Topological Study of Global Quantization in Non-Abelian Gauge Theories

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Abstract

We study some topological aspects of non-abelian gauge theories intimately connected to the Lie algebras of the gauge groups and the homotopy theory in the generalized gauge orbit space. The physics connection to the non-perturbative solution to strong CP problem as originally proposed by the author is also discussed. Some relevant topological formulas are also given and discussed. A result from the physics application is that the usual gauge orbit space on the compactified space can contain at most a $\mathbb{Z}_2$ monopole structure in the SP(2N) gauge theories. Some relevance to the open universe is also discussed. We expect that our results may also be useful to the other studies of non-abelian gauge theories in general.

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1 Introduction

The topological study of field theories has been of fundamental interest since the discovery of Yang-Mills theories\(^1\). In fact, one of the most interesting features of field theories is the topological and non-perturbative aspect in non-abelian gauge theories such as instantons\(^2-3\) and magnetic monopoles\(^4-6\). By considering the possible non-vanishing curvature flux in gauge orbit spaces and its relevant global quantization, a non-perturbative solution to the strong CP problem has been proposed originally\(^7\) by the author. The main purpose of this paper is to study some topological issues intimately connected to this new proposed solution. Some further physics applications will also be given. We expect that our study and the results will be also useful to the other studies of gauge theories, especially to the general study of the topological aspects in non-abelian gauge theories.

It is well known that, in non-abelian gauge theories a \(\theta\) term,

\[
\mathcal{L}_{\theta} = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu}^a F_{\lambda\sigma}^a, \tag{1}
\]

can be added to the lagrangian density of the system due to instanton effects. The \(\theta\) term with an arbitrary value of \(\theta\) can induce CP violation. Especially, such an effective \(\theta\) term in QCD can induce CP violations in strong interactions. Since we will only focus to the topological study, the perturbative effects are not so relevant to our major discussions, the \(\theta\) in this paper may always be regarded physically as denoting the effective vacuum angle. For convenience in discussions relevant to the QCD, the \(\theta\) should be notationally regarded as the usual \(\theta + \text{arg}(\det M)\) with \(M\) being the quark mass matrix when the effects of electroweak interactions are included. It has been an experimental fact that the strong CP violation can be only very small or vanishing. This requires that the values of \(\theta\) can be only very special, for example very small \(\theta \leq 10^{-10}\) or
vanishing\textsuperscript{8}. However, there is no obvious reason why the values of $\theta$ have to be so constraint. This is the well known strong CP problem in elementary particle physics. As it is well known that one of the most interesting approaches to solve the strong CP problem is the assumption of an additional Peccei-Quinn $U(1)_{PQ}$ symmetry\textsuperscript{10}. The spontaneous breaking of the $U(1)_{PQ}$ symmetry implies the existence of scalar particles called axions\textsuperscript{10–11}. The essential idea in this solution is that the existence of axions and their interactions with the other particles will ensure the vanishing of the vacuum angle $\theta$ dynamically. However, the experimental observations have been given more and more constraints on the possible models for axions\textsuperscript{8}. Although the axion approach has been plausible and interesting, the fact that the experiments have not given any support\textsuperscript{8} to the axions is an indication that we may need other mechanism to provide solution to the strong CP problem. Our proposed solution\textsuperscript{7} is essentially due to the topological application to the global understanding of quantum theory.

In this paper, we will especially study some topological results and some physics applications intimately connected to our proposed solution\textsuperscript{7}. We will also give a discussion relevant to the open universe in this approach. This paper will be organized as follows. In next section, we will first give a brief description of the main results in the proposed solution\textsuperscript{7} by using the Dirac quantization condition in the relevant gauge orbit spaces for the vacuum angle $\theta$ in the presence of magnetic monopoles. With this preparation, in section 3, we will then discuss and derive some relevant topological formulas as well as some physics applications. In section 4, we will discuss the relevance to the open universe in our approach. Our conclusions will be given in section 5.
2 Topological quantization in the relevant gauge orbit space for the vacuum angle $\theta$

It is known\textsuperscript{12--13} that gauge theories can be studied in the Schroedinger formulation with constraints of Gauss' law. It can be shown that (See Ref.12, for example) the Hamiltonian equation in the functional space in this formulation gives the same partition functional in the usual definition. In this formulation, it is known that the wave functionals are cross sections of a line bundle on the relevant gauge orbit space. Namely, the existence of a well-defined wave functional physically corresponds to the existence of a cross section in the relevant fiber bundle topologically. The existence of such a cross section in the line bundle requires that if the flux of the field strength or curvature two form through any closed surfaces in the relevant gauge orbit space is non-vanishing, then it must be quantized according to Dirac quantization rule.

Our original approach consists of two main generalizations. The first one is to generalize the concept of gauge orbit space on a compactified space to that on an open space with the existence of magnetic monopoles in order to include the boundary effects consistently. Usually, the physical configuration space is the quotient space of all the well-defined gauge potentials modulo the gauge transformations with continuous gauge functions approaching to the identity element of the gauge group at the spatial infinity. Namely, it is the gauge orbit space on the compactified space. We call this as the usual gauge orbit space\textsuperscript{7}. In the presence of magnetic monopoles with non-vanishing magnetic charges, we must include the space boundary in our physics consideration. The space is topologically a large three-dimensional ball with the boundary which is topologically a two sphere $S^2$. We call the quotient space of all the well-defined gauge potentials restricted on the space boundary $S^2$ modulo all the well-defined
gauge transformations which are only implemented on the space boundary as the restricted gauge orbit space\(^7\). As it is known that\(^{29}\) the well-defined gauge transformations on the entire space boundary are those commutive with the magnetic charges. The physical configuration space is expected to be the union of the usual and the restricted gauge orbit spaces. Physically, one needs to understand this as following. The physical wave functional in the finite space region is required invariant only under the gauge transformations approaching to the identity of the gauge group, or it is a cross section in the relevant line bundle restricted to the usual gauge orbit space. The physical wave functional on the space boundary is required invariant only under the gauge transformations well-defined and implemented only on the space boundary. Namely, it is a cross section in the line bundle restricted to the restricted gauge orbit space. The whole physical wave functional is the product of both.

Our second generalization has been mainly the extension of the method in Ref.\(^{13}\) in Schrodinger formulation for the discussion of an abelian structure inside the non-abelian gauge theories with a Pontryagin or \(\theta\) term. The formalism with different methods have also been used to derive the mass parameter quantization in three-dimensional Yang-Mills theory with a Chern-Simons term\(^{12,13}\). Our approach has been more connected to the method in Ref.\(^{13}\) due to the explicit abelian structure with non-trivial topological properties in the gauge orbit spaces constructed in the Ref.\(^{13}\) is more relevant to our discussion.

Especially, we extended the discussions in Ref.\(^{13}\) for non-abelian theories to the case in the presence of magnetic monopoles. We showed that due to the existence of an abelian field induced by the \(\theta\) term in the presence of magnetic monopoles, there exits non-vanishing flux proportional to \(\theta\) for the corresponding induced field strength in the restricted gauge orbit space. To have a well-defined physical wave functional, Dirac quantization condition ensures that the vacuum
angle $\theta$ must be quantized. We will only give a brief description of the main results here for the consistency of the paper. One of the main equations we obtained is given by

$$\int_{S^2} \mathcal{F} = \frac{\theta}{2\pi^2} \text{tr} \left( \int_{S^2} \{ f \int_{C^1} \delta g g^{-1} \} \right).$$  \hfill (2)

Where the two sphere $S^2$ on the left side of the equation is in the relevant gauge orbit space and the $S^2$ on the right side denotes the space boundary. The $f$ is the monopole field strength 2-form, and $\mathcal{F}$ is the projection of the relevant field strength in the gauge configuration space to the restricted gauge orbit space given by

$$\mathcal{F} = \frac{\theta}{4\pi^2} \int_{S^2} \text{tr} (\delta a \delta a),$$ \hfill (3)

as an integration over the space boundary $S^2$ with $\delta a$ being the relevant parameter differentiation of the gauge potential $a$ in the relevant gauge orbit space.

We derived Eq.(2) through explicit calculation. Topologically, the above equation can be understood as follows. By using the relevant exact homotopy sequence, it is shown that the equation corresponds to the topological result given by

$$\Pi_2(U/G) \cong \Pi_1(G).$$ \hfill (4)

Where $U/G$ is a restricted gauge orbit space with $U$ being the space of induced gauge potentials $A$ on the space boundary $S^2$, and $G$ being the space of gauge functions restricted to the well-defined subgroup of the gauge group $G$ and on the space boundary $S^2$. Namely, the $G$ is the space of continuous gauge transformations which are on the space boundary only. It is well known that these gauge transformations are commutative with the magnetic charges$^{29}$.

For the evaluation of the above homotopy group for the quantization of the $\theta$, we will use the following topological theorem with a proof given in section 3.
Theorem. Let $G$ be the space consisting of all the continuous mappings $g$ from a $S^2$ to a Lie group $G$, then we have

$$\Pi_1(G) \cong \Pi_1(G) \oplus \Pi_3(G). \quad (5)$$

As it is shown that$^7$ for our purpose, the relevant well-defined gauge subgroup commutative with magnetic charges can be regarded as an abelian group. To derive the quantization rule for non-vanishing $\theta$, we only need to consider the $U(1)$ group as given in ref. $7$. Since $\Pi_3(U(1)) = 0$ and $\Pi_1(U(1)) = \mathbb{Z}$, and to obtain the relevant topological number in our discussion corresponding for the $U(1)$ subgroup, we only need to consider the parameter differentiation $\delta g$ for the mappings $g$ which depends only on the relevant parameter for the loop $C^1$. Note that we use the same notation $G$ both for the gauge group and the well-defined gauge subgroup on the space boundary, the meaning is clear in the corresponding discussions. Physically, $\Pi_1(G)$ is non-trivial for the exact gauge group $G$ outside the magnetic monopole is the necessary condition for the existence of magnetic monopoles in a spontaneously broken gauge theory$^{5-6}$. In the realistic case with $G = SU(3) \times U(1)$, this condition is satisfied, when the $SU(3) \times U(1)$ is regarded as the exact gauge symmetry of a spontaneously broken gauge theory, for example $SU(5)$, as it is known that the minimal $SU(5)$ model has monopole solutions$^{31}$.

Now for the quantization of $\theta$ implied, note that the left side of Eq.(2) is the magnetic flux of the monopole in the relevant restricted gauge orbit space. The case of vanishing $\theta$ always ensures the vanishing flux. We are interested in case with the possible non-vanishing vacuum angle $\theta$ in the theory. Using Dirac quantization condition with magnetic monopoles$^{4-6}$, we obtained our quantization rule$^7$ for the $\theta$ given by

$$\theta = 0, \frac{2\pi}{n} \quad (n \neq 0). \quad (6)$$
Where the integer \( n \) is the effective topological charge of the generalized monopole\(^{15} \) which can be written as\(^7 \)
\[
n = -2 < \delta', \beta > = \sum_{i=1}^{r} n_i = -2 \sum_{i=1}^{r} \frac{< \alpha_i, \beta >}{< \alpha_i, \alpha_i >}, \tag{7}
\]
with the magnetic charge up to a conjugate transformation by a group element written in the form of
\[
\int_{S^2} f = G_0 = 4\pi \sum_{i=1}^{r} \beta^i H_i, \tag{8}
\]
Where \( \{ H_i \mid i = 1, 2, ..., r = \text{rank}(G) \} \) form a basis for the Cartan subalgebra\(^{30} \) of the gauge group \( G \) with simple roots \( \alpha_i \) (\( i = 1, 2, ..., r \)). The minus sign is due to our normalization convention for Lie algebra generators \( tr(L_a L_b) = -\frac{1}{2} \delta_{ab} \).

In general, if the well-defined gauge subgroup on the space boundary consists of multiple \( U(1) \) factors giving multiple non-vanishing \( n_i \), the \( \theta \) will be vanishing. As we noted that\(^7 \), up to a conjugate transformation, we can always choose that all the \( n_i \)'s have the same sign if non-vanishing. Physically, we may expect that in the single monopole case, only the fundamental monopoles with \( n = \pm 1 \) are stable.

Therefore\(^7 \), the existence of magnetic monopoles with \( n = \pm 1 \), or \( n \geq 2\pi 10^9 \) as well as others to ensure the vanishing of \( \theta \) can provide solution to the strong CP problem due to the global quantization of the effective \( \theta_{QCD} \). This concludes the brief description of our main results for the quantization of \( \theta \).

### 3 Relevant topological theorems

In this section, we will give a proof to the topological theorem we stated in the section 2. The corresponding generalized theorems will also be given, since it may be useful for the other more general studies of topological applications in physics. As another physics application, we will study the possible monopole structure in the usual gauge orbit space.

7
Let $\mathcal{G} = G^{S^2} = \{ f \mid f : S^2 \to G \text{ continuous} \}$ be the space consisting of all the continuous mappings $f$ from a two sphere $S^2$ to a Lie group $G$, then we will show that the fundamental group of $\mathcal{G}$ is given by $\Pi_1(\mathcal{G}) \cong \Pi_1(G) \oplus \Pi_3(G)$.

Proof: It is well-known that we have the following fibration

$$P : \mathcal{G} \to G.$$  

(9)

Where the projection $P$ is defined by $P(f) = f(x_0)$ with $x_0 \in S^2$ is the base point. The fiber of this fibration is given by

$$P^{-1} = \{ f \mid f \in \mathcal{G} \text{ and } f(x_0) = e \} = \{ f \mid f : (S^2, x_0) \to (G, e) \},$$  

(10)

where $e$ denotes the identity element of the Lie group $G$. Thus, by definition the fiber $P^{-1}$ is the second order loop space of $G$, i.e.

$$P^{-1}(e) = \Omega^2 G.$$  

(11)

Where for simplicity, the base point is not written explicitly since the relevant homotopy groups in our discussions for the loop spaces based on different base points are isomorphic for the Lie groups. The relevant exact homotopy sequence for this fibration is of the form

$$\Pi_N(G) \xrightarrow{\Delta_*} \Pi_{N-1}(\Omega G) \xrightarrow{i_*} \Pi_{N-1}(\mathcal{G}) \xrightarrow{P_*} \Pi_{N-1}(G) \quad (N \geq 1).$$  

(12)

The case of $N=2$ corresponds to our discussions for the fundamental group of $\mathcal{G}$. One can easily see that the fibration $P : \mathcal{G} \to G$ has a cross section $s : G \to \mathcal{G}$ defined by

$$s(g)(x) = g \text{ for any } x \in S^2, g \in G.$$  

(13)

Where a cross section is a continuous mapping $s : G \to \mathcal{G}$ such that $Ps(g) = g$ for any $g \in G$. According to the well-known splitting theorem, the existence of a cross-section of a fiber bundle implies the splitting of the corresponding exact
homotopy sequence. Namely, the homotopy group of the bundle space can be written as the direct sum of the corresponding homotopy groups for the base space and the fiber. Therefore, we have

$$\Pi_N(G) \cong \Pi_N(\Omega^2 G) \oplus \Pi_N(G) \quad (N \geq 1).$$

(14)

Especially in the case of $N=1$, we obtain

$$\Pi_1(G) \cong \Pi_1(G) \oplus \Pi_1(\Omega^2 G).$$

(15)

Now according to the well-known isomorphism relation, we have

$$\Pi_1(\Omega^2 G) = \Pi_2(\Omega G) = \Pi_3(G).$$

(16)

Therefore, the theorem is proved. The corresponding more general theorem can be proved similarly and is given by

**Theorem 1.** Let $G$ be the space consisting of all the continuous mappings from a $S^n \ (n \geq 1)$ to a Lie group $G$, then we have

$$\Pi_N(G) \cong \Pi_N(G) \oplus \Pi_{N+n}(G) \quad (N \geq 1).$$

(17)

There is another relevant theorem which is also useful to our discussions. We will include it here and give a relevant physics application.

**Theorem 2.** Let $G$ be the space consisting of all the continuous mappings from a $S^n \ (n \geq 1)$ to a Lie group $G$, such that a given point in the $S^n$ is always mapped to the identity element of $G$. Then we have

$$\Pi_N(G) \cong \Pi_{N+n}(G) \quad (N \geq 1).$$

(18)

This theorem is essentially due to the isomorphism relation

$$\Pi_N(\Omega^{M+1} G) = \Pi_{N+1}(\Omega^M G).$$

(19)
As a physics application of the theorem 2, we will consider the possible monopole structures in the usual gauge orbit space. Denote by $\mathcal{G}$ all the gauge transformations mapping the spatial infinity to the identity element of the gauge group, namely, they are mappings from the compactified space $S^3$ to the gauge group. The possible monopole structure is determined by the homotopy group given by

$$\Pi_1(\mathcal{G}) \cong \Pi_4(G), \quad (20)$$

due to the theorem 2. For simple and compact gauge group $G$, it is known that the $\Pi_4(G) = Z_2$ is non-vanishing only for $\text{SP}(2N)$ ($\text{SP}(2) \cong \text{SU}(2)$). Therefore, there can be at most a $Z_2$ monopole structure in the usual gauge orbit spaces for $\text{SP}(2N)$ gauge theories. Note that since the relevant homotopy group is finite, it cannot be realized as by the finite flux of the curvature in the gauge orbit space and therefore, it cannot give a finite constraint on the vacuum angle in the usual gauge orbit space. This is consistent with the fact that in the usual gauge theories on the compactified space without axion, the $\theta$ angle is arbitrary.

As a remark in this section, note that the exact homotopy sequences have also been used to the study of global (non-perturbative) gauge anomalies\textsuperscript{16–24} and supersymmetry\textsuperscript{25}, especially in term of the James numbers of Stiefel manifolds.

4 Relevance to the Open Universe

In this section, we will give a brief discussion for the relevance of our results to the topology of the universe. In our discussions, the boundary effects of the magnetic monopoles are explicitly involved, if monopole solution is the true solution to the strong CP problem, then the boundary effects on the topological properties of the universe needs to be considered. In fact, our discussions and
results have important consequences on the structure of the universe.

As we have seen in our discussions and results in the previous sections that for quantizing the vacuum angle $\theta$, a non-vanishing magnetic flux through the space boundary $S^2$ is needed in the presence of magnetic monopoles. This implies that the universe must be open if the monopoles provide the true solution to the strong CP problem, otherwise, the relevant magnetic flux can be only vanishing.

Our discussions in the previous sections are consistent with this physical implication. In the realization of the relevant topological numbers corresponding to $\Pi_1(G)$ for the well-defined gauge subgroup $G$, as we have seen that only gauge functions depending on the parameter for the loop $C^1$ in Eq.(2) are needed. Essentially for the relevant well-defined gauge subgroup this is also the topological number corresponding to the homotopy group $\Pi_1(G)$ for the relevant space of $G$ for the well-defined gauge transformations on the space boundary. This does not mean that the space boundary can be identified as single point. In fact, one can easily see that our result implies that if non-contractable loops $C^1$ exist with $n \neq 0$, then the Dirac quantization condition ensures that the space boundary cannot be continuously contracted into a single point.

This can be seen as follows. Since as we have seen that in the section 2, the projection $\hat{F}$ in the relevant gauge orbit space is given in term of an integration on the space boundary $S^2$, the quantization of the non-vanishing flux on the left side of Eq.(2) by the Dirac quantization condition ensures that the space boundary cannot be contracted continuously into a single point, otherwise the quantized non-vanishing flux corresponding to the non-trivial topological number for the relevant $\Pi_2(U/G)$ can be continuously reduced to zero flux corresponding to the trivial topological number for the $\Pi_2(U/G)$ which is a contradiction to the homotopy inequivalence of the different topological num-
bers. Therefore, if magnetic monopoles provide the solutions to the strong CP problem, the space boundary cannot be continuously contracted into a single point, there exists non-vanishing magnetic flux through the space boundary and consequently the universe must be open. The open universe in the presence of non-vanishing magnetic charges can be regarded as a quantum effect of the global quantization.

5 Conclusions

In this paper, we have studied some topological issues intimately connected to the Lie algebras and homotopy theory. We discussed about the applications to the topological aspects of non-abelian gauge theories with a $\theta$ term. Especially, we have discussed about the non-perturbative solution to the strong CP problem with magnetic monopoles as originally proposed by the author. The vacuum angle can be ensured vanishing by non-vanishing topological charges corresponding to the multiple U(1) factor in the well-defined gauge subgroup on the space boundary. The strong CP problem can also be solved with non-vanishing $\theta$, due to the existence of magnetic monopoles of topological charge $n = \pm 1$ or $|n| \geq 2\pi 10^9$. However, with non-vanishing $\theta$, the CP in strong interactions cannot be exactly conserved since under CP in the relevant cases of our discussions, $n \rightarrow -n$ and the different topological charges $n$ correspond to different monopole sectors or different physical systems. In fact, as we conjectured\(^7\) that the CP violation in weak interactions may be intimately connected to the effects of magnetic monopoles. Since the CP symmetry is violated in the realistic world, we may expect that the strong CP is not exactly conserved. From this consideration, if the strong CP problem is solved due to the magnetic monopoles, then we expect that it is actually solved by the quantization of the non-vanishing vacuum angle.
We note that the $\theta$ term with the existence of magnetic monopoles was first considered relevant to $U_A(1)$ problem and chiral symmetry. It is noted by Witten that t' Hooft and Polyakov monopoles with an arbitrary $\theta$ will carry electric charges proportional to $\theta$. This was also generalized to the case of generalized magnetic monopoles. As we have emphasized, only non-singular magnetic monopole can provide solution to the strong CP problem. For example, it is known that minimal SU(5) model has smooth monopole solutions.

Moreover, we have also derived and discussed about some relevant topological formulas. A result from the physics application is that the usual gauge orbit space on the compactified space can contain at most a $Z_2$ monopole structure in SP(2N) gauge theories. We expect that the topological formulas may also be useful to the other studies of non-abelian gauge theories in general.

We have also discussed about the fact that the universe must be open if the monopoles provide the solution to the strong CP problem. This may be regarded as a quantum effect of the global quantization. Therefore, the fact that the strong CP violations can be only so small may imply the existence of magnetic monopoles and the universe must be open. The understanding of the strong CP problem may provide information for spacetime structure and the structure of our universe.

Acknowledgement: The author would like to express his gratitude to Y. Li, Profs. K. Bardacki, S. Okubo, Y. S. Wu, A. Zee and other people in the Theoretical Physical Group at Lawrence Berkeley Laboratory for valuable discussions.
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