ABELIAN INVARIANTS AND A REDUCTION THEOREM FOR
THE MODULAR ISOMORPHISM PROBLEM

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Abstract. We show that elementary abelian direct factors can be disregarded
in the study of the modular isomorphism problem. Moreover, we obtain four
new series of abelian invariants of the group base in the modular group algebra
of a finite \( p \)-group. Finally, we apply our results to new classes of groups.

Introduction

Given a field \( F \) of positive characteristic \( p \) and a finite \( p \)-group \( G \), the modular
group algebra \( FG \) of \( G \) over \( F \) plays a fundamental role in studying linear repre-
sentations of \( G \) over \( F \). The interplay between the group structure of \( G \) and the
algebra structure of \( FG \) has been thoroughly studied and is yet still not completely
understood. It is obvious that isomorphic finite \( p \)-groups \( G \cong H \) define isomorphic
finite-dimensional algebras \( FG \cong FH \). The modular isomorphism problem (MIP)
asks whether the converse holds. In symbols, this reads

\[ FG \cong FH \implies G \cong H? \]

Over the decades, the last question has been positively answered for finite \( p \)-groups
of small orders and for special classes of them (we refer to \([25, 15, 8]\) for an overview
of most known results, while more recent contributions are \([3, 24, 19, 5, 20]\)).
However, a quite recent breakthrough \([12]\) found the first counterexample to the
modular isomorphism problem for the case \( p = 2 \). Having said that, the modular
isomorphism problem is still open for odd primes and it is far from being solved
in general. For example, as of today, no structural reduction theorems are known.

In this article we prove the following result reducing the modular isomorphism
problem to finite \( p \)-groups free of elementary abelian direct factors.

**Theorem A.** Let \( F \) be a field of positive characteristic \( p \) and let \( G \) and \( H \) be
finite \( p \)-groups. Let, moreover, \( E \) be a finite elementary abelian \( p \)-group. Then
an algebra isomorphism $F[E \times G] \cong F[E \times H]$ implies an algebra isomorphism $FG \cong FH$, in symbols

$$F[E \times G] \cong F[E \times H] \implies FG \cong FH.$$ \hfill (*)

Theorem A is a simplified instance of Theorem 4.1 and is proven in Section 4. One ingredient in the proof of the reduction theorem is that the isomorphism type of the intersection of the socle with the Frattini subgroup of $G$ is an invariant of $FG$. This last invariant can be derived from the following Theorem B; see in particular Corollary 2.8.

Let $Z(G)$ and $\gamma(G)$ denote the center and commutator subgroup of $G$, respectively. Moreover, for a non-negative integer $n$, set

$$\Omega_n(G) = \langle g \in G \mid g^{p^n} = 1 \rangle$$

and

$$\mathfrak{U}_n(G) = \langle g^{p^n} \mid g \in G \rangle.$$ \hfill (**) \hfill (**)

**Theorem B.** Let $F$ be a field of positive characteristic $p$ and let $G$ be a finite $p$-group. Let, moreover, $n$ be a non-negative integer. Then the isomorphism types of the following are invariants of $FG$:

1. $G/\gamma(G)\Omega_n(Z(G))$,
2. $\gamma(G)\Omega_n(Z(G))/\gamma(G)$,
3. $Z(G) \cap \mathfrak{U}_n(G)\gamma(G)$,
4. $Z(G)/Z(G) \cap \mathfrak{U}_n(G)\gamma(G)$.

The search for invariants is, if possible, even more motivated by the discovery of a counterexample to the modular isomorphism problem. Indeed, such invariants provide obstructions to the coexistence of two non-isomorphic group bases in the same modular group algebra.
We apply our results to obtain positive results on the modular isomorphism problem for two new classes of finite 2-groups both of which share certain properties with the known counterexamples; cf. Remark 5.11.

We stress that all our results are independent of the choice of the field, in particular, they are not confined to the case where \( F = \mathbb{F}_p \) is the field of \( p \) elements (note that this is the case for several existing results on the modular isomorphism problem). Moreover, our results would be new also in that framework.

This article is organized as follows. In Section 1 we introduce the notation and recall some standard results. Section 2 is devoted to establishing four new series of abelian invariants of group bases of modular group algebras, the main results being Theorems 2.1 and 2.4. Section 3 deals with elementary decompositions of finite \( p \)-groups from a purely group-theoretic point of view. The reduction, Theorem 4.1, is proved in Section 4 using one of the new abelian invariants and the decomposition from Section 3. Two concrete applications to the modular isomorphism problem, which are independent of each other, are given in Section 5.

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1. Preliminaries and notation

Throughout the whole paper, \( p \) denotes a prime number, \( G \) and \( H \) finite \( p \)-groups and \( F \) a field of characteristic \( p \).

1.1. Groups. Denote by \( \Phi(G) \) the Frattini subgroup of \( G \) and by \( \Psi(G) \) the socle of \( G \), i.e. the subgroup of \( G \) that is generated by central elements of prime order. The set of conjugacy classes of \( G \) is denoted \( \text{cc}(G) \) and for \( g, h \in G \) we write \( [g, h] = g^{-1}h^{-1}gh \) for the commutator of \( g \) and \( h \). The commutator subgroup of \( G \) is denoted by \( \gamma(G) \), while \( Z(G) \) denotes the center of \( G \). Moreover, for every non-negative integer \( n \), we define the following crucial players of this paper:

\[
\begin{align*}
(1) \quad \mathcal{U}_n(G) &= \langle g^{p^n} \mid g \in G \rangle, \\
(2) \quad \mathcal{U}_n^*(G) &= \mathcal{U}_n(G)\gamma(G), \\
(3) \quad \Omega_n(G) &= \{ g \in G \mid g^{p^n} = 1 \}, \\
(4) \quad \Omega_n^*(G) &= \Omega_n(Z(G)).
\end{align*}
\]

For instance, with this notation, we have that

\[
\mathcal{U}_n(G/\gamma(G)) = \mathcal{U}_n^*(G)/\gamma(G), \quad \Phi(G) = \mathcal{U}_1^*(G), \quad \text{and} \quad \Psi(G) = \Omega_1^*(G).
\]

The following result is a direct consequence of the classification of finite abelian \( p \)-groups.
Proposition 1.1. Assume that $G$ and $H$ are abelian. Then $G$ and $H$ are isomorphic if and only if $(|\bar{\Omega}_n(G)|)_{n \geq 0} = (|\bar{\Omega}_n(H)|)_{n \geq 0}$.

Proof. This readily follows from [18, Chapter II, (1.4)]. □

1.2. Algebras. Let $A$ be a finite-dimensional algebra over $F$. Analogously to the notation for groups, we let $\gamma(A) = \sum_{x,y \in A} F(xy - yx)$ denote the commutator subspace of $A$ and $Z(A)$ the center of $A$. We use the following notation for a non-negative integer $n$ and a subspace $X$ of $A$:

\begin{enumerate}
  \item $\bar{\Omega}_n(X) = \{ x \in X | x^p = 0 \}$,
  \item $\Omega_n(X) = \{ x \in X | x^p = 0 \}$.
\end{enumerate}

The codimension of $X$ is $\text{codim} X = \dim A - \dim X$, where the dimensions are taken as vector spaces over $F$.

The following fact on power maps can be found in [22, Chapter 2, Lemma 3.1].

Lemma 1.2. Let $A$ be a finite-dimensional algebra over $F$ and $n$ a non-negative integer. Then

$$(x + y)^p \equiv x^p + y^p \mod \gamma(A)$$

for every pair of elements $x$ and $y$ in $A$.

1.3. Group algebras. The group $H$ is a group base of $FG$ if $H$ consists of units in $FG$ and $H$ is a basis of $FG$. An object $\mathcal{P}(G)$ associated with $G$ is said to be an invariant of $FG$ if $FG \cong FH$ implies that $\mathcal{P}(G) = \mathcal{P}(H)$. A subset $X(FG)$ of $FG$ is said to be canonical in $FG$ if there is a formula $\phi(x)$ in the language of algebras such that $X(FG)$ consists of the elements of $FG$ satisfying $\phi(x)$. Then every algebra isomorphism $FG \to FH$ maps $X(FG)$ to $X(FH)$. The augmentation ideal of $FG$ is

$$\Delta(FG) = \bigoplus_{g \in G \setminus \{1\}} F(g - 1).$$

It is well-known that the augmentation ideal equals the unique maximal ideal in $FG$, cf. [22, Chapter 8, Lemma 1.17]. In particular, $\Delta(FG)$ is canonical in the modular group algebra. Finally, for a conjugacy class $\kappa$ of $G$, we define the class sum of $\kappa$ in $FG$ as

$$\hat{\kappa} = \sum_{g \in \kappa} g.$$
Lemma 1.4. Let $I$ be an ideal of $FG$. Then the following are equivalent:

1. $FG/I$ is commutative.
2. $I \supseteq \gamma(FG)$.
3. $I \supseteq \Delta(F\gamma(G))$.

Proof. To see that (1) implies (2) note that $FG/I$ being commutative implies that $I$ contains all elements of shape $xy - yx$ for elements $x$ and $y$ of $FG$. These elements are exactly the generators of $\gamma(FG)$. Moreover, (2) and (3) are equivalent by [25, Lemma 3.5]. Finally, (3) implies (1) by Lemma 1.3.

Lemma 1.5. Let $I$ be an ideal of $FG$. Let, moreover, $K$ and $L$ be normal subgroups of $G$. Then the following are equivalent:

1. $I \supseteq \Delta(FKL)$.
2. $I \supseteq \Delta(FK)$ and $I \supseteq \Delta(FL)$.

Proof. The implication (1) $\Rightarrow$ (2) is clear. To see that (2) implies (1), note that, for any choice of $x, y \in G$, one has $xy - 1 = (x - 1)(y - 1) + (x - 1) + (y - 1)$. □

Lemma 1.6. The following facts hold in $FG$:

1. $\gamma(FG) \subseteq \Delta(FG)^2$,
2. $Z(FG) = \bigoplus_{\kappa \in \text{cc}(G)} F\hat{\kappa} = \bigoplus_{z \in Z(G)} Fz \bigoplus \bigoplus_{\kappa \in \text{cc}(G), |\kappa| \neq 1} F\hat{\kappa}$,
3. $Z(FG) \cap \gamma(FG) = \bigoplus_{\kappa \in \text{cc}(G), |\kappa| \neq 1} F\hat{\kappa},$
4. $Z(FG) \cap \gamma(FG)$ is an ideal of $Z(FG)$,
5. $Z(FG) = FZ(G) \oplus (Z(FG) \cap \gamma(FG))$.

Proof. (1) Note that, for every $x, y \in FG$, one has $xy - yx = (x - 1)(y - 1) - (y - 1)(x - 1)$. The claim now follows from $FG = F \oplus \Delta(FG)$. Points (2) to (5) are direct consequences of [27, Section III.6]. □

Finally, we state some classical results on the modular isomorphism problem: the invariance of the center due to Ward and Sehgal [27, Chapter III, Theorem 6.6] and the invariance of the Frattini quotient [22, Chapter 14, Lemma 2.7].

Theorem 1.7. The isomorphism type of the center $Z(G)$ is an invariant of $FG$.

Proposition 1.8. The isomorphism type of the Frattini quotient $G/\Phi(G)$ is an invariant of $FG$.

2. Abelian invariants

In this section we present new classes of abelian invariants of the modular group algebra of a finite $p$-group, namely those from Theorem B. This theorem will be obtained from the combination of Theorems 2.1 and 2.4, which are dual to each
other in the following sense. While Theorem 2.1 is a result of filtering the abelianization $G/\gamma(G)$ taking products with the terms of $(\Omega^*_n(G))_{n \geq 0}$, in Theorem 2.4 the center $Z(G)$ is filtered via intersections with the terms of $(\Omega^*_n(G))_{n \geq 0}$.

2.1. Commutator subgroups and socles. We now establish the first pair of abelian invariants from Theorem B. We will do so relying on Proposition 1.1 and identifying canonical ideals in the modular group algebra reflecting properties of the considered group quotients: the analogy is drawn in Figure 2.

**Theorem 2.1.** Let $n$ be a non-negative integer. Then the isomorphism types of the following are invariants of $FG$:

1. $G/\gamma(G)\Omega^*_n(G)$,
2. $\gamma(G)\Omega^*_n(G)/\gamma(G)$.

*Figure 2. Corresponding ideals and subgroups.*

**Lemma 2.2.** Let $m$ and $n$ be non-negative integers. Assume that $G$ is abelian. Then the following equality holds:

$$\Delta(F\theta_m(\Omega_n(G)))FG = \theta_m(\Omega_n(FG))FG.$$

*Proof.* Thanks to Lemma 1.2, it is easy to show that

$$\Delta(F\theta_m(\Omega_n(G)))FG \subseteq \theta_m(\Omega_n(FG))FG.$$

To show the opposite inclusion, take an element $x \in \Omega_n(FG)$. We will show that $x^{p^n} \in \Delta(F\theta_m(\Omega_n(G)))FG$. To this end, choose a complete set $R$ of coset representatives of $\Omega_n(G)$ in $G$ and write

$$x = \sum_{g \in \Omega_n(G)} \sum_{r \in R} \alpha_{gr} gr$$

for some $\alpha_{gr} \in F$.

Observe now that the map $R \rightarrow G$, defined by $r \mapsto r^{p^n}$, is injective. From

$$0 = x^{p^n} = \left( \sum_{g \in \Omega_n(G)} \sum_{r \in R} \alpha_{gr} gr \right)^{p^n} = \sum_{r \in R} \left( \sum_{g \in \Omega_n(G)} \alpha_{gr} \right)^{p^n} r^{p^n},$$
we obtain, for each representative \( r \in R \), that 

\[
\sum_{g \in \Omega_n(G)} \alpha_{gr} = 0.
\]

Hence

\[
x = \sum_{r \in R} \left( \sum_{g \in \Omega_n(G)} \alpha_{gr} g r - \sum_{g \in \Omega_n(G)} \alpha_{gr} g r \right) = \sum_{r \in R} \sum_{g \in \Omega_n(G)} \alpha_{gr} (g - 1) r
\]

and we see that 

\[
x^p = \sum_{r \in R} \sum_{g \in \Omega_n(G)} \alpha_{gr} (g^p - 1) r^p \in \Delta(F \Omega_m(\Omega_n(G))) FG. \]

\[\square\]

**Lemma 2.3.** Let \( m \) and \( n \) be non-negative integers. Then 

\[
\Delta(F \gamma(G) \Omega_m(\Omega_n^*(G))) FG
\]

is the smallest ideal of \( FG \) containing both \( \gamma(FG) \) and \( \Omega_m(\Omega_n(Z(FG))) \).

**Proof.** Let \( I \) be the smallest ideal of \( FG \) containing \( \gamma(FG) \) and \( \Omega_m(\Omega_n(Z(FG))) \) and set \( K = \gamma(G) \Omega_m(\Omega_n^*(G)) \): we show that \( I = \Delta(FK) FG \).

With the aid of Lemmas 1.4 and 1.5, it is easy to show that \( I \supseteq \Delta(FK) FG \).

To prove that \( I \subseteq \Delta(FK) FG \), we will use Lemmas 1.4 and 1.5 to show that \( \Omega_m(\Omega_n(Z(FG))) \subseteq \Delta(FK) FG \). To this end, fix \( x \in \Omega_n(Z(FG)) \): we will prove that \( x^p \in \Delta(FK) FG \). As a consequence of Lemma 1.6(2), write

\[
x = \sum_{z \in Z(G)} \alpha_z z + \sum_{\kappa \in cc(G)} \beta_\kappa \kappa
\]

for some \( \alpha_z, \beta_\kappa \in F \). Lemma 2 ensures that

\[
0 = \sum_{z \in Z(G)} \alpha_z z = \left( \sum_{z \in Z(G)} \alpha_z z \right)^p + \left( \sum_{\kappa \in cc(G)} \beta_\kappa \kappa \right)^p
\]

and so, thanks to Lemma 1.6(2)–(4), we obtain that \( \sum_{z \in Z(G)} \alpha_z z \in \Omega_n(FZ(G)) \).

Observe now that 

\[
x^p = \left( \sum_{z \in Z(G)} \alpha_z z \right)^p + \left( \sum_{\kappa \in cc(G)} \beta_\kappa \kappa \right)^p.
\]

The first summand belongs to \( \Delta(F \Omega_m(\Omega_n^*(G))) FG \), by Lemma 2.2, and the second summand belongs to \( \Delta(F \gamma(G)) FG \), by Lemma 1.6(3)–(4). Now Lemma 1.5 yields that \( x^p \in \Delta(FK) FG \). \[\square\]

We close the present section with the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Assume that \( FG \cong FH \) and let \( n \) be a non-negative integer.

(1) Thanks to Lemma 2.3, the ideal \( \Delta(F \gamma(G) \Omega_n^*(G)) FG \) is canonical in \( FG \) and thus every algebra isomorphism \( FG \to FH \) induces an algebra isomorphism 

\[
FG/\Delta(F \gamma(G) \Omega_n^*(G)) FG \to FH/\Delta(F \gamma(H) \Omega_n^*(H)) FH.
\]
Thanks to Lemma 1.3, the quotient $F G/\Delta(FG)\Omega_n^{\ast}(G)FG$ is isomorphic to $F[G/\gamma(G)\Omega_n^{\ast}(G)]$ and so it follows from Theorem 1.7 that the isomorphism type of $G/\gamma(G)\Omega_n^{\ast}(G)$ is an invariant of $FG$.

(2) Combining Lemmas 1.3 and 2.3 with the fact that $\gamma(G)\Omega_n^{\ast}(G)/\gamma(G)$ is isomorphic to $\gamma(G)\Omega_n^{\ast}(G)/\gamma(G)$, the sequence $((\bar{\Delta}_k(G/\gamma(G))\Omega_n^{\ast}(G)/\gamma(G)))_{k \geq 0}$ is an invariant of $FG$. Proposition 1.1 yields that the isomorphism type of $\gamma(G)\Omega_n^{\ast}(G)/\gamma(G)$ is an invariant of $FG$. This concludes the proof.

2.2. Frattini subgroups and centers. We now establish the second pair of abelian invariants from Theorem B. In analogy to what we did in Section 2.1, we refer the reader to Figure 3 for an overview of the ideals involved in the proof.

Theorem 2.4. Let $n$ be a non-negative integer. Then the isomorphism types of the following are invariants of $FG$:

1. $Z(G) \cap \bar{\Omega}_n^\ast(G)$,
2. $Z(G)/Z(G) \cap \bar{\Omega}_n^\ast(G)$.

$\text{Figure 3. Corresponding ideals and subgroups.}$

Lemma 2.5. Assume that $G$ is abelian. Let $K$ be a subgroup of $G$ and $m$ a non-negative integer. Then

$\Delta(F\bar{\Omega}_m(K)FG)$

is the smallest ideal of $FG$ containing $\bar{\Omega}_m(\Delta(FK)FG)$.

Proof. This is a direct consequence of Lemma 1.2. □

Lemma 2.6. Let $n$ be a non-negative integer. Then

$\Delta(F\bar{\Omega}_n^\ast(G)FG)$

is the smallest ideal of $FG$ containing both $\gamma(FG)$ and $\bar{\Omega}_n(\Delta(FG))$.

$\text{1Technically speaking, we will not consider the algebra } FZ(G), \text{ but rather an isomorphic algebra } Z(FG)/(Z(FG) \cap \gamma(FG)).$
Proof. This follows easily from Lemmas 1.2, 1.4 and 1.5. □

Lemma 2.7. Let $N$ be a subgroup of $G$ containing $\gamma(G)$. Then the following equality holds:

$$Z(FG) \cap \Delta(FN)FG = \Delta([Z(G) \cap N])FZ(G) \oplus \Delta(FG) \cap \gamma(FG)).$$

Proof. Note that $N$ is normal in $G$ because it contains $\gamma(G)$. It is easy to show that the left-hand side contains the right-hand side. To show the opposite inclusion, fix $x \in Z(FG) \cap \Delta(FN)FG$ and, thanks to Lemma 1.6(2), write

$$x = \sum_{z \in Z(G)} \alpha_z z + \sum_{\kappa \in \cc(G), |\kappa| \neq 1} \beta_{\kappa} \hat{\kappa}$$

for some $\alpha_z, \beta_{\kappa} \in F$. By Lemma 1.6(3), the second summand belongs to $Z(FG) \cap \gamma(FG)$; we will show that the first summand belongs to $\Delta(F[Z(G) \cap N])FZ(G)$. Recall that $\Delta(FN)FG$ is the kernel of the natural surjective map $\varepsilon_N: FG \to F[G/N]$ by Lemma 1.3. In particular $\varepsilon_N(x) = 0$ yields

$$\sum_{z \in Z(G)} \alpha_z \varepsilon_N(z) + \sum_{\kappa \in \cc(G), |\kappa| \neq 1} \beta_{\kappa} \varepsilon_N(\hat{\kappa}) = 0.$$ 

Since $N$ contains $\gamma(G)$, each non-central conjugacy class $\kappa$ satisfies $\varepsilon_N(\hat{\kappa}) = 0$. It follows that $\sum_{z \in Z(G)} \alpha_z z = 0$. Choose a complete set $R$ of coset representatives of $Z(G) \cap N$ in $Z(G)$. As the map $R \to G/N$, defined by $r \mapsto rN$ is injective, from

$$0 = \sum_{z \in Z(G)} \alpha_z z = \sum_{r \in R} \sum_{n \in Z(G) \cap N} \alpha_{rn} rN = \sum_{r \in R} \left( \sum_{n \in Z(G) \cap N} \alpha_{rn} \right) rN;$$

we deduce, for each representative $r \in R$, that $\sum_{n \in Z(G) \cap N} \alpha_{rn} = 0$. Hence

$$\sum_{z \in Z(G)} \alpha_z z = \sum_{r \in R} \sum_{n \in Z(G) \cap N} \alpha_{rn} rN = \sum_{r \in R} \left( \sum_{n \in Z(G) \cap N} \alpha_{rn} rN - \sum_{n \in Z(G) \cap N} \alpha_{rn} rN \right)$$

$$= \sum_{r \in R} \sum_{n \in Z(G) \cap N} \alpha_{nr} (n - 1) r$$

and we see that $\sum z \alpha_z z \in \Delta(F[Z(G) \cap N])FZ(G)$.

Note that the sum in (2.1) is direct as consequence of Lemma 1.6(2). □

We close Section 2.2 with the proof of Theorem 2.4.

Proof of Theorem 2.4. Assume that $FG \cong FH$ and let $n$ and $m$ be non-negative integers. We start by showing that the isomorphism type of the quotient group $Z(G)/\mathcal{O}_m(Z(G) \cap \mathcal{O}_n(G))$ is an invariant of $FG$. 

Recall that $Z(FG) \cap \gamma(FG)$ is an ideal of $Z(FG)$ by Lemma 1.6(4). Let $\Theta(FG)$ be the smallest ideal of $Z(FG)$ containing both $\mathcal{U}_m(Z(FG) \cap \Delta(F\mathcal{U}_n^*(G)))FG$ and $Z(FG) \cap \gamma(FG)$ so that
\[
(2.2) \quad \Theta(FG) = \mathcal{U}_m(Z(FG) \cap \Delta(F\mathcal{U}_n^*(G)))FG + (Z(FG) \cap \gamma(FG)).
\]
Since, by Lemma 2.6, the ideal $\Delta(F\mathcal{U}_n^*(G))FG$ is canonical in $FG$ so is the ideal $\Theta(FG)$. Thus every algebra isomorphism $FG \to FH$ induces an algebra isomorphism $Z(FG) / \Theta(FG) \to Z(FH) / \Theta(FH)$.

Thanks to Lemma 2.7, the following equality holds:
\[
(2.3) \quad Z(FG) \cap \Delta(F\mathcal{U}_n^*(G))FG = \Delta(F[Z(G) \cap \mathcal{U}_n^*(G)])FZ(G) + (Z(FG) \cap \gamma(FG)).
\]
We rewrite $\Theta(FG)$ as follows. Substituting (2.3) into (2.2) yields
\[
\Theta(FG) = \mathcal{U}_m(\Delta(F[Z(G) \cap \mathcal{U}_n^*(G)])FZ(G) + (Z(FG) \cap \gamma(FG)))Z(FG)
\]
and expanding the $p^n$-th powers on the right hand side yields
\[
\Theta(FG) = \mathcal{U}_m(\Delta(F[Z(G) \cap \mathcal{U}_n^*(G)])FZ(G))Z(FG) + (Z(FG) \cap \gamma(FG)).
\]
Using $Z(FG) = FZ(G) \oplus (Z(FG) \cap \gamma(FG))$ given in Lemma 1.6(5), we get
\[
\Theta(FG) = \mathcal{U}_m(\Delta(F[Z(G) \cap \mathcal{U}_n^*(G)])FZ(G)) \oplus (Z(FG) \cap \gamma(FG)).
\]
The last sum is direct as the first summand is contained in $FZ(G)$. By Lemma 2.5 we finally obtain
\[
(2.4) \quad \Theta(FG) = \Delta(F[\mathcal{U}_m(Z(G) \cap \mathcal{U}_n^*(G))]FZ(G) \oplus (Z(FG) \cap \gamma(FG)).
\]
Using Lemma 1.6(5) we have
\[
Z(FG) / \Theta(FG) = \frac{FZ(G) \oplus (Z(FG) \cap \gamma(FG))}{\Delta(F[\mathcal{U}_m(Z(G) \cap \mathcal{U}_n^*(G))]FZ(G) \oplus (Z(FG) \cap \gamma(FG))}
\]
and as the first summand of (2.4) is an ideal of $FZ(G)$, we get
\[
Z(FG) / \Theta(FG) \cong FZ(G) / \Delta(F[\mathcal{U}_m(Z(G) \cap \mathcal{U}_n^*(G))]FZ(G).
\]
Thanks to Lemma 1.3, the last quotient is isomorphic to
\[
F[Z(G) / \mathcal{U}_m(Z(G) \cap \mathcal{U}_n^*(G))].
\]
The modular isomorphism problem being positively solved for abelian groups, the isomorphism type of $Z(G) / \mathcal{U}_m(Z(G) \cap \mathcal{U}_n^*(G))$ is an invariant of $FG$ and, $m$ having been chosen arbitrarily, so is the sequence $([\mathcal{U}_k(Z(G) \cap \mathcal{U}_n^*(G))])_{k \geq 0}$.

(2) This is a special case of the last claim, where $m = 0$.

(1) The sequence $([\mathcal{U}_k(Z(G) \cap \mathcal{U}_n^*(G))])_{k \geq 0}$ is an invariant of $FG$ and so Proposition 1.1 yields the isomorphism type of $Z(G) \cap \mathcal{U}_n^*(G)$ is an invariant of $FG$. □
The combination of Theorem 2.1 with Theorem 2.4 is the same as Theorem B, which is therefore now proven. We remark that the proof of Theorem B is inspired by the argument for the invariance of $Z(G) \cap \gamma(G)$ and $Z(G)/Z(G) \cap \gamma(G)$ by Sandling [25, Theorem 6.11].

**Corollary 2.8.** The isomorphism types of the following are invariants of $FG$:

- $G/\gamma(G)$,
- $G/\gamma(G)\Psi(G)$,
- $G/\gamma(G)Z(G)$,
- $\gamma(G)\Psi(G)/\gamma(G)$,
- $\gamma(G)Z(G)/\gamma(G)$,
- $G/\Phi(G)\Psi(G)$,
- $Z(G)$,
- $Z(G)\cap\Phi(G)$,
- $Z(G)\cap\gamma(G)$,
- $Z(G)/Z(G)\cap\Phi(G)$,
- $Z(G)/Z(G)\cap\gamma(G)$,
- $\Psi(G)\cap\Phi(G)$.

**Proof.** Taking $n$ large enough ($n \geq \log_p |G|$, for example) in Theorem B yields the following four invariants:

\[ G/\gamma(G)Z(G), \quad \gamma(G)Z(G)/\gamma(G), \quad Z(G) \cap \gamma(G), \quad Z(G)/Z(G) \cap \gamma(G). \]

Setting $n = 0$ in (1) and (3) of Theorem B yields two classic invariants:

\[ G/\gamma(G), \quad Z(G). \]

Setting $n = 1$ in Theorem B yields four new invariants

\[ G/\gamma(G)\Psi(G), \quad \gamma(G)\Psi(G)/\gamma(G), \quad Z(G) \cap \Phi(G), \quad Z(G)/Z(G) \cap \Phi(G). \]

The remaining two new invariants are obtained from $G/\gamma(G)\Psi(G)$ and $Z(G) \cap \Phi(G)$ by observing that

\[ G/\Phi(G)\Psi(G) \cong \frac{G/\gamma(G)\Psi(G)}{U_1(G/\gamma(G)\Psi(G)/\gamma(G)\Psi(G))} = \frac{G/\gamma(G)\Psi(G)}{U_1(G/\gamma(G)\Psi(G))}, \]

\[ \Psi(G) \cap \Phi(G) = \Omega_1(Z(G)) \cap \Phi(G) = \Omega_1(Z(G) \cap \Phi(G)). \]

We remark that the invariants $Z(G), G/\gamma(G), Z(G) \cap \gamma(G), Z(G)/Z(G) \cap \gamma(G)$ from Corollary 2.8 can be found in Sandling’s survey paper [25, Theorems 6.11, 6.12, 6.7].

**Remark 2.9.** If $G$ has class 2, one can see the isomorphism type of $G/Z(G)$ as an invariant of $FG$ (cf. [25, Theorem 6.23]) from one of the above invariants, namely $G/\gamma(G)Z(G) \cong (G/Z(G))/\gamma(G/Z(G))$.

3. **Elementary decompositions**

In this section we present an elementary group-theoretic result on the decomposition of a finite $p$-group as a direct product of two subgroups, one of which is elementary abelian and maximal with these properties. This turns out to be fundamental for the proof of Theorem A.
Definition 3.1. A decomposition

\begin{equation}
G = T \times U
\end{equation}

of \( G \) into subgroups \( T \) and \( U \) is said to be \textit{elementary} if the following are satisfied:

1. \( T \cap \Phi(G) = 1 \) and \( |T| = |\Psi(G) : \Psi(G) \cap \Phi(G)| \); and
2. \( \Psi(G)U = G \) and \( |U| = |G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| \).

It can be proven that, given a decomposition \( G = T \times U \), Conditions (1) and (2) are equivalent. Moreover, as defined in Definition 3.1, the subgroup \( T \) of \( G \) is maximal with the property that \( T \) is an elementary abelian direct factor of \( G \).

Lemma 3.2. An elementary decomposition \( G = T \times U \) with \( T \) elementary abelian always exists and the subgroups \( T \) and \( U \) are unique up to isomorphism. Furthermore, the following equalities hold:

\[ \Psi(G) = T \times \Psi(U) \quad \text{and} \quad \Phi(G) = \Phi(U). \]

Proof. Since \( \Psi(G) \) is elementary abelian, the subgroup \( \Psi(G) \cap \Phi(G) \) has a complement \( T \) in \( \Psi(G) \). We take such \( T \) and remark that \( T \cap \Phi(G) = 1 \) and \( \Psi(G) = T(\Psi(G) \cap \Phi(G)) \). Since \( G/\Phi(G) \) is also elementary abelian, the subgroup \( \Psi(G)/\Phi(G) \cap \Phi(G) \) has a complement in \( G/\Phi(G) \); we take it to be of the form \( U/\Phi(G) \) with \( U \) containing \( \Phi(G) \). It follows that \( \Psi(G) \Phi(G) \cap U = \Phi(G) \) and, the elements of \( \Phi(G) \) being non-generators, that \( G = \Psi(G)U \); see Figure 4.

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (G) at (0,0) {$G$};
    \node (PsiGPhiG) at (1,1) {$\Psi(G)\Phi(G)$};
    \node (PhiG) at (2,0) {$\Phi(G)$};
    \node (PsiG) at (1,-1) {$\Psi(G)$};
    \node (T) at (0,-2) {$T$};
    \node (PsiGPhiGcapPhiG) at (1,-3) {$\Psi(G) \cap \Phi(G)$};
    \node (1) at (-1,-4) {$1$};
    \node (U) at (3,0) {$U$};
    \draw (G) -- (PsiGPhiG); \draw (PsiGPhiG) -- (PhiG); \draw (PsiGPhiG) -- (PsiG); \draw (PsiG) -- (T); \draw (T) -- (PsiGPhiGcapPhiG); \draw (PsiGPhiGcapPhiG) -- (1); \draw (PsiGPhiGcapPhiG) -- (U); \draw (PsiGPhiG) -- (PsiGPhiGcapPhiG); \draw (PsiGPhiGcapPhiG) -- (PhiG); \draw (PhiG) -- (U);
\end{tikzpicture}
\caption{Subgroups involved in an elementary decomposition.}
\end{figure}

It is now easy to deduce that \( G = TU \) and \( T \cap U = 1 \). Since \( T \) is central, we obtain \( G = T \times U \) and this decomposition is elementary. The uniqueness of \( T \) and \( U \), up to isomorphism, follows from the Krull-Remak-Schmidt theorem. In conclusion, since \( T \) is an elementary abelian direct factor of \( G \), we have \( \Psi(G) = T \times \Psi(U) \) and \( \Phi(G) = \Phi(U) \). \( \square \)
Remark 3.3. Let $G = T \times U$ be an elementary decomposition of $G$ with $T$ elementary abelian. Note that there are natural isomorphisms $G/\Psi(G) \rightarrow U/\Psi(U)$ and $\Phi(G) \rightarrow \Phi(U)$. Furthermore, these are compatible with $p$-th power maps $G/\Psi(G) \rightarrow \Phi(G)$ and $U/\Psi(U) \rightarrow \Phi(U)$, i.e. the following diagram commutes.

\[
\begin{array}{ccc}
G/\Psi(G) & \longrightarrow & U/\Psi(U) \\
\downarrow & & \downarrow \\
\Phi(G) & \longrightarrow & \Phi(U)
\end{array}
\]

Roughly speaking, $G$ and $U$ have the same power structure.

4. A REDUCTION THEOREM

In this section we prove that elementary abelian direct factors can be disregarded in the study of the modular isomorphism problem. More precisely, we prove the following generalization of Theorem A from the Introduction.

Theorem 4.1. Let $G = T \times U$ and $H = S \times V$ be elementary decompositions of $G$ and $H$, with $T$ and $S$ elementary abelian. Then the following are equivalent:

1. $FG \cong FH$.
2. $FT \cong FS$ and $FU \cong FV$.

Until the end of the present section, we write $\Delta = \Delta(FG)$ for the augmentation ideal of $FG$.

Lemma 4.2. Let $K$ and $L$ be normal subgroups of $G$ such that $G$ equals the internal direct product $K \times L$ of $K$ and $L$. Let, moreover, $n$ be a positive integer. Then the following equality holds:

\[
\Delta^n = \Delta(FK)\Delta^{n-1} \oplus \Delta(FL)^n.
\]

Proof. We work by induction on $n$. Assume first that $n = 1$. Then, for each $k \in K$ and $l \in L$, one has $kl - 1 = (k - 1)l + (l - 1)$: this proves $\Delta = \Delta(FK)FG + \Delta(FL)$. The last sum is direct thanks to Lemma 1.3.

Assume now that $n > 1$ and that $\Delta^{n-1} = \Delta(FK)\Delta^{n-2} \oplus \Delta(FL)^{n-1}$. Then, the inclusion $\Delta^n \supseteq \Delta(FK)\Delta^{n-1} \oplus \Delta(FL)^n$ immediately follows from the definition of $\Delta$, while the opposite inclusion is obtained from combining $\Delta^n = \Delta^{n-1}\Delta$, the induction hypothesis, and the case $n = 1$.

Lemma 4.3. The following hold:

1. $\Delta(F\Phi(G))FG \subseteq \Delta^2$.
2. $\Delta(F\Phi(G))FG + \Delta^2 = \Omega_1(Z(FG)) + \Delta^2$.

Proof. (1) This follows from combining Lemmas 1.2, 1.4 and 1.5.
(2) It is easy to show that $\Delta(F \Psi(G))FG + \Delta^2 \subseteq \Omega_1(Z(FG)) + \Delta^2$. To show the converse, it suffices to prove that $\Omega_1(Z(FG)) \subseteq \Delta(F \Psi(G))FG + \Delta^2$. To this end, take an element $x \in \Omega_1(Z(FG))$ and, thanks to Lemma 1.6(2), write

$$x = \sum_{z \in Z(G)} \alpha_z z + \sum_{\kappa \in \text{cc}(G)} \frac{1}{|\kappa| \neq 1} \beta_{\kappa} \hat{\kappa},$$

for scalars $\alpha_z, \beta_{\kappa} \in F$. By Lemma 1.6(1) and (3), the second summand on the right-hand side belongs to $\Delta^2$; we will prove that the first summand belongs to $\Delta(F \Psi(G))FG$. Now, from $x^p = 0$ and Lemma 1.6(3), we derive that

$$\sum_{z \in Z(G)} \alpha_z z \in \Omega_1(FZ(G)).$$

Taking $m = 0$ and $n = 1$ in Lemma 2.2, with $Z(G)$ in the role of $G$, yields that $\Omega_1(FZ(G)) \subseteq \Delta(F \Psi(G))FG$ and thus the claim. □

**Lemma 4.4.** Let $G = T \times U$ be an elementary decomposition of $G$ with $T$ elementary abelian. Let, moreover, $n$ be a positive integer. Define $I = \Delta(FT)FG$. Then the following hold:

1. $\text{codim } I = |G|/|\Psi(G) : \Psi(G) \cap \Phi(G)|$.
2. $I \Delta^{n-1} + \Delta^{n+1} = \Delta(F \Psi(G)) \Delta^{n-1} + \Delta^{n+1}$.

**Proof.** As (1) follows immediately from Lemma 1.3, we show (2). From the fact that $\Psi(U) \subseteq \Phi(U)$ and Lemma 3.2, we derive

$$I + \Delta^2 = \Delta(FT)FG + \Delta^2 = \Delta(FT)FG + \Delta(F \Psi(U))FG + \Delta^2 = \Delta(F \Psi(G))FG + \Delta^2.$$

It now follows from Lemmas 1.5 and 4.3 that

$$I \Delta^{n-1} + \Delta^{n+1} = (I + \Delta^2) \Delta^{n-1} = (\Delta(F \Psi(G))FG + \Delta^2) \Delta^{n-1} = \Delta(F \Psi(G)) \Delta^{n-1} + \Delta^{n+1}. \quad \square$$

**Lemma 4.5.** Let $G = T \times U$ be an elementary decomposition of $G$ with $T$ elementary abelian. Let $I$ be an ideal of $FG$ satisfying

1. $\text{codim } I = |G|/|\Psi(G) : \Psi(G) \cap \Phi(G)|$ and
2. for each integer $n > 0$, one has $I \Delta^{n-1} + \Delta^{n+1} = \Delta(F \Psi(G)) \Delta^{n-1} + \Delta^{n+1}$.

Then one has $FG/I \cong FU$.

**Proof.** In order to prove the lemma, we will show that $FG = I \oplus FU$. To do this, we first prove that, for each positive integer $n$, one has

$$\Delta^n = I \Delta^{n-1} + \Delta(FU)^n. \quad (4.1)$$

Let $c$ be minimal such that $\Delta^c = 0$. Then, for each $n > c$, the equality in (4.1) clearly holds. Assume now that $0 < n \leq c$: we prove (4.1) by induction on $c-n$. If
n = c, we are done because \( I \) is contained in the unique maximal ideal \( \Delta \). Suppose now that \( n < c \) and that \( \Delta^{n+1} = I\Delta^n + \Delta(FU)^{n+1} \). From Lemmas 4.2 and 4.4 and (2) we then obtain

\[
\Delta^n = \Delta^n + \Delta^{n+1}
\]

\[
= \Delta(FT)\Delta^{n-1} + \Delta^{n+1} + \Delta(FU)^n
\]

\[
= \Delta(F \Psi(G))\Delta^{n-1} + \Delta^{n+1} + \Delta(FU)^n
\]

\[
= I\Delta^{n-1} + \Delta^{n+1} + \Delta(FU)^n
\]

\[
= I\Delta^{n-1} + \Delta(FU)^n + \Delta(FU)^n
\]

This completes the proof of (4.1). In particular, from \( n = 1 \), we obtain that \( \Delta = I + \Delta(FU) \) and thus \( FG = I + FU \). The last equality, together with the condition on the codimension of \( I \), yields that \( I \cap FU = 0 \) and hence we have \( FG = I \oplus FU \).

We conclude the current section by proving Theorem 4.1.

**Proof of Theorem 4.1.** Let \( G = T \times U \) and \( H = S \times V \) be elementary decompositions with \( T \) and \( S \) elementary abelian.

For the proof of (2) \( \Rightarrow \) (1), assume that \( FT \cong FS \) and \( FU \cong FV \). It follows from [22, Chapter 1, Lemma 3.4] that

\[
FG = F[T \times U] \cong FT \otimes_F FU \cong FS \otimes_F FV \cong F[S \times V] = FH.
\]

For the proof of (1) \( \Rightarrow \) (2), let \( \varphi : FG \to FH \) be an algebra isomorphism. Note that \( T \cong \Psi(G)/\Psi(G) \cap \Phi(G) \) and \( S \cong \Psi(H)/\Psi(H) \cap \Phi(H) \), so the combination of Theorem 1.7 and Corollary 2.8 yields that \( T \cong S \). Let \( I \) be an ideal of \( FG \) satisfying the conditions from Lemma 4.5; such an ideal exists as a consequence of Lemma 4.4. Now, as a consequence of Lemma 4.3(2), the ideal \( \Delta(F \Psi(G))FG + \Delta^2 \) is canonical in \( FG \) and, thanks to Corollary 2.8, the quantity \( |G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| \) is an invariant of \( FG \). It follows in particular that the ideal \( \varphi(I) \) of \( FH \) also satisfy the conditions from Lemma 4.5 and thus we have that

\[
FU \cong FG/I \cong FH/\varphi(I) \cong FV.
\]

The proof is complete. \( \square \)

5. **Applications**

This section is devoted to particular instances in which our main results can be applied to solve the modular isomorphism problem. Since our aim is to work independently of the choice of the field, in Section 5 we avoid the use of invariants which are only known to hold over the prime field. We remark that, even if working over \( \mathbb{F}_p \) would shorten some arguments, the results here presented would still be new.
5.1. **Infinitely to finitely many.** In the current section we positively solve the modular isomorphism problem for the following infinite family of 2-groups of class 3 by first reducing it to a finite analysis using our Theorem A and subsequently applying old and new invariants, cf. Theorem B.

**Theorem 5.1.** Assume $G$ has class 3 and satisfies $|G: Z(G)| = |\Phi(G)| = 8$. Then the modular isomorphism problem has a positive answer for $G$.

**Lemma 5.2.** Assume $G$ has class 3 and satisfies $|G: Z(G)| = |\Phi(G)| = p^3$. Then one has $|G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| < p^8$.

**Proof.** First, we prove the following inequality, which in fact holds for any $p$-group:

\[
|G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| \leq |G : Z(G) \Phi(G)| \cdot |\Phi(G)|^2.
\]

Consider the chain of subgroups

\[
G \supseteq Z(G) \Phi(G) \supseteq Z(G) \supseteq \Psi(G) \supseteq \Psi(G) \cap \Phi(G) \supseteq 1
\]

and use it to decompose $|G|/|\Psi(G) : \Psi(G) \cap \Phi(G)|$ as

\[
|G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| = |G : Z(G) \Phi(G)| \cdot |Z(G) \Phi(G) : Z(G)| \cdot |Z(G) : \Psi(G)| \cdot |\Psi(G) \cap \Phi(G)|.
\]

One can see that the product of the second and fourth factors is bounded as

\[
|Z(G) \Phi(G) : Z(G)| \cdot |\Psi(G) \cap \Phi(G)|
\]

\[
= |\Phi(G) : Z(G) \cap \Phi(G)| \cdot |\Psi(G) \cap \Phi(G)|
\]

\[
\leq |\Phi(G) : \Psi(G) \cap \Phi(G)| \cdot |\Psi(G) \cap \Phi(G)| = |\Phi(G)|,
\]

and the third factor is bounded as

\[
|Z(G) : \Psi(G)| = |Z(G) : \Omega_1(Z(G))| = |\overline{\Omega}_1(Z(G))| \leq |\overline{\Omega}_1(G)| \leq |\Phi(G)|.
\]

This completes the proof of (5.1).

Now it follows from the assumption that the central quotient $G/Z(G)$ is a non-abelian group of order $p^3$ and thus its minimal number of generators is two. Since

\[
G/Z(G) \Phi(G) \cong \frac{G/Z(G)}{Z(G) \Phi(G)/Z(G)} = \frac{G/Z(G)}{\Phi(G/Z(G))},
\]

we obtain $|G : Z(G) \Phi(G)| = p^2$. Furthermore, one can see that $\overline{\Omega}_1(Z(G)) \neq \Phi(G)$ since $G$ has class 3. Thus the inequality in (5.1) is strict and we obtain

\[
|G|/|\Psi(G) : \Psi(G) \cap \Phi(G)| < p^2 \cdot (p^3)^2 = p^8.
\]

□

In the following result and later in this paper, we denote by $\text{SG}(n, m)$ the $m$-th group of order $n$ in the Small Groups Library [4] of GAP [10].
Lemma 5.3. Assume $G$ has class 3 and satisfies $|G : Z(G)| = |\Phi(G)| = 8$. Let $G = T \times U$ be an elementary decomposition with $T$ elementary abelian. Then $U$ is isomorphic to one of the following groups.

- $U_1 = \text{SG}(32, 9)$
- $U_2 = \text{SG}(32, 10)$
- $U_3 = \text{SG}(32, 11)$
- $U_4 = \text{SG}(32, 13)$
- $U_5 = \text{SG}(32, 14)$
- $U_6 = \text{SG}(32, 15)$
- $U_7 = \text{SG}(64, 97)$
- $U_8 = \text{SG}(64, 108)$
- $U_9 = \text{SG}(64, 118)$
- $U_{10} = \text{SG}(64, 119)$
- $U_{11} = \text{SG}(64, 120)$
- $U_{12} = \text{SG}(64, 124)$
- $U_{13} = \text{SG}(128, 1671)$

Proof. The order of $U$ is bounded by $2^7$, as consequence of Definition 3.1(2) and Lemma 5.2. A quick search with GAP [10], leveraging on Lemma 3.2, yields the claim.

To finish the proof of Theorem 5.1, we collect a number of invariants from the literature known to be valid over any field of characteristic $p$. In the following proposition, we say that a conjugacy class $\kappa$ consists of $p^n$-th powers if $\kappa$ is the conjugacy class of $g^{p^n}$ for some $g \in G$.

Proposition 5.4. Let $n$ be a non-negative integer. The following are invariants of $FG$:

1. the number $k_n(G)$ of conjugacy classes of $G$ consisting of $p^n$-th powers,
2. the number $a_n(G)$ of conjugacy classes of maximal elementary abelian subgroups of $G$ of rank $n$,
3. the number $e(G)$ of elements in a minimal set of generators of the mod-$p$ cohomology ring $H^*(G, F_p)$.

Proof. The first invariant $k_n(G)$ is obtained from an observation of Külshammer [17, Section 1], see also [26, Proposition 1] or [15, Section 2.2]. The second invariant $a_n(G)$ is derived from Quillen’s Stratification Theorem [9, Corollary 8.3.3], to be also found in [15, Section 2.5]. As for the last point, the cohomology ring $H^*(G, F) = \text{Ext}^*_{FG}(F, F)$ of $G$ can be computed directly from $FG$, and hence the number of elements in a minimal set of generators of $H^*(G, F)$ as a graded algebra is also an invariant of $FG$; cf. [7, p. 315]. Since

$$H^*(G, F) \cong H^*(G, F_p) \otimes_{F_p} F$$

by [9, Section 3.4], the number $e(G)$ is an invariant of $FG$.

We conclude the current section by proving Theorem 5.1.

Proof of Theorem 5.1. Assume $FG \cong FH$. First, we show that $H$ also has class 3 and satisfies $|H : Z(H)| = |\Phi(H)| = 8$. Clearly $|G| = |H|$, and $H$ satisfies the last two conditions by Theorem 1.7 and Proposition 1.8. Since $G$ has class 3, we have $Z(G) \supseteq \gamma(G)$ and $|\gamma(G)| > |Z(G) \cap \gamma(G)|$. The same inequality holds for $H$ by Corollary 2.8 and $H$ has class 3 as $|H : Z(H)| = 8$; see also [2, Theorem 2].
Without loss of generality and thanks to Theorem 4.1, we assume that $G$ and $H$ do not have elementary abelian direct factors. Then, as a consequence of Lemma 5.3, we are only concerned with the determination of 13 groups from their modular group algebras. We do so with the aid of the following invariants of modular group algebras, listed in Theorem 1.7 and Proposition 5.4:

- the number $k_n(G)$ of conjugacy classes of $G$ consisting $p^n$-th powers,
- the number $a_n(G)$ of conjugacy classes of maximal elementary abelian subgroups of $G$ of rank $n$,
- the number $e(G)$ of elements in a minimal set of generators of $H^*(G, \mathbb{F}_2)$ as a graded algebra,
- the order of the socle $\Psi(G)$.

| $G$ | $U_1$ | $U_2$ | $U_3$ | $U_4$ | $U_5$ | $U_6$ |
|-----|-------|-------|-------|-------|-------|-------|
| $k_1(G)$ | 4     | 4     | 5     | 5     | 4     | 5     |
| $a_2(G)$ | 0     | 1     | 2     | 1     | 1     | 1     |
| $e(G)$   | 5     | 6     | 6     | 4     | 4     | 5     |

Table 1. Some invariants for the groups from Lemma 5.3 of order 32.

| $G$ | $U_7$ | $U_8$ | $U_9$ | $U_{10}$ | $U_{11}$ | $U_{12}$ |
|-----|-------|-------|-------|-----------|-----------|-----------|
| $k_1(G)$ | 5     | 5     | 6     | 6         | 6         | 6         |
| $a_3(G)$ | 2     | 1     | 2     | 1         | 0         | 0         |
| $|\Psi(G)|$ | 4     | 4     | 4     | 4         | 4         | 2         |

Table 2. Some invariants for the groups from Lemma 5.3 of order 64.

The numbers $k_n(G)$ and $a_n(G)$ can be calculated using MIPConjugacyClassInfo and SubgroupsInfo in ModIsomExt [19], for example. See [13] or [7, Appendix D] for the number $e(G)$. It follows from Tables 1 and 2 that $G \cong H$. □

We remark that our proof of Theorem 5.1 could be slightly shortened with the aid of [21, Lemma 3.7], where the modular isomorphism problem is shown to have a positive solution for groups of order 32 over any field of characteristic 2. Since the techniques are similar and we also deal with groups of order 64, we present here the proof of Theorem 5.1 in its entirety.

**Example 5.5.** As noted in [15], the modular group algebras of the groups $SG(64, 97)$ and $SG(64, 101)$ as well as $SG(64, 108)$ and $SG(64, 110)$ cannot be distinguished by known group-theoretic invariants. Indeed, in [15, Sections 4.1, 4.2]...
ring-theoretic techniques are used to solve the modular isomorphism problem for these groups. Our new invariants from Theorem 2.1 or 2.4 can distinguish the modular group algebras of those groups and in fact they are covered by Theorem 5.1 as it is evident from Lemma 5.3. See also Corollary 5.10.

5.2. Generalizing maximal class. We refer to a group as dihedral if it is a finite 2-group that is generated by precisely two elements of order 2. In particular, the Klein four group is dihedral. In this section, we provide a positive answer to the modular isomorphism problem for the following class of finite 2-groups.

**Theorem 5.6.** Assume that \( G \) has cyclic center \( \mathbb{Z}(G) \) and dihedral central quotient \( G/\mathbb{Z}(G) \). Then the modular isomorphism problem has a positive answer for \( G \).

As we will see from its proof, the last theorem is an instance of the successful application of our new invariants, more specifically \( G/\gamma(G)\mathbb{Z}(G) \); see Corollary 2.8. Moreover, we remark that the class from Theorem 5.6 contains the dihedral group \( D_{2^{1+n}} \) and the generalized quaternion group \( Q_{2^{1+n}} \), whenever \( n \geq 2 \), and the semidihedral group \( S_{2^{1+n}} \), for \( n \geq 3 \). These have presentations

\[
D_{2^{1+n}} = \langle a, b, c \mid a^2 = 1, b^2 = 1, (ab)^{2^n-1} = c, \quad c^2 = [a, c] = [b, c] = 1 \rangle,
\]

\[
Q_{2^{1+n}} = \langle a, b, c \mid a^2 = c, b^2 = c, (ab)^{2^n-1} = c^{1+2^n-1}, \quad c^2 = [a, c] = [b, c] = 1 \rangle,
\]

\[
S_{2^{1+n}} = \langle a, b, c \mid a^2 = 1, b^2 = c, (ab)^{2^n-1} = c^{1+2^n-2}, \quad c^2 = [a, c] = [b, c] = 1 \rangle.
\]

More generally, we define the following three families of 2-groups for each \( m \geq 1 \) and \( n \geq 2 \):

\[
D_{2^m \mid n} = \langle a, b, c \mid a^2 = 1, b^2 = 1, (ab)^{2^{m-1}} = c^{2^m-1}, \quad c^2 = [a, c] = [b, c] = 1 \rangle,
\]

\[
Q_{2^m \mid n} = \langle a, b, c \mid a^2 = c, b^2 = c, (ab)^{2^{m-1}} = c^{2^m-1+2^{n-1}}, \quad c^2 = [a, c] = [b, c] = 1 \rangle,
\]

\[
S_{2^m \mid n} = \langle a, b, c \mid a^2 = 1, b^2 = c, (ab)^{2^{m-1}} = c^{2^m-1+2^{n-2}}, \quad c^2 = [a, c] = [b, c] = 1 \rangle.
\]

The last groups are central extensions of a dihedral group of order \( 2^n \) by a cyclic group of order \( 2^m \). Note that there are exceptional isomorphisms

\[
(5.2) \quad S_{2^m \mid 2} \cong \begin{cases} D_{2^1 \mid 2} & \text{if } m = 1, \\ Q_{2^m \mid 2} & \text{if } m > 1, \end{cases}
\]

given by

\[
S_{2^{1 \mid 2}} \rightarrow D_{2^{1 \mid 2}}, \quad a \mapsto a, \quad b \mapsto ab,
\]

and

\[
S_{2^{m \mid 2}} \rightarrow Q_{2^m \mid 2}, \quad a \mapsto abc^{2m-2-1}, \quad b \mapsto b.
\]

The following theorem allows us to study the family from Theorem 5.6 in terms of the just-defined groups.
Theorem 5.7. Assume that $G$ has cyclic center $Z(G)$ of order $2^m$ and dihedral central quotient $G/Z(G)$ of order $2^n$. Then $G$ is isomorphic to one of $D_{2m|n}$, $Q_{2m|n}$ or $S_{2m|n}$.

Proof. Observe that $m \geq 1$, because finite non-trivial $p$-groups have non-trivial centers, and $n \geq 2$, because $G/Z(G)$ cannot be a non-trivial cyclic group. It follows from the assumptions that there are elements $x, y, z$ generating $G$ and integers $0 \leq \xi, \eta, \zeta < 2^m$ with

$$x^2 = z^\xi, \quad y^2 = z^\eta, \quad (xy)^{2^{n-1}} = z^\zeta, \quad z^{2m} = [x, z] = [y, z] = 1.$$ 

First, we shall replace the generators $x, y, z$ with new generators $a, b, c$ that are relatable via simpler parameters. Let $q_\xi$ and $r_\xi$ be the quotient and remainder of $\xi$ divided by 2 so that

$$\xi = 2q_\xi + r_\xi \quad \text{and} \quad 0 \leq r_\xi < 2.$$

Similarly, define $q_\eta$ and $r_\eta$. Replace the generators $x, y, z$ with

$$a = xz^{-q_\xi}, \quad b = yz^{-q_\eta}, \quad c = z$$

and note that $a, b, c$ satisfy relations of the same form

$$a^2 = c^\alpha, \quad b^2 = c^\beta, \quad (ab)^{2^{n-1}} = c^\gamma, \quad c^{2m} = [a, c] = [b, c] = 1$$

where the parameters’ values range as $0 \leq \alpha, \beta < 2$ and $0 \leq \gamma < 2^m$; we assume, without loss of generality, that $\alpha \leq \beta$. In particular, there are only three possibilities for $(\alpha, \beta)$. Suppose $(\alpha, \beta) = (0, 1)$, namely, $a^2 = 1$ and $b^2 = c$. Then

$$c^\gamma = (ab)^{2^{n-1}} = (ab)^{2^{n-1}} a^2 = a(ba)^{2^{n-1}} a = a(abc^{-1})^{-2^{n-1}} a = a(ab)^{-2^{n-1}} ac^{2^{n-1}} = ac^{-\gamma} ac^{2^{n-1}} = a^2 c^{-\gamma + 2^{n-1}} = c^{-\gamma + 2^{n-1}},$$

which yields $c^{2\gamma - 2^{n-1}} = 1$. The element $c$ belonging to the center, we obtain that $2\gamma - 2^{n-1} \equiv 0 \mod 2^m$ and hence $\gamma \equiv 2^{m-1} + 2^{n-2} \mod 2^m$. A similar argument can be applied, for $(\alpha, \beta) = (1, 1)$, to prove that $\gamma \equiv 2^{m-1} + 2^{n-1} \mod 2^m$ and, for $(\alpha, \beta) = (0, 0)$, to show that $\gamma \equiv 2^{m-1} \mod 2^m$. This completes the proof.

We will prove Theorem 5.6 following the next steps. We will first show that if $FG \cong FH$ and $G$ belongs to the family $\mathcal{X}$ of 2-groups from Theorem 5.6, then so does $H$. Then we will use a number of invariants, to show that any two group bases associated to the same algebra possess the same invariants $(m, n)$; cf. Theorem 5.7. To conclude, we will analyze different (incompatible) properties of the modular group algebras of $D_{2m|n}$, $Q_{2m|n}$, and $S_{2m|n}$ and leverage on Theorem 5.7 to positively solve the modular isomorphism problem for $\mathcal{X}$.

Lemma 5.8. The center $Z(G)$ is cyclic if and only if $|\Psi(G)| = p$.

Proof. This follows from the fact that an abelian $p$-group that has a unique subgroup of order $p$ is cyclic.

Lemma 5.9. The quotient $G/Z(G)$ is dihedral if and only if $|G : \gamma(G)Z(G)| = 4$. 

Proof. We will make use of the following basic fact:

\[ G/\gamma(G)Z(G) \cong (G/Z(G))/\gamma(G/Z(G)). \]

If \( G/Z(G) \) is dihedral, then \( |G : \gamma(G)Z(G)| = 4 \) follows immediately. Suppose that \( |G : \gamma(G)Z(G)| = 4 \). If the central quotient \( G/Z(G) \) is non-abelian, then it cannot be cyclic and is thus isomorphic to the Klein four group \( D_4 \). We assume now that \( G/Z(G) \) is non-abelian. By Taussky’s trichotomy theorem [16, Kapitel III, Satz 11.9], the group \( G/Z(G) \) is isomorphic to the dihedral group \( D_{2n} \) or the semidihedral group \( S_{2n} \), for \( n \geq 2 \), or the semidihedral group \( S_{2n} \), for \( n \geq 3 \). Moreover, [14, Lemma 3.1] ensures that \( Q_{2n} \) and \( S_{2n} \) are incapable of being central quotients, so \( G/Z(G) \) must be dihedral.

For the proof of Theorem 5.6 we will use dimension subgroups, but will avoid presenting here an extensive description of their properties. We refer to [15, Section 1] or [20, Section 2.3] for more details and references. For a positive integer \( n \), the \( n \)-th dimension subgroup \( D_n(G) \) of \( G \) is defined as

\[ D_n(G) = G \cap (1 + \Delta(FG)^n). \]

Jennings showed that the dimension subgroups form a \( p \)-restricted \( N \)-series, i.e. that for any pair of indices \( n \) and \( m \) one has

\[ (D_n(G), D_m(G)) \subseteq D_{n+m}(G) \text{ and } U_1(D_n(G)) \subseteq D_{np}(G). \]

Moreover, the dimension subgroups series, also called the Jennings series, is the fastest descending series of subgroups in \( G \) satisfying (5.3). If \( p = 2 \), for instance, the first few terms are

\[ D_1(G) = G, \quad D_2(G) = \Phi(G), \quad D_3(G) = [\Phi(G), G]U_1(\Phi(G)). \]

Moreover, for every choice of \( n \), the quotient \( D_n(G)/D_{n+1}(G) \) is elementary abelian and can be viewed as a vector space over \( \mathbb{F}_p \). It was also proven by Jennings that, given \( g \in G \), one has \( g^{-1} \in \Delta(FG)^n \) if and only if \( g \in D_n(G) \) and that, if \( g_1, \ldots, g_d \) form a basis of \( D_n(G)/D_{n+1}(G) \), then \( g_1 - 1, \ldots, g_d - 1 \) are linearly independent in \( \Delta(FG)^n/\Delta(FG)^{n+1} \). Moreover, after subtracting 1, any basis of \( D_1(G)/D_2(G) = G/\Phi(G) \) forms a basis of \( \Delta(FG)/\Delta(FG)^2 \).

In what follows we will apply the following identities, holding for \( g, h \in G \):

\[
\begin{align*}
(gh - 1) &= (g - 1) + (h - 1) + (g - 1)(h - 1), \\
(h - 1)(g - 1) &= (g - 1)(h - 1) + (1 + (g - 1)(h - 1) + (g - 1)(h - 1)([h, g] - 1).
\end{align*}
\]

We remark that, the element \([h, g] - 1 \) lying in \( \Delta(FG)^2 \), we have

\[ (h - 1)(g - 1) \equiv (g - 1)(h - 1) + ([h, g] - 1) \mod \Delta(FG)^3. \]

In the upcoming proof of Theorem 5.6 we will sometimes omit to explicitly reference to these identities.
Proof of Theorem 5.6. Assume $FG \cong FH$. It follows from the assumptions that $G$ belongs to the family

$$\mathcal{X} = \{ X \mid X \text{ finite 2-group, } Z(X) \text{ cyclic, } X/Z(X) \text{ dihedral }\}.$$

Lemmas 5.8 and 5.9 and Corollary 2.8 guarantee that $H$ also belongs to $\mathcal{X}$. From Theorem 5.7 we know that $G$ is isomorphic to one of $D_{2m|n}$, $Q_{2m|n}$ or $S_{2m|n}$, for uniquely determined values of $m \geq 1$ and $n \geq 2$. As $|G| = |H|$, we deduce from Theorems 1.7 and 5.7 that $H$ is isomorphic to one of $D_{2m|n}$, $Q_{2m|n}$ or $S_{2m|n}$, too.

Assume first that $m = 1$. Then $G$ and $H$ are 2-groups of maximal class and the modular isomorphism problem for these groups has been positively solved in [6, p. 434] and [1, Theorem 1]; see also [23] for the prime field case.

We now assume that $m > 1$. We write $d(G) = \log_p |G : \Phi(G)|$, which is an invariant by Proposition 1.8. Since $m > 1$, we have $\Phi(D_{2m|n}) = \langle c^2, [b, a] \rangle$, $\Phi(Q_{2m|n}) = \langle c, [b, a] \rangle$ and $\Phi(S_{2m|n}) = \langle c, [b, a] \rangle$, from which we derive

$$d(D_{2m|n}) = 3 \text{ while } d(Q_{2m|n}) = d(S_{2m|n}) = 2.$$

It follows that either $G$ and $H$ are both isomorphic to $D_{2m|n}$ or they are both 2-generated. We assume that neither $G$ nor $H$ is isomorphic to $D_{2m|n}$. Moreover, as $Q_{2m|2} \cong S_{2m|2}$, we assume that $n > 2$ and, for a contradiction, that

$$G = Q_{2m|n} \text{ and } H = S_{2m|n}.$$

To produce the desired contradiction, we will show that $\Omega_1(FG) \subseteq \Delta(FG)^2$, while this is not the case if we replace $FG$ by $FH$.

As $a$ and $b$ span $G/\Phi(G)$, we write a generic element $x$ in $\Omega_1(FG)$ as

$$x \equiv \alpha(a - 1) + \beta(b - 1) \text{ mod } \Delta(FG)^2$$

where $\alpha, \beta \in F$. It follows from Lemma 1.2 and (5.4) that

$$0 = x^2 \equiv (\alpha^2 + \beta^2)(c - 1) + \alpha\beta((a - 1)(b - 1) + (b - 1)(a - 1)), \tag{5.5}$$

$$\equiv (\alpha^2 + \beta^2)(c - 1) + \alpha\beta[b, a] - 1 \text{ mod } \Delta(FG)^3.$$

Set $P = G/D_3(G)$, which is the largest quotient $Q$ of $G$ satisfying $D_3(Q) = 1$. Note that $[a, b]^{2^{m-2}} = c^{2^{m-1}}$ holds in $G$. Since $m > 1$ and $n > 2$, we obtain

$$P \cong \langle a, b, c \mid a^2 = c, b^2 = c, [b, a]^2 = c^2 = [a, c] = [b, c] = 1 \rangle,$$

which is a group of order 16. Then we have

$$D_1(G)/D_2(G) \cong D_1(P)/D_2(P) \cong P/\Phi(P),$$

$$D_2(G)/D_3(G) \cong D_2(P)/D_3(P) \cong \Phi(P)$$

and both $P/\Phi(P)$ and $\Phi(P)$ have rank 2; cf. Figure 5 for a choice of basis. It follows that $c - 1$ and $[b, a] - 1$ are linearly independent modulo $\Delta(FG)^3$. Hence (5.5) yields that $\alpha\beta = \alpha^2 + \beta^2 = 0$, which itself implies $\alpha = \beta = 0$. We have proven that $x \in \Delta(FG)^2$. To conclude, note that, in $FH$, the element $a - 1$ lies
in $\Omega_1(FH)$ but not in $\Delta(FH)^2$, because $a \notin D_2(H) = \Phi(H)$. This shows that $FG \not\cong FH$, a contradiction.

\textbf{Corollary 5.10.} Assume that $G$ satisfies
\[ |G : \gamma(G)Z(G)| = 4 \text{ and } |\Psi(G) \cap \Phi(G)| = 2. \]
Then the modular isomorphism problem has a positive solution for $G$.

\textbf{Proof.} Let $G = T \times U$ be an elementary decomposition of $G$ with $T$ elementary abelian. Then $|U : \gamma(U)Z(U)| = 4$ and Lemma 3.2 yields $|\Psi(U)| = 2$. Lemmas 5.8 and 5.9 show that $U$ has cyclic center and dihedral central quotient. By Theorem A and Theorem 5.6, the modular isomorphism problem has a positive answer. \hfill \square

We conclude the present section and the paper with a note on the similarities between our (positive) examples and the counterexample found in [12].

\textbf{Remark 5.11.} The counterexample to the modular isomorphism problem [12] shares the following properties with our applications:
- $|G : Z(G)| = 8$ and class 3, in Theorem 5.1,
- $G/Z(G)$ dihedral, in Theorem 5.6.

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