Estimating and Controlling for Fairness via Sensitive Attribute Predictors

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February 17, 2023

Abstract

The responsible use of machine learning tools in real world high-stakes decision making demands that we audit and control for potential biases against underrepresented groups. This process naturally requires access to the sensitive attribute one desires to control, such as demographics, gender, or other potentially sensitive features. Unfortunately, this information is often unavailable. In this work we provide a precise characterization of when one can still reliably estimate, and ultimately control, for fairness by using proxy sensitive attributes derived from a sensitive attribute predictor. Specifically, we first provide conditions under which one can provide tight upper bounds to the worst-case fairness violation of a predictor with respect to sensitive attributes that are unobserved. Second, we demonstrate when and how one can provably control such a worst-case unfairness by a new post-processing correction method. These results characterize the cases where controlling for fairness with proxy predicted attributes is – and when is not – optimal. Our results hold under assumptions that are significantly milder than previous works, and we illustrate these results with experiments on synthetic and real datasets.

1 Introduction

Machine learning (ML) algorithms are increasingly used in high stakes prediction applications that can significantly impact society. For example, ML models have been used to detect breast cancer in mammograms [22], inform parole and sentencing decisions [10], and aid in loan approval decisions [26]. While these algorithms often demonstrate excellent overall performance, they can be dangerously unfair and negatively impact under-represented groups [23]. Some of these unforeseen negative consequences can even be fatal when considering under diagnosis biases of deep learning models in chest x-ray diagnosis of diseases, like cancer [21]. Recommendations to ensure that ML systems do not exacerbate societal biases have been raised by several groups, including the White House in a 2016 report on big data, algorithms, and civil rights [17]. It is thus critical to understand how to rigorously evaluate the fairness of ML algorithms and control for unfairness during model development.

These needs have prompted considerable research in the area of Fair ML. While definitions of algorithmic fairness abound [5], common notions of group fairness consider different error rates of a predictor across different groups: males and females, white and non-white, etc. For example, the equal opportunity criterion requires the true positive rate (TPR) be equal across both groups, while
equalized odds requires both TPR and false positive rate (FPR) to be the same across groups [14]. Obtaining predictors that are fair therefore requires enforcing these constraints on error rates across groups during model development, which can be posed as a constrained (or regularized) optimization problem [20, 27, 2, 9, 11]. Alternatively, one can devise post-processing strategies to modify a certain predictor to correct for differences in TPR and FPR [14, 12, 8, 1], or even include data pre-processing steps that ensure that unfair models could not be obtained from such data to begin with [24, 6].

Naturally, all these techniques for estimating or enforcing fairness require access to a dataset with features, $X$, responses, $Y$, and sensitive attributes, $A$. However, in many settings this is difficult or impossible, as datasets often do not include samples that have all these variables. This could be because the sensitive attribute data was withheld due to privacy concerns, which is very common with medical data due to HIPAA federal law requirements, or simply because it was deemed unnecessary to record [25, 28]. A real-world example of this is in the recent Kaggle-hosted RSNA Chest X-Ray Pneumonia Detection Challenge [18]. Even though this dataset of chest x-rays was painstakingly annotated for pneumonia disease by dozens of radiologists, it did not include sensitive attributes (e.g., age, sex, and race), precluding the evaluation of fairness of models developed as part of the challenge. In settings like this, where there is limited or no information on the sensitive attribute of interest, it is still important to be able to accurately estimate the violation of fairness constraints by a ML classifier and to be able to alleviate such biases before deploying it in a sensitive application.

This leads to the natural question, how can one assess and control the fairness of a classifier without having access to sensitive attribute data? In other words, how can we measure and control the fairness violations of a classifier for $Y$ with respect to a sensitive attribute $A$, when we have no data that jointly observes $A$ and $Y$?

1.1 Related Work

Estimating unfairness and, more importantly, developing fair predictors where there is no – or only partial – information about the sensitive attribute has only recently received increasing attention. The recent work by Zhao et al. [29] explores the perspective of employing features that are correlated with the sensitive attribute, and shows that enforcing low correlation with such “fairness related features” can lead to models with lower bias. Although these techniques are promising, they are at the mercy of the presence of meaningful correlations, which one cannot guarantee. An appealing alternative to simply exploring correlations of features with the sensitive attribute is to construct proxy sensitive attributes by means of a second predictor that is trained on a different dataset containing sensitive attribute information [13, 16, 7, 4]. While this can represent a practical solution to the problem above, this must be done with care, as this opens new problems for estimating and controlling for fairness. The work by Prost et al. [19] considers the estimation of fairness in a setting where one develops a predictor for an unobserved covariate, but it does not contemplate predicting the sensitive attribute itself. On the other hand Chen et al. [7] study the sources of error in the estimation of fairness via predicted proxies computed using threshold functions, which are prone to over-estimation.

The closest to our work are the recent results by Kallus et al. [16], and Awasthi et al. [4, 3]. Awasthi et al. [4] make progress in understanding properties of the sensitive attribute classifier, $\hat{A}$, that are desirable for accurate fairness violation estimation of label classifier $\hat{Y}$. Assuming $\hat{Y} \perp \perp \hat{A} \mid (A,Y)$, they demonstrate the counter-intuitive result that given a fixed error budget for the sensitive attribute classifier, the optimal attribute classifier for accurate fairness violation estimation is one that has the most unequal distribution of errors across the subgroups of the sensitive attribute. The result, while interesting, makes a rather strict assumption that is difficult to achieve and furthermore impossible to test for as it assumes one has access to data over $(A,Y)$. As a result, they drop the assumption and present closed form expressions for errors in the estimation of bias. Unfortunately the expressions depend on quantities that cannot be calculated without some data over the entire joint distribution $(X,A,Y)$. 


To this end, Kallus et al. [16] study the identifiability of fairness violations under general assumptions on the distribution and classifiers. They show that, in general, one can rarely obtain point estimates of the fairness violations of a predictor unless there is some common observed data over $A$ and $Y$. Instead, they provide upper and lower bounds of fairness violations under the assumption that one has two datasets, one that is drawn from the marginal over $(X, A)$ and the other drawn from the marginal over $(X, Y, \hat{Y})$ where $\hat{Y}$ are predictions of $Y$ that are already present in the data. In this setting, they are able to provide bounds that depend on the capability of $X$ to predict $A$ or the joint $(Y, \hat{Y})$. Importantly, this results does not allow the modification of the predictor $\hat{Y}$ and, as a result, one cannot hope to employ these bounds to correct for fairness. Neither of the works above study how to provably control for fairness in setting where there is no data that observes both $A$ and $Y$. The work by Awasthi et al. [3] partially addresses this point by studying the control of fairness on data with perturbed sensitive attributes, proving that doing so can indeed be fruitful for optimizing fairness albeit under rather strict assumptions.

Overall, while progress has been made in understanding how to estimate bias in the presence of incomplete sensitive attribute information, these previous results have been limited to simple settings (e.g., having access to some data from the entire distribution, or by making strong assumptions of conditional independence). Importantly, it is still unknown whether useful results can be obtained without access to any sampled data over both $A$ and $Y$, and it is unclear how the sensitive attribute predictors can be deployed in a way that provably enforces fairness constraints with respect to the (unobserved) sensitive attribute.

1.2 Contributions

The contributions of our work can be summarized as follows:

- We demonstrate that, by using proxy sensitive attributes derived from a sensitive attribute predictor, one can obtain computable tight bounds on the worst-case fairness violations of a classifier with respect to true but unobserved sensitive attributes.

- We provide a precise characterization of the classifiers that achieve minimal worst-case fairness violations with respect to unobserved sensitive attributes. Through this characterization, we demonstrate when correcting for fairness with respect to the proxy sensitive attributes can be beneficial, and when instead it can be sub-optimal.

- We provide a simple and practical post-processing technique that yields classifiers that maximize prediction power while achieving optimal minimal worst-case fairness violations with respect to unobserved sensitive attributes.

- We illustrate our results on a series of simulated and real data of increasing complexity.

2 Problem Setting

We work within a binary classification setting and consider a distribution $Q$ over $(\mathcal{X} \times A \times \mathcal{Y})$ where $\mathcal{X} \subseteq \mathbb{R}^n$ is the feature space, $\mathcal{Y} \in \{0, 1\}$ the label space, and $A \in \{0, 1\}$ the sensitive attribute space. Furthermore, and adopting the setting of [4], we consider 2 datasets, $D_1$ and $D_2$. The former is drawn from the marginal over $(\mathcal{X}, A)$ of $Q$ while $D_2$ is drawn from the marginal $(\mathcal{X}, \mathcal{Y})$ of $Q$. In this way, $D_1$ and $D_2$ contain the same set of features, $D_1$ contains sensitive attribute information and $D_2$ contains label information. The drawn samples in $D_1$ and $D_2$ are i.i.d over their respective marginals, and thus different from one another.

Similar to previous work [7, 19, 4], we place ourselves in a demographically scarce regime where there is a designer who has access to $D_1$ to train a sensitive attribute predictor $h : \mathcal{X} \rightarrow A$ and us, who have access to $D_2$ and are given the sensitive attribute classifier $h$ and all computable
We denote \( U \). These quantities lead to a natural corresponding definition of an estimate of \( \hat{A} \) that can be extracted from \( D_1 \). In this setting, our goal is to develop a classifier \( f : X \rightarrow Y \) (from \( D_2 \)) that is fair with respect to \( A \) utilizing \( \hat{A} \). The central idea is to augment every sample in \( D_2 \), \( (x_i, y_i) \), by \( (x_i, y_i, \hat{a}_i) \), where \( \hat{a}_i = h(x_i) \). Intuitively, if the error of the sensitive attribute predictor, denoted herein by \( U = \mathbb{P}(h(X) \neq A) \), is low, we could hope that fairness with respect to the real (albeit unobserved) sensitive attribute can be faithfully estimated. Our goal is to thus estimate the error incurred in measuring and enforcing fairness constraints by means of \( \hat{A} = h(X) \), and potentially alleviate or control for it.

Throughout this work we will focus on equalized odds (EO) as a fairness metric [14], which is one of the most popular fairness metrics and can always be relaxed to consider only equal opportunity. We denote \( \hat{Y} = f(X) \) for simplicity, and for \( i, j \in \{0, 1\} \) define the group conditional probabilities

\[
\alpha_{i,k} = \mathbb{P}_{(X,A,Y)}(\hat{Y} = 1 \mid A = i, Y = j).
\]  

(1)

These probabilities quantify the TPR (when \( j = 1 \)) and FPR (when \( j = 0 \)), for either protected group \( (i = 0 \) or \( i = 1 \). We assume that the rates, \( r_{i,j} = \mathbb{P}(A = i, Y = j) > 0 \) so that these quantities are not undefined. When clear from context, we will omit the variables over which the probability is computed and simply write \( \alpha_{i,j} = \mathbb{P}(\hat{Y} = 1 \mid A = i, Y = j) \). With these conditionals probabilities, we define the true fairness violation of \( f \), \( \Delta(f) \), as the tuple \( \Delta(f) = (\Delta_{\text{TPR}}(f), \Delta_{\text{FPR}}(f)) \), where

\[
\Delta_{\text{TPR}}(f) = \alpha_{1,1} - \alpha_{0,1} \\
\Delta_{\text{FPR}}(f) = \alpha_{1,0} - \alpha_{0,0}.
\]  

(2)

In words, the quantities \( \Delta_{\text{TPR}}(f) \) and \( \Delta_{\text{FPR}}(f) \), respectively quantify the difference in TPR and FPRs among the two protected groups. Throughout this work, we will use \( \Delta(f) \) to refer to both of these quantities simultaneously.

We will also need to characterize the performance of the sensitive attribute classifier, \( h \), via its total error \( U \), which can be decomposed as

\[
U = \mathbb{P}(h(X) \neq A) = U_0 + U_1,
\]  

(3)

where \( U_i = \mathbb{P}(h = i, A \neq i) \), for \( i \in \{0, 1\} \). In a demographically scarce regime, the rates \( r_{i,j} \), and more importantly the quantities of interest, \( \Delta_{\text{TPR}}(f) \) and \( \Delta_{\text{FPR}}(f) \), cannot be computed because samples from \( A \) and \( Y \) are not jointly observed. However, using the sensitive attribute classifier \( h \), we can predict \( \hat{A} \) on \( D_2 \) and instead compute the estimated rates \( \hat{r}_{i,j} = \mathbb{P}(\hat{A} = i, Y = j) \) and the group TPRs and FPRs as

\[
\hat{\alpha}_{i,j} = \mathbb{P}_{(X,\hat{A},Y)}(\hat{Y} = 1 \mid \hat{A} = i, Y = j).
\]  

(4)

These quantities lead to a natural corresponding definition of an estimate of \( \Delta(f) \), denoted as the tuple \( \hat{\Delta}(f) = (\hat{\Delta}_{\text{TPR}}(f), \hat{\Delta}_{\text{FPR}}(f)) \) where

\[
\hat{\Delta}_{\text{TPR}}(f) = \hat{\alpha}_{1,1} - \hat{\alpha}_{1,0} \\
\hat{\Delta}_{\text{FPR}}(f) = \hat{\alpha}_{0,1} - \hat{\alpha}_{0,0}.
\]  

(5)

providing an estimated version of \( \Delta(f) \) with \( \hat{A} \) in lieu of \( A \).

3 Theoretical Results

3.1 Bounding Fairness Violations with Proxy Sensitive Attributes

In this setting, we now present our first result that characterizes the worst-case fairness violation of \( f \) with respect to the true, but unobserved, sensitive attribute \( A \). Importantly, as we will shortly
explain, these guarantees will provide insight into what properties \( f \) must satisfy to have minimal worst-case fairness violations. In turn, these results will lead to a simple post-processing correction method that can correct a pretrained classifier \( f \) into another one, \( \hat{f} \), that has minimal worst-case fairness violations.

Before presenting our results, we first provide the conditions that the classifiers \( h \) and \( f \) should satisfy to achieve non-trivial worst-case (true) fairness violations of \( f \).

**Assumption 1.** For \( i, j \in \{0, 1\} \), the classifiers \( f \) and \( h \) satisfy

\[
\frac{U_i}{\hat{r}_{i,j}} \leq \hat{\alpha}_{i,j} \leq 1 - \frac{U_i}{\hat{r}_{i,j}} \tag{6}
\]

This assumption simply formalizes the natural assumption that the predictive power of \( h \) to predict the sensitive attributes, \( A \), must be better relative to ability of \( f \) to predict the labels, \( Y \). With this in place, we present our main result.

**Theorem 1 (Bounds on \( \Delta(f) \)).** Under Assumption 1, we have that

\[
|\Delta_{\text{TPR}}(f)| \leq B_{\text{TPR}}(f) \overset{\Delta}{=} \max\{|B_1 + C_{0,1}|, |B_1 - C_{1,1}|\}
\]

\[
|\Delta_{\text{FPR}}(f)| \leq B_{\text{FPR}}(f) \overset{\Delta}{=} \max\{|B_0 + C_{0,0}|, |B_0 - C_{1,0}|\}
\]

where

\[
B_j = \frac{\hat{r}_{1,j}}{\hat{r}_{1,j} + U_0 - U_1} \hat{\alpha}_{1,j} - \frac{\hat{r}_{0,j}}{\hat{r}_{0,j} + U_0 - U_1} \hat{\alpha}_{0,j}
\]

\[
C_{i,j} = U_i \left( \frac{1}{\hat{r}_{1,j} + U_0 - U_1} + \frac{1}{\hat{r}_{0,j} + U_0 - U_1} \right).
\]

Furthermore, the upper bounds for \( |\Delta_{\text{TPR}}(f)| \) and \( |\Delta_{\text{FPR}}(f)| \) are tight.

The proof, along with all others in this work, are included in Appendix A.1. We pause here to make a few remarks on this result. First, the bound is tight for settings that depend on the marginal distributions and in these settings one cannot hope to improve the bound further. Second, this result demonstrates that a user in a demographically scarce regime can calculate \( B_{\text{TPR}}(f) \) and \( B_{\text{FPR}}(f) \), the worst-case fairness violations of \( f \), because these quantities are determined by \( \hat{r}_{i,j}, \hat{\alpha}_{i,j} \), and \( U_i \), all of which are all computable. Furthermore, the values of \( \hat{r}_{i,j} \) and \( U_i \) are constant because they rely on the sensitive attribute predictor \( h \) which we cannot modify. Notice that the bounds, however, are linear in the \( \hat{\alpha}_{i,j} \), which we can in fact control as they depend on \( \hat{Y} \). This will prove to be very fruitful moving forward. Third, this result is useful in understanding the specifications needed to develop the sensitive attribute classifier, \( h \); in particular, how much error, \( U = U_0 + U_1 \), one can afford. If the sensitive attribute predictor has low error (small \( U_0 \) and \( U_1 \)), then the true values of \( |\Delta_{\text{TPR}}(f)| \) and \( |\Delta_{\text{FPR}}(f)| \) are approximately bounded by their estimates, \( |\hat{\Delta}_{\text{TPR}}(f)| \) and \( |\hat{\Delta}_{\text{FPR}}(f)| \). Furthermore, if \( B_{\text{TPR}}(f) \) and \( B_{\text{FPR}}(f) \) are low, then as developers of \( f \) we can proceed while having a guarantee on the maximal fairness violation by \( f \) towards a group in \( \mathcal{A} \), even while not observing \( \Delta(f) \). On the other hand, if these bounds are large, this implies a potentially large true bias over the protected group \( A \) by \( f \), and the user should not proceed in deploying the developed model and instead seek other classifier with smaller bounds.

### 3.2 Optimal Worst-Case Fairness Violations

The result above naturally raises the question, what properties should classifiers \( f \) satisfy such that \( B_{\text{TPR}}(f) \) and \( B_{\text{FPR}}(f) \) are minimal? One might intuitively think that classifiers \( f \) that are fair with respect to \( \mathcal{A} \), i.e. the ones satisfying \( \hat{\Delta}_{\text{TPR}}(f) = \hat{\Delta}_{\text{FPR}}(f) = 0 \), might more fair. Yet, since \( B_{\text{TPR}}(f) \)
and $B_{FPR}(f)$ depend on the values $\hat{\alpha}_{i,j}$, are the classifiers $f$ that are fair with respect to $\hat{A}$, the ones that have smallest $B_{TPR}(f)$ and $B_{FPR}(f)$? We now answer this question in the following theorem by considering two classifiers, $f$ and $\bar{f}$, and show what conditions $\bar{f}$ must satisfy so that its worst-case fairness violations are lower than those of $f$.

**Theorem 2** (Minimizers of $B_{TPR}(f)$ and $B_{FPR}(f)$). Let $\bar{f}$ be a classifier with group conditional probabilities, $\hat{\alpha}_{i,j} = P(\bar{Y} = 1 \mid \bar{A} = i, Y = j)$, and corresponding bounds, $B_{TPR}(\bar{f})$ and $B_{FPR}(\bar{f})$. Furthermore, suppose $\bar{f}$ satisfies the following condition.

$$\frac{\hat{\alpha}_{i,j} - \hat{\alpha}_{0,j}}{\hat{\alpha}_{0,j} + U_1 - U_0 \hat{\alpha}_{0,j} - \hat{\alpha}_{1,j}} + \frac{\hat{\alpha}_{i,j} - \hat{\alpha}_{0,j}}{\hat{\alpha}_{0,j} + U_1 - U_0 \hat{\alpha}_{1,j}} \leq \frac{U_0 - U_1}{2} \left( \frac{1}{\hat{\alpha}_{0,j} + U_0 - U_1} + \frac{1}{\hat{\alpha}_{1,j} + U_1 - U_0} \right).$$

(8)

Denote $f$ to be any other classifier for labels $Y$ with corresponding $B_{TPR}(f)$ and $B_{FPR}(f)$, which does not satisfy the equation above. Then, we have that

$$|\Delta_{TPR}(\bar{f})| \leq B_{TPR}(\bar{f}) < B_{TPR}(f)$$

$$|\Delta_{FPR}(\bar{f})| \leq B_{FPR}(\bar{f}) < B_{FPR}(f).$$

(9)

Furthermore, there exist classifiers that satisfy the condition and Assumption 1.

#### 3.3 Controlling Fairness Violations with Proxy Sensitive Attributes

Now that we understand what conditions $f$ must satisfy so that its worst case fairness violations are minimal, what remains is a method to obtain such a classifier. We take inspiration from the post-processing method proposed by Hardt et al. [14], which derives a classifier $\hat{Y} = \hat{f}(X)$ from $\hat{Y} = \hat{f}(X)$ that satisfies equalized odds with respect to a sensitive attribute $A$ minimizing a an expected misclassification loss – *only applicable if one has access to $A$, which is not true in our setting*. Nonetheless, since ours will be a generalization of this idea, we take a couple of paragraphs to review the of Hardt et al. [14]. The method they propose works as follows: given a sample with initial prediction $\hat{Y} = \hat{y}$ and sensitive attribute $A = a$, the derived predictor $\hat{f}$, with group conditional probabilities, $\alpha_{i,j} = P(\hat{Y} = 1 \mid A = i, Y = j)$, predicts $\hat{Y} = 1$ with probability $p_{a,\hat{y}} = P(\hat{Y} = 1 \mid A = a, \hat{Y} = \hat{y})$. The four probabilities $p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}$ can be then calculated so that $\hat{Y}$ satisfies equalized odds and the expected loss between $\hat{Y}$ and labels $Y$, $E[L(\hat{Y}, Y)]$, is minimized. The fairness constraint, along with the objective to minimize the expected loss, give rise to the linear program:

**Equalized Odds Post-Processing** [14]

$$\min_{p_{a,\hat{y}} \in [0,1]} E[L(\hat{Y}, Y)]$$

subject to $\alpha_{0,j} = \alpha_{1,j} \quad j \in \{0,1\}.$

(10)

This is in fact a linear program because since $\hat{Y}$ is derived, then $\hat{Y} \perp \perp Y \mid (A, \hat{Y})$, and thus $E[L(\hat{Y}, Y)]$ and $\alpha_{i,j} = (1 - \alpha_{i,j})p_{i,0} + \alpha_{i,j}p_{i,1}$ are linear with respect to $p_{a,\hat{y}}$.

Returning to our setting, where we do not have access to $A$ but only proxy variables $\hat{A} = h(X)$, we seek classifiers $f$ that are fair with respect to the unknown attribute $A$. Since these are not
Worst-case Fairness Violation Reduction

We begin with a synthetic example that will allow us to showcase different aspects of our results. The data is constructed from 3 features, sensitive attribute \( A \) is modeled as \( \mathcal{N}(\mu, \Sigma) \), where

\[
\mu = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 1 & 0.05 & 0 \\ 0.05 & 1 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}.
\]

The sensitive attribute, \( A \), response \( Y \), and classifier \( f \), are modeled as

\[
\begin{align*}
A &= \mathbb{I}(X_3 + 0.1 \geq 0), \\
Y &= \mathbb{I}[S(X_1 + X_2 + X_3 + c_0(1 - A) + c_1 A) \geq 0.35], \\
f(X; c_1, c_2) &= \mathbb{I}(S(c_1 X_1 + c_2 X_2 + c_3 X_3) \geq 0.35)
\end{align*}
\]

4 Experimental Results

4.1 Synthetic Data

We begin with a synthetic example that will allow us to showcase different aspects of our results. The data is constructed from 3 features, \( X_1, X_2, X_3 \in \mathbb{R} \), sensitive attribute \( A \in \{0, 1\} \), and response \( Y \in \{0, 1\} \). The features are sampled from \( (X_1, X_2, X_3) \sim \mathcal{N}(\mu, \Sigma) \), where
We model the sensitive attribute predictor as
\[
\delta = c
\]
where \( \delta \) is constant. We choose \( c \) so that the sensitive attribute classifier still has the same total error of \( U \approx 0.04 \), however, the errors are distributed unevenly, \( U \approx 0.04 \) and \( U_0 = 0 \). As in the previous experiment we generate classifiers \( f \), and perform the the same correction algorithms

\[\text{expected loss} \approx \frac{1}{2} J_f - 1 \]

where \( J_f \) is the well known Youden's Index.  

\[\text{expected loss} \approx \frac{1}{2} J_f - 1 \]

Indeed, one can show that \( E[L(f, Y)] = -\frac{1}{2}(J_f - 1) \) where \( Y_f \) is the well known Youden’s Index of \( f \).
4.2 Real World Data

We now move to a real and important problem on assisted diagnosis on medical images. In particular, we employ the CheXpert dataset [15]. CheXpert is a large public dataset for chest radiograph interpretation, consisting of 224,316 chest radiographs of 65,240 patients, with labeled annotations for 14 observations (positive, negative, or unlabeled) including cardiomegaly, atelectasis, edema, consolidation, and several others. Each image is also accompanied by the sensitive attribute sex.

We consider a binary classification task in which we aim to learn a classifier \( f \) to predict if an image any annotation for any condition \( Y = 1 \) or does not \( Y = 0 \). We then to wish measure and correct for any fairness violations that \( f \) may exhibit towards the sex attribute assuming we do not have the true sex attribute at the time of measurement and correction. Instead, since the data contains the sex attribute, we aim to learn a sensitive attribute predictor, \( h \), on a withheld subset of the data. To learn both \( f \) and \( h \) we use a DenseNet121 convolutional neural network architecture.

Images are fed into the network with size 320 × 320 pixels. We use the Adam optimizer with default \( \beta \)-parameters of \( \beta_1 = 0.9, \beta_2 = 0.999 \) and learning rate \( 1 \times 10^{-4} \) which is fixed for the duration of the training. Batches are sampled using a fixed batch size of 16 images and we train for 5 epochs. The sex predictor, \( h \), achieves an error of \( U = 0.023 \) with \( U_1 \approx 0.015 \) and \( U_0 \approx 0.008 \). On a separate subset of the data, we generate our predictions \( \tilde{Y} = f' \) and \( \tilde{A} = h \) to yield a dataset over \( (A, Y, \tilde{Y}) \). We utilize the bootstrap method to obtain to generate 500 samples of from this dataset and for each sample, perform the same correction algorithms as before to yield \( f_{\text{fair}} \) and \( f_{\text{opt}} \) and calculate the same metrics as done in the previous experiments.

The results in Fig. 3 show that our proposed correction method performs the best in reducing \( B_{\text{TPR}} \) and \( B_{\text{FPR}} \). Even though \( U \) is very small, since \( U_1 \) is approximately 2 times \( U_0 \), simply correcting for fairness with respect to \( \tilde{A} \) is not the best method in reducing the worst-case fairness violations. In particular, the results in Fig. 3a are noteworthy, as they depict how our proposed correction method, and our bounds, allow the user to certify that the obtained classifier has a fairness violation in TPRs of no more than 0.06, without having access to the true sensitive attributes.
Moreover, the improvement is significant, since before the correction one had $|\Delta \text{TPR}| \lesssim 0.10$. In a high-stakes decision setting, such as this one where the model $f$ could be used to aid in diagnosis, this knowledge could be vital. Naturally, the expected loss is highest for $f_{\text{opt}}$ but that the increase is minimal. We make no claim as to whether this (small) increases in loss are reasonable for this particular problem setting, and the precise trade-offs must be defined in the context of a broader discussion involving policy makers, domain experts and other stakeholders.

5 Discussion and Conclusion

In this paper we address the problem of estimating and controlling any potential fairness violations towards an unobserved sensitive attribute by means of predicted proxy ones. We have shown that one under mild assumptions (easily satisfied in practice, as demonstrated), the worst-case fairness violations, $B_{\text{TPR}}$ and $B_{\text{FPR}}$ have simple closed form solutions that are linear in the estimated group conditional probabilities $\hat{\alpha}_{i,j}$. Furthermore, we give a exact characterization of the properties that a classifier $f$ must have so that $B_{\text{TPR}}$ and $B_{\text{FPR}}$ are indeed minimal. Our characterization demonstrates that, even when the proxy sensitive attributes are highly accurate, simply correcting for fairness with respect to these proxy attributes is not optimal in regards to minimizing the worst-case fairness violations. To this end, we propose a simple post-processing method that can correct a pretrained classifier $f$ to yield an optimally-corrected classifier, $\bar{f}$, with minimal worst-case fairness violations.

Our experiments on both synthetic and real data illustrate our theoretical findings. We show how, even if the proxy sensitive attributes are highly accurate, the smallest imbalance in $U_0$ and $U_1$ renders the naïve correction for fairness with respect to the proxy attributes suboptimal. More importantly, our experiments highlight our general method’s ability effectively control for the worst-case fairness violation of a classifier with minimal decrease in the classifier overall predictive power. On a final observation on our empirical results, the reader might be tempted to believe that the classifier $f_{\text{fair}}$ is better because it has a lower “true” fairness than that of $f_{\text{opt}}$. Unfortunately, these true fairness violations are not identifiable, and all one can hope for is to compute the provided upper bounds, which $f_{\text{opt}}$ minimizes.

The central limitation of our results is Assumption 1. This assumption is indeed met for accurate proxy sensitive attributes (as in the cases illustrated above in chest X-rays) but sometimes this may not be possible. In these scenarios one would still like to know if its possible to control for worst-case fairness violations. We conjecture that one can do away with this assumption and consider less accurate proxy sensitive attributes while resulting in the fairness violations being no longer linear in the group conditional probabilities. As a result, the characterization of the classifiers with minimal worst-case bounds in this setting may be more involved and, as a result, the manner in which one would minimize these violations may prove to be more difficult. Furthermore, while we propose a
simple post-processing correction method that takes a pretrained classifier and yields a corrected
one with minimal worst-case fairness violations, we would like to explore how one could train a
classifier – from scratch – to have minimal violations via the use of constrained optimization. Lastly,
in our setting we assume the sensitive attribute predictor and label classifier are trained on marginal
distributions from the same joint distribution. As a next step, it would be important to understand
how these results extend to settings where these two predictors are trained on marginal distributions
that come from (slightly) different joint distributions. All of this constitutes matter of future work.

Finally, we do not anticipate any negative societal impacts from this work. On the contrary,
our contribution aims to provide better and more rigorous control over potential negative societal
impacts that arise from unfair machine learning algorithms.

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A Appendix

A.1 Proofs

A.1.1 Proof of Theorem 1

We prove the result for \(|\Delta_{\text{TPR}}(f)|\). By the using the rules of conditional probability and the law of total probability, observe

\[
\alpha_{1,1} = \Pr(\hat{Y} = 1 \mid A = 1, Y = 1) = \frac{\Pr(\hat{Y} = 1, A = 1, Y = 1)}{\Pr(A = 1, Y = 1)} \quad (13)
\]

\[
= \frac{\sum_{i \in \{0,1\}} \Pr(\hat{Y} = 1, A = 1, Y = 1, \hat{A} = i)}{\sum_{j \in \{0,1\}} \sum_{i \in \{0,1\}} \Pr(\hat{Y} = j, A = 1, Y = 1, \hat{A} = i)} \quad (15)
\]

\[
= \frac{\sum_{i \in \{0,1\}} \Pr(\hat{Y} = 1, A = 1, Y = 1 \mid \hat{A} = i) \cdot \Pr(\hat{A} = i)}{\sum_{j \in \{0,1\}} \sum_{i \in \{0,1\}} \Pr(\hat{Y} = j, A = 1, Y = 1 \mid \hat{A} = i) \cdot \Pr(\hat{A} = i)} \quad (16)
\]

Similarly,

\[
\alpha_{0,1} = \Pr(\hat{Y} = 1 \mid A = 0, Y = 1) = \frac{\Pr(\hat{Y} = 1, A = 0, Y = 1)}{\Pr(A = 0, Y = 1)} \quad (17)
\]

\[
= \frac{\Pr(\hat{Y} = 1, A = 0, Y = 1) - \Pr(\hat{Y} = 1, A = 1, Y = 1)}{\Pr(A = 0, Y = 1) - \Pr(A = 1, Y = 1)} \quad (19)
\]

\[
= \frac{\Pr(\hat{Y} = 1, Y = 1) - \sum_{i \in \{0,1\}} \Pr(\hat{Y} = 1, A = 1, Y = 1 \mid \hat{A} = i) \cdot \Pr(\hat{A} = i)}{\Pr(Y = 1) - \sum_{j \in \{0,1\}} \sum_{i \in \{0,1\}} \Pr(\hat{Y} = j, A = 1, Y = 1 \mid \hat{A} = i) \cdot \Pr(\hat{A} = i)} \quad (20)
\]

Therefore, it is easily seen that \(\Delta_{\text{TPR}}(f) = \alpha_{1,1} - \alpha_{0,1}\) is a function of the 4 probabilities

\[
\Pr(\hat{Y} = j, A = 1, Y = 1, \hat{A} = i) \quad (21)
\]

that we cannot compute. The Fréchet inequalities tell us

\[
\Pr(\hat{Y} = j, A = 1, Y = 1, \mid \hat{A} = i) \geq \max\{\Pr(\hat{Y} = j, Y = 1, \mid \hat{A} = i) + \Pr(A = 1 \mid \hat{A} = i) - 1, 0\} \quad (22)
\]

\[
\Pr(\hat{Y} = j, A = 1, Y = 1, \mid \hat{A} = i) \leq \min\{\Pr(\hat{Y} = j, Y = 1, \mid \hat{A} = i), \Pr(A = 1 \mid \hat{A} = i)\} \quad (23)
\]

for \(i, j \in \{0,1\}\). Furthermore, it is easy to see that \(\Delta_{\text{TPR}}(f)\) is an increasing function as the 2 probabilities \(\Pr(\hat{Y} = 1, A = 1, Y = 1, \mid \hat{A} = a)\) increase and as the 2 probabilities, \(\Pr(\hat{Y} = 0, A = 1, Y = 1, \mid \hat{A} = a)\), decrease. As a result, \(\Delta_{\text{TPR}}(f)\) is maximal when \(\Pr(Y = 1, A = 1, Y = 1, \mid \hat{A} = a)\) achieve their maximum values and \(\Pr(Y = 0, A = 1, Y = 1, \mid \hat{A} = a)\) achieve their minimum values. On the other hand, \(\Delta_{\text{TPR}}(f)\) is minimal when \(\Pr(Y = 1, A = 1, Y = 1, \mid \hat{A} = a)\) achieve their minimum values and \(\Pr(Y = 0, A = 1, Y = 1, \mid \hat{A} = a)\) achieve their maximum values. We provide the upper bound. Recall that Assumption 1 tells us

\[
\frac{U_i}{\tilde{r}_{i,j}} \leq \hat{\alpha}_{i,j} \leq 1 - \frac{U_i}{\tilde{r}_{i,j}} \quad (24)
\]
Now, we will provide the minimal values of \( P(\hat{Y} = 0, A = 1, Y = 1, | \hat{A} = a) \) under these assumptions. First,

\[
P(\hat{Y} = 0, Y = 1, | \hat{A} = 1) + P(A = 1 | \hat{A} = 1) - 1 = \frac{P(\hat{Y} = 0 | \hat{A} = 1, Y = 1) \hat{r}_{1,1}}{P(\hat{A} = 1)} - \frac{U_1}{P(\hat{A} = 1)} \tag{25}
\]

\[
= \frac{(1 - \hat{\alpha}_{1,1}) \hat{r}_{1,1}}{P(\hat{A} = 1)} - \frac{U_1}{P(\hat{A} = 1)} \tag{26}
\]

\[
\geq \frac{U_1}{P(\hat{A} = 1)} - \frac{U_1}{P(\hat{A} = 1)} = 0 \tag{27}
\]

and second,

\[
P(\hat{Y} = 0, Y = 1, | \hat{A} = 0) + P(A = 1 | \hat{A} = 0) - 1 = \frac{P(\hat{Y} = 0 | \hat{A} = 0, Y = 1) \hat{r}_{0,1}}{P(\hat{A} = 0)} + \frac{U_0}{P(\hat{A} = 0)} - 1 \tag{28}
\]

\[
= \frac{(1 - \hat{\alpha}_{0,1}) \hat{r}_{0,1}}{P(\hat{A} = 0)} + \frac{U_0}{P(\hat{A} = 0)} - 1 \tag{29}
\]

\[
\leq \frac{\hat{r}_{0,1} - U_0}{P(\hat{A} = 0)} + \frac{U_0}{P(\hat{A} = 0)} - 1 \leq 0 \tag{30}
\]

Therefore we have that,

\[
P(\hat{Y} = 0, A = 1, Y = 1, | \hat{A} = 1) \geq \max\{P(\hat{Y} = 0, Y = 1, | \hat{A} = 1) + P(A = 1 | \hat{A} = 1) - 1, 0\} \tag{31}
\]

\[
= P(\hat{Y} = 0, Y = 1, | \hat{A} = 1) + P(A = 1 | \hat{A} = 1) - 1 \tag{32}
\]

\[
P(\hat{Y} = 0, A = 1, Y = 1, | \hat{A} = 0) \geq \max\{P(\hat{Y} = 0, Y = 1, | \hat{A} = 0) + P(A = 1 | \hat{A} = 0) - 1, 0\} \tag{33}
\]

\[
= 0 \tag{34}
\]

Now we provide the maximal values of \( P(\hat{Y} = 1, A = 1, Y = 1, | \hat{A} = a) \). With a slight modification, the assumptions tell us \( \hat{\alpha}_{1,1} \hat{r}_{1,1} \leq \hat{r}_{1,1} - U_1 \) and so

\[
\hat{\alpha}_{1,1} \hat{r}_{1,1} \leq P(\hat{A} = 1) - U_1 \tag{35}
\]

Therefore, we have

\[
P(\hat{Y} = 1, Y = 1, | \hat{A} = 1) = \frac{\hat{\alpha}_{1,1} \hat{r}_{1,1}}{P(\hat{A} = 1)} \tag{36}
\]

\[
\leq \frac{P(\hat{A} = 1) - U_1}{P(\hat{A} = 1)} \tag{37}
\]

\[
= \frac{P(\hat{A} = 1, A = 1)}{P(\hat{A} = 1)} = P(A = 1 | \hat{A} = 1) \tag{38}
\]

Also, by the assumptions we have

\[
P(\hat{Y} = 1, Y = 1, | \hat{A} = 0) = \frac{\hat{\alpha}_{0,1} \cdot \hat{r}_{0,1}}{P(\hat{A} = 0)} \tag{39}
\]

\[
= P(A = 1 | \hat{A} = 0) \tag{40}
\]

Therefore we have that

\[
P(\hat{Y} = 1, A = 1, Y = 1, | \hat{A} = 1) \leq \min\{P(\hat{Y} = 1, Y = 1, | \hat{A} = 1), P(A = 1 | \hat{A} = 1)\} \tag{41}
\]

\[
= P(\hat{Y} = 1, Y = 1, | \hat{A} = 1) \tag{42}
\]

\[
P(\hat{Y} = 1, A = 1, Y = 1, | \hat{A} = 0) \leq \min\{P(\hat{Y} = 1, Y = 1, | \hat{A} = 0), P(A = 1 | \hat{A} = 0)\} \tag{43}
\]

\[
= P(A = 1 | \hat{A} = 0) \tag{44}
\]
We prove the result for \( \Delta_{TPR} \) simply states that all meaning functions that satisfy simply occurs where

\[
B_1 + C_{0,1} = \frac{\hat{r}_{1,1}}{\hat{r}_{0,1} + U_0 - U_1} \hat{\alpha}_{1,1} + \frac{\hat{r}_{0,1}}{\hat{r}_{0,1} + U_1 - U_0} \hat{\alpha}_{0,1} + U_0 \left( \frac{1}{\hat{r}_{1,1} + U_0 - U_1} + \frac{1}{\hat{r}_{0,1} + U_1 - U_0} \right)
\]

One can similarly use the assumptions to derive the lower bound,

\[
B_1 - C_{1,1} = \frac{\hat{r}_{1,1}}{\hat{r}_{0,1} + U_0 - U_1} \hat{\alpha}_{1,1} - \frac{\hat{r}_{0,1}}{\hat{r}_{0,1} + U_1 - U_0} \hat{\alpha}_{0,1} - U_1 \left( \frac{1}{\hat{r}_{1,1} + U_0 - U_1} + \frac{1}{\hat{r}_{0,1} + U_1 - U_0} \right)
\]

and thus \( |\Delta_{TPR}| \leq \max\{|B_1 + C_{0,1}|, |B_1 - C_{1,1}|\} \). One can use same arguments to derive the the bound for \( |\Delta_{FPR}| \).

### A.1.2 Proof of Theorem 1

We prove the result for \( |\Delta_{TPR}(f)| \). Theorem 1, tells us

\[
|\Delta_{TPR}(f)| \leq B_{TPR}(f) \overset{\Delta}{=} \max\{|B_1 + C_{0,1}|, |B_1 - C_{1,1}|\}
\]

Not that \( B_1 \) is linear in \( \hat{\alpha}_{1,1} \) and \( \hat{\alpha}_{0,1} \) and that \( C_{0,1} \) and \( C_{1,1} \) are constants such that \( B_1 + C_{0,1} \geq B_1 - C_{1,1} \) simply because \( B_1 + C_{0,1} \) is the upper bound for \( \Delta_{TPR} \) and \( B_1 - C_{0,1} \) is the lower bound. Since these bounds are affine functions shifted by a constant, then \( \min \max\{|B_1 + C_{0,1}|, |B_1 - C_{1,1}|\} \) simply occurs where

\[
B_1 + C_{0,1} = -B_1 - C_{1,1}
\]

meaning functions that satisfy

\[
2B_1 = -(C_{1,1} + C_{0,1})
\]

have minimal upper bounds on \( \Delta_{TPR} \). This condition is precisely

\[
2 \left( \frac{\hat{r}_{0,1}}{\hat{r}_{0,1} + U_1 - U_0} \hat{\alpha}_{0,1} - \frac{\hat{r}_{1,1}}{\hat{r}_{1,1} + U_0 - U_1} \hat{\alpha}_{1,1} \right) = (U_0 - U_1) \left( \frac{1}{\hat{r}_{1,1} + U_0 - U_1} + \frac{1}{\hat{r}_{0,1} + U_1 - U_0} \right)
\]

This condition simply states that all \( f \) that have this linear relationship between \( \hat{\alpha}_{0,1} \) and \( \hat{\alpha}_{1,1} \) have minimal upper bounds. Now, under Assumption 1, we know

\[
\frac{U_1}{\hat{r}_{1,1}} \leq \hat{\alpha}_{1,1} \leq 1 - \frac{U_1}{\hat{r}_{1,1}}
\]

\[
\frac{U_0}{\hat{r}_{0,1}} \leq \hat{\alpha}_{0,1} \leq 1 - \frac{U_0}{\hat{r}_{0,1}}
\]

Note, that the trivial classifier with \( \hat{\alpha}_{0,1} = \hat{\alpha}_{1,1} = \frac{1}{2} \) always satisfies this condition. These values always satisfy the condition in Theorem 2 because

\[
\left( \frac{\hat{r}_{0,1}}{\hat{r}_{0,1} + U_1 - U_0} - \frac{\hat{r}_{1,1}}{\hat{r}_{1,1} + U_0 - U_1} \right) = (U_0 - U_1) \left( \frac{1}{\hat{r}_{1,1} + U_0 - U_1} + \frac{1}{\hat{r}_{0,1} + U_1 - U_0} \right)
\]

There will exists classifiers with non-trivial performance as long as \( \frac{U_1}{\hat{r}_{1,1}} \) and \( \frac{U_0}{\hat{r}_{0,1}} \) do not equal \( \frac{1}{2} \). This is because the region

\[
\frac{U_1}{\hat{r}_{1,1}} \leq \hat{\alpha}_{1,1} \leq 1 - \frac{U_1}{\hat{r}_{1,1}}
\]

\[
\frac{U_0}{\hat{r}_{0,1}} \leq \hat{\alpha}_{0,1} \leq 1 - \frac{U_0}{\hat{r}_{0,1}}
\]
will not the single point $\hat{\alpha}_{0,1} = \hat{\alpha}_{1,1} = \frac{1}{2}$ but instead a rectangular region in $\mathbb{R}^2$ and since we know the condition for minimal bounds is simply a line that necessarily crosses through $\hat{\alpha}_{0,1} = \hat{\alpha}_{1,1} = \frac{1}{2}$, there will exist other values of $\hat{\alpha}_{0,1}$ and $\hat{\alpha}_{1,1}$ points that satisfy the condition while also being in this region. One can use the same proofs to prove the result for $|\Delta_{\text{FPR}}(f)|$.