An Algebra of Resource Sharing Machines
Unifying Two Flavors of Open Dynamical Systems
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Abstract Dynamical systems are a broad class of mathematical tools used to describe the evolution of physical and computational processes. Traditionally these processes model changing entities in a static world. Picture a ball rolling on an empty table. In contrast, open dynamical systems model changing entities in a changing world. Picture a ball in an ongoing game of billiards. In the literature, there is ambiguity about the interpretation of the “open” in open dynamical systems. In other words, there is ambiguity in the mechanism by which open dynamical systems interact. To some, open dynamical systems are input-output machines which interact by feeding the input of one system with the output of another. To others, open dynamical systems are input-output agnostic and interact through a shared pool of resources.

In this paper, we define an algebra of open dynamical systems which unifies these two perspectives. We consider in detail two concrete instances of dynamical systems — continuous flows on manifolds and non-deterministic automata.

1 Introduction

Classical information and communication theory assumes a one way communication channel between an established sender and receiver. In contrast, physical systems do not have a mechanism for such directed interaction — a property caricatured by the slogan “every action has an equal and opposite reaction.” These observations lead to the following mystery: how does a physical system (such as a transistor or cell) reliably represent an ideal computer (such as a NOT gate or gene regulatory network)? As a first step towards solving this puzzle, we give a general framework for system interaction which captures both undirected and uni-directional communication. These distinct types of interactions are captured by mathematical notions we respectively call resource sharers [Baez et al., 2016; Baez and Pollard, 2017] and machines [Vagner et al., 2014; Schultz et al., 2016; Spivak, 2020; Myers, 2020].

Dynamical systems refer to a broad class mathematical objects which model "things that change." A Turing machine (and more generally, a computer) is a dynamical system; the state of the tape changes according to an algorithm. An electromagnetic field is also a dynamical system; the state of the field changes according to the laws of Gauss, Faraday, and Maxwell. Traditionally, mathematicians and sci-
entists fix a dynamical system and then ask questions about it. What are its equilibrium points? Its orbits? Its entropy? However, in nature dynamical systems do not exist in isolation. For example, the state of a computer is influenced by the network of servers it is connected to and the actions of the user on the keyboard and mouse. Hence, we are interested in open dynamical systems, i.e. those which have a mechanism for interacting with other systems. When dynamical systems interact we say they compose. Resource sharers and machines are two flavors of open dynamical systems with distinct styles of composition.

Inspired by the physical interactions in chemical reaction networks, the authors of [Baez and Pollard, 2017] define a framework for open dynamical systems which compose as resource sharers. Two resource sharers compose by simultaneously affecting and reacting to a shared pool of resources. When resource sharers compose:

1. communication is undirected. Each system may both affect and be affected by the state of the shared pool of resources. Through this medium, they may both affect and be affected by each other.
2. interaction is passive. The communication channel is incidental to the fact that the systems refer to the same resource. The rules for how each system affects and reacts to state of the resource is independent of the action of other systems on the pool.

When people communicate verbally, they are composing as resource sharers where the shared resource is "vibrations in the air space." All participants in a conversation affect and are affected by the changing state of the air between them.

Inspired by the dynamics of computation, the authors of [Vagner et al., 2014] define a framework of open dynamical systems which compose as machines. When two machines compose, one machine is the designated sender and the other is the designated receiver. The sender emits information which directs the evolution of the receiver. In the special case where a system is both the sender and the receiver, this interaction describes feedback. When machines compose:

1. communication is uni-directional. Information travels from the sender to the receiver but not vice versa.
2. interaction is active. The communication channel from sender to receiver is specifically engineered to enable the passing of information. The receiver does not evolve without input from the sender.

When people communicate by passing notes, they are composing as machines where the note plays the role of the engineered communication channel.

The main theorem of this paper (Theorem 5.3) unites these two flavors of composition in a single framework for open dynamical
systems. In Section 2, we define two concrete instances of dynamical systems — continuous dynamical systems and non-deterministic automata — and give examples of each composing as machines and as resource sharers. In Section 3, we exemplify how operad and operad algebras respectively give a syntax and semantics for composition. This formalism will be the main tool we use to define compositions of open dynamical systems. In Section 4, we discuss the established frameworks for composition as resource sharers and as machines in more depth. Finally, in Section 5, we prove our main theorem.

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2 Motivating examples

The goal of this section is to present two examples of dynamical systems where both composition as machines and composition as resource sharers are fruitful methods of gluing dynamical systems together.

2.1 Continuous Dynamical Systems

A continuous dynamical system is defined by a state space $X$ and a vector field $v : X \to TX$. In general, the state space $X$ may be any manifold, but in all of our examples $X$ will be a Euclidean space. The vector field $v$ assigns to each $p \in X$ an arrow $v(p)$ based at $p$.

The data of $(X,v)$ models “things that change” as follows: if the system is at a state $p \in X$, then it will evolve in the direction of $v(p)$. This intuition is particularly poignant if the state space $X$ represents the position of a ball. Then if the ball is at position $p \in X$, it will roll in the direction of the arrow $v(p)$.

For example, the growth of a population of rabbits can be modeled by the vector field $u : \mathbb{R} \to T\mathbb{R}$ defined by

$$u(r) = \beta r \in T_r\mathbb{R},$$

where the state space $\mathbb{R}$ represents a population of rabbits. If there are $r$ rabbits, then the rabbit population grows at a rate of $\beta r$. This vector field may also be denoted by $\dot{r} = \beta r$.

Since we are interested in open continuous dynamical systems, we will generalize to parameterized vector fields $v : X \times I \to TX$. In this more general case, if a ball is in position $p \in X$ and it receives input $i \in I$, then it will roll in the direction of $v(p,i)$.

The growth of a fox population is parameterized by a population of prey that the foxes eat. This system is modeled by the parameterized

$$v : X \times I \to TX$$

2 Continuous dynamical systems often refers to a more general class of systems which are defined by a state space $X$ and a continuous group action $\mathbb{R} \curvearrowright X$ called a flow. However, in this paper we use the term “continuous dynamical systems” to refer to the subclass of systems with flows induced by a vector field.

3 More formally, a vector field $v : X \to TX$ is a section of the tangent bundle $\pi : TX \to X$. So for $p \in X$, $v(p)$ is a vector in the tangent space $T_pX$.

4 For each $p \in X$ there is a unique trajectory $\gamma : \mathbb{R} \to X$ with velocity $\gamma'(t) = v(\gamma(t))$ and initial condition $\gamma(0) = p$. If a ball is dropped at position $p$, then after $t$ time it will be at the position $\gamma(t)$.

5 Given a starting rabbit population $r \geq 0$, the rabbit population will grow according to the unique trajectory $\gamma : \mathbb{R} \to \mathbb{R}$ with velocity $\gamma'(t) = u(\gamma(t))$ and initial condition $\gamma(0) = r$. So after $t$ time, the size of the rabbit population will be $\gamma(t)$.

6 Formally, a parameterized vector field $v : X \times I \to TX$ is a continuous map such that $v(p,i)$ is projection on to the first coordinate, where $\pi : TX \to X$ is the natural projection map.
vector field \( v : \mathbb{R} \times \mathbb{R} \to T\mathbb{R} \) defined by

\[ v(f, e) = \alpha e f \in T_f \mathbb{R} \]

where \( f \) represents the fox population and \( e \) represents the population of prey to be eaten. This continuous dynamical system is equivalently denoted \( \dot{f} = \alpha e f \). In this case, the fox population grows at a rate \( \alpha \) according to the law of mass action.

Now, we will introduce two methods of composing continuous dynamical systems — as machines and as resource sharers.

**First, how do continuous dynamical systems compose as machines?** Recall, the two dynamical systems we have introduced:

\[ \dot{r} = \beta r \]

modeling how a rabbit population grows and

\[ \dot{f} = \alpha e f \]

modeling how a fox population grows parameterized by a population \( e \) of prey for the foxes to eat. Since foxes eat rabbits, we can send the rabbit population \( r \) to the model for fox growth as the parameter \( e \). Doing so composes the two systems. The resulting total system is then given by the vector field \( \mathbb{R}^2 \to T\mathbb{R}^2 \) defined by

\[ \dot{r} = \beta r, \quad \dot{f} = \alpha r f. \]

Figure 1 depicts the composition of these systems as machines.

As a second example (which will be used shortly), we can likewise compose as machines (1) the continuous dynamical system modeling how the fox population declines at a rate \( \delta \) — given by \( \dot{f} = -\delta f \) and (2) the continuous dynamical system modeling how the rabbit population declines as a rate \( \gamma \) parameterized by a population \( h \) of predators that hunt rabbits — given by \( \dot{r} = -\gamma hr \). The resulting total system

\[ \dot{f} = -\delta f, \quad \dot{r} = -\gamma fr \]

describes how both populations decline in synchrony.

Figure 1: This is an example of composing continuous dynamical systems as machines. Examining the left-hand side of the equation, the box on the left (the sender) is filled with a dynamical system modeling how a rabbit population grows. The box on the right (the receiver) is filled with a dynamical system modeling how a fox population grows parameterized by an input population \( e \). The directed wire indicates sending the rabbit population as input to the receiver. The total system (depicted on the right-hand side of the equation) models how the fox and rabbit populations grow in synchrony.
Second, how do continuous dynamical systems compose as resource sharers? Consider the two dynamical systems we have constructed:

\[
\begin{align*}
\dot{r} &= \beta r, \\
\dot{f} &= \alpha r f
\end{align*}
\]

modeling how the rabbit and fox populations grow and

\[
\begin{align*}
\dot{f} &= -\delta f, \\
\dot{r} &= -\gamma f r
\end{align*}
\]

modeling how they decline.

However there are not two separate rabbit populations, one that grows and one that declines. Rather both systems are referring to a shared pool of resources, in this case a population of rabbits. Likewise both systems are referring to a shared population of foxes.

To compose these systems along the shared pools of rabbits and foxes, we add the effects of both systems on the shared resource. The resulting dynamical system is

\[
\begin{align*}
\dot{r} &= \beta r - \gamma f r, \\
\dot{f} &= \alpha r f - \delta f
\end{align*}
\]

known as the Lokta-Volterra predatory-prey model. Figure 2 depicts the composition of these systems as resource sharers.

Figure 2: This is an example of composing continuous dynamical systems as resource sharers. Examining the left-hand side of the equation, the top box is filled with a dynamical system modeling how the rabbit and fox populations grow. The bottom box is filled with a dynamical system modeling how the rabbit and fox populations decline. The undirected wires connecting these boxes indicate that these two systems are composed by identifying their rabbit and fox populations respectively. The resulting dynamical system drawn on the right, is the Lokta-Volterra predator-prey model.

2.2 Non-deterministic Automata

A non-deterministic automaton is a discrete dynamical system with a set of states \( S \) and an update map \( u : S \to \mathcal{P}(S) \).\footnote{\( \mathcal{P} \) denotes that power set monad, so \( \mathcal{P}(S) \) is the set of subsets of \( S \).} For each state \( s \in S \), the set of next possible states is \( u(s) \subseteq S \).

For example, consider an automaton representing a 2-cycle, depicted in Figure 3(a). This automaton has two states and oscillates between them. Formally, \( S = \mathbb{Z}/2\mathbb{Z} \) and

\[
\begin{align*}
u(s) &= \{ s + 1 \mod 2 \}.
\end{align*}
\]

We want to generalize to open automata, in other words automata that are parameterized by some input. These consist of a set of states
$S$, a set of inputs $I$, and an update map $u : S \times I \rightarrow P(S)$. For each state $s \in S$ and input $i \in I$, the set of next possible states is $u(s, i) \subseteq S$.

For example, consider an automaton that takes as input 0s and 1s and adds the number of 1s received modulo 2, depicted in Figure 3(b). Then $S = \mathbb{Z}/2\mathbb{Z}$, $I = \mathbb{Z}/2\mathbb{Z}$, and the update map is

$$v(s, i) = \{s + i \mod 2\}.$$ 

Figure 3: (a) A 2-cycle automaton that oscillates between two states. 
(b) A mod 2 adder automaton that takes as input 0s and 1s and adds the number of 1s received modulo 2.

Now, we will introduce two methods of composing non-deterministic automata — as machines and as resource sharers.

**First, how do automata compose as machines?** By reading off the states, the 2-cycle produces a string of 0s and 1s:

$$...01010101...$$

We can compose the automata from Figure 3 as machines by sending the string generated by the 2-cycle as input to the mod 2 adder. The output of the 2-cycle drives the dynamics of the mod 2 adder to define a total system.⁸

The states of the total system are pairs of states of the individual systems so $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Suppose we are in a state $(r, s)$ of the total system where $r$ belongs to the receiver and $s$ belongs to the sender. We update the total state by the following algorithm:

1. send $s$ (the current state of the sender) to the receiver
2. use the input to update the state of the receiver
3. update the state of the sender

Figure 4: Composing the automata from Figure 3 as machines (left-hand side of the equation) yields a 4-cycle (right-hand side of the equation).

⁸ In Krohn-Rhodes theory, this style of composition is known as the cascade product of automata. See [Krohn and Rhodes, 1965].
Formally, the update map is given by

\[(r, s) \mapsto v(r, s) \times u(s) = \{(r + s \mod 2, s + 1 \mod 2)\}\]

and the total system is a 4-cycle. Figure 4 depicts the composition of these automata as machines.

**SECOND, HOW DO AUTOMATA COMPOSE AS RESOURCE SHARERS?** In the case of automata, “resource sharing” represents “observation sharing” because it aligns automata along a shared observation.

Consider a 4-cycle which emits the parity of its states when observed. We can align two such 4-cycles along the observation of state parity. The states of the total system are pairs of states of the individual 4-cycles which agree on parity. A transition is a pair of transitions of the individual 4-cycles where the domains agree on parity as do the codomains. In this example, the total system consists of two cycles representing the two ways for the two 4-cycles to align along parity: either the states can agree exactly or they can be phase shifted by 2. Figure 5 depicts the composition of these automata as resource sharers.

We restrict our attention to non-deterministic automata because in some instances it is possible for a state in the total system to have no outgoing transitions.

For example, suppose we align along parity a 4-cycle and a 3-cycle. See Figure 6. Consider the state \((2, 2)\) of the total system. The 4-cycle will update to its state 3 while the 3-cycle will update to its state 0. Since these states differ in parity, there is no transition in the total system out of the state \((2, 2)\).

**THE GOAL OF THIS PAPER IS TO GIVE AN OPERAD AND OPERAD ALGEBRA WHICH DESCRIBE COMPOSING DYNAMICAL SYSTEMS AS MACHINES AND AS RESOURCE SHARERS SIMULTANEOUSLY.** We will see the examples of continuous dynamical systems and non-deterministic automata as special cases of this formalism.
3 Operad and Operad Algebras

In Section 2, we graphically represented the composition of continuous dynamical systems and non-deterministic automata. Let’s highlight the patterns in these pictures: (1) dynamical systems filled boxes and (2) composition corresponded to wiring boxes together. This section casually introduces the math behind those pictures, namely operads and operad algebras.

An operad $O$ is much like a category. It has

- a set of types $\text{ob} O$, analogous to objects in a category. Figure 7(a) shows how we might visualize types as boxes.
- for types $s_1, \ldots, s_n, t$, a set of morphisms $O(s_1, \ldots, s_n; t)$, analogous to morphisms in a category. Unlike in a category, an operad morphism may have multiple (but finitely many) types as the domain. Figure 7(b) shows how we might visualize morphisms in an operad as wirings between boxes.

A symmetric monoidal category $(C, \otimes, 1)$ induces an operad $O(C)$ with types $\text{ob} C$ and morphisms

$$O(C)(s_1, \ldots, s_n; t) = C(s_1 \otimes \ldots \otimes s_n, t).$$

All of the operads we will consider are induced by symmetric monoidal categories.

Example 1. Consider the symmetric monoidal category $(\text{FinSet}^{\text{op}}, +, 0)$ and the induced operad $O(\text{FinSet}^{\text{op}})$. A type $M$ in $O(\text{FinSet}^{\text{op}})$ is a finite set. Graphically, a type $M$ is represented by a box with $M$ exposed ports. See Figure 7(a).

A morphism $f : (M_1, \ldots, M_n) \to N$ in $O(\text{FinSet}^{\text{op}})$ is a finite set map $f : N \to M_1 + \ldots + M_n$. Graphically, a morphism $f$ is represented by wiring each port $n \in N$ of the outer box to the port $f(n) \in M_i$ of an inner box. See Figure 7(b).

Operads give a syntax for composition, where a morphism in $O(s_1, \ldots, s_n; t)$ defines a way of composing types $s_1, \ldots, s_n$ such that the

In fact, this section is so casual that we only recite the definition of operad for completeness!

Definition 3.1. An operad $O$ consists of a set of types $\text{ob} O$ and for types $s_1, \ldots, s_n, t \in \text{ob} O$ a set of morphisms $O(s_1, \ldots, s_n; t)$ along with

- for each type $t$, an identity morphism $\text{id}_t \in O(t; t)$
- a substitution map $\circ : O(s_1, \ldots, s_n; t_1) \times O(t_1, \ldots, t_{i-1}, s_1, \ldots, s_n, t_{i+1}, \ldots, t_m; u) \to O(t_1, \ldots, t_{i-1}, s_1, \ldots, s_n, t_{i+1}, \ldots, t_m; u)$
- a symmetry map for each permutation of the domain types satisfying an identity and associativity law.

We refer the reader to Fong and Spivak [2019] Chapter 6 for a helpful exposition of operads and to Leinster [2004] for a complete definition. Note that this definition of an operad historically went by the name colored operad.

Morphisms in an operads are sometimes referred to as operations.
result is of type $t$. Operads allow us to define a many different compositions between the same types. In analogy to the game MadLibs, types are like the parts of speech (noun, verb, adjective) and morphisms are ways of combining the parts of speech. There are many ways to compose two nouns and a verb into a sentence. For example, the MadLibs

\[
\begin{align*}
\text{The } & \underline{\text{noun}} \text{ in the } \underline{\text{noun}} \underline{\text{ed}}. \\
\text{I } & \underline{\text{verb}} \text{ed with joy when I got a } \underline{\text{noun}} \text{ and a } \underline{\text{noun}}. 
\end{align*}
\]

are two of many morphisms $(\text{noun}, \text{noun}, \text{verb}) \to \text{sentence}$ in the MadLibs operad.

To give meaning to a Madlibs, we must fill in each blank with a word of the appropriate type. The analogy continues. To give meaning to an operad, we must fill in each box with an element of the appropriate type. These semantics are given by the following structure.

**Definition 3.2.** Let $O$ be an operad. An $O$-algebra is an operad functor\(^\text{11}\)

\[
F : O \to O(\text{Set}, \times, 1). \text{\(12\)}
\]

An $O$-algebra $F$ gives meaning to the syntax defined by $O$ as follows:

- on types — for a type $t$ of $O$, $F(t)$ is a set and $x \in F(t)$ is an element of type $t$.
- on morphisms — for a morphism $f \in O(s_1, \ldots, s_n; t)$ the set map $Ff : F(s_1) \times \ldots \times F(s_n) \to F(t)$ determines how composing elements of types $s_1, \ldots, s_n$ according to $f$ results in an element of type $t$.

A lax monoidal functor $F : (C, \otimes, 1) \to (\text{Set}, \times, 1)$ induces an $O(C)$-algebra, which we occasionally refer to as a $C$-algebra.

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\(11\) An operad functor $F : O \to O'$ consists of
- a map of types $F : \text{ob}\, O \to \text{ob}\, O'$
- a map of morphisms $F : O(s_1, \ldots, s_n; t) \to O'(F s_1, \ldots, F s_n; F t)$ respecting identity and composition.

Again we refer readers to [Fong and Spivak, 2019 Chapter 6 for a helpful description and to Leinster, 2004] for a complete definition.

\(12\) Recall that $O(\text{Set}, \times, 1)$ is the operad induced by the symmetric monoidal category $(\text{Set}, \times, 1)$. In particular, its types are sets.

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Figure 7: (a) Graphically, we represent a type in an operad by a box with an interface. This box represents 6 in the operad $O(\text{FinSet}^{\text{op}})$.

(b) Graphically, we represent a morphism in $O(s_1, \ldots, s_n; t)$ as a wiring from the boxes corresponding to types $s_1, \ldots, s_n$ to a box with type $t$. The inner boxes correspond to the domain types. The outer box corresponds to the codomain type. This wiring is a morphism $(2, 3) \to 6$ in $O(\text{FinSet}^{\text{op}})$. 
Suppose an operad $O$ can be graphically represented by boxes and wirings. Then we represent an $O$-algebra $F$ as follows. Elements of $F(t)$ fill boxes of type $t$. Let $f \in O(s_1, ..., s_n; t)$ be a wiring. Then the set map $F f$ defines how filling the inner boxes with elements of the appropriate type and composing along the wiring defined by $f$ induces a filling for the outer box.

**Example 2.** Given an alphabet $\Sigma$, there exists a lax monoidal functor $\text{Label}_\Sigma : (\text{FinSet}^{op}, +, 0) \to (\text{Set}, \times, 1)$ defined

- on objects — $M$ maps to $\Sigma^M$, the collection of labelings $\sigma : M \to \Sigma$.
- on morphisms — $f : M \to N$ maps to $f^* : \Sigma^N \to \Sigma^M$ defined by $f^*(\sigma)(m) = \sigma(f(m))$.

Under the induced $O(\text{FinSet}^{op})$-algebra, a box of type $M$ is filled with an element of $\Sigma^M$, i.e. a label in $\Sigma$ for each port $m \in M$. See Figure 8(a).

A finite set map $f : N \to M_1 + ... + M_n$ defines a wiring from inner boxes of types $M_1, ..., M_n$ to an outer box of type $N$. The set map

$$\text{Label}_\Sigma(f) : \text{Label}_\Sigma(M_1 + ... + M_n) \to \text{Label}_\Sigma(N)$$

defines how filling each inner box with a choice in $\text{Label}_\Sigma(M_i)$ and wiring along $f$ results in a labeling of type $N$. See Figure 8(b).

**Example 3 (Syntax for resource sharers).** Consider the symmetric monoidal category $(\text{Cospan}_{\text{FinSet}}, +, 0)$. The induced operad $O(\text{Cospan}_{\text{FinSet}})$ has

- types — finite sets $M$
- morphisms — cospans $M_1 + ... + M_n \to Q \leftarrow N$

In the remainder of this section we will discuss two operads, which define syntaxes for composing dynamical systems as resource sharers and as machines.

**Figure 8:** (a) Suppose $\Sigma = \{a, b\}$. Graphically, a box of type $M$ is filled with a choice of labeling in $\Sigma^M$. Here we have a box of type 3 that is filled with the labeling $(b, a, a) \in \Sigma^3$.

(b) The wiring depicted is an operad morphism $f : (2, 3) \to 6$. Filling the inner boxes with a labeling induces a labeling for the outer box. The label for each port of the outer box is determined by following the wire to a labeled inner port.
As in \(O(\text{FinSet}^{op})\), a type \(M\) is graphically represented by a box with \(M\) exposed ports. To graphically represent a morphism

\[
M_1 + \ldots + M_n \xrightarrow{i} Q \xleftarrow{j} N
\]

we draw an intermediate box with \(Q\) exposed ports. Then we wire the ports from the inner boxes (respectively, outer box) to the intermediate box according to \(i\) (respectively, \(j\)). Often, we do not draw the intermediate box and simply draw the wiring of ports which may include (1) combining many ports irrespective of their origin and (2) terminating ports. See Figure 9.

Since \(\text{Cospan}\)-algebras are 1-equivalent to hypergraph categories \([\text{Fong and Spivak}, 2018]\) and hypergraph categories are input-output agnostic, \(\text{Cospan}_{\text{FinSet}}\) is a sensible syntactic setting for resource sharing.

The operadic setting for machines requires a bit of setup, which is described in more detail in \([\text{Schultz et al.}, 2016]\).

**Definition 3.3.** Let \(C\) be a category. The category of \(C\)-typed finite sets, \(\text{TFS}_C\), has

- objects — pairs \((M \in \text{FinSet}, \tau : M \to \text{ob}\ C)\)
- morphisms — a morphism \(f : (M, \tau) \to (M', \tau')\) is a map \(f : M \to M'\) such that \(f \triangleright \tau' = \tau\).

The category \(\text{TFS}_C\) has a symmetric monoidal product given on objects by

\[
(M, \tau) + (M', \tau') = (M + M', [\tau, \tau'] : M + M' \to \text{ob}\ C)
\]

with monoidal unit \((0, ! : 0 \to \text{ob}\ C)\).

**Definition 3.4.** Let \(C\) be a category. There exists a symmetric monoidal category of \(C\)-wiring diagrams, \(\text{WD}_C\), with

- objects — pairs of \(C\)-typed finite sets \((X^\text{in}, X^\text{out})\)
• morphisms\textsuperscript{14} — pairs $(\phi_{\text{out}} : (X_{\text{out}}^\text{in}) \to (Y_{\text{out}}^\text{in}))$ where 

$$\phi_{\text{out}} : Y_{\text{out}} \to X_{\text{out}}, \quad \phi_{\text{in}} : X_{\text{in}} \to X_{\text{out}} + Y_{\text{in}}$$

are morphisms of $C$-typed finite sets.

The symmetric monoidal structure on $WD_C$ is induced by the symmetric monoidal structure of $TFS_C$.

Example 4 (Syntax for machines). The operad induced by $WD_C$ is an appropriate syntax for composing dynamical systems as machines.

For $C$-typed finite sets

$$X_{\text{in}} = (M_{\text{in}}, \tau_{\text{in}}), \quad X_{\text{out}} = (M_{\text{out}}, \tau_{\text{out}})$$

we interpret the type $(X_{\text{in}}(\tau_{\text{in}}))$ of $O(WD_C)$ as having $M_{\text{in}}$ input wires where the wire $m \in M_{\text{in}}$ carries information of type $\tau_{\text{in}}(m)$. Likewise for the output wires. See Figure 10(a). We interpret morphisms

$$\left(\left(\phi_{\text{out}} : (X_{\text{out}}^\text{in})\right) \cdots (X_{\text{out}}^{n}) \right) \to (Y_{\text{out}}^\text{in})$$

as wiring diagrams like the one shown in Figure 10(b).

4 Previous Work

In this section we discuss two bodies of work, one which introduces resource sharers as a $\text{Cospan}_{\text{FinSet}}$-algebra and one which introduces machines as a $WD_C$-algebra. The motivation for the main theorem of this paper is uniting these two perspectives.

In [Baez and Pollard, 2017], the authors define a hypergraph category $\text{Dynam}$ which describes the composition of continuous dynamical systems as resource sharers. Recall that hypergraph categories are 1-equivalent to $\text{Cospan}$-algebras. Taking the operadic perspective, $\text{Dynam}$ is an $O(\text{Cospan}_{\text{FinSet}})$-algebra. We consider each type $M$ to be a finite set of exposed ports. The set $\text{Dynam}(M)$ consists of triples

$$(S \in \text{FinSet}, v : \mathbb{R}^S \to \mathbb{R}^S, p : S \to M)$$

\textsuperscript{14}The morphisms in this category are often called prisms.

\textsuperscript{15}Recall that $M_{\text{in}}$ is a finite set and $\tau_{\text{in}} : M_{\text{in}} \to \text{ob} C$ assigns to each port $m$ of $M_{\text{in}}$ an object of $C$. Likewise for $M_{\text{out}}$ and $\tau_{\text{out}}$.

Figure 10: (a) A type in $WD_C$ is visualized as a box with a finite number of input and output ports labeled by objects of $C$. This box represents a type in $WD_{\text{Euc}}$ where $\text{Euc}$ is the category of Euclidean spaces.

(b) A morphism in $WD_C$ is visualized as two sets of wires. The purple wires, representing $\phi_{\text{in}}$, feed the inputs to the inner boxes with either (1) outputs of the inner boxes or (2) inputs of the outer box. The orange wires, representing $\phi_{\text{out}}$, feed the outputs of the outer box with outputs of the inner boxes. In this figure, the labeling of the ports has been suppressed however ports connected by a wire must be labeled with the same object of $C$. 

\[R\]

\[R^2, R^{27}\]

(a) (b)
where $v$ is an algebraic vector field\(^{16}\) and $p$ is a map of finite sets. Therefore, under $\text{Dynam}$, boxes with $M$ exposed ports are filled with triples $(S, v, p) \in \text{Dynam}(M)$ where

- $S$ is a finite set of state variables
- $v$ gives the dynamics on the state space $\mathbb{R}^S$
- $p$ assigns a state variable to each exposed port

The details of how $\text{Dynam}$ acts on morphisms in $\text{Cospan}_{\text{FinSet}}$ are subsumed by the discussion following Theorem 5.3. Some intuition is given through the example shown in Figure 2.

In [Schultz et al. 2016], the authors define a $\text{WD}_{\text{Euc}}$-algebra\(^{17}\) $\text{CDS}$\(^{18}\) as follows.

Let $\mathbb{R}^I$ and $\mathbb{R}^O$ be Euclidean spaces. An $(\mathbb{R}^I, \mathbb{R}^O)$ continuous dynamical system is a triple $(\mathbb{R}^S \in \text{Euc}, v : \mathbb{R}^S \times \mathbb{R}^I \to T \mathbb{R}^S, r : \mathbb{R}^S \to \mathbb{R}^O)$ where $v$ is a parameterized vector field.\(^{19}\)

There exists an algebra $\text{CDS} : \text{WD}_{\text{Euc}} \to \text{Set}$ which on objects maps $(X^\text{in}, X^\text{out})$ to the set of $(X^\text{in}, X^\text{out})$ continuous dynamical systems where for a $\text{Euc}$-typed finite set $X = (M, \tau : M \to \text{Euc})$,

$$\widehat{X} = \prod_{m \in M} \tau(m) \in \text{Euc}.$$\(^{20}\)

Therefore, under $\text{CDS}$, boxes of type $(\mathbb{R}^I, \mathbb{R}^O)$ are filled with triples $(\mathbb{R}^S, v, r)$ which we interpret as having

- a state space — the Euclidean space $\mathbb{R}^S$
- dynamics — the vector field $v$ of $\mathbb{R}^S$ parameterized by the input space $\mathbb{R}^I$
- read-out — the map $r$ taking states of $\mathbb{R}^S$ to points in the output space $\mathbb{R}^O$

The details of how $\text{CDS}$ acts on morphisms are subsumed by the discussion following Theorem 5.3. Some intuition is given through the example shown in Figure 2.

Notice the strong similarities between the sets $\text{Dynam}(M)$ and $\text{CDS}((\mathbb{R}^I, \mathbb{R}^O))$. Both contain triples which determine (1) a state space, (2) dynamics, (3) an observation of the state space. However, the two algebras define remarkably different compositions of dynamical systems. In the next section, we give a single algebra capturing both types of composition.

\(^{16}\) A vector field $v$ is algebraic if its components are polynomials.

\(^{17}\) The category $\text{Euc}$ is the full subcategory of $\text{Mfld}$ generated by Euclidean spaces.

\(^{18}\) $\text{CDS}$ stands for “continuous dynamical system.”

\(^{19}\) Recall that for a manifolds $X$ and $Y$, $v : X \times Y \to TX$ is a parameterized vector field on $X$ if the diagram below commutes. The map $TX \to X$ is the natural projection map.

\(^{20}\) Recall that $(X^\text{in}, X^\text{out}) \in \text{ob WD}_{\text{Euc}}$ is a pair of $\text{Euc}$-typed finite sets. We visualize $(X^\text{in}, X^\text{out})$ as a box with a finite number of input and output ports each labeled with a Euclidean space.

\(^{21}\) In general if $C$ has finite products then there exists a functor $(-) : \text{TFS}(C) \to C$ defined by product.
5 Resource Sharing Machines

To define resource sharing machines we will need two mathematical tools, which together define the data of a contravariant dynamical system doctrine. The first tool is a category of lenses whose morphisms capture the data of an open dynamical system with a machine-style interface. The key observation:

The data of an open dynamical system is given by a lens.

is made in [Spivak, 2019] and further explored in [Myers, 2020].

Definition 5.1. For an indexed category $\mathcal{A} : \mathcal{C}^\text{op} \to \text{Cat}$ define

$$\text{Lens}_{\mathcal{A}} := \int_{\mathcal{C}} \mathcal{A}(C)^\text{op},$$

the pointwise opposite of the Grothendieck construction.

Let’s unpack this definition. The category $\text{Lens}_{\mathcal{A}}$ has

- objects — pairs $(I, O)$ for $O \in \text{ob } \mathcal{C}$ and $I \in \text{ob } \mathcal{A}(O)$
- morphisms — lenses

$$\left( f^\# \right) : \left( \begin{array}{c} I \\ O \end{array} \right) \leftrightarrow \left( \begin{array}{c} I' \\ O' \end{array} \right)$$

with $f : O \to O'$ in $\mathcal{C}$ and $f^\# : f^* I' \to I$ in $\mathcal{A}(O)$

The key observation is that for well-chosen indexed categories $\mathcal{A}$, certain lenses

$$\left( \begin{array}{c} u \\ r \end{array} \right) : \left( \begin{array}{c} TS \\ S \end{array} \right) \leftrightarrow \left( \begin{array}{c} I \\ O \end{array} \right)$$

capture the data of an open dynamical systems with

- $S$ — the internal state space
- $O$ — a space of outputs or orientations
- $I$ — a space of contextualized inputs over $O$
- $TS$ — a space of canonical changes over $S$
- $r : S \to O$ — a read-out map
- $u : r^* I \to TS$ — an update map

To see this observation in action, let’s consider two examples. For each example, watch for (1) a choice of indexed category $\mathcal{A}$ and (2) a description of how specific lenses in $\text{Lens}_{\mathcal{A}}$ correspond to open dynamical systems.

Example 5 (non-deterministic automata). Let $\mathcal{P} : \text{Set} \to \text{Set}$ be the powerset monad. Consider the indexed category

$$\text{BiKleisli}(\_ \times \_, \mathcal{P}) : \text{Set}^\text{op} \to \text{Cat}.$$
Unpacking definitions, a lens

\[
\left( \begin{array}{c}
u \\ r \end{array} \right) : \left( \begin{array}{c} S \\ S \end{array} \right) \mapsto \left( \begin{array}{c} 1 \\ O \end{array} \right)
\]

in \( \text{Lens}_{\text{opKleisli}((-) \times (-))} \) consists of

- \( S \) — a set of states of the automaton
- \( O \) — a set of outputs of the automaton
- \( I \) — a set of inputs to the automaton
- \( r : S \rightarrow O \) — a set map assigning an output to each state
- \( u : S \times I \rightarrow TS \) — a set map assigning to a state \( s \) and input \( i \), a set of next possible states

**Example 6** (continuous dynamical systems). Let \( \text{Mfld}_{\text{Sub}} \) be the wide subcategory of \( \text{Mfld} \) whose morphisms are submersions.\(^{26}\)

Consider the indexed category

\[\text{Mfld}_{\text{Sub}}/(-) : \text{Mfld}^{\text{op}} \rightarrow \text{Cat}\]

which takes a manifold \( B \) to the category of submersions over \( B \).\(^{27}\)

Unpacking definitions, a lens

\[
\left( \begin{array}{c}
u \\ r \end{array} \right) : \left( \begin{array}{c} TS \rightarrow S \\ S \end{array} \right) \mapsto \left( \begin{array}{c} I \times O \rightarrow O \\ O \end{array} \right)
\]

in \( \text{Lens}_{\text{Mfld}_{\text{Sub}}/(-)} \) consists of

- \( S \) — a manifold giving the state space of the dynamical system
- \( O \) — a manifold of outputs
- \( I \) — a manifold of inputs
- \( r : S \rightarrow O \) — a continuous map assigning an output to each state of the state space
- \( u : S \times I \rightarrow TS \) — an indexed vector field assigning to a state \( s \) and input \( i \), a vector \( u(s, i) \in T_sS \) indicating a direction in which to evolve

The dynamical systems described above are the special cases where the fiber of inputs over each output \( o \in O \) is constantly \( I \). In the general case, the inputs may vary with the output.\(^{28}\)

As shown in \([\text{Moeller and Vasilakopoulou}, 2018]\), if \( \mathcal{A} : C^{\text{op}} \rightarrow \text{Cat} \) is a symmetric monoidal indexed category then \( \text{Lens}_{\mathcal{A}} \) has a symmetric monoidal structure,\(^{29}\) and thus induces an operad \( O(\text{Lens}_{\mathcal{A}}) \).

**Morphisms of \( \text{Lens}_{\mathcal{A}} \)** define dynamics with a machine-style interface that distinguishes input and output. To define the resource sharing interface for resource sharing machines, we develop a second tool — a section \( T : C \rightarrow \text{Lens}_{\mathcal{A}} \) of the forgetful functor

\[
\left( \begin{array}{c}
1 \\ O \\
\end{array} \right) \otimes \left( \begin{array}{c}
I' \\
O' \\
\end{array} \right) = \left( \begin{array}{c}
I \\
O \otimes O' \\
\end{array} \right)
\]

where \( O \otimes O' \) is the monoidal product in \( C \) and \( I \otimes I' \) is the image of \((I, I') \in \mathcal{A}(O) \times \mathcal{A}(O') \) under the laxator.

---

\(^{25}\) In this example, \( TS \) (the space of canonical changes over \( S \)) is \( S \) itself because a transition in an automaton is a choice of states in \( S \) to transition to.

\(^{26}\) We restrict our attention to submersions because in \( \text{Mfld} \) pullbacks along submersions always exist.

\(^{27}\) \( \text{Mfld}_{\text{Sub}}/B \) is the category with

- objects — submersion \( p : E \rightarrow B \)
- morphisms — commuting triangles

\[\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B & \xrightarrow{c} & B
\end{array}\]

For a continuous map \( f : B \rightarrow B' \), the functor \( \text{Mfld}_{\text{Sub}}/f \) is given by taking pullbacks.

\(^{28}\) More generally, a lens

\[
\left( \begin{array}{c}
u \\ r \end{array} \right) : \left( \begin{array}{c} TS \rightarrow S \\ S \end{array} \right) \mapsto \left( \begin{array}{c} p : I \rightarrow O' \\ O \end{array} \right)
\]

in \( \text{Lens}_{\text{Mfld}_{\text{Sub}}/(-)} \) consists of

- \( S \) — a manifold giving the state space of the dynamical system
- \( O \) — a manifold of outputs
- \( I \) — a manifold of inputs where the fiber \( p^{-1}(o) \subseteq I \) is the space of contextualized inputs for the output \( o \)
- \( r : S \rightarrow O \) — a continuous map assigning an output to each state of the state space
- \( u : S \times O \rightarrow TS \) — an indexed vector field assigning to each state \( s \) and contextualized input \( i \in p^{-1}(r(s)) \), a vector \( u(s, i) \in T_sS \) indicating a direction in which to evolve

\(^{29}\) The monoidal structure for \( \text{Lens}_{\mathcal{A}} \) is given by
\( U : \text{Lens}_A \to C \). Together an indexed category \( A : C^{\text{op}} \to \text{Cat} \)
and a section \( T : C \to \text{Lens}_A \) is the data we need to specify a class of resource sharing machines. We refer to this data as a contravariant dynamical system doctrine\(^3^1\) or simply as a contravariant doctrine.

**Definition 5.2.** A contravariant dynamical system doctrine is an indexed category \( A : C^{\text{op}} \to \text{Cat} \) with a section \( T : C \to \text{Lens}_A \).

Given a contravariant dynamical system doctrine \((A : C^{\text{op}} \to \text{Cat}, T)\), we notate \( T \)

- on objects, by \( S \mapsto \left( \frac{T S}{S} \right) \)
- on morphisms, by \( f \mapsto \left( \frac{T f^*}{f} \right) \)

Given a map of internal state spaces \( f : S \to S' \), we say that

\[ T f^\#: f^*(TS') \to TS \]

pullback transitions over \( S' \) to transitions over \( S \) along \( f \). To see this structure at play in the context of resource sharing, let’s continue with our two examples.

**Example 7** (non-deterministic automata). Recall that the relevant indexed category for non-deterministic automata is

\[ \text{Bikleisli}(\times, P) : \text{Set}^{\text{op}} \to \text{Cat} \]

Define \( T : \text{Set} \to \text{Lens}_{\text{Bikleisli}(\times, P)} \)

- on objects — \( S \) maps to \( \left( \frac{T S}{S} \right) \)
- on morphisms — \( f : S \to S' \) maps to \( \left( \frac{T f^*}{f} \right) \)

The pair \( (\text{Bikleisli}(\times, P), T) \) is a contravariant dynamical system doctrine.

**Example 8** (continuous dynamical systems). Recall that the relevant indexed category for continuous dynamical systems is

\[ \text{Mfld}_{\text{Sub}}(\times, P) : \text{Mfld}^{\text{op}} \to \text{Cat} \]

In order to define a contravariant dynamical system doctrine we must adjust this indexed category.

---

\(^3^0\) \( U : \text{Lens}_A \to C \) is defined
- on objects — \( \left( \frac{\alpha}{\alpha} \right) \mapsto O \)
- on morphisms — \( \left( \frac{\beta}{\beta} \right) \mapsto f \)

\(^3^1\) Contravariant dynamical system doctrines stand in contrast to the (covariant) dynamical system doctrines defined in [Myers, 2020] Definition 1.1.
- A covariant dynamical system doctrine consists of an indexed category \( A : C^{\text{op}} \to \text{Cat} \) along with a section \( T \) of its Grothendieck construction.
- In contrast, a contravariant dynamical system doctrine consists of an indexed category \( A : C^{\text{op}} \to \text{Cat} \) along with a section \( T \) of the pointwise opposite of its Grothendieck construction.
Let $\text{Riem}$ be the category of Riemannian manifolds\(^{32}\) with differentiable maps between them. In particular, a Euclidean space equipped with the standard inner product is an instance of a Riemannian manifold.

Composing $\text{Mfld}_{\text{Sub}}/(-)$ with the functor $\text{Riem} \to \text{Mfld}$ that forgets the Riemannian structure induces a new indexed category

$$
\text{Riem}_{\text{Sub}}/(-) : \text{Riem}^{\text{op}} \to \text{Mfld}^{\text{op}} \xrightarrow{\text{Mfld}_{\text{Sub}}/(-)} \text{Cat}
$$

An object of $\text{Lens}_{\text{Riem}_{\text{Sub}}/(-)}$ is a pair $(\theta^O_{\text{Riem}})$ where $\theta$ is a Riemannian structure on $O$ and $p$ is a submersion of manifolds. Note that $l$ need not be Riemannian.

Define $T : \text{Riem} \to \text{Lens}_{\text{Riem}_{\text{Sub}}/(-)}$ as follows:

- on objects — $(S, g)$ maps to $(S, S')$ where $T \to S$ is the natural projection map
- on morphisms — $f : S \to S'$ maps to

$$
\left( T_f^g \right) : \left( TS \to S \right)_{(S, g)} \cong \left( TS' \to S' \right)_{(S', g')}
$$

where $T_f^g : TS \times_{S'} S \to TS$ is defined as follows. For $x \in S, \tilde{y} \in T_{f(x)}S'$, let

$$
T_f^g(\tilde{y}, x) = ((\theta^S_{\text{Riem}})^{-1} \circ T^S_{\text{Riem}} \circ \theta^{S'}_{f(x)})(\tilde{y}).
$$

This definition is slick but obscures the relationship to resource sharing. To achieve this more earthly goal, let’s restrict our attention to Euclidean spaces.

Let $f : S' \to S$ be a map of finite sets. Then $f$ induces a map of Euclidean spaces $f^* : \mathbb{R}^S \to \mathbb{R}^{S'}$. The slogan for $f^* = \text{add along shared coordinates}$. For each $p \in \mathbb{R}^S$, the tangent space $T_p(\mathbb{R}^S)$ is isomorphic to $\mathbb{R}^S$. We interpret elements of $T_p \mathbb{R}^{S'}$ as maps $v : S \to \mathbb{R}$ which assign to each coordinate $s \in S$ a velocity $v(s)$. Under this interpretation, $T \mathbb{R}^{S'} \times_{\mathbb{R}^S} \mathbb{R}^S \simeq \mathbb{R}^{S'} \times \mathbb{R}^S$ and

$$
T(f^*)(v, p) = \sum_{s' \in f^{-1}(p)} v(s')
$$

justifying the slogan. If two coordinates in $S'$ are identified by $f$, then their velocities are summed in $T(f^*)(v, p)$.

The pair $(\text{Riem}_{\text{Sub}}/(-), T)$ is a contravariant dynamical system doctrine.

We are now ready to use the two tools provided by a contravariant dynamical system doctrine — the indexed category $\mathcal{A} : \text{C}^{\text{op}} \to \text{Cat}$ and the section $T : \text{C} \to \text{Lens}_{\mathcal{A}}$ — to define resource sharing machines.

\(^{32}\) A Riemannian manifold $(M, g)$ is a manifold $M$ equipped with a Riemannian structure $g$. The Riemannian structure allows us to define notions of length and angle on $M$. For our purposes $g$ defines an inner product on the tangent space $T_p M$ for each $p \in M$ that varies continuously with $M$. Importantly, $g$ induces a natural isomorphism between the tangent space $T_p M$ and the cotangent space $T^*_p M$. Let $\theta^{T_p M} : T_p M \to T^*_p M$ denote this isomorphism.
Theorem 5.3. Let $C$ be a cartesian monoidal category with pullbacks. Let $(\mathcal{A} : C^{op} \to \text{Cat}, T : C \to \text{Lens}_{\mathcal{A}})$ be a contravariant dynamical system doctrine such that $\mathcal{A}$ is monoidal and $T$ is oplax. There exists a lax monoidal functor

$$RSM : \text{Lens}_{\mathcal{A}} \times \text{Span}_C \to \text{Set}$$

defined

- on objects — for $(\downarrow O) \in \text{ob} \text{Lens}_{\mathcal{A}}$ and $M \in \text{ob} C$

  $$RSM\left(\begin{bmatrix} I \\ O \end{bmatrix}, M\right) = \left\{\left(\begin{bmatrix} S \\ u \\ r \end{bmatrix}, p\right) | S \in \text{ob} C, \begin{bmatrix} u \\ r \end{bmatrix} \in \text{Lens}_{\mathcal{A}}\left(\begin{bmatrix} TS \\ S \end{bmatrix}, \begin{bmatrix} I \\ O \end{bmatrix}\right), p \in C(S, M)\right\}.$$ 

- on morphisms — For $\left(\begin{bmatrix} f^\# \\ f \end{bmatrix} : (\downarrow O) \leftrightarrow (\downarrow O')\right)$ in $\text{Lens}_{\mathcal{A}}$ and span $M \leftarrow Q \rightarrow M'$ in $\text{Span}_C$, the set map $RSM \left(\begin{bmatrix} f^\# \\ f \end{bmatrix}, M \leftarrow Q \rightarrow M'\right)$ maps the triple $(S, (\downarrow)^u, p) \in RSM\left(\begin{bmatrix} I \\ O \end{bmatrix}, M\right)$ to

  $$\left(\begin{bmatrix} S \times_M Q \\ T(S \times_M Q) \end{bmatrix}, \begin{bmatrix} T(S \times_M Q) \\ S \times_M Q \end{bmatrix}, \begin{bmatrix} u \\ r \end{bmatrix}, \begin{bmatrix} f^\# \\ f \end{bmatrix}, \begin{bmatrix} f^\# \\ f \end{bmatrix}, \begin{bmatrix} f^\# \\ f \end{bmatrix}\right) \in RSM\left(\begin{bmatrix} I' \\ O' \end{bmatrix}, M'\right)$$

where $S \times_M Q$ is the pullback

$$\begin{array}{ccc}
S \times_M Q & \xrightarrow{\beta} & Q \\
\downarrow_{i^\#} & & \downarrow_i \\
S & \xrightarrow{p} & M 
\end{array}$$

and the induced lens is

$$\begin{bmatrix} T(S \times_M Q) \\ S \times_M Q \end{bmatrix} \xrightarrow{i^\#} \begin{bmatrix} TS \\ S \end{bmatrix} \xrightarrow{u} \begin{bmatrix} I \\ O \end{bmatrix} \xleftarrow{f^\#} \begin{bmatrix} I' \\ O' \end{bmatrix}.$$ 

Proof. The functor $T : C \to \text{Lens}_{\mathcal{A}}$ defines a profunctor

$$\text{Lens}_{\mathcal{A}}(T(-), -) : \text{Lens}_{\mathcal{A}} \to C$$

and the inclusion $J : C \to \text{Span}_C$ defines a profunctor

$$\text{Span}_C(-, J(-)) : C \to \text{Span}_C.$$ 

Let $RSM$ be the composition of profunctors

$$\text{Lens}_{\mathcal{A}}(T(-), -) \circ \text{Span}_C(-, J(-)).$$

Lemmas 6.1 and 6.2 show that $RSM$ has the desired behavior on objects and morphisms respectively.
Lastly, we want to define a laxator for RSM. Let $\phi$ be the op-laxator for $T$. Let

\[
\left(\left(\frac{I}{O}, M\right), \left(\frac{I'}{O'}, M'\right)\right) \in \text{ob Lens}_A \times \text{Span}_C.
\]

Define the laxator of RSM so that the pair

\[
\left(\left(S, \left(\frac{u}{r}\right), p\right), \left(S', \left(\frac{u'}{r'}\right), p'\right)\right) \in \text{RSM}\left(\left(\frac{I}{O}, M\right) \times \text{RSM}\left(\left(\frac{I'}{O'}, M'\right)\right)\right)
\]

maps to

\[
\left(S \times S', \phi_{S,S'}; \left(\left(\frac{u}{r}\right) \otimes \left(\frac{u'}{r'}\right)\right), p \times p'\right) \in \text{RSM}\left(\left(\frac{I}{O} \otimes \left(\frac{I'}{O'}\right), M \times M'\right)\right).
\]

The unitor is defined by the choice

\[
(1_C, \eta, \text{id}_1) \in \text{RSM}(1\text{Lens}_A, 1_C)
\]

where $\eta$ is the op-unitor for $T$.

The associativity and unitality conditions follow straightforwardly from the op-associativity and op-unitality conditions for $T$. □

The domain $\text{Lens}_A \times \text{Span}_C$ of the functor RSM defines a syntax for composing open dynamical systems. Loosely, we depict objects $\left(\left(\frac{I}{O}, M\right)\right)$ as boxes with input wires corresponding to $I$, output wires corresponding to $O$, and exposed wires corresponding to $M$. See Figure 11.

A box of this type is filled with an element $\left(S, \left(\frac{u}{r}\right), p\right) \in \text{RSM}\left(\left(\frac{I}{O}, M\right)\right)$ which is interpreted as

- $S$ — a state space or set of states
- $\left(\frac{u}{r}\right)$: $T_S \rightarrow \frac{I}{O}$ — dynamics and a read-out function
- $p: S \rightarrow M$ — the observation of the states from the perspective of the exposed ports

Figure 11: (a) Loosely, this box represents the type $\left(\left(\frac{I}{O}, M\right)\right) \in \text{Lens}_A \times \text{Span}_C$. (b) Such a box may be filled with an element $\left(S, \left(\frac{u}{r}\right), p\right) \in \text{RSM}\left(\left(\frac{I}{O}, M\right)\right)$ representing a choice of states, dynamics, read-out, and observation from exposed ports.
For a morphism
\[
\left( \left( \left( f^R, M \leftarrow Q \rightarrow M' \right) \right), \left( \left( I, M \right) \rightarrow \left( \left( I', O', M' \right) \right) \right) \right)
\]
in \text{Lens}_R \times \text{Span}_C, the set map \( \text{RSM} \left( \left( f^R, M \leftarrow Q \rightarrow M' \right) \right) \) defines
the effects of the machine-style composition given by the lens \( \left( f^R \right) \) and resource sharing given by the span \( M \leftarrow Q \rightarrow M' \) to dynamical
systems in \( \text{RSM} \left( \left( I, O, M \right) \right) \).

In Section 2 we constructed the Lokta-Volterra predator-prey
model as the composition of four simple continuous dynamical sys-
tems. To see Theorem 5.3 in action we will formalize this construction
in the language of the algebra
\[
\text{RSM} : \text{Lens}_{\text{RiemSub}/(-)} \times \text{Span}_{\text{Riem}} \rightarrow \text{Set}
\]
induced by the contravariant dynamical system doctrine consisting of
the indexed category
\[
\text{RiemSub}/(-) : \text{Riem}^{op} \rightarrow \text{Cat}
\]
described in Example 6 and the section \( T : \text{Riem} \rightarrow \text{Lens}_{\text{RiemSub}/(-)} \)
described in Example 8.

Figure 12 defines a syntax for composing four continuous dynami-
cal systems. A box with \( I \) input ports, \( O \) output ports, and \( M \) exposed
ports has type
\[
\left( \left( \mathbb{R}^I \times \mathbb{R}^O \xrightarrow{\pi_2} \mathbb{R}^O \right), \mathbb{R}^M \right) \in \text{ob} \left( \text{Lens}_{\text{RiemSub}/(-)} \times \text{Span}_{\text{Riem}} \right).
\]
In the following we abuse notation by suppressing the submersion in
each object of \( \text{Lens}_{\text{RiemSub}/(-)} \) since they are all given by projection onto
the second coordinate, and instead we write
\[
\left( \left( \mathbb{R}^I \times \mathbb{R}^O \right), \mathbb{R}^M \right) \in \text{ob} \left( \text{Lens}_{\text{RiemSub}/(-)} \times \text{Span}_{\text{Riem}} \right).
\]
The left-most and right-most inner box have 0 input ports, 1 output
port, and 1 exposed port. Therefore, these boxes have type \( \left( \left( \{ \ast \} \times \mathbb{R} \right), \mathbb{R} \right) \).
On the other hand, the middle inner boxes have type $\left( R \times \{ \ast \}, \{ \ast \} \right)$. The outer box has trivial type $\left( \{ \ast \} \times \{ \ast \}, \{ \ast \} \right)$.

The wiring in Figure 12 represents a morphism

$$\left( \left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right) \otimes \left( \mathbb{R} \times \{ \ast \}, \mathbb{R} \right) \otimes \left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right) \otimes \left( \mathbb{R} \times \{ \ast \}, \mathbb{R} \right) \rightarrow \left( \{ \ast \} \times \{ \ast \}, \{ \ast \} \right) \right)$$

in $\text{Lens}_{\text{Riem}_{\text{sub}} / (-)} \times \text{Span}_{\text{Riem}}$. Unwinding definitions, the wiring is given by the pair of morphisms:

1. the lens $\left( \text{id} \times \text{id} \right): \left( \mathbb{R}^2 \times \mathbb{R}^2 \right) \leftrightarrow \left( \{ \ast \} \right)$. The morphisms $\phi^\text{in} = \text{id} \times \text{id}$ and $\phi^\text{out} = \text{id}$ fit into the commutative diagram on below:

2. the span $\mathbb{R}^4 \leftarrow \mathbb{R}^2 \rightarrow \{ \ast \}$.

Now that the syntax is established we can interpret each domain type as a continuous dynamical system. Then composition along the wiring defined above results in an interpretation for the codomain type.

The left-most box in Figure 12 has type $\left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right)$ and hence may be filled with an element $(S, (\pi, p)) \in \text{RSM} \left( \left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right) \right)$. The choice

$$\left( \mathbb{R}, \frac{u}{\text{id}} : \left( \mathbb{R} \right) \right) \leftrightarrow \left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right), \text{id} : \mathbb{R} \rightarrow \mathbb{R}$$

where $u(r) = \beta r \in T_r \mathbb{R}$ models rabbit population growth. As shorthand, we visualize this filling as:

\[
\begin{array}{c}
\vdots \\
\text{id} \\
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\beta r \\
\end{array}
\]

Likewise, we may fill the remaining boxes with open continuous dynamical systems that model fox population growth, rabbit population decline, and fox population decline. The short-hand for these systems fill the inner boxes in the left-hand side of the equation in Figure 13. The set map $\text{RSM} \left( \left( \{ \ast \} \times \mathbb{R}, \mathbb{R} \right) \right)$ maps the quadruple of fillings to the continuous dynamical system

\[
\dot{r} = \beta r - \gamma r f, \quad \dot{f} = ar f - \delta f
\]

the Lokta-Volterra predator-prey model.
6 Proof of Main Theorem

Throughout let

- \( C \) be a cartesian monoidal category with pullbacks
- \( \mathcal{A} : C^{\text{op}} \to \text{Cat} \) be a monoidal indexed category
- \( T : C \to \text{Lens}_\mathcal{A} \) be an oplax section of the forgetful functor with colaxator \( \phi \)

Lemma 6.1. Let \( (I_O) \in \text{ob} \text{Lens}_\mathcal{A} \) and \( M \in \text{ob} \text{Span}_C \). Define \( \text{RSM} \left( \left( \frac{I}{O} \right), M \right) \) to be the set of triples \((S, (u), p)\) with \( S \in \text{ob} C \), \((u) : (TS) \leftrightarrow (I) \) in \( \text{Lens}_\mathcal{A} \), and \( p : S \to M \) in \( C \). Then, \( \text{RSM} \left( \left( \frac{I}{O} \right), M \right) \) is isomorphic to the coend

\[
\int_{S \in C} \text{Lens}_\mathcal{A} \left( \left( \frac{TS}{S} \right), \left( \frac{I}{O} \right) \right) \times \text{Span}_C(M, S).
\]

Proof. For each \( S \in \text{ob} C \) define a set map

\[
\omega_S : \text{Lens}_\mathcal{A} \left( \left( \frac{TS}{S} \right), \left( \frac{I}{O} \right) \right) \times \text{Span}_C(M, S) \to \text{RSM} \left( \left( \frac{I}{O} \right), M \right)
\]

by

\[
\omega_S \left( \left( \frac{u}{r} \right), M \leftrightarrow S' \mapsto S \right) = \left( S', \left( \frac{Tq}{q} \right), \frac{u}{r} \right), p).\)

First we will show that

\[
\omega : \text{Lens}_\mathcal{A} \left( T(-), \left( \frac{I}{O} \right) \right) \times \text{Span}_C(M, -) \Rightarrow \text{RSM} \left( \left( \frac{I}{O} \right), M \right)
\]

is a cowedge of

\[
\text{Lens}_\mathcal{A} \left( T(-), \left( \frac{I}{O} \right) \right) \times \text{Span}_C(M, -) : C^{\text{op}} \times C \to \text{Set}.
\]

Let \( f : S_1 \to S_2 \) in \( C \). The following diagram commutes by unwinding definitions.

Figure 13: On the left-hand side of the equation we fill the syntax defined in Figure 12 with continuous dynamical systems which from left to right correspond to rabbit population growth, fox population growth, rabbit population decline, and fox population decline. On the right-hand side of the equation, the resulting interpretation induced by \( \text{RSM} \) is the Lokta-Volterra predator-prey model.
Next we want to show that $RSM\left(\left(I\right), M\right)$ is the universal cowedge.

Suppose $\alpha : \text{Lens}_A\left(\left(T\right), \left(I\right)\right) \times \text{Span}_C(M, -) \Rightarrow Y$ is any cowedge.

We want to show there exists $h : RSM\left(\left(I\right), M\right) \rightarrow Y$ such that for all $S \in C$, the triangle below commutes.

\[
\text{Lens}_A\left(\left(TS\right), \left(I\right)\right) \times \text{Span}_C(M, S) \xrightarrow{\alpha_S} Y
\]

For $(S, \left(\frac{u}{r}\right), p) \in RSM\left(\left(I\right), M\right)$ define

\[
h\left(S, \left(\frac{u}{r}\right), p\right) = \alpha_S\left(\left(\frac{u}{r}\right), M \xleftarrow{p} S \xrightarrow{id} S\right)
\]  \hspace{1cm} (1)

Then for all \(\left(\left(\frac{u}{r}\right), M \xleftarrow{p} S' \xrightarrow{q} S\right) \in \text{Lens}_A\left(\left(TS\right), \left(I\right)\right) \times \text{Span}_C(M, S)\),

we have

\[\left(h \circ \alpha_S\right)\left(\left(\frac{u}{r}\right), M \xleftarrow{p} S' \xrightarrow{q} S\right) = h\left(S', \left(\frac{Tq^*}{q}\right), \left(\frac{u}{r}\right), p\right)\]

\[= \alpha_{S'}\left(\left(\frac{Tq^*}{q}\right), \left(\frac{u}{r}\right), M \xleftarrow{p} S' \xrightarrow{id} S'\right)\]

\[= \alpha_S\left(\left(\frac{u}{r}\right), M \xleftarrow{p} S' \xrightarrow{q} S\right)\]

where the last line follows from the commuting square induced by $q : S' \rightarrow S$ below.
Lastly we must show that $h$ is unique. Suppose $\tilde{h} : \text{RSM} \left( \left( I_O \right), M \right) \to Y$ also satisfies $\tilde{h} \circ \omega_S = \alpha_S$ for all $S \in \text{ob} C$. For $(S, (u_r), p) \in \text{RSM} \left( \left( I_O \right), M \right)$,

$$\left( S, \left( \frac{u}{r} \right), p \right) = \omega_S \left( \left( \frac{u}{r} \right), M \xleftarrow{p} S \xrightarrow{id} S \right)$$

implies

$$\tilde{h} \left( S, \left( \frac{u}{r} \right), p \right) = \alpha_S \left( \left( \frac{u}{r} \right), M \xleftarrow{p} S \xrightarrow{id} S \right) = h \left( S, \left( \frac{u}{r} \right), p \right).$$

\[\square\]

**Lemma 6.2.** Let $\left( f^* \right) : \left( I_O \right) \leftrightarrow \left( I'_{O'} \right)$ be a morphism in $\text{Lens}_A$ and let $M \xleftarrow{i} Q \xrightarrow{i'} M'$ be a span in $C$. These induce a natural transformation

$$\text{Lens}_A \left( T(-), \left( f^* \right) \right) \times \text{Span}_C(M \xleftarrow{i} Q \xrightarrow{i'} M') : \text{Lens}_A \left( T(-), \left( I_O \right) \right) \times \text{Span}_C(M, -) \Rightarrow \text{Lens}_A \left( T(-), \left( I'_{O'} \right) \right) \times \text{Span}_C(M', -).$$

Let $\text{RSM} \left( \left( I_O \right), M \xleftarrow{i} Q \xrightarrow{i'} M' \right)$ be the set map which takes the triple $(S, (u_r), p) \in \text{RSM} \left( \left( I_O \right), M \right)$ to the triple

$$\left( S \times_M Q, \left( \frac{T\bar{i}}{i} \right) ; \left( \frac{f^*}{f} \right) ; \tilde{p} ; i' \right) \in \text{RSM} \left( \left( I'_{O'} \right), M' \right).$$

Then, $\text{RSM} \left( \left( I_O \right), M \xleftarrow{i} Q \xrightarrow{i'} M' \right)$ is isomorphic to

$$\int_{S \in C} \text{Lens}_A \left( T(-), \left( f^* \right) \right) \times \text{Span}_C(M \xleftarrow{i} Q \xrightarrow{i'} M', -).$$

**Proof.** The transformation

$$\text{Lens}_A \left( T(-), \left( f^* \right) \right) \times \text{Span}_C(M \xleftarrow{i} Q \xrightarrow{i'} M', -)$$

induces a cowedge

$$\alpha : \text{Lens}_A \left( T(-), \left( I_O \right) \right) \times \text{Span}_C(M, -) \Rightarrow \text{RSM} \left( \left( I'_{O'} \right), M' \right)$$

defined such that

$$\alpha_S : \text{Lens}_A \left( \left( \frac{TS}{S} \right), \left( I_O \right) \right) \times \text{Span}_C(M, S) \Rightarrow \text{RSM} \left( \left( I'_{O'} \right), M' \right).$$
maps the pair \((l_u^P : (T^S_S) \leftrightarrow (I^O_I), M \leftarrow S' \rightarrow S)\) to the triple
\[
\left( S' \times M, Q \right) \xrightarrow{T(S' \times M Q)_{T(S')}} \left( T(S' \times M Q) \right)_{T(S')} \xrightarrow{u_f} \left( 1_O \right)_{f^r} \left( I'O' \right), S' \times M Q \rightarrow Q \leftrightarrow M'.
\]

Then
\[
\int_{S \in C} \text{Lens}_A(T(-), (f^#_f)) \times \text{Span}_C(M \leftarrow Q \rightarrow M', -)
\]
is the unique map
\[
\text{RSM}\left(\left( I^O_O \right), M\right) \rightarrow \text{RSM}\left(\left( I'O'_O' \right), M'\right)
\]
such that the diagram below commutes for all \(S \in C,\)
\[
\text{Lens}_A\left(\left( T^S_S \right), \left( I^O_O \right)\right) \times \text{Span}_C(M, S) \xrightarrow{\alpha_S} \text{RSM}\left(\left( I'O'_O' \right), M'\right)
\]
\[
\text{RSM}\left(\left( I^O_O \right), M\right) \xrightarrow{\alpha_S} \text{RSM}\left(\left( I'O'_O' \right), M'\right)
\]
Following Equation \[\Box\] and unwinding definitions,
\[
\int_{S \in C} \text{Lens}_A(T(-), (f^#_f)) \times \text{Span}_C(M, S) \xrightarrow{\alpha_S} \text{RSM}\left(\left( I'O'_O' \right), M'\right)
\]
takes the triple \((S, (u^P), p) \in \text{RSM}\left(\left( I^O_O \right), M\right)\) to
\[
\alpha_S\left(\left( u^P \right), M \leftarrow S \rightarrow S\right) = \text{RSM}\left(\left( f^#_f \right), M \leftarrow Q \rightarrow M'\right)\left( S, (u^P), p \right) \in \text{RSM}\left(\left( I'O'_O' \right), M'\right)
\]
\[
\Box
\]

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