Representing Scott Sets in Algebraic Settings

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1 Introduction

Recall that \( S \subseteq 2^\omega \) is called a Scott set if and only if:

i) \( S \) is a Turing ideal, i.e., if \( x, y \in S \) and \( z \leq_T x \oplus y \), then \( z \in S \), where \( x \oplus y \) is the disjoint union of \( x \) and \( y \);

ii) If \( T \subseteq 2^{<\omega} \) is an infinite tree computable in some element of \( S \), then there is \( f \in S \) an infinite path through \( T \).

Scott sets first arose in the study of completions of Peano arithmetic (PA) and models of PA. Scott [9] shows that the countable Scott sets are exactly the families of sets “representable” in a completion of PA. If \( \mathcal{M} \) is a nonstandard model of Peano arithmetic and \( a \in \mathcal{M} \), let

\[
r(a) = \{n \in \omega : \mathcal{M} \models p_n|a\}
\]

where \( p_0, p_1, \ldots \) is an increasing enumeration of the standard primes. The standard system of \( \mathcal{M} \)

\[
SS(\mathcal{M}) = \{r(a) : a \in \mathcal{M}\}
\]

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is a Scott set. A longstanding and vexing problem in the study of models of arithmetic is whether every Scott set arises as the standard system of a model of Peano arithmetic. The best result is from Knight and Nadel [6].

**Proposition 1.1** If \( S \) is a Scott set and \( |S| \leq \aleph_1 \), then there is a model of Peano Arithmetic with standard system \( S \).

Thus the Scott set problem has a positive solution if the Continuum Hypothesis is true, but the question remains open without additional assumptions. Later in this section, we sketch a proof of Proposition 1.1.

Scott sets also are important when studying recursively saturated structures. We assume that we are working in a computable language \( \mathcal{L} \). We fix a Gödel coding of \( \mathcal{L} \) and say that a set of \( \mathcal{L} \)-formulas is in \( S \) if the corresponding set of Gödel codes is in \( S \).

Let \( T \) be a complete \( \mathcal{L} \)-theory. The following ideas were introduced in [5], [10] and [7].

**Definition 1.2** Let \( S \subseteq 2^{\omega} \). We say that a model \( \mathcal{M} \) of \( T \) is \( S \)-saturated if:

i) every type \( p \in S_n(\emptyset) \) realized in \( \mathcal{M} \) is computable in some element of \( S \);

ii) if \( p(x, \overline{y}) \in S_{n+1}(\emptyset) \) is computable in some element of \( S \), \( \overline{a} \in M^n \) and \( p(x, \overline{a}) \) is finitely satisfiable in \( \mathcal{M} \), then \( p(x, \overline{a}) \) is realized in \( \mathcal{M} \).

If a model is \( S \)-saturated for some \( S \subseteq 2^{\omega} \), then the model is certainly recursively saturated.

**Proposition 1.3** If \( \mathcal{M} \models T \) is recursively saturated, then \( \mathcal{M} \) is \( S \)-saturated for some Scott set \( S \).

If the theory \( T \) has limited coding power, then we can say little about \( S \). For example, an algebraically closed field of infinite transcendence degree will be \( S \)-saturated for every Scott set \( S \). On the other hand, the associated Scott set is unique for many natural examples, such as Peano arithmetic, divisible ordered abelian groups, real closed fields, \( \mathbb{Z} \)-groups (models of \( \text{Th}(\mathbb{Z},+) \)) and Presburger arithmetic (models of \( \text{Th}(\mathbb{Z},+,<) \)).

**Definition 1.4** We say that a theory \( T \) is effective perfect if there is a tree \( (\phi_\sigma : \sigma \in 2^{<\omega}) \) of formulas in \( n \)-free variables computable in \( T \) such that:

i) \( T + \exists \overline{a} \phi_\sigma(\overline{a}) \) is consistent for all \( \sigma \);

ii) if \( \sigma \subseteq \tau \), then \( T \models \phi_\tau(\overline{a}) \rightarrow \phi_\sigma(\overline{a}) \);

iii) \( \phi_\sigma \neg_0(\overline{a}) \land \phi_\sigma \neg_1(\overline{a}) \) is inconsistent with \( T \) for all \( \sigma \).
Proposition 1.5 If $T$ is effectively perfect, then every recursively saturated model of $T$ is $S$-saturated for a unique Scott set $S$.

The theories we will be considering are all effectively perfect. For Peano arithmetic, Presburger arithmetic and $\mathbb{Z}$-groups we can use the formulas $p_n|v$ to find such a tree. For real closed fields we can use $q < v$ for $q \in \mathbb{Q}$ and for ordered divisible abelian groups we can use the binary formulas $mv < nw$ for $m, n \in \mathbb{Z}$.

We now sketch a proof of Proposition 1.1. We first note that for models $\mathcal{M}$ of Peano arithmetic, $\mathcal{M}$ is recursively saturated if and only if $\mathcal{M}$ is $S$-saturated where $S$ is the standard system of $\mathcal{M}$. (For more details see [4]).

Lemma 1.6 If $S$ is a countable Scott set and $T \in S$ is a completion of Peano arithmetic, then there is an $S$-saturated model of $T$.

Proof Sketch Build $\mathcal{M}$ by a Henkin construction. At any stage, we will have a finite tuple $\bar{a}$ and will be committed to $\text{tp}(\bar{a})$ the complete type of $\bar{a}$, where $T \subseteq \text{tp}(\bar{a}) \in S$. At alternating stages, we either witness an existential quantifier or realize a type $p(v, \bar{a}) \in S$, using the join property of Scott sets to compute $p(v, \bar{x}) \cup \text{tp}(\bar{a})$, and using the tree property to find completions. □

Lemma 1.7 Suppose $S_0 \subset S_1$ are countable Scott sets, $T \in S_0$ is a completion of Peano arithmetic, and $\mathcal{M}_0$ and $\mathcal{M}_1$ are countable recursively saturated models of $T$, where $S_i$ is the standard system of $\mathcal{M}_i$. Then there is an elementary embedding of $\mathcal{M}_0$ into $\mathcal{M}_1$.

Proof Sketch Let $a_0, a_1, \ldots$ be a list of the elements of $\mathcal{M}_0$. Suppose we have a partial elementary map $(a_0, \ldots, a_n) \mapsto (b_0, \ldots, b_n)$. If $\text{tp}(a_{n+1}, a_0, \ldots, a_n) = p(v, a_0, \ldots, a_n)$, there is $b \in \mathcal{M}_1$ realizing $p(v, b_0, \ldots, b_n)$, and we can extend the embedding. Box

We can now prove Proposition 1.1. Suppose $|S| = \aleph_1$ and $S$ is the union of an $\omega_1$-chain of countable Scott sets

$$S_0 \subseteq S_1 \subseteq \ldots \subseteq S_\alpha \subseteq \ldots$$

where $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ when $\alpha$ is a limit ordinal. We can build an elementary chain $(\mathcal{M}_\alpha : \alpha < \omega_1)$ where $\mathcal{M}_\alpha$ is recursively saturated with standard
system $S_\alpha$. Then $\bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$ is recursively saturated with standard system $S$.

While we have nothing new to say about the Scott set problem for Peano arithmetic, we show that the analogous problem for recursively saturated models has a positive solution in some related algebraic settings.

For divisible ordered abelian groups, this follows easily from an unpublished result of Harnik and Ressayre. Let $(G, +, <)$ be a divisible ordered abelian group. Define an equivalence relation on $G$ by $g \equiv h$ if and only if there is a natural number $n$ such that $|g| < n|h|$ and $|h| < n|g|$. Let $\Gamma = \{|g|/ \equiv : g \in G\}$, the set of equivalence classes of positive elements. The ordering of $G$ induces an ordering on $\Gamma$. Suppose $S$ is a Scott set and $k_S$ is the set of real numbers computable in some element of $S$ (where we identify a real with its cut in the rationals). It is easy to see that $k_S$ is a real closed field.

**Theorem 1.8 (Harnik–Ressayre)** A divisible ordered abelian group $G$ is $S$-saturated if and only if $\Gamma$ is a dense linear order without endpoints and each equivalence class under $\equiv$ is isomorphic to the ordered additive group of $k_S$.

A complete proof is given in [3].

**Corollary 1.9** For any Scott set $S$, there is an $S$-saturated divisible ordered abelian group.

**Proof** Let $G$ be the set of functions $f : \mathbb{Q} \to k_S$ such that $\{q \in \mathbb{Q} : f(q) \neq 0\}$ is finite. We add elements of $G$ coordinatewise and order $G$ lexicographically. By the Harnik–Ressayre Theorem, $G$ is $S$-saturated. \qed

## 2 Real Closed Fields

In [2], D’Aquino, Knight and Starchenko show that if $\mathcal{M}$ is a nonstandard model of Peano arithmetic with standard system $S$, then the real closure of the fraction field of $\mathcal{M}$ is an $S$-saturated real closed field. Thus, it is natural to ask whether we can find an $S$-saturated real closed field for every Scott set $S$.

**Theorem 2.1** For any Scott set $S$, there is an $S$-saturated real closed field.
The value group of an $S$-saturated real closed field will be an $S$-saturated divisible ordered abelian group. Thus, Corollary 1.9 will also follow from Theorem 2.1.

Theorem 2.1 is a simple induction using the following Lemma.

**Lemma 2.2** Let $S$ be a Scott set. Let $K$ be a real closed field such that every type realized in $K$ is in $S$. Suppose $p(v, \overline{w})$ is a set of formulas in $S$, $\overline{w} \in K$, and $p(v, \overline{w})$ is finitely satisfiable in $K$. Then we can realize $p(v, \overline{w})$ by a (possibly new) element $b$ such that every type realized in $K(b)^{\text{rel}}$ is in $S$.

**Proof** The set of formulas $p(v, \overline{a}) \cup \text{tp}(\overline{a})$ is a consistent partial type in $S$, and, hence, has a completion in $S$. Thus, without loss of generality, we may assume $p(v, \overline{a})$ is a complete type. If $p(v, \overline{a})$ is realized in $K$, then there is nothing to do. If $p(v, \overline{a})$ is not realized in $K$ then it determines a cut in the ordering of $\mathbb{Q}(\overline{a})^{\text{rel}}$ that is not realized in $K$, and, hence, by o-minimality, it determines a unique type over $K$. Let $b$ realize $p(v, \overline{a})$ and let $\overline{c} \in K$. We need to show that $\text{tp}(b, \overline{c})$ is in $S$.

How do we decide whether $K(b)^{\text{rel}} \models \phi(b, \overline{c})$? By o-minimality, $\phi(v, \overline{c})$ defines a finite union of points and intervals with endpoints in $\mathbb{Q}(\overline{c})^{\text{rel}}$. Since $b \not\in K$, $b$ is neither one of the distinguished points nor an endpoint of one of the intervals. There are $\emptyset$-definable Skolem functions $f$ and $g$ such that:

i) $f(\overline{a}) < v < g(\overline{a}) \in \text{tp}(b/\overline{a})$;

ii) $f(\overline{a}) < v < g(\overline{a}) \rightarrow \phi(v, \overline{c})$ or $f(\overline{a}) < v < g(\overline{a}) \rightarrow \neg\phi(v, \overline{c})$.

Given $\text{tp}(b/\overline{a})$ and $\text{tp}(\overline{a}, \overline{c})$ we can computably search and find the decomposition of $\phi(v, \overline{c})$ and $f$ and $g$ as above. We can then decide whether $\phi(b, \overline{c})$ holds. Since $S$ is closed under join and Turing reducibility, $\text{tp}(b, \overline{c})$ is in $S$. \qed

The above argument works for any o-minimal theory $T \in S$.

Every real closed field $K$ has a natural valuation for which the valuation ring is

$$\mathcal{O} = \{ x : |x| < n \text{ for some } n \in \mathbb{N} \}.$$ 

If $K$ is recursively saturated, then the value group is a recursively saturated divisible ordered abelian group. It is natural to ask if every recursively saturated divisible ordered abelian group arises this way.

D’Aquino, Kuhlmann and Lange \[3\] gave a valuation-theoretic characterization of recursively saturated real closed fields. In the following argument we assume familiarity with their results.\[4\]

\[4\]This is essentially our original proof of Theorem 2.1
Proposition 2.3 Let $G$ be a recursively saturated divisible ordered abelian group. There is a recursively saturated real closed field with value group $G$.

**Proof Sketch** Let $S$ be the Scott set of $G$. Start with the field

$$k_S(t^g : g \in G)^{rel}.$$  

This is a real closed field with residue field $k_S$, value group $G$ and all types recursive in $S$. Given $K$ a real closed field $K$ with value group $G$ and all types recursive in $S$ and suppose we have $\overline{\pi} \in K$ and $(f_0, f_1, \ldots)$ a sequence of Skolem functions recursive in $S$ such that $(f_0(\overline{\pi}), f_1(\overline{\pi}), \ldots)$ is pseudo-Cauchy. If the sequence has no pseudo-limit in $K$ it determines a unique type over $K$. Adding a realization $b$ does not change the value group or residue field. As above, every type realized in $K(b)^{rel}$ is in $S$. We can iterate this construction to build the desired real closed field. \hfill $\square$

3 Presburger Arithmetic

In [6] Knight and Nadel proved that for every Scott set $S$ there is an $S$-saturated $\mathbb{Z}$-group, i.e., an $S$-saturated model of $\text{Th}(\mathbb{Z}, +)$. They asked whether the same is true for the theory of $(\mathbb{Z}, +, <)$. This is Presburger arithmetic, which we denote Pr. We answer this question in the affirmative.

**Theorem 3.1** For every Scott set $S$, there is an $S$-saturated model of Presburger arithmetic.

We will consider Presburger arithmetic in the language that includes constants for 0 and 1 and unary predicates $P_n(v)$ for $n = 2, 3, \ldots$ that hold if $n$ divides $v$. We can eliminate quantifiers in this language and the resulting structure is quasi-o-minimal; i.e., any formula $\phi(v, \overline{\pi})$ defines a finite Boolean combination of $0$-definable sets and intervals with endpoints in $\text{dcl}(\overline{a}) \cup \{\pm \infty\}$. We will use this in the following form. (See, for example [8] §3.1 for quantifier elimination and [1] for quasi-o-minimality.)

**Lemma 3.2** i) Any formula $\phi(v, \overline{\pi})$ is equivalent over $\text{tp}(\overline{\pi})$ to a Boolean combination of formulas of the form $v \equiv m \mod n$, $v = \alpha$, $v < \beta$ where $\alpha, \beta$ are in the definable closure of $\overline{a}$ and $m, n \in \mathcal{N}$.

ii) $\text{tp}(b, \overline{\pi})$ is determined by:
• \( \text{tp}(\overline{a}) \);
• the sequence \( b \mod 2, b \mod 3, b \mod 4, \ldots \);
• the cut of \( b \) in the definable closure of \( \overline{a} \).

We obtain Theorem 3.1 by an iterated construction using the following lemma.

**Lemma 3.3** Let \( S \) be a Scott set. Let \( G \models \text{Pr} \) such that every type realized in \( G \) is computable in \( S \). Suppose \( \overline{a} \in G \) and \( p(v, \overline{w}) \) is a complete type in \( S \) such that \( p(v, \overline{a}) \) is finitely satisfiable. Then there is \( H \supseteq G \) such that \( H \models \text{Pr} \), such that \( p(v, \overline{a}) \) is realized in \( G \) and every type realized in \( H \) is in \( S \).

**Proof** If \( p(v, \overline{a}) \) is realized in \( G \), then there is nothing to do, so we assume \( p(v, \overline{a}) \) is not realized in \( G \). Let \( p^-(v, \overline{a}) \) be the partial type describing the cut of \( v \) over the definable closure of \( \overline{a} \), i.e., \( p^- \) consists of all formulas of the form \( mv < \sum n_i a_i \) or \( mv > \sum n_i a_i \) that are in \( p \) where \( m, n_i \in \mathbb{Z} \).

**Case 1:** Suppose \( p^- \) is omitted in \( G \).
Let \( b \) be any realization of \( p \), and let \( H \) be the definable closure of \( G \cup \{ b \} \). It is enough to show that if \( \overline{c} \in G \), then \( \text{tp}(b, \overline{c}) \) is in \( S \). We will show that \( \text{tp}(b, \overline{c}) \) is recursive in \( \text{tp}(b, \overline{a}) \) and \( \text{tp}(\overline{a}, \overline{c}) \). Using only \( \text{tp}(b, \overline{a}) \), we can determine \( b \mod n \) for all \( n \). Thus, we only need to consider formulas of the form \( \alpha < v < \beta \) where \( \alpha, \beta \in \text{dcl}(\overline{a}) \). Since \( p^-(v, \overline{a}) \) is omitted, we can, as in the case of real closed fields, search to find \( \gamma, \delta \in \text{dcl}(\overline{a}) \) such that \( \gamma < b < \delta \) and either \( \alpha \leq \gamma < \delta \leq \beta \), \( \delta < \alpha \) or \( \gamma > \beta \). This can be done recursively in \( \text{tp}(b, \overline{a}) \) and \( \text{tp}(\overline{a}, \overline{c}) \). Thus, every type realized in \( H \) is in \( S \).

**Case 2:** Suppose \( b \in G \) realizes \( p^- \).
Let \( \hat{b} \) be a realization of \( p(v, \overline{a}) \) and let \( q_0(v) \) be the divisibility type of \( \hat{b} - b \), i.e., if \( \hat{b} \equiv m \mod n \) and \( b \equiv l \mod n \) then \( "v \equiv m - l \mod n" \in q_0 \) for \( l, m \in \mathbb{Z} \) and \( n > 1 \).
Let \( q(v) \in S_1(G) \) be the unique type containing
• \( q_0(v) \);
• \( n < v \) for all \( n \in \mathbb{Z} \);
• \( v < g \) for all \( g \in G \) such that \( \mathbb{Z} < g \).
Let $\epsilon$ realize $q$ and let $H$ be the definable closure of $G \cup \{\epsilon\}$. Suppose $\alpha \in \text{dcl}(\overline{a})$ and $b < \alpha$. Since $G \models \text{Pr}$, we have $b + n < \alpha$ for all $n \in \mathbb{Z}$. Thus, $\epsilon < \alpha - b$ and $b + \epsilon < \alpha$. Similarly, if $\alpha < b$, then $\alpha < b + \epsilon$ and, thus, $b + \epsilon$ realizes $p^-(v, \overline{a})$. By the choice of $q_0$, $b + \epsilon$ realizes $p(v, \overline{a})$.

Suppose $\overline{a} \in G$. It suffices to show that $\text{tp}(\epsilon, \overline{a})$ is in $S$. Without loss of generality, we may assume that $\overline{a} = (c_1, \ldots, c_n)$ where all of the $c_i$ are positive infinite and $1, c_1, \ldots, c_n$ are linearly independent over $\mathbb{Q}$. We need to decide the signs of expressions of the form

$$r + s\epsilon + \sum_{i=1}^{n} t_ic_i$$

where $r, s, t_i \in \mathbb{Z}$. Such an expression is positive if and only if

- $\sum t_ic_i > 0$, or
- $\sum t_ic_i = 0$ and $q > 0$, or
- $\sum t_ic_i = s = 0$ and $r > 0$

This can be computed using $\text{tp}(\overline{a})$. Thus $\text{tp}(\epsilon, \overline{a})$ is recursive in $q_0$ and $\text{tp}(\overline{a})$. Hence, every type realized in $H$ is in $S$. \qed

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