BOUNDARY SINGULARITIES FOR WEAK SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial \Omega$. We construct positive weak solutions of the problem $\Delta u + u^p = 0$ in $\Omega$, which vanish in suitable trace sense on $\partial \Omega$, but which are singular at prescribed single points if $p$ is equal or slightly above $\frac{N+1}{N-1}$. Similar constructions are carried out for solutions which are singular on any given embedded submanifold of $\partial \Omega$ of dimension $0 \leq k \leq N - 2$, if $p$ equals or it is slightly above $\frac{N+k}{N-k-1}$, and even on countable families of these objects, dense on a given closed set. The role of this exponent, first discovered by Brezis and Turner [1] for boundary regularity when $p < \frac{N+1}{N-1}$, parallels that of $p = \frac{N}{N-2}$ for interior singularities.

1. Introduction and statement of main results

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial \Omega$. A model of nonlinear elliptic boundary value problem is the classical Lane-Emden-Fowler equation,

\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega \\
u > 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  

(1.1)

where $p > 1$. We are interested in finding solutions to this problem which are smooth in $\Omega$ and equal to 0 almost everywhere on $\partial \Omega$ with respect to surface measure. More precisely, we want to study solutions to problem (1.1) that satisfy the boundary condition in a suitable trace sense, while not necessarily in a continuous fashion.

Following Brezis & Turner [1] and Quittner & Souplet [11], we say that a positive function $u \in C^\infty(\Omega)$ is a very weak solution of problem (1.1) if

$$u, \text{ } u^p \text{dist}(x, \partial \Omega) \in L^1(\Omega)$$

and

$$\int_{\Omega} (u \Delta v + u^p v) \, dx = 0 \quad \text{for all } v \in C^2(\bar{\Omega}) \text{ with } v = 0 \text{ on } \partial \Omega.$$  

From the results in [11], it follows that if $p$ satisfies the constraint

$$1 < p < \frac{N + 1}{N - 1}$$  

(1.2)
then a very weak solution $u$ is actually in $H^1_0(\Omega)$, and it is a weak solution in the usual variational sense:

$$u \in H^1_0(\Omega), \quad \int_{\Omega} (\nabla u \nabla v - u^p v) \, dx = 0 \quad \text{for all } v \in H^1_0(\Omega).$$

Elliptic regularity then yields $u \in C^2(\overline{\Omega})$, so that $u$ solves (1.1) classically. As it is well-known, a constrained minimization procedure involving Sobolev’s embedding implies existence of a weak-variational solution to (1.1) for $1 < p < \frac{N+2}{N-2}$. A natural question is then whether very weak solutions of (1.1) are classical within a broader range of exponents than (1.2). Partially answering this question negatively, Souplet [12] constructed an example of a positive function $a \in L^\infty(\Omega)$ such that Problem (1.1), with $u^p$ replaced by $a(x)u^p$ for $p > \frac{N+1}{N-1}$, has a very weak solution which is unbounded, developing a point singularity on the boundary.

The exponent $p = \frac{N+1}{N-1}$ is thus critical in what concerns to boundary regularity for very weak solutions. The aim of this paper is to construct solutions to Problem (1.1) with prescribed singularities on the boundary. To state an important special case of our main results we need a definition:

**Definition 1.1.** Let $u(x)$ be a function defined in $\Omega$ and $x_0 \in \partial \Omega$. We say that $u(x) \to \ell$ as $x \to x_0$ non-tangentially if

$$\lim_{x \to x_0, \Gamma_\alpha(x_0) \not\ni x} u(x) = \ell \quad \text{for all } \alpha \in \left[0, \frac{\pi}{2}\right),$$

where $\Gamma_\alpha(x_0)$ denotes the cone with vertex $\xi_i$, and angle $\alpha$ with respect to its axis, the inner normal to $\partial \Omega$ at $x_0$.

We have the validity of the following result.

**Theorem 1.1.** There exists a number $p_N > \frac{N+1}{N-1}$ such that if $p$ satisfies

$$\frac{N+1}{N-1} \leq p < p_N,$$

then the following holds: given points $\xi_1, \xi_2, \ldots, \xi_k \in \partial \Omega$, there exists a very weak solution $u$ to problem (1.1) such that $u \in C^2(\Omega \setminus \{\xi_1, \ldots, \xi_k\})$ and

$$u(x) \to +\infty \quad \text{as } x \to \xi_i \text{ non-tangentially, for all } i = 1, \ldots, k.$$

The study of the behavior near an isolated boundary singularity of any positive solution of (1.1) when the exponent $p \geq \frac{N+1}{N-1}$ was recently achieved by Bidaut-Véron-Ponce-Véron in [3].
1.1. **The parallel with \( p = \frac{N+1}{N-2} \) and interior singularities.** The role of the exponent \( p = \frac{N+1}{N-2} \) parallels that of \( p = \frac{N}{N-2} \) for solutions to problem (1.1) with interior singularities. Let us recall that if \( u \in L^p(\Omega) \) is a positive distributional solution of (1.1) and \( 1 < p < \frac{N}{N-2} \), then \( u \) is smooth in \( \Omega \). On the other hand, for \( p \geq \frac{N}{N-2} \), distributional solutions of (1.1) with prescribed interior singularities are built in [7, 9, 10, 8, 5, 6]. Basic cells in those constructions are radially symmetric singular solutions \( u = u(|x|) \) for the equation
\[
\Delta u + u^p = 0.
\]
(1.3)

Whenever \( p > \frac{N}{N-2} \), the function
\[
u_0(|x|) = c_{p,N} |x|^{-\frac{p}{p-1}}, \quad c_{p,N} = \left[ -\frac{2}{p-1} \right]^{\frac{1}{p-1}},
\]
(1.4)
is a explicit singular solution of (1.3) in \( \mathbb{R}^N \setminus \{0\} \). If, in addition, \( \frac{N}{N-2} < p < \frac{N+2}{N-2} \), phase plane analysis for the ODE corresponding to radial solutions of (1.3), yields existence of a singular positive solution \( u_1 \) which connects the behavior of \( u_0 \) near the origin with fast decay at infinity,
\[
u_1(|x|) = c_{p,N} |x|^{-\frac{2}{p-1}}(1 + o(1)) \quad \text{as } x \to 0,
\]
(1.5)
\[
u_1(|x|) = |x|^{-(N-2)/(1 + o(1))} \quad \text{as } |x| \to +\infty,
\]
(1.6)
(note that \( N - 2 > \frac{2}{p-1} \)). The scalings \( u_\lambda(r) = \lambda^{\frac{2}{p-1}} u_1(\lambda r) \) with \( \lambda > 0 \) are then solutions of (1.3) that have the same behavior near the origin but which become very small as \( \lambda \to 0^+ \) on any compact subset of \( \mathbb{R}^N \setminus \{0\} \). Thus, given points
\[\xi_1, \xi_2, \ldots, \xi_k \in \Omega,\]
the function
\[
u_*(x) = \sum_{i=1}^k u_\lambda(|x - \xi_i|)
\]
constitutes a “good approximation” for small \( \lambda > 0 \) to a singular solution of Problem (1.1). Linear theory and perturbation arguments lead to establish the presence of an actual solution to (1.1) near \( \nu_* \), see [8]. When \( p = \frac{N}{N-2} \), a similar construction can be carried out, see [10]. Basic cell \( u_1 \) corresponds in this case to a positive radial solution \( u_1 \) of equation (1.3) in \( B(0,1) \) with
\[
u_1(|x|) = c_N |x|^{-(N-2)} \log(1/|x|)^{-\frac{N-2}{2}} \left(1 + o(1)\right) \quad \text{as } x \to 0.
\]
(1.7)
In this case the scalings \( u_\lambda(x) = \lambda^{\frac{N-2}{2}} u_1(\lambda x) \) have the same behavior as \( u_1 \) at the origin, and they approach zero as \( \lambda \to 0^+ \), uniformly on compact subsets of \( \mathbb{R}^N \setminus \{0\} \).

1.2. **The basic cells: singular solutions on a half-space.** In the construction of the solutions predicted by Theorem 1.1 we will follow a scheme similar to that described above for interior singularities. Basic cells will now be positive solutions of equation (1.3) defined on the half-space,
\[
\mathbb{R}^+_N := \{x = (x_1, \ldots, x_N) / x_N > 0\}
\]
which vanish on its boundary, with a singularity at the origin. Such solutions are of course not radial, and ODE analysis does not apply. Thus, we consider the following two problems:

\[
\begin{aligned}
\Delta u + u^p &= 0 \quad \text{in } \mathbb{R}^N_+ \setminus \{0\} \\
u &> 0 \quad \text{in } \mathbb{R}^N_+ \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+ \setminus \{0\},
\end{aligned}
\]

(1.8)

for \(p > \frac{N+1}{N-1}\), and

\[
\begin{aligned}
\Delta u + u^{\frac{N+1}{N-1}} &= 0 \quad \text{in } B_+ \\
u &> 0 \quad \text{in } B_+ \\
u &= 0 \quad \text{on } \partial \mathbb{R}^N_+ \cap \overline{B}_+ \setminus \{0\},
\end{aligned}
\]

(1.9)

where \(B_+ = \mathbb{R}^N_+ \cap B(0,1)\).

Our purpose is to find families of solutions \(u_\lambda\) with analogous behavior to the radial singular ones previously described. Let us consider first the case \(p > \frac{N+1}{N-1}\). The role of the explicit radial solution \(u_0\) in (1.4) is now played by one found by separation of variables: Let us denote by \(S_+^{N-1}: = \{r = (\theta_1, \ldots, \theta_N) \in S^{N-1} / \theta_N > 0\}\) the half sphere. Looking for a solution of problem (1.8) of the form

\[
u_0(x) = r^{-\frac{N+1}{p-1}} \phi_p(\theta_N), \quad r = |x|, \quad \theta = \frac{x}{|x|},
\]

(1.10)

we arrive at the problem on the half sphere,

\[
\begin{aligned}
(\Delta_{S^{N-1}} + N - 1) \phi_p - \frac{p-1}{p-1} (N - \frac{N+1}{p-1}) \phi_p + \phi_p^p &= 0 \quad \text{in } S_+^{N-1} \\
\phi_p &> 0 \quad \text{in } S_+^{N-1} \\
\phi_p &= 0 \quad \text{on } \partial S_+^{N-1}.
\end{aligned}
\]

(1.11)

Here \(\Delta_{S^{N-1}}\) designates the Laplace-Beltrami operator in \(S^{N-1}\). Since \(N - 1\) is the first eigenvalue of \(-\Delta_{S^{N-1}}\) under Dirichlet boundary conditions, with eigenfunction \(\theta_N\), in the considered range \(N - \frac{N+1}{p-1} > 0\), an application of the mountain pass lemma yields existence of a solution to this problem, provided that, additionally, \(p\) is subcritical in dimension \(N - 1\), namely \(p < \frac{N+1}{N-1}\). When \(p\) tends from above to \(\frac{N+1}{N-1}\), this solution ceases to exist by uniform vanishing. Alternatively, in this regime, a standard application of Crandall-Rabinowitz local bifurcation theorem yields that this solution defines a continuous branch in \(p\) with asymptotic behavior

\[
\phi_p(\theta_N) = c_N (N - \frac{p+1}{p-1})^{\frac{p-1}{p}} \theta_N (1 + o(1)), \quad \text{as } p \downarrow \frac{N+1}{N-1}.
\]

(1.12)

Nevertheless, the function \(u_0\) does not suffice for the construction of approximate profiles for those of Theorem [17] since it is “too large” at infinity. We need an analogue of the radial function \(u_1\) in (1.5)-(1.6), namely one that behaves like \(u_0\) near the origin but having fast decay. A “connection”
between $u_0$ with Poisson’s kernel $x_N/|x|^N$ does indeed exist provided that $p$ is sufficiently close to $\frac{N+1}{N-k}$, as the following result states.

**Proposition 1.1.** There exists a number $p_N > \frac{N+1}{N-k}$, such that for all $\frac{N+1}{N-k} < p < p_N$, there exists a solution $u_1(x)$ to problem (1.1) such that

$$u_1(x) = \frac{1}{x} \phi_p(x_N/|x|) \left(1 + o(1)\right) \quad \text{as } x \to 0,$$

where $\phi_p$ solves (1.11), and

$$u_1(x) = |x|^{-N} x_N \left(1 + o(1)\right) \quad \text{as } |x| \to +\infty.$$

This solution has indeed “fast decay” since $N - 1 > \frac{2}{p-1}$. Observe then that the scalings $u_λ(x) = λ^{\frac{1}{p}} u_1(λx)$ define a family of solutions to Problem (1.8) which have a common, $λ$-independent behavior at the origin, but which vanishes uniformly as $λ \to 0$, on compact subsets of $\mathbb{R}_+^N \setminus \{0\}$.

When $p = \frac{N+1}{N-k}$ there is no solution to problem (1.11) and thus separation of variables fails. On the other hand, we have an exact analogue of the radial solutions $u_1$ in (1.7), as described by the following result.

**Proposition 1.2.** There exists a solution $u_1$ of Problem (1.9) such that

$$u_1(x) = c_N |x|^{-N} \log(1/|x|) \frac{1}{x_N} \left(1 + o(1)\right) \quad \text{as } x \to 0.$$

We observe that in this case the functions $u_λ(x) = λ^{N-1} u_1(λx)$ satisfy that $u_λ(x) \to 0$ uniformly as $λ \to 0^+$ on compact subsets of $\mathbb{R}_+^N \setminus \{0\}$.

### 1.3. Solutions with prescribed singular set: general statements

In reality, the profiles given by the above results can also be used to approximate solutions to Problem (1.1) whose singular set is a $k$ dimensional submanifold of $\partial Ω$ with $1 ≤ k ≤ N - 2$. For instance, if $u_1(x')$, $x' \in \mathbb{R}_+^{N-k}$ is the solution of (1.3) given by Proposition 1.1 for $p$ close from above to $\frac{N-k+1}{N-k-1}$, then $\tilde{u}(x) = u_1(x')$ solves the same problem in $\mathbb{R}_+^N$, now with singular set given by a $k$-dimensional subspace. This is the content of the following result, more general than Theorem 1.1, whose analogue for interior singularities was found in [10, 8].

**Theorem 1.2.** Let $0 ≤ k ≤ N - 2$ and let $p_{N-k}$ be the number given by Proposition 1.1 with $N$ replaced by $N - k$. Given $p$ with

$$\frac{N-k+1}{N-k-1} ≤ p < p_{N-k}$$

and a $k$-dimensional submanifold $S$ embedded in $\partial Ω$, there exist infinitely many (very) weak solutions to problem (1.7) such that $u \in C^2(\overline{Ω} \setminus S)$, and

$$u(x) \to +\infty \quad \text{as } x \to x_0 \text{ non-tangentially, for all } x_0 \in S.$$
When $k = 0$, we agree that $S$ is a finite set of isolated points, so that Theorem 1.1 is recovered. In reality, the solutions found arise as continua, depending on as many real parameters as number of points lie in $S$. When $k \geq 1$, the solutions we construct are infinite dimensional families. The construction actually allows much more: For instance, when $p = \frac{N+1}{N-1}$, the number of points of the singular set can be taken to infinity, to total a dense subset of any given closed set $A$ of $\partial \Omega$, which can be properly called its singular set. In fact, since the solutions we are interested in are smooth in $\Omega$, it is natural to define the singular set of a very weak solution $u$ of (1.1) as the complement in $\partial \Omega$ of the set of points $x \in \partial \Omega$ in a neighborhood of which $u$ is smooth. Observe that, by definition, the singular set of $u$ is a closed subset of $\partial \Omega$. We have the validity of the following general result.

**Theorem 1.3.** Let $0 \leq k \leq N - 2$, and $\frac{N-k+1}{N-2-k} \leq p < p_{N-k}$. Let us consider a nonempty closed subset $A$ of $\partial \Omega$, which contains a sequence of $k$-dimensional embedded submanifolds $S_i$, $i \in \mathbb{N}$, which are also disjoint and satisfy that $S := \cup_i S_i$ is dense in $A$. Then, there exists a positive very weak solution of Problem (1.1) whose singular set is exactly $A$, and such that

$$u(x) \to +\infty \quad \text{as} \quad x \to x_0 \text{ non-tangentially, for all } x_0 \in S,$$

and

$$u(x) \to 0 \quad \text{as} \quad x \to x_0 \text{ non-tangentially, for all } x_0 \in \partial \Omega \setminus S.$$
2.1. Proof of Proposition 1.2. When \( p = \frac{N+1}{N} \), in the language of equation (2.1), problem (1.9) becomes
\[
\begin{cases}
\partial_t^2 \phi + N \partial_t \phi + (\Delta_{SN-1} + N - 1) \phi + \phi^{\frac{N+1}{N-1}} = 0 & \text{in} \ (t_*, \infty) \times S^{N-1}_+ \\
\phi > 0 & \text{in} \ (t_*, \infty) \times S^{N-1}_+ \\
\phi = 0 & \text{on} \ (t_*, \infty) \times \partial S^{N-1}_+.
\end{cases}
\] (2.2)

We allow here \( t_* > 0 \) to be a parameter, which we will choose later to be large. To get a solution of problem (1.9) we actually need \( t_* = 0 \), but this is simply achieved by a translation of \( \phi \) in the \( t \)-variable.

The idea is now to look for a solution of this equation of the form
\[
\phi(t, \theta) = a_N t^{-b_N} \varphi_1(\theta) + \psi(t, \theta),
\] (2.3)
where \( a_N, b_N \) are positive constants to be fixed below, and \( \varphi_1 \) denotes the eigenfunction of \( -\Delta_{S^{N-1}_+} \) associated to the eigenvalue \( N - 1 \) and normalized so that its \( L^2 \)-norm is equal to 1. Explicitly,
\[
\varphi_1(\theta) = \frac{\theta_N}{\left( \int_{S^{N-1}_+} \theta_N^2 d\sigma \right)^{1/2}}.
\]

When substituting the function \( \phi = a_N t^{-b_N} \varphi_1(\theta) \) as an approximation for a solution of equation (2.2), we see that for large \( t \), the main order term in the error created is the function
\[
E(t, \theta) := Na_N b_N t^{-b_N - 1} \varphi_1(\theta) - |a_N t^{-b_N} \varphi_1(\theta)|^{\frac{N+1}{N-1}}.
\]

We make the following choice for the numbers \( a_N \) and \( b_N \):
\[
b_N = \frac{N-1}{2}, \quad a_N = \left[ \frac{2}{N(N-1)} \int_{S^{N-1}_+} \varphi_1^{2N} d\sigma \right]^{-\frac{N-1}{2}}.
\]

This election achieves the \( L^2 \)-orthogonality of \( E \) to \( \varphi_1 \) for all \( t \), namely
\[
\int_{S^{N-1}_+} E(t, \cdot) \varphi_1 d\sigma = 0 \quad \text{for all} \ t > t_*.
\]

We fix these values in what follows. In terms of \( \psi \) in (2.3), equation (2.2) now reads
\[
\partial_t^2 \psi + N \partial_t \psi + (\Delta_{S^{N-1}_+} + N - 1) \psi + b_N (b_N + 1)a_N t^{-b_N - 2} \varphi_1 - Nb_N a_N t^{-b_N - 1} \varphi_1 + |a_N t^{-b_N} \varphi_1 + \psi|^{\frac{N+1}{N-1}} = 0,
\]
\[
\psi = 0 \quad \text{on} \ (t_*, \infty) \times \partial S^{N-1}_+.
\]

We further decompose
\[
\psi(t, \theta) = f_2(t) \varphi_1(\theta) + \psi_1(t, \theta)
\] (2.4)
where \( \psi_1(t, \theta) \) satisfies
\[
\int_{\mathbb{S}_{+}^{N-1}} \psi_1(t, \cdot) \varphi_1 \, d\sigma = 0 \quad \text{for all } t > t_*,
\]
The equation we have to solve then reduces to the coupled system in \((\psi_1, f_2)\) given by
\[
\begin{align*}
(\partial_t^2 + N \partial_t + (\Delta_{S^{N-1}} + N - 1)) \psi_1 &= N_1(\psi_1, f_2) \\
(\partial_t^2 + N \partial_t + \frac{N(N-1)}{2}) f_2 &= N_2(\psi_1, f_2),
\end{align*}
\]
where
\[
N_1(\psi_1, f_2) = \left( \frac{N(N-1)}{2} a_N \varphi_1 - a_N \frac{N+1}{2} \varphi_1 \right) t^{-\frac{N+1}{2}} - \Pi^{\|} \left( a_N \frac{t^{\frac{N}{2}-1} \varphi_1 + \psi_1 + f_2 \varphi_1}{N} - a_N \frac{t^{\frac{N}{2}}}{} \varphi_1 \right),
\]
\[
N_2(\psi_1, f_2) = \frac{N^2-1}{2} a_N t^{-\frac{N+1}{2}} - \int_{\mathbb{S}_{+}^{N-1}} \left( a_N \frac{t^{\frac{N}{2}-1} \varphi_1 + \psi_1 + f_2 \varphi_1}{N} - a_N \frac{t^{\frac{N}{2}}}{} \varphi_1 \right) \psi_1 \, d\sigma.
\]
Here \( \Pi^{\perp} \) denotes the \( L^2 \)-orthogonal projection over the orthogonal complement to \( \varphi_1 \), namely
\[
\Pi^{\perp}(h) = h(t, \theta) - \varphi_1(\theta) \int_{\mathbb{S}_{+}^{N-1}} h(t, \cdot) \varphi_1 \, d\sigma,
\]
and \( \psi \) is given by \((2.3)\). The logic in the resolution of problem \((2.5)\) is simple: we look for a solution \((\psi_1, f_2)\) which is small compared with \( t^{\frac{N}{2}} \varphi_1 \). We will construct inverses to the linear operators defined by the left hand sides of the equations in \((2.5)\) with suitable bounds that allow, for sufficiently large \( t_* \), the resolution of the system via contraction mapping principle. Observe that so far we have not imposed boundary conditions at \( t = t_* \). We will invert the linear operator in \( \psi_1 \), for right hand sides \( L^2 \)-orthogonal to \( \varphi_1 \) for all \( t \), imposing Dirichlet boundary condition at \( t = t_* \). The choice of inverse for the ODE operator in \( f_2 \) will be basically explicit, and will not require imposing boundary conditions. The natural environment to carry out these inversions is \( L^\infty \)-weighted spaces. In the next two lemmas we construct these inverses. Thus we consider the linear problems
\[
\begin{align*}
(\partial_t^2 + N \partial_t + (\Delta_{S^{N-1}} + N - 1)) \psi &= h \quad \text{in } (t_*, \infty) \times \mathbb{S}_{+}^{N-1}, \\
\psi &= 0 \quad \text{on } \partial((t_*, \infty) \times \mathbb{S}^{N-1}),
\end{align*}
\]
for \( h \) such that
\[
\int_{\mathbb{S}_{+}^{N-1}} h(t, \cdot) \varphi_1 \, d\sigma = 0 \quad \text{for all } t > t_*,
\]
and
\[
\left(\partial_t^2 + N \partial_t + \frac{N(N-1)}{2} t \right) f = g \quad \text{in} \quad (t_*, \infty).
\] (2.9)

We have the validity of the following results.

Lemma 2.1. There exists a constant \( c > 0 \) such that the following holds: Given \( \sigma \geq 0 \), there is a \( t_\sigma > 0 \), with \( t_\sigma > 0 \) if \( \sigma > 0 \) and \( t_\sigma = 0 \) if \( \sigma = 0 \), such that, for all \( t_* \geq t_\sigma \), and all \( h \in C^0((t_*, \infty) \times S^{N-1}_\sigma) \) that satisfies (2.8) and \( t^\sigma h \in L^\infty((t_*, \infty) \times S^{N-1}_\sigma) \), there exists a solution \( \psi = T_1(h) \) of problem (2.7), which defines a linear operator of \( h \) and satisfies the estimate
\[
\|t^\sigma \psi\|_{L^\infty} + \|t^\sigma \nabla t_0 \psi\|_{L^\infty} \leq c \|t^\sigma h\|_{L^\infty}.
\]

Lemma 2.2. Given \( \sigma > \frac{N-1}{2} \), there exist numbers \( t_\sigma, c_\sigma > 0 \) such that for all \( t_* > t_\sigma \) and all \( g \in C^0((t_*, \infty)) \) satisfying \( t^\sigma g \in L^\infty((t_*, \infty)) \), there exists a solution \( f = T_2(g) \) of equation (2.9), which defines a linear operator of \( g \) and satisfies the estimate
\[
\|t^\sigma f\|_{L^\infty} \leq c_\sigma \|t^{1+\sigma} g\|_{L^\infty}.
\]

Before proceeding into the proofs of these lemmas, let us conclude the result.

Conclusion of the proof of Proposition 1.2 Let us fix in the above lemmas any number \( \sigma \) such that
\[
\frac{N-1}{2} < \sigma < \frac{N+1}{2}
\]
and \( t_* > t_\sigma \). We obtain a solution of problem (2.9) if \( (\psi_1, f_2) \) solves the fixed point problem
\[
(\psi_1, f_2) = \mathcal{M}(\psi_1, f_2) := (T_1(N_1(\psi_1, f_2)), T_2(N_2(\psi_1, f_2))),
\] (2.10)
in the space of functions
\[
(\psi, f) \in C^0([t_*, \infty) \times S^{N-1}_\sigma) \times C^0([t_*, \infty))
\]
for which the norm
\[
\|\psi, f\|_\mu = \|t^\sigma \psi\|_{L^\infty} + \mu \|t^\sigma f\|_{L^\infty}
\]
is finite. Here \( \mu < 1 \) is a positive number which we will fix later. \( T_1, T_2 \) are the operators predicted by Lemmas 2.1 and 2.2. It is directly checked that we have the pointwise estimates
\[
\begin{align*}
|N_1(\psi_1, f_2)| & \leq A \left[ t^{\frac{N-\sigma}{2}} + t^{-1}|\psi_1| + t^{-1}|f_2| \right], \\
|N_2(\psi_1, f_2)| & \leq A \left[ t^{\frac{N-\sigma}{2}} + t^{-1}|\psi_1| + t^{\frac{N-\sigma}{2}}|f_2|^2 + |f_2|^{\frac{N+1}{2}} \right],
\end{align*}
\] (2.11)
where \( A \) depends only on \( N \), whenever \( \|\psi, f\|_\mu \leq \mu \). It follows that,
\[
\begin{align*}
\|t^\sigma N_1(\psi_1, f_2)\|_{L^\infty} & \leq A \left[ t^{\sigma - \frac{N-\sigma}{2}} + t^{1-\sigma}\|t^\sigma \psi_1\|_{L^\infty} + t^{-1}\|t^\sigma f_2\|_{L^\infty} \right], \\
\|t^{1+\sigma} N_2(\psi_1, f_2)\|_{L^\infty} & \leq A \left[ t^{\sigma - \frac{N-\sigma}{2}} + \|t^\sigma \psi_1\|_{L^\infty} + (t^{1-\frac{N-\sigma}{2}} + t^{1-\frac{N-\sigma}{2}})\|t^\sigma f_2\|_{L^\infty} \right].
\end{align*}
\] (2.12)
These estimates, together with Lemmas 2.1 and 2.2 yield that if \( \mu \) is chosen sufficiently small, depending only on \( \sigma \) and \( N \), and \( t^* \) is taken sufficiently large, then the operator \( \mathcal{M} \) applies the ball \( \|\psi, f\|_\mu \leq \mu \) into itself. A similar estimates shows that, also, \( \mathcal{M} \) is a contraction mapping.
with this norm inside this region. Hence there is a fixed point \((\psi_1, f_2)\) in this ball. The solution obtained this way renders the function

\[
\phi(t, \theta) = a_N t^{-\frac{N-1}{2}} \varphi_1(\theta) + f_2(t) \varphi_1(\theta) + \psi(t, \theta)
\]

positive in \((t_*, +\infty) \times S_{\pm}^{N-1}\), and it is then a solution of problem (2.12). This completes the proof of Proposition 1.2. \(\square\)

Next we carry out the proofs of the lemmas.

**Proof of Lemma 2.1.** Let us consider first the case \(\sigma = 0\), so that \(h\) is bounded. With no loss of generality, we also assume \(t_* = 0\). We see then that problem (2.7) has at most one bounded solution. This can be shown for instance expanding a bounded solution of the equation with \(h = 0\) in eigenfunctions of the Laplace-Beltrami operator with zero boundary conditions on \(S_{\pm}^{N-1}\). The coefficients in this expansion will be functions of \(t\) which correspond to bounded solution of certain homogeneous ODE’s which only have the zero solution. Thus, we only have to prove existence. To do so, let us consider, for any given number \(t_2 > 0\), the problem

\[
\begin{align*}
(\partial^2_t + N \partial_t + (\Delta_{S_{\pm}^{N-1}} + N - 1)) \psi &= h \quad \text{in } (0, t_2) \times S_{\pm}^{N-1} \\
\psi &= 0 \quad \text{on } \partial ((0, t_2) \times S_{\pm}^{N-1}).
\end{align*}
\]

(2.13)

This problem is uniquely solvable since it is just a rephrasing of a Dirichlet problem for the Laplacian in a half-annular region. Let us denote by \(\psi = \psi_{t_2}\) its unique solution. Since, by assumption, \(h(t, \cdot)\) is \(L^2\)-orthogonal to \(\varphi_1\) for all \(t \in (0, t_2)\), so is \(\psi\).

It suffices to check that there exists a constant \(c > 0\) independent of \(t_2 \geq 1\) such that

\[
\|\psi\|_{L^\infty([0, t_2] \times S_{\pm}^{N-1})} \leq c \|h\|_{L^\infty((0, t_2) \times S_{\pm}^{N-1})}.
\]

(2.14)

Indeed, assuming this estimate is already proven, we use elliptic estimates together with Ascoli’s theorem to show that, as \(t_2\) tends to \(\infty\), the sequence of functions \(\psi_{t_2}\) converges uniformly to a function \(\psi\) solution of (2.7) which satisfies

\[
\|\psi\|_{L^\infty((0, \infty) \times S_{\pm}^{N-1})} \leq c \|h\|_{L^\infty((0, \infty) \times S_{\pm}^{N-1})}.
\]

Elliptic estimates then imply that

\[
\|\nabla \psi\|_{L^\infty((0, \infty) \times S_{\pm}^{N-1})} + \|\psi\|_{L^\infty((0, \infty) \times S_{\pm}^{N-1})} \leq c_0 \|h\|_{L^\infty((0, \infty) \times S_{\pm}^{N-1})}.
\]

(2.15)

The orthogonality conditions on \(\psi\) pass certainly to the limit, and existence of a solution with the desired properties thus follows. It remains to prove the uniform estimate (2.14). We argue by contradiction. Since the result is certainly true when \(t_2\) remains bounded, we assume that there exists a sequence \(t_2 = t_{2,i}\) tending to \(\infty\), functions \(h = h_i\) and \(\psi_i\) corresponding solutions to problem (2.13) for which

\[
\|\psi_i\|_{L^\infty([0, t_{2,i}] \times S_{\pm}^{N-1})} = 1 \quad \text{and} \quad \lim_{i \to \infty} \|h_i\|_{L^\infty((0, t_{2,i}) \times S_{\pm}^{N-1})} = 0.
\]

We choose \(t_i \in (0, t_{2,i})\) where \(\|\psi_i\|_{L^\infty([t_{1,i}, t_{2,i}] \times S_{\pm}^{N-1})}\) is achieved and define

\[
\tilde{\psi}_i(t, \theta) = \psi_i(t + t_i, \theta)
\]
Using elliptic estimates together with Ascoli’s theorem, we can extract from \((\tilde{\psi}_i)\) some subsequence which converges uniformly on compact sets to \(\tilde{\psi}\), a bounded solution of

\[
(\partial_t^2 + N \partial_t + (\Delta_{S^{N-1}} + N - 1)) \tilde{\psi} = 0 \tag{2.16}
\]

which is either defined on \([0, \infty) \times S^{N-1}_+\), on \((-\infty, 0] \times S^{N-1}_+\) or on \((-\infty, \infty) \times S^{N-1}_+\). Furthermore,

\[
\|\tilde{\psi}\|_{L^\infty} = 1 \tag{2.17}
\]

with \(\tilde{\psi}\) having 0 boundary data. Furthermore \(\tilde{\psi}(t, \cdot)\) is \(L^2\)-orthogonal to \(\varphi_1\), for all \(t\). Eigenfunction decomposition of \(\tilde{\psi}(t, \cdot)\) for the Laplace-Beltrami operator yields that there is no nontrivial bounded solution of (2.16) and this contradicts (2.17). This completes the proof of the uniform estimate, and thus existence of a unique bounded solution of (2.7) with the desired estimate follows. This solution of course defines a linear operator on bounded \(h\).

To establish the result for \(\sigma > 0\) and \(t_* > 0\) sufficiently large, let us write

\[
h = t^{-\sigma} \tilde{h} \quad \text{and} \quad \psi = t^{-\sigma} \tilde{\psi}
\]

so that \(\tilde{h}\) is bounded. (2.17) reduces to

\[
(\partial_t^2 + N \partial_t + (\Delta_{S^{N-1}} + N - 1)) \tilde{\psi} + \left(\frac{\sigma(s+1)}{s}\right) \tilde{\psi} - \frac{2\sigma}{t} \partial_t \tilde{\psi} = \tilde{h} \tag{2.18}
\]

We can estimate

\[
\left\| \left(\frac{\sigma(s+1)}{s}\right) \tilde{\psi} - \frac{2\sigma}{t} \partial_t \tilde{\psi} \right\|_{L^\infty((t_*, \infty) \times S^{N-1}_+)} \leq \mu \left( \left\| \nabla \tilde{\psi} \right\|_{L^\infty((t_*, \infty) \times S^{N-1}_+)} + \left\| \tilde{\psi} \right\|_{L^\infty((t_*, \infty) \times S^{N-1}_+)} \right)
\]

where \(\mu\) can be taken as small as we wish, after choosing \(t_* \geq t_\sigma\) with \(t_\sigma\) large enough. The resolution of (2.18) with the desired bound then follows from that of (2.7) with \(\sigma = 0\) together with a direct linear perturbation argument. This finishes the proof. \(\square\)

**Proof of Lemma 2.2.** Observe that a right inverse for the operator \(\partial_t^2 + N \partial_t\) on \([t_*, \infty)\) is given by

\[
G(g)(t) = -\int_{t_*}^t e^{-N\zeta} \int_{t_*}^t e^{Ns} g(s) ds \, dt
\]

One checks that

\[
\|t^\sigma G(g)\|_{L^\infty((t_*, +\infty))} \leq \frac{1}{N|\sigma|} \left( 1 - \frac{(\sigma+1)}{Nt_*} \right)^{-1} \|t^{1+\sigma} g\|_{L^\infty((t_*, +\infty))}
\]

provided \(Nt_* - 1 - \sigma > 0\). This follows at once from the computation

\[
\int_{t_*}^t e^{Ns} s^{-\sigma-1} \, ds = \left[ \frac{1}{N} e^{Ns} s^{-\sigma-1} \right]_{t_*}^t + \frac{\sigma+1}{Nt_*} \int_{t_*}^t e^{Ns} s^{-\sigma-2} \, ds \leq \frac{1}{N} e^{Nt} t^{-\sigma-1} + \frac{1+\sigma}{Nt_*} \int_{t_*}^t e^{Ns} s^{-\sigma-1} \, ds
\]
and hence
\[
\left(1 - \frac{\sigma + 1}{N-1}\right) \int_{t_\ast}^t e^{Ns}s^{-\sigma-1} ds \leq \frac{1}{N} e^{Nt} t^{-\sigma-1}.
\]
The result of the lemma then follows from a simple linear perturbation argument, provided \(t_\ast > 0\)
is chosen so that
\[
\frac{N(N-1)}{2} \left\| t^\sigma \frac{1}{t} G(g) \right\|_{L^\infty((t_\ast, +\infty))} \leq \frac{1}{2} \left\| t^{1+\sigma} g \right\|_{L^\infty((t_\ast, +\infty))},
\]
and the result is concluded.

\[\square\]

2.2. Proof of Proposition 1.1. Recall that, when \(p \in \left(\frac{N-1}{N+1}, \frac{N-1}{N}\right)\) the Mountain Pass Lemma
yields the existence of \(\phi_p\), a nontrivial positive solution of (1.11). This solution then induces a
solution \(u_p(x) := |x|^{-\frac{N}{p+1}} \phi_p(x/|x|)\),
of Problem (1.8), for which this time we emphasize its dependence on \(p\). We have to show that
there exists a solution of (1.8) which is asymptotic to \(u_p\) near 0 and it is asymptotic to
\(u_\infty(x) := |x|^{-N} x_N\)
at infinity. Let us consider a smooth cut-off function \(\chi\) which is equal to 0 in \(B_1(0)\) and it is
identically equal to 1 in \(\mathbb{R}^N \setminus B_2(0)\). We will consider the function \(u_p(1-\chi)\) as a first approximation
for the solution we are looking for. Since, we recall, \(\phi_p\) approaches zero uniformly as \(p \downarrow \frac{N+1}{N-1}\),
then the same is true for \(u_{0,p}\) away from the origin. The result of the proposition relies on a
perturbation procedure, and this is the reason why we can only show the for exponents \(p\) close
to \(\frac{N-1}{N}\). To carry out this scheme, we shall build a right inverse for the Laplacian relative to the
following doubly weighted space:

**Definition 2.1.** Given \(\delta, \delta' \in \mathbb{R}\), the space \(L^\infty_{\delta, \delta'}(\mathbb{R}_+^N)\) is defined to be the space of functions
\(u \in L^\infty_{\text{loc}}(\mathbb{R}_+^N)\) for which the following norm
\[
\left\| u \right\|_{L^\infty_{\delta, \delta'}(\mathbb{R}_+^N)} = |||x|^{-\delta} u|||_{L^\infty(B_+(1))} + |||x|^{-\delta'} u|||_{L^\infty(\mathbb{R}_+^N \setminus B_+(1))}
\]
is finite.

Hence \(\delta\) controls the behavior of the function near 0 and \(\delta'\) the behavior of the function near
infinity. Let us consider the problem
\[
\begin{align*}
\Delta u &= |x|^{-2} f & \text{in } \mathbb{R}_+^N \\
u &= 0 & \text{on } \partial \mathbb{R}_+^N \setminus \{0\}.
\end{align*}
\]
(2.19)

We have the validity of the following result.

**Lemma 2.3.** Let \(\delta \in (1-N, 1)\) and \(\delta' \in (-N, 1-N)\) be given. There is a constant \(c > 0\) such
that for each \(f \in L^\infty_{\delta, \delta'}(\mathbb{R}_+^N)\), there exists a solution \(u = G(f)\) of problem (2.19), which defines a
linear operator in \(f\) and can be decomposed as
\[
u = \hat{u} + a\chi u_\infty
\]
Let us observe that \( \delta (N - 2 + \delta) < N - 1 \) precisely when \( \delta \in (1 - N, 1) \). Therefore, we can define \( \varphi_* = \varphi_{N, \delta} \) to be the unique, positive solution of

\[
- (\Delta_{N-1} + \delta (\delta + N - 2)) \varphi_* = 1 \quad \text{in } S^N_{+1}
\]

\[
\varphi_* = 0 \quad \text{on } \partial S^N_{+1}.
\]

A direct computation shows that

\[
- |x|^2 \Delta_R (|x|^\delta \varphi_*) = c|x|^\delta.
\] (2.20)

Assume that \( f \in L^\infty_{\text{loc}} (\mathbb{R}^N_f) \). Given \( r_1 < 1 < r_2 \), we can solve the equation \(|x|^2 \Delta u = f\) in \( \mathbb{R}^N_+ \cap (B_{r_2} - B_{r_1}) \), with 0 boundary conditions. We use the function \( x \mapsto \varphi_* (x) |x|^\delta \) as a barrier to prove the pointwise estimate

\[
|u| \leq c \| f \|_{L^\infty_{\text{loc}} (\mathbb{R}^N_+)} |x|^\delta
\]

in \( \mathbb{R}^N_+ \cap (B_{r_2} - B_{r_1}) \). Furthermore, given \( \delta \in (1 - N, 1) \) we can use the function \( x \mapsto \varphi_* (x) |x|^\delta \) as a barrier to prove the estimate

\[
|u| \leq c \| f \|_{L^\infty_{\text{loc}} (\mathbb{R}^N_+)} |x|^\delta
\]

in \( \mathbb{R}^N_+ \cap (B_{r_2} - B_{r_1}) \).

We use elliptic regularity theory as well as Ascoli’s theorem to pass to the limit as \( r_1 \) tends to 0 and \( r_2 \) tends to \( \infty \). We obtain a solution \( u \) of problem (2.19) which satisfies the pointwise estimates

\[
|u| \leq c \| f \|_{L^\infty_{\text{loc}} (\mathbb{R}^N_+)} |x|^\delta
\]

in \( \mathbb{R}^N_+ \cap B_1 \) \( \setminus \{0\} \), and

\[
|u| \leq c \| f \|_{L^\infty_{\text{loc}} (\mathbb{R}^N_+)} |x|^\bar{\delta}
\]

in \( \mathbb{R}^N_+ \setminus \mathbb{R}^N - B_1 \). Finally, the decomposition of the solution \( u \) at infinity into

\[
u = \hat{u} + a u_{\infty}
\]

where the function \( \hat{u} \) satisfies

\[
|\hat{u}| \leq c \| f \|_{L^\infty_{\text{loc}} (\mathbb{R}^N_+)} |x|^\delta
\]

follows easily from Green’s representation formula. Moreover, we can directly compute the value of \( a \). Indeed, integration of the equation over \( B^*_r := \mathbb{R}^N_+ \cap B_r \) yields, for \( r \) large enough,

\[
\int_{B^*_r} f \, |x|^{-2} \, dx = \int_{\partial B^*_r} \partial_r u \, d\sigma = \int_{S^N_{+1}} (\partial_r \hat{u}) (r \theta) r^{N-1} \, d\sigma - a (N - 1) \int_{S^N_{+1}} \theta_N \, d\sigma
\]

Passing to the limit as \( r \) tends to \( \infty \), we obtain the identity

\[
a (N - 1) \int_{S^N_{+1}} \theta_N \, d\sigma = - \int_{\mathbb{R}^N_+} f \, |x|^{-2} \, dx,
\] (2.21)
and the proof is concluded. □

**Conclusion of the proof of Proposition 1.1.** To find a solution of problem (1.8), we write

$$ u = (1 - \chi) \bar{u}_p + v, $$

where $\chi$ is a smooth cut-off function which is equal to 0 in $B_1$ and identically equal to 0 in $\mathbb{R}^N - B_2$. Let us fix numbers $\delta \in (1 - N, 1)$, $\delta' \in (-N, 1 - N)$ and let $G$ be the operator defined in Lemma 2.3. Then, we obtain a solution with the required properties if $u$ is singular and positive near 0. We now prove that

$$ v = - G \left( |x|^2 \left( \Delta (1 - \chi) \bar{u}_p + |(1 - \chi) \bar{u}_p + v|^p \right) \right) $$

in the space $L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\}) \oplus \text{Span} \{ \chi u_\infty \})$, and $(1 - \chi)u_p + v > 0$.

Let us observe that there exists a constant $c_0 = c(N, \delta, \delta') > 0$ such that

$$ \| |x|^2 \left( \Delta (1 - \chi) \bar{u}_p + |(1 - \chi) \bar{u}_p|^p \right) \|_{L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\})} \leq c_0 \| \phi_p \|_{C^2(S^N_+ - 1)}. $$

Indeed, since $\Delta \bar{u}_p + \bar{u}_p = 0$ we have

$$ \Delta((1 - \chi) \bar{u}_p) + ((1 - \chi) \bar{u}_p)^p = - \Delta \chi \bar{u}_p - 2 \nabla \chi \cdot \nabla \bar{u}_p + (1 - \chi - (1 - \chi)^p) \Delta \bar{u}_p $$

and the estimate follows at once.

On the other hand, if we assume that $p$ is sufficiently close to $\frac{N+1}{N-1}$ from above, we have that

$$ \| \phi_p \|_{C^2(S^N_+ - 1)} \leq 1, $$

and also

$$ \delta > - \frac{2}{p-1} \quad \text{and} \quad \delta' > p(1 - N) + 2. $$

Under these constraints, it is not hard to check the existence of a constant $c = c(N, \delta, \delta') > 0$ such that

$$ \| |x|^2 ((1 - \chi) \bar{u}_p + v_2|^p - |(1 - \chi) \bar{u}_p + v_1|^p) \|_{L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\})} \leq c \| \phi_p \|_{C^2(S^N_+ - 1)} \| v_2 - v_1 \|_{L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\}) \oplus \text{Span} \{ \chi u_\infty \})} $$

for all $v_2, v_1 \in L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\}) \oplus \text{Span} \{ \chi u_\infty \})$ satisfying

$$ \| v_1 \|_{L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\}) \oplus \text{Span} \{ \chi u_\infty \})} \leq 2 c_0 \| \phi_p \|_{C^2(S^N_+ - 1)}. $$

Using estimates (2.23), (2.24) and Lemma 2.3, the existence of a solution to the fixed point problem (2.22) can then be obtained by contraction mapping principle in the ball of radius $2 c_0 \| \phi_p \|_{C^2(S^N_+ - 1)}$ in the space $L_{\delta, \delta'}^\infty((\mathbb{R}^N_+ - \{0\}) \oplus \text{Span} \{ \chi u_\infty \})$, provided that $p$ is chosen larger than (but close enough to) $\frac{N+1}{N-1}$. Let us denote by $v_p$ this fixed point.

Since $\delta > 1 - N$, we have $|v_p| << u_p$ near 0 and hence the solution $u := (1 - \chi) \bar{u}_p + v_p$ is singular and positive near 0. We now prove that $u := (1 - \chi) \bar{u}_p + v_p$ is also positive at infinity. Indeed, the function $v_p$ can be written as

$$ v_p = \tilde{v}_p + a_p \chi u_\infty $$
where, according to formula (2.21), \( a_p \) can be computed as
\[
a_p (N - 1) \int_{S^N} \theta_N \, ds = \int_{\mathbb{R}^N_+} (\Delta (1 - \chi) \bar{u}_p + |1 - \chi| \bar{u}_p + v_p|^p) \, dx = \int_{\mathbb{R}^N_+} |1 - \chi| \bar{u}_p + v_p|^p \, dx > 0.
\]
This implies that \( a_p > 0 \), and by the maximum principle, it is now easy to check that \( u > 0 \) in \( \mathbb{R}^N_+ - \{0\} \). This completes the proof of Proposition 1.1.

2.3. Some open questions. The results of proof of Propositions 1.1 and 1.2 and their parallel with the radial case and \( p \) close to \( \frac{N+1}{N-3} \), to which we recall ODE phase plane analysis applies, lead us naturally to several questions concerning existence of solutions of \( \Delta u + u^p = 0 \) on the punctured half space \( \mathbb{R}^N_+ - \{0\} \) with 0 boundary data. We list some of them next.

**Question 1.** We believe that the solution \( u_1 \) which has been obtained in Proposition 1.1 for \( p \) close to \( \frac{N+1}{N-3} \) should actually exist for all \( p \in (\frac{N+1}{N-1}, \frac{N+2}{N-2}) \).

**Question 2.** When \( p = \frac{N+2}{N-2} \), we believe that there exists a one parameter family of solutions of the form
\[
u(x) = |x|^{\frac{2-N}{2}} v(-\log |x|, \theta)
\]
where \( t \mapsto v(t, \cdot) \) is periodic. This one-parameter family of solution corresponds to the well know periodic solutions for the singular Yamabe problem and also to De launay surfaces in the context of constant mean curvature surfaces.

**Question 3.** When \( p > \frac{N+1}{N-2}, \, N \geq 3 \), we believe that there exists a solution of \( \Delta u + u^p = 0 \) defined on \( \mathbb{R}^N_+ \) which is identically equal to 0 on \( \partial \mathbb{R}^N_+ \) and which is asymptotic to \( u_0 \) in (1.10) at \( \infty \). This solution should correspond to the smooth radially symmetric solution of the same equation which is defined on the whole space and decays like \( |x|^{-\frac{2}{p+2}} \) at infinity, when \( p > \frac{N+2}{N-2} \).

**Question 4.** Are there singular solutions when \( p \geq \frac{N+1}{N-3}, \, N \geq 4 \)? In this regime separation of variables in general fails.

Some partial answer to this question is given in [3].

3. The bounded domain case: proofs of Theorems 1.2 and 1.3

The proof of our main results relies on two basic ingredients: one is the, already established, existence of the “basic cells” given by Propositions 1.1 and 1.2 which we will use to construct approximations to singular solutions. Another important ingredient, on which we elaborate in the next two subsections, is the analysis of invertibility of Laplace’s operator, for right hand sides exhibiting a controlled singular behavior on a given embedded submanifold of \( \partial \Omega \), in the same spirit to that of Lemma 2.3. Then we will use a fixed point scheme analogous to that in the proof of Proposition 1.1.

For notational convenience, we will assume in what follows that \( \Omega \) is actually a subset of \( \mathbb{R}^N \), and that \( S \) is a smooth embedded submanifold of \( \partial \Omega \subset \mathbb{R}^N \) with dimension \( k \). We define
\[
N = n + k.
\]
We start by setting up a suitable description of the space and Laplacian operator in natural coordinates associated to $S$. While the analysis below is done for $k \geq 1$, it applies equally well to the point-singularity case $k = 0$, being actually simpler.

3.1. **Local coordinate system.** In a neighborhood of a point $\bar{p}$ of $S$ let us choose coordinates $y_1, \ldots, y_k$ on $S$. Next we choose sections $E_1, \ldots, E_{n-1}$ of the normal bundle of $S$ in $\partial \Omega$. We can define Fermi coordinates in some tubular neighborhood of $S$ in $\partial \Omega$ by using the exponential map,

$$F(p, (x_1, \ldots, x_{n-1})) = \text{Exp}_{\bar{p}}(\sum_i x_i E_i(p))$$

for $p$ in a neighborhood of $\bar{p} \in S$ and $(x_1, \ldots, x_{n-1})$ in some neighborhood of 0 in $\mathbb{R}^{n-1}$.

In these coordinates, it is well known that the induced metric $g_b$ on $\partial \Omega$ can be expanded as

$$g_b = g_{\mathbb{R}^{n-1}} + g_S + O(|x|)$$

where $g_S$ denotes the induced metric on $S$ and $x = (x_1, \ldots, x_{n-1})$.

Finally, to parameterize a neighborhood of a point of $\partial \Omega$ in $\Omega$, we denote by $E_n$ the normal (inward pointing) vector field about $\partial \Omega$ and again use the exponential map to define

$$G(q, x_n) = q + x_n E_n(q)$$

for $q$ in a neighborhood of $\bar{p}$ in $\partial \Omega$ and $x_n \geq 0$ in some neighborhood of 0.

In these coordinates, it is well known that the Euclidean metric in $\Omega$ can be expanded as

$$g_{\mathbb{R}^{n+k}} = dx_1^2 + g_b + O(x_n).$$

Collecting these two expansions, we conclude that in these coordinates the (Euclidean) Laplacian can be expanded as

$$\Delta = \Delta_{NS} + O(|x|) \nabla^2 + O(1) \nabla$$

where $x = (x_1, \ldots, x_n)$ and $\Delta_{NS} = \Delta_{\mathbb{R}^n} + \Delta_S$ is the Laplace-Betrami operator on $NS$, the normal bundle of $S$ in $\mathbb{R}^n$.

3.2. **Analysis of the Laplacian in weighted spaces.** We want to prove a result in the same spirit as that of Lemma 2.3 in the current setting. To do this, we need to define weighted spaces on $\Omega \setminus S$, which have a controlled blow up rate as $S$ is approached. Unlike those in Lemma 2.3, we choose Hölder spaces, which are more suitable to deal with linear perturbations which are second order operators. Let us define, for sufficiently small $R > 0$, half “balls” and “annuli”

$$B_+(R) := \{(p, x) \in NS_+ / |x| \in (0, R)\}$$

and

$$A_+(R_1, R_2) := \{(p, x) \in NS_+ / |x| \in [R_1, R_2]\}$$

$B_+(R)$ is roughly “half” of a tubular neighborhood of radius $R$ of the manifold $S$, or just a ball in case that $S$ reduces to a single points. We consider the following weighted space of functions defined on $B_+(R) \setminus S$. 
Definition 3.1. The space \( \mathcal{C}_d^{\ell,\alpha}(\bar{B}_+(R) \setminus S) \) is the space of functions \( u \in \mathcal{C}_d^{\ell,\alpha}(\bar{B}_+(R) \setminus S) \) for which the norm
\[
\|u\|_{\mathcal{C}_d^{\ell,\alpha}(\bar{B}_+(R) \setminus S)} = \sup_{r \in (0,R)} r^{-\delta} \|u(\cdot, r \cdot)\|_{\mathcal{C}_d^{\ell,\alpha}(\bar{A}_+(r/2, r))}
\]
is finite.

We consider now the problem
\[
\begin{align*}
\Delta_{NS} u &= |x|^{-2} f \quad \text{in } \bar{B}_+(R) \setminus S \\
u &= 0 \quad \text{on } \partial \bar{B}_+(R) \setminus S.
\end{align*}
\]
(3.2)

We have the validity of the following result.

Lemma 3.1. Assume that \( \delta \in (1-n, 1) \). There exists a constant \( c > 0 \), independent of \( R > 0 \), such that, for each \( f \in \mathcal{C}_d^{0,\alpha}(\bar{B}_+(R) \setminus S) \), there is a solution \( u = G_{\delta,R}(f) \) of problem (3.2) which defines a linear operator of \( f \) and satisfies the estimate
\[
\|u\|_{\mathcal{C}_d^{2,\alpha}(\bar{B}_+(R) \setminus S)} \leq c \|f\|_{\mathcal{C}_d^{0,\alpha}(\bar{B}_+(R) \setminus S)}.
\]

Proof. We only carry out the proof for \( R = 1 \) since the general case follows by scaling. First we solve for each \( r \in (0,1/2) \) the problem
\[
\begin{align*}
\Delta_{NS} u &= |x|^{-2} f \quad \text{in } A_+(r, 1) \\
u &= 0 \quad \text{on } \partial A_+(r, 1)
\end{align*}
\]
(3.3)
and call \( u_r \) its unique solution. Maximum principle employed in a similar way as in Lemma 2.3, taking into account expansion (3.1), yield the a priori bound
\[
|u_r| \leq c \|f\|_{\mathcal{C}_d^{0,\alpha}(\bar{B}_+(1) \setminus S)} |x|^\delta
\]
where \( c = c(n, \delta) > 0 \). Then, elliptic estimates applied on geodesic balls of radius \( r \) centered at distance \( 2r \) from \( S \) give the following bound on the gradient of \( u \)
\[
|\nabla u_r| \leq c \|f\|_{\mathcal{C}_d^{0,\alpha}(\bar{B}_+(1) \setminus S)} |x|^\delta - 1
\]
for some \( c = c(n, \delta) > 0 \). Using Arzela’s theorem, we conclude that, for a sequence of radii tending to 0, the sequence \( u_r \) converges to a function \( u \) which satisfies
\[
|u| \leq c \|f\|_{\mathcal{C}_d^{2,\alpha}(\bar{B}_+(1) \setminus S)} |x|^\delta
\]
and solves (3.2) for \( R = 1 \). Again, elliptic estimates applied on geodesic balls of radius \( r \) centered at distance \( 2r \) from \( S \) yield the bound
\[
\|u\|_{\mathcal{C}_d^{2,\alpha}(\bar{B}_+(1) \setminus S)} \leq c \|f\|_{\mathcal{C}_d^{2,\alpha}(\bar{B}_+(1) \setminus S)}
\]
for some constant \( c = c(n, \delta) > 0 \). Uniqueness of the limit \( u \) is easy to get and we leave it to the reader. The proof is concluded. \( \square \)
Next we will extend the previous result to the entire domain $\bar{\Omega} \setminus S$. To do so, we consider a function
\[ \gamma : \bar{\Omega} \setminus S \rightarrow (0, \infty) \]
smooth, positive, which in the above defined local coordinates coincides with $|x|$ in a neighborhood of $S$ in $\Omega$. This function will play the role of the function $|x|$ defined in $B_+(R) \setminus S$. We define accordingly weighted Hölder spaces as follows.

**Definition 3.2.** We let the space $C^{\ell,\alpha}_\delta(\bar{\Omega} \setminus S)$ be that of functions $u \in C^{\ell,\alpha}_{loc}(\bar{\Omega} \setminus S)$ for which the norm
\[ \|u\|_{C^{\ell,\alpha}_\delta(\bar{\Omega} \setminus S)} = \|u\|_{C^{\ell,\alpha}_\delta(B_+(R) \setminus S)} + \|u\|_{C^{\ell,\alpha}(\Omega \setminus B_+(R/2))} \]
is finite.

We consider now the problem
\[ \begin{align*}
\Delta u &= \gamma^{-2} f \quad \text{in } \Omega \setminus S \\
u &= 0 \quad \text{on } \partial\Omega \setminus S.
\end{align*} \tag{3.4} \]

We have the following result, extension of Lemma 3.1.

**Lemma 3.2.** Assume that $\delta \in (1 - n, 1)$. There exists a constant $c > 0$ such that, for each $f \in C^0(\bar{\Omega} \setminus S)$, there is a solution $u = G_\delta(f)$ of problem (3.4) which defines a linear operator of $f$ and satisfies the estimate
\[ \|u\|_{C^{2,\alpha}_\delta(\bar{\Omega} \setminus S)} \leq c \|f\|_{C^0(\bar{\Omega} \setminus S)}. \]

**Proof.** The proof follows from Lemma 3.1, expansion (3.1) and a linear perturbation argument. First, we claim that the result of Lemma 3.1 remains true in $\bar{B}_+(R) \setminus S$ if the operator $\Delta_{NS}$ is replaced by $\Delta$ and if $R$ is chosen small enough. Indeed, we have from (3.1) and Proposition 3.1
\[ \|f - \gamma^2 (\Delta - \Delta_{NS}) \circ G_{\delta, R}(f)\|_{C^{0,\alpha}_\delta(\bar{B}_+(R) \setminus S)} \leq c R \|f\|_{C^{0,\alpha}_\delta(\bar{B}_+(R) \setminus S)}. \]
The claim follows at once from a perturbation argument, provided that $R$ is fixed small enough.

We denote by $\tilde{G}_{\delta, R}$ the right inverse for $\Delta$ in $\bar{B}_+(R) \setminus S$. We consider a cut-off function $\chi_R$ which is equal to 1 in $B_+(R/2) \setminus S$ and equal to 0 in $\bar{\Omega} \setminus B_+(R)$. We define
\[ \tilde{f} := f - \gamma^2 \Delta(\chi_R u_1), \]
where $u_1 = \tilde{G}_{\delta, R}(f)$. Observe that this function is supported in $\bar{\Omega} \setminus B_+(R/2)$. We have that $\tilde{f} \in C^0(\bar{\Omega})$ and
\[ \|\tilde{f}\|_{C^{0,\alpha}(\bar{\Omega})} \leq c \|f\|_{C^{0,\alpha}(\bar{\Omega})} \]
for some constant $c = c(n, \delta, R) > 0$.

Finally, we can solve
\[ \begin{align*}
\Delta u_2 &= \gamma^{-2} \tilde{f} \quad \text{in } \Omega \\
u_2 &= 0 \quad \text{on } \partial\Omega.
\end{align*} \]
We have the bound
\[ \|u_2\|_{C^{2,\alpha}(\bar{\Omega})} \leq c\|f\|_{C^{0,\alpha}(\bar{\Omega}\setminus S)}. \]
The desired result then follows by letting the solution of (3.4) be \( u = u_1 + u_2 \).

\[ \square \]

### 3.3. Proof of Theorems 1.2 and 1.3

We are now in a position to provide the proof of Theorem 1.2 and Theorem 1.3. The argument goes along the same lines as that in the proof of Proposition 1.1 now with Lemma 3.2 playing the role of Lemma 2.3.

We recall that we are now assuming that \( \Omega \) is a domain in \( \mathbb{R}^N \). We also write \( N = n + k \).

**Proof of Theorem 1.2, case \( p = \frac{n-k}{n+k-1} \)**. We assume that \( S \) is either a finite number of points of \( \partial\Omega \), namely \( k = 0 \), or an embedded \( k \)-dimensional submanifold of \( \partial\Omega \). For all \( \varepsilon > 0 \) small enough, we define
\[ u_\varepsilon := \chi_R \varepsilon^{n-1} u_1(\varepsilon |x|, \varepsilon x_n) \]
where \( u_1 \) is the solution provided by Proposition 1.3 and \( \chi_R \) is a cut-off function which equals 1 in \( B_+(R) \setminus S \) and 0 in \( \Omega \setminus B_+(2R) \). Here we fix \( R > 0 \) sufficiently small and use for \( x \) and \( x_n \) the meanings given in the previous subsections. In particular we have that \( u_\varepsilon \equiv 0 \) on \( \partial\Omega \setminus S \).

The problem we want to solve then reads
\[ \Delta(u_\varepsilon + v) + |u_\varepsilon + v|^{\frac{n+k}{n+1}} = 0 \quad \text{in } \Omega, \]
\[ v = 0 \quad \text{on } \partial\Omega \setminus S, \]
where we also require \( u_\varepsilon + v > 0 \) in \( \Omega \). Let us fix \( \delta \in (1-n, 2-n] \). By virtue of Lemma 3.2 we can rewrite this equation as the fixed point problem
\[ v = -G_\delta \left( \gamma^2(\Delta u_\varepsilon + |u_\varepsilon + v|^{\frac{n+k}{n+1}}) \right). \]

We have the validity of the following fact: there is a constant \( c_0 = c(\delta, \Omega, S) > 0 \) such that
\[ \|\gamma^2(\Delta u_\varepsilon + u_\varepsilon^{\frac{n+k}{n+1}})\|_{C^{0,\alpha}(\bar{\Omega}\setminus S)} \leq c_0 (\log(1/\varepsilon))^{\frac{1-n}{2}}, \]
result that is a consequence of expansion (3.1) and a direct computation using the asymptotic properties of \( u_1 \) in Proposition 1.2.

We restrict our attention to the case where \( \delta \leq 2-n \) since \( \gamma^2(\Delta u_\varepsilon + u_\varepsilon^{\frac{n+k}{n+1}}) \) is bounded by a constant times \( |x|^{2-n} \) near \( S \), and \( \delta \leq 2-n \) guarantees that this function belongs to \( C^{0,\alpha}_\delta(\bar{\Omega}\setminus S) \).

A second estimate we can directly check is the following: Assume that \( \delta \in (1-n, 2-n] \) is fixed. There exists a constant \( c = c(\delta, \Omega, S) > 0 \) such that
\[ \|\gamma^2(|u_\varepsilon + v_2|^{\frac{n+k}{n+1}} - |u_\varepsilon + v_1|^{\frac{n+k}{n+1}})\|_{C^{0,\alpha}(\bar{\Omega}\setminus S)} \leq c (\log(1/\varepsilon))^{-1} \|v_2 - v_1\|_{C^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \]
for all \( v_2, v_1 \in C^{0,\alpha}_\delta(\bar{\Omega}\setminus S) \) satisfying
\[ \|v_i\|_{C^{0,\alpha}_\delta(\bar{\Omega}\setminus S)} \leq 2 c_0 (\log(1/\varepsilon))^{\frac{1-n}{2}}. \]
The above estimates allow an application of contraction mapping principle in the ball of radius 
2c0 \((\log(1/\varepsilon))^{1+\gamma}\) in \(C^{2,\sigma}(\Omega \setminus S)\) to predict existence of a solution to problem \((3.5)\), which we denote by \(v_\varepsilon\).

Since \(\delta > 1 - n\), we have \(|v_\varepsilon| << u_\varepsilon\) near \(S\) and hence the solution \(u := u_\varepsilon + v_\varepsilon\) is singular along \(S\) and is positive near \(S\). The maximum principle then implies that \(u > 0\) in \(\Omega\). This completes the proof of Theorem 1.2 in the case \(p = \frac{n+1}{n-1}\).

The proof of Theorem 1.3 case \(p = \frac{n+1}{n-1}\). This proof uses similar arguments together with an induction process. By assumption, \(A\) is closed and contains a sequence of \(k\)-dimensional submanifolds \(S_i, i \in \mathbb{N}\) such that \(\bigcup_j S_j\) is dense in \(A\). We define inductively the sequence of functions \(u_i\) which are solutions of

\[\Delta u_i + u_i^{\frac{n+1}{n-1}} = 0\]  

(3.6)
in \(\Omega\), satisfy \(u_i = 0\) on \(\partial \Omega \setminus \bigcup_j S_j\) and are singular along \(\bigcup_j S_j\). Assume for example that \(u_{i-1}\) has already been constructed, then, we define

\[u_i = u_{i-1} + \varepsilon_i^{n-1} \chi_{r_i+1} u_1 (\varepsilon_i \text{dist}(\cdot, S_i))\]

where \(u_1\) is the solution provided by Proposition 1.3, \(r_i\) is fixed small enough less than half the distance from \(S_i\) to \(\bigcup_j S_j\) and \(\varepsilon_i > 0\) is small enough. Applying a perturbation argument as above, we can perturb \(u_i\) into a solution \(u_i = \tilde{u}_i + v_i\) of \((3.6)\) for some function \(v_i \in C^{2,\sigma}(\Omega \setminus \bigcup_j S_j)\). Taking \(\varepsilon_i\) small enough, we can ensure that

\[\|u_i - u_{i-1}\|_{L^1(\Omega)} \leq 2^{-i}\]  

(3.7)

and

\[\|\text{dist}(\cdot, \partial \Omega)^2 |u_i - u_{i-1}|^{\frac{n+1}{n-1}}\|_{L^1(\Omega)} \leq 2^{-i}\]  

(3.8)

and

\[\|\tilde{\gamma}^\delta v_i\|_{L^\infty(\Omega)} \leq 2^{-i}\]  

(3.9)

where \(\tilde{\gamma} = \text{dist}(\cdot, S)\) and where \(\delta \in (1 - n, 2 - n)\) is fixed. Clearly \((3.7)\) ensures that the sequence \((u_i)\) converges in \(L^1(\Omega)\) to a function \(u\). Moreover \((3.7)\) and \((3.8)\) imply that \(u\) is a weak solution of \((1.1)\). Finally, \((3.9)\) implies that the nontangential limit of \(u\) at any point of \(S\) is equal to \(+\infty\).

Finally, we observe that, using Proposition 1.1 instead of Proposition 1.2 the results of Theorems 1.2 and 1.3 hold when \(p > \frac{n+1}{n-1}\), is sufficiently close to \(\frac{n+1}{n-1}\). The only difference being that, in the proof of the result corresponding to the one of Theorem 1.3, in addition to the properties \((3.7)\) to \((3.9)\) which ensure the convergence of the sequence of solutions in the appropriate space, we may also ask that the sequence converges in \(W^{1,q}(\Omega)\), for some \(q\) close enough to 1. The proofs are concluded.

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