APPROXIMATE CONTROLLABILITY OF A SOBOLEV TYPE IMPULSIVE FUNCTIONAL EVOLUTION SYSTEM IN BANACH SPACES

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Abstract. In this paper, we investigate the approximate controllability problems of certain Sobolev type differential equations. Here, we obtain sufficient conditions for the approximate controllability of a semilinear Sobolev type evolution system in Banach spaces. In order to establish the approximate controllability results of such a system, we have employed the resolvent operator condition and Schauder’s fixed point theorem. Finally, we discuss a concrete example to illustrate the efficiency of the results obtained.

1. Introduction. Impulsive dynamical systems are characterized by occurrence of an abrupt change in state, which occur at certain time instant over the period of negligible duration. The dynamics of such systems is more complicated than the dynamics of non-impulsive systems. In fact, the theory of impulsive differential equations are arising in many physical phenomena such as population dynamics, electromagnetic waves, mathematical epidemiology (see for instance [2, 20, 21, 51], etc) and has emerged as an important area of investigation. On the other hand, delay differential equations are also emerging in various real-world phenomena, for example, inferred grinding models, automatic control systems, neural networks and reaction-diffusion logistic models with delay, etc. These processes are depending on the history of the system and are appropriately modeled by delay differential equations (cf. [36, 44], etc). In the past, several authors investigated the existence

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results of the impulsive and functional (delay) differential equations, see for example, [3, 4, 24, 25, 33], etc and the references therein.

Controllability is one of the important fundamental concepts for designing and analyzing the control systems. The theory of controllability of the deterministic and stochastic differential systems in infinite dimensional spaces has been studied extensively by several authors with various approaches, see for example, [7, 10, 27, 45], etc and the references therein. In the case of infinite dimensions, two basic notions of controllability, namely exact controllability and approximate controllability have wide range of applications. Exact controllability refers that the system can achieve a desired final state in certain time, while the approximate controllability means that the system can steer into an arbitrary small neighborhood of the final state. It has been observed that the concept of exact controllability in infinite dimensional control systems rarely holds (cf. [52, 53, 57], etc). Moreover, the approximately controllable systems are more prevalent and adequate in applications (see [5, 8, 9, 17, 27, 28, 29, 30, 40], etc). Therefore, it is important to study the problem of approximate controllability for the infinite dimensional control systems.

In the past several years, there have been many developments on the approximate controllability of the deterministic and stochastic impulsive systems with delay, see for instance, [3, 22, 25, 31, 33, 43, 47, 48, 55], etc. In these works, the set of sufficient conditions of the approximate controllability of semilinear systems were established by invoking the so-called resolvent operator condition introduced in [8], whenever the corresponding linear system is approximately controllable.

Sobolev type differential equations occur naturally in the mathematical modeling of many physical phenomena such as thermodynamics, fluid flow through fissured rocks, shear in second order fluids, Kelvin-Voigt model for the non-Newtonian fluid flows, etc (see for example, [1, 6, 13, 32, 35, 39, 41, 50], etc). Recently, some works are reported on the approximate controllability of Sobolev type differential equations via fixed point methods. Kerboua et.al. in [26] examined the approximate controllability problem of Sobolev type fractional order stochastic systems in Hilbert spaces having nonlocal conditions. Mahumudov in [37], investigated the approximate controllability of fractional Sobolev type evolution equations in Banach Spaces by invoking resolvent operator condition and fixed point theorem. Later, in [56], Wang et.al. obtained a set of sufficient conditions for the approximate controllability for a Sobolev type evolution system with nonlocal conditions in Hilbert spaces by employing Schauder’s fixed point theorem. To the best of our knowledge, there is no work yet reported on the approximate controllability of a Sobolev type impulsive functional evolution equation in Banach spaces. Moreover, it appears to the authors that the works available in the literature on the approximate controllability of Sobolev type differential equations in Banach spaces (cf. [14, 54], etc) using resolvent operator condition work only in Hilbert spaces (see the discussions after the resolvent operator definition (3)). In this paper, we examine the approximate controllability of a semilinear Sobolev type impulsive system with finite delay in Banach spaces via resolvent operator condition and Schauder’s fixed point theorem.

Let $X$ be a separable reflexive Banach space having strictly convex dual $X^*$ and $Y$ be a reflexive Banach space such that $X \subset Y$, and $U$ be a separable Hilbert space.
We consider the following semilinear Sobolev type evolution system:

\[
\begin{cases}
  (Ex(t))' = Ax(t) + Bu(t) + f(t, x_t), \quad t \in J = [0, T],
  t \not= \tau_k, \quad k = 1, \ldots, m, \\
  \Delta x|_{t=\tau_k} = I_k(x(\tau_k)), \quad k = 1, \ldots, m, \\
  x(t) = \phi(t), \quad t \in [-h, 0], \quad h > 0,
\end{cases}
\]

(1)

where, \( A : D(A) \subset X \to Y \) and \( E : D(E) \subset X \to Y \) are linear operators and \( \phi(0) \in D(E) \subset X \). The bounded linear operator \( B : U \to Y \), and the control function \( u \in L^2(J; U) \). The nonlinear function \( f : J \times D \to Y \), where

\[ D = \{ \psi : [-h, 0] \to X, \quad \text{\psi}(t) \text{ is piecewise continuous with jump discontinuity at } \bar{t}, \quad \text{satisfying } \psi(\bar{t}^-) = \psi(\bar{t}^+) \}, \]

(2)

dowed with the norm \( \| \psi \|_D = \frac{1}{\pi} \int_{-h}^{0} \| \psi(s) \|_X \, ds \) (see, [23]). The impulsive functions \( I_k : X \to Y \), for \( k = 1, \ldots, m \), and \( \Delta x|_{t=\tau_k} := x(\tau_k^+) - x(\tau_k^-) \) with \( 0 = \tau_0 < \tau_1 < \ldots < \tau_m < \tau_{m+1} = \tau \). Also, for every \( t \in J \), \( x_t \in D \) and define \( x_t(s) := x(t+s), \quad -h \leq s \leq 0 \).

The work [54] investigated the approximate controllability of the impulsive neutral differential inclusions of Sobolev type with infinite delay in Banach spaces. The resolvent operator considered in the paper [54] is well defined only if the state space is a Hilbert space (see (3) below for the resolvent operator defined on Banach spaces). Moreover, the author in [54] assumed that the semigroup generated by the operator \( AE^{-1} \) is compact, which may not hold true in general (cf. [26, 37], etc and see (77) below for such an example of a semigroup). In [14], the authors developed sufficient conditions of the approximate controllability of fractional Sobolev type system via resolvent operator condition in Banach spaces. As discussed earlier, the resolvent operator considered in the article [14] is also well defined only if the state space is a Hilbert space. This also motivated us to consider the system (1) in Banach spaces and establish the sufficient conditions for the approximate controllability using the resolvent operator condition and Schauder’s fixed point theorem in a systematic way.

The rest of the article is organized as follows: In section 2, we provide some basic definitions, assumptions and important results, which are required to develop the approximate controllability results of the Sobolev type control system given in (1). Section 3 is devoted for establishing the approximate controllability of linear control system corresponding to the system (1). Initially, we formulate a linear-quadratic regulator problem to obtain the existence of optimal control (Theorem 3.2), and then derive Lemma 3.4 to get the optimal control in the feedback form. Using this feedback control, we achieve the approximate controllability of the linear control system (see (8) below) by proving Theorem 3.5. In section 4, we establish our main result, that is, the approximate controllability of the Sobolev type impulsive system with finite delay given in (1). In order to establish this result, we first show the existence of a mild solution for the abstract equation defined in (1), by invoking Schauder’s fixed point theorem. Then, we prove the approximate controllability of the system (1), by employing the resolvent operator condition. Finally, we discuss a concrete example to validate the application of the developed theory in section 5.

2. Preliminaries. In the present section, we introduce some fundamental notations, definitions, and also provide important assumptions, which are needed to establish the approximate controllability of the system (1). As discussed in the earlier section, \( X \) denotes a separable reflexive Banach space under the norm \( \| \cdot \|_X \) and
\( \mathcal{X}^* \) denotes it’s topological dual endowed with the norm \( \| \cdot \|_{\mathcal{X}^*} \). The duality pairing between \( \mathcal{X} \) and it’s dual \( \mathcal{X}^* \) is represented by \( \langle \cdot , \cdot \rangle \). Also, \( \mathcal{Y} \) denotes a Banach space with the norm \( \| \cdot \|_{\mathcal{Y}} \) and it’s dual is denoted by \( \mathcal{Y}^* \). The notation \( \mathcal{U} \) stands for a separable Hilbert space (identified with its own dual) endowed with the norm \( \| \cdot \|_{\mathcal{U}} \), and the inner product on this space is denoted by \( \langle \cdot , \cdot \rangle \). The notation \( \mathcal{L}(\mathcal{U}, \mathcal{X}) \) represents the space of all bounded linear operators from \( \mathcal{U} \) into \( \mathcal{X} \) and equipped with the norm \( \| \cdot \|_{\mathcal{L}(\mathcal{U}, \mathcal{X})} \). The space of all bounded linear operators defined on \( \mathcal{X} \) with the operator norm \( \| \cdot \|_{\mathcal{L}(\mathcal{X})} \) is denoted by \( \mathcal{L}(\mathcal{X}) \). A duality mapping \( \mathcal{J} : \mathcal{X} \to 2^{\mathcal{X}^*} \) is defined as

\[
\mathcal{J}[x] = \{ x^* \in \mathcal{X}^* : \langle x, x^* \rangle = \| x \|_{\mathcal{X}}^2 = \| x^* \|_{\mathcal{X}^*}^2 \}, \text{ for all } x \in \mathcal{X}.
\]

Since the space \( \mathcal{X} \) is reflexive, \( \mathcal{X} \) can be renormed such that \( \mathcal{X} \) and \( \mathcal{X}^* \) become strictly convex (Theorem 1.1, [7]). From the strict convexity of \( \mathcal{X}^* \), the mapping \( \mathcal{J} : \mathcal{X} \to \mathcal{X}^* \) becomes single valued as well as demicontinuous (Theorem 1.2, [7]), that is,

\[
x_k \to x \text{ in } \mathcal{X} \text{ implies } \mathcal{J}[x_k] \rightharpoonup^* \mathcal{J}[x] \text{ in } \mathcal{X}^*, \text{ as } k \to \infty.
\]

Let us now give a glimpse on the differentiability of the map \( x \mapsto \frac{1}{2} \| x \|_{\mathcal{X}}^2 \). Let the mapping \( \varphi : \mathcal{X} \to \mathbb{R} \) be defined by \( \varphi(x) = \frac{1}{2} \| x \|_{\mathcal{X}}^2 \). If \( \mathcal{X}^* \) is strictly convex, then \( \varphi \) is Gateaux differentiable, and if \( \mathcal{X}^* \) is uniformly convex, then \( \varphi \) is Fréchet differentiable. In both cases, the derivative is the duality map (see Theorem 2.1, [11]). That is, we have

\[
\langle \partial_\varepsilon \varphi(x), y \rangle = \frac{1}{2} \frac{d}{d\varepsilon} \| x + \varepsilon y \|_{\mathcal{X}}^2 \bigg|_{\varepsilon = 0} = \langle \mathcal{J}[x], y \rangle,
\]

for \( y \in \mathcal{X} \), where \( \partial_\varepsilon \) denotes the Gateaux derivative.

2.1. The operators \( A \) and \( E \). The operators \( A : \text{D}(A) \subset \mathcal{X} \to \mathcal{Y} \) and \( E : \text{D}(E) \subset \mathcal{X} \to \mathcal{Y} \) satisfy the following properties:

(P1) The operator \( A \) is closed.
(P2) \( \text{D}(E) \subset \text{D}(A) \) and the domain \( \text{D}(E) \) is dense in \( \mathcal{X} \).
(P3) The operator \( E \) is bijective and \( E^{-1} : \mathcal{Y} \to \text{D}(E) \) is compact.

The boundedness of the operator \( AE^{-1} : \mathcal{Y} \to \mathcal{Y} \) follows by the conditions (P1)-(P3) and the closed graph theorem. Consequently, using Theorem 1.2, Chapter 1 [45], we infer that the operator \( AE^{-1} \) generates a uniformly continuous semigroup \( \{ T(t) : t \geq 0 \} \) of bounded linear operators from \( \mathcal{Y} \) to itself.

2.2. Resolvent operator and assumptions. To study the approximate controllability of the system (1), let us first define the operators:

\[
\begin{align*}
L_T u := & \int_0^T S(T-t)Bu(t)dt, \\
\Psi_t^0 := & \int_0^T S(T-t)BB^*S^*(T-t)dt = L_T(L_T)^*, \\
R(\lambda, \Psi_t^0) := & (\lambda I + \Psi_t^0 \mathcal{J})^{-1}, \lambda > 0,
\end{align*}
\]

(3)

where \( S(t) = E^{-1}T(t) \). Remember that the product of a compact operator and a bounded operator is compact. Since the operator \( E^{-1} \) is compact and the operator \( T(t) \) is bounded for each \( t \geq 0 \), the operator \( S(t) \) is compact, for each \( t \geq 0 \). Whenever, \( \mathcal{X} \subset \mathcal{Y} \) are separable Hilbert spaces (the duality mapping \( \mathcal{J} \) becomes I,
the identity operator), then one can define the resolvent operator as \( R(\lambda, \Psi_0^T) := (\lambda I + \Psi_0^T)^{-1}, \lambda > 0 \).

A function \( x : [\mu, \sigma] \to X \) is called the normalised piecewise continuous function on the interval \([\mu, \sigma]\), if it is piecewise continuous on \([\mu, \sigma]\), and left continuous on \((\mu, \sigma]\). The space of all normalised piecewise continuous functions defined from \([\mu, \sigma]\) to \(X\) is represented by \(PC([\mu, \sigma]; X)\) endowed with the norm \(\|x\|_{PC} = \sup_{s \in [\mu, \sigma]} \|x(s)\|_X\).

**Definition 2.1.** A mild solution \( x \in PC([-h, T]; X) \) of (1), satisfying \( x(t) = \phi(t) \) on \([-h, 0], \Delta x|_{t=\tau_k} = I_k(x(\tau_k)), k = 1, \ldots, m, \) and the restriction of \( x(\cdot) \) on interval \( J_k, k = 0, \ldots, m, \) is continuous, which is given by

\[
 x(t) = S(t)E\phi(0) + \int_0^t S(t-s)[Bu(s) + f(s, x_s)]ds + \sum_{0<\tau_k<t} S(t-\tau_k)I_k(x(\tau_k)),
\]

where \( J_0 = [0, \tau_1], J_k = (\tau_k, \tau_{k+1}], k = 1, \ldots, m. \)

**Definition 2.2.** The system (1) is said to be approximately controllable on \( J \), if \( \mathcal{R}(T; \phi, u) = X, \) where \( \mathcal{R}(T; \phi, u) \) (reachable set) is defined as

\[
 \mathcal{R}(T; \phi, u) = \{ x(T; \phi, u) : u(\cdot) \in L^2(J; U) \}.
\]

In order to prove the existence of a mild solution and the approximate controllability of the system given in (1), we require the following important assumptions:

**Assumption 2.3.** We impose the following assumptions:

(H_0) For every \( x \in X, \) \( z_\lambda = z_\lambda(x) = \lambda R(\lambda, \Psi_0^T)(x) \to 0 \) as \( \lambda \downarrow 0 \) in strong topology, where \( z_\lambda(x) \) is a solution of the equation

\[
 \lambda z_\lambda + \Psi_0^T J[z_\lambda] = \lambda x.
\]

(H_1) Let us assume that \( \|E^{-1}\|_{\mathcal{L}(Y,X)} = M, \|B\|_{\mathcal{L}(U,Y)} = MB \) and \( \|T(t)\|_{\mathcal{L}(Y)} \leq \tilde{M}, \) for all \( t \in J. \)

(H_2) (i) The function \( f(\cdot, \psi) : J \to Y \) is strongly measurable for each \( \psi \in D \) and \( f(t, \cdot) : D \to Y \) is continuous for a.e. \( t \in J. \)

(ii) For each positive integer \( r, \) there exists a function \( \gamma_r(\cdot) \in L^1(J; \mathbb{R}^+) \) such that

\[
 \sup_{\|\psi\|_D \leq r} \|f(t, \psi)\|_Y \leq \gamma_r(t), \quad \text{for a.e.} \ t \in J,
\]

and

\[
 \liminf_{r \to \infty} \frac{\int_0^T \gamma_r(t)dt}{r} = \beta < \infty.
\]

(H_3) The impulses \( I_k : X \to Y, k = 1, \ldots, m, \) are continuous and satisfy

\[
 \|I_k(x)\|_Y \leq d_k, \quad \text{for all} \ x \in X, \ k = 1, \ldots, m.
\]

**Remark 2.4.** 1. From Assumption 2.3 (H_1), we have

\[
 \|S(t)\|_{\mathcal{L}(X,Y)} = \|E^{-1}T(t)\|_{\mathcal{L}(Y,X)} \leq \|E^{-1}\|_{\mathcal{L}(Y,X)}\|T(t)\|_{\mathcal{L}(Y)} \leq \lambda M \tilde{M}, \quad \text{for all} \ t \in J.
\]

2. If \( X \) is a separable reflexive Banach space, then by using Lemma 2.2 [38], we obtain that for every \( x \in X \) and \( \lambda > 0, \) the equation (5) has a unique solution \( z_\lambda(x) = \lambda (\lambda I + \Psi_0^T J)^{-1}(x) = \lambda R(\lambda, \Psi_0^T)(x) \) and

\[
 \|z_\lambda(x)\|_X = \|J[z_\lambda(x)]\|_X \leq \|x\|_X.
\]
3. Linear control problem. In this section, we discuss the approximate controllability of linear control system corresponding to (1) by using the optimal control $u$. In order to prove the approximate controllability of the linear system, first we formulate an optimal control problem by considering a linear-quadratic regulator problem, consisting of minimizing a cost functional. The cost functional is given by

$$
F(x, u) = \|x(T) - x_T\|_X^2 + \lambda \int_0^T \|u(t)\|_U^2 dt,
$$

where $x(\cdot)$ is a solution of the corresponding linear control system:

$$
\begin{cases}
(Ex(t))' = Ax(t) + Bu(t), & t \in J, \\
x(0) = \phi(0) \in D(E),
\end{cases}
$$

with control $u \in U$, $x(T), x_T \in D(E) \subset X$ and $\lambda > 0$. The class of admissible controls is denoted by

$$
U_{ad} := L^2(J; U),
$$

consisting of the controls $u$. Since $Bu \in L^2(J; Y) \subset L^1(J; Y)$, the system (8) has a unique mild solution $x \in C(J; X)$ given by (see Corollary 2.2, Chapter 4, [45])

$$
x(t) = S(t)E\phi(0) + \int_0^t S(t-s)Bu(s)ds, \ t \in J,
$$

for any $u \in U_{ad}$.

**Definition 3.1 (Admissible class).** The admissible class $\mathcal{A}_{ad}$ for the system (8) is defined as

$$
\mathcal{A}_{ad} := \{(x, u) : x \text{ is a unique mild solution of (8) with the control } u \in U_{ad}\}.
$$

It should be noted that $\mathcal{A}_{ad}$ is a nonempty set as for any $u \in U_{ad}$, there exists a unique mild solution of the system (8). In view of the above definition, we can formulate the optimal control problem as

$$
\min_{(x, u) \in \mathcal{A}_{ad}} F(x, u).
$$

A solution of the problem (10) is called an optimal solution and the existence of an optimal solution is given by the following theorem:

**Theorem 3.2 (Existence of an optimal pair).** Let $\phi(0) \in D(E) \subset X$ be given. Then, there exists at least one optimal pair $(x^0, u^0) \in \mathcal{A}_{ad}$ such that the functional $F(x, u)$ attains its minimum at $(x^0, u^0)$, where $x^0$ is the unique mild solution of the system (8) with the control $u^0$.

**Proof.** Let us first define

$$
R := \inf_{u \in \mathcal{A}_{ad}} F(x, u).
$$

Since, $0 \leq R < +\infty$, we can extract a minimizing sequence $\{u^n\} \in U_{ad}$ such that

$$
\lim_{n \to \infty} F(x^n, u^n) = R,
$$

where $x^n(\cdot)$ is the unique mild solution of the system (8), with the control $u^n \in U_{ad}$ and the initial data $x^n(0) = \phi(0) \in D(E) \subset X$. Note that $x^n(\cdot)$ satisfies

$$
x^n(t) = S(t)E\phi(0) + \int_0^t S(t-s)Bu^n(s)ds, \ t \in J.
$$
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Since \(0 \in \mathcal{V}_{ad}\), without loss of generality, we may assume that \(\mathcal{F}(x^n, u^n) \leq \mathcal{F}(x, 0)\), where \((x, 0) \in \mathcal{V}_{ad}\). Using the definition of \(\mathcal{F}(\cdot, \cdot)\), we easily obtain

\[
\|x^n(T) - x_T\|_X^2 + \lambda \int_0^T \|u^n(t)\|_U^2 dt \leq \|x(T) - x_T\|_X^2 \leq 2(\|x(T)\|_X^2 + \|x_T\|_X^2) < +\infty.
\]

From the above expression, it is clear that, there exists a \(\tilde{C} > 0\), large enough such that

\[
\int_0^T \|u^n(t)\|_U^2 dt \leq \tilde{C} < +\infty.
\]

Using (11), we estimate

\[
\|x^n(t)\|_X \leq \|S(t)E\phi(0)\|_X + \int_0^t \|S(t-s)Bu^n(s)\|_X ds
\]

\[
\leq \|S(t)\|_{\mathcal{L}(Y, X)}\|E\phi(0)\|_Y + \int_0^t \|S(t-s)\|_{\mathcal{L}(Y, X)}\|B\|_{\mathcal{L}(U, Y)}\|u^n(s)\|_U ds
\]

\[
\leq MM\|E\phi(0)\|_Y + MMMBt^{1/2}\left(\int_0^t \|u^n(s)\|_U^2 ds\right)^{1/2}
\]

\[
\leq MM\|E\phi(0)\|_Y + MMMB^{1/2}\tilde{C}^{1/2} < +\infty,
\]

for all \(t \in J\). Since \(L^2(J; X)\) is reflexive, by using the Banach-Alaoglu theorem, we can extract a subsequence \(\{x^{n_k}\}\) of \(\{x^n\}\) such that

\[
x^{n_k} \xrightarrow{w} x^0 \text{ in } L^2(J; X), \quad \text{as } k \to \infty.
\]

We also infer that the sequence \(\{u^n\}\) is uniformly bounded in the space \(L^2(J; U)\) by the relation (13). Since \(L^2(J; U)\) is a Hilbert space (in fact reflexive), once again an application of the Banach-Alaoglu theorem yields the existence of a subsequence \(\{u^{n_k}\}\) of \(\{u^n\}\) such that

\[
u^{n_k} \xrightarrow{w} u^0 \text{ in } L^2(J; U) = \mathcal{W}_{ad}, \quad \text{as } k \to \infty.
\]

Since \(B\) is a bounded linear operator from \(U\) to \(Y\), the above convergence also implies

\[
Bu^{n_k} \xrightarrow{w} Bu^0 \text{ in } L^2(J; Y), \quad \text{as } k \to \infty.
\]

Using the weak convergence given in (16) and the compactness of the operator \((Qf)(\cdot) = \int_0^\cdot S(\cdot-s)f(s)ds : L^2(J; Y) \to C(J; Y)\) (Lemma 3.2, Corollary 3.3, Chapter 3, [34]), we have

\[
\left\|\int_0^t S(t-s)Bu^{n_k}(s)ds - \int_0^t S(t-s)Bu^0(s)ds\right\|_X \to 0, \quad \text{as } k \to \infty,
\]

for all \(t \in J\). Moreover, for all \(t \in J\), we get

\[
\|x^{n_k}(t) - x^0(t)\|_X = \left\|\int_0^t S(t-s)Bu^{n_k}(s)ds - \int_0^t S(t-s)Bu^0(s)ds\right\|_X
\]

\[
\to 0, \quad \text{as } k \to \infty,
\]

where

\[
x^0(t) = S(t)E\phi(0) + \int_0^t S(t-s)Bu(s)ds, \quad t \in J,
\]
is the unique mild solution of the system:
\[
\begin{cases}
    (Ex^0(t))' = Ax^0(t) + Bu^0(t), & t \in J, \\
    x^0(0) = \phi(0) \in D(E).
\end{cases}
\] (19)

Using the continuity in time of \(x^{nk}(\cdot)\) in \(X\), we obtain \(x^{nk} \to x^0\) in \(C(J; X)\), as \(k \to \infty\). Since \(x^0(\cdot)\) is the unique mild solution of (19), the whole sequence \(\{x^n\}\) converges to \(x^0\). Using the fact that \(u^0 \in \mathcal{U}_{\text{ad}}\) and \(x^0\) is the unique mild solution of (19) corresponding to the control \(u^0\), it is immediate that \((x^0, u^0) \in \mathcal{A}_{\text{ad}}\).

Finally, we show that \((x^0, u^0)\) is a minimizer, that is, \(R = \mathcal{F}(x^0, u^0)\). Since the cost functional \(\mathcal{F}(\cdot, \cdot)\) is continuous and convex (see Proposition III.1.6 and III.1.10, [18]) on \(L^2(J; X) \times L^2(J; U)\), it follows that \(\mathcal{F}(\cdot, \cdot)\) is weakly lower semi-continuous (Proposition II.4.5, [18]). That is, for a sequence 
\[(x^n, u^n) \overset{w}{\to} (x^0, u^0) \text{ in } L^2(J; X) \times L^2(J; U),\]
we have 
\[\mathcal{F}(x^0, u^0) \leq \liminf_{n \to \infty} \mathcal{F}(x^n, u^n).\]

Therefore, we obtain 
\[R \leq \mathcal{F}(x^0, u^0) \leq \liminf_{n \to \infty} \mathcal{F}(x^n, u^n) = \lim_{n \to \infty} \mathcal{F}(x^n, u^n) = R,\]
and hence \((x^0, u^0)\) is a minimizer of the problem (10).

\[\square\]

**Remark 3.3.** Since the cost functional defined in (7) is convex, the constraint (8) is linear and \(\mathcal{U}_{\text{ad}} = L^2(J; U)\) is convex, then the optimal control obtained in Theorem 3.2 is unique.

The explicit expression (in the feedback form) for the optimal control \(u\) is given by the following lemma:

**Lemma 3.4.** The optimal control \(u\) satisfying (8) and minimizing the cost functional (7) is given by 
\[u(t) = B^*S^*(T - t)(R(\lambda, \Psi_0^T)p(x(\cdot))) + J, t \in J,\]
where 
\[p(x(\cdot)) = x_T - S(T)E\phi(0).\]

**Proof.** Let \((x, u)\) be an optimal solution of (10) with the control \(u\) and the corresponding trajectory be \(x\). Then, \(\varepsilon = 0\) is the critical point of 
\[I(\varepsilon) = \mathcal{F}(x_{u + \varepsilon w}, u + \varepsilon w),\]
with \(w \in L^2(J; U)\), where \(x_{u + \varepsilon w}\) is the unique mild solution of (8) with respect to the control \(u + \varepsilon w\) satisfying 
\[x_{u + \varepsilon w}(t) = S(t)E\phi(0) + \int_0^t S(t - s)B(u + \varepsilon w)(s)ds, t \in J.\]
(20)

Let us now compute the variation of the cost functional \(\mathcal{F}\) (defined in (7)) as 
\[
\frac{d}{d\varepsilon} \frac{dI(\varepsilon)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left[ \frac{\|x_{u + \varepsilon w}(T) - x_T\|^2_X^2}{2} + \lambda \int_0^T \|u(t) + \varepsilon w(t)\|^2_U dt \right]_{\varepsilon=0} = 2 \left\langle \frac{d}{d\varepsilon} (x_{u + \varepsilon w}(T) - x_T), \frac{d}{d\varepsilon} (x_{u + \varepsilon w}(T) - x_T) \right\rangle.
\]
Theorem 3.5. The following statements are equivalent:

(i) The linear control system (8) is approximately controllable on J.

(ii) If $x^* \in X^*$, we have $B^*S^*(T-t)x^* = 0$, for all $t \in J$, then $x^* = 0$.

(iii) The Assumption $(H_0)$ holds.

The approximate controllability of the linear control system defined in (8) is established in the following theorem (for analogous results in the case of first order linear systems, see [38] for Banach spaces and [8] for Hilbert spaces). Since the linear system under our consideration is Sobolev type, we are providing a proof here.

\[
+ 2\lambda \int_0^T \left( u(t) + \varepsilon w(t), \frac{d}{d\varepsilon}(u(t) + \varepsilon w(t)) \right) dt \bigg|_{\varepsilon = 0} = 2\left\langle J(x(T) - x_T), \int_0^T S(T-t)Bu(t)dt \right\rangle + 2\lambda \int_0^T (u(t), w(t))dt. \tag{21}
\]

Since the first variation of the cost functional is zero, we obtain

\[
0 = \left\langle J(x(T) - x_T), \int_0^T S(T-t)Bu(t)dt \right\rangle + \lambda \int_0^T (u(t), w(t))dt
\]

\[
= \int_0^T \left\langle J(x(T) - x_T), S(T-t)Bu(t) \right\rangle dt + \lambda \int_0^T (u(t), w(t))dt
\]

\[
= \int_0^T (B^*S^*(T-t)J(x(T) - x_T) + \lambda u(t), w(t))dt. \tag{22}
\]

Since $w \in L^2(J; U)$ is an arbitrary element (one can choose $w$ to be $B^*S^*(T-t)J[x(T) - x_T] + \lambda u(t)$), it follows that the optimal control is given by

\[
u(t) = -\lambda^{-1}B^*S^*(T-t)J[x(T) - x_T], \tag{23}\]

for a.e. $t \in J$. It also holds for all $t \in J$, since by the relations (22) and (23), it is clear that $u$ is continuous and belongs to $C(J; U)$. Next, we compute

\[
x(T) = S(T)E\phi(0) - \int_0^T \lambda^{-1}S(T-s)BB^*S^*(T-s)J[x(T) - x_T]ds
\]

\[
= S(T)E\phi(0) - \lambda^{-1}\Psi_0^TJ[x(T) - x_T]. \tag{24}\]

Let us now define

\[
p(x(\cdot)) := x_T - S(T)E\phi(0). \tag{25}\]

Combining (24) and (25), we have the following:

\[
x(T) - x_T = -p(x(\cdot)) - \lambda^{-1}\Psi_0^TJ[x(T) - x_T]. \tag{26}\]

From (26), one can easily obtain that

\[
x(T) - x_T = -\lambda (\lambda I + \Psi_0^TJ)^{-1}p(x(\cdot)) = -\lambda R(\lambda, \Psi_0^T)p(x(\cdot)). \tag{27}\]

Finally, from (23), we have

\[
u(t) = B^*S^*(T-t)J[R(\lambda, \Psi_0^T)p(x(\cdot))], \quad t \in J,
\]

which completes the proof.

The approximate controllability of the linear control system defined in (8) is established in the following theorem (for analogous results in the case of first order linear systems, see [38] for Banach spaces and [8] for Hilbert spaces). Since the linear system under our consideration is Sobolev type, we are providing a proof here.
**Proof. Claim 1:** (i) $\iff$ (iii). First, let us assume that Assumption (H0) holds true, that is, for every $h \in X$, $z_\lambda(h) = \lambda R(\lambda, \Psi^T_0)(h) \to 0$ as $\lambda \downarrow 0$, where $z_\lambda(h)$ is a solution of the equation \((5)\). Since $E \phi(0) \in Y$ and $Bu \in L^1(J; Y)$, then for every $\lambda > 0$ and $x_T \in D(E) \subset X$, there exists a unique mild solution $x_\lambda \in C(J; X)$ for the linear system \((8)\) such that

$$x_\lambda(t) = S(t)E \phi(0) + \int_0^t S(t-s)Bu_\lambda(s)ds, \ t \in J,$$

with

$$u_\lambda(t) = B^*S^*(T-t)J[R(\lambda, \Psi^T_0)p(x(\cdot))], \ \text{and} \ \ p(x(\cdot)) = x_T - S(T)E \phi(0).$$

Using \((28)\), one can easily deduce that

$$x_\lambda(T) = S(T)E \phi(0) + \int_0^T S(T-s)Bu_\lambda(s)ds$$

$$= S(T)E \phi(0) + \Psi^T_0 J[R(\lambda, \Psi^T_0)p(x(\cdot))]$$

$$= x_T - p(x_\lambda(\cdot)) + \Psi^T_0 J[R(\lambda, \Psi^T_0)p(x(\cdot))]$$

$$= x_T - (I + \Psi^T_0 J)R(\lambda, \Psi^T_0)p(x(\cdot)) + \Psi^T_0 J[R(\lambda, \Psi^T_0)p(x(\cdot))]$$

$$= x_T - \lambda R(\lambda, \Psi^T_0)p(x(\cdot)),$$

which holds since $\|S(T)E \phi(0)\|_X \leq M \tilde{M} \|E \phi(0)\|_Y$, $x_T \in D(E) \subset X$. Using Assumption (H0), we get

$$\|x_\lambda(T) - x_T\|_X \leq \|\lambda R(\lambda, \Psi^T_0)(x_T - S(T)E \phi(0))\|_X \to 0, \ \text{as} \ \lambda \downarrow 0.$$ 

Therefore, the system \((8)\) is approximately controllable on $J$, that is, condition (i) holds.

Conversely, we assume that the linear control system \((8)\) is approximate controllable on $J$. Then, for arbitrary $x_T \in D(E) \subset X$, there exists a sequence $\{u^n\}_{n=1}^\infty$ in $\mathcal{U}_{ad} = L^2(J; U)$ such that

$$\|x^n(T) - x_T\|_X \to 0 \ \text{as} \ n \to \infty,$$

where $x^n(\cdot)$ is the unique mild solution of the system \((8)\) with the control $u^n \in \mathcal{U}_{ad}$. For all $n \geq 1$, we have

$$\|x^n(T) - x_T\|_X^2 \leq \|x^n(T) - x_T\|_X^2 + \lambda \int_0^T \|u^n(t)\|_U^2 dt$$

$$\leq \|x^n(T) - x_T\|_X^2 + \lambda \int_0^T \|u^n(t)\|_U^2 dt, \quad (30)$$

where $(x^0, u^0) \in \mathcal{U}_{ad}$ is the optimal pair at which the functional \((7)\) takes its minimum value. For given $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$\|x^n(T) - x_T\|_X < \frac{\varepsilon}{\sqrt{2}}, \ \text{for all} \ n \geq n_0.$$ 

Now, we can choose a $\delta > 0$ sufficiently small such that

$$\lambda \int_0^T \|u^{n_0}(t)\|_U^2 dt < \frac{\varepsilon^2}{2},$$

for all $0 < \lambda < \delta$. Taking $n = n_0$ in \((30)\), we get

$$\|x^0(T) - x_T\|_X^2 \leq \|x^{n_0}(T) - x_T\|_X^2 + \lambda \int_0^T \|u^{n_0}(t)\|_U^2 dt < \varepsilon^2, \quad (31)$$
for all $0 < \lambda < \delta$. Invoking Lemma 3.4, we know that
\[ u^0(t) = B^*S^*(T - t)J[R(\lambda, \Psi^T_0)p(x(\cdot))], \quad t \in J, \]
with
\[ p(x(\cdot)) = x_T - (S(T)E\phi(0)). \]

Since $x^0(\cdot)$ satisfies the equation (28) with the control $u^0$, proceeding in a similar way as in the proof of the equality (29), we deduce that
\[ x^0(T) = x_T - \lambda R(\lambda, \Psi^T_0)p(x(\cdot)). \quad (32) \]

Since $x_T \in D(E) \subset X$ is arbitrary and $D(E)$ is dense in $X$, using the inequality (31) together with the expression (32), we deduce that Assumption $(H_0)$ holds.

**Claim 2:** $(ii) \Rightarrow (i)$. For $x^* \in X^*$ and $u \in L^2(J; U)$, we calculate
\[ ((L_T)^*x^*, u)_{L^2(J; U)} = (x^*, L_T u) = \left( x^*, \int_0^T S(T - t)Bu(t)dt \right) \]
\[ = \int_0^T (x^*, S(T - t)Bu(t))dt = \int_0^T (B^*S^*(T - t)x^*, u(t))dt \]
\[ = (B^*S^*(T - t)x^*, u)_{L^2(J; U)}, \]
and hence $(L_T)^* = B^*S^*(T - t)$. If $B^*S^*(T - t)x^* = 0$ on $J$ implies $x^* = 0$, from the above fact we infer that the operator $\Psi^T_0$ is positive and vice versa. Using Theorem $2.3$ [38], we know that the operator $\Psi^T_0$ is positive if Assumption $(H_0)$ holds. Then, by the Claim 1, we conclude that the linear system (8) is approximately controllable on $J$.

**Claim 3:** $(ii) \Rightarrow (iii)$. We assume that condition $(ii)$ holds true. Consequently, we can obtain that the operator $\Psi^T_0$ is positive. Once again using Theorem $2.3$ [38], we infer that Assumption $(H_0)$ holds. \hfill \square

4. Approximate controllability result for the semilinear system. In this section, we investigate the approximate controllability of the semilinear system given in (1). In order to establish the sufficient conditions of approximate controllability, we first prove that for every $\lambda > 0$ and $x_T \in D(E) \subset X$, the existence of a mild solution of the system (1) for a suitable control function (which is defined in a similar way as in the linear problem)
\[ u_\lambda(t) = B^*S^*(T - t)J[R(\lambda, \Psi^T_0)g(x(\cdot))], \quad (33) \]
where
\[ g(x(\cdot)) = x_T - S(T)E\phi(0) - \int_0^T S(T - s)f(s, \tilde{x}_s)ds - \sum_{k=1}^m S(\tau - \tau_k)I_k(\tilde{x}(\tau_k)), \quad (34) \]
with $\tilde{x} : [-h, T] \to X$ such that $\tilde{x}(t) = \phi(t)$, $t \in [-h, 0]$ and $\tilde{x} = x$ on $J$.

**Theorem 4.1.** If the conditions $(P_1)$-$(P_9)$ and Assumptions $(H_1)$-$(H_3)$ hold true. Then for every $\lambda > 0$, the system (1) with the control (33) has at least one mild solution on $J$, provided
\[ \frac{MMT_\beta}{h} \left( 1 + \frac{M^2MT_\beta^2T}{\lambda} \right) < 1. \quad (35) \]
Proof. Let \( Q = \{ x \in \text{PC}(J; \mathbb{X}) : x(0) = \phi(0) \} \) be the space endowed with the norm \( ||| \cdot |||_{\text{PC}} \). Let us define a set

\[
B_q = \{ x \in Q : ||x||_{\text{PC}} \leq q \},
\]

where \( q \) is a positive constant. For \( \lambda > 0 \), we define an operator \( F_\lambda : Q \to Q \) as

\[
(F_\lambda x)(t) = z(t),
\]

where

\[
z(t) = S(t)E\phi(0) + \int_0^t S(t-s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]ds + \sum_{0 < \tau_k < t} S(t - \tau_k)I_k(\tilde{x}(\tau_k)),
\]

with the control \( u_\lambda \) defined in (33). It is clear from the definition of \( F_\lambda \) that for \( \lambda > 0 \), the fixed point of \( F_\lambda \) is a mild solution of the system (1). We prove that the operator \( F_\lambda \) has a fixed point in the following steps.

**Step (1):** \( F_\lambda(B_q) \subset B_q \), for some \( q \). Let us assume that our claim is not true. That is, there exists a \( \lambda > 0 \) such that for every \( q > 0 \), there exists \( x^q(\cdot) \in B_q \) with

\[
||(F_\lambda x^q)(t)||_\mathbb{X} > q, \text{ for some } t \in J,
\]

where \( t \) may depend upon \( q \). First, we estimate \( ||u_\lambda(t)||_{U} \), by using the expression defined in (33), Remark 2.4 and Assumption 2.3 \((H_1)-(H_3)\) as

\[
||u_\lambda(t)||_U
= ||B^*S^*(T-t)J[R(\lambda, \Psi_0^T)g(x(\cdot))]|_U
\leq 1 \lambda ||B^*||_{\mathcal{L}(Y^*,U)}||S(T-t)^*||_{\mathcal{L}(X^*,Y^*)}||J[R(\lambda, \Psi_0^T)g(x(\cdot))]|_X
\leq M\tilde{M}M_B \lambda ||g(x(\cdot))|_X
\leq \frac{M\tilde{M}M_B \lambda}{\lambda} \left( ||x_T||_X + ||S(T)||_{\mathcal{L}(Y,X)} ||E\phi(0)||_Y + \int_0^T ||S(T-s)||_{\mathcal{L}(Y,X)} ||f(s, \tilde{x}_s)||_Y ds + \sum_{k=1}^{m} ||S(T - \tau_k)||_{\mathcal{L}(Y,X)} ||I_k(\tilde{x}(\tau_k))||_Y \right)
\leq \frac{M\tilde{M}M_B \lambda}{\lambda} \left( N + M\tilde{M} \int_0^T \gamma_q(s) ds + M\tilde{M} \sum_{k=1}^{m} d_k \right)
\leq \frac{M\tilde{M}M_B \lambda}{\lambda} \left( N + M\tilde{M} \int_0^T \gamma_q(s) ds \right),
\]

where \( q' = \frac{T}{\pi} q + ||\phi||_D \) and \( N = ||x_T||_X + M\tilde{M} ||E\phi(0)||_Y + M\tilde{M} \sum_{k=1}^{m} d_k \). Further, using Assumption 2.3 \((H_1)-(H_3)\), we estimate

\[
q < ||(F_\lambda x^q)(t)||_X
= \left| |S(t)E\phi(0) + \int_0^t S(t-s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]ds + \sum_{0 < \tau_k < t} S(t - \tau_k)I_k(\tilde{x}(\tau_k))\right|_X
\leq ||S(t)E\phi(0)||_X + \int_0^t ||S(t-s)[Bu_\lambda(s) + f(s, \tilde{x}_s)]||_X ds
\]
Thus, dividing by $s$ which is a contradiction to (35). Hence, for some $m$:

$$
\text{Step (2): Let us now show that the operator } F_\lambda : B_q \to B_q \text{ is continuous. In order to prove this, we consider a sequence } \{x^n\}_{n=1}^\infty \subset B_q \text{ such that } x^n \to x \text{ in } B_q, \text{ that is,}
$$

$$
\lim_{n \to \infty} \|x^n - x\|_{PC} = 0.
$$

For $s \in J$, we estimate

$$
\|\tilde{x}_s^n - \tilde{x}_s\|_D = \frac{1}{h} \int_{s-h}^{s} \|\tilde{x}_s^n(\theta) - \tilde{x}_s(\theta)\|_X d\theta
= \frac{1}{h} \int_{s-h}^{s} \|x^n(s + \theta) - \tilde{x}(s + \theta)\|_X d\theta
= \frac{1}{h} \int_{s-h}^{s} \|x^n(r) - \tilde{x}(r)\|_X dr.
$$

(39)

If $s < h$, then we can write the above expression as

$$
\|\tilde{x}_s^n - \tilde{x}_s\|_D = \frac{1}{h} \int_{s-h}^{s} \|\tilde{x}_s^n(r) - \tilde{x}(r)\|_X dr + \frac{1}{h} \int_{s-h}^{0} \|x^n(r) - \tilde{x}(r)\|_X dr
\leq \frac{1}{h} \int_{s-h}^{s} \|\tilde{x}_s^n(r) - \tilde{x}(r)\|_X dr + \frac{1}{h} \int_{s-h}^{0} \|x^n(r) - \tilde{x}(r)\|_X dr
= \frac{1}{h} \int_{s-h}^{s} \|x^n(r) - x(r)\|_X dr \leq \frac{T}{h} \|x^n - x\|_{PC} \to 0, \text{ as } n \to \infty.
$$

If $s \geq h$, then by the expression (39), we obtain

$$
\|\tilde{x}_s^n - \tilde{x}_s\|_D \leq \frac{1}{h} \int_{s-h}^{s} \|\tilde{x}_s^n(r) - \tilde{x}(r)\|_X dr \leq \frac{1}{h} \int_{0}^{T} \|x^n(r) - x(r)\|_X dr
\leq \frac{T}{h} \|x^n - x\|_{PC} \to 0, \text{ as } n \to \infty.
$$
Using the above convergence, Assumption 2.3 (H$_2$)-(H$_3$), Remark 2.4 and Lebesgue’s dominant convergence theorem, we deduce that

$$\|R(\lambda, \Psi_{0}^{T})g(x^n(\cdot)) - R(\lambda, \Psi_{0}^{T})g(x(\cdot))\|_\mathcal{X}$$

$$= \frac{1}{\lambda} \|\lambda R(\lambda, \Psi_{0}^{T})g(x^n(\cdot)) - g(x(\cdot))\|_\mathcal{X}$$

$$\leq \frac{1}{\lambda} \|g(x^n(\cdot)) - g(x(\cdot))\|_\mathcal{X}$$

$$\leq \frac{1}{\lambda} \left[ \int_0^T S(t - s) [f(s, \tilde{x}_n) - f(s, \tilde{x})] ds \right]_\mathcal{X}$$

$$+ \left[ \sum_{k=1}^m S(t - \tau_k) [\tilde{I}_k(x^n(\tau_k)) - \tilde{I}_k(\tilde{x}(\tau_k))] \right]_{\mathcal{X}}$$

$$\leq \frac{M \tilde{M}}{\lambda} \left[ \int_0^T \|f(s, \tilde{x}_n) - f(s, \tilde{x})\|_\mathcal{Y} ds + \sum_{k=1}^m \|\tilde{I}_k(x^n(\tau_k)) - \tilde{I}_k(\tilde{x}(\tau_k))\|_\mathcal{Y} \right]$$

$$\to 0 \text{ as } n \to \infty. \quad (40)$$

Thus, $R(\lambda, \Psi_{0}^{T})g(x^n(\cdot)) \to R(\lambda, \Psi_{0}^{T})g(x(\cdot))$ in $\mathcal{X}$ as $n \to \infty$. Since the mapping $\mathcal{J} : \mathcal{X} \to \mathcal{X}^*$ is demicontinuous, it is immediate that

$$\mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x^n(\cdot))] \xrightarrow{\text{w}} \mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x(\cdot))] \text{ as } n \to \infty \text{ in } \mathcal{X}^*. \quad (41)$$

Since the operator $S(t)$ is compact for each $t \geq 0$, the operator $S^*(t)$ is also compact for each $t \geq 0$. Hence, using the above weak convergence and compactness of the operator $S^*(t)$, one can easily find

$$\|S^*(T - t) [\mathcal{J}(R(\lambda, \Psi_{0}^{T})g(x^n(\cdot))) - \mathcal{J}(R(\lambda, \Psi_{0}^{T})g(x(\cdot)))]\|_\mathcal{Y} \to 0 \text{ as } n \to \infty. \quad (42)$$

Using (33) and (42), we easily get

$$\|u^n(\lambda)(t) - u(\lambda)(t)\|_U$$

$$= \|B^* S^*(T - t) [\mathcal{J}(R(\lambda, \Psi_{0}^{T})g(x^n(\cdot))) - \mathcal{J}(R(\lambda, \Psi_{0}^{T})g(x(\cdot)))]\|_U$$

$$\leq \|B^*\|_{L(\mathcal{Y}, U)} \|S^*(T - t) [\mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x^n(\cdot))] - \mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x(\cdot))]]\|_\mathcal{Y}$$

$$\leq M_B \|S^*(T - t) [\mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x^n(\cdot))] - \mathcal{J}[R(\lambda, \Psi_{0}^{T})g(x(\cdot))]]\|_\mathcal{Y}$$

$$\to 0 \text{ as } n \to \infty, \text{ uniformly for all } t \in J. \quad (43)$$

Using (43), Assumption 2.3 (H$_2$)-(H$_3$), Remark 2.4 and Lebesgue’s dominate convergence theorem, we obtain

$$\|(F_\lambda x^n)(t) - (F_\lambda x)(t)\|_\mathcal{X}$$

$$\leq \left[ \int_0^T S(t - s) B[u^n(\lambda)(s) - u(\lambda)(s)] ds \right]_\mathcal{X} + \left[ \int_0^T S(t - s) [f(s, \tilde{x}_n) - f(s, \tilde{x})] ds \right]_\mathcal{X}$$

$$+ \left[ \sum_{0 < \tau_k < t} S(t - \tau_k) [\tilde{I}_k(x^n(\tau_k)) - \tilde{I}_k(\tilde{x}(\tau_k))] \right]_\mathcal{X}$$

$$\leq M \tilde{M} M_B T \sup_{\lambda \in J} \|u^n(\lambda)(t) - u(\lambda)(t)\|_U + M \tilde{M} \int_0^T \|f(s, \tilde{x}_n(s)) - f(s, \tilde{x}_s)\|_\mathcal{Y} ds$$
\[ + M \tilde{M} \sum_{k=1}^{m} \left\| I_k(x^\ast(t_k)) - I_k(\tilde{x}(t_k)) \right\|_Y \]
\[ \to 0 \text{ as } n \to \infty, \quad (44) \]

for each \( t \in J \). Thus, by (44), it follows that the map \( F_\lambda \) is continuous.

**Step (3):** In the final step, we show that \( F_\lambda \), for \( \lambda > 0 \) is a compact operator. For this, first we prove that the image of \( B_q \) under \( F_\lambda \) is equicontinuous. Let \( 0 \leq t_1 \leq t_2 \leq T \) and any \( x \in B_q \), we consider the following estimate:

\[
\begin{align*}
&\| (F_\lambda x)(t_2) - (F_\lambda x)(t_1) \|_X \\
&\leq \| [S(t_2) - S(t_1)]E\phi(0) \|_Y + \int_{t_1}^{t_2} \| S(t_2 - s)B_u(s) + f(s, \tilde{x}_s) \|_Y ds \\
&+ \int_{t_1}^{t_2} \| S(t_2 - s) - S(t_1 - s)B_u(s) + f(s, \tilde{x}_s) \|_Y ds \\
&+ \sum_{0 < t_k < t_1} \| S(t_2 - t_k) - S(t_1 - t_k) \|_Y I_k(\tilde{x}(t_k)) \|_Y \\
&+ \sum_{t_1 < t_k < t_2} \| S(t_2 - t_k) - S(t_1 - t_k) \|_Y I_k(\tilde{x}(t_k)) \|_Y \\
&\leq \| E^{-1} \|_{L(X,Y)} \| T(t_2) - T(t_1) \|_{L(Y)} \| E\phi(0) \|_Y + M \tilde{M} M_B \sup_{t \in J} \| u_\lambda(t) \|_U (t_2 - t_1) \\
&+ M_B \sup_{t \in J} \| u_\lambda(t) \|_U \| E^{-1} \|_{L(X,Y)} \int_{t_1}^{t_2} \| T(t_2 - s) - T(t_1 - s) \|_{L(Y)} ds \\
&+ MM \int_{t_1}^{t_2} \gamma_q(s) ds + \| E^{-1} \|_{L(X,Y)} \int_{t_1}^{t_2} \| T(t_2 - s) - T(t_1 - s) \|_{L(Y)} \gamma_q(s) ds \\
&+ \| E^{-1} \|_{L(X,Y)} \sum_{0 < t_k < t_1} \| T(t_2 - t_k) - T(t_1 - t_k) \|_{L(Y)} d_k + M \tilde{M} \sum_{t_1 < t_k < t_2} d_k \\
&\leq M \| T(t_2) - T(t_1) \|_{L(Y)} \| E\phi(0) \|_Y + M \tilde{M} M_B \sup_{t \in J} \| u_\lambda(t) \|_U (t_2 - t_1) \\
&+ M_B M \sup_{t \in J} T \| u_\lambda(t) \|_U \sup_{s \in [0,t_1]} \| T(t_2 - s) - T(t_1 - s) \|_{L(Y)} \\
&+ M \tilde{M} \int_{t_1}^{t_2} \gamma_q(s) ds + M \sup_{s \in [0,t_1]} \| T(t_2 - s) - T(t_1 - s) \|_{L(Y)} \int_{0}^{t_1} \gamma_q(s) ds
\end{align*}
\]
+ M \sum_{0 < \tau_k < t_1} \|T(t_2 - \tau_k) - T(t_1 - \tau_k)\|_{L(Y)}d_k + M\tilde{M}\sum_{t_1 \leq \tau_k < t_2} d_k. \quad (45)

Clearly, the right hand side of the inequality (45) converges to zero uniformly for \(x \in B_q\) as \(|t_2 - t_1| \to 0\), since the operator \(T(t)\) is continuous in operator topology for each \(t \geq 0\). Therefore, the image of \(B_q\) under \(F_{\lambda}\) is equicontinuous.

Next, we show that for each \(\lambda > 0\), the operator \(F_{\lambda}\) maps \(B_q\) into a relatively compact subset of \(B_q\). In order to do this, we prove that for every \(t \in J\), the set \(V(t) = \{(F_{\lambda}x)(t) : x \in B_q\}\), is relatively compact in \(X\). For \(t = 0\), it is easy to verify that the set \(V(t)\) is relatively compact in \(X\). Let \(0 < t \leq T\) be fixed and for given \(\eta\) with \(0 < \eta < t\), we define

\[
(F_{\lambda}^\eta x)(t) := S(t)E\phi(0) + \int_0^{t-\eta} S(t-s)[Bu_\lambda(s) + f(s, \bar{x}_s)]ds + \sum_{0 < \tau_k < t-\eta} S(t-\tau_k)I_k(\bar{x}(\tau_k))
\]

\[
= E^{-1}T(\eta) \left[ T(t-\eta)E\phi(0) + \int_0^{t-\eta} T(t-s-\eta)[Bu_\lambda(s) + f(s, \bar{x}_s)]ds + \sum_{0 < \tau_k < t-\eta} T(t-\eta - \tau_k)I_k(\bar{x}(\tau_k)) \right]
\]

\[
= E^{-1}T(\eta)g(t-\eta),
\]

where \(u_\lambda(t)\) is defined in (33) and \(y(\cdot)\) is the term appearing inside the parenthesis. Using (37) and Assumption 2.3, one can estimate \(\|y(\cdot)\|_Y\) as

\[
\|y(t-\eta)\|_Y \leq \tilde{M}\|E\phi(0)\|_Y + \frac{M\tilde{M}^2M_2T}{\lambda} \left( N + M\tilde{M}\int_0^T \gamma_\eta(s)ds \right) + \tilde{M} \int_0^T \gamma_\eta(s)ds + \tilde{M} \sum_{k=1}^m d_k < +\infty.
\]

Since the operator \(E^{-1}\) is compact and the operator \(T(t)\) is bounded for \(t \geq 0\), we know that \(E^{-1}T(t)\) is a compact operator for \(t \geq 0\). Therefore, the set \(V_\eta(t) = \{(F_{\lambda}^\eta x)(t) : x \in B_q\}\) is relatively compact in \(X\). Thus, there exists finite \(x_i\)’s, for \(i = 1, \ldots, n\) in \(X\) such that

\[
V_\eta(t) \subset \bigcup_{i=1}^n B(x_i, \varepsilon/2),
\]

for some \(\varepsilon > 0\), where \(B(x_i, \varepsilon/2)\) is an open ball centered at \(x_i\) and of radius \(\varepsilon/2\). We now choose an \(\eta > 0\) such that

\[
\|(F_{\lambda}x)(t) - (F_{\lambda}^\eta x)(t)\|_X
\]

\[
= \left\| \int_{t-\eta}^t S(t-s)[Bu_\lambda(s) + f(s, \bar{x}_s)]ds + \sum_{t-\eta < \tau_k < t} S(t-\tau_k)I_k(\bar{x}(\tau_k)) \right\|_X
\]

\[
\leq M\tilde{M}M_B \int_{t-\eta}^t \|u_\lambda(s)\|_U ds + M\tilde{M} \int_{t-\eta}^t \gamma_\eta(s)ds + \tilde{M} \sum_{t-\eta < \tau_k < t} d_k
\]
the convergence in (43) as
of \( X \)
duality mapping \( J \) continuity given in (41) with uniform continuity in \( B \).

If Remark 4.2.

Consequently

\[
V(t) \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon).
\]

Thus, for each \( t \in J \), the set \( V(t) \) is relatively compact in \( X \).

Hence, by making use of the Arzela-Ascoli theorem, we conclude that the operator \( F_\lambda \) is compact. Then using Schauder’s fixed point theorem, we infer that the operator \( F_\lambda \) has a fixed point in \( B_\eta \), which is a mild solution of the system (1).

**Remark 4.2.** If \( X^* \) is uniformly convex (or if \( X \) is uniformly smooth), then the duality mapping \( J : X \to X^* \) is uniformly continuous on every bounded subset of \( X \) (Theorem 1.2, [7]). Then, with the help of (40), one can replace the weak continuity given in (41) with uniform continuity in \( B_\eta \). Moreover, we can consider the convergence in (43) as

\[
\|u^\lambda_n(t) - u_\lambda(t)\|_U = \|B^*S^*(T - t)J[R(\lambda, \Psi^T_0)g(x^n(\cdot))] - J[R(\lambda, \Psi^T_0)g(x(\cdot))]|_U \leq M_B^M \tilde{M} \|J[R(\lambda, \Psi^T_0)g(x^n(\cdot))] - J[R(\lambda, \Psi^T_0)g(x(\cdot))]|_X \to 0 \quad \text{as} \quad n \to \infty,
\]

uniformly for all \( t \in J \),

using (40) and the uniform continuity of \( J[\cdot] \).

Finally, we establish the approximate controllability result for the system (1). In order to establish this, we impose the following assumption on the nonlinear term \( f(\cdot, \cdot) \):

**Assumption 4.3.** \((H_4)\) The function \( f : J \times D \to Y \) is continuous and uniformly bounded, that is, there exists \( N > 0 \) such that \( \|f(t, \phi)\|_Y \leq N \), for all \((t, \phi) \in J \times D \).

**Theorem 4.4.** Let the conditions \((P_1)-(P_3)\), Assumptions \((H_0)-(H_1),(H_3)-(H_4)\) and the conditions of Theorem 4.1 hold true. Then, the system (1) is approximately controllable.

**Proof.** Let \( x^\lambda(\cdot) \) be a fixed point of the operator \( F_\lambda \), then \( x^\lambda(\cdot) \) is a mild solution of the equation (1) with the control

\[
u_\lambda(t) = B^*S^*(T - t)J[R(\lambda, \Psi^T_0)g(x^\lambda(\cdot))],
\]

where

\[
g(x^\lambda(\cdot)) = x_T - S(T)E\phi(0) - \int_0^T S(T - s)f(s, x^\lambda_s)\,ds - \sum_{k=1}^m S(T - \tau_k)I_k(x^\lambda(\tau_k)).
\]

That is, \( x^\lambda(\cdot) \) satisfies

\[
x^\lambda(t) = S(t)E\phi(0) + \int_0^t S(t - s)[Bu_\lambda(s) + f(s, x^\lambda_s)]\,ds
\]

\[
+ \sum_{0 < \tau_k < t} S(t - \tau_k)I_k(x^\lambda(\tau_k)),
\]

\[
\leq M\tilde{M}B \sup_{\eta \in J}\|u_\lambda(t)\|_U + M\tilde{M} \int_{t-\eta}^t \gamma_q(s)\,ds + M\tilde{M} \sum_{t-\eta < \tau_k < t} d_k \leq \frac{\varepsilon}{2}. \quad (46)
\]
with the control $u^\lambda(\cdot)$ given in (48). It is easy to verify that
\[ x^\lambda(T) = x_T - \lambda R(\lambda, \Psi^T_\theta)g(x^\lambda(\cdot)). \] (49)

Using Assumption 4.3 (H4), we get
\[ \int_0^T \|f(s, \hat{x}_\lambda^s)\|_Y^2 \, ds \leq N^2T, \] (50)
and consequently the sequence \{\(f(\cdot, \hat{x}^\lambda_s) : \lambda > 0\)\} in \(L^2(J; Y)\) is bounded. Thanks to the Banach-Alaoglu theorem, we can find a subsequence relabeled as \{\(f(\cdot, \hat{x}^\lambda_s) : \lambda > 0\)\}, which is weakly convergent to, say \(f(\cdot) \in L^2(J; Y)\). Moreover, by using Assumption 2.3 (H5), we know that the sequence \{\(I_k(x^\lambda_\tau) : \lambda > 0\)\} is bounded in \(Y\), for \(k = 1, \ldots, m\). Once again by invoking the Banach-Alaoglu theorem, we can find subsequences relabeled as \{\(I_k(x^\lambda)(\tau_k) : \lambda > 0\)\}, which are weakly convergent to the pointwise weak limit \(\eta_k \in Y\), for each \(k = 1, \ldots, m\). We now estimate
\[ \|g(x^\lambda(\cdot)) - \omega\|_X \leq \left\| \int_0^T S(T-s)[f(s, \hat{x}_\lambda^s) - f(s)]ds \right\|_X + \sum_{k=1}^m \|S(T - \tau_k)\left[I_k(x^\lambda(\tau_k)) - \eta_k\right]\|_X \to 0 \text{ as } \lambda \to 0^+, \] (51)
where
\[ \omega = x_T - S(T)E\phi(0) - \int_0^T S(T-s)f(s)ds - \sum_{k=1}^m S(T - \tau_k)\eta_k. \]

The first term in the right hand side of the inequality (51) goes to zero using the compactness of the operator \((Qf)(\cdot) = \int_0^T S(\cdot - s)f(s)ds : L^2(J; Y) \to C(J; X)\) (see Lemma 3.2, chapter 3, [34]), and the final term tends to zero using the compactness of the operator \(S(t)\), for \(t \geq 0\).

Finally, we compute \(\|x^\lambda(T) - x_T\|_X\) as
\[ \|x^\lambda(T) - x_T\|_X = \|\lambda R(\lambda, \Psi^T_\theta)g(x^\lambda(\cdot))\|_X \leq \|\lambda R(\lambda, \Psi^T_\theta)\omega\|_X + \|\lambda R(\lambda, \Psi^T_\theta)(g(x^\lambda(\cdot)) - \omega)\|_X, \] (52)
Using the estimate (51) and Assumption 2.3 (H6), we easily obtain
\[ \|x^\lambda(T) - x_T\|_X \to 0 \text{ as } \lambda \to 0^+, \]
which ensures that the system (1) is approximately controllable in \(X\).

5. Application. In this section, we provide an example to validate the results obtained in the previous sections. Before going to the example, we first consider an one dimensional elliptic boundary value problem.

5.1. An elliptic boundary value problem. Let us consider the following elliptic boundary value problem:
\[ \begin{cases} -f''(\xi) + f(\xi) = g(\xi), & 0 < \xi < \pi, \\ f(0) = f(\pi) = 0, \end{cases} \] (53)
where \(g \in L^p([0, \pi], \mathbb{R})\), for \(p \in [2, \infty)\). For any given \(g \in L^p([0, \pi], \mathbb{R})\), the existence and uniqueness of weak solution to the system (53) with the regularity \(f \in W^{2,p}([0, \pi]; \mathbb{R}) \cap W_0^{1,p}([0, \pi]; \mathbb{R})\) (in fact it becomes a strong solution, that is,
equation (53) is satisfied for a.e. \( \xi \in [0, \pi] \) is well known in the literature due to Agmon-Douglis-Nirenberg (see Theorem 9.32 Chapter 9, [12]). For completeness, we provide the energy estimates satisfied by the system (53) and the existence of weak as well as strong solution can be established using a standard Faedo-Galerkin’s approximation technique. The uniqueness easily follows from the linearity of the system (53). For \( p \geq 2 \), multiplying both sides of (53) by \( |f'|^{p-2}f \) and then integrating over \([0, \pi]\), we find

\[
\int_0^\pi |f(\xi)|^p d\xi - \int_0^\pi f''(\xi)|f(\xi)|^{p-2}f(\xi)d\xi = \int_0^\pi g(\xi)|f(\xi)|^{p-2}f(\xi)d\xi. \tag{54}
\]

Using an integration by parts, we estimate

\[
-\int_0^\pi f''(\xi)|f(\xi)|^{p-2}f(\xi)d\xi = -\int_0^\pi \left[(f'(\xi)|f(\xi)|^{p-2}f(\xi)\right)' - f'(\xi)(|f(\xi)|^{p-2}f(\xi))\right]d\xi
= \int_0^\pi f'(\xi)(f(\xi)(|f(\xi)|^2)^{\frac{p-2}{2}})'d\xi
= (p-1) \int_0^\pi |f'(\xi)|^2|f(\xi)|^{p-2}d\xi. \tag{55}
\]

Using the Hölder and Young inequalities, we estimate the term on the right hand side of the equality (54) as

\[
\int_0^\pi g(\xi)|f(\xi)|^{p-2}f(\xi)d\xi \leq \left( \int_0^\pi |g(\xi)|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^\pi |f(\xi)|^p d\xi \right)^{\frac{p-1}{p}}
\leq \frac{1}{2} \int_0^\pi |f(\xi)|^p d\xi + \frac{1}{p} \left( \frac{2(p-1)}{p} \right)^{p-1} \int_0^\pi |g(\xi)|^p d\xi. \tag{56}
\]

Substituting (55) and (56) in (54), we get

\[
\frac{1}{2} \int_0^\pi |f(\xi)|^p d\xi + (p-1) \int_0^\pi |f'(\xi)|^2|f(\xi)|^{p-2}d\xi \leq \frac{1}{p} \left( \frac{2(p-1)}{p} \right)^{p-1} \int_0^\pi |g(\xi)|^p d\xi. \tag{57}
\]

From the above inequality, it is immediate that

\[
\int_0^\pi |f(\xi)|^p d\xi \leq \frac{2}{p} \left( \frac{2(p-1)}{p} \right)^{p-1} \int_0^\pi |g(\xi)|^p d\xi = C'_p \int_0^\pi |g(\xi)|^p d\xi, \tag{58}
\]

where \( C'_p = \frac{2}{p} \left( \frac{2(p-1)}{p} \right)^{p-1} \). Next, we multiply both sides of (53) by \( |f''|^p - 2|f''|f'' \) and then integrate over \([0, \pi]\) to obtain

\[
\int_0^\pi |f''(\xi)|^p d\xi = -\int_0^\pi g(\xi)|f''(\xi)|^{p-2}f''(\xi)d\xi - \int_0^\pi |f''(\xi)|^p - 2|f''(\xi)f' d\xi. \tag{59}
\]

Similar to (56), we estimate the first term from the right hand side of the equality (59) as

\[
\int_0^\pi g(\xi)|f''(\xi)|^{p-2}f''(\xi)d\xi \leq \left( \int_0^\pi |g(\xi)|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^\pi |f''(\xi)|^p d\xi \right)^{\frac{p-1}{p}}
\]
and then integrating over \([0, 20]\)

\[
\int_0^\pi |f''(\xi)|^p d\xi + \frac{1}{p} \left( \frac{4(p - 1)}{p} \right)^{p-1} \int_0^\pi |g(\xi)|^p d\xi.
\]

(60)

Once again using Hölder’s and Young’s inequalities, we calculate the final term from the right hand side of the equality (59) as

\[
\int_0^\pi |f''(\xi)|^{p-2} f''(\xi) f(\xi) d\xi \leq \left( \int_0^\pi |f''(\xi)|^p d\xi \right)^{\frac{p-2}{p}} \left( \int_0^\pi |f(\xi)|^p d\xi \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{4} \int_0^\pi |f''(\xi)|^p d\xi + \frac{1}{p} \left( \frac{4(p - 1)}{p} \right)^{p-1} \int_0^\pi |f(\xi)|^p d\xi
\]

\[
\leq \frac{1}{4} \int_0^\pi |f''(\xi)|^p d\xi + \frac{C''}{p} \left( \frac{4(p - 1)}{p} \right)^{p-1} \int_0^\pi |g(\xi)|^p d\xi,
\]

(61)

where we used (58) also. Using (60) and (61) in (59), we get

\[
\int_0^\pi |f''(\xi)|^p d\xi \leq C'' \int_0^\pi |g(\xi)|^p d\xi,
\]

(62)

where \(C'' = \frac{2(C'_p + 1)}{p} \left( \frac{4(p - 1)}{p} \right)^{p-1} \). Finally, multiplying both sides of (53) by \(|f''|^p f^2\) and then integrating over \([0, \pi]\), we obtain

\[
\int_0^\pi |f''(\xi)|^p|f(\xi)|^2 d\xi - \int_0^\pi f''(\xi) |f'(\xi)|^{p-2} f(\xi) d\xi = \int_0^\pi g(\xi) |f'(\xi)|^{p-2} f(\xi) d\xi.
\]

(63)

An integration by parts yields

\[
- \int_0^\pi f''(\xi) |f'(\xi)|^{p-2} f(\xi) d\xi = \int_0^\pi |f'(\xi)|^p d\xi + (p - 2) \int_0^\pi |f'(\xi)|^{p-2} f''(\xi) f(\xi) d\xi.
\]

(64)

Thus, from (63), it is immediate that

\[
\int_0^\pi |f'(\xi)|^{p-2} |f(\xi)|^2 d\xi + \int_0^\pi |f'(\xi)|^p d\xi = -(p - 2) \int_0^\pi |f'(\xi)|^{p-2} f''(\xi) f(\xi) d\xi
\]

\[
+ \int_0^\pi g(\xi) |f'(\xi)|^{p-2} f(\xi) d\xi.
\]

(65)

Applying the Hölder and Young inequalities, we estimate the first term from the right hand side of the equality (65) as

\[
- (p - 2) \int_0^\pi |f'(\xi)|^{p-2} f''(\xi) f(\xi) d\xi
\]

\[
\leq (p - 2) \left( \int_0^\pi |f'(\xi)|^p d\xi \right)^{\frac{p-2}{p}} \left( \int_0^\pi |f''(\xi)|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^\pi |f(\xi)|^p d\xi \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{4} \int_0^\pi |f'(\xi)|^p d\xi + \frac{2(p - 2)^{p-1}}{p} \left( \frac{4}{p} \right)^{p-1} \left( \int_0^\pi |f''(\xi)|^p d\xi \right)^{\frac{1}{p}} \left( \int_0^\pi |f(\xi)|^p d\xi \right)^{\frac{1}{p}}
\]

\[
\leq \frac{1}{4} \int_0^\pi |f'(\xi)|^p d\xi + \frac{(p - 2)^{p-1}}{p} \left( \frac{4}{p} \right)^{p-1} \left\{ \int_0^\pi |f''(\xi)|^p d\xi + \int_0^\pi |f(\xi)|^p d\xi \right\}
\]
\[
\leq \frac{1}{4} \int_0^\pi |f'(\xi)|^p d\xi + \frac{(p-2)p^{-1}}{p} \left(\frac{4}{p}\right)^{\frac{p-2}{p}} (C'_p + C''_p) \int_0^\pi |g(\xi)|^p d\xi,
\]
where we used (58) and (62). In a similar manner, we estimate the final term from the right hand side of the equality (65) as

\[
\int_0^\pi g(\xi)|f'(\xi)|^{p-2} f(\xi) d\xi
\]
\[
\leq \frac{1}{4} \int_0^\pi |f'(\xi)|^p d\xi + \frac{(p-2)p^{-1}}{p} \left(\frac{4}{p}\right)^{\frac{p-2}{p}} (C'_p + 1) \int_0^\pi |g(\xi)|^p d\xi.
\]
Using (66) and (67) in (65), we obtain

\[
\int_0^\pi |f'(\xi)|^p d\xi \leq C''_p\int_0^\pi |g(\xi)|^p d\xi,
\]
where \(C''_p = \frac{2(p-2)p^{-1}}{p} \left(\frac{4}{p}\right)^{\frac{p-2}{p}} (1 + C'_p + C''_p)\). Combining (58), (62) and (68), we obtain

\[
\int_0^\pi |f(\xi)|^p d\xi + \int_0^\pi |f'(\xi)|^p d\xi + \int_0^\pi |f''(\xi)|^p d\xi \leq C_p \int_0^\pi |g(\xi)|^p d\xi,
\]
where \(C_p = C_p + C''_p + C''''_p\). Hence, from (69), it follows that

\[
\|f\|_{W^{2,p}} \leq C_p\|g\|_{L^p}, \quad \text{and} \quad \|f\|_{W^{1,p}} \leq C_p\|g\|_{L^p},
\]
for all \(p \geq 2\). The existence of a strong solution to the system (53) ensures that

\[
\|f\|_{W^{2,p}} \leq C_p\|f - f''\|_{L^p} \quad \text{and} \quad \|f\|_{W^{1,p}} \leq C_p\|f - f''\|_{L^p},
\]
for all \(p \geq 2\). Furthermore, we have \(\|f - f''\|_{L^p} \leq \|f\|_{W^p} + \|f''\|_{L^p} \leq \|f\|_{W^{2,p}}\) and hence the norms \(\|f - f''\|_{L^p}\) and \(\|f\|_{W^{2,p}}\) are equivalent.

### 5.2. Nonlinear impulsive functional diffusion system.

Let us consider the following impulsive system with delay:

\[
\left\{ \begin{array}{l}
\frac{d}{dt} [y(t, \xi) - y_{\tau_k}(t, \xi)] = y_{\tau_k}(t, \xi) + \eta(t, \xi) + k_0 \cos \left(\frac{2\pi t}{T}\right) \sin(y(t - r, \xi)), \\
\quad \text{for} \ t \in J = [0, T], t \neq \tau_k, k = 1, \ldots, m, \xi \in [0, \pi], k_0 > 0, \\
y(t, 0) = 0 = y(t, \pi), \quad t \in J = [0, T], \\
y(\theta, \xi) = \phi(\theta, \xi), \quad \xi \in [0, \pi], \quad \theta \leq 0, \\
y(\tau_k^+, \xi) = y(\tau_k^-, \xi) = 1_k(y(\tau_k, \xi)), \quad k = 1, \ldots, m, \xi \in [0, \pi],
\end{array} \right.
\]

where \(\phi : [-h, 0] \times [0, \pi] \to \mathbb{R}\) is a piecewise continuous function and \(\eta : J \times [0, \pi] \to [0, \pi]\) is continuous in \(t\). Let \(X_p = W^{1,p}_0([0, \pi] ; \mathbb{R}), \ Y_p = L^p([0, \pi] ; \mathbb{R})\), with \(p \in [2, \infty)\), and \(U = L^2([0, \pi] ; \mathbb{R})\), and the operators \(E_p : D(E_p) \subset X_p \to Y_p\) and \(A_p : D(A_p) \subset X_p \to Y_p\) be defined as

\[
E_p g(\xi) = g(\xi) - g''(\xi), \quad A_p g(\xi) = g''(\xi),
\]
where \(D(E_p) = D(A_p) = W^{2,p}_0([0, \pi] ; \mathbb{R}) \cap W^{1,p}_0([0, \pi] ; \mathbb{R})\). Note that \(C^\infty([0, \pi] ; \mathbb{R}) \subset W^{2,p}([0, \pi] ; \mathbb{R}) \cap W^{1,p}([0, \pi] ; \mathbb{R}) \subset W^{1,p}([0, \pi] ; \mathbb{R})\) and \(\overline{C^\infty([0, \pi] ; \mathbb{R})}^{\| \cdot \|_{W^{1,p}}} = W^{1,p}_0([0, \pi] ; \mathbb{R})\). Thus, taking closure with respect to \(\| \cdot \|_{W^{1,p}}\) in the above inclusion, we get \(D(E_p)\) is dense in \(X_p\) and it is easy to
verify that the operator $A_p$ is closed, which ensures the conditions $(P_1)$ and $(P_2)$. Moreover, the operators $A_p$ and $E_p$ can be explicitly written as

\[ A_p g = \sum_{n=1}^{\infty} -n^2 \langle g, w_n \rangle w_n, \quad g \in D(A_p), \]

\[ E_p g = \sum_{n=1}^{\infty} (1 + n^2) \langle g, w_n \rangle w_n, \quad g \in D(E_p), \]

where $-n^2 (n \in \mathbb{N})$ and $w_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$ are the eigenvalues and the corresponding normalized eigenfunctions of the operator $A_p$, respectively and the duality pairing $\langle g, w_n \rangle := \int_0^\pi g(\xi)w_n(\xi)d\xi$. Furthermore, for any $g \in \mathbb{Y}_p$, we have

\[ E_p^{-1} g = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \langle g, w_n \rangle w_n, \quad (73) \]

\[ A_p E_p^{-1} g = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2} \langle g, w_n \rangle w_n. \quad (74) \]

From the expression (74), we have

\[ A_p E_p^{-1} g = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2} \langle g, w_n \rangle w_n = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \langle g, w_n \rangle w_n - \sum_{n=1}^{\infty} \langle g, w_n \rangle w_n \\
= \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \langle g, w_n \rangle w_n - g. \quad (75) \]

Using (75), we obtain that

\[ \|A_p E_p^{-1} g\|_{L^p} \leq \left\| \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \langle g, w_n \rangle w_n \right\|_{L^p} + \|g\|_{L^p} \\
\leq \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \|\langle g, w_n \rangle\|_p \|w_n\|_{L^p} + \|g\|_{L^p} \\
\leq \sqrt{2\pi} \|g\|_p \sum_{n=1}^{\infty} \frac{1}{n^2} \|w_n\|_{L^p} + \|g\|_{L^p} \\
\leq \left( \frac{\pi^2}{3} + 1 \right) \|g\|_{L^p}. \quad (76) \]

Thus, the operator $A_p E_p^{-1} : \mathbb{Y}_p \to \mathbb{Y}_p$ is bounded. Using Theorem 1.2, Chapter 1, [45], we infer that the operator $A_p E_p^{-1}$ generate a uniformly continuous semigroup $\{T_p(t) : t \geq 0\}$ of bounded linear operators on $\mathbb{Y}_p$. The explicit expression of $T_p(t)$ can be obtained as

\[ T_p(t)g = \sum_{n=1}^{\infty} \exp\left( \frac{-n^2t}{1 + n^2} \right) \langle g, w_n \rangle w_n, \quad \text{for all } g \in \mathbb{Y}_p. \quad (77) \]

From the above expression, it is clear that the semigroup $T_p(t)$ is not compact, as the eigenvalues of compact operators can only accumulate at 0. Moreover, the family of operators $S_p(t) : \mathbb{Y}_p \to \mathbb{X}_p$, for $t \geq 0$ can be written as

\[ S_p(t)g = \sum_{n=1}^{\infty} \frac{1}{1 + n^2} \exp\left( \frac{-n^2t}{1 + n^2} \right) \langle g, w_n \rangle w_n, \quad \text{for all } g \in \mathbb{Y}_p. \quad (78) \]
Next, we show that the operator $E_p^{-1} : \mathcal{Y}_p \to D(E_p)$ is compact. For any $f \in D(E_p)$, we define $\|f\|_{D(E_p)} := \|E_p f\|_{L^p}$. From the inequality (71), we infer that the embedding and (79), we estimate

$$\|f\|_{W^{1,p}} \leq C_p \|E_p f\|_{L^p} = C_p \|f\|_{D(E_p)}.$$  \hfill (79)

Let $\{g_n\}_{n \in \mathbb{N}} \in L^p([0, \pi]; \mathbb{R})$ be a bounded sequence such that $\|g_n\|_{L^p} \leq K$. Then, we have $\|E_p^{-1} g_n\|_{D(E_p)} = \|E_p E_p^{-1} g_n\|_{L^p} = \|g_n\|_{L^p} \leq K$ and hence $\{E_p^{-1} g_n\}_{n \in \mathbb{N}} \in D(E_p)$, is uniformly bounded. From Morrey's inequality (Theorem 9.12, [12]), we infer that the embedding of $W^{1,p}_0([0, \pi]; \mathbb{R}) \subset C^{0,1/2}([0, \pi]; \mathbb{R})$ is continuous. Using this embedding and (79), we estimate

$$\|E_p^{-1} g_n(\xi)\| \leq \|E_p^{-1} g_n\|_{C^{0,1/2}} \leq \|E_p^{-1} g_n\|_{W^{1,p}} \leq C_p \|E_p^{-1} g_n\|_{D(E_p)} \leq C_p K = \tilde{C}_p,$$

for all $\xi \in [0, \pi]$. Moreover, we have

$$|E_p^{-1} g_n(\xi) - E_p^{-1} g_n(\zeta)| \leq C \|E_p^{-1} g_n\|_{W^{1,p}} |\xi - \zeta|^{1/2} \leq C \tilde{C}_p |\xi - \zeta|^{1/2}.$$

Making use of the Arzelá-Ascoli theorem, we can find a subsequence labeled as $\{E_p^{-1} g_n\}_{n \in \mathbb{N}}$, which converges uniformly. Thus, we conclude that $E_p^{-1}$ is compact. Hence, the condition $(P_3)$ holds. Since the operator $E_p^{-1}$ is compact and the operator $T_p(t)$ is bounded, for each $t \geq 0$, the operator $S_p(t)$ is compact, for each $t \geq 0$.

Let us define

$$x(t)(\xi) := y(t, \xi), \quad \text{for } t \in J \text{ and } \xi \in [0, \pi],$$

and the bounded linear operator $B : L^2([0, \pi]; \mathbb{R}) \to \mathcal{Y}_p$ as

$$B(u(t))(\xi) := u(t)(\xi) = \eta(t, \xi), \quad t \in J, \quad \xi \in [0, \pi].$$

Next, we define $f : J \times D \to \mathcal{Y}_p$ as

$$f(t, x_t)(\xi) := k_0 \cos \left(\frac{2\pi t}{T}\right) \sin(x_t), \quad \xi \in [0, \pi],$$

where $k_0$ is some positive constant and $D$ is defined in (2). Clearly, $f$ is continuous and

$$\|f(t, x_t)\|_{L^p} = k_0 \left(\int_0^\pi \left|\cos \left(\frac{2\pi t}{T}\right) \sin(x_t(\zeta))\right|^p d\zeta\right)^{1/p} \leq k_0 \pi \frac{2}{T} \left|\cos \left(\frac{2\pi t}{T}\right)\right| = \gamma_r(t).$$

It should be noted that $\gamma_r(t) = k_0 \pi \frac{2}{T} \left|\cos \left(\frac{2\pi t}{T}\right)\right| \in L^1(J; \mathbb{R}^+)$ and satisfies

$$\lim_{r \to \infty} \int_r^\infty \frac{k_0 \pi \frac{2}{T} \left|\cos \left(\frac{2\pi t}{T}\right)\right| dt}{r} = 0.$$ 

Thus, the above facts ensures the condition $(H_2)$ of Assumption 2.3. Note that here $\beta = 0$ and the condition (35) given in Theorem 4.1 is satisfied. We now consider $I_k : X_p \to \mathcal{Y}_p$, for each $k = 1, \ldots, m$ as

$$I_k((x(\tau_k))(\xi)) := I_k(y(\tau_k, \xi)).$$

Let us choose

$$I_k((x(\tau_k))(\xi)) = \int_0^\pi \rho_k(\xi, \zeta) \cos^2(x(\tau_k)(\zeta)) d\zeta,$$

where $\rho_k \in C([0, \pi] \times [0, \pi]; \mathbb{R})$. The impulses $I_k$, for each $k = 1, \ldots, m$ satisfy condition $(H_3)$ of Assumption 2.3 (see [3] also).

Using the above substitution, the system (72) can be expressed as an abstract form given in (1) satisfying the conditions $(P_1)$-$(P_3)$ and Assumptions 2.3 and 4.3. Now, it remains to show that the corresponding linear system of (1) is approximately
satisfies and \( \gamma \) controllable. In order to prove this, we consider \( B^* S_p^*(T - t)x^* = 0 \), for any \( x^* \in X^* \). Since \( B = I \), then we have
\[
B^* S_p^*(T - t)x^* = 0 \Leftrightarrow S_p^*(T - t)x^* = 0 \Rightarrow x^* = 0,
\]
and hence by Theorem 3.5, we obtain that the linear system corresponding to (1) is approximately controllable. Thus, the condition \((H_0)\) of Assumption 2.3 holds. Finally, by applying Theorem 4.4, we can conclude that the semilinear system (1) (equivalent to the system (72)) is approximately controllable.

**Remark 5.1.** The assumptions of Theorem 4.4 are sufficient conditions only. One can relax some of the conditions and obtain the approximate controllability of the system (1). For instance, one can replace the uniform boundedness condition of Assumption 4.3 given in Theorem 4.4, by the following weaker assumption (see Theorem 3.2, [15]):

\( (C) \) The function \( f : J \times D \to Y \) is continuous and there exists a function \( \sigma \in L^2(J; \mathbb{R}^+) \) such that \( \| f(t, \phi) \|_Y \leq \sigma(t) \), for all \( (t, \phi) \in J \times D \).

Under this condition, we can prove that the system (1) is approximately controllable in the following way. Let \( x^*(\cdot) \) be a mild solution of the equation (1) with the control \( u_\lambda \) defined in (48). Analogous to (50), using the condition \((C)\), we easily deduce that
\[
\left( \int_0^T \| f(s, x^*(s)) \|_Y^2 \, ds \right)^{\frac{1}{2}} \leq \left( \int_0^T \sigma^2(s) \, ds \right)^{\frac{1}{2}} < +\infty,
\]
and consequently the sequence \( \{ f(\cdot, x^*(\cdot)) : \lambda > 0 \} \) in \( L^2(J; \mathcal{Y}) \) is bounded. The rest of the proof can be carried out in a similar way as that of the proof of Theorem 4.4. Hence, the system (1) is approximately controllable under the condition \((C)\) as well.

Next, we provide an example which ensures that the conditions given in Theorem 4.4 is only sufficient but not necessary for approximate controllability of the system (1). For this, let us replace the nonlinear function \( f : J \times D \to \mathcal{Y} \), in the example (72) as
\[
f(t, x_t)(\xi) := \begin{cases} k_0 \left( \frac{\sin(2\pi t)}{t} \right)^{1/4} \cos(x_t), & t \in (0, T], \\ k_0 \cos(x_t), & t = 0, \end{cases}
\]
for \( \xi \in [0, \pi] \), where \( k_0 \) is some positive constant and \( D \) is defined in (2). Clearly, \( f \) is continuous and for \( t \in (0, T] \), we have
\[
\| f(t, x_t) \|_{L^p} = k_0 \left( \int_0^\pi \left( \frac{\sin(2\pi t)}{t} \right)^{1/4} \cos(x_t(\xi)) \, d\xi \right)^{1/p} \leq k_0 \pi^{\frac{1}{4}} \left( \frac{\sin(2\pi t)}{t} \right)^{1/4}.
\]
and \( \gamma(0) := k_0 \pi^{\frac{1}{4}} \). It should be noted that \( \gamma_r(t) \in L^2(J; \mathbb{R}^+) \subset L^1(J; \mathbb{R}^+) \) and satisfies
\[
\liminf_{r \to \infty} \frac{\int_0^T \gamma_r(t) \, dt}{r} = 0.
\]
Thus, the condition \((H_2)\) of Assumption 2.3 is fulfilled. Clearly, the nonlinear function \( f(\cdot, \cdot) \) is not uniformly bounded and hence Assumption \((H_4)\) of (4.3) is not satisfied. Moreover, by the above facts it is clear the nonlinear function \( f(\cdot, \cdot) \)
satisfies the condition (C). Therefore, the system (72) with the nonlinear function defined in (83) is approximately controllable.

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REFERENCES

[1] S. Agarwal and D. Bahuguna, Existence of solutions to Sobolev type partial neutral differential equations, *J. Appl. Math. Stoch. Anal.*, **2006** (2006), 16308, 1–10.
[2] O. Arino, M. L. Habid and R. Bravo de la Parra, A mathematical model of growth of population of fish in the larval stage: Density dependence effects, *Math. Biosci.*, **150** (1998), 1–20.
[3] S. Arora, S. Singh, J. Dabas and M. T. Mohan, Approximate controllability of semilinear impulsive functional differential system with nonlocal conditions, *IMA J. Math. Control Inform.*, (2020).
[4] K. Balachandran and N. Annapoorani, Existence results for impulsive neutral evolution integrodifferential equations with infinite delay, *Nonlinear Anal. Hybrid Syst.*, **3** (2009), 674–684.
[5] K. Balachandran and T. N. Gopal, Approximate controllability of nonlinear evolution systems with time varying delays, *IMA J. Math. Control Inform.*, **23** (2006), 499–513.
[6] K. Balachandran and J. Y. Park, Sobolev type integrodifferential equation with nonlocal condition in Banach spaces, *Taiwanese J. Math.*, **7** (2003), 155–163.
[7] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press, New York, 1993.
[8] A. E. Bashirov and N. I. Mahmudov, On concepts of controllability for deterministic and stochastic systems, *SIAM J. Control Optim.*, **37** (1999), 1808–1821.
[9] W. M. Bian, Approximate controllability of semilinear systems, *Acta Math. Hungar.*, **81** (1998), 41–57.
[10] W. M. Bian, Controllability of nonlinear evolution systems with preassigned responses, *J. Optim. Theory Appl.*, **100** (1999), 265–285.
[11] J. M. Borwein and J. Vanderwerff, Fréchet-Legendre functions and reflexive Banach spaces, *J. Convex Anal.*, **17** (2010), 915–924.
[12] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
[13] H. Brill, A semilinear Sobolev evolution equation in Banach space, *J. Differential Equations*, **24** (1977), 412–425.
[14] Y.-K. Chang, A. Pereira and R. Ponce, Approximate controllability for fractional differential equations of Sobolev type via properties on resolvent operators, *Pract. Calc. Appl. Anal.*, **20** (2017), 963–987.
[15] P. Chen, X. Zhang and Y. Li, Approximate controllability of non-autonomous evolution system with nonlocal conditions, *J. Dyn. Control Syst.*, **26** (2020), 1–16.
[16] R. F. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, New York, 1999.
[17] V. N. Do, A note on approximate controllability of semilinear systems, *Systems Control Lett.*, **12** (1989), 365–371.
[18] I. Ekeland and T. Turnbull, *Infinite Dimensional Optimization and Convexity*, Chicago press, London, 1983.
[19] W. E. Fitzgibbon, Semilinear functional differential equations in Banach spaces, *J. Differential Equations*, **29** (1978), 1–14.
[20] C. Gao, K. Li, E. Feng and Z. Xiu, Nonlinear impulse system of fed-batch culture in fermentative production and its properties, *Chaos Solitons Fractals*, 28 (2006), 271–277.

[21] S. Gao, L. Chen, J. J. Nieto and A. Torres, Analysis of a delayed epidemic model with pulse vaccination and saturation incidence, *Vaccine*, 24 (2006), 6037–6045.

[22] R. K. George, Approximate controllability of non-autonomous semilinear systems, *Nonlinear Anal.*, 24 (1995), 1377–1393.

[23] A. Grudzka and K. Rykaczewski, On approximate controllability of functional impulsive evolution inclusions in a Hilbert space, *J. Optim. Theory Appl.*, 166 (2015), 414–439.

[24] E. Hernández, R. Sakthivel and S. Tanaka Aki, Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differential Equations*, 2008 (2008), 28, 1–11.

[25] J.-M. Jeong and H.-H. Roh, Approximate controllability for semilinear retarded systems, *J. Appl. Math. Anal. Appl.*, 321 (2006), 961–975.

[26] M. Kerboua, A. Debbouche and D. Baleanu, Approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces, *Abstr. Appl. Anal.*, 2013 (2013), 262191, 10pp.

[27] J. Klamka, Constrained controllability of semilinear systems with delays, *Nonlinear Dyn.*, 56 (2009), 169–177.

[28] J. Klamka, Schauder’s fixed point theorem in nonlinear controllability problems, *Control Cybernet.*, 29 (2000), 153–165.

[29] J. Klamka, *Controllability and Minimum Energy Control*, in Series Studies in Systems, Decision and Control, Springer-Verlag, New York, 2019.

[30] J. Klamka, A. Babiarz and M. Niezabitowski, Banach fixed-point theorem in semilinear controllability problems—a survey, *Bull. Polish Acad. Sci. Tech. Sci.*, 64 (2016), 21–35.

[31] J. Klamka, A. Babiarz and M. Niezabitowski, Schauder’s fixed point theorem in approximate controllability problems, *Int. J. Appl. Math. Comput. Sci.*, 26 (2016), 263–275.

[32] K. D. Kucche and M. B. Dhakne, Sobolev type Volterra-Fredholm functional integrodifferential equations in Banach spaces, *Bol. Soc. Parana. Mat.*, 32 (2014), 239–255.

[33] H. Leiva and P. Sundar, Approximate controllability of the Burgers equation with impulses and delay, *Far East J. Math. Sci.*, 102 (2017), 2291–2306.

[34] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser Boston, Boston, 1995.

[35] J. H. Lightbourne and S. M. Rankin, A partial functional differential equation of Sobolev type, *J. Appl. Math. Anal. Appl.*, 93 (1983), 328–337.

[36] A. Lunardi, On the linear heat equation with fading memory, *SIAM J. Math. Anal.*, 21 (1990), 1213–1224.

[37] N. I. Mahmudov, Approximate controllability of fractional Sobolev type evolution equations in Banach Spaces, *Abstr. Appl. Anal.*, 2013 (2013), 502839, 1–9.

[38] N. I. Mahmudov, Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces, *SIAM J. Control Optim.*, 42 (2003), 1604–1622.

[39] N. I. Mahmudov, Existence and approximate controllability of Sobolev type fractional stochastic evolution equations, *Bull. Polish Acad. Sci. Tech. Sci.*, 62 (2014), 205–215.

[40] M. McKibben, A note on the approximate controllability of a class of abstract semilinear evolution equations, *Far East J. Math. Sci.*, 5 (2002), 113–133.

[41] M. T. Mohan, On the three dimensional Kelvin-Voigt fluids: Global solvability, exponential stability and exact controllability of Galerkin approximations, *Evol. Equ. Control Theory*, 9 (2020), 301–339.

[42] K. Naito, Controllability of semilinear control systems dominated by the linear part, *SIAM J. Math. Anal.*, 25 (1987), 715–722.

[43] K. Naito, Approximate controllability for a semilinear control system, *J. Optim. Theory Appl.*, 60 (1989), 57–65.

[44] J. W. Nunziato, On heat conduction in materials with memory, *Quart. Appl. Math.*, 29 (1971), 187–204.

[45] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[46] K. Ravikumar, M. T. Mohan and A. Anguraj, Approximate controllability of a non-autonomous evolution equation in Banach spaces, *Numer. Algebra Control Optim.*, (2020).

[47] R. Sakthivel and E. R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, *Internat. J. Control*, 83 (2010), 387–393.
[48] R. Sakthivel, N. I. Mahmudov and J. H. Kim, Approximate controllability of nonlinear differential systems, *Rep. Math. Phys.*, 60 (2007), 85–96.

[49] A. M. Samoilenko, N. A. Perestyuk and Y. Chapovsky, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[50] R. E. Showalter, Existence and representation theorem for a semilinear Sobolev equation in Banach space, *SIAM J. Math. Anal.*, 3 (1972), 527–543.

[51] S. Tang and L. Chen, Density-dependent birth rate, birth pulses and their population dynamic consequences, *J. Math. Biol.*, 44 (2002), 185–199.

[52] R. Triggiani, Addendum: A note on the lack of exact controllability for mild solutions in Banach spaces, *SIAM J. Control Optim.*, 18 (1980), 98–99.

[53] R. Triggiani, A note on the lack of exact controllability for mild solutions in Banach spaces, *SIAM J. Control Optim.*, 15 (1977), 407–411.

[54] V. Vijayakumar, Approximate controllability results for impulsive neutral differential inclusions of Sobolev type with infinite delay, *Internat. J. Control*, 91 (2018), 2366–2386.

[55] L. Wang, Approximate controllability of delayed semilinear control systems, *J. Appl. Math. Stoch. Anal.*, 2005 (2005), 67–76.

[56] J. Wang, M. Fečkan and Y. Zhou, Approximate controllability of Sobolev type fractional evolution systems with nonlocal conditions, *Evol. Equ. Control Theory*, 6 (2017), 471–486.

[57] E. Zuazua, Controllability and observability of partial differential equations: Some results and open problems, *Handbook of Differential Equations: Evolutionary Equations*, 3 (2007), 527–621.

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