Asymmetric critical $p$-Laplacian problems

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Abstract

We obtain nontrivial solutions for two types of critical $p$-Laplacian problems with asymmetric nonlinearities in a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. For $p < N$, we consider an asymmetric problem involving the critical Sobolev exponent $p^* = Np/(N - p)$. In the borderline case $p = N$, we consider an asymmetric critical exponential nonlinearity of the Trudinger-Moser type. In the absence of a suitable direct sum decomposition, we use a linking theorem based on the $\mathbb{Z}_2$-cohomological index to obtain our solutions.

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critical Sobolev exponent, Trudinger-Moser inequality, linking, \( \mathbb{Z}_2 \)-cohomological index.

1 Introduction

Beginning with the seminal paper of Ambrosetti and Prodi [3], elliptic boundary value problems with asymmetric nonlinearities have been extensively studied (see, e.g., Berger and Podolak [4], Kazdan and Warner [16], Dancer [7], Amann and Hess [2], and the references therein). More recently, Deng [12], de Figueiredo and Yang [9], Aubin and Wang [4], Calanchi and Ruf [6], and Zhang et al. [27] have obtained interesting existence and multiplicity results for semilinear Ambrosetti-Prodi type problems with critical nonlinearities using variational methods.

In the present paper, first we consider the asymmetric critical \( p \)-Laplacian problem

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u + u_+^{p^*-1} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N, N \geq 2, 1 < p < N \), \( p^* = Np/(N-p) \) is the critical Sobolev exponent, \( \lambda > 0 \) is a constant, and \( u_+(x) = \max \{u(x), 0\} \).

We recall that \( \lambda \in \mathbb{R} \) is a Dirichlet eigenvalue of \( -\Delta_p \) in \( \Omega \) if the problem

\[
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.2)

has a nontrivial solution. The first eigenvalue \( \lambda_1(p) \) is positive, simple, and has an associated eigenfunction \( \varphi_1 \) that is positive in \( \Omega \). Problem (1.1) has a positive solution when \( N \geq p^2 \) and \( 0 < \lambda < \lambda_1(p) \) (see Guedda and Véron [15]). When \( \lambda = \lambda_1(p) \), \( t\varphi_1 \) is clearly a negative solution for any \( t < 0 \). Here we focus on the case \( \lambda > \lambda_1(p) \). Our first result is the following.

**Theorem 1.1.** If \( N \geq p^2 \) and \( \lambda > \lambda_1(p) \) is not an eigenvalue of \( -\Delta_p \), then problem (1.1) has a nontrivial solution.
In the borderline case $p = N \geq 2$, critical growth is of exponential type and is governed by the Trudinger-Moser inequality

$$\sup_{u \in W_0^{1,N}(\Omega), \|u\| \leq 1} \int_{\Omega} e^{\alpha_N |u|^{N'}} dx < \infty,$$

(1.3)

where $\alpha_N = N \omega_{N-1}^{1/(N-1)}$, $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$, and $N' = N/(N - 1)$ (see Trudinger [27] and Moser [20]). A natural analog of problem (1.1) for this case is

$$\begin{cases} -\Delta_N u = \lambda |u|^{N-2} u e^{u^{N'}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(1.4)

A result of Adimurthi [1] implies that this problem has a nonnegative and nontrivial solution when $0 < \lambda < \lambda_1(N)$ (see also do Ó [19]). When $\lambda = \lambda_1(N)$, $t\varphi_1$ is again a negative solution for any $t < 0$. Our second result here is the following.

**Theorem 1.2.** If $N \geq 2$ and $\lambda > \lambda_1(N)$ is not an eigenvalue of $-\Delta_N$, then problem (1.4) has a nontrivial solution.

These results complement those in [1, 3, 6, 12, 27] concerning the semilinear case $p = 2$. However, the linking arguments based on eigenspaces of $-\Delta$ used in those papers do not apply to the quasilinear case $p \neq 2$ since the nonlinear operator $-\Delta_p$ does not have linear eigenspaces. Therefore we will use more general constructions based on sublevel sets as in Perera and Szulkin [23]. Moreover, the standard sequence of eigenvalues of $-\Delta_p$ based on the genus does not provide sufficient information about the structure of the sublevel sets to carry out these linking constructions, so we will use a different sequence of eigenvalues introduced in Perera [21] that is based on a cohomological index.

The $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [13] is defined as follows. Let $W$ be a Banach space and let $\mathcal{A}$ denote the class of symmetric subsets of $W \setminus \{0\}$. For $A \in \mathcal{A}$, let $\overline{A} = A/\mathbb{Z}_2$ be the quotient space of $A$ with each $u$ and $-u$ identified, let $f : \overline{A} \to \mathbb{RP}^\infty$ be the classifying map of $\overline{A}$, and let $f^* : H^*(\mathbb{RP}^\infty) \to H^*(\overline{A})$
be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of \( A \) is defined by

\[
i(A) = \begin{cases} 
0 & \text{if } A = \emptyset, \\
\sup \{m \geq 1 : f^*(\omega^{m-1}) \neq 0\} & \text{if } A \neq \emptyset,
\end{cases}
\]

where \( \omega \in H^1(\mathbb{RP}^\infty) \) is the generator of the polynomial ring \( H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[\omega] \).

**Example 1.3.** The classifying map of the unit sphere \( S^{m-1} \) in \( \mathbb{R}^m, m \geq 1 \) is the inclusion \( \mathbb{RP}^{m-1} \subset \mathbb{RP}^\infty \), which induces isomorphisms on the cohomology groups \( H^q \) for \( q \leq m - 1 \), so \( i(S^{m-1}) = m \).

The following proposition summarizes the basic properties of this index.

**Proposition 1.4** (Fadell-Rabinowitz [13]). The index \( i : A \to \mathbb{N} \cup \{0, \infty\} \) has the following properties:

1. **Definiteness:** \( i(A) = 0 \) if and only if \( A = \emptyset \).
2. **Monotonicity:** If there is an odd continuous map from \( A \) to \( B \) (in particular, if \( A \subset B \)), then \( i(A) \leq i(B) \). Thus, equality holds when the map is an odd homeomorphism.
3. **Dimension:** \( i(A) \leq \dim W \).
4. **Continuity:** If \( A \) is closed, then there is a closed neighborhood \( N \in A \) of \( A \) such that \( i(N) = i(A) \). When \( A \) is compact, \( N \) may be chosen to be a \( \delta \)-neighborhood \( N_\delta(A) = \{u \in W : \text{dist}(u, A) \leq \delta\} \).
5. **Subadditivity:** If \( A \) and \( B \) are closed, then \( i(A \cup B) \leq i(A) + i(B) \).
6. **Stability:** If \( SA \) is the suspension of \( A \neq \emptyset \), obtained as the quotient space of \( A \times [-1, 1] \) with \( A \times \{1\} \) and \( A \times \{-1\} \) collapsed to different points, then \( i(SA) = i(A) + 1 \).
7. **Piercing property:** If \( A, A_0 \) and \( A_1 \) are closed, and \( \varphi : A \times [0, 1] \to A_0 \cup A_1 \) is a continuous map such that \( \varphi(-u, t) = -\varphi(u, t) \) for all \( (u, t) \in A \times [0, 1] \), \( \varphi(A \times [0, 1]) \) is closed, \( \varphi(A \times \{0\}) \subset A_0 \) and \( \varphi(A \times \{1\}) \subset A_1 \), then \( i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A) \).
Neighborhood of zero: If $U$ is a bounded closed symmetric neighborhood of 0, then $i(\partial U) = \dim W$.

For $1 < p < \infty$, eigenvalues of problem (1.2) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\int_{\Omega} |u|^p \, dx}, \quad u \in \mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p \, dx = 1 \right\}.$$ 

Let $\mathcal{F}$ denote the class of symmetric subsets of $\mathcal{M}$ and set

$$\lambda_k(p) := \inf_{M \in \mathcal{F}, i(M) \geq k} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.$$ 

Then $0 < \lambda_1(p) < \lambda_2(p) \leq \lambda_3(p) \leq \cdots \to \infty$ is a sequence of eigenvalues of (1.2) and

$$\lambda_k(p) < \lambda_{k+1}(p) \implies i(\Psi_{\lambda_k(p)}) = i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}(p)}) = k,$$

where $\Psi_a = \{ u \in \mathcal{M} : \Psi(u) \leq a \}$ and $\Psi_a = \{ u \in \mathcal{M} : \Psi(u) \geq a \}$ for $a \in \mathbb{R}$ (see Perera et al. [22, Propositions 3.52 and 3.53]). As we will see, problems (1.1) and (1.4) have nontrivial solutions as long as $\lambda$ is not an eigenvalue from the sequence ($\lambda_k(p)$). This leaves an open question of existence of nontrivial solutions when $\lambda$ belongs to this sequence.

We will prove Theorems 1.1 and 1.2 using the following abstract critical point theorem proved in Yang and Perera [26], which generalizes the well-known linking theorem of Rabinowitz [24].

**Theorem 1.5.** Let $\Phi$ be a $C^1$-functional defined on a Banach space $W$ and let $A_0$ and $B_0$ be disjoint nonempty closed symmetric subsets of the unit sphere $S = \{ u \in W : \|u\| = 1 \}$ such that

$$i(A_0) = i(S \setminus B_0) < \infty.$$ 

Assume that there exist $R > r > 0$ and $v \in S \setminus A_0$ such that

$$\sup \Phi(A) \leq \inf \Phi(B), \quad \sup \Phi(X) < \infty,$$
where

\[ A = \{tu : u \in A_0, 0 \leq t \leq R\} \cup \{R \pi((1-t)u + tv) : u \in A_0, 0 \leq t \leq 1\}, \]

\[ B = \{ru : u \in B_0\}, \]

\[ X = \{tu : u \in A, \|u\| = R, 0 \leq t \leq 1\}, \]

and \( \pi : W \setminus \{0\} \to S, u \mapsto u/\|u\| \) is the radial projection onto \( S \). Let \( \Gamma = \{\gamma \in C(X,W) : \gamma(X) \text{ is closed and } \gamma|_A = id_A\} \) and set

\[ c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} \Phi(u). \]

Then

\[ \inf \Phi(B) \leq c \leq \sup \Phi(X), \quad (1.6) \]

in particular, \( c \) is finite. If, in addition, \( \Phi \) satisfies the \((C)_c\) condition, then \( c \) is a critical value of \( \Phi \).

This theorem was stated and proved under the Palais-Smale compactness condition in [26], but the proof goes through unchanged since the first deformation lemma also holds under the Cerami condition (see, e.g., Perera et al. [22, Lemma 3.7]). The linking construction used in the proof has also been used in Perera and Szulkin [23] to obtain nontrivial solutions of \( p \)-Laplacian problems with nonlinearities that cross an eigenvalue. A similar construction based on the notion of cohomological linking was given in Degiovanni and Lancelotti [10]. See also Perera et al. [22, Proposition 3.23].

2 Proof of Theorem 1.1

Weak solutions of problem (1.1) coincide with critical points of the \( C^1 \)-functional

\[ \Phi(u) = \int_{\Omega} \left[ \frac{1}{p} (|\nabla u|^p - \lambda |u|^p) - \frac{1}{p^*} u_+^{p^*} \right] dx, \quad u \in W_0^{1,p}(\Omega). \]
We recall that $\Phi$ satisfies the Cerami compactness condition at the level $c \in \mathbb{R}$, or the $(C)_c$ condition for short, if every sequence $(u_j) \subset W^{1,p}_0(\Omega)$ such that $\Phi(u_j) \to c$ and $(1 + \|u_j\|) \Phi'(u_j) \to 0$, called a $(C)_c$ sequence, has a convergent subsequence. Let

$$S = \inf_{u \in W^{1,p}_0(\Omega) \setminus \{0\}} \frac{\int_\Omega \nabla u^p \, dx}{\left( \int_\Omega |u|^{p^*} \, dx \right)^{p/p^*}}$$

be the best constant in the Sobolev inequality.

**Lemma 2.1.** If $\lambda \neq \lambda_1(p)$, then $\Phi$ satisfies the $(C)_c$ condition for all $c < \frac{1}{N} S^{N/p}$.

**Proof.** Let $c < \frac{1}{N} S^{N/p}$ and let $(u_j)$ be a $(C)_c$ sequence. First we show that $(u_j)$ is bounded. We have

$$\int_\Omega \left[ \frac{1}{p} (|\nabla u_j|^p - \lambda |u_j|^p) - \frac{1}{p^*} u_j^{p^*} \right] \, dx = c + o(1)$$

and

$$\int_\Omega (|\nabla u_j|^{p-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{p-2} u_j v - u_j^{p^*-1} v) \, dx = \frac{o(1) \|v\|}{1 + \|u_j\|^q}, \quad \forall v \in W^{1,p}_0(\Omega).$$

Taking $v = u_j$ in (2.3) and combining with (2.2) gives

$$\int_\Omega u_j^{p^*} \, dx = Nc + o(1),$$

and taking $v = u_j^+$ in (2.3) gives

$$\int_\Omega |
abla u_j^+|^p \, dx = \int_\Omega (\lambda u_j^{p^*} + u_j^{p^*}) \, dx + o(1),$$

so $(u_j^+)$ is bounded in $W^{1,p}_0(\Omega)$. Suppose $\rho_j := \|u_j\| \to \infty$ for a renamed subsequence. Then $\tilde{u}_j := u_j/\rho_j$ converges to some $\tilde{u}$ weakly in $W^{1,p}_0(\Omega)$, strongly in $L^q(\Omega)$ for $1 \leq q < p^*$, and a.e. in $\Omega$ for a further subsequence. Since the sequence $(u_j^+)$ is bounded, dividing (2.2) by $\rho_j^p$ and (2.3) by $\rho_j^{p-1}$, and passing to the limit then gives

$$1 = \lambda \int_\Omega |\tilde{u}|^p \, dx, \quad \int_\Omega |
abla \tilde{u}|^{p-2} \nabla \tilde{u} \cdot \nabla v \, dx = \lambda \int_\Omega |\tilde{u}|^{p-1} \tilde{u} v \, dx \quad \forall v \in W^{1,p}_0(\Omega),$$
respectively. Moreover, since \( \tilde{u}_{j+} = u_{j+}/\rho_j \to 0 \), \( \tilde{u} \leq 0 \) a.e. Hence \( \tilde{u} = t\varphi_1 \) for some \( t < 0 \) and \( \lambda = \lambda_1(\rho) \), contrary to assumption.

Since \((u_j)\) is bounded, so is \((u_{j+})\), a renamed subsequence of which then converges to some \( v \geq 0 \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^q(\Omega) \) for \( 1 \leq q < p^* \) and a.e. in \( \Omega \),

\[
|\nabla u_{j+}|^p \to \mu, \quad u_{j+}^{p^*} \to \nu \tag{2.5}
\]

in the sense of measures, where \( \mu \) and \( \nu \) are bounded nonnegative measures on \( \Omega \) (see, e.g., Folland [14]). By the concentration compactness principle of Lions [17, 18], there exist an at most countable index set \( I \) and points \( x_i \in \Omega, i \in I \) such that

\[
\mu \geq |\nabla v|^p \to \int \varphi_i \delta_{x_i}, \quad \nu = v^{p^*} \to \int \varphi_i \delta_{x_i}, \tag{2.6}
\]

where \( \mu_i, \nu_i > 0 \) and \( \nu_i^{p/p^*} \leq \mu_i/S \). Let \( \varphi : \mathbb{R}^N \to [0,1] \) be a smooth function such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi(x) = 0 \) for \( |x| \geq 2 \). Then set

\[
\varphi_{i,\rho}(x) = \varphi \left( \frac{x-x_i}{\rho} \right), \quad x \in \mathbb{R}^N
\]

for \( i \in I \) and \( \rho > 0 \), and note that \( \varphi_{i,\rho} : \mathbb{R}^N \to [0,1] \) is a smooth function such that \( \varphi_{i,\rho}(x) = 1 \) for \( |x-x_i| \leq \rho \) and \( \varphi_{i,\rho}(x) = 0 \) for \( |x-x_i| \geq 2\rho \). The sequence \((\varphi_{i,\rho} u_{j+})\) is bounded in \( W^{1,p}_0(\Omega) \) and hence taking \( v = \varphi_{i,\rho} u_{j+} \) in (2.3) gives

\[
\int \Omega \left( \varphi_{i,\rho} |\nabla u_{j+}|^p + u_{j+} |\nabla u_{j+}|^{p-2} \nabla u_{j+} \cdot \nabla \varphi_{i,\rho} - \lambda \varphi_{i,\rho} u_{j+}^{p^*} - \varphi_{i,\rho} u_{j+}^{p^*} \right) dx = o(1). \tag{2.7}
\]

By (2.5),

\[
\int \Omega \varphi_{i,\rho} |\nabla u_{j+}|^p dx \to \int \Omega \varphi_{i,\rho} d\mu, \quad \int \Omega \varphi_{i,\rho} u_{j+}^{p^*} dx \to \int \Omega \varphi_{i,\rho} d\nu.
\]

Denoting by \( C \) a generic positive constant independent of \( j \) and \( \rho \),

\[
\left| \int \Omega \left( u_{j+} |\nabla u_{j+}|^{p-2} \nabla u_{j+} \cdot \nabla \varphi_{i,\rho} - \lambda \varphi_{i,\rho} u_{j+}^p \right) dx \right| \leq C \left( \frac{I_j^{1/p}}{\rho} + I_j \right),
\]

8
where
\[ I_j := \int_{\Omega \cap B_{2\rho}(x_i)} u_j^p \, dx \to \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \leq C\rho^p \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*} \, dx \right)^{p/p^*}. \]

So passing to the limit in (2.7) gives
\[ \int_\Omega \varphi_{i,\rho} \, d\mu - \int_\Omega \varphi_{i,\rho} \, d\nu \leq C \left( \left( \int_{\Omega \cap B_{2\rho}(x_i)} v^{p^*} \, dx \right)^{1/p^*} + \int_{\Omega \cap B_{2\rho}(x_i)} v^p \, dx \right). \]

Letting \( \rho \to 0 \) and using (2.6) now gives \( \mu_i \leq \nu_i \), which together with \( \nu_i^{p/p^*} \leq \mu_i/S \) then gives \( \nu_i = 0 \) or \( \nu_i \geq S^{N/p} \). Passing to the limit in (2.4) and using (2.5) and (2.6) gives \( \nu_i \leq Nc < S^{N/p} \), so \( \nu_i = 0 \). Hence \( I = \emptyset \) and
\[ \int_\Omega u_j^p \, dx \to \int_\Omega v^{p^*} \, dx. \] (2.8)

Passing to a further subsequence, \( u_j \) converges to some \( u \) weakly in \( W^{1,p}_0(\Omega) \), strongly in \( L^q(\Omega) \) for \( 1 \leq q < p^* \), and a.e. in \( \Omega \). Since
\[ |u_j^{p^*-1}(u_j - u)| \leq u_j^{p^*} + u_j^{p^*-1} |u| \leq \left( 2 - \frac{1}{p^*} \right) u_j^{p^*} + \frac{1}{p^*} |u|^{p^*} \]
by Young’s inequality,
\[ \int_\Omega u_j^{p^*-1}(u_j - u) \, dx \to 0 \]
by (2.8) and the dominated convergence theorem. Then \( u_j \to u \) in \( W^{1,p}_0(\Omega) \) by a standard argument.

We recall that the infimum in (2.1) is attained by the family of functions
\[ u_\varepsilon(x) = \frac{C_{N,p} \varepsilon^{-(N-p)/p}}{\left[ 1 + \left( \frac{|x|}{\varepsilon} \right)^{(N-p)/p} \right]^{(N-p)/p}}, \quad \varepsilon > 0 \]
when \( \Omega = \mathbb{R}^N \), where \( C_{N,p} > 0 \) is chosen so that
\[ \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} u_\varepsilon^{p^*} \, dx = S^{N/p}. \]
Take a smooth function \( \eta : [0, \infty) \to [0, 1] \) such that \( \eta(s) = 1 \) for \( s \leq 1/4 \) and \( \eta(s) = 0 \) for \( s \geq 1/2 \), and set
\[
u_{\varepsilon, \delta}(x) = \eta\left(\frac{|x|}{\delta}\right) \nu_{\varepsilon}(x), \quad \varepsilon, \delta > 0.
\]

We have the well-known estimates
\[
\int_{\mathbb{R}^N} |\nabla \nu_{\varepsilon, \delta}|^p \, dx \leq S^{N/p} + C\left(\frac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)},
\]
\[
\int_{\mathbb{R}^N} \nu_{\varepsilon, \delta}^p \, dx \geq S^{N/p} - C\left(\frac{\varepsilon}{\delta}\right)^{N/(p-1)},
\]
\[
\int_{\mathbb{R}^N} \nu_{\varepsilon, \delta}^p \, dx \geq \begin{cases} 
\frac{\varepsilon^p}{C} - C\delta^p \left(\frac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)} & \text{if } N > p^2 \\
\frac{\varepsilon^p}{C} \log\left(\frac{\delta}{\varepsilon}\right) - C\varepsilon^p & \text{if } N = p^2,
\end{cases}
\]
where \( C = C(N, p) > 0 \) is a constant (see, e.g., Degiovanni and Lancelotti [11]).

Let \( i, \mathcal{M}, \Psi, \) and \( \lambda_k(p) \) be as in the introduction, and suppose that \( \lambda_k(p) < \lambda_{k+1}(p) \). Then the sublevel set \( \Psi_{\lambda_k(p)} \) has a compact symmetric subset \( E \) of index \( k \) that is bounded in \( L^\infty(\Omega) \cap C^1_{\text{loc}}(\Omega) \) (see [11, Theorem 2.3]). We may assume without loss of generality that \( 0 \in \Omega \). Let \( \delta_0 = \text{dist}(0, \partial \Omega) \), take a smooth function \( \theta : [0, \infty) \to [0, 1] \) such that \( \theta(s) = 0 \) for \( s \leq 3/4 \) and \( \theta(s) = 1 \) for \( s \geq 1 \), set
\[
v_\delta(x) = \theta\left(\frac{|x|}{\delta}\right) v(x), \quad v \in E, \quad 0 < \delta \leq \frac{\delta_0}{2},
\]
and let \( E_\delta = \{ \pi(v_\delta) : v \in E \} \), where \( \pi : W^{1,p}_0(\Omega) \setminus \{0\} \to \mathcal{M}, u \mapsto u/\|u\| \) is the radial projection onto \( \mathcal{M} \).

**Lemma 2.2.** There exists a constant \( C = C(N, p, \Omega, k) > 0 \) such that for all sufficiently small \( \delta > 0 \),
\[\begin{align*}
(i) & \quad \Psi(w) \leq \lambda_k(p) + C\delta^{N-p} \quad \forall w \in E_\delta, \\
(ii) & \quad E_\delta \cap \Psi_{\lambda_{k+1}(p)} = \emptyset,
\end{align*}\]
(iii) \( i(E_\delta) = k, \)

(iv) \( \text{supp } w \cap \text{supp } \pi(u_{\epsilon, \delta}) = \emptyset \ \forall w \in E_\delta, \)

(v) \( \pi(u_{\epsilon, \delta}) \notin E_\delta. \)

Proof. Let \( v \in E \) and let \( w = \pi(v_\delta). \) We have

\[
\int_\Omega |\nabla v_\delta|^p \, dx \leq \int_{\Omega \setminus B_{\delta}(0)} |\nabla v|^p \, dx + C \int_{B_{\delta}(0)} \left( |\nabla v|^p + \frac{|v|^p}{\delta^p} \right) \, dx \leq 1 + C\delta^{N-p}
\]

since \( E \subset M \) is bounded in \( C^1(B_{\delta/2}(0)) \), and

\[
\int_\Omega |v_\delta|^p \, dx \geq \int_{\Omega \setminus B_{\delta}(0)} |v|^p \, dx = \int_\Omega |v|^p \, dx - \int_{B_{\delta}(0)} |v|^p \, dx \geq \frac{1}{\lambda_k(p)} - C\delta^N
\]

since \( E \subset \Psi^{\lambda_k(p)} \), so

\[
\Psi(w) = \frac{\int_\Omega |\nabla v_\delta|^p \, dx}{\int_\Omega |v_\delta|^p \, dx} \leq \lambda_k(p) + C\delta^{N-p}
\]

if \( \delta > 0 \) is sufficiently small. Taking \( \delta \) so small that \( \lambda_k(p) + C\delta^{N-p} < \lambda_{k+1}(p) \) then gives (ii). Since \( E_\delta \subset M \setminus \Psi^{\lambda_{k+1}(p)} \) by (ii),

\[
i(E_\delta) \leq i(M \setminus \Psi^{\lambda_{k+1}(p)}) = k
\]

by the monotonicity of the index and (1.3). On the other hand, since \( E \rightarrow E_\delta, \ v \mapsto \pi(v_\delta) \) is an odd continuous map,

\[
i(E_\delta) \geq i(E) = k.
\]

So \( i(E_\delta) = k. \)

Since \( \text{supp } w = \text{supp } v_\delta \subset \Omega \setminus B_{3\delta/4}(0) \) and \( \text{supp } \pi(u_{\epsilon, \delta}) = \text{supp } u_{\epsilon, \delta} \subset \overline{B_{\delta/2}(0)}, \)

(iv) is clear, and (v) is immediate from (iv). \( \square \)

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1. We have $\lambda_k(p) < \lambda < \lambda_{k+1}(p)$ for some $k \in \mathbb{N}$. Fix $\lambda_k(p) < \lambda' < \lambda$ and $\delta > 0$ so small that the conclusions of Lemma 2.2 hold with $\lambda_k(p) + C\delta^{N-p} \leq \lambda'$, in particular,

$$
\Psi(w) \leq \lambda' \quad \forall w \in E_\delta.
$$

Then take $A_0 = E_\delta$ and $B_0 = \Psi_{\lambda_{k+1}(p)}$, and note that $A_0$ and $B_0$ are disjoint nonempty closed symmetric subsets of $\mathcal{M}$ such that

$$
i(A_0) = i(\mathcal{M} \setminus B_0) = k
$$

by Lemma 2.2 (iii) and (1.3). Now let $R > r > 0$, let $v_0 = \pi(u_{\varepsilon,\delta})$, which is in $\mathcal{M} \setminus E_\delta$ by Lemma 2.2 (v), and let $A$, $B$, and $X$ be as in Theorem 1.3.

For $u \in \Psi_{\lambda_{k+1}(p)}$,

$$
\Phi(ru) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_{k+1}(p)}\right) r^p - \frac{1}{p^*} \frac{S^p}{p} r^{p^*}
$$

by (2.1). Since $\lambda < \lambda_{k+1}(p)$, it follows that $\inf \Phi(B) > 0$ if $r$ is sufficiently small. Next we show that $\Phi \leq 0$ on $A$ if $R$ is sufficiently large. For $w \in E_\delta$ and $t \geq 0$,

$$
\Phi(tw) \leq \frac{tp}{p} \left(1 - \frac{\lambda}{\Psi(w)}\right) \leq -\frac{tp}{p} \left(\frac{\lambda}{\lambda'} - 1\right) \leq 0
$$

by (2.12). Now let $0 \leq t \leq 1$ and set $u = \pi((1-t)w + tv_0)$. Since

$$
\|(1-t)w + tv_0\| \leq (1-t)\|w\| + t\|v_0\| = 1
$$

and since the supports of $w$ and $v_0 \geq 0$ are disjoint by Lemma 2.2 (vi),

$$
|u|^p = \frac{|(1-t)w + tv_0|^p}{\|(1-t)w + tv_0\|^p} \geq (1-t)^p \|w\|^p + t^p \|v_0\|^p \geq \frac{(1-t)^p}{\Psi(w)} \geq \frac{(1-t)^p}{\lambda'}
$$

by (2.12), and

$$
|u^+|^p_{p^*} = \frac{|[(1-t)w + tv_0]^+|_{p^*}^p}{\|(1-t)w + tv_0\|^p_{p^*}} \geq (1-t)^p \|w^+\|_{p^*}^p + t^p \|v_0\|_{p^*}^p \geq t^p \|u_{\varepsilon,\delta}\|_{p^*}^p \geq \frac{t^p}{C}
$$

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by (2.9) and (2.10) if \( \varepsilon \) is sufficiently small, where \( C = C(N, p, \Omega, k) > 0 \). Then
\[
\Phi(Ru) = \frac{R^p}{p} \|u\|^p - \frac{\lambda R^p}{p} |u|^p_p - \frac{R^{p^*}}{p^*} |u_{+}^{p^*}| \leq -\frac{1}{p} \left[ \frac{\lambda}{N} (1 - t)^p - 1 \right] R^p - \frac{t^{p^*}}{C} R^{p^*}.
\]
The last expression is clearly nonpositive if \( t \leq 1 - (\lambda'/\lambda)^{1/p} =: t_0 \). For \( t > t_0 \), it is nonpositive if \( R \) is sufficiently large.

Now we show that \( \sup \Phi(X) < \frac{1}{N} S^{N/p} \) if \( \varepsilon \) is sufficiently small. Noting that
\[ X = \{ \rho \pi((1-t)w + tv_0) : w \in E_\delta, 0 \leq t \leq 1, 0 \leq \rho \leq R \}, \]
let \( w \in E_\delta \), let \( 0 \leq t \leq 1 \), and set \( u = \pi((1-t)w + tv_0) \). Then
\[
\sup_{0 \leq \rho \leq R} \Phi(\rho u) \leq \sup_{\rho \geq 0} \left[ \frac{\rho^p}{p} (1 - \lambda |u|^p_p) - \frac{\rho^{p^*}}{p^*} |u_{+}^{p^*}| \right] = \frac{1}{N} S_u(\lambda)^{N/p}
\]
when \( 1 - \lambda \|u\|^p_p > 0 \), where
\[
S_u(\lambda) = \frac{1 - \lambda |u|^p_p}{|u_{+}^{p^*}|} = \frac{||(1-t)w + tv_0||^p - \lambda ||(1-t)w + tv_0||^p_p}{||(1-t)w + tv_0||^p_{p^*}}
= \frac{(1-t)^p \left( ||w||^p - \lambda |w|^p_p \right) + t^p \left( ||v_0||^p - \lambda |v_0|^p_p \right)}{ \left[ (1-t)^{p^*} ||w_{+}^{p^*} + tv_{0}^{p^*} ||_{p^*} \right]^{p/p^*}}.
\]
Since \( ||w||^p - \lambda |w|^p_p = 1 - \lambda/\Psi(w) \leq 0 \) by (2.12),
\[
S_u(\lambda) \leq \frac{1 - \lambda |v_0|^p_p}{|v_0|^p_{p^*}} = \frac{\|u_{\varepsilon, \delta}||^p_p - \lambda |u_{\varepsilon, \delta}||^p_p}{|u_{\varepsilon, \delta}||^p_{p^*}} \leq \begin{cases} S - \frac{\varepsilon^p}{C} + C\varepsilon^{(N-p)/(p-1)} & \text{if } N > p^2 \\ S - \frac{\varepsilon^p}{C} \log \varepsilon + C\varepsilon^p & \text{if } N = p^2 \end{cases}
\]
by (2.3)–(2.11). In both cases the last expression is strictly less than \( S \) if \( \varepsilon \) is sufficiently small.

The inequalities (1.6) now imply that \( 0 < c < \frac{1}{N} S^{N/p} \). Then \( \Phi \) satisfies the \((C)_c\) condition by Lemma 2.1 and hence \( c \) is a critical value of \( \Phi \) by Theorem 1.5. \( \square \)
3 Proof of Theorem 1.2

Weak solutions of problem (1.4) coincide with critical points of the $C^1$-functional
\[ \Phi(u) = \int_\Omega \left[ \frac{1}{N} |\nabla u|^N - \lambda F(u) \right] dx, \quad u \in W^{1,N}_0(\Omega), \]
where
\[ F(t) = \int_0^t |s|^{N-2} s e^{s^{N'}} ds. \]

First we obtain some estimates for the primitive $F$.

Lemma 3.1. For all $t \in \mathbb{R}$,
\[ F(t) \leq \frac{t^N}{2N} e^{t^{N'}} + \frac{t^N}{N} + C, \quad (3.1) \]
\[ F(t) \leq |t|^{N-1} e^{t^{N'}} + \frac{t^N}{N} + C, \quad (3.2) \]
where $C$ denotes a generic positive constant and $t_- = \max \{ -t, 0 \}$.

Proof. For $t \leq 0$, $F(t) = |t|^N/N$. For $t > 0$, integrating by parts gives
\[ F(t) = \int_0^t s^{N-1} e^{s^{N'}} ds = \frac{t^N}{N} e^{t^{N'}} - \frac{N'}{N} \int_0^t s^{N+N'-1} e^{s^{N'}} ds. \]
For $t \geq (N/N')^{1/N'}$, the last term is greater than or equal to
\[ \frac{N'}{N} \int_{(N/N')^{1/N'}}^{(N/N')^{1/N'}} s^{N+N'-1} e^{s^{N'}} ds \geq \int_0^t s^{N-1} e^{s^{N'}} ds = F(t) - F((N/N')^{1/N'}) \]
and hence
\[ 2F(t) \leq \frac{t^N}{N} e^{t^{N'}} + F((N/N')^{1/N'}). \]

Since $F$ is bounded on bounded sets, (3.1) follows. As for (3.2), $F(t) = (e^{t^2} - 1)/2$ for $t > 0$ if $N = 2$, and
\[ F(t) = \frac{t^{N-N'}}{N'} e^{t^{N'}} - \frac{N - N'}{N'} \int_0^t s^{N-N'-1} e^{s^{N'}} ds \leq t^{N-1} e^{t^{N'}} \]
for $t \geq 1/(N')^{1/(N'-1)}$ if $N \geq 3$. \qed
Proof of Theorem 1.2 will be based on the following lemma.

**Lemma 3.2.** If \( \lambda \neq \lambda_1(N) \) and \( 0 \neq c < \alpha_N^{N-1}/N \), then every \((C)_c\) sequence has a subsequence that converges weakly to a nontrivial critical point of \( \Phi \).

**Proof.** Let \( \lambda \neq \lambda_1(N) \), let \( 0 \neq c < \alpha_N^{N-1}/N \), and let \((u_j)\) be a \((C)_c\) sequence. First we show that \((u_j)\) is bounded. We have

\[
\int_\Omega \left[ \frac{1}{N} |\nabla u_j|^N - \lambda F(u_j) \right] \, dx = c + o(1) \tag{3.3}
\]

and

\[
\int_\Omega \left( |\nabla u_j|^{N-2} \nabla u_j \cdot \nabla v - \lambda |u_j|^{N-2} u_j e^{u_j} v \right) \, dx = \frac{o(1)}{1 + |u_j|} \|v\| \quad \forall v \in W_0^{1,N}(\Omega), \tag{3.4}
\]

in particular,

\[
\int_\Omega \left( |\nabla u_j|^N - \lambda |u_j|^N e^{u_j} \right) \, dx = o(1). \tag{3.5}
\]

Combining (3.5) with (3.3) and (3.1) gives

\[
\int_\Omega u_j e^{u_j} \, dx \leq C, \tag{3.6}
\]

and taking \( v = u_j \) in (3.4) gives

\[
\int_\Omega |\nabla u_j|^N \, dx = \lambda \int_\Omega u_j e^{u_j} \, dx + o(1),
\]

so the sequence \((u_j)\) is bounded in \( W_0^{1,N}(\Omega) \). Passing to a subsequence, \( u_j \) then converges to some \( \hat{u} \geq 0 \) weakly in \( W_0^{1,N}(\Omega) \), strongly in \( L^q(\Omega) \) for \( 1 \leq q < \infty \), and a.e. in \( \Omega \). Then for any \( v \in C_0^\infty(\Omega) \),

\[
\int_\Omega u_j e^{u_j} v \, dx \to \int_\Omega \hat{u} e^{\hat{u}} v \, dx \quad (3.7)
\]

by de Figueiredo et al. [8, Lemma 2.1] and (3.6). Now suppose \( \rho_j := |u_j| \to \infty \). Then \( \tilde{u}_j := u_j/\rho_j \) converges to some \( \tilde{u} \geq 0 \) weakly in \( W_0^{1,N}(\Omega) \), strongly in \( L^q(\Omega) \).
for $1 \leq q < \infty$, and a.e. in $\Omega$ for a further subsequence. Taking $v = u_{j-}$ in (3.4), dividing by $\rho_j^N$, and passing to the limit then gives

$$1 = \lambda \int_{\Omega} \tilde{u}^N dx,$$

so $\tilde{u} \neq 0$. Since the sequence $(u_j)$ is bounded, dividing (3.4) by $\rho_j^{N-1}$ gives

$$\int_{\Omega} |\nabla \tilde{u}|^{N-2} \nabla \tilde{u} \cdot \nabla v = \lambda \int_{\Omega} \tilde{u}^{N-1} v \, dx - \frac{\lambda}{\rho_j^{N-1}} \int_{\Omega} u_j^{N-1} e^{u_j^{N'}} v \, dx + o(1),$$

and passing to the limit using (3.7) gives

$$\int_{\Omega} |\nabla \tilde{u}|^{N-2} \nabla \tilde{u} \cdot \nabla v \, dx = \lambda \int_{\Omega} \tilde{u}^{N-1} v \, dx \quad \forall v \in C_0^\infty(\Omega).$$

This then holds for all $v \in W_0^{1,N}(\Omega)$ by density, so $\tilde{u} = t \varphi_1$ for some $t > 0$ and $\lambda = \lambda_1(N)$, contrary to assumption.

Since the sequence $(u_j)$ is bounded, a renamed subsequence converges to some $u$ weakly in $W_0^{1,N}(\Omega)$, strongly in $L^q(\Omega)$ for $1 \leq q < \infty$, and a.e. in $\Omega$. Since $\int_{\Omega} |u_j|^N e^{u_j^{N'}} \, dx$ is bounded by (3.3), then for any $v \in C_0^\infty(\Omega)$,

$$\int_{\Omega} |u_j|^{N-2} u_j e^{u_j^{N'}} v \, dx \to \int_{\Omega} |u|^{N-2} u e^{u^{N'}} v \, dx$$

by de Figueiredo et al. [8, Lemma 2.1]. So passing to the limit in (3.4) gives

$$\int_{\Omega} \left( |\nabla u|^{N-2} \nabla u \cdot \nabla v - \lambda |u|^{N-2} u e^{u^{N'}} v \right) \, dx = 0.$$  

This then holds for all $v \in W_0^{1,N}(\Omega)$ by density, so $u$ is a critical point of $\Phi$.

Suppose $u = 0$. Then

$$\int_{\Omega} |u_j|^{N-1} e^{u_j^{N'}} \, dx \to 0$$

by de Figueiredo et al. [8, Lemma 2.1] as above, and hence

$$\int_{\Omega} F(u_j) \, dx \to 0$$

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by (3.2) and the dominated convergence theorem, so
\[
\int_{\Omega} |\nabla u_j|^N \, dx \to Nc
\]
by (3.3). Since \( c < \alpha_{N-1}^N / N \), then \( \limsup \|u_j\| < \alpha_{N}^{1/N'} \), so there exists \( \beta > 1/\alpha_{N}^{1/N'} + 1/\gamma = 1 \), then
\[
\int_{\Omega} |u_j|^{N} e^{u_j^{N'}} \, dx \leq \left( \int_{\Omega} |u_j|^{\gamma N} \, dx \right)^{1/\gamma} \left( \int_{\Omega} e^{\alpha_{N}(\beta u_j^{N'})} \, dx \right)^{1/\alpha_{N}\beta N'} \to 0
\]
since \( u_j \to 0 \) in \( L^{\gamma N}(\Omega) \) and the last integral is bounded by (3.3). Then \( u_j \to 0 \) in \( W^{1,N}_0(\Omega) \) by (3.3), so \( \Phi(u_j) \to 0 \), contradicting \( c \neq 0 \).

Let \( i, \mathcal{M}, \Psi, \) and \( \lambda_k(N) \) be as in the introduction, and suppose that \( \lambda_k(N) < \lambda_{k+1}(N) \). Then the sublevel set \( \Psi^{\lambda_k(N)} \) has a compact symmetric subset \( E \) of index \( k \) that is bounded in \( L^{\infty}(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega) \) (see Degiovanni and Lancelotti [11, Theorem 2.3]). We may assume without loss of generality that 0 \( \in \Omega \). For all \( m \in \mathbb{N} \) so large that \( B_{2/m}(0) \subset \Omega \), let
\[
\eta_m(x) = \begin{cases} 
0 & \text{if } |x| \leq 1/2 m^{m+1} \\
2 m^{m} \left( |x| - \frac{1}{2 m^{m+1}} \right) & \text{if } 1/2 m^{m+1} < |x| \leq 1/m^{m+1} \\
(m |x|)^{1/m} & \text{if } 1/m^{m+1} < |x| \leq 1/m \\
1 & \text{if } |x| > 1/m,
\end{cases}
\]
set
\[
v_m(x) = \eta_m(x) \, v(x), \quad v \in E,
\]
and let \( E_m = \{ \pi(v_m) : v \in E \} \), where \( \pi : W^{1,N}_0(\Omega) \setminus \{0\} \to \mathcal{M}, u \mapsto u/\|u\| \) is the radial projection onto \( \mathcal{M} \).

**Lemma 3.3.** There exists a constant \( C = C(N, \Omega, k) > 0 \) such that for all sufficiently large \( m \),
(i) \( \Psi(w) \leq \lambda_k(N) + \frac{C}{m^{N-1}} \) \( \forall w \in E_m \),

(ii) \( E_m \cap \Psi_{\lambda_{k+1}(N)} = \emptyset \),

(iii) \( i(E_m) = k \).

Proof. Let \( v \in E \) and let \( w = \pi(v_m) \). We have

\[
\int |\nabla v_m|^N dx \leq \int_{\Omega \setminus B_{1/m}(0)} |\nabla v|^N dx + \sum_{j=0}^{N} \binom{N}{j} \int_{B_{1/m}(0)} \eta_m^{N-j} |\nabla v|^{N-j} |v|^j |\nabla \eta_m|^j dx.
\]

Since \( E \) is bounded in \( C^1(B_{1/m}(0)) \), \( \nabla v \) and \( v \) are bounded in \( B_{1/m}(0) \). Clearly, \( \eta_m \leq 1 \), and a direct calculation shows that

\[
\int_{B_{1/m}(0)} |\nabla \eta_m|^j dx \leq \frac{C}{m^{N-1}}, \quad j = 0, \ldots, N.
\]

Since \( E_m \subset \mathcal{M} \), it follows that

\[
\int |\nabla v_m|^N dx \leq 1 + \frac{C}{m^{N-1}}.
\]

Next

\[
\int |v_m|^N dx \geq \int_{\Omega \setminus B_{1/m}(0)} |v|^N dx = \int_{\Omega} |v|^N dx - \int_{B_{1/m}(0)} |v|^N dx \geq \frac{1}{\lambda_k(N)} - \frac{C}{m^N}
\]

since \( E \subset \Psi_{\lambda_k(N)} \). So

\[
\Psi(w) = \frac{\int |\nabla v_m|^N dx}{\int |v_m|^N dx} \leq \lambda_k(N) + \frac{C}{m^{N-1}}
\]

if \( m \) is sufficiently large. Taking \( m \) so large that \( \lambda_k(N) + C/m^{N-1} < \lambda_{k+1}(N) \) then gives \((II)\). Since \( E_m \subset \mathcal{M} \setminus \Psi_{\lambda_{k+1}(N)} \) by \((II)\)

\[
i(E_m) \leq i(\mathcal{M} \setminus \Psi_{\lambda_{k+1}(N)}) = k
\]

by the monotonicity of the index and \((13)\). On the other hand, since \( E \to E_m, v \mapsto \pi(v_m) \) is an odd continuous map,

\[
i(E_m) \geq i(E) = k.
\]

So \( i(E_m) = k \). \( \square \)
We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We have $\lambda_k(N) < \lambda < \lambda_{k+1}(N)$ for some $k \in \mathbb{N}$. Fix $\lambda_k(N) < \lambda' < \lambda$ and $m$ so large that the conclusions of Lemma 3.3 hold with $\lambda(N) + C/m^{N-1} \leq \lambda'$, in particular,

$$\Psi(w) \leq \lambda' \quad \forall w \in E_m. \quad (3.8)$$

Then take $A_0 = E_m$ and $B_0 = \Psi_{\lambda_{k+1}(N)}$, and note that $A_0$ and $B_0$ are disjoint nonempty closed symmetric subsets of $M$ such that

$$i(A_0) = i(M \setminus B_0) = k$$

by Lemma 3.3 (iii) and (1.5). Now let $R > r > 0$ and let $A$ and $B$ be as in Theorem 1.5.

First we show that $\inf \Phi(B) > 0$ if $r$ is sufficiently small. Since $e^t \leq 1 + te^t$ for all $t > 0$,

$$F(t) \leq \frac{|t|^N}{N} + t e^{t}N' \quad \forall t \in \mathbb{R},$$

where $\mu = N + N' > N$. So for $u \in \Psi_{\lambda_{k+1}(N)},$

$$\Phi(ru) \geq \int \left[ \frac{r^N}{N} |\nabla u|^N - \frac{\lambda r^N}{N} |u|^N - \lambda r^\mu u^\mu e^{r N'} u^N' \right] dx$$

$$\geq \frac{r^N}{N} \left( 1 - \frac{\lambda}{\lambda_{k+1}(N)} \right) - \lambda r^\mu \left( \int e^{2r N'} u^N' dx \right)^{1/2} |u|^\mu. \quad (3.9)$$

If $2r^{N'} \leq \alpha_N$, then

$$\int e^{2r N'} u^{N'} dx \leq \int e^{\alpha_N u^{N'}} dx,$$

which is bounded by (1.3). Since $W_1^{1,N}(\Omega) \hookrightarrow L^2(\Omega)$ and $\lambda < \lambda_{k+1}(N)$, it follows that $\inf \Phi(B) > 0$ if $r$ is sufficiently small.

Since $e^t \geq 1 + t$ for all $t > 0$,

$$F(t) \geq \frac{|t|^N}{N} + \frac{t}{\mu} \quad \forall t \in \mathbb{R},$$

(3.9)
so for all \( w \in E_m \) and \( t \geq 0 \),

\[
\Phi(tw) \leq \int_{\Omega} \left[ \frac{t^N}{N} |\nabla w|^N - \frac{\lambda^N}{N} |w|^N \right] dx = \frac{t^N}{N} \left( 1 - \frac{\lambda}{\Psi(w)} \right) \leq - \frac{t^N}{N} \left( \frac{\lambda}{N} - 1 \right) \leq 0
\] (3.10)

by (3.8).

Next we show that

\[
\sup_{w \in E_m, s, t \geq 0} \Phi(sw + tv_0) < \frac{\alpha_{N-1}^N}{N}
\]

for a suitably chosen \( v_0 \in M \setminus E_m \). Let

\[
v_j(x) = \begin{cases} 
\left( \log j \right)^{(N-1)/N} & \text{if } |x| \leq 1/j \\
\log |x|^{-1} & \text{if } 1/j < |x| \leq 1 \\
0 & \text{if } |x| > 1.
\end{cases}
\]

Then \( v_j \in W^{1,N}(\mathbb{R}^N) \), \( \|v_j\| = 1 \), and \( |v_j|^N = O(1/\log j) \) as \( j \to \infty \). We take

\[
v_0(x) = \tilde{v}_j(x) := v_j\left( \frac{x}{r_m} \right)
\]

with \( r_m = 1/2m^{m+1} \) and \( j \) sufficiently large. Since \( B_{r_m}(0) \subset \Omega \), \( \tilde{v}_j \in W^{1,N}_0(\Omega) \) and \( \|\tilde{v}_j\| = 1 \). For sufficiently large \( j \),

\[
\Psi(\tilde{v}_j) = \frac{1}{r_m |v_j|^N} \geq \lambda
\]

and hence \( \tilde{v}_j \notin E_m \) by (3.8). For \( w \in E_m \) and \( s, t \geq 0 \),

\[
\Phi(sw + t\tilde{v}_j) = \Phi(sw) + \Phi(t\tilde{v}_j)
\]

since \( w = 0 \) on \( B_{r_m}(0) \) and \( \tilde{v}_j = 0 \) on \( \Omega \setminus B_{r_m}(0) \). Since \( \Phi(sw) \leq 0 \) by (3.10), it suffices to show that

\[
\sup_{t \geq 0} \Phi(t\tilde{v}_j) < \frac{\alpha_{N-1}^N}{N}
\]
for arbitrarily large \( j \). Since \( \Phi(t\tilde{v}_j) \to -\infty \) as \( t \to \infty \) by (3.9), there exists \( t_j \geq 0 \) such that
\[
\Phi(t_j\tilde{v}_j) = \frac{t_j^N}{N} - \lambda \int_{B_{r_m}(0)} F(t_j\tilde{v}_j) \, dx = \sup_{t \geq 0} \Phi(t\tilde{v}_j) \tag{3.11}
\]
and
\[
\Phi'(t_j\tilde{v}_j) \tilde{v}_j = t_j^{N-1} \left( 1 - \lambda \int_{B_{r_m}(0)} t_j^{N'} \tilde{v}_j^{N'} \, dx \right) = 0. \tag{3.12}
\]
Suppose \( \Phi(t_j\tilde{v}_j) \geq \alpha N^{-1}/N \) for all sufficiently large \( j \). Since \( F(t) \geq 0 \) for all \( t \in \mathbb{R} \), then (3.11) gives \( t_j^N \geq \alpha \), and then (3.12) gives
\[
\frac{1}{\lambda} = \int_{B_{r_m}(0)} \tilde{v}_j^N e^{\lambda} \tilde{v}_j^{N'} \, dx \geq \int_{B_{r_m}(0)} \tilde{v}_j^N e^{\alpha N} \tilde{v}_j^{N'} \, dx
\]
\[
= r_m^N \int_{B_{1/2}(0)} v_j^N e^{\alpha N} v_j^{N'} \, dx \geq r_m^N \int_{B_{1/2}(0)} v_j^N e^{\alpha N} v_j^{N'} \, dx = \frac{r_m^N}{N} (\log j)^{N-1},
\]
which is impossible for large \( j \).

Now we show that \( \Phi \leq 0 \) on \( A \) if \( R \) is sufficiently large. In view of (3.10), it only remains to show that \( \Phi(Ru) \leq 0 \) for \( u = \pi((1 - t) w + tv_0) \), \( w \in E_m, 0 \leq t \leq 1 \). Since
\[
\| (1 - t) w + tv_0 \| \leq (1 - t) \| w \| + t \| v_0 \| = 1
\]
and \( w \) and \( v_0 \) are supported on disjoint sets, we have
\[
|u|^N_N = \frac{|(1 - t) w + tv_0|^N_N}{\| (1 - t) w + tv_0 \|^N_N} \geq (1 - t)^N |w|^N_N + t^N |v_0|^N_N \geq \frac{(1 - t)^N}{\Psi(w)} \geq \frac{(1 - t)^N}{\lambda} \tag{3.13}
\]
by (3.8), and
\[
|u_+|^\mu = \frac{|(1 - t) w + tv_0|^\mu}{\| (1 - t) w + tv_0 \|^\mu} \geq (1 - t)^\mu |w_+|^\mu + t^\mu |v_0|^\mu \geq t^\mu |v_0|^\mu. \tag{3.14}
\]
By (3.9), (3.13), and (3.14),
\[
\Phi(Ru) \leq \frac{R^N}{N} \| u \|^N_N - \lambda \frac{R^N}{N} |u|^N_N - \frac{R^\mu}{\mu} |u_+|^\mu \leq - \frac{1}{N} \left[ \frac{\lambda}{\lambda'} (1 - t)^N - 1 \right] R^N - \frac{1}{\mu} |v_0|^\mu t^\mu R^\mu.
\]

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The last expression is clearly nonpositive if \( t \leq 1 - (\lambda' / \lambda)^{1/N} =: t_0 \). For \( t > t_0 \), it is nonpositive if \( R \) is sufficiently large.

The inequalities (1.6) now imply that \( 0 < c < \frac{\alpha_N}{N} \). If \( \Phi \) has no \( (C)_c \) sequences, then \( \Phi \) satisfies the \( (C)_c \) condition trivially and hence \( c \) is a critical value of \( \Phi \) by Theorem 1.5. If \( \Phi \) has a \( (C)_c \) sequence, then a subsequence converges weakly to a nontrivial critical point of \( \Phi \) by Lemma 3.2.

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