The deflection angle of a gravitational source with a global monopole in the strong field limit

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Abstract

We investigate the gravitational lensing effect in the strong field background around the Schwarzschild black hole with extremely small mass and solid deficit angle subject to the global monopole by means of the strong field limit issue. We obtain the angular position and magnification of the relativistic images and show that they relate to the global monopole parameter $\eta$. We discuss that with the increase of the parameter $\eta$, the minimum impact parameter $u_m$ and angular separation $s$ increase and the relative magnification $r$ decreases. We also find that $s$ grows extremely as the increasing parameter $\eta$ becomes large enough. The deflection angle will become larger when the parameter $\eta$ grows. The effect from the solid deficit angle is the dependence of angular position, angular separation, relative magnification and deflection angle on the parameter $\eta$, which may offer a way to characterize some possible distinct signatures of the Schwarzschild black hole with a solid deficit angle associated with the global monopole.

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1. Introduction

The deflection of electromagnetic radiation in a gravitational field leads the phenomena referred as gravitational lensing, while a massive object which brings about the detectable deflection is called a gravitational lens. Certainly gravitational lensing is an important application of general relativity [1]. The basic theory of gravitational lensing has been developed greatly and applied widely [2–7]. As a useful tool in astrophysics, gravitational lensing helps us to investigate the distant stars no matter whether they are bright or dim and to verify gravity theories.
The various topological defects include domain walls, cosmic strings and monopoles subject to the topology of the vacuum manifold produced during the vacuum phase transition in the early Universe [8, 9]. A global monopole is a spherical symmetric gravitational topological defect formed within the phase transition of a system consisting of a self-coupling scalar field triplet whose original global $O(3)$ symmetry is spontaneously broken to $U(1)$ and its gravitational effects lead to the peculiarity like deficit solid angle, which makes all light rays to be deflected by the same angle [10]. When a global monopole lives between an observer and a source, the observer will see two point images separated by an angle related to the deficit angle. More efforts have been contributed to the topics related to the global monopoles. It was shown that the global monopoles provide their relevant contributions to the vacuum polarizations in several kinds of backgrounds [11, 12]. The global monopoles were discussed in the spacetimes with a cosmological constant, and the relations between the models and spacetime topology were obtained [13, 14]. The models of composite monopoles have also been considered and the interactions between them were shown [15].

Gravitational lensing is a useful technique to probe the galaxies and the Universe [5, 6]. The technique can also help us to study dark matter [16, 17] and dark energy [18]. In general, the deflection angle of light passing close to a compact and massive source is expressed in integral forms, so it is difficult to discuss the detailed relations between the angle and the gravitational source. Alternatively we perform the calculation of the integral expressions in the limiting cases such as weak field approximation or strong ones. Most of the approaches of gravitational lensing have been developed in the weak field approximation with the first non-null term in the expansion of the deflection angle [5, 6]. When the light passes very close to a heavy compact body such as a black hole, the amount of the deflection of the light must be quite large because it depends on the mass. In this region the gravitational field is strong and an infinite series of images produced very close to the black hole can be observed. Here the weak field approximation is no longer valid, so strong gravitational lensing has attracted wide attention. For example, strong gravitational lensing was treated in a Schwarzschild black hole [19, 20], a gravitational source with naked singularities [21], a Reissner–Nordström black hole [22], a GMGHS charged black hole [23], a spinning black hole [24, 25], a braneworld black hole [26, 27], an Einstein–Born–Infeld black hole [28], a black hole in Brans–Dicke theory [29], a Barriola–Vilenkin monopole [30], the deformed Horava–Lifshitz black hole [31], etc. According to a COSMOS survey there could be strong-lensing candidates [32, 33]. As mentioned previously, the calculation for an integral expression is a burden, and the method to deal with the derivative in the strong gravitational field cases is necessary. It should be pointed out that Bozza put forward an analytical method which can help us to distinguish the nature of environment around various types of black holes. This method showed that there exists a logarithmic divergence of the deflection angle in the proximity of the photon sphere [34, 35] and has been improved [36–39].

In this paper we shall explore gravitational lensing on the Schwarzschild black hole swallowing a global monopole or equivalently the gravitational source with deficit solid angle. Although the gravitational lens equation for the Barriola–Vilenkin monopole was considered [30], it is necessary to study the influence from the monopole model parameters related to the deficit solid angle on the deflection angle in the case of massive object further and in detail. We wonder whether the deficit solid angle generates the other distinct features, besides it inspires two images mentioned in [10]. We plan to research on the deflection angle for light rays propagating in the background around the massive source with Barriola–Vilenkin monopole by means of Bozza’s device. We plot the dependence of the observational gravitational lensing parameters on the deficit solid angle from the global monopole. We compare the properties
of gravitational lensing in the case of a massive source containing the Barriola–Vilenkin monopole with those of Schwarzschild metrics. At last we list our results.

2. The deflection angle of a massive source with a global monopole

The Lagrangian density describing the simplest model of a global monopole is [9, 10]

\[ \mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \frac{1}{4}\lambda(\phi^a \phi^a - \eta^2)^2. \]  

(1)

Here \( \lambda \) and \( \eta \) are the parameters in this model. Introducing the ansatz of a self-coupling scalar field triplet like \( \phi^a = f(r)\hat{r}^a \) and coupling this matter field with the Einstein equations, a static and spherically symmetric black hole solution was given by the line element

\[ ds^2 = A(r) \, dt^2 - B(r) \, dr^2 - r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \]  

(2)

where

\[ A = B^{-1} = 1 - 8\pi G \eta^2 - \frac{2GM}{r} \]  

(3)

describing the background far from the monopole’s core. \( G \) is the Newton constant. The mass parameter is \( M \sim M_{\text{core}} \) and \( M_{\text{core}} \) is very small. The metric of this black hole has the event horizon

\[ r_h = \frac{2GM}{1 - 8\pi G \eta^2}. \]  

(4)

The parameter \( \eta \) in functions \( A(r) \) and \( B(r) \) will make a great deal of influence on gravitational lensing in the strong field limit.

According to [5, 6, 34], we choose that both the observer and the gravitational source lie in the equatorial plane with conditions \( \theta = \frac{\pi}{2} \). The whole trajectory of the photon is subject to the same plane. On the equatorial plane the metric (2) becomes

\[ ds^2 = A(r) \, dt^2 - B(r) \, dr^2 - C(r) \, d\phi^2 \]  

(5)

where

\[ C(r) = r^2. \]  

(6)

The deflection angle for the electromagnetic ray coming from a remote place can be expressed as a function of the closest approach:

\[ \alpha(r_0) = I(r_0) - \pi \]  

(7)

and

\[ I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{\sqrt{C(r)}\sqrt{\frac{C(r)A(r)}{C(r)A(r) - 1}}} \, dr \]  

(8)

where \( r_0 \) is the minimum distance from the photon trajectory to the source. That the denominator of expression (8) is equal to zero leads to the deflection angle to be divergent, which means that the photon is captured by the source instead of moving towards the observer. We call the largest root of equation \( \frac{C(r)}{C(r)A(r)} = \frac{A(r)}{A(r)} \) the radius of the photon sphere. The closest approach distant \( r_0 \) must be larger than the radius of photon sphere or the photon cannot escape from the gravitational source. The radius of the photon sphere in the global monopole metric can be given by

\[ r_m = \frac{3GM}{1 - 8\pi G \eta^2} \]  

(9)
and is larger than the event horizon \( (4) \). The photon sphere radius \( (9) \) will recover to that of Schwarzschild spacetime as the parameter \( \eta = 0 \). We wonder how the parameter \( \eta \) of the global monopole generates the effects on gravitational lensing. In [34] Bozza put forward an important expression of deflection angle for the metric which is assumed asymptotically flat. Having followed the procedure of [34], in the strong field limit we re-derive the deflection angle of the metric \( (5) \) whose element satisfies \( \lim_{r \to \infty} A(r) \neq 1 \). Here let \( l_0 = \lim_{r \to \infty} A(r) \) and follow the procedure of [34]. We can generalize Bozza’s work to denote the deflection angle in the strong field limit as follows:

\[
\alpha(\theta) = -\bar{a} \ln \left( \frac{\theta D_{OL}}{u_m} - 1 \right) + \bar{b} + O(u - u_m). \tag{10}
\]

In order to apply the expression to the case with a global monopole, we define some functions as follows:

\[
R(z, r_0) = \frac{2\sqrt{B(r)A(r)}}{C(r)A'(r)} \left( l_0 - A(r_0) \right) \sqrt{C(r_0)} \tag{11}
\]

\[
f(z, r_0) = \frac{1}{\sqrt{A(r_0) - [(l_0 - A(r_0))z + A(r_0)] \frac{C(r_0)}{C(r)}}} \tag{12}
\]

\[
f_0(z, r_0) = \frac{1}{\sqrt{\alpha z + \beta z^2}} \tag{13}
\]

\[
I_D(r_0) = \int_0^1 R(0, r_m) f_0(z, r_0) \, dz \tag{14}
\]

\[
I_R(r_0) = \int_0^1 \left[ R(z, r_0) f(z, r_0) - R(0, r_m) f_0(z, r_0) \right] \, dz \tag{15}
\]

where

\[
z = \frac{A(r) - A(r_0)}{l_0 - A(r_0)} \tag{16}
\]

\[
\alpha = \frac{l_0 - A(r_0)}{C(r_0)A'(r_0)} \left( C'(r_0)A(r_0) - C(r_0)A'(r_0) \right) \tag{17}
\]

\[
\beta = \frac{(l_0 - A(r_0))^2}{2C^2(r_0)A^3(r_0)} \left[ 2C_0C_0' A_0^2 + \left( C_0C_0'' - 2C_0'^2 \right) A_0A_0' - C_0C_0'A_0'' \right] \tag{18}
\]

with \( A_0 = A(r_0), C_0 = C(r_0). \) It should be pointed out that our results will recover to be those of [34] by Bozza if \( l_0 = 1 \). In equation (10) \( D_{OL} \) means the distance between the observer and the gravitational lens. By conservation of the angular momentum around the source, the closest approach distance is related to the impact parameter \( u \) defined as \( u = \sqrt{\frac{C(r_0)}{A(r_0)}}. \) Further, the minimum impact parameter corresponding to \( r_0 = r_m \) is certainly \( u_m = u |_{r_0=r_m}. \) The strong field limit coefficients \( \bar{a} \) and \( \bar{b} \) can be expressed as

\[
\bar{a} = \frac{R(0, r_m)}{2\sqrt{\beta_m}} \tag{19}
\]

\[
\bar{b} = -\pi + I_R(r_m) + \bar{a} \ln \frac{2\beta_m}{A(r_m)} \tag{20}
\]
where
\[ \beta_m = \beta \big|_{r_0 = r_m}. \]  
(21)

According to the Barriola–Vilenkin-monopole-type metric (3), here we find \( l_0 = 1 - 8 \pi G \eta^2 \).

According to the discussions above, we apply the device to the case of a massive source involving global monopole with metric (2) to obtain that
\[ R(z, r_0) = 2 \]  
(22)
\[ f(z, r_0) = \left[ -\frac{2GM}{r_0} z^3 - \left( 1 - 8 \pi G \eta^2 - \frac{6GM}{r_0} \right) z^2 + 2 \left( 1 - 8 \pi G \eta^2 - \frac{3GM}{r_0} \right) z \right]^{-\frac{1}{2}} \]  
(23)
\[ f_0(z, r_0) = \left[ - \left( 1 - 8 \pi G \eta^2 - \frac{6GM}{r_0} \right) z^2 + 2 \left( 1 - 8 \pi G \eta^2 - \frac{3GM}{r_0} \right) z \right]^{-\frac{1}{2}}. \]  
(24)

It is clear that the function \( R(z, r_0) \) is regular and the function \( f(z, r_0) \) diverges as \( z \) vanishes.

According to the definition above and equations (19) and (20), we derive the coefficients \( \bar{a}, \bar{b} \) and minimum impact parameter \( u_m \) of the deflection angle:
\[ \bar{a} = \frac{1}{\sqrt{1 - 8 \pi G \eta^2}} \]  
(25)
\[ \bar{b} = \frac{4}{\sqrt{1 - 8 \pi G \eta^2}} \ln(3 - \sqrt{3}) + \frac{1}{\sqrt{1 - 8 \pi G \eta^2}} \ln 6 - \pi \]  
(26)
\[ u_m = \left( \frac{3}{1 - 8 \pi G \eta^2} \right)^\frac{1}{2} GM. \]  
(27)

In the strong field limit the deflection angle of a massive source with deficit angle is
\[ \alpha(\theta) = -\frac{1}{\sqrt{1 - 8 \pi G \eta^2}} \ln \left( \frac{\theta D_{OL}}{3} \right) \frac{1}{GM} - 1 \]  
\[ + \frac{4}{\sqrt{1 - 8 \pi G \eta^2}} \ln(3 - \sqrt{3}) \]  
\[ + \frac{1}{\sqrt{1 - 8 \pi G \eta^2}} \ln 6 - \pi. \]  
(28)

All of the coefficients \( \bar{a}, \bar{b} \) and \( u_m \) are larger than those of the standard Schwarzschild black hole without a global monopole. Expressions (25)–(27) also show that these coefficients become larger as the global monopole parameter \( \eta \) increases. This enables us to distinguish a massive source with a global monopole from a Schwarzschild black hole in virtue of strong field gravitational lensing even if the two gravitational sources have the same mass.

In order to compare our results with the observable evidence, we relate the position and magnification to the strong field limit coefficients. The lens equation in the strong field limit reads [39]
\[ \beta = \theta - \frac{D_{LS}}{D_{OL}} \Delta \alpha \]  
(29)
where \( D_{LS} \) represents the distance between the lens and the source. Here \( D_{OS} = D_{OL} + D_{LS} \). \( \beta \) denotes the angular separation between the source and the lens. \( \theta \) is the angular separation between the lens and the image. Once we subtract all the loops made by the photon, the
offset of the deflection angle is expressed as $\triangle \alpha_n = \alpha(\theta) - 2n\pi$. Because of $u_m \ll D_{OL}$, the position of the $n$th relativistic image can be approximated as

$$\theta_n = \theta_0^n + \frac{u_m e_n (\beta - \theta_0^n) D_{OS}}{\bar{a} D_L D_{OL}} \quad (30)$$

where

$$e_n = e^{\frac{i\pi \eta n}{2}} \quad (31)$$

while the term on the right-hand side of equation (30) is much smaller than $\theta_0^n$, and we introduce $\theta_0^n$ as $\alpha(\theta_0^n) = 2n\pi$. The magnification of the $n$th relativistic image is the inverse of the Jacobian evaluated at the position of the image and is obtained as

$$\mu_n = \frac{1}{\left(\frac{\beta}{\bar{a}D_O} D_{OS}\right)} = \frac{u_m e_n (1 + e_n) D_{OS}}{\bar{a} \beta D_L D^2_{OL}} \quad (32)$$

In the limit $n \rightarrow \infty$, the asymptotic position approached by a set of images of $\theta_\infty$ relates to the minimum impact parameter as

$$u_m = D_{OL} \theta_\infty \quad (33)$$

As an observable the angular separation between the first image and the others is defined as

$$s = \theta_1 - \theta_\infty \quad (34)$$

where $\theta_1$ represents the outermost image in the situation that the outermost one is thought as a single image and all the remaining ones are packed together at $\theta_\infty$. As another observable, the ratio of the flux from the first image and the flux of all the other images is

$$r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n} \quad (35)$$

According to $e^{\frac{\pi}{a}} \gg 1$ and $e^{\frac{i\pi}{\bar{a}}} \sim 1$, the above formulae (34) and (35) can be simplified as

$$s = \theta_\infty e^{\frac{i\pi}{\bar{a}}} \quad (36)$$

$$r = e^{\frac{\pi}{\bar{a}}} \quad (37)$$

It is significant that the strong field limit coefficients such as $\bar{a}$, $\bar{b}$ and $u_m$ are directly connected to the observables such as $r$ and $s$. It is then possible for us to discern a massive object with a global monopole from the standard Schwarzschild black hole by means of strong field gravitational lensing.

3. Numerical estimation of observables in the strong field limit

It is necessary to estimate the numerical values of the coefficients and observables of gravitational lensing in the strong field limit to explore the influence from global monopole existing in the gravitational source. Our results will recover to those of the standard Schwarzschild black hole spacetime as $\eta = 0$ although the mass is tiny. We calculate these coefficients and observables according to the equations above and show their dependence on the model parameter $\eta$ relating to the global monopole in the figures. From figure 1 we discover that the increasing model parameter leads the minimum impact parameter $u_m$ to increase although the source mass may be extremely tiny. Figure 2 exhibits that the two
strong gravitational lensing coefficients $\bar{a}$ and $\bar{b}$ both become larger with the increase of model parameter $\eta$. In the case of a pure Schwarzschild black hole the two coefficients remain constant. These can help us to find whether the metric of the gravitational source has a solid deficit angle subject to the global monopole. Figures 3 and 4 also show that with the increase of parameter $\eta$, the angular separation $s$ becomes larger while the relative magnification $r$ decreases in the case of keeping the total mass constant. It should be pointed out that the angular separation $s$ increases extremely quickly as the parameter $\eta$ becomes sufficiently large. In figure 5 we demonstrate the relation between the deflection angle and the parameter $\eta$ for
Figure 3. The dependence of the angular separation $s$ in the Schwarzschild black hole with a global monopole on the model parameter $x = G\eta^2$ in unit of $\theta_\infty^\circ$; the asymptotic position is approached by a set of images.

Figure 4. The dependence of the relative flux $r$ in the Schwarzschild black hole with a global monopole on the model parameter $x = G\eta^2$.

Various values of $\frac{\partial \Omega_\text{sl}}{\partial M}$ according to equation (28), and the curves are similar. It is evident that the deflection angle becomes much greater when the increasing global monopole parameter $\eta$ is sufficiently large within the region $G\eta^2 \in (0, \frac{1}{\sqrt{2}})$ in figure 5. The model-dependent deflection angle is certainly larger than that of the standard Schwarzschild black hole. In all the figures mentioned above the deviations subject to the global monopole are evident, so strong gravitational lensing is efficient to explore the monopole. The influence from the global monopole is greater, which helps us to identify the nature of the massive source that has the deficit solid angle even when the mass is completely tiny.
4. Summary

In this paper we study gravitational lensing in the strong field limit in the Schwarzschild black hole spacetime with a solid deficit angle owing to a global monopole. The global monopoles contained by a massive source bring about strong influence on the observation. It is found that the global monopole leads the photon sphere radius of gravitational source to be larger. We indicate that in the case of negligible mass the increase of the global monopole model parameter $\eta$ leads the minimum impact parameter $u_m$ to increase and the relative flux $r$ to decrease. We find that the angular separation $s$ becomes larger as the parameter $\eta$ increases and increases extremely quickly as the increasing parameter $\eta$ becomes large enough. It should be pointed out that the deflection angle grows with the increase in the parameter $\eta$. The sufficiently large $\eta$ will lead much more manifest deflection of light. It is interesting that the deficit solid angle shows some other distinct characteristics besides two images presented in [10]. The effect from a solid deficit angle on the astronomical observations is manifest, which provides us a way to probe the spacetime property such as a solid deficit angle resulting from a global monopole. Gravitational lensing in the strong field limit is a potentially powerful tool to research on the nature of various black holes.

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