \min_{1 \leq k \leq n+1} \sum_{j \geq 0} a_{k+j(n+1)} = \sum_{j \geq 1} a_{j(n+1)}. \]
Hence
\[ \rho \geq \left( \sum_{j \geq 1} a_{j(n+1)} \right) \sum_{\lambda^{n+1}=1} \delta_\lambda, \]
and Lemma 1 yields estimate (i).

Proof of estimate (ii). Given a measure \( \rho = \sum_{j \geq 1} a_j \delta_{\lambda_j} \), we take \( k \) and \( \varepsilon \) so that \( \varepsilon k \ll 1 \ll \varepsilon n \) (their values will be chosen at the end of the proof), let \( E = \{\lambda_1, \ldots, \lambda_k\} \), and, denoting by \( E_{+\varepsilon} \) the \( \varepsilon \)-neighbourhood of the set \( E \), note that \( m(E_{+\varepsilon}) \leq 2k\varepsilon \).

Our goal is to construct a polynomial \( P \) of degree at most \( n \) such that
\[ |P(0)| \simeq 1, \max_T |P| \lesssim 1, \text{ and } P \text{ is very small on } E. \]
Then
\[ e_n(\rho)^2 \lesssim \rho(T \setminus E) + \max_E |P|^2. \]
The polynomial \( P \) will be constructed in several steps.

The outer function \( F \). Let \( F = \exp[-m(E_{+\varepsilon})^{-1}(1 \mathbb{1}_{E_{+\varepsilon}} + i \mathbb{1}_{E_{+\varepsilon}})] \), where \( \mathbb{1}_{E_{+\varepsilon}} \) is the indicator function of the set \( E_{+\varepsilon} \), and \( \mathbb{1}_{E_{+\varepsilon}} \) is its harmonic conjugate. Then we have
\begin{enumerate}
  \item \( \sup_T |F| = 1; \)
  \item \( |F(0)| = \exp(\int_T \log |F| \, dm) = \frac{1}{e}; \)
  \item \( \sup_{E_{+\varepsilon}} |F| = \exp(-m(E_{+\varepsilon})^{-1}). \)
\end{enumerate}

The trigonometric polynomial \( q \) well concentrated near the origin. Next, given \( \gamma \in (0, 1) \), we construct a trigonometric polynomial
\[ q(x) = \sum_{|\ell| < n} \hat{q}(\ell) e^{i\ell x} \]
with the following properties:
\begin{enumerate}
  \item \( \hat{q}(0) = 1; \)
  \item \( \int_{-\pi}^{\pi} |q(x)| \, dx \leq C(\gamma); \)
  \item for \( s \geq 1, \int_{-\pi}^{\pi} |q(x)| \, dx \leq C(\gamma) s^{1-\gamma} e^{-s\gamma}. \)
\end{enumerate}
First, we take an entire function $g$ satisfying
\[ \hat{g} \in C_0^\infty(-1,1), \quad \hat{g}(0) = 1, \quad \text{and} \quad |g(x)| \leq C(\gamma)e^{-|x|^{1/\gamma}}. \]

The construction of such entire functions is classical; see for instance [5, Section IVD]. Then, we let $g_n(x) = ng(nx)$, note that the Fourier transform $\hat{g}_n(\xi) = \hat{g}(\xi/n)$ is supported by the interval $(-n,n)$, and consider the periodization of $g_n$
\[ q(x) = \sum_{j \in \mathbb{Z}} g_n(x - 2\pi j) = \sum_{|\ell| < n} \hat{g}(\ell/n)e^{i\ell x} \]
(the second equation is just the Poisson summation formula). The RHS is a trigonometric polynomial of degree less than $n$. It is easy to see that $q$ possesses the properties (A), (B), and (C).

The algebraic polynomial $P$. Take the Laurent polynomial $Q(e^{i\theta}) = q(\theta)$, i.e., $Q(t) = \sum_{|\ell| < n} \hat{g}(\ell)t^\ell$, and set $P = F \ast Q$. This is an algebraic polynomial of degree less than $n$, $|P(0)| = |F(0)| \cdot |\hat{g}(0)| = e^{-1}$, and
\[ \max_T |P| \leq \|F\|_{\infty,T} \cdot \|Q\|_{L^1(m)} \leq C(\gamma). \]

To estimate $\sup_E |P|$, we take $t = e^{i\tau} \in E$, and proceed as follows:
\[ |P(t)| \leq \int_{-\pi}^{\pi} |F(e^{i(\tau-\theta)})| \cdot |q(\theta)| \frac{d\theta}{2\pi} \]
\[ \leq \sup_T |F| \cdot \int_{|\theta| \geq \varepsilon} |q| + \sup_{E + \varepsilon} |F| \cdot \int_{-\pi}^{\pi} |q| \]
\[ \leq C(\gamma)[(\varepsilon n)^{1-\gamma}e^{-(\varepsilon n)^{1/\gamma}} + e^{-m(E+\varepsilon)^{-1}}]. \]

Hence, $\sup_E |P| \leq C(\gamma)[e^{-\frac{1}{2}(\varepsilon n)^{1/\gamma}} + e^{-\frac{1}{2}(\varepsilon k)^{-1}}]$, provided that $\varepsilon n \geq 1$. Thus
\[ e_n(\rho)^2 \lesssim \max_T |P|^2 \rho(T \setminus E) + \max_E |P|^2 \leq C(\gamma)[s_k + e^{-(\varepsilon n)^{1/\gamma}} + e^{-\varepsilon k^{-1}}]. \]

At last, we set $\varepsilon = (k|\log s_k|)^{-1}$, balancing the terms $e^{-\varepsilon k^{-1}}$ and $s_k$, and since $k|\log s_k|^{1+1/\gamma} \lesssim n$, we have $e^{-(\varepsilon n)^{1/\gamma}} \lesssim s_k$. □

Proof of estimate (iii). Here we will use the following lemma:

Lemma 7: ‘lász [4] For any $d \in \mathbb{N}$, there exists a polynomial $H_d$ of degree at most $d$ such that $H_d(0) = 1$, $H_d(1) = 0$, and $\max_T |H_d| \leq 1 + \frac{2}{d}$. 
Note that though more general and precise estimates are known (see, for instance, [6, 1]), the Halász original version suffices for our purposes.

To prove estimate (iii), we fix \( k \leq \frac{1}{2} \frac{n}{k} \) (to be chosen momentarily), let \( d = \lceil \frac{n}{k} \rceil \), and consider the polynomial \( P(z) = \prod_{j=1}^{k} H_d(z \lambda_j) \), where \( H_d \) is the Halász polynomial of degree \( d \) from Lemma 7. Clearly, \( \deg P \leq n \) and \( P(0) = 1 \). Furthermore,

\[
\max_T |P| \leq \left( 1 + \frac{2}{d} \right)^k \leq e^{2k/d} \leq e^{4k^2/n} \quad \text{(since } d \geq \frac{n}{k} - 1 \geq \frac{n}{2k} \text{)}.
\]

Thus,

\[
e_n(\rho)^2 \leq \int_T |P|^2 \, d\rho \leq (\max_T |P|^2) \cdot \sum_{j > k} a_j < e^{8k^2/n} s_k \leq s_k^{1-\sigma},
\]

provided that \( e^{8k^2/n} \leq s_k^{-\sigma} \), that is, \( k^2/(\log s_k^{-1}) \leq \frac{1}{8} \sigma n \).

\[\blacksquare\]

4. Measures with super-exponential decay of \( e_n \)

Here we bring two results, which provide conditions for super-exponential decay of the sequence \( e_n(\rho) \).

**Theorem 8:** Let \( \rho \) be a probability measure on \( \mathbb{T} \), and let \( n \geq 3 \) be an integer.

(A) Suppose that \( e_n(\rho) \leq e^{-\Omega} \) with \( \Omega \geq 16n \log n \). Then there are \( p \leq n \) closed arcs \( I_1, \ldots, I_p \) on \( \mathbb{T} \) such that

\[
\sum_{\ell=1}^{p} \frac{1}{\log |I_\ell|} \leq \frac{n \log n}{\Omega} \quad \text{and} \quad \rho\left( \mathbb{T} \setminus \bigcup_{1 \leq \ell \leq p} I_\ell \right) \leq e^{-\Omega}.
\]

(B) Suppose that there are \( p \leq n/2 \) closed arcs \( I_1, \ldots, I_p \) on \( \mathbb{T} \) such that

\[
\sum_{\ell=1}^{p} \frac{1}{\log |I_\ell|} \leq \frac{n}{2\Omega} \quad \text{and} \quad \rho\left( \mathbb{T} \setminus \bigcup_{1 \leq \ell \leq p} I_\ell \right) \leq e^{-\Omega}.
\]

Then \( e_n(\rho) \leq 2e^{-\frac{1}{2}\Omega} \), provided that \( \Omega \geq 4n \).

Using the logarithmic capacity (which we denote by \( \text{cap} \)) we get upper and lower bounds for \( e_n(\rho) \), which are tighter than the ones given in Theorem 8.
Theorem 9: Let \( \rho \) be a probability measure on \( \mathbb{T} \) and let \( n \geq 2 \) be a positive integer.

(A) Suppose that \( e_n(\rho) \leq e^{-\Omega} \). Then there are \( p \leq n \) closed arcs \( I_1, \ldots, I_p \) on \( \mathbb{T} \) such that
\[
\operatorname{cap} \left( \bigcup_{1 \leq \ell \leq p} I_{\ell} \right) \leq e^{-\frac{1}{2} \Omega} \quad \text{and} \quad \rho \left( \mathbb{T} \setminus \bigcup_{1 \leq \ell \leq p} I_{\ell} \right) \leq e^{-\Omega}.
\]

(B) Suppose that there are \( p \leq n \) closed arcs \( I_1, \ldots, I_p \) on \( \mathbb{T} \) such that
\[
\operatorname{cap} \left( \bigcup_{1 \leq \ell \leq p} I_{\ell} \right) \leq e^{-\frac{\Omega}{2}} \quad \text{and} \quad \rho \left( \mathbb{T} \setminus \bigcup_{1 \leq \ell \leq p} I_{\ell} \right) \leq e^{-\Omega}
\]
with \( \Omega \geq C_1 n \). Then \( e_{Cn}(\rho) \leq e^{-\Omega/4} \). Here \( C \) and \( C_1 \) are positive numerical constants.

Theorem 9 immediately yields a necessary and sufficient condition for super-exponential decay of the sequence \( e_n(\rho) \); cf. [16, Chapter 4].

Theorem 10: Let \( \rho \) be a positive measure on \( \mathbb{T} \). Then the following are equivalent:

(a) the sequence \( e_n(\rho) \) decays super-exponentially, i.e., \( n^{-1} \log e_n(\rho) \to -\infty \) as \( n \to \infty \);

(b) for any positive \( \varepsilon \) and \( A \), there exists \( n_0 \) such that for every \( n \geq n_0 \) there exists a set \( E \subset \mathbb{T} \), which is a union of at most \( n \) arcs, such that
\[
\operatorname{cap}(E) < \varepsilon \quad \text{and} \quad \rho(\mathbb{T} \setminus E) < e^{-An}.
\]

Proof of Theorem 10. (a)\( \Rightarrow \) (b): Suppose that the sequence \( e_n(\rho) \) decays super-exponentially fast and fix \( \varepsilon \) and \( A \). Choose \( A_1 \geq A \) such that \( e^{-A_1/2} \leq \varepsilon \). Then, we choose \( n_0 \) so that \( e_n(\rho) < e^{-A_1 n} \) for \( n \geq n_0 \). Applying part (A) of Theorem 9 with \( \Omega = A_1 n \), we get the set \( E \subset \mathbb{T} \) which is a union of at most \( n \) arcs such that \( \operatorname{cap}(E) < e^{-A_1/2} \leq \varepsilon \) and \( \rho(\mathbb{T} \setminus E) < e^{-A_1 n} \leq e^{-An} \).

(b)\( \Rightarrow \) (a): Given an \( A \geq C_1 \) with \( C_1 \) as in Theorem 9, choose \( \varepsilon \in (0, e^{-A}) \). By hypothesis, for every \( n \geq n_0 \) there exists a set \( E \subset \mathbb{T} \), which is a union of at most \( n \) arcs, such that \( \operatorname{cap}(E) < \varepsilon \) and \( \rho(\mathbb{T} \setminus E) < e^{-An} \). Set \( \Omega = An \). By part (B) of Theorem 9, for \( n \geq n_0 \), we have \( e_{Cn} \leq e^{-\Omega/4} = e^{-(A/4)n} \). Since \( A \) can be chosen arbitrarily large, we conclude that the sequence \( e_n \) decays super-exponentially fast. \( \blacksquare \)
Proof of Theorem 8.

Proof of (A). Here, we will use the classical Boutroux–Cartan lower estimate of monic polynomials outside an exceptional set. We will bring it in the version given by Lubinsky [8, Theorem 2.1].

**Lemma 11 (Boutroux–H. Cartan):** Given a monic polynomial $P$ of degree $n$ and an increasing sequence $0 < r_1 < r_2 < \cdots < r_n$, there exist positive integers $p \leq n$ and $(\lambda_j)_{j=1}^{p}$, $\sum_{j=1}^{p} \lambda_j = n$, and closed disks $(\bar{D}_j)_{j=1}^{p}$ of radii $2r\lambda_j$ such that $\{|P| \leq \prod_{j=1}^{n} r_j\} \subset \bigcup_{j=1}^{p} \bar{D}_j$.

Putting $r_j = \varepsilon j(n!)^{-1/n}$ one gets a more customary version of this lemma [7, Chapter I, Theorem 10], which says that for any monic polynomial $P$ of degree $n$ and any $\varepsilon > 0$, the set $\{|P| < \varepsilon^n\}$ can be covered by at most $n$ closed disks with the sum of radii not exceeding $2\varepsilon$.

Now, turning to the proof of (A), we suppose that $Q$ is an extremal polynomial of degree $n$. Then

$$e^{-2\Omega} \geq e_{n}^2(\rho) \geq e^{-\Omega} \rho\{|Q| \geq e^{-\frac{1}{2}\Omega}\},$$

whence $\rho\{|Q| \geq e^{-\frac{1}{2}\Omega}\} \leq e^{-\Omega}$.

Consider the set $\{|Q| < e^{-\frac{1}{2}\Omega}\}$. Put

$$r_j = \exp\left(-\frac{1}{4} \frac{\Omega}{j \log n}\right), \quad j = 1, 2, \ldots, n,$

and note that

$$\prod_{j=1}^{n} r_j = \exp\left(-\frac{1}{4} \frac{\Omega}{\log n} \sum_{j=1}^{n} \frac{1}{j}\right) > e^{-\frac{1}{2}\Omega}.$$

Then, by the Bourtoux–Cartan estimate, the set $\{|Q| < e^{-\frac{1}{2}\Omega}\}$ can be covered by $p \leq n$ arcs $I_1, \ldots, I_p$ of lengths $|I_{\ell}| = 4r_{m_{\ell}}$, where $\sum_{\ell} m_{\ell} = n$. Observing that

$$4r_{m_{\ell}} < \exp\left(-\frac{1}{4} \frac{\Omega}{m_{\ell} \log n} + 2\right) < \exp\left(-\frac{1}{8} \frac{\Omega}{m_{\ell} \log n}\right) \quad \text{(since $\Omega > 16m_{\ell} \log n$),}$$

we conclude that

$$\sum_{\ell=1}^{p} \frac{1}{\log \frac{1}{|I_{\ell}|}} = \sum_{\ell=1}^{p} \log \frac{1}{4r_{m_{\ell}}} < \sum_{\ell=1}^{p} \frac{8m_{\ell} \log n}{\Omega} = \frac{8n \log n}{\Omega},$$

proving (A).
Proof of (B). Let $z_\ell$ be the center of the arc $I_\ell$, $\ell = 1, 2, \ldots, p$. For each $\ell$ put
\[ m_\ell = \left\lceil \frac{\Omega}{\log |I_\ell|} \right\rceil \]
and note that $\sum_\ell m_\ell \leq \Omega \sum_\ell (\log \frac{1}{|I_\ell|})^{-1} \leq \frac{1}{2} n$. Consider the polynomial
\[ P(z) = \prod_{\ell=1}^{p} (z - z_\ell)^{m_\ell+1} \]
of degree $\sum_\ell m_\ell + p \leq n$. On $I_\ell$ we have
\[ |P| < 2^n |I_\ell|^{m_\ell+1} \leq 2^n \exp \left( \frac{\Omega}{\log |I_\ell|} \cdot \log |I_\ell| \right) = 2^n e^{-\Omega} < e^{-\frac{1}{2} \Omega}. \]

Hence
\[ e_n^2(\rho) \leq \int_T |P|^2 \, d\rho = \left( \int_{\bigcup_\ell I_\ell} + \int_{T \setminus \bigcup_\ell I_\ell} \right) |P|^2 \, d\rho \]
\[ \leq \max_{\ell} |P|^2 + 4^n \rho \left( T \setminus \bigcup_{\ell} I_\ell \right) < e^{-\Omega} + 4^n e^{-\Omega} < 2e^{-\frac{1}{2} \Omega}, \]
proving (B).

Proof of Theorem 9.

Proof of (A). Suppose that $Q$ is an extremal polynomial of degree $n$ for the measure $\rho$. Then $\rho\{|Q| > e^{-\frac{1}{2} \Omega}\} \leq e^{-\Omega}$. Consider the set
\[ E_Q = \{|Q| \leq e^{-\frac{1}{2} \Omega}\} \cap T = \{|Q|^2 \leq e^{-\Omega}\} \cap T. \]

Since $|Q|^2$ is a trigonometric polynomial of degree $2n$, the set $E_Q$ is a union of $p \leq n$ closed arcs. By a basic property of logarithmic capacity (see [14, Theorem 5.5.4]), $\text{cap}(E_Q) \leq e^{-\frac{n}{4} \Omega}$.  

Proof of (B). The proof of (B) needs the following lemma.

**Lemma 12:** Suppose $E \subset T$ is a union of at most $n \geq 14$ arcs. Then there exists a monic polynomial $P$ of degree at most $28n$ with zeros on the unit circle such that
\[ |P| \leq (\text{cap}(E))^n 2^{42n} \]
everywhere on $E$. 

Lemma 12 immediately yields (B). Indeed, for \( n \geq 14 \), \( C = 28 \), and \( C_1 = 80 \) we have
\[
e^{2n\log 2} \leq \int_{T} |P|^2 \, d\rho = \left( \int_{E} + \int_{T \setminus E} \right) |P|^2 \, d\rho \\
\leq \max_{E} |P|^2 + \max_{T} |P|^2 \rho(T \setminus E) \leq e^{2\Omega} 4^{42n} + 4^{28n} e^{-\Omega} \leq e^{-\frac{1}{2} \Omega},
\]
promted that \( 2 \cdot 4^{56n} < e^{\Omega} \). The latter condition holds whenever \( \Omega > 80n \). For \( n < 14 \) we just increase \( C \) and \( C_1 \).

Proof of Lemma 12. Let \( \nu \) be the equilibrium measure of the set \( E = \bigcup_{1 \leq j \leq p} I_j \), \( I_j = \{ e^{i\theta} : \alpha_j \leq \theta \leq \alpha_j' \} \), \( 1 \leq j \leq p \leq n \), and let
\[ U^\nu(e^{i\theta}) = \int_{E} \log |e^{i\theta} - e^{it}| \, d\nu(e^{it}) \]
be its logarithmic potential. We assume that the measure \( \nu \) is normalized by the condition \( \nu(E) = n \). Then
\[ U^\nu|_{E} = n \log \text{cap}(E) \]
(and is \( > n \log \text{cap}(E) \) on \( \mathbb{C} \setminus E \)). We will construct a monic polynomial \( P \) of degree \( 2N \), \( N \leq 14n \), so that \( \log |P| \leq U^\nu + (3 \log 2)N \) everywhere on \( E \).

For this purpose, we will replace the measure \( \nu \) by the sum of point masses \( \sum \delta_{s_j} \). It is well known (see, e.g., [12, Lemma 4.1] or [17, Lemma 3.5]) that \( d\nu(e^{i\theta}) = \varphi(\theta) \, d\theta \), \( e^{i\theta} \in E \), where
\[
(3) \quad \varphi(\theta) = \frac{n}{2\pi} \prod_{j=1}^{p} \frac{|e^{i\theta} - e^{i\beta_j}|}{|e^{i\theta} - e^{i\alpha_j}| \cdot |e^{i\theta} - e^{i\alpha'_j}|}
\]
with a sequence of points \( e^{i\beta_j} \) interlacing with the arcs \( I_j \). Since
\[
\varphi(\theta)^4 = \frac{n^4}{(2\pi)^4} \prod_{j=1}^{p} \frac{(e^{i\theta} - e^{i\beta_j})^2(1 - e^{i\theta - i\beta_j})^2}{(e^{i\theta} - e^{i\alpha_j})(1 - e^{i\theta - i\alpha_j})(e^{i\theta} - e^{i\alpha'_j})(1 - e^{i\theta - i\alpha'_j})}
\]
is a rational function of \( z = e^{i\theta} \) of degree \( 4p \), it has at most \( 8p - 1 \) critical points. Hence \( \varphi' \) has at most \( 8p - 1 \) zeros on \( [0, 2\pi] \). Thus we can represent \( E \) as a union of at most \( 9p - 1 + 4n - 1 \leq 13n - 2 \) arcs \( \Delta'_j \), with disjoint interiors such that \( \int_{\Delta'_j} \varphi \leq 1/4 \) and \( \varphi' \) has a constant sign on \( \Delta'_j \). After that we split the arcs \( \Delta'_j \) of length larger than or equal to \( \pi/8 \) into smaller arcs so that the length of each new arc is less than \( \pi/8 \). Finally, we get \( N \leq 14n \).
arcs $\Delta_j = \{e^{i\theta} : \gamma_j \leq \theta \leq \gamma'_j\}$ with $|\gamma'_j - \gamma_j| < \pi/8$ such that $\int_{\Delta_j} \varphi \leq 1/4$ and $\varphi'$ has a constant sign on $\Delta_j$.

Set
$$P(z) = \prod_{1 \leq j \leq N} (z - e^{i\gamma_j})(z - e^{i\gamma'_j}), \quad \deg P = 2N \leq 28n.$$ We need to show that
$$\log |P(z)| \leq U_\nu(z) + (3 \log 2)N, \quad z \in E.$$ Fix a point $z = e^{i\theta} \in \Delta_j$ at which we will check this bound. Then
$$\log |P(z)| = \log(|z - e^{i\gamma_j}| \cdot |z - e^{i\gamma'_j}|)$$
$$+ \left( \sum_{\text{dist}(z, \Delta_k) \leq \frac{1}{2}, \quad k \neq j} + \sum_{\text{dist}(z, \Delta_k) > \frac{1}{2}} \right) \log(|z - e^{i\gamma_k}| \cdot |z - e^{i\gamma'_k}|).$$ The last sum does not exceed $(\log 4)N$.

If $\text{dist}(z, \Delta_k) \leq \frac{1}{2}, \quad k \neq j$, then $\Delta_k \subset D(z, 1)$, and $\text{dist}(z, \Delta_k) = |z - e^{i\tilde{\gamma}_k}|$, where $\tilde{\gamma}_k$ is one of two points $\gamma_k, \gamma'_k$. Then, recalling that $\nu(\Delta_k) \leq 1/4 < 1$ and using monotonicity of the logarithm function, we see that
$$\log(|z - e^{i\gamma_k}| \cdot |z - e^{i\gamma'_k}|) \leq \log |z - e^{i\tilde{\gamma}_k}|$$
$$\leq \int_{\Delta_k} \log |z - e^{i\tilde{\gamma}_k}| \, d\nu(e^{it}) \leq \int_{\Delta_k} \log |z - e^{it}| \, d\nu(e^{it}).$$ Hence, letting $E_0 = \bigcup_{\text{dist}(z, \Delta_k) \leq \frac{1}{2}, \quad k \neq j} \Delta_k$, $E_1 = E \setminus (E_0 \cup \Delta_j)$, we obtain that
$$\sum_{\text{dist}(z, \Delta_k) \leq \frac{1}{2}, \quad k \neq j} \log(|z - e^{i\gamma_k}| \cdot |z - e^{i\gamma'_k}|)$$
$$\leq \int_{E_0} \log |z - e^{it}| \, d\nu(e^{it})$$
$$\leq \int_{E_0} \log |z - e^{it}| \, d\nu(e^{it})$$
$$+ \int_{E_1 \cap D(z, 1)} \log |z - e^{it}| \, d\nu(e^{it}) + (\log 2)\nu(E_1 \cap D(z, 1))$$
$$+ \int_{E_1 \setminus D(z, 1)} \log |z - e^{it}| \, d\nu(e^{it})$$
$$\leq \int_{E \setminus \Delta_j} \log |z - e^{it}| \, d\nu(e^{it}) + (\log 2)N.$$
That is,
\[ \log |P(z)| \leq \log(|z - e^{i\gamma_j}| \cdot |z - e^{i\gamma'_j}|) + \int_{E \setminus \Delta_j} \log |z - e^{it}| \, d\nu(e^{it}) + (3 \log 2) N. \]

To complete the proof of (4), it remains to show that
\[ \log(|z - e^{i\gamma_j}| \cdot |z - e^{i\gamma'_j}|) < \int_{\Delta_j} \log |z - e^{it}| \, d\nu(e^{it}). \]

To do this, we are going to prove that
\[ 4 \int_{\gamma_j}^{\gamma'_j} \varphi(t) \log \frac{1}{|t - \theta|} \, dt \leq 3 \log \frac{1}{|\theta - \gamma_j| \cdot |\theta - \gamma'_j|} + 2, \quad \gamma_j < \theta < \gamma'_j, \]
with the function \( \varphi \) defined in (3).

First, we verify that (6) yields (5). Since \( d\nu(e^{it}) = \varphi(t) \, dt \), \( \nu(\Delta_j) \leq 1/4 \), and \( \frac{1}{\pi} |\theta - t| \leq |e^{i\theta} - e^{it}| \), \( \theta, t \in \Delta_j \), estimate (6) yields
\[ 4 \int_{\Delta_j} \log \frac{1}{|z - e^{it}|} \, d\nu(e^{it}) \leq 3 \log \frac{1}{|\theta - \gamma_j| \cdot |\theta - \gamma'_j|} + \log \pi + 2, \]
where \( z = e^{i\theta} \). Furthermore, since the length of each arc \( \Delta_j \) does not exceed \( \pi/8 \), we have
\[ |\theta - \gamma_j| \cdot |\theta - \gamma'_j| \leq \frac{1}{4} (\gamma_j - \gamma'_j)^2 \leq \frac{1}{4} \left( \frac{\pi}{8} \right)^2 , \]
and then
\[ \log \frac{1}{|\theta - \gamma_j| \cdot |\theta - \gamma'_j|} \geq \log \left( 4 \cdot \left( \frac{8}{\pi} \right)^2 \right) . \]

Since \( e^2 \cdot \pi^3 < 256 \), the RHS of the last displayed formula is bigger than \( \log \pi + 2 \), which gives us
\[ 4 \int_{\Delta_j} \log \frac{1}{|z - e^{it}|} \, d\nu(e^{it}) < 4 \log \frac{1}{|\theta - \gamma_j| \cdot |\theta - \gamma'_j|} < 4 \log \frac{1}{|z - e^{i\gamma_j}| \cdot |z - e^{i\gamma'_j}|} , \]
which is (5). Thus it remains to verify (6).

Set \( \beta = \theta - \gamma_j \), \( \beta' = \gamma'_j - \theta \), and \( \psi(t) = 4 \varphi(t + \theta) \). Then \( \beta, \beta' \in (0,1) \) and \( \int_{-\beta}^{\beta'} \psi(t) \, dt \leq 1 \). We need to show that
\[ \int_{-\beta}^{\beta'} \psi(t) \log \frac{1}{|t|} \, dt \leq 3 \log \frac{1}{\beta \cdot \beta'} + 2. \]

We assume that \( \psi \) increases on \((-\beta, \beta')\), and set \( \psi_1(x) = \int_0^x \psi(t) \, dt \). Note that the function \( \psi_1 \) is convex, vanishes at the origin, and \( \psi_1(\beta') \leq 1 \), so \( 0 \leq \psi_1(x) \leq x/\beta' \)
on $[0, \beta']$ and $\psi(0) = \psi_1(0) \leq 1/\beta'$. Then, integrating by parts, we get

$$\int_0^{\beta'} \psi(t) \log \frac{1}{t} \, dt = \psi_1(\beta') \log \frac{1}{\beta'} + \int_0^{\beta'} \frac{\psi_1(t)}{t} \, dt \leq \log \frac{1}{\beta'} + 1.$$ If $\beta' < \beta$, then

$$\int_{\beta}^0 \psi(-t) \log \frac{1}{t} \, dt = \left( \int_0^{\beta'} + \int_{\beta'}^\beta \right) \psi(-t) \log \frac{1}{t} \, dt \leq \psi(0) \left( \beta' + \beta' \log \frac{1}{\beta'} \right) + \log \frac{1}{\beta'} < 2 \log \frac{1}{\beta'} + 1,$$

while for $\beta' \geq \beta$, we have

$$\int_0^\beta \psi(-t) \log \frac{1}{t} \, dt \leq \psi(0) \left( \beta + \beta \log \frac{1}{\beta} \right) \leq \log \frac{1}{\beta} + 1.$$ That is,

$$\int_{-\beta}^{\beta'} \log \frac{1}{|t|} \psi(t) \, dt \leq 3 \log \frac{1}{\beta \cdot \beta'} + 2,$$

proving (6) and completing the proof of Lemma 12.

5. Riesz products

Our last results concern a family of singular continuous measures introduced by F. Riesz and called the Riesz products. These measures have a variety of applications in harmonic analysis; see, e.g., [10, §13] and the references therein. Our attention to the Riesz products in the context of this work was attracted by a discussion of Khruschev’s work in [15, Section 2.11].

To define the Riesz products, consider a sequence of probability measures

$$d\rho_n(e^{i\theta}) = \prod_{j=0}^n (1 + \alpha_j \cos(\ell_j \theta)) \frac{d\theta}{2\pi},$$

where $-1 \leq \alpha_j \leq 1$, and $\ell_j$ are positive integers such that $\ell_{j+1} \geq 3 \ell_j$. The sequence of measures $\rho_n$ has a weak limit $\rho$ called the Riesz product. The measure $\rho$ is singular continuous iff

$$\sum_{j=0}^{\infty} \alpha_j^2 = \infty,$$

(otherwise, it is absolutely continuous).
**Theorem 13**: Let \( \rho \) be a Riesz product generated by the sequences \((\alpha_j)\) and \((\ell_j)\), and let \( N_n = \sum_{j=0}^{n} \ell_j \). Then

\[
\prod_{j=0}^{n} \frac{1}{2} \left( 1 + \sqrt{1 - \alpha_j^2} \right) \leq e_{N_n}(\rho)^2 \leq \prod_{j=0}^{n} \left( 1 - \alpha_j^2 \right).
\]

In particular, for \( \alpha_j \to 0 \), we have

\[
2 \log e_{N_n}(\rho) = -\frac{1}{4} \sum_{j=0}^{n} \alpha_j^2 + O \left( \sum_{j=0}^{n} \alpha_j^4 \right),
\]

while, for \( \alpha_j = 1, j \in \mathbb{Z}_+ \), we get

\[
-(n+1) \log 2 \leq 2 \log e_{N_n}(\rho) \leq -(n+1) \log \frac{4}{3}.
\]

**Proof of Theorem 13.** First, we note that the moments of the measures \( \rho \) and \( \rho_n \) coincide up to the order \( N_n = \sum_{j=0}^{n} \ell_j \). So the corresponding orthogonal polynomials (as well as their \( L^2(\rho) \)- and \( L^2(\rho_n) \)-norms) coincide too:

\[
Q_{N_n}(\rho) = Q_{N_n}(\rho_n) \quad \text{and} \quad e_{N_n}(\rho) = e_{N_n}(\rho_n).
\]

**Proof of the lower bound.** The proof is straightforward and uses a familiar integral

\[
\int_{-\pi}^{\pi} \log(1 + \alpha \cos \theta) \frac{d\theta}{2\pi} = \log \left( \frac{1}{2} (1 + \sqrt{1 - \alpha^2}) \right).
\]

Since the measure \( \rho_n \) has a convergent logarithmic integral, by Szegő’s theorem, for every \( k \in \mathbb{N} \), we have

\[
\log e_k(\rho_n) \geq \frac{1}{2} \int_{-\pi}^{\pi} \log \left\{ \prod_{j=0}^{n} (1 + \alpha_j \cos(\ell_j \theta)) \right\} \frac{d\theta}{2\pi}
\]

\[
= \frac{1}{2} \sum_{j=0}^{n} \log \left( \frac{1}{2} (1 + \sqrt{1 - \alpha_j^2}) \right),
\]

whence

\[
\log e_{N_n}(\rho) = \log e_{N_n}(\rho_n) \geq \frac{1}{2} \sum_{j=0}^{n} \log \left( \frac{1}{2} (1 + \sqrt{1 - \alpha_j^2}) \right),
\]

proving the lower bound. \( \blacksquare \)
Proof of the upper bound. Consider the monic polynomial

\[ P_{N_n}(z) := \prod_{j=0}^{n} (z^{l_j} - \alpha_j/2) \]

of degree \( N_n \). Then

\[ e_{N_n}^2(\rho) = e_{N_n}^2(\rho_n) \leq \|P_{N_n}\|_{L^2(\rho_n)}^2 \]

\[ = \int_{-\pi}^{\pi} \prod_{j=0}^{n} |e^{i\ell_j \theta} - \frac{1}{2} \alpha_j|^2 (1 + \alpha_j \cos(\ell_j \theta)) \frac{d\theta}{2\pi} \]

\[ = \int_{-\pi}^{\pi} \prod_{j=0}^{n} \left( 1 - \frac{1}{4} \alpha_j^2 + \frac{1}{8} \alpha_j^3 (e^{i\ell_j \theta} + e^{-i\ell_j \theta}) - \frac{1}{4} \alpha_j^2 (e^{2i\ell_j \theta} + e^{-2i\ell_j \theta}) \right) \frac{d\theta}{2\pi}. \]

Observe that due to the growth condition \( \ell_{j+1} \geq 3\ell_j \), the constant term of the product under the integral sign, and hence the whole integral on the RHS is equal to

\[ \prod_{j=0}^{n} \left( 1 - \frac{\alpha_j^2}{4} \right). \]

This completes the proof of the upper bound.  

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