Non-smooth Modeling and Variable Structure Control of a Class of Hybrid Dynamical Systems

Yasser A. Bin Salamah
Department of Electrical Engineering, King Saud University, Riyadh, Kingdom of Saudi Arabia
E-mail: ybinsalamah@ksu.edu.sa

Abstract. In this work, we propose a modeling formulation and controller design for a class of hybrid dynamical systems. In this formulation, a switching dynamical system is modeled as a dynamical system with discontinuous right hand side. More specifically, the system is transformed to a nonlinear system with discontinuous nonlinearities. Then, a synthesis of feedback linearization and sliding mode control is employed for output tracking control problem. Application and implementation of this approach is illustrated via a chemical process example.

1. Introduction

Hybrid Systems are a class of dynamical systems whose behavior exhibits both continues time and discrete events systems. Applications of hybrid systems include, but not limited to, process control, automotive and transportation systems, manufacturing systems, etc. As a results, researchers have given a considerable attention to the molding [15, 6] and control [2, 3, 5] of Hybrid systems. Switched system is an important class of hybrid systems which can be found in many fields i.e. Process Control [11, 1], air traffic control [13], and power electronics [4]. It is characterized by a set of switching rule and a set of continuous dynamical systems, typically finite, in which each system is described by a set of deferential equations. The switching event can be classified into [12]:

- State dependent versus time dependent.
- Autonomous versus controlled switching.

In this paper, we consider a system with a state dependent switching rule. A simple example can be given by $\dot{x} = x$ when $x < 0$ and $\dot{x} = -x$ when $x > 0$ or equivalently

$$\dot{x} = -x \text{sgn}(x).$$

On the other hand, the above example can be seen as a dynamical system with nonlinear discontinuities (the signum function). Such discontinuities may induce undesirable sliding modes. In this case, conventional control techniques can not be rapidly applied, especially if there is a physical limitation to the system input. Hatipoglu and Ozguner [8, 9] have given substantial consideration to the modeling and control of non-smoothness in the nonlinear systems. Applications to their results found in many industrial applications, see for example [14, 16, 10].
Although many research have been carried out on modeling and control of hybrid and non-smooth systems, there have been few empirical investigation into combining the results from both areas. In this article, we extend the previous works of [8, 9] by showing that: (1) a class of hybrid systems can be modeled as nonlinear systems with discontinuous nonlinearities, thus allowing for (2) a design of robust control methodology for the result system. Accordingly, a synthesis of variable structure and feedback linearization control is deployed to stabilized the system even if undesired sliding mode exists.

The remainder of the paper is organized as follows: The system description and problem statements outlined in section 2. In section 3, the proposed modeling approach and controller design is given. In section 4, a numerical example is given to illustrate the main results.

2. Preliminaries and Problem Statement

2.1. Nonlinear Discontinuities
Inherent discontinuous nonlinearities, such as: backlash, dead-zone, nonlinear friction, etc. are common in many physical systems. [8] show that many discontinuous nonlinearities can be represented as the sum of translated and/or scaled versions of signum functions.

**Example:** Consider the saturation function $Sat(\cdot)$, a well-known non-linearity which can be seen, for example, in magnetizing curve of DC motor. The function is described as:

$$Sat(u, u_{\min}, u_{\max}) = \begin{cases} u_{\min} & u < u_{\min} \\ u & u_{\min} \leq u \leq u_{\max} \\ u_{\max} & u > u_{\max} \end{cases}$$

which can also be rewritten as:

$$Sat(u, u_{\min}, u_{\max}) = \frac{1}{2}(u_{\min} + u_{\max}) - \left| (u - u_{\max}) + \left| (u - u_{\min}) \right| \right|$$

$$= \frac{1}{2}\{u_{\min} + u_{\max} - (u - u_{\max})\text{sgn}(u - u_{\max}) + (u - u_{\min})\text{sgn}(u - u_{\min})\}$$

The reader is referred to [7] for a detailed exposition and more examples. In this work, model the system switching behavior by inducing an intentional nonlinearity in the system model, allowing for the design of robust control for the system.

2.2. System Description and Problem Statement
In this paper, we consider a class of hybrid nonlinear systems given by a set of $N$ Single-Input Single-Output (SISO) subsystems, representing the operational modes. Each mode $i \in I = \{1, 2, ..., N\}$ is defined by set of differential equations. Moreover, for each mode $i$ the operation region is polyhedral defined by matrices $H_i$ and $b_i$ as follows:

$$\Omega_i = \{x \text{ such that } H_i x \geq b\}.$$ 

When the $i$ mode is active, the continuous state evolves according to the following differential equations:

$$S_i : \quad x^{(n)} = f_i(x) + g(x)u \\
y = x$$

(3)
where $x \in \mathbb{R}^n$ is the state, $u, y \in \mathbb{R}$ are the system input and system output, respectively. $f_i(\cdot) : \mathbb{R}^n \to \mathbb{R}$ and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}$ are known smooth and differentiable functions, not necessarily linear.

**Assumption 2.1:** For each $i$ the function $g(.) \neq 0$, for controllability condition over the entire state space.

**Assumption 2.2:** Each set $\Omega_i$ is a convex polyhedron such that $\Omega_i \cap \Omega_i = \phi \ \forall \ i \neq j$ and $\bigcup_{i=1}^n \Omega_i = R^n$. That is, the operation modes does not overlap. System operates in mode $i$ will remains in mode $i$ until it cross the surface $h_j(x)$ and switched to mode $j$.

**Assumption 2.3:** It is assume that the impulse effects is absent. Meaning that, the rest map is the identity and the trajectory is continuous everywhere.

It is desired to formulate the class of hybrid systems described by $3$ as a dynamical system with discontinuos right hand side and to develop a robust control for output tracking using the new formulation.

**Remark 2.1:** Based on the above description and assumption, it is clear that we are considering a state-dependent with uncontrolled switching system.

### 3. Main results

#### 3.1. Non-smooth Modeling for Switching Systems

This section outline the modeling formulation for switching systems described in section 2. In order to model the given switching system (3) as a continues system with discontinuous right hand side we, first, state the definition of signum function, $\text{sgn}(\cdot)$:

$$\text{sgn}(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
-1 & \text{if } x < 0,
\end{cases} \quad (4)$$

Without loss of generality, assume we have a switching system (3) with two operating modes. Then, using the notation of the nonlinear discontinuities, the switching system can be represented as a shifted and scaled version of the signum function as follows:

$$x^{(n)} = \frac{1}{2} + \frac{1}{2} f_1(x) \text{sgn}(h_1(x)) + \frac{1}{2} f_2(x) \text{sgn}(h_2(x)) + g(x)u$$

$$= \frac{1}{2} f_1(x)(1 + \text{sgn}(h_1)) + \frac{1}{2} f_2(x) \text{sgn}(h_2(x)) + g(x)u$$

where

$$h_i(x) = H_i x - b_i.$$ 

Hence, a systrm with $N$ subsystems can be represented as

$$x^{(n)} = \lambda_1(f_1(x)) + \lambda_2(f_2(x)) + ... + \lambda_n(f_n(x)) + g(x)u$$

$$= \sum_{i=1}^N \lambda_i(f_i(x)) + g(x)u \quad (5)$$

where

$$\lambda_i = \frac{1}{2}(1 + \text{sgn}(h_i))$$

This result can be summarized in the following proposition.

**Proposition 1** For the class of Hybrid System described in 3 and satisfies assumptions 1, 2, and 3, if for each operation region (mode) $\Omega_i$ we define a correspondence switching surface defied by $h_i(x) = H_i x - b$, then the system can be modeled as the sum of translated and scaled version of the signum functions 5.
Remark 3.1.1: Although we consider a class of system which has a state dependent switching. The considered switching system can be translated to a class of switched system with switching signal \( \sigma_i = h_i(x) \).

Remark 3.1.2: The switching surfaces defined by the \( h_i \) can induced undesirable sliding mode. This problem will be solved by the control in next section.

3.2. Controller Design
In this section, we overview the controller design procedure for output tracking i.e. the system output \( y = x \) tracks the reference signal \( x_d \). A synthesis of control laws based on feedback linearization and variable structure control is employed in two steps. In the first step, the desired control input \( u_d \) for an output tracking problem is obtained using the feedback linearization. Second, an auxiliary input \( z \) along with switching surface \( s \) are introduced to ensure the convergence of \( u_d \rightarrow u \).

First step:
Let \( x_d \) be the desired state and define the error as

\[
e = x - x_d.
\]

Then, the desired input \( u_d \) can be obtained using the feedback linearization approach as:

\[
u_d = \frac{1}{g(x)} (x_d^{(n)} - K_1 e^{(n-1)} - ... - K_n e - \sum_{i=1}^{n} \lambda_i (f_i(x)))
\] (6)

in which the tracking error dynamics,

\[
e^{(n)} + K_1 e^{(n-1)} + ... + K_n e = 0
\]

can be made asymptotically stable by appropriate choice of \( K_i \).

Second step:
Define the sliding surface:

\[
s(t) = u - u_d
\]

with the auxiliary input \( z \) defined by:

\[
z = \dot{u}
\] (7)

Then,

\[
\dot{s} = \dot{u} - \dot{u}_d = z - \dot{u}_d.
\]

Then, the variable structure law for \( v \) is selected as:

\[
z = -M \text{sgn}(s)
\] (8)

Accordingly, sliding mode existence condition i.e. \( s\dot{s} < 0 \) is ensured if

\[
M > |\dot{u}_d| + \epsilon
\] (9)

for some \( \epsilon > 0 \) and the system will start to slide on the manifold \( s = 0 \) in finite time. As a result \( u \rightarrow u_d \) and hence \( x \rightarrow x_d \).
4. Application to a chemical process example

In this section, we illustrate the application and the implementation of the proposed modeling formulation and controller design to a chemical process. Consider a Continuous Stirred Tank Reactor (CSTR), shown in Figure 1, where an exothermic, first-order, irreversible reaction takes place:

\[ A \xrightarrow{k} B \]

Based on the reaction temperature \( T \), in order to avoid secondary reactions to occur, the Standard Operating Procedure (SOP) requires switching between two operational modes; the first mode is to feed pure \( A \) at a flow rate \( F_1 \), concentration \( C_A^1 \), and temperature \( T_1 \). Similarly, the second mode is to feed pure \( A \) but with different flow rate \( F_2 \), concentration \( C_A^2 \), and temperature \( T_2 \). Since a higher level controller control the switching, the switching between modes considered uncontrollable. The mathematical model for the reaction process is given in [1] by:

\[
\begin{align*}
\dot{C}_A &= \frac{F_i}{V}(C_{Ai} - C_A) - k_i \exp\left(-\frac{E}{RT}\right)C_A \\
\dot{T} &= \frac{F_i}{V}(T_{Ai} - T) + \frac{-\Delta H_r}{\rho c_p}k_i \exp\left(-\frac{E}{RT}\right)C_A + \frac{Q}{\rho c_p V}
\end{align*}
\]  

(10)

where \( C_A \) denotes the concentration of product \( A \), \( T \) denotes the reactor temperature, \( Q \) denotes the rate of input/removal from the reactor, \( V \) denotes the reactor volume, \( k_i, E, \Delta H \) denote the pre-exponential constant, the activation energy, and the enthalpy of reaction, \( c_p, \rho \) denote the heat capacity and density of the fluid in the reactor, respectively. \( i \in \{1, 2\} \) with

\[
mode = \begin{cases} 
1 & \text{when } 0 \leq T < 400K \\
2 & \text{when } T \geq 400K 
\end{cases}
\]  

(11)

The process parameters are given in table 1.

The control objective is to track a desired reaction temperature profile \( T_d(t) \) by manipulating the rate hate input \( Q \) from a given initial condition \((C_{A0}, T_0)\).

4.1. Modeling

For modeling, we define our state in two dimension with \( x_1 = C_A, x_2 = T \) and the control input \( u_1 = 0, u_2 = Q \) and the output \( y = x_2 = T \). Then, following the formulation procedure in section 3, the system (3) can be written as:

\[ \dot{x} = f_i(x) + g(x)u \]
Table 1. The Process Parameters

| Parameter | Value       | Unit      |
|-----------|-------------|-----------|
| $F_1$     | $2.77 \times 10^{-5}$ | m$^3$/s   |
| $F_2$     | $5.5677 \times 10^{-5}$ | m$^3$/s   |
| $C_{A1}$  | 0.577       | kmol/m$^3$|
| $C_{A2}$  | 2           | kmol/m$^3$|
| $T_1$     | 395.3       | K         |
| $T_2$     | 350         | K         |
| $V$       | 0.1         | m$^3$     |
| $k_1$     | $2.00 \times 10^7$ | s$^{-1}$  |
| $k_2$     | $2.00 \times 10^7$ | s$^{-1}$  |
| $\Delta H$ | -400       | kJ/kmol   |
| $R$       | 8.314       | kJ/kmol.K |
| $E$       | $8.314 \times 10^4$ | kJ/kmol   |
| $c_p$     | 0.002       | kJ/kg.K   |
| $\rho$    | 1000        | kg/m$^3$  |

with

\[
 f_i(x) = \begin{bmatrix} f_1^i \\ f_2^i \end{bmatrix} = \begin{bmatrix} \frac{E_i}{V} (C_{Ai} - x_1) - k_i \exp(-\frac{E}{RT}) x_1 \\ \frac{E_i}{V} (T_{Ai} - x_2) + \frac{\Delta H_i}{\rho c_p} k_i \exp(-\frac{E}{RT}) x_1 \end{bmatrix}
\]

and

\[
 g(x) = \begin{bmatrix} g_1^i(x) \\ g_2^i(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\rho c_p V} \end{bmatrix}
\]

In addition, the switching surfaces can be defined as:

\[
 h_1(x_2) = -T + 400 - \epsilon
\]

for some $\epsilon > 0$ and

\[
 h_2(x_2) = T - 400.
\]

Hence, the switched reaction system can be translated to:

\[
 \dot{x} = \lambda_1 f_1(x) + \lambda_2 f_2(x) + g(x)u.
\]

4.2. Control input

In this example, the control input $u_2 = Q$ is the rate of heat input or removal from the reactor. The error is $e = x_2 - x_{2d}$ and the sliding surface $s = u_2 - u_{2d}$. Then, following the procedure in 3.2, let $\dot{u}_2(t) = z(t)$. Where the auxiliary input $z$ is given by:

\[
 z(t) = -M \text{sgn}(u - (\rho c_p V (\dot{x}_{2d} - \lambda_1 f_1(x) - \lambda_2 f_2(x) - K_1 e)))
\]

and we choose $M = 10$ and $K_1 = 1$. Note that, our choice of $M$, ensure local stability. However, for global stability, $M$ should be state-dependent to ensure that the condition in equation 9 is satisfied all time.

\[\text{(12)}\]
4.3. Simulation

We consider an output tracking for two different desired reaction temperature profiles. First, it is desired to follow a temperature profile given by:

\[ x_{2_d}(t) = x_{2,ss}(1 + a_1 \exp(-a_2 t)) \]

where \( T_{ss} \) is the steady state temperature and \( a \) is design factor. Figure 2, shows that the proposed control steers the process temperature to the desired steady state temperature from an initial points \( x_0 = (0.2, 350) \). The system starts initially in mode 1, switched to mode 2 then return to mode 1. In the second case, the process temperature is required to transit from the initial condition to the steady state temperature \( T_{ss} = 375K \) through different steps. Figure 3, shows the system response for each step and the correspondence mode.

Acknowledgments

The author would like to thank Prof. Umit Ozguner and Dr. Arda Kurt for their fruitful discussion and helpful comments on this paper.

5. Conclusion

In this paper, we presented a new modeling and control framework for a class of hybrid dynamical systems. In this framework, a switching dynamical system is modeled as a dynamical system with discontinuous nonlinearities. Specifically, the switched system translated to a dynamical system with sum of shifted and scaled signum functions. Then, a synthesis of feedback linearization and sliding mode control is employed to control the dynamical system. Application and implementation of this approach is illustrated via a switching chemical process.
Figure 3. Top: State trajectory for $x_2$ (Reactor Temperature) when tracking a pre-specified sequence of desired temperature. Bottom: The active mode.

Figure 4. Control input for the first simulation (top) and for the second tracking problem (Bottom).
References

[1] P. D. Christofides and N. El-Farra. Control of nonlinear and hybrid process systems: Designs for uncertainty, constraints and time-delays, volume 324. Springer Science & Business Media, 2005.

[2] M. Dogruel, L. Özcüner, and S. Drakunov. Sliding-mode control in discrete-state and hybrid systems. IEEE Transactions on Automatic Control, 41(3):414–419, 1996.

[3] S. Drakunov, M. Dogruel, and U. Özcüner. Sliding mode control in hybrid systems. In Intelligent Control, 1993., Proceedings of the 1993 IEEE International Symposium on, pages 186–189. IEEE, 1993.

[4] K. Edström, J.-E. Strömberg, U. Söderman, and J. Top. Modelling and simulation of a switched power converter. 1996.

[5] S. Galeani, L. Menini, and A. Potini. Robust trajectory tracking for a class of hybrid systems: an internal model principle approach. IEEE Transactions on Automatic Control, 57(2):344–359, 2012.

[6] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. IEEE Control Systems, 29(2):28–93, 2009.

[7] C. Hatipoğlu. Variable structure control of continuous time systems involving non-smooth nonlinearities. PhD thesis, The Ohio State University, 1998.

[8] C. Hatipoğlu and U. Özcüner. Robust control of systems involving non-smooth nonlinearities using modified sliding manifolds. In American Control Conference, 1998. Proceedings of the 1998, volume 4, pages 2133–2137. IEEE, 1998.

[9] C. Hatipoğlu and Ü. Özcüner. Handling stiction with variable structure control. In Variable structure systems, sliding mode and nonlinear control, pages 143–166. Springer, 1999.

[10] A. Kayihan and F. J. Doyle. Friction compensation for a process control valve. Control engineering practice, 8(7):799–812, 2000.

[11] B. Lennartson, M. Tittus, B. Egardt, and S. Pettersson. Hybrid systems in process control. IEEE control systems, 16(5):45–56, 1996.

[12] D. Liberzon. Switching in systems and control. Springer Science & Business Media, 2012.

[13] G. Meyer and E. W. Aiken. Design of flight vehicle management systems. 1994.

[14] Y. Pan, U. Ozguner, and O. H. Dagci. Variable-structure control of electronic throttle valve. IEEE Transactions on industrial electronics, 55(11):3899–3907, 2008.

[15] K. M. Passino and U. Ozguner. Modeling and analysis of hybrid systems: examples. In Intelligent Control, 1991., Proceedings of the 1991 IEEE International Symposium on, pages 251–256. IEEE, 1991.

[16] C.-Y. Su, Y. Stepanenko, J. Svoboda, and T.-P. Leung. Robust adaptive control of a class of nonlinear systems with unknown backlash-like hysteresis. IEEE Transactions on Automatic Control, 45(12):2427–2432, 2000.