More on homotopy continuation method and discounted zero-sum stochastic game with ARAT structure

A. Dutta\textsuperscript{a,1} and A. K. Das\textsuperscript{b,2}

\textsuperscript{a}Department of Mathematics, Jadavpur University, Kolkata, 700 032, India
\textsuperscript{b}SQC & OR Unit, Indian Statistical Institute, Kolkata, 700 108, India

\textsuperscript{1}Email: aritradutta001@gmail.com
\textsuperscript{2}Email: akdas@isical.ac.in

Abstract

In this paper, we introduce a homotopy function to trace the trajectory by applying modified homotopy continuation method for finding the solution of two-person zero-sum discounted stochastic ARAT game. We show that the algorithm has the higher order of convergence. For the proposed algorithm, the homotopy path approaching the solution is smooth and bounded. Two numerical examples are illustrated to show the effectiveness of the proposed algorithm.

Keywords: Two-person zero-sum stochastic game, discounted ARAT stochastic game, homotopy method, optimal value, optimal stationary strategy.

AMS subject classifications: 91A05, 91A15, 90C33, 90C30, 14F35.

1 Introduction

In this paper, we consider two-person zero-sum discounted stochastic ARAT game. Shapley [37] introduced stochastic game and showed that there exist an optimal value and optimal stationary strategies for a stochastic game with discounted payoff, which depends only on the current state and not on the history. There are many applications of stochastic games like search problems, military applications, advertising problems, the traveling inspector model, and various economic applications. For details see [10]. There are significant research on theoretical as well as computational aspects of stochastic games. For details see [38], [36], [35], [39], [40]. Raghavan et al. [36] studied ARAT(Additive Rewards Additive Transition) games and showed that for a $\beta$-discounted zero-sum ARAT game, the value exists and both players have stationary optimal strategies, which may also be taken as pure strategies. A stochastic

\textsuperscript{1}Corresponding author
game is said to be an Additive Reward Additive Transition game (ARAT game) if
the reward and the transition probabilities satisfy

(i) \( r(s, i, j) = r_1^i(s) + r_2^j(s) \) for \( i \in A_s, j \in B_s, s \in S \).
(ii) \( p_{ij}(s, s') = p_1^i(s, s') + p_2^j(s, s') \) for \( i \in A_s, j \in B_s, (s, s') \in S \times S \).

We denote the matrix \( (p_1^i(s, s'), s, s' \in S, i \in A_s) \) as \( P_1(s) \) where \( S \) is the set of
states. This is a \( m_1(s) \times d \) matrix where \( m_1(s) \) is the cardinality of \( A_s \) and \( d \) is the
cardinality of \( s \). Similarly the matrix \( P_2(s) \) of order \( m_2(s) \times d \) is defined where \( m_2(s) \)
denotes the cardinality of the set \( B_s \). The Shapley equations for state \( s, s' \in S \) can
be stated as

\[
\text{Val}[r(s, i, j) + \beta \sum_{s'} p_{ij}(s, s') v_\beta(s')] = v_\beta(s).
\]

This implies for player I: For any fixed \( j \)

\[
r(s, i, j) + \beta \sum_{s'} p_{ij}(s, s') v_\beta(s') \leq v_\beta(s) \quad \forall \ i.
\] (1.1)

For player II: For any fixed \( i \)

\[
r(s, i, j) + \beta \sum_{s'} p_{ij}(s, s') v_\beta(s') \geq v_\beta(s) \quad \forall \ j.
\] (1.2)

Various approaches have been proposed for solving different classes of stochastic
games. One such approach is to formulate the ARAT game as complementarity prob-
lem. The well-known Lemke’s algorithm solves LCPs when the underlying matrix
class belongs to a particular class. Cottle and Dantzig extended Lemke’s algorithm
to VLCPs. The processability of Lemke’s algorithm and Cottle-Dantzig’s algorithm
is restricted on some classes of matrices. For details see [7], [12]. One sufficient condi-
tion for Lemke-processibility and Cottle-Dantzig processibility is that the underlying
matrix should be both \( E_0 \) and \( R_0 \) matrix [17], [24], [34].

The concept of a class of globally convergent methods, known as the homo-
topy continuation method is used to prove the existence of solutions to many economic
and engineering problems such as systems of nonlinear equations [8], nonlinear opti-
mization problems, fixed point problems, nonlinear programming, game problem and
complementarity problems [43]. In this paper, we introduce a homotopy function
to solve discounted zero sum stochastic ARAT game based on the modified homo-
topy continuation method and establish the higher order global convergence of the
homotopy method.

The paper is organized as follows. Section 2 presents some basic notations and
results which will be used in the next section. In section 3, we propose a new homotopy
function to find the solution of discounted zero sum stochastic ARAT game. We show
that the proposed homotopy function possesses a smooth and bounded homotopy path
to find the solution as the homotopy parameter \( t \) tends to 0. To find the solution
of homotopy function we modify predictor corrector steps to increase the order of convergency of the algorithm. We also find the sign of the positive tangent direction of the homotopy path. Finally, in section 4, we illustrate numerical examples of ARAT stochastic games to present the effectiveness of the introduced homotopy function.

2 Preliminaries

We begin by introducing some basic notations used in this paper. We consider matrices and vectors with real entries. $\mathbb{R}^n$ denotes the $n$ dimensional real space, $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ denote the nonnegative and positive orthant of $\mathbb{R}^n$. We consider vectors and matrices with real entries. Any vector $x \in \mathbb{R}^n$ is a column vector and $x^T$ denotes the row transpose of $x$. $e$ denotes the vector of all 1.

If $A$ is a matrix of order $n$, $\alpha \subseteq \{1, 2, \cdots, n\}$ and $\bar{\alpha} \subseteq \{1, 2, \cdots, n\} \setminus \alpha$ then $A_{\alpha \bar{\alpha}}$ denotes the submatrix of $A$ consisting of only the rows and columns of $A$ whose indices are in $\alpha$ and $\bar{\alpha}$ respectively. $A_{\alpha \alpha}$ is called a principal submatrix of $A$ and $\det(A_{\alpha \alpha})$ is called a principal minor of $A$. In this paper we consider the followings:

$$\mathcal{R} = \{x \in \mathbb{R}^n : x > 0, Ax + q > 0\}$$
$$\bar{\mathcal{R}} = \{x \in \mathbb{R}^n : x \geq 0, Ax + q \geq 0\}$$
$$\mathcal{R}_1 = \mathcal{R} \times \mathbb{R}_{++}^n \times \mathbb{R}_{++}^n$$
$$\bar{\mathcal{R}}_1 = \bar{\mathcal{R}} \times \mathbb{R}_+^n \times \mathbb{R}_+^n.$$
$$\partial \mathcal{R}_1$$ denotes the boundary of $\mathcal{R}_1$.

2.1 Linear Complementarity Problem and its Generalization

The linear complementarity problem [28] is defined as follows: Given square matrix $A \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem is to find $w \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ such that

$$w - Ax = q, w \geq 0, x \geq 0,$$
$$x^T w = 0.$$

This problem is denoted as LCP($q, A$). Several applications of linear complementarity problems are reported in operations research [34], multiple objective programming problems [16], mathematical economics and engineering. For details see [9], [18], [14] and [15]. The linear complementarity problem is well studied in the literature of mathematical programming and arises in a number of applications in operations research, control theory, mathematical economics, geometry and engineering. For recent works on this problem and applications see [6], [25], [31] and [30] and references therein.

In complementarity theory several matrix classes are considered due to the study of theoretical properties, applications and its solution methods. For details see [14], [15], [19], [18], [29] and [27] and references cited therein. The problem of computing the value vector and optimal stationary strategies for structured stochastic games is formulated as a linear complementary problem for discounted and undiscounted zero-sum games. For details see [23], [33] and [26].
2.1.1 Vertical Linear Complementarity Problems

Cottle and Dantzig [5] extended the linear complementarity problem to a problem in which the matrix \( A \) is not a square matrix. The generalization of the linear complementarity problem introduced by them is given below: Consider a vertical block matrix \( A \in \mathbb{R}^{m \times k} (m \geq k) \),

\[
A = \begin{bmatrix}
A_1 \\
A_2 \\
\vdots \\
A_k
\end{bmatrix}
\]

such that \( A_j \in \mathbb{R}^{m_j \times k}, 1 \leq j \leq k \), \( \sum_{j=1}^k m_j = m \). This matrix is called vertical block matrix of type \((m_1, m_2, \ldots, m_k)\) and consider \( q \in \mathbb{R}^m \) where \( m = \sum_{j=1}^k m_j \), the generalized linear complementarity problem is to find \( w \in \mathbb{R}^m \) and \( x \in \mathbb{R}^k \) such that

\[
w - Ax = q, \quad w \geq 0, \quad x \geq 0, \quad (2.3)
\]

\[
x_j \prod_{i} w_{ij}, j = 1, 2, \ldots, k. \quad (2.4)
\]

This generalization is known as vertical linear complementarity problem and denoted by \( \text{VLCP}(q, A) \). For further details see [5]. The vertical block matrix arises naturally in the literature of stochastic games where the states are represented by the columns and actions in each state are represented by rows in a particular block. For details see [20], [22].

An equivalent square matrix \( M \) can be constructed from a vertical block matrix \( A \) of type \((m_1, m_2, \ldots, m_k)\) by copying \( A_j, m_j \) times for \( j = 1, 2, \ldots, k \). Therefore \( M_{pj} = A_{j}, \forall p \in J_j \). \( \text{LCP}(q, M) \) is called as equivalent LCP of \( \text{VLCP}(q, A) \). For more details see [22], [32]. Mohan et al. [22] proposed techniques to convert a VLCP to an LCP and also showed that processibility conditions as well. Mohan et al. [21] formulated zero-sum discounted Additive Reward Additive Transition (ARAT) games as a VLCP.

**Definition 2.1:** [21] A is said to be a vertical block \( E(d) \)-matrix for some \( d > 0 \) if \( \text{VLCP}(d, A) \) has a unique solution \( w = d, z = 0 \).

**Definition 2.2:** [21] A is said to be a vertical block \( R_0 \)-matrix if \( \text{VLCP}(0, A) \) has a unique solution \( w = 0, z = 0 \).

We denote the class of vertical block \( E(d) \) matrices as \( \text{VBE}(d) \) the class of vertical block \( R_0 \) matrices by \( \text{VBR}_0 \).

2.2 Discounted Stochastic Game with the Structure of Additive Reward and Additive Transition

Consider a state space \( S = \{1, 2, \ldots, N\} \). For each \( s \in S \), consider the finite action sets \( A_s = \{1, 2, \ldots, m_s\} \) for Player I and \( B_s = \{1, 2, \ldots, n_s\} \) for Player II. For state \( s \in S \) a reward law \( R(s) = [r(s, i, j)] \) is an \( m_s \times n_s \) matrix whose \((i, j)\)th entry is the payoff from Player II to Player I when Player I chooses an action \( i \in A_s \) and
player II chooses an action \( j \in B_s \), while the game is being played in state \( s \) and the payoff from player I to player II is \( -r(s, i, j) \). Let \( p_{ij}(s, s') \) denotes the probability of a transition from state \( s \) to state \( s' \), given that Player I and Player II choose actions \( i \in A_s \), \( j \in B_s \) respectively. Then transition law is defined by

\[
p = (p_{ij}(s, s') : (s, s') \in S \times S, i \in A_s, j \in B_s).
\]

Let the game be played in stages \( t = 0, 1, 2, \cdots \). At some stage \( t \), the players find themselves in a state \( s \in S \) and independently choose actions \( i \in A_s, j \in B_s \). Player II pays Player I an amount \( r(s, i, j) \) and at stage \( (t + 1) \), the new state is \( s' \) with probability \( p_{ij}(s, s') \). Play continues at this new state. The players guide the game via strategies and in general, strategies can depend on complete histories of the game until the current stage. We are however concerned with the simpler class of stationary strategies which depend only on the current state and not on stages. So for Player I, a stationary strategy \( k \in K_s = \{ k_i(s) | s \in S, i \in A_s, k_i(s) \geq 0, \sum_{i \in A_s} k_i(s) = 1 \} \) indicates that the action \( i \in A_s \) should be chosen by Player I with probability \( k_i(s) \) when the game is in state \( s \).

Similarly for Player II, a stationary strategy \( l \in L_s = \{ l_j(s) | s \in S, j \in B_s, l_j(s) \geq 0, \sum_{j \in B_s} l_j(s) = 1 \} \) indicates that the action \( j \in B_s \) should be chosen with probability \( l_j(s) \) when the game is in state \( s \). Here \( K_s \) and \( L_s \) will denote the set of all stationary strategies for Player I and Player II respectively. Let \( k(s) \) and \( l(s) \) be the corresponding \( m_s \) and \( n_s \) dimensional vectors respectively. Fixed stationary strategies \( k \) and \( l \) induce a Markov chain on \( S \) with transition matrix \( P(k, l) \) whose \((s, s')\)th entry is given by

\[
P_{ss'}(k, l) = \sum_{i \in A_s} \sum_{j \in B_s} p_{ij}(s, s') k_i(s) l_j(s)
\]

and the expected current reward vector \( r(k, l) \) has entries defined by

\[
r_s(k, l) = \sum_{i \in A_s} \sum_{j \in B_s} r(s, i, j) k_i(s) l_j(s) = k^T(s) R(s) l(s).
\]

With fixed general strategies \( k, l \) and an initial state \( s \), the stream of expected payoff to Player I at stage \( t \), denoted by \( v^s_t(k, l), t = 0, 1, 2, \cdots \) is well defined and the resulting discounted payoff is \( \phi^s_\beta(k, l) = \sum_0^\infty \beta^t v^s_t(k, l) \) for a \( \beta \in (0, 1) \), where \( \beta \) is the discount factor. Due to this additive property assumed on the transition and reward functions, the game is called \( \beta \)-discounted zero-sum ARAT(Additive Reward Additive Transition) game. For further details see [13, 36, 11].

**Theorem 2.1:** [10] For ARAT stochastic games

(i) Both players possess \( \beta \) discounted optimal stationary strategies that are pure.

(ii) These strategies are optimal for the average reward criterion as well.

(iii) The ordered field property holds for the discounted as well as the average reward criterion.

Now we observe the following property of the additive components \( P_1 \) and \( P_2 \) of the transition probability matrix \( P \). For details see [21].
Lemma 2.1: If \( p_j^2(s, s') = 0 \) for all \( s' \in S \) and for some \( j \in B(s) \), then \( P_2(s) = 0 \).

Theorem 2.2: \([21]\) Consider the vertical block matrix \( A \) arising from the zero-sum ARAT game. Then \( A \in \text{VBE}(e) \) where \( e \) is the vector each of whose entries is 1.

Theorem 2.3: \([21]\) Consider the vertical block matrix \( A \) arising from zero-sum ARAT game. Then \( A \in \text{VBR}_0 \) if either the condition \((a)\) or the set of conditions \((b)\) stated below is satisfied.

\( (a) \) For each \( s \) and each \( j \in B_s \), \( p_j^2(s, s) > 0 \).

\( (b) \) (i) For each \( s \), the matrix \( P_1(s) \) does not contain any zero column and

(ii) the matrix \( P_2(s) \) is not a null matrix.

2.3 Homotopy Continuation Method

The key idea to solve a system of equations by the homotopy method is to solve \( H(x, t) = 0 \), where \( H : R^m \times [0, 1] \to R^m, x \in R^m, t \in [0, 1] \) is called homotopy parameter. The homotopy method aims to trace the entire path of equilibria in \( H^{-1} = \{(x, t) : H(x, t) = 0 \} \) by varying both \( x \) and \( t \). A parametric path is obtained from a set of functions \( (x(s), t(s)) \in H^{-1} \). When we move along the homotopy path, the auxiliary variable \( s \) either decreases or increases monotonically. Differentiating \( H(x(s), t(s)) = 0 \) with respect to \( s \) we obtain \( \frac{\partial H}{\partial x} x'(s) + \frac{\partial H}{\partial t} t'(s) = 0 \), where \( \frac{\partial H}{\partial x} \) and \( \frac{\partial H}{\partial t} \) are \( n \times n \) Jacobian matrix of \( H \) and \( n \times 1 \) column vector respectively. So this is a system of differential equations in \( n + 1 \) unknowns \( x_i'(s), t'(s) \). This system of differential equations has many solutions for which the solutions of the differential equations differ by monotone transformation of the auxiliary variable \( s \).

Now we state some results on homotopy which will be required in the next section.

Lemma 2.2: \([2]\) Let \( V \subset R^m \) be an open set and \( g : R^m \to R^q \) be smooth. We say \( y \in R^q \) is a regular value for \( g \) if \( \text{Range } Dg(x) = R^q \forall x \in g^{-1}(y) \), where \( Dg(x) \) denotes the \( m \times q \) matrix of partial derivatives of \( g(x) \).

Lemma 2.3: \([11]\) Let \( V \subset R^m, U \subset R^m \) be open sets, and let \( \xi : V \times U \to R^l \) be a \( C^\alpha \) mapping, where \( \alpha > \max\{0, m-l\} \). If \( 0 \in R^l \) is a regular value of \( \xi \), then for almost all \( a \in V, 0 \) is a regular value of \( \xi_a = \xi(a, .) \).

Lemma 2.4: \([11]\) Let \( \xi : V \subset R^m \to R^q \) be \( C^\alpha \) mapping, where \( \alpha > \max\{0, m-q\} \). Then \( \xi^{-1}(0) \) consists of some \( (m-q) \) dimensional \( C^\alpha \) manifolds.

Lemma 2.5: \([11]\) One-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.

Lemma 2.6: \([3]\) Let \( f : R^n \to R^n \) be a sufficiently differentiable function in a neighborhood \( D \) of \( \alpha \), that is a solution of the system \( f(x) = 0 \), whose Jacobian matrix is continuous and nonsingular in \( D \). Consider the iterative method \( x^k = \phi(x^k, y^k), w^k = z^k - f'(y^k)^{-1} f(z^k) \), where \( y^k = x^k - f'(x^k)^{-1} f(x^k) \) and \( z^k = \phi(x^k, y^k) \) is the iteration function of a method of order \( p \). Then for an initial approximation sufficiently close to \( \alpha \), this method has order of convergence \( p + 2 \).
Lemma 2.7: Consider the function $f : \mathbb{R}^n \to \mathbb{R}^n$ and the iterative method $y^k = x^k - f'(x^k)^{-1}f(x^k)$, $z^k = x^k - 2(f'(y^k) + f'(x^k))^{-1}f(x^k)$, $w^k = z^k - f'(y^k)^{-1}f(z^k)$ has 5th order convergence.

3 Main results

In this section, we consider the two-person zero-sum discounted stochastic ARAT game and introduce a homotopy continuation method to find the solution of the discounted zero-sum ARAT game. We state that a pair of strategies $(k^*, l^*)$ is optimal for Player I and Player II in the discounted game if for all $s \in S \phi_s(k, l^*) \leq \phi_s(k^*, l^*) = v^*_s \leq \phi_s(k^*, l)$ for any strategies $k$ and $l$ of Player I and Player II. The number $v^*_s$ is called the value of the game starting in state $s$ and $v^* = (v^*_1, v^*_2, \ldots, v^*_S)$ is called the value vector. To find the optimal strategy of player I and player II of the two-person zero-sum discounted ARAT stochastic game we propose a new function based on the concept of homotopy.

Let $X_1, X_2$ be two topological spaces and $f_1, f_2 : X_1 \to X_2$ be continuous maps. A homotopy from $f_1$ to $f_2$ is a continuous function $H : X_1 \times [0, 1] \to X_2$, defined by $H(x, t) = (1 - t)f_1(x) + tf_2(x)$ satisfying $H(x, 0) = f_1(x), H(x, 1) = f_2(x) \forall x \in X_1$. The value of $t$ will start from 1 and goes to 0 and in this way one can find the solution of the given equation $f_1(x) = 0$ from the solution of $f_2(x) = 0$.

$$H(u, t) = \begin{bmatrix} (1 - t)[(A + A^T)x + y_1 - A^Ty_2] + t(x - x(0)) \\ Y_1x - tY_1(0)x(0) + (1 - t)X(Ax + q) \\ Y_2(Ax + q) - tY_2(0)(Ax(0) + q) \end{bmatrix} = 0 \quad (3.1)$$

where $Y_1 = \text{diag}(y_1), X = \text{diag}(x), Y_2 = \text{diag}(y_2), Y_1(0) = \text{diag}(y_1(0)), Y_2(0) = \text{diag}(y_2(0)), u = (x, y_1, y_2) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+, u(0) = (x(0), y_1(0), y_2(0)) \in \mathcal{R}_1$, and $t \in (0, 1]$.

3.1 Computing Solution of ARAT Stochastic Game based on Homotopy Continuation Method

Now we show that the solution of the proposed homotopy function will give the solution of discounted ARAT stochastic game.

Theorem 3.1: Suppose $u(0) = \{ (u, t) \in \mathbb{R}^n_0 \times (0, 1) : H(u, u(0), t) = 0 \} \subset \mathcal{R}_1 \times (0, 1]$, and $\mathcal{A} = \begin{bmatrix} -\beta P_1 \\ -E + \beta P_2 \end{bmatrix} E - \beta P_1$ and $q = \begin{bmatrix} -r_1^2(s) \\ r_2^2(s) \end{bmatrix}$, where the matrix $P_1 =$

$P_1(s), P_2 = P_2(s)$ and $E = \begin{bmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & e_d \end{bmatrix}$ is a vertical block identity matrix
where \( e_j, 1 \leq j \leq d \), is a column vector of all 1’s. Then the homotopy function \( 3.1 \) solves discounted zero-sum stochastic ARAT game.

**Proof.** Suppose for a zero-sum discounted ARAT game the optimal pure strategy in state \( s \) is \( i_0 \) for Player I and \( j_0 \) for Player II. Then the inequality \( 1.1 \) and the inequality \( 1.2 \) reduces to

\[
r_i^1(s) + r_{j_0}^2(s) + \beta \sum_{s'} p_i^1(s, s') \nu_\beta(s') + \beta \sum_{s'} p_{j_0}^2(s, s') \nu_\beta(s') \leq \nu_\beta(s) \quad \forall i.
\]

(3.2)

\[
r_{i_0}^1(s) + r_j^2(s) + \beta \sum_{s'} p_{i_0}^1(s, s') \nu_\beta(s') + \beta \sum_{s'} p_j^2(s, s') \nu_\beta(s') \geq \nu_\beta(s) \quad \forall j.
\]

(3.3)

The inequalities \( 3.2 \) and \( 3.3 \) yield

\[
r_i^1(s) + \beta \sum_{s'} p_i^1(s, s') \nu_\beta(s') \leq \nu_\beta(s) - \eta_\beta(s) = \xi_\beta(s) \quad \forall i,
\]

where \( \eta_\beta(s) = r_i^1(s) + \beta \sum_{s'} p_{j_0}^2(s, s') \nu_\beta(s') \) and \( \xi_\beta(s) = r_{i_0}^1(s) + \beta \sum_{s'} p_i^1(s, s') \) and \( \eta_\beta(s) + \xi_\beta(s) = \nu_\beta(s) \).

Thus the inequalities are

\[
r_i^1(s) + \beta \sum_{s'} p_i^1(s, s') \xi_\beta(s') - \xi_\beta(s) + \beta \sum_{s'} p_i^1(s, s') \eta_\beta(s') \leq 0 \quad \forall i \in A_s, s \in S
\]

(3.4)

and similarly the inequalities for Player II are

\[
r_j^2(s) + \beta \sum_{s'} p_j^2(s, s') \eta_\beta(s') - \eta_\beta(s) + \beta \sum_{s'} p_j^2(s, s') \xi_\beta(s') \geq 0 \quad \forall j \in B_s, s \in S.
\]

(3.5)

Also for each \( s \), in \( 3.4 \) there is an \( i(s) \) such that equality holds. Similarly, for each \( s \) in \( 3.5 \) there is a \( j(s) \) such that equality holds. Let for \( i \in A_s \),

\[
w_i^1(s) = -r_i^1(s) - \beta \sum_{s'} p_i^1(s, s') \eta_\beta(s') + \xi_\beta(s) - \beta \sum_{s'} p_i^1(s, s') \xi_\beta(s') \geq 0,
\]

(3.6)

and for \( j \in B_s \),

\[
w_j^2(s) = r_j^2(s) - \eta_\beta(s) + \beta \sum_{s'} p_j^2(s, s') \eta_\beta(s') + \beta \sum_{s'} p_j^2(s, s') \xi_\beta(s') \geq 0.
\]

(3.7)

We may assume without loss of generality that \( \eta_\beta(s), \xi_\beta(s) \) are strictly positive. Since there is at least one inequality in \( 3.6 \) for each \( s \in S \) that holds as an equality and one inequality in \( 3.7 \) for each \( s \in S \) that holds as an equality, the following complementarity conditions will hold.

\[
\eta_\beta(s) \prod_{i \in A_s} w_i^1(s) = 0 \quad \text{for} \quad 1 \leq s \leq d
\]

(3.8)

and

\[
\xi_\beta(s) \prod_{j \in B_s} w_j^2(s) = 0 \quad \text{for} \quad 1 \leq s \leq d.
\]

(3.9)
The inequality 3.6 and inequality 3.7 along with the complementarity conditions 3.8 and 3.9 lead to the VLCP(q, A) where the matrix A is of the form

\[
A = \begin{bmatrix}
-\beta P_1 & E - \beta P_1 \\
-E + \beta P_2 & \beta P_2
\end{bmatrix}
\]

and \( q = \begin{bmatrix} -r_1^j(s) \\ r_2^j(s) \end{bmatrix} \),

where the matrix \( P_1 = P_1(s), P_2 = P_2(s) \) and

\[
E = \begin{bmatrix}
e_1 & 0 & 0 & \cdots & 0 \\
0 & e_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & e_d
\end{bmatrix}
\]

is a vertical block identity matrix where \( e_j, 1 \leq j \leq d, \)

is a column vector of all 1’s. Now an equivalent square matrix A can be constructed from the vertical block matrix A of type \((m_1, \ldots, m_c)\) by copying \( A_{ij} \), \( m_j \) times for \( j = 1, 2, \ldots, c \). Therefore \( A_p = A_j \ \forall p \in J_j \) and the LCP(q, A) is the equivalent LCP of VLCP(q, A).

We consider the proposed homotopy function 3.1

\[
H(u, t) = \begin{bmatrix}
(1 - t)[(A + AT)x + q - y_1 - ATy_2] + t(x - x^{(0)}) \\
Y_1x - tY_1^{(0)}x^{(0)} + (1 - t)X(Ax + q) \\
Y_2(Ax + q) - tY_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix} = 0 \tag{3.10}
\]

where \( Y_1 = \text{diag}(y_1) \), \( X = \text{diag}(x) \), \( Y_2 = \text{diag}(y_2) \), \( Y_1^{(0)} = \text{diag}(y_1^{(0)}) \), \( Y_2^{(0)} = \text{diag}(y_2^{(0)}) \), \( u = (x, y_1, y_2) \in R^n_+ \times R^n_+ \times R^n_+ \), \( u^{(0)} = (x^{(0)}, y_1^{(0)}, y_2^{(0)}) \in \mathcal{R}_1 \), and \( t \in (0, 1] \). We denote \( \Gamma_u^{(0)} = \{(u, t) \in R^{3n}_+ \times (0, 1] : H(u, u^{(0)}, t) = 0 \} \subset \mathcal{R}_1 \times (0, 1] \}. \)

For the proposed homotopy function \( t \) varies from 1 to 0. Starting from \( t = 1 \) to \( t \rightarrow 0 \) if we have a smooth bounded curve, then we obtain a finite solution of the equation 3.1 at \( t \rightarrow 0 \).

As \( t \rightarrow 1 \), the equation 3.1 gives the solution \((u^{(0)}, 1)\), and as \( t \rightarrow 0 \), the equation 3.1 gives the solution of the system of following equations:

\[
(A + A^t)x + q - y_1 - A^ty_2 = 0 \\
Y_1x + X(Ax + q) = 0 \\
Y_2(Ax + q) = 0
\]

where \( Y_1 = \text{diag}(y_1) \) and \( Y_2 = \text{diag}(y_2) \). Hence the solution of the homotopy function 3.1 gives the solution of discounted zero-sum ARAT game.

Therefore if the homotopy function 3.1 converges to its solution as the parameter \( t \rightarrow 0 \), we obtain the solution of discounted ARAT stochastic game.

Now we establish the conditions under which the solution exists for the proposed homotopy function 3.1. We prove the following result to show that the smooth curve \( \Gamma_u^{(0)} \) exists for the proposed homotopy function 3.1.

**Theorem 3.2:** Let initial point \( u^{(0)} \in \mathcal{R}_1 \). Then 0 is a regular value of the homotopy function \( H : R^{3n} \times (0, 1] \rightarrow R^{3n} \) and the zero point set \( H^{-1}(0) = \{(u, t) \in \mathcal{R}_1 : H(u, t) = 0 \} \) contains a smooth curve \( \Gamma_u^{(0)} \) starting from \( (u^{(0)}, 1) \).

**Proof.** The Jacobian matrix of the above homotopy function \( H(u, u^{(0)}, t) \) is

\[
DH(u, u^{(0)}, t) = \begin{bmatrix}
\frac{\partial H(u, t)}{\partial u} & \frac{\partial H(u, t)}{\partial u^{(0)}} & \frac{\partial H(u, t)}{\partial t}
\end{bmatrix}.
\]

For all \( u^{(0)} \in \mathcal{R}_1 \) and \( t \in (0, 1], \)
\[
\frac{\partial H(u, t)}{\partial u(0)} = \begin{bmatrix}
-tI & 0 & 0 \\
-tY_1(0) & -tX(0) & 0 \\
-tY_2(0)A & 0 & -tY(0)
\end{bmatrix},
\]

where \( Y(0) = \text{diag}(Ax(0) + q), X(0) = \text{diag}(x(0)), y(0) = Ax(0) + q \).

Now \( \det(\frac{\partial H}{\partial u(0)}) = (-1)^{3n} t^{3n} \prod_{i=1}^{n} x_i(0)^{y_i(0)} \neq 0 \) for \( t \in (0, 1] \). Therefore, 0 is a regular value of \( H(u, u(0), t) \) by the lemma 2.2. By lemma 2.3 and lemma 2.4, for almost all \( u(0) \in \mathcal{R}_1, 0 \) is a regular value of \( H(u, t) \) and \( H^{-1}(0) \) consists of some smooth curves and \( H(u(0), 1) = 0 \). Hence there must be a smooth curve \( \Gamma_u(0) \) starting from \((u(0), 1)\).

Hence by Implicit Function Theorem for every \( t \) sufficiently close to 1, the homotopy function has a unique solution \((u, 1)\) of 3.1 which is smooth in the parameter \( t \) in a neighbourhood of \((u(0), 1)\).

We prove the following result to show that the smooth curve \( \Gamma_u(0) \) for the proposed homotopy function 3.1 is bounded and convergent.

**THEOREM 3.3:** Let \( \mathcal{R} \) be a non-empty set and \( A \in \mathbb{R}^{n \times n} \) a matrix and assume that there exists a sequence of points \( \{u^k\} \subset \Gamma_u(0) \subset \mathcal{R} \times (0, 1] \), where \( u^k = (x^k, y_1^k, y_2^k, t^k) \) such that \( \|x^k\| < \infty \) as \( k \to \infty \) and \( \|y_2^k\| < \infty \) as \( k \to \infty \) and for a given \( u(0) \in \mathcal{R}_1, 0 \) is a regular value of \( H(u, u(0), t) \), then \( \Gamma_u(0) \) is a bounded curve in \( \mathcal{R}_1 \times (0, 1] \).

**Proof.** Note that 0 is a regular value of \( H(u, u(0), t) \) by theorem 3.2. By contradiction we assume that \( \Gamma_u(0) \subset \mathcal{R}_1 \times (0, 1] \) is an unbounded curve. Then there exists a sequence of points \( \{v^k\} \), where \( v^k = (u^k, k^k) \subset \Gamma_u(0) \) such that \( \|(u^k, k^k)\| \to \infty \). As \((0, 1] \) is a bounded set and \( x \) component and \( y_2 \) component of \( \Gamma_u(0) \) is bounded, there exists a subsequence of points \( \{v^k\} \) such that \( x^k \to \bar{x}, y_2^k \to \bar{y}_2, t^k \to \bar{t} \in [0, 1] \) and \( \|y^k\| \to \infty \) as \( k \to \infty \), where \( y^k = \begin{bmatrix} y_1^k \\ y_2^k \end{bmatrix} \). Since \( \Gamma_u(0) \subset H^{-1}(0) \), we have

\[
(1 - t^k)[(A + A^T)x^k + q - y_1^k - A^T y_2^k] + t^k(x^k - x(0)) = 0 \quad (3.11)
\]

\[
Y_1^k x^k - t^k Y_1(0)x(0) + (1 - t^k)X(0)(Ax^k + q) = 0 \quad (3.12)
\]

\[
Y_2^k (Ax^k + q) - t^k Y_2(0)(Ax(0) + q) = 0 \quad (3.13)
\]

where \( Y_1^k = \text{diag}(y_1^k), X^k = \text{diag}(x^k) \) and \( Y_2^k = \text{diag}(y_2^k) \).

Let \( \bar{t} \in [0, 1], \|y_1^k\| = \infty \) and \( \|y_2^k\| < \infty \) as \( k \to \infty \). Then \( \exists i \in \{1, 2, \cdots, n\} \) such that \( y_1^k_i \to \infty \) as \( k \to \infty \). Let \( I_{1y} = \{i \in \{1, 2, \cdots, n\} : \lim_{k \to \infty} y_1^k_i = \infty\} \). When \( \bar{t} \in [0, 1] \), for \( i \in I_{1y} \) we write from equation 3.11

\[
(1 - t^k)[((A + A^T)x^k)_i + q_i - y_1^k_i - (A^T y_2^k)_i] + t^k(x^k_i - x^k_i) = 0
\]

\[
\Rightarrow (1 - t^k)y_1^k_i = (1 - k^k)[((A + A^T)x^k)_i + q_i - (A^T y_2^k)_i] + t^k(x^k_i - x^k_i)
\]

\[
\Rightarrow y_1^k_i = [(t^k((A + A^T)x^k)_i + q_i - (A^T y_2^k)_i) + (1 - k^k)(x^k_i - x^k_i)].
\]

As \( k \to \infty \) right hand side is bounded, but left hand side is unbounded. It contradicts
that $\|y_k^i\| = \infty$. 

When $\bar{t} = 1$, from equation (3.12) we obtain, $x_i^k = \frac{(1-t_k)^i}{y_{ki}}$ for $i \in I_{1y}$. As $k \to \infty$, $x_i^k \to 0$. 

Again from equation (3.11) we obtain $x_i^{(0)} = \frac{(1-t_k)}{x_k}[(A + AT)x_i + q_i - y_i^k - (ATy_2^k)] + x_i^k$ for $i \in I_{1y}$. As $k \to \infty$, we have $x_i^{(0)} = -\lim_{k \to \infty} \frac{(1-t_k)}{x_k}y_{ki}^k \leq 0$. It contradicts that $\|y_k^i\| = \infty$. 

So $\Gamma_u^{(0)}$ is a bounded curve in $\mathcal{R}_1 \times (0, 1]$. 

Therefore the boundedness of the sequences $\{x_k\}$ and $\{y_k^i\}$ guarantee the boundedness of the sequence $\{y_k^i\}$, i.e. the boundedness of the sequence $\{v_k\}$.

**Theorem 3.4**: Suppose the solution set $\Gamma_u^{(0)}$ of the homotopy function $H(u, u^{(0)}, t) = 0$ is unbounded for $t \in [0, 1)$. Then there exists $(\xi, \eta, \zeta) \in \mathbb{R}^n_+$ such that $e^t \xi = 1$, $\xi^t A \xi \leq 0$.

**Proof**. Let assume that the solution set $\Gamma_u^{(0)}$ is unbounded for $t \in [0, 1)$. Then there exists a sequence of points $\{v^k\} \subset \Gamma_u^{(0)} \subset \mathcal{R}_1 \times [0, 1)$, where $v^k = (u^k, t^k) = (x^k, y_1^k, y_2^k, t^k)$ such that $\lim_{k \to \infty} t^k = \bar{t} \in [0, 1)$. Now we consider following two cases.

**Case 1**: $\|y_k^i\| < \infty$ as $k \to \infty$. Since the solution set $\Gamma_u^{(0)}$ is unbounded we consider the following two subcases.

**Subcase (i)** $\lim_{k \to \infty} e^{\bar{t}} x^k = \infty$ : 

Let $\lim_{k \to \infty} \frac{x_k^k}{e^{\bar{t}}x^k} = \xi \geq 0$ and $\lim_{k \to \infty} \frac{y_k^i}{e^{\bar{t}}x^k} = \eta \geq 0$. So it is clear that $e^\bar{t} \xi = 1$. Then dividing by $e^{\bar{t}} x^k$ and taking limit $k \to \infty$ from equations (3.11) (3.12) (3.13) we write

\begin{equation}
(1 - \bar{t})[(A + AT)\xi - \eta] + \bar{t} = 0 \tag{3.14}
\end{equation}

\begin{equation}
\xi, \eta \in \mathbb{R}^n_+, \xi, (A\xi)_i = 0 \quad \forall \ i \tag{3.15}
\end{equation}

From equations (3.14) and (3.15) we write $\eta = (A + AT)\xi + \frac{\bar{t}}{(1-\bar{t})^2}\xi$ and $-\xi^T A \xi = \xi^T \eta$. These two imply that $\xi^T[(A + AT)\xi + \frac{\bar{t}}{(1-\bar{t})^2}\xi] = \xi^T \eta = -\xi^T A \xi$ for $\bar{t} \in [0, 1)$. This implies that $2\xi^T A \xi + \xi^T A T \xi = -\frac{\bar{t}}{(1-\bar{t})^2}\xi \leq 0$ i.e. $\xi^T A \xi \leq 0$ for $\bar{t} \in [0, 1)$.

Specifically for $\bar{t} = 0$, $\xi^T A \xi = 0$ and for $\bar{t} \in (0, 1)$, $\xi^T A \xi < 0$.

**Subcase (ii)** $\lim_{k \to \infty} (1 - t_k)e^t x^k = \infty$ : 

Let $\lim_{k \to \infty} \frac{y_k^i}{(1-t_k)e^t x^k} = \xi' \geq 0$. Then $e^t \xi' = 1$. Let $\lim_{k \to \infty} \frac{y_k^i}{(1-t_k)e^t x^k} = \eta' \geq 0$. Then multiplying the equation (3.11) with $(1 - t_k)$ and dividing by $(1 - t_k)e^t x^k$, multiplying the equation (3.12) with $(1 - t_k)$ and dividing by $((1 - t_k)e^t x^k)^2$ and multiplying the equation (3.13) with $(1 - t_k)$ and dividing by $((1 - t_k)e^t x^k)^2$ and taking limit $k \to \infty$, we write

\begin{equation}
(1 - \bar{t})[\xi' - (1 - \bar{t})\eta'] + \bar{t} \xi' = 0 \tag{3.16}
\end{equation}

\begin{equation}
\xi', \eta' \in \mathbb{R}^n_+, \xi', (A\xi')_i = 0 \quad \forall \ i \tag{3.17}
\end{equation}
Multiplying $(\xi')^T$ in both sides of equation (3.16) we have $(\xi')^T (A + A^T) \xi' - (1 - \bar{t})(\xi')^T \eta' = -\frac{\bar{t}}{(1-\bar{t})} (\xi')^T \xi'$. Now using equation (3.17) we write $(\xi')^T (A + A^T) \xi' + (1 - \bar{t})(\xi')^T A \xi' = -\frac{\bar{t}}{(1-\bar{t})} (\xi')^T \xi'$ for $\bar{t} \in [0, 1)$. Hence $(3 - \bar{t})(\xi')^T A \xi' = -\frac{\bar{t}}{(1-\bar{t})} (\xi')^T \xi'$ which gives $(\xi')^T A \xi' \leq 0$. So we have $(\xi')^T A \xi' \leq 0$ for $\bar{t} \in [0, 1)$. Specifically for $\bar{t} = 0$, $(\xi')^T A \xi' = 0$ and for $\bar{t} \in (0, 1)$, $(\xi')^T A \xi' < 0$.

Case 2: $\lim_{k \to \infty} e^T y_2^k = \infty$. Since the solution set of $\Gamma_u^{(0)}$ is unbounded we consider following two subcases.

Subcase (i) $\lim_{k \to \infty} e^T x^k = \infty$:

Let $\lim_{k \to \infty} \frac{y_k^k}{e^T x^k} = \xi' \geq 0$, $\lim_{k \to \infty} \frac{y_k^k}{e^T y_k^k} = \eta' \geq 0$ and $\lim_{k \to \infty} \frac{y_k^k}{e^T y_k^k} = \zeta' \geq 0$. It is clear that $e^T \xi = 1$. Then dividing by $e^T x^k$ and taking limit $k \to \infty$ from equation (3.11) dividing by $(e^T x^k)^2$ and taking limit $k \to \infty$ from equation (3.12) and (3.13) we write

$$(1 - \bar{t})(A + A^T) \xi - \eta - A^T \eta = 0$$

$$(3.18)$$

$$(\xi, \eta, i, (A\xi)_i = 0 \forall i)$$

$$(3.19)$$

$$(\zeta_i(A\xi)_i = 0 \forall i)$$

$$(3.20)$$

From equation (3.18) we have $\eta + A^T \zeta = (A + A^T)\xi + \frac{\bar{t}}{1-\bar{t}} \xi$ for $\bar{t} \in [0, 1)$. Now multiplying $\xi^T$ in both sides we get $\xi^T(A + A^T)\xi + \frac{\bar{t}}{1-\bar{t}} \xi^T \xi = \xi^T A \xi + \xi^T A \xi$. From equations (3.11) and (3.20) we write $\xi^T(A + A^T)\xi + \frac{\bar{t}}{1-\bar{t}} \xi^T \xi = -\xi^T A \xi$. Hence $\xi^T A \xi + \xi^T(A + A^T)\xi = -\frac{\bar{t}}{1-\bar{t}} \xi^T \xi \leq 0$ for $\bar{t} \in [0, 1)$. Therefore $\xi^T A \xi \leq 0$ for $\bar{t} \in [0, 1)$. Specifically for $\bar{t} = 0$, $\xi^T A \xi = 0$ and for $\bar{t} \in (0, 1)$, $\xi^T A \xi < 0$.

Subcase (ii) $\lim_{k \to \infty} (1 - t^k) e^T x^k = \infty$:

Let $\lim_{k \to \infty} \frac{(1 - t^k)y_k^k}{e^T x^k} = \xi' \geq 0$. Then $e^T \xi' = 1$. Let $\lim_{k \to \infty} \frac{y_k^k}{(1 - t^k)e^T x^k} = \eta' \geq 0$ and $\lim_{k \to \infty} \frac{y_k^k}{(1 - t^k)e^T x^k} = \zeta' \geq 0$. Then multiplying the equation (3.11) with $(1 - t^k)$ and dividing by $(1 - t^k)e^T x^k$, multiplying the equation (3.12) with $(1 - t^k)$ and dividing by $(1 - t^k)e^T x^k)^2$ and multiplying the equation (3.13) with $(1 - t^k)$ and dividing by $(1 - t^k)e^T x^k)^2$ and taking limit $k \to \infty$, we obtain

$$(1 - \bar{t})(A + A^T) \xi_0 - (1 - \bar{t})^2 \eta_0 - (1 - \bar{t})^2 A^T \xi' + \bar{t} \xi' = 0$$

$$(3.21)$$

$$(\xi, \eta_0, i, (A\xi)_i = 0 \forall i)$$

$$(3.22)$$

$$(\zeta_i(A\xi)_i = 0 \forall i)$$

$$(3.23)$$

Multiplying $(\xi')^T$ in both side of equation (3.21) we have $(\xi')^T(A + A^T)\xi' - (1 - \bar{t})(\xi')^T \eta' - (1 - \bar{t})(\xi')^T A \xi' = -\frac{\bar{t}}{(1-\bar{t})} (\xi')^T \xi'$. Now from equations (3.22) and (3.23) we write $(\xi')^T(A + A^T)\xi' + (1 - \bar{t})(\xi')^T A \xi' = (\xi')^T A \xi' + (2 - \bar{t})(\xi')^T \xi' = (3 - \bar{t})(\xi')^T A \xi' = -\frac{\bar{t}}{(1-\bar{t})} (\xi')^T \xi' \Rightarrow (\xi')^T A \xi' = -\frac{\bar{t}}{(1-\bar{t})(3-\bar{t})} (\xi')^T \xi' \leq 0$ for $\bar{t} \in [0, 1)$. Specifically for $\bar{t} = 0$, $(\xi')^T A \xi' = 0$ and for $\bar{t} \in (0, 1)$, $(\xi')^T A \xi' < 0$. 

12
Hence considering all the cases it is proved that the unboundedness of the solution set $\Gamma_u^{(0)}$ of the homotopy function $H(u, u^{(0)}(t)) = 0$ leads to the existence of $(\xi, \eta, \zeta) \in R^3$ such that $e^t \xi = 1$, $\xi^t A\xi \leq 0$ for $t \in [0, 1)$.

\[ \blacksquare \]

**Corollary 3.1:** For $\bar{t} = 1$, the homotopy curve is bounded.

**Proof.** Consider that the homotopy curve is unbounded in the neighbourhood of $\bar{t} = 1$. Then there exists a sequence of points $\{v_k\} \subset \Gamma_u^{(0)} \subset R_1 \times [0, 1)$, where $v_k = (u_k, t_k) = (x_k, y_k, y_2, t_k)$ such that $\lim_{k \to \infty} t_k = \bar{t} = 1$. Now we consider following two cases.

Case 1: $\lim_{k \to \infty} e^t x_k = \infty$.

Case 2: $\lim_{k \to \infty} (1 - t_k)e^t x_k = \infty$.

Case 1. Let $\lim_{k \to \infty} e^t x_k = \infty$ and $\lim_{k \to \infty} \frac{x_k}{e^t x_k} = \xi \geq 0$. Hence $e^t \xi = 1$. If $\|y_k^2\| < \infty$ as $k \to \infty$, then from equation 3.14 we get $\xi = 0$ for $\bar{t} = 1$. If $\lim_{k \to \infty} e^t y_k^2 = \infty$, then from equation 3.18 we get $\xi = 0$ for $\bar{t} = 1$. This contradicts that $e^t \xi = 1$.

Case 2. Let $\lim_{k \to \infty} (1 - t_k) e^t x_k = \infty$ and $\lim_{k \to \infty} \frac{(1 - t_k)x_k}{e^t x_k} = \xi'\geq 0$. Hence $e^t \xi' = 1$. If $\|y_k^2\| < \infty$ as $k \to \infty$, then from equation 3.16 we get $\xi' = 0$ for $\bar{t} = 1$. If $\lim_{k \to \infty} e^t y_k^2 = \infty$, then from equation 3.21 we get $\xi' = 0$ for $\bar{t} = 1$. This contradicts that $e^t \xi = 1$.

Therefore the homotopy curve is bounded for $\bar{t} = 1$.

\[ \blacksquare \]

**Theorem 3.5:** Let $A \in R^{n \times n}$ be a matrix. If the set $R_1$ be nonempty and 0 is a regular value of $H(u, u^{(0)}(t))$, then the homotopy path $\Gamma_u^{(0)} \subset R_1 \times (0, 1]$ is bounded.

**Proof.** Suppose $A \in R^{n \times n}$ is a matrix and there exists a sequence of points $\{v_k\} \subset \Gamma_u^{(0)} \subset R_1 \times (0, 1]$, where $v_k = (x_k, y_k, y_2, t_k)$. Hence by the definition of $R_1 x_k, y_k, y_2$, $A x_k + q > 0$. From corollary 3.1 the homotopy curve is bounded for $t = 1$. Assume that the homotopy curve $\Gamma_u^{(0)} \subset R_1 \times (0, 1]$ is unbounded. Then from theorem 3.4 $(\xi)H A\xi < 0$ for $t \in (0, 1)$. But $Ax_k + q > 0$ implies that $A\xi \geq 0$, where $\xi = \lim_{k \to \infty} \frac{x_k}{e^t x_k} \geq 0$ for $\lim_{k \to \infty} e^{t_k} x_k = \infty$, or $\xi = \lim_{k \to \infty} \frac{(1 - t_k)x_k}{e^t x_k} \geq 0$ for $\lim_{k \to \infty} (1 - t_k) e^t x_k = \infty$. Hence $\xi, A\xi \geq 0$ imply that $\xi H A\xi \geq 0$ for $t \in (0, 1)$, which contradicts that the homotopy path is unbounded for $t \in (0, 1)$. Hence the homotopy curve $\Gamma_u^{(0)} \subset R_1 \times (0, 1]$ is bounded.

\[ \blacksquare \]

Therefore the homotopy curve $\Gamma_u^{(0)}$ is bounded for the parameter $t$ starting from 1 to 0 if the set $R_1$ be nonempty and 0 is a regular value of the homotopy function 3.1.

For an initial point $u^{(0)} \in R_1$ we obtain a smooth bounded homotopy path which leads to the solution of homotopy function 3.1 as the parameter $t \to 0$.

**Theorem 3.6:** For $u^{(0)} = (x^{(0)}, y_1^{(0)}, y_2^{(0)}) \in R_1$, the homotopy equation finds a bounded smooth curve $\Gamma_u^{(0)} \subset R_1 \times (0, 1]$ which starts from $(u^{(0)}, 1)$ and approaches
the hyperplane at \( t = 0 \). As \( t \to 0 \), the limit set \( L \times \{0\} \subset \bar{R}_1 \times \{0\} \) of \( \Gamma_u^{(0)} \) is nonempty and every point in \( L \) is a solution of the following system:

\[
(A + A^T)x + q - y_1 - A^Ty_2 = 0 \\
Y_1x + X(Ax + q) = 0 \\
Y_2(Ax + q) = 0.
\] (3.24)

**Proof.** Note that \( \Gamma_u^{(0)} \) is diffeomorphic to a unit circle or a unit interval \( (0, 1) \) in view of lemma 2.5. As \( \frac{\partial H(u, t^{(0)}_u)}{\partial u^{(0)}} \) is nonsingular, \( \Gamma_u^{(0)} \) is diffeomorphic to a unit interval \( (0, 1) \).

Again \( \Gamma_u^{(0)} \) is a bounded smooth curve by the theorem 3.3. Let \( (\bar{u}, \bar{t}) \) be a limit point of \( \Gamma_u^{(0)} \). Now consider four cases: (i) \((\bar{u}, \bar{t}) \in \mathcal{R}_1 \times \{1\} : As the equation \( H(u, 1) = 0 \) has only one solution \( u^{(0)} \in \mathcal{R}_1 \), this case is impossible.

(ii) \((\bar{u}, \bar{t}) \in \partial \mathcal{R}_1 \times \{1\} : there exists a subsequence of \((u^k, t^k) \in \Gamma_u^{(0)} \) such that \( x_i^k \to 0 \) or \( (Ax^k + q)_i \to 0 \) for \( i \subseteq \{1, 2, \cdots, n\} \). From the last two equalities of the homotopy function 3.11, we have \( y_1^k \to \infty \) or \( y_2^k \to \infty \). Hence it contradicts the boundedness of the homotopy path by the theorem 3.3.

(iii) \((\bar{u}, \bar{t}) \in \partial \mathcal{R}_1 \times (0, 1) : Also impossible followed by the case (ii).

(iv) \((\bar{u}, \bar{t}) \in \mathcal{R}_1 \times \{0\} : The only possible case.

Hence \( \bar{u} = (\bar{x}, \bar{y}_1, \bar{y}_2) \) is a solution of the system 3.24

\[
(A + A^T)x + q - y_1 - A^Ty_2 = 0 \\
Y_1x + X(Ax + q) = 0 \\
Y_2(Ax + q) = 0.
\]

Note that theorem 3.6 establishes the solution of the proposed homotopy function which validates the theorem 3.1. This in turn leads to the solution of discounted ARAT stochastic game.

In this approach the initial point \( u^{(0)} = (x^{(0)}, y_1^{(0)}, y_2^{(0)}) \in \mathcal{R}_1 \) has to be a feasible point. Hence choose the initial point such that \( x^{(0)} > 0, Ax^{(0)} + q > 0 \). Here \((\bar{u}, 0)\) is the solution of the homotopy function 3.11. Therefore \( \bar{u} \in \bar{R}_1 \) is the solution of the system of equations 3.24. Hence \( \bar{Y}_1\bar{x} = 0 \) and \( \bar{X}(A\bar{x} + q) = 0 \), where \( \bar{Y}_1 = \text{diag}(\bar{y}_1) \) and \( \bar{X} = \text{diag}(\bar{x}) \). It is clear that the component \( \bar{x} \) of \( \bar{u} = (\bar{x}, \bar{y}_1, \bar{y}_2) \) provides the solution of discounted ARAT stochastic game.

### 3.2 Tracing Homotopy Path

We trace the homotopy path \( \Gamma_u^{(0)} \subset \mathcal{R}_1 \times (0, 1) \) from the initial point \((u^{(0)}, 1)\) as \( t \to 0 \). To find the solution of the discounted ARAT stochastic game we consider homotopy path along with other assumptions. Let \( s \) denote the arc length of \( \Gamma_u^{(0)} \). We parameterize the homotopy path \( \Gamma_u^{(0)} \) with respect to \( s \) in the following form

\[
H(u(s), t(s)) = 0, \ u(0) = u^{(0)}, \ t(0) = 1.
\] (3.25)
The solution of the equation 3.25 satisfies the initial value problem

\[ \dot{u} = -\frac{\partial}{\partial u} H(u, t)^{-1} \frac{\partial}{\partial t} H(u, t), \quad u(0) = u^{(0)} \]  

(3.26)

From equation 3.1 the choice of \( H \) is \( H(u, t) = (1 - t)f(u) + tg(u) = 0 \), where

\[
\begin{bmatrix}
(A + A^T)x + q - y_1 - A^Ty_2 \\
Y_1x + X(Ax + q) \\
Y_2(Ax + q)
\end{bmatrix}
\text{ and }\begin{bmatrix}
x - x^{(0)} \\
Y_1x - Y_1^{(0)}x^{(0)} \\
Y_2(Ax + q) - Y_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix}
\]

Hence the system 3.26 becomes

\[ \dot{u} = -((1 - t)J_f + tJ_g)^{-1}(g(u) - f(u)), \quad u(0) = u^{(0)} \]

where \( J_f \) and \( J_g \) are Jacobian matrices of the functions \( f \) and \( g \). Hence \( \dot{u} = -\tilde{J}^{-1}\tilde{f} \), where \( \tilde{J} = (1 - t)J_f + tJ_g \) and \( \tilde{f} = g(u) - f(u) \).

The initial value problem 3.26 reduces to

\[ \dot{u} = p(u, t), \quad u(0) = u^{(0)} \text{ where } p(u, t) = -\tilde{J}^{-1}\tilde{f} \]

This problem will be solved by iterative process

\[ u_{(i+1)} = P(u_i, t_i, h_i), \text{ where } h_i = t_{i+1} - t_i. \]

Here \( u_i \) is an approximation of \( u(s) \). \( P(u_i, t_i, h_i) \) is given by

\[
\begin{aligned}
P(u, t, h) &= I_m(u, t, h), \text{ where } I_0(u, t, h) = u \\
\text{and } K_j &= \frac{\partial}{\partial u} H(I_j, t + h)^+H(I_j, t + h) \\
L_j &= I_j - K_j \\
KK_j &= (\frac{\partial}{\partial u} H(L_j, t + h) + \frac{\partial}{\partial u} H(I_j, t + h))^+H(I_j, t + h) \\
LL_j &= I_j - 2*KK_j
\end{aligned}
\]

The next iteration

\[ I_{j+1} = LL_j - \frac{\partial}{\partial u} H(L_j, t + h)^+H(L_j, t + h), \text{ for } j = 0, 1, 2, \ldots, m - 1. \]

Therefore in each step finding the value of \( I_{j+1} \) for \( m \) times we obtain the next iteration and iterative process will continue until the termination criteria is satisfied.
Algorithm 1: Modified Homotopy Continuation Method

**Step 0:** Initialize \((u^{(0)}, t_0)\) and a natural number \(m \in (0, 50)\). Set \(l_0 \in (0, 1)\). Choose \(\epsilon_2 >> \epsilon_3 >> \epsilon_1 > 0\) which are very small positive quantity.

**Step 1:** \(\tau^{(0)} = \xi^{(0)} = (\frac{1}{n_0}) \left[ \begin{array}{c} s^{(0)} \\ -1 \end{array} \right] \) for \(i = 0\), where \(n_0 = \| s^{(0)} \|\) and \(s^{(0)} = (\frac{\partial H}{\partial u}(u^{(0)}, t_0))^{-1} (\frac{\partial H}{\partial t}(u^{(0)}, t_0))\).

For \(i > 0\), \(s^{(i)} = (\frac{\partial H}{\partial u}(u^{(i)}, t_i))^{-1} (\frac{\partial H}{\partial t}(u^{(i)}, t_i))\), \(n_i = \| s^{(i)} \|\), \(\xi^{(i)} = (\frac{1}{n_i}) \left[ \begin{array}{c} s^{(i)} \\ -1 \end{array} \right]\).

If \(\det(\frac{\partial H}{\partial u}(u^{(i)}, t_i)) > 0\), \(\tau^{(i)} = \xi^{(i)}\) else \(\tau^{(i)} = -\xi^{(i)}\), \(i \geq 1\).

Set \(l = 0\).

**Step 2:** (Predictor and corrector point calculation) \((\tilde{u}^{(i)}, \tilde{t}_i) = (u^{(i)}, t_i) + a \tau^{(i)}\), where \(a = l_0^i\). Compute \((\tilde{u}^{(i)}, \tilde{t}_i) = H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i) + H(\tilde{u}^{(i)}, \tilde{t}_i)\) and \((\tilde{u}^{(i)}, \tilde{t}_i) = (\tilde{u}^{(i)}, \tilde{t}_i) - (\tilde{u}^{(i)}, \tilde{t}_i)\).

Now compute \((\hat{u}u^{(i)}, \hat{t}_i) = (H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i) + H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i)) + H(\tilde{u}^{(i)}, \tilde{t}_i)\) and \((\hat{u}u^{(i)}, \hat{t}_i) = (\tilde{u}^{(i)}, \tilde{t}_i) - 2(\tilde{u}^{(i)}, \tilde{t}_i)\).

Compute \((u^{(i+1)}, t_{i+1}) = (\hat{u}u^{(i)}, \hat{t}_i) - H'_{u^{(0)}}(\tilde{u}^{(i)}, \tilde{t}_i) + H(\hat{u}u^{(i)}, \hat{t}_i)\).

Repeat the method from the computation of \((\tilde{u}^{(i)}, \tilde{t}_i)\) to the computation of \((u^{(i+1)}, t_{i+1})\) for \(m\) times. In each step after repeating the computation for \(m\) times, can obtain the value of next iteration. \((u^{(i+1)}, t_{i+1})\)

If \(0 < \|t_{i+1} - t_i\| < 1\), go to step 3. Otherwise if \(m' = \min(a, \|u^{(i+1)}, t_{i+1})\) \(> a_0\), update \(l\) by \(l + 1\), and recompute \((\tilde{t}_i, \tilde{t}_i)\) else go to step 3.

**Step 3:** Determine the norm \(r = \|H(u^{(i+1)}, t_{i+1})\|\). If \(r \leq 1\) and \(u^{(i+1)} > 0\) go to step 5, otherwise if \(a > \epsilon_3\), update \(l\) by \(l + 1\) and go to step 2 else go to step 4.

**Step 4:** If \(|t_{i+1} - t_i| < \epsilon_2\), then if \(|t_{i+1} - \epsilon_2| \), then stop with the solution \((u^{(i+1)}, t_{i+1})\), else terminate (unable to find solution) else \(i = i + 1\) and go to step 1.

**Step 5:** If \(|t_{i+1} - \epsilon_1|\), then stop with solution \((u^{(i+1)}, t_{i+1})\), else \(i = i + 1\) and go to step 1.

Note that in step 2, \(H'_{u^{(0)}}(u, t) = H'_{u^{(0)}}(u, t)^T(H'_{u^{(0)}}(u, t)H'_{u^{(0)}}(u, t))^T)^{-1}\) is the Moore-Penrose inverse of \(H'_{u^{(0)}}(u, t)\).

We prove the following theorem to obtain the positive direction of the proposed algorithm.

**Theorem 3.7:** If the homotopy curve \(\Gamma_{u^{(0)}}^0\) is smooth, then the positive predictor direction \(\tau^{(0)}\) at the initial point \(u^{(0)}\) satisfies \(\det(\frac{\partial H}{\partial u}(u^{(0)}, t), 1) < 0\).

**Proof:** From the equation 3.1 we consider the following homotopy function

\[
H(u, t) = \begin{bmatrix}
(1 - t)[[(A + A^T)x + q - y_1 - A^Tz_2] + t(x - x^{(0)})] \\
Y_1x - tY_1^{(0)}x^{(0)} + (1 - t)X(Ax + q) \\
Y_2(Ax + q) - tY_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix} = 0.
\]
Now \( \frac{\partial H}{\partial u}(u, t) = \)
\[
\begin{bmatrix}
(1 - t)(A + AT) + tI & -(1 - t)I & -(1 - t)AT \\
Y_1 + (1 - t)(Y + XA) & X & 0 \\
Y_2A & 0 & Y
\end{bmatrix}
\begin{bmatrix}
Q \\
-Y_1^{(0)}x^{(0)} - X(Ax + q) \\
-Y_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix},
\]
where \( Q = (x - x^{(0)}) - [(A + AT)x + q - y_1 - A^ty_2] \) and \( Y = \text{diag}(Ax + q) \).

At the initial point \( (u^{(0)}, 1) \)
\[
\frac{\partial H}{\partial u}(u^{(0)}, 1) = \begin{bmatrix}
I & 0 & 0 \\
Y_1^{(0)} & X^{(0)} & 0 \\
Y_2^{(0)}A & 0 & Y^{(0)}
\end{bmatrix}
\begin{bmatrix}
-(A + AT)x^{(0)} + q - y_1^{(0)} - ATy_2^{(0)} \\
-Y_1^{(0)}x^{(0)} - X^{(0)}(Ax^{(0)} + q) \\
-Y_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix}.
\]

Let positive predictor direction be \( \tau^{(0)} = \begin{bmatrix} \kappa \\ -1 \end{bmatrix} = \begin{bmatrix} (Q_1^{(0)})^{(-1)}Q_2^{(0)} \\ -1 \end{bmatrix} \),
where
\[
\begin{align*}
Q_1^{(0)} &= \begin{bmatrix} I & 0 & 0 \\
Y_1^{(0)} & X^{(0)} & 0 \\
Y_2^{(0)}A & 0 & Y^{(0)}
\end{bmatrix}, \\
Q_2^{(0)} &= \begin{bmatrix}
-(A + AT)x^{(0)} + q - y_1^{(0)} - ATy_2^{(0)} \\
-Y_1^{(0)}x^{(0)} - X^{(0)}(Ax^{(0)} + q) \\
-Y_2^{(0)}(Ax^{(0)} + q)
\end{bmatrix}
\end{align*}
\]

and \( \kappa \) is an \( n \times 1 \) column vector.

Hence, \( \det \left[ \frac{\partial H}{\partial u}(u^{(0)}, 1) \right] \)
\[
= \det \begin{bmatrix} Q_1^{(0)} & Q_2^{(0)} \\
(Q_2^{(0)})^t(Q_1^{(0)})^{(-T)} & -1
\end{bmatrix}
= \det \begin{bmatrix} Q_1^{(0)} & Q_2^{(0)} \\
0 & -1 - (Q_2^{(0)})^t(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)}
\end{bmatrix}
= \det(Q_1^{(0)}) \det(-1 - (Q_2^{(0)})^t(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)})
= - \det(Q_1^{(0)}) \det(1 + (Q_2^{(0)})^t(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)})
= - \Pi_{i=1}^{n} x_i^{(0)} y_i^{(0)} \det(1 + (Q_2^{(0)})^t(Q_1^{(0)})^{(-T)}(Q_1^{(0)})^{(-1)}Q_2^{(0)}) < 0.
\]

So the positive predictor direction \( \tau^{(0)} \) at the initial point \( u^{(0)} \) satisfies
\[
\det \left[ \frac{\partial H}{\partial u}(u^{(0)}, 1) \right] < 0. \]

**Remark 3.1:** We conclude from the theorem 3.7 that the positive tangent direction \( \tau \)
of the homotopy path \( \Gamma^{(0)}_u \) at any point \( (u, t) \) be negative and it depends on \( \det(Q_1) \),
where \( Q_1 = \begin{bmatrix} (1-t)(A + A^T) + tI & -(1-t)I & -(1-t)A^T \\ Y_1 + (1-t)(Y + XB) & X & 0 \\ Y_2A & 0 & Y \end{bmatrix} \).

Based on the earlier work the homotopy continuation method to solve the initial value problem 3.26 was formulated with the iterative process as

\[ I_{j+1} = I_j - \frac{\partial}{\partial u} H(I_j, t + h)^+ H(I_j, t + h), \text{ for } j = 0, 1, 2, \cdots, m - 1. \]

For details see [1]. However the proposed modified homotopy continuation method solves homotopy function by solving the initial value problem 3.26 with the following iterative process

\[
\begin{align*}
K_j &= \frac{\partial}{\partial u} H(I_j, t + h)^+ H(I_j, t + h) \\
L_j &= I_j - K_j \\
KK_j &= (\frac{\partial}{\partial u} H(L_j, t + h) + \frac{\partial}{\partial u} H(I_j, t + h))^+ H(I_j, t + h) \\
LL_j &= I_j - 2 * KKK_j \\
I_{j+1} &= LL_j - \frac{\partial}{\partial u} H(L_j, t + h)^+ H(L_j, t + h), \text{ for } j = 0, 1, 2, \cdots, m - 1.
\end{align*}
\]

By this iterative process the proposed homotopy function achieves the order of convergence as \( 5^m - 1 \).

**Theorem 3.8:** Suppose that the homotopy function has derivative, which is lipschitz continuous in a convex neighbourhood \( \mathcal{N} \) of \( c \), where \( c \) is the solution of the homotopy function \( H(u, t) = 0 \), whose Jacobian matrix is continuous and nonsingular and bounded on \( \mathcal{N} \). Then the modified homotopy continuation method has order \( 5^m - 1 \).

**Proof.** By the Implicit Function Theorem ensures the existence of a unique continuous solution \( z(h) \in \mathcal{N} \) of \( \dot{z}(h) = -J^{-1} f, z(0) = u \) and \( h \in (-\delta, \delta) \), for some \( \delta > 0 \). Define \( \beta_j = \| z(h) - I_j(u, h) \|. \) From lemma 2.7, \( \beta_j = O(h^{5^j}) \). Then \( \beta_{j+1} = \| z(h) - I_{j+1} \| \leq K \beta_j^5 \). Hence \( \beta_{j+1} = O(h^{5^{j+1}}) \). By induction method the modified homotopy continuation method has convergence of order \( 5^m - 1 \).

### 3.3 Solving Discounted Zero-Sum Stochastic Game with ARAT Structure

**Example 3.1:** Consider a two player zero-sum discounted ARAT game with \( s = 2 \) states. In each state each player has 2 actions. The transition probabilities are given by

\[
\begin{align*}
p_1^1(1, 1) &= \frac{1}{2}, \quad p_1^1(1, 2) = 0, \\
p_2^1(1, 1) &= \frac{1}{2}, \quad p_2^1(1, 2) = 0, \\
p_1^2(2, 1) &= 0, \quad p_1^2(2, 2) = \frac{1}{2}, \\
p_2^2(2, 1) &= 0, \quad p_2^2(2, 2) = \frac{1}{2}, \\
p_1^3(1, 1) &= \frac{1}{2}, \quad p_1^3(1, 2) = 0, \\
p_2^3(1, 1) &= \frac{1}{2}, \quad p_2^3(1, 2) = 0, \\
p_1^4(1, 1) &= 0, \quad p_1^4(1, 2) = \frac{1}{2}, \\
p_2^4(1, 1) &= 0, \quad p_2^4(1, 2) = \frac{1}{2}, \\
p_1^5(2, 1) &= 0, \quad p_1^5(2, 2) = \frac{1}{2}, \\
p_2^5(2, 1) &= 0, \quad p_2^5(2, 2) = \frac{1}{2}, \\
p_1^6(2, 1) &= \frac{1}{2}, \quad p_1^6(2, 2) = 0, \\
p_2^6(2, 1) &= \frac{1}{2}, \quad p_2^6(2, 2) = 0.
\end{align*}
\]
Note that \( p_{ij}(s, s') = p_i^1(s, s') + p_j^2(s, s') \).

\( P_1 = P_1(s) = ((p_i^1(s, s'), s, s' \in S, i \in A_s)) \) and
\( P_2 = P_2(s) = ((p_j^2(s, s'), s, s' \in S, j \in B_s)) \).

Let the discount factor \( \beta = \frac{1}{2} \).

The reward structure:
\[
\begin{align*}
    r_1^1(1) &= 4, & r_1^1(2) &= 5, \\
    r_2^1(1) &= 3, & r_2^1(2) &= 4, \\
    r_1^2(1) &= 3, & r_1^2(2) &= 6, \\
    r_2^2(1) &= 6, & r_2^2(2) &= 2.
\end{align*}
\]

Note that \( r(s, i, j) = r_i^1(s) + r_j^2(s) \).

Now we solve discounted ARAT game using the proposed homotopy function. The initial point is \( u^{(0)} = (4, 5, 3, 4, 8, 8, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)^T \). As \( t \to 0 \), \( u = (0, 7, 0, 6, 9, 0, 0, 7.33333, 1, 0, 1, 0, 0, 2.33333, 3.33333, 0, 0, 7, 0, 6, 9, 0, 0, 7.33333, 0) \).

Hence the solution of discounted ARAT is \( x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 2.33333 \\ 3.33333 \\ 0 \end{bmatrix} \).

**Example 3.2:** Consider another two player zero-sum discounted ARAT game with \( s = 2 \) states. In each state each player has 2 actions. The transition probabilities are given by
\[
\begin{align*}
p_i^1(1, 1) &= \frac{1}{4}, & p_i^1(1, 2) &= 0, \\
p_i^1(2, 1) &= \frac{1}{4}, & p_i^1(2, 2) &= 0, \\
p_i^2(1, 1) &= 0, & p_i^2(1, 2) &= \frac{1}{2}, \\
p_i^2(2, 1) &= 0, & p_i^2(2, 2) &= \frac{1}{2}, \\
p_j^1(1, 1) &= \frac{2}{3}, & p_j^1(1, 2) &= 0, \\
p_j^1(2, 1) &= 0, & p_j^1(2, 2) &= \frac{1}{3}, \\
p_j^2(1, 1) &= 0, & p_j^2(1, 2) &= \frac{1}{2}, \\
p_j^2(2, 1) &= \frac{2}{7}, & p_j^2(2, 2) &= 0.
\end{align*}
\]

Note that \( p_{ij}(s, s') = p_i^1(s, s') + p_j^2(s, s') \).

\( P_1 = P_1(s) = ((p_i^1(s, s'), s, s' \in S, i \in A_s)) \) and
\( P_2 = P_2(s) = ((p_j^2(s, s'), s, s' \in S, j \in B_s)) \).

Let the discount factor \( \beta = \frac{1}{2} \).

The reward structure:
\[
\begin{align*}
r_1^1(1) &= 4, & r_1^1(2) &= 5, \\
r_2^1(1) &= 3, & r_2^1(2) &= 4, \\
r_1^2(1) &= 3, & r_1^2(2) &= 6, \\
r_2^2(1) &= 6, & r_2^2(2) &= 2.
\end{align*}
\]

Note that \( r(s, i, j) = r_i^1(s) + r_j^2(s) \).

Now we solve discounted ARAT game using the proposed homotopy function. The initial point is \( u^{(0)} = (1, 1, 1, 1, 20, 20, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)^T \). As
\[ t \to 0, u = (0, 9, 0, 6, 7, 0, 0, 7.33333, 1, 0, 1, 0, 2, 3.33333, 0, 0, 9, 0, 6, 7, 0, 0, 7.33333, 0). \]

Hence the solution of discounted ARAT is \[ x = \begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
2 \\
3.33333 \\
0 \\
0
\end{bmatrix}. \]

4 Conclusion

In this paper, we introduce a homotopy continuation method to find the solution of discounted ARAT stochastic game. Mathematically, we obtain the positive tangent direction of the homotopy path. We prove that the smooth curve of the proposed homotopy function is bounded and convergent. We establish that the proposed homotopy functions has \[ 5^m - 1 \] order of convergence. Two numerical examples are illustrated to demonstrate the effectiveness of the proposed homotopy function.

5 Acknowledgment

The author A. Dutta is thankful to the Department of Science and Technology, Govt. of India, INSPIRE Fellowship Scheme for financial support.

References

[1] James P Abbott and Richard P Brent. A note on continuation methods for the solution of nonlinear equations. *The ANZIAM Journal*, 20(2):157–164, 1977.

[2] Shui Nee Chow, John Mallet-Paret, and James A Yorke. Finding zeroes of maps: homotopy methods that are constructive with probability one. *Mathematics of Computation*, 32(143):887–899, 1978.

[3] Alicia Cordero, José L Hueso, Eulalia Martínez, and Juan R Torregrosa. Increasing the convergence order of an iterative method for nonlinear systems. *Applied Mathematics Letters*, 25(12):2369–2374, 2012.

[4] Richard W Cottle and George B Dantzig. Complementary pivot theory of mathematical programming. Technical report, STANFORD UNIV CA OPERATIONS RESEARCH HOUSE, 1967.

[5] Richard W Cottle and George B Dantzig. A generalization of the linear complementarity problem. *Journal of Combinatorial Theory*, 8(1):79–90, 1970.
[6] AK Das, R Jana, and Deepmala. Finiteness of criss-cross method in complementarity problem. In International Conference on Mathematics and Computing, pages 170–180. Springer, 2017.

[7] B Curtis Eaves. The linear complementarity problem. Management science, 17(9):612–634, 1971.

[8] B Curtis Eaves and Romesh Saigal. Homotopies for computation of fixed points on unbounded regions. Mathematical Programming, 3(1):225–237, 1972.

[9] Michael C Ferris and Jong-Shi Pang. Engineering and economic applications of complementarity problems. Siam Review, 39(4):669–713, 1997.

[10] Jerzy Filar and Koos Vrieze. Competitive Markov decision processes. Springer Science & Business Media, 2012.

[11] János Flesch, Frank Thuijsman, and Okko Jan Vrieze. Stochastic games with additive transitions. European Journal of Operational Research, 179(2):483–497, 2007.

[12] CB Garcia. Some classes of matrices in linear complementarity theory. Mathematical Programming, 5(1):299–310, 1973.

[13] Jacob K Goeree and Charles A Holt. Stochastic game theory: For playing games, not just for doing theory. Proceedings of the National Academy of sciences, 96(19):10564–10567, 1999.

[14] R Jana, AK Das, and A Dutta. On hidden z-matrix and interior point algorithm. Opsearch, 56(4):1108–1116, 2019.

[15] R Jana, A Dutta, and AK Das. More on hidden z-matrices and linear complementarity problem. Linear and Multilinear Algebra, 69(6):1151–1160, 2021.

[16] Michael M Kostreva and Malgorzata M Wiecck. Linear complementarity problems and multiple objective programming. Mathematical Programming, 60(1-3):349–359, 1993.

[17] MM KOSTREVA. Direct algorithms for complementarity problems[ph. d. thesis]. 1976.

[18] SR Mohan, SK Neogy, and AK Das. More on positive subdefinite matrices and the linear complementarity problem. Linear Algebra and Its Applications, 338(1-3):275–285, 2001.

[19] SR Mohan, SK Neogy, and AK Das. On the classes of fully copositive and fully semimonotone matrices. Linear Algebra and its Applications, 323:87–97, 01 2001.

[20] SR Mohan, SK Neogy, and T Parthasarathy. Pivoting algorithms for some classes of stochastic games: A survey. International Game Theory Review, 3(02n03):253–281, 2001.
[21] SR Mohan, SK Neogy, T Parthasarathy, and S Sinha. Vertical linear complementarity and discounted zero-sum stochastic games with ar-at structure. *Mathematical programming*, 86(3):637–648, 1999.

[22] SR Mohan, SK Neogy, and R Sridhar. The generalized linear complementarity problem revisited. *Mathematical Programming*, 74(2):197–218, 1996.

[23] Prasenjit Mondal, S Sinha, SK Neogy, and AK Das. On discounted ar–at semi-markov games and its complementarity formulations. *International Journal of Game Theory*, 45(3):567–583, 2016.

[24] Katta G Murty and Feng-Tien Yu. *Linear complementarity, linear and nonlinear programming*, volume 3. Citeseer, 1988.

[25] SK Neogy, R. Bapat, AK Das, and T. Parthasarathy. Mathematical programming and game theory for decision making. 11 2021.

[26] SK Neogy and AK Das. Linear complementarity and two classes of structured stochastic games. *Operations Research with Economic and Industrial Applications: Emerging Trends, eds: SR Mohan and SK Neogy, Anamaya Publishers, New Delhi, India*, pages 156–180, 2005.

[27] SK Neogy and AK Das. On almost type classes of matrices with $Q$-property. *Linear and Multilinear Algebra*, 53(4):243–257, 2005.

[28] SK Neogy and AK Das. Principal pivot transforms of some classes of matrices. *Linear algebra and its applications*, 400:243–252, 2005.

[29] SK Neogy and AK Das. On weak generalized positive subdefinite matrices and the linear complementarity problem. *Linear and Multilinear Algebra*, 61(7):945–953, 2013.

[30] SK Neogy, AK Das, and R Bapat. Optimization models with economic and game theoretic applications. *Annals of Operations Research*, 243, 07 2016.

[31] SK Neogy, AK Das, and R. Bapat. Modeling, computation and optimization. 11 2021.

[32] SK Neogy, AK Das, and Abhijit Gupta. Generalized principal pivot transforms, complementarity theory and their applications in stochastic games. *Optimization Letters*, 6(2):339–356, 2012.

[33] SK Neogy, AK Das, S Sinha, and A Gupta. On a mixture class of stochastic game with ordered field property. In *Mathematical programming and game theory for decision making*, pages 451–477. World Scientific, 2008.

[34] Jong-Shi Pang. Complementarity problems. In *Handbook of global optimization*, pages 271–338. Springer, 1995.
[35] TES Raghavan. Stochastic games—an overview. *Stochastic Games and Related Topics*, pages 1–9, 1991.

[36] Tirukkannamangai ES Raghavan, SH Tijs, and OJ Vrieze. On stochastic games with additive reward and transition structure. *Journal of Optimization Theory and Applications*, 47(4):451–464, 1985.

[37] Lloyd S Shapley. Stochastic games. *Proceedings of the national academy of sciences*, 39(10):1095–1100, 1953.

[38] Sagnik Sinha. *Contribution to the theory of stochastic games*. PhD thesis, Indian Statistical Institute, Delhi, 1989.

[39] Matthew J Sobel. Noncooperative stochastic games. *The Annals of Mathematical Statistics*, 42(6):1930–1935, 1971.

[40] Eilon Solan and Nicolas Vieille. Stochastic games. *Proceedings of the National Academy of Sciences*, 112(45):13743–13746, 2015.

[41] Xiuyu Wang and Xingwu Jiang. A homotopy method for solving the horizontal linear complementarity problem. *Computational and Applied Mathematics*, 33, 04 2013.

[42] Layne T Watson. Globally convergent homotopy methods: a tutorial. *Applied Mathematics and Computation*, 31:369–396, 1989.

[43] Layne T Watson and Raphael T Haftka. Modern homotopy methods in optimization. *Computer Methods in Applied Mechanics and Engineering*, 74(3):289–305, 1989.

[44] X Zhao, S Zhang, and Q Liu. A combined homotopy interior point method for the linear complementarity problem. *Journal of Information and Computational Science*, 7:1589–1594, 07 2010.