A BRATTELI–VERSHIK REPRESENTATION FOR ALL ZERO-DIMENSIONAL SYSTEMS

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Abstract. In this paper, we show that every compact metrizable zero-dimensional homeomorphic system admits a Bratteli–Vershik model that is not trivial. As a corollary, we obtain an analogue of Krieger’s Lemma for compact metrizable zero-dimensional systems. After the first submission of the paper, Downarowicz and Karpel has given the decisive Bratteli–Vershik models, in which they introduce the notion of decisiveness, by which the Vershik map is uniquely determined. The methods we use are inverse system of finite graph coverings. We discuss some link between the Bratteli–Vershik model and the inverse system of finite graph coverings. As an example, we give elementary discussions onto stationary systems.

1. Introduction

In this paper, by a zero-dimensional system, we mean a pair \((X, f)\) of a compact metrizable totally disconnected space \(X\) and a continuous surjective map \(f : X \rightarrow X\). If \(X\) is homeomorphic to the Cantor set, then the zero-dimensional system is said to be a Cantor system. If \(f\) is a homeomorphism, then we explicitly write as homeomorphic zero-dimensional system. For the Cantor essentially minimal homeomorphic systems, Herman, Putnam and Skau \cite{HPS92} derived many of the ordered Bratteli diagrams and some orbit equivalences. Furthermore, for the Cantor essentially minimal homeomorphic systems, they derived relations between some orbit equivalences and the \(K\)-theory of approximately finite \(C^*\)-algebras. In their work, the Bratteli–Vershik model for the essentially minimal Cantor homeomorphisms played the central role. Since then, studies on Bratteli–Vershik model have spread widely and become more involved, to the point where we cannot provide a comprehensive survey within a single paper. Medynets \cite{Med06} showed that every Cantor aperiodic system admits a Bratteli–Vershik model. After the first submission of this paper, we have noticed that Downarowicz and Karpel \cite{DK16, DK17} improved the result showing that a homeomorphic zero-dimensional system \((X, f)\) is Bratteli–Vershikizable if and only if the set of aperiodic points is dense, or its closure misses one periodic orbit. In this paper, we show that every compact metrizable zero-dimensional homeomorphic dynamical system admits some non-trivial Bratteli–Vershik model. Our way to show it, naturally, contains some non-invertible cases. In \cite{Ya09}, Yassawi studied Bratteli–Vershik model for some one-sided substitution systems.

In \(\S\ 2\) we make a brief summary of our previous work on an inverse system of finite graphs and its covering maps, which we call basic graph covering in this paper. In the former papers, we call them just graph covering. Nevertheless, we introduce slightly different kinds of an inverse system of finite graphs and covering maps and we shall have three kinds
of inverse systems of finite graphs and their covering maps, besides the ordered Bratteli diagrams. Therefore, we had to use a different term for the original graph coverings. We think that it leads us to an ambiguity, if we drop the term ‘basic’ and continue to use the term ‘graph covering’ only for the original one.

In § 3 for some study of (compact) zero-dimensional systems, we formulate a new notion of an inverse system of finite graphs and its covering maps. We call them \textit{weighted graph coverings} (see Definitions 3.1 and 3.5). Unlike basic graph coverings, each edge of weighted graphs is weighted with a positive finite integer denoting its length. Every homeomorphic compact zero-dimensional system \((X, f)\) is described by an inverse limit of a weighted graph covering. With a weighted graph covering, we can define a set \(\mathcal{V}_\infty\) (see Notation 3.17). We give a necessary and sufficient condition for \(\mathcal{V}_\infty\) to be a basic set (see Definition 3.18 and Theorem 3.22). Based on weighted graph coverings, without assigning a positive integer for each edge explicitly, we get a rather abstract notion of \textit{flexible graph coverings} (see Definitions 3.2 and 3.5). The lengths of all edges in flexible graph coverings are assigned implicitly by how upper edges wind lower edges successively. These notions of weighted and flexible graph coverings are precisely included in the notion of the Bratteli–Vershik model such that the ordered Bratteli diagrams might have multiple maximal paths and minimal paths.

These relations are described in § 4. The (basic) graph coverings that are defined in our previous paper [S14] bring about some trivial Bratteli–Vershik model for arbitrary zero-dimensional systems. In this trivial model, all paths in the ordered Bratteli diagrams become both maximal and minimal. These kinds of meaningless matters are not intended in this paper. We think that some additional normality conditions have to be introduced in the Bratteli–Vershik models, e.g., we have introduced \textit{closing property} (see Definition 3.21). With this normality condition, we can get that the set of minimal paths \(E_{0,\infty,\min}\) is a basic set (see Theorem 3.20). Weighted and also flexible graph covering models lead us to another desirable property for the Bratteli–Vershik model such that each shorter tower must have a periodic orbit with the least period matching the height of the tower. In evidence, the main Proposition 3.31 can deduce Krieger’s Lemma [Boy84 (2.2) Lemma (Krieger)] (see Corollary 3.28). With an arbitrary sequence of positive integers \(l : l_1 < l_2 < \cdots\), a Bratteli–Vershik model with this condition is said to be \textit{l-periodicity-regulated} (see Definition 4.11). In general, this condition is strictly stronger than closing property. We show that an arbitrary homeomorphic zero-dimensional system admits the Bratteli–Vershik model that satisfy the later normality condition (see Theorem 1.1). These normality conditions are conveyed from weighted graph coverings (see Definitions 3.21 and 3.24). We prove the main Theorem 3.27, which states that every zero-dimensional system can be represented as the inverse limit of a weighted graph covering with \(l\)-periodicity-regulated property. We define a Bratteli–Vershik model for our case and make a link with weighted graph covering. In § 4 we prove Theorem 4.17 that is also the last step of the proof of:

\textbf{Theorem 1.1.} Let \((X, f)\) be a homeomorphic topological dynamical system, where \(X\) is a compact metrizable zero-dimensional set. Let \(l : l_1 < l_2 < \cdots\) be a sequence of positive integers. Then \((X, f)\) admits a \(l\)-periodicity-regulated Bratteli–Vershik model.
Let us explain this theorem. It is well known that in a Bratteli–Vershik model, each $V_n$ decomposes $X$ by finite towers that are linked to each vertices in $V_n$. The condition of $l$-periodicity-regulated tells us that if the tower corresponding $v \in V_n$ has height $l(v) \leq l_n$, then there must exist a periodic orbit of least period $l_p v$. If $(X, f)$ is positively transitive, then for arbitrary sequence of small $\delta_n > 0 \ (n \geq 1)$, we may assume that the decomposition by $V_n$ contains a tower that is $\delta_n$-dense (see Remark 3.32). Additionally, suppose that there exists a closed and open set $U \subseteq X$ and a positive integer $n$ such that $\bigcup_{i=0}^n f^i(U) = X$. Then, Remark 4.35 shows that a basic set can be taken as a subset of $U$.

In [DM08], Downarowicz and Maass have defined topological rank for every Cantor minimal homeomorphism, and showed that if it is $K > 1$, i.e., the system is not an odometer, then it is expansive. Our main result have given a possibility of defining a topological rank for every compact zero-dimensional homeomorphic system. Evidently, our definition of the topological rank is less than or equal to that of [DM08] for the Cantor minimal homeomorphisms. In this regard, in [S17], we have shown that a finite rank compact zero-dimensional homeomorphic system that does not have any infinite odometers is expansive.

In § 5, we give a few examples that concern substitution dynamical systems. With these examples and arguments, we would like to clarify some nature of both the Bratteli–Vershik models and weighted (flexible) graph covering model and their links. With flexible graph coverings, some portion of the well known stationary ordered Bratteli diagrams are transferred to stationary (flexible) graph coverings (see § 5.1).

2. Preliminaries

Let $\mathbb{Z}$ be the set of all integers; $\mathbb{N}$, the set of all non-negative integers; and $\mathbb{N}^+$, the set of all positive integers. For integers $a < b$, the intervals are denoted by $[a, b] := \{ a, a + 1, \ldots, b \}$, and so on. A pair $(X, f)$ of a compact metric space $X$ and a continuous surjection $f : X \to X$ is called a topological dynamical system. A topological dynamical system $(X, f)$ is called a zero-dimensional system if $X$ is totally disconnected. A topological dynamical system $(X, f)$ is homeomorphic if $f$ is a homeomorphism. Let $(X, f)$ be a homeomorphic zero-dimensional system, $h$ be a positive integer, and $U \subseteq X$ be a closed and open set. Then, $\xi := \{ f^i(U) \mid 0 \leq i < h \}$ is called a tower with base $U$ and height $h$ if all of the $f^i(U) \ (0 \leq i < h)$ are mutually disjoint. In this case, we write $\bar{U} := \bigcup \xi$, and use the abbreviation $\bar{U} = \bigcup_{0 \leq i < h} f^i(U)$ to denote a tower of height $h$ and base $U$. The notion of tower plays a central role in our argument. We have described the notion of graph coverings for all zero-dimensional continuous surjections under the naming of the sequences of covers (see [S14 §3]), and have presented a basic observation regarding their ergodic theory (see [S16a]). In [S16a], we used the term “graph covering,” or just “covering,” whereas in this paper, with the notion of the tower, we develop the notion of a finite directed graph with multiple edges, and assign to each edge a positive integer called the “length.” These two kinds of graph coverings have to be termed differently (see Notation 2.1 and Definition 3.8).

In this section, we recall construction of basic graph coverings for general zero-dimensional systems used in [S14, S16a, S15, S16b], and in § 3 we develop a new notion of weighted graph coverings, in which multiple edges between a pair of vertices are permitted, and
each edge is assigned a weight corresponding to its length. In §2 without assigning the lengths of edges explicitly, we also define flexible graph coverings.

**Notation 2.1.** To avoid the ambiguity of these three kinds of graph coverings, in this paper, the original graph covering that are considered in previous papers are called basic graph coverings. Hereafter, in this paper, we do not use an ambiguous simple term “graph covering”. Instead, intentionally, we use the three terms “basic (graph) covering”, “weighted (graph) covering”, and “flexible (graph) covering”, the term ‘graph’ might be omitted.

A pair \( \hat{G} = (\hat{V}, \hat{E}) \) that consists of a finite set \( \hat{V} \) and a relation \( \hat{E} \subseteq \hat{V} \times \hat{V} \) on \( \hat{V} \) can be considered as a directed graph with vertices \( \hat{V} \) and an edge from \( \hat{u} \) to \( \hat{v} \) when \( (\hat{u}, \hat{v}) \in \hat{E} \). In this sense, we herein refer to the finite directed graph \( \hat{G} = (\hat{V}, \hat{E}) \) as a basic graph. We write \( \hat{V} = V(\hat{G}) \) and \( \hat{E} = E(\hat{G}) \). Hereafter, throughout this paper, a basic graph \( \hat{G} \), its vertex set \( \hat{V} \), and edge set \( \hat{E} \) are all described with being checked. Further, a vertex \( \hat{v} \in \hat{V} \) and an edge \( \hat{e} \in \hat{E} \) are also checked.

**Notation 2.2.** In this paper, we assume that a basic graph \( \hat{G} = (\hat{V}, \hat{E}) \) is a surjective relation, i.e., for every vertex \( \hat{v} \in \hat{V} \), there exist edges \((\hat{u}_1, \hat{v}), (\hat{v}, \hat{u}_2) \in \hat{E}\).

For basic graphs \( \hat{G}_i = (\hat{V}_i, \hat{E}_i) \) with \( i = 1, 2 \), a map \( \hat{\varphi} : \hat{V}_1 \to \hat{V}_2 \) is said to be a basic graph homomorphism if, for every edge \((\hat{u}, \hat{v}) \in \hat{E}_1 \), it follows that \((\hat{\varphi}(\hat{u}), \hat{\varphi}(\hat{v})) \in \hat{E}_2 \). In this case, we write \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \). For a basic graph homomorphism \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \), we say that \( \hat{\varphi} \) is edge-surjective if \( \hat{\varphi}(\hat{E}_1) = \hat{E}_2 \). Suppose that a basic graph homomorphism \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \) satisfies the following condition:

\[
(\hat{u}, \hat{v}), (\hat{u}, \hat{v}') \in \hat{E}_1 \text{ implies that } \hat{\varphi}(\hat{v}) = \hat{\varphi}(\hat{v}').
\]

In this case, \( \hat{\varphi} \) is said to be +directional. Suppose that a basic graph homomorphism \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \) satisfies the following condition:

\[
(\hat{u}, \hat{v}), (\hat{u}', \hat{v}) \in \hat{E}_1 \text{ implies that } \hat{\varphi}(\hat{u}) = \hat{\varphi}(\hat{u}').
\]

In this case, \( \hat{\varphi} \) is said to be −directional. A basic graph homomorphism is bidirectional if it satisfies both of the above conditions.

**Definition 2.3.** For basic graphs \( \hat{G}_1 \) and \( \hat{G}_2 \), a basic graph homomorphism \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \) is called a basic cover if it is a +directional edge-surjective graph homomorphism.

Let \( \hat{G}_0 := \{(\hat{v}_0), \{ (\hat{v}_0, \hat{v}_0) \} \} \) be a singleton graph. For a sequence of basic graph covers \( \hat{G}_1 \rightleftarrows \hat{G}_2 \): \( \hat{G}_i \), we attach the singleton graph \( \hat{G}_0 \) at the head. We call a sequence of basic graph covers \( \hat{\varphi} : \hat{G}_0 \rightleftarrows \hat{G}_1 \rightleftarrows \hat{G}_2 \rightleftarrows \cdots \) a basic graph covering, or just a basic covering. In this paper, considering the numbering of Bratteli diagrams, we use this numbering of basic coverings. In the original paper, we used the numbering \( \hat{G}_n \rightleftarrows \hat{G}_{n+1} \).

Let us write basic graphs as \( \hat{G}_i = (\hat{V}_i, \hat{E}_i) \) for \( i \in \mathbb{N} \). We define the inverse limit of \( \hat{G} \) as follows:

\[
\hat{V}_\infty := \{ (\hat{v}_0, \hat{v}_1, \hat{v}_2, \ldots) \in \prod_{i=0}^{\infty} \hat{V}_i \mid \hat{v}_i = \hat{\varphi}_{i+1}(\hat{v}_{i+1}) \text{ for all } i \in \mathbb{N} \}
\]

and

\[
\hat{E}_\infty := \{ (x, y) \in \hat{V}_\infty \times \hat{V}_\infty \mid (\hat{u}_i, \hat{v}_i) \in \hat{E}_i \text{ for all } i \in \mathbb{N} \},
\]

where \( x = (\hat{u}_0 = \hat{v}_0, \hat{u}_1, \hat{v}_2, \ldots), y = (\hat{v}_0, \hat{v}_1, \hat{v}_2, \ldots) \in \hat{V}_\infty \). The sets \( \hat{V}_i \) (\( i \geq 0 \)) are equipped with the discrete topology, and the set \( \prod_{i=0}^{\infty} \hat{V}_i \) is equipped with the product topology.
Notation 2.4. Let $X = V_G$, and let us define a map $f : X \to X$ by $f(x) = y$ if and only if $(x, y) \in E_G$. For each $n \geq 0$, the projection from $X$ to $V_n$ is denoted by $\phi_{X,n}$. For $\hat{v} \in \hat{V}_n$, we define a closed and open set $U(\hat{v}) := \phi_{X,n}^{-1}(\hat{v})$. For a subset $A \subseteq \hat{V}_n$, we define a closed and open set $U(A) := \bigcup_{\hat{v} \in A} U(\hat{v})$.

Notation 2.5. Let $\mathcal{G}$ be a basic covering $\mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_2 \leftarrow \cdots$. Let $X = V_{\mathcal{G}}$, and let us define $f : X \to X$ as above. Then, by [S14] Theorem 3.9, $(X, f)$ is a zero-dimensional system. This zero-dimensional system $(X, f)$ is written as $\lim_{\mathcal{G}}$. By [S14] Lemma 3.5, if all $\phi_n (n > 0)$ are bidirectional, then $f$ is a homeomorphism.

Definition 2.6. Let $\mathcal{G} : \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_2 \leftarrow \cdots$ be a basic covering. A zero-dimensional system $(Y, g)$ admits a basic covering model $\mathcal{G}$, if $(Y, g)$ is topologically conjugate to $\lim_{\mathcal{G}}$. A zero-dimensional system $(Y, g)$ admits a bidirectional basic covering model $\mathcal{G}$ if, in addition, all $\phi_n (n > 0)$ are bidirectional.

The following holds:

Theorem 2.7 (Theorem 3.9 and Lemma 3.5 of [S14]). Every zero-dimensional system $(X, f)$ admits a basic covering model. Every zero-dimensional system $(X, f)$ is homeomorphic if and only if it admits a bidirectional basic covering model.

Remark 2.8. Let $\mathcal{G} : \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_2 \leftarrow \cdots$ be a basic covering, and $\lim_{\mathcal{G}} = (X, f)$. For each $n \geq 0$, the set $\mathcal{U}(\mathcal{G}_n) := \{ U(\hat{v}) \mid \hat{v} \in V(\mathcal{G}_n) \}$ is a closed and open partition such that $U(\hat{v}) \cap f(U(\hat{v})) \neq \emptyset$ if and only if $(\hat{u}, \hat{v}) \in E(\mathcal{G}_n)$. Furthermore, $\bigcup_{n \geq 0} \mathcal{U}_n$ generates the topology of $X$. Conversely, suppose that $\mathcal{U}_n (n \geq 0)$ is a sequence of finite closed and open partitions of a compact, metrizable, zero-dimensional space $X$, that $\bigcup_{n \geq 0} \mathcal{U}_n$ generates the topology of $X$, and that $f : X \to X$ is a continuous surjective map such that, for any $U \in \mathcal{U}_{n+1}$, there exists $U' \in \mathcal{U}_n$ such that $f(U) \subset U'$. Then, we can define a basic covering in natural way (see the discussion after [S14] Theorem 3.9).

Notation 2.9. Let $\hat{G} = (\hat{V}, \hat{E})$ be a basic graph. A sequence of vertices $w = (\hat{v}_0, \hat{v}_1, \ldots, \hat{v}_l)$ of $G$ is said to be a walk of length $l$ if $(\hat{v}_i, \hat{v}_{i+1}) \in \hat{E}$ for all $0 \leq i < l$. We denote $l(w) := l$.

3. Graph coverings weighted by length

In this section, we define the “weighted graphs” and their coverings. We also define the “flexible graphs” and their coverings. We propose also some discussions on the relation between them and basic graph coverings. To distinguish weighted graphs from basic graphs, weighted graphs are denoted as $\overline{G} = (\overline{V}, \overline{E})$, with overlines for sets of vertices, edges, and the graph itself. To distinguish flexible graphs from weighted graphs, flexible graphs are denoted as $\overline{G} = (\overline{V}, \overline{E})$, with large tilda for sets of vertices, edges, and the graph itself. In Notation 3.1, we construct a basic graph $\hat{G} = (\hat{V}, \hat{E})$ from a weighted graph $\overline{G} = (\overline{V}, \overline{E})$.

Definition 3.1. Let $\overline{G} = (\overline{V}, \overline{E})$ be a pair of finite sets such that

- there exists a source map $s : \overline{E} \to \overline{V}$ and a range map $r : \overline{E} \to \overline{V}$ and
- each vertex $v \in \overline{V}$ has edges $e_1, e_2 \in \overline{E}$ such that $s(e_1) = r(e_2) = v$.

Unlike the case of basic graphs, the elements of $\overline{V}$ and $\overline{E}$ may not be overlined, if there is no confusion. Let $l : \overline{E} \to \mathbb{N}^+$ be a map. A pair $(\overline{G}, l)$ is called a weighted graph, with the
vertex set $\tilde{V}$, edge set $\tilde{E}$, and weight map $l : \tilde{E} \to \mathbb{N}^+$. We permit multiple directed edges between each pair of vertices. Usually, we omit $l$ and say that $G$ is a weighted graph.

The weight map is also called the length map.

**Definition 3.2.** Let $\hat{G} = (\hat{V}, \hat{E})$ be a pair of finite sets such that

- there exists a source map $s : \hat{E} \to \hat{V}$ and a range map $r : \hat{E} \to \hat{V}$ and
- each vertex $v \in \hat{V}$ has edges $e_1, e_2 \in \hat{E}$ such that $s(e_1) = r(e_2) = v$.

As with the case of weighted graphs, the elements of $\hat{V}$ and $\hat{E}$ may not be symbolized with large tilde, if there is no confusion. A finite directed graph $\hat{G}$ is called a flexible graph, with the vertex set $\hat{V}$ and the edge set $\hat{E}$. We permit multiple directed edges between each pair of vertices.

**Remark 3.3.** A basic graph $\hat{G} = (\hat{V}, \hat{E})$ can be considered as a weighted graph, i.e., for each $\hat{e} \in \hat{E}$, the length $l(\hat{e}) = 1$ is assigned. On the other hand, because each edge of a flexible graph implicitly has a length that is rarely 1, in many cases, a basic graph should not be considered to be a flexible graph.

A sequence $w = e_1 e_2 \cdots e_k$ ($e_i \in \tilde{E}, i = 1, 2, \ldots, k$) is a walk if $r(e_i) = s(e_{i+1})$ for all $1 \leq i < k$. We denote $\overline{E}(w) := \{ e_i \mid 1 \leq i \leq k \}$ and $\overline{V}(w) := \{ s(e_i) \mid 1 \leq i \leq k \} \cup \{ r(e_i) \}$. We define the range map $\overline{r}(w) := r(e_k)$ and the source map $\overline{s}(w) := s(e_1)$. A walk $w$ is a cycle if $\overline{s}(w) = \overline{r}(w)$. A cycle $w = e_1 e_2 \cdots e_k$ is a circuit if all $s(e_i) (1 \leq i \leq k)$ are mutually distinct. For a flexible graph, a walk $\hat{w}$, $\overline{E}(\hat{w}), \overline{V}(\hat{w})$, the range map, the source map, the cycle, and the circuit are defined similarly. For a walk $w = e_1 e_2 \cdots e_k$ of a weighted graph, the length is defined as $l(w) := \sum_{1 \leq i \leq k} l(e_i)$. For a walk $w = e_1 e_2 \cdots e_k$ both in $G$ and $\hat{G}$, we write $w(\text{first}) = e_1$ and $w(\text{last}) = e_k$. For a weighted graph $\tilde{G} = (\tilde{V}, \tilde{E})$, the set of finite walks is denoted by $\tilde{W}(\tilde{G})$. For a flexible graph $\hat{G} = (\hat{V}, \hat{E})$, the set of finite walks is denoted by $\hat{W}(\hat{G})$.

Let $G_1 = (\overline{V}_1, \overline{E}_1)$ and $G_2 = (\overline{V}_2, \overline{E}_2)$ be weighted graphs, and let $\overline{W}_i = \tilde{W}(\tilde{G}_i)$ $(i = 1, 2)$. A weighted graph homomorphism $\varphi : \overline{G}_1 \to \overline{G}_2$ is a pair of maps $\varphi_V : \overline{V}_1 \to \overline{V}_2$ and $\varphi_E : \overline{E}_1 \to \overline{E}_2$ such that

- $\varphi_V(s(e)) = s(\varphi_E(e))$ for all $e \in \overline{E}_1$,
- $\varphi_V(r(e)) = r(\varphi_E(e))$ for all $e \in \overline{E}_1$, and
- $l(\varphi_E(e)) = l(e)$ for all $e \in \overline{E}_1$.

Note that we have **not** assumed the condition: $\varphi_V(\overline{V}_1) = \overline{V}_2$. We extend $\varphi_E(e_1 e_2 \cdots e_k) = \varphi_E(e_1) \varphi_E(e_2) \cdots \varphi_E(e_k)$ for each finite walk $e_1 e_2 \cdots e_k \in \overline{W}_1$.

For flexible graphs $\hat{G}_1 = (\hat{V}_1, \hat{E}_1)$ and $\hat{G}_2 = (\hat{V}_2, \hat{E}_2)$, a flexible graph homomorphism $\varphi$ satisfies

- $\varphi_V(s(e)) = s(\varphi_E(e))$ for all $e \in \overline{E}_1$ and
- $\varphi_V(r(e)) = r(\varphi_E(e))$ for all $e \in \overline{E}_1$.

**Notation 3.4.** For both weighted graph homomorphisms and flexible graph homomorphisms, we abbreviate as $\varphi(w) := \varphi_E(w)$ for a walk $w$, and $\varphi(v) := \varphi_V(v)$ for a vertex $v$.

A weighted graph homomorphism $\varphi$ is **edge-surjective** if $\bigcup_{e \in \overline{E}_2} \overline{E}(\varphi(e)) = \overline{E}_2$. A flexible graph homomorphism $\varphi$ is **edge-surjective** if $\bigcup_{e \in \overline{E}_1} \hat{E}(\varphi(e)) = \hat{E}_2$. 
Definition 3.5. A weighted graph homomorphism \( \varphi \) is +\textit{directional} if, for every \( e, e' \in \overline{E}_1 \) with \( s(e) = s(e') \), the walks \( w = \varphi(e) \) and \( w' = \varphi(e') \) satisfy \( w(\text{first}) = w'(\text{first}) \). A weighted graph homomorphism \( \varphi \) is -\textit{directional} if, for every \( e, e' \in \overline{E}_1 \) with \( r(e) = r(e') \), the walks \( w = \varphi(e) \) and \( w' = \varphi(e') \) satisfy \( w(\text{last}) = w'(\text{last}) \). A weighted graph homomorphism \( \varphi : \overline{G}_1 \to \overline{G}_2 \) is called a \textit{weighted cover} if it is +directional and edge-surjective. For flexible graph homomorphisms, the +\textit{directionality}, the -\textit{directionality}, the bidirectionality, and the flexible covers are defined similarly.

Definition 3.6. Let \( \varphi : \overline{G}_1 \to \overline{G}_2 \) be a weighted graph cover. An edge \( e \in \overline{E}_2 \) is \textit{constantly covered} by an edge \( e' \in \overline{E}_2 \) if \( \varphi(e') = e \). Note that, in this case, we get \( l(e') = l(e) \).

Definition 3.7. Let \( \varphi : \hat{G}_1 \to \hat{G}_2 \) be a flexible graph cover. An edge \( e \in \hat{E}_2 \) is \textit{constantly covered} by an edge \( e' \in \hat{E}_2 \) if \( \varphi(e') = e \).

Definition 3.8. A sequence of weighted covers \( \overline{G}_i : \overline{G}_0 \xleftarrow{\varphi_1} \overline{G}_1 \xleftarrow{\varphi_2} \overline{G}_2 \xleftarrow{\varphi_3} \cdots \) is said to be a \textit{weighted graph covering} or \textit{weighted covering}. We assume that \( \overline{G}_0 \) is a singleton set \( \{(v_0)\} \) with \( l(v_0) = 1 \) and \( s(v_0) = r(v_0) = v_0 \). For \( m > n \geq 0 \), the composition map \( \varphi_{m,n} := \varphi_{n+1} \circ \varphi_{n+2} \circ \cdots \circ \varphi_m \) is naturally well defined. Note that \( \overline{G} \) may not be bidirectional.

Notation 3.9. We now construct a basic graph from a weighted graph \( \overline{G} = (V, E) \). For each \( e \in \overline{E} \), we form the set \( V(e) \) of vertices \( V(e) := \{ \hat{v}_{e,0} = s(e), \hat{v}_{e,1}, \hat{v}_{e,2}, \ldots, \hat{v}_{e,(l(e)-1)}, \hat{v}_{e,l(e)} = r(e) \} \). Let \( \hat{V} := \bigcup_{e \in \overline{E}} V(e) \). For each \( e \in \overline{E} \), we form the set of edges \( E(e) := \{ (\hat{v}_{e,i}, \hat{v}_{e,i+1}) \mid 0 \leq i < l(e) \} \). Let \( \hat{E} := \bigcup_{e \in \overline{E}} E(e) \). In this way, we have obtained a basic graph \( \hat{G} = (\hat{V}, \hat{E}) \). From this construction, the set \( \overline{V} \) is considered to be a subset of \( \hat{V} \). Note that, if a pair of vertices \( (v, v') \in \overline{V} \times \overline{V} \) have more than one edge with length 1 directed from \( v \) to \( v' \), then they are merged into a single edge \( (v, v') \in \overline{V} \times \overline{V} \).

Lemma 3.10. For a weighted cover \( \varphi : \overline{G}_1 \to \overline{G}_2 \), we can get a basic cover \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \) in a natural way. If \( \varphi \) is bidirectional, then \( \hat{\varphi} \) is bidirectional.

Proof. We have considered \( \overline{V}_1 \subseteq \hat{V}_1 \) and \( \overline{V}_2 \subseteq \hat{V}_2 \). The map \( \hat{\varphi}|_{\overline{V}_1} = \varphi_V : \overline{V}_1 \to \overline{V}_2 \) is well defined and for each \( e \in \overline{E}_1 \), the map \( \hat{\varphi}|_{V(e)} : V(e) \to V_2 \) is defined uniquely, and \( s(e), r(e) \in \overline{V}_1 \) are mapped compatibly with \( \hat{\varphi}|_{\overline{V}_1} \). Because each \( e \in \overline{E}_1 \) is mapped to a walk in \( \overline{G}_2 \), we get a graph homomorphism \( \hat{\varphi} : \hat{G}_1 \to \hat{G}_2 \). Thus, we only need to check the +directionality condition. For each vertex in \( V(e) \setminus \{ s(e), r(e) \} \), this condition is trivial. We only need to check at every \( s(e) \) \( e \in \overline{E}_1 \). Nevertheless, by the +directionality condition for \( \varphi : \overline{G}_1 \to \overline{G}_2 \), this is also obvious. To check the last statement is also obvious.

\[ \square \]

Notation 3.11. In Notation 3.9, we transformed each \( \overline{G}_n = (\overline{V}_n, \overline{E}_n) \) into \( \hat{G}_n = (\hat{V}_n, \hat{E}_n) \). Further, in Lemma 3.10, each \( \varphi_n \) is transformed into a basic cover \( \hat{\varphi}_n \). Thus, from a weighted graph covering \( \overline{G} : \overline{G}_0 \xleftarrow{\varphi_1} \overline{G}_1 \xleftarrow{\varphi_2} \overline{G}_2 \xleftarrow{\varphi_3} \cdots \), we can get a basic graph covering \( \hat{G} : \hat{G}_0 \xleftarrow{\hat{\varphi}_1} \hat{G}_1 \xleftarrow{\hat{\varphi}_2} \hat{G}_2 \xleftarrow{\hat{\varphi}_3} \cdots \).

Definition 3.12. A sequence of flexible covers \( \hat{G} : \hat{G}_0 \xleftarrow{\hat{\varphi}_1} \hat{G}_1 \xleftarrow{\hat{\varphi}_2} \hat{G}_2 \xleftarrow{\hat{\varphi}_3} \cdots \) is said to be a \textit{flexible graph covering} or \textit{flexible covering}. We assume that \( \hat{G}_0 \) is a singleton graph \( \{(v_0)\} \) as well. For \( m > n \geq 0 \), the composition map \( \hat{\varphi}_{m,n} := \hat{\varphi}_{n+1} \circ \hat{\varphi}_{n+2} \circ \cdots \circ \hat{\varphi}_m \)
is naturally well defined. Note that \( \tilde{G} \) may not be bidirectional. We also note that, in the sequence, multiple or even infinite occurrence of the same flexible graph is permitted.

**Remark 3.13.** Let \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \tilde{G}_2 \xleftarrow{\varphi_3} \cdots \) be a flexible graph covering with the singleton graph \( \tilde{G}_0 = (\{ e_0 \}, \{ e_0 \}) \). We consider that \( e_0 \) has length \( l(e_0) = 1 \). Then, all the lengths of all the edges of the graphs shall be fixed by assuming that each \( \varphi_n (n \geq 1) \) preserves the lengths of the walks. Thus, a flexible covering defines a weighted covering uniquely. It is also clear that a weighted covering defines a flexible covering uniquely. If we use the symbol \( \mathcal{G} \) and \( \tilde{G} \), then these graph coverings are in this relation. Each \( \mathcal{G}_n (n \geq 0) \) corresponds to \( \tilde{G}_n (n \geq 0) \).

**Definition 3.14.** Let \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \tilde{G}_2 \xleftarrow{\varphi_3} \cdots \) be a flexible graph covering and \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \), the corresponding weighted graph covering. Then, for each \( n \geq 0 \) and \( \tilde{e} \in \tilde{E}_n \), there exists the corresponding edge \( e \in E_n \) and its length \( l(e) \). We write as \( l(n, e) := l(\tilde{e}) \). We note that it is not a notation that an edge in \( E \) should be overlined, nor an edge in \( \tilde{E} \) should have wide tilde, i.e., we use \( e \in E \) and also \( \tilde{e} \in \tilde{E} \) as well, if there is no confusion.

Fix a weighted covering \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \) or a corresponding flexible covering \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \tilde{G}_2 \xleftarrow{\varphi_3} \cdots \). In Notation 3.11, we have obtained a basic covering \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \). Thus, we have a zero-dimensional system \( \lim \mathcal{G} \).

The zero-dimensional system \( \lim \mathcal{G} \) is also denoted as \( \lim_{\mathcal{G}} \) or \( \lim_{\tilde{G}} \). Thus, these three inverse limits are the same, in other words, \( \lim_{\mathcal{G}} \) is considered to be \( \lim_{\mathcal{G}} \), and \( \lim_{\tilde{G}} \) is considered to be \( \lim_{\mathcal{G}} \).

**Definition 3.15.** A zero-dimensional system \( (Y, g) \) has a weighted graph covering model \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \) or a flexible graph covering model \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \tilde{G}_2 \xleftarrow{\varphi_3} \cdots \), if \( (Y, g) \) is topologically conjugate to \( \lim \mathcal{G} \).

**Notation 3.16.** Hereafter, a definition that is defined for a weighted graph covering is also defined for a flexible graph covering, and a property that is satisfied by a weighted graph covering is also satisfied by a flexible graph covering, if it is possible via the argument in Remark 3.13.

Let \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \) be a weighted covering and \( n(0) = 0 < n(1) < n(2) < \cdots \) be a sequence of positive integers. We get a weighted covering \( \mathcal{G}' : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2' \xleftarrow{\varphi_3} \cdots \) by letting \( \mathcal{G}_i \) for all \( i \geq 1 \), and \( \varphi_i' = \varphi_{n(i-1)+1} \circ \varphi_{n(i-1)+2} \circ \cdots \circ \varphi_{n(i)} \) for all \( i \geq 1 \). This procedure is called telescoping after the telescoping of the Bratteli diagrams. Suppose that a homeomorphic zero-dimensional system \( (Y, g) \) has a weighted graph covering model \( \mathcal{G} \). Then, it is an easy exercise to show that, by telescoping \( \mathcal{G} \), we can get a bidirectional weighted graph covering model.

In [Med06], Medynets defined the notion of the basic set for the Bratteli–Vershik model for aperiodic zero-dimensional systems. We define the same for our case (see Definition 3.13). From a weighted covering \( \mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots \), we can get a basic covering \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \tilde{G}_2 \xleftarrow{\varphi_3} \cdots \). We write \( \lim \tilde{G} = (X, f) \). In Notation 2.4, we have defined a natural projection \( \hat{\varphi}_{x,n} : X \to \hat{V}_n \) such that \( \hat{\varphi}_{n+1} \circ \hat{\varphi}_{x,n+1} = \hat{\varphi}_{x,n} \) for all \( n \geq 0 \). If for \( m > n \geq 0 \), we define \( \hat{\varphi}_{m,n} := \hat{\varphi}_{n+1} \circ \hat{\varphi}_{n+2} \circ \cdots \circ \hat{\varphi}_{m} \), then we get \( \hat{\varphi}_{m,n} \circ \hat{\varphi}_{x,m} = \hat{\varphi}_{x,n} \).

We have also defined \( U(\tilde{v}) = \varphi_{n,x}(\tilde{v}) \), for each \( \tilde{v} \in \tilde{V}_n \). Because we consider that \( \tilde{V}_n \subseteq \hat{V}_n \),
for each \( v \in \overline{V}_n \), we get \( U(v) \) by the above equation. In the same notation, we defined \( U(\overline{V}_n) \) for all \( n \geq 0 \). Thus, we get the following:

Notation 3.17. Suppose that \( \mathcal{G} \) is a weighted covering model with \( \lim \mathcal{G} = (X, f) \). Then, we denote \( \overline{V}_\infty := \bigcap_{n \geq 0} U(\overline{V}_n) \). The argument above can be also applied to \( \mathcal{G} \) via the argument in Remark 3.13, i.e., we define \( \overline{V}_\infty := \overline{V} \).

Definition 3.18. Let \((X, f)\) be a homeomorphic zero-dimensional system. A closed set \( A \in X \) is called a basic set if for every \( x \in X \), the orbit of \( x \) enters \( A \) at most once; and for every \( x \in X \) and open set \( U \supset A \), the positive orbit of \( x \) enters \( U \) at least once.

In general, an orbit of \((X, f)\) may enter the set \( \overline{V}_\infty \) many times. Therefore, we need to define a condition that prevent this. We shall show that in certain condition, the set \( \overline{V}_\infty \) is a basic set (see Definition 3.21 and Theorem 3.22).

Notation 3.19. For \( e \in \overline{E}_n \, (n \geq 1) \), we have constructed a set of vertices \( V(e) := \{ \tilde{v}_{e,0} = s(e), \tilde{v}_{e,1}, \tilde{v}_{e,2}, \ldots, \tilde{v}_{e,l(e)-1}, \tilde{v}_{e,l(e)} = r(e) \} \). In the case of \( l(e) \geq 2 \), we define the base floor \( B(e) := f^{-1}(U(\tilde{v}_{e,l(e)})) \) and the tower \( B(e) := \bigcup_{0 \leq i < l(e)} f^i(B(e)) \) with the height \( l(e) \). It follows that \( f^i(B(e)) = U(\tilde{v}_{e,i}) \) for each \( 0 < i < l(e) \). In the case of \( l(e) = 1 \), because a lot of towers may start from \( U(\overline{V}(e)) \) and a lot of different towers with height 1 may also be included in \( U(\overline{V}(e)) \), we cannot use \( B(e) = U(\overline{V}(e)) \). To get rid of these additional part from \( U(\overline{V}(e)) \), we use the towers by \( \overline{E}_{n+1} \). We set \( v := s(e) \) and \( A(e) := \{ e' \in \overline{E}_{n+1} \mid \overline{E}(\varphi_{n+1}(e')) \ni e \} \). Further, we set \( A_1(e) := \{ e' \in A(e) \mid l(e') = 1 \} \) and \( A_2(e) := \{ e' \in A(e) \mid l(e') \geq 2 \} \). For each \( e' \in A_2(e) \), we write \( \varphi_{n+1}(e') = e'_{1}e'_{2}\ldots e'_{k(e')}, \) and set \( J(e, e') := \{ i \in [1, k(e')] \mid e_i = e \}. \) If \( 1 \in J(e, e') \), then we define \( l(e, e', 1) := 0 \). For each \( 1 < j \in J(e, e'), \) we define \( l(e, e', j) := \sum_{1 \leq i < j} l(e_i) \). Then, \( B(e) := \bigcup_{e\in A_1(e)} U(s(e')) \cup \bigcup_{e\in J(e, e')} f^{l(e, e', j)}(B(e')) \), and define a tower \( B(e) := B(e) \) with the height 1. Now, we get a decomposition \( X = \bigcup_{e\in \overline{E}_n} B(e) \) for each \( n \geq 0 \). The last decomposition is also denoted for \( \overline{G} \), via the argument in Remark 3.13.

Definition 3.20. Let \( \overline{G} : \overline{G}_0 \leftarrow_{\overline{p}_1} \overline{G}_1 \leftarrow_{\overline{p}_2} \overline{G}_2 \leftarrow_{\overline{p}_3} \ldots \) be a weighted covering and \( n \geq 0 \). A sequence \( e_m \in \overline{E}_m \, (m \geq n) \) is an infinite constant covering from \( n \), if \( \varphi_{n,m'}(e_m) = e_m \) for all \( m > m' \). We note that, in this case, \( l(e_m) = l(e_n) \) for all \( m \geq n \). If such a sequence exists over \( e = e_n \in \overline{E}_n \), then we say that \( e \) is infinitely constantly covered. We define the same for flexible covering \( \overline{G} \). We also note that if \( e_m \) is a circuit for some \( m > n \), then for every \( i \in [n, m] \), \( e_i \) is a circuit.

Definition 3.21. A weighted covering \( \overline{G} : \overline{G}_0 \leftarrow_{\overline{p}_1} \overline{G}_1 \leftarrow_{\overline{p}_2} \overline{G}_2 \leftarrow_{\overline{p}_3} \ldots \) has closing property if there exists an infinite constant covering \( e_m \in \overline{E}_m \, (m \geq n) \) from \( n \), then each \( e_m \) are circuits. We define the same for flexible covering \( \overline{G} \).

Theorem 3.22. Let \( \overline{G} : \overline{G}_0 \leftarrow_{\overline{p}_1} \overline{G}_1 \leftarrow_{\overline{p}_2} \overline{G}_2 \leftarrow_{\overline{p}_3} \ldots \) be a weighted covering. It follows that \( \overline{V}_\infty \) is a basic set if and only if \( \overline{G} \) has closing property. The same is true for \( \overline{G} \) and \( \overline{V}_\infty \), via the argument in Remark 3.13.

Proof. Let \( \lim \overline{G} = (X, f) \). By Notation 3.19, we have a decomposition \( X = \bigcup_{e\in \overline{E}_n} U(e) \) for each \( n \geq 0 \). Suppose that for every infinite constant covering \( e_m \, (m \geq n) \) (of some \( e \in \overline{E}_n \) and an \( n \geq 0 \)), every \( e_m \, (m \geq n) \) is a circuit. Let \( x \in X \). Suppose that \( x \) is a fixed point. Then, evidently, there exists a sequence \( v_n \in \overline{V}_n \, (n \geq 0) \) such that \( x \in U(v_n) \) for
all \( n \geq 0 \). Consequently, we get \( x \in \overline{V} \). Next, suppose that \( x \) is a periodic point with least period \( l \geq 2 \). For every \( n \) large enough, there exists a unique edge \( e_n \in \overline{E}_n \) with \( l(e_n) \leq l \) such that \( \overline{U}(e_n) \ni x \). For every sufficiently large \( n \), we get \( \varphi_{n+1}(e_{n+1}) = e_n \). Thus, for sufficiently large \( n \), the sequence \( e_m \ (m \geq n) \) is an infinite constant cover. By the assumption, \( e_m \ (m \geq n) \) are circuits. Thus, the periodic orbit of \( x \) enters \( U(s(e_m)) \) for all \( m \geq n \) exactly once. Finally, suppose that \( x \) is not periodic. For each \( k \geq 0 \), take a unique \( e_k \in \overline{E}_k \) such that \( x \in \overline{U}(e_k) \). It is evident that the sequence \( l(e_k) \ (k \geq 0) \) is not decreasing. Suppose that \( \lim_{k \to \infty} l(e_k) < \infty \). Then, there exists an \( n \geq 1 \) such that the sequence \( e_m \ (m \geq n) \) is an infinite constant covering. By the assumption, \( e_m \ (m \geq n) \) are circuits, and \( x \) becomes a periodic point, a contradiction. Thus, we get \( \lim_{k \to \infty} l(e_k) = \infty \). We have defined \( V(e_k) := \{ \tilde{v}_{e_k,0} = s(e_k), \tilde{v}_{e_k,1}, \tilde{v}_{e_k,2}, \ldots, \tilde{v}_{e_k,l(e_k)-1}, \tilde{v}_{e_k,l(e_k)} = r(e_k) \} \). There exists a \( 0 \leq j(k) < l(e_k) \) with \( x \in U(\tilde{v}_{e_k,j(k)}) \). Let \( a(k) := j(k) \) and \( b(k) := l(e_k) - j(k) \) for each \( k \). Then, both \( a(k) \) and \( b(k) \) with \( k \geq 1 \) are non-decreasing, and \( \lim_{k \to \infty} a(k) \) and/or \( \lim_{k \to \infty} b(k) \) are \( \infty \). Thus, the orbit of \( x \) enters \( \overline{V} \) at most once. Next, suppose that \( U \supseteq \overline{V} \) is an open set. Because each \( U(\overline{V}_i) \ (i \geq 0) \) is also compact, for sufficiently large \( m > 0 \), it follows that \( U(\overline{V}_m) \subseteq U \). Let \( l_m := \max \{ l(e) \mid e \in \overline{E}_m \} \). Then, for each \( x \in X \), it follows that \( f^i(x) \in U(\overline{V}_m) \subseteq U \) for some \( 0 \leq i < l_m \). Thus, it follows that \( X = \bigcup_{0 \leq j < l_m} f^j(U) \). This completes a proof that \( \overline{V} \) is a basic set.

To show the converse, suppose that \( \overline{V} \) is a basic set. We prove by contradiction. Let \( e_m \in \overline{E}_m \ (m \geq n) \) be an infinite constant covering with \( l = l(e_m) \) for all \( m \geq n \). Suppose that there exists an \( N > n \) such that \( e_N \) is not a circuit. Then, for every \( m > N \), it follows that \( e_m \) is not a circuit. We denote as \( \{ y \} = \bigcap_{m \geq N} U(e_m) \). It follows that \( y, f^1(y) \in \overline{V} \) and \( y \neq f^1(y) \), a contradiction. We have proved the converse. The last statement follows via the argument in Remark 3.13.

Suppose that an \( e \in \overline{E}_n \ (n \geq 0) \) is infinitely constantly covered by a sequence \( e_m \ (m \geq n) \). If we write \( \{ x \} = \bigcap_{i \geq n} U(s(e_i)) \), then it follows that \( \bigcap_{i \geq n} \overline{U}(e_i) = \{ x, f(x), f^2(x), \ldots, f^{l(e)}(x) \} \). If, in addition, \( e_m \)'s are circuits, then \( \bigcap_{i \geq n} \overline{U}(e_i) \) is a periodic orbit of least period \( l(e) \).

**Remark 3.23.** In our main result, we show that there exists a weighted graph covering model \( \overline{G} \), in which, as \( n \) increases, the lower towers in \( \overline{G}_n \) has to have periodic orbits. Nevertheless, the notion of closing property discussed above is not strong enough for this purpose. To see this, consider some zero-dimensional system with a fixed point \( p \) such that there exists no periodic orbit of least period 2; and, in addition, there exists a sequence of periodic orbits \( \{ P_n \}_{n>0} \) with \( \lim_{n \to \infty} P_n = \{ p \} \), i.e., the orbits itself converges toward \( p \). We assume that each \( P_m \) has least period 4. In expressing this zero-dimensional system by a weighted graph covering model, as \( n \) increases weighted graph \( \overline{G}_n \) has to separate \( P_m \) for smaller \( m \) compared to \( n \). For discussion, we assume that just \( P_m \ (m \leq n) \) are all separated by towers of level \( n \), and all \( P_{n'} \) for \( n' > n \) are contained by the tower of height 1 that also contains \( p \). In the level \( n \), when \( P_m \ (m \leq n) \) are all separated from \( p \) and also from each other, we have to provide towers that covers \( P_m \ (m \leq n) \). We assume that all the \( P_m \ (m < n) \) are covered by a single tower with height 4, and \( P_n \) is covered by two towers of height 2. And in the next \( n + 1 \), the two parts shall be unified to be covered by a single tower of height 4. In this way, as \( n \) increases, it must happen that there exists an \( e \in \overline{E}_n \) with \( l(e) = 2 \), and yet \( \overline{U}(e) \) does not contain any periodic orbit of least period 2.

With this observation, we give another notion for weighted graph covering model:
Lemma 3.25. Let \( l : l_1 < l_2 < \cdots \) be a sequence of positive integers. A weighted covering \( \overline{G} : \overline{G}_0 \overset{\varphi_1}{\rightarrow} \overline{G}_1 \overset{\varphi_2}{\rightarrow} \overline{G}_2 \overset{\varphi_3}{\rightarrow} \cdots \) is \( l \)-periodicity-regulated, if

- for each \( n \geq 1 \) and for each \( e \in E_n \) with \( l(e) \leq l_n \), \( e \) is infinitely constantly covered by circuits, i.e., there exists a sequence \( e_m \in E_m \) (\( m \geq n \)) such that \( e_n = e \) and \( \varphi_{m,m'}(e_m) = e_m \) for all \( m > m' \geq n \).

This definition is possible for a flexible covering, via the argument in Remark 3.13.

It is evident that by telescoping, \( l \)-periodicity-regulations are preserved. It is also evident that we can make new \( l \) by ‘telescoping’. This notion of \( l \)-periodicity-regulation is stronger than closing property as Remark 3.23 and the next Lemma 3.25 shows. The existence of the \( l \)-periodicity-regulated weighted graph covering model shall be shown in Theorem 3.27.

Lemma 3.26. Let \( l : l_1 < l_2 < \cdots \) be a sequence of positive integers and \( n \geq 0 \). Suppose that \( \overline{G} : \overline{G}_0 \overset{\varphi_1}{\rightarrow} \overline{G}_1 \overset{\varphi_2}{\rightarrow} \overline{G}_2 \overset{\varphi_3}{\rightarrow} \cdots \) is a weighted graph covering that satisfy \( l \)-periodicity-regulated condition. Then, \( \overline{G} \) has closing property. The same is true for \( \overline{G} \) via Remark 3.13.

Proof. Let \( e_m \in E_m \) (\( m \geq n \)) be an infinite constant covering. Take an \( N > n \) such that \( l_N \geq l(e_N) \). Then, for every \( m \geq N \), it follows that \( l_m \geq l_N \geq l(e_N) = l(e_m) \). Thus, \( e_m \) with \( m \geq N \) are all circuits. It follows that all \( e_m \) with \( m \geq n \) are circuits. The last statement follows via the argument in Remark 3.13. \( \square \)

Lemma 3.27. Let \( l : l_1 < l_2 < \cdots \) be a sequence of positive integers and \( n \geq 0 \). Suppose that \( \overline{G} : \overline{G}_0 \overset{\varphi_1}{\rightarrow} \overline{G}_1 \overset{\varphi_2}{\rightarrow} \overline{G}_2 \overset{\varphi_3}{\rightarrow} \cdots \) is a weighted graph covering that satisfy \( l \)-periodicity-regulated condition. Then, it follows that \( \overline{V}_\infty \) is a basic set.

Proof. A proof follows from Theorem 3.22 and Lemma 3.25. \( \square \)

We now present the main theorem:

Theorem 3.27. Let \((X,f)\) be a homeomorphic zero-dimensional system. Then, for every sequence \( l : l_1 < l_2 < \cdots \) of positive integers, \((X,f)\) admits a weighted graph covering model \( \overline{G} : \overline{G}_0 \overset{\varphi_1}{\rightarrow} \overline{G}_1 \overset{\varphi_2}{\rightarrow} \overline{G}_2 \overset{\varphi_3}{\rightarrow} \cdots \) that is \( l \)-periodicity-regulated. It follow that \( \overline{V}_\infty \) is a basic set. The same is true for \( \overline{G} \) and \( \overline{V}_\infty \), via the argument in Remark 3.13.

The main part of the proof is somewhat lengthy, and is given in Proposition 3.31. We firstly present Corollary 3.28 and its proof under the assumption that Theorem 3.27 holds. Secondly, we show the proof of Theorem 3.27 under the assumption that Proposition 3.31 holds. The proof of Proposition 3.31 is given in the end of this section. Now, we start by presenting some devices for the argument. The following argument in this section is only for weighted graphs. The same argument is not possible for flexible graphs. Nevertheless, the conclusion that is related to flexible covering in Theorem 3.27 is also valid via Remark 3.13. A corollary of Theorem 3.27 is an analogue of Krieger’s Lemma (cf. [Boy84] Lemma 2.2) for the zero-dimensional case.

Corollary 3.28. Let \((X,f)\) be a zero-dimensional homeomorphic system, i.e., \( f \) is a homeomorphism. Let \( L \geq 1 \) and \( \varepsilon > 0 \). Then, there exists a closed and open set \( F \) such that \( \hat{F} := \bigcup_{0 \leq i \leq L} f^i(F) \) is a tower, and if \( x \notin \bigcup_{-L \leq i \leq L} f^i(F) \), then there exists a periodic point \( p \) of period \( l \leq L \) such that \( d(x,p) \leq \varepsilon \).
Proof. Let \( l : l_1 < l_2 < \cdots \) be an arbitrary sequence of positive integers. By Theorem 3.27 we can get a weighted graph covering model \( \overline{G} : \overline{G}_0 \xrightarrow{\phi_1} \overline{G}_1 \xrightarrow{\phi_2} \overline{G}_2 \xrightarrow{\phi_3} \cdots \) of \((X, f)\) that is \(l\)-periodicity-regulated. We can assume that \( \lim \overline{G} = (X, f) \) by the conjugating map. Then, there exists an \( n \geq 1 \) such that \( L \leq l_n \). Let \( l = l_n, \overline{G} = \overline{G}_n, \overline{V} = \overline{V}_n \), and \( \overline{E} = \overline{E}_n \). By Notation 3.19 we get a decomposition \( X = \bigcup_{e \in \overline{E}} \overline{B}(e) \). Let \( L' := L + 1 \). Let \( J := \{ e \in \overline{E} \mid l(e) \geq L' \} \), and \( P := \{ e \in \overline{E} \mid l(e) \leq L \} \). Then, \( J \cup P = \overline{E} \). For each \( e \in J \), take an integer \( k(e) \geq 0 \) with \( l(e) - 2L' < L'k(e) \leq l(e) - L' \). For each \( e \in J \), we define \( E(e) := \bigcup \{ f^{L'}(B(e)) \mid 0 \leq i \leq k(e) \} \). We set \( E := \bigcup_{e \in J} E(e) \). Then, for each \( e \in J \), \( f^{p}(E(e)) \) \((0 \leq i \leq L)\) are mutually disjoint, and \( f^{i}(E(e)) \subseteq B(e) \) for all \( 0 \leq i \leq L \). If \( L'k(e) = l(e) - L' \), then we denote \( E(e) := \bigcup_{0 \leq i \leq L} f^{i}(E(e)) \). In this case, we denote \( F(e) := E(e) \). If \( L'k(e) < l(e) - L' \), then we denote as

\[
F(e) := E(e) \cup \{ x \in f^{L'k(e)+L'}(B(e)) \mid f^{i}(x) \notin E \text{ for all } 0 \leq i \leq L \}.
\]

We denote as \( F := \bigcup_{e \in J} F(e) \). Then, \( F \) is closed and open, and we get that \( f^{i}(F) \) \((0 \leq i \leq L)\) are mutually disjoint. If \( x \notin \bigcup_{0 \leq i \leq L} f^{i}(F) \), then \( x \in E(e) \) for some \( e \in P \). Thus, there exists a periodic point \( p \in \overline{E}(e) \) with least period \( L \leq l \) and \( d(x, p) < \varepsilon \). This concludes the proof. \( \square \)

Remark 3.29. In [Boy84], just after the proof of Krieger’s Lemma, Boyle stated that “The simplicity of Krieger’s lemma is perhaps deceptive. The set \( F \), whose definition typically requires many more than \( 2k \) coordinates, incorporates a tremendous amount of combinatorial information.” We joyously think that our proof of Proposition 3.31 that is in the end of this section, might have made some clarification of the “combinatorial information”.

To state Proposition 3.31 we introduce some notions related to the towers. Let \((X, f)\) be a continuous surjective zero-dimensional system. Let \( \widetilde{U} = \bigcup_{0 \leq i < h} f^{i}(U) \) be a tower. Suppose that \( V \cap \widetilde{U} = \emptyset \), and \( \widetilde{V} := \bigcup_{0 \leq i' < h'} f^{i'}(V) \) is another tower. We say that \( \widetilde{V} \) accompanies the tower \( \widetilde{U} \) if \( f^{i}(V) \cap f^{i'}(U) \neq \emptyset \) \((0 \leq i < h, 0 \leq i' < h')\) implies \( f^{i}(V) \subseteq f^{i'}(U) \). We say that a tower \( \widetilde{V} := \bigcup_{0 \leq i < h} f^{i}(V) \) accompanies the tower \( \widetilde{U} := \bigcup_{0 \leq i < h} f^{i}(U) \) properly if \( \widetilde{V} \) accompanies \( \widetilde{U} \) and \( f^{h-1}(V) \cap f^{h}(U) \neq \emptyset \) with \( 0 \leq i < h \) implies \( i = h - 1 \), i.e., if the tower \( \widetilde{V} \) meets the tower \( \widetilde{U} \), then the former always go through the latter to the end of the latter. Note that, when \( h = 1 \), the accompaniments are always proper.

For a weighted graph \( \overline{G} = (\overline{V}, \overline{E}) \), we say that a pair of maps \( B = (B_V, B_E) \) is a base map if

(a) there exists an injective map \( B_V : \overline{V} \to \{ U \mid U \text{ is closed and open and } U \neq \emptyset \} \),
(b) there exists an injective map \( B_E : \overline{E} \to \{ U \mid U \text{ is closed and open and } U \neq \emptyset \} \),
(c) \( B_V(v) \) \((v \in \overline{V})\) are mutually disjoint,
(d) \( B_E(e) \) \((e \in \overline{E})\) are mutually disjoint,
(e) for each \( e \in \overline{E} \), \( B_E(e) \subseteq B_V(s(e)) \) and \( f^{i}(B_E(e)) \subseteq B_V(r(e)) \),
(f) for each \( v \in \overline{V} \), \( B_V(v) = \bigcup_{e \in \overline{E}, s(e) = v} B_E(e) \), and
(g) for each \( e \in \overline{E} \), \( B_E(e) := \bigcup_{0 \leq i < i(e)} f^{i}(B_E(e)) \) is a tower,
(h) \( B_E(e) \) \((e \in \overline{E})\) are mutually disjoint, and
(i) \( X = \bigcup_{e \in \overline{E}} B_E(e) \).
Actually, it is evident that $[b]$ implies $[d]$. We also note that for each $v \in \overline{V}$, we can get $B(v) = \bigcup_{e \in \overline{E}_{f(e)=v}} f^i(e)(B(e))$ with any base map. We write $B : \overline{G} \to (X, f)$. For a singleton weighted graph $\overline{G}_0 = (\{v_0\}, \{e_0\})$ in which $l(e_0) = 1$, the base map $B : \overline{G} \to (X, f)$ is well defined such that $B(v_0) = X$ and $B_E(e_0) = X$. Suppose that there exists a base map $B : \overline{G} \to (X, f)$. Then, there exists a unique natural map $\hat{\Phi} : X \to \hat{V}$. We denote $U(v) := \hat{\Phi}^{-1}(v)$ for all $v \in \overline{V}$. We say that a point $x \in X$ follows a walk $w = e_1e_2 \cdots e_k \in \overline{V}$ if $x \in B_{E}(e_1)$ and, for all $1 \leq i < k$, $f^i_{\overline{E}_{f(e_i)}}(x) \in B_{E}(e_{i+1})$.

Let $\varphi : \overline{G}_1 \to \overline{G}_2$ be a weighted covering map. The base maps $B_i : \overline{G}_i \to (X, f)$ $(i = 1, 2)$ are compatible if, for every $e \in E_1$, all points $x \in B_{1E}(e)$ follow the same walk $\varphi_E(e)$. Even if $B_1$ is compatible with $B_2$, there may be multiple definitions of $B_1$. Nevertheless, according to the arguments in Notation 3.19 if $B_1$ is determined first, then there exists unique $B_2$ that is compatible with $B_1$.

**Lemma 3.30.** Let $B : \overline{G} \to (X, f)$ be a base map with $\overline{G} = (\overline{V}, \overline{E})$. We can choose an arbitrarily small $\varepsilon > 0$ such that, for every vertex $v, v' \in \overline{V}$, every subset $K \subset U(v)$, and every $n \geq 1$ with $f^n(K) \subset U(v')$, the following holds: if $\mathrm{diam}(f^n(K)) \leq \varepsilon$ for all $0 \leq i \leq n$, then all $x \in K$ follow the same walk, independent of the choice of $x \in K$.

**Proof.** We omit a proof. \hfill \Box

**Proposition 3.31.** Let $B : \overline{G} \to (X, f)$ be a base map with $\overline{G} = (\overline{V}, \overline{E})$. Let $\varepsilon > 0$ be arbitrarily small, so as to satisfy the condition of Lemma 3.30. Let $L > 0$ be a positive integer. There exist an integer $r \geq 1$ and a finite set $\{U(j) \mid 1 \leq j \leq r\}$ of closed and open subsets, integers $0 < h_j$ $(1 \leq j \leq r)$, and $v_j, v'_j \in \overline{V}$ $(1 \leq j \leq r)$ that satisfy the following:

(a) for each $1 \leq j \leq r$, $\hat{U}(j) := \bigcup_{0 \leq i < h_j} f^i(U(j))$ is a tower,
(b) $\mathrm{diam}(f^r(U(j))) \leq \varepsilon$ for all $0 \leq i \leq h_j, 1 \leq j \leq r$,
(c) $\hat{U}(j)$ $(1 \leq j \leq r)$ are mutually disjoint,
(d) $X = \bigcup_{1 \leq j \leq r} \hat{U}(j)$,
(e) $U(j) \sqsubseteq U(v_j)$ and $f^{h_j}(U(j)) \sqsubseteq U(v'_j)$ for each $1 \leq j \leq r$, and
(f) for each $1 \leq j \leq r$ with $h_j \leq L$, the tower $\hat{U}(j)$ contains a periodic point with least period $h_j$.

**Remark 3.32.** Let $(X, f)$ be a homeomorphic zero-dimensional system. Let $\Lambda$ be a finite set, $\{B_{\lambda} \mid \lambda \in \Lambda\}$ be mutually disjoint closed and open sets of $X$, and $h_{\lambda}$ $(\lambda \in \Lambda)$ be positive integers. Suppose that there exists a disjoint decomposition $X = \bigcup_{\lambda \in \Lambda} B_{\lambda}$. Let $x \in \bigcup_{\lambda \in \Lambda} B_{\lambda}$ and $L$ be a positive integer such that $x, f(x), f^2(x), \ldots, f^{L-1}(x)$ are mutually distinct and $f^L(x) \in \bigcup_{\lambda \in \Lambda} B_{\lambda}$. Then, we take a small closed and open neighborhood $U_x \ni x$ such that for each $0 \leq i \leq L$, $f^i(U_x)$ are in some $f^i(B_{\lambda})$ $(\lambda \in \Lambda, 0 \leq j < h_{\lambda})$. Cutting off all the $f^i(U_x)$ $(0 \leq i < L)$ from the original decomposition and adding $\bigcup_{i=0}^{L-1} f^i(U_x)$, we get a new decomposition. Suppose that $x \in X$ is positively transitive. Then, it is easy to see that $X$ does not have isolated points and $f^n(x)$ is positively transitive for every $n \in \mathbb{Z}$. Take arbitrarily small $\delta > 0$. We assume that $x \in \bigcup_{\lambda \in \Lambda} B_{\lambda}$ and $\{f^i(x) \mid 0 \leq i < L\}$ is $\delta$-dense in $X$. Then, by the above construction, we get a $\delta$-dense tower $\bigcup_{i=0}^{L-1} f^i(U_x)$. In this way, in Theorem 3.27 for arbitrary sequence $\delta_n > 0$ $(n \geq 1)$, we may assume that the decomposition by towers caused by $\overline{G}_n$ or $\hat{G}_n$ contains a tower that is $\delta_n$-dense in $X$. 
Notation 3.33. For closed sets $K, K' \subset X$, we denote $\text{dist}(K, K') := \min \{ d(x, x') \mid x \in K, x' \in K' \}$.

Lemma 3.34. Let $X$ be a compact zero-dimensional metrizable space with metric $d$. Let $\mathcal{U}$ be a finite partition of $X$ by closed and open sets. Then, there exists some $\varepsilon > 0$ such that, for every set $A_\varepsilon$ of closed sets of $X$ such that $\text{diam}(A) \leq \varepsilon$ for all $A \in A_\varepsilon$, if $\text{dist}(A, A') \leq \varepsilon$ ($A, A' \in A_\varepsilon$), then there exists a $U \in \mathcal{U}$ such that $A \cup A' \subset U$.

Proof. We omit a proof.

Here, we give a proof of Theorem 3.27 under the assumption that Proposition 3.31 holds.

Proof of Theorem 3.27. By Remark 3.13, it is enough to consider the case of weighted graph coverings. Let $\overline{G}_0$ be the singleton graph with $\overline{V} := \{ v_0 \}$, $\overline{E} := \{ e_0 \}$, and $l(e) := 1$. Then, it is obvious that we can obtain the unique base map $B_0 : \overline{G}_0 \to (X, f)$. Let $I : l_1 < l_2 < \cdots$ be a sequence of positive integers. Suppose that we have constructed weighted covering maps $\overline{G}_0 \overset{\varphi_1}{\to} \overline{G}_1 \overset{\varphi_2}{\to} \overline{G}_2 \overset{\varphi_3}{\to} \cdots \overset{\varphi_n}{\to} \overline{G}_n$ and base maps $B_m : \overline{G}_m \to (X, f)$ for all $0 \leq m \leq n$ that are compatible such that, for each $0 \leq m \leq n$ and $e \in \overline{E}_m$ with $l(e) \leq l_n$, there exists a periodic point $p \in B_{m\circ l}(e)$ with least period $l(e)$. We write as $\overline{G} = \overline{G}_n$, $\overline{V} = \overline{V}_n$, $\overline{E} = \overline{E}_n$, and $B = B_n$. Then, we have mutually disjoint towers $B_E(e) \in \overline{E}$ such that $X = \bigcup_{e \in \overline{E}} B_E(e)$. We have selected an arbitrary $l_{n+1} > l_n$. Let $L = l_{n+1}$ and apply Proposition 3.31 with some $\varepsilon = \varepsilon_{n+1} > 0$ that satisfies the condition of Lemma 3.30 for $\overline{G}$ and $B$. Then, we have a finite set $\{ U(j) \mid 1 \leq j \leq r \}$ of closed and open sets, integers $0 < h_j (1 \leq j \leq r)$, and $v_j, v'_j \in \overline{V}(1 \leq j \leq r)$ that satisfies (a) to (f) of Proposition 3.31. On the set $\mathcal{A} := \{ U(j) \mid 1 \leq j \leq r \}$, we construct the equivalence relation $A \sim A'$ that is generated by the relation $A \sim A'$ if and only if $\text{dist}(A, A') \leq \varepsilon$. Let $\{ K_i \mid 1 \leq i \leq r' \}$ be the set of equivalence classes. For each $1 \leq i \leq r'$, we define $U(K_i) := \bigcup_{A \in K_i} A$. Then, we have mutually disjoint closed and open sets $U(K_i) (1 \leq i \leq r')$. By Lemma 3.34, we have $\lim_{\varepsilon \to 0} \max_{1 \leq i \leq r'} \{ \text{diam}(U(K_i)) \} = 0$. In particular, we can choose $\varepsilon$ such that for each $1 \leq i \leq r'$, there exists some $e \in \overline{E}_n$ with $U(K_i) \subset B_E(e)$. We construct $\overline{G}_{n+1}$ and the base map $B_{n+1} : \overline{G}_{n+1} \to (X, f)$ as follows:

- $\overline{V}_{n+1} := \{ U(K_i) \mid 1 \leq i \leq r' \}$,
- $B_{n+1\circ V}(U(K_i)) := U(K_i) \subset X$ for all $1 \leq i \leq r'$,
- $\overline{E}_{n+1} := \{ U(j) \mid 1 \leq j \leq r \}$,
- $B_{n+1} : U(j) \mapsto U(j)$ for all $1 \leq i \leq r$,
- if $e = U(j) \in \overline{E}_{n+1}$, we define $l(e) := h_j$, and
- for each $e = U(j) \in \overline{E}_{n+1}$, $s(e)$ satisfies $U(j) \subset B_{n+1\circ V}(s(e))$, and $r(e)$ satisfies $f(l(e)) \subset B_{n+1\circ V}(r(e))$.

The covering map $\varphi_{n+1} : \overline{G}_{n+1} \to \overline{G}_n$ is defined as follows:

- $\varphi_{n+1\circ V} : \overline{V}_{n+1} \to \overline{V}_n$ is defined such that, if $v = U(K_i)$, then $\varphi_{n+1\circ V}(v)$ is the unique $v \in \overline{V}_n$ such that $U(K_i) \subset B_{\varphi_{n+1\circ V}}(v)$,
- because of the choice of $\varepsilon > 0$ in Lemma 3.30 and Proposition 3.31, the definition of $\varphi_{n+1\circ E}$ is obvious. The compatibility of the base maps $B_n, B_{n+1}$ is evident. The directionality condition is derived from the condition that for each $1 \leq i \leq r'$, there exists some $e \in \overline{E}_n$ with $U(K_i) \subset B_E(e)$. We take $\varepsilon_n (n \geq 1)$ to be strictly decreasing and to satisfy $\varepsilon_n \to 0$ as $n \to \infty$. We obtain a weighted graph covering $\overline{T} : \overline{G}_0 \overset{\varphi_1}{\to} \overline{G}_1 \overset{\varphi_2}{\to} \cdots$.


\( \mathcal{G}_2 \xrightarrow{\varphi_3} \cdots \). From this, we have a basic graph covering \( \tilde{G} : \tilde{G}_0 \xleftarrow{\varphi_1} \tilde{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xrightarrow{\varphi_3} \cdots \), and its inverse limit \( \varprojlim \tilde{G} = (Y, g) \). We wish to show that \( (Y, g) \) is topologically conjugate to \( (X, f) \). For each \( n \geq 0 \), we have shown that there exists a partition \( X = \bigcup_{e \in \mathcal{F}_n} \hat{B}_E(e) \) formed by closed and open sets. Thus, for each \( x \in X \) and each \( n \geq 0 \), there exists a unique \( e \in \mathcal{F}_n \) and a unique \( 0 \leq i < l(e) \) such that \( x \in f^i(\hat{B}_E(e)) \). This determines a unique vertex \( \hat{v}_n(x) = \hat{v}_{e,i} \in \hat{V}_n \) for each \( x \in X \) and each \( n \geq 0 \). Evidently, the sequence \( \psi(x) := (\hat{v}_0(x), \hat{v}_1(x), \hat{v}_2(x), \ldots) \) satisfies \( \varphi_{n+1}(\hat{v}_{n+1}(x)) = \hat{v}_n(x) \) for all \( n \geq 0 \). Thus, \( \psi(x) \in Y \). We have defined a map \( \psi : X \to Y \); obviously, \( \psi \) is a continuous surjective mapping. Because \( \varepsilon_n \to 0 \) as \( n \to \infty \), we have \( \text{diam}(\max_{v \in \hat{V}_n} \varphi_n^{-1}(v)) \to 0 \) as \( n \to \infty \). Thus, \( \psi \) is injective. Finally, it is obvious that \( (\hat{v}_n(x), \hat{v}_n(f(x))) \in \mathcal{E}_n \) for all \( n \geq 0 \), by which the commutativity is implied. Thus, the weighted graph covering model with the \( l \)-periodicity-regulation property has been constructed. By Lemma 3.26, we get \( V_{\infty} \) is a basic set.

At the end of this section, we give a proof of Proposition 3.31.

**Proof of Proposition 3.31.** We proceed by induction to show the following for \( l = 1, 2, \ldots \).

There exist an integer \( r_1 \geq 1 \) and a finite set \( \{ U(j, l) \mid 1 \leq j \leq r_1 \} \) of closed and open subsets, integers \( 0 < h_{j,l} (1 \leq j \leq r_1) \), and \( v_{j,l}, v_{j,l}' \in \overline{V} \) (1 \( \leq j \leq r_1 \)) that satisfy the following:

- (a) \( U(j, l) := \bigcup_{0 \leq i < h_{j,l}} f^i(U(j, l)) \) is a tower for each \( 1 \leq j \leq r_1 \),
- (b) \( \text{diam}(f^i(U(j, l))) \leq \varepsilon \) for all \( 0 \leq i \leq h_{j,l}, 1 \leq j \leq r_1 \),
- (c) the towers \( U(j, l) \) (1 \( \leq j \leq r_1 \)) are mutually disjoint,
- (d) \( X = \bigcup_{1 \leq j \leq r_1} U(j) \),
- (e) \( U(j, l) \subseteq U(v_{j,l}) \) and \( f^{h_{j,l}}(U(j, l)) \subseteq U(v_{j,l}') \) for each \( 1 \leq j \leq r_1 \), and
- (f) for each \( 1 \leq j \leq r_1 \) with \( h_{j,l} \leq l \), the tower \( U(j, l) \) contains a periodic point with least period \( h_{j,l} \).

If this is shown, then setting \( r := r_L, U(j) := U(j, L) \) for \( 1 \leq j \leq r \), and \( h_j := h_{j,L} \) for \( 1 \leq j \leq r \) gives the conclusion.

Proof for \( l = 1 \): First, we show the case in which \( l = 1 \). Define \( K_1 := \{ x \in X \mid f^1(x) = x \} \). Suppose that \( K_1 \neq \emptyset \). It is easy to form a finite set \( \{ K(d, 1) \mid 1 \leq d \leq d_1 \} \) of closed and open sets of the subspace \( K_1 \) such that

- for each \( 1 \leq d \leq d_1 \), there exists some \( v(d) \in \overline{V} \) such that \( K(d, 1) \subseteq U(v(d)) \),
- \( \bigcup \{ K(d, 1) \mid 1 \leq d \leq d_1 \} = K_1 \),
- \( K(d, 1) \) (1 \( \leq d \leq d_1 \)) are mutually disjoint,
- for each \( 1 \leq d \leq d_1 \), \( \text{diam}(K(d, 1)) \leq \varepsilon /2 \).

Therefore, there exist closed and open sets \( W(d, 1) \supseteq K(d, 1) \) (1 \( \leq d \leq d_1 \)) such that

- for each \( 1 \leq d \leq d_1 \), \( W(d, 1), f(W(d, 1)) \subseteq U(v(d)) \),
- for each \( 1 \leq d \leq d_1 \), \( \text{diam}(f(W(d, 1))) \leq \varepsilon \) for \( i = 0, 1 \).

For later use, we rewrite \( t_0 := d_1 \) and \( V(j, 0) := W(j, 1) \) (1 \( \leq j \leq t_0 \)). We also write \( v(j, 0) = v'(j, 0) := v(j) \) (1 \( \leq j \leq t_0 \)). Consider \( V(j, 0) = W(j, 1) \) (1 \( \leq j \leq t_0 \)) to be mutually disjoint towers of height 1. Now, we have obtained towers \( V(j, 0) \) with heights \( h(j, 0) := 1 \) for all \( 1 \leq j \leq t_0 \). Thus, we have a closed and open set \( V_0 := \bigcup_{1 \leq j \leq t_0} V(j, 0) \supseteq K_1 \). Suppose that \( K_1 = \emptyset \). Then, the set of towers is the empty set, and we define
Let \( V_0 := \emptyset \). Let \( V^0 := \bigcup_{v \in V} U(v) \setminus V_0 \) and \( X^0 := X \setminus V_0 \). Then, \( V^0 \) and \( X^0 \) are also closed and open sets. If \( V^0 = \emptyset \), then the proof is complete under the assumption that \( l = 1 \). Therefore, we assume that \( V^0 \neq \emptyset \).

Let \( x \in V^0 \). Suppose that \( x \) is a periodic point with least period \( h(x) \). Then, \( h(x) > 1 \). Take a closed and open set \( V^0 \supseteq V(x) \ni x \) such that \( V(x) \subseteq U(v) \) and \( f^{h(x)}(V(x)) \subseteq U(v) \) for some \( v \in V \) with \( f^{h(x)}(x) = x \in U(v) \), mutually disjoint \( f^i(V(x)) \) \((0 \leq i < h(x))\), and \( \text{diam}(f^i(V(x))) \leq \varepsilon \) for all \( 0 \leq i \leq h(x) \). Next, suppose that \( x \) is not periodic. Take an arbitrarily large integer \( h(x) > 1 \) such that \( f^{h(x)}(x) \in U(v') \) for some \( v' \in V \). We take a closed and open set \( V^0 \supseteq V(x) \ni x \) such that \( V(x) \subseteq U(v) \) and \( f^{h(x)}(V(x)) \subseteq U(v') \) for some \( v, v' \in V \), all \( f^i(V(x)) \) \((0 \leq i < h(x))\) are mutually disjoint, and \( \text{diam}(f^i(V(x))) \leq \varepsilon \) for all \( 0 \leq i \leq h(x) \). Take a finite set \( \{ x_j \mid 1 \leq j \leq n \} \) such that \( \bigcup_{1 \leq j \leq n} V(x_j) = V^0 \).

We write \( h(j) := h(x_j) > 1 \) for each \( 1 \leq j \leq n \). We define \( V(1) := V(x_1) \). When \( V(j) \) is defined for \( 1 \leq j < n \), we define \( V(j) := \bigcup_{0 \leq i < h(j)} f^i(V(j)) \), and

\[
V(j + 1) := V(x_{j+1}) \setminus \bigcup_{1 \leq j' \leq j} V(j').
\]

Finally, when \( V(n) \) is defined, we define \( \tilde{V}(n) := \bigcup_{0 \leq i < h(n)} f^i(V(n)) \). Thus, we have \( \bigcup_{1 \leq j \leq n} \tilde{V}(j) \supseteq V^0 \), and it follows that \( \bigcup_{1 \leq j \leq n} \tilde{V}(j) \supseteq X^0 \). Note that the \( \tilde{V}(j) \) may not be mutually disjoint. Moreover, each \( \tilde{V}(j) \) \((1 \leq j \leq n)\) may not be disjoint with respect to some \( V(j', 0) \) \((1 \leq j' \leq t_0)\).

We begin another induction for \( \alpha = 0, 1, 2, \ldots, n \). Suppose that, for \( 0 \leq \alpha < n \), there exist \( t_\alpha \geq 0 \), bases \( V(j, \alpha) \) \((1 \leq j \leq t_\alpha)\), heights \( h(j, \alpha) \) \((1 \leq j \leq t_\alpha)\), and vertices \( v(j, \alpha), v'(j, \alpha) \in V \) \((1 \leq j \leq t_\alpha)\) such that

\[
\begin{align*}
(1:a) & \quad \tilde{V}(j, \alpha) := \bigcup_{0 \leq i < h(j, \alpha)} f^i(V(j, \alpha)) \text{ is a tower for each } 1 \leq j \leq t_\alpha, \\
(1:b) & \quad \text{diam}(f^i(V(j, \alpha))) \leq \varepsilon \text{ for all } 0 \leq i \leq h(j, \alpha), 1 \leq j \leq t_\alpha, \\
(1:c) & \quad \text{the towers } \tilde{V}(j, \alpha) \text{ are mutually disjoint,} \\
(1:d) & \quad \text{if we define } V_\alpha := \bigcup_{1 \leq j \leq t_\alpha} \tilde{V}(j, \alpha), \text{ then } V_\alpha = \bigcup_{0 \leq i \leq h(j, \alpha)} \tilde{V}(j), \text{ with the convention } V(0) := V_0, \\
(1:e) & \quad V(j, \alpha) \subseteq U(v(j, \alpha)) \text{ and } f^{h(j, \alpha)}(V(j, \alpha)) \subseteq U(v'(j, \alpha)) \text{ for each } 1 \leq j \leq t_\alpha, \text{ and} \\
(1:f) & \quad \text{for each } 1 \leq j \leq t_\alpha \text{ with } h(j, \alpha) = 1, \text{ the tower } \tilde{V}(j, \alpha) \text{ contains a fixed point.}
\end{align*}
\]

We have already shown the case in which \( \alpha = 0 \). Note that, if the case in which \( \alpha = n \) is true, then \( V_n = \bigcup_{0 \leq i \leq n} \tilde{V}(j) = X \).

Let \( \beta = \alpha + 1 \). By the inductive hypothesis for \( \tilde{V}(j) \), \( V_\beta \cap V(\beta) = \emptyset \). For each \( x \in V(\beta) \), let \( 0 < i_1 < i_2 < \cdots < i_n(x) < h(\beta) \) be the maximal sequence such that \( f^{k}(x) \) enters \( V(j, \alpha) \) \((1 \leq j \leq t_\alpha)\). For each \( k \) with \( 1 \leq k \leq n(x) \), let \( j_k \) \((1 \leq j_k \leq t_\alpha)\) be such that \( f^{k}(x) \in V(j_k, \alpha) \). We construct a string \( s(x) := ((i_k, j_k))_{1 \leq k \leq n(x)} \). If \( f^i(x) \notin V(j, \alpha) \) for all \( 0 < i < h(\beta) \) and for all \( 1 \leq j \leq t_\alpha \), then \( s(x) := \emptyset \). We define a set \( S(\beta) := \{ s(x) \mid x \in V(\beta) \} \). Let \( V(s) := \{ x \in V(\beta) \mid s(x) = s \} \) for each \( s \in S(\beta) \). Then, we have a disjoint union \( V(\beta) = \bigcup \{ V(s) \mid s \in S(\beta) \} \) by closed and open sets. Moreover, \( V(\beta) := \bigcup_{0 \leq i < h(\beta)} f^i(V(s)) \) \((s \in S(\beta))\) are towers that are mutually disjoint. It is evident that the towers \( V(s) \) \((s \in S(\beta))\) accompany the towers \( \tilde{V}(j, \alpha) \) \((1 \leq j \leq t_\alpha)\). For each \( s \in S(\beta) \), except when \( \beta = 1 \), the accompaniment may not be proper, i.e., the following may occur:

\[
f^{h(\beta) - 1}(V(s)) \subseteq f^i(V(j, \alpha)) \text{ with } 1 \leq j \leq t_\alpha, 1 \leq i < h(j, \alpha) - 1.
\]
Therefore, roughly speaking, we extend the height of tower $\tilde{V}(s)$ for each $s \in S(\beta)$. If $i \geq 3.1$ does not occur for $s \in S(\beta)$, then we need not change the height $h(\beta)$ of tower $\tilde{V}(s)$, i.e., the base is $V(s)$ and the height is $h(s) := h(\beta)$.

If $i \geq 3.1$ occurs for $s \in S(\beta)$, then we do not change the base $V(s)$, but change the height to $h(s) := h(\beta) + h(j, \alpha) - 1 - i$. Then, $f^{i(h(s) - 1)}(V(s)) \subseteq f^{i(h(\alpha) - 1)}(V(j, \alpha))$ is satisfied. It follows that $\text{diam}(f^i(V(s))) \leq \varepsilon$ for all $0 \leq i < h(s)$, because the extension is part of the tower that has the base $V(j, \alpha)$. The disjointness of $f^i(V(s))$ ($0 \leq i < h(s)$) is also obvious. Now, we cut off the towers $\tilde{V}(s)$ from each $V(j, \alpha)$. Precisely, for each $1 \leq j \leq t_\alpha$, we define $V'(j, \alpha) := V(j, \alpha) \setminus \bigcup_{s \in S(\beta)} \tilde{V}(s)$. We obtain $\tilde{V}'(j, \alpha) := \bigcup_{0 \leq i < h(j, \alpha)} f^i(V'(j, \alpha)) = \tilde{V}(j, \alpha) \setminus \bigcup_{s \in S(\beta)} \tilde{V}(s)$. Now, we have mutually disjoint towers $\{\tilde{V}'(j, \alpha) : 1 \leq j \leq t_\alpha\} \cup \{\tilde{V}(s) \mid s \in S(\beta)\}$. Note that, for all $s \in S(\beta)$, it follows that $h(s) \geq h(\beta) > 1$. We reorder these towers and rewrite the bases $V(j, \beta)$ ($1 \leq j \leq t_\beta$), heights $h(j, \beta)$ ($1 \leq j \leq t_\beta$), and towers $\tilde{V}(j, \beta)$ ($1 \leq j \leq t_\beta$). The existence of vertices $v(j, \beta), v'(j, \beta) \in \tilde{V}$ ($1 \leq j \leq t_\beta$) that satisfy $V(j, \beta) \subseteq U(v(j, \beta))$ and $f^{h(\beta)}(V(j, \beta)) \subseteq \tilde{V}'(j, \beta)$ for each $1 \leq j \leq t_\beta$ is also obvious. If we define $V_\beta := \bigcup_{1 \leq j \leq t_\beta} \tilde{V}(j, \beta)$, then $V_\beta = V_\alpha \cup \tilde{V}(\beta)$. Thus, by induction, we have constructed the mutually disjoint towers $V(j, n)$ ($1 \leq j \leq t_\alpha$) for some $t_\alpha > 0$. Recall that, for each $1 \leq j \leq n$, $h(j) > 1$. Thus, each tower of height 1 has a fixed point. In this way, we have completed a proof of the following statement when $l = 1$:

There exist an integer $r_1 \geq 1$ and a finite set $\{U(j, l) \mid 1 \leq j \leq r_1\}$ of closed and open subsets, integers $0 < h_{j,l}$ ($1 \leq j \leq r_1$), and $v_{j,l}, v'_{j,l} \in \tilde{V}$ ($1 \leq j \leq r_1$) that satisfy the following:

(a) $\check{U}(j, l) := \bigcup_{0 \leq i < h_{j,l}} f^i(U(j, l))$ is a tower for each $1 \leq j \leq r_1$,

(b) $\text{diam}(f^i(U(j, l))) \leq \varepsilon$ for all $0 \leq i < h_{j,l}$, $1 \leq j \leq r_1$,

(c) the towers $\check{U}(j, l)$ ($1 \leq j \leq r_1$) are mutually disjoint,

(d) $X = \bigcup_{1 \leq j \leq r_1} \check{U}(j)$,

(e) $U(j, l) \subseteq U(v_{j,l})$ and $f^{h_{j,l}}(U(j, l)) \subseteq U(v'_{j,l})$ for each $1 \leq j \leq r_1$, and

(f) for each $1 \leq j \leq r_1$ with $h_{j,l} \leq l$, the tower $\check{U}(j, l)$ contains a periodic point with least period $h_{j,l}$.

The inductive step: Suppose that $[a]$ to $[f]$ is satisfied with some $l \geq 1$. Let $L = l + 1$. Let $J := \{j \in [1, r_1] \mid h_{j,l} \leq l\}$. Then, for each $j \in J$, the tower $\check{U}(j, l)$ contains a periodic orbit with least period $h_{j,l}$. We reorder $\{U(j, l) \mid j \in J\}$ as $\{V(j, -1) \mid 1 \leq j \leq t_{-1}\}$. We also reorder the heights $h_{j,l}$ ($j \in J$) to get $h(j, -1)$ ($1 \leq j \leq t_{-1}$), and define the towers $V(j, -1)$ ($1 \leq j \leq t_{-1}$). The inductive hypothesis evidently implies that there exist vertices $v(j, -1), v'(j, -1) \in \tilde{V}$ ($1 \leq j \leq t_{-1}$) such that

$(-1:a)$ $\check{V}(j, -1) := \bigcup_{0 \leq i < h(j, -1)} f^i(V(j, -1))$ is a tower for each $1 \leq j \leq t_{-1}$,

$(-1:b)$ $\text{diam}(f^i(V(j, -1))) \leq \varepsilon$ for all $0 \leq i \leq h(j, -1), 1 \leq j \leq t_{-1}$,

$(-1:c)$ the towers $\check{V}(j, -1)$ ($1 \leq j \leq t_{-1}$) are mutually disjoint,

$(-1:d)$ $V(j, -1) \subseteq U(v(j, -1))$ and $f^{h(j, -1)}(V(j, -1)) \subseteq U(v'(j, -1))$ for each $1 \leq j \leq t_{-1}$, and

$(-1:e)$ for each $1 \leq j \leq t_{-1}$ with $h(j, -1) \leq l$, the tower $\check{V}(j, -1)$ contains a periodic orbit with least period $h(j, -1)$.

Let $V_{-1} := \bigcup_{1 \leq j \leq t_{-1}} \check{V}(j, -1), V_{-1} := \bigcup_{v \in \tilde{V}} U(v) \setminus V_{-1}$, and $X_{-1} := X \setminus V_{-1}$. Note that $\{x \mid \exists i (1 \leq i \leq l), f^i(x) = x\} \subseteq V_{-1}$. Define $K_L := \{x \in X_{-1} \mid f^L(x) = x\}$. Then, $K_L$ is
compact, and \( x \in K_L \) implies that \( f^i(x) \neq x \) for all \( i \) with \( 0 < i < L \). It is easy to make a finite set \( \{ K(d, L) \subset K_L \mid 1 \leq d \leq d_L \} \) of closed and open sets of the subspace \( K_L \) such that

- for each \( 1 \leq d \leq d_L \), there exists a \( v(d) \in \overline{V} \) such that \( K(d, L) = f^L(K(d, L)) \subseteq U(v(d)) \),
- \( \bigcup \{ f^i(K(d, L)) \mid 0 \leq i < L \} = K_L \),
- \( f^1(K(d, L)) (1 \leq d \leq d_L, 0 \leq i < L) \) are mutually disjoint,
- for each \( 1 \leq d \leq d_L \) and \( 0 \leq i \leq L \), \( \text{diam}(f^i(K(d, L)))) \leq \varepsilon/2 \).

We can also assume that, for each \( 1 \leq d \leq d_L \), \( 0 \leq i < L \), \( 1 \leq j \leq t-1 \), and \( 0 \leq i' < h(j, -1) \), it follows that \( f^i(K(d, L)) \cap f^{i'}(V(j, -1)) \neq \emptyset \) implies \( f^i(K(d, L)) \subseteq f^{i'}(V(j, -1)) \), i.e., each tower \( K(d, L) (1 \leq d \leq d_L) \) accompanies all towers \( U(j, -1) \) \((1 \leq j \leq t-1)\). Therefore, there exist closed and open sets \( W(d, L) \supseteq K(d, L) (1 \leq d \leq d_L) \) such that

- for each \( 1 \leq d \leq d_L \), there exists a \( v \in \overline{V} \) such that \( W(d, L), f^L(W(d, L)) \subseteq U(v(d)) \),
- \( \text{diam}(f^i(W(d, 0))) \leq \varepsilon \) for all \( 1 \leq d \leq d_L \) and \( 0 \leq i \leq L \),
- \( W(d, L) := \bigcup_{0 \leq i \leq L} f^i(W(d, L)) (1 \leq d \leq d_L) \) are mutually disjoint towers, and
- for each \( 1 \leq j \leq d_L \), the tower \( W(d, L) \) accompanies all towers \( V(j', -1) (1 \leq j' \leq (t-1)) \).

We note that the accompaniments of towers \( W(d, L) (1 \leq d \leq d_L) \) to \( V(j, -1) (1 \leq j \leq t-1) \) must be proper. We define \( W_0 := \bigcup_{1 \leq d \leq d_L} W(d, L) \) Then, if we take the bases \( V'(j, -1) := V(j, -1) \cap W_0 (1 \leq j \leq t-1) \) and the same heights, we obtain the towers \( V'(j, -1) (1 \leq j \leq t-1) \) such that the towers in the set \( \{ V'(j, -1) \mid 1 \leq j \leq t-1 \} \) are mutually disjoint. We reorder these towers and get towers \( \tilde{V}(j, 0) (1 \leq j \leq t_0) \) with bases \( V(j, 0) (1 \leq j \leq t_0) \) and heights \( h(j, 0) (1 \leq j \leq t_0) \). There also exist vertices \( v(j, 0) = v'(j, 0) \in \overline{V} (1 \leq j \leq t_0) \) such that \( V(j, 0) \subseteq U(v(j, 0)) \) and \( f^h(j, 0)(V(j, 0)) \subseteq U(v'(j, 0)) \) for each \( 1 \leq j \leq t_0 \). Suppose that \( K_L = \emptyset \). Then, we simply define \( t_0 := t-1 \), \( V(j, 0) := V(j, -1) (1 \leq j \leq t_0) \), \( h(j, 0) := h(j, -1) (1 \leq j \leq t_0) \), and the corresponding towers. Note that we have shown the following for the case in which \( \alpha = 0 \):

There exist a finite set of closed and open sets \( V(j, \alpha) (1 \leq j \leq t_0) \), heights \( h(j, \alpha) (1 \leq j \leq t_0) \), and vertices \( v(j, \alpha) \), \( v'(j, \alpha) \in \overline{V} (1 \leq j \leq t_0) \) such that

1. \( \tilde{V}(j, \alpha) := \bigcup_{0 \leq i < h(j, \alpha)} f^i(V(j, \alpha)) \) is a tower for each \( 1 \leq j \leq t_0 \),
2. \( \text{diam}(f^i(V(j, \alpha))) \leq \varepsilon \) for all \( 0 \leq i < h(j, \alpha), 1 \leq j \leq t_0 \),
3. the towers \( \tilde{V}(j, \alpha) (1 \leq j \leq t_0) \) are mutually disjoint,
4. \( V(j, \alpha) \subseteq U(v(j, \alpha)) \) and \( f^{h(j, \alpha)}(V(j, \alpha)) \subseteq U(v'(j, \alpha)) \) for each \( 1 \leq j \leq t_0 \), and
5. for each \( 1 \leq j \leq t_0 \) with \( h(j, \alpha) \leq L \), the tower \( \tilde{V}(j, \alpha) \) contains a periodic point with least period \( \leq h(j, \alpha) \).

We define a closed and open set \( V_0 := \bigcup_{1 \leq j \leq t_0} \tilde{V}(j, 0) \). By the inductive hypothesis, we have \( \{ x \in X \mid f^l(x) = x, 1 \leq l \leq L \} \subset V_0 \). We define \( V^0 := \bigcup_{v \in \overline{V}} U(v) \setminus V_0 \) and \( X^0 := X \setminus V_0 \). Then, \( V^0 \) and \( X^0 \) are also closed and open sets.

Let \( x \in V^0 \). Then, there exists some \( v \in \overline{V} \) such that \( x \in U(v) \). Suppose that \( x \) is a periodic point with least period \( h(x) \). Then, \( h(x) > L \). Take a closed and open set \( V^0 \supseteq V(x) \ni x \) such that \( V(x) \subseteq U(v) \) and \( f^{h(x)}(V(x)) \subseteq U(v) \), \( \tilde{V}(x) := \bigcup_{0 \leq i < h(x)} f^i(V(x)) \) is a
We also define \( \bar{V}(n) := \bigcup_{0 \leq i < h(n)} f^i(V(n)) \). Then, we have \( \bigcup_{1 \leq j \leq n} \bar{V}(j) \supseteq V^0 \). Therefore, it follows that \( \bigcup_{1 \leq j \leq n} \bar{V}(j) \supseteq X^0 \). In the same way as for \( l = 1 \), we note that the \( \bar{V}(j) \) may not be mutually disjoint. Similarly, each \( \bar{V}(j) \) may not be disjoint with respect to \( V(j,0) \) (\( 1 \leq j \leq t_0 \)). As stated before, we have already shown \([2:a][2:e]\) in the above statement for \( \alpha = 0 \). We repeat these with inserting a new \([2:d]\) and we now have \([2:a][2:f]\) as follows:

There exist a finite set of closed and open sets \( V(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)), heights \( h(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)), and vertices \( v(j,\alpha), v'(j,\alpha) \in \bar{V} \) (\( 1 \leq j \leq t_\alpha \)) such that

1. \( \bar{V}(j,\alpha) := \bigcup_{0 \leq i < h(j,\alpha)} f^i(V(j,\alpha)) \) is a tower for each \( 1 \leq j \leq t_\alpha \),
2. \( \text{diam}(f^i(V(j,\alpha))) \leq \varepsilon \) for all \( 0 \leq i \leq h(j,\alpha) \), \( 1 \leq j \leq t_\alpha \),
3. the towers \( \bar{V}(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)) are mutually disjoint,
4. if we define \( \bar{V}_\alpha := \bigcup_{1 \leq j \leq t_\alpha} \bar{V}(j,\alpha) \), then \( \bar{V}_\alpha = \bigcup_{0 \leq j < t_\alpha} \bar{V}(j,\alpha) \) with the convention that \( \bar{V}(0) := \emptyset \),
5. \( V(j,\alpha) \subseteq U(v(j,\alpha)) \) and \( f^{h(j,\alpha)}(V(j,\alpha)) \subseteq U(v'(j,\alpha)) \) for each \( 1 \leq j \leq t_\alpha \), and
6. for each \( 1 \leq j \leq t_\alpha \) with \( h(j,\alpha) \leq L \), the tower \( \bar{V}(j,\alpha) \) contains a periodic orbit with least period \( h(j,\alpha) \).

Evidently, \([2:d]\) is satisfied when \( \alpha = 0 \). Note that the case in which \( \alpha = n \) is true implies \( V_n = \bigcup_{0 \leq j \leq n} \bar{V}(j) = X \). From \([2:f]\) it follows that every tower \( \bar{V}(j,n) \) (\( 1 \leq j \leq t_n \)) with height \( h(j,n) \leq L \) contains a periodic orbit with least period \( h(j,n) \). Thus, we can conclude the inductive step for \( L \), and complete the proof. Therefore, we shall proceed with the inductive step for \( \alpha \).

Suppose that, for \( 0 \leq \alpha < n \), there exist \( t_\alpha > 0 \), bases \( V(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)), heights \( h(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)), and vertices \( v(j,\alpha), v'(j,\alpha) \in \bar{V} \) (\( 1 \leq j \leq t_\alpha \)) such that \([2:a][2:f]\) are satisfied. Let \( \beta = \alpha + 1 \). By the inductive hypothesis for \([2:d]\), \( V_\alpha \cap V(\beta) = \emptyset \). For each \( x \in V(\beta) \), let \( 0 < i_1 < i_2 < \cdots < i_{n(x)} < h(\beta) \) be the maximal sequence such that \( f^{i_k}(x) \) enters some \( V(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)). For each \( k \) with \( 1 \leq k \leq n(x) \), let \( j_k \) (\( 1 \leq j_k < t_\alpha \)) be such that \( f^{i_k}(x) \in V(j_k,\alpha) \). We construct a string \( s(x) := ((i_k,j_k))_{1 \leq k \leq n(x)} \) for each \( x \in V(\beta) \). If \( f^i(x) \notin V(j,\alpha) \) for all \( 0 < i < h(\beta) \) and for all \( 1 \leq j \leq t_\alpha \), then \( s(x) := \emptyset \).

We define a set \( S(\beta) := \{ s(x) \mid x \in V(\beta) \} \). Let \( V(s) := \{ x \in V(\beta) \mid s(x) = s \} \) for each \( s \in S(\beta) \). Then, we have a disjoint union \( V(\beta) = \bigcup \{ V(s) \mid s \in S(\beta) \} \) by closed and open sets. Moreover, \( \bar{V}(s) := \bigcup_{0 \leq i < h(\beta)} f^i(V(s)) \) (\( s \in S(\beta) \)) are towers that are mutually disjoint. It is evident that the towers \( \bar{V}(s) \) (\( s \in S(\beta) \)) accompany \( V(j,\alpha) \) (\( 1 \leq j \leq t_\alpha \)). For each \( s \in S(\beta) \), the accompaniment may not be proper, i.e., the following may occur:

\[
(3.2) \quad f^{h(\beta)-1}(V(s)) \subseteq f^i(V(j,\alpha)) \text{ with } 1 \leq j \leq t_\alpha, 1 \leq i < h(j,\alpha) - 1.
\]
Therefore, roughly speaking, we extend the height of tower $\bar{V}(s)$ for each $s \in S(\beta)$. If (3.2) does not occur for some $s \in S(\beta)$, then we need not change the height of the tower $\bar{V}(s)$, i.e., the base is $V(s)$ and the height is $h(s) := h(\beta)$. If (3.2) occurs for some $s \in S(\beta)$, then we do not change the base $V(s)$, but change the height to $h(s) := h(\beta) + h(j, \alpha) - 1 - i$. Then, $f^h(s) - 1(V(s)) \subseteq f^{h(j, \alpha) - 1}(V(j, \alpha))$ is satisfied. It follows that $\text{diam}(f^i(V(s))) \leq \varepsilon$ for all $0 \leq i < h(s)$, because the extension is part of the tower that has the base $V(j, \alpha)$. The disjointness of $f^i(V(s))$ ($0 \leq i < h(s)$) is also obvious. Now, we cut off the towers $\bar{V}(s)$ from each $V(j, \alpha)$. Precisely, for each $1 \leq j \leq t_\alpha$, we define $V'(j, \alpha) := V(j, \alpha) \setminus \bigcup_{s \in S(\beta)} V(s)$. We obtain $\bar{V}'(j, \alpha) := \bigcup_{0 \leq i < h(j, \alpha)} f^i(V'(j, \alpha)) = \bar{V}(j, \alpha) \setminus \bigcup_{s \in S(\beta)} V(s)$. Now, we have mutually disjoint towers $\{\bar{V}'(j, \alpha) \mid 1 \leq j \leq t_\alpha\} \cup \{\bar{V}(s) \mid s \in S(\beta)\}$. Note that, for all $s \in S(\beta)$, it follows that $h(s) \geq h(\beta) > L$. We reorder these towers and rewrite the bases $V(j, \beta)$ ($1 \leq j \leq t_\beta$), heights $h(j, \beta)$ ($1 \leq j \leq t_\beta$), and towers $\bar{V}(j, \beta)$ ($1 \leq j \leq t_\beta$). If we define $V_{t_\beta} := \bigcup_{1 \leq j \leq t_\beta} \bar{V}(j, \beta)$, then $V_{t_\beta} = V_0 \cup \bar{V}(\beta)$. This concludes the inductive step for $\alpha$, completing a proof. 

Remark 3.35. Suppose that there exists a closed and open set $U \subseteq X$ and a positive integer $n$ such that $\bigcup_{i=0}^n f^i(U) = X$. Then, $U$ intersects with each periodic orbit. We consider a trivial weighted graph $\bar{G} = (\{X\}, e_0)$ with $l(e_0) = 1$ and the trivial base map $B : \bar{G} \to X$. Then, according to the construction of $U(j)$ ($1 \leq j \leq r$) in the above proof of Proposition 3.31, it is easy to see that we can take all $U(j)$’s as subsets of $U$. Thus, by the Theorem 3.21 with the Notation 3.17, we get $\overline{V}_X \subseteq U(\overline{V}_1) \subseteq U$.

4. Link to the Bratteli–Vershik model

In this section, we introduce the well known Bratteli diagrams (see Definition 4.1 for definition). The symbols $V_n$ ($n \geq 0$) are sets of vertices of a Bratteli diagram, and $E_n$ ($1 \leq n$) the sets of edges. These symbols are not used for description of basic graphs that are discussed in §2. The symbols that mean basic graphs are checked like $\tilde{G}$, $\tilde{V}$, $\tilde{v} \in \tilde{V}$.

The Bratteli–Vershik model for the Cantor essentially minimal systems was first described in [HPS92]. In this section, we define a generalized Bratteli–Vershik model. In [HPS92], the Bratteli–Vershik model had unique minimal path and unique maximal path. We extend the notion of Bratteli–Vershik model such that there may be a compact set of minimal paths and a compact set of maximal paths. We have introduced a convention of ‘flexible graph covering model’ that is closely related to both the Bratteli–Vershik model and weighted graph covering model. One of the merits of the Bratteli–Vershik model compared with weighted graph covering model is that in the Bratteli–Vershik model, a finite modification of vertices, edges, and orders are possible. In the case of Bratteli–Vershik models in [HPS92], these operations bring about Kakutani equivalent systems and the isomorphic dimension groups with different distinguished units. In our case of generalized Bratteli–Vershik models, a finite change may even break the possibility of defining continuous Vershik maps. Nevertheless, if it works, then it might be a very valuable transformation. In the discussion using weighted graph covering models, the lengths of edges are fixed, and a finite modification is not possible without exchanging the lengths of the edges totally. For example, we shall face some difficulty in description in discussing stationary weighted graph coverings. With such observation, in Definition 3.12 we have...
defined flexible graph covering model. Of course, via Remark 3.13 the lengths of edges in a graph of a flexible graph covering shall be counted.

In this section, we define the Bratteli–Vershik model and make some links with weighted, flexible, and also basic graph covering models.

4.1. Bratteli–Vershik model.

**Definition 4.1.** A Bratteli diagram is an infinite directed graph \((V, E)\), where \(V\) is the vertex set and \(E\) is the edge set. These sets are partitioned into non-empty disjoint finite sets \(V = V_0 \cup V_1 \cup V_2 \cup \cdots\) and \(E = E_1 \cup E_2 \cup \cdots\), where \(V_0 = \{v_0\}\) is a one-point set. Each \(E_n\) is a set of edges from \(V_{n-1}\) to \(V_n\). Therefore, there exist two maps \(r, s : E \to V\) such that \(r : E_n \to V_n\) and \(s : E_n \to V_{n-1}\) for all \(n \geq 1\), i.e., the range map and the source map, respectively. Moreover, \(s^{-1}(v) \neq \emptyset\) for all \(v \in V\) and \(r^{-1}(v) \neq \emptyset\) for all \(v \in V \setminus V_0\). We say that \(u \in V_{n-1}\) is connected to \(v \in V_n\) if there exists an edge \(e \in E_n\) such that \(s(e) = u\) and \(r(e) = v\).

**Definition 4.2.** Let \((V, E)\) be a Bratteli diagram. For each \(0 \leq n < m\), a sequence of edges \(p = (e_{n+1}, e_{n+2}, \ldots, e_m) \in \prod_{n< i \leq m} E_i\) with \(r(e_i) = s(e_{i+1})\) for all \(n < i < m\) is called a path. A path \(p = (e_{n+1}, e_{n+2}, \ldots, e_m)\) is from a vertex \(v \in V_n\) to a vertex \(v' \in V_m\) if \(v = s(e_{n+1})\) and \(v' = r(e_m)\). We write \(p(i) := e_i\) for \(n < i \leq m\). For each \(n < m\), we define \(E_{n,m} := \{p \in \prod_{n< i \leq m} E_i \mid p\) is a path\}. For \(p = (e_{n+1}, e_{n+2}, \ldots, e_m) \in E_{n,m}\), the source map \(s : E_{n,m} \to E_n\) and the range map \(r : E_{n,m} \to E_m\) are defined by \(s(p) = s(e_{n+1})\) and \(r(p) = r(e_m)\).

For each \(0 \leq n\), an infinite path \(p = (e_{n+1}, e_{n+2}, \ldots)\) is also defined. For each \(0 \leq n\), \(E_{n,\infty}\) denotes the set of all infinite paths from \(V_n\). For \(p = (e_{n+1}, e_{n+2}, \ldots) \in E_{n,\infty}\), the source map \(s : E_{n,\infty} \to V_n\) is defined as \(s(p) = s(e_{n+1})\). For \(p = (e_{n+1}, e_{n+2}, \ldots) \in E_{n,\infty}\), we define \(p(i) := e_i\) for \(n < i\).\(0 \leq n \leq n' < m\) and \(p \in E_{n,m}\), we denote \(p[n', m'] := (p(n' + 1), p(n' + 2), \ldots, p(m')) \in E_{n', m'}\). For \(0 \leq n \leq n'\) and \(p \in E_{n,\infty}\), \(p[n', \infty)\) is also defined.

Particularly, we have defined the set \(E_{0,\infty}\). We consider \(E_{0,\infty}\) with the product topology. By this topology, it is a compact zero-dimensional space.

**Definition 4.3.** Let \((V, E)\) be a Bratteli diagram such that \(V = V_0 \cup V_1 \cup V_2 \cup \cdots\) and \(E = E_1 \cup E_2 \cup \cdots\) are the partitions, where \(V_0 = \{v_0\}\) is a one-point set. Let \(r, s : E \to V\) be the range map and the source map, respectively. We say that \((V, E, \geq)\) is an ordered Bratteli diagram if a partial order \(\geq\) is defined on \(E\) such that \(e, e' \in E\) are comparable if and only if \(r(e) = r(e')\). Thus, we have a linear order on each set \(r^{-1}(v)\) with \(v \in V \setminus V_0\). The edges \(r^{-1}(v)\) are numbered from 1 to \(|r^{-1}(v)|\) and the maximal (resp. minimal) edge is denoted as \(e(v, \text{max})\) (resp. \(e(v, \text{min})\)). Let \(E_{\text{max}}\) and \(E_{\text{min}}\) denote the sets of maximal and minimal edges, respectively.

**Definition 4.4.** Let \((V, E, \geq)\) be an ordered Bratteli diagram. For each \(0 < n < m\) and \(v \in V_m\), the set \(\{p \in E_{n,m} \mid r(p) = v\}\) is linearly ordered by the lexicographic order, i.e., for \(p \neq q \in E_{n,m}\) with \(r(p) = r(q)\), we get \(p < q\) if and only if \(p(k) < q(k)\) with the maximal \(k \in [n+1, m]\) such that \(p(k) \neq q(k)\). For each \(n \geq 0\), suppose that \(p, p' \in E_{n,\infty}\) are distinct cofinal paths, i.e., there exists a \(k > n\) such that \(p(k) \neq p'(k)\), and for all \(l > k\), \(p(l) = p'(l)\). We define the lexicographic order \(p < p'\) if and only if \(p(k) < p'(k)\).
Particularly, we have defined the lexicographic order on $E_{0,\infty}$. This is a partial order and $p, q \in E_{0,\infty}$ is comparable if and only if $p$ and $q$ are cofinal.

Let $(V, E, \geq)$ be an ordered Bratteli diagram. We define

$$E_{0,\infty,\min} := \{ p \in E_{0,\infty} \mid p(k) \in E_{\min} \text{ for all } k \},$$

$$E_{0,\infty,\max} := \{ p \in E_{0,\infty} \mid p(k) \in E_{\max} \text{ for all } k \}.$$

**Definition 4.5.** Let $(V, E, \geq)$ be an ordered Bratteli diagram. For each $p \in E_{0,\infty}\setminus E_{0,\infty,\max}$, there exists the least $p' > p$ with respect to the lexicographic order. We say that $(V, E, \geq)$ admits a continuous Vershik map $\psi : E_{0,\infty} \to E_{0,\infty}$ if $\psi$ is continuous everywhere, $\psi(E_{0,\infty,\max}) = E_{0,\infty,\min}$, and for each $p \in E_{0,\infty}\setminus E_{0,\infty,\max}$, it follows that $\psi(p) = p'$. We note that $\psi$ is surjective and if $|\psi^{-1}(x)| \neq 1$, then $x \in E_{0,\infty,\min}$.

**Definition 4.6.** Let $(X, f)$ be a zero-dimensional system. Let $(V, E, \geq)$ be an ordered Bratteli diagram with a continuous Vershik map $\psi : E_{0,\infty} \to E_{0,\infty}$. Then, $(V, E, \geq, \psi)$ is a Bratteli–Vershik model of $(X, f)$ if $(X, f)$ is topologically conjugate to $(E_{0,\infty}, \psi)$. We also say that $(V, E, \geq, \psi)$ is a Bratteli–Vershik model if $\psi$ is continuous and surjective.

**Notation 4.7.** Let $(V, E, \geq)$ be an ordered Bratteli diagram, $n > 0$, and $v \in V_n$. We abbreviate $P(v) := \{ p \in E_{0,n} \mid r(p) = v \}$. We define $l(v) := |P(v)|$, and we can write $P(v) = \{ p(v, 0) < p(v, 1) < \cdots < p(v, l(v) - 1) \}$. Let $U(v, i) := \{ x = (e_{x,1}, e_{x,2}, \ldots, e_{x,n}) \in E_{0,\infty} \mid (e_{x,1}, e_{x,2}, \ldots, e_{x,n}) = p(v, i) \}$ for all $0 \leq i < l(v)$. We denote that $\bar{U}(v) := \bigcup_{0 \leq i < l(v)} U(v, i)$. Then, for any Vershik map $\psi$, $\bar{U}(v)$ is a tower, i.e., $\psi(U(v, i)) = U(v, i + 1)$ for all $0 \leq i < l(v) - 1$.

Referring Definitions 3.20 and 3.21, we shall define analogies in the Bratteli–Vershik model.

**Definition 4.8.** Let $(V, E)$ be a Bratteli diagram and $n \geq 0$. We say that an infinite path $(e_{n+1}, e_{n+2}, \ldots) \in E_{n,\infty}$ is constant if $|r^{-1}(r(e_i))| = 1$ for all $i > n$. A vertex $v \in V_n$ is infinitely constantly covered by the path $p$ if there exists a constant path $p \in E_{n,\infty}$ with $v = s(p)$.

**Definition 4.9.** A Bratteli–Vershik model $(V, E, \geq, \psi)$ has closing property, if for every constant path $(e_{n+1}, e_{n+2}, \ldots) \in E_{n,\infty}$ with $n \geq 0$, the set $\bigcap_{m \geq n} \bar{U}(s(e_m))$ is a periodic orbit (of least period $l(s(e_{n+1}))$).

The theorem that corresponds to Theorem 3.22 also holds (see Theorem 3.20). Next lemma may clarify the meaning of the above definitions:

**Lemma 4.10.** Let $n \geq 0$ and $(V, E, \geq, \psi)$ be a Bratteli–Vershik model that has closing property. Let $v \in V_n$ with some $n \geq 0$. If $v$ is constantly covered by some infinite path, then $\bar{U}(v)$ contains a periodic orbit of least period $l(v)$.

**Proof.** Let $(e_{n+1}, e_{n+2}, \ldots) \in E_{n,\infty}$ be an infinite path that covers $v$. Then, by closing property, the set $\bigcap_{m \geq n} \bar{U}(s(e_m))$ is a periodic orbit. \qed

**Definition 4.11.** Let $l : l_1 < l_2 < \cdots$ be a sequence of positive integers. We say that a Bratteli–Vershik model $(V, E, \geq, \psi)$ is $l$-periodicity-regulated, if for every $n > 0$ and every $v \in V_n$ with $l(v) \leq l_n$, $\bar{U}(v)$ has a periodic orbit of least period $l(v)$. 
Lemma 4.12. Let \( I = l_1 < l_2 < \cdots \) be a sequence of positive integers. If a Bratteli–Vershik model \((V,E,\geq,\psi)\) is \(I\)-periodicity-regulated, then it has closing property.

Proof. Let \( v \in V_n \) \((n \geq 0)\) and a path \( p_n = (e_{n+1},e_{n+2},\ldots) \in E_{n,\infty}\) with \( s(p_n) = v \) be a constant infinite path. Take an \( m > n \) such that \( l_m \geq l(v) \). Let \( v_m := s(e_{m+1}) \). It is enough to show that \( \bigcap_{l > n} \overline{U}(v_l) \) is a periodic orbit. It follows that \( l(v_m) = l(v) \leq l_m \). By the \( I\)-periodicity-regulation property, for each \( j \geq m \), \( \overline{U}(v_j) \) has a periodic orbit of least period \( l(v_j) = l(v) \). This implies that the set \( \bigcap_{l > n} \overline{U}(v_l) \) is a periodic orbit of least period \( l(v) \). \( \square \)

4.2. Relation to weighted covering models. Let \((X,f)\) be a general homeomorphic zero-dimensional system and \( I = l_1 < l_2 < \cdots \) be a sequence of positive integers. In \([3]\) we have shown that there exists a \(I\)-periodicity-regulated weighted graph covering \( \overline{\mathcal{G}} : \overline{G}_0 \overset{\psi_1}{\rightarrow} \overline{G}_1 \overset{\psi_2}{\rightarrow} \overline{G}_2 \overset{\psi_3}{\rightarrow} \cdots \) with \( \lim \overline{\mathcal{G}} \) being topologically conjugate to \((X,f)\). The base maps \( B_n : \overline{G}_n \rightarrow X \) \((n \geq 0)\) have been defined at the same time. We are now ready to show that \( \overline{\mathcal{G}} \) can be transformed to an Bratteli–Vershik model of \((X,f)\). After we have done this, to show the converse, we have noticed that the next definitions are necessary:

Notation 4.13. Let \((V,E,\geq,\psi)\) be a Bratteli–Vershik model with the Vershik map \( \psi : E_{0,\infty} \rightarrow E_{0,\infty} \). For each \( n \geq 1 \) and \( v \in V_n \), let \( p(v,\text{first}) \in P(v) \) be the minimal path from \( v_0 \) to \( v \) and let \( p(v,\text{last}) \in P(v) \) be the maximal path from \( v_0 \) to \( v \). Denote for each \( v \in V_n \), the closed and open set \( B(v) := \{ p \in E_{0,\infty} \mid p[0,n] = p(v,\text{first}) \} \) and \( l(v) := |P(v)| \). Each \( B(v) \) is said to be the base of \( v \in V_n \) and \( l(v) \) is the height of \( v \in V_n \). We denote as \( \overline{B}(v) := \bigcup_{i=0}^{l(v)-1} \psi^i(B(v)) \), and get a decomposition as \( E_{0,\infty} = \bigcup_{v \in V_n} \overline{B}(v) \).

Definition 4.14. Let \((V,E,\geq,\psi)\) be a Bratteli–Vershik model with the Vershik map \( \psi : E_{0,\infty} \rightarrow E_{0,\infty} \). Among the set \( \{ B(v) \mid v \in V_n \} \) of bases in the level \( n \), define an equivalence relation \( \simeq \) generated by \( B(v) \simeq B(v') \) if there exists a \( v'' \in V_n \) such that \( B(v) \cap \psi^{j(v'')}(B(v'')) \neq \emptyset \) and at the same time \( B(v') \cap \psi^{j(v'')}(B(v'')) \neq \emptyset \). Define \( A_n := \{ B(v) \mid v \in V_n \}/\simeq \). We write as \( A_n = \{ v_{n,1},\ldots,v_{n,a(n)} \} \). Now, for each \( 1 \leq i \leq a(n) \) we get a closed and open set \( U_{n,i} := \bigcup_{B \in v_{n,i}} B \). We say that each \( U_{n,i} \) \((1 \leq i \leq a(n))\) is a cluster of bases of \( V_n \).

Definition 4.15. Let \((V,E,\geq,\psi)\) be a Bratteli–Vershik model with the Vershik map \( \psi : E_{0,\infty} \rightarrow E_{0,\infty} \). The Bratteli–Vershik model has nesting property at the level \( n \) if each cluster of \( V_{n+1} \) is a subset of a base \( B(v) \) for some \( v \in V_n \). We say that a Bratteli–Vershik model has the nesting property if it has the nesting property at the level \( n \) for all \( n \geq 0 \). An equivalent, rather formal, definition of nesting property is as follows: the Bratteli–Vershik model has nesting property at the level \( n \) if and only if for each \( v \in V_{n+1} \), there exists a \( v' \in V_n \) such that \( \psi^{j(v)}(B(v)) \subseteq B(v') \).

Notation 4.16. In a proof of the next theorem, it seems to be confusing to use the term \('edge'\) for both \( \overline{\mathcal{G}} \) and for a Bratteli diagram, even if we use symbols like \( e \in \overline{E} \). Thus, we use the symbol \( p \) for the elements of \( \overline{E} \), and refer to them as \('paths'\) of \( \overline{\mathcal{G}} \), i.e., a path \( p \in \overline{E} \) of \( \overline{\mathcal{G}} \). Nevertheless, the term \('path'\) is also used for the paths of a Bratteli diagram. This convention is applied to the cases in which both the Bratteli–Vershik model and weighted (flexible) graph covering model are concerned in a proof.
Theorem 4.17 (From weighted covering model to BV model). Let $\mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots$ be a weighted graph covering and $\lim \mathcal{G} = (X, f)$. Then, we can construct a Bratteli–Vershik model $(V, E, \preceq, \psi)$ with nesting property, and a homeomorphism $\phi : E_{0,\infty} \to X$ such that $f \circ \phi = \phi \circ \psi$ and $\phi(E_{0,\infty,\min}) = V_\infty$. If $\mathcal{G}$ has closing property, then the Bratteli–Vershik model has closing property. If $\mathcal{G}$ is $l$-periodicity-regulated, then the Bratteli–Vershik model is $l$-periodicity-regulated. The same is true for a flexible graph covering $\mathcal{G}$, via the argument in Remark 3.13.

Proof. Let $\mathcal{G} : \mathcal{G}_0 \xleftarrow{\varphi_1} \mathcal{G}_1 \xleftarrow{\varphi_2} \mathcal{G}_2 \xleftarrow{\varphi_3} \cdots$ be a weighted graph covering and $\lim \mathcal{G} = (X, f)$.

For each $n \geq 0$, $V_n \subset V$ shall be identified with $E_n$. Suppose that $n \geq 0$ and every $p \in E_{n+1} = V_{n+1}$ is written as $\varphi_{n+1}(p) = p_1p_2 \cdots p_k(p)$ with $p_i \in E_n = V_n$ $(1 \leq i \leq k)$. Then, $E_{n+1} \subset E$ shall be identified with the set \( \{(p_1, p, i) \mid \varphi(p) = p_1p_2 \cdots p_k(p), p \in V_{n+1}, p_i \in V_n \ (1 \leq i \leq k(p))\} \). The order of $(p_1, p, i)$ shall be $i$.

As described above, to construct an ordered Bratelli diagram $(V, E, \preceq)$, for each $n \geq 0$, let $V_n := E_n$. Let $n \geq 0$ and $v \in V_{n+1}$. Then, $v$ is a path $v = p \in E_{n+1}$ of $\mathcal{G}_{n+1}$. Therefore, $\varphi(p)$ is a walk in $\mathcal{G}_n$. We write $\varphi(p) = p_1p_2 \cdots p_k(p)$ with $p_i \in E_{n}$ $(1 \leq i \leq k)$. Let $E_{n+1} = \{(p_i, v, i) \mid \varphi(p) = p_1p_2 \cdots p_k(p), v = p \in V_{n+1}, p_i \in V_n \ (1 \leq i \leq k(p))\}$. We define the source map $s((q, v, i)) := q$ and the range map $r((q, v, i)) := v$. The order of each edge $e = (q, v, i)$ is $i$, i.e., if $e_1 = (q_1, v, i_1)$ and $e_2 = (q_2, v, i_2)$, then $e_1 < e_2$ if and only if $i_1 < i_2$. In this way, we have constructed an ordered Bratelli diagram $(V, E, \preceq)$ from a weighted graph covering.

Next, we shall show that there exits a homeomorphism $\phi : E_{0,\infty} \to X$, which will satisfy $f \circ \phi = \phi \circ \psi$. It is easy to check that for each $n \geq 0$ and each $p \in E_n$, the value of $l(p)$ coincides both as $p \in E_n$ and as $p \in V_n$ (see Notation 3.13). In Notation 3.9, for each $n \geq 1$ and each $p \in E_n$, we have formed the set $V(n) := \{e(p) = \{e_1, e_2, \ldots, e_n\} \in E_{0,n} \mid r(e_n) = p\}$, with the lexicographical order. Because $|P(p)| = l(p)$, we can write as $P(p) = \{x_{p,0} < x_{p,1} < x_{p,2} < \cdots < x_{p,l(p)-1}\}$. We define a map $\phi_p : P(p) \to V(n)$ by $\phi_p(x_{p,i}) := \hat{v}_{p,i}$ $(0 \leq i \leq l(p))$. We define a map $\phi_n : \bigcup_{p \in C} P(p) \to \hat{V}_n$ by $\phi|_{P(p)} := \phi_p (p \in C)$. For an arbitrarily fixed $x \in E_{0,n}$, let $C(x) := \{x \in E_{0,n} \mid x[0,n] = x\}$. For an arbitrarily fixed $x = (e_1, e_2, \cdots) \in E_{0,\infty}$, we define $x_n := (e_1, e_2, \ldots, e_n)$ for all $n \geq 1$. Then, for $0 < n < m$, $C(x_n) \supseteq C(x_m)$. We get a commutativity condition $\hat{\psi}_{m,n}(\phi_m(x_m)) = \phi_n(x_n)$. Thus, we get a continuous map $\phi : E_{0,\infty} \to X$. Apparently, this map is surjective. We have to show that this map is injective. Suppose that the map is not injective. Then, there exist $x_1 \neq x_2 \in E_{0,\infty}$ such that $\phi(x_1) = \phi(x_2)$. We write $x_i := (e_{i,1}, e_{i,2}, \ldots)$ $(i = 1, 2)$. We write $x_{i,n} := (e_{i,1}, e_{i,2}, \ldots, e_{i,n}) (i = 1, 2)$. Suppose that one of $x_i$ $(i = 1, 2)$ is not minimal. Then, there exists an $n > 0$ such that one of $x_{1,n}$ $(i = 1, 2)$ is not minimal and $x_{1,n} \neq x_{2,n}$. In this case, it is easy to see that $\phi(x_1) \neq \phi(x_2)$, a contradiction. Suppose that both of $x_i$ $(i = 1, 2)$ are minimal. In this case, if $r(e_{1,n}) = r(e_{2,n})$ (in the Bratelli diagram) for infinitely many $n$, then we get $x_{1,n} = x_{2,n}$ for such $n$. Thus, we get $x_1 = x_2$, a contradiction. Therefore, in this case, there exists an $N > 0$ such that $r(e_{1,n}) \neq r(e_{2,n})$ (in the Bratelli diagram) for all $n \geq N$. Let $p_i,n := r(e_{i,n}) \in E_n = V_n$ for $i = 1, 2$ and for all $n \geq N$. Recall that $p_{i,n}$ $(i = 1, 2)$ are paths of $\mathcal{G}_n$. Therefore, we get $s(p_{i,n}) \in \hat{V}_n$ for $i = 1, 2$ and $n \geq N$. Because $\phi(x_1) = \phi(x_2)$, it follows that $s(p_{1,n}) = s(p_{2,n})$ for all $n \geq N$. By +directionality.
of $\mathcal{G}$, we get $p_{1,n} = p_{2,n}$ for all $n \geq N$, a contradiction. Thus, $\phi$ is bijective, and it is a homeomorphism.

Next, we check that $f \circ \phi = \phi \circ \psi$ is satisfied. We can always define uniquely the Vershik map $\psi$ on the set $E_{0,\infty} \setminus E_{0,\infty,\text{max}}$ by the lexicographic order. From the construction, $f \circ \phi = \phi \circ \psi$ is satisfied on the set $E_{0,\infty} \setminus E_{0,\infty,\text{max}}$. Because $\phi : E_{0,\infty} \to X$ is a homeomorphism, evidently, we can extend the map $\psi$ uniquely on all of $E_{0,\infty}$ such that $f \circ \phi = \phi \circ \psi$ is satisfied. It is easy to check that $\phi(E_{0,\infty,\text{min}}) = V_{\infty}$, consequently, it is easy to check that $\psi(E_{0,\infty,\text{max}}) = E_{0,\infty,\text{min}}$.

Next, we shall check closing property. Suppose that the weighted covering has closing property. Because the homeomorphism $\phi$ commutes with $\psi$ and $f$, all the periodic orbits are preserved by $\phi$. Thus, it follows that the Bratteli–Vershik model has closing property. Suppose that $\mathcal{G}$ is $l$-periodicity-regulated. Let $n \geq 0$ and $v \in V_n$ be such that $l(v) \leq l_n$. Then, $v \in V_n = \overline{E}_n$ is a circuit that is infinitely constantly covered by circuits of $\overline{E}_m$ ($m \geq n$), i.e., a periodic orbit with least period $l(v)$ is included in $U(v)$, as desired. Thus, the Bratteli–Vershik model is $l$-periodicity-regulated. The last statement follows via Remark 4.13.

Now, we give a proof of Theorem 4.11.

Proof of Theorem 4.11. Let $(X,f)$ be a homeomorphic (compact) zero-dimensional system. Let $l : l_1 < l_2 < \cdots$ be an arbitrary sequence of positive integers. In Theorem 3.27, we have constructed a weighted graph covering $\mathcal{G} : G_0 \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_2} G_2 \xrightarrow{\phi_3} \cdots$ that are $l$-periodicity-regulated. Further, the inverse limit $\lim \mathcal{G} = \lim \mathcal{G}$ was topologically conjugate to $(X,f)$. By Theorem 4.17 above, from $\mathcal{G}$, we could have constructed a Bratteli–Vershik model $(V,E,\geq,\psi)$ with the nesting property. Furthermore, because $\mathcal{G}$ could be constructed to be $l$-periodicity-regulated, the Bratteli–Vershik model is $l$-periodicity-regulated. This completes the proof.

To show the converse, the nesting property defined above is important.

Lemma 4.18. Let $(V,E,\geq,\psi)$ be a Bratteli–Vershik model with the Vershik map $\psi : E_{0,\infty} \to E_{0,\infty}$. Then, after sufficient telescopings, we get a Bratteli–Vershik model with nesting property.

Proof. This proof needs the continuity of $\psi$. Fix a metric $d$ on $E_{0,\infty}$. Note that after telescopings, we get a canonical isomorphism on $(E_{0,\infty},\psi)$ and the canonical identification of the metric $d$. Fix $n \geq 1$. Note that the bases $B(v)$ ($v \in V_n$) are mutually disjoint closed sets. Thus, we can get a $\delta$ such that $0 < \delta < \min\{d(x,y) \mid x \in B(v), y \in B(v'), v, v' \in V_n, v \neq v', \}$. Let $\varepsilon > 0$ be such that if $d(x,y) \leq \varepsilon$, then $d(f(x),f(y)) \leq \delta$. It is evident that for sufficiently large $m > n$, each $f^i(B(v))$ ($1 \leq i < l(v)$, $v \in V_n$) has diameter less than $\varepsilon$. Thus, by the continuity of $\psi$, for each base $B(v)$ of $v \in V_n$, $f^i(B(v))$ has diameter less than $\delta$. It follows that $f^i(B(v))$ is contained in some base $B(v)$ ($v \in V_n$).

Telescopings from $n$ to $m$, the new Bratteli–Vershik model has the nesting property at the level $n$. Because $n \geq 0$ is arbitrary, we get the desired result.

Theorem 4.19 (From BV model to weighted covering model). Let $(V,E,\geq,\psi)$ be a Bratteli–Vershik model with nesting property. Then, there corresponds a weighted graph covering model $\mathcal{G} : G_0 \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_2} G_2 \xrightarrow{\phi_3} \cdots$ such that if $\lim \mathcal{G} = (X,f)$, then there exists a homeomorphism $\phi : X \to E_{0,\infty}$ with $\phi \circ f = \psi \circ \phi$ and $\phi(V_{\infty}) = E_{0,\infty,\text{min}}$. If the Bratteli–Vershik model has closing property, then $\mathcal{G}$ has closing property. If the Bratteli–Vershik
model is 1-periodicity-regulated, then \( \overline{G} \) is 1-periodicity-regulated. The same is true for a flexible graph covering \( \overline{G} \), via the argument in Remark 3.13.

Proof. Let \((V, E, \geq, \psi)\) be a Bratteli–Vershik model with nesting property. For each \( n \geq 0 \), \( V_n \subset V \) shall be identified with \( \overline{E}_n \). With this identification, let \( n \geq 0 \), \( v \in V_{n+1} \). Let us write as \( r^{-1}(v) = \{ e_1 < e_2 < \cdots < e_k \} \). Let \( v \) be identified with a path \( p \in \overline{E}_{n+1} \). Then, we shall get an equation \( \varphi_{n+1}(p) = s(e_1)s(e_2)\cdots s(e_k(p)) \), where \( s(e_i) \in V_n = \overline{E}_n \) \((1 \leq i \leq k(p))\).

In this proof, for a weighted graph \( \overline{G} = (\overline{V}, \overline{E}) \), each vertex \( \overline{v} \in \overline{V} \) is overlined. As in the proof for Theorem 4.1.17, for a path \( p \in \overline{E} \), we denote the set of vertices \( V(p) := \{ \tilde{v}_p,0 = s(p), \tilde{v}_p,1, \tilde{v}_p,2, \cdots, \tilde{v}_p,l(p)−1, \tilde{v}_p,l(p) = r(p) \} \subseteq \tilde{V} \). If a symbol \( v \) appears, then it must be a vertex in the Bratteli diagram. For each \( n \geq 1 \) and \( v \in V_n \), let \( p(v, \text{first}) \in P(v) \) be the minimal path from \( v_0 \) to \( v \) in the Bratteli diagram.

Define for each \( v \in V_n \), the closed and open set \( B(v) := \{ p \in E_{0,\infty} | p[0,n] = p(v, \text{first}) \} \).

Then, we define \( \breve{B}(v) := \bigcup_{i=0}^{n−1} \psi^i(B(v)) \), and get a decomposition \( E_{0,\infty} = \bigcup_{v \in V_n} \breve{B}(v) \).

For each \( n \geq 1 \), we define the set \( A_n = \{ V_{n,i} | 1 \leq i \leq a(n) \} \) of clusters (see Definition 4.1.1). By nesting property, for each \( n \geq 1 \) and each cluster \( U_{n+1,i} \) \((1 \leq i \leq a(n+1))\) there exists a base \( B(v) \) \((v \in V_n) \) that contains the cluster. We construct a weighted graph \( \overline{G_n} = (\overline{V}_n, \overline{E}_n) \) as follows: for each \( n \geq 1 \), we identify \( \overline{V}_n := A_n \) with each element \( U(n,i) \) being written as \( \overline{v}_{n,i} \in \overline{V}_n \); and for each \( n \geq 1 \), we identify \( \overline{E}_n := V_n \subset V \). Thus, each \( v \in V_n \) \((n \geq 0)\) is also considered as a path \( p \in \overline{E}_n \). Note that for each \( n \geq 1 \), each \( p = v \in \overline{E}_n \) has a positive integer \( l(v) \) such that each of \( B(v) \) and \( \psi^l(v)(B(v)) \) is contained in a cluster that is identified to an element of \( \overline{V}_n = A_n \). Thus, the source map \( s : \overline{E}_n \to \overline{V}_n \) and the range map \( r : \overline{E}_n \to \overline{V}_n \) are well defined in \( \overline{G_n} = (\overline{V}_n, \overline{E}_n) \). Further, \( l(v) \) is also considered as the length \( l(p) \) of a path \( p = v \in \overline{E}_n = V_n \). We shall construct a cover \( \varphi_{n+1} : \overline{G}_{n+1} \to \overline{G}_n \) for all \( n \geq 0 \). Fix an \( n \geq 0 \). We construct a map from \( \overline{V}_{n+1} \subset V_{n+1} \) into \( \overline{V}_n \subset V_n \) as follows: if \( U(n+1,i) \subseteq U(n,j) \), then \( \varphi_{n+1}(\overline{v}_{n+1,i}) = \overline{v}_{n,j} \).

Next, we have to assign each element \( p = v \in \overline{E}_{n+1} = V_{n+1} \) a walk in \( \overline{G_n} \). To do this, we consider the tower \( \bigcup_{i=0}^{l(v)} \psi^i(B(v)) \). We write as \( r^{-1}(v) = \{ e_1 < e_2 < \cdots < e_{k(v)} \} \) and \( v_i := s(e_i) \in V_n \) for each \( 1 \leq i \leq k(v) \) in the Bratteli diagram. The tower \( \bigcup_{i=0}^{l(v)} \psi^i(B(v)) \) passes through towers \( \bigcup_{i=0}^{l(v)} \psi^i(B(v)) \) for \( 1 \leq i \leq k(v) \) successively. Because we identify \( \overline{V}_n = \overline{E}_n \), we have a walk \( v_1 v_2 \cdots v_{k(v)} = s(e_1)s(e_2)\cdots s(e_{k(v)}) \) in \( \overline{G_n} \). We map \( \varphi_{n+1}(p) = s(e_1)s(e_2)\cdots s(e_{k(v)}) \), we note that \( p = v \).

We have to check the +directionality condition. Nevertheless, this is obvious by the nesting property of the Bratteli–Vershik model.

In this way, we have constructed \( \overline{G} : \overline{G_0} \xrightarrow{\varphi_1} \overline{G}_1 \xrightarrow{\varphi_2} \overline{G}_2 \xrightarrow{\varphi_3} \cdots \). Let \( \lim \overline{G} = (X, f) \).

We need to define a homeomorphism \( \phi : X \to E_{0,\infty} \). To get \((X, f)\), we had constructed a basic graph covering \( G : G_0 \xrightarrow{\varphi_1} \hat{G}_1 \xrightarrow{\varphi_2} \hat{G}_2 \xrightarrow{\varphi_3} \cdots \). Let \( n \geq 0 \) and \( p \in \overline{E}_n = V_n \). Then, we had written as \( V(p) := \{ \hat{v}_{p,0} = s(p), \hat{v}_{p,1}, \hat{v}_{p,2}, \cdots, \hat{v}_{p,l(p)−1}, \hat{v}_{p,l(p)} = r(p) \} \). Because \( p \) has been identified with an \( v \in V_n \) with \( l(p) = l(v) \), for the case in which \( l(p) \geq 2 \), we can assign each \( \hat{v}_{p,i} \) \((1 \leq i \leq l(p)−1)\) to \( \psi^i(B(v)) \subseteq E_{0,\infty} \). Including the case in which \( l(p) = 1 \), the \( \hat{v}_{p,0} \) is assigned to the cluster \( U_{n,j} \) that satisfies \( B(v) \subseteq U_{n,j} \) and the \( \hat{v}_{p,l(p)} \) is assigned to the cluster \( U_{n,j'} \) that satisfies \( \psi^l(v)(B(v)) \subseteq U_{n,j'} \). Thus, for all \( n \geq 0 \), each \( \hat{v} \in \hat{V}_n \) in basic graph \( \hat{G}_n \) is assigned to a closed and open set of \( E_{0,\infty} \), which we denote \( E_{0,\infty}(\hat{v}) \). It is clear that if \( \varphi_n(\hat{v}) = \hat{v}' \) \((n \geq 1)\), then \( E_{0,\infty}(\hat{v}) \leq E_{0,\infty}(\hat{v}') \). We recall that
each $x \in X$ is written as $x = (\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \ldots)$ with $\tilde{v}_n = \tilde{v}_{n-1}$ for all $n \geq 1$. Thus, each $x \in X$ defines a closed set $E_{0,x}(x) := \bigcap_{n \geq 0} E(\tilde{v}_n) \subseteq E_{0,x}$. By the nesting property, it follows that $E_{0,x}(x)$ consists of a single point. We denote $\phi(x) \in E_{0,x}$ in order to satisfy $\{ \phi(x) \} = E_{0,x}(x)$. We have defined a map $\phi : X \to E_{0,x}$. Because $E_{0,x} = \bigcup_{\tilde{v} \in V_n} E_{0,x}(\tilde{v})$ is a disjoint union for each $n \geq 0$, $\phi$ is bijective. Because each $E_{0,x}(\tilde{v})$ ($\tilde{v} \in V_n$, $n \geq 0$) is compact, the continuity of $\phi$ follows. Thus, we get $\phi$ is a homeomorphism. To show the commutativity of $\phi$, it is enough to check for the $x \in X$ such that $\phi(x) \in E_{0,x,\text{max}}$. But, this is shown by the definition of the clusters. Finally, we need to check $\phi(\nabla_x) = E_{0,x,\text{min}}$. From the definition, we get $\phi(\nabla_x) \subseteq E_{0,x,\text{min}}$. Let $x = (\tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \ldots) \notin \nabla_x$. Then, there exists an $n \geq 1$ with $\tilde{v}_n \notin \nabla_n$. Then, it is easy to see that $E_{0,x}(\tilde{v}_n) \in E_{0,x}\setminus E_{0,x,\text{min}}$.

It is straightforward to check that the closing property of the Bratteli–Vershik model brings about the closing property of weighted covering. It is also straightforward to check that the $l$-periodicity-regulation property in the Bratteli–Vershik model brings about the $l$-periodicity-regulation property in weighted covering model. This completes a proof. □

**Theorem 4.20.** A Bratteli–Vershik model $(V, E, \geq, \psi)$ has closing property if and only if the set $E_{0,x,\text{min}}$ is a basic set.

**Proof.** Let $(V, E, \geq, \psi)$ be a Bratteli–Vershik model. Suppose it has closing property. By Lemma 4.18 taking telescopings, we get a Bratteli–Vershik model with nesting property. It is easy to check that, the new Bratteli–Vershik model also has closing property. Thus, Theorem 4.19 implies that the corresponding weighted covering model has closing property.

By Theorem 3.22 it follows that $\nabla_x$ is a basic set. Again, Theorem 4.19 implies that $E_{0,x,\text{min}}$ is a basic set.

To show the converse, suppose that $E_{0,x,\text{min}}$ is a basic set. By Lemma 4.18 taking telescopings, we get a Bratteli–Vershik model $(V', E', \geq, \psi')$ with nesting property. It is obvious that $E_{0,x}$ is preserved identically holding $E_{0,x,\text{min}}$ also identically. Thus, Theorem 4.19 implies that in the corresponding weighted covering model, $\nabla_x$ is a basic set. By Theorem 3.22 it follows that the weighted covering has closing property. By Theorem 4.17 we can reconstruct a Bratteli–Vershik model with closing property. By the proof of Theorems 4.17 and 4.19 we have recovered $(V', E', \geq, \psi')$. Thus, this Bratteli–Vershik model has closing property. Let $(e_{n+1}, e_{n+2}, \ldots) \in E_{n,x}$ be a constant path of $(V, E, \geq, \psi')$. Because $(V', E')$ have been made by telescopings, there exist infinite number of $m > n$ such that $e_m$ are circuits, this implies that all $e_m$ ($m > n$) are circuits. Thus, $(V, E, \geq, \psi)$ has closing property, as desired. □

As we have seen in two theorems above, in making a formal link between the Bratteli–Vershik model and the three kinds of graph coverings discussed here, flexible graph covering model is natural. In § 5 we give an example of making links between flexible graph covering model and the Bratteli–Vershik model for the stationary case. There, we define stationary flexible coverings and compare with the well known stationary ordered Bratteli diagrams. In § 5.3 we give two examples of bidirectional stationary flexible coverings, one of which has two fixed points. It is not true, in general, that the canonically corresponding substitution system is topologically conjugate to the inverse limit of the stationary flexible covering model or related Bratteli–Vershik model. Nevertheless, in the latter case of our examples, there exists canonical isomorphism (see Lemma 5.3). In these discussions, we need the bi-sided array system that had been introduced in DM08. In the following § 4.3
we recall the way of the array system for the natural extension of flexible coverings, not only for stationary ones. Nevertheless, arguments are done in terms of weighted coverings.

4.3. Array systems. In this subsection, following [DM08], we give the notion of the array systems for a little study of stationary flexible coverings that is done in §4. Firstly, we translate flexible covering models to the corresponding weighted covering models. Let $\mathcal{G}: \mathcal{C}_0 \xrightarrow{\phi} \mathcal{C}_1 \xrightarrow{\phi^2} \mathcal{C}_2 \xrightarrow{\phi^3} \ldots$ be weighted covering. We have to recall that to construct the inverse limit, we had translated weighted covering model to basic covering model: $\mathcal{G}: \mathcal{G}_0 \xrightarrow{\phi} \mathcal{G}_1 \xrightarrow{\phi^2} \mathcal{G}_2 \xrightarrow{\phi^3} \ldots$. We had denoted as $\lim \mathcal{G} := \lim \mathcal{G}$. We write $\lim \mathcal{G} = (X, f)$. We recall that $f: X \to X$ is continuous and surjective. By the natural extension of $(X, f)$, we mean a zero-dimensional system $(\hat{X}, \hat{f})$ with $\hat{X} := \{(x_i)_{i \in \mathbb{Z}} \in X^\mathbb{Z} \mid f(x_i) = x_{i+1} \text{ for all } i \in \mathbb{Z}\}$ and a homeomorphism $\hat{f}: \hat{X} \to \hat{X}$ defined by $\hat{f}((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$.

For each $\hat{x} = (x_i)_{i \in \mathbb{Z}} \in \hat{X}$ and $n \geq 0$, there exists a unique $\hat{v} \in V(\mathcal{G}_n)$ such that $x_i \in U(\hat{v})$. We express this $\hat{v}$ as $\hat{v}_{n,i}$. We define a sequence $\hat{x} := (\hat{v}_{n,i})_{n \geq 0, i \in \mathbb{Z}}$. We define $\hat{X} := \{\hat{x} \mid \hat{x} \in \hat{X}\} \subset \prod_{n \geq 0} (V(\mathcal{G}_n))^{\mathbb{Z}}$ and $\hat{x}(n, i) := \hat{v}_{n,i}$ for $n \geq 0$ and $i \in \mathbb{Z}$. Then, $\hat{X}$ is the set of all $y \in \prod_{n \geq 0} (V(\mathcal{G}_n))^{\mathbb{Z}}$ such that if we write as $y = y(n, i)$ with $n \geq 0$, $i \in \mathbb{Z}$, then the followings are satisfied: $y(n, i) \in V(\mathcal{G}_n)$, $\varphi_{n+1}(y(n+1, i)) = y(n, i)$ for all $n \geq 0, i \in \mathbb{Z}$, and $(y(n, i), y(n, i + 1)) \in E(\mathcal{G}_n)$ for all $n \geq 0$ and $i \in \mathbb{Z}$. This is a closed condition, and we get $\hat{X}$ is a compact metrizable zero-dimensional space. Clearly, $\hat{f}$ is bijective. Thus, we get $(\hat{X}, \hat{f})$ is a homeomorphism zero-dimensional system. We denote $\hat{x}[n] := (v_{n,i})_{i \in \mathbb{Z}}$, the $n$th path. Because every $\hat{x}[n]$ is an infinite walk of $\mathcal{G}_n$, every segment $([\hat{x}[n]][a, b] = (\hat{x}(n, a), \hat{x}(n, a + 1), \ldots, \hat{x}(n, b))$ is a finite walk of $\mathcal{G}_n$. In this paper, the homeomorphic zero-dimensional system $(\hat{X}, \hat{f})$ is called an array system generated by $\mathcal{G}$. Let $\hat{x} \in \hat{X}$, $n \geq 0$ and $i \in \mathbb{Z}$. Now, we consider three cases:

Case 1: Suppose that there exists an $e \in \overline{\mathcal{E}}_n$ such that $l(e) > 1$ and $(\hat{x}(n, i), \hat{x}(n, i + 1)) \in E(e)$. In this case, such an $e$ is unique and we can denote $\hat{x}(n, i) := e$. Once an $e \in \overline{\mathcal{E}}_n$ appears with $l(e) > 1$, then the $e$ continues at least $l(e)$ times.

Case 2: Suppose that $(\hat{x}(n, i), \hat{x}(n, i + 1)) \in E(e)$ for an $e \in \overline{\mathcal{E}}_n$ with $l(e) = 1$ and that $(\hat{x}(n + 1, i), \hat{x}(n + 1, i + 1)) \in E(e_{n+1})$ and $l(e_{n+1}) > 1$. Then, by taking a factor map $\varphi_{n+1,n}(e_{n+1})$ in $\mathcal{G}$, we can determine a unique $e \in \overline{\mathcal{E}}_n$ in a suitable position of the walk $\varphi_{n+1,n}(e_{n+1})$. We can denote as $\hat{x}(n, i) := e$.

Case 3: Suppose that $(\hat{x}(n + 1, i), \hat{x}(n + 1, i + 1)) \in E(e_{n+1})$ and $l(e_{n+1}) = 1$. We note that $e_{n+1}$ are not identified uniquely. Nevertheless, $\hat{x}(n + 1, i) = v_{n+1} \in \overline{\mathcal{V}}_{n+1}$ is identified uniquely. Thus, by $+\text{directionality condition}$, $e = \varphi_{n+1,n+1}(e_{n+1})$ is identified uniquely. We denote $\hat{x}(n, i) := e$.

Thus, we have defined $\hat{x}(n, i)$ for all $n \geq 0$ and all $i \in \mathbb{Z}$.

Following [DM08], we make an $n$-cut in each $\hat{x}[n]$ just before each $i$ with $\hat{x}(n, i) = s(\hat{x}(n, i))$ (see Figure 4). We define as $\hat{X} := \{\hat{x} \mid x \in X\}$. Because there exists a continuous bijective factor map from $\hat{X}$ onto $\hat{X}$, $\hat{X}$ is a compact metrizable zero-dimensional space. We define the shift map $\hat{f}: \hat{X} \to \hat{X}$ that shifts left. Thus, we get a homeomorphic zero-dimensional system $(\hat{X}, \hat{f})$, which we call an array system generated by $\mathcal{G}$. If it is necessary to distinguish the beginning of the edge, then it can be done by changing $\hat{x}(n, i) = e$ into $\hat{x}(n, i) = \hat{e}$ for all the $i$’s with $\hat{x}(n, i) = s(\hat{x}(n, i))$.

Remark 4.21. In two cases of our examples that are given in §4 it is easy to see that for each $n \geq 1$, the numbers of continuations of the same $e \in \overline{\mathcal{E}}_n$ with $l(e) > 1$ are bounded.
Let \( n = 1 \) are cut everywhere, we do not need this change.

Therefore, with the information that the infinite continuations of the same symbol of length 1 are cut everywhere, we do not need this change.

The next theorem is obvious:

**Theorem 4.22.** Let \( \overline{G} : G_0 \xrightarrow{p_1} G_1 \xrightarrow{p_2} G_2 \xrightarrow{p_3} \cdots \) be a weighted covering. We write as \( \lim \overline{G} = (X, \tilde{f}) \). Then, the natural extension \((\tilde{X}, \tilde{f})\) defined above is topologically conjugate to the array systems \((\bar{X}, \bar{f})\) and also \((\bar{X}, \bar{f})\).

We recall the notation of the singleton weighted graph \( G_0 = (\{ e_0 \}, \{ e_0 \}) \). By this, for each \( \tilde{x} \in \bar{X} \), \( \tilde{x}[0] = (\ldots, e_0, e_0, e_0, \ldots) \) that is cut everywhere. Therefore, for each \( \tilde{x} \in \bar{X} \) and \( n \geq 0 \), there exists a unique sequence \( \tilde{x}[n] := (\ldots, \tilde{x}(n, -2), \tilde{x}(n, -1), \tilde{x}(n, 0), \tilde{x}(n, 1), \ldots) \) of edges of \( G_n \) that is separated by the cuts. For integers \( s < t \), we denote \( (\tilde{x}[n])[s, t] := (\tilde{x}(n, s), \tilde{x}(n, s + 1), \tilde{x}(n, s + 2), \ldots, \tilde{x}(n, t)) \). For an interval \( [n, m] \) with \( m > n \geq 0 \), the combination of rows \( \tilde{x}[n'] \) with \( n \leq n' \leq m \) is denoted as \( \tilde{x}[n, m] \). For each \( \tilde{x} \in \bar{X} \), we get an infinite combination \( \tilde{x}[0, \infty) \in \bar{X} \) of rows \( \tilde{x}[n] \) for all \( 0 \leq n < \infty \) (see Figure 2). Note that for \( m > n \geq 0 \), if there exists an \( m \)-cut at position \( i \) (just before position \( i \)), then there exists an \( n \)-cut at position \( i \) (just before position \( i \)).

**Notation 4.23.** For each edge \( e \in \overline{E}_n \), if we write \( \varphi_n(e) = a_1a_2\cdots a_w(e) \) with \( a_j \in \overline{E}_{n-1} \) \((1 \leq j \leq w(e))\) as a walk in \( G_{n-1} \), then each \( a_j \) determines a walk of \( G_{n-2} \) similarly. Thus, we can determine a set of circuits arranged in a square form as in Figure 3. This form is said to be the \( n \)-symbol and denoted by \( e \). For \( 0 \leq m < n \), the projection \( e[m] \) that is a finite sequence of circuits of \( \overline{G}_m \) is also defined.

It is clear that \( \tilde{x}[n] = \tilde{x}'[n] \) implies that \( \tilde{x}[0, n] = \tilde{x}'[0, n] \). If \( x \neq x' \) \((x, x' \in X)\), then there exists an \( n > 0 \) with \( x[n] \neq x'[n] \). For \( x, x' \in X \), we say that the pair \((x, x')\) is \( n \)-compatible if \( x[n] = x'[n] \). If \( x[n] \neq x'[n] \), then we say that \( x \) and \( x' \) are \( n \)-separated.
We recall that if there exists an \( n \)-cut at position \( k \) (just before the position \( k \)), then there exists an \( m \)-cut at position \( k \) (just before the position \( k \)) for all \( 0 \leq m \leq n \). The set \( X_n := \{ \bar{x}[n] \mid x \in X \} \) is a two-sided subshift of a finite set \( E_n \cup \{ e \mid e \in E_n \} \). The factoring map is denoted by \( \pi_n : \bar{X} \to \bar{X}_n \), and the shift map is denoted by \( \bar{f}_n : \bar{X}_n \to \bar{X}_n \). For \( m > n \geq 0 \), the factoring map \( \pi_{m,n} : \bar{X}_m \to \bar{X}_n \) is defined by \( \pi_{m,n}(x[m]) = \bar{x}[n] \) for all \( x \in X \).

5. Substitution systems and the stationary systems

In this section, we try to show how weighted coverings and flexible coverings, are related to the Bratteli–Vershik models much more concretely. To do this, we have picked up the substitution systems and have made elementary analyses that are related to the stationary ordered Bratteli diagrams (see Definition 5.3) and also stationary flexible coverings (see Definition 5.22). This is an example of discussion using our weighted graph coverings in relation with the stationary ordered Bratteli diagrams.

In [DHS99], Durand, Host, and Skau had shown the relation between the Bratteli diagrams and the primitive substitution systems. And, in [BKM09], Bezuglyi, Kwiatkowski, and Medynets extended the results to aperiodic substitution systems. We, in this paper, shall not study in such depth. We treat only stationary ordered Bratteli diagrams that are instantly made up by substitutions. Thus, the stationary ordered Bratteli diagrams may not even have the continuous Vershik maps. Even if it had, the Vershik map has to be extended to the natural extension and also factored to the first line \( \pi \bar{X}_1, \bar{f}_1 \) to produce the intended substitution system. Our concern lies in whether weighted graph covering models can be discussed analogously to the Bratteli–Vershik models, or not.

We had to start an elementary study from the very beginning. We have produced many obvious definitions and little lemmas (see from Definition 5.1 to Proposition 5.21). Clearly, the secret of a substitution is hidden behind the Vershik maps. When we discuss using stationary flexible covering model, at first glance, it may seem that we shall be lead to many analogous definitions and lemmas. But, it was false. There exist many trivial and significant differences between the usage of the two models. It seems that, by the usage of stationary flexible covering models, some definitions and lemmas are simplified (cf. Definitions 5.5 and 5.14 and Definition 5.24). In addition, some portion of the elementary link between substitutions and the stationary ordered Bratteli diagrams might have become obvious, e.g., stationary flexible coverings always have continuous inverse limits. In return for this convenience, many substitutions cannot be transferred to the stationary flexible covering models at least instantly.

5.1. Stationary systems. Firstly, in this section, we recall a definition of stationary ordered Bratteli diagrams. To simplify the later notations, we introduce the following notation:

**Definition 5.1.** A quadruplet \((\mathcal{V}, \mathcal{E}, s, r)\) of the set of vertices \( \mathcal{V} \), the set of edges \( \mathcal{E} \), the source map \( s \), and the range map \( r \) is said to be a "mono-graph" if \( r : \mathcal{E} \to \mathcal{V} \) and \( s : \mathcal{E} \to \mathcal{V} \) are maps such that \( r^{-1}(v) \neq \emptyset \) for all \( v \in \mathcal{V} \). A mono-graph \((\mathcal{V}, \mathcal{E}, s, r)\) is surjective if, in addition, \( s^{-1}(v) \neq \emptyset \) for all \( v \in \mathcal{V} \).

**Definition 5.2.** A quintuplet \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) is an ordered mono-graph, if \((\mathcal{V}, \mathcal{E}, s, r)\) is a mono-graph and \( \geq \) is a partial order on \( \mathcal{E} \) such that \( e, e' \in \mathcal{E} \) is comparable if and only if...
e, e' ∈ r⁻¹(v) for some v ∈ V and edges in r⁻¹(v) are linearly ordered, i.e., the elements of r⁻¹(v) are numbered from 1 to |r⁻¹(v)|. For each v ∈ V, the maximal (resp. minimal) edge in r⁻¹(v) is denoted as e(v, max) (resp. e(v, min)). The notations e(v, max) and e(v, min) are synchronized with the same in an ordered Bratteli diagram (V, E, ≥) (cf. Definition 5.3).

We recall the definition of:

**Definition 5.3** (Stationary ordered Bratteli diagram). Let (V, E, s, r, ≥) be an ordered surjective mono-graph and n(v) (v ∈ V) be a sequence of positive integers. An ordered Bratteli diagram (V, E, ≥) is called a *stationary ordered Bratteli diagram* generated by (V, E, s, r, ≥, n(v) (v ∈ V)) if Vₙ is a copy of V for all n ≥ 1, Eₙ is a copy of E for all n ≥ 2, and there exist exactly n(v) edges from v₀ to v ∈ V₁ that are linearly ordered for all v ∈ V₁. If we do not consider n(v) (v ∈ V), then we say that a stationary ordered Bratteli diagram is generated by a mono-graph (V, E, s, r, ≥). For v ∈ V and n ≥ 1, the copy of v is denoted as (v, n) ∈ Vₙ, for e ∈ E and n ≥ 2, the copy of e is denoted as (e, n), for each (e, n) ∈ Eₙ (n ≥ 2), s(e, n) := (s(e), n - 1) and r(e, n) := (r(e), n), and the elements of E₁ is denoted as (v, 1, i) (v ∈ V, 1 ≤ i ≤ n(v)) with r(v, 1, i) = v for all (v, 1, i) ∈ E₁. The map o : V \ V₀ → V is defined as o(v, n) := v, and the map o : E \ E₁ → E is defined as o(e, n) := e. For every (v, n) or (e, n + 1) with v ∈ V, e ∈ E, and n ≥ 1, r⁻¹((v, n)) might be written as r⁻¹(v, n), s((e, n + 1)) might be written as s(e, n + 1), the set of paths from v₀ to (v, n) might be written as P(v, n), and so on.

**Definition 5.4.** If a stationary ordered Bratteli diagram (V, E, ≥) admits a continuous Vershik map ψ : E₀,∞ → E₀,∞, then the Bratteli–Vershik model (V, E, ≥, ψ) is also called stationary.

We are going to simplify the stationary ordered Bratteli diagrams by taking telescopings. By telescopings, we get a stationary ordered Bratteli diagram in which the maximal and also the minimal paths are all ‘straight’ as in the next:

**Definition 5.5.** Let (V, E, ≥) be a stationary ordered Bratteli diagram generated by an ordered surjective mono-graph (V, E, s, r, ≥). For 1 ≤ n < m ≤ ∞, a path p ∈ Eₙ,m is straight if there exists an eₚ ∈ E such that o(p(i)) = eₚ for all n < i ≤ m. Consequently, for a straight path p, we denote as v_p ∈ V the unique element in V such that o(s(p(i))) = v_p for all n < i ≤ m. We note that o(r(p(i))) = v_p for all n < i ≤ m. For 2 ≤ n < ∞, p ∈ E₀,n is said to be straight if p[1, n] is straight. A p ∈ E₀,∞ is said to be straight if p[1, ∞) is straight. We also say that e ∈ E is straight if s(e) = r(e). If p ∈ E₀,∞,max is straight and pass through (v, i) ∈ V_i for all i ≥ 1, then we denote p_max(v) := p and v_p := v. If p ∈ E₀,∞,min is straight and pass through (v, i) ∈ V_i for all i ≥ 1, then we denote p_min(v) := p and v_p := v.

A result of this simplification is seen in Lemma 5.8. In a mean while, we continue to define related notions:

**Definition 5.6.** Let (V, E, ≥) be a stationary ordered Bratteli diagram generated by (V, E, s, r, ≥, n(v) (v ∈ V)), n ≥ 1, and 0 < K ∈ Z. Then, by identifying V with Vₙ and also Vₙ⁺K, we define a set of edges E^K from Eₙ⁻¹,n⁺K. The source map, the range map, and the partial order are from Eₙ⁻¹,n⁺K. Note that E^K is independent of n ≥ 1.
Lemma 5.7. Let \( (V, E, \geq) \) be a stationary ordered Bratteli diagram generated by \( (V, \mathcal{E}, s, r, \geq, n(v) \ (v \in V)) \) and \( K \geq 1 \). Then, the stationary ordered Bratteli diagram obtained by telescoping from 1 + iK to 1 + (i + 1)K for all \( i \geq 0 \) is isomorphic to the stationary ordered Bratteli diagram generated by \( (V, \mathcal{E}^K, s, r, \geq, n(v) \ (v \in V)) \).

Proof. We omit a proof. \( \square \)

Lemma 5.8. Let \( (V, \mathcal{E}, s, r, \geq) \) be an ordered surjective mono-graph. Then, there exists a \( K \geq 1 \) such that in the stationary ordered Bratteli diagram generated by \( (V, \mathcal{E}^K, s, r, \geq) \), all \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \) are straight.

Proof. For each \( (v, n) \in V_n \ (n \geq 2) \), there exists a unique maximal edge in \( r^{-1}(v, n) \).

Because \( n \geq 2 \) is arbitrary, we get \( |E_{0, \infty, \text{max}}| \leq |V| < \infty \). In the same way, we get \( |E_{0, \infty, \text{min}}| \leq |V| < \infty \). For each \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \), it is easy to find a \( K(p) > 0 \) with \( o(p(i + K(p))) = o(p(i)) \) for all \( i \geq 2 \). We define \( K := \prod_{p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}}} K(p) \).

Further simplification is implied in the next:

Lemma 5.9. Let \( (V, \mathcal{E}, s, r, \geq) \) be an ordered surjective mono-graph. Suppose that all \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \) are straight. Then, for each \( v \in V \), there exist an \( n_{v, \text{max}} > 1 \) and \( p \in E_{0, \infty, \text{max}} \) such that the maximal path in \( P(v, n_{v, \text{max}}) \cup E_{0, n} \) pass the vertex \( (v_p, 1) \in V_1 \).

For the minimality, the same holds for every \( v \in V \) with an \( n_{v, \text{min}} > 1 \), a minimal path \( p \in E_{0, \infty, \text{min}} \), and \( (v_p, 1) \in V_1 \).

Proof. We omit a proof. \( \square \)

Remark 5.10. Let \( (V, \mathcal{E}, s, r, \geq) \) be an ordered surjective mono-graph such that all \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \) are straight. Take \( N := \max_{v \in V} \max(n_{v, \text{max}}, n_{v, \text{min}}) \). The ordered surjective mono-graph \( (V, \mathcal{E}^N, s, r, \geq) \) generates a stationary ordered Bratteli diagram such that all \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \) are straight and for all \( v \in V_n \) with \( n \geq 2 \), there exist both maximal path \( p' \) and minimal path \( p'' \) such that \( s(e(v, \text{max})) = v_{p'} \) and \( s(e(v, \text{min})) = v_{p''} \).

From Lemmas 5.8 and 5.9 and Remark 5.10 we immediately get:

Lemma 5.11. Let \( (V, \mathcal{E}, s, r, \geq) \) be an ordered surjective mono-graph. Then, there exists a \( K > 0 \) such that the ordered surjective mono-graph \( (V, \mathcal{E}^K, s, r, \geq) \) generates an ordered Bratteli diagram that satisfy

(a) each \( p \in E_{0, \infty, \text{max}} \cup E_{0, \infty, \text{min}} \) is straight and

(b) for each \( v \in V_n \) with \( n \geq 2 \), there exist both \( p' \in E_{0, \infty, \text{max}} \) and \( p'' \in E_{0, \infty, \text{min}} \) such that \( s(e(v, \text{max})) = v_{p'} \) and \( s(e(v, \text{min})) = v_{p''} \).

Proof. We omit a proof. \( \square \)

From the above lemma we get the notation:

Notation 5.12. We denote as \( \mathcal{E}_{\text{max}} := \{ e \in \mathcal{E} \mid e \text{ is maximal in } r^{-1}(r(e)) \} \) and \( \mathcal{E}_{\text{min}} := \{ e \in \mathcal{E} \mid e \text{ is minimal in } r^{-1}(r(e)) \} \). We denote as \( \mathcal{E}_{\text{max},1} := \{ e \in \mathcal{E}_{\text{max}} \mid s(e) = r(e) \} \) and \( \mathcal{E}_{\text{min},1} := \{ e \in \mathcal{E}_{\text{min}} \mid s(e) = r(e) \} \). We denote as \( \mathcal{V}_{\text{max},1} := \{ s(e) \mid e \in \mathcal{E}_{\text{max},1} \} \) and \( \mathcal{V}_{\text{min},1} := \{ s(e) \mid e \in \mathcal{E}_{\text{min},1} \} \).

From Lemma 5.11 we immediately get the following:
Lemma 5.13. Let \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) be an ordered surjective mono-graph. Then, there exists a \(K > 0\) such that the ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}^K, s, r, \geq)\) satisfies that for every \(v \in \mathcal{V}\), \(s(e(v, \text{max})) \in \mathcal{V}_{\text{max}, 1}\) and \(s(e(v, \text{min})) \in \mathcal{V}_{\text{min}, 1}\).

Finally, we get an end of our way of simplification by defining:

Definition 5.14. An ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) is straight if all \(p \in E_{0, x, \text{max}} \cup E_{0, x, \text{min}}\) are straight, and for every \(v \in \mathcal{V}\), both of \(s(e(v, \text{max})) \in \mathcal{V}_{\text{max}, 1}\) and \(s(e(v, \text{min})) \in \mathcal{V}_{\text{min}, 1}\) are satisfied. For a straight ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}, s, r, \geq)\), we denote as \(\max(v) := s(e(v, \text{max})) \in \mathcal{V}_{\text{max}, 1}\) and \(\min(v) := s(e(v, \text{min})) \in \mathcal{V}_{\text{min}, 1}\).

Definition 5.15. A stationary ordered Bratteli diagram \((V, E, \geq)\) is straight if it is generated by a straight ordered surjective mono-graph. The numbers \(n(v) (v \in \mathcal{V})\) are arbitrary positive integers. If a straight stationary ordered Bratteli diagram \((V, E, \geq)\) admits a Bratteli–Vershik model \((V, E, \geq, \psi)\), then we call it a straight stationary Bratteli–Vershik model.

From the discussion above, we get:

Proposition 5.16. Let \((V, E, \geq)\) be a stationary ordered Bratteli diagram generated by \((\mathcal{V}, \mathcal{E}, s, r, \geq, n(v) (v \in \mathcal{V}))\). Then, there exists a \(K > 0\) such that by telescoping from \(1 + iK\) to \(1 + (i + 1)K\) for all \(i \geq 0\), we get a straight stationary ordered Bratteli diagram.

Notation 5.17. Let \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) be a mono-graph and \(e \in \mathcal{E}\). Suppose that the order of \(e\) is numbered by \(i\) with \((1 \leq i \leq |r^{-1}(r(e))|)\), i.e., \(e\) is not maximal. Then, the next edge \(e' \in r^{-1}(r(e))\) that is numbered by \(i + 1\) is denoted as \(e(\text{next})\).

Definition 5.18. We say that a straight ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) has continuity condition if there exists a surjective map \(\psi : \mathcal{V}_{\text{max}, 1} \to \mathcal{V}_{\text{min}, 1}\) such that for all \(e \in \mathcal{E}\setminus\mathcal{E}_{\text{max}}, \psi(\max(s(e))) = \min(s(e(\text{next})))\), or equivalently, there exists a surjective map \(\psi : E_{0, x, \text{max}} \to E_{0, x, \text{min}}\) such that for all \(e \in \mathcal{E}\setminus\mathcal{E}_{\text{max}}, v = s(e), \) and \(v' = s(e(\text{next})),\) it follows that \(\psi(p_{\text{max}}(\max(v))) = \min(\text{min}(v'))\).

Lemma 5.19. A straight ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}, s, r, \geq)\) generates a Bratteli–Vershik model if and only if it has continuity condition, i.e., there exists a surjective map \(\psi : \mathcal{V}_{\text{max}, 1} \to \mathcal{V}_{\text{min}, 1}\) such that for all \(e \in \mathcal{E}\setminus\mathcal{E}_{\text{max}},\) it follows that \(\psi(\max(s(e))) = \min(s(e(\text{next}))).\)

Proof. Let \((V, E, \geq)\) be a straight stationary ordered Bratteli diagram that is generated by a straight ordered surjective mono-graph \((\mathcal{V}, \mathcal{E}, s, r, \geq)\). Suppose that there exists a continuous Vershik map \(\psi : E_{0, x} \to E_{0, x}\). Let \(e \in \mathcal{E}\setminus\mathcal{E}_{\text{max}}\) and \(v, v_1 \in \mathcal{V}\) be such that \((v, n) = s(e, n + 1)\) and \((v_1, n + 1) = r(e, n + 1)\) for an arbitrarily large \(n\). Then, there exists a straight maximal path \(p_{\text{max}}(\max(v))\). Because the minimal path \(\psi(p_{\text{max}}(\max(v)))\) is straight, there exists a \(v_2 \in \mathcal{V}_{\text{min}, 1}\) such that \(\psi(p_{\text{max}}(\max(v))) = \min(v_2)\). Let \(p\) be the concatenation of the maximal path from \(v_0\) to \((v, n), (e, n + 1),\) and some other \(p[n + 1, x) \in E_{n + 1, x}\). Because \(p[0, n]\) is maximal, we get \(p[0, n - 1] = p_{\text{max}}(\max(v))[n - 1]\). By the continuity of \(\psi : E_{0, x} \to E_{0, x}\), for sufficiently large \(n, \psi(p)[0, 2] = \min(v_2)[0, 2]\). Thus, \(\psi(\max(s(e))) = \psi(\max(v)) = v_2\). On the other hand, because \(p[0, n]\) is maximal, we get \(\psi(p)(n + 1) = (e(\text{next}), n + 1)\). Thus, we get \(\psi(p)\) is a concatenation of \(p_{\text{min}}(v_2)[0, n - 1],\)
the minimal edge from \((v_2, n-1)\) to \((s(e(\text{next})), n), (e(\text{next}), n+1), \) and \(p[n+1, \infty)\). Thus, we get \(\min(s(e(\text{next}))) = v_2\). It follows that \(\psi(\max(s(e))) = v_2 = \min(s(e(\text{next})))\) for every \(e \in \mathcal{E}|\mathcal{E}_{\text{max}}\). Because \(\psi(E_{0, \infty, \text{max}}) = E_{0, \infty, \text{min}}\), the surjectivity of \(\psi: \mathcal{V}_{\text{max}} \rightarrow \mathcal{V}_{\text{min}}\) is obtained.

To show the converse, let \((V, E, \geq)\) be an straight stationary ordered Bratteli diagram that is generated by a straight ordered surjective mono-graph \((V, \mathcal{E}, s, r, \geq)\) and a sequence \(n(v) (v \in V)\) of positive integers. Suppose that there exists a map \(\psi: E_{0, \infty, \text{max}} \rightarrow E_{0, \infty, \text{min}}\) such that for all \(e \in \mathcal{E}\), \(\psi(\max(s(e))) = \psi(\min(s(e(\text{next}))))\). We define a Vershik map \(\psi: E_{0, \infty} \rightarrow E_{0, \infty}\) by the lexicographic order on \(E_{0, \infty}\). We need to check that \(\psi\) is continuous at every \(p \in E_{0, \infty, \text{max}}\). Fix \(p \in E_{0, \infty, \text{max}}\). Because the mono-graph is straight, \(p\) is straight. Let \(e_p \in \mathcal{E}_{\text{max}, 1}\) be such that \(e_p = (i)\) for all \(i \geq 2\). We can also define a \(v_p = s(e_p) = r(e_p) \in \mathcal{V}_{\text{max}, 1}\). Define \(C_m(p) := \{ x \in E_{0, \infty} \mid x[0, m] = p[0, m] \}\) for each \(m \geq 2\). Then, the sequence \(C_m(p) (m \geq 2)\) is a system of neighbourhood of \(p\). Fix a large \(m\). Take an arbitrary \(x \in C_m(p)\). Suppose that \(x\) is maximal. Then, because of straightness, it follows that \(x = p\). Suppose that \(x\) is not maximal. Let \(k \geq 1\) be the least number such that \(x(m+k) \notin E_{\text{max}}\). It follows that \(x[0, m+k-1] = \max(s(e)) = v_p\) and \(p[0, m+k-2] = p[0, m+k-2]\). Now, \(\psi(x)(m+k) = (e(\text{next}), m+k)\) and \(\psi(x)(0, m+k-1)\) is minimal. Thus, \(\psi(x)[0, m+k-1] = \psi(p)[0, m+k-2] = \psi(p)(0, m+k-2) = \psi(p)(0, m+k-2)\). It follows that \(s(e) = v'\), i.e., \(\psi(p)(v')) = p[0, m+k-2]\). This implies the continuity of \(\psi\) at \(p\).

Lemma 5.20. Let \((V, \mathcal{E}, s, r, \geq)\) be a straight ordered surjective mono-graph that generates a Bratteli–Vershik model \((V, E, \geq, \psi)\). Suppose that for \(p \in E_{0, \infty}, n \geq 1, v \in \mathcal{V}_{\text{max}, 1}\), and \(v' \in \mathcal{V}_{\text{min}, 1}\), both \(p[0, n] = p_{\text{max}}(v)[0, n]\) and \(\psi(p)[0, n] = p_{\text{min}}(v')[0, n]\) are satisfied. Then, it follows that \(\psi(v) = v'\), i.e., \(\psi(p_{\text{max}}(v')) = p_{\text{min}}(v')\).

Proof. By Lemma 5.19 the mono-graph satisfies the continuity condition. Take \(p \in E_{0, \infty}, n \geq 1, v \in \mathcal{V}_{\text{max}, 1}\), and \(v' \in \mathcal{V}_{\text{min}, 1}\) such that \(p[0, n] = p_{\text{max}}(v)[0, n]\) and \(\psi(p)[0, n] = p_{\text{min}}(v')[0, n]\). If \(p \in E_{0, \infty, \text{max}}\), then \(\psi(p) \in E_{0, \infty, \text{min}}\). It follows that \(p = p_{\text{max}}(v)\) and \(\psi(p) = p_{\text{min}}(v')\). Thus, there is nothing to prove. Suppose that \(p\) is not maximal. Then, there exists the least \(m > n\) such that \(e = o(p(m)) \in \mathcal{E}|\mathcal{E}_{\text{max}}\). Let \(v_1 = s(e)\) and \(v'_1 = s(e(\text{next}))\).

Case 1: \(m \geq n + 2\). Because \(p[0, m-1] = \max(s(e)) = v_1\), by the straightness, we get \(p[0, m-2] = p[0, m-2]\). By the Vershik map, we get \(\psi(p)(m) = e(\text{next})\) and \(\psi(p)[0, m-1] = p[(v_1, m-1), \text{first})\). By straightness, we get \(\psi(p)[0, m-2] = p_{\text{min}}(v'_1)[m-2]\). By the continuity condition, \(p_{\text{min}}(v'_1)\) with the Vershik map \(\psi\). Combining the above two, we get \(\psi(p)[0, m-2] = \psi(p_{\text{max}}(v_1))[0, m-2]\). Because \(m \geq n + 2\), we get at least \(\psi(p)[0, n] = \psi(p_{\text{max}}(v_1))[0, n]\). From the assumption, we get \(\psi(p)[0, n] = p_{\text{min}}(v')[0, n]\). Combining the above two, we get \(\psi(p_{\text{max}}(v_1))[0, n] = p_{\text{min}}(v')[0, n]\). Because \(v_1 = s(e) = o(p(n+1))\), we get \(\max(v_1) = v\). Combining above two, we get \(\psi(p_{\text{max}}(v))[0, n] = p_{\text{min}}(v')[0, n]\), i.e., \(\psi(v) = v'\).

Case 2: \(m = n + 1\). In this case, we have assumed that \(e = o(p(n+1)) \in \mathcal{E}|\mathcal{E}_{\text{max}}\) and yet by the assumption of the proposition, we get \(s(p(n+1)) = (v, 1) \in V_1\). Thus, we
get $s(e) = v \in V_{\max,1}$. In the same way, we get $s(e(\text{next})) = v' \in V_{\min,1}$. We note that even if $e \in E \setminus E_{\max}$, it may happen that $s(e) \in V_{\max,1}$. The continuity condition: $\psi(\max(s(e))) = \min(s(e(\text{next})))$ implies that $\psi(\max(v)) = \min(v')$, i.e., $\psi(v) = v'$ as desired. □

**Proposition 5.21.** Suppose that $(V, E, \geq, \psi)$ is a straight stationary Bratteli–Vershik model. Then, it has the nesting property (at all the levels $\geq 0$).

**Proof.** Let $(V, E, s, r, \geq)$ be a straight ordered surjective mono-graph that generates a straight stationary Bratteli–Vershik model $(V, E, \geq, \psi)$. We only need to check the nesting property at the levels $n \geq 1$. Let $n \geq 1$. By Lemma 5.19, the mono-graph satisfies the continuity condition. It follows that for all $e \in E \setminus E_{\max}$, $\psi(p_{\max}(\max(s(e)))) = p_{\min}(\min(s(e(\text{next}))))$. For each $v \in V$, we consider $(v, n+1) \in V_{n+1}$, the minimal path from $v_0$ to $(v, n+1)$ is denoted as $p((v, n+1), \text{first})$. Define $B(v, n+1) := \{ x \in E_{0,1} : x[0, n+1] = p((v, n+1), \text{first}) \}$ for each $v \in V$. To examine nesting property it is sufficient to show that $\psi^j(B(v, n+1)) \subseteq B(\psi(\max(v)), n)$. Suppose that $p \in \psi^j((v, n+1))^{-1}(B(v, n+1))$ and $\psi(p) \in B(v_1, n+1)$. In particular, we get $p[0, n] = p_{\max}(\max(v))[0, n]$ and $\psi(p)[0, n] = p_{\min}(\min(v_1))[0, n]$. By Lemma 5.20, we get $\psi(\max(v)) = \min(v_1)$. It follows that $\psi(p) \in B(\min(v_1), n) = B(\psi(\max(v)), n)$, as desired. □

Finally, in this subsection, as an analogue of the stationary Bratteli–Vershik models, we introduce the following definition. Instead of ordered surjective mono-graphs for the stationary Bratteli diagram, we introduce a flexible self-cover: we say that a flexible cover $\varphi : \tilde{G} \to \tilde{G}$ is a flexible self-cover.

**Definition 5.22 (Stationary flexible graph covering).** Suppose that there exists a flexible graph $\tilde{G} = (\tilde{V}, \tilde{E})$ and a flexible self-cover $\varphi : \tilde{G} \to \tilde{G}$. Suppose that there exists a sequence of positive integers $n(e) (e \in \tilde{E})$. Define $\tilde{G}_0$ to be the singleton graph $\{ \{ v_0 \}, \{ e_0 \} \}$, $\tilde{G}_n := \tilde{G}$ for all $n \geq 1$, $\varphi_n := \varphi$ for all $n \geq 2$, and $\varphi_1$ to be the unique natural homomorphism such that for each $e \in \tilde{G}_1$, $\varphi_1(e) = e_0^{\varphi_1(n(e))} := e_0 e_1 \cdots e_0$. We say that $\tilde{G} : \tilde{G}_0 \xrightarrow{\varphi_1} \tilde{G}_1 \xrightarrow{\varphi_2} \tilde{G}_2 \xrightarrow{\varphi_3} \cdots$ is a stationary flexible graph covering generated by a flexible self-cover $\varphi : \tilde{G} \to \tilde{G}$ and a sequence $n(e) (e \in \tilde{E})$.

**Remark 5.23.** We remark that the stationary ordered Bratteli diagrams might not have the continuous Vershik maps (see the example after [Mci08, Proposition 2.5.]). On the other hand, by the $+$directionality condition of a flexible cover $\varphi : \tilde{G} \to \tilde{G}$, stationary flexible graph coverings always give continuous zero-dimensional systems.

As in the case of stationary ordered Bratteli diagrams generated by a surjective ordered mono-graph, we present some analysis on straightness for stationary flexible graph coverings. Let $\varphi : \tilde{G} \to \tilde{G}$ be a flexible cover and $G = (\tilde{V}, \tilde{E})$. Because $\varphi : \tilde{V} \to \tilde{V}$ is a map, for each $v \in \tilde{V}$ there exists a positive integer $K(v)$ such that the sequence $v, \varphi(v), \varphi^2(v), \ldots$ is eventually periodic with least period $K(v)$. We write $K(\tilde{G}, \varphi) := \bigcup_{v \in \tilde{V}} K(v)$. Then, we get a flexible cover $\varphi' : \tilde{G} \to \tilde{G}$ defined by $\varphi' := \varphi^{K(\tilde{G}, \varphi)}$. For each $v \in \tilde{V}$, the sequence $v, \varphi'(v), \varphi'^2(v), \ldots$ is eventually constant. Let $\lim v \in \tilde{V}$ be the constant vertex, i.e., there exists an $n$ such that $\varphi^m(v) = \lim v$ and $\varphi'(\lim v) = \lim v$. If we take sufficiently
large $L > 0$, then we get for all $v \in \hat{V}$, it follows that $\varphi'^L(v) = \lim v$. Thus, by taking $K := L \cdot K(\hat{G}, \varphi)$ and defining $\varphi' := \varphi^K$, it follows that for all $v \in \hat{V}$, $\varphi'(v) = \lim v$ and $\varphi'(\lim v) = \lim v$. Next, we need to consider the straightness for the edges. First, we assume that $\varphi(v) = \lim v$ for all $v \in \hat{V}$. Set $e = s(e)$. Then, by +directionality of $\varphi$, we obtain unique $\varphi(e) \in s^{-1}(\varphi(v)) = s^{-1}(\lim v)$ that is independent of the choice of $e \in s^{-1}(v)$. With the condition $\varphi(\lim v) = \lim v$, we apply $\varphi$ onto all $s^{-1}(\lim v)$, and get unique $\lim f e \in s^{-1}(\lim v)$, namely, we denote $\lim f := \varphi(\varphi(e)(\text{first}))(\text{first})$. Thus, $\lim f e$ is uniquely determined by $\lim v \in \hat{V}$. Thus, by taking $\varphi^2$ instead of $\varphi$, we assume that for each $e \in \hat{E}$, $\varphi(e)(\text{first}) = \lim f e$. In particular, we get $\varphi(\lim f e)(\text{first}) = \lim f e$. This is the corresponding notion of elements of $\mathcal{V}_{\min,1}$ in the stationary ordered surjective mono-graphs.

We also consider the counterparts of elements of $\mathcal{V}_{\max,1}$. Let $v \in \hat{V}$ and $e = r^{-1}(v)$. If we consider only bidirectional flexible self-covers, then the same argument is possible and we get $\lim f e$ as well. Suppose that the bidirectionality condition might not hold. We define a map $\rho : \hat{E} \to \hat{E}$ by $\rho(e) = \varphi(e)(\text{last})$ for all $e \in \hat{E}$. Then, the sequence $e, \rho(e), \rho^2(e), \ldots$ is eventually periodic and the periodic edges are in $r^{-1}(\lim r(e))$. Let $K'(e) \geq 1$ be the least period. By defining $K' := \prod_{e \in \hat{E}} K'(e)$ and taking sufficiently large $L' \geq 1$, we define $\bar{\varphi} := \varphi^{L'K'}$. We define $\rho : \hat{E} \to \hat{E}$ by $\rho(e) : = \varphi(e)(\text{last})$ for all $e \in \hat{E}$. Then, for each $e \in \hat{E}$, the sequence $\bar{\rho}(e), \bar{\rho}^2(e), \ldots$ is constant from the beginning. This constant edge is denoted as $\lim f e$. For each $e \in \hat{E}$, we get $\varphi(e)(\text{last}) = \lim f e$ and we get $\varphi(\lim f e)(\text{last}) = \lim f e$. We note that for each $v \in \hat{V}$, the set $\{ \lim f e \mid e \in \hat{E} \text{ and } r(e) = \lim v \}$ might have more than one elements.

**Definition 5.24.** A flexible self-cover $\varphi : \hat{G} \to \hat{G}$ is straight, if for each $v \in \hat{V}$, there exists $\lim v \in \hat{V}$ such that $\varphi(v) = \lim v$ and $\varphi(\lim v) = \lim v$; and for each $e \in \hat{E}$, there exist $\lim f e \in s^{-1}(\lim s(e))$ and $\lim f e \in r^{-1}(\lim r(e))$ such that $\varphi(e)(\text{first}) = \lim f e$, $\varphi(e)(\text{last}) = \lim f e$, $\varphi(\lim f e)(\text{first}) = \lim f e$, and $\varphi(\lim f e)(\text{last}) = \lim f e$. We define as $\lim \hat{V} := \{ \lim v \mid v \in \hat{V} \}$, $\lim \hat{E} := \{ \lim f e \mid e \in \hat{E} \}$, and $\lim \hat{E} := \{ \lim f e \mid e \in \hat{E} \}$. By the +directionality condition, for each $v \in \lim \hat{V}$, there exists a unique $e \in \lim \hat{E}$ with $s(e) = v$.

A stationary flexible graph covering is straight if it is generated by a straight flexible self-cover.

**Remark 5.25.** Let $\hat{\mathcal{G}} := \hat{G}_0 \overset{\varphi_1}{\leftarrow} \hat{G}_1 \overset{\varphi_2}{\leftarrow} \hat{G}_2 \overset{\varphi_3}{\leftarrow} \cdots$ be a stationary flexible covering. Then, there exists a $K > 0$ such that, by telescoping from $1 + (i + 1)K$ to $1 + iK$ for each $i \geq 0$, we get a straight stationary flexible covering.

**Remark 5.26.** Let $\hat{\mathcal{G}} := \hat{G}_0 \overset{\varphi_1}{\leftarrow} \hat{G}_1 \overset{\varphi_2}{\leftarrow} \hat{G}_2 \overset{\varphi_3}{\leftarrow} \cdots$ be a stationary flexible covering generated by a straight flexible self-cover $\varphi : \hat{G} \to \hat{G}$. Consider the corresponding weighted covering $\mathcal{G} := \mathcal{G}_0 \overset{\varphi_1}{\leftarrow} \mathcal{G}_1 \overset{\varphi_2}{\leftarrow} \mathcal{G}_2 \overset{\varphi_3}{\leftarrow} \cdots$. Suppose that $e \neq e' \in \lim \hat{E}$ satisfy $r(e) = r(e')$ and $e_n, e'_n \in \mathcal{T}_n$ are copies of them. Because $e, e' \in \lim \hat{E}$, we get $\varphi_n(e_n)(\text{last}) = e_n$ and $\varphi_{n+1}(e'_{n+1})(\text{last}) = e'_n$ for all $n \geq 1$. If $l(e_n) = l(e'_n) = 1$ for all $n \geq 1$, then we can conclude that $s(e_n) \neq s(e'_n)$ for all $n \geq 1$. To see this, we employ a proof by contradiction. Because $l(e_n) = l(e'_n) = 1$ for all $n \geq 1$, it follows that $\varphi_{n+1}(e_{n+1}) = e_n$ and $\varphi_{n+1}(e'_{n+1}) = e'_n$ for all $n \geq 1$. Suppose that $s(e_n) = s(e'_n)$ for some $n$. Then, because $e_n, e'_n$ are copies of $e, e'$, we get $s(e_n) = s(e'_n)$ for all $n \geq 1$. Let $v_n = s(e_n) = s(e'_n)$ for all $n \geq 1$ and apply...
+directionality condition at \( v_n \) for \( n \geq 2 \). Then, we get \( e_{n-1} = e'_{n-1} \) for all \( n \geq 2 \), a contradiction.

5.2. **Substitutions.** In this subsection, according to Durand, Host, and Skau [DHS99] and also Bezgulyi, Kwiatkowski, and Medynets [BKM09], we follow a standard way of introduction for the theory of substitution systems. An alphabet is a finite \( \geq 2 \) set of symbols called letters. If \( A \) is an alphabet, a word on \( A \) is a finite sequence of letters; \( A^+ \) is the set of words. For a word \( u = u_1u_2 \cdots u_n \in A^+ \), \( |u| = n \) is the length of \( u \). We set \( A^* := A^+ \cup \{ \emptyset \} \), in which \( \emptyset \) is the empty word of length 0. Given a word \( u = u_1u_2 \cdots u_m \) and an interval \([i, j] \subseteq [1, m]\), we write \( u[i, j] \) to denote the word \( u_iu_{i+1} \cdots u_j \). We extend this notation in the obvious way to infinite intervals. Suppose that \( v = v_1v_2 \cdots v_m \) is a word. Then, a word \( u = u_1u_2 \cdots u_n \) is a factor of \( v \) if there exists \([i, j] \subseteq [1, m] \) such that \( u = v[i, j] \). If \( u \) is a factor of \( v \), then we write \( u \prec v \). For any two words \( u = u_1u_2 \cdots u_n, v = v_1 \cdots v_m \in A^* \), we denote \( uv := u_1 \cdots u_nv_1 \cdots v_m \). Evidently, \( |uv| = |u| + |v| \).

**Definition 5.27.** We say that a map \( \sigma : A \to A^+ \) is a substitution. Let \( \sigma : A \to A^+ \) be a substitution. Then, by concatenation, we extend as \( \sigma : A^+ \to A^+ \). We define \( A_1 := \{ a \in A \mid |\sigma^n(a)| \to \infty \text{ as } n \to \infty \} \). In this paper, we assume that \( A_1 \neq \emptyset \).

It is natural to make a link between a mono-graph and a substitution.

**Definition 5.28.** Let \((V, E, s, r, \geq)\) be an ordered mono-graph, and let \( \iota : A \to V \) be a bijection. For each \( v \in V \), we can define a sequence \( v_1v_2 \cdots v_k \) of elements of \( V \) with \( k = \max \{ r^{-1}(v) \} \) such that: if \( r^{-1}(v) \) is written as \( \{ e_1 < e_2 < \cdots < e_k \} \), then \( v_i = s(e_i) \) for all \( 1 \leq i \leq k \). Let \( \iota(a) = v \) and \( \iota(u_i) = v_i \). Then, we define as \( \sigma(a) = u_1u_2 \cdots u_k \). This is defined for each \( a \in A \). If \( A_1 \neq \emptyset \), then we get a substitution \( \sigma \) called substitution read on the mono-graph.

**Notation 5.29.** Let \((V, E, \geq)\) be a stationary ordered Bratteli diagram generated by an ordered surjective mono-graph \((V, E, s, r, \geq)\) and a sequence \( n(v) \) \((v \in V)\). With a bijection \( \iota : A \to V \), we also define \( \iota_n : A \to V_n \) \((n \geq 1)\).

By this definition, from an ordered surjective mono-graph that generates a Bratteli diagram with \( |E_{0, \infty}| = \infty \), we can make a substitution. Conversely, suppose that \( \sigma : A \to A^+ \) is a substitution in which for every letter \( u \in A \), there exists an \( a \in A \) such that \( u \) appears in \( \sigma^n(a) \) for some \( n \geq 1 \). Then, we can construct an ordered surjective mono-graph. Nevertheless, the ordered Bratteli diagram may not admit continuous Vershik map. By Lemma 5.19, we can see that there exist a lot of substitutions which do not bring about Bratteli–Vershik models in this static way.

For flexible self-covers, analogously, we define:

**Definition 5.30.** Let \( \tilde{G} = (\tilde{V}, \tilde{E}) \) be a flexible graph and \( \varphi : \tilde{G} \to \tilde{G} \), a flexible self-cover. Let \( \iota : A \to \tilde{E} \) be a bijection. For each \( e \in \tilde{E} \), we have a walk \( \varphi(e) = e_1e_2 \cdots e_k \) with \( e_i \in \tilde{E} \) \((1 \leq i \leq k)\). Let \( \iota(a) = e \) and \( \iota(u_i) = e_i \). Then, we get \( \sigma(a) = u_1u_2 \cdots u_k \). This is defined for each \( a \in A \). If \( A_1 \neq \emptyset \), then we get a substitution \( \sigma \) called substitution read on the flexible self-cover \( \varphi : \tilde{G} \to \tilde{G} \).

**Definition 5.31.** Let \( \sigma : A \to A^+ \) be a substitution with a letter \( a \in A \) such that the length \( |\sigma^n(a)| \to \infty \) as \( n \to \infty \). We denote by \( \mathcal{L}(\sigma) \) the language of \( \sigma \), i.e., \( \mathcal{L}(\sigma) \) is
the set of all words on $A$ which are factors of $\sigma^n(a)$ for some $a \in A$ and some $n \geq 1$. Denote by $X_\sigma$ the subshift of $A^\mathbb{Z}$ associated to this language, i.e., the set of all $x \in A^\mathbb{Z}$ whose every finite factor belongs to $\mathcal{L}(\sigma)$. It follows that $X_\sigma$ is non-empty closed set in $A^\mathbb{Z}$, and invariant under the shift; we denote by $T_\sigma$ the restriction of the shift to $X_\sigma$, i.e., $(T_\sigma(x)_i) = x_{i+1}$ $(i \in \mathbb{Z})$ for all $x \in X_\sigma \subseteq A^\mathbb{Z}$. The homeomorphic zero-dimensional system $(X_\sigma, T_\sigma)$ is called the substitution dynamical system associated to $\sigma$.

**Definition 5.32.** Let $\varphi : \tilde{G} \to \tilde{G}$ be a flexible self-cover, and $\sigma : A \to A^+$, the substitution read on flexible self-cover. We assume that $A_1 \neq \emptyset$. Define the sequence $n(e) = 1$ $(e \in \tilde{E})$. Suppose that the stationary flexible covering $\tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \cdots$ generated by $\varphi : \tilde{G} \to \tilde{G}$ and the sequence $n(e) = 1$ $(e \in \tilde{E})$ has the inverse limit $\varprojlim \tilde{G} = (X, f)$. Each walk $w$ on $\tilde{G}_1$ or on $\tilde{G}$ is considered to be a word by the identification $\iota$, i.e., we define $\iota^{-1}(w) \in A^+$. Further, for each $\bar{x}[1] \in \tilde{X}_1$ and $s < t$, we define $\iota^{-1}(\bar{x}[1][s, t]) \in A^+$. We also define $\iota^{-1}(\bar{x}[1]) \in A^2$.

**Lemma 5.33.** Let $\varphi : \tilde{G} \to \tilde{G}$ be a flexible self-cover, and $\sigma : A \to A^+$, the substitution read on flexible self-cover. We assume that $A_1 \neq \emptyset$. Define the sequence $n(e) = 1$ $(e \in \tilde{E})$. Suppose that the stationary flexible covering $\tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \cdots$ generated by $\varphi : \tilde{G} \to \tilde{G}$ and the sequence $n(e) = 1$ $(e \in \tilde{E})$ has the inverse limit $\varprojlim \tilde{G} = (X, f)$. Then, there exists a canonical injection $(X_\sigma, T_\sigma) \to (\tilde{X}_1, \bar{f}_1)$.

**Proof.** We recall that there exists a corresponding weighted graph covering $\mathcal{G} : \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \mathcal{G}_2 \leftarrow \cdots$. Because $n(e) = 1$ $(e \in \tilde{E})$, each edge in $\mathcal{G}_1$ has length 1. Thus, $\tilde{X}_1$ is a sequence of elements of $\tilde{E}$. We show that for every word $w \in \mathcal{L}_\sigma$, there exists an $\bar{x}[1] \in \tilde{X}_1$ and $s < t$ such that $w = \iota^{-1}(\bar{x}[1][s, t])$. Then, for every $x \in X_\sigma$ and $s < t$, there exists an $\bar{x}_1 \in \tilde{X}_1$ and $s' < t'$ such that $x[s, t] = \bar{x}_1[s', t']$, concluding that $X_\sigma \subseteq \tilde{X}_1$. Let $w \in \mathcal{L}_\sigma$. Then, there exists an $a \in A$ and an $n \geq 1$ such that $w$ is a factor of $\sigma^n(a)$. Take the $e = \iota(a)$ from $\tilde{E}_{n+1} = \tilde{E}$. We get corresponding $e_{n+1} \in T_{n+1}$. We consider the $e_{n+1}$ as an $(n+1)$-symbol. Then, we get $e_{n+1}[1] = \sigma^n(a)$ with the identification $\iota$. By edge surjectivity of $\varphi_i$ $(i \geq 1)$, it is obvious that there exists an $\bar{x} \in \tilde{X}$ such that $\bar{x}[n+1]$ contains an edge corresponding the $\iota(a)$. Then, $\bar{x}[1]$ contains $w$ as desired. \hfill $\Box$

**Notation 5.34.** In the case $n(e) = 1$ for all $e \in \tilde{E}$, we identify $\mathcal{G}_1$ with $\tilde{G}$. Thus, in this case, each $\bar{x}[1] \in \tilde{X}_1$ can be considered to be a biinfinite walk in $\tilde{G}$ such that each $e \in \tilde{E}$ has length 1. In general case, for all $n \geq 1$ and $\bar{x} \in \tilde{X}$, the sequence $\bar{x}[n]$ can be considered to be a sequence of elements of $\tilde{E}$.

**Notation 5.35.** The injection that we have obtained by Lemma 5.33 is also denoted as $\iota: (X_\sigma, T_\sigma) \to (\tilde{X}_1, \bar{f}_1)$.

We consider an equivalent condition for $\iota$ to be bijective (cf. Proposition 5.32). The next lemma is somewhat trivial. Nevertheless, in order to carry out our intension that this section is for understanding three kinds of graph coverings, we would like to describe in detail.

**Lemma 5.36.** Let $\bar{G} : \bar{G}_0 \leftarrow \bar{G}_1 \leftarrow \bar{G}_2 \leftarrow \cdots$ be a straight stationary flexible graph covering generated by a straight flexible self-cover $\varphi : \bar{G} \to \bar{G}$ and a sequence $n(e) = 1$ for all $e \in \tilde{E}$. For every $v \in \lim \bar{V}$ and for every pair $(e_1, e_2) \in (\lim_1 \tilde{E}) \times (\lim_1 \tilde{E})$ with $r(e_1) = v = s(e_2)$, there exists an $\bar{x}[1] \in \tilde{X}_1$ such that $e_1(e_2)$ is a factor of $\bar{x}[1]$. 

Proof. From the flexible graph covering \( \tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \ldots \), we get a weighted graph covering \( \tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \tilde{G}_3 \leftarrow \ldots \). From weighted graph covering \( \tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \tilde{G}_3 \leftarrow \ldots \), we get a basic graph covering \( \tilde{G} : \tilde{G}_0 \leftarrow \tilde{G}_1 \leftarrow \tilde{G}_2 \leftarrow \tilde{G}_3 \leftarrow \ldots \).

We have defined as \( \lim \tilde{G} = \lim \tilde{G} = \lim \tilde{G} \), which we denote \((X, \tilde{f})\). From \((X, \tilde{f})\), we have considered the natural extension \((\tilde{X}, \tilde{f})\). Finally, from \((\tilde{X}, \tilde{f}), \tilde{G}\), and \(\tilde{G}\) (or \(\tilde{G}\)), we have constructed \((\tilde{X}, \tilde{f})\). Let us write as \(\tilde{G}_n = (\tilde{V}_n, \tilde{E}_n)\) for each \(n \geq 0\). For each \(n \geq 1\), there exist \(v_n \in \tilde{V}_n\) and \(e_{n,1}, e_{n,2} \in \tilde{E}_n\), which are the copies of \(v \in V\) and \(e_1, e_2 \in \tilde{E}_n = \tilde{E}\) respectively. Each of \(e_{n,1}, e_{n,2}\) has length \(\geq 1\). Let \(l_{n,1} = l(e_{n,1})\) and \(l_{n,2} = l(e_{n,2})\). In Notations \(\text{3.10}\) and \(\text{3.11}\) and Lemma \(\text{3.10}\) we have transformed \(\tilde{G}\) into \(\tilde{G}\) a basic graph covering. We have constructed vertices of basic graph \(\tilde{G}_n = (\tilde{V}_n, \tilde{E}_n)\) as follows: for each \(e \in \tilde{E}_n\), we have constructed \(V(e) := \{\bar{v}_{\bar{e},i,0} = s_{\bar{e}}(v), \bar{v}_{\bar{e},i,1}, \bar{v}_{\bar{e},i,2}, \ldots, \bar{v}_{\bar{e},i,l(e)-1}, \bar{v}_{\bar{e},i,l(e)} = r(e)\}\). For each \(e \in \tilde{E}_n\), we have produced the edges of basic graph as \(E(e) := \{((e_{i-1}, e_{i+1}) \mid 0 \leq i < l(e)\}\).

Thus, each \(e_{n,1}, e_{n,2}\) can be described as a walk \((\bar{v}_{n,i,0} = s(e_{n,i}), \bar{v}_{n,i,1}, \bar{v}_{n,i,2}, \ldots, \bar{v}_{n,i,l_{n,i}-1}, \bar{v}_{n,i,l_{n,i}} = r(e_{n,i}))\) in \(\tilde{G}_n\) for \(i = 1, 2\). Because \(v \in \lim \tilde{V}\), \((e_1, e_2) \in (\lim \tilde{E}) \times (\lim \tilde{E})\), and \(r(e_1) = v = s(e_2)\), we get \(\varphi_{e_{n,1}}(v_{n,1}) = v_n, \bar{v}_{n,1,i,n,1} = v_n = \bar{v}_{n,2,0}, (\varphi_{e_{n,2}}(v_{n,2,1}))(\text{last}) = e_{n,1}\), and \(\varphi_{e_{n,1}}(e_{n,2}))(\text{first}) = e_{n,2}\). Thus, we can define three pints \(p_1, p_2, p_3 \in X\) as \(p_1 = (\bar{v}_0, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{l_{n,1}-1}, \bar{v}_{l_{n,2}, l_{n,2}-1}, \bar{v}_{l_{n,3}, l_{n,3}-1}, \ldots)\), \(p_2 = (\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots)\), and \(p_3 = (\bar{v}_0, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \ldots)\). Because \(\varphi_{e_{n,1}}(v_{n,1}) = v_n\) and \(\varphi_{e_{n,2}}(v_{n,2,1})\) are edges of \(\tilde{G}_n\) for all \(n \geq 1\), we get \(f(p_1) = p_2\) and \(f(p_2) = p_3\). Because \(v_0\) is also considered to be a vertex in \(\tilde{V}_n\), in any levels \(n\), all \(p_1, p_2, p_3\) are not passed by a unique tower. In the form of the array system \((\bar{X}, \bar{f})\) (see \(\S 3.3\)), intuitively, there seem to be an infinite cut between \(p_1\) and \(p_2\). Concretely, in \(\bar{X}\), we can define \(\bar{x} \in \bar{X}\) such that \(\bar{x}[n][l_{n,1}, l_{n,1}-1] = (\bar{v}_{n,1,0}, \bar{v}_{n,1,1}, \bar{v}_{n,1,2}, \ldots, \bar{v}_{n,1,l_{n,1}-1})\) and \(\bar{x}[n][l_{n,2}, l_{n,2}-1] = (\bar{v}_{n,2,0}, \bar{v}_{n,2,1}, \bar{v}_{n,2,2}, \ldots, \bar{v}_{n,2,l_{n,2}-1})\). Even if \(l_{n,i} = 1\) for some \(i = 1, 2\), by the definition of \(\bar{x}\), we can conclude that \(\bar{x}(1, -1) = e_{i,1}\) and \(\bar{x}(1, 0) = e_{i,2}\). Thus, we get \(\bar{x}[-1, 0] = e_{1,2}\) by the identification \(\tilde{G} = \tilde{G}_1\). \(\square\)

Suppose that \(\iota_x\) is surjective. Then, for every such \(v, e_1, e_2\) as in the above lemma, if we define \(\iota_x(a) = e_1 (i = 1, 2)\), it follows that \(\iota(a_1 a_2) \in \varphi_{\iota_x}(\sigma)\). In addition, if \(\iota_x(1) = e_1, \iota_x(1, s + 1) = e_2\), then for any \(L_1 > 0\) and \(L_2 > 0\), it follows that the block \(\iota_x[1][s - L_1, s + 1 + L_2] \subseteq \varphi_{\iota_x}(\sigma)\) by the identification with \(\iota\), i.e., there exists an \(a \in A\) and \(n \geq 1\) such that \(\iota^{-1}(\iota_x[1][s - L_1, s + 1 + L_2]) < \sigma^n(a)\).

**Lemma 5.37.** Let \(\varphi : \tilde{G} \rightarrow \tilde{G}\) be a straight flexible self-cover, and \(\sigma : A \rightarrow A^+\), the substitution read on flexible self-cover. We assume that \(A_1 \neq \varnothing\). Suppose that \(\iota_x\) is surjective. Then, it follows that for every \(v \in \lim \tilde{V}\), \(e_1 \in \lim \tilde{E}\), and \(e_2 \in \lim \tilde{E}\) with \(r(e_1) = v = s(e_2)\), there exist an \(e \in \tilde{E}\) and an \(n > 0\) such that \(e_1 e_2\) is a sub-walk of \(\varphi^n(e)\).

**Proof.** Let \(v \in \lim \tilde{V}\), \(e_1 \in \lim \tilde{E}\), and \(e_2 \in \lim \tilde{E}\) with \(r(e_1) = v = s(e_2)\). For every \(n > 0\), there exists a copy walk \(e_{n,1} e_{n,2}\) on \(\tilde{G}_n\) of \(e_{1,2}\). For all \(m \geq n > 0\), it follows that \(\varphi_{e_{n,1}}(e_{m,1})(\text{last}) = e_{n,1}\) and \(\varphi_{e_{n,2}}(e_{m,2})(\text{first}) = e_{n,2}\). As in the proof of Lemma \(\text{5.30}\), it follows that there exists an \(\bar{x} \in \bar{X}\) such that \(\bar{x}[1][l_{n,1}, l_{n,1}-1] = (e_{n,1}, e_{n,2})\). Let \(\iota(a_1) = e_1\) and \(\iota(a_2) = e_2\). Because of surjectivity of \(\iota_x\), we get \(a_1 a_2 \in \varphi_{\iota_x}(\sigma)\), i.e., there exists an \(a \in A\) and \(n \geq 0\) with \(a_1 a_2 < \sigma^n(a)\). Let \(\iota(a) = e\). Then, we get \(e_1 e_2\) is a sub-walk of \(\varphi^n(e)\). \(\square\)

The condition in the above lemma is not a sufficient condition for \(\iota_x\) to be surjective.
Definition 5.38. Let $\varphi : G \to \tilde{G}$ be a straight flexible self-cover. A finite sequence $(v_1, v_2, \ldots, v_k)$ with $v_i \in \text{lim } \tilde{V}$ ($1 \leq i \leq k$) is constant if there exists a sequence $e_1, e_2, \ldots, e_{k-1} \in \tilde{E}$ such that $v_i = s(e_i)$ and $r(e_i) = v_{i+1}$ for all $1 \leq i < k$, and the walk $w := (e_1, e_2, \ldots, e_{k-1})$ satisfies $\varphi(w) = w$. Note that, the above $e_i$’s do not change its length, even when they are realised in $\overline{G_n}$ for $n > 1$. We also make a convention that a sequence $(v)$ (of length 1) with $v \in \text{lim } \tilde{V}$ is also constant in the above definition.

Remark 5.39. Let $\varphi : G \to \tilde{G}$ be a straight flexible self-cover and $(v_1, v_2, \ldots, v_k)$ be a constant sequence. By definition, there exists a walk $w = (e_1, e_2, \ldots, e_{k-1})$ on $\tilde{G}$ such that $v_i = s(e_i)$ and $r(e_i) = v_{i+1}$ for all $1 \leq i < k$ and $\varphi(w) = w$. Because of +directionality condition, such walk $w$ has to be unique.

Definition 5.40. Let $(v_1, v_2, \ldots, v_k)$ be a constant sequence. Let $w = (e_1, e_2, \ldots, e_{k-1})$ be the unique walk on $\tilde{G}$ with $v_i = s(e_i)$ and $r(e_i) = v_{i+1}$ for all $1 \leq i < k$ and $\varphi(w) = w$. Then, we say that $(v_1, v_2, \ldots, v_k)$ is overlapped, if for every $e_0 \in \text{lim}_l \tilde{E}$ with $r(e_0) = v_1$ and unique $e_k \in \text{lim}_l \tilde{E}$ with $s(e_k) = v_k$, there exists an $e \in \tilde{E}$ and $n \geq 1$ such that $\varphi^n(e)$ has a sub-walk $w' = (e_0, e_1, \ldots, e_k)$.

Lemma 5.41. Let $\varphi : G \to \tilde{G}$ be a straight flexible self-cover. Suppose that $\iota_\infty$ is surjective. Then, every constant sequence $(v_1, v_2, \ldots, v_k)$ is overlapped.

Proof. Let $(v_1, v_2, \ldots, v_k)$ be a constant sequence. In Lemma 5.37 we have already proved the case in which $k = 1$. We assume $k \geq 2$. Let $w = (e_1, e_2, \ldots, e_{k-1})$ be the unique walk on $\tilde{G}$ such that $v_i = s(e_i)$ and $r(e_i) = v_{i+1}$ for all $1 \leq i < k$ and $f(w) = w$. Because $\varphi(w) = w$, it follows that $\varphi(e_i) = e_i$ for all $1 \leq i < k$. Take an arbitrary $e_0 \in \text{lim}_l \tilde{E}$ and the unique $e_k \in \text{lim}_l \tilde{E}$ such that $r(e_0) = v_1$ and $s(e_k) = v_k$. Then, for all $n > 0$, there exist the copies $e_{n,i}$ of $e_i$ for all $0 \leq i \leq k$. It follows that $\varphi_{n+1}(e_{n+1,i}) = e_{n,i}$ for all $1 \leq i \leq k-1$, $\varphi_{n+1}(e_{n+1,0})$ (last) $= e_{n,0}$, and $\varphi_{n+1}(e_{n+1,k})$ (first) $= e_{n,k}$. Let $w_n = e_{n,1}e_{n,2}\cdots e_{n,k-1}$. Obviously, there exists an $\bar{x} \in \tilde{X}$ and sequences $\cdots s_2 \leq s_1 < s_0 < t_0 < t_1 < t_2 \leq \cdots$ such that for each $n > 0$, $\bar{x}[n][s_0,t_0] = w_n$, $\bar{x}[n][s_0,s_0-1] = e_{n,0}e_{n,0}\cdots e_{n,0}$, and $\bar{x}[n][t_0+1,t_1] = e_{n,k}e_{n,k}\cdots e_{n,k}$. Evidently, it follows that $\bar{x}[1] \in \tilde{X}_1$. Now, because the assumption that $\iota_\infty$ is surjective, all the finite sub-blocks of $\iota_\infty^{-1}(\bar{x}[1])$ are in $\mathcal{L}(\sigma)$, especially, we get $\iota^{-1}(\bar{x}[1][s_1,t_1]) \in \mathcal{L}(\sigma)$. Thus, there exists an $a \in A$ and $n > 0$ such that $\iota^{-1}(\bar{x}[1][s_1,t_1]) < \sigma^n(a)$. Let $e = \iota(a)$. Obviously, it follows that $\varphi^n(e)$ overlaps $(v_1, v_2, \ldots, v_k)$.

Proposition 5.42. Let $\varphi : G \to \tilde{G}$ be a straight flexible self-cover. The map $\iota_\infty$ is bijective if and only if all the constant sequences are overlapped.

Proof. By Lemmas 5.33 and 5.41 and Notation 5.35 we only need to show that $\iota_\infty$ is surjective if every constant sequence $(v_1, v_2, \ldots, v_k)$ is overlapped. Thus, suppose that every constant sequence $(v_1, v_2, \ldots, v_k)$ is overlapped. To show that $\iota_\infty$ is surjective, take an arbitrary $\bar{x}[1] \in \tilde{X}_1$ and $s < t$. We need to show that there exists an $e \in \tilde{E}$ and $n > 0$ such that $\bar{x}[1][s,t]$ is a sub-walk of $\varphi^n(e)$. We consider an array system $\bar{x}$. We say that there exists an infinite cut at position $i$ if for all $n \geq 0$, there exist $n$-cut at position $i$ (just before the position $i$). Suppose that there exist no infinite cut in $(s,t)$. Then, there exists an $n > 0$ such that there exist no $n$-cut in $(s,t)$. We find that $(e_n, e_{n}, \ldots, e_n) = \bar{x}[n][s,t]$ for some $e_n \in \tilde{E}_n$. Let $e_n$ be a copy of $e \in \tilde{E}$. Then, it is evident that $\varphi^{n-1}(e)$ contains $\bar{x}[1][s,t]$.
as a sub-block. Suppose that there exists only one infinite cut in \((s, t]\). Let \(i_0 \in (s, t]\) be the position of the infinite cut. There exists an \(n > 0\) such that there exists no \(n\)-cut in \((s, i_0 - 1) \cup [i_0 + 1, t]\). We find that \((e_{n,1}, e_{n,2}, \ldots, e_{n,2n}) = \bar{x}[n][s, t]\) for some cut \(i_0\) with \(e_{n,1}, e_{n,2} \in \bar{E}_n\). Let \(e_{n,1}, e_{n,2}\) be the copy of \(e_1(n), e_2(n) \in \bar{E}\) respectively. For every \(m\) with \(m > n\), it follows that \(\varphi(e_1(m))\) (last) = \(e_1(m - 1)\) and \(\varphi(e_2(m))\) (first) = \(e_2(m - 1)\). Thus, by straightness of the flexible self-cover, there exist \(e_1, e_2 \in \bar{E}\) such that \(e_{m,1}'s\) are copies of \(e_1 \in \lim_i \bar{E}\) and \(e_{m,2}'s\) are copies of \(e_2 \in \lim_j \bar{E}\). Further, we get \(v = \tau(e_1) = s(e_2) \in \lim \bar{V}\) and the sequence \((v)\) of length 1 is a constant sequence. By the overlapping property, we find that there exists an \(e \in \bar{E}\) and an \(n' > 0\) such that \(\varphi^{n'}(e)\) contains \(e_1e_2\). Now, we get \(\varphi^{n' + n - 1}(e)\) contains \(\bar{x}[1][s, t]\). Finally, suppose that there exist at least 2 positions of infinite cuts in \((s, t]\). Let \(i_1\) be the first position and \(i_2\), the last position of the infinite cut in \((s, t]\). Let \(n > 1\) be such that for all \(m \geq n\) the positions of \(m\)-cuts in \((s, t]\) is equal to the positions of the infinite cuts in \((s, t]\). Then, we find that there exist \(e_{n,1}, e_{n,2} \in \bar{E}_n\) such that \((e_{n,1}, e_{n,2})\) (first cut) \(\bar{x}[n][i_1, i_2 - 1]\), (last cut) \(e_{n,2}, e_{n,2}, \ldots, e_{n,2}) = \bar{x}[n][s, t]\). From \(\bar{x}[n][i_1, i_2 - 1]\) with \(m \geq n\), we can get a walk that is denoted as \(w_m\). Thus, we get \(\varphi_m(m)(w_m) = w_{m'}\) for all \(m > m' \geq n\) and \(\varphi_m(m)(w_m) = \bar{x}[1][i_1, i_2 - 1]\) for all \(m \geq n\). From this and by the assumption that the \(m\)-cuts do not change for \(m \geq n\), each edge that consists of \(w_m\) is mapped to a single edge of \(w_{m'}\) for \(m > m' \geq n\). By straightness of the flexible self-cover, all \(w_m\) \((m \geq n)\) are identically copies of \(w\) for some fixed walk \(w\) on \(\bar{G}\); particularly, we also get \(\varphi(w) = w\). Let \(e_{n,i}\) \((i = 1, 2)\) be copies of \(e_i\) \((i = 1, 2)\). Then, by straightness of the flexible self-cover, it follows that \(e_1 \in \lim_i \bar{E}\) and \(e_2 \in \lim_j \bar{E}\). Thus, by overlapping property, there exists an \(e \in \bar{E}\) and \(n' > 0\) such that \(\varphi^{n'}(e)\) contains a sub-walk \(e_1 \bar{w} e_2\). It follows that \(\varphi^{n' + n - 1}(e)\) has a sub-walk \(\bar{x}[1][s, t]\), as desired. 

### 5.3 Examples

In the previous section, we get a condition that can represent some substitution dynamical systems as \((\bar{X}_1, \bar{f}_1)\) of some stationary flexible graph covering. In this section, we give two substitution systems that satisfy this condition. The Fibonacci sequence is made by the substitution \(\tau(a) = ab\) and \(\tau(b) = a\) (cf. for example [Fog22, Definition 2.6.1.]). To describe by flexible self-cover \(\varphi : \tilde{G} \to \tilde{G}\), let \(\tilde{V} = \{v\}\), \(\tilde{E} = \{e_a, e_b\}\), and \(\phi(e_a) = e_a e_b\), \(\phi(e_b) = e_a\). It is easy to check that \(\phi\) is +directional. Unfortunately, this is not straight. We consider \(\phi^2(e_a) = \phi(e_a e_b) = e_a e_a e_b e_a\) and \(\phi^2(e_b) = \phi(e_a) = e_a e_b\), and define \(\varphi := \phi^2\). We get \(\varphi(e_a) = e_a e_b e_a\) and \(\varphi(e_b) = e_a e_b\) (see Figure 5.42). It is easy to check that \(\varphi : \bar{G} \to \bar{G}\) is straight. It is evident that \(\lim \bar{V} = \{v\}\), \(\lim_i \bar{E} = \{e_a, e_b\}\), and \(\lim_j \bar{E} = \{e_a\}\). Because \(\varphi^n(e_x) \neq e_x\) for \(n > 0\) and \(x = a, b\), the only constant sequence is \((v)\). To apply Proposition 5.42, we have to check that both \(e_a e_a\) and \(e_b e_a\) appear in \(\varphi^n(e)\) for some \(e \in \bar{E}\) and \(n > 0\). The \(e_b e_a\) has already appeared in \(\varphi(e_a)\). We compute as \(\varphi^2(e_a) = \varphi(e_a e_b e_a) = e_a e_b e_b e_a = e_b e_a = e_b e_a e_b e_b e_a\). Thus, \(e_a e_a\) has also appeared. The related substitution read is \(\sigma(a) = aba\), \(\sigma(b) = ab\). In this way, we have shown that \(e_a : (X_\sigma, T_\sigma) \to (\bar{X}_1, \bar{f}_1)\) is an isomorphism. Because the powers \(\varphi^i\) are not bidirectional, for the stationary flexible graph covering \(\tilde{G} : \tilde{G}_0 \xleftarrow{e_1} \tilde{G}_1 \xleftarrow{e_2} \tilde{G}_2 \xleftarrow{e_3} \cdots\) generated by \(\varphi\) and the sequence \(n(e_a) = n(e_b) = 1\), the inverse limit \(\lim \tilde{G}\) which we denote \((X, f)\) is not a homeomorphism. Thus, \((X, f)\) is not topologically conjugate to \((\bar{X}, \bar{f})\) nor \((\bar{X}_1, \bar{f}_1)\), which is topologically conjugate to \((X_\sigma, T_\sigma)\) by Proposition 5.42. We do not study much deeper, e.g., the coincidence of \((\bar{X}, \bar{f})\) with \((\bar{X}_1, \bar{f}_1)\) or \((X_\sigma, T_\sigma)\) is not studied here. For
Thus, this straight flexible self-cover satisfies the overlapping property. From this fact, see that just can see to consider walks: the constant sequences are only\[ \lim_{r \to \infty} \frac{\phi}{\hat{\phi}} = \frac{\phi}{\hat{\phi}} \]the stationary flexible covering condition is easy to check. Because of bidirectionality condition, if we construct exist six edges as Figure 5. And, \[
abla \{ v_l, v_m, v_r \} \]. There exist six edges as \( \hat{E} = \{ e_a, e_b, e_c, e_d, e_e, e_f \} \). The substitution read on \( \varphi \) is written as in Figure 5. And, \( \varphi \) is defined as \( \varphi(e_a) = e_a, \varphi(e_b) = e_a e_b e_d e_e, \varphi(e_c) = e_d e_a e_c e_a, \varphi(e_d) = e_d e_c e_d e_f, \varphi(e_e) = e_f e_c, \) and \( \varphi(e_f) = e_f \). The bidirectionality condition and the straightness condition is easy to check. Because of bidirectionality condition, if we construct the stationary flexible covering \( \hat{G} : \hat{G}_0 \xrightarrow{\varphi_1} \hat{G}_1 \xrightarrow{\varphi_2} \hat{G}_2 \xrightarrow{\varphi_3} \cdots \) and its inverse limit \[ \lim_{r \to \infty} \hat{G} = (X, f) \], then we get \( (X, f) \) is isomorphic to the natural extension \( (\hat{X}, \hat{f}) \) and to the array system \( (\hat{X}, \hat{f}) \).

With this definition, we get \[ \lim_{r \to \infty} \hat{V} = \hat{V} = \{ v_l, v_m, v_r \}, \lim_{r \to \infty} \hat{E} = \{ e_a, e_c, e_d, e_f \} \), and \[ \lim_{r \to \infty} \hat{E} = \{ e_a, e_c, e_d, e_f \} \]. The edges \( e \in \hat{E} \) that satisfy \( \varphi(e) = e \) are \( e_a, e_f \). It follows that the constant sequences are only \( (v_l, v_1, \ldots, v_l) \) with the length \( \geq 1 \), \( (v_m) \), and \( (v_r, v_r, \ldots, v_r) \) with the length \( \geq 1 \). To check the overlapping property, we only need to consider walks: \[ e_a e_a \cdots e_a \]for \( (v_l, v_1, \ldots, v_l) \) with the length \( \geq 1 \), \( e_c e_d \) for \( (v_m) \), and \[ e_f e_f \cdots e_f \]for \( (v_r, v_r, \ldots, v_r) \) with the length \( \geq 1 \). To check \( e_a e_a \cdots e_a \), we compute
\[
\varphi^2(e_b) = e_a e_a e_b \cdots, \varphi^4(e_b) = e_a e_a e_a e_a e_a e_b \cdots, \text{ and so on. To check } e_c e_d, \text{ we just can see } \varphi(e_d) = e_d e_c e_d e_f. \text{ To check } e_f e_f \cdots e_f, \text{ we compute } \varphi^n(e_a) = e_f e_f \cdots e_f e_c. \]
Thus, this straight flexible self-cover satisfies the overlapping property. From this fact, \( (X_\sigma, T_\sigma) \) is isomorphic to \( (\hat{X}, \hat{f}) \). From the calculation \( \sigma^n(a) \) and \( \sigma^n(f) \), we easily see that \( (X_\sigma, T_\sigma) \) has fixed points \( (\ldots, a, a, a, \ldots) \) and \( (\ldots, f, f, f, \ldots) \). If we construct \( \overline{G} : \overline{G}_0 \xrightarrow{\varphi_1} \overline{G}_1 \xrightarrow{\varphi_2} \overline{G}_2 \xrightarrow{\varphi_3} \cdots \) from \( \hat{G} \), then the copies \( e_{n,b}, e_{n,c}, e_{n,d}, e_{n,e} \) on \( \overline{G}_n \) of \( e_b, e_c, e_d, e_e \) for \( n > 0 \) have lengths \( \to \infty \) for \( n \to \infty \).

\[
\sigma(a) = aba, \ \sigma(b) = ab
\]
\[
e_a = \iota(a) \quad e_b = \iota(b)
\]

**Figure 4.** Self-cover for the Fibonacci sequence.

\[
\sigma(a) = a, \ \sigma(b) = ab, \ \sigma(c) = dec, \quad \sigma(d) = def, \ \sigma(e) = fe, \ \sigma(f) = f
\]

**Figure 5.** Self-cover with fixed points.

The next example is a straight flexible self-cover \( \varphi : \hat{G} \to \hat{G} \) that constructs a substitution system with two fixed points. See Figure 5. The vertices of the graph \( \hat{G} = (\hat{V}, \hat{E}) \) consists of left vertex \( v_l \), middle vertex \( v_m \), and right vertex \( v_r \), i.e., \( \hat{V} = \{ v_l, v_m, v_r \} \). Figure 5. Self-cover with fixed points.

topological conjugacy between substitution systems and stationary Bratteli diagrams, we refer the reader to [BKM09] [DHS99] [Ya09].
For this example, it is possible to conclude that $(\hat{X}, \hat{f})$ is isomorphic to $(\hat{X}_1, \hat{f}_1)$ as the next lemma shows. By this, we can conclude that $\lim \hat{G} = (X, f)$ is isomorphic to $(X_\sigma, T_\sigma)$.

**Lemma 5.43.** According to flexible self-cover in Figure 6, it follows that $(X_\sigma, T_\sigma)$ is isomorphic to $(\hat{X}, \hat{f})$.

**Proof.** We show a proof by showing that for all $n \geq 1$, from any $\tilde{x}[n] \in \tilde{X}_n$, we can recover $\tilde{x}[n + 1]$ in a unique way. Because the way we use here is the same for all $n \geq 1$, we only show the case in which $n = 1$. Let $x_1 = \tilde{x}[1]$. Firstly, suppose that only $e_a$ appears on $x_1$. Then, clearly we find unique $\tilde{x} \in \tilde{X}$. The same holds true in the case that only $e_f$ appears on $x_1$. Secondly, suppose that $e_b$ appeared in $x_1$ at the position $i$. Then, it follows that $x_1[i - 1, i + 2] = e_a e_b e_d e_c$. Thus, it follows that $\tilde{x}[2][i - 1, i + 2] = e_2 b e_2 e_2 e_2 b$. Again, we get an occurrence of a copy of $e_b$ in $\tilde{x}[2]$ in $[i - 1, i + 2]$. From which, we can clarify all the $\tilde{x}[2][i - 4, i + 2 + 8]$. Further, we get $\tilde{x}[3][i - 5, i + 10] = e_3 b \cdots e_3 b$. In this way, we can identify all of $\tilde{x}[2]$. Thirdly, suppose that $e_c$ appeared in $x_1$. Because it must be the form $e_d e_c e_c e_a$, with the identical argument, we can identify the unique $\tilde{x} \in \tilde{X}$. Thus, from here, we assume that $x_1$ does not contain either $e_b$ nor $e_c$. In this case, $e_a$ does not appear neither. Thus, from here, we need to consider the case in which there appear only $e_d$, $e_c$ and $e_f$. In this case, in $\tilde{x}[2]$, only $e_2 d$, $e_2 e$ and $e_2 f$ appears. In $x_1$, the edge $e_c$ appears in the forms $e_f e_c$ or $e_d e_c e_d e_f$. This form can easily be identified by looking just the left side of $e_c$. To decode this, firstly, we decode all the $e_d e_c e_d e_f$, to get $e_d e_d e_d e_d$ at the same position in $\tilde{x}[2]$. It is easily checked that another possibility does not exist. All the other occurrence of $e_c$ is remained in the form $e_f e_c$. Thus, these occurrences are decoded in $\tilde{x}[2]$ as $e_c e_c$. Thus, we could have decoded into $\tilde{x}[2]$ uniquely at the positions on which $e_c$ appear. There does not remain any one of $e_d$. The remaining $e_f$ in $\tilde{x}[1]$ is decoded as $e_f$ in $\tilde{x}[2]$ at the same position. Thus, from $x_1$ we could have recovered $\tilde{x}[2]$ uniquely, as desired.

In §4, we have made some link between the Bratteli–Vershik model and our graph covering models. Nevertheless, it might be the case that the satisfactory normality condition on Bratteli–Vershik models should not be $L$-periodicity-regulation property. We believe to attain such perception, a different approach is needed, even if it uses graph coverings. In §5, we only could give two examples for the better understanding of our graph coverings. Of course, there remains a lot to do on this topic. It might be the case that there exist a lot of substitutions that do not come from the way of flexible self-covers (see Definition 5.22). Further, we could not clarify the substitution dynamical systems that are canonically identical with $(\hat{X}_1, \hat{f}_1)$. The topological rank that had been found for homeomorphic Cantor minimal systems by Downarowicz and Maass in [DM08] can be extended for all homeomorphic zero-dimensional systems. In this regard, we have not yet solved the next:

**Question.** Does there exist a Cantor minimal homeomorphism whose topological ranks differ.

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