DATA-DRIVEN STABILIZATION OF
SISO FEEDBACK LINEARIZABLE SYSTEMS

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ABSTRACT. In this paper we propose a methodology for stabilizing single-input single-output feedback linearizable systems when no system model is known and no prior data is available to identify a model. Conceptually, we have been greatly inspired by the work of Fliess and Join on intelligent PID controllers (e.g., [FJ09, FJ13]) and the results in this paper provide sufficient conditions under which a modified version of their approach is guaranteed to result in asymptotically stable behavior. One of the key advantages of the proposed results is that, contrary to other approaches to controlling systems without a model (or with a partial model), such as reinforcement learning, there is no need for extensive training neither large amounts of data. Technically, our results draw heavily from the work of Nesic and co-workers on observer and controller design based on approximate models [AN04, NT04]. Along the way we also make connections with other well established results such as high-gain observers and adaptive control. Although we focus on the simple setting of single-input single-output feedback linearizable systems we believe the presented results are already theoretically insightful and practically useful, the last point being substantiated by experimental evidence.

1. Introduction

1.1. Motivation. This paper was motivated by two initially independent lines of inquiry: the thought-provoking work of Fliess and Join on intelligent PID controllers [FJ09, FJ13], and the growing impact that machine learning, in particular deep learning, recently had in a wide variety of engineering problems [LBH15, Sch15]. Curiously, the techniques of Fliess and Join can be seen as a method to transform sensor measurement data into control inputs with minimal reliance on plant models. Therefore, we can interpret intelligent PID controllers as data-driven controllers and this is the view espoused in this work.

1.2. Contribution. The main contribution of this paper is to identify a class of nonlinear systems for which a modified version of intelligent PID controllers can guarantee asymptotic stability. This is by no means the largest class of such systems, but a large enough class to make the technical contribution of this paper relevant to applications as illustrated by the experimental results presented in Section 10. Moreover, the techniques used to prove the results are also of interest as they rely on an apparently unrelated line of work by Nesic and co-workers [AN04, NT04] on state estimation and control based on approximate models. In particular, we show in this paper how the results in [AN04, NT04] can be used to provide a formal justification for the working assumption upon which the work of Fliess and Join [FJ09, FJ13] relies: the sampling rate can be made high enough so that the relevant signals can be considered constant in between sampling instants.

Although the use of learning techniques has been surging within the control community, learning has always been an integral part of the scientific discipline of control. Classical bodies of work within control, such as a...
system identification [Liu99, vOdM12] and adaptive control [NA89, IS12], are essentially learning techniques
tailored to the needs of control. The results in this paper make connections with, and some times have been
inspired by, such classical results. We shall expose several of these connections throughout the paper although
readers with a different background may see other connections that have eluded the authors. Yet, it matters
to highlight the advantages of the results in this paper over other learning techniques for control. First,
the proposed data-driven controllers do not require large amounts of data neither lengthy offline or online
training. In this sense, they are much closer to adaptive control than to techniques based on reinforcement
learning [SB18] or deep learning [GBC16]. However, contrary to most work on adaptive control that relies
on linearly parameterized models (for the plant or controller), the proposed data-driven controllers require
no assumption on the functions to be learned other than smoothness\footnote{Smoothness is assumed for simplicity, most technical results only require a small number of derivatives.} This follows from a key insight in the
work of Fliess and Join: rather than learning a nonlinear model of the plant, described by a few nonlinear
functions, it suffices to learn the value of these functions at the current state. Hence, we always work on small
finite-dimensional spaces and, for this reason, only need small amounts of data. A further advantage of the
proposed data-driven controllers is that its users only need to use linear control techniques, an observation
that justifies the well crafted title of [FJ09]. Finally, the results in this paper should be regarded as a design
methodology since its key steps can be performed by resorting to different techniques. To show feasibility of
the approach and ease of use, we propose a specific technique for each step although it should be clear these
are by no means unique or even the best. We shall return to this point in more detail in Section 5 where we
provide an outline of the proposed data-driven control methodology.

We would be remiss if we would not give due importance to the limitations of the proposed data-driven control
methodology: it can be quite sensitive to measurement noise. This is a consequence of the need to estimate
derivatives of sensed signals. While we leave a detailed study of how to best handle noise for future work, the
experimental results in Section 10 already offer evidence that the proposed data-driven methodology can be
practically useful despite the aforementioned limitation.

1.3. Related work. As previously stated, the results in this paper were directly inspired by the work of
Fliess and Join on intelligent PID controllers. We regard the papers [FJ09, FJ13] as entry points into this
literature since the number of papers on this topic has been growing over the last ten years. The main
contributions with respect to this line of work are: 1) to rigorously formalize the idea that signals can be
treated as constant in between sampling times provided the sampling rate is high enough; 2) to identify a class
of nonlinear systems for which this type of data-driven controllers are guaranteed to result in asymptotically
stable behavior. This was accomplished by: 1) proposing several modifications to intelligent PID controllers;
2) a feedback linearizability assumption; 3) and leveraging the work of Nesic and co-workers on estimation and
control based on approximate models. Moreover, we also address the case where the control gain is unknown
whereas it is assumed to be known in the intelligent PID literature.

Two recent papers [CLD19, PT20], inspired by behavioral techniques, have also proposed data-driven control
techniques. It is shown, in both cases, that the proposed controllers can be used with nonlinear systems even
though they were developed for linear systems. The key requirement is that the mismatch between the linear
and nonlinear models is small. A similar idea is used in this paper: by suitably increasing the sampling rate a
point-wise linear approximation suffices for control. For this reason the authors suspect it may be possible to
combine these different perspectives to obtain even stronger results. The use of behavioral techniques for the
development of data-driven control techniques is not recent and had been advocated before, see [MR08, MR09].
However the algorithms proposed in this earlier work are better suited for offline computation as they require
several complex matrix operations. All the aforementioned papers, as well as [BKP19], rely on acquiring
enough sufficiently informative data to produce control inputs (see [vWETC19] for a discussion on how much
informative data is required for different control tasks). This requires that enough experiments are conducted
using persistently exciting inputs. In contrast, no prior data or persistency of excitation is required for the
results in this paper.
The previous observation sets the current paper apart from much work on data-driven control, not all cited in this paper. There is, yet, another distinguishing feature with respect to other data-driven control approaches that directly work with unknown nonlinear functions, see, e.g., [JJ17] and the literature cited therein. The data-driven controllers proposed in this paper only learn the value of a few nonlinear functions at the current state whereas the previously cited literature seeks to directly learn nonlinear functions or approximations thereof.

Preliminary versions of the results in this paper appeared in the conference publications [TMGA17, TF19]. While in [TMGA17] the control gain is assumed to be known this assumption was dropped in [TF19]. However, the results in [TF19] rely on a persistency of excitation assumption that is difficult to verify in practice. In this paper we do not assume the control gain to be known (although we assume the knowledge of upper and lower bounds) neither persistency of excitation.

2. Notation

2.1. Miscellania. The natural numbers, including zero, are denoted by $\mathbb{N}$, the real numbers by $\mathbb{R}$, the non-negative real numbers by $\mathbb{R}_0^+$, and the positive real numbers by $\mathbb{R}^+$. If $c : \mathbb{R} \to \mathbb{R}^n$ is a function of time, we denote its first time derivative by $\dot{c}$. When higher time derivatives are required, we use the notation $c^{(k)}$ defined by the recursion $c^{(1)} = \dot{c}$ and $c^{(k+1)} = (c^{(k)})^{(1)}$. The Lie derivative of a function $h : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, given by $\frac{\partial h}{\partial f}$, is denoted by $L_f h$.

Given a symmetric matrix $Q$ we denote by $\lambda_{\min}(Q)$ its smallest eigenvalue and by $\lambda_{\max}(Q)$ its largest eigenvalue.

2.2. Big O notation. Consider a function $f : \mathbb{R}_0^+ \times \mathcal{S} \to \mathbb{R}^n$ with $\mathcal{S} \subseteq \mathbb{R}^n$. We will use the notation $f(t, x) = O_x(T)$ to denote the existence of constants $M, T \in \mathbb{R}^+$ so that for all $t \in [0, T]$ and $x \in \mathcal{S}$ we have $\|f(t, x)\| \leq MT\|x\|$ with $\|x\|$ denoting the 2-norm of $x$. The following rules apply to this notation where the equalities below are to be used to replace the left-hand side with the right-hand side:

$$O_x(T^2) = O_x(T), \quad (O_x(T))^2 = O_x(T^2), \quad T O_x(T) = O_x(T^2), \quad g(x) O_x(T) = O_x(T).$$

The subscript $x^2$ in $O_x(T^2)$ indicates we are squaring the norm, i.e., $O_x(T^2)$ denotes the upper bound $MT^2\|x\|^2$. Moreover, the function $g$ is assumed to have bounded norm, i.e., there exists $b \in \mathbb{R}^+$ so that $\|g(x)\| \leq b$ for all $x \in \mathcal{S}$. To illustrate the use of these equalities, consider the equality $f(t, x) = O_x(T^2)$ which is defined by $\|f(t, x)\| \leq MT^2\|x\|$. If we take $T \leq 1$, we have the bound $T^2 \leq T$ that enables us to conclude $\|f(t, x)\| \leq MT\|x\|$, i.e., $f(t, x) = O_x(T)$. Using the above rules we can directly replace $f(t, x) = O_x(T^2)$ with $f(t, x) = O_x(T)$.

3. Models

We consider an unknown single-input single-output nonlinear system described by:

\begin{align}
\dot{x} &= f(x) + g(x)u \\
y &= h(x) + d,
\end{align}

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions and we denote by $y \in \mathbb{R}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $d \in \mathbb{R}$, the output, state, input, and measurement noise, respectively. We make the assumption that the output function $h$ has relative degree $n$, i.e., this system is feedback linearizable. This means that $L_i L_j h(x) = 0$ for $i = 0, \ldots, n - 2$ and $L_y L_{\bar{j}}^{-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$. Since the function $L_y L_{\bar{j}}^{-1} h$ is continuous and never zero, its sign is constant. We will assume the sign of $L_y L_{\bar{j}}^{-1} h$ to be known and, without
loss of generality, take it to be positive. Moreover, we assume the knowledge of two constants \( \beta, \bar{\beta} \in \mathbb{R}^+ \) such that:

\[
(3.3) \quad \beta \leq L_y L_f^{-1} h(x) \leq \bar{\beta},
\]

for all \( x \in \mathbb{R}^n \). Knowledge of the sign of \( L_y L_f^{-1} h \) is not a strong assumption beyond \( L_y L_f^{-1} h \neq 0 \). A simple input/output experiment can be performed to infer the sign of \( L_y L_f^{-1} h \). Knowledge of the constants \( \beta \) and \( \bar{\beta} \) is a stronger assumption that can be justified whenever some structural information about the system is available. For example, if a point mass with mass \( m \) moves in a one dimensional line by the action of a force \( u \), we have the model \( m \dot{x}^{(2)} = u \) where \( x \) denotes the mass’ position that is measured by a sensor, i.e., \( y = h(x) = x \). In this case we have \( L_y L_f^{-1} h = 1/m \) and thus knowledge of the bounds \( \underline{\beta} \) and \( \bar{\beta} \) requires some prior knowledge about the range of possible mass values.

With the objective of presenting the results in its most understandable form, we assume \( n = 2 \) throughout this paper, although all the results hold for arbitrary \( n \in \mathbb{N} \). This will enable us to preform all the necessary computations explicitly and without the need for distracting bookkeeping.

Invoking the feedback linearizability assumption, we can rewrite the unknown dynamics in the coordinates \( (z_1, z_2) = \Psi(x) = (h(x), L_f h(x)) \):

\[
(3.4) \quad \dot{z}_1 = z_2
\]

\[
(3.5) \quad \dot{z}_2 = \alpha(z) + \beta(z) u
\]

\[
(3.6),
\]

where \( \alpha = L_f^2 h \circ \Psi^{-1} \) and \( \beta = L_y L_f h \circ \Psi^{-1} \). We note that \( f, g, \) and \( h \) are unknown and thus so are \( \alpha \) and \( \beta \). This form of the dynamics has the advantage of using the two scalar valued functions \( \alpha \) and \( \beta \) to describe the full dynamics, independently of the value of \( n \). We exploit this observation by learning the values of these functions online.

System \( (3.4)-(3.6) \) will be controlled using piecewise constant inputs for a sampling time \( T \in \mathbb{R}^+ \). This means that inputs \( u : [0, T] \rightarrow \mathbb{R} \) satisfy the following equality for all \( k \in \mathbb{N} \):

\[
u(kT) = u(kT + \tau), \quad \forall \tau \in [0, T].
\]

It will be convenient to use \( u \) to denote an input only defined on \([0, T]\). Since the curve \( u \) is constant on the interval \([0, T]\), we identify it with the corresponding element of \( \mathbb{R} \).

The solution of \( (3.4)-(3.5) \) is denoted by \( F^e_t(z, u) = (F^e_{t,1}(z, u), F^e_{t,2}(z, u)) \), for \( t \in [0, T] \), and satisfies \( F^e_0(z, u) = z \). The superscript “\( e \)” reminds us that this is an exact solution. In the next section we discuss approximate solutions.

4. Approximate models

In this section we develop an approximate solution of \( (3.4)-(3.5) \) based on the well known Taylor’s theorem that we now recall.

**Theorem 4.1** ([Rud76]). Let \( c : \mathcal{I} \rightarrow \mathbb{R}^n \) be an \( n \) times differentiable function where \( \mathcal{I} \subseteq \mathbb{R} \) is an open and connected set. For any \( t, \tau \in \mathcal{I} \) such that \( \tau + t \in \mathcal{I} \) we have:

\[
c(\tau + t) = c(\tau) + c^{(1)}(\tau) t + \frac{c^{(2)}(\tau)}{2} t^2 + \ldots + \frac{c^{(n-1)}(\tau)}{(n-1)!} t^{n-1} + \frac{c^{(n)}(\tau')}{n!} t^n,
\]

for some \( \tau' \in [\tau, \tau + t] \).

Applying this result to \( F^e_{\tau+t,1} \) we obtain:

\[
F^e_{\tau+t,1}(z, u) = F^e_{\tau,1}(z, u) + \left( F^e_{\tau,1} \right)^{(1)}(z, u) t + \left( F^e_{\tau,1} \right)^{(2)}(z, u) \frac{t^2}{2} + \left( F^e_{\tau,1} \right)^{(3)}(z, u) \frac{t^3}{3!}.
\]
If we only retain the first three terms we obtain an approximate solution with an approximation error given by the magnitude of the (neglected) fourth term. The following result provides a bound for the approximation error in a form useful for the results derived in this paper.

**Proposition 4.2.** Let \( D \subset \mathbb{R}^3 \) be a compact set. Then, there exist \( T \in \mathbb{R}^+ \) and \( M \in \mathbb{R}^+ \) such that:

\[
(F_{\tau,1}^e)^{(3)}(z,u) \frac{t^3}{3!} \leq MT^3 \| (z,u-u_0) \|, 
\]

for all \((z,u) \in D\), all \(t, \tau' \in [0,T]\), and where \(u_0 = -\alpha(0)\beta^{-1}(0)\).

Using the \( O \) notation, this result states that:

\[
(F_{\tau,1}^e)^{(3)}(z,u) \frac{t^3}{3!} = O_{(z,u-u_0)}(T^3). 
\]

**Proof.** Since (3.4)-(3.5) is a smooth differential equation (recall that inputs are constant), solutions exist for all \( \tau \in [0,T_{z,u}] \) where \( [0,T_{z,u}] \) is the maximal interval for which the solution \( F_{\tau,1}^e(z,u) \) exists. The function \( (z,u) \rightarrow T_{z,u} \) is lower semi-continuous and, given that \((z,u)\) belongs to the compact set \( D\), it achieves its minimum on \( D\). Let \( T \in \mathbb{R}^+ \) be smaller than \( \min_{(z,u) \in D} T_{z,u} \). By definition of \( T \), for any \((x,u) \in D\) solutions exist on the interval \([0,T]\). Consider now the function \( (F_{\tau,1}^e)^{(3)} \) and note it is continuously differentiable, by assumption, and thus Lipschitz continuous on \( D \times \{\tau'\} \) for each fixed \( \tau' \in [0,T] \). Hence, by definition of Lipschitz continuity we have:

\[
(F_{\tau,1}^e)^{(3)}(z,u) - (F_{\tau,1}^e)^{(3)}(z',u') \leq L(\tau') \| (z,u) - (z',u') \| 
\]

for all \((z,u),(z',u') \in D\) and all \( \tau' \in [0,T] \). Noting that, according to (3.4)-(3.5), \( F_{\tau,1}^e(0,u_0) = 0 \) for \( u_0 = -\alpha(0)\beta^{-1}(0) \) and all \( \tau' \in [0,T] \), we conclude that \( (F_{\tau,1}^e)^{(3)}(0,u_0) = 0 \). Using this equality in (4.2) we obtain:

\[
(F_{\tau,1}^e)^{(3)}(z,u) \leq L(\tau') \| (z,u) - (0,u_0) \| = L(\tau') \| (z,u-u_0) \|, 
\]

by setting \( z' = 0 \) and \( u' = -u_0 \). If we now take \( M = \frac{1}{3} \max_{\tau' \in [0,T]} L(\tau') \) we obtain the desired inequality. Note that \( M \) is well defined since \( L \) is continuous and \([0,T]\) compact. \( \square \)

Based on Proposition 4.2, we can write the exact solution \( F_{\tau}^c \) of (3.4)-(3.5) valid for all \( t \in [0,T] \), as:

\[
F_{\tau,1}^e(z,u) = z_1 + z_2t + (\alpha(z) + \beta(z)u) \frac{t^2}{2} + O_{(z,u-u_0)}(T^3) 
\]

\[
F_{\tau,2}^e(z,u) = z_2 + (\alpha(z) + \beta(z)u)t + O_{(z,u-u_0)}(T^2). 
\]

By setting \( t \) equal to \( T \), the previous model provides a family of discrete-time approximate models indexed by \( T \):

\[
z_1(k+1) = F_{T,1}^a(z,u) 
\]

\[
z_2(k+1) = F_{T,2}^a(z,u). 
\]

\[
F_{T,1}^a(z,u) \overset{df}{=} z_1(k) + z_2(k)T + (\alpha(k) + \beta(k)u(k)) \frac{T^2}{2} 
\]

\[
F_{T,2}^a(z,u) \overset{df}{=} z_2(k) + (\alpha(k) + \beta(k)u(k))T, 
\]

where \( z(k), \alpha(k), \) and \( \beta(k) \) denote the value of \( z, \alpha(z), \) and \( \beta(z) \) at time \( kT, k \in \mathbb{N} \), respectively. The superscript “\( a \)” emphasizes the fact that \( z \) is the solution of an approximate model.

\[\text{\footnote{Although \( t \in [0,T] \)}, solutions are not altered by changing the input on a zero measure set.}\]
5. A DATA-DRIVEN CONTROL DESIGN METHODOLOGY

In this section we summarize the proposed data-driven control design methodology that is presented in detail in Section 6 through Section 8. The design will be based on different approximate models, all of which based on \((4.4)\). We start by observing that the model \((4.4)\) is affine and thus all the design techniques described in this paper only require knowledge of linear systems theory.

The affine nature of the model \((4.4)\) suggests that we can use the preliminary controller:

\[
u(k) = \beta^{-1}(z(k))(-\alpha(z(k)) + v(k)),\]

where \(v\) is a new input, to cancel the effect of the nonlinear functions \(\alpha\) and \(\beta\) provided that \(z(k)\) and the values of \(\alpha\) and \(\beta\) at the current state \(z(k)\) are known. Note that it suffices to know the values of \(\alpha\) and \(\beta\) at \(z(k)\) and there is no need to learn the functions \(\alpha\) and \(\beta\). After this preliminary controller it is easy to design a controller stabilizing the resulting linear system with input \(v\). Any controller design technique can be employed provided the dissipation inequality (8.1) in Section 8 holds. Moreover, we also show in Section 8 that a linear controller suffices for this purpose.

By considering \(\alpha\) and \(\beta\) to be constant functions in \((4.4)\) we obtain an observable linear system by formally treating \(\alpha + \beta u\) as a new state \(z_3\) and using the measurement equation \(y = z_1\). Hence, any technique to reconstruct the state of an observable linear system can be employed provided the reconstruction error is of order \(T\), see (6.6) in Section 6. In this section we propose to reconstruct the state by directly solving the equation \(Y = Oz\) where \(Y\) is a sequence of measurements and \(O\) is the observability matrix of the aforementioned observable linear system.

Once an estimate of \(z_3\) is obtained, we formally treat \(z_3\) as an observation and seek to reconstruct \(\alpha\) and \(\beta\) from the measurement equation \(z_3 = \alpha + \beta u\). This is not possible unless a persistency of excitation is placed on the input \(u\). Rather than assuming persistency of excitation, we note this type of problem has been extensively studied in adaptive control \([\text{NA89}, \text{IS12}]\) and it is known that any choice of parameters \(\alpha\) and \(\beta\) that satisfies the measurement equation \(z_3 = \alpha + \beta u\) suffices for control purposes. Therefore, we propose a very simple linear observer to estimate \(\alpha\) and \(\beta\) in Section 7. Once again, any other technique for estimating \(\alpha\) and \(\beta\) can be employed provided it leads to the dissipation inequality (7.4) in Section 7.

6. STATE ESTIMATION

For state estimation purposes it is convenient to formally treat \(\alpha(k) + \beta(k)u(k)\), in the family of approximate models \((4.4)\), as the state \(z_3\) to obtain:

\[
\begin{align*}
z_1(k+1) &= z_1(k) + z_2(k)T + z_3(k)\frac{T^2}{2} \\
z_2(k+1) &= z_2(k) + z_3(k)T \\
z_3(k+1) &= z_3(k).
\end{align*}
\]

Note that this approximate model states that \(z_3\) is constant although \((F_{t,1})^{(2)}\) will, in general, not be so. Equality (6.3) follows from applying Proposition 4.2 to \((F_{t,1})^{(2)}\) and dropping the error term \(O_{(z,u-u_0)}(T)\).

Since (6.1)-(6.3) is a linear model, it can be written in the form:

\[
z(k+1) = Az(k), \quad y(k) \overset{\text{def.}}{=} z_1(k) = Cz(k).
\]

\[\text{Formally justifying this design assumption is one of the purposes of the results in Section 9.}\]
Moreover, it can be easily checked that \( A \) is invertible and we thus denote by \( O \) the observability matrix for the pair \( (A^{-1}, C) \) which allows us to write:

\[
Y(k) \overset{\text{def.}}{=} \begin{bmatrix}
y(k) \\
y(k-1) \\
\vdots \\
y(k-\rho + 1)
\end{bmatrix} = O z(k),
\]

where \( \rho \in \mathbb{N}, \rho \geq n+1 \), is the number of measurements that will be used for state estimation. The estimate \( \hat{z}(k) \) of the state vector \( z(k) \) can then be obtained by solving this equation via least-squares:

\[
\hat{z}(k) = (O^T O)^{-1} O^T Y(k).
\]

Given that equalities \((6.1),(6.2),\) and \((6.3)\) only hold up to \( O(z,u-u_0)(T^3), O(z,u-u_0)(T^2), \) and \( O(z,u-u_0)(T) \), respectively, we can easily establish the equality \( z = \hat{z} + O(z,u-u_0)(T) \). If we introduce the estimation error \( e_z \), defined by \( e_z = z - \hat{z} \), it follows that:

\[
e_z = O(z,u-u_0)(T).
\]

The control scheme proposed in Section 8 only depends on the preceding equality. Hence, we can replace least-squares estimation with any other estimation technique leading to \((6.6)\).

Remark 6.1. In \([RJ09]\) it is shown that the algebraic techniques proposed in \([MLF07]\), and used in \([FJ09,FJ13]\) to estimate derivatives of a measured signal, can be interpreted as estimating the state of the state-space linear model governing the signals \( y \) satisfying \( y^{(3)} = 0 \). If we denote the constructability Gramian of this linear model by \( W_{cn} \) and its state-transition matrix by \( \Phi \), the estimate is given by the well known expression (see (3.9), page 250, \([AM97]\)):

\[
W_{cn}^{-1} \int_{t_0}^{t_1} \Phi^T(\tau,t_1)O^T y(\tau) d\tau.
\]

Equality \((6.5)\) can be seen as the discrete-time analogue of this finite-time estimation technique.

Remark 6.2. The matrix \((O^T O)^{-1} O^T\) contains terms of the form \( T^{-1} \) on its second row and terms of the form \( T^{-2} \) on its third row. Hence, it can be conceptually understood as a linear high-gain observer with finite-time convergence and where \( T \) plays the role of the parameter \( \varepsilon \) used in \([Kha17]\). Similarly to high-gain observers, the estimate provided by \((6.5)\) can be very sensitive to measurement noise. This can be mitigated by using more samples for estimation so as to “average out” noise, i.e., by increasing \( \rho \). Contrary to high-gain observers, however, we do not need to explicitly worry about the peaking phenomenon when computing the estimate since it is not computed recursively. As mentioned before, \((6.5)\) could be replaced with an high-gain observer or even the more recent low-power high-gain observers \([AMPT18]\). Which specific estimation technique works better in practice, and in the context of the results in this paper, is an important problem that we leave for future research.

7. Parameter estimation

We now consider the problem of estimating \( \alpha \) and \( \beta \), i.e., estimating the \textit{values} of the functions \( \alpha \) and \( \beta \) at the current state and not the functions themselves. This important conceptual difference, first suggested in the work of Fliess and Join \([FJ09,FJ13]\), explains why we do not need large quantities of data neither lengthy training.

Since \( z_3 = \alpha + \beta u \) we regard \( z_3 \) as a measurement and build an observer for \( \pi = (\alpha, \beta) \). Applying Proposition 4.2 to \( \pi \circ F_T^T(z,u) \) provides:

\[
\alpha(T) = \alpha(0) + O(z,u-u_0)(T), \quad \beta(T) = \beta(0) + O(z,u-u_0)(T),
\]

and results in the approximate model:

\[
\alpha(k+1) = \alpha(k), \quad \beta(k+1) = \beta(k),
\]
obtained by dropping the $O(T)$ terms.

To simplify the presentation we will assume the state $z$ to be known and consider the observer:

$$
\hat{\pi}(k+1) = \hat{\pi}(k) + \Gamma \sqrt{T} \Phi(k) (z_3(k) - \Phi^T(k) \hat{\pi}(k)),
$$

$$
\Gamma = \begin{bmatrix} \gamma_\alpha & 0 \\ 0 & \gamma_\beta \end{bmatrix},
$$

where $\gamma_\alpha, \gamma_\beta \in \mathbb{R}^+$ are observer gains, $\hat{\pi}$ is the estimate of $\pi$, and $\Phi$ is defined by:

$$
\Phi(k) = \begin{bmatrix} 1 \\ u(k) \end{bmatrix}.
$$

If we define the observer error as:

$$
e_\pi = \pi - \hat{\pi},
$$

we obtain the error dynamics:

$$
e_\pi(k+1) = G_\pi^T(e_\pi(k))
$$

(7.3)

This suggests using $V_\pi(e_\pi) = e_\pi^T P_\pi e_\pi$, with $P_\pi = \Gamma^{-1}$, as a Lyapunov function, so as to obtain:

$$
V_\pi(G_\pi^T(e_\pi)) - V_\pi(e_\pi) = e_\pi^T (I - \Gamma \sqrt{T} \Phi \Phi^T) P (I - \Gamma \sqrt{T} \Phi \Phi^T) e_\pi - e_\pi^T P e_\pi \\
= -2 \sqrt{T} e_\pi^T \Phi \Phi^T e_\pi + T e_\pi^T (\Phi \Phi^T) \Phi \Phi^T e_\pi \\
= -2 \sqrt{T} e_\pi^T \Phi \Phi^T e_\pi + (\Phi \Phi^T) e_\pi^T \Phi \Phi^T e_\pi,
$$

where we used associativity of matrix multiplication and the fact that $\Phi^T \Gamma \Phi$ is a scalar. We can now choose $\lambda_\pi \in \mathbb{R}^+$ and $T_0 \in \mathbb{R}^+$ satisfying:

$$
\lambda_\pi < 2, \quad T_0 = \frac{(2 - \lambda_\pi)^2}{N^2},
$$

where $N$ is an upper bound for $\Phi^T \Gamma \Phi$ whenever $u$ ranges in a compact set\footnote{The proof of the main results will show why we can assume the input to range on a compact set.} to conclude\footnote{The inequality $-2 \sqrt{T} + T N \leq -\lambda_\pi \sqrt{T}$ holds for all $T \in [0, T_0]$.} that for all $T \in [0, T_0]$ we have:

(7.4)

$$
V_\pi(G_\pi^T(e_\pi)) - V_\pi(e_\pi) \leq -\lambda_\pi \sqrt{T} e_\pi^T \Phi \Phi^T e_\pi.
$$

Only this inequality will play a role in the proof of the main results. Hence, any estimation approach leading to a similar inequality can be used.

Remark 7.1. Note that we cannot use an algebraic estimation technique, as in Section 6, since the model defined by (7.1) and by the output equation $z_3 = \alpha + \beta u$ is not observable unless we impose a persistency of excitation assumption on the input $u$.

As the state $z_3$ cannot be measured, we will use the proposed observer with the estimate $\hat{z}_3$ described in Section 6. This leads to the equations:

(7.5)

$$
\hat{\alpha}(k+1) = \hat{\alpha}(k) - \gamma_\alpha \sqrt{T} (\hat{\alpha}(k) + u(k) \hat{\beta}(k)) + \gamma_\alpha \sqrt{T} \hat{z}_3(k)
$$

(7.6)

$$
\hat{\beta}(k+1) = \hat{\beta}(k) - \gamma_\beta \sqrt{T} u(k)(\hat{\alpha}(k) + u(k) \hat{\beta}(k)) + \gamma_\beta \sqrt{T} u(k) \hat{z}_3(k).
$$

It is known from adaptive control, e.g., [IS12, NA89], that this kind of observer is not robust to measurement errors in the sense that it is not possible to guarantee boundedness of the parameter estimates. Hence, we now
use the stronger assumption that bounds for $\beta$ are known and modify the parameter estimator by projecting the estimate for $\beta$ on the set $[\underline{\beta}, \overline{\beta}]$. This is achieved by replacing (7.6) with:

$$\hat{\beta}(k + 1) = P \left( \hat{\beta}(k) - \gamma_{\beta} \sqrt{T} u(k)(\hat{\alpha}(k) + u(k)\hat{\beta}(k)) + \gamma_{\beta} \sqrt{T} u(k)\hat{z}_3(k) \right).$$

where $P$ is the projection defined by:

$$P(r) = \begin{cases} \frac{\beta}{r} & \text{if } r < \beta \\ r & \text{if } \beta \leq r \leq \overline{\beta} \\ \frac{1}{\beta} & \text{if } r > \overline{\beta}. \end{cases}$$

The following sequence of equalities and inequalities:

$$|e_\beta| = |\beta - P(\hat{\beta})| = |P(\beta) - P(\beta)| \leq |\beta - \beta|,$$

which is a consequence of convexity of $[\underline{\beta}, \overline{\beta}]$ implying $|P(r) - P(s)| \leq |r - s|$, shows that by replacing (7.6) with (7.7), inequality (7.4) still holds.

Establishing that observer (7.5), (7.7) can be used for stabilization despite the use of the approximate model for the dynamics in its design and despite the use of the estimate $\hat{z}_3$ will be done in the proof of the main results in Section 8.

8. Controller design

If we assume the parameters $\alpha$ and $\beta$ to be known, we can design a family of controllers (parameterized by $T$) for the family of approximate models (4.4)-(4.4) with the objective of asymptotically stabilizing the origin in the following specific sense: there exists a symmetric and positive definite matrix $P_z$ and constants $\lambda_z, T_0 \in \mathbb{R}^+$ so that $V_z(z) = z^TP_zz$ satisfies:

$$V_z(z(k + 1)) - V_z(z(k)) \leq -\lambda_z T \|z(k)\|^2 + O_{\|z, u\|^2}(T^2),$$

for all $T$ in the interval $[0, T_0]$. The main results in this paper will only require this inequality, hence, any family of controllers leading to (8.1) can be used. Strikingly, we can achieve this inequality with the very simple family of controllers which is independent of $T$:

$$u = \beta^{-1}(-\alpha + Kz),$$

where $K$ is a suitable matrix. We note that the approximate model (4.4)-(4.4) can be written as:

$$z(k + 1) = Az(k) + B\alpha(k) + B\beta(k)u(k),$$

where the matrices $A$ and $B$ are of the form:

$$A = I + A_1T, \quad B = B_1T + B_2T^2.$$

Since $(A_1, B_1)$ is a controllable pair, there exists a controller $u = Kz$ and a symmetric and positive definite matrix $P_z$ so that:

$$(A_1 + B_1K)^TP_z + P_z(A_1 + B_1K) = -Q,$$

for some symmetric and positive definite matrix $Q$. Using this controller we have:

$$(A + BK)z = (I + (A_1 + B_1K)T + B_2KT^2)z.$$

Computing $V_z((A + BK)z) - V_z(z)$ provides:

$$V_z((A + BK)z) - V_z(z) = z^T(A + BK)^TP_zz - z^TP_zz$$

$$= z^T((A_1 + B_1K)T)^TP_zz + z^TP_z((A_1 + B_1K)T)z$$

$$+ O_z(T^2) + O_{z^2}(T^3) + O_{z^3}(T^4)$$

$$\leq -\lambda_{\min}(Q)T \|z\|^2 + O_{\|z, u\|^2}(T^2),$$
which is the desired inequality (8.1).

Since neither $\alpha$ and $\beta$ nor $z$ are exactly known, we use instead the control law:

$$u(k) = \bar{\beta}^{-1}(k) (-\hat{\alpha}(k) + K\hat{z}(k)),$$

obtained by replacing the unknown quantities by its estimates. This control law, in combination with the observer (7.5), (7.7), defines a dynamic controller. In order to fully specify this controller, we need to describe its operation during the initial transient of $\rho - 1$ steps during which enough measurements are collected to produce the first state estimate according to (6.5). We simply choose a fixed sequence of inputs $u_0^\ast, u_1^\ast, \ldots, u_{\rho-2}^\ast$ to be used during this transient. Although different sequences will lead to different transients, the results in Section 9 are independent of this choice.

The main results in the next section explain why such dynamic controller works despite being designed for an approximate model while assuming knowledge of the exact values of the parameters and states in its design.

9. Main results

9.1. The noise-free scenario. It is pedagogically convenient to start with the noise-free scenario, i.e., $d = 0$ in (3.2), as it allows us to expose the key ideas in a simpler manner. Notwithstanding the absence of noise, the proofs of the main results in this section are quite long and for this reason can be found in the Appendix. The authors hope its length does not hide the simple idea upon which it rests: we can formally justify the use of approximate models for observer and controller design by using the framework developed by Arcak and Nesic in [AN04] for observer design based on approximate models and by Nesic and Teel in [NT04] for controller design based on approximate models. Although we have not used Immersion and Invariance techniques in this paper, the Lyapunov techniques used in [AKO08] to prove convergence of adaptive observers without persistency of excitation underly our analysis of the observer for this paper, the Lyapunov techniques used in [AKO08] to prove convergence of adaptive observers without persistency of excitation underly our analysis of the observer for this paper, the Lyapunov techniques used in [AN04] for observer design based on approximate models and by Nesic and Teel in [NT04] for controller design based on approximate models. Although we have not used Immersion and Invariance techniques in this paper, the Lyapunov techniques used in [AKO08] to prove convergence of adaptive observers without persistency of excitation underly our analysis of the observer for this paper.

Theorem 9.1. Consider an unknown nonlinear system of the form (3.1), (3.2) where the output function $h$ has relative degree $2$ and assume the existence of two constants $\overline{\beta}, \underline{\beta} \in \mathbb{R}^+$ satisfying $\underline{\beta} \leq L_gL_fh(x) \leq \overline{\beta}$ for all $x \in \mathbb{R}^2$. In the absence of measurement noise, i.e., $d = 0$, for any compact set $S \subset \mathbb{R}^2$ of initial conditions containing the origin in its interior there exists a time $T^* \in \mathbb{R}^+$ and a constant $b \in \mathbb{R}^+$ (both depending on $S$) so that for any sampling time $T \in [0, T^*]$ the controller (8.5) using the parameter estimates provided by the observer (7.5), (7.7) and the state estimate (6.5) renders the closed-loop trajectories bounded, i.e., $\|\hat{z}(k)\| \leq b$ and $\|\hat{\alpha}(k), \hat{\beta}(k)\| \leq b$ for all $k \in \mathbb{N}$, and $\|x(t)\| \leq b$ for all $t \in \mathbb{R}_0^+$. Moreover:

$$\lim_{t \to \infty} x(t) = 0.$$

As is typical in adaptive control there is no guarantee the parameter estimates converge to the true values although $\Phi^T e_\pi$ converges to zero. A suitable persistency of excitation assumption on $\Phi$, i.e., on the input $u$, can be derived so that convergence of $\Phi^T e_\pi$ to zero implies convergence of $e_\pi$ to zero.

Although the previous result only claims that trajectories converge to the origin, it can be readily applied to trajectory tracking problems by considering convergence to zero of the error between the real trajectory and the trajectory to be tracked.

9.2. The noisy scenario. In the presence of essentially bounded measurement noise, the state estimation error is now given by:

$$e_z = O(z, u-u_0)(T) + O_\pi(T^{-2}),$$

where $d$ is the noise bound. This expression show the tension between choosing a small sampling time to render the approximate models adequate and choosing a large sampling time to reduce the effect of noise.
Nevertheless, the projection (7.8) is used in (7.7) to ensure that $\hat{\beta}$ remains bounded and this will force all the other signals to remain bounded while the state converges to a ball centered at the origin provided the noise bound $n$ is not too large.

As with the noise-free case, the proof of the following result can be found in the Appendix.

**Theorem 9.2.** Consider an unknown nonlinear system of the form

$$(3.1) \quad (3.2)$$

where the output function $h$ has relative degree 2, assume the existence of two constants $\beta, \bar{\beta} \in \mathbb{R}^+$ satisfying $\beta \leq L_g L_f h(x) \leq \bar{\beta}$ for all $x \in \mathbb{R}^2$, and assume the noise $d$ to be essentially bounded, i.e., the exists a constant $\bar{d} \in \mathbb{R}^+$ satisfying $\bar{d} = \text{ess sup}_{t \in \mathbb{R}^+} \|d(t)\|$. For any compact set $S \subset \mathbb{R}^2$ of initial conditions containing the origin in its interior there exists a time $T^* \in \mathbb{R}^+$ (depending on $S$), and constants $b_1, b_2, b_3 \in \mathbb{R}^+$ (depending on $S$ and $T^*$) so that for any sampling time $T \in [0, T^*]$ if $\bar{d} \leq b_1$ the controller (8.5) using the parameter estimates provided by the observer (7.5), (7.7) and the state estimate (6.5) renders the closed-loop trajectories bounded, i.e., $\|\hat{z}(k)\| \leq b_2$ and $\|\hat{\alpha}(k), \hat{\beta}(k)\| \leq b_2$ for all $k \in \mathbb{N}$, and $\|x(t)\| \leq b_2$ for all $t \in \mathbb{R}^+$. Moreover:

$$\limsup_{t \to \infty} \|x(t)\| \leq 2b_3(\bar{\beta} - \beta) + b_3 \bar{d} T^{-5/4}.$$ 

Note that when $\beta$ is constant and known, i.e., $\bar{\beta} = \beta$, as commonly assumed in the intelligent PID literature, the state converges to a ball centered at the origin and whose radius only depends on the noise level $\bar{d}$ and sampling time $T$.

### 10. Experimental evaluation

In this section we report on an experimental evaluation of the proposed data-driven controller to regulate the altitude of a quad-copter. The experiments were performed on a Bitcraze Crazyflie 2.1 and an Optitrack Prime 17W motion capture system was used to measure the quad-copter’s altitude during the experiments.

**10.1. Experimental setup.** The Crazyflie 2.1 is a small open source modular quad-copter designed by Bitcraze AB [Cra] equipped with an IMU based on a 3-axis accelerometer and gyroscope. The baseline firmware for the Crazyflie includes a PID based flight controller. We separated this controller into attitude and altitude controllers, keeping the former and replacing the latter with a data-driven controller.

To provide the data-driven controller with altitude measurements we used eight Optitrack Prime 17W cameras [Opt] distributed on three sides along the top of a roughly cubic area. The cameras have a refresh rate of up to 360Hz and provide coordinate and pose measurements by triangulating a set of markers placed on the quad-copter. Whereas the PID controller regulating attitude receives measurements from the IMU and the motion capture system, the data-driven controller only receives altitude measurements from the motion capture system.

A qualitative view of the measurement noise, when the quad-copter is static on the floor, is presented in Figure 1. The real altitude corresponds to the location of the markers on top of the quad-copter. We observe the noise typically has a magnitude of 1 mm, i.e., $\bar{d} = 1$, although there are occasional valleys in the noise signal corresponding to instants where the motion capture system loses track of some of the markers.

**10.2. Model.** To obtain a single-input single-output system we kept the PID controller regulating attitude and restricted the quad-copter’s motion to a vertical line. Therefore, assuming perfect attitude regulation, the quad-copter’s motion can be described by:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\frac{g}{m} + \frac{1}{m} u_{tr},$$

$$y = x_1,$$
where $x_1$ denotes altitude, $g$ is gravity’s constant, and the input $u_{tr}$ represents the thrust created by the propellers rotation. The thrust is commanded by a PWM signal and the relation between the commanded signal $u$ and the exerted thrust $u_{tr}$ is well described by an affine map $u_{tr}(u) = \sigma_0 + \sigma_1 u$. This results in the dynamics:

$$\dot{x}_1 = \frac{\sigma_0 - g}{m} + \frac{\sigma_1}{m} u,$$

from which we can infer the relative degree of $y$ to be 2 with $\alpha(x) = \frac{\sigma_0 - g}{m}$ and $\beta(x) = \frac{\sigma_1}{m}$. Since our results apply to the case where $\alpha$ and $\beta$ are functions, rather than constants, we emulate in software the function:

$$(10.1) \beta(x) = \frac{\sigma_1}{m} - x_1^4,$$

i.e., when the data-driven controller requests the input $u$, we create a PWM signal using $u - \frac{m}{\sigma_1} x_1^4 u$.

10.3. Data-driven controller and its implementation. The quad-copter receives altitude measurements from the motion capture system and uses them for state estimation using (6.5) with $\rho = 5$. This choice of $\rho$ mitigates the effects of the measurement noise that can be appreciated in Figure 1. The resulting state estimate is then fed to the observer (7.5), (7.7) with $\gamma_\alpha = 0$ and $\gamma_\beta = 0$ initialized with $\hat{\alpha}(0) = -5$ and $\hat{\beta}(0) = 1$. Moreover, we use the bounds $\underline{\beta} = 1$ and $\overline{\beta} = 30$ in (7.7). Controller (8.5) was used with $K = [-6, -5]$ so as to place the eigenvalues of $A_1 + B_1 K$ at $-2$ and $-3$. For the initial transient we use the sequence of inputs $0.6, 1.0, 0.0, 1.0$. The overall data-driven controller was executed with a sample time of $T = 0.0028s$ which corresponds to the maximal rate at which the motion capture system provides data.

10.4. Experiments. Figure 2 shows how the data-driven controller regulates altitude. In the top horizontal panel on the left, the desired set points are displayed in red and in blue we can see the quad-copter’s trajectory. In the bottom horizontal panel on the right we can observe altitude steady-state errors around 1cm, even though $\hat{\alpha}$ and $\hat{\beta}$ do not converge to the correct values, in accordance with Theorem 9.2. We can also observe that most of the learning happens immediately after set-point changes, a consequence of the latter inducing changes in the input and thus producing more informative measurements. The peaks and valleys in the estimates observed at the rightmost end of the plots are due to the quad-copter’s landing since the model (10.1) is no longer valid, e.g., inputs resulting in a thrust smaller than gravity’s force do not produce net motion since the quad-copter has landed. As mentioned before, and is well known in adaptive control, the proposed observer suffers of

---

$^9$We only use PWM values up to 90% so as to leave some control authority for the attitude controller.
Figure 2. Experimental results including the quad-copter’s altitude and desired altitude, tracking error, and estimates for \( \alpha \) and \( \beta \).

Figure 3. Rescaled version of the plots on the left of Figure 2.

Parameter drift due to measurement noise although the estimates are guaranteed to remain bounded. This can be better appreciated in Figure 3 where we present the same data as in the left panels of Figure 2 but with a smaller range for the vertical axis.

Although the estimates of \( \alpha \) and \( \beta \) do not converge to the correct values, the proof of Theorems 9.1 and 9.2 established that \( \lim_{k \to \infty} \Phi^T(u(k))e_\pi(k) = 0 \) is sufficient to ensure \( \lim_{t \to \infty} x(t) = 0 \). Figure 4 shows the temporal evolution of \( \Phi^T e_\pi \) and we can observe that this quantity does converge, modulo the effect of noise, to close to zero. The offset between the “average” value of \( \Phi^T e_\pi \) and zero is caused by our inability to accurately compute \( \alpha \) and \( \beta \) as these depend on several quad-copter parameters that are not easy to measure such as the constants \( \sigma_0 \) and \( \sigma_1 \).
In summary, the experimental results show that, in spite of measurements errors, the proposed data-driven controller can successfully regulate altitude. Due to lack of persistency of excitation, the estimates of $\alpha$ and $\beta$ do not converge to the correct values but they remain bounded as expected.

11. Conclusions

There are several important questions that were left unaddressed in this paper. Feedback linearizability was convenient to construct the technical arguments but one can easily see extensions to partially feedback linearizable systems with well behaved zero dynamics. Identifying the largest class of systems to which the results in this paper (or suitable generalizations thereof) apply is a worthwhile endeavor.

Similarly, investigating which state estimation, parameter estimation, and controller design techniques result in better performance in the context of the proposed data-driven methodology would make the results more useful in practical applications. In particular, investigating how to best mitigate the effect of measurement noise would be especially important.

Finally, there were connections with existing results in high-gain observers, adaptive control, and potentially other areas, as well as with the recent papers [CLD19, PT20]. All of these deserve to be better understood.

Appendix

Proof of Theorem 9.1. The proof will be based on the feedback linearized form (3.4)-(3.6) of the dynamics rather than the original nonlinear form (3.1)-(3.2). This results in no loss of generality since both systems are related by the diffeomorphism $\Psi$ that satisfies $\Psi(0) = 0$. For simplicity, we will denote the set $\Psi(S)$ simply by $S$. Since $\Psi$ is an homeomorphism, $\Psi(S)$ is still a compact set. In the same spirit, rather than working with the estimate $\hat{\pi}$ we will work with $e_\pi$. We will see that boundedness of $e_\pi$ implies boundedness of $\hat{\pi}$.

The initial transient: the state estimate $\hat{\pi}$ requires $\rho$ samples to be collected. To simplify the argument we consider the case where $\rho = 3$ which leads to an initial (fixed) sequence of $\rho - 1 = 2$ inputs $u_0^*, u_1^*$ used at time $k = 0$ and $k = 1$. This corresponds to an initial transient that must be analyzed separately.

By applying Proposition 4.2 to the compact set $\mathcal{D} = S \times \{u_0^*\}$ we conclude the existence of a time $T_0$ so that trajectories are well defined for all $T \in [0, T_0]$ and for all initial conditions in $S$. We regard $T_0$ as the time elapsed during the first time step under input $u_0^*$. The set of points reached under all these trajectories and
for all $T \in [0,T_0]$ is denoted by $Z_0$. We can repeat this argument, using $Z_0$ as the set of initial conditions (and assuming the initial time to be zero) and the input $u_T^*$ to conclude the existence of a time $T_1$ so that trajectories are well defined for all $T \in [0,T_1]$ and for all initial conditions in $Z_0$. By taking $T_2 = \min\{T_0,T_1\}$ we conclude that solutions are well defined for the sequence of inputs $u_T^*,u_T^*$ where each input is applied for $T_2$ units of time. Let now $Z$ be the set of points reached under all the trajectories with initial conditions in $S$ and that result by applying $u_T^*$ for $T$ units of time, followed by applying $u_T^*$ for $T$ units of time with $T$ ranging through all the values in the set $[0,T_2]$. This set will be used several times in the remainder of the proof.

At time step $k = 2$ both state and parameter estimates are already available. Let us denote by $E$ the set of possible parameter estimation errors $e_\pi = \pi - \hat{\pi}$ at time $k = 2$ depending on the different initial conditions. Since the estimates $\hat{\pi}$ are continuous functions of the initial condition $z(0)$ that belongs to the compact set $\mathcal{S}$, $E$ is a bounded set. Consider also the set $R$ defined as the smallest sub-level set of $W = V_\pi + V_\hat{\pi}$ that contains $Z \times E$ where $V_\pi$ is defined by $V_\pi(e_\pi) = e_\pi^T P_\pi e_\pi$ (see Section 7) and $V_\hat{\pi}$ is the Lyapunov function satisfying (8.1). Our objective is to show that $R$ is an invariant set.

**Existence of solutions one step beyond the transient:** we first show that it is possible to continue the solutions from $R$ by employing again Proposition 4.2. For future use, we define the projections $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$, $\pi_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, and $\pi_E : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\pi_1(z_1,z_2) = z_1$, $\pi_2(z,e_\pi) = z$, and $\pi_E(z,e_\pi) = e_\pi$. The controller (8.5) is a function of $\hat{z}$ and $\hat{\pi}$. However, since $\hat{\pi} = \pi - e_\pi$ and both $\pi$ and $\hat{z}$ are functions of $z$, we can regard the controller as a smooth function of $z$ and $e_\pi$. We can thus consider the set of inputs $U \subset \mathbb{R}$ defined by all the inputs obtained via (8.5) when $z$ ranges in $\pi_2(R)$, $e_\pi$ ranges in $\pi_E(R)$, and $\hat{z}$ is given by (6.5) with $Y(2)$, defined in (6.4), ranging in $(\pi_1 \circ \pi_2(R))^P$, the $P$-fold Cartesian product of $\pi_1 \circ \pi_2(R)$. By taking its closure, if needed, we can assume the set $\pi_2(R) \times U$ to be compact and apply Proposition 4.2 to obtain a time $T_3$ ensuring that solutions starting at $\pi_2(R)$ exist for all $T \in [0,T_3]$. Moreover, Proposition 4.2 ensures the existence of a constant $M$ for which the bound (4.1) holds and, as a consequence, the approximate model (4.4)-(4.4) is valid for any solution with initial condition in $R$ and any input in $U$. If $T_2 < T_3$ we proceed by only considering sampling times in $[0,T_2]$ and note that none of the conclusions reached so far change. If $T_2 > T_3$, we can use sampling times in $[0,T_3]$ while noting that all the reached conclusions remain valid by redefining $Z$ to be the set of points reached for any time in $[0,T_3]$ (if the conclusions holds for the (non-strictly) larger $Z$ set they also hold for the (non-strictly) smaller $Z$ obtained by reducing $T_2$ to $T_3$).

**Invariance of the set $R$:** we can now establish invariance of $R$ by computing $W(F_T^*(z,u),G_T^*(e_\pi)) - W(z,e_\pi)$, in several steps, with $u$ given by (8.5), $G_T^*$ denoting the exact dynamics of the parameter error $e_\pi$, and for all $(z,e_\pi) \in R$. In the first step we establish that the evolution of $V_\pi$ under $F_T^*$ equals the evolution of $V_\pi$ under $F_T^*$ up to $O(T^2)$ terms. In order to do so, we recall that $F_T^*(z,u)$ can be expressed as:

$$F_T^*(z,u) = F_T^a(z,u) + O(z,u-u_0)(T^2)$$

$$= Az + B(\alpha + \beta u) + O(z,u-u_0)(T^2)$$

$$= (I + A_1 T)z + (B_1 T + B_2 T^2)(\alpha + \beta u) + O(z,u-u_0)(T^2).$$

\[\text{(11.1)}\]

\[\text{Note that division by } \hat{\beta} \text{ in (8.5) creates no problems since the projection (7.8) in (7.7) ensures that } \hat{\beta} \text{ is always lower bounded by } 2.\]
We then have:
\[
V_z(F_T^z(z, u)) - V_z(z) = (F_T^z(z, u) + O_{z, u - u_0}(T^2))^T P_z(F_T^z(z, u) + O_{z, u - u_0}(T^2)) - z^T P_z z
\]
\[
= V_z(F_T^z(z, u)) - V_z(z) + 2O_{(z, u - u_0)}^T(z, u)P_z F_T^z(z, u) + O_{(z, u - u_0)}^2(z, u)
\]
\[
+ 2O_{(z, u - u_0)}(T^2)P_z O_{(z, u - u_0)}(T^2)
\]
\[
\leq V_z(F_T^z(z, u)) - V_z(z) + 2O_{(z, u - u_0)}^T(z, u)P_z F_T^z(z, u) + O_{(z, u - u_0)}(T^4)
\]
\[
= V_z(F_T^z(z, u)) - V_z(z) + 2O_{(z, u - u_0)}^T(z, u)P_z A z
\]
\[
+ 2O_{(z, u - u_0)}(T^2)P_z B(\alpha + \beta u) + O_{(z, u - u_0)}(T^4)
\]
\[
= V_z(F_T^z(z, u)) - V_z(z) + O_{(z, u - u_0)}(T^2) + O_{(z, u - u_0)}(T^3) + O_{(z, u - u_0)}(T^4)
\]
\[
= V_z(F_T^z(z, u)) - V_z(z) + O_{(z, u - u_0)}(T^2),
\]
(11.2)

where we used \(\|z\| \leq \|(z, u - u_0)\|\) and boundedness of \(\alpha, \beta,\) and \(u\) in virtue of \((z, u)\) belonging to the compact set \(\pi_Z(R) \times U\), to obtain the fourth equality.

We now consider the term \(\|(z, u - u_0)\|\) in more detail:
\[
\|(z, u - u_0)\| \leq \|z\| + \|\beta^{-1}(-\hat{\alpha} + Kz) - u_0\|
\]
\[
= \|z\| + \left\| \frac{1}{\beta - e_\beta}(-\hat{\alpha} + Kz) - u_0 \right\|
\]
\[
= \|z\| + \left\| \left( \frac{1}{\beta} + \frac{e_\beta}{\beta(\beta - e_\beta)} \right)(-\hat{\alpha} + Kz) - u_0 \right\|
\]
\[
= \|z\| + \left\| \left( \frac{1}{\beta} + \frac{e_\beta}{\beta} \right)(-\hat{\alpha} + Kz) - u_0 \right\|
\]
\[
\leq \|z\| + \|\beta^{-1}(-\alpha + Kz) - u_0\| + \|\beta^{-1}(e_\alpha - Ke_z) + \beta^{-1}e_\beta \beta^{-1}(-\hat{\alpha} + Kz)\|
\]
\[
\leq \|z\| + \|\beta^{-1}(-\alpha + Kz) - u_0\| + \|-\beta^{-1}Ke_z + \beta^{-1}e_\alpha + \beta^{-1}e_\beta u\|
\]
\[
\leq \|z\| + \|\beta^{-1}(-\alpha + Kz) - u_0\| + \|\beta^{-1}Ke_z\| + \|\beta^{-1}F T e_x\|.
\]
(11.3)

Noting the function \(\beta^{-1}(z)(-\alpha(z) + Kz)\) is Lipschitz continuous (with Lipschitz constant \(L\)) on \(\pi_Z(R)\), and that it produces the value \(u_0\) at \(z = 0\), we conclude that:
\[
\|\beta^{-1}(-\alpha + Kz) - u_0\| \leq L\|z\|.
\]

The preceding sequence of inequalities, and boundedness of \(\beta^{-1}\) on \(\pi_Z(R)\), lead to:
\[
O_{z, u - u_0}(T) = O_T(z) + O_{\epsilon z}(T) + O_{\Phi T e_x}(T).
\]

In fact, we go further and eliminate the dependence on \(\epsilon_z\) as follows. According to (6.6) the estimation error \(\epsilon_z\) satisfies:
\[
\epsilon_z = O_{z, u - u_0}(T) = O_T(z) + O_{\epsilon z}(T) + O_{\Phi T e_x}(T).
\]

If we denote by \(M\) the largest of the constants appearing in the \(O\) terms, we can rewrite the previous equality as:
\[
\|\epsilon_z\| \leq M\|z\|T + M\|\epsilon_z\|T + M\|\Phi T e_x\|T
\]
\[
\Leftrightarrow\|\epsilon_z\|(1 - MT) \leq M\|z\|T + M\|\Phi T e_x\|T
\]
\[
\Leftrightarrow\|\epsilon_z\| \leq \frac{MT}{1 - MT}\|z\| + \frac{MT}{1 - MT}\|\Phi T e_x\|.
\]

We now note that for all \(T \in [0, T_4]\), \(T_4 = (2M)^{-1}\), we have:
\[
\frac{MT}{1 - MT} \leq 2MT,
\]
and we can simplify the bound on the norm of $e_z$ to:
\[
\|e_z\| \leq 2MT\|z\| + 2MT\|\Phi^T \hat{e}_\pi\|,
\]
which leads to:
\[
O_{e_z}(T) = O_z(T^2) + O_{\Phi^T \hat{e}_\pi}(T^2),
\]
and finally to:
\[
O_{(z,u-u_0)}(T) = O_z(T) + O_{\Phi^T \hat{e}_\pi}(T).
\]

Combining the previous bounds [11.2] and [11.6] we obtain:
\[
V_z(F_T^2(z,u)) = V_z(F_T^2(z,u)) + O_z(T^2) + O_{(\Phi^T \hat{e}_\pi)^2}(T^2),
\]
(11.7)

establishing that the decrease of $V_z$ imposed by $F_T^2$ equals the decrease imposed by $F_T^2$ up to $O(T^2)$ terms. In the second step we show that $V_z(F_T^2(z,u)) = V_z(z)$, when using the controller (8.5), is negative definite up to $O(T^2)$ terms. The approximate dynamics, using the controller (8.5), are given by:
\[
F_T^2(z,u) = Az + B\alpha + B\hat{\beta}\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= Az + B\alpha + B\left(\hat{\beta} + e_\beta\right)\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= Az + B\alpha + B(-\hat{\alpha} + K\hat{z}) + Be_\beta\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= Az + B\alpha + B(-\hat{\alpha} + e_\alpha + Kz - Ke_z) + Be_\beta\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= Az + BKz - BKe_z + Be_\alpha + Be_\beta\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= Az + BKz - BKe_z + Be_\alpha + Be_\beta\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= (A + BK)z - BKe_z + Be_\alpha + Be_\beta\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= (A + BK)z - BKe_z + B\hat{\beta}\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= (A + BK)z - BKe_z + B\hat{\beta}\hat{\beta}^{-1}(-\hat{\alpha} + K\hat{z})
\]
\[
= (A + BK)z - BKe_z + B\Phi^T \hat{e}_\pi.
\]

By using an argument similar to the one used to establish (8.1) and (11.2), we conclude the existence of a constant $\lambda_z \in \mathbb{R}^+$ so that the following inequality holds:
\[
V_z(F_T^2(z,u)) - V_z(z) \leq -\lambda_z T\|z\|^2 + O_{e_z}(T) + O_{(\Phi^T \hat{e}_\pi)^2}(T).
\]
(11.8)

Moreover, by using (11.5) we obtain the following inequality that no longer depends on $e_z$:
\[
V_z(F_T^2(z,u)) - V_z(z) \leq -\lambda_z T\|z\|^2 + O_z(T^2) + O_{(\Phi^T \hat{e}_\pi)^2}(T).
\]

We now analyze the effect of using the estimate $\hat{z}$ when implementing the observer (7.5), (7.7) and also the effect of using the approximate model $G_T^e$ defined in (7.3), instead of the exact model $G_T^e$. By redoing the analysis in Section 7 with the parameter estimation error defined as in (7.2) and using the observer (7.5), (7.7) we obtain:
\[
V_\pi(G_T^e(e_\pi)) - V_\pi(e_\pi) \leq -\lambda_{\pi\nu}\sqrt{T}e_\pi^T \Phi^T \hat{e}_\pi + \sqrt{T}O_{(z,u-u_0)^2}(T^2),
\]
(11.9)

where the term $\sqrt{T}O_{(z,u-u_0)^2}(T^2)$ accounts for the estimation error $e_z$ as well as the modeling error stemming from using the approximate model. The second inequality is obtained from the first by using (11.6) and $\sqrt{T}O(T^2) = O(T^2)$.

We now put the three intermediate steps, (11.7), (11.8), and (11.9), together:
\[
W(F_T^2(z,u), G_T^e(e_\pi)) - W(z,e_\pi) \leq -\lambda_z T\|z\|^2 - \lambda_{\pi\nu}\sqrt{T}\|\Phi^T \hat{e}_\pi\|^2 + O_{z^2}(T^2) + O_{(\Phi^T \hat{e}_\pi)^2}(T)
\]
\[
\leq -\lambda_z T\|z\|^2 - \lambda_{\pi\nu}\sqrt{T}\|\Phi^T \hat{e}_\pi\|^2 + MT\|z\|^2 + MT\|\Phi^T \hat{e}_\pi\|^2,
\]
where $M \in \mathbb{R}^+$ is the largest constant stemming from the definition of the $O$ terms. If we choose $\lambda \in \mathbb{R}^+$ and $T_5 \in \mathbb{R}^+$ satisfying:
\[
\lambda < \min\{\lambda_z, \lambda_\pi\},
\]
\[
T_5 < \min\left\{\frac{1}{M}(\lambda_z - \lambda), \frac{1}{M^2}(\lambda_\pi^2 - 2\lambda_\pi \lambda + \lambda^2)\right\},
\]
it follows that for all $T \in [0, T_6]$ we have:
\[
(11.10) \quad W(F_T^\pi(z, u), G_T^\pi(e_\pi)) - W(z, e_\pi) \leq -\lambda T \|z\|^2 - \lambda \sqrt{T} \|\Phi^T e_\pi\|^2.
\]
Therefore, for any $T \in [0, T_6]$, $T_6 = \min\{T_1, \ldots, T_5\}$, we have that $R$ remains invariant. By noting that trajectories remain in $R$ for any time in $[0, T_6]$ we conclude that we can apply the same argument to establish that trajectories remain in $R$ for any number of time steps since we only assumed that inputs were generated based on output measurements that remained in $\pi_1 \circ \pi_Z(R)$. Compactness of $R$ establishes that trajectories are bounded and thus there exists a constant $b_1 \in \mathbb{R}^+$ so that $\|e_\pi(k)\| \leq b_1$ and $\|z(k)\| \leq b_1$ for all $k \in \mathbb{N}$. Since $\pi$ is a smooth function of $z$ and $\pi$ is bounded, boundedness of $e_\pi = \pi - \hat{\pi}$ implies boundedness of the estimate $\hat{\pi}$, i.e., there exists a constant $b_2 \in \mathbb{R}^+$ so that $\|\hat{\pi}(k)\| \leq b_2$ for all $k \in \mathbb{N}$. Moreover, (11.10) enables the use of LaSalle’s invariance principle to conclude that trajectories will converge to the largest invariant set contained in the set defined by the equality:
\[
\|z\|^2 + \|\Phi^T e_\pi\|^2 = 0.
\]
In particular, $z$ will converge to the origin. Invoking Theorem 1 in [NTS99], combined with invariance of $R$ and smoothness of the dynamics, we conclude that the solutions of (3.1), when using the controller (8.5) with the parameter estimates provided by the observer (7.5), (7.7) and the state estimate (6.5), are bounded, i.e., there exists a constant $b_3 \in \mathbb{R}^+$ so that $\|x(t)\| \leq b_3$ and, moreover, \(\lim_{t \to \infty} x(t) = 0\). Hence, by taking $b = \max\{b_1, b_2, b_3\}$ we conclude the proof.

**Proof of Theorem 9.2** In the presence of noise, the estimation error is given by $e_\pi = O(z, u - w_0)(T) + O_G(T^-2)$. The proof follows the same arguments of the proof of Theorem 9.1 while accounting for the effect of $e_\pi$. Therefore, we shall describe only the required modifications.

We first note that to establish boundedness of all the signals it is sufficient to establish the existence of a sub-level set $R$ of $W$ that is forward invariant and satisfies $S \subseteq \pi_Z(R)$. We define $R$ to be the smallest sub-level set of $W$ containing the set:
\[
(11.11) \quad (B_{r_1 + r_2}(0) \cup S) \times B_{r_1 + r_2}(0) \subset \mathbb{R}^2 \times \mathbb{R}^2,
\]
where $B_{r_1 + r_2}(0) \subset \mathbb{R}^2$ denotes the Euclidean ball centered at zero and of radius $r_1 + r_2$ with $r_1 = 2(\beta - \beta)$ and $r_2 \in \mathbb{R}^+$ a fixed constant. Note that $S \subseteq B_{r_1 + r_2}(0) \cup S \subseteq \pi_Z(R)$.

Consider any point $(z, e_\pi)$ in the boundary of $R$. The norm of $(z, e_\pi)$ is no smaller than the norm of any point in the boundary of the set (11.11) and any such point has norm at least $\sqrt{r_1^2 + r_2^2}$. Hence:
\[
\|z\|^2 + |e_\alpha|^2 + |e_\beta|^2 = \|z\|^2 + \|e_\pi\|^2 = r_1^2 + r_2^2 \implies \|z\|^2 + |e_\alpha|^2 \geq r_1^2 + r_2^2 - |e_\beta|^2 = r_1^2 + r_2^2 - 4(\beta - \beta)^2 = r_1^2 + r_2^2 - r_1^2 = r_2^2,
\]
where we used the fact that the projection (7.5) in (7.7) ensures the bound $|e_\beta| \leq 2(\beta - \beta)$. Therefore, for any point $(z, e_\pi)$ on the boundary of $R$ we have:
\[
-\|z\|^2 - \|\Phi^T e_\pi\|^2 = -\|z\|^2 - |e_\alpha|^2 - u^2 |e_\beta|^2 \leq -\|z\|^2 - |e_\alpha|^2 \leq -r_2^2.
\]

(11.12)
Propagating the effect of the estimation error through the computation of \( W(F_T^r(z, u), G_T^r(e_\pi)) - W(z, e_\pi) \) in the proof of Theorem 9.1 we obtain, for sufficiently small \( T \) and \( \lambda = \min\{\lambda_2, \lambda_3\} \):

\[
W(F_T^r(z, u), G_T^r(e_\pi)) - W(z, e_\pi) \leq -\lambda T \|z\|^2 - \lambda_2 \sqrt{T} \|\Phi^T e_\pi\|^2 + O_T(T^{-2})
\]

\[
\leq -\lambda \sqrt{T} \|z\|^2 - \lambda \sqrt{T} \|\Phi^T e_\pi\|^2 + M \bar{d}^2 T^{-2}.
\]

Using (11.12) in the previous inequality leads to:

\[
W(F_T^r(z, u), G_T^r(e_\pi)) - W(z, e_\pi) \leq -\lambda \sqrt{T} \bar{r}_2^2 + M \bar{d}^2 T^{-2}.
\]

Hence, if:

\[
d \leq b_1 \overset{\text{def.}}{=} r_2 \bar{T}^{\frac{5}{2}} \sqrt{\frac{\lambda}{M}},
\]

we conclude that \( R \) is invariant thereby showing that all signals remain bounded, i.e., if we define \( b_2' \) as the radius of the smallest ball containing \( R \) we conclude that \( \|\hat{z}(k)\| \leq b_2' \) and \( \|\hat{\alpha}(k), \hat{\beta}(k)\| \leq b_2' \) for all \( k \in \mathbb{N} \). By using arguments by using arguments similar to those employed in the proof of Theorem 9.1 there exists a constant \( b_2'' \) so that \( \|x(t)\| \leq b_2'' \) for all \( t \in \mathbb{R} \) and we can define \( b_2 \) to be \( \max\{b_2', b_2''\} \).

Moreover, trajectories will converge to the smallest sub-level set of \( W \) containing the ball of radius \( r_1 + r_3 \) centered at zero where \( r_3 \) is the smallest real number satisfying \( -\lambda \sqrt{T} \bar{r}_2^2 + M \bar{d}^2 T^{-2} \leq 0 \), i.e., \( r_3 = \bar{d} T^{-\frac{1}{2}} \left(\frac{M}{\lambda}\right)^{\frac{1}{2}} \).

Since said sub-level set is contained in the ball centered at the origin and of radius \((r_1 + r_3)\lambda_{\max}(P_w)/\lambda_{\min}(P_w)\)

where \( P_w \) is the matrix defining the quadratic Lyapunov function \( W(z, e_\pi) = w^T P_w w, \ w = (z, e_\pi) \), the result is proved by taking \( b_3 \) to be:

\[
\max \left\{ \frac{\lambda_{\max}(P_w)}{\lambda_{\min}(P_w)}, \left(\frac{\lambda}{M}\right)^{-\frac{1}{2}} \frac{\lambda_{\max}(P_w)}{\lambda_{\min}(P_w)} \right\}.
\]

□

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