1. INTRODUCTION

Model Predictive Control (MPC) (Mayne, 2014) is a widely-used optimization-based control method, which is able to handle general nonlinear constrained systems. For nominal MPC schemes, which are assuming that an actual deterministic model of the system is available, rigorous theoretical guarantees (such as recursive feasibility, constraint satisfaction and stability) are well established in the literature (Rawlings et al., 2017). Robust and stochastic MPC (RMPC and SMPC, respectively) have been developed to ensure these properties despite uncertainties in the model and/or external disturbances (Kouvaritakis and Cannon, 2016). While RMPC generally assumes that uncertainties lie in bounded sets, SMPC can additionally incorporate stochastic descriptions. This enables SMPC to enforce chance constraints, which are constraints that allow for a given probability of violation.

In many domains, stochastic models for complex phenomena, e.g., loads or failures in electrical power grids, are well-established, yet these phenomena often arise in already nonlinear control problems. In order to tackle such problems, we propose an SMPC framework for nonlinear systems with rigorous theoretical guarantees. Existing SMPC approaches for nonlinear systems (Schildbach et al., 2014) suffer from a tremendous amount of online computation. Our method on the other hand is able to consider nonlinear systems under general disturbances at the price of only a limited increase in online computational demand over nominal MPC scheme.

Related work

Mesbah (2016) summarizes the current state of the art of SMPC and notes that there is a lack of efficient algorithms for nonlinear systems that are able to consider general probabilistic uncertainty descriptions. In this work, we aim to provide such an algorithm in tradition of tube-based approaches to SMPC, which are among the most efficient methods.

Tube-based solutions to propagate uncertainty were first proposed for RMPC (Chisci et al., 2001) for linear systems. This has later been extended to nonlinear systems using class $K$ functions or Lipschitz constants (Pin et al., 2009). Such approaches were shown to be conservative, especially for longer prediction horizons, and are often difficult to implement for nonlinear systems. This method was extended by Santos et al. (2019) to the stochastic case. The authors were able to precompute a constraint backoff for the chance constraints enforced at the first step, since they only considered the additive disturbances. From there, the uncertainty could be propagated as in any RMPC approach, as they eliminated the uncertainty early in the approach. In this article, we consider general uncertainty, that may also depend on the current state and input, hence we have to consider uncertainty in the online optimization.

Villanueva et al. (2017) proposed to compute the tube fully online employing min-max-differential-inequalities. This leads to a complexity increase over nominal MPC in the number of states squared. A middle ground in online complexity is achieved in Köhler et al. (2019), where, by using sublevel sets of an incremental Lyapunov function (ILF) as tube, the authors reduce the conservatism significantly, while only requiring a single additional state and constraint over nominal MPC. Our method on the other hand just introduces a single constraint for each chance constraint probability considered, enabling stochastic disturbances.

Inspired by these result, we propose an extension of the computationally efficient framework by Köhler et al. (2019) to SMPC, which is additionally able to consider stochastic disturbances and chance constraints.

Notation

The quadratic norm with respect to a positive definite matrix $Q > 0$ is denoted by $\|x\|_Q^2 = x^\top Q x$, the minimal and maximal eigenvalue of $Q$ are denoted by $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$, respectively.
2. PRELIMINARIES

2.1 Problem setup

We consider a nonlinear stochastic discrete-time system

\[ x(t + 1) = f(x(t), u(t), d(t)) \]

with time \( t \in \mathbb{N} \), state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^m \), and bounded independent identically distributed (i.i.d.) random variables \( d(t) \in \mathcal{D} \) as disturbance. The nominal prediction model is chosen by certainty-equivalence as

\[ x^+ = f(x, u) := E_x[ f(x, u, d)] \]

Thus, the system can be decomposed into

\[ x^+ = f_u(x, u, d) := f(x, u) + d_w(x, u) \]

with the model mismatch \( d_w \) as random variable.

Firstly, we enforce hard state and input constraints

\[ (x(t), u(t)) \in \mathcal{Z}_R \]

with some compact nonlinear constraint set \( \mathcal{Z}_R = \{ (x, u) \in \mathbb{R}^{n+m} | g_j(x, u) \leq 0, j = 1, \ldots, q_f \} \subseteq \mathbb{R}^{n+m} \).

Secondly, we impose nonlinear individual chance constraints (ICC) on an output at the next time step, i.e.,

\[ \mathbb{P}_t[h_j(x(t+1), u(t+1)) \leq 0] \geq p_j, j = 1, \ldots, q_p \]

with a probability level \( p_j \in (0,1) \). The set of all probability levels used by at least one of the ICCs is denoted

\[ \mathcal{P} := \{ p_j | j = 1, \ldots, q_p \} \]

Instead of requiring the exact cumulative distribution function, we make use of a lower bound thereof, which may be easier to obtain in practice.

Assumption 1. The random variable \( d_w \) (3) has a known probability distribution \( p_w(x, u) \) with compact finite support \( \mathcal{W}(x, u) \) for all \( (x, u) \in \mathcal{Z}_R \). Hence, for any \( \epsilon \in [0,1] \), there exists a scalar function \( \tilde{w} : \mathcal{Z}_R \to \mathbb{R}_+ \) that satisfies

\[ \mathbb{P}[ \| d_w(x,u) \| \leq \tilde{w}(x,u)] \geq \epsilon \]

with \( \tilde{w}(x,u) \) finite for all \( x \) and \( u \). Furthermore, \( \tilde{w} \) satisfies the following monotonicity property:

\[ \forall (x, u) \in \mathcal{Z}_R, 0 \leq c_1 \leq c_2 \leq 1 : \tilde{w}^{c_1}(x, u) \leq \tilde{w}^{c_2}(x, u) \]

This uncertainty description encompasses additive, multiplicative and more general nonlinear disturbances or unmodeled nonlinearities.

We assume that \( f(0,0) = 0 \) and that the constraints satisfy

\[ 0 \in \text{int}(\mathcal{Z}_R) \cap \{ (x, u) \in \mathbb{R}^{n+m} | h_j(x, u) \leq 0, j = 1, \ldots, q_h \} \]

since we consider the problem of stabilizing the origin. Further, the control objective is to minimize the open-loop cost \( J_N \) of the predicted state and input sequence, with

\[ J_N(x_{[t], u_{[t]}}) = \sum_{k=0}^{N-1} \ell(x_{k|t}, u_{k|t}) + V_f(x_{N|t}) \]

where the stage cost \( \ell \) and terminal cost \( V_f \) (defined in Sec. 3.3) are positive definite.

2.2 Local incremental stabilizability

In order to provide the theoretical guarantees for the constraint backoff and robust stability, it is assumed, similarly to Köhler et al. (2019), that the system is locally incrementally stabilizable. For completeness, we restate the required assumptions and adapt them, where necessary, to accommodate the chance constraints.

Assumption 2. (Köhler et al., 2019, Ass. 2) There exist a control law \( \kappa : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \), an incremental Lyapunov function (ILF) \( V_f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \), which is continuous in the first argument and satisfies \( V_f(z, v) = 0 \) for all \( (z, v) \in \mathcal{Z}_R \), and parameters \( c_{\delta, t}, c_{\delta, u}, \delta_{\text{loc}}, \kappa_{\text{max}} > 0, \rho(\cdot) \in (0,1) \), such that the following properties hold for all \( (x, z, v) \in \mathbb{R}^n \times \mathbb{R}^m \) with \( V_f(x, z, v) \leq \delta_{\text{loc}} \), and all \( (x^+, z^+, v^+) \in \mathbb{R}^n \times \mathbb{Z}_R \):

\[ c_{\delta, t} \| x - z \|^2 \leq V_f(x, z, v) \leq c_{\delta, u} \| x - z \|^2, \]

\[ \| \kappa(x, z, v) - v \| \leq \kappa_{\text{max}} V_f(x, z, v), \]

\[ V_f(x^+, z^+, v^+) \leq \rho V_f(x, z, v), \]

with \( x^+ = f(x, \kappa(x, z, v)) \), and \( z^+ = f(z, v) \).

The ILF will be used to construct the stochastic tube later on, yet we only require its existence and knowledge of the scalar parameters, but not the functions \( V_f, \kappa \) themselves. In particular, we exploit the fact that the ILF provides an upper bound on the achievable contraction rate between two trajectories, e.g., between the predicted trajectory of the MPC scheme and the closed-loop trajectory.

The following assumptions enable us to compute scalar bounds that relate the nonlinear constraints (4) and (5) to the level sets of the ILF \( V_f \).

Assumption 3. (Köhler et al., 2019, Ass. 3) The stage cost \( \ell : \mathcal{Z}_R \to \mathbb{R}_+ \geq 0 \) satisfies

\[ \ell(r) \geq \alpha_\ell(\| r \|), \]

\[ \ell(r) - \ell(r) \leq \alpha_\ell(\| r \|), \forall r \in \mathcal{Z}, r \in \mathbb{R}^{n+m}, \]

with \( \alpha_\ell, \alpha_c \in \mathbb{K}_\infty \). Furthermore, for any \( \rho \in (0,1) \), we have \( \alpha_{c\rho}(c) := \sum_{k=0}^{\infty} \alpha_c(\rho^k c) \in \mathbb{K}_\infty \).

Assumption 4. There exist local Lipschitz constants \( L_i^R, L_j^P \) such that

\[ g_i(r) - g_i(r) \leq L_i^R \| r - r \|, \quad i = 1, \ldots, q_R, \]

\[ h_i(r) - h_i(r) \leq L_i^P \| r - r \|, \quad i = 1, \ldots, q_P, \]

holds for all \( r \in \mathcal{Z}_R \) and all \( r \in \mathbb{R}^{n+m} \) with \( \| r - r \| \leq \frac{\delta}{c_{\delta, i}} \).

Proposition 5. Suppose that Ass. 2–4 hold, then there exist constants \( l_{ij}^R \geq 0, i = 1, \ldots, q_R, c_{ij}^R \geq 0, j = 1, \ldots, q_P \), and a function \( \alpha_c \in \mathbb{K}_\infty \) such that the following inequalities hold for all \( (x, z, v) \in \mathbb{R}^n \times \mathbb{R}^m \) with \( V_f(x, z, v) \leq c^2 \) and any \( c \in [0, \delta_{\text{loc}}] : \)

\[ \ell(x, \kappa(x, z, v)) - \ell(z, v) \leq \alpha_c(c), \]

\[ g_j(x, \kappa(x, z, v)) - g_j(z, v) \leq c_{ij}^R \cdot c, \]

\[ h_j(x, \kappa(x, z, v)) - h_j(z, v) \leq c_{ij}^P \cdot c. \]

Proof. For the proof of the first part, i.e., (17) and (18), see Köhler et al. (2019, Prop. 1). Equation (19) is derived analogously to (18).
This proposition will allow us to relate the constraints to the tube, we construct in the next sections.

2.3 Efficient uncertainty description

Additionally, for the tube construction, we need to consider how uncertainty propagation affects the ILF. A computationally efficient way, proposed by Köhler et al. (2019), is to describe the uncertainty in terms of the ILF. As we not only consider bounded disturbances, but also stochastic uncertainties, we need an revised construction.

**Assumption 6.** Consider the disturbance bound \( \tilde{w} \), the incrementally stabilizing feedback \( k \) and the ILF \( V_{\delta} \) from Ass. 1, and 2. For any \( c \in [0, 1] \), there exists a function \( w_{\delta}^{c} : \mathbb{Z} \times \mathbb{R}_{0}^{n} \rightarrow \mathbb{R}_{0} \), such that for any point \( (x, z, v) \in \mathbb{R}^{n} \times \mathbb{Z} \) with \( V_{\delta}(x, z, v) \leq c^2 \), and any \( c \in [0, \delta_{oc}] \), we have

\[
\tilde{w}(x, \kappa(x, z, v)) \leq w_{\delta}^{c}(z, v, c).
\]

Furthermore, \( w_{\delta}^{c} \) satisfies the following monotonicity properties: Firstly, for any point \( (x, z, v) \in \mathbb{R}^{n} \times \mathbb{Z} \) such that \( V_{\delta}(x, z, v) \leq (c_1 - c_2)^2 \) with constants \( 0 \leq c_2 \leq c_1 \leq \delta_{oc} \), we have

\[
w_{\delta}^{c_1}(x, \kappa(x, z, v), c_2) \leq w_{\delta}^{c_2}(z, v, c_1).
\]

Secondly, for any constant \( 0 \leq c_1 \leq c_2 \leq 1 \), we have

\[
w_{\delta}^{c_1}(x, \kappa(x, z, v), c) \leq w_{\delta}^{c_2}(z, v, c).
\]

This assumption establishes \( w_{\delta}^{c} \) as an \( \epsilon \)-upper bound on the uncertainty that can occur at a state \( x \) of an incrementally stabilized trajectory in a neighborhood of a point \( (z, v) \in \mathbb{Z}_{R} \), where the neighborhood is given by \( V_{\delta}(x, z, v) \leq c^2 \). Based thereupon, we can bound the increase of the ILF due to the disturbance in the next time step with probability \( \epsilon \).

**Proposition 7.** Let Ass. 1, 2, and 6 hold. Then, there exists a function \( w_{\delta}^{c} : \mathbb{Z} \times \mathbb{R}_{0}^{n} \rightarrow \mathbb{R}_{0} \), such that for any point \( (x, z, v) \in \mathbb{R}^{n} \times \mathbb{Z} \) with \( V_{\delta}(x, z, v) \leq c^2 \), any \( c \in [0, \delta_{oc}] \), any \( (z^+, v^+) \in \mathbb{Z} \) with \( z^+ = f(z, v) \), and disturbance \( d_w \) as random variable, we have

\[
\mathbb{P}[V_{\delta}(z^+ + d_w(x, \kappa(x, z, v)), z^+, v^+)] \leq (w_{\delta}^{c}(z, v, c))^2 \geq \epsilon.
\]

Furthermore, \( w_{\delta}^{c} \) satisfies the same monotonicity properties as \( w_{\delta} \), i.e., (21) and (22) hold for \( w_{\delta}^{c} \).

**Proof.** The proof follows trivially from the assumptions, by setting \( \tilde{w}_{\delta}^{c}(z, v, c) = \sqrt{w_{\delta}^{c}(z, v, c)} \).

The function \( w_{\delta}^{c} \) can be constructed similarly as in Köhler et al. (2019), an example there of is given in Sec. 4. In the absence of chance constraint, we could now construct the tube as in Köhler et al. (2019). In fact, by setting \( \epsilon = 1 \), this reduces to the same considerations and results. This is how we will implement the hard constraints (4). For the chance constraints (5), however, additional consideration are required, in order to ensure closed-loop constraint satisfaction, which are discussed later in Sec. 3.2.

3. STOCHASTIC MODEL PREDICTIVE CONTROL FRAMEWORK

This section presents the proposed stochastic MPC framework for nonlinear uncertain systems. The overall scheme is introduced in Sec. 3.1. In Sec. 3.2 the constraint backoff for the chance constraints are discussed. The theoretical analysis in Sec. 3.4 uses the terminal ingredients described in Sec. 3.3.

3.1 Proposed nonlinear MPC scheme

The basic idea of our scheme is similar to Köhler et al. (2019). Therein, the authors proposed to indirectly characterize the tube as sublevel sets of the ILF \( V_{\delta} \) (Ass. 2) by an online predicted tube size \( s^i \). Then, this tube size is used to tighten the state and input constraints ensuring robust constraint satisfaction. In this work, this is extended to allow for ICCs and stochastic uncertainties. Here, satisfaction is ensured by designing a constraint backoff. This depends on the state \( x \) and the input \( u \) to accommodate the \( (x, u)\)-dependence of the disturbance, as well as on the robust tube size \( s^i \), in order to account for uncertainty accumulated over the previous steps of the prediction. This backoff introduces additional probabilistic tube sizes \( s^p \).

This lead to the deterministic optimization problem

\[
V_N(x(t)) = \min_{u_{|t}, w_{|t}, x_{|t}} J_N(x_{|t}, u_{|t})
\]

\[
\text{s.t. } x_{|t+1} = f(x_{|t}, u_{|t}),
\]

\[
x_{|t+1} = f(x_{|t}, u_{|t}),
\]

\[
s_{|t+1} = 0,
\]

\[
w_{|t+1} = w_{|t} + w_{|t},
\]

\[
\tilde{w}_{|t+1} = \tilde{w}_{|t} + \tilde{w}_{|t},
\]

\[
h_{|t} = h_{|t} + h_{|t},
\]

\[
g_{|t} = g_{|t} + g_{|t},
\]

\[
s_{|t} \leq s
\]

\[
w_{|t} \leq s_{|t} \leq w,
\]

\[
(x_{|t+1}, y_{|t}) \in \mathcal{F},
\]

\[
i = 1, \ldots, q_R, \quad j = 1, \ldots, q_P,
\]

\[
k = 0, \ldots, N - 1, \quad p \in \mathcal{P} \cup \{1\},
\]

which is to be solved at each time instant. The solution of (24) are optimal trajectories for the state \( x_{|t} \), the input \( u_{|t} \), the tube sizes \( s_{|t} \), the disturbance bounds \( w_{|t} \), and the value function \( V_N \). The terminal ingredients \( V_f, \mathcal{F}, s, \) and \( w \) are introduced in Sec. 3.3.

The first portion of the resulting optimal input sequence is applied to the system, resulting in the closed-loop system is given by

\[
x(t + 1) = f(x(t), u(t), d(t)), \quad u(t) := u_{0|t}.
\]

3.2 Chance Constraints

For the sake of simplicity, we will consider in this section without loss of generality only single ICCs

\[
\mathbb{P}[h(x(t + 1), u(t + 1)) \leq 0] \geq p.
\]

In the literature, the chance constraints are commonly handled by so-called constraint backoffs. This idea originates in linear MPC with additive stochastic disturbances (van Hessem and Bosgra, 2002). There, one can simply backoff the constraint, by enforcing at least a precomputed constant distance from the constraint boundary. In this work, however, we consider nonlinear systems with general
Fig. 1. Illustration of the idea behind the proposed incremental backoff (Thm. 8). The robust and stochastic tubes are shown in orange and blue, respectively.

disturbances, where the required backoff not only becomes state-dependent, but also intractable to compute.

Using the sublevel sets of the ILF, we can construct a tube around the prediction, which contains the disturbed closed-loop trajectory with at least probability $p$. The size of this tube will be used as our backoff.

The idea of the tube construction is illustrated in Fig. 1. Starting off, we begin with the prediction (black). Using Prop. 7 for $\epsilon = 1$, a robust tube (orange) can be constructed around this prediction, inside which the true state will certainly lie. This tube is constituted by the sublevel set

$$ S_{k|t} = \left\{ x \in \mathbb{R}^n \mid V(x, x_{k|t}, u_{k|t}) \leq S_{k|t} \right\}. $$

If one assumes that the previous time step was without disturbance, i.e., $d_w = 0$, then a contraction $\rho$ of the robust set (Ass. 2) is reached by the incremental stabilization $\kappa$. Thus, we obtain an inner tube (green) lacking the influence of the last disturbance with the sets $S_{k+1|t} = \rho S_{k|t}$.

Using Prop. 7 for $\epsilon = p \in (0, 1)$, the disturbance is added to the tube. Thereby, we obtain an $\epsilon$-likely tube, indicate by the blue error bars. This tube confines the state with a probability greater than $\epsilon$ at each time step in the sets

$$ S_{k+1|t} = S_{k|t} + \left\{ x \in \mathbb{R}^n \mid V(x, x_{k|t}, u_{k|t}) \leq S_{k|t} \right\} \quad (28) $$

By this construction, we can employ the size $s^p$ of the sublevel sets of $V$, i.e., the size of our tube, as our backoff to ensure ICC satisfaction formally by the following theorem.

**Theorem 8.** Suppose that Ass. 1, 2, 4, and 6 hold, then for all $k \geq 0$ with $(x_{k-1|t+1}, x_{k|t}, u_{k|t}) \in \mathbb{R}^n \times Z$ such that $V(x_{k-1|t+1}, x_{k|t}, u_{k|t}) \leq (s_{k|t})^2$, the deterministic constraints (24) imply the nonlinear ICCs (5).

This theorem will be proven jointly with Thm. 10 in Sec. 3.4.

### 3.3 Terminal ingredients

By using the minimal bound on the uncertainty $\bar{w}_{\text{min}}$ and the maximal tube size $s$

$$ \omega_{\text{min}} = \inf_{(x,u) \in \mathbb{R}^n} \bar{w}_{\text{min}}(x, u), \quad s = \sqrt{\delta_{\text{loc}}}, $$

we capture the desired properties of the terminal ingredients in the following assumption.

**Assumption 9.** There exist a terminal controller $k_f : \mathbb{R}^n \to \mathbb{R}^m$, a terminal cost function $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a terminal set $X_f \subset \mathbb{R}^{n+1}$, and a constant $\bar{\omega} \in \mathbb{R}_{\geq 0}$ such that the following holds for all $(x, s) \in X_f$, all $d_w \in \mathbb{R}^n$, all $w \in [\omega_{\text{min}}, \bar{w}]$, and all $s^+ \in [0, \rho^N w + \bar{w}_{\text{dis}}(x, k_f(x), s)]$, such that $V_f(x^+ + d_w, x^+, k_f(x^+)) \leq \rho^N w^2$ with $x^+ = f(x, k_f(x))$:

$$ V_f(x) - \ell(x, k_f(x)) \geq V_f(x^+), \quad (30a) $$

$$ (x^+ + d_w, s^+) \in X_f, \quad (30b) $$

$$ \bar{w}_{\text{dis}}(x, k_f(x), s) \leq \bar{\omega}, \quad (30c) $$

$$ g_t(x, k_f(x)) + c^T s \leq 0, \quad (30d) $$

$$ \rho s - \rho^N w + \bar{w}_{\text{dis}}(x, k_f(x), s) = \beta^p j \quad (30e) $$

$$ h_j(x^+, k_f(x^+)) + c^T p^j \leq 0, \quad (30f) $$

$$ s \leq s^+, \quad (30g) $$

with $i = 1, \ldots, q_R$ and $j = 1, \ldots, q_P$. Furthermore, the terminal cost $V_f$ is continuous on the compact set $X_{f,x} = \{ x \mid [0, s](x, s) \in X_f \}$, i.e., there exists a function $\alpha_f \in K_{\infty}$ such that

$$ V_f(z) \leq V_f(x) + \alpha_f(\|x - z\|), \forall x, z \in X_{f,x}. \quad (31) $$

These technical conditions are similar to the standard conditions in nominal MPC for the augmented state $(x, s)$ and input $(u, w, w^p)$. Details on constructive satisfaction can be found in Köhler et al. (2019).

The only extension to the robust case, is the inclusion of the ICC in the construction of the terminal set, i.e., (30). This ensures that also the ICCs are satisfied by the terminal controller, using the same backoff technique as just described in Sec. 3.2. For the construction of the terminal set, these constraints are treated similar to the hard constraint (30d).

#### 3.4 Theoretical analysis

In the following theorem, we provide guarantees on the closed-loop properties of the proposed MPC scheme.

**Theorem 10.** Let Ass. 1–4, 6, and 9 hold, and suppose that (24) is feasible at $t = 0$. Then (24) is recursively feasible, the constraints (4), (5) are satisfied and the origin is practically asymptotically stable for the resulting closed loop system.

**Proof of Thm. 10.** The proof is based on an extension of the main idea behind Köhler et al. (2019, Thm. 1), as such we will refer to their results, whenever it is possible. This will enable us to focus on handling the chance constraints, as the impact of the hard constraints is equivalent.

The core idea is to use the control law $\kappa$ from Ass. 2 to construct a candidate solution, ensuring recursive feasibility and bounding the cost increase.

1. **Candidate Solution:** For convenience, define

$$ u^*_{N|t} = k_f(x^*_{N|t}), \quad u^*_{N+1|t} = k_f(x^*_{N+1|t}), \quad (32a) $$

$$ x^*_{N+1|t} = f(x^*_{N|t}, u^*_{N|t}), \quad (32b) $$

where $\delta_{\text{loc}}$ is the terminal controller, using the same backoff technique
Consider the adapted candidate solution, i.e.,
\[ x_{0|t+1} = f(x_{0|t}, u_{0|t}, d(t)) \] (33a)
\[ u_{k|t+1} = \kappa(x_{k|t+1}, u_{k|t+1}, u_{k+1|t+1}) \] (33b)
\[ x_{k+1|t+1} = f(x_{k+1|t+1}, u_{k+1|t+1}) \] (33c)
\[ s_{k+1|t+1} = p_{k}s_{k|t+1} + w_{k+1}p \] (33d)
\[ w_{k+1|t} = w_{0|k+1}^* \] (33e)
with \( k = 0, \ldots, N - 1 \) and \( p \in \mathcal{P} \). As in Köhler et al. (2019, eq. 17), we obtain using Prop. 23 with \( \epsilon = 1 \) and repeatedly applying Ass. 2 (12) that for \( k = 0, \ldots, N \)
\[ V_0(x_{k|t+1}, x_{k+1|t+1}, u_{k+1|t}) \leq \rho^2[w_{0|k+1}]^2 \leq \delta_{loc} \] (34)
Thus, the candidate and previous optimal solution stay in the region \( V_0(x, r, s) \leq \delta_{loc} \), for which we have a local incremental Lyapunov function \( V_0 \) by Ass. 2.

II. Tube Dynamics: From Köhler et al. (2019, Proof of Thm. 1, Part II, eq. 18-19), we have the inequalities
\[ s_{k+1|t} \leq s_{k|t}^* - \rho^k w_{0|t}^* \] (35)
\[ w_{k+1|t} \leq w_{k|t}^* \] (36)
for \( k = 0, \ldots, N - 1 \). Analogously to the derivation of (36), we can show
\[ w_{k+1|t} \leq w_{k|t}^* \] (37)
This enables us to consider the general case of \( s_{k|t}^* \), yielding that for all \( p \in \mathcal{P} \cup \{1\} \) the inequality
\[ s_{k+1|t} \leq s_{k|t}^* - \rho^k w_{0|t}^* \] (38)
holds for all \( k = 0, \ldots, N - 1 \) and \( j = 1, \ldots, q_p \), since
\[ s_{0|t+1} = 0 \] (24b)
\[ s_{k+1|t} \leq \rho s_{k|t+1} + w_{k|t+1} \] (35)
\[ s_{k+1|t} \leq \rho s_{k|t+1} + w_{k|t+1} \] (37)
\[ s_{k+1|t} \leq \rho s_{k|t+1} + w_{k|t+1} \] (36)
\[ s_{k+1|t} \leq \rho s_{k|t+1} + w_{k|t+1} \] (24d)
\[ s_{k+1|t} \leq c_{k+1} \rho s_{k+1}^* w_{0|t}^* \] (37)
\[ s_{k+1|t} \leq c_{k+1} \rho s_{k+1}^* w_{0|t}^* \] (37)

III. Satisfaction of Hard Constraints, Terminal Constraints, and Tube Bounds: The constraints (24g-i) are satisfied by the candidate solution (33) as shown in Köhler et al. (2019, Proof for Thm. 1, Part III–V)

IV. Satisfaction of Deterministic ICC Replacement: In the following, we show that the deterministic constraints (24f) used in place of the ICCs (5) hold for \( k = 0, \ldots, N - 1 \). For \( k = 0, \ldots, N - 2 \), we have
\[ h_j(x_{k+1|t+1}, u_{k+1|t+1}) + s_{k+1|t+1} \leq h_j(x_{k+2|t+1}, u_{k+2|t+1}) + c_{k+1} \rho s_{k+1|t+1} + s_{k+2|t+1} \] (38)
\[ h_j(x_{k+2|t+1}, u_{k+2|t+1}) + s_{k+2|t+1} \leq 0 \] (24f)
The terminal condition (24i) ensures constraint satisfaction for \( k = N - 1 \) with
\[ h_j(x_{N|t+1}, u_{N|t+1}) + c_{N}^p s_{N|t+1} \leq h_j(x_{N|t+1}, u_{N|t+1}) + c_{N}^p s_{N|t+1} \] (30f)
\[ h_j(x_{N|t+1}, u_{N|t+1}) + c_{N}^p s_{N|t+1} \leq 0 \] (30f)

v. Practical Stability: As shown in Köhler et al. (2019, Proof of Thm. 1, Part VI), there exist \( \alpha^- \), \( \alpha^+ \), \( \alpha_w \in \mathbb{K}_\infty \) such that
\[ \alpha^- (\|x(t)\|) \leq V_N(x(t)) \leq \alpha^+ (\|x(t)\|) \] (39)
\[ V_N(x(t + 1)) - V_N(x(t)) \leq -\alpha^- (\|x(t)\|) + \alpha_w (\alpha_w) \] (40)
Thus, the closed-loop is practically asymptotically stable.

vi. Closed-looped Chance Constraint Satisfaction: Since (24f) implies the ICCs (5) by Thm. 8, the ICCs are satisfied in closed-loop with at least the specified probability.

Proof of Thm. 8. Again, we consider just a single ICC (26). By Prop. 5, \( V_0(x_{k|t+1}, x_{k+1|t}, u_{k+1|t}) \leq c^2 \) implies
\[ h(x_{k|t+1}, u_{k|t+1}) - h(x_{k+1|t}, u_{k+1|t}) \leq c \] (41)
By Ass. 2 (12), we have \( V_0(x_{k+1|t}, u_{k+1|t}) \leq (\rho s_{k|t})^2 \) for \( d_w = 0 \). Using Prop. 7, we can bound the additional increase of \( V_0 \) due to the \( d_w \neq 0 \). Together, this yields
\[ \mathbb{P}[c = \sqrt{V_0(x_{k|t+1}, x_{k+1|t}, u_{k+1|t})} \leq (\rho s_{k|t})^2 + w_{k|t}^*] \geq p \] (42)
Then, given that \( V_0(x_{k-1|t}, u_{k|t}) \leq (\rho s_{k|t})^2 \), substituting (42) into (41) yields
\[ \mathbb{P}[h(x_{k+1|t}, u_{k+1|t}) \leq h(x_{k-1|t}, u_{k+1|t}) + c \rho s_{k+1|t}] \geq p \] (43)
hence, we obtain that
\[ h(x_{k|t+1}, u_{k|t}) \leq c + c \rho s_{k+1|t} \] (44)
\[ \mathbb{P}[h(x_{k|t+1}, u_{k|t}) \leq 0] \geq p \iff (26) \] (45)

3.5 Discussion

In the following, we discuss various properties of the proposed SMPC scheme, as well as relations to other existing MPC schemes for uncertain systems.

Remark 11. Compared to a nominal MPC scheme, \( s_p \) and \( w^p \) augment the state and the input, resp., for each \( p \in \mathcal{P} \cup \{1\} \). Thus, the online computational demand of solving (24) is comparable to a nominal MPC scheme with \( n + 1 + |\mathcal{P}| \) states, \( m + 1 + |\mathcal{P}| \) inputs, and additional \( 1 + |\mathcal{P}| \) nonlinear constraints for each time step. Correspondingly, while it is possible to have multiple probability levels \( p_j \) for the ICCs, using the same \( p \) for all ICCs will be computationally significantly cheaper.

Remark 12. The disturbance bounds (24e) could also be stated, as equality constraints, which is the case considered for the candidate solution. Yet, by using inequality constraints, while we still achieve at least the required constraint tightening, we obtain more freedom for the optimization. Furthermore, allowing tightening beyond the required amount has no impact on the result due to optimality. In particular, one common form for the function \( u^S \) employs online maximization over a finite set of values, and hence is easily restated as inequality constraints, thereby alleviating the need for additional optimizations within the constraints.

Remark 13. In absence of any ICCs or equivalently, if all ICCs have to be fulfilled with certainty, i.e., \( P = \{1\} \), the proposed SMPC scheme trivially reduces to the RMPC scheme proposed by Köhler et al. (2019).

Remark 14. The SMPC method in Santos et al. (2019) for additive uncertainty is based on a \( K \)-function or as
a special case on a Lipschitz constant $L$. Therein, the authors employed the inverse of empirical cumulative distributions $\hat{F}_W$ as uncertainty description, which can be equivalently used in our method by setting $\hat{w}^p = \hat{F}_W^{-1}(p)$. In particular, the result on Lipschitz constants $L$ are contained as a special case in our framework with $V_\delta(x, z, v) = \|x - z\|^2$, $\kappa(x, z, v) = v$, and $\rho = L$. Similarly, the use of a $\mathcal{K}$-function $\sigma$ is also a special of our scheme using the same $V_\delta$ and $\kappa$ by rewriting $\rho$ and $\hat{w}_\delta$ in terms of $\sigma$. The main difference is that we use a backoff for the ICCs depending on the prediction in order to consider state- and input-dependent disturbances, whereas Santos et al. (2019) can employ constant backoff as the current state has no impact on the considered additive stationary stochastic uncertainty. Overall, the proposed framework using ILF is typically less conservative than either one of these choices (Köhler et al., 2018).

**Remark 15.** While other methods (e.g., Santos et al. (2019)), often require prestabilization in order to limit the tube growth for unstable system. In our method, this is not necessary as the tube size depends on the controller $\kappa$ in ILF, which is efficiently able to handle instability.

### 4. NUMERICAL SIMULATION

A widely used benchmark case study in the SMPC literature is the DC-DC-converter regulation problem. The discrete-time dynamics translated to the origin are described by Lazar et al. (2008), including their parameters, as

$$ x^+ = \begin{bmatrix} x_1^+ \\ x_2^+ \\ v^+ \end{bmatrix} = \begin{bmatrix} x_1 + \alpha x_2 + \left( \beta - \frac{v}{L} \right) u \\ \frac{\delta}{(\frac{\beta}{L}+\gamma)} x_2 + (1-\frac{\delta}{\beta}) v x_1 \end{bmatrix}. \quad (45) $$

We consider a (possibly time-varying) parameter uncertainty in $\theta = [\alpha, \delta]$ with a Gaussian distribution with $\Sigma_\theta = 0.1 I_{2 \times 2}$ variance truncated to a maximal deviation of $1.6 \sigma$. The system is subject to hard input constraint $|u| \leq 0.2$, and the electric power is chance constrained by $P_v(x_1, x_2) \leq 2 \geq 0.8$. Using an ILF $V_\delta(x, z, v) = \|x - z\|^2$ and controller $\kappa(x, z, v) = K(x - z) + v$ a contraction rate $\rho \approx 0.82$ can be achieved. The disturbance bounds $\hat{w}_\delta^p(x, u, c) = \|P_{\theta} \left( \frac{\theta(x, u)}{[\alpha; \delta]} \right) \Sigma_\theta^p \|_2 (p) + L_u c$ is derived analogously to (Köhler et al., 2019, Prop. 3) with Lipschitz constant $L_u^\theta \approx 0.15$, $L_u^{0.0} \approx 0.06$ and $P[\|\theta\|_{\Sigma_\theta}^p \leq \epsilon(p)] \geq p$.

For a standard quadratic cost in state and input, stability and constraints satisfaction is achieved. For this example, however, we shall consider $\ell(x, u) = u^2$, which voids the stability claim as Ass. 3 is violated. We exploit that inherent instability of the system (45) will drive the system into the constraint. Thereby, we can more easily study the chance constraint satisfaction. In the same vein, we drop the terminal constraints required for the stabilization.

In Fig. 2, we can see one evolution of the closed-loop under the proposed SMPC without stabilization. In 95% of the 14000 simulated steps the power constraint is satisfied. While the constraint is not active in all of the considered state (cf. Fig. 2), this is admittedly still rather conservative compared to the required 80% satisfaction. Yet, this is significantly less conservative than approximating the disturbance as additive would be. Likewise, as stated in Rem. 14, using Lipschitz constants would yield a more conservative result. With our approach, however, the conservatism could be further reduced by using a less conservative disturbance bound $\hat{w}_\delta$ at the price of additional computational complexity. Therefore, our method can be tune to the desired comprise between conservatism and complexity. At the same time with the fairly conservative, but simple, bound presented in this example, we achieved tighter satisfaction than existing methods.

### 5. CONCLUSION

We have proposed a nonlinear SMPC framework based on incremental stabilizability for nonlinear systems incorporating general state- and input-dependent uncertainty descriptions. The scheme can ensure satisfaction of individual chance constraints and hard constraints, as well as recursive feasibility. By using a specially designed growing tube, we achieve this with only a small computation cost increase over nominal MPC. We have demonstrated the performance gains of the proposed framework over RMPC.

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