A Second Order IMEX Method for Radiation Hydrodynamics

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Abstract. Radiation hydrodynamics problem is multphysics field coupled and challenge in mathematics. A new scheme, IMEX, is put forward in solving RHD equations when the explicit method have an extremely small time step and implicit method have a complicated and high cost solver. RHD equations are solving by splitting the explicit operator and implicit operator with IMEX scheme. The results of Riemann problems are plotted and the scheme shows the effect to lower the computation cost.

1. Introduction
When we try to solve the equations as Radiation hydrodynamics (RHD), it is so challenge in numerical calculation because of the strongly multi-physics field coupled equations. In the radiation transport, implicit scheme is applied to solve the equations with the characteristic velocity equal to the ray velocity in case of limit of the Courant number. However, explicit method is used to solve the fluid solution [1] [2] [3]. Thus there are two main challenges to solve RHD equations: a) highly-nonlinear algorithm and b) large disparities between temporal and spatial scales in the multi-physics field, compared with the single physics phenomenon.

The method introduced focuses on the difference of temporal scale in the multi-physics field [2] [3], and is mainly applied on the radiation hydrodynamics equations [4].

2. Philosophy of IMEX
The philosophy of the method is:

- Using an explicit scheme on the larger temporal scale
- Using an implicit scheme on the smaller temporal scale

Thus, the method is named as IMEX, which combine the IMplicit scheme and the Explicit scheme [5] [6]. The method has some advantages, for example, the method can be computational less expensive than the purely implicit method and can have a larger temporal scale than explicit method [7] [8].

When we use IMEX method, the general equation [9] can be expressed as:

$$\frac{du}{dt} = E(u) + I(u)$$

(1)

E (U) is the explicit component and the I(U) is the implicit component.

2.1. BDF2 scheme
If we consider the application of the BDF2 scheme [10] to discretize the equation (1)
\[
\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = 2E(u^n) - E(u^{n-1}) + I(u^{n+1})
\]  (2)

This scheme performs well but when \(I(u) \rightarrow 0\), it plays poor treatment of the explicit operator and a property IMEX should be based on good explicit and implicit operator performance. It is necessary to design such a scheme corresponding to this requirement.

### 2.2. BDF2-RK2 scheme

When \(I(u) \equiv 0\), the scheme should degrade to a second order, explicit RK2 scheme,

\[
\begin{aligned}
    v_{n+1} &= u^n + \Delta t \cdot E(u^n) \\
    u_{n+1} &= u^n + \Delta t \cdot \left[ E(u^n) + E(v^{n+1}) \right] / 2
\end{aligned}
\]  (3)

When \(E(u) \equiv 0\), the scheme should degrade to BDF2 scheme,

\[
\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = I(u^{n+1})
\]  (4)

To satisfy these two requirement, the IMEX scheme should be expressed as:

\[
\begin{aligned}
    v_{n+1} &= u^n + \Delta t \left[ E(u^n) + I(v^{n+1}) \right] \\
    I(u^{n+1}) &= \frac{3}{2} RK(u, v, n) - \frac{1}{2} RK(u, v, n - 1) \\
    RK(u, v, n) &= \frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{2} [E(u^n) + E(v^{n+1})]
\end{aligned}
\]  (5)

Without any precision loss, the first equation in the (6) can be rewritten as

\[
v_{n+1} = u^n + \Delta t [E(u^n) + I(u^{n+1})]
\]  (6)

### 3. Radiation Hydrodynamics Equations

The radiation model in the coupled equations is the non-relativistic Euler equations with a grey P1 radiation treatment in non-dimensional form

\[
\begin{aligned}
    \frac{\partial}{\partial t} \rho + \frac{\partial (\rho u)}{\partial x} &= 0 \\
    \frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} &= -\mathcal{P} S_F \\
    \frac{\partial (\rho E)}{\partial t} + \frac{\partial (\rho u E + p)}{\partial x} &= -\mathcal{P} S_F \\
    \frac{\partial F_r}{\partial t} + \frac{1}{3} \frac{\partial F_r}{\partial x} &= S_E \\
    \frac{\partial F_r}{\partial t} + \frac{1}{3} \frac{\partial F_r}{\partial x} &= C S_F
\end{aligned}
\]  (7)

The expression of some operator in equations (7) is

\[
\begin{aligned}
    S_E &= \xi \sigma_t \Phi + \sigma_t v F_{r0} \\
    S_F &= -\sigma_t F_{r0} + \sigma_t \frac{v}{r} \Phi \\
    \Phi &= T^4 - E_r \\
    F_{r0} &= F_r + 4 \frac{v}{3} E_r
\end{aligned}
\]  (8)

\(E_r\) is the radiation energy, \(F_r\) is the radiation flux, and \(\sigma_t\) describes the degree of the opacity.

The non-dimensional parameters in equations (7) and (8) is

\[
\mathcal{P} = \frac{\alpha_r T_0^4}{\rho_{\infty} a_{\infty}} , \quad C = \frac{c}{a_{\infty}}
\]  (9)

\(\alpha_r\) is Stefan coefficient, \(c\) is light speed, and \(a_{\infty}\) is reference acoustic speed.
4. Implementation

4.1. Newton-Raphson Iteration
For the implicit component, Newton-Raphson iteration is applied:

\[
\begin{align*}
\frac{d_{k+1}^{n+1} - d_n}{\Delta t} + R(u_{k+1}^{n+1}) &= 0 \\
\frac{d_{k+1}^{n+1} - d_n}{\Delta t} + R(u_{k+1}^{n+1} + \Delta u) &= 0 \\
\Delta u \left( \frac{i}{\Delta t} - \frac{\partial R(u)}{\partial u} \bigg|_{u_{k+1}^{n+1}} \Delta u \right) &= - \left( \frac{d_{k+1}^{n+1} - d_n}{\Delta t} + R(u_{k+1}^{n+1}) \right)
\end{align*}
\]

In the equations (10), \( \hat{I} \) is a unit matrix. In the iterations, when \( k \) is increasing, \( \Delta u \) is approaching to 0.

4.2. Splitting the explicit and implicit component
If we directly compute the equations (7), equations (8) and equations (9) without splitting the explicit and implicit component, we have to solve the following equations:

\[
\begin{align*}
\frac{\partial (\rho v + E_F r)}{\partial t} + \frac{\partial (E + 1/2 \rho E_r)}{\partial x} &= 0 \\
\frac{\partial (\rho v^2 + v P)}{\partial t} + \frac{\partial E}{\partial x} &= 0
\end{align*}
\]

In equations (11), the temporal scale of the hydrodynamics part has large difference with that of the radiation part. Thus the explicit operator is used in hydrodynamic component and the implicit operator is used in the radiation component. According to the equations (5), the explicit operator and the implicit operator as:

\[
E(u) = \begin{pmatrix}
\frac{\partial (\rho v)}{\partial x} \\
\frac{\partial (\rho v^2)}{\partial x} \\
\frac{\partial (\rho E + P v)}{\partial x} \\
0 \\
\end{pmatrix}, \quad I(u) = \begin{pmatrix}
0 \\
\rho S_F \\
\rho S_E \\
-C_F \rho E_r - S_E \\
\frac{1}{3} C_F \rho E_r - C S_F
\end{pmatrix}
\]

In the equations (12), \( E(u) \) is the explicit operator and \( I(u) \) is the implicit operator.

4.3. Roe Solver
Roe solver is selected as the Godunov solver for the equations (5), and the one dimension solver for hydrodynamics is

\[
E(u) = \frac{1}{2} (F_L + F_R) - \frac{1}{2} \left[ |\lambda|_3 \Delta \rho + C_1 \right] \\
|\lambda|_3 \Delta (\rho u_{avg}) + C_1 u_{avg} + C_2 N_x \\
|\lambda|_3 \Delta (\rho E) + C_1 H_{avg} + C_2 u_{avg}
\]

And,

\[
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix}
\rho \\
\rho v \\
\rho E
\end{pmatrix}
\]

The implicit solver can be expressed as

\[
F_{Roe}^{Rad} = \frac{1}{2} (F_L^{Rad} + F_R^{Rad}) - \frac{1}{2} \sqrt{3} \frac{\Delta E_r}{\Delta F_r}
\]
In the equation (15),
\[
\begin{align*}
\left( \frac{\Delta E_r}{\Delta F_r} \right) &= \left( \frac{(E_r)_R - (E_r)_L}{(F_r)_R - (F_r)_L} \right) \\
(16)
\end{align*}
\]

After linearization, the radiation part can use the implicit solver.

5. Case

Two cases can be applied to observe the result of the numerical scheme. The first case is “Diffusion Limit”, which means high opacity, bad light transmission and a lot of refraction and reflection of the radiation. The second case is called “Streaming Limit”, which means low opacity, good light transmission, few obstacles for radiation transmission. In the second case, fluid and radiation is coupled weakly, and thus the energy of the radiation spreads much faster than that of the fluid.

5.1. Diffusion Limit

Under the case of diffusion limit, the Riemann problem is described as
\[
\begin{align*}
\rho_L &= 1, U_L = 0, P_L = 1 \\
\rho_R &= 0.125, U_R = 0, P_R = 0.1 \\
\mathbb{C} &= 1000, \mathbb{P} = 0.001, \sigma_t &= 1000 \\
\Delta x &= 1E10^{-3}, t_{total} = 5E10^{-3}
\end{align*}
\]

And the result is

(a) Density                           (b) Velocity
(c) Pressure                      (d) Internal Energy
5.2. Streaming Limit

Under the case of Streaming limit, the Riemann problem is described as

\[ \rho_L = 1, U_L = 0, P_L = 1 \]
\[ \rho_R = 0.125, U_R = 0, P_R = 0.1 \]
\[ \mathbb{C} = 10, \mathbb{P} = 0.001, \sigma_t = 1 \]
\[ \Delta x = 1E10^{-3}, t_{\text{total}} = 5E10^{-3} \]

And the result is
6. Conclusion

The cases above demonstrate the there is the tight coupling under the diffusion limit of the RHD equations but the loose coupling under the streaming limit.

For the pure explicit method, there are two characteristic speeds: the radiation characteristic speed, which is the light speed, and the fluid characteristic speed, which is the acoustic speed. In the coupling process, the light speed is the leading speed.

In the problem of diffusion limit, this scheme iterates 16667 steps, and the average time step is $3.00 \times 10^{-7}$ second, and therefore the characteristic speed is $3.33 \times 10^4$ m/s. As the characteristic speed is light speed in the explicit scheme, the time step increase remarkably. In the problem of the streaming limit, the characteristic speed is $3.00 \times 10^5$ m/s, which also leads to a significant time step increase.

If we use a pure implicit method, it costs too much on the Roe solver for the fluid equations because it is much more difficult to calculate the linearized matrix.

This method might be used in the equations for implicit scheme and meanwhile costs less than the implicit scheme. When the implicit operator or the explicit operator incline to 0, the method still has stable performance. In the future, the IMEX method can be combine with the other high precious method such as DG, to solve the RHD equations.

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