A REWRITING LOGIC APPROACH TO STOCHASTIC AND SPATIAL
CONSTRAINT SYSTEM SPECIFICATION AND VERIFICATION

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Abstract. This paper addresses the issue of specifying, simulating, and verifying reactive systems in rewriting logic. It presents an executable semantics for probabilistic, timed, and spatial concurrent constraint programming —here called stochastic and spatial concurrent constraint systems (SSCC)— in the rewriting logic semantic framework. The approach is based on an enhanced and generalized model of concurrent constraint programming (CCP) where computational hierarchical spaces can be assigned to belong to agents. The executable semantics faithfully represents and operationally captures the highly concurrent nature, uncertain behavior, and spatial and epistemic characteristics of reactive systems with flow of information. In SCC, timing attributes —represented by stochastic duration— can be associated to processes, and exclusive and independent probabilistic choice is also supported. SMT solving technology, available from the Maude system, is used to realize the underlying constraint system of SCC with quantifier-free formulas over integers and reals. This results in a fully executable real-time symbolic specification that can be used for quantitative analysis in the form of statistical model checking. The main features and capabilities of SCC are illustrated with examples throughout the paper. This contribution is part of a larger research effort aimed at making available formal analysis techniques and tools, mathematically founded on the CCP approach, to the research community.

Key words and phrases: Reactive systems, constraint systems, concurrent constraint programming, rewriting logic, probabilistic rewrite theories, Maude, real-time, statistical model checking, rewriting logic semantics.
1. Introduction

Reactive systems are a broad class of concurrent systems used to meet today’s application demands in, e.g., cloud-based clusters with hundreds of processors, 100% uptime systems with milliseconds time response, and petabytes of information produced by social networks in very short periods of time. According to the Reactive Manifesto [5], a reactive system is a responsive, resilient, elastic, and message driven system that is almost the standard of many real-world applications these days. Responsive systems are about establishing reliable upper bounds so that consistent behavior builds user confidence because of rapid response times to external stimuli, also requiring that such stimuli from the unpredictable environment is dealt with properly. Resilience refers to a property that all mission-critical systems must meet: responsiveness to failure. This means that even when a failure happens, the system must continue to be responsive and meet the user requirements. Elasticity is a key feature of reactive systems that, under varying workload, can increase or decrease allocated resources. Message driven means that component interaction within a system relies on asynchronous message passing allowing, for instance, non-blocking communication in which components only consume resources while active.

The importance and proliferation of reactive systems in today’s world is central to this research. From an abstract standpoint, the reason why reactive systems are of special interest is that they can illustrate the complexity and nature of today’s ubiquitous computing where many agents execute and share information in a distributed configuration. Although this characterization is widely generic, it gives key insights on the importance and inherent complexity of reactive systems and justifies why they are gaining ground, both as an industrial paradigm and as a subject of research. However, giving mathematical foundations and formal meaning to reactive computation is a serious challenge since traditional mathematical models of computation do not single out important aspects of these systems such as information flow and hierarchical agent structures. Furthermore, without the required mathematical scaffolding, quantitative analysis useful to understand or predict, e.g., the behavior, reliability, and responsiveness—which are key attributes of correctness and utility—is well beyond the reach of any serious practitioner or industry. Therefore, the question of how to specify, simulate, and verify a reactive system is an important one, especially in the presence of quantitative demands.

Key features of reactive systems have been sufficiently addressed in the context of concurrent constraint programming (CCP) [49], a well-established process model for concurrency based upon the shared-variables communication model. Its basic intuitions arise mostly from logic; in fact, CCP processes can be interpreted both as concurrent computational entities and logic specifications (e.g., process composition can be seen as parallel execution and as conjunction). In CCP, agents can interact by posting (or telling) partial information in a medium such as a centralized store. Partial information is represented by constraints on the shared variables of the system. The other way in which agents can interact is by querying (or asking) about partial information entailed by the store. This provides the synchronization mechanism of the model: asking agents are suspended until there is enough information in the store to answer their query. As other mature models
of concurrency, CCP has been extended to capture aspects such as mobility [15, 20], and—most prominently—probabilistic [18, 17, 39, 3] and temporal [47, 46, 9, 34, 39, 7] reactive computation, where processes can be constrained also by probabilistic choice, unit delays, and time-out conditions.

Due to their centralized notion of store, all the previously-mentioned extensions are unsuitable for today’s systems where information and processes can be spatially distributed among certain groups of agents. Examples of these systems include agents posting and querying information in the presence of spatial hierarchies for sharing information and knowledge, such as friend circles and shared albums in social networks, or shared folders in cloud storage. Recently, the authors of [29] enhanced and generalized the theory of CCP to systems with spatial distribution of information in the novel spatial constraint system (SCS), where computational hierarchical spaces can be assigned to belong to agents. In SCS, each space may have CCP processes and other sub-spaces, and processes can post and query information in their given space (i.e., locally) and may as well move from one space to another.

This work is part of a larger research effort aimed at making available formal analysis techniques and tools, mathematically founded on the CCP approach, to the reactive systems community. The goal of the present paper is to introduce an executable semantics for probabilistic, timed, and spatial concurrent constraint systems, here called stochastic and spatial concurrent constraint systems (SSCC). Towards this endeavor, rewriting logic [31] is used as a semantic framework to faithfully represent and operationally capture the highly concurrent nature, uncertain behavior, and spatial and epistemic spirit of reactive systems in a robust and executable specification. This specification, by being executable in the Maude system [8], is amenable to automatic algorithmic quantitative analysis in the form of statistical model checking.

The SSCC executable rewriting logic semantics unifies and extends previous efforts by the research community to capture phenomena of reactive systems in the CCP mathematical framework. The notions of timed behavior and probabilistic choice, and non-determinism in tcc [47] and pntcc [39], respectively, can be well expressed in SSCC. The notion of processes with stochastic duration, as proposed in [3], is also expressible in the proposed semantics. In SSCC, the underlying notion of hierarchical and distributed store found in SCC [29] is faithfully modeled by local spaces, where inconsistent local stores need not propagate their inconsistencies towards the global store. Previous efforts by the authors to bring executable semantics and reachability analysis to SCC using rewriting logic in [43]—and its subsequent extension to real-time in [40]—are transparently subsumed as sub-models in SSCC, and further extended with stochastic features and support for statistical model checking. This latter claim means that the SSCC executable semantics presented in this work can also be used for reachability analysis and LTL model checking of reactive systems, although these features are not part of the present exposition. The reader is referred to [43, 40] for an explanation and examples of how these features can be supported by a subset of the rewriting logic semantics presented in this work.
In the **SSCC** rewriting logic semantics, flat configurations of object-like terms encode the hierarchical structure of spaces, and equational and rewrite rules axiomatize the concurrent computational steps of processes. Time attributes are associated to process-store interaction with stochastic duration, as well as to process mobility in the space structure, by means of maps from agents to probability distribution functions. These choices can be interpreted to denote, as previously mentioned, upper bounds in the execution time of the given operations. Furthermore, exclusive and independent non-determinism, which complement the existing form of non-deterministic choice in CCP, can be parametric on probabilistic choice, modeling the fact that processes can sometimes execute due to interaction with the external environment. The underlying constraint system of **SSCC** is materialized with the help of the rewriting modulo SMT [41] approach, with constraints being quantifier-free formulas over Boolean, integer, and real valued shared variables, and information entailment queried as semantic inference and automatically delivered by the SMT-based decision procedures. The proposed automatic analysis of quantitative properties for **SSCC** relies on the PVeStA statistical model checker [2]. Given a probabilistic rewrite theory, such as the one presented in this paper, PVeStA is used to simulate its execution while automatically evaluating expected values of any numerical expression or path expression encoded in the QuaTEx language [1]. This is achieved automatically by performing enough Monte Carlo simulations to meet an error threshold and still be meaningful for statistical inference. The proposed approach is used to compute the expected execution time of a reactive system. Of course, many other quantitative measures could be computed by following an approach similar to the one presented in this paper.

This work can be seen as yet another interesting use of rewriting logic as a semantic framework. The support in rewriting logic for real-time systems [37], probabilistic systems [1], and open systems [41], make of the **SSCC** stochastic and spatial rewriting logic semantics a symbolic and fully executable specification in Maude [8] for constraint systems exhibiting discrete and dense linear timing constraints. The executable semantics can be used to specify and simulate the exchange of information in a very general setting and perform automated quantitative analysis for a broad class of reactive systems. All in all, the **SSCC** semantics contributed by this work could become an important test bed to formally model and assess quantitative attributes of today’s reactive systems.

Finally, this paper is a substantial extension of the conference paper [40] in the following ways:

- The **SSCC** executable rewriting logic semantics supports non-deterministic execution with exclusive (one process among many is chosen) and independent (some processes among many are chosen) choice, complementing the previous development in which only non-deterministic parallel execution was present.
- The notion of processes with stochastic duration is also developed in this work; previously, processes could only be assigned a constant duration.
- In terms of formal analysis, the semantics proposed in this work can be used for probabilistic simulation and quantitative analysis, extending the reachability and LTL model checking analysis previously offered in [40].
Extensive experimentation has been developed on a case study where a process randomly searches for specific information, encoded as a constraint, through a hierarchy of agents' spaces. Moreover, probabilistic simulation examples have been added.

Outline. The rest of the paper is organized as follows. Section 2 collects preliminary notions on concurrent constraint programming, rewriting logic, rewriting modulo SMT, and probabilistic rewrite theories. Section 3 presents stochastic and spatial concurrent constraint programming and Section 5 its rewriting logic semantics. Section 6 presents some examples on probabilistic simulation with the rewriting logic semantics and Section 7 presents a case study on quantitative analysis. Section 8 presents related work and Section 9 concludes the paper. B contains the complete SSCC specification presented in Section 5 and the code of the experiments presented in Section 6.

2. Preliminaries

The development of this paper relies on notions of process calculi, concurrent constraint programming, and rewriting logic; all within the general scope of concurrency theory. Concurrency theory is the field of theoretical computer science concerned with the fundamental aspects of systems consisting of multiple computing agents that interact among each other. This covers a vast variety of systems including reactive systems.

2.1. Concurrent Constraint Programming and Constraint Systems. Concurrent constraint programming (CCP) [48, 45, 49] (see a survey in [36]) is a model for concurrency that combines the traditional operational view of process calculi with a declarative view based on logic. This allows CCP to benefit from the large set of reasoning techniques of both process calculi and logic. Under this paradigm, the conception of store as valuation in the von Neumann model is replaced by the notion of store as constraint and processes are seen as information transducers.

The CCP model of computation makes use of ask and tell operations instead of the classical read and write. An ask operation tests if a given piece of information (i.e., a constraint as in $\text{temperature} > 23$) can be deduced from the store. The tell operations post constraints in the store, thus augmenting/refining the information in it. A fundamental issue in CCP is then the specification of systems by means of constraints that represent partial information about certain variables. The state of the system is specified by the store (i.e., a constraint) that is monotonically refined by processes adding new information.

The basic constructs (processes) in CCP are: (1) the tell($c$) agent, which posts the constraint $c$ to the store, making it available to the other processes. Once a constraint is added, it cannot be removed from the store (i.e., the store grows monotonically). And (2), the ask process $c \rightarrow P$, which queries if $c$ can be deduced from the information in the current store; if so, the agent behaves like $P$, otherwise, it remains blocked until more information is added to the store. In this way, ask processes define a reactive synchronization mechanism based on entailment of constraints. A basic CCP process language usually adds parallel
composition \((P \parallel Q)\) combining processes concurrently, a hiding operator for local variable definition, and potential infinite computation by means of recursion or replication. This notion of parallelism will be revisited in the light of probabilistic choice.

The CCP model is parametric in a constraint system (CS) specifying the structure and interdependencies of the partial information that processes can query (ask) and post (tell) in the shared store. The notion of constraint system can be given by using first-order logic. Given a signature \(\Sigma\) and a first-order theory \(\Delta\) over \(\Sigma\), constraints can be thought of as first-order formulae over \(\Sigma\). The (binary) entailment relation \(\vdash\) over constraints is defined for any pair of constraints \(c\) and \(d\) by \(c \vdash d\) iff the implication \(c \Rightarrow d\) is valid in \(\Delta\).

An algebra-based representation of CS is used in the present work.

Definition 1 Constraint Systems. A constraint system (CS) \(C\) is a complete algebraic lattice \((\text{Con}, \sqsubseteq)\). The elements of \(\text{Con}\) are called constraints. The symbols \(\sqcup\), true and false will be used to denote the least upper bound (lub) operation, the bottom, and the top element of \(C\), respectively.

In Definition 1, a CS is characterized as a complete algebraic lattice. The elements of the lattice, the constraints, represent (partial) information. A constraint \(c\) can be viewed as an assertion (or a proposition). The lattice order \(\sqsubseteq\) is meant to capture entailment of information: \(d \sqsubseteq c\), alternatively written \(c \sqsupseteq d\), means that the assertion \(c\) represents as much information as \(d\). Thus \(d \sqsubseteq c\) may be interpreted as saying that \(c \vdash d\) or that \(d\) can be derived from \(c\). The least upper bound (lub) operator \(\sqcup\) represents join of information and thus \(c \sqcup d\) is the least element in the underlying lattice above \(c\) and \(d\), asserting that both \(c\) and \(d\) hold. The top element represents the lub of all, possibly inconsistent, information, hence it is referred to as false. The bottom element true represents the empty information.

2.2. Order-sorted Rewriting Logic in a Nutshell. Rewriting logic [31] is a general semantic framework that unifies a wide range of models of concurrency. Language specifications can be executed in Maude [8], a high-performance rewriting logic implementation and benefit from a wide set of formal analysis tools available to it, such as an LTL model checker and an inductive theorem prover. The reader is referred to [31, 8, 33, 44] for an in-depth treatment of the topics discussed next.

2.2.1. Rewriting Logic. A rewriting logic specification or rewrite theory is a tuple \(\mathcal{R} = (\Sigma, E \uplus B, \mathit{R})\) where:

- \((\Sigma, E \uplus B)\) is an order-sorted equational theory with \(\Sigma = (S, \leq, F)\) a signature with finite poset of sorts \((S, \leq)\) and a set of function symbols \(F\) typed with sorts in \(S\); \(E\) is a set of \(\Sigma\)-equations, which are universally quantified Horn clauses with atoms that are \(\Sigma\)-equations \(t = u\) with \(t, u\) terms of the same sort; \(B\) is a set of structural axioms — disjoint from the set of equations \(E\) — (e.g., associativity, commutativity, identity) such that there exists a matching algorithm modulo \(B\) producing a finite number of \(B\)-matching substitutions or failing otherwise; and
• $R$ a set of universally quantified conditional rewrite rules of the form
  
  \[ t \to u \text{ if } \bigwedge_i \phi_i \]

  where $t, u$ are $\Sigma$-terms of the same sort and each $\phi_i$ is a $\Sigma$-equality.

  Given $X = \{X_s\}_{s \in S}$, an $S$-indexed family of disjoint variable sets with each $X_s$ countably infinite, the set of terms of sort $s$ and the set of ground terms of sort $s$ are denoted, respectively, by $T_{\Sigma}(X)_s$ and $T_{\Sigma,s}$; similarly, $T_{\Sigma}(X)$ and $T_{\Sigma}$ denote, respectively, the set of terms and the set of ground terms. The expressions $T_E(X)$ and $T_E$ denote the corresponding order-sorted $\Sigma$-term algebras. All order-sorted signatures are assumed preregular [16], i.e., each $\Sigma$-term $t$ has a unique least sort $ls(t) \in S$ s.t. $t \in T_{\Sigma}(X)_{ls(t)}$. It is also assumed that $\Sigma$ has nonempty sorts, i.e., $T_{\Sigma,s} \neq \emptyset$ for each $s \in S$. Many-sorted equational logic is the special case of order-sorted equational logic when the subsort relation $\leq$ is restricted to be the identity relation over the sorts.

  An equational theory $E = (\Sigma, E \uplus B)$ induces the congruence relation $=_E$ on $T_{\Sigma}(X)$ (or simply $=_{E \uplus B}$) defined for $t, u \in T_{\Sigma}(X)$ by $t =_E u$ if and only if $E \vdash t = u$, where $E \vdash t = u$ denotes $E$-provability by the deduction rules for order-sorted equational logic in [32]. For the purpose of this paper, such inference rules, which are analogous to those of many-sorted equational logic, are even simpler thanks to the assumption that $\Sigma$ has nonempty sorts, which makes unnecessary the explicit treatment of universal quantifiers.

  The expressions $T_E(X)$ and $T_E$ (also written $T_{\Sigma/E \uplus B}(X)$ and $T_{\Sigma/E \uplus B}$) denote the quotient algebras induced by $=_E$ on the term algebras $T_{\Sigma}(X)$ and $T_{\Sigma}$, respectively; $T_{\Sigma/E \uplus B}$ is called the initial algebra of $(\Sigma, E \uplus B)$.

  A rewrite theory $R = (\Sigma, E \uplus B, R)$ induces a rewrite relation $\to_R$ on $T_{\Sigma}(X)$ (sometimes denoted also as $\to_{R/E \uplus B}$) defined for every $t, u \in T_{\Sigma}(X)$ by $t \to_R u$ if and only if there is a rule $(l \to r \text{ if } \phi) \in R$ and a substitution $\theta : X \to T_{\Sigma}(X)$ satisfying $t =_{E \uplus B} l\theta, u =_{E \uplus B} r\theta$, and $E \vdash \phi\theta$. The tuple $\mathcal{T}_R = (T_{\Sigma/E \uplus B}, \to_R)$ is called the initial reachability model of $R$ [6].

2.2.2. Admissible Rewrite Theories. Appropriate requirements are needed to make an equational theory $E$ admissible, i.e., executable in rewriting languages such as Maude [8]. In this paper, it is assumed that the equations $E$ can be oriented into a set of (possibly conditional) sort-decreasing, operationally terminating, and confluent rewrite rules $\overline{E}$ modulo $B$ (denoted by $\to_{E/B}$ and equivalent to $=_{B \uplus E = B}$). The rewrite system $\overline{E}$ is sort decreasing modulo $B$ if and only if for each $(t \to u \text{ if } \gamma) \in \overline{E}$ and substitution $\theta : ls(t\theta) \geq ls(u\theta)$ if $(\Sigma, B, \overline{E}) \vdash \gamma\theta$. The system $\overline{E}$ is operationally terminating modulo $B$ [11] if and only if there is no infinite well-formed proof tree in $(\Sigma, B, \overline{E})$ (see [30] for terminology and details).

  Furthermore, $\overline{E}$ is confluent modulo $B$ if and only if for all $t, t_1, t_2 \in T_{\Sigma}(X)$, if $t \to_{E/B}^* t_1$ and $t \to_{E/B}^* t_2$, then there is $u \in T_{\Sigma}(X)$ such that $t_1 \to_{E/B}^* u$ and $t_2 \to_{E/B}^* u$. The term $t \downarrow_{E/B} \in T_{\Sigma}(X)$ denotes the $E$-canonical form of $t$ modulo $B$ so that $t \to_{E/B}^* t \downarrow_{E/B}$ and $t \downarrow_{E/B}$ cannot be further reduced by $\to_{E/B}$. Under sort-decreasingness, operational termination, and confluence, the term $t \downarrow_{E/B}$ is unique up to $B$-equality.
For a rewrite theory $\mathcal{R}$, the rewrite relation $\rightarrow_{\mathcal{R}}$ is undecidable in general, even if its underlying equational theory is admissible, unless conditions such as coherence [51] are given (i.e., whenever rewriting with $\rightarrow_{\mathcal{R}/E \cup B}$ can be decomposed into rewriting with $\rightarrow_{E/B}$ and $\rightarrow_{R/B}$). A key goal of [41] was to make the relation $\rightarrow_{\mathcal{R}}$ both decidable and symbolically executable when $E$ decomposes as $E_0 \uplus B_1$, representing a built-in theory $E_0$ for which formula satisfiability is decidable and $B_1$ has a matching algorithm.

2.2.3. Rewriting Logic Semantics. The rewriting logic semantics of a language $\mathcal{L}$ is a rewrite theory $\mathcal{R}_{\mathcal{L}} = (\Sigma_{\mathcal{L}}, E_{\mathcal{L}} \uplus B_{\mathcal{L}}, R_{\mathcal{L}})$ where $\rightarrow_{\mathcal{R}_{\mathcal{L}}}$ provides a step-by-step formal description of $\mathcal{L}$’s observable run-to-completion mechanisms. The conceptual distinction between equations and rules in $\mathcal{R}_{\mathcal{L}}$ has important consequences that are captured by rewriting logic’s abstraction dial [33]. Setting the level of abstraction in which all the interleaving behavior of evaluations in $\mathcal{L}$ is observable, corresponds to the special case in which the dial is turned down to its minimum position by having $E_{\mathcal{L}} \uplus B_{\mathcal{L}} = \emptyset$. The abstraction dial can also be turned up to its maximal position as the special case in which $R_{\mathcal{L}} = \emptyset$, thus obtaining an equational semantics of $\mathcal{L}$ without observable transitions. The rewriting logic semantics presented in this paper is faithful in the sense that such an abstraction dial is set at a position that exactly captures the interleaving behavior of the concurrency model.

2.2.4. Probabilistic Rewrite Theories. In a probabilistic rewrite theory [1], rewrite rules can have the more general form $l(x) \rightarrow r(x, y)$ if $\phi(x)$ with probability $y := \pi(x)$

Because the pattern $r(x, y)$ on the right-hand side may have new variables $y$, the next state specified by such a rule is not uniquely determined: it depends on the choice of an additional substitution $\rho$ for the variables $y$. In this case, the choice of $\rho$ is made according to the family of probability functions $\pi_\theta$: one for each matching substitution $\theta$ of the variables $x$. Therefore, a probabilistic rewrite theory can express both non-deterministic and probabilistic behavior of a concurrent system. At any given point of execution of a probabilistic rewrite theory many different rules can be enabled. Once a matching substitution $\theta$ has been chosen for one of these rules, the choice of the substitution $\rho$ is made probabilistically according to the probability distribution function $\pi_\theta$.

2.2.5. Real time. Time sampling strategies offer alternatives to assign the time that a rewrite step needs to be applied. For instance, the maximal time sampling strategy advances time by the maximum possible time elapse and tries to advance time by a user-given time value in tick rules having other forms. There are two different kinds of tick rule applications that the maximal strategy can treat: (i) ticks from states from which time can only advance up to a certain maximal time, and (ii) ticks from states from which time can advance by any amount. Here, the tick rule is the second one and the maximal time sampling strategy handles it by advancing time by a user-given time value.
A time-robust system is one where from any given state time can advance either by: (i) any amount, (ii) any amount up to (and including) a specific instant in time, or (iii) not at all. Advancing time is not affected unless in a specific state time is advanced all the way to the specific bound in time given in (ii). An instantaneous rewrite rule can only be applied at specific times, namely, when the system has advanced time by the maximal possible amount.

A time-robust system may have Zeno paths, those are paths where the sum of the durations of an infinite number of tick steps is bounded. It is necessary to differentiate between Zeno paths forced by the specification and Zeno paths that are due to bad choices in the tick increments. The intuition in the second type of Zeno behavior does not reflect realistic behaviors in the system and therefore is not simulated by the maximal time sampling strategy. The paths of the system that do not exhibit this unrealistic kind of Zeno behavior are called timed fair paths.

For systems satisfying time-robustness and tick-stabilizing properties unbounded and time-bounded LTL (excluding the next operator) model checking using the maximum time elapsed strategy is complete [38].

2.2.6. Maude, PMaude, and PVeStA. Maude [8] is a language and system based on rewriting logic. It supports order-sorted equational and rewrite theory specifications in functional and system modules, respectively. Admissibility of functional and system modules can be checked with the help of the Maude Formal Environment (MFE) [13, 12], an executable formal specification in Maude with tools to mechanically verify such properties. The MFE includes the Maude Termination Tool, the Maude Sufficient Completeness Checker, the Church-Rosser Checker, and the Maude Inductive Theorem Prover. All this tools are available at http://maude.lcc.uma.es/MFE.

PMaude [1] is both a language for specifying probabilistic rewrite theories and an extension of Maude supporting the execution of such theories by discrete-event simulation. PMaude can capture the dynamics of various elements of a system by stochastic real-time: computation and message-passing between entities of a system may take some positive real-valued time that can be distributed according to some continuous probability distribution function. Time associated to computation and message passing can also be zero, indicating instant transitions and synchronous communication. In general, PMaude supports discrete probabilistic choice as found in discrete-time Markov chains and stochastic continuous-time as found in continuous-time Markov chains.

A specification in PMaude without unquantified non-determinism is a key requirement for the statistical model checking analysis. Intuitively, non-existence of unquantified non-determinism means that non-deterministic choice during the simulation of a probabilistic rewrite theory $\mathcal{R}$ is exclusively due to probabilistic choice and not to concurrent transitions firing simultaneously at different parts of a system state. Under this assumption (and the admissibility assumptions on $\mathcal{R}$), a one-step computation with $\rightarrow_{\mathcal{R}}$ represents a single step in a discrete-event simulation of a specification written in PMaude. For details about one-step computation and sufficient conditions for absence of unquantified non-determinism
in PMaude, we refer the interested reader to [1]. The rewriting logic semantics \texttt{SSCC} in Section 5 is assumed to be admissible. Freeness of unquantified non-determinism is achieved by using a scheduler that is deterministic (in the sense that its behavior is deterministic relative to a given seed for random number generation).

Once a probabilistic system has been modeled in PMaude, various quantitative properties of the system can be specified by using the \textit{Quantitative Temporal Expressions} language (QuaTEx) [1] and queried with the help of the PVeStA statistical model checker [2]. The reader is referred to [1] for additional details about QuaTEx syntax and semantics, how other logics such as the \textit{Probabilistic Computation Tree Logic} (PCTL) [26] can be encoded in it, and the mechanisms used by PVeStA for statistical evaluation of QuaTEx expressions. Formally, given a probabilistic model \(M\), an expectation QuaTEx formula of the form \(E[\text{Exp}]\) —with \text{Exp} a QuaTEx expression—, and bounds \(\alpha\) and \(\delta\), PVeStA approximates the value of \(E[\text{Exp}]\) within a \((1-\alpha)100\%\) confidence interval and with size at most \(\delta\). This is done by generating a large enough number \(n\) of random sample values \(x_1, x_2, \ldots, x_n\) of \text{Exp}, computed from \(n\) independent Monte Carlo simulations of \(M\) [2].

PVeStA is implemented in Java 1.6 and it is available at \url{http://maude.cs.uiuc.edu/tools/pvesta/}.

\textbf{2.3. SMT Solving.} Satisfiability Modulo Theories (SMT) studies methods for checking satisfiability of first-order formulas in specific models. The SMT problem is a decision problem for logical formulas with respect to combinations of background theories expressed in classical first-order logic with equality. An SMT instance is a formula \(\phi\) (typically quantifier free, but not necessarily) in first-order logic and a model \(\mathcal{T}\), with the goal of determining if \(\phi\) is satisfiable in \(\mathcal{T}\).

In this work, the representation of the constraint system is based on SMT solving technology. Given a many-sorted equational theory \(\mathcal{E}_0 = (\Sigma_0, E_0)\) and a set of variables \(X_0 \subseteq X\) over the sorts in \(\Sigma_0\), the formulas under consideration are in the set \(QF_{\Sigma_0}(X_0)\) of quantifier-free \(\Sigma_0\)-formulas: each formula being a Boolean combination of \(\Sigma_0\)-equation with variables in \(X_0\) (i.e., atoms). The terms in \(T_{E_0}\) are called \textit{built-ins} and represent the portion of the specification that will be handled by the SMT solver (i.e., semantic data types). In this setting, an SMT instance is a formula \(\phi \in QF_{\Sigma_0}(X_0)\) and the initial algebra \(T_{E_0^+}\), where \(\mathcal{E}_0^+\) is a decidable extension of \(\mathcal{E}_0\) such that

\[
\phi \text{ is satisfiable in } T_{E_0^+} \iff (\exists \sigma : X_0 \rightarrow T_{\Sigma_0}) T_{E_0} \models \phi \sigma.
\]

Many decidable theories \(\mathcal{E}_0^+\) of interest are supported by SMT solvers satisfying this requirement (see [41] for details). In this work, the Maude alpha 118 release, which integrates Yices2 [14] and CVC4 [4], is used for reachability analysis with SMT constraints.

\textbf{3. Stochastic and Spatial Concurrent Constraint Systems}

In this section we present the stochastic and spatial concurrent constraint (\texttt{SSCC}) calculus and illustrate the main features of the language.
3.1. **Spatial Constraints.** The authors of [29] extended the notion of CS to account for distributed and multi-agent scenarios where agents have their own space for local information and computation. In [19, 22, 23, 25, 21] CS are further extended to model mobile behaviour and reason about beliefs, lies, and group epistemic behaviour inspired by social networks.

Locality and Nested Spaces. Each agent \(i\) has a space function \([\cdot]_i\) from constraints to constraints (recall that constraints can be viewed as assertions). Applying the space function \([\cdot]_i\) to a constraint \(c\) gives us a constraint \([c]_i\) that can be interpreted as an assertion stating that \(c\) is a piece of information that resides within a space attributed to agent \(i\). An alternative epistemic interpretation of \([c]_i\) is an assertion stating that agent \(i\) believes \(c\) or that \(c\) holds within the space of agent \(i\) (but it may or may not hold elsewhere). Both interpretations convey the idea that \(c\) is local to agent \(i\). Following this intuition, the assertion \([\cdot]_i\) is a hierarchical spatial specification stating that \(c\) holds within the local space the agent \(i\) attributes to agent \(j\). Nesting of spaces such as in \([\cdots [c]_{i_m} \cdots]_{i_2}]_{i_1}\) can be of any depth.

Parallel Spaces. A constraint of the form \([c]_i \sqcup [d]_j\) can be seen as an assertion specifying that \(c\) and \(d\) hold within two parallel/neighboring spaces that belong to agents \(i\) and \(j\). From a computational/concurrency point of view, it is possible to think of \(\sqcup\) as parallel composition; from a logic point of view, \(\sqcup\) corresponds to conjunction.

The notion of an \(n\)-agent spatial constraint system is formalized in Definition 2.

**Definition 2** Spatial Constraint system [29]. An \(n\)-agent spatial constraint system (\(n\)-SCS) \(C\) is a CS \((\text{Con}, \sqsubseteq)\) equipped with \(n\) self-maps \([\cdot]_1, \ldots, [\cdot]_n\) over its set of constraints \(\text{Con}\) satisfying for each function \([\cdot]_i : \text{Con} \rightarrow \text{Con}\):

**S.1:** \([\text{true}]_i = \text{true}\), and

**S.2:** \([c \sqcup d]_i = [c]_i \sqcup [d]_i\) for each \(c, d \in \text{Con}\).

Property S.1 in Definition 2 requires space functions to be strict maps (i.e., bottom preserving) where an empty local space amounts to having no knowledge. Property S.2 states that space functions preserve (finite) 

Mobility plays a key role in distributed systems. Following the algebraic approach, under certain conditions it is possible to provide each agent \(i\) with an extrusion function \(\uparrow_i : \text{Con} \rightarrow \text{Con} [25, 19]\). The expression \(\uparrow_i c\) within a space context \([\cdot]_i\) means that \(c\) must be posted outside of agent's \(i\) space.

More precisely, given a space function \([\cdot]_i\), the extrusion function \(\uparrow_i\) of agent \(i\) is the **right inverse** of \([\cdot]_i\). Such function exists if and only if \([\cdot]_i\) is surjective [25]. By right
inverse of \([\cdot]_i\), we mean a function \(\uparrow_i: Con \rightarrow Con\) such that \([\uparrow_i c]_i = c\). The computational interpretation of \(\uparrow_i\) is that of a process being able to extrude any \(c\) from the space \([\cdot]_i\). The extruded information \(c\) may not necessarily be part of the information residing in the space of agent \(i\). For example, using properties of space and extrusion functions we shall see that \([d \sqcup \uparrow_i c]_i = [d]_i \sqcup c\) specifying that \(c\) is extruded (while \(d\) is still in the space of \(i\)).

The extruded \(c\) could be inconsistent with \(d\) (i.e., \(c \sqcup d = false\)), it could be related to \(d\) (e.g., \(c \sqsubseteq d\)), or simply unrelated to \(d\). From an epistemic perspective, we can use \(\uparrow_i\) to express utterances by agent \(i\) and such utterances could be intentional lies (i.e., inconsistent with their beliefs), informed opinions (i.e., derived from the beliefs), or simply arbitrary statements (i.e., unrelated to their beliefs). One can then think of extrusion/utterance as the right inverse of space/belief.

We can now recall the notion of spatial constraint system with extrusion.

**Definition 3** Spatial Constraint System with Extrusion [25, 19]. An \(n\)-agent spatial constraint system with extrusion (n-SCSE) is an \(n\)-SCS \(C\) equipped with \(n\) self-maps \(\uparrow_1, \ldots, \uparrow_n\) over \(Con\), written \((C, \uparrow_1, \ldots, \uparrow_n)\), such that each \(\uparrow_i\) is the right inverse of \([\cdot]_i\).

**Agent Views.** Let us recall the notion of agent view.

**Definition 4** Agent View [29]. The agent \(i\)'s view of \(c\), \(c^i\), is given by \(c^i = \bigsqcup \{d \mid [d]_i \sqsubseteq c\}\).

Intuitively, \(c^i\) represents all the information the agent \(i\) may see or have in \(c\). For example if \(c = [d]_i \sqcup [e]_j\) then agent \(i\) sees \(d\), so \(d \sqsubseteq c^i\).

### 3.2. Spatial Concurrent Constraint Programming with Probabilistic Choice

This section presents the syntax of \(SSCC\) and main intuition behind its constructs. The operational semantics of \(SSCC\) will be given in Section 5.

**Definition 5** \(SSCC\) Processes. Let \(C = (Con, \sqsubseteq)\) be a constraint system, \(A\) a set of \(n\)-agents, and \(V\) an infinite countable set of variables. Let \((C, [\cdot]_1, \ldots, [\cdot]_n, \uparrow_1, \ldots, \uparrow_n)\) be an \(n\)-SCS and consider the following EBNF-like syntax:

\[
P \ ::= \ 0 \mid \text{tell}(c) \mid \text{ask}(c) \rightarrow P \mid P \parallel P \mid [P]_i \mid \uparrow_i (P) \mid x \mid \mu x. P \mid \bigoplus_j (P, q)_j \mid \bigodot_j (P, q)_j
\]

where \(c \in Con, i \in A, x \in V, j\) belongs to a finite set of indexes \(J\), and \(q \in [0, 1]\). An expression \(P\) in the above syntax is a process if and only if every variable \(x\) in \(P\) occurs in the scope of an expression of the form \(\mu x.P\). The set of processes of \(SSCC\) is denoted by \(Proc\).

The \(SSCC\) calculus can be thought of as a shared-spaces model of computation. Each agent \(i \in A\) has a computational space of the form \([\cdot]_i\), possibly containing processes and other agents’ spaces. The basic constructs of \(SSCC\) are \(tell, ask,\) and \(parallel\) composition, and they are defined as in standard CCP [49]. A process \(\text{tell}(c)\) running in an agent \(i \in A\) adds \(c\) to its local store \(s_i\), making it available to other processes in the same space. This addition, represented as \(s_i \sqcup c\), is performed even if the resulting constraint is inconsistent. The
process \( \text{ask}(c) \rightarrow P \) running in space \( i \) may execute \( P \) if \( c \) is entailed by \( s_i \), i.e., \( c \subseteq s_i \). The process \( P \parallel Q \) specifies the parallel execution of processes \( P \) and \( Q \); given \( I = \{i_1, \ldots, i_n\} \), the expression \( \prod_{i \in I} P_i \) is used as a shorthand for \( P_{i_1} \parallel \ldots \parallel P_{i_m} \). A construction of the form \([P]_i\) denotes a process \( P \) running within the agent \( i \)'s space. Any information that \( P \) produces is available to processes that lie within the same space. The process \( \uparrow_i (P) \) denotes that process \( P \) runs outside the space of agent \( i \) and the information posted by \( P \) resides in the store of the parent of agent \( i \). Unbounded behaviour is specified using recursive definitions of the form \( \mu x.P \) whose behaviour is that of \( P[\mu x.P/x] \), i.e., \( P \) with every free occurrence of \( x \) replaced with \( \mu x.P \). We assume that recursion is ask guarded: i.e., for every \( \mu x.P \), each occurrence of \( x \) in \( P \) occurs under the scope of an ask process. For simplicity we assume an implicit “\( \text{ask}(true) \rightarrow \) ” in unguarded occurrences of \( X \).

The last two processes represent exclusive and independent probabilistic choice, which are part of the main contribution of this work. The pair \((P, q)_j\) is a shorthand for \((P_j, q_j)\) and represents a process \( P_j \) with probability \( q_j \) to be scheduled for execution. The probability that \( P_j \) is not scheduled is given by \( 1 - q_j \). The \( \bigoplus_j (P, q)_j \) process represents the exclusive choice of some process \( P_k \) with probability \( q_k \) where \( k \in J \) and \( \sum_{j \in J} q_j = 1 \). The \( \bigodot_j (P, q)_j \) process represents the independent choice (possibly none) of some processes \( P_{j_1}, P_{j_2}, \ldots, P_{j_k} \) each one with probability \( q_{j_1}, q_{j_2}, \ldots, q_{j_k} \), respectively, where \( j_h \in K \) (with \( 1 \leq h \leq k \)), for some (possibly empty) \( K \subseteq J \). Once a choice is made, \( \bigodot_j (P, q)_j \) evolves to the parallel composition of the chosen processes \( \prod_{k \in K} P_k \) and the remaining processes (i.e., those indexed by \( J \setminus K \)) are precluded.

**Example 1.** Consider the processes \( P = \text{tell}(c) \) and \( Q = \text{ask}(c) \rightarrow \text{tell}(d) \).

- The process \([P]_i \parallel [Q]_i\), by the above intuitions, have the effect that the constraints \( c \) and \( d \) are added to store of agent \( i \).
- A similar behavior is achieved by the process \([P]_i \parallel [Q]_i\), which also produces \( c \cup d \) in the store of agent \( i \) (note that \( [c \cup d]_i \) is equivalent to \( [c]_i \cup [d]_i \) by Property S.2 in Definition 2).
- In contrast, the process \([P]_i \parallel [Q]_i\), with \( i \neq j \), does not necessarily add \( d \) to the space of agent \( i \) because \( c \) is not made available for agent \( i \); likewise in \( P \parallel [Q]_i \), \( d \) is not added to the space of agent \( i \).
- Consider \([P]_i \parallel [\uparrow_j (Q)]_j\). In this case, because of extrusion, both \( c \) and \( d \) will be added to store of agent \( i \). However, \([P]_i \parallel [\uparrow_j (Q)]_j\) with \( i \neq j \), adds \( c \) to the space of agent \( i \) within the space of agent \( j \), but \( c \) is not made available for agent \( j \), therefore \( d \) could not be added to its space. Note that in \([P]_i \parallel [\uparrow_i (Q)]_j\), the constraint \( c \) is added to the space of agent \( i \), but since \( Q \) cannot be extruded in \([\uparrow_i (Q)]_j\), \( d \) is not added neither to the space of \( i \) nor \( j \).
- Consider the process \(([P]_i, q_1) \circ ([Q]_i, q_2)\) and a random sampling \( p = 0.5 \). If \( q_1 = q_2 = 0.7 \), the store of agent \( i \) is modified with \( c \) and \( d \) because \( q_1 \geq p \) and \( q_2 \geq p \). If \( q_1 = 0.7 \) and \( q_2 = 0.4 \) however, \( c \) is added to the store but \( Q \) cannot be executed within the space of agent \( i \) because \( q_1 \geq p \) but \( q_2 \nleq p \).
- For exclusive choice, consider \( ([P]_i, q_1) \oplus ([P]_j, q_2) \) and a random sampling \( p = 0.5 \). In this case, the exclusive choice in \( ([P]_i, q_1) \oplus ([P]_j, q_2) \) determines which agent, either \( i \) or \( j \), is going to add to its store, but only one of them is able to add such a constraint. If \( q_1 = 0.7 \) and \( q_2 = 0.3 \), \( c \) is added to the store of agent \( i \).

3.3. Configurations. As usual configuration are used to represent the state of the system. A configuration is a pair of the form \( \langle P; c \rangle \in \text{Proc} \times \text{Con} \), where \( P \) is a process and \( c \) is the spatial distribution of information available to it.

Example 2. Consider the constraint \( d \) below and its tree-like structure depicted in Figure 1.

\[
d \stackrel{\text{def}}{=} \ (y = 1) \cup \left[ x = 3 \cup [y = 3]_j \right]_i \cup \left[ x > 0 \cup [x = 42]_k \right] \cup \left[ x < 42 \right]_j \cup [y > 0]_k.
\]

Each node in such a tree corresponds to the information (constraint) contained in an agent’s space. Edges define the spatial hierarchy of agents. For example, the configuration \( \langle \text{ask } x = 42 \rightarrow [P]_j ; d \rangle \) is a deadlock, while \( \langle [[\text{ask } x = 42 \rightarrow P]_k]_j ; d \rangle \) can evolve to \( \langle [[P]_k]_j ; d \rangle \) since \( x = 42 \sqsubseteq d^k \) (see Def.4).

![Figure 1. A spatial hierarchy of processes.](image_url)

3.4. Timed Processes in sscC. In these systems time is conceptually divided into discrete or continuous intervals. In a time interval, the processes may receive a stimulus from the environment and react computing and responding to it. Time can be thought of as a sequence of time slots. The processes make their internal transitions in a given time unit. When the current unit ends the processes that are pending to finish their transitions are passed to the next time unit. In this system there is no time limit, i.e., the processes make their internal transitions until no further transition can be done. Note that the processes remained at the end of the execution are only \textit{ask} processes. They are the processes for which the store may not have enough information to derive their condition.
Modeling real-time involves a scheduler to manage the processes preemption based on the time that they require to complete their transitions. Thereby, a process makes its transitions when no other process has a lower time out.

This approach uses a time-unit $T$ in which transitions can be done. If a process cannot be reduced because there is not enough information, it waits to complete its reduction, if it is possible, in the next time-unit. Accessing the store has a time penalty, namely, $tell$ processes have a cost for modifying the store and $ask$ processes have a for querying the store. Mobility implies a cost for changing spaces, for $[\cdot]_i$ this cost represents the time of getting in the space of agent $i$, and for $\uparrow_i (\cdot)$ is the time of leaving the space of agent $i$. This time is given by probability distribution functions ($\alpha$, $\mu$, $\phi$ and $\rho$, respectively) in each space and for each one of the processes, therefore each process could have different times within each space.

**Example 3.** As an example, consider the tree-like structures depicted in Figure 2. They correspond to hierarchical computational spaces of, e.g., virtual containerization (i.e., virtual machines inside other virtual machines). Each one of these spaces is endowed with an agent identifier (either root or a natural number) and a local store (i.e., a constraint), and the processes can be executed and spawned concurrently inside any space, with the potential to traverse the structure, querying and posting information locally, and even creating new spaces. The SCCP calculus enables the formal modeling of such scenarios and of transitions that can lead from an initial system state (e.g., Figure 2a) to a final state (e.g., Figure 2b) by means of an operational semantics [29].

The initial state is represented by the constraint $d$ specifying the store of each space. The final state is reached after execution of the configuration $C$:

$$
\begin{align*}
&d \overset{\text{def}}{=} (W = 9) \cup [X \geq 11]_0 \cup [true \cup [Y > 5]_0]_1 \cup [true]_2, \\
&C \overset{\text{def}}{=} ((ask X > 2 \rightarrow \uparrow_0 ((tell(Y < 10))_0]_1) \_0 \| (tell(Z \neq 10))_2, root, d, 0).
\end{align*}
$$

The configuration $C$ reduces the container system $d$ to the state in Figure 2b using the time functions:

$$
\begin{align*}
&\alpha \overset{\text{def}}{=} \{(\text{root}, 0.1), (\text{0.root}, 0.15), (\text{1.root}, 0.15), (\text{2.root}, 0.15), (\text{0.1.root}, 0.2)\}, \\
&\mu \overset{\text{def}}{=} \{(\text{root}, 0.05), (\text{0.root}, 0.1), (\text{1.root}, 0.1), (\text{2.root}, 0.1), (\text{0.1.root}, 0.15)\}, \\
&\phi \overset{\text{def}}{=} \{(\text{root}, 0.5), (\text{0.root}, 0.7), (\text{1.root}, 0.65), (\text{2.root}, 0.6), (\text{0.1.root}, 0.8)\}, \\
&\rho \overset{\text{def}}{=} \{(\text{root}, 0.5), (\text{0.root}, 0.65), (\text{1.root}, 0.5), (\text{2.root}, 0.6), (\text{0.1.root}, 1)\}.
\end{align*}
$$

According to the time functions the configuration $C$ reaches the final state in 2.6 time units.

It is important to note that the timing attributes associated to the processes in Example 3 represent constant time. However, as it will be explained in Section 5, processes can be associated with probability distribution functions to represent their timing behavior.
4. Structural Operational Semantics

This section introduces the structural operational semantics (SOS) of \texttt{sscc}. The SOS shows how the system evolves over time. The inference rules specify the transitions of the processes and the conditions that must hold for their execution.

There are some processes in \texttt{sscc} that involve consumption of time, such resource is represented as delay in their execution. This idea comes from previous works intended to represent timed processes (see –). Processes including probabilistic choice are supported by an auxiliary function, which samples a random variable with uniform distribution in order to select, accordingly to exclusive or independent choice, the processes to be executed.

Time consumption is associated to processes that interact with the store (querying or adding information) and involve mobility (spatial and extrusion processes). The time consumed for each process is computed by the function $\Delta : \text{Con} \rightarrow \text{Rat}$, which assigns time to the process accordingly to the size of the store where it will be executed.

Consider $\Delta(d) = m_d$ for every $d \in \text{Con}$. The SOS of \texttt{sscc} is defined as follows:
The inference rules of \textit{SSCC} represent the execution time of a process as a delay in its execution. The function $\eta : \text{Proc} \times \text{Rat} \rightarrow \text{Proc}$ is used to represent timed processes as delayed processes: $\eta(P, m)$ is a process that mimics $P$ but is delayed $m$ units of time. When $m = 0$, the process $P$ is launched for execution.

$$\eta(P, m) = \begin{cases} P & \text{if } m = 0 \\ \eta(P, m-t) & \text{if } m \geq t \end{cases}$$

In the definition above, $t$ is the execution time of the process with the least execution time amongst all the processes listed for execution.

The function $\eta_s : \text{Con} \times \text{Rat} \rightarrow \text{Con}$ is defined analogously. The constraint $\eta_s(c, m)$ represents that the constraint $c$ will not be added to the store until $m$ units of time.

$$\eta_s(c, m) = \begin{cases} c & \text{if } m = 0 \\ \eta_s(c, m-t) & \text{if } m \geq t \end{cases}$$

The functions $\eta$ and $\eta_s$ are defined over the rational numbers. Now, we must verify that $\eta(P, 0)$ and $\eta_s(c, 0)$ can be reached, i.e., they are well-defined. Recall that $m$ in $\eta(P, m)$ and $\eta_s(c, m)$ is decreased $t$ units of time per execution, where $t$ is the execution time of the process with the least execution time. This time $t$ is extracted from each process in the list of pending processes, then $P$ is the process with the least time we have $m = t$. This guarantees that every delayed process will be executed once its associated execution time passes.

Notice that delayed constraints are related to \textit{tell} and \textit{spatial} processes. Both of them reflect a delay to add a constraint to the store. However, the latter exists because a delayed process within a given space cannot modifies the local store until its execution time passes.
The function $I : \text{Proc} \rightarrow J$ returns the indexes of the chosen processes to be executed after the exclusive or independent choice.

The inference rules in the SOS of \texttt{SSCC} describe the transitions of the processes. Intuitively, they can be understood as follows (recall that $m_d = \Delta(d)$):

\textbf{T-Tell}: Adds the constraint $c$ to the store through the delayed constraint $\eta_s(c, m_d)$, i.e., adding the constraint is delayed $m_d$ units of time.

\textbf{T-Ask}: If $d$ entails $c$ then it behaves as the process $P$ with a delay of $m_d$ units of time.

\textbf{T-Delay}: If the store $d$ is not strong enough to derive $c$, the process $\text{ask } c \rightarrow P$ is delayed $m_d$ units of time.

\textbf{T-Spatial}: If the process $P$ with the information that agent $i$ has in store $d$ (i.e., $d^i$) evolves to $P'$ with store $d'$, then such a transition can be made within the space of agent $i$ and, the local store of $i$ is modified with $[d']_i$, both with delay $m_d$.

\textbf{T-Extrusion}: This rule specifies how a process $P$ leaves the space of agent $i$. It must be known the target space, i.e. $j$’s space. The delay $m_d$ represents the cost of leaving the space of agent $i$.

\textbf{Parallel}: The process $P \parallel Q$ evolves to $P' \parallel Q$ if a transition can be made from $\langle P; d \rangle$ to $\langle P'; d' \rangle$.

\textbf{Exclusive}: Given a set of processes $\{(P_j, q_j) \mid j \in J\}$, the function $I$ chooses uniformly an index $k \in J$ of a process $P_k$ with probability $q_k$. Such process $P_k$ is executed and the others are precluded.

\textbf{Independent}: Given a set of processes $\{(P_j, q_j) \mid j \in J\}$, the function $I$ chooses uniformly a subset of indexes $K \subseteq J$ of processes $P_k$ with probability $q_k$ for each $k \in K$. These processes are executed as the parallel composition $\Pi_{k \in K} P_k$ and the others are precluded.

As mentioned before, the processes that consume time are those that model mobility and interact with the store. For that reason the rules Parallel, Exclusive and Independent are not affected for the delay $m_d$.

An implementation of the structural operational semantics of \texttt{SSCC}, using rewriting logic semantics, is presented in the next section. It provides a formal description of the system language including its main features of time and probability.

## 5. Rewriting Logic Semantics

This section presents $\mathcal{R}$, a rewriting logic semantics for \texttt{SSCC}. Section 5.1 includes the key details of the state infrastructure and Section 5.2 presents the computational rules of the semantics. The reader is suggested to consult on need basis A, which details aspects of the constraint system implementation via SMT technology, and explains the main features of the timing and probabilistic infrastructure, including the ‘tick rule’ behind the real-time behavior of the specification. B contains the complete Maude specification of $\mathcal{R}$. This section assumes familiarity with the Maude language [8].
5.1. **System States.** The rewriting logic semantics for **SSCC** represents the tree-like structure of the hierarchical spaces as a flat configuration of object-like terms. The hierarchical relationships among spaces in **SSCC** are captured by using common prefixes as part of an agent’s name. In an observable state, each agent’s space is represented by a set of terms: some encoding the state of execution of all its processes and exactly one object representing its local store.

The object-based system is represented by the Maude object [8] predefined by the CONFIGURATION module imported in **SSCC-STATE** in including mode. The objects and configuration are defined using the Maude object syntax as follows:

```
subsorts Nat Aid < Oid .
ops agent process simulation : -> Cid .
```

A system state is represented by a configuration of objects containing the setup of each one of the agents in the system. A **Configuration** is a multiset of objects with set union denoted by juxtaposition and identity **none**. There are two types of object identifiers (as **Oid**): agent identifiers (as **Aid**), for identifying agents and their hierarchical structure, and natural numbers (as **Nat**), for some additional identification used internally in **R**. There are three types of class identifiers (as **Cid**), namely, for agents, processes, and a simulation object. This object-based representation allows to define a set of attributes, which represent the characteristics of the objects as follows:

```plaintext
--- agents attributes
op store :_ : Boolean -> Attribute [ctor] .
op set :_ : Set{Nat} -> Attribute [ctor] .
--- processes attributes
op UID :_ : Nat -> Attribute [ctor] .
op command :_ : SSCCCmd -> Attribute [ctor] .
--- simulation attributes
op gtime :_ : Time -> Attribute [ctor] .
op pqueue :_ : Heap{2Tuple} -> Attribute [ctor] .
op pend :_ : Heap{2Tuple} -> Attribute [ctor] .
op nextID :_ : Nat -> Attribute [ctor] .
op flag :_ : Bool -> Attribute [ctor] .
op counter :_ : Nat -> Attribute [ctor] .
op tTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op aTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op sTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op eTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op factor :_ : PosRat -> Attribute [ctor] .
```

Each agent has two attributes, namely, its store (attribute **store**) and a set of its predecessors in the hierarchy structure (attribute **set**); and each process has two attributes: a universal identifier (used internally for execution purposes, attribute **UID**) and the command (i.e., **SSCC** process, attribute **command**) that it is executing. The attributes of the simulation object include the global time (attribute **gtime**); the priority queue of system commands to be processed as ordered by time-to-execution (attribute **pqueue**); the collection of pending
commands, i.e., ask commands that are waiting for its guarding constraint to become active (attribute \texttt{pend}); the counter for assigning the next internal identifier when spawning a new process (attribute \texttt{nextID}); a flag that is on whenever a tick rule needs to be applied (attribute \texttt{flag}); a seed for the sample of random variables (attribute \texttt{counter}); a collection of maps containing the stochastic expression for the time it takes to process certain commands relative to the space where they are executed (attributes \texttt{tTM}, \texttt{aTM}, \texttt{sTM}, and \texttt{eTM}); and a multiplicative factor for the time it takes to process an ask command (attribute \texttt{factor}). The \texttt{factor} attribute has been added to support the fact that querying a store can depend on the size of its constraint: the bigger it is, the longer it takes. However, if this feature is not of importance in a specific application, this attribute can be set to 1. The sort \texttt{Time}, as it is often the case in Real-time Maude [37], can be used to represent either discrete or dense linear time.

Agent and process objects use a qualified name (sort \texttt{Aid}) to identify to which agent’s space each one belongs; this sort is defined in module \texttt{AGENT-ID}. The hierarchical structure of spaces in \texttt{SSCC} is a tree-like structure where the root space is identified by constant \texttt{root}. Any other qualified name corresponds to a dot-separated natural numbers list (sort \texttt{Nat}), organized from left to right and including the constant \texttt{root} at the end. For instance, \texttt{3.1.root} denotes that agent 3 is child of agent 1 and, at the same time agent 1 is child of \texttt{root}.

\begin{verbatim}
op root : -> Aid . 
op _._ : Nat Aid -> Aid .
\end{verbatim}

The commands available in the \texttt{SSCC} model are defined in module \texttt{SSCC-SYNTAX}, each one of sort \texttt{SSCCCmd}. Note that the syntax of each command is very close to the actual syntax in the \texttt{SSC} model, e.g., constructs of the form \texttt{P \parallel Q} in \texttt{SSC} are represented in the syntax of \texttt{SSCCCmd} by terms of the form \texttt{P \parallel Q}.

\begin{verbatim}
op 0 : -> SSCCCmd . 
op tell : Boolean -> SSCCCmd . 
op ask->_ : Boolean SSCCCmd -> SSCCCmd . 
op _||_ : SSCCCmd SSCCCmd -> SSCCCmd [assoc comm gather (e E) ] . 
op _in_ : SSCCCmd Nat -> SSCCCmd . 
op _out_ : SSCCCmd Nat -> SSCCCmd . 
op V : Nat -> SSCCCmd . 
op mu : Nat SSCCCmd -> SSCCCmd . 
op exc : List{SSCCCmd} List{Float} -> SSCCCmd . 
op ind : List{SSCCCmd} List{Float} -> SSCCCmd .
\end{verbatim}

The argument of command \texttt{tell} is a formula (as \texttt{Boolean}), to be added to the agent’s constraint. Command \texttt{ask->_} has two arguments: a formula (as \texttt{Boolean}) and a program (as \texttt{SSCCCmd}). The program is executed if the formula is entailed by the agent’s constraint. Both arguments of command \texttt{_||_} are programs (as \texttt{SSCCCmd}). The arguments of the command \texttt{_in_} are a program (as \texttt{SSCCCmd}) and a natural number (as \texttt{Nat}) representing the identifier of a children (related to an agent \texttt{Aid}) where the program is moved. The arguments of the command \texttt{_out_} are a program (as \texttt{SSCCCmd}) and a natural number (as
Nat, related to an agent Aid, this command moves a program from an agent’s space to its parent. Command V has as argument a natural number identifying a process local name. The arguments of command mu are a natural number for the variable process to be replaced, and the program to be replaced in (as SSCCCmd). Commands exc and ind codify exclusive and independent probabilistic choice; the arguments of the commands exc and ind are a list of programs and a list of probabilities (as Float between 0 and 1) of the same size, each probability is related to a program.

5.1.1. Time functions. The real-time behavior in \( \mathcal{R} \) associates timing behavior to those commands that interact with stores (i.e., tell and ask commands) and to commands that involve mobility among the space structure of the system (i.e., \( [\_\_] \_ \) and \( \uparrow (\_\_) \)). More precisely, tell and ask commands take time when posting in and querying from the store, respectively. Moving the execution of a command inside an agent and extruding from a space can also take up time, i.e., spatial and extrusion commands can take time. Such duration is represented by maps from an agent identifier (as Aid) to a stochastic expression (as StExp), i.e., each agent has its own time functions. Maps tTM (for tell), aTM (for ask), sTM (for \( [\_\_] \_ \)), and eTM (for \( \uparrow (\_\_) \)) can be accessed using the getTimeCmd function. Stochastic expressions include probability distribution functions to sample the time for the corresponding command. For example, tTM[i] denotes the stochastic expression to be sampled in order to get the time it takes to execute a tell command inside the agent’s \( i \) space.

\[
\begin{align*}
\text{op fTime : Map{Aid, StExp} Aid Nat -> Tuple{Time, Nat<}}. \\
\text{eq fTime(TM, L, N) = if hasMapping(TM, L) then eval(TM[L], N) else eval(Norm(1.0, 0.2), N) fi}.
\end{align*}
\]

\[
\begin{align*}
\text{op getTimeCmd : SSCCCmd Aid Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Nat -> Tuple{Time, Nat<}}. \\
\text{eq getTimeCmd(tell(B1), L, TMt, TMs, TMe, N) = fTime(TMt, L, N) \ldotp} \\
\text{eq getTimeCmd(C1 in I1, L, TMt, TMs, TMe, N) = fTime(TMs, L, N) \ldotp} \\
\text{eq getTimeCmd(C1 out I1, L, TMt, TMs, TMe, N) = fTime(TMe, L, N) \ldotp} \\
\text{eq getTimeCmd(C1, L, TMt, TMs, TMe, N) = (0, N) \ldotp \text{wise} \ldotp} \\
\end{align*}
\]

The probability distribution functions available for the executable specification of \( \text{SSCC} \) include the exponential, Weibull, normal/Gauss, Gamma, \( \chi^2 \), Erlang, F, geometric, Pascal, Pareto, and uniform functions [52].

5.1.2. Probabilistic Choice Functions. There are auxiliary functions supporting the probabilistic choice of processes.

Function getProb is used for sampling from the uniform distribution function for the probabilistic choice.

\[
\begin{align*}
\text{op getProb : Nat -> Tuple{Float<, Nat<}}. \\
\text{eq getProb(N) = evalF(Unif(0.0, 1.0), N) \ldotp}
\end{align*}
\]
The `exclusive` function takes as arguments the list of processes for the probabilistic choice (as `List{SSCCCmd}`), the list of its probabilities (as `List{Float}`), the counter of the internal identifier of processes (as `Nat`), the priority queue of the system (as `Heap{2Tuple}`), the seed for random sampling (as `Nat`), the maps of the time functions (as `Map{Aid, StExp}`), and the agent identifier of the agent where the process is executed (as `Aid`). It returns the process selected to be executed, the next internal identifier to be applied, the priority queue updated with the selected process, and the new seed for random sampling in the overall system.

```
op exclusive : List{SSCCCmd} List{Float} Nat Heap{2Tuple} Nat Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Aid -> Tuple{List, Nat, Heap, Nat}.

c eq exclusive(C, Q, N, P, N1, TMt, TMs, TMe, L) = (C, N + 1, HO, N')
  if (T, N') := getTimeCmd(C, L, TMt, TMs, TMe, N1) /
    HO := insert(((T, N)), P) .

c eq exclusive(C NeLC, Q Q1 LF, N, P, N1, TMt, TMs, TMe, L)
  = if Q' <= Q
    then (C, N + 1, HO, N')
    else exclusive(NeLC, (Q + Q1) LF, N, P, N'', TMt, TMs, TMe, L)
  fi
  if (Q', N'') := getProb(N1) /
    (T, N'') := getTimeCmd(C, L, TMt, TMs, TMe, N') /
    HO := insert(((T, N)), P) .
```

Similarly, the `independent` function takes as arguments the list of processes for the probabilistic choice (as `List{SSCCCmd}`), the list of its probabilities (as `List{Float}`), the list of previous selected processes (as `List{SSCCCmd}`), the counter of the internal identifier of processes (as `Nat`), the priority queue of the system (as `Heap{2Tuple}`), the seed for random sampling (as `Nat`), the maps of the time functions (as `Map{Aid, StExp}`), and the agent identifier of the agent where the process is executed (as `Aid`). It returns a list of the processes selected to be executed, the next internal identifier to be applied, the priority queue updated with the selected processes, and the new seed for random sampling.

```
op independent : List{SSCCCmd} List{Float} List{SSCCCmd} Nat Heap{2Tuple} Nat Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Aid -> Tuple{List, Nat, Heap, Nat}.

eq independent(nil, nil, LC', N, P, N1, TMt, TMs, TMe, L) = (LC', N, P, N1) .

eq independent(C LC, Q LF, LC', N, P, N1, TMt, TMs, TMe, L)
  = if Q' <= Q
    then independent(LC, LF, LC', C, N + 1, HO, N', TMt, TMs, TMe, L)
    else independent(LC, LF, LC', N, P, N', TMt, TMs, TMe, L)
  fi
  if (Q', N') := getProb(N1) /
    (T, N'') := getTimeCmd(C, L, TMt, TMs, TMe, N') /
    HO := insert(((T, N)), P) .
```

### 5.2. System Transitions

The transitions in the rewriting logic semantics of `SSCC` comprise both *invisible* (given by equations) and *observable* transitions (given by rules).

There are two invisible transitions, namely, one for removing a `O` command from a configuration and another one to join the contents of two stores of the same space (i.e., two stores with the same `Aid`).

```
eq < L0 : process | command : 0, Atts > = none .
eq < L0 : agent | store : B0 > < L0 : agent | store : B1 > = < L0 : agent | store : (B0 and B1) > .
```
The second type of (invisible) transitions is important because when a new process is spawned in an agent's space, a store with the empty constraint (i.e., true) is created for that space. If such a space existed before, then the idea is that the newly created store is subsumed by the existing one. Note that neither of the invisible transitions take time, i.e., they are instantaneous and axiomatize structural properties of commands.

There are nine observable transitions, i.e., rewrite rules, that capture the concurrent behavior in \( \mathcal{R} \); they are explained in the rest of this section. It is important to observe that rules such as [exclusive] and [independent] are probabilistic in nature. However, the syntax of probabilistic rewrite rules is slightly modified with respect to the one presented in Section 2 by encapsulating probabilistic sampling is some auxiliary functions presented before. Furthermore, the stochastic behavior of processes is determined also by auxiliary functions that sample probability distribution functions to assign a time (i.e., duration) to processes. Finally, in most of the rules, the counter in the simulation object is updated with a new value computed in the conditions. This is because more than one probabilistic choice could be performed internally by the auxiliary functions in the conditions; in these cases, the new counter represents the next counter that can be used after all of these samplings have been made.

Rule [tell]. It defines the semantics of \( \text{tell}(\_\_\_) \) processes. Once a process of this type is the next to be executed (as indicated by the priority queue in the simulation object with time 0), its constraint is placed in the corresponding store and the process terminates.

\[
\begin{align*}
\text{r1 [tell]} : \\
&< L0 : \text{agent | store : } B0 > \\
&< L0 : \text{process | UID : I0, command : } \text{tell } (B1) > \\
&< I : \text{simulation | pqueue : } T(Ra,((Ti, I0)), Le, Ri), \text{flag : false, pend : P, Atts} > \\
\Rightarrow < L0 : \text{agent | store : } (B0 \text{ and } B1) > \\
&< I : \text{simulation | pqueue : } T(Ra,((Ti, I0)), Le, Ri), \text{flag : true, pend : P, Atts} > .
\end{align*}
\]

Rule [ask]. It defines the semantics of \( \text{ask}(\_\_\_) \rightarrow \_\_ \) commands when their guards are entailed by the constraint in the current store. Note that the semantic consequence relation of the constraint system is queried by asking the SMT solver. When the guard \( B1 \) is entailed by the constraint \( B0 \) in the local store \( L0 \), the command \( C1 \) is moved into the priority queue of the system (attribute \text{pqueue}). The time of the command \( C1 \) includes the time function \( aTM \) for querying the local store \( B0 \) and a factor of its size (attribute \text{factor}). Note that the matching conditions (i.e., the ones of the form \( \_ := \_ \)) are used in a similar way as “let” commands in functional programming languages.

\[
\begin{align*}
\text{c1 [ask]} : \\
&< L0 : \text{agent | store : } B0 > \\
&< L0 : \text{process | UID : I0, command : } (\text{ask } B1 \rightarrow C1) > \\
&< I : \text{simulation | pqueue : } T(Ra,((Ti, I0)), Le, Ri), \text{flag : false, pend : P, nextID : N, counter : N1, tTM : TMT, aTM : TMa, sTM : TMs, eTM : TMe, factor : alpha, Atts} > \\
\Rightarrow < L0 : \text{agent | store : } B0 > \\
&< L0 : \text{process | UID : N, command : C1} > .
\end{align*}
\]
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : true, pend : H0, nextID : (N + 1), counter : N3, tTM : TMt, aTM : TMa, sTM : TMs, eTM : TMe, factor : alpha, Atts >

if entails(B0, B1) /
(T0, N2) := getTimeCmd(C1, L0, TMt, TMs, TMe, N1) /
(T1, N3) := fTime(TMa, L0, N2) /
S := size(B0) /
H0 := insert(((T0 plus (T1 plus (S * alpha)), N)), P) .

Rule [delay]. It defines the semantics of ask(_) → _ commands when their guards are not entailed by the constraint in the current store. Similar to the case handled by the rule [ask], the semantic consequence relation of the constraint system is queried by asking the SMT solver. When the guard B1 is not entailed by the constraint B0 in the local store L0, the ask command is delayed and placed in the priority queue for pending ask commands, where it will remain “locked” until the [tick] rule executes again (see A.3).

crl [delay] :
< L0 : agent | store : B0 > < L0 : process | UID : I0, command : (ask B1 -> C1) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), pend : P, Atts >
==> < L0 : agent | store : B0 > < L0 : process | UID : I0, command : (ask B1 -> C1) >
< I : simulation | pqueue : merge(Le, Ri), pend : insert(((Ti, I0)),P), Atts >
if not(entails(B0, B1)) .

Rule [parallel]. It implements the semantics for parallel composition of processes by spawning the two processes in the current space by creating a new object in the configuration for each of the two commands. These two commands are assigned a time-to-execution and added to the system’s scheduler.

crl [parallel] :
< L0 : process | UID : I0, command : (C0 || C1) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
==> < L0 : process | UID : N, command : C0 >
< L0 : process | UID : (N + 1), command : C1 >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 2), flag : true, pend : H0, counter : N3, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
if (T0, N2) := getTimeCmd(C0, L0, TMt, TMs, TMe, N1) /
(T1, N3) := getTimeCmd(C1, L0, TMt, TMs, TMe, N2) /
H0 := insert(((T0, N)), insert(((T1, N + 1)), P)) .

Rule [recursion]. It defines the semantics of the μ._._ recursion commands. Operationally, for a command μ(N0,C0) ready for execution, it replaces all appearances of the variable command V(N0) within command C0 with μ(N0, C0) using the auxiliary function replace. This creates the effect of a recursive call.

crl [recursion] :
< L0 : process | UID : I0, command : μ(N0, C0) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
==> < L0 : process | UID : N, command : replace(N0, C0, μ(N0, C0)) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 1), flag : true, pend : H0, counter : N2, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
if (T0, N2) := getTimeCmd(C0, L0, TMt, TMs, TMe, N1) \ /
H0 := insert(((T0, N)), P).

Rules [extrusion] and [space]. They define the semantics of space navigation given by commands ↑_(_)_ and [_[_]]. The [extrusion] rule executes the extrusion command C0 out NO by executing the C0 command in the parent’s space L0 of the current space NO. The [space] rule executes the command C0 in NO by creating a new space for agent NO inside the current space (denoted by NO.L0) with an empty store (i.e., true), in addition to the execution of the C0 command within the new agent’s space. Note that if such a space already exists, the empty store will be subsumed by it thanks to the invisible rule for merging two stores corresponding to the same agent, as presented above. If the time functions are not defined for the new agent in the initial state, then they are inherited from its nearest ancestor for future computation; otherwise, they are assigned to a default distribution function.

crl [extrusion]:
< NO . L0 : process | UID : I0, command : (C0 out NO) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1,
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
==> < NO . L0 : process | UID : N, command : C0 >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : true, pend : H0, nextID : (N + 1), counter : N2,
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
if (T0, N2) := getTimeCmd(C0, NO . L0, TMt', TMs', TMe', N1) \ /
H0 := insert(((T0, N)), P).

crl [space]:
< NO . L0 : process | UID : I0, command : (C0 in NO) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1,
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
==> < NO . L0 : agent | store : true >
< NO . L0 : process | UID : N, command : C0 >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : true, pend : H0, nextID : (N + 1), counter : N2,
tTM : TMt', sTM : TMs', eTM : TMe', Atts >
if TMT' := insert(NO . L0, get-ancestor(TMt, NO . L0), TMt) \ /
TMa' := insert(NO . L0, get-ancestor(TMa, NO . L0), TMa) \ /
TMs' := insert(NO . L0, get-ancestor(TMs, NO . L0), TMs) \ /
TMe' := insert(NO . L0, get-ancestor(TMe, NO . L0), TMe) \ /
(T0, N2) := getTimeCmd(C0, NO . L0, TMt', TMs', TMe', N1) \ /
H0 := insert(((T0, N)), P).

Rule [exclusive]. It implements the semantics of the ⊕_ exclusive probabilistic choice command. This rule executes a single command from a given list of commands LC with their corresponding probabilities LF. For each command in the list, a random variable with uniform distribution is sampled by calling the auxiliary function exclusive. Internally, if the sampled probability is greater or equal to the probability associated to the corresponding command, then it is selected for execution; otherwise, the next command is evaluated with the corresponding cumulative probability. In this function, it is assumed that the sum of
the probabilities of the list must be equal to 1. The last command of the list is executed if no other one is executed.

crl [exclusive]:
< L0 : process | UID : I0, command : exc(LC, LF) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1,
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
=> genCommands(LC', N, L0)
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N', flag : true, pend : P0, counter : N'',
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
if (LC', N', P0, N'') := exclusive(LC, LF, N, P, N1, TMt, TMs, TMe, L0) .

Rule [independent]. It implements the semantics of the \( \bigodot \) independent probabilistic choice command. This rule executes a subset of a given list of commands LC with their corresponding probabilities LF with the help of the auxiliary function independent. Each command in the list is chosen or dropped by using a random variable with uniform distribution. In the end, all, some, or none of the given commands in the list can be selected for execution.

crl [independent]:
< L0 : process | UID : I0, command : ind(LC, LF) >
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1,
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
=> genCommands(LC', N, L0)
< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N', flag : true, pend : P0, counter : N'',
tTM : TMt, sTM : TMs, eTM : TMe, Atts >
if (LC', N', P0, N'') := independent(LC, LF, nil, N, P, N1, TMt, TMs, TMe, L0) .

This section concludes by introducing an example illustrating the rewriting logic semantics developed for SSCC.

**Example 4.** The initial state (a) in Figure 2 can be represented in the SSCC semantics as follows:

< root : agent | store : V === 9 >
< 0 . root : agent | store : I >= 11 >
< 1 . root : agent | store : true >
< 2 . root : agent | store : true >
< 0 . 1 . root : agent | store : Y > 5 >

In this syntax, the hierarchical task assignment system is modeled, where agents are workers and processes are tasks to be assigned. Mobility in this system is modeled by tasks flowing between workers. Each worker has a unique representation with a unique qualified name (i.e., Aid). This representation is useful to identify the time taken by each one of the processes. Consider the following process R:

\[ P_{11} := \text{ind}((\text{tell}(A === 1) \text{ in } 1) \ (\text{tell}(B === 1) \text{ in } 2) \ (\text{tell}(C === 1) \text{ in } 3) \ (\text{tell}(D === 1) \text{ in } 4), 0.5 \ 0.5 \ 0.5 \ 0.5) \]

\[ P_{12} := (\text{tell}(Y === 25) \ || \ \text{ask}(Y > 2) -> (\text{tell}(Y > 2) \text{ out } 2)) \text{ in } 2 \]
The hierarchical system for process $R$ is represented in Figure 3. This configuration is encoded as follows:

\[
\begin{align*}
P_1 &:= \text{esc}((P_1 \parallel \text{tell}(Y === 5) \parallel (\text{ask}(Y > 2) \rightarrow (\text{tell}(Y > 2) \text{ out 1}))) \text{ in 1}) \parallel P_2, \ 0.60 \ 0.40) \\
P_2 &:= \text{ask}(Y > 2) \rightarrow (\text{tell}(X === 15) \parallel \text{ask}(X >= 10) \rightarrow (\text{tell}(X >= 10) \text{ out 1}) \\
P &:= (P_1 \parallel P_2) \text{ in 1} \\
Q_1 &:= (\text{tell}(Z === 9) \parallel \text{ask}(Z < 15) \rightarrow (\text{tell}(Z < 15) \text{ out 3}) \text{ in 3} \\
Q_2 &:= (\text{tell}(W === 25) \parallel \text{ask}(W > 0) \rightarrow (\text{tell}(W > 0) \text{ out 4}) \text{ in 4} \\
Q_3 &:= \text{ask}(Z < 15 \ \text{and} \ W > 0) \rightarrow (\text{tell}(V === 67) \parallel \text{ask}(V < 100) \rightarrow (\text{tell}(V < 100) \text{ out 2}) \\
Q &:= (Q_1 \parallel Q_2 \parallel Q_3) \text{ in 2} \\
R &:= P \parallel Q \parallel (\text{ask}(X >= 10 \ \text{and} \ V < 100) \rightarrow (\text{tell}(U === 50) \parallel \text{ask}(U < 55) \rightarrow \text{tell(DONE)}))
\end{align*}
\]

The global time of the system (attribute $\text{gtime}$) starts at zero. The random seed for the probability distributions sampling (attribute $\text{counter}$) is $N$ and the multiplicative factor (attribute $\text{factor}$) is $1/2$, i.e., the $\text{[ask]}$ rule will consider half of the size of the local store of the agent. The size of an agent's store is proportional to the number of atomic quantifier-free formulas in it plus its Boolean connectives. For example, the size of the formula $W > 0 \ \text{and} \ Z < 15$, with $W$ and $Z$ integers, is 3, therefore the $\text{[ask]}$ includes 1.5 time units. Time functions ($\alpha, \mu, \phi, \rho$) are defined as follows, respectively:

\[
\begin{align*}
\text{Figure 3. Example. Initial state of the system.}
\end{align*}
\]
The time functions of the example are normal distributions with the usual median and variance parameters, respectively. This expression is defined for the root agent and the other agents will inherit it. For example, querying the store at the root agent uses a normal distribution function with parameters $\mu = 1.2$ and $\sigma^2 = 0.2$.

The system may evolve as detailed below. Note that the final state depends on the random sampling of the time functions and the [exclusive] and [independent] rules.

Figure 3 depicts the exclusive process $P_1$ in the space of agent $1.root$. In the final state encoded above, the agent $1.1.root$ was created as a result of executing the $P_1$ process. Agents $2.4.1.1.root$, $3.4.1.1.root$, and $4.4.1.1.root$ were independently from process $P_1$.

6. Model Simulation

Some properties of reactive systems, e.g., fault-tolerance and consistency, can be tested with the rewriting logic semantics $\mathcal{R}$ of SSCE. In particular, this section uses Maude’s search command to test for consistency, fault-tolerance, and knowledge inference examples. Note, however, that since the rewrite theory $\mathcal{R}$ is probabilistic in nature, the results in this section can change if the seed selected for executing the experiments is different. The reader is referred to Section 7 for an example of quantitative analysis with $\mathcal{R}$ in the form of statistical model checking.

This section uses a SSCE simplified version of the system presented in Example 4, where:

$$
P_1 := \text{ask}(W > 1) \rightarrow (\text{tell}(Y === 32) \mid \text{ask}(Y > 9) \rightarrow (\text{tell}(Y > 9) \text{ out 2}))
$$

$$
P_2 := \text{ask}(Y > 2) \rightarrow (\text{tell}(X === 15) \mid \text{ask}(X >= 10) \rightarrow (\text{tell}(X >= 10) \text{ out 1}))
$$

$$
P := (P_1 \mid P_2) \text{ in 2}
$$

$$
Q := \text{ask}(X >= 10) \rightarrow (\text{tell}(U === 50) \mid \text{ask}(U < 55) \rightarrow \text{tell}(DONE))
$$

$$
R := P \mid Q
$$
The process \( P_1 \) posts some information in the space of agent \( 2.1 \).root and waits until it has enough information to infer \( W > 1 \). Once it has enough information, it posts \( Y > 9 \) in the space of its ancestor. Process \( P \) executes \( P_1 \) and asks if \( Y > 2 \) is known to be true in the space of agent \( 1 \).root. Once the agent has gained enough information, it posts \( X \geq 10 \) in the space of its ancestor. Process \( R \) executes \( P \) and asks if \( X \geq 10 \) is known to be true in the space of agent root. Once this agent has gained enough information, it posts \text{DONE} \) in its current space. Note that probabilities are used only for sampling the execution time of processes, and that exclusive and independent operators are not used.

Consistency and Fault-tolerance. Consistency is the property that ensures a local failure does not propagate to the entire system; fault tolerance ensures a system to continue operating properly in the event of a failure. In \text{SSCC}, this means that if a store becomes inconsistent, it is not the case that such an inconsistency spreads to the entire system. Even though, inconsistencies can appear in other stores due to some unrelated reasons.

Queries such as these ones, can be implemented with the help of \( R \) and the rewriting modulo SMT approach by using Maude’s \text{search} \ command. As an example, consider the following \text{search} \ command:

```maude
search in APMAUDE :
< root : agent | store : (X < 5), set : (empty) >
< root : process | UID : 1, command : R >
< 1 : simulation | gtime : 0, pqueue : T(1,((0,1)),empty,empty), pend : empty, nextID : 2, flag : false,
  counter : 13, tTM : ((root) \mapsto \text{Norm}(1.0, 0.2)), aTM : ((root) \mapsto \text{Norm}(1.2, 0.2)),
  sTM : ((root) \mapsto \text{Norm}(0.5, 0.2)), eTM : ((root) \mapsto \text{Norm}(0.5, 0.2)), factor : 1/2 >
flg(true, 1000.0)
=>* < L0 : agent | store : B0:Boolean, set : SN:Set{Nat} > C:Config
such that check-unsat(B0:Boolean) .
```

Note that a store is inconsistent if it is unsatisfiable, thereby checking whether a store is inconsistent is accomplished with the function \text{check-unsat}. This command finds 20 reachable states where there is at least one inconsistent store. Therefore, even though the inconsistency appears, the system continues evolving until no more processes can be performed. It is possible to verify that there are states with consistent and inconsistent stores at the same time by slightly modifying the search command.

Solution 1 (state 341)

```maude
states: 342 rewrites: 276047 in 1888ms cpu (1889ms real) (146211 rewrites/second)
C:Config --> flg(true, 1.0e+3)
< root : process | UID : 3, command : Q >
< 1 : simulation | gtime : 16.22, pqueue : T(1,(0.98,21),empty,empty), pend : T(1,(0,3),empty,empty),
  nextID : 22, counter : 26, flag : true, tTM : ..., aTM : ..., sTM : ..., eTM : ..., factor : 1/2 >
< 1 . root : agent | store : (Y:Integer > 9 and X:Integer === (15).Integer), set : 2 >
< 2 . 1 . root : agent | store : (Y:Integer > 9 and X:Integer === (15).Integer), set : 2 >
Y:Integer === (32).Integer, set : empty >
L0 --> root
B0 --> X:Integer < (5).Integer and X:Integer >= (10).Integer
SN:Set{Nat} --> (1).NzNat
```
Knowledge Inference. It refers to acquiring new knowledge from existing facts. In the setting of $\mathcal{R}$, this means at some point an agent has gained enough information to infer –from the rules of first-order logic– new facts. As an example, consider the following search command:

```
search in APMAUDE :
  < root : agent | store : true, set : (empty) >
  < root : process | UID : 1, command : R >
  < 1 : simulation | gtime : 0, queue : T(1,(0,1)), empty, empty), pend : empty, nextID : 2, flag : false,
           counter : 13, tTM : ((root) |-> Norm(1.0, 0.2)), aTM : ((root) |-> Norm(1.2, 0.2)),
           sTM : ((root) |-> Norm(0.5, 0.2)), eTM : ((root) |-> Norm(0.5, 0.2)), factor : 1/2 >
  flg(true, 1000.0)
  =>* < L0 : agent | store : B0:Boolean, set : SN:Set{Nat} > C:Config
  such that entails(B0:Boolean, gen-var("X") > 9).
```

This command finds a reachable state from the given initial state, in which some store logically implies $Y > 9$. This query finds the following solutions:

Solution 1 (state 211)

```
states: 212 rewrites: 241358 in 1156ms cpu (1160ms real) (208787 rewrites/second)
C:Config --> flg(true, 1.0e+3)
  < root : agent | store : true, set : 1 >
  < root : process | UID : 3, command : Q >
  < 1 : simulation | gtime : 4.41, queue : T(1,(1.03,14)), empty, empty),
     pend : T(2,(0,3), T(1,(0,7),empty,empty)), nextID : 15,
     counter : 18, flag : true, tTM : ..., aTM : ..., sTM : ..., eTM : ..., factor : 1/2 >
  < 1 . root : agent | store : true, set : 2 >
  < 1 . root : process | UID : 7, command : P2 >
  < 2 . 1 . root : process | UID : 13, command : (ask Y:Integer > 9 -> (tell(Y:Integer > (9).Integer) out 2)) >
  L0 --> 2 . 1 . root
  B0 --> W:Integer === (5).Integer and Y:Integer === (32).Integer
  SN:Set{Nat} --> (empty).Set{Nat}
```

Same Knowledge. Same knowledge refers to different agents gaining, at some point, the same knowledge. In the case of the particular constraint system implemented in this manuscript, it means having two stores with logically equivalent stores. As an example, consider the following Maude search command, querying for two stores having the same information when they are non-empty:

```
search in APMAUDE :
  < root : agent | store : true, set : (empty) >
  < root : process | UID : 1, command : R >
  < 1 : simulation | gtime : 0, queue : T(1,(0,1)), empty, empty), pend : empty, nextID : 2, flag : false,
     counter : 13, tTM : ((root) |-> Norm(1.0, 0.2)), aTM : ((root) |-> Norm(1.2, 0.2)),
     sTM : ((root) |-> Norm(0.5, 0.2)), eTM : ((root) |-> Norm(0.5, 0.2)), factor : 1/2 >
  flg(true, 1000.0)
```
Note that it is never the case that there are two stores with the same information, which agrees with the following output of Maude.

No solution.
states: 363 rewrites: 306427 in 4824ms cpu (4859ms real) (63521 rewrites/second)

It is important to emphasize, as pointed out in the opening of this section, that the fact that no solution can be found, does not mean that actually no solution exists: this reachability queries are dependent on the seed used for the pseudo-random number generator. That is, the reachability queries presented in this section can be seen as mechanisms for testing, but not as a fully-fledged approached for reachability analysis.

7. **Statistical Model Checking of a Random Search on a Hierarchy of Spaces: A Case Study**

This section presents a case study on a process that performs a random search on a hierarchy of spaces, which illustrates both the use of the SSOC rewriting logic semantics and how statistical model checking can be performed with the help of the PVeStA statistical model checker.

7.1. **Description of the Case Study.** Social networks relate agents sharing information with each other. Such interactions could create, e.g., friend circles, social forums, and debates. A discussion among agents could result in a far-from-desirable-scenario in which damaging comments are posted. This could ultimately affect users in the social network. Therefore, the possibility to detect such unwelcome posts can be seen as a mechanism to improve fair interaction among users in the social network.

The case study is about a robot trying to find inappropriate posts in a social network, by randomly visiting the spaces associated to each user. As such, the social network is represented in the SSOC model as a hierarchy of spaces and the information posted by the users, distributed in the hierarchy of spaces, as constraints. The overall idea is that the robot will explore the network looking for posts containing unwanted information. The evolution of the robot in the space hierarchy is decided at random by visiting the spaces adjacent to the one it is with equal probability (i.e., it chooses from the parent space –if it exists– and the children spaces uniformly at random). If such a post is found, a constraint is added to the corresponding store representing the fact that a warning message is posted. To keep the case study simple, the robot will stop once it finds the first unwanted post. If more posts were to be found, a copy of the robot process could be spawn again.
7.2. **Formal Specification.** The behavior of the robot is specified by two rewrite rules. Each one of these rules implements a macro command called \texttt{watch} that is defined on top of the SSCC specification and, as such, does not add any new expressive power to the model. Specifically, the command \texttt{watch}(C, B) indicates that if the store in which the constraint B is entailed (e.g., an offensive message is found), the command C is the action to be executed by the robot.

The \texttt{[search]} rule handles the \texttt{watch} command when the undesired post is not found in the current space.

\begin{verbatim}
    crl [search] :
      < L0 : agent | store : B0, set : SN >
      < L0 : process | UID : I0, command : watch(C0, B1) >
      < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N,
        flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
    =>
      < L0 : agent | store : B0, set : SN >
      < L0 : process | UID : N, command : exc(LC, LF) >
      < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 1),
        flag : true, pend : H0, counter : N'', tTM : TMt, sTM : TMs, eTM : TMe, Atts >
      if not(entails(B0, B1))
      \/
      LC := command-list(L0, SN, watch(C0, B1))
      \/
      P(LF, N'') := prob-list(size(LC), N1)
      \/
      (T0, N'') := getTimeCmd(exc(LC, LF), L0, TMt, TMs, TMe, N'')
      \/
      H0 := insert(((T0, N)), P) .
\end{verbatim}

The \texttt{[search]} rule interprets the command \texttt{watch}(C0, B1) into an exclusive command \texttt{exc}(LC, LF) when the constraint B1 is not entailed by the store in the space where the robot is. The exclusive command uniformly at random chooses the next space the robot will visit, by either leaving the current space or going to a children space. If no such a space exists, then it will stay in its current space.

The rule \texttt{[found]} executes the command C0 specified in \texttt{watch}(C0, B1) if the constraint B1 is entailed by the store of the space the robot is in, i.e., an unwanted message is found.

\begin{verbatim}
    crl [found] :
      < L0 : agent | store : B0, set : SN >
      < L0 : process | UID : I0, command : watch(C0, B1) >
      < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N,
        flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
    =>
      < L0 : agent | store : B0, set : SN >
      < L0 : process | UID : N, command : C0 >
      < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 1),
        flag : true, pend : H0, counter : N2, tTM : TMt, sTM : TMs, eTM : TMe, Atts >
      if entails(B0, B1)
      \/
      (T0, N2) := getTimeCmd(C0, L0, TMt, TMs, TMe, N1)
      \/
      H0 := insert(((T0, N)), P) .
\end{verbatim}

In this case study, the command C0 represents posting a warning message (i.e., posting a specific constraint to the local store). However, the rule can still be used if other commands
were to be used. For instance, if the goal is to keep searching for unwanted messages, the command \texttt{C0} could correspond to \texttt{watch(C0, B1) || tell('...')}, where the command \texttt{tell} marks the current store as containing an unwanted message.

7.3. Using \texttt{PVeStA}. The initial state of the \texttt{SSCC} system is created by the \texttt{initState} function, where the seed for sampling (\texttt{N} as \texttt{Nat}) is included. Besides the initial structure and the process (including the robot process), a \texttt{flg} message is placed in the initial configuration for indicating whether the simulation will continue (as \texttt{Bool}) and the maximum time for the simulation (as \texttt{Float}).

--- init state
\begin{verbatim}
op initState : Nat -> Configuration .
op initState : -> Configuration .
rl initState => initState(counter) .

eq initState(N)
  = ... flg(false, 0.0) .
\end{verbatim}

The property to be checked is the expected time it takes the robot find an unwanted message. This is achieved with the help of the \texttt{val(_,_)} function: it takes as arguments a unique natural number to identify the property (as \texttt{Nat}) and the configuration of the system (as \texttt{Configuration}). The function returns the corresponding value of the property (as \texttt{Float}). The \texttt{QuaTEx} expression uses the \texttt{val(_,_)} function to calculate $E[Exp]$.

\begin{verbatim}
op val : Nat Configuration -> Float .
eq val(1, Conf) = getExcecutionTime(Conf) .
\end{verbatim}

The expected execution time of the sample is taken from the \texttt{gtime} argument of the \texttt{simulation} object in the configuration. Note that such a duration is parametric on the random sampling of the stochastic expressions defined in the time functions (e.g., $\alpha, \mu, \phi, \rho$) defined as part of the initial state of the system.

--- execution time
\begin{verbatim}
op getExcecutionTime : Configuration -> Float .
eq getExcecutionTime(< I : simulation | gtime : T, Atts > Conf)
  = float(T) .
\end{verbatim}

Finally, given the unique identifier of the property assigned by the \texttt{val(_,_)} function, the \texttt{QuaTEx} expression to be checked is defined as

```
extecTime ( ) = s.rval( 1 ) ;
```

where \texttt{s.rval( 1 )} command connects \texttt{PVeStA} with the probabilistic system. Then, the expression is evaluated with the command

```
eval E[ # execTime ( ) ] ;
```

which output is the expected value of the property.
7.4. **Experimentation.** The setup of the experiments is the following: a hierarchical structure of spaces is generated at random (up to some given depth), with constraints and processes. The robot process is included in the initial configuration of processes. Note that other processes can concurrently modify the contents of any store, so that it makes sense that the robot can revisit a space.

The experiment consisted in computing the expected value of the time it takes the robot to find an unwanted post in different hierarchical configurations. In particular, these configurations are generated from depth 5 to depth 13 at random. The results are presented in Table 1; the experimentation was performed in a cluster with a front-end with 16 cores and 32 GB of RAM, and four workers with 64 cores and 64 GB of RAM each. The experiment with height 13 did not finish because of memory issues.

| Hierarchy (depth) | Spaces (count) | Samples (count) | RAM (GB) | Exec. time (sec) | Expected value (units) |
|-------------------|----------------|-----------------|----------|------------------|------------------------|
| 5                 | 11             | 2400            | 6.60     | 175.19           | 38.88                  |
| 6                 | 23             | 1800            | 22.60    | 770.27           | 201.94                 |
| 7                 | 42             | 300             | 6.00     | 282.45           | 324.11                 |
| 8                 | 48             | 1800            | 12.90    | 605.51           | 111.74                 |
| 9                 | 65             | 300             | 12.20    | 895.12           | 676.60                 |
| 10                | 81             | 600             | 10.60    | 1024.89          | 303.05                 |
| 11                | 97             | 300             | 12.50    | 1541.57          | 716.75                 |
| 12                | 115            | 300             | 57.50    | 11607.78         | 3312.83                |
| 13                | 144            | –               | 120.00*  | 25200.00*        | –                      |

**Table 1.** Expected values computed by the robot case study. From left to right, the columns represent the height of the search tree, the number of spaces in the tree, the amount of memory RAM used for the execution, the run-time of the execution, and expected time it takes the robot find an unwanted post.

It is important to note that PVeStA performs different number of runs to compute the expected value of the robot finding the unwanted message, depending on the height of the tree and the number of nodes. On a separate note, this experimentation suggests that PVeStA uses a significant amount of memory, even if the number of spaces is small.

8. **Related Work**

Distributed information is a central notion in systems with hierarchical structure, where agents hold spaces with data and processes. S. Knight et al. [29] extend the theory of CCP
with spatial distribution of information in the \textit{spatial constraint system} (SCS). Computational hierarchical spaces can be assigned to belong to agents, and each space may have CCP processes and other (sub) spaces. Processes can post and query information in their given space (i.e., locally) and may move from one space to another. M. Guzman et al. [20] develop the theory of spatial constraint systems to provide a mechanism for specifying the mobility of information or processes from one space to another. In [24], spatial constraint systems are used as an abstract representation of modal logics. This is useful to characterize the notion of normality for self-maps in a constraint system. As a counterpart of normal modal operator, it is shown that a self-map is normal if and only if it preserves finite suprema. Also, this abstraction is used to derive right inverse operators for modal languages such as Kripke spatial constraint systems. D. Gilbert and C. Palamidessi [15] propose a different approach to characterize process mobility using labeled transition systems.

Other efforts have focused on extending concurrent constraint programming with temporal behavior. For example, V. Saraswat, R. Jagadeesan, and V. Gupta [47] present a timed asynchronous computation model and propose an implementation using loop-free deterministic finite automata, a declarative framework for reactive systems where time is represented as discrete time units. They also present a non-deterministic version of CCP in [46], extending the model to express strong time-outs and preemption. Jagadeesan et al. [27] propose a policy algebra in the timed concurrent constraint programming paradigm that uses a form of default constraint programming and reactive computing to deal with explicit denial, inheritance, overriding, and history-sensitive access control. M. Nielsen et al. [34] introduce a model of temporal concurrent constraint programming, adding the capability of modeling asynchronous and nondeterministic timed behavior. Additionally, they propose a proof system for linear-temporal properties. G. Sarria and C. Rueda [50] present an extension of \texttt{ntcc} for specifying and modeling real-time behavior; their operational semantics supports resources and limited time, and define a denotational semantics. As an application, the formal modeling of a music improvisation example is implemented in this language. F. de Boer et al. [9] define a timed extension of CCP with more expressive power than CCP.

Extensions of CCP with probability have also been explored. In [18], discrete random variables are introduced with a given probability distribution. A new operator defines the probability distribution of a random variable that is used to select whether a process is executed or precluded. Random variables may be constrained, and thus inconsistencies may arise between the chosen values of random variables and constraints in the store. Those inconsistencies can cause some system runs to be precluded. In [39], an operational semantics is proposed by using probabilistic automata, with the final goal of extending \texttt{tcc} [47] with probabilistic and non-deterministic choices for processes. A probabilistic choice operator is defined to select processes guarded by constraints and determine which one will be executed. When a process is chosen for execution, the other processes are blocked. Also, the notion of \textit{probabilistic eventualty} is introduced to model the possible delay of a process \textit{P} to be executed. If \( r \) represents the probability for executing \( P \) in the current time unit, the closer \( r \) is to 1, the greater the probability of executing \( P \) will be. Analogously, \( 1 - r \)
denotes the probability of delaying the execution of \( P \). In such a case, the given probability distribution modifies \( r \) in a recursive call used to reserve \( P \) for the next time unit. In [28] labeled continuous time Markov chains are used to provide the semantics of stochastic processes. If \( x \) and \( y \) are states, and \( a \) is a label to move from \( x \) to \( y \), the exponential distribution governs the duration of the transition from \( x \) to \( y \) with label \( a \). Conditional probability is used to govern the transition \( x \overset{a}{\rightarrow} y \): it is the probability that \( x \) makes the transition to \( y \) by an \( a \)-transition.

The inclusion of stochastic information for processes has also been proposed. J. Aranda et al. [3] associate a random variable to each computation for determining its time duration: given a set of competing actions, the fastest action is executed (i.e., the one with the shortest duration). Based on CCP, D. Chiarugi et al. [7] implement a technique for the stochastic simulation of biochemical reactions with non-Markovian behavior. V. Gupta et al. [17] describe a stochastic concurrent constraint language for the description and programming of concurrent probabilistic systems. In this language, programs encode probability distributions over sets of objects. Also, structural operational semantics and denotational semantics are provided.

Finally, in the realm of rewriting logic, some executable semantics in the Maude system have been proposed for CCP-based models. M. Romero and C. Rocha [42] present a symbolic rewriting logic semantics of the spatial modality of CCP with extrusion based on the work in [29, 20]. More recently, M. Romero and C. Rocha [43] have proposed a symbolic rewriting logic semantics of the spatial modality of CCP with extrusion and real-timing behavior. Somewhat related, P. Degano et al. [10] provide a rewriting logic semantics for Milner’s CCS with interleaving behavior. Additionally, a set of axioms is defined for a logical characterization of the concurrency of CCS processes.

9. Concluding Remarks

This paper has presented a rewriting logic semantics for \texttt{SSCC}, a probabilistic, timed, and spatial concurrent constraint model. It is fully executable in the Maude system. The intended models of \texttt{SSCC} are spatially-distributed multi-agent reactive systems that may have different computing capabilities, and be subject to real-time requirements and probabilistic choice. In this setting, time attributes are associated to process-store interaction, as well as to process mobility in the space structure, by means of maps from agents to probability distribution functions. Details about the underlying constraint system have been given as materialized with the help of rewriting modulo SMT and real-time behavior with the help of Real-Time Maude. Furthermore, examples of quantitative analysis based on statistical model checking have been given to illustrate key features of \texttt{SSCC}.

Future work includes the study of a structural operational semantics for \texttt{SSCC} and proofs of correspondence between this new development and the rewriting logic semantics contributed by this work. Also, algebraic properties of \texttt{SSCC} need to be explored and, if possible, establish relationships with previous extensions of CCP with time, probabilities, and spaces. Finally, new case studies need to be developed for \texttt{SSCC}.
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Appendix A. Scaffolding

The rewriting logic semantics is specified in Maude as a collection of modules and module operations presented in B.

A.1. Leftist Heap. The run-to-completion time of commands is simulated with the help of a leftist heap that keeps track of all the active commands that are waiting for the global timer to advance. The processes in the heap are represented by pairs \((N, t)\) where \(N\) is a unique identifier and \(t\) is its execution time. In any initial state of computation, all processes are added to the heap and they are sorted with respect to their execution time. A process is executed when its execution time is the least time of all the processes that are pending to complete their transitions. When a process \((N, t)\) is at the root of the heap, it is removed and the execution time of the remaining processes in the heap is reduced \(t\) units. When execution ends, the heap will be empty or will contain only processes that cannot make any further transition.

A leftist heap [35] is a heap-ordered binary tree that satisfies the leftist property: the rank of any left child is at least as large as the rank of its right sibling. It is implemented as a parametrized container in the functional module \texttt{LEFTIST-HEAP}\{X :: STRICT-TOTAL-ORDER\}. Admissible sets of elements are strict totally ordered sets. Heaps are constructed from the constant \texttt{empty} and by means of the constructor operator written as \texttt{T(_,_,_,_)}. For a given heap \texttt{T(R,_,_,_)} , \(E\) is a node element, \(R\) is the rank of \(E\); and \(L\) and \(R\) are the left and right siblings of \(E\), respectively. Auxiliary operations include \texttt{isEmpty}, \texttt{rank}, and \texttt{makeT}, which are used to verify whether a heap is empty, compute the rank of a given heap, and create a heap out of two heaps, respectively.

\begin{verbatim}
op isEmpty : Heap{X} -> Bool .
op rank : Heap{X} -> Nat .
op makeT : XElt Heap{X} Heap{X} -> NeHeap{X} .

eq isEmpty(empty) = true .

eq isEmpty(T(Ra,E,L,R)) = false .

eq rank(empty) = 0 .

eq rank(T(Ra,E,L,R)) = Ra .

eq makeT(E,L,R) = if rank(L) >= rank(R)
          then T(rank(R) + 1,E,L,R)
          else T(rank(L) + 1,E,R,L)
        fi .
\end{verbatim}

Heap operations are defined as follows:

\begin{verbatim}
op findMin : NeHeap{X} -> X\$Elt .
op deleteMin : NeHeap{X} -> Heap{X} .
op insert : X\$Elt Heap{X} -> NeHeap{X} .
op merge : Heap{X} Heap{X} -> Heap{X} .
\end{verbatim}
eq findMin(T(Ra,E,L,R)) = E .
eq deleteMin(T(Ra,E,L,R)) = merge(L,R) .
eq insert(E,L) = merge(T(1,E,empty,empty),L) .
eq merge(empty, L) = L .
eq merge(L, empty) = L .
eq merge(T(Ra,E,L,R),T(Ra',E',L',R'))
  = if (E < E')
    then makeT(E,L,merge(R,T(Ra',E',L',R')))
    else makeT(E',L',merge(T(Ra,E,L,R),R'))
  fi .

The length of the left spine of any node is always at least as large as the length of its right spine because of the condition \((E < E')\) in equation for \text{merge}. Operations \text{insert} and \text{deleteMin} are based on the merge operation; \text{findMin} returns the element at the top of a non-empty heap.

A.2. SMT Solver. SMT solving technology available from the current version of Maude is used to realize the underlying constraint system of \text{SSCC}. The sort \text{Boolean} (available in the current version of Maude from the \text{INTEGER} module) defines the data type used to represent \text{SSCC}'s constraints. Terms of sort \text{Boolean} are quantifier-free formulas built from variables ranging over the Booleans and integers, and the usual function symbols. The current version of Maude is integrated with the CVC4 [4] and Yices2 [14] SMT solvers, which can be queried via the meta-level. In this semantics, queries to the SMT solvers are encapsulated by functions \text{check-sat} and \text{check-unsat}:

\begin{verbatim}
op check-sat : Boolean -> Bool .
op check-unsat : Boolean -> Bool .
eq check-sat(B)
  = metaCheck(["INTEGER"], upTerm(B)) .
eq check-unsat(B)
  = not(check-sat(B)) .
\end{verbatim}

The function invocation \text{check-sat}(B) returns true only if \(B\) is satisfiable. Otherwise, it returns false if it unsatisfiable or undefined if the SMT solver cannot decide. Note that function invocation \text{check-unsat}(B) returns true only if \(B\) is unsatisfiable. Therefore, the rewriting logic semantics of \text{SSCC} instantiates the constraint system \((\text{Con}, \sqsubseteq)\) by having quantifier-free formulas as the constraints \text{Con} and semantic validity (w.r.t. the initial model of the theory queried in the SMT solver) as the entailment relation \(\sqsubseteq\). More precisely, if \(\Gamma\) is a finite set of terms of sort \text{Boolean} and \(\phi\) is term of sort \text{Boolean}, the following equivalence holds:

\[ \Gamma \sqsubseteq \phi \iff \text{check-unsat} \left( \left( \bigwedge_{\gamma \in \Gamma} \gamma \right) \land \neg \phi \right). \]

In order to make a direct relation between the entailment relation \(\sqsubseteq\) and the Maude syntax, the operator \text{entails} is defined as follows:
This operator returns true if \( C_2 \) can be derived from \( C_1 \).

A.3. **The tick rule.** A real-time rewrite theory is a rewrite theory where some rules, called tick rules, model time elapse in the system [37]. Here the tick rule is defined as follows:

\[
crl \text{[tick]} : \\
\text{< I : simulation | pqueue : P, gtime : T, flag : true, pend : P0, Atts > } \\
\Rightarrow \text{< I : simulation | pqueue : merge(delta(deleteMin(P),T0),P0), gtime : (T plus T0),} \\
\text{flag : false, pend : empty, Atts > } \\
\text{if } T0 := p1(findMin(P)) .
\]

where the auxiliary operation \( \text{delta} \) reduces \( T0 \) units the execution time of every command in the heap \( P \):

\[
\begin{align*}
\text{op delta : Heap(2Tuple) Time -> Heap(2Tuple) .} \\
\text{eq } \text{delta(\text{empty},T')} &= \text{empty .} \\
\text{eq } \text{delta}(T(N,((T1, I)),P,P0),T') &= T(N,((T1 \text{ monus } T', I))\text{,delta}(P,T'),\text{delta}(P0,T')) .
\end{align*}
\]

When the \([\text{tick}]\) rule is fired, the global time \( T \) is incremented in \( T0 \) units, where \( T0 \) is the minimum time present in the priority queue \( P \), which is modified by removing the process with the minimum execution time. It also adds the pending commands in heap \( P0 \) to the priority queue \( P \). The pending commands are querying commands that, although they have been activated already for execution, have not yet been able to execute because their guard has not been entailed by the current state of the corresponding local stores. The tick rule puts all these pending process back in the main queue \( P \), so that their guards can be checked again and be executed or put back in the pending queue. Figure 4 depicts the possible transitions that an ask command can take between being in the priority queue, in the pending queue, and finally executing.

**Appendix B. Rewriting Logic Semantics of sScc**

This appendix includes the specification in Maude of Real-Time sScc.
--- agent identifier
fmod AGENT-ID is
pr EXT-BOOL .
pr NAT .
sort Aid .

op root : -> Aid .
op _._ : Nat Aid -> Aid .

vars L L0 L1 L2 : Aid .
vars N N0 N1 N2 : Nat .

--- auxiliary operations
op is-prefix? : Aid Aid -> Bool .
eq is-prefix?(root, L) = true .
eq is-prefix?(N . L, root) = false .
eq is-prefix?(N0 . L0, N1 . L1) = (N0 . L0 == N1 . L1) or-else is-prefix?(NO . L0, L1) .

op is-son? : Aid Aid -> Bool .
eq is-son?(root, L) = false .
eq is-son?(N . L, L0) = (L == L0) .

op sizeAid : Aid -> Nat .
eq sizeAid(root) = 1 .
eq sizeAid(N . L) = 1 + sizeAid(L) .
endfm

fmod SMT-UTIL is
inc INTEGER .
pr CONVERSION .
pr META-LEVEL .

op check-sat : Boolean -> Bool .
op check-unsat : Boolean -> Bool .
op entails : Boolean Boolean -> Bool .
eq check-sat(B:Boolean) = metaCheck(['INTEGER], upTerm(B:Boolean)) .
eq check-unsat(B:Boolean) = not(check-sat(B:Boolean)) .
eq entails(C1:Boolean, C2:Boolean)
  = check-unsat(C1:Boolean and not(C2:Boolean)) .

--- some Boolean identities
eq B:Boolean and true = B:Boolean .
eq B:Boolean and false = false .
eq B:Boolean or true = true .
eq B:Boolean or false = B:Boolean .
eq true and B:Boolean = B:Boolean .
eq false and B:Boolean = false .
eq true or B:Boolean = true .
eq false or B:Boolean = B:Boolean .
eq not((true).Boolean) = (false).Boolean .
eq not((false).Boolean) = (true).Boolean .
endfm

fmod 2-TUPLE{X :: STRICT-TOTAL-ORDER, Y :: STRICT-TOTAL-ORDER} is
  sort Tuple{X, Y} .
  op ((_,_)) : X\$Elt Y\$Elt -> Tuple{X, Y} [ctor] .
  op p1_ : Tuple{X, Y} -> X\$Elt .
  op p2_ : Tuple{X, Y} -> Y\$Elt .
  eq p1(A:X\$Elt, B:Y\$Elt) = A:X\$Elt .
  eq p2(A:X\$Elt, B:Y\$Elt) = B:Y\$Elt .

  op _<_ : Tuple{X, Y} Tuple{X, Y} -> Bool .
  eq A:Tuple{X, Y} < B:Tuple{X, Y} = p1(A:Tuple{X, Y}) < p1(B:Tuple{X, Y}) .
endfm

fmod PAIR{X :: TRIV, Y :: TRIV} is
  sort Pair{X, Y} .
  op P : X\$Elt Y\$Elt -> Pair{X, Y} [ctor] .
  op p1_ : Pair{X, Y} -> X\$Elt .
  op p2_ : Pair{X, Y} -> Y\$Elt .
  eq p1(P(A:X\$Elt, B:Y\$Elt)) = A:X\$Elt .
  eq p2(P(A:X\$Elt, B:Y\$Elt)) = B:Y\$Elt .
endfm

fmod 4-TUPLE{W :: TRIV, X :: TRIV, Y :: TRIV, Z :: TRIV} is
  sort Tuple{W, X, Y, Z} .
  op ((_,_,_,_)) : W\$Elt X\$Elt Y\$Elt Z\$Elt -> Tuple{W, X, Y, Z} [ctor] .
  op p1_ : Tuple{W, X, Y, Z} -> W\$Elt .
  op p2_ : Tuple{W, X, Y, Z} -> X\$Elt .
  op p3_ : Tuple{W, X, Y, Z} -> Y\$Elt .
  op p4_ : Tuple{W, X, Y, Z} -> Z\$Elt .
  eq p1(A:W\$Elt, B:X\$Elt, C:Y\$Elt, D:Z\$Elt) = A:W\$Elt .
  eq p2(A:W\$Elt, B:X\$Elt, C:Y\$Elt, D:Z\$Elt) = B:X\$Elt .
  eq p3(A:W\$Elt, B:X\$Elt, C:Y\$Elt, D:Z\$Elt) = C:Y\$Elt .
  eq p4(A:W\$Elt, B:X\$Elt, C:Y\$Elt, D:Z\$Elt) = D:Z\$Elt .
endfm
fmod LEFTIST-HEAP(X :: STRICT-TOTAL-ORDER) is
  protecting NAT .
  sort Heap(X) NeHeap(X) .
  subsort NeHeap(X) < Heap(X) .

  op empty : -> Heap(X) .
  op T(_,_,_,_) : Nat X\$Elt Heap(X) Heap(X) -> NeHeap(X) .

  vars L L' R R' : Heap(X) .
  vars E E' : X\$Elt .
  vars Ra Ra' : Nat .

  op isEmpty : Heap(X) -> Bool .
  eq isEmpty(empty) = true .
  eq isEmpty(T(Ra,E,L,R)) = false .

  op rank : Heap(X) -> Nat .
  eq rank(empty) = 0 .
  eq rank(T(Ra,E,L,R)) = Ra .

  op makeT : X\$Elt Heap(X) Heap(X) -> NeHeap(X) .
  eq makeT(E,L,R)
  = if rank(L) >= rank(R)
  then T(rank(R) + 1,E,L,R)
  else T(rank(L) + 1,E,R,L)
  fi.

  op merge : Heap(X) Heap(X) -> Heap(X) .
  eq merge(empty, L) = L .
  eq merge(L, empty) = L .
  eq merge(T(Ra,E,L,R),T(Ra',E',L',R'))
  = if (E < E')
  then makeT(E,L,merge(R,T(Ra',E',L',R')))
  else makeT(E',L',merge(T(Ra,E,L,R),R'))
  fi.

  op insert : X\$Elt Heap(X) -> NeHeap(X) .
  eq insert(E,L) = merge(T(1,E,empty,empty),L) .

  op findMin : NeHeap(X) -> X\$Elt .
  eq findMin(T(Ra,E,L,R)) = E .

  op deleteMin : NeHeap(X) -> Heap(X) .
  eq deleteMin(T(Ra,E,L,R)) = merge(L,R) .
endfm

view Time from STRICT-TOTAL-ORDER to POSRAT-TIME-DOMAIN is
sort Elt to Time.
endv

view 2Tuple from STRICT-TOTAL-ORDER to 2-TUPLE(Time, Nat<) is
  sort Elt to Tuple(Time, Nat<).
endv

fmod STOCHASTIC-EXPRESSION is
  pr MGDISTRIBUTIONS.
  pr POSRAT-TIME-DOMAIN.
  pr 2-TUPLE(Time, Nat<).
  pr 2-TUPLE(Float<, Nat<).

  sort StochasticExpression.
  subsort Time < StochasticExpression.

  vars F L M SG SH SC ND DD P LB UB : Float.
  vars S LI MI SGI SHI SCI NDI DDI PI LBI UBI : Nat.
  vars LPR MPR SGPR SHPR SCPR NDPR DDPR PPR LBPR UBPR : PosRat.
  var T : Time.

  ---- NORMAL/GAUSS DISTRIBUTION
  op Norm : -> StochasticExpression.
  op Norm : Float Float -> StochasticExpression.
  eq Norm = Norm(0.0, 1.0).

  ---- EXPONENTIAL DISTRIBUTION
  op Exp : Float -> StochasticExpression.

  ---- UNIFORM DISTRIBUTION
  op Unif : Float Float -> StochasticExpression.

  ---- GAMMA DISTRIBUTION
  op Gam : Float Float -> StochasticExpression.

  ---- WEIBULL DISTRIBUTION
  op Weib : Float Float -> StochasticExpression.

  ---- CHI-SQUARE DISTRIBUTION
  op Chi : Float -> StochasticExpression.

  ---- LOG-NORMAL DISTRIBUTION
  op Log : Float Float -> StochasticExpression.

  op eval : StochasticExpression Int -> Tuple(Time, Nat<).
  eq eval(Norm(M, SG), S) = (if normDistr(M, SG, S) >= 0.0
    then rat(normDistr(M, SG, S))
    else 0
    fi, s S).
eq \text{eval}(\text{Exp}(L), S) = \begin{cases} \text{if } \text{expDistr}(L, S) \geq 0.0 \\
\quad \text{then } \text{rat}(\text{expDistr}(L, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(\text{Unif}(LB, UB), S) = \begin{cases} \text{if } \text{unifDistr}(LB, UB, S) > 0.0 \\
\quad \text{then } \text{rat}(\text{unifDistr}(LB, UB, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(\text{Gam}(SH, SC), S) = \begin{cases} \text{if } \text{gammaDistr}(SH, SC, S) > 0.0 \\
\quad \text{then } \text{rat}(\text{gammaDistr}(SH, SC, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(\text{Weib}(SC, SH), S) = \begin{cases} \text{if } \text{weibDistr}(SC, SH, S) > 0.0 \\
\quad \text{then } \text{rat}(\text{weibDistr}(SC, SH, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(\text{Chi}(SH), S) = \begin{cases} \text{if } \text{chiSDistr}(SH, S) > 0.0 \\
\quad \text{then } \text{rat}(\text{chiSDistr}(SH, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(\text{Log}(M, SG), S) = \begin{cases} \text{if } \text{logNormDistr}(M, SG, S) > 0.0 \\
\quad \text{then } \text{rat}(\text{logNormDistr}(M, SG, S)) \\
\quad \text{else } 0 \\
\end{cases}, s S.

eq \text{eval}(T, S) = (T, S).

op \text{evalF} : \text{StochasticExpression} \rightarrow \text{Tuple} <\text{Float}, \text{Nat}>.

op \text{evalF}(\text{Unif}(LB, UB), S) = (\text{unifDistr}(LB, UB, S), s S).

endfm

--- commands syntax

fmod SCCP-SYNTAX is
  pr INTEGER.
  pr AGENT-ID.

sort SCCPCmd.

op 0 : -> SCCPCmd.

op tell : Boolean -> SCCPCmd.

op ask,-> : Boolean SCCPCmd -> SCCPCmd.

op _||_ : SCCPCmd SCCPCmd -> SCCPCmd [assoc comm gather (e E)].
op _in_ : SCCPCmd Nat -> SCCPCmd.
op _out_ : SCCPCmd Nat -> SCCPCmd.
op V : Nat -> SCCPCmd.
op mu : Nat SCCPCmd -> SCCPCmd.
endfm

view SCCPCmd from TRIV to SCCP-SYNTAX is
  sort Elt to SCCPCmd.
endv

view Aid from TRIV to AGENT-ID is
  sort Elt to Aid.
endv

fmod SCCP-SYNTAX-EXT is
  pr SCCP-SYNTAX.
  pr LIST{SCCPCmd}.
  pr LIST{Float}.
  pr ALIST{Aid}.
  op exc : List{SCCPCmd} List{Float} -> SCCPCmd.
  op ind : List{SCCPCmd} List{Float} -> SCCPCmd.
  op watch : SCCPCmd Boolean -> SCCPCmd.
endfm

view Aid from TRIV to AGENT-ID is
  sort Elt to Aid.
endv

view StExp from TRIV to STOCHASTIC-EXPRESSION is
  sort Elt to StochasticExpression.
endv

--- state syntax
mod SCCP-STATE is
  pr SCCP-SYNTAX-EXT.
  inc CONFIGURATION.
  pr STOCHASTIC-EXPRESSION.
  pr LEFTIST-HEAP{2Tuple}.
  pr MAP{Aid, StExp}.
  pr SET{Nat} * (op _,_ to _,_;) .
  sort Sys.
  subsorts Nat Aid < Oid.
  ops agent process simulation : -> Cid.
  op store :_ : Boolean -> Attribute [ctor].
op set :_ : Set{Nat} -> Attribute [ctor] .
op UID :_ : Nat -> Attribute [ctor] .
op command :_ : SCCPCmd -> Attribute [ctor] .
op gtime :_ : Time -> Attribute [ctor] .
op pqueue :_ : Heap{2Tuple} -> Attribute [ctor] .
op pend :_ : Heap{2Tuple} -> Attribute [ctor] .
op nextID :_ : Nat -> Attribute [ctor] .
op counter :_ : Nat -> Attribute [ctor] .
op flag :_ : Bool -> Attribute [ctor] .
op timeMap :_ : Map{Aid, StExp} -> Attribute [ctor] .
op tTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op aTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op sTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op eTM :_ : Map{Aid, StExp} -> Attribute [ctor] .
op factor :_ : PosRat -> Attribute [ctor] .

view List from TRIV to LIST{SCCPCmd} is
  sort Elt to List{SCCPCmd} .
endv

view Heap from TRIV to LEFTIST-HEAP{2Tuple} is
  sort Elt to Heap{2Tuple} .
endv

view Map from TRIV to MAP{Aid, StExp} is
  sort Elt to Map{Aid, StExp} .
endv

view FList from TRIV to LIST{Float} is
  sort Elt to List{Float} .
endv

--- transitions
mod SCCP is
  inc SCCP-STATE .
  pr SMT-UTIL .
  pr 4-TUPLE{List, Nat, Heap, Nat} .
  pr PAIR{FList, Nat} .

vars N N0 N1 N2 N3 N4 N5 S N’ N’’ N2’ I I0 I1 Ia : Nat .
vars M M’ : NzNat .
vars SN, SN’ : Set{Nat} .
vars In1 In2 In3 : Integer .
vars Q Q’ Q1 Q1’ : Float .
vars alpha beta : PosRat .
vars B1 B1’ : Bool .
vars T T’ T0 T1 Ti : Time .
vars L L0 L1 : Aid .
vars B0 B1 : Boolean .
vars C C0 C1 C2 : SCCPCmd .
vars H H' H0 H0' H1 H1' : NeHeap(2Tuple) .
vars P P' P0 P0' Le Ri : Heap(2Tuple) .
var Atts : AttributeSet .
var X : Configuration .
var O : Object .
var LC LC' : List(SCCPCmd) .
var NeLC NeLC' : NeList(SCCPCmd) .
var LF LF' F1 : List(Float) .
vars TM TM' Tma Tmt TMa' TMs' TMe' : Map(Aid, StExp) .

--- time delta
op delta : Heap(2Tuple) Time -> Heap(2Tuple) .
eq delta(empty, T') = empty .
eq delta(T(N,((T1, I)),P,P0),T') = T(N,((T1 monus T', I)),delta(P,T'),delta(P0,T')) .

--- auxiliary operations
op get-ancestor : Map(Aid, StExp) Aid -> StochasticExpression .
eq get-ancestor(TM, root)
  = if $hasMapping(TM, root)
    then TM[root]
    else Norm(1.0, 0.2)
  fi .
eq get-ancestor(TM, N . L)
  = if $hasMapping(TM, N . L)
    then TM[N . L]
    else get-ancestor(TM, L)
  fi .

op size : Boolean -> Nat .
eq size(not B0) = 1 + size(B0) .
eq size(B0 and B1) = 1 + size(B0) + size(B1) .
eq size(B0 xor B1) = 1 + size(B0) + size(B1) .
eq size(B0 or B1) = 1 + size(B0) + size(B1) .
eq size(B0 implies B1) = 1 + size(B0) + size(B1) .
eq size(B0 == B1) = 1 + size(B0) + size(B1) .
eq size(B0 /= B1) = 1 + size(B0) + size(B1) .
eq size(In1 < In2) = 1 .
eq size(In1 <= In2) = 1 .
eq size(In1 > In2) = 1 .
eq size(In1 >= In2) = 1 .
eq size(In1 == In2) = 1 .
eq size(In1 /= In2) = 1 .
eq size(B0) = 1 [owise] .

op replace : Nat SCCPCmd SCCPCmd -> SCCPCmd .
eq replace(N, 0, C) = 0 .
eq replace(N, tell (B), C) = tell (B) .
eq replace(N, ask B -> C, C) = ask B -> replace(N, CO, C) .
eq replace(N, C0 || C1, C) = replace(N, C0, C) || replace(N, C1, C).

eq replace(N, exc(LC, LF), C) = exc(repalceInList(N, LC, C), LF).

eq replace(N, ind(LC, LF), C) = ind( replaceInList(N, LC, C), LF).

eq replace(N, C0 in N0, C) = replace(N, C0, C) in N0.

eq replace(N, C0 out N0, C) = replace(N, C0, C) out N0.

eq replace(N, mu(N0, C0), C) = mu(N0, C0).

eq replace(N, V(N0), C) = if (N0 == N) then C else V(N0) fi.

op replaceInList : Nat List{SCCPCmd} SCCPCmd -> List{SCCPCmd}.

eq replaceInList(N, C0 LC, C) = replace(N, C0, C) LC.

eq replaceInList(N, nil, C) = nil.

--- get time from maps

op fTime : Map{Aid, StExp} Aid Nat -> Tuple{Time, Nat<}.

eq fTime(TM, L, N) = if hasMapping(TM, L) then eval(TM[L], N) else eval(Norm(1.0, 0.2), N) fi.

op getTimeCmd : SCCPCmd Aid Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Nat -> Tuple{Time, Nat<}.

eq getTimeCmd(tell(B1), L, TMt, TMs, TMe, N) = fTime(TMt, L, N).

eq getTimeCmd(C1 in I1, L, TMt, TMs, TMe, N) = fTime(TMs, L, N).

eq getTimeCmd(C1 out I1, L, TMt, TMs, TMe, N) = fTime(TMe, L, N).

eq getTimeCmd(C1, L, TMt, TMs, TMe, N) = (0, N) [owise].

op getProb : Nat -> Tuple{Float<, Nat<}.

eq getProb(N2) = evalF(Unif(0.0, 1.0), N2).

--- exclusive and independent parallel functions

op exclusive : List{SCCPCmd} List{Float} Nat Heap{2Tuple} Nat Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Aid -> Tuple{List, Nat, Heap, Nat}.

ceq exclusive(C, Q, N, P, N1, TMt, TMs, TMe, L) = (C, N + 1, HO, N') if (T, N') := getTimeCmd(C, L, TMt, TMs, TMe, N1) \ HO := insert(((T, N)), P).

ceq exclusive(C NeLC, Q Q1 LF, N, P, N1, TMt, TMs, TMe, L) = if Q' <= Q then (C, N + 1, HO, N''') else exclusive(NeLC, (Q + Q1) LF, N, P, N''', TMt, TMs, TMe, L) fi if (Q', N') := getProb(N1) \ (T, N'') := getTimeCmd(C, L, TMt, TMs, TMe, N') \ HO := insert(((T, N)), P).

op independent : List{SCCPCmd} List{Float} List{SCCPCmd} Nat Heap{2Tuple} Nat Map{Aid, StExp} Map{Aid, StExp} Map{Aid, StExp} Aid -> Tuple{List, Nat, Heap, Nat}.

eq independent(nil, nil, LC', N, P, N1, TMt, TMs, TMe, L) = (LC', N, P, N1).

ceq independent(C LC, Q LF, LC', N, P, N1, TMt, TMs, TMe, L)
= if $Q' \leq Q$
  then independent($LC$, $LF$, $LC'$, $N + 1$, $H0$, $N''$, $TMt$, $TMs$, $TMe$, $L$)
  else independent($LC$, $LF$, $LC'$, $N$, $P$, $N''$, $TMt$, $TMs$, $TMe$, $L$)
fi
if $(Q', N') := \text{getProb}(N1) \land (T, N'') := \text{getTimeCmd}(C, L, TMt, TMs, TMe, N') \land H0 := \text{insert}(((T, N)), P)$.

op genCommands : List{(SCCPCmd)} Nat Aid -> Configuration.
eq genCommands(nil, N, L) = none.
eq genCommands(C LC, N, L) = < L : process | UID : N, command : C > genCommands(LC, N + 1, L).

op command-list : Aid Set{Nat} SCCPCmd -> List{(SCCPCmd)}.
eq command-list(N, L, SN, CO) = (CO out N) $command-list(SN, CO)$.
eq command-list(root, SN, CO) = $command-list(SN, CO)$.

op $command-list : Set{Nat} SCCPCmd -> List{(SCCPCmd)}.
eq $command-list(N1 ; SN), CO) = (CO in N1) $command-list(SN, CO)$.
eq $command-list(empty, CO) = nil$.

op prob-list : Nat Nat -> Pair{(FList, Nat)}.
eq prob-list(N, N1) = $prob-list(N, N1, nil, 0.0)$.

op $prob-list : Nat Nat List{Float} Float -> Pair{(FList, Nat)}.
ceq $prob-list(N, N1, F1, Q1) = $prob-list(N, N', Q F1, Q1 + Q)
if $(Q, N') := \text{getProb}(N1)$.
eq $prob-list(0, N1, F1, Q1) = P($normalize(F1, Q1), N1)$.

op $normalize : List{Float} Float -> List{Float}.
eq $normalize(F1, Q1) = (Q / Q1) $normalize(F1, Q1)$.
eq $normalize(nil, Q1) = nil$.

--- non-observable concurrent transitions

eq < L0 : process | command : 0, Atts > = none.
eq < L0 : agent | store : B0, set : SN > < L0 : agent | store : B1, set : SN' >
  = < L0 : agent | store : (B0 and B1), set : (SN ; SN') >.

--- observable concurrent transitions

rl [tell]:
  < L0 : agent | store : B0, set : SN > < L0 : process | UID : IO, command : tell (B1) >
  < I : simulation | pqueue : T(Ra,((Ti, IO)), Le, Ri), flag : false, Atts >
=> < L0 : agent | store : (B0 and B1), set : SN >
  < I : simulation | pqueue : T(Ra,((Ti, IO)), Le, Ri), flag : true, Atts >.

rl [tell-set]:
  < L0 : agent | store : B0, set : SN > < L0 : process | UID : IO, command : tell (N) >
  < I : simulation | pqueue : T(Ra,((Ti, IO)), Le, Ri), flag : false, Atts >
=> < L0 : agent | store : B0, set : (SN ; N) >
  < I : simulation | pqueue : T(Ra,((Ti, IO)), Le, Ri), flag : true, Atts >.
**crl [parallel]**:

< L0 : process | UID : I0, command : (C0 || C1) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

=> < L0 : process | UID : I0, command : C0 >

< L0 : process | UID : (N + 1), command : C1 >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 2), flag : true, pend : H0, counter : N3, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

if (T0, N2) := getTimeCmd(C0, L0, TMt, TMs, TMe, N1) /
(T1, N3) := getTimeCmd(C1, L0, TMt, TMs, TMe, N2)
\/
H0 := insert(((T0, N)), insert(((T1, N + 1)), P)) .

crl [exclusive] :

< L0 : process | UID : I0, command : exc(LC, LF) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

=> genCommands(LC', N, L0)

if (LC', N', P0, N'') := exclusive(LC, LF, N, P, N1, TMt, TMs, TMe, L0) .

crl [independent] :

< L0 : process | UID : I0, command : ind(LC, LF) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

=> genCommands(LC', N, L0)

if (LC', N', P0, N'') := independent(LC, LF, nil, N, P, N1, TMt, TMs, TMe, L0) .

crl [space] :

< L0 : agent | store : B0, set : SN >

< L0 : process | UID : I0, command : (C0 in NO) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

=> < L0 : agent | store : B0, set : (SN ; N0) >

< NO . L0 : agent | store : true, set : empty >

< NO . L0 : process | UID : N, command : CO >

< L0 : process | UID : (N + 1), command : tell(NO) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : true, pend : H1, nextID : (N + 2), counter : N3, tTM : TMt, sTM : TMs, eTM : TMe, Atts >

if TMt' := insert(NO . L0, get-ancestor(TMt, NO . L0), TMt)
\/
TMt' := insert(NO . L0, get-ancestor(TMt, NO . L0), TMt)
\/
TMt' := insert(NO . L0, get-ancestor(TMt, NO . L0), TMt)
\/
TMt' := insert(NO . L0, get-ancestor(TMt, NO . L0), TMt)
\/
(TM0, N2) := getTimeCmd(C0, NO . L0, TMt', TMt', TMt', TMt', N1) \/
H0 := insert(((T0, N)), P)
\/
(TM1, N3) := getTimeCmd(tell(NO), L0, TMt', TMt', TMt', N2) \/
H1 := insert(((T1, (N + 1))), H0) .

crl [extrusion] :

< NO . L0 : process | UID : I0, command : (C0 out NO) >

< I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1,
\[ tTM : TMt, sTM : TMs, eTM : TMe, Atts \]
\[ \Rightarrow < L0 : process | UID : N, command : C0 > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : true, pend : H0, nextID : (N + 1), counter : N2, tTM : TMt, sTM : TMs, eTM : TMe, Atts > \]
\[ \text{if } (T0, N2) := \text{getTimeCmd}(C0, L0, TMt, TMs, TMe, N1) \land H0 := \text{insert}(((T0, N)), P) . \]

crl [ask] :
\[ < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : I0, command : (ask B1 \rightarrow C1) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : false, pend : P, nextID : N, counter : N1, tTM : TMt, aTM : TMa, sTM : TMs, eTM : TMe, factor : alpha, Atts > \]
\[ \Rightarrow < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : N, command : C1 > \]
\[ < I : simulation | pqueue : merge(Le, Ri), flag : false, pend : P, Atts > \]
\[ \Rightarrow < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : I0, command : (ask B1 \rightarrow C1) > \]
\[ < I : simulation | pqueue : merge(Le, Ri), flag : false, pend : H0, Atts > \]
\[ \text{if } \neg \text{entails}(B0,B1) \land (T0, N2) := \text{getTimeCmd}(C1, L0, TMt, TMs, TMe, N1) \land (T1, N3) := \text{ftime}(TMa, L0, N2) \land S := \text{size}(B0) \land H0 := \text{insert}(((T0, B0, N3), P)) . \]

crl [delay] :
\[ < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : I0, command : (ask B1 \rightarrow C1) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), flag : false, pend : P, Atts > \]
\[ \Rightarrow < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : N, command : C1 > \]
\[ < I : simulation | pqueue : merge(Le, Ri), flag : false, pend : H0, Atts > \]
\[ \text{if } \neg \text{entails}(B0,B1) \land H0 := \text{insert}(((Ti, I0)), P) . \]

crl [recursion] :
\[ < L0 : process | UID : I0, command : mu(N0, C0) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts > \]
\[ \Rightarrow < L0 : process | UID : N, command : replace(N0, CO, mu(N0,C0)) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 1), flag : true, pend : H0, counter : N2, tTM : TMt, sTM : TMs, eTM : TMe, factor : alpha, Atts > \]
\[ \text{if } (T0, N2) := \text{getTimeCmd}(C0, L0, TMt, TMs, TMe, N1) \land \text{size}(B0) \land H0 := \text{insert}(((T0, N)), P) . \]

crl [search] :
\[ < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : I0, command : watch(C0, B1) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, tTM : TMt, sTM : TMs, eTM : TMe, Atts > \]
\[ \Rightarrow < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : N, command : exc(LC, LF) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : (N + 1), flag : true, pend : H0, counter : N', tTM : TMt, sTM : TMs, eTM : TMe, Atts > \]
\[ \text{if } \neg \text{entails}(B0,B1) \land LC := \text{command-list}(L0, SN, watch(C0, B1)) \land P(LF, N') := \text{prob-list}(size(LC), N1) \land (T0, N') := \text{getTimeCmd}(exc(LC, LF), L0, TMt, TMs, TMe, N') \land H0 := \text{insert}(((T0, N)), P) . \]

crl [found] :
\[ < L0 : agent | store : B0, set : SN > \]
\[ < L0 : process | UID : I0, command : watch(C0, B1) > \]
\[ < I : simulation | pqueue : T(Ra,((Ti, I0)), Le, Ri), nextID : N, flag : false, pend : P, counter : N1, \]
APMAUDE is

pr SCCP * (sort Configuration to Config).
pr COUNTER.

var Atts : AttributeSet.
vars H P : Heap(2Tuple).
vars I N : Nat.
var B : Bool.
var L LO : Aid.
var C BO : Boolean.
vars T TO : Time.
var R : Float.
vars Conf X : Config.
var PO : Heap(2Tuple).
vars V1 V2 : AList(Aid).
var Obj : Object.

---- used by Quatex
op flg : Bool Float -> Config. ---- a flag delimiting execution rounds

op tick : Config -> Config.
op tick2 : Config -> Config.
eq tick(Conf) = tick2(Conf).
eq tick2( Conf flg(B, R) ) = Conf flg(true, R + 500.0).

op val : Nat Config -> Float.
eq val(1, Conf) = getExcecutionTime(Conf).

----- tick rule
crl [tick]:
  < I : simulation | pqueue : P, gtime : T, flag : true, pend : PO, Atts >
  flg(true, R) Conf
=> < I : simulation | pqueue : merge(delta(deleteMin(P),TO),PO), gtime : (T plus TO), flag : false,
    pend : empty, Atts >
  flg(float(T plus TO) < R, R) Conf
if TO := pi(findMin(P))
  [print "tick = T"].

---- init state
op initState : Nat -> Config .

op initState : -> Config .

rl initState => initState(counter) .

eq initState(N) = < root : agent | store : true >
< root : process | UID : 1, command : ( ( exc( ( ( ind( ( ( tell(A:Integer === 1) ) in 1 )
( ( tell(B:Integer === 1) ) in 2 ) ( ( tell(C:Integer === 1) ) in 3 ) ( ( tell(D:Integer === 1) )
in 4 ), 0.5 0.5 0.5 0.5 ) ) in 4 ) || tell(Y:Integer === 5) || ask(Y:Integer > 2) ->
( tell(Y:Integer > 2) out 1 ) ) in 1 ) ( ( tell(Y:Integer === 25) || ask(Y:Integer > 2) ->
( tell(Y:Integer > 2) out 2 ) ) in 2 ), 0.60 0.40 ) || ask (Y:Integer > 2) ->
( tell(X:Integer === 15) || ask(X:Integer >= 10) -> ( tell(X:Integer >= 10) out 1 ) ) ) in 1 ) ||
( ( ( tell(Z:Integer === 9) || ask(Z:Integer < 15) -> ( tell(Z:Integer < 15) out 3 ) ) in 3 ) ||
( ( tell(W:Integer === 25) || ask(W:Integer > 0) -> ( tell(W:Integer > 0) out 4 ) ) in 4 ) ||
ask (Z:Integer < 15 and W:Integer > 0) -> ( tell(V:Integer === 67) || ask(V:Integer < 100) ->
( tell(V:Integer < 100) out 2 ) ) ) in 2 ) || ( ask (X:Integer >= 10 and V:Integer < 100) ->
( tell(U:Integer === 50) || ask(U:Integer < 55) -> tell(DONE:Boolean) ) ) >
< 1 : simulation | gtime : 0, pqueue : T1,((0,1)),empty,empty), pend : empty, nextID : 19, flag : false,
counter : N, tTM : ((root) |-> Norm(1.0, 0.2)), aTM : ((root) |-> Norm(1.2, 0.2)),
sTM : ((root) |-> Norm(0.5, 0.2)), eTM : ((root) |-> Norm(0.5, 0.2)), factor : 1/2 >
flg(false, 0.0).

--- pvesta functions
--- execution time

op getExcecutionTime : Config -> Float .

eq getExcecutionTime(< 1 : simulation | gtime : T, flag : false, pqueue : empty, Atts > Conf) = float(T) .
endm

execTime ( ) = { s.rval( 1 ) } ;
eval E[ # execTime ( ) ] ;