Uniting Bose-Einstein condensates in optical resonators

D. Jaksch\textsuperscript{1}, S.A. Gardiner\textsuperscript{1,2} K. Schulze\textsuperscript{3}, J.I. Cirac\textsuperscript{1}, and P. Zoller\textsuperscript{1}

\textsuperscript{1} Institut f"ur Theoretische Physik, Universit"at Innsbruck, A–6020 Innsbruck, Austria. and
\textsuperscript{2} Institut f"ur Physik, Universit"at Potsdam, D–14469 Potsdam, Germany

The relative phase of two initially independent Bose-Einstein condensates can be laser cooled to unite the two condensates by putting them into a ring cavity and coupling them with an internal Josephson junction. First, we show that this phase cooling process already appears within a semiclassical model. We calculate the stationary states, find regions of bistable behavior and suggest a Ramsey-type experiment to measure the build up of phase coherence between the condensates. We also study quantum effects and imperfections of the system.

PACS numbers: 03.75.Fi, 74.50.+r, 42.50.-p

During recent years Bose-Einstein condensates (BEC) of Alkali atoms and Hydrogen have been produced and studied extensively in the laboratory \cite{1}. In view of potential applications, such as the generation of bright beams of coherent matter waves (atom laser), a central goal has been the formation of condensates with a number of atoms as large as possible. It is thus of particular interest to study a scenario where this goal is achieved by uniting two (or more) independently grown condensates to form one large single condensate. Physically speaking, two independently formed condensates are characterized by a random relative phase of their macroscopic wave functions. A “fusion” of two condensates thus amounts to locking the relative phase in a dissipative process. Below we will study such a mechanism in the context of optical Cavity QED \cite{2}. In our schemes two condensates in different internal atomic states are coupled by lasers in a Raman configuration (internal Josephson effect \cite{3}), and in addition to a lossy optical cavity, which provides an effective zero temperature reservoir. As a result, we obtain a damping mechanism for the relative condensate phase. A physical picture behind this cooling mechanism can be given by establishing a formal analogy of the dynamical equations describing this “laser cooling” of the relative phase, and the recently discussed cavity assisted laser cooling of the motional degrees of freedom of atoms or molecules \cite{3}.

We consider two equally large, independently produced BECs, where the atoms of the individual condensates are the same species but in two different hyperfine states \cite{1} and \cite{2}. The elements of the one-body density matrix are given by $\rho_{kl}(x,x') = \langle \psi_k^*(x') \psi_l(x) \rangle$. Here, $\psi_k(x)$ is a bosonic field operator which annihilates a particle at position $x$ and in hyperfine state $|k\rangle$, and $k,l \in \{1,2\}$. The one-body density matrix of these two independent condensates has vanishing off diagonal elements $\rho_{12}(x,x') = 0$, and two dominant eigenvalues of approximately $N/2$, where $N$ is the total particle number. The diagonal terms can then be written as $\rho_{kk}(x,x') \approx N \varphi_k(x) \varphi_k(x')/2$, with $\varphi_k(x)$ the wave function of the condensate in state $|k\rangle$. There is thus no phase-coherence between the two different condensates and any attempt to measure a relative condensate phase $\Phi$ will produce a random result \cite{4}. Here we study the possibility of building up and locking the phase $\Phi$ by (i) an internal atomic Josephson junction (JJ) \cite{5} transferring atoms between the two initial condensates, and (ii) coupling the atoms to a dissipative ring cavity \cite{6}. The final result is that for experimentally relevant parameters the phase $\Phi$ develops towards a stationary value on a time scale of a few trap oscillation periods. This is reflected by one dominant eigenvalue of approximately $N$ of $\rho_{kl}(x,x')$. The corresponding condensate mode may be an electronic superposition state. In this sense we have thus joined two condensates together to form one larger condensate.

The internal JJ \cite{5} is realized by coupling the two hyperfine states $\{|1\rangle,|2\rangle\}$ by a Raman transition with detuning $\Delta$ and effective Rabi frequency $\Omega$ (cf. Fig. \ref{fig}). We assume $\Omega$ to be real, positive, and $x$ independent, i.e. there is negligible momentum transfer to the conden-

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1}
\caption{a) Experimental setup: Two initially independent condensates 1 and 2 are trapped in a ring cavity and coupled to the cavity mode $c$ as well as to the two lasers $\Omega$ and $\Omega_1$. b) Level structure: The Raman lasers $\Omega$ transfer atoms from $|1\rangle$ to $|2\rangle$. A laser $\Omega_1$ drives the transition between particles in $|1\rangle$ and an auxiliary level $|3\rangle$. The cavity couples the levels $|3\rangle$ and $|2\rangle$ with coupling strength $g_c$.}
\end{figure}
sates due to the Raman lasers. In second quantized form the Hamiltonian \( H_{\text{BEC}} = H_1 + H_2 + H_L \) is given by (with \( \hbar = 1 \))

\[
H_k = \int dx \, \psi_k^\dagger \left( -\frac{\nabla^2}{2m} + V + \sum_l \frac{u_{kl}}{2} \psi_l^\dagger \psi_l \right) \psi_k, \quad (1a)
\]

\[
H_L = \int dx \left[ \Delta \psi_1^\dagger \psi_1 + \left( \frac{\Omega}{2} \psi_1^\dagger \psi_2 + \text{h.c.} \right) \right], \quad (1b)
\]

where \( \psi_{1,2} \equiv \psi_{1,2}(x) \), \( V \equiv V(x) \) the trapping potential; we have suppressed \( k \) for notational convenience. We assume the trapping potential to be harmonic with frequency \( \omega \) and the mass of the atoms to be \( m \). Two-particle interactions are characterized by \( u_{kk} = 4\pi a_k/m \) where \( a_k \) is the s-wave scattering length of atoms in condensate \( k \) and by \( u_{12} = u_{21} = 4\pi a_{12}/m \) with \( a_{12} \) the interspecies s-wave scattering length. We restrict ourselves to the case where \( a_1 a_2 > a_{12}^2 \), i.e. to two stable strongly overlapping condensates \( 1 \).

The ring cavity mode couples the state \( |2 \rangle \) to an auxiliary excited atomic state \( |3 \rangle \), with detuning \( \delta_1 \) and coupling strength \( g_1(x) \), which decays at a rate \( \kappa \). We also add a classical laser field driving the \( 1 \) to \( 3 \) transition with Rabi frequency \( \Omega_1(x) \) and detuning \( \delta_1 \) (cf. Fig. 1). Adiabatically eliminating the internal state \( |3 \rangle \) (which requires \( \delta_1 \gg \Omega_1(x) \) and \( \delta_1 \gg g_1(x) \), with \( |C \rangle \) the square root of the number of photons in the cavity) we obtain the following master equation with \( H = H_{\text{BEC}} + H_C \):

\[
\dot{\rho} = -i[H, \rho] + \frac{\kappa}{2} \left( 2c \rho c^\dagger - c^\dagger c \rho - \rho c^\dagger c \right), \quad (2)
\]

where \( c \) is the annihilation operator of a photon in the cavity mode, \( \nu = \delta_c - \delta_1 - \Delta \) is the effective cavity detuning, and

\[
H_C = \int dx \left[ g(x) c \Psi_2^\dagger + \text{h.c.} \right] + \nu c^\dagger c. \quad (3)
\]

The effective coupling of the condensates to the cavity is given by \( g(x) = \Omega_1(x) g_1(x)/2\delta_1 \). In a ring cavity the spatial dependences of \( \Omega_1(x) \) and \( g_1(x) \) can be made to almost cancel each other. There is thus negligible momentum transfer to the atoms due to the coupling to the cavity, and we may set \( g(x) = g \), where \( g \) is \( x \)-independent, real, and positive.

We derive equations of motion for the operators \( \psi_{k}(x) \) and \( c \). First we study laser cooling of the phase \( \Phi \) in a semiclassical model before returning to the full quantum model below. In the semiclassical model we assume \( N \gg 1 \) and replace the operators \( \psi_{k}(x) \) and \( c \) by the c-numbers \( \Psi_k(x) \) and \( C \), respectively. We define \( C_1 = C, C_2 = C^* \), denote the Kronecker delta by \( \delta_{kl} \) and find

\[
\dot{\Psi}_k = -i \left( -\frac{\nabla^2}{2m} + V + \sum_l u_{kl} |\Psi_l|^2 \right) \Psi_k
- i \left( gC_k + \frac{\Omega}{2} \right) \Psi_{l\neq k} - i \Delta \delta_{kl} \Psi_k, \quad (4a)
\]

\[
\dot{C} = -i \int dx \, g \Psi_1^* \Psi_1 - \left( i\nu + \frac{\kappa}{2} \right) C. \quad (4b)
\]

In terms of this semiclassical framework the initial independence of the two condensates is modelled by a random initial phase, i.e. \( \rho_{\text{in}}(x, x') \approx \langle \Psi_1^*(x') \Psi_k(x) \rangle \). Here \( \langle \ldots \rangle \) is the stochastic average over relative phases \( \Phi \) which are initially equally distributed between \( [0, 2\pi] \).

Apart from the relative phase the initial conditions for solving Eqs. (4) are assumed to be identical.

For analytical calculations our numerical studies (see below) suggest the two mode ansatz (with \( \varphi(x) \) a fixed and normalized wave function) \( \Psi_k(x) = \sqrt{N_k} \varphi(x) \exp(\mp i\Phi_k) \), where \( N_k \) is the expectation value of the number of particles in state \( |k\rangle \), and \( \Phi_k \) is an \( x \)-independent total phase. Within this semiclassical two mode model we find:

\[
\dot{\varphi} = -2\sqrt{N_1 N_2} \text{Im} \{ \alpha \}, \quad (5a)
\]

\[
\dot{P}_\Phi = -2\sqrt{N_1 N_2} Im \{ \alpha \}, \quad (5b)
\]

\[
\dot{C} = -i \sqrt{N_1 N_2} g \varphi \varphi^* - \left( i\nu + \frac{\kappa}{2} \right) C, \quad (5c)
\]

where we have defined \( \Phi = \Phi_1 - \Phi_2 \), \( P_\Phi = (N_2 - N_1)/2 \) and \( \alpha = (gC + \Omega/2) \exp(-i\Phi) \). The chemical potentials \( \mu_k \) are defined by

\[
\mu_k = \int dx \varphi \left( -\frac{\nabla^2}{2m} + V + \sum_l u_{kl} |\varphi|^2 \right), \quad (6)
\]

To make contact with the resistively shunted junction model, often used in describing JJs we set \( u_{11} = u_{22} \), and assume \( N \gg 2|P_\Phi| \) and \( NU \gg |\Omega, g C| \), which can be enforced by adjusting the laser parameters correspondingly. Here \( U = (u_{11} - u_{12}) \int dx |\varphi(x)|^4 \). We further simplify Eqs. (3) by adiabatically eliminating the cavity mode \( C \) to second order, valid when \( \kappa \gg \{ g\sqrt{N}, UN \} \), and derive an intuitively appealing damping equation for a fictitious particle moving along a coordinate \( \Phi \) (with \( M = 1/2U \)),

\[
M \ddot{\Phi} + a_r \dot{\Phi} + \frac{\Omega N}{2} \sin(\Phi) = F_d. \quad (7)
\]

The particle moves in a potential \( V(\Phi) = \Omega N \cos(\Phi)/2 + F_d \Phi \) where \( F_d = g^2 N^2 \kappa/(\kappa^2 + 4\nu^2) \) is a constant force. This potential has minima at \( \Phi = (2n + 1)\pi - \arcsin(\Omega_c/\Omega) \) with integer \( n \) for \( \Omega > \Omega_c = 2F_d/N \).

Close to a minimum \( V(\Phi) \) is harmonic with frequency \( \omega^2 = UN \Omega^2 - \Omega_\kappa^2 \). The motion of the particle in \( V(\Phi) \) is damped with a friction coefficient \( a_r = \nu g^2 N^2 \kappa/(2(\kappa^2 + 4\nu^2))^2 \). From Eq. (3) we easily find the damping time scale of the relative phase \( \Phi \), which is given by \( \tau = 2M/a_r \). Note that cooling of the phase towards a minimum of the potential occurs only for \( a_r > 0 \) which requires a cavity detuning of \( \nu > 0 \) and that both \( F_d \) and \( a_r \) are induced by the cavity damping rate \( \kappa \).
If $NU \leq \{\Omega, gC\}$ Eq. (11) no longer holds. However, from Eqs. (11) we can still find an approximate expression for $\tau$ which has a minimum at $\Omega = \Omega_m = [(4r^2 - \kappa^2 + 2\sqrt{\kappa^4 + 4r^2\kappa^2 + 16\omega^2})/12]^{1/2}$. This case corresponds to the Rabi-oscillation limit (3).

We now determine the stationary solutions of our semiclassical two mode model [Eqs. (3)] which are characterized by $C = 0$, $P_b = 0$ and by $\Phi = 0$. Writing $C = |C|\exp(i\Phi_c)$ we find the following conditions:

$$2gN_1N_2\sin(\Phi - \Phi_c) = \kappa|C|,$$  \hspace{1cm} (8a)
$$-gN_1N_2\cos(\Phi - \Phi_c) = \nu|C|,$$  \hspace{1cm} (8b)
$$-\Omega\sqrt{N_1N_2}\sin\Phi = \kappa|C|^2,$$  \hspace{1cm} (8c)
$$\sqrt{N_1N_2}(\mu_1 + \Delta - \mu_2) = (N_1 - N_2)\text{Re}\{\alpha\}.$$  \hspace{1cm} (8d)

We set the macroscopic wave function $\varphi(x) = \varphi_0(x)$ where $\varphi_0(x)$ is found from numerically solving Eqs. (4) for $N_1 = N_2$ with all the lasers turned off ($\Omega = g = \Delta = 0$) for the ground state. Thus we find $\mu_1 - \mu_2 = U(N_2 - N_1)$. As shown in Fig. 2 we obtain one stable (unstable) branch $A$ (B) of stationary solutions with $N_1 = N_2$ for $\Omega > \Omega_c$. For $\Omega < \Omega_T$, where $\Omega_T^2 = \Omega_T^2 + \nu^2 \nu g^2 / (\nu^2 + \kappa^2 + 4\omega^2)$, there exist stationary solutions with $N_1 \neq N_2$ where the solutions on branch C (D) with $N_1 < N/2$ ($N_1 > N/2$) are stable (unstable). The solutions of branches A and B can also be found from the Josephson model Eq. (1) while branches C and D are beyond the range of validity of Eq. (1). The stability of the stationary solutions is checked by a linear stability analysis of Eqs. (4). We also numerically evolve Eqs. (4) with initial using the two mode ansatz where $N_1$, $N_2$, $\Phi$, are found from Eqs. (3) and $\varphi(x) = \varphi_0(x)$. In numerically solving Eq. (4) we restrict ourselves to one spatial dimension (3). For solutions on branch A these initial conditions are exact stationary solutions of Eqs. (4) and we find them to be numerically stable. The phase $\Phi$ always evolves towards its stationary value. Therefore solutions on branch A are best suited for uniting two condensates. On branch C with $N_1 \neq N_2$ which is less interesting for the purpose of laser cooling of the phase $\Phi$ the validity of the two mode ansatz depends on the strength of the two particle interaction. For $UN < \omega$ we find good agreement between the numerics and our analytical results while for $UN > \omega$ instabilities in the evolution of Eqs. (4) emerge which do not appear in the linear stability analysis of Eqs. (4).

A possible experimental scenario to monitor the locking of the relative phase $\Phi$ is a Ramsey experiment. After time $t$ cooling is stopped and a $\pi/2$ pulse with phase $\Phi_R$ is applied to the condensates [3]. This pulse effectively transforms the relative phase $\Phi$ into a difference in occupation numbers. For a well defined phase $\Phi$, by varying $\Phi_R$ we expect to observe maximum fringe visibility $v = 1$, whereas for an undefined, effectively random phase $v = 0$. In terms of the one particle density matrix $\rho_{kl}(x, x')$ the visibility is given by $v(x) = 2|\rho_{12}(x, x)|/|\rho_{11}(x, x) + \rho_{22}(x, x)|$ at position $x$. Figure 3a shows the visibility $v = v(x = 0)$ in the center of a one dimensional trap found by numerically solving the semiclassical model [Eqs. (4)] against the cooling time $t$. As expected $v$ does not go to 1 in the bistable region while it tends to 1 for $\Omega > \Omega_T$. The results agree very well with the semiclassical two mode model Eqs. (4).

So far we have studied a semiclassical version of a Josephson–cavity model, where the atoms and the cavity mode are described by c-number fields. We will investigate now quantum effects within a two-mode model of a JJ [3]. This is obtained from the Hamiltonian (3) with the replacement $\psi_{kl}(x) = b_k \varphi(x)$ where $b_k$ are destruction operators of a particle in electronic state $|k\rangle$, and $\varphi(x)$ a (fixed) spatial mode function. We define an operator describing the imbalance in the particle numbers $p_\phi = (b_2^\dagger b_2 - b_1^\dagger b_1)/2$, and its canonical conjugate $-\phi$, where $\phi$ has the meaning of a relative phase operator $\phi(\tau)$.

Projection on the eigenstates of the phase operator, $|\phi\rangle$, gives the quantum phase distribution $p(\phi) = \text{tr}\{|\phi\rangle \langle \phi|\rho\}$. We note that the eigenstates of the phase operator $|\phi\rangle \sim \sum_m e^{-im\phi}|N/2 - m\rangle_1|N/2 + m\rangle_2$ are entangled states of particles in the first and second internal states (3). For small particle numbers and low occupation of the cavity mode, the master equation (4) can be solved directly by quantum Monte Carlo techniques, and the time evolution of the phase distribution can thus be calculated exactly. As an example, Fig. 3b shows the evolution of the phase distribution $p(\phi)$ for $N = 50$ atoms, $u_{11} = u_{22}$, and initially independent condensates, i.e. $p(\phi) = 1/2\pi$, obtained from a simulation. During the cooling process $p(\phi)$ builds up around the semiclassical stationary solution.
tion for $\Phi$ on the time scale predicted by the semiclassical model, and one finds excellent agreement between the Monte Carlo simulation and the semiclassical model, even for a small number of particles.

The quantum limits of the steady state distribution of the phase can be discussed within a quantum phase model. For $U N \gg \{\Omega, g_c\}$ the Hamiltonian simplifies to

$$H = -U \frac{\partial^2}{\partial \phi^2} - i \Delta \frac{\partial}{\partial \phi} + \frac{\Omega N}{2} \cos \phi + \frac{Ng}{2} (\cos \phi \mp \text{h.c.}) + \nu \phi \text{c.c.}.$$ (9)

which describes the motion of a fictitious quantum particle in a $\cos \phi$ phase-potential coupled to a (damped) harmonic oscillator representing the cavity mode. In this picture, cavity assisted phase locking can be understood as “cooling of the motion” of this fictitious particle in the phase potential due to coupling to a damped harmonic oscillator. In fact, the master equation (9) with Hamiltonian (1) is identical to the master equation which has been derived for laser cooling of single atoms or molecules in a high-Q cavity moving in a trapping potential (4). Analytical expressions for the width of the steady state phase distributions can be given by expanding the Hamiltonian (9) around the semiclassical steady state solutions of the relative phase between two initially independent condensates without loss of particles.

We thank R. Ballagh, R. Dum, P. Horak, and H. Ritsch for discussions. Research supported by the Austrian and European Science Foundation, and by the TMR networks ERB-FMRX-CT96-0087 and ERBFMRX-CT96-0002.

[1] E.A. Cornell, J.R. Ensher, C.E. Wieman, cond-mat/9903109; W. Ketterle, D.S. Durfee, D.M. Stamper-Kurn, cond-mat/9904034; W. Ketterle, Phys. Today 52, 30 (1999); F. Dalfovo, et al., Rev. Mod. Phys. 71, 463 (1999).

[2] see: P.W.H. Pinkse et al., Nature 404, 365 (2000); C.J. Hood, et al., Science 287, 1447 (2000); M.G. Moore and P. Meystre, Phys. Rev. A 59, R1754 (1999).

[3] W. Zwerger, Phys. Rev. B 35, 4737 (1987); A.J. Leggett, Rev. Mod. Phys., in press; J.I. Cirac et al., Phys. Rev. A 57, 1208 (1998); P. Öhberg and S. Stenholm, ibid. 59, 3890 (1999); S. Raghavan, A. Smerzi, and V.M. Kenkre, ibid. 60, R1787 (1999); J. Williams, et al., ibid. 59, R31 (1999); C. Menotti et al., ibid. 63, 023601 (2001); P.B. Blaike, R.J. Ballagh, and C.W. Gardiner, J. Opt. B 1, 378 (1999); T.-L. Ho, Phys. Rev. Lett. 81, 742 (1998); A. Sinatra and Y. Castin, Eur. Phys. J. D 8, 319 (2000).

[4] G. Hechenblaikner et al., Phys. Rev. A 58, 3030 (1998); P. Möbusmann, et al., Phys. Rev. Lett. 82, 3791 (1999); C.J. Hood, et al., ibid 80, 4157 (1998); V. Vuletić and S. Chu, ibid. 84, 3757 (2000).

[5] Y. Castin and R. Dum, Phys. Rev. A 57, 3008 (1998).

[6] D.S. Hall et al., Phys. Rev. Lett. 81, 1543 (1998).

[7] Intrinsinc damping effects are presently being studied by F. Sols, private communication.

[8] D.S. Petrov, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. 85, 3745 (2000); K. Bongs et al., Phys. Rev. A 63 031602(R) (2001).

[9] The frequency $\tilde{\omega}^2$ is shifted from the previously de-
fined value by $4g^2UN^2\nu/(\kappa^2 + 4\nu^2)$.

[10] S.A. Gardiner et al., cond-mat/0011341.