Some remarks on exact wormhole solutions

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Exact wormhole solutions, while eagerly sought after, often have the appearance of being overly specialized or highly artificial. A case for the possible existence of traversable wormholes would be more compelling if an abundance of solutions could be found. It is shown in this note that for many of the wormhole geometries in the literature, the exact solutions obtained imply the existence of large sets of additional solutions.

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I. INTRODUCTION

One of the fundamental problems in the general theory of relativity is finding exact solutions of the Einstein field equations [1]. An example is the Schwarzschild solution, which proved to be a major milestone. Similarly, finding exact solutions of wormhole geometries has occupied many researchers. The difference is that in this setting exact solutions are hardly a virtue if they have the appearance of being artificial or, at best, overly specialized. It is shown in this note that exact solutions may be used to show the existence of entire families of solutions, which in turn helps strengthen the case for the possible existence of traversable wormholes.

II. WORMHOLE SOLUTIONS

A wormhole may be defined as a handle or tunnel in the spacetime topology linking different universes or widely separated regions of our own universe [9]. A general line element is the following:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$  (1)

where $\Phi(r) \to 0$ and $\Lambda(r) \to 0$ as $r \to \infty$, usually referred to as asymptotic flatness. We also assume that $\Phi$ and $\Lambda$ have continuous derivatives in their respective domains. The function $\Phi = \Phi(r)$ is called the redshift function, which must be everywhere finite to prevent an event horizon. The function $\Lambda = \Lambda(r)$ is related to the shape function $b(r) = r(1 - e^{-2\Lambda(r)})$, that is, $e^{2\Lambda(r)} = 1/[1 - b(r)/r]$. The minimum radius $r = r_0$ is the throat of the wormhole, where $b(r_0) = r_0$. Also needed is $b'(r_0) < 1$, usually referred to as the flare-out condition. As a consequence, $\Lambda(r)$ has a vertical asymptote at $r = r_0$; $\lim_{r \to r_0^+} \Lambda(r) = +\infty$. For the redshift function we also have the additional requirement that $\Phi'(r_0)$ be finite. Finally, to hold a wormhole open, the null energy condition must be violated [9].

The components of the Einstein tensor in the orthonormal frame are listed next [4].

$$G_{tt} = \frac{2}{r} e^{-2\Lambda(r)} \Lambda'(r) + \frac{1}{r^2} \left(1 - e^{-2\Lambda(r)}\right),$$  (2)

$$G_{rr} = \frac{2}{r} e^{-2\Lambda(r)} \Phi'(r) - \frac{1}{r^2} \left(1 - e^{-2\Lambda(r)}\right),$$  (3)

$$G_{\theta\theta} = G_{\phi\phi} = \frac{e^{-2\Lambda(r)}}{r^2} \left(\Phi''(r) - \Phi'(r)\Lambda'(r) + [\Phi'(r)]^2 + \frac{1}{r} \Phi'(r) - \frac{1}{r} \Lambda'(r)\right).$$  (4)

The case that will be examined in some detail is a wormhole supported by phantom energy, whose equation of state (EoS) is $p = -K\rho$, $K > 1$. For this case, $\rho + p = \rho(1 - K) < 0$, in violation of the null energy condition.

From the Einstein field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ and Eqs. (2) and (3), we now get

$$\frac{1}{8\pi} \left[\frac{2}{r} e^{-2\Lambda(r)} \Phi'(r) - \frac{1}{r^2} \left(1 - e^{-2\Lambda(r)}\right)\right] = -K \frac{1}{8\pi} \left[\frac{2}{r} e^{-2\Lambda(r)} \Lambda'(r) + \frac{1}{r^2} \left(1 - e^{-2\Lambda(r)}\right)\right].$$

Solving for $\Lambda'(r)$, we have

$$\Lambda'(r) = \frac{1}{K} \left[-\Phi'(r) - \frac{1}{2r} \left(e^{2\Lambda(r)} - 1\right)(K - 1)\right].$$  (5)

Observe that $\lim_{r \to r_0^+} \Lambda(r) = -\infty$. Otherwise $\Lambda'(r)$ is continuous in the interval $(r_0, \infty)$.

The continuity assumptions play a critical role: thus for $a > r_0$, the integral $\int_a^b e^{\Lambda(r)} dr$ exists since $\Lambda(r)$ is continuous. Now from the line element [4], the proper radial distance $\ell(r)$ from the throat $r = r_0$ to any point away from the throat is given by

$$\ell(r) = \int_{r_0}^r e^{\Lambda'(r')} dr'.$$  (6)
For an “arbitrary” $\Lambda(r)$, this integral is not likely to be finite, given the asymptotic behavior of $\Lambda(r)$. For certain choices of $\Phi(r)$, however, $\ell(r)$ does exist, thereby producing a traversable wormhole. Examples are the exact solutions for $\Lambda(r)$ obtained by choosing the following form of EoS: $\Phi(r) \equiv \text{constant}$ and $\Phi(r) = \ln r$.

Next, let us suppose that $\Lambda_1(r)$ is one of the above exact solutions [corresponding to some $\Phi(r)$], so that

$$\ell_1(r) = \int_{r_0}^{r} e^{\Lambda_1(r')} dr'$$

is finite. Given that $\Phi'(r)$ is finite by assumption, we deduce from Eq. (5) that

$$q_1(r) = \Lambda'_1(r) + \frac{1}{K} \ln \frac{1}{2r} \left( e^{2\Lambda_1(r)} - 1 \right) (K - 1)$$

(7)

is also finite. To generate a new solution, we let $\epsilon = \epsilon(r)$ be a function with a continuous derivative satisfying the conditions $\epsilon(r_0) = \epsilon'(r_0) = 0$ and $\epsilon(r) \to 0$ as $r \to \infty$. Define $\Lambda_2(r) = \Lambda_1(r) + \epsilon(r)$. Then by Eq. (7)

$$q_2(r) = \Lambda'_2(r) + \frac{1}{K} \ln \frac{1}{2r} \left( e^{2\Lambda_2(r)} - 1 \right) (K - 1)$$

(8)

is finite. Now consider the equation

$$\Lambda'_2(r) - \frac{1}{K} \left[ -\Phi'(r) + \eta'(r) - \frac{1}{2r} \left( e^{2\Lambda_2(r)} - 1 \right) (K - 1) \right]$$

(9)

for some finite differentiable function $\eta = \eta(r)$. Writing Eq. (9) in the form

$$\Lambda'_2(r) + \epsilon'(r) = \frac{1}{K} \left[ -\Phi'(r) + \eta'(r) - \frac{1}{2r} \left( e^{2\Lambda_2(r)} + 2\epsilon(r) - 1 \right) (K - 1) \right]$$

(10)

we see that, by comparison to Eq. (5), $\eta'(r)$ must satisfy the conditions $\eta'(r_0) = 0$ and $\eta(r) \to 0$ as $r \to \infty$, thereby retaining the asymptotic flatness. The function $\eta'(r)$ is defined only implicitly in Eq. (10), but since we are strictly interested in the existence of new solutions, it is sufficient to note that the formal relationship $\eta'(r) = -\Phi'(r) - Kq_2(r)$ implies that $\eta'(r)$ exists and is sectionally continuous, so that $\eta = \eta(r)$ exists, as well.

We conclude that $\Lambda_2(r)$ is a solution to Eq. (5) corresponding to some finite redshift function $\Phi(r) + \eta(r)$. Moreover,

$$\ell_2(r) = \int_{r_0}^{r} e^{\Lambda_2(r')} dr'$$

(11)

is finite since $\epsilon(r_0) = 0$. The result is the existence of an entire family of solutions, one for each $\epsilon = \epsilon(r)$.

Similar arguments can be used to obtain new classes of solutions from exact solutions of wormholes supported by Chaplygin and generalized Chaplygin gas, modified Chaplygin gas, and even van der Waals quintessence. For the case of a generalized Chaplygin gas, the EoS is $p = -A/r^\alpha$, $0 < \alpha \leq 1$, and where $A$ is a constant. After substituting and solving for $\Lambda'(r)$, we get

$$\Lambda'(r) = \frac{1}{2(8\pi)^{1+1/\alpha}} \left[ r^{-2\alpha} e^{2\Lambda(r)} - 1 \right] - \frac{1}{2r} \left( e^{2\Lambda(r)} - 1 \right).$$

(12)

According to Ref. [9], there is an exact solution for $\Phi' \equiv 0$, making $\ell(r)$ finite. In analogous manner, we now let $\Lambda_1(r)$ be an exact solution of Eq. (12). Then $\Lambda_2(r) + \epsilon(r)$ is a new solution, corresponding to some $\Phi(r) + \eta(r)$. (Assume that $\epsilon(r)$ and $\eta(r)$ satisfy the same conditions as before.) The resulting proper distance $\ell_2(r)$ is again finite.

An example that illustrates the problem of generalizing an exact solution even better comes from the modified Chaplygin gas model, whose EoS is $p = A\rho - B/r^\alpha$, $0 < \alpha \leq 1$, and where $A$ and $B$ are constants. This time we begin with $\Lambda(r)$ and determine $\Phi(r)$.

According to Ref. [2], one may choose $b(r) = r_0 + d(r - r_0)$, where $d$ must be less than unity to meet the flare-out condition. The shape function $b(r)$ determines $\Lambda(r)$, as well as $\Phi(r)$ [2]:

$$\Phi(r) = \phi_0 + \frac{1 + Ad}{2(1 - d)} \ln |r - r_0| - \frac{1}{2} \ln r$$

$$-\frac{32\pi^2 B}{d(1 - d)} \ln |r - r_0|$$

$$-\frac{32\pi^2 B}{d(1 - d)} \left[ \frac{1}{4} (r - r_0)^4 + 4r_0^3 (r - r_0)$$

$$+ 3r_0^2 (r - r_0)^2 + \frac{4}{3} r_0 (r - r_0)^3 \right].$$

(13)

This $\Phi(r)$ is not finite unless $A$ has the (corrected) value

$$A = \frac{1}{d^2} (64\pi^2 Br_0^4 - d).$$

(14)

Remark: Since $A$ is part of the EoS, it would be more realistic to state the condition as follows: determine $d$ ($0 < d < 1$), if it exists, so that Eq. (14) is satisfied.

For such a choice of $d$, we find that

$$\Phi(r) = \phi_0 - \frac{1}{2} \ln r - \frac{32\pi^2 B}{d(1 - d)} \left[ \frac{1}{4} (r - r_0)^4$$

$$+ 4r_0^3 (r - r_0) + 3r_0^2 (r - r_0)^2 + \frac{4}{3} r_0 (r - r_0)^3 \right],$$

(15)

which has a well defined and continuous derivative. If we now substitute Eqs. (2) and (3) in the EoS and solve for
\[ \Phi'(r) = \frac{1}{2r} \left( e^{2\Lambda(r)} - 1 \right) \]
\[ + A \left[ \Lambda'(r) + \frac{1}{2r} \left( e^{2\Lambda(r)} - 1 \right) \right] \]
\[ - \frac{(8\pi)^{1+\alpha} B(\frac{d}{r}) e^{2\Lambda(r)}}{\left[ e^{-2\Lambda(r)} \Lambda'(r) + \frac{1}{r^2} (1 - e^{-2\Lambda(r)}) \right]^\alpha}. \tag{16} \]

Since the left side is well behaved (due to the choice of \( d \)), the right side is also well behaved in the sense of being finite and sectionally continuous. As before, we can replace \( \Lambda(r) \) by \( \Lambda(r) + \epsilon(r) \), where, once again, \( \epsilon = \epsilon(r) \) has a continuous derivative with \( \epsilon(r_0) = \epsilon'(r_0) = 0 \) and \( \epsilon(r) \to 0 \) as \( r \to \infty \). Now denote the left side by \( \Phi_1'(r) \). Since \( \Phi_1'(r) \) is also sectionally continuous, \( \Phi_1(r) \) will exist, again showing the existence of an entire family of solutions.

### III. CONCLUSION

We have shown in this note that exact solutions of certain wormhole geometries, such as traversable wormholes supported by phantom energy, Chaplygin and generalized Chaplygin gas, or modified Chaplygin gas, imply the existence of entire families of additional solutions. Having an abundance of solutions is highly desirable since exact solutions often have the appearance of being overly specialized or even artificial.

From a physical standpoint, the proliferation of solutions, given the various equations of state, would greatly increase the probability of wormholes forming naturally. In particular, it is argued in Ref. [5] that the EoS had very likely crossed the “phantom divide” some time in the past, so that wormholes could have formed naturally. In approaching the present dark-energy phase, such wormholes would have formed event horizons, thereby becoming black holes or (possibly) quasi-black holes. This outcome provides at least a possible explanation for the abundance of black holes and the complete lack of wormholes.

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