Can the correlated stability conjecture be saved?

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Abstract

Correlated stability conjecture (CSC) proposed by Gubser and Mitra \cite{1, 2} linked the thermodynamic and classical (in)stabilities of black branes. In \cite{3} it was shown that the thermodynamic instabilities, specifically the negative specific heat, indeed result in the instabilities in the hydrodynamic spectrum of holographically dual plasma excitations. Counter-examples of CSC were presented in the context of black branes with scalar hair undergoing a second-order phase transition \cite{4, 5}. The latter translationary invariant horizons have scalar hair, raising the question whether the asymptotic parameters of the scalar hair can be appropriately interpreted as additional charges leading to a generalization of the thermodynamic stability criterion. In this paper we show that the generalization of the thermodynamic stability criterion of this type cannot save CSC. We further present a simple statistical model which makes it clear that thermodynamic and dynamical (in)stabilities generically are not correlated.

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1 Generalized CSC

A standard claim in classical thermodynamics\(^1\) is that a system is thermodynamically stable if the Hessian \(\mathbb{H}^E_{s,Q_A}\) of the energy density \(E = E(s,Q_A)\) with respect to the entropy density \(s\) and charges \(Q_A \equiv \{Q_1, \ldots, Q_n\}\), i.e.,
\[
\mathbb{H}^E_{s,Q_A} \equiv \begin{pmatrix}
\frac{\partial^2 E}{\partial s^2} & \frac{\partial^2 E}{\partial s \partial Q_B} \\
\frac{\partial^2 E}{\partial Q_A \partial s} & \frac{\partial^2 E}{\partial Q_A \partial Q_B}
\end{pmatrix},
\]
(1.1)
does not have negative eigenvalues. In the simplest case \(n = 0\), i.e., no conserved charges, the thermodynamic stability implies that
\[
0 < \frac{\partial^2 E}{\partial s^2} = \frac{T}{c_v},
\]
(1.2)
that is the specific heat \(c_v\) is positive. In the context of gauge theory/string theory correspondence \([6]\) black holes with translationary invariant horizons in asymptotically anti-de-Sitter space-time are dual (equivalent) to equilibrium thermal states of certain strongly coupled systems. Thus, the above thermodynamic stability criterion should be directly applicable to black branes as well. The correlated stability conjecture (CSC) asserts that it is only when the Hessian (1.1) for a given black brane geometry is positive, the spectrum of on-shell excitations in this background geometry is free from tachyons \([1,2]\).

In the simplest case, i.e., the absence of the chemical potentials, one can trivially identify the classical instabilities of the thermodynamically unstable system \([3]\). Indeed, since the speed of sound waves squared in this case is \(c_s^2 = \frac{\rho}{c_v}\), the thermodynamic instability of the system \((c_v < 0)\) immediately implies that the hydrodynamic (sound) modes are classically unstable. There is no simple argument implying that thermodynamic stability of the system is enough to secure its classical stability; moreover,

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\(^1\)Assuming that the temperature is positive.
the instability link with the sound waves does not work in strongly coupled R-charged $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma [5] — here, the speed of sound is always fixed to a conformal value $c_s^2 = \frac{1}{3}$ even though there is an equilibrium branch with $c_v < 0$.

In [4,5] it has demonstrated that, at least for a canonical interpretation of the black brane thermodynamics, the CSC is violated in the case of black branes with scalar hair that undergo a continuous phase transition. The dual gauge theory picture makes such violation almost self-evident. Indeed, in the vicinity of a continuous phase transition the condensate does not noticeably modify the thermodynamics, and thus should not affect the thermodynamic stability of the system. On the other hand, the phase of the system with the higher free energy is expected to be classically unstable. The condensation of the tachyon should bring the system to the equilibrium phase with the lowest free energy.

The important qualifier for the above counter-examples is the *canonical interpretation* of the corresponding black brane thermodynamics. Specifically, the black branes considered have scalar hair and in the proper boundary (field theoretic) thermodynamic interpretation one has to keep non-normalizable coefficients of the scalars fixed. The reason for this is that these non-normalizable coefficients are dual to mass-scales in the boundary field theory. In thermodynamic stability analysis one naturally would like to keep microscopic mass scales in the field theory fixed. If one abandons the gauge/gravity analogy and considers black branes as thermal systems in higher dimensional general relativity, the motivation for keeping the asymptotic scalar hair parameters fixed is removed. It is an interesting question as to whether these parameters might be treated as generalized charges in the context of thermodynamic stability of translationary invariant horizons in such a way that the CSC is validated\(^2\). We argue here that CSC generalizations of these type are false.

In the next section we present a simple statistical model in which the generalized thermodynamic and the dynamical (in)stabilities are not correlated. In section 3 we show that the exotic hairy black branes discussed in [5,7,8] while classically unstable, are thermodynamically stable in the generalized manner outlined above. In both cases the classical instabilities we identify are long-wavelength, provided, in the statistical model in section 2, $\Lambda \ll T$, and for exotic hairy black branes one stays close to the phase transition.

\(^2\)We would like to thank Barak Kol for raising this possibility.
2 Counter-example to generalized CSC in statistical physics

Consider a Landau-Ginsburg model with the following free energy density functional

\[ F = -T^4 + \Lambda^4 + \frac{1}{2} \left( \nabla \phi(\vec{x}) \right)^2 - \frac{1}{2} \Lambda^2 \phi(\vec{x})^2 + \frac{1}{4} \phi(\vec{x})^4, \]  

where \( \Lambda \) is a mass-scale, and \( \phi(\vec{x}) \) is a dynamical scalar field. For any temperature \( T \), there are three equilibrium states of the system: one unstable \( u \) and two degenerate stable ones \( s \),

\[
\begin{align*}
\langle \phi(\vec{x}) \rangle \bigg|_{\text{unstable}} & = 0, \quad \Rightarrow \quad F_u = -T^4 + \Lambda^4, \\
\langle \phi(\vec{x}) \rangle \pm \bigg|_{\text{stable}} & = \pm \Lambda, \quad \Rightarrow \quad F_{s}^\pm = -T^4 + \frac{3}{4} \Lambda^4.
\end{align*}
\]

In what follows we focus on the unstable equilibrium. Here, the energy density \( E_u \) is given by

\[
E_u = \frac{3}{2^{8/3}} s^{4/3} + \frac{1}{3} \Lambda^4, \quad (2.3)
\]

where \( s \) is the entropy density. It is straightforward to see that whether or not we treat the scale \( \Lambda \) as a generalized charge \( Q_A \) in the context of the thermodynamic stability (see (1.1)), this classically unstable equilibrium is thermodynamically stable. In other words, both Hessians \( H_{E_u}^s \) and \( H_{E_u}^{s,\Lambda} \) are positive.

Notice that since the energy \( E_s \) of the stable equilibrium is

\[
E_s = \frac{3}{2^{8/3}} s^{4/3} + \frac{3}{4} \Lambda^4, \quad (2.4)
\]

any other definition of the generalized charge \( Q_A = f(\Lambda) \) would imply that the two equilibria \( u \) and \( s \) are simultaneously either thermodynamically stable or not\(^3\). It might be possible to define a generalized charge \( Q_A \) which depends both on \( s \) and \( \Lambda \), i.e., \( Q_A = f(s, \Lambda) \), so that \( s \) state is thermodynamically stable while the \( u \) state is thermodynamically unstable — it is not clear to us how to make such a definition universally for all statistical systems.

The model (2.1) is probably the simplest example which clearly demonstrates that the thermodynamic and the dynamical (in)stabilities of the system do not generically correlate. Since black holes with translationary invariant horizons in asymptotically anti-de-Sitter space-time are dual (albeit sometimes in a purely phenomenological way)

\(^3\)Clearly, we need to restrict definition of \( Q_A \) so that \( s \) equilibrium is thermodynamically stable.
to some strongly coupled field theory, one expects that it should be possible to construct a counter-example of generalized CSC as well. In the next section we show that the generalized CSC$^4$ is violated for the exotic hairy black branes introduced in [7].

3 Counter-example to generalized CSC in gravity

Consider the following effective (3+1)-dimensional gravitational action [7]:

$$S_4 = \frac{1}{2\kappa^2} \int d^4\sqrt{-\gamma} \left[ R + 6 - \frac{1}{2} (\nabla \phi)^2 + \phi^2 - \frac{1}{2} (\nabla \chi)^2 - 2\chi^2 - g\phi^2\chi^2 \right], \quad (3.1)$$

where $g$ is a coupling constant$^5$. This effective action admits asymptotically AdS$_4$ hairy black brane solutions with translationary invariant horizon. Specifically, the background geometry takes form

$$ds_4^2 = -c_1(r)^2 dt^2 + c_2(r)^2 \left[ dx_1^2 + dx_2^2 \right] + c_3^2 dr^2, \quad \phi = \phi(r), \quad \chi = \chi(r). \quad (3.2)$$

We find it convenient to introduce a new radial coordinate $x$ as follows

$$1 - x \equiv \frac{c_1(r)}{c_2(r)}, \quad (3.3)$$

so that $x \to 0$ corresponds to the boundary asymptotic, and $y \equiv 1 - x \to 0$ corresponds to a regular Schwarzschild horizon asymptotic. Further introducing

$$c_2(x) = \frac{a(x)}{(2x - x^2)^{1/3}}, \quad (3.4)$$

the equations of motion obtained from (3.1), with the background ansatz (3.2), imply

$$a = \alpha \left( 1 - \frac{1}{40} p_1^2 x^{2/3} - \frac{1}{18} p_1 p_2 x + O(x^{4/3}) \right),$$

$$\phi = p_1 x^{1/3} + p_2 x^{2/3} + \frac{3}{20} p_1^2 x + O(x^{4/3}), \quad (3.5)$$

$$\chi = \chi_4 \left( x^{4/3} + \left( \frac{1}{7} g - \frac{3}{70} \right) p_1^2 x^2 + O(x^{7/3}) \right),$$

near the boundary $x \to 0_+$, and

$$a = \alpha \left( a_0^h + \frac{a_1^h}{y^2} + O(y^4) \right), \quad \phi = p_0^h + O(y^2), \quad \chi = c_0^h + O(y^2), \quad (3.6)$$

$^4$The violation of the canonical CSC in this system is shown in [5].

$^5$In numerical analysis we set $g = -100$. 
near the horizon $y = 1 - x \to 0$. Apart from the overall scaling factor $\alpha$ (which is related to the temperature), the background is uniquely specified with 3 UV coefficients $\{p_1, p_2, \chi_4\}$ and 4 IR coefficients $\{a_0^h, a_1^h, p_0^h, c_0^h\}$.

It is straightforward to compute the temperature $T$ and the entropy density $s$ of the black brane solution (3.2):

$$\left(\frac{8\pi T}{\alpha}\right)^2 = \frac{6(a_0^h)^3(6 - 2(\chi_0^h)^2 + (p_0^h)^2 - g(p_0^h)^2(c_0^h)^2)}{3a_1^h + a_0^h},$$

$$\hat{s} \equiv \frac{384}{c} s = 4\pi\alpha^2 (a_0^h)^2,$$

where

$$c = \frac{192}{\kappa^2},$$

is the central charge of the UV fixed point. The free energy density $\mathcal{F}$ and the energy density $\mathcal{E}$ are given by

$$\hat{\mathcal{F}} \equiv \frac{384}{c} \mathcal{F} = \alpha^3 \left(2 - \frac{1}{6}p_1p_2 - \frac{(a_0^h)^3}{2} \sqrt{\frac{6a_0^h(6 - 2(\chi_0^h)^2 + (p_0^h)^2 - g(p_0^h)^2(c_0^h)^2)}{3a_1^h + a_0^h}}\right),$$

$$\hat{\mathcal{E}} \equiv \frac{384}{c} \mathcal{E} = \alpha^3 \left(2 - \frac{1}{6}p_1p_2\right).$$

Lastly, we identify $\Lambda$,

$$\Lambda \equiv p_1 \alpha,$$

with the mass scale of the dual (boundary) field theory. Notice that the scalar field $\chi$ can not have a non-zero non-normalizable coefficient as the latter would destroy the asymptotic $AdS_4$ geometry — near the boundary, the non-normalizable mode of $\chi$ behaves as$^6 \chi \sim x^{-1/3}$.

For a given set of $\{\alpha, p_1\}$ there is a discrete set of the remaining parameters

$$\{p_2, \chi_4, a_0^h, a_1^h, p_0^h, c_0^h\}$$

characterizing black brane solutions. One of these solutions has $\{\chi_4, c_0^h\} = \{0, 0\}$ and describes the black brane without the condensate of the $\chi$ field. All the other solutions have $\{\chi_4, c_0^h\} \neq \{0, 0\}$ and describe the ”exotic black branes” [7]. In was shown in [5] that all the exotic black branes contain a tachyonic quasinormal mode,

$^6$Further details of the hairy black brane solutions can be found in [7].
and thus are dynamically unstable. In the remainder of this section we show that exotic black branes are not only thermodynamically stable in a canonical way \cite{7}, they are thermodynamically stable in a generalized way as well, with $\Lambda$ being treated as a generalized charge.

Given a dataset \( \{p_1, p_2, \chi_4, a_0^h, a_1^h, p_0^h, c_0^h\} \) for each of the discrete branches of the black brane solutions we can construct parametric dependence of $\frac{\hat{\mathcal{E}}}{s^{3/2}}$ versus $\frac{\Lambda}{s^{1/2}}$, i.e., the function $(x, \mathcal{G}(x))$ such that

$$\hat{\mathcal{E}} = \frac{s^{3/2}}{s^{1/2}} \mathcal{G} \left( \frac{\Lambda}{s^{1/2}} \right). \tag{3.12}$$

Given (3.7)-(3.11) we have

$$\frac{\Lambda}{s^{1/2}} = \frac{p_1}{2\pi^{1/2}a_0^h}, \tag{3.13}$$

$$\frac{\hat{\mathcal{E}}}{s^{3/2}} = \frac{12 - p_1p_2}{48\pi^{3/2}(a_0^h)^3}. \tag{3.14}$$

Figure 1 presents the function $(x, \mathcal{G}(x))$ for the black branes without the condensate of the $\chi$ scalar (the red points), and with the condensate of the $\chi$ scalar (purple points).

The following fits to $\mathcal{G}(x)^{\text{red}}$ and $\mathcal{G}(x)^{\text{purple}}$ are indistinguishable with a naked eye
Figure 2: (Colour online) Canonical $\frac{\dot{\hat{s}}}{s^{1/2}} \frac{\partial^2 \hat{E}}{\partial \hat{s}^2} > 0$ (left plot) and the generalized $\det \left( \frac{\partial^2 \hat{E}}{\partial \hat{s}^2} \right) > 0$ (right plot) thermodynamic stability criteria for the dynamically stable (red curves) and the dynamically unstable (purple curves) hairy black branes.

from the data points in Figure 1:

\[
G(x)^{\text{red}} = -0.0448955 + 0.000128216 \, x + 0.0316168 \, x^2 + 0.0212735 \, x^3, \\
G(x)^{\text{purple}} = -0.0458244 - 0.0130892 \, x + 0.0721953 \, x^2 + 0.06829850585 \, x^3. \tag{3.15}
\]

We are now ready to analyze the canonical and the generalized thermodynamic stability criterion for the hairy black branes.

- In the canonical case we require that the Hessian

\[
\mathbb{H}^E_{\hat{s}}, \tag{3.16}
\]

be positive, which translates into

\[
0 < \frac{\dot{\hat{s}}}{s^{1/2}} \frac{\partial^2 \hat{E}}{\partial \hat{s}^2} = \left\{ \frac{3}{4} \, G(x) - \frac{3}{4} \, x \, G'(x) + \frac{1}{4} \, x^2 \, G''(x) \right\} \bigg|_{x = \frac{\Lambda}{s^{1/2}}}. \tag{3.17}
\]

- In the generalized case, the scale $\Lambda$ is treated as one of the charges $Q_A$; thus, the thermodynamic stability criterion becomes the positivity of the Hessian

\[
\mathbb{H}^E_{\hat{s}, \Lambda}, \tag{3.18}
\]

which in addition to (3.17) requires that

\[
0 < \det \left( \mathbb{H}^E_{\hat{s}, \Lambda} \right) = \left\{ \frac{3}{4} \, G''(x) \, G(x) + \frac{1}{4} \, x \, G''(x) \, G'(x) - (G'(x))^2 \right\} \bigg|_{x = \frac{\Lambda}{s^{1/2}}}. \tag{3.19}
\]
The results of the stability analysis (3.17) and (3.19) are presented in Figure 2. Much like in the simple statistical model of section 2 both the canonical and the generalized thermodynamic stability criteria characterize the hairy black branes (with or without the scalar condensate $\chi$) as being stable. As established in [5], the hairy black branes with the non-zero condensate of $\chi$ are dynamically unstable. Thus, we conclude that generalizing the thermodynamic stability criterion to include (in an appropriate manner) the asymptotic coefficients of scalar fields sourcing the black branes in an asymptotically anti-de-Sitter space-time can not validate the “Correlated Stability Conjecture”.

Much like in the statistical model in section 2, it is clear that, once one is sufficiently close to the transition (so that the tachyon condensate contribution to the thermodynamics is negligible), any redefinition of the generalized charge $Q_A = f(\Lambda)$ would change the thermodynamic stability of both classically stable and unstable phases in identical manner. Thus, insisting that the classically stable phase is thermodynamically stable (in a generalized way) as well would imply that the classically unstable phase is also thermodynamically stable. From this perspective our counter-example of the CSC conjecture is robust with respect to a definition of the generalized charge\textsuperscript{7} $Q_A$.

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\textsuperscript{7}As emphasized in section 2 we assume that $Q_A$ depends only on the microscopic scale(s) of the theory.
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