THE RECTIFIED $n$-HARMONIC MAP FLOW WITH APPLICATIONS TO HOMOTOPY CLASSES

MIN-CHUN HONG

Abstract. We introduce a rectified $n$-harmonic map flow from an $n$-dimensional closed Riemannian manifold to another closed Riemannian manifold. We prove existence of a global solution, which is regular except for a finite number of points, of the rectified $n$-harmonic map flow and establish an energy identity for the flow at each singular time. Finally, we present two applications of the rectified $n$-harmonic map flow to minimizing the $n$-energy functional and the Dirichlet energy functional in a homotopy class.

1. Introduction

Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold without boundary, and let $(N, h)$ be another $m$-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^L$). The $n$-energy functional $E_n(u; M)$ of a map $u : (M, g) \to (N, h)$ is defined by

$$E_n(u; M) = \frac{1}{n} \int_M |\nabla u|^n \, dv.$$ 

A map $u$ from $M$ to $N$ is said to be an $n$-harmonic map if $u$ is a critical point of the $n$-energy functional; i.e. it satisfies

$$\text{div} \left[ |\nabla u|^{n-2} \nabla u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u) = 0 \quad \text{in} \, M,$$

where $A$ is the second fundamental form of $N$.

When $n = 2$, an $n$-harmonic map is a harmonic map. The fundamental question on harmonic maps, asked by Eells and Sampson [8] (see also [9]), is whether a given smooth map $u_0$ can be deformed to a harmonic map in its homotopy class $[u_0]$. Eells and Sampson [8] answered the question for the case that the sectional curvature of $N$ is non-positive by introducing the heat flow for harmonic maps. In order to solve the Eells-Sampson question, it is very important to establish global existence of the harmonic map flow. When $n = 2$, Struwe [25] proved global existence of the weak solution to the harmonic map flow, where the solution is smooth except for a finite set of singularities. Chang, Ding and Ye [1] constructed a counter-example that the harmonic map flow blows up at finite time. Ding and Tian [7] established the energy identity of the harmonic map flow at each blow-up time through a finite number of harmonic maps on $S^2$ (called bubbles). Qing and Tian [21] proved that as $t \to \infty$, there is no neck between a limit map $u_\infty$ and bubbles. Therefore, a given map $u_0$ can be deformed into a splitting sum of finite harmonic maps.

When $n > 2$, Chen and Struwe [4] showed global existence of a weak solution of the harmonic map flow, in which the weak solution is partially regular and has a complicated singular set. In general, it is difficult to apply the harmonic map flow to investigate the Eells-Sampson question. Motivated by the Eells-Simpson question,
it is interesting to ask whether a given map $u_0 \in C^\infty(M, N)$ can be deformed to an $n$-harmonic map in the homotopy class $[u_0]$. Related to this question, Hungerbuhler [18] investigated the $n$-harmonic map flow in the following equation:

$$
\frac{\partial u}{\partial t} = \text{div} \left[ |\nabla u|^{n-2} \nabla u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u)
$$

with initial value $u_0$, and generalized the result of Struwe [25] to prove that there exists a global weak solution $u : M \times [0, +\infty) \to N$ of (2) such that $u \in C^{1,\alpha}(M \times (0, +\infty) \setminus \{\Sigma_k \times T_k\}_{k=1}^L)$ for a finite number of singular times $\{T_k\}_{k=1}^L$ and a finite number of singular closed sets $\Sigma_k \subset M$ for $k = 1, \ldots, L$ with an integer $L$, depending only on $M$ and $u_0$. Chen, Cheung, Choi, Law [2] constructed a counter-example to show that the flow (2) blows up at finite time for $n = 3$. However, it has been an open question whether the singular set $\Sigma_k$ of the $n$-flow (2) at a singular time $T_k$ is finite. Without the finiteness of the singular set $\Sigma_k$, it is difficult to control the loss of the energy at the singular time $T_k$. In order to overcome this difficulty, we introduce a rectified $n$-harmonic map flow in the following equation:

$$
(1 + a|\nabla u|^{n-2}) \frac{\partial u}{\partial t} = \text{div} \left[ |\nabla u|^{n-2} \nabla u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u)
$$

with initial value $u(0) = u_0$ for a constant $a \in (0, 1]$. In particular, for $n = 2$, it is also the harmonic map flow.

In this paper, we firstly prove:

**Theorem 1.** For a constant $a \in (0, 1]$, there exists a global weak solution $u : M \times [0, +\infty) \to N$ of (3) with initial value $u_0$ in which there are finite times $\{T_k\}_{k=1}^L$ and finite singular points $\{x^{i,k}\}_{j=1}^{l_k}$ such that $u$ is regular in $M \times (0, +\infty) \setminus \{(x^{1,k}, \ldots, x^{l_k,k}) \times T_k\}_{k=1}^L$ and $\nabla u \in L^\infty(M \times (0, +\infty) \setminus \{(x^{1,k}, \ldots, x^{l_k,k}) \times T_k\}_{k=1}^L)$ in the following sense:

$$
u \in C^{0,\alpha}_{\text{loc}}(M \times (0, +\infty) \setminus \{(x^{1,k}, \ldots, x^{l_k,k}) \times T_k\}_{k=1}^L),$$

$$\nabla u \in L^\infty(M \times (0, +\infty) \setminus \{(x^{1,k}, \ldots, x^{l_k,k}) \times T_k\}_{k=1}^L).$$

As $t \to T_k$, $u(x, t)$ strongly converges to $u(x, T_k)$ in $W^{1,n+1}_{\text{loc}}(M \setminus \{(x^{1,k}, \ldots, x^{l_k,k})\})$.

Theorem 1 generalized the result of Struwe [25]. For the proof of Theorem 1, one of key ideas is to obtain an $\varepsilon$-regularity estimate by improving the delicate proof of Hungerbuhler in [18] for the case of $a = 0$ based on a variant of Moser’s iteration. Since the term $|\nabla u|^{n-2} \partial_t u$ in the flow (3) causes an extra difficulty, we have to carry out much more complicated analysis to obtain the boundedness of $|\nabla u|$ (see Lemma 2.4). We would like to point out that the Hölder continuity of the solution of the flow (3) is good enough to keep the topology of the solution in the homotopy class before the blow-up time.

Secondly, we prove

**Theorem 2.** Let $u : M \times [0, +\infty) \to N$ be a solution of (3) with initial value $u_0$ in Theorem 1. Let $T_k$ be the above singular time. Then, there are a finite number of $n$-harmonic maps $\{\omega_{i,k}\}_{i=1}^{m_k}$ (also called bubbles) on $S^n$ such that

$$
\lim_{t \to T_k} E_n(u(t); M) = E_n(u(\cdot, T_k); M) + \sum_{i=1}^{m_k} E_n(\omega_{i,k}, S^n).
$$

Theorem 2 generalized the result of Ding-Tian [7] from two-dimensional case to $n$-dimensional cases. For establishing the energy identity, Wang and Wei [28]
proved an energy identity for approximate \( n \)-harmonic maps by reducing multiple bubbles to a single bubble. In order to make proofs more clear, we give a new detailed procedure of bubble-neck decomposition based on the idea of Ding-Tian [7] and then prove the energy identity.

Next, we will present some applications of the related \( n \)-flow to minimizing the \( n \)-energy functional in a homotopy class \([u_0]\). When \( n = 2 \), Lemaire [19] and Schoen-Yau [24] established results of harmonic maps by minimizing the Dirichlet energy in a homotopy class under the topological condition \( \pi_2(N) = 0 \). In [22], Sacks and Uhlenbeck established many existence results of minimizing harmonic maps in their homotopy classes by introducing the ‘Sacks-Uhlenbeck functional’. Recently, the author and Yin [17] introduced the Sacks-Uhlenbeck flow on Riemannian surfaces to provide a new proof of the energy identity of a minimizing sequence in a homotopy class \([u_0]\). A similar result for the Yang-Mills \( \alpha \)-flow on 4-manifolds has been obtained by the author, Tian and Yin [15]. Extending the idea in [17] with applications of a rectified \( n \)-flow, we prove:

**Theorem 3.** For a homotopy class \([u_0]\), let \( \{u_k\}_{k=1}^{\infty} \) be a minimizing sequence of \( E_n \) in the homotopy class \([u_0]\) and \( u \) the weak limit in \( W^{1,n}(M, N) \). Then, there is a finite set \( \Sigma \) of singular points in \( M \) so that as \( k \to \infty \), \( u_k \) converges strongly to \( u \) in \( W^{1,n}_{loc}(M\setminus \Sigma, N) \) and there are a finite number of \( n \)-harmonic maps \( \{\omega_i\}_{i=1}^l \) on \( S^{n-1} \) such that

\[
\lim_{k \to \infty} E_n(u_k; M) = E_n(u_\infty; M) + \sum_{i=1}^l E_n(\omega_i, S^{n-1}).
\]

If \( \pi_n(N) = 0 \), the singular set \( \Sigma \) is empty and there is a minimizing map of the \( n \)-energy functional in the homotopy class \([u_0]\).

We would like to point out that Duzaar and Kuwert [6] studied the decomposition of a minimizing sequence of the \( n \)-energy functional in a homotopy class \([u_0]\) with \( N = S^1 \), which can be used to prove an energy identity for the minimizing sequence. Our proof is completely different from one in [6]. By a modification of the above \( n \)-harmonic flow, we follow the idea of the \( \alpha \)-flow [17] to rectify a new minimizing sequence \( \{u_{\varepsilon_k}\}_{k=1}^{\infty} \), having the same weak limit of the minimizing sequence \( \{u_k\}_{k=1}^{\infty} \) in the same homotopy class. More precisely, we extend the idea in [12] to approximate the \( n \)-energy functional \( E_n(u) \) by

\[
E_{n,\varepsilon}(u) = \int_M \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{n} |\nabla u|^n + \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) dv
\]

for a small constant \( \varepsilon > 0 \). Then we consider a modified gradient flow for the functional (4) in the following equation:

\[
(1 + a|\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1}) \frac{\partial u}{\partial t} = \text{div} \left[ (\varepsilon + |\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1}) \nabla u \right] + \varepsilon + |\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1}) A(u)(\nabla u, \nabla u)
\]

(5)

for a small constant \( a > 0 \). Choose the map \( u_k \) as an initial value, there is a sequence of \( \varepsilon = \varepsilon_k \to 0 \) such that the flow (5) has a unique global smooth solution \( u_{\varepsilon_k}(t) \) of (5) on \( M \times [0, \infty) \) with \( u_{\varepsilon_k}(0) = u_k \). There is a \( \tilde{t} \in [\frac{1}{4}, 1] \) such that \( u_{\varepsilon_k}(\tilde{t}) \) is a new minimizing sequence in the homotopy class \([u_0]\) and converges, up-to a subsequence, to the same limit map \( u \) in \( W^{1,n+1}_{loc}(M\setminus \{x_1, ..., x_1\}) \).
Furthermore, in order to prove the existence of a harmonic map in a given homotopy class $[u_0]$, it is a nature way to minimize the Dirichlet functional in the homotopy class. Indeed, there were successful results for $n = 2$, which were mentioned above ([19], [24] and [22]). In higher dimensions, it is very challenging to minimize the Dirichlet functional in a homotopy class. White [29] showed that if $d$ is the greatest integer strictly less than $p$, a homotopy equivalence is well defined for neighboring maps after restriction to the $d$-skeleton of $M$ and there exists a minimizer of the $p$-energy $E_p(u; M) = \frac{1}{p} \int_M |\nabla u|^p$ with prescribed $d$-homotopy type. White [29] raised an open problem about the partial regularity of the minimum solution of the $p$-energy with prescribed $d$-homotopy type. In particular, even for $p = 2$, the partial regularity theory of Schoen-Uhlenbeck [22] (also Giaquinta-Giusti [11]) on an energy minimizing map $u$ in $W^{1, 2}(M, N)$ cannot be applied since the Sobolev space $W^{1, 2}(M, N)$ cannot be approximated by smooth maps and a minimizing map of the Dirichlet in $W^{1, 2}(M, N)$ is not in the homotopy class.

Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence of the $p$-energy $E_p$ in the homotopy class $[u_0]$ for $2 \leq p \leq n$ and let $u \in W^{1, p}(M, N)$ be the weak limit of the minimizing sequence. Related to the above White problem, it is a very interesting problem whether the limit map $u$ is a weakly $p$-harmonic map and partial regular. Motivated by recent results of [12] and [16], we answer the question partially by applying a modified $n$-flow and prove:

**Theorem 4.** Let $p$ be a number with $2 \leq p \leq n$. Assume that $N$ is a homogeneous Riemannian manifold without boundary. For a given homotopy class $[u_0]$, let $\{u_i\}_{i=1}^{\infty}$ be a minimizing sequence of the $p$-energy $E_p(u; M)$ in the homotopy class $[u_0]$. Then, there is a subsequence of $\{u_i\}_{i=1}^{\infty}$ such that $u_i$ weakly converges to a weak $p$-harmonic map $u$. Moreover, $u$ belongs to $C^{1, p}(M \setminus \Sigma, N)$ for a closed singular set $\Sigma \subset M$ and $\mathcal{H}^n-p(\Sigma) < \infty$, where $\mathcal{H}^n-p$ denotes the Hausdorff measure.

For proving Theorem 4, we employ a perturbation of the $p$-energy functional in the homotopy class and its gradient flow. This kind perturbation of the Dirichlet functional was used by Uhlenbeck in [27] to reprove Eells-Sampson’s result, and was employed by Giaquinta, the author and Yin [12] for proving partial regularity of the relaxed functional of harmonic maps and also by the author and Yin [17] for proving partial regularity of the relaxed functional of bi-harmonic maps.

The paper is organised as follows. In Section 2, we establish some basic estimates and global existence of weak solutions to the rectified $n$-flow. In Section 3, we prove the energy identity at a singular time and finish a proof of Theorem 1. In Section 4, we prove Theorem 3. In Section 5, we finish a proof of Theorem 4.

### 2. Some estimates and global existence

In local coordinates, the Riemannian metric $g$ on $M$ can be represented by

$$g = g_{ij} dx^i \otimes dx^j$$

with a positive definite symmetric $n \times n$ matrix $(g_{ij})$. The volume element $dv$ of $(M; g)$ is defined by

$$dv = \sqrt{|g|} dx \quad \text{with} \quad |g| = \det (g_{ij}).$$
The rectified $n$-harmonic map flow

The gradient norm $|\nabla u|$ is given by

$$|
abla u(x)|^2 = \sum_{\alpha, \beta, i, j} g^{ij}(x) h_{\alpha \beta}(u(x)) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j},$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix of $(g_{ij})$. A smooth map $u$ from $M$ to $N$ is called an $n$-harmonic map if it satisfies

$$\nabla u = 0.$$

Moreover, we have the following local energy's inequality:

$$\int_M \left( \frac{\epsilon}{2} |
abla u(s)|^2 + \frac{1}{n} |\nabla u(s)|^n \right) dv + \int_0^s \int_M (1 + a|\nabla u|^{n-2}) \left( \frac{\partial u}{\partial t} \right)^2 dv dt$$

$$= \int_M \frac{\epsilon}{2} |\nabla u_0|^2 + \frac{1}{n} |\nabla u_0|^n dv.$$

Moreover, we have the following local energy's inequality:

**Lemma 2.1.** Let $u(t)$ be a smooth solution to the flow (9) in $M \times [0, T)$ with initial value $u(0) = u_0$. Then for each $s$ with $0 < s < T$, we have

$$\int_{B_{2R}(x_0)} e_\epsilon(u)(\cdot, \tau) dV + \int_s^\tau \int_M (1 + a|\nabla u|^{n-2}) |\partial_t u|^2 dv dt$$

$$\leq \int_{B_{2R}(x_0)} e_\epsilon(u)(\cdot, s) dV + \frac{C(\tau - s) \epsilon}{R^2} \int_M e_\epsilon(u_0) dv.$$
and

\begin{equation}
\int_{B_{R}(x_0)} e_\varepsilon(u)(\cdot, s) \, dv - \int_{B_{2R}(x_0)} e_\varepsilon(u)(\cdot, \tau) \, dv \\
\leq C \int_s^\tau \int_M (1 + a|\nabla u|^n-2)|\partial_\tau u|^2 \, dv \, dt \\
+ C \left( \frac{(\tau - s)}{R^2} \int_M e_\varepsilon(\varphi_0) \, dv \right) \int_s^\tau \int_M (1 + |\nabla u|^n-2)|\partial_\tau u|^2 \, dv \, dt \right)^{1/2}. 
\end{equation}

Proof. Let \( \varphi \) be a cut-off function with support in \( B_{2R}(x_0) \) and \( \varphi \equiv 1 \) on \( B_R(x_0) \) with \( |\nabla \varphi| \leq C/R \). Then

\[ \frac{d}{dt} \int_M \varphi^2 e_\varepsilon(u) \, dv = \int_M \varphi^2 \left( \varepsilon + |\nabla u|^n-2 \right) \nabla u \cdot \nabla \left[ \frac{1}{\partial t} \right] \, dv \\
= - \int_M \varphi^2 (1 + a|\nabla u|^n-2) \left| \frac{\partial u}{\partial t} \right|^2 \\
+ \int_M \varphi (\varepsilon + |\nabla u|^n-2) \nabla u \cdot \nabla \varphi \, \frac{\partial u}{\partial t} \, dv. \]

(11) follows from integrating in \( t \) over \( [s, \tau] \) and using Young’s inequality. Similarly, we have (12). \( \square \)

Lemma 2.3. Let \( u : M \to N \) be a smooth solution to the flow equation (9). Set \( e_\varepsilon(u) = \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{n} |\nabla u|^n \). There is a small constant \( \varepsilon_0 > 0 \) such that if the inequality

\[ \sup_{0 \leq t \leq T} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^n \, dv \, dt \leq \varepsilon_0 \]

holds for some positive \( R_0 \), then we have

\begin{equation}
\int_0^T \int_{B_{R_0}(x_0)} |\nabla u|^{n+2} + |\nabla^2 u|^2 \varepsilon + |\nabla u|^n \, dv \, dt \\
\leq C(1 + TR_0^{-2})E_\varepsilon(u_0),
\end{equation}

where the constant \( C \) does not depend on \( \varepsilon \) and \( u \).

Proof. In a neighborhood of each point \( x_0 \in M \), we can choose an orthonormal frame \( \{e_i\}_i \). We denote by \( \nabla_i \) the first covariant derivative with respect to \( e_i \) and by \( \nabla^2_{ij} \) the second covariant derivatives of \( u \) and so on.

Let \( \phi \) be a cut-off function with support in \( B_{2R_0}(x_0) \) such that \( \phi = 1 \) in \( B_{R_0}(x_0) \), \( |\nabla \phi| \leq CR_0^{-1} \) and \( |\phi| \leq 1 \) in \( B_{2R_0}(x_0) \). Multiplying (3) by \( \phi^n \Delta u \), we have

\begin{equation}
\int_{B_{2R_0}(x_0)} \left( \varepsilon + |\nabla u|^n \right) \nabla_k \left( \nabla_k u \right) \phi^n \, dv \\
- \int_{B_{2R_0}(x_0)} \left( 1 + a |\nabla u|^n \right) \left( \partial_\tau u, \Delta u \right) \phi^n \, dv \\
= \int_{B_{2R_0}(x_0)} \left( \varepsilon + |\nabla u|^n \right) A(u) \left( \nabla u, \nabla u \right) \phi^n \, dv
\end{equation}
In order to estimate the first term of the left-hand side of (14), it follows from the well-known Ricci identity that
\[ \nabla_k \nabla_l \left( (\varepsilon + |u|^{-2}) \nabla u \right) = \nabla_l \nabla_k \left( (\varepsilon + |u|^{-2}) \nabla u \right) + R_M \left( (\varepsilon + |u|^{-2}) \nabla u \right) \]
with the Riemannian curvature \( R_M \).

Then, integrations by parts twice yield that
\[
\int_{B_{2R_0}(x_0)} \left\langle \nabla_k (\varepsilon + |u|^{-2}) \nabla_k u, \Delta u \right\rangle \phi^n \, dv = \int_{B_{2R_0}(x_0)} \left\langle \nabla_l (\varepsilon + |u|^{-2}) \nabla_k u, \nabla_k \nabla_l u \right\rangle \phi^n \, dv - \int_{B_{2R_0}(x_0)} \left\langle \nabla_k (|u|^{-2}) \nabla_k u, \nabla_l \phi^n \right\rangle \, dv + \int_{B_{2R_0}(x_0)} \left\langle \nabla_l (\varepsilon + |u|^{-2}) \nabla_k u, \nabla_k \phi^n \right\rangle \, dv + \int_{B_{2R_0}(x_0)} \left\langle R_M \left( (\varepsilon + |u|^{-2}) \nabla_k u \right), \nabla_l \phi^n \right\rangle \, dv \\
\geq \frac{3}{4} \int_{B_{2R_0}(x_0)} (\varepsilon + |u|^{-2}) |\nabla^2 u|^2 \phi^n \, dv + \frac{n - 2}{2} \int_{B_{2R_0}(x_0)} (\varepsilon + |u|^{-2}) |\nabla|\nabla u||^2 \phi^n \, dv - C \int_{B_{2R_0}(x_0)} (\varepsilon + |u|^{-2}) |\nabla u|^2 \phi^{n-2} (\phi^2 + |\phi|^2) \, dv.
\]

In order to estimate the second term of the left-hand side of (14), it follows from integrating by parts and using Young’s inequality that
\[
\int_{B_{2R_0}(x_0)} \left\langle (1 + a|u|^{-2}) \partial_t u, \Delta u \right\rangle \phi^n \, dv = \frac{d}{dt} \int_{B_{2R_0}(x_0)} \left( \frac{1}{2} \nabla u^2 + \frac{a}{n} |u|^n \phi^n \right) \, dv - a \int_{B_{2R_0}(x_0)} \nabla_k (|u|^{-2}) \partial_t u \nabla_k \phi \, dv - n \int_{B_{2R_0}(x_0)} (1 + a|u|^{-2}) \partial_t u \nabla_k u \phi^{n-1} \nabla_k \phi \, dv \\
\leq \frac{d}{dt} \int_{B_{2R_0}(x_0)} \left( \frac{1}{2} \nabla u^2 + \frac{a}{n} |u|^n \phi^n \right) \, dv + C \int_{B_{2R_0}(x_0)} (1 + a|u|^{-2}) |\partial_t u|^2 \phi^n \, dv + \frac{n - 2}{4} \int_{B_{2R_0}(x_0)} |\nabla u|^{n-2} |\nabla (|\nabla u|)|^2 \phi^n \, dv + C \int_{B_{2R_0}(x_0)} (1 + a|u|^{-2}) |\nabla u|^2 |\nabla \phi| \phi^{n-2} \, dv.
Combining above inequalities, we obtain
\begin{equation}
\frac{d}{dt} \int_{B_{2R_0}(x_0)} \left( \frac{1}{2} |\nabla u|^2 + \frac{a}{n} |\nabla u|^n \right) \phi^n \, dv \\
+ \frac{1}{2} \int_{B_{2R_0}(x_0)} |\nabla^2 u|^2 (\varepsilon + |\nabla u|^{n-2}) \phi^n \, dv \\
\leq C \int_{B_{2R_0}(x_0)} |\nabla u|^2 (\varepsilon |\nabla u|^2 + |\nabla u|^n) \phi^n \, dv \\
+ C \int_{B_{2R_0}(x_0)} (1 + |\nabla u|^n) \phi^{n-2} (\phi^2 + |\nabla \phi|^2) \, dv \\
+ C \int_{B_{2R_0}(x_0)} (1 + a |\nabla u|^{n-2}) |\partial_t u|^2 \phi^n \, dv.
\end{equation}

By applying the Hölder and Sobolev inequalities, we have
\begin{align*}
\int_0^T \int_{B_{2R_0}(x_0)} |\nabla u|^{n+2} \phi^n \, dv \, dt \\
\leq \left( \sup_{0 \leq t \leq T} \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv \right)^{\frac{2}{n+2}} \int_0^T \left( \int_{B_{2R_0}(x_0)} |\nabla u|^{\frac{2n}{n+2}} \phi^\frac{n}{n+2} \, dv \right)^{\frac{n+2}{n}} \, dt \\
\leq C \varepsilon_0^{\frac{2}{n+2}} \int_0^T \int_{B_{2R_0}(x_0)} \|\nabla (|\nabla u|^{n/2} \phi^{n/2})\|^2 \, dv \, dt \\
\leq C \varepsilon_0^{\frac{2}{n+2}} \int_0^T \int_{B_{2R_0}(x_0)} (|\nabla^2 u|^2 |\nabla u|^{n-2} \phi^n + \frac{1}{R_0^2} |\nabla u|^n) \, dv \, dt.
\end{align*}

Integrating (15) in $t$ over $[0, T]$, choosing $\varepsilon_0$ sufficiently small and Lemma 2.1, we have
\begin{align*}
\int_0^T \int_{B_{2R_0}(x_0)} |\nabla u|^{n+2} + |\nabla^2 u|^2 (\varepsilon + |\nabla u|^{n-2}) \, dv \, dt \\
\leq C \int_{B_{2R_0}(x_0)} \left( \frac{1}{2} |\nabla u|^2 + \frac{a}{n} |\nabla u|^n \right)(x, 0) \, dv \\
+ C(1 + \frac{1}{R_0^2}) \int_0^T E(x; B_{2R_0}(x_0)) \, dt \\
\leq C(1 + T + \frac{T}{R_0^2}) E(x_0).
\end{align*}

This proves the claim. \qed

For $R > 0$ and $x_0 = (x_0, t_0) \in M \times (0, \infty)$, we denote
\[ P_R(x_0) = \{ z = (x, t) : |x - x_0| < R, t_0 - R^2 < t \leq t_0 \} \cdot \]

**Lemma 2.4.** Let $u_\varepsilon$ be a smooth solution to (9). For any $\beta \geq 1$, there exists a positive constant $\varepsilon_1$ (depending on $\beta$) such that if for some $R_0$ with $0 < R_0 < \min\{\varepsilon_1, \frac{1}{\beta^2}\}$ the inequality
\[ \sup_{t_0 - 4R_0^2 \leq t \leq t_0} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^n \, dv \leq \varepsilon_1 \]
holds, then we have

\[
\int_{t_0 - R_0^2}^{t_0} \int_{B_R_0(x_0)} |\nabla u|^{n+2+\beta} + |\nabla^2 u|^2 (\varepsilon + |\nabla u|^{n-2+\beta}) \, dv \, dt \\
\leq CR_0^4 + C \int_{t_0 - 4R_0^2}^{t_0} \int_{B_{2R_0}(x_0)} (1 + R_0^{-2}) |\nabla u|^{n+\beta} \, dv \, dt,
\]

where the constant \( C \) does not depend on \( \varepsilon, u \) and \( a \).

**Proof.** In a neighborhood of each point \( x_0 \in M \), we still denote by \( \nabla_i \) the first covariant derivative with respect to \( e_i \) and by \( \nabla_{ji}^2 u \) the second covariant derivatives of \( u \) and so on.

Let \( \phi = \phi(x, t) \) be a cut-off function with support in \( B_{R_0}(x_0) \times [t_0 - 4R_0^2, t_0 + 4R_0^2] \) such that \( \phi = 1 \) in \( B_{R_0}(x_0) \times [t_0 - R_0^2, t_0 + R_0^2] \), \(|\nabla \phi| \leq C/R_0\), \(|\partial_t \phi| \leq \frac{1}{R_0} \) and \(|\phi| \leq 1 \) in \( B_{R_0}(x_0) \times [t_0 - 4R_0^2, t_0 + 4R_0^2] \).

Multiplying (9) by \( \phi^n |\nabla u|^{\beta} \partial_t u \) and integrating by parts, we have

\[
\int_{P_2R_0(x_0, t_0)} (1 + a |\nabla u|^{n-2}) |\nabla u|^{\beta} |\partial_t u|^2 \phi^n \, dv \, dt \\
= \int_{P_2R_0(x_0, t_0)} (\langle |\nabla u|^{n-2} \rangle \nabla u, |\nabla u|^{\beta} \nabla_k (\partial_t u) \rangle \phi^n \, dv \, dt \\
- \int_{P_2R_0(x_0, t_0)} (\langle |\nabla u|^{n-2} \rangle \nabla_k u, \beta |\nabla u|^{\beta-1} \nabla_k (|\nabla u|) \partial_t u \rangle \phi^n \, dv \, dt \\
- \int_{P_2R_0(x_0, t_0)} (\langle |\nabla u|^{n-2} \rangle \nabla_k u, |\nabla u|^{\beta} \partial_t u \rangle \nabla_k \phi^n \, dv \, dt \\
+ \int_{P_2R_0(x_0, t_0)} (\langle |\nabla u|^{n-2} \rangle A(u)(\nabla u, \nabla u), |\nabla u|^{\beta} \partial_t u \rangle \phi^n \, dv \, dt \\
\leq \int_{B_{R_0}(x_0, t_0)} \left( \frac{\varepsilon}{2 + \beta} |\nabla u|^{2+\beta} + \frac{1}{n + \beta} |\nabla u|^{n+\beta} \right) \phi^n (\cdot, t_0) \, dv \\
+ \int_{P_2R_0(x_0, t_0)} \left( \frac{\varepsilon}{2 + \beta} |\nabla u|^{2+\beta} + \frac{1}{n + \beta} |\nabla u|^{n+\beta} \right) n \phi^{n-1} \partial_t \phi \, dv \, dt \\
+ \frac{a}{2} \int_{P_2R_0(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta} |\partial_t u|^2 \phi^n \, dv \, dt \\
+ \frac{\beta^2}{2a} \int_{P_2R_0(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta-2} |\nabla_k u \nabla_k (|\nabla u|) |^2 \phi^n \, dv \, dt \\
+ C \int_{P_2R_0(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta+1} \phi^{n-1} (\phi |\nabla u| + |\nabla \phi|) \partial_t u \, dv \, dt.
\]
Then it follows from using Young’s inequality and (17) that

\begin{align*}
(17) \quad a^2 \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^\beta |\partial_t u|^2 \phi^n \, dv \, dt \\
\leq \beta^2 \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^{\beta-2} |\nabla_k u \nabla_k (|\nabla u|)|^2 \, dv \, dt \\
+ Ca \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^{\beta} \phi^{n-1} (|\phi| |\nabla u|^2 + |\nabla u| |\nabla \phi|) |\partial_t u| \, dv \, dt \\
+C a \frac{1}{\beta} \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^{\beta+2} \phi^{n-1} |\partial_t \phi| \, dv \, dt.
\end{align*}

Then it follows from using Young’s inequality and (17) that

\begin{align*}
(18) \quad -a(n-2) \int_{P_{2R_0}(x_0, t_0)} \langle |\nabla u|^{n-3} \nabla_i (|\nabla u|) \partial_t u, |\nabla u|^{\beta} \nabla_i u \rangle \phi^n \, dv \, dt \\
\leq (n-2) \int_{P_{2R_0}(x_0, t_0)} |\nabla u|^{n-2+\beta} \left( \frac{a^2 |\partial_t u|^2}{2\beta} \right) + \frac{\beta}{2} |\nabla_i (|\nabla u|) \nabla_i u|^2 |\nabla u|^{-2} \phi^n \, dv \, dt \\
\leq \beta(n-2) \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^{\beta-2} |\nabla_i (|\nabla u|) \nabla_i u|^2 \phi^n \, dv \, dt \\
+C a \frac{1}{\beta} \int_{P_{2R_0}(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})|\nabla u|^{\beta+1} \phi^{n-1} [(|\phi| |\nabla u| + |\nabla \phi|) |\partial_t u| + \frac{1}{\beta} |\partial_t \phi|] \, dv \, dt.
\end{align*}

Multiplying (3) by $\phi^n \nabla \cdot (|\nabla u|^{\beta} \nabla u)$ and integrating by parts, we have

\begin{align*}
(19) \quad &\int_{P_{2R_0}(x_0, t_0)} \langle \nabla_k (\varepsilon + |\nabla u|^{n-2}) \nabla_k u, \nabla_i (|\nabla u|^{\beta} \nabla_i u) \rangle \phi^n \, dv \, dt \\
= &\int_{P_{2R_0}(x_0, t_0)} \langle (1 + a |\nabla u|^{n-2}) \partial_t u, \nabla_i (|\nabla u|^{\beta} \nabla_i u) \rangle \phi^n \, dv \, dt \\
- &\int_{P_{2R_0}(x_0, t_0)} \langle (\varepsilon + |\nabla u|^{n-2}) A(u) (\nabla u, \nabla u), \nabla_i (|\nabla u|^{\beta} \nabla_i u) \rangle \phi^n \, dv \, dt \\
= &\int_{P_{2R_0}(x_0, t_0)} \langle (1 + a |\nabla u|^{n-2}) \partial_t u, \nabla_i (|\nabla u|^{\beta} \nabla_i u) \rangle \phi^n \, dv \, dt \\
+ &\int_{P_{2R_0}(x_0, t_0)} \langle \nabla_i [(\varepsilon + |\nabla u|^{n-2}) A(u) (\nabla u, \nabla u)]_i, |\nabla u|^{\beta} \nabla_i u \rangle \phi^n \, dv \, dt \\
+ &\int_{P_{2R_0}(x_0, t_0)} \langle (\varepsilon + |\nabla u|^{n-2}) A(u) (\nabla u, \nabla u), |\nabla u|^{\beta} \nabla_i u \nabla_i (\phi^n) \rangle \, dv \, dt.
\end{align*}

The second term of the right-hand side of (19) is a good one, but we need to analyze the first term of the right-hand side of (19). In order to estimate the term, using equation (9), we note that

$$a |\partial_t u| \leq C(|\nabla^2 u| + |\nabla u|^2).$$
Then, by integrating by parts, we have

\begin{equation}
\int_{P_{2R_0}(x_0,t_0)} \langle (1 + a|\nabla u|^{n-2}) \partial_t u, \nabla_i (|\nabla u|^\beta \nabla_l u) \rangle \phi^n \, dv \, dt \\
= -a(n-2) \int_{P_{2R_0}(x_0,t_0)} \langle |\nabla u|^{n-3} \nabla_l (|\nabla u|) \partial_t u, |\nabla u|^\beta \nabla_l u \rangle \phi^n \, dv \, dt \\
- \int_{P_{2R_0}(x_0,t_0)} \langle (1 + a|\nabla u|^{n-2}) \nabla_i (\partial_t u), |\nabla u|^\beta \nabla_l u \rangle \phi^n \, dv \, dt \\
- \int_{P_{2R_0}(x_0,t_0)} \langle (1 + a|\nabla u|^{n-2}) \partial_t u, |\nabla u|^\beta \nabla_l u \rangle \nabla_l (\phi^n) \, dv \, dt.
\end{equation}

\begin{align*}
\leq & \beta(n-2) \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta-2} \nabla_l (|\nabla u|) \nabla_l u|^2 \phi^n \, dv \, dt \\
& - \int_{B_{2R_0}(x_0,t_0)} \left( \frac{1}{2 + \beta} |\nabla u|^{2+\beta} + \frac{a}{n+\beta} |\nabla u|^{n+\beta} \right) \phi^n(\cdot,t_0) \, dv \\
& + \int_{P_{2R_0}(x_0,t_0)} \left( \frac{1}{2 + \beta} |\nabla u|^{2+\beta} + \frac{a}{n+\beta} |\nabla u|^{n+\beta} \right) \phi^{n-1} \partial_t \phi \, dv \, dt \\
& - \int_{P_{2R_0}(x_0,t_0)} \langle (1 + a|\nabla u|^{n-2}) \partial_t u, |\nabla u|^\beta \nabla_l u \rangle \nabla_l (\phi^n) \, dv \, dt \\
& + C \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta+1} \phi^{n-2} \left[ \frac{1}{\beta^2} |\nabla \phi|^2 + \frac{1}{\beta^2} |\partial_t \phi| \right] \, dv \, dt \\
& + \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta} \left( \frac{1}{4} |\nabla^2 u|^2 + C |\nabla u|^4 \right) \phi^n \, dv \, dt.
\end{align*}

Integrating by parts twice and using the Ricci formula, we estimate the first term of the left-hand side of (18) to obtain that

\begin{equation}
\int_{P_{2R_0}(x_0,t_0)} \langle \nabla_k ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), \nabla_l (|\nabla u|^\beta \nabla_l u) \rangle \phi^n \, dv \, dt \\
= \int_{P_{2R_0}(x_0,t_0)} \langle \nabla_l ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), \nabla_k (|\nabla u|^\beta \nabla_l u) \rangle \phi^n \, dv \, dt \\
+ \int_{P_{2R_0}(x_0,t_0)} \langle R_M \# ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), |\nabla u|^\beta \nabla_l u \rangle \phi^n \, dv \, dt \\
+ \int_{P_{2R_0}(x_0,t_0)} \langle \nabla_l ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), |\nabla u|^\beta \nabla_l u \rangle \nabla_k \phi^n \, dv \, dt \\
- \int_{P_{2R_0}(x_0,t_0)} \langle \nabla_k ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), |\nabla u|^\beta \nabla_l u \rangle \nabla_l \phi^n \, dv \, dt.
\end{equation}
Moreover, we note that

\[
\int_{P_{2R_0}(x_0,t_0)} \langle \nabla_i ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), \nabla_k (|\nabla u|^2 \nabla_i u) \rangle \phi^n \, dv \, dt
\]

\[
= \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^2 |\nabla^2 u|^2 \phi^n \, dv \, dt
\]

\[
+(n-2+\beta) \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta} |\nabla (|\nabla u|)|^2 \phi^n \, dv \, dt
\]

\[
+\beta(n-2) \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{|\nabla l|} |\nabla_i | |\nabla_i u|^2 \phi^n \, dv \, dt
\]

and

\[
\int_{P_{2R_0}(x_0,t_0)} \langle \nabla_i ((\varepsilon + |\nabla u|^{n-2}) \nabla_k u), |\nabla u|^\beta \nabla_i u \rangle \nabla_k \phi^n \, dv \, dt
\]

\[
= \int_{P_{2R_0}(x_0,t_0)} \langle \nabla_i (|\nabla u|^{n-2}) \nabla_k u, |\nabla u|^\beta \nabla_i u \rangle \nabla_k \phi^n \, dv \, dt
\]

\[
+\frac{1}{2} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta} |\nabla_k | |\nabla u|^2 \nabla_k \phi^n \, dv \, dt
\]

\[
\leq \frac{n-2+\beta}{2} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{\beta} |\nabla (|\nabla u|)|^2 \phi^n \, dv \, dt
\]

\[
+\frac{C}{n-2+\beta} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{|\nabla l|} |\nabla_i | |\nabla_i u|^2 \phi^n \, dv \, dt
\]

Combining (19)-(22) with (23), we have

\[
\frac{1}{2} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla^2 u|^2 \phi^n \, dv \, dt
\]

\[
+\frac{(n-2+\beta)}{2} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla (|\nabla u|)|^2 \phi^n \, dv \, dt
\]

\[
+\int_{B_{2R_0}(x_0)} \left( \frac{1}{2+\beta} |\nabla u|^{2+\beta} + \frac{1}{n+\beta} |\nabla u|^{n+\beta} \right) \phi^n \, dv \, dt
\]

\[
\leq C \int_{P_{2R_0}(x_0,t_0)} \left( \frac{1}{2+\beta} |\nabla u|^{2+\beta} + \frac{1}{n+\beta} |\nabla u|^{n+\beta} \right) \phi^n \, dv \, dt
\]

\[
+ C \frac{1}{\beta} \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^{|\nabla l|} |\nabla_i | |\nabla_i u|^2 \phi^n \, dv \, dt
\]

\[
+ C \int_{P_{2R_0}(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) (|\nabla u|^{2+\beta} + |\nabla u|^{4+\beta}) \phi^n \, dv \, dt
\]

where the third term of the right hand side of (19),(20) and the last term of the right-hand side of (22) are canceled by using equation (9).
On the other hand, by using the Hölder and Sobolev inequalities, we have
\[
\int_{t_0-4R_0}^{t_0} \int_{B_{2R_0}(x_0)} |\nabla u|^{n+2+\beta} \phi^n dv \, dt \\
\leq \int_{t_0-4R_0}^{t_0} \left( \int_{B_{2R_0}(x_0)} |\nabla u|^n dv \right) \left( \int_{B_{2R_0}(x_0)} |\nabla u|^{n+2+\beta} \phi^n dv \right)^{\frac{n}{n+2}} dt \\
\leq C \varepsilon \int_{t_0-4R_0}^{t_0} \int_{B_{2R_0}(x_0)} |\nabla |\nabla u|^{n+2+\beta} + (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla^2 u|^2 \phi^n dv \, dt \\
\leq C \int_{B_{2R_0}(x_0)} (1 + |\nabla \phi|^2 + |\partial_t \phi|) |\nabla u|^{n+\beta} dv \, dt.
\]

Choosing \( \varepsilon_1 \) (depending on \( \beta \) here) sufficiently small yields
\[
\int_{t_0-4R_0}^{t_0} \int_{B_{2R_0}(x_0)} (|\nabla u|^{n+2+\beta} + (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla^2 u|^2) \phi^n dv \, dt \\
\leq C \int_{B_{2R_0}(x_0)} (1 + |\nabla \phi|^2 + |\partial_t \phi|) |\nabla u|^{n+\beta} dv \, dt.
\]

This proves our claim. \( \square \)

**Lemma 2.5.** Let \( u_\varepsilon \) be a smooth solution to (9). There exists a positive constant \( \varepsilon_0 < i(M) \) such that if for some \( R_0 \) with \( 0 < R_0 < \min\{\varepsilon_0, \frac{1}{M} \} \) the inequality
\[
\sup_{t_0-4R_0 \leq t < t_0} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^n dv \leq \varepsilon_0
\]
holds, we have
\[
\sup_{P_{R_0}(x_0,t_0)} |\nabla u_\varepsilon|^n \leq CR_0^{-n},
\]
where \( C \) is a constant independent of \( \varepsilon \) and \( R_0 \).

**Proof.** Let \( \phi = \phi(x,t) \) be a cut-off function with support in \( B_R(x_0) \times [t_0-R^2,t_0+R^2] \) such that \( \phi = 1 \) in \( B_R(x_0) \times [t_0 - \rho^2, t_0 + \rho^2] \), \( |\nabla \phi| \leq \frac{\rho}{R^2}, |\partial_t \phi| \leq \frac{1}{(R^2-\rho)}, \) and \( \phi \leq 1 \) in \( B_R(x_0) \times [t_0 - R^2, t_0 + R^2] \). From the proof of Lemma 2.4, we have

\[ \begin{align*}
\int_{P_R(x_0,t_0)} &\left( \varepsilon + |\nabla u|^{n-2} |\nabla u|^\beta |\nabla^2 u|^2 \phi^n \right) dv \, dt \\
&+ \frac{(n-2+\beta)}{2} \int_{P_R(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla (|\nabla u|)|^2 \phi^n dv \, dt \\
&+ \sup_{t_0-R^2 \leq s \leq t_0} \int_{B_R(x_0)} \left( \frac{1}{2+\beta} |\nabla u|^{2+\beta} + \frac{1}{n+\beta} |\nabla u|^{n+\beta} \phi^{n-1} |\partial_t \phi| dv \right) ds \\
&\leq C \int_{P_R(x_0,t_0)} \left( \frac{1}{2+\beta} |\nabla u|^{2+\beta} + \frac{1}{n+\beta} |\nabla u|^{n+\beta} \phi^{n-1} |\partial_t \phi| dv \right) \, dt \\
&+ C \int_{P_R(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2}) |\nabla u|^\beta |\nabla^2 u|^2 \phi^n dv \, dt \\
&+ C \int_{P_R(x_0,t_0)} (\varepsilon + |\nabla u|^{n-2})(|\nabla u|^{2+\beta} + |\nabla u|^{4+\beta}) \phi^n dv \, dt.
\end{align*} \]
Lemma 2.6. Let $u_\varepsilon : M \to N$ be a smooth solution to the equation (9). There is a small constant $\varepsilon_0 > 0$ such that if the inequality
\[ \sup_{t_0 - T \leq t < t_0} \int_{B_{2R_0}(x_0)} |\nabla u_\varepsilon|^n \, dv \, dt \leq \varepsilon_0 \]
holds for some positive $R_0$, then $\|u_\varepsilon\|_{C^{0, \alpha}(P_{R_0}(z_0))}$ is uniformly bounded in $\varepsilon$. 

Using Hölder’s and Sobolev’s inequalities, we have
\[
\int_{t_0 - \rho^2}^{t_0} \int_{B_\rho(x_0)} |\nabla u|^{(n+\beta)(1+\frac{2+\beta}{2n})} \, dv \, dt \\
\leq \int_{t_0 - \rho^2}^{t_0} \left( \int_{B_R(x_0)} |\nabla u|^\beta \phi^n \, dv \right)^{\frac{\beta}{\phi}} \left( \int_{B_R(x_0)} |\nabla u|^{\frac{n(n+\beta)}{n-2}} \phi^{\frac{n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \, dt \\
\leq C \sup_{t_0 - R^2 \leq t \leq t_0} \left( \int_{B_R(x_0)} |\nabla u|^\beta \phi^n \, dv \right) \int_{t_0 - R^2}^{t_0} \int_{B_R(x_0)} |\nabla u|^{\frac{n(n+\beta)}{n-2}} \phi^{\frac{n}{n-2}} \, dv \, dt \\
\leq C \left( \int_{P_R(x_0, t_0)} (|\nabla u|^{2+\beta} + |\nabla u|^{n+\beta}) \phi^{n-2}(|\partial_\theta \phi| + |\phi|^2) \, dv \, dt \\
+ \beta \int_{P_R(x_0, t_0)} (\varepsilon + |\nabla u|^{n-2})(|\nabla u|^{2+\beta} + |\nabla u|^{n+\beta}) \phi^n \, dv \, dt \right)^{1+2/n} \\
\]

Next, we follow [18] to process a Moser’s iteration. Set $R = R_k = R_0(1+2^{-k})$, $\rho = R_{k+1} = R_0(1+2^{-1-k})$, $\beta = \beta_k = \theta_k(1-2n) + n-2$ and $\theta = 1+2/n$ with $d_0 > 2n$.

$d_k = n + \beta_k + 2 = \theta_k(1-2n) + 2n, \ d_{k+1} = (n + \beta_k) \left( 1 + \frac{2\beta_k + 2}{n\beta_k + n} \right) = \theta d_k - 4$.

Then
\[
\int_{P_{k+1}} |\nabla u|^{d_{k+1}} \, dv \, dt \leq C4^{d_k} \left( \int_{P_k} (1 + |\nabla u|^{d_k}) \right)^{\frac{d_k}{\phi}} \\
\]

Set
\[
I_k = \left( \int_{P_k} 1 + |\nabla u|^{d_k} \right)^{\frac{1}{d_k}} \\
\]

Applying an iteration, we have
\[
I_{k+1} \leq C^{\frac{1}{\theta+3} + \frac{1}{2n}} I_k \leq C^{\frac{1}{\theta+3} + \frac{1}{2n}} I_0 \leq C I_0. \\
\]

Therefore, noting $d_k = \theta_k(1-2n) + 2n$ for all $k \geq 1$, we have
\[
\left( \int_{P_{R_0}} |\nabla u|^{d_k+1}(1-2n) \right)^{\frac{1}{d_k+1}(1-2n)} \leq C \left( \int_{P_{R_0}} 1 + |\nabla u|^{d_{k+1}} \right)^{\frac{1}{d_{k+1}}(1-2n)} \leq C(u_0, R_0). \\
\]

This implies that $|\nabla u|$ is bounded in $P_{R_0}$. 

\qed
Proof. Using the above Lemma 2.5, $|\nabla u_\varepsilon|$ is bounded by a constant $C$. By a similar proof of the local energy inequality, we have

$$\int_{P_R(z_0)} (\varepsilon + |\nabla u|^{n-2})^2 \|\partial_t u\|^2 dv \leq C \sup_{t_0-R^2 \leq t \leq t_0} E_\varepsilon(u(t); B_{2R}(x_0)) \leq CR^n.$$

Set $u_{z_0,R} = \int_{P_R(z_0)} u(x,t) \, dz$. By a variant of the Sobolev-Poincare inequality, we have

$$\int_{P_R(z_0)} |u - u_{z_0,R}|^2 \, dz \leq C \left[R^2 \int_{P_R(z_0)} |\nabla u|^2 \, dz + R^4 \int_{P_R(z_0)} |\partial_t u|^2 \, dz\right] \leq CR^{n+4}$$

for all $R \leq R_0/2$. This implies that $u(x,t)$ is Hölder continuous near $z_0$.

Theorem 5. There exists a local regular solution $u : M \times [0, T_0) \to N$ of (3) with initial value $u_0$ satisfying

\begin{align}
(26) \quad \int_0^{T_0} \int_M (|\nabla u|^{n+2} + |\nabla^2 u|^2 |\nabla u|^{n-2}) \, dv \, dt \\ \leq CE_n(u_0) + C(1 + T_0R_0^{-2})E_n(u_0).
\end{align}

Proof. Since $u_0 \in W^{1,n}(M,N)$ can be approximated by maps in $C^\infty(M,N)$, we can assume that $u_0$ is smooth without loss of generality. Note that equation (9) is equivalent to

\begin{align}
\frac{\partial u_{a,\varepsilon}^{i\beta}}{\partial t} &= \frac{1}{(1 + a|\nabla u_{a,\varepsilon}|^{n-2})} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[ (\varepsilon + |\nabla u_{a,\varepsilon}|^{n-2}) g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u_{a,\varepsilon}^{i\beta} \right] \\
&\quad + \frac{\varepsilon + |\nabla u_{a,\varepsilon}|^{n-2} A^{i\beta}(a,\varepsilon) (\nabla u_{a,\varepsilon}, \nabla u_{a,\varepsilon})}{(1 + a|\nabla u_{a,\varepsilon}|^{n-2})} \\
&\quad + \sum_{i,k,a} b_{ij}^{a\beta}(\nabla u_{a,\varepsilon}) \frac{\partial^2 u_{a,\varepsilon}^{\alpha\beta}}{\partial x_i \partial x_j} + f(u_{a,\varepsilon}, \nabla u_{a,\varepsilon}),
\end{align}

where

$$b_{ij}^{a\beta}(\nabla u_{a,\varepsilon}) = \frac{\varepsilon + |\nabla u_{a,\varepsilon}|^{n-2}}{(1 + a|\nabla u_{a,\varepsilon}|^{n-2})} \left( g^{ij} \delta^{\alpha\beta} + \frac{(n-2)|\nabla u_{a,\varepsilon}|^{n-2} \partial_{x_i} u_{a,\varepsilon}^{\alpha} \partial_{x_j} u_{a,\varepsilon}^{\beta}}{\varepsilon + |\nabla u_{a,\varepsilon}|^{n-2}} \right)$$

satisfies the elliptic condition. For a smooth map $u_0$ and for a fixed parameter $\varepsilon$, there is a local smooth solution $u_{a,\varepsilon}$ to the rectified gradient flow (9) with smooth initial value $u_0$ in $[0, T_{a,\varepsilon})$ for a maximal existence time $T_{a,\varepsilon}$. In order to prove the local existence of (3), we need to show that there is a uniform constant $T_0 > 0$ such that $T_{a,\varepsilon} \geq T_0$ for all $\varepsilon > 0$. Since $T_{a,\varepsilon}$ is the maximal existence time of the smooth solution $u_{a,\varepsilon}$ of the flow (9), it follows from using the local energy inequality that there are sufficiently small $R_0 > 0$ and $\delta > 0$ such that for $t \leq \delta R_0^2$, we have

\begin{align}
(28) \quad \int_{B_{2R_0}(x_0)} e_\varepsilon(u_{a,\varepsilon})(\cdot, t) \, dv \leq \int_{B_{4R_0}(x_0)} e_\varepsilon(u_0) \, dv + \frac{Ct}{R_0} \int_M e_\varepsilon(u_0) \, dV < \varepsilon_0.
\end{align}
For a fixed $\varepsilon > 0$, let $\tilde{T}$ be a constant, depending on $u_0$, such that $\tilde{T} \leq T_{a, \varepsilon}$ for all $a \in [0, 1]$. If $\tilde{T} \leq \delta R_0^2$, then it follows from using Lemma 2.5 that $\nabla u_{a, \varepsilon}$ is bounded in $M \times [0, \tilde{T}]$ by the norm $\|\nabla u_0\|_{L^p(M)}$ and hence $f(u_{a, \varepsilon}, \nabla u_{a, \varepsilon})$ is bounded. By the PDE theory, $\nabla u_{a, \varepsilon}(x, t)$ is continuous in $a \in [0, 1]$ for any $t \leq \tilde{T} < T_{\varepsilon}$. For any $\delta > 0$, there is a $\eta > 0$ such that for any two $a, a_0 \in [0, 1]$ with $|a - a_0| < \eta$, we have

$$|b_{ij}^{\alpha \beta}(\nabla u_{a, \varepsilon})(x, t) - b_{ij}^{\alpha \beta}(\nabla u_{a_0, \varepsilon})(x, t)| \leq \delta.$$  

We assume that $\nabla u_{a_0, \varepsilon}(x, t)$ is Hölder continuous in $M \times [\frac{\tilde{T}}{4}, \tilde{T}]$, with its Hölder norm depending only on the bound of $\nabla u_{a_0, \varepsilon}(x, t)$. In fact, this is known for $a_0 = 0$ (see [18]). Noting

$$\frac{\partial u_{a, \varepsilon}}{\partial t} - b_{ij}^{\alpha \beta}(\nabla u_{a, \varepsilon}) \frac{\partial^2 u_{a, \varepsilon}}{\partial x_i \partial x_j} = \left(b_{ij}^{\alpha \beta}(\nabla u) - b_{ij}^{\alpha \beta}(\nabla u_{a_0, \varepsilon})\right) \frac{\partial^2 u_{a, \varepsilon}}{\partial x_i \partial x_j} + f(u_{a, \varepsilon}, \nabla u_{a, \varepsilon}),$$

we apply the $L^p$-estimate to obtain that

$$\int_{P_{\tilde{T}/2}(x, \tilde{T})} \frac{\partial u_{a, \varepsilon}}{\partial t} |p| dx dt + \int_{P_{\tilde{T}/2}(x, \tilde{T})} |\nabla^2 u_{a, \varepsilon}|^p dx dt \leq C\delta \int_{P_{\tilde{T}/2}(x, \tilde{T})} |\nabla^2 u_{a, \varepsilon}|^p dx dt + C \int_{P_{\tilde{T}/2}(x, \tilde{T})} (|f(u_{a, \varepsilon}, \nabla u_{a, \varepsilon})|^p + |u_{a, \varepsilon}|^p) dv dt.$$  

By a covering argument od $M$ and choosing $\delta$ sufficiently small with $C\delta < \frac{1}{4}$, we have

$$\int_{M \times [\frac{\tilde{T}}{4}, \tilde{T}]} \frac{\partial u_{a, \varepsilon}}{\partial t} |p| dx dt + \frac{1}{2} \int_{M \times [\frac{\tilde{T}}{4}, \tilde{T}]} |\nabla^2 u_{a, \varepsilon}|^p dv dt \leq C(\tilde{T}) \int_{M \times [0, \tilde{T}]} (|f(u_{a, \varepsilon}, \nabla u_{a, \varepsilon})|^p + |u_{a, \varepsilon}|^p) dv dt.$$  

By the Sobolev imbedding theorem of parabolic version, $\nabla u_{a, \varepsilon}$ is continuous up to the time $\tilde{T}$ for all $a \in [0, \tilde{T}]$ and therefore $u_{a, \varepsilon}$ is smooth across $\tilde{T} \geq \delta R_0^2$ for all $a \in [0, 1]$. Therefore, for each fixed $\varepsilon > 0$, there is a smooth solution of the flow (9) in $[0, T_0]$ with $T_0 \geq \delta R_0^2$ satisfying

$$\int_0^{T_0} \int_M |\nabla u_{a, \varepsilon}|^{n+2} + |\nabla^2 u_{a, \varepsilon}|^2 (\varepsilon + |\nabla u_{a, \varepsilon}|^{n-2}) dv dt \leq CE_{a, \varepsilon}(u_0) + (1 + T_0 R_0^2) E_{u_0}(u_0).$$

As $\varepsilon \to 0$, $u_{a, \varepsilon}$ converges to a map $u$, which satisfies the heat equation (3) using Lemmas 2.3-2.5.  

3. Energy identity and neck-bubble decompositions

In this section, let $u(x, t)$ be a regular solution of the rectified $n$-flow (3) in $M \times [0, T_1)$. Consider now a sequence of $\{u(x, t_i)\}$ as $t_i \to T_1 \leq \infty$. Then they have uniformly bounded energy; i.e. $E_n(u(t_i)); M) \leq E_n(u_0; M)$. As $t_i \to T_1$, $u(x, t_i)$ converges to a map $w_\gamma$ strongly in $W^{1,n+1}_{loc}(M \setminus \{x^1, \ldots, x^l\})$ with finite integer $l$. At each singularity $x^l$, there is a $R_0 > 0$ such that there is no other
singularity inside $B_{R_0}(x^j)$. Moreover, there is a constant $\varepsilon_0 > 0$ such that each singular point $x^j$ for $j = 1, \ldots, l$ is characterized by the condition
\[
\liminf_{i \to \infty} E_n(u_i; B_R(x^j)) \geq \varepsilon_0
\]
for any $R \in (0, R_0]$. Then there is a $\Theta > 0$ such that as $t_i \to T_1$
\[
(32) \quad |\nabla u(x, t)|^n dv \to \Theta \delta_{x^j} + |\nabla u_{R_1}|^n dv,
\]
where $\delta_{x^j}$ denotes the Dirac mass at the singularity $x^j$.

In order to establish the energy identity of the sequence $\{u(x, t_i)\}_{i=1}^\infty$, we need to get the neck-bubble decomposition. We recall the removable singularity theorem of n-harmonic maps [5] and the gap theorem: there is a constant $\varepsilon_g > 0$ such that if $u$ is a n-harmonic map on $S^n$ satisfying $\int_{S^n} |\nabla u|^2 < \varepsilon_g$, then $u$ is a constant on $S^n$. For completeness, we give a detailed proof on constructing the bubble-neck decomposition by following the idea of Ding-Tian [7].

**Step 1.** To find a maximal (top) bubble at the level one (first re-scaling).

Since $u(x, t_i) \to u_{R_1}$ regularly in $B_{R_0}(x^j)$ away from $x^j$, where $u_{R_1}$ is a map in $W^{1,n}(M, N)$. Since $x^j$ is a concentration point, we find such that as $t_i \to T_1$,
\[
\max_{x \in B_{R_0}(x^j), 0 \leq t \leq t_i} |\nabla u(x, t)| \to \infty, \quad r_{i,1} = \frac{1}{\max_{x \in B_{R_0}(x^j), 0 \leq t \leq t_i} |\nabla u(x, t)|} \to 0.
\]

In the neighborhood of the singularity $x^j$, we define the rescaled map
\[
\tilde{u}_i(x, \tilde{t}) := u_i(x_j + r_{i,1} \tilde{x}, t_i + (r_{i,1})^2 \tilde{t}).
\]

Then $\tilde{u}_i(x, t)$ satisfies
\[
(33) \quad 1 + \frac{n-2}{2} A_\alpha(\tilde{u}) = \text{div} \left( |\nabla \tilde{u}|^{n-2} \nabla \tilde{u} \right) + \int_{B_{R_0}(x^j), 0 \leq t \leq t_i} \frac{|\partial u|}{\partial t} \nabla u_{R_1} | |\nabla u_{R_1}|^n dv dt \to 0.
\]

Therefore, there is a $\tilde{t} \in (-1, 0)$ such that
\[
(35) \quad \int_{B_{R_0}(x^j), 0 \leq t \leq t_i} \frac{|\partial u|}{\partial t} \nabla u_{R_1} | |\nabla u_{R_1}|^n dv \to 0.
\]

Using Lemma 2.2, it can be shown that as $i \to \infty$,
\[
(36) \quad |\nabla u(x, t_i) + r_{i,1}^2 \tilde{t}_i)|^n dv \to \Theta \delta_{x^j} + |\nabla u_{R_1}|^n dv.
\]

For simplicity, we set
\[
u_i(x) := u_i(x, t_i + r_{i,1}^2 \tilde{t}_i) \quad \text{for} \quad x \in B_{R_0}(x^j), \quad \tilde{u}_i(\tilde{x}) := u(x^j + r_{i,1} \tilde{x}, t_i + r_{i,1}^2 \tilde{t}_i).
\]

Since $|\nabla \tilde{u}_i(\tilde{x})| \leq 1$ for all $\tilde{x} \in B_{R_0 r_{i,1}}(0)$, $\tilde{u}_i$ sub-converges to an n-harmonic map $\omega_{1,j}$ locally in $C^{1,\alpha}(\mathbb{R}^n, N)$ as $i \to \infty$, and $\omega_{1,j}$ can be extended to an n-harmonic map on $S^n$ (see [5]) and is nontrivial due to (36). We call $\omega_{1,j}$ the first bubble at the singularity $x^j$, which satisfies
\[
(37) \quad E_n(\omega_{1,j}; \mathbb{R}^n) = \lim_{R \to \infty} \lim_{i \to \infty} E_n(\tilde{u}_i; B_R(0)) = \lim_{R \to \infty} \lim_{i \to \infty} E_n(u_i; B_{R r_{i,1}}(x^j)).
\]
Step 2. To find out new bubbles at the second level (second re-scaling).

Assume that for a fixed small constant $\varepsilon > 0$ (to be chosen later), there exist two positive constants $\delta$ and $R$ with $R r_{i,1} < 4\delta \leq R_0$ such that for all $\alpha$ sufficiently close to 1, we have

$$\int_{B_{2r} \setminus B_r(x^i)} |\nabla u(t_i)|^n dV \leq \varepsilon$$

for all $r \in (\frac{R r_{i,1}}{2}, 2\delta)$.

If (38) is true, it follows from (2.2) and (37) that

$$\lim_{i \to \infty} E_n(u; B_{Rr}(x^i)) = E_n(u_{T_i}; B_{R_0}(x^i)) + E_n(\omega_{1,j}; \mathbb{R}^n)$$

$$+ \lim_{R \to \infty} \lim_{\delta \to 0} \lim_{i \to \infty} E_n(u; B_\delta \setminus B_{R r_{i,1}}(x^i)).$$

In this case, this means that there is only single bubble $\omega_{1,j}$ around $x^i$.

If the assumption (38) is not true, then for any two constants $R$ and $\delta$ with $R r_{i,1} < 4\delta \leq R_0$, there is a number $r_{i,2} \in (\frac{R r_{i,1}}{2}, 2\delta)$ such that

$$\lim_{i \to \infty} \int_{B_{2r_{i,2}} \setminus B_{r_{i,2}}(x^i)} |\nabla u_i|^{n} > \varepsilon.$$

If $\lim_{i \to \infty} r_{i,2} \neq 0$, it can be ruled out by choosing $\delta$ sufficiently small in (38), so we assume that $\lim_{i \to \infty} r_{i,2} = 0$. If $\lim_{i \to \infty} \frac{r_{i,1}}{r_{i,2}} \leq C$ for a finite constant $C$, it is not a problem since $\tilde{u}_i$ converges regularly to $\omega_{1,\infty}$ locally in $\mathbb{R}^n$. Therefore, we can assume that $\lim_{i \to \infty} \frac{r_{i,2}}{r_{i,1}} = \infty$. Since there might be many different numbers $r_{i,2} \in (\frac{R r_{i,1}}{2}, 2\delta)$ satisfying (39), we must classify these numbers. For any two numbers $r_{i,2}$ and $\tilde{r}_{i,2}$ in $(\frac{R r_{i,1}}{2}, 2\delta)$ satisfying (39), they can be classified in different groups by the following properties:

$$\lim_{i \to \infty} \frac{r_{i,2}}{\tilde{r}_{i,2}} = +\infty$$

$$\lim_{i \to \infty} \frac{r_{i,2}}{\tilde{r}_{i,2}} = 0;$$

$$0 < a \leq \lim_{i \to \infty} \frac{r_{i,2}}{\tilde{r}_{i,2}} < b$$

for finite constants $a$ and $b$. We say that $r_{i,2}$ and $\tilde{r}_{i,2}$ are in the same group if they satisfy (41). Otherwise, they are in different groups if they satisfy (40).

Since there is a uniformly bounded energy $\leq K$ for some constant $K$ and $\varepsilon$ is a fixed constant, the above different groups of $r_{i,2}$ satisfying (39) must be finite, so that we can choose a number $r_{i,2}$ in the smallest group satisfying (39), but $r_{i,2}$ is not in the group of $r_{i,1}$; i.e. $\lim_{i \to \infty} \frac{r_{i,2}}{r_{i,1}} = \infty$ and $\lim_{i \to \infty} r_{i,2} = 0$. There might be many other numbers $\tilde{r}_{i,2}$ satisfying (39) in the same group of $r_{i,2}$. Because of (41), $\tilde{r}_{i,2}/r_{i,2}$ are bounded as $i \to \infty$, so these numbers $\tilde{r}_{i,2}$ can be ruled out by the following procedure: Set

$$\tilde{u}_{2,i}(\tilde{x}) = u_i(x^i + r_{i,2}\tilde{x}).$$

Passing to a subsequence, $\tilde{u}_{2,i}$ converges to a $\omega_2$ locally in $\mathbb{R}^n$ away from a finite concentration set of $\{\tilde{u}_{2,i}\}$. Those numbers $\tilde{r}_{i,2}$ in the same group $r_{i,2}$ have been handled out. If $\omega_2$ is non-trivial, then $\omega_2$ is a new bubble, which is different from the bubble $\omega_1$. We must point out that the above bubble connection $\omega_2$ might be trivial, called a ‘ghost bubble’. In this case, there is at least a concentration point $p \in B_2 \setminus B_1$ of $\{\tilde{u}_{2,i}\}$ due to (39). At each concentration point $p$ of $\tilde{u}_{2,i}$, we can
repeat the procedure in Step 1; i.e. at each concentration point $p$ of $\tilde{u}_{2,i}$, there are sequences $x^p_i \to p$ and $\lambda^p_i \to 0$ such that

$$\tilde{u}_{2,i}(x^p_i + \lambda^p_i x) \to \omega_{2,p}$$

up-to gauge transformation, where $\omega_{2,p}$ is a $n$-harmonic map on $\mathbb{R}^n$. Note that $\tilde{u}_{2,p,\infty}$ is also a bubble for the sequence $\{u_i(x^j + r_i \lambda^j x)\}$.

Set $x^2_i = x_j + r_{1,2} x^p_i$. If $p \neq 0$, then

$$\frac{|x^j - x^2_i|}{r^2_i} = \frac{r_{1,2}}{r_{1,1}} |x^p_i| \to \infty \text{ as } i \to \infty.$$

Therefore, the bubble $\omega_{2,p}$ at $p \neq 0$ is different from the bubble $\omega_1$.

Since $\lim_{i \to \infty} \frac{r_{1,1}}{r_{1,2}} = 0$ and $\omega_1$ is a bubble limiting map for the sequence $\{u_i(x^j + r^j x) = u_{2,i}(\frac{1}{r^j} x)\}$, then $p = 0$ is also a concentration point of $\tilde{u}_{2,i}$ on $\mathbb{R}^n$. Like Step 1, there is a small $R_0^0 > 0$ such that the ball $B_{R_0^0}(0)$ contains only the isolated concentration point $0$ of $\tilde{u}_{2,i}$. Similarly to Step 1, we choose

$$\lambda^0_i = \frac{1}{\max_{\bar{B}_{R_0^0}(0)} |\tilde{u}_{2,i}|} \geq \frac{r_{1,2}}{r_{1,1}}$$

the bubble $\omega_{2,0}$ is chosen as the maximal (top) bubble for $\tilde{u}_{2,i}$ at $p = 0$. Therefore the bubble $\omega_{2,0}$ must be the same bubble $\omega_1$. We can keep it there without a problem.

Next, we continue the above procedure for possible new multiple bubbles at each blow-up point $p$ again. Since there is a uniform bound $K$ for $E_n(u_i; M)$ and each non-trivial bubble on $S^n$ costs at least $\epsilon_g$ of the energy by the gap theorem, the above process must stop after finite steps.

**Step 3.** To find out all multiple bubbles.

Let $r_{1,3}$ be in the second small group of numbers satisfying (39) with $\lim_{i \to \infty} \frac{r_{1,1}}{r_{1,2}} = \infty$ and $\lim_{i \to \infty} r_{1,3} = 0$. Set

$$\tilde{u}_{3,i}(\bar{x}) = u_i(x^j + r_{1,3} \bar{x}).$$

Passing to a subsequence, $\tilde{u}_{3,i}$ converges locally to a $\omega_3$ away from a finite concentration set of $\{\tilde{u}_{3,i}\}$ on $\mathbb{R}^n$. Then we can repeat the argument of Steps 1-2. All bubbles produced by $\tilde{u}_{3,i}$, except for those concentrated in 0, are different from Steps 1-2. By induction, we can find out all bubbles in all cases of the finite different groups. The above process must stop after finite steps by the gap theorem.

In summary, at each group level $k$, the blow-up happens, there are finitely many blow-up points and bubbles on $\mathbb{R}^n$. At each level $k$ and each bubble point $p_{k,l}$, there are sequences $\bar{x}^{k,l}_i \to p_{k,l}$ and $r_{i,k} \to 0$ with $\lim_{i \to \infty} \frac{r_{1,1}}{r_{1,k-1}} = \infty$ such that passing to a subsequence, $\tilde{u}_{i,k,l}(x) = u_i(x^{k,l}_i + r_{i,k} x)$ converges to $\omega_{k,l}$, where $\omega_{k,l}$ is an $n$-harmonic map in $\mathbb{R}^n$, where $x^{k,l}_i = x^j + r_{i,k} \bar{x}^{k,l}_i$.

In conclusion, there are finite numbers $r_{i,k}$, finite points $x^{k,l}_i$, positive constants $R_{k,l}$, $\delta_{k,l}$ and a finite number of non-trivial $n$-harmonic maps $\omega_{k,l}$ on $\mathbb{R}^n$ such that
Moreover, at each neck region $B_{b_{k,i}} \setminus B_{R_{k,i}, r_{i,k}}(x_{i}^{k,i})$ in (42), for all $i$ sufficiently large, we have
\begin{equation}
\int_{B_{2r} \setminus B_{r}(x_{i}^{k,i})} |\nabla \tilde{u}_{k,i}|^{n} dV \leq \varepsilon
\end{equation}
for all $r \in \left(\frac{R_{k,i}r_{i,k}}{4}, 2\delta_{k,i}\right)$, where $\varepsilon$ is a fixed constant to be chosen sufficiently small.

Proof of Theorem 1. Wei-Wang [28] proved an energy identity of a sequence of regular approximated $n$-harmonic maps $u_{i}$ in $W^{1,n}(M, N) \cap C^{0}(M, N)$, whose tension fields $h_{i}$ are bounded in $L^{n/n-1}(M)$. Let $u_{i}(x) = u(x, t_{i})$ satisfy the equation (3). In this case, $h_{i} := (1 + a|\nabla u_{i}|^{n-2})\partial u_{i}$, which is bounded in $L^{n/n-1}(M)$. In fact, using Hölder’s inequality, we have
\begin{equation}
\int_{M} \left( |\nabla u_{i}|^{n-2} |\partial u_{i}| \right)^{\frac{n}{n-1}} \leq \left( \int_{M} |\nabla u_{i}|^{n} \right)^{\frac{n-2}{n}} \left( \int |\nabla u_{i}|^{n-2} |\partial u_{i}|^{2} \right)^{\frac{1}{2}} \leq C.
\end{equation}
Under the condition (43), we can apply Theorem B of [28] to prove
\begin{equation}
\lim_{R_{k,i} \rightarrow \infty} \lim_{\delta_{k,i} \rightarrow 0} \lim_{i \rightarrow \infty} E_{n}(\tilde{u}_{k,i}; B_{b_{k,i}} \setminus B_{R_{k,i}, r_{i,k}}(x_{i}^{k,i})) = 0.
\end{equation}
Therefore, the energy identity follows from (42). \hfill \square

4. Minimizing the $n$-energy functional in homotopy classes

In this section, we will present some applications of the related $n$-flow to minimizing the $n$-energy functional in a given homotopy class and give a proof of Theorem 3. For a map $u : M \rightarrow N$, we recall the functional
\begin{equation}
E_{n,\varepsilon}(u, M) = \int_{M} e_{n,\varepsilon}(u) dv,
\end{equation}
where we set $e_{n,\varepsilon}(u) = \frac{\varepsilon}{g} |\nabla u|^{2} + \frac{1}{n} |\nabla u|^{n} + \frac{\varepsilon}{n+1} |\nabla u|^{n+1}$.

Let $u_{i} \in C^{\infty}(M, N)$ be a minimizing sequence of the $n$-energy in a homotopy class $[u_{0}]$. Since a minimizing sequence $u_{i}$ does not satisfy any equation, we cannot have a good tool to use. Following an idea of the $\alpha$-harmonic map flow [17], we introduce a modified gradient flow for the functional (7) in the following:
\begin{equation}
(1 + a|\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1}) \frac{\partial u}{\partial t} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_{i}} \left( \varepsilon + |\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1} \right) g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_{j}} u + (\varepsilon + |\nabla u|^{n-2} + \varepsilon |\nabla u|^{n-1}) A(u)(\nabla u, \nabla u)
\end{equation}
with initial value \( u(0) \) for a small constant \( a > 0 \). Since the minimizing sequence \( u_i \) is smooth, there is a sequence \( \varepsilon_i \) with \( \varepsilon_i \to 0 \) such that
\[
\lim_{i \to \infty} E_{n, \varepsilon_i}(u_i, M) = \lim_{i \to \infty} E_n(u_{\varepsilon_i}, M) = \inf_{u \in [u_0]} E_n(u, M).
\] Choosing \( u(0) = u_i \) to be initial values, there is a sequence of \( \varepsilon = \varepsilon_i \to 0 \) such that the flow (44) has a unique global smooth solution \( u_{\varepsilon_i}(x, t) \) on \( M \times [0, \infty) \) with \( u_{\varepsilon_i}(0) = u_i \).

By (44), we have the energy identity
\[
E_{n, \varepsilon_i}(u_{\varepsilon_i}(s), M) + \int_0^s \int_M (1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}) |\frac{\partial u_{\varepsilon_i}}{\partial t}|^2 \, dv \, dt
= E_{n, \varepsilon_i}(u_i, M)
\]
for each \( s > 0 \). This implies that
\[
\lim_{i \to \infty} \int_M \frac{\varepsilon_i}{n} |\nabla u_{\varepsilon_i}(s)|^{n+1} \, dv = 0,
\]
and
\[
\lim_{i \to \infty} \int_0^s \int_M (1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}) |\partial_t u_{\varepsilon_i}|^2 \, dv \, dt = 0.
\]
Hence the sequence \( \{u_{\varepsilon_i}(s)\}_{i=1}^\infty \) for each \( s > 0 \) is also a minimizing sequence in the homotopic class \([u_0]\).

**Lemma 4.1.** Let \( \rho, R \) be two constants with \( \rho < R \leq 2\rho \). For any \( x_0 \) with \( B_{2\rho}(x_0) \subset M \) and for any two \( s, \tau \in [0, T) \), we have
\[
\int_{B_{\rho}(x_0)} e_{n, \varepsilon_i}(u_{\varepsilon_i}(. , s)) \, dv - \int_{B_{\rho}(x_0)} e_{n, \varepsilon_i}(u_{\varepsilon_i}(. , \tau)) \, dv
\leq C \int_s^\tau \int_M (1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}) |\partial_t u_{\varepsilon_i}|^2 \, dv \, dt
+ C \frac{(\tau - s)}{(R - \rho)^2} \int_s^\tau \int_M e_{n, \varepsilon_i}(u_i) \, dv \int_s^\tau \int_M (1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}) |\partial_t u_{\varepsilon_i}|^2 \, dv \, dt)^{1/2}.
\]

*Proof.* Let \( \phi \) be a cut-off function in \( B_R(x_0) \) such that \( \phi = 1 \) in \( B_{\rho} \) and \( |\nabla \phi| \leq C/(R - \rho) \). The required result follows from multiplying (44) by \( \phi \partial_t u_{\varepsilon_i} \). \( \Box \)

We can repeat the same steps of Lemma 2.5 to obtain

**Lemma 4.2.** There exists a positive constant \( \varepsilon_0 < i(M) \) such that if for some \( R_0 \) with \( 0 < R_0 < \min\{\varepsilon_0, \frac{n}{i(M)}\} \) the inequality
\[
\sup_{t_0 - 4R_0^2 \leq t < t_0} \int_{B_{2R_0}(x_0)} |\nabla u_{\varepsilon_i}|^n \, dv \leq \varepsilon_0
\]
holds, we have
\[
\|\nabla u_{\varepsilon_i}\|_{L^\infty(B_{R_0} (x_0))} \leq C(R_0)
\]
where \( C \) is a constant independent of \( \varepsilon \) and depends on \( R_0 \).

Now we complete a proof of Theorem 3.
Proof of Theorem 3. For a minimizing sequence \( u_i \) of the \( n \)-energy in the homotopy class, let \( u \) be the weak limit of \( \{u_i\}_{i=1}^{\infty} \) in \( W^{1,n}(M) \). Set

\[
\Sigma_0 = \bigcap_{R>0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \liminf_{i \to \infty} \int_{B_R(x_0)} |\nabla u_i|^n \, dx \geq \varepsilon_0 \right\}
\]

for a small constant \( \varepsilon_0 > 0 \). It is known that \( \Sigma_0 \) is a set of finite points. For the above sequence \( \{u_{\varepsilon_i}(s)\}_{i=1}^{\infty} \), we set

\[
\Sigma_s = \bigcap_{R>0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \liminf_{i \to \infty} \int_{B_R(x_0)} |\nabla u_{\varepsilon_i}(\cdot, s)|^n \, dx \geq \varepsilon_0 \right\},
\]

which is also finite. Applying (47-48) to Lemma 4.1, we obtain that \( \Sigma_0 = \Sigma_s \) for all \( s > 0 \) (see a similar argument to one in \([15]\)). By using Lemmas 4.1-4.2, \( |\nabla u_{\varepsilon_i}(x, s)| \leq C(R) \) on \( P_R(x_0, s) \) for each \( x_0 \in M \setminus \Sigma \) with \( B_R(x_0) \subset M \). By this result, we know that \( u(x, t) \) is a weak solution to the flow (44). Since \( u_i(x, t) \) converges weakly to \( u(x, t) \) in \( W^{1,2}(M \times [0, 1]) \), \( u(\cdot, t) \equiv u(\cdot, 0) = u \). Then \( u(x, t) \) is an \( n \)-harmonic map from \( M \) to \( N \) independent of \( t \in [0, 1] \). By the regularity result on \( n \)-harmonic maps, \( u \) is a smooth map on \( M \).

For any \( x_0 \in M \setminus \Sigma \), there is a constant \( R > 0 \) such that \( B_R(x_0) \subset M \setminus \Sigma \). Note that \( u_{\varepsilon_i}(\tau) \) converges strongly to \( u \) in \( W^{1,n}(B_R(x_0)) \). As \( i \to \infty \), we apply Lemma 5.1 to obtain that

\[
\int_{B_{\rho}(x_0)} \frac{1}{n} |\nabla u|^n \leq \liminf_{i \to \infty} \int_{B_{\rho}(x_0)} \frac{1}{n} |\nabla u_i|^n \, dv \leq \limsup_{i \to \infty} \int_{B_{\rho}(x_0)} e_{\varepsilon_i}(u_i) \, dv
\]

\[
\leq \limsup_{i \to \infty} \int_{B_R(x_0)} e_n(e_{\varepsilon_i}(u_{\varepsilon_i})(\cdot, \tau)) \, dv = \int_{B_R(x_0)} \frac{1}{n} |\nabla u|^n \, dv
\]

for any \( R \) with \( \rho < R \). Letting \( R \to \rho \), we have

\[
\int_{B_{\rho}(x_0)} \frac{1}{n} |\nabla u|^n = \lim_{i \to \infty} \int_{B_{\rho}(x_0)} \frac{1}{n} |\nabla u_i|^n \, dv.
\]

This implies that \( u_i \) converges strongly to \( u \) in \( W^{1,n}(B_{\rho}(x_0)) \) and hence strongly in \( W^{1,n}_{loc}(M \setminus \Sigma) \).

Next, we use a similar proof of Sacks-Uhlenbeck \([22]\) to show that \( \Sigma_0 = \Sigma_s = \emptyset \) if \( \pi_n(N) = 0 \). Let \( \{u_{\varepsilon_i}(s)\}_{i=1}^{\infty} \) be the above sequence. It is known that \( u_{\varepsilon_i}(s) \) converges to \( u \) strongly in \( W^{1,n+1}_{loc}(M \setminus \Sigma) \). Without loss of generality, we assume that there is one singularity \( x^i \) in \( \Sigma_s \). Let \( \eta(r) \) be a smooth cutoff function in \( \mathbb{R} \) with the property that \( \eta \equiv 1 \) for \( r \geq 1 \) and \( \eta \equiv 0 \) for \( r \leq 1/2 \). For some \( \rho > 0 \), we define a new sequence of maps \( v_i : M \to N \) such that \( v_i \) is the same as \( u_i \) outside \( B_\rho(x_1) \), and for \( x \in B_\rho(x_1) \),

\[
v_i(x) = \exp_{u(x)} \left( \frac{|x|}{\rho} \exp_{u(x)}^{-1} \circ u_{\varepsilon_i}(x, s) \right),
\]

where \( \exp \) is the exponential map on \( N \). Note that \( v_i \equiv u \) on \( B_{\rho/2}(x_1) \) and \( v_i \equiv u_{\varepsilon_i}(s) \) outside \( B_\rho(x_1) \) and that \( u_{\varepsilon_i}(s) \) converges to \( u \) on \( B_\rho(x_1) \setminus B_{\rho/2}(x_1) \) strongly in \( W^{1,n+1} \) and thus in \( C^\beta \) for some \( \beta > 0 \). Hence for sufficiently large \( i \), \( v_i(B_\rho(x_1) \setminus B_{\rho/2}(x_1)) \) lies in a small neighborhood of \( u(x_1) \), where \( \exp_{u(x)}^{-1} \) is a well defined smooth map (if \( \rho \) is small). Since \( F(y) = \exp_{u(x)} \left( \frac{|y|}{\rho} \exp_{u(x)}^{-1} y \right) \) is a smooth
map from a neighborhood of \( u(x_1) \) into itself, we have

\[
\int_{B_r \setminus B_{r/2}(x_1)} |\nabla (v_i - u)|^n \, dv = \int_{B_r \setminus B_{r/2}(x_1)} |\nabla (F \circ u_{\varepsilon_i}(s) - F \circ u)|^n \, dv \\
\leq C \int_{B_r \setminus B_{r/2}(x_1)} |\nabla (u_{\varepsilon_i}(s) - u)|^n \, dv \to 0
\]
as \( i \to \infty \). It implies that

\[
\|v_i - u\|_{W^{1,n}(M)} \to 0
\]
as \( i \to \infty \).

Since \( \pi_n(N) \) is trivial, \( v_i \) is in the same homotopy class as \( u_{\varepsilon_i}(s) \). Since \( u_{\varepsilon_i}(s) \) is a minimizing sequence of \( E_{n,\varepsilon_i} \) and \( u_{\varepsilon_i}(s) \) converges weakly to \( u \) in \( W^{1,n} \), we have

\[
E_n(u) \leq \limsup_{i \to \infty} E_{n,\varepsilon_i}(u_{\varepsilon_i}(s)) \leq \limsup_{i \to \infty} E_{n,\varepsilon_i}(v_i) = E_n(u),
\]
which implies that \( u_{\varepsilon_i}(s) \) converges to \( u \) strongly in \( W^{1,n}(M, N) \), which means that there is no energy concentration; i.e. \( \Sigma_0 = \Sigma_s = \emptyset \).

\[\square\]

5. Minimizing the \( p \)-energy functional in homotopy classes

For a small \( \varepsilon > 0 \), we introduce a perturbation of the \( p \)-energy functional by

\[
E_{p,\varepsilon}(u; M) = \int_M \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{p} |\nabla u|^p + \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \, dv.
\]

The Euler-Lagrange equation associated to this functional is

\[
\nabla \cdot \left( [\varepsilon + |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1}] \nabla u \right) \\
+ [\varepsilon + |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1}] A(u) (\nabla u, \nabla u) = 0.
\]

The gradient flow for the above equation is

\[
\frac{\partial u}{\partial t} = \text{div} \left( [\varepsilon + |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1}] \nabla u \right) \\
+ (\varepsilon + |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1}) A(u) (\nabla u, \nabla u)
\]

with initial value \( u(0) = u_0 \) in \( M \). If the initial map \( u_0 \) is smooth, there is a global smooth solution to (52) by using proofs in [18] and [10].

Without loss of generality, we assume \( g_{ij} = \delta_{ij} \). Then we have
Lemma 5.1. Let \( u \) be a solution of the equation (52). Then for all \( \rho \leq R \) with \( B_R(x_0) \subset \Omega \), we have

\[
\rho^{p-n} \int_{B_\rho(x_0)} \left[ \frac{\varepsilon}{2} \nabla u^2 + \frac{1}{p} |\nabla u|^p + \frac{1}{n+1} \varepsilon |\nabla u|^{n+1} \right] dx \\
+ \frac{n(p-2)}{2p} \int_R^\rho r^{p-1-n} \int_{B_r} \varepsilon |\nabla u|^2 \, dx \, dr \\
+ \int_{B_R \setminus B_\rho(x_0)} \left[ \frac{1}{2} |\partial_r u|^2 + \frac{1}{n+1} \varepsilon |\nabla u|^{n-1} |\partial_r u|^2 \right] r^{p-n} \, dx \\
= R^{p-n} \int_{B_R(x_0)} \left[ \frac{\varepsilon}{2} \nabla u^2 + \frac{1}{p} |\nabla u|^p + \frac{1}{n+1} \varepsilon |\nabla u|^{n+1} \right] dx \\
+ \frac{n+1-p}{n+1} \int_R^\rho \int_{B_\rho(x_0)} r^{p-1-n} \varepsilon |\nabla u|^{n+1} \, dx \, dr \\
+ \int_R^\rho \int_{B_r} r^{p-1-n} \left< \frac{\partial u}{\partial t}, x_i \nabla_i u \right> \, dx \, dr.
\]

Proof. Without loss of generality, we assume that \( x_0 = 0 \). Multiplying (52) by \( x_i \nabla_i u \), we have

\[
\int_{B_\rho} \left< \frac{\partial u}{\partial t}, x_i \nabla_i u \right> - \left< div \left( \varepsilon |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1} \right) \nabla u, x_i \nabla_i u \right> \, dx = 0.
\]

Integration by parts yields that

\[
\int_{B_\rho} \left< \frac{\partial u}{\partial t}, x_i \nabla_i u \right> \, dx - \frac{1}{r} \int_{\partial B_\rho} \left( \varepsilon |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1} \right) x_i \nabla_i u^2 \, d\omega \\
= - \int_{B_\rho} \left( \varepsilon |\nabla u|^2 + |\nabla u|^p + \varepsilon |\nabla u|^{n+1} \right) + \frac{1}{2} \left( \varepsilon |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1} \right) x_i \nabla_i (|\nabla u|^2) \, dx \\
= \int_{B_\rho} \left( \frac{n-2}{2} \varepsilon |\nabla u|^2 + \frac{(n-p)}{p} |\nabla u|^p - \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) \, dx \\
- \frac{r}{n} \int_{\partial B_\rho} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{p} |\nabla u|^p + \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) \, d\omega.
\]

Multiplying by \( r^{p-1-n} \) both sides of the above identity, we have

\[
\frac{d}{dr} \left[ r^{p-n} \int_{B_r} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{p} |\nabla u|^p + \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) \right] \\
- \frac{n(p-2)}{2p} r^{p-1-n} \int_{B_r} \varepsilon |\nabla u|^2 \, dx + r^{p-1-n} \int_{B_r} \frac{\varepsilon(n+1-p)}{n+1} |\nabla u|^{n+1} \, dx \\
= -r^{p-1-n} \int_{B_r} \left( \frac{n-2}{2} \varepsilon |\nabla u|^2 + \frac{(n-p)}{p} |\nabla u|^p - \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) \, dx \\
+ \frac{r^p}{n} \int_{\partial B_r} \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{p} |\nabla u|^p + \frac{\varepsilon}{n+1} |\nabla u|^{n+1} \right) \, d\omega \\
= r^{p-n} \int_{B_r} \left( \varepsilon |\nabla u|^{p-2} + \varepsilon |\nabla u|^{n-1} \right) |\partial_r u|^2 \, d\omega - \int_{B_r} r^{p-1-n} \left< \frac{\partial u}{\partial t}, x_i \nabla_i u \right> \, dx.
\]

Then integrating with respect to \( r \) from \( \rho \) to \( R \) yields the result. \( \square \)
Lemma 5.2. Let \( u_i \in C^\infty(M, N) \) be a minimizing sequence in the homotopy class \([u_0]\). Then there is a sequence of \( \varepsilon_i \to 0 \) and solutions \( u_{\varepsilon_i} \) of equation (52) with initial value \( u_i \) such that \( u_{\varepsilon_i}(t) \) for all \( t \in [0, \infty) \) is also a minimizing sequence in the same homotopy class. Moreover, there is a uniform \( \tilde{t} \in [1/2, 1] \) such that

\[
\lim_{i \to \infty} \int_M \frac{\varepsilon_i}{n+1} |\nabla u_{\varepsilon_i}|^{n+1}(\cdot, \tilde{t}) \, dv + \lim_{i \to \infty} \int_M |\partial_t u_{\varepsilon_i}(\cdot, \tilde{t})|^2 \, dv = 0.
\]

Proof. Since the minimizing sequence \( u_i \) is smooth, there is a sequence \( \varepsilon_i \to 0 \) such that

\[
E_{p, \varepsilon_i}(u_i) \leq E_p(u_{\varepsilon_i}) + \frac{1}{i},
\]

which implies

\[
\lim_{i \to \infty} E_{p, \varepsilon_i}(u_i) = \lim_{i \to \infty} E_p(u_{\varepsilon_i}) = \inf_{v \in [u_0]} E_p(v).
\]

Then there is a unique solution \( u_{\varepsilon_i}(x, t) \) to the flow (52) with initial value \( u_{\varepsilon_i}(0) = u_i \). Similar to Lemma 2.1, we have

\[
E_{\varepsilon_i}(u_{\varepsilon_i}(\cdot, \tau)) + \int_0^\tau \int_M |\partial_t u_{\varepsilon_i}|^2 \, dvdt = E_{\varepsilon_i}(u_{\varepsilon_i}).
\]

This implies that \( u_{\varepsilon_i}(x, \tau) \) for \( \tau \) is a minimizing sequence of \( E \) in the homotopy class \([u_0]\), which yields

\[
\lim_{i \to \infty} \int_M \frac{\varepsilon_i}{n+1} |\nabla u_{\varepsilon_i}|^{n+1}(\cdot, \tau) \, dv + \int_0^{\tau} \int_M |\partial_t u_{\varepsilon_i}|^2 \, dvdt = 0.
\]

Then there is a uniform \( \tilde{t} \in [1/2, 1] \) such that

\[
\lim_{i \to \infty} \int_M |\partial_t u_{\varepsilon_i}(\cdot, \tilde{t})|^2 \, dv = 0.
\]

□

Proof of Theorem 4. Let \( u_i \in C^\infty(M, N) \) be a minimizing sequence in a homotopy class and \( u \) its weak limit. If \( N \) is a homogeneous manifold, we claim that \( u \) is a weak \( p \)-harmonic map from \( M \) and \( N \).

Let \( X = (X^1, \ldots, X^L) \) be a Killing vector on \( N \subset \mathbb{R}^L \) as in Hélein [14] and \( u = (u^1, \ldots, u^L) \in N \). Let \( \varphi \) be a cut-off function compactly supported in \( M \). Since \( u_{\varepsilon} \) is a solution of (52), we use \( \varphi X(u) \) as a testing vector to get

\[
\int_M (\nabla_k (\varphi X(u_{\varepsilon})), (\varepsilon + |\nabla u_{\varepsilon}|^{p-2} + \varepsilon |\nabla u_{\varepsilon}|^{n-1}) \nabla_k u_{\varepsilon}) dv = - \int_M \langle \varphi X(u_{\varepsilon}), \partial_t u_{\varepsilon} \rangle.
\]

Since \( X \) is a Killing vector, it implies that \( \sum_{l=1}^L \nabla_k u_{\varepsilon}^l \nabla_m X^l \nabla_k u_{\varepsilon}^m = 0 \), so

\[
\int_M (\nabla_k \varphi X(u_{\varepsilon})), (\varepsilon + |\nabla u_{\varepsilon}|^{p-2} + \varepsilon |\nabla u_{\varepsilon}|^{n-1}) \nabla_k u_{\varepsilon}) dv = - \int_M \langle \varphi X(u_{\varepsilon}), \partial_t u_{\varepsilon} \rangle.
\]

Let \( u \) be the weak limit of \( u_{\varepsilon_i} \) in \( W^{1,p}(M \times [0, 1]) \) by passing to a subsequence if necessary. By a compact result in [3], \( |\nabla u_{\varepsilon_i}|^{p-2} \nabla u_{\varepsilon_i} \) converges weakly to \( |\nabla u|^{p-2} \nabla u \) in \( L^p \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \). Since \( u_{\varepsilon_i} \) converges to \( u \) strongly in \( L^p \) and \( X \) is a smooth vector on \( N \), \( X(u_{\varepsilon_i}) \) converges to \( X(u) \) strongly in \( L^p \). Letting \( \varepsilon_i \) go to zero in equation (55) and noting (54), we have

\[
\int_M \nabla_k \varphi X(u), |\nabla u|^{p-2} \nabla_k u d\mu = 0,
\]
which implies
\[ \int_M \langle \nabla_k (\varphi X(u)), |\nabla u|^p - 2 \nabla_k u \rangle d\mu = 0 \]
due to the fact that \( X \) is a Killing vector. Since \( N \) is a homogeneous space, we apply the construction of a Killing field \( \{X_j\} \) by Helein [14] and choose \( \varphi_j \) to obtain that
\[ \sum_j \varphi_j X_j(u) \]
is any compactly supported vector field (along \( u \)). This implies that \( u \) is a weak \( p \)-harmonic map. We know that \( u \) is a weak solution to the \( p \)-harmonic map flow. It follows from (54) that \( u \) is a map independent of \( t \in [0, 1] \). Since \( u_{\varepsilon_i}(x, t) \) converges weakly to \( u(x, t) \) in \( W^{1,2}(M \times [0, 1]) \). Hence, \( u(\cdot, t) \equiv u(\cdot, 0) \) is a (weakly) \( p \)-harmonic map from \( M \) to \( N \).

We define
\[ \Sigma = \bigcap_{R > 0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \liminf_{\varepsilon_i \to 0} \frac{1}{R^{n-p}} \int_{B_R(x_0)} |\nabla u_{\varepsilon_i}|^p \, dx \geq \varepsilon_0 \right\} \]
for a sufficiently small constant \( \varepsilon_0 \). Then, \( H^{n-p}(\Sigma) < +\infty \). For any \( x_0 \notin \Sigma \) with \( B_{R_0}(x_0) \subset M \setminus \Sigma \), for each \( y \in B_{R_0/2}(x_0) \) and for each \( \rho \in (0, R_0/2) \), we have
\[ \rho^{p-n} \int_{B_\rho(y)} |\nabla u|^p \, dM \leq \lim_{\varepsilon_i \to 0} \rho^{p-n} \int_{B_\rho(y)} |\nabla u_{\varepsilon_i}|^p \, dM < \varepsilon_0 \]
for a sufficiently small constant \( \varepsilon_0 > 0 \). Since \( u \) is a weakly \( p \)-harmonic map satisfying (56), it follows from a similar proof of stationary \( p \)-harmonic maps into homogenous manifolds (see [26]) that \( u \) belongs to \( C^{1,\alpha}_{loc}(M \setminus \Sigma) \).

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MIN-CHUN HONG, DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF QUEENSLAND, BRISBANE, QLD 4072, AUSTRALIA
E-mail address: hong@maths.uq.edu.au