Vive la différence II.
The Ax-Kochen isomorphism theorem

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Abstract
We show in §1 that the Ax-Kochen isomorphism theorem [AK] requires the continuum hypothesis. Most of the applications of this theorem are insensitive to set theoretic considerations. (A probable exception is the work of Moloney [Mo].) In §2 we give an unrelated result on cuts in models of Peano arithmetic which answers a question on the ideal structure of countable ultraproducts of \( \mathbb{Z} \) posed in [LLS]. In §1 we also answer a question of Keisler and Schmerl regarding Scott complete ultrapowers of \( \mathbb{R} \).

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§1 of this paper owes its existence to Annalisa Marcja’s hospitality in Trento, July 1987; van den Dries’ curiosity about Kim’s conjecture; and the willingness of Hrushovski and Cherlin to look at §3 of [326] through a glass darkly. §2 of this paper owes its existence to a question of G. Cherlin concerning [LLS]. This paper was prepared with the assistance of the group in Arithmetic of Fields at the Institute for Advanced Studies, Hebrew University, during the special year on Arithmetic of Fields, 1991-92. Publ. 405.
Introduction

In a previous paper [Sh326] we gave two constructions of models of set theory in which the following isomorphism principle fails in various strong respects:

(Iso 1) If $\mathcal{M}$, $\mathcal{N}$ are countable elementarily equivalent structures and $\mathcal{F}$ is a nonprincipal ultrafilter on $\omega$, then the ultrapowers $\mathcal{M}^*, \mathcal{N}^*$ of $\mathcal{M}$, $\mathcal{N}$ with respect to $\mathcal{F}$ are isomorphic.

As is well known, this principle is a consequence of the continuum hypothesis. Here we will give a related example in connection with the well-known isomorphism theorem of Ax and Kochen. In its general formulation, that result states that a fairly broad class of henselian fields of characteristic zero satisfying a completeness (or saturation) condition are classified up to isomorphism by the structure of their residue fields and their value groups. The case that interests us here is:

(Iso 2) If $\mathcal{F}$ is a nonprincipal ultrafilter on $\omega$, then the ultraproducts

$$
\prod_p \mathbb{Z}_p/F \text{ and } \prod_p \mathbb{F}_p[[t]]/F
$$

are isomorphic.

Here $\mathbb{Z}_p$ is the ring of $p$-adic integers and $\mathbb{F}_p$ is the finite field of order $p$. It makes no difference whether we work in the fraction fields of these rings as fields, in the rings themselves as rings, or in the rings as valued rings, as these structures are mutually interpretable in one another. In particular, the valuation is definable in the field structure (for example, if the residual characteristic $p$ is greater than 2 consider the property: “$1 + px^2$ has a square root”). We show that such an isomorphism cannot be obtained from the axioms of set theory (ZFC). As an application we may mention that certain papers purporting to prove the contrary need not be refereed.

Of course, the Ax-Kochen isomorphism theorem is normally applied as a step toward results which cannot be affected by set-theoretic independence results. One exception is found in the work of Moloney [Mo] which shows that the ring of convergent real-valued sequences on a countable discrete set has exactly 10 residue domains modulo prime ideals, assuming the continuum hypothesis. This result depends on the general theorem of Ax and Kochen which lies behind the isomorphism theorem for ultraproducts, and also on an explicit construction of a new class of ultrafilters based on the continuum hypothesis. It is very much an open question to produce a model of set theory in which Moloney’s result no longer holds.

Our result can of course be stated more generally; what we actually show here may be formulated as follows.

Proposition A

It is consistent with the axioms of set theory that there is an ultrafilter $\mathcal{F}$ on $\omega$ such that for any two sequences of discrete rank 1 valuation rings $(\mathcal{R}_i^n)_{n=1,2,...}$ ($i = 1, 2$) having countable residue fields, any isomorphism $F : \prod_n \mathcal{R}_1^n/\mathcal{F} \rightarrow \prod_n \mathcal{R}_2^n/\mathcal{F}$ is an ultraproduct of isomorphisms $F_n : \mathcal{R}_1^n \rightarrow \mathcal{R}_2^n$ (for a set of $n$ contained in $\mathcal{F}$). In particular most of the pairs $\mathcal{R}_1^n$, $\mathcal{R}_2^n$ are isomorphic.

In the case of the rings $\mathbb{F}_p[[t]]$ and $\mathbb{Z}_p$, we see that (Iso 2) fails.
From a model theoretic point of view this is not the right level of generality for a problem of this type. There are three natural ways to pose the problem:

1. Characterize the pairs of countable models $\mathcal{M}$, $\mathcal{N}$ such that for some ultrafilter $\mathcal{F}$ in some forcing extension, $\prod \mathcal{M}^\omega / \mathcal{F} \not\cong \prod \mathcal{N}^\omega$;

2. Characterize the pairs of countable models $\mathcal{M}$, $\mathcal{N}$ with no isomorphic ultrapowers in some forcing extension;

3. Write $\mathcal{M} \leq \mathcal{N}$ if in every forcing extension, whenever $\mathcal{F}$ is an ultrafilter on $\omega$ such that $\mathcal{N}^\omega / \mathcal{F}$ is saturated, then $\mathcal{M}^\omega / \mathcal{F}$ is also saturated. Characterize this relation.

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This is somewhat like the Keisler order [Ke, Sh-a or Sh-c Chapter VI] but does not depend on the fact that the ultrafilter is regular. We can replace $\aleph_0$ here by any cardinal $\kappa$ satisfying $\kappa^{<\kappa} = \kappa$.

However the set theoretic aspects of the Ax-Kochen theorem appear to have attracted more interest than the two general problems posed here. We believe that the methods used here are appropriate also in the general case, but we have not attempted to go beyond what is presented here.

With the methods used here, we could try to show that for every $\mathcal{M}$ with countable universe (and language), if $P_3$ is the partial order for adding $\aleph_3$-Cohen reals then we can build a $P_3$-name for a non principal ultrafilter $\mathcal{F}$ on $\omega$, such that in $V^P \mathcal{M}^\omega / \mathcal{F}$ resembles the models constructed in [Sh107]; we can choose the relevant bigness properties in advance (cf. Definition 1.5, clause (5.3)). This would be helpful in connection with problems (1,2) above.

In §2 of this paper we give a result on cuts in models of Peano Arithmetic which has previously been overlooked. Applied to $\omega_1$-saturated models, our result states that some cut does not have countable cofinality from either side. As we explain in §2, this answers a question on ideals in ultrapowers of $\mathbb{Z}$ which was raised in [LLS]. The result has nothing to do with the material in §1, beyond the bare fact that it also gives some information about ultraproducts of rings over $\omega$.

The model of set theory used for the consistency result in §1 is obtained by adding $\aleph_3$-cohen reals to a suitable ground model. There are two ways to get a “suitable” ground model. The first way involves taking any ground model which satisfies a portion of the GCH, and extending it by an appropriate preliminary forcing, which generically adds the name for an ultrafilter which will appear after addition of the cohen reals. The alternative approach is to start with an L-like ground model and use instances of diamond (or related weaker principles) to prove that a sufficiently generic name already exists in the ground model. That was the method used in §3 of [Sh326], which is based in turn on [ShHL162], which has still not appeared as of this writing. However the formalism of [ShHL162], though adequate for certain applications, turns out to be slightly too limited for our present use. More specifically, there are continuity assumptions built into that formalism which are not valid here and cannot easily be recovered. The difficulty, in a nutshell, is that
a union of ultrafilters in successively larger universes is not necessarily an ultrafilter in the universe arising at the corresponding limit stages, and it can be completed to one in various ways.

We intend to include a more general version of [ShHL162] in [Sh482]. However as our present aim is satisfied by any model of set theory with the stated property, we prefer to emphasize the first approach here. So the family $\text{App}$ defined below will be used as a forcing notion for the most part. However we will also take note of some matters relevant to the more refined argument based on a variant of [ShHL162]. For those interested in such refinements, we summarize [ShHL162] in an appendix, as well as a version closer to the form we intend to present in [Sh482]. In addition the exposition in [Sh326, §3] includes a very explicit discussion of the way such a result may be used to formalize arguments of the type given here, in a suitable ground model (in the second sense).

0. Obstructing the Ax-Kochen isomorphism.

Discussion

We will prove Proposition A as formulated in the introduction. We begin with a few words about our general point of view. In practice we do not deal directly with valuation rings, but with trees. If one has a structure with a countable sequence of refining equivalence relations $E_n$ (so that $E_{n+1}$ refines $E_n$) then the equivalence classes carry a natural tree structure in which the successors of an $E_n$-class are the $E_{n+1}$-classes contained in it. Each element of the structure gives rise to a path in this tree, and if the equivalence relations separate points then distinct elements give rise to distinct paths. This is the situation in the valuation ring of of a valued field with value group $\mathbb{Z}$, where we have the basic family of equivalence relations: $E_n(x, y) \iff v(x - y) \geq n$. (Or better: $E(x, y; z) =: "v(x - y) \geq v(z)"$.) Of course an isomorphism of structures would induce an isomorphism of trees, and our approach is to limit the isomorphisms of such trees which are available.

The main result for trees.

We consider trees as structures equipped with a partial ordering and the relation of lying at the same level of the tree. We will also consider expansions to much richer languages. We use the technique of [Sh326, §3] to prove:

**Proposition B**

*It is consistent with the axioms of set theory that there is a nonprincipal ultrafilter $\mathcal{F}$ on $\omega$ such that for any two sequences of countable trees $(T^i_n)_{n=1,2,...}$ for $i = 1, 2$, with each tree $T^i_n$ countable with $\omega$ levels, and with each node having at least two immediate successors, if $T^i = \prod_n T^i_n/\mathcal{F}$, then for any isomorphism $F: T^1 \simeq T^2$ there is an element $a \in T^1$ such that the restriction of $F$ to the cone above $a$ is the restriction of an ultraproduct of maps $F_n: T^1_n \to T^2_n$.*

Proposition B implies Proposition A.

Given an isomorphism $F$ between ultraproducts $R^1$, $R^2$ modulo $\mathcal{F}$ of discrete valuation rings $R^i_n$, we
may consider the induced map $F_+$ on the tree structures $T^1$, $T^2$ associated with these rings, as indicated above. We then find by Proposition B that on a cone of $T^1$, $F_+$ agrees with an ultraproduct of maps $F_{+,n}$ between the trees $T^i_n$ associated with the $R^i_n$. On this cone $F$ is definable from $F_+$, in the following sense: $F(x) = y$ iff for all $n$, $F_+(a \mod \pi^n_i) \equiv b \mod \pi^n_i$, where $\pi_i$ generates the maximal ideal of $R^1$ and we identify $R^i/\pi^n_i$ with the $n$-th level of $T^i$. (This is expressed rather loosely; in the notation we are using at the moment, one would have to take $n$ as a nonstandard integer. After formalization in an appropriate first order language it will look somewhat different.) Furthermore $F$ is definable in $(R^1, R^2)$ from its restriction to this cone: the cone corresponds to a principal ideal $(a)$ of $R^1$ and $F(x) = F(ax)/F(a)$. Summing up, then, there is a first order sentence valid in $(R^1, R^2; F_+)$ (with $F_+$ suitably interpreted as a parametrized family of maps $R^1/\pi^n_1 \rightarrow R^2/\pi^n_2$) stating that an isomorphism $F : R^1 \rightarrow R^2$ is definable in a particular way from $F_+$; so the same must hold in most of the pairs $(R_{1,n}, R_{2,n})$, that is, for a set of indices $n$ which lies in $\mathcal{F}$. In particular in such pairs we get an isomorphism of $R^1$ and $R^2$.

Context

We concern ourselves solely with Proposition B in the remainder of this section. For notational convenience we fix two sequences $(T^i_n)_{n<\omega}$ of trees ($i = 1$ or 2) in advance, where each tree $T^i_n$ is countable with $\omega$ levels, no maximal point, and no isolated branches. The tree $T^i_n$ is considered initially as a model with two relations: the tree order and equality of level. Although we fix the two sequences of trees, we can equally well deal simultaneously with all possible pairs of such sequences, at the cost of a little more notation.

As explained in the introduction, we work in a cohen generic extension of a suitable ground model. This ground model is assumed to satisfy $2^{\aleph_n} = \aleph_{n+1}$ for $n = 0, 1, 2$. If we use the partial order $App$ defined below as a preliminary forcing, prior to the addition of the cohen reals, then this is enough. If we wish to avoid any additional forcing then we assume that the ground model satisfies $\diamondsuit_S$ for $S = \{\delta < \aleph_3 : \text{cof} \delta = \aleph_2\}$, and we work with $App$ directly in the ground model using the ideas of [ShHL162]. The second alternative requires more active participation by the reader.

Let $\mathbf{P}$ be cohen forcing adding $\aleph_3$ cohen reals. An element $p$ of $\mathbf{P}$ is a finite partial function from $\aleph_3 \times \omega$ to $\omega$. For $\mathcal{A} \subseteq \aleph_3$, and $p \in \mathbf{P}$, let $p\upharpoonright \mathcal{A}$ denote the restriction of $p$ to $\mathcal{A} \times \omega$ and $\mathbf{P}\upharpoonright \mathcal{A} = \{p\upharpoonright \mathcal{A} : p \in \mathbf{P}\}$. Let $\varepsilon_{\beta}$ be the $\beta^{th}$ cohen real. The partial order $App$ is defined below.

We will deal with a number of expansions of the basic language of pairs of trees. For a forcing notion $\mathbf{Q}$ and $G$ $\mathbf{Q}$-generic over $V$, we write $G(T^1_n, T^2_n)$ for the expanded structure in which for every $k$, every sequence $(r^k_n)_{n<\omega}$ of $k$-place relations $r^k_n$ on $(T^1_n, T^2_n)$ is represented by a $k$-place relation symbol $R$ (i.e., $R_{(r^k_n,n<\omega)}$); that is, $R$ is interpreted in $(T^1_n, T^2_n)$ by the relation $r^k_n$. This definition takes place in $V[G]$. In $V$ we will have names for these relations and relation symbols. We write $\mathbf{Q}(T^1_n, T^2_n)$ for the corresponding collection of names. In practice $\mathbf{Q}$ will be $\mathbf{P}\upharpoonright \mathcal{A}$ for some $\mathcal{A} \subseteq \omega_3$ and in this case we write $\mathcal{A}(T^1_n, T^2_n)$.

Typically we will have certain subsets of each $T^i_n$ singled out, and we will want to study the ultraproduct of these sets, so we will make use of the predicate whose interpretation in each $T^i_n$ is the desired set. We would prefer to deal with $\mathbf{P}(T^1_n, T^2_n)$, but this is rather large, and so we have to pay some attention to
matters of timing.

Definition

As in [Sh326], we set up a class $App$ of approximations to the name of an ultrafilter in the generic extension $V[\mathcal{P}]$. In [Sh326] we emphasized the use of the general method of [ShHL162] to construct the name $\mathcal{F}$ of a suitable ultrafilter in the ground model. Here we emphasize the alternative and easier approach, forcing with $App$. However we include a summary of the formalism of [ShHL162], and a related formalism, in an appendix at the end.

The elements of $App$ are triples $q = (A, \mathcal{F}, \varepsilon)$ such that:

1. $A$ is a subset of $\aleph_3$ of cardinality $\aleph_1$;
2. $\mathcal{F}$ is a $\mathcal{P}|A$-name of a nonprincipal ultrafilter on $\omega$, called $\mathcal{F}|A$;
3. $\varepsilon = (\varepsilon_\alpha : \alpha \in A)$, with each $\varepsilon_\alpha \in \{0, 1\}$, and $\varepsilon_\alpha = 0$ whenever $\text{cof} \alpha < \aleph_2$;
4. For $\beta \in A$ we have: $[\mathcal{F} \cap \{q : q \text{ a } \mathcal{P}|(A \cap \beta)\text{-name of a subset of } \omega\}]$ is a $\mathcal{P}|(A \cap \beta)$-name;
5. If $\text{cof} \beta = \aleph_2$, $\beta \in A$, $\varepsilon_\beta = 1$ then $\mathcal{P}|A$ forces the following:
   5.1 $x_\beta/\mathcal{F}$ is an element of $(\prod_{n < \omega} T_n^1/\mathcal{F}|A)^{\mathcal{P}|A}$ whose level is above all levels of elements of the form $x/\mathcal{F}$ for $x$ a $\mathcal{P}|(A \cap \beta)$-name;
   5.2 $x_\beta$ induces a branch $B$ on $(\prod_{n} T_n^1)^{\mathcal{P}|(\mathcal{A} \cap \beta)}/[\mathcal{F}|(A \cap \beta)]$ which has elements in every level of that tree (such a branch will be called full) and which is a $\mathcal{P}|(A \cap \beta)$-name (and not just forced to be equal to one);
   5.3 The branch $B$ intersects every dense subset of $(\prod_{n} A^{\mathcal{A} \cap \beta}(T_n^1, T_n^2)/(\mathcal{F}|(A \cap \beta)))^{\mathcal{P}|(\mathcal{A} \cap \beta)}$.

Note in (5.3) that the dense subset under consideration will have a $\mathcal{P}|(A \cap \beta)$-name, and also that by L"{o}s’ theorem a dense subset of the type described extends canonically to a dense subset in any larger model. The notion of “bigness” alluded to in the introduction is given by (5.3).

We write $q_1 \leq q_2$ if $q_2$ extends $q_1$ in the natural sense. We say that $q_2 \in App$ is an end extension of $q_1$, and we write $q_1 \leq_{\text{end}} q_2$, if $q_1 \leq q_2$ and $\mathcal{A}^{q_2} \setminus \mathcal{A}^{q_1}$ follows $\mathcal{A}^{q_1}$. Here we have used the notation: $q = (\mathcal{A}^{q}, \mathcal{F}^{q}, \varepsilon^{q})$.

Remark

The following comments bear on the version based on the method of [ShHL162]. In this setting, rather than examining each $x_\beta$ separately, we would really group them into short blocks $X_\beta = (x_\beta + \zeta : \zeta < \aleph_2)$, for
β divisible by \( \aleph_2 \). Then our assumptions on the ground model \( V \) allow us to use the method of \([ShHL162]\) to construct the name \( \mathcal{F} \) in \( V \). One of the ways \( \diamondsuit_{\mathcal{S}} \) would be used is to “predict” certain elements \( p_\delta \in P|\delta \) and certain \( P|\delta \)-names of functions \( F_\delta \) which amount to guesses as to the restriction to a part of \( \prod_n T_n^1 \) of (the name of) a function representing some isomorphism \( F \) modulo \( \mathcal{F} \). As we indicated at the outset, we intend to elaborate on these remarks elsewhere.

**Lemma**

If \( (q_\zeta)_{\zeta<\xi} \) is an increasing sequence of at most \( \aleph_1 \) members of \( \text{App} \) such that \( q_{\zeta_1} \leq_{\text{end}} q_{\zeta_2} \) for \( \zeta_1 < \zeta_2 \), then we can find \( q \in \text{App} \) such that \( \mathcal{A}^q = \bigcup_\zeta \mathcal{A}^{\zeta_1} \) and \( q_\zeta \leq_{\text{end}} q \) for \( \zeta < \xi \).

**Proof:**

We may suppose \( \xi > 0 \) is a limit ordinal. If \( \text{cof}(\xi) > \aleph_0 \) then \( \bigcup_{\zeta<\xi} q_\zeta \) will do, while if \( \text{cof}(\xi) = \aleph_0 \) then we just have to extend \( \bigcup_\zeta \mathcal{F}^{\zeta_1} \) to a \( P|\bigcup_\zeta \mathcal{A}^{\zeta_1} \)-name of an ultrafilter on \( \omega \), which is no problem. (cf. \([Sh326, 3.10]\)).

**Lemma**

Suppose \( \varepsilon = 1 \), \( q \in \text{App} \), \( \gamma > \sup \mathcal{A}^q \), and \( B \) is a \( P|\mathcal{A}^q \)-name of a branch of \( (\prod_n T_n^1/\mathcal{F}^q)^{V[P|\mathcal{A}^q]} \).

Then:

1. We can find an \( r \in \text{App} \) with \( \mathcal{A}^r = \mathcal{A}^q \cup \{\gamma\} \), and a \( (P|\mathcal{A}^r) \)-name \( \bar{x} \) of a member of \( \prod_n T_n^c/\mathcal{F}^r \) which is above \( B \).
2. We can find an \( r \in \text{App} \) with \( q \leq_{\text{end}} r \) and \( \mathcal{A}^r = \mathcal{A}^q \cup [\gamma, \gamma + \omega_1) \), and a \( (P|\mathcal{A}^r) \)-name \( B' \) of a full branch extending \( B \), which intersects every definable dense subset of \( (\prod_n \mathcal{A}^r T_n^c)^{V[P|\mathcal{A}^r]/\mathcal{F}^r} \).
3. In (2) we can ask in addition that any particular type \( p \) over \( \prod \mathcal{A}^r (T_n^1, T_n^2)/\mathcal{F}^q \) (in \( V[P|\mathcal{A}^q] \)) be realized in \( (\prod_n \mathcal{A}^r T_n^c)^{V[P|\mathcal{A}^q]/\mathcal{F}^r} \).

**Proof:**

1. Make \( x_\gamma \) realize the required type, and let \( \varepsilon_{-\gamma} = 0 \).
2. We define \( r_\zeta = r|(\mathcal{A}^q \cup [\gamma, \gamma + \zeta)) \) by induction on \( \zeta \leq \omega_1 \). For limit \( \zeta \) use 1.7 and for successor \( \zeta \) use part (1). One also takes care, via appropriate bookkeeping, that \( \mathcal{F}^{\zeta_1} \) should intersect every dense definable subset of \( (\prod_n \mathcal{A}^r T_n^c)^{V[P|\mathcal{A}^r]/\mathcal{F}^r} \) by arranging for each such set to be met in some specific \( (\prod_n \mathcal{A}^c T_n^c/\mathcal{F}^r)^{V[P|\mathcal{A}^c]} \) with \( \zeta < \aleph_1 \).
3. We can take \( \alpha \in [\gamma, \gamma + \omega_1) \) with \( \text{cof} \alpha \neq \aleph_2 \) and use \( x_\alpha \) to realize the type.

**Lemma**

Suppose \( q_0, q_1, q_2 \in \text{App} \), \( q_0 = q_2|\beta \), \( q_0 \leq q_1 \), \( \mathcal{A}^{q_1} \subseteq \beta \).

1. If \( \mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0} = \{\beta\} \) and \( \varepsilon_\beta^{q_0} = 0 \), then there is \( q_3 \in \text{App} \), \( q_3 \geq q_1, q_2 \) with \( \mathcal{A}^{q_3} = \mathcal{A}^{q_1} \cup \mathcal{A}^{q_2} \).
2. Suppose \( \mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0} = \{\beta\} \), \( \text{cof} \beta = \aleph_2 \), \( \varepsilon_\beta^{q_0} \neq 0 \), and in particular \( \sup \mathcal{A}^{q_0} < \beta \). Assume that \( B_1 \) is a \( P|\mathcal{A}^{q_0} \)-name of a full branch of \( (\prod_n T_n^2/\mathcal{F}^{q_0})^{V[P|\mathcal{A}^{q_0}]} \) intersecting every dense subset of this tree which is definable in \( (\prod_n \mathcal{A}^{q_1} T_n^1, T_n^2)/\mathcal{F}^{q_0})^{V[P|\mathcal{A}^{q_1}]} \), such that \( B_1 \) contains the branch \( B_0 \) which \( x_\beta \)
induces according to \( q_2 \). Then there is \( q_3 \geq q_1, q_2 \) with \( A^{q_3} = A^{q_1} \cup \{ \beta \} \), such that according to \( q_3 \), \( x_\beta \) induces \( B_1 \) on \( (\prod T^n_{q_1} / F | A^{q_1})^{V[P |A^{q_1}]} \).

3. If \( A^{q_2} \setminus A^{q_0} = \{ \beta \} \), \( \text{cof } \beta = \aleph_2, \varepsilon_\beta^q = 1 \), and \( \sup A^{q_1} < \gamma < \beta \) with \( \text{cof } \gamma \neq \aleph_2 \), then there is \( q_3 \in \text{App} \) with \( q_1 \leq q_3, q_2 \leq q_3, A^{q_3} = A^{q_1} \cup A^{q_2} \cup [\gamma, \gamma + \omega_1) \).

4. There are \( q_3 \in \text{App} \), \( q_1, q_2 \leq q_3 \), so that \( A^{q_1} \setminus A^{q_1} \cup A^{q_2} \) has the form \( \bigcup (\{ \gamma, \gamma + \omega_1 \}) : \zeta \in A^{q_2} \setminus A^{q_0}, \text{cof } \zeta = \aleph_2 \} \) where \( \gamma_\zeta \) is arbitrary subject to \( \sup(A^{q_2} \setminus A^{q_0}) < \gamma_\zeta < \zeta \).

5. Assume \( \delta_1 < \aleph_2, \beta < \aleph_3 \), that \( (p_i)_{i<\delta} \) is an increasing sequence from \( \text{App} \), and that \( q \in \text{App} | \beta \) satisfies:

\[
\text{For } i < \delta_1 : p_i | \beta \leq q.
\]

Then there is an \( r \in \text{App} \) with \( q \leq r_\text{end} r \) and \( p_i \leq r \) for all \( i < \delta_1 \).

6. Assume \( \delta_1, \delta_2 < \aleph_2, (\beta_j)_{j<\delta_2} \) is an increasing sequence with all \( \beta_j < \aleph_3 \), that \( (p_i)_{i<\delta_1} \) is an increasing sequence from \( \text{App} \), and that \( q_j \in \text{App} | \beta_j \) satisfy:

\[
\text{For } i < \delta_1, j < \delta_2 : p_i | \beta_j \leq q_j; \quad \text{For } j < j' < \delta_2 : q_j \leq r_\text{end} q_j'.
\]

Then there is an \( r \in \text{App} \) with \( p_i \leq r \) and \( q_j \leq r_\text{end} r \) for all \( i < \delta_1 \) and \( j < \delta_2 \).

Proof:

1. The proof is easy and is essentially contained in the proofs following. (One verifies that \( F^{q_1} \cup F^{q_2} \) generates a proper filter in \( V[P | (A^{q_1} \cup A^{q_2})] \).

2. Let \( \mathcal{A}_i = A^{q_i} \) and let \( F_i = F^{q_i} \) for \( i = 1, 2 \), and \( \mathcal{A}_3 = \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}_1 \cup \{ \beta \} \). The only nonobvious part is to show that in \( V[P | A_3] \) there is an ultrafilter extending \( F_1 \cup F_2 \) which contains the sets:

\[
\{ n : \text{for } i, x \in B_1 \text{ a } P | A_1 \text{-name.} \}
\]

If this fails, then there is some \( p \in P | A_3 \), a \( P | A_1 \)-name \( q \) of a member of \( F_1 \), a \( P | A_2 \)-name \( b \) of a member of \( F_2 \), and some \( x \in B_1 \) such that \( p \not\models \text{"\( q \cap b = 0^n \" \)} \) where \( \epsilon = \{ n : x(n) \leq x_\beta(n) \} \). Let \( p_i = p | \mathcal{A}_i \) for \( i = 0, 1, 2 \), and let \( H^0 \subseteq P | A_0 \) be generic over \( V \), with \( p_0 \in H^0 \).

Let:

\[
A^1_n[H^0] = \{ y \in T^n_{q_1} : \text{For some } p'_1, p_1 \leq p'_1 \in P | A_1, p'_1 | A_0 \in H^0 \text{ and } p'_1 \not\models \text{"\( x(n) \leq y, \) and } n \in \epsilon \" \}.
\]

Then \( A^1_n \) is a \( P | A_0 \)-name. Let \( A^1 = (\prod_n A^1_n/F | A_0)^{V[P |A_0]} \). Now \( A^1 \) is not necessarily dense in \( (\prod_n T^n_{q_1} / F | A_0)^{V[P |A_0]} \), but the set

\[
A^* = \{ y \in (\prod_n T^n_{q_1} / F | A_0)^{V[P |A_0]} : y \in A^1, \text{ or } y \text{ is incompatible in the tree with all } y' \in A^1 \}
\]

is dense, and it is definable, hence not disjoint from \( B_0 \). Fix \( y \in A^* \cap B_0 \). As \( x \in B_1 \), \( x \) and \( y \) cannot be forced to be incompatible, and thus \( y \in A^1 \).
The following sets are in $\mathcal{F}^{V[H^0]}$:

\[ A = \{ n : \text{for some } p_1', p_1 \leq p_1' \in \mathcal{P} | A_1, p_1' | A_0 \in H^0 \text{ and } p_1' \models "x(n) \leq y(n), \text{ and } n \in q" \}. \]

\[ B = \{ n : \text{for some } p_2', p_2 \leq p_2' \in \mathcal{P} | A_2, p_2' | A_0 \in H^0 \text{ and } p_2' \models "y(n) \leq x_\beta(n), \text{ and } n \in b" \}. \]

For example, $A$ is a subset of $\omega$ in $V[H^0]$ which is in $\mathcal{F}^{q_1}$. As the complement of $A$ cannot be in $\mathcal{F}^{q_0}$, $A$ must be.

Now for any $n \in A \cap B$ we can force $n \in q \cap b \cap c$ by amalgamating the corresponding conditions $p_1', p_2'$.

3. Let $B_0$ be the $\mathcal{P} | A^{q_0}$-name of the branch which $x_\beta$ induces. By 1.8 (2) there is $q_1, A^{q_1} = A^{q_0} \cup \gamma, \gamma + \omega_1), q_1 \leq q_1' \in \mathcal{P}$ and there is a $\mathcal{P} | A^{q_1}$-name $B_1 \supseteq B_0$ of an appropriate branch for $q_1'$. Now apply part (2) to $q_0, q_1, q_2$.

4. As in [Sh326, 3.9(2)], by induction on the order type of $(A^{q_2} \setminus A^{q_0})$, using (3).

5, 6. Since (6) includes (5), it suffices to prove (6); but as we go through the details we will treat the cases corresponding to (5) first. We point out at the outset that if $\delta_2$ is a successor ordinal or a limit of uncountable cofinality, then we can replace the $q_j$ by their union, which we call $q$, setting $\beta = \sup_j j$, so all these cases can be treated using the notation of (5).

We will prove by induction on $\gamma < \omega_2$ that if all $\beta_j \leq \gamma$ and all $p_i$ belong to $\mathcal{P} | \gamma$, then the claim (6) holds for some $r$ in $\mathcal{P} | \gamma$.

We first dispose of most of the special cases which fall under clause (5). If $\delta_1 = \delta_0 + 1$ is a successor ordinal it suffices to apply (4) to $p_{\delta_0}$ and $q$. So we assume for the present that $\delta_1$ is a limit ordinal. In addition if $\gamma = \beta$ we take $r = q$, so we will assume $\beta < \gamma$ throughout.

The case $\gamma = \gamma_0 + 1$, a successor.

In this case our induction hypothesis applies to the $p_i | \gamma_0, q, \beta, \gamma_0$, yielding $r_0$ in $\mathcal{P} | \gamma_0$ with $p_i | \gamma_0 \leq r_0$ and $q \leq_{\text{end}} r_0$. What remains to be done is an amalgamation of $r_0$ with all of the $p_i$, where dom $p_i \subseteq \text{dom} r_0 \cup \{ \gamma_0 \}$, and where one may as well suppose that $\gamma_0$ is in dom $p_i$ for all $i$. This is a slight variation on 1.9 (1 or 3) (depending on the value of $\varepsilon^{p_i}$, which is independent of $i$).

The case $\gamma$ a limit of cofinality greater than $\aleph_1$.

Since $\delta_1 < \aleph_2$ there is some $\gamma_0 < \gamma$ such that all $p_i$ lie in $\mathcal{P} | \gamma_0$ and $\beta < \gamma_0$, and the induction hypothesis then yields the claim.

The case $\gamma$ a limit of cofinality $\aleph_1$.

Choose $\gamma_j$ a strictly increasing and continuous sequence of length at most $\omega_1$ with supremum $\gamma$, starting with $\gamma_0 = \beta$. By induction choose $r_j \in \mathcal{P} | \gamma_j$ for $i < \omega_1$ such that:

(0) \hspace{1cm} r_0 = q;

(1) \hspace{1cm} r_j \leq_{\text{end}} r_{j'} \text{ for } j < j' < \omega_1;
At successor stages the inductive hypothesis is applied to \( p_i|\gamma_j \leq r_j \) for \( i < \delta_1 \) and \( j < \omega_1 \).

At limit stages \( j \) we apply the inductive hypothesis to \( p_i|\gamma_j, r_j, \gamma_j, \) and \( \gamma_{j+1} \). At limit stages \( j \) we apply the inductive hypothesis to \( p_i|\gamma_{j'}, r_{j'}, \gamma_{j'}, \) for \( j' < j \), \( \gamma_j \) for \( j' < j \), and \( \gamma_j \); and here (6) is used, inductively.

Finally let \( r = \bigcup r_j \).

We now make an observation about the case of (5) that we have not yet treated, in which \( \gamma \) has cofinality \( \omega \). In this case we can use the same construction used when \( \gamma \) has cofinality \( \aleph_1 \), except for the last step (where we set \( r = \bigcup r_j \), above). What is needed at this stage would be an instance of (6), with the \( r_j \) in the role of the \( q_j \) and \( \delta_2 = \omega \).

This completes the induction for the cases that fall under the notation of (5), apart from the case in which \( \gamma \) has cofinality \( \omega \), which we reduced to an instance of (6) with the same value of \( \gamma \) and with \( \delta_2 = \omega \). Accordingly as we deal with the remaining cases we may assume \( \delta_2 = \omega \). In this case \( q = \bigcup q_j \) is a well-defined object, but not necessarily in \( App \), as the filter \( F^q \) is not necessarily an ultrafilter (there are reals generated by \( P \)[dom \( q \)] which do not come from any \( P \)[dom \( q_j \)]).

We distinguish two cases. If \( \beta := \sup \beta_j \) is less than \( \gamma \), then induction applies, delivering an element \( r_0 \in App|\beta \) with \( p_i|\beta \leq r_0 \) and all \( q_j \leq \text{end} r_0 \). This \( r_0 \) may then play the role of \( q \) in an application of 1.9 (5).

In some sense the main case (at least as far as the failure of continuity is concerned) is the remaining one in which \( \beta = \gamma \). Notice in this case that although \( p_i|\beta_j \leq q_j \) it does not follow that \( p_i|\beta \leq q \) (for the reason mentioned above: \( p_i|\beta \) includes an ultrafilter on part of the universe, while the filter associated with \( q \) need not be an ultrafilter). All that is needed at this stage is an ultrafilter containing all \( F^{p_i} \cup F^q \). As this is a directed system of filters, it suffices to check the compatibility of each such pair, as was done in 1.9 (2).

Construction, first version.

We force with \( App \) and the generic object gives us the name of an ultrafilter in \( V[App][P] \). The forcing is \( \aleph_2 \)-complete by 1.9 (5). We also claim that it satisfies the \( \aleph_3 \)-chain condition, and hence does not collapse cardinals and does not affect our assumptions on cardinal arithmetic. (Subsets of \( \aleph_2 \) are added, but not very many.) In particular \( (\prod T^t_n, T^t_n, /\mathcal{F}^t)^V[P[A]^t] \) is a \( P[A]^t \)-name, not dependent on forcing with \( App \).

We now check the chain condition. Suppose we have an antichain \( \{q_\alpha\} \) of cardinality \( \aleph_3 \) in \( App \), where for convenience the index \( \alpha \) is taken to vary over ordinals of cofinality \( \aleph_2 \). We claim that by Fodor’s lemma, we may suppose that the condition \( q_\alpha|\alpha \) is constant. One application of Fodor’s lemma allows us to assume that \( \gamma = \sup(A^\text{on} \cap \alpha) \) is constant. Once \( \gamma \) is fixed, there are only \( \aleph_2 \) possibilities for \( q_\alpha|\gamma \); by our assumptions on the ground model, and a second application of Fodor’s lemma allows us to take \( q_\alpha|\gamma \) to be constant.

Now fix \( \alpha_1 \) of cofinality \( \aleph_2 \) (or more accurately, in the set of indices which survive two applications of
Fodor’s lemma), and let \( q_1 = q_{\alpha_1}, \beta = \sup A^\alpha \), and take \( \alpha_2 > \beta \) of cofinality \( \aleph_2 \). We find that \( q_2 =: q_{\alpha_2} \) and \( q_1 \) are compatible, by 1.9 (4), and this is a contradiction.

Construction, second version.

If we wish to apply the method of [ShHL162] (over a suitable ground model) and build the name of our ultrafilter in the ground model, we proceed as follows. For \( \alpha \leq \aleph_3 \) we choose \( G^\alpha \subseteq \text{App} | \alpha \), directed under \( \leq \), inductively as in [Sh326, §3], making all the commitments we can; more specifically, take \( \mathcal{N} \sim (\mathcal{H}(\beth_{\omega+1}), \in) \) of cardinality \( \aleph_2 \) with \( \delta \in \mathcal{N} \), \( \aleph_2 \subseteq \mathcal{N} \), \( \mathcal{N} \) is \((< \aleph_2)\)-complete, and the oracle associated with \( \mathcal{Q}_S \) belongs to \( \mathcal{N} \), and make all the commitments known to \( \mathcal{N} \). Then \( G^\alpha \) is in the ground model but behaves like a generic object for \( \text{App} | \alpha \) in \( V[\mathcal{P} | \alpha] \), and in particular gives rise to a name \( \mathcal{F}^\alpha \).

The lengthy discussion in [Sh326 §3] is useful for developing intuition. Here we will just note briefly that what is called a commitment here is really an isomorphism type of commitment, in a more conventional sense; this is a device for compressing \( \aleph_3 \) possible commitments into a set of size \( \aleph_2 \).

The formalism is documented in the appendix to this paper, but as we have said it has to be adapted to allow weaker continuity axioms. Compare paragraphs A1 and A6 of the appendix. The axioms in the appendix have been given in a form suitable to their application to the proof of the relevant combinatorial theorem, rather than in the form most convenient for verification. 1.9 above represents the sort of formulation we use when we are actually verifying the axioms.

We will now add a few details connecting 1.9 with the eight axioms of paragraph A6. The first three of these are formal and it may be expected that they will be visibly true of any situation in which this method would be applied. The fourth axiom is the so-called amalgamation axiom which has been given in a slightly more detailed form in 1.9 (4). The last four axioms are various continuity axioms, which are instances of 1.9 (5). We reproduce them here:

5’. If \( (p_i)_{i < \delta} \) is an increasing sequence in App of length less than \( \lambda \), then it has an upper bound \( q \).

6’. If \( (p_i)_{i < \delta} \) is an increasing sequence of length less than \( \lambda \) of members of \( \text{App} | (\beta + 1) \), with \( \beta < \lambda^+ \) and if \( q \in \text{App} | \beta \) satisfies \( p_i | \beta \leq q \) for all \( i < \delta \), then \( \{p_i : i < \delta\} \cup \{q\} \) has an upper bound \( r \) in \( \text{App} \) with \( q \leq r \).

7’. If \( (\beta_j)_{j < \delta} \) is a strictly increasing sequence of length less than \( \lambda \), with each \( \beta_j < \lambda^+ \), and \( p \in \text{App} \), \( q_i \in \text{App} | \beta_i \), with \( p | \beta_j \leq q_j \), and \( p_j | \beta_j = p_j \) for \( j < j' < \delta \), then \( \{p\} \cup \{q_j : i < \delta\} \) has an upper bound \( r \) with all \( q_j \leq r \).

8’. Suppose \( \delta_1, \delta_2 \) are limit ordinals less than \( \lambda \), and \( (\beta_j)_{j < \delta_2} \) is a strictly increasing continuous sequence of ordinals less than \( \lambda^+ \). Let \( I(\delta_1, \delta_2) := (\delta_1 + 1) \times (\delta_2 + 1) - \{(\delta_1, \delta_2)\} \). Suppose that for \( (i, j) \in I(\delta_1, \delta_2) \) we have \( p_{ij} \in \text{App} | \beta_i \) such that

\[
\begin{align*}
  i \leq i' & \implies p_{ij} \leq p_{i'j} \\
  j \leq j' & \implies p_{ij} = p_{i'j} | \beta_j;
\end{align*}
\]

Then \( \{p_{ij} : (i, j) \in I(\delta_1, \delta_2)\} \) has an upper bound \( r \) in \( \text{App} \) with \( r | \beta_j = p_{i\delta_2} \) for all \( j < \delta_2 \).

The first three are visibly instances of 1.9 (5). In the case of axiom (8’) we set \( p_i = p_{i, \delta_2} \) for \( i < \delta_1 \).
and \( q_j = p_{i,j} \) for \( j < \delta_2 \). Then \( p_i|\beta_j = p_{i,j} \leq q_j \), so 1.9 (5) applies and yields \((8')\).

Lemma

Suppose \( \delta < \aleph_3 \), \( \text{cof}(\delta) = \aleph_2 \), and \( H^\delta \subseteq P|\delta \) is generic for \( P|\delta \). Then in \( V[H^\delta] \) we have:

\[
\prod_n^{\delta} (T_n^1, T_n^2)/\mathcal{F}[H^\delta] \text{ is } \aleph_2\text{-compact.}
\]

Proof:

Similar to 1.8 (2). We can use some \( \tilde{x}_\beta \) with \( \beta \) of cofinality less than \( \aleph_2 \) to realize each type. In the forcing version, this means \( \text{App} \) forces our claim to hold since it can’t force the opposite. In the alternative approach, what we are saying is that the commitments we have made include commitments to make our claim true. As \( 2^{\aleph_1} = \aleph_2 \) in \( V[H^\delta] \) we can “schedule” the commitments conveniently, so that each particular type of cardinality \( \aleph_1 \) that needs to be considered by stage \( \delta \) in fact appears before stage \( \delta \).

Killing isomorphisms

We begin the verification that our filter \( \mathcal{F} \) satisfies the condition of Proposition B. We suppose therefore that we have a \( P \)-name \( \tilde{x} \) and a condition \( p^* \in P \) forcing:

\[ \text{"} \tilde{x} \text{ is a map from } \prod_n T_n^1 \text{ onto } \prod_n T_n^2 \text{ which represents an isomorphism modulo } \mathcal{F}. \text{"} \]

We then have a stationary set \( S \) of ordinals \( \delta < \aleph_3 \) of cofinality \( \aleph_2 \) which satisfy:

(a) \( p^* \in P|\delta \).

(b) For every \( P|\delta \)-name \( \tilde{x} \) for an element of \( \prod_n T_n^1 \), \( F(\tilde{x}) \) is a \( P|\delta \)-name.

(c) Similarly for \( F^{-1} \).

If we are using our second approach, over an \( L \)-like ground model:

(d) At stage \( \delta \) of the construction of the \( G^\alpha \), the diamond “guessed” \( p^\delta = p^* \) and \( F_\delta = F|^\delta \).

(In this connection, recall that the guesses made by diamond influence the choice of “commitments” made in the construction of the \( G^\delta \).) Let \( y^* =: F(\tilde{x}_\delta) \). Then:

\[
(*_{y^*}) \quad p^* \Vdash \{ \text{"} y^* \text{ induces a branch in } (\prod_n T_n^2/\mathcal{F})^{V[\mathcal{P}|\delta]} \text{ which is the image under } F_\delta \text{ of the branch which } \tilde{x}_\delta \text{ induces on } (\prod_n T_n^1/\mathcal{F})^{V[\mathcal{P}|\delta]} \text{."} \}
\]

Now we come to one of the main points. We claim that there is some \( q^* \in G \) with the following property:

Given \( q_1 \in G^\delta \) with \( q^*|\delta \leq q_1 \) and \( P|\mathcal{A}^\delta \)-names \( (\tilde{x}, y) \) with \( \tilde{x} \in \prod T_n^1, y \in \prod T_n^2 \), then for any \( q_\delta^* \in \text{App} \) with \( q_1, q^* \leq q_\delta^* \) and \( q_\delta^*|\delta \in G^\delta \), \( p^* \) forces:

\[
(\dagger)_{\delta} \quad \text{"} \text{If } y = F(\tilde{x}) \text{ then } x \leq \tilde{x}_\delta \text{ iff } y \leq y^*, \text{ and if } y \text{ and } F(\tilde{x}) \text{ are incomparable, then } x \leq \tilde{x}_\delta \text{ implies } y \not\leq y^*. \text{"} \]
Notice here that \( q'_1 \) need not be in \( G \).

The reason for this depends slightly on which of the two approaches to the construction of \( G \) we have taken. In a straight forcing approach, we may say that some \( q^* \in G \) forces \((*)_{q^*}\), and this yields \((\dagger)_\delta\). In the second, pseudo-forcing, approach we find that our “commitments” include a commitment to falsify \((*)_{q^*}\) if possible; as we did not do so, at a certain point it must have been impossible to falsify it, which again translates into \((\dagger)_\delta\).

We now fix \( q^* \) satisfying \((\dagger)_\delta\), and we set \( q_0 = q^*_\dagger \). At this stage, \((\dagger)_\delta\) gives some sort of local definition of \( F^\dagger \), on a cone in \( (\prod^\dagger T_n^1/F^\dagger)^{V[P]} \) determined by \( q_0 \). The next result allows us to put this definition in a more useful form (and this is nailed down in 1.15). One may think of this as an elimination of quantifiers.

**Lemma**

**Suppose that:**

\[
\begin{align*}
(1) & \quad q_0, q_1, q_2, q_3 \text{ are in } App \text{ with } q_0 = q_2\upharpoonright \beta_0 \leq q_1 \leq q_3, \text{ and } q_2 \leq q_3.

(2) & \quad q_0 \leq r_0 \in App \text{ with } A_0^q \subseteq A_0^\omega \subseteq \beta_0.
\end{align*}
\]

Let \( A_i = A_0^q \) for \( i = 0, 1, 2, 3 \), and suppose that

\[
\begin{align*}
(3) & \quad f_0 \text{ is a } P\upharpoonright A_0^\omega\text{-name of a map from } (\prod_n T_n^1, T_n^2)^{V[P]A_0} \text{ to } (\prod_n T_n^1, T_n^2)^{V[P]A_0}
\end{align*}
\]

representing a partial elementary embedding of

\[
\begin{align*}
(\prod_n A_0(T_n^1, T_n^2)/F[A_0])^{V[P]A_0} \text{ into } (\prod_n A_0(T_n^1, T_n^2)/F[A_0])^{V[P]A_0}
\end{align*}
\]

which is equal to the identity on \((\prod_n T_n^1, T_n^2)/F[A_0])^{V[P]A_0}.

Then there is an \( r \in App \) with:

\[
q_2 \leq r; \quad r_0 \leq r; \quad A_3 \subseteq A_0^r; \quad A_0^r \cap \beta_0 = A_0^\omega;
\]

and there is a \( P \)-name \( f \) of a function from \((\prod_n T_n^1, T_n^2)^{V[P]A_3}\text{ into } (\prod_n T_n^1, T_n^2)^{V[P]A_3}\) representing an elementary embedding of \( A_2(\prod_n T_n^1, T_n^2/F[A_3])^{V[P]A_3} \text{ into } A_2(\prod_n T_n^1, T_n^2/F[A_3])^{V[P]A_3} \) which is the identity on \((\prod_n T_n^1, T_n^2/F[A_2])^{V[P]A_2}\).

**Proof:**

It will be enough to get \( f \) as a partial elementary embedding, as one may then iterate 1.8 (3) \( \aleph_1 \) times.

We may suppose \( \beta_0 = \inf (A_3 - A_0^\omega) \). Let \( A_3 \setminus \beta_0 = (\beta_i)_{i<\xi} \) enumerated in increasing order. We will construct two increasing sequences, one of names \( f_i \) and one of elements \( r_i \in App \), indexed by \( i \leq \xi \), such that our claim holds for \( f_i, q_2\upharpoonright \beta_i, q_3\upharpoonright \beta_i, r_i \), and in addition \( A_0^\omega \subseteq \beta_i \). At the end we take \( r = r_\xi \) and \( f = f_\xi \).
The case \( i = 0 \)

Initially \( r_0 \) and \( f_0 \) are given.

The limit case

Suppose first that \( i \) is a limit ordinal of cofinality \( \aleph_0 \), and let \( A = \bigcup_{j < i} A^j \). In this case \( \bigcup_{j < i} \mathcal{F}^j \) is not an ultrafilter in \( V|P[A] \) and the main point will be to prove that there is a \( P|A \)-name for an ultrafilter \( \mathcal{F}_i \) extending \( \mathcal{F}^{q_2|\beta_1} \), and \( \bigcup_{j < i} \mathcal{F}^j \), such that

\[
\begin{cases}
\text{The map } f_i \text{ defined as the identity on } (\prod_n(T_n^1, T_n^2))^{\mathcal{F}[P|(A_2 \cap \beta_1)]} \text{ and as } \bigcup_{j < i} f_j \\
\text{on the latter’s domain is a partial elementary map from } \\
(\prod_n(A_2 \cap \beta_i)(T_n^1, T_n^2) / \mathcal{F})(A_3 \cap \beta_i))^{\mathcal{F}[P|(A_3 \cap \beta_i)]} \text{ into } (\prod_n(A_2 \cap \beta_i)(T_n^1, T_n^2) / \mathcal{F}^j)^{\mathcal{F}[P[A]}.}
\end{cases}
\]

So it will suffice to find \( \mathcal{F}_i \) making (\( \ast \)) true. This means we must check the finite intersection property for a certain family of (names of) sets. Suppose toward a contradiction that we have a condition \( p \in P | A \) forcing “\( q \cap b \cap c = \emptyset \),” where:

(A) \( g \) is a \( P|A^\gamma \)-name for a member of \( \mathcal{F}^\gamma \)

(B) \( b \) is a \( P|A^{q_2|\beta_i} \)-name for a member of \( \mathcal{F}^{q_2|\beta_i} \)

(C) \( c \) is the name of a set of the form: \( \{ n : (T_n^1, T_n^2) \models \varphi(x(n), f_j(y)(n)) \} \).

(C1) \( x, y \) are finite sequences from \( (\prod_n(T_n^1, T_n^2))^{\mathcal{F}[P|(A_2 \cap \beta_1)]} \) and \( (\prod_n(T_n^1, T_n^2))^{\mathcal{F}[P|(A_3 \cap \beta_i)]} \) respectively.

(C2) \( \varphi \) is a \( P|A^{q_2|\beta_i} \)-name for a formula in the language of \( \prod_n A^{q_2|\beta_i} \), \( (T_n^1, T_n^2) \)

(C3) \( \varphi(x, y) \) holds in \( A_2 \cap \beta_i) \), \( (\prod_n(T_n^1, T_n^2) / \mathcal{F})(A_3 \cap \beta_i))^{\mathcal{F}[P|(A_3 \cap \beta_i)]} \).

Here \( j < i \) arises as the supremum of finitely many values below \( i \). As \( x \) can be absorbed into the language, we will drop it.

Now let \( H \) be generic for \( P|(A_2 \cap \beta_j) \) with \( p|(A_2 \cap \beta_j) \in H \), and define:

\[ A_n = \{ u : \text{for some } p_2 \geq p|(A_2 \cap \beta_i) \text{ with } p_2|(A_2 \cap \beta_j) \in H, \quad p_2 \models \text{"}n \in b \text{ and } (T_n^1, T_n^2) \models \varphi(u).\text{"} \} \]

\( A_n \) is a \( P|(A_2 \cap \beta_i) \)-name of a subset of \( T_n^2 \). Take \( A_n \) as a relation in \( \prod(A^{q_2|\beta_1}(T_n^1, T_n^2)). \) By hypothesis \( \{ n : (T_n^1, T_n^2) \models \varphi(y(n)) \} \in \mathcal{F}^{q_2|\beta_1} \), and this set is contained in the set \( \varphi' = \{ n : y(n) \in A_n \} \), which belongs to \( V[P|(A_3 \cap \beta_j)] \). Therefore \( \varphi' \in \mathcal{F}^{q_2|\beta_j} \), and applying \( f_j \), we find:

\[ \{ n : f_j(y)(n) \in A_n \} \in \mathcal{F}^j. \]

Hence we may suppose that \( p \) forces: for \( n \in q \), \( f_j(y)(n) \in A_n \). But then any element of \( q \) can be forced by an extension of \( p \) to lie in \( b \cap c \), by amalgamating appropriate conditions over \( A_2 \cap \beta_j \).

Limits of larger cofinality are easier.
The successor case

Suppose now that \( i = j + 1 \). We may suppose that \( \beta_j \in A_2 \) as otherwise there is nothing to prove. If \( \varepsilon_{\beta_j}^q = 0 \) we argue as in the previous case. So suppose that \( \varepsilon_{\beta_j}^q = 1 \). In particular \( \beta_j \) has cofinality \( \aleph_2 \).

Using 1.8 (3) repeatedly, and the limit case, we can find \( B, q_1', r', f' \) such that:

\[
\begin{align*}
(1) \quad q_3|\beta_j \leq \text{end } q_1'; & \quad A^{q_1'} \subseteq \beta_j; \\
(2) \quad r_j \leq \text{end } r'; & \quad A^{r'} \subseteq \beta_j;
\end{align*}
\]

\[
\begin{cases}
\quad f' \text{ is a map from } \prod_n (T^1_n, T^2_n)^V[A^{q_1'}] \text{ onto } \prod_n (T^1_n, T^2_n)^V[A^{r'}] \text{ representing an isomorphism} \\
\quad \text{of } (\prod_n (T^1_n, T^2_n)/\mathcal{F}^{q_1'})^V[A^{q_1'}] \text{ with } (\prod_n (T^1_n, T^2_n)/\mathcal{F}^{r'})^V[A^{r'}] \text{ extending } f;
\end{cases}
\]

\[
B \text{ is a } P[A^{q_1'}] \text{-name of a branch of } (\prod_n T^1_n/\mathcal{F}^{q_1'})^V[A^{q_1'}] \text{ which is sufficiently generic;}
\]

\[
f''[B] \text{ is a } P[A^{r'}] \text{-name of a branch of } (\prod_n T^1_n/\mathcal{F}^{r'})^V[A^{r'}] \text{ which is sufficiently generic.}
\]

Let \( q_3' \) satisfy \( q_3|\beta_i \leq q_3', q_1' \leq \text{end } q_3' \), with \( A^{q_1'} \subseteq \beta_i \) such that according to \( q_3' \) the vertex \( x_{\beta_j} \) lies above \( B \) (using 1.9(2)). We intend to have \( r_i \) put \( x_{\beta_j} \) above \( f''[B] \) (to meet conditions (5.2, 5.3) in the definition of \( A_{\text{App}} \)), while meeting our other responsibilities. As usual the problem is to verify the finite intersection property for a certain family of names of sets. Suppose therefore toward a contradiction that we have a condition \( p \in P \) forcing "\( q \cap b \cap c \cap d = \emptyset \)," where

\( a \) is a \( P[A^{r'}] \)-name of a member of \( \mathcal{F}^{r'} \);

\( b \) is a \( P[A^{q_2^1 \cap \beta_i}] \)-name of a member of \( \mathcal{F}^{q_2^1 \cap \beta_i} \);

\( c \) is the name of a set of the form \( \{ n : (T^1_n, T^2_n) \models \varphi(x_{\beta_j}(n), f''(y)(n)) \} \)

\( d \) is \( \{ n : T^1_n = \varphi(n) < x_{\beta_j}(n) \} \)

where in connection with \( c \) we have:

\[
y \in (\prod_n (T^1_n, T^2_n)^V[A^{q_1'}],
\]

\( \varphi(x_{\beta_j}, y) \) is defined and holds in \( (\prod_n A^{q_2^1 \cap \beta_i} (T^1_n, T^2_n)/\mathcal{F}^{q_1'}))^V[A^{q_1'}] \),

and we have absorbed some parameters occurring in \( \varphi \) into the expanded language which is associated with \( V[P(A_2 \cap \beta_j)] \) as individual constants, while in connection with \( d \) we have:

\( x \) is a \( P[A^{q_1'}] \)-name for a member of \( f''[B] \).
Let $H^* \subseteq P$ be generic over $V$ with $H \subseteq H^*$ and $p \in H^*$. Set $H = H^* \downarrow A^{q_1[\beta]}$, $H_1 = H^* \downarrow A^{q_1}'$, and $H_3 = H^* \downarrow A^{q_2}_1$. In $V[H]$ we define:

$A_n^1 := \{(x, u) : \text{For some } p_1 \in P[A], \text{ with } p_1 \geq p[A] \text{ and } p_1[A^{q_1[\beta]}] \in H, \text{ and hence }$ \n $\exists n \in q, x(n) = x, f'(y)(n) = u \}$

$p_1$ forces: \[ n \in q, x(n) = x, f'(y)(n) = u \}

$A_n^2 := \{(x^*, u) : \text{For some } p_2 \in P[A_2 \cap \beta_i] \text{ with } p_2 \geq p[A_2 \cap \beta_i] \text{ and } p_2[A_2 \cap \beta_j] \in H, \text{ and hence }$ \n $\exists n \in h, x_{\beta_j}(n) = x^*, \text{ and } \varphi(x^*, u) \}$

$p_2$ forces: \[ n \in h, x_{\beta_j}(n) = x^*, \text{ and } \varphi(x^*, u) \}

In $V[H]$ there is no $n$ satisfying:

\[ \exists x, x^*, u \ (x, u) \in A_n^1 \land (x^*, u) \in A_n^2 \land x < x^*. \]

Otherwise we could extend $p$ by amalgamating suitable conditions $p_1, p_2$, to force such an $n$ into $q \cap h \cap \varphi \cap d$.

For $n < \omega$ and $u \in T_n^1$ let

$A_n^2(u) := \{x \in T_n^1 : (x, u) \in A_n^2 \}$

$A_n^3(u) := \{x \in T_n^1 : \text{Either } (x, u) \in A_n^2 \lor \text{there is no } x' \text{ above } x \text{ in } T_n^1 \text{ for which } (x', u) \in A_n^2 \}$

Then $A_n^2(u)$ is dense in $T_n^1$, and hence so is $A_3 := \prod A_n^3 / \mathcal{F}^{q_1[\beta]}[H]$. Let $T = (T_1, T_2, A, A_3)$ be the ultraproduct $(\prod_n(T_n^1, T_n^2, A_n^2, A_n^3) / \mathcal{F}^{q_1}[H], T_1)$ with $\varphi[x, y] \in \prod A_n^3(\mathcal{F}^{q_1}[H])$. Now $\varphi[x, y]$ holds in $\prod A_n^3(T_n^1, T_n^2, A_n^2, A_n^3) / \mathcal{F}^{q_1}[H]$.

Then $A_n^3(y)$ is dense in $A_3$.

Let $\mathcal{A} = (T_1, T_2, A^3) \subseteq \mathcal{A}_1$ be the ultrapower $(\prod_n(T_n^1, T_n^2, A_n^2, A_n^3) / \mathcal{F}^{q_1}[H], T_1)$ with $\varphi[x, y] \in \prod A_n^3(\mathcal{F}^{q_1}[H])$.

Then $\varphi[x, y] \in \prod A_n^3(\mathcal{F}^{q_1}[H])$.

For $z \in A_3(y) \cap B[H_1], \text{ as } z < x_{\beta_j}$, we have also $z \in A_2(y[H_1]) \cap B[H_1]$. Hence in $V[H_1]$ we have:

\[ A_2^3(y) \cap B[H_1] \text{ is unbounded in } B[H_1] \]

and hence $A_2^3(f'(y)) \cap f'(B)[H^* \downarrow A']$ is unbounded in $f'(B)[H^* \downarrow A']$, and we can find $\zeta \in A_2^3(f'(y[H_3])) \cap f'(B)[H^* \downarrow A']$ with $\zeta < z$ in $\prod_n T_n^1 / \mathcal{F}[H^* \downarrow A']$.

In particular for some $n \in q[H^*], \text{ we have } x(n) \in H^* \text{ and } \zeta(n) \in H^* \text{ in } T_n^1$ and $\zeta(n) \in A_2^3(y(n))$. Letting $x = x(n)[H_1], x^* = z(n)[H_1], \text{ and } u = f'(y(n))[H \downarrow A']$, we find that (*) holds in $V[H]$, a contradiction. \hfill \Box

Weak definability

**Proposition**

Let $\delta < \kappa_3$ be an ordinal of cofinality $\kappa_2$ satisfying conditions 1.13 (a-d). Suppose $q_1, q_2 \in G, q_2 | \delta = q_0 \leq q_1, A^{q_1} \subseteq \delta, \delta \in A^{q_2}, y^* \text{ is a } P[A^{q_2}]-\text{name of an element of } \prod_n T_n^2, \text{ and } \varepsilon_0^{q_2} = 1$. Suppose
further that \( x', x'' \) and \( y', y'' \) are \( P, A^{\emptyset}_1 \)-names, \( p \in P \), \( p_i = p|A^{\emptyset}_i \) (\( i = 1, 2 \)), and:

\[
p_1 \models "x', x'' \in \prod_n T^1_n, \) and \( y', y'' \in \prod_n T^2_n," \quad \text{and} \quad \text{p}_2 \models "F(x_\delta) = y^*"
\]

\[
p_1 \models "\text{The types of} (x', y') \) and \( (x'', y'') \) \) over \( \{x/F : x \in P, A^{\emptyset}_1 \)-name of a member of \( \prod_n A^{\emptyset}_0 (T^1_n, T^2_n)\}
\]

in the model \( (\prod_n A^{\emptyset}_0 (T^0_n, T^1_n)/\mathcal{F} q_i)^V[P|A^{\emptyset}_0] \) are equal."

Then the following are equivalent.

1. There is \( r^0 \in \text{App} \) such that \( q_1, q_2 \leq r^0 \), \( r^0|\delta \in G^\delta \), and

\[
p \models "\prod_n T^1_n/F^{r^0} \models (x'/F^{r^0} < x_\delta/F^{r^0}) \) and \( \prod_n T^2_n/F^{r^0} \models (y'/F^{r^0} < y^*/F^{r^0})."
\]

2. There is \( r^1 \in \text{App} \) such that \( q_1, q_2 \leq r^1 \), \( r^1|\delta \in G^\delta \) and

\[
p \models "\prod_n T^1_n/F^{r^1} \models (x''/F^{r^1} < x_\delta/F^{r^1}) \) and \( \prod_n T^2_n/F^{r^1} \models (y''/F^{r^1} < y^*/F^{r^1})."
\]

Proof:

It suffices to show that (1) implies (2). Take \( H^\delta \subseteq P|\delta \) generic over \( V \) with \( p_1 \in H^\delta \), and suppose that \( r^0 \) is as in (1). Let \( r_0 = r^0|\delta \) and let \( f_0 \) be the extension of the identity map on \( (\prod T^k_n)^V[P|A^{\emptyset}_0] \) by:

\[
f_0(x') = x'', \quad f_0(y') = y''.
\]

Writing \( \beta_0 = \delta \) and taking \( q_3 \) provided by 1.9 (4), we recover the assumptions of 1.13, which produces a certain \( r \) in \( \text{App} \), an end extension of \( r_0 \); here we may easily keep \( r|\delta \in G^\delta \) (cf. 1.12). It suffices to take \( r^1 = r \).

Definability.

We claim now that \( F \) is definable on a cone by a first order formula. For a stationary set \( S_0 \) of \( \delta < \aleph_3 \) of cofinality \( \aleph_2 \), we will have conditions (a-d) of 1.13 which may be expressed as follows:

Both \( F|(P|\delta - \text{names}) \) and \( F^{-1}|(P|\delta - \text{names}) \) are \( P|\delta \)-names;

When working with \( \diamond_S \):

\( \diamond_S \) guessed the names of these two restrictions and also guessed \( p^* \) correctly;

and hence for suitable \( y_\delta \) and \( q_\delta^* \) we have the corresponding conditions \( (\star)_y \) and \( (\ddagger)_\delta \) (with \( q_\delta^* \) in place of \( q^* \)). By Fodor’s lemma, on a stationary set \( S_1 \subseteq S_0 \) we have \( q_0 = q_\delta^*|\delta \) is constant, and also the isomorphism type of the pair \( (q_\delta^*, y_\delta) \) over \( A^{\emptyset}_0 \) is constant.

So for \( \delta \) in \( S_1 \), we have the following two properties, holding for \( x' \) in \( V[P|\delta] \) and \( y' = F(x') \), by \( (\ddagger)_\delta \) and 1.15 respectively:

1. The decision to put \( x' \) below \( x_\delta \) implies also that \( y' \) must be put below \( y_* \); and

2. This decision is determined by the type of \( (x', y') \) in \( \prod A^{\emptyset}_0 (T^1_n, T^2_n)/F V[H|P|\delta, H] \).
As $S_1$ is unbounded below $\aleph_3$ this holds generally.

This gives a definition by types of the isomorphism $F$ above the branch in $\prod T_n^1/\mathcal{F}^V[P;A^0]$ which the condition $q^*_{\delta}$ says that the vertex $\bar{x}_3$ induces there (using 1.9 (2)), and this branch does not depend on $\delta$. Note that this set contains a cone, and the image of this cone is a cone in the image. Now by $\aleph_2$-saturation of $\prod A^0_n(T_n^1, T_n^2)/\mathcal{F}^V[P;A]$ we get a first order definition on a smaller cone; this last step is written out in detail in the next paragraph. This proves Proposition B.

**Lemma (true definability)**

Let $M$ be a $\lambda$-saturated structure, and $A \subseteq M$ with $|A| < \lambda$. Let $(D_1; <_1)$, $(D_2; <_2)$ be $A$-definable trees in $M$; that is, the partial orderings $<_i$ are linear below each node. Assume that every node of $D_1$ or $D_2$ has at least two immediate successors. Let $F : D_1 \rightarrow D_2$ be a tree isomorphism which is type-definable in the following sense:

\[
[f(x) = y \& \text{tp}(x, y/A) = \text{tp}(x', y'/A)] \implies f(x') = y'.
\]

Then $f$ is $A$-definable, on some cone of $D_1$.

Before entering into the proof, we note that we use somewhat less information about $F$ (and its domain and range) than is actually assumed; and this would be useful in working out the most general form of results of this type (which will apply to some extent in any unsuperstable situation). We intend to develop this further elsewhere, as it would be too cumbersome for our present purpose.

The proof may be summarized as follows. If a function $F$ is definable by types in a somewhat saturated model, then on the locus of each 1-type, it agrees with the restriction of a definable function. If $F$ is an automorphism and the locus of some 1-type separates the points in a definable set $C$ in an appropriate sense, then $F$ can be recovered, definably, on $C$. Finally, in sufficiently saturated trees of the type under consideration, some 1-type separates the points of a cone. Details follow.

**Proof:**

If we replace $M$ by a $\lambda$-saturated elementary extension, the definition of $F$ by types continues to work (and the extension is an elementary extension for the expansion by $F$). In particular, replacing $|M|$ by a more saturated structure, if necessary, but keeping $A$ fixed, we may suppose that $\lambda > |T|, |A|, \aleph_0$.

We show first:

1. There is a 1-type $p$ defined over $A$ such that its set of realizations $p[D_1]$ is dense in a cone of $D_1$, i.e., for some $a$ in $D_1$ we require that any element above $a$ lies below a realization of $p$. For any 1-type $p$ over $A$, if $p[D_1]$ does not contain a cone of $D_1$ then by saturation there is some $\varphi \in p$ with:

\[
\forall a \exists b > a \neg \exists x > b \varphi(x)
\]

So if (1) fails we may choose one such formula $\varphi_p$ for each 1-type $p$ over $A$, and then it is consistent (hence true) that we have a wellordered increasing sequence $a_p$ (in the tree ordering) such that for each 1-type $p$,
above $a_p$ we have:

$$\neg \exists x > a_p \varphi_p(x)$$

By saturation there is a further element $a$ above all $a_p$ (either by increasing $\lambda$ or by paying attention to what we are actually doing) and we have arranged that there is no 1-type left for it to realize. As this is improbable, (1) holds. We fix a 1-type $p$ and an element $a_0$ in $D_1$ so that the realizations of $p$ are dense in the cone above $a_0$. It is important to note at this point that the density implies that any two distinct vertices above $a_0$ are separated by the realizations of $p$ in the sense that there is a realization of $p$ lying above one but not the other (here we use the immediate splitting condition we have assumed in the tree $D_1$).

Let $a$ realize the type $p$, and let $q$ be the type of $a, F(a)$ over $A$. If $b$ is any other realization of $p$, then there is an element $c$ with $b, c$ realizing $q$, and hence $F(b) = c$; thus $p$ determines $q$ uniquely. Furthermore each realization $a$ of $p$ determines a unique element $b$ such that $a, b$ realizes $q$, and hence by saturation there is a formula $\varphi(x, y) \in q$ so that $\varphi(x, y) \Rightarrow \exists z \varphi(x, z)$. Hence $p \cup \{\varphi\} \in q$.

Now the following holds in $M$:

$$p(x) \cup p(x') \cup \{\varphi(x, y), \varphi(x', y')\} \Rightarrow (x < x' \iff y < y')$$

and hence for some formula $\alpha(x) \in p$ the same holds with $p$ replaced by $\alpha$. We may suppose $\varphi(x, y) \Rightarrow \alpha(x)$ and conclude that $\varphi(x, y)$ defines a partial isomorphism $f$. Let $B$ be \{ $a > a_0 : \exists y \varphi(a, y)$\}. $f$ coincides with $F$ on the set of realizations of $p$ above $a$, and the action of $F$ on this set determines its action on the cone above $a$ by density (or really by the separation condition mentioned above), so $f$ coincides with $F$ on $B$. Furthermore the action of $F$ on $B$ determines its action on the cone above $a_0$ definably, so $F$ is definable above $a$.

The definition $\varphi^*(x, y)$ of $F$ on the cone above $a$ obtained in this manner may easily be written down explicitly:

$$\forall x', y' [\varphi(x', y') \Rightarrow (x < x' \iff y < y')]$$

For the application in 1.16 we take $\lambda = \aleph_2$.

Remark

Proposition

$P$ forces: In $\prod_n T_n^1/\mathcal{F}$ ($\mathcal{F} = \mathcal{F}(\aleph_3)$), every full branch is an ultraproduct of branches in the original trees $T_n^1$.

Proof (in brief):

One can follow the line of the previous argument, or derive the result from Proposition B. Following the line of the previous argument we argue as follows. If $B$ is a $P$-name for such a branch, then for a stationary
set of ordinals \( \delta < \aleph_2 \) of cofinality \( \aleph_3 \) of \( \tilde{B} \cap (\prod_n T^1_n/F)^{\mathcal{P} | \delta} \) will be a full branch and a \( \mathcal{P} | \delta \)-name, guessed correctly by \( \diamond_S \). We tried to make a commitment to terminate this branch, but failed, and hence for some \( q^* \) and \( y^* \) witnesses to the failure, we were unable to omit having \( q^* \delta \in G^\delta \) where \( q^* \) is essentially the support of \( \text{"y* is a bound"} \). Using 1.14 one shows that the branch was definable at this point by types in \( \aleph_1 \) parameters, and by \( \aleph_2 \)-compactness we get a first order definition, which by Fodor's lemma can be made independent of \( \delta \).

Filling in the details in the foregoing argument constitutes an excellent, morally uplifting exercise for the reader. However the more pragmatic reader may prefer the following derivation of the proposition from Proposition B.

In the first place, we may replace the trees \( T^1_n \) in the proposition above by the universal tree of this type, which we take to be \( T = Z^{<\omega} \) (writing \( Z \) rather than \( \omega \) for the sake of the notation used below). Now apply Proposition B to the pair of sequences \( (T^1_n), (T^2_n) \) in which \( T^i_n = T \) for all \( i, n \). Using the model of ZFC and the ultrafilter referred to in Proposition B, suppose \( B \) is a full branch of \( T^* = \prod T^2_n/F \), and let \( Z^* = Z^{<\omega}/F, \ N^* = N^{<\omega}/F \). For each \( i \in N^* \) let \( B_i \) be the \( i \)-th node of \( B \); this is a sequence in \( (Z^*)^{[0,i]} \) which is coded in \( N^* \). Define an automorphism \( f_B \) of \( T^* \) whose action on the \( i \)-th level is via addition of \( B_i \) (pointwise addition of sequences). Applying Proposition B and Loś’ theorem to this automorphism, we see that \( f_B \) is the ultraproduct of addition maps corresponding to various branches of \( T \), and that \( B \) is the ultraproduct of these branches.

Corollary

It is consistent with ZFC that \( R^{<\omega}/F \) is Scott-complete for some ultrafilter \( F \).

Here \( R^{<\omega}/F \) is called Scott-complete if it has no proper dedekind cut \( (A, B) \) in which \( \inf(b - a : a \in A, b \in B) = 0 \) in \( R^{<\omega}/F \). 1.18 is sufficient for this by [KeSc, Prop. 1.3]. This corollary answers Question 4.3 of [KeSc, p. 1024].

Remark

The predicate “at the same level” may be omitted from the language of the trees \( T^1_n \) throughout as the condition on \( Z^\delta \) that uses this (the “full branch” condition) follows from the “bigness” condition: meeting every suitable dense subset.

GARBAGE HEAP: From 1.9.

5. Assume \( \delta < \aleph_2 \), that \( (q_i)_{i<\delta} \) is an increasing sequence from \( App \), that \( (\beta_i)_{i<\delta} \) is a strictly increasing sequence of ordinals, and that \( (p_i)_{i<\delta} \) satisfies:

\[
\text{For } i < \delta: \ q_i | \beta \leq p_i \in App | \beta_i; \quad \text{For } i < j < \delta: \ p_i \leq_{\text{end}} p_j.
\]

Then there is an \( r \in App \) with \( p_i \leq_{\text{end}} r \) and \( q_i \leq r \) for all \( i < \delta \). If each \( q_i \) belongs to \( App_{\sup \beta_i} \), then \( r \) may be taken to have domain \( \bigcup_i (\text{dom } q_i \cup \text{dom } p_i) \).
5. We will prove by induction on $\gamma < \omega_2$ that if $p_i, q_i \in \text{App}|\gamma$ and for all $i$ we have $\beta_i \leq \gamma$, then the
claim holds (with $r$ in $\text{App}|\gamma$). If $\delta = \delta_0 + 1$ is a successor ordinal it suffices to apply (4) to $q_{\delta_0}$ and $p_{\delta_0}$,
with $\beta = \beta_{\delta_0}$. So we assume throughout that $\delta$ is a limit ordinal. In particular $\beta_i < \gamma$ for all $i$.

The case $\gamma = \gamma_0 + 1$, a successor.

In this case our induction hypothesis applies to the $q_i|\gamma_0$, the $p_i$, the $\beta_i$, and $\gamma_0$, yielding $r_0$ in
$\text{App}|\gamma_0$ with $p_i, q_i|\gamma_0 \leq r_0$ (and with a side condition on the domain if all $q_i|\gamma_0$ lie in $\text{App}|(\sup \beta_i)$). What
remains then is an amalgamation of $r_0$ with all of the $q_i$, where $\text{dom} q_i \subseteq \text{dom} r \cup \{\gamma_0\}$, and where one may
as well suppose that $\gamma_0$ is in $\text{dom} q_i$ for all $i$. This is a slight variation on 1.9 (2,3) (depending on the value
of $\epsilon_{\xi_i}$, which is independent of $i$).

The case $\gamma$ a limit of cofinality greater than $\aleph_1$.

Since $\delta < \aleph_2$ there is some $\gamma_0 < \gamma$ such that all $p_i, q_i \in \text{App}|\gamma_0$ and all $\beta_i < \gamma_0$, and the induction
hypothosis then yields the claim.

The case $\gamma$ a limit of cofinality $\aleph_1$.

If $\gamma = \sup \beta_i$ then $r = \bigcup p_i$ suffices. Assume therefore that $\gamma_0 := \sup \beta_i < \gamma$. By the induc-
tion hypothesis applied to $q_i|\beta_i$, $p_i$, and $\gamma_0$, we have $r_0 \in \text{App}|\gamma_0$ with $q_i|\gamma_0, p_i \leq r_0$ and $\text{dom} r_0 =
\bigcup_i (\text{dom} q_i|\gamma_0 \cup \text{dom} p_i)$.

Choose $\gamma_i^*$ a strictly increasing and continuous sequence of length $\omega_1$ with supremum $\gamma$, starting with
$\gamma_0^* = \gamma_0$. By induction choose $r_i \in \text{App}|\gamma_i^*$ for $i < \omega_1$ such that:

(1) $r_i \leq \text{end} \ r_j$ for $i < j < \omega_1$;

(2) $q_j|\gamma_i^* \leq r_i$ for $j < \delta$ and $i < \omega_1$.

Here for each $i$ the inductive hypothesis is applied to $q_j|\gamma_i^*$, $r_i$, and $\gamma_i$.

The case $\gamma$ a limit of cofinality $\aleph_0$.

End of Garbage Heap
Appendix

Omitting types

In §1 we made (implicit) use of the combinatorial principle developed in [ShHL162]. In the context of this paper, this is a combinatorial refinement of forcing with $App$, which gives (in the ground model) a $P_3$-name $\mathcal{F}$ for a filter with the required properties in a $P_3$-generic extension. We now review this material. Our discussion overlaps with the discussion in [Sh326], but will be more complete in some technical respects and less complete in others. We begin in sections A1-A5 by presenting the material of [Sh162] as it was summarized in [Sh326]. However the setup of [Sh162] can be (and should be) tailored more closely to the applications, and we will present a second setup which is more convenient in sections A6-A10. One could take the view that the axioms given in section A6 below should supercede the axioms given in section A1, and one should check that the proofs of [Sh162] work with these new axioms. Since this would be awkward in practice, we take a different route, showing that the two formalisms are equivalent.

After dealing with this technical point, we will not explain in any more detail the way this principle is applied, as that aspect is dealt with at great length in a very similar context in [Sh326]. For the reader who is not familiar with [Sh162] the discussion in the appendix to [Sh326] should be more useful than the present discussion.

Uniform partial orders

We review the formalism of [Sh162].

With the cardinal $\lambda$ fixed, a partially ordered set $(P, <)$ is said to be standard $\lambda^+$-uniform if $P \subseteq \lambda^+ \times P_\lambda(\lambda^+)$ (we refer here to subsets of $\lambda^+$ of size strictly less than $\lambda$), has the following properties (if $p = (\alpha, u)$ we write $\text{dom} p$ for $u$, and we write $P_\alpha$ for $\{p \in P : \text{dom} p \subseteq \alpha\}$):

1. If $p \leq q$ then $\text{dom} p \subseteq \text{dom} q$.
2. For all $p \in P$ and $\alpha < \lambda^+$ there exists a $q \in P$ with $q \leq p$ and $\text{dom} q = \text{dom} p \cap \alpha$; furthermore, there is a unique maximal such $q$, for which we write $q = p|\alpha$.
3. (Indiscernibility) If $p = (\alpha, v) \in P$ and $h : v \rightarrow v' \subseteq \lambda^+$ is an order-isomorphism onto $V'$ then $(\alpha, v') \in P$. We write $h[p] = (\alpha, h[v])$. Moreover, if $q \leq p$ then $h[q] \leq h[p]$.
4. (Amalgamation) For every $p, q \in P$ and $\alpha < \lambda^+$, if $p|\alpha \leq q$ and $\text{dom} p \cap \text{dom} q = \text{dom} p \cap \alpha$, then there exists $r \in P$ so that $p, q \leq r$.
5. For all $p, q, r \in P$ with $p, q \leq r$ there is $r' \in P$ so that $p, q \leq r'$ and $\text{dom} r' = \text{dom} p \cup \text{dom} q$.
6. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than $\lambda$, then it has a least upper bound $q$, with domain $\bigcup_{i<\delta} \text{dom} p_i$; we will write $q = \bigcup_{i<\delta} p_i$, or more succinctly: $q = p_{<\delta}$.
7. For limit ordinals $\delta$, $p|\delta = \bigcup_{\alpha<\delta} p|\alpha$.
8. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than $\lambda$, then $(\bigcup_{i<\delta} p_i)|\alpha = \bigcup_{i<\delta} (p_i|\alpha)$.

It is shown in [ShHL162] that under a diamond-like hypothesis, such partial orders admit reasonably generic objects. The precise formulation is given in A5 below.
Density systems

Let \( \mathcal{P} \) be a standard \( \lambda^+ \)-uniform partial order. For \( \alpha < \lambda^+ \), \( \mathcal{P}_\alpha \) denotes the restriction of \( \mathcal{P} \) to \( p \in \mathcal{P} \) with domain contained in \( \alpha \). A subset \( G \) of \( \mathcal{P}_\alpha \) is an admissible ideal (of \( \mathcal{P}_\alpha \)) if it is closed downward, is \( \lambda \)-directed (i.e. has upper bounds for all small subsets), and has no proper directed extension within \( \mathcal{P}_\alpha \).

For \( G \) an admissible ideal in \( \mathcal{P}_\alpha \), \( \mathcal{P}/G \) denotes the restriction of \( \mathcal{P} \) to \( \{ p \in \mathcal{P} : p|\alpha \in G \} \).

If \( G \) is an admissible ideal in \( \mathcal{P}_\alpha \) and \( \alpha < \beta < \lambda^+ \), then an \((\alpha, \beta)\)-density system for \( G \) is a function \( D \) from pairs \((u, v)\) in \( P_\lambda(\lambda^+) \) with \( u \subseteq v \) into subsets of \( \mathcal{P} \) with the following properties:

(i) \( D(u, v) \) is an upward-closed dense subset of \( \{ p \in \mathcal{P}/G : \text{dom } p \subseteq v \cup \beta \} \);
(ii) For pairs \((u_1, v_1), (u_2, v_2)\) in the domain of \( D \), if \( u_1 \cap \beta = u_2 \cap \beta \) and \( v_1 \cap \beta = v_2 \cap \beta \), and there is an order isomorphism from \( v_1 \) to \( v_2 \) carrying \( u_1 \) to \( u_2 \), then for any \( \gamma \) we have \((\gamma, v_1) \in D(u_1, v_1)\) iff \((\gamma, v_2) \in D(u_2, v_2)\).

An admissible ideal \( G' \) (of \( \mathcal{P}_\gamma \)) is said to meet the \((\alpha, \beta)\)-density system \( D \) for \( G \) if \( \gamma \geq \alpha \), \( G' \geq G \) and for each \( u \in P_\lambda(\gamma) \) there is \( v \in P_\lambda(\gamma) \) containing \( u \) such that \( G' \) meets \( D(u, v) \).

The genericity game

Given a standard \( \lambda^+ \)-uniform partial order \( \mathcal{P} \), the genericity game for \( \mathcal{P} \) is a game of length \( \lambda^+ \) played by Guelfs and Ghibellines, with Guelfs moving first. The Ghibellines build an increasing sequence of admissible ideals meeting density systems set by the Guelfs. Consider stage \( \alpha \). If \( \alpha \) is a successor, we write \( \alpha^- \) for the predecessor of \( \alpha \); if \( \alpha \) is a limit, we let \( \alpha^- = \alpha \). Now at stage \( \alpha \) for every \( \beta < \alpha \) an admissible ideal \( G_\beta \) in some \( \mathcal{P}_\beta \) is given, and one can check that there is a unique admissible ideal \( G_{\alpha^-} \) in \( \mathcal{P}_{\alpha^-} \) containing \( \bigcup_{\beta < \alpha} G_\beta \) (remember A 3.1(5)) [Lemma 1.3, ShHL 162]. The Guelfs now supply at most \( \lambda \) density systems \( D_\iota \) over \( G_{\alpha^-} \) for \( (\alpha, \beta_i) \) and also fix an element \( g_\alpha \) in \( \mathcal{P}/G_{\alpha^-} \). Let \( \alpha' \) be minimal such that \( g_\alpha \in \mathcal{P}_{\alpha'} \) and \( \alpha' \geq \sup \beta_i \). The Ghibellines then build an admissible ideal \( G_{\alpha''} \) for \( \mathcal{P}_{\alpha'} \) containing \( G_{\alpha^-} \) as well as \( g_\alpha \), and meeting all specified density systems, or forfeit the match; they let \( G_{\alpha''} = G_{\alpha'} \cap \alpha'' \) when \( \alpha \leq \alpha'' < \alpha' \). The main result is that the Ghibellines can win with a little combinatorial help in predicting their opponents’ plans.

For notational simplicity, we assume that \( G_\delta \) is an \( \aleph_2 \)-generic ideal on \( \text{App}[\delta] \), when \( \text{cof } \delta = \aleph_2 \), which is true on a club in any case.

\( D_{\lambda} \)

The combinatorial principle \( D_{\lambda} \) states that there are subsets \( Q_\alpha \) of the power set of \( \alpha \) for \( \alpha < \lambda \) such that \( |Q_\alpha| < \lambda \), and for any \( A \subseteq \lambda \) the set \( \{ \alpha : A \cap \alpha \in Q_\alpha \} \) is stationary. This follows from \( \diamondsuit_\lambda \) or inaccessibility, obviously, and Kunen showed that for successors, \( D \) and \( \diamondsuit \) are equivalent. In addition \( D_{\lambda} \) implies \( \lambda^{<\lambda} = \lambda \).

A general principle

Theorem

Assuming \( D_{\lambda} \), the Ghibellines can win any standard \( \lambda^+ \)-uniform \( \mathcal{P} \)-game.
This is Theorem 1.9 of [ShHL 162].

Uniform partial orders revisited

We introduce a second formalism that fits the setups encountered in practice more closely. In our second version we write “quasiuniform” rather than “uniform” throughout as the axioms have been weakened slightly.

With the cardinal \( \lambda \) fixed, a partially ordered set \( (\mathcal{P}, <) \) is said to be **standard \( \lambda^+ \)-quasiuniform** if \( \mathcal{P} \subseteq \lambda^+ \times P_\lambda(\lambda^+) \) has the following properties (if \( p = (\alpha, u) \) we write \( \text{dom} p \) for \( u \), and we write \( \mathcal{P}_\alpha \) for \( \{ p \in \mathcal{P} : \text{dom} p \subseteq \alpha \} \)):

1'. If \( p \leq q \) then \( \text{dom} p \subseteq \text{dom} q \).

2'. For all \( p, q \in \mathcal{P} \) and \( \alpha < \lambda^+ \) there exists a \( q \in \mathcal{P} \) with \( q \leq p \) and \( \text{dom} q = \text{dom} p \cap \alpha \); furthermore, there is a unique maximal such \( q \), for which we write \( q = p|\alpha \).

3'. (Indiscernibility) If \( p = (\alpha, v) \in \mathcal{P} \) and \( h: v \to v' \subseteq \lambda^+ \) is an order-isomorphism onto \( V' \) then \( (\alpha, v') \in \mathcal{P} \). We write \( h[p] = (\alpha, h[\epsilon]) \). Moreover, if \( q \leq p \) then \( h[q] \leq h[p] \).

4'. (Amalgamation) For every \( p, q \in \mathcal{P} \) and \( \alpha < \lambda^+ \), if \( p|\alpha \leq q \) and \( \text{dom} p \cap \text{dom} q = \text{dom} p \cap \alpha \), then there exists \( r \in \mathcal{P} \) so that \( p, q \leq r \).

5'. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \), then it has an upper bound \( q \).

6'. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \) of members of \( \mathcal{P}_{\beta+1} \), with \( \beta < \lambda^+ \) and if \( q \in \mathcal{P}_\beta \) satisfies \( p_i|\beta \leq q \) for all \( i < \delta \), then \( \{ p_i : i < \delta \} \cup \{ q \} \) has an upper bound in \( \mathcal{P} \).

7'. If \( (\beta_i)_{i<\delta} \) is a strictly increasing sequence of length less than \( \lambda \), with each \( \beta_i < \lambda^+ \), and \( \mathcal{P} \), \( p_i \in \mathcal{P}_{\beta_i} \), with \( q|\beta_i \leq p_i \), then \( \{ p_i : i < \delta \} \cup \{ q \} \) has an upper bound.

8'. Suppose \( \xi, \zeta \) are limit ordinals less than \( \lambda \), and \( (\beta_i)_{i<\zeta} \) is a strictly increasing continuous sequence of ordinals less than \( \lambda^+ \). Let \( I(\xi, \zeta) := (\zeta + 1) \times (\xi + 1) - \{(\xi, \zeta)\} \). Suppose that for \( (i, j) \in I(\xi, \zeta) \) we have \( p_{ij} \in \mathcal{P}_{\beta_i} \) such that

\[
 i \leq i' \implies p_{ij} = p_{ij}|\beta_i; \\
 j \leq j' \implies p_{ij} \leq p_{ij}'.
\]

Then \( \{ p_{ij} : (i, j) \in I(\xi, \zeta) \} \) has an upper bound in \( \mathcal{P} \).

Density systems revisited

Let \( \mathcal{P} \) be a standard \( \lambda^+ \)-quasiuniform partial order. A subset \( G \) of \( \mathcal{P}_\alpha \) is a **quasiadmissible ideal** (of \( \mathcal{P}_\alpha \)) if it is closed downward and is \( \lambda \)-directed (i.e. has upper bounds for all small subsets). For \( G \) a quasiadmissible ideal in \( \mathcal{P}_\alpha \), \( \mathcal{P}/G \) denotes the restriction of \( \mathcal{P} \) to \( \{ p \in \mathcal{P} : p|\alpha \in G \} \).

If \( G \) is a quasi-admissible ideal in \( \mathcal{P}_\alpha \) and \( \alpha < \beta < \lambda^+ \), then an \( (\alpha, \beta) \)-density system for \( G \) is a function \( D \) from sets \( u \) in \( P_\lambda(\lambda^+) \) into subsets of \( \mathcal{P} \) with the following properties:

(i) \( D(u) \) is an upward-closed dense subset of \( \mathcal{P}/G \);

(ii) For pairs \( (u_1, v_1) \) and \( (u_2, v_2) \) with \( u_1, u_2 \) in the domain of \( D \), and \( v_1, v_2 \in P_\lambda(\lambda^+) \) with \( u_1 \subseteq v_1 \)

\[
 u_2 \subseteq v_2, \text{ if } u_1 \cap \beta = u_2 \cap \beta \text{ and } v_1 \cap \beta = v_2 \cap \beta, \text{ and there is an order isomorphism from } v_1 \text{ to } v_2 \text{ carrying } u_1 \text{ to } u_2, \text{ then for any } \gamma \text{ we have } \gamma \in D(u_1) \text{ iff } \gamma \in D(u_2).
\]
For $\gamma \geq \alpha$, a quasiadmissible ideal $G'$ of $\mathcal{P}_\alpha$ is said to meet the $(\alpha, \beta)$-density system $D$ for $G$ if $G' \supseteq G$ and for each $u \in P_\alpha(\gamma)$ $G'$ meets $D(u, v)$.

The genericity game revisited

Given a standard $\lambda^+$-quasuniform partial order $\mathcal{P}$, the genericity game for $\mathcal{P}$ is a game of length $\lambda^+$ played by Guelfs and Ghibellines, with Guelfs moving first. The Ghibellines build an increasing sequence of admissible ideals meeting density systems set by the Guelfs. Consider stage $\alpha$. If $\alpha$ is a successor, we write $\alpha^-$ for the predecessor of $\alpha$; if $\alpha$ is a limit, we let $\alpha^- = \alpha$. Now at stage $\alpha$ for every $\beta < \alpha$ an admissible ideal $G_\beta$ in some $\mathcal{P}_{\beta'}$ is given. The Guelfs now supply at most $\lambda$ density systems $D_i$ over $G_{\alpha^\varepsilon}$ for $(\alpha, \beta_i)$ and also fix an element $g_\alpha$ in $\mathcal{P}/G_{\alpha^\varepsilon}$. Let $\alpha'$ be minimal such that $g_\alpha \in \mathcal{P}_{\alpha'}$ and $\alpha' \geq \sup \beta_i$. The Ghibellines then build an admissible ideal $G_{\alpha'}$ for $\mathcal{P}_{\alpha'}$ containing $\bigcup_{\beta < \alpha} G_\beta$ as well as $g_\alpha$, and meeting all specified density systems, or forfeit the match; they let $G_{\alpha''} = G_{\alpha'} \cap \alpha''$ when $\alpha \leq \alpha'' < \alpha'$. The main result is that the Ghibellines can win with a little combinatorial help in predicting their opponents’ plans.

**Theorem**

Assuming $Dl\lambda$, the Ghibellines can win any standard $\lambda^+$-uniform $\mathcal{P}$-game.

We will show this is equivalent to the version given in [ShHL162].

The translation

To match up the uniform and quasiuniform settings, we give a translation of the quasiuniform setting back into the uniform setting; there is then an accompanying translation of density systems and of the genericity game. So we assume that the standard $\lambda^+$-quasuniform partial order $\mathcal{P}$ is given and we will define an associated partial ordering $\mathcal{P}'$.

The set of elements of $\mathcal{P}'$ is the set of sequences $p = (p_{ij}, \beta_i)_{i < \zeta, j < \xi}$ such that:

(a) $\zeta, \xi < \lambda$; $\beta_i$ is strictly increasing;

(b) $p_{ij} = p_{i'i'}|\beta_i$, and $\beta_i \in \text{dom} p_{i'i'}$, for $i < i'$;

(c) $p_{ij} < p_{ij'}$ for $j < j'$;

(d) If $\alpha = \delta + \alpha' \in \text{dom} p_{ij}$ with $\alpha' < \lambda$ and $\delta$ is divisible by $\lambda$ and of cofinality less than $\lambda$, then $\delta \cap \text{dom} p_{ij}$ is unbounded in $\delta$.

For $p \in \mathcal{P}'$ let $\text{dom} p = \{ \delta + n : \exists i, j \text{ dom} p_{ij} \cap [(\delta + \varepsilon_\delta + n)\lambda, (\delta + \varepsilon_\delta + n + 1)\lambda) \neq \emptyset \}$, where $\delta$ is a limit ordinal or 0 and where $\varepsilon_\delta$ is 0 if $\text{cof} \delta = \lambda$, and is 1 otherwise. We can represent the elements of $\mathcal{P}'$ naturally by codes of the type used in §A1, so that the domain as defined here is the domain in the sense of this coding as well.

Now we define the order on $\mathcal{P}'$. For $p, q \in \mathcal{P}'$ we have the associated ordinals (such as $\zeta^p$), and the elements $p_{ij}, q_{ij}$ of $\mathcal{P}$. We say $p \leq q$ if one of the following occurs:

1. $p = q$;

2. $\zeta^p = \zeta^q$, $\beta_i^p = \beta_i^q$ for $i < \zeta^p$, and there is $j' < \xi^q$ such that $p_{ij} \leq q_{ij'}$ for all $i < \zeta^p$ and $j < \xi^p$. 


3. \( \xi^p = \xi^q \) and there is \( i' < \zeta^q \) such that \( p_{ij} \leq q_{ij} \) for all \( i < \zeta^p \) and \( j < \zeta^p \).

4. There are \( i', j' \) such that \( p_{ij} \leq q_{ij'} \) for all \( i < \zeta^p \) and \( j < \zeta^q \).

The first thing to be checked is that this is transitive. We will refer to relations of the type described in (2-4) above as vertical, horizontal, or planar respectively. The equality relation may be considered as being of all three types. With regard to transitivity, if \( p \leq q \leq r \), then if both of the inequalities involved are horizontal, or both are vertical, we have an inequality \( p \leq r \) of the same type; and otherwise we have a planar inequality \( p \leq r \).

We do not insist on asymmetry; if one wishes to have a partial order in the strict sense then it will be necessary to factor out an equivalence relation.

Properties (A1.1-4)

We claim that if \( P \) is a partial order with properties 1'-8' of §A6, then the associated partial ordering \( P' \) enjoys properties 1-8 of §A1. The first four properties were assumed for \( P \); we have to check that they are retained by \( P' \).

1. If \( p \leq q \) then \( \text{dom } p \subseteq \text{dom } q \).

\textit{Proof}:

If \( p \leq q \) then \( \bigcup \text{dom } p_{ij} \leq \bigcup \text{dom } q_{ij'} \) by (1) applied to \( P \) and hence (1) holds for \( P' \) by applying the definition of \( \text{dom } \) in \( P' \).

2. For all \( p \in P' \) and \( \alpha < \lambda^+ \) there exists a \( q \in P' \) with \( q \leq p \) and \( \text{dom } q = \text{dom } p \cap \alpha \); furthermore, there is a unique maximal such \( q \), for which we write \( q = p|\alpha \).

\textit{Proof}:

Let \( \alpha' = \alpha \cdot \lambda \), \( \zeta' = \{ i : \beta_i^p < \alpha' \} \), and \( p_{ij}' = p_{ij}|\alpha' \) for \( i < \zeta' \). Set \( p|\alpha = (p_{ij}', \beta_i)_{i < \zeta', j < \zeta_p} \).
3. (Indiscernibility) If \( p = (\alpha, v) \in P' \) and \( h : v \to v' \subseteq \lambda^+ \) is an order-isomorphism onto \( V' \) then \( (\alpha, v') \in P' \). We write \( h[p] = (\alpha, h[v]) \). Moreover, if \( q \leq p \) then \( h[q] \leq h[p] \).

4. (Amalgamation) For every \( p, q \in P' \) and \( \alpha < \lambda^+ \), if \( p|\alpha \leq q \) and \( \text{dom} p \cap \text{dom} q = \text{dom} p \cap \alpha \), then there exists \( r \in P' \) so that \( p, q \leq r \).

Property (A1.5)

We consider the fifth property:

5. For all \( p, q, r \in P' \) with \( p, q \leq r \) there is \( r' \in P' \) so that \( p, q \leq r' \) and \( \text{dom} r' = \text{dom} p \cup \text{dom} q \).

Properties (A1.6-8)

The last three properties are:

6. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \), then it has a least upper bound \( q \), with domain \( \bigcup_{i<\delta} \text{dom} p_i \); we will write \( q = \bigcup_{i<\delta} p_i \), or more succinctly: \( q = p_{<\delta} \).

7. For limit ordinals \( \delta \), \( p|\delta = \bigcup_{\alpha<\delta} p|\alpha \).

8. If \( (p_i)_{i<\delta} \) is an increasing sequence of length less than \( \lambda \), then \( (\bigcup_{i<\delta} p_i)|\alpha = \bigcup_{i<\delta} (p_i|\alpha) \).

Application

In our application we identify \( \text{App} \) with a standard \( \aleph_2 \)-uniform partial order via a certain coding. We first indicate a natural coding which is not quite the right one, then repair it.

First Try

An approximation \( q = (A, F, \mathcal{g}) \) will be identified with a pair \((\tau, u)\), where \( u = A \), and \( \tau \) is the image of \( q \) under the canonical order-preserving map \( h : A \leftrightarrow \text{otp}(A) \). One important point is that the first parameter \( \tau \) comes from a fixed set \( T \) of size \( \aleph_1 = \aleph_2 \); so if we enumerate \( T \) as \((\tau_\alpha)_{\alpha<\aleph_2}\) then we can code the pair \((\tau_\alpha, u)\) by the pair \((\alpha, u)\). Under these successive identifications, \( \text{App} \) becomes a standard \( \aleph_2 \)-uniform partial order, as defined in §A1. Properties 1, 2, 4, 5, and 6 are clear, as is 7, in view of the uniformity in the iterated forcing \( P \), and properties 3, 8 were, stated in 1.7 and 1.9 (4).

1. This part will change

The difficulty with this approach is that in this formalism, density systems cannot express nontrivial information: any generic ideal meets any density system, because for \( q \leq q' \) with \( \text{dom} q = \text{dom} q' \), we will have \( q = q' \); thus \( D(u, u) \) will consist of all \( q \) with \( \text{dom} q = u \), for any density system \( D \).

So to recode \( \text{App} \) in a way that allows nontrivial density systems to be defined, we proceed as follows.

Second Try

Let \( \iota : \aleph_2^+ \leftrightarrow \aleph_2^+ \times \aleph_2 \) be order preserving where \( \aleph_2^+ \times \aleph_2 \) is ordered lexicographically. Let \( \pi : \aleph_2^+ \times \aleph_2 \rightarrow \aleph_2^+ \) be the projection on the first coordinate. First encode \( q \) by \( \iota[q] = (\iota[A], \ldots) \), then encode \( \iota[q] \) by \((\tau, \pi[A])\), where \( \tau \) is defined much as in the first try – a description of the result of collapsing \( q \).
into $\text{otp} \pi[A] \times \aleph_2$, after which $\tau$ is encoded by an ordinal label below $\aleph_2$. The point of this is that now the domain of $q$ is the set $\pi[A]$, and $q$ has many extensions with the same domain. After this recoding, $\text{App}$ again becomes a $\aleph_2^+$-uniform partial ordering, as before. We will need some additional notation in connection with the indiscernibility condition. It will be convenient to view $\text{App}$ simultaneously from an encoded and a decoded point of view. One should now think of $q \in \text{App}$ as a quadruple $(u, A, \tilde{F}, \varepsilon \varepsilon \varepsilon)$ with $A \subseteq u \times \aleph_2$. If $h: u \leftrightarrow v$ is an order isomorphism, and $q$ is an approximation with domain $u$, we extend $h$ to a function $h_*$ defined on $A^q$ by letting it act as the identity on the second coordinate. Then $h[q]$ is the transform of $q$ using $h_*$, and has domain $v$.

For notational simplicity, we assume that $G_\delta$ is an $\aleph_2$-generic ideal on $\text{App}|\delta$, when $\text{cof} \delta = \aleph_2$ which is true on a club in any case.

2. Does this remark go anywhere?
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