Polynomial bounds for chromatic number.
I. Excluding a biclique and an induced tree

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Abstract

Let $H$ be a tree. It was proved by Rödl that graphs that do not contain $H$ as an induced subgraph, and do not contain the complete bipartite graph $K_{t,t}$ as a subgraph, have bounded chromatic number. Kierstead and Penrice strengthened this, showing that such graphs have bounded degeneracy. Here we give a further strengthening, proving that for every tree $H$, the degeneracy is at most polynomial in $t$. This answers a question of Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak.
1 Introduction

The Gyárfás-Sumner conjecture [6, 15] asserts:

1.1 Conjecture: For every forest $H$, there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $H$-free graph $G$.

(We use $\chi(G)$ and $\omega(G)$ to denote the chromatic number and the clique number of a graph $G$, and a graph is $H$-free if it has no induced subgraph isomorphic to $H$.) One attractive feature of this conjecture is that it is best possible in a sense: for every graph $H$ that is not a forest, there is no function $f$ as in 1.1 (because, as shown by Erdős [4], there are graphs with arbitrarily large chromatic number and girth). The conjecture has been proved for some special families of trees (see, for example, [3, 7, 8, 9, 11, 12, 13]) but remains open in general.

A class $C$ of graphs is $\chi$-bounded if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G$ that is an induced subgraph of a member of $C$ (see [14] for a survey). Thus the Gyárfás-Sumner conjecture asserts that the class of all $H$-free graphs is $\chi$-bounded, for every forest $H$. For some $\chi$-bounded classes, the function $f$ can be taken to be polynomial, and it remains open whether for every forest $H$, there is a polynomial $f$ that satisfies 1.1. (Indeed, Esperet [5] made the even stronger conjecture that, for every $\chi$-bounded class, $f$ can always be chosen to be a polynomial, but this has recently been shown to be false [2].)

The complete bipartite graph with parts of cardinality $s, t$ is denoted by $K_{s,t}$. Let us define $\tau(G)$ to be the largest $t$ such that $G$ contains $K_{t,t}$ as a subgraph (not necessarily induced). It was proved by Rödl (mentioned in [10], and see also [8]) that the analogue of the Gyárfás-Sumner conjecture is true if we replace $\omega(G)$ by $\tau(G)$. Explicitly:

1.2 For every forest $H$, there is a function $f$ such that $\chi(G) \leq f(\tau(G))$ for every $H$-free graph $G$.

This has the same attractive feature that the result is best possible (in the same sense).

This result was strengthened by Kierstead and Penrice. Let us say a graph $G$ is $d$-degenerate (where $d \geq 0$ is an integer) if every nonnull subgraph has a vertex of degree at most $d$; and the degeneracy $\partial(G)$ of $G$ is the smallest $d$ such that $G$ is $d$-degenerate. Then $\chi(G) \leq \partial(G) + 1$, and so the following result of Kierstead and Penrice [9] is a strengthening of 1.2:

1.3 For every forest $H$, there is a function $f$ such that $\partial(G) \leq f(\tau(G))$ for every $H$-free graph $G$.

What about the analogue of Esperet’s question: do 1.2 and 1.3 remain true if we require $f$ to be a polynomial in $\tau(G)$? This question was raised by Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak in [1], and they proved it when $H$ is a path, that is:

1.4 For every path $H$, there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every $H$-free graph $G$.

In this paper we answer the question completely. Our main result is:

1.5 For every forest $H$, there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every $H$-free graph $G$.

We also look at a related question: what can we say about $\chi(G)$ and $\partial(G)$ if $G$ is $H$-free and does not contain $K_{s,t}$ as a subgraph? More exactly, if $H, s$ are fixed, how do $\chi(G)$ and $\partial(G)$ depend on $t$? We will show that the dependence is in fact linear in $t$: 1
1.6 For every forest $H$ and every integer $s > 0$, there exists $c > 0$ such that for every graph $G$ and every integer $t > 0$, if $G$ is $H$-free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) \leq ct$.

We also prove a weaker result, that under the same hypotheses, $\chi(G) \leq ct$, and for this the bound on $c$ is a small function of $s, H$.

Finally, there is a second pretty theorem in the paper [1] of Bonamy, Bousquet, Pilipczuk, Rzążewski, Thomassé and Walczak:

1.7 Let $\ell$ be an integer; then there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every graph $G$ with no induced cycle of length at least $\ell$.

We give a new proof of this, simpler than that in [1].

In this paper, all graphs are finite and have no loops or parallel edges. We denote by $|H|$ the number of vertices of a graph $H$. If $X \subseteq V(G)$, we denote the subgraph of $G$ induced on $X$ by $G[X]$. We use “$G$-adjacent” to mean adjacent in $G$, and “$G$-neighbour” to mean a neighbour in $G$, and so on.

2 Producing a path-induced rooted tree.

We will prove 1.5 in this section and the next. We need to show that if a graph $G$ has degeneracy at least some very large polynomial in $t$ (independent of $G$), and does not contain $K_{t,t}$ as a subgraph, then it contains any desired tree as an induced subgraph. We will show this in two stages: in this section we will show that $G$ contains a large (with degrees a somewhat smaller polynomial in $t$) “path-induced” tree, and in the next section we will convert this to the desired induced tree. “Path-induced” means that each path of the tree starting at the root is an induced path of $G$; so we should be talking about rooted trees. Let us say this carefully.

A rooted tree $(H, r)$ consists of a tree $H$ and a vertex $r$ of $H$ called the root. A rooted subtree of $(H, r)$ means a rooted tree $(J, r)$ where $J$ is a subtree of $H$ and $r \in V(J)$. The height of $(H, r)$ is the length (number of vertices) of the longest path of $H$ with one end $r$. If $u, v \in V(H)$ are adjacent and $u$ lies on the path of $H$ between $v, r$, we say $v$ is a child of $u$ and $u$ is the parent of $v$. The spread of $H$ is the maximum over all vertices $u \in V(H)$ of the number of children of $u$. (Thus the spread is usually one less than the maximum degree.) Let $H$ be a subgraph of $G$ (not necessarily induced), where $(H, r)$ is a rooted tree. We say that $(H, r)$ is a path-induced rooted subgraph of $G$ if every path of $H$ with one end $r$ is an induced subgraph of $G$.

Let $\zeta, \eta \geq 1$. The rooted tree $(H, r)$ is $(\zeta, \eta)$-uniform if

- every vertex with a child has exactly $\zeta$ children;
- every vertex with no child is joined to $r$ by a path of $H$ of length exactly $\eta$.

We need two lemmas:

2.1 Let $k, \zeta, \eta \geq 1$ with $\zeta \geq 2$, and let $(H_1, r_1), \ldots, (H_k, r_k)$ be $(k\zeta^{n+1}, \eta)$-uniform rooted trees, each a subgraph of a graph $G$, such that $r_i \notin V(H_j)$ for all distinct $i, j \in \{1, \ldots, k\}$. Then for $1 \leq i \leq k$ there is a $(\zeta, \eta)$-uniform rooted subtree $(H'_i, r_i)$ of $(H_i, r_i)$, such that the trees $H'_1, \ldots, H'_k$ are pairwise vertex-disjoint.
Proof. Choose \( j \leq k \) maximum such that there are \((\zeta, \eta)\)-uniform rooted subtrees \((H'_i, r_i)\) of \((H_i, r_i)\) for \(1 \leq i \leq j\), such that the trees \(H'_1, \ldots, H'_j\) are pairwise vertex-disjoint. Let \( X = V(H'_1) \cup \cdots \cup V(H'_j) \). Thus \( |X| \leq j\zeta^{\eta+1} \), since each \( H'_i \) has
\[ 1 + \zeta + \zeta^2 + \cdots + \zeta^\eta \leq \zeta^{\eta+1} \]
vertices (here we use that \( \zeta \geq 2 \)). Suppose that \( j < k \). Then each vertex of \((H_{j+1}, r_{j+1})\) with a child has at least \((k - j)\zeta^{\eta+1} \geq \zeta^{\eta+1} \geq \zeta \) children not in \( X \); and since \( r_{j+1} \notin X \), it follows that there is a \((\zeta, \eta)\)-uniform rooted subtree \((H'_{j+1}, r_{j+1})\) of \((H_{j+1}, r_{j+1})\) vertex-disjoint from \( X \), contrary to the maximality of \( j \). Thus \( j = k \), and this proves 2.1.

Let \((T, r)\) be a rooted tree, where \( T \) is a subgraph of \( G \). For \( t > 0 \), a vertex \( u \in V(G) \) is \( t\)-bad for \((T, r)\) if there is a vertex \( v \in V(T) \) such that \( u \) is distinct from and \( G\)-adjacent to more than \( d(1 - 1/t) \) children of \( w \), where \( d \) is the number of children of \( w \). We will often use the following:

2.2 Let \( t, \eta \geq 1 \) and \( \zeta \geq 2 \) be integers. Let \((T, r)\) be a \((t\zeta, \eta)\)-uniform rooted tree, where \( T \) is a subgraph of \( G \); and let \( u \in V(G) \setminus V(T) \). If \( u \) is not \( t\)-bad for \((T, r)\), then there is a \((\zeta, \eta)\)-uniform rooted subtree \((S, r)\) of \((T, r)\) such that \( u \) has no \( G\)-neighbour in \( V(S) \) except possibly \( r \).

We omit the proof, which is clear. The second lemma is:

2.3 Let \( t, \eta \geq 1 \) and \( \zeta \geq 2 \) be integers, where \( t \) divides \( \zeta \). Let \( G \) be a graph that does not contain \( K_{t, t} \) as a subgraph, and let \((T, r)\) be a \((\zeta, \eta)\)-uniform rooted tree, where \( T \) is a subgraph of \( G \). Then fewer than \( \zeta^\eta \) vertices in \( V(G) \) are \( t\)-bad for \((T, r)\).

Proof. There are \( \zeta^\eta/(\zeta - 1) \) vertices in \( V(T) \) that have children (since \( \zeta \geq 2 \)). Let \( w \in V(T) \) with \( \zeta \) children, and let \( C_w \) be the set of its children in \((T, r)\). Suppose that there are \( t \) distinct vertices \( u_1, \ldots, u_t \) in \( V(G) \) such that each is \( G\)-nonadjacent to more than \( |C_w|/t \) vertices of \( C_w \), and hence to at least \( |C_w|/1/t + 1 \) such vertices, since \( t \) divides \( |C_w| \).

It follows that each \( u_i \) is equal or \( |C_w|/t - 1 \) \( G\)-nonadjacent to at most \( |C_w|/t - 1 \) vertices of \( C_w \), and so at most \( t(|C_w|/t - 1) \) vertices of \( C_w \) belong to or have a \( G\)-non-neighbour in \( \{u_1, \ldots, u_t\} \). Consequently at least \( t \) vertices in \( C_w \) are \( G\)-adjacent to all of \( u_1, \ldots, u_t \), contradicting that \( G \) does not contain \( K_{t, t} \) as a subgraph. Thus there are at most \( t - 1 \leq \zeta - 1 \) vertices in \( V(G) \) with more than \( |C_w|/(t - 1)/t \) \( G\)-neighbours in \( C_w \). So the number of vertices in \( V(G) \) that are \( t\)-bad for \((T, r)\) is at most \( \zeta - 1 \) times the number of vertices of \( T \) that have children, and so smaller than \( \zeta^\eta \). This proves 2.3.

The main result of this section is the following:

2.4 Let \( \eta > 0 \) be an integer and let \( c = (\eta + 1)! \). Let \( \zeta \geq 2 \), and let \((H, r)\) be a rooted tree of height at most \( \eta \), and spread at most \( \zeta \). Let \( t \geq 1 \) be an integer, and suppose that the graph \( G \) does not contain \( K_{t, t} \) as a subgraph, and does not contain a rooted tree isomorphic to \((H, r)\) as a path-induced rooted subgraph. Then \( d(G) \leq (\zeta t)^c \).

Proof. We may assume that \( t \geq 2 \). We proceed by induction on \( \eta \). If \( \eta = 1 \), it follows that \( G \) has maximum degree at most \( \zeta - 1 \), since it does not contain \((H, r)\) as a path-induced rooted subgraph; and so \( d(G) \leq \zeta - 1 \leq (\zeta t)^c \) as required. So we may assume that \( \eta \geq 2 \), and the result holds for all
rooted trees with height less than \( \eta \). Let \( c' = \eta! \) and \( \zeta' = t\zeta^{\eta+1} \). Let us say a limb is a \((\zeta', \eta - 1)\)-uniform rooted tree that is a path-induced rooted subgraph of \( G \).

(1) For each vertex \( u \), there are at most \( \zeta - 1 \) \( G \)-neighbours \( v \) of \( u \) with the property that there is a limb \((J, v)\) of \( G \) such that \( u \notin V(J) \) and \( u \) is not \( t \)-bad for \((J, v)\).

Suppose there are \( \zeta \) such vertices \( v_1, \ldots, v_\zeta \), and let the corresponding limbs be \((J_i, v_i)\) for \( 1 \leq i \leq \zeta \). By 2.2, for \( 1 \leq i \leq \zeta \), there is a \((\zeta^{\eta+1}, \eta - 1)\)-uniform rooted subtree \((J'_i, v_i)\) of \((J_i, v_i)\), such that \( u \) has no neighbour in \( V(J'_i) \) except \( v_i \). By 2.1, there is a \((\zeta, \eta - 1)\)-uniform rooted subtree \((H'_i, r_i)\) of \((J'_i, r_i)\) for \( 1 \leq i \leq \zeta \), such that the trees \( H'_1, \ldots, H'_\zeta \) are pairwise vertex-disjoint. But then adding \( u \) to the union of these trees gives a \((\zeta, \eta)\)-uniform rooted tree, and it is path-induced in \( G \), and contains a rooted induced subgraph isomorphic to \((H, r)\), a contradiction. This proves (1).

Let \( P \) be the set of vertices \( v \) of \( G \) such that there is a limb with root \( v \), and let \( Q = V(G) \setminus P \). For each \( v \in P \), there is at least one limb with root \( v \); select one, and call it \((J_v, v)\). For each edge \( e \) with at least one end in \( P \), select one such end, and call it the head of \( e \).

- Let \( A \) be the set of all edges with both ends in \( Q \);
- Let \( B \) be the set of all edges \( uv \) of \( G \) with head \( v \), such that \( u \notin V(J_v) \), and \( u \) is not \( t \)-bad for \((J_v, v)\);
- Let \( C \) be the set of all edges \( uv \) of \( G \) with head \( v \), such that \( u \notin V(J_v) \), and \( u \) is \( t \)-bad for \((J_v, v)\);
- Let \( D \) be the set of all edges \( uv \) of \( G \) with head \( v \), such that \( u \in V(J_v) \).

Thus every edge of \( G \) belongs to exactly one of \( A, B, C, D \). Since \( G[Q] \) does not contain a limb, the inductive hypothesis implies that \( \partial(G[Q]) \leq (\zeta t)^{c'} \). Consequently

\[
|A| \leq (\zeta t)^{c'} |Q| \leq (\zeta t)^{c'} |G|.
\]

By (1), for each vertex \( u \in V(G) \), there are at most \( \zeta - 1 \) edges \( uv \in B \) with head \( v \); and so

\[
|B| \leq (\zeta - 1) |G|.
\]

For each \( v \in P \), there are at most \( \zeta^{\eta-1} \) edges \( uv \in C \) with head \( v \) by 2.3, and so

\[
|C| \leq \zeta^{\eta-1} |P| \leq \zeta^{\eta-1} |G|.
\]

For each \( v \in P \), since \((J_v, v)\) is path-induced, there are at most \( \zeta' \) edges \( uv \in D \) with head \( v \), and so

\[
|D| \leq \zeta' |P| \leq \zeta' |G|.
\]

Summing, we deduce that

\[
|E(G)| \leq \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta-1} + \zeta' \right) |G|,
\]
and so some vertex of \( G \) has degree at most \( 2 \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta - 1} + \zeta' \right) \). Since this also holds for every non-null induced subgraph of \( G \), we deduce that
\[
\partial(G) \leq 2 \left( (\zeta' t)^{c'} + (\zeta - 1) + \zeta^{\eta - 1} + \zeta' \right).
\]

We recall that \( \zeta' = t\zeta^{\eta + 1} \) and \( c = (\eta + 1)c' \); and so
\[
\partial(G) \leq 2 \left( c^{(\eta + 1)c'} t^{c'} + (\zeta - 1) + \zeta^{\eta - 1} + \zeta^{\eta + 1} t \right) 
\leq 2c \left( t^{c'} + 1 + t^{\eta - 1} + t \right) 
\leq 8c t^{c'} \leq \zeta c t^{c'}
\]
(since \( c \geq c' + 3 \) and \( t \geq 2 \)). This proves 2.4.

We remark that 2.4 implies 1.4, and a strengthening:

2.5 If \( H \) is a path, and \( t \geq 1 \) is an integer, and \( G \) is \( H \)-free and does not contain \( K_{t,t} \) as a subgraph, then \( \partial(G) \leq (2t)^{|H|!} \).

**Proof.** Let \( \zeta = 2 \), and \( \eta = |E(H)| = |H| - 1 \). Let \( r \) be one end of \( H \). Then \( G \) does not contain \((H, r)\) as a path-induced rooted subgraph, and so \( \partial(G) \leq (2t)^{|H|!} \) by 2.4. This proves 2.5.

3 Growing a tree

If \((T, r)\) is a rooted tree and \( v \in V(T) \), the **height of \( v \)** in \((T, r)\) is the number of edges in the path between \( v, r \); and so the height of \((T, r)\) is the largest of the heights of its vertices. Let \((T, r)\) be a rooted tree, and let \((S, r)\) be a rooted subtree. The graph obtained from \( T \) by deleting all the edges of \( S \) is disconnected, and each of its components contains a unique vertex of \( S \); for each \( v \in V(S) \), let \( T_v \) be the component that contains \( v \in V(S) \). We call the rooted tree \((T_v, v)\) the **decoration of \( S \) at \( v \) in \( T \)**.

Let \( G \) be a graph, let \((S, r)\) be a rooted tree, and let \( \zeta \geq 2 \) and \( \eta \geq 1 \). We say that \((S, r)\) is \( (\zeta, \eta) \)-decorated in \( G \) if \( S \) is an induced subgraph of \( G \) with height at most \( \eta - 1 \), and there is a rooted tree \((T, r)\) with the following properties:

- \((S, r)\) is a rooted subtree of \((T, r)\), and \((T, r)\) is a path-induced rooted subgraph of \( G \);
- for each \( u \in V(S) \) and \( v \in V(T) \setminus V(S) \), if \( u, v \) are \( G \)-adjacent then they are \( T \)-adjacent;
- for each \( v \in V(S) \), the decoration of \( S \) at \( v \) in \( T \) is \((\zeta, \eta - h)\)-uniform, where \( h \) is the height of \( v \) in \((S, r)\).

Thus, informally, \( T \) is obtained from \( S \) by attaching to \( S \) uniform trees rooted at each vertex of \( S \). Note that \( T \) is only required to be path-induced: the various uniform trees that are attached to \( S \) might have edges between them.

In view of 2.4, if we have a graph \( G \) with huge degeneracy that does not contain \( K_{t,t} \), then it contains a \((\zeta, \eta)\)-uniform rooted tree \((T, r)\) as a path-induced rooted subgraph; and consequently
there is a one-vertex rooted tree $(S, r)$ that is $(\zeta, \eta)$-decorated in $G$. The next result shows that if we start with $\zeta$ large enough, then by reducing $\zeta$ we can grow $S$ into any larger tree that we wish, and that will prove 1.5.

3.1 Let $\eta, t \geq 1$ and $\zeta \geq 2$ be integers, let $G$ be a graph that does not contain $K_{t,t}$ as a subgraph, and let $(S', r)$ be a $(\zeta', \eta)$-decorated rooted tree in $G$, where $\zeta' \geq (\zeta t)^{\eta} |S'| + \zeta t$. Let $p \in V(S')$ with height in $(S', r)$ less than $\eta$. Then there is a $G$-neighbour $q$ of $p$, with $q \in V(G) \setminus V(S')$, and with no other $G$-neighbour in $V(S')$, such that, if $S$ denotes the tree obtained from $S'$ by adding $q$ and the edge $pq$, then $(S, r)$ is a $(\zeta, \eta)$-decorated rooted tree in $G$.

Proof. For each $v \in V(S')$, let $h(v)$ denote the height of $v$ in $(S', r)$. Since $(S', r)$ is $(\zeta', \eta)$-decorated in $G$, it follows that $S'$ is an induced subgraph of $G$, and there is a rooted tree $(T', r)$ such that

- $(S', r)$ is a rooted subtree of $(T', r)$, and $(T', r)$ is a path-induced rooted subgraph of $G$;
- for each $u \in V(S')$ and $v \in V(T') \setminus V(S')$, if $u, v$ are $G$-adjacent then they are $T'$-adjacent;
- for each $v \in V(S')$, the decoration of $S'$ at $v$ in $T'$ is $(\zeta', \eta - h(v))$-uniform.

For each $v \in V(S')$, let $(T_v, v)$ be the decoration of $S'$ at $v$ in $T'$. Since $T_p$ is $(\zeta', \eta - h(p))$-uniform, and $h(p) \leq \eta$, it follows that $p$ has $\zeta'$ children in $(T_p, p)$. We need to select one of these children, say $q$, to add to $S'$, forming $S$. Any one of them would make a larger induced tree when added to $S'$, so $(S', r)$ is $(\zeta, \eta)$-decorated. But in order to make the new rooted tree $(\zeta, \eta)$-decorated, we will delete from $T'$ all vertices of $T'$ that are $G$-adjacent and not $T'$-adjacent to $q$, and doing so must not destroy too much of $T'$.

For each $v \in V(S')$, let $(S_v, v)$ be a $(t\zeta, \eta - h(v))$-uniform rooted subtree of $(T_v, v)$. By 2.3, there are fewer than $(t\zeta)^{t - h(v)} \leq (t\zeta)^{\eta}$ vertices not in $V(S_v)$ that are $t$-bad for $(S_v, v)$, and so there are fewer than $(t\zeta)^{|S'|}$ children of $p$ in $(T_p, p)$ that are $t$-bad for one of the rooted trees $(S_v, v)$ $(v \in V(S'))$. Also, since $(S_p, p)$ is path-induced, every $G$-neighbour of $p$ in $V(S_p)$ is an $S_p$-neighbour of $p$; so there are only $t\zeta$ children of $p$ in $(T_p, p)$ that belong to $V(S_p)$. Since $\zeta' \geq (\zeta t)^{\eta} |S'| + \zeta t$, there is a child $q$ of $p$ in $(T_p, p)$ that is $t$-bad for none of the trees $(S_v, v)$ $(v \in V(S'))$ and does not belong to $V(S_p)$.

Let $Q$ be the component containing $q$ of the graph obtained from $T'$ by deleting $V(S)$; thus $(Q, q)$ is $(\zeta', \eta - h(p) - 1)$-uniform, and so we may choose a $(\zeta, \eta - h(p) - 1)$-uniform rooted subtree $(R_v, v)$ of $(Q, q)$. Note that $q$ has no neighbours in $V(Q)$ except its neighbours in $T'$, since $(T', r)$ is path-induced. Since $q$ is not $t$-bad for any of the rooted trees $(S_v, v)$ $(v \in V(S'))$, it follows by 2.2 that for each $v$ there is a $(\zeta, \eta - h(v))$-uniform rooted subtree $(R_v, v)$ of $(S_v, v)$ such that $q$ has no $G$-neighbour in $V(R_v)$ except possibly $v$, and $q$ is $G$-adjacent to $v$ if and only if they are $T'$-adjacent (that is, $v = p$), since $v \in V(S')$ and $(S', r)$ is $(\zeta', \eta)$-decorated. Let $S$ be the tree induced on $V(S') \cup \{q\}$, and let $T$ be the union of $T'$, the trees $R_v$ $(v \in V(S') \cup \{q\})$ and the edge $pq$. Then $S$ satisfies the theorem, because the tree $T$ exists. This proves 3.1.

We deduce 1.5, which we restate in a strengthened form:

3.2 Let $\eta, t \geq 1$ and $\zeta \geq 2$. For every rooted tree $(H, s)$ with height at most $\eta$ and spread at most $\zeta$, let $c = (\eta + 3)! |H|$; then $\partial(G) \leq (|H| \zeta t)^c$ for every $H$-free graph $G$ that does not contain $K_{t,t}$ as a subgraph.
Proof. Choose $\eta \geq 1$ and $\zeta \geq 2$ such that $(H,s)$ has height at most $\eta$ and spread at most $\zeta$. Let $H$ have $k$ vertices. Define $\zeta_k = \zeta$, and for $i = k-1,k-2,\ldots,1$ let $\zeta_i = k(t\zeta_{i+1})^\eta$. Thus $\zeta_i \geq i(t\zeta_{i+1})^\eta + t\zeta_{i+1}$.

Let $G$ be an $H$-free graph that does not contain $K_{s,t}$ as a subgraph. Suppose that $G$ contains a one-vertex rooted tree that is $(\zeta_1,\eta)$-decorated in $G$. Choose a maximal rooted subtree $(F,s)$ of $(H,s)$ such that there is a rooted subtree $(S,r)$ of $G$, isomorphic to $(F,s)$, such that $(S,r)$ is $(\zeta_i,\eta)$-decorated in $G$, where $i = |F|$. By 3.1, $i = k$; and so $G$ contains an induced subgraph isomorphic to $H$, a contradiction.

Thus $G$ contains no one-vertex rooted tree that is $(\zeta_1,\eta)$-decorated in $G$. Hence $G$ contains no $(\zeta_1,\eta)$-uniform rooted tree as a path-induced rooted subgraph, and so by 2.4 (applied with $(H,r)$ replaced by a $(\zeta_1,\eta)$-uniform rooted tree), $\partial(G) \leq (\zeta_1 t)^d$ where $d = (\eta + 1)!$.

Now $\zeta_k = \zeta$, and $\zeta_{k-1} = k(t\zeta)^\eta$. For all $i$ with $1 \leq i \leq k-2$, $\zeta_{i+1} \geq k\zeta i^\eta$, and so $\zeta_i = k(t\zeta_{i+1})^\eta \leq \zeta_i^{\eta+1}$. Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq (k(t\zeta)^\eta)^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So $\partial(G) \leq (k\zeta t)^c$ where $c = (k-2)(\eta + 1)^2(\eta + 1)! + (\eta + 1)! \leq (\eta + 3)!k$. This proves 3.2.

Now $\zeta_k = \zeta$, and $\zeta_{k-1} = (k-1)\zeta \eta i^\eta + \zeta t$. For all $i$ with $1 \leq i \leq k-2$, $\zeta_{i+1} \geq i(\eta+1)^\eta$, and so $\zeta_i = i(\eta+1)^\eta \leq \zeta_i^{\eta+1}$. Consequently

$$\zeta_1 \leq \zeta_{k-1}^{(k-2)(\eta+1)} \leq (k\zeta \eta i^\eta)^{(k-2)(\eta+1)} \leq (k\zeta t)^{(k-2)(\eta+1)^2}.$$

So $\partial(G) \leq (k\zeta t)^c$ where $c = (k-2)(\eta + 1)^2(\eta + 1)! + (\eta + 1)! \leq (\eta + 3)!k$. This proves 3.2.

4 Excluding $K_{s,t}$

In this section we prove 1.6, and before that we prove a weaker statement, with $\partial(G)$ replaced by $\chi(G)$. For the latter we need the following lemma:

4.1 Let $J$ be a digraph such that every vertex has outdegree at most $k$. Then the undirected graph underlying $J$ has chromatic number at most $2k + 1$.

Proof. Let $G$ be the undirected graph underlying $J$. Since every subgraph of $G$ has the property that its edges can be directed so that it has outdegree at most $k$, it follows that every such subgraph $H$ has at most $k|H|$ edges; and therefore (if it is non-null) has a vertex of degree at most $2k$. Consequently $G$ is $2k$-degenerate, and so is $(2k + 1)$-colourable. This proves 4.1.

We use 4.1 to prove the following (which we include here because the proof gives a relatively small constant $c$, although the fact that some $c$ exists follows from 1.6):

4.2 Let $H$ be a tree and $s \geq 1$ an integer, and let $c = (2s|H|)^{s+|H|}$. Then for every $H$-free graph $G$ and every integer $t \geq 1$, if $G$ does not contain $K_{s,t}$ as a subgraph then $\chi(G) \leq ct$. 

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Proof. We will prove this by induction on $|H|$ (for the same value of $s$). Let $H$ be a tree and $s \geq 0$ an integer, and suppose the theorem holds for all smaller trees and the same value of $s$. We may assume that $|H| \geq 3$, since the theorem is true for trees with at most two vertices; let $p \in V(H)$ have degree one, and let $q$ be its $H$-neighbour. Let $H'$ be obtained by deleting $p$ from $H$. Let $c' = (2s|H'|)^{s+|H'|}$. We observe that

$$(1)\ c \geq \max\ ((|H| - 2)^{s-1}, (s-1)(|H| - 2), (2(s-2)(|H| - 2) + 1)c' + 1).$$

Let $t \geq 1$ be an integer, and let $G$ be an $H$-free graph not containing $K_{s,t}$ as a subgraph. We will show that $\chi(G) \leq ct$. Suppose that this is false, and choose a minimal induced subgraph $G'$ of $G$ with $\chi(G') > ct$. It follows that every vertex of $G'$ has degree at least $ct$ (since $c$ is an integer).

Let $v \in V(G')$. We say a subset $X \subseteq V(G') \setminus \{v\}$ is a $v$-bag if there is an isomorphism from $H'$ to $G[X \cup \{v\}]$ that maps $q$ to $v$. (Thus each $v$-bag has cardinality $|H| - 2$.)

Let $v \in V(G')$, and suppose that there are $s - 1$ pairwise disjoint $v$-bags, say $X_1, \ldots, X_{s-1}$. Since $G$ is $H$-free, every $G$-neighbour $u$ of $v$ either belongs to $X_i$ or has a $G$-neighbour in $X_i$, for $1 \leq i \leq s - 1$. In particular, every $G$-neighbour $u$ of $v$ not in $X_1 \cup \cdots \cup X_{s-1}$ has a $G$-neighbour in each of $X_1, \ldots, X_{s-1}$. But for each choice of $x_i \in X_i$ ($1 \leq i \leq s - 1$) there are at most $t - 1$ $G$-neighbours of $v$ $G$-adjacent to each of $x_1, \ldots, x_{s-1}$ (since they are also all adjacent to $v$, and $G$ has no $K_{s,t}$ subgraph). Consequently there are at most $(t - 1)(|H| - 2)^{s-1}$ $G$-neighbours of $v$ not in $X_1 \cup \cdots \cup X_{s-1}$; and hence

$$(s - 1)(|H| - 2) + (t - 1)(|H| - 2)^{s-1} > ct.$$

Since $ct = c + c(t - 1)$, and $(s - 1)(|H| - 2) < c$, and $(t - 1)(|H| - 2)^{s-1} \leq c(t - 1)$, this contradicts (1); so there is no such choice of $X_1, \ldots, X_{s-1}$.

Choose an integer $r$ maximum such that there are $r$ pairwise disjoint $v$-bags, say $X_1, \ldots, X_r$. Consequently $r \leq s - 2$. Let $Y_v = X_1 \cup \cdots \cup X_r$; then from the maximality of $r$, $X \cap Y_v \neq \emptyset$ for every $v$-bag $X$. Moreover $|Y_v| \leq (s - 2)(|H| - 2)$.

Let $J$ be the digraph with vertex set $V(G')$ in which every vertex in $Y_v$ is $J$-adjacent from $v$, for each $v \in V(G')$. Thus $J$ has maximum outdegree at most $(s - 2)(|H| - 2)$, and so by 4.1, the undirected graph $J'$ underlying $J$ has chromatic number at most $2(s - 2)(|H| - 2) + 1$; and so $V(G') = V(J')$ can be partitioned into $2(s - 2)(|H| - 2) + 1$ sets each of which is a stable set of $J'$. Let $Z$ be one of these sets. Then $G[Z]$ is $H'$-free (because otherwise there would be a vertex $v \in Z$, and a subset $X \subseteq Z \setminus \{v\}$, and an isomorphism from $H'$ to $G[X \cup \{v\}]$ mapping $q$ to $v$, and hence with $X \cap Y_v \neq \emptyset$; but no vertex of $Y_v$ belongs to $Z$, since $Z$ is stable in $J'$). From the inductive hypothesis, $\chi(Z) \leq c't$, and hence

$$ct < \chi(G) = \chi(G') \leq (2(s-2)(|H| - 2) + 1)c't$$

contrary to (1). This proves 4.2.

To prove 1.6, we will need the following strengthening of 1.3, also proved in [9]:

4.3 For every forest $H$, and every integer $s > 0$, there is a tree $S$ such that for every $H$-free graph $G$, if $G$ contains $S$ as a subgraph, then $G$ contains $K_{s,s}$ as a subgraph.

Now we prove 1.6, which we restate:
4.4 For every forest $H$ and every integer $s > 0$, there exists $c > 0$ such that for every graph $G$ and every integer $t > 0$, if $G$ is $H$-free and does not contain $K_{s,t}$ as a subgraph, then $\partial(G) < ct$.

Proof. Let $S$ be as in 4.3, and let $c = |S|^s$; we will show that $c$ satisfies the theorem. Let $t > 0$ be an integer, and let $G$ be an $H$-free graph that does not contain $K_{s,t}$ as a subgraph. Suppose that $\partial(G) \geq ct$, and choose $G$ minimal with these properties: then every vertex of $G$ has degree at least $ct$.

(1) Let $R$ be a tree. If every vertex of $G$ has degree at least $t|R|^s$, then $G$ contains a subgraph $T$ isomorphic to $R$, and $V(T)$ can be ordered as $\{t_1, \ldots, t_n\}$, such that for $1 \leq i \leq n$, $t_i$ is $G$-adjacent to at most $s - 1$ of $t_1, \ldots, t_{i-1}$.

We prove this by induction on $|R|$. We may assume that $|R| > 1$; let $p \in V(R)$ have degree one in $R$, and let $q$ be its $R$-neighbour. Let $R'$ be obtained from $R$ by deleting $p$. From the inductive hypothesis, $G$ contains a subgraph $T'$ isomorphic to $R'$, and its vertex set can be ordered as $\{t_1, \ldots, t_{n-1}\}$, such that for $1 \leq i \leq n - 1$, $t_i$ is $G$-adjacent to at most $s - 1$ of $t_1, \ldots, t_{i-1}$. Choose $v \in V(T')$ such that some isomorphism from $R'$ to $T'$ maps $q$ to $v$. If some $G$-neighbour $u$ of $v$ does not belong to $V(T')$ and has at most $s - 1$ $G$-neighbours in $V(T')$, then we may set $t_n = u$ as required; so we may assume that every $G$-neighbour $u$ of $v$ in $G$ either belongs to $V(T')$ or has at most $s$ $G$-neighbours in $V(T')$.

Let $X \subseteq V(T')$ with $|X| = s$. If there are at least $t$ vertices in $V(G)$ that are $G$-adjacent to every vertex in $X$, then $G$ contains $K_{s,t}$ as a subgraph, a contradiction. So for each such $X$, there are at most $t - 1$ vertices in $V(G)$ that are $G$-adjacent to every vertex in $X$. Since there are at most $|R|^s$ choices of $X$, there are at most $(t - 1)|R|^s$ vertices in $V(G) \setminus V(T')$ that have at least $s$ $G$-neighbours in $V(T')$. Consequently $v$ has at most $(t - 1)|R|^s$ $G$-neighbours not in $V(T')$. But it has at most $|R'|$ $G$-neighbours in $V(T')$ and so the degree of $v$ in $G$ is at most $(t - 1)|R|^s + |R'| < t|R|^s$. This proves (1).

Each vertex of $G$ has degree at least $ct = t|S|^s$; let us apply (1) taking $R = S$. We deduce that $G$ contains a subgraph $T$ isomorphic to $S$, and its vertex set can be ordered as $\{t_1, \ldots, t_n\}$, such that for $1 \leq i \leq n$, $t_i$ is $G$-adjacent in $G$ to at most $s - 1$ of $t_1, \ldots, t_{i-1}$. By 4.3, $G[V(T)]$ contains $K_{s,s}$ as a subgraph. Choose $i$ maximum such that $t_i$ belongs to this subgraph; then $t_i$ is $G$-adjacent to at least $s$ vertices that are earlier in the ordering, a contradiction. This proves 4.4.

5 Long holes

There is another result in the paper by Bonamy et al. [1]:

5.1 Let $\ell \geq 2$ be an integer; then there exists $c > 0$ such that $\partial(G) \leq \tau(G)^c$ for every graph $G$ with no induced cycle of length at least $\ell$.

In this section we give a simpler proof of this result.

Let $\eta, \ell \geq 1$ be integers. We say a rooted tree $(H, r)$ is $(t, \eta)$-tapering if $(H, r)$ has height $\eta$, and every vertex $v \in V(H)$ of height $i < \eta$ has exactly $t^{\eta-i}$ children. For each $v \in V(H)$, let $h(v)$ be its height in $(H, r)$.

Let $G$ be a graph. A map $\phi$ from $V(H)$ into $V(G)$ is a $(t, \eta)$-infusion of $(H, r)$ into $G$ if
• for all distinct $u, v \in V(H)$, if $u, v \in V(H)$ are $H$-adjacent then $\phi(u), \phi(v)$ are distinct and $G$-adjacent;

• for each $u \in V(H)$, if $v, w$ are distinct children of $u$ in $(H, r)$, then $\phi(v) \neq \phi(w)$;

• for every path $P$ of $H$ with one end $r$, the vertices $\phi(v)$ ($v \in V(P)$) are all distinct; and

• for every path $P$ of $H$ with one end $r$, and for all distinct $u, v \in V(P)$, $\phi(u), \phi(v)$ are $G$-adjacent if and only if $u, v$ are $H$-adjacent.

Let $\phi$ be a $(t, \eta)$-infusion into $G$. We define $V(\phi) = \{\phi(v) : v \in V(H)\}$, and we define the root of $\phi$ to be $\phi(r)$. We say $u \in V(G)$ is $t$-bad for $\phi$ if there exists $v \in V(H)$ with $h(v) < \eta$, such that $u$ is distinct from and $G$-adjacent to $\phi(w)$ for more than $(t - 1)t^{\eta-h(v)-1}$ children $w$ of $v$ in $(H, r)$. Then we have:

5.2 Let $t, \eta \geq 1$ be integers, let $(H, r)$ be a $(t, \eta)$-tapering rooted tree, let $G$ be a graph not containing $K_{t,t}$ as a subgraph, and let $\phi$ be a $(t, \eta)$-infusion of $(H, r)$ into $G$. There are at most $t^{\eta^2}$ vertices in $G$ that are $t$-bad for $\phi$.

The proof is like that for 2.3, using that $H$ has at most $t^{\eta^2}$ vertices that have children, and we omit it.

The next result strengthens 1.7:

5.3 Let $\eta \geq 2$ be an integer, and let $G$ be a graph with no induced cycle of length more than $\eta$. For every integer $t \geq 1$, if $G$ does not contain $K_{t,t}$ as a subgraph then $\partial(G) \leq t^{7\eta^2}$.

**Proof.** Let $t \geq 1$ be an integer, and let $G$ be a graph with no induced cycle of length more than $\eta$ that does not contain $K_{t,t}$. We may assume that $t \geq 2$. Let $(H, r)$ be a $(t, \eta)$-tapering rooted tree (not necessarily contained in $G$).

(1) If $u \in V(G)$ and $v_i$ is a $G$-neighbour of $u$ for $1 \leq i \leq t^\eta$, all distinct, and for each $i$ there is a $(t, \eta)$-infusion of $(H, r)$ into $G$ with root $v_i$, such that $u \notin V(\phi_i)$, and $u$ is not $t$-bad for $\phi_i$, then there is a $(t, \eta)$-infusion of $(H, r)$ into $G$, with root $u$.

Let $(H', r)$ be a $(t, \eta - 1)$-tapering rooted subtree or $(H, r)$. It follows (analogously to 2.2) that for $1 \leq i \leq t^\eta$, there is a $(t, \eta - 1)$-infusion $\phi'_i$ of $(H', r)$ into $G$ such that $u$ has no $G$-neighbour in $V(\phi'_i)$ except $v_i$. Let us number the components of $H \setminus \{r\}$ as $H_1, \ldots, H_{t^\eta}$. Let $\psi(r) = v$, and for $1 \leq i \leq t^\eta$ and each $v \in V(H_i)$, define $\psi(v) = \phi'_i(w)$ where $w$ is the parent of $v$ in $(H, r)$. Then $\psi$ is a $(t, \eta)$-infusion of $(H, r)$ into $G$, with root $v$. This proves (1).

In these circumstances we say that $\psi$, constructed as in the proof of (1), is derived from the sequence $(\phi_i : 1 \leq i \leq t^\eta)$.

If $P$ is a path of $H$ with length $\eta$ and one end $r$, and $\phi$ is a $(t, \eta)$-infusion of $(H, r)$ into $G$, then $\phi$ maps $P$ to an induced path $\phi(P)$ of $G$ with length $\eta$ and with one end the root of $\phi$. We call $\phi(P)$ a column of $\phi$. We observe that if $\psi$ is derived from $(\phi_i : 1 \leq i \leq t^\eta)$ as above, then for every column $Q$ of $\psi$, there is a column $Q'$ of one of $\phi_i$ ($1 \leq i \leq t^\eta$), say of $\phi'$, such that $Q \setminus \psi(r)$ is a subpath of $Q'$. Let us call $(\phi', Q')$ a shift of $(\phi, Q)$.
Let $\mathcal{A}_t$ be the set of all $(t, \eta)$-infusions of $(H, r)$ into $G$. Inductively for $i > 1$, let $\mathcal{A}_i$ be the set of all $(t, \eta)$-infusions $\phi$ such that for some choice of $\phi_1, \phi_2, \ldots, \phi_{i-1} \in \mathcal{A}_{i-1}$, $\phi$ is derived from the sequence $(\phi_j : 1 \leq j \leq t\eta)$. Thus $\mathcal{A}_i \subseteq \mathcal{A}_{i-1}$ for each $i$. There are two cases: either $\mathcal{A}_i$ is empty for some $i$, or it remains nonempty for all values of $i$. Suppose first that $\mathcal{A}_i$ is nonempty for all $i$, and let $\mathcal{A}$ be the intersection of all the sets $\mathcal{A}_i$ ($i \geq 1$). Choose $\phi_1 \in \mathcal{A}$, and let $Q_1$ be a column of $\phi_1$. Since $\phi_1$ is derived from some members of $\mathcal{A}$, there exists $\phi_2 \in \mathcal{A}$ with root $u_2$, and a column $Q_2$ of $\phi_2$, such that $(\phi_2, Q_2)$ is a shift of $(\phi_1, Q_1)$. Similarly we can choose an infinite sequence $(\phi_i, Q_i)$ ($i = 1, 2, 3 \ldots$) such that each $\phi_i \in \mathcal{A}$ and each $(\phi, Q_i)$ is a shift of its predecessor. Let $v_i$ be the root of $\phi_i$ for each $i$. Then $v_i, v_{i+1}, \ldots, v_{i+n}$ are the vertices in order of $Q_i$ for each $i$; and so form an induced path of $G$. Since $G$ is finite, there exists $j > 0$ such that $v_j$ is adjacent to one of $v_1, \ldots, v_{j-2}$; choose a minimum such value of $j$, and choose $i \leq j-2$ maximum such that $v_i, v_j$ are adjacent. Then $\{v_i, \ldots, v_j\}$ induces a cycle of $G$ of length more than $\eta$, a contradiction.

So the second case holds, that is, $\mathcal{A}_i$ is empty for some $i$. Choose $k$ minimum such that $\mathcal{A}_{k+1} = \emptyset$. For $1 \leq i \leq k$ let $X_i$ be the set of all vertices $v$ such that $v$ is the root of a member of $\mathcal{A}_i$ and not the root of any member of $\mathcal{A}_{i+1}$. Thus the sets $X_1, \ldots, X_k$ are pairwise disjoint. Let $X_0$ be the set of vertices that are not the root of any member of $\mathcal{A}_1$; so the sets $X_0, \ldots, X_k$ form a partition of $V(G)$. For each edge $e$ of $G$ with an end in one of $X_1, \ldots, X_k$, choose $i$ maximum such that $e$ has an end in $X_i$, let $v$ be an end of $e$ in $X_i$, and call $v$ the head of $e$. For each $v \in X_i$, choose $\phi_v \in \mathcal{A}_i$ with root $v$. (Thus $\phi_v \notin \mathcal{A}_{i+1}$ from the definition of $X_i$.)

- Let $A$ be the set of all edges of $G$ with both ends in $X_0$;
- Let $B$ be the set of all edges $uv$ with head $v$ such that $u \notin V(\phi_v)$ and $u$ is not bad for $\phi_v$;
- Let $C$ be the set of all edges $uv$ with head $v$ such that $u \notin V(\phi_v)$ and $u$ is bad for $\phi_v$;
- Let $D$ be the set of all edges $uv$ with head $v$ such that $u \in V(\phi_v)$.

Since there is no $(t, \eta)$-infusion of $(H, r)$ into $G[X_0]$, it follows that $G[X_0]$ does not contain a $(\zeta, \eta)$-uniform tree as a path-induced rooted subgraph, where $\zeta = t\eta$, and so $\partial(G[X_0]) \leq (\zeta t)(\eta+1)!$ from 2.4.

Hence
$$|A| \leq (\zeta t)(\eta+1)!|G|.$$  

For each $u \in V(G)$, with $u \in X_i$ say, there do not exist $t\eta$ neighbours $v$ of $u$ such that $uv$ has head $v$ and belongs to $B$, since there is no $(t, \eta)$-infusion of $(H, r)$ with root $u$ that is derived from members of $\mathcal{A}_i$. Hence
$$|B| \leq t\eta|G|.$$  

For each $v \in V(G)$, there are at most $t\eta$ neighbours $u$ of $v$ such that the edge $uv$ has head $v$ and belongs to $C$, by 5.2; so
$$|C| \leq t\eta^2|G|.$$  

Finally, for each $v \in V(G)$, there are at most $t\eta$ neighbours $u$ of $v$ such that the edge $uv$ has head $v$ and belongs to $D$; so
$$|D| \leq t\eta|G|.$$  

Summing, we obtain
$$|E(G)| \leq \left((t^{\eta+1})(\eta+1)! + t\eta + t\eta^2 + t\eta^3\right)|G| \leq \left(t^{(\eta+2)!} + t\eta^2\right)|G| \leq t\eta^2/2.$$  

Consequently $\partial(G) \leq t\eta^2$. This proves 5.3. 

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