Comments on Witten Invariants
of 3-Manifolds for SU(2) and $Z_m$

S. Kalyana Rama and Siddhartha Sen
School of Mathematics, Trinity College, Dublin 2, Ireland
Email : kalyan,sen@maths.tcd.ie

ABSTRACT. The values of the Witten invariants, $I_W$, of the lens space $L(p, 1)$ for SU(2) at level $k$ are obtained for arbitrary $p$. A duality relation for $I_W$ when $p$ and $k$ are interchanged, valid for asymptotic $k$, is observed. A method for calculating $I_W$ for any group $G$ is described. It is found that $I_W$ for $Z_m$, even for $m = 2$, distinguishes 3-manifolds quite effectively.
Recently a class of topological invariants associated with 3-manifolds have been discovered following Witten’s work on the Jones polynomial within the framework of a Chern-Simons gauge theory with the gauge group SU(2) \[1, 2, 3\]. In this paper we describe different procedures for calculating the Witten invariants \(I_W(M; G)\) associated with a compact, closed, orientable three dimensional manifold \(M\) for \(G\) an arbitrary semi simple Lie group or a finite abelian group. Remarkably, to our surprise, we find that the invariant \(I_W(M; Z_m)\) is a powerful invariant even for \(m = 2\). It can distinguish between manifolds which have the same fundamental group or even manifolds which are of the same homotopy type. The explicit form of \(I_W\) for the group SU(2) has been considered for a few special manifolds, notably for lens spaces \[4\]-\[7\].

There are two standard methods for constructing \(M\), an arbitrary closed orientable 3-manifold \[8\]. The first is the method of surgery which was used by Witten in \[1\] and the second is that of Heegard splitting which we briefly recall. Let \(h_g\) denote a handle body of genus \(g\) with boundary \(\Sigma_g\), a two dimensional Riemannian surface of genus \(g\). Let \(M_g\) denote the mapping class group of \(\Sigma_g\) and let \(\zeta \in M_g\). One can obtain an \(M\), which we sometime denote by \(M(g)\), by gluing together two copies \(h_{1g}\) and \(h_{2g}\) of \(h_g\) by identifying the surface \(\Sigma_{1g}\) with \(\zeta \Sigma_{2g}\). It is known that in this way one can generate all closed 3-manifolds \(M\) by choosing an appropriate genus \(g\) and \(\zeta \in M_g\). For example, for \(g = 1\), one obtains all lens spaces \(L(p,q)\) by this method.

It is known \[8\] further that the group \(M_g\) can be generated by Dehn twists \(\zeta_i\) around the cycles \(C_i\), \(i = 1, 2, \ldots, 2g + 1\) shown in figure 1. The Dehn twists \(\zeta_i\) satisfy a set of relations given in \[9\]. Furthermore, Heegard splitting can also be specified as follows \[8\]. Take two copies of \(h_g\). On one of them let \(d_i\), \(i = 1, 2, \ldots, g\) be the curves shown in figure 1 and on the other let \(\delta_i\), \(i = 1, 2, \ldots, g\) be a given set of \(g\) non intersecting curves. Then a 3-manifold \(M\) can be obtained by gluing the boundaries of these two \(h_g\’s\) such that the curves \(d_i\) are stitched along the curves \(\delta_i\), in any order. From the above descriptions of Heegard splitting it is obvious that for a given \(\zeta \in M_g\) only its action on the curves \(d_i\), in any order, is important. This implies that there are many equivalent choies for the element \(\zeta\) that produces a given \(M\).

We first give a representation of \(\zeta\) on the space of curves \(\{C_i\}\). This is constructed by the action of the Dehn twists on \(C_i\) and further using the braiding relations of \(\zeta_i’\)s to fix some constants. The representation so constructed automatically satisfies the remaining relations for \(\zeta_i\). We give
these representations for \( g = 2 \) and 3 explicitly. Our construction can be easily generalised for \( g > 3 \) as well.

The advantage of this construction is that given the curves \( \delta_i \) we can give a presentation of the fundamental group of \( \pi_1(M) \) in terms of generators and relations between them. For example, for the case of lens spaces \( L(p, q) \), i.e. \( M(1) \), the manifolds \( L(p, q) \) for a given \( p \) but arbitrary \( q \) (< \( p \)) all have the same \( \pi_1 \), namely \( \mathbb{Z}_p \). Moreover it is easier and more intuitive to specify the curves \( \delta_i \).

The Brieskorn manifolds \( \Sigma(p, q, r) \) considered by Freed and Gompf in [6] have the fundamental group \( \pi_1(\Sigma) \) of finite order \( \frac{4}{pqr}(p^{-1} + q^{-1} + r^{-1} - 1)^{-2} \) if \( p^{-1} + q^{-1} + r^{-1} > 1 \) and are examples of 3-manifolds which have a simple surgery description. Other values of \( p, q, r \) correspond to \( \pi_1 \) of infinite order. Furthermore manifolds with \( H_1 = 0 \) (i.e. homology 3-spheres) can be constructed if \( p, q, r \) are relatively prime [10]. We will briefly consider manifolds of type \( \Sigma(p, q, r) \) and their generalisations — the so called Seifert manifolds [6] — to illustrate the nature and power of the Witten invariant.

To obtain a representation of \( \zeta_i \), we first pictorially follow its action on the curves \( \{C_i\} \). Thus e.g. \( \zeta_2 \) can be written as, with \( \vec{C} = (C_1, C_2, \ldots, C_{2g+1}) \),

\[
\zeta_2 \vec{C} = (C_1 + \epsilon C_2, C_2, C_3 + \tilde{\epsilon} C_2, \ldots)
\]

(1)

where \( \epsilon, \tilde{\epsilon} = \pm 1 \) and the dots denote the identity action on the corresponding elements. The braiding relations between \( \zeta_i \) are used to obtain relations, if any, between \( \epsilon \)'s and \( \tilde{\epsilon} \)'s. For \( g = 3 \) one obtains

\[
\begin{align*}
\zeta_1 \vec{C} & = (C_1, \epsilon_1 C_1 + C_2, C_3, C_4, C_5, C_6, C_7) \\
\zeta_2 \vec{C} & = (C_1 - \epsilon_1 C_2, C_2, \tilde{\epsilon}_1 C_2 + C_3, C_4, C_5, C_6, C_7) \\
\zeta_3 \vec{C} & = (C_1, C_2 - \tilde{\epsilon}_1 C_3, C_3, \epsilon_2 C_3 + C_4, C_5, C_6, C_7) \\
\zeta_4 \vec{C} & = (C_1, C_2, C_3 - \epsilon_2 C_4, C_4, \tilde{\epsilon}_2 C_4 + C_5, C_6, \epsilon_0 C_4 + C_7) \\
\zeta_5 \vec{C} & = (C_1, C_2, C_3, C_4 - \epsilon_2 C_5, C_5, \epsilon_3 C_5 + C_6, C_7) \\
\zeta_6 \vec{C} & = (C_1, C_2, C_3, C_4, C_5 - \epsilon_3 C_6, C_6, C_7) \\
\zeta_7 \vec{C} & = (C_1, C_2, C_3, C_4 - \epsilon_0 C_7, C_5, C_6, C_7)
\end{align*}
\]

(2)

The \( \epsilon \)'s and \( \tilde{\epsilon} \)'s in the above equation can be \( \pm 1 \) corresponding to the independent choices for the orientation of the curves \( C_i \). The \( \zeta_i \)'s for \( g = 2 \) can be obtained from equation (1) by first totally omitting \( \zeta \)'s and \( C \)'s with
In terms of the standard generators expansion of $p_i$, it is known that $M$ for any given $\zeta$, the contractible cycle $C_i$, fundamental group lens spaces $L_i$ of the mapping class group $C_i$ one can choose $p_i$ since $M$ of the manifolds $\zeta_i$. Note that $\delta_i$ which generates these representations for $\zeta_i$'s given above satisfy the following relations (see \text{[3]})

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1} \quad i = 1, 2, 3, 4$$

$$(\zeta_1 \zeta_2 \zeta_3)^4 = \zeta_5 \delta^{-1} \zeta_5 \delta$$

(3)

where $\delta = \zeta_4 \zeta_2 \zeta_2 \zeta_2 \zeta_3 \zeta_4$. The first equation above gives the braiding relations used to construct the representations given in equation (2).

**EXAMPLE**: $S^3$. The following choices for $\delta_i$ specify $S^3$ \text{[3]}: (i) $(\delta_1, \delta_2) = (C_2, C_4)$ and (ii) $(\delta_1, \delta_2) = (C_2 + C_3, C_4 - C_3)$. The corresponding $\zeta \in M_2$ which generates these $\delta_i$'s are $\zeta(i) = (\zeta_1 \zeta_2 \zeta_1)^{-\epsilon_1} \zeta_2 \zeta_3 \zeta_4$ and $\zeta(ii) = \zeta(i) \zeta_5^{-\epsilon_2}$ respectively. Note that $\zeta \in M_2$ and $\zeta_1 \zeta_2 \zeta_3 \zeta_2 \zeta_3 \zeta_4 \zeta \in M_2$ generate the same $M$ since $\zeta_1$, $\zeta_3$, and $\zeta_5$ do not affect the curves $\delta_i$.

Let us now consider the features discussed above for the case $g = 1$. (It is known that $M(1)$ are the lens spaces $L(p_2, p_1)$). Now, the generators of the mapping class group $M_1$ are the Dehn twists $\zeta_i$ around the curves $C_i$, $i = 1, 2$. Thus a representation of $\zeta_i$ would be

$$\zeta_1 = (C_1, \epsilon_1 C_1 + C_2)$$

$$\zeta_2 = (C_1 - \epsilon_1 C_2, C_2).$$

(4)

(In terms of the standard generators $S$ and $T$, $\zeta_1 = T$ and $\zeta_2 = -STS$). The manifolds $M(1)$ can also be specified by the curve $\delta_1$. Thus in general $\zeta_1 = p_1 C_1 + p_2 C_2$. Using the fact that $\zeta$ and $\zeta_1^{\epsilon_1}$ give the same manifold, one can choose $p_1 < p_2$. This is precisely the condition one has for the lens spaces $L(p_2, p_1)$, i.e. $p_1 < p_2$. We know for $M(1)$, i.e. $L(p_2, p_1)$, that the fundamental group $\pi_1(L(p_2, p_1)) = \mathbb{Z}_{p_2}$ is independent of $p_1$, the coefficient of the contractible cycle $C_1$ in $\delta_1$. Moreover for $M(1)$ we know the generators $\zeta$ for any given $p_1$ and $p_2$ : for $p_1 = 1$, $\zeta = ST^{p_2}S$ and for $p_1 \neq 1$, $\zeta = S \prod_i (T^{a_i} S)$ where $a_i$ are the integers appearing in the continued fraction expansion of $\frac{p_2}{p_1}$ \text{[8]}

$$\frac{p_2}{p_1} = a_n - (a_{n-1} - \cdots (a_3 - (a_2 - (a_1)^{-1})^{-1})^{-1})^{-1}.$$

(5)

An analogous method for $g > 1$, i.e. a method to express $\zeta$ in terms of $\zeta_i$ for the given set of integers $p_i$ and $q_i$, $i = 1, \ldots, 5$ does not exist \text{[11]}. 

3
We now turn to methods of describing $I_W(M(g))$. In order to calculate $I_W(M(g))$ it is necessary to represent the geometrical picture given above in terms of operators in the Hilbert space $H_g$ associated with the handle body $h_g$. This is done by representing the geometrical operators $\zeta_i$ by operators $O_i$ on $H_g$. Witten’s invariant is then a certain matrix element involving the diffeomorphism operations used when $\Sigma_1 g$ and $\Sigma_2 g$ are identified. Symbolically, for example $I_W(M(1)) = \langle v_0, \prod_i S^{a_i} T^i v_0 \rangle$ where $v_0$ is the vacuum element in the Hilbert space $H_g$ [1].

Now consider the example of $g = 1$. For the group SU(2) and level $k$ the basis for the (finite dimensional) Hilbert space is labelled by an integer $i \leq \frac{k}{2}$. The representation of the mapping class group generators $S$ and $T$ are given by

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(2i+1)(2j+1)}{k+2}\right)$$
$$T_{ij} = \delta_{ij} t_j$$

(6)

where $t_j = e(\Delta_j - \frac{c}{2r})$, $e(a) = e^{2\pi\sqrt{-1}a}$ and $\Delta_j = \frac{j(j+1)}{k+2}$. Using this, we can write down the invariant of the manifolds $M_1$. For example, since $L(p, 1)$ is generated by $\zeta = ST^p S$ (geometrically this corresponds to the identification $\delta_1 = C_1 + pC_2$ noted before) the corresponding invariant is given by [1]

$$I_W(L(p, 1)) = \frac{2}{k+2} \left[ \sum_{n=0}^{k+1} \sin^2 \left(\frac{\pi n}{k+2}\right) \sum_{n=0}^{k+1} \exp\left(i\frac{\pi pn^2}{2(k+2)}\right) \right].$$

(7)

Similar expressions for $L(p, q)$ can be found in [3, 6]. The above expression can be evaluated exactly for any $p$ and $k$. Following straightforwardly the method given in [1], one gets the following result for the above $I_W$. Let $(x, y)$ denote the highest common factor between the integers $x$ and $y$. Define $\omega$ by $(p\omega + 2) = 2Nr$ where $r = k + 2$ and $N$ is some integer (such an $\omega$ is guaranteed to exist if $(p, 2r)$ divides $(2, 2r) = 2$). For $p$ odd, let $(p, 4r) = t$. Then

$$I_W = \sqrt{\frac{t}{2r}}$$
$$I_W = \sqrt{\frac{2}{r}} |\sin(\frac{\pi}{4r} p\omega^2)|$$

(8)
respectively for $t > 1$ and $t = 1$. For $p$ even, let $(\frac{p}{2} \cdot 2r) = t$. Then

\[ I_W = \frac{t}{2r} \tilde{G}(\frac{p}{2t}, 0, \frac{2r}{t}) \]

\[ I_W = \sqrt{\frac{2}{r}} \]

\[ I_W = \frac{2}{\sqrt{r}} |\sin(\frac{\pi}{4r} p \omega^2)| \]

\[ I_W = 1 + \frac{(-1)^{pr/2}}{\sqrt{r}} |\sin(\frac{\pi}{4r} p \omega^2)| \]  \hspace{1cm} (9)

respectively for $t > 2$, $t = 2$ (r even), $t = 2$ (r odd) and $t = 1$, where $\tilde{G}(a, b, l) = \sum_{n=0}^{l-1} \exp(i \frac{2π}{r}(an^2 + bn))$. By using the above expressions and the properties of the function $\tilde{G}$ given in [7], $I_W(L(p, 1))$ can be evaluated explicitly for any $p$ and $k$. 

For large $r$, we can evaluate $I_W$ in equation (7) asymptotically using Fourier transformation. First let $\tilde{G}(a, b, l) = \sum_{n=0}^{l-1} f(n)$. Noting that $f(x)$ can be expressed in terms of its Fourier transform $F(\kappa)$ as

\[ f(x) = \int d\kappa e(\kappa x) F(\kappa), \]  \hspace{1cm} (10)

the summation in $\tilde{G}(a, b, l)$ can be carried out to give

\[ \tilde{G}(a, b, l) = \int d\kappa \frac{1 - e(\kappa l)}{1 - e(\kappa)} F(\kappa) \]  \hspace{1cm} (11)

where $F(\kappa) = \sqrt{\frac{1}{2\pi}} e(-\frac{(b^2 + kl)^2}{4al})$. Since [8] can be expressed as (see [7]) $I_W = \frac{1}{4r} (\tilde{G}(p, 0, 4r) - \tilde{G}(p, 4, 4r))$, we can write after a few simple steps

\[ I_W(L(p, 1)) \approx \sqrt{\frac{2}{r}} \frac{1}{\sqrt{p}} \int_0^\infty d\kappa \frac{1}{1 - e(\kappa)} \frac{1}{l} c\left(-\frac{\kappa^2 r}{p}\right) \sin^2 \frac{2\pi \kappa}{p} \]  \hspace{1cm} (12)

in the limit $r \to \infty$. The integrand has poles for integer $\kappa$ and the corresponding residue is periodic in $\kappa$ with period $p$. Hence, we can write

\[ I_W(L(p, 1)) \approx \frac{1}{4} \sqrt{\frac{2}{r}} \frac{4}{\sqrt{p}} \sum_{\kappa=0}^{p-1} c\left(-\frac{\kappa^2 r}{p}\right) \sin^2 \frac{2\pi \kappa}{p} \]  \hspace{1cm} (13)
Furthermore, since the summand remains the same under $\kappa \to p - \kappa$, the range of summation can be halved and the above asymptotic formula then corresponds exactly to that given in [8]. As noted there, the summand $\frac{1}{\sqrt{p}} \sin^2 \frac{2\pi \kappa}{p}$, which can be considered as the coefficient of $\frac{1}{\sqrt{r}}$ in the large $r$ asymptotic expansion of $I_W(L(p, 1))$, is equal to the Ray-Singer torsion. Moreover we note an interesting duality relation: the above asymptotic expansion for large $r$ of $I_W(L(p, 1); r)$ is very similar to the expression for $I_W(L(2r, 1); \frac{p}{2})$, the Witten invariant for the lens space $L(2r, 1)$ at level $\frac{p}{2}$. However the significance of this relation eludes us.

A few comments are in order. First, for $SU(2)$ the large $r$ limit of the Turaev-Viro invariant (conjectured to be equal to $|I_w|^2$ — see [3, 12, 11, 13] and references there) is expected to be related to the small cosmological constant limit of the partition function of the respective manifold with Einstein-Hilbert action [14]. Thus evaluating $I_W(M)$, either exactly or asymptotically for as large a class of manifolds as possible is of some interest. Secondly, we note that in the large $r$ ($\to \infty$) limit, one loses some information in $I_W(L(p, 1)) \approx \frac{4}{\sqrt{p}} \sin^2 \frac{2\pi \kappa}{p}$. In this limit $I_W$ is zero for both $L(2, 1)$ and $L(1, 1)$ unlike Ray-Singer torsion which for $L(p, 1)$ is $\frac{4}{\sqrt{p}} \sin \frac{\pi \kappa}{p}$ (see, for instance, [8]) and is zero only for $L(1, 1)$. We further observe that for a compact connected semi simple Lie group $G$ the expressions for the mapping class group elements $S$ and $T$ are available which can be readily used to obtain lens space invariants for the group $G$ [3].

We now turn to the manifolds $M(g)$ obtained by Heegard splitting on genus $g$. In order to construct the most general 3-manifold obtainable from the genus $g$ Heegard splitting and to determine $I_W(M(g))$ an analogous Hilbert space procedure is necessary. A convenient set of basis vectors for $H_g$ needs to be constructed. We first note that the set of basis vectors is not unique and hence one must be able to relate any two sets of basis vectors. This is achieved, as we will see, by using fusion and braiding matrices of the associated RCFT. Next, given the Hilbert space and a set of basis vectors in it, the geometrical operators $\zeta_i$ need to be represented as operators in the Hilbert space. Once these operators are known the Witten invariants $I_W(M(g))$ can be determined.

Note that a Riemannian surface $\Sigma_g$ of genus $g$ can be constructed from $(2g - 2)$ trinions (spheres with three holes, see figure 2) by gluing them along their holes. However, the trinions can be put together in many ways; each will
correspond to one particular basis in the Hilbert space of \( \Sigma_g \). Representing
a trinion by a trivalent vertex (three edges meeting at a vertex, figure 2), a
graph can be associated to each basis. For example, \( \Sigma_2 \) can be represented
in two different ways with the corresponding graphs as shown in figure 3. The graphs corresponding to different bases are related by the “fusion” move shown in figure 4 [13].

In the language of rational conformal field theory (RCFT) any two bases
will be related through fusion and braiding matrices. Given an RCFT with
primary fields labelled by \( i, j, k, \ldots \) and with fusion rules \( N^k_{ij} \) one can con-
struct chiral vertices \( \Phi^k_{ij} \) [16]. Each such chiral vertex can be associated with
a trinion and hence with a trivalent vertex of the graph of \( \Sigma_g \), its edges being
labelled by the corresponding \( i, j \) and \( k \). Note that the labels of the edges
of a graph are restricted such that the edges meeting at a trivalent vertex
have labels \( (i, j, k) \) compatible with the fusion rules of the associated primary
fields \( (i, j, k) \) of the RCFT under consideration. Then fusion \( (F_{ij}^{j_1j_2j_3j_4}) \) and
braiding \( (B_{ij}^{j_1j_2j_3j_4}) \) matrices which relate different bases can be written as
shown in figure 5 (see e.g. [17]). The braiding matrix can be expressed in
terms of fusion matrix as follows (see, for example, [17]):

\[
B_{ij}^{j_1j_2j_3j_4} = (-1)^{j_1+j_4-i-j}e\left(\frac{1}{2}(\Delta_{j_1} + \Delta_{j_4} - \Delta_i - \Delta_j)\right)F_{ij}^{j_1j_2j_3j_4}
\]

(14)

where \( \Delta_i \) is the conformal weight of the primary field labelled \( i \).

Now we can use the RCFT techniques to write down the representa-
tions of \( \zeta_i \), the Dehn twists around the curves \( C_i \)'s. First let us choose for \( \Sigma_g \) a
basis whose graph and the labelling are as shown in figure 6. \( \zeta_1(\zeta_5) \) can be
easily represented as \( T_{a_1}(T_{a_5}) \), the twist matrix. \( T_a \) will act diagonally on the
edge labelled \( a \), with an eigenvalue which is a pure phase. \( \zeta_3 \) can be written
easily in the dual basis obtained by “fusion” moves. In terms of the original
basis, it will be of the form \([F^{-1} T F]\). To write down \( \zeta_2 \) and \( \zeta_4 \) we need the
switching matrix \( \sigma(k) \) [18, 19, 20, 17] which interchanges the two cycles of a
torus with a hole labelled \( k \), as shown in figure 7. An explicit expression for
the \( (il)^{th} \) element of the switching matrix \( \sigma \), given first in [20], is as follows:

\[
\sigma(k)_{il} = \sum_q e(\Delta_q - \Delta_i - \Delta_l)S_{0q}B_{il}^{kqi}
\]

(15)

where \( j \) labels the edge corresponding to the puncture on the torus and other
quantities are defined before. It can be easily seen, along the lines of [17],

7
that when \( j = 0 \) the switching matrix becomes

\[
\sigma(0)_{il} = S_{il}
\]  

(16)

where \( S_{il} \) is defined in equation (3).

Thus using the switching matrix \( \sigma \) we can switch \( C_2 \) cycle to the one isomorphic to \( C_1 \), the Dehn twist around which is represented by a twist matrix \( T \). Thus \( \zeta_2(\zeta_4) \) can be represented as \( \sigma T \sigma (= -T^{-1} \sigma T^{-1}) \). This will give a representation for \( \zeta_i \in \mathcal{M}_2 \). Such a representation can also be written down for any genus \( g \) by noting that the switching matrix \( \sigma(jk) \) for a torus with two holes labelled \( j \) and \( k \) can be expressed in terms of \( \sigma(j') \) using fusion rules \([18, 20]\).

Using the above representation, one can construct a topological invariant \( I_W(M(g)) \) for an \( M \) generated by \( \zeta_1 \in \mathcal{M}_g \). One starts with the vacuum element of a basis (a graph with trivial labels for all its edges) denoted by \( I \) in the Hilbert space \( \mathcal{H}_g \). Acting on this basis with the operator \( R_\zeta \) in the Hilbert space \( \mathcal{H}_g \) representing the mapping class group element \( \zeta \in \mathcal{M}_g \) and then taking its inner product with the vacuum, one obtains \( i.e \ < I| R_\zeta | I > \), an invariant of the manifold. The topological nature of this invariant has been proved for \( SU(2) \) in \([17]\) and can be proved for other groups also along the same lines. This process of taking the inner product closely parallels the construction of \( M_3(\zeta) \) by Heegard splitting. Furthermore, the normalisation of this invariant is fixed so as to have the factorisation property \( I(M \# M') = I(M)I(M') \) where \( M \# M' \) denotes the connected sum of the manifolds \( M \) and \( M' \). The fact that \( I(M) \) is a topological invariant when constructed in this way using RCFT methods can be shown along the lines given in \([17]\).

The Dehn twists \( \zeta_i, i = 1, 2, \ldots, 5 \) can be represented, with \( R_i \equiv R_\zeta_i \), as follows:

\[
\begin{align*}
R_1 &= \delta_{a_1a_1'} \delta_{a_2a_2'} \delta_{bb'} t_{a_1} \\
R_2 &= \delta_{a_2a_2'} \delta_{bb'} e(-\frac{c}{12}) \sum_k e(\Delta_k) \sigma(0)_{0k} B_{a_1a_1'}^{a_2a_2} \\
R_3 &= \delta_{a_1a_1'} \delta_{a_2a_2'} \sum_B F_{B}^{a_2a_1a_1a_2} t_B F_{B}^{a_1a_1a_2a_2} \\
R_4 &= \delta_{a_1a_1'} \delta_{bb'} e(-\frac{c}{12}) \sum_k e(\Delta_k) \sigma(0)_{0k} B_{a_2a_2'}^{a_1a_1} \\
R_5 &= \delta_{a_1a_1'} \delta_{a_2a_2'} \delta_{bb'} t_{a_2}.
\end{align*}
\]  

(17)
In $S_{ij}$ and $t_a$ above, $i,j$ and $a$ are multi index labels in the case of arbitrary $G$. The fusion and braiding matrices will then contain parameters specifying different couplings of representations that are permitted. For example, for SU(3) these would include the so called $F$ and $D$ type couplings \cite{21}. The fusion and braiding matrices $F_{b_b}^{a_1a_2a_3}$, $B_{aa'}^{bka}$ are represented in figure 5. A systematic procedure to determine these quantities for arbitrary compact Lie group $G$ is given in \cite{18, 19}. With the help of these expressions $I_W(M)$ can be constructed. The method outlined clearly extends to the genus $g$ case.

For the group SU(2) $S_{ij}$ and $t_a$ are given in equation (6) and the fusion and braiding matrices can be expressed in terms of quantum 6-j symbols as in, for example, \cite{17}. Using these expressions, the invariant of a manifold $M$, for a $\zeta \in M_g$ expressed in terms of $\zeta_i$'s, can be written down analogous to (7) and, for the group SU(2), evaluated asymptotically using the properties of 6-j symbols.

We now observe that we can get all $M(g-1)$ specified by the curves $(\delta_1, \delta_2, \cdots, \delta_{g-1})$ as a subclass of $M(g)$ specified by the curves $\delta_i$ as follows:

$$(\delta_1, \delta_2, \cdots, \delta_{g-1}, \delta_g) = (\delta_1, \delta_2, \cdots, \delta_{g-1}, C_{2g})$$

or any of the equivalent configuration. In fact it can be easily seen that the above assignment corresponds to $M_3(g) = M_3(g-1) \# S^3 = M_3(g-1)$. With the normalisation of the invariants that we adapt as in \cite{17}, this equality follows quite trivially for the invariants also (since, in this normalisation, $I_W(S^3) = 1$).

This can be specifically seen as follows for $g = 2$. In this case the manifolds $M(1)$ are specified by $\delta_1 (= p_1 C_1 + p_2 C_2)$. The corresponding specification for $\delta_2$ is given by $(\delta_1, \delta_2) = (\delta_1, C_4)$. Let $\zeta \in M_2$ denote the corresponding generator. Then it can be easily seen that $\zeta = \zeta_{1,2} \zeta_4 \zeta_5 \zeta_4$ where $\zeta_{1,2}$ is a function of $\zeta_1$ and $\zeta_2$ only. First note that in calculating the invariant of the manifold, $\zeta$ acts on the graph of $\Sigma_2$ (see figure 6) with vacuum labels. Hence from the expressions for $R_4$ and $R_5$ given in equation (17) and from the properties of the braiding matrices $B$ (see \cite{4}, for example), it follows that the elements $\zeta_4 \zeta_5 \zeta_4$ in $\zeta$ act trivially. Thus one gets

$$< I | R_{\zeta} | I > = < I | R_{\zeta_{1,2}} | I > .$$

Now it is easy to see that the above expression reduces to the corresponding expression calculated for $I_W(M(1))$. The key is to note that (i) the edge
labelled \( b \) (in figure 6) has its label unchanged by the action of \( \zeta \) and hence has always the vacuum label; (ii) this is the same label as on the puncture of the torus used in the switching matrix \( \sigma \) needed to calculate \( \zeta_2 \in \mathcal{M}_2 \); and (iii) therefore, the switching matrix \( \sigma(j) \) involved in calculating \( R_{\zeta_2} \) has \( j = 0 \). Furthermore \( \sigma(0)_{ij} = S_{ij} \) as we noted before. Since \( S_{ij} \) is the representation of the generator \( S \) of the modular group of transformations acting on torus, it follows that \( I_W(M(2)) = I_W(M(1)) \) where \( M(2) \) and \( M(1) \) are specified by the curves \( (\delta_1, \delta_2) \) and \( (\delta_1) \) respectively.

The absolute value of the invariant \( I_W \) of a manifold \( M \) generated by \( \zeta \in \mathcal{M}_2 \) should be same as that for a manifold \( M' \) generated by \( \zeta_1^a \zeta_3^a \zeta_5^b \zeta_1^b \zeta_3^b \zeta_5^b \) \( \in \mathcal{M}_2 \). The action of \( \zeta_1 \) and \( \zeta_5 \) for \( SU(2) \) are just pure phases and the above equality then is obvious. However, for \( \zeta_3 \) also this equality follows since \( \zeta_3 \sim F^{-1} TF \) and \( F|I > ( \text{and similarly} < I|F) \) is trivial.

We now turn to some explicit computations. We note that the Witten invariants \( I_W \) can also be evaluated for the case of finite groups, say, \( Z_m \) (see also \[22\]). Even for such a group \( I_W \) is capable of distinguishing different manifolds, in particular distinguishing the Poincare manifold \( X_P \) from the 3-sphere \( S^3 \), as we shall show below. For details, see \[6, 18, 8\]. We illustrate our general discussions by evaluating \( I_W \) using the Heegard decomposition approach.

Following \[18\] we observe that the \( S \) and \( T \) matrices for the group \( Z_m \) can be represented as

\[
S_{ij} = S_{00} e(\Delta_i + \Delta_j - \Delta_{i+j}) \\
T_{ij} = \delta_{ij} e(\Delta_i - \psi) \equiv \delta_{ij} t_i
\]

where \( \Delta_i = (\frac{i^2}{2m} + \frac{i}{2} (\frac{1}{m} - \frac{1}{2})) \) and \( i = 0, 1, 2, \ldots, m-1 \). In the above equation \( \psi \) is a phase to be chosen such that the \( S \) and \( T \) matrices obey \( S^2 = (ST)^3 = \mathcal{C} \) where \( \mathcal{C} \) is a charge conjugation matrix. This relation further gives \( S_{00} = \frac{1}{\sqrt{m}} \) for the group \( Z_m \). However in the following an explicit form of \( \psi \) will not be relevant. The fusion matrix for \( Z_m \) turns out to be \( F_{ab}^{cdef} = 1 \) with the labels \( a, b, \ldots \) satisfying suitable decomposition conditions. Using these expressions the braiding matrix \( B \) and the switching matrix \( \sigma(k)_{ij} \) can also be calculated easily. Furthermore, the representations for \( R_{\zeta_i}, \zeta_i \in \mathcal{M}_2 \) can also be written down as given in equation (17). Note that \( S_{0j} = S_{00} \) and \( I_W(S^3) = S_{00} \) for the group \( Z_m \).
The invariant for the lens spaces \( L(p,1) \) and \( L(p_1p_2-1,p_2) \) turn out to be as given below, where the lens space \( L(p_1p_2-1,p_2) \) is generated by \( ST^{p_1}ST^{p_2}S \in M_1 \) as explained above the equation (3):

\[
I_W(L(p,1)) = S_{00}^2 |\sum_{i=0}^{m-1} e(p\Delta_i)|
\]

\[
I_W(L(p_1p_2-1,p_2)) = S_{00}^3 |\sum_{a,b} e(\phi)|
\]

with \( \phi = (p_1\Delta_a + p_2\Delta_b + \Delta_a + \Delta_b - \Delta_{a+b}) \). Note that even for \( G = Z_m \), \( I_W(L(p,q)) \neq I_W(L(p,1)) \). Recall \( L(p,q) \neq L(p',q) \) if \( p \neq p' \) since \( \pi_1(L(p,q)) = Z_p \). Also, \( L(p,q) \) is homeomorphic to \( L(p,q') \) if \( qq' = \pm 1 \mod p \). They are of the same homotopy type if \( qq' = \pm m^2 \mod p \). Thus for instance,

\[
L(7,1) = L(7,6)
\]

\[
L(7,2) = L(7,4).
\]

(22)

But \( L(7,1) \neq L(7,2) \) although they are of the same homotopy type. Even with \( G = Z_2 \) \( I_W \) can distinguish \( L(7,1) \) from \( L(7,2) \) as can be seen from table 1 where \( I_W \) values for the group \( SU(2) \) are taken from [3]. We further calculate \( I_W \) for the Poincare manifold \( X_P \) denoted as the Seifert space \( X(-2,3,5) \), following [3]. A general Seifert space is denoted by \( X(r_1,r_2,\ldots,r_n) \), \( r_i = \frac{p_i}{q_i} \) and its invariant is given by

\[
I_W(X) = \sum_{\alpha_i,\beta}(S_{0\beta})^{1-n} \prod_{i=1}^{n}[(M_i)_{0\alpha_i}S_{\alpha_i\beta}]
\]

(23)

where \( M_i \in M_1 \) are the generators of the lens spaces \( L(p_i,q_i) \). Thus for the Poincare manifold \( X_P \) the invariant is given by

\[
I_W(X_P) = \sum_{\alpha_1,\alpha_2,\alpha_3,\beta} S_{0\beta}^{-2}(ST^{-2}S)_{0\alpha_1}(ST^3S)_{0\alpha_2}(ST^5S)_{0\alpha_3}S_{\alpha_1\beta}S_{\alpha_2\beta}S_{\alpha_3\beta}
\]

(24)

which becomes

\[
I_W(X_P) = S_{00} \sum_{\alpha_i,\alpha_i,\beta} t_{a_1}^{-2}t_{a_2}^3t_{a_3}^5 \prod_{i=1}^{3}(S_{\alpha_i\alpha_i}S_{\alpha_i\beta})
\]

(25)
This invariant thus distinguishes between the Poincare manifold $X_P$ and the 3-sphere $S^3$. Thus the Witten invariant $I_W$, even for the group $Z_m$ is capable of distinguishing between different manifolds rather efficiently.

In view of the remarkable ability of finite groups (even $Z_2$) to distinguish 3-manifolds topologically, a few comments are in order. The “finite abelian group” structure considered in this paper can arise in two different ways. First, as solutions to the set of conditions which define a rational conformal field theory. This reduces, because of the simple nature of the fusion rules for the abelian groups, to a group cohomology problem which was solved by Moore and Seiberg in [18]. Secondly, the representation matrices for elements of the mapping class group similar to the ones used in our calculation also arise by considering a Chern-Simons gauge theory with gauge group U(1). Thus the finite abelian groups considered in this paper are not the ones considered in [22]. In these works a Chern-Simons gauge theory with a finite gauge group is considered.

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FIGURE CAPTIONS

Figure 1: Genus $g$ Riemannian surface $\Sigma_g$ showing the curves $C_i$ and $d_i$.

Figure 2: Trinion (a sphere with three holes) and its graph, a trivalent vertex.

Figure 3: $\Sigma_2$ with two different graphs.

Figure 4: The “fusion” move connecting different graphs.

Figure 5: Fusion and Braiding matrices $F_{ij}^{j_1j_2j_3j_4}$ and $B_{ij}^{j_1j_2j_3j_4}$ respectively relating two different labelled bases.

Figure 6: A labelled basis for $\Sigma_2$ used in the paper.

Figure 7: Switching matrix $\sigma(k)$.

| $L(5,1)$ | $0.5(1+i)$ | 0.707 | 0.5i |
|----------|------------|-------|------|
| $L(5,2)$ | 0.707i     | −0.707| 0.5  |
| $L(7,1)$ | 0.5(1−i)  | 0.707i| 0.354(1−i) |
| $L(7,2)$ | 0.707     | 0.707i| 0.354(−1+i) |
| $L(25,4)$| −0.707    | 0.707 | 0.5  |
| $L(25,9)$| 0.5(−1+i) | 0.707 | 0.5  |

| $X_p$ | 0 |   |   |
|-------|---|---|---|
| $S^4$ | 0.707 | 0.707 | 0.5 |

Table 1: $I_W$ for various manifolds for $Z_2$ and $SU(2)_k$, $k = 1, 2$. 

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