Weyl anomaly of conformal higher spins on six-sphere

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Abstract

This paper is a sequel to arXiv:1309.0785 where we computed the Weyl anomaly $a$ (Euler density or logarithmic divergence on $S^d$) coefficient for higher-derivative conformal higher spin field in $d = 4$ and shown that it matches the expression found in arXiv:1306.5242 by a “holographic” method from a ratio of massless higher spin determinants in AdS$_5$. Here we repeat the same computation in on 6-sphere and demonstrate that the result matches again the one following from AdS$_7$. We also discuss explicitly similar matching in the $d = 2$ case.

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1 Introduction

This paper continues the investigation \cite{1} of quantum conformal higher spin (CHS) models \cite{2,3} with higher-derivative flat-space action $\int d^d x \phi_s \partial^{2s+d-4} \phi_s$ ($\partial_s$ is transverse traceless symmetric rank $s$ tensor projector). Generalising this action to curved background is a highly non-trivial problem, but as was argued in \cite{1}, at least in the case of a conformally flat Einstein background (i.e. $(A)dS_d$ or $S^d$), the corresponding Weyl-covariant $2s+d-4$ derivative kinetic operator should factorize into product of standard 2nd-derivative operators.\footnote{Here we shall always assume that $d$ is even ($d$ was called $D$ in \cite{1}).}

Explicitly, the partition function of a conformal higher spin $s$ field on a $d$-dimensional sphere of unit radius can be written as \cite{1}

$$Z_s(S^d) = \prod_{k=0}^{s-1} \left( \frac{\det [- \nabla^2 + k - (s-1)(s+d-2)]_{k \perp}}{\det [- \nabla^2 + s - (k+1)(k+d-2)]_{s \perp}} \right)^{1/2} \times \prod_{k'=-\frac{1}{2}(d-4)}^{s-1} \left( \frac{1}{\det [- \nabla^2 + s - (k'+1)(k'+d-2)]_{s \perp}} \right)^{1/2}, \quad (1.1)$$

or, equivalently, as

$$Z_s(S^d) = \prod_{k=0}^{s-1} \left( \det [- \nabla^2 + k - (s-1)(s+d-2)]_{k \perp} \right)^{1/2} \times \prod_{k'=-\frac{1}{2}(d-4)}^{s-1} \left( \det [- \nabla^2 + s - (k'+1)(k'+d-2)]_{s \perp} \right)^{-1/2}, \quad (1.2)$$

where the 2nd-order differential operator $(-\nabla^2 + M^2)_{k \perp}$ is defined on transverse traceless symmetric rank $k$ tensors. The first line in (1.1) is the contribution of the “partially-massless” modes (with residual gauge invariance and thus “ghost” numerators) while the second corresponds to extra “massive” modes present for $d \neq 4$ (see \cite{1} and refs. there).

This representation allows one to compute the CHS partition function on $S^d$ using standard (e.g., $\zeta$-function) techniques, and, in particular, to find the coefficient of the logarithmic UV divergence or the a-coefficient of the Euler density term in the corresponding Weyl anomaly.

Remarkably, the arguments in \cite{4,1} suggest that $Z_s(S^d)$ in (1.1) should have also a “holo-graphic” representation in terms of the ratio of determinants of the standard (second-derivative) massless higher spin $s$ operators with alternate boundary conditions in euclidean AdS$_{d+1}$:

$$\frac{Z_{s_0}^{-}(AdS_{d+1})}{Z_{s_0}^{+}(AdS_{d+1})} = Z_s(S^d), \quad (1.3)$$

$$Z_{s_0}(AdS_{d+1}) = \left( \frac{\det [- \nabla^2 + (s-1)(s+d-2)]_{s-1 \perp}}{\det [- \nabla^2 - s + (s-2)(s+d-2)]_{s \perp}} \right)^{1/2}. \quad (1.4)$$

Here AdS$_{d+1}$ and its boundary $S^d$ are assumed to have unit radius. The subscripts $\pm$ indicate the different boundary conditions.\footnote{These correspond to dimensions $\Delta_+ = s + d - 2$, $\Delta_- = 2 - s$ for the “physical” denominator and $\Delta_+ = s + d - 1$, $\Delta_- = 1 - s$ for the “ghost” numerator \cite{1} (see also section 4).}
Let us note that while motivated by the AdS/CFT \[5, 6, 4\], the relation (1.3) is essentially “kinematical” in nature (i.e. it does not rely on any non-renormalization and should be true for any \(d\) belonging to a class of bulk-boundary determinant relations like the one discussed in \[7\]. One should thus be able to prove it by starting from the one-loop path integral in AdS\(_{d+1}\) and “integrating out” the values of the fields in the interior points of AdS\(_{d+1}\). As in the scalar case \[8, 9, 10\] one should pay special attention to regularization. Indeed, the AdS\(_{d+1}\) side of (1.3) is IR divergent while the \(S^d\) side is UV divergent. The logarithm of partition function \(Z_{s0}\) on AdS\(_{d+1}\) is proportional to its volume which for even \(d\) has the following regularized value \[9\] (we shall keep track of logarithmic divergences only):

\[
\Omega(AdS_{d+1}) = \frac{2(-1)^{\frac{d}{2}}\pi^{\frac{d}{2}}}{\Gamma(\frac{d+1}{2})}\ln L + \ldots .
\] (1.5)

where \(L \to \infty\) is IR cutoff. The free energy on \(S^d\) of radius \(r\) has the following structure

\[
F = -\ln Z = \frac{1}{2}\ln \det (-\nabla^2 + M^2) = -B_d \ln (Lr) + \ldots ,
\] (1.6)

\[
B_d = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{g} b_d = \frac{1}{(4\pi)^{d/2}} \Omega(S^d) b_d,
\]

\[
\Omega(S^d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d+1}{2})},
\] (1.7)

where \(b_d\) is the integrated Seeley coefficient (often called also \(a_{d/2}\)) of the operator \(-\nabla^2 + M^2\) and \(L \to \infty\) is UV (heat kernel) cutoff. In the case when the classical theory is conformally invariant \(B_d\) represents the integrated Weyl anomaly (see \[11, 12, 13, 14\] and refs. there). The total coefficient of \(\ln L\) term in \(\ln Z_s\) can be found by summing the \(B_d\)-coefficients for the operators in (1.2).

Identifying the IR cutoff in the AdS\(_{d+1}\) bulk and the UV cutoff at the \(S^d\) boundary the first check of (1.3) is the matching of the coefficients of the \(\ln L\) terms. Following \[4\] let us call \(a_s\) the coefficient of the IR singular term in the AdS\(_{d+1}\) free energy in (1.3). Comparing to the \(S^d\) expression (1.6) we should get\[4\]

\[
B_d^{(s)} = -a_s.
\] (1.8)

Equivalently, \(a_s\) should be the coefficient of the \(\ln r\) term in free energy on \(S^d\).

In the case of \(d = 4\) the coefficient of the IR divergent term in the l.h.s. of (1.3) was found in to be \[4\]

\[
a_s = \frac{1}{180}\nu_s^2(14\nu_s + 3), \quad \nu_s = s(s + 1).
\] (1.9)

The same expression was also obtained directly from the spin \(s\) CHS partition function (11) on \(S^4\) as (minus) the value of the total Weyl anomaly coefficient \(B_4^{(s)}\) \[11\]

\[
b_4 = -a_s R^* R^* = -24a_s, \quad B_4^{(s)} = \frac{1}{(4\pi)^{2}}\frac{8\pi^2}{3} b_4 = -4a_s = -a_s,
\] (1.10)

\[
B_4^{(s)} = -a_s = -\frac{s^2(s+1)^2}{180}(14s^2 + 14s + 3).
\] (1.11)

\[3\] The minus sign in the relation between \(B_d^{(s)}\) and \(a_s\) is due to the canonical minus sign in (1.6) or the definition of \(a_s\) in \[4\] so that it has the same sign as the \(a\)-coefficient in the trace anomaly. It is also sensitive to the order of the signs or the power of the l.h.s. of (1.3).
Our aim here will be to perform a further non-trivial test of the relation (1.3) by considering the \( d = 6 \) case (and also the \( d = 2 \) case, see Appendix). The case of \( d = 6 \) is of interest in view of the AdS\(_7\)/CFT\(_6\) duality and also because the structure of the CHS partition function (1.1) changes for \( d \neq 4 \). We shall first consider the r.h.s. of (1.3), i.e. find the coefficient \( B_6 \) (1.7) of logarithmically divergent term in \( F = -\ln Z_s \) in (1.2) on \( S^6 \).

In general, the local Weyl anomaly coefficient has the following structure in \( d = 6 \) [12, 13, 15] \(^4\)

\[
b_6 = a E_6 + \sum_{i=1}^{3} c_i I_i + \nabla_m J^m, \quad E_6 = -\epsilon_6 \epsilon_6 \epsilon_{RRR},
\]

(1.12)

where \( I_1 \sim C(\nabla^2 + ...)C, \ I_{2,3} \sim CCC \) contain powers of the Weyl tensor \( C \). Then for a unit-radius sphere \( S^6 \)

\[
b_6(S^6) = a E_6 = -\tfrac{8!}{7} a, \quad B_6(S^6) = \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{16\pi^2}{15} b_6 = \frac{1}{60} b_6 = -96a \equiv -a.
\]

(1.13)

For a conformally coupled scalar \( \hat{\Delta} = -\nabla^2 + \frac{d-2}{4(d-1)} R \)

\[
a_0 = -\frac{5}{9}, \quad B_6^{(0)} = -a_0 = \frac{1}{756}.
\]

(1.14)

As we shall find below, for a conformal higher spin field in \( d = 6 \) the total value of \( B_6 \) corresponding to (1.2) (generalizing (1.14) to any \( s \geq 0 \)) is

\[
B_6^{(s)} = -a_s = -\frac{(s+1)(s+2)^2}{151200} (22s^6 + 198s^5 + 671s^4 + 1056s^3 + 733s^2 + 120s - 50) \\
= -\frac{1}{18900} \nu_s \left( 88\nu_s^{3/2} - 110\nu_s - 4\nu_s^{1/2} + 1 \right), \quad \nu_s = \frac{1}{4}(s + 1)^2(s + 2)^2.
\]

(1.15)

Like in the \( d = 4 \) expression (1.9) here \( \nu_s \) stands for the number of dynamical degrees of a spin \( s \) CHS field in \( d = 6 \). Specialising the general expression [4] for the coefficient of the IR divergent part of the AdS\(_{d+1}\) side of (1.3) to the case of \( d = 6 \) we will also show that it indeed matches (1.15) according to (1.8).

We shall start in section 2 with a general discussion of the values of the \( \zeta \)-function and the logarithmic UV divergence coefficient \( B_d \) for a massive higher spin operator \( (-\nabla^2 + M^2)_{s} \) on \( S^d \), and then specialise to the cases \( d = 4 \) and \( d = 6 \). In section 3 we shall apply the resulting expression for \( B_6 \) to the operators appearing in (1.2) to obtain eq. (1.15). In section 4 we shall rederive (1.15) as the coefficient of the IR divergence of the ratio of the AdS\(_7\) massless spin \( s \) partition functions in (1.3). Section 5 will contain concluding remarks. In Appendix we shall consider the \( d = 2 \) case of (1.3) and demonstrate explicitly that the AdS\(_3\) expression for \( a_s \) matches the coefficient \( B_2 \) of the UV divergence in the \( d = 2 \) conformal higher-spin partition function (1.3), thus providing another check of (1.3),(1.8).

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\(^4\)In contrast to [13] here we do not include \( \frac{1}{(4\pi)^{\frac{d-2}{2}}} \) in the definition of \( b_d \).

\(^5\)Here \( R_{mnkl} = g_{mnl} g_{nk} - g_{mn} g_{lk} \), \( R = d(d-1) = 30 \). The (minus) Euler density \( E_6 \) is equal to \( -\frac{16}{75} R^3 \) on a conformally flat background.
2 $\zeta$-function and $B_d$ coefficient for spin $s$ operators on $S^d$

To compute $B_d$ we shall use the known solution of the spectral problem for the 2nd-order operator $\hat{\Delta}_{s\perp}$ defined on symmetric traceless transverse tensors of rank $s$ on $S^d$

$$\hat{\Delta}_{s\perp}(M^2) \equiv (-\nabla^2 + M^2)_{s\perp}, \quad \hat{\Delta}_{s\perp}(\phi_n) = \lambda_n(\phi_n). \quad (2.1)$$

The eigen-values and their degeneracy are given by [16, 17, 18]

$$\lambda_n = (n + s)(n + s + d - 1) - s + M^2, \quad n = 0, 1, 2, ..., \quad (2.2)$$

$$d_n = g_s \frac{(n + 1)(n + 2s + d - 2)(2n + 2s + d - 1)(n + s + d - 3)!}{(d - 1)! (n + s + 1)!}, \quad (2.3)$$

$$g_s = \frac{(2s + d - 3)(s + d - 4)!}{(d - 3)! s!}. \quad (2.4)$$

Here $g_s \equiv g_s^{(d)}$ is the number of components of the symmetric traceless transverse rank $s$ tensor in $d$ dimensions.

$$g_s \equiv N_{s\perp} = N_s - N_{s-1}, \quad N_s = \frac{(2s + d - 2)(s + d - 3)!}{(d - 2)! s!}, \quad g_s^{(d)} = N_{s^{(d-1)}}, \quad (2.5)$$

where $N_s \equiv N_s^{(d)}$ is the number of symmetric traceless rank $s$ tensor components. The number of dynamical components of a massless spin $s$ field is (cf. (1.4))

$$\mu_s = N_{s\perp} - N_{s-1\perp} = \frac{(2s + d - 4)(s + d - 5)!}{(d - 4)! s!}, \quad \mu_s^{(d)} = g_s^{(d-1)} = N_s^{(d-2)}. \quad (2.6)$$

Note also that the number of dynamical degrees of freedom of a conformal higher spin $s$ field is (cf. (1.2))

$$\nu_s = [s + \frac{1}{2}(d - 4)]N_{s\perp} - \sum_{k=0}^{s-1} N_{k\perp} = \frac{(d - 3)(2s + d - 2)(2s + d - 4)(s + d - 4)!}{2(d - 2)! s!}, \quad (2.7)$$

$$\nu_s = \frac{(2s + d - 2)(s + d - 4)}{2(d - 2)} \mu_s. \quad (2.8)$$

The $\zeta$-function corresponding to the operator (2.1) is defined by

$$\zeta_{\hat{\Delta}_{s\perp}}(z) = \sum_{n=0}^{\infty} \frac{d_n}{\lambda_n z}. \quad (2.9)$$

In general, it is $B_d$ and not $\zeta_{\hat{\Delta}}(0)$ that governs the scale dependence of log det $\hat{\Delta}$ in (1.6). Note that the definition of $\zeta$ we use here requires summation over all modes, including the zero ones. Then
while for the operator $\hat{\Delta}_s$ defined on differentially unconstrained tensors one has $\zeta_{\hat{\Delta}_s}(0) = B_d[\hat{\Delta}_s]$, this is not so in general for $\hat{\Delta}_{s\perp}$: $\zeta_{\hat{\Delta}_{s\perp}}(0)$ turns out to be equal to $B_d[\hat{\Delta}_{s\perp}]$ in \cite{17} for the operator \cite{21} only up to the contribution of the zero modes of the operator related to the change of variables from an unconstrained tensor $\phi_s$ to its transverse part. In the case of $d = 4$ and $s \leq 2$ the reason for this was explained in \cite{20} to define the operators acting on constrained (transverse) tensors one decomposes the field into its transverse and gradient parts but that introduces $N$ additional zero modes of the Jacobian of the change of variables. Since these modes were not present for the original unconstrained operator one finds $B_d[\hat{\Delta}_{s\perp}] = \zeta_{\hat{\Delta}_{s\perp}}(0) - N$.

In more detail, starting with path integral over symmetric traceless tensor $\phi_s$ we may change the variables to transverse symmetric traceless rank $s$ tensor $\phi_{s\perp}$ and symmetric traceless rank $s - 1$ tensor $\varphi_{s - 1}$
\begin{equation}
\phi_s = \phi_{s\perp} + K\varphi_{s - 1} , \quad \nabla \cdot \phi_{s\perp} = 0 , \quad (K\varphi_{s - 1})_{m_1...m_s} = \nabla_{(m_s} \varphi_{m_1...m_{s - 1})} - \frac{s - 1}{2(s - 2) + d} g_{(m_s m_{s - 1})} \nabla^n \varphi_{m_1...m_{s - 2})n} .
\end{equation}

Then $\det K$ will appear as the Jacobian. The zero modes of $K$ are rank $s - 1$ conformal Killing tensors and their number is dimension of $(s - 1, 2, 0, ..., 0)$ representation of $SO(d + 1, 1)$ \cite{22}
\begin{equation}
k_{s - 1,d} = (2s + d - 4)(2s + d - 3)(2s + d - 2) \frac{(s + d - 4)! (s + d - 3)!}{s! (s - 1)! d! (d - 2)!} .
\end{equation}

Thus
\begin{equation}
B_d[\hat{\Delta}_{s\perp}] = \zeta_{\hat{\Delta}_{s\perp}}(0) - N , \quad N = \text{dim ker} K = k_{s - 1,d} .
\end{equation}

It should be noted that this subtlety is absent if one considers instead of $S^d$ the non-compact euclidean $H^d = \text{AdS}_d$ background: then the corresponding $\zeta_{\hat{\Delta}_{s\perp}}(0)$-function defined according to \cite{18} matches $B_d[\hat{\Delta}_{s\perp}]$ \cite{16}.

In what follows we shall be interested in the two special cases: the familiar $d = 4$ case (to compare to the results of \cite{11} which were found directly from the general expression for $B_4$, i.e. without using the spectrum on $S^4$) and the new $d = 6$ one. One finds from \cite{2.2}–\cite{2.4}, \cite{2.12}
\begin{align}
d = 4 : \quad & \lambda_n = n^2 + (2s + 3)n + s(s + 2) + M^2 , \quad g_4(s) = 2s + 1 , \quad d_n = \frac{1}{6} g_4(n + 1)(n + 2s + 2)(2n + 2s + 3) , \quad k_{s - 1,4} = \frac{1}{12}(2s + 1)s^2(s + 1)^2 , \\
d = 6 : \quad & \lambda_n = n^2 + (2s + 5)n + s(s + 4) + M^2 , \quad g_6(s) = \frac{1}{6}(s + 1)(s + 2)(2s + 3) , \quad d_n = \frac{1}{120} g_6(n + 1)(n + s + 2)(n + s + 3)(n + 2s + 4)(2n + 2s + 5) , \quad k_{s - 1,6} = \frac{1}{3360}(2s + 3)s(s + 1)^3(s + 2)^3(s + 3) .
\end{align}

Note that in $d = 6$ the number of symmetric traceless tensor components is (see \cite{2.5})
\begin{equation}
N_s = \frac{1}{12}(s + 1)(s + 2)^2(s + 3) ; \quad \text{the number of transverse components is } N_{s\perp} = g_s = \frac{1}{6}(s +
\footnote{The difference between $B_4$ and $\zeta(0)$ was pointed out also in \cite{21}.}
\footnote{Here the zero modes are non-normalizable and effectively drop out of $\zeta_{\hat{\Delta}_{s\perp}}(0)$ on $H^d$ defined as in \cite{18}.}
1)\( (s + 2)(2s + 3) \); the number of dynamical degrees of freedom of a massless spin \( s \) field \((2.6)\) is \( \mu_s = (s + 1)^2 \); the number of dynamical degrees of freedom of a conformal spin \( s \) field \((2.8)\) is \( \nu_s = \frac{1}{4}(s + 1)^2(s + 2)^2 \).

Let us now consider the computation of the corresponding values of \( \zeta_{\Delta_{s\perp}}(0) \) in \( d = 4 \) and \( d = 6 \).

### 2.1 \( \mathbf{d = 4 \ case} \)

The computation of \( \zeta_{\Delta_{s\perp}}(z) \) in \( d = 4 \) was discussed in \([23]\). First, we write \((2.9)\) as

\[
\zeta_{\Delta_{s\perp}}(z) = \frac{1}{3}(2s + 1) \sum_{k = s + \frac{3}{4}}^{\infty} \frac{k[k^2 - (s + \frac{1}{2})^2]}{k^2z(1 - \frac{k^2}{k^2})^z}, \quad h^2 = s + \frac{9}{4} - M^2.
\]

Then using that

\[
(1 - \frac{h^2}{k^2})^z = \sum_{m=0}^{\infty} c_m(z) \frac{h^{2m}}{k^{2m}}, \quad c_m(z) = \frac{(z + m - 1)!}{m! (z - 1)!},
\]

we get

\[
\zeta_{\Delta_{s\perp}}(z) = \frac{1}{3}(2s + 1) \sum_{m=0}^{\infty} c_m(z) h^{2m} \left[ \zeta_R(2z + 2m - 3, s + \frac{3}{2}) - (s + \frac{1}{2})^2 \zeta_R(2z + 2m - 1, s + \frac{3}{2}) \right],
\]

where \( \zeta_R(z, b) \equiv \sum_{n=0}^{\infty} (n + b)^{-z} \). To find the limit \( z \to 0 \) we need to use that the terms with \( m = 1, 2 \) may have a pole as \( \zeta_R(x, b) = \frac{1}{x-1} - \psi(b) + ... \). Then we end up with

\[
\zeta_{\Delta_{s\perp}}(0) = \frac{1}{3}(2s + 1) \left[ \zeta_R(-3, s + \frac{3}{2}) - (s + \frac{1}{2})^2 \zeta_R(-1, s + \frac{3}{2}) - \frac{1}{4}(s + \frac{9}{4} - M^2)(2s^2 + s - \frac{7}{4} + M^2) \right],
\]

where \( \zeta_R(-1, b) = -\frac{1}{2}b^2 + \frac{1}{2}b - \frac{1}{12} \) and \( \zeta_R(-3, b) = -\frac{1}{4}b^4 + \frac{1}{4}b^3 - \frac{1}{4}b^2 + \frac{1}{120} \). Finally,

\[
\zeta_{\Delta_{s\perp}}(0) = \frac{1}{180}(2s + 1) \left[ 15M^4 + 30(s^2 - 2)M^2 + 58 - 10s - 70s^2 + 15s^4 \right].
\]

Then using \((2.13), (2.16)\) we get

\[
B_4[\hat{\Delta}_{s\perp}(M^2)] = \zeta_{\Delta_{s\perp}}(0) - k_{s-1,4}
\]

\[
= \frac{1}{180}(2s + 1) \left[ 15M^4 + 30(s^2 - 2)M^2 + 58 - 10s - 85s^2 - 30s^3 \right].
\]

Taking into account \((1.10)\) this matches the expression for \( a[\hat{\Delta}_{s\perp}(M^2)] \) which was found \([1]\) directly from the standard algorithm for \( B_4 \) \([28]\) and using that \([1]\)

\[
det \hat{\Delta}_{s\perp}(M^2) = \frac{\det \hat{\Delta}_{s}(M^2)}{\det \hat{\Delta}_{s-1}(M^2 - 2s - d + 3)},
\]

\[
B_d[\hat{\Delta}_{s\perp}(M^2)] = B_d[\hat{\Delta}_{s}(M^2)] - B_d[\hat{\Delta}_{s-1}(M^2 - 2s - d + 3)].
\]
Applying (2.25) to find the total $B_4$ or a coefficient (1.10) corresponding to the $d = 4$ CHS partition (1.1) one ends up with (1.11) [1].

Let us note that the same expression (2.25) can be found also by considering instead of $S^4$ the non-compact $H^4$ (euclidean AdS$_4$) background. Indeed, the local expressions for the coefficient $b_4$ in (1.7) should match since it depends on the square of the curvature while $R(S^4) = - R(H^4)$ (one should also change the sign of the $M^2$ term as it enters as $M^2 \epsilon$, $R = d(d-1) \epsilon$, $\epsilon = \pm 1$).

Computing the corresponding value of $\zeta_{\Delta_{s,\perp}}(0)$ as in [24] (where its “un-integrated” value was found) and taking into account [18] that the regularized volume of $H^4$ is [1] $\Omega(H^4) = \frac{4 \pi^2}{3}$ while $\Omega(S^4) = \frac{8 \pi^2}{3}$ we conclude that $B_4$ and $\zeta_{H^4}(0)$ should be equal up to the factor of 2 coming from the ratio of the two volumes. Explicitly, given the operator $\hat{\Delta}_{s,\perp}$ corresponding values of the massless higher spin theory in AdS$_4$.

Let us note that the same conclusion applies also for the zero-mode terms in (2.28),(2.16) given by (cf. (1.4))

$$
\sum_{s=0}^{\infty} (k_{s-1,4} - k_{s-2,4}) = \sum_{s=1}^{\infty} \frac{1}{6} (s^2 + 5 s^4)
$$

This is equivalent to the expression obtained in [25] using (2.20). It was found there that the $\zeta$-function regularized sum of the values of $\zeta_{H^4}(0)$ over all massless spins $s > 0$ plus the $s = 0$ (scalar) contribution vanishes [12]. Let us note that the same conclusion applies also for the corresponding values of the massless higher spin $\zeta$-function computed on $S^4$ or dS$_4$: the sum over the zero-mode terms in (2.28),(2.16) given by (cf. (1.4))

$$
\sum_{s=1}^{\infty} (k_{s-1,4} - k_{s-2,4}) = \sum_{s=1}^{\infty} \frac{1}{6} (s^2 + 5 s^4)
$$

vanishes separately when $\zeta$-function regularized.

\[11\] In general, for even-dimensional case one has $\Omega(H^{2n}) = \pi^{n-\frac{1}{2}} \Gamma(-n + \frac{1}{2})$ [9].

\[12\] Note that the $s = 0$ value of $B_4^{(s=0)}$ in (2.32) is not equal to the conformal scalar contribution $-\frac{1}{90}$ but is twice this value (the reason is that here the “ghost” contribution $-B_4[\hat{\Delta}_{s-1,\perp}(M_{s-1,4})]$ does not vanish for $s = 0$ and effectively doubles the “physical” mode contribution). Regularizing the sum $\sum_{s=1}^{\infty} B_4^{(s=0)}$ with $\zeta$-function gives $-\frac{2}{90} \zeta(0) = \frac{1}{90}$ which cancels against the separate massless scalar contribution [4].
2.2 \( d = 6 \) case

According to (2.13) we should have the following relation between \( B_6 \) in (1.6) and the corresponding \( \zeta \)-function on \( S^6 \)

\[
B_6[\Delta_{s\perp}] = \zeta_{\Delta_{s\perp}}(0) - k_{s-1,6},
\]

(2.33)

where \( k_{s-1,6} \) is given in (2.19). The computation of the \( \zeta_{\Delta_{s\perp}}(0) \) in \( d = 6 \) uses (2.17),(2.18) and follows the same lines as in \( d = 4 \). The counterpart of (2.20) is

\[
\zeta_{\Delta_{s\perp}}(z) = \frac{1}{60}g_s \sum_{k=s+\frac{5}{2}}^{\infty} \frac{k(k^2 - \frac{1}{4})[k^2 - (s + \frac{3}{2})^2]}{(k^2 - h^2)^2}, \quad h^2 = s + \frac{25}{4} - M^2, \quad (2.34)
\]

and using (2.21) we get

\[
\zeta_{\Delta_{s\perp}}(z) = \frac{1}{60}g_s \sum_{m=0}^{\infty} c_m(z)h^{2m} \left[ \zeta_R(2z + 2m - 5, s + \frac{5}{2}) - (s^2 + 3s + \frac{5}{2})\zeta_R(2z + 2m - 3, s + \frac{5}{2}) + \frac{1}{4}(s + \frac{5}{2})^2\zeta_R(2z + 2m - 1, s + \frac{5}{2}) \right] \quad (2.35)
\]

Taking the limit \( z \rightarrow 0 \) gives (cf. (2.23))

\[
\zeta_{\Delta_{s\perp}}(0) = \frac{1}{60}g_s \left[ \zeta_R(-5, s + \frac{5}{2}) - (s^2 + 3s + \frac{5}{2})\zeta_R(-3, s + \frac{5}{2}) + \frac{1}{4}(s + \frac{5}{2})^2\zeta_R(-1, s + \frac{5}{2}) \right. \\
\left. + \frac{1}{6}h^6 - \frac{1}{4}(s^2 + 3s + \frac{5}{2})h^4 + \frac{1}{8}(s + \frac{5}{2})^2h^2 \right] \quad (2.36)
\]

As a result,

\[
\zeta_{\Delta_{s\perp}}(0) = \frac{(s+1)(s+2)(2s+3)}{453600} \left[ -210M^6 - 315M^4(s^2 + s - 10) + 630M^2(s^3 + 8s^2 + 8s - 24) + 22780 - 17514s - 15288s^2 + 2940s^3 + 945s^4 + 105s^6 \right] \quad (2.37)
\]

Then eq.(2.33) implies that (cf. (2.25))

\[
B_6[\Delta_{s\perp}(M^2)] = \frac{(s+1)(s+2)(2s+3)}{453600} \left[ -210M^6 - 315M^4(s^2 + s - 10) + 630M^2(s^3 + 8s^2 + 8s - 24) + 22780 - 17514s - 15288s^2 + 2940s^3 + 945s^4 + 105s^6 \right]. \quad (2.38)
\]

In particular, in the case of the conformal scalar \( s = 0 \), \( M^2 = \frac{d-2}{4(d-1)}R = \frac{1}{4}d(d - 2) = 6 \) we get \( B_6 = \frac{1}{750} \), i.e. the standard value (1.14).

It should be possible of course to find (2.38) directly from the general expression (2.28) for the \( b_6 \) heat kernel coefficient of a 2nd-order differential operator in curved space, but in the arbitrary spin \( s \) case in \( d = 6 \) this computation appears to be more involved than the one based on the \( \zeta \)-function on \( S^6 \) presented here.
3 \( B_6^{(s)} \) coefficient in conformal spin \( s \) partition function on \( S^6 \)

Let us now apply the general expression (2.38) to find the \( B_6^{(s)} \) coefficient corresponding to the CHS partition function (1.2) on \( S^6 \). Explicitly, in \( d = 6 \) we get

\[
Z_s(S^6) = \prod_{k=0}^{s-1} \left[ \det \hat{\Delta}_{k \perp}(M_{k,s}^2) \right]^{1/2} \prod_{k' = -1}^{s-1} \left[ \det \hat{\Delta}_{k \perp}(M_{s,k'}^2) \right]^{-1/2}, \quad M_{k,m}^2 = k - (m - 1)(m + 4). \tag{3.1}
\]

Using (2.38) we find for the total anomaly coefficient (cf. (1.11))

\[
B_6^{(s)} = \sum_{k' = -1}^{s-1} B_6[\hat{\Delta}_{s \perp}(s - (k' - 1)(k' + 4))] - \sum_{k = 0}^{s-1} B_6[\hat{\Delta}_{k \perp}(k - (s - 1)(s + 4))] \nonumber
\]

\[
= - \frac{(s+1)^2(s+2)^2}{151200} (22s^6 + 198s^5 + 671s^4 + 1056s^3 + 733s^2 + 120s - 50) . \tag{3.2}
\]

Let us note that the \( k = s - 1, \ k' = s - 1 \) terms in (3.1) represent the partition function of massless spin \( s \) field on \( S^6 \) (or dS\(_6\)) which is the same as the AdS\(_6\) one in (1.4) up to the sign of the dimensionless mass parameters: on \( S^6 \) we have

\[
M_{s,s-1}^2 = -s^2 + 6 , \quad M_{s-1,s}^2 = -s^2 - 2s + 3 . \tag{3.3}
\]

We find for the contribution of this massless spin \( s \) factor (cf. (2.32))

\[
B_6^{(s0)} = B_6[\hat{\Delta}_{s \perp}(M_{s,s-1}^2)] - B_6[\hat{\Delta}_{s-1 \perp}(M_{s-1,s}^2)] \nonumber
\]

\[
= - \frac{(s+1)^2}{15120} (63s^6 + 378s^5 + 847s^4 + 868s^3 + 378s^2 + 28s - 20) . \tag{3.4}
\]

For \( s = 0 \) this equals to the conformal scalar value (1.14) as in this case \( B_6[\hat{\Delta}_{s-1 \perp}(M_{s-1,s}^2)] \) vanishes.

4 \( a_s \) coefficient in ratio of massless spin \( s \) partition functions in AdS\(_7\)

Let us now show that exactly the same expression (3.2) appears as a coefficient of the IR divergent term in the ratio of the massless spin \( s \) partition functions in AdS\(_7\) in the l.h.s. of eq.(1.3). We shall first review the general expression for this coefficient found in [2] and then apply it to the case of \( d = 6 \).

Starting with a mass \( m \) spin \( s \) operator in AdS\(_{d+1}\) of unit radius (\( \epsilon = -1 \))

\[
\hat{\Delta}(M^2)_{s \perp} = (-\nabla^2 + M^2 \epsilon)_{s \perp} , \quad M^2 = -m^2 + s - (s - 2)(s + d - 2) , \tag{4.1}
\]

one finds that the powers of near-boundary asymptotics of the corresponding solutions are \( \gamma_\pm = \Delta_\pm - s \) where [29]

\[
\Delta_\pm(m) = \frac{1}{2} d \pm \sqrt{m^2 + (s + \frac{1}{2} d - 2)^2} = \frac{1}{2} d \pm \sqrt{\frac{1}{4} d^2 + 3s - 4 - M^2} , \tag{4.2}
\]

\[
\Delta_+ \equiv \Delta_+(0) = s + d - 2 , \quad \Delta_- \equiv \Delta_-(0) = 2 - s , \quad \Delta_- = d - \Delta_+ . \tag{4.3}
\]
These $\Delta_\pm$ apply to the physical (spin $s$) part of (1.4) while for the “ghost” (spin $s-1$) part of (1.4) $\Delta'_+ = s + d - 1$, $\Delta'_- = 1 - s$ [4]. As discussed in the Introduction, the partition function of a constant-mass operator on AdS$_{d+1}$ is proportional to its volume which for even $d$ is IR divergent (see (1.5)). Calling the coefficient of the ln $L$ term in the corresponding free energy $F = \frac{1}{2} \ln \det \Delta_{s,1}(M^2)$ as $a_s(\Delta)$ where $\Delta = \Delta_+$ in (1.2) one finds that [4]

$$\delta a_s(\Delta) \equiv a_s(\Delta) - a_s(d - \Delta)$$

$$= -\frac{2g_s(d+1)}{\pi d!} \int^\Delta_1 dx (x - \frac{1}{2}d)(x + s + 1)(x - s + d + 1)\Gamma(x - 1)\Gamma(d - 1 - x)\sin(\pi x)$$

(4.4)

where $g_s(d+1)$ is the same as $g_s$ in (2.4) with $d \to d + 1$. Then the coefficient $a_s$ corresponding to the ratio of the partition functions appearing in the l.h.s. of (1.3) can be found as

$$a_s = \delta a_s(\Delta_+) - \delta a_{s-1}(\Delta'_+) = \delta a_s(s + d - 2) - \delta a_{s-1}(s + d - 1)$$

(4.5)

The special cases of $d = 2$ and $d = 4$ were already discussed in [4]. Doing the integral in (4.4) gives [3]

$$d = 2 : \quad \delta a_s(\Delta) = \frac{2}{3}(\Delta - 1) [3s^2 - (\Delta - 1)^2]$$

(4.6)

$$d = 4 : \quad \delta a_s(\Delta) = \frac{(s+1)^2}{180} (\Delta - 2)^3 [5(s+1)^2 - 3(\Delta - 2)^2]$$

(4.7)

Using these expressions in (4.5) leads to (here for $d = 2$ $s \geq 2$ and $a_0 = \frac{1}{3}$, $a_1 = \frac{1}{3}$)

$$d = 2 : \quad a_s = \frac{3}{2} + 4s(s - 1)$$

(4.8)

$$d = 4 : \quad a_s = \frac{s^2(s+1)^2}{180} (14s^2 + 14s + 3)$$

(4.9)

Thus in $d = 4$ one finds $a_s$ in (1.9) that matches $B_4^{(s)}$ (1.11) derived in [4] directly from (1.1) (see also section 2.1).

The $d = 2$ coefficient (4.8) (rescaled by $-3$) was interpreted in [4] as the central charge $c_s = -2[1 + 6s(s-1)]$ ($s \geq 2$) of the first-order bc-ghost system with weights $s$ and $1 - s$ corresponding to spin $s$ W-gravity field [30, 31, 32]. In Appendix we shall demonstrate that the AdS$_3$ prediction (4.8) matches the $B_2$ anomaly coefficient for the $d = 2$ case of the conformal higher spin partition function (1.2).

Let us now consider the $d = 6$ case. Computing the integral in (4.4) we get (cf. (4.6), (4.7))

$$\delta a_s(\Delta) = \frac{(s+1)(s+2)^2(3s+1)^2}{453600} (\Delta - 3)^3 \left[ -35(s+2)^2 + 21[(s+2)^2 + 1](\Delta - 3)^2 - 15(\Delta - 3)^4 \right]$$

(4.10)

Let us recall again that the normalization of $a_s$ in (4.4) is such that it is the coefficient of the logarithm of the radius of $S^d$, i.e. it is equal to minus the corresponding value of $B_d$: in the case of $d = 6$ for $s = 0$, $\Delta = \frac{d}{2} + 1 = 4$ eq. (4.10) gives $-\frac{1}{756} = -B_6^{(0)}$ (cf. (1.13)).

Applying (4.10) to the case of (4.5) we find

$$d = 6 : \quad a_s = \delta a_s(s + 4) - \delta a_{s-1}(s + 5)$$

$$= \frac{(s+1)^2(s+2)^2}{151200} (22s^6 + 198s^5 + 671s^4 + 1056s^3 + 733s^2 + 120s - 50)$$

(4.11)

This is the same expression as in (1.15), i.e. it matches the expression (3.2) for $-B_6^{(s)}$ found above directly from the CHS partition function on $S^6$.

In $d = 2$ one finds from (2.4) that $g_s^{(d+1)} = 2$ for $s \geq 1$ and $1$ for $s = 0$. 

13
5 Concluding remarks

To summarize, in this paper we have shown the agreement (1.8) between the UV divergence coefficient $B_6^{(s)} (3.2)$ of the conformal higher spin partition function on $S^6$ and the IR divergence coefficient $a_s$ in the ratio of massless higher spin partition functions with alternate boundary conditions on AdS$_7$. Together with the corresponding $d = 4$ results of [4, 11] this provides a non-trivial test of the relation (1.3). We also demonstrate a similar matching in the $d = 2$ case in Appendix below.

In $d = 4$ the sum of the anomaly coefficients $a_s$ in (1.9) over all spins $s$ vanishes [4] when computed using the standard $\zeta$-function prescription. The same is true also for the sum of the $s \geq 1$ massless spin $s$ divergence coefficients in (2.32) plus the $s = 0$ conformal scalar contribution [25]. In the $d = 6$ case we discussed here the corresponding sums of the coefficients in (3.2) and in (3.4) do not appear to vanish. This may not be surprising since in $d = 6$ there is no a priori reason to sum over all spins with weight one and, moreover, to consider only totally symmetric traceless tensor representation [14].

In general, it would be interesting also to study the $d = 6$ conformal higher spin partition function on other backgrounds, e.g., on Ricci-flat one as in $d = 4$ case in [1]. The corresponding covariant and Weyl-invariant CHS action should have the structure $\int d^6x \sqrt{g} \phi_s (\nabla^2 \phi_s + ...)\phi_s = \int d^6x \sqrt{\mathcal{G}} C_{2s} (\nabla^2 + ...)C_{2s}$, where the rank $2s$ tensor $C_{2s}$ is a gauge-covariant CHS field strength $C_{2s} \sim P_s \nabla^s \phi_s + ...$. This action is known explicitly only for lowest values of the spin. For $s = 1$ the field strength $C_2$ is the antisymmetric tensor and the 2nd order Weyl-covariant operator $(\nabla^2 + ...)$ acting on it can be found, e.g., in [33]. For $s = 2$ the field strength $C_4$ is the same as the Weyl tensor and the corresponding Weyl-covariant operator $(\nabla^2 + ...)$ is the same that appears in the $I_1 \sim C(\nabla^2 + ...)C$ term in the trace anomaly (1.12) [12] (see also [33]). The “minimal” $d = 6$ Weyl gravity action $\int d^6x \sqrt{\mathcal{G}} I_1$ (which can be expressed in terms of Ricci tensor $I_1 \sim R_{ab} (\nabla^2 + ...)R_{mn}$ [12]) admits an equivalent representation [34] in terms of a collection of fields with ordinary (2nd-derivative) kinetic terms. Such an ordinary-derivative description of the CHS field with any spin $s$ and in any even dimension $d$ is known in flat space [19], and, following the $s = 2$ example [34], it may serve as a starting point for constructing a covariant CHS $s \geq 2$ actions in generic curved backgrounds.

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14For example, one may include also the self-dual 2-form field which contributes $221/210$ to the Weyl anomaly coefficient $B_6$ on $S^6$ [15].

15As was shown in [34], the other two Weyl invariants $I_2, I_3 \sim CCC$ may also be expressed in terms of fields of the “ordinary-derivative” formulation but that leads to higher than second derivative terms and that may be considered as an argument for $I_1$ as the natural conformal spin 2 action in $d = 6$ provided one uses the ordinary-derivative formulation of [34] as a starting point.
Appendix:
Partition function and $B_2$ coefficient of conformal higher spins on $S^2$

Here we shall show that the AdS$_3$ prediction for $a_s$ [4,6] is indeed the same (1.8) as the logarithmic UV divergence coefficient $B_2$ in the conformal higher-spin partition function (1.1) specialised to the $d = 2$ case.

Naively, the $d = 2$ limit of the conformal higher spin action should start with a $\partial^{2s+d-4} = \partial^{2s-2}$ term ($s \geq 2$). However, the $d = 2$ case of the CHS theory is special – here the number of components $N_s$ (2.5) of a symmetric traceless rank $s$ tensor is $s$-independent: $N_s = 2$ for $s \geq 1$ ($N_s = 1$ for $s = 0$). Then the number of the corresponding transverse components $g_s = N_s$ (2.3) vanishes for $s \geq 2$: $N_{s\perp} = N_s - N_{s-1} = 0$ ($N_{1\perp} = 1$). Equivalently, a symmetric rank tensor CHS field $\phi_s$ can be completely gauged away by a combination of the gradient gauge symmetry (generalized reparametrizations) and the algebraic gauge symmetry (generalized Weyl symmetry), i.e. there is no non-trivial gauge-invariant field strength $C_{2s} \sim P_s \partial^s \phi_s$ (this is an $s \geq 3$ generalization of the fact of the absence of Weyl tensor in $d = 2$).

Thus the classical $d = 2$ CHS action is trivial (a familiar fact for $s = 2$ or gravity in $d = 2$). Still, non-zero contributions to the corresponding partition function may come from the gauge-fixing or ghost sector. Indeed, the number of dynamical degrees of freedom of a CHS field in $d = 2$ as following from the general expression in (2.7) is $\nu_s = -2$ (again, a well-known result for $d = 2$ gravity with trivial Einstein term action). More precisely, the CHS action in the path integral for the partition function in a background covariant harmonic gauge ($\nabla \cdot \phi_s = 0$) will have actually a non-trivial $\phi_s \partial^{2s-2} \phi_s + ...$ kinetic term but it will be coming solely from the gauge-fixing term. Thus, despite the triviality of the classical gauge-invariant CHS action, the corresponding partition function will still contain “physical” determinants of spin $s$ operators coming from the gauge-fixing term.

Indeed, the $d = 2$ limit of the CHS partition function (1.2) is found to be

$$Z_s(S^2) = \prod_{k=0}^{s-1} \left[ \frac{\det \hat{\Delta}_{k\perp}(k - s(s - 1))}{\det \hat{\Delta}_{s\perp}(s - k'(k' - 1))} \right]^{1/2} \prod_{k'=1}^{s-1} \left[ \frac{\det \hat{\Delta}_{k\perp}(k - s(s - 1))}{\det \hat{\Delta}_{s\perp}(s - k'^2 + k')} \right]^{1/2} . \quad (A.1)$$

Using (2.26) this may be written explicitly in terms of unconstrained operators as ($\hat{\Delta}_{-\perp} \equiv 1$)

$$Z_s(S^2) = \prod_{k=0}^{s-1} \left[ \frac{\det \hat{\Delta}_k(k - s^2 + s)}{\det \hat{\Delta}_{-\perp}(k - s^2 - s + 1)} \right]^{1/2} \prod_{k'=1}^{s-1} \left[ \frac{\det \hat{\Delta}_{-\perp}(1 - s - k'^2 + k')}{\det \hat{\Delta}_s(k'^2 - k')} \right]^{1/2} . \quad (A.2)$$

Given an operator $\hat{\Delta}_k(M^2) = -\nabla^2 + M^2$ defined on unconstrained symmetric traceless rank $k$ tensor the corresponding Seeley coefficient (1.7) in the free energy (1.6) on unit-radius $S^2$ (with curvature $R = d(d - 1) = 2$) is

$$B_2[\hat{\Delta}_k(M^2)] = \frac{1}{4\pi} \Omega(S^2) b_2 = b_2 , \quad b_2 = N_k \left( \frac{1}{6} R - M^2 \right) , \quad k \geq 1 : \quad b_2 = 2 \left( \frac{1}{3} - M^2 \right) , \quad k = 0 : \quad b_2 = \frac{1}{3} - M^2 . \quad (A.3)$$

\footnote{For example, for $s = 2$ the two components of the traceless rank 2 tensor $\phi_2$ (or $h_{mn}$ fluctuation of metric) will enter as $(\nabla^m h_{mn})^2 \sim h_+ + \nabla^2 h_{--} + ...$}
Applying (2.27) we find
\begin{align*}
k \geq 2 : \quad & B_2[\hat{\Delta}_{k+1}(M^2)] = B_2[\hat{\Delta}_k(M^2)] - B_2[\hat{\Delta}_{k-1}(M^2 - 2k + 1)] = -4k + 2, \quad (A.5) \\
& B_2[\hat{\Delta}_{1\perp}(M^2)] = -2 - M^2, \quad B_2[\hat{\Delta}_{0\perp}(M^2)] = B_2[\hat{\Delta}_0(M^2)] = \frac{1}{3} - M^2. \quad (A.6)
\end{align*}
Then the total $B_2$ coefficient in free energy (1.6) corresponding to (A.1) is $(s \geq 2)$
\begin{align*}
B_2^{(s)} &= \sum_{k'=0}^{s-1} B_2[\hat{\Delta}_{s\perp}(s-k'(k'-1))] - \sum_{k=0}^{s-1} B_2[\hat{\Delta}_{k\perp}(k-s(s-1))] \\
&= B_2[\hat{\Delta}_{s\perp}(s)] - B_2[\hat{\Delta}_{1\perp}(1-s(s-1))] - B_2[\hat{\Delta}_{0\perp}(s(s-1))] - 4 \sum_{k=2}^{s-1} (s-k) \\
&= -\frac{2}{3} - 4s(s-1). \quad (A.7)
\end{align*}
In the conformal 2d vector $s = 1$ case (corresponding to the Schwinger $\int F\partial^{-2}F = \int A_{m\perp}^2$ action) we get from (A.1) $Z_1 = [\det \hat{\Delta}_0(0)]^{1/2}$ and thus $B_2^{(1)} = -\frac{1}{2}$. This matches the expression for $a_s$ (4.5) found from AdS$_3$, in line with the $d = 4$ and $d = 6$ tests of (1.3), (1.8) discussed above.

The $d = 2$ CHS model discussed here is, of course, closely related to spin $s$ W-gravity model [31]: both have the same linearized symmetries – generalized reparametrizations and Weyl transformations for spin $s$ field. The resulting conformal anomaly coefficient (A.7) is indeed equivalent to the quantum W-gravity anomaly given solely by the corresponding bc ghost contribution to the central charge $c_{gh} = -2(1 + 6s^2 - 6s)$ [31] [32] [17]. What is remarkable about the above derivation of this result from the CHS partition function (A.1) is that it illustrates that the $d = 2$ case, while somewhat degenerate (having trivial classical action), can still be viewed as a limit of $d$-dimensional conformal higher spin theory (which itself may then be interpreted as a natural $d > 2$ generalization of W-gravity) [18].

\[17\] In standard normalization with $c = 1$ for a real scalar one has $B_2 = \frac{1}{16\pi^2} \int d^2x \sqrt{-g}R$ or $B_2 = \frac{1}{4}c$ on $S^2$.

\[18\] It may be useful also to comment on a special nature of the $d = 2$ case regarding the structure of induced actions. Starting with a matter Lagrangian coupled to a CHS field and integrating out the matter field one, in general, gets a local logarithmically divergent term proportional to Weyl-invariant CHS action. In $d = 2$ this term is trivial which is related to the fact that in $d = 2$ the trace anomaly does not contain a Weyl-invariant B-type part [13] and is consistent with the vanishing of a gauge-invariant CHS action. The induced action will contain, of course, also finite non-local terms which, being anomalous, are not Weyl-invariant and thus are not candidates for a “critical” (i.e. fully symmetric) CHS action. Indeed, the induced actions for W-gravity spin $s$ field $\varphi_s$ discussed in [31] [35] may be written (generalizing the $s = 2$ Polyakov induced $d = 2$ gravity action) as $\int d^2x \left( R_{2s} \partial^{-2}R_{2s} + \ldots \right)$. Here $R_{2s} = \partial^\nu \varphi_s + \ldots$ is the higher spin curvature, which is invariant under the generalized reparametrizations but not under the generalized Weyl transformations.
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