Low-complexity modeling of partially available second-order statistics via matrix completion

Armin Zare, Yongxin Chen, Mihailo R. Jovanović, and Tryphon T. Georgiou

Abstract

State statistics of linear systems satisfy certain structural constraints that arise from the underlying dynamics and the directionality of input disturbances. In the present paper we study the problem of completing partially known state statistics. The dynamical interaction between state variables is known while the directionality of input excitation is uncertain. Thus, the goal of the inverse problem that we formulate is to identify the dynamics and directionality of input excitation so as to explain the observed sample statistics. In particular, we seek to explain the data with the least number of possible input disturbance channels. This can be formulated as a rank minimization problem, and for its solution, we employ a convex relaxation based on the nuclear norm. The resulting optimization problem can be cast as a semidefinite program and solved efficiently using general-purpose solvers for small- and medium-size problems. We develop a customized alternating minimization algorithm to solve the problem for large-scale systems. We provide an example to illustrate that identified colored-in-time stochastic disturbances represent an effective means for explaining available second-order state statistics.

Index Terms

Alternating minimization algorithm, convex optimization, disturbance dynamics, low-rank approximation, matrix completion problems, nuclear norm regularization, structured covariances.

I. INTRODUCTION

Motivation for this work stems from control-oriented modeling of systems with very large number of degrees of freedom. Indeed, dynamics governing many physical systems are prohibitively complex for purposes of control design and optimization. Thus, it is common practice to investigate low-dimensional models that preserve the essential dynamics. To this end, stochastically driven linearized models often represent an effective option that is also capable of explaining observed statistics. Further, such models are well-suited for analysis and synthesis using tools from modern robust control.

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An example that illustrates the point is the modeling of fluid flows. In this, the Navier-Stokes equations are prohibitively complex for control design. On the other hand, linearization of the equations around mean-velocity profile in the presence of stochastic excitation has been shown to qualitatively replicate structural features of shear flows [1]–[7]. However, it has also been recognized that a simple white-in-time stochastic excitation cannot reproduce important statistics of the fluctuating velocity field [8], [9]. In this paper, we consider stochastic linear models but depart from the white-in-time restriction on stochastic disturbances. Our objective is to identify low-complexity disturbance models that account for partially available second-order state statistics. Further, we are interested in large-scale systems with many degrees of freedom.

Herein, we formulate a covariance completion problem for linear systems with uncertain disturbance dynamics. The complexity of the disturbance model is quantified by the number of input channels. In fact, we show that the number of input channels relates to the rank of a certain matrix which reflects the directionality of input disturbances and correlation structure of excitation sources. We use the nuclear norm as a surrogate for rank [10]–[17].

The resulting optimization problem is convex. It can be cast as a semidefinite program (SDP), readily solvable by standard software for small- and medium-size problems. We exploit the problem structure and develop an efficient customized Alternating Minimization Algorithm (AMA), which is suitable for large-scale problems. AMA works as a proximal gradient for the dual problem. The solution to this covariance completion problem gives rise to a class of linear filters that realize colored-in-time disturbances which account for the observed state statistics.

Our presentation is organized as follows. We summarize key results regarding the structure of state covariances and its relation to the power spectrum of input processes in Section II. We characterize admissible signatures for matrices that parametrize disturbance spectra and formulate the covariance completion problem in Section III. Section IV develops an efficient optimization algorithm for solving this problem in large dimensions. To highlight the theoretical and algorithmic developments we provide an example in Section V. We conclude with remarks and future directions in Section VI.

II. LINEAR STOCHASTIC MODELS AND STATE STATISTICS

We now discuss algebraic conditions that state covariances of linear time-invariant (LTI) systems satisfy. For white-in-time stochastic inputs state statistics satisfy an algebraic Lyapunov equation. A similar algebraic characterization holds for LTI systems driven by colored stochastic processes [18], [19]. This characterization provides the foundation for the covariance completion problem that we study in this paper.

Consider a linear time-invariant system

\[ \dot{x} = Ax + Bu \]  

where \( x(t) \in \mathbb{C}^n \) is a state vector, \( u(t) \in \mathbb{C}^m \) is a zero-mean stationary stochastic input, \( A \in \mathbb{C}^{n \times n} \) is Hurwitz, \( B \in \mathbb{C}^{n \times m} \) is a full-column-rank matrix with \( m \leq n \), and \( (A, B) \) is a controllable pair. Let \( X \) be the steady-state
covariance of the state vector of system \( (1) \), \( X = \lim_{t \to \infty} \mathcal{E} \{ x(t)x(t)^* \} \), with \( \mathcal{E} \) being the expectation operator. We next review key results and provide new insights into the following questions:

(i) What is the algebraic structure of \( X \)? In other words, given a positive definite matrix \( X \), under what conditions does it qualify to be the steady-state covariance of \( (1) \)?

(ii) Given the steady-state covariance \( X \) of \( (1) \), what can be said about the power spectra of input processes that are consistent with these statistics?

A. Algebraic constraints on admissible covariances

The steady-state covariance matrix \( X \) of the state vector in \( (1) \) satisfies [18], [19]

\[
\begin{bmatrix}
AX + XA^* & B \\
B^* & 0
\end{bmatrix} = \begin{bmatrix}
0 & B \\
B^* & 0
\end{bmatrix}.
\] (2a)

An equivalent characterization is that there is a solution \( H \in \mathbb{C}^{n \times m} \) to the equation

\[
AX + XA^* = -BH^* - HB^*.
\] (2b)

Either of these conditions, together with the positive definiteness of \( X \), completely characterize state covariances of linear dynamical systems driven by white or colored stochastic processes [18], [19]. When the input \( u \) is white noise with covariance \( W \), \( X \) satisfies the algebraic Lyapunov equation

\[
AX + XA^* = -BWB^*.
\]

In this case, \( H \) in (2b) is determined by \( H = BW/2 \) and the right-hand-side \(-BWB^*\) is sign-definite. In fact, except for this case when the input is white noise,

\[
Z := -(AX + XA^*) \quad \text{(3a)}
\]

\[
= BH^* + HB^* \quad \text{(3b)}
\]

may have both positive and negative eigenvalues. Additional discussion on the structure of \( Z \) is provided in Section III-A.

B. Power spectrum of input process

For stochastically-driven linear systems the state statistics can be obtained from knowledge of the system model and the input statistics. Herein, we are interested in the converse: starting from the steady-state covariance \( X \) and the system dynamics \( (1) \), we want to identify the power spectrum of the input process \( u \). As illustrated in Fig. 1a

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we seek to construct a filter which, when driven by white noise, produces a suitable stationary input \( u \) to (1) so that the state covariance is \( X \). Next, we characterize a class of filters with degree at most \( n \).

Consider the linear filter given by

\[
\dot{\xi} = (A - BK)\xi + Bw
\]

\[ u = -K\xi + w \tag{4b} \]

where \( w \) is a white stochastic process with covariance \( \Omega \succ 0 \) and

\[
K = \frac{1}{2} \Omega B^*X^{-1} - H^*X^{-1}. \tag{4c}
\]

The power spectrum of \( u \) is determined by

\[
\Pi_{uu}(\omega) = \Psi(j\omega)\Omega \Psi(j\omega)^*
\]

where

\[
\Psi(s) = I - K(sI - A + BK)^{-1}B
\]

is the transfer function of the filter (4). To verify this, consider the cascade connection shown in Fig. 1a with state

Fig. 1: (a) A cascade connection of an LTI system with a linear filter that is designed to account for the sampled steady-state covariance matrix \( X \); (b) An equivalent feedback representation of the cascade connection in (a).
space representation

\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = \begin{bmatrix}
A & -BK \\
0 & A - BK
\end{bmatrix} \begin{bmatrix}
x \\
\xi
\end{bmatrix} + \begin{bmatrix}
B \\
B
\end{bmatrix} w
\]

\[x = \begin{bmatrix}
I & 0
\end{bmatrix} \begin{bmatrix}
x \\
\xi
\end{bmatrix}.\]

It is easy to see that this realization is not controllable and therefore not minimal. After eliminating the uncontrollable modes, it is seen that the transfer function from the white-in-time \(w\) to \(x\) is \((sI - A + BK)^{-1}B\). Thus, the corresponding algebraic Lyapunov equation in conjunction with (4c) yields

\[(A - BK)X + X(A - BK)^* + B \Omega B^* \]
\[= AX + XA^* + B \Omega B^* - BKX - XK^*B^* \]
\[= AX + XA^* + BH^* + HB^* \]
\[= 0.\]

This shows that (4) generates a process \(u\) that is consistent with \(X\).

C. Stochastic control interpretation

The class of power spectra described by (4) is closely related to the covariance control problem, or the covariance assignment problem, studied in [20], [21]. To illustrate this, let us consider

\[\dot{x} = Ax + Bv + Bw \quad (5a)\]

where \(w\) is again white with covariance \(\Omega\); see Fig. [11]. In the absence of a control input \((v = 0)\), the steady-state covariance satisfies the Lyapunov equation

\[AX + XA^* + B \Omega B^* = 0.\]

A choice of a non-zero process \(v\) can be used to assign different values for \(X\). Indeed, for

\[v = -Kx \quad (5b)\]

and \(A - BK\) Hurwitz, \(X\) satisfies

\[(A - BK)X + X(A - BK)^* + B \Omega B^* = 0. \quad (6)\]

It is easy to see that any \(X > 0\) satisfying (6) also satisfies (2b) with \(H = -XK^* + B \Omega / 2\). Conversely, if \(X > 0\) satisfies (2b), for \(K = \Omega B^* X^{-1} - H^* X^{-1}\), then \(X\) also satisfies (6) and \(A - BK\) is Hurwitz. Thus, the following statements are equivalent:
• A matrix $X \succ 0$ qualifies as the stationary state covariance of (5a) via a suitable choice of state-feedback (5b).

• A matrix $X \succ 0$ is a state covariance of (1) for some stationary stochastic input $u$.

To clarify the connection between $K$ and the corresponding modeling filter for $u$, let

$$u = -Kx + w.$$  

(7a)

Substitution of (5b) into (5a) yields

$$\dot{x} = (A - BK)x + Bw$$

(7b)

$$= Ax + Bu$$

which coincides with (1). Thus, $X$ can also be achieved by driving (1) with $u$ given by (7a). The equivalence of (1) and (7) is evident.

In general, there is more than one choice of $K$ that yields a given feasible $X$. A criterion for the selection of an optimal feedback gain $K$, can be to minimize

$$\lim_{t \to \infty} \mathbb{E}\{v(t)^*v(t)\}.$$  

It turns out that this optimality criterion relates to information theoretic notions of distance (Kullback-Leibler divergence) between corresponding models with and without control [22]–[24]. Based on this criterion, the optimal feedback gain $K$ can be obtained by minimizing $\text{trace}(KXK^*)$, subject to the linear constraint (6). This choice of $K$ characterizes an optimal filter of the form (7). We return to this in Section V where we provide an illustrative example.

III. COVARIANCE COMPLETION AND MODEL COMPLEXITY

In Section II we presented the structural constraints on the state covariance $X$ of an LTI system. We also proposed a method to construct a class of linear filters that generate the appropriate input process $u$ to account for the statistics in $X$. In many applications, the dynamical generator $A$ in (1) is known. On the other hand, the observed state statistics often originate from disturbances that are difficult to model directly. To complicate matters, the state statistics may be only partially known. Thus, we now develop a framework for completing unknown elements of $X$ and, thereby, obtain information about input disturbances to (1).

For colored-in-time input $u$ that enters into the state equation through the identity matrix, condition (2a) is trivially satisfied. Indeed, any sample covariance $X$ can be generated by a linear model (1) with $B = I$. Thus, a disturbance input $u$ that excites all degrees of freedom in the original system can trivially account for the observed statistics and provides no useful information about the underlying physics.
In our setting, the structure and size of the matrix \(B\) in (1) is not known \textit{a priori}, which means that the direction of the input disturbances are not given. In most physical systems, disturbance can directly excite only a limited number of directions in the state space. For instance, in mechanical systems where inputs represent forces and states represent position and velocity, disturbances can only enter into the velocity equation. Hence, it is of interest to identify a disturbance model that involves a small number of input channels. This requirement can be formalized by restricting the input to enter into the state equation through a matrix \(B \in \mathbb{C}^{n \times m}\) with \(m < n\). Thus, our objective is to identify matrices \(B\) and \(H\) in (2b) to reproduce a partially known \(X\) while striking an optimal balance with the complexity of the model; the complexity is reflected in the rank of \(B\), i.e., the number of input channels.

A. The signature of \(Z\)

As mentioned in Section II, the matrix \(Z\) in (3) is not necessarily positive semidefinite. However, it is not arbitrary. We next examine admissible values of the \textit{signature} on \(Z\), i.e., the number of positive, negative, and zero eigenvalues. In particular, we show that the number of positive and negative eigenvalues of \(Z\) impacts the number of input channels in the state equation (1).

There are two sets of constraints on \(Z\) arising from (3a) and (3b), respectively. The first one is a standard Lyapunov equation with Hurwitz \(A\) and a given Hermitian \(X \succ 0\). The second provides a link between the signature of \(Z\) and the number of input channels in (1).

First, we study the constraint on the signature of \(Z\) arising from (3a) which we repeat here,

\[AX +XA^* = -Z.\]  

The unique solution to this Lyapunov equation, with Hurwitz \(A\) and Hermitian \(X\) and \(Z\), is given by

\[X = \int_0^\infty e^{At} Z e^{A^*t} dt.\]

Lyapunov theory implies that if \(Z\) is positive definite then \(X\) is also positive definite. However, the converse is not true. Indeed, for a given \(X \succ 0\), \(Z\) obtained from (8) is not necessarily positive definite. Clearly, \(Z\) cannot be negative definite either, otherwise \(X\) obtained from (9) would be negative semidefinite. We can thus conclude that (9) does in fact introduce a constraint on the signature of \(Z\). In what follows, the signature is defined as the triple

\[\text{In}(Z) = (\pi(Z), \nu(Z), \delta(Z))\]

where \(\pi(Z), \nu(Z),\) and \(\delta(Z)\) denote the number of positive, negative, and zero eigenvalues of \(Z\), respectively.

Several authors have studied constraints on signatures of \(A\), \(X\), and \(Z\) that are linked through a Lyapunov equation [25–27]. Typically, such studies focus on the relationship between the signature of \(X\) and the eigenvalues
of $A$ for a given $Z \succeq 0$. In contrast, [28] considers the relationship between the signature of $Z$ and eigenvalues of $A$ for $X \succ 0$ and we make use of these results.

Let $\{\lambda_1, \ldots, \lambda_l\}$ denote the eigenvalues of $A$, $\mu_k$ denote the geometric multiplicity of $\lambda_k$, and

$$
\mu(A) := \max_{1 \leq k \leq l} \mu_k.
$$

The following result is a special case of [28, Theorem 2].

**Proposition 1:** Let $A$ be Hurwitz and let $X$ be positive definite. For $Z = -(AX + XA^*)$,

$$
\pi(Z) \geq \mu(A).
$$

(10)

To explain the nature of the constraint $\pi(Z) \geq \mu(A)$, we first note that $\mu(A)$ is the least number of input channels that are needed for system (1) to be controllable. Now consider the decomposition

$$
Z = Z_+ - Z_- \tag{11}
$$

where $Z_+, Z_-$ are positive semidefinite matrices, and accordingly $X = X_+ - X_-$ with $X_+, X_-$ denoting the solutions of the corresponding Lyapunov equations. Clearly, unless the above constraint (11) holds, $X_+$ cannot be positive definite. Hence, $X$ cannot be positive definite either. Interestingly, there is no constraint on $\nu(Z)$ other than

$$
\pi(Z) + \nu(Z) \leq n
$$

which comes from the dimension of $Z$.

To study the constraint on the signature of $Z$ arising from (3b), we begin with a lemma, whose proof is provided in the appendix.

**Lemma 1:** For a Hermitian matrix $Z$ decomposed as

$$
Z = S + S^*\tag{12}
$$

the following holds

$$
\pi(Z) \leq \text{rank}(S).
$$

Clearly, the same bound applies to $\nu(Z)$, that is,

$$
\nu(Z) \leq \text{rank}(S).
$$

The importance of these bounds stems from our interest in decomposing $Z$ into summands of small rank. A
decomposition of $Z$ into $S + S^*$ allows us to identify input channels and power spectra by factoring $S = BH^*$. The rank of $S$ coincides with the rank of $B$, that is, with the number of input channels in the state equation. Thus, it is of interest to determine the minimum rank of $S$ in such a decomposition and this is given in the following proposition.

**Proposition 2:** For a Hermitian matrix $Z$ having signature $(\pi(Z), \nu(Z), \delta(Z))$,

$$\min \{ \text{rank}(S) | Z = S + S^* \} = \max \{ \pi(Z), \nu(Z) \}.$$  

**Proof:** The proof is provided in the appendix.

We can now summarize the bounds on the number of positive and negative eigenvalues of the matrix $Z$ defined by (3). By combining Proposition 1 with Lemma 1 we show that these upper bounds are dictated by the number of inputs in the state equation (1).

**Proposition 3:** Let $X > 0$ denote the steady-state covariance of the state $x$ of a stable linear system (1) with $m$ inputs. If $Z$ satisfies the Lyapunov equation (8), then

$$0 \leq \nu(Z) \leq m$$
$$\mu(A) \leq \pi(Z) \leq m.$$  

**Proof:** From Section II a state covariance $X$ satisfies

$$AX + XA^* = -BH^* - HB^*.$$  

Setting $S = BH^*$,

$$Z = BH^* + HB^* = S + S^*.$$  

From Lemma 1

$$\max\{\pi(Z), \nu(Z)\} \leq \text{rank}(S) \leq \text{rank}(B) = m.$$  

The lower bounds follow from Proposition 1.

**B. Decomposition of $Z$ into $BH^* + HB^*$**

Proposition 2 expresses the possibility to decompose the matrix $Z$ into $BH^* + HB^*$ with $S = BH^*$ of minimum rank equal to $\max \{ \pi(Z), \nu(Z) \}$. Here, we present an algorithm that achieves this objective. Given $Z$ with signature
\[(\pi(Z), \nu(Z), \delta(Z))\], we can choose an invertible matrix \(T\) to bring \(Z\) into the following form

\[
\hat{Z} := ZZ^* = 2 \begin{bmatrix}
I_{\pi} & 0 & 0 \\
0 & -I_{\nu} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

where \(I_{\pi}\) and \(I_{\nu}\) are identity matrices of dimension \(\pi(Z)\) and \(\nu(Z)\) [29, pages 218–223]. We first present factorization of \(Z\) for \(\pi(Z) \leq \nu(Z)\). With

\[
\hat{S} = \begin{bmatrix}
I_{\pi} & -I_{\pi} & 0 & 0 \\
I_{\pi} & -I_{\pi} & 0 & 0 \\
0 & 0 & -I_{\nu-\pi} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

we clearly have \(\hat{Z} = \hat{S} + \hat{S}^*\). Furthermore, \(\hat{S}\) can be written as \(\hat{S} = \hat{B}\hat{H}^*\), where

\[
\hat{B} = \begin{bmatrix}
I_{\pi} & 0 \\
I_{\pi} & 0 \\
0 & I_{\nu-\pi} \\
0 & 0
\end{bmatrix}, \quad \hat{H} = \begin{bmatrix}
I_{\pi} & 0 \\
-I_{\pi} & 0 \\
0 & -I_{\nu-\pi} \\
0 & 0
\end{bmatrix}.
\]

In case \(\nu(Z) = \pi(Z)\), \(I_{\nu-\pi}\) and the corresponding row and column are empty. Finally, the matrices \(B\) and \(H\) are determined by \(B = T^{-1}\hat{B}\) and \(H = T^{-1}\hat{H}\).

Similarly, for \(\pi(Z) > \nu(Z)\), \(Z\) can be decomposed into \(BH^* + HB^*\) with \(B = T^{-1}\hat{B}\), \(H = T^{-1}\hat{H}\), and

\[
\hat{B} = \begin{bmatrix}
I_{\pi-\nu} & 0 \\
0 & I_{\nu} \\
0 & I_{\nu} \\
0 & 0
\end{bmatrix}, \quad \hat{H} = \begin{bmatrix}
I_{\pi-\nu} & 0 \\
0 & I_{\nu} \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

Note that both \(B\) and \(H\) are full column rank matrices.

C. Covariance completion problem

Given the dynamical generator \(A\) and partially observed state correlations, it is desired to obtain a low-complexity model for the disturbance that can explain the observed entries of \(X\). Here the complexity is reflected by the number of input channels, i.e., the rank of the input matrix \(B\). Thus, it is desired to minimize the rank of \(S\), which is equal to the rank of \(B\).

The rank is a non-convex function of the matrix and the problem of rank minimization is difficult. Recent
advances have demonstrated that the nuclear-norm (i.e., the sum of the singular values)
\[ \|S\|_* := \sum_{i=1}^{n} \sigma_i(S) \]
represents a good proxy for rank minimization [10]–[17]. We thus formulate the following matrix completion
problem: Given a Hurwitz \(A\) and the matrix \(G\), determine matrices \(X = X^*\) and \(Z = S + S^*\) from the solution to
\[
\begin{align*}
\text{minimize}_{S,X} & \quad \|S\|_* \\
\text{subject to} & \quad AX +XA^* + S + S^* = 0 \\
& \quad (CX^*) \circ E - G = 0 \\
& \quad X \succeq 0.
\end{align*}
\]
(14)

In the above, \(A, G \in \mathbb{C}^{n \times n}\) are problem data, while \(S, X \in \mathbb{C}^{n \times n}\) are optimization variables. The entries of \(G\) represent partially known second-order statistics which reflect output correlations provided by numerical simulations or experiments of the underlying physical system. The symbol \(\circ\) denotes elementwise matrix multiplication and the matrix \(E\) is the structural identity,
\[
E_{ij} = \begin{cases} 1, & \text{if } G_{ij} \text{ is available} \\ 0, & \text{if } G_{ij} \text{ is unavailable.} \end{cases}
\]

The constraint set in (14) represents the intersection of the positive semidefinite cone and two linear subspaces. These are specified by the Lyapunov-like constraint, which is imposed by the linear dynamics, and the linear constraint which relates \(X\) with the available statistics in \(G\). The steady-state covariance matrix of the output \(z(t) = Cx(t)\) is
\[
\lim_{t \to \infty} E(z(t)z(t)^*) = CX^*.
\]

As shown in Proposition 2, minimizing the rank of \(S\) is equivalent to minimizing \(\max\{\pi(Z), \nu(Z)\}\). Given \(Z\), there exist matrices \(Z_+ \succeq 0\) and \(Z_- \succeq 0\) with \(Z = Z_+ - Z_-\) such that \(\text{rank}(Z_+) = \pi(Z)\) and \(\text{rank}(Z_-) = \nu(Z)\). Furthermore, any such decomposition of \(Z\) satisfies \(\text{rank}(Z_+) \geq \pi(Z)\) and \(\text{rank}(Z_-) \geq \nu(Z)\). Thus, instead of (14), we can alternatively consider the following convex optimization problem, which aims at minimizing \(\max\{\pi(Z), \nu(Z)\}\),
\[
\begin{align*}
\text{minimize}_{X,Z_+,Z_-} & \quad \max\{\text{trace}(Z_+), \text{trace}(Z_-)\} \\
\text{subject to} & \quad AX +XA^* + Z_+ - Z_- = 0 \\
& \quad (CX^*) \circ E - G = 0 \\
& \quad X \succeq 0, \ Z_+ \succeq 0, \ Z_- \succeq 0.
\end{align*}
\]
(15)

Both (14) and (15) can be solved efficiently using standard SDP solvers [30], [31] for small- and medium-size
In Section IV, we develop an efficient customized algorithm which solves the following covariance completion problem

\[
\begin{align*}
\text{minimize} & \quad -\log \det(X) + \gamma \|Z\|_* \\
\text{subject to} & \quad AX + XA^* + Z = 0 \\
& \quad (CXC^*) \circ E - G = 0.
\end{align*}
\]

This algorithm is well suited for large-scale problems and the solution provides an upper bound to the objective of (15) because

\[
\|Z\|_* = \text{trace}(Z_+) + \text{trace}(Z_-) \\
\geq \max\{\text{trace}(Z_+), \text{trace}(Z_-)\}.
\]

In recent work [32], [33], we considered (CC) in the absence of the logarithmic barrier function. However, the corresponding semidefinite $X$ is not suitable for synthesizing the input filter as explained in Section II-B.

IV. CUSTOMIZED ALGORITHM FOR SOLVING THE COVARIANCE COMPLETION PROBLEM

We begin this section by bringing (CC) into a form which is convenient for alternating direction methods. We then study the optimality conditions, formulate the dual problem, and develop a customized Alternating Minimization Algorithm (AMA) for (CC). The alternating minimization algorithm allows us to exploit the respective structure of the logarithmic barrier function and the nuclear norm, thereby leading to an efficient implementation that is well-suited for large systems.

We note that AMA was originally developed by Tseng [34] and its enhanced variants have been recently presented in [35], [36] and used, in particular, for estimation of sparse Gaussian graphical models.

In (CC), $\gamma$ determines the importance of the nuclear norm relative to the logarithmic barrier function. The convexity of (CC) follows from the convexity of the objective function

\[
J_p(X, Z) := -\log \det(X) + \gamma \|Z\|_*
\]

and the convexity of the constraint set. Problem (CC) can be equivalently expressed as follows,

\[
\begin{align*}
\text{minimize} & \quad -\log \det(X) + \gamma \|Z\|_* \\
\text{subject to} & \quad AX + BZ - C = 0,
\end{align*}
\]
where the constraints are now given by

\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} X + \begin{bmatrix}
I \\
0
\end{bmatrix} Z - \begin{bmatrix}
0 \\
G
\end{bmatrix} = 0.
\]

Here, \(A_1, A_2 : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}\) are linear operators, with

\[
A_1(X) := AX + XA^*,
\]

\[
A_2(X) := (CXC^*) \circ E
\]

and

\[
A := \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}, \quad B := \begin{bmatrix}
I \\
0
\end{bmatrix}, \quad C := \begin{bmatrix}
0 \\
G
\end{bmatrix}.
\]

### A. Optimality conditions and the dual problem

By splitting \(Z\) into positive and negative definite parts,

\[
Z = Z_+ - Z_-, \quad Z_+ \succeq 0, \quad Z_- \succeq 0
\]

we can cast \((CC-1)\) as an SDP,

\[
\begin{align*}
\text{minimize} & \quad - \log \det (X) + \gamma (\text{trace} (Z_+) + \text{trace} (Z_-)) \\
\text{subject to} & \quad A_1(X) + Z_+ - Z_- = 0 \\
& \quad A_2(X) - G = 0 \\
& \quad Z_+ \succeq 0, \quad Z_- \succeq 0.
\end{align*}
\]

Fig. 2: The saturation and soft-thresholding operators are related via identity (20).
To derive the dual of the *primal* problem (P), we introduce the Lagrangian

\[ \mathcal{L}(X, Z_{\pm}; Y_1, Y_2, \Lambda_{\pm}) = -\log \det (X) + \gamma \text{trace} (Z_+ + Z_-) - \langle \Lambda_+, Z_+ \rangle - \langle \Lambda_-, Z_- \rangle + \langle Y_1, A_1(X) + Z_+ - Z_- \rangle + \langle Y_2, A_2(X) - G \rangle \]

where Hermitian matrices \( Y_1, Y_2, \) and \( \Lambda_{\pm} \preceq 0 \) are dual variables, and \( \langle \cdot, \cdot \rangle \) represents the standard inner product \( \langle M_1, M_2 \rangle := \text{trace}(M_1^*M_2) \). By minimizing \( \mathcal{L} \) with respect to primal variables \( X \) and \( Z_{\pm} \), we arrive at the Lagrangian *dual* of (P)

\[
\begin{align*}
\text{maximize} & \quad \log \det \left( A_1^\dagger(Y_1) + A_2^\dagger(Y_2) \right) - \langle G, Y_2 \rangle + n \\
\text{subject to} & \quad \|Y_1\|_2 \leq \gamma
\end{align*}
\]

(D)

where the adjoints of the operators \( A_1 \) and \( A_2 \) are given by

\[ A_1^\dagger(Y) = A^* Y + YA \]
\[ A_2^\dagger(Y) = C^*(E \circ Y) C. \]

The dual problem (D) is a convex optimization problem with variables \( Y_1, Y_2 \in \mathbb{C}^{n \times n} \) and the objective function \( J_d(Y_1, Y_2) \). These variables are dual feasible if the constraint in (D) is satisfied. This constraint is obtained by minimizing the Lagrangian with respect to \( Z_+ \) and \( Z_- \), which leads to

\[ \gamma I - \Lambda_+ + Y_1 \succeq 0, \quad Z_+ \succeq 0 \]
\[ \gamma I - \Lambda_- - Y_1 \succeq 0, \quad Z_- \succeq 0. \]

Because of the positive semi-definiteness of the dual variables \( \Lambda_+ \) and \( \Lambda_- \), we also have that

\[ Y_1 + \gamma I \succeq \Lambda_+ \succeq 0 \]
\[ -Y_1 + \gamma I \succeq \Lambda_- \succeq 0, \]

which results in

\[ -\gamma I \preceq Y_1 \preceq \gamma I \iff \|Y_1\|_2 \leq \gamma. \]

On the other hand, minimization of \( \mathcal{L} \) with respect to \( X \) yields

\[ X^{-1} = A_1^\dagger(Y_1) + A_2^\dagger(Y_2) \succ 0. \]  

(16)

In the case of primal and dual feasibility, any dual feasible pair \( (Y_1, Y_2) \) gives a lower bound on the optimal value \( J^*_p \) of the primal problem (P). The alternating minimization algorithm of Section [IV-B] can be interpreted as a proximal gradient algorithm on the dual problem and is developed to achieve sufficient dual ascent and satisfy (16).
B. Alternating Minimization Algorithm (AMA)

The logarithmic barrier function in (CC) is strongly convex over any compact subset of the positive definite cone [37]. This makes it well-suited for the application of AMA, which requires strong convexity of the smooth part of the objective function.

The augmented Lagrangian associated with (CC-1) is given by

\[ L_\rho(X, Z; Y_1, Y_2) = -\log \det (X) + \gamma \|Z\|_* + \langle Y, A(X) + BZ - C \rangle + \frac{\rho}{2} \|A(X) + BZ - C\|_F^2 \]

where

\[ Y := [Y_1, Y_2]^*, \quad Y_1 = Y_1^*, \quad Y_2 = Y_2^* \]

are Lagrange multipliers, \( \rho \) is a positive scalar, and \( \| \cdot \|_F \) is the Frobenius norm.

AMA follows a sequence of iterations,

\[
X^{k+1} := \arg\min_X L_0(X, Z^k, Y^k) \quad (17a) \\
Z^{k+1} := \arg\min_Z L_\rho(X^{k+1}, Z, Y^k) \quad (17b) \\
Y^{k+1} := Y^k + \rho (A X^{k+1} + B Z^{k+1} - C) \quad (17c)
\]

which terminate when the duality gap

\[ \Delta_{\text{gap}} := -\log \det (X^{k+1}) + \gamma \|Z^{k+1}\|_* - J_d(Y^{k+1}) \]

and the primal residual

\[ \Delta_p := \|A X^{k+1} + B Z^{k+1} - C\|_F \]

are sufficiently small, i.e., \( |\Delta_{\text{gap}}| \leq \epsilon \), and \( \Delta_p \leq \epsilon \). In the \( X \)-minimization step (17a), AMA minimizes the Lagrangian \( L_0 \) to obtain a closed form expression for \( X^{k+1} \). This step is followed by a \( Z \)-minimization step (17b) in which the augmented Lagrangian \( L_\rho \) is minimized with respect to \( Z \). Finally, the Lagrange multiplier, \( Y \), is updated based on the primal residual with the step size \( \rho \).

In contrast to the Alternating Direction Method of Multipliers [38], which minimizes the augmented Lagrangian \( L_\rho \) in both \( X \)- and \( Z \)-minimization steps, AMA updates \( X \) via minimization of the standard Lagrangian \( L_0 \). As shown below, in [18], use of AMA leads to a closed-form expression for \( X^{k+1} \). Another differentiating aspect of AMA is that it works as a proximal gradient on the dual function. This allows us to select the step size \( \rho \) in order to achieve sufficient ascent.
1) Solution to the X-minimization problem (17a): At the $k$th iteration of AMA, minimizing the Lagrangian $L_0$ with respect to $X$ for fixed $\{Z^k, Y^k\}$ yields

$$X^{k+1} = (A^\dagger (Y^k))^{-1}.$$  

(18)

2) Solution to the Z-minimization problem (17b): For fixed $\{X^{k+1}, Y^k\}$, the augmented Lagrangian $L_0$ is minimized with respect to $Z$,

$$\text{minimize}_{Z} \gamma \|Z\|_\ast + \frac{\rho}{2} \|Z - V^k\|^2_F.$$  

(19)

where

$$V^k := - (A_1 (X^{k+1}) + (1/\rho) Y_1^k).$$

The solution to (19) is obtained by singular value thresholding $[39]$. For this purpose we first compute the singular value decomposition of the symmetric matrix

$$V^k = U \Sigma U^*$$

where $\Sigma$ is the diagonal matrix of the singular values $\sigma_i$. The solution to the Z-minimization problem gives

$$Z^{k+1} = S_{\gamma/\rho}(V^k)$$

where the soft-thresholding operator $S_\tau$ is defined as

$$S_\tau(V^k) := U S_\tau(\Sigma) U^*,$$

$$S_\tau(\Sigma) = \text{diag}((\sigma_i - \tau)_+)$$

and $a_+ = \max\{a, 0\}$. Thus, the optimality condition in (17b) is satisfied by applying the soft-thresholding operator $S_{\gamma/\rho}$ on the singular values of the matrix $- (A_1 (X^{k+1}) + (1/\rho) Y_1^k)$.

3) Lagrange multiplier update: The expressions for $X^{k+1}$ and $Z^{k+1}$ can be used to bring (17c) into the following form

$$Y_1^{k+1} = T_{\gamma} (Y_1^k + \rho A_1 (X^{k+1}))$$

$$Y_2^{k+1} = Y_2^k + \rho (A_2 (X^{k+1}) - G).$$

For Hermitian matrix $M$ with singular value decomposition $M = U \Sigma U^*$, $T_\tau$ is the saturation operator,

$$T_\tau(M) := U T_\tau(\Sigma) U^*$$

$$T_\tau(\Sigma) = \text{diag} (\min (\max (\sigma_i, -\tau), \tau))$$

which restricts the singular values of $M$ between $-\tau$ and $\tau$. As illustrated in Fig. 2 the saturation and soft-thresholding operators are related via

$$M = T_\tau(M) + S_\tau(M).$$  

(20)
This relation guarantees dual feasibility of the update, i.e., \( \| Y^{k+1} \|_2 \leq \gamma \) at each iteration, and justifies the choice of stopping criteria in ensuring primal feasibility of the solution.

4) Choice of step-size for the dual update \((17c)\): We follow an enhanced variant of AMA \([36]\) which utilizes an adaptive Barzilai-Borwein step \([40]\) in \((17b)\) and \((17c)\) to guarantee sufficient dual ascent and positive definiteness of \( X \). Our numerical experiments indicate that this provides substantial acceleration relative to the use of a fixed step-size. Since the standard Barzilai-Borwein step-size may not always satisfy the feasibility or the sufficient ascent conditions, we employ a backtracking procedure to determine an appropriate step-size.

At the \( k \)th iteration of AMA, an initial step-size,

\[
\rho_{k,0} = \frac{\langle Y^{k+1} - Y^k, Y^{k+1} - Y^k \rangle}{\langle Y^{k+1} - Y^k, \nabla J_d(Y^k) - \nabla J_d(Y^{k+1}) \rangle},
\]

is adjusted through a backtracking procedure to guarantee positive definiteness of the subsequent iterate of \((17a)\) and sufficient ascent of the dual function,

\[
A^\dagger(Y^{k+1}) \succ 0
\]

\[
J_d(Y^{k+1}) \geq J_d(Y^k) + \langle \nabla J_d(Y^k), Y^{k+1} - Y^k \rangle - \frac{1}{2\rho_k} \| Y^{k+1} - Y^k \|_F^2. \tag{21b}
\]

Here, \( \nabla J_d \) is the gradient of the dual function. Condition \((21a)\) guarantees the positive definiteness of \( X^{k+1} \), cf. \((18)\), and the right hand side of \((21b)\) is a local quadratic approximation of the dual objective around \( Y^k \).

5) Computational complexity: The \( X \)-minimization step in AMA involves a matrix inversion, which takes \( O(n^3) \) operations. Similarly, since the \( Z \)-minimization step amounts to a singular value decomposition, it requires \( O(n^3) \) operations. Therefore, the total computational cost for a single iteration of our customized algorithm is also \( O(n^3) \).

In contrast, the worst-case complexity of standard SDP solvers is \( O(n^6) \).

Our customized AMA is summarized in Algorithm 1.

V. COMPUTATIONAL EXPERIMENTS

We provide an example to demonstrate the utility of our modeling and optimization framework. This is based on a stochastically-forced mass-spring-damper (MSD) system. Stochastic disturbances are generated by a low-pass filter,

\[
\dot{\zeta} = -\zeta + d \tag{22a}
\]
**Algorithm 1 Customized Alternating Minimization Algorithm**

**input:** $A$, $G$, $\gamma > 0$, tolerance $\epsilon$, and backtracking constant $\beta \in (0, 1)$.

**initialize:** $k = 0$, $\rho_{0,0} = 1$, $\Delta_{\text{gap}} = \Delta_p = 2\epsilon$, $Y_2^0 = O_{n \times n}$, and choose $Y_1^0$ such that $A_1^*(Y_1^0) = (\gamma/\|Y_1^0\|_2)I_{n \times n}$.

**while:** $|\Delta_{\text{gap}}| > \epsilon$ or $\Delta_p > \epsilon$,

$X^{k+1} = (A^*(Y^k))^{-1}$

compute $\rho_k$: Largest feasible step in $\{\beta^j \rho_{k,j}\}_{j=0,1,...}$ such that $Y^{k+1}$ satisfies (21)

$Z^{k+1} = \arg\min_Z L_{\rho_k}(X^{k+1}, Z, Y^k)$

$Y^{k+1} = Y^k + \rho_k \left(AX^{k+1} + BZ^{k+1} - C\right)$

$\rho_{k,0} = \langle Y^{k+1} - Y^k, Y^{k+1} - Y^k \rangle / \langle Y^{k+1} - Y^k, \nabla J_d(Y^k) - \nabla J_d(Y^{k+1}) \rangle$

$\Delta_{\text{gap}} = -\log \det(X^{k+1}) + \gamma \|Z^{k+1}\|_* - J_d(Y^{k+1})$

$\Delta_p = \|AX^{k+1} + BZ^{k+1} - C\|_F$

$k = k + 1$

**endwhile**

**output:** $\epsilon$-optimal solutions, $X^{k+1}$ and $Z^{k+1}$.

---

where $d$ represents a zero-mean unit variance white process. The state space representation of the MSD system is given by

**MSD system:**

$$\dot{x} = Ax + B_\zeta \zeta$$

(22b)

where the state vector $x = [p^* v^*]^*$, contains position and velocity of masses. Accordingly, the state and input matrices are

$$A = \begin{bmatrix} O & I \\ -T & -I \end{bmatrix}, \quad B_\zeta = \begin{bmatrix} 0 \\ I \end{bmatrix}$$
Fig. 4: Performance of the customized AMA for the MSD system with 50 masses, $\gamma = 2.2$, and $\epsilon = 10^{-3}$. (a) The dual objective function $J_d(Y_1, Y_2)$ of (CC); (b) the duality gap, $|\Delta_{gap}|$; and (c) the primal residual, $\Delta_p$.

where $O$ and $I$ are zero and identity matrices of suitable sizes, and $T$ is a symmetric tridiagonal Toeplitz matrix with 2 on the main diagonal and $-1$ on the first upper and lower sub-diagonals.

The steady-state covariance of system (22) can be found as the solution to the Lyapunov equation

$$\tilde{A}\Sigma + \Sigma \tilde{A}^* + \tilde{B}\tilde{B}^* = 0$$

where

$$\tilde{A} = \begin{bmatrix} A & B \\ O & -I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xz} \\ \Sigma_{zx} & \Sigma_{zz} \end{bmatrix}.$$ 

The matrix $\Sigma_{xx}$ denotes the state covariance of the MSD system, partitioned as,

$$\Sigma_{xx} = \begin{bmatrix} \Sigma_{pp} & \Sigma_{pv} \\ \Sigma_{vp} & \Sigma_{vv} \end{bmatrix}.$$ 

We assume knowledge of one-point correlations of the position and velocity of masses, i.e., we assume knowledge of the diagonal elements of matrices $\Sigma_{pp}$, $\Sigma_{vv}$, and $\Sigma_{pv}$. Thus, in order to account for these available statistics, we seek a state covariance $X$ of the MSD system which agrees with the available statistics whose structure is displayed in Fig. [3].

For 50 masses we use the alternating minimization algorithm of Section [IV] to solve (CC). Figure 4a illustrates the monotonic increase of the dual objective function. The absolute value of the duality gap, $|\Delta_{gap}|$, and the primal residual, $\Delta_p$ are displayed in Fig. 4 thereby demonstrating convergence of our customized algorithm.
Recall that in (CC), $\gamma$ determines the importance of the nuclear norm relative to the logarithmic barrier function. While larger values of $\gamma$ yield solutions with lower rank they may fail to provide reliable completion of the “ideal” state covariance $\Sigma_{xx}$. Figure 5 illustrates this tradeoff by showing the relative error in matching $\Sigma_{xx}$ as a function of $\gamma$. Minimum error is achieved with $\gamma = 1.26$ and for larger values the error gradually increases. However, this value of $\gamma$ does not yield a low-rank input correlation $Z$. For $\gamma = 2.2$ reasonable matching is obtained (82.7% matching) and the resulting $Z$ displays a clear-cut in its singular values with 62 of them being nonzero; see Fig. 6.

The spectrum of $Z$ contains 50 positive and 12 negative eigenvalues. Based on Proposition 2, $Z$ can be decomposed into $BH^* + HB^*$, where $B$ has 50 independent columns. In other words, the identified $X$ can be explained by driving the state-space model with 50 stochastic inputs $u$,

$$\dot{x} = Ax + Bu.$$
The algorithm presented in Section III-B is used to decompose $Z$ into $BH^* + HB^*$. For the identified input matrix $B$, the design parameter $K$ is then chosen to satisfy the optimality criterion described in Section II-C. This yields the optimal filter (7) that generates the stochastic input $u$. We use this filter to validate our approach as explained next.

We conduct linear stochastic simulations of system (7b) with zero-mean unit variance input $w$. Figure 7 shows the time evolution of the state variance of the MSD system. Since proper comparison requires ensemble-averaging, we have conducted twenty stochastic simulations with different realizations of the stochastic input $w$ to (7b). The variance, averaged over all simulations, is given by the thick black line. Even though the responses of individual simulations differ from each other, the average of twenty sample sets asymptotically approaches the correct steady-state variance.

One-point correlations of the position and velocity of masses are displayed in Fig. 8. We see that the averaged output of twenty stochastic simulations (red circles) agrees well with true profiles (black lines). The recovered covariance matrix of mass positions, $X_{pp}$, resulting from the ensemble-averaged simulations for $\gamma = 2.2$ is shown in Fig. 9b. We observe close correspondence with the true covariance $\Sigma_{pp}$ in Fig. 9a.

VI. CONCLUDING REMARKS

We are interested in explaining partially known second-order statistics that originate from experimental measurements or simulations using stochastic linear models. This is motivated by the need for control-oriented models of systems with large number of degrees of freedom, e.g., turbulent fluid flows. In our setup, the linearized approximation of the dynamical generator is known whereas the nature and directionality of disturbances that
can explain partially observed statistics are unknown. We thus formulate the problem of identifying appropriate stochastic input that can account for the observed statistics and is consistent with the linear dynamics.

This inverse problem is cast as a convex optimization problem. Nuclear norm minimization is utilized to identify noise parameters of low rank and to complete unavailable covariance data. To efficiently solve covariance completion problems of large size we develop a customized alternating minimization algorithm which works as a proximal gradient on the dual problem. Based on the solution of the optimization problem, a class of linear filters is obtained to realize appropriate colored-in-time excitation that accounts for the observed state statistics.

Our ongoing effort is directed towards application of the developed framework for control-oriented modeling of turbulent flows [41]. These models will be used to design distributed flow control strategies for improving efficiency of fluid flow systems.
Proof of Lemma \[7\]

Without loss of generality, let us consider $Z$ of the following form

$$Z = 2 \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as in \[12\]. Given any $S$ that satisfies $Z = S + S^*$ we can decompose it into

$$S = M + N$$

with $M$ Hermitian and $N$ skew-Hermitian. It is easy to see that

$$M = \frac{1}{2} Z = \begin{bmatrix} I_\pi & 0 & 0 \\ 0 & -I_\nu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By partitioning $N$ as

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix},$$

we have

$$S = \begin{bmatrix} I_\pi + N_{11} & N_{12} & N_{13} \\ N_{21} & -I_\nu + N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix}.$$

Clearly,

$$\text{rank}(S) \geq \text{rank}(I_\pi + N_{11}).$$

Since $N_{11}$ is skew-Hermitian, all its eigenvalues are on the imaginary axis. This implies that all the eigenvalues of $I_\pi + N_{11}$ have real part 1 and therefore $I_\pi + N_{11}$ is a full rank matrix. Hence, we have

$$\text{rank}(S) \geq \text{rank}(I_\pi + N_{11}) = \pi(Z)$$

which completes the proof.
Proof of Proposition 2

The inequality

$$\min \{ \text{rank}(S) \mid Z = S + S^* \} \geq \max \{ \pi(Z), \nu(Z) \}$$

follows from Lemma 1. To establish the proposition we need to show that the bounds are tight, i.e.,

$$\min \{ \text{rank}(S) \mid Z = S + S^* \} \leq \max \{ \pi(Z), \nu(Z) \}.$$ 

Given $Z$ in (12), for $\pi(Z) \leq \nu(Z)$, $Z$ can be written as

$$Z = 2 \begin{bmatrix} I_\pi & 0 & 0 & 0 \\ 0 & -I_\pi & 0 & 0 \\ 0 & 0 & -I_{\nu-\pi} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

By selecting $S$ in the form (13) we conclude that

$$\text{rank}(S) = \text{rank}\left( \begin{bmatrix} I_\pi & -I_\pi \\ I_\pi & -I_\pi \end{bmatrix} \right) + \text{rank}(I_{\nu-\pi})$$

$$= \pi(Z) + \nu(Z) - \pi(Z) = \nu(Z).$$

Therefore

$$\min \{ \text{rank}(S) \mid Z = S + S^* \} \leq \nu(Z).$$

Similarly, for the case $\pi(Z) > \nu(Z)$,

$$\min \{ \text{rank}(S) \mid Z = S + S^* \} \leq \pi(Z).$$

Hence,

$$\min \{ \text{rank}(S) \mid Z = S + S^* \} \leq \max \{ \pi(Z), \nu(Z) \}$$

which completes the proof.
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