ON THE AVERAGE SENSITIVITY OF THE WEIGHTED SUM FUNCTION

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Abstract. In this paper we obtain the bound on the average sensitivity of the weighted sum function. This confirms a conjecture of Shparlinski. We also compute the weights of the weighted sum functions and show that they are almost balanced.

1. Introduction

A weighted sum function, also known as laced Boolean function, is defined in terms of certain weighted sums in the residue ring modulo a prime. Explicitly, it can be defined as follows [11]. Let $n$ be a positive integer and $p$ is the least prime number that is no less than $n$. For $X = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_p^n$, we define $s(X)$ by the least positive integer of $\sum_{k=1}^{n} k x_k \text{ mod } p$, i.e.,

\[ s(X) \equiv \sum_{k=1}^{n} k x_k \text{ (mod } p) \quad 1 \leq s(X) \leq p. \]

We define that

\[ f(X) = \begin{cases} x_{s(X)}, & 1 \leq s(X) \leq n; \\ x_1, & \text{otherwise}. \end{cases} \]

This function was first studied by P. Savický and S. Žák [11] in their study of read-once branching programs. It has also been used for several more complexity theory applications by M. Sauerhoff [12, 13]. For instance, in [13] a certain modification of the same function has been used to prove that quantum read-once branching programs are exponentially more powerful than classical read-once branching programs.

For a given input $X = (x_1, x_2, \ldots, x_n)$, the sensitivity $\sigma_s f$ on input $X$ is defined to be the number of bits such that flipping one of them will change the value of the function. Explicitly,

\[ \sigma_{s,X}(f) = \sum_{i=1}^{n} \left| f(X) - f(X^{(i)}) \right|, \]

where $X^{(i)} = (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$ denotes the vector obtained from $X$ by flipping the $i$-th coordinate. The sensitivity $\sigma_s f$ of $f(X)$ is the maximum of $\sigma_{s,X}(f)$ on input $X$ over $\mathbb{Z}_p^n$. The average sensitivity $\sigma_{av}(f)$ is defined to be the sensitivity average on all inputs, i.e.,

\[ \sigma_{av}(f) = 2^{-n} \sum_{X \in \mathbb{Z}_p^n} \sum_{i=1}^{n} \left| f(X) - f(X^{(i)}) \right|. \]

This work is supported by the National Science Foundation of China (11001170).
Sensitivity, and more generally, block sensitivity are important measures of complexity of Boolean functions. It recently draws some attention. For instance, Rubinstein and Bernasconi showed large gaps between the average sensitivity and the average block sensitivity [10, 2], Bernasconi, Dam and Shparlinski gave the average sensitivity of testing square-free numbers [3], Boppana considered the average sensitivity of bounded-depth circuits [4] and Shi gave that the average sensitivity as a lower bound of approximation polynomial degree, and thus can serve as a lower bound of quantum query complexity [14]. For more details we refer to [5].

In [15] Shparlinski studied the average sensitivity of laced Boolean function and gave a lower bound by obtaining a nontrivial bound on the Fourier coefficients of the laced boolean function via exponential sums methods. He also gave two conjectures on the bounds of Fourier coefficients and the average sensitivity of laced Boolean functions respectively. Explicitly he conjectured that the average sensitivity of laced Boolean functions on \( n \) variables should be at least \((\frac{1}{2} + o(1))n\). He proved in the same paper that this value is greater than \( \gamma n \), where \( \gamma \approx 0.135 \) is a constant.

Recently an explicit formulas for the average sensitivity of laced Boolean functions was given by D. Canright, etc. in [6] for the case \( p = n \) by using some counting formulas of the subset sums over prime fields given by Li and Wan [7]. Equivalently they proved Shparlinski’s conjecture for the case \( p = n \). They also showed further experimental evidence for the above conclusion on the average sensitivity.

As we have known by the Prime Number Theorem, \( p = n + o(n) \), see [1] for the best known result about gaps between prime numbers. Thus, the gap between the present result and the expected result is quite large. In this paper, we completely settled this problem. In fact, we proved that the average sensitivity of laced Boolean functions is \((\frac{1}{2} + o(1))n\).

We also compute the weights of the laced Boolean functions. Some explicit formulas were given by D. Canright, S. Gangopadhay, S. Maitra and P. Stanica in [6] when \( p - n \leq 3 \) by the using the same counting formulas of the subset sums over prime fields given in [7]. In this paper we extend their result for all the general case. We show that for a laced Boolean function \( f \), the weight of \( f \) should be \( 2^{n-1}(1+o(1)) \) and hence \( f \) is asymptotically balanced function.

This paper is organized as follows. In Section 2 we present a sieve formula and we prove the main results in Section 3.

2. A DISTINCT COORDINATE SIEVING FORMULA

Our method is to evaluate a special exponential sum via a new approach. The starting point is a new sieving formula discovered in [8], which significantly improves the classical inclusion-exclusion sieve in many interesting cases. We cite it here without proof. For the details and some related applications we refer to [8, 9].

Let \( D \) be a finite set, and let \( D^k = D \times D \times \cdots \times D \) be the Cartesian product of \( k \) copies of \( D \). Let \( X \) be a subset of \( D^k \). Define

\[
\overline{X} = \{(x_1, x_2, \ldots, x_k) \in X \mid x_i \neq x_j, \forall i \neq j\}. \tag{2.1}
\]

Let \( f(x_1, x_2, \ldots, x_k) \) be a complex valued function defined over \( X \). Many problems arising from coding theory, additive number theory and number theory are reduced to evaluate the summation

\[
F = \sum_{x \in \overline{X}} f(x_1, x_2, \ldots, x_k). \tag{2.2}
\]
Note that if we let \(f(x_1, x_2, \ldots, x_k) \equiv 1\), then \(F\) is just the number of elements in \(\mathbb{X}\).

Let \(S_k\) be the symmetric group on \(\{1, 2, \cdots, k\}\). Each permutation \(\tau \in S_k\) factorizes uniquely (up to the order of the factors) as a product of disjoint cycles and each fixed point is viewed as a trivial cycle of length 1. Two permutations in \(S_k\) are conjugate if and only if they have the same type of cycle structure (up to the order). Let \(C_k\) be the set of conjugacy classes of \(S_k\) and note that \(|C_k| = p(k)\), the partition function. For a given \(\tau \in S_k\), let \(l(\tau)\) be the number of cycles of \(\tau\) including the trivial cycles. Then we can define the sign of \(\tau\) to be \(\text{sign}(\tau) = (-1)^{k-l(\tau)}\). For a given permutation \(\tau = (i_1 i_2 \cdots i_{a_1})(j_1 j_2 \cdots j_{a_2}) \cdots (l_1 l_2 \cdots l_{a_s})\) with \(1 \leq a_i, 1 \leq i \leq s\), define
\[
X_{\tau} = \{(x_1, \ldots, x_k) \in X, x_{i_1} = \cdots = x_{i_{a_1}}, \ldots, x_{j_1} = \cdots = x_{j_{a_2}}\}. \tag{2.3}
\]
Each element of \(X_{\tau}\) is said to be of type \(\tau\). Thus \(X_{\tau}\) is the set of all elements in \(X\) of type \(\tau\). Similarly, for \(\tau \in S_k\), we define
\[
F_{\tau} = \sum_{x \in X_{\tau}} f(x_1, x_2, \ldots, x_k).
\]

Now we can state our sieve formula. After that we will give one corollary for the use of our proof. We remark that there are many other interesting corollaries of this formula. For interested reader we refer to [8].

**Theorem 2.1.** We have
\[
F = \sum_{\tau \in S_k} \text{sign}(\tau) F_{\tau}.
\]

Note that the symmetric group \(S_k\) acts on \(D^k\) naturally by permuting coordinates. That is, for given \(\tau \in S_k\) and \(x = (x_1, x_2, \ldots, x_k) \in D^k\), we have
\[
\tau \circ x = (x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(k)}).
\]

Before stating a useful corollary, we first give two definitions.

**Definition 2.2.** A subset \(X\) in \(D^k\) is said to be symmetric if for any \(x \in X\) and any \(\tau \in S_k\), \(\tau \circ x \in X\).

**Definition 2.3.** Let \(X \subseteq D^k\) and assume \(X\) is symmetric. A complex-valued function \(f(x_1, x_2, \ldots, x_k)\) defined over \(X\) is called normal on \(X\) if for any two \(S_k\)-conjugate elements \(\tau\) and \(\tau'\) in \(S_k\) (thus \(\tau\) and \(\tau'\) have the same type), we have
\[
\sum_{x \in X_{\tau}} f(x_1, x_2, \ldots, x_k) = \sum_{x \in X_{\tau'}} f(x_1, x_2, \ldots, x_k).
\]

**Remark:** If \(f(x_1, x_2, \ldots, x_k)\) is a symmetric function and \(X\) is symmetric, then \(f(x_1, x_2, \ldots, x_k)\) must be normal on \(X\).

**Corollary 2.4.** Let \(C_k\) be the set of conjugacy classes of \(S_k\). If \(f\) is normal on \(X\), then we have
\[
F = \sum_{\tau \in C_k} (-1)^{k-l(\tau)} C(\tau) F_{\tau},
\]
where \(C(\tau)\) is the number of permutations conjugate to \(\tau\).

For the purpose of our proof, we will also need the following combinatorial formula. For simplicity we omit the proof and it can be found in [8].
Lemma 2.5. Let \( N(c_1, c_2, \ldots, c_k) \) be the number of permutations in \( S_k \) of type \((c_1, c_2, \ldots, c_k)\), that is,
\[
N(c_1, c_2, \ldots, c_k) = \frac{k!}{1^{c_1}c_1!2^{c_2}c_2!\cdots k^{c_k}c_k!},
\]
and define the generating function
\[
C_k(t_1, t_2, \ldots, t_k) = \sum_{\sum w_i = k} N(c_1, c_2, \ldots, c_k)t_1^{c_1}t_2^{c_2}\cdots t_k^{c_k}.
\]
If \( t_1 = t_2 = \cdots = t_k = q \), then we have
\[
C_k(q, q, \ldots, q) = \sum_{\sum w_i = k} N(c_1, c_2, \ldots, c_k)q^{c_1}q^{c_2}\cdots q^{c_k} = (q + k - 1)_k
\]

3. Subset sums on smooth subsets

Let \( D \subseteq \mathbb{Z}_p \) be a nonempty subset of cardinality \( n \). An additive character \( \chi : \mathbb{Z}_p \to \mathbb{C}^* \) is a homomorphism from \( \mathbb{Z}_p \) to the non-zero complex numbers \( \mathbb{C}^* \). We define the Fourier bias of \( D \) to be
\[
\Phi(D) = \max_{\chi \not= \chi_0} \left| \sum_{a \in D} \chi(a) \right|.
\]
Suppose that we have known \( \Phi(D) \). Let \( N \) be the number of solutions of the equation
\[
x_1 + x_2 + \cdots + x_k = b, x_i \in D, x_i \neq x_j, i \neq j.
\]
In the following theorem we will give an asymptotic bound on \( N \) when \( \Phi(D) \) is small compared to \( n = |D| \).

Theorem 3.1. Let \( N \) be the number of solutions of the equation
\[
x_1 + x_2 + \cdots + x_k = b, x_i \in D, x_i \neq x_j, i \neq j.
\]
Then we have
\[
\frac{N}{k!} \geq \frac{1}{p} \binom{n}{k} - \binom{\Phi(D) + k - 1}{k}.
\]

Proof. Let \( X = D^k = D \times D \times \cdots \times D \) be the Cartesian product of \( k \) copies of \( D \). Let \( \overline{X} = \{(x_1, x_2, \ldots, x_k) \in D^k \mid x_i \neq x_j, \forall i \neq j\} \). It is clear that \( |X| = n^k \) and \( |\overline{X}| = (n)_k \). Note that we have defined that for a permutation \( \tau \in S_k \), \( X_\tau \) consists of the elements \( x \in X \) of type \( \tau \).

Let \( G \) be the group of additive characters of \( \mathbb{Z}_p \) and \( \chi_0 \) be the trivial character. Then we deduce that
\[
N = \frac{1}{p} \sum_{(x_1, x_2, \ldots, x_k) \in \overline{X}} \sum_{\chi \in G} \chi(x_1 + x_2 + \cdots + x_k - b)
\]
\[
= \frac{1}{p} \sum_{\chi \in G} \sum_{(x_1, x_2, \ldots, x_k) \in \overline{X}} \chi(x_1 + x_2 + \cdots + x_k - b)
\]
\[
= \frac{(n)_k}{p} + \frac{1}{p} \sum_{\chi \not= \chi_0} \sum_{(x_1, x_2, \ldots, x_k) \in \overline{X}} \chi(x_1)\chi(x_2)\cdots \chi(x_k)\chi^{-1}(b)\]
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\[
= \frac{(n)_k}{p} + \frac{1}{p} \sum_{\chi \neq \chi_0} \chi^{-1}(b) \sum_{(x_1, x_2, \ldots, x_k) \in X} \prod_{i=1}^{k} \chi(x_i).
\]

For given \( \chi \neq \chi_0 \), let \( f_{\chi}(x) = f_{\chi}(x_1, x_2, \ldots, x_k) = \prod_{i=1}^{k} \chi(x_i) \), and for given \( \tau \) let

\[
F_{\tau}(\chi) = \sum_{x \in X_{\tau}} f_{\chi}(x) = \sum_{x \in X_{\tau}} \prod_{i=1}^{k} \chi(x_i).
\]

Obviously \( X \) is symmetric. It is also easy to check that \( f_{\chi}(x_1, x_2, \ldots, x_k) \) is normal on \( X \). Thus by applying Corollary 2.4, we deduce

\[
N = \frac{(n)_k}{p} + \frac{1}{p} \sum_{\chi \neq \chi_0} \chi^{-1}(b) \sum_{\tau \in C_k} \text{sign}(\tau) C(\tau) F_{\tau}(\chi)
\]

where \( C_k \) is the set of conjugacy classes of \( S_k \), \( C(\tau) \) is the number of permutations conjugate to \( \tau \), and \( F_{\tau}(\chi) = \sum_{x \in X} \prod_{i=1}^{k} \chi(x_i) \).

For given \( \tau \in C_k \), assume \( \tau \) is of type \((c_1, c_2, \ldots, c_k)\), where \( c_i \) is the number of \( i \)-cycles in \( \tau \) for \( 1 \leq i \leq k \). Note that \( \sum_{i=1}^{k} ic_i = k \) and thus we deduce

\[
F_{\tau}(\chi) = (\sum_{a \in D} \chi(a))^{c_1} (\sum_{a \in D} \chi^2(a))^{c_2} \cdots (\sum_{a \in D} \chi^k(a))^{c_k}
\]

\[
= \prod_{i=1}^{k} (\sum_{a \in D} \chi^i(a))^{c_i}.
\]

By the definition of \( \Phi(D) \) we have \( F_{\tau}(\chi) \leq (\Phi(D))^{\sum_{i=1}^{k} c_i} \) and thus

\[
N \geq \frac{(n)_k}{p} - \frac{1}{p} \sum_{\chi \neq \chi_0} \sum_{\tau \in C_k} C(\tau)(\Phi(D))^{\sum_{i=1}^{k} c_i}
\]

\[
= \frac{(n)_k}{p} - \frac{p-1}{p} \sum_{\sum i_c i = k} \frac{k!}{1^{c_1}2^{c_2}c_2! \cdots k^{c_k}c_k!}(\Phi(D))^{\sum_{i=1}^{k} c_i}
\]

\[
= \frac{(n)_k}{p} - \frac{p-1}{p}(\Phi(D) + k - 1)_k. \quad \square
\]

The last equality is from Lemma 2.5 and the proof is complete.

If \( n = p - c \), where \( c \) is a fixed constant, then one checks that \( \Phi(D) \leq c \) and thus we get a clean and better bound, which was first found by [7] by using elementary counting method.

**Corollary 3.2.** If \( n = p - c \), where \( c \) is a fixed constant, then we have

\[
\frac{N}{k!} \geq \frac{1}{p} \binom{p-c}{k} - \binom{c+k-1}{k}.
\]

Similarly we have
Corollary 3.3. If \( n = p - o(p) \), then \( \Phi(D) \leq o(p) \) and thus we have
\[
\frac{N}{k!} \geq \frac{1}{p} \left( \binom{p-o(p)}{k} - \binom{o(p)+k-1}{k} \right).
\]

Corollary 3.4. Let \( n = p - o(p) \). Let \( N(b, D) \) be the number of subsets in \( D \) which sum to \( b \). Then we have
\[
N(b, D) = \frac{2^n}{p} (1 + o(1)).
\]

We say a subset \( D \subseteq A \) is smooth if \( \Phi(D) = O(\sqrt{n \log |A|}) \), that is, for every nontrivial additive character \( \chi, |\sum_{a \in D} \chi(a)| = O(\sqrt{n \log |A|}) \).

Corollary 3.5. Let \( D \subseteq \mathbb{Z}_p \) and \( \epsilon \) be a positive constant. If \( |D| = \log^{1+\epsilon} p \) and \( D \) is smooth, then there are two constants \( c_1 \) and \( c_2 \) such that if \( c_1 \frac{\log p}{\log \log p} \leq k \leq c_2 n \), then each element \( \mathbb{Z}_p \) can be written to be a \( k \)-subset sum in \( D \).

Proof. The proof is left to the reader. \( \Box \)

4. Average sensitivity

The average sensitivity \( \sigma_{av}(f) \) of an \( n \)-variate Boolean function \( f(x_1, x_2, \ldots, x_n) \) is defined as
\[
\sigma_{av}(f) = 2^{-n} \sum_{X \in \mathbb{Z}_2^n} \left| \sum_{i=1}^{n} f(X) - f(X^{(i)}) \right|
\]
where \( X^{(i)} \) is the vector obtained from \( X \) by flipping its \( i \)th coordinate. In [15] the author asked the following question.

For given \( X = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}_2^n \), we define \( s(X) \) by the least positive integer of \( \sum_{k=1}^{n} k x_k \mod p \), i.e.,
\[
s(X) = \sum_{k=1}^{n} k x_k \mod p, 1 \leq s(X) \leq p.
\]

Following [11] we define the so called laced Boolean function
\[
f(X) = \begin{cases} x_{s(X)}, & 1 \leq s(X) \leq n; \\ x_1, & \text{otherwise}. \end{cases}
\]

We first compute the weight of \( f(X) \). We have the following theorem, which significantly improve the results given by [6].

Theorem 4.1. Let \( f(X) \) be defined as above. Then we have
\[
wt(f) = 2^{n-1} (1 + o(1)).
\]
In other words, \( f(X) \) is an asymptotically balanced function.

Proof. Let \( A = \{0, n+1, n+2, \ldots, p-1\} \) and \( D = \mathbb{Z}_p \setminus A \). By applying Corollary [6] we have
\[
wt(f) = \sum_{X \in \mathbb{Z}_2^n} f(X) = \sum_{s=1}^{n} \sum_{X \in \mathbb{Z}_2^n : s(X) = s, x_s = 1} 1 + \sum_{s=n+1}^{p} \sum_{X \in \mathbb{Z}_2^n : s(X) = s, x_s = 1} 1
\]
\[
= \sum_{s=1}^{n} N(0, D \setminus \{s\}) + \sum_{s=n+1}^{p} N(s-1, D \setminus \{1\})
\]
In \cite{Shparlinski} Shparlinski studied $\sigma_{av}(f)$ and raised the following conjecture:

**Conjecture 4.2.** Is it true that for the function given by (1) we have

$$\sigma_{av}(f) \geq \left(\frac{1}{2} + o(1)\right) n?$$

In the same paper Shparlinski gave a lower bound by obtaining a nontrivial bound on the Fourier coefficients of the laced boolean function via exponential sums methods. He proved in the same paper that this value is greater than $\gamma n$, where $\gamma \approx 0.135$ is a constant.

Recently an explicit formulas for the average sensitivity of laced Boolean functions was given by D. Canright, etc. in \cite{Canright} for the case $p = n$ by using some counting formulas of the subset sums over prime fields given in \cite{Shparlinski}. Equivalently they proved Shparlinski’s conjecture for $p = n$. They also showed further experimental evidence for the above conclusion on the average sensitivity.

We will prove this conjecture now. In fact we obtain a stronger result.

**Theorem 4.3.** Let $\sigma_{av}(f)$ be the average sensitivity of the laced Boolean function. Then

$$\sigma_{av}(f) = \left(\frac{1}{2} + o(1)\right) n.$$ 

**Proof.** Let $A = \{0, n + 1, n + 2, \ldots, p - 1\}$ and $D = Z_p \setminus A$. Since we have the symmetry between the bits 1 and 0, for simplicity we just need to consider the number of bit changes from 0 to 1. Thus we have

$$2^{n-1}\sigma_{av}(f) = \sum_{X \in Z_p^n} \sum_{i=1}^{n} \left| f(X) - f(X^{(i)}) \right|$$

$$= \sum_{s \in D} \sum_{X \in Z_p^n, s(X) = s, x_s = 1}^{n} \left| 1 - f(X^{(i)}) \right| + \sum_{s \in D} \sum_{X \in Z_p^n, s(X) = s, x_s = 0}^{n} \left| 0 - f(X^{(i)}) \right|$$

$$+ \sum_{s \in A} \sum_{X \in Z_p^n, s(X) = s, x_s = 0}^{n} \left| 1 - f(X^{(i)}) \right| + \sum_{s \in A} \sum_{X \in Z_p^n, s(X) = s, x_s = 1}^{n} \left| 0 - f(X^{(i)}) \right|$$

$$= \sum_{s=1}^{p} \sum_{i=1}^{n} \sum_{X \in Z_p^n, s(X) = s, x_s = 0, x_s+i = 1}^{n} 1 + \sum_{s=1}^{p} \sum_{i=1}^{n} \sum_{X \in Z_p^n, s(X) = s, x_s = 1, x_s+i = 1}^{n} 1$$

$$+ \sum_{i=1}^{n} \sum_{s=1}^{p} \sum_{X \in Z_p^n, s(X) = s, x_s = 0, x_s+i = 0}^{n} 1 + \sum_{i=1}^{n} \sum_{s=1}^{p} \sum_{X \in Z_p^n, s(X) = s, x_s = 1, x_s+i = 1}^{n} 1$$

Recall that from the Prime Number Theorem we have $p = n + o(n)$. We notice that for simplicity we may assume that in the 4 summations, $s + i (\mod p) \in D$, otherwise the summations is bounded by $o(1)$ respectively (for instance: we only
need to replace \( x_{s+i} \) to \( x_i \) and thus can be omitted. Thus we have

\[
2^{n-1} \sigma_{av}(f) = \sum_{i=1}^{n} \sum_{s=1}^{n} N(0, D\{i, s + i, s\}) + \sum_{i=1}^{n} \sum_{s=1}^{n} N(0, D\{i, s - i, s\}) + \sum_{i=1}^{n} \sum_{s=n+1}^{p} N(0, D\{i, s + i, 1\}) + \sum_{i=1}^{n} \sum_{s=n+1}^{p} N(0, D\{i, s - i, 1\})
\]

\[
\approx \sum_{i=1}^{n} \sum_{s=1}^{n} \frac{2^{n-2}}{p} (1 + o(1)) + \sum_{i=1}^{n} \sum_{s=n+1}^{p} \frac{2^{n-2}}{p} (1 + o(1))
\]

\[
\approx n 2^{n-2} (1 + o(1)) + o(n) 2^{n-2} (1 + o(1))
\]

Thus we have

\[
\sigma_{av}(f) = \left(\frac{1}{2} + o(1)\right) n. \quad \square
\]

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