SPIN CALOGERO PARTICLES AND BISPECTRAL SOLUTIONS OF THE MATRIX KP HIERARCHY

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ABSTRACT. Pairs of $n \times n$ matrices whose commutator differ from the identity by a matrix of rank $r$ are used to construct bispectral differential operators with $r \times r$ matrix coefficients satisfying the Lax equations of the Matrix KP hierarchy. Moreover, the bispectral involution on these operators has dynamical significance for the spin Calogero particles system whose phase space such pairs represent. In the case $r = 1$, this reproduces well-known results of Wilson and others from the 1990’s relating (spinless) Calogero-Moser systems to the bispectrality of (scalar) differential operators. This new class of pairs $(L, \Lambda)$ of bispectral matrix differential operators is different than those previously studied in that $L$ acts from the left, but $\Lambda$ from the right on a common $r \times r$ eigenmatrix.

1. INTRODUCTION

1.1. Background. Let

$$CM_n = \{(X, Z) \in M_{n \times n} | \text{rank } ([X, Z] - I) = 1\}$$

be the set of pairs of complex $n \times n$ matrices whose commutator differs from the identity by a matrix of rank one. This space arises naturally in the study of the integrable Calogero-Moser-Sutherland particle system [vDV00, Pol06]. In particular, the eigenvalues of the time dependent matrix $X + itZ^{i-1}$ move according to the $i$th Hamiltonian of this integrable hierarchy and even allows the continuation of the dynamics through collisions [KKS78, Wil98].

The KP hierarchy is the collection of nonlinear partial differential equations

$$\frac{\partial}{\partial t_i} L = [(L^i), L], \quad i = 1, 2, 3, \ldots$$

for a monic pseudo-differential operator $L$ of order one whose coefficients are scalar functions depending on the time variables $t_i$ [SS83, SW85]. If the coefficients of $L$ are further assumed to be rational functions of $t_1$ which vanish as $t_1 \to \infty$, then the solutions can be written in terms of the matrices in $CM_n$ and the poles move according to the dynamics of the Calogero-Moser-Sutherland system [AMM77, Kri79, Shi94, Wil98]. This was interpreted as a special case of a more general relationship between “rank one conditions” and the KP hierarchy in [GK06].

Although it seems at first to be quite different in nature, having no obvious dynamical interpretation, the bispectral problem [HK98] turns out to be another aspect of this relationship between the KP hierarchy and the Calogero-Moser-Sutherland particle system. As originally formulated in [DG86], the bispectral problem seeks to find scalar coefficient ordinary differential operators $L$ and $\Lambda$ in

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the variables $x$ and $z$ respectively such that there is a common eigenfunction $\psi(x, z)$ satisfying the eigenvalue equations

\begin{align}
L\psi &= p(z)\psi \\
\Lambda\psi &= \pi(x)\psi
\end{align}

for non-constant functions $p$ and $\pi$. As it turns out, if one additionally requires the operator $L$ to commute with another ordinary differential operator of relatively prime order, the solutions to the bispectral problem are exactly the same as the rational solutions to the KP hierarchy mentioned above [Wil93]. (Specifically, up to trivial renormalizations, the bispectral operators are the ordinary differential operators that commute with the pseudo-differential operator $\mathcal{L}$ with the identification $x = t_1$.) Moreover, the bispectral property for these operators is a manifestation of the involution on $\text{CM}_n$ given by $(X, Z) \mapsto (Z^\top, X^\top)$ which linearizes the dynamics of the particle system [Kas95, Wil98].

In [Wil98] and the conference proceedings [Wil00], Wilson suggests that the correspondence should generalize naturally to the case in which the “rank one condition” (1) is replaced with a “rank $r$ condition”. Indeed, various authors have demonstrated that a similar relationship exists between the matrices whose commutator differs from the identity by a matrix of rank $r$ ($1 < r \leq n$), the “spin generalization” of the integrable particle system, and matrix generalizations of the KP hierarchy. In particular, the spin generalized system [GH84] was shown to be related to the matrix KP equation in [KBBT95] and to the multi-component KP hierarchy in [BZN07], and rather general rank $r$ conditions were shown to produce solutions to the matrix potential KP hierarchy in [DMH07]. None of these, however, has specifically addressed the question of whether and how the results relating to bispectrality generalize to the matrix case.

1.2. Outline. Section 1.3 will introduce a version of the bispectral problem in which matrix differential operators act on a common eigenmatrix from opposite sides. The main result of this paper will be to demonstrate that this formulation allows for the generalization of the results on bispectrality to the spin version of the particle system and the matrix KP hierarchy.

Section 2 introduces the generalization of (1) to the case of arbitrary rank and relates it to the dynamics of the spin Calogero particle system. Special attention is paid to the block decompositions of the associated operators corresponding to the generalized eigenspaces of the matrix $Z$.

A wave function and pseudo-differential operator are constructed from a choice of $n \times n$ matrices satisfying the rank $r$ condition in Section 3. This $r \times r$ matrix pseudo-differential operator is shown to satisfy the Lax equation of the KP hierarchy (2). A key component of the proof is the explicit construction of an $rrn$-dimensional space of finitely supported distributions in the spectral parameter which annihilate the wave function.

Several obvious group actions on the space of matrices satisfying the rank $r$ condition are investigated in Section 4 with emphasis on their effect on the corresponding KP solution. Of special interest is the bispectral involution which has the effect of exchanging variables and transposing the wave function.

The main theorem is the construction in Section 5 of commutative rings of matrix differential operators in $x$ and $z$ and the demonstration that they have the KP wave function as a common eigenfunction.
A final section contains closing remarks and lists problems for future research on this topic.

1.3. **Notation and Matrix Bispectrality.** We will make use of the notation $M_L$ and $M_R$ to distinguish between the cases in which the operator $M$ is acting from the left or the right, respectively. So, for instance, if $M$ and $P$ are both $r \times r$ matrices, then

$$[M, P] = (M_L - M_R)P.$$  

Similarly, if

$$L = \sum_{i=0}^{N} M_i(x) \partial_x^i$$

is an ordinary differential operator in $x$ of degree $N$ with coefficients $M_i(x)$ that are $r \times r$ matrices and $\psi(x)$ is an $r \times r$ matrix function we define

$$L_L(\psi) = L(\psi) = \sum_{i=0}^{N} M_i(x) \left( \frac{\partial^i}{\partial x^i} \psi(x) \right),$$

as usual. However, the operator can also act from the right

$$L_R(\psi) = \sum_{i=0}^{N} \left( \frac{\partial^i}{\partial x^i} \psi(x) \right) M_i(x).$$

Equivalently, if we denote by $L^\top$ the differential operator with coefficients $M_i^\top$ that are the ordinary matrix transpose of the coefficients of $L$, we can say

$$L_R(\psi) = (L^\top(\psi^\top))^\top.$$  

**Definition 1.1.** A bispectral triple $(L, \Lambda, \psi)$ consists of a differential operator $L$ in $x$ as in (4), a differential operator $\Lambda$ in the variable $z$ also having $r \times r$ matrix coefficients, and an $r \times r$ matrix function $\psi(x, z)$ of $x$ and $z$ satisfying the equations

$$L_L(\psi) = p(z)\psi \quad \text{and} \quad \Lambda_R(\psi) = \pi(x)\psi,$$

where $p(z)$ and $\pi(x)$ are non-constant, scalar eigenvalues.

This seems to be a natural matrix generalization of the scalar bispectral problem for differential operators considered in [DG86]. However, we note that this differs from the matrix generalization previously considered by Zubelli [Zub90] in which both operators acted from the same side, and also from the “bundle bispectrality” considered by Sakhnovich-Zubelli [SZ01] where the operators $L$ and $\Lambda$ were allowed to depend on both variables. (Here we are interested only in the case that $L$ is independent of $z$ and $\Lambda$ is independent of $x$.)

In the rest of the paper we will use the notation $I_k$ for the $k \times k$ identity matrix. Also we will abuse notation by using the symbol $I$ to denote the identity transformation on many different vector spaces whenever its use should make it clear which is intended.
2. Spin Calogero Matrices

Let $\text{sCM}_n^r$ be the the set of 4-tuples of matrices $(X, Z, A, B)$ such that the $n \times n$ commutator $[X, Z]$ differs from the identity by the rank $r$ matrix $BA$:

$$\text{(7)} \ \text{sCM}_n^r = \{(X, Z, A, B) \mid X, Z \in M_{n \times n}, \ A, B^T \in M_{r \times n}, \ [X, Z] - I = BA \neq 0\}.$$  

This space arises naturally in the description of generic initial conditions of the Spin Calogero particles, as we will see below. More importantly, the dynamics linearizes there (when one considers the phase space to be $\text{sCM}_n^r$ modulo the action of $\text{GL}(n)$ to be described in the section on symmetries).

Let $q_i$ ($1 \leq i \leq n$) be the distinct positions of $n$ particles on the complex plane, $\dot{q}_i$ their momenta, and $f_{ij} = \beta_j \alpha_i$ be their “spins” represented as the products of $n$ column $r$-vectors $\alpha_i$ and $n$ row $r$-vectors $\beta_j$ subject to the constraint $f_{ii} = -1$.

We associate to this data the matrices $(X, Z, A, B) \in \text{sCM}_n^r$ in the form

$$X_{ij} = q_i \delta_{ij} \quad Z_{ij} = q_i \delta_{ij} + (1 - \delta_{ij}) \frac{f_{ij}}{q_i - q_j} \quad A = (\alpha_1 \cdots \alpha_n) \quad B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}.$$  

The dynamics of the eigenvalues of $X + itZ^{r-1}$ are governed by the Hamiltonian $H = \text{tr}Z^r$. This is the spin Calogero system [GH84, Kri79]. In the special case $r = 1$, this reduces to the more famous (spinless) Calogero-Moser-Sutherland particle system [vDV00].

2.1. Block Decomposition. For a fixed choice of $(X, Z, A, B) \in \text{sCM}_n^r$ we get a decomposition of $V = \mathbb{C}^n$ into generalized eigenspaces of $Z$:

$$V = \bigoplus_{\lambda} V_\lambda \quad V_\lambda = \{v \in \mathbb{C}^n \mid (Z - \lambda I)^k v = 0 \text{ for some } k \geq 0\}.$$  

The restriction of $Z$ to $V_\lambda$ will be denoted by

$$Z_\lambda = \lambda I + N_\lambda$$  

where $I$ is the identity operator on $V_\lambda$ and $N_\lambda$ is nilpotent. Rational expressions in $zI - Z_\lambda$ below will always be interpreted by expansion in positive powers of the nilpotent $N_\lambda$.

We will utilize subscripts $\lambda$ and $\mu$ which will run over the eigenvalues of $Z$ to similarly denote the blocks of other linear operators associated to this decomposition of $\mathbb{C}^n$. Specifically, $A_\lambda : V_\lambda \rightarrow \mathbb{C}^r$ will be the restriction of the map $A, B_\lambda : \mathbb{C}^r \rightarrow V_\lambda$ will be the map $B$ followed by projection onto $V_\lambda$, and for a linear operator $M$ from $\mathbb{C}^r$ to itself (such as $X$) $M_{\lambda, \mu}$ will be the block corresponding to the map from $V_\mu$ to $V_\lambda$.

The $\text{sCM}$ condition (7) involves the commutator $[X, Z] = (Z_R - Z_L)(X)$. Interestingly, although the operator $Z_R - Z_L = -\text{ad}(Z)$ is not invertible, its “off-diagonal” action is invertible which allows us to solve for $X_{\lambda \mu}$ when $\mu \neq \lambda$.

**Lemma 2.1:** Let $\lambda \neq \mu$ be generalized eigenvalues of $Z$. Then

$$X_{\mu \lambda} = [(Z_{\lambda})_R - (Z_{\mu})_L]^{-1} B_\mu A_\lambda = \sum_{k \geq 0} \frac{((N_\lambda)_R - (N_\mu)_L)^k}{(\lambda - \mu)^{k+1}} B_\mu A_\lambda.$$
Proof: Since

\[(Z_R - Z_L)_{\mu \lambda} = (\lambda - \mu)I_{\mu \lambda} + (N_{\lambda})_R - (N_{\mu})_L\]

differs from the a nonzero multiple of the identity by a nilpotent matrix, it is invertible. Specifically, we may invert it in general using

\[(Z_R - Z_L)^{-1}_{\mu \lambda} = \sum_{k \geq 0} \frac{((N_{\lambda})_R - (N_{\mu})_L)^k}{(\lambda - \mu)^{k+1}}.\]

Applying this when solving \([X, Z] - I = BA\) for any off-diagonal block of \(X\) yields the claimed formula.

It will later be necessary to evaluate residues of matrix functions written in terms of the blocks \(Z_{\lambda}\). For this purpose the following “obvious” lemma will be useful.

For convenience we introduce notation for “divided derivatives”:

\[f[k] = \frac{1}{k!} \frac{d^k f}{dz^k}.\]

Lemma 2.2: Let \(f(z)\) be a rational function that is regular at \(z = \lambda\), then

\[\text{Res}_{z=\lambda} \left( \frac{f(z)}{(zI - Z_{\lambda})^{k+1}} \right) = f[k](Z_{\lambda}).\]

Proof:

\[\text{Res}_{z=\lambda} \left( \frac{f(z)}{(zI - Z_{\lambda})^{k+1}} \right) = \text{Res}_{z=\lambda} \left( f(z) \frac{(-\partial_z)^k}{k!} \frac{1}{zI - Z_{\lambda}} \right)\]

\[= \text{Res}_{z=\lambda} \left( f[k](z) \frac{1}{zI - Z_{\lambda}} \right) = \text{Res}_{z=\lambda} \left( f[k](z) \sum_{s \geq 0} \frac{N_s^\lambda}{(z - \lambda)^{s+1}} \right)\]

\[= \text{Res}_{z=\lambda} \left( f[k](z) \sum_{s \geq 0} \left[ \frac{(-\partial_z)^s}{s!} \frac{N_s^\lambda}{z - \lambda} \right] \right) = \text{Res}_{z=\lambda} \left( \sum_{s \geq 0} \frac{\partial_z^s f[k](z)N_s^\lambda}{z - \lambda} \right)\]

\[= \text{Res}_{z=\lambda} \left( f[k](z + N_{\lambda}) \frac{1}{z - \lambda} \right) = f[k](Z_{\lambda}).\]

Of course, in the above lemma \(f[k](Z_{\lambda})\) is a matrix and may not commute with other matrices appearing. So, one needs a little care in applying the Lemma 2.2.

3. Matrix KP Hierarchy

Let \(\nu = (X, Z, A, B) \in s\text{CM}_n^r\) and associate to it the wave function \(\psi_\nu\) depending on the spectral parameter \(z\) and the times \(\vec{t} = (t_1, t_2, t_3, \ldots)\):

\[(8) \quad \psi_\nu(\vec{t}, z) = \gamma(\vec{t}, z) \left( I_r + A \hat{X}^{-1}(zI_n - Z)^{-1} B \right),\]

where\(^1\)

\[\hat{X} = \hat{X}(\vec{t}) = \sum_{i=1}^{\infty} it_i Z_i^{\mu - 1} - X, \quad \text{and} \quad \gamma(\vec{t}, z) = \exp \left( \sum_{i=1}^{\infty} t_i z_i \right).\]

\(^1\)Note that the dependence on \(t_i\) in \(-\hat{X}\) is such that its eigenvalue dynamics are governed by the \(i^{th}\) spin Calogero Hamiltonian.
If \( q_\nu(z) = \det(zI - Z) \) is the characteristic polynomial of \( Z \) then the wave function (8) can be multiplied by \( q_\nu \) and an exponential so as to yield a polynomial in \( z \) with coefficients that are rational in the times

\[
K(\vec{t}, z) = \gamma^{-1}(\vec{t}, z)\psi_\nu(\vec{t}, z)q_\nu(z).
\]

Indeed, \( q_\nu(z)(zI - Z)^{-1} \) is the classical adjoint of \( zI_n - Z \) is and hence polynomial in \( z \). Letting \( \partial = \frac{\partial}{\partial z} \) be the differential operator in \( x = t_1 \), we note that the ordinary differential operator \( K_\nu = K(\vec{t}, \partial) \) satisfies

\[
\psi_\nu(\vec{t}, z) = \frac{1}{q_\nu(z)}K_\nu \gamma(\vec{t}, z).
\]

The main goal of this section is to prove that the pseudo-differential operator

\[
\mathcal{L}_\nu = K_\nu \circ \partial \circ K_\nu^{-1}
\]

is a solution to the matrix KP hierarchy in that it satisfies the Lax equation (2). As in [Wil93] (see also [Kas95, SW85]), the proof will involve identifying finitely supported distributions in \( z \) that annihilate the function \( \psi_\nu \).

3.1. Conditions satisfied by \( \psi_\nu \). Consider a generalized eigenvalue \( \lambda \) of \( Z \) with multiplicity \( \ell \) and use the notation of Section 2.1 to denote by \( Z_\lambda, X_\lambda, A_\lambda, \) etc. the blocks of the operators \( X, Z, A \) and \( B \). Let \( v \in \mathbb{C}^{r \ell + \ell} \) have the decomposition

\[
v = \begin{pmatrix}
v_0 \\
v_1 \\
v_2 \\
\vdots \\
v_{r-1} \\
w
\end{pmatrix}
\]

where \( v_i \in \mathbb{C}^r \) and \( w \in \mathbb{C}^\ell \) and define the distribution \( c_{\nu, \lambda} \) taking \( r \times r \) matrix functions of \( z \) to \( r \) component constant vectors by the formula

\[
c_{\nu, \lambda}(f(z)) = \text{Res}_{z=\lambda} \left( f(z) \cdot \left( A_\lambda(zI - Z_\lambda)^{-1}w + \sum_{i=0}^{\ell-1} (z - \lambda)^i v_i \right) \right).
\]

In this section we will show that there are \( r \ell \) linearly independent distributions of this form satisfying \( c_{\nu, \lambda}(\psi_\nu(\vec{t}, z)) = 0 \). Consequently, by running through all of the eigenvalues of \( Z \) we obtain in this manner an \( rm \)-dimensional space of conditions satisfied by the wave function. Indeed, if \( \{\lambda_i\} \) are the generalized eigenvalues of the \( n \times n \) matrix \( Z \) with multiplicities \( \{\ell_i\} \), then \( n = \sum \ell_i \).

Consider the \( r \times (r \ell + \ell) \) matrix

\[
\Gamma_\lambda = \begin{pmatrix}
B_\lambda & N_\lambda B_\lambda & N_\lambda^2 B_\lambda & \cdots & N_\lambda^{r-1} B_\lambda & -X_\lambda & X_\lambda & \cdots & X_\lambda & \cdots
\end{pmatrix}.
\]

Lemma 3.1: If \( v \in \ker \Gamma_\lambda \) then \( c_{\nu, \lambda}(\psi_\nu(\vec{t}, z)) = 0 \) for all values of the variables \( \vec{t} \).

Proof: Note first that \( \psi_\nu(\vec{t}, z) \) has the block decomposition

\[
\psi_\nu(\vec{t}, z) = \gamma(z, t) \left( I - \sum_{\kappa, \mu} A_{\kappa}(X^{-1})_{\kappa\mu}(zI - Z_\mu)^{-1}B_\mu \right)
\]

where again the sum is taken over all (not necessarily distinct) pairs of generalized eigenvalues \( \kappa \) and \( \mu \) of \( Z \).
Now, we wish to use Lemma 2.2 to expand the residue in (11) where \(f(z)\) is replaced by (12). It will be convenient to introduce the abbreviation
\[
C_\mu = - \sum_n A_n \tilde{X}_{n\mu}^{-1},
\]
so that we have
\[
\psi_\nu(\vec{t}, z) = \gamma(\vec{t}, z) \left( I + \sum_\mu C_\mu (z I - Z_\mu)^{-1} B_\mu \right)
\]
and
\[
A_\lambda = - \sum_\mu C_\mu \tilde{X}_{\mu\lambda}.
\]

The various contributions to the residue are usefully organized according to dependence on \(C_\mu\). First of all, there is the contribution independent of \(C_\mu\). It is given by
\[
\text{(A)} \quad \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) A_\lambda (z I - Z_\lambda)^{-1} w \right) = A_\lambda \gamma(\vec{t}, Z_\lambda) w.
\]
Making use of Lemma 2.1 one finds that the contributions containing \(C_\mu\) for \(\mu \neq \lambda\) are
\[
\text{(B)} \quad \sum_{\mu \neq \lambda} \text{Res}_{z=\lambda} \left( (z I - Z_\mu)^{-1} B_\mu A_\lambda (z I - Z_\lambda)^{-1} w \gamma(\vec{t}, z) \right) = C_\mu \tilde{X}_{\mu\lambda} \gamma(\vec{t}, Z_\lambda) w.
\]
Next we turn to the terms involving \(C_\lambda\). The first one is
\[
\text{(C)} \quad \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-1} B_\lambda \sum_{i=0}^{\ell-1} (z - \lambda)^i v_i \right)
\]
\[
= C_\mu \sum_{i,j=0}^{\ell-1} \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{i+1} N_\lambda^i B_\lambda (z - \lambda)^j v_j \right)
\]
\[
= C_\lambda \gamma(\vec{t}, Z_\lambda) \sum_{i=0}^{\ell-1} N_\lambda^i B_\lambda v_i.
\]
The other term linear in \(C_\lambda\) is
\[
\text{(D)} \quad \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-1} B_\lambda \gamma(\vec{t}, Z_\lambda) \right)
\]
\[
= C_\lambda \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-1} (X_{\lambda\lambda}, Z_\lambda) - I \right) (z I - Z_\lambda)^{-1} w
\]
\[
= - C_\lambda \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-2} w \right)
\]
\[
+ C_\lambda \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-1} (X_{\lambda\lambda}, Z_\lambda) - z \right) (z I - Z_\lambda)^{-1} w
\]
\[
= - C_\lambda \gamma(\vec{t}, Z_\lambda) w - \lambda \text{Res}_{z=\lambda} \left( \gamma(\vec{t}, z) (z I - Z_\lambda)^{-1} X_{\lambda\lambda} w \right)
\]
\[
+ C_\lambda \text{Res}_{z=\lambda} \left( X_{\lambda\lambda} (z I - Z_\lambda)^{-1} \gamma(\vec{t}, z) \right)
\]
\[
= - C_\lambda \gamma(\vec{t}, Z_\lambda) w - C_\lambda \gamma(\vec{t}, Z_\lambda) X_{\lambda\lambda} w + C_\lambda \gamma(\vec{t}, Z_\lambda) X_{\lambda\lambda} w
\]
\[
= C_\lambda \tilde{X}_{\lambda\lambda} \gamma(\vec{t}, Z_\lambda) w - C_\lambda \gamma(\vec{t}, Z_\lambda) X_{\lambda\lambda} w.
\]
Since \( v \in \ker \Gamma_\lambda \) is equivalent to the statement
\[
\ell^{-1} \sum_{k=0}^{\ell-1} N_\lambda^k B_\lambda v_k - X_{\lambda\lambda} w = 0,
\]
we see that (C) cancels against the second term in (D). So, combining all four terms gives
\[
(A) + (B) + (C) + (D) = A_\lambda \gamma(\vec{t}, Z_\lambda) w + \sum_{\mu \neq \lambda} C_\mu \bar{X}_{\mu\lambda} \gamma(\vec{t}, Z_\lambda) w + C_\lambda \bar{X}_{\lambda\lambda} \gamma(\vec{t}, Z_\lambda) w.
\]
Applying (14) shows that this is equal to zero as required.

**Lemma 3.2:** The distributions \( c_{v,\lambda} \) for \( v \in \ker \Gamma_\lambda \) form an \( r\ell \)-dimensional space.

**Proof:** Note that the map \( \Omega : v \in \ker \Gamma_\lambda \mapsto c_{v,\lambda} \) is itself a linear map. What we need to prove, therefore is that
\[
\dim \ker \Gamma_\lambda - \dim \ker \Omega = r\ell.
\]
A vector \( v \) clearly does not lie in the kernel of \( \Omega \) if \( v_i \neq 0 \) for any \( i \). The dimension of the kernel of \( \Omega \) is therefore equal to the dimension of the space of vectors \( w \) with the property that \( X_{\lambda\lambda} w = 0 \) and \( A_\lambda (zI - Z_\lambda)^{-1} w = 0 \). In fact, we will show that the only such \( w \) is the zero vector (and hence that \( \dim \ker \Omega = 0 \)). Beginning with the fact that
\[
[(zI - Z_\lambda), X_{\lambda\lambda}] - B_\lambda A_\lambda = I.
\]
Multiplying by \( (zI - Z_\lambda)^{-1} \) on the right, applying both sides of the resulting equation to \( w \) and then multiplying by \( (zI - Z_\lambda)^{-1} \) on the left gives us that
\[
X_{\lambda\lambda}(zI - Z_\lambda)^{-1} w = (zI - Z_\lambda)^{-2} w.
\]
Expanding both sides of this equation in terms of powers of \( (z - \lambda) \) and equating like powers gives us that
\[
X_{\lambda\lambda} N_\lambda^k w = k N_\lambda^{k-1} w, \quad \text{for } k > 0.
\]
Since \( N_\lambda \) is nilpotent, for a sufficiently large \( k \) the left-hand side is equal to zero. But the equation then tells us that \( N_\lambda^{k-1} w \) is then also equal to zero, which again means that the left hand side would be zero for a smaller value of \( k \). Repeating this process until \( k = 1 \) we find that \( w = 0 \).

A similar argument shows that \( \dim \ker \Gamma_\lambda = r\ell \). Considering instead the vectors \( w \) such that \( w^T \Gamma_\lambda = 0 \) implies that \( w^T X = w^T (zI - Z_\lambda)^{-1} B_\lambda = 0 \) and the same process reveals that \( w = 0 \) so that \( \Gamma_\lambda \) has rank \( \ell \). Consequently, its kernel has dimension \( (r\ell + \ell) - \ell = r\ell \).

**3.2. The Kernel of \( K_\nu \).** The results of the previous section on the distributions annihilating \( \psi_\nu \) give us information about the kernel of the matrix ordinary differential operator \( K_\nu \) defined in (10):

**Corollary 3.3:** Let \( c_{v,\lambda} \) be as in Lemma 3.1. Then the \( r \) component vector valued function
\[
\phi_{c,\lambda}(\vec{t}) = c_{v,\lambda} \left( \begin{array}{c} \gamma(\vec{t}, z) \\ q_\nu(z) \end{array} \right)
\]
is in the kernel of the operator \( K_\nu \).
Proof: Since $c_{v,\lambda}$ commutes with multiplication and differentiation in $x = t_1$, we have
\[ K_\nu \phi_{v,\lambda} = c_{v,\lambda} \left( K_\nu \gamma(t, z) q_\nu^{-1}(z) \right) = c_{v,\lambda} (\psi_\nu) = 0, \]
by (8) and Lemma 3.1. □

In fact, the entire kernel of $K_\nu$ is spanned by functions of this form, and as a consequence they satisfy certain useful linear differential equations.

Corollary 3.4: If $\phi(t)$ is a vector in the kernel of $K_\nu$ then it is a linear combination of the $\phi_{v,\lambda}(t)$, and so satisfies the equation
\[ \frac{\partial^k}{\partial t_1^k} \phi(t) = \frac{\partial}{\partial t_k} \phi(t). \]

Proof: By making use of all of the eigenvalues of $Z$, Corollary 3.3 gives us $rn$ linearly independent vector functions in the kernel of the $n$th order ordinary differential operator with $r \times r$ matrix coefficients. Since they are linearly independent (those corresponding to the same eigenvalue are linearly independent by Lemma 3.2 and those corresponding to different eigenvalues cannot be linearly dependent due to the factor of $e^{x\lambda}$) this accounts for the entire kernel of $K_\nu$.

Note that $\gamma(t, z)$ trivially satisfies
\[ \frac{\partial^k}{\partial t_1^k} \gamma(t, z) = z^k \gamma(t, z) = \frac{\partial}{\partial t_k} \gamma(t, z). \]

Now, the proof here is elementary because differentiation in $t_1$ commutes with the residue, multiplication by functions of $z$ and matrix multiplication in the definition of $\phi_{v,\lambda}$ and applies by linearity to the entire kernel. □

3.3. The Lax Equation. Now we come to main point of this section. If the $\nu$ moves according the spin Calogero dynamics the wave function $ψ_\nu$ depends on the time variables $\vec{t}$, and this produces a solution of the matrix KP hierarchy. More precisely:

Theorem 3.5: The pseudo-differential operator $L_\nu = K_\nu \circ \partial \circ K_\nu^{-1}$ satisfies
\[ \frac{\partial}{\partial t_i} L_\nu = [(L_\nu^i)_+, L_\nu]. \]

Proof: First, we note that the (pseudo)-differential operator $(L_\nu^i)_- \circ K_\nu$ is actually a differential operator since
\[ (L_\nu^i)_- \circ K_\nu + (L_\nu^i)_+ \circ K_\nu = K_\nu \circ \partial^i \]
and therefore $(L_\nu^i)_- \circ K_\nu = -(L_\nu^i)_+ \circ K_\nu + K_\nu \circ \partial^i$.

Now, let $\phi(x)$ be a vector function in the kernel of the operator $K_\nu$. Then applying $\frac{\partial}{\partial t_i}$ to the equality $K_\nu \phi = 0$ and using Corollary 3.4 we find
\[ 0 = \frac{\partial}{\partial t_i} \circ K_\nu(\phi) = (K_\nu)(t_i, \phi) + K_\nu(\phi_{t_i}) = (K_\nu)(t_i, \phi) + \partial^i(\phi) + L_\nu(\phi) = (K_\nu)(t_i, \phi) + (L_\nu^i)_+ \circ K_\nu(\phi) + (L_\nu^i)_- \circ K_\nu(\phi) \]
However, since $\phi$ is in the kernel of $K_\nu$, we know that $(L^i_\nu)_+ \circ K_\nu(\phi) = 0$. Then the last displayed equality gives us that the entire kernel of $K_\nu$ is in the kernel of the ordinary differential operator $(K_\nu)_i + (L^i_\nu)_- \circ K_\nu$. Since this operator has order strictly less than $n$, it can only have such a large kernel if it is the zero operator and we conclude

$$(K_\nu)_i = -(L^i_\nu)_- \circ K_\nu.$$ 

Using this we find that

$$(L_\nu)_i = (K_\nu)_i \circ \partial \circ K_\nu^{-1} - K_\nu \circ \partial \circ K_\nu^{-1} \circ (K_\nu)_i \circ K^{-1}$$

$$= -(L^i_\nu)_- \circ K_\nu \circ \partial \circ K_\nu^{-1} + K_\nu \circ \partial \circ K_\nu^{-1} \circ (L^i_\nu)_- \circ K_\nu \circ K_\nu^{-1}$$

$$= [L_\nu, (L^i_\nu)_-] = [(L^i_\nu)_+, L_\nu].$$

\[ \square \]

4. Symmetries

The symmetry $X \mapsto X + cZ^j$ of $sCM^r_n$ induces the integrable dynamics of both the particle system and the wave equations of the matrix KP hierarchy. Here are some other symmetries and how they affect the KP Lax operator.

4.1. Action of $GL(n)$. For any $G \in GL(n)$ we define $S_G$ acting on $sCM^r_n$ by

$$S_G : sCM^r_n \rightarrow sCM^r_n$$

$$(X, Z, A, B) \mapsto (GXG^{-1}, GZG^{-1}, AG^{-1}, GB).$$

The Lax operator $L_\nu$ is unaffected by this action: $L_{S_G(\nu)} = L_\nu$.

Since this symmetry does not affect the corresponding dynamical objects from the previous sections, it makes sense to consider $sCM^r_n$ modulo this group action as the phase space of the spin Calogero particle dynamics as well as the corresponding matrix KP solutions.

4.2. Action of $GL(r)$. Clearly, if we have $G \in GL(r)$ then we can conjugate solutions to the matrix KP hierarchy to get solutions that are technically different, but not very different. This also manifests itself as a group action on the level of $sCM^r_n$. Let $G \in GL(r)$ then if

$$s_G : sCM^r_n \rightarrow sCM^r_n$$

$$(X, Z, A, B) \mapsto (X, Z, GA, BG^{-1})$$

one finds that $L_{s_G(\nu)} = GL_\nu G^{-1}$.

4.3. Changing $r$. There is an easy way to take an $r \times r$ solution and turn it into an $R \times R$ solution for $r < R$. Let $a$ be an $R \times r$ and $b$ an $r \times R$ matrix such that $ba = I$ is the $r \times r$ identity matrix. Defining $U_{a,b} : sCM^r_n \rightarrow sCM^R_n$ by

$$U_{a,b}(X, Z, A, B) = (X, Z, aA, Bb).$$

one then has

$$L_{U_{a,b}(\nu)}(\vec{t}) = aL_\nu(\vec{t})b.$$
4.4. The Bispectral Involution. Finally, we have an important discrete symmetry whose effect on the KP solution will be the subject of the next section:

\[ b : sCM^r_n \rightarrow sCM^r_n \]

\[ \nu = (X, Z, A, B) \mapsto \nu^b = (Z^\top, X^\top, B^\top, A^\top) \]

The significance of this symmetry on the KP solution is most easily seen by looking at the wave function \( \psi_\nu(x, z) \) as a function of \( x = t_1 \) and \( z \) only (setting all of the other times equal to zero). As in the case \( r = 1 \), it involves an exchange of \( x \) and \( z \), but when \( r > 1 \) one must also take the transpose of the function:

\[ \psi_\nu(x, z) = \psi_\nu^\top(z, x). \]

5. Bispectrality

Let \( \nu \in sCM^r_n \) and \( q_\nu(z) = \text{det}(zI - Z) \). In this section, since the dynamics are not significant, we will consider \( x = t_1 \) and \( t_i = 0 \) for \( i > 1 \). Thus, for instance, we will write \( \psi_\nu(x, z) \) for \( \psi_\nu((x, 0, 0, \ldots, z)) \) and \( \gamma(x, z) = e^{xz} \). In the next section we will associate two commutative rings of ordinary differential operators (one acting in \( x \) and one acting in \( z \)) to the choice of \( \nu \in sCM^r_n \) and then in the following section we will demonstrate that \( \psi_\nu(x, z) \) is a common eigenfunction for the operators in the rings. In particular, any operator from each of the rings along with \( \psi_\nu \) form a bispectral triple as in Definition 1.1.

5.1. Commutative Rings of Matrix Differential Operators. Associate to \( \nu \in sCM^r_n \) the ring \( R_\nu \subset \mathbb{C}[z] \) defined by the property that the polynomials preserve the conditions annihilating \( \psi_\nu(\vec{t}, z) \) from Lemma 3.1:

\[ R_\nu = \{ p \in \mathbb{C}[z] | c_{\nu,\lambda}(\psi_\nu) = 0 \Rightarrow c_{\nu,\lambda}(p(z)\psi_\nu) = 0 \} . \]

**Lemma 5.1:** The ring \( R_\nu \) is non-empty. In particular,

\[ q_\nu^\top(z)\mathbb{C}[z] \subset R_\nu. \]

**Proof:** Note that \( q_\nu(z)\psi_\nu(\vec{t}, z) = K_\nu \gamma(\vec{t}, z) \) is non-singular in \( z \). Then the claim follows from the fact that \( c_{\nu,\lambda}(q_\nu(z)f(z)) = 0 \) for any non-singular function \( f \). \( \square \)

By substituting the pseudo-differential operator \( L_\nu \) into these polynomials, we associate a commutative ring of pseudo-differential operators

\[ \mathcal{R}_\nu = \{ p(\mathcal{L}_\nu) | p \in R_\nu \} \]

to \( \nu \). However, as the next lemma demonstrates, these are in fact differential operators.

**Lemma 5.2:** If \( p \in R_\nu \) then \( L = p(\mathcal{L}_\nu) \) is a differential operator (as opposed to a general pseudo-differential operator) satisfying the eigenvalue equation

\[ L\psi_\nu(x, z) = p(z)\psi_\nu(x, z). \]

**Proof:** Since the leading coefficient of \( K_\nu \) is a nonsingular matrix (in fact, it is the identity matrix because of the form of \( \psi_\nu \)), it is sufficient to show that the kernel of \( K_\nu \) is contained in the kernel of \( K_\nu \circ p(\partial) \) because then we know that this ordinary differential operator factors as \( L \circ K_\nu \) for some ordinary differential operator \( L \) which meets all of the other criteria.
So, now let $\phi(x)$ be a function in the kernel of $K_\nu$. By Lemma 3.3 we know that $\phi(x)$ is a linear combination of functions of the form (16). However,

$$K_\nu \circ p(\partial) c_{\nu,\lambda}(\gamma(x,z)q_{\nu}^{-1}(z)) = c_{\nu,\lambda}(p(z)\psi_\nu(x,z)) = 0$$

by (10) and the definition of $R_\nu$. 

We will also associate a commutative ring of ordinary differential operators in $z$ to $\nu$. Applying the procedure above to the point $\nu = (Z^T, X^T, B^T, A^T) \in sCM^r$, we have another commutative ring $R_{\nu}^*$ of differential operators in $x$. We convert them to differential operators in $z$ by simply replacing $x$ with $z$, $\partial_x$ with $\partial_z$ and transposing the coefficients:

$$R_{\nu}^* = \{ L^\top(z, \partial_z)|L(x, \partial_x) \in R_\nu \}.$$ 

5.2. Common Eigenfunction. Our main result is the observation that $\psi_\nu(x,z)$ is a common eigenfunction for the differential operators in the rings $R_\nu$ and $R_{\nu}^*$ satisfying eigenvalue equations of the form (6):

**Theorem 5.3:** Let $p \in R_\nu$ and $\pi \in R_{\nu}^*$, then there exist ordinary differential operators $L(x, \partial_x) \in R_\nu$ and $\Lambda(z, \partial_z) \in R_{\nu}^*$ such that

$$L\psi_\nu(x,z) = p(z)\psi_\nu(x,z) \quad \text{and} \quad \Lambda_\nu\psi_\nu(x,z) = \pi(x)\psi_\nu(x,z).$$

**Proof:** The first equation follows from Lemma 5.2. Similarly, it follows from Lemma 5.2 that there is a differential operator $Q(x, \partial_x)$ with the property that

$$Q\psi_\nu(x,z) = \pi(z)\psi_\nu(x,z).$$

Exchanging the roles of $x$ and $z$ in this equation, taking the transpose (see ), and applying (5) and (17) results in the second equation of the claim. 

6. Example

For the sake of clarity, we briefly illustrate the main ideas with an example.

Consider $\nu = (X, Z, A, B) \in sCM^2$ where

$$X = \begin{pmatrix} 0 & 0 & 0 \\
-1 & 0 & -1 \\
1 & 0 & 2 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\
-2 & 0 \end{pmatrix}.$$ 

Then

$$\psi_\nu(x,z) = e^{xz} \left( I + \begin{pmatrix} -\frac{2x^2+3x^2+2-2}{(x-2)x^2} & \frac{1}{(x-2)x^2} \\
\frac{1}{(x-2)x^2} & \frac{1}{(x-2)x^2} \end{pmatrix} \right).$$

This can be written as $\psi_\nu = K_\nu e^{xz}/q_\nu(z)$ where $q_\nu(z) = \det(zI - Z) = z^3$ and

$$K = \begin{pmatrix} \frac{2(x-1)}{(x-2)x} & 0 \\
0 & \partial + \begin{pmatrix} \frac{3-2x}{(x-2)x} & \frac{1}{(x-2)x} \\
\frac{1}{(x-2)x} & \frac{1}{(x-2)x} \end{pmatrix} \partial^2 + \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} \partial^3 \end{pmatrix}$$

is an ordinary differential operator.
To find the conditions satisfied by \( \psi_{\nu}(x, z) \), we note that the kernel of

\[
\Gamma_0 = \begin{pmatrix}
0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 & -2
\end{pmatrix}
\]

is made up of vectors of the form

\[
v = \begin{pmatrix} c_1 & c_2 & c_3^2 & c_4 & c_5 & c_6 & 3c_1 + c_2 & c_8 & -c_1 - c_2 \end{pmatrix}^T.
\]

Hence, we conclude that \( c_{\nu,0}(\psi_{\nu}(x, z)) = 0 \) where

\[
c_{\nu,0}(f(z)) = \text{Res}_{z=0} \left( f(z) \left( c_5 z^2 + \frac{1}{2} c_2 z + c_1 + \frac{c_8}{z} \right) \right).
\]

Moreover, \( c_{\nu,0}(p(z)\psi_{\nu}) = 0 \) whenever \( p \in z^4\mathbb{C}[z] = R_{\nu} \). Consequently, we can find an ordinary differential operator having any of these polynomials as its eigenvalue. In particular, solving

\[
K_\nu \circ \partial^4 = L \circ K_\nu
\]

for \( L \) we find

\[
L = \begin{pmatrix}
\frac{8(3x^4-18x^3+50x^2-63x+30)}{(x-2)^4 x^4} & \frac{4(23x^3-93x^2+138x-72)}{(x-2)^4 x^5} \\
\frac{8(2x^3-5x^2+9x-6)}{(x-2)^4 x^4} & \frac{4(2x^3-8x^2+27x^2-42x+24)}{(x-2)^4 x^5}
\end{pmatrix} \partial
\]

\[
+ \begin{pmatrix}
\frac{4(6x^3-27x^2+49x-30)}{(x-2)^4 x^4} & \frac{4(13x^3-35x^2+20)}{(x-2)^4 x^5} \\
\frac{4(2x^3-5x^2+6x+13x-10)}{(x-2)^4 x^5} & \frac{4(2x^3-6x^2+13x-10)}{(x-2)^4 x^5}
\end{pmatrix} \partial^2 + \partial^4
\]

which satisfies \( L\psi_{\nu} = z^4 \psi_{\nu} \).

Of course, we can follow this same procedure beginning with another element of sCM\(_2^3\). In particular, if we begin with

\[
\nu^b = (Z^T, X^T, B^T, A^T)
\]

instead then the differential operator we produce will be

\[
Q = \begin{pmatrix}
\frac{-4(16x^2+65x+90)}{x^6} & \frac{80 x+216}{4x^2+21x+36} & \frac{4(16x^2+65x+90)}{x^5} & \frac{-8(10 x+27)}{4x^2+21x+36} \\
\frac{8x+12}{x^4} & \frac{4(2x+3)}{x^3} & \frac{4(4x^2+21x+36)}{x^5}
\end{pmatrix} \partial
\]

\[
+ \begin{pmatrix}
\frac{-2(12x^2+65x+90)}{x^4} & \frac{40 x+108}{8x^2+42x+72} \\
\frac{6}{x^2} & \frac{14x+24}{x^3}
\end{pmatrix} \partial^2 + \begin{pmatrix}
\frac{34x+60}{x^6} & \frac{-4(2x+9)}{14x+24} \\
0 & \frac{x^6}{14x+24}
\end{pmatrix} \partial^3
\]

\[
+ \begin{pmatrix}
4 - \frac{12}{x} \\
0
\end{pmatrix} \partial^4 + 4 \partial^5 + \partial^6.
\]

The function \( \psi_{\nu,\nu} \) is an eigenfunction for this operator satisfying \( Q\psi_{\nu,\nu}(x, z) = (4z^4-4z^5+z^6)\psi_{\nu,\nu}(x, z) \).

More interestingly, since \( \psi_{\nu}(x, z) = \psi_{\nu}^T(z, x) \), if we transpose the matrix coefficients on \( Q \), replace \( x \) with \( z \) and \( \partial = \partial_x \) with \( \partial_z \), we get a differential operator \( \Lambda \)
in the variable $z$. This operator applied to $\psi_\nu(x, z)$ (the wave function computed earlier) from the right satisfies
\[
\Lambda R \psi_\nu(x, z) = (4x^4 - 4x^5 + x^6)\psi_\nu(x, z),
\]
demonstrating bispectrality.

7. Conclusions and Comments

The main results of the present paper can be viewed as another step in addressing the “bispectral problem” of F.A. Grünbaum [HK98, DG86], seeking operators satisfying eigenvalue equations of the form (3). In [DG05], the authors considered the case in which one of the operators is a second order difference operator with matrix coefficients and the two operators act on matrix eigenfunctions from different directions. However, bispectrality for matrix differential operators has only been studied with both operators acting from the left [Zub90]. Here we consider the case (6) in which the operators are $r \times r$ matrix differential operators acting from different directions. Since our construction conveniently reproduces the results of Wilson’s seminal paper [Wil93] in the special case $r = 1$, this particular formulation of the bispectral problem appears to be the correct one for generalizing those results to the case of matrix differential operators. However, the method of proof and especially the explicit formulation of the “conditions” satisfied by the wave function above are novel even for $r = 1$.

In [Wil93], it was shown that the bispectral operators associated to $\text{sCM}_n^1$ are in fact the only bispectral scalar ordinary differential operators which commute with operators of relatively prime order up to obvious renormalizations and changes of variable. By Lemma 5.1 it follows that the differential operators produced by the construction in this paper also all have the property that they commute with other differential operators of relatively prime order.

In addition, this paper can be seen as contributing to the literature establishing a link between bispectrality and duality in classical and quantum integrable systems. (See, for instance, [FGNR00, Hai07, Kas95, Kas00, Kas01, Wil98].) Again, the main results of the present paper for the spin Calogero system in the case $r = 1$ reproduce results previously presented for the spinless case in [Kas95, Wil98].

Some questions arise naturally which we have not pursued. There are additional commuting Hamiltonians for the spin Calogero system [GH84] and corresponding isospectral deformations for the multi-component KP hierarchy [BtK95], but their relationship to bispectrality has not been explored here. We have not looked at the algebro-geometric implications of the rings $R_\nu$. Certainly as in the case $r = 1$ [Wil93, Wil93], these contain operators of relatively prime order are isomorphic to the coordinate rings of rational curves with only cuspidal singularities. However, whether there is any further algebro-geometric significance such as was found in [BW02] or whether every commutative ring of matrix ordinary differential operators with these properties is necessarily bispectral have not been considered. These questions, along with the obvious question of what other matrix differential operators satisfy equations of the form (6) will hopefully be addressed in future papers.

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2These were called “rank one” operators in that context, but we will avoid that terminology here to avoid confusion with the rank $r$ which is something different.
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