INTEGRABLE REDUCTIONS OF THE DRESSING CHAIN

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Abstract. In this paper we construct a family of integrable reductions of the dressing chain, described in its Lotka-Volterra form. For each \(k, n \in \mathbb{N}\) with \(n \geq 2k + 1\) we obtain a Lotka-Volterra system \(LV_b(n, k)\) on \(\mathbb{R}^n\) which is a deformation of the Lotka-Volterra system \(LV(n, k)\), which is itself an integrable reduction of the \(2m+1\)-dimensional Bogoyavlenskij-Itoh system \(LV(2m+1, m)\), where \(m = n - k - 1\). We prove that \(LV_b(n, k)\) is both Liouville and non-commutative integrable, with rational first integrals which are deformations of the rational first integrals of \(LV(n, k)\). We also construct a family of discretizations of \(LV_b(n, 0)\), including its Kahan discretization, and we show that these discretizations are also Liouville and superintegrable.

1. Introduction. The dressing chain is an integrable Hamiltonian system, which was constructed in [21] as a fixed point of compositions of Darboux transformations of the Schrödinger operator. It was shown in [7] that after a simple linear transformation it becomes a Lotka-Volterra system which is a deformation of the Bogoyavlenskij-Itoh system [12, 4]. For the integrable reductions of the dressing chain which we will study here, the latter formulation is the most convenient; also, we will use many results from [19, 14, 6, 7], which are all written in that formulation.

For integers \(n\) and \(k\), satisfying \(n \geq 2k + 1\), the Hamiltonian system \(LV(n, k)\) has as its phase space \(\mathbb{R}^n\), which we equip with its natural coordinates \(x_1, \ldots, x_n\). It has as Hamiltonian \(H\) the sum of these coordinates, \(H := x_1 + \cdots + x_n\), and as Poisson structure a quadratic Poisson structure, with brackets

\[
\{x_i, x_j\} := A_{i,j}^{(n,k)} x_i x_j , \quad \text{where} \quad A_{i,j}^{(n,k)} = \begin{cases} 
+1 & \text{if } i + n > j + k , \\
-1 & \text{if } i + n \leq j + k ,
\end{cases}
\]
when $1 \leq i < j \leq n$. The Hamiltonian vector field $X_H$ has the form

$$
\dot{x}_i = \sum_{j=1}^{n} A_{i,j}^{(n,k)} x_i x_j , \quad 1 \leq i \leq n .
$$

These systems, for $k = 0$, were introduced in [19] where their Liouville and superintegrability was established with rational first integrals. For $n = 2k + 1$ one recovers the Bogoyavlenskij-Itoh system [1] whose deformation, which is the dressing chain [21], was constructed in [7]. We denote this deformation by $LV_b(2k+1, k)$. The observation that $LV(n, 0)$ can be obtained by a reduction from a Bogoyavlenskij-Itoh system $LV(2n-1, n-1)$ led us to study the more general case $LV(n, k)$ with $k > 0$, since these systems can be obtained by a similar reduction from a Bogoyavlenskij-Itoh system $LV(2m+1, m)$, with $m := n-k-1$. The systems $LV(n, k)$ for $k > 0$, were studied in detail in [6], where their Liouville and non-commutative integrability was proven (see Definition 4.1), again with rational first integrals. The same reduction can be applied to the deformed systems $LV_b(2m+1, m)$, leading to Hamiltonian systems, which we will denote by $LV_b(n, k)$. The Hamiltonian vector field $X_H$ now has the form

$$
\dot{x}_i = \sum_{j=1}^{n} \left( A_{i,j}^{(n,k)} x_i x_j + B_{i,j}^{(n,k)} \right) , \quad 1 \leq i \leq n ,
$$

where all entries $b_{i,j}$ of the skew-symmetric matrix $B^{(n,k)}$, satisfying $|i-j| \notin \{m, m+1\}$, are zero and the other entries are arbitrary parameters. Setting in these systems all deformation parameters equal to zero, one recovers $LV(n, k)$. A natural question, studied here, is the integrability of $LV_b(n, k)$ for all $n$ and $k$ with $n \geq 2k+1$. For $LV_b(2k+1, k)$ the answer is known [21, 7]: $LV_b(2k+1, k)$ is Liouville integrable with polynomial first integrals which are deformations of the first integrals of $LV(2k+1, k)$.

The main result of this paper is that $LV_b(n, k)$ is on the one hand Liouville integrable, with rational first integrals which are deformations of the first integrals of $LV(n, k)$, and is on the other hand non-commutative integrable, with such first integrals. See Theorem 3.4 for the case of $(n, 0)$ and Theorem 4.8 for the case of $(n, k)$ with $k > 0$. In order to establish these results, we need to construct the deformed first integrals and show that they have the desired involutivity properties: independence is in fact quite automatic and is proven by a simple deformation argument.

Surprisingly, the construction of the deformed first integrals from the undeformed ones is very simple, and is the same for all first integrals of $LV(n, k)$ that were constructed in [6]: from such a first integral $F$ of $LV(n, k)$ we obtain a first integral $F^b$ of $LV_b(n, k)$ by setting $F^b := e^{D_b} F = F + D_b F + \frac{D_b^2}{2} F + \cdots$, where

$$
D_b = \sum_{1 \leq i \leq k+1} b_{i,i+m} \frac{\partial^2}{\partial x_i \partial x_{i+m}} - \sum_{1 \leq i \leq k} b_{i,i+m+1} \frac{\partial^2}{\partial x_i \partial x_{i+m+1}} , \quad (1)
$$

where we recall that $m = n-k-1$. Notice that $H^b = H$ because $e^{D_b}$ acts on linear polynomials as identity.

The proof that we get in this way first integrals and that they are in involution when the corresponding undeformed first integrals are in involution needs however extra work, as it does not follow directly from their definition. In the case of $LV_b(n, 0)$, studied in Section 3, there is only one deformation parameter $\beta := b_{1,n}$ and the above action of $e^{D_b}$ on the rational first integrals of $LV(n, 0)$ which were
constructed in [19] can be equivalently described as the pullback of a birational map, which we introduce. Moreover, we show that this map is a Poisson map between the deformed and undeformed systems (Proposition 3.2). This yields the integrability results for LV_k(n, 0), since apart from the Hamiltonian, all constructed first integrals are rational; the fact that these rational first integrals are in involution with the Hamiltonian, i.e. are first integrals, can in this case be shown by direct computation (Proposition 3.1).

When k > 0 the above idea can also be used, but some care has to be taken because there are now 2k + 1 deformation parameters, and they can be added one by one, upon decomposing \( D_b = \sum_{p=1}^{2k+1} D_{(p)} \), but in order to be able to view at each step the action of \( e^{D_{(p)}} \) on the rational first integrals as the pullback by some Poisson map, one has to add the parameters in a very specific order. The reason for this is that in this process the form of the rational first integrals at each step is very important. With this, one gets that the deformed rational first integrals of LV_k(n, k) are in involution (second part of Theorem 4.2). This system has \( k + 1 \) independent polynomial first integrals, which are by construction in involution, because they are restrictions to a Poisson submanifold of the involutive first integrals of LV_k(2m + 1, m), but they also have to be shown to be in involution with the rational first integrals. This is again done by using the above Poisson maps, but since these maps do not produce the deformed polynomial first integrals, some extra arguments which are again very much dependent on the particular structure of the first integrals, are needed. In the end, this proves Theorem 4.8 which says that the deformed systems LV_k(n, k) are both Liouville and non-commutatively integrable.

Several, a priori different, discretizations of the dressing chain LV_k(2m + 1, m) have been constructed and studied in the literature [2, 18, 9, 8]. We will construct in the final section of this paper a class of discretizations of the deformed Lotka-Volterra systems LV_k(n, k), with k = 0. In order to construct them, we start from the compatibility conditions of a linear problem associated with the Lax operator of the dressing chain. Upon reducing these conditions to LV_k(n, 0), as in the continuous case, we can easily solve the compatibility conditions, and hence construct the discrete maps explicitly. We prove that these discrete maps preserve the Poisson structure, the Hamiltonian and all rational first integrals of LV_k(n, 0). These discretizations are therefore both Liouville and superintegrable. We show that the Kahan discretization of LV_k(n, 0) is a particular instance of the discretizations that we construct, thereby showing that the Kahan map of LV_k(n, 0) arises as the compatibility condition of a linear system. For k > 0 the reduction can also be performed, leading to an integrable discretization of LV_k(n, k), but the proof is rather long and complicated, so it will not be given here. It is worthwhile pointing out that when k > 0 the Kahan discretization of LV_k(n, k) is not a particular case of this discretization.

The structure of the paper is as follows. We construct in Section 2 the systems LV_k(n, k) as Poisson reductions of the systems LV_k(2m + 1, m), with m := n − k − 1, and we show that the inherited Poisson structure \( \Pi_b^{(n,k)} \) is a deformation of the Poisson structure \( \Pi(n,k) \) of LV(n, k). In Section 3 we construct rational first integrals of LV_k(n, 0) as deformations of the first integrals of LV(n, 0), which were constructed in [19]. We show by using a Poisson map, which we also construct, that half of these first integrals are in involution, establishing both the Liouville and superintegrability of LV_k(n, 0). We also give explicit solutions for this system. In Section 4 we treat the more complicated case of k > 0, where we prove again Liouville integrability,
and also non-commutative integrability. In this case we use $2k + 1$ Poisson maps, which are composed in a very specific order to obtain the results. In Section 5 we construct a family of discrete maps for $LV_b(n, 0)$ as compatibility conditions for a linear system, associated to the Lax operator of $LV_b(n, 0)$, and show their Liouville and superintegrability. We show that the Kahan discretization of $LV_b(n, 0)$ is a particular case and deduce from this the Liouville and superintegrability of the Kahan discretization.

2. The Hamiltonian systems $LV_b(n, k)$. In this section, we construct the polynomial Hamiltonian systems $LV_b(n, k)$. Recall that $n$ and $k$ stand for two arbitrary integers satisfying $n \geq 2k + 1$. We construct them as reductions of the deformed Bogoyavlenskij-Itoh systems, which we introduced in [7]; in the notation of the present paper, the latter systems are the systems $LV_b(2m + 1, m)$, where $m := n - k - 1$.

2.1. The deformed Bogoyavlenskij-Itoh systems. We first recall the Bogoyavlenskij-Itoh systems $LV(2m + 1, m)$, which have first been introduced by O. Bogoyavlenskij [3, 4] and Y. Itoh [12], and their deformations $LV_b(2m + 1, m)$, which we constructed in [7]. In both cases, the phase space of the system is $\mathbb{R}^{2m+1}$, which is equipped with its natural coordinates $x_1, \ldots, x_{2m+1}$. Since many formulas are invariant under a cyclic permutation of these coordinates, we view the index of $x$ as being taken modulo $2m + 1$, i.e., we set $x_{2m+\ell+1} = x_{\ell}$ for all $\ell \in \mathbb{Z}$. The Poisson structure $\Pi^m$ of $LV(2m + 1, m)$ is constructed from the skew-symmetric Toeplitz matrix\(^1\) $A^m$ whose first row is given by

$$(0, 1, 1, \ldots, 1, -1, -1, \ldots, -1).$$

It leads to a quadratic Poisson structure $\Pi^m$ on $\mathbb{R}^{2m+1}$, upon defining the Poisson brackets

$$\{x_i, x_j\}^m := A^m_{ij} x_i x_j, \quad 1 \leq i, j \leq 2m + 1.$$  

As Hamiltonian we take $H := x_1 + x_2 + \cdots + x_{2m+1}$, the sum of all coordinates. The corresponding Hamiltonian system is given by

$$\dot{x}_i = x_i \sum_{j=1}^m (x_{i+j} - x_{i-j}), \quad 1 \leq i \leq 2m + 1. \quad (2)$$

It is called the Bogoyavlenskij-Itoh system, and is denoted by $LV(2m + 1, m)$. Given any real skew-symmetric matrix $B^m$ of size $2m + 1$, define

$$\{x_i, x_j\}_B^m := A^m_{ij} x_i x_j + B^m_{ij}, \quad 1 \leq i, j \leq 2m + 1. \quad (3)$$

These brackets define a Poisson structure, denoted $\Pi_B^m$, if and only if all entries $b_{i,j} := B_{ij}^m$ of $B^m$, with $|j - i| \notin \{m, m + 1\}$ are zero (see [7, Prop. 3]). Under this condition on $B^m$, we can consider the Hamiltonian system on $\mathbb{R}^{2m+1}$ with the same Hamiltonian $H$ and Poisson structure $\Pi_B^m$. It is given by

$$\dot{x}_i = x_i \sum_{j=1}^m (x_{i+j} - x_{i-j}) + b_{i,i+m} - b_{i,-m,i}, \quad 1 \leq i \leq 2m + 1. \quad (4)$$

\(^1\)Later on, the matrix $A^m$, and similarly the matrix $B^m$ and the Poisson structure $\Pi^m$, will have two superscripts; in that notation, $A^m$ is written as $A^{(2m+1,m)}$, and similarly for $B^m$ and $\Pi^m$. 

It is called the deformed Bogoyavlenskij-Itoh system, and is denoted by \( \text{LV}_b(2m + 1, m) \). It is clear that setting all parameters \( b_{i,j} \) equal to zero, one recovers \( \text{LV}(2m + 1, m) \). A Lax equation (with spectral parameter) for (4) is given by

\[
(X + \lambda^{-1} \Delta + \lambda M)' = [X + \lambda^{-1} \Delta + \lambda M, D - \lambda M^{m+1}],
\]

where for \( 1 \leq i, j \leq 2m + 1 \) the \((i, j)\)-th entry of the matrices \( X \) and \( M \), and of the diagonal matrices \( \Delta \) and \( D \), is given by

\[
X_{i,j} := x_{i} \delta_{i,j+m}, \quad \Delta_{i,j} := b_{i+m,j} \delta_{i,j}, \quad M_{i,j} := \delta_{i+1,j}, \quad D_{i,j} := -\delta_{i,j}(x_{i} + x_{i+1} + \cdots + x_{i+m}),
\]

and the indices of \( b, x \) and \( \delta \) are considered modulo \( 2m + 1 \). It generalizes Bogoyavlenskij’s Lax equation, which can be recovered from it by putting all \( b_{i,j} \) equal to zero, i.e., by setting \( \Delta = 0 \).

2.2. The reduced systems. The systems \( \text{LV}_b(n, k) \), with \( n > 2k + 1 \), are obtained by reduction from \( \text{LV}_b(2m + 1, m) \), where \( m := n - k - 1 > k \). Consider the submanifold \( N_n \) of \( \mathbb{R}^{2m+1} \), defined by \( x_{n+1} = x_{n+2} = \cdots = x_{2m+1} = 0 \). It is a linear space of dimension \( n \) which we identify with \( \mathbb{R}^n \) and on which we take the restrictions of \( x_1, x_2, \ldots, x_n \) as coordinates (without changing the notation).

**Proposition 2.1.** The submanifold \( N_n \) of \( \mathbb{R}^{2m+1} \) is a Poisson submanifold of \( (\mathbb{R}^{2m+1}, \Pi^m_b) \) if and only if the entries of the skew-symmetric matrix \( B^m \) satisfy \( b_{i,j} = 0 \) whenever \( n + 1 \leq i \leq 2m + 1 \).

**Proof.** The submanifold \( N_n \) is a Poisson submanifold of \( (\mathbb{R}^{2m+1}, \Pi^m_b) \) if and only if all Hamiltonian vector fields \( X_F := \{ \cdot, F \}_b \) are tangent to \( N_n \) at all points of \( N_n \).

This is equivalent to the vanishing of \( X_F[x_i] = \{ x_i, F \}_b \) at all points of \( N_n \), for any \( i \) with \( n + 1 \leq i \leq 2m + 1 \). For such \( i \), by the derivation property of the Poisson bracket, we have

\[
X_F[x_i] = \{ x_i, F \}_b = \sum_{j=1}^{2m+1} \frac{\partial F}{\partial x_j} (A_{i,j}^m x_j + b_{i,j}),
\]

which equals \( \sum_{j=1}^{2m+1} \frac{\partial F}{\partial x_j} b_{i,j} \) on \( N_n \). This clearly vanishes, for all functions \( F \) on \( \mathbb{R}^{2m+1} \), if and only if \( b_{i,j} = 0 \) for all \( j \). \( \square \)

Assuming that \( B^m \) verifies the assumptions of Proposition 2.1, \( N_n \simeq \mathbb{R}^n \) is a Poisson submanifold of \( (\mathbb{R}^{2m+1}, \Pi^m_b) \), and we can restrict \( \Pi^m_b \) (as given by (3)) to \( \mathbb{R}^n \), giving a Poisson structure \( \Pi^m_{b}(n,k) \), with associated Poisson bracket

\[
\{ x_i, x_j \}_{b}^{(n,k)} := A_{i,j}^{(n,k)} x_i x_j + B_{i,j}^{(n,k)}, \quad 1 \leq i, j \leq n, \tag{6}
\]

where \( A^{(n,k)} \) and \( B^{(n,k)} \) denote the \( n \times n \) matrices obtained from \( A^m \) and \( B^m \) by removing its last \( 2m + 1 - n \) rows and columns. Said differently, \( A^{(n,k)} \) denotes the skew-symmetric \( n \times n \) Toeplitz matrix whose first row is given by

\[
(0, 1, 1, \ldots, 1, -1, -1, \ldots, -1). \tag{m=n−k−1 \atop k}
\]

Thus, all upper triangular entries of the skew-symmetric matrix \( A^{(n,k)} \) are \( \pm 1 \) and \( A_{i,j}^{(n,k)} = 1 \) if and only if \( n + i > k + j \). Also, \( B^{(n,k)} \) is the skew-symmetric \( n \times n \)
matrix whose uppertriangular entries \( b_{i,j} := B^{(n,k)}_{i,j} \) with \( j - i \not\in \{m, m+1\} \) are zero. So, when \( k > 0 \), the first line of \( B^{(n,k)} \) is given by

\[
(0, 0, \ldots, 0, b_{1,m+1}, b_{1,m+2}, 0, 0, \ldots, 0),
\]

while for \( k = 0 \) it has the form \( (0, 0, \ldots, 0, b_{1,n}) \). We define \( \text{LV}_k(n, k) \) to be the Hamiltonian system with \( \Pi^{(n,k)}_b \) as Poisson structure and \( H = x_1 + x_2 + \cdots + x_n \) as Hamiltonian. Explicitly, the Hamiltonian vector field \( \mathcal{X}_H \) of \( \text{LV}_k(n, k) \) is given by

\[
\dot{x}_i = \sum_{j=1}^{n} \left( A^{(n,k)}_{i,j} x_j + B^{(n,k)}_{i,j} \right), \quad 1 \leq i \leq n.
\] (7)

Setting \( B^{(n,k)} = 0 \), one recovers the Hamiltonian system \( \text{LV}(n, k) \), in particular its Poisson structure \( \Pi^{(n,k)} \), which was constructed and studied in [6]. Therefore, the system \( \text{LV}_k(n, k) \) is a deformation of the system \( \text{LV}(n, k) \).

We show in the following proposition that the above matrices \( B^{(n,k)} \) are the only ones for which the brackets given by (6) define a Poisson bracket (on \( \mathbb{R}^n \)).

**Proposition 2.2.** Suppose that \( B = (b_{i,j}) \) is a skew-symmetric \( n \times n \) matrix. Then the brackets, given by

\[
\{x_i, x_j\}_b := A^{(n,k)}_{i,j} x_i x_j + b_{i,j}, \quad 1 \leq i, j \leq n,
\]

define a Poisson structure \( \Pi_b \) on \( \mathbb{R}^n \) if and only if all uppertriangular entries \( b_{i,j} \) of \( B \), with \( j - i \not\in \{m, m+1\} \) are zero. The rank of \( \Pi_b \) is \( 2 \left\lfloor \frac{n}{2} \right\rfloor \).

**Proof.** Let us denote by \( \Pi \) the Poisson structure defined by \( A^{(n,k)} \), and let us denote the derived Poisson bracket by \( \{\cdot, \cdot\} \). The constant Poisson bracket defined by \( B \) is denoted by \( \{\cdot, \cdot\}_B \). We know already from Proposition 2.1 that if all uppertriangular entries \( b_{i,j} \) of \( B \), with \( j - i \not\in \{m, m+1\} \) are zero, then \( \Pi_b \) is the restriction of a Poisson structure to a Poisson submanifold, hence it is a Poisson structure. We therefore only need to show that if one of these entries \( b_{i,j} \) with \( i < j \) is non-zero, then \( \Pi_b \) is not a Poisson structure. Suppose first that \( 0 < j - i < m \) and \( i \neq 1 \). Then

\[
\frac{\partial}{\partial x_{i-1}} \left[ \{ x_{i-1}, x_i \}_B + \{ x_i, x_{i-1} \}_B + \{ x_j, x_{i-1} \}_B \right] = A^{(n,k)}_{i-1,j} b_{i,j} + A^{(n,k)}_{i-1,1} b_{i,j} = 2b_{i,j} \neq 0,
\]

so that \( \Pi_b \) does not satisfy the Jacobi identity. When \( 0 < j - i < m \) and \( i = 1 \) it suffices to replace in the above computation \( i - 1 \) by \( j + 1 \) to arrive at the same conclusion. Finally, when \( j - i > m + 1 \) one replaces in the above computation \( i - 1 \) by \( i + 1 \) to arrive again at the same conclusion.

The rank of \( \Pi \) is the rank of \( A^{(n,k)} \), which is equal to \( n \) when \( n \) is even and \( n - 1 \) when \( n \) is odd. Since \( \Pi_b \) is obtained by adding constants to the quadratic structure \( \Pi \), its rank is at least the rank of \( A^{(n,k)} \). However, the rank of \( \Pi_b \) is even and bounded by \( n \), so \( \Pi_b \) and \( \Pi \) have the same rank, which is \( 2 \left\lfloor \frac{n}{2} \right\rfloor \).

The proposition implies that from the above reduction process we get all possible deformations of \( \text{LV}(n, k) \) obtained by adding to \( \Pi^{(n,k)} \) a constant Poisson structure.
3. The Liouville and superintegrability of LVₜ(n, 0). We construct in this section enough independent first integrals for the Hamiltonian system LVₜ(n, 0) to prove its superintegrability and then select from them enough first integrals in involution to prove its Liouville integrability. Notice that since the phase space of LVₜ(n, 0) is Rⁿ and since the Poisson structure on it has rank n or n − 1, depending on whether n is even or odd, we need to provide n − 1 independent first integrals to prove superintegrability and \([\frac{n+1}{2}]\) independent first integrals (including the Hamiltonian) in involution to prove Liouville integrability. Throughout the section, n is fixed, and k = 0 also, so we will drop from the notations the label (n, 0), except in the statements of the propositions and the theorem.

3.1. First integrals. We first write down the equations for the vector field \(X_H\) where we recall that \(H = x_1 + x_2 + \cdots + x_n\) and that the Poisson structure \(\Pi_b = \Pi^{(n,0)}_b\) is defined by (6); the matrix \(B = B^{(n,0)}\) has all entries equal to zero, except for \(b_1n = -b_{n,1}\), which we will denote in this section by \(\beta\). Also, the skew-symmetric matrix \(A = A^{(n,0)}\) has all its upper triangular entries equal to (plus!) 1. Therefore, \(X_H\) is given by

\[
\begin{align*}
\dot{x}_1 &= x_1(x_2 + x_3 + \cdots + x_n) + \beta, \\
\dot{x}_i &= x_i(-x_1 - \cdots - x_{i-1} + x_{i+1} + \cdots + x_n), \quad 1 < i < n, \\
\dot{x}_n &= x_n(-x_1 - x_2 - \cdots - x_{n-1}) - \beta.
\end{align*}
\]

We construct the first integrals of this system as deformations of the first integrals of LV(n, 0), which were constructed in [19, Prop. 3.1]. We first recall the formulas for these first integrals. For \(1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor\), the following functions \(F_\ell = F^{(n,0)}_\ell\) are first integrals of LV(n, 0):

\[
F_\ell := \begin{cases} 
(x_1 + x_2 + \cdots + x_{2\ell-1}) \frac{x_2 x_3 x_4 \cdots x_{n-1}}{x_{2\ell} x_{2\ell+1} \cdots x_n}, & \text{if } n \text{ is odd}, \\
(x_1 + x_2 + \cdots + x_{2\ell}) \frac{x_2 x_4 x_5 \cdots x_n}{x_{2\ell+1} x_{2\ell+2} \cdots x_{n-1}}, & \text{if } n \text{ is even}.
\end{cases}
\]

More first integrals were constructed by using the anti-Poisson involution \(\iota\) on \(\mathbb{R}^n\), defined by

\[
\iota(a_1, a_2, \ldots, a_n) := (a_n, a_{n-1}, \ldots, a_1),
\]

which leaves \(H\) invariant, \(\iota^*H := H \circ \iota = H\), so that the rational functions \(G_\ell := \iota^*F_\ell\) (\(\ell = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor\)) are also first integrals of LV(n, 0). This yields exactly \(n - 1\) different first integrals, because when \(n\) is even, all \(F_\ell\) and \(G_\ell\) are different, except for \(F_{n/2} = H = G_{n/2}\), and when \(n\) is odd, all \(F_\ell\) and \(G_\ell\) are different, except for \(F_{(n+1)/2} = H = G_{(n+1)/2}\) and \(F_1 = G_1\). We recall also that the functions \(F_\ell\) are pairwise in involution, just like the functions \(G_\ell\), and that all these functions are independent, which accounts for the Liouville and superintegrability of LV(n, 0).

In order to construct from these first integrals of LV(n, 0) first integrals of LVₜ(n, 0) we use the constant coefficient differential operator \(\mathcal{D}_b\), which we define by

\[
\mathcal{D}_b = \beta \frac{\partial^2}{\partial x_1 \partial x_n}.
\]

Notice that \(\iota^*\) and \(\mathcal{D}_b\) commute, \(\iota^* \circ \mathcal{D}_b = \mathcal{D}_b \circ \iota^*\). As said, we use the operator \(\mathcal{D}_b\) to define some first integrals of LVₜ(n, 0): we define for \(1 \leq \ell \leq \lfloor \frac{n+1}{2} \rfloor\) the functions

\[

\]
We have used that when the operator $\mathcal{D}_b$ is applied twice to $F_\ell$, the result is zero. This follows from the fact that the variables $x_1$ and $x_n$ appear linearly in $F_\ell$ (and hence are absent in $\mathcal{D}_b F_\ell$), as is clear from (9). Explicit formulas for the rational functions $F^b_\ell$, with $1 \leq \ell \leq \left[\frac{n-1}{2}\right]$ are given by

$$F^b_\ell := F^{(n,0),b}_\ell$$

and similarly for $G^b_\ell$. Also, for $\ell = \left[\frac{n+1}{2}\right]$ the above definitions (11) amount to $F^b_\ell = G^b_\ell = H$.

**Proposition 3.1.** For $\ell = 1, \ldots, \left[\frac{n+1}{2}\right]$, the rational functions $F^{(n,0),b}_\ell$ and $G^{(n,0),b}_\ell$ are first integrals of $LV_0(n,0)$.

**Proof.** Since $i$ is an anti-Poisson map which leaves the Hamiltonian $H$ invariant, it suffices to show that the rational functions $F^b_\ell$ are first integrals of $LV_0(n,0)$. We do this for odd $n$, the case of even $n$ being completely analogous. Let $1 \leq \ell \leq \left[\frac{n-1}{2}\right]$. To prove that $F^b_\ell$ is a first integral of (8) we show that its logarithmic derivative $(\log(F^b_\ell))' = \frac{\dot{F}^b_\ell}{F^b_\ell}$ is zero. Thanks to the particular form of the vector field (8), one easily obtains the following two formulas:

$$\left(\log \left( x_1 + x_2 + \cdots + x_{2\ell-1} + \frac{\beta}{x_n} \right) \right)' = x_2\ell + x_{2\ell+1} + \cdots + x_n + \frac{\beta}{x_n},$$

$$\left(\log \left( \frac{x_{2\ell+1}}{x_{2\ell}} \cdots \frac{x_n}{x_{n-1}} \right) \right)' = -x_{2\ell} - x_{2\ell+1} - \cdots - x_n - \frac{\beta}{x_n}.$$

Summing them up, we find $(\log(F^b_\ell))' = 0$, and hence that $\dot{F}^b_\ell = 0$. \qed

### 3.2. Involutivity

We now show that the first integrals $F^b_\ell$ of $LV_0(n,0)$ are in involution. For doing this, observe by comparing (9) and (12) that formally $F^b_\ell$ can be obtained from $F_\ell$ by replacing $x_1$ with $x_1 + \beta/x_n$. Said differently, if we denote by $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ the birational map defined for $(a_1, \ldots, a_n)$ with $a_n \neq 0$ by

$$\sigma(a_1, \ldots, a_n) := \left( a_1 + \frac{\beta}{a_n}, a_2, \ldots, a_n \right),$$

then $\sigma^* F_\ell = F^b_\ell$.

**Proposition 3.2.** The birational map $\sigma : \left( \mathbb{R}^n, \{\cdot, \cdot\}_b^{(n,0)} \right) \to \left( \mathbb{R}^n, \{\cdot, \cdot\}_b^{(n,0)} \right)$, defined by (13), is a Poisson map.

**Proof.** It suffices to show that $\{\sigma^* x_i, \sigma^* x_j\}_b = \sigma^* \{x_i, x_j\}$ for $1 \leq i < j \leq n$. Since $\sigma^* x_i = x_i$ for $i > 1$, this is obvious when $1 < i < j$. We therefore only need to verify the formula for $i = 1$ and $j > 1$. If $1 < j < n$ then

$$\{\sigma^* x_1, \sigma^* x_j\}_b = \left\{ x_1 + \frac{\beta}{x_n}, x_j \right\}_b = \left( x_1 + \frac{\beta}{x_n} \right) x_j = \sigma^* (x_1 x_j) = \sigma^* \{x_1, x_j\}.$$
where we have used, in the second equality, that \( \{x_n, x_j\}_b = - \{x_j, x_n\} = -x_j x_n \), with a minus sign because \( j < n \). If \( j = n \) then

\[
\{\sigma^* x_1, \sigma^* x_n\}_b = \{x_1 + \frac{\beta}{x_n}, x_n\}_b = \{x_1, x_n\}_b = x_1 x_n + \beta = \sigma^* (x_1 x_n) = \sigma^* \{x_1, x_n\}.
\]

\[\square\]

**Corollary 3.3.** The rational functions \( F^{(n,0),b}_\ell \) defined in (12) are in involution with respect to the Poisson bracket \( \{\cdot, \cdot\}_b^{(n,0)} \). Similarly, the rational functions \( G^{(n,0),b}_\ell \) are in involution.

**Proof.** Let \( 1 \leq \ell, \ell' \leq \big[ \frac{n-1}{2} \big] \). Then, according to Proposition 3.2,

\[
\{F^{b}_{\ell'}, F^{b}_{\ell''}\}_b = \{\sigma^* F_\ell, \sigma^* F_{\ell'}\}_b = \sigma^* \{F_\ell, F_{\ell'}\} = 0,
\]

where we have used in the last step that the functions \( F_\ell \) of \( LV(n,0) \) are in involution \([19, \text{Prop. 3.2}]\). The fact that the functions \( G^{b}_\ell \) are also in involution follows from the fact that \( \sigma \) is an anti-Poisson map of \((\mathbb{R}^n, \Pi_b)\).

Notice that although \( \sigma \) is a birational Poisson isomorphism, it is not an isomorphism between the Hamiltonian systems \( LV(n,0) \) and \( LV_b(n,0) \) because \( \sigma^* H \neq H \). In particular, (3.3) does not imply that the rational functions \( F^{b}_\ell \) are in involution with the Hamiltonian \( H \), i.e., that they are first integrals of \( LV_b(n,0) \); this requires a separate proof, which has been given in Proposition 3.1 above.

### 3.3. Integrability.

We now prove the Liouville and superintegrability of \( LV_b(n,0) \).

As we will see, the main result that remains to be proven is that the \( n - 1 \) constructed first integrals, to wit the Hamiltonian, the rational functions \( F^{b}_\ell \) (with \( \ell = 1, \ldots, \big[ \frac{n-1}{2} \big] \)) and the rational functions \( G^{b}_\ell \) (with \( \ell = 1, \ldots, \big[ \frac{n-1}{2} \big] \) when \( n \) is even and \( \ell = 2, \ldots, \big[ \frac{n-1}{2} \big] \) when \( n \) is odd) are independent, i.e. have independent differentials on an open dense subset of \( \mathbb{R}^n \). Since these functions are rational, it suffices to show that their differentials are independent in at least one point of \( \mathbb{R}^n \).

To see this, we use the fact that the undeformed functions \( H, F_\ell \) and \( G_\ell \) are independent at some point \( P \) (see [19]). Since the deformed functions depend polynomially on the deformation parameter \( \beta \), they will still be independent at \( P \) for \( \beta \) in a small interval, centered at zero. Notice that if we rescale all variables by a factor \( \lambda \neq 0 \) and rescale \( \beta \) by a factor \( \lambda^2 \) all these functions also get multiplied by a non-zero factor. It follows that the differentials of the deformed functions are independent at \( P \) for all values of \( \beta \).

**Theorem 3.4.** For any \( n \), the Hamiltonian system \( LV_b(n,0) \) is superintegrable, with first integrals the rational functions \( F^{(n,0),b}_\ell \) and \( G^{(n,0),b}_\ell \). Moreover, it is Liouville integrable with rational functions \( F^{(n,0),b}_\ell \), where \( \ell = 1, \ldots, \big[ \frac{n-1}{2} \big] \); also, it is Liouville integrable with rational functions \( G^{(n,0),b}_\ell \), where \( \ell = 1, \ldots, \big[ \frac{n+1}{2} \big] \).

**Proof.** Recall that a superintegrable system on an \( n \)-dimensional manifold is a vector field (Hamiltonian or not), which has \( n - 1 \) independent first integrals. As we have constructed precisely this number of independent first integrals for \( LV_b(n,0) \), we have proven its superintegrability. For Liouville integrability of a Hamiltonian vector field on an \( n \)-dimensional Poisson manifold of rank \( 2r \) we need \( n - r \) independent first integrals which are in involution. Here, the rank of the Poisson structure
Example 3.5. For \( n = 4 \) and \( k = 0 \) the matrices \( A^{(4,0)} \) and \( B^{(4,0)} \) are given by

\[
A^{(4,0)} = \begin{pmatrix}
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1 \\
-1 & -1 & 0 & 1 \\
-1 & -1 & -1 & 0
\end{pmatrix}, \quad B^{(4,0)} = \begin{pmatrix}
0 & 0 & 0 & \beta \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\beta & 0 & 0 & 0
\end{pmatrix}.
\]

The corresponding system \( LV_b(4,0) \) is given by the formulas

\[
\begin{align*}
\dot{x}_1 &= x_1(x_2 + x_3 + x_4) + \beta, \\
\dot{x}_2 &= x_2(-x_1 + x_3 + x_4), \\
\dot{x}_3 &= x_3(-x_1 - x_2 + x_4), \\
\dot{x}_4 &= x_4(-x_1 - x_2 - x_3) - \beta,
\end{align*}
\]

and besides the Hamiltonian \( H = x_1 + x_2 + x_3 + x_4 \) it has two more independent rational first integrals \( F^b \) and \( G^b \), namely

\[
F^b = \frac{(x_1 + x_2)x_4 + \beta}{x_3} \quad \text{and} \quad G^b = \hat{\iota} F^b = \frac{(x_4 + x_3)x_1 + \beta}{x_2},
\]

where \( \hat{\iota} \) is the anti-Poisson map defined in (10). The above three functions give the superintegrability of the system \( LV_b(4,0) \). The rank of the Poisson structure is 4 and each one of the pairs \((H, F^b)\) and \((H, G^b)\) provide the Liouville integrability of \( LV_b(4,0) \).

3.4. Explicit solutions. The Hamiltonian vector field \( \chi_H \) of \( LV(n,0) \) can be explicitly integrated in terms of elementary functions, as was first shown in [14]. We show that such an integration can also be done for (8), the Hamiltonian vector \( \chi_H \) of \( LV_b(n,0) \). This is most easily done by introducing some linear coordinates on \( \mathbb{R}^n \): for \( i = 0, 1, \ldots, n \), let \( u_i := x_1 + x_2 + \cdots + x_i \) and notice that \( u_0 = 0 \) and \( u_n = H \). It is clear that everything can be easily expressed in terms of the coordinates \( u_i \) by substituting \( u_i - u_{i-1} \) for \( x_i \) (\( i = 1, \ldots, n \)). As we will see, this simplifies some of the formulas (for \( \chi_H \), for example) and makes others more complex (the rational integrals, for example). For the proposition which follows, the formulas are the simplest when expressed in the \( u_i \) coordinates.

First, we need to express \( LV_b(n,0) \) in terms of the coordinates \( u_1, \ldots, u_n \). The simplest way to do this is to first compute \( \Pi_b \) in terms of these coordinates. Since for \( i < j \), \( \{x_i, x_j\}_b = x_ix_j \), except that \( \{x_1, x_n\} = x_1x_n + \beta \), we get

\[
\begin{align*}
\{u_i, u_j\}_b &= u_i(u_j - u_i), & 1 \leq i < j < n, \\
\{u_i, u_n\}_b &= u_i(u_n - u_i) + \beta, & 1 \leq i < n.
\end{align*}
\]

Since \( H = u_n \), we can compute \( \chi_H \) as \( \{\cdot, u_n\}_b \), which takes in view of the above formulas the following simple, decoupled form:

\[
\begin{align*}
\dot{u}_i &= u_i(H - u_i) + \beta, & i = 1, 2, \ldots, n - 1, \\
u_n &= 0,
\end{align*}
\]

which can easily be integrated, for any initial condition. We describe the integration in a geometrical language, which will be useful when we use it in Section 5. For any
point $P \in \mathbb{R}^n$, we can consider the integral curve of $X_H$, starting from $P$, which we will denote by $\gamma_P$. Usually, the domain of an integral curve is taken to be an interval, but in the present case we will take it to be all of $\mathbb{R}$ minus a discrete subset. On the one hand, it is natural to do this because in the case of $X_H$ the solutions are precisely defined on such a set. On the other hand, the systems $LV_b(n,k)$ can equally be defined on a complex phase space $\mathbb{C}^n$ and then the integral curves, with complex time, are defined for all of $\mathbb{C}$, minus a discrete subset; the domain of the real integral curves which we consider is just the real part of this complex subset. Since it is convenient to express the integral curves in terms of coordinates (here the $u_i$ coordinates) we will write, once $P$ has been fixed, $u_i(t)$ for $u_i(\gamma_P(t))$. The $u$-coordinates of $P$ will be denoted $(P_1, \ldots, P_n)$, so $P_i = u_i(P)$ for $i = 1, \ldots, n-1$ and $P_n = u_n(P) = H(P)$.

**Proposition 3.6.** Let $P$ be any point of $\mathbb{R}^n$ and let $\gamma_P$ denote the integral curve of (15), which is $LV_b(n,0)$, expressed in the $u_i$ variables, starting from $P$. Denote by $h$ the value of the Hamiltonian at $P$, i.e., $h = H(P) = u_n(P)$. Let $\Delta_0$ be a square root of $h^2 + 4\beta$, which may be real or imaginary. Then, for $i = 1, 2, \ldots, n-1$,

$$u_i(t) = \begin{cases} P_i, & \text{if } P_i^2 - P_i h - \beta = 0, \\ \frac{h}{2} + \frac{2P_i^2 - h}{2 + (2P_i^2 - h)}, & \text{if } \Delta_0^2 = h^2 + 4\beta = 0, \\ \frac{(h + \Delta_0)(h - \Delta_0 - 2P_i) - (h + \Delta_0 - 2P_i)(h - \Delta_0)e^{-t\Delta_0}}{2(h - \Delta_0 - 2P_i)(h - \Delta_0 + 2P_i)e^{-t\Delta_0}}, & \text{otherwise.} \end{cases}$$

(16)

Obviously, $u_n(t) = H$ is constant.

4. The Liouville and non-commutative integrability of $LV_b(n,k)$. In this section, we generalize the results of Section 3 on the integrability of $LV_b(n,0)$ to the case of $LV_b(n,k)$, where $k \in \mathbb{N}$ satisfies $2k + 1 < n$, but is otherwise arbitrary. We do not treat here the case of $n = 2k + 1$ because we have already established the Liouville integrability of $LV_b(2k+1,k)$ in [7]. We show in this section that if $1 < 2k + 1 < n$ then $LV_b(n,k)$ is on one hand Liouville integrable, and on the other hand is non-commutatively integrable of rank $k + 1$. We start by recalling the definition of non-commutative integrability (see [17, 15]), which we specialize to $\mathbb{R}^n$.

**Definition 4.1.** Let $\Pi$ be a Poisson structure on $\mathbb{R}^n$, with associated Poisson bracket $\{\cdot, \cdot\}$. Let $F = (f_1, \ldots, f_s)$ be an $s$-tuple of functions on $\mathbb{R}^n$, where $2s \geq n$ and set $r := n - s$. Suppose the following:

1. The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$:
   $$\{f_i, f_j\} = 0, \quad 1 \leq i \leq r \text{ and } 1 \leq j \leq s;$$

2. For $P$ in a dense open subset of $\mathbb{R}^n$:
   $$\text{d}f_1(P) \wedge \cdots \wedge \text{d}f_s(P) \neq 0 \quad \text{and} \quad X_{f_1}|_P \wedge \cdots \wedge X_{f_s}|_P \neq 0.$$

Then the triplet $(\mathbb{R}^n, \Pi, F)$ is called a non-commutative integrable system of rank $r$.

The classical case of a Liouville integrable system corresponds to the particular case where $r$ is half the (maximal) rank of $\Pi$; this implies that all the functions $f_1, \ldots, f_s$ are pairwise in involution. The case of a superintegrable system corresponds to $r = 1$; in this case, setting $H = f_1$, condition (1) just means that $X_H$ has
$n - 1$ first integrals, while the second condition in (2) is trivially satisfied: superintegrability means, as recalled in the previous section, that $\mathcal{X}_H$ has $n - 1$ independent first integrals.

In order to establish Liouville and non-commutative integrability in Section 4.3 below, we first construct a set of polynomial first integrals for $LV_b(n, k)$, which are pairwise in involution, and then we construct a set of rational first integrals for $LV_b(n, k)$, which are also pairwise in involution. This will be done in the two subsections which follow. Throughout the section, we suppose that $1 < 2k + 1 < n$ and that $B = B^{(n, k)}$ is a skew-symmetric $n \times n$ matrix such that (6) defines a Poisson structure on $\mathbb{R}^n$. Recall that this means that the uppertriangular entries $b_{i,j}$ of $B$ with $j - i \notin \{m, m + 1\}$ are zero.

4.1. The polynomial first integrals. Recall from Section 2.2 that the systems $LV_b(n, k)$ are obtained by reduction from the systems $LV_b(2m + 1, m)$, where $m := n - k - 1 > k$, where the last inequality comes from our assumption $n > 2k + 1$. Recall also that in order to do this reduction, one supposes that the last $2m + 1 - n$ rows and columns of the $(2m + 1) \times (2m + 1)$ matrix $B$ are zero, so that $B$ can be viewed as an $n \times n$ matrix by removing these zero rows and columns (see Proposition 2.1). Since $LV_b(2m + 1, m)$ is Liouville integrable, with $m + 1$ independent polynomial first integrals, whose formulas are recalled below, we obtain by reduction a set of first integrals of $LV_b(n, k)$, which are automatically in involution with respect to the reduced Poisson structure, which is by definition the Poisson structure $\Pi^{(n, k)}_b$ of $LV_b(n, k)$. One has however to be careful with the independence of the reduced first integrals, for example some of these reduced first integrals are zero! Moreover, since $n > 2k + 1$, more first integrals are needed for integrability, as we will see.

Let us first recall the formulas for the polynomial first integrals of $LV_b(2m + 1, m)$. One method of constructing them is as coefficients of the characteristic polynomial of the Lax operator $L(\lambda) := X + \lambda^{-1} \Delta + \lambda M$, which we recalled in (5). It is a classical fact that the coefficients of the characteristic polynomial of a Lax operator yield first integrals for any Lax equation in which the operator appears [15, Sect. 12.2.5]. For $L(\lambda)$, the following expansion of its characteristic polynomial was obtained in [7, Prop. 8]:

$$\det(L(\lambda) - \mu \text{Id}) = \lambda^{2m+1} + \frac{1}{\lambda^{2m+1}} \prod_{j=1}^{2m+1} (b_{j+m,j} - \lambda \mu) + \sum_{i=0}^{m} (\lambda \mu)^{m-i} K_i^b. \quad (17)$$

Thus, the polynomials $K_i^b$ in this expansion are first integrals of $LV_b(2m + 1, m)$. Setting the deformation parameters equal to zero, one recovers the first integrals, $K_i$, of $LV(2m + 1, m)$ which were first constructed by Bogoyavlenskij [4] and Itoh [13]. We construct $k + 1$ first integrals of $LV_b(n, k)$ by setting, for $i = 0, \ldots, k$,

$$K_i^{(n, k), b} := K_i^b \big|_{\mathbb{R}^n} := K_i^b \big|_{x_{n+1} = x_{n+2} = \cdots = x_{2m+1} = 0}, \quad (18)$$

where the notation introduced by the latter equality is a convenient shorthand. By construction, these polynomials are first integrals of $LV_b(n, k)$ and they are in involution. Also, $K_0^b = H$ and $K_i^b$ is of degree $2i + 1$ for $i = 0, 1, \ldots, k$.

We give an alternative description of the first integrals (18) as deformations of the polynomial first integrals of $LV(n, k)$. On the one hand, this description will be important for showing the independence of these first integrals, and on the other hand it will provide information about the structure of these first integrals, which we will use to prove some of their properties (involutivity, for example).
It was shown in [7, Prop. 9] that the first integrals $K^b_i$ of $\text{LV}_b(2m+1, m)$ can be obtained using the operator $D_b$, defined by

$$D_b := \sum_{1 \leq i \leq 2m+1} b_{i,i+m} \frac{\partial^2}{\partial x_i \partial x_{i+m}} ,$$

by the following formula, valid for $i = 0, 1, \ldots, m$:

$$K^b_i = e^{D_b} K_i = K_i + D_b K_i + \frac{1}{2!} D_b^2 K_i + \cdots + \frac{1}{i!} D_b^i K_i .$$

We have used in the last step that $\deg K_i = 2i + 1$ (see Formula (22) below). Let us show that $D_b$ commutes with restriction to $\mathbb{R}^n$. Let $F$ be a smooth or rational function on $\mathbb{R}^{2m+1}$. In view of the conditions on $B = B^{(n,k)}$ (see Proposition 2.2), the operator $D_b$ is given by

$$D_b := \sum_{1 \leq i \leq k+1} b_{i,i+m} \frac{\partial^2}{\partial x_i \partial x_{i+m}} - \sum_{1 \leq i \leq k} b_{i,i+m+1} \frac{\partial^2}{\partial x_i \partial x_{i+m+1}} ,$$

and we see that $D_b$ does not involve derivation with respect to any of the variables $x_{n+1}, x_{n+2}, \ldots, x_{2m+1}$ (recall that $n = m+k+1$), so $D_b$ commutes with restriction to the subspace $\mathbb{R}^n$ of $\mathbb{R}^{2m+1}$, which is defined by $x_{n+1} = x_{n+2} = \cdots = x_{2m+1} = 0$. It follows that

$$K^{(n,k),b}_i = K_i \big|_{\mathbb{R}^n} = (e^{D_b} K_i) \big|_{\mathbb{R}^n} = e^{D_b} K_i \big|_{\mathbb{R}^n} = e^{D_b} K^{(n,k)}_i$$

$$= K^{(n,k)}_i + D_b K^{(n,k)}_i + \frac{1}{2!} D_b^2 K^{(n,k)}_i + \cdots + \frac{1}{i!} D_b^i K^{(n,k)}_i .$$

This shows that the polynomial first integrals $K^{(n,k),b}_i$ of $\text{LV}_b(n, k)$ are deformations of the first integrals $K^{(n,k)}_i$ of $\text{LV}(n,k)$. Notice also that (21) implies that $K^{(n,k),b}_i = 0$ when $i > k$ since $K^{(n,k)}_i = 0$ when $i > k$ (see the comments after Proposition 3.3 in [6]). That is the reason why we restricted $i$ in (18) to $i = 0, 1, \ldots, k$ rather than $i = 0, 1, \ldots, m$.

For later use, we quickly recall from [6] a combinatorial formula for $K^{(n,k)}_i$. Let $m = (m_1, m_2, \ldots, m_{2i+1})$ be a $2i+1$-tuple of integers, satisfying $1 \leq m_1 < m_2 < \cdots < m_{2i+1} \leq n$. We view these integers as indices of the rows and columns of $A^{(n,k)}$: we denote by $A^{(n,k)}_m$ the square submatrix of $A^{(n,k)}$ of size $2i + 1$, corresponding to rows and columns $m_1, m_2, \ldots, m_{2i+1}$ of $A^{(n,k)}$, so that

$$(A^{(n,k)}_m)_{s,t} = (A^{(n,k)})_{m_s,m_t} \text{, \ for } s,t = 1, \ldots, 2i + 1 .$$

Letting

$$S^{(n,k)}_i := \left\{ m \mid A^{(n,k)}_m = A^{(2i+1,i)} \right\} ,$$

the first integral $K^{(n,k)}_i$ is given by

$$K^{(n,k)}_i = \sum_{m \in S^{(n,k)}_i} x_{m_1} x_{m_2} \cdots x_{m_{2i+1}} .$$

One immediate consequence is that every variable $x_j$ has degree at most one in $K^{(n,k)}_i$ and also in $K^{(n,k),b}_i$. 
4.2. The rational first integrals. We now construct a set of rational first integrals of \(L/V(n,k)\). In order to follow some of the more technical arguments in this subsection, the reader is advised to already take a look at Section 4.4 below, where explicit formulas for a few examples are given.

Recall that we assume in this section that \(k > 0\) and that \(n > 2k + 1\). We define the rational first integrals of \(L/V(n,k)\) as deformations of the rational first integrals of \(L(n,k)\), which were first constructed in [6]. We first recall the definition of the latter first integrals as pullbacks of the rational first integrals \(F_{\ell}^{(n-2k,0)}\) and \(G_{\ell}^{(n-2k,0)}\) of \(L(n-2k,0)\), which we recalled in Section 3.1. Consider the polynomial map \(\phi_k : \mathbb{R}^n \to \mathbb{R}^{n-2k}\), defined by

\[
\phi_k(a_1,a_2,\ldots,a_n) := a_1a_2\ldots a_k(a_{k+1},a_{k+2},\ldots,a_{n-k})a_{n-k+1}\ldots a_n a_n^2.
\]

If we denote the standard coordinates on \(\mathbb{R}^n\) by \(x_1,\ldots,x_n\), and on \(\mathbb{R}^{n-2k}\) by \(y_1,\ldots,y_{n-2k}\), then \(\phi^*y_i = x_1x_2\ldots x_kx_{k+1}\ldots x_{n-k+1}\ldots x_n\) for \(i = 1,\ldots,n-2k\). It was shown in [6] that for \(\ell = 1,2,\ldots \left[\frac{n+1}{2}\right] - k\), the rational functions \(F_{\ell}^{(n,k)} := \phi_k^*F_{\ell}^{(n-2k,0)}\) and \(G_{\ell}^{(n,k)} := \phi_k^*G_{\ell}^{(n-2k,0)}\) are first integrals of \(L(n,k)\) and that the first integrals \(F_{\ell}^{(n,k)}\) are pairwise in involution with respect to \(\{\cdot,\cdot\}^{(n,k)}\), just like the first integrals \(G_{\ell}^{(n,k)}\). Setting \(s := 2\ell - 1\) when \(n\) is odd and \(s := 2\ell\) when \(n\) is even, so that \(s\) and \(n\) have the same parity, one computes easily from (9) that

\[
F_{\ell}^{(n,k)} = \prod_{i=1}^k(x_i x_{n+1-i})\sum_{j=1}^s x_{j+k} \prod_{t=1}^{\frac{n-s-k}{2}} \frac{x_{s+k+2t}}{x_{s+k+2t-1}}.
\]

When \(\ell = \left[\frac{n+1}{2}\right] - k\), the last product in this expression reduces to 1 and so \(F_{\ell}^{(n,k)}\) is actually a polynomial function. Since some of the arguments below which depend on the structure of the first integrals fail for the polynomial first integrals, we will exclude these first integrals in this section, and so we will throughout this section only consider the rational first integrals \(F_{\ell}^{(n,k)}\), with \(\ell = 1,2,\ldots \left[\frac{n+1}{2}\right] - k\). Notice also that every variable or its inverse appears precisely once in (23), with the \(k\) variables \(x_1,\ldots,x_k\) and the \(k + 1\) variables \(x_{n-k},\ldots,x_n\) appearing linearly. This property will be important in what follows.

We construct the rational first integrals of \(L/V(n,k)\) as deformations of the first integrals \(F_{\ell}^{(n,k)}\) of \(L(n,k)\) by using the operators \(D_b\) (see (19) or (20)): for \(\ell = 1,2,\ldots,\left[\frac{n-1}{2}\right] - k\), we set

\[
F_{\ell}^{(n,k),b} := e^{D_b}F_{\ell}^{(n,k)} = e^{D_b}\left(\phi_k^*F_{\ell}^{(n-2k,0)}\right),
\]

and similarly for \(G_{\ell}^{(n,k),b}\). The present subsection is devoted to the proof of the following theorem, which says that the deformed first integrals \(F_{\ell}^{(n,k),b}\) are first integrals of \(L/V(n,k)\) which are pairwise in involution.

**Theorem 4.2.** For \(1 \leq \ell \leq \ell' \leq \left[\frac{n-1}{2}\right] - k\),

\[
\left\{F_{\ell}^{(n,k)}, H\right\}_b^{(n,k)} = 0 \quad \text{and} \quad \left\{F_{\ell}^{(n,k),b}, F_{\ell'}^{(n,k),b}\right\}_b^{(n,k)} = 0.
\]

The same result holds for the rational functions \(G_{\ell}^{(n,k),b}\).

For the proof of Theorem 4.2, we need some extra notation. Since throughout this subsection \(n\) and \(k\) are fixed, we will until the rest of the subsection drop \((n,k)\)
from the notation, writing $F^b_\ell$ for $F^{(n,k),b}_\ell$, writing $B$ for $B^{(n,k)}$, and so on. We will need a specific ordering of the entries of the matrix $B$. Therefore, we label the parameters of $B$ with single indices as follows (recall that $B$ is skew-symmetric and that its non-zero uppertriangular entries are at positions $(i,j)$ with $j - i = n - k - 1$ or $j - i = n - k$):

$$B = \begin{pmatrix} \cdots & 0 & b_{2k+1} & -b_{2k} & 0 & \cdots & 0 \\ \cdots & 0 & b_{2k-1} & -b_{2k-2} & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & b_3 & -b_2 & \ddots \\ \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \end{pmatrix}$$

Expressed in terms of a formula,

$$b_p := (-1)^{p+1} b_{k+1-[\frac{k}{2}]n-[\frac{k-1}{2}]}. $$

For $1 \leq p \leq 2k + 1$, we denote by $B^{(p)}$ the matrix obtained from $B$ by setting the parameters $b_{p+1}, b_{p+2}, \ldots, b_{2k+1}$ equal to zero. We also set $B^{(0)}$ equal to the zero matrix. So $B^{(1)}$ contains only the parameter $b_1$ and $B^{(2k+1)} = B$. The corresponding Poisson structure, which can be obtained from $\Pi_b = \Pi^{(n,k)}_b$ by setting the parameters $b_{p+1}, b_{p+2}, \ldots, b_{2k+1}$ equal to zero, is denoted by $\Pi_b$. In particular, $\Pi_b = \Pi_{(2k+1)}$. The associated Poisson bracket is denoted by $\{\cdot, \cdot\}$. We also associate to $p$ the following constant coefficient differential operator

$$D^{(p)} := b_p \frac{\partial^2}{\partial x_{k+1-[\frac{k}{2}]n-[\frac{k-1}{2}]}}. $$

It is clear that $D_b = D^{(2k+1)} + D^{(2k)} + \cdots + D^{(2)} + D^{(1)}$ and, since these operators commute, that $e^{D^{(p)}} = e^{D^{(2k+1)}} \circ e^{D^{(2k)}} \circ \cdots \circ e^{D^{(2)}} \circ e^{D^{(1)}}$. We could of course consider any alternative order of the operators, but we will use the above one for some reason which will become clear later. Finally, for any smooth or rational function $F$ on $\mathbb{R}^n$ we set $F^{(p)} := e^{D^{(p)}} F^{(p-1)}$ for $1 \leq p \leq 2k + 1$, and $F^{(0)} := F$. With this notation, $F^b_\ell = F^{(2k+1)}_\ell$.

A crucial fact which we will use is that the partially deformed integral $F^{(p)}_\ell$ is the pullback of $F^{(p-1)}_\ell$ by the birational Poisson map $\sigma_p$, defined for $1 \leq i \leq n$ by

$$\sigma_p(x_i) := \begin{cases} x_i + \frac{b_p}{x_{i+n-k+1-\frac{k}{2}}} \delta_{i,k+1-\frac{k-1}{2}} & \text{if } p \text{ is odd,} \\ x_i + \frac{b_p}{x_{i+n-k+1-\frac{k}{2}}} \delta_{i,n+1-\frac{k}{2}} & \text{if } p \text{ is even.} \end{cases} \quad (25)$$

Notice that

$$\sigma_p \left(x_{n-[\frac{n-1}{2}]} x_{k+1-[\frac{k}{2}]}\right) = x_{n-[\frac{n-1}{2}]} x_{k+1-[\frac{k}{2}]} + b_p, \quad (26)$$

independently of whether $p$ is even or odd.

**Proposition 4.3.** $F^{(p)}_\ell = \sigma_p F^{(p-1)}_\ell$ for $1 \leq p \leq 2k + 1$ and $1 \leq \ell \leq \left[\frac{n-1}{2}\right] - k$. 

Proof. We first show that if \( 0 \leq p \leq 2k \) then
\[
F^{(p)}_\ell = x_n - \left[ \frac{x}{2} \right] x_k - \left[ \frac{x}{2} - 1 \right] E_1 + E_2, \tag{27}
\]
where \( E_1 \) is independent of \( x_n - \left[ \frac{x}{2} \right] \) and of \( x_k - \left[ \frac{x}{2} - 1 \right] \), while \( E_2 \) is independent of \( x_n - \left[ \frac{x}{2} \right] \) when \( p \) is odd and is independent of \( x_k - \left[ \frac{x}{2} - 1 \right] = x_{k+1} - \frac{x}{2} \) when \( p \) is even. We do this for \( p \) odd, i.e., we show that when \( p \) is odd,
\[
F^{(p)}_\ell = x_n - \frac{x}{2} x_k - \frac{x}{2} E_1 + E_2,
\]
where \( E_1 \) is independent of \( x_n - \frac{x}{2} \) and of \( x_k - \frac{x}{2} \), while \( E_2 \) is independent of \( x_n - \frac{x}{2} \). According to the explicit formula (23), we can write \( F_\ell = F_\ell^{(0)} \) as
\[
F_\ell = x_n - \frac{x}{2} x_k - \frac{x}{2} F_\ell + F_\ell'',
\]
where \( F_\ell' \) and \( F_\ell'' \) are independent of \( x_n - \frac{x}{2} \) and of \( x_k - \frac{x}{2} \). Then,
\[
F^{(p)}_\ell = e^{D(p)} F_\ell^{(p-1)} = (e^{D(p)} F_\ell)^{(p-1)} = (F_\ell + D(p) F_\ell)^{(p-1)},
\]
where we have used in the last step that \( D^2(p) F_\ell = 0 \), which is a consequence of the fact that \( F_\ell \) depends linearly on \( x_{k+1} - \frac{x}{2} \) and on \( x_n - \frac{x}{2} \), which are the variables with respect to which \( D(p) \) differentiates (see (24)). Since, moreover, \( F_\ell'' \) is independent of \( x_n - \frac{x}{2} \),
\[
F^{(p)}_\ell = \left( x_n - \frac{x}{2} x_k - \frac{x}{2} F_\ell' + F_\ell'' + b_p x_k - \frac{x}{2} \frac{\partial F_\ell'}{\partial x_{k+1} - \frac{x}{2}} \right)^{(p-1)}.
\]
Since \( D(1), \ldots, D(p-1) \) do not involve the variables \( x_n - \frac{x}{2} \) and \( x_k - \frac{x}{2} \),
\[
F^{(p)}_\ell = x_n - \frac{x}{2} x_k - \frac{x}{2} F_\ell^{(p-1)} + \left( F_\ell'' + b_p x_k - \frac{x}{2} \frac{\partial F_\ell'}{\partial x_{k+1} - \frac{x}{2}} \right)^{(p-1)}
\]
\[
= x_n - \frac{x}{2} x_k - \frac{x}{2} E_1 + E_2,
\]
where \( E_1 \) is independent of \( x_n - \frac{x}{2} \) and of \( x_k - \frac{x}{2} \), and \( E_2 \) is independent of \( x_n - \frac{x}{2} \). This shows our claim when \( p \) is odd. The proof in case \( p \) is even is similar. We use the obtained formula (27) to show that \( F^{(p)}_\ell = \sigma^p F^{(p-1)}_\ell \) for any \( p \). According to (27), we can write
\[
F^{(p-1)}_\ell = x_n - \left[ \frac{x}{2} - 1 \right] x_k - \left[ \frac{x}{2} - 1 \right] E_1 + E_2,
\]
where \( E_1 \) is independent of \( x_n - \left[ \frac{x}{2} - 1 \right] \) and of \( x_k - \left[ \frac{x}{2} - 1 \right] \), while \( E_2 \) is independent of \( x_n - \left[ \frac{x}{2} - 1 \right] = x_{n+1} - \frac{x}{2} \) when \( p \) is even and is independent of \( x_{k+1} - \left[ \frac{x}{2} \right] = x_{k+1} - \frac{x}{2} \) when \( p \) is odd. Therefore, on the one hand,
\[
F^{(p)}_\ell = e^{D(p)} F^{(p-1)}_\ell = F^{(p-1)}_\ell + D(p) \left( x_n - \left[ \frac{x}{2} - 1 \right] x_{k+1} - \left[ \frac{x}{2} \right] E_1 + E_2 \right)
\]
\[
= F^{(p-1)}_\ell + b_p \frac{\partial^2}{\partial x_{k+1} - \left[ \frac{x}{2} \right] \partial x_n - \left[ \frac{x}{2} - 1 \right]} \left( x_n - \left[ \frac{x}{2} - 1 \right] x_{k+1} - \left[ \frac{x}{2} \right] E_1 + E_2 \right)
\]
\[
= F^{(p-1)}_\ell + b_p E_1,
\]
while on the other hand,

\[
\sigma_p^*(F_{\ell}^{(p-1)}) = \sigma_p^* \left( x_{n-[\frac{p-1}{2}]k+1-[\frac{p-1}{2}]n} E_1 + E_2 \right)
\]

\[
\overset{(*)}{=} \sigma_p^* \left( x_{n-[\frac{p-1}{2}]k+1-[\frac{p-1}{2}]n} \right) E_1 + E_2
\]

\[
\overset{(26)}{=} \left( x_{n-[\frac{p-1}{2}]k+1-[\frac{p-1}{2}]n} + b_p \right) E_1 + E_2 = F_{\ell}^{(p-1)} + b_p E_1,
\]

which shows that \( F_{\ell}^{(p)} = \sigma_p^* F_{\ell}^{(p-1)} \). We have used in (*) that \( \sigma_p^*(E_1) = E_1 \) and that \( \sigma_p^*(E_2) = E_2 \), which hold because \( E_1 \) and \( E_2 \) are independent of the only variable which is not fixed by \( \sigma_p^* \).

We next show that the maps \( \sigma_p \) are Poisson maps with respect to the appropriate Poisson structures on \( \mathbb{R}^n \).

**Proposition 4.4.** For \( 1 \leq p \leq 2k+1 \) the birational map

\[
\sigma_p : \left( \mathbb{R}^n, \left\{ \cdot, \cdot \right\}_{(p)} \right) \rightarrow \left( \mathbb{R}^n, \left\{ \cdot, \cdot \right\}_{(p-1)} \right)
\]

defined by (25), is a Poisson map.

**Proof.** We give the proof in case \( p \) is even. Then (25) simplifies to

\[
\sigma_p^*(x_i) = \begin{cases} 
  x_i + \frac{b_p}{x_{i+k-n}} & \text{if } i = n + 1 - \frac{p}{2}, \\
  x_i & \text{if } i \neq n + 1 - \frac{p}{2}.
\end{cases}
\]

We need to show that \( \sigma_p^* \{x_i, x_j\}_{(p-1)} = \{\sigma_p^*(x_i), \sigma_p^*(x_j)\}_{(p)} \) for all \( 1 \leq i < j \leq n \). Notice that if \( (i, j) \neq (k+1 - \frac{p}{2}, n+1 - \frac{p}{2}) \) then \( x_i, x_j \}_{(p)} = \{x_i, x_j\}_{(p-1)} \) and otherwise \( \{x_i, x_j\}_{(p)} = \{x_i, x_j\}_{(p-1)} - b_p \) (recall that \( p \) is even). It follows that, if \( i, j \neq n + 1 - p/2, \)

\[
\sigma_p^* \{x_i, x_j\}_{(p-1)} = \{x_i, x_j\}_{(p)} = \{\sigma_p^*(x_i), \sigma_p^*(x_j)\}_{(p)},
\]

as was to be shown. Suppose now that \( 1 \leq i = n + 1 - \frac{p}{2} < j \). Notice that, in this case, \( \{x_i, x_j\}_{(p)} = \{x_i, x_j\}_{(p-1)} = +x_i x_j \), with a plus sign. Then

\[
\sigma_p^* \{x_i, x_j\}_{(p-1)} = \sigma_p^*(x_i, x_j) = \left( x_i + \frac{b_p}{x_{i+k-n}} \right) x_j,
\]

while

\[
\{\sigma_p^*(x_i), \sigma_p^*(x_j)\}_{(p)} = \left\{ x_i + \frac{b_p}{x_{i+k-n}}, x_j \right\}_{(p)} = x_i x_j + \frac{b_p}{x_{i+k-n}} x_j,
\]

where we have used that \( \{x_{i+k-n}, x_j\}_{(p)} = -x_{i+k-n} x_j \), with a minus sign because \( i < j \). Finally, suppose that \( 1 \leq i < j = n+1-\frac{p}{2} \) and notice that \( j+k-n = k+1-\frac{p}{2} \).
Then,

\[
\{\sigma_p^* (x_i), \sigma_p^* (x_j)\}_{(p)} = \left\{ x_i, x_j + \frac{b_p}{x_j + k - n}\right\}_{(p)} = \left\{ x_i, x_j + \frac{b_p}{x_j + k - n}\right\}_{(p)} = \left\{ x_i, x_j + \frac{b_p}{x_j + k - n}\right\}_{(p)} = \left\{ x_i, x_j + \frac{b_p}{x_j + k - n}\right\}_{(p)}
\]

while

\[
\sigma_p^* (x_i, x_j)_{(p-1)} = \sigma_p^* (\pm x_i x_j) = \pm x_i \left(x_j + \frac{b_p}{x_j + k - n}\right),
\]

where the + sign corresponds to the case \( i > k + 1 - \frac{p}{2} \) and the – sign to the case \( i \leq k + 1 - \frac{p}{2} \). Clearly, this gives the same result as in (28). This shows that \( \sigma_p \) is a Poisson map.

Propositions 4.3 and 4.4 imply, in that order, that for any \( p = 1, \ldots, 2k + 1 \), and for any \( 1 \leq \ell \leq \ell' \leq \left[ \frac{n-1}{2} \right] - k \),

\[
\left\{ F_{\ell}^{(p)}, F_{\ell'}^{(p)} \right\}_{(p)} = \left\{ \sigma_p^* F_{\ell}^{(p-1)}, \sigma_p^* F_{\ell'}^{(p-1)} \right\}_{(p)} = \sigma_p^* \left\{ F_{\ell}^{(p-1)}, F_{\ell'}^{(p-1)} \right\}_{(p-1)}
\]

and so, since the undeformed rational first integrals \( F_{\ell} = F_{\ell}^{(0)} \) are pairwise in involution, an easy induction shows that their deformations \( F_{\ell}^b = F_{\ell}^{(2k+1)} \) are in involution as well. This shows the second part of Theorem 4.2.

We will next show that that the deformed first integrals \( F_{\ell}^b \) are first integrals of \( LV_\ell(n, k) \). To do this, we first prove the following lemma.

**Lemma 4.1.** Let \( 1 \leq \ell \leq \left[ \frac{n-1}{2} \right] - k \) and denote \( s := 2\ell - 1 \) when \( n \) is odd and \( s := 2\ell \) when \( n \) is even. If \( i \notin \{ k + 1, k + 2, \ldots, k + s \} \) then \( \{ x_i, F_{\ell}^b \}_{b} = 0 \).

**Proof.** We prove by induction on \( p \) that \( \{ x_i, F_{\ell}^{(p)} \}_{(p)} = 0 \), for \( p = 0, 1, \ldots, 2k + 1 \).

Since \( F_{\ell}^b = F_{\ell}^{(2k+1)} \) this proves the statement. For \( p = 0 \) this amounts to showing that \( \{ x_i, F_{\ell} \}_{b} = 0 \), which was done in [6, Prop. 3.1]. Let \( 1 \leq p \leq 2k + 1 \) and assume that the property is true for \( p - 1 \). If \( i \) is such that \( \sigma_p^* (x_i) = x_i \), then by Propositions 4.3 and 4.4,

\[
\{ x_i, F_{\ell}^{(p)} \}_{(p)} = \left\{ \sigma_p^* (x_i), \sigma_p^* F_{\ell}^{(p-1)} \right\}_{(p)} = \sigma_p^* \left\{ x_i, F_{\ell}^{(p-1)} \right\}_{(p-1)} = 0,
\]
where we used the induction hypothesis in the last step. Suppose now that \( \sigma_p^*(x_i) \neq x_i \). If \( p \) is even, this means that \( i = n + 1 - \frac{p}{2} \), and so

\[
\{ x_i, F^{(p)}_\ell \} = \sigma_p^* \left( x_i - \frac{b_p}{x_i + k - n} \right), \sigma_p^* F^{(p-1)}_\ell \right\} \]

\[
= \sigma_p^* \left( x_i - \frac{b_p}{x_i + k - n} \right), F^{(p-1)}_\ell \right\} = 0,
\]

where we used again the induction hypothesis (twice): we could do so because \( i + k - n \leq k \) so that \( i + k - n \notin \{k + 1, k + 2, \ldots, k + s\} \). If \( \sigma_p^*(x_i) \neq x_i \) and \( p \) is odd, then \( i = k + 1 - \frac{p-1}{2} \), so that

\[
\{ x_i, F^{(p)}_\ell \} = \sigma_p^* \left( x_i - \frac{b_p}{x_i + n - k - 1} \right), \sigma_p^* F^{(p-1)}_\ell \right\} \]

\[
= \sigma_p^* \left( x_i - \frac{b_p}{x_i + n - k - 1} \right), F^{(p-1)}_\ell \right\} = 0,
\]

as before.

Using the lemma and the fact that \( F_\ell \) is a first integral of \( LV(n,k) \) (see [6]), we show that \( F^{(p)}_\ell \) is a first integral of \( LV_b(n,k) \). As before, we show by induction that \( \{ F^{(p)}_\ell, H \} = 0 \), the case of \( p = 0 \) already being established. Suppose that

\[
\{ F^{(p-1)}_\ell, H \} = 0 \text{ for some } p \geq 1. \]

Then

\[
\{ F^{(p)}_\ell, H \} = \{ \sigma_p^* F^{(p-1)}_\ell, \sigma_p^* \left( H - \frac{b_p}{x_i} \right) \} \]

\[
= \sigma_p^* \left( F^{(p-1)}_\ell, H - \frac{b_p}{x_i} \right) = 0,
\]

since \( t = k + 1 - \frac{p-1}{2} \leq k \) when \( p \) is even and \( t = n - \frac{p-1}{2} \geq n - k \) when \( p \) is odd; in either case, \( t \notin \{k + 1, k + 2, \ldots, k + s\} \), which proves the last equality. We conclude that \( \{ F^{(p)}_\ell, H \} = 0 \), which is the first statement of Theorem 4.2.

4.3. Integrability. We have now most ingredients to state and prove the Liouville and non-commutative integrability of \( LV_b(n,k) \), where we recall that \( k > 0 \) and \( 2k + 1 < n \). Since in this subsection \( (n,k) \) is fixed, we will again drop \( (n,k) \) from the notation, except in the statement of the propositions and of the theorem. We have constructed in Section 4.1 a set of polynomial first integrals for \( LV_b(n,k) \) and in Section 4.2 a set of rational first integrals \( F_\ell \). We first show that these polynomial first integrals are in involution with these rational first integrals. To do this, we need a property of the polynomial first integrals, which we first define.

**Definition 4.5.** A polynomial function \( K \) on \( \mathbb{R}^n \) is said to be \( (n,k) \)-admissible if

1. \( K \) is of degree at most one in each of its variables \( x_j \);
2. \( K \) can be written (uniquely) as \( K = LK' + K'' \), where \( K'' \) is independent of \( x_{k+1}, \ldots, x_{n-k} \) and \( L \) is the sum of these variables, \( L = x_{k+1} + x_{k+2} + \cdots + x_{n-k} \).

A key property of the polynomial Hamiltonians is that they are \( (n,k) \)-admissible:
Proposition 4.6. For \( i = 0, 1, \ldots, k \), the polynomial first integral \( K_i^{(n,k),b} \) is \((n,k)\)-admissible.

Proof. As said, we write in the proof \( K_i \) for \( K_i^{(n,k)} \) and \( K_i^b \) for \( K_i^{(n,k),b} \). The fact that \( K_i \) is \((n,k)\)-admissible follows from the following observation made in [6, Cor. 3.5(4)]: if we denote for \( m \in S_i^{(n,k)} \) by \( m' \) the vector \( m' \) with its middle entry \( m'_{i+1} \) replaced by \( m'_{i+1} \), then \( m' \in S_i^{(n,k)} \) when \( k < m'_{i+1} < n - k + 1 \). According to the formula (22) for \( K_i \) this means that when some term of \( K_i \) contains a variable \( x_j \) with \( k + 1 \leq j \leq n - k \), then it contains also a similar term with \( x_j \) replaced by any \( x_l \) with \( k + 1 \leq l \leq n - k \). Considering the sum of these substitutions yields a polynomial which is divisible by \( L \). Therefore, \( K_i \) is \((n,k)\)-admissible. Let us show that if for some substitutions yields a polynomial which is divisible by \( K_i \). We can write \( K_i^{(p-1)} = LK' + K'' \), where \( K' \) and \( K'' \) are independent of \( x_{k+1}, \ldots, x_{n-k} \), and \( D(p) \) differentiates with respect to the variables \( x_{k+1} - [\frac{j}{2}] \) and \( x_n - [\frac{n-1}{2}] \). When \( p \neq 1 \) and \( p \neq 2k + 1 \), these variables are outside the range \( k + 1, \ldots, n - k \), hence \( D(p) K_i^{(p-1)} = LD(p) K' + D(p) K'' \), with \( D(p) K' \) and \( D(p) K'' \) independent of \( x_{k+1}, \ldots, x_{n-k} \), so that \( K_i^{(p)} \) is \((n,k)\)-admissible. For \( p = 1 \),

\[
K_i^{(1)} = e^{D(1)} K_i^{(0)} = K_i^{(0)} + b_1 \frac{\partial^2}{\partial x_{k+1} \partial x_n} (LK' + K'') = K_i^{(0)} + LD(1) K' + D(1) K'' + b_1 \frac{\partial K'}{\partial x_n},
\]

showing that \( K_i^{(1)} \) is also \((n,k)\)-admissible. The proof for \( p = 2k + 1 \) is very similar, since \( D(2k+1) \) differentiates with respect to the variables \( x_1 \) and \( x_{n-k} \).

We are now ready to show that every polynomial integral is in involution with every rational integral.

Proposition 4.7. For \( \ell = 1, 2, \ldots, \left[ \frac{n-1}{2} \right] - k \) and \( i = 0, 1, \ldots, k \),

\[
\{ F_\ell^{(n,k),b}, K_i^{(n,k),b} \}^{(n,k)}_{b} = 0.
\]

Proof. In view of Proposition 4.6, we can write \( K_i^{(p)} = LK' + K'' \) where \( K' \) and \( K'' \) are independent of \( x_{k+1}, \ldots, x_{n-k} \). Using Lemma 4.1 twice,

\[
\{ F_\ell^{b}, K_i^{b} \} = \{ F_\ell^{b}, LK' + K'' \} = \{ F_\ell^{b}, LK' \} = K' \{ F_\ell^{b}, L \}.
\]

The Hamiltonian \( H \) is of course also \((n,k)\)-admissible, \( H = L + H'' \), with \( H'' \) independent of the variables \( x_{k+1}, \ldots, x_{n-k} \). Using that \( F_\ell^{b} \) is a first integral (Theorem 4.2) and Lemma 4.1, we conclude that

\[
\{ F_\ell^{b}, K_i^{b} \} = K' \{ F_\ell^{b}, H - H'' \} = -K' \{ F_\ell^{b}, H'' \} = 0.
\]

Combining the results obtained in this section, we can state and prove the main theorem on the integrability of the systems \( LV_b(n,k) \), with \( n \geq 2k + 1 \). We denote, in that order, by \( H_1^{(n,k),b}, H_2^{(n,k),b}, \ldots, H_{n-k-2}^{(n,k),b} \) the following first integrals:

\[
F_1^b = G_1^b, F_2^b, \ldots, F_{p-1}^b, G_2^b, \ldots, G_{p-1}^b, F_p^b = G_p^b, \text{ when } n - k \text{ is even},
\]

\[
F_1^b, \ldots, F_{p-1}^b, G_1^b, \ldots, G_{p-1}^b, F_p^b = G_p^b, \text{ when } n - k \text{ is odd},
\]

where \( p := \left[ \frac{n-k}{2} \right] \).
Theorem 4.8. Consider the system $LV_b(n,k)$, where $n \geq 2k + 1$.

(1) When $n > 2k + 1$, $LV_b(n,k)$ is non-commutative integrable of rank $k+1$, with first integrals

$$H = K_0^{(n,k),b}, K_1^{(n,k),b}, \ldots, K_k^{(n,k),b}, H_1^{(n,k),b}, H_2^{(n,k),b}, \ldots, H_{n-2k-2}^{(n,k),b}.$$ 

The first $k+1$ functions of this list have independent Hamiltonian vector fields and are in involution with every function of the complete list.

(2) $LV_b(n,k)$ is Liouville integrable with first integrals

$$H = K_0^{(n,k),b}, K_1^{(n,k),b}, \ldots, K_k^{(n,k),b}, H_1^{(n,k),b}, H_2^{(n,k),b}, \ldots, H_{s-1}^{(n,k),b},$$

where $s := \left\lfloor \frac{n+1}{2} \right\rfloor - k$.

Proof. We first consider (1). We have already checked the first item of Definition 4.1, namely that the $k+1$ polynomials $K_i$ are first integrals of $LV(n,k)$, and are in involution with both the polynomial and rational first integrals (Section 4.1 and Proposition 4.7). We need to check the second item which says that the differentials of these first integrals are independent on a dense open subset of $\mathbb{R}^n$, and similarly for the Hamiltonian vector fields associated to the polynomial first integrals. To do this, we use the fact that the undeformed first integrals have this property, as they define a non-commutative integrable system of rank $k+1$ (see [6, Theorem 1.1]). Since all first integrals are rational functions and since the Poisson structure is polynomial, it suffices to prove that the differentials (resp. vector fields) are independent at some point. The argument is the same as the one used in Section 3.3 to derive the independence of the first integrals of $LV_b(n,0)$ from the independence of the first integrals of $LV(n,0)$: since the property is true at some point $P$ when all parameters are zero, it is still true on a neighborhood of $P$ for small values of the parameters; by rescaling the variables and parameters, one finds that at $P$ the property is true for all values of the parameters. This proves (1). We now consider (2), the Liouville integrability. Since the rank of $\Pi_b$ is $n$ when $n$ is even and $n-1$ when $n$ is odd, we need $n/2$ independent first integrals in involution when $n$ is even and $\frac{n+1}{2}$ when $n$ is odd. Clearly, the above list in (2) contains $k+s = \frac{n+1}{2}$ functions, which is the right number, we know that they are pairwise in involution, and by the above argument they are independent. So they define a Liouville integrable system.

4.4. Examples. For explicitness, we give below two examples, $LV_b(4,1)$, which is the smallest new system with $k > 0$ and $LV_b(7,1)$, where one can see some non-trivial examples of the first integrals which we consider.
Example 4.9. For $n = 4$ and $k = 1$ the matrices $A^{(4,1)}$ and $B^{(4,1)}$ are given by

$$A^{(4,1)} = \begin{pmatrix} 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}, \quad B^{(4,1)} = \begin{pmatrix} 0 & 0 & b_3 & -b_2 \\ 0 & 0 & 0 & b_1 \\ -b_3 & 0 & 0 & 0 \\ b_2 & -b_1 & 0 & 0 \end{pmatrix}. $$

The corresponding system $\text{LV}_b(4,1)$ is given by

$$\dot{x}_1 = x_1(x_2 + x_3 - x_4) + b_3 - b_2, \quad \dot{x}_2 = x_2(-x_1 + x_3 + x_4) + b_1, \quad \dot{x}_3 = x_3(-x_1 - x_2 + x_4) - b_3, \quad \dot{x}_4 = x_4(x_1 - x_2 - x_3) + b_2 - b_1.$$ 

Besides the Hamiltonian $H = x_1 + x_2 + x_3 + x_4$, it has an additional polynomial integral $K^b_1 = (x_1x_4 + b_2)(x_2 + x_3) + b_3x_4 + b_1x_1$ which is easily seen to be a $(4,1)$-admissible polynomial. The above two polynomials give the Liouville integrability of the system $\text{LV}_b(4,1)$ which coincides in this case with the non-commutative integrability of rank $k + 1 = 2$ just like in all $\text{LV}_b(2k + 2, k)$ systems.

Example 4.10. We now consider the case $n = 7$ with $k = 1$. The matrix $A := A^{(7,1)}$ is the skew-symmetric Toeplitz matrix with first line $(0, 1, 1, 1, 1, 1, -1)$ and $B := B^{(7,1)}$ is the skew-symmetric matrix whose only non-zero upper triangular entries are $b_{1,6} = b_3$, $b_{1,7} = -b_2$ and $b_{2,7} = b_1$. The corresponding system $\text{LV}_b(7,1)$ is given by the equations

$$\dot{x}_i = \sum_{j=1}^{7} (A_{i,j}x_j + b_{i,j}), \quad \text{for } i = 1, 2, \ldots, 7.$$ 

Besides the Hamiltonian $H = x_1 + x_2 + \cdots + x_7$, the system $\text{LV}_b(7,1)$ has one more independent polynomial first integral $K^b_1$, given by

$$K^b_1 = (x_2 + x_3 + x_4 + x_5 + x_6)(x_1x_7 + b_2) + b_3x_7 + b_1x_6,$$

which is a $(7,1)$-admissible polynomial. It has also three rational first integrals given by

$$F^b_1 = \frac{(x_1x_7 + b_2)x_2x_4x_6 + b_3x_2x_4x_7 + b_1x_1x_4x_6 + b_3b_1x_4}{x_3x_5},$$

$$F^b_2 = \frac{(x_1x_7 + b_2)(x_2 + x_3 + x_4)x_6 + b_3(x_2 + x_3 + x_4)x_7 + b_1(x_1x_6 + b_3)}{x_5},$$

and $G^b_2 = \tau^* F^b_2$. The rank of the Poisson structure $\Pi_b^{(7,1)}$ is 6 and $F^b_1$ is a Casimir, invariant under $\tau^*$. It can be seen that the above first integrals are obtained from the undeformed ones (obtained by setting the parameters equal to zero), by applying on the operator $e^{D_b}$, which now becomes

$$e^{D_b} = \left( I + b_3 \frac{\partial^2}{\partial x_1 \partial x_6} \right) \left( I + b_2 \frac{\partial^2}{\partial x_2 \partial x_7} \right) \left( I + b_1 \frac{\partial^2}{\partial x_1 \partial x_7} \right).$$

The system $\text{LV}_b(7,1)$ is non-commutative integrable of rank 2 with first integrals $H, K^b_1, F^b_1, F^b_2, G^b_2$ and is also Liouville integrable with first integrals $H, K^b_1, F^b_1, F^b_2$ or $H, K^b_1, F^1_1, G^b_2$. 


5. Discretization of $LV_b(n,0)$. In this section we construct a family of discretizations of $LV_b(n,0)$. They are obtained from a discrete zero curvature condition, which is the compatibility condition of a linear system $L\Psi = \lambda \Psi, \tilde{\Psi} = N\Psi$, where $L$ is the Lax matrix of $LV(n,0)$, which appears in (5). We prove that an important class of these discretizations, which includes the Kahan (also called Kahan-Hirota-Kimura) discretization of $LV(n,0)$ has the following integrability properties: it has the rational first integrals of $LV(n,0)$ as invariants, and so it is both Liouville and superintegrable; also it has an invariant measure.

Throughout this section, $(n, k) = (n, 0)$ is fixed and so we will drop $(n, 0)$ from the notation for the invariants, the Poisson structure, and so on. Also, since we have in this case only one parameter $b_{1,n}$, we will denote it by $\beta$, as we did in Section 3.

5.1. Preliminaries. We first recall a few basic definitions and properties of discrete maps and their integrability. By a discrete map of $\mathbb{R}^n$ we mean an algebra homomorphism $\Phi : \mathbb{R}(x_1, x_2, \ldots, x_n) \to \mathbb{R}(x_1, x_2, \ldots, x_n)$, where $x_1, \ldots, x_n$ are as elsewhere in this paper the natural coordinates on $\mathbb{R}^n$. Such a map is the pullback of a unique rational map $\phi : \mathbb{R}^n \to \mathbb{R}^n$, i.e., for any rational function $F$, one has $\Phi(F) = \phi^*(F) = F \circ \phi$. We will also use the convenient abbreviations $\tilde{F}$ for $\Phi(F)$. Similarly, for a matrix $F = (p_{i,j})$ whose entries are rational functions of $\mathbb{R}^n$, we will write $\tilde{F}$ for the matrix $(\tilde{p}_{i,j})$.

When $\mathbb{R}^n$ is equipped with a Poisson structure $\Pi$, then saying that $\Phi$ is a homomorphism of Poisson algebras is tantamount to saying that $\phi$ is a Poisson map; we will simply say that $\Phi$ preserves the Poisson structure $\Pi$. Also, on $\mathbb{R}^n$ we have a natural $n$-form, $dx_1 \wedge \cdots \wedge dx_n$ which allows us to identify rational measures with rational $n$-forms and with rational functions. We will say that $\Phi$ is measure preserving, with preserved measure $F$, if it preserves the $n$-form $F dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ in the sense that

$$F dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = \tilde{F} d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge \cdots \wedge d\tilde{x}_n.$$ 

A rational function $F$ is called an invariant of $\Phi$ if $\tilde{F} = F$. We also recall the definition of an integrable map [20].

**Definition 5.1.** Suppose that $\Phi$ is a discrete map of $\mathbb{R}^n$.

1. $\Phi$ is Liouville integrable if there exist $n - r$ functionally independent invariants of $\Phi$, which are in involution with respect to a Poisson structure $\Pi$, where $r$ is half the rank of $\Pi$.
2. $\Phi$ is superintegrable if it has $n - 1$ functionally independent invariants and is measure preserving.

5.2. Discrete maps from a linear problem. Recall that $LV_b(n,0)$ is by definition a reduction of $LV_b(2m + 1, m)$, with $m := n - 1$, and that

$$L = X + \lambda^{-1} \Delta + \lambda M$$

is the square matrix of size $2m + 1$, where

$$X_{i,j} := x_i \delta_{i,j+m} , \quad \Delta_{i,j} := b_{i+m,j} \delta_{i,j} , \quad M_{i,j} := \delta_{i+1,j} ,$$

and the indices of $x, b$ and $\delta$ are considered modulo $2m + 1$. We consider the compatibility conditions of the linear system

$$L\Psi = \lambda \Psi, \tilde{\Psi} = N\Psi,$$

where $L$ is the Lax matrix of $LV_b(2m + 1, m)$, recalled above, and $\Psi$ is an $n$-dimensional vector whose entries are rational functions on $\mathbb{R}^{2m+1}$. Recall that $\tilde{\Psi}$ is
the vector $\Psi$ with $\Phi$ applied to its entries. The $(2m + 1) \times (2m + 1)$ matrix $N$ is defined as

$$N = D - \lambda K,$$

(32)

where $K_{i,j} := \delta_{i,j+m}$ and $D_{i,j} := d_i \delta_{i,j}$ for some functions $d_i$ that will be determined from the compatibility condition of (31), which reads $\hat{L}N = NL$. Since $N$ is invertible, it means that $\hat{L} = N L N^{-1}$ and therefore the coefficients of the characteristic polynomial of $L$, which are rational functions on $\mathbb{R}^{2m+1}$, are invariants of $\Phi$. The above ansatz for $N$ was taken so that $N L N^{-1}$ equals $L$ at the entries with constant values. Therefore, the compatibility condition $\hat{L}N = NL$ reduces to a system of equations for the $\tilde{x}_i$ and $d_i$ variables, which we make explicit in the following proposition:

**Proposition 5.2.** The compatibility condition $\hat{L}N = NL$ of the linear system (31) is equivalent to the following system of equations:

$$d_{i+1} - d_i + x_{m+1+i} - \tilde{x}_i = 0 \quad \text{and} \quad d_{m+1+i} \tilde{x}_i - d_i x_i + b_{i,m+1+i} - b_{m+i,i} = 0,$$

(33)

for $i = 1, 2, \ldots, 2m + 1 = 2n - 1$.

**Proof.** Notice first that the equations we get from the first line of $\hat{L}N = NL$ are

$$d_2 - d_1 + x_{m+2} - \tilde{x}_1 = 0 \quad \text{and} \quad d_{m+2} \tilde{x}_1 - d_1 x_1 + b_{1,m+2} - b_{m+1,1} = 0,$$

which is (33) for $i = 1$. Because of the form of the matrices appearing in $\hat{L}N = NL$ the other equations are obtained by shifting all indices by 1, and the result follows. \hfill $\square$

We now reduce these equations to $\text{LV}_b(n, 0)$, by setting $x_i = \tilde{x}_i = 0$ for $i = n + 1, n + 2, \ldots, 2m + 1$ and $b_{i,m+i} = 0$ for $i = 2, 3, \ldots, 2m + 1$, where we recall that $m = n - 1$ and that we denote the single parameter $b_{1,n}$ of $\text{LV}(n, 0)$ as $b_{1,n} = \beta$. The system (33) is then transformed to the following one:

$$\begin{align*}
\tilde{x}_i &= d_{i+1} - d_i, \quad i = 1, \ldots, n - 1, \\
\tilde{x}_n &= x_1 + d_{n+1} - d_n, \\
x_{i+1} &= d_{n+i} - d_{n+i+1}, \quad i = 1, \ldots, n - 1, \\
d_{n+i} \tilde{x}_{i+1} &= d_{i+1} x_{i+1}, \quad i = 1, \ldots, n - 2, \\
d_{n+1} \tilde{x}_1 &= d_1 x_1 - \beta, \\
d_1 \tilde{x}_n &= d_n x_n + \beta,
\end{align*}$$

(34)

where the first three equation are instances of the first equation in (33) and the last three equations of the second one. Before solving the above system, we recall from Section 3.4 the alternative coordinates $u_1, \ldots, u_n$ for $\mathbb{R}^n$, in which the system $\text{LV}_b(n, 0)$ completely separates. They are defined by $u_i = \sum_{j=1}^i x_j$ for all $i = 0, 1, 2, \ldots, n$. For $i = n$, $u_n$ is just the Hamiltonian, $u_n = H = x_1 + x_2 + \ldots + x_n$.

**Proposition 5.3.** For any rational function $R \in \mathbb{R}(x_1, \ldots, x_n)$, different from the $n$ functions $u_i - H$, with $i = 1, \ldots, n$, the reduced compatibility equations (34) have a unique solution for $\tilde{x}_1, \ldots, \tilde{x}_n$ and for $d_2, \ldots, d_{2n-1}$, with $d_1 = R$. It is given by

$$\begin{align*}
\tilde{x}_i &= d_{i+1} - d_i, \quad i = 1, 2, \ldots, n - 1, \\
\tilde{x}_n &= x_1 + d_{n+1} - d_n,
\end{align*}$$
while the second equation says that this constant value is equal to
\[ \Phi \]

The first equation in (36) says that
\[ d_i = \frac{\mathcal{R}(\mathcal{R} + H) - \beta}{\mathcal{R} + H - u_i}, \quad i = 2, 3, \ldots, n, \]
\[ d_{n+i} = \mathcal{R} + H - u_i, \quad i = 1, 2, \ldots, n - 1. \]  \hspace{1cm} (35)

Proof. We first show how the third and fourth equations in (35) are derived from the equations in (34). The last equation is obtained from the third equation in (34):
\[ H - u_i = \sum_{j=i+1}^{n} x_j = \sum_{j=i+1}^{n} (d_{n+j-1} - d_{n+j}) = d_{n+i} - \mathcal{R}, \]
where we have used that, by periodicity, \( d_{2n} = d_{2n+2} = d_1 = \mathcal{R} \). In order to derive the third equation in (35), one first uses the first three equations in (34) to substitute \( \tilde{x}_i \) (\( i = 1, \ldots, n \)) in (34), to obtain, in that order,
\[ d_{i+1}d_{n+i} = d_{i+2}d_{n+1+i}, \quad i = 1, \ldots, n - 2, \]
\[ \mathcal{R}(x_1 + d_{n+1}) = d_2d_{n+1} + \beta. \]  \hspace{1cm} (36)

The first equation in (36) says that \( d_{i+1}d_{n+i} \) is independent of \( i \) for \( i = 1, \ldots, n - 2 \), while the second equation says that this constant value is equal to \( \mathcal{R}(x_1 + d_{n+1}) - \beta \),
\[ d_{i+1}d_{n+i} = \mathcal{R}(x_1 + d_{n+1}) - \beta = \mathcal{R}(\mathcal{R} + H) - \beta, \]  \hspace{1cm} (37)
for \( i = 1, \ldots, n - 2 \). By our assumption on \( \mathcal{R} \), the \( d_{n+i} = \mathcal{R} + H - u_i \) with \( i = 1, \ldots, n - 1 \) are all different from zero, so that we can divide (37) by \( d_{n+i} \). It yields the third equation in (35). This shows that (35) is the only possible solution for (34) with \( d_1 = \mathcal{R} \). That it is indeed a solution is easily verified by substituting the formulas (35) in (34).

We now define a discrete map using the solution of (34) given by (35). Let \( \mathcal{R} \) be a rational function, with \( \mathcal{R} \neq u_i - H \) for all \( i = 1, 2, \ldots, n - 1 \), and let \( \Phi_{\mathcal{R}} \) be the discrete map \( x_i \mapsto \tilde{x}_i \), defined by the formulas
\[ \tilde{x}_1 = \frac{x_1\mathcal{R} - \beta}{\mathcal{R} + H - x_1}, \]
\[ \tilde{x}_i = x_i \frac{\mathcal{R}(\mathcal{R} + H) - \beta}{(\mathcal{R} + H - u_i)(\mathcal{R} + H - u_{i-1})}, \quad i = 2, 3, \ldots, n - 1, \]
\[ \tilde{x}_n = \frac{x_n(\mathcal{R} + H) + \beta}{\mathcal{R} + x_n}. \]  \hspace{1cm} (38)

Using the first \( n \) equations in (34), we get \( \tilde{u}_i = d_{i+1} - \mathcal{R} \) and therefore the map is given in terms of the coordinates \( u_i \) by \( \tilde{u}_n = u_n \) and
\[ \tilde{u}_i = \frac{u_i\mathcal{R} - \beta}{\mathcal{R} + H - u_i}, \quad i = 1, 2, \ldots, n - 1. \]  \hspace{1cm} (39)

Since the discrete map \( \Phi_{\mathcal{R}} \) is by construction isospectral, it has the coefficients of the characteristic polynomial of the Lax matrix \( L \) as invariants. However, as we noted just after equation (21), we get in this way only one invariant, namely the Hamiltonian \( H \). We show in the next proposition that \( \Phi_{\mathcal{R}} \) also preserves the rational first integrals of \( LV_\beta(n, 0) \). By \( \Phi_{\mathcal{R}} \) we will denote the rational map underlying \( \Phi_{\mathcal{R}} \).

**Proposition 5.4.** Let \( P \) be any point of \( \mathbb{R}^n \) for which \( Q := \phi_{\mathcal{R}}(P) \) is defined. Then \( Q \) belongs to the integral curve of the continuous system \( LV_\beta(n, 0) \) starting at \( P \). In particular, the discrete map \( \Phi_{\mathcal{R}} \) preserves all the first integrals of \( LV_\beta(n, 0) \).
Proof. Since $\Phi_R$ is a discrete map and the first integrals of $LV_b(n,0)$ are rational functions, it suffices to show that for a generic $P$ of $\mathbb{R}^n$ for which $Q := \phi_R(P)$ is defined, $Q$ belongs to the integral curve of the continuous system $LV_b(n,0)$ starting at $P$.

We use the notation of the proof of Proposition 3.6: we denote by $\gamma_P$ the integral curve of (15) starting from $P$, and we write $u_i(t) = u_i(\gamma_P(t))$. We denote by $h$ the value of the Hamiltonian at $P$ and by $\Delta_0$ a square root of $h^2 + 4\beta$, which may be real or imaginary. Also, let $r_0$ denote the value of $R$ evaluated at $P$ and $Q = (Q_1, Q_2, \ldots, Q_n)$.

It is clear from the above that we only need to show that for each $P$ such that $Q$ is defined there exists a $t$, depending only on $P$, such that $Q_i = u_i(t)$ for all $i = 1, 2, \ldots, n - 1$. It is also clear that we may consider our system $LV_b(n,0)$ living on $\mathbb{C}^n$ and therefore the integral curves are defined on all of $\mathbb{C}$ minus a discrete set (see Section 3.4 for details and comments).

We only need to consider the case that $\Delta_0 \neq 0$. In this case the solution of $LV_b(n,0)$, for

$$t = -\frac{\ln \left( \frac{h + 2r_0 + \Delta_0}{h + 2r_0 - \Delta_0} \right)}{\Delta_0},$$

gives $Q_i = u_i(t)$ for all $i$, as can be seen by comparing the formulas (39) and the explicit solution of $LV_b(n,0)$ given in Proposition 3.6.

5.3. Integrable discretization of $LV_b(n,0)$. For a general rational function $R$, the discrete map $\Phi_R = \phi_R$ cannot be expected to have any integrability properties. We establish in this subsection a few results under the assumption that $R$ is a first integral of $LV(n,0)$, or under the stronger hypothesis that $R$ depends on $H$ only. We first prove that, under these conditions, $\Phi_R$ is birational.

**Proposition 5.5.** Suppose that $R$ is a first integral of $LV_b(n,0)$. Then $\phi_R$ is a birational map, so that $\Phi_R$ is an algebra automorphism of $\mathbb{R}(x_1, \ldots, x_n)$.

**Proof.** Let $R$ be as announced, so that $\tilde{R} = R$, in view of Proposition 5.4. If we exchange $x_i$ and $\tilde{x}_{n+1-i}$ as well as $d_i$ and $d_{2n+1-i}$ in the equations (34) we get the same set of equations: the first and third equations are permuted, as well as the last ones, while the other two are unchanged. Since we know from Proposition (5.3) that given $d_1 := \tilde{R}$ the reduced compatibility equations (34) have a unique solution for $\tilde{x}_1, \ldots, \tilde{x}_n$ and for $d_2, \ldots, d_{2n-1}$, in terms of $x_1, \ldots, x_n$, this means given $d_1 := R = \tilde{R}$, they also have a unique solution for $x_1, \ldots, x_n$ and for $d_2, \ldots, d_{2n-1}$, in terms of $\tilde{x}_1, \ldots, \tilde{x}_n$. Therefore, the map $\phi_R$, defined by the solutions of the system (34), is birational and so $\Phi_R$ is an algebra automorphism of $\mathbb{R}(x_1, \ldots, x_n)$.

We now prove that $\Phi_R$ is, under the same assumption on $R$, measure preserving.

**Proposition 5.6.** Suppose that $R$ is a first integral of $LV_b(n,0)$. Then the discrete map $\Phi_R$ preserves the rational $n$-form

$$\Omega_b := \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{x_1 x_2 \cdots x_n + \beta x_2 x_3 \cdots x_{n-1}}.$$

**Proof.** We need to show that

$$\frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{x_1 x_2 \cdots x_n + \beta x_2 x_3 \cdots x_{n-1}} = \frac{d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge \cdots \wedge d\tilde{x}_n}{x_1 \tilde{x}_2 \cdots \tilde{x}_n + \beta \tilde{x}_2 \tilde{x}_3 \cdots \tilde{x}_{n-1}}.$$


Since the coordinate change between the coordinates \( u_i \) and \( x_i \) have triangular form, and since the functions \( \tilde{u}_i \) depend in the same way on the \( \tilde{x}_i \), i.e., \( \tilde{u}_i = \sum_{j=1}^{n} \tilde{x}_j \), we have that

\[
\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)} = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(u_1, u_2, \ldots, u_n)} = 1,
\]

where the above determinants are the Jacobian determinants of these two transformations. This implies that we need to show that

\[
\frac{du_1 \wedge du_2 \wedge \cdots \wedge du_n}{x_1 x_2 \ldots x_n + \beta x_2 x_3 \ldots x_{n-1}} = \frac{d\tilde{u}_1 \wedge d\tilde{u}_2 \wedge \cdots \wedge d\tilde{u}_n}{\tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n + \beta \tilde{x}_2 \tilde{x}_3 \ldots \tilde{x}_{n-1}}.
\]

We assume for the moment that \( \mathcal{R} \) is any rational function such that \( \Phi_{\mathcal{R}} \) is well defined and we denote by \( \mathcal{R}_{\tilde{u}_j} \) the partial derivatives \( \frac{\partial \mathcal{R}}{\partial \tilde{u}_j} \) and by \( \tilde{u}_{i,j} \) the partial derivatives \( \frac{\partial \tilde{u}_i}{\partial u_j} \). The explicit formulas (39) for \( \tilde{u}_i \) and (15) for \( u_i \) give, for any \( i = 1, 2, \ldots, n \) and any \( j \notin \{i, n\} \), that

\[
\tilde{u}_{i,j} = \frac{\mathcal{R}_{\tilde{u}_j} u_i}{(\mathcal{R} + H - u_i)^2}, \quad \tilde{u}_{i,i} = \frac{\mathcal{R} \dot{u}_i + \mathcal{R} H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_i)^2}, \quad (40)
\]

and \( \tilde{u}_{n,j} = \delta_{n,j} \), since \( \tilde{u}_n = u_n \). We do not display the formulas for the derivatives \( \tilde{u}_{i,n} \) \( (i = 1, \ldots, n - 1) \) since we do not need them in this proof. Differentiating (39) and rearranging, we get

\[
d\tilde{u}_i = \sum_{j=1}^{n} \tilde{u}_{i,j} du_j, \quad i = 1, 2, \ldots, n.
\]

Taking the wedge product of the above \( n \) equations we obtain

\[
d\tilde{u}_1 \wedge d\tilde{u}_2 \wedge \cdots \wedge d\tilde{u}_n = \det(U) \ du_1 \wedge du_2 \wedge \cdots \wedge du_n,
\]

where \( U = (\tilde{u}_{i,j}) \). The last line of \( U \) is the vector \((0, 0, \ldots, 1)\) and therefore expanding the determinant \( \det(U) \) with respect to the last line we get \( \det(U) = \det(V) \) where \( V \) is the minor of \( U \) obtained by removing its last row and last column. According to the first formula in (40) the determinant of the \((n-1) \times (n-1)\) matrix \( V \) has the following form:

\[
V = \frac{1}{S} \det \begin{pmatrix}
\mathcal{R}_1 \dot{u}_1 + R & \mathcal{R}_2 \dot{u}_2 & \mathcal{R}_3 \dot{u}_3 & \cdots & \mathcal{R}_n \dot{u}_{n-1} \\
\mathcal{R}_1 \dot{u}_1 & \mathcal{R}_2 \dot{u}_2 + R & \mathcal{R}_3 \dot{u}_3 & \cdots & \mathcal{R}_n \dot{u}_{n-1} \\
\mathcal{R}_1 \dot{u}_1 & \mathcal{R}_2 \dot{u}_2 & \mathcal{R}_3 \dot{u}_3 + R & \cdots & \mathcal{R}_n \dot{u}_{n-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mathcal{R}_1 \dot{u}_1 & \mathcal{R}_2 \dot{u}_2 & \mathcal{R}_3 \dot{u}_3 & \cdots & \mathcal{R}_n \dot{u}_{n-1} + R
\end{pmatrix},
\]

where \( S = \prod_{i=1}^{n-1} (\mathcal{R} + H + u_i)^2 \) and \( R = \mathcal{R} H + \mathcal{R}^2 - \beta \). The above matrix is written as \( W + R I_{n-1} \), where \( I_{n-1} \) is the \((n-1) \times (n-1)\) identity matrix and \( W \) has \( n \) equal lines. This means that \( W \) has only two eigenvalues \( \lambda_1 \) and \( \lambda_2 \). The first one is \( \lambda_1 = \sum_{j=1}^{n-1} \mathcal{R}_j \dot{u}_j \), which is of multiplicity 1 and the other one is \( \lambda_2 = 0 \) of multiplicity \( n - 2 \). In the particular case where \( \mathcal{R} \) is a first integral of \( LV_b(n, 0) \), the eigenvalue \( \lambda_1 \) reduces to zero (since \( \dot{u}_n = 0 \)). This shows that, in that case,

\[
\det(V) = \frac{R^{n-1}}{S} = \frac{(\mathcal{R} H + \mathcal{R}^2 - \beta)^{n-1}}{\prod_{i=1}^{n-1} (\mathcal{R} + H + u_i)^2}.
\]
Therefore, what we need to show is that
\[
\prod_{i=1}^{n-1} \frac{\mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_i)^2} = \frac{\tilde{x}_1 \tilde{x}_2 \ldots \tilde{x}_n + \beta \tilde{x}_2 \tilde{x}_3 \ldots \tilde{x}_{n-1}}{x_1 x_2 \ldots x_n + \beta x_2 x_3 \ldots x_{n-1}}.
\]
A comparison with the explicit formulas (38) gives that
\[
\frac{x_2 x_3 \ldots x_{n-1}}{\tilde{x}_2 \tilde{x}_3 \ldots \tilde{x}_{n-1}} \frac{(\mathcal{R}H + \mathcal{R}^2 - \beta)^{n-1}}{\prod_{i=1}^{n-1} (\mathcal{R} + H - u_i)^2} = \frac{\mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_1)(\mathcal{R} + H - u_{n-1})}.
\]
To complete the proof, it remains to be shown that
\[
\frac{\tilde{x}_1 \tilde{x}_n + \beta}{x_1 x_n + \beta} = \frac{\mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_1)(\mathcal{R} + H - u_{n-1})}.
\]
This can be done by substituting the formulas for \(\tilde{x}_1\) and \(\tilde{x}_n\), given in (38). \(\square\)

In order to preserve the Poisson structure, one needs stronger conditions on \(\mathcal{R}\), as given in the following proposition:

**Proposition 5.7.** Suppose that \(\mathcal{R}\) is a rational function of the Hamiltonian \(H\). Then the map \(\Phi_\mathcal{R}\) preserves the Poisson structure \(\Pi_b\).

**Proof.** We give the proof using the coordinates \(u_i\) (see Section 3.4, in particular the formulas (14) for the Poisson structure in terms of these coordinates). According to (14) we need to show that \(\{\tilde{u}_i, \tilde{u}_j\}_b = \tilde{u}_i(\tilde{u}_j - \tilde{u}_i)\) and that \(\tilde{u}_\ell, \tilde{u}_n\) depends only on \(u_n\) for \(1 \leq i < j < n\) and \(1 \leq \ell < n\).

According to (39) \(\tilde{u}_i\) depends only on \(u_i\) and \(H\), so we give only the non-zero derivatives \(\tilde{u}_{i,i}\) and \(\tilde{u}_{i,n}\) for \(i < n\). They are (for \(\tilde{u}_{i,i}\) see the second formula in (40)):
\[
\tilde{u}_{i,i} = \frac{\mathcal{R}H + \mathcal{R}^2 - \beta}{(\mathcal{R} + H - u_i)^2} \quad \text{and} \quad \tilde{u}_{i,n} = \frac{\mathcal{R}H u_i H - \mathcal{R}^2 u_i^2 + \mathcal{R}H \beta - \mathcal{R} u_i + \beta}{(\mathcal{R} + H - u_i)^2},
\]
where \(\mathcal{R}_H = \frac{\partial \mathcal{R}}{\partial H}\). We then have, for all \(i, j\) and \(\ell\) as above,
\[
\{\tilde{u}_i, \tilde{u}_j\}_b = \tilde{u}_{i,i}\tilde{u}_{j,j} + \tilde{u}_{i,j}\tilde{u}_{j,i} - \tilde{u}_{i,n}\tilde{u}_{j,n} \quad \text{and} \quad (u_i - u_j)(\beta - \mathcal{R} u_i)(\mathcal{R}H + \mathcal{R}^2 - \beta) = \tilde{u}_{i,j}(\tilde{u}_j - \tilde{u}_i),
\]
\[
\{\tilde{u}_\ell, H\}_b = \frac{(\mathcal{R}H + \mathcal{R}^2 - \beta)(Hu_\ell - u_\ell^2 + \beta)^2}{(\mathcal{R} + H - u_\ell)^2} = \tilde{u}_\ell(H - \tilde{u}_\ell) + \beta,
\]
which establishes the required equalities, since \(\tilde{u}_n = u_n = H\). \(\square\)

The above propositions 5.4–5.7 lead to the following theorem.

**Theorem 5.8.** Let \(\mathcal{R}\) be a rational function, depending on the Hamiltonian \(H\) only. Then the discrete map \(\Phi_\mathcal{R}\) of \(LV_b(n,0)\) has the following properties:

1. It is birational;
2. It preserves the Poisson structure \(\Pi_b\);
3. It is measure preserving: it preserves the volume form \(\Omega_b\);
4. It is Liouville integrable with \(H\) and the rational functions \(F^b_\ell\) as invariants;
5. It is superintegrable with \(H\) and the rational functions \(F^b_\ell\) and \(G^b_\ell\) as invariants.

Under the weaker hypothesis that \(\mathcal{R}\) depends only on the invariants of \(LV_b(n,0)\), items (1), (3) and (5) still hold, but (2) and (4) may not hold.
5.4. Kahan discretization of $LV_b(n,0)$. In this subsection we consider the Kahan discretization of the systems $LV_b(n,0)$. We show that the Kahan map is of the form $\Phi_R$, for a specific choice of the rational function $R$, depending on the Hamiltonian $H$ only, and so all integrability properties that we have seen in Theorem 5.8 hold for the Kahan map as well.

We first define the Kahan map for $LV_b(n,0)$. Since the Kahan discretization commutes with any linear change of variables, we can do the Kahan discretization in the $u_i$ coordinates, instead of the $x_i$ coordinates, i.e., apply it on the vector field (15). Following the recipe [5], we obtain for the Kahan discretization with step size $2\epsilon$ the following system of equations:

$$\bar{u}_i - u_i = \epsilon u_i (H - \bar{u}_i) + \epsilon \bar{u}_i (H - u_i) + 2\beta, \quad i = 1, 2, \ldots, n - 1,$$

where we have used that $H = u_n$. Since $H$ is a linear first integral of $LV_b(n,0)$, it is an invariant for the Kahan map. The system (42) is diagonal with solution

$$\bar{u}_i = \frac{(1 + \epsilon H)u_i + 2\epsilon \beta}{1 - \epsilon H + 2\epsilon u_i}, \quad i = 1, 2, \ldots, n - 1,$$

and $\bar{u}_n = u_n$. This defines the Kahan map. Comparing the formulas (43) and (39) it is clear that the Kahan map is of the form $\Phi_R$, with

$$R = \frac{1 + \epsilon H}{2\epsilon}.$$  

Notice that $R$ depends on $H$ only. Therefore, we get by Theorem 5.8 the following results on the Kahan discretization of $LV_b(n,0)$, which generalize the results on the integrability of the Kahan discretization of $LV(n,0)$, which were first established in [19]:

**Theorem 5.9.** The Kahan map of $LV_b(n,0)$ has the following properties:

1. It is birational;
2. It preserves the Poisson structure $\Pi_b$;
3. It is measure preserving: it preserves the volume form $\Omega_b$;
4. It is Liouville integrable with $H$ and the rational functions $F^b_\ell$ as invariants;
5. It is superintegrable with $H$ and the rational functions $F^b_\ell$ and $G^b_\ell$ as invariants.

As a byproduct of our analysis, we find that the Kahan map of $LV_b(n,0)$ arises as the compatibility conditions of a linear system. It would be interesting to see if there are other examples where the Kahan map is of this form, as it links the Kahan map to isospectrality, so it may have non-trivial applications to the study of the integrability of the Kahan map of other integrable systems.

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