Assessing the Effects of the Uncertainty in Reheating Energy Scale on Primordial Spectrum and CMB

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The details of reheating energy scale \(\rho_{reh}\) is largely uncertain today, independent of inflation models. This would induce uncertainty in predicting primordial spectrum. Such uncertainty could be very large, especially for spectra with large running \(n_s\). We find that for some inflation models with a large \(d\ln n_S(k)/d\ln k\), \(\rho_{reh}\) could be highly restricted by current CMB observations.

The cosmic microwave background radiation (CMBR) anisotropy for the \(l\)-th multipole \(C_l\) by definition is related to the angular correlation function\[1\]:

\[
\langle \Delta(\hat{n}_1)\Delta(\hat{n}_2) \rangle \equiv \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)C_l P_l[\cos(\hat{n}_1 \cdot \hat{n}_2)], \tag{1}
\]

and for scalar temperature modes

\[
C_l^S \equiv \frac{2\pi}{l(l+1)} C_l = \frac{4\pi}{(2l+1)} \int \frac{dk}{k} T_l^2(k) P_S(k), \tag{2}
\]

where \(T_l(k)\) is the transfer function and \(P_S(k)\) is the primordial spectrum from inflation. The scalar spectrum index is defined as \(n_s(k) \equiv 1 + d\ln P_S(k)/d\ln k\).

Usually one can specify the \(k\) modes during inflation basing on the e-folds number \(N(k)\) before the end of inflation\[2\]:

\[
N(k) = 62 - \ln \frac{k}{a_0H_0} - \ln \frac{10^{16}GeV}{V(k)^{1/4}} + \ln \frac{V(k)^{1/4}}{V_{end}^{1/4}} - \frac{1}{3} \ln \frac{V_{end}^{1/4}}{\rho_{reh}^{1/4}}, \tag{3}
\]

where \(V(k), V_{end}\) denote the inflaton potential at \(k = aH\) and at the end of inflation respectively, \(\rho_{reh}\) is the energy density when reheating ends, resuming a standard big bang evolution. Commonly one takes \(a_0 = 1\), where the subscript '0' denotes the present value. For a given model of inflation (which ends naturally), \(P_S(a/a_{end})\) or \(P_S(N)\) can be exactly formulated, however \(N(k)\) cannot be specified accurately in the lack of details of reheating. Consequently, we are unable to tell precisely when a mode \(k\) leaves out horizon, and there exists an uncertainty in predicting the primordial spectrum and its effects on \(C_l\). In this paper we study in detail two effects related to the uncertainty of \(N(k)\): the normalization of inflaton's potential and the effects on the spectrum of CMBR.

Commonly one assumes \(n_s = constant\) without using specific inflation models when fitting cosmological parameters to observations. An analysis by Wang et al.\[3\] gave that current observations indicated \(n_s = 0.91^{+0.15}_{-0.07}\). However, for a general inflation model, it does not give exactly \(n_s = constant\). Sometimes significant scale-dependence of \(n_s(k)\) is possible and allowed by current data\[4\]. It is often important to relate primordial spectra from specific inflation models with CMBR observations for constraining model parameters, making distinctive predictions and excluding some certain inflation models.

When applying \(k\) to \(P_S\), a typical method is to specify a mode around the COBE pivot scale \(k_{COBE} \approx 7.0a_0H_0\). The three undecided energy scales \(V(k_{COBE}), V_{end}^{1/4}\) and \(\rho_{reh}^{1/4}\) give a typical range of \(N(k_{COBE}) = 40 \sim 60\). It is
even possible that \( N(k_{CMB}) = 0 \sim 60 \) if there exists thermal inflation. The fit to CMB data can give an estimation of \( V(k_{CMB}) \) for specific inflation models. However the reheating energy scale cannot be restricted stringently by observations today. Phenomenologically \( \rho_{reh}^{1/4} \) can be in the range of \( 1 MeV \sim 10^{16} GeV \), which based on Eq.3, gives rise to an uncertainty of 15 in \( N_{CMB} \).

If the slow rolling(SR) parameter \( \epsilon \) is small enough, the effect of changing \( N(k_{CMB}) \) is exactly like a horizontal translation in the \( P_S(k) - k \) picture. When the primordial spectrum is scale invariant, the translation changes nothing. As long as \( P_S(k) \) is tilted, choosing \( a_1H_1 \) or \( a_2H_2 \) as \( k_{CMB} \) would globally change the value of \( P_S(k) \). When \( n_s(k) \) is scale independent, different choice of \( k_{CMB} \) will change \( P_S(k) \) to \( cP_S(k) \), where \( c \) is a constant, acting also like a vertical shift of \( P_S(k) \) in the \( P_S(k) - k \) picture. In Fig.1 we show the effect of choosing different \( N_{CMB} \). Assuming the Hubble constant \( H \) remains the same on the shown scale, although \( P_S(N) \) is exactly known, \( N_{CMB} = 50 \) or 60 will give different value for \( P_S(k) \). Assuming constant \( n_s \) and no gravitational waves, Bunn and White’s fitting to COBE observation gives 4:

\[
|\delta_H(k_{CMB})| = 1.94 \times 10^{-5} \times \Omega_0^{-0.785 - 0.05 \ln \Omega_0} \exp[-0.95(n_s - 1) - 0.169(n_s - 1)^2],
\]

with an uncertainty less than 10% , where

\[
\delta_H(k) = \frac{2g(\Omega_0)}{\Omega_0} P_S^{1/2}(k)
\]

and

\[
g(\Omega_0) = \frac{5}{2} \Omega_0 \left( \frac{1}{70} + \frac{209\Omega_0}{140} - \frac{\Omega_0^2}{140} + \Omega_0^{4/7} \right)^{-1},
\]

with less than 5% uncertainty for \( g(\Omega_0)/\Omega_0 \). \( P_S(k_{CMB}) \) has been stringently restricted for given \( n_s \) and \( \Omega_0 \). On the other hand, taking single-field inflation model \( V = V_0 f(\phi/\mu) \) as a example, one has

\[
P_S(k) \propto \frac{V^3}{V_0^2} \propto V_0.
\]

For different \( N_{CMB} \), \( V_0 \) has to be chosen differently to match the COBE normalization. For \( n_s = 0.84 \), \( \Delta N = 15 \) the uncertainty on the amplitude of \( V_0 \) is about \( e^{15(1-0.84)} \approx 11 \).

In some sense the uncertainty of \( V_0 \) matters little, since current inflation theories can not physically give exact value of \( V_0 \). \( V_0 \) can even be used as a free parameter to be normalized by CMB and LSS observations. For \( n_s = constant \), the effects caused by the uncertainty of \( N_{CMB} \) will be fully cancelled by normalization. However, in the case of \( dn_s/d\ln k \neq 0 \), even if \( V_0 \) is best used, two normalized primordial spectra \( P_S(k) \) with different choosing of \( N(k_{CMB}) \) still cannot fully overlap.

Now let us study the effects on CMB. For simplicity we have fixed the cosmological parameters as \( h = .64, \Omega_\Lambda = 0.66, \Omega_0 h^2 = 0.020 \) and \( \Omega_k = 0.05 \) in the following studies.

Firstly when the running of \( n_S \) is very small, the uncertainty of \( \rho_{reh} \) cannot bring forth observable effects in CMB. In Fig.2 we give an example

\[
P_S(k) = A(k/k_*)^{0.91 - 1 - 0.001 \ln(k/k_*)},
\]
where the constant $A$ is to be normalized. $C_{\text{ol}}$ denotes $C_l$ when setting $N_{\text{COBE}} = N_*$. For assessing the difference, we have used the cosmic variance values, where

$$\Delta \hat{C}_l = \sqrt{\frac{2}{2l+1}} \hat{C}_l.$$  \hfill (9)

The two specified lines stand for choosing $N'_{\text{COBE}} = N_* + 15$ and $N_* - 15$ respectively. They are both within the cosmic variance of $C_{\text{ol}}$ for $l \lesssim 2000$ and are hard to distinguish from $C_{\text{ol}}$. We’ve also checked the effect of choosing different $n_*$ in Eq.8, and find the result remains the same for $0.84 < n_* < 1.06$. In this sense, for many inflation models in which $dn_*/d\ln k$ are very small, such as chaotic and natural inflation, the effect caused by the uncertainty of reheating temperature is negligible, provided $V_0$ can be normalized.

Secondly we rebuild a primordial spectrum with a constant running $dn_*/d\ln k = -0.022$:

$$P_S(k) = A(k/k_*)^{1.014 - 0.011 \ln(k/k_*)}. \hfill (10)$$

$n_S$ runs from 1.06 to 0.84 on the scale relevant to CMB and LSS observations ($k = 3 \times 10^{-4} \sim 6 \ hMpc^{-1}$) when choosing $k_*=k_{\text{COBE}}$. $P_S(k)$ with constant index $n_S=0.91^{+0.15}_{-0.07}$ is allowed by current observations, phenomenologically such a running should also satisfy current observations. In Fig.3, $C_{\text{ol}}$ stands for $C_l$ when choosing $k_{\text{COBE}} = k_*$, and the two dashed lines are its cosmic variance limits. One can see $C_{\text{ol}}$ is hardly distinguishable from the constant index spectrum case in which $n_S = 0.97$. As the figure also shows, one e-fold of uncertainty in $N_{\text{COBE}}$ can not make difference out of cosmic variance if $A$ can be best normalized. And it is obvious that if $N_{\text{COBE}}$ varies larger than 10, i.e. when $n_S$ is globally larger than 1.06 or less than 0.84, this spectrum will be excluded by current observations.

Significant scale dependent primordial spectrum can explain the tentatively observed feature at $k \sim 0.05 Mpc^{-1}$, and in some degree solve the small scale problems of the $CDM$ model on dark halo densities and dwarf satellites, but it also can be easily excluded by current observations if $N(k)$ changes. The location of bump on $P_S$ may be shifted to different $k$ by the uncertainty of $N_{\text{COBE}}$, 15 e-fold of uncertainty can make the bump out of the range of CMB and LSS observation scale, being fully unobsorable, or to any other mode where $k \neq 0.05 Mpc^{-1}$, making the primordial spectrum disagree with the LSS and CMB observations. In the BSI model which has been used to solve the $CDM$ puzzle on small scales, one requires a red-tilted $P_S(k)$ which is near scale invariant at scale $k = 3 \times 10^{-4} \sim 6 \ hMpc^{-1}$ and damps severely at larger $k$, however the location of such feature is still sensitive to $\rho_{\text{reh}}$.

Now let us study a specific inflation model considered in ref. \cite{16}:

$$V(\phi) = V_0(1 + \cos \frac{\phi}{f} + \delta \cos \frac{N\phi}{f}). \hfill (11)$$

For the calculations here we’ve set $N = 300$, $\delta = 5 \times 10^{-5}$ and $f = 0.4 M_{Pl}$. Slow rolling is well satisfied for $N > 50$, using the SR parameter $\epsilon$:

$$\epsilon = \frac{M_{Pl}^2}{16\pi} \left( \frac{V_\phi}{V} \right)^2, \hfill (12)$$

where $V_\phi \equiv \partial V/\partial \phi$. One can easily get

$$P_S(k) = \frac{8}{3 M_{Pl}} V(k)/\epsilon(k). \hfill (13)$$
With the background values of the cosmological parameters mentioned above, COBE normalization predicts

\[ V(k)^{\frac{1}{2}} \approx 6.4 c^{\frac{1}{2}} \exp[-0.95(n_{S} - 1)/2 - 0.169(n_{S} - 1)^2/2] \times 10^{16} \text{GeV} \]

(14)

around \( k_{\text{COBE}} \). For \( 0.84 < n_{S} < 1.06 \) the exponential factor is negligible. The normalization to full current CMB and LSS data differs little with the COBE normalization. In Fig.4 the Hubble constant \( H \) remains almost constant for \( N > 40 \) and it changes no more than one percent on the full shown scale, rendering \( d\ln k \approx dN \). \( V(N) \) is also almost constant for \( N > 40 \) and \( V^{1/4}(N > 40)/V^{1/4}_{\text{end}} \approx 2.0 \), disregarding the exact values of \( V_{0} \). In the current CMB and LSS observation scale, the minimum value of \( k \) is about \( k_0 = H_0 \), the maximum possible value \( N(k_0) \) is

\[ N(k_0) \approx 62 - \ln\left(\frac{10^{16} \text{GeV}}{6.4 \times 10^{14} \text{GeV}}\right) + \ln 2 \approx 60.0 \]

(15)

where \( \epsilon \sim 10^{-8} \). The corresponding spectrum index is \( n_s(k_0) \approx 0.96 \). As one can see from Fig.4, the primordial spectrum with a certain range where \( N(k_0) > 60.0 \) should also fit current CMB data well, but the details of this model has excluded the possibility. In the figure, \( N = 52.6 \) corresponds to \( n_s = 0.84 \). It is also obvious that for \( n_s(k_0) \leq 0.84 \), the model is excluded by current observations, and \( n_s(k_0) > 0.84 \) gives \( \rho_{\text{reh}}^{1/4} > 10^{9} \text{GeV} \). To avoid too many gravitinos, one requires \( \rho_{\text{reh}}^{1/4} < 10^{10} \text{GeV} \), i.e. \( N(k_0) < 56.5 \), \( n_s(k_0) \leq 0.91 \). Phenomenologically when thermal inflation follows, \( N(k_0) \) can be much smaller, however, such spectra cannot satisfy current CMB data, as shown in Fig.5. Only a small range between the two dotted lines may be acceptable. In this range the spectrum has a large running \( |dS/d\ln k| \sim 0.01 \), such an \( n_s \) cannot be excluded by current CMB data, as can be seen from Fig.6 that it performs exactly like the constant \( n_s = 0.9 \) case. Meanwhile such a tilted spectrum exhibits the possibility of solving the CDM puzzle on small scales.

In summary, the uncertainties of \( \rho_{\text{reh}} \) has induced some difficulty in formulating \( P_S(k) \) from inflation model. This led to additional degeneracy in the inflation parameter \( V_0 \) which otherwise could be highly restricted for specific inflation models\[19\]. For models in which \( dn_s/d\ln k \) is small enough, the uncertainty is negligible. However, for models where the spectrum is significant scale-dependent, the difference induced by reheating can be very large as shown in this paper. Furthermore, for some specific inflation models which have large running \( n_s \), current observations provide a constraint on the reheating temperature. Before concluding we should point that some aspects relevant to the uncertainty have been partly mentioned in Refs\[4, 7, 20, 21\], but our paper is different from those mentioned above since we mainly focus on the uncertainty of reheating and on restriction of \( \rho_{\text{reh}} \).

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FIG. 1: Left: the primordial spectrum $P_S$ as a function of the e-fold number $N$; Right: $P_S$ as a function of the wavenumber $k$. The figure illustrates the effect on $P_S(k)$ when choosing different $N_{COBE}$.

FIG. 2: The ratio of different CMB angular power spectra $C_l$ relative to $C_{60}$. $C_{60}$ stands for $n_S(k_{COBE}) = 0.91$ and $dn_S/d\ln k = -0.002$, the region between the two dashed lines are allowed by its cosmic variance.
**FIG. 3:** $C_{\text{ot}}$ stands for $n_S(k_{\text{COBE}}) = 1.014$ and $dn_S/d\ln k = -0.022$.

**FIG. 4:** The SR parameter $\epsilon$, spectrum $P_S$, scalar spectra index $n_S$ and its running $dn_S/d\ln k$ with $N = 300$, $\delta = 5 \times 10^{-5}$ and $f = 0.4M_P$ for $V(\phi) = V_0(1 + \cos \frac{\phi}{f} + \delta \cos \frac{N\phi}{f})$, the region within the two dashed lines is roughly allowed by the model and current observations for $k_0 = H_0$. 
FIG. 5: CMB angular power spectra for choosing $N(k_{\text{COBE}})$ differently, assuming the existence of thermal inflation some time after the inflation in Eq. 11. From left top to bottom, the lines stand for $N(k_{\text{COBE}}) = 37$, 27 and 32 respectively.

FIG. 6: $C_{01}$ stands for $N(k_0) \approx 60$ for the $P_S(N)$ shown in Fig. 4.