 PRODUCTS OF GENERAL MENGER SPACES

PIOTR SZEWCZAK AND BOAZ TSABAN

Abstract. We study products of general topological spaces with Menger’s covering property, and its refinements based on filters and semifilters. To this end, we extend the projection method from the classic real line topology to the Michael topology. Among other results, we prove that, assuming the Continuum Hypothesis, every productively Lindelöf space is productively Menger, and every productively Menger space is productively Hurewicz. None of these implications is reversible.

1. Introduction

Let $[N]^\infty$ be the set of all infinite subsets of the set $N$ of natural numbers. For sets $a, b \in [N]^\infty$, we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A semifilter $[4]$ is a set $S \subseteq [N]^\infty$ such that, for each set $s \in S$ and each set $b \in [N]^\infty$ with $s \subseteq^* b$, we have $b \in S$. For a semifilter $S$, let $S^+ := \{ a \in [N]^\infty : a^c \notin S \}$. For all sets $a \in S$ and $b \in S^+$, the intersection $a \cap b$ is infinite.

By space we mean a topological space. Let $S$ be a semifilter. A space $X$ is $S$-Menger [21][19] if for each sequence $U_1, U_2, \ldots$ of open covers of the space $X$, there are finite sets $F_1 \subseteq U_1$, $F_2 \subseteq U_2$, … such that $\{ n \in N : x \in \bigcup F_n \} \in S$ for all points $x \in X$. A space is Menger [14][11] if it is $[N]^\infty$-Menger [11]. Let $cF := ([N]^\infty)^+$, the (semi)filter of cofinite subsets of $N$. A space is Hurewicz [11] if it is $cF$-Menger. Restricting the definition of $S$-Menger spaces to countable open covers, we obtain the definition of countably $S$-Menger space. This makes it possible to extend our investigations beyond the Lindelöf realm.

Proposition 1.1. Let $S$ be a semifilter. A space $X$ is $S$-Menger if and only if it is Lindelöf and countably $S$-Menger. \hfill $\square$

We identify each set $a \in [N]^\infty$ with its increasing enumeration. Thus, for a natural number $n$, $a(n)$ is the $n$-th smallest element of the set $a$. We identify the Cantor space $\{0,1\}^N$ with the family $P(N)$ of all subsets of the set $N$. We denote by $[N]^\leq\infty$ the family of finite subsets of $N$, so that $P(N) = [N]^\infty \cup [N]^\leq\infty$. Since we identify every set $a \in [N]^\infty$ with its increasing enumeration, we have $[N]^\infty \subseteq N^N$. The topology of the space $[N]^\infty$ (a subspace of the Cantor space $P(N)$) coincides with the subspace topology induced by the Baire space $N^N$.

Let $S$ be a semifilter. For elements $a, b \in [N]^\infty$, we write $a \leq_S b$ if $\{ n \in N : a(n) \leq b(n) \} \in S$. Then $a \notin_S b$ if and only if $b <_{S^+} a$. We write $a \leq^* b$ if $a \leq_{cF} b$, $a \leq^\infty b$ if $a \leq_{[N]^\infty} b$, and $a \leq b$ if $a(n) \leq b(n)$ for all natural numbers $n$.

A map $\Psi$ from a space $X$ into $[N]^\infty$ is upper continuous if the sets $\{ x \in X : \Psi(x)(n) \leq m \}$ are open for all natural numbers $n$ and $m$. In particular, continuous functions are upper continuous.

Let $X$ and $Y$ be spaces. A set-valued map $\Phi : X \to Y$ is compact-valued upper semicontinuous (cusco) if, for each point $x \in X$, the set $\Phi(x)$ is a nonempty compact subset of the space $Y$, and for every open set $V \subseteq Y$, the set $\Phi^{-1}[V] := \{ x \in X : \Phi(x) \subseteq V \}$ is open in

2010 Mathematics Subject Classification. 54D20, 03E17.
the space $X$. The image of a set $X$ under a set-valued map $\Phi$ is the set $\Phi[X] := \bigcup_{x \in X} \Phi(x)$. The following observation can be proved using earlier methods \cite[Theorem 7.3]{16}.

**Proposition 1.2.** Let $X$ be a space, and $S$ be a semifilter. The following assertions are equivalent:

1. The space $X$ is countably $S$-Menger.
2. Every upper continuous image of the space $X$ in $[\mathbb{N}]^\infty$ is $\leq_S$-bounded.
3. Every cusco image of the space $X$ in $[\mathbb{N}]^\infty$ is $\leq_S$-bounded.

For a semifilter $S$, let $b(S)$ be the minimal cardinality of a $\leq_S$-unbounded subset of $[\mathbb{N}]^\infty$.

**Corollary 1.3.**

1. Every space of cardinality smaller than $b(S)$ is countably $S$-Menger.
2. The discrete space of cardinality $b(S)$ is not countably $S$-Menger.

2. Countable covers

Let $S$ be a semifilter. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq b(S)$ is an $S$-scale \cite[Definition 4.1]{19} if, for each element $b \in [\mathbb{N}]^\infty$, there is an element $c \in [\mathbb{N}]^\infty$ such that $b \leq_S c$ and $c \not\leq_S x$ for all but less than $b(S)$ elements $x \in X$. For every semifilter $S$, $S$-scales provably exist \cite[Lemma 2.9]{21}. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq b(S)$ is a cofinal $S$-scale \cite[Definition 6.1]{19} if for each element $b \in [\mathbb{N}]^\infty$, $b \leq_S x$ for all but less than $b(S)$ elements $x \in X$. By filter we mean a semifilter that is closed under finite intersections. If $F$ is a filter, then there is a cofinal $F$-scale if and only if $b(F) = b(F^+)$ \cite[Corollary 6.3]{19}.

Let $P$ and $Q$ be properties of spaces. A space $X$ satisfies $(P, Q)^x$ if for each space $Y$ with the property $P$, the product space $X \times Y$ has the property $Q$. Define $P^x := (P, P)^x$. A space is productively $P$ if it satisfies $P^x$.

2.1. General semifilters. Let $\kappa$ be an uncountable cardinal number. A space $X$ is $\kappa$-Lindelöf if every open cover of $X$ has a subcover of cardinality smaller than $\kappa$. A space $X$ with $|X| \geq \kappa$ is $\kappa$-concentrated on a set $D \subseteq X$ if $|X \setminus U| < \kappa$ for all open sets $U$ containing $D$.

**Theorem 2.1.** Let $S$ be a semifilter with a cofinal $S$-scale.

$$(b(S)\text{-Concentrated}, b(S)\text{-Lindelöf})^x \subseteq \text{countably } S\text{-Menger}.$$ 

For the proof of Theorem 2.1 and for later discussions, we introduce several notions and auxiliary results. Let $X \subseteq [\mathbb{N}]^\infty$. The Michael topology \cite[15] on the set $X \cup [\mathbb{N}]^{<\infty} \subseteq P(\mathbb{N})$ is the one where the points of the set $X$ are isolated, and the neighborhoods of the points of the set $[\mathbb{N}]^{<\infty}$ are those induced by the Cantor space topology on $P(\mathbb{N})$.

Let $\kappa$ be an uncountable cardinal number. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \kappa$ is $\kappa$-unbounded if $|X| \geq \kappa$, and the cardinality of every $\leq$-bounded subset of the set $X$ is smaller than $\kappa$. A standard argument \cite[Lemma 2.3]{19} implies the following result.

**Lemma 2.2.** Let $\kappa$ be an uncountable cardinal number, and $X$ be a subset of $[\mathbb{N}]^\infty$. The set $X$ is $\kappa$-unbounded if and only if the space $X \cup [\mathbb{N}]^{<\infty}$, with the Michael topology, is $\kappa$-concentrated on the set $[\mathbb{N}]^{<\infty}$. 

**Definition 2.3.** A real space is a subspace of the Cantor space $P(\mathbb{N})$. 

Proposition 2.4. Let \( \kappa \) be an uncountable cardinal number, and \( S \) be a semifilter. Let \( X \subseteq [\kappa]^{<\infty} \) be a real space containing a \( \kappa \)-unbounded set \( Y \). For the Michael topology on the set \( Y \cup [\kappa]^{<\infty} \), the product space \( X \times (Y \cup [\kappa]^{<\infty}) \) is not \( \kappa \)-Lindelöf.

Proof. Let \( X \) be a real space containing a \( \kappa \)-unbounded set \( Y \). By Lemma 2.2 the space \( Y \cup [\kappa]^{<\infty} \) with the Michael topology is \( \kappa \)-concentrated. The diagonal set \( \{ (y, y) : y \in Y \} \) is a closed discrete subset of the product space \( X \times (Y \cup [\kappa]^{<\infty}) \), of cardinality at least \( \kappa \). Thus, the space \( X \times (Y \cup [\kappa]^{<\infty}) \) is not \( \kappa \)-Lindelöf. \( \Box \)

Lemma 2.5. Let \( S \) be a semifilter with a cofinal \( S \)-scale. Every \( \leq_S \)-unbounded set in \([\kappa]^{<\infty}\) contains a \( b(S) \)-unbounded set.

Proof. Let \( Y \) be a \( \leq_S \)-unbounded set in \([\kappa]^{<\infty}\), and \( \{ d_\alpha : \alpha < b(S) \} \) be a cofinal \( S \)-scale in \([\kappa]^{<\infty}\). For each ordinal number \( \alpha < b(S) \), let \( y \) be a \( \leq_S \)-bound of the set \( \{ y_\alpha : \alpha < b(S) \} \). There is an element \( y_\alpha \in Y \) such that \( y, d_\alpha <_S y_\alpha \). It follows that \( y_\alpha \notin \{ y_\beta : \beta < \alpha \} \). The set \( \{ y_\alpha : \alpha < b(S) \} \) is \( b(S) \)-unbounded: Let \( b \in [\kappa]^{<\infty} \). For each ordinal number \( \alpha < b(S) \) such that \( b \leq_S d_\alpha \), since \( d_\alpha <_{S^+} y_\alpha \), we have \( b <_{S^+} y_\alpha \). As the set \( \{ d_\alpha : \alpha < b(S) \} \) is a cofinal \( S \)-scale, we have \( b <_{S^+} y_\alpha \) for all but less than \( b(S) \) ordinal numbers \( \alpha < b(S) \). \( \Box \)

Proof of Theorem 2.7 Assume that a space \( X \) is not countably \( S \)-Menger. By Proposition 1.2 there is in \([\kappa]^{<\infty}\) a \( \leq_S \)-unbounded cusco image \( Y \) of \( X \). By Lemma 2.5 the set \( Y \) contains a \( b(S) \)-unbounded set \( Z \). By Proposition 2.4 for the Michael topology on the set \( Z \cup [\kappa]^{<\infty} \), the product space \( Y \times (Z \cup [\kappa]^{<\infty}) \) is not \( b(S) \)-Lindelöf. Since \( Y \) is a cusco image of the space \( X \), the product space \( X \times (Z \cup [\kappa]^{<\infty}) \) is not \( b(S) \)-Lindelöf. \( \Box \)

Definition 2.6. Let \( S \) be a semifilter. A space \( X \cup [\kappa]^{<\infty} \) is a (cofinal) \( S \)-scale space if the set \( X \) is a (cofinal) \( S \)-scale in \([\kappa]^{<\infty}\), and \( X \cup [\kappa]^{<\infty} \) has the Michael topology.

Remark 2.7. Since moving to a coarser topology on a space provides a continuous image of that space, all results for (cofinal) \( S \)-scale spaces are also true when using coarser topologies, for example, the Cantor space topology.

A semifilter \( S \subseteq [\kappa]^{<\infty} \) is meager if \( S \) is a meager subset of \([\kappa]^{<\infty}\). The proof of the next theorem is a literal repetition of proofs of earlier, analogous results [19] Theorems 5.4 and 6.5.

Proposition 2.8.

1. Let \( S \) be a meager semifilter. Every \( S \)-scale space is productively countably Hurewicz.
2. Every cofinal \( cF \)-scale space is productively countably \( S \)-Menger, for all semifilters \( S \).

Every product of a \( \mathfrak{d} \)-concentrated real space and a Hurewicz space that is hereditarily Lindelöf is Menger [22] Theorem 4.6]. The following theorem generalizes that. For a semifilter \( S \), let \( \chi(S) \) be the minimal cardinality of a basis for \( S \).

Theorem 2.9. Let \( S \) be a semifilter with \( \chi(S) < \text{cf}(b(S^+)) \). Let \( X \) be a Tychonoff \( b(S^+) \)-concentrated space, and \( Y \) be an \( S \)-Menger space. Then the product space \( X \times Y \) is countably Menger.

Proof. We use the projection method, introduced in an earlier work [22]. Let \( C \) be a compactification of the space \( X \), and \( Q \) be a countable subset of \( X \) on which the space \( X \) is concentrated. Let \( U_1, U_2, \ldots \) be countable open covers of the product space \( X \times Y \), by sets open in the product space \( C \times Y \). The product space \( Q \times Y \) is a countable union of
Menger spaces, and thus Menger. Let $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ be finite sets such that the set $U := \bigcup_n \mathcal{F}_n$ contains the set $Q \times Y$. Since the set $(C \times Y) \setminus U$ is closed in the space $C \times Y$, it is $S$-Menger. Thus, the projection $H$ of the set $(C \times Y) \setminus U$ in $C$ is an $S$-Menger set disjoint of the set $Q$.

**Lemma 2.10.** Let $S$ be a semifilter. Let $X$ be an $S$-Menger subset of some space, and $G$ a $G_\delta$ set containing $X$. There are closed sets $C_\alpha$, for the ordinal numbers $\alpha < \chi(S)$, such that $X \subseteq \bigcup_{\alpha < \chi(S)} C_\alpha \subseteq G$.

**Proof.** Let $G := \bigcap_n G_n$, where every set $G_n$ is open in $C$, and $G_{n+1} \subseteq G_n$ for all natural numbers $n$. Fix a base of cardinality $\chi(S)$ for the semifilter $S$. Let $n$ be a natural number. For each point $x \in X$, there is an open neighborhood $U^n_x$ of $x$ such that $U^n_x \subseteq G_n$. The family $U_n := \{ U^n_x : x \in X \}$ is an open cover of $X$.

Since the space $X$ is $S$-Menger, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that

$$I_x := \{ n \in \mathbb{N} : x \in \bigcup \mathcal{F}_n \} \in S$$

for all points $x \in X$.

Fix a point $x \in X$. Let $J$ be a member of the basis of $S$ such that $J \subseteq I_x$. Let

$$K := \bigcap_{n \in J} \bigcup \mathcal{F}_n.$$ 

Since the set $J$ is infinite and $G_{n+1} \subseteq G_n$ for all natural numbers $n$, we have $K \subseteq G$. The union of these sets $K$ is as required. \hfill \Box

By Lemma 2.10 there are closed sets $C_\alpha$ such that $H \subseteq \bigcup_{\alpha < \chi(F)} C_\alpha \subseteq C \setminus Q$. Since the set $X$ is $b(S^+)$-concentrated on $Q$, we have $|X \cap C_\alpha| < b(S^+)$ for all ordinal numbers $\alpha < \chi(S)$, and since $\chi(S) < \text{cf}(b(S^+))$, we have

$$\left| X \cap \bigcup_{\alpha < \chi(F)} C_\alpha \right| = \left| \bigcup_{\alpha < \chi(F)} X \cap C_\alpha \right| < b(S^+).$$

By Proposition 1.12 the union of less than $b(S^+)$ spaces that are countably $S$-Menger is countably Menger. Thus, the product space $(X \cap \bigcup_{\alpha < \chi(F)} C_\alpha) \times Y$ is countably Menger, and there are finite sets $\mathcal{F}_1' \subseteq \mathcal{U}_1, \mathcal{F}_2' \subseteq \mathcal{U}_2, \ldots$ such that the family $\bigcup_n \mathcal{F}_n'$ covers this product space. Then the family $\bigcup_n \mathcal{F}_n \cup \mathcal{F}_n'$ is an open cover of the product space $X \times Y$. \hfill \Box

2.2. **Superfilters.** A **superfilter** is a semifilter $S$ such that, for all sets $a, b \in [\mathbb{N}]^\infty$, $a \cup b \in S$ implies $a \in S$ or $b \in S$. A semifilter $S$ is a superfilter if and only if $S = F^+$ for some filter $F$.

For a superfilter $S$, every $S$-scale is a cofinal $S$-scale. In particular, every superfilter $S$ has a cofinal $S$-scale [19 Proposition 6.6].

**Theorem 2.11.** Let $S$ be a superfilter.

$$((\text{cofinal}) \ S\text{-scale space, } b(S)\text{-Lindel"of})^\times \subseteq \text{countably } S\text{-Menger}.$$ 

**Proof.** For superfilters, we have the following improvement of Lemma 2.6

**Lemma 2.12.** Let $S$ be a superfilter. Every $\leq_S$-unbounded set in $[\mathbb{N}]^\infty$ contains a (necessarily, cofinal) $S$-scale.
Proof. The proof is similar to that of Lemma 2.5. For a superfilter \( S \), the intersection of each element of \( S \) and each element of \( S^+ \) belongs to the set \( S \). Thus, if \( b \leq_S d_\alpha < y_\alpha \), then \( b < S y_\alpha \).

The rest of the proof is similar to that of Theorem 2.1 using Lemma 2.12.

Both Theorems 2.1 and 2.11 imply the following result.

Corollary 2.13. Let \( S \) be a superfilter.

\[(b(S)\text{-Concentrated, } b(S)\text{-Lindel"of})^\times \subseteq \text{countably } S\text{-Menger}. \]

2.3. Filters. Recall that a filter \( F \) has a cofinal \( F \)-scale if and only if \( b(F) = b(F^+) \).

Theorem 2.14. Let \( F \) be a filter.

1. (\( F \)-scale space, \( b(F) \)-Lindel"of)\(^\times \) \( \subseteq \) countably \( F^+ \)-Menger.
2. If \( b(F) = b(F^+) \), then:
   a. (cofinal \( F \)-scale space, \( b(F) \)-Lindel"of)\(^\times \) \( \subseteq \) countably \( F^+ \)-Menger.
   b. ((cofinal) \( F^+ \)-scale space, \( b(F^+) \)-Lindel"of)\(^\times \) \( \subseteq \) countably \( F \)-Menger.

Proof. The following lemma constitutes the combinatorial skeleton of the theorem.

Lemma 2.15. Let \( F \) be a filter.

1. Every \( \leq_{F^+} \)-unbounded set in \([N]^\infty \) contains an \( F \)-scale.
2. If \( b(F) = b(F^+) \), then:
   a. Every \( \leq_{F^+} \)-unbounded set contains a cofinal \( F \)-scale.
   b. Every \( \leq_{F^+} \)-unbounded set contains a (cofinal) \( F^+ \)-scale.

Proof. (1) Let \( Y \) be a \( \leq_{F^+} \)-unbounded set in \([N]^\infty \), and \( \{b_\alpha : \alpha < b(F)\} \) be a \( \leq_F \)-unbounded set in \([N]^\infty \). For each ordinal number \( \alpha < b(F) \), there is an element \( c \in [N]^\infty \) such that \( \{b_\beta, y_\beta : \beta < \alpha\} \subseteq c \). Pick an element \( y_\alpha \in Y \) such that \( c < F y_\alpha \). Since \( F \) is a filter, we have

\[ \{b_\beta, y_\beta : \beta < \alpha\} < F y_\alpha. \]

The set \( \{y_\alpha : \alpha < b(F)\} \) is an \( F \)-scale: Let \( b \in [N]^\infty \). As the set \( \{b_\alpha : \alpha < b(F)\} \) is \( \leq_F \)-unbounded, there is an ordinal number \( \alpha < b(F) \) such that \( b < F b_\alpha \). For each ordinal number \( \beta < b(F) \) that is greater than \( \alpha \), we have \( b < F b_\alpha < F y_\beta \).

(2)(a) Let \( Y \) be a \( \leq_{F^+} \)-unbounded set in \([N]^\infty \), and \( \{d_\alpha : \alpha < b(F)\} \) be a cofinal \( F \)-scale in \([N]^\infty \). For each ordinal number \( \alpha < b(F) \), let \( y_\alpha \) be an element of \( Y \) with \( d_\alpha < F y_\alpha \). The set \( \{y_\alpha : \alpha < b(F)\} \) is a cofinal \( F \)-scale: Let \( b \in [N]^\infty \). For each ordinal number \( \alpha < b(F) \) such that \( b < F d_\alpha \), we have \( b < F d_\alpha < F y_\alpha \), and since \( F \) is a filter, \( b < F y_\alpha \). The set \( \{y_\alpha : \alpha < b(F)\} \) is \( \leq_F \)-unbounded, and thus its cardinality is not smaller than \( b(F) \).

(2)(b) The proof is similar to that of Lemma 2.14. Here, since \( F \) is a filter, \( b \leq_F d_\alpha < F y_\alpha \) implies \( b < F y_\alpha \).

The proof of the theorem is now similar to that of Theorem 2.1 using Lemma 2.15.

Following the proof of earlier, analogous results \cite[Theorem 5.3, Theorem 6.2(2)]{19}, we obtain the following result.

Proposition 2.16. Let \( F \) be a filter.

1. Every \( F \)-scale space satisfies \((\text{countably } F\text{-Menger, countably } F^+\text{-Menger})^\times \).
2. Every cofinal \( F \)-scale space is productively countably \( F \)-Menger.
Theorem 2.17. Let $F$ be a filter.

1. $(b(F)\text{-Concentrated}, b(F)\text{-Lindelöf})^\times \subseteq \text{countably } F^+\text{-Menger}.$
2. $(\text{countably } F^+\text{-Menger}, b(F)\text{-Lindelöf})^\times \subseteq \text{countably } F^+\text{-Menger}.$
3. If $b(F) = b(F^+)$, then $(\text{countably } F^+\text{-Menger}, b(F^+)\text{-Lindelöf})^\times \subseteq \text{countably } F\text{-Menger}.$

Proof. (1) Every $F$-scale is $b(F)$-unbounded [19, Proposition 4.3]. Apply Lemma 2.2 and Theorem 2.14(1).

(2) By Proposition 2.16(1), every $F$-scale space is countably $F^+$-Menger. Apply Theorem 2.14(1).

(3) Apply Theorem 2.14(2). \hfill \Box

We only consider nonprincipal ultrafilters. An ultrafilter is simultaneously a filter and a superfilter.

Corollary 2.18. Let $U$ be an ultrafilter.

1. $(b(U)\text{-Concentrated}, b(U)\text{-Lindelöf})^\times \subseteq \text{countably } U\text{-Menger}.$
2. $(\text{countably } U\text{-Menger}, b(U)\text{-Lindelöf})^\times \subseteq \text{countably } U\text{-Menger}.$
3. Every $U$-scale space is productively countably $U$-Menger.

Proof. (1) Apply any of Theorem 2.1, Theorem 2.14, Theorem 2.17(1), or Corollary 2.13.

(2) Apply Theorem 2.17(2) or (3).

(3) Apply $U^+ = U$ and Proposition 2.16(1) or (2). \hfill \Box

Let $F$ be a filter. By Lemmata 2.12 and 2.15, every $\leq F^+$-unbounded set contains a (cofinal) $F^+$-scale and an $F$-scale. However, in general, a $\leq F$-unbounded set may not contain an $F$-scale; this may be the case for the filter $cF$, even if there is a cofinal $cF$-scale, as we now explain.

A Luzin set is an uncountable real space whose intersection with each meager set is countable. In particular, every Luzin set is $\leq_{cF}$-unbounded.

Proposition 2.19. No Luzin subset of $[\mathbb{N}]^\infty$ contains a $cF$-scale.

Proposition 2.19 follows from the following Lemma.

Lemma 2.20. Every $cF$-scale is meager in $[\mathbb{N}]^\infty$.

Proof. Let $Y \subseteq [\mathbb{N}]^\infty$ be a $cF$-scale. Fix a cofinite set $b \in [\mathbb{N}]^\infty$. There is a set $c \in [\mathbb{N}]^\infty$ such that $b \leq^* c$ and $c \leq^* y$ for all but less than $b$ elements $y \in Y$. The set $Z := \{ y \in Y : c \leq^* y \}$ is a countable union of sets of the form $\{ y \in Y : c' \leq y \}$, with $c' \in [\mathbb{N}]^\infty$ a cofinite set.

Fix a cofinite set $c' \in [\mathbb{N}]^\infty$. The set $X := \{ x \in [\mathbb{N}]^\infty : c' \leq x \}$ is nowhere dense: Let $U$ be an open subset of $[\mathbb{N}]^\infty$. Let $a \in U$ be a cofinite set. There is a natural number $n$ such that $a(n) < c'(n)$. The open set

$$\{ x \in [\mathbb{N}]^\infty : x(1) = a(1), \ldots, x(n) = a(n) \}$$

is a subset of the set $U$ disjoint of the set $X$.

Since $|Y \setminus Z| < b$, the remainder $Y \setminus Z$ is $\leq^*$-bounded, and thus meager. \hfill \Box

Remark 2.21. Let $V$ be a model of set theory satisfying the Continuum Hypothesis. In $V$, let $\kappa$ be a cardinal number with uncountable cofinality, and $C_\kappa$ be the Cohen forcing notion adding $\kappa$ Cohen reals. In the extended model $V^{C_\kappa}$, we have

$$\mathbb{N} = b = b(cF) \leq \mathfrak{d} = b(cF^+) = c = \kappa,$$
and the canonical set of generic Cohen reals is a Luzin set of cardinality \( \kappa \) \([5, \text{Lemma 8.2.6}]\). By Proposition 2.19, this shows that, for the filter \( F := cF \), neither \( b(F^+) < b(F) \) (taking \( \kappa = \aleph_1 \)) nor \( b(F) = b(F^+) \) (taking \( \kappa = \aleph_2 \)) imply that every \( \leq_F \)-unbounded set in \([\mathbb{N}]^\infty\) contains an \( F \)-scale.

Let \( F \) be a filter. By Lemma 2.5, if \( b(F) = b(F^+) \), then every \( \leq_F \)-unbounded set contains a \( b(F) \)-unbounded set. The assumption \( b(F) = b(F^+) \) is not redundant. Here too, a consistent counterexample exists for the filter \( F := cF \).

Example 2.22. Let \( \kappa \) be a cardinal number. A \( \kappa \)-scale in \([\mathbb{N}]^\infty\) is a \( \leq^* \)-unbounded set \( \{ a_\alpha : \alpha < \kappa \} \) in \([\mathbb{N}]^\infty\) such that \( a_\alpha \leq^* a_\beta \) for all ordinal numbers \( \alpha < \beta < \kappa \). Blass \([6]\) points out that, by a theorem of Hechler \([9]\), it is consistent that, for example, \( b = \aleph_1 \) and there is an \( \aleph_2 \)-scale \( X = \{ a_\alpha : \alpha < \aleph_2 \} \). Since the cardinal number \( \aleph_2 \) is regular, every subset of the \( \aleph_2 \)-scale \( X \) of cardinality \( b \) is \( \leq^* \)-bounded (indeed, by some member of the set \( X \)). In particular, the set \( X \) does not contain a \( b \)-unbounded set.

2.4. Countably Menger and countably Hurewicz spaces. Let \( b := b(cF) \) and \( d := b([\mathbb{N}]^\infty) \). Applying the results of the earlier subsections to the filter \( cF \) and the superfilter \([\mathbb{N}]^\infty = cF \) + , we obtain the following results.

Theorem 2.23.

(1) Every \( d \)-concentrated Tychonoff space satisfies \((\text{Hurewicz}, \text{countably Menger})^*\).

(2) \((\text{Concentrated}, \text{Lindel"of})^* \subseteq \text{countably Menger}\).

(3) \((\text{Concentrated}, \text{d-Lindel"of})^* \subseteq \text{countably Menger}\). In particular, \((\text{countably Menger}, \text{d-Lindel"of})^* \subseteq \text{countably Menger}\).

(4) If \( b = d \), then \((\text{d-Concentrated}, \text{d-Lindel"of})^* \subseteq \text{countably Hurewicz and, in particular, (countably Menger, d-Lindel"of)}^* \subseteq \text{countably Hurewicz}\).

Proof. (1) Apply Theorem 2.9.

(2) Apply Theorem 2.17(1).

(3) Apply Theorem 2.13. It is a simple, folklore observation that every \( d \)-concentrated space is countably Menger.

(4) Apply any of the theorems 2.14(2) or 2.1 □

3. General covers

3.1. General semifilters. A space is concentrated if it is \( \aleph_1 \)-concentrated. Theorem 2.1 and Proposition \([1, \text{Theorem 1.1}\) imply the following result.

Theorem 3.1. Let \( S \) be a semifilter with a cofinal \( S \)-scale, and \( b(S) = \aleph_1 \).

\((\text{Concentrated, Lindel"of})^* \subseteq S\text{-Menger}\). □

Theorem 3.2. Assume that \( b = \aleph_1 \).

(1) Let \( S \) be a meager semifilter. Every \( S \)-scale space is productively Hurewicz.

(2) Every cofinal \( cF \)-scale space is productively \( S \)-Menger, for all semifilters \( S \).

Proof. (1) Let \( X \cup [\mathbb{N}]^{<\infty} \) be an \( S \)-scale space, and \( Y \) be a Hurewicz space. By Proposition 2.8(1), the product space \((X \cup [\mathbb{N}]^{<\infty}) \times Y \) is countably Hurewicz.

Lemma 3.3. Every Tychonoff concentrated space satisfies \((\text{Hurewicz, Lindel"of})^*\).
Proof. This is a routine application of the projection method [22], see the proof of Theorem 2.9. □

As the semifilter $S$ is meager, we have $b(S) = b = \aleph_1$ [21 Corollary 2.27]. Being an $S$-scale, the set $X$ is $b(S)$-unbounded. By Lemma 2.2, the space $X \cup [\mathbb{N}]^{<\infty}$ is concentrated. Therefore, the space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is also Lindelöf, and thus Hurewicz.

(2) Let $X \cup [\mathbb{N}]^{<\infty}$ be a cofinal cF-scale space, and $Y$ be an $S$-Menger space. By Proposition 2.8(2), the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is countably $S$-Menger.

**Lemma 3.4.** Let $S$ be a semifilter, and $X$ be a subset of $[\mathbb{N}]^{<\infty}$ such that every $\leq_S$-bounded subset of the set $X$ is countable. The space $X \cup [\mathbb{N}]^{<\infty}$, with the Michael topology, satisfies $(S\text{-Menger}, \text{Lindelöf})^\times$.

Proof. Let $\leq_\mathbb{R}$ be the usual order on the real line, and

$$C := (P(\mathbb{N}) \times \{0\}) \cup ([\mathbb{N}]^{<\infty} \times \{-1, 1\})$$

be a space with the order topology generated by the lexicographic order on $C$. The space $C$ is the Dedekind compactification of the space $P(\mathbb{N})$ with the Michael topology. Since the subspace $(X \cup [\mathbb{N}]^{<\infty}) \times \{0\}$ of the space $C$ is homeomorphic to the space $X \cup [\mathbb{N}]^{<\infty}$, it suffices to prove that the space $(X \cup [\mathbb{N}]^{<\infty}) \times \{0\}$ satisfies $(S\text{-Menger}, \text{Lindelöf})^\times$.

Let $Y$ be an $S$-Menger space, and $\mathcal{U}$ be an open cover of the product space

$$((X \cup [\mathbb{N}]^{<\infty}) \times \{0\}) \times Y,$$

by sets open in the product space $C \times Y$. The product space $([\mathbb{N}]^{<\infty} \times \{0\}) \times Y$ is Lindelöf. Thus, there is a countable family $\mathcal{U} \subseteq \mathcal{U}$ such that the set $U := \bigcup \mathcal{U}'$ contains $([\mathbb{N}]^{<\infty} \times \{0\}) \times Y$. Since the set $(C \times Y) \setminus U$ is closed in the space $C \times Y$, it is $S$-Menger. Thus, the projection $H$ of the set $(C \times Y) \setminus U$ in $[\mathbb{N}]^{<\infty} \times \{-1, 0, 1\}$ is $S$-Menger, too.

The map $f : C \to [\mathbb{N}]^{<\infty}$ such that $f(x, y) = x$ for all elements $(x, y) \in C$, is continuous. Thus, the set $f[H]$ is $S$-Menger. Therefore, there is a $\leq_S$-bound $b$ for the set $f[H]$ in $[\mathbb{N}]^{<\infty}$. Since the set $\{ x \in X : x \leq_S b \}$ is countable, the set $X \cap f[H]$ is countable, and the set $(X \times \{0\}) \cap H$ is countable, too. Thus, the product space $((X \times \{0\}) \cap H) \times Y$ is Lindelöf.

Let $\mathcal{U}'' \subseteq \mathcal{U}$ be a countable cover of $((X \times \{0\}) \cap H) \times Y$. The family $\mathcal{U}' \cup \mathcal{U}''$ is a countable cover of the space $((X \cup [\mathbb{N}]^{<\infty}) \times \{0\}) \times Y$. □

Since $\mathfrak{d} = \aleph_1$, every uncountable subset of the cofinal cF-scale $X$ is dominating in $[\mathbb{N}]^{<\infty}$. Thus, each $\leq_S$-bounded subset of $X$ is countable. Apply Lemma 3.4 and Proposition 1.1. □

Concentration is necessary in Lemma 3.3 since the product of an uncountable space and a Hurewicz space need not be Lindelöf. Indeed, an uncountable discrete space is not Lindelöf.

Theorem 3.2(2) is a generalization of an earlier result [10 Theorem 6.2], from hereditarily Lindelöf spaces to general spaces.

Assuming the Continuum Hypothesis, every productively Lindelöf metric space is $\sigma$-compact [1]. By Theorem 3.2(2) and Remark 2.7, we have the following corollary.

**Corollary 3.5.** Assume the Continuum Hypothesis. Let $S$ be a semifilter, and $X$ be a cofinal cF-scale in $[\mathbb{N}]^{<\infty}$. The real space $X \cup [\mathbb{N}]^{<\infty}$ is productively $S$-Menger, but not productively Lindelöf. □
3.2. Superfilters. We have the following corollary of theorems 2.11 and 3.1.

**Corollary 3.6.** Let $S$ be a superfilter with $b(S) = \aleph_1$.

1. ((cofinal) $S$-scale space, Lindelöf)$^\times \subseteq S$-Menger.
2. (Concentrated, Lindelöf)$^\times \subseteq S$-Menger.

3.3. Filters. By Theorem 2.14(2) and Proposition 1.1, we have the following theorem.

**Theorem 3.7.** Let $F$ be a filter such that $b(F^+) = \aleph_1$.

1. (cofinal) $F^+$-scale space, Lindelöf $\subseteq F^+$-Menger.
2. ((cofinal) $F^+$-scale space, Lindelöf)$^\times \subseteq F^+$-Menger.

**Theorem 3.8.** Let $F$ be a filter such that $b(F) = \aleph_1$. Every cofinal $F$-scale space is productively $F$-Menger.

*Proof.* Let $X \cup [N]^{<\infty}$ be a cofinal $F$-scale space, and $Y$ be an $F$-Menger space. By Proposition 2.16(2), the product space $(X \cup [N]^{<\infty}) \times Y$ is countably $F$-Menger. Apply Lemma 3.4 and Proposition 1.1.

**Proposition 3.9.** Let $U$ be an ultrafilter such that $b(U) = \aleph_1$.

1. ((cofinal) $U$-scale space, Lindelöf)$^\times \subseteq U$-Menger.
2. (Concentrated, Lindelöf)$^\times \subseteq U$-Menger.
3. (U-Menger, Lindelöf)$^\times \subseteq U$-Menger.
4. Every $U$-scale space is productively $U$-Menger.

*Proof.* (1) Apply Corollary 3.6(1) or Theorem 3.7.
(2) Apply one of the corollaries 2.18(1) or 3.6.
(3) Apply (1).
(4) Apply Theorem 3.8.

3.4. Hurewicz, Menger, and Lindelöf spaces. Let $\kappa$ be a cardinal number. A real space of cardinality at least $\kappa$ is $\kappa$-Luzin if the cardinalities of its intersections with meager sets are all smaller than $\kappa$. A Luzin set is an $\aleph_1$-Luzin set. If $\kappa \in \{\text{cf}(\mathfrak{d}), \mathfrak{d}\}$, then for every $\kappa$-Luzin set $L$, there is a $\mathfrak{d}$-concentrated real space $Y$ such that the product space $L \times Y$ is not Menger [19, Corollary 2.11]. It is unknown whether, for every Luzin set $L$, there is a Menger real space $Y$ such that the product space $L \times Y$ is not Menger. We obtain a positive resolution for general spaces.

**Theorem 3.10.** Let $\kappa$ be an uncountable cardinal number.

1. For every $\kappa$-Luzin set $L$, there is a $\kappa$-concentrated space $Y$ such that the product space $L \times Y$ is not $\kappa$-Lindelöf.
2. No Luzin space is productively Menger.

*Proof.* (1) Apply Proposition 2.4.
(2) Apply (1).

We apply the results of the previous sections to Menger and Hurewicz spaces.

**Proposition 3.11.** Every Tychonoff concentrated space satisfies $(\text{Hurewicz, Menger})^\times$.

*Proof.* Apply Theorem 2.23(1), Lemma 3.3 and Proposition 1.1.

**Proposition 3.12.** Assume that $\mathfrak{d} = \aleph_1$. 
(1) $(\text{cofinal} \mathbb{N}^\infty, \text{Lindelöf})^\times \subseteq \text{Hurewicz},$
(2) $(\text{Concentrated}, \text{Lindelöf})^\times \subseteq \text{Hurewicz},$
(3) $(\text{cofinal} \mathcal{cF}\text{-scale space}, \text{Lindelöf})^\times \subseteq \text{Menger},$
(4) $(\text{Hurewicz}, \text{Lindelöf})^\times \subseteq \text{Menger}.$

Proof. (1) Apply Theorem 3.7(2).
(2) Apply (1) or Theorem 3.1.
(3) Apply Theorem 3.7(1).
(4) Apply (3) and Theorem 3.2(1). □

Assume, for this paragraph, that $\mathfrak{d} = \aleph_1$. Aurichi and Tall [2, Theorem 23] improved earlier results by proving that every productively Lindelöf space is Hurewicz (later, Tall [20, Section 3] and Repovš and Zdomskyy [17, Theorems 1.1 and 1.2] proved the same result using weaker hypotheses). It was later shown that, in the realm of hereditarily Lindelöf spaces, every productively Lindelöf space is productively Hurewicz and productively Menger [16, Theorem 8.2]. In our previous paper, we proved that in that realm, every productively Menger space is productively Hurewicz [19, Theorem 4.8]. We obtain an improved result in the general realm.

Theorem 3.13. Assume that $\mathfrak{d} = \aleph_1$.

$$\text{Lindelöf}^\times \subseteq (\text{Menger}, \text{Lindelöf})^\times = \text{Menger}^\times \subseteq \text{Hurewicz}^\times.$$ 

Proof. $(\text{Menger}, \text{Lindelöf})^\times \subseteq \text{Menger}^\times$: Let $X$ be a space. Assume that there is a Menger space $M$ such that the product space $X \times M$ is not Menger. By Proposition 3.12(3), there is a cofinal $\mathcal{cF}$-scale space $Y$ such that the product space $(X \times M) \times Y$ is not Lindelöf. By Theorem 3.2, the space $M \times Y$ is Menger. In summary, the product of the space $X$ and the Menger space $M \times Y$ is not Lindelöf.

$(\text{Menger}, \text{Lindelöf})^\times \subseteq \text{Hurewicz}^\times$: Let $X$ be a space. Assume that there is a Hurewicz space $H$ such that the product space $X \times H$ is not Hurewicz. By Proposition 3.12(2), there is a concentrated space $Y$ such that the product space $(X \times H) \times Y$ is not Lindelöf. By Proposition 3.11, the space $H \times Y$ is Menger. Thus, the product of the space $X$ and the Menger space $H \times Y$ is not Lindelöf. □

In the realm of hereditarily Lindelöf spaces, if $b = \mathfrak{d}$, then every productively Menger real space is productively Hurewicz [19, Theorem 4.8]. It is unknown whether, when $b = \mathfrak{d}$, these classes provably coincide for real spaces [19, Problem 6.9].

Proposition 3.14.

(1) Assume that $b = \aleph_1 < \mathfrak{d}$. There is a hereditarily Lindelöf productively Hurewicz space that is not productively Menger.

(2) It is consistent with the Continuum Hypothesis that there is a productively Hurewicz space that is not productively Menger.

Proof. (1) We view $P(\mathbb{N})$ as a subset of the real line. Let $\leq_\mathbb{R}$ be the usual order on the real line. For points $a, b \in P(\mathbb{N})$, let $[a, b] := \{ x \in P(\mathbb{N}) : a \leq_\mathbb{R} x <_\mathbb{R} b \}$, and $(a, b) := \{ x \in P(\mathbb{N}) : a <_\mathbb{R} x \leq_\mathbb{R} b \}$. The Sorgenfrey topology (Sorgenfrey* topology) [18] on the set $P(\mathbb{N})$ is the topology generated by the sets $[a, b]$ and $(a, b)$, for $a, b \in P(\mathbb{N})$.

Let $X \cup [\mathbb{N}]^{<\infty}$ be a $\mathcal{cF}$-scale space such that the open neighborhoods of the points from the set $X$ are as in the space $X \cup [\mathbb{N}]^{<\infty}$ with the Sorgenfrey topology. Let $Y$ be a subset of $X$ with $|Y| = \aleph_1$. Equip $Y$ with the Sorgenfrey* topology. Since the space $Y$ is Lindelöf and
|Y| < d, it is Menger. The diagonal set \( \{(y, y) : y \in Y\} \) is a closed and discrete subset of the product space \((X \cup \mathbb{N}^{<\infty}) \times Y\). The space \((X \cup \mathbb{N}^{<\infty}) \times Y\) is not Lindelöf, and thus not Menger. By Theorem 3.2(1) and Remark 2.7 the space \(X \cup \mathbb{N}^{<\infty}\) is productively Hurewicz.

(2) We use the following lemma, pointed out to us by A. Miller. We sketch a proof that assumes familiarity with the method of forcing.

**Lemma 3.15.** It is consistent with the Continuum Hypothesis that some cF-scale is Menger.

**Proof.** Let \(V\) be a model of the Continuum Hypothesis. Let \(X\) be a cF-scale in \(V\). Extend \(V\) generically by adding \(\aleph_1\) Cohen reals. Every Borel map \(X \to \mathbb{N}^{\mathbb{N}}\) is coded in an intermediate extension, and is thus \(\leq^{\infty}\)-bounded by the next added Cohen reals. It follows that, in the extension, the set \(X\) is a Menger space, even with respect to countable Borel covers. \(\square\)

Let \(X \subseteq \mathbb{N}^{\mathbb{N}}\) be a cF-scale with Menger’s property. Equip the space \(Y := X \cup \mathbb{N}^{<\infty}\) with the Michael topology. By Theorem 3.2(1) and Remark 2.7 the space \(Y\) is productively Hurewicz. The product space \(Y \times X\) contains the closed discrete subset \(\{(x, x) : x \in X\}\), and is thus not Lindelöf. In particular, the product is not Menger. \(\square\)

**Corollary 3.16.** It is consistent with the Continuum Hypothesis that all inclusions in Theorem 3.13 are strict. \(\square\)

### 4. Countably compact spaces

A space is *countably compact* if every countable open cover of this space has a finite subcover. Let \(S\) be a semifilter. Countably compact spaces are countably \(S\)-Menger. Every compact space is productively \(S\)-Menger and productively countably \(S\)-Menger. In contrast, a result of Frolik [10, §3.1.6], applied to a discrete space of cardinality \(d\), implies that there are two countably compact spaces whose product is not countably Menger.

A *P-space* is a space where all \(G_\delta\) sets are open. Using results of Galvin [8, page 157] and Alster [1, Theorem 1], Babinkostova, Scheepers, and Pansera [3, Lemma 18, Theorem 23] proved that every Lindelöf P-space is productively Lindelöf, productively Menger, and productively Hurewicz.

**Proposition 4.1.** Assume that \(d = \aleph_1\). There are a Lindelöf P-space \(X\) and a first countable, countably compact space \(Y\) such that the product space \(X \times Y\) is not countably Menger.

**Proof.** Let \(X := \aleph_1 + 1\), where the neighborhoods of the point \(\aleph_1 \in X\) are those of the ordinal topology on the ordinal number \(\aleph_1 + 1\), and the remaining points are isolated. The space \(X\) is a Lindelöf P-space. The space \(Y := \aleph_1\), with the ordinal topology, is first countable and countably compact.

The diagonal set \(\{(\alpha, \alpha) : \alpha < \aleph_1\}\) is a closed discrete subset of the product space \(X \times Y\). Since \(d = \aleph_1\), the diagonal set is not countably Menger. \(\square\)

**Definition 4.2.** Let \(X\) and \(Y\) be spaces. A set-valued map \(\Phi : X \Rightarrow Y\) is *countably compact-valued upper semicontinuous (ccusco)* if it satisfies the definition of cusco map, with compact replaced by countably compact.

Using arguments similar to those used for cusco maps [23, Lemma 1], we obtain the following result.

**Proposition 4.3.** Let \(S\) be a semifilter. The class of countably \(S\)-Menger spaces is preserved by ccusco maps. \(\square\)
Let $S$ be a semifilter. Let $\text{add}(S\text{-Menger})$ be the minimal number of $S$-Menger subspaces of $[\mathbb{N}]^\infty$ whose union is not $S$-Menger. For a filter $F$, we have $\text{add}(F\text{-Menger}) = b(F)$.

**Lemma 4.4.** Let $S$ be a semifilter. Every space that is a union of less than $\text{add}(S\text{-Menger})$ countably $S$-Menger spaces is countably $S$-Menger. □

A space is *sequential* if every subset that is closed under limits of sequences is closed. In particular, every first countable space is sequential. By Proposition 4.1, the product of countably compact spaces with countably Menger spaces need not be countably Menger. Item (2) of the following results implies that every product of countably compact spaces with countably Menger sequential spaces is countably Menger.

**Proposition 4.5.** Let $S$ be a semifilter.

1. Every space that is a union of less than $\text{add}(S\text{-Menger})$ compact spaces is productively countably $S$-Menger.
2. Every product of a union of less than $\text{add}(S\text{-Menger})$ countably compact spaces and a countably $S$-Menger sequential space is countably $S$-Menger.

Proof. (1) Apply Proposition 4.3 and Lemma 4.4.
(2) Let $X$ be a countably compact space, and $Y$ be a countably $S$-Menger sequential space. The projection $p: X \times Y \to Y$ is a closed map onto $Y$ [7, Theorem 3.10.7]. Thus, the inverse map $p^{-1}: Y \Rightarrow X \times Y$ is ccusco. By Proposition 4.3, the products space $X \times Y$ is countably $S$-Menger. Apply Lemma 4.4. □

5. Comments and open problems

Assume the Continuum Hypothesis. Consider all potentially provable assertions

$$(A, B)^x \subseteq C,$$

for $A, B, C \in \{\text{Concentrated, Lindel"of, Menger, Hurewicz}\}$. Some of these assertions fail for obvious reasons. Some others hold for trivial reasons. We also have the following observation.

**Proposition 5.1.**

1. Assuming the Continuum Hypothesis, we have $(\text{Hurewicz, Menger})^x \not\subseteq \text{Hurewicz}$.
2. $(\text{Concentrated, Menger})^x \not\subseteq \text{Concentrated}$.

Proof. (1) By Proposition 3.11 every Luzin set satisfies $(\text{Hurewicz, Menger})^x$. No Luzin set is Hurewicz [12, page 196, footnote 1].

(2) Consider an uncountable non-concentrated compact space. □

This shows that all provable results of the above-considered type are included in items (2) and (4) of Proposition 3.12.

The following problem is motivated by Theorem 3.13.

**Problem 5.2.** Assume the Continuum Hypothesis.

1. Does $(\text{Hurewicz, Lindel"of})^x \subseteq \text{Menger}^x$?
2. And if not, does $(\text{Hurewicz, Lindel"of})^x \subseteq \text{Hurewicz}^x$?

A *scale* is a $\leq^*$-increasing sequence $\{s_\alpha : \alpha < \delta\} \subseteq [\mathbb{N}]^\infty$ that is $\leq^*$-unbounded in $[\mathbb{N}]^\infty$. If $S$ is a scale then, in the realm of hereditarily Lindelöf spaces, the real space $S \cup [\mathbb{N}]^{<\infty}$ is productively Hurewicz and productively Menger [16, Theorem 6.2]. A positive solution of the following problem implies the same assertion for general spaces.
Problem 5.3. Assume that $b = d$. Is every real space of cardinality smaller than $b$ provably productively Hurewicz? Productively Menger?

Proposition 3.14 and Lemma 3.15 motivate the following problem. A positive solution of its second item implies a positive solution of its first item.

Problem 5.4.

1. Does the Continuum Hypothesis imply the existence of a productively Hurewicz space that is not productively Menger?
2. Does the Continuum Hypothesis imply that some $cF$-scale is Menger?

An uncountable subspace of the real line is Sierpiński if the cardinalities of its intersections with Lebesgue measure zero sets are all countable. Every Sierpiński set is Hurewicz [13, Theorem 2.10]. Assuming the Continuum Hypothesis, there is a Sierpiński set whose square is not Menger [13]. The following problem is also open in the realm of hereditarily Lindelöf spaces [19, Problem 6.8]. The solution for general spaces may turn out different.

Problem 5.5. Does the Continuum Hypothesis imply that no Sierpiński set is productively Hurewicz?

Acknowledgments. We thank Arnold Miller for Lemma 3.15 and Franklin Tall for pointing out some useful references. The research of the first named author was supported by an Eltuda 2 grant, Polish National Science Center, UMO-2014/12/T/ST1/00627.

References

[1] K. Alster, On the class of all spaces of weight not greater than $\omega_1$ whose Cartesian product with every Lindelöf space is Lindelöf, Fundamenta Mathematicae 129 (1988), 133–140.
[2] L. Aurichi, F. Tall, Lindelöf spaces which are indestructible, productive, or D, Topology and its Applications 159 (2012), 331–340.
[3] L. Babinkostova, B. Pansera, M. Scheepers, Weak covering properties and selection principles, Topology and its Applications 160 (2013), 2251–2271.
[4] T. Banakh, L. Zdomskyy, The Coherence of Semifilters: a Survey, in: Selection Principles and Covering Properties in Topology (L. Kočinac, ed.), Quaderni di Matematica 18, Seconda Universita di Napoli, Caserta, 2006, 53–99.
[5] T. Bartoszyński, H. Judah, Set Theory: On the structure of the real line, A. K. Peters, Massachusetts: 1995.
[6] A. Blass, Answer to Reference request: The consistency of a tall tower in $\mathbb{N}^\mathbb{N}$, Mathoverflow question 239316, May 2016.
[7] R. Engelking, General topology, Heldermann, Berlin, 1989.
[8] J. Gerlits, Zs. Nagy, Some properties of $C(X)$, I, Topology and its Applications 14 (1982), 151–161.
[9] S. Hechler, On the existence of certain cofinal subsets of $\omega^\omega$, in: Axiomatic set theory (editor T. Jech), Proceedings of Symposia in Pure Mathematics XIII, Part II, American Mathematical Society, 1974, 155–173.
[10] Z. Frolic, Generalizations of compactness and the Lindelöf property, Czech Mat. J. 9 (1959), 172–217.
[11] W. Hurewicz, Über eine Verallgemeinerung des Borelschen Theorems, Mathematische Zeitschrift 24 (1925), 401–421.
[12] W. Hurewicz, Über Folgen stetiger Funktionen, Fundamenta Mathematicae 9 (1927), 193–204.
[13] W. Just, A. Miller, M. Scheepers, P. Szeptycki, The combinatorics of open covers II, Topology and its Applications 73 (1996), 241–266.
[14] K. Menger, Einige Überdeckungssätze der Punktmengenlehre, Sitzungsberichte der Wiener Akademie 133 (1924), 421–444.
[15] E. Michael, The product of a normal space and a metric space need not be normal, Bulletin of the American Mathematical Society 69 (1963), 375–378.
[16] A. Miller, B. Tsaban, L. Zdomskyy, Selective covering properties of product spaces, Annals of Pure and Applied Logic 165 (2014), 1034–1057.

[17] D. Repovš, L. Zdomskyy, Productively Lindelöf spaces and the covering property of Hurewicz, Topology and its Applications 169 (2014), 16–20.

[18] R. Sorgenfrey, On the topological product of paracompact spaces, Bulletin of the American Mathematical Society 53 (1947), 631–632.

[19] P. Szewczak, B. Tsaban, Products of Menger spaces: a combinatorial approach, Annals of Pure and Applied Logic 168 (2017), 1–18.

[20] F. Tall, Productively Lindelöf spaces may all be D, Canadian Mathematical Bulletin 56 (2013), 203–212.

[21] B. Tsaban, L. Zdomskyy, Scales, fields, and a problem of Hurewicz, Journal of the European Mathematical Society 10 (2008), 837–866.

[22] B. Tsaban, L. Zdomskyy, Additivity of the Gerlits–Nagy property and concentrated sets, Proceedings of the American Mathematical Society 142 (2014), 2881–2890.

[23] L. Zdomskyy, A semifilter approach to selection principles, Commentationes Mathematicae Universitatis Carolinae 46 (2005), 525–539.