STDP-based associative memory formation and retrieval

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Abstract

Spike-timing-dependent plasticity (STDP) is a biological process in which the precise order and timing of neuronal spikes affect the degree of synaptic modification. While there has been numerous research focusing on the role of STDP in neural coding, the functional implications of STDP at the macroscopic level in the brain have not been fully explored yet. In this work, we propose a neurodynamical model based on STDP that renders storage and retrieval of a group of associative memories. We showed that the function of STDP at the macroscopic level is to form a “memory plane” in the neural state space which dynamically encodes high dimensional data. We derived the analytic relation between the input, the memory plane, and the induced macroscopic neural oscillations around the memory plane. Such plane produces a limit cycle in reaction to a similar memory cue, which can be used for retrieval of the original input.

Mathematics Subject Classification 92-10

1 Introduction

Spike-timing-dependent plasticity (STDP), as a synaptic modification rule according to the order of pre- and post-synaptic spiking within a critical time window, has been demonstrated in the nervous systems over a wide range of species from insects to humans. STDP is considered to be critical for understanding the cognitive mechanisms such as learning of temporal sequences (Blum and Abbott 1996; Rao and Sejnowski 2001), formation of associative memory (Tsodyks 2002; Szatmáry and Izhikevich 2010) and manipulation of existing memory (Han et al. 2009; Ramirez et al. 2013; Redondo et al. 2014). Despite such progress and findings, the question still remains
open as to how STDP affects the distributed process of information at the macroscopic level in the brain.

Modeling macroscopic brain activity with nonlinear dynamical systems facilitates understanding of brain functions (Kelso 1995; Globus 1995; Breakspear 2017). The hypothesis of storing memory in a form of an attractor of the dynamics is now accepted with substantial supporting evidence (Wills et al. 2005; Rolls 2007; Tsodyks 1999; Stringer et al. 2005; Rennó-Costa et al. 2014; Rolls 2010). However, it is still unclear how specific trajectories of neural states could emerge through neural plasticity.

In this work, we propose that a neurodynamical function of STDP is related to storage and retrieval of associative memories at a macroscopic scale. When the system is excited by a repeating sequence, STDP creates an anti-symmetric set of directed connections that trigger neural oscillations in the neural state space. While the neural state space is extremely high dimensional, the oscillations are confined in a two-dimensional plane which we call memory plane. Such memory plane can act as a generator of a limit cycle in reaction to an external input for retrieval. That is, once the system converges under a sequential memory input and forms the corresponding memory plane, it produces a limit cycle in reaction to a similar memory cue, which can be used for retrieval of the original input.

The presence and the function of such planar memory structure in the neural state space have caught attention in Susman et al. (2019), where it has been proposed that STDP can store transient inputs as imaginary-coded memories. In this work, we formalized the concept of the memory plane and the retrievability of neural states to analyze how data is effectively stored in the neural state space. We derived the analytic relation between the input, the memory plane, and the induced macroscopic neural oscillations around the memory plane. This enables us to understand the functional role of STDP in terms of neurodynamical systems and view the macroscopic neural oscillations in the brain as circulations across the memory representations. The analytic results in this paper suggest an alternative method to store and retrieve high-dimensional and strongly associated data sets in analogue devices. In our separate work (Yoon and Kim 2022), we proposed a practical encoding algorithm based on the analysis done in this article to store associate image/text data sets into retrievable neural states.

2 Model setups

2.1 Main model

Our work follows the framework of standard firing-rate models (Dayan and Abbott 2003; Susman et al. 2019). We set the differential equation for the neural state as

\[ \dot{x} = -x + W \phi(x) + b(t), \] (1)

where \( x = [x_1 \cdots x_N]^T \in \mathbb{R}^N \) is the state of \( N \) neuronal nodes and \( W = (W_{ij}) \in \mathbb{R}^{N \times N} \) is a connectivity matrix with \( W_{ij} \) corresponding to the strength of synaptic connection from node \( j \) to \( i \). Here \( \phi(x) = \tanh(x) \) is a saturating transfer function, and \( b(t) \) represents the incoming stream of external inputs.
We propose the STDP mechanism based on Kempter et al. (1999) as

\[
\dot{W}_{ij}(t) = -\gamma W_{ij}(t) + \rho \left( \int_{0}^{\infty} K(s)\phi(x_j(t-s))\phi(x_i(t)) \, ds \right)_{\text{pre- to post- firing}}
+ \int_{0}^{\infty} K(-s)\phi(x_j(t))\phi(x_i(t-s)) \, ds \right)_{\text{post- to pre- firing}},
\]

where \( K \) is a temporal kernel characterizing the intensity of neural weight updates from inter-spike timing. The parameters \( \gamma \) and \( \rho \) are the decaying rate of homeostatic plasticity and the learning rate, respectively.

There have been numerous researches focusing on the effect of kernel structure on the neural dynamics, especially by investigating the effective inter-spike timing and intensity (Rubin et al. 2001; Feldman 2012; Luz and Shamir 2014; Zappacosta et al. 2018). Here, we adopt an anti-symmetric parameterized kernel \( K(s) = \text{sgn}(s)K^+(|s|) \)

where

\[
K^+(s; \tau, \vartheta) := \frac{\vartheta}{\tau \vartheta} s^{\vartheta-1} e^{-(s/\vartheta)^\vartheta}, \quad \tau > 0, \quad \vartheta > 1.
\]

The parameter \( \tau \) represents the effective inter-spike timing and \( \vartheta \) represents the degree of concentration around \( \tau \). Figure 1 shows the plot of \( K(s, \tau, \vartheta) \) for different values of \( \vartheta \) for a fixed \( \tau \). For detailed properties of this function, see Appendix A.

Now Eq. (2) can be concisely expressed in the following multi-dimensional form

\[
\begin{cases}
\dot{x} = -x + W\phi(x) + b(t) \\
\dot{W} = -\gamma W + \rho \int_{0}^{\infty} K^+(s; \tau, \vartheta) \left( \phi(x)\phi(x_s)^\top - \phi(x_s)\phi(x)^\top \right) \, ds,
\end{cases}
\]

where the subscript on \( x \) indicates the amount of time delay, i.e., \( x_s(t) = x(t-s) \). This Eq. (4) will be referred to as the full model throughout this text.

### 2.2 Storage and retrieval schemes

Let \( m_1, \ldots, m_n \in \mathbb{R}^N \), and \( m_i \) be memory representations that are associated with one another (for example, they could be \( n \) images of the same object viewed from \( n \) different angles). We assume that such related information are sequentially provided to the system (4) through \( b(t) \). Then the system stores input stream into an internal structure by driving the synaptic connectivity \( W(t) \) to a certain constant \( W^* \). Considering that the brain activity shows rhythmic synchrony, we assume that the input stream \( b(t) \) takes a oscillatory form carrying \( m_i \), as

\[
b(t; m_1, \ldots, m_n) = \sum_{i=1}^{n} \sin(\omega t - \xi_i) m_i, \quad 0 \leq \xi_1 < \cdots < \xi_n < \pi,
\]
Fig. 1 Plot of full anti-symmetric STDP kernels $K(s; \tau, \vartheta)$, with respect to different values of concentration parameter $\vartheta$. Here, $\tau$ is fixed as $\frac{\pi}{3} \approx 1.0472$, and the black dash-dotted curve ($s^*, K^+(s^*)$) (or $(-s^*, -K^+(-s^*; \tau, \vartheta))$) indicates the maximum (or minimum) of $K(s; \tau, \vartheta)$, where $s^* = \arg\max_s K(s; \tau, \vartheta)$ (or $s^* = \arg\min_s K(s; \tau, \vartheta)$). It asymptotically approaches to $s = \tau$ (or $s = -\tau$), as $\vartheta \to \infty$.

where $\omega$ is the frequency of neural oscillations and $\xi_i, i = 1, \ldots, n$ is the phase of each oscillation. Later we will show that this harmonic type input signal forms of a 2-dimensional manifold in the neural state space, which eventually brings the synaptic connectivity $W(t)$ to a certain constant matrix $W^*$ under suitable conditions.

Once convergence of $W(t)$ is established, for retrieval, the system suppresses further synaptic change (i.e., $\gamma = \rho = 0$) as

$$\dot{x} = -x + W^* \phi(x) + b_c(t). \quad (6)$$

Here $b_c(t)$ is the cue input in the form of

$$b_c(t; m_c) = \sin \omega t \ m_c, \quad m_c \in \mathbb{R}^N. \quad (7)$$

We are interested in how the original representations $m_1, \ldots, m_n$ can be revived from the neural activity $x(t)$, especially when $m_c \in \mathbb{R}^N$ is close to one of the memory representations. Figure 2a, b illustrate the setup for storage and retrieval process on the systems (4) and (6).

### 3 Analysis on a simplified model

In this section, we present theoretical results on our model (4) under two simplifying assumptions; (i) The level of the neural activity is regular in a limited range and...
Fig. 2 Description of the associative memory process for storage and retrieval of sensory input information. **a** Storage phase: The STDP-based system processes the memory representations \( \{m_i\}_{i=1}^n \) and the connectivity matrix \( W(t) \) converges to a constant connectivity \( W^* \) as a result. **b** Retrieval phase: A memory cue input \( b_c(t) \) triggers the retrieval of the original inputs through the connectivity \( W^* \) acquired in the storage phase therefore the saturating transfer function can be linearly approximated as \( \phi(x) \approx x \). (ii) The weight of STDP is infinitely concentrated at a timing \( \tau \), that is, \( K^+(s; \tau, \vartheta) \to \delta(s-\tau) \) with \( \vartheta \to \infty \). The resulted model is

\[
\begin{align*}
\dot{x} &= -x + W^* \phi(x) + b_c(t) \\
\dot{W} &= -\gamma W + \rho \left( x x^T \tau - x^T x \tau \right),
\end{align*}
\]  

(8)

where the fact that \( \int_0^\infty \delta(s-\tau) \phi(x_s(t)) ds = \phi(x_\tau(t)) \approx x_\tau(t) \) has been directly used. Analogously, the dynamics for retrieval phase (6) is also simplified into

\[
\dot{x} = -x + W^* x + b_c(t; m_c).
\]  

(9)

Although the analysis in Sect. 3 applies to the simplified system (8), numerical simulations in Sect. 4 will show that the original full system (4) also closely reproduces the theoretical results in a suitable range of the parameters.

### 3.1 Robust storage by STDP

We start with the results on the storage phase. One can confirm that the oscillating sensory input in Eq. (5) always resides in a plane in \( \mathbb{R}^N \), a memory plane, which is defined as in the following lemma.
Lemma A \( \mathbf{b}(t) \) is periodic and embedded in a plane \( S := \text{Span}\{\mathbf{u}, \mathbf{v}\} \) where

\[ u = -\Psi \sin \xi \quad \text{and} \quad v = \Psi \cos \xi. \tag{10} \]

Here \( \Psi = [m_1 \cdots m_n] \in \mathbb{R}^{N \times n} \), \( \sin \xi = [\sin \xi_1 \cdots \sin \xi_n]^\top \in \mathbb{R}^n \), and \( \cos \xi = [\cos \xi_1 \cdots \cos \xi_n]^\top \in \mathbb{R}^n \).

Now, we give a theorem which asserts the existence of the periodic solutions \((x^*(t), W^*)\) of the simplified system (8) in terms of the memory plane \( S \).

**Theorem 1 (Periodic Solution with Steady Connectivity)** The system (8) under input (5) has a periodic neural solution \( x^*(t) \) with a constant connectivity matrix \( W^* \), where \( x^*(t) \in S \) for all \( t \), and \( W^* \in \bigwedge^2(S) \). \tag{11}

Here, \( \bigwedge^2(S) \) indicates an exterior power of \( S \), which corresponds to a 1-dimensional set of anti-symmetric matrices in the form of \( \alpha(\mathbf{u}\mathbf{v}^\top - \mathbf{v}\mathbf{u}^\top) \) for any vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( S \). The proof and the exact analytic form of such \((x^*(t), W^*)\) can be found in Appendix B2, and it reveals that Lemma A provides a firm background for the existence of such solution, emphasizing the importance of oscillatory dynamics of input \( \mathbf{b}(t) \). Also, the uniqueness is validated throughout a wide range of parameters.

Along with the guaranteed existence of such orbit \((x^*(t), W^*)\) stated in Theorem 1, we investigate its stability by performing the analysis on the Maximal Lyapunov Exponent (MLE) (Sprott and Sprott 2003; Sandri 1996; Farmer 1982). Setting \( x(t) = x^*(t) + \delta x(t) \) and \( W(t) = W^* + \delta W(t) \), from Eq. (8) we get a variational equation describing the evolution of \((\delta x, \delta W)\) (see Eq. (92) in Appendix C1). It allows us to compute the maximal rate of exponential growth of \((\delta x, \delta W)\), which is the MLE (for detailed computational methodology, refer to Appendix C2). Figure 3 shows the color map of numerically estimated MLE of the found periodic solution \((x^*(t), W^*)\) of the system (8). For the regions showing negative values of MLE, one can assure that such solution is an attractor, thus consequently achieving a robust learning for any types of input of the form (5). In contrast, on the unstable regions showing non-negative values of MLE, the system’s trajectory fails to converge and results in an unbounded growth. The stable regions are relatively large and roughly confined by \( \omega \geq 0.5 \) and \( \gamma \geq \alpha \rho \) with \( \alpha \approx 0.7 \), and the shape of stable/unstable regions are hardly affected by the size of the system \( N \). On the other hand, the unstable region tends to grow larger if the size of the input \( \mathbf{b}(t) \) (or analogously, the number of memory representations \( n \)) increases.

Figure 4 illustrates the convergence of the neural activity toward a periodic orbit \( x^*(t) \) on memory plane \( S \), validated possible by above analysis. Analysis in Appendix B1 shows that if \( \tau = \frac{\pi}{2\omega} \), then the norm of the converging \( W^* \) on the 1-dimensional set \( \bigwedge^2(S) \) is maximized and therefore achieves the most efficient learning. Note that the memory plane \( S \) does not necessarily contain the memory representations \( m_1, \ldots, m_n \) in general. However, we show in the next section that \( S \) is likely located close to the memory representations in the high dimensional neural state space.
3.2 Associative retrieval by a memory cue

Now we provide the analysis on the retrieval phase. We propose that the convergent synaptic connectivity $W^*$ acquired from the storage phase effectively contains the information of a whole set of memory representations $\{m_i\}_{i=1}^{n}$ and leads to periodic retrieval of them.

Let us define a retrievable subspace $\mathcal{M} := \text{Span}\{m_i\}_{i=1}^{n}$ with respect to a set of memory representations $\{m_i\}_{i=1}^{n}$. A neural state $x \in \mathbb{R}^N$ is said to be retrievable with respect to $\{m_i\}_{i=1}^{n}$, if $x \in \mathcal{M} \setminus \{0\}$. Note that the memory plane $S$ is a subset of the retrievable subspace $\mathcal{M}$ (see Eq. (10)). In Sect. 4 that deals with numerical simulations, we suggest a specific way to disentangle a retrievable neural state $x$ and retrieve individual memory representations $m_1, \ldots, m_n$.

The following theorem states that for some appropriately chosen memory cue representation $m_c$, there is a specific moment $t = t^*$ at which the corresponding neural state $x(t)$ becomes retrievable.
For the intuitive graphical understanding, the retrieval subspace $\mathcal{M}$ and the memory plane $S$ are visualized as a plane and an embedded line, respectively. (i) Case $m_c \in \mathcal{M}$ (good cue): $x^r_\ast(t) \in \mathcal{M}$ for all $t$.
(ii) Case $m_c \in S_{\perp} \cap \mathcal{M}^c$ (relevant cue): $x^r_\ast(t)$ is retrievable at $t = t^\dagger$ as in Theorem 2. $m_c$ from any appropriately perturbed $m_i$ will be in this case. (iii) Case $m_c \in S_{\perp} \cap \mathcal{M}^c$ (wrong cue): $x^r_\ast(t)$ never becomes retrievable.

**Theorem 2** (PERIODIC RETRIEVAL) For any non-zero cue $m_c$, the solution of Eq. (9) under input (7) asymptotically approaches to some periodic solution $x^r_\ast(t)$. Especially if $m_c \notin S_{\perp}$, then $x^r_\ast(t)$ has a nontrivial intersection with $S$ on every $t = t^\dagger > 0$ where

$$t^\dagger = \frac{1}{\omega} \tan^{-1} \omega + \frac{\pi}{\omega}, \quad n \in \mathbb{Z},$$

thus being periodically retrievable on $t = t^\dagger$.

Above theorem states that the retrieval dynamics $x_r(t)$ is attracted to a limit cycle $x^r_\ast(t)$, and in fact, when especially given a cue $m_c \notin S_{\perp}$, the limit cycle $x^r_\ast(t)$ intersects the memory plane $S$ on the timing $t = t^\dagger$. The condition for retrieval can be extended further: the proximity of $m_c$ to $S$ and $\mathcal{M}$ determines the retrievability of $x^r_\ast(t)$ as follows.

(i) Case $m_c \in \mathcal{M}$ (good cue): $x^r_\ast(t) \in \mathcal{M}$ for all $t$.
(ii) Case $m_c \in S_{\perp} \cap \mathcal{M}^c$ (relevant cue): $x^r_\ast(t)$ is retrievable at $t = t^\dagger$ as in Theorem 2.
(iii) Case $m_c \in S_{\perp} \cap \mathcal{M}^c$ (wrong cue): $x^r_\ast(t)$ never becomes retrievable.

Figure 5 gives a graphical illustration about dependence of the retrieval dynamics on the memory cue. More details about the retrievability conditions including the proof of Theorem 2 can be found in Appendix B3.

As above results imply, the retrieval performance indeed directly depends on the angle between the cue $m_c$ and $S$, so the chance for good and relevant cues increase if the memory plane $S$ is formed near the memory representations $m_1, \ldots, m_n$. Introducing the mean cosine similarity $\langle \cos \theta_i \rangle_i$ where $\theta_i$ that represents the angle between each $m_i$ and $S$, the next theorem tells that one can choose the optimal sequential phase for storing input $\xi_1, \ldots, \xi_n$ in Eq. (5) in terms of $\langle \cos \theta_i \rangle_i$. 

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**Theorem 3 (Optimal Choice for $\xi_i$)** Suppose $\{m_i\}_{i=1}^n$ with $m_i \in \mathbb{R}^N$ are mutually orthogonal vectors of the same magnitude. Then the maximum of $\langle \cos \theta_i \rangle_i$ can be attained with the distribution of $\xi_i$ as

$$
\xi_i = (i - 1)\frac{\pi}{n} + \alpha, \quad i = 1, \ldots, n, \quad 0 \leq \alpha < \frac{\pi}{n},
$$

with the maximum value $\sqrt{\frac{2}{n}}$.

The theorem suggests a uniform sequences of phase $\xi_i$ for the sequential input of representations in order to naturally maximize the expected performance of retrieval.

### 4 Numerical simulations

This section intends to provide two numerical tests about the full model (4). Firstly, we show that the both full and simplified model produce similar dynamics, implying that the analytic results for the simplified model obtained in Sect. 3 largely remains valid for the full model. In Sect. 4.2, we combine the full model (4) with a proper encoding method and show that it is then able to store and retrieve actual high dimensional data.

We have used the modified Euler’s method to simulate delay differential equations throughout numerical tests.

#### 4.1 Comparison of full and simplified models

Figure 6 compares the synaptic dynamics of the full (4) and simplified (8) model during the storage phase. In Fig. 6a, one can find that under suitable parameter choices, the neural activity $x(t)$ of the full model driven by the periodic input rapidly converges to a limit cycle embedded in the memory plane $S$, as expected from the analysis on simplified model in Sect. 3. This induces the convergence in the connectivity matrix too, that is, making $W(t)$ slowly converge to $W^*$. These observations suggest that the theoretical predictions acquired through the analysis on the simplified model effectively approximates the qualitative dynamics of the full model. Figure 6d shows the convergence of the full model according to the concentration parameter $\vartheta$. We regard $\vartheta > 3$ works for effective approximation.

#### 4.2 Tests for associative retrieval

In this section, we numerically demonstrate the storage and retrieval process on the full model (4). The performed task is to sequentially store a group of associative images $f_i$, $i = 1, \ldots, n$ and then restore them from a partial or noisy cue. In order to encode each image $f_i$ to a memory representation $m_i$, we adopt the tensor product representation method suggested in Smolensky (1990): we label the inputs by binding with a specific tagging vector $r$ through tensor product as

$$
m_i = f_i \otimes r,
$$

(14)
Fig. 6 Comparison between the dynamics of the full and simplified model. The parameters $\gamma = \rho = 0.5$, $\omega = 1.5$, $\tau = \frac{\pi}{2\omega} = \frac{\pi}{3}$, $\vartheta = 5$, and the optimal value of $\xi_i = (i - 1) \frac{\pi}{n}$ (i.e., $n$-evenly sequenced points on $[0, \pi]$) are used. a Plot of $\|\text{Proj}_S x(t)\| / \|x(t)\|$, the measured overlap of the neural activity $x(t)$ on the plane $S$. Reaching 1 indicates the perfect embedding. b Plot of $\|W(t) - W^*\|_F / \|W^*\|_F$. c Shape of the full STDP kernel function $K(s; \tau, \vartheta)$ with $\tau = \frac{\pi}{3}$, $\vartheta = 5$. d Plot of $\|W - W^*\|_F / \|W^*\|_F$ according to the kernel’s concentration parameter $\vartheta$. $W$ represents the convergent value of $W(t)$ in the full model. Similar results are reproduced for a wide range of $\omega$.

where $r \in \mathbb{R}^d$ is a vector chosen for each $m_i$ from the orthonormal set $\{r_j\}_{j=1}^m$ called tagging vectors. Use of the tagging vectors can be justified from the fact that the cognitive systems do not simply receive external inputs in a passive way, but rather actively pose them in the neural state space on acceptance. For example, the tagging vectors can be assigned to each input image to distinctively indicate and memorize the angle at which the image is taken.

For $x$ in a retrievable neural state, the tagging vectors are used to retrieve the original information $f_i$, $i = 1, \ldots, n$ as

$$x \cdot r_i = \left( \sum_{j=1}^{n} c_j m_j \right) \cdot r_i$$

$$= \left( \sum_{j=1}^{n} c_j (f_j \otimes r_j) \right) \cdot r_i$$

$$= \sum_{j=1}^{n} c_j f_j (r_j \cdot r_i)$$

$$= c_i f_i. \quad (15)$$

We first demonstrate an associative memory task of storing four $48 \times 48$ grayscale images containing snapshots of a 3D modeled car, as shown in Fig. 7a. The four snapshots in Fig. 7b are taken from the model in Fig. 7a at different angles. Throughout this section, we use an orthonormal set of tagging vectors $\{r_i\}$ to distinctively indicate
Fig. 7  a An image of a 3D-modeled hatchback-styled car. b 48 × 48 pixel grayscale snapshots taken from the front, side, rear, and top. All images were rendered using a free-licensed 3D object from Turbosquid.

Fig. 8  Group associative retrieval from a contaminated cue input. a The noisy cue is used for retrieval of the original images. b Snapshot of the retrieved images at the farthest point (red dot) from memory plane $S$. c Snapshot of the retrieved images on $t = t^\dagger$, i.e., at the intersection (green dot) of the orbit and the memory plane $S$. Here, the contrast between the object and the background are not same as the original images, since a uniform threshold $\sigma = 0.005$ was used for drawing the retrieved images—that is, any element of $\mathbf{x} \cdot \mathbf{r}_j$ having value outside $[-\sigma, +\sigma]$ is developed to a pixel of just pure black or white. One can obtain an original(either identical or exactly inverted) image by using an optimal value of $\sigma$ for each retrieved image (color figure online).

Each memory items to be stored. In this example, four arbitrary orthonormal vectors in $\mathbb{R}^4$ has been used to tag given four different angles (front, side, rear, top) respectively. The storage phase was proceeded by using the full model (4), with the same parameters described in the caption of Fig. 6.

Figure 8 shows the result of the retrieval phase. A contaminated cue is delivered to the retrieving system (6) through the cue input (7). The neural activity tends to converge to a certain periodic orbit as predicted in Theorem 2. Figure 8b, c shows the reconstructed images which are obtained from $\mathbf{x}(t)$ when is converges to a periodic orbit. On the time $t^\dagger = \frac{1}{\omega} \tan^{-1} \omega + n \frac{\pi}{\omega}$, $n \in \mathbb{Z}$, a perfectly denoised snapshot of the original images are recovered as predicted in Theorem 2 in Fig. 8c. On the other hand, when $t \neq t^\dagger$, relatively noisy images are obtained as in Fig. 8b. Note that the images can be retrieved simultaneously through (15) once the neural state becomes retrievable. The retrieved images continuously oscillate, developing week/strong and positive/negative images in turns. Such flashing patterns are generally different from image to image and are affected by the sequential order of the memory representations in Eq. (5) in the storage phase.
In the second numerical experiment, we perform group associative retrieval tasks using human face images. The four images of a boy in Fig. 9a are tagged by $r_i$ to indicate four different facial expressions (neutral, happy, angry, sad). Similarly, the five images of a female from different perspective are tagged by $r_i$ indicating the corresponding angles (0 to 180 degree). In the retrieval phase, we used specific tagging vectors to access the necessary information selectively. Figure 10 shows that the target images are well retrieved in the harmonically oscillating contrast as expected. Further on above numerical results, our separate work (Yoon and Kim 2022) deals with the demonstration of the effect in the retrieval performance according to the contamination level of the cue, along with storing multiple stream of associative memories.

5 Discussion

Due to recent advances in large-scale neural measurements and neural computational models, there have been substantial progress in understanding how computations are performed by neural population dynamics in the high-dimensional neural state space (Paninski and Cunningham 2018; Cunningham and Byron 2014). A number of researches have found evidences from neurophysiological data that transient dynamics
occurs near a confined subspace (manifold) during the cognitive processes in the brain (Chaudhuri et al. 2019; Kim et al. 2017; Sadler et al. 2014).

As for the computational model of neural information process, recurrent neural networks (RNNs) is the most widely used framework. While the major body of work using RNNs apply tools from dynamical systems theory to discover the optimized network structure to perform a task (Sussillo and Barak 2013; Schaeffer et al. 2020), there have been an alternative approach to develop algorithm to construct networks that produce dynamics with a predefined manifold (Beiran et al. 2021; Darshan and Rivkind 2022; Pollock and Jazayeri 2020; Claudi and Branco 2022). In Chaisangmongkon et al. (2017), RNNs navigate around categorical memories stored in the form of attracting fixed points and saddle points, before it finally settles and yields the final result.

In this work, we extended the idea proposed in Susman et al. (2019) to the manifold encoding of associative memory. To the best of the authors’ knowledge, this is the first model that handles multiple associative data sets in one unified geometric structure, the memory plane. The association of memory therefore can occur not just for two items, but for series of several items continuously in time. Moreover, the proposed memory model is not a phenomenological one, but based on a theoretical analysis of STDP. The flexibility of the model facilitates manipulation of the composite information by combining each item with a different tag, and our simulation confirmed that the model can effectively deal with high-dimensional data in a scalable way. This suggests that the STDP-based neural circuits can handle data with high precision, and further provides a strong support for the dynamical system’s approach to information process in the neural systems.

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Appendix A: Properties of the STDP kernel function

Here we show some important properties of the function $K^+(s; \tau, \vartheta)$, which is the restriction of $K(s; \tau, \vartheta)$ on $0 \leq s < \infty$.

**Proposition** Consider the following function $K^+(s; \tau, \vartheta)$ as follows:

$$K^+(s; \tau, \vartheta) := \frac{\vartheta}{\tau^\vartheta} s^{\vartheta-1} e^{-\left(\frac{s}{\vartheta}\right)\tau}, \quad s, \tau > 0, \quad \vartheta > 1,$$

where $\tau$ and $\vartheta$ are parameters. Then we have the followings:

a. $\int_0^\infty K^+(s; \tau, \vartheta) \, ds = 1$, regardless of $\tau$ and $\vartheta$.

b. Maximum of $K^+(s; \cdot, \cdot)$ occurs near at $s = \tau$ whenever $\vartheta \gg 1$.

c. $K^+(s; \tau, \vartheta)$ behaves as a $\tau$-shifted Dirac-$\delta$ function $\delta(s - \tau)$ as $\vartheta \to \infty$.

**Proof** a. Note that $s, \tau > 0$ and $\vartheta > 1$, thus we may perform substitution of variable $\left(\frac{s}{\vartheta}\right)^\vartheta = u$ with same integration range $[0, \infty)$. Such transformation directly yields $\int_0^\infty e^{-u} \, du = 1$, which immediately shows the result.
b. Performing differentiation \( \frac{\partial}{\partial s} K^+(s; \tau, \vartheta) \) yields

\[
\frac{\partial}{\partial s} K^+(s; \tau, \vartheta) = \frac{\partial}{\partial s} \left( \frac{\vartheta}{\tau^{\vartheta}} s^{\vartheta-1} e^{-\left(\frac{\tau}{\vartheta}\right)^{\vartheta}} \right) = -\vartheta s^{\vartheta-2} - 2\tau^{\vartheta} e^{-\left(\frac{\tau}{\vartheta}\right)^{\vartheta}} \left( \vartheta s^{\vartheta} - \frac{\tau^{\vartheta}}{\vartheta} + \tau^{\vartheta} \right).
\]

\[
(17)
\]

\[
\text{Note that the coefficients at the left side of the bracketed term (*) in (18) is always non-zero. Also, } K^+(0; \tau, \vartheta) = 0 \text{ and } \lim_{s \to \infty} K^+(s; \tau, \vartheta) = 0, \text{ thus the maximum of the positive, and differentiable real-valued function } K^+(s) \text{ only occurs on some } s^* \text{ satisfying } (*) = 0. \text{ Such } s^* \text{ is found to be unique in } \mathbb{R}^+, \text{ and therefore}
\]

\[
s^* = \tau \left( 1 - \frac{1}{\vartheta} \right)^{-\frac{1}{\vartheta}}.
\]

\[
(19)
\]

Now, one directly reads that \( s^* \to \tau \) as \( \vartheta \to \infty \) at a fast, exponential order, thus if \( \vartheta \gg 1 \), then one can approximate \( s^* \approx \tau \).

c. Whenever \( s \neq \tau \), we easily see that \( \lim_{\vartheta \to \infty} K^+(s; \tau, \vartheta) = 0 \), but only when \( s = \tau \), then \( \lim_{\vartheta \to \infty} K^+(\tau; \tau, \vartheta) = \infty \). Thus by statement a and b, we get \( \lim_{\vartheta \to \infty} K^+(s; \tau, \vartheta) = \delta(t - \tau) \).

Appendix B: Proofs of the theoretical results

B1: Proof of Lemma A

Lemma A \( b(t; m_1, \ldots, m_n) \) is periodic and embedded in a plane \( S := \text{Span}\{u, v\} \) where

\[
u = -\Psi \sin \xi \quad \text{and} \quad v = \Psi \cos \xi.
\]

Here \( \Psi = [m_1 | \cdots | m_n] \in \mathbb{R}^{N \times n}, \sin \xi = [\sin \xi_1 \cdots \sin \xi_n]^\top \in \mathbb{R}^n, \text{ and } \cos \xi = [\cos \xi_1 \cdots \cos \xi_n]^\top \in \mathbb{R}^n.\)

Proof Let \( b(t; m_1, \ldots, m_n) = [b_1(t) \cdots b_N(t)]^\top \), and \( m_i = [m_{i1} \ldots m_{in}]^\top \). Then, each component of \( b(t) \) satisfies

\[
b_j(t) = \sum_{i=1}^{n} m_{ij} \sin(\omega t - \xi_i)
\]

\[
= \sum_{i=1}^{n} m_{ij} (\sin \omega t \cos \xi_i - \cos \omega t \sin \xi_i)
\]

\(\square\) Springer
Thus if we introduce

\[
\begin{align*}
    \mathbf{u} &= -\left[\sum_{i=1}^{n} m_{i1} \sin \xi_i \cdots \sum_{i=1}^{n} m_{IN} \sin \xi_i \right]^T, \\
    \mathbf{v} &= \left[\sum_{i=1}^{n} m_{i1} \cos \xi_i \cdots \sum_{i=1}^{n} m_{IN} \cos \xi_i \right]^T,
\end{align*}
\]

then this choice of \( \mathbf{u}, \mathbf{v} \) can be represented in alternate form of \( \mathbf{u} = -\Psi \sin \xi \) and \( \mathbf{v} = \Psi \cos \xi \) where \( \Psi, \sin \xi, \) and \( \cos \xi \) are defined as in the theorem statement, and guarantees

\[
b(t; m_1, \ldots, m_n) = \cos \omega t \mathbf{u} + \sin \omega t \mathbf{v}
\]

by Eq. (21). Therefore \( b(t; m_1, \ldots, m_n) \) is periodic and embedded in plane \( \text{Span}[\mathbf{u}, \mathbf{v}] \).

\section*{B2: Proof of Theorem 1}

\textbf{Theorem 1 (Periodic Solution with Steady Connectivity)} The system (8) under input (5) has a periodic neural solution \( x^*(t) \) with a constant connectivity matrix \( W^* \), where

\[
x^*(t) \in S \text{ for all } t, \quad \text{and} \quad W^* \in \bigwedge^2(S).
\]

\textbf{Proof} First, let us define the operator ‘\( \wedge \)’ to indicate the wedge product of vectors, as

\[
\mathbf{a} \wedge \mathbf{b} := \mathbf{a} \mathbf{b}^\top - \mathbf{b} \mathbf{a}^\top.
\]

Then the definition of the space \( \bigwedge^2(S) \) with respect to a set \( S \) introduced in the main text can be restated as \( \bigwedge^2(S) := \{\alpha(s_1 \wedge s_2) \mid s_i \in S\} \). Now we introduce some important lemmas for step-by-step proof, based on finding the consistent conditions for such type of solution to exist.

\textbf{Lemma 1} If \( (x^*(t), W^*) \) is a periodic solution of the system (8) with steady synapses \( W^* \), then

\[
x^*(t) \in S, \quad \forall t, \quad \text{and} \quad W^* \in \bigwedge^2(S),
\]

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where $S$ a 2-dimensional subspace in $\mathbb{R}^N$. Furthermore, $(x^*(t), W^*)$ must satisfy
\[
x^*(t) \wedge x^*(t - \tau) = \frac{\gamma}{\rho} W^*, \quad \forall t,
\]  
(27)

i.e., which is a constant matrix.

**Proof** Suppose that $(x^*(t), W^*)$ is a solution of system (8). Then, substituting $(x^*(t), W^*)$ directly in the equation of $W$, we get
\[
O = -\gamma W^* + \rho \left( x^* \wedge x^*_t \right),
\]  
(28)

and this directly yields Eq. (27). $\frac{\gamma}{\rho} W^* = x^*(t) \wedge x^*(t - \tau)$, which should be a constant matrix for all $t$. Furthermore, the wedge product $x^*(t) \wedge x^*(t - \tau)$ defines a rank-2 anti-symmetric matrix, therefore $W^* \in \wedge^2(S)$ for some 2-dimensional subspace $S \subset \mathbb{R}^N$. Therefore, we finally see that $x^*(t)$ and the delayed term $x^*(t - \tau)$ should always be in $S$ in order to maintain the wedge product $x^*(t) \wedge x^*(t - \tau) \in \wedge^2(S)$ for all $t$, thus we finally deduce $x^*(t) \in S$ for all $t$, proving (26).

Based on above observations, we suppose that $W^* \in \wedge^2(S)$ for some arbitrary 2-dimensional subspace $S \subset \mathbb{R}^N$. This assumption leads to another useful lemma:

**Lemma 2** Let $W^*$ be some fixed matrix in $\wedge^2(S)$, where $S \subset \mathbb{R}^N$ is an arbitrary 2-dimensional subspace of $\mathbb{R}^N$. Consider a reduced system of (8) of only having the evolution of $x(t)$ while fixing $W(t) = W^*$, i.e.,
\[
\dot{x} = -x + W^* x + b(t; m_1, \ldots, m_n).
\]  
(29)

Then, for a given input signal $b(t; m_1, \ldots, m_n)$, there exists a unique globally stable periodic solution of Eq. (29).

**Proof** Let the arbitrarily given initial condition of Eq. (29) as $x(0)$, and denote the solution with $x_{W^*}$. Observe that from the variation of parameters formula,
\[
x_{W^*}(t) = e^{(-I+ W^*)} x(0) + \int_0^t e^{(-I+ W^*)} b(t' ; m_1, \ldots, m_n) dt'
\]  
(30)

\[
= e^{-t} e^{W^*} x(0) + \int_0^t e^{-t'} \left( \cos(\omega(t - t')) e^{W^*} u + \sin(\omega(t - t')) e^{W^*} v \right) dt'.
\]  
(31)

where Eq. (23) from Lemma A, and the fact that $e^{(-I+ W^*)} x = e^{W^*} e^{-tI} x = e^{-t} e^{W^*} x$ by the commutativity between $I$ and $W^*$ has been used at Eqs. (30)–(31). Furthermore, $W^* \in \wedge^2(S)$ for some $S$, so the matrix exponential terms of $e^{W^*} u$ (or $e^{W^*} v$) in Eq. (31) can be alternatively expressed as
\[
e^{W^*} u = u_\Delta + \cos(\lambda^* t) \bar{u} + \sin(\lambda^* t) \bar{u}_\Lambda,
\]  
(32)

where $\bar{u} = \text{Proj}_\Delta u$, and $u_\Delta = u - \bar{u}$, and $\bar{u}_\Lambda \in S$ satisfying $\|\bar{u}_\Lambda\| = \|\bar{u}\|$ with $\bar{u}_\Lambda \perp \bar{u}$, following the orientation of the flow $e^{W^*} u$ (for a graphical illustration, refer to Fig. 11). Here, $\pm \lambda i$ ($\lambda \in \mathbb{R}$) stands for the only pair of imaginary eigenvalue of $W^*$.
Using the principle of Eq. (32) into Eq. (31), we get

$$\mathbf{x}_W(t) = e^{-t} \left( \mathbf{x}(0)_{\Delta} + \cos(\lambda^* t)\mathbf{x}(0) + \sin(\lambda^* t)\mathbf{x}(0)_\wedge \right)$$

$$+ \int_0^t e^{-t'} \cos(\omega(t - t'))(\mathbf{u}_\Delta + \cos(\lambda^* t')\tilde{\mathbf{u}} + \sin(\lambda^* t')\tilde{\mathbf{u}}_\wedge) dt'$$

$$+ \int_0^t e^{-t'} \sin(\omega(t - t'))(\mathbf{v}_\Delta + \cos(\lambda^* t')\tilde{\mathbf{v}} + \sin(\lambda^* t')\tilde{\mathbf{v}}_\wedge) dt'$$

$$= e^{-t} \mathbf{x}(0)_{\Delta} + e^{-t} \cos(\lambda^* t)\mathbf{x}(0) + e^{-t} \sin(\lambda^* t)\mathbf{x}(0)_\wedge$$

$$+ \int_0^t e^{-t'} \cos(\omega(t - t'))\mathbf{u}_\Delta dt' + \int_0^t e^{-t'} \cos(\lambda^* t') \cos(\omega(t - t'))\tilde{\mathbf{u}} dt'$$

$$+ \int_0^t e^{-t'} \sin(\lambda^* t') \cos(\omega(t - t'))\tilde{\mathbf{u}}_\wedge dt'$$

$$+ \int_0^t e^{-t'} \sin(\omega(t - t'))\mathbf{v}_\Delta dt' + \int_0^t e^{-t'} \cos(\lambda^* t') \sin(\omega(t - t'))\tilde{\mathbf{v}} dt'$$

$$+ \int_0^t e^{-t'} \sin(\lambda^* t') \sin(\omega(t - t'))\tilde{\mathbf{v}}_\wedge dt'. \quad (33)$$

Now let us define the time integral terms in Eq. (33) as follows:

$$\int_0^t e^{-t'} \cos(\omega(t - t')) dt' = \Psi_c(t),$$

$$\int_0^t e^{-t'} \sin(\omega(t - t')) dt' = \Psi_s(t),$$

$$\int_0^t e^{-t'} \cos(\lambda^* t') \cos(\omega(t - t')) dt' = \Psi_{cc}(t).$$
Lemma I, then one can tell that assuming the integration interval of the 6 terms in Eq. (1) as 0, \(\infty\) instead of [0, t]. For example,

\[
\int_0^t e^{-t'} \cos(\lambda^* t') \sin(\omega(t - t')) \, dt' = \Psi_{sc}(t),
\]

\[
\int_0^t e^{-t'} \sin(\lambda^* t') \cos(\omega(t - t')) \, dt' = \Psi_{cs}(t), \quad \text{and}
\]

\[
\int_0^t e^{-t'} \sin(\lambda^* t') \sin(\omega(t - t')) \, dt' = \Psi_{ss}(t).
\]

Then Eq. (33) can be simply expressed as

\[
x_{W^*}(t) = e^{-t} x(0)_\Delta + e^{-t} \cos(\lambda^* t) x(0)_\Lambda + e^{-t} \sin(\lambda^* t) x(0)_\Lambda \\
+ \Psi_c(t) u_\Delta + \Psi_{cc}(t) \bar{u} + \Psi_{sc}(t) \bar{u}_\Lambda + \Psi_s(t) v_\Delta + \Psi_{cs}(t) \bar{v} + \Psi_{ss}(t) \bar{v}_\Lambda,
\]

and this the solution of \(x_{W^*}(t)\) for any \(x(0)\).

Now, we introduce the ‘\(\sim\)’ notation to indicate the improper integral made by assuming the integration interval of the 6 terms in Eq. (1) as 0, \(\infty\) instead of [0, t]. For example,

\[
\tilde{\Psi}_c(t) = \int_0^\infty e^{-t'} \cos(\omega(t - t')) \, dt'.
\]

Then one can immediately read that each \(\tilde{\Psi}_\alpha(t), \alpha \in \{c, s, cc, sc, cs, ss\}\), is smooth, \(\frac{2\pi}{\omega}\)-periodic, thus always well-defined. Further, one can show that each \(\tilde{\Psi}_\alpha(t)\) is asymptotically equivalent to \(\Psi_\alpha(t)\) respectively, since one can always have

\[
|\tilde{\Psi}_\alpha(t) - \Psi_\alpha(t)| < \int_0^\infty e^{-t'} \, dt' = e^{-t},
\]

where the inequality comes from the fact that the combinations of the product of trigonometric functions are less than 1. Thus for any \(\epsilon > 0\), if \(t > \ln(1/\epsilon)\), then \(|\tilde{\Psi}_\alpha(t) - \Psi_\alpha(t)| < \epsilon\) for every \(\alpha\), so for any \(x \in \mathbb{R}^N\), \(\tilde{\Psi}_\alpha(t)x\) is actually a limit cycle of \(\Psi_\alpha(t)x\).

Therefore, now one straightforwardly get that \(x_{W^*}(t)\) has a limit cycle \(\tilde{x}_{W^*}\), where

\[
\tilde{x}_{W^*}(t) = \tilde{\Psi}_c(t) u_\Delta + \tilde{\Psi}_{cc}(t) \bar{u} + \tilde{\Psi}_{sc}(t) \bar{u}_\Lambda + \tilde{\Psi}_s(t) v_\Delta + \tilde{\Psi}_{cs}(t) \bar{v} + \tilde{\Psi}_{ss}(t) \bar{v}_\Lambda,
\]

and it is unique through the entire system, since such limit cycle is independent from the choice of \(x(0)\). Thus Eq. (38) is a unique, globally stable periodic solution of system (29), completing the proof. \(\Box\)

We have now specified the only possible form of periodic \(x\), i.e., \(\tilde{x}_{W^*}\) with \(W^* \in \bigwedge^2(S)\) for some 2-dimensional plane \(S\). If such periodic orbit satisfies Eq. (27) in Lemma I, then one can tell that \(\tilde{x}_{W^*}\) is a consistent form to also be the periodic of system (8). From this, we directly read that \(u_\Delta = v_\Delta = 0\) in Eq. (38), since the statement (26) of Lemma I indicates that \(\tilde{x}_{W^*}(t) \subset S\) for all \(t\). Therefore, the plane
S generating $W^*$ must coincide with $\text{Span}[u, v]$, for consistency. Thus from now on, we now analogously set $S$ as the memory plane $\text{Span}[u, v]$, and also naturally $\bar{u} = u$ and $\bar{v} = v$.

We still have to quantitatively check Eq. (27), so one has to directly compute the exact form of $\tilde{\Psi}_{cs}, \tilde{\Psi}_{ss}, \tilde{\Psi}_{cs}$, and $\tilde{\Psi}_{ss}$. First, observe that

$$\tilde{\Psi}_{cs}(t) = \int_0^\infty e^{-t'} \cos(\lambda^* t') \sin(\omega(t - t')) dt'$$

$$= \frac{i}{4} \int_0^\infty e^{-t'} \left( e^{i\lambda^* t'} + e^{-i\lambda^* t'} \right) \left( -e^{i\omega(t-t')} + e^{-i\omega(t-t')} \right) dt'$$

$$= \frac{i}{4} \left( \frac{e^{i((\lambda^* - \omega)t' + \omega t')}}{-1 + i(\lambda^* - \omega)} + \frac{e^{i((\lambda^* + \omega)t' + \omega t')}}{-1 + i(\lambda^* + \omega)} \right)_{t' = \infty} - \frac{e^{-i((\lambda^* + \omega)t' + \omega t')}}{-1 - i(\lambda^* + \omega)} - \frac{e^{-i((\lambda^* - \omega)t' + \omega t')}}{-1 - i(\lambda^* - \omega)}$$

$$= \frac{1}{2} \left( \frac{\sin \omega t - (\omega + \lambda^*) \cos \omega t}{1 + (\omega + \lambda^*)^2} + \frac{\sin \omega t - (\omega - \lambda^*) \cos \omega t}{1 + (\omega - \lambda^*)^2} \right),$$

and similarly,

$$\tilde{\Psi}_{ss}(t) = \int_0^\infty e^{-t'} \sin(\lambda^* t') \sin(\omega(t - t')) dt'$$

$$= \frac{1}{2} \left( \frac{\cos \omega t + (\omega + \lambda^*) \sin \omega t}{1 + (\omega + \lambda^*)^2} - \frac{\cos \omega t + (\omega - \lambda^*) \sin \omega t}{1 + (\omega - \lambda^*)^2} \right).$$

Here, introducing notations

$$\Phi_{\omega, \pm \lambda} = 1 + (\omega \pm \lambda)^2$$

and $$\vartheta_{\omega, \pm \lambda} = \tan^{-1}(\omega \pm \lambda),$$

then Eqs. (39) and (40) simplifies into

$$\begin{cases}
\tilde{\Psi}_{cs}(t) = \frac{1}{2} \left( \frac{\sin(\omega t - \vartheta_{\omega, +\lambda^*})}{\sqrt{\Phi_{\omega, +\lambda^*}}} + \frac{\sin(\omega t - \vartheta_{\omega, -\lambda^*})}{\sqrt{\Phi_{\omega, -\lambda^*}}} \right), \\
\tilde{\Psi}_{ss}(t) = \frac{1}{2} \left( \frac{\cos(\omega t - \vartheta_{\omega, +\lambda^*})}{\sqrt{\Phi_{\omega, +\lambda^*}}} - \frac{\cos(\omega t - \vartheta_{\omega, -\lambda^*})}{\sqrt{\Phi_{\omega, -\lambda^*}}} \right),
\end{cases}$$

where the elementary fact that $\sin t \pm b \cos t = \sqrt{a^2 + b^2} \sin(t \pm \phi)$ with $\phi = \tan^{-1}(b/a)$, and $a \cos t \pm b \sin t = \sqrt{a^2 + b^2} \cos(t \mp \phi)$ with $\phi = -\tan^{-1}(b/a)$ has been used. Let’s denote

$$\tilde{\Psi}_{cs}(t) = \Gamma_{1,[\omega, \lambda^*]}(t), \quad \text{and} \quad \tilde{\Psi}_{ss}(t) = \Gamma_{2,[\omega, \lambda^*]}(t).$$

Note that since $$\int_0^\infty e^{-t'} \cos(\lambda^* t') \sin(\omega(t + \frac{\pi}{2\omega} - t')) dt' = \int_0^\infty e^{-t'} \cos(\lambda^* t') \cos(\omega(t - t')) dt',$$ we see that $\tilde{\Psi}_{cs}(t + \frac{\pi}{2\omega}) = \tilde{\Psi}_{cs}(t)$ and similarly $\tilde{\Psi}_{ss}(t + \frac{\pi}{2\omega}) = \tilde{\Psi}_{ss}(t)$. Thus
one can write
\[
\tilde{\Psi}_{cc}(t) = \Gamma_{1,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega}), \quad \text{and} \quad \tilde{\Psi}_{sc}(t) = \Gamma_{2,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega}). \tag{44}
\]

Now we are ready to try Eq. (27). Using \(\tilde{x}_W^*(t)\) as \(x^*(t)\) and substituting all \(\tilde{\Psi}_a(t)\) in Eq. (38) with expressions in \(\Gamma_{1,[\omega,\lambda^*]}(t)\) and \(\Gamma_{2,[\omega,\lambda^*]}(t)\), one gets
\[
\tilde{x}_W^*(t) \wedge \tilde{x}_W^*(t - \tau) = \left( \Gamma_{1,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega}) u + \Gamma_{2,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega}) u \wedge \\
+ \Gamma_{1,[\omega,\lambda^*]}(t)v + \Gamma_{2,[\omega,\lambda^*]}(t)v_{\wedge} \right) \\
\wedge \left( \Gamma_{1,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega} - \tau) u + \Gamma_{2,[\omega,\lambda^*]}(t + \frac{\pi}{2\omega} - \tau) u \wedge \\
+ \Gamma_{1,[\omega,\lambda^*]}(t - \tau)v + \Gamma_{2,[\omega,\lambda^*]}(t - \tau)v_{\wedge} \right). \tag{45}
\]

Following statement (27) in Lemma I, this should be a i) constant matrix, ii) which is in \(\wedge^2(S)\), for consistency. Fortunately, i) is guaranteed by the following key property of harmonics: for any \(\omega, \lambda, \text{ and } \delta,\)
\[
\Gamma_{i,[\omega,\lambda]}(t) \Gamma_{j,[\omega,\lambda]}(t - \delta) - \Gamma_{i,[\omega,\lambda]}(t - \delta) \Gamma_{j,[\omega,\lambda]}(t) \quad \text{is a constant.} \tag{46}
\]
Above can be checked by direct computation with using the following trigonometric identity
\[
\sin(\omega t - \vartheta_1) \sin(\omega t - \vartheta_2 - \tau) - \sin(\omega t - \vartheta_1 - \tau) \sin(\omega t - \vartheta_2) = \sin\omega\tau \sin(\vartheta_1 - \vartheta_2). \tag{47}
\]
Then after some lengthy computation, one finally deduces that
\[
\tilde{x}_W^*(t) \wedge \tilde{x}_W^*(t - \tau) = \frac{\sin\omega\tau}{4} \left[ \left( \frac{1}{\Phi_{\omega,-\lambda^*}} - \frac{1}{\Phi_{\omega,\lambda^*}} \right) (u_{\wedge} \wedge u + v_{\wedge} \wedge v) \right. \\
+ \left( \frac{1}{\Phi_{\omega,-\lambda^*}} + \frac{1}{\Phi_{\omega,\lambda^*}} \right) + \frac{2\cos(\theta_{\omega,-\lambda^*} - \theta_{\omega,\lambda^*})}{\Phi_{\omega,-\lambda^*} \Phi_{\omega,\lambda^*}} \right) (v \wedge u) \\
+ \left( \frac{1}{\Phi_{\omega,-\lambda^*}} + \frac{1}{\Phi_{\omega,\lambda^*}} \right) - \frac{2\cos(\theta_{\omega,-\lambda^*} - \theta_{\omega,\lambda^*})}{\Phi_{\omega,-\lambda^*} \Phi_{\omega,\lambda^*}} \right) (v_{\wedge} \wedge u_{\wedge}) \\
\left. + \frac{2\sin(\theta_{\omega,-\lambda^*} - \theta_{\omega,\lambda^*})}{\Phi_{\omega,-\lambda^*} \Phi_{\omega,\lambda^*}} (v_{\wedge} \wedge u + v_{\wedge} \wedge u_{\wedge}) \right]. \tag{48}
\]
which is a constant in \(t\) as expected. It seems quite complicated, but it is actually a matrix in \(\wedge^2(S)\) thus satisfying condition ii), since \(u, u_{\wedge} \text{ and } v, v_{\wedge}\) are all in \(S\), and the space \(\wedge^2(S)\) generated by \(S\) is in fact one-dimensional. This expression can be further simplified by using the fact that the following matrix \(R = (v \wedge u)/\|u\|\|v\| \sin\theta)\), where \(\theta\) stands for the angle between \(u\) and \(v\), is an unitary matrix in \(\wedge^2(S)\) that
performs and orthogonal rotational transform in the orientation of $W^*$ on $S$. Thus denoting $\|u\| = \eta_1$, $\|v\| = \eta_2$, and $u^Tv/(\|u\|\|v\|) = \mu$, then from $u_\wedge = Ru$ and $v_\wedge = Rv$, one gets

$$u_\wedge = \frac{1}{\sqrt{1 - \mu^2}} (-\mu u + \frac{\eta_1}{\eta_2} v), \quad v_\wedge = \frac{1}{\sqrt{1 - \mu^2}} (-\frac{\eta_2}{\eta_1} u + \mu v).$$

(49)

The above directly leads to the following scaling relations for the six wedge product terms in (48) as:

$$\begin{align*}
\mathbf{u}_\wedge \wedge \mathbf{u} &= \frac{\eta_1}{\eta_2 \sqrt{1 - \mu^2}} (v \wedge u) \\
\mathbf{v}_\wedge \wedge \mathbf{v} &= \frac{\eta_2}{\eta_1 \sqrt{1 - \mu^2}} (v \wedge u) \\
\mathbf{v}_\wedge \wedge \mathbf{u}_\wedge &= v \wedge u
\end{align*}$$

(50)

Furthermore, since $\lambda^*$ was the magnitude of the only pair of imaginary eigenvalue of $W^* \in \wedge^2(S)$, so one can immediately write

$$W^* = \lambda^* \mathbf{R} = \lambda^* \frac{\mathbf{v} \wedge \mathbf{u}}{\eta_1 \eta_2 \sqrt{1 - \mu^2}}.$$ 

(51)

Now simplifying Eq. (48) only in terms of $\mathbf{v} \wedge \mathbf{u}$ using Eq. (50), and substituting the result into Eq. (27) (i.e., $\mathbf{O} = -\gamma W^* + \rho (xW^* \wedge xW^*)$) alongside with Eq. (51), then we get the following consistency equation only with respect to $\lambda^*$:

$$\mathbf{O} = \left[ -\frac{\gamma \lambda^*}{\eta_1 \eta_2 \sqrt{1 - \mu^2}} + \frac{\rho \sin \omega \tau}{\Phi_{\omega, -\lambda^*} \Phi_{\omega, +\lambda^*}} \left( \lambda^* + \frac{(\eta_1^2 + \eta_2^2) \omega}{\eta_1 \eta_2 \sqrt{1 - \mu^2}} + \omega^2 + 1 \right) \right] (v \wedge u).$$

This directly implies that the value of $\lambda^* \in \mathbb{R} \setminus \{0\}$, which $\pm \lambda^*i$ were the eigenvalue of $W^*$ must satisfy the algebraic equation $h(\lambda) = 0$, where

$$h(\lambda) = \frac{\lambda \Phi_{\omega, -\lambda} \Phi_{\omega, +\lambda}}{(\eta_1 \eta_2 \sqrt{1 - \mu^2}) (\lambda^2 + \omega^2 + 1) + (\eta_1^2 + \eta_2^2) \omega \lambda} - \frac{\rho \sin \omega \tau}{\gamma}.$$ 

(52)

Summing up the results, we have: for given $\mathbf{u}$, $\mathbf{v}$ and the parameters $\omega$, $\rho$, $\gamma$, if there exists a real root $\lambda_0$ of the algebraic equation $h(\lambda) = 0$, then for each $\lambda_0$, there exists a unique periodic orbit with steady synapses of system (8), say, $(x^*(t), W^*)$, where

$$\begin{align*}
x^*(t) &= \Gamma_{1, [\omega, \lambda_0]}(t + \frac{\pi}{2\omega}) u + \Gamma_{2, [\omega, \lambda_0]}(t + \frac{\pi}{2\omega}) u_\wedge + \Gamma_{1, [\omega, \lambda_0]}(t)v + \Gamma_{2, [\omega, \lambda_0]}(t)v_\wedge \\
W^* &= \frac{\lambda_0}{\eta_1 \eta_2 \sqrt{1 - \mu^2}} (v \wedge u).
\end{align*}$$

(53)

Since each $\Gamma_{i, [\omega, \lambda_0]}(t)$ are periodic functions and $u$, $v$, $u_\wedge$, and $v_\wedge$ are all in $S$, we complete the proof of Theorem 1.
Note that the expression of \( x^*(t) \) in Eq. (53) can be further expressed in terms with only \( u \) and \( v \), using Eq. (49). We reorganize the full statements of Theorem 1 with straightforwardly expanding every abbreviated notations as follows:

**Theorem 1 (Periodic Solution with Steady Connectivity)** Let the vectors \( u \), \( v \) are given as the form in Lemma A from input (5) (i.e., \( u = -\Psi \sin \xi, v = \Psi \cos \xi \)), and let \( \|u\| = \eta_1, \|v\| = \eta_2 \), and \( \frac{u^\top v}{\|u\|\|v\|} = \mu \). If \( \lambda_0 \) is the real root of algebraic equation \( h(\lambda) = 0 \), where

\[
h(\lambda) = \frac{\lambda^5 + (-2\omega^2 + 2)\lambda^3 + (\omega^4 + 2\omega^2 + 1)\lambda}{(\eta_1\eta_2\sqrt{1-\mu^2}) (\lambda^2 + \omega^2 + 1) \sqrt{\frac{\eta_1}{\eta_2} (\frac{\eta_1}{\eta_2} - 2\lambda^2 + \omega^2 + 1)}} - \frac{\rho \sin \omega \tau}{\gamma}, \tag{54}
\]

then for each \( \lambda_0 \), the system (8) under input (5) has a unique periodic neural solution \( x^*(t) \) with a constant connectivity matrix \( W^* \). Specifically,

\[
\begin{align}
x^*(t) &= \left( \frac{\Gamma_1[\lambda_0, \omega]}{\eta_1\eta_2\sqrt{1-\mu^2}} (t + \frac{\pi}{2\omega}) \right) \left( \frac{\Gamma_2[\lambda_0, \omega]}{\sqrt{1-\mu^2}} (t + \frac{\pi}{2\omega}) \right) u \\
&+ \frac{\mu}{\eta_1\eta_2\sqrt{1-\mu^2}} \left( \frac{\eta_1}{\eta_2} \Gamma_2[\lambda_0, \omega] (t + \frac{\pi}{2\omega}) \right) v \\
W^* &= \frac{\lambda_0}{\eta_1\eta_2\sqrt{1-\mu^2}} (u^\top v - uv^\top), \tag{55}
\end{align}
\]

where \( \Gamma_1[\lambda, \omega](t) \) and \( \Gamma_2[\lambda, \omega](t) \) are \( \frac{2\pi}{\omega} \)-periodic functions in \( t \) given by:

\[
\begin{align}
\Gamma_1[\lambda, \omega](t) &= \frac{1}{2} \left( \frac{\sin\left(\omega t - \tan^{-1}(\omega + \lambda)\right)}{\sqrt{\lambda^2 + 2\omega\lambda + \omega^2 + 1}} + \frac{\sin\left(\omega t - \tan^{-1}(\omega - \lambda)\right)}{\sqrt{\lambda^2 - 2\omega\lambda + \omega^2 + 1}} \right) \tag{56}
\end{align}
\]

\[
\begin{align}
\Gamma_2[\lambda, \omega](t) &= \frac{1}{2} \left( \frac{\cos\left(\omega t - \tan^{-1}(\omega + \lambda)\right)}{\sqrt{\lambda^2 + 2\omega\lambda + \omega^2 + 1}} - \frac{\cos\left(\omega t - \tan^{-1}(\omega - \lambda)\right)}{\sqrt{\lambda^2 - 2\omega\lambda + \omega^2 + 1}} \right). \tag{57}
\end{align}
\]

**Remarks**

1. The solubility of periodic neural solution \( x^*(t) \) is not directly shown, but automatically guaranteed by the principle of Lemma II. Still, it can be straightforwardly shown by direct computation (refer to Yoon 2022).

2. The global uniqueness of such periodic solution \( (x^*(t), W^*) \) can be assured by checking if the algebraic equation \( h(\lambda) = 0 \) has only one real root. It seems to be confirmed that having a unique real root for almost all reasonable parameter families (\( \omega, \rho, \gamma \)), however, explicitly obtaining the exact conditions for strict uniqueness still have to be explored. Figure 12 shows the plot of the number of real roots \( \#(\lambda_0 \in \mathbb{R}) \), for some forcibly found parameter regions exhibiting non-uniqueness.

3. One can see that from the shape of \( h(\lambda) = 0 \), the absolute value of the real root \( \lambda_0 \) is maximized when the condition

\[
\sin \omega \tau = 1 \tag{58}
\]
Fig. 12 Figure of the non-uniqueness regions of $\lambda_0$ (i.e., region of $#(\lambda_0 \in \mathbb{R}) > 1$, plotted dark gray). Such regions don’t seem to be affected by the ratio of $\rho$ and $\gamma$, or $\omega$ and $\tau$, but only by the ratios of $\eta_1$, $\eta_2$, and $\mu$. a Plot of non-uniqueness region for $(\eta_1, \eta_2) \in [0.5, 10] \times [0.5, 10]$ with parameters $\omega = 1$, $\tau = \pi/2$, $\rho/\gamma = 1$, and $\mu = 0$. b Plot of non-uniqueness region for $(\mu, \eta_2) \in [0, 1] \times [0.5, 10]$ with parameters $\omega = 1$, $\tau = \pi/2$, $\rho/\gamma = 1$, and $\eta_1 = 1$. This plot is still identically drawn if one replaces $\eta_2$ with $\eta_1$ (color figure online).

holds (i.e., $\tau = \frac{\pi}{2\omega}$) for a given value of $\rho$ and $\gamma$. Thus such value of $\tau$ guarantees the maximum norm of $W^*$, allowing the system to achieve the most efficient learning on the storage phase.

B3: Proof of Theorem 2 and the Retrievability Conditions

**Theorem 2** (PERIODIC RETRIEVAL) For any non-zero cue $m_c$, the solution of Eq. (9) under input (7) asymptotically approaches to some periodic solution $x_r^*(t)$. Especially if $m_c \notin S_\perp$, $x_r^*(t)$ becomes periodically retrievable at $t = t^\dagger > 0$ where

$$t^\dagger = \frac{1}{\omega} \tan^{-1} \frac{\omega}{n \pi} + n \pi, \quad n \in \mathbb{Z}.$$  \hfill (59)

**Proof** Since the simplified retrieval system is equivalent to the reduced system in Eq. (29), as in Lemma II in Appendix B2, we can again obtain explicit solution of $x_r(t)$ using the variational formula by only replacing the storage input $b(t; m_1, \ldots, m_n)$ with the cue input $b_c(t; m_c) = \sin \omega t m_c$, i.e.,

$$x_r(t) = e^t(W^* - I)x_r(0) + \int_0^t e^{t'}(W^* - I)\sin(\omega(t - t'))m_c \, dt'$$

$$= e^{-t}e^tW^*x_r(0) + \int_0^t e^{-t'}\sin(\omega(t - t'))e^{t'}W^*m_c \, dt'.$$  \hfill (60)

Similarly, we use that

$$e^{t'W^*}m_c = m_c^{\triangle} + \cos(\lambda^* t)\overline{m}_c + \sin(\lambda^* t)\overline{m}_c^{\wedge}.$$  \hfill (61)
where \( \overline{m}_c = \text{Proj}_S m_c \), and \( m_{cΔ} = m_c - \overline{m}_c \), and \( \overline{m}_{c∧} \in S \) satisfying \( \| \overline{m}_{c∧} \| = \| \overline{m}_c \| \) with \( \overline{m}_{c∧} \perp \overline{m}_c \), following the orientation of the flow \( e^{tW_m} m_c \). Also, \( ±\lambda i \ (\lambda \in \mathbb{R}) \) stands for the only pair of imaginary eigenvalue of \( W^* \). Then the RHS of Eq. (60) becomes

\[
x_r(t) = e^{-t} e^{tW^*} x_r(0) + \int_0^t e^{-t'} \sin(\omega(t - t')) \overline{m}_c \, dt'
+ \int_0^t e^{-t'} \left( \cos(\lambda^* t') \sin(\omega(t - t')) \overline{m}_c + \sin(\lambda^* t') \sin(\omega(t - t')) \overline{m}_{c∧} \right) \, dt'.
\]

(62)

Here we use the results of Lemma II, directly having that \( x_r(t) \) always approaches a unique stable limit cycle regardless of \( x_r(0) \), say, \( x^*_r(t) \), being

\[
x^*_r(t) = \frac{1}{\sqrt{\omega^2 + 1}} \sin \left( \omega t - \tan^{-1} \omega \right) m_{cΔ} + \Gamma_1,\omega^* (t) \overline{m}_c + \Gamma_2,\omega^* (t) \overline{m}_{c∧},
\]

(63)

where the facts that

\[
\int_0^\infty e^{-t'} \sin(\omega(t - t')) \, dt' = \frac{1}{\sqrt{\omega^2 + 1}} \sin \left( \omega t - \tan^{-1} \omega \right),
\]

(64)

\[
\int_0^\infty e^{-t'} \cos(\lambda^* t') \sin(\omega(t - t')) \, dt' = \Gamma_1,\omega^* (t),
\]

(65)

\[
\int_0^\infty e^{-t'} \sin(\lambda^* t') \sin(\omega(t - t')) \, dt' = \Gamma_2,\omega^* (t)
\]

(66)

has been used.

To show the periodic retrieval, let \( \mathcal{M} \) be the retrievable subspace with respect to the representation set \( \{\overline{m}_i\}_{i=1}^n \), and consider the case that \( m_c \notin S_\perp \). Then \( \overline{m}_c, \overline{m}_{c∧} \in \mathcal{M} \) and \( \overline{m}_{c∧} \perp \overline{m}_c \in S = \text{Span}(u, v) \), one can directly see that also \( \overline{m}_c, \overline{m}_{c∧} \in \mathcal{M} \setminus \{0\} \). Now, we claim that the term \( \Gamma_1,\omega^* (t) \overline{m}_c + \Gamma_2,\omega^* (t) \overline{m}_{c∧} \) in Eq. (63) always lies in \( \mathcal{M} \setminus \{0\} \). This can be shown from the following alternate expressions of \( \Gamma_1,\omega^* (t) \) and \( \Gamma_2,\omega^* (t) \):

\[
\Gamma_1,\omega^* (t) = \sqrt{\alpha^2 + \beta^2} \sin(\omega t + \Delta_1) \quad \text{and} \quad \Gamma_2,\omega^* (t) = \sqrt{\alpha^2 + \beta^2} \sin(\omega t + \Delta_2),
\]

(67)

where

\[
\alpha = \frac{1}{2} \left( \frac{\cos \theta_{ω^*,\omega^*} + \cos \theta_{ω^*,ω^*}}{\sqrt{\Phi_{ω^*,ω^*} + \sqrt{\Phi_{ω^*,ω^*}}}} \right) \quad \text{and} \quad \Delta_1 = \tan^{-1} \left( -\frac{\beta}{\alpha} \right),
\]

\[
\beta = \frac{1}{2} \left( \frac{\sin \theta_{ω^*,\omega^*} + \sin \theta_{ω^*,ω^*}}{\sqrt{\Phi_{ω^*,ω^*} + \sqrt{\Phi_{ω^*,ω^*}}}} \right) \quad \text{and} \quad \Delta_2 = \tan^{-1} \left( \frac{\alpha}{\beta} \right).
\]

(68)

From Eq. (67), one can read that the only condition making \( \Gamma_1,\omega^* \) and \( \Gamma_2,\omega^* \) to vanish simultaneously is \( \Delta_1 = \Delta_2 \), and the bijective property of the arctangent function.

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function implies
\[ -\frac{\beta}{\alpha} = \frac{\alpha}{\beta} \iff \alpha^2 + \beta^2 = 0, \]  
(69)

thus yielding \( \alpha = \beta = 0 \), which only is a pointless triviality.

From this, one can assure that \( x^*_{\bar{c}}(t) \) must belongs to \( \mathcal{M} \setminus \{0\} \) on instances that making the coefficient of \( m_{c\Delta} \) in Eq. (63) to vanish. Denoting such time \( t \) as \( t = t^\dagger \), we derive that such \( t^\dagger \) must satisfy \( \omega t^\dagger - \theta = n\pi, n \in \mathbb{Z} \), that is,
\[ t^\dagger = \frac{1}{\omega} \tan^{-1} \omega + n \frac{\pi}{\omega}, \quad n \in \mathbb{Z}, \quad (t^\dagger > 0), \]
(70)
which yields Eq. (12) indicating periodic retrieval. This proves that \( x^*_{\bar{c}}(t) \) with \( t = t^\dagger \) is always retrievable unless \( m_{c} \notin S_\perp \). Besides, if \( m_{c} \in \mathcal{M} \), then one can easily see \( m_{c\Delta} \in \mathcal{M} \) thus \( x^*_{\bar{c}}(t) \in \mathcal{M} \setminus \{0\} \) for all \( t \), so always being retrievable.

**Remarks** On the other hand, considering the case when \( m_{c} \in S_\perp \), first suppose that also \( m_{c} \in \mathcal{M} \). Then, \( \overline{m}_{c}, \overline{m}_{c\lambda} = 0 \), but \( m_{c\Delta} \neq 0 \). Thus \( x^*_{\bar{c}}(t) \in \mathcal{M} \) for all \( t \), but especially only on \( t = t^\dagger, x^*_{\bar{c}}(t) = 0 \). In contrary, if \( m_{c} \notin \mathcal{M} \), then also \( \overline{m}_{c}, \overline{m}_{c\lambda} = 0 \), and even \( m_{c\Delta} \notin \mathcal{M} \), therefore \( x^*_{\bar{c}}(t) \notin \mathcal{M} \setminus \{0\} \) for all \( t \), so never becoming retrievable.

Based on the above observations, the retrievability conditions given in the main text can be further refined as follows:

1. If \( m_{c} \in S_\perp \cap \mathcal{M} \), then only for \( t = t^\dagger, x^*_{\bar{c}}(t) \in \mathcal{M} \setminus \{0\} \), and in fact \( x^*_{\bar{c}}(t^\dagger) \in S \).
2. If \( m_{c} \in S_\perp \cap \mathcal{M} \), then \( \forall t > 0, x^*_{\bar{c}}(t) \in \mathcal{M} \setminus \{0\} \), and in fact \( x^*_{\bar{c}}(t^\dagger) \in S \).
3. If \( m_{c} \in S_\perp \cap \mathcal{M} \), then \( \forall t > 0, x^*_{\bar{c}}(t) \notin \mathcal{M} \setminus \{0\} \) and in fact \( x^*_{\bar{c}}(t^\dagger) = 0 \).
4. If \( m_{c} \in S_\perp \cap \mathcal{M} \), then \( \forall t > 0 \) except \( t = t^\dagger, x^*_{\bar{c}}(t) \in \mathcal{M} \setminus \{0\} \), and in fact \( x^*_{\bar{c}}(t^\dagger) = 0 \).

Lastly, one can also directly show that \( x^*_{\bar{c}}(t) \) is a solution of Eq. (9) (Yoon 2022).

**B4: Proof of Theorem 3**

**Theorem 3** (Optimal Choice for \( \xi_i \)) Suppose \( \{m_i\}_{i=1}^n \) with \( m_i \in \mathbb{R}^N \) are mutually orthogonal vectors of the same magnitude. Then the maximum value of \( \langle \cos \theta_i \rangle_i \) is \( \sqrt{\frac{\pi}{n}} \) and can be attained with the the the distribution of \( \xi_i \) as
\[ \xi_i = (i - 1) \frac{\pi}{n} + \alpha, \quad i = 1, \ldots, n, \quad 0 \leq \alpha < \frac{\pi}{n}. \]
(71)

**Proof** One can directly use \( \cos \theta_i = \frac{\| \text{Proj}_{S(\xi_1, \ldots, \xi_n)} m_i \|}{\| m_i \|} \) where \( S(\xi_1, \ldots, \xi_n) \) denotes the memory plane determined with the choice of \( \{\xi_i\}_{i=1}^n \). Firstly, one can generally observe that for any \( m \in \mathbb{R}^N \) and \( S = \text{Span}\{u, v\} \),
\[ \text{Proj}_S m = \text{Proj}_{\text{Span}\{u, v\}} m = \frac{m^\top u}{\|u\|} u + \frac{m^\top u_\perp}{\|u_\perp\|} u_\perp, \]
(72)
where $u^\wedge \in \text{Span}(u, v)$ satisfying $u^\wedge \perp u$ and $\|u^\wedge\| = \|u\|$, in which can be specifically expressed as in Eq. (49). Thus substituting it into above equation yields

$$
\text{Proj}_S m_i = \frac{1}{1 - \mu^2} \left( \frac{m_i^\top u}{\eta_1^2} - \mu \frac{m_i^\top v}{\eta_1 \eta_2} \right) u + \frac{1}{1 - \mu^2} \left( \frac{m_i^\top v}{\eta_2^2} - \mu \frac{m_i^\top u}{\eta_1 \eta_2} \right) v,
$$

(73)

where $\eta_1 = \|u\|$, $\eta_2 = \|v\|$ and $\mu = u^\top v / (\|u\| \|v\|)$. Besides, from the fact

$$
\|\text{Proj}_S m_i\|^2 = m_i \cdot \text{Proj}_S m_i,
$$

(74)

for $u, v$ defined as in Lemma A, one can directly read that

$$
\|\text{Proj}_S(\xi_1, \ldots, \xi_n) m_i\| = \sqrt{\frac{\left( \frac{m_i^\top u}{\eta_1} \right)^2 + \left( \frac{m_i^\top v}{\eta_2} \right)^2 - 2 \mu \frac{(m_i^\top u)(m_i^\top v)}{\eta_1 \eta_2}}{1 - \mu^2}}.
$$

(75)

Let $\|m_i\| = l$. Since $u = -\Psi \sin \xi = \sum_{i=1}^n \sin \xi_i m_i$, and $v = \Psi \cos \xi = \sum_{i=1}^n \cos \xi_i m_i$, the orthogonality of $\{m_i\}_{i=1}^n$ guarantees $m_i^\top u = -\sin \xi_i$, and $m_i^\top v = \cos \xi_i$. Further, one can easily verify that

$$
\eta_1 = l^2 \sum_{j=1}^n \sin^2 \xi_j, \quad \eta_2 = l^2 \sum_{j=1}^n \cos^2 \xi_j \quad \text{and} \quad \mu = -\frac{\sum_{j=1}^n \sin \xi_j \cos \xi_j}{\sqrt{\left(\sum_{j=1}^n \sin^2 \xi_j\right) \left(\sum_{j=1}^n \cos^2 \xi_j\right)}}.
$$

So substituting these expressions into Eq. (75) and completing tedious simplification procedure, we finally deduce

$$
\frac{\|\text{Proj}_S(\xi_1, \ldots, \xi_n) m_i\|}{\|m_i\|} = \sqrt{\frac{\sum_{j=1}^n \sin^2 (\xi_j - \xi_i)}{\sum_{j, k=1}^{n} \sum_{j > k} \sin^2 (\xi_j - \xi_k)}}.
$$

(76)

Note that this value does not depend on $l = \|m_i\|$. Now, consider the following double summation $\sum_{i, j=1}^n \sin^2 (\xi_j - \xi_i)$. This is exactly the sum with respect to $i$ performed to the squared numerator of the last term in Eq. (76). Moreover, $\sin^2 (\xi_j - \xi_i) = \sin^2 (\xi_i - \xi_j)$ and is zero when $j = i$, thus we read that

$$
\sum_{i, j=1}^n \sin^2 (\xi_j - \xi_i) = 2 \sum_{i, j=1}^n \sin^2 (\xi_j - \xi_i),
$$

(77)
which the term \( \sum_{i,j=1}^{n} \sin^2(\xi_j - \xi_i) \) is identical the squared denominator of the last term in Eq. (76). This directly leads to the following strong result:

\[
\sum_{i=1}^{n} \left( \frac{\| \text{Proj}_{S(\xi_1, \ldots, \xi_n)} \mathbf{m}_i \|}{\| \mathbf{m}_i \|} \right)^2 = 2. \tag{78}
\]

From this, by Cauchy-Schwarz inequality, the maximum of \( \langle \cos \theta_i \rangle_i = \frac{1}{n} \sum_{i=1}^{n} \frac{\| \text{Proj}_{S(\xi_1, \ldots, \xi_n)} \mathbf{m}_i \|}{\| \mathbf{m}_i \|} \) is achieved with value \( \sqrt{\frac{2}{n}} \) when each \( \cos \theta_i \) is equal and \( \frac{\| \text{Proj}_{S(\xi_1, \ldots, \xi_n)} \mathbf{m}_i \|}{\| \mathbf{m}_i \|} \) is \( \sqrt{\frac{2}{n}} \) for all \( i = 1, \ldots, n \), so proving Eq. (71).

However, finding the possible distributions of \( \xi_i \) achieving the maximum is quite difficult, but we claim that such distribution exists, and one family of those are given as in (71). To show this, first suppose that each \( \xi_i \) is chosen as (71) but with zero shifts, i.e., \( \alpha = 0 \), and denote such values with \( \xi_i \). We first verify that \( \mathbf{u} \perp \mathbf{v} \) in this case. Observe that when \( n \) is even,

\[
\mathbf{u}^\top \mathbf{v} = -\sum_{i=1}^{n} \sin \xi_i \cos \xi_i = -\sin \xi_1 \cos \xi_1 - \sum_{i=2}^{n} \sin \xi_i \cos \xi_i
\]

\[
= 0 - \sum_{i=2}^{n/2} \left[ \sin \xi_i \cos \xi_i + \sin \xi_{n-i+2} \cos \xi_{n-i+2} \right] + \sin \xi_{n/2+1} \cos \xi_{n/2+1}
\]

\[
= -\sum_{i=2}^{n/2} \left[ \sin \xi_i \cos \xi_i + \sin(\pi - \xi_i) \cos(\pi - \xi_i) \right] + \sin \frac{\pi}{2} \cos \frac{\pi}{2} = 0, \tag{79}
\]

and similarly when \( n \) is odd,

\[
\mathbf{u}^\top \mathbf{v} = -\sum_{i=1}^{n} \sin \xi_i \cos \xi_i
\]

\[
= -\sin \xi_1 \cos \xi_1 - \sum_{i=2}^{(n+1)/2} \left[ \sin \xi_i \cos \xi_i + \sin \xi_{n-i+2} \cos \xi_{n-i+2} \right]
\]

\[
= 0 - \sum_{i=2}^{(n+1)/2} \left[ \sin \xi_i \cos \xi_i + \sin(\pi - \xi_i) \cos(\pi - \xi_i) \right] = 0. \tag{80}
\]

Therefore, \( \mathbf{u} \perp \mathbf{v} \), so simply considering a \( \mu = 0 \) case in Eq. (75), we have

\[
\frac{\| \text{Proj}_{S(\xi_1, \ldots, \xi_n)} \mathbf{m}_i \|}{\| \mathbf{m}_i \|} = \frac{\sin^2 \xi_i + \cos^2 \xi_i}{\sum_{j=1}^{n} \sin^2 \xi_j + \sum_{j=1}^{n} \cos^2 \xi_j}, \quad i = 1, \ldots, n. \tag{81}
\]
Here, one can even show that

$$\sum_{j=1}^{n} \sin^2 \bar{\xi}_j = \sum_{j=1}^{n} \cos^2 \bar{\xi}_j = \frac{n}{2}$$

(82)

by observing the following: From Riemann integral,

$$\Delta \sum_{j=1}^{n} \sin^2 \bar{\xi}_j \approx \int_0^\pi \sin^2 \theta \, d\theta = \frac{\pi}{2}, \quad \Delta \sum_{j=1}^{n} \cos^2 \bar{\xi}_j \approx \int_0^\pi \cos^2 \theta \, d\theta = \frac{\pi}{2}$$

(83)

as \(n \to \infty\) where \(\Delta = \frac{\pi}{n}\) being the interval between each sampling point \(\bar{\xi}_j\).

However, by the symmetry of functions \(\cos^2 \theta\) and \(\sin^2 \theta\) on interval \([0, \pi]\) and the arithmetically sequenced property of \(\bar{\xi}_j\), one can luckily confirm that the approximation (83) is actually an equality for all \(n\). Thus we finally have

$$\sum_{j=1}^{n} \sin^2 \bar{\xi}_j = \sum_{j=1}^{n} \cos^2 \bar{\xi}_j = \frac{\pi}{2\Delta} = \frac{\pi}{2 \cdot \frac{\pi}{n}}$$

(84)

which yields Eq. (82). Therefore, we can now write Eq. (81) simply as

$$\frac{\|\text{Proj}_{S(\bar{\xi}_1, \ldots, \bar{\xi}_n)} m_i\|}{\|m_i\|} = \sqrt{\frac{2(\sin^2 \bar{\xi}_i + \cos^2 \bar{\xi}_i)}{n}} = \sqrt{\frac{2}{n}}, \quad i = 1, \ldots, n.$$  

(85)

This indicates that the value of \(\|\text{Proj}_{S(\bar{\xi}_1, \ldots, \bar{\xi}_n)} m_i\|\) is constant throughout every \(i = 1, \ldots, n\) with value \(\sqrt{2/n}\), so such set of \(\{\bar{\xi}_i\}_{i=1}^{n}\) (i.e., in Eq. (71) with \(\alpha = 0\)) can achieve \(\frac{1}{n} \max_{\bar{\xi}_1, \ldots, \bar{\xi}_n} \|\text{Proj}_{S(\bar{\xi}_1, \ldots, \bar{\xi}_n)} m_i\|\) = \(\sqrt{2/n}\).

Lastly, for the remaining \(\alpha \neq 0\) case, i.e., \(0 < \alpha < \frac{\pi}{n}\), let’s denote \(\bar{\xi}_i + \alpha = \xi_i^*\), and \(b(t) = \sum_{i=1}^{n} \sin(\omega t - \xi_i) m_i\) with \(b(\xi_1, \ldots, \xi_n)(t)\), i.e., the input orbit generated by \(\{\xi_i\}_{i=1}^{n}\). Then, one can easily see that \(b(\xi_1^*, \ldots, \xi_n^*)(t) = b(\xi_1, \ldots, \xi_n)(t + \frac{\alpha}{\omega})\) for any \(t\), so the orbit of \(b(\xi_1^*, \ldots, \xi_n^*)\) and \(b(\xi_1, \ldots, \xi_n)\) is actually identical thus sharing the same plane, i.e., \(S(\xi_1^*, \ldots, \xi_n^*) \equiv S(\xi_1, \ldots, \xi_n)\) from Lemma A. Thus, one must have \(\langle \cos \theta_i(\xi_1^*, \ldots, \xi_n^*) \rangle_i = \langle \cos \theta_i(\xi_1, \ldots, \xi_n) \rangle_i = \sqrt{\frac{2}{n}}\) which implies that \(\xi_i^* = \bar{\xi}_i + \alpha = \frac{\pi}{n}(i-1) + \alpha\) also achieves the maximum of \(\langle \cos \theta_i \rangle_i\). 

\(\square\)
Appendix C: Stability analysis of the periodic solution \((x^*(t), W^*)\)

C1: Derivation of the variational equation

First, rewriting the original system (8) into a general form, then

\[
\begin{align*}
\dot{x} &= f(x, W) \\
\dot{W} &= G(x, x_\tau, W)
\end{align*}
\quad \text{where} \quad f(x, W) = -x + Wx + b(t), \quad G(x, x_\tau, W) = -\gamma W + \rho \left( xx_\tau^\top - x_\tau x^\top \right). \tag{86}
\]

Considering deviation \(x(t) = x^*(t) + \delta x(t)\) and \(W(t) = W^* + \delta W(t)\) from reference trajectory \((x^*, W^*)\), we have

\[
\begin{align*}
\dot{x}^* + \delta \dot{x} &= f(x^* + \delta x, W^* + \delta W) \\
\dot{W}^* + \delta \dot{W} &= G(x^* + \delta x, x_\tau^* + \delta x_\tau, W^* + \delta W).
\end{align*}
\tag{87}
\]

Now, applying first-ordered Taylor expansion on \((x^*, W^*)\) to each RHS and using \(\dot{x}^* = f(x^*, W^*)\) and \(\dot{W}^* = G(x^*, x_\tau^*, W^*) = 0\), we get

\[
\begin{align*}
\delta \dot{x} &= \left. \frac{\partial f}{\partial x} \right|_{(x^*, W^*)} \delta x + \left. \frac{\partial f}{\partial W} \right|_{(x^*, W^*)} : \delta W \\
\delta \dot{W} &= \left. \frac{\partial G}{\partial x} \right|_{(x^*, x_\tau^*, W^*)} \delta x + \left. \frac{\partial G}{\partial x_\tau} \right|_{(x^*, x_\tau^*, W^*)} \delta x_\tau + \left. \frac{\partial G}{\partial W} \right|_{(x^*, x_\tau^*, W^*)} : \delta W,
\end{align*}
\tag{88}
\]

where : is used for the double dot product notation. Now computing each tensor-represented Jacobians, firstly we immediately see \(\left. \frac{\partial f}{\partial x} \right|_{(x^*, W^*)} = -I + W\), therefore

\[
\left. \frac{\partial f}{\partial x} \right|_{(x^*, W^*)} \delta x = (-I + W^*) \delta x. \tag{89}
\]

For the remaining ones, observe that \(\left. \frac{\partial f}{\partial W} \right|_{(x^*, W^*)}\) is a third-order tensor and each element can be found by

\[
\left( \left. \frac{\partial f}{\partial W} \right|_{(x^*, W^*)} \right)_{ijk} = \left. \frac{\partial f_i}{\partial W_{jk}} \right|_{(x^*, W^*)} = \frac{\partial}{\partial W_{jk}} \left( -x_i + \sum_l W_{il} x_l + b_i(t) \right) = \delta_{ij} \delta_{kl} x_l = \delta_{ij} x_k. \tag{90}
\]

Thus if write \(e^i\) as the \(i\)-th coordinate Euclidean canonical vector (i.e., \((e^i)_j = \delta_{ij}\)), then one can have

\[
\left. \frac{\partial f}{\partial W} \right|_{(x^*, W^*)} \delta W &= \sum_{i,j,k} \delta_{ij} x_k^* (e^i \otimes e^j \otimes e^k) : \sum_{l,m} \delta W_{lm} (e^l \otimes e^m) \\
&= \sum_{i,j,k,l,m} \delta_{ij} x_k^* \delta_{jl} \delta_{km} \delta W_{lm} e^i = \sum_{i,j,k} \delta_{ij} x_k^* \delta W_{jk} e^i.
\]
\[
\sum_{i,k} \delta W_{ik} x_k^i e^i = \delta W x^*. \quad (91)
\]

By similar computations, for remaining terms one can easily verify that \( \frac{\partial g}{\partial x} \bigg|_{(x^*,x_t^*,W^*)} \cdot \delta x \) \( = \rho \left( x x_t^* \tau - x_t^* \delta x^T \right) \), \( \frac{\partial g}{\partial x} \bigg|_{(x^*,x_t^*,W^*)} \cdot \delta x_t^* \) \( = \rho \left( x^* x_t^* \tau - x_t^* x^* \right) \), and \( \frac{\partial g}{\partial W} \bigg|_{(x^*,x_t^*,W^*)} \cdot \delta W = -\gamma \delta W \). Therefore summing up the results, we finally get

\[
\begin{align*}
\delta x &= (-I + W^*) \delta x + \delta W x^* \\
\delta \dot{W} &= -\gamma \delta W + \rho \left( x x_t^* \tau - x_t^* \delta x^T + x^* \delta x_t^* - x_t^* x^* \right),
\end{align*}
\quad (92)
\]

and this is the variational equation (which is also a Delay Differential Equation (DDE); it includes a delayed term \( \delta x_t^* \) of \( \delta x \)) that has been referred in the main text. \( \square \)

**C2: Computational method for estimating maximal Lyapunov exponent of DDEs**

The method of computation directly follows Farmer (1982). Since our variational Eq. (92) is a DDE, it is infinite dimensional - the method for MLE estimation slightly differs from the conventional methods of finite dimensional ODEs (Sandri 1996). First, the DDE (92), say, \( \dot{U} = F(U, U_t) \), where \( U \in \mathbb{R}^{N+N^2} \) represents the collection of all components of \( \delta x \) and \( \delta W \), can be approximated with some conjugate discrete finite dimensional map

\[
\tilde{F} : \mathbb{R}^{N+N^2} \times \cdots \times \mathbb{R}^{N+N^2} \rightarrow \mathbb{R}^{N+N^2} \times \cdots \times \mathbb{R}^{N+N^2},
\quad (93)
\]

having variables \( \tilde{U}^n \in \mathbb{R}^{N+N^2}, n = 1, \ldots, d \), which

\[
(\tilde{U}^1, \ldots, \tilde{U}^{d-1}, \tilde{U}^d) = (U(t - (d - 1)\Delta t), \ldots, U(t - \Delta t), U(t)), \quad (\Delta t = \frac{\tau}{d - 1}),
\quad (94)
\]

so that the each iteration \( \tilde{U}(k + 1) = \tilde{F}(\tilde{U}(k)) \) for \( \tilde{U} \) represents the mapping of \( U = (\tilde{U}^1, \ldots, \tilde{U}^d) \) on time \( t \) to \( t + \tau + \Delta t \). As the initial choice of \( \tilde{U} \) is given by sampled discrete points on \( t \in [-\tau, 0] \), this map starts to generate the approximated solution on interval \( \tau, \tau + \Delta t, [\tau + 2\Delta t, 2\tau + 2\Delta t] \) and so on.

The discrete map \( \tilde{F} \) conjugate to \( F \) can be found by any convenient integration techniques. Simply, for example, Euler-method takes

\[
\begin{align*}
\tilde{U}^1(k + 1) &= \tilde{U}^d(k) + F(\tilde{U}^d(k), \tilde{U}^1(k)) \Delta t, \\
\text{and for } 1 < i \leq d; \quad \tilde{U}^i(k + 1) &= \tilde{U}^{i-1}(k + 1) + F(\tilde{U}^{i-1}(k + 1), \tilde{U}^i(k)) \Delta t.
\end{align*}
\quad (95)
Now, setting $\tilde{U}(0)$ containing all of the discrete-sampled initial data of each $\delta \tilde{x}_i$, $\delta \tilde{W}_{ij} \in \mathbb{R}^d$ and obtaining the evolution of $\tilde{U}$ for each step, then the rate of exponential growth of universal deviation (the collection of every deviations)

$$[\tilde{U}(k) = [\delta \tilde{x}_1(k); \cdots; \delta \tilde{x}_N(k); \delta \tilde{W}_{11}(k); \cdots; \delta \tilde{W}_{NN}(k)] \in \mathbb{R}^{d(N+N^2)}$$ (96)

where ‘;’ denotes the vertical concatenation, is estimated by directly computing the value

$$\lambda_{\text{max}} = \lim_{K \to \infty} \frac{1}{K(\tau + \Delta t)} \sum_{k=1}^{K} \ln \left( \frac{\|\tilde{U}(k)\|}{\|\tilde{U}(k-1)\|} \right).$$ (97)

This value $\lambda_{\text{max}}$, turns out to be the maximal rate of exponential evolution of the universal deviation and in fact is the MLE, and its convergence as $K \to \infty$ is well known (Farmer 1982).

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