GYSIN-$(\mathbb{Z}/2\mathbb{Z})^d$-FUNCTORS

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Abstract

Let $d \geq 1$ be an integer and $K_d$ be a contravariant functor from the category of subgroups of $(\mathbb{Z}/2\mathbb{Z})^d$ to the category of graded and finite $\mathbb{F}_2$-algebras. In this paper, we generalize the conjecture of G. Carlsson [C3], concerning free actions of $(\mathbb{Z}/2\mathbb{Z})^d$ on finite CW-complexes, by suggesting, that if $K_d$ is a Gysin-$(\mathbb{Z}/2\mathbb{Z})^d$-functor (that is to say, the functor $K_d$ satisfies some properties, see 2.2), then we have:

$$(C_d) : \sum_{i \geq 0} \dim_{\mathbb{F}_2}(K_d(0))^i \geq 2^d$$

We prove this conjecture for $1 \leq d \leq 3$ and we show that, in certain cases, we get an independent proof of the following results (for $d = 3$ see [C4]):

If the group $(\mathbb{Z}/2\mathbb{Z})^d$, $1 \leq d \leq 3$, acts freely and cellularly on a finite CW-complex $X$, then

$$\sum_{i \geq 0} \dim_{\mathbb{F}_2}H^i(X; \mathbb{F}_2) \geq 2^d.$$ 

1. Introduction.

Since the work of Paul A. Smith around 1938 [Sm] (see also [MB]) known as "Smith theory" the following problem has been posed.

$(\mathcal{P}_{d,k})$: Which group $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly on a product of $k$ spheres?

The case $k = 1$, which is easy, was proved since 1935 ([Sm], [M] and [MTW]); the result is that, if $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely on the sphere $S^n$ then $d \leq 1$.

The case $k = 2$ has been proved by A.Heller [He] in 1959 using a combinatorial method which, apparently, doesn’t extend to the case of three spheres (see [DV]). The result is that, if $(\mathbb{Z}/2\mathbb{Z})^d$ acts freely and cellularly on the product of two spheres $S^{n_1} \times S^{n_2}$, then $d \leq 2$.

Some works concerning the problem $(\mathcal{P}_{d,k})$ (such as [Co]) allow to have a generalization. The following statement has been conjectured by Benson-Carlson [BC]:

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(C_{d,S}): The group \((\mathbb{Z}/2\mathbb{Z})^{d+1}\) doesn't act freely and cellurally on a product of \(d\) spheres, \(d \geq 1\).

The conjecture \((C_{3,S})\) was proved by G. Carlsson in 1987 [C4].

Among other works concerning the conjecture \((C_{d,S})\), we can cite [AB], [C2], [Han], [OY] and [R]. Carlsson’s work [C2] concerns the case where the spheres have the same dimension and the action of the group \((\mathbb{Z}/2\mathbb{Z})^{d}\) on homology is trivial. The work of Adem-Browder [AB] concerns the case of the group \((\mathbb{Z}/p\mathbb{Z})^{d}\), \(p\) an odd prime.

In the middle of 1980s the conjecture \((C_{d,S})\) was generalized by G. Carlsson [C3] (and S. Halperin [Hal] for Torus) who suggest the following ”Halperin-Carlsson” Conjecture (it is also called ”toral rank conjecture” in some literature).

\((C_{d,X}): \) Let \(X\) be a finite CW-complex on which the group \((\mathbb{Z}/2\mathbb{Z})^{d}\) acts freely and cellurally . Then, \(\sum_{i \geq 0} \dim_{\mathbb{F}_2} H^i(X; \mathbb{F}_2) \geq 2^d\)

In this paper we generalize the conjecture \(C_{d,X}\) in the following sense which will be more precise in paragraph 2.2. Let’s call a Gysin-\((\mathbb{Z}/2\mathbb{Z})^{d}\)-functor a contravariant functor \(K_{(\mathbb{Z}/2\mathbb{Z})^{d}}\), or \(K_{d}\) for simplicity, from the category of subgroups of \((\mathbb{Z}/2\mathbb{Z})^{d}\) to the category of graded, finite and unitary \(\mathbb{F}_2\)-algebras such that:

- For every subgroup \(W\) of \((\mathbb{Z}/2\mathbb{Z})^{d}\), the graded, finite and unitary \(\mathbb{F}_2\)-algebra \(K_{d}(W)\) is non trivial and is an \(H^*(W; \mathbb{F}_2)\)-algebra,

- For every subgroup \(W\) of \((\mathbb{Z}/2\mathbb{Z})^{d}\) and for every \(U\) a subgroup of \(W\) of codimension one, there exist an exact sequence of \(H^*(W; \mathbb{F}_2)\)-modules of the form:

\[
\ldots \longrightarrow K_{d}(W)^{*-1} \overset{i}{\longrightarrow} K_{d}(W)^* \overset{\psi}{\longrightarrow} K_{d}(U)^* \overset{\pi^*}{\longrightarrow} K_{d}(W)^* \overset{\tau}{\longrightarrow} \ldots
\]

where

- \(i: U \hookrightarrow W\) is the inclusion,
- \(K_{d}(U)\) is an \(H^*(W; \mathbb{F}_2)\)-algebra via \(i^*: H^*(W; \mathbb{F}_2) \rightarrow H^*(U; \mathbb{F}_2)\),
- \(K_{d}(W)\) is an \(H^*(W/U; \mathbb{F}_2)\)-algebra via \(\pi^*: H^*(W/U; \mathbb{F}_2) \rightarrow H^*(W; \mathbb{F}_2)\), \(\pi: W \rightarrow W/U\) is the natural projection,
- \(H^*(W/U; \mathbb{F}_2) \simeq \mathbb{F}_2[t]\).

We propose the following conjecture:
\( (C_d) \): Let \( K_d \) be a Gysin-(\( \mathbb{Z}/2\mathbb{Z} \))\(^d\)-functor, then:
\[
\sum_{i \geq 0} \dim_{\mathbb{F}_2}(K_d(0))^i \geq 2^d.
\]

The conjecture \( C_d \) implies the conjecture \( C_{d,X} \) because if \( X \) is a finite CW-complex on which the group \( (\mathbb{Z}/2\mathbb{Z})^d \) acts freely and cellulyarly, then the functor \( K_d \) defined by \( K_d(W) = H^*_W(X; \mathbb{F}_2) \) is a Gysin-(\( \mathbb{Z}/2\mathbb{Z} \))\(^d\)-functor whose 0\(^{th}\)-term is \( H^*(X; \mathbb{F}_2) \).

The aim of this paper is to prove, in certain cases, the conjecture \( C_d \) for \( 1 \leq d \leq 3 \).

The paper is structured as follows. In the second paragraph we fix the notations and we give some properties of Gysin-(\( \mathbb{Z}/2\mathbb{Z} \))\(^d\)-functors. The third paragraph will concern the proof of the main result of this paper.

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2. **On Gysin-(\( \mathbb{Z}/2\mathbb{Z} \))\(^d\)-functors**

In this paragraph we fix some notations, introduce the Gysin-(\( \mathbb{Z}/2\mathbb{Z} \))\(^d\)-functors and give some of their properties.

2.1. **Notations.** Let \( V \) be an elementary abelian 2-group that is, a group isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^d \), \( d \geq 1 \); the integer \( d \) is called the rank of \( V \) and will be denoted by \( d = rk(V) \). The mod. 2 cohomology of \( V \) will be simply denoted \( H^*V \). Let’s recall that \( H^*V \) is a polynomial algebra over \( \mathbb{F}_2 \) on \( d \) generators \( t_i \), \( 1 \leq i \leq d \), of degree one.

We denote by \( (t)^k_0 = \mathbb{F}_2[t]/<t^{k+1}> \) where \( <t^s> \), \( s \in \mathbb{N} \), is the ideal of \( \mathbb{F}_2[t] \) of elements of degree \( \geq s \).

Let \( X \) be a CW-complex. Throughout this paper, the action of \( V \) on \( X \) will be considered cellulyar (see [TD] Chap. II, Sect. 1 for the notion of equivariant CW-complexes).

2.2. **Gysin-\( V \)-functors.** Let \( V \) be an elementary abelian 2-group of rank \( \geq 1 \). The set \( W \) of subgroups of \( V \) is ordered by inclusion and then can be considered as a category. Let \( \mathbb{K}_f \) be the category of graded, finite and unitary \( \mathbb{F}_2 \)-algebras; we denote by \( H^*V-\mathbb{K}_f \) the category of graded, finite and unitary \( H^*V-\mathbb{F}_2 \)-algebras. An object of this category is a graded, finite and unitary \( \mathbb{F}_2 \)-algebra \( K \) equipped with a map of graded unitary \( \mathbb{F}_2 \)-algebras \( H^*V \otimes K \to K \).
Definition 2.2.1. A Gysin-$V$-functor is a contravariant functor

$$\mathcal{K}_V : \mathcal{W} \ni \mathbb{K}_f, W \mapsto \mathcal{K}_V(W) = K_W$$

such that:

(i) For every subgroup $W$ of $V$, the algebra $K_W$ is a non trivial object of the category $H^*W.\mathbb{K}_f$.

(ii) For every subgroup $W$ of $V$ and for every subgroup $U$ of $W$ of codimension one, there exist an exact sequence of $H^*W$-modules of the form:

$$G(U, W) : \ldots \rightarrow (K_W)^{*-1} \xrightarrow{t} (K_W)^* \xrightarrow{\mathcal{K}_V(i)} (K_U)^* \xrightarrow{\psi} (K_W)^{*-t} \rightarrow \ldots$$

where

- $i : U \hookrightarrow W$ is the inclusion,
- $K_U$ is an $H^*W$-algebra via $i^* : H^*W \rightarrow H^*U$,
- $H^*(W/U) \cong \mathbb{F}_2[t]$ and $t : K_W \rightarrow K_W$ is the $H^*(W/U)$-structure of $K_W$ via the morphism $\pi^* : H^*(W/U) \rightarrow H^*W$ induced by the projection $\pi : W \rightarrow W/U$.

2.2.2. Vocabularies and notations

2.2.2.1. The exact sequence $G(U, W)$ will be called the Gysin sequence associated to the subgroups $U$ and $W$ of $V$ ($U \subseteq W$ of codimension one).

2.2.2.2. When the structures and morphisms are fixed, a Gysin-$V$-functor will be simply denoted $\mathcal{K}_V = \{K_W, W \text{ subgroup of } V\}$.

Remark 2.2.3. Let $\tilde{H}^*(W/U)$ be the augmentation ideal of $H^*(W/U)$. Since $K_W$ is an $H^*(W/U)$-module via $\pi^* : H^*(W/U) \rightarrow H^*W$ where $\pi : W \rightarrow W/U$ is the projection; we denote

$$\overline{K_W}^{W/U} = K_W/\tilde{H}^*(W/U).K_W$$

$$= \mathbb{F}_2 \otimes \left. K_W \right|_{H^*(W/U)} = Tor^H_{W/U} \left. K_W, \mathbb{F}_2 \right|_{H^*(W/U)}$$

The previous Gysin sequence $G(U, W)$ induces a short exact sequence of $H^*U$-modules:

$$\overline{G}(U, W) : 0 \rightarrow \overline{K_W}^{W/U} \xrightarrow{\mathcal{K}_V(i)} K_U \xrightarrow{\psi} \tau^{W/U}(K_W) \rightarrow 0$$

where
One can construct various examples of Gysin-V-functors; some of them are purely algebraic examples and the other comes from topology.

2.2.4. Examples

Example 2.2.4.1. Let $K_0 = \langle \iota, x_1, x_2, x_4, y_4, x_5 \rangle$ be the graded, finite and unitary $\mathbb{F}_2$-algebra generated by six generators: $\iota$ of degree zero, $x_i$ of degree $i$, $i = 1, 2, 4, 5$ and $y_4$ of degree 4. These generators satisfy the following relations:

$$
\begin{cases}
    x_j^2 = y_4^2 = 0, & j = 1, 2, 4, 5, \\
    x_1 x_4 = x_1 y_4 = 0, \\
    x_2 x_4 = x_2 x_5 = 0, \\
    x_2 y_4 = x_1 x_5.
\end{cases}
$$

In $K_0$ the elements $x_1 x_2$ and $x_1 x_5$ are non trivial. As an $\mathbb{F}_2$-vector space $K_0 = \langle \iota, x_1, x_2, x_1 x_2, x_4, y_4, x_5, x_1 x_5 \rangle$.

Let $H^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[t]$ . We consider the graded, finite and unitary $\mathbb{F}_2[t]$-$\mathbb{F}_2$-algebra $K_{\mathbb{Z}/2\mathbb{Z}} = \langle \mu, t, z_1, z_2 \rangle$ generated by four generators: $\mu$ of degree zero, $t$ and $z_1$ of degree one and $z_2$ of degree two. In $K_{\mathbb{Z}/2\mathbb{Z}}$ we have the relations:

$$
\begin{cases}
    z_1^2 = z_2^2 = 0, \\
    t^5 \mu = t^4 z_1 = t^4 z_2 = 0.
\end{cases}
$$

In $K_{\mathbb{Z}/2\mathbb{Z}}$ the elements $z_1 z_2$, $t^4 \mu$, $t^3 z_1$, $t^3 z_2$ and $t^3 z_1 z_2$ are non trivial. We then have:

$$
\begin{cases}
    \overline{K_{\mathbb{Z}/2\mathbb{Z}}} = \langle \mu, z_1, z_2, z_1 z_2 \rangle \text{ as an } \mathbb{F}_2\text{-vector space,} \\
    \tau^{\mathbb{Z}/2\mathbb{Z}}(K_{\mathbb{Z}/2\mathbb{Z}}) = \{ t^4 \mu, t^3 z_1, t^3 z_2, t^3 z_1 z_2 \} \text{ as an } \mathbb{F}_2\text{-vector space.}
\end{cases}
$$

Consider the following sequence of $\mathbb{F}_2$-vector spaces

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K_{\mathbb{Z}/2\mathbb{Z}}} & \longrightarrow & K_0 = \langle \iota, x_1, x_2, x_1 x_2, x_4, y_4, x_5, x_1 x_5 \rangle & \longrightarrow & \tau^{\mathbb{Z}/2\mathbb{Z}}(K_{\mathbb{Z}/2\mathbb{Z}}) = \{ t^4 \mu, t^3 z_1, t^3 z_2, t^3 z_1 z_2 \} & \longrightarrow & 0
\end{array}
$$

where

- $\sigma(\mu) = \iota$, $\sigma(z_1) = x_1$, $\sigma(z_2) = x_2$ and $\sigma(z_1 z_2) = x_1 x_2$.
- $\psi(\iota) = \psi(x_1) = \psi(x_2) = \psi(x_1 x_2) = 0$
- $\psi(x_4) = t^4 \mu$, $\psi(y_4) = t^3 z_1$, $\psi(x_5) = t^3 z_2$ and $\psi(x_1 x_5) = t^3 z_1 z_2$. 
We verify that this sequence is exact and, by definition, that \( \mathcal{K}_{Z/2Z} = \{ K_0, K_{Z/2Z} \} \) is a Gysin-\(\mathbb{Z}/2\mathbb{Z}\)-functor with \( \mathcal{K}_{Z/2Z}(i) = \sigma, i : \{ 0 \} \hookrightarrow \mathbb{Z}/2\mathbb{Z} \) is the inclusion.

**Example 2.2.4.2.** Let \( V \) be an elementary abelian 2-group and let \( X \) be a finite CW-complex on which the group \( V \) acts freely. For every subgroup \( W \) of \( V \), we denote by \( X_{hW} = EW \times_W X \) the Borel construction which is the quotient of \( EW \times X \) by the diagonal action of \( W \). Here \( EW \) is a contractible space on which \( W \) acts freely; \( BW = EW/W \) is a classifying space of \( W \).

The mod.2 cohomology of the space \( X_{hW} \), \( H^*(X_{hW}) = H^*_W X \), is called the mod.2 equivariant cohomology of \( X \). We denote by \( \pi_W : X_{hW} \to BW \) the map induced by \( X \to \{ * \} \). It is clear that \( H^*_W X \) is a graded \( H^*W \)-module (resp. \( H^*V \)-module) via \( \pi_W^* : H^*W \to H^*_W X \) (resp. via \( i^* : H^*V \to H^*W \), where \( i : W \hookrightarrow V \) is the natural inclusion). We verify that \( H^*_W X \) is an object of the category \( H^*W \mathbb{K}_f \) that is a graded, finite and unitary \( \mathbb{F}_2 \)-algebra equipped with a map of graded unitary \( \mathbb{F}_2 \)-algebras \( H^*W \otimes H^*_W X \to H^*_W X \).

The contravariant functor \( \mathcal{K}_V : W \twoheadrightarrow \mathbb{K}_f, W \mapsto \mathcal{K}_V(W) = H^*_W X \) is a Gysin-V-functor because:

(i) For every subgroup \( W \) of \( V \), \( H^*_W X \) is a non trivial object of the category \( H^*W \mathbb{K}_f \).

(ii) Let \( W \subset V \) be a subgroup and let \( U \subset W \) be a subgroup of codimension 1. The inclusion \( i : U \hookrightarrow W \) induces the following two sheets covering: \( W/U \cong \mathbb{Z}/2\mathbb{Z} \rightarrow X_{hW} \xrightarrow{\pi_W} X_{hW} \) with \( B(\pi) \circ \pi_W : X_{hW} \to BW \to B(W/U) \) as a classifying map, \( \pi : W \to W/U \) is the natural projection.

Let \( H^*(W/U) \cong \mathbb{F}_2[t] \), we also denote by \( t \) the non trivial element \( (B(\pi) \circ \pi_W)^*(t) \) of \( H^*_{W} X \). The Gysin exact sequence associated to the previous covering is the following exact sequence of \( H^*W \)-modules:

\[
\ldots \longrightarrow H^{i-1}_W X \overset{t}{\longrightarrow} H^i_W X \overset{i^*}{\longrightarrow} H^i_U X \overset{tr}{\longrightarrow} H^i_W X \overset{t}{\longrightarrow} \ldots
\]

where \( tr \) is the the transfer \( ([Sp], \mathbb{Z}) \) and for \( x \in H^*_W X \), \( t x = (B(\pi) \circ \pi_W)^*(t) \sim x \).

This shows that \( \mathcal{K}_V = \{ K_W = H^*_W X, W \text{ a subgroup of } V \} \) is a Gysin-V-functor. This example comes from “topology” via the equivariant cohomology of a free action of \( V \) on a finite CW-complex \( X \).

2.3. Some properties of Gysin-V-functors.

Let’s recall some definitions and fix some notations. Let \( E \) be a finite graded \( \mathbb{F}_2 \)-vector space.

- We denote by \( \| E \| \) the norm of \( E \) which is the maximum of the set \( \{ k \in \mathbb{N}, E^k \neq 0 \} \).

- Let \( V \) be an elementary abelian 2-group. If \( E \) is an \( H^*V \)-module and \( x \in E \), we denote by \( \langle x \rangle_V \) the sub-\( H^*V \)-module of \( E \) generated by the element \( x \).
Definitions 2.3.1. (i) A finite graded $\mathbb{F}_2$-vector space $E$ is called:

(i-a) **connected** if $E^0 \cong \mathbb{Z}/2\mathbb{Z}$.

(i-b) **bi-connected** if:

\[
\begin{align*}
E \text{ is connected: } E^0 & \cong \mathbb{Z}/2\mathbb{Z}, \\
\text{and } & \\
E^{\|E\|} & \cong \mathbb{Z}/2\mathbb{Z}.
\end{align*}
\]

(ii) A Gysin-$V$-functor $K_V = \{K_W, W \text{ subgroup of } V\}$ is **connected** (resp. **bi-connected**) if $K_0$ is **connected** (resp. $K_0$ is **bi-connected**).

(iii) A finite CW-complex $X$ is **bi-connected** if $H^*X$ is **bi-connected**.

We have the following property of Gysin-$V$-functors.

**Lemma 2.3.2.** Let $V$ be an elementary abelian 2-group and let $K_V = \{K_W, W \text{ subgroup of } V\}$ be a bi-connected Gysin-$V$-functor. Then, for each subgroup $W$ of $V$, the graded finite $\mathbb{F}_2$-algebra $K_W$ is bi-connected and we have $\|K_W\| = \|K_0\|$.

**Proof.** The proof is by induction on the rank of the subgroup of $V$. Let $K_V$ be a bi-connected Gysin-$V$-functor and let $U \subseteq V$ be a subgroup of rank one. The Gysin exact sequence of graded $\mathbb{F}_2$-vector spaces:

\[
\overline{G}(0, U) : 0 \longrightarrow \overline{K_U} \xrightarrow{K_V(i)} K_0 \xrightarrow{\psi} \tau^U(K_U) \longrightarrow 0
\]
shows that:

**2.3.2.1.** In degree zero, we have: $(P_0) : \mathbb{Z}/2\mathbb{Z} \cong (K_0)^0 \cong (\overline{K_U})^0$.

Since $(\tau^U(K_U))^0 \subseteq (K_U)^0 = (\overline{K_U})^0$, then $(\overline{K_U})^0 = 0$ implies $(\tau^U(K_U))^0 = 0$. This contradicts the equality $(P_0)$. Then we deduce that: $(\tau^U(K_U))^0 = 0$ and $\mathbb{Z}/2\mathbb{Z} \cong (\overline{K_U})^0 \cong (K_U)^0$. This shows that $K_U$ is connected.

**2.3.2.2.** In degree $\|K_0\|$, we have: $(P_{\|K_0\|}) : \mathbb{Z}/2\mathbb{Z} \cong (K_0)^{\|K_0\|} \cong (\overline{K_U})^{\|K_0\|} \oplus (\tau^U(K_U))^{\|K_0\|}$.

Since $K_U$ is a graded finite $H^*U$-module, we have: $\|K_U\| = \|\tau^U(K_U)\|$. The following inequalities follow: $\|\overline{K_U}\| \leq \|K_U\| = \|\tau^U(K_U)\| \leq \|K_0\|$. We deduce from $(P_{\|K_0\|})$ that:

\[
(\overline{K_U})^{\|K_0\|} = 0 \text{ and } (\tau^U(K_U))^{\|K_0\|} \cong \mathbb{Z}/2\mathbb{Z}.\]

This shows that: $\|K_U\| = \|\tau^U(K_U)\| \geq \|K_0\|$. So, we have the equality: $\|K_U\| = \|\tau^U(K_U)\| = \|K_0\|$.

We proved that $K_U$ is bi-connected and $\|K_U\| = \|K_0\|$.

The lemma holds by induction on the rank of subgroups of $V$ using the same method. □
Let $E$ be a graded finite $\mathbb{F}_2$-vector space. We denote by $d(E) = \sum_{i \geq 0} \dim_{\mathbb{F}_2} E^i$ the (total) dimension of $E$. We have:

**Proposition 2.3.3.** Let $V$ be an elementary abelian 2-group and $\mathcal{K}_V = \{ K_W, W \text{ subgroup of } V \}$ be a Gysin-$V$-functor. Then, the dimension of $K_0$ is even: $d(K_0) \equiv 0 \pmod{2}$.

**Proof.** Let $U \subset V$ be a subgroup of rank one, then the Gysin exact sequence

$$\overline{G}(0, U) : 0 \longrightarrow K_U \longrightarrow \mathcal{K}_V(i) \longrightarrow K_0 \xrightarrow{\psi} \tau^U(K_U) \longrightarrow 0$$

Shows that $d(K_0) = d(K_U^U) + d(\tau^U(K_U))$. The proposition 2.3.3 is a consequence of the following lemma. \n
**Lemma 2.3.4.** Let $U$ be an elementary abelian 2-group of rank one and let $M$ be a graded finite $H^*U$-module. We have: $d(K_U^U) = d(\tau^U(M))$.

**Proof.** The proof is by induction on the dimension of the finite $\mathbb{F}_2$-vector space $K_U^U$. \n
**Remark 2.3.5.** Here is an example of application of the previous lemma. Let $U_i, i = 1, 2$, be an elementary abelian 2-group of rank one, let $V = U_1 \oplus U_2$ and let $H^*U_i \cong \mathbb{F}_2[t_i], i = 1, 2$. Let $M$ be a graded finite $H^*V$-module, $x_1$ and $x_2$ two elements of $M$ such that the finite $\mathbb{F}_2[t_i]$-modules $M_U^{U_2}$ and $\tau^{U_2}(M)$ are monogenic:

\[
\begin{cases}
M_U^{U_2} \cong (t_1)^{k_1} x_1, \\
\tau^{U_2}(M) \cong (t_1)^{k_2} x_2.
\end{cases}
\]

In this case we have $d(k_1) = k_1 + 1$ and $d(k_2) = k_2 + 1$. The lemma 2.3.4, applied for the $H^*U_2$-module $M$, implies the equality: $k_1 = k_2$.

2.4. **On the extension of Gysin-$V$-functors.** Let $V$ be an elementary abelian 2-group, $V'$ a subgroup of $V$ and $\mathcal{K}_V = \{ K_W, W \text{ subgroup of } V \}$ be a Gysin-$V$-functor. Then, $\mathcal{K}_{V'} = \{ K_W, W \text{ subgroup of } V' \}$ is a Gysin-$V'$-functor called a "sub-Gysin-functor" of $\mathcal{K}_V$. We say also that $\mathcal{K}_V$ is an extension of the Gysin-$V'$-functor $\mathcal{K}_{V'}$.

It is interesting to know when a Gysin-$V$-functor extends because this question is related to the extension of a free action of the group $V$ on a finite CW-complex. We have.

**Proposition 2.4.1.** Let $V$ be an elementary abelian 2-group and let $\mathcal{K}_V = \{ K_W, W \text{ subgroup of } V \}$ be a Gysin-$V$-functor such that $\overline{K}_V \cong \mathbb{Z}/2\mathbb{Z}$, then $\mathcal{K}_V$ doesn’t extend.
Proof. Suppose that the Gysin-V-functor $\mathcal{K}_V$ extends to $\mathcal{K}_H$, $H = V \oplus \mathbb{Z}/2\mathbb{Z}$ then, the Gysin exact sequence of graded finite $H^*V$-modules:

$$\overline{G}(V, H) : 0 \to \overline{K_H}^{H/V} \xrightarrow{\mathcal{K}_H(i)} K_V \xrightarrow{\psi} \tau^{H/V}(K_H) \to 0$$

Shows that

$$\left(K_V\right)^0 \cong \left(\overline{K_H}^{H/V}\right)^0 \oplus \left(\tau^{H/V}(K_H)\right)^0$$

Since $\left(\tau^{H/V}(K_H)\right)^0 \subseteq \left(K_H\right)^0$ and $\left(K_V\right)^0 \cong \mathbb{Z}/2\mathbb{Z}$ because $\overline{K_V} \cong \mathbb{Z}/2\mathbb{Z}$, we deduce that

$$\text{Im} \left(\left(\overline{K_H}^{H/V}\right)^V \xrightarrow{\mathcal{K}_H(i)} \overline{K_V}^V \right) \cong \mathbb{Z}/2\mathbb{Z} \text{ is an epimorphism and } \tau^{H/V}(K_H)^V = 0.$$

This leads to a contradiction because the $H^*(H)$-module $K_H$ is non trivial and finite. \qed

**Proposition 2.4.2.** Let $V$ be an elementary abelian 2-group, $\mathcal{K}_V$ a bi-connected Gysin-V-functor and $\iota_V$ the unit of the $\mathbb{F}_2$-algebra $K_V$: $(K_V)^0 \cong \mathbb{Z}/2\mathbb{Z} = \langle \iota_V \rangle$.

If the norm of $K_V$ is equal to the norm of its sub-$H^*V$-module generated by $\iota_V$:

$$\| K_V \| = \| \langle \iota_V \rangle_V \|, \quad \text{then } \mathcal{K}_V \text{ doesn't extend.}$$

**Proof.** Let $\mathcal{K}_V = \{ K_W, W \text{ subgroup of } V \}$ be a bi-connected Gysin-V-functor and suppose that $\mathcal{K}_V$ extends to $\mathcal{K}_H$, where $H = V \oplus \mathbb{Z}/2\mathbb{Z}$. By the lemma 2.3.2, the Gysin-H-functor $\mathcal{K}_H$ is bi-connected and we have: $\| K_V \| = \| K_H \|$ and $(K_H)^0 \cong \mathbb{Z}/2\mathbb{Z} = \langle \iota_H \rangle$. Since the map $\mathcal{K}_H(i) : K_H \to K_V$, induced by the inclusion of $V$ in $H$, is a map of unitary-(connected)-$\mathbb{F}_2$-algebras then, $\mathcal{K}_H(i)(\iota_V) = \iota_V$.

Let’s denote by $j : \langle \iota_H \rangle_H \hookrightarrow K_H$ the natural inclusion. We have the following commutative diagram whose second line is the short Gysin exact sequence $\overline{G}(V, H)$ of $H^*V$-modules.

$$\begin{array}{c}
0 \to \overline{K_H}^{H/V} \xrightarrow{\mathcal{K}_H(i)} K_V \xrightarrow{\psi} \tau^{H/V}(K_H) \to 0 \\
\text{Im}(j^{H/V}) \downarrow \quad \uparrow \quad \langle \iota_V \rangle_V \\
0 \to \overline{K_H}^{H/V} \xrightarrow{\mathcal{K}_H(i)} K_V \xrightarrow{\psi} \tau^{H/V}(K_H) \to 0
\end{array}$$

This shows that the sub-$H^*V$-module $\text{Im}(j^{H/V})$ of $\overline{K_H}^{H/V}$ is isomorphic to sub-$H^*V$-module $\langle \iota_V \rangle_V$ of $K_V$ ($\mathcal{K}_H(i) : \text{Im}(j^{H/V}) \to \langle \iota_V \rangle_V$ is an isomorphism). This implies the inequality between norms:

$$\| \overline{K_H}^{H/V} \| \geq \| \text{Im}(j^{H/V}) \| = \| \langle \iota_V \rangle_V \| = \| K_V \|.$$

Since $\overline{K_H}^{H/V}$ is a sub-$H^*V$-module of $K_V$, we have that $\| \overline{K_H}^{H/V} \| \leq \| K_V \|$. So we have the
equality: $\| K_{H/H}^{H/V} \|=\| K_V \|$. The Gysin exact sequence $\mathcal{G}(V, H)$ of $H^*V$-modules:

$$0 \rightarrow K_{H/H}^{H/V} \xrightarrow{K_{H/(i)}} K_V \xrightarrow{\psi} \tau_{H/V}(K_H) \rightarrow 0$$

implies the following isomorphism: $(K_V)^{||K_V||} \cong (K_H^{H/V})^{||K_V||} \oplus (\tau_{H/V}(K_H))^{||K_V||}$. Since the Gysin-$V$-functor $K_V$ is bi-connected, $(K_V)^{||K_V||} \cong \mathbb{Z}/2\mathbb{Z}$, and $\| K_{H/H}^{H/V} \|=\| K_V \|$, we deduce from the previous isomorphism:

$$\begin{cases}
(i) \ (K_H^{H/V})^{||K_V||} \cong (K_V)^{||K_V||} \cong \mathbb{Z}/2\mathbb{Z}, \\
(ii) \ (\tau_{H/V}(K_H))^{||K_V||} = 0.
\end{cases}$$

By proposition 2.3.2, $\| K_V \|=\| K_H \|$; since $K_H$ is a graded, finite and non trivial $H^*(H/V)$-module, then $(\tau_{H/V}(K_H))^{||K_V||} \neq 0$. This contradicts the point (ii).

3. The main result

Let $V$ be an elementary abelian 2-group of rank $d$ and $K_V = \{ K_W, \ W \text{ subgroup of } V \}$ be a Gysin-$V$-functor. Let’s denote $d(K_0) = \Sigma_{i \geq 0} \dim_{\mathbb{F}_2}(K_0)^i$ the total dimension of the graded finite $\mathbb{F}_2$-vector space $K_0$.

The main result of this paper is to show, in certain cases, that $d(K_0)$ is related to the rank of the group $V$, as suggested by the conjecture $(C_d)$, $d(K_0) \geq 2^{rk(V)}$.

More precisely, we have:

**Theorem 3.1.** Let $V$ be an elementary abelian 2-group and $K_V = \{ K_W, \ W \text{ subgroup of } V \}$ be a Gysin-$V$-functor. Then,

(i) For $rk(V) = 1$, $d(K_0) \geq 2$ so the conjecture $(C_1)$ holds.

(ii) For $rk(V) = 2$, if the Gysin-$V$-functor $K_V$ is connected, we have the inequality: $d(K_0) \geq 4$, so the conjecture $(C_2)$ holds.

(iii) For $rk(V) = 3$, if the Gysin-$V$-functor $K_V$ is bi-connected, we have the inequality: $d(K_0) \geq 8$, so the conjecture $(C_3)$ holds.

As an application of this theorem we get an independent proof of the results concerning $(C_{d,X})$ for $d \leq 3$. 
**Proposition 3.2.** Let $V$ be an elementary abelian 2-group and let $X$ be a finite CW-complex on which the group $V$ acts freely. Then,

(i) For $rk(V) = 1$, we have: $d(H^*X) \geq 2$.

(ii) For $rk(V) = 2$ and $X$ connected, we have: $d(H^*X) \geq 4$.

(iii) For $rk(V) = 3$ and $X$ bi-connected, we have: $d(H^*X) \geq 8$.

**Proof.**

Let $V$ be an elementary abelian 2-group and let $X$ be a finite CW-complex on which the group $V$ acts freely. By the example 2.2.4.2, the contravariant functor $K_V : W \mapsto \mathbb{K}_f$, $W \mapsto K_V(W) = H^*_W X$ is a Gysin-$V$-functor whose 0th-term $K_0 = K_V(0) = H^*_X$.

Let $S^n$, $n \geq 1$, be the standard unit sphere in $\mathbb{R}^{n+1}$, then the product $S^{n_1} \times ... \times S^{n_k}$, $k \geq 1$, is a bi-connected CW-complex. By the proposition 3.2, if an elementary abelian 2-group $V$, $1 \leq rk(V) \leq 3$, acts freely on a product of $k$ spheres then, $k \geq rk(V)$.

3.1. **proof of theorem 3.1.**

To prove theorem 3.1 we consider the following three cases:

3.1.1. **The case $rk(V) = 1$.**

The proposition 2.3.3 shows that if $K_V = \{K_W, W \text{ subgroup of } V\}$ is a Gysin-$V$-functor, $rk(V) = 1$, then $d(K_0) \equiv 0 \pmod{2}$. This implies that $d(K_0) \geq 2$ because the graded $\mathbb{F}_2$-vector space $K_0$ is not trivial.

3.1.2. **The case $rk(V) = 2$.**

Let $K_V = \{K_W, W \text{ subgroup of } V\}$ be a Gysin-$V$-functor, $rk(V) = 2$, and suppose that $d(K_0) < 4$. Since $d(K_0) \equiv 0 \pmod{2}$ (see proposition 2.3.3) and $K_0$ non trivial, we deduce that $d(K_0) = 2$.

Let $U \subseteq V$ be a subgroup of rank one and consider the short exact sequence of graded $\mathbb{F}_2$-vector spaces associated to the couple ($\{0\} \subseteq U$) of subgroups of $V$

$$
\mathcal{G} \{\{0\}, U\} : \quad 0 \longrightarrow \overline{K_U^U} \quad \xrightarrow{\kappa_V(i)} \quad K_0 \quad \xrightarrow{\psi} \quad \tau^U(K_U) \longrightarrow 0
$$

$i : \{0\} \hookrightarrow U$ denotes the inclusion. This shows that: $d(K_0) = 2 = d(\overline{K_U^U}) + d(\tau^U(K_U))$. The lemma 2.3.4 implies that: $d(\overline{K_U^U}) = 1$. Since the Gysin-$V$-functor $K_V$ is connected, we have: $\mathbb{Z}/2\mathbb{Z} \cong (K_U)^0 \cong (\overline{K_U^U})^0 \cong \overline{K_U^U}$.

The proposition 2.4.1 shows that, in this case, the Gysin-$U$-functor $K_U = \{K_W, W \text{ subgroup of } U\}$ can not extend to $K_V$. This leads to a contradiction.
3.1.3. The case \( \text{rk}(V) = 3 \).

Let \( \mathcal{K}_V = \{K_W, W \text{ subgroup of } V\} \) be a bi-connected Gysin-\( V \)-functor, \( \text{rk}(V) = 3 \), and suppose that \( d(K_0) < 8 \). Since \( d(K_0) \equiv 0 \pmod{2} \) (see proposition 2.3.3) and \( K_0 \) non trivial, then we have three possibility: \( d(K_0) = 2 \), \( d(K_0) = 4 \) and \( d(K_0) = 6 \). We will show that the three cases \( d(K_0) = 2 \), \( d(K_0) = 4 \) and \( d(K_0) = 6 \) are impossible. Let \( U_i, 1 \leq i \leq 3 \), be a rank one subgroup of \( V \) such that: \( V \cong U_1 \oplus U_2 \oplus U_3 \).

3.1.3.1 The case \( d(K_0) = 2 \) is impossible by the previous case 3.1.2. We proved in 3.1.2 that if \( d(K_0) = 2 \), then \( K_0 \) can’t be the 0th-term of a Gysin-\( E \)-functor with \( E \) an elementary abelian 2-group of rank 2 and a fortiori of rank \( \geq 2 \).

3.1.3.2 Suppose that \( d(K_0) = 4 \). The Gysin exact sequence of graded finite \( \mathbb{F}_2 \)-vector spaces

\[
\overline{G}([0], U_1): 0 \longrightarrow K_{U_1}^{U_1} \xrightarrow{\mathcal{K}_V(i_1)} K_0 \xrightarrow{\psi} \tau_{U_1}(K_{U_1}) \longrightarrow 0
\]

\( (i_1 : \{0\} \hookrightarrow U_1 \) is the inclusion), shows that \( d(K_0) = 4 = d(\overline{K_{U_1}}^U) + d(\tau_{U_1}(K_{U_1})) \). The lemma 2.3.4 implies that \( d(\overline{K_{U_1}}^U) = 2 \), that is: \( \overline{K_{U_1}}^U \cong \langle \overline{1}, \overline{g_1} \rangle \) is the \( \mathbb{F}_2 \)-vector space generated by two generators \( \overline{1} \) and \( \overline{g_1} \) where \( i_1 \in (K_{U_1})^0 \cong \mathbb{Z}/2\mathbb{Z} \) is the unit and \( g_1 \in (K_{U_1})^{k_1} \), \( k_1 \geq 1 \).

Since \( \mathcal{K}_{U_1} \) is a sub-Gysin-functor of \( \mathcal{K}_{U_1 \oplus U_2} \) whose 0th-term \( K_0 \) is bi-connected, then by 2.4.2, the norm of \( K_{U_1} \) is bigger than the norm of the sub-\( H^*U_1 \)-module generated by \( i_1 \). We have:

\[
\| K_{U_1} \| = \| \langle g_1 \rangle_{U_1} \|
\]

\[
> \| \langle i_1 \rangle_{U_1} \|
\]

This shows, in particular, that we have an isomorphism of \( H^*U_1 \)-modules:

\[
K_{U_1} \cong \langle i_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1}
\]

The Gysin exact sequence \( \overline{G}(U_1, U_1 \oplus U_2) \) of \( H^*U_1 \)-modules

\[
0 \longrightarrow \overline{K_{U_1 \oplus U_2}} \xrightarrow{\mathcal{K}_{U_1 \oplus U_2}(j_1)} K_{U_1} \cong \langle i_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1} \xrightarrow{\psi} \tau_{U_2}(K_{U_1 \oplus U_2}) \longrightarrow 0
\]

\( j_1 : U_1 \hookrightarrow U_1 \oplus U_2 \) denotes the natural inclusion, shows that:

\[
\begin{align*}
(i) \quad \overline{K_{U_1 \oplus U_2}} & \cong \langle i_1 \rangle_{U_1}, \\
(ii) \quad \tau_{U_2}(K_{U_1 \oplus U_2}) & \cong \langle \psi(g_1) \rangle_{U_1}.
\end{align*}
\]

The point (i) implies that \( \overline{K_{U_1 \oplus U_2}} \cong \mathbb{Z}/2\mathbb{Z} \) and the proposition 2.4.1 shows the contradiction since the Gysin-\( V \)-functor \( \mathcal{K}_V \) extends \( \mathcal{K}_{U_1 \oplus U_2} \) (\( V \cong U_1 \oplus U_2 \oplus U_3 \)).
3.1.3.3 Suppose that $d(K_0) = 6$. To show a contradiction, in this case, we will analyse the graded, finite and unitary $H^*W$-$\mathbb{F}_2$-algebras $K_W$ for $W = U_1$ and $W = U_1 \oplus U_2$.

**I1. Informations on $K_{U_1}$.**

By the same previous method, using the Gysin exact sequence
\[ G(\{0\}, U_1) : \quad 0 \longrightarrow \overline{K_{U_1}} \overset{K_{\mathcal{V}(i_1)}}{\longrightarrow} K_0 \longrightarrow \psi \longrightarrow \tau^{U_1}(K_{U_1}) \longrightarrow 0, \]
we show that $d(\overline{K_{U_1}}) = 3$ that is: $\overline{K_{U_1}} \cong \langle \overline{t_1}, \overline{g_1}, \overline{g_2} \rangle$ is the $\mathbb{F}_2$-vector space generated by three generators $\overline{t_1}$, $\overline{g_1}$ and $\overline{g_2}$: $t_1 \in (K_{U_1})^0 \cong \mathbb{Z}/2\mathbb{Z}$ is the unit, $g_1 \in (K_{U_1})^{k_1}$, $k_1 \geq 1$ and $g_2 \in (K_{U_1})^{k_2}$, $k_2 \geq 1$.

Since the bi-connected $U_1$-Gysin functor $K_{U_1}$ extends, then by proposition 2.4.2, the norm of the graded finite $\mathbb{F}_2$-vector space $K_{U_1}$ is reached as the norm of a sub-$\mathbb{F}_2$-vector space generated by a generator different of $t_1$, for example $g_1$. We have: $\| \langle t_1 \rangle_{U_1} \| < \| K_{U_1} \| = \| \langle g_1 \rangle_{U_1} \|$. We verify then that we have a short exact sequence of $H^*U_1$-modules of the form:
\[ (E(U_1)) : \quad 0 \longrightarrow \langle t_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1} \longrightarrow K_{U_1} \longrightarrow C_{U_1} \longrightarrow 0 \]
where $C_{U_1}$ is a graded finite monogenic $H^*U_1$-module generated by the element $g_2$.

In refers to 2.1, let $H^*U_i \cong \mathbb{F}_2[t_i]$, $i = 1, 2$, the polynomial algebra over $\mathbb{F}_2$ on one generator $t_i$ of degree one, $< t^s >$, $s \in \mathbb{N}$, be the ideal of $\mathbb{F}_2[t]$ of elements of degree $\geq s$ and $(t)^k = \mathbb{F}_2[t]/ < t^{k+1} >$.

With these notations we have:

**I1.1** $\langle t_1 \rangle_{U_1} \cong \langle t_1 \rangle^0_{U_1} t_1$, $n_1 \geq 1$.

**I1.2** $C_{U_1} \cong \langle t_1 \rangle_{U_1}^0 g_2$ with $l_1 \leq n_1$ because, in the graded finite unitary $\mathbb{F}_2$-algebra $K_{U_1}$, we have: $g_2 = t_1.g_2$. This implies that: $t_1^s g_2 = (t_1^s t_1).g_2$, $s \in \mathbb{N}$.

**I1.3 Remark.** In I1.1 the integer $n_1$ is $\geq 1$ because if not $n_1 = 0$ which means that: $t_1.t_1 = 0$. This implies that $t_1 \in \tau^{U_1}(K_{U_1})^0$. Since $(K_0)^0 \cong (\overline{K_{U}})^0 \oplus \tau^{U_1}(K_{U_1})^0$, we get a contradiction with $K_0$ connected: $(K_0)^0 \cong \mathbb{Z}/2\mathbb{Z}$.

**I2. Informations on $K_{U_1 \oplus U_2}$.**

Let $t_{1,2}$ be the unit of the graded $\mathbb{F}_2$-algebra $K_{U_1 \oplus U_2}$ and consider the short exact sequence of $H^*(U_1 \oplus U_2)$-modules:
\[ (E(U_1 \oplus U_2)) : \quad 0 \longrightarrow \langle t_{1,2} \rangle_{U_1 \oplus U_2} \longrightarrow K_{U_1 \oplus U_2} \longrightarrow C_{U_1 \oplus U_2} \longrightarrow 0 \]
We have the following commutative diagram, (D), of $H^*U_1$-modules:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Im}(j^{U_2}) & \rightarrow & \langle \iota_1 \rangle_{U_1} \oplus \langle g_1 \rangle_{U_1} & \rightarrow & \langle \psi(g_1) \rangle_{U_1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & K_{U_1 \oplus U_2} & \rightarrow & K_{U_1} & \rightarrow & \tau^{U_2}(K_{U_1 \oplus U_2}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C_{U_1 \oplus U_2} & \rightarrow & C_{U_1} & \rightarrow & Q & \rightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

where $i_1 : U_1 \hookrightarrow U_1 \oplus U_2$ is the natural inclusion.

Note that the morphism $\langle \psi(g_1) \rangle_{U_1} \rightarrow \tau^{U_2}(K_{U_1 \oplus U_2})$ is injective because

\[
\| \langle \psi(g_1) \rangle_{U_1} \| = \| K_{U_1} \|
\]

\[
= \| K_{U_1 \oplus U_2} \|, \text{ by lemma 2.3.2}
\]

\[
= \| \tau^{U_2}(K_{U_1 \oplus U_2}) \|, \text{ because the graded } H^*(U_2) \text{-module } K_{U_1 \oplus U_2} \text{ is finite.}
\]

This shows that the graded finite $H^*(U_1 \oplus U_2)$-module $C_{U_1 \oplus U_2}$ is monogenic generated by an element $\xi \in K_{U_1 \oplus U_2}$. Since the bi-connected Gysin-$(U_1 \oplus U_2)$-functor is the restriction of the Gysin-$V$-functor, $(V = U_1 \oplus U_2 \oplus U_3)$, then by proposition 2.4.2, we have:

\[
\| K_{U_1 \oplus U_2} \| = \| \langle \xi \rangle_{U_1 \oplus U_2} \| > \| (t_{1,2})_{U_1 \oplus U_2} \|.
\]

This implies an isomorphism of $H^*(U_1 \oplus U_2)$-modules: $K_{U_1 \oplus U_2} \cong (t_{1,2})_{U_1 \oplus U_2} \oplus \langle \xi \rangle_{U_1 \oplus U_2}$.

By analysing the previous diagram (D), we verify that:

I2.1 $\text{Im}(j^{U_2}) \cong (t_{1,2})_{U_1 \oplus U_2} \cong \langle t_1 \rangle_{U_1} \cong (t_1)_0 n_1$, $n_1 \geq 1$, (see I1.1).

I2.2 $\overline{\langle \xi \rangle_{U_1 \oplus U_2}}^{U_2} \cong \overline{\langle \xi \rangle_{U_1 \oplus U_2}}^{U_2}$ and $\langle \psi(g_1) \rangle_{U_1} \cong \tau^{U_2}(\langle \xi \rangle_{U_1 \oplus U_2})$

I2.3 $Q \cong \tau^{U_2}(\langle t_{1,2} \rangle_{U_1 \oplus U_2}) \cong (t_1)_{0}^{m_1} \psi(g_2)$, $m_1 \in \mathbb{N}$, as a graded finite monogenic $H^*U_1$-module (see notations 2.1).
I3. The contradiction.

The last line of the previous diagram (D), which is an exact sequence of graded finite monogenic $H^*U_1$-modules, can now be written, using I1.2, as follows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C_{U_1 \oplus U_2}^{U_2} & \rightarrow & C_{U_1} & \rightarrow & \tau^{U_2}(\langle t_{1,2} \rangle_{U_1 \oplus U_2}) & \rightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
& & (t_1)_0^l g_2 & & (t_1)_0^m \psi(g_2) & & & & \\
\end{array}
\]

So we get: \( \| C_{U_1 \oplus U_2}^{U_2} \| = \| C_{U_1} \| > \| \tau^{U_2}(\langle t_{1,2} \rangle_{U_1 \oplus U_2}) \| \).

This is equivalent to: \( \| (t_1)_0^l g_2 \| = l_1 + k_2 > \| (t_1)_0^m \psi(g_2) \| = m_1 + k_2 \), where \( k_2 \) is the degree of \( g_2 \).

We have then, \( l_1 > m_1 \).

In conclusion, we have:

\[
\begin{array}{l}
(\langle t_{1,2} \rangle_{U_1 \oplus U_2}) \cong (t_1)_0^m \, t_1, \text{ see I2.1,}
\\
\tau^{U_2}(\langle t_{1,2} \rangle_{U_1 \oplus U_2}) \cong (t_1)_0^m \psi(g_2),
\\
m_1 < n_1, \text{ because } m_1 < l_1 \leq n_1, \text{ see I1.2}
\end{array}
\]

The lemma 2.3.4 (see also the remark 2.3.5) shows the equality of dimensions:

\[
d(\langle t_{1,2} \rangle_{U_1 \oplus U_2}^{U_2}) = n_1 = d(\tau^{U_2}(\langle t_{1,2} \rangle_{U_1 \oplus U_2})) = m_1,
\]

so the contradiction.

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