ON $L_p$-BRUNN-MINKOWSKI TYPE AND $L_p$-ISOPERIMETRIC TYPE INEQUALITIES FOR GENERAL MEASURES

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ABSTRACT. In 2011 Lutwak, Yang and Zhang extended the definition of the $L_p$-Minkowski combination ($p \geq 1$) introduced by Firey in the 1960s from convex bodies containing the origin in their interiors to all measurable subsets in $\mathbb{R}^n$, and as a consequence, extended the $L_p$-Brunn-Minkowski inequality to the setting of all measurable sets. In this paper, we present a functional extension of their $L_p$-Minkowski combination and prove the $L_p$-Borell-Brascamp-Lieb type inequalities. In particular, we extend the $L_p$-Brunn-Minkowski inequality for measurable sets to the class of Borel measures on $\mathbb{R}^n$ having $(\frac{1}{s})$-concave densities, with $s \geq 0$; that is, we show that, for any pair of Borel sets $A, B \subset \mathbb{R}^n$, any $t \in [0, 1]$ and $p \geq 1$, one has

$$\mu((1-t)A +_p tB)^{\frac{1}{n}} \geq (1-t)\mu(A)^{\frac{1}{n}} + t\mu(B)^{\frac{1}{n}},$$

where $\mu$ is a measure on $\mathbb{R}^n$ having a $(\frac{1}{s})$-concave density for $0 \leq s < \infty$. Additionally, we prove functional $L_p$ version of Minkowski’s first inequality, $L_p$ isoperimetric inequalities for general measures, and a functional counterpart of the Gardner-Zvavitch conjecture.

Keywords: $L_p$-Brunn-Minkowski inequality, Borell-Brascamp-Lieb type inequalities, functional inequalities, mixed volumes

1. INTRODUCTION

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space equipped with the usual norm $\| \cdot \|$ and its usual inner product structure $\langle \cdot, \cdot \rangle$. For a function $f : \mathbb{R}^n \to \mathbb{R}_+$ we denote its support by $\text{supp}(f)$. For measurable subsets $A, B \subset \mathbb{R}^n$ and $\alpha, \beta > 0$ the Minkowski combination of the sets $A$ and $B$ with respect to the constants $\alpha$ and $\beta$ is defined by

$$\alpha A + \beta B = \{ \alpha x + \beta y : x \in A, y \in B \}.$$

A result involving the Minkowski combination is the famed Brunn-Minkowski inequality, which asserts that, for any $t \in [0, 1]$ and for any measurable sets $A, B \subset \mathbb{R}^n$ such that the sets $(1-t)A+tB$ is also measurable, one has

$$|(1-t)A + tB|_{n}^{1/n} \geq (1-t)|A|_{n}^{1/n} + t|B|_{n}^{1/n}. \quad (1)$$

If, in addition, $A$ and $B$ are taken to be convex sets, then equality in (1) holds if and only if $A$ and $B$ are homothetic. Here $| \cdot |_n$ denotes the $n$-dimensional Lebesgue measure on $\mathbb{R}^n$. The Brunn-Minkowski inequality has lead to a rich theory which links probability, analysis, and convex geometry (for example, see the monographs [1, 23, 43] and the survey [17]). Many generalizations of the Brunn-Minkowski inequality have been studied in the cases of measures (see [5, 14, 18, 24, 28, 33]) and in the functional setting (see [11, 26, 35]).

Let $\alpha \in [-\infty, \infty]$. We say that a Borel measure $\mu$ on $\mathbb{R}^n$ is $\alpha$-concave if, for any Borel sets $A, B \subset \mathbb{R}^n$ and any $t \in [0, 1]$, one has

$$\mu((1-t)A + tB) \geq M_{\alpha}^{t}(\mu(A), \mu(B)).$$
where, for $a, b \geq 0$,

$$M^t_\alpha(a, b) = \begin{cases} 
((1-t)a^\alpha + tb^\alpha)^{1/\alpha} & \text{if } \alpha \neq 0, \pm \infty \\
 a^{1-t}b^t & \text{if } \alpha = 0 \\
 \max\{a, b\} & \text{if } \alpha = \infty \\
 \min\{a, b\} & \text{if } \alpha = -\infty,
\end{cases}$$

if $ab > 0$, and $M^t_\alpha(a, b) = 0$ if $ab = 0$. In the case when $\alpha = 0$ the measure $\mu$ is often referred to as a log-concave measure. Similarly, we say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $\alpha$-concave if, for any $x, y \in \mathbb{R}^n$ and every $t \in [0, 1]$, one has

$$f((1-t)x + ty) \geq M^t_\alpha(f(x), f(y)).$$

Moreover, when $\alpha = -\infty$, such functions are referred to as quasi-concave functions. Moreover, a bounded function $f : \mathbb{R}^n \to \mathbb{R}_+$ is quasi-concave if and only if its super-level sets

$$C_r(f) = \{x \in \mathbb{R}^n : f(x) \geq r\|f\|_\infty\}$$

are convex for every $0 \leq r \leq 1$.

A seminal extension of the Brunn-Minkowski inequality is the Borell-Brascamp-Lieb inequality, studied by Borell in [5] and by Brascamp and Lieb in [11].

**Theorem 1** (Borell-Brascamp-Lieb inequality). Let $\alpha$ be such that $\alpha \geq -\frac{1}{n}$ and $t \in [0, 1]$. Suppose that $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ is a triple of measurable functions that satisfy the condition

$$h((1-t)x + ty) \geq M^t_\alpha(f(x), g(y))$$

for every $x, y \in \mathbb{R}^n$. Then the following inequality holds true:

$$\int_{\mathbb{R}^n} h(x) dx \geq M^t_{\gamma\alpha} \left( \int_{\mathbb{R}^n} f(x) dx, \int_{\mathbb{R}^n} g(x) dx \right), \quad \gamma = \frac{\alpha}{1 + n\alpha}.$$

The case $\alpha = 0$ is referred to as the Prékopa-Leindler inequality and was explored by Prékopa in [35] and Leindler in [26].

Theorem 1 implies, in particular, that measures having $\alpha = \left(\frac{1}{n}\right)$-concave density must be $\left(\frac{1}{n+s}\right)$-concave for $s \leq -n$ and $s > 0$. The minimal function satisfying the condition (2) is the so-called $(t, s)$-supremal convolution, that is, the measurable function $h_{t,s} : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$h_{t,s}(z) = \sup_{z = (1-t)x + ty} M^{\frac{s}{t}}_\alpha(f(x), g(y))$$

$$= \sup_{z = (1-t)x + ty} \left[(1-t)f(x)^\frac{t}{s} + tg(y)^\frac{t}{s}\right]^s,$$

where $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are arbitrary measurable functions. Alternatively, we can express the $(t, s)$-supremal convolution via an operation of addition and scalar multiplication. Given two measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}_+$, we define $f \oplus_s g$ by

$$[f \oplus_s g](x) = \sup_{z = x + y} (f(x)^{1/s} + g(y)^{1/s})^s,$$

where the supremum is taken over all way to write $z = x + y$, with $x \in \text{supp}(f)$ and $y \in \text{supp}(g)$. Moreover, given a positive scalar $\alpha$ we define $\times_s$ by

$$[\alpha \times_s f](x) = \alpha^s f(x/\alpha).$$
Therefore, we really have $h_{t,s} = (1 - t) \times_s f \oplus_s t \times_s g$. Moreover, the function $h_{t,s}$ is $(\frac{1}{s})$-concave whenever $f, g$ are as well. Finally, whenever $f = 1_A$ and $g = 1_B$ are characteristic functions of measurable sets $A, B \subset \mathbb{R}^n$ with $(1 - t)A + tB$ also measurable, then

$$h_{t,s}(z) = 1_{(1-t)A+tB}(z).$$

In this sense, the $(t, s)$-supremal convolution extends naturally the Minkowski convex combination to the functional setting.

In [16] Firey extended the Minkowski combination in the setting of convex bodies $K^n_o$ (convex, compact subsets of $\mathbb{R}^n$ with the origin in their interiors), which has been named as the $L_p$-Minkowski convex combination (see, for example, [43]). Let $p \in [1, \infty)$ and $t \in [0, 1]$. Then, for any convex bodies $K, L \subset K^n_o$, and $\alpha, \beta > 0$, the $L_p$-Minkowski combination of $K$ and $L$ with respect to $\alpha$ and $\beta$ is defined to be the convex body $\alpha \cdot_p K + \beta \cdot_p L$ whose support function is given by

$$h_{\alpha \cdot_p K + \beta \cdot_p L}(x) = (\alpha h_K(x)^p + \beta h_L(x)^p)^{1/p},$$

where, for a convex body $K$, $h_K(x) = \max_{y \in K} \langle x, y \rangle$ is the support function of $K$. Note that when $p = 1$, equation (5) becomes the usual Minkowski convex combination. In [16] an extension of the Brunn-Minkowski inequality, often referred to as the $L_p$-Brunn-Minkowski inequality, was established: for any convex bodies $K, L \in K^n_o$, and for any $t \in [0, 1]$, one has

$$|(1 - t) \cdot_p K + t \cdot_p L|_n \geq M^n_p(|K|_n, |L|_n),$$

with equality only when $K$ and $L$ are dilates of one another.

This study, ignited by Firey, has been widely pushed forward by Lutwak in [29, 30]. Additionally, the famous log-Brunn-Minkowski conjecture of Böröczky, Lutwak, Yang, and Zhang [7] has been regarded as a result of Firey’s generalization. For more advances of the $L_p$-Brunn-Minkowski theory see also [6, 8, 9, 10, 12, 14, 25, 31, 32, 36, 40, 41, 42, 44, 45, 47, 48].

In [31] Lutwak, Yang, and Zhang extended the definition (5) to the collection of all measurable subsets of $\mathbb{R}^n$ that coincides with original definition due to Firey when the sets involved are convex bodies containing the origin in their interiors. Let $p \in [1, \infty)$ and $t \in [0, 1]$. Given non-empty measurable subsets $A$ and $B$ of $\mathbb{R}^n$, and $\alpha, \beta > 0$, the $L_p$-Minkowski combination of $A$ and $B$ with respect to $\alpha$ and $\beta$ is defined by

$$\alpha \cdot_p A + \beta \cdot_p B = \left\{ \alpha^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x + \beta^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} y : x \in A, y \in B, 0 \leq \lambda \leq 1 \right\}.$$

As a consequence of this extension of the $L_p$-Minkowski convex combination, Lutwak, Yang, and Zhang extended the inequality (6) to the setting of all non-empty measurable sets.

**Theorem 2** (Lutwak-Yang-Zhang). Let $p \in [1, \infty)$ and $t \in [0, 1]$. For any non-empty measurable subsets $A, B$ of $\mathbb{R}^n$ such that $(1 - t) \cdot_p A + t \cdot_p B$ is also measurable, one has

$$|(1 - t) \cdot_p A + t \cdot_p B|_n \geq M^n_p(|A|_n, |B|_n).$$

In addition, for $p > 1$, if $A, B \in K^n_o$, then equality occurs in (8) only when $A$ and $B$ are dilates of one another.

The first result of this paper is the following theorem which extends inequality (8) to the case of measures having $(\frac{1}{s})$-concave densities, with $0 \leq s < \infty$.

**Theorem 3.** ($L_p$-Brunn-Minkowski inequality for measures with $(1/s)$-concave densities) Let $p \in [1, \infty)$, $t \in [0, 1]$ and $s \in [0, \infty)$. Let $\mu$ be a measure given by $d\mu(x) = \phi(x)dx$, where
\( \phi : \mathbb{R}^n \rightarrow \mathbb{R}_+ \) is a \( \left( \frac{1}{\epsilon} \right) \)-concave function on its support. Then, for any Borel subsets \( A \) and \( B \) in \( \mathbb{R}^n \), one has
\[
\mu((1 - t) \cdot_p A +_p t \cdot_p B) \geq M^+_{n+s}(\mu(A), \mu(B)).
\]

We note that the interesting case of the above theorem is when \( 0 \leq s < \infty \) due to the fact that the case \( s = \infty \) follows immediately from the Borell-Brascamp-Lieb inequality (3) and the inclusion (see [31])
\[
(1 - t) \cdot_p A +_p t \cdot_p B \supset (1 - t)A + tB.
\]

Therefore, we develop the following improved \( L_p \)-Brunn-Minkowski inequality for the case \( s = \infty \), but before stating the result, we require the following definition from [37]. We say that set \( A \subset \mathbb{R}^n \) is weakly unconditional if, for every \( x = (x_1, \ldots, x_n) \in A \) and every \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n \), one has
\[
(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in A.
\]

**Theorem 4.** Let \( p \in (1, \infty), t \in [0, 1], \) and \( \mu = \mu_1 \times \cdots \times \mu_n \) be a product measure on \( \mathbb{R}^n \), where, for each \( i = 1, \ldots, n \), \( \mu_i \) is a measure on \( \mathbb{R} \) having a quasi-concave density with maximum at the origin. Then for any weakly unconditional measurable sets \( A, B \subset \mathbb{R}^n \), such that \((1 - t) \cdot_p A +_p t \cdot_p B \) is also measurable, one has
\[
\mu((1 - \lambda) \cdot_p A +_p \lambda \cdot_p B)^\frac{\varepsilon}{n} \geq (1 - \lambda)\mu(A)^\frac{\varepsilon}{n} + \lambda\mu(B)^\frac{\varepsilon}{n}.
\]

It is curious to know whether the above theorem can be proved, up to some constant independent of the dimension and the measure and sets involve, without the assumption of weak unconditional-ity on the measure and sets involved. We partially answer this question with the following theorem.

**Theorem 5.** Let \( p \geq 1, t \in [0, 1], \) and \( \mu \) be a log-concave measure on \( \mathbb{R}^n \). Then there exists a universal constant \( C > 1 \) such that, for any \( K, L \in K_{\infty}(o) \), one has
\[
\mu((1 - t) \cdot_p K +_p t \cdot_p L)^\frac{\varepsilon}{n} \geq \frac{1}{Cp} \left[ (1 - t)\mu(K)^\frac{\varepsilon}{n} + t\mu(L)^\frac{\varepsilon}{n} \right].
\]

In the same spirit, the above theorems hold true for functional counterparts (see Theorem 6, Theorem 7, and Theorem 12 below), in general, by taking functions of the form \( \phi \cdot 1_A \), where \( \phi \) is the density of an appropriately selected measure \( \mu \), and \( 1_A \) denotes the characteristic function of an appropriately selected Borel set \( A \).

The organization of the paper is as follows. In Section 2 we present \( L_p \)-versions of the Borell-Brascamp-Lieb inequalities from which we establish Theorem 3 and Theorem 4. In Sections 3 and 4 of these functional inequalities, we establish the aforementioned functional inequalities. In Section 5, detailed proof of the \( L_p \) functional counterparts of Minkowski’s first inequality for functions and \( L_p \)-isoperimetric type inequalities for measures are presented. Finally, in Section 6 we establish a functional counterpart of the Gardner-Zvavitch conjecture (see [15, 18, 20, 24], for example) and prove a partial case.

### 2. \( L_p \)-Borell-Brascamp-Lieb type inequalities

In this section we establish some \( L_p \)-Borell-Brascamp-Lieb type inequality for \( s \in [0, +\infty) \) (see Theorem 6) and an \( L_p \)-Prékopa-Leindler type inequality (Theorem 7), each of which, respectively, leads to Theorem 3 and Theorem 4.
Theorem 6. Let \( p \in [1, \infty) \), \( 0 \leq s < \infty \), and \( t \in [0, 1] \). Let \( f, g, h: \mathbb{R}^n \to \mathbb{R}_+ \) be a triple of bounded integrable functions. Suppose, in addition, that this triple satisfies the condition

\[
   h \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{\mu - 1}{p}} x + t^\frac{1}{p} \lambda^{\text{supp}(g)} y \right) \geq \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{\mu - 1}{p}} f(x)^{\frac{1}{p}} + t^\frac{1}{p} \lambda^{\text{supp}(g)} \right)^s
\]

for every \( x \in \text{supp}(f), y \in \text{supp}(g) \) and every \( \lambda \in [0, 1] \). Then the following integral inequality holds:

\[
   \int_{\mathbb{R}^n} h(x) \, dx \geq M^*_{\frac{p}{n+s}} \left( \int_{\mathbb{R}^n} f(x) \, dx, \int_{\mathbb{R}^n} g(x) \, dx \right).
\]

We postpone the proof of Theorem 6 until Section 3.

The minimal function \( h \) which satisfies the condition (12) shall be referred to as the \( L_p \)-supremal convolution; that is, the function \( h_{p,t,s}: \mathbb{R}^n \to \mathbb{R}_+ \) defined by

\[
   h_{p,t,s}(z) = \sup_{0 \leq \lambda \leq 1} \left( \sup_{z = (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{\mu - 1}{p}} x + t^{\frac{1}{p}} \lambda^{\text{supp}(g)} y} \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{\mu - 1}{p}} f(x)^{\frac{1}{p}} + t^{\frac{1}{p}} \lambda^{\text{supp}(g)} \right]^s \right),
\]

where the supremums combined together represent the supremum taken over

\[
   z \in (1 - t) \cdot \text{supp}(f) + t \cdot \text{supp}(g)
\]
as in definition (7), and we set \( h_{p,t,s}(z) = 0 \) otherwise. Note that this definition is well defined whenever \( f, g: \mathbb{R}^n \to \mathbb{R}_+ \) are measurable functions and \( s \in [-\infty, \infty] \); the limits \( s = 0, \pm \infty \) should be understood in the usual sense. Moreover, when \( p = 1 \), \( h_{p,t,s} \) recovers the usual \((t, s)\)-supremal convolution appearing in (4).

Alternatively, we can work with the following definition.

We require a few definitions. One is a different interpretation of the \( L_p \)-(t, s)-supremal convolution introduced above.

Definition 1. Let \( p \in [1, \infty), s \in [0, \infty] \), and \( \alpha, \beta > 0 \). Given non-negative functions \( f, g: \mathbb{R}^n \to \mathbb{R}_+ \), we define the addition \( \oplus_{p,s} \) of \( f \) and \( g \) as follows

\[
   [f \oplus_{p,s} g](z) = \sup_{0 \leq \lambda \leq 1} \left( \sup_{z = (1 - \lambda)^{(p-1)/p} x + \lambda^{(p-1)/p} y} \left( 1 - \lambda \right)^{(p-1)/p} f(x)^{1/p} + \lambda^{(p-1)/p} g(y)^{1/p} \right)^s,
\]

where the second supremum is taken over all ways to write \( z = (1 - \lambda)^{(p-1)/p} x + \lambda^{(p-1)/p} y \) with \( x \in \text{supp}(f) \) and \( y \in \text{supp}(g) \).

Additionally, given any scalar \( \alpha > 0 \), we define a scalar multiplication \( \times_{p,s} \) as

\[
   (\alpha \times_{p,s} f)(x) = \alpha^{s/p} f \left( \frac{x}{\alpha^{1/p}} \right).
\]

More generally, given \( \alpha, \beta > 0 \), we define the \((L_p, s)\)-combination of \( f \) and \( g \) with respect to the constants \( \alpha \) and \( \beta \) by \( \alpha \times_{p,s} f \oplus_{p,s} \beta \times_{p,s} g \).

Therefore, when \( \alpha = 1 - t \) and \( \beta = t \) for some \( t \in [0, 1] \), we have that

\[
   h_{p,t,s} = (1 - t) \cdot p_s f \oplus_{p,s} t \cdot p_s g.
\]

We begin with the following proposition.
**Proposition 1.** Let $p > 1$, $t \in [0, 1]$, $s \in [-\infty, \infty]$. Then we have that
\[ (1 - t) \cdot_{p,s} f \oplus_{p,s} t \cdot_{p,s} g \]
is $\left(\frac{1}{s}\right)$-concave whenever $f, g : \mathbb{R}^n \to \mathbb{R}_+$ are as well.

The proof follows the ideas of [4].

**Proof.** First, assume that $s \neq \pm \infty$. For each $0 \leq \lambda \leq 1$, we set
\[ u_\lambda(x, y) = \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} f(x)^{\frac{1}{2}} + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} g(y)^{\frac{1}{2}} \right]^s, \quad x \in \text{supp}(f), \quad y \in \text{supp}(g), \]
and for $z \in (1 - t)^p \text{supp}(f) + t \cdot p \text{supp}(g)$, let
\[ m(z) = [(1 - t) \cdot_{p,s} f \oplus_{p,s} t \cdot_{p,s} g](z) \]
\[ = \sup_{0 \leq \lambda \leq 1} \left( \sup_{x=(1-t)^{\frac{1}{p}} (1-\lambda)^{\frac{1}{p}}} \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} f(x)^{\frac{1}{2}} + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} g(y)^{\frac{1}{2}} \right]^s \right), \]
and $m(z) = 0$ otherwise.

For each fixed $\lambda \in [0, 1]$, we claim that the functions $u_\lambda$ is $\left(\frac{1}{s}\right)$-concave on $\text{supp}(f) \times \text{supp}(g)$.

Set $(x, y) = (1 - t')(x_1, y_1) + t'(x_2, y_2)$ for some $t' \in [0, 1]$, $x_1, x_2 \in \text{supp}(f)$ and $y_1, y_2 \in \text{supp}(g)$. Then we have
\[ u_\lambda(x, y) = \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} f((1 - t')x_1 + t'x_2)^{\frac{1}{2}} + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} ((1 - t')y_2 + t'y_2)^{\frac{1}{2}} \right]^s \]
\[ \geq \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} [(1 - t')f(x_1)^{\frac{1}{2}} + t'f(x_2)^{\frac{1}{2}}] + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} [(1 - t')g(y_1)^{\frac{1}{2}} + t'g(y_2)^{\frac{1}{2}}] \right]^s \]
\[ = [(1 - t') \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} f(x_1)^{\frac{1}{2}} + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} g(y_1)^{\frac{1}{2}} \right) \]
\[ + t' \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} f(x_2)^{\frac{1}{2}} + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} g(y_2)^{\frac{1}{2}} \right) ]^s \]
\[ = \left[ (1 - t')u_\lambda(x_1, y_1)^{\frac{1}{2}} + t'u_\lambda(x_2, y_2)^{\frac{1}{2}} \right]^s, \]
which means exactly that $u_\lambda$ is $\left(\frac{1}{s}\right)$-concave on its support for any fixed $\lambda \in [0, 1]$, as claimed.

Now fix a decomposition $z = (1 - t')z_1 + t'z_2$, with $z_1, z_2$ belonging to $(1 - t) \cdot_{p,s} \text{supp}(f) + t \cdot_{p,s} \text{supp}(g)$. Using truncation, if necessary, we may assume that all the functions involved are bounded. Then, given $\varepsilon > 0$, choose $x_1, x_2 \in \text{supp}(f)$, $y_1, y_2 \in \text{supp}(g)$, and $0 \leq \lambda \leq 1$ such that
\[ z_1 = (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} x_1 + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} y_1, \quad z_2 = (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p - 1}{p}} x_2 + t^{\frac{1}{p}} \lambda^{\frac{p - 1}{p}} y_2, \]
and such that
\[ m(z_1) \leq u_\lambda(x_1, y_1) + \varepsilon, \quad m(z_2) \leq u_\lambda(x_2, y_2) + \varepsilon. \]
Since each function $u_\lambda$ is $\left(\frac{1}{s}\right)$-concave, by setting $x = (1 - t')x_1 + t'x_2$ and $y = (1 - t')y_1 + t'y_2$, we see that
\[ u_\lambda(x, y) \geq M_x^{\frac{1}{s}}(u_\lambda(x_1, y_1), u_\lambda(x_2, y_2)) \geq M_x^{\frac{1}{s}}((h(z_1) - \varepsilon)_+, (h(z_2) - \varepsilon)_+). \]
Finally, we note that $(1 - t')x + t'y = (1 - t')z_1 + t'z_2 = z$, which implies that $u_\lambda(x, y) \leq h(z)$, completing the proof. The case $s = \pm \infty$ follows analogously.

Some remarks are in order.

**Remark 1.** Let $f, g : \mathbb{R}^n \to \mathbb{R}_+$ be measurable functions, $p \in [1, \infty)$, and $t \in [0, 1]$. 
(a) For any \( p > 1 \), one has \( h_{p,t,s}(z) \geq h_{1,t,s}(z) := h_{t,s}(z) \).

Moreover, one may apply Theorem 1 to obtain the following inequality (which is weaker than inequality (13))

\[
\int_{\mathbb{R}^n} h_{p,t,s}(x)dx \geq M_{\frac{1}{p},1} \left( \int_{\mathbb{R}^n} f(x)dx, \int_{\mathbb{R}^n} g(x)dx \right),
\]

whenever \( \frac{1}{s} = -\frac{1}{n} \).

(b) Let \( f = 1_A \) and \( g = 1_B \) be characteristic functions of Borel sets \( A, B \subset \mathbb{R}^n \), respectively. The \( L_p^{-}(t,s) \)-supremal convolution for \( p \geq 1 \) extends the \( L_p \)-Minkowski convex combination to the functional setting in the sense that

\[
h_{p,t,s}(z) = \sup_{0 \leq \lambda \leq 1} \left( \sup_{z=(1-t)\frac{t}{p} (1-\lambda) \frac{1}{p} A + t \lambda \frac{1}{p} y} \left[ (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} A(x) + t \frac{1}{p} \lambda (y) \right] \right)^{\frac{s}{p}}
\]

\[
= \sup_{0 \leq \lambda \leq 1} \left( \sup_{z=(1-t)\frac{t}{p} (1-\lambda) \frac{1}{p} A + t \lambda \frac{1}{p} y, x \in A, y \in B} \left[ (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} A(x) + t \frac{1}{p} \lambda (y) \right] \right)^{\frac{s}{p}}
\]

\[
= 1_{(1-t)A + tB}(z).
\]

To check the third equality, using (a), we see that

\[
h_{t,p,s}(z) \geq h_{t,1,s}(z) = 1_{(1-t)A + tB}(z) = 1.
\]

In order to check that \( h_{p,t,s}(z) \leq 1 \), it is enough to check that the function \( w(t,\lambda) = (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} + t \frac{1}{p} \lambda \frac{1}{p} \) is bounded from above by 1; this can be seen by checking values of \( w \) on the boundary of \([0,1] \times [0,1]\) and checking its derivative or using the Hölder inequality directly as

\[
\left[ (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} + t \frac{1}{p} \lambda \frac{1}{p} \right] \leq [(1-t) + t]^{1/p} [(1-\lambda) + \lambda]^{1/p} = 1.
\]

We are now prepared to prove the \( L_p \)-Borell-Brascamp-Lieb inequality for measures having \((1/s)\)-concave densities assuming that Theorem 6 holds true.

**Proof of Theorem 3.** Consider the functions

\[
f = 1_A \phi, \quad g = 1_B \phi, \quad h = 1_{(1-t)A + tB} \phi
\]

for Borel sets \( A \) and \( B \). We need to show that the above triple of functions satisfy condition (12) in order to give the proof of this theorem. Let \( z \in (1-t)\cdot \text{supp}(f) + pt \cdot \text{supp}(g) \) be arbitrary. Then \( z = (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} x + t \frac{1}{p} \lambda \frac{1}{p} y \) for some \( x \in \text{supp}(f), y \in \text{supp}(g) \), and some \( \lambda \in [0,1] \). Our goal is to check that

\[
h(z)^{\frac{t}{p}} \geq (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} f(x)^{\frac{t}{p}} + t \frac{1}{p} \lambda \frac{1}{p} g(y)^{\frac{t}{p}}
\]

\[
= (1-t)^{\frac{t}{p}} (1-\lambda)^{\frac{1}{p}} \phi(x)^{\frac{t}{p}} + t \frac{1}{p} \lambda \frac{1}{p} \phi(y)^{\frac{t}{p}}.
\]
Since \((1 - t)\frac{1}{p} (1 - \lambda) \frac{p-1}{p} \in [0, 1]\), using the concavity of \(\phi^\lambda\), we see that \(h\) must satisfy

\[
h(z)^{1/s} = h \left( \left(1 - t\right)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} y \right) \phi^{\lambda}
\]

\[
= \phi \left( \left(1 - t\right)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} y \right)^{\frac{1}{s}} \lambda^{\frac{p-1}{p}} \left( \left(1 - t\right)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} y \right)
\]

\[
\geq (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} \phi(x) + \left(1 - (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} \right) \phi \left( \frac{t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}}}{1 - (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}}} \cdot y \right)
\]

where, we have used the fact that

\[
\frac{t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}}}{1 - (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}}} \leq 1 \iff (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} \leq 1,
\]

as desired. Hence, applying Theorem 6 to the triple \(f, g, h\), we conclude that

\[
\mu((1 - t) \cdot_p A + t \cdot_p B) = \int_{\mathbb{R}^n} h(x) dx
\]

\[
\geq M^\phi \left( (\int_{\mathbb{R}^n} f(x) dx, \int_{\mathbb{R}^n} g(x) dx) \right)
\]

\[
= M^\phi (\mu(A), \mu(B)),
\]

completing the proof.

In order to establish Theorem 4, we require a Prékopa-Leindler type inequality. We begin with the following definition inspired by the work [37]. A function \(f : \mathbb{R}^n \to \mathbb{R}_+\) is weakly unconditional if, for any \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and any \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n\), one has

\[
f(\varepsilon x) = f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \geq f(x_1, \ldots, x_n) = f(x).
\]

Our goal is to establish the following Prékopa-Leindler type inequality.

**Theorem 7.** Let \(p > 1, t \in [0, 1]\), and \(\mu = \mu_1 \times \cdots \times \mu_n\) be a product measure on \(\mathbb{R}^n\), where, for each \(i = 1, \ldots, n\), \(\mu_i\) is a measure on \(\mathbb{R}\) having a quasi-concave density \(\phi : \mathbb{R} \to \mathbb{R}_+\) with maximum at origin. Let \(f, g, h : \mathbb{R}^n \to \mathbb{R}_+\) be a triple of measurable functions, with \(f, g\) weakly unconditional and positively decreasing, that satisfy the condition

\[
h((1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} x + t^{1/p} \lambda^{(p-1)/p} y) \geq f(x)^{(1-t)^{1/p}(1-\lambda)^{(p-1)/p}} g(y)^{t^{1/p} \lambda^{(p-1)/p}}
\]

for every \(x \in \text{supp}(f), y \in \text{supp}(g)\), and every \(0 < \lambda < 1\). The the following integral inequality holds:

\[
\int_{\mathbb{R}^n} h d\mu \geq \sup_{0 < \lambda < 1} \left\{ \left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} f^{\frac{(1-t)}{p}} d\mu \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g^{\frac{1}{p}} d\mu \right)^{\lambda} \right\}.
\]
As an immediate consequence of the above theorem, we get the following lemma.

**Lemma 1.** Let \( p \in (1, \infty), t \in [0, 1] \), and \( \mu = \mu_1 \times \cdots \times \mu_n \) be a product measure on \( \mathbb{R}^n \), where, for each \( i = 1, \ldots, n \), \( \mu_i \) is a measure on \( \mathbb{R} \) having a quasi-concave density with maximum at the origin. Then, for any weakly unconditional measurable sets \( A, B \subset \mathbb{R}^n \), such that \((1-t)_p A +_p t_\cdot_p B\) is also measurable, one has

\[
\mu((1-t)_p A +_p t_\cdot_p B) \geq \sup_{0 < \lambda < 1} \left\{ \left[ \left( \frac{1-t}{1-\lambda} \right)^{1-\lambda} \left( \frac{t}{\lambda} \right)^\lambda \right]^{\frac{\mu}{\mu(B)}} \mu(A)^{1-\lambda} \mu(B)^\lambda \right\}.
\]

**Proof.** Consider the functions \( f = 1_A, g = 1_B \), and \( h = 1_{(1-t)_p A +_p t_\cdot_p B} \). It is easy to check that the functions \( f \) and \( g \) are weakly unconditional, and that the the triple \( f, g, h \) satisfies the assumption (17) of Theorem 7, which yields the desired conclusion. \( \square \)

**Proof of Theorem 4.** We may assume that \( t \in (0, 1) \). Suppose that \( \mu(A) \mu(B) > 0 \) and set

\[
1 - \lambda = \frac{(1-t)\mu(A)^{\frac{\mu}{\mu(B)}}}{(1-t)\mu(A)^{\frac{\mu}{\mu(B)}} + t\mu(B)^{\frac{\mu}{\mu(B)}}},
\]

then

\[
\frac{1-t}{1-\lambda} = \frac{(1-t)\mu(A)^{\frac{\mu}{\mu(B)}} + t\mu(B)^{\frac{\mu}{\mu(B)}}}{\mu(A)^{\frac{\mu}{\mu(B)}}}, \quad \frac{t}{\lambda} = \frac{(1-t)\mu(A)^{\frac{\mu}{\mu(B)}} + t\mu(B)^{\frac{\mu}{\mu(B)}}}{\mu(B)^{\frac{\mu}{\mu(B)}}}.
\]

Consequently, applying Lemma 1, we see that

\[
\left[ \left( \frac{1-t}{1-\lambda} \right)^{1-\lambda} \left( \frac{t}{\lambda} \right)^\lambda \right]^{\frac{\mu}{\mu(B)}} \mu(A)^{1-\lambda} \mu(B)^\lambda = \left[ (1-t)\mu(A)^{\frac{\mu}{\mu(B)}} + t\mu(B)^{\frac{\mu}{\mu(B)}} \right]^{\frac{\mu}{\mu(B)}},
\]

which is the desired inequality in this case.

Without loss of generality, suppose now that \( \mu(B) = 0 \). Using the fact that \( A, B \) are weakly unconditional, we see that \( 0 \in B \), and moreover, \((1-t)_p A +_p t_\cdot_p B \supset (1-t)_p A \). Let \( \phi \) be a quasi-concave function with maximum at the origin which is the density of \( \mu \). Then

\[
\mu((1-t)_p A +_p t_\cdot_p B) \geq \mu((1-t)_p A) = \int_{(1-t)^{1/p} A} \phi(y) dy = (1-t)^{\frac{\mu}{p}} \int_A \phi((1-t)^{1/p} x) dx = (1-t)^{\frac{\mu}{p}} \int_A \phi( (1-t)^{\frac{1}{p} x_1}, \ldots, (1-t)^{\frac{1}{p} x_n} ) dx \geq (1-t)^{\frac{\mu}{p}} \int_A \phi(x_1, \ldots, x_n) dx = (1-t)^{\frac{\mu}{p}} \mu(A).
\]

Consequently, we see that

\[
\mu((1-t)_p A +_p t_\cdot_p B) \frac{\mu}{\mu(B)} \geq (1-t)\mu(A)^{\frac{\mu}{\mu(B)}} + t\mu(B)^{\frac{\mu}{\mu(B)}}.
\]

\( \square \)
3. PROOF OF THEOREM 6

The proof of Theorem 6 is adopted from the ideas of Klartag in [21] (see also [39]). The main idea is to replace the functions with bodies of revolution and apply inequality (8) to the bodies in order to derive the inequality (13) in Theorem 6. The proof is split into two cases: the case when \( s \) is a positive integer, and then the case when \( s \) is a positive rational number in lowest terms, and then by approximation.

We begin with the following proposition.

**Proposition 2.** Let \( p \in [1, \infty), s \in \mathbb{N}, \) and \( t \in [0, 1]. \) Let \( f, g, h : \mathbb{R}^n \to \mathbb{R}_+ \) be a triple of bounded integrable functions. Suppose, in addition, that this triple satisfies the condition

\[
(19) \quad h \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x + t^{\frac{p}{p-1}} \lambda^{\frac{p-1}{p}} y \right) \geq \left[ (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} f(x)^{\frac{1}{p}} + t^{\frac{p}{p-1}} \lambda^{\frac{p-1}{p}} g(y)^{\frac{1}{p}} \right]^s
\]

for every \( x \in \text{supp}(f), y \in \text{supp}(g) \) and every \( \lambda \in [0, 1]. \) Then the following integral inequality holds:

\[
(20) \quad \int_{\mathbb{R}^n} h(x) dx \geq M_p^{\frac{s}{p+s}} \left( \int_{\mathbb{R}^n} f(x) dx, \int_{\mathbb{R}^n} g(x) dx \right).
\]

Fix \( s \in \mathbb{N}. \) Given any bounded, integrable function \( w : \mathbb{R}^n \to \mathbb{R}_+, \) consider the body of revolution

\[
(21) \quad A_s(w) = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^s : x \in \text{supp}(w), \|y\| \leq w(x)^{\frac{1}{s}} \right\}.
\]

We remark that \( A_s(w) \) is a convex body if and only if \( w^{\frac{1}{s}} \) is a concave function on its support. Additionally, using Fubini’s theorem, we see that

\[
|A_s(w)|_{n+s} = |B_2^s|_s \int_{\mathbb{R}^n} w(x) dx,
\]

where \( B_2^s \) denotes the closed Euclidean unit ball in \( \mathbb{R}^s. \)

**Lemma 2.** Let \( p \geq 1, s \in \mathbb{N}, \) and \( t \in [0, 1]. \) Suppose that \( f, g, h : \mathbb{R}^n \to \mathbb{R}_+ \) are bounded integrable functions that satisfy the condition (19) of Proposition 2. Then

\[
(22) \quad A_p := (1 - t)^{\frac{1}{p}} A_s(f) +_p t^\cdot_p A_s(g) \subset A_s(h) \subset A_p(h),
\]

where \( h_{p,t,s} \) is the \( L_p^{-}(s, t) \)-supremal convolution introduced in equation (14).

**Proof.** The second inclusion appearing in (22) follows immediately as \( h \geq h_{p,t,s} \) by assumption (19). Therefore, we need only to establish the first inclusion. Let \( z \in A_p. \) Then \( z \) is of the form

\[
z = (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} z_1 + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} z_2
\]

for some \( z_1 \in A_s(f), z_2 \in A_s(g) \) and \( \lambda \in [0, 1]. \) Write \( z_1 = (x_1, y_1) \) with \( x_1 \in \text{supp}(f) \) and \( y_1 \in \mathbb{R}^s \) satisfying \( \|y_1\| \leq f(x_1)^{\frac{1}{s}}. \) Similarly, \( z_2 = (x_2, y_2) \) with \( x_2 \in \text{supp}(g) \) and \( y_2 \in \mathbb{R}^s \) and \( \|y_2\| \leq g(x_2)^{\frac{1}{s}}. \) Then, we have

\[
z = ((1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} x_1 + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} x_2, (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} y_1 + t^{\frac{1}{p}} \lambda^{\frac{p-1}{p}} y_2) = (\tilde{z}, \overline{z})
\]
where \( \tilde{z} = (1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} x_1 + t^{1/p} \lambda (p-1)/p x_2 \) belongs to \( (1 - t) \cdot p \supp(f) + p \cdot t \cdot p \supp(g) \), and with

\[
\| \tilde{z} \| = \|(1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} y_1 + t^{1/p} \lambda (p-1)/p y_2 \|
\leq (1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} \| y_2 \| + t^{1/p} \lambda (p-1)/p \| y_2 \|
\leq (1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} f(x_1)^{1/s} + t^{1/p} \lambda (p-1)/p g(x_2)^{1/s}
\leq h_{p,t,s}((1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} x_1 + t^{1/p} \lambda (p-1)/p x_2)^{1/s}
\leq h_{p,t,s}(\tilde{z})^{1/s},
\]

which yields the desired inclusion. \(\square\)

**Proof of Proposition 2.** Let \( f, g, h \) satisfy the condition (19). Then, combining the set inclusions (22) appearing in Lemma 2 with the \( L_p \)-Brunn-Minkowski inequality (8), we conclude that

\[
|A_s(h)|_{n+s} \geq |A_s(h_{p,t,s})|_{n+s} \geq M^t_{1/s}(|A_s(f)|_{n+s}, |A_s(g)|_{n+s}).
\]

Finally, combining the equalities

\[
|A_s(h)|_{n+s} = |B^s_2|_s \int_{\mathbb{R}^n} h, \quad |A_s(h_{p,t,s})|_{n+s} = |B^s_2|_s \int_{\mathbb{R}^n} h_{p,t,s},
\]

with

\[
|A_s(f)|_{n+s} = |B^s_2|_s \int_{\mathbb{R}^n} f, \quad |A_s(g)|_{n+s} = |B^s_2|_s \int_{\mathbb{R}^n} g,
\]

the inequality (23) directly yields inequality (20), as desired. \(\square\)

The next goal is to establish Theorem 6 in the case when \( s = \ell/m \) is a positive rational number in lowest terms, \( \ell, m \in \mathbb{N} \). Given any integrable function \( w: \mathbb{R}^n \rightarrow \mathbb{R}_+ \) and positive integer \( m \), consider the function \( \tilde{w}: (\mathbb{R}^n)^m \rightarrow \mathbb{R}_+ \) defined as the product of \( m \)-independent copies of \( w \), i.e.,

\[
\tilde{w}(x) = \tilde{w}(x_1, \ldots, x_m) := \prod_{i=1}^m w(x_i).
\]

Using the independent structure of the function \( \tilde{w} \), Fubini’s theorem yields

\[
\int_{(\mathbb{R}^n)^m} \tilde{w}(x) dx = |B^m_2|_m \left( \int_{\mathbb{R}^n} w(x) dx \right)^m.
\]

Moreover, we see that

\[
\supp(\tilde{w}) = \supp(w) \times \cdots \times \supp(w) =: \supp(w)^m,
\]

where, for a subset \( A \) of \( \mathbb{R}^n \), \( A^m = A \times \cdots \times A \subset (\mathbb{R}^n)^m \) denotes the Cartesian product of \( m \) copies of \( A \).

We require a few additional lemmas before returning to the proof of Theorem 6, the first of which establishes a relation between the \( L_p \)-Minkowski convex combination and Cartesian product as follows.

**Lemma 3.** Let \( p \in [1, \infty), m \in \mathbb{N}, \text{ and } \alpha, \beta > 0 \). Then, for any Borel sets \( A, B \subset \mathbb{R}^n \), one has

\[
[\alpha \cdot p A + \beta \cdot p B]^m \supset \alpha \cdot p A^m + \beta \cdot p B^m.
\]
Proof. For each \( \lambda \in [0, 1] \), set \( \alpha_{\lambda,p} = \alpha^{1/p} (1 - \lambda)^{(p-1)/p} \) and \( \beta_{\lambda,p} = \beta^{1/p} \lambda^{(p-1)/p} \). Observe that

\[
\alpha_{\lambda,p} A^m + \beta_{\lambda,p} B^m
= \{ \alpha_{\lambda,p}(x_1, \ldots, x_m) + \beta_{\lambda,p}(y_1, \ldots, y_m) : x_i \in A, y_i \in B, \text{ for all } 1 \leq \lambda \leq 1 \}
\]

\[
\subset \{ \alpha_{\lambda,p} x + \beta_{\lambda,p} y : x \in A, y \in B, \text{ for all } 0 \leq \lambda \leq 1 \}
\]

as desired.

Our next key lemma concerns, in some sense, the concavity conditions of the functions \( \tilde{w} \) defined above. Before we state the lemma, we will need the following version of Hölder’s inequality (see [19]).

**Lemma 4.** Given \( m \in \mathbb{N} \) and sequences of real numbers \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_m\} \), one has

\[
\left| \prod_{i=1}^m a_i \right| + \left| \prod_{i=1}^m b_i \right| \leq \left( \prod_{i=1}^m (|a_i|^m + |b_i|^m) \right)^{1/m}.
\]

As a result of the previous lemma, we obtain the following Corollary.

**Corollary 1.** Let \( t, \lambda \in [0, 1], p \in [1, \infty) \), and \( s = \ell/m \), with \( \ell, m \in \mathbb{N} \) in lowest terms. Given \( u, w : \mathbb{R}^n \to \mathbb{R}_+ \), and \( (x_1, \ldots, x_m), (y_1, \ldots, y_m) \in (\mathbb{R}^n)^m \), one has

\[
\prod_{i=1}^m \left[ (1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} u(x_i)^{1/s} + t^{1/p} \lambda^{(p-1)/p} w(y_i)^{1/s} \right]^{1/m}
\]

\[
\geq (1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} \prod_{i=1}^m u(x_i)^{1/\ell} + t^{1/p} \lambda^{(p-1)/p} \prod_{i=1}^m w(y_i)^{1/\ell}.
\]

**Proof.** The results follows immediately by taking the sequences \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_m\} \), with

\[
a_i = [(1 - t)^{1/p} (1 - \lambda)^{(p-1)/p}]^{1/m} u(x_i)^{1/\ell}, \quad b_i = [t^{1/p} \lambda^{(p-1)/p}]^{1/m} w(y_i)^{1/\ell}, \quad i = 1, \ldots, m
\]

in Lemma 4. □

Set \( s = \ell/m \), with \( \ell, m \in \mathbb{N} \) in lowest terms. Next, we consider bodies of revolution akin to those appearing in equation (21). Given any bounded integrable function \( w : \mathbb{R}^n \to \mathbb{R}_+ \) that is not identically zero, define

\[
B_s(w) = A_s(\tilde{w}) = \{ (x, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell : x \in \text{supp}(\tilde{w}), \| y \| \leq \tilde{w}(x)^{1/\ell} \}
\]

\[
= \left\{ (x_1, \ldots, x_m, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell : x_i \in \text{supp}(w), i = 1, \ldots, m, \| y \| \leq \prod_{i=1}^m w(x_i)^{1/\ell} \right\}.
\]

Notice that \( B_s(w) \) is a convex body if and only if the function \( \tilde{w} \) is \( (1/\ell) \)-concave on its support. Moreover, we have

\[
|B_s(w)|_{nm+\ell} = |B_s|^\ell \left( \int_{\mathbb{R}^n} w(x) dx \right)^m.
\]

We require one final lemma before we prove Theorem 6.
Lemma 5. Let $p \in [1, \infty)$, $s = \ell/m$, with $\ell, m \in \mathbb{N}$, and $t \in [0, 1]$. Let $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ be a triple satisfying the assumptions of Theorem 6, and let $h_{p,t,s}$ denote the $L_{p}^{-}(t,s)$-supremal convolution of $f$ and $g$. Then

$$(27) \quad B_p := (1 - t) \cdot_p B_s(f) +_p t \cdot_p B_s(g) \subset B_s(h_{p,t,s}) \subset B_s(h).$$

Proof. The second inclusion appear in (27) follows immediately since $h \geq h_{p,t,s}$. Therefore, it is enough to show that $B_p \subset B_s(h_{p,t,s})$.

We begin by remarking that $B_p$ is the set of all $(x, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell$ such that

$$x \in (1 - t) \cdot_p \text{supp}(\tilde{f}) +_p t \cdot_p \text{supp}(\tilde{g})$$

and

$$(28) \quad \|y\| \leq \sup \left\{ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} \tilde{f}(x_1) + t^{1/p} \lambda^{(p-1)/p} \tilde{g}(x_2) \right\}^{1/\ell},$$

where the supremum is taken twice: once over all ways to write $x = (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} x_1 + t^{1/p} \lambda^{(p-1)/p} x_2$ and once over all $0 \leq \lambda \leq 1$. That is, $B_p = A_{\ell}(h_{p,t,s})$.

Now, using definition (25) together with Lemma 3, we see that

$$B_s(h_{p,t,s})$$

$$= \left\{ (z, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell : z \in \text{supp}(h_{p,t,s}), \|y\| \leq h_{p,t,s}(z)^{1/\ell} \right\}$$

and

$$(29) \quad = \left\{ (z, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell : z \in [(1 - t) \cdot_p \text{supp}(\tilde{f}) +_p t \cdot_p \text{supp}(\tilde{g})]^m, \|y\| \leq h_{p,t,s}(z)^{1/\ell} \right\} \supset \left\{ (z, y) \in (\mathbb{R}^n)^m \times \mathbb{R}^\ell : z \in (1 - t) \cdot_p \text{supp}(\tilde{f}) +_p t \cdot_p \text{supp}(\tilde{g}), \|y\| \leq h_{p,t,s}(z)^{1/\ell} \right\}.$$

Therefore, to complete the proof, it is enough to compare the last equality appearing in (29) with the condition appearing in (28). For every $z \in (1 - t) \cdot_p \text{supp}(\tilde{f}) +_p t \cdot_p \text{supp}(\tilde{g})$ consider

$$\sup \left\{ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} \tilde{f}(x_1) + t^{1/p} \lambda^{(p-1)/p} \tilde{g}(y) \right\}^{1/\ell}$$

$$= \sup \left\{ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} \prod_{i=1}^{m} f(x_i)^{1/\ell} + t^{1/p} \lambda^{(p-1)/p} \prod_{i=1}^{m} g(y_i)^{1/\ell} \right\},$$

where the supremum is taken twice: once over all ways to write

$$z = (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} x + t^{1/p} \lambda^{(p-1)/p} y,$$

with $x \in \text{supp}(\tilde{f})$ and $y \in \text{supp}(\tilde{g})$ and once over all $0 \leq \lambda \leq 1$. With this, Corollary 1 implies that

$$\sup \left\{ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} \tilde{f}(x_1) + t^{1/p} \lambda^{(p-1)/p} \tilde{g}(y) \right\}^{1/\ell} \leq \sup \left\{ \prod_{i=1}^{m} \left[ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} u(x_i)^{1/s} + t^{1/p} \lambda^{(p-1)/p} w(y_i)^{1/s} \right]^{1/m} \right\}^{1/m}$$

$$\leq \prod_{i=1}^{m} \sup \left\{ \left[ (1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} u(x_i)^{1/s} + t^{1/p} \lambda^{(p-1)/p} w(y_i)^{1/s} \right]^{1/m} \right\}^{1/m}$$

$$= \prod_{i=1}^{m} h_{p,t,s}(1 - t)^{1/p}(1 - \lambda)^{(p-1)/p} x + t^{1/p} \lambda^{(p-1)/p} y \right\}^{1/(sm)}$$

$$= h_{p,t,s}(z)^{1/\ell}.$$
Consequently, if
\[ \|y\| \leq \sup \left\{ (1 - t)^{1/r}(1 - \lambda)^{(p-1)/r} \bar{f}(x_1) + t^{1/p} \lambda^{(p-1)/r} \bar{g}(x_2)^{1/\ell} \right\}, \]
that is, if \((x, y) \in B_p\), then \(\|y\| \leq h_{p,t,s}(x)^{1/\ell}\), so that \((x, y) \in B_s(h_{p,t,s})\), as desired. \(\square\)

We are now in a position to prove Theorem 6.

**Proof of Theorem 6.** Let \(s = \ell/m\) be a positive rational number in lowest terms and consider the bodies \(B_s(f), B_s(g), B_s(h)\). By applying Lemma 5, inequality (8), and using equality (26) we see that

\[
|B_s^\ell| \left( \int_{\mathbb{R}^n} h(x) dx \right)^m = |B_s(h)|_{nm+\ell} \geq M \frac{m}{n+m+\ell} (|B_s(f)|_{nm+\ell}, |B_s(g)|_{nm+\ell}) = |B_s^\ell| M \frac{m}{n+m+\ell} \left( \int_{\mathbb{R}^n} f(x) dx, \int_{\mathbb{R}^n} g(x) dx \right) = |B_s^\ell| M \frac{m}{n+m+\ell} \left( \int_{\mathbb{R}^n} f(x) dx, \int_{\mathbb{R}^n} g(x) dx \right),
\]

which is the desired result. The case of general \(s \geq 0\) follows from standard approximation. \(\square\)

4. PROOF OF THEOREM 7

In this section we prove Theorem 7 by induction on the dimension \(n\). We will make use of the next lemma first.

**Lemma 6.** Let \(t, \lambda \in [0, 1]\) and \(\mu\) be a measure on \(\mathbb{R}\) having a quasi-concave density \(\phi: \mathbb{R} \to \mathbb{R}_+\) with maximum at \(0\). Then, for any measurable sets \(A, B \subset \mathbb{R}^n\), such that \((1 - t)^{1/p}(1 - \lambda)^{\frac{1}{p-1}} A + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} B\) is also measurable, one has

\[
\mu \left( (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} A + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} B \right) \geq (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} \mu(A) + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} \mu(B).
\]

In particular,

\[
\mu((1 - t)^{1/p} A + t^{1/p} B) \geq \sup_{0 \leq \lambda \leq 1} \left[ (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} \mu(A) + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} \mu(B) \right].
\]

**Proof.** For \(r \in [0, 1]\), we set \(C_r(\phi) = \{ x \in \mathbb{R} : \phi(x) \geq r \| \phi \|_{\infty} \}\). We begin by showing that

\[
\left[ (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} A + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} B \right] \cap C_r(\phi)
\supset (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} [A \cap C_r(\phi)] + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} [B \cap C_r(\phi)]
\]

for any measurable sets \(A\) and \(B\).

Let \(z \in (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} [A \cap C_r(\phi)] + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} [B \cap C_r(\phi)]\). Then \(z\) is of the form

\[
z = (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} x + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} y
\]

for some \(x \in A \cap C_r(\phi)\) and \(y \in B \cap C_r(\phi)\). It is clear that \(z \in (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} A + t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} B\). Using the quasi-concavity of \(\phi\), we see that \((1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{1}{p-1}} x, t^{\frac{1}{p}} \lambda^{\frac{1}{p-1}} y \in C_r(\phi)\) if \(x, y \in C_r(\phi)\).
We observe that, again using definition of $C_r(\phi)$,
\[
\phi(z) = \phi \left( (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} x + t^\frac{1}{p} \lambda \frac{p-1}{r} y \right) \\
= \phi \left\{ (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} x \right. \\
+ \left[ 1 - (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} \right] \left( \frac{t^\frac{1}{p} \lambda \frac{p-1}{r}}{1 - (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r}} \cdot y \right) \right\}
\]
\[
\geq \min_{x \in A \cap C_r(\phi), y \in B \cap C_r(\phi)} \left\{ \phi(x), \phi \left( \frac{t^\frac{1}{p} \lambda \frac{p-1}{r}}{1 - (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r}} \cdot y \right) \right\}
\]
\[
\geq \min_{x \in A \cap C_r(\phi), y \in B \cap C_r(\phi)} \{ \phi(x), \phi(y) \}
\]
\[\geq r \|\phi\|_\infty,
\]
where we have used the fact that
\[
\frac{t^\frac{1}{p} \lambda \frac{p-1}{r}}{1 - (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r}} \leq 1
\]
from Hölder’s inequality and the fact that $\phi$ is quasi-concave with maximum at the origin.

Therefore, by applying Fubini’s theorem together with the one-dimensional Brunn-Minkowski inequality, we see that
\[
\mu \left( (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} A + t^\frac{1}{p} \lambda \frac{p-1}{r} B \right)
\geq \|\phi\|_\infty \int_0^1 \left| \left[ (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} A + t^\frac{1}{p} \lambda \frac{p-1}{r} B \right] \cap C_r(\phi) \right| \, dr
\geq \|\phi\|_\infty \int_0^1 \left| \left[ (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} [A \cap C_r(\phi)] + t^\frac{1}{p} \lambda \frac{p-1}{r} [B \cap C_r(\phi)] \right] \right| \, dr
\geq \left( (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} \mu(A) + t^\frac{1}{p} \lambda \frac{p-1}{r} \mu(B) \right),
\]
as claimed. Since $\lambda$ was selected arbitrarily, the final result also holds true, i.e.,
\[
\mu(\left( (1-t)^\frac{1}{p} A + t \cdot \frac{1}{p} B \right) \geq \sup_{0 \leq \lambda \leq 1} \left[ (1-t)^\frac{1}{p} (1-\lambda) \frac{p-1}{r} \mu(A) + t^\frac{1}{p} \lambda \frac{p-1}{r} \mu(B) \right].
\]

\[\square
\]

**Proof of Theorem 7.** The proof of the theorem is by induction on the dimension $n$. Without loss of generality we may assume that $f, g$ are bounded with $0 < \|f\|_\infty, \|g\|_\infty < \infty$ and $t \in (0, 1)$. Fix $\lambda \in (0, 1)$ arbitrarily.

Assume that $n = 1$. Multiplying the assumptions and conclusion of the theorem by constants $c_f, c_g, c_h$ with
\[
c_h = \sup_{0 \leq \lambda \leq 1} \left( c_f (1-t)^{1/p} (1-\lambda)^{(p-1)/p} (c_g)^{(1/p) \lambda (p-1)/p} \right),
\]
by taking $c_f = \|f\|_\infty^{-1}, c_g = \|g\|_\infty^{-1}$, and therefore
\[
c_h = \sup_{0 \leq \lambda \leq 1} \left( \|f\|_\infty^{-1} (1-t)^{1/p} (1-\lambda)^{(p-1)/p} \right)^{(1/p) \lambda (p-1)/p} \left( \|g\|_\infty^{-1} \right)^{-t^{1/p} \lambda (p-1)/p}.
\]
We assume that \( \|f\|_\infty = \|g\|_\infty = 1 \) without loss of generality. We claim that for any \( 0 \leq r \leq 1 \),
\[
\{ h \geq r \} \supset (1 - t)^{\frac{1}{p}} \frac{\mu_1 - 1}{p} \{ f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \} + t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} \{ g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \}.
\]
Indeed, for any \( x \in \left\{ f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \right\} \) and \( y \in \left\{ g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \right\} \), the hypothesis (17) of the theorem implies that
\[
h((1 - t)^{1/p}(1 - \lambda)^{(p - 1)/p}, x + t^{1/p} \lambda^{(p - 1)/p} x) \geq f(x) (1 - t)^{1/p}(1 - \lambda)^{(p - 1)/p} g(y) t^{1/p} \lambda^{(p - 1)/p} x \geq (f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}}(x))^{1 - \lambda} \lambda^{\frac{\mu_1 - 1}{p}} (g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}}(y))^{\lambda} \geq r^{1 - \lambda + \lambda} = r.
\]
Therefore, using Fubini’s theorem together with inequality (30) of Lemma 6, we see that
\[
\int_R h d\mu \geq \int_0^\infty \mu(\{ h \geq r \}) dr \\
\geq \int_0^1 \mu(\{ h \geq r \}) dr \\
\geq (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{\mu_1 - 1}{p}} \int_0^1 \mu \left( \{ f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \} \right) dr \\
+ t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} \int_0^1 \mu \left( \{ g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}} \geq r \} \right) dr \\
= (1 - \lambda) \left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}} \left( \int_R f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}}(x) d\mu(x) \right) + \lambda \left( \frac{1}{\lambda} \right)^{\frac{1}{p}} \left( \int_R g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}}(x) d\mu(x) \right) \\
\geq \left[ \left( \frac{1 - t}{1 - \lambda} \right)^{1 - \lambda} \lambda^{\frac{1}{p}} \left( \int_R f\left( \frac{1 - t}{1 - \lambda} \right)^{\frac{1}{p}}(x) d\mu(x) \right) \right]^{1 - \lambda} \left( \int_R g\left( \frac{1}{1 - \lambda} \right)^{\frac{1}{p}}(x) d\mu(x) \right)^\lambda,
\]
where, in the last step, we have used the arithmetic-geometric means inequality. This completes the proof in the case when \( n = 1 \).

Assume that the conclusion of the theorem holds in dimension \( n - 1 \) for some \( n \geq 2 \). Let \( \lambda \in [0, 1], y_0, y_1 \in \mathbb{R} \) be arbitrary. Set \( y_2 = (1 - t)^{1/p}(1 - \lambda)^{(p - 1)/p} y_0 + t^{1/p} \lambda^{(p - 1)/p} y_1 \). Consider the function \( f_{y_0}, g_{y_1}, h_{y_2} : \mathbb{R}^{n-1} \to \mathbb{R}_+ \) defined by
\[
f_{y_0}(x) = f(y_0, x), \quad g_{y_1}(x) = g(y_1, x), \quad h_{y_2}(x) = h(y_2, x).
\]
The functions and \( f_{y_0} \) and \( g_{y_1} \) are weakly unconditional and positively decreasing.

Let \( x_0, x_1 \in \mathbb{R}^{n-1} \) be arbitrary points belonging to the supports of \( f_{y_0} \) and \( g_{y_1} \), respectively, and set \( x_2 = (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{\mu_1 - 1}{p}} x_0 + t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} x_1 \). Using the hypothesis (17) placed on the triple of functions \( f, g, h \), we have that
\[
h_{y_2}(x_2) = h((1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{\mu_1 - 1}{p}} y_0 + t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} y_1, (1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{\mu_1 - 1}{p}} x_0 + t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} x_1) \\
\geq f(y_0, x_0)(1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{\mu_1 - 1}{p}} g(y_1, x_1) t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}} \\
= f_{y_0}(x_0)(1 - t)^{\frac{1}{p}}(1 - \lambda)^{\frac{\mu_1 - 1}{p}} g_{y_1}(x_1) t^{\frac{1}{p}} \lambda^{\frac{\mu_1 - 1}{p}}.
Consequently, as all quantities involved were selected arbitrarily, the triple of functions $f_{y_0}$, $g_{y_1}$, $h_{y_2}$ satisfy the hypothesis of the theorem in dimension $n - 1$, and so by the inductive hypothesis, we see that

$$\int_{\mathbb{R}^{n-1}} h_{y_2} d\bar{\mu} \geq \left[ \left( \frac{1 - t}{1 - \lambda} \right)^{1 - \lambda} \left( \frac{t}{\lambda} \right)^{\lambda} \right]^{(n-1) \cdot \frac{1}{p}} \left( \int_{\mathbb{R}^{n-1}} f_{y_0} \left( \frac{t}{1 - \lambda} \right)^{\frac{1 - \lambda}{p}} d\bar{\mu} \right)^{1 - \lambda} \left( \int_{\mathbb{R}^{n-1}} g_{y_1} \left( \frac{t}{\lambda} \right)^{\frac{1 - \lambda}{p}} d\bar{\mu} \right)^{\lambda},$$

where $\bar{\mu} = \mu_1 \times \cdots \times \mu_{n-1}$.

Now define the functions $F, G, H : \mathbb{R} \rightarrow \mathbb{R}^+$ in the following

$$H(y_2) = \left[ \left( \frac{1 - t}{1 - \lambda} \right)^{1 - \lambda} \left( \frac{t}{\lambda} \right)^{\lambda} \right]^{-(n-1) \cdot \frac{1}{p}} \int_{\mathbb{R}^{n-1}} h_{y_2}(x) d\bar{\mu}(x),$$

$$F(y_0) = \int_{\mathbb{R}^{n-1}} f_{y_0} \left( \frac{t}{1 - \lambda} \right)^{\frac{1 - \lambda}{p}} (1 - \lambda)^{\frac{1}{p}} d\bar{\mu}(x), \quad G(y_1) = \int_{\mathbb{R}^{n-1}} g_{y_1} \left( \frac{t}{\lambda} \right)^{\frac{1 - \lambda}{p}} (1 - \lambda)^{\frac{1}{p}} d\bar{\mu}(x).$$

It can be seen that $F$ and $G$ are weakly unconditional and positively decreasing functions as the functions $f_{y_0} \left( \frac{t}{1 - \lambda} \right)^{\frac{1 - \lambda}{p}}$ and $g_{y_1} \left( \frac{t}{\lambda} \right)^{\frac{1 - \lambda}{p}}$ remain weakly unconditional and positively decreasing. Then, as all quantities were selected arbitrarily, we see that

$$H \left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{1 - \lambda}{p}} y_0 + t^{\lambda} \lambda^{\frac{1 - \lambda}{p}} y_1 \right) \geq F(y_0)^{1 - \lambda} G(y_1)^{\lambda}$$

$$= F(y_0)^{\left( (1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{1 - \lambda}{p}} \right)^{\frac{1}{p}}} G(y_1)^{\left( t^{\lambda} \lambda^{\frac{1 - \lambda}{p}} \right)^{\frac{1}{p}}}.$$

Therefore, applying the one dimensional case, we see that

$$\int_{\mathbb{R}} H(y_2) d\mu_n(y_2) \geq \left[ \left( \frac{1 - t}{1 - \lambda} \right)^{1 - \lambda} \left( \frac{t}{\lambda} \right)^{\lambda} \right]^{\frac{1}{p}} \left( \int_{\mathbb{R}} \left( F(y_0)^{\left( t^{\lambda} \lambda^{\frac{1 - \lambda}{p}} \right)^{\frac{1}{p}}} \right) d\mu_1(y_0) \right)^{1 - \lambda}$$

$$\times \left( \int_{\mathbb{R}} \left( G(y_1)^{\left( t^{\lambda} \lambda^{\frac{1 - \lambda}{p}} \right)^{\frac{1}{p}}} \right) d\mu_1(y_1) \right)^{\lambda}.$$
that is,
\[
\int_{\mathbb{R}^n} h(x) d\mu(x) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} h_{y_2}(x) d\bar{\mu}(x) d\mu_2(y_2)
\geq \left[ \frac{1 - t}{1 - \lambda} \right]^{1-\lambda} \left( \frac{t}{\lambda} \right)^{\frac{n}{p}} \left( \int_{\mathbb{R}^n} F(y_0) d\mu_1(y_0) \right)^{1-\lambda} 
\times \left( \int_{\mathbb{R}} G(y_1) d\mu_n(y_1) \right)^\lambda 
= \left[ \frac{1 - t}{1 - \lambda} \right]^{1-\lambda} \left( \frac{t}{\lambda} \right)^{\frac{n}{p}} \left( \int_{\mathbb{R}^n} f^{\left( \frac{1}{1-\lambda} \right)}(x) d\mu(x) \right)^{1-\lambda} 
\times \left( \int_{\mathbb{R}^n} g^{\left( \frac{1}{1-\lambda} \right)}(x) d\mu(x) \right)^\lambda.
\]

Since all quantities involved were selected arbitrarily, the desired inequality follows.

\[\square\]

We noticed by using a similar proof, we can obtain the following theorem for the Lebesgue measure \( \mu \) in Theorem 7 which removes the weak unconditional condition on the functions involved.

**Theorem 8.** Let \( p > 1 \) and \( t \in [0, 1] \). Let \( f, g, h : \mathbb{R}^n \to \mathbb{R}_+ \) be a triple of measurable functions that satisfy the condition
\[
h((1 - t)^{1/p} (1 - \lambda)^{(p-1)/p} x + t^{1/p} \lambda^{(p-1)/p} y) \geq f(x)^{(1-t)^{1/p} (1-\lambda)^{(p-1)/p}} g(y)^{t^{1/p} \lambda^{(p-1)/p}}
\]
for every \( x \in \text{supp}(f), y \in \text{supp}(g), \) and every \( 0 < \lambda < 1 \). The following integral inequality holds:
\[
\int_{\mathbb{R}^n} h dx \geq \sup_{0 < \lambda < 1} \left\{ \left[ \frac{1 - t}{1 - \lambda} \right]^{\frac{n}{p}} \left( \int_{\mathbb{R}^n} f^{\left( \frac{1}{1-\lambda} \right)} dx \right)^{1-\lambda} \left( \int_{\mathbb{R}^n} g^{\left( \frac{1}{1-\lambda} \right)} dx \right)^\lambda \right\}
\]

5. **Applications**

5.1. **\( L_p \)-Minkowski’s first type inequalities.** In this section we will explore \( L_p \) versions of the Minkowski’s first inequality for non-negative functions, which gives rise to \( L_p \)-Minkowski first type inequalities for certain classes of measures (as discussed in the introduction). We require the following definition.

**Definition 2.** Let \( p \in [1, \infty), s \in [0, \infty], \) and \( F : \mathbb{R}_+ \to \mathbb{R} \) be an invertible differentiable function. We say that a measure \( \mu \) on \( \mathbb{R}^n \) is \( F(t) \)-concave with respect to the \( (L_p, s) \)-combination of functions belonging to some class \( \mathcal{A} \) of bounded non-negative \( \mu \)-integrable functions if, for any members \( f, g \) belonging to \( \mathcal{A} \) and any \( t \in [0, 1], \) one has that
\[
(31) \quad \int_{\mathbb{R}^n} \left[ ((1 - t) x_{p, s} f) \oplus_{p, s} (t x_{p, s} g) \right] d\mu \geq F^{-1} \left( (1 - t) F \left( \int_{\mathbb{R}^n} f d\mu \right) + t F \left( \int_{\mathbb{R}^n} g d\mu \right) \right).
\]

We require the following definition inspired by the works of Colesanti and Fraga in [13] and of Klartag in [21].
**Definition 3.** Let $\mu$ be a Borel measure on $\mathbb{R}^n$, $p \geq 1$, and $s \in [0, \infty]$. We define the $L^p_\mu$-surface area of a $\mu$-integrable function $f: \mathbb{R}^n \rightarrow \mathbb{R}_+$ with respect to a $\mu$-integrable function $g$ by

$$S_{\mu,p,s}(f, g) := \lim_{\varepsilon \to 0^+} \inf \frac{\int_{\mathbb{R}^n} f \oplus_{p,s} (\varepsilon \times_{p,s} g) d\mu - \int_{\mathbb{R}^n} f d\mu}{\varepsilon}.$$ 

When $\mu$ is the Lebesgue measure on $\mathbb{R}^n$ we will simply write $S_{p,s}$.

The main result of this section is the following theorem, which is a generalization of Theorem 3.8 in [27] (see also Theorem 4.1 in [46]).

**Theorem 9.** Let $p \in [1, \infty)$, $s \in [0, \infty]$, and $F: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable invertible function. Assume, first that $F(t)$-concave with respect to some class, $A$, of non-negative bounded $\mu$-integrable functions and the $(L^p_\mu, s)$-combination. Then the following inequality holds for any members $f, g$ of the class $A$:

$$S_{\mu,p,s}(f, g) \geq S_{\mu,p,s}(f, f) + \frac{F(\int_{\mathbb{R}^n} f d\mu) - F(\int_{\mathbb{R}^n} g d\mu)}{F'(\int_{\mathbb{R}^n} f d\mu)}.$$ 

In particular, when $\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} g d\mu$, we obtain the following isoperimetric type inequality:

$$S_{\mu,p,s}(f, g) \geq S_{\mu,p,s}(f, f).$$

**Proof.** Let $f, g \in A$. Since $f, g$ are $\mu$-integrable, without loss of generality, we may assume that $f, g$ are compactly supported.

Assume, first that $s \neq 0, +\infty$. According to the assumption (31), for any $\varepsilon > 0$ sufficiently small, we may write

$$\int_{\mathbb{R}^n} f \oplus_{p,s} (\varepsilon \times_{p,s} g)(x) d\mu(x) = \int_{\mathbb{R}^n} \left\{ \left[ (1 - \varepsilon) \times_{p,s} \left( \frac{1}{(1 - \varepsilon)^{s/p}} f \right) \right] \oplus_{p,s} (\varepsilon \times_{p,s} g) \right\} (x) d\mu(x) \geq F^{-1} \left\{ (1 - \varepsilon) F \left( (1 - \varepsilon)^{-s/p} \int_{\mathbb{R}^n} f(x) d\mu(x) \right) + \varepsilon F \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right) \right\}.$$ 

Define the function

$$G_{F,\mu,s,p}(\varepsilon) := F^{-1} \left[ (1 - \varepsilon) F \left( (1 - \varepsilon)^{-s/p} \int_{\mathbb{R}^n} f(x) d\mu(x) \right) + \varepsilon F \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right) \right].$$

When $\varepsilon = 0$ the equality above holds, and we have $G_{F,\mu,s,p}(0) = \int_{\mathbb{R}^n} f(x) d\mu(x)$. Therefore, we see that

$$S_{\mu,p,s}(f, g) \geq \lim_{\varepsilon \to 0^+} \frac{\int_{\mathbb{R}^n} f \oplus_{p,s} (\varepsilon \times_{p,s} g)(x) d\mu(x) - \int_{\mathbb{R}^n} f(x) d\mu(x)}{\varepsilon} \geq \lim_{\varepsilon \to 0^+} \frac{G_{F,\mu,s,p}(\varepsilon) - G_{F,\mu,s,p}(0)}{\varepsilon} = G_{F,\mu,s,p}'(0).$$
What remains is to complete $G'_{F, \mu, s, p}(0)$. Next, we notice that

$$
\frac{d}{d\varepsilon} \left[ (1 - \varepsilon)^{-s/p} \int_{\mathbb{R}^n} f(x) d\mu(x) \right] \bigg|_{\varepsilon=0} \\
= \frac{d}{d\varepsilon} \left[ \int_{\mathbb{R}^n} ((1 - \varepsilon)^{-1} \times_{p,s} f)(x) d\mu(x) \right] \bigg|_{\varepsilon=0} \\
= \lim_{\varepsilon \to 0^+} \frac{\int_{\mathbb{R}^n} \left[ (1 + \varepsilon + \varepsilon^2 + \cdots) \times_{p,s} f \right](x) d\mu(x) - \int_{\mathbb{R}^n} f(x) d\mu(x)}{\varepsilon} \\
= S_{\mu, p, s}(f, f).
$$

Finally, using the fact that, for any invertible differentiable function $F: \mathbb{R}_+ \to \mathbb{R}$, one has

$$
F^{-1}(a)' = \frac{1}{F'(F^{-1}(a))},
$$

we see that

$$
S_{\mu, p, s}(f, g) \geq G'_{F, \mu, s, p}(0) \\
= \frac{F \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right) - F \left( \int_{\mathbb{R}^n} \left[ (1 - \varepsilon) \times_{p,s} f \right](x) d\mu(x) \right) |_{\varepsilon=0}}{F' \left( \int_{\mathbb{R}^n} f d\mu \right)} \\
+ \frac{(1 - \varepsilon)S_{\mu, p, s} \cdot F' \left( \int_{\mathbb{R}^n} \left[ (1 - \varepsilon)^{-1} \times_{p,s} f \right](x) d\mu(x) \right) |_{\varepsilon=0}}{F' \left( \int_{\mathbb{R}^n} f d\mu \right)} \\
= S_{\mu, p, s}(f, f) + \frac{F \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right) - F \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right)}{F' \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)},
$$

as desired.

The arguments for $s = 0, \infty$ follow in the same manner.

We obtain a natural corollary of Theorem 9 for Lebesgue measure $\mu$ and for $s = \infty$ respectively.

**Corollary 2.** (1) Let $p \in [1, \infty)$, $s \in [0, \infty)$. Then, for any bounded integrable functions $f, g: \mathbb{R}^n \to \mathbb{R}_+$, one has

$$
S_{p, s}(f, g) \geq S_{p, s}(f, f) + \frac{\left( \int_{\mathbb{R}^n} f(x) dx \right)^{\frac{s}{n+s}} - \left( \int_{\mathbb{R}^n} g(x) dx \right)^{\frac{s}{n+s}}}{\frac{s}{n+s} \left( \int_{\mathbb{R}^n} f(x) dx \right)^{\frac{n+s}{n+s}}}. 
$$

In particular, when $\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} g(x) dx > 0$, we obtain the following isoperimetric type inequality:

$$
S_{p, s}(f, g) \geq S_{p, s}(f, f).
$$

(2) Let $p \in [1, \infty)$. When $s = +\infty$, and $\mu$ is any log-concave measure on $\mathbb{R}^n$, the following inequality holds:

$$
S_{\mu, p, \infty}(f, g) \geq S_{\mu, p, \infty}(f, f) + \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right) \log \left[ \frac{\int_{\mathbb{R}^n} g(x) d\mu(x)}{\int_{\mathbb{R}^n} f(x) d\mu(x)} \right].
$$

If, in addition, $\int_{\mathbb{R}^n} f d\mu = \int_{\mathbb{R}^n} g d\mu > 0$, then we obtain the following isoperimetric type inequality:

$$
S_{\mu, p, \infty}(f, g) \geq S_{\mu, p, \infty}(f, f).
$$
5.2. \( L_p \) isoperimetric type inequalities for general measures. In [29, 30] Lutwak studied the \( L_p \) versions of mixed volumes (see also [43] for more details) with regard to the volume functional, which lead to the definition of \( L_p \) surface area of convex bodies. Here we seek to establish the \( L_p \) versions of Minkowski’s first inequality for some classes of measures.

We begin with a definition for measure. Let \( p \geq 1 \) and \( F: \mathbb{R}_+ \to \mathbb{R} \) be a strictly increasing invertible differentiable function. We say that a non-negative measure \( \mu \) on \( \mathbb{R}^n \) (that is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^n \)) is \( F(t) \)-concave with respect to the \( L_p \)-Minkowski combination on some class of Borel sets if, for all Borel sets \( A, B \subset \mathbb{R}^n \) belonging to this class, and every \( t \in [0, 1] \), one has

\[
F(\mu((1-t) A +_p t B)) \geq (1-t)F(\mu(A)) + tF(\mu(B)).
\]

Following the definitions in [28] (see also [27, 37, 46]): given a measure \( \mu \) on \( \mathbb{R}^n \) which is \( F(t) \)-concave with respect to some class of Borel sets and members \( A, B \subset \mathbb{R}^n \) belonging to this class, we define the \( L_p \)-\( \mu \)-surface area of the set \( A \) with respect to the set \( B \) by

\[
V_{p,\mu}(A, B) = F'(1)\lim_{t \to 0^+} \frac{\mu(A +_p t B) - \mu(A)}{t}, \quad M_{p,\mu}(A) = \frac{1}{F'(1)} \frac{d}{dt} \bigg|_{t=0} \mu(\varepsilon A).
\]

Before stating the main result of this section, we first require the following lemma.

**Lemma 7.** Let \( p \geq 1 \), \( F: \mathbb{R}_+ \to \mathbb{R} \) be a strictly increasing differentiable function and \( \mu \) be a measure on \( \mathbb{R}^n \) that is absolutely continuous with respect to the Lebesgue measure. Suppose that \( \mu \) is \( F(t) \)-concave with respect to some class \( A \subset K_{(o)}^n \) and closed with respect to the \( L_p \)-Minkowski combination of its members. Then, for any members \( A, B \in A \), the maps

\[
\varepsilon \mapsto F(\mu(\varepsilon \cdot A)), \quad \varepsilon \mapsto F(\mu(A +_p \varepsilon \cdot B))
\]

are concave on \([0, \infty)\).

**Proof.** Let \( t \in [0, 1] \) and \( \varepsilon_1, \varepsilon_2 \geq 0 \). Since all sets involved are convex bodies containing the origin, it means \( A +_p [(1-t)\varepsilon_1 + t\varepsilon_2] \cdot B \in K_{(o)}^n \). Therefore, we may use the support function definition of the \( L_p \)-Minkowski combination. Notice that

\[
h_{A +_p [(1-t)\varepsilon_1 + t\varepsilon_2] \cdot B} = [h^p_A + [(1-t)\varepsilon_1 + t\varepsilon_2] h^p_B]^\frac{1}{p} = [(1-t) (h^p_A + \varepsilon_1 h^p_B) + t (h^p_A + \varepsilon_2 h^p_B)]^\frac{1}{p} = [(1-t) (h_{A +_p \varepsilon_1 \cdot B})^p + t (h_{A +_p \varepsilon_2 \cdot B})^p]^\frac{1}{p} = h_{(1-t)t[A +_p \varepsilon_1 \cdot B] + t[A +_p \varepsilon_2 \cdot B]}.
\]

Therefore, we see that

\[
A +_p [(1-t)\varepsilon_1 + t\varepsilon_2] \cdot B = (1-t) \cdot [A +_p \varepsilon_1 \cdot B] + t \cdot [A +_p \varepsilon_2 \cdot B].
\]

Using the fact that \( \mu \) is \( F(t) \)-concave with respect to the given class of convex bodies containing the origin, we obtain

\[
F(\mu(A +_p [(1-t)\varepsilon_1 + t\varepsilon_2] \cdot B)) = F(\mu((1-t) \cdot [A +_p \varepsilon_1 \cdot B] + t \cdot [A +_p \varepsilon_2 \cdot B])) \geq (1-t)F(\mu(A +_p \varepsilon_1 \cdot B)) + tF(\mu(A +_p \varepsilon_2 \cdot B)).
\]

In the same spirit, the proof of the other inequality assertion follows obviously.

We are now prepared to establish the following isoperimetric type inequality.
**Theorem 10.** Let $p \geq 1$, $F: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a strictly increasing differentiable function and $\mu$ be a measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure. Suppose that $\mu$ is $F(t)$-concave with respect to some class $\mathcal{A} \subset \mathcal{K}^n_{(0)}$ and closed with respect to the $L_p$-Minkowski combination of its members. Then for any $A, B \in \mathcal{A}$ and such that $\mu((1 - t) \cdot A + t \cdot B))$ is finite, one has

$$V_{p,F}(A, B) + F'(1)M_{p,F}^\mu(A) \geq \frac{F'(1)[F(\mu(B)) - F(\mu(A))]}{F'(\mu(A))} + \mu(A),$$

with equality only if and only if $A = B$.

The proof of the theorem follows from the ideas of the proof of the classical isoperimetric inequality (see [43, Theorem 7.2.1]).

**Proof.** Consider the function $f: [0, 1] \rightarrow \mathbb{R}_+$ defined by

$$f(t) = F(\mu((1 - t) \cdot A + t \cdot B)) - [(1 - t)F(\mu(A)) + tF(\mu(B))].$$

Since $\mu$ is $F(t)$-concave with respect to a class of convex bodies containing the origin and closed with respect to the $L_p$-Minkowski combination, the functions $f$ is concave and such that $f(0) = f(1) = 0$. Therefore, the right-derivative of $f$ at $t = 0$ exists (cf. [38, Theorem 23.1]) and, moreover,

$$\left.\frac{d^+}{dt}\right|_{t=0} f(t) \geq 0,$$

with equality if and only if $f(t) = 0$ for all $t \in [0, 1]$, i.e., if and only if inequality (33) holds for all $t \in [0, 1]$.

As

$$\left.\frac{d^+}{dt}\right|_{t=0} f(t) = F'(\mu(A)) \cdot \left.\frac{d^+}{dt}\right|_{t=0} \mu((1 - t) \cdot A + t \cdot B) + F(\mu(A)) - F(\mu(B)),$$

it suffices only to compute the right derivative at 0 of $\mu((1 - t) \cdot A + t \cdot B)$.

To this end, we notice that, by Lemma 7, the one-sided derivative of $\mu((1 - t) \cdot A)$ and $\mu(A + t \cdot B)$ at $t = 1$ and $t = 0$, respectively, exist. Set $g(r, s) = \mu(r \cdot A + s \cdot B))$. Then we have

$$\left.\frac{d^+}{dt}\right|_{t=0} \mu((1 - t) \cdot A + t \cdot B) = \left.\frac{d^+}{dt}\right|_{t=0} g\left(1 - t, \frac{t}{1 - t}\right) = \left.\frac{d^+}{dt}\right|_{t=0} \mu((1 - t) \cdot A) + \left.\frac{d^+}{dt}\right|_{t=0} \mu((1 + t) \cdot B) = M_{p,F}^\mu(A) - \frac{1}{F'(1)}M_{p,F}(A, B).$$

Therefore,

$$\left.\frac{d^+}{dt}\right|_{t=0} f(t) = F'(\mu(A)) \left[M_{p,F}(A) - \frac{1}{F'(1)}M_{p,F}(A) + \frac{1}{F'(1)}V_{p,F}(A, B)\right] + F(\mu(A)) - F(\mu(B)).$$

Combing the above inequality together with the fact that $\left.\frac{d^+}{dt}\right|_{t=0} f(t) \geq 0$ yields the inequality part of the theorem. The equality follows from the characterization of the equality case of $\mu((1 - t) \cdot A + t \cdot B)$ mentioned above. \qed
We draw the following consequences of Theorem 10. The first follows by combining Theorem 10 with Theorem 3 by taking the class of all convex bodies \( K_{(o)}^n \) and the functions \( F(t) = t^{\frac{2}{n+2}} \), when \( 0 \leq s < \infty \).

**Corollary 3.** Let \( p \in [1, \infty) \), \( t \in [0, 1] \), and \( s \in [0, \infty) \). Let \( \mu \) be a measure given by \( d\mu(x) = \phi(x)dx \), where \( \phi : \mathbb{R}^n \to \mathbb{R} \) is a \( \left( \frac{t}{s} \right) \)-concave function on its support. Then, for any convex bodies \( A, B \subset \mathbb{R}^n \) containing the origin such that \((1 - t) \cdot_p A + t \cdot_p B \) has finite \( \mu \)-measure, one has

\[
V_{p, \frac{2}{n+2}}^\mu(A, B) + \frac{p}{n+s} M_{p, \frac{n+2}{n+2}}^\mu(A) \geq \mu(B)^{\frac{n}{n+2}} \mu(A)^{1-\frac{n}{n+2}},
\]

with equality if and only if \( A = B \).

Next we take the function \( F(t) = t^\frac{2}{n} \) in Theorem 10 and combine this with Theorem 4

**Corollary 4.** Let \( p > 1 \), \( t \in [0, 1] \), and \( \mu = \mu_1 \times \cdots \times \mu_n \) be a product measure on \( \mathbb{R}^n \), where, for each \( i = 1, \ldots, n \), \( \mu_i \) is a measure on \( \mathbb{R} \) having a quasi-concave density \( \phi : \mathbb{R} \to \mathbb{R}_+ \) with maximum at origin. Then, for any weakly unconditional convex bodies \( A, B \) such that \((1 - t) \cdot_p A + t \cdot_p B \) has finite \( \mu \)-measure, one has

\[
V_{p, \frac{n}{n+2}}^\mu(A, B) + \frac{n}{p} M_{p, \frac{n+2}{n+2}}^\mu(A) \geq \mu(B)^{\frac{n}{n+2}} \mu(A)^{1-\frac{n}{n+2}},
\]

with equality if and only if \( A = B \).

6. A FUNCTIONAL COUNTERPART OF THE GARDNER-ZVAVITCH CONJECTURE

The study of Borell-Brascamp-Lieb type inequalities, and their connections to isoperimetric type problems lead to the following problem, which can be seen as a functional counterpart of the Gardner-Zvavitch conjecture (see [18]):

**Conjecture 1** (Gardner-Zvavitch Conjecture). Let \( \gamma_n \) denote the standard Gaussian measure on \( \mathbb{R}^n \) having density

\[
\phi(x) = \frac{1}{(2\pi)^{n/2}} e^{-x^2/2}.
\]

Is it true that for any convex bodies \( K, L \in K_{(o)}^n \), one has

\[
\gamma_n((1 - t)K + tL)^{\frac{n}{2}} \geq (1 - t)\gamma_n(K)^{\frac{n}{2}} + t\gamma_n(L)^{\frac{n}{2}}?
\]

The above question was recently shown to be true in [15] when \( K \) and \( L \) are taken to be origin symmetric. However, it was shown in [34] that, in general, the above conjecture fails without this symmetry assumption. It is curious to know whether or not for general convex bodies containing the origin, the Gardner-Zvavitch conjecture holds true up to some universal constant. This can be seen by the following Corollary of Theorem 5.

**Corollary 5.** Let \( t \in [0, 1] \). There exists some absolute constant \( C > 1 \) such that, for any \( K, L \in K_{(o)}^n \),

\[
\gamma_n((1 - t)K + tL)^{\frac{n}{2}} \geq \frac{1}{C} \left( (1 - t)\gamma_n(K)^{\frac{n}{2}} + t\gamma_n(L)^{\frac{n}{2}} \right).
\]

Motivated by the advances on the Gardner-Zvavitch conjecture, we ask the following functional counterpart, and answer it in some special cases.
**Conjecture 2** (Functional Gardner-Zvavitch conjecture). Let $p \in [1, \infty)$. Let $f, g: \mathbb{R}^n \to \mathbb{R}_+$ be centered integrable functions, and $\mu$ be measure on $\mathbb{R}^n$, and $t \in (0, 1)$. Assume that $h: \mathbb{R}^n \to \mathbb{R}_+$ is a measurable function such that

$$h((1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{n-1}{p}}x + t\lambda^{\frac{n-1}{p}}y) \geq f(x)(1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{n-1}{p}}g(y)$$

holds for all $x, y \in \mathbb{R}^n$ and for all $0 < \lambda < 1$. Does there exist an absolute constant $C \geq 1$ such that following inequality hold:

$$\int_{\mathbb{R}^n} h(x)d\mu(x) \geq \frac{1}{C^n} \left( (1-t) \left( \int_{\mathbb{R}^n} f(x)d\mu(x) \right)^{p/n} + t \left( \int_{\mathbb{R}^n} g(x)d\mu(x) \right)^{p/n} \right)^{n/p}$$

We answer the above conjecture in the following particular case when $p = 1$ and $C = 1$.

**Theorem 11.** Let $\mu = \mu_1 \times \cdots \times \mu_n$ be a product measure, where $\mu_i$ is a measure on $\mathbb{R}$ having quasi-concave function $\phi$ such that $\phi(0) = \|\phi\|_{\infty}$ for all $i \in \{1, \cdots, n\}$, and let $t \in (0, 1)$.

Suppose that $f, g: \mathbb{R}^n \to \mathbb{R}_+$ are measurable functions with each having maximum at the origin, and satisfying $\|f\|_{\infty} = \|g\|_{\infty}$, and whose super-level sets can be written as products of Borel subsets of $\mathbb{R}$ containing the origin. Then, for any measurable function $h: \mathbb{R}^n \to \mathbb{R}_+$ satisfying

$$h((1-t)x + ty) \geq \min\{f(x), g(y)\}$$

holds for every $x, y \in \mathbb{R}^n$ and any $t \in [0, 1]$, and with maximum at the origin, one has

$$\int_{\mathbb{R}^n} h(x)d\mu(x) \geq \left[ (1-t) \left( \int_{\mathbb{R}^n} f(x)d\mu(x) \right)^{1/n} + t \left( \int_{\mathbb{R}^n} g(x)d\mu(x) \right)^{1/n} \right]^n.$$

**Proof.** By replacing $f$ with $f/\|f\|_{\infty}$ and $g$ with $g/\|g\|_{\infty}$, we assume that $\|f\|_{\infty} = \|g\|_{\infty} = 1$ without loss of generality. Let $H_r, F_r, G_r$ denote the super level sets of $h, f, g$, respectively, that is.

$$H_r = \{x: h(x) \geq r\}, \quad F_r = \{x: f(x) \geq r\}, \quad G_r = \{x: g(x) \geq r\}$$

By the conditions that $f, g, h$ satisfy, one sees that, for all $0 \leq r < 1$,

$$H_r \supset (1-t)F_r + tG_r.$$

Now, using the fact that each $F_r, G_r$ are coordinate boxes, one can write

$$F_r = \prod_{i=1}^n I_{i,r} \quad G_r = \prod_{i=1}^n J_{i,r}$$

for some Borel sets $I_{i,r}, J_{i,r} \subset \mathbb{R}$ all containing the origin; consequently,

$$(1-t)F_r + tG_r = \prod_{i=1}^n [(1-t)I_{i,r} + tJ_{i,r}].$$

Using the fact that $\mu$ is a product measure together with Fubini’s theorem, we have that

$$\int_{\mathbb{R}^n} h(x)d\mu(x) \geq v \int_0^1 \mu(H_r)dr$$

$$\geq \int_0^1 \mu((1-t)F_r + tG_r)dr$$

$$= \int_0^1 \prod_{i=1}^n \mu_i((1-t)I_{i,r} + tJ_{i,r})dr.$$
Using Lemma 6, we have that, for each \( i = 1, \ldots, n \),
\[
\mu_i((1-t)I_{i,r} + tJ_{i,r}) \geq (1-t)\mu_i(I_{i,r}) + t\mu_i(J_{i,r}),
\]
and so
\[
\int_{\mathbb{R}^n} h(x) d\mu(x) \geq \int_0^1 \left( \prod_{i=1}^n [(1-t)\mu_i(I_{i,r}) + t\mu_i(J_{i,r})] \right) dr
\]
\[
= \prod_{i=1}^n \left( (1-t) \int_0^1 \mu_i(I_{i,r}) dr + t \int_0^1 \mu_i(J_{i,r}) dr \right),
\]
where, in the last step, we have used independence of the product. Finally, applying Minkowski’s inequality, one has:
\[
\left[ \int_{\mathbb{R}^n} h(x) d\mu(x) \right]^{1/n} \geq \left[ \prod_{i=1}^n \left( (1-t) \int_0^1 \mu_i(I_{i,r}) dr + t \int_0^1 \mu_i(J_{i,r}) dr \right) \right]^{1/n}
\]
\[
\geq (1-t) \left( \prod_{i=1}^n \int_0^1 \mu_i(I_{i,r}) dr \right)^{1/n} + t \left( \prod_{i=1}^n \int_0^1 \mu_i(J_{i,r}) dr \right)^{1/n}
\]
\[
= (1-t) \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)^{1/n} + t \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right)^{1/n},
\]
as desired. \( \square \)

In particular, due to the nature of the \( L_p-(t, s) \)-supremal convolution, we also obtain the following Corollaries.

**Corollary 6.** Let \( p \geq 1 \). Let \( \mu = \mu_1 \times \cdots \times \mu_n \) be a product measure, where \( \mu_i \) is a measure on \( \mathbb{R} \) having a quasi-concave density \( \phi \) such that \( \phi(0) = \|\phi\|_\infty \), and let \( t \in (0, 1) \). Suppose that \( f, g : \mathbb{R}^n \to \mathbb{R}_+ \) are measurable functions with each having maximum at the origin, and such that \( \|f\|_\infty = \|g\|_\infty \), and whose super-level sets can be written as a product of Borel subsets of \( \mathbb{R} \) each containing the origin. Then, for any measurable function \( h : \mathbb{R}^n \to \mathbb{R}_+ \) satisfying
\[
h \left( (1-t)^{1/p}(1-\lambda)^{(p-1)/p} x + t^{1/p} \lambda^{(p-1)/p} y \right) \geq (1-t)^{1/p}(1-\lambda)^{(p-1)/p} f(x) + t^{1/p} \lambda^{(p-1)/p} g(y)
\]
holds for every \( x \in \text{supp}(f), y \in \text{supp}(g) \), and every \( 0 \leq \lambda \leq 1 \), and such that \( h \) attains its maximum at the origin, one has
\[
\int_{\mathbb{R}^n} h(x) d\mu(x) \geq \left( (1-t) \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)^{1/n} + t \left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right)^{1/n} \right)^n.
\]

Combining the above corollary with Theorem 9, we obtain the next result.

**Corollary 7.** Let \( p \in [1, \infty) \) and \( s \in [0, \infty] \). Let \( \mu = \mu_1 \times \cdots \times \mu_n \) be a product measure, where \( \mu_i \) is a measure on \( \mathbb{R} \) having a quasi-concave density \( \phi \) such that \( \phi(0) = \|\phi\|_\infty \), and let \( t \in (0, 1) \). Suppose that \( f, g : \mathbb{R}^n \to \mathbb{R}_+ \) are measurable functions with each having maximum at the origin, and such that \( \|f\|_\infty = \|g\|_\infty \), and whose super-level sets can be written as products of Borel subsets of \( \mathbb{R} \) containing the origin. Then one has
\[
S_{\mu,p,s}(f, g) \geq S_{\mu,p,s}(f, f) + \frac{\left( \int_{\mathbb{R}^n} g(x) d\mu(x) \right)^{1/n} - \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)^{1/n}}{\frac{1}{n} \left( \int_{\mathbb{R}^n} f(x) d\mu(x) \right)^{\frac{n}{p}-1}}.
\]
If ∫_R^n f(x)dµ(x) = ∫_R^n g(x)dµ(x) > 0, we have the following isoperimetric type inequality

S_{µ,p,s}(f, g) ≥ S_{µ,p,s}(f, f).

Our next result concerns the following class of log-concave functions on R^n:

L^n = \{ f : R^n → R^+ : f(0) = ∥f∥∞, 0 < ∫ f < ∞, f is log-concave \}.

The result is the following theorem.

**Theorem 12.** Let p ≥ 1, t ∈ [0, 1], and f, g ∈ L^n. Then

\[
\left(\frac{1}{\|(1 - t) ∗_{p,∞} f ∗_{p,∞} t ∗_{p,∞} g\|_∞} \int_{R^n} (1 - t) ∗_{p,∞} f ∗_{p,∞} t ∗_{p,∞} g dx\right)^\frac{1}{p} ≥ \frac{1}{C^p} \left[ (1 - t) \left(\frac{1}{∥f∥_∞} \int_{R^n} f(x) dx\right)^\frac{1}{p} + t \left(\frac{1}{∥g∥_∞} \int_{R^n} g(x) dx\right)^\frac{1}{p}\right]
\]

where C > 1 is an absolute constant.

We are now in a position to prove Theorem 5.

**Proof of Theorem 5.** Let ϕ denote the density of the measure µ consider the functions f = ϕ · 1_K and g = ϕ · 1_L. Then

h = (1 - t) ∗_{p,∞} f ∗_{p,∞} t ∗_{p,∞} g = ϕ · 1_{(1 - t)K + tL}.

Then by applying inequality (34) to the triple of functions f, g, h, we obtain inequality (11), as desired.

□

Before proceeding to the proof of Theorem 12, we must introduce a notion of convex bodies associated to log-concave functions belonging to the class L^n. Let q > 0 and f ∈ L^n. Following [2, 3] we consider the following critical sets

K_q(f) = \{ x ∈ R^n : \left(\frac{1}{∥f∥_∞} \int_0^∞ qf(rx)r^{q-1}dr\right)^\frac{1}{q} ≤ 1 \}.

It was shown in [2] that K_q(f) is a convex body containing the origin for any q > 0 and any f ∈ L^n, and whose radial function ρ_{K_q(f)} : S^{n-1} → R^+ is given by

\[
ρ_{K_q(f)}(u) = \left(\frac{1}{∥f∥_∞} \int_0^∞ qf(ru)r^{q-1}dr\right)^\frac{1}{q}.
\]

Moreover, when q = n we see that:

\[
|K_n(f)|_n = \frac{1}{∥f∥_∞} \int_{R^n} f(x) dx.
\]
Indeed, integrating in polar coordinates, we see that
\[ |K_n(f)|_n = \int_{K_n(f)} dx = n|B_2^n| n \int_{S^{n-1}} \int_0^{{\rho K_n(f)}} r^{n-1} dr d\nu \]
\[ = |B_2^n| n \int_{S^{n-1}} \rho K_n(f)(u)^n du \]
\[ = |B_2^n| n \int_{S^{n-1}} \frac{n}{\|f\|_\infty} \int_0^\infty f(r u) r^{n-1} dr du \]
\[ = \frac{1}{\|f\|_\infty} \int_{\mathbb{R}^n} f(x) dx, \]
as claimed.

For \( q > 0 \) and \( f \in \mathcal{L}^n \), we consider the level-set
\[ L_n(f) = \{ x \in \mathbb{R}^n : f(x) \geq e^{-n} \|f\|_\infty \}. \]

We make use of the following lemma, originally due to Klartag and Milman in [22].

**Lemma 8.** Let \( f \in \mathcal{L}^n \). The following set inclusion holds
\[ K_n(f) \subset L_n(f) \subset C \cdot L_n(f), \]
where \( C > 1 \) is some universal constant.

**Proof of Theorem 12.** Let \( f, g \in \mathcal{L}^n \) and set
\[ h = (1 - t) \cdot p, \infty f \oplus p, \infty t \cdot p, \infty g \in \mathcal{L}^n, \]
and consider the bodies \( K_n(f), K_n(g), \) and \( K_n(h) \) associated to \( f, g, \) and \( h \), respectively. We notice that, if we can show the inclusion
\[ C \cdot K_n(h) \supset (1 - t) \cdot p, K_n(f) + p t \cdot K_n(g), \tag{37} \]
then, by combining the identity (36) together with the \( L_p \)-Brunn-Minkowski inequality (13), it would imply that
\[ \left( \frac{1}{\|h\|_\infty} \int_{\mathbb{R}^n} h(x) dx \right)^{\frac{1}{p}} = |K_n(h)|_p^{\frac{1}{p}} \]
\[ \geq \frac{1}{C_p} \left( (1 - t) |K_n(f)|_p^{\frac{1}{p}} + t |K_n(g)|_p^{\frac{1}{p}} \right) \]
\[ = \frac{1}{C_p} \left[ (1 - t) \left( \frac{1}{\|f\|_\infty} \int_{\mathbb{R}^n} f(x) dx \right)^{\frac{1}{p}} + t \left( \frac{1}{\|g\|_\infty} \int_{\mathbb{R}^n} g(x) dx \right)^{\frac{1}{p}} \right]. \]
Therefore, we need only to establish the inclusion (37).

Using Lemma 8, to establish the desired inclusion, it suffices to show that
\[ (1 - t) \cdot p, L_n(f) + p t \cdot L_n(g) \subset L_n(h). \]
Set \( \bar{f} = f/\|f\|_\infty \) and \( \bar{g} = g/\|g\|_\infty \). Let \( z \in (1 - t) \cdot p, L_n(f) + p t \cdot L_n(g) \). Then there exist \( x \in \text{supp}(f), y \in \text{supp}(g), \) and \( 0 \leq \lambda \leq 1 \) such that
\[ z = (1 - t) \frac{1}{p} (1 - \lambda) \frac{p - 1}{p} x + t \frac{1}{p} \lambda \frac{p - 1}{p} y, \quad \bar{f}(x) \geq e^{-n}, \quad \bar{g}(y) \geq e^{-n}. \]
Using these conditions, we see that
\[ \bar{f}(x)^{(1 - t) \frac{1}{p} (1 - \lambda) \frac{p - 1}{p}} \bar{g}(y)^{t \frac{1}{p} \lambda \frac{p - 1}{p}} \geq e^{-n} \left[ (1 - t) \frac{1}{p} (1 - \lambda) \frac{p - 1}{p} + t \frac{1}{p} \lambda \frac{p - 1}{p} \right] \geq e^{-n}, \]
where, in the last step, we have used Hölder’s inequality to conclude that
\[(1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}} + t \frac{1}{p} \lambda^{\frac{p-1}{p}} \leq 1.\]
Therefore, we see that \(z \in L_n((1 - t)^{\frac{1}{p}} \cdot t, \infty, \tilde{f} \oplus \tilde{g}).\)
Finally, to complete the proof, we observe that,
\[
h(z) = \sup_{0 \leq \lambda \leq 1} \left[ \sup_{(1 - t)^{\frac{1}{p}} x + t \lambda^{\frac{1}{p}} y} \left( f(x)^{(1 - t)^{\frac{1}{p}} (1 - \lambda)^{\frac{p-1}{p}}} g(y)^{t \lambda^{\frac{p-1}{p}}} \right) \right] \geq e^{-n} \sup_{0 \leq \lambda \leq 1} \left( \|f\|_{(1 - t)^{\frac{1}{p}} \cdot (1 - \lambda)^{\frac{p-1}{p}}} \cdot \|g\|_{t \lambda^{\frac{p-1}{p}}} \right) = e^{-n} \|h\|_{\infty}.
\]
Hence, the inclusion (37) holds, and the proof is complete.

\[\square\]

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