The procedure for the complete updating of immersive sets in one cutting plane method

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Abstract. In order to improve the practical implementation of the previously proposed cutting plane method, we have developed a procedure for updating immersive polyhedral sets in it. These sets successively approximate the epigraphs of some auxiliary functions that are being built at each step of the method. Each of the approximating sets is obtained by cutting off the current iterative point from the previous set by the support hyperplane. In connection with this, additional linear inequalities that form approximating sets are accumulated with an increase in the number of steps. This accumulation of additional constraints leads to an increase in the complexity of solving linear programming problems, which are set up in the method for finding iterative points. The procedure for updating approximating sets developed and included in the method allows periodically discarding accumulating cutting planes, which greatly simplifies the solution of auxiliary problems. This procedure is based on the introduced criterion for estimating the quality of the approximation of epigraphs. At those iterations, where the quality of the approximation becomes good enough, the mentioned updates occur. The convergence of the cutting plane method with the updating procedure included in it is proved. The implementation of the method is discussed.

1. Introduction
In practice, convex programming problems with nonsmooth objective functions often arise, in particular, in the design of technical systems (e.g., [1]). In this case, the constraint region of the original problem could be either polyhedral or of a general type. In the first case, it is convenient for the solution to use cutting methods with approximation of the epigraph of the objective function (e.g., [2 - 4]). The convenience lies in the fact that the iterative points in such methods are constructed by solving linear programming problems and, in addition, at each step it is possible to estimate the closeness of the current value of the objective function to the optimal value. But, if the constraint area has a general form, then the named methods are practically not used because of the complexity of the auxiliary problems of constructing approximations.

According to the references [5, 6], the authors proposed cutting methods that differ from those mentioned above in that they use the approximation of epigraphs not of an objective function, but of auxiliary functions built on the basis of external fines. The usage of such auxiliary functions made it possible to construct iterative points in the methods [5, 6] by solving linear programming problems, despite the general form of the admissible set.

However, the methods [5, 6] have the following significant disadvantage. From step to step, due to the cutting planes the number of additional constraints in the construction of iterative points increases...
unlimitedly, which affects the complexity of solving the auxiliary problems. The cutting method proposed here is close to the method [6], but in contrast to it is free from said disadvantage.

2. Problem statement

The minimization problem for a convex function \( f(x) \) defined on an \( n \)-dimensional Euclidean space \( R_n \) on a convex bounded closed set \( D \subset R_n \) is solved.

Assume that \( X^* = \text{Argmin} \{ f(x) : x \in D \} \), \( f^* = \min \{ f(x) : x \in D \} \), \( X^*_e = \{ x \in D : f(x) \leq f^* + e \} \), where \( e > 0 \), \( \text{epi}(g, U) = \{ (x, γ) \in R_{n+1} : x \in U, γ \geq g(x) \} \), \( U \subseteq R_n \), \( g(x) \) – the function defined in \( R_n \), \( W(z, Q) = \{ a \in R_{n+1} : \|a\| = 1, <a, u - z> \leq 0 \forall u \in Q \} \), where \( z \in R_{n+1}, Q \subset R_{n+1}, K = \{0, 1, \ldots \} \).

3. The cutting plane method

The proposed method of solving this problem generates sequences of approximations \( \{y_i\} \), \( i \in K \), and \( \{x_k\}, k \in K \), as follows.

We choose convex functions \( P_i(x), i \in K \), such that

\[
P_i(x) = 0 \forall x \in D, i \in K, 0 < P_i(x) \leq P_{i+1}(x) \forall x \in D, i \in K,
\]

and set

\[
F_i(x) = f(x) + P_i(x), i \in K.
\]

Select a point \( v \) from the interior of set \( \text{epi}(f, D) \). A convex bounded closed set \( G \subset R_n \) is constructed such that \( D \subset G \), and the convex closed set \( M_0 \subset R_{n+1} \) such that \( M \neq R_{n+1} \) and \( \text{epi}(F_0, R_n) \subset M_0 \). Define a numerical sequence \( \{ε_k\}, k \in K \), satisfying conditions

\[
ε_k \geq 0 \forall k \in K, ε_k \to 0, k \to \infty.
\]

Set \( i = 0, k = 0 \).

1. The point \( u_i = (y_i, γ_i) \in R_{n+1} \), is sought as the solution of the following problem:

\[
\min \{ γ : (x, γ) \in M_i, x \in R \}.
\]

If \( u \in \text{epi}(f, D) \), then \( y_i \in X^*_e \), and the process is over. If the relations \( γ_i \in D, f(y_i) - γ_i \leq e \), hold, then the inclusion \( y_i \in X^*_e \) holds.

2. Find \( v_i \in R_{n+1} \), as the intersection point of the interval \([v, u_i]\) with the boundary of the set \( \text{epi}(F_i, R_n) \).

3. A vector \( a_i \in W(v_i, \text{epi}(F_i, R_n)) \) is constructed.

4. Let \( T_i = \{ u \in R_{n+1} : <a, u - v_i> \leq 0 \} \),

\[
M_{i+1} = Q_i \cap T_i
\]

where \( Q_i = M_i \), if the inequality

\[
\|v_i - u_i\| \geq ε_k
\]

holds, and \( Q_i = M_0 \), if

\[
\|v_i - u_i\| < ε_k.
\]

5. If inequality (4) holds, then let \( i_k = i \),

\[
x_k = y_{i_k}.
\]

the value of \( k \) is incremented by one and we move to the next item. Otherwise, step 6 is immediately executed.

6. The value of \( i \) is incremented by one and we go to step 1.
Let us make some remarks concerning the method.

To begin with, we show that the admissible set of the problem (1) for each \(i \in K\) is nonempty. Namely, the point \(u^* = (x^*, f^*)\), where \(x^* \in X^*\), satisfies the constraints of this problem. Since \(x^* \in G\), then it suffices to prove the inclusion

\[
u^* \in M_i\]

for all \(i \in K\). Indeed, in view of the equality \(F(x^*) = f(x^*)\), \(i \in K\), the inclusion (6) is valid for \(i = 0\) by the choice of the set \(M_0\). Now let the inclusion (6) be satisfied for some \(i = l \geq 0\). In view of the fact that \(u^* \in \text{epi}(F_i, R_i)\), we have the inequality \(<a_i, u^* - v^* \leq 0\). Consequently, \(u^* \in T_i\). In addition, the inclusion \(u^* \in M_i\) holds by the induction hypothesis. But according to step 4 of the method, \(Q = M_i\) or \(Q = M_0\), i.e. in any case \(u^* \in Q_i\). From this and the equation (2), it follows that \(u^* \in M_{i+1}\), and the assertion is proved.

The inclusion (6) implies the following inequality

\[
g_i \leq f_i\]

(7)
on the basis of which, in particular, it is easy to prove the following optimality criterion.

Theorem 1. Let \(u_i \in \text{epi}(f, D)\) for some \(i \in K\). Then \(y_i \in X^*\).

Here we note that if the relations \(y_i \in D_i, f(y_i) \leq g_i + \varepsilon\) hold for the point \(u_t = (y_i, g_i)\), then, by the inequality (7), \(y_i \in X^*\).

Further, if \(G\) and \(M_0\) are chosen as polyhedral sets, then, taking into account the equality (2), the problems of constructing the points \(u_t\) for all \(i \in K\) are linear programming problems.

The methods for specifying penalty functions \(P_i(x)\), \(i \in K\), could be found, for example, by reference [7], and the method of constructing generalized support vectors \(a_i\) in the paper [8].

As noted earlier, the proposed method allows periodic updates of the sets \(M_i\), approximating the epigraphs of auxiliary functions, by discarding the accumulating additional constraints. Let us explain this in more detail.

Suppose that in the preliminary step of the method the sequence \(\{v_k\}\) is chosen so that \(v_k = 0\) for at least one \(k \geq 0\). Then, beginning with a certain number \(i' \geq 0\), for the points \(u_i, v_i\) the inequalities (3) will be satisfied. This means that for \(i \geq i'\) step 5 of the method will stop executing. As a result, only the sequence \(\{y_i\}\) will be constructed from the sequences \(\{v_i\}, \{x_i\}\), and if \(v_0 = 0\), then none of the points \(x_i\) will be fixed. As was proved in [6], the sequence \(\{y_i\}\) constructed in this way has the following property: any of its limit points belongs to the set \(X^*\).

At the same time, note that if for all \(i \geq i'\) condition (3) is satisfied, then

\[
M_{i+i} = M_i \cap T_i \forall i = i'.
\]

In this case, beginning with the number \(i'\), the number of additional constraints that define the cutting sets \(T_i\) increases unlimitedly step by step. Thus, for all \(i \geq i' \geq 0\), no updates of the sets \(M_i\) occur.

Suppose now that the sequence \(\{v_k\}\) is chosen with the condition

\[
ev_k > 0 \forall k \in K.
\]

(8)

As will be shown below, in the case (8), for the construction of \(\{u_k\}\), for each \(k \in K\) the number \(i = i_k\) will be fixed, for which the inequality (4) and the equality (5) are satisfied. For such numbers \(i = i_k\) according to step 4 of the metod \(Q_{i_k} = M_0 \cap M_{i_{k+1}} = M_0 \cap T_{i_k}\). Thus, updating of approximating sets occurs on iterations with the numbers \(i = i_k\).

4. Convergence of the method

Let us proceed to justification of the convergence of the proposed method. First of all, we prove that for each \(k \in K\) there is a number \(i = i_k\) for which the equality (5) holds. This proves the existence of a sequence \(\{v_k\}, k \in K\).
Lemma 1. If the sequences \( \{u_i\}, \{v_i\}, i \in K \), are constructed by the method with the condition (8), then for each \( k \in K \) exists a number \( i=k \in K \), for which inequality (4) holds.

Proof. 1) Let \( k=0 \). Let us prove the existence of a number \( i_0 \) such that

\[
\| v_{i_0} - u_{i_0} \| < e_0.
\]

Assume the opposite, i.e. for all \( i \in K \)

\[
\| v_i - u_i \| \geq e_0.
\]

(9)

Select a convergent subsequence \( \{u_i\}, i \in K' \subset K \), from the sequence \( \{u_i\}, i \in K \), and fix \( i', i'' \in K' \) such that \( i'' > i' \). According to the assumption (9) and step 4 of the method \( M_{i,j} = M_i \cap T_j \), for all \( i \in K \).

Therefore, \( M_{i,j} \subset M_{i,j+1} \) and \( M_{i,j} \in W(v_i, M_{i,j}) \). Then, taking into account the inclusion \( u_{i,j} \in M_{i,j} \), we have \( <a_{i,j}, u_{i,j} - v_i > \leq 0 \). But \( v_i = u_i + J_i(v - u_i) \), where \( J_i \in [0, 1], i \in K \). Hence,

\[
<a_{i,j}, u_{i,j} - v_i > \geq \| a_{i,j} \| = 1
\]

(10)

According to the preliminary step of the method \( v \in \text{int} \text{epi}(f, D) \), and according to step 2 \( v \notin \text{epi}(f, D) \) for all \( i \in K \). Therefore, we could choose (for example, [9], Lemma 1) such number \( \delta > 0 \) that

\[
<a_{i,j}, v - v_i > \leq -\delta \ \forall \ i \in K.
\]

Hence, taking into account that \( v - v_i = v - u_i - J_i(v - u_i) = (1 - J_i)(v - u_i), i \in K \), and \( 0 < 1 - J_i \leq 1 \), the inequality \( <a_{i,j}, v - v_i > \leq -\delta, i \in K \), holds. Then from the inequality (10) and the equality \( \| a_{i,j} \| = 1 \) we have

\[
\| u_{i,j} - u_{i,j} \| \geq J_i \delta.
\]

This and the convergence of the subsequence \( \{u_i\}, i \in K' \), imply the equality \( \lim_{i \in K'} J_i = 0 \). But \( v_i - u_i = J_i(v - u_i), i \in K' \), and the sequence \( \{\| v - u_i \|\}, i \in K' \), is bounded. Consequently,

\[
\lim_{i \in K'} \| v_i - u_i \| = 0,
\]

(11)

which contradicts the assumption (9). Thus, the existence of \( i_0 \) is proved.

2) Now let \( k = l \), where \( l \geq 0 \), and the existence of the number \( i_l \) is proved. Let us prove then that there exists an index \( i_{l+1} \geq i_l + 1 \), for which

\[
\| v_{i_{l+1}} - u_{i_{l+1}} \| < e_{i_{l+1}}.
\]

Thus the lemma will be proved. Assume the opposite, i.e. that for all \( i \geq i_l + 1 \) the inequality \( \| v_i - u_i \| \geq e_{i_{l+1}} \) holds. As in the first part of the lemma’s proof, it is easy to prove the inconsistency of this inequality, since according to step 4 of the method in the equality (2) \( Q_i = M_i \), for all \( i \geq i_l + 1 \), and for the convergent subsequence \( \{u_i\}, i \in K' \), equation (11) is satisfied. The lemma is proved.

Lemma 1 shows that under condition (8) for the sequence \( \{u_i\}, i \in K \), constructed by the method, its subsequence \( \{u_{i_k}\}, k \in K \), is also fixed. In view of this remark, we form the following auxiliary assertion.

Lemma 2. Let \( \{u_{i_k}\}, k \in K' \subset K \), be a convergent subsequence of the sequence \( \{u_i\}, i \in K \). Then the following inclusion holds for the limit point \( \overline{u} \) of this subsequence:

\[
\overline{u} \in \text{epi}(f, D).
\]

Proof. Extract from the sequence \( \{v_{i_k}\}, k \in K' \), a convergent subsequence \( \{v_{i_k}\}, k \in K'' \subset K' \), with the limit point \( \overline{v} \) and show that

\[
\overline{v} \in \text{epi}(f, D).
\]

(12)
Then the assertion of the lemma will be proved, since from the inequality (4) with allowance for the choice of \( \{e_i\}, k \in K \), it follows that \( \lim_{t \in K''} \| v_{i_k} - u_{i_k} \| = 0 \).

Let \( v_{i_k} = (w_k, \sigma_k), k \in K'' \), where \( w_k, \bar{w} \in R_n, \sigma, \bar{\sigma} \in R_1 \). According to the choice of points \( v_i \) in step 2 of the method \( F_{i_k}(w_k) \leq \sigma_k, k \in K'' \), or

\[
f(w_k) + P_{i_k}(w_k) \leq \sigma_k, k \in K''.
\] (13)

Taking into account the inequality \( P_{i_k}(w_k) \geq 0 \), it follows that \( f(w_k) \leq \sigma_k, k \in K'' \), and hence \( f(\bar{w}) \leq \bar{\sigma} \).

Thus, to prove the inclusion (12), it remains to show that

\[
\bar{w} \in D.
\] (14)

Since the sequences \( \{\sigma_i\}, \{f(w_i)\}, k \in K'' \), are bounded, the non-negative sequence \( \{P_{i_k}(w_k)\}, k \in K'' \), is also bounded by the assertion (13). And therefore

\[
P_{i_k}(w_k) \leq \alpha < +\infty \forall k \in K''.
\] (15)

Suppose now that the inclusion (14) does not hold. Then \( \lim_{t \in K''} P_{i_k}(\bar{w}) = +\infty \) by the choice of the sequence \( \{P_i(x)\}, i \in K \). Therefore, \( P_{i_k}(\bar{w}) > \alpha \) for some \( r \in K'' \). Select a neighborhood \( \omega \) of the point \( \bar{w} \) such that \( P_{i_k}(x) > \alpha \) for all \( x \in \omega \). Then, taking into account that \( P_i(x) > P_i(y) \) for all \( i \in K, x \in R_n \), we have

\[
P_{i_k}(x) > \alpha \forall x \in \omega, k \geq r, k \in K''.
\] (16)

We choose a number \( k' \) such that \( k' \in K'', k' \geq r, w_k \in \omega \). Then, in view of the statement (16), the inequality \( P_{i_k}(w_k) \geq \alpha \) is valid, which contradicts the assertion (15). Thus, the inclusion (14) is proved, and hence the inclusion (12) is proved too. Justification of the lemma is complete.

As already noted above, in the case (8), along with \( \{u_i\}, i \in K \), will be constructed the basic sequence of approximations \( \{x_i\}, k \in K \). State the following main result for it.

Theorem 2. Any limit point of the sequence \( \{x_k\}, k \in K \), belongs to the set \( X' \).

Proof. Let \( \{(x_i, y_{i_k})\}, k \in K' \subset K \), be a convergent subsequence of the sequence \( \{(x_i, y_{i_k})\}, k \in K \), and let \( (\bar{x}, \bar{y}) \) be its limit point. By (5), it follows that

\[
u_{i_k} = (x_i, y_{i_k}), k \in K'.
\]

And by Lemma 2 implies the inclusion \( (\bar{x}, \bar{y}) \in epi(f, D) \). Hence,

\[
\bar{x} \in D, \bar{y} \geq f(\bar{x})
\]

But according to the inequality (7) \( \bar{y} \leq f' \). Therefore \( f(\bar{x}) \geq f' \) and at the same time \( f(\bar{x}) \leq \bar{y} \leq f' \). Consequently, \( f(\bar{x}) = f' \), and the theorem is proved.

Numerical experiments showed that in the process of solving test problems many of the iterative points \( y_i \) fell into the domain \( D \). Especially in the case when the set \( G \) well approximated a set of constraints. At such iterations, there is an estimate of the closeness of values \( f(y_i) \) to the optimal value \( f' \), because in view of the inequality (7) \( y_i \leq f' \leq f(y_i) \).

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