A novel 8-parameter integrable map in $\mathbb{R}^4$

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Abstract. We present a novel 8-parameter integrable map in $\mathbb{R}^4$. The map is measurepreserving and possesses two functionally independent 2-integrals, as well as a measure-preserving 2-symmetry.

1. Introduction

Discrete integrable systems have attracted a lot of attention in recent years [7]. One of the reasons for this comes from physics: many physical models include discreteness at a fundamental level. Another reason for the increased interest in discrete integrable systems comes from mathematics: in several instances it turns out that discrete integrable systems are more fundamental than continuous (i.e. non-discrete) ones. Prime examples are (i) Integrable partial difference equations (PΔEs), where a single PΔE yields (through the use of vertex operators) an entire infinite hierarchy of integrable partial differential equations [13]; (ii) Discrete Painlevé equations, where the Sakai classification is much richer in the discrete case than in the continuous one [12]; (iii) Darboux polynomials, where in the discrete case unique factorization of the so-called co-factors can be used (which does not exist in the continuous (additive) case) [2, 3].

In this Letter we will be interested in autonomous integrable ordinary difference equations (or maps). Much interest was generated by the discovery of the 18-parameter integrable QRT map in $\mathbb{R}^2$ ([6, 10, 11]). For some other examples in higher dimensions, cf. e.g. Chapter 6 of [7].

A special aspect of the maps we consider in this Letter is that they are an example of integrable maps arising as discretisations of ordinary differential equations (ODEs). Earlier examples of this arose using the Kahan discretisation of first-order quadratic ODEs (cf. [5] and references therein), by the discretisation of ODEs of order 1 and arbitrary degree using polarisation methods [1], and by the methods in [8] for the discretisation of of ODEs of order $o$ and degree $o + 1$, cf. also [9].

In section 3 we present a novel integrable 8-parameter map in $\mathbb{R}^4$. This map generalizes a 5-parameter map in $\mathbb{R}^4$ found earlier in [4] to the inhomogeneous case, and because the derivation of the novel map may be somewhat mysterious if the reader is unfamiliar with the previous map and its derivation, we summarise the latter in section 2.
2. What went before

In [4] Celledoni, McLachlan, McLaren, Owren and Quispel introduced a novel integrable map in \( \mathbb{R}^4 \). It was constructed as follows.

The authors considered the homogeneous quartic Hamiltonian

\[
H = aq^4 + 4bq^3p + 6cq^2p^2 + 4dpq^3 + ep^4,
\]

where \( a, b, c, d \) and \( e \) are 5 arbitrary parameters.

This gave rise to an ordinary differential equation (ODE)

\[
\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = f_3 \left( \begin{pmatrix} q \\ p \end{pmatrix} \right),
\]

where the cubic vector field \( f_3 \) is defined by

\[
f_3 \left( \begin{pmatrix} q \\ p \end{pmatrix} \right) = \begin{pmatrix} 4bq^3 + 12cq^2p + 12dpq^2 + 4ep^3 \\ -4aq^3 - 12bq^2p - 12cq^2p - 4dp^3 \end{pmatrix}.
\]

Defining \( x := \begin{pmatrix} q \\ p \end{pmatrix} \), and introducing the timestep \( h \), the vector field (2) was then discretized:

\[
\frac{x_{n+2} - x_n}{2h} = F_3(x_n, x_{n+1}, x_{n+2}),
\]

where \( F_3 \) was defined using polarization, i.e.

\[
F_3(x_n, x_{n+1}, x_{n+2}) := \frac{9}{2}f_3 \left( \begin{pmatrix} x_n + x_{n+1} + x_{n+2} \\ 3 \end{pmatrix} \right) - 4f_3 \left( \begin{pmatrix} x_n + x_{n+1} \\ 2 \end{pmatrix} \right)
\]

\[
- \frac{4}{3}f_3 \left( \begin{pmatrix} x_n + x_{n+2} \\ 2 \end{pmatrix} \right) - \frac{4}{3}f_3 \left( \begin{pmatrix} x_{n+1} + x_{n+2} \\ 2 \end{pmatrix} \right)
\]

\[
+ \frac{1}{6}f_3(x_n) + \frac{1}{6}f_3(x_{n+1}) + \frac{1}{6}f_3(x_{n+2}).
\]

By construction, the rhs of (5) is linear in \( x_{n+2} \) and \( x_n \) for cubic vector fields, i.e. \( \Delta \) represents a birational map (see [4]), and it was shown that this map possesses two functionally independent 2-integrals (recall that a 2-integral of a map \( \phi \) is defined to be an integral of \( \phi \circ \phi \)):

\[
I(q_n, q_{n+1}, q_{n+1}) = q_{n+1}q_{n+1} - p_{n+1}q_{n+1}
\]

\[
I(q_{n+1}, q_{n+1}, q_{n+1}, p_{n+1}) = q_{n+1}q_{n+1} - p_{n+1}q_{n+1},
\]

where \( q_{n+2} \) and \( p_{n+2} \) should be eliminated from (7) using \( \Delta \).

Note that (6) above does not depend on the parameters \( a, b, c, d, e \) (in contrast to (7), which will depend on the parameters once expressed in \( q_n, q_{n+1}, p_n, p_{n+1} \)).

The map (5) also preserves the measure

\[
\frac{dq_n \wedge dp_n \wedge dq_{n+1} \wedge dp_{n+1}}{1 + 4h^2 \Delta_1},
\]

where

\[
\Delta_1 = \begin{vmatrix} c & d & p_n^2 & p_{n+1}^2 \\ d & e & p_n & p_{n+1} \end{vmatrix} + \begin{vmatrix} b & c & p_n q_n^2 & q_{n+1} q_{n+1} \\ a & d & q_n & p_{n+1} \end{vmatrix} + \begin{vmatrix} a & c & p_n q_n q_{n+1} & q_{n+1} \\ b & c & q_n & q_{n+1} \end{vmatrix}.
\]

Erratum: In eqs (4.1) of [4], \( 1 - 4h^2 \Delta \) should read \( 1 + 4h^2 \Delta \). Their \( \Delta \) is our \( \Delta_1 \).
Finally, the map (4) is invariant under the scaling symmetry group
\[ x_n \to a^{(-1)^n} x_n. \] (10)

3. A novel 8-parameter integrable map in \( \mathbb{R}^4 \)

We now generalise the treatment of section 2 to the non-homogeneous Hamiltonian
\[ H = a q^4 + 4 b q^3 p + 6 c q^2 p^2 + 4 d q p^3 + e p^4 + \frac{1}{2} \rho q^2 + \sigma q p + \frac{1}{2} \tau p^2, \] (11)
where \( a, b, c, d, e, \rho, \sigma \) and \( \tau \) are 8 arbitrary parameters.

This gives rise to an ODE
\[ \frac{d}{dt} \left( \begin{array}{c} q \\ p \end{array} \right) = f_3 \left( \begin{array}{c} q \\ p \end{array} \right) + f_1 \left( \begin{array}{c} q \\ p \end{array} \right), \] (12)
where the cubic part of the vector field, \( f_3 \), is again given by (5), whereas the linear part \( f_1 \) is given by
\[ f_1 \left( \begin{array}{c} q \\ p \end{array} \right) = \left( \begin{array}{c} \sigma q + \tau p \\ -\rho q - \sigma p \end{array} \right). \] (13)

We now discretise the cubic part resp. the linear part of the vector field in different ways:
\[ \frac{x_{n+2} - x_n}{2h} = F_3(x_n, x_{n+1}, x_{n+2}) + F_1(x_n, x_{n+2}), \] (14)
where \( F_3 \) is again defined by (5), but \( F_1 \) is defined by a kind of midpoint rule:
\[ F_1(x_n, x_{n+2}) = f_1 \left( \frac{x_n + x_{n+2}}{2} \right). \] (15)

It follows that equation (14) again defines a birational map, and, importantly, it again preserves the scaling symmetry (10). (Indeed the latter is the primary reason we use the discretization (15)).

Two questions thus remain:
(i) Does eq (14) preserve two 2-integrals?
(ii) Is eq (14) measure-preserving?

The answer to both these questions will turn out to be positive.

We actually had numerical evidence several years ago that the map (14) (or at least a special case of it) was integrable. However it has taken us until now to actually find closed-form expressions for the preserved measure and for the 2-integrals.

A first clue to the identity of a possible 2-integral of (14) came when we were carrying out experimental mathematical computations (in the sense of [1]) to find “discrete Darboux polynomials” for the map (14) (cf. [2] and [3]). This gave a hint that a possible quadratic 2-integral \( I(q_n, p_n, q_{n+1}, p_{n+1}) \) generalising (6), might exist for the map (14).

However, the mathematical complexity of the general 8-parameter map (14) was too great to carry out these computations for a completely general quadratic 2-integral in four variables with all 8 parameters symbolic.

Our process of discovery thus proceeded in two steps:

Step 1: Taking all parameters \( a, b, c, d, e, \rho, \sigma, \tau \) and \( h \) to be random integers, and assuming the 2-integral was an arbitrary quadratic function in four variables (with coefficients to be determined),
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we computed the 2-integral for a large number of random choices of the integer parameters. In each case, it turned out that the same six coefficients in the quadratic function were zero, i.e. the 2-integral always had the form

$$I(q_n, p_n, q_{n+1}, p_{n+1}) = Aq_nq_{n+1} + Bp_np_{n+1} + Cq_np_{n+1} + Dp_qn_{n+1},$$

(16)

where $A, B, C,$ and $D$ depended on the parameters in a way as yet to be determined.

Step 2: Now taking all parameters $a, b, c, d, e, \rho, \sigma, \tau$ and $h$ symbolic, and assuming the 2-integral $I$ had the special quadratic form (16), we found

$$I(q_n, p_n, q_{n+1}, p_{n+1}) = (h\sigma + 1)q_np_{n+1} + (h\sigma - 1)p_qn_{n+1} + h\rho q_n q_{n+1} + h\tau p_n p_{n+1}.$$  

(17)

Notes:

(i) The 2-integral (17) is invariant under the scaling symmetry group (10).

(ii) Equation (17) reduces to equation (6) for $h = 0$.

(iii) Like equation (6), equation (17) does not explicitly depend on the parameters $a, b, c, d, e$.

Once we had the putative equation (17), it was not difficult to verify using symbolic computation that $I(q_n, p_n, q_{n+1}, p_{n+1})$ and $I(q_{n+1}, p_{n+1}, q_{n+2}, p_{n+2})$ are indeed functionally independent 2-integrals of (14).

The map (14) preserves the measure

$$\frac{dq_n \wedge dp_n \wedge dq_{n+1} \wedge dp_{n+1}}{1 + 4h^2(\Delta_1 + \Delta_2)},$$

(18)

where the quartic function $\Delta_1$ is given by (9) and the quadratic function $\Delta_2$ is given by

$$\Delta_2 = \frac{1}{2} \left( \begin{array}{cccc} a & b & c & d \\ \sigma & \tau & \sigma & \rho \\ \rho & \sigma & \tau \end{array} \right) q_nq_{n+1} + \frac{1}{2} \left( \begin{array}{cccc} c & b & d & e \\ \sigma & \tau & \sigma & \rho \\ \rho & \sigma & \tau \end{array} \right) p_np_{n+1}.$$  

(19)

Finally, the map (14) is again invariant under the scaling symmetry group (10).

Theorem The birational map defined by (14) is integrable.

Proof The proof of integrability is identical to the proof in [4]. The second iterate of the map defined by (14) has a one-dimensional measure-preserving symmetry group. The map thus descends to a measure-preserving map on the three-dimensional quotient. The two integrals of the second iterate of the map are invariant under the symmetry and therefore also pass to the quotient. This yields a three-dimensional measure-preserving map with two integrals, which is thus integrable.

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