ON A SYMMETRY OF MÜGER’S CENTRALIZER FOR THE DRINFELD DOUBLE OF A SEMISIMPLE HOPF ALGEBRA

SEBASTIAN BURCIU

ABSTRACT. In this paper we prove a formula that relates Müger’s centralizer in the category of representations of a factorizable Hopf algebra to the notion of Hopf kernel of a representation of the dual Hopf algebra. Using this relation we obtain a complete description for Müger’s centralizer of some fusion subcategories of the fusion category of finite dimensional representations of a Drinfeld double of a semisimple Hopf algebra.

1. INTRODUCTION AND MAIN RESULTS

Müger has introduced in [22] the notion of centralizer of a fusion subcategory of a braided fusion category. One of the most remarkable features of this notion is that the centralizer of a nondegenerate fusion subcategory of a modular category is a categorical complement of the nondegenerate subcategory. This principle is applied in many classification results of fusion categories, see for example [8, 9, 12].

Despite its importance, in general it is a difficult task to give a concrete description for the centralizer of all fusion subcategories of a given fusion category. Only few cases are known in the literature. For instance, in the same aforementioned paper, [22], Müger described the centralizer of all fusion subcategories of the category of finite dimensional representations of a Drinfeld double of a finite abelian group. More generally, for the category of representations of a (twisted) Drinfeld double of an arbitrary finite group a similar formula was then given in [23]. For the braided center of Tambara-Yamagami categories, this centralizer was described by computing completely the $S$-matrix of the modular category in [13].

In this paper we study some properties of Müger’s centralizer for the category of representations of a semisimple factorizable Hopf algebra. A formula that relates Müger’s centralizer to the Hopf kernel of representations of the dual Hopf algebra is proven in Theorem 4.8. Then we specialize these results to the category of representations of Drinfeld doubles of semisimple Hopf
algebras. This allows us to obtain a certain symmetry of Müger’s centralizer for these categories. This symmetry can also be viewed as a generalization of the above mentioned results for the (twisted) Drinfeld double of a finite group.

Normal Hopf subalgebras of a semisimple Drinfeld double were studied in [7]. In loc.cit. the author shows that if $K, L$ are two normal Hopf subalgebras of semisimple Hopf algebra $A$ such that $[K, L] = 0$ and $[(A//K)^*, (A//L)^*] = 0$ then $B(K, L) := (A//K)^{\text{cop}} \triangleright L$ is a normal Hopf subalgebra of $D(A)$. In this paper we denote by $\mathcal{D}(K, L) := \text{Rep}(D(A)//B(K, L))$ the fusion subcategory of $\text{Rep}(D(A))$ obtained by taking the normal quotient $B(K, L)$.

Our first main result is the following symmetry for the centralizer of $\mathcal{D}(K, L)$:

**Theorem 1.1.** Let $A$ be a semisimple Hopf algebra and $K, L$ be two normal Hopf subalgebras of $A$ such that $B(K, L)$ is a normal Hopf subalgebra of $D(A)$. Then

$$\mathcal{D}(K, L)' = \mathcal{D}(L, K)$$

as fusion subcategories of $\text{Rep}(D(A))$.

Let $A$ be a semisimple Hopf algebra and $K$ be a Hopf subalgebra of the Drinfeld double $D(A)$. We denote by $\mathcal{D}(K)$ the fusion subcategory of $\text{Rep}(D(A))$ whose objects are those $D(A)$-representations that receive a trivial $K$-action.

Our second main result gives a description for the centralizer of $\mathcal{D}(K)$:

**Theorem 1.2.** Let $A$ be a semisimple Hopf subalgebra and $K$ be a normal Hopf subalgebra of $A$. Then

$$\mathcal{D}(K)' = < K >$$

where $< K >$ is the fusion subcategory of $\text{Rep}(D(A))$ generated by the $D(A)$-module $K$. Here $K$ is regarded as $D(A)$-module via the action

$$(f \triangleleft a)x = a_1xS(a_2) \leftarrow S^{-1}f$$

where $a \in A$, $f \in A^*$ and $x \in K$.

**Organization of the paper.** In Section 2 we recall the definition of Müger’s centralizer of a subcategory of a modular fusion category as well as its basic properties that are used throughout this paper. Section 3 presents some results concerning the lattice of fusion subcategories of the category of representation of a semisimple Hopf algebra. We recall here Brauer’s theorem for kernels of representations, as presented in [6]. In Section 4 we prove Theorem 4.8 that relates the kernel of a corepresentation and Müger’s centralizer in the category of representations of a factorizable semisimple Hopf algebra. Theorem 1.2 and Theorem 1.1 for are proven in Section 5.
We work over an algebraically closed field $k$ of characteristic zero. We use Sweedler’s notation for comultiplication with the sigma symbol dropped and the notation $S$ for the antipode. All the other Hopf algebra notations are those used in [21].

Acknowledgments. This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-RU-TE-2012-3-0168.

2. Preliminaries

In this section we recall some preliminary results that are further needed throughout the paper.

General conventions on fusion categories. As usually, by a fusion category we mean a $k$-linear semisimple rigid tensor category $\mathcal{C}$ with finitely many isomorphism classes of simple objects, finite dimensional spaces of morphisms, and such that the unit object of $\mathcal{C}$ is simple. We refer the reader to [11] for a general theory of such categories.

For a fusion category $\mathcal{C}$ we denote by $\Lambda_\mathcal{C}$ the set of isomorphism classes of simple objects of $\mathcal{C}$ and by $\mathcal{O}(\mathcal{C})$ the class of all objects of $\mathcal{C}$. Recall that the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$ is the free $\mathbb{Z}$-module generated by $\Lambda_\mathcal{C}$ with the multiplication induced by the tensor product in $\mathcal{C}$. The Grothendieck ring $K_0(\mathcal{C})$ is a based unital ring (see for example [14] for definition of based rings).

Recall [11] that there is a unique algebra homomorphism $\text{FPdim} : K_0(\mathcal{C}) \to k^*$ such that $\text{FPdim}([X]) > 0$ for any simple object $X$ of $\mathcal{C}$. By definition, the Frobenius-Perron dimension of $\mathcal{C}$ is given by $\text{FPdim}(\mathcal{C}) := \sum_{X \in \Lambda_\mathcal{C}} \text{FPdim}(X)^2$.

By a fusion subcategory of a fusion category $\mathcal{C}$ we understand a full replete tensor subcategory of $\mathcal{C}$. For two fusion subcategories $\mathcal{D}$ and $\mathcal{E}$ of $\mathcal{C}$ we denote by $\mathcal{D} \vee \mathcal{E}$ the smallest fusion subcategory of $\mathcal{C}$ containing both $\mathcal{D}$ and $\mathcal{E}$ as fusion subcategories. If $\mathcal{E} \subseteq \mathcal{D}$ are fusion subcategories of a fusion category $\mathcal{C}$ such that $\text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{E})$ then clearly $\mathcal{D} = \mathcal{E}$.

2.1. Modular fusion categories. Recall that a braided tensor category $\mathcal{C}$ is a tensor category equipped for all $X, Y \in \mathcal{O}(\mathcal{C})$ with natural isomorphisms $c_{X, Y} : X \otimes Y \to Y \otimes X$ satisfying the hexagon axiom, see for example [1, 15]. A twist on a braided fusion category $\mathcal{C}$ is a natural automorphism $\theta : \text{id}_\mathcal{C} \to \text{id}_\mathcal{C}$ satisfying $\theta_1 = \text{id}_1$ and

\begin{equation}
\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y)c_{Y, X}c_{X, Y}.
\end{equation}
A braided fusion category is called *premodular* or *ribbon* if it has a twist satisfying $\theta_{X^*} = \theta_X$ for all $X \in \mathcal{O}(\mathcal{C})$.

Recall that the entries of the $S$-matrix, $S = \{s_{X,Y}\}$ of a premodular category are defined as the quantum trace $s_{X,Y} := tr_q(c_{YX}c_{XY})$, see [28]. A premodular category $\mathcal{C}$ is called *modular* if the above $S$-matrix is nondegenerate.

**Centralizers in braided fusion categories.** Let $\mathcal{C}$ be a modular fusion category and $\mathcal{K}$ be a fusion subcategory of $\mathcal{C}$. Müger introduced the notion of *centralizer of $\mathcal{K}$* as the fusion subcategory $\mathcal{K}'$ of $\mathcal{C}$ generated by all simple objects $X$ of $\mathcal{C}$ satisfying

\[(2.2) \quad c_{X,Y}c_{Y,X} = \text{id}_{X \otimes Y}\]

for all objects $Y \in \mathcal{O}(\mathcal{K})$ (see [22]).

In the case of a ribbon category $\mathcal{C}$ the condition of Equation (2.2) is equivalent to

\[(2.3) \quad s_{X,Y} = \text{FPdim}(X)\text{FPdim}(Y)\]

for all objects $Y \in \mathcal{O}(\mathcal{K})$. Note that in general

\[(2.4) \quad |s_{X,Y}| \leq \text{FPdim}(X)\text{FPdim}(Y)\]

by [22, Proposition 2.5]. In the situation of Equation (2.2) we say that the objects $X$ and $Y$ centralize each other.

**Properties of the centralizer.** If $\mathcal{C}$ is a modular fusion category and $\mathcal{K}$ a fusion subcategory of $\mathcal{C}$ then by [22, Theorem 3.2] one has $\mathcal{K}'' = \mathcal{K}$ and

\[(2.5) \quad \text{FPdim}(\mathcal{K})\text{FPdim}(\mathcal{K}') = \text{FPdim}(\mathcal{C}).\]

Moreover by [22, Theorem 4.2] it follows that if $\mathcal{K}$ is also a modular category then $\mathcal{C} \cong \mathcal{K} \boxtimes \mathcal{K}'$. Let $\mathcal{D}$ and $\mathcal{E}$ be fusion subcategories of a modular fusion category $\mathcal{C}$. Then one has

\[(2.6) \quad (\mathcal{D} \vee \mathcal{E})' = \mathcal{D}' \cap \mathcal{E}' \text{ and } (\mathcal{D} \cap \mathcal{E})' = \mathcal{D}' \vee \mathcal{E}',\]

see [22].

3. **On the lattice of fusion subcategories of $\text{Rep}(A)$**

Let $A$ be a finite dimensional semisimple Hopf algebra over $\mathbb{k}$. Then $A$ is also cosemisimple and $S^2 = \text{id}$, see [20]. The character ring $C(A)$ of $A$ is a semisimple $\mathbb{k}$-subalgebra of $A^*$ and it has a vector space basis given by the set $\text{Irr}(A)$ of irreducible characters of $A$, see [29]. Moreover, $C(A) = \text{Cocom}(A^*)$, the space of cocommutative elements of $A^*$. By duality, the character ring of $A^*$ is a semisimple $\mathbb{k}$-subalgebra of $A$ and $C(A^*) = \text{Cocom}(A)$. If $M$ is an $A$-representation with character $\chi$ then
\[ M^* \] is also an \( A \)-representation with character \( \chi^* = \chi \circ S \). This induces an involution \( \text{ad}^* : C(A) \to C(A) \) on \( C(A) \). Let also \( m_A(\chi, \mu) \) be the usual multiplicity form on \( C(A) \). Recall that if \( M \) and \( N \) are \( A \)-representations affording characters \( \chi, \mu \in C(A) \) respectively, then \( m_A(\chi, \mu) \) is defined by 
\[ m_A(\chi, \mu) := \dim_k \text{Hom}_A(M, N). \]
We will use the notation \( G(A) \) for the set of grouplike elements of \( A \).

For any right \( A \)-comodule \( M \) with comodule structure \( \rho : M \to A \otimes M \) denote by \( C_M \) the subcoalgebra of coefficients of \( M \). Recall that \( C_M \) is the smallest subcoalgebra \( C \subset A \) with the property that \( \rho(M) \subset C \otimes M \), see [19]. If \( d \in C(A^*) \) is the character of \( M \) as an \( A \)-comodule then \( C_M \) is also denoted by \( C_d \).

A left coideal subalgebra of \( A \) is a left coideal \( L \) of \( A \) (i.e a vector subspace with \( \Delta(L) \subset A \otimes L \)) which is also a unitary subalgebra of \( A \). A left coideal subalgebra \( L \) of \( A \) is called normal if it is closed under the left adjoint action \( \text{ad}_L \) of \( A \) on itself. Recall that \( \text{ad}_L(x)(a) = x_1 a S(x_2) \) for all \( a, x \in A \). If \( L \) is a left normal coideal subalgebra of \( A \) then \( AL^+ = L^+ A \) is a Hopf ideal and let \( A/\!\!/L := A/AL^+ \) be the Hopf algebra quotient. Let also \( \pi_L : A \to A/\!\!/L \) be the canonical Hopf projection.

It is known that any fusion subcategory of \( A \) is of the form \( \text{Rep}(A/\!\!/L) \) for some normal left coideal subalgebra \( L \) of \( A \) (see for instance [7].)

3.1. Hopf kernels of representations of semisimple Hopf algebras.
Let \( A \) be any semisimple Hopf algebra and \( M \) a finite dimensional left \( A \)-representation affording a character \( \chi \). By [4, Proposition 1.2] one has that \( |\chi(d)| \leq \chi(1) \epsilon(d) \) for any \( d \in \text{Irr}(A^*) \). Moreover, [4, Remark 1.3] implies that \( \chi(d) = \epsilon(d) \chi(1) \) if and only if the subcoalgebra \( C_d \) acts trivially on \( M \).

Recall [4] that the Hopf kernel \( \text{HKer}_A(M) \) is defined as Hopf subalgebra of \( A \) generated by the set of all characters \( d \in \text{Irr}(A^*) \) such that \( \chi_M(d) = \chi_M(1) \epsilon(d) \), where \( \chi_M \) is the character associated to \( M \). It follows that the Hopf kernel \( \text{HKer}_A(M) \) of \( M \) coincides to the largest Hopf subalgebra of \( A \) that acts trivially on \( M \).

Let \( K \) be a normal Hopf subalgebra of \( A \). Since \( K = \bigoplus_{x \in \text{Irr}(K^*)} C_x \) it follows from above that
\[
(3.1) \quad \text{Rep}(A/\!\!/K) = \cap_{x \in \text{Irr}(K^*)} \text{Rep}(\text{HKer}_A(x^*)).
\]

3.2. Brauer’s theorem for left kernels of representations. Let \( M \) be an \( A \)-module and let \( \text{L Ker}_A(M) \) be the left kernel of \( M \). Recall [6] that \( \text{L Ker}_A(M) \) is defined by:
\[
(3.2) \quad \text{L Ker}_A(M) = \{ a \in A \mid a_1 \otimes a_2 m = a \otimes m, \text{ for all } m \in M \}
\]
Then by [6] it follows that \( \text{L Ker}_A(M) \) is the largest left coideal subalgebra of \( A \) that acts trivially on \( M \). It is also a normal left coideal subalgebra and obviously \( \text{H Ker}_A(M) \subset \text{L Ker}_A(M) \).

Next theorem generalizes a well known result of Brauer in the representation theory of finite groups.

**Theorem 3.3.** [6, Theorem 4.2.1]. Suppose that \( M \) is a finite dimensional module over a semisimple Hopf algebra \( A \). Then
\[
< M > = \text{Rep}(A//\text{L Ker}_A(M))
\]
where \(< M >\) is the fusion subcategory of \( \text{Rep}(A) \) generated by \( M \).

Suppose that \( L \) is a normal left coideal subalgebra of a Hopf algebra \( A \) and let \( B := A//L \) be the quotient Hopf algebra of \( A \) via \( \pi_L : A \to B \). Then under the dual map \( \pi_L^* \) it follows that \( B^* \) is a Hopf subalgebra of \( A^* \) that can be identified with:
\[
\pi_L^*(B^*) = \{ f \in A^* | f(al) = f(a)e(l) \ \text{for all} \ a \in A, \ l \in L \}.
\]

**Proposition 3.6.** Let \( B \) and \( B' \) be two Hopf subalgebras of a semisimple Hopf algebra \( A \). If \( C(A^*) \) is commutative then
\[
\dim_k(< B, B' >) = \frac{(\dim_k B)(\dim_k B')}{\dim_k(B \cap B')}.
\]

**Proof.** Let \( \Lambda_B, \Lambda_{B'} \) be the idempotent integrals of \( B \) and \( B' \) respectively. Since \( C(A^*) \) is commutative one has that \( \Lambda_B \Lambda_{B'} = \Lambda_{B'} \Lambda_B \). This implies that \( \Lambda_B \Lambda_{B'} = \Lambda_{< B, B' >} \). The equality now follows counting the multiplicity of 1 in both sides of the previous equality. \( \square \)

**Remark 3.7.** Alternatively the statement follows from [9, Lemma 3.5] applied to \( \mathcal{C} = \text{Rep}(A^*) \).

If \( S \) and \( R \) are two left coideal subalgebras of \( A \) denote by \( SR \) the vector subspace of \( A \) generated by elements of the type \( sr \) with \( s \in S, r \in R \). Clearly \( SR \) is a coideal of \( A \) but not a coideal subalgebra unless \( SR = RS \). Denote by \(< S, R >\) the left coideal subalgebra of \( A \) generated by \( S \) and \( R \).

**Theorem 3.8.** Let \( L \) and \( K \) be two normal coideal subalgebras of \( A \). Then the following equalities hold in \( A^* \):

1. \((A//L)^* \cap (A//K)^* = (A//LK)^*\).
2. \(< (A//L)^*, (A//K)^* > = (A//(L \cap K))^*\).

If the Grothendieck ring \( \mathcal{G}_0(A) \) is commutative then
\[
\dim_k(LK) = \frac{\dim_k(L \cap K)}{(\dim_k L)(\dim_k K)}.
\]

**Proof.** 1.) It follows directly from Equation (3.5).
2) Using Equation (3.5) it is easy to see that
\[ < (A//L)^*, (A//K)^* > \subseteq (A//(L \cap K))^* . \]

On the other hand, from Takeuchi’s correspondence [27] between Hopf subalgebras of \( A^* \) and coideal subalgebras of \( A \) it follows that there is a normal left coideal subalgebra \( M(L, K) \) of \( A \) such that
\[ (3.10) \quad < (A//L)^*, (A//K)^* > = (A//((L \cap K))^* . \]

Since \( < (A//L)^*, (A//K)^* > \subseteq (A//M(L, K))^* \) it follows that \( M(L, K) \subseteq L \cap K \), which implies that \( \dim_k < (A//L)^*, (A//K)^* > \geq \dim_k (A//((L \cap K))^* \). Then Equation (3.10) implies the equality
\[ < (A//L)^*, (A//K)^* > = (A//((L \cap K))^* . \]

Suppose now that \( G_0(A) \) is commutative. Using the previous proposition it follows that\[ \dim_k < (A//L)^*, (A//K)^* > = \dim_k (A//M(L, K))^* \].

Since \( < (A//L)^*, (A//K)^* > \subseteq (A//((L \cap K))^* \) the equality follows. \( \square \)

For a normal left coideal subalgebra \( L \) of \( A \) it follows that \( M \in \text{Rep}(A//L) \) if and only if \( L \) acts trivially on \( M \). Recall that \( L \) acts trivially on \( M \) if and only if \( x m = \epsilon(x)m \) for all \( x \in L \) and all \( m \in M \).

**Corollary 3.11.** Let \( L \) and \( K \) be two normal left coideal subalgebras of a semisimple Hopf algebra \( A \). Then one has the following:

1) \( \text{Rep}(A//L) \vee \text{Rep}(A//K) = \text{Rep}(A//L \cap K) \).
2) \( \text{Rep}(A//L) \cap \text{Rep}(A//K) = \text{Rep}(A//LK) \).

### 4. Kernels and centralizers in factorizable Hopf algebras

Recall that a Hopf algebra \( A \) is called *quasitriangular* if \( A \) admits an \( R \)-matrix, i.e. an element \( R \in A \otimes A \) satisfying the following properties:

1) \( R \Delta(x) = \Delta^{\text{cop}}(x)R \) for all \( x \in A \).
2) \( (\Delta \otimes \text{id})(R) = R^1 \otimes r^1 \otimes R^2 r^2 \).
3) \( (\text{id} \otimes \Delta)(R) = R^1 r^1 \otimes r^2 \otimes R^2 \).
4) \( (\text{id} \otimes \epsilon)(R) = 1 = (\epsilon \otimes \text{id})(R) \).

Here \( R = r = R^1 \otimes R^2 = r^1 \otimes r^2 \).

If \( (A, R) \) is a quasitriangular Hopf algebra then the category of representations is a braided fusion category with the braiding given by
\[ c_{M, N} : M \otimes N \to N \otimes M, \quad m \otimes n \mapsto R_{21}(n \otimes m) \]
for all \( M, N \in \mathcal{O}(\text{Rep}(A)) \) (see [17]). Recall that \( R_{21} := R^2 \otimes R^1 \).
A quasitriangular Hopf algebra \((A, R)\) is called \textit{factorizable} if and only if the linear map

\[(4.2) \quad \phi_A : A^* \to A, \ f \mapsto (\text{id} \otimes f)(R_{21}R)\]

is an isomorphism of vector spaces. In this situation, following [26, Lemma 2.2] \(\phi_A\) maps the character ring \(C(A)\) onto the center of \(Z(A)\) of \(A\). Moreover by [26, Theorem 2.1] one has that

\[(4.3) \quad \phi_A(f\chi) = \phi_A(f)\phi_A(\chi)\]

for all \(f \in A^*\) and \(\chi \in C(A)\). Thus \(\phi_A|_{C(A)} : C(A) \to Z(A)\) is an isomorphism of \(k\)-algebras.

4.1. \textbf{On the \(S\)-matrix for a factorizable Hopf algebra.} Let \(A\) be a semisimple factorizable Hopf algebra. By [10] (see also [18]) one has that the \(S\)-matrix \((s_{ij})\) of the modular tensor category \(\text{Rep}(A)\) is given by

\[(4.4) \quad s_{ij} = \chi_i(\phi_A(\chi_j^*))\]

Moreover it is not difficult to see that for all \(1 \leq i, j \leq s\), one has that \(s_{ij} = s_{ji}\), and \(s_{ij} = s_{i^*j^*}\) and \(s_{ij^*} = s_{ji^*}\) (cf. [1, 26]). Note that in this case inequality (2.4) becomes

\[(4.5) \quad |s_{ij}| \leq \chi_i(1)\chi_j(1).\]

4.2. \textbf{Central primitive idempotents of the character ring.} Let \(A\) be a semisimple Hopf algebra and \(\chi_1, \ldots, \chi_s\) be the irreducible characters of \(A\). Let also \(e_1, \ldots, e_s\) be their associated central primitive central idempotents in \(A\) and let \(E_j := \phi_A^{-1}(e_j)\) for all \(1 \leq j \leq s\). Since the map \(\phi_A\) from Equation (4.2) is an algebra isomorphism between \(C(A)\) and \(Z(A)\) it follows that \(\{E_j\}_{j=1}^s\) is a set of primitive central idempotents in the character ring \(C(A)\) of \(A\).

\textbf{Proposition 4.6.} Let \(A\) be a semisimple Hopf algebra and \(W\) be an irreducible right \(A^*\)-comodule with associated character \(d \in \text{Irr}(A^*)\). Then for any primitive central idempotent \(E_j \in C(A)\) one has that \(E_j(d) \in \mathbb{Z}_{\geq 0}\) and \(\sum_{j=1}^s E_j(d) = \epsilon(d)\).

\textbf{Proof.} Note that \(W\) can also be regarded as a left \(A^*\)-module. Its character, \(\hat{d} : A^* \to k\), is given by evaluation at \(d\), i.e., \(\hat{d}(f) = f(d)\) for all \(f \in A^*\). Since \(C(A)\) is a semisimple subalgebra of \(A^*\) it follows that \(W\) can also be regarded as a left \(C(A)^\ast\)-module.

Let \(M_j\) be the associated simple \(C(A)^\ast\)-module corresponding to the central primitive idempotent \(E_j \in C(A)\). Then

\[E_j(d) = \hat{d}(E_j) = m_{C(A)}(M_j, W \downarrow_{C(A)}^A) \dim_k M_j \in \mathbb{Z}_{\geq 0}\]
where \( m_{C(A)}(M_j, W \downarrow_{C(A)}^A) \) coincides to the multiplicity of the simple left \( C(A) \)-module \( M_j \) associated to \( E_j \) inside \( W \downarrow_{C(A)}^A \). Clearly \( \sum_{j=1}^s m_{C(A)}(M_j, W \downarrow_{C(A)}^A) \dim_k M_j = \dim_k W = \epsilon(d). \)

For a subset \( X \in \text{Irr}(A) \) we denote by \( \langle \chi \mid \chi \in X \rangle \) the fusion subcategory of \( \text{Rep}(A) \) generated by the irreducible modules \( M \) whose characters satisfy \( \chi_M \in X \).

**Remark 4.7.** Note that for any Hopf subalgebra \( L \subset A \) one has \( \text{Rep}(L^*) \subset \text{Rep}(A^*) \) via the canonical projection \( A^* \to L^* \).

**Theorem 4.8.** Let \( A \) be a semisimple factorizable Hopf algebra and \( d \in \text{Irr}(A^*) \). Then one has the following equality:

\[
\text{Rep}(\text{HKer}_{A^*}(d)^*)' = \langle \chi_j \mid E_j(d) \neq 0 \rangle .
\]

**Proof.** Using the above remark, since \( \text{HKer}_{A^*}(d) \) is a Hopf subalgebra of \( A^* \) it follows that \( \text{Rep}(\text{HKer}_{A^*}(d)^*) \) is a fusion subcategory of \( \text{Rep}(A) \). Moreover, by its definition one has that \( \chi \in \text{Rep}(\text{HKer}_{A^*}(d)^*) \) if and only if \( \chi(d) = \chi(1)\epsilon(d) \).

Using Equation (4.4) and [26, Remark 3.4] one can write that

\[
(4.9) \quad \chi_i = \sum_{j=1}^s \frac{s_{ij}}{s_{0j}} E_j.
\]

for all \( 1 \leq i \leq s \). On the other hand Proposition 4.6 and inequality (4.5) gives that

\[
|\chi_i(d)| = \left| \sum_{j=1}^s \frac{s_{ij}}{s_{0j}} E_j(d) \right| \leq \sum_{j=1}^s \frac{s_{ij}}{s_{0j}} |E_j(d)| \leq \chi_i(1)\epsilon(d)
\]

for any character \( d \in \text{Irr}(A^*) \). Therefore one has that \( \chi_i(d) = \epsilon(d)\chi_i(1) \) if and only if \( s_{ij} = \chi_i(1)\chi_j(1) \) for all \( j \) with \( E_j(d) \neq 0 \). It follows that \( \chi_i \in \text{HKer}_{A^*}(d) \) if and only if \( \chi_i \) centralize any \( \chi_j \) with the property that \( E_j(d) \neq 0 \). This completes the proof of the theorem. \( \square \)

**Corollary 4.10.** Let \( A \) be a factorizable Hopf algebra and \( K \) a normal Hopf subalgebra \( K \) of \( A \). If \( \Lambda_K \in K \) is the idempotent integral of \( K \) then

\[
(4.11) \quad \text{Rep}(A/K)' = \langle \chi_j \mid E_j(\Lambda_K) \neq 0 \rangle .
\]

**Proof.** By Equation (3.1) one has \( \text{Rep}(A/K) = \cap_{x\in\text{Irr}(K^*)} \text{Rep}(\text{HKer}_{A^*}(x)^*) \). Therefore

\[
\text{Rep}(A/K)' = \vee_{x\in\text{Irr}(K^*)} \text{Rep}(\text{HKer}_{A^*}(x)^*)' = \vee_{x\in\text{Irr}(K^*)} < \chi_j \mid E_j(x) \neq 0 > .
\]

\( \square \)
By [19, Proposition 4.1] note that the idempotent integral of $K$ satisfies

\[(4.12) \quad \Lambda_K = \frac{\sum_{x \in \text{Irr}(K^*)} \epsilon(x)x}{\text{dim } K}\]

Then Proposition 4.6 shows that $E_j(\Lambda_K) \neq 0$ if and only if $E_j(x) \neq 0$ for some $x \in \text{Irr}(K^*)$. Thus

\[(4.13) \quad \text{Rep}(A/\!/K)' = \langle \chi_j \mid E_j(\Lambda_K) \neq 0 \rangle .\]

### 4.3. Kernels of grouplike elements

Let $A$ be a semisimple factorizable Hopf algebra and $g \in G(A)$ be a grouplike element. Then Theorem 1.1 implies that

\[(4.14) \quad \text{Rep}(\text{HKer}_A^*(g)^*)' = \langle \chi_g \rangle ,\]

where the character $\chi_g \in \text{Irr}(A)$ is defined as follows. First note that by Proposition 4.6 there is a unique central primitive idempotent $E_g \in C(A)$ that does not vanish on $g$. Then the corresponding irreducible character of the primitive idempotent $\phi_A(E_g)$ is denoted by $\chi_g$.

Note that by [3, Equation (1.3)] the idempotent $\phi_A(E_g)$ is a scalar multiple of

\[(4.15) \quad F_g = \sum_{i=1}^s \chi_i^*(g) \chi_i .\]

### 5. Kernels and centralizers in semisimple quantum doubles

In this section we will describe the Müger centralizer of some fusion subcategories of $\text{Rep}(D(A))$ for any semisimple Hopf algebra $A$. In particular we will prove the two main results mentioned in the introduction.

Recall that the Drinfeld double $D(A)$ of a Hopf algebra $A$ is defined by $D(A) \cong A^{\text{cop}} \otimes A$ as coalgebras with the multiplication given by

\[(5.1) \quad (g \triangleright a)(f \triangleright b) = \sum g(a_1 \to f \leftarrow S^{-1}a_3) \triangleright a_2 b ,\]

for all $a, b \in A$ and $f, g \in A^*$. Moreover its antipode is given by $S(f \triangleright h) = S^{-1}(h)S(f)$. Recall that $(a \leftarrow f \rightarrow b)(x) = f(bra)$ for all $a, b \in A$ and $f \in A^*$. It is well known that $D(A)$ is a semisimple Hopf algebra if and only if $A$ is a semisimple Hopf algebra [21].

If $A$ is a semisimple Hopf algebra and $\mathcal{C} = \text{Rep}(A)$ then $\mathcal{Z}(\mathcal{C}) \simeq \text{Rep}(D(A))$ where $D(A)$ is its quantum double [17].
Moreover, the Drinfeld double \( D(A) \) is a quasitriangular Hopf algebra with \( R \)-matrix: 
\[
R = \sum_{i=1}^{n} (\epsilon \triangleright b_i) \otimes (b_i^* \triangleright 1)
\]
where \( \{b_i\} \) is a vector basis for \( A \) and \( \{b_i^*\} \). In this case one has 
\[
R_{21} R = \sum_{i,j=1}^{n} (b_j^* \triangleright b_i) \otimes (\epsilon \triangleright b_j)(b_i^* \triangleright 1)
\]
and by \([10, Lemma 1.1]\) \( D(A) \) is a factorizable Hopf algebra. The linear map \( \varphi_{D(A)} : D(A)^* \to D(A) \) is given by
\[
f \mapsto \sum_{i,j=1}^{n} (b_j^* \triangleright b_i) f[(\epsilon \triangleright b_j)(b_i^* \triangleright 1)].
\]

5.1. **Left Kernels of normal Hopf subalgebras.** Recall \([30]\) that \( A \) can
be regarded as a \( D(A) \)-module via the action
\[
(f \triangleright a) b = (a_1 b S(a_2)) \leftarrow S f
\]
for all \( a, b \in A \) and all \( f \in A^* \). Then note that any normal Hopf subalgebra \( K \) of \( A \) can be regarded as \( D(A) \)-submodule of \( A \).

**Proposition 5.3.** Let \( K \) be a normal Hopf subalgebra of \( A \). Then
\[
L\text{Ker}_{D(A)}(K) \supseteq (A//K)^* \triangleright L\text{Ker}_A(K),
\]
where \( L\text{Ker}_A(K) \) is the left kernel of \( K \) regraded as a \( A \)-module via the
adjoint action.

**Proof.** From the module structure of Equation (5.2) it can easily be seen that
the left coideal subalgebra \((A//K)^* \triangleright L\text{Ker}_A(K)\) of \( D(A) \) acts trivially
on \( K \). Indeed \((A//K)^* \triangleright L\text{Ker}_A(K)\) acts trivially on \( K \) due to Equation (3.5). Since
\( L\text{Ker}_{D(A)}(K) \) is the largest coideal subalgebra of \( D(A) \) that acts trivially on
the \( D(A) \)-module \( K \) the inclusion \((A//K)^* \triangleright L\text{Ker}_A(K) \subseteq L\text{Ker}_{D(A)}(K)\)
follows. \( \square \)

5.2. **Primitive central idempotents in** \( C(A) \). There is another realiza-
tion (see \([16, Proposition 6.3]\)) of \( A \) as a \( D(A) \)-module on the vector space \( A^* \).
This realization coincides with the trivial \( A \)-module induced to \( D(A) \)
and has the following structure:
\[
(f \triangleright a) g = f(a_1 \rightarrow g \leftarrow S(a_2))
\]
for all \( f, g \in A^* \) and \( a \in A \). Then by \([16]\) each homogenous \( D(A) \)-component
of \( A^* \) can be described as \( A^* E_i \) for some central primitive idempotent \( E_i \in C(A) \), see [16]. Moreover \( \{E_i\}_{i=1,s} \) is a complete set of central orthogonal
primitive idempotents of \( C(A) \).

Recall by \([5, Proposition 3.1]\) that the Fourier transform \( \mathcal{F} : A \to A^* \)
given by \( a \mapsto a \mapsto t \) is a morphism of \( D(A) \)-modules where \( t \in A^* \) is the
idempotent integral of \( A^* \).
Let $A = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ be the decomposition of $A$ into homogenous $D(A)$-components and suppose that $\mathcal{F}(V_i) = A^*E_i$. Define the linear functionals $p_i \in A^*$ by the relation that $p_i$ coincides to $\epsilon_A$ on $V_i$ and vanishes on any other $V_j$ with $i \neq j$. Then by [5, Theorem 5.13] it follows that $p_{V_i} = S(E_i)$ if $\mathcal{F}(V_i) = A^*E_i$.

**Lemma 5.4.** Suppose that $K$ is a normal Hopf subalgebra of a semisimple Hopf algebra $A$ and $d \in \text{Irr}(K^*)$ is a cocommutative element of $K$. With the above notations if $p_{V_i}(d) \neq 0$ then $V_i \subset K$.

**Proof.** If $K$ is a normal Hopf subalgebra of $A$ then $K$ is a full isotopic submodule of $A$, see [5, Proposition 5.4]. Thus without loss of generality one may suppose that as homogenous $D(A)$-components one has the decomposition $K = V_1 \oplus \ldots \oplus V_r$ and $A = K \oplus V_{r+1} \oplus \ldots \oplus V_s$ for some $1 \leq r \leq s$. Then clearly if $p_{V_i}(d) \neq 0$ then $1 \leq i \leq r$ and therefore $V_i \subset K$. \hfill $\Box$

**5.3. Proof of Theorem 1.2.** Now we are ready to prove Theorem 1.2.

**Proof.** Let $\chi_1, \ldots, \chi_l$ be the irreducible characters of $D(A)$ and $\tilde{e}_1, \ldots, \tilde{e}_l \in \mathcal{Z}(D(A))$ be their associated central idempotents in $D(A)$. Denote also by $\tilde{E}_j = \phi_{D(A)}^{-1}(\tilde{e}_j)$ the primitive central idempotents of $D(A)$.

Similar to Equation (3.1) one has that $\mathcal{D}(K) = \cap_{d \in \text{Irr}(K^*)}\text{Rep}(\text{HKer}_{D(A)^*}(d)^*)$. Thus

$$\mathcal{D}(K)' = (\cap_{d \in \text{Irr}(K^*)}\text{Rep}(\text{HKer}_{D(A)^*}(d)^*))' = \vee_{d \in \text{Irr}(K^*)}\text{Rep}(\text{HKer}_{D(A)^*}(d)^*)'$$

Since $D(A)$ is a factorizable Hopf algebra, Theorem 4.8 implies that

$$\mathcal{D}(K)' = \vee_{d \in \text{Irr}(K^*)}(\chi_j : \tilde{E}_j(d) \neq 0) = \chi_j : E_j(d) \neq 0 \text{ for some } d \in \text{Irr}(K^*) \] .$$

Let $\tilde{E}_j$ be a primitive central idempotent of $C(D(A))$ such that $\tilde{E}_j(d) \neq 0$ for some $d \in \text{Irr}(K^*)$. It follows that $\tilde{E}_j \in D(A)^*$ has a nonzero restriction to $A$. Therefore by [16, Theorem 6.3] $\tilde{E}_j \cdot A^{D(A)} = E_j$ where $E_j$ is a central idempotent of $C(A)$ and $\chi_j$ is the character of any simple $D(A)$-submodule of the homogenous component $A^*E_j$. It follows by the above arguments that $S(E_j) = p_{V_j}$. Then the previous lemma implies that $\chi_j$ is the character of a simple $D(A)$-submodule $V_j$ of $A$ with $V_j \subset K$. Thus:

$$\mathcal{D}(K)' = \vee_{d \in \text{Irr}(K^*)}(\chi_j : E_j(d) \neq 0 \text{ for some } d \in \text{Irr}(K^*)) = \vee_{V_j \subset K}(\chi_j : V_j \subset K) = <K>.$$

\hfill $\Box$

It is well known that $D(A) \cong D(A^{* \text{cop}})^{\text{cop}}$ as Hopf algebras via $f \mapsto a \mapsto f$ (see for instance [25, Theorem 3]). By duality this implies the following result:
Corollary 5.5. If $L$ is a normal Hopf subalgebra of $A$ then

$$\mathcal{D}((A//L)^*)' = <(A//L)^*>,$$

where $< (A//L)^*>$ denotes the fusion subcategory of $\text{Rep}(D(A))$ generated by $(A//L)^*$, regarded as a $D(A)$-module via the above isomorphism $D(A) \cong D(A^{* \text{cop}})^{\text{cop}}$. 

Next proposition is the dual version of the Proposition 5.3.

**Proposition 5.6.** Let $L$ be a normal Hopf subalgebra of $A$. Then

$$\text{LKer}_{D(A)}((A//L)^{* \text{cop}}) \supseteq \text{LKer}_{A^*}((A//L)^*) \bowtie L$$

5.4. **Proof of Theorem 1.1.**

**Proof.** Suppose that $K$ and $L$ are normal Hopf subalgebras of $A$. Using Theorem 1.2 and the previous corollary it follows that:

$$\mathcal{D}(K, L)' = (\mathcal{D}((A//K)^*))' = \mathcal{D}((A//K)^*)' \bigvee \mathcal{D}(L)' = <(A//K)^*> \bigvee <L>$$

Theorem 3.3 implies that

$$\mathcal{D}(K, L)' = \text{Rep}(D(A)//\text{LKer}_{D(A)}(A//K)^*) \bigvee \text{Rep}(D(A)//\text{LKer}_{D(A)}(L))$$

Then Corollary 3.11 gives that

$$\mathcal{D}(K, L)' = \text{Rep}(D(A)//(\text{LKer}_{D(A)}(A//K)^* \bigcap \text{LKer}_{D(A)}(L)))$$

Proposition 5.3 and its dual version, Proposition 5.6, imply that

$$\text{LKer}_{D(A)}((A//K)^*) \bigcap \text{LKer}_{D(A)}(L) \supseteq ((A//K)^* \bowtie \text{LKer}_{A}(K)) \bigcap ((A//L)^* \bowtie \text{LKer}_{A^*}((A//L)^*))$$

Note that since $B(K, L) = (A//K)^* \bowtie L$ is a normal Hopf subalgebra of $D(A)$ it follows by [7, Theorem 4.5] that the pairs $(L, K)$ and $((A//L)^*, (A//K)^*)$ are commuting pairs of Hopf subalgebras of $A$ and $A^*$ respectively. Since $L$ commutes elementwise with $K$ it follows that $L$ acts trivially on the $A$-representation $K$ and therefore $\text{LKer}_{A}(K) \supseteq L$. Similarly $\text{LKer}_{A^*}((A//L)^*) \supseteq (A//K)^*$. These two inclusions imply that:

$$((A//K)^* \bowtie \text{LKer}_{A}(K)) \cap ((A//L)^* \bowtie \text{LKer}_{A^*}((A//L)^*)) = (A//K)^* \bowtie L$$

and $\text{Rep}(D(A)//(\text{LKer}_{D(A)}(A//K)^* \bigcap \text{LKer}_{D(A)}(L))) \subseteq \mathcal{D}(K, L)$. Hence one can conclude that $\mathcal{D}(K, L)' \subseteq \mathcal{D}(L, K)$. 

On the other hand, note that
\[ \text{FPdim } \mathcal{D}(K, L) = \frac{(\dim_k A)(\dim_k L)}{\dim_k K}. \]
Thus
\[ \text{FPdim } (\mathcal{D}(K, L)' = \frac{(\dim_k A)^2}{\text{FPdim } \mathcal{D}(K, L)} = \frac{(\dim_k A)(\dim_k K)}{\dim_k L} = \text{FPdim } \mathcal{D}(L, K). \]
This implies that \( \mathcal{D}(K, L)' = \mathcal{D}(L, K). \)

**Corollary 5.9.** If \( K \) is a normal commutative Hopf subalgebra of a semisimple Hopf algebra \( A \) then \( \mathcal{D}(K)' \subseteq \mathcal{D}(K) \).

**Proof.** Note that \( K \) regarded as \( D(A) \)-representation via Equation (5.2) satisfies \( K \in \mathcal{D}(K) \) since \( K \) is a commutative Hopf algebra. Therefore Theorem 1.1 implies that \( \mathcal{D}(K)' = \langle K \rangle \subseteq \mathcal{D}(K) \). \( \square \)

5.5. **Normal Lagrangian subcategories of the category of representation of a Drinfeld double.** We call a fusion subcategory of \( \text{Rep}(A) \) normal if it is of the type \( \text{Rep}(A//L) \) for a normal Hopf subalgebra \( L \) of \( A \). This definition agrees with the definition of a normal fusion subcategory from \([2]\). Let \( \mathcal{C} \) be a premodular category with braiding \( c \) and twist \( \theta \). According to \([8]\), a fusion subcategory \( \mathcal{E} \) of \( \mathcal{C} \) is called isotropic if \( \theta \) restricts to the identity on \( \mathcal{E} \), i.e., if \( \theta(X) = \text{id}_X \) for all \( X \in \mathcal{E} \). Moreover, an isotropic subcategory \( \mathcal{E} \) is called Lagrangian if \( \mathcal{E}' = \mathcal{E} \). Recall also that a fusion category \( \mathcal{C} \) is called hyperbolic if it has a Lagrangian subcategory and \( \mathcal{C} \) is called anisotropic category if it has no non-trivial isotropic subcategories.

**Theorem 5.10.** Suppose that \( F, G \) are finite groups and \( A \) is an abelian extension fitting the extension
\[
\begin{array}{ccc}
k & \rightarrow & k^G \\
& \rightarrow & A \\
& \rightarrow & kF \\
& \rightarrow & k
\end{array}
\]
Then \( \mathcal{D}(k^G, k^G) \) is a Lagrangian subcategory of \( \mathcal{D}(A) \).

**Proof.** Note that \( A^* \) fits the exact sequence
\[
\begin{array}{ccc}
k & \rightarrow & k^F \\
& \rightarrow & A^* \\
& \rightarrow & k^G \\
& \rightarrow & k
\end{array}
\]
Since \( A/k^G \cong k^F \) it follows that \( B(k^G, k^G) \) is a normal Hopf subalgebra of \( \mathcal{D}(A) \). Moreover, by Theorem 1.1 one has that \( \mathcal{D}(k^G, k^G)' = \mathcal{D}(k^G, k^G) \). Let \( \pi : \mathcal{D}(A) \rightarrow \mathcal{D}(A)/B(k^G, k^G) \) be the canonical Hopf projection.

Using \([17, \text{Proposition XIV.2}]\) note that the twist structure on \( \mathcal{D}(A) \) is given by the left multiplication by the Drinfeld element \( u \) associated to \( A \), see also \([10]\).
On the other hand, in our case, the Drinfeld element of $D(A)$ can be written as
\[ u = \sum_{x \in F, a \in G} S(q_x \# a)(p_a \# x) \]
where \( \{q_x\} \) and \( \{p_a\} \) are the dual group element bases in \( k^F \) respectively \( k^G \). It follows that \( \pi(u) = \sum_{a, x} \pi_2(q_x \# a)\pi_1(p_a \# x) = 1 \), which shows that \( \pi \) acts as identity on each object of \( \mathcal{D}(k^G, k^G) \).

\[ \square \]

**Remark 5.13.** Note that [8, Theorem 4.5] shows that in the situation of an abelian extension \( A \) one has that Rep\((D(A))\) is a hyperbolic modular category, i.e. it is braided tensor equivalent to the center \( Z(\mathcal{V}ec_G) \) for some finite group \( G \) and some \( \omega \in H^3(G, k^*) \). In this way one can recover [24, Theorem 1.3].

Recall the subcoalgebra of coefficients \( C_M \) associated to a any right \( A \)-comodule \( M \). By duality, any left \( A \)-module \( V \) with associated character \( \chi \in C(A) \) can be regarded as a right \( A^* \)-module and one can associate to it its subcoalgebra of coefficients \( C_{\chi} \subset A^* \).

### 5.6. Normal Hopf subalgebras of \( D(A) \)

Let \( L \) be a normal Hopf subalgebra of \( A \). Recall that an irreducible character \( \alpha \) of \( L \) is called \( A \)-stable if there is a character \( \chi \in \text{Rep}(A) \) such that \( \chi \downarrow_L^A = \frac{\chi(1)}{\alpha(1)} \alpha \). Such a character \( \chi \in \text{Irr}(A) \) is said to seat over the character \( \alpha \in \text{Irr}(L) \). The set of all irreducible \( A \)-characters seating over \( \alpha \) is denoted by \( \text{Irr}(A|\alpha) \). Denote by \( G_{st}^A(L) \) the set of all \( A \)-stable linear characters of \( L \). Clearly \( G_{st}^A(L) \) is a subgroup of the group of grouplike elements \( G(L^*) \) of the dual Hopf algebra of \( L \).

Suppose that \( K \) and \( L \) are two normal Hopf subalgebras of \( A \) and let \( G \) be a finite group that can be simultaneously embedded in \( G_{st}^A(K) \) and \( G_{st}^A((A//L)^*) \) via the embeeddings \( \psi_1 : G \hookrightarrow G_{st}^A(K) \) and respectively \( \psi_2 : G \hookrightarrow G_{st}^A((A//L)^*) \). Let \( B(K, L, G, \mathcal{X}, \psi_1, \psi_2) \) be the subcoalgebra of \( D(A) \) defined by
\[
B(K, L, G, \mathcal{X}, \psi_1, \psi_2) = \bigoplus_{x \in G} C_{\psi_1(x)\uparrow_K^A} \bowtie C_{\psi_2(x)\downarrow_{(A//L)^*}^A}. \tag{5.14}
\]

Recall by [7, Theorem 4.2] that any normal Hopf subalgebra of \( D(A) \) is of the type \( B(K, L, \mathcal{X}, \psi) \) where \( K, L \) are normal Hopf subalgebras of \( D(A) \) and \( \mathcal{X} \) is a finite group satisfying the above properties, for more details see [7, Subsection 3.2].

Let \( \mathcal{D}(K, L, \mathcal{X}, \psi) \) be the category of representations of the quotient Hopf algebra \( D(A)//B(K, L, \mathcal{X}, \psi) \).
Proposition 5.15. Let $A$ be a semisimple Hopf algebra. Then $\text{Rep}(D(A))$ has a normal Lagrangian fusion subcategory if and only if $A$ is an abelian extension Hopf algebra.

Proof. Suppose that $\mathcal{D}(K, L, \mathcal{X}, \psi)$ is a Lagrangian fusion subcategory of the category $\text{Rep}(D(A))$. In particular one has that $\mathcal{D}(K, L, \mathcal{X}, \phi)' = \mathcal{D}(K, L, \mathcal{X}, \psi)$. Note that the Hopf algebra inclusion $B(K, L) \subseteq B(K, L, \mathcal{X}, \psi)$ induces an inclusion of fusion subcategories $\mathcal{D}(K, L) \supseteq \mathcal{D}(K, L, \mathcal{X}, \psi)$. Thus $\mathcal{D}(L, K) = \mathcal{D}(K, L)' \subseteq \mathcal{D}(K, L, \mathcal{X}, \phi)' = \mathcal{D}(K, L, \mathcal{X}, \phi)$. This inclusion implies that $B(L, K) \supseteq B(K, L, \mathcal{X}, \phi)$. Since $B(K, L) \subseteq B(K, L, \mathcal{X}, \psi)$ this shows that $B(L, K) \supseteq B(K, L)$, and therefore $K = L$. By [7, Theorem 4.3] one has that $[K, L] = 0$, i.e. $K$ is a commutative Hopf algebra. Similarly $(A / \langle K \rangle)^*$ is commutative and therefore $A$ is an abelian extension. \hfill \Box

It would be an interesting question to determine Müger’s centralizer for any normal fusion subcategory of $\text{Rep}(D(A))$ and to decide if the centralizer is also a normal fusion subcategory.

References

1. B. Bakalov and A. Jr. Kirillov, Lectures on Tensor categories and modular functors, vol. 21, Univ. Lect. Ser., Amer. Math. Soc, Providence, RI, 2001.
2. A. Bruguériès and S. Natale, Exact sequences of tensor categories, Internat. Math. Res. Not. 24 (2011), 5644–5705.
3. S. Burciu, On some representations of the Drinfeld double, J. Algebra 296 (2006), 480–504.
4. Normal Hopf subalgebras of semisimple Hopf Algebras, Proc. Amer. Math. Soc. 137 (2009), no. 12, 3969–3979.
5. On normal Hopf algebras of semisimple Hopf algebras, Alg. Rep. Th. (2011).
6. Kernels of representations and coideal subalgebras of Hopf algebras, Glasgow Math. J. 54 (2012), 107–119.
7. On normal Hopf subalgebras of semisimple Drinfeld doubles, preprint (2013).
8. V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, Group-theoretical properties of nilpotent modular categories, arXiv:0704.0195v2.
9. V. Drinfeld, S. Gelaki, D. Nikshych, and V. Ostrik, On braided fusion categories I, Sel. Math. New Ser. 16 (2010), 1–119.
10. P. Etingof and S. Gelaki, Some properties of finite dimensional semisimple Hopf algebras, Math. Res. Lett. 5 (1988), 191–197.
11. P. Etingof, D. Nikshych, and V. Ostrik, On fusion categories, Annals of Mathematics 162 (2005), 581–642.
12. P. Etingof, D. Nikshych, and V. Ostrik, Weakly group-theoretical and solvable fusion categories, Adv. Math. 226 (2011), no. 1, 176–205.
13. S. Gelaki, D. Naidu, and D. Nikshych, Centers of graded fusion categories, Alg. Numb. Th. (2009), no. 3, 959–990.
14. S. Gelaki and D. Nikshych, Nilpotent fusion categories, Adv. Math. 217 (2008), no. 3, 1053–1071.
15. A. Joyal and R. Street, Braided tensor categories, Adv. Math. 102 (1993), 20–78.
16. Y. Kashina, Y. Sommerhäuser, and Y. Zhu, Higher Frobenius-Schur indicators, vol. 181, Mem. Am. Math. Soc., Am. Math. Soc., Providence, RI, 2006.
17. C. Kassel, Quantum groups, Graduate Texts in Mathematics, Springer Verlag, 1995.
18. A. Jr. Kirillov and V. Ostrik, On a $q$-analogue of the McKay correspondence and the ADE classification of $sl_2$ conformal field theories, Adv. Math. 171 (2002), no. 2, 183–227.
19. R. G. Larson, Characters of Hopf algebras, J. Alg. 17 (1971), 352–368.
20. R. G. Larson and D. E. Radford, Finite dimensional cosemisimple Hopf Algebras in characteristic zero are semisimple, J. Alg. 117 (1988), 267–289.
21. S. Montgomery, Hopf algebras and their actions on rings, vol. 82, CBMS Regional Conference Series in Mathematics, Amer. Math. Soc, Providence, RI, 1993.
22. M. Müger, On the structure of modular categories, Proc. Lond. Math. Soc. 87 (2003), 291–308.
23. D. Naidu, D. Nikshych, and S. Witherspoon, Fusion subcategories of representation categories of twisted quantum doubles of finite groups, Internat. Math. Res. Not. 22 (2009), 4183–4219.
24. S. Natale, On group theoretical Hopf algebras and exact factorizations of finite groups, J. Alg. 270 (2003), no. 1, 199–211.
25. D. E. Radford, Minimal Quasitriangular Hopf Algebras, J. Algebra 157 (1993), 285–315.
26. H. J. Schneider, Some properties of factorizable Hopf algebras, Proc. Amer. Math. Soc. 129 (1990), 1891–1898.
27. M. Takeuchi, A correspondence between Hopf ideals and Hopf subalgebras, manuscripta Math. 7 (1972), 251 – 270.
28. V. Turaev, Crossed group-categories, Arab. J. Sc. Eng. 33, no. 2C (2008), 484–503.
29. Y. Zhu, Hopf algebras of prime dimension., Int. Math. Res. Not. 1 (1994), 53–59.
30. ______, A commuting pair in Hopf algebras, Proc. Amer. Math. Soc. 125 (1997), 2847–2851.