On sums of arithmetic functions involving the greatest common divisor

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Abstract

Let gcd($d_1, \ldots, d_k$) be the greatest common divisor of the positive integers $d_1, \ldots, d_k$, for any integer $k \geq 2$, and let $\tau$ and $\mu$ denote the divisor function and the Möbius function, respectively. For an arbitrary arithmetic function $g$ and for any real number $x > 5$ and any integer $k \geq 3$, we define the sum

$$S_{g,k}(x) := \sum_{n \leq x} \sum_{d_1 \cdots d_k = n} g(\text{gcd}(d_1, \ldots, d_k))$$

In this paper, we give asymptotic formulas for $S_{\tau,k}(x)$ and $S_{\mu,k}(x)$ for $k \geq 3$.

1 Introduction and main results

Let gcd($d_1, \ldots, d_k$) be the greatest common divisor of the integers $d_1, \ldots, d_k$ for any integer $k \geq 2$, and let $\mu$ and $\tau$ denote the Möbius function and the divisor function, respectively. We recall that the Dirichlet convolution of two arithmetic functions $f$ and $g$ is defined by $f \ast g(n) = \sum_{d|n} f(d)g(n/d)$ for all positive integers $n$. The arithmetic functions $1(n)$ and id$(n)$ are defined by $1(n) = 1$ and id$(n) = n$ respectively. The function $\tau_k$ is the $k$-factors Piltz divisor function given by $1 \ast 1 \ast \cdots \ast 1$. In case of $k = 2$, we write $\tau_2 = \tau$. For an arbitrary arithmetic function $g$, we define the sum

$$f_{(g,k)}(n) := \sum_{d_1 \cdots d_k = n} g(\text{gcd}(d_1, \ldots, d_k)).$$

In [4], Krätzel, Nowak and Tóth gave asymptotic formulas for a class of arithmetic functions, which describe the value distribution of the greatest common divisor function. Typically, they are generated by a Dirichlet series whose analytic behavior is determined by the factor $\zeta^2(s)\zeta(2s - 1)$, where $\zeta(s)$ is the Riemann zeta-function. In regards to the above formula, they proved that

$$f_{(g,k)}(n) = \sum_{a^k b = n} \mu * g(a)\tau_k(b).$$

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This identity was used to establish asymptotic formulas for $f_{(g,k)}(n)$ for specific choices of $g$ such as the identity function $id$ and the sum of divisors function $\sigma = id * 1$. It was shown that

$$f_{(id,k)}(n) = \sum_{a^k b = n} \phi(a) \tau_k(b), \quad f_{(\sigma,k)}(n) = \sum_{a^k b = n} a \tau_k(b),$$

and that, for $\Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{f_{(id,k)}(n)}{n^s} = \frac{\zeta^k(s) \zeta(k s - 1)}{\zeta(k s)}, \quad \sum_{n=1}^{\infty} \frac{f_{(\sigma,k)}(n)}{n^s} = \zeta^k(s) \zeta(k s - 1).$$

In case of $g = \delta := \mu * 1$ (i.e., $\delta(1) = 1$ or $\delta(n) = 0$ else), the function

$$f_{(\delta,k)}(n) = \sum_{d_1 \cdots d_k = n \atop \gcd(d_1 \cdots d_k) = 1} 1$$

has been considered in [2].

Now, we define the sum

$$S_{g,k}(x) := \sum_{n \leq x} f_{(g,k)}(n) = \sum_{a^k b \leq x} \mu * g(a) \tau_k(b)$$

In two cases $g = \tau$ and $g = \mu$, it is easy to see that

$$S_{\tau,k}(x) = \sum_{m \leq x^{1/k}} \sum_{\ell \leq x/m^k} \tau_k(\ell), \quad (1)$$

and that

$$S_{\mu,k}(x) = \sum_{m \leq x^{1/k}} \mu * \mu(m) \sum_{n \leq x/m^k} \tau_k(n). \quad (2)$$

In this paper, we give asymptotic formulas for $S_{\tau,k}(x)$ and $S_{\mu,k}(x)$ for any integer $k \geq 3$. We have the following results.

**Theorem 1.** For any real number $x > 5$, we have

$$S_{\tau,3}(x) = \frac{\zeta(3)}{2} x \log^2 x + \zeta(3) \left( 3\gamma - 1 + 3 \frac{\zeta'(3)}{\zeta(3)} \right) x \log x \quad (3)$$

$$+ \zeta(3) \left( 3\gamma^2 - 3\gamma + 3\gamma_1 + 1 + 3(3\gamma - 1) \frac{\zeta'(3)}{\zeta(3)} + \frac{9}{2} \frac{\zeta''(3)}{\zeta(3)} \right) x + O \left( x^{\frac{43}{96} + \varepsilon} \right),$$

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and

\[ S_{r,4}(x) = \frac{\zeta(4)}{6} x \log^3 x + \zeta(4) \left( 2\gamma - \frac{1}{2} + 2 \frac{\zeta'(4)}{\zeta(4)} \right) x \log^2 x \]
\[ + \zeta(4) \left( 6\gamma^2 - 4\gamma + 4\gamma_1 + 1 + 8 \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta'(4)}{\zeta(4)} + 8 \frac{\zeta''(4)}{\zeta(4)} \right) x \log x \]
\[ + \zeta(4) \left( 12\gamma\gamma_1 + 4\gamma^3 - 6\gamma^2 + 4(\gamma - \gamma_1 + \gamma_2) - 1 + 4(6\gamma^2 - 4\gamma + 4\gamma_1 + 1) \frac{\zeta'(4)}{\zeta(4)} \right) x \]
\[ + 16\zeta(4) \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta''(4)}{\zeta(4)} + 2 \frac{\zeta(3)(4)}{3} \zeta(4) \right) x + O \left( x^{\frac{3}{2}+\epsilon} \right) , \]

where \( \gamma \) is the Euler constant, \( \gamma_1 \) and \( \gamma_2 \) are the Laurent-Stieltjes constants, (see Section 2 below for details). Here the function \( \zeta^{(k)}(s) \) is the \( k \)-th derivative of the Riemann zeta-function \( \zeta(s) \) with respect to \( s \).

**Theorem 2.** Under the hypotheses of Theorem 1, we have

\[ S_{\mu,3}(x) = \frac{1}{2\zeta(3)} x \log^2 x + \frac{1}{\zeta(3)} \left( 3\gamma - 1 - 6 \frac{\zeta'(3)}{\zeta(3)} \right) x \log x \]
\[ + \frac{1}{\zeta(3)} \left( 3\gamma^2 - 3\gamma + 3\gamma_1 + 1 - 6(3\gamma - 1) \frac{\zeta'(3)}{\zeta(3)} - 9 \frac{\zeta''(3)}{\zeta(3)} + 27 \left( \frac{\zeta'(3)}{\zeta(3)} \right)^2 \right) x \]
\[ + O \left( x^{\frac{3}{2}+\epsilon} \right) , \]

and

\[ S_{\mu,4}(x) = \frac{1}{6\zeta^2(4)} x \log^3 x + \frac{1}{\zeta^2(4)} \left( 2\gamma - \frac{1}{2} - 4 \frac{\zeta'(4)}{\zeta(4)} \right) x \log^2 x \]
\[ + \frac{16}{\zeta^2(4)} \left( 6\gamma^2 - 4\gamma + 4\gamma_1 + 1 \right) - \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta''(4)}{\zeta(4)} + 3 \left( \frac{\zeta'(4)}{\zeta(4)} \right)^2 \right) x \log x \]
\[ + \frac{1}{\zeta^2(4)} \left( 12\gamma\gamma_1 + 4\gamma^3 - 6\gamma^2 + 4(\gamma - \gamma_1 + \gamma_2) - 1 + 8(6\gamma^2 - 4\gamma + 4\gamma_1 + 1) \frac{\zeta'(4)}{\zeta(4)} \right) x \]
\[ + \frac{32}{\zeta^2(4)} \left( 2\gamma - \frac{1}{2} \right) \left( 3 \left( \frac{\zeta'(4)}{\zeta(4)} \right)^2 - \frac{\zeta''(4)}{\zeta(4)} \right) x \]
\[ - \frac{64}{3\zeta^2(4)} \left( 12 \left( \frac{\zeta'(4)}{\zeta(4)} \right)^3 - 9 \frac{\zeta'(4)}{\zeta(4)} \cdot \frac{\zeta''(4)}{\zeta(4)} + \frac{\zeta(3)(4)}{\zeta(4)} \right) x + O \left( x^{\frac{3}{2}+\epsilon} \right) . \]

The following theorem states asymptotic formulas of \( S_{r,k}(x) \) and \( S_{\mu,k}(x) \) for any integer \( k \geq 5 \).

**Theorem 3.** Let \( P_{g,k}(u) \) be a polynomial in \( u \) of degree \( k-1 \) depending on \( g \). For \( g = \tau \), \( g = \mu \), and \( k \geq 5 \), we have

\[ S_{g,k}(x) = x P_{g,k}(\log x) + E_{g,k}(x) \]
where
\[
E_{g,k}(x) = O\left(x^{\frac{4k-4}{4k} + \varepsilon}\right) \quad (5 \leq k \leq 8),
\]
\[
E_{g,9}(x) = O\left(x^{\frac{35}{34} + \varepsilon}\right),
\]
\[
E_{g,10}(x) = O\left(x^{\frac{11}{10} + \varepsilon}\right),
\]
\[
E_{g,11}(x) = O\left(x^{\frac{7}{16} + \varepsilon}\right),
\]
\[
E_{g,k}(x) = O\left(x^{\frac{k-2}{k+2} + \varepsilon}\right) \quad (12 \leq k \leq 25),
\]
\[
E_{g,k}(x) = O\left(x^{\frac{k+1}{k+1} + \varepsilon}\right) \quad (26 \leq k \leq 50),
\]
\[
E_{g,k}(x) = O\left(x^{\frac{31k-98}{32k} + \varepsilon}\right) \quad (51 \leq k \leq 57),
\]
\[
E_{g,k}(x) = O\left(x^{\frac{2k-34}{3k} + \varepsilon}\right) \quad (k \geq 58),
\]
for any small number \(\varepsilon > 0\).

2 Auxiliary results

Before going into the proof of Theorems, we recall that the Laurent expansion of the Riemann zeta-function at its pole \(s = 1\) is given by
\[
\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \gamma_k (s-1)^k.
\]
Here the constants \(\gamma_k\) are often called the Laurent-Stieltjes constants and it is known that
\[
\gamma_n = \frac{(-1)^n}{n!} \lim_{M \to \infty} \left(\sum_{m=1}^{M} \frac{(\log m)^n}{m} - \frac{(\log M)^{n+1}}{(n+1)}\right),
\]
for all \(n \geq 0\) with \(\gamma_0 = \gamma = 0, 577 \cdots\) being the Euler-Mascheroni constant.

We define the error term \(\Delta_k(x)\) of the Piltz divisor problem by
\[
\Delta_k(x) := \sum_{n \leq x} \tau_k(n) - x P_{k-1}(\log x), \quad (8)
\]
where \(P_{k-1}(t)\) is a polynomial of degree \(k - 1\) in \(t\). Notice that the coefficients of \(P_{k-1}\) may be evaluated by using
\[
P_{k-1}(\log x) = \text{Res}_{s=1} \zeta^k(s) \frac{x^{s-1}}{s}. \quad (9)
\]
From Eqs. (8) and (9), one can calculate explicitly the coefficients of \(P_{k-1}\) as functions of the Laurent-Stieltjes constants. For more details, see [2 Chapter 13].

In order to prove our main results, it will be necessary to give some lemmas.
Lemma 1. We have
\[
\sum_{n \leq x} \tau_3(n) = (b_1 \log^2 x + b_2 \log x + b_3) x + \Delta_3(x),
\]
where \( \Delta_3(x) \ll x^{43/96 + \varepsilon} \), and
\[
b_1 = \frac{1}{2}, \quad b_2 = 3\gamma - 1, \quad b_3 = 3\gamma^2 - 3\gamma + 3\gamma_1 + 1.
\]
Furthermore, we have
\[
\sum_{n \leq x} \tau_4(n) = (c_1 \log^3 x + c_2 \log^2 x + c_3 \log x + c_4) x + \Delta_4(x)
\]
where \( \Delta_4(x) \ll x^{1/2 + \varepsilon} \), and
\[
c_1 = \frac{1}{6}, \quad c_2 = 2\gamma - \frac{1}{2}, \quad c_3 = 6\gamma^2 - 4\gamma + 4\gamma_1 + 1,
\]
\[
c_4 = 12\gamma_1 + 4\gamma^3 - 6\gamma^2 + 4(\gamma - \gamma_1 + \gamma_2) - 1.
\]
Proof. The proof of Eqs. (10) and (11) can be found in [3] and [2, Theorems 13.2], respectively.

Lemma 2. Let \( \alpha_k \) be the infimum of numbers \( a_k \) such that \( \Delta_k(x) \ll (x^{a_k + \varepsilon}) \) for any small \( \varepsilon > 0 \). Then
\[
\alpha_k \leq \frac{3k - 4}{4k} \quad (5 \leq k \leq 8),
\]
\[
\alpha_9 \leq \frac{35}{54},
\]
\[
\alpha_{10} \leq \frac{41}{60},
\]
\[
\alpha_{11} \leq \frac{7}{10},
\]
\[
\alpha_k \leq \frac{k - 2}{k + 2} \quad (12 \leq k \leq 25),
\]
\[
\alpha_k \leq \frac{k - 1}{k + 4} \quad (26 \leq k \leq 50),
\]
\[
\alpha_k \leq \frac{31k - 98}{32k} \quad (51 \leq k \leq 57),
\]
\[
\alpha_k \leq \frac{7k - 34}{7k} \quad (k \geq 58).
\]
Proof. The proof of this lemma can be found in [2, Theorem 13.2].
Lemma 3. For any real number $x > 5$ and any integer $k \geq 3$, we have

$$
\sum_{n \leq x^{1/k}} \frac{1}{n^k} = \zeta(k) + \frac{1}{1-k} x^{1/k} + O(x^{-1}).
$$

(12)

Furthermore, we have

$$
\sum_{n \leq x^{1/k}} \frac{\log n}{n^k} = -\zeta'(k) + \frac{1}{k(1-k)} x^{1/k} \log x - \frac{1}{(1-k)^2} x^{1/k} + O(x^{-1} \log x),
$$

(13)

$$
\sum_{n \leq x^{1/k}} \frac{\log^2 n}{n^k} = \zeta''(k) + \frac{1}{(1-k)k^2} x^{1/k} \log^2 x - \frac{2}{k(1-k)^2} x^{1/k} \log x
$$

$$
+ \frac{2}{k(1-k)^3} x^{1/k} + O(x^{-1} \log^2 x),
$$

(14)

$$
\sum_{n \leq x^{1/k}} \frac{\log^3 n}{n^k} = -\zeta^{(3)}(k) + \frac{1}{(1-k)k^3} x^{1/k} \log^3 x - \frac{3}{k^2(1-k)^2} x^{1/k} \log^2 x
$$

$$
+ \frac{6}{k(1-k)^3} x^{1/k} \log x + \frac{6}{(1-k)^4} x^{1/k} + O(x^{-1} \log^3 x).
$$

(15)

Proof. Eq. (12) is given in [2, Eq. (14.40)]. Let $z$ be any large real number, and let $r$ be any positive integer. Then we have

$$
\sum_{n \leq z} \frac{\log^r n}{n^s} = (-1)^r \zeta^{(r)}(s) - \lim_{y \to \infty} \sum_{z < n \leq y} \frac{\log^r n}{n^s}
$$

(16)

for any real number $s > 1$. Using the fact that

$$
\sum_{n \leq z} \log n = z \log z - z + O(\log z)
$$

and the partial summation, we get

$$
\lim_{y \to \infty} \sum_{z < n \leq y} \frac{\log n}{n^s} = -\frac{1}{1-s} z^{1-s} \log z + \frac{1}{(1-s)^2} z^{1-s} + O(z^{-s} \log z).
$$

(17)

Taking in the above formula $z = x^{1/k}$, $s = k$ and substituting Eq. (17) into Eq. (16) with $r = 1$, we complete the proof of Eq. (13).

Similarly to the above, we use the following formula, [5, Theorem 6.11],

$$
\sum_{n \leq z} \log^2 n = z \log^2 z - 2z \log z + 2z + O(\log^2 z)
$$

with $r = 2$.
and the partial summation to get
\[
\lim_{y \to \infty} \sum_{z < n \leq y} \frac{\log^2 n}{n^s} = -\frac{1}{1-s} z^{1-s} \log^2 z
\]
\[
+ \frac{2}{(1-s)^2} z^{1-s} \log z - \frac{2}{(1-s)^3} z^{1-s} + O(z^{-s} \log^2 z).
\]
(18)

Taking \( z = x^{1/k} \) and \( s = k \) in the latter formula and substituting Eq. (18) into Eq. (16) with \( r = 2 \), we get Eq. (14). It remains to prove Eq. (15). Notice that
\[
\sum_{n \leq z} \log^3 n = [z] \log^3 z - 3 \int_1^z \frac{[t] \log^2 t}{t} dt
\]
\[
= z \log^3 z - 3z \log^2 z + 6z \log z - 6z + O(\log^3 z).
\]

Using the partial summation, we obtain that
\[
\lim_{y \to \infty} \sum_{z < n \leq y} \frac{\log^3 n}{n^s} = -\frac{1}{1-s} z^{1-s} \log^3 z + \frac{3}{(1-s)^2} z^{1-s} \log^2 z
\]
\[
- \frac{6}{(1-s)^3} z^{1-s} \log z + \frac{6}{(1-s)^4} z^{1-s} + O(z^{-s} \log^3 z).
\]
(19)

Combining Eqs. (19) and (16) with \( z = x^{1/k} \), \( s = k \) and \( r = 3 \), we get the desired conclusion.

Now, for any large real number \( x > 5 \), we define
\[
M(x) := \sum_{n \leq x} \mu * \mu(n)
\]
It is known that \( M(x) \) can be estimate by, see [1, Eq. (4.11)],
\[
M(x) = O(x \varepsilon(x)),
\]
(20)

where
\[
\varepsilon(x) = \exp \left( -C \left( \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}} \right) \right)
\]
(21)
with \( C \) being a positive constant. Under the above hypotheses, we are ready to state the following result.

**Lemma 4.** For any real number \( x > 5 \) and any integers \( k \geq 3 \), we have
\[
\sum_{n \leq x^{1/k}} \frac{\mu * \mu(n)}{n^k} = \frac{1}{\zeta^2(k)} + O(x^{1/k} \varepsilon(x)),
\]
(22)
and, for any positive integer,

\[
\sum_{n \leq x^k} \frac{\mu \ast \mu(n) \log^r n}{n^k} = M_{k,r}(k) + O \left( x^{\frac{1-r}{k}} \varepsilon(x) \right) \tag{23}
\]

where \(\varepsilon(x)\) is given by [21]. Here \(M_{k,r}(k)\) are certain constants depending on the Riemann zeta-function. Moreover, we have

\[M_{k,1}(k) = 2 \frac{\zeta'(k)}{\zeta^3(k)}, \quad M_{k,2}(k) = 2 \frac{3(\zeta'(k))^2 - \zeta''(k)\zeta(k)}{\zeta^4(k)},\]

\[M_{k,3}(k) = 2 \frac{12 \left( \frac{\zeta'(k)}{\zeta(k)} \right)^3 + 3 \frac{\zeta'(k)}{\zeta(k)} \cdot \frac{\zeta''(k)}{\zeta(k)} - \frac{\zeta(3)(k)}{\zeta(k)}}{\zeta^2(k)}.
\]

Proof. We recall that, for any large real number and any \(s > 1\),

\[
\sum_{n \leq z} \frac{\mu \ast \mu(n)}{n^s} = \frac{1}{\zeta^2(s)} - \lim_{y \to \infty} \sum_{z < n \leq y} \frac{\mu \ast \mu(n)}{n^s}. \tag{24}
\]

By Eq. (20), the partial summation and letting \(y \to \infty\), we obtain

\[
\lim_{y \to \infty} \sum_{z < n \leq y} \frac{\mu \ast \mu(n)}{n^s} = O \left( z^{1-s} \varepsilon(z) \right). \tag{25}
\]

Substituting (25) into (24) with \(z = x^1/k\) and \(s = k\), we get Eq. (22).

Similarly to the above, we prove Eq. (23). Notice that

\[
\sum_{n \leq z} \frac{\mu \ast \mu(n) \log^r n}{n^s} = \sum_{n=1}^{\infty} \frac{\mu \ast \mu(n) \log^r n}{n^s} - \lim_{y \to \infty} \sum_{z < n \leq y} \frac{\mu \ast \mu(n) \log^r n}{n^s}, \tag{26}
\]

and that

\[
\sum_{n=1}^{\infty} \frac{\mu \ast \mu(n) \log^2 n}{n^s} = 2 \frac{\zeta'(s)}{\zeta^3(s)}, \quad \sum_{n=1}^{\infty} \frac{\mu \ast \mu(n) \log^2 n}{n^s} = 2 \frac{3(\zeta'(s))^2 - \zeta''(s)\zeta(s)}{\zeta^4(s)},
\]

\[
\sum_{n=1}^{\infty} \frac{\mu \ast \mu(n) \log^2 n}{n^s} = 2 \frac{12 \left( \frac{\zeta'(s)}{\zeta(s)} \right)^3 + 3 \frac{\zeta'(s)}{\zeta(s)} \cdot \frac{\zeta''(s)}{\zeta(s)} - \frac{\zeta(3)(s)}{\zeta(s)}}{\zeta^2(s)}.
\]

Again by (20) and using the partial summation, we find that

\[
\lim_{y \to \infty} \sum_{z < n \leq y} \frac{\mu \ast \mu(n) \log^r n}{n^s} = O \left( z^{1-s} \varepsilon(z) \log^r z \right) = O \left( z^{1-s} \varepsilon(z) \right) \tag{27}
\]

Substituting (27) into Eq. (26) with \(z = x^{1/k}\) and \(s = k\), Eq. (23) is proved. \(\square\)
3 Proofs

3.1 Proof of Theorem 1

In case of $k = 3$, we substitute Eq. (10) into Eq. (1) and then we use Eqs. (12), (13), and (14) of Lemma 3 to obtain

$$S_{r,3}(x) = (b_1 \log^2 x + b_2 \log x + b_3) x \sum_{n \leq x} \frac{1}{n^3}$$

$$- (6b_1 \log x + 3b_2) x \sum_{n \leq x} \frac{\log n}{n^3} + 9b_1 x \sum_{n \leq x} \frac{\log^2 n}{n^3} + \sum_{n \leq x} \Delta_3 \left( \frac{x}{n^3} \right)$$

$$= \frac{\zeta(3)}{2} x \log^2 x + \zeta(2) \left( 3\gamma - 1 + 3 \frac{\zeta'(3)}{\zeta(3)} \right) x \log x$$

$$+ \zeta(3) \left( 3\gamma^2 - 3\gamma + 3\gamma_1 + 1 + 3(3\gamma - 1) \frac{\zeta'(3)}{\zeta(3)} + \frac{9 \zeta''(3)}{2 \zeta(3)} \right) x$$

$$- \frac{1}{8} (12\gamma^2 - 30\gamma + 12\gamma_1 + 19) x^{1/3} + \sum_{m \leq x} \Delta_3 \left( \frac{x}{m^3} \right) + O \left( \log^2 x \right).$$

Now, we use the estimate, see [3],

$$\Delta_3(x) \ll x^{\frac{43}{96}} + \varepsilon$$

to get

$$\sum_{n \leq x} \Delta_3 \left( \frac{x}{n^3} \right) \ll x^{\frac{43}{96}} + \varepsilon \sum_{n \leq x} \frac{1}{n^{43/96}} \ll x^{\frac{43}{96}} + \varepsilon,$$

for any small number $\varepsilon > 0$. This completes the proof of Eq. (3).

In the case of $k = 4$, we substitute Eq. (11) into Eq. (1)

$$S_{r,4}(x) = (c_1 \log^3 x + c_2 \log^2 x + c_3 \log x + c_4) x \sum_{n \leq x} \frac{1}{n^4}$$

$$- 4 \left( 3c_1 \log^2 x + 2c_2 \log x + c_3 \right) x \sum_{n \leq x} \frac{\log n}{n^4}$$

$$+ 4^2 \left( 3c_1 \log x + c_2 \right) x \sum_{n \leq x} \frac{\log^2 n}{n^4} - 4^2 c_1 x \sum_{n \leq x} \frac{\log^3 n}{n^4} + \sum_{n \leq x} \Delta_4 \left( \frac{x}{n^4} \right),$$

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and use Lemma 3 to obtain

\begin{align*}
S_{\tau,4}(x) &= \frac{\zeta(4)}{6} x \log^3 x + \zeta(4) \left( 2\gamma - \frac{1}{2} + 2 \frac{\zeta'(4)}{\zeta(4)} \right) x \log^2 x \\
&\quad + \zeta(4) \left( 6\gamma^2 - 4\gamma + 4\gamma_1 + 1 + 8 \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta'(4)}{\zeta(4)} + 8 \frac{\zeta''(4)}{\zeta(4)} \right) x \log x \\
&\quad + \zeta(4) \left( 12\gamma_1 + 4\gamma^3 - 6\gamma^2 + 4(\gamma - \gamma_1 + \gamma_2) - 1 + 4(6\gamma^2 - 4\gamma + 4\gamma_1 + 1) \frac{\zeta'(4)}{\zeta(4)} \right) x \\
&\quad + 16\zeta(4) \left( \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta''(4)}{\zeta(4)} + \frac{2}{3} \frac{\zeta(3)}{\zeta(4)} \right) x + \rho x^{\frac{1}{2}} + \sum_{m \leq x^{\frac{1}{2}}} \Delta_4 \left( \frac{x}{m^4} \right) + O \left( \log^3 x \right),
\end{align*}

with \( \rho \) being a computable constant. By Lemma 2, we have

\[ \sum_{m \leq x^{\frac{1}{2}}} \Delta_4 \left( \frac{x}{m^4} \right) \ll x^{\frac{1}{2} + \varepsilon} \sum_{m \leq x^{\frac{1}{2}}} \frac{1}{m^2} \ll x^{\frac{1}{2} + \varepsilon}. \]

Therefore Eq. (4) is proved.

### 3.2 Proof of Theorem 2

In much the same way as in Subsection 3.1, we prove Eqs. (5) and (6). First, in the case of \( k = 3 \), we substitute Eqs. (10), (22), and (23) into Eq. (2) with \( r = 1, 2 \) to obtain

\begin{align*}
S_{\mu,3}(x) &= \frac{1}{2\zeta^2(3)} x \log^2 x + \frac{1}{\zeta^2(3)} \left( 3\gamma - 1 - 6 \frac{\zeta'(3)}{\zeta(3)} \right) x \log x \\
&\quad + \frac{1}{\zeta^2(3)} \left( 3\gamma^2 - 3\gamma + 3\gamma_1 + 1 - 6(3\gamma - 1) \frac{\zeta'(3)}{\zeta(3)} - 9 \frac{\zeta''(3)}{\zeta(3)} + 27 \left( \frac{\zeta'(3)}{\zeta(3)} \right)^2 \right) x \\
&\quad + \sum_{n \leq x^{\frac{1}{3}}} \mu \ast \mu(n) \Delta_3 \left( \frac{x}{n^3} \right) + O \left( x^{1/3} \varepsilon(x) \right).
\end{align*}

Again we use \( \Delta_3(x) \ll x^{\frac{43}{48} + \varepsilon} \) to get

\[ \sum_{n \leq x^{\frac{1}{3}}} \mu \ast \mu(n) \Delta_3 \left( \frac{x}{n^3} \right) \ll x^{\frac{43}{48} + \varepsilon} \sum_{n \leq x^{\frac{1}{3}}} \frac{\tau(n)}{n^{\frac{11}{48}}} \ll x^{\frac{43}{48} + \varepsilon}. \]

This completes the proof of Eq. (3).

Second, for the case of \( k = 4 \) and \( r = 1, 2, 3 \), we substitute Eqs. (11), (22) and (23)
substituting into Eq. (2) to obtain

\[
S_{\mu,A}(x) = \frac{1}{6\zeta^2(4)} x \log^3 x + \frac{1}{\zeta^2(4)} \left( 2\gamma - \frac{1}{2} - \frac{4\zeta'(4)}{\zeta(4)} \right) x \log^2 x + \frac{16}{\zeta^2(4)} \left( \frac{6\gamma^2 - 4\gamma + 4\gamma_1 + 1}{16} - \left( 2\gamma - \frac{1}{2} \right) \frac{\zeta'(4)}{\zeta(4)} - \frac{\zeta''(4)}{\zeta(4)} + 3 \left( \frac{\zeta'(4)}{\zeta(4)} \right)^2 \right) x \log x + \frac{1}{\zeta^2(4)} \left( 12\gamma_1 + 4\gamma^3 - 6\gamma^2 + 4(\gamma - \gamma_1 + \gamma_2) - 1 - 8(6\gamma^2 - 4\gamma + 4\gamma_1 + 1) \frac{\zeta'(4)}{\zeta(4)} \right) x
\]

\[
+ \frac{32}{\zeta^2(4)} \left( 2\gamma - \frac{1}{2} \right) \left( 3 \left( \frac{\zeta'(4)}{\zeta(4)} \right)^2 - \frac{\zeta''(4)}{\zeta(4)} \right) x - \frac{64}{3\zeta^2(4)} \left( 12 \frac{\zeta'(4)}{\zeta(4)} - 9 \frac{\zeta'(4)}{\zeta(4)} \cdot \frac{\zeta''(4)}{\zeta(4)} + \frac{\zeta'(3)(4)}{\zeta(4)} \right) x + \sum_{m \leq x} \mu * \mu(m) \Delta_4 \left( \frac{x}{m^4} \right)
\]

\[
+ O \left( x^{1/4} 3(x) \right).
\]

Using \( \Delta_4(x) \ll x^{\frac{1}{2} + \epsilon} \), we get

\[
\sum_{m \leq x} \mu * \mu(m) \Delta_4 \left( \frac{x}{m^4} \right) \ll x^{\frac{1}{2} + \epsilon} \sum_{m \leq x} \frac{\tau(m)}{m^2} \ll x^{\frac{1}{2} + \epsilon}.
\]

This completes the proof of Eq. (6).

### 3.3 Proof of Theorem 3

By the following formula and the partial summation

\[
\sum_{n \leq z} \log^r n = z \log^r z + d_1 z \log^{r-1} z + d_2 z \log^{r-2} z + \cdots + d_r z + O \left( \log^r z \right),
\]

we find that

\[
\lim_{y \rightarrow \infty} \sum_{z \leq n \leq y} \frac{\log^r n}{n^s} = -\frac{z^{1-s}}{1-s} \sum_{j=0}^{r} h_j(s) \log^j z,
\]

where \( h_j(s) \) and \( d_j \) \((1 \leq j \leq r)\) are computable constants. By recalling Eq. (16)

\[
\sum_{n \leq z} \frac{\log^r n}{n^s} = (-1)^r \zeta^{(r)}(s) - \lim_{y \rightarrow \infty} \sum_{z \leq n \leq y} \frac{\log^r n}{n^s}
\]

and taking \( s = k \) and \( z = x^{1/k} \), we deduce

\[
\sum_{n \leq x^{1/k}} \frac{\log^r n}{n^k} = (-1)^r \zeta^{(r)}(k) + \frac{x^{\frac{1-r}{k}}}{1-k} \sum_{j=0}^{r} \frac{h_j(k)}{k^j} \log^j x
\]

(30)
with $0 \leq r \leq k$. Combining Eqs. (1), (8) and (30), we obtain

$$S_{\tau,k}(x) = \sum_{n \leq x^{1/k}} \left( \frac{x}{n^k} P_{k-1} \left( \log \frac{x}{n^k} \right) + \Delta_k \left( \frac{x}{n^k} \right) \right)$$

$$= x \sum_{n \leq x^{1/k}} \frac{1}{n^k} \sum_{r=0}^{k-1} q_r \log^r \frac{x}{n^k} + \sum_{n \leq x^{1/k}} \Delta_k \left( \frac{x}{n^k} \right)$$

$$= x \sum_{r=0}^{k-1} q_r \sum_{\ell=0}^r \binom{r}{\ell} (\log x)^{r-\ell} k^\ell \sum_{n \leq x^{1/k}} \frac{\log^\ell n}{n^k} + \sum_{n \leq x^{1/k}} \Delta_k \left( \frac{x}{n^k} \right)$$

$$= x \sum_{r=0}^{k-1} q_r \sum_{\ell=0}^r (-k)^{\ell} x^\ell \sum_{n \leq x^{1/k}} \Delta_k \left( \frac{x}{n^k} \right) + O \left( x^{1/k} \log^k x \right)$$

$$= xP_{\tau,k}(\log x) + E_{\tau,k}(x),$$

where

$$E_{\tau,k}(x) = \sum_{n \leq x^{1/k}} \Delta_k \left( \frac{x}{n^k} \right) + O \left( x^{1/k} \log^k x \right),$$

and $P_{\tau,k}(u)$ is a polynomial in $u$ of degree $k - 1$ depending on the derivative of the Riemann zeta-function. From Lemma 2 with $k \geq 5$, we have

$$\sum_{n \leq x^{1/k}} \Delta_k \left( \frac{x}{n^k} \right) \ll x^{\alpha_k + \epsilon} \sum_{n \leq x^{1/k}} \frac{1}{n^{k\alpha_k + \epsilon}} \ll x^{\alpha_k + \epsilon},$$

where we used the fact that $k\alpha_k > 2$. This completes the proof of Theorem 3 in the case $g = \tau$.

Similar arguments apply to the case $g = \mu$. From Eqs. (2), (22) and (23), we get

$$S_{\mu,k}(x) = x \sum_{r=0}^{k-1} q_r \sum_{\ell=0}^r \binom{r}{\ell} (\log x)^{r-\ell} k^\ell \sum_{n \leq x^{1/k}} \mu \cdot \mu(n) \frac{\log^\ell n}{n^k} + \sum_{n \leq x^{1/k}} \mu \cdot \mu(n) \Delta_k \left( \frac{x}{n^k} \right)$$

$$= x \sum_{r=0}^{k-1} q_r \sum_{\ell=0}^r \binom{r}{\ell} (\log x)^{r-\ell} k^\ell \sum_{m \leq x^{1/k}} \mu \cdot \mu(m) M_{k,\ell}(k) + \sum_{n \leq x^{1/k}} \mu \cdot \mu(n) \Delta_k \left( \frac{x}{n^k} \right) + O \left( x^{1/k} \log^k x \right)$$

$$= xP_{\mu,k}(\log x) + E_{\mu,k}(x),$$

where

$$E_{\mu,k}(x) = \sum_{n \leq x^{1/k}} \mu \cdot \mu(n) \Delta_k \left( \frac{x}{n^k} \right) + O \left( x^{1/k} \log^k x \right),$$

and $M_{k,0}(x) = 1/\zeta^2(2)$. From Lemma 2, the above sums can be estimated by

$$x^{\alpha_k + \epsilon} \sum_{n \leq x^{1/k}} \frac{\tau(n)}{n^{k\alpha_k + \epsilon}} \ll x^{\alpha_k + \epsilon}.$$
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