On Three-point Functions
in the $AdS_4/CFT_3$ Correspondence

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Abstract

We calculate planar, tree-level, non-extremal three-point functions of operators belonging to the $SU(2) \times SU(2)$ sector of ABJM theory. First, we generalize the determinant representation, found by Foda for the three-point functions of the $SU(2)$ sector of $\mathcal{N} = 4$ SYM, to the present case and find that, up to normalization factors, the ABJM result factorizes into a product of two $\mathcal{N} = 4$ SYM correlation functions. Secondly, we treat the case where two operators are heavy and one is light and BPS, using a coherent state description of the heavy ones. We show that when normalized by the three-point function of three BPS operators the heavy-heavy-light correlation function agrees, in the Frolov-Tseytlin limit, with its string theory counterpart which we calculate holographically.
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1 Introduction

Unlike what is the case for the $AdS_5 \times S^5$-correspondence not much is known about three-point functions of its $AdS_4 \times CP^3$ cousin. Planar three-point functions of scalar chiral primaries were calculated at strong coupling more than 10 years ago using M-theory on $AdS_4 \times S^7$ [1]. More recently, strong coupling results were obtained for the case of two giant gravitons and one tiny graviton, all BPS [2]. These three-point functions all show an explicit dependence on the 't Hooft coupling constant and hence are not protected like their $AdS_5 \times S^5$ counterparts [3, 4]. Perhaps for this reason, little effort has been put into studying the corresponding three-point functions at weak coupling. Weak coupling three-point functions only make sense for operators with well-defined conformal dimensions i.e. for operators which are eigenstates of the dilatation operator of the field theory. Scalar chiral primary operators belong to this category. Their two-point functions are protected. One can hence immediately proceed with the calculation of three-point functions of such operators. A number of tree-level results for three-point functions of scalar chiral primaries, including operators dual to giant gravitons, can be found in the references [2, 6]. Furthermore, it has been shown that the one-loop correction to any $n$-point function of scalar chiral primaries vanishes due to colour combinatorics [7] but apart from that there are no results on higher loop corrections to correlation functions neither of chiral primaries nor of more general operators.

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1 In [5] certain three-point functions involving two (non-BPS) semi-classical string states and the dilaton field were presented.
2 It is expected that $n$-point correlation functions of BPS operators involving space-time points with light like separation are related to $n$-sided light like polygonal Wilson loops and to scattering
In the present paper we initiate the study of three-point functions of scalar operators which are not necessarily chiral primaries. More precisely, we will be concerned with planar, non-extremal tree-level three-point functions of a class of operators belonging to the $SU(2) \times SU(2)$ sub-sector. On the field theory side we will exploit the integrability of the spectral problem \cite{10} to represent each operator as a Bethe eigenstate of an integrable spin chain and then generalize the construction invented for $\mathcal{N} = 4$ SYM by Escobedo et al. \cite{11} and by Foda \cite{12}. In addition, we will consider a case where two of these operators are large and one is small and BPS, and calculate the corresponding three-point function in a coherent state approach \cite{13, 14}. The latter three-point function we also determine holographically from string theory using the method developed in \cite{15}. Somewhat surprisingly, if we normalize by dividing the result by the three-point function of three chiral primaries with matching charges we obtain the same expression on the string theory and the gauge theory side.

The organization of our paper is as follows. We start by giving a precise characterization of the operators we wish to consider in section 2. Subsequently, in section 3 we sketch the derivation of the three-point functions of these operators in the Foda approach \cite{12}. After that we specialize to the case of two large and one small BPS operator and determine the three-point function first from the gauge theory perspective in section 4.1 and secondly from the string theory perspective in section 4.2. Section 5 contains our conclusion. The details of the Foda approach are given in appendix A and in appendix B we have collected the necessary background material on type IIA strings on $AdS_4 \times CP^3$.

2 Three-point functions in the $SU(2) \times SU(2)$ sector of ABJM theory

The field theory which enters the $AdS_4 \times CP^3$ correspondence \cite{16} is an $\mathcal{N} = 6$, $U(N)_k \times U(N)_{-k}$ superconformal Chern-Simons theory. The theory has a ’t Hooft expansion with the ’t Hooft coupling constant given by $\lambda = N/k$. Furthermore, it contains two pairs of chiral superfields transforming in a bi-fundamental representation of $U(N) \times U(N)$. There is also an $SU(2) \times SU(2)$ R-symmetry which has been shown to be enhanced to $SU(4)$.

The scalar sector of the field theory, ABJM theory, consists of two complex scalars $Z_1, Z_2$ which transform in the $N \times \bar{N}$ representation of $U(N) \times U(N)$ and two complex scalars $W_1, W_2$ which transform in the $\bar{N} \times N$ representation. The scalars can be grouped as it is the case in $\mathcal{N} = 4$ SYM \cite{8}. Similar relations are argued to hold for more general classes of operators and for theories in general dimensions \cite{9}.
into multiplets of the R-symmetry group SU(4)

\[ \mathcal{Z}^a = (Z_1, Z_2, \bar{W}_1, \bar{W}_2), \quad \bar{\mathcal{Z}}_a = (\bar{Z}_1, \bar{Z}_2, W_1, W_2), \] (2.1)

with \( \mathcal{Z}^a \) transforming in the fundamental representation and \( \bar{\mathcal{Z}}_a \) in the anti-fundamental representation of SU(4). The conformal dimension of all the scalars is \( \Delta = 1/2 \).

A gauge invariant single trace operator containing only scalars is made by combining the scalars \( \mathcal{Z}^a \) with the scalars \( \bar{\mathcal{Z}}_a \) in an alternating way. Such operators are of the form \([10]\)

\[ \mathcal{O} = C^{a_1 b_2 \cdots b_n}_{a_2 a_3 \cdots a_n} \text{Tr}(\mathcal{Z}^{a_1} \bar{\mathcal{Z}}_{b_1} \cdots \mathcal{Z}^{a_n} \bar{\mathcal{Z}}_{b_n}). \] (2.2)

The bare dimension of this operator is \( n \). Chiral primary operators are operators for which the tensor \( C^{a_1 b_2 \cdots b_n}_{a_2 a_3 \cdots a_n} \) is symmetric in upper as well as lower indices and, in addition, is traceless when tracing over one upper and one lower index. The spectral problem of ABJM theory is believed to be integrable \([10, 17, 18]\) in much the same way as the spectral problem of \( \mathcal{N} = 4 \) SYM \([19, 20]\). The dilatation operator of the theory constitutes the Hamiltonian of an integrable spin chain and the operators with well-defined conformal dimensions are the eigenstates of this Hamiltonian. In particular, the scalar operators like (2.2) have the interpretation of a spin chain state of length \( 2n \) with the spins in the odd sites transforming in the fundamental and the spins in the even sites in the anti-fundamental representation of SU(4).

Among the possible sub-sectors of ABJM theory we are interested in the SU(2) × SU(2) sector. This sector is obtained by considering operators made out of 2 scalars among \( \mathcal{Z}^a \) and 2 scalars among \( \bar{\mathcal{Z}}_a \) in Eq.(2.1) transforming in two separate SU(2) subgroups of SU(4). If for instance we consider the scalars \( Z_{1,2} \) and \( W_{1,2} \), the single-trace operators are of the form

\[ \mathcal{O} = C_{i_1 i_2 \cdots i_J}^{j_1 j_2 \cdots j_J} \text{Tr}(Z_{i_1} W_{j_1} \cdots Z_{i_J} W_{j_J}). \] (2.3)

When restricted to the SU(2) × SU(2) sub-sector the dilatation operator becomes the Hamiltonian of two decoupled ferromagnetic XXX\(1/2\) Heisenberg spin chains, one living at the even sites and the other one living at odd sites with the two chains being related only through the momentum constraint \([10]\).

In the table below we describe the field content of the three operators of SU(2) × SU(2) type which enter the planar, non-extremal, tree-level three-point functions we are interested in. \(^3\)

Here we have indicated which fields are to be considered vacua and which are to be considered excitations in the interpretation of each operator as a state of two coupled XXX\(1/2\) spin chains. We have in mind the situation depicted in figure [1] with site

\[^3\]There exist another class of such three-point functions which have trivial factorization properties \([21]\).
| Operator | Vacuum odd | Excitation odd | Vacuum even | Excitation even |
|----------|-------------|----------------|-------------|-----------------|
| $\mathcal{O}_1$ | $(J - J_1) Z_1$ | $J_1 Z_2$ | $(J - J_2) W_1$ | $J_2 W_2$ |
| $\mathcal{O}_2$ | $(J_1 + J_2) Z_2$ | $(J - J_1 - J_2) Z_1$ | $(J_2 + J_2) W_2$ | $(J - J_2 - J_1) W_1$ |
| $\mathcal{O}_3$ | $j_2 W_2$ | $j_1 \bar{Z}_1$ | $j_2 Z_2$ | $j_1 W_1$ |

Table 1: The field content of our operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ of $SU(2) \times SU(2)$ type having a non-vanishing planar, non-extremal three-point function. The notation $J_1 Z_2$ means that the number of $Z_2$-fields is $J_1$. It is understood that the number of fields of any type can not be negative.

number one being at the left end of each operator. When we contract the three operators at the planar level all vacuum fields from $\mathcal{O}_3$ are contracted with vacuum fields in $\mathcal{O}_2$ and all excitations of $\mathcal{O}_3$ are contracted with $\mathcal{O}_1$. This means that only a term in $\mathcal{O}_3$ for which all vacuum fields are to the left of all excitations can contribute to the three-point function. Notice also that for contractions involving $\mathcal{O}_1$ we connect even sites to even sites and odd sites to odd sites. For the contractions between $\mathcal{O}_2$ and $\mathcal{O}_3$, however, odd sites get connected to even sites and vice versa. We have illustrated the possible contractions in figure 1. Dashed lines are fields corresponding to excitations and solid lines are fields corresponding to vacua. The results that we present will be structure constants $C_{123}$ appearing in the three-point functions

$$
\langle O_1(x)O_2(y)O_3(z) \rangle = \frac{1}{N} \frac{C_{123}}{|x - y|^{2(\Delta_1 + \Delta_2 - \Delta_3)}|x - z|^{2(\Delta_1 + \Delta_3 - \Delta_2)}|y - z|^{2(\Delta_2 + \Delta_3 - \Delta_1)}},
$$

of unit normalized operators, i.e. operators whose two-point functions fulfill

$$
\langle \bar{O}_i(x)O_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta}}.
$$

3 The Foda approach

An elegant representation of three-point functions of the $SU(2)$-sector of $\mathcal{N} = 4$ SYM was found by Foda [12]. Here, we will generalize this representation to the $SU(2) \times SU(2)$ sector of ABJM theory. The key idea of Foda was to map various parts of the three-point function onto already known sums over states for a statistical mechanical lattice model, namely the 6-vertex model. The starting point of Foda’s approach is to consider the operators as spin chain eigenstates as produced by the algebraic Bethe ansatz. In this picture any given eigenstate is obtained from a unit normalized reference state (vacuum), which we will take to be all spins up, by acting with an appropriate series of spin-flipping or lowering operators. In this picture the structure constant corresponding to the three-point function appearing in figure 1 can be written as the
following inner product between Bethe states

\[ C_{123} = N_{123} (r\langle O_3 | \otimes l\langle O_2 | ) | O_1 \rangle, \]  

(3.1)

where the subscripts \( l \) and \( r \) refer to the left and right part respectively and where \( N_{123} \) is a normalization constant. In order to arrive at (3.1) we have exploited the fact that the inner product between two vacuum states is equal to one. Now \( |O_1 \rangle \) is a Bethe eigenstate but \( r\langle O_3 | \otimes l\langle O_2 | \) is not.

Figure 1: The possible contractions between \( O_1, O_2 \) and \( O_3 \). The full lines represent vacua and the dashed lines represent excitations. The two different colours illustrate fields in the two different spin chains.

In the case of the \( SU(2) \)-sector of \( N = 4 \) SYM the equivalent of the expression (3.1) could be expressed in terms of known quantities for the 6-vertex model. More precisely, the contractions between \( |O_1 \rangle \) and \( |O_3 \rangle \) gave rise to a factor which could be identified as a so-called domain wall partition function of the 6-vertex model (i.e. the partition function of the model with all initial arrows pointing upwards and all final arrows pointing downwards.). What remained was also a quantity which was well-known in the language of the 6-vertex model, namely another special type of partition function which could be expressed in terms of a so-called Slavnov inner product. In the following we will generalize this construction to the \( SU(2) \times SU(2) \) sector of ABJM theory.

In the ABJM case the operators \( O_1, O_2 \) and \( O_3 \) are to be viewed as algebraic Bethe Ansatz eigenstates of the \( SU(2) \times SU(2) \) spin chains and hence must be obtained from a reference state by acting with a number of spin-flipping operators. In order to derive these spin-flipping operators one first has to construct the necessary \( R \)-matrices and then form the monodromy matrix. The construction of the four \( R \)-matrices which are necessary for the full \( SU(4) \) spin chain was carried out in [10]. From these \( R \)-matrices one can form two monodromy matrices, one pertaining to the even sites of the spin chain and the other one to the odd sites of the spin chain. Consequently, one also
gets two sets of lowering operators $B_e$ and $B_o$ where the subscripts $e$ and $o$ refer to even and odd respectively. When we constrain to the $SU(2) \times SU(2)$ sub-sector, two of the four $R$-matrices trivialize and the remaining two become the $R$-matrices of two independent $SU(2)$ spin chains, one living on odd sites and one living on even sites. Similarly, the two monodromy matrices simply become the monodromy matrices of two independent $SU(2)$ spin chains and finally the lowering operators $B_e$ and $B_o$ become the usual $SU(2)$ spin flipping operators for even and odd sites respectively. The two spin-flipping operators $B_o$ and $B_e$ depend on rapidity variables $\{u_o\}$ and $\{u_e\}$ and in order to obtain an eigenstate both sets of rapidities $\{u_o\}$ and $\{u_e\}$ have to satisfy the $SU(2)$ Bethe equations. The only connection between the two sets of rapidities $\{u_o\}$ and $\{u_e\}$ is that they are related via the momentum constraint which says that the total momentum of all excitations should vanish and reflects the fact that the corresponding single trace operator of ABJM theory should be invariant when one or more pairs of fields are cyclically displaced. Apart from this constraint we thus effectively have for each operator two non-interacting $SU(2)$ spin chains. In the following we will denote the rapidity variables corresponding to the operator $O_1$ as $(\{w_o\}, \{w_e\})$, the rapidity variables corresponding to $O_2$ as $(\{v_o\}, \{v_e\})$, and the rapidity variables corresponding to $O_3$ as $(\{w_o\}, \{w_e\})$.

Now we can map the elements of each of the two independent $R$-matrices, the one of the even sites and the one of the odd sites, into the vertex weights of two independent 6-vertex models. In this way our three-point function effectively decouples into two $SU(2)$ three-point functions. Following the procedure of Foda [12] we can furthermore easily express the ABJM three-point functions in terms of special partition functions of the 6-vertex model. More precisely, the decoupling properties imply that we can write our ABJM three-point function as follows

$$C_{123} = N_{123} Z_{j_1}(\{w_o\}) S[J, J_1, J - J_1 - j_1](\{u_o\}, \{v_o\}) \times$$

$$Z_{j_1}(\{w_e\}) S[J, J_2, J - J_2 - j_1](\{u_e\}, \{v_e\}).$$  \hspace{1cm} (3.2)

Here the $Z$’s are domain wall partition functions and the $S$’s are Slavnov inner products. Both types of quantities can be expressed as determinants. The normalization constant $N_{123}^{ABJM}$ takes the form

$$N_{123} = \frac{\sqrt{J(j_1 + j_2)(J + j_2 - j_1)}}{\sqrt{N_{1o}N_{1e}N_{2o}N_{2e}N_{3o}N_{3e}}}. \hspace{1cm} (3.3)$$

The quantities in the denominator are the Gaudin norms (i.e. the norms of the eigenstates of the algebraic Bethe ansatz) for the odd and even parts of the three Bethe states. These norms can also be expressed as determinants. Finally the factor in the
decoupling is not complete since the cyclicity properties are different for single trace operators in $\mathcal{N} = 4$ SYM and in ABJM theory.
numerator takes into account the cyclic nature of the three operators. In App. A we will make the arguments of the present section more precise.

4 Two heavy and one light operator

4.1 The coherent state approach

In this section we wish to calculate a three-point function of the type considered above in the limit where the two operators, $O_1$ and $O_2$, are much longer than the operator $O_3$. In order to simplify the presentation we now restrict ourselves to the following special case:

$$j_1 = j_2 = j, \quad J_1 = J_2. \quad (4.1)$$

The operator $O_3$ then has length $4j$ and $O_1$ and $O_2$ are both of the same length, namely $2J$. The limit we will be considering is the following

$$1 \ll j \ll J_1, J. \quad (4.2)$$

We can represent the long operators $O_1$ and $O_2$ as coherent states in a $SU(2) \times SU(2)$ spin chain [13, 14]. The way in which we contract the fields is the same as depicted in figure 1 but we have to deal with the periodicity of the spin chains in a different way. Let us define the first site in $O_1$ for which the corresponding field is contracted with a field in $O_2$ to be site number $2k + 2j - 1$ of the spin chain corresponding to $O_1$. Similarly, let us define the site in the operator $O_2$ to which this field is contracted to be site number $2k + 2j - 1$ of the spin chain corresponding to $O_2$. This in particular means that in $O_1$ as well as in $O_2$ the fields at the sites $2k - 1, 2k, \ldots, 2k + 2j - 2$ are contracted with $O_3$.

To take into account all possible contractions we then have to sum over $k$ from $k = 1$ to $k = J$. We can represent $O_1$ in the following manner

$$O_1 = \ldots (u_o^{(2k-1)} \cdot Z)(u_e^{(2k)} \cdot W)(u_o^{(2k+1)} \cdot Z)(u_e^{(2k+2)} \cdot W) \ldots \quad (4.3)$$

where the sub-scripts $o$ and $e$ refer to quantities describing the spin chains at odd and even sites respectively. The vectors $u_o = (u^1_o, u^2_o)$ and $u_e = (u^1_e, u^2_e)$ belong to $\mathbb{C}^2$ and are unit normalized, i.e. $\bar{u}_o^{(p)} \cdot u_o^{(p)} = \bar{u}_e^{(p)} \cdot u_e^{(p)} = 1$ and finally $Z = (Z_1, Z_2), W = (W_1, W_2)$. With a similar notation we can write $O_2$ as

$$O_2 = \ldots (\bar{v}_o^{(2k-1)} \cdot Z)(\bar{v}_e^{(2k)} \cdot W)(\bar{v}_o^{(2k+1)} \cdot Z)(\bar{v}_e^{(2k+2)} \cdot W) \ldots \quad (4.4)$$

5 Notice the difference to $\mathcal{N} = 4$ SYM that not the full length but half the length of the operators appear.

6 After the preparation of this manuscript we learned that a factorization formula of the same type as (3.2) was proposed (but not substantiated) in [21].

7 The general case is not more complicated, but the notation becomes quite cumbersome.
where $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$ and $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2)$. In order for $\mathcal{O}_1$ and $\mathcal{O}_2$ to be eigenstates of the two loop dilatation operator, $u_o^{(p)} \equiv u_o(\pi p/J)$ must be periodic in $p$ with period $2J$ and fulfill the equations of motion of the Landau-Lifshitz sigma model and similarly for $u_e$, $v_o$, and $v_e$.\footnote{Notice that in the present context we are free to choose which fields are considered vacua and which are considered excitations.}

The third operator, $\mathcal{O}_3$, is built from $j$ of each of the fields $Z_1$, $W_1$, $Z_2$ and $W_2$. We will now furthermore assume that $\mathcal{O}_3$ is BPS which implies that it must be a sum over all possible orderings of the fields with equal weight. However, only one ordering of the fields contributes to the planar three-point function, i.e.

$$\mathcal{O}_3 = N_3 \text{Tr}((Z_1 W_1)^j (\bar{Z}_2 \bar{W}_2)^j) + \text{irrelevant terms}, \quad (4.5)$$

where $N_3$ is a normalization constant which ensures that the two-point function of the operator is unit normalized, cf. Eq.(2.5). More precisely,

$$N_3 = \frac{(j!)^2}{\sqrt{(2j)!((2j - 1)!}}} \quad (4.6)$$

We can now calculate the planar tree-level three-point function of our three operators. The contractions involving $\mathcal{O}_3$ give rise to the factor

$$\prod_{m=k}^{k+j-1} (2m-1)\pi \bar{u}_o (2m\pi J) \bar{v}_o \left( \frac{2m\pi}{J} \right) \bar{u}_e \left( \frac{2m\pi}{J} \right), \quad (4.7)$$

and each contraction between $\mathcal{O}_2$ and $\mathcal{O}_1$ gives rise to a factor of $u_o \cdot \bar{v}_o$ or $u_e \cdot \bar{v}_e$. Therefore we can write the three point function as\footnote{Here we use the notation of \cite{11} that each circle in the superscript represents an operator appearing in the three-point function. Filled circles correspond to non-BPS operators and empty circles correspond to BPS ones.}

$$C^{\bullet \bullet \bullet} = N_3 B \sum_{k=1}^{J} \prod_{m=k}^{k+j-1} \frac{(2m-1)\pi \bar{u}_o (2m\pi J) \bar{v}_o \left( \frac{2m\pi}{J} \right) \bar{u}_e \left( \frac{2m\pi}{J} \right)}{(u_o^{(2m-1)} \cdot \bar{v}_o^{(2m-1)}) (u_e^{(2m)} \cdot \bar{v}_e^{(2m)})), \quad (4.8)$$

where

$$B = \prod_{m=1}^{J} (u_o^{(2m-1)} \cdot \bar{v}_o^{(2m-1)}) (u_e^{(2m)} \cdot \bar{v}_e^{(2m)}), \quad (4.9)$$

which is the overlap between the operators $\mathcal{O}_1$ and $\mathcal{O}_2$.

We now assume that $u_o$, $u_e$, $v_o$ and $v_e$ are slowly varying, i.e. $u_o^{(p)} - u_o^{(p-2)} \sim 1/J$ and similarly for $u_e$, $v_o$ and $v_e$. There is no similar condition relating $u_o$ and $u_e$ or relating $v_o$ and $v_e$. Then we can approximate $u_o^{(p)} \equiv u_o(\frac{2p}{J})$ with a continuous field $u_o(\sigma)$ where $\sigma$ is likewise continuous and belongs to the interval $[0, 2\pi]$, and similarly for
\( \mathbf{u}, \mathbf{v} \) and \( \mathbf{v} \). The statement that \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are eigenstates of the two-loop dilatation operator now translates into the statement that \( \mathbf{u}_o, \mathbf{u}_e, \mathbf{v}_o \) and \( \mathbf{v}_e \) obey the continuum Landau-Lifshitz equations of motion.

Due to the fact that the \( \mathbf{u} \)'s and \( \mathbf{v} \)'s vary slowly and in addition that \( j \ll J \) we can now equate all factors in the product over \( m \). Therefore our three-point function reduces to

\[
C_{\bullet\bullet\bullet} = N_3 B \sum_{k=1}^{J} \left( \frac{u_o^1 \left( \frac{2k-1}{J} \right)}{u_o^{(2k-1)} \cdot \bar{v}_o^{(2k-1)}} \cdot \frac{u_e^1 \left( \frac{2k}{J} \right)}{u_e^{(2k)} \cdot \bar{v}_e^{(2k)}} \right)^j
\]

\[
\rightarrow N_3 BJ \int_0^{2 \pi} \frac{d\sigma}{2 \pi} \left( \frac{u_o^1(\sigma)u_e^2(\sigma)\bar{v}_o^2(\sigma)\bar{v}_e^2(\sigma)}{(u_o(\sigma) \cdot \bar{v}_o(\sigma))(u_e(\sigma) \cdot \bar{v}_e(\sigma))} \right)^j, \tag{4.10}
\]

We now choose \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) so similar that

\[
\mathbf{v}(a)(\sigma) \approx \mathbf{u}(a)(\sigma) + \delta \mathbf{u}(a), \tag{4.11}
\]

where \( \delta \mathbf{u}(a) \) is of order \( j/J \). A procedure for implementing this choice at the level of Bethe roots was given in [13]. Then, as shown in [13, 14] we get in the limit \( j/J \rightarrow 0 \) that \( B = 1 \) and our three-point function can be written as

\[
C_{\bullet\bullet\bullet} = N_3 J \int_0^{2 \pi} \frac{d\sigma}{2 \pi} \left( u_o^1(\sigma)u_e^2(\sigma)\bar{v}_o^2(\sigma)\bar{v}_e^2(\sigma) \right)^j. \tag{4.12}
\]

We have observed that one obtains an interesting match with string theory if one considers the following quantity

\[
r_{\lambda \ll 1} = \left. \frac{C_{\bullet\bullet\bullet}}{C_{\circ\circ\circ \circ}} \right|_{\lambda \ll 1}, \tag{4.13}
\]

where \( C_{\circ\circ\circ \circ} \) is the three-point correlation function coefficient for three chiral primaries with the same charges as the operators considered in the numerator.

We can compute three point functions of three chiral primaries by considering a limit of (3.2) where all the rapidities go to infinity. In [23] it was shown how to perform this limit for operators in \( \mathcal{N}=4 \) SYM theory and the same strategy can be applied in the present case. Adapting the procedure of [23] to our operators in the \( SU(2) \times SU(2) \) of ABJM theory we find

\[
C_{\circ\circ\circ \circ} = J \sqrt{2J} \frac{(J - J_1 + j)!J_1!((J - j)!)^2 j!^2}{(J!)^2(J - J_1)!((J_1 - j)!(2j)!}. \tag{4.14}
\]

Note that, apart from a different normalization, this is precisely the square of the result of [23] for operators in the \( SU(2) \) sector of \( \mathcal{N}=4 \) SYM theory. Taking the limit \( J, J_1 \rightarrow \infty \) keeping \( J - J_1 \) large, we have

\[
C_{\circ\circ\circ \circ} \sim N_3 J s^j, \tag{4.15}
\]
where we have defined the quantity \( s = \frac{J_1(J+J_1)}{J^2} \).

Using the result \((4.12)\), we then compute

\[
\lambda \ll 1 = \frac{1}{s^j} \int_0^{2\pi} \frac{d\sigma}{2\pi} (u_1^1(\sigma)u_1^1(\sigma)\bar{u}_2^2(\sigma)\bar{u}_2^2(\sigma))^j. \tag{4.16}
\]

We will show in the next section that for the ratio \( \lambda \gg 1 \) at strong coupling we obtain the same result.

### 4.2 The holographic approach

Here we compute the holographic three-point function dual to the correlator of two heavy and one light operator considered in Sec. 4.1 using the prescription of [15]. The procedure for this computation has already been outlined in [13, 14, 24] for type IIB string theory on \( \text{AdS}_5 \times S^5 \) and can be easily generalized to type IIA string theory on \( \text{AdS}_4 \times \mathbb{C}P^3 \) using the results of Ref.s [1, 2, 25].

Our convention and notation for the \( \text{AdS}_4 \times \mathbb{C}P^3 \) background for type IIA string theory are explained in appendix [B]. Here to parametrize the two two-spheres associated to the two \( SU(2) \) sectors contained in \( \mathbb{C}P^3 \) we use two complex vectors \( U_e(\tau,\sigma) = (U_1^e, U_2^e) \) and \( U_o(\tau,\sigma) = (U_1^o, U_2^o) \).

With this parametrization the results of this section will be directly comparable with the ones of Sec. 4.1.

The prescription of [15, 13, 14] gives in our case

\[
C^{\cdots} = a_j \lambda^3 \int_{-\infty}^{\infty} d\tau e^{\tau^2} \int_0^{2\pi} \frac{d\sigma}{2\pi} Y \left[ \frac{3}{\kappa^2 \cosh^2 \frac{\tau}{\kappa}} - \frac{1}{\kappa^2} - \frac{1}{\kappa^2} \frac{1}{\kappa^2} \frac{1}{\kappa^2} \frac{1}{\kappa^2} \right], \tag{4.17}
\]

where we already implemented the gauge choice \((B.12)\), we introduced the Euclidean time \( \tau_e \) and we defined

\[
a_j = \sqrt{\pi(4j+1)} \frac{2^{2j}(2j+1)!}{j^{12}}, \quad Y = (U_1^e \bar{U}_2^o U_2^e \bar{U}_2^o)^j. \tag{4.18}
\]

To compare with the result of Sect. 4.1 we take the Frolov-Tseytlin limit \([26, 27]\) which in our notation reads \([14, 22, 28]\)

\[
\kappa \to 0, \quad \frac{1}{\kappa^2} \partial\tau U_e,o \text{ fixed}, \quad \partial_{\sigma} U_e,o \text{ fixed}. \tag{4.19}
\]

A subclass of solutions that can be mapped to coherent spin chain states at weak coupling is given by considering the parametrization \( U_{e,o}(\sigma,\tau) = e^{i\tau/\kappa} u_{e,o}(\sigma,\tau) \) with the condition \( \bar{u}_e \cdot u_e = 1 \) and similarly for \( u_o \). The limit \((4.19)\) becomes

\[
\kappa \to 0, \quad \frac{1}{\kappa^2} \partial\tau u_{e,o} \text{ fixed}, \quad \partial_{\sigma} u_{e,o} \text{ fixed}. \tag{4.20}
\]

---

\(^{10}\)Note that in App. [B] we use a different parametrization for the two two-spheres. The two parametrizations are related by a coordinate transformation.

\(^{11}\)Note that the unconventional powers of \( \kappa \) are due to a rescaling of the time coordinate (see App. [B]).
The functions $u_{e,o}$ are solutions of the Landau Lifshitz equations of motion derived from the action (B.15) and satisfy the Virasoro condition $\bar{u}_e \cdot \partial_\sigma u_e + \bar{u}_o \cdot \partial_\sigma u_o = 0$. Note that in our notation, the energy that one computes using the action (B.15) goes as $E - J \sim \mathcal{O}(\lambda/J^2)$. This is due to the rescaling of $t$ in (B.7). This rescaling has the effect that the gauge constant $\kappa \sim \sqrt{\lambda/J}$. This implies that the expansion in powers of $\kappa$ on the string side parallels the expansion in powers of $\lambda/J^2$ that one has on the gauge theory side.

In the limit (4.20), Eq. (4.17), to leading order, gives

$$ C^{\bullet\bullet\bullet} = \lambda^{1/4} \sqrt{\pi (4j - 1)} \frac{2^{j+2}(2j+1)!}{j!} \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma \left( u_e^1 \bar{u}_e^2 u_o^1 \bar{u}_o^2 \right)^j \frac{1}{\kappa^2 \cosh^2(2j+2) \tau_e}. $$

(4.21)

For $\kappa \to 0$, the integrand peaks around $\tau_e = 0$ and the $\tau$-integral can thus be evaluated (see [13, 14] for more details on this point). The result reads

$$ \int_{-\infty}^{\infty} \frac{d\tau_e}{\kappa^2 \cosh^{2j+2}(\frac{\tau_e}{\kappa})} = \frac{2^{2j+1} (j!)^2}{\kappa (2j+1)!}. $$

(4.22)

Using that $\kappa = \frac{\sqrt{\lambda}}{J\pi\sqrt{2}}$ (see App. B) we obtain

$$ C^{\bullet\bullet\bullet} = J^{3/4} 2^{j+1} \sqrt{\pi} \sqrt{4j - 1} \int_0^{2\pi} \frac{d\sigma}{2\pi} (u_e^1 \bar{u}_e^2 u_o^1 \bar{u}_o^2)^j. $$

(4.23)

The expression for the holographic three-point function for the chiral primaries with the same charges as the operators considered in Sec. 4.1 can be computed using Ref. [2]. We get

$$ C^{\circ\circ\circ} = \frac{\lambda^{1/4} 2^{j+1} \sqrt{\pi}}{\sqrt{4j - 1}} \left( \frac{2j+1}{(2j+1)(J-j)!} \right) \left( \frac{J - J_1 + j}{(J - J_1)!} \right) \left( \frac{J_1}{(J - J_1)!} \right). $$

(4.24)

Note that this expression differs from (4.14) which is valid at weak coupling. In particular the dependence on the coupling is very different, showing explicitly that the three-point function for three chiral primaries in ABJM theory is not a protected quantity.

In the limit $J, J_1 \to \infty$ with $J - J_1$ large we have

$$ C^{\circ\circ\circ} = \frac{\lambda^{1/4} 2^{j+1}}{\sqrt{\pi}} J^{3/4} \sqrt{4j + 1}. $$

(4.25)

\[12\] Note that, following the notation of Ref. [2], in our case $p = j - j$. Moreover, from Appendix A of [2] we have $n_6 = j$, $n_1 = n_2 = p = J - j$, $n_3 = j$. Note also that in our notation $\gamma_1 = \gamma_2 = 2j$, $\gamma_3 = 2J - 2j$ and $\gamma = 2J + 2j$ where we used that the relation between our notation and $J_1, J_2$ and $J_3$ in [2] is that $(J_1/2)_{\text{there}} = (J_2/2)_{\text{there}} = J_{\text{our}}$ and $(J_3/2)_{\text{there}} = 2J_{\text{our}}$. 


We can now compute the ratio between Eq. (4.23) and Eq. (4.25) and compare it with the corresponding quantity (4.16) at weak coupling. We find

\[ r_{\lambda \gg 1} = \frac{C_{\lambda \lambda \lambda \lambda}}{C_{\lambda \lambda \lambda \lambda}} \approx \frac{1}{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (u^1_\sigma \bar{u}^1_\sigma u^1_0 \bar{u}^1_0)^j. \] (4.26)

It is easy to see that to leading order we have

\[ r_{\lambda \gg 1} = r_{\lambda \ll 1}. \] (4.27)

Note that we have that \( r_{\lambda \gg 1} = r_{\lambda \ll 1} \) only in the limit \( J, J_1 \to \infty \) which is the regime for which also the nice matching of Ref. [13] was observed.

5 Conclusion

We have seen that the Foda approach to three-point functions generalizes in a straightforward manner to the \( SU(2) \times SU(2) \) sector of ABJM theory. Obviously a much more challenging project would be to extend the approach to the full \( SU(4) \) sector. While the approach of Escobedo et al. has been extended to the \( SO(6) \) sector of \( \mathcal{N} = 4 \) SYM [29] the Foda approach has so far resisted generalization, except for the one presented in this paper and the one of [30] where it was generalized to spin-1 chains of relevance for certain structure constants in QCD. Another interesting line of investigation would be to include loop corrections. For ABJM theory three-point functions of chiral primaries are in general not protected so even considering just such operators would provide valuable new information. Some progress on the inclusion of loop corrections in the case of \( \mathcal{N} = 4 \) SYM was recently achieved in [14, 31, 32, 33, 34].

In addition, we made the observation that for certain cases involving two large and one small and BPS operator one gets agreement between field and string theory for three-point functions measured relative to three-point functions of chiral primaries, to leading order in a large-spin limit. It would be interesting to investigate if this agreement persists beyond the limit considered. For this purpose it would be useful to find a way to extract the large-spin limit of the heavy-heavy-light correlator from the Foda approach [33]. Apart from allowing more directly for a systematic large-spin expansion this would also shed light on the connection between the two different approaches employed in the present work.

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13The large-spin limit of the heavy-heavy-heavy correlator was extracted from the Foda approach in [23].
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A Details of the Foda approach

As mentioned in the introduction the single trace scalar operators of ABJM theory can be viewed as states of a spin chain of even length where the variables on the even sites transform in the fundamental of an $SU(4)$ and the variables at the odd sites transform in the anti-fundamental of an $SU(4)$. The dilatation operator of ABJM theory then acts as a Hamiltonian for this spin chain and is conjectured to be integrable. At the lowest loop order (two loops) this Hamiltonian can be studied by standard techniques of integrable models [10]. Hence one can introduce the $R$-matrix, a monodromy matrix and a transfer matrix. For the alternating $SU(4)$ spin chain one needs a total of four $R$-matrices [10]

\[ R_{ab} : V_a \otimes V_b \rightarrow V_a \otimes V_b, \quad R_{ab}(u_a) = u_a I_a \otimes I_b + \eta P_{ab}, \quad (A.1) \]

\[ R_{\bar{a}b} : V_{\bar{a}} \otimes V_b \rightarrow V_{\bar{a}} \otimes V_b, \quad R_{\bar{a}b}(u_{\bar{a}}) = u_{\bar{a}} I_{\bar{a}} \otimes I_b + \eta P_{\bar{a}b}, \ \ (A.2) \]

\[ R_{\bar{a}b} : V_a \otimes V_{\bar{b}} \rightarrow V_a \otimes V_{\bar{b}}, \quad R_{\bar{a}b}(u_{\bar{a}}) = u_{\bar{a}} I_a \otimes I_{\bar{b}} + K_{\bar{a}b}, \ \ (A.3) \]

\[ R_{\bar{a}b} : V_{\bar{a}} \otimes V_{\bar{b}} \rightarrow V_{\bar{a}} \otimes V_{\bar{b}}, \quad R_{\bar{a}b}(u_{\bar{a}}) = u_{\bar{a}} I_{\bar{a}} \otimes I_{\bar{b}} + K_{\bar{a}b}. \]

Here $V_a$ and $V_{\bar{a}}$ are the vector spaces of the fundamental and anti-fundamental representation respectively. The operator $I$ is the identity operator, $P$ is the permutation, and $K$ is the $SU(4)$ trace. Furthermore, $u_e$ and $u_o$ are spectral parameters and $\eta$ is the shift which we will later take to be equal to $i/2$. From these $R$-matrices one constructs two monodromy matrices, one for sites of the fundamental representation and one for sites of the anti-fundamental representation

\[ M_a(u_o) = R_{a1}(u_o)R_{a1}(u_o)...R_{aJ}(u_o)R_{aj}(u_o), \quad (A.2) \]

\[ M_{\bar{a}}(u_{\bar{a}}) = R_{\bar{a}1}(u_{\bar{a}})R_{\bar{a}1}(u_{\bar{a}})...R_{\bar{a}J}(u_{\bar{a}})R_{\bar{a}\bar{J}}(u_{\bar{a}}). \quad (A.3) \]

Specializing to the $SU(2) \times SU(2)$ sector the trace operator $K$ does not contribute and the two $R$-matrices $R_{ab}$ and $R_{\bar{a}b}$ become proportional to the identity. The $R$-matrices $R_{ab}$ and $R_{\bar{a}b}$ each become the $R$-matrix of an $SU(2)$ spin chain. We can now generalize the system to an inhomogeneous one where the $R$-matrices depends on the particular
This leads to the following expression for the non-trivial $R$-matrices:\[14\]

$$R_{ab}(u_o, z_o) = [u_o - z_o] \begin{pmatrix}
[u_o - z_o + \eta] & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & [u_o - z_o + \eta]
\end{pmatrix}
≡ [u_o - z_o] \mathcal{R}_{ab}, \quad (A.4)$$

$$R_{\bar{a}\bar{b}}(u_e, z_e) = [u_e - z_e] \begin{pmatrix}
[u_e - z_e + \eta] & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & [u_e - z_e + \eta]
\end{pmatrix}
≡ [u_e - z_e] \mathcal{R}_{\bar{a}\bar{b}}, \quad (A.5)$$

The remaining two are

$$R_{a\bar{b}}(u_o, z_e) = [u_o - z_e] I, \quad (A.6)$$
$$R_{\bar{a}b}(u_e, z_o) = [u_e - z_o] I. \quad (A.7)$$

Here the parameters $z_e$ and $z_o$ are also denoted as quantum rapidities. There is one for each site of the spin chain and it is natural to divide them into two groups, $\{z_o\}$ and $\{z_e\}$, corresponding to respectively the odd and the even sites. As shown in [12] it is convenient to keep these parameters arbitrary in the course of the derivation and only take the homogeneous limit where all $z$'s are identical at the end.

Now, the expressions (A.2) and (A.3) for the monodromy matrices turn into

$$M_a(u_{a_0}, \{z_o, z_e\}, J) = \left( \prod_{i=1}^{J} [u_o - z_i] [u_{a_0} - z_i] \right) \mathcal{R}_{a1}(u_{a_0}, z_{1o}) \ldots \mathcal{R}_{aJ}(u_{a_0}, z_{J0}), \quad (A.8)$$
$$M_{\bar{a}}(u_{a_e}, \{z_o, z_e\}, J) = \left( \prod_{i=1}^{J} [u_e - z_i] [u_{a_e} - z_i] \right) \mathcal{R}_{\bar{a}1}(u_{a_e}, z_{1e}) \ldots \mathcal{R}_{\bar{a}J}(u_{a_e}, z_{Je}). \quad (A.9)$$

Notice that (as usual) the indices $a$ and $\bar{a}$ refer to auxiliary spaces. We see that up to trivial pre-factors we get one monodromy matrix which only involves $\mathcal{R}$-matrices with fundamental indices and one monodromy matrix which only involves $\mathcal{R}$-matrices with anti-fundamental indices. Our model has hence decoupled completely into two $SU(2)$ models and we can easily construct the eigenstates of the full $SU(2) \times SU(2)$ model by means of eigenstates of the two $SU(2)$ models. (Of course we have to bear in mind that we are only interested in eigenstates which have cyclic symmetry when viewed as

\[14\] Here $R_{ab}$ is expressed in the basis $(|\uparrow_a\rangle \otimes |\uparrow_b\rangle, |\uparrow_a\rangle \otimes |\downarrow_b\rangle, |\downarrow_a\rangle \otimes |\uparrow_b\rangle, |\downarrow_a\rangle \otimes |\downarrow_b\rangle)$ and similarly for the other three.
Let us write \( M_a(u_o, \{z_o, z_e\}, J) \) in the following way

\[
M_a(u_o, \{z_o, z_e\}, J) = \left( \begin{array}{c}
A_o(u_o, \{z_o, z_e\}, J) \\
B_o(u_o, \{z_o, z_e\}, J) \\
C_o(u_o, \{z_o, z_e\}, J) \\
D_o(u_o, \{z_o, z_e\}, J)
\end{array} \right)_a
\]

and similarly for \( M_\pi(u_e, \{z_o, z_e\}, J) \). Then we define the reference state \( |\uparrow_z N\rangle \) as all spins up, i.e., \(|\uparrow_z N\rangle = |\uparrow_{z_{1o}}\rangle \otimes |\uparrow_{z_{1e}}\rangle \otimes \cdots \otimes |\uparrow_{z_{Jo}}\rangle \otimes |\uparrow_{z_{Je}}\rangle \) and from the usual constructions of the algebraic Bethe ansatz for the \( SU(2) \) spin chain it follows that we can create an eigenstate with respectively \( j_1 \) spins at even sites flipped and \( j_2 \) spins at odd sites flipped as follows

\[
\prod_{i=1}^{j_1} B_e(u_{i_e}, \{z_o, z_e\}, J) \prod_{i=1}^{j_2} B_o(u_{i_o}, \{z_o, z_e\}, J)|\uparrow_z N\rangle.
\]

This transition amplitude can be understood as a domain wall partition function for a vertex model as shown in figure 2. Here a vertical blue line represents an odd spin chain site and an vertical red line an even spin chain site. Furthermore, each blue horizontal

\[\text{Figure 2: A domain wall partition function.}\]
line represents a (normalized) spin-flipping operator $B_o$ and each red horizontal line represents a normalized spin-flipping operator $B_e$. We start with all spins pointing up and after application of $2J$ spin-flipping operators (the horizontal lines) we end with a configuration with all spins pointing down. If we ignore the prefactors in front of the $\mathcal{R}$’s in the monodromy matrices this quantity can be mapped onto a domain wall partition function of a vertex model with the vertices shown in figure 3 and the following weights

$$a[u_i, z_j] = \frac{u_i - z_j + \eta}{u_i - z_j}, \quad c[u_i, z_j] = \frac{\eta}{u_i - z_j}, \quad (A.13)$$

$$b[u_i, z_j] = d[u_{ei}, z_{o_j}] = d'[u_{oi}, z_{e_j}] = 1. \quad (A.14)$$

In particular, the weights of all the mixed (red-blue) vertices are equal to one. This means that the partition function of the model factorizes into a partition function of a red model and a partition function of (an identical) blue model. Each of these models can be identified as a usual 6-vertex model. Summarizing we get for the transition amplitude in (A.12)

$$Z_{2J}(\{u_o, u_e\}_J, \{z_o, z_e\}_J) = Z_J(\{u_o\}_J, \{z_o\}_J)Z_J(\{u_e\}_J, \{z_e\}_J), \quad (A.15)$$
where $Z_J(\{u\}_J, \{z\}_J)$ is a domain wall partition function of the 6-vertex model on a lattice of size $J \times J$ connecting an initial state with all arrows pointing upwards to a final state with all arrows pointing down.

Another object of interest for the calculation of three-point functions is the Slavnov scalar product defined for a single $SU(2)$ spin chain as

$$S[\{u\}_{N_1}, \{v\}_{N_2}, \{z\}_J] = \langle \downarrow z_{N_3,J} | \prod_{i=1}^{N_2} C(u_i, \{z\}_J) \prod_{j=1}^{N_1} B(v_j, \{z\}_J) \uparrow z_j \rangle,$$  \hspace{1cm} (A.16)

where

$$\langle \downarrow z_{N_3,J} | = \langle \downarrow z_1 | \otimes \cdots \otimes \langle \downarrow z_{N_3} | \otimes \langle \uparrow z_{N_3+1} | \otimes \cdots \otimes \langle \uparrow z_J |,$$

with $N_3 = N_1 - N_2 > 0$. In the special case where $N_1 = N_2$, $u_i = v_i$ and $z_i = i/2$ for $i = 1, \ldots, N_1$ the Slavnov scalar product reduces to the Gaudin norm,

$$N(\{u\}) = S[\{u\}_N, \{u\}_N, i/2].$$ \hspace{1cm} (A.17)

Generalizing the construction of Foda, a three-point function of the type we are interested in can, up to a normalization factor, be expressed as the partition function of the lattice depicted appearing in the upper part of figure 4. Again, since the weights of all vertices of mixed type are equal to one the function factorizes into a red (even) contribution times a blue (odd) contribution. Each term is equal to the partition function which one encounters when calculating three point functions of $N = 4$ SYM and which was already determined by Foda who found that it could be written as a product of a Slavnov inner product and a domain wall partition function both evaluated in the homogeneous limit $z_{i_0}, z_{i_k} \rightarrow i/2$. The domain wall partition function comes from the lower left corner of the lattice while the remaining part constitutes a Slavnov scalar product. For simplicity we have depicted a case where we have the same number of excitations on the odd and the even lattice but the result holds in the general case as well. Again, it is a simple consequence of the decoupling of the two lattices. In order that the Bethe eigenstates which enter the three-point functions be normalized to unity we must divide the result by the Gaudin norm for each operator. In addition we must multiply by a factor which cures the fact that the presentation of our three-point function as in figure 4 fails to take into account the cyclicity of the ABJM operators. For this final factor one does not have a similar complete decoupling into a product of two factors. This is due to the alternating nature of the ABJM operators which implies that we only have cyclicity (in the horizontal direction) for the combined red-blue model and not for the red and blue model alone. Collecting everything one gets the expression (3.2) for the three-point function.
Figure 4: The decoupling of the three-point function into two parts.

B Type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ and its $SU(2) \times SU(2)$ sigma model limit.

The holographic dual of ABJM theory is given by type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ [16] with metric

$$ds^2 = \frac{R^2}{4} \left( - \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \hat{\Omega}_2^2 \right) + R^2 ds^2_{\mathbb{CP}^3}, \quad (B.1)$$

where for the moment we leave the $\mathbb{CP}^3$ part of the metric unspecified and where

$$\frac{R^2}{l_s^2} = \sqrt{2^5 \pi^2 \lambda}, \quad (B.2)$$

with $\lambda = N/k$ and with string coupling constant and Ramond-Ramond four-form field strength given by

$$g_s = \left( \frac{2^5 \pi^2 N}{k^5} \right)^{\frac{1}{4}}, \quad F_{(4)} = \frac{3R^3}{8} \epsilon_{AdS_4}. \quad (B.3)$$
In the regime $\lambda \gg 1$ and $N \ll k^5$, this is a valid background for type IIA string theory [10].

We are interested in zooming in to the $SU(2) \times SU(2)$ sector of type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. This can be achieved by taking a limit of small momenta which was first found in [27] (see also [35, 28, 22, 14]). How to do this for type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ is explained in detail in [22] and the relevant part of the metric becomes

$$ds^2 = -\frac{R^2}{4}dt^2 + R^2 \left[ \frac{1}{8} d\Omega_2^2 + \frac{1}{8} d\Omega_2'^2 + (d\delta + \omega)^2 \right],$$

with $R$ given in (B.2) and where

$$d\Omega_2^2 = d\theta_1^2 + \cos^2 \theta_1 d\varphi_1^2, \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2 \theta_2 d\varphi_2^2,$$

$$\omega = \frac{1}{4}(\sin \theta_1 d\varphi_1 + \sin \theta_2 d\varphi_2), \quad \delta = \frac{1}{4}(\phi_1 + \phi_2 - \phi_3 - \phi_4)$$

We see that the coordinates $(\theta_i, \varphi_i), i = 1, 2$, parametrize two two-spheres corresponding to the two $SU(2)$ sectors. For later convenience, the two two-spheres can also be written in terms of two unit vectors fields $\vec{n}_{1,2}$ given by

$$\vec{n}_i = (\cos \theta_i \cos \varphi_i, \cos \theta_i \sin \varphi_i, \sin \theta_i).$$

We now introduce the angular momenta $L_1$ and $L_2$ in one $SU(2)$ and $L_3$ and $L_4$ in the other $SU(2)$ with the condition $L_1 + L_2 + L_3 + L_4 = 0$. As explained in [22] the $SU(2) \times SU(2)$ sector is obtained by considering states for which $\Delta - L_1 - L_2$ is small, where $\Delta$ is the energy. This can be implemented as a sigma-model limit with the following coordinate transformation

$$\tilde{t} = \lambda' t, \quad \chi = \delta - \frac{1}{2} t,$$

where $\lambda' = \lambda/J^2, J \equiv L_1 + L_2$ and so that

$$\tilde{H} \equiv i\partial_{\tilde{t}} = \frac{(\Delta - J)}{\lambda'}, \quad 2J = -i\partial_{\chi},$$

We see that sending $\lambda' \to 0$, one has that $\Delta - J \to 0$ which means that we keep the modes of the $SU(2) \times SU(2)$ sector dynamical, while the other modes become non-dynamical and decouple in this limit.

Using (B.7), the type IIA metric becomes

$$ds^2 = R^2 \left[ \left( \frac{1}{\lambda'} d\tilde{t} + d\chi + \omega \right) (d\chi + \omega) + \frac{1}{8} d\Omega_2^2 + \frac{1}{8} d\Omega_2'^2 \right].$$

The bosonic sigma-model Lagrangian and Virasoro constraints are

$$\mathcal{L} = -\frac{1}{2} G_{\mu \nu} h^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x^\nu,$$
\[ G_{\mu\nu}(\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu) = 0, \]  
(B.11)

with \( G_{\mu\nu} \) being the metric \( [B.9] \). \( h^{\alpha\beta} = \sqrt{-\det \gamma_{\alpha\beta}} \) with \( \gamma_{\alpha\beta} \) being the world-sheet metric.

Our gauge choice is
\[ \tilde{t} = \kappa \tau, \]  
(B.12)

\[ 2\pi p_- = \frac{\partial L}{\partial \dot{x}^-} = \text{const.}, \quad \frac{\partial L}{\partial \sigma} = 0. \]  
(B.13)

Moreover, the constant \( \kappa \) can also be determined from
\[ 2J = P_\chi = \int_0^{2\pi} d\sigma p_\chi = \frac{R^2 \kappa}{2\lambda'} = \frac{2\pi \sqrt{2\lambda \kappa}}{\lambda'}. \]  
(B.14)

We see that \( \kappa = \frac{\sqrt{\lambda}}{\pi \sqrt{2}} \). Thus \( \kappa \to 0 \) for \( \lambda' \to 0 \). Moreover, from (B.8) we have that the right energy scale is given by \( \tilde{\tau} = \kappa \tau \). This means that the quantity that we keep fixed in the limit \( \kappa \to 0 \) is \( \dot{x}^\mu = \partial_{\tilde{\tau}} x^\mu \).

Proceeding as in [22], we can then solve the Virasoro constraints and the gauge conditions order by order in \( \kappa \). This actually corresponds, on the gauge theory side, to an expansion in powers of \( \lambda' \). Here we skip the various steps and report the final result for the action to leading order
\[ I = \frac{J}{4\pi} \sum_{i=1}^{2} \int d\tilde{t} \int_0^{2\pi} d\sigma \left[ \sin \theta_i \dot{\varphi}_i = \pi^2 (\tilde{n}_i)^2 \right], \]  
(B.15)

\[ \sum_{i=1}^{2} \int_0^{2\pi} d\sigma \sin \theta_i \varphi'_i = 0, \]  
(B.16)

where the last expression gives the momentum constraint.

We see that, up to the perturbative order we are interested in, by taking the \( SU(2) \times SU(2) \) sigma-model limit we obtain two Landau-Lifshitz models added together \( [B.15] \), one for each \( SU(2) \), which are related only through the momentum constraint \( [B.16] \) [22]. This is moreover consistent with results on the gauge theory side.

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