A new time relaxation model with iterative modified Lavrentiev regularization method is studied. The aim of the relaxation term is to drive the unresolved fluctuations in a computational simulation to zero exponentially faster by an appropriate and often problem-dependent choice of its time relaxation parameter; together with iterative modified Lavrentiev regularization, the model will give a better approximation through de-convolution with fewer steps to compute. The goal of this paper herein is to understand how this time relaxation term acts to truncate solution scales and to use this understanding to give some helpful insight into parameter selection.

1. Introduction

Direct numerical simulation of a 3D turbulent flow typically is usually not considered as computationally feasible and desire, since it often requires a spatial mesh size of \( N_{dof}^{NSE} \approx O(Re^{9/4}) \) points per time step [H69]. However the largest structures in the flow (containing most of the flow’s energy) are commonly sought since they are responsible for much of the mixing and most of the flow’s momentum transport. Hence, various numerical regularization for truncating the small scales and turbulence models of the large scales are used for simulations seeking to predict flow averages instead.

In this paper, one such improved model is studied: a time relaxation with the iterative modified Lavrentiev regularization introduced as a numerical regularization. This regularization method aims to truncate the small scales in a solution without altering the solution’s large scales. A number of useful properties were already discussed in [MZ18]. The effect of the time relaxation combined with iterative modified Lavrentiev regularization is the main focus of this paper, and an optimal choice of the time relaxation parameter \( \chi \) is also discussed.

To introduce the Time Relaxation term, when added to the Navier-Stokes equations, a continuum model has to be considered; let the spatial domain be \( \Omega = (0, L)^3 \) and suppose periodic with zero mean boundary conditions.
are imposed on $\partial \Omega$:

\[(1)\quad \phi(x + L e_j, t) = \phi(x, t) \quad \text{and} \quad \int_{\Omega} \phi(x, t) \, dx = 0 \quad \text{for} \quad \phi = u, p, f, u_0\]

A local spatial filter associated with a length-scale $\delta$ is selected out of many possible choices, e.g., \cite{BIL05}, \cite{JO4}, and \cite{SO1}. In this paper, only the differential filter is considered, \cite{G86} (related to a Gaussian, e.g., \cite{G00}): given a $L$-periodic $\phi(x)$, its average $\bar{\phi}$ is the unique $L$-periodic solution of the following:

\[(2)\quad -\delta^2 \Delta \bar{\phi} + \bar{\phi} = \phi, \quad \text{in} \quad \Omega\]

This filtering operator is defined as $G : G\phi = \bar{\phi}$ and $A : A\phi = -\delta^2 \Delta \phi + \phi$ as the inverse of $G$ for convenient usage throughout this paper. The $N^{th}$ iterative modified Lavrentiev regularization operator $D_N$ (the special $N = 0$ case is also included) is defined compactly by:

\[
D_0\bar{\phi} := ((1 - \alpha)G + \alpha I)^{-1}\bar{\phi}
\]

\[
D_N\bar{\phi} := \left(\sum_{j=0}^{N} (I - D_0G)^j\right)D_0\bar{\phi}, \quad \text{for} \ N = 1, 2, 3, \ldots
\]

Section 2.1 will give more details on this de-convolution operator. The (bounded) operator $D_N$ is an approximation to the (unbounded) inverse of the filter $G$ in the sense that for very smooth functions and as $\delta \to 0$:

\[\phi = D_N\bar{\phi} + O(\alpha^{N+1}\delta^{2N+2})\]

e.g., \cite{MZ18}. The model I consider is to find the $L$-periodic (with zero mean) velocity and pressure satisfying:

\[
(3)\quad \begin{array}{c}
\begin{aligned}
u_t + u \cdot \nabla u + \nabla p - \nu \Delta u + \frac{\chi}{\delta}(u - D_N(\bar{u})) &= f, \quad \text{in} \quad \Omega \times (0, T) \\
u(x, 0) &= u_0(x), \quad \text{in} \quad \Omega \\
\n\n\n\n\n\n\end{aligned}
\end{array}
\]

The Time Relaxation term $\chi$ will be specified later. The term $u - D_n(\bar{u})$ is a generalized fluctuation included to drive fluctuations below $O(\delta)$ to zero rapidly as $t \to \infty$ without affecting the order of accuracy of the model’s solution $u$ as an approximation to the resolved scales ($\geq O(\delta)$).

To use Time Relaxation, the relaxation $\chi$ must be chosen. Analytical guidance concerning its appropriate choice with respect to other problem parameters is essential. Our work herein has been greatly inspired by \cite{LN07}.

1.1. **Summary of Results.** The results are presented in the following sections with full details. I give here an overview of the main results of this paper keeping notation as simple and transparent as possible and describing only the most interesting cases.
Section 3 reviews the analytic framework of the space-periodic problem. Using simple energy estimates, I show that the component of the solution of (3) fluctuating below $O(\delta)$ must $\to 0$ in $L^2(\Omega \times (0, T))$ as $\chi \to \infty$. This result follows directly from the continuum equations (3) and validates the relaxation term as a general computational strategy but it sheds no light into parameter selection or the details of how scales are truncated by the relaxation term. In section 4, I demonstrate those details by developing a similarity theory for (3) following the $K^{-41}$ theory of the Navier-Stokes equations. There are several interesting cases, but the most important consequences for practical computing is the following predicted optimal scaling of the relaxation parameter which forces the model's micro-scale $\eta_{\text{model}} = O(\delta)$:

$$(4) \quad \chi \simeq \frac{U}{L^2} \delta^{\frac{1}{3}} (1 + \frac{1}{\alpha})^{N+1}$$

Note the $\chi \to \infty$ as $\alpha \to 0$ as required in the analytic estimates of Section 4. For this value of the relaxation term the consistency error of the relaxation term is:

$$|\frac{\chi}{\delta} (u - D_N \bar{u})| = O(\chi \alpha^{N+1} \delta^{2N+1}) = O((1 + \alpha)^{N+1} \delta^{2N+\frac{2}{3}})$$

Note that this consistency error $\to 0$ as either $\alpha$ or $\delta \to 0$, so the solution from the Time Relaxation plus iterative modified Lavrentiev regularization will converge to the true solution of NSE.

2. Preliminaries

The De-convolution problem is central in both image processing [BB98] and turbulence modelling in large eddy simulation [BIL05, Geu97, LL03, LL05, LL07]. The basic problem in approximate de-convolution is: given $\bar{u}$, then find a better approximation of $u$. In other words, solve the following equation for an approximation which is appropriate for the application at hand:

$$Gu = \bar{u}, \quad \text{solve for } u$$

For most filtering operators, $G$ is symmetric and positive semi-definite. Typically, $G$ is not invertible. Thus, this de-convolution problem is ill posed.

2.1. The Modified Iterative Tikhonov-Lavrentiev Deconvolution.

The Modified Tikhonov-Lavrentiev Deconvolution was first studied in [MZ18]. So for each $N = 0, 1, \ldots$, it computes an approximate solution $u_N$ to the above deconvolution by $N$ steps of a fixed point iteration [BB98, MZ18]. Rewrite the above de-convolution equation as the fixed-point problem:

*given* $\bar{u}$ *solve* $u = ((1 - \alpha)G + \alpha I)^{-1} u + ((1 - \alpha)G + \alpha I)^{-1} (\bar{u} - Gu)$ *for* $u$

The de-convolution approximation is then computed as follows.

$$((1 - \alpha)G + \alpha I)u_0 = \bar{u}$$

$$(5) \quad ((1 - \alpha)G + \alpha I)(u_n - u_{n-1}) = \bar{u} - Gu_{n-1}, \quad \text{for } n = 1, 2, \ldots$$
Although $G$ is non-invertible, $((1 - \alpha)G + \alpha I)$ will be invertible for appropriate choice of $0 < \alpha < 1$. And in [MZ18], it can be shown that there is an optimal stopping $N$, which gives the best approximation to $u$. And the deconvolution problem is easier to compute, and the operator $((1 - \alpha)G + \alpha I)$ is symmetric. So convergence as $N \to \infty$ can be expected.

**Definition 2.1.** The $N^{th}$ iterative modified Lavrentiev regularization operator $D_N : L^2(\Omega) \to L^2(\Omega)$ is the map $D_N : \bar{u} \to u_N$, or $D_N(\bar{u}) = u_N$. $H_N$ denotes the map $H_N : L^2(\Omega) \to L^2(\Omega)$ by $H_N(\phi) := D_N G\phi = D_N(\phi)$.

By eliminating the intermediate steps, it is easy to find an explicit formula for the $N^{th}$ deconvolution operator $D_N$:

\begin{equation}
D_N\phi = \left(\sum_{j=0}^{N} (I - D_0 G)^j \right) D_0 \phi, \quad \text{where } D_0 = ((1 - \alpha)G + \alpha I)
\end{equation}

The consistency error $e_n = u - u_N = u - D_N \bar{u}$ of $D_N$ as an approximate inverse of $G$ is shown to be $O(\alpha^{N+1} \delta^{2N+2})$ in [MZ18].

3. **Energy Estimates**

Recall that I impose the zero men condition $\int_{\Omega} \phi \, dx = 0$ on $\phi = u, p, f$, and $u_0$. We can thus expand the fluid velocity in a Fourier series

\[ u(x, t) = \sum_k \hat{u}(k, t) e^{-i k \cdot x}, \quad k = \frac{2\pi n}{L} \text{ is the wave number and } n \in \mathbb{Z}^3 \]

The Fourier coefficients are given by

\[ \hat{u}(k, t) = \frac{1}{L^3} \int_{\Omega} u(x, t) e^{-i k \cdot x} \, dx. \]

Magnitudes of $k$ and $n$ are defined by:

\[ |n| = |n_1|^2 + |n_2|^2 + |n_3|^2 \frac{1}{2}, \quad |k| = \frac{2\pi |n|}{L}, \]

\[ |n|_\infty = \max \{|n_1|, |n_2|, |n_3|\}, \quad |k|_\infty = \frac{2\pi |n|_\infty}{L} \]

The length-scale of the wave number $k$ is defined by $l = \frac{2\pi}{|k|_\infty}$. Parseval’s equality implies that the energy in the flow can be decomposed by wave number as follows. For $u \in L^2(\Omega)$,

\[ \frac{1}{L^3} \int_{\Omega} \frac{1}{2} |u(x, t)|^2 \, dx = \sum_k \frac{1}{2} |\hat{u}(k, t)|^2 = \sum_k \left( \sum_{|k|=k} \frac{1}{2} |\hat{u}(k, t)|^2 \right), \]

where $k = \frac{2\pi n}{L}$ is the wave number and $n \in \mathbb{Z}^3$.

Let $< \cdot >$ denote a long time averaging (e.g., [R95]),

\begin{equation}
<\phi>(x) := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(x, t) \, dt.
\end{equation}
**Definition 3.1.** The kinetic energy distribution functions are defined by

\[ E(k, t) = \frac{L}{2\pi} \sum_{|k|=k} \frac{1}{2} |\hat{u}(k, t)|^2 \quad \text{and} \quad E(k) := \langle E(k, t) \rangle. \]

Parseval’s equality thus can be re-written as the following:

\[ \frac{1}{L^3} \int_\Omega \frac{1}{2} |u(x, t)|^2 \, dx = \frac{2\pi}{L} \sum_k E(k, t) \quad \text{and} \quad \langle \frac{1}{L^3} \int_\Omega \frac{1}{2} |u(x, t)|^2 \, dx \rangle = \frac{2\pi}{L} \sum_k E(k). \]

**Lemma 3.2.** Define the bounded linear operator \( H_N : L^2(\Omega) \rightarrow L^2(\Omega) \) by \( H_N \phi = D_N G \phi \). Then, \( H_N \) and \( I - H_N \) are both symmetric, positive semi-definite operators on \( L^2_0(\Omega) \). For \( u \in L^2_0(\Omega) \),

\[ \int_{\Omega} (u - H_N u) \cdot u \, dx \geq 0, \quad \int_{\Omega} (H_N u) \cdot u \, dx \geq 0. \]

**Proof.** Both \( H_N \) and \( D_N \) are functions of the symmetric positive semi-definite operator \( G \), so symmetry is immediate and positivity is easily established in the periodic case by a direct calculation using Fourier series. To begin, expand \( u(x, t) = \sum_k \hat{u}(k, t) e^{-i k \cdot x} \), where \( k = \frac{2\pi n}{L} \) is the wave number and \( n \in \mathbb{Z}^3 \). With the corresponding \( \hat{G}(k) = \frac{1}{1 + \delta^2 k^2} \) and \( \hat{D}_0(k) = \frac{1 + \delta^2 k^2}{1 + \alpha \delta^2 k^2} \), and direct calculation using Parseval’s equality

\[ \frac{1}{2L^3} \int_{\Omega} (H_N u) \cdot u \, dx = \frac{2\pi}{L} \sum_k \hat{H}_N(k) E(k, t), \]

where \( \hat{H}_N(k) = \frac{1}{1 + z^2} \sum_{j=0}^N (1 - \frac{1}{1 + z^2})^j \), where \( z = \sqrt{\alpha \delta} k \)

The expression for \( \hat{H}_N(k) \) can be simplified by summing the geometric series. Thus giving

\[ \hat{H}_N(k) = 1 - (\frac{z^2}{1 + z^2})^{N+1}, \quad z = \sqrt{\alpha \delta} k \]

Since \( z \) is real, \( 0 \leq \frac{z^2}{1 + z^2} \leq 1 \), and \( 0 \leq 1 - \frac{z^2}{1 + z^2} \leq 1 \). Thus I have shown

\[ 0 \leq \int_{\Omega} (H_N u) \cdot u \, dx \leq \int_{\Omega} |u|^2 \, dx. \]

Similarly, it can be shown \( 0 \leq 1 - \hat{H}_N(k) \leq 1 \) and

\[ 0 \leq \int_{\Omega} (u - H_N u) \cdot u \, dx \leq \int_{\Omega} |u|^2 \, dx, \]

which completes the proof. \( \square \)
It is insightful to plot the transfer function \( \hat{H}_N(k) = 1 - (\frac{z}{1+z})^N \) for a few values of \( N \). I do so for \( N = 5, 10, 100 \) (see Fig. ??).

Examining these graphs, I observe that \( H_N(u) \) is very close to \( u \) for the low frequencies/largest solution scales and that \( H_N(u) \) attenuates small scales/high frequencies. The breakpoint between the flow frequencies and high frequencies is somewhat arbitrary.

**Remark 3.3.** By the above lemma and energy estimate, and the model’s relaxation term thus extracts energy from resolved scales. Thus, I can define an energy dissipation rate induced by time relaxation for (refsec1:model) as the following:

\[
\varepsilon_{\text{model}}(u)(t) := \frac{1}{L^3} \int_\Omega \chi \delta(u - H_N u) \cdot u \, dx.
\]

The models kinetic energy is the same for the Euler equations:

\[
E_{\text{model}}(u)(t) := \frac{1}{L^3} \frac{1}{2} ||u(t)||^2.
\]

4. A Similarity Theory of Time Relaxation with Modified Iterative Tikhonov-Lavrentiev Deconvolution

I consider now the Navier-Stokes equations with timer relaxation at a high enough Reynolds number and large enough relaxation coefficient that viscous dissipation is negligible. It would be natural to ask whether this new Time Relaxation model actually truncates scales.

First I want to find the model’s equivalent of the large scales’ Reynolds number of the Navier-Stokes equations. Recall the Reynolds number for the Navier-Stokes equations is the ratio of non-linearity to viscous terms acting on the largest scales:

\[
\text{for the NSE: } Re \simeq \frac{|u \cdot \nabla u|}{|\nu \nabla u|} \simeq \frac{U \frac{H}{L} U}{\nu \frac{H}{L} U} = \frac{UL}{\nu}.
\]

The NSE’s Reynolds number with respect to the smallest scales is obtained by replacing the large scales velocity and length by their small scales equivalent as in \( Re_{\text{small}} = \frac{u_{\text{small}}}{\nu U} \). To process I must find the physically appropriate and mathematically analogous quantity for the new Time Relaxation model. Again, this derivation is under the assumption that the viscous dissipation is negligible compared to dissipation caused by Time Relaxation. Proceeding analogously, it is similar that the ratio of the non-linearity to dissipative effects should be the analogous quantity, and it should be:

\[
Re_N \simeq \frac{|u \cdot \nabla u|}{\frac{H}{L}(u - H_N u)}.
\]
Keeping in mind that for large scales $\delta \ll L$ and $0 < \alpha < 1$, then I can simplify the previous equation (10)

$$Re_N \simeq \frac{|u \cdot \nabla u|}{\left| \frac{1}{\delta} (u - H_n u) \right|}$$

$$\simeq \frac{\chi \left( \frac{\alpha \delta^2}{L^2} \right)^{N+1} (1 + \frac{\alpha \delta^2}{L^2})^{-N+1} U}{UL^{2N+1} \chi^{\alpha N+1} \delta^{2N+1}}$$

$$= \frac{UL^{2N+1}}{\chi^{\alpha N+1} \delta^{2N+1}} (1 + \frac{\alpha \delta^2}{L^2})^{-N+1}$$

This parameter definition can also be obtained by non-dimensionalization.

**Definition 4.1.** The non-dimensionalization time relaxation parameter for the new Time Relaxation model is:

$$Re_N = \frac{UL^{2N+1}}{\chi^{\alpha N+1} \delta^{2N+1}}, \quad N = 0, 1, 2, \ldots$$

Next I must form the small scales parameters which measure the ratio of non-linearity to dissipation on the smallest persistent eddies. Let $u_{small}$ denote a characteristic velocity of the smallest persistent eddies and let $\eta_{model}$ denote the length scale associated with them. Then exactly as in equation (10), I calculate:

$$Re_{N-small} \simeq \frac{|u_{small} \cdot \nabla u_{small}|}{\left| \frac{1}{\delta} (u_{small} - H_n u_{small}) \right|}$$

$$\simeq \frac{u_{small}^{2N+1}}{\chi^{\alpha N+1} \delta^{2N+1}} \left( \frac{\alpha \delta^2}{\eta_{model}^2} \right)^{N+1} (1 + \frac{\alpha \delta^2}{\eta_{model}^2})^{-N+1} u_{small}$$

$$= \frac{u_{small}^{2N+1}}{\chi^{\alpha N+1} \delta^{2N+1}} (1 + \frac{\alpha \delta^2}{\eta_{model}^2})^{-N+1}$$

It is no longer reasonable to assume any order relationship between $\delta$ and $\eta_{model}$.

**Definition 4.2.** Let $u_{small}$ and $\eta_{model}$ denote, respectively, a characteristic velocity and length of smallest persistent structures in the flow. The non-dimensionalized parameter associated with the smallest persistent scales of new Time Relaxation model is

$$Re_{N-small} = \frac{u_{small}^{2N+1}}{\chi^{\alpha N+1} \delta^{2N+1}} (1 + \frac{\alpha \delta^2}{\eta_{model}^2})^{-N+1}$$

In order to find out what $\chi$ is in terms of $\delta$ and $\alpha$, the following calculation is based upon two principles:

$$Re_{N-small} = O(1) \quad \text{at length - scale } \eta_{model}$$
and statistical equilibrium in the form energy input at large scales = dissipation at small scales. As in the Navier-Stokes equations, the new Time Relaxation term’s energy cascade is halted by dissipation caused by the time relaxation effects grinding down eddies exponentially fast when \( \text{Re}_N - \text{small} = O(1) \) at length-scale \( \eta_{\text{model}} \). The rate of energy input, at the large scales, is \( O \left( \frac{U^3}{L} \right) \), exactly as in the Navier-Stokes case. The dissipation at the smallest resolved scales, estimated carefully, is

\[
dissipation \text{ at small scales} \approx \frac{\chi}{\delta} (u - H_N(u))u
\]

\[
\approx \frac{\chi}{\delta} (-\alpha \delta^2)^{N+1} (D_0 G)^{N+1}(\Delta^{N+1}) u
\]

\[
\approx \frac{\chi}{\delta} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}} \right)^{N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{-(N+1)} u_{\text{small}}^2
\]

These two conditions hence give the following pair of equations

\[
\frac{u_{\text{small}}^2}{\chi_{\eta_{\text{model}}}^{N+1}} (1 + \frac{\alpha \delta^2}{\eta_{\text{model}}} )^{N+1} \approx 1
\]

\[
\frac{U^3}{L} \approx \frac{\chi}{\delta} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}} \right)^{N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{-(N+1)} u_{\text{small}}^2
\]

The characteristic velocity of the smallest eddies, \( u_{\text{small}} \) can be solved in terms of the other parameters

\[
u_{\text{small}} \approx \frac{\chi_{\alpha}^{N+1} \delta^{2N+1}}{\eta_{\text{model}}^{2N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{N+1}}
\]

Inserting this \( u_{\text{small}} \) to the second equation yields the following equation determining the model’s micro-scale

\[
(13) \quad \frac{U^3}{L} \approx \frac{\chi}{\delta} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}} \right)^{N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{-(N+1)} \left[ \frac{\chi_{\alpha}^{N+1} \delta^{2N+1}}{\eta_{\text{model}}^{2N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{N+1}} \right]^2
\]

However in order to determine the model’s micro-scale, we have to discuss three different cases: \( \delta < \eta_{\text{model}}, \delta > \eta_{\text{model}}, \) and \( \delta = \eta_{\text{model}}. \) And the third case will provide the most important insight into practical computation.

**Case 1 (Fully Resolved).** In this case \( \delta < \eta_{\text{model}} \) and I also have \( 0 < \alpha < 1, \) so that \( 1 + \frac{\alpha \delta^2}{\eta_{\text{model}}} \approx 1. \) And the equation (13) reduces to

\[
\frac{U^3}{L} \approx \frac{\chi}{\delta} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}} \right)^{N+1}\left[ \frac{\chi_{\alpha}^{N+1} \delta^{2N+1}}{\eta_{\text{model}}^{2N+1}(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}})^{N+1}} \right]^2
\]

Solving for \( \eta_{\text{model}} \), I have the following

\[
\eta_{\text{model}} = \left( \frac{\chi^2 L}{U^3} \right)^{\frac{1}{2N+1}} \alpha^{\frac{1}{N+1}} \delta^{\frac{1}{N+1}}
\]

**Case 2 (Under Resolved).** In this case \( \delta > \eta_{\text{model}}, \) there can be three different
situations $\alpha < \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$, $\alpha > \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$, and $\alpha = \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$. When $\alpha$ is small enough, $\eta_{\text{model}}$ will be the same from the fully resolved case; When $\alpha > \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$, $1 + \frac{\alpha \delta^2}{\eta_{\text{model}}^2} \simeq \frac{\alpha \delta^2}{\eta_{\text{model}}^2}$, the equation (13) can be simplified as the following

$$\frac{U^3}{L} \simeq \frac{\chi}{\delta} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}^2} \right)^{N+1} \left( \frac{\alpha \delta^2}{\eta_{\text{model}}^2} \right)^{-(N+1)} \left( \frac{\chi \alpha^{N+1} \delta^{2N+1}}{\eta_{\text{model}}^2 \left( \frac{\eta_{\text{model}}}{\delta} \right)^N} \right)^2$$

After further simplification, I get

$$\eta_{\text{model}} = \sqrt{\frac{U^3 \delta^3}{\chi^3 L}}$$

I do not, however, have a very appealing interpretation on how I can use this $\eta_{\text{model}}$ for future prediction of turbulent flow. And if $\alpha = \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$, $1 + \frac{\alpha \delta^2}{\eta_{\text{model}}^2} \simeq 2$. I can determine the choice of relaxation parameter that enforces $\alpha = \left( \frac{\eta_{\text{model}}}{\delta} \right)^2$. Setting $\alpha \delta^2 = \eta_{\text{model}}$ and solving for $\chi$ gives

$$\chi = \frac{U}{L^2} \left( \frac{\delta}{\alpha} \right)^{\frac{3}{2}}$$

The consistency error of the relaxation term (evaluated for smooth flow fields) is, for this scaling of relaxation parameter,

$$|\frac{\chi}{\delta}(u - D_N \bar{u})| = O(\chi \alpha^{N+1} \delta^{2N+1}) = O(\alpha^{N+\frac{1}{2}} \delta^{2N+\frac{1}{2}})$$

Even though the ratio $\frac{\delta}{\alpha}$ might be greater than 1, the consistency error is still going to 0, for appropriate choice of $\alpha$ and $\delta$.

**Case 3** *(Perfectly Resolved)*. In this case $\delta = \eta_{\text{model}}$, so that $1 + \frac{\alpha \delta^2}{\eta_{\text{model}}^2} = 1 + \alpha$. Equation (13) can be simplified as follows

$$\frac{U^3}{L} \simeq \frac{\chi^3}{\delta} \left( \frac{\alpha}{1 + \alpha} \right)^{3N+3}$$

Solving for $\chi$ gives

$$\chi \simeq \frac{U}{L^3} \left( \frac{\delta}{\alpha} \right)^{\frac{3}{2}} (1 + \frac{1}{\alpha})^{N+1}$$

The associated consistency error with this particular choice of $\chi$ gives the following estimate

$$|\frac{\chi}{\delta}(u - D_N \bar{u})| = O(\chi \alpha^{N+1} \delta^{2N+1}) = O((1 + \alpha)^{N+1} \delta^{2N+\frac{1}{2}})$$

As $\alpha \rightarrow 0$, $\chi \rightarrow \infty$, and the consistency error is going to $O(\delta^{2N+\frac{7}{3}})$, which will still go to zero as $\delta \rightarrow 0$. 

4.1. Interpreting the assumption that viscous dissipation is negligible. My assumption that viscous dissipation is negligible compared to dissipation caused by time relaxation holds when the Kolmogorov micro-scale for the Navier-Stokes equations is very small compared to the model’s micro-scale induced by the Time Relaxation term. It is because that by the $K-41$ theory, viscous dissipation is considered negligible at scales above the Kolmogorov micro-scale. Since the model’s micro-scale is indeed above the Kolmogorov micro-scale (for computational practical purpose), with high enough Reynolds number and large enough time relaxation parameter, it is possible that the Time Relaxation term dominates the viscosity, forcing the latter to be negligible.

The first interpretation of “large enough” is that $\eta_{\text{model}} \gg \eta_{\text{Kolmogorov}}$. When $\eta_{\text{model}} \gg \eta_{\text{Kolmogorov}}$, I can consider the case when $\delta \leq \eta_{\text{model}}$, and have the following results

$$\left(\frac{\chi^3 L}{U^3}\right)\frac{1}{6N+4} \alpha^{\frac{1}{2} + \frac{1}{6N+4}} \delta^{1 - \frac{1}{6N+4}} = \eta_{\text{model}} > \eta_{\text{Kolmogorov}} = Re^{-\frac{3}{4}}$$

Hence I have the following lower bound for $\chi$

$$\chi > (Re^{-\frac{3}{4}}L)^{2N+\frac{5}{4}} \frac{U}{L^{\frac{3}{4}}} \alpha^{-(N+1)} \delta^{-(2N+1)}$$

In the typical case of $\delta \gg \eta_{\text{Kolmogorov}}$ and $\chi$ large, it will place almost no constraint upon the time relaxation parameter.

The second interpretation is that at $\eta_{\text{model}} = \eta_{\text{Kolmogorov}}$, $Re_{\text{small}} \gg Re_{\text{N-small}}$ and $Re_{\text{small}} = \frac{\eta_{\text{small}} \eta_{\text{Kolmogorov}}}{\nu}$; this also gives a mild condition on $\chi$:

$$\chi > \nu \left(\frac{\delta}{\eta_{\text{model}}}ight)^{-2N} \delta^{-1} \alpha^{-(N+1)} \left(1 + \frac{\alpha \delta^2}{\eta_{\text{model}}^2}\right)^{N+1}$$

5. Conclusions and Open Problems

This Time Relaxation with iterative modified Lavrentiev regularization possesses an energy cascade that truncates that energy spectrum at a point that depends upon the global velocity $U$, the global length scale $L$, the de-convolution parameter $\alpha$, and the filtering radius $\delta$. This Time Relaxation term does not dissipate appreciable energy for the resolved scales of the flow for $N$ large enough. The action of this time relaxation term is to induce a micro-scale, analogous to the Kolmogorov micro-scale in the turbulence, and to trigger decay of eddies at the model’s own micro-scale. The extra dissipation at the cut-off length scale induced by time relaxation must reduce the number of degrees of freedom needed (per time step) for a 3D turbulent flow simulation. With proper choice of $\chi$, this extra dissipation will also balance the transfer of energy to those scales from the flow’s power input and thus prevent a non-physical accumulation of energy around the cut-off length scale as well as force the model’s micro-scale to coincide with the averaging radius $\delta$ and de-convolution parameter $\alpha$. 
From equation (14), \( \chi \simeq \frac{L}{\delta^2} (1 + \frac{1}{\alpha})^{N+1} \), the model’s micro-scale is \( \delta \) and the number of degrees of freedom (per time step) needed for a 3D turbulent flow simulation with the model (3) is

\[
N_{dof} \simeq \left( \frac{L}{\delta} \right)^3, \quad \text{independent of } Re!
\]

This leads to a huge computational speedup using (3) over a DNS of

\[
\left( \frac{N_{dof}^{NSE}}{N_{dof}} \right)^\frac{1}{4} \simeq \left( \frac{Re^2}{L^3 \delta^{-3}} \right) = \left( \frac{\delta}{L} \right)^4 Re^3
\]

This Time Relaxation combined with iterative modified Lavrentiev regularization provided a faster and yet cheaper way to compute a de-convoluted solution, with consistency error \( |u - D(\bar{u})| = O(\alpha^{N+1}\delta^{2N+2}) \), providing better accuracy. The above value of \( \chi \) is derived for fully developed turbulent flow.
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Department of Applied Mathematics & Statistics, Johns Hopkins University, Baltimore, MD
E-mail address: mzhong5@jhu.edu