NONEXISTENCE OF STEADY WAVES WITH NEGATIVE VORTICITY

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ABSTRACT. We prove that no two-dimensional Stokes and solitary waves exist when the vorticity function is negative and the Bernoulli constant is greater than a certain critical value given explicitly. In particular, we obtain an upper bound $F \lesssim \sqrt{2}$ for the Froude number of solitary waves with a negative constant vorticity, sufficiently large in absolute value.

1. INTRODUCTION

We consider the classical water wave problem for two-dimensional steady waves with vorticity on water of finite depth. We neglect effects of surface tension and consider a fluid of constant (unit) density. Thus, in an appropriate coordinate system moving along with the wave, stationary Euler equations are given by

\begin{align*}
(u - c)u_x + vu_y &= -P_x, \\
(u - c)v_x + vv_y &= -P_y - g, \\
u_x + v_y &= 0,
\end{align*}

which holds true in a two-dimensional fluid domain

\[ 0 < y < \eta(x). \]

Here $(u, v)$ are components of the velocity field, $y = \eta(x)$ is the surface profile, $c$ is the wave speed, $P$ is the pressure and $g$ is the gravitational constant. The corresponding boundary conditions are

\begin{align*}
v &= 0 \quad \text{on } y = 0, \\
v &= (u - c)\eta_x \quad \text{on } y = \eta, \\
P &= P_{\text{atm}} \quad \text{on } y = \eta.
\end{align*}

It is often assumed in the literature that the flow is irrotational, that is $v_x - u_y$ is zero everywhere in the fluid domain. Under this assumption components of the velocity field are harmonic functions, which allows to apply methods of complex analysis. Being a convenient simplification it forbids modeling of non-uniform currents, commonly occurring in nature. In the present paper we will consider rotational flows, where the vorticity function is defined by

\[ \omega = v_x - u_y. \]

Throughout the paper we assume that the flow is unidirectional, that is

\[ u - c > 0 \quad \text{everywhere in the fluid.} \]

This forbids the presence of stagnation points and gives an advantage of using the partial hodograph transform.

In the two-dimensional setup relation (1.1e) allows to reformulate the problem in terms of a stream function $\psi$, defined implicitly by relations

\[ \psi_y = u - c, \quad \psi_x = -v. \]

This determines $\psi$ up to an additive constant, while relations (1.1d), (1.1f) force $\psi$ to be constant along the boundaries. Thus, by subtracting a suitable constant, we can always assume that

\[ \psi = m, \quad y = \eta; \quad \psi = 0, \quad y = 0. \]
Here $m$ is the mass flux, defined by

$$m = \int_0^\eta (u - c) dy.$$ 

In what follows we will use non-dimensional variables proposed by Keady & Norbury [KN78], where lengths and velocities are scaled by $(m^2/g)^{1/3}$ and $(mg)^{1/3}$ respectively; in new units $m = 1$ and $g = 1$. For simplicity we keep the same notations for $\eta$ and $\psi$.

Taking the curl of Euler equations (1.1a)-(1.1c) one checks that the vorticity function $\omega$ defined by (1.2) is constant along paths tangent everywhere to the relative velocity field $(u-c,v)$; see [Con11] for more details. Having the same property by the definition, stream function $\psi$ is strictly monotone by (1.3) on every vertical interval inside the fluid region. These observations together show that $\omega$ depends only on values of the stream function, that is

$$\omega = \omega(\psi).$$

This property and Bernoulli’s law allow to express the pressure $P$ as

$$P - P_{\text{atm}} + \frac{1}{2} |\nabla \psi|^2 + y + \Omega(\psi) - \Omega(1) = \text{const},$$

(1.4)

where

$$\Omega(\psi) = \int_0^\psi \omega(p) dp$$

is a primitive of the vorticity function $\omega(\psi)$. Thus, we can eliminate the pressure from equations and obtain the following problem:

$$\Delta \psi + \omega(\psi) = 0 \quad \text{for } 0 < y < \eta,$$

(1.5a)

$$\frac{1}{2} |\nabla \psi|^2 + y = r \quad \text{on } y = \eta,$$

(1.5b)

$$\psi = 1 \quad \text{on } y = \eta,$$

(1.5c)

$$\psi = 0 \quad \text{on } y = 0.$$  

(1.5d)

Here $r > 0$ is referred to as Bernoulli’s constant. Another constant of motion known as the flow force is given by

$$S = \int_0^\eta (\psi_y^2 - \psi_x^2 - y + \Omega(1) - \Omega(\psi) + r) dy.$$  

(1.6)

This constant is important in several ways; for instance, it plays the role of the Hamiltonian in spatial dynamics; see [BM92]. The flow force constant is also involved in a classification of steady motions; see [Ben95].

1.1. **Stream solutions.** Laminar flows or shear currents, for which the vertical component $v$ of the velocity field is zero play an important role in the theory of steady waves. Let us recall some basic facts about stream solutions $\psi = U(y)$ and $\eta = d$, describing shear currents. It is convenient to parameterize the latter solutions by the relative speed at the bottom. Thus, we put $U_y(0) = s$ and find that $U = U(y; s)$ is subject to

$$U'' + \omega(U) = 0, \quad 0 < y < d; \quad U(0) = 0, \quad U(d) = 1.$$  

(1.7)

Our assumption (1.3) implies $U' > 0$ on $[0;d]$, which puts a natural constraint on $s$. Indeed, multiplying the first equation in (1.7) by $U'$ and integrating over $[0;y]$, we find

$$U'^2 = s^2 - 2\Omega(U).$$

This shows that the expression $s^2 - 2\Omega(p)$ is positive for all $p \in [0;1]$, which requires

$$s > s_0 = \sqrt{\max_{p\in[0,1]} 2\Omega(p)}.$$
On the other hand, every \( s > s_0 \) gives rise to a monotonically increasing function \( U(y; s) \) solving (1.5) for some unique \( d = d(s) \), given explicitly by
\[
d(s) = \int_0^1 \frac{1}{\sqrt{s^2 - 2\Omega(p)}}.\]
This formula shows that \( d(s) \) monotonically decreases to zero with respect to \( s \) and takes values between zero and
\[
d_0 = \lim_{s \to s_0^+} d(s).\]
The latter limit can be finite or not. For instance, when \( \omega = 0 \) we find \( s_0 = 0 \) and \( d_0 = +\infty \). On the other hand, when \( \omega = -b \) for some positive constant \( b \neq 0 \), then \( s_0 = 0 \) but \( d_0 < +\infty \).

We note that our main theorem is concerned with the case \( d_0 < +\infty \).

Every stream solution \( U(y; s) \) determines the Bernoulli constant \( R(s) \), which can be found from the relation (1.5c). This constant can be computed explicitly as
\[
R(s) = \frac{1}{2}s^2 - \Omega(1) + d(s).
\]
As a function of \( s \) it decreases from \( R_0 \) to \( R_c \) when \( s \) changes from \( s_0 \) to \( s_c \) and increases to infinity for \( s > s_c \). Here the critical value \( s_c \) is determined by the relation
\[
\int_0^1 \frac{1}{(s^2 - 2\Omega(p))^{3/2}} dp = 1.
\]
The constants \( R_0 \) and \( R_c \) are of special importance for the theory. For example, it is proved in [KKL15] that \( r > R_c \) for any steady motion other than a laminar flow. In the present paper we will consider the water wave problem (1.5) for \( r > R_0 \), provided \( R_0 < +\infty \). The latter is true, for instance, for a negative constant vorticity.

For any \( r \in (R_c, R_0] \) there are exactly two solutions \( s_-(r) < s_+(r) \) to the equation
\[
R(s) = r,
\]
while for \( r > R_0 \) one finds only one solution \( s = s_+(r) \). The laminar flow corresponding to \( s_-(r) \) is called subcritical and it’s depth is denoted by \( d_-(r) = d(s_-(r)) \). The other flow, with \( s = s_+(r) \) is called supercritical and it’s depth is \( d_+(r) = d(s_+(r)) \). According to the definition, we have
\[
d_+(r) < d_-(r).
\]
The flow force constants corresponding to flows with \( s = s_{\pm} \) are denoted by \( S_{\pm}(r) \).

It was recently proved in [KLW20] that all solitary waves are supported by supercritical depths \( d_+(r) \) and the corresponding flow force constant equals to \( S_+(r) \); here \( r \) is the Bernoulli constant of a solitary wave.

### 1.2. Formulations of main results.

Just as in [KKL15] we split the set of all vorticity functions into three classes as follows:

1. \( \max_{p \in [0,1]} \Omega(p) \) is attained either at an inner point of \((0,1)\) or at an end-point, where \( \omega \) attains zero value;
2. \( \Omega(p) < 0 \) for all \( p \in (0,1) \) and \( \omega(0) \neq 0 \);
3. \( \Omega(p) < \Omega(1) \) for all \( p \in [0,1) \) (and so \( \omega(1) \neq 0 \)).

The first class can be characterized by relations \( R_0 = +\infty \) and \( d_0 = +\infty \), while \( R_0, d_0 < +\infty \) for all vorticity functions that belong to the second and third classes. Our main result states

**Theorem 1.1.** Let \( \omega \in C^r([0,1]) \) satisfies conditions (ii) or (iii). Then there exist no Stokes waves with \( r \geq R_0 - \Omega(1) \). Furthermore, there are no solitary waves with \( r \geq R_0 \).

A part of the statement, when \( \omega \) is subject to (iii) was proved in [KKL15], where it was shown that no steady waves exist for \( r \geq R_0 \) (under condition (iii)). We note that there is no analogues statement for irrotational waves. A typical example of a vorticity function satisfying condition (ii) is a negative constant vorticity \( \omega(p) = -b, b > 0 \). It is known (see [Wah09]) that vorticity distributions of this type give rise to Stokes waves over flows with internal stagnation points, that exist for all Bernoulli constants \( r \geq R_0 \). Furthermore, a recent study [KKL20]...
shows that there exist continuous families of such Stokes waves that approach a solitary wave in the long wavelength limit. The latter solitary wave has \( r > R_0 \) and rides a supercritical unidirectional flow (corresponding to one of stream solutions \( U(y; s) \) with \( s > s_c \)) but has a near-bottom stagnation point on a vertical line passing through the crest. Thus, even so there are no unidirectional waves for \( r > R_0 \), there exist Stokes and solitary waves with \( r > R_0 \) violating assumption (1.3). These considerations show that the statement of Theorem 1.1 is sharp in a certain sense. On the other hand, inequality \( r \geq R_0 - \Omega(1) \) is not sharp and probably can be improved further. However it is not clear if one can omit completely the term \(-\Omega(1)\) from the bound on the Bernoulli constant.

Inequality \( r \leq R_0 \) for solitary waves puts a natural upper bound for the Froude number

\[
F^2(s) = \left( \int_0^d (U_y(y; s))^{-2} dy \right)^{-1}.
\]

It is well known that for irrotational solitary waves \( F < \sqrt{2} \); see [Sta47], [KP74]. Furthermore, the bound \( F < 2 \) for rotational waves with a negative vorticity was obtained in [Whe15]. For small negative vorticity distributions inequality \( 1 < F(s) < 2 \) is stronger than \( R_c < R(s) < R_0 \). However, already for \( \omega(p) = -1 \) the inequality \( R_c < R(s) < R_0 \) is equivalent to \( 1 < F(s) \lesssim \sqrt{2} \), which is significantly better than \( F < 2 \).

2. Preliminaries

2.1. Reformulation of the problem. Under assumption (1.3) we can apply the partial hodograph transform introduced by Dubreil-Jacotin [DJ34]. More precisely, we present new independent variables

\[
q = x, \quad p = \psi(x, y),
\]

while new unknown function \( h(q, p) \) (height function) is defined from the identity

\[
h(q, p) = y.
\]

Note that it is related to the stream function \( \psi \) through the formulas

\[
\psi_x = -\frac{h_q}{h_p}, \quad \psi_y = \frac{1}{h_p},
\]

where \( h_p > 0 \) throughout the fluid domain by (1.3). An advantage of using new variables is in that instead of two unknown functions \( \eta(x) \) and \( \psi(x, y) \) with an unknown domain of definition, we have one function \( h(q, p) \) defined in a fixed strip \( S = \mathbb{R} \times [0, 1] \). An equivalent problem for \( h(q, p) \) is given by

\[
\left( \frac{1 + h_q^2}{2h_p^2} + \Omega \right)_p - \left( \frac{h_q}{h_p} \right)_q = 0 \quad \text{in } S, \tag{2.2a}
\]

\[
\frac{1 + h_q^2}{2h_p^2} + h = r \quad \text{on } p = 1, \tag{2.2b}
\]

\[
h = 0 \quad \text{on } p = 0. \tag{2.2c}
\]

The wave profile \( \eta \) becomes the boundary value of \( h \) on \( p = 1 \):

\[
h(q, 1) = \eta(q), \quad q \in \mathbb{R}.
\]

Using (2.1) and Bernoulli’s law (1.4) we recalculate the flow force constant \( S \) defined in (1.6) as

\[
S = \int_0^1 \left( \frac{1 - h_q^2}{h_p^2} - h - \Omega + \Omega(1) + r \right) h_p \, dp. \tag{2.3}
\]
Laminar flows defined by stream functions $U(y; s)$ correspond to height functions $h = H(p; s)$ that are independent of horizontal variable $q$. The corresponding equations are
\[
\frac{1}{2H_p^2} + \Omega = 0, \quad H(0) = 0, \quad H(1) = d(s), \quad \frac{1}{2H_p^2(1)} + H(1) = R(s).
\]
Solving equations for $H(p; s)$ explicitly, we find
\[
H(p; s) = \int_0^p \frac{1}{\sqrt{s^2 - 2\Omega(\tau)}} d\tau.
\]
Given a height function $h(q, p)$ and a stream solution $H(p; s)$, we define
\[
w^{(s)}(q, p) = h(q, p) - H(p; s).
\]
This notation will be frequently used in what follows. In order to derive an equation for $w^{(s)}$, we first write (2.2a) in a non-divergence form as
\[
\frac{1}{h_p^2}h_{pp} - \frac{2}{h_p^2}h_{qp} + h_{qq} - \omega(p)h_p = 0.
\]
Now using our ansats (2.4), we find
\[
\frac{1 + h^2}{h_p^2}w^{(s)}_{pp} - \frac{2}{h_p^2}w^{(s)}_{qp} + w^{(s)}_{qq} - \omega(p)w^{(s)}_p + \frac{(w^{(s)}_q)^2}{h_p^2}H_{pp} - \frac{w^{(s)}_q}{h_p^2}H_{pp} = 0.
\]
Thus, $w^{(s)}$ solves a homogeneous elliptic equation in $S$ and is subject to a maximum principle; see [Vit07] for an elliptic maximum principle in unbounded domains. The boundary conditions for $w^{(s)}$ can be obtained directly from (2.2b) and (2.2c) by inserting (2.4) and using the corresponding equations for $H$. This gives
\[
\frac{(w^{(s)}_q)^2}{2h_p^2} - \frac{w^{(s)}_q}{2h_p^2}H_p + w^{(s)} = r - R(s) \quad \text{for } p = 1,
\]
\[
w^{(s)} = 0 \quad \text{for } p = 0.
\]
Concerning the regularity, we will always assume that $\omega \in C^\gamma([0; 1])$ and $h \in C^{2,\gamma}(\overline{S})$, where $C^{2,\gamma}(\overline{S})$ is the usual subspace of $C^2(\overline{S})$ (all partial derivatives up to the second order are bounded and continuous in $S$) of functions with Hölder continuous second-order derivatives with a finite Hölder norm, calculated over the whole strip $S$. The exponent $\gamma \in (0; 1)$ will be fixed throughout the paper. The Bernoulli constant $r$ will remain unchanged and we will often omit it from notations, such as $s_\pm$ or $S_\pm$. Furthermore, in many formulas such as (2.6a), we will omit the dependence on $s$ in the notation for $H$, while the right choice of $s$ will be clear from the context.

2.2. Auxiliary functions $\sigma$ and $\kappa$. For a given $r > R_c$ and $s > s_0$ we define
\[
\sigma(s; r) = \int_0^1 \left( \frac{1}{2H_p^2(p; s)} - H(p; s) - \Omega(p) + \Omega(1) + r \right) H_p(p; s) dp.
\]
This expression coincides with the flow force constant for $H(p; s)$, but with the Bernoulli constant $R(s)$ replaced by $r$. We also note that
\[
\sigma(s_\pm(r); r) = S_\pm(r).
\]
The key property of $\sigma(s; r)$ is stated below.

**Lemma 2.1.** For a given $r \geq R_0$ the function $s \mapsto \sigma(s; r)$ decreases for $s \in (s_0, s_+(r))$ and increases to infinity for $s \in (s_+(r), +\infty)$. 


Proof. Because

\[ H_p(p; s) = \frac{1}{\sqrt{s^2 - 2Ω(p)}}, \quad \partial_s H_p(p; s) = -sH_p^3(p; s), \]

we can compute the derivative

\[
\sigma_s(s; r) = \int_0^1 \left( \frac{1}{2H_p^2(p; s)} - H(p; s) - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p; s) dp \\
+ \int_0^1 \left( -\frac{\partial_s H_p(p; s)}{H_p^2(p; s)} - \partial_s H(p; s) \right) H_p(p; s) dp \\
= \int_0^1 \left( -\frac{1}{2H_p^2(p; s)} - \Omega(p) + \Omega(1) + r \right) \partial_s H_p(p; s) dp - d(s)d'(s) \\
= \int_0^1 \left( -\frac{1}{2} s^2 + \Omega(1) + r \right) \partial_s H_p(p; s) dp - d(s) \int_0^1 \partial_s H_p(p; s) dp \\
= -s(r - R(s)) \int_0^1 H_p^3(p; s) dp.
\]

Finally, because \( R(s) < r \) for \( s_0 < s < s_+ (r) \) and \( R(s) > r \) for \( s > s_+ (r) \) we obtain the statement of the lemma. \( \square \)

Our function \( \sigma_s(s; r) \) and it’s role is similar to the function \( \sigma(h) \) introduced by Keady and Norbury in \([KN75]\). The main purpose of the latter is to be used for a comparison with the flow force constant \( S \).

The following function will be also involved in our analysis.

\[ \kappa(s; r) = 2(S - \sigma(s; r)) - (r - R(s))^2. \] (2.8)

A direct computation gives

\[
\partial_s \kappa(s; r) = -2\partial_s \sigma(s; r) + 2(r - R(s))R'(s) \\
= 2s(r - R(s)) \int_0^1 H_p^3(p; s) dp + 2(r - R(s))(s + d'(s)) \\
= 2s(r - R(s)).
\]

Thus, we obtain

**Lemma 2.2.** For a given \( r \geq R_0 \) the function \( s \mapsto \kappa(s; r) \) increases for \( s \in (s_0, s_+(r)) \) and decreases to minus infinity for \( s \in (s_+(r), +\infty) \).

Properties of functions \( \sigma \) and \( \kappa \) will be used in what follows.

### 2.3. Flow force flux functions.

Our aim is to extract some information by comparing the flow force constant \( S \) (of a given solution with the Bernoulli constant \( r \geq R_0 \)) to \( \sigma(s; r) \) for different values of \( s \geq s_0 \). For this purpose we first compute the difference

\[
S - \sigma(s; r) = \int_0^1 \left( \frac{1 - (w_p^{(s)})^2}{2h_p^2} - w^{(s)} - \frac{1}{2H_p^2} \right) H_p dp \\
+ \int_0^1 \left( \frac{1 - (w_p^{(s)})^2}{2h_p^2} - h - \Omega + \Omega(1) + r \right) w_p^{(s)} dp \\
= \int_0^1 \left( \frac{(w_p^{(s)})^2}{2h_p^2H_p^2} - \frac{(w_p^{(s)})^2}{2h_p} + w^{(s)} H_p \right) dp \\
+ \int_0^1 \left( -\frac{1}{2H_p^2} - w^{(s)} - H - \Omega + \Omega(1) + r \right) w_p^{(s)} dp.
\]

Now using the identity

\[-\Omega(p) + \Omega(1) + R(s) = \frac{1}{2H_p^2} + H(1)\]
Proposition 2.4. Let
\[ 2(S - \sigma(s; r)) = 2(r - R(s))w^{(s)}(q, 1) - (w^{(s)}(q, 1))^2 + \int_0^1 \left( \frac{(w^{(s)}_q)^2}{h_p} - \frac{(w^{(s)}_p)^2}{h_p} \right) \, dp. \]

Let us define the (relative) flow force flux function \( \Phi^{(s)} \) by setting
\[ \Phi^{(s)}(q, p) = \int_0^p \left( \frac{(w^{(s)}_p(q, p'))^2}{h_p(q, p') - (H_p(p'; s))^2} \right) \, dp'. \]  \hfill (2.9)

An analog (partial case with \( s = s_\omega(r) \)) of this function was recently introduced in \cite{KLW20}. The same computation as in \cite{KLW20} gives
\[ \Phi_q^{(s)} = -w_q^{(s)} \left( 1 + \frac{(w_q^{(s)})^2}{h_p^2} - \frac{1}{h_p^2} \right), \quad \Phi_p^{(s)} = \frac{(w_p^{(s)})^2}{h_p^2} - \frac{(w_q^{(s)})^2}{h_p}. \]  \hfill (2.10)

A surprising fact about \( \Phi^{(s)} \) is that it solves a homogeneous elliptic equation as stated in the next proposition.

**Proposition 2.3.** There exist functions \( b_1, b_2 \in L^\infty(S) \) such that
\[ \frac{1 + h_q^2}{h_p^2} \Phi_{pp}^{(s)} - 2 \frac{h_q}{h_p} \Phi_{qp}^{(s)} + \Phi_{qq}^{(s)} + b_1 \Phi_q^{(s)} + b_2 \Phi_p^{(s)} = 0 \text{ in } S. \]  \hfill (2.11)

Furthermore, \( \Phi^{(s)} \) satisfies the boundary conditions
\[ \Phi^{(s)} = 2(S - \sigma(s; r)) - 2(r - R(s))w^{(s)}(q, 1) - (w^{(s)}(q, 1))^2 \quad \text{for } p = 1, \]  \hfill (2.12a)
\[ \Phi^{(s)} = 0 \quad \text{for } p = 0. \]  \hfill (2.12b)

In the irrotational case \( b_1, b_2 = 0 \) and (2.11) is equivalent to the Laplace equation.

For the proof we refer to \cite{KLW20}. We also note that \( \Phi^{(s)} \in C^{2, \gamma}(S) \), provided \( h \in C^{2, \gamma}(S) \) and \( \omega \in C^{\gamma}([0, 1]) \).

The next proposition explains the meaning of the auxiliary function \( \kappa(s; r) \).

**Proposition 2.4.** Let \( h \in C^{2, \gamma}(S) \) be a solution to (2.2) with \( r > R_c \). Assume that the flow force flux function \( \Phi^{(s)} \) for some \( s > s_0 \) satisfies \( \inf_{q \in \mathbb{R}} \Phi^{(s)}(q; 1) \leq 0 \). Then
\[ \inf_{q \in \mathbb{R}} \Phi^{(s)}(q; 1) = \kappa(s; r), \]

where \( \kappa(s; r) \) is defined by (2.3).

**Proof.** First, we assume that the infimum is attained at some point \( (q_0, 1) \), where \( \Phi_q^{(s)}(q_0; 1) = 0 \). Differentiating the boundary condition (2.12a), we find
\[ \Phi_q^{(s)}(q_0, 1) = 2w_q^{(s)}(q_0)(w^{(s)}(q_0, 1) - (r - R(s) = 0. \]  \hfill (2.13)

Because \( \Phi^{(s)} \) attains its global minimum at \( (q_0, 1) \), then the maximum principle and the Hopf lemma give \( \Phi_p^{(s)}(q_0, 1) < 0 \). In particular, we find that \( w_q^{(s)}(q_0, 1) \neq 0 \) by the second formula (2.10). Thus, we necessarily obtain
\[ w^{(s)}(q_0, 1) = (r - R(s)). \]

Using this equality in (2.10), we conclude \( \Phi^{(s)}(q_0, 1) = \kappa(s; r) \) as required.

Now we assume that the infimum is attained over a sequence \( \{q_j\}_{j=1}^\infty \) accumulating at the positive infinity. Passing to a subsequence, if necessary, we can assume that
\[ \lim_{j \to +\infty} \Phi_q^{(s)}(q_j, 1) = 0, \quad \lim_{j \to +\infty} \Phi_p^{(s)}(q_j, 1) \leq 0. \]  \hfill (2.14)

There are two possibilities:
\[ (i) \lim_{j \to +\infty} w_q^{(s)}(q_j, 1) = 0 \text{ and (ii) } \lim_{j \to +\infty} w_q^{(s)}(q_j, 1) \neq 0. \]
In the first case relations in (2.14) give
\[
\lim_{j \to +\infty} w^{(s)}(q_j, 1) = \lim_{j \to +\infty} w_{p}^{(s)}(q_j, 1) = 0,
\]
which then require
\[
\lim_{j \to +\infty} u^{(s)}(q_j, 1) = r - R(s)
\]
by the Bernoulli equation (2.6a). In this case \(\inf_S \Phi^{(s)} = \kappa(s; r)\) as desired. The remaining option (ii) provides with a subsequence \(\{q_{j_k}\}\) such that \(\lim_{k \to +\infty} w^{(s)}(q_{j_k}, 1) = r - R(s)\), which follows from the first relation in (2.14) and (2.13). Thus, we find again that \(\inf_S \Phi^{(s)} = \kappa(s; r)\), which completes the proof. \(\square\)

3. Proof of Theorem 1.1

Assume that the vorticity function \(\omega\) satisfies condition (ii) of the theorem. In this case \(d_0, R_0 < +\infty, s_0 = 0\) and
\[
\inf_{s > s_0} H_p(0; s) = +\infty.
\]
(3.1)

First we prove the claim about solitary waves. Thus, we assume that there exists a solitary wave solution \(h\) with \(r \geq R_0\). Choosing \(s = s_+(r)\), we put
\[
w(q, p) = h(q, p) - H(p; s_+(r)).
\]
It follows from Theorem 1 in [KKL15] that \(w(q, 1) > 0\) for all \(q \in \mathbb{R}\). Now because for a supercritical solitary wave \(S = \sigma(s_+(r); r)\) and the relation (2.12a) is then reduced to
\[
\Phi^{(s_+(r))} = (w^{(s)})^2,
\]
we find that \(\Phi^{(s_+(r))}\) is strictly positive along the top boundary. On the other hand, we can choose \(s \in (s_0, s_+(r))\) sufficiently small so that \(w_{p}^{(s)}(q_0, 0) = 0\) for some \(q_0 \in \mathbb{R}\), which follows from (3.1). Then the corresponding flow force flux function \(\Phi^{(s)}\) must attain negative values somewhere along the top boundary, because otherwise \(\Phi_p^{(s)}(q, 0) > 0\) for all \(q \in \mathbb{R}\) by the Hopf lemma, leading to a contradiction with \(w_p^{(s)}(q_0, 0) = 0\) in view of the second formula (2.10). Since \(\Phi^{(s)}\) depends smoothly on \(s\), by the continuity we can find \(s_* \in (s_0, s_+(r))\) for which \(\inf_{q \in \mathbb{R}} \Phi^{(s_*)}(q, 1) = 0\).

By Proposition 2.1 we obtain \(\kappa(s_*; r) = 0\) so that \(S > \sigma(s_*; r)\). Now Lemma 2.1 gives \(\sigma(s_*; r) > \sigma(s_+(r); r)\) and then \(S > \sigma(s_+(r); r)\), which can not be true for a supercritical solitary wave.

Now we consider the case of a Stokes wave \(h\) for some \(r \geq R_0\). Our aim is to show that \(r < R_0 - \Omega(1)\). We start by proving

Lemma 3.1. There exists \(s_* \in (s_0, s_+(r))\) such that \(S < \sigma(s_*; r)\).

Proof. Let \(q_t < q_c\) be coordinates for some adjacent trough and crest respectively, so that \(h(q, 1)\) is monotonically increasing on the interval \((q_t, q_c)\). By (3.1) we can choose a stream solution \(H(p; s_*)\) with \(s_* \in (s_0, s_+(r))\) such that \(h_p(q_*, 0) = H_p(0; s_*)\) for some \(q_* \in (q_t, q_c)\). For the function
\[
w^{(s)}(q, p) = h(q, p) - H(p; s_*)
\]
we consider the zero level set
\[
\Gamma = \{(q, p) \in (q_t, q_c) \times (0, 1) : w^{(s)}(q, p) = 0\}
\]
inside the rectangle \(Q = (q_t, q_c) \times (0, 1)\). We claim that \(\Gamma\) is a graph \(\{(f(p), p) , p \in (0, 1)\}\) of some function \(f \in C^{2,\gamma}([0, 1])\) such that \(f(0) = q_*\) and \(f(1) \in (q_t, q_c)\). Thus, the curve \(\Gamma\) connects a point on the bottom with the surface. To explain this fact we need to recall some properties of Stokes waves. Let \(Q_l, Q_r, Q_t\) and \(Q_b\) be the left, right, top and bottom boundaries of \(Q\), excluding corner points. Then the following properties are true:

(a) \(w_q^{(*)} > 0\) on \(Q\), while \(w_q^{(*)} = 0\) on \(Q_l, Q_r\) and \(Q_b\);
(b) \(w_q^{(*)} > 0\) on \(Q_t\);
(c) \(w_{qq}^{(*)} < 0\) on \(Q_r\);
(d) $w_q^{(s)} > 0$ on $Q_b$.

First of all, (a) guarantees that $\Gamma$ (if not empty) is locally the graph of a function as desired. We only need to show that it connects $Q_t$ and $Q_b$. Note that $w^{(s)}$ attains a unique zero value at some point $(q_t, 1)$ on $Q_t$. Otherwise, we would find that $w^{(s)}(q, 0)$ has a constant sign by the Hopf lemma, contradicting to the equality $w^{(s)}_p(q, 0) = 0$. Thus, $\Gamma$ bifurcates locally from $(q_t, 1)$ inside $Q$. On the other hand, (d) shows that $\Gamma$ also bifurcates inside $Q$ from $(q_s, 0)$ on the bottom. Now it is easy to see that these two curves must be connected with each other. Indeed, relations (b) and (c) and inequalities $w^{(s)}_p(q_t, 0) < 0 < w^{(s)}_p(q_s, 0)$ guarantee that $w^{(s)}$ has constant sign on the vertical sides $Q_t$ and $Q_r$. In particular, $w^{(s)}$ is strictly negative on $Q_t$ and positive on $Q_r$. Thus, $\Gamma$ can not approach sides $Q_t$ and $Q_r$ and must connect $Q_t$ and $Q_b$ as desired.

Now we can prove that $\Phi^{(s)}(q_t, 1) < 0$ and then $\mathcal{S} < \sigma(s_\ast; r)$ by (2.12a), since $w^{(s)}(q_t, 1) = 0$. For that purpose we compute $\Phi^{(s)}(q_t, 1)$ by changing a contour of integration as follows:

$$
\Phi^{(s)}(q_t, 1) = \int_0^1 \Phi^{(s)}_p(q_t, p) \, dp = \int_\Gamma (\Phi^{(s)}_p, -\Phi^{(s)}_q) \cdot n \, dl,
$$

where $dl$ is the length element and $n = (n_1, n_2)$ is the unit normal to $\Gamma$ with $n_1 > 0$ (because $\Gamma$ is the graph of $f(p)$). Note that $n$ is proportional with $(w^{(s)}_q, w^{(s)}_p)$ along $\Gamma$ and is oriented in the same way. Therefore, $(\Phi^{(s)}_p, -\Phi^{(s)}_q) \cdot n$ has the same sign as

$$
(\Phi^{(s)}_p, -\Phi^{(s)}_q) \cdot (w^{(s)}_q, w^{(s)}_p) = - \left( \frac{w^{(s)}_p^2}{h^2 p^2} + \frac{w^{(s)}_q^2}{h^2 p^2} \right) w^{(s)}_q < 0,
$$

(3.2) which is a matter of a straightforward computation based on (2.10). To see that we first rewrite $\Phi^{(s)}_q$ as

$$
\Phi^{(s)}_q = u_q^{(s)} \left( \frac{\Phi^{(s)}_p}{h_p} + \frac{2w^{(s)}_p}{h_p^2 p} \right).
$$

Using this formula we compute

$$
(\Phi^{(s)}_p, -\Phi^{(s)}_q) \cdot (w^{(s)}_q, w^{(s)}_p) = \Phi^{(s)}_p w^{(s)}_q - u_q^{(s)} w^{(s)}_p \left( \frac{\Phi^{(s)}_p}{h_p} + \frac{2w^{(s)}_p}{h_p^2 p} \right)
$$

$$
= u_q^{(s)} \left( \frac{h_p \Phi^{(s)}_p}{h_p} - \frac{2(w^{(s)}_p)^2}{h_p^2 p} \right).
$$

It is left to use formula (2.10) for $\Phi^{(s)}_p$ to conclude (3.2). Thus, $(\Phi^{(s)}_p, -\Phi^{(s)}_q) \cdot n$ is negative along $\Gamma$ and then $\Phi^{(s)}(q_t, 1) < 0$. The lemma is proved.

Using Lemma [3.7] it is easy to complete the proof of the theorem. Indeed, for all $s \in (s_0, s_\ast)$ we have $\mathcal{S} < \sigma(s_\ast; r)$, while at the every crest we have $\Phi^{(s)}(q_c, 1) > 0$, because of (2.2a) and that $w^{(s)}_q(q_c, p) = 0$ for all $p \in [0, 1]$. Thus, the boundary condition (2.12a) then implies

$$
w^{(s)}(q_c, 1) > 2(r - R(s)),
$$

which is true for all $s \in (s_0, s_\ast)$. Here we used the fact that $w^{(s)}(q_c, 1) > 0$, which was proved in [KKL15]. Passing to the limit $s \to s_0$, we find

$$
\eta(q_c) > d_0 + 2(r - R_0).
$$

Finally, because $\eta(q_c) < r$ by (2.2b) and $R_0 = d_0 - \Omega(1)$, we obtain

$$
r < R_0 - \Omega(1),
$$

which finishes the proof of the theorem.
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