GENERALIZED ITERATED WREATH PRODUCTS OF CYCLIC GROUPS AND ROOTED TREES CORRESPONDENCE

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ABSTRACT. Consider the generalized iterated wreath product $Z_{r_1} \wr Z_{r_2} \wr \ldots \wr Z_{r_k}$. The number of irreducible representations of the generalized iterated wreath product satisfies a certain recursion, which coincides with the number of certain rooted trees. We give the dimension of irreducible representations of the generalized iterated wreath product and literature's fastest FFT upper bound estimate to date.

1. Introduction

One may think of a cyclic group as rotational symmetries of a regular polygon while the generalized wreath product $Z_{r_1} \wr Z_{r_2} \wr \ldots \wr Z_{r_k}$ are those automorphisms generated by cyclic shifts of nodes of a certain rooted tree. With applications on functions on rooted trees, pixel blurring, and nonrigid molecules in molecular spectroscopy in mind, we generalize [OOR04]. We show that the irreducible representations of the generalized iterated wreath product $W(\vec{r}_k)$ are indexed by a family of trees, which we call $\vec{r}_k$-trees. These are labeled complete $\vec{r}_k$-ary trees of height $k$.

1.1. Background and Notation. Throughout this paper, we will fix $\vec{r} = (r_1, r_2, r_3, \ldots) \in \mathbb{N}^\omega$, a positive integral vector. We denote by $\vec{r}_k := (r_1, r_2, \ldots, r_k)$ the $k$-length vector found by truncating $\vec{r}$. We say that $\mathcal{R}$ is a traversal for $G$ if $\mathcal{R} \subset \hat{G}$ that contains one irreducible for each isomorphism class.

Definition 1.1. Let $W(\vec{r}_k)$ be the (generalized) $k$-th $\vec{r}$-symmetric wreath product, where

$$W(\vec{r}_0) = \{0\} \text{ and } W(\vec{r}_k) = W(\vec{r}_{k-1}) \wr Z_{r_k}.$$ 

Note that $W(\vec{r}_1) = Z_{r_1}$, $W(\vec{r}_2) = Z_{r_1} \wr Z_{r_2}$, and $W(\vec{r}_3) = Z_{r_1} \wr \ldots \wr Z_{r_k}$. Throughout this paper, we will be considering the chain of groups given in Definition 1.1.

For a group $G$, we denote by $\hat{G}$ the set of irreducible representations of $G$. We say that $\mathcal{R}$ is a traversal for $G$ if $\mathcal{R} \subset \hat{G}$ that contains one irreducible for each isomorphism class. Thus note that

$$\sum_{\rho \in \mathcal{R}} \dim(\rho)^2 = |G|.$$ 

We refer the reader to [Ker71] for some background on representations of permutations groups.

2. Irreducible representations of iterated wreath products

We use $\text{Ind}^G_H \sigma$ to denote the induced representation of $G$ from a subgroup $H \leq G$ and representation $\sigma \in \hat{H}$.

Theorem 2.1. Suppose that $\mathcal{R} = \{\rho_1, \ldots, \rho_h\}$ is a traversal for $G$. The irreducible representations given by

$$\left\{ \text{Ind}^G_{\hat{G} \wr Z_{r_k}} (\rho_{i_1} \otimes \rho_{i_2} \otimes \ldots \otimes \rho_{i_d} \otimes \sigma) \right\} \left\{ d | \sigma \in \hat{Z_{r_k/d}}, i_1 = \min\{i_1, \ldots, i_d\} \right\}$$

form a traversal of $G \wr Z_{r_k}$.

Proof. First notice that the set $\{\rho^\alpha = \rho_1^{\alpha_1} \otimes \ldots \otimes \rho_h^{\alpha_h} : \alpha \vdash n\}$ forms a traversal for $G^k$. We will be able to recursively apply Clifford’s theory: we consider the subgroup $(W(\vec{r}_{k-1}))^{r_k} \leq W(\vec{r}_k)$, since it is true $W(\vec{r}_k) = (W(\vec{r}_{k-1}))^{r_k} \rtimes Z_{r_k}$.

Suppose $\{\rho_i\}$ is a complete set of irreducible representations of $W(\vec{r}_{k-1})$. Then $(W(\vec{r}_{k-1}))^{r_k}$-irreducible representations are of the form $\rho_1 \otimes \ldots \otimes \rho_{r_k}$, where $\rho_i$’s are not necessarily distinct. The cyclic
group \( \mathbb{Z}_{r_k} \) acts on \((W(\vec{r}_{k-1}))^{r_k}\) by cyclic translations, which induces an \( \mathbb{Z}_{r_k} \)-action on the representation space of \((W(\vec{r}_{k-1}))^{r_k}\).

This group action gives a decomposition of the representation space into a disjoint union of orbits. Let \( \mathcal{O}_i \) be an orbit of the representation space. Fix a representative \( \sigma_i \in \mathcal{O}_i \) and let \( G_i \leq \mathbb{Z}_{r_k} \) be the corresponding inertia factor of \( \sigma_i \). Then \((W(\vec{r}_{k-1}))^{r_k})/G_i\) is an inertia group of \((W(\vec{r}_{k-1}))^{r_k}\). We know that every irreducible representation of \((W(\vec{r}_{k-1}))^{r_k}\) can be extended trivially to an irreducible representation of \((W(\vec{r}_{k-1}))^{r_k})/G_i\). We tensor this extension with any irreducible representation of \( G_i \) to yield an irreducible representation of \((W(\vec{r}_{k-1}))^{r_k})/G_i\). Finally, we induce this tensored extension from \((W(\vec{r}_{k-1}))^{r_k})/G_i\) to \((W(\vec{r}_{k}))^{r_k}\) to obtain an irreducible representation, and it is a classical fact that every irreducible representation of \((W(\vec{r}_{k}))^{r_k}\) is obtained in this way.

\[ \Box \]

### 2.1. Number of irreducible representations

Following the exact same argument as given in [OOR04], we find the following recursion for the number of irreducible representations for \((W(\vec{r}_{k}))^{r_k}\).

**Theorem 2.2.** The number \( M_k(\vec{r}_{k}) \) of irreducible representations of \((W(\vec{r}_{k}))^{r_k}\) satisfies the recursion

\[
M_k(\vec{r}_{k}) = \frac{1}{r_k} \sum_{d|r_k} f(d)d^2 = \frac{1}{r_k} \sum_{d|r_k} \mu(c/d)M_{k-1}(\vec{r}_{k-1})^{r_k/c} d^2, \tag{1}
\]

where \( M_1(\vec{r}_1) = r_1 \).

### 3. Bijection between the branching diagram for generalized iterated wreath products and rooted trees

#### 3.1. \( \vec{r}_{k} \)-trees

We restate the following definitions from [OOR04]. A *rooted tree* is a connected simple graph without cycles with a distinguished vertex called the *root*. A vertex is at *level* \( j \) if the distance from it to the root is \( j \). If \( x \) is a vertex at level \( j \) that is connected to vertex \( y \) at level \( j + 1 \), then \( y \) is said to be a *child* of \( x \) and \( x \) is the *parent* of \( y \). The *branching factor* of a vertex is its number of children. A *leaf* is a vertex with branching factor zero.

If \( x \) is a vertex of a rooted tree \( T \), then the *subtree of \( T \) with root \( x \) is the connected component containing \( x \) of the forest obtained by removing the edge between \( x \) and its parent. A *subtree* of \( T \) is a subtree with root \( y \) for some vertex \( y \) of \( T \), and the *maximal subtrees* of \( T \) are the subtrees obtained by removing the root of \( T \), together with the edges between the root and its children (at level 1).

A *labeled* rooted tree is a rooted tree whose vertices have been labeled using the elements of some set. If \( H \) is a subgroup of the group of automorphisms of a rooted tree \( T \), then the action of \( H \) on \( T \) induces an action of \( H \) on the labellings of \( T \). We say that two labellings of \( T \) are equivalent with respect to \( H \) if they are in the same orbit under the action of \( H \).

**Definition 3.1.** The complete \( \vec{r}_{k} \)-ary tree of height \( k \) is the rooted tree whose leaves are all at level \( k \), and whose vertices that are not leaves at level \( l \) (\( l < k \)) all have branching factor \( r_{k-l} \).

**Definition 3.2.** An \( \vec{r}_{0} \)-tree of height 0 is a vertex labeled with an integer from \( \{1, \ldots, r_k\} \). An \( \vec{r}_{k+1} \)-tree of height \( k+1 \) is a labeled complete \( \vec{r}_{k+1} \)-ary tree \( T \) of height \( k+1 \) whose maximal subtrees are \( \vec{r}_{k} \)-trees of height \( k \), and whose root is labeled with an integer from \( \{r_k/d_k, 2(r_k/d_k), \ldots, d_k(r_k/d_k)\} \), where \( \mathbb{Z}_{d_k} \) is the stabilizer of the labeled maximal subtrees of \( T \) under cyclic permutation.

Two \( \vec{r}_{k} \)-trees are equivalent if they are in the same orbit under the action of \((W(\vec{r}_{k}))^{r_k}\).

**Proposition 3.3.** There exists a one-to-one correspondence between \( \vec{r}_{k} \)-trees of height \( k \) and irreducible representations of \((W(\vec{r}_{k+1}))^{r_k}\).

**Proof.** Let \( z_1 \) be a generator of \( \mathbb{Z}_{r_1} \) and let \( w_1 \) be a primitive \( r_1^{th} \) root of unity. For \( i = 1, \ldots, r_1 \), let \( \rho_i \) be the irreducible representation of \( \mathbb{Z}_{r_1} \) such that \( \rho_i(z_1) = w_1^i \). If \( \mathbb{Z}_{d_i} \) is a subgroup of \( \mathbb{Z}_{r_1} \), then \( z_1^{r_1/d_i} \) is a generator for \( \mathbb{Z}_{d_i} \), and a complete set of irreducible representations of \( \mathbb{Z}_{d_i} \) are those \( \rho_i \downarrow \mathbb{Z}_{d_i} \), where \( i = 1, \ldots, d_1 \). So we may view the labels of an \( \vec{r}_{k} \)-tree as corresponding to irreducible representations, where the label \( i(r_1/d_1) \) corresponds to the representations \( \rho_i \) with appropriate stabilizer.

Thus by construction, we have a bijection between \( \vec{r}_{k} \)-trees of height \( k \) and the irreducible representation of \((W(\vec{r}_{k+1}))^{r_k}\). In particular, the maximal subtrees of an \( \vec{r}_{k} \)-tree of height \( k \) correspond to irreducible representations of \((W(\vec{r}_{k}))^{r_k}\). We then tensor these representations to give a representation for...
Let in [Roc90], we state its main result: obstacle through strategies in [Rad68] and [RSR69]. Since the fastest algorithm to date in literature is the Cooley-Tukey algorithm given in [CT65] fails for cyclic groups. Operations needed to compute all Fourier transforms for any complex-valued function \( f \in C \) companion tree \( \vec{r} \) copies of the same \( \vec{r} \)-

\[ d_k(r_k) = \frac{1}{r_k} \sum_{d \mid r_k} \mu(c/d) h_{k-1}(r_{k-1})^{r k/c} d^2, \]

where \( h_0(r_0) = r_k \).

**4. Degrees of Irreducible Representations**

Following the discussion in [OOR04], we define for any \( r_k \)-tree \( T \) the companion tree \( C_T \).

**Definition 4.1.** Fix an \( r_k \)-tree \( T \). Define the companion tree \( C_T \) to be the \( r_k \)-tree that has the following labels: for a vertex \( v \) of \( T \), let \( T_v \) be the subtree of \( T \) rooted at \( v \). Let \( C : V_{C_T} \rightarrow \mathbb{N} \), where

\[ C(v) = \left\{ T_u^{W(r_k-1)|u|} \mid u \text{ is a child of } v \right\}, \]

the \( r_{k-1} \)-trees rooted at children of \( v \), where \( k - \ell \) is the level of the vertex \( v \) and we disregard equivalent copies of the same \( r_{k-1} \)-tree.

**Proposition 4.2.** Let \( \rho \) be an irreducible representation of \( W(r_k) \) associated to \( r_k \)-tree \( T \) with companion tree \( C_T \). Then the dimension \( d_\rho \) of \( \rho \) is given by

\[ d_\rho = \prod_v C(v). \quad (2) \]

**5. Fast Fourier Transforms, Adapted Bases and Upper Bound Estimates**

Any efficient algorithm for applying a discrete Fourier transform is a fast Fourier transform. Since Cooley-Tukey algorithm given in [CT65] fails for cyclic groups \( C_n \) when \( n \) is prime, we overcome this obstacle through strategies in [Rad68] and [RSR09]. Since the fastest algorithm to date in literature is in [Roc90], we state its main result:

**Theorem 5.1.** Let \( K \subseteq G \) be a normal subgroup such that \( G/K \) is abelian. Let \( \{ \eta_i \}_{i \in I} \) be a complete set of representatives for the irreducible representations of \( K \) under the action of \( G \). Let \( H_{\eta_i} \) be a subgroup of \( G \) containing \( K \) such that \( \eta_i \) extends to \( H_{\eta_i} \) but no further. Then the number of operations needed to compute all Fourier transforms for any complex-valued function \( f \) defined on \( G \) is at most

\[ \frac{|G|}{|K|} \cdot T(K) + \sum_{i \in I} m_{\eta_i} \left( \frac{|G|}{|H_{\eta_i}|} \right)^2 d^2 + |\Delta(\eta_i)| \left( \frac{|G|}{|K|} \right) d_{\eta_i} + d_{\eta_i} O\left( \frac{|G|}{|K|} \log \frac{|H_{\eta_i}|}{|K|} \right), \]

where \( m_{\eta_i} \) is the number of irreducible representations of \( G \) induced by extensions of \( \eta_i \), \( \alpha \) is the exponent of matrix multiplication, and \( O \) is the universal FFT constant.

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