Coarse-grained Hydrodynamics of turbulent superfluids: HVBK vs HST.
Comment to discussion section of INT 2019-1a program.

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In the comment I develop a critical analysis of the use of the HVBK method for the study of three-dimensional turbulent flows of superfluids. The conception of the vortex bundles forming the structure of quantum turbulence is controversial and does not justify the use of the HVBK method. In addition, this conception is counterproductive, because it gives incorrect ideas about the structure of the vortex tangle as a set of bundles containing parallel lines. The only type of dynamics of vortex filaments inside these bundles is possible, namely, Kelvin waves running along the filaments. At the same time, as shown in numerous numerical simulations, a vortex tangle consists of a set of entangled vortex loops of different sizes and having a random walk structure. These loops are subject to large deformations (due to highly nonlinear dynamics), they reconnect with each other and with the wall, split and merge, creating a lot of daughter loops. They also bear Kelvin waves on them, but the latter have little impact.

I also propose and discuss an alternative variant of study of three-dimensional turbulent flows, in which the vortex line density \( L(r, t) \) is not associated with \( \nabla \times \mathbf{v}_s \), but it is an independent variable described by a separate equation.

I. INTRODUCTION

In the comment, I discuss the problems of construction of coarse-grained hydrodynamics of turbulent flows of superfluids. In particular, I develop a critical analysis of the use of the HVBK method for the study of three-dimensional flows of superfluids. The conception of the vortex bundles forming the structure of quantum turbulence is also critically discussed. I also propose an alternative variant in which the vortex line density \( L(r, t) \) is not associated with \( \nabla \times \mathbf{v}_s \), but it is an independent and equipollent variable described by a separate equation.
II. COARSE-GRAINED HYDRODYNAMICS OF TURBULENT SUPERFLUIDS.

In presence of the vortex filaments the two-fluid hydrodynamics of the superfluid helium should be modernized and be represented as follows:

\[ \rho_n \partial_t v_n + \rho_n (v_n \cdot \nabla) v_n = -\frac{\rho_n}{\rho} \nabla p_n - \rho_s s \nabla T + F_{mf} + \eta \nabla^2 v_n, \]  

(1)

\[ \rho_s \partial_t v_s + \rho_s (v_s \cdot \nabla) v_s = -\frac{\rho_s}{\rho} \nabla p_s + \rho_s s \nabla T - F_{mf}, \]  

(2)

We assume that the motion of both components is incompressible, \( \nabla \cdot v_n = 0, \nabla \cdot v_s = 0 \), and where \( v_n \) and \( v_s \) are the coarse-grained velocity of the normal and superfluid component (averaged over a small volume \( V \)), \( p_n \) and \( p_s \) are the effective pressures acting on the normal and the superfluid component (\( \nabla p_n = \nabla p + (\rho_s/2) \nabla w^2 \) and \( \nabla p_s = \nabla p - (\rho_n/2) \nabla w^2 \)), \( p \) is the total pressure, \( s \) is the entropy, \( T \) is the absolute temperature, \( \eta \) is the dynamic viscosity of the normal component, and \( v = |v_n - v_s| \). The effects of the vortices on the two components (normal and superfluid) are described by the friction force exerted by the superfluid component on the normal component \( F_{mf} \).

The normal component reacts to a moving vortex by producing the "microscopic" mutual friction force \( f_{mf} \) (see e.g. [1]). Starting from the microscopical viewpoint, it is possible to find the macroscopical expression of the mutual friction force \( F_{mf} \). When this force is averaged over all vortices inside the small volume \( V \), then the following expression of \( F_{mf} \) is obtained

\[ F_{mf} = [f_{mf}]_{av} = \mathcal{L} < f_{MF} >= \alpha \rho_s \kappa \mathcal{L} < s' \times (s' \times (v_{ns} - v_i)) > + \alpha' \rho_s \kappa \mathcal{L} < s' \times (v_{ns} - v_i) >. \]  

(3)

In this equation \( s' \) the tangent vectors of the vortex filaments composing the vortex tangle, \( \alpha, \alpha' \) are temperature-dependent dimensionless mutual friction parameters.

Equations (1) and (2) are coarse-grained equations, hence the inclusion of the effects of the vortex lines requires a high vortex line density per unit volume. These equations had been written and discussed for long term, recent variant coinciding with stated above can be found in ([2],[3])

The above written equations are the point of consensus among physicists. Disagreements begin with the question how to treat variable \( \mathcal{L} \). There are two way to treat the variable \( \mathcal{L}(r,t) \). They are the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) model (see e.g., book by Khalatnikov, 1965) and the Hydrodynamics of superfluid turbulence model (Nemirovskii & Lebedev, 1983 [5],[6]).
III. HVBK APPROACH FOR ROTATING SUPERFLUIDS

The Hall-Weinen-Bekarevich-Khalatnikov (HVBK) model (see, for example, the book Khalatnikov [4]) became the basis for the mathematical formalism of the hydrodynamics of rotating superfluid fluids. As is well known (see [7]), in a vessel rotating with an angular velocity $\Omega$, appears a regular array of vortex filaments with a density $n = 2\Omega/\kappa$. Such a distribution of vortices creates an average coarse-grained superfluid velocity $\langle \mathbf{v}_s \rangle$, which satisfies the condition of solid rotation $\langle \mathbf{v}_s \rangle = \Omega \times \mathbf{r}$. The vorticity filed $\omega$ is $\omega = 2\Omega$. As a result, the density of the vortex filaments in this case can be related to the vorticity field by the following relation:

$$\nabla \times \mathbf{v}_s = \kappa \mathbf{L} \quad (4)$$

Due to the smallness of the quantum of circulation $\kappa$, even a relatively weak rotational speed produces a high density of vortex lines. Thus, it is possible to construct a coarse-grained hydrodynamics of hydrodynamic equations, which the average contribution of many individual vortex lines and incorporate their contribution to the macroscopic evolutionary equations for superfluid and normal He II velocities.

Combining (3) with (4) one get expression for average coarse-grained mutual friction $F_{mf}^{(HVBK)}$

$$F_{mf}^{(HVBK)} = \rho_s \omega \times [\omega \times (\mathbf{v} - \tilde{\beta} \nabla \times \hat{\omega})] + \rho_s \alpha' \omega \times (\mathbf{v} - \tilde{\beta} \nabla \times \hat{\omega}), \quad (5)$$

where $\omega = \nabla \times \mathbf{v}_s$ is the averaged superfluid vorticity, $\hat{\omega} = \omega / |\omega|$. Thus, the question of elimination of the vortex line density is resolved with the use of the Feynman rule, it allows to study various problems of coarse-grained dynamics of rotating superfluids (see, e.g. book be Sonin [8]).

IV. HVBK APPROACH FOR THREE-DIMENSIONAL CASE

The HVBK model is a fruitfull and elegant approach, however it is principally assigned for rotating superfluids. Nevertheless this approach, which uses relations $\nabla \times \mathbf{v}_s = \kappa \mathbf{L} \quad (4)$, and the force $F_{mf}^{(HVBK)} \quad (5)$ is widely used for numerical and analytical study of coarse-grained hydrodynamic problems of turbulent superfluid in three-dimensional situations.

My categorical position is that I strongly disagree with using HVBK approach in three-dimensional cases. In my sight there is no way to apply it to three-dimensional hydrodynamics. Anticipating objections from the readers I would like to discuss usual arguments in favor of the use HVBK method. Probably, the only argument is the belief that vortex tangle consists of the so
called vortex bundles, which include all vortex filaments, existing in the vortex tangle (otherwise the use of the Feynman rule, which relates VLD $\mathcal{L}$ to the coarse-grained vorticity $\nabla \times \mathbf{v}_s$ is invalid). But in fact, there are no evidences of the existence of these structures.

There are few papers (see, e.g. [9], [10]), where the authors claim of the existence of the bundles. In fact, they only demonstrated how, with the help of statistical analysis, one can get a small polarization (prevailing of one direction over the other) in the vortex tangle. But, firstly, it is just a statistical effect and, secondly, under no circumstances this small polarization allows to use the Feynman rule (4), which is crucially needed in the HVBK equations. To use this ansatz, it is necessary that all filaments in the vortex tangle are involved in the rotation. But if the polarization is partial, then there is a lot of free (randomly orientated) vortex filaments that contribute to mutual friction and do not contribute to the Feynman rule (4). In addition, these chaotic lines interact with polarized lines, thereby destroying the polarization and, correspondingly, the quasi-bundle structure.

There are other examples of observation of vortex bundles (in numerical works), when they are artificially prepared structure, or are initiated by eddies of normal component (see, e.g. [11], [11]). However, there are works in the literature, in which it is stated that even if the vortex bundles are artificially created, they can be destroyed rather soon. For instance, G. Volovik [12]) have shown that at low temperatures, where the mutual friction is small, the existence of the bundles is impossible. They should melt, changing into a highly irregular structure. Other example is a series of numerical simulations by Kivotides [13], [14], [15], who studied the exact (not HVBK) dynamics of quantum vortices in the turbulent flows (at finite temperature) and concluded that the results do not show that a turbulent normal-fluid with a Kolmogorov energy spectrum induces superfluid vortex bundles in the superfluid.

There are many physical mechanisms that result in the destruction of the regular vortex bundle structure. It is the interaction (and/or reconnections) of random free lines (which are not involved into rotation) with the regular lines, composing the bundles. Numerous reconnections between lines inside bundles, or between neighboring bundles definitely violate any regular structure. The long-range interaction between vortex filaments in the bundle and the "external" vortices also destroy regular array due to the action of tidal forces.

One more objection to using HVBK approach is that the Feynman rule (4) is applicable only for stationary "thermodynamics" situations. It is not clear its validity for transient processes that take place in highly fluctuating turbulent flows.

Resuming, I can state that HVBK is definitely incorrect in the three-dimensional case, therefore,
works using this approach are questionable, and the corresponding results are not reliable.

V. OTHER WAYS TO TREAT THE VORTEX LINE DENSITY

The main goal in HVBK approach was to rid out of vortex line density $L$ in equations of motions (1) - (3).

I think that the question of "elimination" the vortex line density $L(r, t)$ should be principally resolved in different way. We have not to "eliminate" it, but on the contrary, to include it into consideration. It is necessary to consider the three-field problem, in which there are three independent and equipollent variables - velocities $v_n$ and $v_s$, and vortex line density $L(r, t)$. In this way, however, we need an additional independent equation for the temporal and spacial evolution of quantity $L(r, t)$.

This is particular task that require a lot of effort. Derivation of such equation, certainly, depends on the type of flow of superfluid, e.g. counterflow, co-flow, flow past objects, unsteady rotation, etc. In fact, so far the corresponding equation exists only for case of counterflow, this is the famous Vinen equation, (or some modernized versions of this equation). Although, there are some problems with this equation (see e.g. Ch. IV in [16]), it works well for hydrodynamic (e.g. acoustic or engineering) problems. It is important that the construction of a theory of the evolution of three fields ($v_n$, $v_s$ and $L(r, t)$) is not an automatic addition of Vinen equation to the two-fluid Hydrodynamics. It is a more involved procedure, since all hydrodynamic equations change in the presence of the vortex tangle. This self-consistent procedure was implemented in [5], it is called Hydrodynamics of Superfluid Turbulence (HST).

The Vinen equation reflects the fact that the vortex line density $L(r, t)$ growth due to the relative velocity $v_n - v_s$ and attenuates, probably due to the cascade like breaking down of vortex loops, described by Feynman [7]. That is good guideline how to develop an appropriate theory, for any flow, attracting, of course, some auxiliary speculations.

There was one more, crude way, which was applied on the early stages of research on the superfluid turbulence. It was the use the Gorter - Mellink formula for mutual friction which immediately follows, from the Vinen theory

$$F_{mf} \propto A(T)(v_n - v_s)^2(v_n - v_s).$$

Here $A(T)$ is the Gorter - Mellink. This replacement also resolves the problem of "elimination"
the vortex line density $L(r,t)$. I think that the ansatz $L \propto (v_n - v_s)^2$, used in Eq. (6), not worse than $\nabla \times v_s = \kappa L$, which is used in HVBK approach.

VI. WHERE IS IT FROM?

It is somewhat mysterious question how did the idea of the using a pure rotational HVBK approximation for a three-dimensional turbulent flows arise? I analyzed a large mass of literary sources. The most frequent references are to the papers by Sonin [17] and [18]. But these links are absolutely irrelevant, since the authors definitely wrote that they work with rotating helium. Probably, the one of the first papers in which the use of this method for three-dimensional turbulent flows is discussed is the work of Holm [19]. It is ridiculous, however, that he started paper with the text "Recent experiments establish the Hall-Vinen-Bekarevich-Khalatnikov (HVBK) equations as a leading model for describing superfluid Helium turbulence. See Nemirovskii and Fiszdon [1995] and Donnelly [1999] for authoritative reviews."

But R. Donnelly [20] discussed HVBK approach namely for rotating helium. As for my (with W. Fiszdon) paper [6] on superfluid turbulence, then, firstly, there was no mention on HVBK theory at all, and, secondly, I generally opposed to this method for three-dimensional quantum turbulence.

I can conclude that the origin of the idea of the using a pure rotational HVBK approximation for a three-dimensional turbulent flows is rather vague.

VII. CONCLUSION

So, I described two approaches how to investigate turbulent flow in superfluids - HVBK and HST. In the first one, the quantity $L$ (which determines mutual friction) is straightforwardly excluded from equations of motion (1) and (2) with the use of Feynman rule (4). In the second approach, the variable $L(r,t)$ is considered as additional variable with the according equation of evolution.

We have argued that the HVBK is only suitable for rotating cases and fails in three-dimensional situations. And the commonly used vortex bundle model, which justifies the use of this method, is inconsistent.

Moreover, in my opinion, the vortex bundle model is counterproductive because it gives incorrect ideas about the structure of the vortex tangle as a set of bundles containing parallel lines. The
only type of dynamics of vortex filaments inside these bundles is possible, namely, the Kelvin waves running along the filaments.

Whilst, as shown by numerous numerical simulations, the vortex tangle consists of a set of entangled vortex loops of different sizes and having a random walk structure. These loops are subject to large deformations (due to highly nonlinear dynamics), they reconnect with each other and with the wall, split and merge, creating a lot of daughter loops. They also bear Kelvin waves on them, but the latter have little impact. This is a rather rich and diverse dynamics, completely different from the monotonous dynamics of Kelvin’s waves inside of bundles.

I consider that the introduction of three independent fields, a normal component velocity $v_n$, superfluid components velocity $v_s$ and a set of vortex lines $\{s(\xi, t)\}$ is the only correct approach, and not only for the problems of coarse-grained hydrodynamics of turbulent flow of superfluids.

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