Uniform approximations by Fourier sums on classes of generalized Poisson integrals

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Abstract

We find asymptotic equalities for exact upper bounds of approximations by Fourier sums in uniform metric on classes of $2\pi$–periodic functions, representable in the form of convolutions of functions $\varphi$, which belong to unit balls of spaces $L_p$, with generalized Poisson kernels. For obtained asymptotic equalities we introduce the estimates of remainder, which are expressed in the explicit form via the parameters of the problem.

Key words: Fourier sums, generalized Poisson integrals, asymptotic equality.

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1. Introduction. Let $L_p$, $1 \leq p < \infty$, be the space of $2\pi$–periodic functions $f$ summable to the power $p$ on $[0, 2\pi)$, in which the norm is given by the formula 
$$
\|f\|_p = \left( \frac{2\pi}{\int_0^{2\pi} |f(t)|^p dt} \right)^{\frac{1}{p}};
$$
$L_\infty$ be the space of measurable and essentially bounded $2\pi$–periodic functions $f$ with the norm $\|f\|_\infty = \text{ess sup}_t |f(t)|$; $C$ be the space of continuous $2\pi$–periodic functions $f$, in which the norm is specified by the equality $\|f\|_C = \max_t |f(t)|$.

Denote by $C^{\alpha,r,\beta}_{\beta,p}$, $\alpha > 0$, $r > 0$, $1 \leq p \leq \infty$, the set of all $2\pi$–periodic functions $f$ representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$
f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \in B^0_p, \quad (1)
$$

with fixed generated kernels

$$
P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos(kt - \frac{\beta \pi}{2}), \quad \beta \in \mathbb{R}.
$$

The kernels $P_{\alpha,r,\beta}(t)$ are called generalized Poisson kernels. For $r = 1$ and $\beta = 0$ the kernels $P_{\alpha,r,0}(t)$ are usual Poisson kernels of harmonic functions.

For any $r > 0$ the classes $C^{\alpha,r}_{\beta,p}$ belong to set of infinitely differentiable $2\pi$–periodic functions $D^\infty$, i.e., $C^{\alpha,r}_{\beta,p} \subset D^\infty$ (see, e.g., [1, p. 128], [2]). For $r \geq 1$ the classes $C^{\alpha,r}_{\beta,p}$ consist of functions $f$, admitting a regular extension into the strip $|\text{Im } z| \leq c$, $c > 0$ in the complex plane (see, e.g., [1, p. 141]), i.e., are the classes of analytic functions. For
Begin with the classes $C^{\alpha,r}_{\beta,p}$ consisting of functions regular on the whole complex plane, i.e., of entire functions (see, e.g., [1, p. 131]). Besides, it follows from the theorem 1 in [3] that for any $r > 0$ the embedding holds $C^{\alpha,r}_{\beta,p} \subset J_1/r$, where $J_a, a > 0$, are known Gevrey classes

$$J_a = \left\{ f \in D_{\infty} : \sup_{k \in \mathbb{N}} \left\| f^{(k)} \right\|_{C, k!^a}^{1/k} < \infty \right\}.$$

Approximation properties of classes of generalized Poisson integrals $C^{\alpha,r}_{\beta,p}$ in metrics of spaces $L_s, 1 \leq s \leq \infty$, were considered in [4]–[12] from the viewpoint of order or asymptotic estimates for approximations by Fourier sums, best approximations and widths.

In the present paper we obtain asymptotic equalities as $n \to \infty$ for the quantities

$$E_n(C^{\alpha,r}_{\beta,p})_C = \sup_{f \in C^{\alpha,r}_{\beta,p}} \left\| f(\cdot) - S_{n-1}(f; \cdot) \right\|_{C, r > 0, \alpha > 0, 1 \leq p \leq \infty},$$

where $S_{n-1}(f; \cdot)$ are the partial Fourier sums of order $n - 1$ for a function $f$.

Approximation by Fourier sums on other classes of differentiable functions in uniform metric were investigated in works [1], [13]–[17].

Nikol’skii [14, p. 221] considered the case $r = 1, p = \infty$ and established that following asymptotic equality is true

$$E_n(C^{\alpha,1}_{\beta,\infty})_C = e^{-\alpha n} \left( \frac{8}{\pi^2} K(e^{-\alpha}) + O(1)n^{-1} \right),$$

where

$$K(q) := \int_0^q \frac{dt}{\sqrt{1 - q^2 \sin^2 t}}, q \in (0, 1),$$

is a complete elliptic integral of the first kind, and $O(1)$ is a quantity uniformly bounded in parameters $n$ and $\beta$.

Later, the equality (3) was clarified by Stechkin [18, p. 139], who established the asymptotic formula

$$E_n(C^{\alpha,1}_{\beta,\infty})_C = e^{-\alpha n} \left( \frac{8}{\pi^2} K(e^{-\alpha}) + O(1) \frac{e^{-\alpha}}{(1 - e^{-\alpha})n} \right), \quad \alpha > 0, \beta \in \mathbb{R},$$

where $O(1)$ is a quantity uniformly bounded in all analyzed parameters.

In work [10] for $r = 1$ and arbitrary values of $1 \leq p \leq \infty$ for quantities $E_n(C^{\alpha,r}_{\beta,p})_C, \alpha > 0, \beta \in \mathbb{R}$, the following equality was established

$$E_n(C^{\alpha,1}_{\beta,p})_C = e^{-\alpha n} \left( \frac{2}{\pi^{1 + 1/p}} \| \cos t \|_{p'} K(p', e^{-\alpha}) + O(1) \frac{e^{-\alpha}}{n(1 - e^{-\alpha})s(p)} \right), \quad \alpha > 0, \beta \in \mathbb{R},$$

where $p' = \frac{p}{p-1},$

$$s(p) := \begin{cases} 1, & p = \infty, \\ 2, & p \in [1, 2) \cup (2, \infty), \\ -\infty, & p = 2, \end{cases}$$
\[
K(p', q) := \frac{1}{2^{1-p'}} \left\| (1 - 2q \cos t + q^2)^{-\frac{1}{2}} \right\|_{2^{p'}}, \quad q \in (0, 1),
\]
and \(O(1)\) is a quantity uniformly bounded in \(n, p, \alpha\) and \(\beta\). For \(p = \infty\), by virtue of the known equality \(K(1, q) = K(q)\), the estimate (5) coincides with the estimate (4).

Note that for \(p = 2\) and \(r = 1\) formula (5) becomes the equality

\[
E_n(C_{\beta, 2}^{\alpha, 1}C) = \frac{1}{\sqrt{\pi(1-e^{-2\alpha})}} e^{-\alpha n}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N},
\]
(see [10]). Moreover, it follows from [19] that for \(p = 2\) and \(r > 0\) for the quantities \(E_n(C_{\beta, 2}^{\alpha, r})\) the equalities take place

\[
E_n(C_{\beta, 2}^{\alpha, r}) = \frac{1}{\sqrt{\pi}} \left( \sum_{k=n}^{\infty} e^{-2\alpha k^r} \right)^{\frac{1}{2}}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{6}
\]

In the case of \(r > 1\) and \(p = \infty\) the asymptotic equalities for the quantities \(E_n(C_{\beta, p}^{\alpha, r})\), \(\alpha > 0, \quad \beta \in \mathbb{R}\), were obtained by Stepanets [20, Chapter 3, Section 9], who showed that for any \(n \in \mathbb{N}\)

\[
E_n(C_{\beta, \infty}^{\alpha, r}) = \left( \frac{4}{\pi} + \gamma_n \right) e^{-\alpha n^r}, \tag{7}
\]
where

\[
|\gamma_n| < 2 \left( 1 + \frac{1}{\alpha r n^{r-1}} \right) e^{-\alpha n^{r-1}}.
\]

Later Telyakovskii [6] established the asymptotic equality

\[
E_n(C_{\beta, \infty}^{\alpha, r}) = \frac{4}{\pi} e^{-\alpha n^r} + O(1) \left( e^{-\alpha(2n+1)\rho} + \left( 1 + \frac{1}{\alpha r(n+2)^r} \right) e^{-\alpha(n+2)^r} \right), \tag{8}
\]
where \(O(1)\) is a quantity uniformly bounded in all analyzed parameters. Formula (8) contains more exact estimate of remainder in asymptotic decomposition of the quantity \(E_n(C_{\beta, p}^{\alpha, r})\) comparing with the estimate (7).

For \(r > 1\) and for arbitrary values of \(1 \leq p \leq \infty\) the asymptotic equalities for the quantities \(E_n(C_{\beta, p}^{\alpha, r})\), \(\alpha > 0, \quad \beta \in \mathbb{R}\), are found in [10] and have the form

\[
E_n(C_{\beta, p}^{\alpha, r}) = e^{-\alpha n^r} \left( \frac{\|\cos t\|_{2^{p'}}}{\pi} + O(1) \left( 1 + \frac{1}{\alpha r n^{r-1}} \right) e^{-\alpha n^{r-1}} \right), \tag{9}
\]
where \(O(1)\) is a quantity uniformly bounded in all analyzed parameters. For \(p = \infty\) the formula (9) follows from (7) and (8).

Concerning the case \(0 < r < 1\), except the presented above case \(p = 2\), asymptotic equalities for quantities \(E_n(C_{\beta, p}^{\alpha, r})\), \(\alpha > 0, \quad \beta \in \mathbb{R}\), were known only for \(p = \infty\) due to the work of Stepanets [21], who showed that

\[
E_n(C_{\beta, \infty}^{\alpha, r}) = \frac{4}{\pi^2} e^{-\alpha n^r} \ln n^{1-r} + O(1) e^{-\alpha n^r}, \tag{10}
\]
where \(O(1)\) is a quantity uniformly bounded in \(n\) and \(\beta\).
In case of $0 < r < 1$ and $1 \leq p < \infty$ the following order estimates for quantities $E_n(C^{\alpha,r}_{\beta,p})$, $\alpha > 0$, $\beta \in \mathbb{R}$, hold (see, e.g., [8], [11])

$$E_n(C^{\alpha,r}_{\beta,p}) \lesssim e^{-\alpha n^r} \frac{1-r}{n^p}. \quad (11)$$

We remark that for $0 < r < 1$ and $1 \leq p < \infty$ Fourier sums provide the order of best approximations of classes $C^{\alpha,r}_{\beta,p}$, $\alpha > 0$, $\beta \in \mathbb{R}$, in uniform metric, i.e. (see, e.g., [11], [12])

$$E_n(C^{\alpha,r}_{\beta,p}) \lesssim E_n(C^{\alpha,r}_{\beta,p}) \lesssim e^{-\alpha n^r} \frac{1-r}{n^p},$$

where

$$E_n(C^{\alpha,r}_{\beta,p}) = \sup_{f \in C^{\alpha,r}_{\beta,p}} \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f - t_{n-1}\|_C,$$

and $\mathcal{T}_{2n-1}$ is the subspace of all trigonometric polynomials $t_{n-1}$ of degree not higher than $n - 1$.

Besides, as follows from Temlyakov’s work [8] for $2 \leq p < \infty$ quantities of approximations by Fourier sums realize order of the linear widths $\lambda_{2n}$ (definition of $\lambda_m$ see, e.g., [22, Chapter 1, Section 1.2]) of the classes $C^{\alpha,r}_{0,p}$, i.e.

$$\lambda_{2n}(C^{\alpha,r}_{0,p}, C) \asymp E_n(C^{\alpha,r}_{\beta,p}).$$

In this paper we establish asymptotically sharp estimates of the quantities $E_n(C^{\alpha,r}_{\beta,p})$, $\alpha > 0$, $\beta \in \mathbb{R}$, for any $0 < r < 1$ and $1 \leq p \leq \infty$. In particular, it is proved, that for $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $1 < p \leq \infty$ as $n \to \infty$ the following asymptotic equality takes place

$$E_n(C^{\alpha,r}_{\beta,p}) = e^{-\alpha n^r} \frac{1-r}{n^p} \left( \frac{\| \cos t \|_{L^p}^p}{\pi^{1+\frac{1}{p}}(\alpha r)^\frac{1}{p}} \left( \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{1}{p}}} \right)^\frac{1}{p} + O(1) \left( \frac{1}{n^{1-r(p'-1)}} + \frac{1}{n^r} + \frac{1}{n^{\frac{1}{p}}} \right) \right), \quad (12)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and $O(1)$ is a quantity uniformly bounded with respect to $n$ and $\beta$. Herewith, in this paper we found the estimates for remainder in $(12)$, which are expressed via the parameters of the problem $\alpha, r, p$ in the explicit form and that can be used for practical application.

2. Formulation of main results. For arbitrary $\nu > 0$ and $1 \leq s \leq \infty$ assume

$$J_s(\nu) := \left\| \frac{1}{\sqrt{t^2 + 1}} \right\|_{L_s[0,\nu]}, \quad (13)$$

where

$$\|f\|_{L_s[a,b]} = \begin{cases} \left( \int_a^b |f(t)|^s \, dt \right)^\frac{1}{s}, & 1 \leq s < \infty, \\ \text{ess sup}_{t \in [a,b]} |f(t)|, & s = \infty. \end{cases}$$
Also for $\alpha > 0$, $r \in (0, 1)$ and $1 \leq p \leq \infty$ we denote by $n_0 = n_0(\alpha, r, p)$ the smallest integer $n$ such that

$$\frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \leq \left\{ \begin{array}{ll} \frac{1}{\pi^r}, & p = 1, \\ \frac{1}{(3\alpha)^r}, & 1 < p < \infty, \\ \frac{1}{(3\alpha)^r}, & p = \infty, \end{array} \right. \quad (14)$$

where $\chi(p) = p$ for $1 \leq p < \infty$ and $\chi(p) = 1$ for $p = \infty$.

With the notations introduced above, the main result of this paper is formulated in the following statement:

**Theorem 1.** Let $0 < r < 1$, $1 \leq p \leq \infty$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then for $n \geq n_0(\alpha, r, p)$ the following estimate is true

$$E_n(C^\alpha_{\beta,p}) = e^{-\alpha n^r} n^{\frac{1-r}{r}} \left( \frac{\| \cos t \|^p_{r'}}{\pi^{(1+r)^{-1}} (\alpha r)^{\frac{1}{r'}}} \right)^{\frac{1}{p'}} \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} + 
\gamma(1)_{n,p} \left( \frac{1}{\pi \alpha r} \right)^{\frac{1}{p'}} \left( \frac{\chi(p)}{n^{1-r}} \right)^{\frac{1}{p'}} \left( \frac{1}{n^{1-r}} \right), \quad (15)$$

where $\frac{1}{p'} + \frac{1}{p'} = 1$, and the quantity $\gamma(1)_{n,p} = \gamma(1)_{n,p}(\alpha, r, \beta)$ is such that $|\gamma(1)_{n,p}| \leq (14\pi)^2$.

Now we present some corollaries of theorem 1.

For $1 < p < \infty$ theorem 1 yields the following statement:

**Theorem 2.** Let $0 < r < 1$, $1 \leq p < \infty$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then for $1 < p < \infty$ and $n \geq n_0(\alpha, r, p)$ the following estimate is true

$$E_n(C^\alpha_{\beta,1}) = e^{-\alpha n^r} n^{1-r} \left( \frac{1}{\pi \alpha r} \right)^{\frac{1}{p'}} \left( \frac{\chi(p)}{n^{1-r}} \right)^{\frac{1}{p'}} \left( \frac{1}{n^{1-r}} \right), \quad (17)$$

where $\frac{1}{p'} + \frac{1}{p'} = 1$, and the quantity $\gamma(2)_{n,p} = \gamma(2)_{n,p}(\alpha, r, \beta)$ is such that $|\gamma(2)_{n,p}| \leq (14\pi)^2$.

**Proof of the theorem 2.** According to theorem 1 the following estimate is true for all $1 < p < \infty$, $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $n \geq n_0(\alpha, r, p)$

$$E_n(C^\alpha_{\beta,p}) = e^{-\alpha n^r} n^{\frac{1-r}{r}} \left( \frac{\| \cos t \|^p_{r'}}{\pi^{(1+r)^{-1}} (\alpha r)^{\frac{1}{r'}}} \right)^{\frac{1}{p'}} \int_0^\infty \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}} + 
\gamma(1)_{n,p} \left( \frac{1}{\pi \alpha r} \right)^{\frac{1}{p'}} \left( \frac{\chi(p)}{n^{1-r}} \right)^{\frac{1}{p'}} \left( \frac{1}{n^{1-r}} \right), \quad (18)$$
where \( \frac{1}{p} + \frac{1}{p'} = 1 \), and the quantity \( \gamma_n^{(1)}_{\alpha,p} = \gamma_n^{(1)}(\alpha,p) \) is such that \( |\gamma_n^{(1)}| \leq (14\pi)^2 \).

By applying the Lagrange theorem, for \( n \geq n_0(\alpha,r,p) \) we obtain

\[
\left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'}} - \left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'} - 1} \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \leq \frac{1}{p'} \int_0^{\infty} \frac{dt}{(t + 1)^{\frac{1}{p'}}} \int_0^{\infty} \frac{dt}{(t + 1)^{\frac{1}{p'}}} = \frac{1}{p'} (p' - 1) \left( 1 - \left( \frac{\pi n^{1-r}}{\alpha r} + 1 \right)^{1-p'} \right) \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} \leq \frac{1}{p'} (p' - 1) \left( 1 - \left( \frac{27\pi^4 p^2}{p - 1} + 1 \right)^{1-p'} \right) \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} = \frac{1}{p' - 1} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} \left( p - 1 \right) \left( \frac{p^2}{p - 1} + 1 \right)^{1-p'} \left( \frac{1}{\pi n^{1-r}} \right)^{p'-1} < 2.
\]

It can be shown that

\[
\frac{p - 1}{p} \left( 1 - \left( \frac{27\pi^4 p^2}{p - 1} + 1 \right)^{1-p} \right)^{-\frac{1}{p}} < 2.
\]

As follows from (19)

\[
\left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'}} = \left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'}} + \Theta_{\alpha,r,p,n}^{(1)}(\frac{\alpha r}{\pi n^{1-r}})^{p'-1} \left( \frac{\alpha r}{\pi n^{1-r}} \right)^{p'-1} < 2.
\]

From relations

\[
\left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'}} \left( \int_0^{\infty} \frac{dt}{(t^2 + 1)^{\frac{1}{2}}} \right)^{\frac{1}{p'}} < \left( 1 + \int_1^{\infty} \frac{dt}{t^{p'}} \right)^{\frac{1}{p'}} < p^{\frac{1}{p'}}
\]

and formulas (18) and (20) we obtain (16).

Formula (17) can be obtained from the equality (15) as consequence of substitution \( p = 1 \) and elementary transformations. Theorem 2 is proved.

The following statement follows from the theorem 2 in the case \( p = 2 \).

**Corollary 1.** Let \( 0 < r < 1, \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then for \( n \geq n_0(\alpha,r,2) \) the following estimate is true

\[
\mathcal{E}_n(C_{\beta,2}^{\alpha,r}) C = \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1}{p'}} \left( 1 + \gamma_n^{(1)}(\frac{1}{\alpha r} n^r + \sqrt{\alpha r}) \right),
\]

(22)
where the quantity $\gamma_n^{(1)} = \gamma_n^{(1)}(\alpha, r, \beta)$ is such that $|\gamma_n^{(1)}| \leq 392\pi^{\frac{5}{2}}$.

**Proof of the corollary 1.** Indeed, setting $p = p' = 2$ in the equality (16), we obtain for $n \geq n_0(\alpha, r, 2)$

$$
\mathcal{E}_n(C_{\beta, 2}^\alpha) = e^{-\alpha n} n^{\frac{1-r}{2}} \left( \frac{\| \cos t \|_2}{\pi \frac{3}{2} (\frac{1}{3})^{\frac{1}{2}}} \left( \int_0^\infty \frac{dt}{t^2 + 1} \right)^{\frac{1}{2}} +
\right.

\left. + \gamma_n^{(2)}(\frac{\sqrt{\alpha r}}{n^{1-r}} + \sqrt{\frac{2}{\alpha r} \frac{1}{n^{1-r}}} + \sqrt{\frac{\alpha r}{n^{1-r}}}) \right) =

\frac{e^{-\alpha n}}{\sqrt{2\pi \alpha r} n^{\frac{1-r}{2}}} \left( 1 + \gamma_n^{(2)}(\frac{\sqrt{2 \alpha r}}{n^{1-r}} + \frac{1}{\alpha r} \frac{1}{n^{1-r}} + \frac{\alpha r}{n^{1-r}}) \right).
$$

(23)

According to (14) for $n \geq n_0(\alpha, r, 2)$

$$\frac{\sqrt{\alpha r}}{n^{1-r}} \leq \frac{1}{2(3\pi)^{\frac{1}{2}}},$$

therefore

$$\left( \frac{\alpha r}{n^{1-r}} + \sqrt{\frac{2}{\alpha r} \frac{1}{n^{1-r}}} + \sqrt{\frac{\alpha r}{n^{1-r}}} \right) \leq \sqrt{2} \left( \frac{1}{\alpha r} \frac{1}{n^{1-r}} + \sqrt{\frac{\alpha r}{n^{1-r}}} \right).$$

(24)

From (23) and (24) we have (22). Corollary 1 is proved.

However, it is possible to obtain more exact estimate than (22) on the basis of equality (6). Namely, for $\alpha > 0$, $r \in (0, 1)$, $\beta \in \mathbb{R}$ and $n \geq n_0(\alpha, r, 2)$ the following estimate is true

$$
\mathcal{E}_n(C_{\beta, 2}^\alpha) = e^{-\alpha n} n^{\frac{1-r}{2}} \left( 1 + \gamma_n^{(2)}(\frac{1}{2\alpha r} \frac{1}{n^{1-r}} + \frac{\alpha r}{n^{1-r}}) \right),
$$

(25)

where the quantity $\gamma_n^{(2)} = \gamma_n^{(2)}(\alpha, r)$ is such that $|\gamma_n^{(2)}| \leq \sqrt{\frac{54\pi^3}{54\pi^3 - 1}}$. In order to prove (25) we use the following estimate, which will be useful in what follows.

Let $\gamma > 0$, $r > 0$, $m \geq 1$ and $\delta \in \mathbb{R}$. Then for $m \geq \left( \frac{14|\delta + 1 - r|}{\gamma r} \right)^{\frac{1}{2}},$ the estimate takes place

$$
\int_m^\infty e^{-\gamma t^r} t^\delta dt = \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta + 1 - r} \left( 1 + \Theta_{\gamma, m} \left( \frac{\Delta + 1 - r}{\gamma r} \frac{1}{m^r} \right) \right), \quad |\Theta_{\gamma, m}| \leq \frac{14}{13}.
$$

(26)

Indeed, integrating by parts, we obtain

$$
\int_m^\infty e^{-\gamma t^r} t^\delta dt = \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta + 1 - r} + \frac{\Delta + 1 - r}{\gamma r} \int_m^\infty e^{-\gamma t^r} t^{-r+\delta} dt.
$$

(27)

Since

$$
\int_m^\infty e^{-\gamma t^r} t^{-r+\delta} dt = \frac{\Theta_{\gamma, m}}{m^r} \int_m^\infty e^{-\gamma t^r} t^\delta dt, \quad 0 < \Theta_{\gamma, m} < 1,
$$

(28)
by virtue of (27) for $m \geq \left(\frac{14|\delta+1-r|}{\gamma r}\right)^{\frac{1}{2}}$ we have

$$\int_{m}^{\infty} e^{-\gamma t^r} t^\delta dt \leq \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta+1-r} + \frac{1}{14} \int_{m}^{\infty} e^{-\gamma t^r} t^\delta dt,$$

whence

$$\int_{m}^{\infty} e^{-\gamma t^r} t^\delta dt \leq \frac{14e^{-\gamma m^r}}{13\gamma r} m^{\delta+1-r}. \quad (29)$$

The estimate (26) follows from (27)–(29).

From the equality (6) and relation

$$\int_{n}^{\infty} \xi(u) du < \sum_{j=n}^{\infty} \xi(j) < \int_{n}^{\infty} \xi(u) du + \xi(n), \quad (30)$$

which takes place for any positive and decreasing function $\xi(u), u \geq 1$, such that $\int_{n}^{\infty} \xi(u) du < \infty$, we get

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r}) = \frac{1}{\sqrt{\pi}} \left( \int_{n}^{\infty} e^{-2\alpha t^r} dt + \Theta_{\alpha,r,n}^{(1)} e^{-2\alpha n^r} \right)^{\frac{1}{2}} < 1. \quad (31)$$

In order to estimate the integral $\int_{n}^{\infty} e^{-2\alpha t^r} dt$ it suffices to use the equality (26) for $\gamma = 2\alpha, \delta = 0, m = n$ and $r \in (0,1)$. Then, taking into account that $n_0(\alpha, r, 2) > \left(\frac{7(1-r)}{\alpha r}\right)^{\frac{1}{2}}$, for $n \geq n_0(\alpha, r, 2)$ from (26) and (31) we get

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r}) = \frac{1}{\sqrt{\pi}} \left( \frac{e^{-2\alpha n^r}}{2\alpha r} n^{1-r} \left( 1 + \Theta_{\alpha,r,n}^{(1)} \right) \right)^{\frac{1}{2}} =$$

$$= \frac{e^{-\alpha n^r}}{\sqrt{2\pi \alpha r}} n^{\frac{1-r}{2}} \left( 1 + \Theta_{\alpha,r,n}^{(2)} \right)^{\frac{1}{2}}, \quad |\Theta_{\alpha,r,n}^{(2)}| \leq 2. \quad (32)$$

Since for $n > n_0(\alpha, r, 2)$

$$\left| \left( 1 + \Theta_{\alpha,r,n}^{(2)} \right)^{\frac{1}{2}} - 1^{\frac{1}{2}} \right| \leq$$

$$\leq \frac{1}{\sqrt{1 - \left( \frac{1}{\alpha n^r} + \frac{2\alpha r}{n^{1-r}} \right)}} \left( \frac{1}{2\alpha r} \frac{1}{n^{1-r}} + \frac{\alpha r}{n^{1-r}} \right) \leq \sqrt{\frac{54\pi^3}{54\pi^3 - 1}} \left( \frac{1}{2\alpha r} \frac{1}{n^{1-r}} + \frac{\alpha r}{n^{1-r}} \right),$$

then (25) follows from (32).

In the case of $p = \infty$ theorem 1 allows to clarify the asymptotic equality (10).
We set \( n_1 = n_1(\alpha, r) \) be the smallest number \( n \) such that
\[
\frac{1}{\alpha r n^r} \left( 1 + \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) \right) + \frac{\alpha r}{n^{1-r}} \leq \frac{1}{(3\pi)^3}. \tag{33}
\]

The following assertion takes place.

**Theorem 3.** Let \( 0 < r < 1, \alpha > 0 \) and \( \beta \in \mathbb{R} \). Then for \( n \geq n_1(\alpha, r) \) the following estimate is true
\[
\mathcal{E}_n(C_{\alpha, r, \beta}^l)_{C} = \frac{4}{\pi^2} e^{-\alpha n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \gamma_{n, \infty}^{(2)} e^{-\alpha n^r}, \tag{34}
\]
where the quantity \( \gamma_{n, \infty}^{(2)} = \gamma_{n, \infty}^{(2)}(\alpha, r, \beta) \) is such that \( |\gamma_{n, \infty}^{(2)}| \leq 20\pi^4 \).

**Proof of the theorem 3.** From definitions \((33)\) and \((14)\) it follows that
\[
n_1(\alpha, r) > n_0(\alpha, r, \infty). \tag{35}
\]
So, applying the equality \((15)\) for \( p = \infty \left( p' = 1 \right) \), we get for \( n \geq n_1(\alpha, r) \)
\[
\mathcal{E}_n(C_{\alpha, r}^{\alpha, r, \beta})_{C} = e^{-\alpha n^r} \left( \frac{4}{\pi^2} \int_{0}^{\pi} \frac{dt}{\sqrt{t^2 + 1}} + \gamma_{n, \infty}^{(1)} \left( \frac{1}{\alpha r n^r} \int_{0}^{\pi} \frac{dt}{\sqrt{t^2 + 1}} + 1 \right) \right). \tag{36}
\]
Since
\[
\int_{0}^{\pi} \frac{dt}{\sqrt{t^2 + 1}} = \int_{1}^{\pi} \frac{dt}{t} + \left( \int_{0}^{1} \frac{dt}{\sqrt{t^2 + 1}} - \int_{1}^{\pi} \frac{dt}{t} \right) = \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta_{\alpha, r, n}^{(3)}, \quad 0 < \Theta_{\alpha, r, n}^{(3)} < 1,
\]
by virtue of \((35)\) and \((36)\) for \( n \geq n_1(\alpha, r) \)
\[
\mathcal{E}_n(C_{\alpha, r}^{\alpha, r, \beta})_{C} = e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \frac{4}{\pi^2} \Theta_{\alpha, r, n}^{(3)} + \gamma_{n, \infty}^{(1)} \left( \frac{1}{\alpha r n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta_{\alpha, r, n}^{(3)} + 1 \right) \right). \tag{37}
\]
The results of our calculations show that for \( n \geq n_1(\alpha, r) \)
\[
\frac{4}{\pi^2} \Theta_{\alpha, r, n}^{(3)} + |\gamma_{n, \infty}^{(1)}| \left( \frac{1}{\alpha r n^r} \ln \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta_{\alpha, r, n}^{(3)} + 1 \right) \leq 20\pi^4, \quad \tag{38}
\]
and therefore, in view of \((37)\) and \((38)\) we obtain \((34)\). Theorem 3 is proved.

**3. Proof of the theorem 1.** According to \((1)\) and \((2)\) we have
\[
\mathcal{E}_n(C_{\alpha, r, \beta}^{\alpha, r, \beta})_{C} = \frac{1}{\pi} \sup_{\varphi \in B_p} \left\| \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} P_{\alpha, r, \beta}^{(n)}(x-t)\varphi(t)dt \right\|_{C}, \quad 1 \leq p \leq \infty, \tag{39}
\]
where

$$P_{\alpha,r,\beta}^{(n)}(t) := \sum_{k=n}^{\infty} e^{-\alpha k r} \cos \left( kt - \frac{\beta \pi}{2} \right), \quad 0 < r < 1, \quad \alpha > 0, \quad \beta \in \mathbb{R}. \quad (40)$$

Taking into account the invariance of the sets $B_{p}^{0}$, $1 \leq p \leq \infty$, under shifts of the argument, from (39) we conclude that

$$\mathcal{E}_{n}(C_{\beta,p}^{0}) = \frac{1}{\pi} \sup_{\varphi \in B_{p}^{0}} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}^{(n)}(t) \varphi(t) dt. \quad (41)$$

On the basis of the duality relation (see, e.g., [22, Chapter 1, Section 1.4])

$$\sup_{\varphi \in B_{p}^{0}} \int_{-\pi}^{\pi} P_{\alpha,r,\beta}^{(n)}(t) \varphi(t) dt = \inf_{\lambda \in \mathbb{R}} \| P_{\alpha,r,\beta}^{(n)}(t) - \lambda \|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (42)$$

In order to find the estimate for the quantity $\inf_{\lambda \in \mathbb{R}} \| P_{\alpha,r,\beta}^{(n)}(t) - \lambda \|_{p'}$ we use the following assertion, proof of which will be presented later.

**Lemma 1.** Let $1 \leq s \leq \infty$, $2\pi$–periodic functions $g(t)$ and $h(t)$ have finite derivatives and satisfy the conditions:

$$r(t) := \sqrt{g^2(t) + h^2(t)} \neq 0, \quad (43)$$

$$M := \sup_{t \in \mathbb{R}} \frac{\sqrt{(g'(t))^2 + (h'(t))^2}}{\sqrt{g^2(t) + h^2(t)}} < \infty. \quad (44)$$

Then for the function

$$\phi(t) = g(t) \cos(nt + \gamma) + h(t) \sin(nt + \gamma), \quad \gamma \in \mathbb{R}, \quad n \in \mathbb{N}, \quad (45)$$

for all numbers $n \geq \begin{cases} 4\pi s M, & 1 \leq s < \infty, \\ 1, & s = \infty, \end{cases}$ the following estimates take place

$$\| \phi \|_{s} = \| r \|_{s} \left( \frac{\| \cos t \|_{s}}{(2\pi)^{\frac{s}{2}}} + \delta_{s,n}^{(1)} \frac{M}{n} \right), \quad (46)$$

$$\inf_{\lambda \in \mathbb{R}} \| \phi(t) - \lambda \|_{s} = \| r \|_{s} \left( \frac{\| \cos t \|_{s}}{(2\pi)^{\frac{s}{2}}} + \delta_{s,n}^{(2)} \frac{M}{n} \right), \quad (47)$$

$$\sup_{h \in \mathbb{R}} \frac{1}{2} \| \phi(t + h) - \phi(t) \|_{s} = \| r \|_{s} \left( \frac{\| \cos t \|_{s}}{(2\pi)^{\frac{s}{2}}} + \delta_{s,n}^{(3)} \frac{M}{n} \right), \quad (48)$$

where

$$|\delta_{s,n}^{(i)}| < 14\pi, \quad i = 1,3. \quad (49)$$

We represent the function $P_{\alpha,r,\beta}^{(n)}(t)$, which is defined by formula (40), in the form

$$P_{\alpha,r,\beta}^{(n)}(t) = g_{\alpha,r,n}(t) \cos \left( nt - \frac{\beta \pi}{2} \right) + h_{\alpha,r,n}(t) \sin \left( nt - \frac{\beta \pi}{2} \right), \quad (50)$$
where
\[ g_{\alpha,r,n}(t) := \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \cos kt, \] (51)
\[ h_{\alpha,r,n}(t) := -\sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \sin kt. \] (52)

Let us show, that for functions \( g_{\alpha,r,n} \) and \( h_{\alpha,r,n} \) the following conditions are satisfied
\[ \sqrt{g_{\alpha,r,n}(t)}^2 + h_{\alpha,r,n}(t)^2 \neq 0 \] (53)
and
\[ M_n = M_n(\alpha; r) := \sup_{t \in \mathbb{R}} \sqrt{\left(g'_{\alpha,r,n}(t)\right)^2 + \left(h'_{\alpha,r,n}(t)\right)^2} < \infty. \] (54)

Since, for arbitrary \( \alpha > 0, \ 0 < r < 1 \) the sequence \( \{e^{-\alpha(k+n)r}\}_{k=0}^{\infty} \) is convex downwards, then (see, e.g., [23, Chapter 10, Section 2])
\[ \frac{1}{2}e^{-\alpha n^r} + \sum_{k=1}^{\infty} e^{-\alpha(k+n)^r} \cos kt \geq 0, \] and
\[ \sqrt{g_{\alpha,r,n}(t)}^2 + h_{\alpha,r,n}(t)^2 \geq \frac{1}{2}e^{-\alpha n^r} > 0. \] (55)

Further, since
\[ g'_{\alpha,r,n}(t) = -\sum_{k=1}^{\infty} ke^{-\alpha(k+n)^r} \sin kt, \] (56)
\[ h'_{\alpha,r,n}(t) = -\sum_{k=1}^{\infty} ke^{-\alpha(k+n)^r} \cos kt, \] (57)
it is clear that
\[ \sqrt{(g'_{\alpha,r,n}(t))^2 + (h'_{\alpha,r,n}(t))^2} < \sum_{k=1}^{\infty} ke^{-\alpha(k+n)^r} < \infty. \] (58)

On the basis of (55) and (58), the functions \( g_{\alpha,r,n}(t) \) and \( h_{\alpha,r,n}(t) \) satisfy the conditions (53) and (54). Therefore, setting in lemma 1 \( g(t) = g_{\alpha,r,n}(t) \), \( h(t) = h_{\alpha,r,n}(t) \), \( s = p' \) and \( \gamma = -\frac{\beta\pi}{2} \), we get that for
\[ n \geq \begin{cases} 4\pi p'M_n, & 1 \leq p' < \infty, \\ 1, & p' = \infty, \end{cases} \] (59)
the estimate takes place
\[ \inf_{\lambda \in \mathbb{R}} \| P_{\alpha,r,\beta}^{(n)}(t) - \lambda \|_{p'} = \left\| \sqrt{(g_{\alpha,r,n}(t))^2 + (h_{\alpha,r,n}(t))^2} \right\|_{p'} \left( \frac{\| \cos t \|_{p'}}{(2\pi)^{\frac{1}{p'}}} + \delta_n \frac{M_n}{n} \right), \] (60)
where $\frac{1}{p'} + \frac{1}{p'} = 1$, quantity $M_n$ is defined by equality (54), and the quantity $\delta_n^{(1)}(\alpha, r, \beta, p)$ is such that $|\delta_n^{(1)}| < 14 \pi$.

Setting

$$P_{\alpha, r, n}(t) := g_{\alpha, r, n}(t) - ih_{\alpha, r, n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{ikt},$$

we have

$$\sqrt{(g'_{\alpha, r, n}(t))^2 + (h'_{\alpha, r, n}(t))^2} = |P'_{\alpha, r, n}(t)|$$

and therefore

$$M_n = \sup_{t \in \mathbb{R}} \frac{|P'_{\alpha, r, n}(t)|}{|P_{\alpha, r, n}(t)|}.$$  \hfill (62)

Then, by virtue of the formulas (41), (42), (60) and (61), for all numbers $n$, which satisfy the condition (59), the estimate holds

$$\mathcal{E}_n(C_{\alpha, r, \beta, p}) = \|P_{\alpha, r, n}(t)\|_{p'} \left(\frac{\|\cos t\|_{p'}}{2^{1/p_1} \pi^{1/p_1}} \right) + \frac{\delta_n^{(2)}}{n}, 1 \leq p \leq \infty,$$

where $M_n$ is defined by equality (62), and for the quantity $\delta_n^{(2)} = \delta_n^{(2)}(\alpha, r, \beta, p)$ is such that $|\delta_n^{(2)}| < 14$.

Since

$$|P_{\alpha, r, n}(t)|^2 = P_{\alpha, r, n}(t) \widetilde{P}_{\alpha, r, n}(t),$$

where

$$\widetilde{P}_{\alpha, r, n}(t) = g_{\alpha, r, n}(t) + ih_{\alpha, r, n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{-ikt},$$

by expanding the product $P_{\alpha, r, n} \widetilde{P}_{\alpha, r, n}$ in the Fourier series (see, e.g., [23, Chapter 1, Section 23]), we get

$$P_{\alpha, r, n}(t) \widetilde{P}_{\alpha, r, n}(t) = \left(\sum_{k=0}^{\infty} e^{-\alpha(k+n)r} e^{ikt}\right) \left(\sum_{k=-\infty}^{0} e^{-\alpha(-k+n)r} e^{ikt}\right) =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} e^{-\alpha(j+n)r} e^{-\alpha(j+|k|+n)r} e^{ikt} =$$

$$= \sum_{j=n}^{\infty} e^{-2\alpha j'} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j'} e^{-\alpha(j+k)r} \cos kt. \hfill (65)$$

Let convert the sum $\sum_{j=n}^{\infty} e^{-2\alpha j'} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j'} e^{-\alpha(j+k)r} \cos kt$ with a help of Poisson summation formula.
Assertion 1 [24, Chapter 2, Section 2.8]. Let continuous function \( \phi(x) \) be a function of bounded variation in the interval \((0, \infty)\), \( \lim_{x \to \infty} \phi(x) = 0 \) and

\[
\int_0^\infty \phi(t) dt < \infty.
\]

Then the following equality takes place

\[
\sqrt{a} \left( \frac{\phi(0)}{2} + \sum_{k=1}^{\infty} \phi(ka) \right) = \sqrt{\frac{2\pi}{a}} \left( \frac{\Phi_c(0)}{2} + \sum_{k=1}^{\infty} \Phi_c\left(\frac{2\pi k}{a}\right) \right), \quad a > 0,
\]

where \( \Phi_c(x) \) is the Fourier cosine transform of the function \( \phi(x) \) of the form

\[
\Phi_c(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \phi(u) \cos xu \, du.
\]

Let fix \( t \in [-\pi, \pi] \), \( \alpha > 0 \), \( r \in (0, 1) \) and set

\[
\phi(x) = 2 \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha (j+x)^r} \cos xt, \quad x \geq 0
\]

and \( a = 1 \). One can easily check that all conditions of the assertion 1 are satisfied, and therefore, according to (66) we obtain

\[
\sum_{j=n}^{\infty} e^{-2\alpha j^r} + 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha (j+k)^r} \cos kt =
\]

\[
= 2 \int_0^\infty \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha (j+u)^r} \cos ut \, du +
\]

\[
+ 4 \sum_{k=1}^{\infty} \int_0^\infty \sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha (j+u)^r} \cos ut \cos 2\pi k \, du =
\]

\[
= Q_n(t) + R_n(t),
\]

where

\[
Q_n(t) = Q_n(\alpha; r; t) := 2 \sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^\infty e^{-\alpha (j+u)^r} \cos ut \, du,
\]

\[
R_n(t) = R_n(\alpha; r; t) :=
\]

\[
= 2 \sum_{k=1}^{\infty} \sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^\infty e^{-\alpha (j+u)^r} \left( \cos((t - 2\pi k)u) + \cos((t + 2\pi k)u) \right) \, du.
\]
Hence, as a consequence of (64), (65) and (67)
\[ |\mathcal{P}_{\alpha,r,n}(t)|^2 = Q_n(t) + R_n(t). \] (70)

Denote by \( n_2 = n_2(\alpha, r, p) \) the smallest number \( n \) such that
\[ \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \leq \frac{1}{14}, \] (71)
where
\[ \chi(p) = \begin{cases} p, & 1 \leq p < \infty, \\ 1, & p = \infty, \end{cases} \]
and let us show that for the quantity \( Q_n(t) \) for \( n \geq n_2(\alpha, r, p) \) and arbitrary \( t \in [-\pi, \pi] \) the following estimate takes place
\[ Q_n(t) = \frac{e^{-2\alpha r}}{t^2 + (\alpha rn^{r-1})^2} \left( 1 + \Theta_{\alpha,r,n}^{(4)}(t) \left( \frac{1 - r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad |\Theta_{\alpha,r,n}^{(4)}(t)| < 5. \] (72)

Integrating by parts, we find
\[ \int e^{-\alpha(j+u)^r} \cos utdu = \]
\[ = e^{-\alpha(j+u)^r} \frac{-\alpha(j+u)^{r-1} \cos ut + t \sin ut}{t^2 + (\alpha(j+u)^{r-1})^2} + \alpha(1 - r) \times \]
\[ \times \int e^{-\alpha(j+u)^r} (j + u)^{r-2} \frac{(\alpha(j+u)^{r-1})^2 - t^2}{(t^2 + (\alpha(j+u)^{r-1})^2)^2} \cos ut - 2t\alpha(j+u)^{r-1} \sin ut \] du.

Hence, we obtain the equality
\[ \int_0^\infty e^{-\alpha(j+u)^r} \cos utdu = \frac{\alpha r^{r-1}}{t^2 + (\alpha r^{r-1})^2} e^{-\alpha j^r} + \alpha(1 - r) \times \]
\[ \times \int_0^\infty e^{-\alpha(j+u)^r} (j + u)^{r-2} \frac{(\alpha(j+u)^{r-1})^2 - t^2}{(t^2 + (\alpha(j+u)^{r-1})^2)^2} \cos ut - 2t\alpha(j+u)^{r-1} \sin ut \] du. \] (73)

It is easy to verify that
\[ \left| \int_0^\infty e^{-\alpha(j+u)^r} (j + u)^{r-2} \frac{(\alpha(j+u)^{r-1})^2 - t^2}{(\alpha(j+u)^{r-1})^2 + t^2)^2} \cos ut - 2t\alpha(j+u)^{r-1} \sin ut \right| \]
\[ \leq \int_0^\infty e^{-\alpha(j+u)^r} (j + u)^{r-2} \left( \frac{1}{t^2 + (\alpha(j+u)^{r-1})^2} + \frac{2t\alpha(j+u)^{r-1}}{(t^2 + (\alpha(j+u)^{r-1})^2)^2} \right) \] du \leq \]
\[ \leq 2 \int_0^\infty e^{-\alpha(j+u)^r} \frac{(j + u)^{r-2}}{t^2 + (\alpha(j+u)^{r-1})^2} \] du. \] (74)
For fixed $\alpha > 0$, $r \in (0, 1)$ and $t \in [-\pi, \pi]$ the function $\frac{e^{-\alpha r}}{t^2 + (\alpha \cos^2 t)}$, $v \geq 1$ decreases. Besides, according to (29), for $\delta = 0$, $\gamma = \alpha$, $m = j$, $f \geq n_2(\alpha, r, p)$ the estimate takes place

$$
\int_0^\infty e^{-\alpha(j+u)^r} \frac{(j + u)^{r-2}}{t^2 + (\alpha(j + u)^{r-1})^2} du \leq \frac{j^{r-2}}{t^2 + (\alpha j^{r-1})^2} \int_0^\infty e^{-\alpha(j+u)^r} du = \frac{j^{r-2}}{t^2 + (\alpha j^{r-1})^2} \int_j^\infty e^{-\alpha u^r} du \leq \frac{14}{13} \frac{e^{-\alpha j^r}}{\alpha j(t^2 + (\alpha j^{r-1})^2)}. \tag{75}
$$

It follows from relations (73)–(75) that for $j \geq n_2(\alpha, r, p)$

$$
\int_0^\infty e^{-\alpha(j+u)^r} \cos^2(tu) du = \frac{\alpha j^{r-1}}{t^2 + (\alpha j^{r-1})^2} e^{-\alpha j^r} (1 + \Theta^{(5)}_{\alpha, r, j}(t) \frac{1 - r}{\alpha r} j^{r}), \quad |\Theta^{(5)}_{\alpha, r, j}(t)| \leq \frac{28}{13}. \tag{76}
$$

Therefore, taking into account (68), for $n \geq n_2(\alpha, r, p)$ we have

$$
Q_n(t) = 2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j^r} j^{r-1}}{t^2 + (\alpha j^{r-1})^2} \left(1 + \Theta^{(6)}_{\alpha, r, j}(t) \frac{1 - r}{\alpha r} \frac{1}{n^r}\right), \quad |\Theta^{(6)}_{\alpha, r, j}(t)| \leq \frac{28}{13}. \tag{77}
$$

Further, let us find bilateral estimates for the quantities $\sum_{j=n}^\infty \frac{e^{-2\alpha j^r} j^{r-1}}{t^2 + (\alpha j^{r-1})^2}$ for $n \geq n_2(\alpha, r, p)$. It can be shown that for fixed $\alpha > 0$, $r \in (0, 1)$ and $t \in [-\pi, \pi]$ the function $\xi(u) = \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha u^r)^2}$ decreases for $u \geq n_2(\alpha, r, p)$. Therefore, on basis of (30)

$$
2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j^r} j^{r-1}}{t^2 + (\alpha j^{r-1})^2} = 2\alpha \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha u^r)^2} du + \Theta^{(7)}_{\alpha, r, n}(t) \frac{\alpha r}{t^2 + (\alpha n^{r-1})^2} e^{-2\alpha n^r} n^{r-1}, \quad 0 \leq \Theta^{(7)}_{\alpha, r, n}(t) \leq 2. \tag{78}
$$

Integrating by parts, we have

$$
2\alpha \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha u^r)^2} du = \frac{e^{-2\alpha n^r} n^{r-1}}{t^2 + (\alpha n^{r-1})^2} + 2(\alpha r)^2(1 - r) \int_n^\infty \frac{e^{-2\alpha u^r} u^{2r-3}}{(t^2 + (\alpha u^r)^2)^2} du. \tag{79}
$$

Since

$$
(\alpha r)^2 \int_n^\infty \frac{e^{-2\alpha u^r} u^{2r-3}}{(t^2 + (\alpha u^r)^2)^2} du \leq \int_n^\infty \frac{e^{-2\alpha u^r} u^{-1}}{t^2 + (\alpha u^r)^2} du \leq \int_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha u^r)^2} du
$$
\[
\leq \frac{1}{n^r} \int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du,
\]

(80)

it follows from (79) that for \( n \geq n_2(\alpha, r, p) \) the following inequalities are true

\[
\int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du \leq \frac{1}{2\alpha r} \frac{e^{-2\alpha u r}}{t^2 + (\alpha u^{r-1})^2} + \frac{1 - r}{\alpha r} \int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du \leq \frac{1}{2\alpha r} \frac{e^{-2\alpha u r}}{t^2 + (\alpha u^{r-1})^2} + \frac{1}{14} \int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du.
\]

Hence, for \( n \geq n_2(\alpha, r, p) \)

\[
\int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du \leq \frac{7}{13\alpha r} \frac{e^{-2\alpha u r}}{t^2 + (\alpha u^{r-1})^2}.
\]

(81)

From (79)–(81) for \( n \geq n_2(\alpha, r, p) \) we arrive at the following estimate

\[
2\alpha r \int_0^\infty \frac{e^{-2\alpha u} u^{r-1}}{t^2 + (\alpha u^{r-1})^2} du =
\]

\[
= \frac{e^{-2\alpha u r}}{t^2 + (\alpha u^{r-1})^2} \left(1 + \Theta_{a,r,n}^{(9)}(t) \frac{1 - r}{\alpha r} \frac{1}{n^r} \right), \quad 0 < \Theta_{a,r,n}^{(9)}(t) \leq \frac{14}{13}.
\]

It follows from formulas (78) and (82) that

\[
2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j r} j^{r-1}}{t^2 + (\alpha j^{r-1})^2} =
\]

\[
= \frac{e^{-2\alpha u r}}{t^2 + (\alpha u^{r-1})^2} \left(1 + \Theta_{a,r,n}^{(9)}(t) \left(\frac{1 - r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\right) \right),
\]

(83)

where \( n \geq n_2(\alpha, r, p) \) and \( 0 < \Theta_{a,r,n}^{(9)}(t) \leq 2. \)

In view of (77) and (83) for all \( n \geq n_2(\alpha, r, p) \) we obtain (72). In particular, it follows from formulas (71) and (72) that

\[
Q_n(t) > 0, \quad t \in [-\pi, \pi], \quad n \geq n_2(\alpha, r, p).
\]

(84)

Let us find upper estimate for the quantity \( R_n(t) \) of the form (69). Denote by \( \mathfrak{M} \) the set of all convex downwards, continuous functions \( \psi(t) > 0, \quad t \geq 1, \) such that \( \lim_{t \to \infty} \psi(t) = 0. \) The following assertion takes place.
Lemma 2. Let $\psi \in M$. Then

$$0 < \int_0^\infty \psi(\tau + u) \cos vudu \leq \frac{\pi}{v^2}|\psi'(\tau)|, \quad \forall v \in \mathbb{R} \setminus \{0\}, \quad \tau \geq 1. \quad (85)$$

Proof of lemma 2. We use the scheme of the proof of the estimate (2.4.31) from the work [25, p. 93]. Let, e.g., consider the case $v > 0$. Using the method of integration by parts, we have

$$\int_0^\infty \psi(\tau + u) \cos vudu = -\frac{1}{v} \int_0^\infty \psi'(\tau + u) \sin vudu. \quad (86)$$

We set

$$I(x) = I(\psi; \tau; v; x) := -\int_x^\infty \psi'(j + u) \sin vudu, \quad x \geq 0, \quad v > 0, \quad \tau \in \mathbb{N}.$$ 

The function $I(x)$, obviously, is continuous for every fixed $v$, and on every interval between the consecutive zeros $u_m = \frac{\pi m}{v}$ and $u_{m+1} = \frac{\pi (m+1)}{v}$ of the function $\sin vu$ has one simple zero $x_m$. Existence of zeros $x_m$ of the function $I(x)$ is a consequence of the Leibniz theorem on alternating series, and uniqueness of zero $x_m$ on the interval $(u_m, u_{m+1})$ follows from the equality

$$\text{sign } I'(x) = -\text{sign } \sin xv, \quad x \in (u_m, u_{m+1}) \quad m \in \mathbb{Z}_+.$$ 

Let $x_0$ be the zero closest from the right to the point $x = 0$. It is obvious that

$$0 \leq x_0 \leq \frac{\pi}{v}.$$ 

Taking into account this fact and also monotone decreasing of the function $-\psi'(t)$ on the interval $[1, \infty)$, we have

$$-\frac{1}{v} \int_0^\infty \psi'(\tau + u) \sin vudu = \frac{1}{v} \int_0^{x_0} |\psi'(\tau + u)| \sin vudu \leq$$

$$\leq \frac{x_0}{v} \int_0^{\frac{\pi}{v}} |\psi'(\tau + u)| du \leq \frac{\pi}{v^2} |\psi'(\tau)|. \quad (87)$$

For $v > 0$ inequality (85) follows from the formulas (86) and (87). For $v < 0$ the proof of inequality (85) is analogous. Lemma 2 is proved.

Setting in inequality (85) $v = t \pm 2\pi k, \quad k \in \mathbb{N}, \quad \tau = j$, we obtain that for arbitrary $\psi \in M$ and $t \in [-\pi, \pi]$

$$0 < \sum_{k=1}^\infty \sum_{j=n}^\infty \int_0^\infty \psi(j + u) \left(\cos((t - 2\pi k)u) + \cos((t + 2\pi k)u)\right) du \leq$$

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\[
\leq \pi \sum_{k=1}^{\infty} \left( \frac{1}{(t - 2k\pi)^2} + \frac{1}{(t + 2k\pi)^2} \right) \sum_{j=n}^{\infty} \psi(j)|\psi'(j)| \leq \\
\leq \pi \sum_{k=1}^{\infty} \left( \frac{1}{(\pi - 2k\pi)^2} + \frac{1}{(\pi + 2k\pi)^2} \right) \psi(n) \left( |\psi'(n)| + \int_{n}^{\infty} |\psi'(u)| du \right) = \\
= \frac{1}{\pi} \sum_{k=1}^{\infty} \left( \frac{1}{(2k - 1)^2} + \frac{1}{(2k + 1)^2} \right) \psi(n) \left( |\psi'(n)| + \psi(n) \right) = \\
= \left( \frac{\pi}{4} - \frac{1}{\pi} \right) \psi(n) \left( |\psi'(n)| + \psi(n) \right), \quad n \in \mathbb{N}.
\]

Setting in (88) \( \psi(t) = e^{-ar^r}, \ 0 < r < 1, \ \alpha > 0, \) we get that for the function \( R_n(t) \) of the form (69) the following estimate takes place
\[
0 < R_n(t) \leq \left( \frac{\pi}{2} - \frac{2}{\pi} \right) e^{-2an^r} (\frac{\alpha r}{n^{1-r}} + 1) \leq \left( \frac{\pi}{2} - \frac{2}{\pi} \right) \frac{15}{14} e^{-2an^r} < \frac{\pi}{3} e^{-2an^r},
\]
where \( n \geq n_2(\alpha, r, p). \)

By virtue of (70)
\[
|P_{a, r, n}(t)| = \sqrt{Q_n(t) + R_n(t)},
\]
and therefore, taking into account (84) and (89), we have
\[
\|P_{a, r, n}\|_{p'} = \|\sqrt{Q_n}\|_{L_{p'}[-\pi, \pi]} + \Theta_{a, r, n, p}^{(2)} e^{-an^r}, \quad 1 \leq p' \leq \infty,
\]
where \( \Theta_{a, r, n, p}^{(2)} < \frac{2\pi^2}{3} \) and \( n \geq n_3(\alpha, r, p). \)

Let us show, that for \( 1 \leq p' \leq \infty, \ \frac{1}{p'} + \frac{1}{p'} = 1, \) and \( n \geq n_2(\alpha, r, p) \) the estimate is true
\[
\|P_{a, r, n}\|_{p'} = e^{-an^r} n^{-\frac{1}{p'}} \left( \frac{2\pi}{\alpha r} \right)^{\frac{1}{p'}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \\
+ \Theta_{a, r, p, n}^{(3)} \left( \frac{1 - r}{(\alpha r)^{1+\frac{1}{p'}}} J_{p'} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right),
\]
where
\[
|\Theta_{a, r, p, n}^{(3)}| \leq \left\{ \begin{array}{ll}
\frac{\pi^2}{14}, & 1 \leq p' < \infty, \\
\frac{13}{14}, & p' = \infty.
\end{array} \right.
\]

Since, on the basis of estimate (72) for \( n \geq n_2(\alpha, r, p) \) and \( 1 \leq p' \leq \infty \)
\[
\left| \left( 1 + \Theta_{a, r, n}^{(10)}(t) \left( \frac{1 - r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)^{\frac{1}{p'}} - 1 \right) \right| \leq \\
\leq \frac{5}{2} \sqrt{1 - 5 \left( \frac{1 - r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)} \leq \frac{5\sqrt{7}}{3\sqrt{2}} \left( \frac{1 - r}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right)
\]
we get
\[ \sqrt{Q_n(t)} = \frac{e^{-\alpha r}}{\sqrt{t^2 + (\alpha r n^{r-1})^2}}(1 + \Theta^{(10)}_{\alpha r n}(t)\left(\frac{1 - r}{\alpha r} n^{1-r} + \frac{\alpha r}{n^{1-r}}\right)), \quad |\Theta^{(10)}_{\alpha r n}(t)| \leq \frac{5\sqrt{7}}{3\sqrt{2}} \] (94)

For \(1 \leq p' < \infty\) from (94) we have
\[ \left\| \sqrt{Q_n} \right\|_{L^{p'}[-\pi,\pi]} = e^{-\alpha r} \left(\int_{-\pi}^{\pi} \frac{dt}{(t^2 + (\alpha r n^{r-1})^2)^{\frac{1}{p'}}}\right)^{\frac{1}{p'}} (1 + \Theta^{(4)}_{\alpha r p n}(1 - r \frac{1}{\alpha r} n^{1-r} + \frac{\alpha r}{n^{1-r}})) = \]
\[ = 2^{\frac{1}{p'}} e^{-\alpha r} \left(\frac{n^{1-r}}{\alpha r}\right)^{\frac{1}{p'}} J'_{\nu'}(\pi n^{1-r}) (1 + \Theta^{(4)}_{\alpha r p n}(1 - r \frac{1}{\alpha r} n^{1-r} + \frac{\alpha r}{n^{1-r}})), \]
\[ \text{where } |\Theta^{(4)}_{\alpha r p n}| \leq \frac{5\sqrt{7}}{3\sqrt{2}}, \quad \text{and } J'_{\nu'}(\pi n^{1-r}) \text{ is defined by equality } (13). \]

Combining (91) and (95), we obtain that for \(1 \leq p' < \infty\) the following relation takes place
\[ \left\| P_{\alpha r n} \right\|_{p'} = e^{-\alpha r} n^{\frac{1-r}{p'}} \left(\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p'}}} J'_{\nu'}(\pi n^{1-r}) \right) + \]
\[ + \Theta^{(4)}_{\alpha r p n} \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p'}}} J'_{\nu'}(\pi n^{1-r}) \left(1 - r \frac{1}{\alpha r} n^{1-r} + \frac{\alpha r}{n^{1-r}}\right) + \Theta^{(2)}_{\alpha r p n}(\pi n^{1-r}) \] (96)

However, for all \(n > n_2(\alpha, r, p)\)
\[ \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p'}}} J'_{\nu'}(\pi n^{1-r}) \frac{\alpha r}{n^{1-r}} < \frac{1}{n^{1-r}}, \quad 1 \leq p' < \infty. \] (97)

Indeed, taking into account (13) and (71), for all \(1 < p' < \infty\) and \(n \geq n_2(\alpha, r, p)\) we find
\[ \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p'}}} J'_{\nu'}(\pi n^{1-r}) \frac{\alpha r}{n^{1-r}} \frac{n^{1-r}}{p'} = \left(\frac{2\alpha r}{n^{1-r}}\right)^{\frac{1}{p'}} J'_{\nu'}(\pi n^{1-r}) < \]
\[ < \left(\frac{2\alpha r}{n^{1-r}}\right)^{\frac{1}{p'}} \left(\int_{0}^{\infty} \frac{dt}{(t^2 + 1)^{\frac{p'}{2}}}\right)^{\frac{1}{p'}} < \left(\frac{2\alpha r}{n^{1-r}}\right)^{\frac{1}{p'}} \left(1 + \int_{1}^{\infty} \frac{dt}{t^{\nu'}}\right)^{\frac{1}{p'}} = \]
\[ = \left(\frac{2\alpha r}{n^{1-r}}\right)^{\frac{1}{p'}} < \left(\frac{1}{7}\right)^{\frac{1}{p'}} < 1, \] (98)

and for \(p' = 1\) and \(n \geq n_2(\alpha, r, p)\), taking into account decreasing on the interval \([e, \infty)\) of the function \(\frac{\ln u}{u^2}\), we have
\[ \frac{2\alpha r}{n^{1-r}} J_1(\pi n^{1-r}) = \frac{2\alpha r}{n^{1-r}} \left(\int_{0}^{\pi} \frac{dt}{\sqrt{t^2 + 1}}\right) < \frac{2\alpha r}{n^{1-r}} \left(1 + \int_{1}^{\pi} \frac{dt}{\sqrt{t^2 + 1}}\right) < \]
\[ < \frac{2\alpha r}{n^{1-r}} + \frac{2\alpha r}{n^{1-r}} \ln(\pi n^{1-r}) \leq \frac{2}{14} + \frac{2\pi \ln 14}{14\pi} < 1. \] (99)
Formulas (98) and (99) prove (97). For $1 \leq p' < \infty$ estimate (92) follows from (96) and (97).

Let us verify validity of the estimate (92) for $p' = \infty$. It follows from (61) and (30) that

$$\|P_{\alpha,r,n}\|_{\infty} = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} = \int_{n}^{\infty} e^{-\alpha r t} dt + \Theta^{(11)}_{\alpha,r,n} e^{-\alpha n r}, \ |\Theta^{(11)}_{\alpha,r,n}| \leq 1. \quad (100)$$

Setting in formula (26) $\gamma = \alpha$, $\delta = 0$ and $m = n$, from (100) we obtain that for arbitrary $n \geq n_{2}(\alpha, r, p)$

$$\|P_{\alpha,r,n}\|_{\infty} = \frac{e^{-\alpha n r}}{\alpha r} n^{1-r} \left(1 + \Theta^{(12)}_{\alpha,r,n} \left(\frac{1 - r}{\alpha r} - \frac{\alpha r}{n^{1-r}}\right)\right), \quad (101)$$

where $|\Theta^{(12)}_{\alpha,r,n}| \leq \frac{14}{13}$. For $p' = \infty$ the validity of (92) follows from (101) and the equality $J_{\infty}(\frac{\pi n^{1-r}}{\alpha r}) = 1$.

To complete the proof of theorem 1 it suffices to find the upper estimate of the quantity $M_{n}$ in formula (63). It is clear that

$$M_{n} = \sup_{t \in \mathbb{R}} \left|\frac{P'_{\alpha,r,n}(t)}{P_{\alpha,r,n}(t)}\right|^{2} = \max \left\{ \sup_{|t| \leq \frac{\alpha r}{\alpha r - n^{1-r}}} \left|\frac{P'_{\alpha,r,n}(t)}{P_{\alpha,r,n}(t)}\right|, \sup_{\frac{\alpha r}{\alpha r - n^{1-r}} \leq |t| \leq \pi} \left|\frac{P'_{\alpha,r,n}(t)}{P_{\alpha,r,n}(t)}\right| \right\}. \quad (102)$$

In view of formulas (71) and (72) and the fact that $R_{n}(t) > 0$ for $n \geq n_{2}(\alpha, r, p)$ we obtain

$$\left|P_{\alpha,r,n}(t)\right|^{2} > Q_{n}(t) > \frac{9}{14 t^{2}} \frac{e^{-2\alpha n r}}{(\alpha r)^{r-1}}. \quad (103)$$

It directly follows from (61) that

$$\left|P_{\alpha,r,n}(t)\right| \leq \sum_{k=0}^{\infty} e^{-\alpha(k+n)r}, \quad \left|P'_{\alpha,r,n}(t)\right| \leq \sum_{k=1}^{\infty} k e^{-\alpha(k+n)r}. \quad (104)$$

By virtue of (101) for $n \geq n_{2}(\alpha, r, p)$ we have

$$\left|P_{\alpha,r,n}(t)\right| \leq \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} < \frac{14}{13} \frac{e^{-\alpha n r} n^{1-r}}{\alpha r}. \quad (105)$$

The function $t e^{-\alpha r t}$ is monotone decreasing for $t > (\alpha r)^{-\frac{1}{r}}$. Therefore, according to (30), for $n \geq n_{2}(\alpha, r, p)$ the following estimate takes place

$$\sum_{k=1}^{\infty} e^{-\alpha(k+n)r} k = \sum_{k=n}^{\infty} e^{-\alpha k r} k - n \sum_{k=n}^{\infty} e^{-\alpha k r} \leq$$

$$\leq e^{-\alpha n r} n + \int_{n}^{\infty} e^{-\alpha r t} dt - n \int_{n}^{\infty} e^{-\alpha r t} dt. \quad (106)$$
According to (26) \( \gamma = \alpha, \delta = 1, \) \( m = n, \) and also \( \gamma = \alpha, \delta = 0, m = n, \) from (104) and (106) we have

\[
|P'_{\alpha,r,n}(t)| \leq e^{-\alpha r} \left( \frac{42}{13} \left( \frac{1}{\alpha r} \right)^2 + n \right), \quad n \geq n_2(\alpha, r, p). \tag{107}
\]

In view of (103), (105) and (107) for \( n \geq n_2(\alpha, r, p) \) we arrive at the estimate

\[
\sup_{|t| \leq \frac{\pi}{\alpha r}} \frac{|P'_{\alpha,r,n}(t)| |P_{\alpha,r,n}(t)|}{|P_{\alpha,r,n}(t)|^2} \leq \frac{14}{9} e^{2\alpha r} \sup_{|t| < \frac{\pi}{\alpha r}} |P'_{\alpha,r,n}(t)| |P_{\alpha,r,n}(t)| (t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2) \leq \frac{5488}{507} \left( \frac{n^{1-r}}{\alpha r} + \alpha r n^r \right).	ag{108}
\]

Applying the Abel transformation to the function \( P_{\alpha,r,n}(t) \) for \( 0 < |t| \leq \pi, \) and taking into account the inequality

\[
\left| \sum_{j=0}^{\infty} e^{ijt} \right| \leq \frac{\pi}{|t|}, \quad 0 < |t| \leq \pi,
\]

we get

\[
|P_{\alpha,r,n}(t)| = \left| \sum_{k=0}^{\infty} (e^{-\alpha (k+n)^r} - e^{-\alpha (k+n+1)^r}) \sum_{j=0}^{k} e^{ijt} \right| \leq \frac{\pi}{|t|} e^{-\alpha r}.	ag{109}
\]

By analogy, for \( 0 < |t| \leq \pi \)

\[
|P'_{\alpha,r,n}(t)| = \left| \sum_{k=0}^{\infty} (e^{-\alpha (k+n)^r} k - e^{-\alpha (k+n+1)^r} (k+1)) \sum_{j=0}^{k} e^{ijt} \right| \leq \frac{\pi}{|t|} \sum_{k=0}^{\infty} |e^{-\alpha (k+n)^r} k - e^{-\alpha (k+n+1)^r} (k+1)| \leq \frac{\pi}{|t|} \left( \sum_{k=0}^{\infty} k (e^{-\alpha (k+n)^r} - e^{-\alpha (k+n+1)^r}) + \sum_{k=0}^{\infty} e^{-\alpha (k+n+1)^r} \right) = \tag{110}
\]

According to (105) and (110)

\[
|P'_{\alpha,r,n}(t)| \leq \frac{2\pi}{|t|} \sum_{k=0}^{\infty} e^{-\alpha (k+n+1)^r} \leq \frac{28\pi}{13|t|} e^{-\alpha r} n^{1-r} \tag{111}
\]

In view of (103), (109) and (111) we obtain the estimate

\[
\sup_{\frac{\alpha r}{n^{1-r}} \leq |t| \leq \pi} \frac{|P'_{\alpha,r,n}(t)| |P_{\alpha,r,n}(t)|}{|P_{\alpha,r,n}(t)|^2} \leq
\]

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\[
\leq \frac{14}{9} e^{2\alpha r} \sup_{n \leq |t| \leq \pi} |P'_{\alpha,r,n}(t)||P_{\alpha,r,n}(t)| \left( t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2 \right) \leq
\]
\[
\leq \frac{392\pi^2 n^{1-r}}{117} \sup_{n \leq |t| \leq \pi} \frac{t^2 + \left( \frac{\alpha r}{n^{1-r}} \right)^2}{t^2} \leq \frac{784 \pi^2 n^{1-r}}{117}. \tag{112}
\]

Combining (102), (108) and (112), we arrive at the estimate
\[
M_n \leq \frac{784 \pi^2}{117} \left( \frac{n^{1-r}}{\alpha r} + \alpha n^r \right), \quad n \geq n_2(\alpha, r, p). \tag{113}
\]

It follows from conditions (14) and (71) that \( n_0(\alpha, r, p) \geq n_2(\alpha, r, p) \) for arbitrary \( 1 \leq p \leq \infty \). It means that estimates (92) and (113) are true also for \( n \geq n_0(\alpha, r, p) \). Let us show that for \( n \geq n_0(\alpha, r, p) \) the condition (59) is satisfied. This is obvious for \( p' = \infty \). For \( 1 \leq p' < \infty \) by virtue of (113), we have
\[
4\pi M_n p' \leq \frac{3136 \pi^3}{117} \left( \frac{n^{1-r}}{\alpha r} + \alpha n^r \right) p' < 27 \pi^3 \left( \frac{n^{1-r}}{\alpha r} + \alpha \chi(p)n^r \right) p'. \tag{114}
\]

According to (14) and (114) for any \( n \geq n_0(\alpha, r, p) \) the following inequality is true
\[
4\pi p'M_n \leq n,
\]
which is equivalent to (59) for \( 1 \leq p' < \infty \).

By using formulas (63), (92) and (113) for \( n \geq n_0(\alpha, r, p) \) we arrive at the estimate
\[
\mathcal{E}_n(C_{\beta,n})_C =
\]
\[
= e^{-\alpha r} n^{\frac{1-r}{p}} \left( \frac{2^\frac{1}{p}}{(\alpha r)^\frac{1}{p}} J_{\alpha r} \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \Theta^{(3)}_{\alpha,r,p,n} \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{\alpha r} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right) \times
\]
\[
\times \left( \frac{\| \cos t \|_{p'}}{2 \pi^{\frac{1}{p}+\frac{1}{p'}}} + \delta^{(3)}_{n} \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right), \quad 1 \leq p \leq \infty, \tag{115}
\]
where for \( \Theta_{\alpha,r,p,n}^{(3)} \) the estimate (93) takes place, and \( |\delta^{(3)}_{n}| < \frac{10076 \pi^2}{117} \).

For \( n \geq n_0(\alpha, r, p) \) the following inequality holds
\[
|\delta^{(3)}_{n}| \frac{2^\frac{1}{p}}{(\alpha r)^\frac{1}{p}} J_{\alpha r} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \left( \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) <
\]
\[
< \frac{21952 \pi^2}{117} \left( \frac{1}{(\alpha r)^{1+\frac{1}{p}}} \frac{\pi n^{1-r}}{\alpha r} \frac{1}{n^r} + \frac{1}{n^{1-r}} \right), \tag{116}
\]
which follows from (97) for \( 1 \leq p' < \infty \), and it is obvious for \( p' = \infty \). Besides, according to (93) and (14) for \( n \geq n_0(\alpha, r, p) \)
\[
|\Theta_{\alpha,r,p,n}^{(3)}| \left( \frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{\alpha r} \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \left( \| \cos t \|_{p'} + |\delta^{(3)}_{n}| \left( \frac{1}{\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \right) \right) <
\]

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In view of formulas (115)–(117) we arrive at (15). Theorem 1 is proved.

4. Proof of lemma 1. It is obvious that for \(1 \leq s \leq \infty\)

\[
\inf_{\lambda \in \mathbb{R}} \| \phi(t) - \lambda \|_s \leq \| \phi \|_s,
\]

\[
\frac{1}{2} \| \phi(t + \frac{\pi}{n}) - \phi(t) \|_s \leq \sup_{h \in \mathbb{R}} \frac{1}{2} \| \phi(t + h) - \phi(t) \|_s
\]

and

\[
\sup_{h \in \mathbb{R}} \frac{1}{2} \| \phi(t + h) - \phi(t) \|_s \leq \inf_{\lambda \in \mathbb{R}} \| \phi(t) - \lambda \|_s.
\]

Hence, in order to proof lemma it suffices to verify the validity of formula (46) and relation

\[
\frac{1}{2} \| \phi(t + \frac{\pi}{n}) - \phi(t) \|_s \geq \| r \|_s \left( \frac{\| \cos t \|_s}{(2\pi)^{\frac{1}{2}}} - 14\pi \frac{M}{n} \right).
\]

(118)

First, we consider the case \(1 \leq s < \infty\). Let verify the validity of equality (46). Setting

\[
\phi_k(t) = g\left(\frac{k\pi}{n}\right) \cos(nt + \gamma) + h\left(\frac{k\pi}{n}\right) \sin(nt + \gamma), \quad k = -n + 1, n,
\]

we get

\[
\| \phi \|_s = \left( \sum_{k = -n + 1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k-1)\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \left( \sum_{k = -n + 1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k-1)\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} + \Theta^{(1)}(\frac{n}{\pi}) \left( \sum_{k = -n + 1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k-1)\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}}, \quad |\Theta^{(1)}| \leq 1.
\]

(120)

Let us find the estimate of first term in (120). It is obvious, that according to (119)

\[
\left( \sum_{k = -n + 1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k-1)\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \left( \sum_{k = -n + 1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k-1)\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \left( \sum_{k = -n + 1}^{n} r^s \left(\frac{k\pi}{n}\right) \int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \left| \cos \left( nt + \gamma - \text{arg} \left( g\left(\frac{k\pi}{n}\right) + ih\left(\frac{k\pi}{n}\right) \right) \right) \right|^s dt \right)^{\frac{1}{s}} = \]
\[
(r(t))^s = \left( \sum_{k=-n+1}^{n} r^s \left( \frac{k \pi}{n} \right) \left[ \frac{1}{n} \pi \cos t | \right] dt \right)^{\frac{1}{s}} = \left[ \sum_{k=-n+1}^{n} r^s \left( \frac{k \pi}{n} \right) \right]^{\frac{1}{s}},
\]
(121)

where \( r(t) \) is defined by formula (43), and \( i \) is imaginary unit.

Let us show that for any collection of points \( \xi_k, k = -n+1, n \), such that \( \frac{(k-1) \pi}{n} \leq \xi_k \leq \frac{k \pi}{n} \), for \( n \geq 4 \pi sM \) the following estimate is true

\[
\left( \sum_{k=-n+1}^{n} r^s(\xi_k) \frac{\pi}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(2)} \frac{M}{n} \right), \quad |\Theta_n^{(2)}| \leq 4.
\]
(122)

Indeed, since

\[
\sum_{k=-n+1}^{n} r^s(\xi_k) \frac{\pi}{n} = \pi \int_{-\pi}^{\pi} r^s(t) dt + \Theta_n^{(3)} \frac{\pi \sqrt{r^s}}{n}, \quad |\Theta_n^{(3)}| \leq \pi,
\]
and under the condition

\[
n \geq \frac{2 \pi \sqrt{r^s}}{\pi} \left( \int_{-\pi}^{\pi} r^s(t) dt + \Theta_n^{(3)} \frac{\pi \sqrt{r^s}}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(4)} \frac{\pi \sqrt{r^s}}{n s \|r\|_s^s} \right), \quad |\Theta_n^{(4)}| \leq 2,
\]
(124)

hence

\[
\left( \sum_{k=-n+1}^{n} r^s(\xi_k) \frac{\pi}{n} \right)^{\frac{1}{s}} = \|r\|_s \left( 1 + \Theta_n^{(4)} \frac{\pi \sqrt{r^s}}{n s \|r\|_s^s} \right), \quad |\Theta_n^{(4)}| \leq 2.
\]
(125)

It is easy to verify that

\[
\sqrt{r^s} = s \int_{-\pi}^{\pi} r^{s-1}(t)|r'(t)| dt \leq s \|r\|_s \|r'(t)\|_\infty,
\]
(126)

\[
\left| \frac{r'(t)}{r(t)} \right| = \left| \frac{g(t)g'(t) + h(t)h'(t)}{r^2(t)} \right| \leq \frac{|g'(t)| + |h'(t)|}{r(t)} \leq 2M, \quad t \in \mathbb{R},
\]
(127)

therefore

\[
\frac{\sqrt{r^s}}{\|r\|_s^s} \leq s \left\| \frac{r'(t)}{r(t)} \right\|_\infty \leq 2 s M.
\]
(128)

By virtue of (128), for \( n \geq 4 \pi sM \) the condition (123) is satisfied. Therefore, according to (125), the estimate (122) takes place. Setting in (122) \( \xi_k = \frac{k \pi}{n}, k = -n+1, n, \) in view of
(121) we obtain
\[
\left( \sum_{k=-n+1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} |\phi_k(t)|^s dt \right)^{\frac{1}{s}} = \|r\|_s \left( \left\| \frac{\cos t}{(2\pi)^{\frac{1}{s}}} \right\|_s + \Theta_n^{(5)} M_n \right), \quad |\Theta_n^{(5)}| \leq 4. \tag{129}
\]

Let us find upper estimate of the second term in (120). On the basis of (45) and (119)
\[
\phi(t) - \phi_k(t) =
\]
\[
= \left( r(t) - r\left(\frac{k\pi}{n}\right) \right) \left( g\left(\frac{k\pi}{n}\right) \cos(nt + \gamma) + \frac{h\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \sin(nt + \gamma) \right) +
\]
\[
+ r(t) \left( g\left(\frac{t}{r(t)}\right) - g\left(\frac{k\pi}{n}\right) \right) \cos(nt + \gamma) + \left( h\left(\frac{t}{r(t)}\right) - h\left(\frac{k\pi}{n}\right) \right) \sin(nt + \gamma) \right), \tag{130}
\]
therefore
\[
\left( \sum_{k=-n+1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}} \leq I_n^{(1)} + I_n^{(2)}, \tag{131}
\]
where
\[
I_n^{(1)} := \left( \sum_{k=-n+1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} \left| r(t) - r\left(\frac{k\pi}{n}\right) \right|^s \left( |\cos(nt + \gamma)| + |\sin(nt + \gamma)| \right)^s dt \right)^{\frac{1}{s}},
\]
\[
I_n^{(2)} := \left( \sum_{k=-n+1}^{n} \int_{\frac{k\pi}{n}}^{\frac{(k+1)\pi}{n}} r^s(t) \left( \left| g\left(\frac{t}{r(t)}\right) - g\left(\frac{k\pi}{n}\right) \right| \cos(nt + \gamma) +
\right.
\]
\[
\left. + \left| h\left(\frac{t}{r(t)}\right) - h\left(\frac{k\pi}{n}\right) \right| \sin(nt + \gamma) \right)^s dt \right)^{\frac{1}{s}}.
\]
Using obvious inequality
\[
|\cos t| + |\sin t| \leq \sqrt{2}, \tag{132}
\]
Lagrange theorem and relation (127), we have
\[
I_n^{(1)} \leq \sqrt{2} \left( \sum_{k=-n+1}^{n} \max_{\frac{k\pi}{n} \leq t \leq \frac{(k+1)\pi}{n}} \left| r(t) - r\left(\frac{k\pi}{n}\right) \right|^s \right)^{\frac{1}{s}} \leq
\]
\[
\leq \frac{\sqrt{2}\pi}{n} \sup_{t \in \mathbb{R}} \left| r'(t) \right| \left( \sum_{k=-n+1}^{n} \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r^s(t) \right)^{\frac{1}{s}}. \tag{133}
\]
It is follows from (122), (127) and (133), that for \( n \geq 4\pi sM \)

\[
I_n^{(1)} \leq 2\sqrt{2\pi} \frac{M}{n} (1 + \frac{4M}{n}) \|r\|_s \leq 2\sqrt{2\pi} \frac{M}{n} \left(1 + \frac{1}{n}\right) \|r\|_s = \frac{2\sqrt{2M(1 + \pi)}}{n} \|r\|_s. \tag{134}
\]

It is easy to see that

\[
I_n^{(2)} \leq \left( \sum_{k=-n+1}^{n} \left( \max_{\frac{k\pi}{n} \leq t \leq \frac{k\pi}{n} + \frac{\pi}{n}} \left\{ \left| \frac{g(t)}{r(t)} - \frac{g\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right| \cos(nt + \gamma) \right\} + \left| \frac{h(t)}{r(t)} - \frac{h\left(\frac{k\pi}{n}\right)}{r\left(\frac{k\pi}{n}\right)} \right| \sin(nt + \gamma) \right) \right)^{\frac{1}{s}} \tag{135}
\]

For any \( t_1, t_2 \in \mathbb{R} \) such that \( |t_1 - t_2| \leq \frac{\pi}{n} \) the following inequalities take place

\[
\left| \frac{g(t_1)}{r(t_1)} - \frac{g(t_2)}{r(t_2)} \right| \leq \frac{3\pi M}{n}, \tag{136}
\]

\[
\left| \frac{h(t_1)}{r(t_1)} - \frac{h(t_2)}{r(t_2)} \right| \leq \frac{3\pi M}{n}. \tag{137}
\]

Indeed, by virtue of Lagrange theorem, taking into account (44) and (127), we have

\[
\left| \frac{g(t_1)}{r(t_1)} - \frac{g(t_2)}{r(t_2)} \right| \leq \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{g'(t)r(t) - g(t)r'(t)}{r^2(t)} \right| \leq \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{g'(t)}{r(t)} \right| + \frac{\pi}{n} \sup_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \leq \frac{3\pi M}{n}. \tag{138}
\]

By analogy, we prove the inequality (137). In view of (132), (136), (137) and (135) we obtain

\[
I_n^{(2)} \leq \frac{3\sqrt{2\pi} M}{n} \|r\|_s, \quad n \in \mathbb{N}. \tag{139}
\]

Combining (131), (134) and (139), we arrive at the estimate

\[
\left( \sum_{k=-n+1}^{n} \frac{\pi}{n} \int_{\frac{k\pi}{n}}^{\frac{k\pi}{n} + \frac{\pi}{n}} |\phi(t) - \phi_k(t)|^s dt \right)^{\frac{1}{s}} \leq \sqrt{2(5\pi + 2)} \|r\|_s \frac{M}{n}, \quad n \geq 4\pi sM. \tag{140}
\]

By comparing estimates (120), (129) and (140) we conclude that for \( n \geq 4\pi sM \)

\[
\|\phi\|_s = \|r\|_s \left( \frac{\|\cos t\|_s + \delta(1)}{(2\pi)^{\frac{1}{s}} s,n} \right), \quad |\delta(1)| \leq \sqrt{2(5\pi + 2)} + 4, \quad 1 \leq s < \infty. \tag{141}
\]

Further, we prove the relation (118) for \( 1 \leq s < \infty \). In view of definition (45)

\[
|\phi(t + \frac{\pi}{n}) - \phi(t)| =
\]

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Thus, the validity of formula (118) is established for $1 \leq s < \infty$. Let us prove the relation (46) for $s = \infty$. Consider a function $\phi^*(t)$ such that

$$\phi^*(t) = \phi_k^*(t), \quad \frac{(k - 1)\pi}{n} \leq t \leq \frac{k\pi}{n}, \quad k = -n + 1, n,$$
where
\[ \phi_k^*(t) = g(t_k^*) \cos(nt + \gamma) + h(t_k^*) \sin(nt + \gamma), \] (146)
and points \( t_k^*, \ t_k^* \in \left[\frac{(k-1)\pi}{n}, \frac{k\pi}{n}\right] \) are chosen from the condition
\[ r(t_k^*) = \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r(t). \]
For the function \( \phi^*(t) \) the following equality takes place
\[ \|\phi^*\|_\infty = \|r\|_C. \] (147)
Indeed,
\[
\|\phi^*\|_\infty = \max_{-n+1 \leq k \leq n} \quad \text{ess sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} |\phi^*(t)| = \\
= \max_{-n+1 \leq k \leq n} r(t_k^*) \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right| = \\
= \max_{-n+1 \leq k \leq n} r(t_k^*) \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| \cos \left( nt + \gamma - \arctan\left( \frac{g(t_k^*) + ih(t_k^*)}{r(t_k^*)} \right) \right) \right| = \\
= \max_{-n+1 \leq k \leq n} r(t_k^*) \|\cos t\|_C = \|r\|_C.
\]
It is obvious that in view of (147) we obtain
\[ \|\phi\|_\infty = \|\phi^*\|_\infty + \Theta_n^{(6)} \|\phi - \phi^*\|_\infty = \|r\|_C + \Theta_n^{(6)} \|\phi - \phi^*\|_\infty, \quad |\Theta_n^{(6)}| \leq 1. \] (148)
Let us find upper estimate for the quantity \( \|\phi - \phi^*\|_\infty \). By virtue of (45) and (146), for any \( t \in \left[\frac{(k-1)\pi}{n}, \frac{k\pi}{n}\right] \) the following equality takes place
\[
|\phi(t) - \phi_k^*(t)| = \left| (r(t) - r(t_k^*)) \left( \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right) + \\
r(t) \left( \left( \frac{g(t)}{r(t)} - \frac{g(t_k^*)}{r(t_k^*)} \right) \cos(nt + \gamma) + \left( \frac{h(t)}{r(t)} - \frac{h(t_k^*)}{r(t_k^*)} \right) \sin(nt + \gamma) \right) \right|.
\] (149)
By using (132), the Lagrange theorem and inequality (127), we get
\[
\text{ess sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| (r(t) - r(t_k^*)) \left( \frac{g(t_k^*)}{r(t_k^*)} \cos(nt + \gamma) + \frac{h(t_k^*)}{r(t_k^*)} \sin(nt + \gamma) \right) \right| \leq \\
\leq \sqrt{2} \text{ess sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \left| r(t) - r(t_k^*) \right| \leq \frac{\sqrt{2}\pi}{n} \text{sup}_{t \in \mathbb{R}} \left| \frac{r'(t)}{r(t)} \right| \|r\|_C \leq \frac{2\sqrt{2}\pi M}{n} \|r\|_C. \] (150)
Further, it follows from (132), (136) and (137) that
\[
\text{ess sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r(t) \left( \left| \frac{g(t)}{r(t)} - \frac{g(t_k^*)}{r(t_k^*)} \right| \cos(nt + \gamma) + \left| \frac{h(t)}{r(t)} - \frac{h(t_k^*)}{r(t_k^*)} \right| \sin(nt + \gamma) \right) \leq 
\]
\[ \|\phi - \phi^*\|_\infty = \max_{-n+1 \leq k \leq n} \text{ess sup}_{\frac{k}{n} \leq t \leq \frac{k+1}{n}} |\phi(t) - \phi_k^*(t)| \leq 5\sqrt{2\pi} \frac{M}{n}\|r\|_C, \quad n \in \mathbb{N}. \] (152)

It follows from (148), (147) and (152) that

\[ \|\phi\|_\infty = \|r\|_C \left(1 + \delta^{(1)}_{\infty,n} \frac{M}{n}\right), \quad |\delta^{(1)}_{\infty,n}| \leq 5\sqrt{2\pi}. \] (153)

Let us prove inequality (118) for \( s = \infty \). By using the inequality (143) for \( s = \infty \), by applying Lagrange theorem, formulas (127) and (153), we obtain

\[
\frac{1}{2}\|\phi(t + \frac{\pi}{n}) - \phi(t)\|_\infty \geq \|\phi\|_\infty - \frac{1}{2\sqrt{2}} \left(\|r(t + \frac{\pi}{n}) - r(t)\|_\infty + 3\pi\|r\|_C \frac{M}{n}\right) \geq \|\phi\|_\infty - \frac{1}{2\sqrt{2}} \left(\frac{n}{\pi} \sup_{t \in \mathbb{R}} |r'(t)| \|r\|_C + 3\pi\|r\|_C \frac{M}{n}\right) > \|r\|_C \left(1 - \frac{15\pi M}{\sqrt{2} n}\right).
\]

Lemma 1 is proved.

**Remark 1.** In proving of lemma 1 we established more exact, than (49) estimates of quantities \( \delta^{(i)}_{s,n}, i = 1, 3 \). Namely, we showed that for \( n \geq \left\{ \begin{array}{ll}
4\pi s M, & 1 \leq s < \infty, \\
1, & s = \infty,
\end{array} \right. \) the following estimates hold

\[
|\delta^{(1)}_{s,n}| \leq \left\{ \begin{array}{ll}
\sqrt{2}(5\pi + 2) + 4, & 1 \leq s < \infty, \\
5\sqrt{2\pi}, & s = \infty,
\end{array} \right.
\]

\[ -\frac{15\pi + 6}{\sqrt{2}} - 4 \leq \delta^{(2)}_{s,n} \leq \sqrt{2}(5\pi + 2) + 4, \quad i = 2, 3, 1 \leq s < \infty,
\]

\[ -\frac{15\pi}{\sqrt{2}} \leq \delta^{(3)}_{s,n} \leq 5\sqrt{2\pi}, \quad i = 2, 3, s = \infty.
\]
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