Dimension of Alexandrov Topologies

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Abstract

We prove that the Krull dimension of an Alexandrov space of finite height can be characterised with the specialisation preorder of its associated $T_0$-space.

A topological space is called Alexandrov, if the intersection of an arbitrary family of closed subsets is closed. It is well known that an Alexandrov topology on a set $X$ is determined by the specialisation preorder $\leq$ given as:

$$x \leq y :\iff x \in \text{cl}\{y\},$$

and it is well known that the specialisation preorder is a partial order if and only if $X$ is a $T_0$-space.

A topological space $X$ is irreducible, if it is not the union of two proper, non-empty, closed subsets $B, C$:

$$X = B \cup C \Rightarrow X = B \text{ or } X = C.$$  

For a topological space $X$, there is the Krull dimension, defined as the supremum of all lengths $n$ of chains

$$X_0 \subset X_1 \subset \cdots \subset X_n$$

with proper inclusions of non-empty closed irreducible subsets. The Krull dimension of the empty space $\emptyset$ is defined as $-1$. A space is called finite dimensional, if its Krull dimension is a finite number.

For an Alexandrov space $X$, there is the height, defined as the supremum of all lengths $n$ of chains

$$\text{cl}\{x_0\} \subset \cdots \subset \text{cl}\{x_n\}$$

with proper inclusions. In case $X$ is a $T_0$-Alexandrov space, then the height coincides with the supremum of all lengths of chains

$$x_0 \leq \cdots \leq x_n$$

where $x_i \neq x_{i+1}$ for $i = 0, \ldots, n-1$. An Alexandrov space is said to be of finite height, if its height is a finite number.

The height and the Krull dimension of an Alexandrov space are related, because of the following Lemma which holds true in any topological space:
Lemma 1. Let $X$ be a topological space. Then $\text{cl} \{x\}$ is irreducible for any $x \in X$.

Proof. Notice that $\text{cl} \{x\}$ is the smallest closed subset of $X$ containing $x$. Hence, if

$$\text{cl} \{x\} = B \cup C$$

is the union of closed subsets $B, C$ of $\text{cl} \{x\}$ in the subspace topology with $B \not\subseteq C$ and $C \not\subseteq B$, then first of all, $B$ and $C$ are also closed in $X$, because $\text{cl} \{x\}$ is closed in $X$, and closedness is transitive: $F$ closed subset of $G$, and $G$ closed subset of $H$ implies that $F$ is a closed subset of $H$. Consequently, $x$ cannot lie in both $B$ and $C$, because $\text{cl} \{x\}$ is the smallest closed subset of $X$ containing $a$. Hence, we may assume that $x \in B$ and $x \notin C$. But then $x$ is contained in the proper subset $B$ of $\text{cl} \{x\}$ which is closed in $X$. This cannot be, because $\text{cl} \{x\}$ is the smallest closed subset of $X$ containing $a$. This proves that $\text{cl} \{x\}$ is irreducible.

If the height of an Alexandrov space is finite, then there is a simple characterisation of irreducible subsets.

Lemma 2. A closed subset $A$ of an Alexandrov space $X$ of finite height is irreducible if and only if

$$A = \text{cl} \{a\},$$

i.e. $A$ is the closure of a point $a \in A$.

Proof. In Lemma 1 it was shown that $\text{cl} \{a\}$ is irreducible.

Let the closed set $A$ be irreducible. If $A \neq \text{cl} \{x\}$ for any $x \in A$, then

$$A = \text{cl} (A \setminus \text{cl} \{x\}) \cup \text{cl} \{x\}$$

is the union of two proper non-empty closed subsets. Hence, for all $x \in A$:

$$A = \text{cl}(A \setminus \text{cl} \{x\}),$$

because $A \neq \text{cl} \{x\}$ and $A$ is irreducible. Since $x \in A$, it follows that the open set $U_x := \{u \in X \mid x \leq u\}$ has a non-empty intersection with $A \setminus \text{cl} \{x\}$. In other words, there exists $y \in A \setminus \text{cl} \{x\}$ such that $y \geq x$. Now $A \setminus \text{cl} \{x\}$ contains a maximal element $a$ such that $a \geq x$. It is maximal in the sense that for all $b \in A \setminus \text{cl} \{x\}$ with $b \geq a$ it holds true that $\text{cl} \{b\} = \text{cl} \{a\}$. Otherwise there would be an infinite ascending chain

$$\text{cl} \{a\} \subset \text{cl} \{b\} \subset \text{cl} \{c\} \subset \ldots$$

with strict inclusions, where $a, b, c, \ldots \in A \setminus \text{cl} \{x\}$. This contradicts the finiteness of the height of $X$. So, from

$$A = \text{cl}(A \setminus \text{cl} \{a\}),$$

it follows that there is $b \in A \setminus \text{cl} \{a\} \subseteq A \setminus \text{cl} \{x\}$ such that $b \geq a$. Hence, by the maximality property of $a$, it follows that $\text{cl} \{a\} = \text{cl} \{b\}$ which cannot be, as otherwise $b \in A \setminus \text{cl} \{b\}$. 

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Let $X$ be a topological space which is Alexandrov with specialisation pre-order $\leq$. There is a natural equivalence relation $\sim$ on $X$:

$$x \sim y :\Leftrightarrow x \leq y \text{ and } y \leq x,$$

and let $X_0 := X/\sim$ be the Kolmogorov quotient. It is a $T_0$-space, its induced specialisation preorder is a partial order. There is a canonical map $\pi: X \rightarrow X_0$ which takes each $x \in X$ to its equivalence class. It has the important property:

$$x \leq y \iff \pi(x) \leq \pi(y) \quad (1)$$

**Lemma 3.** The map $\pi: X \rightarrow X_0$ induces a bijection between sets

$$\{\text{cl}\{x\} \mid x \in X\} \cong \{\text{cl}\{x_0\} \mid x_0 \in X_0\} \quad (2)$$

**Proof.** First, observe that $\pi(\text{cl}\{x\}) = \text{cl}\{\pi(x)\}$. Namely, $y \in \text{cl}\{x\}$ implies $\pi(y) \in \text{cl}\{\pi(x)\}$ by continuity, and $\pi(y) \in \text{cl}\{\pi(x)\}$ implies $\pi(y) \leq \pi(x)$, from which it follows by (1) that $y \leq x$, i.e. $x \in \text{cl}\{y\}$.

Secondly, observe that $\pi^{-1}(\text{cl}\{\pi(x)\}) = \text{cl}\{x\}$. Here, the inclusion $\supseteq$ is clear, because the left hand side is closed and contains $x$. For the other inclusion $\subseteq: y \in \pi^{-1}(\text{cl}\{\pi(x)\})$ implies $\pi(y) \leq \pi(x)$, hence $y \leq x$ by (1), i.e. $y \in \text{cl}\{x\}$.

By those two observations, we have established the bijection (2).

**Theorem 1.** An Alexandrov space $X$ is finite dimensional if and only if it is of finite height. In this case, the Krull dimension and the height of $X$ coincide. Furthermore, this number equals the height and the Krull dimension of the Kolmogorov quotient of $X$.

**Proof.** Clearly, if $X$ is of finite dimension, then $X$ is of finite height. The converse assertion is an immediate consequence of Lemma 2 from which it also follows that the Krull dimension and height coincide, if they are finite. The last assertion follows from the fact that the height of $X$ equals the height of $X_0$, and that the Krull dimension of $X$ equals the Krull dimension of $X_0$. This latter statement follows from Lemma 3.

$\square$