Chern-Simons supergravities, with a twist

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Abstract

We discuss noncommutative extensions of Chern-Simons (CS) supergravities in odd dimensions. The example of $D = 5$ CS supergravity, invariant under the gauge supergroup $SU(2,2|N)$, is worked out in detail. Its noncommutative version, with a $\star$-product associated to an abelian Drinfeld twist, is found to exist only for $N = 4$. 
1 Introduction

Chern-Simons (CS) supergravities [1, 2, 3] offer an interesting alternative to standard supergravities for at least two reasons:

- supersymmetry is realized as a gauge symmetry, part of a gauge supergroup $G$ under which the CS Lagrangian is invariant up to a total derivative. The superalgebra closes off-shell by construction.
- the gauge supergroup contains the (anti)-De Sitter superalgebra, so that the theory is translation-invariant and does not have dimensionful coupling constants. Group contraction can be used to recover the Poincaré superalgebra, and the corresponding Poincaré supergravity.

Both these features can be relevant for a consistent quantization of the theory [2]. CS gravities and supergravities live only in odd dimensions $D = 2n - 1$, and contain, besides the usual Einstein-Hilbert term and its supersymmetrization, also a cosmological term (in the uncontracted version) and higher powers of the curvature 2-form $R$ up to order $n - 1$. Note also that CS gravities are a particular example of Lovelock gravities [4], with at most second order field equations for the metric.

In this paper we present a noncommutative (NC) extension of five dimensional Chern-Simons supergravity. We find that noncommutativity requires a particular value of $N$ in the gauge supergroup $SU(2, 2|N)$, namely $N = 4$, for which the supergroup becomes nonsimple, with a central $U(1)$. The gauge potentials of $SU(2, 2|4)$ are the fields appearing in the noncommutative action: the vielbein $V^a$, four gravitini $\psi_i$, the spin connection $\omega^{ab}$, the $SU(4)$ gauge field $a^i j$, and an extra $U(1)$ gauge field $b$ (necessary for the NC extension).

Physical motivations for studying (super)gravity on noncommutative spacetime, reformulated as a field theory on ordinary spacetime but with a deformed $\star$-product between fields, have been extensively discussed in the last two decades. The possibility of encoding quantum properties in the texture of spacetime, and obtain deformations of gauge and gravity theories (invariant under deformed symmetries), is one among the interesting applications of noncommutative geometry to physics. Comprehensive reviews can be found for example in refs. [5, 6, 7, 8, 9, 10, 11].

The $D = 5$ NC Chern-Simons action studied here is invariant under the full supergroup noncommutative transformations (or $\star$-transformations), in particular under the four $\star$-supersymmetries. It provides an example of locally $\star$-supersymmetric noncommutative theory in $D = 5$. (A $D = 3$ locally supersymmetric theory was constructed in [12], see also ref. [13] for a $D = 4$ noncommutative supergravity).

Four-dimensional noncommutative gravities based on topological actions ` à la Mac-Dowell-Mansouri have been considered in [15, 16, 17, 18, 19], and in [14] for $OSp(1|4)$ noncommutative supergravity. One of the early works on higher dimensional Chern-Simons actions in noncommutative spaces is ref. [20].

The paper is organized as follows. Section 2 briefly recalls the definition of Chern-Simons forms and some of their properties. In Section 3 we discuss their noncommutative extensions. Section 4 deals with noncommutative $D = 5$ CS supergravity, and Section 5 contains some conclusions and outlook. In Appendix A
we discuss in some detail the gauge variation of noncommutative CS forms, with
the explicit example of $D = 3$. Finally in Appendix B we collect conventions and
useful formulas for $D = 5$ gamma matrix algebra.

2 Chern-Simons forms

By definition a CS Lagrangian $L_{CS}^{(2n-1)}$ is a $(2n - 1)$-form whose exterior derivative
yields a gauge invariant $2n$-form. In the present note we concentrate on the case

$$dL_{CS}^{(2n-1)} = STr(R^n)$$

(2.1)

where $R^n \equiv R \wedge R \wedge \cdots \wedge R$ ($n$ times), the curvature 2-form $R$ being defined
as $R = d\Omega - \Omega \wedge \Omega$, and $L_{CS}^{(2n-1)}$ contains (exterior products of) the $G$
gauge potential one-form $\Omega$ and its exterior derivative. The supertrace $STr$ is taken on
some representation of the supergroup $G$.

Thus the CS action is related to a topological action in $2n$ dimensions via Stokes
d=\int_{\partial M} L_{CS}^{(2n-1)} = \int_M STr(R^n)$$

(2.2)

Gauge transformations are defined by

$$\delta_\varepsilon \Omega = d\varepsilon - \Omega \varepsilon + \varepsilon \Omega, \quad \Rightarrow \quad \delta_\varepsilon R = -R\varepsilon + \varepsilon R$$

(2.3)

so that $STr(R^n)$ is manifestly gauge invariant. Then (2.1) implies that the gauge
variation of $L_{CS}^{(2n-1)}$ is closed, and hence locally exact:

$$\delta_\varepsilon L_{CS}^{(2n-1)} = d\alpha^{(2n-2)}(\Omega, R, \varepsilon)$$

(2.4)

We conclude that the Chern-Simons action is gauge invariant

$$\delta_\varepsilon \int L_{CS}^{(2n-1)} = 0$$

(2.5)

with suitable boundary conditions.

The CS Lagrangian is given in terms of $\Omega$ and $d\Omega$ (or $R$) by the following
expressions [21, 22]:

$$L_{CS}^{(2n-1)} = n \int_0^1 STr[\Omega(td\Omega - t^2\Omega^2)^{n-1}]dt = n \int_0^1 t^{n-1} STr[\Omega(R + (1-t)\Omega^2)^{n-1}]dt$$

(2.6)

For example:

$$L_{CS}^{(3)} = STr[R\Omega + \frac{1}{3}\Omega^3]$$

(2.7)

$$L_{CS}^{(5)} = STr[R^2\Omega + \frac{1}{2}R\Omega^3 + \frac{1}{10}\Omega^5]$$

(2.8)

$$L_{CS}^{(7)} = STr[R^3\Omega + \frac{2}{5}R^2\Omega^3 + \frac{1}{5}R\Omega^2R\Omega + \frac{1}{5}R\Omega^5 + \frac{1}{35}\Omega^7]$$

(2.9)
In the following $L^{(2n-1)}_{CS}$ is always considered as a function of $\Omega$ and $R$. Its gauge variation is easily computed using (2.3). For example:

$$\delta_\varepsilon L^{(5)}_{CS} = STr[R^2\varepsilon + \frac{1}{2}R(d\varepsilon\Omega^2 + \Omega d\varepsilon\Omega + \Omega^2 d\varepsilon) + \frac{1}{2}d\varepsilon\Omega^4]$$  \hspace{1cm} (2.10)$$

The general rule to obtain the variation is simple: just replace in $L^{(2n-1)}_{CS}$ each $\Omega$ factor in turn by $d\varepsilon$ (the terms with undifferentiated $\varepsilon$ cancel out because of the cyclicity of $STr$). Using now

$$d\Omega = R + \Omega \Omega, \quad dR = \Omega R - R \Omega \quad \text{(Bianchi identity)} \hspace{1cm} (2.11)$$

one recognizes that

$$\delta_\varepsilon L^{(5)}_{CS} = d STr[R^2\varepsilon + \frac{1}{2}R(\varepsilon\Omega^2 - \Omega \varepsilon\Omega + \Omega^2 \varepsilon) + \frac{1}{2}\varepsilon\Omega^4] \hspace{1cm} (2.12)$$

This leads to the general recipe

$$\delta_\varepsilon L^{(2n-1)}_{CS} = d(j_\varepsilon L^{(2n-1)}_{CS}) \hspace{1cm} (2.13)$$

where $j_\varepsilon$ is a contraction acting selectively on $\Omega$, i.e.

$$j_\varepsilon \Omega = \varepsilon, \quad j_\varepsilon R = 0 \hspace{1cm} (2.14)$$

with the graded Leibniz rule $j_\varepsilon(\Omega\Omega) = j_\varepsilon(\Omega)\Omega - \Omega j_\varepsilon(\Omega) = \varepsilon\Omega - \Omega \varepsilon$ etc.

## 3 Noncommutative CS actions

The preceding discussion holds for a generic supergauge connection $\Omega$, and relies only on the (graded) cyclicity of the supertrace. As such, it can be extended without effort to construct noncommutative Chern-Simons actions, where the noncommutativity is controlled by an abelian twist. This amounts to a deformation of the exterior product:

$$\tau \wedge_* \tau' \equiv \sum_{n=0}^{\infty} \binom{i}{n} \theta^{A_1 B_1} \cdots \theta^{A_n B_n} (\ell_{X_{A_1}} \cdots \ell_{A_n} \tau) \wedge (\ell_{B_1} \cdots \ell_{B_n} \tau')$$

$$= \tau \wedge \tau' + \frac{i}{2} \theta^{AB} (\ell_A \tau) \wedge (\ell_B \tau') + \frac{1}{2!} \binom{i}{2} \theta^{A_1 B_1} \theta^{A_2 B_2} (\ell_{A_1} \ell_{A_2} \tau) \wedge (\ell_{B_1} \ell_{B_2} \tau') + \cdots$$  \hspace{1cm} (3.1)$$

where $\theta^{AB}$ is a constant antisymmetric matrix, and $\ell_A$ are Lie derivatives along commuting vector fields $X_A$. This noncommutative product is associative due to $[X_A, X_B] = 0$. If the vector fields $X_A$ are chosen to coincide with the partial derivatives $\partial_\mu$, and if $\tau, \tau'$ are 0-forms, then $\tau \star \tau'$ reduces to the well-known Moyal-Groenewold product \[{23}\]. A short review on twisted differential geometry can be found for example in \[{18}\].
However, the supertrace is not cyclic for twisted products. What is (graded) cyclic is the \(\text{integrated} \) supertrace:

\[
\int \text{STr}(\tau \wedge \star \tau') = (-1)^{\text{deg}(\tau)\text{deg}(\tau')} \int \text{STr}(\tau' \wedge \star \tau) + \text{boundary terms}
\] (3.2)

This is sufficient to prove the \(\star\)-gauge invariance of the noncommutative Chern-Simons action. Indeed if we denote by \(L_{CS*}^{(2n-1)}\) the noncommutative Chern-Simons Lagrangian, obtained by substituting \(\star\)-exterior products to ordinary exterior products, then

\[
\delta_\star \int L_{CS*}^{(2n-1)} = \int d(j_\varepsilon L_{CS*}^{(2n-1)}) = 0
\] (3.3)

for suitable boundary conditions. This is because the variation formula (2.13) holds, but only under integration, also in the \(\star\)-deformed case, with \(\star\)-gauge transformations given by:

\[
\delta_\star \varepsilon \Omega = d\varepsilon - \Omega \star \varepsilon + \varepsilon \star \Omega, \quad \Rightarrow \quad \delta_\star R = -R \star \varepsilon + \varepsilon \star R
\] (3.4)

For example the \(D=5 \star\)-Chern-Simons action reads

\[
\int L_{CS*}^{(5)} = \int \text{STr}[R \wedge R \wedge \Omega + \frac{1}{2} R \wedge R \wedge \Omega \wedge \Omega + \frac{1}{10} \Omega \wedge R \wedge R \wedge R \wedge R \wedge \Omega]
\] (3.5)

and is invariant under the \(\star\)-gauge variations (3.4).

4 \(D=5\) noncommutative CS supergravity

The relevant supergroup for \(D=5\) CS supergravity is \(SU(2,2|N)\) (For a group-geometric construction of standard \(D=5\) supergravity see for ex. [24], p. 755). We begin by writing the noncommutative connection and curvature supermatrices. The gauge connection 1-form is given by:

\[
\Omega \equiv \begin{pmatrix} \Omega^\alpha_\beta & \psi^a_j \\ -\psi^i_{\bar{\beta}} & A^i_j \end{pmatrix}, \quad \Omega^\alpha_\beta \equiv \left(\frac{1}{4} \omega^{ab}_a \gamma_{ab} - \frac{i}{2} V^a \gamma_a + \frac{i}{4} b I\right)^\alpha_\beta, \quad A^i_j = \frac{i}{N} b \delta^i_j + a^i_j
\] (4.1)

where the bosonic \(U(2,2)\) subgroup is gauged by the 1-forms \(\omega^{ab}\) (spin connection), \(V^a\) (vielbein) and \(b\) (\(U(1)\) gauge field); the antihermitian matrix-valued 1-forms \(a^i_j\) \((i,j = 1...N)\) gauge the \(SU(N)\) bosonic subgroup; finally the \(N\) gravitino 1-form fields \(\psi_j\) gauge the \(N\) supersymmetries. The Dirac conjugate is defined as \(\bar{\psi} = \psi^\dagger \gamma_0\). The NC connection coincides in fact with the commutative one: indeed for generic \(N\) no extra fields are needed (contrary to the case of \(D=4\) NC (super)gravity [18, 13]). This is due to the fact that the \(D=5\) gamma matrices \(\gamma^a, \gamma^{ab}\) and the identity matrix span a complete basis for \(4 \times 4\) matrices.

The corresponding curvature supermatrix 2-form is

\[
\mathbf{R} = d\Omega - \Omega \wedge \Omega \equiv \begin{pmatrix} R + \psi_i \wedge \psi^j \Sigma_j - \Sigma_i F^i_j + \bar{\psi} \wedge \psi_j \end{pmatrix}
\] (4.2)
Immediate algebra yields the components of the $U(2,2)$ curvature $R$:

\[
R^{ab} = d\omega^{ab} - \frac{1}{2} \omega_c^{[a} \wedge_a \omega^{b]c} + \frac{1}{2} V^{[a} \wedge_a V^{b]} + \frac{i}{4} \epsilon^{cde} (\omega^{cd} \wedge_e V^e + V^e \wedge_e \omega^{cd}) - \frac{i}{4} (\omega^{ab} \wedge b + b \wedge \omega^{ab})
\]

\[
R^a = dV^a - \frac{1}{2} (\omega^a_b \wedge b - b \wedge \omega^a_b) + \frac{i}{8} \epsilon^{bcde} \omega^{bc} \wedge \omega^{de} - \frac{i}{4} (V^a \wedge b + b \wedge V^a)
\]

\[
r = db - \frac{i}{2} \omega^{ab} \wedge \omega_{ab} - iV^a \wedge V^a - \frac{i}{4} b \wedge b
\]

A direct consequence of the curvature definition \((4.2)\) is the Bianchi identity

\[
dR = -R \wedge \Omega + \Omega \wedge_R R
\]

which becomes, on the supermatrix entries

\[
d\Sigma = -R \wedge \psi + \Omega \wedge \Sigma - \Sigma \wedge \psi + A \wedge F,
\]

\[
d\tilde{\Sigma} = -\tilde{\Sigma} \wedge \tilde{\psi} + \Omega \wedge \Sigma \wedge F
\]

### 4.1 $SU(2,2|N)$ gauge transformations

The NC gauge transformations \((3.4)\) close on the $\ast$-Lie algebra:

\[
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_1 \ast \epsilon_2 - \epsilon_2 \ast \epsilon_1}
\]

In the case at hand the $SU(2,2|N)$ gauge parameter is given by the supermatrix

\[
\epsilon \equiv \begin{pmatrix}
\epsilon^a_{\beta} & \epsilon^a_{\beta} \\
-\bar{\epsilon}^a_{\beta} & \eta^a_{\bar{i} j}
\end{pmatrix}, \quad \epsilon^a_{\beta} \equiv \frac{1}{4} \epsilon^{abc} \gamma_{ab} - \frac{i}{2} \epsilon^a \gamma_a + \frac{i}{4} \epsilon_D^a, \quad \eta^a_{\bar{i} j} = \frac{i}{N} \epsilon \delta^a_{\bar{i} j} + \epsilon^a_{\bar{i} j}
\]

and the NC gauge variations on the block entries of $\Omega$ read

\[
\delta \Omega = d\epsilon - \Omega \ast \epsilon + \epsilon \ast \Omega + \psi \ast \bar{\epsilon} + \epsilon \ast \bar{\psi}
\]

\[
\delta \psi_i = d\epsilon_i - \Omega \ast \epsilon_i + \epsilon_i \ast \bar{\psi} + \psi \ast \bar{\psi}_i + \epsilon \ast \psi_i
\]

\[
\delta \bar{\psi}_i = d\bar{\epsilon}_i + \bar{\epsilon} \ast \Omega - \bar{\psi}_i \ast \bar{\psi} + \psi \ast \bar{\psi}_i - \epsilon \ast \bar{\psi}_i
\]

\[
\delta A^i = d\eta^i_j - A^i_k \ast \eta^k_j + \eta^i_j \ast A^k_j + \psi \ast \bar{\psi}_i - \bar{\epsilon} \ast \psi_i
\]
On the \( \Omega \) component fields the gauge variations take the form

\[
\delta \omega^{ab} = d \varepsilon^{ab} - \omega^{[a} \varepsilon^{b]c} + \varepsilon_c^{[a} \omega^{b]c} + V^{[a} \varepsilon^{b]} + \varepsilon^{[b} V^{a]} + \frac{1}{2}(\bar{\psi} \gamma^{ab} \varepsilon - \bar{\varepsilon} \gamma^{ab} \psi) \\
+ \frac{i}{4} \varepsilon_{cde}(\omega^{cd} \varepsilon - \varepsilon^{c} \omega^{cd} + V^{c} \varepsilon^{cd} - \varepsilon^{cd} V^{e}) - \frac{i}{4}(\omega^{ab} \varepsilon - \bar{\varepsilon} \omega^{ab}) \\
- \frac{i}{4}(\bar{\varepsilon} \varepsilon^{ab} - \varepsilon^{ab} \varepsilon) \tag{4.20}
\]

\[
\delta V^a = d \varepsilon^a - \frac{1}{2}(\omega^{ab} \varepsilon^b + \varepsilon^b \omega^{ab}) + \frac{1}{2}(V^b \varepsilon^{ab} + \varepsilon^{ab} V^b) - i(\bar{\psi} \gamma^a \varepsilon - \bar{\varepsilon} \gamma^a \psi) \\
+ \frac{i}{8} \varepsilon_{bcde}(\omega^{bc} \varepsilon^{de} - \varepsilon^{de} \omega^{bc}) + \frac{i}{4}(\varepsilon^a b - b \varepsilon^a + \varepsilon V^a - V^a \varepsilon) \tag{4.21}
\]

\[
\delta \psi_i = d \omega^i - \Omega \gamma^i + \varepsilon_j A^j_i + \left( \frac{1}{4} \varepsilon^{ab} \gamma_{ab} - \frac{i}{2} \varepsilon^a \gamma_a + \frac{i}{4} \bar{\varepsilon} \right) \psi_i - \frac{i}{N} \psi_i \varepsilon - \psi_j \varepsilon^j_i \tag{4.22}
\]

with \( \Omega \) and \( A^j_i \) given in (4.11). On the \( b \) field we find, respectively from \( \delta \Omega \) and \( \delta A^j_i \)

\[
\delta b = d \varepsilon - i(\bar{\psi} \varepsilon - \bar{\varepsilon} \psi) + \frac{i}{2}(\varepsilon^{ab} \omega^{ab} - \omega^{ab} \varepsilon^{ab}) + i(\varepsilon_a V^a - V^a \varepsilon_a) \\
+ \frac{i}{4}(\varepsilon * b - b * \varepsilon) \tag{4.23}
\]

\[
\delta b = d \varepsilon - i(\bar{\psi} \varepsilon - \bar{\varepsilon} \psi) + i(\varepsilon_j a^j_i - \varepsilon^j_i a^i_j) \\
+ \frac{i}{N}(\varepsilon * b - b * \varepsilon) \tag{4.24}
\]

We see that the \( b \) gauge variations are consistent only if \( N = 4 \) (otherwise the variation of \( b \) under \( U(1) \) in the second line of (4.23) would not agree with the second line of (4.24)). Thus a noncommutative extension of Chern-Simons \( D = 5 \) supergravity exists only for \( N = 4 \). In this case the supergroup \( SU(2,2|4) \) is not simple anymore and the \( U(1) \) gauged by the \( b \) field becomes a central extension. Note that in the commutative limit \( \varepsilon * b - b * \varepsilon \) vanishes, and no condition on \( N \) arises.

Consider now the \( U(1) \) gauge variation of the gravitini, cf. (4.22):

\[
\delta \psi_i = \frac{i}{4} \varepsilon \psi_i - \frac{i}{N} \psi_i \varepsilon \tag{4.25}
\]

For \( N = 4 \) we see that in the commutative limit the gravitini become uncharged with respect to this \( U(1) \), but remain charged in the noncommutative setting.

### 4.2 The action

Substituting \( R \) and \( \Omega \) into the Chern-Simons action (3.5), we obtain the noncommutative CS action invariant under the \( SU(2,2|4) \) gauge variations of the preceding subsection. The result is

\[
\int Str(L_{CS}^{(5)}) = \int L_{U(2,2)} + L_A + L_{fermi} \tag{4.26}
\]
with

\[ L_{U(2,2)} = Tr[R \wedge_* R \wedge_* \Omega + \frac{1}{2} R \wedge_* \Omega^3 + \frac{1}{10} \Omega^5] \quad (4.27) \]

\[ L_A = -Tr[F \wedge_* F \wedge_* A + \frac{1}{2} F \wedge_* A^3 + \frac{1}{10} A^5] \quad (4.28) \]

\[ L_{fermi} = \frac{3}{2} \bar{\psi} \wedge_* (R \wedge_* \Sigma + \Sigma \wedge_* F) + \frac{3}{2} \bar{\Sigma} \wedge_* (R \wedge_* \psi + \psi \wedge_* F) + \bar{\psi} \wedge_* \psi \wedge_* (\bar{\psi} \wedge_* \Sigma + \Sigma \wedge_* \psi) \quad (4.29) \]

where \( \Omega^3 \equiv \Omega \wedge_* \Omega \wedge_* \Omega \) etc. In the commutative limit it reproduces the action discussed in refs. [25, 1, 2]. The \( b \) field kinetic term has two contributions, from the \( R \wedge_* R \wedge_* \Omega \) and the \( F \wedge_* F \wedge_* A \) terms, and is proportional to:

\[ (\frac{1}{16} - \frac{1}{N^2})(db \wedge_* db \wedge_* b) \quad (4.30) \]

which vanishes for \( N = 4 \), as in the commutative case. The essential difference is that the gravitini retain here a nonvanishing \( U(1) \) charge.

Since the \( \star \)-product contains the imaginary unit in its definition, it is necessary to check the reality of the NC action. We first observe that the supermatrix connection \( \Omega \) satisfies a reality condition:

\[ \Omega^\dagger = -\Gamma_0 \Omega \Gamma_0, \quad \Gamma_0 \equiv \left( \begin{array}{cc} \gamma_0 & 0 \\ 0 & I \end{array} \right) \quad (4.31) \]

due to \( \gamma_{ab} \) being \( \gamma_0 \) antihermition (i.e. \( \gamma_0^\dagger = -\gamma_0 \gamma_{ab} \gamma_0 \) etc), while \( I \) and \( \gamma_a \) are \( \gamma_0 \)-hermitian. Noting that \( \Gamma_0^2 = 1 \), and that the \( \Gamma_0 \)-antihermition of \( \Omega \) implies \( \Gamma_0 \)-antihermition of \( R \), one easily proves that the NC action \( (3.5) \) or \( (4.26) \), multiplied by \( i \), is real.

We can obtain a slightly more explicit form for \( \int L_{U(2,2)} \) by splitting the \( U(2,2) \) connection in its “Lorentz + rest” parts as

\[ \Omega = \omega + V, \quad \omega \equiv \frac{1}{4} \omega^{ab} \gamma_{ab}, \quad V \equiv -\frac{i}{2} V^a \gamma_a + \frac{i}{4} bI \quad (4.32) \]

and correspondingly the \( U(2,2) \) curvature as

\[ R = \mathcal{R} + T - V \wedge_* V, \quad \mathcal{R} \equiv d\omega - \omega \wedge_* \omega, \quad T \equiv dV - \omega \wedge_* V - V \wedge_* \omega \quad (4.33) \]

Then we find, after some integrations by parts and use of the Bianchi identities \( (4.11)-(4.13) \):

\[ \int L_{U(2,2)} = 3 \int Tr[\mathcal{R} \wedge_* \mathcal{R} \wedge_* V \wedge_* V - \frac{2}{3} \mathcal{R} \wedge_* V^3 + \frac{1}{5} V^{*5} + \frac{1}{2}(T \wedge_* R + R \wedge_* T) \wedge_* V + \frac{1}{3} T \wedge_* T \wedge_* V - \frac{1}{2} T \wedge_* V^3] + \int Tr[\mathcal{R} \wedge_* \mathcal{R} \wedge_* \omega \wedge_* \omega + \frac{1}{2} \mathcal{R} \wedge_* \omega^3 + \frac{1}{10} \omega^{*5}] \quad (4.34) \]
The last line is the integral of the noncommutative Lorentz CS form \( L_{\text{Lorentz}} \). Its integrated derivative gives the integrated NC Pontryagin 6-form:

\[
\int dL_{\text{Lorentz}} = \int Tr[\mathcal{R} \wedge \sigma \mathcal{R} \wedge \sigma \mathcal{R}]
\] (4.35)

This 6-form \( Tr[\mathcal{R} \wedge \sigma \mathcal{R} \wedge \sigma \mathcal{R}] \) vanishes in the commutative limit, so that the ordinary \( L_{\text{Lorentz}} \) is closed and its integral becomes a boundary term, vanishing for suitable boundary conditions. Thus the last line of (4.34) is absent in the commutative limit, but gives a nonvanishing contribution in the NC action.

5 Conclusions and outlook

We have constructed a noncommutative version of Chern-Simons supergravity in five dimensions. The theory is invariant under the \( \star \)-gauge transformations of the supergroup \( SU(2, 2|4) \). The geometric generalization of the Seiberg-Witten map [26] developed in refs. [27, 28] can now be applied to this NC action: the result is a classical higher-derivative deformation of \( SU(2, 2|4) \) Chern-Simons supergravity in \( D = 5 \), where the zero order (in \( \theta \)) term coincides with the classical action of refs. [25, 1, 2]. All higher-order (in \( \theta \)) corrections are separately gauge invariant under the ordinary \( SU(2, 2|4) \) gauge transformations. This work is in progress and will be reported in a separate paper.

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A On the gauge variation of NC Chern-Simons forms

We have argued in Section 3 that the integrated gauge variation of the NC Chern-Simons form vanishes for suitable boundary conditions since it is equal to a surface term.

This implies that the gauge variation is a total derivative \( d\alpha(\Omega, R, \varepsilon) \). That it must be so is also clear from the fact that all computations are similar to the commutative ones, and these are based on the graded cyclicity of the (super)trace, which holds in the noncommutative case up to total derivatives.

As an exercise, we determine here the explicit expression of \( \alpha(\Omega, R, \varepsilon) \) for the \( D = 3 \) noncommutative Chern-Simons form.

We note that the expression (3.1) for the \( \star \)-exterior product can be rewritten in terms of a bidifferential operator \( \Delta \):

\[
\tau \wedge \star \tau' = \tau \wedge \tau' + \frac{i}{2} \theta^{AB}(l_A \tau) \wedge (l_B \tau') + \frac{1}{2!} \left( \frac{i}{2} \right)^2 \theta^{A_1 B_1} \theta^{A_2 B_2} (l_{A_1} l_{A_2} \tau) \wedge (l_{B_1} l_{B_2} \tau') + \cdots
\]
\[ e^\Delta(\tau, \tau') \] (A.1)

where powers of \( \Delta \) are defined as
\[
\Delta^n(\tau, \tau') \equiv \left( \frac{i}{2} \right)^n \theta^{A_1B_1} \cdots \theta^{A_nB_n} (l_{A_1} \cdots l_{A_n}) \wedge (l_{B_1} \cdots l_{B_n}) \] (A.2)
\[
\Delta^0(\tau, \tau') \equiv \tau \wedge \tau' \] (A.3)

and \( \ell_A \) are Lie derivatives along commuting vector fields \( X_A \). From the definition (A.1) one finds the following expression for the \( \star \)-commutator (when at least one of the forms is of even rank, so that \( \tau \wedge \tau' = \tau' \wedge \tau \)):
\[
\tau \wedge \star \tau' - \tau' \wedge \star \tau = 2 \theta^{AB} i_A \left[ \frac{\sinh \Delta}{\Delta} (\tau, l_B \tau') \right], \quad (A.4)
\]
a generalization of the \( \star \)-commutator formulas of ref.s [29, 30]. If \( \tau \wedge \tau' \) is a form of maximal degree, the Lie derivative in (A.4) can be replaced by \( di_A \), where \( i_A \) is the contraction along the vector field \( X_A \). Moreover, under trace the formula holds also for matrix-valued forms:
\[
\text{Tr}(\tau \wedge \star \tau' - \tau' \wedge \star \tau) = d \text{Tr} \left[ 2 \theta^{AB} i_A \left[ \frac{\sinh \Delta}{\Delta} (\tau, l_B \tau') \right] \right] \equiv d \text{Tr}[C(\tau, \tau')] \] (A.5)

This is the relevant formula for cyclic reorderings in noncommutative Chern-Simons forms, which are traces of maximal (odd) degree forms: any splitting inside the trace involves always one even form.

Consider now the \( D = 3 \) noncommutative CS form
\[
L^{(3)}_{\text{CS}} = \text{STr}[R \wedge \star \Omega + \frac{1}{3} \Omega \wedge \star \Omega \wedge \star \Omega] \] (A.6)

We can compute its variation under the gauge transformations (2.3) and find:
\[
\delta L^{(3)}_{\text{CS}} = \frac{1}{3} d \text{Tr} \left[ 3 R \star \varepsilon + \varepsilon \star \Omega \wedge \star \Omega - \Omega \star \varepsilon \wedge \star \Omega + \Omega \wedge \star \varepsilon \wedge \star \Omega \times \varepsilon \right.
\]
\[
+ 2 C(\varepsilon, R \wedge \star \Omega) + C(\varepsilon, \Omega^3) - C(\Omega \wedge \star R, \varepsilon)
\]
\[
- C(\Omega, R \star \varepsilon) - C(\Omega \times \varepsilon, R) - C(\Omega \star \varepsilon \wedge \star \Omega, \Omega) \right] \] (A.7)

In the commutative limit, the first line reduces to \( \delta L^{(3)}_{\text{CS}} \), while the second and third line vanish.

**B  Gamma matrices in \( D = 5 \)**

We summarize in this Appendix our gamma matrix conventions in \( D = 5 \).
\[
\eta_{ab} = (1, -1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2 \eta_{ab}, \quad [\gamma_a, \gamma_b] = 2 \gamma_{ab}, \] (B.1)
\[
\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 = -1, \quad \varepsilon_{01234} = \varepsilon_{01234} = 1, \] (B.2)
\[
\gamma_a \dagger = \gamma_0 \gamma_a \gamma_0, \] (B.3)
\[
\gamma_a^T = C \gamma_a C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C \] (B.4)
B.1 Useful identities

\begin{align}
\gamma_a \gamma_b &= \gamma_{ab} + \eta_{ab} \\
\gamma_{abc} &= \frac{1}{2} \epsilon_{abcde} \gamma^{de} \\
\gamma_{abcd} &= -\epsilon_{abcde} \gamma^e \\
\gamma_{ab} \gamma_{c} &= \eta_{bc} \gamma_a - \eta_{ac} \gamma_b + \frac{1}{2} \epsilon_{abcde} \gamma^{de} \\
\gamma_{c} \gamma_{ab} &= \eta_{ac} \gamma_{b} - \eta_{bc} \gamma_{a} + \frac{1}{2} \epsilon_{abcde} \gamma^{de} \\
\gamma^{ab} \gamma_{cd} &= -\varepsilon^{ab}_{\ cde} \gamma^e - 4\delta^{[a}_{[c} \gamma^{b]}_{d]} - 2\delta^{ab}_{cd} 
\end{align}

where \( \delta^{ab}_{cd} \equiv \frac{1}{2}(\delta^a_d \delta^b_c - \delta^b_d \delta^a_c) \), \( \delta^{rse}_{abc} \equiv \frac{1}{3!}(\delta^r_a \delta^s_b \delta^e_c + 5 \text{ terms}) \), and indices antisymmetrization in square brackets has total weight 1.

B.2 Noncommutative \( D = 5 \) Fierz identities

\[ \psi \wedge_\star \bar{\chi} = -\frac{1}{4} (-1)^{pq} [(\bar{\chi} \wedge_\star \psi) J + (\bar{\chi} \wedge_\star \gamma^a \psi) \gamma_a - \frac{1}{2} (\bar{\chi} \wedge_\star \gamma^{ab} \psi) \gamma_{ab}] \]

where \( \psi \) is a spinor \( p \)-form and \( \chi \) is a spinor \( q \)-form.

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