Abstract

Charge transport in tunnel junctions between singlet anisotropically paired superconductors is investigated theoretically making use of the Eilenberger equations for the quasiclassical Green functions. For specularly reflecting tunnel barrier plane we have found and described analytically characteristic singular points of the I-V curves which are specific for the case of anisotropic pairing. All four terms for the electric current are examined. Two of them describe ac Josephson effect and two correspond to the quasiparticle current (the last term occurs only for a time-dependent voltage). Various momentum dependences of the order parameter in the bulk of superconductors and different crystal orientations are considered. The results of numerical calculations for the total I-V curves are presented for some particular cases.

I. INTRODUCTION

Theoretical description of the Josephson and quasiparticle currents through a tunnel junction was developed many years ago for the case of s-wave isotropic superconductors. In particular, the total tunnel current under the externally applied time-dependent voltage was investigated in detail microscopically making use of tunneling Hamiltonian method [1,2] (see also [3] and references therein). Now it becomes clear that in the case of anisotropically paired superconductors the measurements of the Josephson effect and the quasiparticle current provide an important information about the structure of the superconducting order parameters from both sides of the junction and about the proximity and some other specific surface superconducting effects at the tunnel barrier plane [4,5]. The investigations in this field are of great interest and Josephson and quasiparticle tunneling experiments studying the anisotropic structure of superconducting order parameter for various high-temperature superconductors have attracted now much attention (see, for example, [6,7]).

One of the characteristic peculiarities of microscopic description of charge transport through tunnel junctions in anisotropically paired superconductors is, in fact, the failure of the tunneling Hamiltonian method applied to this case [12]. This approach, being usually quite suitable for tunnel junctions between s-wave isotropic superconductors, contains ambiguities in the case of anisotropic pairing due to substantial momentum dependence of the matrix elements describing tunneling between superconductors. One can show that the
choice of the tunneling matrix elements as being independent on the momentum direction (standard for the s-wave case) leads to confusing results for anisotropically paired superconductors. Furthermore, an unambiguous choice of this dependence cannot be done within this method. This leads to the necessity of making use of a microscopic description of tunneling between such superconductors based on matching of the electron propagators at the tunnel barrier, since this approach leaves no space for ambiguity.

Below we develop a microscopic study of the singular points on current-voltage characteristics for ac Josephson effect and for quasiparticle current of tunnel junctions between anisotropically paired superconductors under the externally applied (and, generally speaking, time-dependent) voltage. In our microscopic approach we are based on the Eilenberger equations for quasiclassical electron propagators combined with corresponding boundary conditions and the microscopic expression for the tunnel current. In the particular case of s-wave isotropic superconductors our results coincide with those found by Larkin and Ovchinnikov [1]. They obtained general expressions and described the singular points for functions $I_m(V)$ ($m = 1, 2, 3, 4$), which enter the relation for the total tunnel current (see below, for example, Eqs. (6)-(7)). Here $I_{1,2}(V)$ describe the amplitudes of two terms in the expression for the Josephson current, which are reduced for time-independent voltage to $I_1(V) \sin(\chi_1 - \chi_2 + 2eVt/\hbar) + I_2(V) \cos(\chi_1 - \chi_2 + 2eVt/\hbar)$. Further, $I_{3,4}(V)$ describe the quasiparticle current. The function $I_4(V)$ occurs only in the case of time-dependent voltage while for permanent voltage the quasiparticle current is entirely reduced to the only term $I_3(V)$. It was shown, in particular, that at $|eV| = \Delta_1 + \Delta_2$ the singular parts of functions $I_{1,4}(V)$ diverge logarithmically, while the functions $I_{2,3}(V)$ undergo a jump. Analogous singularities were found for time-dependent voltage oscillating with the frequency $\hbar\omega = \Delta_1 + \Delta_2$. The singularity of $I_1$ is known as Riedel singularity [3]. It is closely associated with the corresponding singularity in the density of states for superconductors at $\hbar\omega = \Delta$.

It is clear that Riedel’s anomalies themselves must be washed out in the case of strongly anisotropic pairing when the density of states doesn’t have the divergence. Nevertheless, as we show below, some new characteristic singularities appear even in the presence of nodes in the order parameter on the Fermi surface. The characteristic behaviors of the I-V curves turn out to be strongly dependent upon the crystal orientations of the superconductors, which, in particular, govern the spatial dependence of the order parameters near the barrier plane. But even in the simplest case of uniform distribution of the superconducting order parameter on both sides of a tunnel barrier the anomalies become strongly modified as compared to the isotropic case. The point is that for anisotropically paired singlet superconductors the functions $I_m(V)$ become also depending upon the momentum direction on the Fermi surface and enter the expression for the tunnel current as corresponding integrands. For obtaining the current-voltage relation in the vicinities of singular points one should carry out the integration over the momentum directions. We show that the nonanalytical behavior of the I-V curves takes place only for the voltages $|eV|$ which are equal to the values of the expressions $||\Delta_2(\hat{p}_2)|| \pm |\Delta_1(\hat{p}_1)||$ at their extremal points. Here $\hat{p}_1$ is the direction of the incident momentum and $\hat{p}_2$ – of the transmitted momentum, which is directly connected with $\hat{p}_1$ and the shapes of the Fermi surfaces. All the expressions are considered, for example, as functions of $\hat{p}_1$. Further, the singular behaviors of the I-V curves at these extrema turn out to be strongly dependent upon the type of the extremal points. This was pointed out earlier in [7], where the characteristic behaviors of the I-V curves have been determined
for the important particular case of the quasiparticle current at low temperatures, for the crystalline orientations when there is no surface pair breaking.

The situation becomes more complicated for the crystal orientations, for which the spatial dependence of the order parameter near the barrier plane plays an important role. The possibility for existence of quasiparticle bound states located in the vicinity of barrier plane is of importance in this case. At least two factors may be associated with such kind of effects. The former is the zero energy quasiparticle state which occurs at the barrier plane under certain general conditions [8,14–17]. The latter is the appearance of additional quasiparticle bound surface states due to the spatial dependence of the order parameter, which is suppressed by the surface [8]. We discuss below these possibilities and respective consequences for the current-voltage characteristics.

II. MICROSCOPIC EXPRESSION FOR THE TUNNEL ELECTRIC CURRENT

We consider a tunnel junction with transparency coefficient $D \ll 1$ and specularly reflecting barrier plane between two clean singlet superconductors. The external voltage $V(t) = \Phi_2(t) - \Phi_1(t)$ is assumed to be applied to the junction. Let the normal to the junction barrier plane be directed along the $x$-axis: $n \parallel Ox$. Then the microscopic expression for a tunnel current density in the first order of the barrier transparency $D$ may be represented in the form [18]:

$$j_x = -\frac{1}{8\pi^3} \int_{v_x > 0} \frac{d^2 S_1 \cdot v_x}{4\pi \cdot v_f} S_p \, \tau_3 D(\mathbf{p}_1) \left( \hat{g}_1^R \cdot \hat{g}_2^K + \hat{g}_1^K \cdot \hat{g}_2^A - \hat{g}_2^R \cdot \hat{g}_1^K - \hat{g}_2^K \cdot \hat{g}_1^A \right)_o (t, t) .$$  \hspace{1cm} (1)

Here and below we put $\epsilon = 1$ as well as $\hbar, c = 1$.

The retarded, advanced and Keldysh quasiclassical matrix propagators are taken in Eq. (1) at the junction barrier plane and must be calculated in zeroth order in junction transparency (i.e. for impenetrable barrier). They depend upon the respective momentum directions $\mathbf{p}_1$ and $\mathbf{p}_2$. Index 1 (2) labels the left (right) half space with respect to the boundary plane, and $v_x$ is the Fermi velocity component along the normal to the plane interface $n$. The integration in Eq. (1) is carried out over the part of the Fermi surface with $v_x > 0$. The relation between the incident and transmitted Fermi momenta (that is between $\mathbf{p}_1$ and $\mathbf{p}_2$) is as follows. The components parallel to the specular plane interface are equal to each other, while the values of normal components are determined by the condition, that $\mathbf{p}_1$ and $\mathbf{p}_2$ lie on respective Fermi surfaces. Naturally, in the particular case of identical superconductors with the spherical or cylindrical Fermi surfaces (provided the cylindrical axes are parallel to the barrier plane) the total incident and transmitted momenta are equal to each other $\mathbf{p}_1 = \mathbf{p}_2$.

Notations used in Eq. (1) are in correspondence with the following example:

$$(\hat{g}_1^R \cdot \hat{g}_2^K)_o (t, t) = \int_{-\infty}^t dt_1 \hat{g}_1^R (t, t_1) \cdot \hat{g}_2^K (t_1, t) .$$  \hspace{1cm} (2)
Here $\hat{g}_l^{R,A,K}(t, t_1) = \hat{S}_l(t) \hat{g}_l^{R,A,K}(t - t_1) \hat{S}_l^+(t_1)$ ($l = 1, 2$), where
\begin{equation}
\hat{S}_l(t) = \begin{pmatrix}
e^{i\chi_l(t)/2} & 0 \\
0 & e^{-i\chi_l(t)/2}
\end{pmatrix}, \quad \chi_l(t) = \chi_l - 2 \int^t \Phi_l(t') dt'
\end{equation}
and $\chi_l$ is the phase of the order parameter of $l$-th superconductor at the boundary plane for the case of zero electric potential $\Phi_l$.

Below we consider only singlet types of anisotropic pairing, for which matrix propagators may be represented in the form
\begin{equation}
\hat{g} = \begin{pmatrix} g & f \\ f^+ & -g \end{pmatrix}.
\end{equation}

For further calculations it is also important that nonequilibrium effects, as a rule, are not essential in tunnel junctions. The voltage $V$ simply shifts the Fermi levels in the electrodes relative each other by the value $V$. Besides, for finding the current in the first approximation with regard to transparency, the calculation of the Green functions may be performed for impenetrable half spaces (disregarding the transmissions of electrons through the junction). Under the conditions we consider the distribution functions of electrons remain the equilibrium ones and the effect of electric potential results only in the appearance of corresponding spatial independent terms in phases of superconducting order parameters and Green functions on both banks of the junction (see, for instance, Eq. (3)). Then the following relationship is valid
\begin{equation}
\hat{g}^K(\omega) = (\hat{g}^R(\omega) - \hat{g}^A(\omega)) \tanh \left( \frac{\omega}{2T} \right),
\end{equation}

According to general symmetries of the propagators one has $\hat{g}^A = \hat{\tau}_3 (\hat{g}^R)^\dagger \hat{\tau}_3$, where $\hat{\tau}_3$ is the third Pauli matrix. We suppose that the complex phase of the order parameter within the superconducting half space with impenetrable boundary is constant in the coordinate and momentum spaces. Then this phase is just the quantity $\chi_l$ in the Eq. (3) and the Green functions further must be calculated for the real values of the order parameter, though its sign may depend on the momentum direction. Under these conditions one has $f^{R,A}(\hat{p}, x, \omega) = -f^{+,R,A}(\hat{p}, x, \omega)$, $g^{R,A}(\hat{p}, x, \omega) = g^{R,A}(\hat{p}, x, \omega)$, $f^{R,A}(\hat{p}, x, -\omega) = f^{R,A\ast}(\hat{p}, x, \omega)$, $g^{R,A}(\hat{p}, x, -\omega) = -g^{R,A\ast}(\hat{p}, x, \omega)$. We take into account as well that a propagator taken at the impenetrable boundary for the incoming momentum $\hat{p}$ is equal to the one taken for the outgoing momentum $\hat{p}$, and $v_x(\hat{p}) = -v_x(\hat{p})$. Then one can get that for the time-dependent voltage $V(t) = V_0 + a \cos(\omega_0 t)$ the expression (11) for the tunnel current acquires the form
\begin{align*}
J_x &= \sum_{n=-\infty}^{\infty} J_n \left( \frac{a}{\omega_0} \right) \left\{ j_1(V_0 + \omega_n) \sin \left( \chi_1 - \chi_2 + 2V_0 t + \frac{a}{\omega_0} \sin (\omega_0 t) + \omega_n t \right) + \\
&+ j_2(V_0 + \omega_n) \cos \left( \chi_1 - \chi_2 + 2V_0 t + \frac{a}{\omega_0} \sin (\omega_0 t) + \omega_n t \right) \right\} + \\
&+ \sum_{n=-\infty}^{\infty} J_{n+1} \left( \frac{a}{\omega_0} \right) \left\{ j_1(V_0 + \omega_{n+1}) \sin \left( \chi_1 - \chi_2 + 2V_0 t + \frac{a}{\omega_0} \sin (\omega_0 t) + \omega_{n+1} t \right) + \\
&+ j_2(V_0 + \omega_{n+1}) \cos \left( \chi_1 - \chi_2 + 2V_0 t + \frac{a}{\omega_0} \sin (\omega_0 t) + \omega_{n+1} t \right) \right\}
\end{align*}
$$+ j_3 (V_0 + \omega_n) \cos \left( \omega_n t - \frac{a}{\omega_0} \sin (\omega_0 t) \right) + j_4 (V_0 + \omega_n) \sin \left( \omega_n t - \frac{a}{\omega_0} \sin (\omega_0 t) \right) \right). \quad (6)$$

The following notations are introduced here:

$$j_m (V) = \int_{v_{x1} > 0} \frac{d^2 S_{v_{x1}}}{(2\pi)^3} \frac{v_{x1} D (\hat{p}_1)}{v_{f1}} I_m (V, \hat{p}_1), \quad m = 1, 2, 3, 4, \quad (7)$$

$$I_1 (V, \hat{p}_1) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi^2} \tanh \left( \frac{\omega}{2T} \right) Im \left( f_1^R (\omega - V) \left( f_2^{+R} (\omega) + f_2^{R*} (\omega) \right) + (1 \leftrightarrow 2) \right), \quad (8)$$

$$I_2 (V, \hat{p}_1) = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi^2} \tanh \left( \frac{\omega}{2T} \right) Re \left( f_1^R (\omega - V) \left( f_2^{+R} (\omega) + f_2^{R*} (\omega) \right) + (1 \leftrightarrow 2) \right), \quad (9)$$

$$I_3 (V, \hat{p}_1) = \int_{-\infty}^{\infty} \frac{d\omega}{\pi^2} \left( \tanh \left( \frac{\omega - V}{2T} \right) - \tanh \left( \frac{\omega}{2T} \right) \right) Im g_1^R (\omega) \cdot Im g_2^R (\omega - V), \quad (10)$$

$$I_4 (V, \hat{p}_1) = -\int_{-\infty}^{\infty} \frac{d\omega}{\pi^2} \tanh \left( \frac{\omega}{2T} \right) \left( Reg_1^R (\omega - V) Im g_2^R (\omega) + Reg_2^R (\omega - V) Im g_1^R (\omega) \right). \quad (11)$$

The quantities $I_{1,4}(V)$ are even functions on $V$ and $I_{2,3}$ - odd functions. Note, that the interchanges $f \leftrightarrow f^+$ in the expressions (8), (9) lead to the same result for the Josephson current after the integration over the momentum directions in (7). Eqs.(7)-(11) are in correspondence with those obtained by Larkin and Ovchinnikov \cite{1} in consideration of s-wave isotropic superconductors.

In the case of permanent voltage through the junction one should let $a = 0$ in Eq.(6). Then one gets:

$$j_x (V) = j_1 (V) \sin (\chi_1 - \chi_2 + 2Vt) + j_2 (V) \cos (\chi_1 - \chi_2 + 2Vt) + j_3 (V). \quad (12)$$

It follows from Eqs.(12) and (6), that the specific singular points on the voltage-current characteristic for the tunnel current for both time-dependent and permanent voltages through the junction are defined by the singular points of function $j_m (V)$. If corresponding singular points are $V = V_m$, then in the case of constant voltage the specific points on the current-voltage characteristics are $V_{1,2,3}$, while for the current (1) peculiarities appear for the values $V_{m,n} = V_m - n\omega_0 \quad (m = 1, 2, 3, 4; \quad n = 0, \pm 1, \pm 2, \ldots)$. For large enough values $n \quad ((a/\omega_0) \lesssim n)$ the amplitude of the current becomes quite small (according to respective behavior of the Bessel functions).
III. QUASICLASSICAL GREEN FUNCTIONS AT THE IMPENETRABLE BOUNDARY PLANE

As it follows from previous section, for the calculation of the tunnel electric current one should consider the retarded electron propagators for the impenetrable superconducting half space and then find the values of the propagators at the boundary plane. The total analytical solution of this problem is quite complicated and hasn’t yet been obtained for any particular pairing potential leading to an anisotropically paired superconductivity (excluding the particular orientation for which there is no surface pair breaking and the order parameter doesn’t manifest a spatial dependence). The problem may be essentially simplified if one is interested only in the singular points on the I-V curves and consequently the singular points of the propagators taken at the barrier plane. For the consideration of this problem we make use of the Eilenberger equations for the retarded quasiclassical propagators, which may be written for the case of superconductors with singlet pairing as follows (further we omit the superscript for the retarded propagators)

\[
\begin{aligned}
(2\omega + iv_x \partial_x) f(\hat{\mathbf{p}}, x, \omega) + 2\Delta(\hat{\mathbf{p}}, x) g(\hat{\mathbf{p}}, x, \omega) &= 0 \\
(2\omega - iv_x \partial_x) f^+(\hat{\mathbf{p}}, x, \omega) - 2\Delta^*(\hat{\mathbf{p}}, x) g(\hat{\mathbf{p}}, x, \omega) &= 0 \\
v_x \partial_x g(\hat{\mathbf{p}}, x, \omega) - \Delta(\hat{\mathbf{p}}, x) f^+(\hat{\mathbf{p}}, x, \omega) - \Delta^*(\hat{\mathbf{p}}, x) f(\hat{\mathbf{p}}, x, \omega) &= 0.
\end{aligned}
\] (13)

For the sake of definiteness let us assume that the superconductor occupies a half space \( x > 0 \).

Apart from a self-consistency equation for \( \Delta(\hat{\mathbf{p}}, x) \), Eqs.(13) have to be supplemented by a normalization condition

\[ g^2 + f \cdot f^+ = -\pi^2 , \] (14)

and boundary conditions for quasiclassical propagators. For the specularly reflecting impenetrable surface one gets

\[ g(\hat{\mathbf{p}}, \omega) = g(\hat{\mathbf{p}}, \omega) \big|_{x=0} \ , \ f(\hat{\mathbf{p}}, \omega) = f(\hat{\mathbf{p}}, \omega) \big|_{x=0} \ , \ f^+(\hat{\mathbf{p}}, \omega) = f^+(\hat{\mathbf{p}}, \omega) \big|_{x=0} . \] (15)

Here \( \hat{\mathbf{p}} \) is the direction of incident momentum and \( \hat{\mathbf{p}} \) - the direction of reflected momentum.

The behaviors of the propagators in the depth of the superconductor

\[ g(\hat{\mathbf{p}}, \omega) \big|_{x=\infty} = \frac{-\pi \omega}{\sqrt{|\Delta_\infty(\hat{\mathbf{p}})|^2 - \omega^2}} , \ f \big|_{x=\infty} = -f^+(\hat{\mathbf{p}}, \omega) \big|_{x=\infty} = \frac{\pi \Delta_\infty(\hat{\mathbf{p}})}{\sqrt{|\Delta_\infty(\hat{\mathbf{p}})|^2 - \omega^2}} , \] (16)

must be also taken account of as the additional condition for the solutions of Eqs.(13).

Supposing that one can choose the gap function \( \Delta \) to be real within the superconducting half space with impenetrable boundary (i.e. in the absence of a current across the junction), we define

\[ f_1 = \frac{1}{2} \left( f - f^+ \right) , \ f_2 = \frac{1}{2} \left( f + f^+ \right) . \] (17)

The Eqs.(13), (14), being applied to \( f_{1,2} \), take the form
\begin{align}
2\omega f_1 + iv_x \partial_x f_2 + 2\Delta g &= 0 \\
f_2 &= -i\frac{v_x}{2\omega} \partial_x f_1 \\
\partial_x g &= -i\frac{2\Delta}{v_x} f_2 , \tag{18}
\end{align}

\begin{align}
g^2 + f_2^2 - f_1^2 &= -\pi^2 . \tag{19}
\end{align}

The boundary conditions at \( x = 0 \) for functions \( f_{1,2} \) are the same as (15), while in the depth of the superconductor we have

\begin{align}
f_1(\hat{p}, \omega) |_{x=0} = \frac{\pi \Delta_\infty(\hat{p})}{\sqrt{|\Delta_\infty(\hat{p})|^2 - \omega^2}} , \quad f_2 |_{x=\infty} = 0 . \tag{20}
\end{align}

The representations for \( g \) and \( f \) taken on the boundary, which turn out to be useful from the point of view of the consideration of singular parts of the propagators, may be derived from these equations. For obtaining the representations it is convenient to introduce the following function

\begin{align}
\tilde{\Delta}(\hat{p}, \omega) = \frac{\int_0^\infty \Delta(\hat{p}, x)f_2(\hat{p}, x, \omega)dx}{\int_0^\infty f_2(\hat{p}, x, \omega)dx} . \tag{21}
\end{align}

Since from the second and third equations of (18) one easily obtains

\begin{align}
g(\infty) - g(0) = -\frac{2i}{v_x} \int_0^\infty \Delta(\hat{p}, x)f_2(\hat{p}, x, \omega)dx , \quad f_1(\infty) - f_1(0) = 2\frac{i\omega}{v_x} \int_0^\infty f_2(\hat{p}, x, \omega)dx \tag{22}
\end{align}

the quantity \( \tilde{\Delta}(\hat{p}, \omega) \) may be written also in the form

\begin{align}
\tilde{\Delta}(\hat{p}, \omega) = -\omega \frac{g(\infty) - g(0)}{f_1(\infty) - f_1(0)} . \tag{23}
\end{align}

Here and below we denote

\begin{align}
g |_{x=0} = g(0) , \quad g |_{x=\infty} = g(\infty) , \quad f |_{x=0} = f(0) , \quad f |_{x=\infty} = f(\infty) .
\end{align}

After the substitution of the expressions (16), (20) for the propagators in the depth of the superconductor into the Eq.(23), it reduces to the following relation between \( g(0) \), \( f_1(0) \) and \( \Delta(\hat{p}, \omega) \):

\begin{align}
g(0) = \frac{\pi}{\omega} \frac{\tilde{\Delta}(\hat{p}, \omega)\Delta_\infty(\hat{p}) - \omega^2}{\sqrt{|\Delta_\infty(\hat{p})|^2 - \omega^2}} - \frac{\tilde{\Delta}(\hat{p}, \omega)}{\omega} f_1(0) . \tag{24}
\end{align}

Entirely analogous relation may be written also for the momentum direction \( \hat{p} \), after that the boundary conditions (15) allow us to write down the following representations for \( g(0) \) and \( f_1(0) \) separately
Indeed, since the function $f$ the normalization condition (19) the relation $g$ parts of $g$ for the quantity $\tilde{\Delta}(\hat{\mathbf{p}}, \omega)$ in the vicinity of the point $\tilde{\Delta}(\hat{\mathbf{p}}, \omega)$ we find that the function $\Delta$ limit cases in Eqs.(25), (26).

One can see from Eq.(25) that candidates for the singular points of the propagator $g(0)$ are $\omega = 0, \pm|\Delta_\infty(\hat{\mathbf{p}})|$, and, generally speaking, the singularities of functions $\Delta(\hat{\mathbf{p}}, \omega)$, $\Delta(\hat{\mathbf{p}}, \omega)$. Analogously, one finds from Eq.(24) that candidates for the singular points $f_1(0)$ are $\omega = \pm|\Delta_\infty(\hat{\mathbf{p}})|$. As far as the solutions of the equality $\tilde{\Delta}(\hat{\mathbf{p}}, \omega) = \Delta(\hat{\mathbf{p}}, \omega)$ are concerned, we note that the consideration of Eq.(24) for the momentum direction $\hat{\mathbf{p}}$ doesn’t result then in the independent relation as compared to Eq.(24) for the momentum direction $\hat{\mathbf{p}}$. Thus, some additional information is needed for the consideration of these limiting cases in Eqs.(25), (26).

Let us consider firstly singular parts of the propagators $g_s$, $f_s$, $f_s^+$ taken at the boundary in the vicinity of the point $\omega = 0$. Taking the low-frequency limit $\omega \to 0$ in Eqs.(25), (26), we find that the function $f_1(0)$ has no singularity at $\omega = 0$, opposite to the propagator $g(0)$ which turns out to have the pole at this point (under the condition that $\Delta_\infty(\hat{\mathbf{p}})$ and $\Delta_\infty(\hat{\mathbf{p}})$ have opposite signs):

$$g_s(0) = \frac{1}{2\pi\Delta_\infty(\hat{\mathbf{p}})} \left( sgn(\Delta_\infty(\hat{\mathbf{p}})) - sgn(\Delta_\infty(\hat{\mathbf{p}})) \right) = \frac{B_g(\hat{\mathbf{p}})}{\omega}. \quad (27)$$

It is remarkable that in addition to this relation one may find also the explicit expression for the quantity $\tilde{\Delta}(\hat{\mathbf{p}}, 0)$ through the inhomogeneous distribution of the order parameter. Indeed, since the function $f_1$ has no singularity at $\omega = 0$ (see Eq.(18)), we can derive from the normalization condition (14) the relation $g_s = \pm i f_2$ (when $\omega \to 0$) between the singular parts of $g$ and $f_2$. Then from the last equation of the system (18) we get the equation for the singular part of $f_2$: $\partial_x f_2 + \frac{2\Delta}{v_s} f_2 = 0$. The solution of this equation which satisfies the conditions (24), (25), is

$$f_2(\hat{\mathbf{p}}, x, \omega) = f_2(\hat{\mathbf{p}}, 0, \omega) \exp \left( -\frac{2sgn(\Delta_\infty(\hat{\mathbf{p}}))}{|v_x|} \int_0^x \Delta(\hat{\mathbf{p}}, x') dx' \right). \quad (28)$$

Substituting this solution into Eq.(21) we obtain

$$\tilde{\Delta}(\hat{\mathbf{p}}, 0) = \frac{1}{2} \frac{|v_x|sgn(\Delta_\infty(\hat{\mathbf{p}}))}{\int_0^\infty \exp \left( -\frac{2sgn(\Delta_\infty(\hat{\mathbf{p}}))}{|v_x|} \int_0^x \Delta(\hat{\mathbf{p}}, x') dx' \right) dx}. \quad (29)$$

Note, that the determination of sign in Eq.(28) allows to fix sign in the relation between the zero frequency singular parts of the quasiclassical propagators in the close vicinity of the boundary:

$$f_s(\hat{\mathbf{p}}, x, \omega) = f_s^+(\hat{\mathbf{p}}, x, \omega) = -i sgn(v_x \Delta_\infty(\hat{\mathbf{p}})) g_s(\hat{\mathbf{p}}, x, \omega). \quad (30)$$
Thus, Eqs. (27), (29) and (30) provide the quite general description for the zero frequency singular parts of the propagators taken at the boundary. If one is interested in only the singular points of the I-V curve, one can separate the problem of the solution of the self-consistency equation for particular pairing potentials for other investigations. The latter problem being an important part of the total theoretical description of the I-V curve for the tunnel junction, is very cumbersome and obviously includes large numerical investigations within the framework of any microscopic model for the pairing potential.

Passing to the other candidates for the singular points \( \omega = \pm |\Delta_\infty(\hat{p})|, \pm |\Delta_\infty(\hat{p})| \), we find firstly one important relationship for the function \( \tilde{\Delta}(\hat{p}, \omega) \) at this frequency. For this purpose the asymptotic behavior of the function \( f_2(\hat{p}, x, \omega) \) at \( x \to \infty \) is of interest, which follows from Eqs. (18)–(20):

\[
f_2(\hat{p}, x, \omega) \propto \exp \left( -\frac{2\sqrt{|\Delta_\infty(\hat{p})|^2 - \omega^2}}{|v_x|} x \right). \tag{31}
\]

One can see from Eqs. (31), (21) that in the limit \( |\omega| \to |\Delta_\infty(\hat{p})| \) the main contribution to the integrals in (21) comes from the depth of the superconductor, where \( \Delta(\hat{p}, x) \) is equal to its bulk value. Then one obtains the relationship

\[
\tilde{\Delta}(\hat{p}, \omega) \to \Delta_\infty(\hat{p}) \quad \text{for} \quad |\omega| \to |\Delta_\infty(\hat{p})|. \tag{32}
\]

Now one can consider the behaviors of \( g \) and \( f_1 \) in the vicinity of the points \( |\omega| \to |\Delta_\infty(\hat{p})|, |\Delta_\infty(\hat{p})| \). Taking the limit \( |\omega| \to |\Delta_\infty(\hat{p})| \) or \( |\Delta_\infty(\hat{p})| \) in Eqs. (25), (26), one can find with the help of (32) the cancellation of divergences and, in the first approximation, the appearance of the square root nonanalytical behavior of the form \( \sqrt{\Delta_\infty^2(\hat{p}) - \omega^2} \) (or \( \sqrt{\Delta_\infty^2(\hat{p}) - \omega^2} \)) for \( g(0) \) and \( f_1(0) \) (and therefore \( f(0) \)) at these points. The exclusions may represent the orientations for which \( \tilde{\Delta}(\hat{p}, \omega) = \Delta(\hat{p}, \omega) \), when strictly speaking Eqs. (25), (26) themselves do not provide more information than (24). It is well known that if this condition holds for all momentum orientations, there is no surface pair breaking and the divergences in the propagators at \( |\omega| \to |\Delta_\infty(\hat{p})| \) occur in this particular case:

\[
g^R(\hat{p}, \omega) = -\frac{\pi \omega}{\sqrt{|\Delta(\hat{p})|^2 - \omega^2}}, \quad f^R(\hat{p}, \omega) = \frac{\pi \Delta(\hat{p})}{\sqrt{|\Delta(\hat{p})|^2 - \omega^2}}. \tag{33}
\]

At last one should discuss the possibility for existence of quasiparticle bound states with nonzero energy located near the boundary. They may appear, for instance, due to the spatial dependence of the order parameter which is suppressed at the boundary. The bound state may be interpreted as a bound state in the "potential well" formed by the order parameter \[8\]. Since the quasiparticle bound state corresponds to the pole in the quasiclassical propagators, we simply add below the pole-like term to the singular parts of the propagators. If the propagators \( g(0), f_1(0) \) have similar poles describing localized states with nonzero energy, then it follows from Eq. (24) and the boundary conditions \[15\], that at the point of the pole \( \Delta(\hat{p}, \omega) = \Delta(\hat{p}, \omega) \).
Taking into account the results obtained above, the nonanalytical terms of \( g(0) \) and \( f(0) \) may be written in the form
\[
g_s^R (\hat{p}, \omega) |_{x=0} = \frac{B_g (\hat{p})}{\omega + i\delta} + \frac{Q_g (\hat{p})}{\omega - h(\hat{p}) \cdot sgn(\omega) + i\delta} + C (\hat{p}, \omega) sgn(\omega) \sqrt{\Delta^2 (\hat{p})} - \omega^2 + C (\hat{p}, \omega) sgn(\omega) \sqrt{\Delta^2 (\hat{p})} - \omega^2 + \cdots ,
\]
\[
f_s^R (\hat{p}, \omega) |_{x=0} = \frac{iB_f (\hat{p})}{\omega + i\delta} + \frac{iQ_f (\hat{p})}{\omega - h(\hat{p}) \cdot sgn(\omega) + i\delta} + E (\hat{p}, \omega) \sqrt{\Delta^2 (\hat{p})} - \omega^2 + E (\hat{p}, \omega) \sqrt{\Delta^2 (\hat{p})} - \omega^2 + \cdots , \quad \delta \to +0 .
\]

The following relationships take place: \( B_f (\hat{p}) = -sgn(v_x \Delta_\infty (\hat{p})) B_g (\hat{p}) = -B_f (-\hat{p}) \), \( Q_g (\hat{p}) = |Q_f (\hat{p})| \), \( Q_f (\hat{p}) = -Q_f^* (-\hat{p}) \), \( h(-\hat{p}) = h(\hat{p}) \).

Below we are interested only in the values of functions \( Q_{g,f}(\hat{p}) \) near the poles \( \omega = \pm h(\hat{p}) \). In considering positive and negative poles one should take into account that quantities \( Q_g (\hat{p}, \omega) \), \( ReQ_f (\hat{p}, \omega) \) are even functions, while \( ImQ_f (\hat{p}, \omega) \) is an odd function of \( \omega \). We do not indicate further the explicit energy dependence of functions \( Q_{g,f} \), using only their values at the positive pole.

Since for \(|\omega| < min (|\Delta_\infty (\hat{p})|, |\Delta_\infty (\hat{p})|) \) the quasiparticle density of states in the continuum is zero for fixed momentum direction, it is natural to demand \( ImC, ImQ_s = 0 \) under this condition. In the particular case \( \Delta (\hat{p}, x) = -\Delta (\hat{p}, x) \), when \( \Delta (x = 0) \), \( f_1 (0) = 0 \), one may obtain from the Eilenberger equation \( Re f(0) = 0 \) for the frequencies \(|\omega| < |\Delta_\infty (\hat{p})| \).

Then for this frequency interval one has \( Re (E (\hat{p}, \omega) + E (\hat{p}, \omega)) = 0 \).

As we show below, the location and type of singular points on the I-V characteristics of the tunnel current associated with the quasiparticle bound states, are determined, in particular, by the extremal and nonanalytical points of the function \( h(\hat{p}) \). From the boundary conditions for the propagators and the parity of pairing we get \( h(\hat{p}) = h(-\hat{p}) \). Taking into account also that for \( \hat{p} \parallel \mathbf{n} \) it follows \( \hat{p} = -\hat{p} \), we find that the function \( h(\hat{p}) \) (as well as the total propagator \( g(0) \)) must have an extremal value for the direction \( \hat{p} \) along the normal to the boundary \( \mathbf{n} \). Analogously, we get that the function \( Q_f (\hat{p}) \) takes purely imaginary value for the direction \( \hat{p} \parallel \mathbf{n} \). In particular, it follows from here for the the orientations \( \Delta (\hat{p}, x) = -\Delta (\hat{p}, x) \), that \( Q_{f,g} (\hat{p}) = 0 \) for the momentum direction \( \hat{p} \parallel \mathbf{n} \). The other characteristic points of the function \( h(\hat{p}) \) are the momentum directions where the quasiparticle state bound to the boundary disappears.

The results of numerical calculations of quantities \( B_g (\phi), Q_g (\phi) = |Q_f (\phi)|, ReQ_f (\phi) \) and \( h(\phi) \) are presented in Fig. 1. We consider a tetragonal superconductor with cylindrical Fermi surface and impenetrable specularly reflecting interface at \( x = 0 \) (the cylindrical axis \( z \) is parallel to the boundary plane). The pairing interaction is supposed to have the particular form \( V(\phi, \phi') = 2V \cos(2\phi - 2\phi_0) \cos(2\phi' - 2\phi_0) \), which results in a d-wave order parameter. Here \( \phi \) is the azimuthal angle in \( xy \)-plane, which is counted from the direction of the normal to the boundary. Angle \( \phi_0 \) describes orientation of the crystalline \( x_0 \)-axis with respect to the normal to the boundary. Then the order parameter has the form: \( \Delta (\phi, x) = \Delta (x) \cos(2\phi - 2\phi_0) \), where \( \Delta (x) \) has to be evaluated self-consistently. The angle \( \phi \)
defines the direction of incoming momentum along the quasiparticle trajectory. We choose \( \phi_o = \pi/9 \), \( T = 0.45T_c \), \( \Delta_0/(2T) = 2 \), where \( \Delta_0 \equiv \Delta(x = \infty) \), \( T_c \) is the critical temperature. 

For the surface to lattice orientation \( \phi_o = \pi/9 \) the order parameter is suppressed in the vicinity of the boundary up to the value \( \Delta(x = 0) = 0.28\Delta_0 \). Quantities \( B_g(\phi) \), \( Q_g(\phi) \), \( \text{Re}Q_f(\phi) \) are normalized in Fig. 1 to the value \( \pi\Delta_0 \), while the quantity \( h(\phi) \) – to the value \( \Delta_0 \). One can see that the bound states with nonzero energy \( \pm h(\phi) \) exist only in the narrow angular region \( \phi \in (0.095, 0.095) \) in the vicinity of the normal to the boundary. The mid-gap states exist within two other wider regions of momentum directions, where \( B_g(\phi) \neq 0 \). 

It follows from Fig. 1, that the maximal value of the function \( h(\phi) \) is \( h_m = 0.7\Delta_0 \) and the value of \( h(\phi) \) at the edge points is \( h_{ed} = 0.63\Delta_0 \). It is worth noting that for the momentum directions \( \hat{p} \) (\( \phi_{ed} = \pm 0.095 \)), where the bound states with nonzero quasiparticle energy disappear, the following relation holds \( h_{ed} = \min(|\Delta_{\infty}(\hat{p})|, |\Delta_{\infty}(\hat{p})|) \). This means that discrete bound levels combine with the continuum spectrum at the momentum directions \( \phi_{ed} \). Quantities \( C(\hat{p}, \omega) \), \( E(\hat{p}, \omega) \) (or \( C(\hat{p}, \omega) \), \( E(\hat{p}, \omega) \) - depending on what quantity is smaller among \( |\Delta_{\infty}(\hat{p})|, |\Delta_{\infty}(\hat{p})| \)), which are present in Eqs. (34), (35), diverge at the momentum directions \( \phi_{ed} \) and the energy \( |\omega| = h_{ed} \).

IV. CURRENT-VOLTAGE CHARACTERISTICS FOR THE JOSEPHSON AND QUASIPARTICLE CURRENTS

The peculiarities of the I-V curves are due to the singular points of functions \( I_m(V, \hat{p}_1) \). If these functions don’t depend upon the momentum directions, their singular behavior with the variable \( V \) directly describes the respective behavior of the I-V curves, according to Eqs. (3) – (11). In contrast to this case, for anisotropically paired superconductors the integration over momentum directions greatly modifies the singularities of the current–voltage characteristics. Then the singular points of functions \( j_m(V) \) describe the singularities on the I-V curves. The other distinctive feature of the anisotropically paired superconductors which essentially influences the behavior of the Josephson effect as well as the quasiparticle current, is their sensitivity to the inhomogeneities and interfaces. The suppression of the anisotropic order parameter at the boundary results in the specific surface superconducting effects. At least two important features must be taken into account in this context. The former is associated with the possibility for the opposite signs of the bulk order parameter taken for the directions of the incident and reflected momenta. This results in the zero energy quasiparticle bound states at the specular boundary. The latter appears if the additional quasiparticle bound states localized near the barrier plane occur due to the particular form of the spatial dependence of the order parameter (see, [13, 18, 20, 14, 17]).

A. Crystal Orientations with no Surface Pair Breaking

Specific features of current-voltage characteristics for the tunnel electric current for anisotropically paired superconductors differ from the s-wave isotropic ones even in the case of conventional boundary conditions, when the surface doesn’t suppress the superconducting order parameter. This holds if the values of the order parameter taken for the incident and reflected momenta are equal to each other for all momentum directions. The
consideration of this particular case is just the subject of current section. Let us consider the behavior of functions $j_m(V)$, which determine the current-voltage curves, for the cases when the electron propagators at the impenetrable plane have the form (33) coinciding with their bulk expressions. Substituting these propagators into Eqs. (8)-(11), we get

$$I_1(V, \hat{p}_1) = -\Delta_1(\hat{p}_1)\Delta_2(\hat{p}_2) \int_{-\infty}^{\infty} d\omega \tanh\left(\frac{\omega}{2T}\right) \left(\frac{\Theta(|\Delta_1(\hat{p}_1)| - |\omega - V|)}{\sqrt{\Delta_1(\hat{p}_1)^2 - (\omega - V)^2}} \times \frac{\Theta(|\omega| - |\Delta_2(\hat{p}_2)|)}{\sqrt{\omega^2 - |\Delta_2(\hat{p}_2)|^2}} + \frac{\Theta(|\omega| - |\Delta_1(\hat{p}_1)|)}{\sqrt{\omega^2 - |\Delta_1(\hat{p}_1)|^2}} \frac{\Theta(|\Delta_2(\hat{p}_2)| - |\omega + V|)}{\sqrt{|\Delta_2(\hat{p}_2)|^2 - (\omega + V)^2}}\right), \quad (36)$$

$$I_2(V, \hat{p}_1) = \Delta_1(\hat{p}_1)\Delta_2(\hat{p}_2) \int_{-\infty}^{\infty} d\omega \left(\tanh\left(\frac{\omega}{2T}\right) - \tanh\left(\frac{\omega + V}{2T}\right)\right) \times \frac{\text{sgn}(\omega) \text{sgn}(\omega + V) \Theta(|\omega| - |\Delta_1(\hat{p}_1)|) \Theta(|\omega + V| - |\Delta_2(\hat{p}_2)|)}{\sqrt{\omega^2 - |\Delta_1(\hat{p}_1)|^2} \sqrt{(\omega + V)^2 - |\Delta_2(\hat{p}_2)|^2}}, \quad (37)$$

$$I_3(V, \hat{p}_1) = \int_{-\infty}^{\infty} d\omega \left(\tanh\left(\frac{\omega - V}{2T}\right) - \tanh\left(\frac{\omega}{2T}\right)\right) \frac{|\omega + V\Theta(|\omega| - |\Delta_1(\hat{p}_1)|)|}{\sqrt{\omega^2 - |\Delta_1(\hat{p}_1)|^2}} \times \frac{\Theta(|\omega - V| - |\Delta_2(\hat{p}_2)|)}{\sqrt{(\omega - V)^2 - |\Delta_2(\hat{p}_2)|^2}}, \quad (38)$$

$$I_4(V, \hat{p}_1) = -\int_{-\infty}^{\infty} d\omega \tanh\left(\frac{\omega}{2T}\right) (\omega - V)|\omega| \left(\frac{\Theta(|\Delta_1(\hat{p}_1)| - |\omega - V|)}{\sqrt{\Delta_1(\hat{p}_1)^2 - (\omega - V)^2}} \times \frac{\Theta(|\omega| - |\Delta_2(\hat{p}_2)|)}{\sqrt{\omega^2 - |\Delta_2(\hat{p}_2)|^2}} + \frac{\Theta(|\omega| - |\Delta_1(\hat{p}_1)|)}{\sqrt{\omega^2 - |\Delta_1(\hat{p}_1)|^2}} \frac{\Theta(|\Delta_2(\hat{p}_2)| - |\omega - V|)}{\sqrt{|\Delta_2(\hat{p}_2)|^2 - (\omega - V)^2}}\right). \quad (39)$$

The singularities of functions $I_m(V, \hat{p}_1)$ appear after the integration over the frequency only if two square roots (multiplied by each other in the denominators of the integrands) are equal to zero simultaneously. It is possible for certain values of frequency and voltage and the singularities turn out to be located at momentum-dependent points $|V| = ||\Delta_2| \pm |\Delta_1||$. It follows from Eqs. (36)-(39) that the expressions for singular parts of $I_m(V, \hat{p}_1)$ are as follows

$$I_1 = \frac{1}{2} \sqrt{\Delta_1 \Delta_2} \text{sgn}(\Delta_1 \Delta_2) \left\{ \tanh\left(\frac{|\Delta_1|}{2T}\right) + \tanh\left(\frac{|\Delta_2|}{2T}\right) \right\} \times \ln |V| - |\Delta_1| - |\Delta_2| - \pi \tanh\left(\frac{|\Delta_1|}{2T}\right) - \tanh\left(\frac{|\Delta_2|}{2T}\right) \Theta(|V| - ||\Delta_2| - |\Delta_1||) \right\}, \quad (40)$$
\[ I_2 = \frac{1}{2} \sqrt{\Delta_1 \Delta_2} sgn(\Delta_1 \Delta_2) sgn(V) \left\{ sgn (|\Delta_1| - |\Delta_2|) \left[ \tanh \left( \frac{|\Delta_1|}{2T} \right) - \right. \right. \right. \\
- \left. \left. \left. \tanh \left( \frac{|\Delta_1| + |V| sgn (|\Delta_2| - |\Delta_1|)}{2T} \right) \right] \ln ||V| - ||\Delta_2| - |\Delta_1||| + \right. \right. \right. \\
+ \pi \left[ \tanh \left( \frac{|\Delta_1|}{2T} \right) + \tanh \left( \frac{|\Delta_2|}{2T} \right) \right] \Theta (||V| - |\Delta_1| - |\Delta_2|) \left\} \right. , \] (41)

\[ I_3 = \frac{1}{2} \sqrt{\Delta_1 \Delta_2} sgn(V) \left\{ sgn (|\Delta_1| - |\Delta_2|) \left[ \tanh \left( \frac{|\Delta_1|}{2T} \right) - \right. \right. \right. \\
- \left. \left. \left. \tanh \left( \frac{|\Delta_1| + |V| sgn (|\Delta_2| - |\Delta_1|)}{2T} \right) \right] \ln ||V| - ||\Delta_2| - |\Delta_1||| - \right. \right. \right. \\
- \pi \left[ \tanh \left( \frac{|\Delta_1|}{2T} \right) + \tanh \left( \frac{|\Delta_2|}{2T} \right) \right] \Theta (||V| - |\Delta_1| - |\Delta_2|) \left\} \right. , \] (42)

\[ I_4 = -\frac{1}{2} \sqrt{\Delta_1 \Delta_2} \left\{ \left[ \tanh \left( \frac{|\Delta_1|}{2T} \right) + \tanh \left( \frac{|\Delta_2|}{2T} \right) \right] \ln ||V| - |\Delta_1| - |\Delta_2|| + \right. \right. \right. \\
+ \pi \left| \tanh \left( \frac{|\Delta_1|}{2T} \right) - \tanh \left( \frac{|\Delta_2|}{2T} \right) \right| \Theta (||V| - ||\Delta_2| - |\Delta_1||) \right\} . \] (43)

According to Eqs. (3), (4), for obtaining the current-voltage relation in the vicinities of singular points one should carry out the integration of expressions (40)-(43) over the momentum directions and consider the correspondent expressions for \( j_m(V) \). It follows from this integration that nonanalytical behavior of the I-V curves takes place only for the values \(|V|\) in the close vicinities of the extremal points of the expressions \(||\Delta_2(\hat{p}_2)\| \pm |\Delta_1(\hat{p}_1)|\|\), which are considered, for example, as functions of \( \hat{p}_1 \). The characteristic behaviors of the I-V curves near these extrema turn out to be strongly dependent upon the type of the extremal point. Comparing the expressions (40)-(43) one finds the same singular behaviors for pairs of functions \( j_1(V), j_4(V) \) and \( j_2(V), j_3(V) \) (disregarding the differences in signs for the moment). Due to this fact it is sufficient to describe, for example, only the singular points for \( j_1(V) \) and \( j_3(V) \).

It is convenient further to examine the singular points for the conductance \( G = dj_x/dV \). It can be shown, that only the derivatives of \( \Theta \)- and logarithmic functions with respect to the voltage are associated with nonanalytical behavior of \( G(V) \). In the former case one obtains \( \delta \)-function, which in fact reduces the integration over the Fermi-surface to the integration over the line on the surface. Corresponding terms \( \tilde{G}_{1,3}(V) \) of functions \( G_{1,3}(V) = dj_{1,3}/dV \) may be represented as follows:

\[ \tilde{G}_1 = -sgn(V) \int dl \frac{K^-(\hat{p}_1)}{|\nabla_{\hat{p}_1} (|\Delta_1| - |\Delta_2|)|} , \quad \tilde{G}_3 = -\int dl \frac{K^+(\hat{p}_1) sgn(\Delta_1 \Delta_2)}{|\nabla_{\hat{p}_1} (|\Delta_1| + |\Delta_2|)|} . \] (44)
Here \( l \) is the local coordinate along the line \(|V| = ||\Delta_1(p_1)| - |\Delta_2(p_2)||\) on the Fermi surface (sign plus corresponds to \( \hat{G}_3 \) and minus – to \( \hat{G}_1 \)). The functions \( K^\pm \) are determined as follows:

\[
K^\pm(p_1) = \frac{\pi}{2(2\pi)^3} \left| \tanh \left( \frac{\Delta_1(p_1)}{2T} \right) \pm \tanh \left( \frac{\Delta_2(p_2)}{2T} \right) \right| \sqrt{\Delta_1 \Delta_2} \text{sgn}(\Delta_1 \Delta_2) \frac{\text{sgn}(\Delta_1 \Delta_2)(m_1 D)}{V_f} \tag{45}
\]

We consider below various types of extrema and get respective behaviors of \( G_m(V) \). Let the function \(||\Delta_2| - |\Delta_1||\) take maximal or minimal value at the point \( p_1 = p_0 \) on the Fermi surface and in the vicinity of this point has the form

\[
||\Delta_2| - |\Delta_1|| = a \pm (b\hat{p}_1^2 + c\hat{p}_2^2), a, b, c > 0 . \tag{46}
\]

Here \( \hat{p}_1, \hat{p}_2 \) are the local orthogonal coordinates in the vicinity of the point \( p_0 \). Since the function \(||\Delta_2| - |\Delta_1||\) comes in the expressions for \( j_{1,4} \) as the argument of \( \Theta \)-function and in the formulae for \( j_{2,3} \) as an argument of the logarithmic function, we obtain two different singular behaviors near the value \(|V| = a\):

\[
\delta G_1 \left|_{|V|=a} = \text{sgn}(\Delta_1 \Delta_2) \right|_{\hat{p}_1=\hat{p}_0} \delta G_4 \mid_{|V|=a} = \mp \frac{\pi}{\sqrt{bc}} K^- (\hat{p}_0) \tag{47}
\]

\[
G_2 = \text{sgn}(\Delta_1 \Delta_2) \mid_{\hat{p}_1=\hat{p}_0} G_3 = \pm \frac{1}{\sqrt{bc}} K^- (\hat{p}_0) \ln ||V| - a| \tag{48}
\]

The notation for a jump of the conductance \( \delta G \mid_{|V|=a} = G(||V| > a) - G(||V| < a) \) is introduced here.

The singularities coming from the maximal and minimal values of the quantity \(|\Delta_2| + |\Delta_1|\), when one has \(|\Delta_2| + |\Delta_1| = a \pm (b\hat{p}_1^2 + c\hat{p}_2^2), a, b, c > 0\), are described analogously:

\[
\delta G_2 \mid_{|V|=a} = -\text{sgn}(\Delta_1 \Delta_2) \mid_{\hat{p}_1=\hat{p}_0} \delta G_3 \mid_{|V|=a} = \mp \frac{\pi}{\sqrt{bc}} K^+ (\hat{p}_0) \tag{49}
\]

\[
G_1 = -\text{sgn}(\Delta_1 \Delta_2) \mid_{\hat{p}_1=\hat{p}_0} G_4 = \pm \frac{1}{\sqrt{bc}} K^+ (\hat{p}_0) sgn(V) \ln ||V| - a| \tag{50}
\]

In the case of a saddle point of the function \(||\Delta_2| - |\Delta_1||\) one has near this point

\[
||\Delta_2| - |\Delta_1|| = a + b\hat{p}_1^2 - c\hat{p}_2^2, a, b, c > 0 . \tag{51}
\]

The respective singular parts of the conductance read

\[
G_1 = \text{sgn}(\Delta_1 \Delta_2) \mid_{\hat{p}_1=\hat{p}_0} G_4 = \frac{1}{\sqrt{bc}} K^- (\hat{p}_0) sgn(V) \ln ||V| - a| \tag{52}
\]

\[
G_2 = \text{sgn}(\Delta_1 \Delta_2) \mid_{\hat{p}_1=\hat{p}_0} G_3 = \frac{2}{\pi \sqrt{bc}} sgn(||V| - a) K^- (\hat{p}_0) \ln^2 ||V| - a| \tag{53}
\]
Analogously, for the saddle point

\[ |\Delta_2| + |\Delta_1| = a + b\tilde{p}_1^2 - c\tilde{p}_2^2, \quad a, b, c > 0 \]  

(54)

we get

\[ G_2 = -sgn(\Delta_1\Delta_2) \big|_{\tilde{p}_1=\tilde{p}_0} G_3 = -\frac{1}{\sqrt{bc}} K^+ (\tilde{p}_0) \ln |a - |V|| \]  

(55)

\[ G_1 = -sgn(\Delta_1\Delta_2) \big|_{\tilde{p}_1=\tilde{p}_0} G_4 = \frac{2sgn(V)sgn(|V| - a) K^+ (\tilde{p}_0)}{\pi\sqrt{bc}} \ln^2 |V - a| \]  

(56)

The jumps in conductance described by Eqs. (47), (49) correspond, of course, to the kinks on the I-V curves. Logarithmic divergences in conductance described by Eqs. (48), (50), (52) and (55), result in step-like points on the I-V curves (note that on both sides of these singular points G has the same sign). At last the terms containing logarithm squared in Eqs. (53), (56) describe the cusps (beak-like points) on the current-voltage characteristics. They appear in the case of saddle points of the functions \(|\Delta_2| \pm |\Delta_1||\) after the integration of logarithmic singularities for \(I_m\).

Let now the quantities \(|\Delta_2| \pm |\Delta_1||\) have the extremal values on some line \(\tilde{l}\) on the Fermi-surface, rather than on isolated points as it was suggested above. Then, for example, in the vicinities of saddle points of the functions \(|\Delta_2| - |\Delta_1||\) one has

\[ ||\Delta_2| - |\Delta_1|| = a \pm b\tilde{p}_1^2, \quad a, b > 0, \]  

(57)

where \(\tilde{p}_1\) is the local coordinate on the Fermi-surface orthogonal to the extremal line \(\tilde{l}\).

In this case the inverse square root singularities in the conductance appear from one side of the voltage value \(|V| = a\):

\[ G_1 = sgn(\Delta_1\Delta_2) \big|_{\tilde{l}} G_4 = -\frac{sgn(V)}{\sqrt{||V| - a|}} |V - a| K^- (\tilde{p}_0) \ln^2 |V - a| \]  

(58)

\[ G_2 = sgn(\Delta_1\Delta_2) \big|_{\tilde{l}} G_3 = \pm \frac{\Theta (\mp (|V| - a))}{\sqrt{||V| - a|}} \int_{|\Delta_2| - |\Delta_1|| = a} K^- (\tilde{p}_0) \tilde{l} \]  

(59)

These singularities correspond to the vertical slope of the I-V curve from one side of the voltage value \(|V| = a\) for each of four terms presented in the total expression for the tunnel current. For example, in the case of maximum on the line \(\tilde{l}\) there are the vertical slope of the curves \(j_1(V)\) and \(j_4(V)\) at \(|V| = a\) from side \(|V| < a\), and of the curves \(j_2(V), j_3(V)\) from side \(|V| > a\).

Analogously, in the case of maximal or minimal value of the quantity \(|\Delta_1| + |\Delta_2|\) on the line \(\tilde{l}\), when near this line one has

\[ |\Delta_1| + |\Delta_2| = a \pm b\tilde{p}_1^2, \quad a, b > 0, \]  

(60)
the singular behavior of the conductance is described as follows:

$$G_2 = -sgn(\Delta_1 \Delta_2) \left| G_3 \right| = \frac{\Theta (\pm (|V| - a))}{\sqrt{|V| - a}} \int_{|\Delta_1| + |\Delta_2| = a} d\bar{\Gamma} \frac{K^+}{2\sqrt{b}}$$ (61)

$$G_1 = -sgn(\Delta_1 \Delta_2) \left| G_4 \right| = \mp \frac{sgn(V)}{\sqrt{|V| - a}} \Theta (\mp (|V| - a)) \int_{|\Delta_1| + |\Delta_2| = a} d\bar{\Gamma} \frac{K^+}{\sqrt{b}}$$ (62)

We have considered above different superconductors or at least different crystal orientations from both sides of the junction, when not only sum $|\Delta_1| + |\Delta_2|$ may have extremal points or lines on the Fermi surface, but the difference $|\Delta_1| - |\Delta_2|$ also depend upon the momentum direction and may have extrema. As a result one can’t take directly the limit $\Delta_1 = \pm \Delta_2 = \Delta$ in the expressions written above for singularities, which are associated with extrema of the difference $|\Delta_1| - |\Delta_2|$. For this particular case quantities $G_{1,4}$ have not singular point at $V = |\Delta_1| - |\Delta_2| = 0$, while for $G_{2,3}$ in the vicinity $V = 0$ (more exactly, for $|V| \ll T$) one gets instead of Eqs. (63), (64), (65)

$$G_2 = \pm G_3 = \pm \frac{\ln |V|}{2T} \int_{v_{x1} > 0} \frac{d^2 S_1}{(2\pi)^3} |v_{f1}| D(\hat{\mathbf{p}}_1) \frac{|\Delta|}{\cosh^2 \frac{|\Delta|}{2T}}$$ (63)

For isotropic s-wave superconductors this term is exponentially small at low temperatures $T \ll \Delta$. In contrast to this case, for anisotropically paired superconductors the expression in Eq.(63) manifests power law temperature behavior. For instance, for the line of nodes of the order parameter (when in its vicinity $|\Delta(\hat{\mathbf{p}})| = b|\mathbf{p}_1|$) it follows from Eq.(63) at low temperatures $G_{2,3} \propto T \ln |V|$.

The results of numerical calculations for $j_m(v)$ ($v = V/\Delta_o$) for the case when there is no surface pair breaking from both sides of the tunnel barrier are shown in Figs. 2, 3. In Figure 2 the junction between isotropically and anisotropically paired superconductors is considered: $\Delta_1 = \Delta_0 \cos (2\phi)$, $\Delta_2 = \Delta_0/2 = \text{const}$. Here $\phi$ is the azimuth angle in $xy$ - plane of a tetragonal superconductor (axis z is parallel to the boundary plane). The Fermi surface is assumed to be cylindrical for anisotropically paired superconductor (with d-wave pairing). We let for the barrier transparency $D \propto \cos^2 (\phi)$ and $\Delta_0/(2T) = 0.5$. In this case the singular points on the I-V curves are only the points of maximum values of the quantities $|\Delta_1(\phi)| \pm |\Delta_2|$, as the minimum corresponds to the zero value of $\Delta_1$. Note that for $\Delta_2 = 2\Delta_0$ and for the same $\Delta_1(\phi)$ only minima of the quantities $|\Delta_1(\phi)| - |\Delta_2|$ would be of importance. In Figure 3 the case of two identical anisotropically paired superconductors is described: $\Delta_1 = \Delta_2 = \Delta_0 \cos (2\phi)$. All functions $j_m(v)$ are normalized to the value $|j_1(0)|$.

**B. Crystal Orientations with Surface Pair Breaking from One Side of the Barrier Plane**

Let us discuss now the tunnel junction between two anisotropically paired superconductors, considering a gradual change of the crystalline orientation of one of them relative to
the barrier plane and retaining the condition \( \Delta_2(\hat{p}) = \Delta_2(\hat{p}) \) to be fulfilled only for the second superconductor. According to Eqs. (33), (34), there are no square root divergences of the propagators for the first superconductor, taken on the barrier plane for an intermediate crystal orientation. Hence, the singularities, which were found in the previous subsection, must begin to become smooth and subsequently, for large enough deviations from the initial orientation, will disappear. At the same time, as it was already mentioned above, some new characteristic singular points on the I-V curves appear in this case. One kind of them turns out to be associated with the existence of regions on the Fermi surface with the opposite signs of the order parameter \( \Delta_{1,\infty}(\hat{p}) \). Then the zero energy quasiparticle bound state occurs at the boundary plane. Other singularities appear if the additional quasiparticle bound states localized near the barrier plane and retaining the condition \( \Delta_2(\hat{p}) = \Delta_2(\hat{p}) \) to be fulfilled only for the second superconductor. According to Eqs. (33), (34), there are no square root divergences of the propagators for the first superconductor, taken on the barrier plane for an intermediate crystal orientation. Hence, the singularities, which were found in the previous subsection, must begin to become smooth and subsequently, for large enough deviations from the initial orientation, will disappear. At the same time, as it was already mentioned above, some new characteristic singular points on the I-V curves appear in this case. One kind of them turns out to be associated with the existence of regions on the Fermi surface with the opposite signs of the order parameter \( \Delta_{1,\infty}(\hat{p}) \). Then the zero energy quasiparticle bound state occurs at the boundary plane. Other singularities appear if the additional quasiparticle bound states localized near the barrier plane appear due to the particular form of the spatial dependence of the order parameter \( \hat{q} \). As it is seen from Eqs. (33), (34), the terms in propagators containing the factor \( 1/\omega \) are of importance for the former case. For the latter case the contribution to the current from additional poles of the propagators has to be taken into account. So, we let the singular parts of the propagators for the first superconductor be represented by Eqs. (33), (34), and for the second superconductor we use Eq. (33).

It is very essential that in the case when zero energy bound state takes place only from one side of the junction (while from the other side there is no surface pair breaking), the respective singular contribution comes only to the quasiparticle current, not to the Josephson effect. It can be shown under the same conditions as are supposed to be fulfilled above. The point is that for the singular parts of the propagators, associated with the mid-gap state, one has the following property \( f_s(\hat{p}) = f_s^+(\hat{p}) \). Opposite to this equality, the relation \( f(\hat{p}) = -f^+(\hat{p}) \) holds for the orientations when there is no surface pair breaking. Due to this reason the corresponding singular parts of \( j_1, j_2 \) reduce to zero. It is not the case for the bound states with nonzero energy, since the function \( Q_f(\hat{p}) \) is complex in contrast to the real function \( B_f(\hat{p}) \) \( (B_f^+ = B_f^* = B_f, Q_f^+ = Q_f^*) \).

Substituting the expressions (34), (33), (33) into Eqs. (10), (11), we get for \( I_{3,4} \) the singular points which appear after the integration over \( \omega \):

\[
I_3(V, \hat{p}_1) = -B_{g1}(\hat{p}_1) \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta (|V| - |\Delta_2(\hat{p}_2)|)}{\sqrt{V^2 - |\Delta_2(\hat{p}_2)|^2}}
- \frac{Q_{g1}(\hat{p}_1) \sqrt{|\Delta_2|} sgn(V)}{\sqrt{2}} \left\{ \left( \tanh \left( \frac{h_1}{2T} \right) + \tanh \left( \frac{|\Delta_2|}{2T} \right) \right) \frac{\Theta (|V| - h_1 - |\Delta_2|)}{\sqrt{|V| - h_1 - |\Delta_2|}} +
+ \left| \tanh \left( \frac{|\Delta_2|}{2T} \right) - \tanh \left( \frac{h_1}{2T} \right) \right| \frac{\Theta ((|V| - |\Delta_2|) sgn(|\Delta_2|) h_1) sgn(|\Delta_2| - h_1))}{\sqrt{|V| - |\Delta_2| - h_1 - |\Delta_2| - h_1}} \right\}, \quad (64)
\]

\[
I_4(V, \hat{p}_1) = B_{g1}(\hat{p}_1) \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta (|\Delta_2(\hat{p}_2)| - |V|)}{\sqrt{|\Delta_2(\hat{p}_2)|^2 - V^2}}
+ \frac{Q_{g1}(\hat{p}_1) \sqrt{|\Delta_2|}}{\sqrt{2}} \left\{ \left( \tanh \left( \frac{h_1}{2T} \right) + \tanh \left( \frac{|\Delta_2|}{2T} \right) \right) \frac{\Theta (h_1 + |\Delta_2| - |V|)}{\sqrt{h_1 + |\Delta_2| - |V|}} +
+ \left| \tanh \left( \frac{|\Delta_2|}{2T} \right) - \tanh \left( \frac{h_1}{2T} \right) \right| \frac{\Theta ((|V| - |\Delta_2|) sgn(|\Delta_2|) h_1) sgn(|\Delta_2| - h_1))}{\sqrt{|V| - |\Delta_2| - h_1 - |\Delta_2| - h_1}} \right\},
\]

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\[
+ \left( \tanh \left( \frac{|\Delta_2|}{2T} \right) - \tanh \left( \frac{h_1}{2T} \right) \right) \Theta \left( \left| |V| - |\Delta_2| - h_1 \right| \right) \frac{\sgn(h_1 - |\Delta_2|)}{\sqrt{\left| |V| - |\Delta_2| - h_1 \right|}} \right) . \tag{65}
\]

The nonanalytical terms of square root type presented in Eqs. (34), (35), are omitted in Eqs. (64), (65), since in this case they result in jumps or divergences only for the derivatives of the conductance, not for the current or for the conductance itself.

Further integration over the Fermi surface may lead to different kinds of the singular points on the I-V curve, since there are various possibilities for the behaviors of the order parameter \(\Delta_2\) and function \(h_1\) with the momentum and types of the corresponding extrema on the Fermi surface. We consider firstly the singular points of the I-V curves coming from the terms of \(1/\omega\)-form in the expressions for the propagators of the first superconductor and confine ourselves by two examples. In the particular case of isotropic \(s\)-wave second superconductor there are the inverse square root singular point on the current-voltage characteristic:

\[
j_3 = - \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta \left( |V| - |\Delta_2| \right)}{\sqrt{V^2 - |\Delta_2|^2}} \int_{v_{x1} > 0} \frac{d^2S_1}{(2\pi)^3 v_f} \frac{v_{x1}}{DBg_1(\hat{p}_1)} , \tag{66}
\]

\[
j_4 = \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta \left( |\Delta_2| - |V| \right)}{\sqrt{|\Delta_2|^2 - V^2}} \int_{v_{x1} > 0} \frac{d^2S_1}{(2\pi)^3 v_f} \frac{v_{x1}}{DBg_1(\hat{p}_1)} . \tag{67}
\]

So, the quantity \(j_3\) diverges at \(|V| = \Delta_2\) from the side \(|V| > \Delta_2\), and the quantity \(j_4\) from the side \(|V| < \Delta_2\). In the case \(\Delta_2 = 0\) (S-N junction) we get from Eq. (66) that the conductance \(G_3 \propto \left( T \cosh^2(V/2T) \right)^{-1}\). Then for small voltage \(|V| \ll T\) it follows \(G_3 \propto 1/T\) and we obtain the zero-bias anomaly of the conductance at low enough temperatures (see also \[21\]). The divergence of \(G_3\) in the zero-temperature limit takes place only in the idealized system without taking account of the factors, which broaden delta-peaks in the quasiparticle density of states.

Further, if the anisotropic order parameter \(\Delta_2\) has an extremal line on the Fermi surface and in its vicinity one gets

\[
|\Delta_2| = a \pm b\tilde{p}_1^2, \quad a, b > 0 , \tag{68}
\]

then the kind of the singular point depends on the behavior of \(B_{g1}(\tilde{p}_1)\) near this line. If this function doesn’t vanish in the vicinity of the extremal line, the logarithmic divergences take place for \(j_3\) for the line of maxima and for \(j_4\) for the line of minima:

\[
j_3, j_4 \propto \sqrt{\frac{a}{b}} \tanh \left( \frac{a}{2T} \right) \ln ||V| - a| . \tag{69}
\]

In the case of vanishing of the function \(B_{g1}(\tilde{p}_1)\) on the line, when in its vicinity this function has the form

\[
B_{g1}(\tilde{p}_1) = \beta|\tilde{p}_1| , \tag{70}
\]

Further integration over the Fermi surface may lead to different kinds of the singular points on the I-V curve, since there are various possibilities for the behaviors of the order parameter \(\Delta_2\) and function \(h_1\) with the momentum and types of the corresponding extrema on the Fermi surface. We consider firstly the singular points of the I-V curves coming from the terms of \(1/\omega\)-form in the expressions for the propagators of the first superconductor and confine ourselves by two examples. In the particular case of isotropic \(s\)-wave second superconductor there are the inverse square root singular point on the current-voltage characteristic:

\[
j_3 = - \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta \left( |V| - |\Delta_2| \right)}{\sqrt{V^2 - |\Delta_2|^2}} \int_{v_{x1} > 0} \frac{d^2S_1}{(2\pi)^3 v_f} \frac{v_{x1}}{DBg_1(\hat{p}_1)} , \tag{66}
\]

\[
j_4 = \tanh \left( \frac{V}{2T} \right) \frac{|V| \Theta \left( |\Delta_2| - |V| \right)}{\sqrt{|\Delta_2|^2 - V^2}} \int_{v_{x1} > 0} \frac{d^2S_1}{(2\pi)^3 v_f} \frac{v_{x1}}{DBg_1(\hat{p}_1)} . \tag{67}
\]

So, the quantity \(j_3\) diverges at \(|V| = \Delta_2\) from the side \(|V| > \Delta_2\), and the quantity \(j_4\) from the side \(|V| < \Delta_2\). In the case \(\Delta_2 = 0\) (S-N junction) we get from Eq. (66) that the conductance \(G_3 \propto \left( T \cosh^2(V/2T) \right)^{-1}\). Then for small voltage \(|V| \ll T\) it follows \(G_3 \propto 1/T\) and we obtain the zero-bias anomaly of the conductance at low enough temperatures (see also \[21\]). The divergence of \(G_3\) in the zero-temperature limit takes place only in the idealized system without taking account of the factors, which broaden delta-peaks in the quasiparticle density of states.

Further, if the anisotropic order parameter \(\Delta_2\) has an extremal line on the Fermi surface and in its vicinity one gets

\[
|\Delta_2| = a \pm b\tilde{p}_1^2, \quad a, b > 0 , \tag{68}
\]

then the kind of the singular point depends on the behavior of \(B_{g1}(\tilde{p}_1)\) near this line. If this function doesn’t vanish in the vicinity of the extremal line, the logarithmic divergences take place for \(j_3\) for the line of maxima and for \(j_4\) for the line of minima:

\[
j_3, j_4 \propto \sqrt{\frac{a}{b}} \tanh \left( \frac{a}{2T} \right) \ln ||V| - a| . \tag{69}
\]

In the case of vanishing of the function \(B_{g1}(\tilde{p}_1)\) on the line, when in its vicinity this function has the form

\[
B_{g1}(\tilde{p}_1) = \beta|\tilde{p}_1| , \tag{70}
\]
one-sided vertical slope for functions $j_{3,4}$ appears at $|V| = a$:

$$G_3 \propto \frac{\sqrt{a}}{b} \tanh \left( \frac{a}{2T} \right) \frac{\Theta(|V| - a)}{\sqrt{|V| - a}}, \quad G_4 \propto \frac{\sqrt{a}}{b} \tanh \left( \frac{a}{2T} \right) \frac{\Theta(a - |V|)}{\sqrt{a - |V|}}. \quad (71)$$

Now let us examine the singular points coming from the terms describing the additional pole in the propagators for the first superconductor. In contrast to the contribution from the midgap states, even in the case of surface pair breaking from one side of the barrier plane the quasiparticle states bound to the tunnel barrier with nonzero energy contribute not only to the quasiparticle current but also to the Josephson effect. It takes place due to the nonzero values of $\text{Im}Q_f(\hat{p})$, while $\text{Im}B_f(\hat{p}) = 0$. Positions of respective singular points on the I-V curves turn out to be associated with the extremal points on the Fermi surface of quantities $h_1 \pm |\Delta_2|$. Considering, for example, the line of extrema $l$ for the quantity

$$h_1 + |\Delta_2| = a \pm bp_1^2, \quad a, b > 0 \quad (72)$$

we find the logarithmic divergences and jumps for $j_m$

$$j_1, -j_1 = M_{f,g}^+ \left\{ \begin{array}{c} \ln |V| - a \\ \pi \Theta(|V| - a) \end{array} \right\} - j_2, j_3 = M_{f,g}^- sgn(V) \left\{ \begin{array}{c} \pi \Theta(a - |V|) \\ \ln |V| - a \end{array} \right\}. \quad (73)$$

Here the upper (down) line in all four expressions corresponds to the upper (down) sign in Eq. (72). We introduce the notations:

$$M_{f,g}^\pm = \frac{1}{16\sqrt{2\pi}^3} \int_{v_{z1} > 0} dv_{z1} D \sqrt{\frac{|\Delta_2|}{b}} Q_{g1}(\hat{p}_1) \left| \tanh \left( \frac{h_1}{2T} \right) \pm \tanh \left( \frac{|\Delta_2|}{2T} \right) \right|, \quad (74)$$

$$M_{f,g}^\pm = \int_{v_{z1} > 0} dv_{z1} D \sqrt{\frac{sgn(\Delta_2)}{b} \frac{|\Delta_2|}{16\sqrt{2}\pi}} Q_{f1}(\hat{p}_1) \left| \tanh \left( \frac{h_1}{2T} \right) \pm \tanh \left( \frac{|\Delta_2|}{2T} \right) \right|. \quad (75)$$

If the function $||\Delta_2| - h_1|$ has the line of extrema

$$||\Delta_2| - h_1| = a \pm bp_1^2, \quad a, b > 0 \quad (76)$$

then one gets

$$j_{1,4} = M_{f,g}^- \{ -\pi \Theta(\pm (|\Delta_2| - h_1)) \ln |V| - a + \pi \Theta(|\Delta_2| - h_1)) \Theta(a - |V|) \}, \quad (77)$$

$$j_{2,3} = M_{f,g}^- sgn(V) \{ \Theta(\mp (|\Delta_2| - h_1)) \ln |V| - a \pm \pi \Theta(|\Delta_2| - h_1)) \Theta(a - |V|) \}. \quad (78)$$

So, for positive (negative) value of $|\Delta_2| - h_1$ the function $j_{2,3}$ ($j_{1,4}$) has logarithmic divergence only for the line of maxima of $||\Delta_2| - h_1|$, while $j_{1,4}$ ($j_{2,3}$) has such divergence only for the line of minima of the same quantity.
As it was mentioned above, the function $h(\hat{p})$ may manifest the nonanalytical behavior, for example, at the momentum direction, for which the bound state at the boundary disappears. Taking this possibility into account, we suppose now that the function $h_1 + |\Delta_2|$ has near some line on the Fermi surface the following nonanalytical form

$$h_1 + |\Delta_2| = a + (b\Theta(\hat{p}) + c\Theta(-\hat{p}))\tilde{p}, \quad a > 0.$$  

(79)

Particular cases $b$ (or $c$) $\rightarrow \infty$ just correspond to the absence of the bound state for $\tilde{p} > 0$ (or $\tilde{p} < 0$) on this line. Then one-sided vertical slope for functions $j_m$ appears at $|V| = a$:

$$G_1, -G_4 = \Theta(a - |V|)sgn(V)P^+_{f,g}, \quad G_2, -G_3 = \Theta(|V| - a)\frac{1}{\sqrt{|V| - a}}P^+_{f,g}. \quad (80)$$

We introduce the notations:

$$P_g = \int d\bar{\ell} \left( \frac{1}{b} - \frac{1}{c} \right) \frac{v_{x1}}{v_{f1}} D\sqrt{|\Delta_2|} \frac{Q_g(\hat{p}_1)}{8\sqrt{2}\pi^3} \left| \tanh \left( \frac{h_1}{2T} \right) \right| \mp \tanh \left( \frac{|\Delta_2|}{2T} \right), \quad (81)$$

$$P_f = \int d\bar{\ell} \left( \frac{1}{b} - \frac{1}{c} \right) \frac{v_{x1}}{v_{f1}} D\sqrt{|\Delta_2|} \frac{sgn(\Delta_2)}{8\sqrt{2}\pi^3} Im Q_f(\hat{p}_1) \left| \tanh \left( \frac{h_1}{2T} \right) \right| \mp \tanh \left( \frac{|\Delta_2|}{2T} \right) \quad (82)$$

If the function $|h_1 - |\Delta_2||$ has the form (79) near some line, then also one-sided vertical slope for functions $j_m$ appears at $|V| = a$:

$$G_{1,4} = sgn(V)sgn(|\Delta_2| - h_1)\frac{\Theta(a - |V|) (|\Delta_2| - h_1))}{\sqrt{(a - |V|) sgn(|\Delta_2| - h_1)}} P^-_{f,g}, \quad (83)$$

$$G_{2,3} = -\frac{\Theta(|V| - a) (|\Delta_2| - h_1))}{\sqrt{(|V| - a) sgn(|\Delta_2| - h_1)}} P^-_{f,g}. \quad (84)$$

The Eqs. (84), (83) are not appropriate for direct consideration of the case $\Delta_2 = 0$ for a finite voltage, since the voltage is supposed to lie near the corresponding singularity for each pole-like term. For S-N junction it follows from Eqs. (83), (84), (85), that in the case of extremal point of the quantity $h_1$ ($h_1 = a \pm b\tilde{p}_1^2$, $a, b > 0$) there is the low-temperature anomaly in the conductance for $|V| = a$: $G_{3,4} \propto 1/\sqrt{b\tilde{T}}$.

The results of numerical calculations for $j_m(v)$ ($v = V/\Delta_0$) for the case when there is surface pair breaking from one side of the tunnel barrier are shown in Figs. 4.1, 4.2. The tunnel junction between $d$-wave and isotropic $s$-wave superconductors is considered under the conditions: $\Delta_{1\infty} = \Delta_0 \cos(2\phi - 2\phi_o)$, $\Delta_2 = 0.2\Delta_0 = const$. For the $d$-wave superconductor we choose the same parameters as earlier (see Fig. 1): $\phi_o = \pi/9, T = 0.45T_{c1}, \Delta_0/(2T) = 2$. Here $T_{c1}$ is the critical temperature for the superconductor with $d$-wave pairing. We let for the barrier transparency $D \propto \cos^2(\phi)$.

There are the inverse square root singular points on the curves $j_3, j_4$ for $V = \Delta_2 = 0.2\Delta_0$. For $V = h_m - \Delta_0 = 0.5\Delta_0$ there are logarithmic divergences of $j_1, j_4$ and jumps of $j_2, j_3$. For $V = h_m + \Delta_2 = 0.9\Delta_0$ there are logarithmic divergences of $j_2, j_3$ and jumps of $j_1, j_4$. At voltages $V = h_{ed} \pm \Delta_2 = 0.83\Delta_0, 0.43\Delta_0$ there are kinks on the I-V curves, though some of them are feebly marked. All functions $j_m(v)$ are normalized to the value $j_1(0)$.
C. Crystal Orientations with Surface Pair Breaking for both Superconductors

Let now the both superconductors have the intermediate orientations relative to the boundary. Then one should make use the expressions (34), (35) for the singular parts of the propagators on both banks of the barrier plane. Substituting these expressions into Eqs. (8)–(11), we obtain several kinds of the singularities. They come from the multiplications of the terms with poles by each other and from the products of the pole-like term and the square root nonanalytical term, represented in Eqs. (34), (35). All these terms result, for example, in the following singular part of \( I_3 \)

\[
I_3 = \frac{1}{\pi} \tanh \left( \frac{V}{2T} \right) B_{g_1} (\hat{p}_1) \left( \text{Re} C_2 (\hat{p}_2) \right) \sqrt{V^2 - \Delta_{2\infty}^2 (\hat{p}_2) \Theta (|V| - |\Delta_{2\infty} (\hat{p}_2)|) +}
\]

\[
\text{Im} C_2 (\hat{p}_2) \sqrt{\Delta_{2\infty}^2 (\hat{p}_2) - V^2 \Theta (|\Delta_{2\infty} (\hat{p}_2)| - |V|) - \frac{\pi}{2} Q_{g_2} (\hat{p}_2) \delta (|V| - h_2 (\hat{p}_2)) +}
\]

\[
+ (\hat{p}_2 \rightarrow \hat{p}_2)) + \frac{1}{\pi} \left\{ \left( \tanh \left( \frac{h_1 (\hat{p}_1) + V}{2T} \right) - \tanh \left( \frac{h_1 (\hat{p}_1)}{2T} \right) \right) Q_{g_1} (\hat{p}_1) \right. \cdot
\]

\[
\left. \left( -\frac{\pi}{4} Q_{g_2} (\hat{p}_2) \delta (|h_1 (\hat{p}_1) + V| - h_2 (\hat{p}_2)) + \text{Re} C_2 (\hat{p}_2) \sqrt{(h_1 (\hat{p}_1) + V)^2 - \Delta_{2\infty}^2 (\hat{p}_2)} \cdot \Theta \left( |h_1 (\hat{p}_1) + V| - |\Delta_{2\infty} (\hat{p}_2)| \right) + \text{Im} C_2 (\hat{p}_2) \sqrt{\Delta_{2\infty}^2 (\hat{p}_2) - (h_1 (\hat{p}_1) + V)^2} \cdot \Theta \left( |\Delta_{2\infty} (\hat{p}_2)| - |h_1 (\hat{p}_1) + V| - (V \rightarrow -V) \right) \right. \cdot
\]

\[
\left. (\hat{p}_2 \rightarrow \hat{p}_2) \right) \right\} + (1 \leftrightarrow 2) , \quad (85)
\]

Analogues singularities appear in the expressions for \( I_{1,2,4} \).

Here we have omitted the joint contribution from the zero frequency singularities of the propagators from both sides of the junction. For finite voltages this contribution is equal to zero in the idealized system in question, though it may be of importance for low voltages less or comparable with the width of the broaden delta-peak in the density of states. For dc Josephson effect this term leads to the low temperature anomaly in the critical current and to the possibility of \( 0 - \pi \) phase transition \[22,23\].

Further integration over the momentum direction is performing below for the particular cases of the lines of extrema for the order parameter \( |\Delta_{2\infty} (\hat{p}_2)| \) or for quantities \( |h_1 \pm |\Delta_{2\infty}|, h_2, |h_1 \pm h_2| \). Let, for example, the order parameter \( |\Delta_{2\infty} (\hat{p}_2)| \) or the quantity \( |h_1 \pm |\Delta_{2\infty}| \) have the line of extrema of the form

\[
|\Delta_{2\infty}| , |h_1 \pm |\Delta_{2\infty}| = a \pm bp^2 , \quad a, b > 0 , \quad (86)
\]

and the quantity \( B_1(\hat{p}_1) \), or \( |\Delta_{2\infty}|, Q_1(\hat{p}_1) \) is not equal to zero on this line. Then the terms in the conductance \( G_{1,2,4} \) acquire the following logarithmic singularities

\[
\text{sgn}(V) G_1, G_2, \text{sgn}(V) G_4 \propto \sqrt{\frac{a}{b}} \ln ||V| - a| . \quad (87)
\]

At the same time the logarithmic divergence \[87\] of \( G_3 \) appears only in the case of the line of maxima, due to the relation \( \text{Im} C = 0 \) under the condition \( |\omega| < \min (|\Delta (\hat{p})|, |\Delta_\infty (\hat{p})|) \).
For the particular orientation, when one has $\Delta_{2\infty}(\hat{p}_1) = -\Delta_{2\infty}(\hat{p}_2)$, the logarithmic divergence of $G_4$ remains only for the line of minima. Note, that the square root behavior of the propagators of the form $\sqrt{\Delta_{2\infty}(\hat{p}) - \omega^2}$ result here in the logarithmic divergences of the conductance. For some other cases these terms may result in the kinks for a conductance or in the divergences of its derivatives. We do not consider these kinds of singularities here.

If there is the line of extrema of $h_2$

$$h_2 = a \pm b\tilde{p}^2, \quad a, b > 0,$$  

then the following inverse square root divergences appear on the I-V curves:

$$j_1, j_4 = \frac{\pm 1}{16\pi^3} \tanh\left(\frac{|V|}{2T}\right) \frac{\Theta(\pm(a - |V|))}{\sqrt{|V| - a}} \int_{v_{x1} > 0} dv_{x1} B_{f,g1}(\hat{p}_1) ReQ_{f,g2}(\hat{p}_2) \frac{D}{\sqrt{b}},$$  

$$j_2, j_3 = \frac{1}{8\pi^3} \tanh\left(\frac{V}{2T}\right) \frac{\Theta(\pm(|V| - a))}{\sqrt{|V| - a}} \int_{v_{x1} > 0} dv_{x1} B_{f,g1}(\hat{p}_1) ReQ_{f,g2}(\hat{p}_2) \frac{D}{\sqrt{b}}.$$  

In this section coefficients $B_{f2}, Q_{f2}^*$ originate from the corresponding expression for $f^+$ for second superconductor, while $B_{f1}, Q_{f1}$ come from the expression for $f$ for the first superconductor.

In the case of nonanalytical behavior of the function $h_2$ near some line one has, for example,

$$h_2 = a + (b\Theta(\tilde{p}) + c\Theta(-\tilde{p})) \tilde{p}, \quad a > 0.$$  

Then the logarithmic divergences of $j_{1,4}$ occur at $|V| = a$:

$$j_1, j_4 = \frac{\ln|V| - a}{8\pi^4} \tanh\left(\frac{|V|}{2T}\right) \int_{v_{x1} > 0} dv_{x1} DB_{f,g1}(\hat{p}_1) ReQ_{f,g2}(\hat{p}_2) \left(\frac{1}{c} - \frac{1}{b}\right),$$  

and the following jumps appear in $j_{2,3}$ at $|V| = a$:

$$j_2, j_3 = \frac{1}{8\pi^3} \tanh\left(\frac{V}{2T}\right) \int_{v_{x1} > 0} dv_{x1} DB_{f,g1}(\hat{p}_1) ReQ_{f,g2}(\hat{p}_2) \left(\frac{\Theta(|V| - a)}{b}\right) +$$

$$+ \frac{\Theta((a - |V|)c)}{|c|}.$$  

At last, let us consider the case of the line of extrema of the quantity $|h_1 - h_2|$:

$$|h_1 - h_2| = a \pm b\tilde{p}^2, \quad a, b > 0.$$  

Then one gets divergences on the I-V curves of the inverse square root types:

$$j_1, j_4 = \frac{\mp \Theta(\pm(a - |V|))}{16\pi^3 \sqrt{|V| - a}} \int_{v_{x1} > 0} dl \sqrt{b} N_{f,g}.$$  

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where we introduce

\[ N_{f,g} = \frac{v_{s1}}{v_{f1}} DRe \left( Q_{f,g}Q'_{f,g} \right) \left| \tanh \left( \frac{h_1}{2T} \right) - \tanh \left( \frac{h_2}{2T} \right) \right| \].

In the case when the quantity \(|h_1 - h_2|\) near some line behaves like (91), the logarithmic divergences of \(j_{1,4}\) appear at \(|V| = a\):

\[ j_1, -j_4 = \frac{\ln |V| - a}{8\pi^4} \int_{v_{s1} > 0} dN_{f,g} \left( \frac{1}{b} - \frac{1}{c} \right), \]

and the following jumps take place in \(j_{2,3}\) at \(|V| = a\):

\[ j_2, -j_3 = \frac{sgn(V)}{8\pi^3} \int_{v_{s1} > 0} dN_{f,g} \left( \frac{\Theta ((|V| - a) b)}{|b|} + \frac{\Theta ((a - |V|) c)}{|c|} \right) \]

The results for the line of extrema of the quantity \(h_1 + h_2\) follow from (94)-(99) after the substitution \(h_2 \rightarrow -h_2, j_1 \rightarrow -j_3, Q'_{f,g} \rightarrow Q_{f,g}\). One should note, that in addition to the presented in this subsection singular points there are the analogous ones which may be described by the same equations (80)-(83) after the interchange \(1 \leftrightarrow 2\).

The results of numerical calculations for \(j_m(v) (v = V/\Delta_0)\) for the case when there is surface pair breaking from both sides of the tunnel barrier are shown in Figs. 5.1, 5.2. The tunnel junction between two identical \(d\)-wave superconductors is considered for the particular case of "mirror" junction, when the barrier is a reflection-symmetry plane of the superconducting electrodes: \(\Delta_{1\infty}(\hat{p}_1) = \Delta_{2\infty}(\hat{p}_1) = \Delta_0 \cos(2\phi - 2\phi_o)\). As earlier we let (see Fig. 1) \(\phi_o = \pi/9, T = 0.45T_c, \Delta_0/(2T) = 2, D \propto \cos^2(\phi)\). There are the inverse square root singular points on the curves \(j_{1,2,3,4}\) at voltage \(V = 2h_m = 1.4\Delta_0\). For \(V = 2h_{ed} = 1.26\Delta_0\) there are logarithmic divergences of \(j_1, j_4\) and jumps of \(j_2, j_3\). At voltage \(V = 0.635\Delta_0\) curves \(j_1, j_2\) have kink-like behavior, which may be associated with the interplay between the contributions of mid-gap states from one side of the junction and of continuous quasiparticle spectrum from another side: the function \(\min (|\cos(2\phi - 2\phi_o)|, |\cos(2\phi + 2\phi_o)|)\) has maximal value 0.635 for the direction \(\phi = \pm 0.79\) at which \(B_f \neq 0\).

V. CONCLUSIONS

As we have shown above, there may be a large variety of different types of nonanalytical points on the I-V curves for tunnel junctions in anisotropically paired superconductors. The singular behavior differ essentially from that which is characteristic for case of s-wave isotropic superconductors. The quality of the barrier plane may have a great influence on the manifestations of the singular points in question. For example, instead of the divergences of the current, which are obtained for the idealized system, there are peaks, whose practical finite magnitudes are sensitive to the values of elastic and inelastic scattering processes,
roughness of the barrier plane, and even the finite value of the junction transparency. The inelastic scattering processes are common for removing the singularities of the tunnel current in s-wave isotropic and in anisotropically paired superconductors. The elastic scattering processes are pair breaking just for the anisotropically paired superconductors. Besides, these factors together with the quality of the barrier plane and the finite value of transparency may broaden the delta-peaks in quasiparticle density of states and, subsequently, wash out the corresponding peaks in the tunnel current. Nevertheless, the characteristic behavior of the I-V curves considered above is observable under the certain realistic conditions and may be employed as a sensitive test for identifying the anisotropic types of pairing in the superconductors, in particular, different signs of the order parameter on the Fermi surface.

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Figure 2

current $j_m$

voltage $v$

$j_1$, $j_2$, $j_3$, $j_4$
Figure 3
Figure 4.1
Figure 4.2
Figure 5.1
Figure 5.2