GENERIC IDENTITIES FOR FINITE GROUP ACTIONS

PIOTR GRZESZCZUK

Dedicated to the memory of my friend Jeffrey Bergen

Abstract. Let $G$ be a finite group of order $n$, and $Z_G = \mathbb{Z}\langle \zeta_{i,g} \mid g \in G, \ i = 1, 2, \ldots, n \rangle$ be the free generic algebra, with canonical action of $G$ according to $(\zeta_{i,g})^x = \zeta_{i,x^{-1}g}$. It is proved that there exists a positive integer $\upsilon(G)$ such that for any $g_1, g_2, \ldots, g_n \in G$

$$v(G) \cdot \zeta_{1,g_1} \cdot \zeta_{2,g_2} \cdots \zeta_{n,g_n} = \sum_{i=1}^{N} \gamma_i a_i \text{tr}_G(b_i)c_i,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_N$ are integers, and $a_i, b_i, c_i$ are monomials in $\zeta_{i,g}$ such that $\deg(b_i) > 0$ and $\deg(a_i) + \deg(b_i) + \deg(c_i) = n$. As a consequence, if $R$ is a ring (not necessarily unital) acted on by $G$, then the product $v(G) \cdot R^n$ is contained in the ideal $\langle \text{tr}_G(R) \rangle$ generated by all traces $\text{tr}_G(r) = \sum_{g \in G} r^g$, $r \in R$. This gives the best possible nilpotence bound in Bergman-Isaacs theorem for finite group actions on non-commutative rings, which was a long standing problem. The main result was obtained by transferring the problem to certain family of Cayley graphs, and estimating their minimal eigenvalues by the clique numbers. It is proved that the clique number $\omega(\Gamma)$ of any $k$-regular graph $\Gamma$ admits the Delsarte upper bound $\omega(\Gamma) \leq \lfloor 1 - k/\lambda_{\min} \rfloor$.

\section{Introduction}

Let a finite group $G$ acts on a non-commutative ring $R$ so that we have a group homomorphism $G \to \text{Aut}(R)$, $r \mapsto r^g$. Then we can form the fixed subring $R^G = \{ r \in R \mid r^g = r \ \text{for all} \ g \in G \}$.

A natural way to construct fixed points of the action is to use the trace map $\text{tr}_G: R \rightarrow R$ defined by $\text{tr}_G(r) = \sum_{g \in G} r^g$. The image $T = \text{tr}_G(R)$ is an ideal of $R^G$. One of the most fundamental results in the theory of fixed rings is the following theorem of G.M. Bergman and I.M. Isaacs [2].

Theorem. Let $G$ be a finite group of automorphisms of the ring $R$ with no additive $|G|$-torsion. If $T = \text{tr}_G(R)$ is nilpotent of index $d$, then $R$ is nilpotent of index at most $f(|G|)^d$, where $f(m) = \prod_{k=1}^{m} ((m\choose{k}) + 1)$. In particular if $\text{tr}_G(R) = 0$, then $R^{f(|G|)} = 0$.

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Bergman-Isaacs theorem is extremely useful and has been the basic tool in theory of finite group actions on non-commutative rings for a long time. If the acting group $G$ is solvable it is known that the best possible nilpotence bound is $|G|$ (see [2], [9]). There were other proofs of Bergman-Isaacs theorem, but none of them yield better information on the bound. In a nice short paper [10] E.R. Puczyłowski proved that if the group $G$ acts on the ring $R$ and $I$ is a $G$-stable one-sided ideal such that the trace $\text{tr}_G(I)$ is contained in the Jacobson radical $J(R^G)$, then $|G| \cdot I \subseteq J(R)$. Then using certain ring extensions he concluded that $|G| \cdot I$ is nilpotent. It is interesting that the arguments in Puczyłowski’s paper are purely ring theoretical with almost invisible role of the group. In the other paper [11] D. Quinn proved that if $|G| \cdot R = R$, then $R$ is fully integral of degree $m = m(|G|)$ over the fixed ring. The Bergman-Isaacs theorem is a consequence of Quinn’s result but with much bigger nilpotence bound.

In this paper we propose a graph theory approach to estimation of the nilpotence bound in the Bergman-Isaacs theorem. With any group $G$ we associate a family $\mathcal{G} = \{\mathcal{G}_m(G) \mid m > 1\}$ of Cayley graphs, with the Cartesian power $G^m$ as a set of vertices and an extended diagonal as a symmetric set defining the edges. These graphs have many interesting properties. In particular, in [1] we prove that each member of $\mathcal{G}$ determines the group $G$. It means that for any given groups $G$ and $H$, and an integer $m > 1$ if the graphs $\mathcal{G}_m(G)$ and $\mathcal{G}_m(H)$ are isomorphic, then $G$ and $H$ are isomorphic as well. Therefore, it seems that the family $\mathcal{G}$ contains many interesting combinatorial invariants of the group $G$, which can be used also in other contexts. In the problem under consideration, the clique number of graphs from the family $\mathcal{G}$ turned out to be an effective invariant. Controlling the value of a clique number is easy, but it provides an unexpected information about the spectrum of the adjacency matrix of $\mathcal{G}_m(G)$. This is sufficient for solving our problem. Because there is a quite large margin for modifying the symmetric sets defining graphs $\mathcal{G}_m(G)$, we expect that this approach can lead to many other nontrivial connections of such graphs with the structure of the group $G$.

We will work with the generic model defined by D.S. Passman in [8] (c.f. [9], chapter 6). For a given finite group $G$ and some index set $I$ let

$$Z_G = \mathbb{Z}\langle \zeta_{i,g} \mid g \in G, i \in I \rangle$$

$$Q_G = \mathbb{Q}\langle \zeta_{i,g} \mid g \in G, i \in I \rangle$$

be the free algebras without 1 over the ring of integers $\mathbb{Z}$ and the field $\mathbb{Q}$, respectively. The group $G$ acts naturally on the left side on $Z_G$ and $Q_G$ permuting the variables according to the formula $(\zeta_{i,g})^z = \zeta_{i,z^{-1}g}$. The algebra $Z_G$ has a nice universal property saying that for any ring $A$ acted upon by $G$ and for given elements $a_i \in A$ ($i \in I$) the map

$$\theta: \zeta_{i,g} \mapsto a_i^g$$

extends to a $G$-homomorphism of rings $\theta: Z_G \to A$. Furthermore, for a sufficiently large set $I$ this map can be made a surjection and in this case $\theta(\text{tr}_G(Z_G)) = \text{tr}_G(A)$.

For any positive integer $m$ let $Q_G(m)$ be the linear span over $\mathbb{Q}$ of all monomials in $\zeta_{i,g}$ of degree $m$. For monomials $a = \zeta_{i_1,g_1} \cdots \zeta_{i_{k-1},g_{k-1}}$, $b = \zeta_{i_k,g_k} \cdots \zeta_{i_{l-1},g_{l-1}}$, $c = \zeta_{i_l,g_l} \cdots \zeta_{i_m,g_m}$ we have
(1.1) \[ a \text{tr}_G(b)c = \sum_{h \in G} \zeta_{i_1, g_1} \cdots \zeta_{i_k, g_k} \cdots \zeta_{i_{l-1}, h_{g_{k-1}}} \cdots \zeta_{i_{l-1}, h_{g_{l-1}}} \zeta_{i_l, g_l} \cdots \zeta_{i_m, g_m}. \]

Let \( \mathcal{T} \) be the set of all elements \( a \text{tr}_G(b)c \), where \( a, b, c \) are monomials in \( \zeta_{i,g} \) such that \( \deg(b) \geq 1 \) and \( \deg(a) + \deg(b) + \deg(c) = m \), and let \( T_G(m) \) be the linear span of \( \mathcal{T} \) over \( \mathbb{Q} \). We call \( T_G(m) \) a trace subspace. In light of Bergman-Isaacs theorem the following question seems natural.

**Question 1.1.** Is there a positive integer \( m = m(G) \) such that \( T_G(m) = Q_G(m) \)?

If \( G = \{e, g\} \) is cyclic of order 2, the question is easy and the answer is \( m = 2 \). The identities like

\[
2 \zeta_{1,e} \zeta_{2,e} = (\zeta_{1,e} \zeta_{2,e} + \zeta_{1,e} \zeta_{2,g}) + (\zeta_{1,e} \zeta_{2,e} + \zeta_{1,g} \zeta_{2,e}) - (\zeta_{1,e} \zeta_{2,g} + \zeta_{1,g} \zeta_{2,e})
\]

show that \( T_G(2) = Q_G(2) \).

We answer Question 1.1 positively by proving the following

**Theorem 4.1.** For a given group \( G \) if \( m \geq |G| \), then \( T_G(m) = Q_G(m) \).

Below we will discuss some consequences of this result. In particular, it implies that any monomial in \( Q_G \) of degree \( m \geq |G| \) can be expressed as a linear combination with rational coefficients of elements of the form \( a \text{tr}_G(b)c \) which are homogeneous polynomials in \( \zeta_{i,g} \) of degree \( m \) and length \( |G| \). It allows us to obtain some generic identities in the integral generic algebra \( Z_G \). Restricting the index set to \( I = \{1, 2, \ldots, |G|\} \) in Section 4 we obtain the following

**Theorem 4.2.** Let \( G \) be a finite group of order \( n \) and let \( \mathbb{Z}\langle \zeta_{i,g} \mid g \in G, \ i = 1, 2, \ldots, n \rangle \) be the integral generic algebra. Then there exists an integer \( v(G) > 0 \) divisible by \( n \), and such that for any \( g_1, g_2, \ldots, g_n \in G \)

\[
v(G) : \zeta_{i_1, g_1} \zeta_{i_2, g_2} \cdots \zeta_{i_n, g_n} = \sum_{i=1}^{N} \gamma_i a_i \text{tr}_G(b_i)c_i,
\]

where \( \gamma_1, \gamma_2, \ldots, \gamma_N \) are integers, and \( a_i, b_i, c_i \) are monomials in \( \zeta_{i,g} \) such that \( \deg(b_i) > 0 \) and \( \deg(a_i) + \deg(b_i) + \deg(c_i) = n \).

Notice that by the universal property of the algebra \( Z_G \) it follows that if the ring \( R \) is acted on by a finite group \( G \), then

\[
v(G) \cdot R^{|G|} \subseteq \langle \text{tr}_G(R) \rangle,
\]

where \( \langle X \rangle \) denotes the ideal of \( R \) generated by \( X \). Clearly this gives the best possible bound in Bergman-Isaacs theorem, which was a long standing problem.

First, we will briefly describe the idea of the proof. The relations (1.1) defining the trace subspace \( T_G(m) \) can be interpreted as linear equations in the vector space \( Q_G(m) \). With
any sequence \((i_1, i_2, \ldots, i_m)\) of indices one can associate the system of \(\binom{m+1}{2}|G|^m\) linear equations in \(|G|^m\) monomials of the form \(\omega = \zeta_{i_1, g_1} \zeta_{i_2, g_2} \cdots \zeta_{i_m, g_m}\) with coefficients from the set \(\{0, 1\}\). The aim is to prove that this system has a solution. Then any monomial \(\omega\) can be expressed as a linear combination of elements from \(\mathcal{F}\). It appears that the matrix of this system determines a Cayley graph \(\mathcal{G}_m(G) = \text{Cay}(G^m, \mathcal{S})\) of the Cartesian power \(G^m\) with a symmetric set \(\mathcal{S}\) canonically determined by \(G\) and \(m\). We transfer our problem to the question on estimation of minimal eigenvalues of graphs \(\mathcal{G}_m(G)\). As one can expect, since the graphs from \(\mathcal{G}\) contain all informations about the group \(G\), examining their spectra requires non-standard methods. Our analysis of the spectrum of \(\mathcal{G}_m(G)\) for small values of \(m\) \((m = 3, 4, 5)\) and for groups of small order allowed us to formulate a hypothesis that the ratio \(\lambda_{\text{max}}/|\lambda_{\text{min}}|\) of the largest and the smallest eigenvalues of the adjacency matrix of \(\mathcal{G}_m(G)\) is close to the maximal size of a clique of \(\mathcal{G}_m(G)\), which is equal \(\max(|G|, m+1)\), for \(m > 2\) (see \([1]\)). Note that testing our problem for the smallest non abelian simple group \(A_5\) goes beyond our technical capabilities. This requires examining the spectrum of a matrix of huge size \(60^{60} \times 60^{60}\), which is greater than \(2^{354} \times 2^{354}\). Fortunately, our attention was drawn by the result obtained by P. Delsarte \([3]\). He proved that if the graph \(\Gamma\) is \(k\)-regular and strongly regular, then the clique number \(\omega(\Gamma) \leq 1 - k/\lambda_{\text{min}}\). It was later extended by C. Godsil \([4]\) to distance regular graphs. This type of estimation for the clique number of an arbitrary regular graph turns out to be sufficient to solve our problem. And indeed, extending the ideas of \([3]\) and \([4]\) we are able we obtain in Section \(3\) the Delsarte bound for arbitrary regular graphs.

**Theorem 3.7.** The clique number \(\omega(\Gamma)\) of any \(k\)-regular graph \(\Gamma\) satisfies the inequality

\[
\omega(\Gamma) \leq \left\lfloor 1 - \frac{k}{\lambda_{\text{min}}} \right\rfloor,
\]

where \(\lambda_{\text{min}}\) is the smallest eigenvalue of the adjacency matrix of \(\Gamma\).

We expect that this bound for the clique number, applied to certain regular graphs associated with the family \(\mathcal{G}_m(G)\) (after modification of the symmetric set \(\mathcal{S}\)) will allow us to characterize other invariants of finite groups.

2. Reduction to Cayley graphs

Let \(G\) be a finite group and \(Q_G = \mathbb{Q}\langle \zeta_{i,g} \mid g \in G, i \in I \rangle\) be the free generic algebra over the field \(\mathbb{Q}\), defined in the introduction. For any positive integer \(m\) let \(Q_G(m)\) be the linear span over \(\mathbb{Q}\) of all monomials in \(\zeta_{i,g}\) of degree \(m\) and let \(T_G(m)\) be the trace subspace of \(Q_G(m)\). The aim of this section is to establish a reduction of Question \([1]\) to a question concerning spectra of suitable Cayley graphs.

First we will recall the notion of the Cayley graph. Let \(X\) be a group and \(S = S^{-1}\) be a symmetric subset of \(X^\times = X \setminus \{e\}\). The Cayley graph, denoted by \(\text{Cay}(X, S)\), is the graph whose vertex set is \(X\) and two vertices \(g, h\) are adjacent (we denote this by \(g \sim h\)) if \(hg^{-1} \in S\). Equivalently, \(g \sim h\) if and only if \(h = s \cdot g\) for some \(s \in S\).

The definition of the graph \(\mathcal{G}_m(G) = \text{Cay}(G^m, \mathcal{S})\) is based on the choice of the set \(\mathcal{S}\). The vertex set of \(\mathcal{G}_m(G)\) is the Cartesian power \(G^m\). For \(x \in G^x\) and \(1 \leq k < l \leq m + 1\),
we denote by $x_{[k,l]}$ the element
\[
\underbrace{(e, e, \ldots, e, x, x, \ldots, x, e, e, \ldots, e)}_{k-1 \text{ times} \quad \overbrace{\vdots}^{l-k \text{ times}}}
\]
of $G^m$. By $G[k,l]$ we denote the set of all elements $x_{[k,l]}$, where $x \in G^\omega$, and call it an
interval. The symmetric set $S$ is the union of all intervals:
\[
S = \bigcup_{1 \leq k < l \leq m+1} G[k,l].
\]
Thus if $g = (g_1, \ldots, g_m)$, $h = (h_1, \ldots, h_m)$ are two vertices of $G_m(G)$, then
\[
g \sim h \iff h = x_{[k,l]} \cdot g \text{ for some } x \in G^\omega \text{ and } 1 \leq k \leq l \leq m + 1.
\]
It is easy to see that the degree of any vertex $g \in G^m$ is equal to the size of $S$ which is \((m+1)(|G| - 1)\). Thus the graph $G_m(G)$ is regular.

For a fixed sequence $i = (i_1, i_2, \ldots, i_m)$ of elements of $I$ (not necessarily different) let
\[
\Omega_i = \{ \zeta_{i_1, g_1} \zeta_{i_2, g_2} \cdots \zeta_{i_m, g_m} \mid g_1, g_2, \ldots, g_m \in G \}.
\]
It is clear that $|\Omega_i| = |G|^m$. Notice that each monomial $\omega \in \Omega_i$ determines in a natural
way \((m+1)\) elements of $T_G(m)$. Indeed, each partition
\[
\{1, 2, \ldots, m\} = \{1, 2, \ldots, k - 1\} \cup \{k, k + 1, \ldots, l - 1\} \cup \{l, l + 1 \ldots, m\},
\]
where $1 \leq k < l \leq m + 1$ determines the element $a \text{tr}_G(b)c$, where $a = \zeta_{i_1, g_1} \cdots \zeta_{i_{k-1}, g_{k-1}}$, $b = \zeta_{i_k, g_k} \cdots \zeta_{i_{l-1}, g_{l-1}}$, $c = \zeta_{i_l, g_l} \cdots \zeta_{i_m, g_m}$ and $\omega = abc$.

We will interpret the identity
\[
\sum_{h \in G} \zeta_{i_1, g_1} \cdots \zeta_{i_{k-1}, g_{k-1}} \zeta_{i_k, h g_k} \cdots \zeta_{i_{l-1}, h g_{l-1}} \zeta_{i_l, g_l} \cdots \zeta_{i_m, g_m} = a \text{tr}_G(b)c
\]
as a linear equation in the set of variables $\Omega_i$. Our aim is to express any $\omega \in \Omega_i$ using the
elements of $T_G(m)$.

Therefore, since $a \text{tr}_G(b^c) = a \text{tr}_G(b)c$ for $x \in G$, the set $\Omega_i$ determines the system of
\((m+1)|G|^{m-1}\) linear equations in $|G|^m$ variables $\omega \in \Omega_i$. Let $B$ be the matrix of this system
(with respect to a fixed order of elements of $\Omega_i$ and equations \((2.2)\)). Clearly each entry of $B$ is either 0 or 1. Furthermore, each its row has exactly $|G|$ entries equal to 1, and in each of
its column the number 1 appears exactly \((m+1\times2)\)-times. Thus
\[
B^T B = \binom{m + 1}{2} I + A,
\]
where $A$ is a symmetric $|G|^m \times |G|^m$ matrix.

We will see that $A$ is the adjacency matrix of the Cayley graph $G_m(G)$. It is clear that
there is one to one correspondence
\[
\zeta_{i_1, g_1} \zeta_{i_2, g_2} \cdots \zeta_{i_m, g_m} \mapsto (g_1, g_2, \ldots, g_m)
\]
between elements of $\Omega_i$ and $G^m$. Let $\omega = \zeta_{i_1, g_1} \zeta_{i_2, g_2} \cdots \zeta_{i_m, g_m}$ and $\omega' = \zeta_{i_1, h_1} \zeta_{i_2, h_2} \cdots \zeta_{i_m, h_m}$. Clearly in the matrix $B^T B$ the $(\omega, \omega')$-position is equal to the product of the $\omega$-th column
by the $\omega'$-th column of $B$. Notice that the $\omega$-th column of $B$ has entries equal to 1 precisely
in the rows corresponding to equations (2.2). On the other hand two different monomials \( \omega \) and \( \omega' \) can appear simultaneously in at most one equation of the form (2.2). Therefore the product of the \( \omega \)-th column and the \( \omega' \)-th column is 1 only in the case when \( \omega \) and \( \omega' \) appear in the same equation, so when \( g \sim h \) in the graph \( \mathcal{G}_m(G) \). It means that \( A \) is the adjacency matrix of the graph \( \mathcal{G}_m(G) \).

By the Cauchy-Binet formula \( \det(B^TB) = \sum_i (\det B_i)^2 \), where the sum runs over all \( |G|^m \times |G|^m \) submatrices of \( B \). Thus if \( \det(B^TB) \neq 0 \), \( \det B_i \neq 0 \) for some submatrix \( B_i \). In this case the system of equations (2.2) has a unique solution, and therefore \( T_G(m) = R_G(m) \).

Let \( \lambda_{\min} \) be the smallest eigenvalue of \( A \). Since the matrix \( B^TB \) is positive semi-definite, by formula (2.3), it follows that

\[
\lambda_{\min} \geq -\binom{m+1}{2}.
\]

Therefore \( \det(B^TB) \neq 0 \) if and only if \( \lambda_{\min} > -\binom{m+1}{2} \).

The above discussion can be summarized as follows.

**Corollary 2.1.** If the smallest eigenvalue \( \lambda_{\min} \) of the adjacency matrix of the graph \( \mathcal{G}_m(G) \) satisfies the inequality

\[
\lambda_{\min} > -\binom{m+1}{2}
\]

then \( T_G(m) = Q_G(m) \).

In general, there is no simple explicit formula for computing eigenvalues of a Cayley graph. When the symmetric set \( S \subset G^\times \) is normal (that is \( s^g \in S \) for all \( s \in S, \ g \in G \)) the spectrum of \( \text{Cay}(G, S) \) can be computed explicitly in terms of the complex character values (see [12]). Namely, if \( \text{Irr}(G) = \{\chi_1, \ldots, \chi_t\} \) is the set of all irreducible characters of \( G \), then for \( j = 1, \ldots, t \)

\[
(2.4) \quad \lambda_j = \frac{1}{\chi_j(e)} \sum_{s \in S} \chi_j(s)
\]

are all eigenvalues of \( \text{Cay}(G, S) \). Moreover the multiplicity of \( \lambda_j \) is equal to \( \sum_{k: \chi_k = \lambda_j} \chi_k(e)^2 \).

But the symmetric set \( S \) of \( \mathcal{G}_m(G) \) is normal only in the case when the group \( G \) is abelian. Thus the above formulas for \( \lambda_j \) can be applied to abelian groups only. It is well known (see [7], Theorem 4.21) that if \( \text{Irr}(G) = \{\chi_1, \ldots, \chi_s\} \), then the set of all irreducible characters of \( G^m \) consists of the characters \( \chi = \chi_{i_1} \times \cdots \times \chi_{i_m} \) defined as:

\[
\chi(g_1, \ldots, g_m) = \chi_{i_1}(g_1) \cdots \chi_{i_m}(g_m),
\]

where \( \chi_{i_1}, \ldots, \chi_{i_m} \in \text{Irr}(G) \). Furthermore, if \( G \) is abelian then irreducible characters are linear and form the group \( \hat{G} = \text{Hom}(G, \mathbb{C}^\times) \cong G \). Let \( \chi_1 = 1 \) be the identity of \( \hat{G} \). The orthogonality relations for characters yield that if \( \chi_{j_1}, \ldots, \chi_{j_k} \in \hat{G} \), then

\[
(2.5) \quad \sum_{g \in G^\times} \chi_{j_1}(g) \cdots \chi_{j_k}(g) = \begin{cases} |G| - 1 & \text{if } \chi_{j_1} \cdots \chi_{j_k} = 1 \\ -1 & \text{if } \chi_{j_1} \cdots \chi_{j_k} \neq 1 \end{cases}
\]
Proposition 2.2. Let $G$ be a finite abelian group. If $m \geq |G|$, then each eigenvalue $\lambda$ of $G^m(G)$ satisfies the inequality

$$\lambda > -\left(\frac{m+1}{2}\right)$$

Therefore $T_G(m) = Q_G(m)$ for $m \geq |G|$.

Proof. Take $\chi = \chi_i_1 \times \cdots \times \chi_i_m \in \text{Irr}(G^m)$. Since $G$ is abelian, $\chi(e) = \chi_i_1(e) \cdots \chi_i_m(e) = 1$. According to (2.4) and (2.5) the eigenvalue $\lambda$ corresponding to $\chi$ is equal:

$$\lambda = \sum_{1 \leq k < l \leq m+1} \sum_{g \in G^x} \chi_i_k(e) \cdots \chi_i_{k-1}(e) \chi_i_k(g) \cdots \chi_i_{l-1}(g) \chi_i_l(e) \cdots \chi_i_m(e)$$

$$= \sum_{1 \leq k < l \leq m+1} \sum_{g \in G^x} \chi_i_k(g) \cdots \chi_i_{l-1}(g) = -\left(\frac{m+1}{2}\right) + n_{\chi}|G|,$$

where $n_{\chi}$ is the number of pairs $(k, l)$ such that $1 \leq k < l \leq m+1$ and $\chi_{i_k} \cdots \chi_{i_{l-1}} = 1$. If $m \geq |G|$, then the pigeonhole principle implies that some elements of the sequence

$$\chi_{i_1}, \chi_{i_1} \chi_{i_2}, \ldots, \chi_{i_1} \chi_{i_2} \cdots \chi_{i_m}$$

must be equal, that is $\chi_{i_k} \cdots \chi_{i_{l-1}} = 1$ for some $k < l$. Therefore $n_{\chi} > 0$, and hence $\lambda > -\left(\frac{m+1}{2}\right)$. $\square$

Unfortunately, the general case goes beyond this scheme. Therefore, in the next section we will deal with the problem of determining upper bounds for clique numbers of regular graphs in terms of their spectra.

3. Regular graphs and their clique numbers

Throughout this section all graphs are assumed to be finite, undirected, without loops or multiple edges. For a graph $\Gamma$ we let $V = V(\Gamma)$ and $E = E(\Gamma)$ denote the vertex and the edge sets, respectively. Whenever there is an edge between two vertices $x$ and $y$, we say that $x$ is adjacent to $y$, or that $x$ and $y$ are neighbours. The distance in the graph between two vertices $x$ and $y$ is denoted by $d(x, y)$, and is given by the length of the shortest path between $x$ and $y$. The diameter of a connected graph is the the number $D = \max_{x,y \in V} d(x, y)$.

The complement $\overline{\Gamma}$ of a graph $\Gamma$ has the same vertex set as $\Gamma$, while vertices $x$ and $y$ are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in $\Gamma$. Recall that the degree of a vertex $x \in V$, denoted by $\delta(x)$, is the number of neighbours of $x$. A graph in which every vertex has equal degree $k$ is called $k$-regular or shortly regular. For a matrix $A$ with rows and columns indexed by the vertex set $V$ by $A(x, y)$ we denote the entry corresponding to $(x, y)$-position.

The edge set $E(\Gamma)$ can be identified with a subset of the set $\mathcal{P}_2(V)$ of all 2-element subsets of $V$. Notice that any partition

$$(3.6) \quad \mathcal{P}_2(V) = E_1 \cup E_2 \cup \cdots \cup E_m$$
determines a family of graphs $\Gamma_i = (V, E_i)$. For $i = 1, \ldots, m$ we let $A_i$ denote the adjacency matrix of $\Gamma_i$, that is

$$A_i(x, y) = \begin{cases} 1, & \text{if } x \text{ and } y \text{ are adjacent in } \Gamma_i \\ 0, & \text{if } x \text{ and } y \text{ are not adjacent in } \Gamma_i \end{cases}$$

A natural example is a partition determined by the distance function in a connected graph $\Gamma = (V, E)$. Namely, for $i = 1, 2, \ldots, D$ the edge set $E_i$ consists of $x, y \in V$ such that $d(x, y) = i$. We will call it the distance partition.

Put $A_0 = I$ for the identity matrix and $A = \{A_0, A_1, \ldots, A_m\}$. Note that

$$A_0 + A_1 + A_2 + \cdots + A_m = J,$$

where $J$ is a matrix with every entry 1. The set $A$ generates the algebra $\mathbb{R}[A] \subseteq M_n(\mathbb{R})$ over the field of real numbers $\mathbb{R}$ and the algebra $\mathbb{C}[A] \subseteq M_n(\mathbb{C})$ over the field of complex numbers, where $v = \#V$ is the cardinality of the vertex set. We call these algebras the algebras associated with the partition. Notice that $\mathbb{C}[A]$ has a natural structure of a $\mathbb{Z}_2$-graded ring $\mathbb{C}[A] = \mathbb{C}[A]_0 \oplus \mathbb{C}[A]_1$, where $\mathbb{C}[A]_0 = \mathbb{R}[A]$ and $\mathbb{C}[A]_1 = i\mathbb{R}[A]$.

For a given graph $\Gamma = (V, E)$ the partition

$$\mathcal{P}_2(V) = E_1 \cup E_2 \cup \cdots \cup E_m$$

is said to be commuting $\Gamma$-partition, if $\Gamma_1 = \Gamma$ and the adjacency matrices $A_j$ of graphs $\Gamma_j = (V, E_j)$ commute. Then the associated algebras $\mathbb{R}[A]$ and $\mathbb{C}[A]$ are commutative and consist of symmetric matrices.

Now we will prove our key observation

**Proposition 3.1.** Every regular graph $\Gamma$ admits a commuting $\Gamma$-partition.

**Proof.** Suppose that $\Gamma = (V, E)$ is $k$-regular with the adjacency matrix $A \in M_n(\mathbb{R})$. We will prove that the partition $\mathcal{P}_2(V) = E \cup \overline{E}$ determines two commuting adjacency matrices. For $a, b \in V$, since $A = A^T$, we have

$$A\overline{A}(a, b) = \sum_{x \in V} A(a, x)\overline{A}(x, b) = \sum_{x \in V} A(a, x)\overline{A}(b, x)$$

$$= \#\{x \mid A(a, x) = \overline{A}(b, x) = 1\} = \#\{x \mid A(a, x) = 1 \text{ and } A(b, x) = 0\}$$

$$= \delta(a) - \#\{x \mid A(a, x) = A(b, x) = 1\}.$$

By the same reason we obtain $\overline{A}A(a, b) = \delta(b) - \#\{x \mid A(a, x) = A(b, x) = 1\}$. Since our graph is regular, we have $\delta(a) = \delta(b)$ and hence $A\overline{A} = \overline{A}A$. \hfill $\square$

**Remark 3.2.** 1. It is easy to see that the converse to Proposition 3.1 is true, i.e. the commuting $\Gamma$-partitions characterize only regular graphs. Indeed, if the matrices $A_i$ and $A_j$ commute, then $A = A_1$ and $J = \sum_{i=0}^m A_i$ commute. Notice that for any two vertices $a, b \in V$, $AJ(a, b) = \delta(a)$ and $JA(a, b) = \delta(b)$. Therefore the graph $\Gamma$ is regular.
2. There are large classes of regular graphs $\Gamma = (V, E)$ admitting a finer commuting $\Gamma$-partition than $E \cup E$. For instance this property is satisfied by distance partition connected with distance regular graphs \[5\]. For such graphs there exist non negative integers $a_i, b_i, c_i$ such that

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}$$

for $i = 1, 2, \ldots, D$. From this recurrence it follows that there exist polynomials $p_i$ such that $A_i = p_i(A)$. Thus the algebra $\mathbb{R}[\mathcal{A}]$ is commutative.

3. The distance regular graphs are a special case of so-called association schemes. The algebra $\mathbb{R}[\mathcal{A}]$ corresponding to the association scheme is called the Bose-Mesner algebra \[3\]. It is commutative and $(m + 1)$-dimensional with the canonical linear basis $\mathcal{A} = \{A_0, A_1, \ldots, A_m\}$. Thus $A_iA_j \in \text{span}_\mathbb{R}\{A_0, A_1, \ldots, A_m\}$ for all $i, j$.

The next interesting example is provided by Cayley graphs with normal symmetric subsets.

**Proposition 3.3.** Let $\Gamma = \text{Cay}(G, S)$ be the connected Cayley graph, where $G$ is a finite group and the symmetric subset $S$ of $G^\times$ is closed under conjugations. Then the algebra $\mathbb{R}[\mathcal{A}]$ determined by the distance partition is commutative.

**Proof.** For $g \in G$ and a positive integer $r$ let

$$S(g, r) = \{x \in G \mid d(g, x) = r\}$$

be a sphere with the center $g$ and the radius $r$. We will prove that for any vertices $g, h \in G$ and positive integers $i, j$

(3.7) $$\#(S(g, i) \cap S(h, j)) = \#(S(g, j) \cap S(h, i)).$$

First notice that if $x \sim y$, then $s = xy^{-1} \in S$ and

$$x^{-1} = y^{-1}s^{-1} = (y^{-1}s^{-1}y)y^{-1} = (s^{-1})yy^{-1}$$

Since $S$ is closed under conjugations, we obtain that $x \sim y$ if and only if $x^{-1} \sim y^{-1}$. Thus if

$$x = a_0 \sim a_1 \sim \cdots \sim a_i = y$$

is a path in $\Gamma$ connecting $x$ and $y$, then

$$x^{-1} = a_0^{-1} \sim a_1^{-1} \sim \cdots \sim a_i^{-1} = y^{-1}$$

is a path in $\Gamma$ connecting $x^{-1}$ and $y^{-1}$ of the same length. This yields that $d(x, y) = d(x^{-1}, y^{-1})$ for all $x, y \in G$.

Consider the map $\varphi : S(g, i) \cap S(h, j) \to G$ given by $\varphi(x) = gx^{-1}h$. It is clear that $\varphi$ is injective. Notice that for a given $g \in G$ the right transfer $x \mapsto xg$ is an automorphisms of the graph $\Gamma$. Thus, in particular $d(x, y) = d(xg, yg)$ for all $g, x, y \in G$. Next notice

$$d(h, \varphi(x)) = d(h, gx^{-1}h) = d(e, gx^{-1}) = d(x, g) = i$$

and

$$d(g, \varphi(x)) = d(g^{-1}, \varphi(x)^{-1}) = d(g^{-1}, h^{-1}xg^{-1}) = d(e, h^{-1}x) = d(x^{-1}, h^{-1}) = d(x, h) = j.$$
It means that \( \varphi(x) \in S(g, j) \cap S(h, i) \), and hence \( \#(S(g, i) \cap S(h, j)) \leq \#(S(g, j) \cap S(h, i)) \). Symmetrically, the map \( x \mapsto h^{-1}xg^{-1} \) provides an injection from \( S(g, j) \cap S(h, i) \) into \( S(g, i) \cap S(h, j) \). Therefore both sets have the same cardinality.

Since \( A_i(g, x)A_j(x, h) = 1 \) if and only if \( d(g, x) = i \) and \( d(x, h) = j \), that is \( x \in S(g, i) \cap S(h, j) \), \( \text{(3.7)} \) forces

\[
A_iA_j(g, h) = \sum_{x \in G} A_i(g, x)A_i(x, h) = \#(S(g, i) \cap S(h, j)) = \#(S(g, j) \cap S(h, i))
\]

\[
= \sum_{x \in G} A_j(g, x)A_i(x, h) = A_jA_i(g, h).
\]

Consequently, \( A_iA_j = A_jA_i \) for all \( i, j \) and hence the algebra \( \mathbb{R}[A] \) is commutative. \( \square \)

Now we will present the basic properties of algebras \( \mathbb{C}[A] \) and \( \mathbb{R}[A] \) associated to commuting \( \Gamma \)-partitions, where \( \Gamma = (V, E) \) is a regular graph. For matrices \( X, Y \in M_v(\mathbb{C}) \) we let \( X \odot Y \) denote the entrywise product, that is

\[
(X \odot Y)(a, b) = X(a, b)Y(a, b)
\]

and let \( \Sigma(X) \) be the sum of all elements of the matrix \( X \), that is

\[
\Sigma(X) = \sum_{(a, b) \in V \times V} X(a, b)
\]

Let \( \langle A, B \rangle = \text{Tr}(A^T B) \) be the inner product in \( M_v(\mathbb{C}) \) and let \( \|X\| = \sqrt{\langle X, X \rangle} \) be the norm of \( X \). Notice that for \( X, Y \in M_v(\mathbb{C}) \)

\[
\langle X, Y \rangle = \Sigma(X \odot Y)
\]

Below we collect fundamental properties of \( \mathbb{C}[A] \).

**Proposition 3.4.** Let \( \Gamma = (V, E) \) be a regular graph with a fixed commuting \( \Gamma \)-partition and an associated commutative algebra \( \mathbb{C}[A] \). Then

1. \( A_i \odot A_j = \delta_{ij}A_i \);
2. the set \( A = \{A_0, A_1, \ldots, A_m\} \) is linearly independent over \( \mathbb{C} \);
3. \( \sum_{i=0}^{m} A_i = J \), where \( J \) is the \( v \times v \) matrix with all entries equal to 1;
4. each matrix \( X \in \mathbb{C}[A] \) is symmetric, that is \( X^T = X \);
5. the algebra \( \mathbb{C}[A] \) is isomorphic to the product \( \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C} \).

**Proof.** The points (1)-(4) are straightforward. For (5) notice that \( \mathbb{C}[A] \) is reduced, i.e it does not contain nilpotent elements. Indeed, if \( X \) is such that \( X^2 = 0 \), then since \( \mathbb{C}[A] \) is a commutative subalgebra of \( M_v(\mathbb{C}) \) and \( XX = XX^T \), we may apply the non-degenerate inner product

\[
\langle XX, XX \rangle = \text{Tr}(XX^2) = 0,
\]

to conclude that \( XX = 0 \). This forces that \( \langle X, X \rangle = 0 \) and then \( X = 0 \). By Wedderburn theorem the algebra \( \mathbb{C}[A] \) being commutative, reduced and finite dimensional over \( \mathbb{C} \) is a direct product of finitely many copies of \( \mathbb{C} \). \( \square \)
The above implies that $\mathbb{C}[\mathcal{A}]$ contains a family $\{E_1, E_2, \ldots, E_n\}$ of minimal idempotents in $\mathbb{C}[\mathcal{A}]$ such that

\begin{equation}
E_1 + E_2 + \cdots + E_n = I \quad \text{and} \quad E_i E_j = \delta_{ij} E_i.
\end{equation}

We will prove that all $E_i$ are real matrices. This follows from the following general fact.

**Lemma 3.5.** Let $R$ be a finite dimensional commutative algebra over a field $F$, with a ring automorphism $\sigma$ of order 2 such that the norm $N(r) = r^\sigma r$ is non-degenerate (that is $N(r) = 0$ if and only if $r = 0$). Then each $F$-linear basis $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ of $R$ consisting of orthogonal idempotents is contained in $R^\sigma$, where $R^\sigma$ is the fixed subring for $\sigma$.

**Proof.** For a fixed idempotent $e_k$ there exist scalars $\alpha_j \in F$ such that $e_k^\sigma = \sum_{i=1}^n \alpha_i e_i$. The orthogonality relation yields $e_k^\sigma e_i = \alpha_k e_i$. On the other hand for any $k$, $f_k = e_k^\sigma e_k \in R^\sigma$ and $f_k$ is an idempotent of $R$ such that $f_k = \alpha_k e_k$. This forces that $\alpha_k^2 = \alpha_k$. Thus either $\alpha_k = 0$ or $\alpha_k = 1$. By assumption the norm $N$ is non-degenerate, so $\alpha_k \neq 0$. Consequently $e_k = f_k \in R^\sigma$. \qed

It is clear that the complex conjugation $X \mapsto \overline{X}$ is an automorphism of $\mathbb{C}[\mathcal{A}]$ of order 2 with $\mathbb{R}[\mathcal{A}]$ as a fixed subring and with a non-degenerate norm. Thus in light of Lemma 3.5 the family $\{E_1, E_2, \ldots, E_n\}$ is contained in $\mathbb{R}[\mathcal{A}]$. Consequently $\{E_1, E_2, \ldots, E_n\}$ is an $\mathbb{R}$-linear basis of $\mathbb{R}[\mathcal{A}]$. The algebra $\mathbb{R}[\mathcal{A}]$ consists of symmetric matrices and it is equipped with the real inner product

$$\langle X, Y \rangle = \text{Tr}(XY).$$

Since $$\langle E_i, E_j \rangle = \text{Tr}(E_i E_j) = \text{Tr}(\delta_{ij} E_i) = \delta_{ij} \|E_i\|^2$$
we have that $\{E_1, E_2, \ldots, E_n\}$ is an orthogonal basis of $\mathbb{R}[\mathcal{A}]$ with respect to the inner product.

In particular, each matrix $X \in M_n(\mathbb{R})$ has an orthogonal projection onto $\mathbb{R}[\mathcal{A}]$ given by the formula

\begin{equation}
\hat{X} = \sum_{i=1}^n \frac{\langle X, E_i \rangle}{\langle E_i, E_i \rangle} E_i
\end{equation}

It is also clear that for $i \neq j$

$$\langle A_i, A_j \rangle = \Sigma(A_i \circ A_j) = 0,$$

so the set $\{A_0, A_1, \ldots, A_m\}$ is orthogonal. It can be completed by some $B_{m+1}, \ldots, B_{n-1}$ to an orthogonal basis of $\mathbb{R}[\mathcal{A}]$. Hence there is a second formula for the projection of $X$ onto $\mathbb{R}[\mathcal{A}]$

\begin{equation}
\hat{X} = \sum_{i=1}^m \frac{\langle X, A_i \rangle}{\langle A_i, A_i \rangle} A_i + \sum_{i=m+1}^{n-1} \frac{\langle X, B_i \rangle}{\langle B_i, B_i \rangle} B_i.
\end{equation}
Proof. According to (3.8) we can write

\[ c_x = \begin{cases} 
1, & \text{if } x \in C \\
0, & \text{if } x \notin C 
\end{cases} \]

Then \( C = c^T c \) is a \( v \times v \) matrix such that

\[ C(x, y) = \begin{cases} 
1, & \text{if } x \in C \text{ and } y \in C \\
0, & \text{if } x \notin C \text{ or } y \notin C 
\end{cases} \]

Notice that \( \|C\| = \sqrt{\langle C, C \rangle} = \sqrt{\Sigma(C \circ C)} = c. \) Furthermore, for any \( x \in \mathbb{R}^v \)

\[ xCx^T = x(c^T c)x^T = (xc^T)(cx^T) = (xc^T)^2 \geq 0. \]

Hence the matrix \( C \) is positive semi-definite.

**Proposition 3.6.** Let \( \Gamma = (V, E) \) be a regular graph with a fixed commuting \( \Gamma \)-partition and an associated commutative algebra \( \mathbb{R}[A] \). Suppose that \( C \) is a matrix associated with a \( c \)-element clique \( C \) of \( \Gamma \). Then there exist parameters \( \alpha, \beta \in \mathbb{R} \) and a matrix \( B \in \text{span}_\mathbb{R}\{B_{m+1}, \ldots, B_{n-1}\} \) such that

1. the orthogonal projection \( \hat{C} \) of \( C \) onto \( \mathbb{R}[A] \) has the form \( \hat{C} = \alpha I + \beta A + B \),
2. \( \text{Tr}(B) = 0, \Sigma(B) = 0, \)
3. \( \frac{\Sigma(\hat{C})}{\text{Tr}(\hat{C})} = c, \)
4. the matrix \( \hat{C} \) is positive semi-definite.

**Proof.** According to (3.8) we can write

\[ \text{Tr}(\hat{C}) = \langle \hat{C}, I \rangle = \sum_{i=1}^n \langle \hat{C}, E_i \rangle = \sum_{i=1}^n \langle \sum_{j=1}^n \langle C, E_j \rangle E_j, E_i \rangle = \sum_{i=1}^n \langle C, E_i \rangle = \langle C, I \rangle = \text{tr}(C) = c. \]

Since \( C \circ A_i = 0 \) for \( i \geq 2 \), by (3.10) it follows that

\begin{equation} 
(3.11) \quad \hat{C} = \frac{\langle C, I \rangle}{\langle I, I \rangle} I + \frac{\langle C, A \rangle}{\langle A, A \rangle} A + B = \frac{c}{v} I + \frac{\langle C, A \rangle}{\langle A, A \rangle} A + B,
\end{equation}

where \( B \in \text{span}_\mathbb{R}\{B_{m+1}, \ldots, B_{n-1}\} \). Thus we can put \( \alpha = \frac{c}{v} \) and \( \beta = \frac{\langle C, A \rangle}{\langle A, A \rangle} \). Since \( \text{Tr}(A) = 0 \), we obtain \( \text{Tr}(\hat{C}) = c + \text{Tr}(B) \), so \( \text{Tr}(B) = 0 \). Next we compute the sum \( \Sigma(\hat{C}) \).

\[ \Sigma(\hat{C}) = \langle \hat{C}, J \rangle = \langle \hat{C}, \sum_{j=0}^m A_j \rangle = \sum_{j=0}^m \langle \hat{C}, A_j \rangle = \sum_{j=0}^m \langle \sum_{i=0}^m \frac{\langle X, A_i \rangle}{\langle A_i, A_i \rangle} A_i, A_j \rangle + \sum_{j=0}^m \langle \sum_{i=m+1}^{m+n-1} \frac{\langle X, B_i \rangle}{\langle B_i, B_i \rangle} B_i, A_j \rangle = \sum_{i=0}^m \langle C, A_i \rangle = \langle C, J \rangle = \Sigma(C) = c^2. \]
Notice that $\langle C, A \rangle = \Sigma(C \circ A) = c(c - 1)$ and $\langle A, A \rangle = \Sigma(A \circ A) = \Sigma(A)$, so by (3.11) we see that $\Sigma(B) = 0$.

It remains to check that $\hat{C}$ is positive semi-definite. To this end take $x \in \mathbb{R}^v$. First notice that for any $i = 1, 2, \ldots, n$

$$xE_i x^T = x(E_i x x^T) = (x E_i)(x E_i)^T = \|x E_i\|^2_v \geq 0,$$

where $\|x\|_v = \sqrt{\sum_i x_i^2}$ is the Euclidean norm in $\mathbb{R}^v$. We also have

$$\langle c^T c, E_i \rangle = \text{Tr}(c^T c E_i) = \text{Tr}(E_i (c^T c) E_i) = \langle c E_i, c E_i \rangle_v = \|c E_i\|_v^2 \geq 0.$$

Finally combining these with (3.9) yields

$$x \hat{C} x^T = \sum_{i=1}^m \frac{\langle c^T c, E_i \rangle}{\|E_i\|^2_v} x E_i x^T \geq 0$$

and the result follows.

We are able to obtain the main result of this section.

**Theorem 3.7.** The clique number $\omega(\Gamma)$ of any $k$-regular graph $\Gamma$ satisfies the inequality

$$\omega(\Gamma) \leq 1 - \left\lfloor \frac{k}{\lambda_{\min}} \right\rfloor,$$

where $\lambda_{\min}$ is the smallest eigenvalue of the adjacency matrix of $\Gamma$.

**Proof.** In light of Proposition 3.6 we can take a commutative algebra $\mathbb{R}[\mathcal{A}]$ associated to some commuting $\Gamma$-partition. Suppose $\Gamma$ contains a clique $C$ of size $c$. Let $\mathcal{R}$ be the set of all matrices $M \in \mathbb{R}[\mathcal{A}]$ such that

- $M = \alpha I + \beta A + B$ for some real parameters $\alpha$ and $\beta$,
- $\text{Tr}(B) = 0$, $\Sigma(B) = 0$,
- $M$ is positive semi-definite.

By Proposition 3.6 the family $\mathcal{R}$ is non empty and

$$c \leq \sup_{M \in \mathcal{R}} \frac{\Sigma(M)}{\text{Tr}(M)} = \sup_{M \in \mathcal{R}} \frac{\alpha v + \beta k v}{\alpha v} = \sup_{M \in \mathcal{R}} \left(1 + k \frac{\beta}{\alpha}\right).$$

Notice that values of the function $\Theta(\alpha, \beta) = 1 + k \frac{\beta}{\alpha}$ depend only on the ratio $\frac{\beta}{\alpha}$. Moreover, if the matrix $M$ belongs to $\mathcal{R}$, then $|\beta|^{-1} M \in \mathcal{R}$. Thus to maximize the value of $\Theta(\alpha, \beta)$ we may assume that either $\beta = -1$ or $\beta = 1$. First suppose that $\beta = -1$. Since $M = \alpha I - A + B$ is positive semi-definite, all eigenvalues of $M$ must be non negative. But $A$ and $B$ commute, so they have common eigenvectors for any pair of respective eigenvalues. Thus if $\lambda$ and $\mu$ are eigenvalues of $A$ and $B$ respectively, then $\alpha - \lambda + \mu$ is an eigenvalue of $M$. Let $\lambda_{\max}$ and $\mu$ be the largest and smallest eigenvalues of $A$ and $B$ respectively. Since $\text{Tr}(B) = 0$, we have $\mu \leq 0$. Thus $\alpha - \lambda_{\max} + \mu \geq 0$ implies that $\alpha \geq \lambda_{\max} - \mu \geq \lambda_{\max} > 0$. In this case $\Theta(\alpha, -1) = 1 - \frac{k}{\alpha} < 1$. 

Let $\beta = 1$. By the same reason as above, since $M = \alpha I + A + B$ is positive semi-definite, we have $\alpha + \lambda_{\min} + \mu \geq 0$. Again since $\mu \leq 0$, we have $\alpha \geq -\lambda_{\min}$ and $\Theta(\alpha, 1) \leq 1 - \frac{k}{\lambda_{\min}}$.

We conclude therefore that in both cases $c \leq 1 - \frac{k}{\lambda_{\min}}$. This finishes the proof. \hfill \Box

4. The main result

In this last short section we prove the main result and we discuss some of its consequences.

**Theorem 4.1.** For a given finite group $G$ if $m \geq |G|$, then $T_G(m) = Q_G(m)$.

**Proof.** By Theorem 3.7 if $\omega$ is the size of a clique in the graph $G_m(G)$, then

$$\lambda_{\min} \geq \frac{k}{1 - \omega}.$$  

Notice that if $m \geq |G|$ and $g \neq e$, then the set

$$\mathcal{C} = \{e\} \cup \{g_{i,j} \mid j = 2, 3, \ldots, m + 1\}$$

is an $(m + 1)$-element clique in $G_m(G)$. Since $k = \frac{(m+1)}{2}(|G| - 1)$, we obtain for $m \geq |G|$ and $\omega = m + 1$

$$\lambda_{\min} \geq \frac{(m+1)(|G| - 1)}{1 - \omega} = \frac{(m+1)(|G| - 1)}{m} > \left(\frac{m+1}{2}\right).$$

By Corollary [2.1] this finishes the proof. \hfill \Box

In light of Theorem 4.1 it is reasonable to restrict the index set to $I = \{1, 2, \ldots, n\}$.

**Theorem 4.2.** Let $G$ be a finite group of order $n > 1$ and let $\mathbb{Z}\langle\zeta_{i,g} \mid g \in G, i = 1, 2, \ldots, n\rangle$ be the integral free generic algebra. Then there exists a positive integer $v(G)$ divisible by $n$, and such that for any $g_1, g_2, \ldots, g_n \in G$

$$v(G) \cdot \zeta_{1,g_1} \zeta_{2,g_2} \cdots \zeta_{n,g_n} = \sum_{i=1}^{N} \gamma_i \cdot a_i \text{tr}_G(b_i)c_i,$$

where $\gamma_1, \gamma_2, \ldots, \gamma_N$ are integers, and $a_i, b_i, c_i$ are monomials in $\zeta_{i,g}$ such that $\deg(b_i) > 0$ and $\deg(a_i) + \deg(b_i) + \deg(c_i) = n$.

**Proof.** Recall that the vector space $Q_G(n)$ is spanned by all monomials $\zeta_{i_1,g_1} \zeta_{i_2,g_2} \cdots \zeta_{i_n,g_n}$, where $g_1, g_2, \ldots, g_n \in G$. By Theorem 4.1 $Q_G(n)$ has the spanning set $\mathcal{T}$ consisting of elements of the form $a \text{tr}_G(b)c$ such that

$$(4.12) \quad a = \zeta_{i_1,g_1} \cdots \zeta_{i_k,g_{k-1}}, \quad b = \zeta_{i_k,g_k} \cdots \zeta_{i_l-1,g_{l-1}}, \quad c = \zeta_{i_l,g_l} \cdots \zeta_{i_n,g_n}$$

where $1 \leq k < l \leq n + 1$.

Let $Q^n$ be the vector space spanned by the vertex set of $G_n(G)$. Clearly, it is an $n^n$-dimensional vector space over $\mathbb{Q}$, with basis $\{g \mid g \in G^n\}$. Notice that the mapping

$$\iota_n : (g_1, g_2, \ldots, g_n) \mapsto \zeta_{i_1,g_1} \zeta_{i_2,g_2} \cdots \zeta_{i_n,g_n}$$

provides a natural isomorphism between vector spaces $Q^n$ and $Q_G(n)$. Moreover any graph automorphism $\varphi \in \text{Aut}(G_m(G))$ induces an automorphism $\widehat{\varphi} = \iota_n \varphi \iota_n^{-1}$ of $Q_G(n)$.  


The important thing is that \( \hat{\varphi} \) preserves the spanning set \( \mathcal{T} \). To see this, notice that if \( a, b, c \) are as in (1.12), then
\[
\iota_n^{-1}(a \text{tr}_G(b)c) = \sum_{x \in G} x_{[k,l]}g
\]
Observe that a maximal \( n \)-element clique around the vertex \( g \) of \( G_n(G) \) is of the form
\[
\mathcal{C} = \{x_{[k,l]}g \mid x \in G\}, \quad 1 \leq k < l \leq n + 1 \quad (\text{c.f. [1]}).
\]
The image \( \varphi(\mathcal{C}) \) should be a maximal \( n \)-element clique around \( \varphi(g) \), as well. As a consequence, there exist \( 1 \leq s < t \leq n + 1 \) such that \( \varphi(\mathcal{C}) = \{x_{[s,t]}\varphi(g) \mid x \in G\} \). Now it is clear that
\[
\varphi\iota_n^{-1}(a \text{tr}_G(b)c) = \sum_{x \in G} \varphi(x_{[k,l]}g) = \sum_{x \in G} x_{[s,t]}\varphi(g)
\]
and hence \( \hat{\varphi}(a \text{tr}_G(b)c) \in \mathcal{T} \).

Fix \( g_1, g_2, \ldots, g_n \in G \) and take a minimal positive integer \( v \) such that
\[
(4.13) \quad v \cdot \zeta_{1,g_1}\zeta_{2,g_2} \ldots \zeta_{n,g_n} = \sum_{j=1}^N \gamma_j \cdot a_j \text{tr}_G(b_j)c_j
\]
where \( \gamma_1, \gamma_2, \ldots, \gamma_N \) are integers, and \( a_j \text{tr}_G(b_j)c_j \in \mathcal{T} \). Since the right transfer \( T_h : x \mapsto xh \) is an automorphism of \( G_n(G) \), we may apply the automorphism \( \hat{T}_g^{-1}h \) to both sides of (4.13) to get an analogous identity for \( \zeta_{1,h_1}\zeta_{2,h_2} \ldots \zeta_{n,h_n} \) (where \( h = (h_1, h_2, \ldots, h_n) \in G^n \)) with the same coefficient \( v \). Consequently, the coefficient \( v = \varphi(G) \) does not depend on the sequence \( g_1, g_2, \ldots, g_n \in G \).

Notice that the coefficient \( v(G) \) is divisible by the order \( |G| \). To see this, it is enough to consider the trivial action of \( G \) on \( \mathbb{Z} \). Then the mapping \( \theta : \zeta_{i,g} \mapsto 1 \) has an extension to the \( G \)-homomorphism \( \theta : Z_G \to \mathbb{Z} \). Applying \( \theta \) to both sides of the identity (4.13) yields
\[
v(G) = \sum_{j=1}^N \gamma_j \cdot |G| = |G| \cdot \sum_{j=1}^N \gamma_j.
\]

**Example 4.3.** In this example, we present a generic identity for the cyclic group \( G = \{e, g, g^2\} \) of order 3. It has been determined by inverting a suitable \( 27 \times 27 \) submatrix \( B_i \) of the matrix of the system of equations (2.2).

\[
\begin{align*}
9 \cdot \zeta_{1,e}\zeta_{2,e}\zeta_{3,e} &= 3 \cdot \zeta_{1,e}\zeta_{2,e}\text{tr}[\zeta_{1,e}] + 5 \cdot \zeta_{1,e}\text{tr}[\zeta_{2,e}]\zeta_{3,e} - 3 \cdot \zeta_{1,e}\text{tr}[\zeta_{2,e}\zeta_{1,e}] + 4 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{1,e}] \\
- &- 5 \cdot \zeta_{1,e}\zeta_{3,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}] + 4 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] - 2 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] \\
- &- 4 \cdot \text{tr}[\zeta_{1,e}\zeta_{3,g}\zeta_{2,e}] - 5 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + 3 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + 2 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] \\
+ &+ 3 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + 5 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] - 5 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + 5 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] \\
- &- 2 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] - 3 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] + 5 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] - 4 \cdot \text{tr}[\zeta_{1,e}\zeta_{2,e}\zeta_{3,g}] \\
- &- \zeta_{1,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}] + 5 \cdot \zeta_{1,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}] - \zeta_{1,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}] - 3 \cdot \zeta_{1,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}] \\
- &- 4 \cdot \zeta_{1,g}\text{tr}[\zeta_{2,e}\zeta_{3,g}]
\end{align*}
\]
Remark 4.4. Unfortunately, we were not able to find the exact value of the \( \upsilon(G) \) factor. Applying the famous Hadamard’s inequality

\[
| \det(A) | \leq \prod_{i=1}^{k} \| a_i \|
\]

(where \( A \) is a \( k \times k \) real matrix with rows \( a_1, a_2, \ldots, a_k \in \mathbb{R}^k \)) to a suitable \( n \times n \) submatrix \( B_i \) of the matrix \( B \) of the system \((2.2)\), gives us \( | \det(B_i) | \leq (\sqrt{n})^n \). Since \( \upsilon(G) \) is a divisor of \( \det(B_i) \), one obtains that \( \frac{\upsilon(G)}{n} \) is a positive integer and

\[
n \leq \upsilon(G) \leq (\sqrt{n})^n.
\]

Notice that from Theorem 4.1 it follows that for any \( m \geq n \) in the generic algebra \( \mathbb{Z} \langle \zeta_{i,g} \mid g \in G, i = 1, 2, \ldots, m \rangle \) the identity of the form

\[
\upsilon_m(G) \cdot \zeta_{1,g_1} \zeta_{2,g_2} \ldots \zeta_{m,g_m} = \sum_{i=1}^{N_m} \gamma_{i,m} a_{i,m} \text{tr}_G(b_{i,m}) c_{i,m},
\]

is satisfied (where \( \upsilon_m(G) \) is a positive integer, possibly smallest). It is also clear that \( \upsilon_m(G) \leq \upsilon(G) = \upsilon_n(G) \), because an identity for \( m \) can be obtained from the identity for \( n \) by multiplication by some \( \zeta_{i,g} \)'s on the right side. Of course, the sequence \( \{ \upsilon_m(G) \}_{m \geq n} \) must stabilize from a certain point \( m_0 \), that is \( \upsilon_m(G) = \upsilon_0 \) for \( m \geq m_0 \). On the other hand from the Bergman-Isaacs Theorem it follows that for sufficiently large \( m \) there are generic identities with factors \( \upsilon_m(G) \) dividing the power \( n^{f(m)} \). Therefore the smallest factor \( \upsilon_0 \) is a divisor of some power \( n^{k} \). We conjecture that \( \upsilon(G) \) is also a divisor of \( n^{k} \). \( \square \)

Finally, we will apply Theorem 4.2 to a concrete ring \( R \) acted on by a finite group \( G \). For any sequence \( r_1, r_2, \ldots, r_n \in R \), where \( n = |G| \), the mapping \( \theta: \zeta_{i,g} \mapsto r_i^g \) has an extension to a \( G \)-homomorphism of rings

\[
\theta: \mathbb{Z} \langle \zeta_{i,g} \mid g \in G, i = 1, 2, \ldots, n \rangle \rightarrow R
\]

By Theorem 4.2 \( \upsilon(G) \cdot r_1 r_2 \ldots r_n \in \langle \text{tr}_G(R) \rangle \). Therefore we have

**Corollary 4.5.** If \( G \) is a finite group of automorphisms of the ring \( R \), then

\[
\upsilon(G) \cdot R^{|G|} \subseteq \langle \text{tr}_G(R) \rangle.
\]

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Faculty of Computer Science, Białyce University of Technology, Wiejska 45A, 15-351 Białyce, Poland

E-mail address: p.grzeszczuk@pb.edu.pl