The Critical Exponents Of The Matrix Valued 
Gross-Neveu Model

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Abstract
We study the large $N$ limit of the matrix valued Gross-Neveu model in $2 < d < 4$ dimensions. The method employed is a combination of the approximate recursion formula of Polyakov and Wilson with the solution to the zero dimensional large $N$ counting problem of Makeenko and Zarembo. The model is found to have a phase transition at a finite value for the critical temperature and the critical exponents are approximated by $\nu = 1/(2(d-2))$ and $\eta = d-2$. We test the validity of the approximation by applying it to the usual vector models where it is found to yield exact results to leading order in $1/N$. 

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1 Introduction

In spite of more than twenty years of efforts, we are still lacking a quantitative understanding of the large \( N \) limit of matrix valued field theories in more than two dimensions. A solution to this problem would be extremely welcome because of its twofold application to gauge theory on one side (as pioneered by 't Hooft in [1]), and string theory and random surfaces on the other (see [2] for a review of the solution to lower dimensional systems and for a list of references).

This state of affairs is to be compared with our successful understanding of vector valued models [3]. The difference between these two problems is more technical than conceptual; for vector models, where the number of fields grows like \( N \), we can use saddle point techniques, whereas this is not directly possible if the number of fields grows like \( N^2 \). This difficulty has been circumvented in dimension \( d \leq 1 \) [4] and, for gauge theory, \( d = 2 \) [5] but the naive extension to higher dimensions faces formidable computational problems, although recently some progress has been made in this direction [6] by pursuing the master field approach [7].

In a previous paper [8], we have proposed an approximation to the \( d \) dimensional problem that combines the known solution to the corresponding zero dimensional large \( N \) problem and the Polyakov-Wilson recursion formula [9], [10]. We have applied this approximation to the three dimensional hermitian matrix model and found the values for the critical exponents to be \( \eta = 0.20 \) and \( \nu = 0.67 \).

In this paper we extend our investigation to the study of the matrix valued Gross-Neveu model in \( 2 < d < 4 \) dimensions, described by the action

\[
S = \int d^d x \; \text{tr} \left( \bar{\Psi} \not\partial \Psi + \frac{u_0}{2N} \bar{\Psi} \Psi \bar{\Psi} \Psi \right).
\]  

The action has the same form as the one for the original model [11] with the crucial difference that the fields are now arranged into a matrix rather than a column vector. This means that the field content is, ignoring the spin indices, \( \bar{\Psi}^i(x), \Psi^j(x) \), \( (i, j, k, l = 1, \ldots, N) \). It might be disturbing to some people to have fermionic fields living in fractional dimensions but it is possible to make sense of them with the usual prescriptions of first performing the calculations for the Dirac algebra and then analytically continue in \( d \). Each element of the matrix can be assumed to be a two-component complex spinor but their Dirac algebra will never be written out explicitly since, as we shall see, the only relevant properties are the anticommuting nature of the fields and the existence of a discrete \( \mathbb{Z}_2 \) symmetry preventing the occurrence of a bare mass term. This symmetry can be viewed either as “chirality” near two dimensions or “parity” near three dimensions:

\[
\begin{align*}
    d = 2 & : & x \to x & \Psi \to \gamma_{\text{ch}} \Psi & \bar{\Psi} \to -\bar{\Psi} \gamma_{\text{ch}} \\
    d = 3 & : & x \to -x & \Psi \to \Psi & \bar{\Psi} \to -\bar{\Psi}.
\end{align*}
\]  

Our main results are the following. The fermionic matrix model has the same qualitative phase structure as the ordinary vector model, with a phase transition
at some critical value for the inverse temperature $u^*$ for $2 < d < 4$, with $u^* \to 0$ as $d \to 2$. The universal quantities at criticality are however quite different, showing that the two models are not in the same universality class. In particular, to leading order in $1/N$, the “basic” critical exponents for the matrix model within our approximation are

$$\nu = \frac{1}{2(d-2)}, \quad \eta = d-2,$$

(to be compared with $\nu = 1/(d-2)$ and $\eta = 0$ for the vector model).

As we will argue towards the end of the paper, it might even be that these exponents are closer to the actual value than one would generally expect from an approximation of this kind. This is related to the existence of a small parameter (the thickness of the integration shell in momentum space), under which the recursion formula for the fermionic model is better behaved than its bosonic analogue.

The paper is organized as follows. In section 2 we review the nature of the approximation and show how it can be applied to various models. We shall try to avoid too many repetitions; for more details see [8]. In section 3 we apply our approximation to the solvable vector model and compare the results with the exact solution. We show that in a particular limiting case it is possible to obtain exact results for the exponents. A similar phenomenon has been found for the bosonic case in [12] and we shall compare the two cases. Section 4 is devoted to the study of the matrix model. The recursion formula is obtained and the critical properties of the system are investigated. In particular, the various critical exponents are computed for $2 < d < 4$ within this approximation. In section 5 we conclude with a discussion of our results and some future projects. The relevant results for the corresponding zero dimensional vector and matrix theories are summarized in appendix A and B respectively.

2 The nature of the approximation.

The approximation we proposed in [8] is a combination of the Polyakov-Wilson approximate recursion formula [9] [10] and the solution to the appropriate zero dimensional large $N$ counting problem (in this particular case [13]). To perform wilsonian renormalization on a (euclidean) field theory one introduces an ultraviolet cut-off $\Lambda$, that we shall always set to one by a choice of scale, and then integrates over the Fourier components of the fields having momenta $\rho < p < 1$ for some $\rho \in]0,1[$. By rescaling the fields and the momenta so that the new action has the same form as the original one, one obtains a recursion relation for the coupling constants of the theory describing the critical behavior of the model.

In many cases the integration of the fast Fourier modes cannot be performed exactly. The approximate recursion formula is perhaps the simplest, albeit uncontrolled, way to proceed beyond perturbation theory. It amounts to calculating all

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2We shall see later that perhaps, in some special cases, there is a controlling small parameter.
Feynman graphs by setting the incoming momenta to zero and approximating the propagators and the loop integrals by two constants, $b$ and $c$ respectively. It is clear that this reduces the integration problem to a zero dimensional counting problem, where the dimensionality of space $d$ appears only parametrically through the rescaling of fields and momenta at the last step of the renormalization\footnote{One could also assume that the constant $c$ depends on the dimensionality of space as in $[8]$, but it turns out that the universal quantities are independent of this choice, so we shall ignore this dependence altogether.}. An important difference between the fermionic and the bosonic case is that in the former we have to allow for $b$ to be complex; the actual phase of $b$ having to be determined a posteriori by the requirement that the critical temperature be real positive.

In its original formulation $[9] [10]$, this approximation was performed for a single component bosonic field $\Phi(x)$. One wrote an integral recursion relation for a function $v(\phi)$ of a real variable $\phi$ as the approximation to the full effective potential $V[\Phi]$. This formulation has the advantage of keeping track of the renormalization of all the ultralocal terms of the kind $\int d^d x \, \Phi(x)^n$, but it has the drawback of ruling out wave function renormalization from the onset (see however $[14]$) and excluding a large class of important diagrams such as the “setting sun” depicted in fig. 1.

The power of combining the recursion formula with the large $N$ limit is that the integral for $v(\phi)$ can be performed explicitly ($\phi$ is now a vector or matrix valued object) reducing the integral recursion relation to an algebraic one that is \textit{analytic} at the gaussian point because of the nice convergence properties of large $N$ integrals. All the leading diagrams in $1/N$ are at least \textit{counted} properly, including the “setting sun” and its friends. Wave function renormalization is easy to introduce in this scheme; one simply isolates the contributing diagrams, i.e., those diagrams with two external legs not joining at the same vertex, and approximates each of them as:

\begin{align*}
D(q^2) &= D(0) + q^2 D'(0) + \cdots \approx D(0) - q^2 b D(0) + \cdots \quad \text{for bosons} \\
D(\ell) &= D(0) + \ell D'(0) + \cdots \approx D(0) - i \ell b D(0) + \cdots \quad \text{for fermions},
\end{align*}

where $b$ is the constant replacing the propagator as before. Replacing the derivative by a constant is in the same spirit as the rest of the approximation since the derivative “adds” a propagator to the integral of the Feynman diagram. It is also the correct substitution on dimensional grounds.
In [8] these approximations were applied to the hermitian matrix model
\[ S[\Phi] = \frac{1}{2} \int d^d x \ tr \left( (\nabla \Phi)^2 + r_0 \Phi^2 \right) + \frac{u_0}{N} \int d^d x \ tr \Phi^4, \quad (5) \]
yielding the approximate recursion relations\[ (l = 0, 1, \cdots) \]
\[
\begin{align*}
    r_{l+1} &= 4 \frac{r_l + 2(1 + r_l)f_{r_0}(g)}{1 + 2f_q^2(g)} \\
    u_{l+1} &= 2^{4-d} u_l \frac{1 + f_{u_0}(g)}{(1 + 2f_q^2(g))^2},
\end{align*}
\quad (6)
\]
in terms of three known functions \( f_{r_0}(g), f_{u_0}(g) \) and \( f_q^2(g) \) (mass, coupling constant and wave function renormalization respectively) depending only on the coupling constant of the zero dimensional matrix model [4]
\[ s(\phi) = tr \left( \frac{1}{2} \phi^2 + \frac{g}{N} \phi^4 \right). \quad (7) \]
The relation between \( g \) and the physical quantities \( r_l \) and \( u_l \) was found by analyzing the topology of the diagrams, (i.e., by counting the relative number of loops, propagators and vertices), and it was found that
\[ g(u_l, r_l) = \frac{cu_l}{(1 + r_l)^2}. \quad (8) \]

By studying [8] it was possible to estimate the critical exponents for the three dimensional hermitian matrix model to be \( \nu = 0.67 \) and \( \eta = 0.20 \).

There are two main differences between the bosonic case of [8] and the fermionic one treated here, irrespective of whether they are vector or matrix models. The first difference is that the symmetry [4] prevents a bare mass for the fermions\[ 5 \]. The second is that the four fermions interaction is irrelevant at the gaussian point. The outcome of this is that the analogue of the pair of equations (3) is a single recursion relation for \( u_l \) and the approximate renormalization group flow takes place on a line rather than on a plane (see fig. [4]) for which the gaussian point is attractive. For both vector and matrix models, we will show that there is another fixed point \( u^* \) that is repulsive, hence corresponding to a phase transition, and compute the critical exponents at that point. In a language perhaps more familiar to particle physicists we could call the gaussian point infrared stable and \( u^* \) ultraviolet stable. There is no possibility of confusion as long as we are dealing with a one dimensional flow.

In [12] a similar calculation was performed for the bosonic vector and matrix models where the thickness \( 1 - \rho \) of the integration shell was kept arbitrary. It was

\[ ^4 In [8] we had set \( b = 1/(1 + r) \) and \( \rho = 1/2 \).

\[ ^5 This symmetry is of course spontaneously broken and the fermions acquire a mass, but all we need here is that no Feynman graph gives rise to a term \( \approx \Psi \Psi \) after integration. \]
found that the critical exponents were mildly dependent on the unphysical value of $\rho$; an indication that some non-universal quantities had “sneaked in” during the approximation. By comparison with the bosonic vector model, it was then argued that the exponents might become exact in the limit $\rho \to 0$. We shall see that a similar phenomenon occurs here as well, but for the opposite limit $\rho \to 1$. We try to give an explanation of this curious fact in section 4 and 5.

3 Application to the vector model.

To test our technique and to gain some more insights, let us begin by considering the original model [11] with vector valued fermions $\bar{\Psi}_i, \Psi_j, (i, j = 1, \ldots, N)$:

$$ S = \int d^d x \left( \bar{\Psi} \not\! \partial \Psi + \frac{u_0}{2N} (\bar{\Psi} \Psi)^2 \right). \quad (9) $$

This model is very well understood and its critical exponents have been extensively studied [15]. To leading order they are $\eta = 0$ and $\nu = 1/(d - 2)$. The discrete chiral symmetry is always broken for $d = 2$, where the model is asymptotically free and the fermions are massive. For $2 < d < 4$, the model has a phase transition at some finite value of the temperature, above which chiral symmetry is restored.

Let us split the field $\Psi$ into the sum of $\Psi = \Psi_s + \Psi_f$ carrying only slow ($0 < \rho < p$) and fast ($\rho < p < 1$) Fourier components and define, as usual,

$$ S[\Psi_s + \Psi_f] = S[\Psi_s] + \sigma[\Psi_s, \Psi_f] + S[\Psi_f]. \quad (10) $$

The term $\sigma[\Psi_s, \Psi_f]$ is the part of $S$ that does not factorize under the decomposition, namely,

$$ \sigma[\Psi_s, \Psi_f] = \frac{u_0}{N} \int d^d x (\bar{\Psi}_s \Psi_s)(\bar{\Psi}_f \Psi_f) + \cdots, \quad (11) $$

where the dots represent terms like, for instance, $(\bar{\Psi}_s \Psi_f)(\bar{\Psi}_s \Psi_f)$ that would give subleading contribution in $1/N$ after integrating out the fast modes. This fact should be very familiar to the large $N$ experts given the index structure of the “cactus” diagrams (see fig. 2).

We then define a new action $S'$ obtained by integrating out the fast modes:

$$ e^{S'[\Psi_s]} = \int \mathcal{D}\Psi_f e^{S[\Psi_s + \Psi_f]} = e^{S[\Psi_s]} \int \mathcal{D}\Psi_f e^{\sigma[\Psi_s, \Psi_f] + S[\Psi_f]}, \quad (12) $$

and perform the integral to leading order in $1/N$, keeping the terms $O(\Psi_s^4)$ to all orders in perturbation theory, to obtain

$$ S'[\Psi_s] = S[\Psi_s] + \frac{u_0^2}{2N} \int d^d x d^d y (\bar{\Psi}_s \Psi_s)(x)(\bar{\Psi}_s \Psi_s)(y) \times \left( < \frac{1}{N}(\Psi_s \Psi_f)(x)(\Psi_f \Psi_f)(y) >_{\text{conn}} - < \frac{1}{N} \bar{\Psi}_s(x) \Psi_f(y) >_{\text{conn}}^2 \right), \quad (13) $$

To keep the notation simple, we indicate only $\Psi$ and not $\bar{\Psi}$ but it should always be understood that they are independent and subject to the same treatment.

7The use of the measure $\exp(S)$ instead of $\exp(-S)$ is allowed for Grassmann integrals.
Figure 2: Among these three simple diagrams, only the first one is leading. The thin external lines represent the slow fields, the thick internal one s the fast fields. The only vertex allowed between the two in a leading diagram is \((\bar{\Psi}_s \Psi_s)(\bar{\Psi}_f \Psi_f)\). Of course, there can be any number of vertices with four internal lines.

where the connected Green functions are defined with the full measure \(\exp(S[\Psi_f])\) to all orders in \(u_0\). Eq. (13) is the exact correction to the four fermions term to leading order in \(1/N\). We claim that if we approximate (13) by making the assumption listed in section 2 we obtain:

\[
S'()[\Phi_s] = \int d^d x \left( \bar{\Phi}_s \Phi_s + \frac{u_0}{2N} \frac{\Gamma_4(g_0)}{g_0}(\bar{\Phi}_s \Phi_s)^2 \right),
\]

(14)

where \(\Gamma_4\) is the amputated one particle irreducible four point function for the analogous zero dimensional vector model \(s(\psi) = \bar{\psi}\psi + g/(2N)(\bar{\psi}\psi)^2\) (see appendix A), and the physical coupling constant \(u_0\) is related to the zero dimensional one by

\[
g_0 = u_0 c b^2.
\]

(15)

A very explicit way to go from (13) to (14) is to make the approximation directly in (13) and then use the zero dimensional Schwinger-Dyson equation

\[
\Gamma_4(g) = g + g^2(C_4(g) - C_2(g)).
\]

(16)

One can check directly the validity of (14) expression by expanding \(S'\) to the first few orders in perturbation theory. It should be clear however that, after setting the external momenta to zero, the only surviving diagrams are the one particle irreducible ones\(^8\) and that the relative powers \(L, P\) and \(V\) of loops, propagators and vertices are controlled by the following topological formulas valid for the four point function:

\[
2V = 2 + P \quad \text{and} \quad V - P + L = 1.
\]

(17)

Solving for \(P\) and \(L\) we see that a diagram with \(V\) vertices gives a contribution proportional to

\[
u_0^V \times b^{2(V-1)} \times c^{V-1} = u_0 g_0^{V-1}
\]

(18)

as properly encoded in the power series of (14).

\(^8\)All internal propagators vanish for \(q < \rho\).
The final step is to rescale coordinates and fields so that action (14) has the same cut-off in dimensionless units. Since no wave function renormalization appears to leading order for vector models, the rescaling is the “classical” one: \( x \rightarrow \rho^{-1} x \) and \( \Psi_s \rightarrow \rho (d-1)/2 \Psi \) to get the new coupling constant

\[
u_1 = \rho^{d-2} u_0 \times \frac{\Gamma_4(g_0)}{g_0}.
\]

(19)

It is actually more convenient to multiply both sides by \( cb^2 \) and express everything in terms of \( g_l = u_l cb^2 \), \((l = 0, 1, 2, \cdots)\). Thus, the expression for the approximate recursion formula for the vector valued Gross-Neveu model becomes

\[
g_{l+1} = \rho^{d-2} \Gamma_4(g_l).
\]

(20)

The behavior of (20) is shown in fig. 3. There are two fixed points; the gaussian one and \( g_l = g_{l+1} = g^* \). What is important is that \( g^* \) is unstable under \( g_l \rightarrow g_{l+1} \) thus defining a phase transition. This is consistent with the fact that the four fermions term is irrelevant at the origin and makes the renormalization group flow look as shown in fig. 4 for the physical coupling constant \( u \).

The fact that the critical value of \( g \) is negative forces us to choose \( b \) to be purely imaginary in order to have a positive critical inverse temperature \( u^* \). There is no danger in doing this as \( b \) does not have any direct physical meaning. One can solve (20) explicitly for the critical point \( g^* \) and compute the exponent \( \nu \) at that point by linearization [9]:

\[
u = -\frac{\log \rho}{\log(\rho^{d-2} \Gamma_4'(g^*))} = -\frac{\log \rho}{\log(2 - 3\rho^{2-d} + 2\rho^{4-2d})}.
\]

(21)

The exponent \( \nu \) is always positive and independent on the unphysical quantities \( b \) and \( c \) but still depends on the (also unphysical) parameter \( \rho \). We must admit that for, say, \( \rho = 1/2 \), formula (21) does not do a very good job, giving, for instance, \( \nu = 1/2 \) at \( d = 3 \). However, as we let the integration shell become thinner, i.e., \( \rho \rightarrow 1 \), formula (21) becomes exact, i.e., \( \nu \rightarrow 1/(d-2) \). This is the analogue phenomenon found in [12] for the bosonic case. Its occurrence for vector models is not too surprising. For vector models, the leading value for \( \nu \) is obtained by a one loop calculation, the “gap equation”. If there is a limiting value for \( \rho \) for which the critical coupling \( u^* \) becomes small, the two calculations are bound to coincide. If anything, this observation provides a useful check of our calculation. Also, the fact that for bosons [12] the exact value is reached as \( \rho \rightarrow 0 \) rather that \( \rho \rightarrow 1 \) is a direct expression of the fact that, there, \((\Phi^2)^2\) is relevant.

In the fermionic case, we can think of \( \rho \) as a sort of “control” parameter\(^9\) for the (generically uncontrolled) approximate recursion formula, in the sense that by choosing a very thin integration shell in momentum space, we can keep the critical

\(^9\)To be precise, the control parameter is the thickness of the integration shell \( 1 - \rho \).
Figure 3: Plot of eq. (24). For clarity, we plot the difference $\rho^{d-2}\Gamma_4(g) - g$ vs. $g$ for $d = 3$ and $\rho = 0.9$. The function $\Gamma_4$ is real for all $g > -1/4$; the range shown is $-0.13 < g < +0.1$.

Figure 4: The qualitative behavior of the renormalization group flow for both vector and matrix valued fermionic models. The arrows represent the change in the physical coupling $u_l$ under iteration $l \rightarrow l + 1$. 
point near the origin and get exact results for the exponents. Of course, for a thin shell the critical point is kept near the origin in the bosonic case as well, but, there, the exact exponents are recovered in the opposite limit because of the different scaling dimension of the interaction; the interpretation of \( \rho \) as a control parameter becomes then more problematic. We now move to the case of matrix valued fields, where we shall see that the difference between the bosonic and fermionic case is even more significant.

### 4 The matrix valued Gross-Neveu model.

The model of real interest in this paper is described by the action (1) presented in the introduction. Since the solution to the corresponding zero dimensional problem is known [13], there are no obstacles to extending our technique to this system. We can still decompose \( \Psi \) and \( \bar{\Psi} \) into fast and slow modes, as before, with the assumption that all fields are matrix valued. We can still write equation (10), but now

\[
\sigma[\Psi_s, \Psi_f] = \frac{u_0}{2N} \int d^dx \, \text{tr} \left( 2\bar{\Psi}_s\Psi_f\bar{\Psi}_f\Psi_f + 2\bar{\Psi}_f\Psi_s\bar{\Psi}_f\Psi_f + 2\bar{\Psi}_s\Psi_f\bar{\Psi}_f\Psi_s + 2\bar{\Psi}_f\Psi_s\bar{\Psi}_f\Psi_s + 2\bar{\Psi}_s\Psi_f\bar{\Psi}_f\Psi_f + 2\bar{\Psi}_f\Psi_s\bar{\Psi}_f\Psi_s \right). \tag{22}
\]

Although we will never need this in practice, it should be mentioned that some of the terms in (22) do not contribute to leading order: the fifth and sixth terms give rise to diagrams that are subleading in \( 1/N \) and the last two terms do not contribute to the four point function because they only appear in one particle reducible diagrams that vanish by momentum conservation. Hence, only the first four terms in (22) survive. One could perform the integral over \( \Psi_f \) and find the form of \( S' \) in terms of the exact connected Green functions. We skip this tedious calculation because it is a repetition of what was done in section 3 and because a detailed calculation for the bosonic case is presented in [8]. Instead, we shall argue directly from the diagrams what the answer should be.

We need to compute coupling constant and wave function renormalization\(^{10}\). Coupling constant renormalization is essentially the same story as for the vector model; by setting the external momenta to zero, we select the one particle irreducible diagrams, i.e., \( u_0 \to u_0\Gamma_4(g_0)/g_0 \) where now \( \Gamma_4 \) is the vertex function of the fermionic matrix model, computed in appendix B. The relation \( g_0 = u_0cb^2 \) is unchanged because the topological formulas (17) are model independent\(^{11}\).

As for wave function renormalization, we need to isolate the contributing diagrams and apply the approximation (1). We let \( q \approx 0 \) and analyze those diagrams

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\(^{10}\)Contrary to the vector model, wave function renormalization appears to leading order in the matrix model.

\(^{11}\)In (17), the integer \( L \) refers to the number of momentum loops, not color loops.
Figure 5: The full contribution to wave function renormalization comes from diagram B. Diagrams like A should be ruled out by symmetry considerations but they already vanish in the zero dimensional matrix theory. In this figure, dark circles represent the connected Green functions and the light circle the irreducible four point vertex.

that retain some dependence on $q$ in this limit. Those are all the one particle irreducible diagrams with two external legs not connected at the same vertex, as shown in fig. B. All the diagrams that are not one particle irreducible vanish by momentum conservation because the internal propagators are non zero only for $\rho < q < 1$, and those diagrams where the two external lines connect at the same vertex (fig. A) do not have any dependence on $q$. Recalling the symmetry (2), we expect these diagrams to vanish anyway in the exact solution, together with the $q$ independent part of all other two point functions.

When using the zero dimensional results, the symmetry (2) must be imposed “by hand”\(^{12}\). However, for the case of the matrix model we encounter a surprise: diagram A vanishes already in zero dimensions because of the color algebra. This fact can be easily checked to one loop in perturbation theory:

$$< \bar{\Psi}_i \Psi_k | \text{tr} \bar{\Psi} \Psi \bar{\Psi} \Psi > |_0 = N\delta_i^j \delta_k^l (1 + 1 - 1 - 1) = 0,$$

(23)

and it is related to another curious property of the fermionic matrix model described in appendix B. As a result, the Schwinger-Dyson equations allow us to write the contribution of diagram B in the much simpler form:

$$1 - gC_2^2(g)\Gamma_4(g) = 2 - \Gamma_2(g),$$

(24)

\(^{12}\)There is a zero dimensional $\mathbb{Z}_4$ discrete symmetry $\Psi \rightarrow \Psi$, $\bar{\Psi} \rightarrow -\bar{\Psi}$, but this is not what we want because, for instance, the term $\text{tr} (\bar{\Psi} \Psi \bar{\Psi} \Psi)$ is not invariant under this symmetry.
yielding the integral over $\Psi_f$

$$S'[\Psi_s] = \int d^d x \, \text{tr} \left( (2 - \Gamma_2(g_0)) \bar{\Psi}_s \partial \Psi_s + \frac{u_0 \Gamma_4(g_0)}{2N} \bar{\Psi}_s \Psi_s \bar{\Psi}_s \Psi_s \right).$$  \hspace{1cm} (25)$$

The relative powers of $u_0$, $b$ and $c$ can be checked as before; the topological formulas that apply to the two point function are

$$2V = 1 + P \quad \text{and} \quad V - P + L = 1,$$  \hspace{1cm} (26)$$
yielding a contribution for a diagram with $V$ vertices proportional to $u_0 V c^V b^{2V-1}$. Multiplying by the extra propagator $b$, to approximate the derivative as in (4), gives a contribution to wave function renormalization proportional to $u_0 V c^V b^{2V-1}$.

The renormalization group recursion relations are obtained as before by rescaling $x \rightarrow \rho^{-1} x$ and $\Psi_s \rightarrow \rho^{(d-1)/2} (2 - \Gamma_2(g_0))^{-1/2} \Psi$ and redefining the new coupling

$$u_1 = \rho^{d-2} u_0 \times \frac{\Gamma_4(g_0)}{g_0 (2 - \Gamma_2(g_0))}.$$  \hspace{1cm} (27)$$

Better yet, we multiply (27) by $c b^2$ to obtain, for $l = 0, 1 \cdots$,

$$g_{l+1} = \rho^{d-2} \frac{\Gamma_4(g_l)}{2 - \Gamma_2(g_l)} \equiv \rho^{d-2} R(g_l),$$  \hspace{1cm} (28)$$

where the last equation defines $R(g)$.

The form of $R(g)$ can be found after a fair amount of tedious algebra to be

$$R(g) = \frac{g \beta(\beta + 3)(\beta + 6)^2}{(\beta + 4)^3 (2\beta^2 + 9\beta + 6)},$$  \hspace{1cm} (29)$$

where $\beta(g)$ is the solution to $g^2 \beta^4 - \beta^2 - 8\beta - 12 = 0$, $\beta(0) = -2$ necessary to satisfy the one cut ansatz to the Riemann-Hilbert problem as described in appendix [4].

By setting $g_l = g_{l+1} = g^*$ we see that (28) admits two purely imaginary solutions $g^* = \pm i\tilde{g}^*$. This is fairly easy to show: the function $R(g)$ is odd ($\beta$ is even) and real near the origin, where it has a power series $R(g) = g - g^2 + O(g^5)$. Hence, for $\rho \rightarrow 1^-$ we obtain $g^{*2} \approx 1 - \rho^{2-d} < 0$ leading to two imaginary solutions. This can be seen more generally by setting $R(i\tilde{g}) = i\tilde{R}(\tilde{g})$. By virtue of $R(g)$ being real and odd, $\tilde{R}(\tilde{g})$ is also real and odd and eq. (28) becomes

$$\tilde{g}_{l+1} = \rho^{d-2} \tilde{R}(\tilde{g}_l).$$  \hspace{1cm} (30)$$

The plot of (30) is shown in fig. [3] for the region in which $\tilde{R}$ is real. One can see that there are only two $Z_2$ symmetric non trivial solution, both of which are unstable. The requirement that $u^*$ be real positive forces us to fix $b^2$ to be purely imaginary and to discard one of the two solution as unphysical. The renormalization group
Figure 6: Plot of eq. (30). For clarity, we plot the difference $\rho^{d-2}\tilde{R}(\tilde{g}) - \tilde{g}$ vs. $\tilde{g}$ for $d = 3$ and $\rho = 0.9$. The function $\tilde{R}$ is real for $|\tilde{g}| < 0.211$. 
flow involving the gaussian point and \( u^* \) is qualitatively the same as the one depicted in fig. 4.

There are other solutions to (28) in the whole complex plane but the above solution is the only one that can be really trusted by virtue of reducing to the gaussian when the integration shell becomes thinner (\( \rho \to 1 \)). In fact we see that, just as for the fermionic vector model in section 3, the parameter \( \rho \) provides a way of controlling the approximation by keeping the critical point near the gaussian one for thin integration shells. The situation is more favorable than in the bosonic case [12] because it is in this region that the exponents for the fermionic vector model become exact. It is possible to linearize (28) numerically and extract the exponent \( \nu \). However, as we have argued that the results should be reliable for \( \rho \approx 1 \), it is enough to consider the first non trivial term in the Taylor expansion near the origin:

\[
g_{l+1} \approx \rho^{d-2}(g_l - g_l^3). \tag{31}
\]

To illustrate why we expect a universal behavior different from the one of the vector model, let us recall that in the vector case the Taylor expansion of (24) contains a quadratic term: \( g_{l+1} \approx \rho^{d-2}(g_l - g_l^2) \). In the most general case

\[
g_{l+1} \approx \rho^{d-2}(g_l + A g_l^n), \tag{32}
\]

one obtains

\[
\nu \approx -\frac{\log \rho}{\log(n - (n - 1)\rho^{d-2})} \to \frac{1}{(n - 1)(d - 2)} \text{ as } \rho \to 1. \tag{33}
\]

For \( n = 2 \) we recover the exponent for the vector model and for \( n = 3 \) we prove our first claim on the matrix model: \( \nu = 1/(2(d - 2)) \).

In a similar way, it is possible to evaluate the wave function renormalization at the critical point and to extract the exponent

\[
\eta = -\frac{\log(2 - \Gamma_2(g^*))}{\log \rho} \to d - 2 \text{ as } \rho \to 1. \tag{34}
\]

If we are willing to be optimistic, we can use the combined values of \( \nu \) and \( \eta \) to compute the other magnetic exponents from the well known scaling relations:

\[
\alpha = \frac{3d - 8}{2(d - 2)}, \quad \beta = \frac{1}{2}, \quad \gamma = \frac{4 - d}{2(d - 2)} \quad \text{and} \quad \delta = \frac{2}{d - 2}. \tag{35}
\]

It should be possible to test the validity of some of these results by a computer simulation.

5 Conclusions.

We have shown how to compute the critical exponents for the matrix valued Gross-Neveu model by using a combination of the approximate recursion formula and
the zero dimensional large $N$ limit. The success of this technique in predicting the
exponents for the ordinary vector models (both bosonic and fermionic) gives us more
confidence in the approximation. The recursion relation for the fermionic models is
even better behaved that for the bosonic ones; the thickness of the integration shell
controls the distance of the critical point from zero.

Since we have used a Taylor series expansion to obtain the values of the criti-
cal exponents, one might be tempted to say that a perturbative calculation should
suffice. This is not the case because the existence of a non trivial fixed point can
never be established by perturbation theory alone; one must rely on some non per-
turbative treatment such as the one used in this paper. However, it is true that for
a certain range of the parameter $\rho$, the fixed point falls near the origin and then
there is no harm in doing a Taylor expansion. In a sense, the situation is similar
that in the $\epsilon$ expansion [16] where the controlling parameter $\epsilon$ can be chosen in such
a way that the fixed point also falls near the origin, thus allowing one to expand
in powers of the coupling constant. The parameter $\rho$ plays a role similar to $\epsilon$ but
only for fermionic models. One crucial difference is that $\epsilon$ is a physical parameter
(essentially, the dimension of space) whereas $\rho$ is not. It would be interesting to
develop a self consistent way to show that is is possible to neglect higher powers of
$\Psi$ from the exact renormalization group equations.

Two directions worth exploring in the future are the connection with randomly
triangulated surfaces and the application of similar methods to gauge theories. In
both cases, the simple approximations used in this paper need to be modified for
many reasons; the critical points studied here and in [8] are not the ones correspond-
ing to the continuum limit of the string partition function and gauge symmetry is
not preserved in the presence of a sharp cut-off. A solution to either of these prob-
lems, allowing one to carry out the spirit of the approximation for these systems,
would be extremely welcome.

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A The zero dimensional fermionic vector model.

In this appendix we quickly review the relevant results for the fermionic vector
model. We will be brief but try to be self contained; for more details see [13]. The
great simplification for the vector models is that their Schwinger-Dyson equations
can be solved exactly in a closed form, contrary to the matrix model case where,
even after imposing the Schwinger-Dyson equations, one is still left with some un-
dertermined correlators. Hence, for the vector model, we can obtain the quantities
we need without doing any real work!
For any quantity $\mathcal{O}$ we define

$$< \mathcal{O} > = \frac{\int d\bar{\psi} d\psi \mathcal{O} e^{\bar{\psi} \psi + \frac{\beta}{2\pi} (\bar{\psi} \psi)^2}}{\int d\bar{\psi} d\psi e^{\bar{\psi} \psi + \frac{\beta}{2\pi} (\bar{\psi} \psi)^2}}. \quad (36)$$

It is a simple matter to check that the first two Schwinger-Dyson equations read

$$< \frac{1}{N} \bar{\psi} \psi > + g < \frac{1}{N^2} (\bar{\psi} \psi)^2 > - 1 = 0$$

$$< \frac{1}{N^2} (\bar{\psi} \psi)^2 > + g < \frac{1}{N^3} (\bar{\psi} \psi)^3 > - \left( 1 - \frac{1}{N} \right) < \frac{1}{N} \bar{\psi} \psi > = 0. \quad (37)$$

Eq. (37) is exact, i.e., valid for all finite $N$'s. Now we define the connected correlation functions by cluster decomposition:

$$< \tilde{\psi}^i \psi^j > \quad = \quad < \tilde{\psi}^i \psi^j >_{\text{conn.}}$$

$$< \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l > \quad = \quad < \tilde{\psi}^i \psi^j >_{\text{conn.}} < \tilde{\psi}^k \psi^l >_{\text{conn.}} - \text{cross term}$$

$$+ < \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l >_{\text{conn.}}$$

$$< \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l \tilde{\psi}^m \psi^n > \quad = \quad < \tilde{\psi}^i \psi^j >_{\text{conn.}} < \tilde{\psi}^k \psi^l >_{\text{conn.}} < \tilde{\psi}^m \psi^n >_{\text{conn.}} \pm \text{five c.t.}$$

$$+ < \tilde{\psi}^i \psi^j >_{\text{conn.}} < \tilde{\psi}^k \psi^l \tilde{\psi}^m \psi^n >_{\text{conn.}} \pm \text{eight c.t.}$$

$$+ < \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l \tilde{\psi}^m \psi^n >_{\text{conn.}}. \quad (38)$$

For our purposes, we need to keep track of terms of order $1/N$ in (38) and this can be done by introducing three quantities $C_2(g), \tilde{C}_2(g)$ and $C_4(g)$ through

$$< \tilde{\psi}^i \psi^j >_{\text{conn.}} = \delta^i_j \left( C_2(g) + \frac{1}{N} \tilde{C}_2(g) \right) + O(\frac{1}{N^2})$$

$$< \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l >_{\text{conn.}} = \left( \delta^i_j \delta^k_l - \delta^k_j \delta^i_l \right) \frac{1}{N} C_4(g) + O(\frac{1}{N^2})$$

$$< \tilde{\psi}^i \psi^j \tilde{\psi}^k \psi^l \tilde{\psi}^m \psi^n >_{\text{conn.}} = O(\frac{1}{N^2}). \quad (39)$$

Summing over the color indices in (38) and substituting into (37) we obtain, after keeping careful track of all powers of $1/N$,

$$C_2 + gC_2^2 - 1 = 0$$

$$\tilde{C}_2 + g(2C_2 \tilde{C}_2 + C_4 - C_2^2) = 0$$

$$2C_2 \tilde{C}_2 + C_4 - C_2^2 + 3gC_2(C_2 \tilde{C}_2 + C_4 - C_2^2) - \tilde{C}_2 + C_2 = 0. \quad (40)$$

Eq. (40) forms a closed system of equations that can be solved for the quantities that we need:

$$C_2(g) = \frac{\sqrt{1+4g} - 1}{2g}$$

$$C_4(g) = \frac{(1 + 2g - \sqrt{1+4g})^2}{2g^2(1 + 4g - \sqrt{1+4g})}. \quad (41)$$
By a decomposition similar to (38) it can be immediately checked that the relations between the connected Green functions $C_2$ and $C_4$ and the vertex functions $\Gamma_2$ and $\Gamma_4$ are the usual ones:

\[
\Gamma_2(g) = C_2(g)^{-1} = \frac{1 + \sqrt{1 + 4g}}{2},
\]

\[
\Gamma_4(g) = C_4(g)C_2(g)^{-4} = \frac{g}{2} \left( 1 + \frac{1}{\sqrt{1 + 4g}} \right),
\]

(42)

where the only unusual thing is, perhaps, the relative sign between $C_4$ and $\Gamma_4$ that is dictated by the choice of the measure $\exp(+S)$. The vertex function $\Gamma_4$ is the one used in the computation for the vector model in section 3.

B The zero dimensional fermionic matrix model.

We now move on to the fermionic matrix model for which we have to do a bit more work since the Schwinger-Dyson equations do not close. For any quantity $\mathcal{O}$ we now define

\[
< \mathcal{O} > = \frac{\int d\psi \; d\bar{\psi} \; \mathcal{O} \; e^{\text{tr} \left( \bar{\psi} \psi + g N \bar{\psi} \psi \right)}}{\int d\psi \; d\bar{\psi} \; e^{\text{tr} \left( \bar{\psi} \psi + g N \bar{\psi} \psi \right)}}.
\]

(43)

The only Schwinger-Dyson equation we shall need is

\[
< \frac{1}{N^2} \text{tr} \left( \bar{\psi} \psi \right) + g < \frac{1}{N^3} \text{tr} \left( \bar{\psi} \psi \bar{\psi} \psi \right) > -1 = 0.
\]

(44)

The trick of cluster decomposition of appendix A will not work here because the connected components are of the same order as the disconnected ones. Instead, we follow [13] and define a density $\rho(\mu)$

\[
< \frac{1}{N^{k+1}} \text{tr} \left( (\bar{\psi} \psi)^k \right) > = \int d\mu \; \rho(\mu) \mu^k,
\]

(45)

and a generating function $\omega(z)$

\[
\omega(z) = < \frac{1}{N} \text{tr} \left( \frac{1}{z - \frac{\psi \psi}{N}} \right) > = \int d\mu \; \frac{\rho(\mu)}{z - \mu}.
\]

(46)

We shall not repeat the argument here but it is well known [13] that the density $\rho(\mu)$ is the same as the density for the Penner model [17], i.e., it has support on a finite complex arch on which it satisfies

\[
\frac{1}{\lambda} - \frac{1}{2} - \frac{g\lambda}{2} = \varphi \int d\mu \; \frac{\rho(\mu)}{\lambda - \mu}.
\]

(47)

\[\text{A matrix with anticommuting elements cannot be diagonalized and } \rho \text{ no longer has the meaning of eigenvalue density.}\]
The one cut solution to the Riemann-Hilbert problem (47) is:

\[ \omega(z) = \frac{1}{z} - \frac{1}{2} - \frac{gz}{2} + \left( \frac{g}{2} - \frac{1}{\beta z} \right) \sqrt{z^2 + \alpha z + \beta^2}, \]  

where \( \alpha \) and \( \beta \) are determined by the asymptotic condition \( \omega(z) \to 1/z \) at infinity. The quantity \( \beta \) is the same as the one used in section 4. After some tedious algebra, it is possible to eliminate \( \alpha \) from the problem and to write the two point function as

\[ < \frac{1}{N^2} \text{tr} (\bar{\psi}\psi) >= -\frac{\beta (\beta + 4)}{\beta + 6}, \]  

where \( \beta \) is the solution to \( g^2 \beta^4 - \beta^2 - 8 \beta - 12 = 0 \) with the initial condition \( \beta = -2 \) at \( g = 0 \) necessary to ensure the proper gaussian limit. The four point function can be read off from (44).

To conclude, we wish to show how all this is related to the connected Green functions \( C_2 \) and \( C_4 \) and the vertex functions \( \Gamma_2 \) and \( \Gamma_4 \). This is achieved by simple cluster decomposition:

\[ < \bar{\psi}_j^i \psi_l^k > = < \bar{\psi}_j^i \psi_l^k >_{\text{conn.}} \]
\[ < \bar{\psi}_j^i \psi_l^k \bar{\psi}_m^p \psi_q^q > = < \bar{\psi}_j^i \psi_l^k >_{\text{conn.}} < \bar{\psi}_m^p \psi_q^q >_{\text{conn.}} \]
\[ - < \bar{\psi}_j^i \psi_q^q >_{\text{conn.}} < \bar{\psi}_m^p \psi_l^k >_{\text{conn.}} \]
\[ + < \bar{\psi}_j^i \psi_l^k \bar{\psi}_m^p \psi_q^q >_{\text{conn.}} \]  

and by the usual definition

\[ < \bar{\psi}_j^i \psi_l^k >_{\text{conn.}} = \delta^i_j \delta^k_i C_2(g) + O\left( \frac{1}{N} \right) \]
\[ < \bar{\psi}_j^i \psi_l^k \bar{\psi}_m^p \psi_q^q >_{\text{conn.}} = \left( \delta^i_j \delta^k_q \delta^p_m \delta^m_n - \delta^i_q \delta^j_m \delta^p_m \delta^k_n \right) \frac{1}{N} C_4(g) + O\left( \frac{1}{N^2} \right). \]  

Substituting (51) into (50) and taking the trace, we see that the disconnected piece drops out because of the antisymmetry in the color indices yielding

\[ C_2(g) \equiv < \frac{1}{N^2} \text{tr} (\bar{\psi}\psi) >= -\frac{\beta (\beta + 4)}{\beta + 6} \]
\[ C_4(g) \equiv < \frac{1}{N^3} \text{tr} (\bar{\psi}\psi\bar{\psi}\psi) >= \frac{\beta^2 + 5\beta + 6}{g(\beta + 6)}. \]  

The result for \( C_4 \) is rather curious and applies only to fermionic matrix models; for all other models a disconnected term \( \approx C_2^2 \) would survive after tracing.

Finally, by a decomposition similar to (50), we see that the vertex functions are again related to \( C_2 \) and \( C_4 \) as in appendix A:

\[ \Gamma_2(g) = C_2(g)^{-1} \quad \text{and} \quad \Gamma_4(g) = C_4(g)C_2(g)^{-4}. \]  

These are the quantities used in section 4 to obtain the renormalization group equations.
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