Decays \( h \rightarrow e_a e_b, \) \( e_b \rightarrow e_a \gamma, \) and \((g-2)_e,\mu\) in a 3-3-1 model with inverse seesaw neutrinos

T.T. Hong,\(^1\) N.H.T. Nha,\(^2\) T. Phong Nguyen,\(^2\) L. T. T. Phuong,\(^1,\)§ and L.T. Hue\(^{3,4,∗∗}\)

\(^1\)An Giang University, VNU - HCM, Ung Van Khiem Street, Long Xuyen, An Giang 88000, Vietnam
\(^2\)Department of Physics, Can Tho University, 3/2 Street, Can Tho, Vietnam
\(^3\)Subatomic Physics Research Group, Science and Technology Advanced Institute, Van Lang University, Ho Chi Minh City 70000, Vietnam
\(^4\)Faculty of Applied Technology, School of Engineering and Technology, Van Lang University, Ho Chi Minh City 70000, Vietnam

Abstract

We will show that the 3-3-1 model with new heavy right handed neutrinos as \( SU(3)_L \) singlets can explain simultaneously the lepton flavor violating decays of the SM-like Higgs boson, charged lepton flavor violating decays \( e_b \rightarrow e_a \gamma \), and the electron \((g-2)_e\) anomalies under recent experimental data. The discrepancy of \((g-2)_\mu\) predicted by the model under consideration and that of the standard model can reach \(10^{-9}\). The decay rates of the standard model-like Higgs boson \( h \rightarrow \tau e, \tau \mu \) can reach the values of \(\mathcal{O}(10^{-4})\).

PACS numbers:
I. INTRODUCTION

The experimental evidence of neutrino oscillation [1-5] confirms that the lepton flavor number is violated in the neutral lepton sector. This is a great motivation to search for many lepton flavor violating (LFV) processes, namely the promoting ones we will focus on in this work are the LFV decays of the charged leptons $e_b \rightarrow e_a \gamma$ and the standard model-like (SM-like) Higgs boson (LFVH) $h \rightarrow e^+_ae^-$. The charged lepton flavor violating (cLFV) decays $e_b \rightarrow e_a \gamma$ are constrained by experiments as follows [6, 7]:

$$\text{Br}(\tau \rightarrow \mu \gamma) < 4.4 \times 10^{-8}, \quad \text{Br}(\tau \rightarrow e \gamma) < 3.3 \times 10^{-8}, \quad \text{Br}(\mu \rightarrow e \gamma) < 4.2 \times 10^{-13}. \quad (1)$$

Upcoming sensitivities will be orders of $10^{-9}$ and $10^{-14}$ for decays $\tau \rightarrow \mu \gamma, e \gamma$ [8, 9] and $\mu \rightarrow e \gamma$ [10], respectively. LFVH decays have been investigated in many models beyond the standard model (BSM). On the other hand, the latest experimental constraints are: $\text{Br}(h \rightarrow \tau^+\mu^-) < 2.5 \times 10^{-3}$ [11], $\text{Br}(h \rightarrow \tau^+e^-) < 4.7 \times 10^{-3}$ [12], and $\text{Br}(h \rightarrow \mu^+e^-) < 6.1 \times 10^{-5}$ [13]. The future experimental sensitivities may be $1.4 \times 10^{-4}$, $1.6 \times 10^{-4}$, and $1.2 \times 10^{-5}$, respectively [14]. The small upper bounds of the cLFV branching rates prefer the explanation that they come from loop corrections relevant to LFV sources, including ones available in the neutral lepton sector. For models consisting of these necessary tree level couplings to accommodate neutrino oscillation data such as the Zee model [15], constraints on the LFV sources such as Yukawa couplings are very strict [16, 17]. Therefore, new scalar masses must not be heavier than 300 GeV in order to explain successfully the recent $(g-2)$ data [16, 18], while the LFVH decay rates are small [17, 19].

To explain the neutrino oscillation data, the BSMs with the general seesaw (GSS) mechanism also result in LFV decays. But the versions adding only heavy seesaw neutrinos type-I predict suppressed LFV rates that are much smaller than the upcoming experimental sensitivities [20, 21]. In contrast, the models with only new inverse seesaw (ISS) neutrinos can predict large LFV rates. In addition, LFVH rates may be large in the regions satisfying constraints of $\text{Br}(e_b \rightarrow e_a \gamma)$ [22-25]. On the other hand, LFVH rates may be smaller when other constraints are considered [26, 27]. In the supersymmetric (SUSY) versions of these models with new LFV sources from superparticles, LFVH rates may reach large order of $O(10^{-5})$ [20, 28, 30]. LFVH decays were also addressed with other experimental data in many other non-SUSY extensions of the SM [37-66]. Many BSM predict that the strong constraints of cLFV decay rates $\text{Br}(e_b \rightarrow e_a \gamma)$ give small LFVH ones, or suppressed $(g-2)_\mu$.  

2
Unless there is some specific condition of the appearance of very light new bosons, the above cLFV constraints will result in small new one-loop contributions to anomalous magnetic moments (AMMs) of charged leptons \((g - 2)_{e_a}/2 \equiv a_{e_a}\), in contrast with recent experimental data. Namely, the 4.2\(\sigma\) deviation between standard model (SM) prediction \(68\), combined contributions from previous works \(69–94\), and muon experiments \(95, 96\) is
\[
\Delta a_{\mu}^{\text{NP}} \equiv a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = (2.51 \pm 0.59) \times 10^{-9}.
\] (2)

This result is slightly inconsistent with the latest one, which calculated the hadronic vacuum polarization for the SM prediction based on the lattice QCD approach, giving a combined value reported in Refs. \(77, 78, 97\) closer to the experimental data. This value was shown to fit with other experimental data such as global electroweak fits \(98–100\).

Regarding the electron anomaly, a 1.6\(\sigma\) discrepancy between SM and experiment was reported \(101\)
\[
\Delta a_{e}^{\text{NP}} \equiv a_{e}^{\text{exp}} - a_{e}^{\text{SM}} = (4.8 \pm 3.0) \times 10^{-13}.
\] (3)

The recent studies of cLFV decays in the regions satisfying the AMM data were done in some specific models such as SUSY with largest \(\text{Br}(h \rightarrow \tau\mu) \sim \mathcal{O}(10^{-4})\) \(102\). Other BSM containing leptoquarks can explain the large \(\Delta a_{\mu}^{\text{NP}} \sim \mathcal{O}(10^{-9})\) \(103\).

Recent work has discussed an extension of the 3-3-1 with right-handed neutrinos \(104–106\), named the 3-3-1 model with inverse seesaw neutrinos (331ISS) \(107\), with the aim of giving an explanation of both the \((g - 2)_{\mu}\) data and the neutrino oscillation data through the ISS mechanism. The model needs new \(SU(3)_L\) gauge singlets including three neutral leptons \(X_{aR}\) and a new singly charged Higgs boson \(h^\pm\) to accommodate all the experimental data of neutrino oscillation, the cLFV bounds in Eq. (1) and the \(\Delta a_{\mu}\) in 1\(\sigma\) deviation given in Eq. (2). Although cLFV and/or LFVH decays were investigated previously with promoting predictions for the 331ISS \(108, 111\), the AMM data was not included. Our aim in this work is filling this gap. We note that other 3-3-1 models \(112–115\) constructed previously can accommodate the \((g - 2)_{\mu}\) data only when they are extended such as adding new vector-like fermions, or/and scalars \(116, 120\). But none of them paid attention to the correlations between LFVH decays and \((g - 2)_{e_a}\) anomalies.

Our paper is organized as follows. In Sec. II we discuss the necessary ingredients of a 331ISS model for studying LFVH decays and how the ISS mechanism works to generate active neutrino masses and mixing consistent with current experimental data. In Sec. III
we present all couplings needed to determine the one-loop contributions to the LFVH decay amplitudes of the SM-like Higgs boson, cLFV decays, and \((g-2)_{e_a}\). In section IV, we provide detailed numerical illustrations and discussions. Section V contains our conclusions. Finally, the appendix lists all of the analytic formulas expressing one-loop contributions to LFVH decay amplitudes calculated in the unitary gauge.

II. THE 331ISS MODEL FOR TREE-LEVEL NEUTRINO MASSES

A. Particle content and lepton masses

We summarize the particle content of the 331ISS model in this section. We ignore the quark sector irrelevant in our work, which was discussed previously \([121, 122]\). We also ignore many detailed calculations presented in Ref. \([107]\). The electric charge operator defined by the gauge group \(SU(3)_L \times U(1)_X\) is

\[
Q = T_3 - T_8 + X,
\]

where \(T_3, T_8\) are diagonal \(SU(3)_L\) generators. Each lepton family consists of an \(SU(3)_L\) triplet \(L_aL = (\nu_a, e_a, N_a)^T\) and a right-handed charged lepton \(e_aR \sim (1, -1)\) with \(a = 1, 2, 3\). The 331ISS model contains three neutral leptons \(X_aR \sim (1, 0)\), \(a = 1, 2, 3\), and a singly charged Higgs boson \(\sigma^\pm \sim (1, \pm 1)\). There are three Higgs triplets \(\rho = (\rho_1^+, \rho_0^0, \rho_2^+)^T \sim (3, \frac{2}{3})\), \(\eta = (\eta_1^0, \eta_2^-)^T \sim (3, -\frac{1}{3})\), and \(\chi = (\chi_1^0, \chi_2^-)^T \sim (3, -\frac{1}{3})\). The vacuum expectation values (vev) for generating all tree-level quark masses and leptons are

\[
\langle \rho \rangle = (0, v_1^2, 0)^T, \quad \langle \eta \rangle = (v_1^2, 0, 0)^T \quad \text{and} \quad \langle \chi \rangle = (0, 0, w_2^2)^T.
\]

Two neutral Higgs components have zero vevs because of their non-zero generalized lepton numbers \([107]\) corresponding to a new global symmetry \(U(1)_L\) \([122]\).

In the 331ISS, nine gauge bosons get masses through the covariant kinetic Lagrangian of the Higgs triplets, \(\mathcal{L}^H = \sum_{H=x, \eta, \rho} (D_\mu H)^\dagger (D^\mu H)\), where \(D_\mu = \partial_\mu - ig \sum_{a=1}^8 W^a_{\mu} T^a - ig_X T^9 X \mu\), \(a = 1, 2, \ldots, 8\), and \(T^9 \equiv \frac{I_3}{\sqrt{6}}\) and \(\frac{1}{\sqrt{6}}\) for (anti)triplets and singlets \([123]\). There are two pairs of singly charged gauge bosons, denoted as \(W^\pm\) and \(Y^\pm\), defined as

\[
W^\pm_\mu = \frac{W_\mu^1 \pm i W_\mu^2}{\sqrt{2}}, \quad Y^\pm_\mu = \frac{W_\mu^6 \pm i W_\mu^7}{\sqrt{2}},
\]

with the respective masses \(m_W^2 = \frac{g^2}{4} (v_1^2 + v_2^2)\) and \(m_Y^2 = \frac{g^2}{4} (w^2 + v_1^2)\). The breaking pattern of the model is \(SU(3)_L \times U(1)_X \to SU(2)_L \times U(1)_Y \to U(1)_Q\), leading to the matching
condition that $W^\pm$ are the SM gauge bosons. As a consequence, we have

$$v_1^2 + v_2^2 \equiv v^2 = (246 \text{GeV})^2, \quad \frac{g_X}{g} = \frac{3\sqrt{2}s_W}{\sqrt{3 - 4s_W^2}}, \quad gs_W = e,$$  

(5)

where $e$ and $s_W$ are, respectively, the electric charge and sine of the Weinberg angle. Similarly to the Two Higgs Doublet Models (2HDM), we use the parameter

$$t_\beta \equiv \tan \beta = \frac{v_2}{v_1},$$  

(6)

which leads to $v_1 = v\beta$ and $v_2 = v\beta$. 

The Yukawa Lagrangian generating lepton masses is:

$$\mathcal{L}_Y^\nu = - h_{ab}^\nu \bar{L}_a \rho e_{bR} + h_{ab}^\nu \epsilon_{ijk} (L_a) i (L_b) j \rho_k^* - y_{ab}^\nu \bar{X}_{bR} \chi^\dagger L_a - \frac{1}{2} (\mu_X)_{ab} \bar{X}_{aR} (X_{bR})^c$$

$$- Y_{ab}^c (X_{aR})^c e_{bR} \sigma^+ + \text{H.c.},$$  

(7)

where $a, b = 1, 2, 3$. The first term generates charged lepton masses as $m_{e_a} \equiv \frac{\overline{h_{ab}^\nu} v_1}{\sqrt{2}} \delta_{ab}$, with the assumption that the flavor states are also physical.

In the basis $n'_L = (\nu_L, N_L, (X_R)^c)^T$, Lagrangian in Eq. (7) generates a neutrino mass term written in terms of the total $9 \times 9$ mass matrix consisting of nine $3 \times 3$ sub-matrices $M_{ab}$, namely

$$-\mathcal{L}_\text{mass}' = \frac{1}{2} (n'_L)^c \mathcal{M}' n'_L + \text{H.c.}, \quad \text{where} \quad \mathcal{M}' = \begin{pmatrix}
\mathcal{O}_3 & m_D^T & \mathcal{O}_3 \\
\mathcal{O}_3 & m_D & \mathcal{O}_3 \\
\mathcal{O}_3 & M_R & \mathcal{O}_3
\end{pmatrix},$$  

(8)

where $(n'_L)^c = (\nu_L)^c, (N_L)^c, (X_R)^c)^T$, $(M_R)_{ab} \equiv \overline{y_{ab}} \frac{v}{\sqrt{2}}$, and $(m_D^T)_{ab} = -(m_D)_{ab} \equiv \sqrt{2} h_{ab}^\nu v_1$ with $a, b = 1, 2, 3$. The matrix $\mu_X$ in Eq. (7) is symmetric, and can be considered as a diagonal matrix without loss of generality.

The mass matrix $\mathcal{M}'$ is diagonalized by a $9 \times 9$ unitary matrix $U'$

$$U'^T \mathcal{M}' U' = \tilde{M}' = \text{diag}(m_{n_1}, m_{n_2}, ..., m_{n_9}) = \text{diag}(\tilde{m}_\nu, \tilde{M}_N),$$

(9)

where $m_{n_i} (i = 1, 2, ..., 9)$ are masses corresponding to the physical states $n_{iL}$. The two mass matrices $\tilde{m}_\nu = \text{diag}(m_{n_1}, m_{n_2}, m_{n_3})$ and $\tilde{M}_N = \text{diag}(m_{n_4}, m_{n_5}, ..., m_{n_9})$ consist of the masses of the active $n_{aL} (a = 1, 2, 3)$ and extra neutrinos $n_{IL} (I = 1, 2, ..., 6)$, respectively. The following approximation solution of $U'$ is valid for any specific seesaw mechanisms,

$$U' = \Omega \left( \begin{array}{c}
U_{\text{PMNS}} \\
\mathcal{O}_{3 \times 6} \\
\mathcal{O}_{6 \times 3}
\end{array} \right) V, \quad \Omega \simeq \begin{pmatrix}
I_3 - \frac{1}{2} RR^\dagger & R \\
-R^\dagger & I_6 - \frac{1}{2} R^\dagger R
\end{pmatrix},$$

(10)
where $R$, $V$ are $3 \times 6$, $3 \times 6$ matrices, respectively. All entries of $R$ must satisfy $|R_{ii}| \ll 1$, so that all ISS relations can be derived perturbatively.

The relations between the flavor and mass eigenstates are

$$n'_L = U^\nu n_L, \quad (n'_L)^c = U^{\nu*} (n_L)^c \equiv U^{\nu*} n_R,$$

where $n_L \equiv (n_{1L}, n_{2L}, ..., n_{9L})^T$, and the Majorana states are $n_i = (n_{iL}, n_{iR})^T$.

The ISS relations are

$$R_2^* = m_D^T M_{R}^{-1}, \quad R_1^* = -R_2^* \mu_X (M_{R}^T)^{-1} \simeq O_3,$$

$$m_\nu = R_2^* \mu_X R_2^\dagger = U_{\text{PMNS}}^* m_\nu U_{\text{PMNS}}^\dagger = m_D^T M_{R}^{-1} \mu_X (M_R^{-1})^T m_D,$$

$$V^* \tilde{M}_N V^\dagger = M_N + \frac{1}{2} M_N R^\dagger R + \frac{1}{2} R^T R^* M_N. \quad (14)$$

From experimental data of $m_\nu$, we can determine all independent parameters in $m_D$ and three entries of $M^{-1} \equiv M_{R}^{-1} \mu_X (M_R^{-1})^T$ \cite{109,121}. Namely, the Dirac mass matrix has the antisymmetric form

$$m_D = z e^{i\alpha_{23}} \times \tilde{m}_D,$$

where $\alpha_{23} \equiv \arg[h_{32}^\nu]$, $\tilde{m}_D$ is an antisymmetric matrix with $(\tilde{m}_D)_{23} = 1$, and

$$z = \sqrt{2} |\nu_1| |h_{32}^\nu| = \sqrt{2} |\nu_1| |h_{23}^\nu| \equiv z_0 c_\beta$$

is a positive and real parameter. Eq. \cite{13} gives $(m_\nu)_{ij} = [m_D^T M^{-1} m_D]_{ij}$ for all $i, j = 1, 2, 3$, leading to six independent equations. Solving three of them with $i \neq j$, the non-diagonal entries of $M^{-1}$ are functions of $M_{ii}^{-1}$ and $x_{12,13}$. Inserting these functions into the three remaining relations with $i = j$, we obtain

$$(\tilde{m}_D)_{32} = \frac{(m_\nu)_{13}^2 - (m_\nu)_{11} (m_\nu)_{33}}{(m_\nu)_{13} (m_\nu)_{23} - (m_\nu)_{12} (m_\nu)_{33}}, \quad (\tilde{m}_D)_{21} = \frac{(m_\nu)_{12} (m_\nu)_{13} - (m_\nu)_{11} (m_\nu)_{23}}{(m_\nu)_{13} (m_\nu)_{23} - (m_\nu)_{12} (m_\nu)_{33}}, \quad (17)$$

and $\text{Det}[m_\nu] = 0$. From $M^{-1} = M_{R}^{-1} \mu_X (M_{R}^{-1})^T$ we derive that three parameters of the matrix $\mu_X$ as certain but lengthy functions of $(z e^{i\alpha_{23}})$, all entries of $M_R$ and $m_\nu$. While $m_\nu$ are fixed by experiments, all entries of $M_R$ are free parameters. We will fix $\alpha_{23} = 0$, because it is absorbed into the $\mu_X$.

In the limit that $|R_2| \ll 1$, the heavy neutrino masses can be determined approximately based on Eq. \cite{14}, namely

$$V^* \tilde{M}_N V^\dagger \simeq M_N. \quad (18)$$
We define the reduced matrix $M_R \equiv z \tilde{M}_R$, $(\tilde{M}_R)_{ij} \equiv k_{ij}$, provided that $R_2^* = -\tilde{m}_D/\tilde{M}_R$.

The matrix $M_R$ is always diagonalized by two unitary transformations $V_{L,R}$ [124]:

$$V_L^T M_R V_R = z \times \hat{k} = z \times \text{diag}(\hat{k}_1, \hat{k}_2, \hat{k}_3),$$

where all $\hat{k}_{1,2,3}$ are always positive and $\hat{k}_a \gg 1$ so that all ISS relations are valid. Therefore, $M_R$ is expressed in terms of $\hat{k}$ and $V_{L,R}$. Then the matrix $V$ in Eq. (14) can be found approximately as follows

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} V_R & iV_R \\ V_L & -iV_L \end{pmatrix} \rightarrow V^T M_N V = z \times \begin{pmatrix} \hat{k} & \mathcal{O}_{3 \times 3} \\ \mathcal{O}_{3 \times 3} & \hat{k} \end{pmatrix}.$$  \hspace{1cm} (20)

As a consequence, for any qualitatively estimations we use the approximation that heavy neutrinos masses are $m_{n_a+3} = m_{n_a+6} \simeq z \hat{k}_a$ with $a=1,2,3$; $R_1 \simeq \mathcal{O}_3$; and

$$U^\nu \simeq \begin{pmatrix} \left( I_3 - \frac{1}{2} R_2 R_2^\dagger \right) U_{\text{PMNS}} & \frac{1}{\sqrt{2}} R_2 V_L \\ \mathcal{O}_3 & -R_2^\dagger U_{\text{PMNS}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} R_2 V_L \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \hspace{1cm} (21)$$

We have checked and confirmed that the above approximations give numerical results consistent with those discussed in Ref. [107]. Therefore, these approximate formulas will be used in this work. $m_\nu$ is chosen as the input with $3\sigma$ neutrino oscillation data to fix $\tilde{m}_D$.

The free parameters $z_0$ and $\hat{k}_{1,2,3}$, $V_R$ will be scanned in the valid ranges to construct the total neutrino mixing matrix $U^\nu$ defined in Eq. (21). Because

$$R_2 V_L = \tilde{m}_D^\dagger V_R^\dagger \hat{k}^{-1}, \quad R_2 R_2^\dagger = \tilde{m}_D^\dagger V_R^\dagger \hat{k}^{-2} \tilde{m}_D,$$

which do not depend explicitly on $V_L$, it affects weakly on all relevant processes. We will fix $V_L = I_3$ from now on.

Lagrangian for quark masses were discussed previously [121, 122]. Here, we just recall that the Yukawa couplings of the top quark must satisfy the perturbative limit $h_{33}^u < \sqrt{4\pi}$, leading to a lower bound of $v_2$: $v_2 > \frac{\sqrt{2} m_t}{\sqrt{4\pi}}$. Combined with the relations in Eqs. (5) and (6), the lower bound of $t_\beta$ is $t_\beta \geq 0.3$. The upper bound of $t_\beta$ can be derived from the tau mass, $m_\tau = h_{33}^3 \times v c_\beta \sqrt{2} \rightarrow h_{33}^3 = m_\tau \sqrt{2}/(v c_\beta) < \sqrt{4\pi}$, leading to a rather weak upper bound $t_\beta = \sqrt{1/c_\beta^2 - 1} \leq 346$. 


B. Higgs bosons

The Higgs potential used here respect the new lepton number defined in Ref. [122], namely
\[
V_h = \sum_S \left[ \mu_S^2 S^\dagger S + \lambda_S \left( S^\dagger S \right)^2 \right] + \lambda_{12} (\eta^\dagger \eta)(\rho^\dagger \rho) + \lambda_{13} (\eta^\dagger \eta) (\chi^\dagger \chi) + \lambda_{23} (\rho^\dagger \rho) (\chi^\dagger \chi) \\
+ \tilde{\lambda}_{12} (\eta^\dagger \rho) (\rho^\dagger \eta) + \tilde{\lambda}_{13} (\eta^\dagger \chi) (\chi^\dagger \eta) + \tilde{\lambda}_{23} (\rho^\dagger \chi) (\chi^\dagger \rho) + \sqrt{2} \omega f (\epsilon_{ijk} \eta^i \rho^j \chi^k + \text{h.c.}) \\
+ \sigma^+ \sigma^- \left[ \mu_\sigma^2 + \sum_S \lambda_S^\sigma S^\dagger S \right] + \left[ f_\eta (\rho^\dagger \eta) \sigma^+ + f_\chi (\rho^\dagger \chi) \sigma^+ + \text{h.c.} \right],
\]
(23)

where \( f \) is a dimensionless parameter, \( f_\eta, \chi \) are mass dimensional, \( S = \eta, \rho, \chi \). These three trilinear couplings softly break the general lepton number \( \mathcal{L} \). For simplicity, we fix \( f_\chi = 0 \) by applying a suitable discrete symmetry. The last line in Eq. (23) contains all additional terms couplings with new charged Higgs singlets compared with Higgs potential considered in previous works [107]. They do not affect the squared mass matrices of both neutral CP-odd and CP-even Higgs bosons. The minimum conditions of the Higgs potential as well as the identification of the SM-like Higgs boson have previously been discussed in detail [33, 125], hence we just list the necessary results here. The model contains three pairs of singly charged Higgs bosons \( h_{1,2,3}^\pm \) and two Goldstone bosons \( G_{W,Y}^\pm \) of the singly charged gauge bosons \( W^\pm \) and \( Y^\pm \), respectively. In the limit of \( f_\eta = 0 \), the singly charged Higgs masses are \( m_{h_1}^2 = \left( \frac{\lambda_{12} v^2}{2} + \frac{f v^2}{s_\rho c_\rho} \right), m_{h_2}^2 = \left( v^2 c_\beta^2 + w^2 \right) \left( \frac{\lambda_{23}}{2} + ft_\theta \right) \) and \( m_{G_W}^2 = m_{G_Y}^2 = 0 \) [125]. The mass of the Higgs singlet \( \sigma \equiv h_3^\pm \) is a function of \( \mu_\sigma^2 \) and \( \lambda_\sigma^\sigma \). With \( f_\eta \neq 0 \) considered in this work, the relations between the original and mass eigenstates of the charged Higgs bosons are
\[
\begin{pmatrix}
\eta^\pm \\
\rho_1^\pm \\
\sigma^\pm
\end{pmatrix}
= \begin{pmatrix}
-s_\beta & c_\alpha c_\beta & s_\alpha c_\beta \\
c_\beta & c_\alpha s_\beta & s_\alpha s_\beta \\
0 & -s_\alpha & c_\alpha
\end{pmatrix}
\begin{pmatrix}
G_w^\pm \\
h_1^\pm \\
h_2^\pm
\end{pmatrix},
\begin{pmatrix}
\tilde{\rho}_2^\pm \\
\tilde{\chi}^\pm
\end{pmatrix}
= \begin{pmatrix}
-s_\theta & c_\theta \\
c_\theta & s_\theta
\end{pmatrix}
\begin{pmatrix}
G_Y^\pm \\
h_3^\pm
\end{pmatrix},
\]
(24)

where \( t_\theta = v_1/w \), and
\[
f = \frac{c_\beta s_\beta \left( 2 c_\alpha^2 m_{h_1}^2 + 2 s_\alpha^2 m_{h_2}^2 - \tilde{\lambda}_{12} v^2 \right)}{2 \omega}, \quad f_\eta = \frac{\sqrt{2} c_\alpha s_\alpha (m_{h_2}^2 - m_{h_1}^2)}{v},
\]
\[
\mu_\sigma^2 = \frac{1}{2} \left( 2 c_\alpha^2 m_{h_2}^2 - v^2 (c_\beta^2 \lambda_2^\eta + s_\alpha^2 \lambda_\sigma^\sigma) + 2 s_\alpha^2 m_{h_1}^2 - \lambda_\sigma^\sigma v^2 \right).
\]
(25)

These result are consistent with Refs. [123, 125, 126] in the limits of \( s_\alpha = 0, \pm 1 \). The results given in Eqs. (24) and (25) obtained by solving the following \( 3 \times 3 \) squared mass matrix in
the basis \((\eta^\pm, \rho^\pm_1, \sigma^\pm)\):

\[
M_c^2 = \begin{pmatrix}
\frac{f_\omega^2}{t_\beta} + \frac{1}{2}c_\beta^2 \tilde{\lambda}_1 v^2 & f_\omega^2 + \frac{1}{2}c_\beta \tilde{\lambda}_2 s_\beta v^2 & \frac{c_\beta f_\lambda v}{\sqrt{2}} \\
f_\omega^2 + \frac{1}{2}c_\beta \tilde{\lambda}_1 s_\beta v^2 & ft_\beta \omega^2 + \frac{1}{2}\tilde{\lambda}_2 s_\beta^2 v^2 & \frac{f_\nu s_\beta v}{\sqrt{2}} \\
\frac{c_\beta f_\lambda v}{\sqrt{2}} & \frac{f_\nu s_\beta v}{\sqrt{2}} & v^2 \left( \frac{c_\beta^2 \lambda_2^2 + s_\beta^2 \lambda_1^2}{2} + \frac{\lambda_\sigma^2 \omega^2}{2} + \mu^2 \right)
\end{pmatrix}.
\tag{26}
\]

We will find out that the Higgs masses \(m_{h_1^\pm}\) and the mixing angle \(\alpha\) are functions of the Higgs parameters in the Higgs potential.

The model contains five CP-odd neutral scalar components included in the five neutral Higgs bosons \(\eta_0^0 = (v_2 + R_1 + iI_1)/\sqrt{2}\), \(\rho^0 = (v_1 + R_2 + iI_2)/\sqrt{2}\), \(\chi_0^0 = (\omega + R_3 + iI_3)/\sqrt{2}\), \(\eta_2^0 = (R_4 + iI_4)/\sqrt{2}\), and \(\chi_1^0 = (R_5 + iI_5)/\sqrt{2}\). Three of them are Goldstone bosons of the neutral gauge bosons \(Z, Z',\) and \(X^0\). The two remaining are physical states with masses

\[
m_{a_1}^2 = (s_\beta^2 v^2 + \omega^2) \left( ft_\beta^2 + \frac{1}{2} \tilde{\lambda}_{13} \right), \quad m_{a_2}^2 = f \left( \frac{\omega^2}{c_\beta s_\beta} + c_\beta s_\beta v^2 \right).
\tag{27}
\]

As a consequence, the parameter \(f\) must satisfies \(f > 0\).

Considering the CP-even scalars, there are two sub-matrices \(2 \times 2\) and \(3 \times 3\) for masses of these Higgs bosons in two bases \((\eta_0^0, \chi_0^0)\) and \((\eta_1^0, \rho_1^0, \chi_1^0)\), namely

\[M_{0,3}^2 = \begin{pmatrix}
\frac{c_\beta f_\lambda^2}{s_\beta^2} + 2s_\beta^2 \lambda_1 v^2 & c_\beta s_\beta \lambda_2 v^2 - \omega^2 f \omega(s_\beta \lambda_{13} - c_\beta f)v \\
c_\beta s_\beta \lambda_2 v^2 - \omega^2 f \frac{s_\beta f_\omega^2}{c_\beta} + 2c_\beta^2 \lambda_2 v^2 \omega(c_\beta \lambda_{23} - s_\beta f)v \\
\omega(s_\beta \lambda_{13} - c_\beta f)v \omega(c_\beta \lambda_{23} - s_\beta f)v 2\lambda_3 \omega^2 + c_\beta s_\beta v^2
\end{pmatrix},
\tag{28}
\]

The matrix \(M_{0,2}^2\) has one zero value and \(m_{h_4}^2 = \left( \frac{f_\lambda}{\tilde{\lambda}_3} + \frac{\tilde{\lambda}_4}{2} \right)(s_\beta^2 v^2 + \omega^2)\) corresponding to one Goldstone boson of \(X^0\) and a heavy neutral Higgs boson \(h_4^0\) with mass at the \(SU(3)_L\) breaking scale. On the other hand, we see that \(\text{Det}[M_{0,3}^2] \neq 0\) but \(\text{Det}[M_{0,3}^2]_{v=0} = 0\), which implies that there is at least one Higgs boson mass at the electroweak scale that can be identified with the SM-like Higgs boson. In particular, it can be proved that

\[
C_1^h M_{0,3}^2 C_1^{hT}|_{v=0} = \text{diag}(0, 2\lambda_3 w^2, f w^2/(s_\beta c_\beta)), \quad C_1^h = \begin{pmatrix} s_\beta & c_\beta & 0 \\ -c_\beta & s_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix},
\tag{29}
\]

and \(C_1^h M_{0,3}^2 C_1^{hT} \equiv M_{0,3}^2\) satisfying:

\[
(M_{0,3}^2)_{11} = 2v^2 (c_\beta^4 \lambda_2 + c_\beta^2 \lambda_{12} s_\beta^2 + \lambda_1 s_\beta^4),
\]

9
(M_{0,3}^2)_{22} = 2c_\beta^2 s_\beta^2 v^2(\lambda_1 - \lambda_{12} + \lambda_2) + \frac{f_\omega^2}{c_\beta s_\beta},
(M_{0,3}^2)_{33} = fc_\beta s_\beta v^2 + 2\lambda_3 \omega^2,
(M_{0,3}^2)_{12} = (M_{0,3}^2)_{21} = c_\beta s_\beta v^2 \left(s_\beta^2(\lambda_{12} - 2\lambda_1) - c_\beta^2(\lambda_{12} - 2\lambda_2)\right),
(M_{0,3}^2)_{13} = (M_{0,3}^2)_{31} = v\omega \left(-2fc_\beta s_\beta + c_\beta^2 \lambda_{23} + \lambda_{13} s_\beta^2\right),
(M_{0,3}^2)_{32} = (M_{0,3}^2)_{23} = v\omega \left(fc_\beta^2 - fs_\beta^2 + c_\beta s_\beta (\lambda_{23} - \lambda_{13})\right). \quad (30)

Therefore, there is a unitary transformation $C_2^h$ with $(C_2^h)_{ij} \sim O(v/w)$ ($i \neq j$) so that $C_2^h M_{0,3}^2 C_2^{hT} = \text{diag}\left(m_{h_1^0}^2, m_{h_2^0}^2, m_{h_3^0}^2\right)$ and $m_{h_i^0}^2 \sim O(v^2)$ [109, 127, 128]. Hence $h_1^0$ is identified with the SM-like Higgs boson found at the LHC, namely $h_1^0 \equiv h$. For simplicity, we will fix $C_2^h = I_3$ in this work, and use the relations $(\eta_1^0, \rho_1^0, \lambda_1^0) = C_1^{hT}(h_1^0, h_2^0, h_3^0)$ in our numerical investigation, where only $\eta_1^0 \sim R_1$ and $\rho_1^0 \sim R_2$ give contributions to $h_1^0$, namely

$$R_1 = s_\beta h_1^0 - c_\beta h_2^0, \quad R_2 = c_\beta h_1^0 + s_\beta h_2^0.$$ \quad (31)

This assumption leads to a consequence that the $m_{h_i^0}$ is independent with the Higgs self-couplings relating with one-loop decays $h_1^0 \rightarrow e_a e_b$, as it will be seen as follows:

$$-\mathcal{L}_h = V_h = \sum_{i,j=1}^3 -g_{h_{ij}}h_1^0 h_i^+ h_j^- + \ldots,$$ \quad (32)

where non-zero $g_{h_{ij}} = g_{h_{ji}}$ are

$$
g_{h11} = -v c_\alpha \left[2c_\beta^2 s_\beta^2(\lambda_1 - \lambda_{12} + \lambda_2) + \lambda_{12} + \bar{\lambda}_{12} \right] + t_\alpha^2 \left(c_\beta^2 \lambda_1^\sigma + s_\beta^2 \lambda_1^\sigma\right) + \frac{2s_\alpha^2 (m_{h_1^+}^2 - m_{h_2^+}^2)}{v^2},
$$
g_{h22} = -v c_\alpha \left[t_\alpha \left(2c_\beta^2 s_\beta^2(\lambda_1 - \lambda_{12} + \lambda_2) + \lambda_{12} + \bar{\lambda}_{12} \right) + (c_\beta^2 \lambda_2^\sigma + s_\beta^2 \lambda_2^\sigma) \right] - \frac{2s_\alpha^2 (m_{h_1^+}^2 - m_{h_2^+}^2)}{v^2},
$$
g_{h12} = -c_\alpha s_\alpha v \left[2s_\beta^2 c_\beta(\lambda_1 - \lambda_{12} + \lambda_2) - s_\beta^2 \lambda_1^\sigma - c_\beta^2 \lambda_2^\sigma + \lambda_{12} - \frac{(c_\beta^2 - s_\beta^2)(m_{h_1^+}^2 - m_{h_2^+}^2)}{v^2}\right],
$$
g_{h33} = -v \left[c_\beta^2 \left(2c_\beta^2 \lambda_2 + s_\beta^2(\lambda_{23} + \bar{\lambda}_{23})\right) + s_\beta^2 (c_\beta^2 \lambda_2 + \lambda_{13} s_\sigma) + c_\beta s_\beta^2 \left(2f_\beta + c_\beta \bar{\lambda}_{23}\right)\right]. \quad (33)
$$

In the next section, we will derive all of the remaining couplings giving one-loop contributions of decays mentioned in this work.
III. COUPLINGS AND ANALYTIC FORMULAS

A. Decays $e_b \to e_a \gamma$ and $(g-2)e_a$

The couplings of charged gauge bosons giving one-loop contributions to LFV amplitudes are:

$$L_{V\pm ff} = \frac{g}{\sqrt{2}} \sum_{a=1}^{3} \sum_{i=1}^{9} \pi_i^a \gamma^\mu P_L e_a \left[ U^{\nu*}_{ai} W^{\pm} + U^{\nu*}_{(a+3)i} Y^{\pm}_\mu \right] + \text{h.c.}, \quad (34)$$

All the calculation steps to derive these couplings were presented in Ref. [109]. From now on, we always choose that $m_{e_b} > m_{e_a}$, equivalently $b > a = 1, 2, 3$, to define the decays $e_b \to e_a \gamma$. One-loop form factors from charged gauge bosons are [129]:

$$c_{(ab)R}(W) = \frac{eg^2}{32\pi^2 m_W^2} \sum_{i=1}^{9} U^{\nu}_{ai} U^{\nu*}_{bi} \tilde{F}_V(x_{W,i}),$$

$$c_{(ab)R}(Y) = \frac{eg^2}{32\pi^2 m_Y^2} \sum_{i=1}^{9} U^{\nu}_{(a+3)i} U^{\nu*}_{(b+3)i} \tilde{F}_V(x_{Y,i}), \quad (35)$$

where $x_{v,i} = m_{v_i}^2/m_{v^*}^2$, $v = W, Y$;

$$\tilde{F}_V(x) = -\frac{10 - 43x + 78x^2 - 49x^3 + 4x^4 + 18x^3 \ln(x)}{24(x-1)^4}; \quad (36)$$

$e = \sqrt{4\pi\alpha_{em}}$ being the electromagnetic coupling constant; and $g = e/s_W$.

The Yukawa couplings of charged Higgs bosons with leptons are defined by

$$L^\text{enh} = -\frac{g}{\sqrt{2} m_W} \sum_{k=1}^{3} \sum_{a=1}^{3} \sum_{i=1}^{9} h_\kappa^a \pi_i \left( \lambda_{ai}^{L,k} P_L + \lambda_{ai}^{R,k} P_R \right) e_a + \text{h.c.}, \quad (37)$$

where

$$\lambda_{ai}^{R,1} = m_{e_a} c_\alpha t_\beta V^{\nu*}_{ai} - \sum_{c=1}^{3} \frac{v Y^{\sigma}_{ca} s_\alpha}{\sqrt{2}} U^{\nu*}_{(c+6)i}, \quad \lambda_{ai}^{L,1} = c_\alpha s_\beta z_0 e^{i\alpha z_3} \sum_{c=1}^{3} (\tilde{m}_D)_{ac} U^{\nu}_{(c+3)i},$$

$$\lambda_{ai}^{R,2} = m_{e_a} s_\alpha t_\beta V^{\nu*}_{ai} + \sum_{c=1}^{3} \frac{v Y^{\sigma}_{ca} c_\alpha}{\sqrt{2}} U^{\nu*}_{(c+6)i}, \quad \lambda_{ai}^{L,2} = s_\alpha s_\beta z_0 e^{i\alpha z_3} \sum_{c=1}^{3} (\tilde{m}_D)_{ac} U^{\nu}_{(c+3)i},$$

$$\lambda_{ai}^{R,3} = \frac{m_{e_a} c_\beta U^{\nu*}_{(a+3)i}}{c_\beta}, \quad \lambda_{ai}^{L,3} = c_\beta z_0 3 \left[ -e^{i\alpha z_3} (\tilde{m}_D)_{ac} U^{\nu}_{ci} + t^2_\beta (\tilde{M}_R)_{ac} U^{\nu}_{(c+6)i} \right]. \quad (38)$$

The interactions given in Eqs. (34) and (37) also give tree and loop contributions to the lepton flavor conserved decay $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. Regarding the gauge couplings given in Eq.
the couplings of $Y^\pm$ with active neutrinos are zeros because $U_{ν(c+3)1}^ν = U_{ν(c+3)2}^ν = 0$, the difference of the couplings of $W$ with active neutrinos and charged leptons between the SM and the 331ISS model under consideration is $|\frac{1}{2}(R_2 R_+^2 U)_{ab}| \ll 1$. Regarding the Higgs boson contributions, only $\lambda_{ai}^{L,3}$ may give large contributions to the decay amplitude $μ^- → e^-ν_μν_μ$, because the remaining couplings are always proportional to $g m_μ t_β / m_W \ll 1$ or $U_{ν(c+3)2}^ν U_{ν(c+3)1}^ν = 0$. Assuming $t_θ = 0$ for very large $SU(3)_L$ scale $w ≫ v$, we have a crude approximation that $|\lambda_{ai}^{L,3}| \leq z_0$. The large values of $|\lambda^{L,3}|$ appear because $h_3^± ∼ ρ_2^±$, which has couplings with active neutrinos $\overline{ν_a} (ν_β L)^c ν_2^- \sim h_ν^ν \sim (m_D)_{ab}$ derived from the second term in Lagrangian [7]. Based on the well-known formulas of the partial decay width $Γ(μ → 3e)$ at tree level given in the Zee-Babu model [142], the coupling $λ^L$ leads to a deviation of the decay width of the decay $μ^- → e^-ν_μν_μ$ between the 331ISS model and the SM as follows:

\[
|δΓ^{331ISS}(μ^- → e^-ν_μν_μ)| ≡ \left| \frac{Γ^{331ISS}(μ^- → e^-ν_μν_μ)}{Γ^{SM}(μ^- → e^-ν_μν_μ)} - 1 \right| \lesssim \left[ \frac{|λ_{ai}^{L,3}|^2}{4m_μ^2 h_3^±} \right]^2 \lesssim \left[ \frac{|z_0|^2}{4m_μ^2 h_3^±} \right]^2 \leq 10^{-6}. \tag{39} \]

The constraint is derived from the mean life time of muon [132]. The derivation of the formula (39) is summarized as follows. The total amplitude is $iM = iM_W + iM_{h^±}$, where $M_W$ and $M_{h^±}$ are the contributions from $W$ and charged Higgs bosons, respectively. In the low energy limit we have

\[
M_W \simeq M_{SM} \sim g^2 \frac{2m_μ^2}{m_W^2} [\overline{ν_μ}γ^μ P_L u_μ] [\overline{ν}_eγ_μ P_L v_e],
\]

\[
M_{h^±} \sim \frac{g^2}{2m_μ^2 m_{h^±}^2} \times [\overline{ν_μ} \left( λ^L P_L + λ^R P_R \right) u_μ] [\overline{ν}_e \left( λ^L R_P + λ^R R_P L \right) v_e].
\]

Now it can be proved that $|M|^2 = |M_W|^2 + |M_{h^±}|^2$ because $M_W^∗ M_{h^±}$ has an odd number of the gamma matrices in the trace and $m_e, m_ν_μ, m_ν_e \simeq 0$, leading to $M_W^∗ M_{h^±} = 0$.

In the numerical investigation, we will choose $m_{h^±} ≥ z_0 \times 10^5$ to accommodate the constraint (39). Now we can assume the approximation that $Γ^{331ISS}(μ^- → e^-ν_μν_μ) \simeq Γ^{SM}(μ^- → e^-ν_μν_μ)$. This approximation for calculating the cLFV decay rates is consistent with many works published recently [140, 141].

The one-loop form factors are [129]:

\[
c_{(ab)R}(h^±_k) = \frac{eg^2}{32π^2 m_W^2 m_{eμ} m_{h^±}^2} \sum_{i=1}^{9} \left[ λ_{ai}^{L,k} λ_{bi}^{R,k} m_{n_i} F_H (x_{k,i}) \right]
\]
where \( b \geq a, x_{k,i} = m_{n_i}^2/m_{h_k^+}^2 \), and the one-loop functions \( F_H(x) \) and \( \tilde{F}_H(x) \) are
\[
F_H(x) = -\frac{1 - x^2 + 2x \ln(x)}{4(x - 1)^3}, \quad \tilde{F}_H(x) = -\frac{-1 + 6x - 3x^2 - 2x^3 + 6x^2 \ln(x)}{24(x - 1)^4}.
\]
The total one-loop contributions to the cLFV amplitude \( e_b \to e_a \gamma \) and \( \Delta a_{e_a}^{331ISS} \) are
\[
c_{(ab)R} = \sum_{x=W,Y} c_{(ab)R}(x) + \sum_{k=1}^{3} c_{(ab)R}(h_k^\pm),
\]
\[
c_{(ba)R} = (c_{(ab)R}[a \leftrightarrow b]) \times \frac{m_{e_a}}{m_{e_b}}.
\]
The second line of Eq. (42) is derived from the equality that \( c_{(ab)R}(x) = (c_{(ab)R}(x)[b \leftrightarrow a]) \times (m_{e_a}/m_{e_b}) \) for all \( x = W,Y,h_1^\pm,2,3 \). The formulas for the contributions to \( a_{e_a} \) are:
\[
a_{e_a} = -\frac{4m_{e_a}^2}{e} \text{Re}[c_{(aa)R}] = -\frac{4m_{e_a}^2}{2\pi^2 e^2} \text{Re}[c_{(aa)R}'], \quad c_{(ab)R}' = c_{(ab)R} \times \left(\frac{eg^2}{32\pi^2 m_W^2}\right)^{-1}.
\]
One-loop contributions from heavy neutral Higgs bosons are very suppressed, hence they are ignored here. The deviation of \( a_{e_a} \) between predictions by the two models 331ISS and SM are
\[
\Delta a_{e_a} = \Delta a_{e_a}^{331ISS} \equiv a_{e_a} - a_{e_a}^{SM}(W),
\]
where \( a_{e_a}^{SM}(W) = 5g^2m_\mu^2/(96\pi^2m_W^2) \) is the SM’s prediction \([130]\). In this work, \( \Delta a_{e_a} \) will be considered as new physics (NP) predicted by the 331ISS, used to compare with experimental data in numerical investigations.

The branching ratios of the cLFV processes are \([129]\)
\[
\text{Br}(e_b \to e_a \gamma) \simeq \frac{6\alpha_{em}}{\pi} \left( |c_{(ab)R}'|^2 + |c_{(ba)R}'|^2 \right) \text{Br}(e_b \to e_a \nu_a \bar{\nu}_b),
\]
where \( G_F = 1/(\sqrt{2}v^2) \), consistent with previous results \([109,126]\) for 3-3-1 models.

The formulas of \( U^\nu \) given in Eq. \([21]\) results in approximate expressions of \( c_{(ab)R} \) and \( c_{(ba)R} \) with \( b \geq a \) as follows:
\[
c_{(ab)R}'(W) = -\frac{5}{12} \left[ \delta_{ab} - (\tilde{m}_D^\dagger V_R^k\tilde{m}_D)_{ab} \right] + \sum_{e=1}^{3} (\tilde{m}_D^\dagger V_R^k\tilde{m}_D)_{ae} (\tilde{m}_D^T V_R^k\tilde{m}_D)_{be} F_V(x_{W,e}'),
\]
\[
c_{(ab)R}'(Y) = \frac{m_{W}^2}{m_Y^2} \sum_{e=1}^{3} (V_R^*)_{ae} V_R_{be} F_V(x_{Y,e}'),
\]

\[ c'_{(ab)}(h_1^\pm) = \frac{2^3}{m_{h_1^\pm}^2} \sum_{e=1}^3 (\tilde{m}_D V_R^e)_{ae} \left\{ \tilde{c}_\alpha s_\beta (\tilde{m}_D V_R \hat{k}^{-1})_{be} - \frac{u s_\alpha s_\beta}{4m_b} \left[ Y_\sigma T V_R \right]_{be} \right\} \hat{k_e} F_H(x'_{e,1}) \]

\[ + \frac{2^3}{m_{h_1^\pm}^2 z_0^3} \sum_{e=1}^3 (\tilde{m}_D V_R)^e (\tilde{m}_D V_R)_{be} \tilde{F}_H(x'_{e,1}) \]

\[ + \frac{1}{24} \left\{ \frac{m_{e_a}^2 c_{a\beta} t_\beta}{m_{h_1^\pm}^2} \delta_{ab} + \frac{m_{e_a} u s_\alpha c_{a\beta}}{\sqrt{2} m_{h_1^\pm}^2} \left[ \frac{m_{e_a}}{m_{e_b}} (R_2 Y^\sigma)_{ab} + \left( Y_\sigma^\dagger R_2^\dagger \right)_{ab} \right] \right\} \]

\[ + \frac{3}{m_{h_1^\pm}^2} \tilde{F}_H(x'_{e,1}) c_\alpha^2 \left\{ \frac{m_{e_a}^2 t_\beta}{m_{h_1^\pm}^2} \left[ (R_2 V_L)_{ae} (R_2 V_L^*)_{be} \right] + \frac{u t_\alpha^2 m_{e_a}}{m_{e_b} m_{h_1^\pm}^2} \left[ (Y_\sigma^T V_R)_{ae} (Y_\sigma^T V_R^*)_{be} \right] \right\} \]

\[ - \frac{m_{e_a} u s_\alpha c_{a\beta}}{m_{h_1^\pm}^2} \left[ \frac{m_{e_a}}{m_{e_b}} (R_2 V_L)_{ae} (Y_\sigma T V_R^*)_{be} + (Y_\sigma^T V_R)_{ae} (R_2 V_L^*)_{be} \right] \right\}, \tag{46} \]

\[ c'_{(ab)}(h_2^\pm) = c'_{(ab)}(h_1^\pm) \left[ m_{h_1^\pm} \rightarrow m_{h_2^\pm}, c_\alpha \rightarrow s_\alpha, s_\alpha \rightarrow -c_\alpha \right], \]

\[ c'_{(ab)}(h_3^\pm) = \frac{2^3}{m_{h_3^\pm}^2} \left\{ \sum_{e=1}^3 \left[ (\tilde{m}_D^* \tilde{m}_D V_R \hat{k}^{-1})_{ae} (V_R^*)_{be} \hat{k_e} F_H(x'_{e,3}) \right] - \frac{1}{24} (\tilde{m}_D^* \tilde{m}_D)_{ab} \right\} \]

\[ + \frac{3}{m_{h_3^\pm}^2} \left[ (\tilde{m}_D^* \tilde{m}_D V_R \hat{k}^{-1})_{ae} (V_R^*)_{be} \hat{k_e} F_H(x'_{e,3}) \right] \]

where equalities in Eq. (22) were used. In addition, we ignore the minor contributions proportional to \( R_2^\dagger R_2 \), \( R_2 R_2^\dagger \). Because only two terms relating to \( R_2 Y^\sigma \) and \( R_2 R_2^\dagger \) depend on \( V_L \), but give small one-loop contributions to \( \Delta a_{e_a} \), we fix \( V_L = I_3 \) without loss of generality.

The above expressions of \( c_{(ab)}R \) and \( c_{(ba)}R \) given in Eq. (46) give some interesting properties. First, all terms are proportional to \( 1/m_{h_1^\pm}^2 \), hence large \( |\Delta a_{e_a}| \) corresponding to large \( |c_{(aa)}R| \) will prefer small \( m_{h_1^\pm}^2 \). In contrast, experimental constraints on cLFV decay rates require small \( |c_{(ab)}R| \) and \( |c_{(ba)}R| \), hence \( m_{h_1^\pm}^2 \) should be large. It is easy to get small \( Br(e_b \rightarrow e_a \gamma) \) with enough large \( m_{h_1^\pm} \), but difficult to get large \( |\Delta a_{e_a}| \). Previously numerical investigation has showed another situation [107], where small \( m_{h_1^\pm}^2 \) are needed for large \( \Delta a_\mu \), and the destructive correlations between particular terms in \( c_{(ab)}R \) and \( c_{(ba)}R \) must appear to result in small \( Br(e_b \rightarrow e_a \gamma) \). The structure of the mass Dirac matrix \( \tilde{m}_D \) strongly affects these destructive correlations. As we will see, the antisymmetric property of \( \tilde{m}_D \) and the neutrino oscillation data fix a certain form of \( \tilde{m}_D \), namely the fixed values considered in this work are \( (\tilde{m}_D)_{32} = -(\tilde{m}_D)_{23} = 1, -(\tilde{m}_D)_{12} = (\tilde{m}_D)_{21} \approx 0.613, \) and \( -(\tilde{m}_D)_{13} = (\tilde{m}_D)_{31} \approx 0.357, \) and \( (\tilde{m}_D)_{11} = (\tilde{m}_D)_{22} = (\tilde{m}_D)_{33} = 0 \). They do not support
large absolute values of the diagonal entries relating to \( c_{(aa)R} \). Therefore, the simple case of \( V_R = I_3 \), degenerate values of heavy neutrino masses \( \hat{k}_{11} = \hat{k}_{22} = \hat{k}_{33} \), and \( Y_σ = O_{3 \times 3} \) will give \( c_{(ab)R} \sim \tilde{m}_D \tilde{m}_D^* \). As a result, constraints on cLFV decays always exclude the regions of parameter space predicting large \((g - 2)_e, \mu\). This conclusion is completely consistent with the numerical results reported in Ref. [107]. In addition, the presence of \( σ^± \) and non-zero Yukawa coupling matrix \( Y_σ \) is necessary to explain 1σ range of \((g - 2)_μ\) obtained by experiment. Additionally, the formulas given in Eq. (46) explain explicitly that large \( Δa_μ \) also needs large \( z_0 \). And, large \( t_β \) and non-zero \( Y_σ \) support more strong destructive correlations to guarantee that \((e_b \rightarrow e_a)\) satisfies the current constraints.

Finally, We emphasize that the \((g - 2)_e\) data and LFVH decays were not discussed previously for the 331ISS model. Our numerical investigation showed that large \((g - 2)_e\) requires nonzero values of \( s_α \), which was not considered in Ref. [107]. In addition, large values of \( Y_σ^{22,33,23,32} \) should be investigated carefully because they may result in too large \( Br(h \rightarrow τμ) \) that may be excluded by the experimental constraints.

B. Decays \( h_1^0 \rightarrow e_a e_b \)

The Yukawa couplings \( h_1^0 \) is , namely

\[
L_{Y_{h_1}}^{h_1^0ff} = -\frac{g}{2m_W} h_1^0 \left[ \frac{1}{2} \sum_{i,j=1}^{9} \overline{n}_i \left( \lambda_{ij}^0 P_L + \lambda_{ij}^{0*} P_R \right) n_j + m_{e_a} \overline{e}_a e_a \right],
\]

where

\[
\lambda_{ij}^0 = \sum_{c=1}^{3} \left( U_{ci}^\nu U_{cj}^{\nu*} m_{n_i} + U_{ci}^{\nu*} U_{cj}^\nu m_{n_j} \right),
\]

is a symmetric coefficient \( λ_{ij}^0 = λ_{ji}^0 \) corresponding to the Feynman rules given in Ref. [124]. All of the Feynman rules for couplings involved in LFV processes at one-loop level are listed in Table I where we used \( s_θ = \frac{g v_1}{2m_Y} \). We focus on the limit of tiny \( t_θ \simeq s_θ = 0 \), and the suppressed deviation of the SM-like Higgs mixing mentioned previously [109, 127]. Namely, they will be fixed to be zeros in the numerical calculations.

The effective Lagrangian and partial decay width of the decay \( h_1^0 \rightarrow e_a^+ e_b^- \) are

\[
L_{LFVH}^{h_1^0} = h_1^0 \left( Δ_{(ab)L} \overline{e_a} P_L e_b + Δ_{(ab)R} \overline{e_a} P_R e_b \right) + H.c.,
\]

\[
Γ(h_1^0 \rightarrow e_a e_b) = Γ(h_1^0 \rightarrow e_a^+ e_b^-) + Γ(h_1^0 \rightarrow e_a^- e_b^+) = \frac{m_{h_1^0}}{8π} \left( |Δ_{(ab)L}|^2 + |Δ_{(ab)R}|^2 \right),
\]

15
where the scalar factors \( \Delta_{(ab)\ L,R} \) are loop contributions in this work. In the unitary gauge, the one-loop Feynman diagrams contributing to \( \Delta_{(ab)\ L,R} \) are shown in Fig. 1. The valid

\[
\begin{array}{|c|c|}
\hline
\text{Vertex} & \text{Coupling} \\
\hline
h_{10}^{\pm}e_{a}e_{a} & -igme_{a}^{2}m_{W} \\
\hline
h_{10}^{|ni}n_{j} & -ig2m_{W} \left( \lambda_{ij}^{0}P_{L} + \lambda_{ij}^{0*}P_{R} \right) \\
\hline
h_{10}^{\pm}e_{b}e_{a}n_{i} & -ig2m_{W} \left( \lambda_{bi}^{L,k}P_{L} + \lambda_{bi}^{R,k}P_{R} \right), -ig2m_{W} \left( \lambda_{bi}^{L,k*}P_{R} + \lambda_{bi}^{R,k*}P_{L} \right) \\
\hline
W_{\mu}^{+}\pi_{i}e_{b}, W_{\mu}^{-}\pi_{a}n_{i} & \frac{ig}{\sqrt{2}}U_{\alpha i}^{*}\gamma_{\mu}P_{L}, \frac{ig}{\sqrt{2}}U_{\alpha i}^{\nu}\gamma_{\mu}P_{L} \\
\hline
Y_{\mu}^{+}\pi_{i}e_{b}, Y_{\mu}^{-}\pi_{a}n_{i} & \frac{ig}{\sqrt{2}}U_{(a+3)i}^{*}\gamma_{\mu}P_{L}, \frac{ig}{\sqrt{2}}U_{(a+3)i}^{\nu}\gamma_{\mu}P_{L} \\
\hline
h_{3}^{0}h_{1}^{0}Y_{\mu}^{+}, h_{3}^{0}Y_{\mu}^{+}h_{1}^{|0} & \frac{i}{2}g_{c}\beta_{e}(p_{+} - p_{0})^{\mu}, -\frac{i}{2}g_{c}\beta_{e}(p_{-} - p_{0})^{\mu} \\
\hline
h_{0}^{0}W_{\mu}^{+}W_{\nu}^{-} & igm_{W}g^{\mu\nu} \\
\hline
h_{0}^{0}Y_{\mu}^{+}Y_{\nu}^{-} & igc_{b}g_{b}m_{Y}g^{\mu\nu} \\
\hline
\end{array}
\]

TABLE I: Feynman rules for one-loop contributions to \((g-2)\) anomalies, \(e_{b} \rightarrow e_{a}\gamma\), and \(h_{1}^{0} \rightarrow e_{a}e_{b}\) in the unitary gauge. \(p_{0}\) and \(p_{\pm}\) are the incoming momenta of \(h_{1}^{0}\) and \(h_{3}^{0}\), respectively.

Here \(V^{\pm} = W^{\pm},\ Y^{\pm};\ k,l = 1,2,3\).

\[
\Delta_{(ab)\ L,R} = \sum_{i=1,5,7,8} \Delta_{(ab)\ L,R}^{(i)W} + \sum_{i=1}^{10} \Delta_{(ab)\ L,R}^{(i)Y}
\]

(50)
where the analytic forms of $\Delta_{(ab)LR}^{(i)W}$ and $\Delta_{(ab)LR}^{(i)Y}$ are shown in the Appendix. There are numbers of tiny one-loop contributions, which we will ignore in the numerical calculations. They are calculated using the unitary gauge with the same techniques given in Refs. [25, 109]. The contributions from diagrams (2), (3), and (5) with $Y^\pm$ exchanges have suppressed factors $c_\beta m^3_W/m^3_Y$. The one-loop contributions from diagram (6) are suppressed with heavy singly charged Higgs bosons, which we checked consistently with the result mentioned in Refs. [67, 109].

**IV. NUMERICAL DISCUSSION**

In this work, we will use the neutrino oscillation data given in Refs. [132, 139]. The standard form of the lepton mixing matrix $U_{PMNS}$ is the function of three angles $\theta_{ij}$, one Dirac phase $\delta$ and two Majorana phases $\alpha_1$ and $\alpha_2$ [134], namely

$$U^{PDG}_{PMNS} = f(s_{12}, s_{13}, s_{23, \delta}) \times \text{diag}(1, e^{i\alpha_1}, e^{i\alpha_2}),$$

$$f(s_{12}, s_{13}, s_{23, \delta}) \equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
c_{13} & 0 & s_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-s_{13}e^{i\delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},$$  \tag{51}

where $s_{ij} \equiv \sin \theta_{ij}$, $c_{ij} \equiv \cos \theta_{ij} = \sqrt{1 - s_{ij}^2}$, $i, j = 1, 2, 3$ ($i < j$), $0 \leq \theta_{ij} < 90$ [Deg.] and $0 < \delta \leq 360$ [Deg.]. The Majorana phases are chosen in the range $-180 \leq \alpha_i \leq 180$ [Deg.]. For numerical investigation, we choose a benchmark corresponding to the normal order of the neutrino oscillation data as the input to fix $\tilde{m}_D$ that $s_{12}^2 = 0.32$, $s_{23}^2 = 0.547$, $s_{13}^2 = 0.0216$, $\Delta m^2_{21} = 7.55 \times 10^{-5} [eV^2]$, $\Delta m^2_{32} = 2.424 \times 10^{-3} [eV^2]$, $\delta = 180$ [Deg], and $\alpha_1 = \alpha_2 = 0$. Consequently, the reduced Dirac mass matrix $\tilde{m}_D$ is fixed as

$$\tilde{m}_D = \begin{pmatrix}
0 & 0.613 & 0.357 \\
-0.613 & 0 & 1 \\
-0.357 & -1 & 0
\end{pmatrix}.$$  \tag{52}

The best-fit point for the normal (inverted) order is $\delta = -1.89^{+0.7}_{-0.58}(-1.38^{+0.48}_{-0.54}) \neq 180$ deg. [139], which rules out the value 180 deg. at 95% confidence level. But it is still allowed in 3$\sigma$ range. The other quantities corresponding to the best-fit point are $s_{23}^2 = 0.53$, $\Delta m^2_{21} = 7.53 \times 10^{-5} [eV^2]$, $\Delta m^2_{32} = 2.45 \times 10^{-3} [eV^2]$, leading to a new $\tilde{m}_D$ with
\( (\tilde{m}_D)_{12} = 0.546 e^{0.18i} \) and \( (\tilde{m}_D)_{13} = 0.453 e^{-0.23i} \). The existence of the non-zero CP violation \( \delta \neq 180 \text{ deg.} \) will lead to the complex values of the two entries of \( \tilde{m}_D \) instead of the real ones given in Eq. (52). These imagine parts result in non-zero values of \( \text{Im}[c_{(ab)R}] \), which is enough to give large \( \text{Br}(\mu \rightarrow e\gamma) > 4.2 \times 10^{-13} \) in many regions of the parameter space, even when \( \text{Re}[c_{(ab)R}] \) = 0. Therefore, many very complicated relations between parameters must be satisfied to guarantee that all Im and Re parts contributing to these cLFV decays satisfy the experimental constraints. In this work, the limit that \( \delta = 180 \text{ [Deg]} \) is fixed for simplicity.

The mixing matrix \( V_R \) is parameterized using the same formulas given in Eq. (51), \( V_R = f(s_{12}^r, s_{13}^r, s_{23}^r, 0) \) with \( |s_{ij}^r| \leq 1 \). The remaining free parameters are scanned in the following ranges:

\[
\hat{k}_{1,2,3} \geq 5, \ 600 \ [\text{GeV}] \leq m_{h_{1,2}^\pm} \leq 1500 \ [\text{GeV}], \ s_\alpha |\leq 1, \ \text{max}|[Y^\sigma_{ij}]| \leq 1.5,
\]

\[
t_\beta \in [30, 70], \ 400 \ [\text{GeV}] \leq z \leq 1200 \ [\text{GeV}],
\]

and \( m_{h_{1,2}^\pm} = 40 \text{ TeV} \), so that the decay width of \( \mu^- \rightarrow e^-\overline{\nu}_e\nu_\mu \) is consistent with that predicted by the SM. In addition, the collected points satisfy that max\(|(R_2 R_2^\dagger)_{ab}| < 10^{-3} \) with all \( a, b = 1, 2, 3 \). This constraints also satisfies many other recent experimental results such as electroweak precision tests, cLFV decays [143–146]. The experimental parameters are \( G_F = 1.663787 \times 10^{-5} \ [\text{GeV}^{-2}], \ g = 0.652, \ \alpha_{em} = e^2/(4\pi) = 1/137, s_W^2 = 0.231, \ m_e = 5 \times 10^{-4} \ [\text{GeV}], \ m_\mu = 0.105 \ [\text{GeV}], \ m_\tau = 1.776 \ [\text{GeV}], \) and \( m_W = 80.385 \ [\text{GeV}], \) \( \text{Br}(\mu \rightarrow e\overline{\nu}_e\nu_\mu) \approx 1, \ \text{Br}(\tau \rightarrow e\overline{\nu}_e\nu_\tau) \approx 0.1782, \) and \( \text{Br}(\tau \rightarrow \mu\overline{\nu}_\mu\nu_\tau) \approx 0.1739. \) We note that the upper bounds of \( m_{h_{1,2}^\pm} \) and \( t_\beta \) based on the previous work to accommodate large values of \( \Delta a_\mu. \) Chosen scanning range of \( t_\beta \) also satisfies the perturbative limit mentioned above.

We comment here the results obtained previously in Ref. [107], where large \( t_\beta \geq 50 \) and small values of singly charged Higgs bosons \( h_k^\pm (k = 1, 2) \) are required for large \((g-2)_\mu \) satisfying \( 1\sigma \) experimental data of \((g-2)_\mu \) and all constraints from cLFV decays \( e_b \rightarrow e_a\gamma. \) But only the case of \( s_\alpha = 0 \) and non-zero \( Y^\sigma_{22,33,23,32} \) was mentioned. Our numerical investigation shows that this case results in small \( \Delta a_e \) which cannot satisfy \( 1\sigma \) range of the experimental data given in Eq. (3). Without \( \sigma^\pm, \) we obtain two maximal values of \( \Delta a_e \) that \( \Delta a_e \lesssim 2.5 \times 10^{-14} \) and \( 1.5 \times 10^{-14} \) for the NO and IO schemes, respectively. Hence, determining the regions of parameter space giving large \( \Delta a_e \) will be very interesting.

Because of the above reasons, we focus on the regions of parameter space giving large
\( \Delta a_e \) that satisfies the 1\( \sigma \) experimental data of \((g - 2)_e \) as well as all current constraints of cLFV decay rates \( \text{Br}(e_b \to e_a \gamma) \). The investigation shows that the 1\( \sigma \) range of \( \Delta a_e \in [1.8 \times 10^{-13}, 7.8 \times 10^{-13}] \) can be obtained easily in a wide range of the parameter space, for example the following fixed values of \( z_0 = 500 \text{ GeV} \), \( t_\beta = 50 \), and \( s_\alpha = 0.5 \), and scanning the remaining parameters, we have a benchmark point that \( m_{h^\pm} = 814.8 \text{ GeV}, m_{h^0} = 771.5 \text{ GeV}, m_{h^1} = m_{h^2} = 2.152 \text{ TeV}, m_{h^3} = m_{h^4} = 4.365 \text{ TeV}, m_{h_5} = m_{h_6} = 3.156 \text{ TeV}, s_{12} = -0.075, s_{13} = -0.565, s_{23} = -0.063, \) and \( 0 \leq |Y'_\alpha|^2 \leq 0.293 \), which results in the following allowed values of the relevant physical processes: \( \Delta a_e = 4.243 \times 10^{-13}, \Delta a_\mu = 1.019 \times 10^{-9}, \)

\( \text{Br}(\mu \to e\gamma) = 2.95 \times 10^{-13}, \text{Br}(\tau \to e\gamma) = 6.18 \times 10^{-9}, \text{Br}(\tau \to \mu\gamma) = 3.52 \times 10^{-8}, \)

\( \text{Br}(h_1^0 \to \mu e) = 1.59 \times 10^{-7}, \text{Br}(h_1^0 \to \tau e) = 6.56 \times 10^{-6}, \) and \( \text{Br}(h_1^0 \to \tau\mu) = 2.7 \times 10^{-4}. \)

We list here other interesting benchmark points of the parameter space corresponding to large \( t_\beta = 60 \), that satisfy 1\( \sigma \) range of \((g - 2)_e, \Delta a_\mu \geq 0.6 \times 10^{-9}, \) and all current LFV upper bounds. For other large \( t_\beta \) values, the results are the same.

1. A benchmark point giving large \( \text{Br}(h_1^0 \to \tau e) \sim O(10^{-5}) \):

\[
\{z_0[\text{GeV}], t_\beta, s_\alpha\} = \{867.7, 60, 0.460\}, \quad \{s_{12,13,23}\} = \{0.377, 0.556, -0.907\}, \\
\{m_{h_{1,2}^\pm}[\text{TeV}]\} = \{0.974, 0.918\}, \quad \{m_{4,5,6} = m_{7,8,9}[\text{TeV}]\} = \{3.32, 5.265, 3.341\},
\]

\[
Y' = \begin{pmatrix}
0.015 & 0.006 & -0.013 \\
-0.044 & 0.047 & -0.108 \\
0.003 & -0.183 & 0.063
\end{pmatrix}.
\]

The corresponding values of \( \Delta a_{e,\mu} \) and LFV decay rates are

\( \Delta a_e = 5.89 \times 10^{-13}, \Delta a_\mu = 1.077 \times 10^{-9}, \)

\[
\text{Br}\{\mu \to e\gamma, (\tau \to e\gamma, (\tau \to \mu\gamma)\} = \{8.31 \times 10^{-14}, 1.28 \times 10^{-8}, 4.04 \times 10^{-8}\}
\]

\( \text{Br}(h_1^0 \to \{\mu e, \tau e, \tau\mu\}) = \{5.9 \times 10^{-7}, 5.91 \times 10^{-5}, 5.18 \times 10^{-4}\}. \)

2. There exists benchmark point that allows large \( \text{Br}(h_1^0 \to \tau e) \sim O(10^{-5}), \) but small \( \text{Br}(h_1^0 \to \tau\mu) < O(10^{-7}): \)

\[
\{z_0[\text{GeV}], t_\beta, s_\alpha\} = \{478.5, 60, 0.993\}, \quad \{s_{12,13,23}\} = \{0.629, -0.867, -0.818\}, \\
\{m_{h_{1,2}^\pm}[\text{TeV}]\} = \{0.997, 0.864\}, \quad \{m_{4,5,6} = m_{7,8,9}[\text{TeV}]\} = \{2.994, 4.092, 2.378\},
\]
between the mixing angles our numerical code is not still smart enough to collect these points. Because of the special to control. This is because of the large number of free parameters in the 331ISS that data and cLFV constraints are very wide. But the regions allowed large (∆%

\[ Y^\sigma = \begin{pmatrix} 0.069 & 0.164 & -0.085 \\ -0.076 & 0.058 & -0.199 \\ 0.074 & -0.180 & -0.086 \end{pmatrix}, \]

\[ \Delta a_e = 4.67 \times 10^{-13}, \quad \Delta a_\mu = 0.998 \times 10^{-9}, \]

\[ \text{Br}\{(\mu \rightarrow e\gamma), (\tau \rightarrow e\gamma), (\tau \rightarrow \mu\gamma)\} = \{2.196 \times 10^{-13}, 3.523 \times 10^{-9}, 3.547 \times 10^{-8}\}, \]

\[ \text{Br}(h_1^0 \rightarrow \{\mu e, \tau e, \tau \mu\}) = \{5.93 \times 10^{-6}, 1.88 \times 10^{-5}, 6.56 \times 10^{-8}\}. \]

3. There exists a benchmark point predicting large \( \text{Br}(h \rightarrow e\mu) \sim \mathcal{O}(10^{-5}) \), which is close to the experimental constraint:

\[ \{z_0[\text{GeV}], t_\beta, s_\alpha\} = \{1019.5, 60, 0.848\}, \quad \{s_{12,13,23}^r\} = \{0.11, -0.89, -0.822\}, \]

\[ \{m_{h_{1,2}^0}[\text{TeV}]\} = \{0.671, 0.622\}, \quad \{m_{4,5,6} = m_{7,8,9}[\text{TeV}]\} = \{6.533, 9.657, 4.414\}, \]

\[ Y^\sigma = \begin{pmatrix} 0.079 & 0.189 & -0.113 \\ -0.094 & -0.061 & -0.210 \\ 0.079 & -0.241 & -0.059 \end{pmatrix}, \]

\[ \Delta a_e = 3.19 \times 10^{-13}, \quad \Delta a_\mu = 0.917 \times 10^{-9}, \]

\[ \text{Br}\{(\mu \rightarrow e\gamma), (\tau \rightarrow e\gamma), (\tau \rightarrow \mu\gamma)\} = \{2.37 \times 10^{-13}, 2.80 \times 10^{-9}, 3.07 \times 10^{-8}\}, \]

\[ \text{Br}(h_1^0 \rightarrow \{\mu e, \tau e, \tau \mu\}) = \{1.85 \times 10^{-5}, 1.05 \times 10^{-4}, 1.48 \times 10^{-3}\}. \]

It is noted that large \( \text{Br}(h \rightarrow e\mu) \) requires both large \( z_0 \) and \( \text{Br}(h \rightarrow \tau \mu) \sim \mathcal{O}(10^{-3}) \) which may be excluded by planned experiments. In this case, the numerical results show that \( \text{Br}(h \rightarrow \tau \mu) < \mathcal{O}(10^{-4}) \) will lead to \( \text{Br}(h \rightarrow e\mu) < \mathcal{O}(10^{-6}) \), which is still smaller than the planned experimental sensitivity.

From our numerical investigation, we found that the regions allowing 1σ range of \( \Delta a_e \) data and cLFV constraints are very wide. But the regions allowed large \((g - 2)_\mu\) are difficult to control. This is because of the large number of free parameters in the 331ISS that our numerical code is not still smart enough to collect these points. Because of the special form of \( \hat{m}_D \) that require the non-degenerate matrix \( \hat{k} \) and the strong destructive correlations between the mixing angles \( s_{ab}^r \) and the entries of \( Y^\sigma \) in order to get small \( \text{Br}(e_b \rightarrow e_a\gamma) \), in the regions allow large \( \Delta a_\mu \). There may exist some certain relations between these parameters for collecting more interesting points allowing large \((g - 2)_\mu\) at 1σ experimental range. We will determine them in a future work.

20
Finally, we comment some properties of the current $Z$ boson decay data which may put useful constraints on the parameter space of the 331ISS model. In the limit of $v/w \to 0$, equivalently $t_\theta = 0$, the couplings of $Z$ boson with all other SM particles. We can see that all masses of the new heavy neutrinos appearing in the collected points we showed above as the numerical results are much larger than the $Z$ boson masses. Therefore, $Z$ do have not any new tree level decays $Z \to n_IN_j$ with at least a new heavy neutrino $n_I$ ($I > 3$). In addition, all masses of the new heavy particles predicted by the 331ISS models are heavier than the $Z$ boson masses, therefore the invisible decays of the $Z$ boson in this case is the same as that in the SM and the 2HDM discussed in Ref. [67]. We therefore conclude that the current $Z$ boson decay data affects weakly the allowed region of the parameters space we focus on this work.

There is another cLFV decay mode $Z \to e^+_ae_b^-$ discussed in detailed in 2HDM [67], which is still invisible in the regions predicting large $Br(h \to e_a e_b)$ and satisfying all the constraints of $cLFV$ decays $Br(e_b \to e_a \gamma)$. Therefore, this decay channel will not change significantly the allowed regions of parameters discussed in this work. On the other hand, the interesting topic we will focus on is that when the experimental sensitivities are improved, both $cLFV$ decays of $\mu^- \to e^- \nu_e \nu_\mu$ and $Z \to e_a e_b$ may give more significant constraints on those mentioned in this work.

V. CONCLUSION

In this work, we have constructed the analytic formulas for one-loop contributions to the LFV decays of the SM-like Higgs boson $h^0_1 \to e_a e_b$ in the 331ISS model. We also give analytic formulas to explain qualitatively the results of large $(g - 2)_\mu$ reported previously. Numerical tests were used to confirm the consistency between the two calculations. We introduced a new parameterization of the heavy neutrino mass matrix to reduce the number of free parameters used to investigate $(g - 2)_{e,\mu}$ anomalies, LFV decays $e_b \to e_a \gamma$, and $h^0_1 \to e_a e_b$. Our numerical investigation shows that the model can predict easily the 1$\sigma$ range of experimental data for $(g - 2)_e$ and satisfy simultaneously the cLFV constrains $Br(e_b \to e_a \gamma)$. But we only obtained the regions of parameter space that give largest values of $\Delta a_\mu \simeq 10^{-9}$, which rather smaller than the lower bound of 1$\sigma$ range reported recently. The reason is that the recent numerical code used in our investigation only works in the
limit of small $\max[|y^a|] < 0.25$. In these regions of the parameter space, the largest values of $\text{Br}(h_1^0 \to \tau e)$ and $\text{Br}(h_1^0 \to \tau \mu)$ are order of $\mathcal{O}(10^{-4})$ and $10^{-3}$, respectively. In addition, large $\text{Br}(h_1^0 \to \tau \mu)$ predicts large $\text{Br}(h_1^0 \to \mu e) \sim \mathcal{O}(10^{-5})$, which is close to the recent experimental bounds. The regions with large $|Y^r|_{ab}$ may be more interesting, which is our future work, where many other LFV processes such as $Z \to e_b e_a$, $e_b \to e_c e_d e_f$, and the $\mu - e$ conversion in nuclei will be discussed together.

Acknowledgments

We are grateful Prof. Martin Hofericher, and Dr. Mukesh Kumar for their communications. We would like to thank the referee for reminding us of the important contribution of the singly charged Higgs bosons to the decay $\mu^- \to e^- \overline{\nu}_e \nu_\mu$, which significantly changes our numerical results. This research is funded by An Giang University under grant number 21.01.TB. L. T. Hue is grateful to Van Lang University.

Appendix A: Form factors of LFVH in the unitary gauge

The one-loop contributions here are calculated using the notations of Passarino-Veltman (PV) functions \[135, 136\] given in Ref. \[27\], consistent with LoopTools \[137\], see a detailed discussion in Refs. \[33, 138\]. The PV functions used in this work are defined as follows: $B^{(i)}_\mu \equiv B^{(i)}_1 \times (-1)^ip_{i\mu}$ with $i = 1, 2$, and $C^{\mu} \equiv \sum_{i=1}^2 (-1)^ip_{i\mu} \times C_i$. As mentioned in Ref. \[27\], the two $B^{(1)}_1$ and $C_1$ have opposite signs with those introduced in Ref. \[33\]. They come from the signs of $p_{1,2}$ in the internal momenta $(k - p_1)$ and $(k + p_2)$ shown in Fig. 1, which $p_1$ has an opposite sign, which is different from the standard notation of $k + p_1$ defined in LoopTools. The PV-functions used in our formulas are: $B^{(i)}_{0,1} = B_{0,1}(p_1^2; M_0^2, M_1^2)$, \[C_{0,1,2} = C_{0,1,2}(p_1^2, (p_1 + p_2)^2, p_2^2; M_0^2, M_1^2, M_2^2),\] and $B^{(12)}_0 = B_0((p_1 + p_2)^2; M_1^2, M_2^2)$. In the below, when the external momenta are fixed as $p_1^2 = m_{e_a}^2$, $p_2^2 = m_{e_b}^2$, and $(p_1 + p_2)^2 = m_{h_1}^2$, we use the simpler notations as follows $C_{0,1,2}(p_1^2, m_{h_0}^2, p_2^2; M_0^2, M_1^2, M_2^2) \equiv C_{0,1,2}(M_0^2, M_1^2, M_2^2)$, $B^{(i)}_{0,1}(M_0^2, M_2^2) = B_{0,1}(p_1^2, M_0^2, M_2^2)$, and $B_0(m_{h_1}^2; M_1^2, M_2^2) = B_0^{(12)}(M_1^2, M_2^2)$.

The analytic expressions $\Delta^{(iW)}_{L,R} \equiv \Delta^{(iW)}_{(ab)L,R}$ for one-loop contributions from the diagram
(i) in Fig. 1 are

$$\Delta_L^{(1)W} = \frac{g^3 m_a}{64 \pi^2 m_W^5} \sum_{i=1}^9 U_{ai}^{\nu} U_{bi}^{\nu} \left\{ m_{n_i} \left( B_0^{(1)} + B_0^{(2)} + B_1^{(1)} \right) + m_b B_1^{(2)} - \left( 2m_W^2 + m_{h_1}^2 \right) m_{n_i} C_0 \right. \\
- \left[ m_{n_i} \left( 2m_W^2 + m_{h_1}^2 \right) + 2m_W^2 \left( 2m_W^2 + m_{a}^2 - m_{b}^2 \right) \right] C_1 \\
- \left[ 2m_W^2 \left( m_{a}^2 - m_{h_1}^2 \right) + m_{b}^2 m_{h_1}^2 \right] C_2 \right\},$$

$$\Delta_R^{(1)W} = \frac{g^3 m_b}{64 \pi^2 m_W^5} \sum_{i=1}^9 U_{ai}^{\nu} U_{bi}^{\nu} \left\{ m_{n_i} \left( B_0^{(1)} + B_0^{(2)} + B_1^{(2)} \right) + m_b^2 B_1^{(1)} - \left( 2m_W^2 + m_{h_1}^2 \right) m_{n_i} C_0 \right. \\
- \left[ m_{n_i} \left( 2m_W^2 + m_{h_1}^2 \right) + 2m_W^2 \left( 2m_W^2 - m_{a}^2 + m_{b}^2 \right) \right] C_2 \\
- \left[ 2m_W^2 \left( m_{b}^2 - m_{h_1}^2 \right) + m_{a}^2 m_{h_1}^2 \right] C_1 \right\},$$

$$\Delta_L^{(7+8)W} \times \frac{g^3 m_a m_b}{64 \pi^2 m_W^5 \left( m_{b}^2 - m_{a}^2 \right)} \\
\times \sum_{i=1}^9 U_{ai}^{\nu} U_{bi}^{\nu} \left\{ 2m_{n_i} \left( B_0^{(2)} - B_0^{(1)} \right) + \left( 2m_W^2 + m_{n_i}^2 \right) \left( B_1^{(2)} - B_1^{(1)} \right) + m_b^2 B_1^{(2)} - m_a B_1^{(1)} \right\},$$

$$\Delta_R^{(7+8)W} = \frac{m_a}{m_b} \Delta_L^{(7+8)W},$$

where $B_{0,1} = B_{0,1}^{(k)}(m_{n_i}, m_W)$ and $C_{0,1,2} = C_{0,1,2}(m_{n_i}, m_W, m_W^2)$.

$$\Delta_L^{(5)W} = \frac{g^3 m_a}{64 \pi^2 m_W^5} \sum_{i,j=1}^9 U_{ai}^{\nu} U_{bj}^{\nu} \left\{ D_{ij} \left[ -m_{n_i} B_0^{(12)} + m_{n_j} B_1^{(1)} + m_{b}^2 m_W^2 C_0 \right. \\
+ \left( 2m_W^2 (m_{n_i}^2 + m_{n_j}^2) + 2m_{n_i}^2 m_{n_j} + m_{a}^2 m_{n_j}^2 - m_{b}^2 m_{n_i}^2 \right) C_1 \right\} \\
+ D_{ij}^* m_{n_i} m_{n_j} \left[ -B_0^{(12)} + B_1^{(1)} + m_W^2 C_1 + \left( 4m_W^2 + m_{n_i}^2 + m_{n_j}^2 - m_{a}^2 - m_{b}^2 \right) C_1 \right]\right\},$$

$$\Delta_R^{(5)W} = \frac{g^3 m_b}{64 \pi^2 m_W^5} \sum_{i,j=1}^9 U_{ai}^{\nu} U_{bj}^{\nu} \left\{ D_{ij} \left[ -m_{n_i} B_0^{(12)} + m_{n_j} B_1^{(2)} + m_{b}^2 m_W^2 C_0 \right. \\
+ \left( 2m_W^2 (m_{n_i}^2 + m_{n_j}^2) + 2m_{n_i}^2 m_{n_j} - m_{a}^2 m_{n_j}^2 - m_{b}^2 m_{n_i}^2 \right) C_2 \right\} \\
+ D_{ij}^* m_{n_i} m_{n_j} \left[ -B_0^{(12)} + B_1^{(2)} + m_W^2 C_0 + \left( 4m_W^2 + m_{n_i}^2 + m_{n_j}^2 - m_{a}^2 - m_{b}^2 \right) C_2 \right]\right\},$$

where $D_{ij} = \sum_{c=1}^3 U_{ci}^{\nu} U_{cj}^{\nu}$, $B_0^{(12)} = B_0^{(12)}(m_{n_i}, m_{n_j})$, $B_1^{(1)} = B_1^{(1)}(m_W, m_{n_i})$, $B_1^{(2)} = B_1^{(2)}(m_W, m_{n_j})$, and $C_{0,1,2} = C_{0,1,2}(m_{n_i}, m_{n_j}, m_{n_j})$. The analytic expressions $\Delta_L^{(i)Y} \equiv \Delta_{(ab)L,R}^{(i)Y}$ with $i = 4, 6, 9, 10$, are

$$\Delta_L^{(1)Y} = \frac{g^3 m_a c_b s_b}{64 \pi^2 m_W^5} \sum_{i=1}^9 U_{(a+3)i}^{\nu} U_{(b+3)i}^{\nu} \left\{ m_{n_i} \left( B_0^{(1)} + B_0^{(2)} + B_1^{(1)} \right) + m_b B_1^{(2)} \right. \\
- \left( 2m_Y^2 + m_{h_1}^2 \right) m_{n_i} C_0 - \left[ 2m_Y^2 (2m_Y^2 + m_{a}^2 - m_{b}^2) + m_{n_i} (2m_Y^2 + m_{h_1}^2) \right] C_1 \right\}.$$
\[
\Delta_R^{(1)Y} = \frac{g^3 m_b \beta Q}{64 \pi^2 m_Y^2} \sum_{i=1}^9 U_{(a+3)i}^{\nu*} U_{(b+3)i}^{\nu} \left\{ m_{n_i}^2 \left[ B_0^{(1)} + B_1^{(2)} \right] + m_a B_1^{(1)} - m_Y^2 \right\} C_2
\]

\[
\Delta_L^{(7+8)Y} = \frac{g^3 m_a m_b^2}{64 \pi^2 m_W m_Y^2 (m_b^2 - m_a^2)} \sum_{i=1}^9 U_{(a+3)i}^{\nu*} U_{(b+3)i}^{\nu} \times \left[ m_{n_i}^2 \left( B_0^{(2)} - B_1^{(1)} \right) + \left( 2 m_Y^2 + m_{n_i}^2 \right) \left( B_1^{(2)} - B_1^{(1)} \right) - m_a B_1^{(1)} + m_b B_1^{(2)} \right],
\]

\[
\Delta_R^{(7+8)Y} = \frac{m_a}{m_b} \Delta_L^{(7+8)Y},
\]

where \( B_{0,1} = B_{0,1}^{(k)}(m_{n_i}^2, m_Y^2) \) and \( C_{0,1,2} = C_{0,1,2}(m_{n_i}^2, m_Y^2, m_{n_j}^2) \). One-loop contributions from diagram 5 are

\[
\Delta_L^{(5)Y} = \frac{g^3 m_a}{64 \pi^2 m_W m_Y^2} \times \sum_{i,j=1}^9 U_{(a+3)i}^{\nu} U_{(b+3)j}^{\nu*} \left\{ D_{ij} \left[ -m_{n_i}^2 B_0^{(12)} + m_{n_j}^2 B_1^{(1)} + m_{n_j}^2 m_W C_0 \right.ight.
\]

\[
+ \left. \left( 2 m_Y^2 (m_{n_i}^2 + m_{n_j}^2) + 2 m_{n_i}^2 m_{n_j}^2 - m_a^2 m_{n_j} - m_b^2 m_{n_i} \right) C_1 \right],
\]

\[
+ D_{ij}^* m_{n_i} m_{n_j} \left[-B_0^{(12)} + B_1^{(1)} + m_W^2 C_0 + \left( 4 m_Y^2 + m_{n_i}^2 + m_{n_j}^2 - m_a^2 - m_b^2 \right) C_1 \right]\} ,
\]

\[
\Delta_R^{(5)Y} = \frac{g^3 m_b}{64 \pi^2 m_W m_Y^2} \times \sum_{i,j=1}^9 U_{(a+3)i}^{\nu} U_{(b+3)j}^{\nu*} \left\{ D_{ij} \left[ -m_{n_i}^2 B_0^{(12)} + m_{n_j}^2 B_1^{(2)} + m_{n_j}^2 m_W C_0 \right.ight.
\]

\[
+ \left. \left( 2 m_Y^2 (m_{n_i}^2 + m_{n_j}^2) + 2 m_{n_i}^2 m_{n_j}^2 - m_a^2 m_{n_j} - m_b^2 m_{n_i} \right) C_2 \right],
\]

\[
+ D_{ij}^* m_{n_i} m_{n_j} \left[-B_0^{(12)} + B_1^{(2)} + m_W^2 C_0 + \left( 4 m_Y^2 + m_{n_i}^2 + m_{n_j}^2 - m_a^2 - m_b^2 \right) C_2 \right]\} ,
\]

where \( B_0^{(12)} = B_0^{(12)}(m_{n_i}^2, m_{n_j}^2), B_1^{(1)} = B_1^{(1)}(m_Y^2, m_{n_i}^2), B_1^{(2)} = B_1^{(2)}(m_Y^2, m_{n_j}^2), \) and \( C_{0,1,2} = C_{0,1,2}(m_Y^2, m_{n_i}^2, m_{n_j}^2) \).

\[
\Delta_L^{(2)Y} = -\frac{g^3 m_a c_q \beta}{64 \pi^2 m_W m_Y^2} \sum_{i=1}^9 U_{(a+3)i}^{\nu} \times \left\{ \lambda_{b_i}^L m_{n_i} \left[ B_0^{(1)} + B_1^{(1)} + \left( m_Y^2 + m_{h_3}^2 - m_{h_0}^2 \right) C_0 - \left( m_Y^2 - m_{h_3}^2 + m_{h_0}^2 \right) C_1 \right] \right.
\]

\[
- \lambda_{b_i}^R m_b \left[ 2 m_Y^2 C_1 + \left( m_Y^2 + m_{h_3}^2 - m_{h_0}^2 \right) C_2 \right] \} .
\]
\[ \Delta^{(2)}_{R} = \frac{g^2 c_o c_3}{64 \pi^2 m_Y m_Y} \sum_{i=1}^{9} U_{(a+3)i}^\nu \]

\[ \times \left\{ \lambda_{b_i}^{L_1} m_{b_i} m_{n_i} \left[ 2m_Y^2 C_0 + \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) C_2 \right] \right. \\
+ \lambda_{b_i}^{R_1} \left[ m_{b_i}^2 B_0^{(1)} + m_{b_i}^2 B_1^{(1)} - m_{n_i}^2 \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) C_0 \right] \\
+ \left[ 2m_Y^2 \left( m_{h_3^+}^2 - m_{h_3^0}^2 \right) - m_a^2 \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) \right] \left( C_1 - 2m_Y^2 m_Y^2 C_2 \right) \}
\]

where \( B_k^{(1)} = B_k^{(1)}(m_Y^2, m_{n_i}^2) \ (k = 0, 1) \) and \( C_{0.1.2} = C_{0.1.2}(m_{n_i}^2, m_Y^2, m_{h_3^0}^2) \),

\[ \Delta^{(3)}_{L} = \frac{g^2 c_o c_3}{64 \pi^2 m_Y m_Y} \sum_{i=1}^{9} U_{(b+3)i}^{\nu*} \]

\[ \times \left\{ \lambda_{a_i}^{L_1} m_{a_i} m_{n_i} \left[ 2m_Y^2 C_0 + \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) C_1 \right] \right. \\
+ \lambda_{a_i}^{R_1} \left[ m_{a_i}^2 B_0^{(2)} + m_{a_i}^2 B_1^{(2)} - m_{n_i}^2 \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) C_0 \right] \\
- \left[ 2m_Y^2 m_Y^2 C_1 + \left[ 2m_Y^2 \left( m_{h_3^+}^2 - m_{h_3^0}^2 \right) - m_a^2 \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) \right] \right] C_2 \}
\]

\[ \Delta^{(3)}_{R} = \frac{g^2 c_o c_3}{64 \pi^2 m_Y m_Y} \sum_{i=1}^{9} U_{(b+3)i}^{\nu*} \]

\[ \times \left\{ \lambda_{a_i}^{L_1} m_{a_i} \left[ \left( m_Y^2 + m_{h_3^+}^2 - m_{h_3^0}^2 \right) C_1 + 2m_Y^2 C_2 \right] \right. \\
- \lambda_{a_i}^{R_1} \left( m_Y^2 + m_{h_3^+}^2 - m_{h_3^0}^2 \right) C_0 - \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) C_0 \right. \\
+ \lambda_{a_i}^{L_1} \left[ 2m_Y^2 \left( m_{h_3^+}^2 - m_{h_3^0}^2 \right) - m_a^2 \left( m_Y^2 - m_{h_3^+}^2 + m_{h_3^0}^2 \right) \right] C_2 \}
\]

where \( B_k^{(2)} = B_k^{(2)}(m_Y^2, m_{n_i}^2) \) and \( C_{0.1.2} = C_{0.1.2}(m_{n_i}^2, m_Y^2, m_{h_3^0}^2) \),

\[ \Delta^{(4)h_{k,l}^\pm}_{L} = \frac{g^2 h_{k,l}}{32 \pi^2 m_Y} \sum_{i=1}^{9} \left[ -\lambda_{a_i}^{R_1} \lambda_{b_i}^{L_1} m_{a_i} m_{n_i} C_0 + \lambda_{a_i}^{L_1} \lambda_{b_i}^{L_1} m_{a_i} m_{n_i} C_1 + \lambda_{a_i}^{R_1} \lambda_{b_i}^{R_1} \lambda_{b_i}^{R_1} m_{a_i} m_{n_i} C_2 \right] \\
\]

\[ \Delta^{(4)h_{k,l}^\pm}_{R} = \frac{g^2 h_{k,l}}{32 \pi^2 m_Y} \sum_{i=1}^{9} \left[ -\lambda_{a_i}^{L_1} \lambda_{b_i}^{L_1} m_{a_i} m_{n_i} C_0 + \lambda_{a_i}^{R_1} \lambda_{b_i}^{R_1} \lambda_{b_i}^{R_1} m_{a_i} m_{n_i} C_1 + \lambda_{a_i}^{L_1} \lambda_{b_i}^{L_1} m_{a_i} m_{n_i} C_2 \right] \\
\]

where \( \{k, l\} = \{1, 2\}, \{2, 1\}, \{1, 1\}, \{2, 2\}, \{3, 3\} \), \( g_{h_{21}} = g_{h_{12}} \) and \( C_{0.1.2} = C_{0.1.2}(m_{n_i}^2, m_{h_3^0}^2, m_{h_3^0}^2) \),

\[ \Delta^{(6)h_{k}^\pm} = \frac{g^2}{64 \pi^2 m_Y} \sum_{i,j=1}^{9} \left\{ \lambda_{0}^{R_1} \lambda_{a_i}^{L_1} \lambda_{a_j}^{L_1} \lambda_{b_j}^{L_1} \left( B_0^{(12)} + m_{h_3^0}^2 C_1 + m_{h_3^0}^2 C_2 \right) \right. \\
+ \lambda_{a_i}^{R_1} \lambda_{b_j}^{R_1} m_{b_i} m_{n_j} C_2 + \lambda_{a_i}^{L_1} \lambda_{b_j}^{L_1} m_{a_i} m_{n_j} C_1 \right. \\
+ \lambda_{a_i}^{L_1} \lambda_{b_j}^{R_1} m_{a_i} m_{n_j} C_0 + \lambda_{a_i}^{R_1} \lambda_{b_j}^{L_1} m_{a_i} m_{n_j} C_0 + \lambda_{a_i}^{L_1} \lambda_{b_j}^{L_1} m_{a_i} m_{n_j} (C_0 + C_2) \right. \\
+ \lambda_{a_i}^{L_1} \lambda_{b_j}^{L_1} m_{a_i} m_{n_j} (C_0 + C_1) + \lambda_{a_i}^{L_1} \lambda_{b_j}^{L_1} m_{a_i} m_{n_j} (C_0 + C_1 + C_2) \right. \}
\]
\[ \Delta^{(6)h^\pm}_k = \frac{g^3}{64\pi^2m_W^3} \sum_{i,j=1}^9 \left\{ \lambda^0_{ij} \left[ \lambda^{L,k}_{ai} \lambda^{R,k}_{bi} \left( B^{(12)}_0 + m^2_{h^\pm_1} C_0 + m^2_a C_1 + m^2_b C_2 \right) + \lambda^{L,k}_{ai} \lambda^{L,k}_{bj} m_a m_n C_2 + \lambda^{R,k}_{ai} \lambda^{R,k}_{bj} m_a m_n C_1 \right] + \lambda^k_{ij} \left[ \lambda^{L,k}_{ai} \lambda^{R,k}_{bj} m_a m_n C_0 + \lambda^{L,k}_{ai} \lambda^{L,k}_{bj} m_a m_n C_0 + \lambda^{R,k}_{ai} \lambda^{L,k}_{bj} m_a m_n C_0 + \lambda^{R,k}_{ai} \lambda^{R,k}_{bj} m_a m_n C_0 \right] \right\}, \]

where \( k = 1, 2, 3, B^{(12)}_0 = B^{(12)}_0(m^2_n, m^2_a), \) and \( C_{0,1,2} = C_{0,1,2}(m^2_n, m^2_a, m^2_{h^\pm_1}), \)

\[ \Delta^{(9+10)h^\pm}_L = \frac{g^3}{64\pi^2m_W^3(m^2_a - m^2_b)} \times \sum_{i=1}^9 \left[ m_a m_b m_n \lambda^{L,k}_{ai} \lambda^{R,k}_{bi} \left( B^{(1)}_0 - B^{(2)}_0 \right) - m_n \lambda^{R,k}_{ai} \lambda^{L,k}_{bi} \left( m^2_b B^{(1)}_0 - m^2_a B^{(2)}_0 \right) \right], \]

\[ \Delta^{(9+10)h^\pm}_R = \frac{g^3}{64\pi^2m_W^3(m^2_a - m^2_b)} \times \sum_{i=1}^9 \left[ m_a m_b m_n \lambda^{R,k}_{ai} \lambda^{R,k}_{bi} \left( B^{(1)}_0 - B^{(2)}_0 \right) + m_n \lambda^{L,k}_{ai} \lambda^{L,k}_{bi} \left( m^2_b B^{(1)}_0 - m^2_a B^{(2)}_0 \right) \right], \]

where \( k = 1, 2, 3, B^{(k)}_{0,1} = B^{(k)}_0(m^2_n, m^2_{h^\pm_1}). \) The details to derive the above formulas of \( \Delta^{(i)}_{L,R} \) were shown in Refs. [23] [121], and hence we do not present them in this work. We note that the scalar functions \( \Delta^{(1)W}_{L,R} \) and \( \Delta^{(1,2,3)Y}_{L,R} \) include parts that do not depend on \( m_n, \) and therefore they vanish because of the Glashow-Iliopoulos-Maiani mechanism.

The divergent cancellation in the total \( \Delta_{L,R} \) is shown as follows.

\[ \text{div} \left[ \Delta^{(1)W}_L \right] = m_a \Delta_\epsilon \times \frac{3}{2} \times \sum_{i=1}^9 U^\nu_{ai} U^\nu_{bi} m^2_{n_i}, \]

\[ \text{div} \left[ \Delta^{(5)W}_L \right] = m_a \Delta_\epsilon \times \sum_{i,j=1}^9 U^\nu_{ai} U^\nu_{bj} \left( -D^\nu_{ij} m^2_{n_j} - \frac{1}{2} D_{ij} m^2_{n_i} \right), \]

\[ \text{div} \left[ \Delta^{(7+8)W}_L \right] = \text{div} \left[ \Delta^{(4)Y}_L \right] = \text{div} \left[ \Delta^{(7+8)Y}_L \right] = 0, \]

\[ \text{div} \left[ \Delta^{(1)Y}_L \right] = m_a \Delta_\epsilon \times \left( \frac{3s^2_w}{2c_w^2} \right) \sum_{i=1}^9 U^\nu_{(a+3)i} U^\nu_{(b+3)i} m^2_{n_i}, \]

\[ \text{div} \left[ \Delta^{(2)Y}_L \right] = m_a \Delta_\epsilon \times \left( -\frac{c_\theta s^2_w}{2c_w^2} \right) \sum_{i=1}^9 U^\nu_{(a+3)i} \lambda^{L,1}_{bi} m_{n_i}, \]
\[
\text{div} \left[ \Delta^{(3)Y}_L \right] = \Delta_c \times \left( \frac{c_\theta s_\beta^2}{c_\beta} \right) \sum_{i=1}^{9} U^{\nu*}_{(a+3)i} L_{a1}^{R1} m_{n_i}^2,
\]

\[
\text{div} \left[ \Delta^{(5)Y}_L \right] = m_a \Delta_c \times \frac{s_\theta^2}{c_\beta} \sum_{i,j=1}^{9} U^{\nu*}_{(a+3)i} U^{\nu}_{(b+3)j} \left( -D_{ij} m_{n_j}^2 - \frac{1}{2} D_{ij} m_{n_i}^2 \right),
\]

\[
\text{div} \left[ \Delta^{(6)Y^+_{h^+}}_L \right] = m_a \Delta_c \times \sum_{i,j=1}^{9} U^{\nu*}_{(a+3)i} \lambda_{ij}^{L,k} \lambda_{bj}^{L,k},
\]

\[
\text{div} \left[ \Delta^{(9+10)Y^+_{h^+}}_L \right] = -m_a \Delta_c \times \sum_{i=1}^{9} U^{\nu*}_{(a+3)i} \lambda_{bi}^{L,k} m_{n_i},
\]

(A1)

where: \( \text{div} B^{(1)}_0 = \text{div} B^{(1)}_1 = -2 \text{div} B^{(2)}_1 = -2 \text{div} B^{(2)}_2 = \Delta_c \) and \( 1/m_Y = s_\theta/(c_\beta m_W) \).

Easy to see that \( \text{div} \left[ \Delta^{(1)W}_L \right] + \text{div} \left[ \Delta^{(5)W}_L \right] = \text{div} \left[ \Delta^{(6)Y^+_{h^+}}_L \right] + \text{div} \left[ \Delta^{(9+10)Y^+_{h^+}}_L \right] = 0 \) and the sum of the remaining divergent parts is zero in case we are focusing on investigating \( c_\theta = 1 \).

[1] Y. Fukuda et al. [Super-Kamiokande], Phys. Rev. Lett. 81 (1998), 1562-1567 [arXiv:hep-ex/9807003 [hep-ex]].
[2] S. Fukuda et al. [Super-Kamiokande], Phys. Rev. Lett. 86 (2001), 5651-5655 [arXiv:hep-ex/0103032 [hep-ex]].
[3] S. Fukuda et al. [Super-Kamiokande], Phys. Rev. Lett. 86 (2001), 5656-5660 [arXiv:hep-ex/0103033 [hep-ex]].
[4] Q. R. Ahmad et al. [SNO], Phys. Rev. Lett. 89 (2002), 011301 [arXiv:nucl-ex/0204008 [nucl-ex]].
[5] Q. R. Ahmad et al. [SNO], Phys. Rev. Lett. 89 (2002), 011302 [arXiv:nucl-ex/0204009 [nucl-ex]].
[6] B. Aubert et al. [BaBar], Phys. Rev. Lett. 104, 021802 (2010) [arXiv:0908.2381 [hep-ex]].
[7] A. M. Baldini et al. [MEG], Eur. Phys. J. C 76, no.8, 434 (2016) [arXiv:1605.05081 [hep-ex]].
[8] E. Kou et al. [Belle-II], PTEP 2019, no.12, 123C01 (2019) [erratum: PTEP 2020, no.2, 029201 (2020)] [arXiv:1808.10567 [hep-ex]].
[9] T. Aushev, W. Bartel, A. Bondar, J. Brodzicka, T. E. Browder, P. Chang, Y. Chao, K. F. Chen, J. Dalseno and A. Drutskoy, et al. [arXiv:1002.5012 [hep-ex]].
[10] A. M. Baldini et al. [MEG II], Eur. Phys. J. C 78, no.5, 380 (2018) [arXiv:1801.04688 [physics.ins-det]].

[11] A. M. Sirunyan et al. [CMS], JHEP 06 (2018), 001.

[12] G. Aad et al. [ATLAS], Phys. Lett. B 800, 135069 (2020) [arXiv:1907.06131 [hep-ex]].

[13] [ATLAS], ATLAS-CONF-2019-037.

[14] Q. Qin, Q. Li, C. D. Lü, F. S. Yu and S. H. Zhou, Eur. Phys. J. C 78 (2018) no.10, 835 [arXiv:1711.07243 [hep-ph]].

[15] A. Zee, Phys. Lett. B 93, 389 (1980) [erratum: Phys. Lett. B 95, 461 (1980)] doi:10.1016/0370-2693(80)90349-4

[16] R. K. Barman, R. Dcruz and A. Thapa, JHEP 03, 183 (2022) [arXiv:2112.04523 [hep-ph]].

[17] J. Herrero-García, T. Ohlsson, S. Riad and J. Wirén, JHEP 04 (2017), 130.

[18] D. Sabatta, A. S. Cornell, A. Goyal, M. Kumar, B. Mellado and X. Ruan, Chin. Phys. C 44, no.6, 063103 (2020) [arXiv:1909.03969 [hep-ph]].

[19] A. Vicente, Front. in Phys. 7, 174 (2019) [arXiv:1908.07759 [hep-ph]].

[20] E. Arganda, A. M. Curiel, M. J. Herrero and D. Temes, Phys. Rev. D 71 (2005), 035011.

[21] X. Marcano and R. A. Morales, Front. in Phys. 7, 228 (2020) doi:10.3389/fphy.2019.00228 [arXiv:1909.05888 [hep-ph]].

[22] A. Ilakovac, Phys. Rev. D 62 (2000), 036010.

[23] E. Arganda, M. J. Herrero, X. Marcano and C. Weiland, Phys. Rev. D 91 (2015) no.1, 015001.

[24] E. Arganda, M. J. Herrero, X. Marcano, R. Morales and A. Szynkman, Phys. Rev. D 95 (2017) no.9, 095029.

[25] N. H. Thao, L. T. Hue, H. T. Hung and N. T. Xuan, Nucl. Phys. B 921, 159-180 (2017), arXiv:1703.00896 [hep-ph].

[26] G. Hernández-Tomé, J. I. Illana and M. Masip, Phys. Rev. D 102, no.11, 113006 (2020) [arXiv:2005.11234 [hep-ph]].

[27] T. P. Nguyen, T. T. Thuc, D. T. Si, T. T. Hong and L. T. Hue, PTEP 2022, 023 (2022), arXiv:2011.12181 [hep-ph].

[28] A. Brignole and A. Rossi, Phys. Lett. B 566 (2003), 217-225.

[29] A. Brignole and A. Rossi, Nucl. Phys. B 701 (2004), 3-53.

[30] J. L. Diaz-Cruz, JHEP 05 (2003), 036.
[31] P. T. Giang, L. T. Hue, D. T. Huong and H. N. Long, Nucl. Phys. B 864 (2012), 85-112.
[32] M. Arana-Catania, E. Arganda and M. J. Herrero, JHEP 09 (2013), 160 [erratum: JHEP 10 (2015), 192].
[33] L. T. Hue, H. N. Long, T. T. Thuc and T. Phong Nguyen, Nucl. Phys. B 907 (2016), 37, arXiv:1512.03266 [hep-ph].
[34] E. Arganda, M. J. Herrero, R. Morales and A. Szynkman, JHEP 03 (2016), 055.
[35] E. Arganda, M. J. Herrero, X. Marcano and C. Weiland, Phys. Rev. D 93 (2016) no.5, 055010.
[36] M. Zeleny-Mora, J. L. Díaz-Cruz and O. Félix-Beltrán, [arXiv:2112.08412 [hep-ph]].
[37] B. Yang, J. Han and N. Liu, Phys. Rev. D 95 (2017) no.3, 035010.
[38] H. K. Guo, Y. Y. Li, T. Liu, M. Ramsey-Musolf and J. Shu, Phys. Rev. D 96 (2017) no.11, 115034.
[39] M. Aoki, S. Kanemura, K. Sakurai and H. Sugiyama, Phys. Lett. B 763 (2016), 352-357.
[40] K. Cheung, W. Y. Keung and P. Y. Tseng, Phys. Rev. D 93 (2016) no.1, 015010.
[41] K. Huitu, V. Keus, N. Koivunen and O. Lebedev, JHEP 05 (2016), 026.
[42] C. H. Chen and T. Nomura, Eur. Phys. J. C 76 (2016) no.6, 353.
[43] C. F. Chang, C. H. V. Chang, C. S. Nugroho and T. C. Yuan, Nucl. Phys. B 910 (2016), 293-308.
[44] W. Altmannshofer, S. Gori, A. L. Kagan, L. Silvestrini and J. Zupan, Phys. Rev. D 93 (2016) no.3, 031301.
[45] Y. Omura, E. Senaha and K. Tobe, Phys. Rev. D 94 (2016) no.5, 055019.
[46] A. Lami and P. Roig, Phys. Rev. D 94 (2016) no.5, 056001.
[47] D. Das and A. Kundu, Phys. Rev. D 92 (2015) no.1, 015009.
[48] A. Crivellin, G. D’Ambrosio and J. Heeck, Phys. Rev. Lett. 114 (2015), 151801.
[49] M. D. Campos, A. E. Cárcamo Hernández, H. Päs and E. Schumacher, Phys. Rev. D 91 (2015) no.11, 116011.
[50] Y. Omura, E. Senaha and K. Tobe, JHEP 05 (2015), 028.
[51] L. de Lima, C. S. Machado, R. D. Matheus and L. A. F. do Prado, JHEP 11 (2015), 074.
[52] J. Heeck, M. Holthausen, W. Rodejohann and Y. Shimizu, Nucl. Phys. B 896 (2015), 281-310.
[53] I. Doršner, S. Fajfer, A. Greljo, J. F. Kamenik, N. Košnik and I. Nišandžić, JHEP 06 (2015),
[54] X. G. He, J. Tandean and Y. J. Zheng, JHEP 09 (2015), 093.
[55] A. Dery, A. Efrati, Y. Nir, Y. Soreq and V. Susič, Phys. Rev. D 90 (2014), 115022.
[56] A. Celis, V. Cirigliano and E. Passemar, Phys. Rev. D 89 (2014), 013008.
[57] A. Falkowski, D. M. Straub and A. Vicente, JHEP 05 (2014), 092.
[58] R. Harnik, J. Kopp and J. Zupan, JHEP 03 (2013), 026.
[59] P. S. Bhupal Dev, R. Franceschini and R. N. Mohapatra, Phys. Rev. D 86 (2012), 093010.
[60] A. Goudelis, O. Lebedev and J. h. Park, Phys. Lett. B 707 (2012), 369-374.
[61] J. L. Diaz-Cruz and J. J. Toscano, Phys. Rev. D 62 (2000), 116005.
[62] J. G. Korner, A. Pilaftsis and K. Schilcher, Phys. Rev. D 47 (1993), 1080-1086.
[63] A. Pilaftsis, Z. Phys. C 55 (1992), 275-282.
[64] A. Pilaftsis, Phys. Lett. B 285 (1992), 68-74.
[65] G. Blankenburg, J. Ellis and G. Isidori, Phys. Lett. B 712 (2012), 386-390.
[66] A. E. Cárcamo Hernández, E. Cataño Mur and R. Martinez, Phys. Rev. D 90, no.7, 073001 (2014) [arXiv:1407.5217 [hep-ph]].
[67] D. Jurčiukonis and L. Lavoura, JHEP 03, 106 (2022) [arXiv:2107.14207 [hep-ph]].
[68] T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C. M. Carloni Calame, M. Cè and G. Colangelo, et al. Phys. Rept. 887, 1-166 (2020) [arXiv:2006.04822 [hep-ph]].
[69] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Phys. Rev. Lett. 109, 111808 (2012) [arXiv:1205.5370 [hep-ph]].
[70] T. Aoyama, T. Kinoshita and M. Nio, Atoms 7, no.1, 28 (2019)
[71] A. Czarnecki, W. J. Marciano and A. Vainshtein, Phys. Rev. D 67, 073006 (2003) [erratum: Phys. Rev. D 73, 119901 (2006)] [arXiv:hep-ph/0212229 [hep-ph]].
[72] C. Gnendiger, D. Stöckinger and H. Stöckinger-Kim, Phys. Rev. D 88, 053005 (2013) [arXiv:1306.5546 [hep-ph]].
[73] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, Eur. Phys. J. C 77, no.12, 827 (2017) [arXiv:1706.09436 [hep-ph]].
[74] A. Keshavarzi, D. Nomura and T. Teubner, Phys. Rev. D 97, no.11, 114025 (2018) [arXiv:1802.02995 [hep-ph]].
[75] G. Colangelo, M. Hoferichter and P. Stoffer, JHEP 02, 006 (2019) [arXiv:1810.00007 [hep-}
[76] M. Hoferichter, B. L. Hoid and B. Kubis, JHEP 08, 137 (2019) [arXiv:1907.01556 [hep-ph]].
[77] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, Eur. Phys. J. C 80, no.3, 241 (2020) [erratum: Eur. Phys. J. C 80, no.5, 410 (2020)] [arXiv:1908.00921 [hep-ph]].
[78] A. Keshavarzi, D. Nomura and T. Teubner, Phys. Rev. D 101, no.1, 014029 (2020) [arXiv:1911.00367 [hep-ph]].
[79] A. Kurz, T. Liu, P. Marquard and M. Steinhauser, Phys. Lett. B 734, 144-147 (2014) [arXiv:1403.6400 [hep-ph]].
[80] K. Melnikov and A. Vainshtein, Phys. Rev. D 70, 113006 (2004) [arXiv:hep-ph/0312226 [hep-ph]].
[81] P. Masjuan and P. Sanchez-Puertas, Phys. Rev. D 95, no.5, 054026 (2017) [arXiv:1701.05829 [hep-ph]].
[82] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP 04, 161 (2017) [arXiv:1702.07347 [hep-ph]].
[83] M. Hoferichter, B. L. Hoid, B. Kubis, S. Leupold and S. P. Schneider, JHEP 10, 141 (2018) [arXiv:1808.04823 [hep-ph]].
[84] A. Gérardin, H. B. Meyer and A. Nyffeler, Phys. Rev. D 100, no.3, 034520 (2019) [arXiv:1903.09471 [hep-ph]].
[85] J. Bijnens, N. Hermansson-Truedsson and A. Rodríguez-Sánchez, Phys. Lett. B 798, 134994 (2019) [arXiv:1908.03331 [hep-ph]].
[86] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub and P. Stoffer, JHEP 03, 101 (2020) [arXiv:1910.13432 [hep-ph]].
[87] T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung and C. Lehner, Phys. Rev. Lett. 124, no.13, 132002 (2020) [arXiv:1911.08123 [hep-lat]].
[88] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera and P. Stoffer, Phys. Lett. B 735, 90-91 (2014) [arXiv:1403.7512 [hep-ph]].
[89] V. Pauk and M. Vanderhaeghen, Eur. Phys. J. C 74, no.8, 3008 (2014) [arXiv:1401.0832 [hep-ph]].
[90] I. Danilkin and M. Vanderhaeghen, Phys. Rev. D 95, no.1, 014019 (2017) [arXiv:1611.04646 [hep-ph]].
[91] F. Jegerlehner, Springer Tracts Mod. Phys. 274, pp.1-693 (2017)
[92] M. Knecht, S. Narison, A. Rabemananjara and D. Rabetiarivony, Phys. Lett. B 787, 111-123 (2018) [arXiv:1808.03848 [hep-ph]].

[93] G. Eichmann, C. S. Fischer and R. Williams, Phys. Rev. D 101, no.5, 054015 (2020) [arXiv:1910.06795 [hep-ph]].

[94] P. Roig and P. Sanchez-Puertas, Phys. Rev. D 101, no.7, 074019 (2020) [arXiv:1910.02881 [hep-ph]].

[95] B. Abi et al. [Muon g-2], Phys. Rev. Lett. 126, no.14, 141801 (2021) [arXiv:2104.03281 [hep-ex]].

[96] G. W. Bennett et al. [Muon g-2], Phys. Rev. D 73 (2006), 072003 [arXiv:hep-ex/0602035 [hep-ex]].

[97] S. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato and K. K. Szabo, et al. Nature 593 (2021) no.7857, 51-55 [arXiv:2002.12347 [hep-lat]].

[98] A. Crivellin, M. Hoferichter, C. A. Manzari and M. Montull, Phys. Rev. Lett. 125, no.9, 091801 (2020) [arXiv:2003.04886 [hep-ph]].

[99] A. Keshavarzi, W. J. Marciano, M. Passera and A. Sirlin, Phys. Rev. D 102, no.3, 033002 (2020) [arXiv:2006.12666 [hep-ph]].

[100] G. Colangelo, M. Hoferichter and P. Stoffer, Phys. Lett. B 814, 136073 (2021) [arXiv:2010.07943 [hep-ph]].

[101] L. Morel, Z. Yao, P. Cladé and S. Guellati-Khélifa, Nature 588, no.7836, 61-65 (2020)

[102] Z. N. Zhang, H. B. Zhang, J. L. Yang, S. M. Zhao and T. F. Feng, Phys. Rev. D 103 (2021) no.11, 115015 [arXiv:2105.09799 [hep-ph]].

[103] S. Baek and K. Nishiwaki, Phys. Rev. D 93 (2016) no.1, 015002 [arXiv:1509.07410 [hep-ph]].

[104] R. Foot, H. N. Long and T. A. Tran, Phys. Rev. D 50 (1994) no.1, R34-R38 [arXiv:hep-ph/9402243 [hep-ph]].

[105] H. N. Long, Phys. Rev. D 54 (1996), 4691-4693 [arXiv:hep-ph/9607439 [hep-ph]].

[106] H. N. Long, Phys. Rev. D 53 (1996), 437-445 [arXiv:hep-ph/9504274 [hep-ph]].

[107] L. T. Hue, H. T. Hung, N. T. Tham, H. N. Long and T. P. Nguyen, Phys. Rev. D 104 (2021) no.3, 033007 [arXiv:2104.01840 [hep-ph]].

[108] H. B. Zhang, T. F. Feng, S. M. Zhao, Y. L. Yan and F. Sun, Chin. Phys. C 41 (2017) no.4, 043106 [arXiv:1511.08979 [hep-ph]].
[109] T. P. Nguyen, T. T. Le, T. T. Hong and L. T. Hue, Phys. Rev. D 97 (2018) no.7, 073003 [arXiv:1802.00429 [hep-ph]].

[110] A. E. Cárcamo Hernández, L. T. Hue, S. Kovalenko and H. N. Long, Eur. Phys. J. Plus 136 (2021) no.11, 1158 [arXiv:2001.01748 [hep-ph]].

[111] H. T. Hung, N. T. Tham, T. T. Hieu and N. T. T. Hang, PTEP 2021 (2021) no.8, 083B01 [arXiv:2103.16018 [hep-ph]].

[112] J. C. Montero, F. Pisano and V. Pleitez, Phys. Rev. D 47 (1993), 2918-2929 [arXiv:hep-ph/9212271 [hep-ph]].

[113] M. Singer, J. W. F. Valle and J. Schechter, Phys. Rev. D 22 (1980), 738

[114] P. H. Frampton, Phys. Rev. Lett. 69 (1992), 2889-2891

[115] F. Pisano and V. Pleitez, Phys. Rev. D 46 (1992), 410-417 [arXiv:hep-ph/9206242 [hep-ph]].

[116] M. Lindner, M. Platscher and F. S. Queiroz, Phys. Rept. 731 (2018), 1-82 [arXiv:1610.06587 [hep-ph]].

[117] A. S. De Jesus, S. Kovalenko, F. S. Queiroz, C. Siqueira and K. Sinha, Phys. Rev. D 102 (2020) no.3, 035004 [arXiv:2004.01200 [hep-ph]].

[118] Á. S. de Jesus, S. Kovalenko, F. S. Queiroz, C. A. de S. Pires and Y. S. Villamizar, Phys. Lett. B 809 (2020), 135689 [arXiv:2003.06440 [hep-ph]].

[119] A. E. C. Hernández, D. T. Huong and I. Schmidt, Eur. Phys. J. C 82 (2022) no.1, 63 [arXiv:2109.12118 [hep-ph]].

[120] L. T. Hue, K. H. Phan, T. P. Nguyen, H. N. Long and H. T. Hung, Eur. Phys. J. C 82, no.8, 722 (2022) [arXiv:2109.06089 [hep-ph]].

[121] S. M. Boucenna, J. W. F. Valle and A. Vicente, Phys. Rev. D 92 (2015) no.5, 053001 [arXiv:1502.07546 [hep-ph]].

[122] D. Chang and H. N. Long, Phys. Rev. D 73 (2006), 053006 [arXiv:hep-ph/0603098 [hep-ph]].

[123] A. J. Buras, F. De Fazio, J. Girrbach and M. V. Carlucci, JHEP 02 (2013), 023 [arXiv:1211.1237 [hep-ph]].

[124] H. K. Dreiner, H. E. Haber and S. P. Martin, Phys. Rept. 494 (2010), 1-196 [arXiv:0812.1594 [hep-ph]].

[125] L. Ninh and H. N. Long, Phys. Rev. D 72 (2005), 075004 [arXiv:hep-ph/0507069 [hep-ph]].

[126] L. T. Hue, L. D. Ninh, T. T. Thuc and N. T. T. Dat, Eur. Phys. J. C 78 (2018) no.2, 128 [arXiv:1708.09723 [hep-ph]].
[127] H. Okada, N. Okada, Y. Orikasa and K. Yagyu, Phys. Rev. D 94 (2016) no.1, 015002 [arXiv:1604.01948 [hep-ph]].

[128] H. T. Hung, T. T. Hong, H. H. Phuong, H. L. T. Mai and L. T. Hue, Phys. Rev. D 100 (2019) no.7, 075014 [arXiv:1907.06735 [hep-ph]].

[129] A. Crivellin, M. Hoferichter and P. Schmidt-Wellenburg, Phys. Rev. D 98 (2018) no.11, 113002 [arXiv:1807.11484 [hep-ph]].

[130] F. Jegerlehner and A. Nyffeler, Phys. Rept. 477 (2009), 1-110 [arXiv:0902.3360 [hep-ph]].

[131] A. Denner, S. Heinemeyer, I. Puljak, D. Rebuazzi and M. Spira, Eur. Phys. J. C 71 (2011), 1753 [arXiv:1107.5909 [hep-ph]].

[132] P. A. Zyla et al. [Particle Data Group], PTEP 2020 (2020) no.8, 083C01

[133] K. Abe et al. [T2K], Nature 580 (2020) no.7803, 339-344 [erratum: Nature 583 (2020) no.7814, E16] [arXiv:1910.03887 [hep-ex]].

[134] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D 98 (2018) no.3, 030001

[135] G. ’t Hooft and M. J. G. Veltman, Nucl. Phys. B 44 (1972), 189-213

[136] A. Denner and S. Dittmaier, Nucl. Phys. B 734 (2006), 62-115 [arXiv:hep-ph/0509141 [hep-ph]].

[137] T. Hahn and M. Perez-Victoria, Comput. Phys. Commun. 118 (1999), 153-165 [arXiv:hep-ph/9807565 [hep-ph]].

[138] K. H. Phan, H. T. Hung and L. T. Hue, PTEP 2016 (2016) no.11, 113B03 [arXiv:1605.07164 [hep-ph]].

[139] K. Abe et al. [T2K], Nature 580, no.7803, 339-344 (2020) [erratum: Nature 583, no.7814, E16 (2020)] [arXiv:1910.03887 [hep-ex]].

[140] K. Enomoto, S. Kanemura, K. Sakurai and H. Sugiyama, Phys. Rev. D 100, no.1, 015044 (2019) [arXiv:1904.07039 [hep-ph]].

[141] H. B. Camara, R. G. Felipe and F. R. Joaquim, JHEP 05, 021 (2021) [arXiv:2012.04557 [hep-ph]].

[142] M. Nebot, J. F. Oliver, D. Palao and A. Santamaria, Phys. Rev. D 77, 093013 (2008) [arXiv:0711.0483 [hep-ph]].

[143] E. Fernandez-Martinez, J. Hernandez-Garcia and J. Lopez-Pavon, JHEP 08, 033 (2016) [arXiv:1605.08774 [hep-ph]].

[144] N. R. Agostinho, G. C. Branco, P. M. F. Pereira, M. N. Rebelo and J. I. Silva-Marcos, Eur.
[145] A. M. Coutinho, A. Crivellin and C. A. Manzari, Phys. Rev. Lett. 125, no.7, 071802 (2020) [arXiv:1912.08823 [hep-ph]].

[146] C. A. Manzari, A. M. Coutinho and A. Crivellin, PoS LHCP2020, 242 (2021) [arXiv:2009.03877 [hep-ph]].