Retraction

Hoop Conjecture, Minimal Length and Black Hole Formation in the Asymptotically Safe Scenario of Quantum Gravity

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S Basu acknowledges that this paper was submitted to the IC-MSQUARE 2012 proceedings without the knowledge of, or consultation with, the co-author and apologises accordingly.

In addition, this paper has been found to have substantial overlap with a previously published paper, [1], and has therefore been retracted.

Reference
[1] Basu S and Mattingly D 2010 Asymptotic safety, asymptotic darkness, and the hoop conjecture in the extreme UV Physical Review D 82 124017
Hoop Conjecture, Minimal Length and Black Hole Formation in the Asymptotically Safe Scenario of Quantum Gravity

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Abstract. Black hole dominance, assumed to be a fairly ubiquitous feature of any theory of quantum gravity, amounts to that any observer trying to perform a localized experiment on ever smaller length scales will ultimately be thwarted by the formation of a trapped surface within the spatial domain of the experiment. The argument based on Thorne’s hoop conjecture, conjointly leads to a fundamental length scale in physics. Black hole dominance also suggests that ordinary field theory cannot be used to describe quantum gravity in the extreme UV, contrary to implications of asymptotic safety. We re-examine black hole dominance in an asymptotically safe scenario, in the presence of higher curvature terms with running couplings, by modifying a proof of Thorne’s hoop conjecture. We find that the proof falls apart, and along with it, so does the argument for the obligatory formation of a trapped surface inside the domain of the experiment. However, neither do we find a contrary proof that local trapped surfaces do not form. Instead in this approach whether an observer can perform local measurements in arbitrary small regions of spacetime depends on the specific values of the couplings near the UV fixed point. In this sense there is no all pervasive local version of the minimal length argument. However, we argue that one trapped surface must still form outside an experiment, when the domain of this experiment is localized to scales much smaller than the Planck length. This enshrouding horizon then prevents any information from reaching observers at infinity, thus retaining a vestige of “asymptotic darkness” for them.

1. Introduction

Black hole dominance (or asymptotic darkness), of the spectrum of particle states near the Planck scale, often invoked to argue against the non-renormalizability of gravity. A renormalizable field theory at a certain scale is a perturbation of a CFT by relevant operators with finite couplings. But in \(d\)-dimensions, black hole thermodynamics would yield an energy scaling \(\rho_{\text{BH}} = \frac{e^{E/d}}{d-2}\) in black hole states, while the density of states of a CFT scales as \(\rho_{\text{CFT}} = e^{E/d-1}\) [1]. Strictly speaking proving the conjecture requires a quantum theory of gravity. However, the argument rests on Thorne’s hoop conjecture (HC) [2], and quantum mechanics: If an amount of energy or matter \(E_s\) compressed into a region of characteristic size \(L\), a trapped surface of size \(L\) forms if \(L < R_S = 2GE\). If \(L\) scales according to \(L \propto 1/E\), then a trapped surface forms for any experiment at energies \(E > E_{\text{Planck}}\). The HC is unproven. However it finds

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strong support from numerical work on particle scattering [3] which vindicates the Aichelburg-Sexl result [4], and other work on Planck scale scattering [5, 6].

Asymptotic darkness also implies that the Planck length is the minimum observable length [7] for any observer. Consider simple General Relativity and a static, spherically symmetric experiment. The solution at radii $r > L$ is Schwarzschild, which has trapped surfaces at every radius from $L$ out to $2GL^{-1}$. Therefore there is no observer at radius $L$ to infinity that could receive any signal from the experiment once it is scaled down past $E_{\text{Planck}}$, which becomes a fundamental measurement cutoff for any observer. Thus, using HC, one can argue for black hole dominance, and the appearance of a minimal length. We query, first, if this continues to be true in asymptotic safety scenarios of quantum gravity, where gravity is supposed to be a field theory based upon the metric all the way to the extreme UV. Skeptics object that the hoop conjecture is an artefact of IR gravity, and hence black holes may not form at all in the UV. But notice that if one ramps the energy scale of an experiment, so that $E \gg E_{\text{Planck}}$, a large black hole will eventually form; large black holes have low horizon curvature scales, and therefore IR gravity should be a perfectly valid description. Hence, momentarily postponing the debate of what happens in the far UV, within asymptotic safety, compatibility with GR, must be restored in the low energy limit, for a similar experiment. The UV/IR connection implied by the formation of large black holes, has long been discussed as a distinctive feature of high scale quantum gravity [8, 9], and recently led to ideas of UV “classicalization” of non-renormalizable theories [10].

We now revisit and parallel the usual minimum length arguments by incorporating one feature, that of running couplings implied by asymptotic safety, into a proof due to Bizon et al [11] for a minimum length under certain conditions. We frequently refer to the authors’ research article [12] for certain details we have omitted here.

2. Hoop Theorem for Spherical Symmetry

Black holes can form in myriad ways, so black hole dominance is a rather vague statement. The hoop conjecture, implies that a black hole will form, regardless of the details, if we concentrate enough matter into a small volume of space. So to be concrete, we choose a specific behavior for gravity, spacetime topology, and matter distribution as follows. 1. We assume gravity is asymptotically safe, 2. the spacetime topology allows a foliation by spacelike surfaces, $\Sigma_t$, labeled by a time function $t$, and outward normal $n^a$. 3. A spherical ball of matter $\Omega$, of proper radius $L$ and density $\rho$ on a maximal slice $\Sigma_0$ with boundary $\partial \Omega$ and spacelike unit normal $r^a (r \cdot n = 0)$. 4. The system is momentarily stationary. The reason for these choices are the following. First the event horizon is a global object, and its sharp location in quantum gravity is prima facie suspect. This motivates reformulating black hole thermodynamics, and the hoop conjecture in terms of local trapping/apparent horizons [11, 13, 14, 15, 16]. Spherical symmetry and stationarity are used to simplify the problem, but note that the momentarily stationary assumption intuitively matches the idea of scaling a static system down in some frame.

The proof uses the Hamiltonian constraint of General Relativity, which for our purposes is written as

$$H_\kappa - K^2 - K_{ab}K^{ab} = \kappa_0^{-1} \left( H_\kappa + \frac{1}{2} \rho \right); \quad \rho = T_{m}^{ab}n_{a}n_{b}; \quad \kappa_0^{-1} = 16\pi G. \quad (1)$$

Here $H_\kappa$ denotes the contribution from the higher curvature terms, which appear in a general truncation of the 4d effective action for gravity a la asymptotic safety [20, 21, 22]

$$I = \int d^4x \sqrt{-g} \left( -\Lambda(\mu) + \kappa_0(\mu)R + \kappa_1(\mu)R^2 + \kappa_2(\mu)R_{ab}R^{ab} + L_m \ldots \right). \quad (2)$$
The coupling constants, \( \kappa_I \) (\( I = 1, 2, 3, \ldots \)) of the higher derivative terms in eq. 2 are implicitly absorbed in \( H_\kappa \) above. One could question the use of the classical Hamiltonian constraint in quantum gravity, but we underscore that we are asking how asymptotic safety modifies the standard minimum length argument, which is based on just such a semiclassical picture, and as a first attempt, we apply the predictions of Asymptotic Safety to an otherwise mathematically precise argument.

To proceed we choose a standard gauge adapted to spherical symmetry for the spatial metric on \( \Sigma_0 \), namely \( q_{ab} = \varphi^4 \delta_{ab} \). This in turn implies that the 3-Ricci scalar is \( -8\varphi^{-5}\nabla^2 \varphi \). Further for a slice where everything is momentarily stationary, the extrinsic curvature \( K_{ab} = 0 \); these restrictions on the canonical data \((q_{ab}, K_{ab})\) simplifies the constraint equation 1 to

\[
-8\varphi^{-5}\nabla^2 \varphi = \kappa_0^{-1} \left( H_\kappa + \frac{1}{3} \rho \right)
\]

This implies that \( \nabla^2 \varphi \leq 0 \) (or \( \frac{d\varphi}{dr} \leq 0 \)) if the higher curvature satisfy the positive energy condition as well as the matter contribution [17, 18, 19]. Now, any 2-sphere inside \( \Omega \) is not a trapped surface if the expansion \( \theta_r = r^a_{\alpha} > 0 \). with \( r^a_{\alpha} = (r^2 \varphi^6 - \varphi^2) \), this means

\[
\theta_r = \partial_r (r \varphi^2) > 0 ,
\]

Integrating eq. 3 over \( \Omega \) and simplifying for spherical symmetry obtains

\[
-32\pi \int_0^{r_\Omega} dr \varphi \partial_r (r \varphi^2) = \kappa_0^{-1} \int_\Omega d\nu H_\kappa .
\]

where \( E = \int q dx^a dx^b T_{ab} n_a n_b \) is the proper energy of matter inside \( \Omega \). A partial integration of the term on the left hand side then yields

\[
16\pi L - 16\pi r \partial_r (r \varphi^2) \bigg|_0^{r_\Omega} + 16\pi \int_0^{r_\Omega} dr \varphi \partial_r (r \varphi^2) = \kappa_0^{-1} \left( \frac{E}{2} + \int_\Omega d\nu H_\kappa \right) .
\]

where \( L = \int_0^{r_\Omega} \varphi^2 dr \) is the proper radius of \( \Omega \). If there are no trapped surfaces in \( \Omega \) and \( \partial \Omega \) is not a trapped surface, then using \( \frac{d\varphi}{dr} \leq 0 \) \( \theta_r > 0 \), as above, all terms except the first on the left side of eq. 6 are negative. Therefore we have

\[
E \approx \frac{3}{2} \kappa_0^{-1} \left( \frac{E}{2} + \int_\Omega d\nu H_\kappa \right) .
\]

as the condition for no trapped surfaces in \( \Omega \).

For the case of pure General Relativity, i.e with \( H_\kappa = 0 \) and no running couplings, eq. 7 implies that as we increase \( E \) and therefore decrease \( L \approx \frac{E}{3} \), a trapped surface must eventually form in \( \Omega \). The left hand side decreases and the right side increases, thereby violating the inequality at some point. This happens when \( E = 32\pi \kappa_0 L \approx 32\pi \kappa_0 E^{-1} \), i.e at \( E \approx \sqrt{ \frac{3}{2} \kappa_0 } \).

2.1. Implications of Asymptotic Safety
The reader is referred to Sec. IIIA of Ref.[12] for details on the behavior of running coupling constants. We will here summarize the essential point. In the far UV the behavior of the running Newton constant is dominated by the Non-Gaussian fixed point \( g_* \) and given by [23]

\[
\frac{1}{16\pi \kappa_0 (E)} = G(E) \approx \frac{g_*}{E^2}
\]
Momentarily confining ourselves to the Einstein Hilbert+matter truncation, we find from eq. 7 that there are no trapped surfaces in Ω if

\[ L > \frac{g_\ast}{2E^2} E \]  

Using the scaling \( L \sim \frac{1}{E} \), factors of \( E \) drop out from both sides, and gives

\[ g_\ast < 2 \]  

for no trapped surfaces to be found in \( \Omega \). This shows that the small numbers arising from order of magnitude estimates/scaling arguments etc, there is some value of coefficients for which the standard minimum length arguments fail. Interestingly, in many truncations have been found to be close to 2. We quote here the values obtained by Benedetti et al [24], who consider truncations with minimally coupled scalar field and find

\[ g_\ast = 0.860 \quad \text{(For Einstein-Hilbert truncation)} \]  

\[ g_\ast = 2.279 \quad \text{(For } R^2 + C^2 \text{ truncation)} \]  

Consider now the impact of higher curvature terms with running couplings on our argument. Returning to eq. (7) we find that no trapped surfaces form inside \( \Omega \) if

\[ L > \frac{g_\ast}{E^2} \left( \frac{E}{\sqrt{2}} + \int \kappa dv \right) \]  

With \( L \) replaced by \( E^{-1} \) eq. (13) becomes

\[ E^{-1} > \frac{g_\ast}{E^2} \left( 1 + \int \tilde{k}_I dv \right) \]  

We now observe that the integral involving the higher curvature terms is dimensionally an energy and so the integral must be a combination of \( E \) multiplied by the dimensionless coefficients \( \tilde{k}_I \). We therefore have,

\[ E^{-1} > g_\ast \left( 1 + \int E^{-1} \tilde{k}_I dv \right) \]  

where \( \int \tilde{k}_I dv \) is a dimensionless number whose details depend on both the unknown \( \kappa_I \) and the precise solution for the metric in the presence of the higher curvature terms with our given stress tensor. It suffices for use at it is a linear function of \( \tilde{k}_I \). In asymptotic safety the \( \tilde{k}_I \) coefficients remain finite as \( E \to \infty \) and hence \( E^{-1} \) is also finite. Therefore as \( E \to \infty \) there is no proof that a trapped surface is formed for each side scales the same way with \( E \), and so the argument for a minimum length no longer necessarily holds (at least in this formulation). Whether a trapped surface can still form is sensitive to truncation details, coefficients and the actual value of \( g_\ast \).

3. The Infrared Limit and General Relativity

While the question of whether or not a trapped surface forms inside \( \Omega \) for Planck sized experiments becomes muddled with asymptotic safety, in the limit of energy \( E \gg E_P \) there still must be a trapped surface at large radius, as this corresponds to a large black hole with low horizon curvature. To reach this spacetime picture, we can analyze the “horizon” structure which emerges by considering a background distribution of matter with total energy \( E_B \) in \( \Omega \), and ask when a probe experiment at a length scale \( L_E \) (or equivalent energy scale \( E_E \)) encounters a trapped surface. Since we are away from the fixed point dominated regime we now use a specific form of \( G(E) \) [25, 26],

\[ G(E) = \frac{g_\ast}{E^2 + g_\ast M^2_P} \]  

\[ \text{(16)} \]
return to equation (7), but modify the scenario. We define

\[ F(E_E) = E_E - G(E_E) \left( \frac{E_E + E_B}{2} + \int dvH_\kappa \right) \]  

(17)

. If \( F(E) > 0 \), then there is no trapped surface in the region. Assume that \( g_s \) is such that there are no trapped surface in \( \Omega \) as \( E_E \to \infty \). Given this, as we take \( E_E \) towards the IR eventually \( F(E_E) \) may cross zerosignifying that at some radius \( L_E \), there is a trapped surface. As we move towards even lower energies, \( F \) may cross zero again. Hence there may be in general multiple horizons. We can locate these horizons by setting \( F(E) = 0 \). A straightforward algebra \[12\], then gives, in the case of \( E_B \gg E_{pl} \), two roots,

\[ L_{E+} = \frac{E_B}{M_{pl}} \]  

(18)

\[ L_{E-} = \frac{1 - g_s E_B J^2}{g_s E_B J^2} \]  

(19)

where \( J_E = 1/2 + I_E(\kappa_i) \) and \( J_B = 1/2 + I_B(\kappa_i) \) encapsulate the effect of the higher curvature terms. \( L_{E+} \) is \( R \approx G E_B \), i.e. it corresponds to the Schwarzschild radius for a large black hole. To get an intuitive picture, set the higher curvature terms to zero. In that case \( J_E = J_B = 1/2 \), the inner horizon location becomes \( L_{E-} = (2 - g_s)/(g_s E_B) \). In the previous section, for just one experiment, our limit was roughly \( g_s \), so we must further decrease \( g_s \) to avoid the minimum length argument. Here we have the additional background energy \( E_B \) in a region \( \Omega \) to \( E_E \), so we must further decrease \( g_s \) to avoid a trapped surface forming in the domain of experiment. In other words, if we chose \( E_E = E_B \) for example, then there is twice the energy density as we had in the previous section in a region of size \( L_E = L_B \) so \( g_s \) would now need to be less than one to prevent formation of a trapped surface. If we consider \( g_s \ll 1 \), such as this bound is satisfied, then we see that the inner horizon is reached when \( L_{E-} \approx (E_B/g_s E_B) \). Thus the following picture emerges: We have a background energy density \( E_B \gg E_{pl} \) in a region of radius \( L_B = 1/E_B \). We probe the system at length scales smaller than \( L_B \) and we see no trapped surface. As we approach the length scale \( (2 - g_s)/g_s \) we see a trapped surface form. This behavior persists until we reach the large radius \( G E_B \), at which point we return to see classical physics outside an ordinary Schwarzschild horizon. Thus in this picture local observers performing local measurements see no trapped surfaces, but they cannot transmit any measurements to asymptotic observers, for whom, there is still a minimum length.

In sum therefore, our findings indicate that within asymptotic safety, running of couplings conspire such that the absence of forming a trapped surface at the scale of a high energy experiment occurs. While trapped surfaces form is a matter of delicate details about truncation and values of fixed points. Hence, it is of general interest to explore such issues with realistic matter content. Significantly compatibility with General Relativity in the IR can be restored. The tension between local measurements and asymptotic information which emerges in this discussion warrants caution about blanket statements regarding the status of minimum length within Asymptotic Safety.

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