ON THE REFINED STRICHARTZ ESTIMATES
CIPRIAN DEMETER

Abstract. We present a slightly simpler proof of the multilinear refined Strichartz estimate from [6], and prove a slightly more general linear refined Strichartz estimate. Our arguments seek to clarify the connection between these estimates, refined decoupling and tube incidences.

1. Introduction
Let $E$ be the extension operator associated with the paraboloid. More precisely, if $f : [-1, 1]^{n-1} \rightarrow \mathbb{C}$, let for $x = (x_1, \ldots, x_n)$

$$Ef(x) = \int f(\xi_1, \ldots, \xi_{n-1})e(\xi_1 x_1 + \ldots + \xi_{n-1} x_{n-1} + (\xi_1^2 + \ldots + \xi_{n-1}^2)x_n) d\xi_1 \ldots d\xi_{n-1}.$$ 

Let us first recall the classical Strichartz estimate.

Theorem 1.1 ([9]). For each $f : [-1, 1]^{n-1} \rightarrow \mathbb{C}$ we have

$$\|Ef\|_{L^{2(n+1)}_{t,x}(\mathbb{R}^n)} \lesssim \|f\|_2.$$ 

The following result (in fact, its equivalent formulation in Corollary 1.3 below) was proved in [6] when $n = 2$. The proof extends to $n \geq 3$, as observed in [7]. This result proved instrumental in the recent resolution of Carleson’s problem on pointwise convergence of the solution of Schrödinger’s equation to the initial data, see [6] and [5].

The notation $\lesssim$ will signal the existence of implicit constants of the order $O((\log R)^{O(1)})$, and $|\cdot|$ will refer to the cardinality of finite sets.

Theorem 1.2. Let $\Omega_1, \ldots, \Omega_n$ be transverse cubes in $[-1, 1]^{n-1}$ with diameter $\sim 1$, in the sense that no hyperplane in $\mathbb{R}^{n-1}$ simultaneously intersects all $\Omega_i$. For $1 \leq i \leq n$, let $f_i : \Omega_i \rightarrow \mathbb{C}$. Let $Q$ be a collection of $N$ pairwise disjoint cubes $q$ in $[0, R]^n$ with side length $R^{1/2}$.

Then there is a subcollection $Q' \subset Q$ such that, writing $S = \bigcup_{q \in Q'} q$, we have

$$|Q| \lesssim |Q'|$$

and for each $\epsilon > 0$

$$(\prod_{i=1}^n \|Ef_i\|_{L^{2(n+1)}_{t,x}(S)})^{1/n} \lesssim \epsilon^N R^{\frac{n-1}{n+1}} R^\epsilon (\prod_{i=1}^n \|f_i\|_2)^{1/n}. \quad (2)$$

Note that the exponent $\frac{2(n+1)}{n-1}$ is the same in both theorems. The gain $N^{-\frac{n-1}{n+1}}$ in Theorem 1.2 comes at the cost of replacing the domain of integration $\mathbb{R}^n$ with with a restricted collection of cubes.

The author is partially supported by the Research NSF grant DMS-1800305.
We say that a variable quantity is essentially constant if its value is always in some interval \([v, 2v]\), for some fixed \(v > 0\). Theorem 1.2 admits the following corollary, which on closer inspection may in fact be seen to be an equivalent reformulation of the former, using pigeonholing as explained later in this note.

**Corollary 1.3** (Multilinear refined Strichartz estimate). Let \(\Omega_1, \ldots, \Omega_n\) be transverse cubes in \([-1, 1]^{n-1}\), with diameter \(\sim 1\). For \(1 \leq i \leq n\), let \(f_i : \Omega_i \to \mathbb{C}\). Let \(Q\) be a collection of \(N\) pairwise disjoint cubes \(q\) in \([0, R]^n\) with side length \(R^{1/2}\), such that for each \(1 \leq i \leq n\), the quantity \(\|Ef_i\|_{L^{2(n+1)/n}_{n-1}((\cup_q \in Q)q)}\) is essentially constant in \(q\). Then

\[
\prod_{i=1}^{n} \|Ef_i\|_{L^{2(n+1)/n}_{n-1}((\cup_q \in Q)q)}^{1/n} \lesssim N^{-\frac{n-1}{n(n+1)}} R^\epsilon \prod_{i=1}^{n} \|f_i\|_2^{1/n}.
\]

The proofs in [6] and [7] of this result are as follows. First, a linear refined Strichartz estimate is obtained for families of cubes \(q\) in \([0, R]^n\) with side length \(R^{1/2}\), that we recall below. The proof of the refined decoupling from [8], that we recall below. The proof of the refined decoupling from [8] uses very similar ideas to the proof of the linear Strichartz estimate from [6] and [7] (in particular rescaling from \(R\) to \(R^{1/2}\) and the \(l^2\) decoupling from [2]), but it is conceptually easier, in the sense that it relies less on creating structure, and consequently, it uses less pigeonholing.

In Section 2 we present a slightly different argument for Theorem 1.2, one that does not rely on a linear refined Strichartz estimate. Instead, it will use the refined decoupling from [8], that we recall below. The proof of the refined decoupling from [8] uses very similar ideas to the proof of the linear Strichartz estimate from [6] and [7] (in particular rescaling from \(R\) to \(R^{1/2}\) and the \(l^2\) decoupling from [2]), but it is conceptually easier, in the sense that it relies less on creating structure, and consequently, it uses less pigeonholing.

In Section 4 we repeat this argument to prove a slightly more general linear refined Strichartz estimate. For reader’s convenience, in the last section we recall (in the simplest case \(n = 2\) the way the bilinear refined Strichartz estimate solves the Carleson problem.

A tube \(T\) is a cylinder in \(\mathbb{R}^n\) with radius \(R^{1/2}\) and length \(R\). We write \(R^\epsilon T\) to denote the cylinder with radius \(R^{1/2+\epsilon}\) and length \(R\), centered at the same point as, and having the same direction as \(T\). We will say that the tubes \(T_1, \ldots, T_n\) with directions specified by unit vectors \(n_1, \ldots, n_n\) are transverse if the volume of the parallelepiped determined by these vectors has volume \(\gtrsim n\).

We recall the wave packet decomposition for \(Ef\) on \([0, R]^n\), where \(f : [-1, 1]^{n-1} \to \mathbb{C}\)

\[
Ef(x) = \sum_{T \in \mathcal{T}_R(f)} F_T(x) + O(R^{-100n})\|f\|_2, \quad x \in [0, R]^n.
\]

The tubes in the collection \(\mathcal{T}_R(f)\) intersect \([0, R]^n\). Here \(F_T = (Ef_T)1_{[0, R]^n}\), and \(|F_T|\) is approximately equal to \(w_T 1_T\). In particular, \(\|F_T\|_p \sim w_T R^{\frac{n+1}{mp}}\). We call \(w_T\) the weight of \(F_T\). Moreover, the functions \(F_T\) are almost orthogonal and

\[
\|f\|_2 \sim (\sum_{T \in \mathcal{T}_R(f)} \|F_T\|_2^2)^{1/2}.
\]

See for example Chapter 2 in [3].
We now recall the two key results that will be used in this note.

**Theorem 1.4** (Refined decoupling, [8]). Let $Q$ be a collection of pairwise disjoint cubes $q$ in $[0, R]^n$ with side length $R^{1/2}$. Let $\epsilon > 0$. Assume that each $q$ intersects at most $M$ fat tubes $R^T$, with $T \in \mathcal{T}_R(f)$, for some $1 \leq M \lesssim R^{n+1-\epsilon}$. Then for each $2 \leq p \leq \frac{2(n+1)}{n-1}$ we have

$$\|E_f\|_{L^p(\bigcup_{q \in Q} q)} \lesssim \epsilon R^{O(\epsilon)} M^{\frac{1}{2}-\frac{1}{p}} \left( \sum_{T \in \mathcal{T}_R(f)} \|F_T\|_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{p}}.$$ 

This result is a refinement of the decoupling proved in [2], which has the factor $R^{\frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right)}$ in place of $M^{\frac{1}{2}-\frac{1}{p}}$.

We will also use the following multilinear Kakeya inequality from [1], in essentially the same way that has been used in [6] and [7].

**Theorem 1.5** (Multilinear Kakeya). Let $T_1, \ldots, T_n$ be transverse families of tubes in $\mathbb{R}^n$. This means that $T_1, \ldots, T_n$ are transverse whenever $T_i \in \mathcal{T}_i$.

Given $M_1, \ldots, M_n \geq 1$, let $Q'$ be a collection of pairwise disjoint $R^{1/2}$-cubes $q$, with each $q$ intersecting $\sim M_i$ tubes from $T_i$, for each $1 \leq i \leq n$. Then

$$|Q'| \lesssim \epsilon R^\left( \prod_{i=1}^n \frac{|T_i|}{\prod_{i=1}^n M_i} \right)^{\frac{1}{n-1}}.$$ \hfill (3)

**2. Proof that Theorem 1.4 and Theorem 1.5 imply Theorem 1.2**

We may assume that for each $q \in Q$ and each $1 \leq i \leq n$ we have

$$R^{-10^m} \|f_i\|_2 \leq \|E f_i\|_{L^{\frac{2(n+1)}{n-1}}(q)},$$ \hfill (4)

as the cubes $q$ failing to satisfy this requirement will produce a very small contribution to the left hand side of (2), so they can be harmlessly added to $Q'$.

We use the wave packet decomposition for each $f_i$ on $[0, R]^n$

$$E f_i(x) = \sum_{T_i \in \mathcal{T}_i} F_{T_i}(x) + O(R^{-100n}) \|f_i\|_2, \quad x \in [0, R]^n,$$

with $F_{T_i} = Ef_{T_i}$. We denote by $w_{T_i}$ the weight of $F_{T_i}$. We can partition $\mathcal{T}_i$ into $\lesssim 1$ collections $\mathcal{T}_{i,l}$ such that $w_{T_i}$ is essentially constant for $T_i \in \mathcal{T}_{i,l}$ and such that

$$E f_i(x) = \sum_{j \geq 1} \sum_{T_i \in \mathcal{T}_{i,l}} F_{T_i}(x) + O(R^{-100n}) \|f_i\|_2, \quad x \in [0, R]^n.$$ \hfill (5)

Because of (4) and (5), for each $q \in Q$ and each $1 \leq i \leq n$ there is some $j_i \lesssim 1$ such that

$$\|E f_i\|_{L^{\frac{2(n+1)}{n-1}}(q)} \lesssim \sum_{T_i \in \mathcal{T}_{i,l}} \|F_{T_i}\|_{L^{\frac{2(n+1)}{n-1}}(q)}.$$ \hfill (6)
We partition \( \mathcal{Q} \) into collections in such a way that all \( q \) in each given collection are assigned the same \( n \)-tuple \((j_1, \ldots, j_n)\). Since there are \( \lesssim 1 \) such collections, we may use pigeonholing to pick one, call it \( \mathcal{Q}^* \), such that

\[
|\mathcal{Q}| \lesssim |\mathcal{Q}^*|. \tag{7}
\]

Call \( T_{i,j_i} = T_i \), with \((j_1, \ldots, j_n)\) being the \( n \)-tuple associated with \( \mathcal{Q}^* \), and assume the weight of each \( F_{T_i} \) with \( T_i \in T_i \) is \( \sim w_i \). We perform one last dyadic pigeonholing.

Fix \( \epsilon > 0 \). We distinguish the cubes \( q \) in \( \mathcal{Q}^* \) according to how many fat tubes \( R^c T \) with \( T \in T_{i,j_i} \) intersect \( q \). We may thus find a collection \( \mathcal{Q}' \subset \mathcal{Q}^* \) such that

\[
|\mathcal{Q}'| \lesssim |\mathcal{Q}^*| \tag{8}
\]

and such that for each \( q \in \mathcal{Q}' \) and each \( 1 \leq i \leq n \) we have

\[
M_i \leq |\{ T_i \in T_i : R^c q \cap T_i \neq \emptyset \}| \leq 2M_i
\]

for some dyadic \( M_i \). Note that (4) combined with the Schwartz-type decay of \( F_{T_i} \) away from \( T_i \) shows that \( M_i \) cannot be zero.

Using Theorem 1.3 (and (7), (5)) we get

\[
N \lesssim |\mathcal{Q}'| \lesssim R^{O(\epsilon)} \left( \prod_{i=1}^{n} \left| \frac{T_i}{M_i} \right| \right)^{\frac{1}{n}}. \tag{9}
\]

Let \( S = \bigcup_{q \in \mathcal{Q}'q} \). Note that (3) implies that

\[
\left( \prod_{i=1}^{n} \| E_{f_i} \|_{L^{2(n+1)}} \right)^{1/n} \lesssim \left( \prod_{i=1}^{n} \sum_{T_i \in T_i} \| F_{T_i} \|_{L^{2(n+1)}} \right)^{1/n}.
\]

Using Theorem 1.4 we can dominate the right hand side by

\[
R^n \left( \prod_{i=1}^{n} M_i \right)^{\frac{1}{n(n+1)}} \left( \prod_{i=1}^{n} \sum_{T_i \in T_i} \| F_{T_i} \|_{L^{2(n+1)}} \right)^{\frac{1}{n}}.
\]

Recall that for each \( p \geq 1 \)

\[
\| F_{T_i} \|_{p} \sim w_i |T_i|^{1/p} \sim w_i R^{\frac{n+1}{2p}},
\]

so the above expression is

\[
\sim R^{\frac{n+1}{2}+\epsilon} \left( \prod_{i=1}^{n} M_i \right)^{\frac{1}{n(n+1)}} \left( \prod_{i=1}^{n} w_i \right)^{\frac{1}{2}} \left( \prod_{i=1}^{n} |T_i| \right)^{\frac{1}{2(n+1)}}.
\]

It remains to show that

\[
R^{\frac{n-1}{2}} \left( \prod_{i=1}^{n} M_i \right)^{\frac{1}{n+1}} \left( \prod_{i=1}^{n} w_i \right)^{\frac{1}{2}} \left( \prod_{i=1}^{n} |T_i| \right)^{\frac{n-1}{2(n+1)}} \lesssim \epsilon R^{O(\epsilon)} N^{-\frac{n-1}{n(n+1)}} \left( \prod_{i=1}^{n} \| f_i \|_{2} \right)^{1/n}.
\]

Using orthogonality we write

\[
\| f_i \|_{2} \gtrsim \left\| \sum_{T_i \in T_i} f_{T_i} \|_{2} \sim w_i |T_i|^{1/2} R^{\frac{n-1}{4}}.
\]
It thus suffices to show that
\[
\left( \prod_{i=1}^{n} M_i \right)^{\frac{1}{n(n+1)}} \left( \prod_{i=1}^{n} |T_i| \right)^{\frac{n}{2(n+1)}} \lesssim_{\epsilon} R^{O(\epsilon)} N_{-}^{\frac{n-1}{n+1}} \left( \prod_{i=1}^{n} |T_i| \right)^{\frac{1}{2n}}.
\]
After rearranging it, this is the same as (9).

3. Reversing the implication?

One may wonder if the multilinear refined Strichartz estimate (Corollary 1.3) implies the refined decoupling in Theorem 1.4, or at least some multilinear version of it. What we ask for is a more or less direct argument, like the one for the reverse implication described in the previous section. The answer appears to be “no”. To illustrate the relative strength of the latter compared to the former result, we point out below that the implication under question does hold if the collection \( Q \) satisfies the saturation condition (S2) (see the statement of the following theorem). In light of (3), this condition reads (we use \( \approx \) to hide arbitrarily small \( R^{\epsilon} \) factors)

\[
\left( \prod_{i=1}^{n} |T_i| \right)^{\frac{1}{n+1}} \approx |Q|,
\]
and means that a significant fraction of the transverse incidences between tubes occur at the cubes from \( Q \).

Let \( Q_1, \ldots, Q_n \) be transverse cubes in \([-1, 1]^{n-1}\) and let \( f_i : Q_i \to \mathbb{C} \). Consider the wave packet decomposition on \([0, R]^n\)

\[
Ef_i(x) = \sum_{T_i \in T_R(f_i)} F_{T_i}(x) + O(R^{-100n})\|f_i\|_2, \quad x \in [0, R]^n.
\]
A rather immediate computation shows that Corollary 1.3 implies the following result.

**Theorem 3.1** (Saturated multilinear refined decoupling). Let \( p = \frac{2(n+1)}{n-1} \). Assume that for each \( i \) and each \( T_i \in T_R(f_i) \), the weight of \( F_{T_i} \) is \( \sim \) 1. Let \( Q \) be a collection of pairwise disjoint cubes \( q \) in \([0, R]^n\) with side length \( R^{1/2} \). Let \( \epsilon > 0 \). Assume that each \( q \) intersects \( \sim M_i \) fat tubes \( R^{\epsilon}T_i \), with \( T_i \in T_R(f_i) \), for each \( 1 \leq i \leq n \). Assume that

(S1): For each \( i \), \( \|Ef_i\|_{L^p(Q)} \) is essentially constant in \( q \)

(S2):

\[
\left( \prod_{i=1}^{n} |T_i| \right)^{\frac{1}{n+1}} \lesssim_{\epsilon} R^{\epsilon} |Q|.
\]

Then

\[
\left( \prod_{i=1}^{n} \|Ef_i\|_{L^p(Q \in \cup Q_i)} \right)^{\frac{1}{p}} \lesssim_{\epsilon} R^{O(\epsilon)} \left( \prod_{i=1}^{n} M_i^{\frac{1}{p'} - \frac{1}{p}} \right)^{\frac{1}{p'}} \left( \prod_{i=1}^{n} \sum_{T_i \in T_R(f_i)} \|F_{T_i}\|_{L^p(R^n)} \right)^{\frac{1}{p'}}.
\]
4. Refined decoupling implies linear refined Strichartz estimate

Recall that the directions of the tubes arising in our wave packet decompositions -let us call them admissible- are given by the normal vectors to the paraboloid over $[-1,1]^{n-1}$. Thus, the angles between the directions of admissible tubes and the vertical axis $x_n$ are at most $C_n \pi$, with $C_n < \frac{1}{2}$. In other words, admissible tubes are never close to being horizontal.

**Definition 4.1.** Let $D_n$ be a fixed parameter depending only on the dimension $n$. A collection $Q$ of pairwise disjoint $R^{1/2}$-cubes in $[0, R]^n$ is said to be almost horizontal if each admissible tube intersects at most $D_n$ cubes in $Q$.

The collection of cubes intersecting a horizontal hyperplane $x_n = \text{const}$ is almost horizontal. But so is the collection of cubes intersecting the graph of a smooth function $\phi : [0, R]^{n-1} \to [0, R]$ with $\|\nabla \phi\|_{L^\infty} \lesssim C_n, D_n, 1$.

We prove the following result, using the refined decoupling in Theorem 1.4. The reader should compare this with Theorem 1.1.

**Theorem 4.2** (Linear refined Strichartz estimate). Let $f : [-1,1]^n \to \mathbb{C}$. Let $Q$ be a collection of pairwise disjoint $R^{1/2}$-cubes $q$ in $[0, R]^n$. Assume the quantity $\|Ef\|_{L^{2(n+1)}(\bigcup_{q \in Q} q)}$ is essentially constant in $q$. We partition $Q$ into almost horizontal collections $Q_j$. Let $\sigma$ be the cardinality of the smallest among these collections.

Then

$$\|Ef\|_{L^{2(n+1)}(\bigcup_{q \in Q} q)} \lesssim \sigma^{-\frac{1}{n+1}} R^n \|f\|_2.$$  (10)

The implicit constant is independent of the number of collections $Q_j$.

The case of this theorem when each of the almost horizontal collections is in fact horizontal (the cubes touch a fixed horizontal hyperplane) was proved in [6]. Let us briefly sketch an argument here that relies solely on Theorem 1.4.

**Proof.** The proof is very similar to the one in Section 2, but it uses a trivial incidence bound, rather than the multilinear Kakeya estimate. Using pigeonholing we may assume that all the wave packets of $Ef$ have weight $\sim 1$, and that each $q \in Q'$ (for some $Q' \subset Q$ with $|Q| \lesssim |Q'|$) intersects roughly $M$ fat tubes $R'T$ with $T \in T_R(f)$, for some fixed $M \geq 1$.

Then the essentially constant property combined with refined decoupling implies that

$$\|Ef\|_{L^{2(n+1)}(\bigcup_{q \in Q} q)} \lesssim \|Ef\|_{L^{2(n+1)}(\bigcup_{q \in Q'} q')} \lesssim M^{\frac{1}{n+1}} |T_R(f)|^{\frac{1}{n+1}}.$$  

Comparing this with (10) and using that $\|f\|_2 \sim R^{\frac{n-1}{2}} |T_R(f)|^{1/2}$, it remains to prove that

$$M \sigma \lesssim |T_R(f)|.$$  (11)

It is immediate that $Q'$ contains an almost horizontal collection of size $\gtrless \sigma$, call it $Q^*$. Indeed, since

$$\sum_j |Q' \cap Q_j| = |Q'| \gtrless |Q| = \sum_j |Q \cap Q_j|$$

there must be some $j$ with $|Q' \cap Q_j| \gtrless |Q \cap Q_j| \geq \sigma$. We let $Q^* = Q' \cap Q_j$. 
Recall that each \( q \in \mathcal{Q}^\ast \) intersects roughly \( M \) tubes, and that each tube can intersect at most \( D_n \) cubes in \( \mathcal{Q}^\ast \). It follows that
\[
M|\mathcal{Q}^\ast| \lesssim D_n |\mathcal{T}_R(f)|.
\]

The estimate (11) is now immediate. \( \square \)

5. CARLESON’S PROBLEM

Carleson’s problem \[3\] in the plane about the pointwise convergence of the solution to Schrödinger’s equation to initial data is equivalent to the following theorem.

**Theorem 5.1.** Let \( f_1, f_2 \) be supported on disjoint intervals in \([-1, 1]\). Consider a collection of \( \sim R \) unit squares \( \omega \) in \([0, R]^2\) with pairwise disjoint projections onto the \( x \) axis, and let \( S \) be their union. Then
\[
\| (E_1 E_2)^{1/2} \|_{L^2(S)} \lesssim R^{1/4+\epsilon} (\| f_1 \|_2 \| f_2 \|_2)^{1/2}.
\]

Let us see how Theorem 5.1 implies this. Pigeonholing, we may assume that \( \| (E_1 E_2)^{1/2} \|_{L^2(\omega)} \) is essentially constant in \( \omega \), and that there are either 0 or \( \sim \lambda \) squares \( \omega \) inside each \( R^{1/2} \)-square \( q \) (we take \( q \) from a partition of \([0, R]^2\)). The value of \( \lambda \) will be irrelevant. Let \( \mathcal{Q} \) be the collection of \( R^{1/2} \)-squares containing the squares \( \omega \). Assume it has size \( N \). Apply Theorem 1.2 to find \( \mathcal{Q}' \subset \mathcal{Q} \) such that \( N \lesssim |\mathcal{Q}'| \) and
\[
\| (E_1 E_2)^{1/2} \|_{L^6(\cup_{q \in \mathcal{Q}'})} \lesssim R^{1/6} N^{-1/6} (\| f_1 \|_2 \| f_2 \|_2)^{1/2}.
\]

Note that because of the essentially constant assumption and the \( \lambda \) uniformity we have
\[
\| (E_1 E_2)^{1/2} \|_{L^2(S)} \lesssim \| (E_1 E_2)^{1/2} \|_{L^2(\cup_{q \in \mathcal{Q}'\cup \cup_{q \in \mathcal{Q}}} \omega)}.
\]

Let us denote by \( M \) the cardinality of \( \{ \omega \subset q : q \in \mathcal{Q}' \} \). Recall that \( M \leq R \). Using Hölder, the right hand side is dominated by
\[
M^{1/3} \| (E_1 E_2)^{1/2} \|_{L^6(\cup_{q \in \mathcal{Q}'\cup \cup_{q \in \mathcal{Q}}} \omega)}.
\]

This is trivially dominated by
\[
M^{1/3} \| (E_1 E_2)^{1/2} \|_{L^6(\cup_{q \in \mathcal{Q}'})}.
\]

Note also that
\[
M \approx \lambda N, \text{ so } N \gtrsim MR^{-1/2}.
\]

Combining these leads to the desired estimate
\[
\| (E_1 E_2)^{1/2} \|_{L^2(S)} \lesssim R^{1/6} M^{1/3} N^{-1/6} (\| f_1 \|_2 \| f_2 \|_2)^{1/2} \lesssim R^{1/4+\epsilon} (\| f_1 \|_2 \| f_2 \|_2)^{1/2}.
\]

**References**

[1] Bennett, J., Carbery, A. and Tao, T. *On the multilinear restriction and Kakeya conjectures*, Acta Math. 196 (2006), no. 2, 261-302

[2] Bourgain, J. and Demeter, C. *The proof of the \( l^2 \) Decoupling Conjecture*, Annals of Math. 182 (2015), no. 1, 351-389.

[3] Demeter, C. *Fourier restriction, decoupling and applications*, Cambridge University Press, 2020

[4] Carleson, L. *Some analytic problems related to statistical mechanics* In: Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), pp. 5-45. Lecture Notes in Math., vol. 779. Springer, Berlin
[5] Du, X and Zhang R \textit{Sharp }$L^2$\textit{ estimate of Schrödinger maximal function in higher dimensions}, Ann. of Math. (2) 189 (2019), no. 3, 837-861

[6] Du, X, Guth, L and Li, X \textit{A sharp Schrödinger maximal estimate in }$\mathbb{R}^2$\textit{ Ann. of Math. (2) 186 (2017), no. 2, 607-640}

[7] Du, X, Guth, L, Li, X and Zhang, R \textit{Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates}, Forum Math. Sigma 6 (2018), e14, 18 pp

[8] Guth, L., Iosevich, A., Ou, Y. and Wang, H. \textit{On Falconer’s distance set problem in the plane}, to appear in Invent. Math.

[9] Strichartz, R. S. \textit{Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations}, Duke Math. J. 44 (1977), no. 3, 705-714.

Department of Mathematics, Indiana University, Bloomington IN

E-mail address: demeterc@indiana.edu