A finite number of defining relations and a UCE theorem of the elliptic Lie algebras and superalgebras with rank $\geq 2$

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Abstract
In this paper, we give a finite number of defining relations satisfied by a finite number of generators for the elliptic Lie algebras and superalgebras $\mathfrak{g}_R$ with rank $\geq 2$. Here the $R$’s denote the reduced and non-reduced elliptic root systems with rank $\geq 2$. We also show that if $\mathcal{L}$ is an extended affine Lie algebra (EALA) whose non-isotropic roots form the $R$, then there exists a natural homomorphism $\mathcal{F}: \mathfrak{g}_R \to \mathcal{L}$, which also give a universal central extension (UCE) surjective map from $[\mathfrak{g}_R,\mathfrak{g}_R]$ to the core of $\mathcal{L}$. (More precisely, we take a $\bar{\mathfrak{g}}_R$ instead of the $\mathfrak{g}_R$.)

Introduction
In 1985, K. Saito [S] introduced the notion of the (reduced and non-reduced) extended affine root systems. In this paper, we also call them the SEARS’s. Let $R$ be an SEARS. Let $V := R R$ and $V^0 := \{v \in V | s_\alpha(v) = v \text{ for all } \alpha \in R\}$, where $s_\alpha$ denotes the reflection with respect to an $\alpha$. Let $m := \dim V^0$ and $l := \dim V - m$. We say that the $m$ is the nullity of the $R$ and say that the $l$ is the rank of the $R$. The $R$ is reduced if $\mathbb{R} \alpha \cap R = \{\alpha, -\alpha\}$ for all $\alpha \in R$. We notice that if $m = 0$, the $R$ is a finite root system, and that if $m = 1$, the $R$ is an affine root system. If $m = 2$, the $R$ is called an elliptic root system (or ERS for short); see also Subsec. 1.3. (The notation $V$ and $V^0$ is used only in Introduction.)

In 1997, B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola [AABGP] introduced and studied root systems defined by different axioms from those of the reduced SEARS’s. They also studied Lie algebras $\mathcal{L} = (\mathcal{L}, (, ), \mathcal{H})$ associated with their root systems. The $\mathcal{L}$ is called the extended affine Lie
algebra (EALA for short) (see [AABGP]). In 2002, S. Azam [A] showed that there exists a natural one-to-one correspondence between their root systems and the reduced SEARS’s.

Let $R$ be an ERS. Let $\tilde{\pi} : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0$ be a natural projective map. The $R$ is called \textit{simply-laced} if $l \geq 2$ and $\tilde{\pi}(R)$ is a simply-laced finite root system. In 2000, K. Saito and D. Yoshii [SY] studied a Lie algebra $g_R$ whose non-isotropic roots form a simply-laced ERS $R$, showed that the $g_R$ is isomorphic to the (2-variable) toroidal Lie algebra [MRY] of type ADE, and gave a Serre-type theorem for the $g_R$ (see also the next paragraph). We also notice the results in [Mi1], [Mi2], [T] for the $g_R$ with the $\tilde{\pi}(R)$ of type A.

In this paper, for every ERS $R$ with $l \geq 2$, we give a Serre-type theorem for a Lie algebra or superalgebra $g_R$ having the property that the system formed by its non-isotropic roots is isomorphic to the $R$ (see Theorems 2.1 and 6.2); in other words, we give a finite number of defining relations of the $g_R$ satisfied by Chevalley generators. The $g_R$ is not a Lie algebra but a Lie superalgebra if and only if the $R$ is not reduced. (In the text, the $g_R$ shall be denoted as the $g_D$.) We call them the \textit{elliptic Lie (super)algebra}. If $R$ is simply-laced, our Serre-type theorem (Theorem 6.2) coincides with K. Saito and D. Yoshii’s Serre-type theorem [SY].

Let $R$ be an ERS. If there exists a one dimensional subspace $G_R$ of $\mathcal{V}^0$ such that $G_R \cap ZR \neq \{0\}$ and the $\pi(R)$ is a reduced affine root system, where $\pi : \mathcal{V} \to \mathcal{V}/G_R$ is the natural projection, then we call the $R$ the \textit{reduced marked ERS}. (More precisely, in the text, we call the pair $(R, G)$ the reduced marked ERS (see Subsec. 1.3), where $G = G_R + \sqrt{-1}G_R$.) If the $R$ is a reduced marked ERS with $l \geq 2$, then the Serre-type theorem for the $g_R$ has already been given by the author [Ya2] in 2004.

In the process of giving the Serre-type theorem, we also give a Saito-type classification theorem of all the ERS’s $R$ with $l \geq 2$ (see Theorems 1.2 and 2.2). To prove the classification theorem, we use K. Saito’s classification theorem [S] of the reduced marked ERS’s, and use the $g_R$ to show that each $R$ in the classification theorem really exists.

We also give universality theorems of the $g_R$ with $l \geq 2$. (see Theorems 5.1, 5.2 and 5.4). Especially, we see that if $R$ is a reduced ERS with $l \geq 2$ and if there exists a basis $\{\delta, a\}$ of $\mathcal{V}^0$ such that $(\alpha + \mathcal{V}^0) \cap R = \alpha + Z\delta + Za$ for every $\alpha \in R$, then the $D^1(g_R) := [g_R, g_R]$ is isomorphic to the universal central extension (UCE for short) of the Lie algebra $f_R \otimes \mathbb{C}[t_{1,1}^\pm, t_{2,1}^\pm]$, where the $f_R$ is a finite dimensional simple Lie algebra whose root system is isomorphic to the $\tilde{\pi}(R)$, i.e., the $D^1(g_R)$ is a (2-variable) toroidal Lie algebra in the sense
of [MRY] (see Corollary 5.2). We also treat the quantum tori elliptic Lie algebras studied in [BGR] (see Subsec. 5.2). We show that if an \((L, ( , ), H)\) is an EALA whose non-isotropic roots form an ERS \(R\) with \(l \geq 2\), there exists a natural homomorphism \(F : g_R \rightarrow L\) (More precisely, we take \(\bar{g}_R = \bar{g}_D\) instead of the \(g_R\)). It turns out that the \(F|_{\text{D}^1(g_R)} : \text{D}^1(g_R) \rightarrow F(\text{D}^1(g_R))\) is a UCE, and the image \(F(\text{D}^1(g_R))\) is the core of the \((L, ( , ), H)\) (see Corollary 5.2) (see [AABGP, Chap. I Definition 2.20] for the term).

This paper is organized as follows. Notice that we always assume \(l \geq 2\).

In §1, we discuss the ERS's \(R = \text{R}(k, g)\). In §2 we give a definition of the elliptic Lie algebras and superalgebras \(g_R = g_D\) and give Theorem 2.1, which states a root space decomposition of the \(g_R\), and Theorem 2.2, which give a Saito-type classification of all the ERS's \(R\) with \(l \geq 2\). In §3, we give a proof of Theorem 2.1 after supposing Lemma 2.1. In §4, we give a proof of Lemma 2.1. In §5, we give Theorem 5.1, which states that the \(g_R\) is the maximal ones among the Lie (super)algebras having the root space \(R\) and satisfying some additional conditions, and give Theorem 5.3 which is a UCE theorem of the \(\text{D}^1(g_R)\). In §6, we give Theorem 6.2, which is a natural extension of K. Saito and D. Yoshii's Serre-type theorem [SY] of the simply-laced elliptic Lie algebras.

1 Preliminary

1.1 Pre-elliptic base system

In this paper, we set \(Z_+ := \{r \in \mathbb{Z} | r \geq 0\}\), i.e., \(Z_+ = \{0\} \cup \mathbb{N}\). We also set \(Z_- := \{r \in \mathbb{Z} | r \leq 0\}\).

Let \(l\) be a fixed positive integer. Throughout this paper, we assume \(l \geq 2\). Let \(E\) be an \(l + 4\)-dimensional \(C\)-vector space. Let \(J : E \times E \rightarrow C\) be a non-degenerate symmetric bilinear form. Let \(E^\times := \{x \in E | J(x, x) \neq 0\}\). If \(x \in E^\times\), we call \(x\) a non-isotropic element, let \(x^\vee := \frac{2x}{J(x, x)}\) and define \(s_x \in \text{GL}(E)\) by \(s_x(y) = y - J(x^\vee, y)x\). Let \(\{\alpha_0, \ldots, \alpha_l, \Lambda_\delta, a, \Lambda_\delta\}\) be a basis of \(E\) satisfying the following.

(B1) The \((l + 1) \times (l + 1)\)-matrix \(A := (J(\alpha_i^\vee, \alpha_j))_{0 \leq i, j \leq l}\) is an affine type generalized Cartan matrix [K1], [K2]. Then the \(A\) is called \(A^{(1)}_l (l \geq 2), B^{(1)}_l (l \geq 3), C^{(1)}_l (l \geq 2), D^{(1)}_l (l \geq 4), E^{(1)}_6 (l = 6, 7, 8), F^{(1)}_4 (l = 4), G^{(1)}_2 (l = 2), A^{(2)}_{2l} (l \geq 2), A^{(2)}_{2l-1} (l \geq 3), D^{(2)}_{l+1} (l \geq 2), E^{(2)}_6 (l = 4)\) or \(D^{(3)}_4 (l = 2)\). (See
Let $\Pi := \{\alpha_0, \ldots, \alpha_t\}$. Let $W$ be the subgroup of $\text{GL}(\mathcal{E})$ generated by $s_\alpha$ ($\alpha \in \Pi$), i.e., $W$ is the affine Weyl group.

For a subset $S$ of $\Pi$, let $W_S$ be the subgroup of $W$ generated by $s_\alpha$ ($\alpha \in S$). Let $l(w)$ be the length of $w \in W$ with respect to $s_\alpha$ ($\alpha \in \Pi$).

**Lemma 1.1.** Let $\alpha, \beta \in \Pi$ and $w \in W$ be such that $w(\alpha) = \beta$ and $l(w) > 0$. Then there exists a $\gamma \in \Pi \setminus \{\alpha\}$ and a $w' \in W_{\{\alpha, \gamma\}}$ such that $w'(\alpha) \in \{\alpha, \gamma\}$ and $l(w') + l(w(w')^{-1}) = l(w)$.

This can be proved by a well-known argument (see [J] Proof of Proposition 8.20).

A function $f : \Pi \to \mathbb{C}$ is called $W$-invariant if $f(\alpha) = f(\beta)$ for every $(\alpha, \beta) \in \Pi \times \Pi$ with $\beta = w(\alpha)$ for some $w \in W$. By Lemma 1.1 we see the following.

**Lemma 1.2.** Keep the notation as above. A function $f : \Pi \to \mathbb{C}$ is $W$-invariant if and only if $f(\alpha) = f(\beta)$ for every $(\alpha, \beta) \in \Pi \times \Pi$ with $J(\alpha^\vee, \beta^\vee) = J(\alpha, \beta) = -1$.

Let $k : \Pi \to \mathbb{N}$ be a $W$-invariant function such that G.C.D.$\{k(\alpha) | \alpha \in \Pi\} = 1$. Let $g : \Pi \to 2^\mathbb{Z}$ be a $W$-invariant function, where $2^\mathbb{Z}$ is the power set of $\mathbb{Z}$, i.e., the set of the subsets of $\mathbb{Z}$. We call a quintuple $\mathcal{D} = (\mathcal{E}, \Pi, a, k, g)$ of such $\mathcal{E}$, $\Pi$, $a$, $k$ and $g$ a pre-elliptic base system (PEBS for short). If $g(\alpha) = 0$ for every $\alpha$, the $g$ is also denoted by $0$.

For $x \in \mathcal{E}$ and a subset $B$ of $\mathbb{C}$, let $Bx := \{bx \in \mathcal{E} | b \in B\}$; moreover, for a subset $X$ of $\mathcal{E}$, let $BX := \sum_{x \in X} Bx$. (If $B$ is an empty set $\emptyset$, then $BX = \emptyset$.) For subsets $S$ and $T$ of $\mathcal{E}$, let $S + T := \{x + y \in \mathcal{E} | x \in S, y \in T\}$; if $T = \{x\}$, let $x + S := T + S$. (If $S = \emptyset$, $S + T = \emptyset$.)
Let $\mathcal{D}$ be a PEBS. Let

$$R(k, g) := \bigcup_{w \in \mathcal{W}} w\left(\bigcup_{\alpha \in \Pi} \left((\alpha + Zk(\alpha)a) \cup (2\alpha + g(\alpha)k(\alpha)a)\right)\right).$$

Then

$$R(k, g) \subset \left((\mathbb{Z}+\Pi+Za) \cup (\mathbb{Z}_-\Pi+Za)\right) \setminus Za.$$  

Lemma 1.3. Keep the notation as above. Let $S$ be a subset of $\Pi$. Let $\lambda \in (\mathbb{Z}S) \cap \mathcal{E}^\times$. Then there exists a $w \in \mathcal{W}_S$ such that

$$w(\lambda) \in \left(\bigcup_{\alpha \in S} Z\alpha\right) \cup \left(\mathbb{Z}S \setminus (\mathbb{Z}_+S \cup Z_-S)\right).$$

Proof. By the definition of $\Pi$ in (B1), there exists an $e \in \mathbb{C} \setminus \{0\}$ such that $eJ(\alpha_i, \alpha_j) \in \mathbb{R}$ ($0 \leq i, j \leq l$), and $eJ(\alpha_i, \alpha_i) > 0$ ($0 \leq i \leq l$). Then the symmetric bilinear form $(eJ)_{\mathbb{R}^\Pi \times \mathbb{R}^\Pi} : \mathbb{R}^\Pi \times \mathbb{R}^\Pi \rightarrow \mathbb{R}$ is semipositive definite.

For $\mu = \sum_{\alpha \in \Pi} b_\alpha \alpha \in \mathbb{R}^\Pi$ with $b_\alpha \in \mathbb{R}$, let $ht(\mu) := \sum_{\alpha \in \Pi} b_\alpha \in \mathbb{R}$.

Let $\lambda$ be as in the statement. We may assume $\lambda \in (\mathbb{Z}+S \cup \mathbb{Z}^-S) \setminus \{0\}$. Moreover we may assume $\lambda \in \mathbb{Z}_+S$. We use an induction on $ht(\lambda) \in \mathbb{N}$. If $ht(\lambda) = 1$, then $\lambda \in S$. We assume $ht(\lambda) > 1$. We may assume $\lambda \notin E\alpha$ for any $\alpha \in S$. Since $\lambda \in \mathcal{E}^\times$, $eJ(\lambda, \lambda) > 0$. Hence there exists an $\alpha \in S$ such that $eJ(\lambda, \alpha) > 0$. Notice that $s_\alpha(\lambda) = \lambda - \frac{2eJ(\lambda, \alpha)}{eJ(\alpha, \alpha)}\alpha$. If $s_\alpha(\lambda) \notin \mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}^-S)$, then $s_\alpha(\lambda) \in \mathbb{Z}_+S$ and $ht(s_\alpha(\lambda)) = ht(\lambda)$. This completes the proof. \hfill $\Box$

For a subset $S$ of $\Pi$, let

$$R(k, g)_S := R(k, g) \cap ((\oplus_{\alpha \in S} \mathbb{C}\alpha) \oplus \mathbb{C}a).$$

Lemma 1.4. Keep the notation as above. Then

$$R(k, g)_S = \bigcup_{w \in \mathcal{W}_S} w\left(\bigcup_{\alpha \in S} \left((\alpha + Zk(\alpha)a) \cup (2\alpha + g(\alpha)k(\alpha)a)\right)\right).$$

In particular, for $\alpha \in \Pi$, we have

$$R(k, g)_{\{\alpha\}} = \bigcup_{e \in \{1, -1\}} \left((e\alpha + Zk(\alpha)a) \cup (2e\alpha + g(\alpha)k(\alpha)a)\right).$$

Proof. If $|S| = 1$, then the lemma follows immediately from the definition. From this, together with Lemma 1.3 and Lemma 1.3, the lemma for a general $S$ follows; notice that $w(a) = a$ for all $w \in \mathcal{W}_S$. \hfill $\Box$
1.2 Elliptic and quasi-elliptic base systems

Here we introduce the notions of an elliptic base system (EBS for short) and a quasi-elliptic base system (QEBS for short). In Theorem 2.2, we shall show that these notions are equivalent. In Theorem 1.2, we shall show how the EBS’s are associated with the elliptic root systems.

Let \( \mathcal{D} \) be an PEBS with \( l \geq 2 \). For \( \alpha \in \Pi \), let 
\[
\Pi_c(\alpha) := \{ \beta \in \Pi | \beta \neq \alpha, J(\beta, \alpha) \neq 0 \}.
\]

Define a subset \( \Pi^B \) of \( \Pi \) by 
\[
\Pi^B := \{ \alpha \in \Pi | \forall \beta \in \Pi_c(\alpha), J(\alpha^\vee, \beta) = -2 \}.
\]

We call \( \mathcal{D} \) an an quasi elliptic base system (QEBS for short) if the following hold.

(KG1) If \( \alpha \in \Pi, \beta \in \Pi_c(\alpha) \) and \( J(\beta^\vee, \alpha) = -1 \), then \( \frac{k(\beta)}{k(\alpha)} \in \mathbb{Z} \) and \( J(\alpha^\vee, \beta) \frac{k(\alpha)}{k(\beta)} \in \mathbb{Z} \).

(KG2) \( g(\alpha) = \emptyset \) if \( \alpha \notin \Pi^B \).

(KG3) If \( \alpha \in \Pi^B \) and \( \beta \in \Pi_c(\alpha) \), then \( g(\alpha) \frac{k(\alpha)}{k(\beta)} = \emptyset, \mathbb{Z}, 2\mathbb{Z} \) or \( 2\mathbb{Z} + 1 \).

We call a PEBS \( \mathcal{D} \) an elliptic base system (EBS for short) if the following holds.
\[
\forall \alpha \in R(k, g), \quad s_\alpha(R(k, g)) = R(k, g).
\]

Lemma 1.5. Let \( \mathcal{D} \) a PEBS with \( l \geq 2 \). If \( \mathcal{D} \) is an EBS, then it is also a QEBS.

Proof. The axiom (KG1) follows from Lemma 1.4 and the following (cf. [S] Proof of (6.1) Assertion]).
\[
\left\{ \begin{array}{l}
\quad s_\alpha s_{\alpha + mk(\alpha)} a(\beta) = \beta + m J(\alpha^\vee, \beta) k(\alpha)a,
\quad s_\beta s_{\beta + mk(\beta)} a(\alpha) = \alpha + mk(\beta)a
\end{array} \right.
\]

for \( m \in \mathbb{Z} \).

If \( \beta \in \Pi_c(\alpha) \), then \( \mathbb{Z} \ni J((2\alpha)^\vee, \beta) = \frac{J(\alpha^\vee, \beta)}{2} \). Hence, if \( g(\alpha) \neq \emptyset \), then \( \alpha \notin \Pi^B \), which implies the axiom (KG2).

Let \( \alpha \in \Pi^B \). Assume \( g(\alpha) \neq \emptyset \). Let \( n \in \mathbb{Z} \) be such that \( 2\alpha + nk(\alpha)a \in R(k, g) \). Then
\[
s_\alpha s_{\alpha + nk(\alpha)a}(2\alpha + nk(\alpha)a) = 2\alpha + (n + 4)k(\alpha)a
\]
and
\[ s_\alpha s_{2\alpha + nk(\alpha)} a(2\alpha + nk(\alpha)a) = 2\alpha - nk(\alpha)a. \]
Hence \( g(\alpha) = \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1, 4\mathbb{Z} \) or \( 4\mathbb{Z} + 2 \). Let \( \beta \in \Pi_c(\alpha) \). Then
\[ s_\beta s_{\beta + mk(\beta)} a(2\alpha + nk(\alpha)a) = 2\alpha + (nk(\alpha) - 2mk(\beta))a. \]
Hence \( g(\alpha) = \mathbb{Z}, 2\mathbb{Z} \) or \( 2\mathbb{Z} + 1 \) if \( k(\beta) = k(\alpha) \). Moreover
\[ s_\alpha s_{2\alpha + nk(\alpha)} a(\beta + mk(\beta))a = \beta + (mk(\beta) - nk(\alpha))a. \]
Hence \( g(\alpha) = 2\mathbb{Z}, 4\mathbb{Z} \) or \( 4\mathbb{Z} + 2 \) if \( k(\beta) = 2k(\alpha) \). This implies the axiom (KG3) and completes the proof. \( \square \)

Converse of Lemma 1.5 shall be given in Theorem 2.2.

If \( D(\mathcal{E}, \Pi, a, k, 0) \) is a QEBS, i.e., \( g = 0 \), then it is called a special QEBS (SQEBS for short).

### 1.3 Elliptic root systems

Keep the notation in §1. Notice that \( l \geq 2 \). For a subset \( S \) of \( \mathcal{E} \), let \( S^\perp := \{ x \in \mathcal{E} | \forall y \in S, J(x, y) = 0 \} \). Following [S] (and [SY]), we say that a subset \( R \) of \( \mathcal{E} \) is an elliptic root system (ERS for short) of rank \( l \) if it satisfies the following.

(SER1) \( \forall x, \forall y \in \mathbb{R} R, J(x, x)J(y, y) \in \mathbb{R}_+ \),
(SER2) \( \dim_{\mathbb{C}}(\mathbb{C} R \cap R^\perp) = 2 \),
(SER3) \( \dim_{\mathbb{C}} \mathbb{C} R = l + 2 = \text{rank}_{\mathbb{Z}} \mathbb{Z} R \),
(SER4) \( \forall \alpha \in R, s_\alpha(R) = R \),
(SER5) \( \forall \alpha, \forall \beta \in R, J(\alpha^\vee, \beta) \in \mathbb{Z} \),
(SER6) \( \text{If } R = R_1 \cup R_2, R_2 \subseteq (R_1)^\perp, \text{then } R_1 \neq \emptyset \text{ or } R_2 \neq \emptyset \),

where \( \mathbb{R}_+ = \{ x \in \mathbb{R} | x \geq 0 \} \).

Let \( R \) be an ERS. We call the \( \mathcal{E} \) for the \( R \) the base space. A one dimensional subspace \( G \) of \( \mathbb{C} R \cap R^\perp \) is called a marking line if \( G \cap \mathbb{Z} R \neq \{0\} \). The pair \( (R, G) \) of the above \( R \) and \( G \) is called a marked elliptic root system (MERS for short). An MERS \( (R, G) \) is called a reduced marked elliptic root system (RMERS for short) if
\begin{equation}
\forall \alpha, \forall \beta \in R, 2\alpha - \beta \notin G.
\end{equation}

By [S], we have the following.

\[ 7 \]
Theorem 1.1 ((6.4) of [S]). If \((R,G)\) be an RMERS, then there exists an SQEBS \(\mathcal{D}(\mathcal{E},\Pi,a,k,0)\) such that \(R = R(k,0)\), \(G = \mathbb{C}a\) and \(\mathcal{E}\) is the base space of \(R\). If \(\mathcal{D}(\mathcal{E},\Pi,a,k,0)\) is an SQEBS, then \((R(k,0),\mathbb{C}a)\) is an RMERS; in particular, \(\mathcal{D}(\mathcal{E},\Pi,a,k,0)\) is an EBS.

In [S], we do not need the assumption \(l \geq 2\) for Theorem 1.1.

Theorem 1.2. (1) If \(\mathcal{D}\) is an EBS, then \((R(k,g),\mathbb{C}a)\) is an MERS.

(2) Let \((R,G)\) be an MERS. Then there exists an EBS \(\mathcal{D}\) such that \(R = R(k,g)\), \(G = \mathbb{C}a\) and \(\mathcal{E}\) is the base space of \(R\).

Proof. The statement (1) is clear. We prove the statement (2). Let 

\[ R' := \{ \alpha \in R \mid \frac{\alpha}{2} \notin R + G \} \]

Let \(R'' := R \setminus R'\). We have

\[(1.5) \quad \frac{1}{2} R'' \subset R' + G \]

because, if \(\alpha \in R''\) is such that \(\frac{\alpha}{2} \notin (R' + G)\), then \(\frac{\alpha}{4} \in R + G\) and \(J(\alpha^\vee, \frac{\alpha}{4}) = \frac{1}{2} \notin \mathbb{Z}\), contradiction.

We show

\[(1.6) \quad ZR' = ZR.\]

Clearly \(ZR' \subset ZR\) holds. Let \(\beta \in R''\). By (1.5), \(\beta = 2\alpha + x\) for some \(\alpha \in R'\) and some \(x \in G\). Notice that

\[(1.7) \quad \forall \gamma \in R, \quad \sigma_\gamma(R') = R' \quad \text{and} \quad \sigma_\gamma(R'') = R''.\]

It follows that \(R' \ni \sigma_\alpha \sigma_\beta(\alpha) = \sigma_\alpha(\alpha - \beta) = -\alpha - (\beta - 4\alpha) = 3\alpha - \beta = \alpha - x\). Hence \(x \in ZR'\). Hence \(ZR \subset ZR'\), as desired.

By (1.6), we see that the \((R',G)\) is an RMERS whose base space is \(\mathcal{E}\). By Theorem 1.1, there exists an SQEBS \(\mathcal{D}(\mathcal{E},\Pi,a,k,0)\) such that \(R(k,0) = R'\) and \(G = \mathbb{C}a\). It follows from Lemma 1.4 that

\[(1.8) \quad (\alpha + G) \cap R = (\alpha + G) \cap R' = (\alpha + G) \cap R(k,0) = \alpha + Zk(\alpha)a\]

for \(\alpha \in \Pi\). To complete the proof, it suffices to show that

\[ \forall \alpha \in \Pi, \quad R'' \cap (2\alpha + G) \subset 2\alpha + Zk(\alpha)a. \]

Let \(\beta \in R'' \cap (2\alpha + G)\). Let \(x := \beta - 2\alpha\). Then \(\sigma_\alpha \sigma_\beta(\alpha) = 3\alpha - \beta = \alpha - x\). By (1.8), we have \(x \in Zk(\alpha)a\), as desired. \(\Box\)
2 Elliptic Lie algebras and superalgebras

2.1 Definition with generators and relations

Let \( \mathcal{D} \) be a QEBS with \( l \geq 2 \). For \( \alpha \in \Pi \), let

\[
c(\alpha) := \begin{cases} 
2 & \text{if } g(\alpha) = \mathbb{Z} \text{ or } 2\mathbb{Z} + 1, \\
1 & \text{otherwise}, 
\end{cases}
\]

and let

\[
\alpha^* := c(\alpha)\alpha + k(\alpha)a \in \mathcal{E}.
\]

For \( \alpha \in R(k, g) \), let

\[
p(\alpha) := \begin{cases} 
1 & \text{if } 2\alpha \in R(k, g), \\
0 & \text{otherwise}.
\end{cases}
\]

Let

\[
\mathcal{A} := \{ (\alpha, \beta, y) \in \Pi \times \Pi \times \mathbb{N} \mid \alpha \neq \beta, J(\alpha, \beta^\vee) = -1, k(\alpha)y = k(\beta) \}.
\]

For a subset \( T \) of the \( \mathcal{E} \), let \( -T := \{ -t \in \mathcal{E} \mid t \in T \} \). Let \( \Pi^* := \{ \alpha^* \mid \alpha \in \Pi \} \) and \( \mathcal{B}_+ := \Pi \cup \Pi^* \). Let

\[
\mathcal{B} := \mathcal{B}_+ \cup -\mathcal{B}_+.
\]

Let \( (\mathcal{B} \times \mathcal{B})' := \{ (\mu, \nu) \in \mathcal{B} \times \mathcal{B} \mid \mu \neq \nu, \mu + \nu \neq 0 \} \). For \( (\mu, \nu) \in (\mathcal{B} \times \mathcal{B})' \), let

\[
x_{\mu, \nu} := \begin{cases} 
1 - J(\mu^\vee, \nu) & \text{if } J(\mu^\vee, \nu) < 0, \\
0 & \text{if } J(\mu^\vee, \nu) \geq 0.
\end{cases}
\]

Keep the notation as above. Let \( \mathfrak{g}_\mathcal{D} = \mathfrak{g}_\mathcal{D}(\mathcal{E}, \Pi, a, k, g) \) be the Lie superalgebra defined by generators

\[(2.1) \quad h_\sigma (\sigma \in \mathcal{E}), E_\mu (\mu \in \mathcal{B})\]

with parities

\[(2.2) \quad p(h_\sigma) = 0, p(E_\mu) = p(\mu)\]

and the following defining relations.

\[(SR1) \quad xh_\sigma + yh_\tau = h_{x\sigma + y\tau} \quad \text{if } x, y \in \mathbb{C} \text{ and } \sigma, \tau \in \mathcal{E},\]
\[(SR2)\quad [h_\sigma, h_\tau] = 0 \quad \text{if } \sigma, \tau \in \mathcal{E},\]

\[(SR3)\quad [h_\sigma, E_\mu] = J(\sigma, \mu)E_\mu \quad \text{if } \sigma \in \mathcal{E} \text{ and } \mu \in \mathcal{B},\]

\[(SR4)\quad [E_\mu, E_{-\mu}] = h_\mu \quad \text{if } \mu \in \mathcal{B}_+,\]

\[(SR5)\quad (\text{ad}E_\mu)^{\nu_\mu}E_\nu = 0 \quad \text{if } (\mu, \nu) \in (\mathcal{B} \times \mathcal{B})',\]

\[(SR6)\quad c(\alpha)(\text{ad}E_{\alpha^*})^yE_\beta = (\text{ad}E_\alpha)^{c(\alpha)y}E_\beta \quad \text{if } (\alpha, \beta, y) \in \mathcal{A},\]

\[(SR7)\quad (-1)^{c(\alpha)+1}c(\alpha)(\text{ad}E_{-\alpha^*})^yE_{-\beta} = (\text{ad}E_{-\alpha})^{c(\alpha)y}E_{-\beta} \quad \text{if } (\alpha, \beta, y) \in \mathcal{A},\]

\[(SR8)\quad (\text{ad}E_\alpha)^i(\text{ad}E_{\alpha^*})^{y-i}E_\beta = 0 \quad \text{if } (\alpha, \beta, y) \in \mathcal{A} \text{ and } 1 \leq i \leq y-1,\]

\[(SR9)\quad (\text{ad}E_{-\alpha})^i(\text{ad}E_{-\alpha^*})^{y-i}E_{-\beta} = 0 \quad \text{if } (\alpha, \beta, y) \in \mathcal{A} \text{ and } 1 \leq i \leq y-1.\]

We call the \( \mathfrak{g}_D \) the elliptic Lie (super)algebra. In Introduction, the \( \mathfrak{g}_D \) is also denoted as \( \mathfrak{g}_R \), where \( R = R(k, g) \). For \( \mu \in \mathcal{E} \), let \( \mathfrak{g}_{D,\mu} := \{ X \in \mathfrak{g}_D | h_\sigma, [h_\sigma, X] = J(\sigma, \mu)X \} \). Define the sub Lie superalgebra \( \mathfrak{h}_D \) of \( \mathfrak{g}_D \) by \( \mathfrak{h}_D := \{ h_\sigma \in \mathfrak{g}_{D,0} | \sigma \in \mathcal{E} \}. \)

**Lemma 2.1.** Keep the notation as above. Then \( h_\sigma \neq 0 \) for \( \sigma \in \mathcal{E} \setminus \{0\} \). In particular, \( \dim \mathfrak{h}_D = l + 4 \).

Proof of the lemma shall be given in Subsec. 4.2.

### 2.2 Main theorem

We see that there exists a unique \( \delta \in \mathbb{Z}_+ \Pi \) such that \( \mathbb{Z}_+ \delta = \{ \lambda \in \mathbb{Z}_+ \Pi | J(\Lambda_\delta, \delta) = 0 \} \). By \( \mathbb{Z}_+ \Pi \), we have \( J(\Lambda_\delta, \delta) = 1 \).

Let \( \mathbb{Z}_+^{2'} := \mathbb{Z}_+^2 \setminus \{(0,0)\}. \)

**Theorem 2.1.** Let \( \mathcal{D} = \mathcal{D}(\mathcal{E}, \Pi, a, k, g) \) be a QEBS with \( l \geq 2 \). Then we have

\[ \mathfrak{g}_D = \mathfrak{h}_D \bigoplus \left( \bigoplus_{\nu \in R(k, g)} \mathfrak{g}_{D,\nu} \right) \bigoplus \left( \bigoplus_{(m,n) \in \mathbb{Z}_+^{2'}} \mathfrak{g}_{D,m\delta+na} \right). \]

Moreover \( \mathfrak{h}_D = \mathfrak{g}_{D,0} \), and \( \dim \mathfrak{g}_{D,\alpha} = 1 \) for all \( \alpha \in R(k, g) \).

Proof of the theorem shall be given in Subsec. 3.2.

From now until the end of Subsec. 2.2, we suppose that we have proved Theorem 2.1. Let \( \nu \in R(k, g) \). Let \( E_{\pm\nu} \in \mathfrak{g}_{D,\pm\nu} \) be such that \( [E_{\nu}, E_{-\nu}] = h_{\nu\nu} \).
By Theorem 2.1, $E'_\pm\nu$ are locally nilpotent since $|J(\sigma + r\nu, \sigma + r\nu)| \rightarrow +\infty$ as $r \rightarrow +\infty$, where $\sigma \in \mathcal{E}$. Hence we can define $n_\nu = n_{E'_\nu} \in \text{Aut}(g_D)$ by (2.3)

$$n_\nu = \begin{cases} \exp \text{ad}E'_\nu \exp \text{ad}(-E'_{-\nu}) \exp \text{ad}E'_\nu & \text{if } p(\nu) = 0, \\ \exp(\frac{1}{4}\text{ad}[E'_\nu, E'_\nu]) \exp(\frac{1}{4}\text{ad}[E'_{-\nu}, E'_{-\nu}]) \exp(\frac{1}{4}\text{ad}[E'_\nu, E'_\nu]) & \text{if } p(\nu) = 1. \end{cases}$$

**Theorem 2.2.** Let $D$ be a PEBS with $l \geq 2$. Then $D$ is an EBS if and only if it is a QEBS.

**Proof.** The ‘only-if’-part follows from Lemma 1.5. Here we prove the ‘if’-part. Recall $n_\nu (\nu \in R(k, g))$ from (2.3). Notice that $n_\nu(g_{D,\lambda}) = g_{D,s_\nu(\lambda)}$. Then, by Theorem 2.1, we see that $D$ is an EBS. □

### 3 Proof of Theorem 2.1

In this section, we suppose that we have proved Lemma 2.1.

#### 3.1 Rank one and two subsystems

Let $S$ be a finite subset of $R(k, g)$. Assume that the elements of the $S$ are linearly independent and that the square matrix $A_S := (J(\alpha^\vee, \beta))_{\alpha, \beta \in S}$ is an affine type generalized Cartan matrix in the sense of [K1, §4.8]. Then we call the $S$ the affine type subset of $R(k, g)$. Let $EWS$ be the subgroup of GL($\mathcal{E}$) generated by $s_\mu (\mu \in S)$. Let $S^{\text{odd}} := \{ \alpha \in S | p(\alpha) = 1 \}$ and let

$$R(k, g)^S := \bigcup_{w \in EWS} w \left( \bigcup_{\alpha \in S \setminus S^{\text{odd}}} \{ \alpha \} \bigcup_{\alpha \in S^{\text{odd}}} \{ \alpha, 2\alpha \} \right).$$

Then we see that $R(k, g)^S$ is the affine type (real) root system with the base $S$; $R(k, g)^S$ is reduced if and only if $S^{\text{odd}} = \emptyset$. For the pair $(A_S, S^{\text{odd}})$ of the above $A_S$ and $S^{\text{odd}}$, we define the Dynkin diagram $\Gamma(A_S, S^{\text{odd}})$ in the same manner as in [K2]. If $\Gamma(A_S, S^{\text{odd}})$ is called $X$ in the tables [K2, Tables 1-4], we say that the name of $S$ is $X$. The following two lemmas follow from (KG1-3), Lemma 1.4 and the well-known fact [Ma, Appendixes 1-2].

**Lemma 3.1.** Let $D$ be a QEBS with $l \geq 2$. Let $\alpha \in \Pi$. Then $\{ \alpha, -\alpha^* \}$ is an affine type subset of $R(k, g)$, and we have $R(k, g)_{(\alpha)} = R(k, g)^{\{\alpha, -\alpha^*\}}$. Moreover, letting $X$ be the name of the $\{\alpha, -\alpha^*\}$, we have one of the following cases.
Lemma 3.2. Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Let $\alpha, \beta \in \Pi$ be such that $J(\alpha, \beta^\vee) = -1$. Then $g(\beta) = \emptyset$ and there exists a unique $\gamma \in R(k, g)_{\{\alpha, \beta\}} \cap (-a + \mathbb{Z}. \Pi)$ such that $\{\alpha, \beta, \gamma\}$ is an affine type subset of the $R(k, g)_{\{\alpha, \beta\}}$. Moreover we have $R(k, g)^{\{\alpha, \beta, \gamma\}} = R(k, g)_{\{\alpha, \beta\}}$. Furthermore, letting $Y$ be the name of the $\{\alpha, \beta, \gamma\}$ and letting $Jkg := \{J(\alpha^\vee, \beta), k(\beta), g(\alpha)\}$, we have one of the following cases.

(i) $Jkg = \{-1, 1, 0\}$, $\gamma = s_\alpha(-\beta^\ast)$, $Y = A_2^{(1)}$.
(ii) $Jkg = \{-2, 1, 0\}$, $\gamma = s_\alpha(-\beta^\ast)$, $Y = C_2^{(1)}$.
(iii) $Jkg = \{-3, 1, 0\}$, $\gamma = s_\beta s_\alpha(-\beta^\ast)$, $Y = G_2^{(1)}$.
(iv) $Jkg = \{-2, 2, 0\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = D_3^{(2)}$.
(v) $Jkg = \{-3, 3, 0\}$, $\gamma = s_\alpha s_\beta(-\alpha^\ast)$, $Y = D_4^{(3)}$.
(vi) $Jkg = \{-2, 1, 2\mathbb{Z} + 1\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = A_4^{(2)}$.
(vii) $Jkg = \{-2, 1, 2\mathbb{Z}\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = B_2^{(4)}(0, 2)$.
(viii) $Jkg = \{-2, 1, 2\mathbb{Z}\}$, $\gamma = s_\alpha(-\beta^\ast)$, $Y = A_2^{(2)}(0, 3)$.
(ix) $Jkg = \{-2, 2, 2\mathbb{Z}\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = C_2^{(3)}(3)$.
(x) $Jkg = \{-2, 2, 4\mathbb{Z} + 2\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = A_4^{(3)}(0, 4)$.
(xi) $Jkg = \{-2, 2, 4\mathbb{Z}\}$, $\gamma = s_\beta(-\alpha^\ast)$, $Y = A_4^{(4)}(0, 4)$.

3.2 Embedded rank two affine (super)algebras

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Let $\mu, \nu \in \mathcal{B}_+$ be such that $J(\mu^\vee, \nu) = -1$. Then we have

\[
\begin{align*}
(3.1) & \quad n_{x E_\mu}(E_{\pm \nu}) = \mp x^{\pm 1}[E_{\mp \mu}, E_{\pm \nu}], \\
& \quad n_{x E_\mu}(E_{\pm \mu}) = (\pm x) \mp J(\nu, \nu) \frac{1}{(-J(\nu, \nu)!)^2}(\text{ad} E_{\pm \nu}) J(\nu, \nu) E_{\pm \mu}
\end{align*}
\]

for $x \in \mathbb{C} \setminus \{0\}$ (see (2.3)).

From now until the end of the paper, we assume that for every $\mu \in \mathcal{B}$, $n_\mu$ always denotes $n_{E_\mu} \in \text{Aut}(\mathfrak{g}_D)$ and we let $X \sim Y$ mean that there exists a $z \in \mathbb{C} \setminus \{0\}$ such that $X = zY$. 

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Lemma 3.3. Keep the notation as in Lemma 3.2. Let $T := \{\alpha, \beta, \gamma\}$. Assume $\gamma = s_{\nu_1} \cdots s_{\nu_r} (-\mu^*)$ to be the same expression as in Lemma 3.2 (i)-(xi). Set

\[(3.2) \quad E_{\pm \gamma} := (\pm 1)^{p(\mu^*)} n_{\nu_1} \cdots n_{\nu_r} (E_{\mp \mu^*}).\]

Then we have

\[(3.3) \quad \begin{cases} (\text{ad} E_{\pm \mu})^{1-J(\mu^*, \nu)} E_{\pm \nu} = 0 & \text{if } \mu, \nu \in T \text{ with } \mu \neq \nu, \\ [E_{\mu}, E_{-\nu}] = \delta_{\mu \nu} h_{\mu \nu} & \text{if } \mu, \nu \in T. \end{cases}\]

**Proof.** If the name of $T$ is neither $A_4^{(2)}$ nor $B^{(1)}(0, 2)$, the equalities (3.3) is proved in a similar way to [Ya2, §2.3]. We assume that the name of $T$ is $A_4^{(2)}$ or $B(0, 2)^{(1)}$. It is clear that $[E_{\gamma}, E_{-\gamma}] = h_{s_{\beta}(-\alpha^*) \gamma} = h_{\gamma \gamma}$. We have

\[E_{\gamma} = n_{\beta} (E_{-\alpha^*}) \sim [E_{-\beta}, [E_{-\beta}, E_{-\alpha^*}]] \sim [E_{-\beta}, [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]].\]

It is clear that $[E_{\gamma}, E_{-\beta}] = 0$. We have

\[\begin{align*}
[E_{\gamma}, E_{-\alpha}] & \sim [[E_{-\beta}, E_{-\alpha}], [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]] \\
& \sim [[E_{-\beta}, E_{-\alpha}], [E_{-\beta}, E_{-\alpha^*}]] \\
& \sim n_{\beta} ([E_{-\alpha}, [E_{-\beta}, E_{-\alpha^*}]]) \\
& \sim n_{\beta} ([E_{-\alpha}, [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]]) = 0.
\end{align*}\]

Similarly we have $[E_{\gamma}, E_{-\beta}] = 0$ and $[E_{-\gamma}, E_{\alpha}] = 0$. Since the $E_{\lambda}$ ($\lambda \in T \cup -T$) are locally nilpotent, we have all the equalities in (3.3). \(\Box\)

For a subset $S$ of $\Pi$, let $g^S_D$ be the sub Lie superalgebra of $g_D$ generated by $h_\sigma$ ($\sigma \in \mathcal{E}$) and $E_{\mu}, E_{-\mu}, E_{\mu^*}, E_{-\mu^*}$ ($\mu \in S$). For $\nu \in \mathcal{E}$, let $g^S_{D, \nu} := g^S_D \cap g_{D, \nu}$. Let $m_S := \min \{k(\alpha) | \alpha \in S\}$.

**Lemma 3.4.** Let $D$ be a QEBS with $l \geq 2$. Let $S$ be a subset of $\Pi$ such that $|S| = 1$ or 2. Then we have

\[(3.4) \quad g^S_D = h_D \bigoplus \left( \bigoplus_{\lambda \in R(k, g)_S} g^S_{D, \lambda} \right) \bigoplus \left( \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} g^S_{D, nm \alpha} \right).\]

Moreover we have $\dim g^S_{D, \lambda} = 1$ for $\lambda \in R(k, g)_S$. Furthermore

\[(3.5) \quad g^S_{D, nm \alpha} = g^{(\alpha)}_{D, nm \alpha} + g^{(\beta)}_{D, nm \alpha}\]

if $S = \{\alpha, \beta\}$. 

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Proof. We first assume $|S| = 1$ and $S = \{\alpha\}$. Recall the affine subset $U := \{\alpha, -\alpha^*\}$ from Lemma 3.1. It follows from Lemma 2.1 that $E_{\pm\mu} \neq 0$ for $\mu \in U$, since $[E_{\mu}, E_{-\mu}] = h_{\mu^*} \neq 0$. Since $E_{\pm\mu}$ ($\mu \in U$) satisfy the Serre relations (see (SR5)), the lemma follows from the well-known argument in the proof of [K1 Corollary 5.12] (see also [K2 Proposition 1.6]).

Assume $|S| = 2$, and $S = \{\alpha, \beta\}$. If $J(\beta^\vee, \alpha) = 0$, the lemma follows from the same argument as above. Assume $J(\beta^\vee, \alpha) = -1$. Recall $\gamma$ and $E_{\pm\gamma}$ from Lemma 3.3, especially recall (3.2). Let $T := \{\alpha, \beta, \gamma\}$. Notice that $E_{\pm\gamma} \in g^S_D$. Let $g^{(T)}_D$ be the sub Lie superalgebra of $g^S_D$ generated by $h_D$ and $E_{\pm \omega}$ ($\omega \in T$). Then $E_{\pm\mu} = n_{\mu_1} \cdots n_{\mu_l} E_{\pm\gamma} \in g^{(T)}_D$. Let $\rho \in \{\alpha, \beta, \gamma\}$ be such that $\rho \neq \mu$. By (SR6) and (3.1), we have $n_{\mu} E_{\pm\rho} = n_{\mu} E_{\pm\rho}$. Hence $E_{\pm\rho} = n_{\mu} E_{\pm\rho} \in g^{(T)}_D$. Hence $g^{(T)}_D = g^S_D$. By Lemma 3.3, $E_{\pm\omega}$ ($\omega \in T$) satisfy the Serre relations (3.3). Using the well-known argument in the proof of [K1 Corollary 5.12] again, we have (3.4) for the $S$. The equality (3.5) follows from the fact that $g^S_D$ is generated by $g^{(\alpha)}_{D, \lambda}$ ($\lambda \in R(k, g)_{(\alpha)}$) and $g^{(\beta)}_{D, \nu}$ ($\nu \in R(k, g)_{(\beta)}$) (See also Lemma 3.4). This completes the proof. \[\square\]

Keep the notation as above. Let $S$ be a subset of $\Pi$. Let $ED_S := ZS + Zm_S a$. Define the subsets $ED_{S,+}$ and $ED_{S,-}$ of the $ED_S$ by $ED_{S,\pm} := (Z_{\pm}S + Zm_S a) \setminus Zm_S a$. Define the sub Lie superalgebras $n^S_D$, $n^S_D$, $l^S_D$ and $l^S_D$ of the $g^S_D$ by

$$n^S_D := \bigoplus_{\lambda \in ED_{S,\pm}} g^S_{D, \lambda}$$

and

$$l^S_D := \bigoplus_{n \in Z_{\pm} \setminus \{0\}} g^S_{D, nm, a}.$$  

Lemma 3.5. Let $S$ be a subset of $\Pi$. Then the following hold.

1. $g^S_D = n^S_D \oplus l^S_D \oplus h_D \oplus l^S_D \oplus n^S_D$.

2. The $n^S_D$ is generated by $n^{(\alpha)}_D$ with $\alpha \in S$. The $n^S_D$ is generated by $n^{(\alpha)}_D$ with $\alpha \in S$.

3. $l^S_D = \sum_{\alpha \in S} l^{(\alpha)}_D$, $l^S_D = \sum_{\alpha \in S} l^{(\alpha)}_D$.

4. dim $g^S_{D, \lambda} = 1$ for $\lambda \in R(k, g)_{(\alpha)}$ if $\alpha \in \Pi$.

Proof. If $|S| = 1$ or 2, the lemma follows from Lemma 3.4. Assume $|S| \geq 3$. Notice that the $g^S_D$ is generated by $h_D$ and $n^{(\gamma)}_D$, $n^{(\gamma)}_D$ with $\gamma \in S$. 

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Assume $\alpha, \beta \in S$ with $\alpha \neq \beta$. Notice that

\[(3.6) \quad (\mathbb{Z}_+\alpha + \mathbb{Z}_-\beta + \mathbb{Z}_0) \cap R(k, g)_S = \emptyset.\]

From Lemma 3.4 and (3.6) when $J(\alpha^\vee, \beta) \neq 0$, or from the defining relations (SR1-9) when $J(\alpha^\vee, \beta) = 0$, it follows that $[n^{(\alpha),+}_D, n^{(\beta),-}_D] = \{0\}$ and $[n^{(\beta),+}_D, t^{(\alpha),+}_D + t^{(\alpha),-}_D] \subset n^{(\beta),+}_D \cap n^{(\beta),-}_D$. Then the lemma follows from this fact and Lemmas 2.1 and 3.4.

Proof of Theorem 2.1 The theorem follows from Lemmas 1.3, 1.4 and 3.5 and from the existence of the $n_\gamma$’s with $\gamma \in \Pi$. \qed

4 Proof of Lemma 2.1

4.1 Contragredient Lie superalgebra

Here we first recall the definition of the contragredient Lie superalgebras [K2]. Let $\bar{I}$ be a finite set. Let $\bar{I}\text{odd}$ be a subset of $\bar{I}$. Define a map $\bar{p} : \bar{I} \to \{0, 1\}$ by $\bar{p}(i) = 1$ $(i \in \bar{I}\text{odd})$ and $\bar{p}(j) = 0$ $(j \in \bar{I} \setminus \bar{I}\text{odd})$. Let $\bar{A} := (\bar{a}_{ij})_{i,j \in \bar{I}}$ be an $\bar{I} \times \bar{I}$ matrix. Let $\bar{\mathfrak{y}}$ be the 2$|\bar{I}|$-dimensional $\mathbb{C}$-vector space. Let $\bar{\mathfrak{y}}^*$ be the dual space of $\bar{\mathfrak{y}}$. Let $\{\bar{\alpha}_i, \bar{\gamma}_i (i \in \bar{I})\}$ be a basis of $\bar{\mathfrak{y}}$. Let $\{\bar{h}_i, \bar{t}_i (i \in \bar{I})\}$ be a basis of $\bar{\mathfrak{y}}^*$. We assume that $\bar{\alpha}_i(\bar{h}_j) = \bar{a}_{ij}, \bar{\alpha}_i(\bar{t}_j) = \bar{\gamma}_i(\bar{h}_j) = 0$ and $\bar{\gamma}_i(\bar{t}_j) = 0$. Let $\bar{\Pi} := \{\bar{\alpha}_i (i \in \bar{I})\}$. Let $\bar{\Pi}^\vee := \{\bar{h}_i (i \in \bar{I})\}$. For the datum $\bar{\mathfrak{d}} := (\bar{\mathfrak{a}}, \bar{\mathfrak{I}}, \bar{\mathfrak{I}}^\text{odd}, \bar{\Pi}, \bar{\Pi}^\vee)$, we define a Lie superalgebra $\bar{\mathfrak{g}}'_{\bar{\mathfrak{d}}} := \bar{\mathfrak{g}}'(\bar{\mathfrak{a}}, \bar{\mathfrak{I}}^\text{odd})$ by generators

\[\bar{h}'_i, \bar{p}'_i, \bar{E}'_i, \bar{F}'_i (i \in \bar{I})\]

with parities

\[p(\bar{h}'_i) = p(\bar{p}'_i) = 0, \quad p(\bar{E}'_i) = p(\bar{F}'_i) = \bar{p}(i)\]

and defining relations

\[
\begin{cases}
[\bar{h}'_i, \bar{h}'_j] = [\bar{h}'_i, \bar{p}'_j] = [\bar{p}'_i, \bar{p}'_j] = 0, \\
[\bar{h}'_i, \bar{E}'_j] = \bar{a}_{ij}\bar{E}'_j, [\bar{p}'_i, \bar{E}'_j] = \delta_{ij}\bar{E}'_j, [\bar{h}'_i, \bar{F}'_j] = -\bar{a}_{ij}\bar{F}'_j, [\bar{p}'_i, \bar{F}'_j] = -\delta_{ij}\bar{F}'_j, \\
[\bar{E}'_i, \bar{F}'_j] = \delta_{ij}\bar{h}'_i.
\end{cases}
\]

Let $\bar{\mathfrak{y}}'$ be the sub Lie superalgebra of $\bar{\mathfrak{g}}'_{\bar{\mathfrak{d}}}$ generated by $\bar{h}'_i, \bar{p}'_i$. Let $\mathfrak{r}$ be the ideal of $\bar{\mathfrak{g}}'_{\bar{\mathfrak{d}}}$ which is maximal among the ones $\mathfrak{r}'$ such that $\mathfrak{r}' \cap \bar{\mathfrak{y}}' = \{0\}$. We denote by $\bar{\mathfrak{g}}_{\bar{\mathfrak{d}}} = \bar{\mathfrak{g}}(\bar{\mathfrak{a}}, \bar{\mathfrak{I}}^\text{odd})$ the quotient Lie superalgebra $\bar{\mathfrak{g}}'_{\bar{\mathfrak{d}}}/\mathfrak{r}$. In this paper, we
call the $\mathfrak{G}_D$ the contragredient Lie superalgebra. Let $\bar{\pi} : \mathfrak{G}_D^I \to \mathfrak{G}_D$ be a natural projective map. Notice that $\dim\bar{\pi}(\mathfrak{H}_D) = 2|\mathcal{I}|$. By abuse of notation, we shall also denote $\pi(\mathfrak{H}_D)$ and $\pi(\mathfrak{F}_i)$ by $\mathfrak{H}_i$, $\mathfrak{F}_i$ and $\mathfrak{F}_i$, respectively. We shall also denote $\pi(E_i)$ and $\pi(F_i)$ by $E_i$ and $F_i$, respectively.

Keep the notation as above. Let $\mathcal{I}^\text{pos} := \{ i \in \mathcal{I} | a_{ii} \neq 0 \}$. Let $\mathcal{I}^\text{null} := \mathcal{I} \setminus \mathcal{I}^\text{pos}$. Define the square matrix $A^\text{pos}$ by $A^\text{pos} := (a_{ij})_{i,j \in \mathcal{I}^\text{pos}}$. We say that $\mathfrak{D}$ is a handy datum if the following hold.

(HD1) If $i \in \mathcal{I}^\text{pos}$, then $a_{ii} = 2$.
(HD2) If $i, j \in \mathcal{I}^\text{pos}$ with $i \neq j$ and $0 \leq |a_{ij}| \leq |a_{ji}|$, then $(a_{ij}, a_{ji})$ is $(0, 0), (-1, -1), (-1, -2)$ or $(-1, -3)$.
(HD3) If $i, j \in \mathcal{I}^\text{null}$ with $i \neq j$, then $(a_{ij}, a_{ji}) = (0, 0)$ or $(2, 2)$.
(HD4) If $i \in \mathcal{I}^\text{pos}$ and $j \in \mathcal{I}^\text{null}$, then $(a_{ij}, a_{ji}) = (0, 0), (-1, -1)$ or $(-1, -2)$.
(HD5) If $i \in \mathcal{I}^\text{pos}$ and $j \in \mathcal{I}^\text{null}$, then $(a_{ij}, a_{ji}) = (-1, -1)$ if and only if there exists an $r \in \mathcal{I}^\text{null} \setminus \{ j \}$ such that $a_{ir} \neq 0$ and $a_{jr} \neq 0$.
(HD6) $\mathcal{I}^\text{null} \subset \mathcal{I}^\text{odd}$.
(HD7) If $i \in \mathcal{I}^\text{pos} \cap \mathcal{I}^\text{odd}$ and $j \in \mathcal{I}^\text{null}$, then $(a_{ij}, a_{ji}) = (0, 0)$.
(HD8) If $i \in \mathcal{I}^\text{pos} \cap \mathcal{I}^\text{odd}$, $j \in \mathcal{I}^\text{pos} \setminus \{ i \}$ and $a_{ij} \neq 0$, then $j \not\in \mathcal{I}^\text{odd}$ and $(a_{ij}, a_{ji}) = (-2, -1)$.
(HD9) If $i \in \mathcal{I}^\text{null}$, then there exists a unique $j \in \mathcal{I}^\text{null}$ such that $i \neq j$ and $a_{ij} \neq 0$.
(HD10) There exist $\varepsilon_i \in \mathbb{C} \setminus \{ 0 \} (i \in \mathcal{I})$ such that $D^{-1}\bar{A}$ is a symmetric matrix, where $D$ is the diagonal matrix $(\delta_{ij}\varepsilon_i)$.

Assume $\mathfrak{D}$ to be a handy datum. Then there exists a nondegenerate symmetric bilinear form $J : \mathfrak{H}_D^* \times \mathfrak{H}_D^* \to \mathbb{C}$ such that $J(\bar{a}_i, \bar{a}_j) = \varepsilon_i^{-1}\bar{a}_{ij}$, $J(\bar{\gamma}_i, \bar{\gamma}_j) = \varepsilon_i^{-1}\delta_{ij}$, $J(\bar{\gamma}_i, \bar{\gamma}_j) = 0$ For $\sigma \in \mathfrak{H}_D^*$, let $h_{(\sigma)} \in \mathfrak{H}_D^*$ be such that $\tau(h_{(\sigma)}) = J(\tau, \sigma)$ for all $\tau \in \mathfrak{H}_D^*$. Then $\bar{h}_i = \varepsilon_i\bar{h}_{(\bar{a}_i)}$ and $\bar{t}_i = \varepsilon_i\bar{h}_{(\bar{a}_i)}$.

**Lemma 4.1.** Let $\mathfrak{D}$ be a handy datum. Then the following hold for $\mathfrak{S}(\bar{A}, \mathcal{I}^\text{odd})$.

1. $(\text{ad}E_i)^{1-a_{ij}}E_j = 0$ for $i \in \mathcal{I}^\text{pos}$ and $j \in \mathcal{I} \setminus \{ i \}$.
2. $[\bar{E}_i, \bar{E}_j] = 0$ if $a_{ij} = 0$. In particular it follows that if $a_{ii} = 0$, then $[\bar{E}_i, \bar{E}_j] = 0$ and $(\text{ad}E_i)^2X = 0$ for any homogeneous element $X$ of $\mathfrak{G}$.
3. $[E_j, [E_i, E_j, E_m]] = 0$ if $a_{jj} = 0$ and $-a_{ji} = a_{jm} \neq 0$.
4. $[[E_i, E_j], E_r] = [[E_i, E_r], E_j]$ if $a_{ii} = 2, a_{jj} = a_{rr} = 0, a_{ij} = a_{ji} = a_{ir} = a_{ri} = -1$ and $a_{jr} = a_{rj} = 2$.
5. The same formulas as (1)-(4) with $\bar{E}_i$’s in place of $E_i$’s hold.
(6) There exists a super-symmetric invariant form \( J : \mathfrak{G} \times \mathfrak{G} \to \mathbb{C} \) such that \( J(\bar{h}_{(\sigma)}, \bar{h}_{(\tau)}) = J(\sigma, \tau) \) and \( J(\bar{E}_i, \bar{F}_j) = \delta_{ij} \bar{e}_i \). (By abuse of notation, we use the same symbol \( J \) for the bilinear forms on \( \mathfrak{G} \) and \( \mathfrak{F}^* \).)

This can be checked directly (see also [Ya1, Proposition 6.7.1]).

Let \( \mathfrak{D} \) be a handy datum. Let \( \mathbb{C}[t, t^{-1}] \) be the Laurent polynomial algebra. Let

\[
\mathcal{L}(\mathfrak{D}) := \mathfrak{G}(\bar{A}, \bar{I}^{\text{odd}}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}v \oplus \mathbb{C}w.
\]

We view \( \mathcal{L}(\mathfrak{D}) \) as a Lie superalgebra in the following way. The parity of \( X \otimes t^n \) is the same as the one of \( X \) for a homogeneous element \( X \) of \( \mathfrak{G}(\bar{A}, \bar{I}^{\text{odd}}) \); the parities of \( v \) and \( w \) are 0. The Lie super bracket of \( \mathcal{L}(\mathfrak{D}) \) is given by

\[
[X \otimes t^m + a_1 v + b_1 w, Y \otimes t^n + a_2 v + b_2 w] = [X, Y] \otimes t^{m+n} + m \delta_{m+n,0} J(X, Y) v + b_1 n Y \otimes t^n - b_2 m X \otimes t^m
\]

for homogeneous elements \( X, Y \) of \( \mathfrak{G}(\bar{A}, \bar{I}^{\text{odd}}) \). We shall also denote by \( \bar{J} \) the invariant form on \( \mathcal{L}(\mathfrak{D}) \) defined by

\[
\bar{J}(X \otimes t^m + a_1 v + b_1 w, Y \otimes t^n + a_2 v + b_2 w) = \delta_{m+n,0} J(X, Y) + a_1 b_2 + b_1 a_2.
\]

### 4.2 Unfolding

Let \( \mathcal{D} \) be a QEBS with \( l \geq 2 \). We define a map \( k^\vee : \Pi \to \{1, 2, 3, 4\} \) by the following.

- (KV1) If \( J(\beta^\vee, \alpha) = -1 \) and \( g(\alpha) = 0 \) or \( \mathbb{Z} \), then \( k^\vee(\alpha) = \frac{k(\beta)}{k(\alpha)} k^\vee(\beta) \).
- (KV2) If \( J(\alpha^\vee, \beta) = -2 \), \( k(\alpha) = 2k(\beta) \) and \( g(\alpha) = 2\mathbb{Z} \), then \( k^\vee(\alpha) = 2k^\vee(\beta) \).
- (KV3) If \( J(\alpha^\vee, \beta) = -2 \), \( k(\alpha) = k(\beta) \) and \( g(\alpha) = 2\mathbb{Z} + 1 \) or \( 2\mathbb{Z} \), then \( k^\vee(\alpha) = k^\vee(\beta) = 2 \).
- (KV4) If \( J(\alpha^\vee, \beta) = -2 \) and \( g(\alpha) = 4\mathbb{Z} \) or \( 4\mathbb{Z} + 2 \), then \( k^\vee(\alpha) = 3 \) and \( k^\vee(\beta) = 2 \).
- (KV5) \( k^\vee(\alpha) = 4k^\vee(\beta) \) if and only if \( 4k(\alpha) = k(\beta) \).

For the above \( \mathcal{D} \), we define a handy datum \( \mathfrak{D}_D = (\bar{A}_D, \bar{I}_D, \bar{I}^{\text{odd}}_D, \bar{\Pi}_D, \bar{\Pi}^*_D) \) in the following way. Let

\[
\bar{I}_D := \{ (\alpha, x) \in \Pi \times \{1, 2, 3, 4\} | 1 \leq x \leq k^\vee(\alpha) \}.
\]
We define a square matrix $\bar{A}_D = (\bar{a}_{(\alpha,x),(\beta,y)})_{(\alpha,x),(\beta,y) \in I_D}$ in the following way.

(AD1) If $J(\alpha^\vee, \beta) = 0$, then $\bar{a}_{(\alpha,x),(\beta,y)} = 0$.

(AD2) Let $\alpha \in \Pi$. If $g(\alpha) = \emptyset$ or $\mathbb{Z}$ then $\bar{a}_{(\alpha,x),(\beta,y)} = 2\delta_{xy}$. If $g(\alpha) = 2\mathbb{Z} + 1$, then $\bar{a}_{(\alpha,x),(\beta,y)} = 3\delta_{xy} - 1$. If $g(\alpha) = 2\mathbb{Z}$, then $\bar{a}_{(\alpha,x),(\beta,y)} = 2 - 2\delta_{xy}$. If $g(\alpha) = 4\mathbb{Z}$ or $4\mathbb{Z} + 2$, then $\bar{a}_{(\alpha,1),(\alpha,1)} = \bar{a}_{(\alpha,3),(\alpha,3)} = \bar{a}_{(\alpha,1),(\alpha,2)} = \bar{a}_{(\alpha,2),(\alpha,1)} = 0$,

$\bar{a}_{(\alpha,2),(\alpha,2)} = \bar{a}_{(\alpha,1),(\alpha,3)} = \bar{a}_{(\alpha,3),(\alpha,1)} = 2$, $\bar{a}_{(\alpha,2),(\alpha,3)} = -1$ and $\bar{a}_{(\alpha,3),(\alpha,2)} = -2$

(AD3) Assume $J(\beta^\vee, \alpha) = -1$. Then

\[
(\bar{a}_{(\alpha,x),(\beta,y)}, \bar{a}_{(\beta,y),(\alpha,x)}) = \begin{cases} 
(0, 0) & \text{if } k(\alpha) = 4, k^\vee(\beta) = 2 \text{ and } x - y \notin 2\mathbb{Z}, \\
(0, 0) & \text{if } k^\vee(\beta) \leq k(\alpha) \leq \frac{3}{2} k^\vee(\beta) \text{ and } x \neq y, \\
(-2, -1) & \text{if } 2 \leq k(\alpha) \leq 3, k^\vee(\beta) = 2, \bar{a}_{(\alpha,x),(\alpha,x)} = 0 \\
 & \text{and } x = y, \\
(-1, -1) & \text{if } g(\alpha) = 2\mathbb{Z} + 1 \text{ and } x = y, \\
\left( \frac{k(\alpha)}{k^\vee(\beta)} J(\alpha^\vee, \beta), -1 \right) & \text{otherwise.}
\end{cases}
\]

Let $\bar{E}_{(\alpha,x)} := \bar{F}_{(\alpha,x)} \in \mathfrak{G}_{\mathbb{D}_D}$, and notice that the $\bar{h}_{(\alpha,x)} \in \mathfrak{G}_{\mathbb{D}_D}$ is not necessarily $\bar{h}_{(\alpha,x)}$.

**Lemma 4.2.** Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Then there exists a unique homomorphism $\pi_D : \mathfrak{g}_D \rightarrow \mathfrak{L}(\mathscr{D}_D)$ satisfying the following properties:

(PD1)

\[
\pi_D(E_{\pm \alpha}) = \begin{cases} 
\frac{k(\alpha)}{2} \sum_{x=1}^{\mathcal{D}} \bar{E}_{\pm(\alpha,x)} & \text{if } g(\alpha) = \emptyset, \mathbb{Z} \text{ or } 2\mathbb{Z}, \\
\sqrt{2} \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } g(\alpha) = 2\mathbb{Z} + 1, \\
\sqrt{2} \bar{E}_{\pm(\alpha,2)} \pm \frac{1}{\sqrt{2}} [\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,3)}] & \text{if } g(\alpha) = 4\mathbb{Z} + 2, \\
\bar{E}_{\pm(\alpha,1)} \pm [\bar{E}_{\pm(\alpha,3)}, \bar{E}_{\pm(\alpha,2)}] & \text{if } g(\alpha) = 4\mathbb{Z},
\end{cases}
\]
\[ \pi_D(E_{\pm \alpha^*}) = \begin{cases} 
\sum_{x=1}^{k^\vee(\alpha)} \zeta_\alpha x^{(2x-1-k^\vee(\alpha))} [\tilde{E}_{\pm(\alpha,x)}, \tilde{E}_{\pm(\alpha,x)}] \otimes t^\pm k(\alpha) & \text{if } g(\alpha) = \emptyset \text{ or } 2\mathbb{Z}, \\
\pm \frac{1}{4} \sum_{x=1}^{k^\vee(\alpha)} \zeta_\alpha x^{(2x-1-k^\vee(\alpha))} [\tilde{E}_{\pm(\alpha,1)}, \tilde{E}_{\pm(\alpha,2)}] \otimes t^{\pm 1} & \text{if } g(\alpha) = \mathbb{Z}, \\
\sqrt{-1} [\tilde{E}_{\pm(\alpha,1)}, \tilde{E}_{\pm(\alpha,2)}] \otimes t^{\pm 1} & \text{if } g(\alpha) = 2\mathbb{Z} + 1, \\
(\tilde{E}_{\pm(\alpha,1)} + \sqrt{-1} [\tilde{E}_{\pm(\alpha,3)}, \tilde{E}_{\pm(\alpha,2)}]) \otimes t^{\pm 1} & \text{if } g(\alpha) = 4\mathbb{Z} + 2, \\
\sqrt{-2}[\tilde{E}_{\pm(\alpha,2)} + \frac{1}{2} [\tilde{E}_{\pm(\alpha,1)}, \tilde{E}_{\pm(\alpha,3)}]] \otimes t^{\pm 1} & \text{if } g(\alpha) = 4\mathbb{Z}, 
\end{cases} \]

where \( \zeta_\alpha := \exp(\frac{\pi}{k^\vee(\alpha)}) \).

(PD2) There exists a \( \kappa \in \mathbb{C} \setminus \{0\} \) such that \( J(\pi_D(h_\mu), \pi_D(h_\nu)) = \kappa J(\mu, \nu) \) for \( \mu, \nu \in \mathcal{E} \).

(PD3) \( \pi_D(h_a) = \kappa v \), \( \pi_D(h_{\Lambda a}) = w \) and \( \pi_D(h_{\Lambda_0}) = J(\Lambda_{\delta}, \alpha_0) \sum_{i=1}^{k^\vee(\alpha_0)} \tilde{l}_{(\alpha_0,i)} \).

In particular, \( \pi_D(h_\sigma) \neq 0 \) for all \( \sigma \in \mathcal{E} \).

This can be proved directly by using Lemma 4.1.

Proof of Lemma 2.1. The lemma follows immediately from Lemma 4.2. \( \square \)

Keep the notation as in Lemma 4.2. We shall also denote by \( J \) the invariant form on \( g_D \) defined by \( \frac{1}{\kappa} \tilde{J} \).

5 Invariant form and universal central extension (UCE)

5.1 Invariant form and a universal property

Let \( \mathcal{D} \) be a QEBS with \( l \geq 2 \). Following the notation in \([S]\), we say that \( \mathcal{D} \) is \( A_l^{(1,1)} \) if the \((l + 1) \times (l + 1)\)-matrix \( A = (J(\alpha^\vee, \beta))_{\alpha, \beta \in \Pi} \) is \( A_l^{(1,1)} \) (see also Subsec. 1.1 for the name \( A_l^{(1)} \)). Notice that if \( \mathcal{D} \) is \( A_l^{(1,1)} \), then \( p(\alpha) = 0 \) and \( g(\alpha) = \emptyset \) for all \( \alpha \in \Pi \).

Theorem 5.1. Let \( \mathcal{D} \) be a QEBS with \( l \geq 2 \). Assume that \( \mathcal{D} \) is not \( A_l^{(1,1)} \). Let \( g_D' \) be a Lie superalgebra satisfying the following conditions.

1. \( g_D' \) is a Lie superalgebra.
2. \( g_D' \) is a Lie superalgebra.
3. \( g_D' \) is a Lie superalgebra.
4. \( g_D' \) is a Lie superalgebra.
5. \( g_D' \) is a Lie superalgebra.
6. \( g_D' \) is a Lie superalgebra.
7. \( g_D' \) is a Lie superalgebra.
8. \( g_D' \) is a Lie superalgebra.
9. \( g_D' \) is a Lie superalgebra.
10. \( g_D' \) is a Lie superalgebra.

Let \( g_D' \) be a Lie superalgebra satisfying the following conditions.

1. \( g_D' \) is a Lie superalgebra.
2. \( g_D' \) is a Lie superalgebra.
3. \( g_D' \) is a Lie superalgebra.
4. \( g_D' \) is a Lie superalgebra.
5. \( g_D' \) is a Lie superalgebra.
6. \( g_D' \) is a Lie superalgebra.
7. \( g_D' \) is a Lie superalgebra.
8. \( g_D' \) is a Lie superalgebra.
9. \( g_D' \) is a Lie superalgebra.
10. \( g_D' \) is a Lie superalgebra.
(UI1) $g'_D$ includes $h_D$ as a sub Lie superalgebra.
(UI2)
\[ g'_D = h_D \bigoplus \left( \bigoplus_{\nu \in R(k, g)} g'_{D, \nu} \right) \bigoplus \left( \bigoplus_{(m, n) \in \mathbb{Z}^2} g'_{D, m\delta + na} \right), \]
and $\dim g'_{D, \nu} = 1$ for $\nu \in R(k, g)$, where $g'_{D, \sigma} := \{ X \in g'_D \mid [h, X] = \sigma(h)X \ (h \in h_D) \}$. 
(UI3) The $g'_D$ is generated by $h_D$ and $g'_{D, \nu}$ with $\nu \in R(k, g)$.
(UI4) There exists an invariant form $J'$ on $g'_D$ such that $J'(h_\sigma, h_\tau) = J(\sigma, \tau)$ ($\sigma, \tau \in E$) and such that $\ker J' \subset \bigoplus_{(m, n) \in \mathbb{Z}^2} g'_{D, m\delta + na}$.

Then there exists an epimorphism $\eta : g_D \to g'_D$ such that $\eta(h_\sigma) = h_\sigma$ ($\sigma \in E$) and $J' \circ (\eta \times \eta) = J$.

Proof. Using the same argument as in the proof of [K1, Theorem 2.2], we can choose non-zero elements $E'_\rho$ of $g'_{D, \rho}$ ($\rho \in R(k, g)$) so that $[E'_\rho, E'_\rho] = h_\rho$. By comparing (3.1) with the equalities in (SR6-7), we can normalize the elements $E'_\mu$'s with $\mu \in \mathcal{B}$ so that they and the $h_\sigma$'s ($\sigma \in E$) satisfy the relations (SR1-9). Then the theorem follows from Theorem 2.1 and the existence of the $n_\mu$'s with $\mu \in \Pi \cup \Pi^*$.

Corollary 5.1. Let $D = D(\mathcal{E}, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. Assume $\alpha \in \Pi^B$ to be such that $g(\alpha) = 4\mathbb{Z}$. Let $D' = D(\mathcal{E}, \Pi, a, k, g')$ be the QEBS obtained from the $D$ by replacing $g$ by $g'$ such that $g'(\alpha) = 4\mathbb{Z} + 2$ and $g'(\beta) = g(\beta)$ ($\beta \neq \alpha$). Then there exists an isomorphism $\xi : g_D \to g'_D$ such that $\xi(h_\alpha) = h_\alpha^*$, $\xi(h_\beta) = h_\beta$ ($\beta \neq \alpha$), $\xi(h_a) = h_a$, $\xi(h_{\Lambda_\alpha}) = h_{\Lambda_\alpha}$ and $\xi(h_{\Lambda_a}) = h_{\Lambda_a - \Lambda_a}$, where $\Lambda_\alpha \in \mathbb{C}\Pi \oplus \mathbb{C}a$ is such that $J(\Lambda_\alpha, \Lambda_\alpha) = 0$ and $J(\Lambda_\alpha, \gamma) = \delta_{\alpha \gamma}$ for $\gamma \in \Pi$.

This can be proved easily by using Theorem 5.1.

5.2 A Lie algebra with the quantum tori

Here we recall a Lie algebra studied in [BGK]. Let $q \in \mathbb{C} \setminus \{0\}$. Let $\mathbb{C}_q = \mathbb{C}_q[s^{\pm 1}, t^{\pm 1}]$ be the $\mathbb{C}$-algebra defined by generators $s^{\pm 1}$, $t^{\pm 1}$ and defining relations $ts = qst$. Let $M_{l+1}(\mathbb{C}_q)$ be the $\mathbb{C}_q$-algebra of the $(l + 1) \times (l + 1)$-matrices over $\mathbb{C}_q$. Let $\tilde{M}_{l+1}(\mathbb{C}_q) := M_{l+1}(\mathbb{C}_q) \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. We
regard $\hat{\mathfrak{M}}_{l+1}(\mathbb{C}_q)$ as a $\mathbb{C}$-Lie algebra by
\[
[s^{x_1}t^{x_2}E_{ij}, s^{y_1}t^{y_2}E_{mn}] = s^{x_1+y_1}t^{x_2+y_2}(\delta_{jm}q^{x_2y_1}E_{in} - \delta_{im}q^{x_1y_2}E_{mj}) + \delta_{x_1+y_1,o}\delta_{x_2+y_2,0}q^{x_2y_1}(x_1c_1 + x_2c_2),
\]
and
\[
[c_i, s^{x_1}t^{x_2}E_{mn}] = 0, \quad [d_i, s^{x_1}t^{x_2}E_{mn}] = x_is^{x_1}t^{x_2}E_{mn}
\]
and
\[
[c_i, c_j] = [c_i, d_j] = [d_i, d_j] = 0.
\]

We define a symmetric invariant form $J_q$ on $\hat{\mathfrak{M}}_{l+1}(\mathbb{C}_q)$ by
\[
\begin{align*}
\bar{J}_q(s^{x_1}t^{x_2}E_{ij}, s^{y_1}t^{y_2}E_{mn}) &= \delta_{x_1+y_1,o}\delta_{x_2+y_2,0}q^{x_2y_1}, \\
\bar{J}_q(s^{x_1}t^{x_2}E_{ij}, c_i) &= \bar{J}_q(s^{x_1}t^{x_2}E_{ij}, d_i) = 0, \\
\bar{J}_q(c_i, c_j) &= \bar{J}_q(d_i, d_j) = 0, \quad \bar{J}_q(c_i, d_j) = \delta_{ij}.
\end{align*}
\]

Let $\mathcal{D}$ be $A_l^{(1,1)}$. Let $\mathfrak{g}_D^0$ be the Lie algebra defined be the generators (2.1) and the defining relations obtained from the ones in (SR1-9) by replacing $[E_{\pm\alpha_0^0}, E_{\pm\alpha_1}] = [E_{\pm\alpha_0}, E_{\pm\alpha_1}]$ of (SR6-7) with
\[
q^{\pm 1}[E_{\pm\alpha_0^0}, E_{\pm\alpha_1}] = [E_{\pm\alpha_0}, E_{\pm\alpha_1}].
\]
Then there exists a unique homomorphism $\pi_D^q : \mathfrak{g}_D^0 \to \hat{\mathfrak{M}}_{l+1}(\mathbb{C}_q)$ such that $\pi_D^q(E_{\alpha_0}) = E_{ii+1}$, $\pi_D^q(E_{-\alpha_1}) = tE_{ii+1}$, $\pi_D^q(E_{-\alpha_0}) = t^{-1}E_{ii+1}$ ($1 \leq i \leq l$), $\pi_D^q(E_{\alpha_1}) = sE_{ii+1}$, $\pi_D^q(E_{-\alpha_1}) = stE_{ii+1}$, $\pi_D^q(E_{-\alpha_0}) = s^{-1}E_{ii+1}$, $\pi_D^q(E_{-\alpha_0}) = q^{-1}\pi_D^q(h_{\Lambda_0}) = d_1$ and $\pi_D^q(h_{\Lambda_0}) = d_2$. We see that similar results to Theorems 5.1 and 5.2 also hold for $\mathfrak{g}_D^0$ with $\pi_D^q$. Especially we have the following.

**Theorem 5.2.** Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Assume that $\mathcal{D} = A_l^{(1,1)}$. Let $\mathfrak{g}_D^0$ be a Lie superalgebra satisfying the same conditions as the (UI1-4) in Theorem 7.1. Then there exist a $q \in \mathbb{C} \setminus \{0\}$ and an epimorphism $\eta : \mathfrak{g}_D^0 \to \mathfrak{g}_D^0$ such that $\eta(h_\sigma) = h_\sigma$ ($\sigma \in \mathcal{E}$) and $J' \circ (\eta \times \eta) = J$.

Using [AABGP] Chap. I Theorem 1.29(d), by Theorems 5.1 and 5.2 we have the following.
Theorem 5.3. Let $D = D(E, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. We assume that $g(\alpha) = 0$ or $2\mathbb{Z} + 1$ for every $\alpha \in \Pi$. We also keep the notation in [AABGP, Chap. I §1]. Let $(\mathcal{L}, (, ), \mathcal{H})$ be an extended affine Lie algebra (EALA) in the sense of [AABGP, Chap. I Definition 1.33]. Assume that there exists an isometric monomorphism $\varphi : E \to \mathcal{H}$ such that the set $\varphi(R(k, g))$ coincides with the set of the non-isotropic roots of the $\mathcal{L}$. Then there exists a homomorphism $F : \bar{g}_D \to \mathcal{L}$ such that $F(h_\sigma) = t_{\varphi(\sigma)} (\sigma \in E)$ and $F(g_\mu) = \mathcal{L}_{\varphi(\mu)} (\mu \in R(k, g))$, where we let $\bar{g}_D := g^q_D$ for some $q \in \mathbb{C} \setminus \{0\}$ if $D = A_{i}^{(1,1)}$, and, otherwise, we let $\bar{g}_D := g_D$. (See [AABGP, Chap. I (1.2) and (1.8)] for the symbols $\mathcal{L}_{\varphi(\mu)}$ and $t_{\varphi(\sigma)}$.)

5.3 Universal central extension (UCE)

We first recall the definition of the universal central extension (UCE for short) of a Lie superalgebra. See [K] for more detail. Let $a = a_0 \oplus a_1$ be a Lie superalgebra. Let $D^1(a) := [a, a]$. We say that $a$ is perfect if $D^1(a) = 0$. We say that a Lie superalgebra epimorphism $P : u = u_0 \oplus u_1 \to a$ is a central extension if $[\ker P, u] = \{0\}$ and $\ker P = (u_0 \cap \ker P) \oplus (u_1 \cap \ker P)$ (we do not assume $\ker P \subset u_0$). We say that a central extension $V : u \to a$ is a UCE if $u = D^1(u)$ and if for any central extension $W : b \to a$, there exists a homomorphism $M : u \to b$ such that $W \circ M = V$. Notice that if $P : u \to a$ is a central extension and if $x, y \in a$ are homogeneous elements, then there exists a unique $z \in u$ such that $z \in [P^{-1}\{x\}, P^{-1}\{y\}]$; we denote the $z$ by $N(P, x, y)$. Notice that if $x \in a_i$ and $y \in a_j$, then $N(P, x, y) \in u_{i+j}$.

The following lemma seems to be trivial, but we give it to use it in the proof of Theorem 5.4.

**Lemma 5.1.** Let $a = a_0 \oplus a_1$ be a Lie superalgebra such that $a = \text{Der}(a) \oplus \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n, k_i \in a_0, [k_i, k_j] = 0$, and $D^1(a) = \oplus_{x \in \mathbb{C}^n} a'_x$, where $a'_x = \{X \in D^1(a) | [k_i, X] = x_iX\}$ (here $x = (x_1, \ldots, x_n)$). Assume that $a$ is presented by generators $k_i \ (1 \leq i \leq n)$, $a_p \in D^1(a) \ (p \in P)$ with $a_p \in a'_x(p)$ for some $x(p) = (x(p)_1, \ldots, x(p)_n) \in \mathbb{C}^n$ and defining relations $f_t = 0 \ (t \in T)$ and $[k_i, a_p] = x(p)_i a_p, [k_i, k_j] = 0$, where $f_t$'s are assumed to be expressed only by the elements $a_p \ (p \in P)$ and to be homogeneous with respect to the $a_p$'s. Then $D^1(a)$ is also presented by the generators $a_p \ (p \in P)$ and the defining relations $f_t = 0 \ (t \in T)$.

**Proof.** Let $c := \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n$. Let $b$ be the Lie superalgebra generated by the generators $b_p \ (p \in P)$ and the defining relations $g_t = 0 \ (t \in T)$,
where \( g_t \) is expressed by replacing \( a_p \) of \( f_t \) with \( b_p \). For \( x \in \mathbb{C}^n \), let \( b_x \) be the subspace of \( b \) spanned by the elements \((\text{ad}b_{p_1})\cdots(\text{ad}b_{p_{r-1}})b_{p_r}\) with \( x(p_1) + \cdots + x(p_r) = x \). Then \( b = \oplus_{x \in \mathbb{C}^n} b_x \). We can define a Lie superalgebra \( \mathcal{D} = b \oplus \mathcal{C} \) by \([b + \sum y_i k_i, b' + \sum y'_i k_i] = [b, b'] + (\sum y_i x'_i) b' - (\sum y'_i x_i) b\), where \( b \in b_x \) and \( b' \in b_{x'} \). We see that there exists an isomorphism \( \Phi: \mathcal{D} \to \mathcal{A} \) such that \( \Phi(b_p) = a_p \) and \( \Phi(k_i) = k_i \).

Now we give a UCE theorem.

**Theorem 5.4.** Let \( \mathcal{D} \) be a QEBS with \( l \geq 2 \). Recall the \( \mathfrak{g}_D \) from Theorem 5.3. Let \( \varrho: \mathcal{D}^1(\mathfrak{g}_D) \to \mathcal{A} \) be a central extension. Then the \( \varrho \) is a UCE.

**Proof.** We first assume that \( \mathfrak{g}_D = \mathfrak{g}_D \). Let \( f: b \to \mathcal{A} \) be a central extension. For \( \mu \in \mathcal{B}_+ \), let

\[
\varrho_\mu := N(f, \varrho(E_\mu), \varrho(E_{-\mu})) \quad \text{and} \quad E'_\mu := N(f, \varrho(\pm \frac{1}{2} h_{\mu'}) \varrho(E_{\pm\mu})).
\]

By Lemma 5.1 it suffices to show that these elements of the \( b \) satisfy the equalities in the (SR1-9).

We first show the (SR4). Let \( \mu \in \mathcal{B}_+ \). Notice that

\[
\{f(E'_{\pm\mu})\} = \{f([f^{-1}(\{\varrho(\pm \frac{1}{2} h_{\mu'})\}), f^{-1}(\{\varrho(E_{\pm\mu})\})])\} = \{\varrho(E_{\pm\mu})\}.
\]

Hence we have

\[
(5.1) \quad f(E'_{\pm\mu}) = \varrho(E_{\pm\mu}) \quad \text{and} \quad E'_\pm \in f^{-1}(\{\varrho(E_{\pm\mu})\}),
\]

which implies the (SR4), as desired.

We show the (SR2-3). For \( \mu \in \mathcal{B}_+ \), by (5.1) and the (SR4), we have \( f(h'_{\mu'}) = [f(E'_\mu), f(E'_{-\mu})] = \varrho(h_{\mu'}) \) and, by (5.1), we have \( E'_\pm = [\pm \frac{1}{2} h'_{\mu'}, E'_{\pm\mu}] \). Hence, for \( \mu, \nu \in \mathcal{B}_+ \), we have

\[
[h'_{\mu'}, E'_{\pm\mu}] = [h'_{\mu'}, [\pm \frac{1}{2} h'_{\mu'}, E'_{\pm\mu}]] = \pm \frac{1}{2} [h'_{\mu'}, h'_{\mu'}, E'_{\pm\mu}] = J(\nu', \pm \mu) E'_{\pm\mu},
\]

and \([h'_{\mu'}, h'_{\mu'}] = [h'_{\mu'}, [E'_{\mu'}, E'_{-\mu}]] = 0 \), as desired.

For \( \mu \in \mathcal{B}_+ \), let \( h'_{(-\mu)'} := -h'_{\mu'} \).

We show the (SR5-9). For \( (\mu, \nu) \in (\mathcal{B} \times \mathcal{B})' \) and \( y, z \in \mathbb{C} \), by (SR3), we have

\[
0 = [yh'_{\mu'} + zh'_{\mu'} (\text{ad} E'_\mu)^{\nu', \varrho} E'_{\nu'}] = J(y\mu' + z\nu', x_{\mu, \nu} \mu + \nu)(\text{ad} E'_\mu)^{\nu', \varrho} E'_{\nu'},
\]

where \( \varrho \) is expressed by replacing \( a_p \) of \( f_t \) with \( b_p \). For \( x \in \mathbb{C}^n \), let \( b_x \) be the subspace of \( b \) spanned by the elements \((\text{ad}b_{p_1})\cdots(\text{ad}b_{p_{r-1}})b_{p_r}\) with \( x(p_1) + \cdots + x(p_r) = x \). Then \( b = \oplus_{x \in \mathbb{C}^n} b_x \). We can define a Lie superalgebra \( \mathcal{D} = b \oplus \mathcal{C} \) by \([b + \sum y_i k_i, b' + \sum y'_i k_i] = [b, b'] + (\sum y_i x'_i) b' - (\sum y'_i x_i) b\), where \( b \in b_x \) and \( b' \in b_{x'} \). We see that there exists an isomorphism \( \Phi: \mathcal{D} \to \mathcal{A} \) such that \( \Phi(b_p) = a_p \) and \( \Phi(k_i) = k_i \).
which implies \((\text{ad}E'_\mu)^{x_{\mu,\nu}}E'_\nu = 0\). Hence we have the (SR5). Similarly we have (SR6-9).

Finally we show the (SR1). For \(\alpha \in \Pi\), let

\[ h'_{a,\alpha} := \frac{J(\alpha,\alpha)c(\alpha)}{2k(\alpha)}(c(\alpha)h'_{(\alpha^*)\nu} - h'_{\alpha^\nu}). \]

It suffices to show that \(h'_{a,\alpha} = h'_{a,\beta}\) for all \(\alpha, \beta \in \Pi\). By (SR3-5), we see that for each \(\mu \in B\), \(E'_\mu\) is locally nilpotent, so \(n_\mu = n_{E'_\mu} \in \text{Aut}(b)\) can be defined in the same way as in (2.3). Let \((\alpha, \beta, y) \in A\). By the (SR6-7), we have \(n_\alpha E'_{\pm \beta} = n_\alpha E'_{\pm \beta^*}\). Hence \(n_\alpha h'_{\beta^\nu} = n_\alpha h'_{(\beta^*)\nu}\). Since \(n_\alpha h'_{\beta^\nu} = h'_{\beta^\nu} - J(\alpha^*,\beta^\nu) = h'_{\beta^\nu} - c(\alpha)J(\alpha,\beta^\nu)h'_{(\alpha^*)\nu}\) and \(n_\alpha h'_{(\beta^*)\nu} = h'_{(\beta^*)\nu} - J(\alpha,\beta^\nu)h'_{\alpha^\nu}\), we have \(h'_{a,\alpha} = h'_{a,\beta}\), as desired.

We see that the case where \(D = A_l^{(1,1)}\) and \(\mathfrak{g}_D = \mathfrak{g}_D^2\), with \(q \neq 1\) can also be treated similarly. \(\square\)

Keep the notation as above. Denote by \(\tilde{\pi}_D\) the homomorphism \(\pi_D : \mathfrak{g}_D \to \mathfrak{L}(\mathfrak{D}_D)\) in Lemma 4.2 if \(\mathfrak{g}_D = \mathfrak{g}_D\); and, otherwise, let \(\tilde{\pi}_D\) denote the homomorphism \(\pi_D^q : \mathfrak{g}_D^q \to M_l+1(\mathbb{C}_q)\). Let \(\varpi : \text{Im}\tilde{\pi}_D \to \text{Im}\tilde{\pi}_D/\pi_D(\mathfrak{ch}_3 \oplus \mathfrak{ch}_a)\) be the natural projection map. Define the epimorphism \(\tilde{\pi}_D : D^1(\mathfrak{g}_D) \to (\varpi \circ \tilde{\pi}_D)(\text{Der}(\mathfrak{g}_D))\) by \(\tilde{\pi}_D = (\varpi \circ \tilde{\pi}_D)|_{\text{Der}(\mathfrak{g}_D)}\). By Theorem 5.3 we see that the \(\tilde{\pi}_D\) is a UCE. In particular, we have the following.

**Corollary 5.2.** (1) The \(\tilde{\pi}_D\) is a UCE.

(2) Keep the notation as in Theorem 5.3. Then the \(\mathcal{F}|_{D^1(\mathfrak{g}_D)} : D^1(\mathfrak{g}_D) \to \mathcal{F}(D^1(\mathfrak{g}_D))\) is a UCE. (Notice that the \(\mathcal{F}(D^1(\mathfrak{g}_D))\) is the core of the EALA \(\mathcal{L}\) in the sense of [AABGP, Chap. I Definition 2.20].)

**Proof.** The statement (1) follows from the argument in the paragraph above the corollary. The statement (2) also follows from Theorem 5.3. \(\square\)

By Corollary 5.2 we see the following. If \(g(\alpha) = \emptyset\) for all \(\alpha \in \Pi\) and if the Cartan matrix \(A\) is \(X_l^{(1)}\) for some \(X = A, \ldots, G\), then \(\text{Der}(\mathfrak{g}_D)\) is isomorphic to the (2-variable) toroidal Lie algebra in the sense of [MRY]. If the \(A = A_l^{(1)}\), i.e., \(D = A_l^{(1,1)}\), then \(\text{Der}(\mathfrak{g}_D^q)\) is isomorphic to the Steinberg Lie algebra \(\text{st}_{l+1}(\mathbb{C}_q)\) (see [BGK] for the term and symbol).

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6 Elliptic root base

6.1 Weakening the condition of (SR5)

Here we show that some of the (SR5) are not necessary. Let $D = D(\mathcal{E}, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. Let $(\Pi \times \Pi)' := \{((\alpha, \beta) \in \Pi \times \Pi | \alpha \neq \beta\}$. Define a subset $(\Pi \times \Pi)^{\sharp}$ of $(\Pi \times \Pi)'$ by

$$(\Pi \times \Pi)^{\sharp} := \{((\alpha, \beta) \in (\Pi \times \Pi)' \mid k(\alpha) = k(\beta), J((\alpha^*)^\vee, \beta) = -1\}
\cup \{((\alpha, \beta) \in (\Pi \times \Pi)' \mid k(\alpha) < k(\beta), J(\beta^\vee, \alpha) = -1\}.$$ 

For a subset $\mathcal{X}$ of $(B \times B)'$, let $PM(\mathcal{X}) := \{(\varepsilon_1 \mu, \varepsilon_2 \nu) \mid (\mu, \nu) \in \mathcal{X}, \varepsilon_1, \varepsilon_2 \in \{1, -1\}\}$. Define a subset $(B \times B)^{\sharp}$ of $(B \times B)'$ by

$$(B \times B)^{\sharp} := \{(\mu, \nu) \in (B \times B)' \mid J(\mu, \nu) = 0\}
\cup PM(\cup_{(\alpha, \beta) \in (\Pi \times \Pi)^{\sharp}} \{(\alpha, \beta), (\beta, \alpha), (\alpha^*, \beta), (\beta, \alpha^*), (\alpha, \beta^*)\})
\cup PM(\cup_{\alpha \in \Pi} \{(\alpha, \alpha^*), (\alpha^*, \alpha)\}).$$

Then we consider the following.

(SR5') $(\text{ad}E_\mu)^{\pm \nu} E_\nu = 0$ if $(\mu, \nu) \in (B \times B)^{\sharp}$.

**Theorem 6.1.** Let $D$ be a QEBS with $l \geq 2$. Let $g_{D}^{\sharp}$ be the Lie superalgebra defined by the same generators as in $(2.2)$ with the same parities as in $(2.2)$ and by the defining relations (SR1-4), (SR5'), (SR6-9). Define the homomorphism $\Theta : g_{D}^{\sharp} \rightarrow g_D$ by $\Theta(h_{\sigma}) = h_{\sigma}$ $(\sigma \in \mathcal{E})$ and $\Theta(E_\mu) = E_\mu$ $(\mu \in B)$. Then $\Theta$ is an isomorphism.

**Proof.** Let $(\alpha, \beta) \in (\Pi \times \Pi)^{\sharp}$ and let $T := \{\alpha, \alpha^*, \beta\}$. Let $(g_D^{\sharp})^{(T)}$ be the sub Lie superalgebra of $g_D^{\sharp}$ generated by the $h_{\sigma}$ $(\sigma \in \mathcal{E})$ and $E_\mu$ $(\mu \in T \cup -T)$. Then we see that for each $\mu \in T \cup -T$, the $E_\mu$ is a locally nilpotent element of $(g_D^{\sharp})^{(T)}$, so $n_{\mu} \in \text{Aut}((g_D^{\sharp})^{(T)})$ can be defined in the same way as in $(2.3)$. By (SR6-7), we have $E_{\pm \beta^*} = n_{\alpha}^{-1}n_{\alpha}E_{\pm \beta^*} = n_{\alpha}^{-1}n_{\alpha^*}E_{\pm \beta} \in (g_D^{\sharp})^{(T)}$. It follows from (SR5') and (SR6-7) that

$$(6.1) \quad [E_{\pm \alpha}, [E_{\pm \alpha^*}, E_{\pm \beta}]] = (\text{ad}E_{\pm \alpha})^{c(\alpha)+1}E_{\pm \beta^*} = 0 \quad \text{if } k(\alpha) = k(\beta).$$

By (SR5'), we have

$$(6.2) \quad [E_{\pm \alpha}, E_{\pm \beta^*}] = 0.$$
It follows from (6.1) and (SR1-4,5′,8-9), that

\begin{equation}
\begin{aligned}
[E_{\pm \alpha^*}, E_{\pm \beta^*}] & \sim n_\alpha^{-1}([\text{ad}_{E_{\pm \alpha}}] J(\alpha^*, \alpha^*) E_{\pm \alpha^*}, (\text{ad}_{E_{\pm \alpha}}) - J((\alpha^*)^\vee, \beta^*) E_{\pm \beta^*}] \\
& \sim n_\alpha^{-1}(\text{ad}_{E_{\pm \alpha}}) J(\alpha^*, \alpha^*)(\text{ad}_{E_{\pm \alpha}}) - J((\alpha^*)^\vee, \beta^*)^{-1} E_{\pm \beta^*} = 0.
\end{aligned}
\end{equation}

Since the $E_{\pm \alpha^*}$ and $E_{\pm \beta^*}$ are locally nilpotent in $(\mathfrak{g}_R)^{(T)}$, by (6.2) and (6.3), we see that $\text{ad}_{E_{\mu}} E_{\nu} = 0$ for $(\mu, \nu) \in PM(\{(\beta^*, \alpha), (\alpha^*, \beta), (\beta^*, \alpha^*)\})$. This completes the proof. \(\square\)

We notice that there are also redundant equalities in (SR5′), which can be seen by the calculations in [Ya2, §2.3] and in Proof of Lemma 3.3. We also notice that the condition of the equality $[E_{\alpha^*}, E_{-\beta^*}] = 0$ of [Ya2, (S4)] can be weakened to be the one that $\alpha, \beta \in \Pi_{af}$ with $I(\alpha, \beta) = 0$, which can be seen the same calculations as in (6.3).

### 6.2 Relations for the elliptic root basis

Let $D = D(\mathcal{E}, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. Here by extending the notion given in [S], we introduce an elliptic root basis of $R(k, g)$ for the $D$. Recall the $\delta \in \mathbb{Z}_+ \Pi$ from Subsec. 2.2. For $\alpha \in \Pi$, let $x_\alpha \in \mathbb{Z}_+ (\alpha \in \Pi)$ be the coefficient of $\delta$, i.e., $\delta = \sum_{\alpha \in \Pi} x_\alpha \alpha$. Let $m_\alpha := \frac{c(\alpha) I(\alpha, \alpha)}{\delta(\alpha)}$. Let $m_{\max} := \max\{m_\alpha | \alpha \in \Pi\}$ and $\Pi_{\max} := \{\alpha | m_\alpha = m_{\max}\}$. For a subset $S$ of $\Pi$, let $S^* := \{\alpha^* | \alpha \in S\}$. Let $\Gamma(R, G) := \Pi \cup \Pi_{\max}$ (where $R$ and $G$ denote $R(k, g)$ and $\mathbb{C} a$ respectively). We call the $\Gamma(R, G)$ the elliptic root basis of $R(k, g)$. For a subset $S$ of $\Pi$, let $\Gamma(R, G; S) := \Gamma(R, G) \cap (S \cup S^*)$. Recall the Lie superalgebra $\mathfrak{g}_D$ from Subsec. 2.1.

**Theorem 6.2.** Let $D$ be a QEBS with $l \geq 2$. Let $\mathfrak{g}_D^{\Gamma(R, G)}$ be the Lie superalgebra defined by the generators

\begin{equation}
h_\sigma (\sigma \in \mathcal{E}), E_\mu, E_{-\mu} (\mu \in \Gamma(R, G))
\end{equation}

with the same parities as (6.2) and the following defining relations.

\begin{enumerate}
\item[(TSRi)] $(1 \leq i \leq 4, i = 5′, 6 \leq i \leq 9)$ The same relations as the ones among those in (SRi) expressed only by the same symbols as in (6.4).
\item[(TSR10)] $[E_{\pm \alpha^*}, [E_{\pm \alpha}, E_{\pm \beta}]] = 0,$ if $\Gamma(R, G; \{\alpha, \beta\}) = \{\alpha, \alpha^*, \beta\}$ and $\frac{J(\alpha^*, \beta)}{J(\beta^*, \alpha)} = c(\alpha),$
\end{enumerate}

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that the elements $\xi = \xi_1, \xi_2, \xi_3, \xi_4$ are the same as the ones in (SR6-7). Then we see that the $\xi$ is surjective since $\xi(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2$.

We only need to check that the equalities in (SR5-6) are satisfied. This can be shown as follows. (See Subsec. 3.2 for the notation $\sim$)

where we assume the $\alpha, \beta$, $\gamma$ are distinct. Then there exists a unique isomorphism $\Xi : D^{\Gamma(R,G)} \rightarrow gb^D$ such that $\Xi(h_\sigma) = h_\sigma$ and $\Xi(E_\mu) = E_\mu$ ($\mu \in B^{\Gamma(R,G)}$), where we set $B^{\Gamma(R,G)} := \Gamma(R,G) \cup -\Gamma(R,G)$.

Proof. If the $D$ is an SQEBS, then the theorem can be proved by the same argument as that for [Ya2, Theorem 4.1] (See also the last paragraph in Subsec. 6.1).

Assume the $D$ not to be an SQEBS. Then, for each $\beta \in \Pi \setminus \Pi_{\max}$, there exists a unique $\alpha_\beta \in \Pi \setminus \{\beta\}$ such that $J(\beta, \alpha_\beta) \neq 0$, and we see that $\alpha_\beta \in \Pi_{\max}$ and $J(\beta, \alpha_\beta) = 1$ or 2. By Theorems 2.1 and 6.1 we see that the homomorphism $\Xi$ in the statement exists. By the same argument as in the proof of Theorem 3.1 we see that for every $(\mu, \nu) \in ((B \times B') \cap (B^{\Gamma(R,G)} \times B^{\Gamma(R,G)}))$, the equality $(adE_\mu) x_{\mu, \nu} E_\nu = 0$ holds in $g^D_\Gamma$. Hence the elements $E_\mu (\mu \in B^{\Gamma(R,G)})$ of $g^D_\Gamma$ are locally nilpotent, so we can define $n_\mu = n_{E_\mu} \in \text{Aut}(g^D_\Gamma)$ in the same way as in (2.3). For each $\beta \in \Pi \setminus \Pi_{\max}$, let $E_{\pm \beta^*} := n_{\alpha_\beta}^{-1} n_{\alpha_{\beta}^*} E_{\pm \beta} \in B^{\Gamma(R,G)}$, where we notice that the $E_{\pm \beta^*}$ are also locally nilpotent and that the equalities $n_{\alpha_\beta} E_{\pm \beta^*} = n_{\alpha_\beta}^* E_{\pm \beta}$ are the same as the ones in (SR6-7). Then we see that the $\Xi$ is surjective since $\Xi(E_\mu) = E_\mu$ for all $\mu \in B$. To show that the $\Xi$ is injective, it suffices to show that the elements $h_\sigma (\sigma \in E)$, $E_\mu (\mu \in B)$ of $g^D_\Gamma$ satisfy the equalities in (SR1-4, 5, 6-9). As mentioned above, the equalities in (SR6-7) are satisfied. We only need to check that the equalities in (SR5) are satisfied. This can be shown as follows. (See Subsec. 3.2 for the notation $\sim$.)

1. Assume that $S = \{\alpha, \beta\} \subset \Pi$ with $J((\alpha^*)^\gamma, \beta) = -1$ and $\Gamma(R,G; S) = \{\alpha, \alpha^*, \beta\}$. Notice that $\alpha_\beta = \alpha$. Then we have

$$[E_{\pm \alpha}, E_{\pm \beta}] \sim n_{\beta}^{-1} [E_{\pm \alpha}, n_{\alpha^*} E_{\pm \beta}]$$

$$\sim n_{\alpha}^{-1} [E_{\pm \alpha}, [E_{\pm \alpha^*}, E_{\pm \beta}]] = 0 \text{ (by (TSR10-11))},$$
and

\[ [E_{\pm \alpha}, E_{\pm \alpha^*}] \sim n_{\alpha}^{-1}[(\text{ad} E_{\pm \alpha})c(\alpha) E_{\pm \beta}, [E_{\pm \alpha^*}, E_{\pm \beta}]] = n_{\alpha}^{-1}[(\text{ad} E_{\pm \alpha})c(\alpha)^{-1}([E_{\pm \alpha}, E_{\pm \beta}], [E_{\pm \alpha^*}, E_{\pm \beta}]) (\text{by } \text{TSR10-11})]
\]

\[ \sim \begin{cases} 0 (\text{by } \text{TSR11}) & \text{if } c(\alpha) = 1, \\ n_{\alpha}^{-1}[E_{\pm \alpha}, n_{\beta}[E_{\pm \alpha}, [E_{\pm \alpha^*}, E_{\pm \beta}]] = 0 (\text{by } \text{TSR10}) & \text{if } c(\alpha) = 2, \end{cases} \]

Since the \( E_{\pm \mu} \)'s (\( \mu \in S \cup S^* \)) are locally nilpotent, they also satisfy the equalities in (SR5') other than the above ones.

(2) Assume that \( S = \{\alpha, \beta\} \subset \Pi \) with \( J(\beta^\vee, \alpha) = -1, J(\alpha^\vee, \beta) = -2, c(\alpha) = 1 \) and \( \Gamma(R, G; S) = \{\alpha, \alpha^*, \beta\} \). Notice that \( \alpha \beta = \alpha \) and \( 2k(\alpha) = k(\beta) \).

Then, by (TSR8-9), we have

\[ [E_{\mp \alpha}, E_{\pm \beta^*}] \sim n_{\alpha}^{-1}[E_{\mp \alpha}, [E_{\pm \alpha^*}, [E_{\pm \alpha^*}, E_{\pm \beta}]]] = 0, \]

and

\[ [E_{\pm \beta}, E_{\pm \beta^*}] \sim n_{\alpha}^{-1}[(\text{ad} E_{\pm \alpha})^2 E_{\pm \beta}, (\text{ad} E_{\pm \alpha^*})^2 E_{\pm \beta}] = n_{\alpha}^{-1}[(\text{ad} E_{\pm \alpha})^2 E_{\pm \beta}, E_{\pm \beta}] = n_{\alpha}^{-1}[(\text{ad} E_{\pm \alpha})^2 E_{\pm \beta}, E_{\mp \alpha}, E_{\pm \beta}] = 0. \]

Then we can use the same argument as in (1).

(3) Assume that \( S = \{\alpha, \beta, \gamma\} \subset \Pi \) with \( J(\beta^\vee, \alpha) < 0, J(\gamma^\vee, \beta) < 0, J(\alpha^\vee, \gamma) = 0 \), and assume that \( \Gamma(R, G; S) = \{\alpha, \beta, \beta^*, \gamma\} \) or \( \{\alpha, \alpha^*, \beta, \beta^*, \gamma\} \). Then we can use the same argument as in [Ya2 (2)-(3) of Proof of Proposition 4.2].

\[ \Box \]

**Appendix**

As mentioned in the text, especially in Theorem 1.1, K. Saito [S] (see also [SY]) introduced the notion of the ERS, and showed that every RMERS is realized as the \( R(k, 0) \) for some SEBS \( D(E, \Pi, a, k, 0) \). However, there exists a reduced ERS which is not realized as an RMERS. As mentioned in Introduction, the authors of [AABGP] introduced the notion of extended affine root systems (EARS for short), which is different from the reduced SEARS's introduced in [S]. It is known (see [A]) that there exists a natural
one-to-one correspondence between the reduced SEARS’s and the EARS’s. Here we also use the terminology and notation in [AABGP] Construction 2.32 and Theorem 2.37. By Theorems 1.2 and 2.2 we see that if an EARS has the nullity equal to two and the rank equal to or more than 2, the corresponding reduced ERS is realized as $R(k, g)$ for some QEBS $D = D(E, \Pi, a, k, g)$ with $g(\alpha) = \emptyset$ or $2\mathbb{Z} + 1$ ($\alpha \in \Pi$). Let $D$ be a QEBS with $l \geq 2$ such that the name of $A(= A_H)$ is $D_{l+1}^{(2)}$. Notice that there exist $\varepsilon_i \in E$ $(1 \leq i \leq l)$ such that $J(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $\alpha_0 = \delta - \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(2 \leq i \leq l - 1)$ and $\alpha_l = \varepsilon_l$. Then the corresponding EARS is $R(X, S, L, E)$ is such that

$$X = \begin{cases} B_l & \text{if } g(\alpha_0) = g(\alpha_l) = \emptyset \\ BC_l & \text{otherwise}, \end{cases}$$

and $S = ((2\mathbb{Z} + 1)\delta + \mathbb{Z}k(\alpha_0)a) \cup (2\mathbb{Z}\delta + \mathbb{Z}k(\alpha_l)a)$, $L = 2\mathbb{Z}\delta + \mathbb{Z}k(\alpha_i)a$ $(2 \leq i \leq l - 1)$ and $E = ((4\mathbb{Z} + 2)\delta + g(\alpha_0)a) \cup (4\mathbb{Z}\delta + g(\alpha_l)a)$. Here if $g(\alpha_0) = \emptyset$, then $(4\mathbb{Z} + 2)\delta + g(\alpha_0)a = \emptyset$; if $g(\alpha_l) = \emptyset$, then $(4\mathbb{Z} + 2)\delta + g(\alpha_l)a = \emptyset$. (Strictly speaking, in [AABGP], $R(X, S, L, \emptyset)$ is denoted as $R(X, S, L)$).

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