Compact bordered Riemannian surfaces as vibrating membranes: an estimate à la Hersch-Yang-Yau-Fraser-Schoen

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Abstract. We try to present an estimate relating the first Dirichlet and Neumann eigenvalues of a compact bordered Riemannian surface.

1 Introduction

In a recent paper Fraser-Schoen [3] took advantage of Ahlfors’ conformal representation of compact bordered Riemann(ian) surfaces Σ over the disc to obtain a Steklov eigenvalue estimate in terms of their topological invariants (the genus \( p \geq 0 \) and the number of contours \( r \geq 1 \), i.e. boundary components). We do not need to recall here the rich history antedating the result of Ahlfors 1950 [1] (presented already in Spring 1948 at Harvard), except for saying that this (pre)history is (surprisingly?) confined to the schlicht case (i.e., \( p = 0 \)) which involves primarily a contribution of Riemann 2 and subsequently Schottky 1877, Bieberbach 1925, Grunsky 1937–41. In the present note we have attempted to use the same method (as Fraser-Schoen) to get a similar estimate for the classical vibrating membrane problem (e.g., Poisson 1829, Helmoltz 1862, Clebsch 1862, Lord Rayleigh (=J. W. Strutt) 1894–96, Weyl 1911, Courant 1918, Faber–Krahn 1923–24, etc.):

\[
-\Delta u = \lambda u,
\]

where \( \Delta \) denotes the Laplacian (of Beltrami 1867) attached to the Riemannian metric. As the nature of the question seems to impose it one must not focalize on the fixed membrane (under Dirichlet boundary condition \( u = 0 \) on \( \partial \Sigma \)) nor on the free membrane problem (under the Neumann boundary condition \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Sigma \), where \( n \) is the normal to the boundary \( \partial \Sigma \)), but rather more consider both problems in some natural symbiosis (suggested by the Pythagorean geometry of the sphere \( x_1^2 + x_2^2 + x_3^2 = 1 \)). Then the sought for estimate becomes very straightforward (indeed completely parallel to Hersch 1970 [6]). To picture out

1Compare, Nehari, 1950, Trans. AMS, p. 258.
2A Riemann’s Nachlass worked out by H. Weber, compare the historical interrogations raised by Bieberbach in [2].
3For accurate references we refer the interested reader to the bibliography in [4], which is by far not exhaustive, e.g., a serious omission is R. Courant, Conformal mapping of multiply connected domains, Duke Math. J. 5 (1939), 814–823.
4Precise references are given in Kuttler–Sigillito, Eigenvalues of the Laplacian in two dimensions, SIAM Review 26 (1984), 163–193.
the right historical perspective as a commutative diagram, recall that Yang-Yau 1980 [2] Prop., p. 58] generalized the first estimate of Hersch [6] Inequality (1), p. 1645] for Riemannian metrics on the sphere to arbitrary closed (oriented) surfaces, whereas the present note tries to achieve the same goal regarding the second estimate of Hersch [6] Inequality (2), p. 1646] involving bordered surfaces topologically equivalent to the disc. Hoping that the understanding of the (newcomer) author is trustful, the key trick seems to use as “isoperimetric” model not the flat round disc but rather the (north) hemisphere of the (unit) sphere, which “sounds” better.

Numerical justification: Indeed, comparing the quantity $\lambda_1 A$ (where $\lambda_1$ is the first Dirichlet eigenvalue, and $A = \text{area}$) we get for the disc $\lambda_2 \pi \approx 5.783 \pi$ (where $\lambda_2 \approx 2.404825576$ is the first positive zero of the Bessel function $J_0$), while for the hemisphere we have $2 \cdot 2 \pi \approx 4 \pi$ which has a gravest fundamental tone (than the planar disc). In contradistinction for the free membrane problem the quantity $\mu_1 A$ ($\mu_1$=first nonzero Neumann eigenvalue) has now to be maximized for a “good sounding”! We find for the disc $\mu_1 A = \mu_2 \pi \approx 3.390 \cdot \pi$, where $\mu_2 \approx 1.8411837813$ is the first positive zero of the Bessel function $J'_1$, while for the hemisphere we have $2 \cdot 2 \pi$ which is larger (hence “better”).

2 An inequality extending the one of Hersch as a bordered avatar of the one by Yang-Yau

Proposition 2.1 Let $\Sigma = \Sigma_{p,r}$ be a compact bordered Riemannian orientable surface of genus $p$ with $r$ contours of total area $A$. Denote by $\lambda_1$ the first Dirichlet eigenvalue for the problem (1), and by $\mu_1 \leq \mu_2$ the first two non-zero Neumann eigenvalues. Assume the existence of a conformal mapping $f: \Sigma \to D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ to the disc of degree $d$. Then we have the inequality

$$\left(\frac{1}{\lambda_1} + \frac{1}{\mu_1} + \frac{1}{\mu_2}\right) \frac{1}{A} \geq \frac{1}{d} \frac{4}{\pi}.$$  

(2)

Remark 2.2 Ahlfors [1 §4, pp. 122–133] showed that for such a Riemann surface there is always a holomorphic branched covering to the disc $D^2$ of degree $\leq r + 2p$, whereas the present author modestly improved the degree bound as being $\leq r + p$ (compare [4]). Hence the degree $d$ involved in inequality (2) can be taken as $r + p$. Of course for some particularized Riemann surfaces one can hope to be more economical.

Remark 2.3 In the case where the topology is simple $\Sigma \approx D^2$ then by the Riemann mapping theorem we may choose $d = 1$ and inequality (2) turns into an equality for the (unit) hemisphere $H = S^2 \cap \{x_3 \geq 0\}$ as in this case $\lambda_1 = \mu_1 = \mu_2 = 2$. (This is of course already observed in Hersch [6].)

Proof. We merely have to follow the idea of conformal transplantation of Pólya-Szegő (1951), as elaborated subsequently by Hersch 1970 [6] and Yang-Yau 1980 [7], conjointly with the variational characterization of eigenvalues (Poincaré 1890, Rayleigh 1894, Fischer 1905, Ritz 1908, Courant 1920, Pólya-Schiffer 1954, Hersch 1961 [3] [6]. So let $f: \Sigma \to D^2$ be our conformal mapping.

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[1] Accurate references as on p. 100 of C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, 1980.

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As it will be soon apparent it is more convenient to work with the (north) hemisphere (instead of the flat disc)

\[ H = S^2 \cap \{ x_3 \geq 0 \} \quad \text{of} \quad S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \} . \]

The first (non-zero) eigenvalues \( \lambda_1 \) and \( \mu_1 \) admits a variational characterization as the absolute minimizers of the Rayleigh quotient:

\[
R[u] = \frac{\int_{\Sigma} |\nabla u|^2 \, dv}{\int_{\Sigma} u^2 \, dv} ,
\]

where in the Neumann case orthogonality to the constant functions (eigenfunctions for \( 0 = \mu_0 \)) imposes the extra side-condition \( \int_{\Sigma} u \, dv = 0 \). Likewise Hersch established in \([5]\) a variational characterization for sums of reciprocals of eigenvalues. In our situation this gives:

\[
\frac{1}{\lambda_1} + \frac{1}{\mu_1} + \frac{1}{\mu_2} = \max( R[u_1]^{-1} + R[v_1]^{-1} + R[v_2]^{-1} ) ,
\]

where \( u_1 \) satisfies the Dirichlet and \( v_1, v_2 \) the Neumann boundary condition. The method is to pull-back (transplant via \( f \)) the best functions on the target to get competitive trial functions at the source. So on the hemisphere \( H \subset \mathbb{R}^3 \ni (x_1, x_2, x_3) \) we consider the ambient coordinate functions: \( x_3 \) verifying the Dirichlet condition, and \( x_1, x_2 \) verifying the Neumann condition. The pull-backs \( x_i \circ f \) are eligible for the variational principle, since after post-composing \( f \) by a suitable conformal automorphism of the hemisphere we may balance the center of gravity \( G = (\int_{\Sigma} x_1 \circ f \, dv, \int_{\Sigma} x_2 \circ f \, dv) \in \mathbb{R}^2 \) so as to make it coincide with the origin \((0, 0)\). This involves a topological argument initiated by Szegö 1954 (later Weinberger 1956), which in our setting is (brilliantly) exposed in Hersch 1970 \([6, \text{Point 2., p. 1646}]\)). We thus arrive at the inequality:

\[
\frac{1}{\lambda_1} + \frac{1}{\mu_1} + \frac{1}{\mu_2} \geq \frac{\int_{\Sigma} (x_3 \circ f)^2 \, dv}{\int_{\Sigma} |\nabla (x_3 \circ f)|^2 \, dv} + \frac{\int_{\Sigma} (x_1 \circ f)^2 \, dv}{\int_{\Sigma} |\nabla (x_1 \circ f)|^2 \, dv} + \frac{\int_{\Sigma} (x_2 \circ f)^2 \, dv}{\int_{\Sigma} |\nabla (x_2 \circ f)|^2 \, dv} .
\]

Each of the integrals occurring in the denominators \( \int_{\Sigma} |\nabla (x_i \circ f)|^2 \, dv \) are equal to \( d \int_{\Sigma} |\nabla x_i|^2 \, dv = \frac{4\pi}{3} \) (by conformal invariance of the Dirichlet integrand, compare Yang-Yau \([7]\) Lemma, p.59, (ii))). Adding up the numerators, we obtain, as \( \sum_{i=1}^3 (x_i \circ f)^2 \equiv 1 \) (\( f \) taking values in the unit sphere), finally \( \int_{\Sigma} dv = A \). This complete the proof of the proposed inequality \([4]\). \( \blacksquare \)

One can also use merely the simple variational characterization of the first eigenvalues to get first

\[
\mu_1 \int_{\Sigma} (x_i \circ f)^2 \, dv \leq \int_{\Sigma} |\nabla (x_i \circ f)|^2 \, dv \quad \text{(for} \quad i = 1, 2 \text{)}
\]

and likewise

\[
\lambda_1 \int_{\Sigma} (x_3 \circ f)^2 \, dv \leq \int_{\Sigma} |\nabla (x_3 \circ f)|^2 \, dv
\]
which added up (after multiplying by $\lambda_1$ the first two inequalities and by $\mu_1$ the last one) lead to the following estimate involving only $\lambda_1$ and $\mu_1$:

$$\lambda_1 \mu_1 A \leq d \frac{4\pi}{3} (2\lambda_1 + \mu_1).$$

The latter inequality can of course also be deduced from inequality (2) by using the trivial inequation $\mu_1 \leq \mu_2$ (to eliminate $\mu_2$).

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