Edge Double-Critical Graphs

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Abstract: Problem statement: The vertex double-critical conjecture that the only vertex double-critical graph is the complete graph has remained unresolved for over forty years. The edge analogue of this conjecture has been proved. Approach: It was observed that if the chromatic number decreases by two upon the removal of a 2-matching, then the 2-matching comprises four vertices which determine an induced subgraph isomorphic to the complete graph on four vertices. This observation was generalized to t-matchings. Results: In this note, it has been shown that the only edge double-critical graph is the complete graph. Conclusion/Recommendations: An alternate proof that the only edge double-critical graph is the complete graph has been obtained. Moreover, the result has been obtained independently.

Key words: Chromatic number, critical clique, k-matching

INTRODUCTION

The graphs considered in this study are finite, undirected and simple. For a given graph G, the vertex and edge sets of G are denoted by V(G) and E(G), respectively. The order of G, denoted by n = |V(G)|, is the cardinality of V(G). An r-clique is a complete subgraph of order r and is denoted by K_r. A subset M of E(G) is said to be independent whenever no two edges in M share a common vertex. In case |M| = k, the set M is called a k-matching. For a subset X of V(G), the subgraph of G induced by X is denoted by G\[X\]. All vertex colorings are proper, i.e., a partition of V(G) into independent subsets of V(G) called color classes. Lastly, \(\chi(G)\) denotes the chromatic number of G and is the minimum cardinality of a partition of V(G) determined by a proper vertex coloring of G.

A graph G is said to be vertex double-critical provided \(\chi(G-v) = \chi(G)-2\) for every adjacent pair of vertices u, v. This definition arises out of its relation to the Erdos-Lovasz Tihany Conjecture. A special case of this conjecture is that the only vertex double-critical graph is the complete graph; it is often referred to as the Erdos-Lovasz double-critical conjecture. Relations to FTTMs and the inertia tensor of a tetrahedron as defined in (Ahmad et al., 2010; Tonon, 2005), respectively are also being investigated.

Edge double-critical graphs: It is now shown that K_n is the only edge double-critical graph. First, some notational conventions and a required definition are given. Let \(M_2 = \{e_1, e_2, \ldots, e_t\}\) be a set of t edges in E(G) and set \(e_i = u_iv_i\) for i = 1, 2, ..., t. Next, define \(M' = \{u_1, v_1\} \cup \{u_2, v_2\} \cup \ldots \cup \{u_t, v_t\}\). Clearly, \(M_t\) is a t-matching when |\(M_t\)| = 2t.

Definition 1: Let G be a graph which contains 2-matchings. Then G is called edge double-critical whenever \(\chi(G-M_2) = \chi(G)-2\) for every 2-matching M_2.

Necessarily, an edge double-critical graph is connected. An important observation is given in the following lemma.

Lemma 1: Let \(M_2 = \{e_1, e_2\}\) be a 2-matching such that \(\chi(G-M_2) = \chi(G)-2\). Then G[M'\(M_2\)] \(\cong K_4\).

Proof: Set k = \(\chi(G)\) and let \(e_i = u_iv_i\) for i = 1, 2. Consider any (k-2)-coloring of G-M_2, the colors being from among \(\{c_1, c_2, \ldots, c_{k-2}\}\). Then u_1 and v_1 must be colored the same since otherwise there would exist a (k-2)-coloring of G-e_2. A similar argument shows that u_2 and v_2 must be colored the same, necessarily using a different color from that used for u_1 and v_1.
Theorem 2: The main result of this note.

Lemma 1 and Corollary 1 together set the stage for because G is connected, it would follow that u, v ∈ E(G) resulting in a coloring of G using fewer than k colors. A similar argument shows that u ∈ V(G), v ∈ V(G) ∈ E(G-M2). Consequently, G[M2] ≅ K4.

Theorem 1: Let t ≥ 1. If χ(G-Mt) = χ(G)-t, then Mt is a t-matching of G. Moreover, G[Mt] ≅ K2t.

Proof: Let k = χ(G). The result is trivial for t = 1. Let t ≥ 2 and consider a subset Mt of E(G) such that χ(G-Mt) = k-t. Because [Mt] = t, it follows that Mt is a t-matching as incident edges can decrease the chromatic number of a graph by at most one upon their removal. Observe now that for all pairs i, j with i ≠ j, χ(G-ei-ej) = k-2. By setting Mt = {ei, ej} and applying Lemma 1, G[Mt] ≅ K4. Hence, G[Mt] ≅ K2t.

Proposition 1: Every t-matching in K2t is critical.

Proof: The proof is by induction on t. For t = 1, the result is trivial. Since χ(K2-M1) = χ(C4) = 2 for every 2-matching M of K4, Proposition 1 holds for t = 2. Now, inductively assume that Proposition 1 holds for t = 1, 2, ..., t'-1. Let Mt be any t'-matching in K2t. Notice that K2t can be written as K2t = K2 + K2(t'-1). Moreover, it can be assumed, without loss of generality, that the single edge in the K2 term is in the t'-matching Mt. Consequently, Mt can be written as Mt = M1 ⋃ M(t-1), where M1 is the t-matching in the K2 term and M(t-1) is a (t'-1)-matching in the K2(t'-1) term. By the inductive hypothesis, χ(K2(t'-1) - M(t-1)) = t'-1. Therefore:

K2t - Mt = (K2 + K2(t'-1)) - (M1 ⋃ M(t-1)) = (K2 - M2) + (K2(t'-1) - M(t-1)) = E2 + (K2(t'-1) - M(t-1))

Hence, χ(K2t-Mt) = 1 + (t'-1) = t'.

Corollary 1: Every matching in Kn, n ≥ 2, is critical.

Lemma 1 and Corollary 1 together set the stage for the main result of this note.

Theorem 2: G is edge double-critical if and only if G ≅ K3, provided n ≥ 4.

Proof: If G ≅ K3, where n ≥ 4, then by Corollary 1, every 2-matching in K3 is critical. Thus, G is edge double-critical. Conversely, let G by a connected, edge double-critical graph. Take any u, v ∈ V(G) and suppose to the contrary that uv /∈ E(G). Then N(u) = N(v) = {wu, v} for some vertex wu, v ∈ V(G). Otherwise, because G is connected, it would follow that u, v ∈ M2v for some 2-matching M2. Since G is edge double-critical, G[M2] ≅ K4 by Lemma 1. This implies that uv /∈ E(G), contrary to our supposition. Next, observe that N(z) = {wu, v} for every vertex z /∈ N(u) = {wu, v}. Else, by using exactly the same argument as above, we would be forced to conclude that z ∈ N(u) = {wu, v}, which is clearly not possible by the choice of z. The above argument leads to the conclusion that G is a star. But such a graph is known not to be edge double-critical because of the absence of 2-matchings in any star. Hence, uv /∈ E(G) so that G ≅ K3.

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