Random-Receiver Quantum Communication

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We introduce the task of random-receiver quantum communication, in which a sender transmits a quantum message to a receiver chosen from a list of n spatially separated parties. The choice of receiver is unknown to the sender, but is known by the n parties, who coordinate their actions by exchanging classical messages. In normal conditions, random-receiver quantum communication requires a noiseless quantum communication channel from the sender to each of the n receivers. In contrast, we show that random-receiver quantum communication can take place through entanglement-breaking channels if the order of such channels is controlled by a quantum bit that is accessible through quantum measurements. Notably, this phenomenon cannot be mimicked by allowing free quantum communication between the sender and any subset of k < n parties.

Introduction: The transmission of quantum messages from a sender to a receiver is the cornerstone of quantum communication. When the identity of the receiver is known, this task can be achieved with a reliable quantum communication channel between the sender and the receiver. But what if the identity of the receiver is unknown?

Here we introduce a scenario, which we call random-receiver quantum communication: a sender A is connected to n spatially separated parties \( \{ B_i \}_{i=1}^n \) through n communication channels \( \{ C_i \}_{i=1}^n \), as in Figure 1. The sender wants to transmit a quantum message to the x-th party, for some \( x \in \{1, \ldots, n\} \). However, the identity of such party (i.e. the value of \( x \)) is unknown to the sender. This scenario could arise, for example, in delegated quantum computation, where a client sends inputs to a server, asking it to perform a desired quantum computation on them. Here the receiver \( B_x \) could be one of the n servers, and the sender may not know in advance which server is available to perform the desired computation. In this situation, the sender has to delocalize the message, and send it to all the n servers, in such a way that the available one can retrieve the message and operate on it. In the following, we will assume that the n parties know the value of \( x \) (for example, because they have communicated classically among each other) and cooperate in order to let the message reach party \( B_x \). To coordinate their actions, the parties are allowed to exchange classical messages. We say that a communication protocol is successful if it works for all values of \( x \in \{1, \ldots, n\} \).

Random-receiver quantum communication is related to the task of quantum summoning [1, 2], where a quantum message has to be revealed at a given set of spacetime points. The crucial difference, however, is that summoning includes limits on the exchange of signals among the n parties induced by the structure of the underlying spacetime. In random-receiver quantum communication, classical communication among the n parties is permitted, while quantum communication is forbidden.

To introduce the task of random-receiver quantum communication, we first consider the simple scenario where the quantum message is a generic state of a quantum bit (qubit), and all the channels from the sender to the receivers are noiseless. To transmit the quantum state \( \psi = \alpha |0\rangle + \beta |1\rangle \), the sender can encode it in the generalized Greenberger-Horne-Zeilinger (GHZ) state \( |\psi_n\rangle := \alpha |0\rangle^{\otimes n} + \beta |1\rangle^{\otimes n} \) and send such state to the n receivers. To let party \( B_x \) retrieve the message, each of the other \((n-1)\) parties performs a measurement on
the Fourier basis \{\{+, -\}\}, \{\pm\} := ((|0\rangle \pm |1\rangle) / \sqrt{2}, collapsing the state of party \(B_x\) to \(|\phi_s\rangle := \alpha |0\rangle + (-1)^s \beta |1\rangle\), where \(s := \sum_{y \neq x} o_y\) is the sum of the measurement outcomes, \(o_y\) being the measurement outcome obtained by the \(y\)-th party. Finally, the \(n - 1\) parties communicate their outcomes to \(B_x\), who performs the correction operation \(Z^s\), with \(Z := |0\rangle\langle 0| - |1\rangle\langle 1|\). It is easy to see that party \(B_x\) eventually receives the quantum state \(|\psi\rangle\) without any error. All together, this protocol requires 1 qubit of quantum communication from the sender to each receiver.

Now, suppose that the quantum channels are noisy. For protocols involving a single round of classical communication to the chosen receiver, we show that perfect random-receiver quantum communication is possible only if each of the channels \((C_i)_{i=1}^n\) can transfer at least one qubit without errors. This result implies that the simple noiseless protocol presented above is optimal in terms of quantum communication. Moreover, the result shows that random-receiver quantum communication cannot take place if some of the channels \((C_i)_{i=1}^n\) are entanglement-breaking. As it turns out, this impossibility of random-receiver quantum communication with entanglement-breaking channels holds not only for one-way protocols, but also for protocols involving arbitrarily many rounds of local operations and classical communication (LOCC) (see Appendix C2).

In contrast with the above observations, in the following we will show that random-receiver quantum communication can take place when multiple entanglement-breaking channels are applied in a superposition of alternative orders. Suppose that the quantum communication between the sender and the \(i\)-th receiver takes place through two channels \(A_i\) and \(B_i\), and that the order of application of the two channels is entangled with the state of a control qubit, which we call the order qubit. In this scenario, illustrated in Figure 2a, we show that perfect random-receiver quantum communication is possible even if all the channels \((A_i, B_i)_{i=1}^n\) are entanglement-breaking, provided that the order qubit is accessible through measurements, and that the outcome of a binary measurement is sent to the chosen receiver. In other words, the indefiniteness of the order enables \(n\)-party random-receiver quantum communication using only entanglement-breaking channels and one bit of classical communication to one of the parties.

Remarkably, this phenomenon cannot be reproduced in a scenario where the the order of the channels is definite and the sender can send quantum data to one of the parties, as illustrated in Figure 2b. In other words, the access to the qubit that determines the order is a more powerful resource than the noiseless transmission of quantum data from the sender to one of the parties. In fact, we prove an even stronger result: classical communication of the measurement outcomes on the order qubit is a more powerful resource than noiseless quantum communication to \(n - 1\) parties. Achieving random-receiver quantum communication in the scenario of Figure 2b requires at least one qubit of noiseless quantum communication to each of the \(n\) parties. Our results show that the order qubit can unlock quantum communication to a randomly chosen receiver. The unlocking takes place thanks to the correlations between the order qubits and the output of the noisy channels connecting the sender to the receivers. In contrast, any noiseless quantum communication channel from the sender to a given receiver does not establish correlations with the output of the other receivers. As a consequence, the only way to achieve random-receiver quantum communication through the addition of noiseless communication is to have one noiseless communication channel for each of the \(n\) receivers.

**Conditions for random-access quantum communication.**

We first show that the noiseless protocol provided in the introduction is optimal among one-way protocols, that is, protocols consisting of a single round of classical communication to the chosen receiver.

**Theorem 1.** Every one-way protocol for random-receiver communication of a \(d\)-dimensional quantum message requires each of the channels \((C_i)_{i=1}^n\) to have a quantum capacity of at least \(\log d\) qubits.

The proof is provided in Appendix A. In particular, Theorem 1 implies that random-receiver quantum communication cannot take place when some of the channels \((C_i)_{i=1}^n\) are entanglement-breaking. We recall that entanglement-breaking channels are of the measure-and-prepare form \(C(\rho) = \sum_j \text{Tr}[M_j \rho] \rho_j\), where \((M_j)\) is a quantum measurement and \(\{\rho_j\}\) is a set of output states \([3]\). Entanglement-breaking channels are the prototype of channels with zero quantum capacity, and therefore they cannot achieve random-receiver quantum communication.

In the rest of the paper, we will focus on the scenario where all channels are entanglement-breaking, and ask which additional resources should be added in order to enable random-receiver quantum communication. In the basic model of Figure 1, we replace each entanglement-breaking channel \(C_i\) with a new channel \(C_i \otimes S_i\), where \(S_i\) is an additional channel from the sender to the \(i\)-th receiver. For simplicity, we assume that each side-channel \(S_i\) acts on a quantum system of dimension \(d\), equal to the dimension of the quantum message. In this setting, we prove that random-receiver communication is possible if and only if each side-channel is noiseless.

**Theorem 2.** Random-receiver quantum communication with entanglement-breaking channels \((C_i)_{i=1}^n\) and side-channels \((S_i)_{i=1}^n\) is possible if and only if all side-channels are noiseless.

The proof is provided in Appendix B. In particular, Theorem 2 shows that random-receiver quantum com-
Alice encodes the unknown qubit state \( |\psi\rangle \) in an \( n \)-partite GHZ state and sends through the noisy channels. The order qubit controls the order in which the subsystems pass through noisy channels, i.e., either \( \{A_i\} \) before \( \{B_i\} \) or \( \{B_i\} \) before \( \{A_i\} \). The order qubit has been prepared in \( |+\rangle \) state, which is a superposition of these two orderings. Finally, the order qubit is measured and depending on the outcome the spatially separated Bobs apply local operations to establish GHZ correlation among themselves. Now even with LOCC the unknown state can be perfectly retrieved at any of the randomly chosen Bob’s lab.

Figure 2: Random-receiver quantum communication task with indefinite and definite ordering of noisy channels. (a) Perfect protocol with indefinite order: Alice encodes the unknown qubit state \( |\psi\rangle \) in a \( n \)-partite GHZ state and sends through the noisy channels. The order qubit controls the order in which the subsystems pass through noisy channels, i.e., either \( \{A_i\} \) before \( \{B_i\} \) or \( \{B_i\} \) before \( \{A_i\} \). The order qubit has been prepared in \( |+\rangle \) state, which is a superposition of these two orderings. Finally, the order qubit is measured and depending on the outcome the spatially separated Bobs apply local operations to establish GHZ correlation among themselves. Now even with LOCC the unknown state can be perfectly retrieved at any of the randomly chosen Bob’s lab. (b) Even if Alice shares \( n-1 \) noiseless channels with \( (n-1) \) Bobs and the noisy channels \( A_1, B_1 \) in a fixed order with the other Bob, the task cannot be perfectly accomplished.

Random-receiver quantum communication through the quantum SWITCH. Let \( \mathcal{A} := \bigotimes_{i=1}^n \mathcal{A}_i \) and \( \mathcal{B} := \bigotimes_{i=1}^n \mathcal{B}_i \) be two quantum channels, describing the noise experienced by the data transmitted by a sender to \( n \) receivers. The action of the channels \( \mathcal{A} \) and \( \mathcal{B} \) in a superposition of two alternative orders is described by the quantum SWITCH, a higher-order map that transforms the pair of channels \( (\mathcal{A},\mathcal{B}) \) into a new quantum channels \( \mathcal{S}(\mathcal{A},\mathcal{B}) \), involving a control qubit that determines the order of application of channels \( \mathcal{A} \) and \( \mathcal{B} \). In its simplest version, the quantum SWITCH produces the channel \( \mathcal{S}(\mathcal{A},\mathcal{B}) \) with Kraus operators

\[
\mathcal{S}_R := A_j B_k \otimes |0\rangle \langle 0| + B_k A_j \otimes |1\rangle \langle 1|,
\]

where \( \{A_j\} \) and \( \{B_k\} \) are Kraus representations for channels \( \mathcal{A} \) and \( \mathcal{B} \), respectively. It is easy to verify that the definition of channel \( \mathcal{S}(\mathcal{A},\mathcal{B}) \) is independent of the choice of Kraus representations. When the order qubit is initialized in the state \( \omega \) we use the shorthand \( \mathcal{S}_\omega(\mathcal{A},\mathcal{B}) (\rho) := \mathcal{S}(\mathcal{A},\mathcal{B}) (\rho \otimes \omega) \), and we call \( \mathcal{S}_\omega(\mathcal{A},\mathcal{B}) \) the switched channel.

When \( \mathcal{A} \) and \( \mathcal{B} \) are products of Pauli channels, the switched channel has the simple expression

\[
\mathcal{S}_\omega(\mathcal{A},\mathcal{B}) = p_+ \mathcal{C}_+ \otimes \omega_+ + p_- \mathcal{C}_- \otimes \omega_-,
\]

where \( (p_+,p_-) \) are two probabilities, \( \omega_+ := \omega \) and \( \omega_- := Z \omega Z \) are states of the order qubit, and \( (\mathcal{C}_+,\mathcal{C}_-) \) are two quantum channels (see Appendix C for the explicit expression). In the following, we will focus on the case where all channels \( \{A_i\}_{i=1}^n \) and \( \{B_i\}_{i=1}^n \) are equal to the Pauli channel \( \mathcal{N}_{XY} \), defined by \( \mathcal{N}_{XY}(\rho) = 1/2(X\rho X + Y\rho Y) \). This channel is entanglement-breaking and therefore cannot directly transmit quantum information. However, we will see that the use of this channel in a superposition of orders achieves perfect quantum communication to a randomly chosen receiver.

For simplicity, we illustrate the idea for \( n = 2 \). First, the sender encodes the message \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) into the state \( |\psi_2\rangle := \alpha |0\rangle \langle 0| + \beta |1\rangle \langle 1| \), as in the noiseless protocol. Then, the sender sends the two qubits to receivers 1 and 2, using the channels \( \mathcal{A} = \mathcal{N}_{XY} \otimes \mathcal{N}_{XY} \) and \( \mathcal{B} = \mathcal{N}_{XY} \otimes \mathcal{N}_{XY} \) in a superposition of orders. When the order qubit is initialized in the state \( |+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \), the channels \( \mathcal{C}_{\pm} \) in Eq. (2) are

\[
\mathcal{C}_+(\rho) = \frac{\rho + (Z \otimes Z)\rho(Z \otimes Z)}{2},
\]

\[
\mathcal{C}_-(\rho) = \frac{(I \otimes Z)\rho(I \otimes Z) + (Z \otimes I)\rho(Z \otimes I)}{2},
\]

and the probabilities \( p_{\pm} \) are both equal to 1/2. The output states of the order qubit are either \( \omega_+ = |+\rangle \langle +| \)}
Theorem 2 implies that random-receiver quantum communication is impossible even if one adds any number \( k < n \) of qubit side-channels. In short, the mere access to the order qubit is a more powerful resource than the access to \((n - 1)\) qubit side-channels. 

Discussion. Quantum communication with the assistance of the quantum SWITCH is similar to quantum communication with classical assistance from the environment [4–7]. In both cases, the access to a measurement outcome unlocks some quantum information that would be inaccessible otherwise. The analogy goes even further, because the quantum SWITCH of two Pauli channels \( A \) and \( B \) is an extension of the quantum channel \( AB \), that is, the channel that arises when channels \( A \) and \( B \) are applied in cascade in a definite causal order. Precisely, the channel \( AB \) can be obtained from the switched channel \( S_\omega(A, B) \) by discarding the order system. From this point of view, the order qubit is indeed part of the environment of the channel \( AB \), and quantum communication with the assistance of the SWITCH is a special case of quantum communication with classical assistance from the environment. The key difference is that, in the case of the quantum SWITCH, only a small part of the environment needs to be accessible, while in the other examples of quantum communication with the assistance of environment it is generally assumed that the whole environment be accessible.

Another class of communication protocols that exhibit similarities with the quantum SWITCH are the communication protocols using controlled operations before and after the communication channels [8]. Like the quantum SWITCH, these protocols use a control qubit, which determines the choice of operations performed on the input and output of the communication channels. The key difference with the quantum SWITCH is that such protocols generally transfer information to the control system in a way that bypasses the original channels [9, 10]. In contrast, in all the protocols considered in the literature, the quantum SWITCH does not deposit information into the order qubit. For protocols involving Pauli channels, this feature is evident from Eq. (2), where the states \( \omega_{\pm} \) of the order qubit are independent of the message, and so are the probabilities \( p_{\pm} \) (see Appendix C for the explicit expression).

We observe that, if we allow arbitrary controlled operations before and after the noisy channels, then protocols for random-receiver quantum communication with entanglement-breaking channels can be constructed also in the causally ordered scenario. This is because controlled operations can be used (i) to transfer information directly from the message to the control qubit, bypassing the noisy channels \( A \) and \( B \), and (ii) to generate the generalized GHZ state \( |\alpha|0^\otimes n + |\beta|1^\otimes n \) from the state of the control qubit, evading the locality restriction that affects the receivers. An example of protocol that achieves random-receiver communication through controlled operations in a definite causal order is presented in Appendix E.

The possibility of random-receiver quantum communication through controlled operations in a definite order can be interpreted in two ways. On the one hand, controlled operations can generate entanglement among the \( n \) receivers, and therefore appear to be too powerful to be interesting in the problem of random-receiver quantum communication, where locality in space is an essential feature of the problem. On the other hand, controlled operations have some similarity with the quantum SWITCH, which can be regarded as a controlled SWAP operation in time. Controlled SWAP operations are a special subset of the set of all controlled operations, and one may wonder whether this special subset can reproduce the features of the quantum SWITCH. Interestingly, the answer is negative: in Appendix E we show that no controlled routing of the inputs and outputs of channels \( A = B = N_{XY}^\otimes \) permits random-receiver quantum communication for odd \( n \).

Photonic simulation of the random-receiver quantum communication task. Quantum-SWITCH has recently been simulated in several photonic setups [11–14]. For instance, in the scheme of Ref.[13] photon’s transverse spatial mode behaves as the target system evolving under two quantum operations whose relative order is controlled by photon’s polarization degrees of freedom (DOF). For implementing random-receiver quantum communication through quantum SWITCH in photonic
setup we require more than two DOFs to be considered at a time with one of them playing the role of order system. In the present context we assume that the sender possesses advanced optical devices that allow her to apply any joint (entangled) quantum operation on multiple DOFs of the photon, whereas the receiver can address each DOF individually. This assumption effectively mimics the scenario of random-receiver quantum communication with different DOFs playing the role of different spatially separated Bobs. In which DOF the quantum information has to be reproduced is decided at a later time after the DOFs evolve through noisy processes. Multiple DOFs of photon, such as polarization, spatial-mode, orbit-angular-momentum, time-bin and frequency have already been addressed simultaneously in different photonic experiments [15–18]. The proposed random-receiver quantum communication task thus welcomes an inquisitive conglomeration of presently available quantum optical devices to demonstrate a novel information theoretic advantage of indefinite causal order. 

Conclusions. Coherent control of orders/paths of quantum process has gained much of recent interests as it finds useful applications in quantum communication tasks [19–21]. To what extent these advantages are specific to superpositions of causal orders, rather than being generic to other forms of coherent superpositions of communication protocols, is currently a matter of debate [8–10]. In particular, the advantage of coherent control of orders in time over that of paths in space is achieved under the distinct role of external and internal degrees of freedoms in communication task. In this regard the present work is quite important. Here we have introduced a novel generalization of quantum communication task and established advantage of indefinite ordering of quantum processes over coherently controlled processes with fixed order. Importantly, this advantage implies that access to a qubit system, controlling the order of quantum processes, is a more powerful resource than $(n − 1)$ qubit side-channels for any natural number $n > 1$. Present study also opens up potential use of indefinite causal order in distributed protocols, such as multipartite quantum state transfer, quantum network, and entanglement distribution [22, 23] which have enormous practical relevance in the emerging new technology of quantum internet [24, 25].

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Appendix A: Proof of Theorem 1

The proof uses a general result, expressed in terms of the following definition: for a generic quantum channel $C_L$ of a generic quantum system $L$, we say that $C$ can transmit a $d$-dimensional quantum system in a one-way protocol if there exists an encoding channel $E_{LR} : L(H_L) \rightarrow L(H_L \otimes H_R)$, a measurement a measurement $(P_j)_j$ on system $R$, and a set of local operations $(D_j)_j$ on system $L$, such that $\sum_j \text{Tr}_L[(D_j(C_L \otimes P_j)E_{LR}(\rho))] = \rho$ for every state $\rho \in \text{St}(C^d)$.

**Proposition 1.** If channel $C_L$ can transmit the state of a $d$-dimensional quantum system in a one-way protocol, then channel $C_L$ has quantum capacity of at least $\log d$.

**Proof.** Defining $E_j := \text{Tr}_L[(I_R \otimes P_j)E_{LR}]$, we obtain the equivalent condition $\sum_j D_j C_R E_j = I_d$. This condition is satisfied if and only if each term in the sum is proportional to the identity map, namely $D_j C_R E_j = p_j I_S$ for some probability distribution $(p_j)_j$. Since $F_j$ and $C_L$ are trace-preserving, this condition implies that $E_j' := E_j/p_j$ is trace-preserving. Since the condition $D_j C_R E_j' = I_d$ holds, the channel $C_R$ permits a perfect transmission of a $d$-dimensional system, and therefore has a quantum capacity of at least $\log d$ qubits.

**Proof of Theorem 1.** Suppose that there exists a one-way protocol for random-receiver quantum communication using channels $(C_i)_{i=1}^n$, and that the protocol can successfully transfer a $d$-dimensional quantum system to any of the receivers $(B_i)_{i=1}^n$. Let $E : L(C^d) \rightarrow L(H_1 \otimes \cdots \otimes H_n)$ be the encoding channel used in the protocol. For every $x \in \{1, \ldots, n\}$, let $(M^{(x)}_i)$ be the measurement performed by the $(n-1)$ parties other than party $x$, and let $(B^{(x)}_i)$ be the conditional operations performed by party $x$. We can then regard systems $B_x$ and $\otimes_{y \neq x} B_y$ as systems $L$ and $R$ in Proposition 1, with encoding channel $E_{LR} := (F_1 \otimes \cdots \otimes F_n)E$ with $E_y := C_y$ for $y \neq x$ and $E_x := I_x$. Applying Proposition 1, we then obtain that channel $C_x$ must have a capacity of at least $\log d$ qubits. Since $x$ is an arbitrary number in $\{1, \ldots, n\}$, every channel in the set $(C_i)_{i=1}^n$ must have a capacity of at least $\log d$.

Appendix B: Proof of Theorem 2

The proof uses a generalization of Proposition 1 to arbitrary separable protocols. For a generic quantum channel $S_L$ on a generic quantum system $L$, we say that $C$ can transmit a $d$-dimensional quantum system in a separable protocol if there exists a system $R$, an encoding channel $E_{LR} : L(H_L) \rightarrow L(H_L \otimes H_R)$, and a separable channel $D = \sum_j L_j \otimes R_j$ where $L_j : L(H_L) \rightarrow L(C^d)$ and $R_j : L(H_R) \rightarrow C$ are completely positive maps for every $j$, such that

$$D(S_L \otimes I_R)E = I_d,$$

where $I_R$ is the identity channel on system $R$.

**Proposition 2.** If the input of channel $S_L$ is a $d$-dimensional quantum system and $S_L$ can transfer the state of a $d$-dimensional quantum system in a separable protocol, then $S_L$ is a unitary channel.
Proof. Defining $\mathcal{E}_i := (\mathcal{I}_L \otimes \mathcal{R}_j)\mathcal{E}_{LR}$, we can rewrite Eq. (B1) as $\sum_{i} L_i S_i \mathcal{E}_i = \mathcal{I}_d$. This condition is satisfied if and only if each term in the sum is proportional to the identity map, namely $D(C_R)\mathcal{E}_j = p_j \mathcal{I}_S$ for some probability distribution $(p_j)$. Using Kraus representations for the maps $L_i$, $S_i$, and $\mathcal{E}_j$, we obtain the condition $F_{jk} S_i E_{jm} = \lambda_{jkln} I_d$ for suitable coefficients $\lambda_{jkln}$ satisfying the normalization condition $\sum_{jkln} |\lambda_{jkln}|^2 = 1$. Due to the normalization condition, there must exist values of the indices $\langle j, k, l, m \rangle$ such that $\lambda_{jkln} \neq 0$. For these values, the condition $F_{jk} S_i E_{jm} = \lambda_{jkln} I_d$ implies that the operator $F_{jk}$ is invertible, and that one has $S_i E_{jm} = \lambda_{jkln} F_{jk}^{-1}$. Multiplying by $F_{jk}$ on both sides of the equation, we then obtain

$$S_i E_{jm} F_{jk} = \lambda_{jkln} I_d.$$  
(B2)

Now, define the completely positive map $\tilde{E}$ by $\tilde{E}(\rho) := \sum_{i,j,k,m} F_{jm} F_{jk} \rho F_{jk}^\dagger F_{jm}^\dagger$. Eq. (B2) implies the relation $S_i \tilde{E} = \mathcal{I}_d$. Since $S_i$ and $\mathcal{I}_d$ are trace-preserving, also $\tilde{E}$ must be trace-preserving. Hence, $\tilde{E}$ is an invertible quantum channel and $S_i$ is its inverse. Since the input and output systems of these two channels have the same dimension, this means that both channels must be unitary.

Proof of Theorem 2. Suppose that there exists a general LOCC protocol for random-receiver quantum communication using channels $(C_i^n)_{i=1}^n$ and side-channels $(S_i^n)_{i=1}^n$ acting on $d$-dimensional quantum systems. Let $\mathcal{E} : L(C^d) \rightarrow L(H'_1 \otimes \cdots \otimes H'_n)$ be the encoding channel used in the protocol, with $H'_i := H_B \otimes H_{S_i}$, where $H_i$ is the output of channel $C_i$ and $S_i = C_i^d$ is the output of the side-channel $S_i$.

For every $x \in \{1, \ldots, n\}$, we can regard systems $S_x$ and $B_x \otimes \bigotimes_{y \neq x} (B_y \otimes S_y)$ as systems $L$ and $R$ in Proposition 2, with channel $S_i := S_x$ and encoding channel $\mathcal{E}_{LR} := (F_{1} \otimes \cdots \otimes F_{n})\mathcal{E}$ with $E_{y} := C_y \otimes S_x$ for $y \neq x$ and $E_{x} := C_{B_x} \otimes I_{S_x}$. The original LOCC protocol can be regarded as a special case of separable protocol with respect to the bipartition $(L, R)$. Applying Proposition 1, we then obtain that channel $S_x$ must be unitary. Since $x$ is an arbitrary element of $\{1, \ldots, n\}$, every channel in the set $(S_i^n)_{i=1}^n$ must be unitary.

Appendix C: Switching products of Pauli channels

Let $\mathcal{E} := \bigotimes_{k=1}^2 \mathcal{E}_k$ be the product of two Pauli channels, given by $\mathcal{E}_1 \equiv \{ \sqrt{p_0}I, \sqrt{p_1}X, \sqrt{p_2}Y, \sqrt{p_3}Z \}$ and $\mathcal{E}_2 \equiv \{ \sqrt{q_0}I, \sqrt{q_1}X, \sqrt{q_2}Y, \sqrt{q_3}Z \}$, $\sum_{i=0}^3 p_i^2 = \sum_{i=0}^3 q_i^2 = 1$, respectively. If two instances of the quantum channel $\mathcal{E}$ are combined through the quantum SWITCH, the resulting channel is

$$S_{\omega_c}^{(2)}(\mathcal{E}, \mathcal{E})[\rho_{B_{1} \cdots B_{n}}] = C_+ (\rho_{B_{1} \cdots B_{n}}) \otimes \omega_c + C_- (\rho_{B_{1} \cdots B_{n}}) \otimes Z \omega_c Z,$$
(C1)

where,

$$C_+ (\rho_{B_{1}B_{2}}) = \left( \sum_{i=0}^3 p_i^2 q_i^2 + p_0^2 (1 - q_0^2) + p_1^2 (1 - q_1^2) + p_2^2 (1 - q_2^2) + p_3^2 (1 - q_3^2) \right) [\rho_{B_{1}B_{2}}]$$

$$+ 2q_0 \sum_{i=0}^3 p_i^2 (q_1 (I \otimes X) [\rho_{B_{1}B_{2}}] (I \otimes X) + q_2 (I \otimes Y) [\rho_{B_{1}B_{2}}] (I \otimes Y) + q_3 (I \otimes Z) [\rho_{B_{1}B_{2}}] (I \otimes Z))$$

$$+ 2p_0 \sum_{i=0}^3 q_i^2 (p_1 (X \otimes I) [\rho_{B_{1}B_{2}}] (X \otimes I) + p_2 (Y \otimes I) [\rho_{B_{1}B_{2}}] (Y \otimes I) + p_3 (Z \otimes I) [\rho_{B_{1}B_{2}}] (Z \otimes I))$$

$$+ 4(p_{0}p_{1}q_{0}q_{1} + p_{2}p_{3}q_{2}q_{3}) (X \otimes X) [\rho_{B_{1}B_{2}}] (X \otimes X) + 4(p_{0}p_{2}q_{0}q_{2} + p_{1}p_{3}q_{1}q_{3}) (Y \otimes Y) [\rho_{B_{1}B_{2}}] (Y \otimes Y)$$

$$+ 4(p_{0}p_{3}q_{0}q_{3} + p_{2}p_{1}q_{2}q_{1}) (Z \otimes Z) [\rho_{B_{1}B_{2}}] (Z \otimes Z) + 4(p_{0}p_{1}q_{0}q_{1} + p_{2}p_{3}q_{2}q_{3}) (X \otimes Y) [\rho_{B_{1}B_{2}}] (X \otimes Y)$$

$$+ 4(p_{0}p_{1}q_{0}q_{1} + p_{2}p_{3}q_{2}q_{3}) (X \otimes Z) [\rho_{B_{1}B_{2}}] (X \otimes Z) + 4(p_{0}p_{2}q_{0}q_{2} + p_{1}p_{3}q_{1}q_{3}) (Y \otimes X) [\rho_{B_{1}B_{2}}] (Y \otimes X)$$

$$+ 4(p_{0}p_{2}q_{0}q_{2} + p_{1}p_{3}q_{1}q_{3}) (Y \otimes Z) [\rho_{B_{1}B_{2}}] (Y \otimes Z) + 4(p_{0}p_{3}q_{0}q_{3} + p_{2}p_{1}q_{2}q_{1}) (Z \otimes X) [\rho_{B_{1}B_{2}}] (Z \otimes X)$$

$$+ 4(p_{0}p_{3}q_{0}q_{3} + p_{2}p_{1}q_{2}q_{1}) (Z \otimes Y) [\rho_{B_{1}B_{2}}] (Z \otimes Y);$$
(C2)
\[ C_-(\rho_{B_1B_2}) = 2^3 \sum_{l=0}^3 p_l^2 (q_1 q_2 (I \otimes Z) |\rho_{B_1B_2} |(I \otimes Z) + q_2 q_3 (I \otimes X) |\rho_{B_1B_2} |(I \otimes X) + q_3 q_1 (I \otimes Y) |\rho_{B_1B_2} |(I \otimes Y)) \]
\[ + 2^3 \sum_{l=0}^3 q_l^2 (p_1 p_2 (Z \otimes I) |\rho_{B_1B_2} |(Z \otimes I) + p_2 p_3 (X \otimes I) |\rho_{B_1B_2} |(X \otimes I) + p_3 p_1 (Y \otimes I) |\rho_{B_1B_2} |(Y \otimes I)) \]
\[ + 4(p_2 p_3 q_0 q_1 + p_0 p_1 q_2 q_3) (X \otimes X) |\rho_{B_1B_2} |(X \otimes X) + 4(p_1 p_3 q_0 q_2 + p_2 p_1 q_1 q_3) (Y \otimes Y) |\rho_{B_1B_2} |(Y \otimes Y) \]
\[ + 4(p_1 p_2 q_0 q_3 + p_0 p_3 q_1 q_2) (Z \otimes Z) |\rho_{B_1B_2} |(Z \otimes Z) + 4(p_0 p_1 q_1 q_2 + p_2 p_3 q_0 q_3) (X \otimes Z) |\rho_{B_1B_2} |(X \otimes Z) \]
\[ + 4(p_0 p_1 q_1 q_3 + p_2 p_3 q_0 q_2) (X \otimes Y) |\rho_{B_1B_2} |(X \otimes Y) + 4(p_1 p_3 q_0 q_1 + p_0 p_2 q_2 q_3) (Y \otimes X) |\rho_{B_1B_2} |(Y \otimes X) \]
\[ + 4(p_1 p_2 q_0 q_3 + p_0 p_3 q_1 q_2) (Y \otimes Z) |\rho_{B_1B_2} |(Y \otimes Z) + 4(p_1 p_2 q_0 q_1 + p_0 p_3 q_2 q_3) (Z \otimes X) |\rho_{B_1B_2} |(Z \otimes X) \]
\[ + 4(p_1 p_2 q_0 q_1 + p_0 p_3 q_2 q_3) (Z \otimes Y) |\rho_{B_1B_2} |(Z \otimes Y). \] (C3)

For \( E_1 = E_1 = \mathcal{N}_{XY} \), putting \( p_0 = p_3 = q_0 = q_3 = 0 \) and \( p_1 = p_2 = q_1 = q_2 = \frac{1}{\sqrt{2}} \) in Eqs.(C2-C3) we obtain
\[ C_+ (\rho_{B_1B_2}) = \rho_{B_1B_2} + Z \otimes Z (\rho_{B_1B_2}) Z \otimes Z, \] (C4)
\[ C_- (\rho_{B_1B_2}) = I \otimes Z (\rho_{B_1B_2}) I \otimes Z + Z \otimes I (\rho_{B_1B_2}) Z \otimes I. \] (C5)

Now considering \( \omega_c = |+\rangle_c \langle +| \), after proper normalization Eq.(C1) becomes,
\[
S_{|+\rangle_c}^{(2)} (E, E) [\rho_{B_1B_2}] = \frac{1}{4} \left[ I \otimes Z (\rho_{B_1B_2}) I \otimes Z + Z \otimes I (\rho_{B_1B_2}) Z \otimes I \right] \otimes \mathbb{P}^{(\pm)}_c \]
\[ + \frac{1}{4} [\rho_{B_1B_2} + Z \otimes Z (\rho_{B_1B_2}) Z \otimes Z] \otimes \mathbb{P}^{(\pm)}_c, \] (C6)
where, \( \mathbb{P}^{(\pm)} := |\pm\rangle \langle \pm| \). Upon measuring the order system in \( \{\mathbb{P}^{(\pm)}\} \) basis, the conditional states of \( B_1B_2 \) read as,
\[ ' - 1' \text{ outcome} \rightarrow \frac{1}{2} \left[ I \otimes Z (\rho_{B_1B_2}) I \otimes Z + Z \otimes I (\rho_{B_1B_2}) Z \otimes I \right], \]
\[ ' + 1' \text{ outcome} \rightarrow \frac{1}{2} [\rho_{B_1B_2} + Z \otimes Z (\rho_{B_1B_2}) Z \otimes Z]. \] (C7)

Appendix D: Random-receiver quantum communication for \( n \) receivers

When \( E = F := \otimes_{k=1}^n \mathcal{N}_{XY} \) the switched quantum distribution channel from Alice to \( n \) Bobs read as,
\[ S^{(n)}_{\omega_c} [\rho_{B_1...B_n}] := \sum_{i_1j_1,...,i_nj_n=0} G_{i_1j_1...i_nj_n} (\rho_{B_1...B_n} \otimes \omega_c) G_{i_1j_1...i_nj_n}^\dagger, \] (D1)

where, \( G_{i_1j_1...i_nj_n} := \otimes_{k=1}^n E_{i_k} F_{j_k} \otimes |0\rangle_c \langle 0| + \otimes_{k=1}^n F_{j_k} E_{i_k} \otimes |1\rangle_c \langle 1| \);

with, \( E_0 = F_0 = \frac{1}{\sqrt{2}} X \) and, \( E_1 = F_1 = \frac{1}{\sqrt{2}} Y \).

Consider now an individual term in the summation of the right hand side of Eq.(D1). The sign of the coherence term of the order qubit will be flipped , i.e., \( \omega_c \rightarrow Z \omega_c Z \) if \( \otimes_{k=1}^n (i_k \oplus j_k) = 1 \), and whenever \( \otimes_{k=1}^n (i_k \oplus j_k) = 0 \) it will remain invariant, i.e., \( \omega_c \rightarrow \omega_c \). Furthermore we will use the facts that \( XX = YY = I \) and \( XY = iZ \) and \( YX = -iZ \) in our following analysis.

Case-I: Order bit invariant terms \( \otimes_{k=1}^n (i_k \oplus j_k) = 0 \). In this case we have the following terms:
\( i_k = j_k, \forall k \); which will result the term \( G_{i_1j_1...i_nj_n} \) of the form \( I_{B_1...B_n} \otimes (|0\rangle_c \langle 0| + |1\rangle_c \langle 1|) \) at right hand side of Eq.(D1).

(ii) For even number of cases (say \( 2m \)) the indices \( i_k \)'s are different than the corresponding \( j_k \)'s and for the other cases they are equal. For a given \( m \in \{0,1,...,|\frac{n}{2}|\} \) this will result terms \( G_{i_1j_1...i_nj_n} \) of the form \( (-1)^m (Z_{1,...,Z_{2m}}, I_{2m+1,...,I_q} \otimes (|0\rangle_c \langle 0| + |1\rangle_c \langle 1|), \) where \( (Z_1,...,Z_p, I_{p+1},...,I_q) \) denotes term with \( Z \) acting on \( p \) among \( q \) numbers of state and identity acting on rest. Number of such terms be \( \binom{q}{p} \frac{q!}{p!(q-p)!}. \)

Case-II: Order bit flipped terms \( \otimes_{k=1}^n (i_k \oplus j_k) = 1 \). In this case we have the following terms:
(i) For odd number of cases (say $2m+1$) the indices $i_k$’s and the corresponding $j_k$’s are $i_k = j_k = 0$ and for the other cases they are equal. For a given $m \in \{0, 1, ..., \frac{n-1}{2}\}$ this will result terms $G_{i_1j_1..i_nj_n}$ of the form $i \times (-1)^m(Z_1, ..., Z_{2m+1}, \mathbb{I}_{2m+2}, ..., \mathbb{I}_n) \otimes (|0\rangle_c (0|-1)_c (1))$.

(ii) $2m+1$ numbers of $i_k = j_k = 1$ and rests are equal. For a given $m \in \{0, 1, ..., \frac{n-1}{2}\}$ this will result terms $G_{i_1j_1..i_nj_n}$ of the form $-i \times (-1)^m(Z_1, ..., Z_{2m+1}, \mathbb{I}_{2m+2}, ..., \mathbb{I}_n) \otimes (|0\rangle_c (0|-1)_c (1))$.

Combining these all together we finally have,

$$S^{(n)}_\omega [\rho_{B_1..B_n}] := \frac{1}{2^n} \sum_{m=0}^{\lceil \frac{n}{2} \rceil} \left[ (Z_1, ..., Z_{2m}, \mathbb{I}_{2m+1}, ..., \mathbb{I}_n) \rho_{B_1B_2..B_n} (Z_1, ..., Z_{2m}, \mathbb{I}_{2m+1}, ..., \mathbb{I}_n) \right] \otimes \omega_c$$

$$+ \frac{1}{2^n} \sum_{m=0}^{\lceil \frac{n}{2} \rceil} \left[ (Z_1, ..., Z_{2m+1}, \mathbb{I}_{2m+2}, ..., \mathbb{I}_n) \rho_{B_1B_2..B_n} (Z_1, ..., Z_{2m+1}, \mathbb{I}_{2m+2}, ..., \mathbb{I}_n) \right] \otimes Z\omega_c Z.$$  \hspace{1cm} (D2)

Suppose that initial state of order system is $\omega_c = |+\rangle \langle +|$. In that case after the evolution of switched channel if depending on the outcome of Pauli $X$ measurement on order system if one of the receivers apply suitable local unitary correction on his subsystem, then the final outcome state reads as,

$$\frac{1}{2^n} \sum_{m=0}^{\lceil \frac{n}{2} \rceil} \left[ (Z_1, ..., Z_{2m}, \mathbb{I}_{2m+1}, ..., \mathbb{I}_n) \rho_{B_1B_2..B_n} (Z_1, ..., Z_{2m}, \mathbb{I}_{2m+1}, ..., \mathbb{I}_n) \right].$$  \hspace{1cm} (D3)

In the present context the input state is the generalized GHZ state $|\psi\rangle_{B_1..B_n} = a|0 \cdots 0\rangle_{B_1..B_n} + \beta|1 \cdots 1\rangle_{B_1..B_n}$ which is invariant under local unitary $Z$ operation by any even number of parties. Thus the state gets distributed perfectly among $n$ receivers.

To reproduce the qubit information at one of the receivers’ lab they will follow a LOCC protocol. Suppose that the qubit state needs to be reproduced at $i^{th}$ Bob. All other Bobs will perform Pauli $X$ measurement on their respective subsystems and inform the measurement results $x_k \in \{1, -1\}, \forall k \in \{1, ..., n\} \& k \neq i$. Depending on this information $i^{th}$ Bob will apply $Z$ unitary correction on his part if $\Pi_{k:k \neq i}x_k$ is $-1$, otherwise he does nothing.

### Appendix E: Random-receiver quantum communication with controlled operations in a definite order

In this section we discuss the use of controlled operations in a definite causal order for random-receiver quantum communication. First, we show that if arbitrary controlled operations are allowed, it is easy to construct protocols that achieve random-receiver quantum communication through entanglement-breaking channels. Then, we show that, if the controlled operations are restricted to be $\text{SWAP}$ operations, then random-receiver quantum communication cannot be achieved for any odd $n$.

#### 1. Protocol for random-receiver quantum communication using arbitrary controlled operations

The following protocol permits random-receiver quantum communication through the channels $A = B = N^{\otimes n}_{XY}$. The protocol starts with the message encoded in the state of the first qubit. First, the sender applies a CNOT gate to the message and the control qubit, initialized in the state $|+\rangle$. As a result, the message and the control qubit end up in the state $\left[ |0\rangle (a|0\rangle + \beta|1\rangle) + |1\rangle (a|1\rangle + \beta|0\rangle) \right]/\sqrt{2}$. Second, the sender prepares $n-1$ qubits in the state $|0\rangle$. The sender sends the message and the other $n-1$ qubits through the channel $AB = (N^2_{XY})^{\otimes n}$, which collapses the overall state a classical mixture of the states $|0\rangle_0^{\otimes (n-1)} (a|0\rangle + \beta|1\rangle)$ and $|1\rangle_0^{\otimes (n-1)} (a|1\rangle + \beta|0\rangle)$. Third, CNOT gates are applied to the control qubit and to the additional $n-1$ qubits, producing either the state $|0\rangle (a|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n})$ or the state $|1\rangle (a|1\rangle^{\otimes n} + \beta|0\rangle^{\otimes n})$.

Finally, the first receiver measures the first qubit in the computational basis, and, if the outcome is $1$, all the other receivers perform the bit flip measurement $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ on the remaining qubits and on the control qubit. In this way, the remaining qubits end up in the generalized GHZ state $a|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$, and random-receiver quantum communication can be achieved as in the noiseless protocol presented in the introduction.

Note that this protocol uses the control qubit to bypass the noisy channels $A$ and $B$, as one can see from the fact that, after the channels $A$ and $B$ have acted, all the information about the message is on the control qubit.
addition, the protocol freely generates entanglement between the \( n \) receivers after the channels \( A \) and \( B \) have acted. The entanglement generation is achieved by the \( n - 1 \) CNOT gates applied in the last step of the protocol. In general, these CNOTs cannot be implemented by the receivers, due to their spatial separation. Hence, they must be regarded as performed by a third party other than the receivers.

Note that the presence of entangling operations between the control and each receiver is essential in the above protocol. More generally, entangling operations are necessary in any protocol that achieves random-receiver quantum communication through entanglement-breaking channels in a definite order. Any such protocol needs to bypass the entanglement-breaking channels by encoding quantum information in the control qubit. An equivalent condition can be obtained by introducing an additional reference system at the receiver’s end: in order to achieve perfect quantum communication, the protocol must transform a maximally entangled state of the input and the reference into a maximally entangled state of the control and the reference. After the action of an entanglement-breaking channel, the control and the reference have no correlation with the qubits at the receivers’ locations. Hence, no quantum information can be transferred back from the control to the receivers without the use of entangling operations.

2. No GHZ state generation with controlled SWAP operations

Arbitrary controlled operations appear to be a too broad set for the problem of random-receiver quantum communication, in that they allow a complete transfer of information to the control qubit and violate the locality restrictions among the receivers. On the other hand, the quantum SWITCH can be regarded as a controlled SWAP operation in time: it swaps the order of quantum systems appearing in a given time sequence, putting the inputs/outputs of channel \( A \) either before or after the inputs/outputs of channel \( B \). One may then ask if controlled SWAP operations in space can reproduce the same features when the channels \( A \) and \( B \) are arranged in a fixed sequential order, say with \( A \) acting before \( B \).

Here we show that the answer is negative, in the following sense: suppose that the quantum channels \( A \) and \( B \) are placed in a fixed order, and that their inputs and outputs undergo controlled permutations, as in Figure 3. For odd \( n \), we show that, no matter which controlled permutations are chosen, one of the receivers will remain in a fixed state, independent of the quantum message from the sender.

![Figure 3: Random-receiver quantum communication task with controlled SWAP operations in space.](image)

The argument is simple. The output of channel \( N_{XY}^\otimes n \) is a mixture of states in the computational basis, of the form \( \otimes_{j=1}^n |b_j\rangle \), with \( b_j \in \{0, 1\} \) for every \( j \). Perfect quantum communication through the entanglement-breaking channel
is possible only if, for every state appearing in the mixture, the initial message has been perfectly transferred to the control qubit.

Let us focus on one specific state in the mixture, say $|b\rangle = \bigotimes_{j=1}^{n} |b_j\rangle$, and let us denote by $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$ the state of the control qubit conditional to the state $\bigotimes_{j=1}^{n} |b_j\rangle$. The state $|b\rangle|\psi'\rangle$ then undergoes a controlled operation $W$, becoming the state $W|b\rangle|\psi'\rangle = \alpha' \bigotimes_{i=1}^{n} |b_{\pi(i)}\rangle \otimes |0\rangle + \beta' \bigotimes_{i=1}^{n} |b_{\sigma(i)}\rangle \otimes |1\rangle$, where $\pi$ and $\sigma$ are two permutations. Equivalently, the state can be rewritten as $W|b\rangle|\psi'\rangle = \alpha' \bigotimes_{i=1}^{n} |b_i\rangle \otimes |0\rangle + \beta' \bigotimes_{i=1}^{n} |b_{\tau(i)}\rangle \otimes |1\rangle$ for some suitable permutation $\tau$. When $n$ is odd, there exists at least one value of $j$ such that $b_j = \tau(b_j)$. Hence, the $j$-th system ends up in the state $|b_j\rangle$, which has no dependence on the coefficients $\alpha'$ and $\beta'$, and therefore on the initial message.

In summary, the $n$ output systems and the control end up in a mixture of pure states, one of which is a product between a fixed state of system $j$ and the remaining systems. Hence, the quantum message cannot be transferred perfectly to the $j$-th receiver, since with some non-zero probability, the state of the $j$-th system will be independent of the message.