Abstract

We explore the representation theory of Renner monoids associated to classical groups and their Hecke algebras. In Cartan type $A_n$, the Hecke algebra is a natural deformation of the rook monoid algebra, and its representation theory has been studied extensively by Solomon and Halverson, among others. It is known that the character tables are block upper triangular, i.e. $M = AY = YB$ for some matrices $A$ and $B$. We compute the $A$ and $B$ matrices in Cartan type $B_n$ by using the results of Li, Li, and Cao in [9] to pursue analogous results to those of Solomon in [13]. We then compute some type $B_n$ Hecke algebra character values by using the same $B$ matrix as in the monoid case.

1 Introduction

Let $M$ be the Zariski closure in $\text{Mat}_n(F)$ of a connected reductive group $G$. Such a variety $M$ has the structure of a monoid; we call such monoids reductive monoids. The representation theory of $M$ is governed by that of its Renner monoid $R = \overline{N_G(T)}/T$. In this paper, we discuss the representation theory of Renner monoids and their Hecke algebras.

The type $A_n$ Renner monoid, the rook monoid, is the set of all partial permutation matrices ($0 - 1$ matrices with at most one 1 in each row and column) of size $n$ [12]. The type $B_n$ Renner monoid, the symplectic rook monoid $R Sp_{2n}$, is realized as a submonoid of the rook monoid. We describe $R Sp_{2n}$ further in Section 2.2.

A Renner monoid, $R$, has a nested sequence of conjugacy classes of idempotents. Any representation $(\pi, V)$ of $R$ has a minimal conjugacy class $C$ of idempotents on which it is nonzero. For $e$ an idempotent in $C$, we consider the natural restriction of $\pi$ onto the subset $E$ of $R$ fixed under multiplication by $e$. As $E$ is a group with identity $i$, we can study $\pi|_E$ as a representation of a finite group. The key insight in the representation theory of Renner monoids is that every irreducible $R$-representation arises uniquely in this way, and that irreducible representations of $R$ correspond bijectively to the irreducible representations of this nested sequence of subsemigroups.

Munn [10] showed the following in type $A_n$: the irreducible representations of $R_n$, correspond to the irreducible representations of $S_r, 0 \leq r \leq n$. If $\rho$ is a representation of $S_r$, then the corresponding representation $\rho^*$ of $R_n$ is equal to $\rho \oplus 0$ upon restriction to $S_r$, and 0 upon restriction to $S_s, s < r$, where in both cases the 0 denotes a 0-isotypic representation. Thus if $M$ is the character table of $R_n$, and $M_r$ is the character table of $S_r$, we have that $M$ is block-upper-triangular and of the form

$$M = \begin{bmatrix}
M_0 & \cdots & * \\
0 & \cdots & M_1 \\
& & \ddots & \vdots \\
& & & M_n
\end{bmatrix}.$$ 

Li, Li, and Cao gave analogues to these results for $R Sp_{2n}$ [9], and for general $R$ [8]. For $R Sp_{2n}$, the set of subsemigroups is $(B_n) \cup \{S_r : 0 \leq r \leq n\}$, where $B_n$ here denotes the group $B_n$. Thus the representations
of $\text{RSp}_{2n}$ are in bijective correspondence with representations of its group of units and of symmetric groups that are parabolic subgroups of $B_n$.

Solomon [13] recast Munn’s results on the rook monoid in terms of what he called the A and B matrices associated to $R_n$. Orthogonality of characters tells us that $M$ is an invertible matrix, and so if we define the block diagonal matrix

$$Y := \begin{bmatrix} M_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_0 \end{bmatrix},$$

there exist matrices $A, B$ such that $M = AY = YB$. For the rook monoid, $A$ involves binomial coefficients, and $B$ is a 0-1 matrix governed by well-known Pieri rules. The entries of $B$ correspond to multiplicities of restriction from one subsemigroup to a smaller one.

In this paper, we find the $A$ and $B$ matrices for the symplectic rook monoid (Sections 3 and 4). We then use these matrices for two main tasks. In Section 4.1 we prove a Pieri-type rule for type restriction from one subsemigroup to a smaller one.

2 Preliminaries

We begin with important preliminary information including the definitions of type $A_n$ and $B_n$ Renner monoids. We then give a matrix description of the symplectic Renner monoid. Finally, we recall an important result for decomposing the character tables of finite inverse semigroups, which is our main line of attack in the paper.

2.1 Essential Definitions

Let $M$ be a finite reductive monoid, let $G$ be its group of units, and let $R$ be its Renner monoid.

- In the case of type $A_{n-1}$, the Renner monoid is the rook monoid, $R_n$. This is also the set of all one-to-one partial injective transformations $\sigma$ from the set of $n$ elements to itself. In other words, $R_n$ is isomorphic to the submonoid of $\text{Mat}_n(F)$ consisting of partial permutation matrices, i.e. matrices with at most one 1 per row and column, and all the other entries being 0.

- The Renner monoid of type $B_n$ is called either the symplectic Renner monoid or the symplectic rook monoid, and denoted $\text{RSp}_{2n}$. Let $N = \{1, 2, \ldots, 2n - 1, 2n\}$. As in $[3]$, let $\theta : N \to N$ be the involution given by $\theta(t) = 2n + 1 - t$. A subset $P$ of $N$ is admissible if, for all $t \in N$, then $\theta(t) \notin N$ or if $P = N, \emptyset$. Define $\text{RSp}_{2n}$ to be the monoid of all injective partial transformations on $N$ that send admissible sets to admissible sets. We give a matrix description of this monoid in Section 2.2.

- The Weyl group $W$ of $G$ embeds as the group of units of $R$. In type $A_{n-1}$, this corresponds to $S_n$. In type $B_n$, this corresponds to the group of full rank admissible injective transformations on $N$.

- The generic Hecke algebra of a Renner monoid $R$ is the $\mathbb{C}$-algebra $\mathcal{H}(R)$ given by generators and relations by Godelle [4, Corollary 1.31].
Let $\Omega_n = \mathcal{S}\mathcal{P}_n \cup \bigcup_{r=0}^n \mathcal{P}_r$ be the union of double (also referred to as “signed”) partitions of total size $n$ and the partitions of $r$ for all $0 \leq r \leq n$. This is an indexing set for the collections of Munn classes and irreducible representations of the symplectic rook monoid. We use the standard ordering as discussed in Li, Li, and Cao’s paper [9].

### 2.2 Matrix Description for the Symplectic Renner Monoid

Li, Li, and Cao [9] define the symplectic Renner monoid of type $B_n$ as

$$R = \{ A \in R_n \mid APA^t = A^tPA = 0 \} \cup W$$

for

$$p = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}, \quad J_m = \begin{pmatrix} 0 & \cdots & 1 \\ \cdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix},$$

where $n = 2m$, $J_m$ is the $m \times m$ anti-diagonal matrix of 1’s, and $R_n$ is the rook monoid, a submonoid of $\text{Mat}_n$. We will take their definition of the symplectic Weyl group as the collection of “injective partial transformations of $n$ that map all admissible sets of $n$ to admissible sets” ([9], p.841). With the knowledge that $\{1, \ldots, n\}$ is an admissible set, we see that elements of the Weyl group are full rank, and also elements of the rook monoid, so that the Weyl group is a submonoid of $S_n \subseteq R_n$. With this, we establish:

**Proposition 2.2.1.**

$$R\mathcal{S}p_n \equiv \{ A \in R_n \mid A^tJ_nA = AJ_nA^t = 0 \quad \text{or} \quad A^tJ_nA = AJ_nA^t = J_n \}$$

**Proof.** We first show that

$$M \in \{ A \in R_n \mid A^tPA = APA^t = 0 \} \iff M \in \{ A \in R_n \mid A^tJ_nA = AJ_nA^t = 0 \}$$

Indeed, note that $J_nA^t = (b_{ij})$ is still an element of $R_n$ by nature of having at most one entry in every row and column that is equal to 1, so that the condition that

$$AJ_nA^t = 0 \iff \forall i, j, \quad (AJ_nA^t)_{ij} = c_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = 0$$

means that there is at most one non-zero summand (since all summands must be non-negative) in the evaluation of $c_{ij}$, forcing that summand to be 0. Clearly, if $d_{ij} := (APA^t)_{ij}$, then $c_{ij} = \pm d_{ij}$, so that $c_{ij} = 0 \iff d_{ij} = 0$. Thus $AJ_nA^t = 0 \iff APA^t = 0$, and the non-full rank components of these sets coincide.

It suffices to show that

$$W = \{ A \in R_n \mid A^tJ_nA = AJ_nA^t = J_n \}$$

Let $\overline{t} = n + 1 - i$. For $A \in W$, we have that

$$A(k) = i_k \iff A(\overline{k}) = \overline{i_k}$$

for if not, then using our first definition of the symplectic rook monoid, $\{k, A^{-1}(\overline{t})\}$ would be an admissible set mapped to a non-admissible set. Yet note that

$$A^tJ_nA = AJ_nA^t = J_n \iff AJ_n = J_nA \iff J_nAJ_n = A$$

having used the fact that $J_n^{-1} = J_n$ and $A^t = A^{-1}$ which follows because $A \in W$ must be full rank and hence orthogonal. However, $J_nAJ_n = A$ is exactly the condition that $A(k) = i_k \iff A(\overline{k}) = \overline{i_k}$, as $J_n$ is
the permutation corresponding to \((1 \, T)(2 \, 7) \cdots (m \, m)\) and hence conjugating a permutation matrix by \(J_n\) makes it so that \(k \mapsto \overline{A(k)}\), thus

\[
A = J_n A J_n \iff A(k) = \overline{A(k)} \iff \overline{A(k)} = A(k)
\]

so every \(A \in W\) satisfies \(A = J_n A J_n\). Similarly, matrices \(A \in R_n\) satisfying \(A = J_n A J_n\) are full rank and map admissible sets to admissible sets. For if not, then there would exist \(k, s \neq \overline{k}\) s.t. \(A(k) = i_k, \ A(s) = i_{\overline{k}}\) contradicting the condition that \(J_n A J_n = A\). 

2.3 Decomposing Character Tables of Finite Inverse Semigroups

Steinberg \([13]\) tells us that the character table of any finite inverse semigroup is block upper-triangular. Using this, Solomon decomposes the character table of \(R_n\) into the product of a block diagonal matrix and a much simpler block-upper-triangular matrix \([13]\). For the rook monoid, Solomon finds matrices \(A\) and \(B\) such that the character table \(M = \text{AY} = \text{YB}\) where \(Y = \text{diag}(X_1, \ldots, X_0)\) for \(X_i\) the character tables of the groups \(S_i, 0 \leq i \leq n\). We consider \(S_0\) to be the trivial group.

We find similar results in the case of \(\text{RSp}_{2n}\). In \([6]\) Proposition 4.2, Li, Li, and Cao show that a similar decomposition holds for type \(B_n\): the character table \(M\) is block upper triangular with diagonal blocks equal to \(Z_n, X_n, X_{n-1}, \ldots, X_0\), where \(Z_n\) is the character table of \(B_n\). Writing \(M = \text{AY} = \text{YB}\), where \(Y = \text{diag}(Z_n, X_n, X_{n-1} \ldots, X_0)\), we can write

\[
A = \begin{pmatrix} \text{Id} & U \\ 0 & T \end{pmatrix}, \quad B = \begin{pmatrix} \text{Id} & V \\ 0 & L \end{pmatrix}
\]

In Section 2.3 we compute \(T\), \(L\), and \(U\), and in Section 4.1 we compute \(V\), thereby computing \(M\) in two different ways.

3 Decomposing the Character Table of \(\text{RSp}_{2n}\): The \(A\) Matrix

In this Section, we compute the \(A\) matrix of \(\text{RSp}_{2n}\) as defined in Section 2.3. This is an analogous result to Solomon’s Proposition 3.5 in \([13]\). Let \(M = \text{M}(\text{RSp}_{2n})\) denote the character table of \(\text{RSp}_{2n}\), and recall that the rows of \(M\) (and by convention, our related matrices, \(A\) and \(B\)) are labelled by Munn classes of elements of \(\text{RSp}_{2n}\), and the columns are labelled by the irreducible representations, both of which are indexed by \(Q_n\).

Because the left half of \(A\) is given in Section 2.3, we only need consider the values of irreducible representations indexed by elements of \(\bigcup_{r=0}^{2n} \mathcal{P}_r\). These irreducible representations arise from irreducible representations of \(S_r\), and hence are indexed by a partition \(\lambda \vdash s\) for \(0 \leq s \leq n\). Call the character of such a representation \(\chi^\lambda_s\). Now let \(\alpha \in Q_n\), and let \(\sigma\) be a representative of the Munn class indexed by \(\alpha \in Q_n\). For \(r \in \{1, \ldots, n, 2n\}\), let \(\sigma_{\lambda} \in S_r\) be the “conjugate” of \(\sigma\), and let \(C(\tau, \sigma) := C(\tau, \sigma)\) as on p. 849 of \([3]\). Note that we have made the dependence of \(C(\tau)\) on \(\sigma\) more explicit by writing \(C(\tau, \sigma)\). Intuitively, we can realize \(\sigma\) as a matrix and decompose it the sum of a nilpotent matrix, \(N\), and a rank \(r\) matrix, \(M\), so that \(\sigma = M + N\). Then \(C(\tau, \sigma)\) represents all of the admissible subsets which consist of some cycles of the permutation corresponding to \(M\) (which is \(\sigma^o\) in the notation of \([3]\)).

The value of \(\chi^\lambda_s(\sigma)\) is independent of which representative of the Munn class is chosen by Theorem 4.3 of \([3]\), and it corresponds to the matrix entry \(M_{\alpha, \lambda}\) using our index notation

\[
M_{\alpha, \lambda} = \chi^\lambda_s(\sigma) = \sum_{k \in C(\tau, \sigma)} \chi^\lambda_s(\sigma_k)
\]
Partition \( C(r, \sigma) \) into a disjoint union as follows:

\[
C(r, \sigma) = \bigcup_{\mu \vdash r} \{ K \in C(r, \sigma) \mid K = \{ i_1, \ldots, i_{m(1)}, j_1, \ldots, j_{m(2)}, \ldots, i_1, \ldots, i_{m(e)} \} \}
\]

\[
\text{s.t. } \mu = (m(1), m(2), \ldots, m(e))
\]

\[
=: \bigcup_{\mu \vdash r} S_{\mu, \sigma}
\]

i.e. we organize the \( K \) by the cycle types of \( S_r \) that they imitate (again, see p. 849 of [3] for reference) and then define that collection of \( K \)'s to be \( S_{\mu, \sigma} \). Given a fixed \( \mu \vdash r \), we know that

\[
\forall K \in S_{\mu, \sigma}, \quad \sigma_{\mu} := \sigma_K = (1 \cdots m(1)) (m(1) + 1 \cdots m(1) + m(2)) \cdots \left( \sum_{i=1}^{e-1} m(i) \right) + 1 \cdots \left( \sum_{i=1}^{e} m(i) \right)
\]

The independence of \( \sigma_{\mu} \) from the specific set \( K \) is immediate from having the same cycle type and by Corollary 4.1 of [3]. From this, we can write

\[
M_{\alpha, \lambda} = \chi_\lambda^*(\sigma) = \sum_{K \in C(r, \sigma)} \chi_\lambda(\sigma_K) = \sum_{\mu \vdash r} \sum_{K \in S_{\mu, \sigma}} \chi_\lambda(\sigma_K) = \sum_{\mu \vdash r} |S_{\mu, \sigma}| \chi_\lambda(\sigma_{\mu})
\]

We claim

**Lemma 3.0.1.** For \( M = AY \), let \( A_{\alpha, \lambda} \) be the matrix entry under the indexing convention in which rows are labelled by Munn classes of elements of \( \text{RSp}_{2n} \) and the columns are labelled by the irreducible representations. For \( \alpha \in Q_n \), \( \sigma \) a representative of the Munn class indexed by \( \alpha \), and \( \lambda \in P_r \) for some \( 0 \leq r \leq n \),

\[
A_{\alpha, \lambda} = |S_{\lambda, \sigma}|
\]

**Proof.** Note that

\[
(AY)_{\alpha, \lambda} = \sum_{\beta \in Q_n} A_{\alpha, \beta} Y_{\beta, \lambda} = \sum_{\beta \in P_r \subseteq Q_n} A_{\alpha, \beta} Y_{\beta, \lambda}
\]

Because \( \beta \notin P_r \implies Y_{\beta, \lambda} = 0 \) due to \( Y \) being block diagonal. Now note that \( P_r \) can be indexed by the various \( \mu \vdash r \), so let \( \omega_{\mu} \) be a representative of the Munn class indexed by \( \mu \). Then

\[
\sum_{\beta \in P_r \subseteq Q_n} A_{\alpha, \beta} Y_{\beta, \lambda} = \sum_{\beta \in P_r \subseteq Q_n} A_{\alpha, \beta} Y_{\beta, \lambda} = \sum_{\mu \vdash r} A_{\alpha, \mu} Y_{\mu, \lambda} = \sum_{\mu \vdash r} A_{\alpha, \mu} \chi_\lambda(\omega_{\mu})
\]

yet from before, we have

\[
(AY)_{\alpha, \lambda} = M_{\alpha, \lambda} = \sum_{\mu \vdash r} |S_{\mu, \sigma}| \chi_\lambda(\omega_{\mu})
\]

So using linear independence of characters of \( \chi_\lambda \) for \( \lambda \vdash r \), we see that \( A_{\alpha, \mu} = A_{\alpha, \beta} = |S_{\mu, \sigma}| \) for \( \beta \in P_r \) corresponding to the partition \( \mu \vdash r \) with \( 0 \leq r \leq n \). \( \square \)

In the case that \( \alpha \in P_q \) for \( 0 \leq q \leq n \), a representative \( \sigma \in C_\alpha \), where \( C_\alpha \) denotes the Munn class corresponding to \( \alpha \), has rank less than or equal to \( n \). In particular, let \( N_{\sigma} \) represent the \( r \times r \) matrix corresponding to \( \sigma \) as an element of \( S_r \). Then from our knowledge of the Munn classes of \( \text{RSp}_{2n} \), we know that

\[
N = \begin{pmatrix} N_{\sigma} & 0 \\ 0 & 0 \end{pmatrix} \in \text{RSp}_{2n}
\]

is the matrix form for a valid representative of \( C_\alpha \). Moreover, because \( t \leq n \), we see that \( N \) has domain and range equal to \( \{1, \ldots, t\} \), which is makes it an admissible map on \( \{1, \ldots, 2n\} \). Hence any set, \( S \subseteq I^0(\sigma) \)
consisting of some cycles of \( \sigma^n \) (see [9, p. 849]) will automatically be admissable. In particular, this means that

\[
A_{\alpha, \lambda} = |S_{\lambda, \sigma}| = \left( \frac{\alpha}{\lambda} \right) = \prod_{i=1}^{s} \left( \frac{\alpha_i}{\lambda_i} \right)
\]

which is the same \( A \)-matrix entry as in [13, p. 321]. Note that though \( \alpha \) and \( \lambda \) are partitions of \( t, r \leq n \), the notation of \( \alpha = (\alpha_i) \) does not describe \( \alpha \) in terms of its components, but rather \( \alpha_i \) represents the number of components in the partition which are equal to \( i \). For example, if \( \alpha = (a_1, a_2, \ldots, a_4) \) for \( a_i \in \mathbb{Z}^+ \) is the partition, then

\[
\alpha_i := |\{ k \text{ s.t. } a_k = i \}|
\]

**Example 3.0.2.** Let \( \alpha \in \mathcal{P}_3 \) be \( \alpha = (1, 2, 2) \), then \( \alpha_1 = 1, \alpha_2 = 2, \) and \( \alpha_3 = 0 \) for all other \( i \).

The above shows that the \( T \) block of the \( A \) matrix for the symplectic rook monoid is the \( A \) matrix for the rook monoid of size \( n \), i.e. \( T_{\alpha, \lambda} = \left( \frac{\alpha}{\lambda} \right) = |S_{\lambda, \sigma}| \) for the \( \alpha, \lambda \) described above. Moreover, this means the character table of the symplectic rook monoid has the following form

\[
M(\text{RSp}_{2n}) = \begin{pmatrix} M(B_n) & * \\ 0 & M(R_n) \end{pmatrix}
\]

where \( M(\cdot) \) denotes the character table of the correspond monoid as a block matrix in our larger matrix. This is in accordance with Proposition 4.2 of Li, Li, and Cao [9]. We know from our initial description of the \( Y \) and \( B \) matrices that

\[
B = \begin{pmatrix} \text{Id} & V \\ 0 & L \end{pmatrix}, \quad Y = \begin{pmatrix} Z_n & 0 \\ 0 & Y(R_n) \end{pmatrix} \implies YB = \begin{pmatrix} Z_n & Z_nV \\ 0 & Y(R_n)L \end{pmatrix} = M = \begin{pmatrix} M(B_n) & * \\ 0 & M(R_n) \end{pmatrix}
\]

where \( Y(R_n) \) denotes the \( Y \) matrix of \( R_n \). Comparing the lower right block and noting that \( Y(R_n) \) is invertible, we get that the \( L \) block of \( B = B(\text{RSp}_{2n}) \) must correspond to the \( B(R_n) \). \( U \) is determined by the values of \( |S_{\mu, \lambda}| \) for \( \mu \in \text{SP}_m \) and \( \lambda \in \cup_{t=0}^{m} \mathcal{P}_t \).

### 3.1 Determining the \( \mathcal{U} \) Block for \( \text{RSp}_{2n} \)

To determine \( \mathcal{U} \), we consider elements of the Weyl group as described in Propositions 1.4.1 (p. 21), 3.4.2 (p. 92 – 93), and 3.4.7 (p. 96 – 97) of [9]. In particular, we look at representatives of the form

\[
w_{\gamma, \delta} = b_{m_1, e_1}^{-} \cdots b_{m_i, e_i}^{-} \cdot b_{m_{i+1}, e_{i+1}}^{+} \cdots b_{m_r, e_r}^{+}
\]

where \( \delta = (e_1, \ldots, e_l) \) and \( \gamma = (e_{l+1}, \ldots, e_r) \) consist of a decreasing partition and increasing partition of some \( r \) and \( n-r \) respectively, so that \( |\gamma| + |\delta| = \sum_{i=1}^{r} e_i = n \). We also stipulate that \( m_1 = 0, m_{i+1} = m_i + e_i \) for \( i = 1, \ldots, r-1 \), and \( m_r + e_r = n \) as on p. 93 of [9]. We have that

\[
b_{m,e}^{+} = s_{m+1} s_{m+2} \cdots s_{m+e-1} \in W_n
\]

\[
b_{m,e}^{-} = t_m s_{m+1} s_{m+2} \cdots s_{m+e-1} \in W_n
\]

where \( 0 \leq m < n \) and \( s_m \) is the permutation \( (m, m+1) \in S_n \). \( s_{m+1} s_{m+2} \cdots s_{m+e-1} \) can then be realized as a matrix in \( S_n \) cyclically permuting some elements and fixing others, and this permutation can be realized in \( \text{Mat}_n \). For \( 0 \leq m \leq n-1 \), \( t_m \) is the identity matrix in \( \text{Mat}_n \) except \( t_{m+1, m+1} = -1 \). To formulate these representatives in \( \text{Mat}_{2n} \), as indicated in Section 2.2, we describe the correspondence between the signed matrices of \( -1 \)’s, \( 0 \)’s, and \( 1 \)’s in \( \text{Mat}_n \) and unsigned matrices of \( 0 \)’s and \( 1 \)’s in \( \text{Mat}_{2n} \).
Correspondence between signed and unsigned matrices

Let signed $n \times n$ matrices be demarcated by $\{\cdots\}_n$ and unsigned matrices of size $2n \times 2n$ be demarcated by $\{\cdots\}_{2n}$. Given a signed matrix $M$, then the corresponding permutation matrix contained in $\text{Mat}_{2n}$, call it $M'$, is determined by the following rules

$$(M)_{ij} = 1 \iff (M')_{ij} = 1 = (M')_{\bar{i}\bar{j}} = 1$$

$$(M)_{ij} = -1 \iff (M')_{i\bar{j}} = 1 = (M')_{\bar{i}j}$$

Representatives of Weyl Group

We claim

**Lemma 3.1.1.** The $b_{m,e}^+$ blocks adhere to the following correspondence between elements of $\text{Mat}_n$ and $\text{Mat}_{2n}$

$$b_{m,e}^+ = \sum_{s_{m+1} \cdots s_{m+e-1}} (m+1, m+2, \ldots, m+e-1, \overline{m+e})_{2n}$$

$$b_{m,e}^- = \sum_{t_{m}^{s_{m+1}} \cdots s_{m+e-1}} (m+1, m+2, \ldots, m+e-1, \overline{m+e})_{2n}$$

**Proof.** Looking at the rules given in the previous section, the image of $b_{m,e}^+$ in $\text{Mat}_{2n}$ is justified. The image of $b_{m,e}^-$ can be explained as follows: consider the image of $b_{m,e}^-$ and that of $b_{m,e}^+$ as matrices in $\text{Mat}_{2n}$, denoted by $M^-$ and $M^+$. These matrices are very similar, in particular, the $m + 2$ through $m + e - 1$ coincide. However $M_{m+1,m+e}^+ = 1 = M_{m+1,m+e}^-$ while $M_{m+1,m+e}^- = 1 = M_{m+1,m+e}^-$, which comes from the effect of multiplying by $t_m$. This means that instead of mapping $m + e$ to $m + 1$ under the action of $b_{m,e}^-$, we have that $m + e$ is mapped to $\overline{m+1}$. From here, the sequence of shifts, $\overline{m+1} \rightarrow \overline{m+2} \rightarrow \cdots \rightarrow \overline{m+e}$ occurs, as

$$M_{m+1, m+e}^- = \cdots = M_{m+1, m+e-1}^- = M_{m+2, m+e}^- = \cdots = M_{m+1, m+e-1}^{\overline{m+e}} = 1$$

holds. However, $b_{m,e}^-(\overline{m+e}) = m+1$, as $M_{m+1, m+e}^- = 1$ which completes the permutation of $(m+1, \ldots, m+e, \overline{m+1}, \ldots, \overline{m+e})$. This fully justifies the matrix description of the blocks. \qed

**Example 3.1.2.** Consider

$$b_{0.3}^- = t_0 s_1 s_2 = t_0(1 2)(2 3) = t_0(1 2 3)$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow (1 2 3 6 5 4) = (1 2 3 \overline{7} \overline{5} \overline{3})$$

With this, we note that the representatives $\{w_{r, \delta}\}$ consist of blocks of disjoint signed permutations, i.e. each block affects subset of $n$ distinct from other blocks. This because $m_i + e_i = m_{i+1}$, $e_r + m_r = n$, and the fact that each block acts on $(m_1 + 1, \ldots, m_i + e_i)$. Now recall the set

$$C(r) = C(r, \sigma) = \{K \subseteq \Pi^n(\sigma) \mid K \text{ admissible and consists of all the elements of some cycles of } \sigma \text{ with } |K| = r \}$$
for which we sum over in the evaluation

$$\chi_{\lambda}(\sigma) = \sum_{\kappa \in C(\tau, \sigma)} \chi_{\lambda}(\kappa)$$

from [3, p. 848-9]. We note that no admissible set in $C(\tau)$ can contain a cycle induced by a block $b_{i,e}^+$, as such a cycle contains both $m+i$ and $m+i$ for $1 \leq i \leq e$. Thus the cycles from which an admissible $K \in C(\tau)$ can be created are those coming from only the $b_{i,e}^+$ blocks. Now we can determine the $A$ matrix for $RSp_{2n}$.

**Theorem 3.1.3.** For $M$, the character table of $RSp_{2n}$, we have

$$M = AY, \text{ s.t. } A = \begin{pmatrix} \text{Id} & U \\ 0 & T \end{pmatrix}$$

Let $\lambda \vdash \tau$ with $\lambda = (\lambda(1), \ldots, \lambda(e))$ in terms of components. Furthermore, let $\lambda_i$ be the number of components of $\lambda$ such that $\lambda(k) = i$ and let

$$S_{\lambda,\sigma} = \{K \in C(\tau, \sigma) \mid K = \{i_1, \ldots, i_{\lambda(1)}, j_1, \ldots, j_{\lambda(2)}, \ldots, l_1, \ldots, l_{\lambda(e)}\}$$

as in the first part of Section 3. Then

$$U_{\alpha,\lambda} = |S_{\lambda,\sigma}| = 2^{\Sigma_i \lambda_i} \cdot \prod_{i=1}^{2n} \binom{\gamma_i}{\lambda_i}$$

where $\alpha = (\gamma, \delta) \in S^2_n$, $w_{\gamma,\delta}$ is the representative of the Weyl group corresponding to $\alpha$, $\sigma$ is the realization of $w_{\gamma,\delta}$ as a permutation in $Mat_{2n}$, and $\gamma_i$ is the number of components of $\gamma$ such that $\gamma(k) = i$.

Proof. Looking back at the definition of $w_{\gamma,\delta}$ in terms of the $b_{i,e}^+$, and our bolded remark about $b_{i,e}^+$ blocks, we see that the only admissible $i$-cycles in $\sigma$ come from the $b_{i,e}^+$ blocks, whose lengths are indexed by the components of $\gamma$ (see p. 93 in [3]). Moreover, $\gamma_i$ is equal to the number of $\{e_k\}_{k=1}^i$ such that $e_k = i$. Then

$$|S_{\lambda,\sigma}| = \prod_{i=1}^{2n} \binom{\gamma_i}{\lambda_i} \cdot 2^{\lambda_1} = 2^{\Sigma_i \lambda_i} \cdot \prod_{i=1}^{2n} \binom{\gamma_i}{\lambda_i}$$

because if there are $\gamma_i$ components of $\gamma$ of size $i$, then we have $\binom{\gamma_i}{\lambda_i} \cdot 2^{\lambda_1}$ ways of choosing admissible $i$-cycles. This is because an admissible $i$-cycle can come from the collection of cycles acting on $\{1, \ldots, n\}$ or from the disjoint collection of cycles acting $\{n+1, \ldots, 2n\} = \{\pi, \ldots, \tau\}$. Note that such cycles come in pairs as shown by the image of $b_{i,e}^+$ in $Mat_{2n}$, and so for every chosen $i$ cycle, we can choose that cycle or its admissible conjugate. This determines the values of $|S_{\lambda,\sigma}|$ for $\alpha \in S^2_n$ and $\lambda \vdash \tau$, completing our description of the $U$ block matrix and hence all of the $A$ matrix.

4 Restricting Monoid Representations to Group Representations and Decomposing Character Tables

In this section, we decompose the character table of $RSp_{2n}$ into $YB$ where $Y$ is the same block diagonal matrix as in the previous section. We determine the $B$ matrix by restricting the monoid representations to the group of units $B_n$, using also Solomon’s computation of the $B$ matrix of the rook monoid [13, Proposition 3.11].

**Definition 4.0.1.** Given groups $G$ and $H$, and corresponding representations $V_G$ and $V_H$, we define the box tensor representation $V_G \boxtimes V_H$ to be the representation of $G \times H$ with the action $(g, h) \cdot (v_1 \boxtimes v_2) = gv_1 \boxtimes hv_2$. 
We can now explain the restriction of an $RSp_{2n}$ representation to $B_n$, in analogue to Solomon’s Corollary 3.15 for type $A_n$. Solomon shows that given $\chi^*$ an irreducible representation of $R_n$ corresponding to a partition of $k$, the restriction $\chi^*|_{S_n} = \text{Ind}_{S_{k\times S_{n-k}}}^{S_{k}\times B_{n-k}}(\chi \boxtimes \eta_{n-k})$, where $\eta_{n-k}$ is the trivial representation. We show that similarly

**Theorem 4.0.2.** Let $\chi$ be a character $S_k$ for $k \geq 1$, and $\chi^*$ the associated character of $RSp_{2n}$. Then

$$\chi^*|_{B_n} = \text{Ind}_{S_{k}\times B_{n-k}}^{B_n}(\chi \boxtimes \eta_{n-k})$$

**Proof.** This is an application of the general character formula from [3, Theorem 4.1]. Adopting their notation, let $e$ be an idempotent element of $RSp_{2n}$, the group $W^*(e) = \{ \alpha \in \Delta, s_\alpha e = e s_\alpha \neq e \}$ where $\Delta$ is the lattice of roots of the Weyl group $W_n$. Then given an irreducible character $\chi$ of $W^*(e)$, the character formula states

$$\chi^*(\sigma) = \sum_{K \in \mathcal{F}(e), K \sigma = K} \chi(\mu_k \sigma \mu_k^-)$$

where $\mathcal{F}(e) = w \cdot [r], w \in W$, for $[r] = \{1, \ldots, r\}$.

When setting $e = e_r$ we have $W^*(e)$ is generated by elements which involve the first $r$ indices. By examining the Coxeter diagrams for type $B_n$ Weyl groups, the only generators which involve the first $r$ indices are the transpositions $(k \ k+1)$ for $k \leq r$. However, $(r \ r+1)e_r = e_r(r \ r+1)$, so $W^*(e) = \{(1 \ 2), \ldots, (r-1 \ r)\} = S_r$.

Because $K \in \mathcal{F}(e)$, $|K| = r$ and there exists a Weyl group element $w$ that restricts to $\mu_k$, so that $\mu_k \sigma \mu_k^-$ is the restriction of $w_\sigma w^{-1}$ to $[r]$. Therefore, $\chi^*(\sigma) = \sum_{K \in \mathcal{F}(e), K \sigma = K} \chi(\mu_k \sigma \mu_k^-)$ summing over all $|K| = r$ with $w_k \sigma w_k^{-1} \in S_r \times W_{n-r}$. The elements $w_k$ are a set of coset representatives $B_n/(S_r \times B_{n-r})$, which implies

$$\chi^*(\sigma) = \sum_{K \in \mathcal{F}(e), K \sigma = K} \chi(\mu_k \sigma \mu_k^-) = \text{Ind}_{B_r \times B_{n-r}}^{B_n}(\chi \boxtimes \eta_{n-r})(\sigma)$$

as desired. □

**Remark 4.0.3.** Note that this proof extends to more general Renner monoids, such as the Renner monoid associated to the group $D_n$. For $r < n-2$, we still have that $W^*(e_r) = \{(1 \ 2), \ldots, (r-1 \ r)\} = S_r$, which yields the result of the theorem for characters $\chi$ of $S_k$ for $k < n-2$. Likewise, the same proof yields Solomon’s result for type $A_n$.

### 4.1 Analogue of Pieri Rule for Type $B_n$ Weyl Group

We’ll now determine an analogue of the Pieri rule for $B_n$. We first highlight two facts from [3].

**Proposition 4.1.1.** [3, Lemma 6.1.3] Let $n \geq 1$ and $k, l \geq 0$ be integers such that $n = k + l$. Let $(\lambda_1, \lambda_2)$ and $(\mu_1, \mu_2)$ be pairs of partitions with $|\lambda_1| + |\lambda_2| = k$ and $|\mu_1| + |\mu_2| = l$. Then, using the diagonal embedding $B_k \times B_l \subseteq B_n$, we have

$$\text{Ind}_{B_k \times B_l}^{B_n}(X_{\lambda_1, \lambda_2} \boxtimes X_{\mu_1, \mu_2}) = \sum_{(v_1, v_2)} c_{\lambda_1, \mu_1}^{v_1} c_{\lambda_2, \mu_2}^{v_2} X_{v_1, v_2}$$

where the sum runs over all pairs of partitions $(v_1, v_2)$ for which $|v_i| = |\lambda_i| + |\mu_i|$ for $i = 1, 2$.

**Proposition 4.1.2.** [3, Lemma 6.1.4] Let $n \geq 1$ and consider the subgroup $S_n \subset B_n$. Let $\nu \vdash n$ and $\chi_\nu \in \text{Irr}(S_n)$ be the corresponding irreducible character. Then

$$\text{Ind}_{S_n}^{B_n} \chi_\nu = \sum_{\lambda \vdash n} c_{\lambda, \nu}^\lambda X_{\lambda, \nu}$$
We can now derive a more explicit formula for \( \text{Ind}_{B_n \times B_1}^{B_n}(\chi_\nu \boxtimes \eta_l) \) for a fixed partition \( \nu \vdash k \). Using transitivity of induction along with 4.2.1 and 4.2.2 we have

\[
\begin{align*}
\text{Ind}_{S_k \times B_1}^{B_n}(\chi_\nu \boxtimes \eta_l) &= \text{Ind}_{B_k \times B_1}^{B_n}(\chi_\nu \boxtimes \eta_l) \\
&= \text{Ind}_{B_k \times B_1}^{B_n} \left( \sum_{\lambda, \mu}^{\nu} c_{\lambda, \mu}^\nu \chi_\lambda \otimes \eta_l \right) \\
&= \sum_{\lambda, \mu}^{\nu} c_{\lambda, \mu}^\nu \text{Ind}_{B_k \times B_1}^{B_n}(\chi_\lambda \boxtimes X_{\nu_1, \emptyset}) \\
&= \sum_{\lambda, \mu}^{\nu} c_{\lambda, \mu}^\nu \sum_{\nu_1 + \nu_2 = n} \text{Ind}_{B_k \times B_1}^{B_n}(\chi_{\nu_1} \boxtimes \chi_{\nu_2}, \emptyset)
\end{align*}
\]

where \([l]\) denotes a horizontal strip of size \( l \). Note that \( c_{\lambda, \mu}^{\nu_2} = 0 \) unless \( \mu = \nu_2 \) in which case it is equal to 1. This sum reduces to

\[
\begin{align*}
\text{Ind}_{S_k \times B_1}^{B_n}(\chi_\nu \boxtimes \eta_l) &= \sum_{\lambda, \mu}^{\nu} c_{\lambda, \mu}^\nu \sum_{\nu_1 + \nu_2 = n} \text{Ind}_{B_k \times B_1}^{B_n}(\chi_{\nu_1} \boxtimes \chi_{\nu_2}, \emptyset) \\
&= \sum_{\gamma, \mu}^{\nu} c_{\gamma, \mu}^\nu \sum_{\nu_1 + \nu_2 = n} \text{Ind}_{B_k \times B_1}^{B_n}(\chi_{\nu_1} \boxtimes \chi_{\nu_2}, \emptyset)
\end{align*}
\]

The last line comes from swapping the order of summation and noting that \( c_{\lambda, \mu}^{\nu_1} \) is 1 if \( \gamma - \lambda \) is a horizontal strip and 0 otherwise by [3, p. 182].

Thus we have found:

**Proposition 4.1.3.** For \( n \geq 1 \) and integers \( k, l \) such that \( k + l = n \) we have that

\[
\begin{align*}
\text{Ind}_{S_k \times B_1}^{B_n}(\chi_\nu \boxtimes \eta_l) &= \sum_{\gamma, \mu}^{\nu} c_{\gamma, \mu}^\nu \sum_{\nu_1 + \nu_2 = n} \text{Ind}_{B_k \times B_1}^{B_n}(\chi_{\nu_1} \boxtimes \chi_{\nu_2}, \emptyset) \\
&= \sum_{\gamma, \mu}^{\nu} c_{\gamma, \mu}^\nu \sum_{\nu_1 + \nu_2 = n} \text{Ind}_{B_k \times B_1}^{B_n}(\chi_{\nu_1} \boxtimes \chi_{\nu_2}, \emptyset)
\end{align*}
\]

Now we can determine the B matrix for type \( B_n \).

**Corollary 4.1.4.** In the matrix decomposition of

\[
M(\text{RSp}_{2n}) = YB,
\]

with

\[
B = \begin{pmatrix}
\text{Id} & V \\
0 & L
\end{pmatrix},
\]

10
we have

\[ V_{(\gamma, \mu), \nu} = \sum_{\lambda \text{ partition s.t. } \gamma - \lambda \text{ horiz strip of size } l} c_{\overline{\nu}, \lambda, \mu} \]

where \((\gamma, \mu) \in S^p n\), \(\nu \in P_r, 0 \leq r \leq n\); and \(L\) is the B matrix of \(R_n\) with coefficients

\[ L_{\gamma, \mu} = \begin{cases} 1 & \gamma - \mu \text{ is a horizontal strip} \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** The values \(L_{\gamma, \mu}\) are given by \([13]\) when both \(\gamma\) and \(\mu\) are partitions, so we consider the case where \(\lambda \in S^{2n}_n, \mu \in P_r, 0 \leq r \leq n\). Then if \(M = M(RSP_{2n})\), we have

\[ M_{\lambda, \mu} = \sum_{\gamma \in \mathcal{Q}_n} Y_{\lambda, \gamma} B_{\gamma, \mu} \]

so if \(\lambda\) corresponds to the Munn class \(C_{\lambda}\) and \(\chi_{\mu}\) is the character of \(RSP_{2n}\) corresponding to \(\mu\), then the restriction

\[ (\text{Res}_{B_n}^{RSP_{2n}} \chi_{\mu})(C_{\lambda}) = \chi_{\mu}(C_{\lambda}) = \sum_{\gamma \in S^{2n}_n} Y_{\lambda, \gamma} B_{\gamma, \mu} \]

so

\[ \text{Res}_{B_n}^{RSP_{2n}} \chi_{\mu} = \sum_{\gamma \in S^{2n}_n} (\text{Res}_{B_n}^{RSP_{2n}} \chi_{\gamma}) B_{\gamma, \mu} = \sum_{\gamma \in S^{2n}_n} \psi_{\gamma} B_{\gamma, \mu} \]

where the \(\psi_{\gamma} := \text{Res}_{B_n}^{RSP_{2n}} \chi_{\gamma}\) are the irreducible characters of \(B_n\). Thus we see that \(B_{\gamma, \mu}\) is the multiplicity of \(\psi_{\gamma}\) in \(\text{Res}_{B_n}^{RSP_{2n}} \chi_{\mu}\).

By Proposition 4.0.2,

\[ \text{Res}_{B_n}^{RSP_{2n}} \chi_{\mu} = \text{Ind}_{S_z \times B_{n-r}}^B \left( \phi_{\mu} \boxtimes \eta_{n-r} \right), \]

where \(\phi_{\mu} := \text{Res}_{S_z}^{RSP_{2n}} \chi_{\mu}\), and so Proposition 4.1.3 gives the desired result. \(\square\)

### 4.2 A Formula of Group Characters

We can use our calculation of the \(A\) and \(B\) matrices to determine a formula for group characters, in analogue to Corollary 3.14 of \([13]\). Let \(z_{\alpha}\) be the size of the centralizer of an element in \(\alpha \in \mathcal{Q}_n\), and let \(W\) be the diagonal matrix \((\delta_{\alpha, \beta} z_{\alpha})_{\alpha, \beta}\). Then \(YY^T = W\) by the column orthogonality relations of character tables. For the next result, we use the notation \(\chi_{\lambda} := \chi(w_\lambda)\) for any element \(w_\lambda\) in the conjugacy class indexed by \(\lambda\).

**Corollary 4.2.1.** Let \(\lambda = (\lambda_1, \lambda_2) \vdash n, 0 \leq r \leq n\). Then

\[ \sum_{\substack{\alpha = (\alpha_1, \alpha_2) \vdash n \\ \beta \vdash \tau}} z_{\alpha}^{-1} 2 \sum_{i=1}^{2n} \prod_{\mu} c_{\overline{\nu}, \lambda, \mu} = \sum_{\gamma \vdash \lambda, \gamma \vdash \tau} c_{\overline{\nu}, \lambda, \mu}, \]

where for a partition \(\gamma, \gamma_i\) denotes the number of parts of \(\gamma\) of size \(i\).

**Proof.** We show that both sides are equal to \(B_{\gamma, \mu}\). The equality for the right side is given in Corollary 4.1.3. To get the left hand side, note that \(B = Y^{-1} A Y = Y^T W^{-1} A Y\), so

\[ B_{\lambda, \mu} = \sum_{\alpha} (Y^T W^{-1})_{\lambda, \alpha} (AY)_{\alpha, \mu} \]

\[ = \sum_{\alpha} Y^T_{\alpha, \lambda} z_{\alpha}^{-1} (AY)_{\alpha, \mu} \]

\[ = \sum_{\alpha} \sum_{\beta} z_{\alpha}^{-1} Y_{\alpha, \lambda} A_{\alpha, \beta} Y_{\beta, \mu}, \]

and the definition of \(Y\) along with Theorem 3.1.3 gives the result. \(\square\)
5 Characters of Hecke Algebras

Our method for computing the character tables for Renner monoids has been the decomposition

\[ M = AY = YB. \]

In this section, we consider the Hecke algebra version of this decomposition,

\[ M_q = Y_q B_q. \]

B encodes multiplicities of restrictions, which will cause \( B_q = B \), and so we can compute the character table (which contains some, but not all character values) of any Hecke algebra for which \( B \) and \( Y_q \) are known. Dieng, Halverson, and Poladian [2] have computed the character table of \( \mathcal{H}(\mathfrak{R}_n) \); we compute the character table of \( \mathcal{H}(\mathfrak{RSp}_{2n}) \).

5.1 Character Values on Standard Elements of \( \mathcal{H}(\mathcal{R}) \): The B Matrix

Let \( \mathcal{R} \) be a Renner monoid, and define \( \Lambda, W, W(e), W^*(e), W_*(e) \) to be as in [4]. Let \( \mathcal{H} \) be the generic Hecke algebra of \( \mathcal{R} \) in the sense of [4, Definition 1.30] (with all parameters equal). For \( e \in \Lambda \), consider the set \( W_e := e W^*(e) \). Notice that elements of \( W^*(e) \) commute with \( e \), and thus that \( W_e = W^*(e)e = e W^*(e)e \).

Also, suppose that \( w, w' \in W^*(e) \). Then \( ew = we = ew' = w'e \), so \( e = eww^{-1} = ew'w^{-1} \), so \( w'w^{-1} \in W_*(e) \). Since \( W(e) = W^*(e) \times W_*(e) \), this means that \( w = w' \). Therefore, \( W_e \) is a group with identity \( e \), and is isomorphic to the parabolic subgroup \( W^*(e) \). Moreover, this means that \( \mathcal{H}(W^*(e)) \) embeds in \( \mathcal{H} \) via \( T_w \mapsto T_{ew} \).

Each \( W^*(e) \cong W_e \) indexes a block of the character table of \( \mathcal{R} \). By the orthogonality of Renner monoid character tables, given a character \( \chi \) of \( \mathcal{R} \),

\[
\text{Res}^\mathcal{R}_{W_e} \chi = \sum_{\chi' \in \text{Irr}(W_e)} \dim \text{Hom}(\text{Res}^\mathcal{R}_{W_e} \chi, \chi') \chi' = \sum_{\chi' \in \text{Irr}(W_e)} B_{\chi' \chi},
\]

where \( B_{\chi' \chi} \) denotes the entry of the B matrix of \( \mathcal{R} \) with row corresponding to \( \chi' \) and column corresponding to \( \chi \).

By the Putcha-Tits deformation theorem [12, p 347], \( \mathcal{H} \) has the same representation theory as \( \mathcal{R} \). Every irreducible character \( \chi \) of \( \mathcal{R} \) corresponds to an irreducible character of \( \mathcal{H} \), and we write \( \chi_* \) for this character. Thus we have the analogous equation in the Hecke algebra case:

\[
\text{Res}^\mathcal{H}_{\mathcal{H}(W_*)} \chi_* = \sum_{\chi'_* \in \text{Irr}(\mathcal{H}(W_*))} B_{\chi'_* \chi_*} \chi'_*.
\]

If we take an element \( h \in \mathcal{H}(W_e) \subset \mathcal{H} \), then

\[
\chi_*(h) = \text{Res}^\mathcal{H}_{\mathcal{H}(W_e)} \chi_*(h) = \sum_{\chi'_* \in \text{Irr}(\mathcal{H}(W_*))} B_{\chi'_* \chi_*}(h) \chi'_*.
\]

This formula depends on the particular embedding of \( \mathcal{H}(W^*(e)) \) into \( \mathcal{H} \); any such embedding will give a correct formula, but it may not be apparent how a given \( h \in \mathcal{H} \) relates to such an embedding.

The rows of the character table for \( \mathcal{R} \) are indexed by Munn classes; we want to create a Hecke algebra character table indexed by these same Munn classes. We do this by choosing row representatives, which we call “standard elements” such that the Hecke algebra character table becomes the monoid character table under the specialization \( q \to 1 \). Moreover, we want these representatives to be of the form \( T_{rc} \), where \( rc \) is an element of the Munn class \( c \). This construction is done in the group case in [3, Section 8.2].

Let \( c \) be a Munn class of \( \mathcal{R} \). Notice that from the above formula, \( h \in \mathcal{H} \) will be a standard element if it is a standard element in \( \mathcal{H}(W_e) \). By [3, Definition 8.2.9], we can take \( h = T_{ew} \), where \( w \in W^*(e) \) is an element of minimal length in its conjugacy class (note that because \( W^*(e) \) is a parabolic subgroup, this element will indeed be contained in \( W^*(e) \)). With this definition, the above work shows:
Theorem 5.1.1. Let $R$ be a Renner monoid with $B$ matrix $B$. Let $Y$ be the block diagonal matrix $\text{diag}(J_n, \ldots, J_0)$, where the $J_n$ are the Hecke algebra character tables of the groups $W^\ast(e)$, $e \in \Lambda$, ordered by inclusion (i.e. $Y$ is the Hecke algebra version of the $Y$ matrix above). Then the character table of $\mathcal{H}(R)$ is $M = YB$.

5.2 Non-Standard Elements

Ideally, a character table computation would be accompanied by a protocol for calculating the character values of $T_r$ for all $r \in R$. Our results only give the character values for linear combinations of $T_{ew}$, $w \in W^\ast(e)$, $e$ an idempotent of $\mathcal{H}$. This is a substantial part of $\mathcal{H}$, but not nearly a complete description. An approach to this problem could possibly be to simulate [3, Section 8.2], but this appears to be significantly more complicated than in the group case.

5.3 The Type B case

We computed the $B$ matrix of $\text{RSp}_{2n}$ in Corollary 4.1.4. The Hecke algebra character tables for types A and $B$ are well-known; see for instance, [3, Sections 10.2-10.3]. Let $Y = \text{diag}(K_n, H_n, H_{n-1}, \ldots, H_1, H_0)$, where $K_n$ denotes the Hecke algebra character table of $B_n$, and $H_r$ denotes the Hecke algebra character table of $S_r$. Therefore, we can use Theorem 5.1.1 to compute the character table $M(\mathcal{H}(\text{RSp}_{2n}) = YB$ of $\mathcal{H}(\text{RSp}_{2n})$.

Example 5.3.1. In the case $R = \text{RSp}_4$, we obtain the following. Note that when we send $q \mapsto 1$ in $M$, we obtain the character table of $\text{RSp}_4$.

\[
Y = \begin{pmatrix}
1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\
q & q-1 & -1 & q & -1 & 0 & 0 & 0 \\
q^2 & -2q^2 & 1 & q^4 & q^2 & 0 & 0 & 0 \\
-1 & q-1 & -1 & q & q & 0 & 0 & 0 \\
-q & 0 & 1 & q^2 & -q & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
M(\mathcal{H}(\text{RSp}_{2n})) = YB = \begin{pmatrix}
(1^2,0) & (1,1) & (0,1^2) & (2,0) & (0,2) & (2) & (1^2) & (1) & (0) \\
1 & 2 & 1 & 1 & 1 & 4 & 4 & 4 & 1 \\
q & q-1 & -1 & q & -1 & 2q-2 & 2q-2 & 2q-2 & 3q-1 \\
q^2 & -2q^2 & 1 & q^4 & q^2 & q^4-q^2 & q^4-q^2 & q^4 & q^4 \\
-1 & q-1 & -1 & q & q & 3q-1 & q-3 & 2q-2 & q \\
-q & 0 & 1 & q^2 & -q & q^2-q & -q+1 & q^2-q & q^2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & q & -1 & q-1 & q \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}.
\]
Example 5.3.2. The following is the character table of \( S_4(R_2) \).

\[
\begin{pmatrix}
1 & 1 & 2 & 1 \\
q & -1 & q - 1 & q \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Under the decomposition \( M = AY \), we get

\[
A = \begin{pmatrix}
1 & 0 & 2 & 1 \\
0 & 1 & q - 1 & q \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

so \( A \) depends on \( q \), which justifies our \( B \) matrix strategy. We are, of course, able to calculate the \( A \) matrix once we have computed the character table.

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7 Appendix

7.1 A Note on the Definition $RSp_{2n}$

Li, Li, and Cao [8, Corollary 2.3] suggest an equivalent classification of the symplectic rook monoid as:

$$RSp_n = \{ A \in R_n \mid APA^t = A^tPA = 0 \text{ or } P \}$$

for $P$, $J_m$, and $n = 2m$ as in section [2.2]. Yet note that for $m = 1$, the symplectic rook monoid is the entirety of the rook monoid (see [3, p. 842]):

$$RSp_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

However note that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \implies APA^t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

indicating that the description given in the corollary is not accurate, and indeed, there is no immediate proof of this proposition in the same paper. Our Proposition [2.2.1] gives an alternate classification, which we show to be equivalent to

$$\{ A \in R_n \mid A^tJ_nA = A_1J_nA^t = 0 \quad \text{or} \quad A^tJ_nA = A_1J_nA^t = J_n \}.$$ 

Note that the alternate classification and the above both adhere to the size formula,

$$|RSp_{2n}| = 2^n n! + \sum_{i=0}^{n} \binom{n}{i} 2^i$$

so we believe that Theorem 2.2.1 is a correction of [9, Corollary 2.3]. Note that this does not pose a significant problem for that paper, as the authors mostly use their definition in terms admissible sets.

7.2 Character Table for $RSp_6$

Below we present the character table for the symplectic rook monoid, as calculated via the method of Li, Li, and Cao [9].

|     | C1 | C2 | C3 | C4 | C5 | C6 | C7 | C8 | C9 | C10 | (1^3) | (21) | (3) | (1^2) | (2) | (1) | (0) |
|-----|----|----|----|----|----|----|----|----|----|-----|-------|-----|----|-------|----|----|----|
| W1  | 1  | 1  | 1  | 1  | 2  | 2  | 3  | 3  | 3  | 8   | 16   | 8   | 12  | 12    | 6  | 1  |
| W2  | 1  | 1  | 1  | 1  | 2  | 2  | -1 | -1 | -1 | 1   | 0    | 0   | 0   | 0     | 0  | 2  | 1  |
| W3  | 1  | 1  | -1 | 1  | 1  | 1  | 1   | -1 | 1   | 0   | 0   | 2   | -2  | 0     | 1  |    |    |
| W4  | 1  | -1 | -1 | 1  | 0  | 0   | -1 | -1 | -1 | 1   | 0   | 2   | -2  | 2     | 1  |    |    |
| W5  | 1  | -1 | -1 | 1  | 0  | 0   | -1 | -1 | -1 | 1   | 0   | 4   | 4   | 4     | 1  |    |    |
| W6  | 1  | -1 | -1 | 1  | 0  | 0   | -1 | -1 | -1 | 1   | 0   | -4  | 2   | -2    | 2  | 1  |
| W7  | 1  | -1 | 1  | -1 | 1  | -1 | 0   | 0   | 0   | 0   | 0   | 0   | 0    | 0   | 1  |
| W8  | 1  | 1  | 1  | 1  | -1 | -1 | 0   | 0   | 2   | -2  | 0   | 0   | 0    | 1  |    |    |
| W9  | 1  | 1  | -1 | 1  | -1 | 0   | 0   | 1   | -1 | -1 | 0   | 0   | 0    | 0   | 0   | 2  |
| W10 | 1  | -1 | 1  | -1 | -1 | -1 | 0   | 0   | 2   | -2  | 0   | 0   | 0    | 0   | 1  |

Note that this table is symmetric under certain permutations of the columns, indicating the invariance of the character under these operations.
where \( C_i \) stands for the \( i \)th irreducible character indexed by a signed partition, \((\lambda, \mu)\), such that \(|\lambda| + |\mu| = n\) and \(W_i\) stands for the \( i \)th Munn class of the Weyl group, with the same indexing convention. By \( \emptyset \), we know that the Munn classes and irreducible representations of \( RSp_{2n} \) are both indexed by elements of \( Q_n \).

In this case, the Munn class representatives for the Weyl group are

\[
\{W_1, W_2, W_3\} = \{(0 0 0 0 0 0), (0 0 0 0 0 1), (0 0 0 1 0 0)\}
\]

\[
\{W_4, W_5, W_6\} = \{(0 0 0 1 0 0), (0 0 1 0 0 0), (0 0 0 1 0 0)\}
\]

\[
\{W_7, W_8, W_9\} = \{(0 0 0 0 0 1), (0 0 0 0 1 0), (0 0 0 1 0 0)\}
\]

\[
W_{10} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\{W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}\} = \\
\{(1 \ | \ 1^3), (1 \ | \ 1^2), (2 \ | \ 1), (0 \ | \ 1, 2), (1^2 \ | \ 1), (2, 1 \ | \ 0), (0 \ | \ 3), (1 \ | \ 2), (0 \ | \ 1^3)\}
\]

where the correspondence between \( 2n \times 2n \) matrices and signed partitions (also known as pairs of partitions) of \( n \) is explained in Section 2.2.

For our decomposition of \( M = YB = AY \) with \( Y \) the block diagonal matrix of character tables of \( B_3, S_3, \ldots \)
Given $S_2$, $S_1$, and $S_0$, we have

$$Y = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & -2 & 0 & 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 2 & 0 & -1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},$$

$$B = Y^{-1}M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},$$
and

\[
A = MY^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 12 & 0 & 6 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

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