A geometric realization of silting theory for gentle algebras

Wen Chang · Sibylle Schroll

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Abstract
A gentle algebra gives rise to a dissection of an oriented marked surface with boundary into polygons and the bounded derived category of the gentle algebra has a geometric interpretation in terms of this surface. In this paper we study silting theory in the bounded derived category of a gentle algebra in terms of its underlying surface. In particular, we show how silting mutation corresponds to the changing of graded arcs and that in some cases silting mutation results in the interpretation of the octahedral axioms in terms of the flipping of diagonals in a quadrilateral as in the work of Dyckerhoff and Kapranov (J Eur Math Soc 20(6):1473–1524, 2018) in the context of triangulated surfaces. We also show that the cutting of the underlying surface along a curve without self-intersections corresponds to silting reduction. This is analogous to the Calabi–Yau reduction of surface cluster categories as shown by Marsh and Palu (Proc Lond Math Soc (3) 108(2):411–440, 2014) and Qiu and Zhou (Compos Math 153(9):1779–1819, 2017).

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1 Introduction
Gentle algebras are classical objects in the representation theory of associative algebras. They were introduced in the 1980s as a generalization of iterated tilted algebras of type $A_n$ [3], and affine type $\tilde{A}_n$ [4]. Remarkably, they play a role in many other areas of mathematics such as in the context of dimer models [9, 11], Lie algebras [23], cluster theory [1, 17], discrete derived categories [15, 43] and in connection with homological mirror symmetry [10, 14, 24, 33]. In [33, 36] it is shown that any gentle algebra gives rise to a dissection of an oriented marked surface.

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Wen Chang
changwen161@163.com
Sibylle Schroll
schroll@math.uni-koeln.de

1 School of Mathematics and Statistics, Shaanxi Normal University, Xi’an 710062, China
2 Department of Mathematics, Universität zu Köln, Weyertal 86-90, 50931 Cologne, Germany
3 Institutt for matematiske fag, NTNU, 7491 Trondheim, Norway
surface with boundary, and the objects and morphisms in the derived category of this gentle algebra can be interpreted on the surface model. This geometric interpretation led to the construction of a complete derived invariant \[2, 35\], building on a geometric interpretation of a derived invariant by Avella-Alaminos and Geiss in \[33, 36\].

Since their inception, gentle algebras have been a constant object of study and much about their representation theory is known. For example, the indecomposable objects and the Auslander-Reiten sequences in their module category are described in \[4, 18, 44\]. A geometric model of the module category of a gentle algebra is given in \[19\]. The indecomposable objects in their derived category are classified in terms of so-called homotopy string and homotopy band objects in \[8\] (the terminology homotopy strings and bands is due to \[12\]). A basis of the morphism space between indecomposable objects is given in \[6\] in terms of string and band combinatorics. Gentle algebras also have nice homological properties, for example, they are Gorenstein \[28\], the class of gentle algebras is closed under tilting-cotilting equivalences \[41\] and also under derived equivalences \[42\].

Silting objects in triangulated categories were introduced in \[32\]. They play an important role for triangle equivalences, as shown, for example, in terms of generalised Morita equivalences in \[29\]. Recently, there has been a renewed interest in silting objects in connection with cluster theory and, in particular, with \(\tau\)-tilting theory, see for example the surveys \[7, 16\]. An important development of silting theory, in particular, in connection or analogy with mutation in cluster theory is made in \[5\]. In this paper the notion of silting mutation at silting subcategories of triangulated categories is introduced, thereby giving a systematic way of constructing new silting subcategories from a given silting subcategory. Furthermore, in analogy with the notion of Calabi–Yau reduction in cluster categories, the notion of a silting reduction of a triangulated category by a pre-silting subcategory is introduced. The connection between Calabi–Yau reduction and silting reduction is made precise in \[26, 27\].

In this paper, we study the silting theory of the derived category of gentle algebras in terms of their geometric models. More precisely, we show that in analogy with mutation of surface cluster algebras, the silting mutation at a silting subcategory generated by an indecomposable object, is given by changing exactly one graded curve in a dissection of the surface into polygons. In terms of silting reduction we show that in analogy of with Calabi–Yau reduction of surface cluster categories, the silting reduction at a pre-silting subcategory generated by an indecomposable object associated to a curve on the surface corresponds to cutting the surface along this curve. This connection of mutation in terms of exchanging curves and reduction in terms of cutting surfaces establishes a geometric analogy with cluster theory and triangulations of surfaces more generally: Mutation of (surface) cluster algebras corresponds to flipping arcs in an ideal triangulation of a surface \[22\], the octahedral axiom in a triangulated category associated with a triangulated surface has been interpreted as flipping the diagonal in a quadrilateral on a triangulated surface \[21\], and Calabi–Yau reduction of cluster categories has an interpretation in terms of cutting surfaces \[34, 39\]. Furthermore, flips on decorated marked disks are used in \[30\] to construct groupoid structures on pure braid groups and in \[31, 38, 40\] flips on general decorated marked surfaces appear in the context of 3-Calabi–Yau triangulated categories. Finally, we note that a geometric surface model for discrete derived algebras is given in \[13\], and mutation is studied in terms of this geometric model.

We now summarise the main results of this paper. For this we briefly set up some notations, which are defined in more detail in Sect. 2. For a gentle algebra \(A\), let \((S, M, \Delta_A)\) be the associated surface of the bounded derived category \(D^b(A)\) as constructed in \[36\] and where \(S\) is a compact oriented surface, \(M\) a set of marked points both in the boundary and in the interior of \(S\) and \(\Delta_A\) the admissible dissection of \((S, M)\) induced by \(A\). Then the set of iso-
morphism classes of indecomposable objects in the bounded homotopy category $K^b(projA)$ of complexes of finitely generated projective modules is in bijection with graded curves (up to homotopy) connecting marked points in the boundary, which we refer to as graded arcs, and homotopy classes of certain graded closed curves together with indecomposable representations over $k[x, x^{-1}]$. Denote by $P_\gamma$ the object in $K^b(projA)$ corresponding to a graded arc $\gamma$ under this bijection. Recall from [2] that any basic silting object in $K^b(projA)$ corresponds to an admissible dissection $\Delta$ of graded arcs, where the arcs satisfy certain grading conditions. We denote this silting object by $P_\Delta$. In Sect. 4, given a graded arc $\gamma$ in $\Delta$, we define the left and right mutation $\mu_{\gamma}^\pm(\Delta)$ of $\Delta$, which is obtained from $\Delta$ by removing the graded arc $\gamma$ and replacing it with another uniquely defined graded arc in $(S, M)$. Then we show that $\mu_{\gamma}^\pm(\Delta)$ is again an admissible dissection of $(S, M)$, and that $P_{\mu_{\gamma}^\pm(\Delta)}$ is a silting object in $K^b(projA)$. More precisely, we show the following.

**Theorem 1** (Proposition 4.3, Theorem 4.4) Let $A$ be a gentle algebra with associated surface model $(S, M, \Delta_A)$ and let $P_\Delta$ be a silting object in $K^b(projA)$ with associated graded admissible dissection $\Delta$. Then for each graded arc $\gamma$ in $\Delta$, the object $P_{\mu_{\gamma}^\pm(\Delta)}$ is a silting object in $K^b(projA)$.

Furthermore, $P_{\mu_{\gamma}^\pm(\Delta)}$ is the silting mutation of $P_\Delta$ at the indecomposable direct summand $P_\gamma$, that is the following diagram is commutative

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\mu_{\gamma}^\pm} & \Delta \\
\downarrow & & \downarrow \\
P_\Delta & \xrightarrow{\mu_{P_\gamma}^\pm} & P_{\mu_{\gamma}^\pm(\Delta)}
\end{array}
\]

As a consequence, we determine when the silting mutation of a tilting object is again a tilting object. We do this by using the combinatorics of the surface in Proposition 4.6, and then in Corollary 4.7 we restate the result in purely in algebraic terms without reference to the surface.

In order to describe the silting reduction at a pre-silting subcategory, let $\gamma$ be a graded arc in $(S, M, \Delta_A)$ with two different endpoints such that $P_\gamma$ is a pre-silting object in $K^b(projA)$. Given such an arc $\gamma$, we cut the surface $(S, M)$ along $\gamma$ and denote by $(S_\gamma, M_\gamma)$ the corresponding cut surface (see Definition 5.1). Now set $\mathcal{P} = \text{add}(P_\gamma)$. Following [5], the silting reduction of the category $K^b(projA)$ with respect to $\mathcal{P}$ is the Verdier quotient $\mathcal{Z}_\mathcal{P} = K^b(projA)/\text{thick}\mathcal{P}$ which is again a triangulated category. Denote by $\overline{\mathcal{Z}_\mathcal{P}}$ its orbit category with respect to the shift functor. In analogy with Calabi–Yau reduction of surface cluster algebras [34, 39], the following theorem shows that $(S_\gamma, M_\gamma)$ is a geometric realization of $\overline{\mathcal{Z}_\mathcal{P}}$.

**Theorem 2** (Theorem 5.17) Let $\gamma$ be a graded arc in $(S, M, \Delta_A)$ with two different endpoints such that $P_\gamma$ is a pre-silting object in $K^b(projA)$ and set $\mathcal{P} = \text{add}(P_\gamma)$. Then the cut surface $(S_\gamma, M_\gamma)$ is a geometric model of the orbit category $\overline{\mathcal{Z}_\mathcal{P}}$ of the silting reduction of $K^b(projA)$ with respect to $\mathcal{P}$. More precisely,

1. the isomorphism classes of indecomposable objects in $\overline{\mathcal{Z}_\mathcal{P}}$ are in bijective correspondence with the set of arcs and pairs consisting of a gradable closed curve in $(S_\gamma, M_\gamma)$ and a non-zero element in the base field $k$.
2. the dimensions of the morphism spaces in $\overline{\mathcal{Z}_\mathcal{P}}$ are equal to the number of oriented intersections of the corresponding arcs and closed curves in $(S_\gamma, M_\gamma)$.
Note that if the two endpoints of $\gamma$ coincide, then the cut surface will have a $\circ$-point in the interior, thus corresponding to an infinite dimensional gentle algebra and the bijective correspondence of indecomposable objects in the bounded derived category and curves in the surface has not been established in this case.

The paper is organised as follows. We begin by recalling some background on marked surfaces, derived categories of gentle algebras, and silting theory in Sect. 2. In Sect. 3, we introduce the notion of a distinguished triangle on the surface, and we prove two key lemmas, which will be frequently used throughout the paper. In Sects. 4 and 5 we respectively study silting mutations and silting reductions in the derived category of gentle algebras as well as their geometric interpretations.

2 Preliminaries

In this paper, all algebras will be assumed to be over a base field $k$ which is algebraically closed. The algebraic closure of $k$ is not a strictly necessary condition, however, it ensures that an indecomposable $k[x, x^{-1}]$-module is uniquely determined by its dimension up to isomorphism. This simplifies the correspondence of band objects and closed curves as recalled in Sect. 2.1. Arrows in a quiver are composed from left to right as follows: for arrows $a$ and $b$ we write $ab$ for the path from the source of $a$ to the target of $b$. We adopt the convention that maps are also composed from left to right, that is if $f: X \to Y$ and $g: Y \to Z$ then $fg: X \to Z$. In general, we consider left modules. We denote by $\mathbb{Z}$ the set of integer numbers, and by $\mathbb{Z}^*$ the set of non-zero integer numbers.

2.1 Marked surfaces

In this subsection we recall some concepts about marked surfaces from [36] and [2].

Definition 2.1 [2, Definition 1.7] A pair $(S, M)$ is called a marked surface if

- $S$ is an oriented surface with non-empty boundary $\partial S$;
- $M = M^{\circ} \cup M^* \cup P^*$ is a finite set of marked points, where the elements in $M^\circ \cup M^*$ are in $\partial S$, and the elements in $P^*$, which will be called punctures, are in the interior of $S$. Each connected component of $\partial S$ is required to contain at least one marked point in $M^\circ \cup M^*$, and the points $\circ$ and $\bullet$ are alternating on such a component. The elements of $M^\circ$ and $M^* \cup P^*$ will be respectively represented by symbols $\circ$ and $\bullet$.

On the surface, all curves are considered up to homotopy. We will call the orientation of the surface the clockwise orientation, and we call anti-clockwise orientation the opposite orientation. When drawing surfaces in the plane, we will do so that locally, the orientation of the surface becomes the clockwise orientation of the plane.

Definition 2.2 [36, Definition 1.19] Let $(S, M)$ be a marked surface.

- An $\circ$-arc is a non-contractible curve, with endpoints in $M^\circ$.
- A loop is an $\circ$-arc which starts and ends at the same $\circ$-point.
- An $\bullet$-arc is a non-contractible curve, with endpoints in $M^* \cup P^*$.
- An infinite-arc is a non-contractible infinite curve which either starts or ends at a $\circ$-marked point and wraps around a single puncture in $P^*$ infinitely many times in the anti-clockwise direction or is a curve wrapping such that on either end it wraps infinitely many times around a puncture in $P^*$ in the anti-clockwise direction.
• An admissible arc is an o-arc or an infinite-arc.
• A closed curve is a non-contractible curve in the interior of $S$ whose starting point and ending point are equal. A closed curve is called primitive if it is not a non-trivial power of a closed curve in the fundamental group of $S$. For a closed curve $\gamma$ and a positive integer $n$, denote by $\gamma^n$ the closed curve on $S$, which is the $n$-th power of $\gamma$ in the fundamental group of $S$.

For admissible arcs and closed curves in a marked surface $(S, M)$, we now introduce the notion of oriented intersections and oriented intersection numbers. We will use these to recall in Sect. 2.2 some results from [36] on the morphisms in the derived category of a gentle algebra. By abuse of notation we will say that an infinite arc wrapping infinitely many times around a $*$-puncture is an arc connecting to the corresponding $*$-point. Furthermore, all intersections of curves are required to be transversal.

**Definition 2.3** Let $\alpha$ and $\beta$ be two admissible arcs or primitive closed curves on $(S, M)$ which intersect at a point $p$. An oriented intersection from $\alpha$ to $\beta$ at $p$ is the anticlockwise angle from $\alpha$ to $\beta$ based at $p$ such that the angle is in the interior of the surface with the convention that if the intersection is in the interior of the surface then the opposite angles are considered equivalent.

Denote by $\text{Int}_S(\alpha, \beta)$ the set of oriented intersections from $\alpha$ to $\beta$ in $S$. Let $\sharp\text{Int}_S(\alpha, \beta)$ be the number of elements in $\text{Int}_S(\alpha, \beta)$. We define the oriented intersection number $|\text{Int}_S(\alpha, \beta)|$ from $\alpha$ to $\beta$ in $S$ as follows.

$$|\text{Int}_S(\alpha, \beta)| = \begin{cases} \infty & \text{if } \exists \bullet \in \text{Int}_S(\alpha, \beta); \\ \sharp\text{Int}(\alpha, \alpha) + 1 & \text{if } \alpha = \beta \text{ is an admissible arc;} \\ \sharp\text{Int}(\alpha, \beta) & \text{otherwise.} \end{cases}$$

**Definition 2.4** [2, Definition 1.9] A collection of $o$-arcs $\{\gamma_1, \ldots, \gamma_r\}$ is admissible if the only possible intersections of these arcs are at the endpoints, and each subsurface enclosed by the arcs contains at least one $*$-point from $M^* \cup P^*$. A maximal admissible collection of $o$-arcs is called an admissible $o$-dissection. The notion of admissible $*$-dissection is defined in a similar way.

Denote by $g$ the genus of $S$ and by $b$ the number of connected components of $\partial S$. It is proved in [2], see also [37], that an admissible collection of $o$-arcs is an admissible $o$-dissection if and only if it contains exactly $|M^*| + |P^*| + b + 2g - 2$ arcs if and only if each subsurface enclosed by the arcs is a polygon which contains exactly one $*$-point from $M^* \cup P^*$.

Given an admissible $o$-dissection $\Delta$, it is shown in [2], see also [36, 37], that there exists a unique admissible $*$-dissection $\Delta^*$ (up to homotopy) such that each arc of $\Delta^*$ intersects exactly one arc of $\Delta$. We call $\Delta^*$ the dual $*$-dissection of $\Delta$. For a fixed admissible $o$-dissection $\Delta$ and its dual $*$-dissection $\Delta^*$, unless otherwise stated, all curves on the surface are always assumed to be in minimal position with respect to both these dissections.

**Definition 2.5** [36, Definition 2.10] [2, Definition 2.4] A graded curve $(\gamma, f)$ is an admissible arc or a closed curve $\gamma$, together with a function

$$f : \gamma \cap \Delta^* \longrightarrow \mathbb{Z},$$

where $\gamma \cap \Delta^*$ is the totally ordered set of intersection points of $\gamma$ with $\Delta^*$, where the order is induced by the direction of $\gamma$. The function $f$ is defined as follows: If $p$ and $q$ are in $\gamma \cap \Delta^*$ and $q$ is the direct successor of $p$, then $\gamma$ enters a polygon enclosed by $*$-arcs of $\Delta^*$ via $p$.
and leaves it via \( q \). If the point \( \circ \) in this polygon is to the left of \( \gamma \), then \( f(q) = f(p) + 1 \); otherwise, \( f(q) = f(p) - 1 \). Furthermore, if \( \gamma \) is a closed curve and if \( \gamma \cap \Delta^* = \{p_0, \ldots, p_n\} \), where the index is considered modulo \( n \), then \( f \) needs to satisfy \( f(p_0) = f(p_{n+1}) \).

Note that if \( \gamma \) is an admissible arc, then there always exists a grading, while if \( \gamma \) is a closed curve, then there may not exist a grading of \( \gamma \) since there might not be a function \( f \) satisfying \( f(p_0) = f(p_{n+1}) \). In the geometric language of winding numbers (see \([2, \text{Section 1]}\) or \([24, 33]\)), a closed curve is gradable if and only if its winding number with respect to the admissible dissection is zero. We also note that any grading on \( \gamma \) is completely determined by its value on a single element in \( \gamma \cap \Delta^* \). If a grading \( f \) on \( \gamma \) exists, then the map \( f[n] : l \mapsto f(l) - n \) with \( n \in \mathbb{Z} \) is also a grading on \( \gamma \), and all gradings on \( \gamma \) are of this form. We call \([1]\) the shift of the grading \( f \). As we will see in Theorem 2.9 (1), the grading shift on curves corresponds to the shift functor in the derived category.

**Lemma 2.6** Let \( \gamma \) be a primitive closed curve, and let \( n \) be any positive integer. Then \( \gamma \) is gradable if and only if its power \( \gamma^n \) is gradable. Moreover, there is a one-to-one correspondence between the gradings \( f \) on \( \gamma \) and the gradings \( f^n \) on \( \gamma^n \) with \( f^n[1] = f[1]^n \).

**Proof** Denote by \( \gamma \cap \Delta^* = \{p_0, \ldots, p_m\} \), where \( p_i = \gamma \cap \gamma_i^* \) for some initial \( \bullet \)-arc \( \gamma_i^* \) in \( \Delta^* \), \( 0 \leq i \leq m \). Then we may assume \( \gamma^n \cap \Delta^* = \{p_0^1, \ldots, p_m^1, \ldots, p_0^n, \ldots, p_m^n\} \), where \( p_j^i = \gamma^n \cap \gamma_i^* \) for any \( 0 \leq i \leq m, 1 \leq j \leq n \). Suppose \( \gamma \) is gradable and let \( f : \gamma \cap \Delta^* \to \mathbb{Z} \) be a grading on it. Let 

\[
 f^n : \gamma^n \cap \Delta^* \to \mathbb{Z}
\]

be the function such that \( f^n(p_j^i) = f(p_i) \) for any \( 0 \leq i \leq m, 1 \leq j \leq n \). Then note that \( f^n \) is a well-defined grading on \( \gamma^n \), and this gives a one-to-one correspondence between the gradings on \( \gamma \) and the gradings on \( \gamma^n \). Furthermore, it directly follows from the definitions that \( f^n[1] = f[1]^n \). \( \square \)

### 2.2 Derived categories

We recall in this section the derived category of a gentle algebra, and its geometric realization given in \([36]\) using a graded marked surface.

For a finite dimensional algebra \( A \), it is well known that the (bounded) derived category \( \mathcal{D}^b(A) \) is triangle equivalent to the homotopy category \( K^{-,b}(\text{proj} A) \) of complexes of projective \( A \)-modules bounded on the right and bounded in homology. In the following, we will not distinguish these two categories, and view the perfect derived category \( K^b(\text{proj} A) \) as a full subcategory of \( \mathcal{D}^b(A) \). The derived category of a gentle algebra is well-studied. In particular, the authors in \([8]\) classified the indecomposable objects in the category in terms of (homotopy) string objects and (homotopy) band objects. The morphisms between these objects are explicitly described in \([6]\).

More recently, a one-to-one correspondence between graded marked surfaces \((\mathcal{S}, M, \Delta_A)\) and gentle algebras \( A \) has been established in \([36, 37]\), where \( \Delta_A \) is the admissible dissection uniquely determined by \( A \). A complete description of the indecomposable objects in \( \mathcal{D}^b(A) \) was given in terms of graded curves in \([36, \text{Theorem 2.12}]\), and the morphisms are described as graded oriented intersections in \([36, \text{Theorem 3.3}]\). We adopt the notation from \([36]\), where we denote the indecomposable object corresponding to a graded curve \((\gamma, f)\) by \( P(\gamma, f) \). The precise correspondence is recalled below. Note, however, that in \([36]\) all closed curves considered were assumed to be primitive. In contrast, in this paper, we allow closed curves which are not necessarily primitive.
Theorem 2.7 [36, Theorem 2.12] Let $A$ be a gentle algebra associated with a graded marked surface $(S, M, \Delta_A)$. Then

1. the isomorphism classes of indecomposable string objects $P_{(\gamma, f)}$ in $\mathcal{D}^b(A)$ are in bijection with graded admissible arcs $(\gamma, f)$ on $S$;
2. the isomorphism classes of indecomposable band objects $P_{(\gamma, f, \lambda)}$ in $\mathcal{D}^b(A)$ are in bijection with triples $(\gamma, f, \lambda)$, where $(\gamma, f)$ is a graded closed curve on $S$ and $\lambda \in k^*$;
3. under the correspondences in (1) and (2), the indecomposable objects in the perfect category $K^b(\text{proj} A)$ correspond to the graded $\circ$-arcs and graded closed curves.

Proof Statement (1) correspond to [36, Theorem 2.12] (1).

For the statement (2), by [36, Theorem 2.12] (2), the isomorphism classes of the indecomposable band objects $P_{(\gamma, f, M)}$ in $\mathcal{D}^b(A)$ are in bijection with triples $(\gamma, f, M)$, where $(\gamma, f)$ is a graded closed curve on $S$ with $\gamma$ primitive and $M$ is an isomorphism class of indecomposable $k[x, x^{-1}]$-module. Note that since $k$ is algebraically closed, for any positive integer $n$ and any non-zero $\lambda \in k$, there exists, up to isomorphism, only one indecomposable $n$-dimensional $k[x, x^{-1}]$-module with eigenvalue $\lambda$. On the other hand, Lemma 2.6 shows that any grading $f$ over a primitive closed curve $\gamma$ naturally induces a grading $f^n$ over $\gamma^n$, and this gives a one-to-one correspondence between the gradings on $\gamma$ and the gradings on $\gamma^n$. Therefore any indecomposable band object $P_{(\gamma, f, M)}$ can be parameterized by a triple $(\gamma^n, f^n, \lambda)$, where $(\gamma^n, f^n)$ is a graded closed curve, $n$ is the dimension of the indecomposable $k[x, x^{-1}]$-module $M$ whose eigenvalue is $\lambda$. So we have the stated bijection.

Statement (3) is true because an object in $\mathcal{D}^b(A)$ is of infinite global dimension if and only if it is of the form $P_{(\gamma, f)}$ with $\gamma$ an infinite arc. \qed}

Remark 2.8 We adopt the convention that if a result or proof does not depend on the scalar $\lambda$, we will omit it and we will write $P_{(\gamma, f)}$ instead of $P_{(\gamma, f, \lambda)}$.

Now we recall how the oriented intersection number of graded curves relates to the dimension of the morphism space between the associated indecomposable objects.

Before giving the precise statement of the correspondence of morphisms and intersections in the theorem below, let us recall the following from [36]. Each intersection of two curves on the boundary (i.e. at a $\circ$-marked point) gives rise to exactly one morphism between the corresponding indecomposable objects, while each intersection in the interior (not at a $\bullet$-puncture) of $S$ gives rise to two morphisms. More precisely, let $P_{(\gamma_1, f_1)}$ and $P_{(\gamma_2, f_2)}$ be two indecomposable objects in $\mathcal{D}^b(A)$ corresponding to admissible arcs or primitive closed curves $\gamma_1$ and $\gamma_2$, where $f_1$ and $f_2$ are their respective gradings.

Assume first that $\gamma_1$ and $\gamma_2$ intersect at a $\circ$-point $p$ on the boundary as on the left side of Fig. 1, where for $i \in \{1, 2\}$, $q_i$ is the intersection in $\gamma_i \cap \Delta^\bullet$ which is nearest to $p$. Then there is an oriented intersection $\alpha$ from $\gamma_1$ to $\gamma_2$ at $p$. If $f_1(q_1) = f_2(q_2)$, then there is a morphism from $P_{(\gamma_1, f_1)}$ to $P_{(\gamma_2, f_2)}$ induced by $\alpha$. Note that there is no morphism from $P_{(\gamma_2, f_2)}$ to $P_{(\gamma_1, f_1)}$ for any $j \in \mathbb{Z}$ arising from this intersection at $p$.

Assume now that $\gamma_1$ and $\gamma_2$ intersect at some interior point $p$ (the black bullet) of the surface, which is not a $\bullet$-puncture. Suppose that the local configuration in $S$ is as on the right side of Fig. 1. Then up to identification of opposite angles, there are two oriented intersections at $p$, one is $\alpha$ from $\gamma_1$ to $\gamma_2$, and another one is $\beta$ from $\gamma_2$ to $\gamma_1$. If $f_1(q_1) = f_2(q_2)$, then there is one morphism from $P_{(\gamma_1, f_1)}$ to $P_{(\gamma_2, f_2)}$ induced by $\alpha$, and one morphism from $P_{(\gamma_2, f_2)}$ to $P_{(\gamma_1, f_1)}[-1]$ (or to $P_{(\gamma_1, f_1)}[-1]$ depending on the position of the corresponding $\circ$-marked point) induced by $\beta$.

Theorem 2.9 [36, Theorem 3.3] Let $A$ be a gentle algebra associated with a graded marked surface $(S, M, \Delta_A)$. Then
Fig. 1 Oriented intersections of graded curves on the surface give rise to homomorphisms of associated objects in the derived category

1. The shift of any indecomposable object corresponds to the shift of the corresponding graded curve, that is

\[ P_{(\gamma, f)}[1] = P_{(\gamma, f[1])}, \]

where \((\gamma, f)\) is a graded admissible arc or a graded closed curve.

2. For any two indecomposable objects \(P_{(\gamma_1, f_1)}\) and \(P_{(\gamma_2, f_2)}\) such that \(\gamma_1\) and \(\gamma_2\) are admissible arcs or primitive closed curves, if we denote

\[ \bigoplus_{i=-\infty}^{\infty} \text{Hom}_{D^b(A)}(P_{(\gamma_1, f_1)}, P_{(\gamma_2, f_2)[i]}) \]

by

\[ \text{Hom}_{D^b(A)}^*(P_{(\gamma_1, f_1)}, P_{(\gamma_2, f_2)}), \]

then

\[ \dim \text{Hom}_{D^b(A)}^*(P_{(\gamma_1, f_1)}, P_{(\gamma_2, f_2)}) \]

\[ = \begin{cases} |\text{Int}(\gamma_1, \gamma_2)| + 2 & \text{if } \gamma_1 = \gamma_2 \text{ is a closed curve, and} \\ |\text{Int}(\gamma_1, \gamma_2)| & \text{otherwise.} \end{cases} \]

Proof If \((\gamma, f)\) is a graded admissible \(\alpha\)-arc or a graded primitive closed curve, then the first statement follows from [36] (c.f the paragraph after [36, Definition 2.4]). Now let \((\gamma, f) = (\gamma^\mu, f^\mu)\) for some primitive graded closed curve \((\gamma', f')\). On the one hand, note that \(P_{(\gamma, f)}\) and \(P_{(\gamma', f')}\) are in the same homogenous tube of \(D^b(A)\), thus \(P_{(\gamma, f[1])}\) and \(P_{(\gamma', f'[1])}\) are in the same homogenous tube. On the other hand, \(P_{(\gamma', f'[1])}\) and \(P_{(\gamma^m, f'[1])}\) are in the same homogenous tube. Furthermore, \(f[1] = f^m[1] = f'[1]^m\) by Lemma 2.6, so \(P_{(\gamma^m, f'[1])} = P_{(\gamma, f[1])}\). Therefore \(P_{(\gamma, f[1])}\) and \(P_{(\gamma, f[1])}\) are in the same homogenous tube at the same height. So they coincide, that is \(P_{(\gamma, f)[1]} = P_{(\gamma, f[1])}\).

Recall the definition of the oriented intersection number \(|\text{Int}_S(\gamma_1, \gamma_2)|\) of two arcs or primitive closed curves \(\gamma_1\) and \(\gamma_2\) from Definition 2.3. Then statement (2) corresponds to [36, Theorem 3.3 (1)] if \(\gamma_1\) and \(\gamma_2\) are not the same closed curve. We note, in particular, that if \(\gamma_1\) and \(\gamma_2\) are the same admissible curve then the identity map is not induced by any self-intersection of the associated curve. If \(\gamma_1 = \gamma_2 = \gamma\) are the same closed curve, then the statement follows from [36, Theorem 3.3 (2)], which says that neither the identity map of \(P_{(\gamma, f)}\) nor the map from \(P_{(\gamma, f)}\) to \(P_{(\gamma, f)[1]}\), arising from the Auslander-Reiten triangle, correspond to a self-intersection of \(\gamma\).
2.3 Silting objects

In the following sections, we recall some background on the silting theory of a general triangulated category and for the bounded derived categories of gentle algebras, in particular.

Let $T$ be a triangulated category. We call a full subcategory $\mathcal{P}$ in $T$ pre-silting if $\text{Hom}_T(\mathcal{P}, \mathcal{P}[i]) = 0$ for all $i > 0$. It is silting if in addition $T = \text{thick}\mathcal{P}$. An object $P$ of $T$ is said to be pre-silting if $\text{add}\mathcal{P}$ is a pre-silting subcategory and silting if $\text{add}\mathcal{P}$ is a silting subcategory. We always assume that pre-silting objects as well as silting objects are basic.

Note that for any gentle algebra $A$, the category $K^b(\text{proj} A)$ always has silting objects, while the derived category $D^b(A)$ contains silting objects if and only if $\text{gl.dim} A < \infty$, that is, $D^b(A)$ contains silting objects exactly if it is triangle equivalent to $K^b(\text{proj} A)$. The silting objects in $K^b(\text{proj} A)$ are completely described as follows.

Theorem 2.10 [2, Theorem 3.2] Let $(S, M, \Delta_A)$ be a graded marked surface. Let $X$ be a basic silting object in $K^b(\text{proj} A)$. Then $X$ is isomorphic to a direct sum $\bigoplus_{(\gamma, f) \in (\Delta, f)} P(\gamma, f)$, where $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ is an admissible $o$-dissection of $(S, M)$.

Note that for an arbitrary admissible $o$-dissection $\Delta$, there currently is no characterisation of when there exists a grading $f$ such that $(\Delta, f)$ gives rise to a silting object. So we introduce the following.

Definition 2.11 Let $(S, M, \Delta_A)$ be a graded marked surface. Let $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ be an admissible $o$-dissection and let $f = \{f_{\gamma_1}, \ldots, f_{\gamma_n}\}$ be a set of gradings of the arcs in $\Delta$. We call $(\Delta, f)$ a silting dissection if

$$ P(\Delta, f) = \bigoplus_{(\gamma, f_{\gamma}) \in (\Delta, f)} P(\gamma, f_{\gamma}) $$

is a silting object in $K^b(\text{proj} A)$.

2.4 Silting mutations

It is shown in [5] that it is always possible to mutate silting objects. In the following we recall the definition of (irreducible) silting mutations.

Proposition-Definition 2.12 [5] Let $M$ be a (basic) silting object in a triangulated category $T$. Let $X$ be an indecomposable summand of $M$: $M = M' \oplus X$. We define $N_X$ as the mapping cone in the distinguished triangle

$$ X \xrightarrow{a} M_X \rightarrow N_X \rightarrow M[1], $$

where $a$ is a minimal left $\text{add}(M')$-approximation. We call $\mu_+(M) = M' \oplus N_X$ the left silting mutation of $M$ at $X$. The triangle (2.1) is called the exchange triangle of the left silting mutation. The right silting mutation $\mu_-(M)$ and its exchange triangle are defined dually. Then $\mu_+(M)$ and $\mu_-(M)$ are both silting objects in $T$.

2.5 Silting reductions

Let $T$ be a triangulated category with a pre-silting subcategory $\mathcal{P}$. We call the Verdier quotient $T/\text{thick}\mathcal{P}$ the silting reduction of $T$ with respect to $\mathcal{P}$, where $\text{thick}\mathcal{P}$ is the smallest
subcategory of \( \mathcal{T} \) containing \( \mathcal{P} \), which is closed under direct summands. The following result shows that under some mild conditions we can realize this Verdier quotient as a subfactor category of \( \mathcal{T} \) introduced in [25]. For this, in the notation above, we define the following subcategories of \( \mathcal{T} \)

\[
\mathcal{P}^\perp[>0] = \{ M \in \mathcal{T} : \text{Hom}_\mathcal{T}(M, P[i]) = 0 \text{ for } \forall \ i > 0, \text{ and } \forall \ P \in \mathcal{P} \},
\]

\[
\mathcal{P}^\perp[<0] = \{ M \in \mathcal{T} : \text{Hom}_\mathcal{T}(P[i], M) = 0 \text{ for } \forall \ i < 0, \text{ and } \forall \ P \in \mathcal{P} \}.
\]

Let

\[
\mathcal{Z} = \mathcal{P}^\perp[>0] \cap \mathcal{P}^\perp[<0],
\]

then \( \mathcal{P} \) is a subcategory of \( \mathcal{Z} \). We denote by \( \mathcal{Z}_\mathcal{P} \) the additive quotient of \( \mathcal{Z} \) by \( \mathcal{P} \).

**Theorem 2.13** [25, Theorem 4.2] [26, Lemma 3.5, Theorem 3.6] Let \( \mathcal{T} \) be a triangulated category and let \( \mathcal{P} \) be a pre-silting subcategory of \( \mathcal{T} \).

(1) As a subfactor category of \( \mathcal{T} \), \( \mathcal{Z}_\mathcal{P} \) has the structure of a triangulated category with respect to the following shift functor and triangles:

- For \( X \in \mathcal{Z} \), we take a triangle

\[
X \xrightarrow{\iota_X} P_X \rightarrow X(1) \rightarrow X[1]
\]

with a (fixed) left \( \mathcal{P} \)-approximation \( \iota_X \). Then (1) gives a well-defined auto-equivalence of \( \mathcal{Z}_\mathcal{P} \), which is the shift functor of \( \mathcal{Z}_\mathcal{P} \).

- For a triangle \( X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z \overset{h}{\rightarrow} X[1] \) with \( X, Y, Z \in \mathcal{Z} \), take the following commutative diagram of triangles:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\iota_X} & & \downarrow{a} \\
X & \xrightarrow{\iota_X} & P_X \\
\end{array}
\quad
\begin{array}{ccc}
& & Z \\
& \downarrow{a} & \\
& & X[1] \\
\end{array}
\quad
\begin{array}{ccc}
& & X[1] \\
Z & \xrightarrow{h} & X[1] \\
\end{array}
\]

Then we have a complex \( \overline{\rho} : \mathcal{Z}_\mathcal{P} \xrightarrow{\sim} \mathcal{T}/\text{thick}\mathcal{P} \) obtained in this way.

(2) We assume the following conditions are satisfied:

(\( P_1 \)) \( \mathcal{P} \) is a functorially finite subcategory of \( \mathcal{T} \);

(\( P_2 \)) \( \text{Hom}_\mathcal{T}(X, \mathcal{P}[i]) = 0 = \text{Hom}_\mathcal{T}(\mathcal{P}, X[i]) \) for \( \forall \ X \in \mathcal{T} \) and \( i \gg 0 \).

Let \( \rho : \mathcal{T} \rightarrow \mathcal{T}/\text{thick}\mathcal{P} \) be the canonical projection functor. Then the composition \( \mathcal{Z} \subset \mathcal{T} \xrightarrow{\rho} \mathcal{T}/\text{thick}\mathcal{P} \) of natural functors induces an equivalence of triangulated categories:

\[
\bar{\rho} : \mathcal{Z}_\mathcal{P} \xrightarrow{\sim} \mathcal{T}/\text{thick}\mathcal{P}.
\]

Note that the two conditions (\( P_1 \)) and (\( P_2 \)) in the above theorem are satisfied for any pre-silting subcategory \( \mathcal{P} \) (arising from a pre-silting object) in the perfect derived category \( \mathcal{T} = K^b(\text{proj}A) \) of a gentle algebra \( A \). In the following we will identify \( \mathcal{Z}_\mathcal{P} \) and \( \mathcal{T}/\text{thick}\mathcal{P} \) through the equivalence \( \bar{\rho} \), and also view \( \mathcal{Z}_\mathcal{P} \) as the silting reduction of \( \mathcal{T} \) with respect to \( \mathcal{P} \).
For a triangulated category $\mathcal{T}$, we denote by $\overline{\mathcal{T}}$ its \textit{orbit category} with respect to the shift functor. Denote by $\overline{X}$ an object in $\overline{\mathcal{T}}$ with representative $X$. Then for any two objects $\overline{X}$ and $\overline{Y}$, we have

$$\text{Hom}_{\mathcal{T}}(\overline{X}, \overline{Y}) = \text{Hom}^\bullet_{\mathcal{T}}(X, Y) := \bigoplus_{i=-\infty}^{\infty} \text{Hom}_{\mathcal{T}}(X, Y[i]),$$

where $X$ and $Y$ are any fixed representatives of $\overline{X}$ and $\overline{Y}$ respectively. Note that $\mathcal{T}$ is an additive category whose morphism space might be infinite dimensional. However, if $\mathcal{T} = K^b(\text{proj}A)$ then $\text{Hom}_{\mathcal{T}}(\overline{X}, \overline{Y})$ is finite dimensional for any $\overline{X}, \overline{Y} \in \overline{\mathcal{T}}$.

### 3 Distinguished triangles on the surface

Let $A$ be a gentle algebra associated with a graded marked surface $(S, M, \Delta_A)$. We will denote by $\mathcal{D}^b(A)$ its bounded derived category. Throughout this paper, we will keep these notation. We begin by introducing the notion of a distinguished triangle on the surface, which will play a central role throughout the remainder of the paper.

**Proposition-Definition 3.1** [distinguished triangle on the surface] \textit{Let $\alpha$ and $\beta$ be two $\circ$-arcs with a common endpoint $q_1$. Let $\gamma$ be the smoothing of $\alpha$ and $\beta$ at $q_1$, where $\beta$ and $\gamma$ share a common endpoint $q_2$, and $\gamma$ and $\alpha$ share a common endpoint $q_3$, see Fig. 2. Then $\alpha$ is the smoothing of $\beta$ and $\gamma$ at $q_2$, and $\beta$ is the smoothing of $\gamma$ and $\alpha$ at $q_3$.}

We call the (generalized) triangle formed by $\alpha, \beta$ and $\gamma$ a \textit{distinguished triangle on the marked surface} and we denote it by $\Delta(\alpha, \beta, \gamma)$.

**Proof** Since $\gamma$ is the smoothing of $\alpha$ and $\beta$, there exists neither a $\bullet$-puncture nor a boundary component in the interior of $\Delta(\alpha, \beta, \gamma)$. The arc obtained by smoothing $\beta$ and $\gamma$ at $q_2$ is homotopic to $\alpha$. Similarly $\beta$ is homotopic to the smoothing of $\gamma$ and $\alpha$ at $q_3$. \qed

**Remark 3.2** (1) We will refer to a usual triangle on the surface as a \textit{real triangle}, that is, a triangle such that the sides are all simple and intersect only in marked $\circ$-points $q_i, 1 \leq i \leq 3$, and the curve obtained from smoothing the crossings of the sides is contractible. Then a real triangle is a distinguished triangle. However, note that the sides of a distinguished triangle on the surface might self-intersect or intersect with each other in other points than the $q_i, 1 \leq i \leq 3$, and that the $q_i$ may coincide with each other. So a distinguished triangle is not necessarily a real triangle.

**Fig. 2** Distinguished triangle on the surface
The name distinguished triangle on the surface is motivated by the following observation: We will see in Lemmas 3.3 and 3.4 that for any distinguished triangle on the surface there are gradings on the arcs corresponding to the sides of the distinguished triangle such that the distinguished triangle on the surface gives rise to a distinguished triangle in the derived category.

For the remainder of the paper, we fix the following notation. Let $\gamma$ be an $\alpha$-arc with endpoints $q_1$ and $q_2$, denote by $s_1^\alpha$ the intersection in $\gamma \cap \Delta^*_A$ which is nearest to $q_i$, see the examples in Fig. 2. Note, in order for the notation to be well-defined in the case of a loop $\gamma$, we treat the unique endpoint of a loop as two distinct endpoints.

**Lemma 3.3** Assume that we have a distinguished triangle on the surface as in Fig. 2. Let $f_\alpha$, $f_\beta$ and $f_\gamma$ be any gradings on $\alpha$, $\beta$ and $\gamma$, respectively. Then

$$f_\alpha(s_1^\alpha) - f_\alpha(s_{q_1}^\alpha) + f_\beta(s_{q_2}^\beta) + f_\gamma(s_{q_3}^\gamma) - f_\gamma(s_{q_3}^\gamma) - f_\gamma(s_{q_3}^\gamma) = 1.$$  

**Proof** Assume that $\delta, \delta' \in \Delta^*_A$ are $\bullet$-arcs such that $s_1^\alpha = \alpha \cap \delta$ and $s_{q_1}^\beta = \beta \cap \delta'$. Then there are two cases.

Case I. $\delta \neq \delta'$. Since $\gamma$ is the smoothing of $\alpha$ and $\beta$ at $q_1$, $\gamma$ intersects $\delta$ and $\delta'$ successively. Denote the respective intersection points by $p$ and $q$. Since $\gamma$ is the smoothing of $\alpha$ and $\beta$, we have the following equalities

$$f_\alpha(s_{q_1}^\alpha) - f_\alpha(s_1^\alpha) = f_\gamma(s_{q_3}^\gamma) - f_\gamma(p);$$

$$f_\beta(s_{q_1}^\beta) - f_\beta(s_{q_1}^\beta) = f_\gamma(s_{q_1}^\gamma) - f_\gamma(q).$$

On the other hand, going from $q$ to $p$ along $\gamma$, we note that $q_1$ is at the right side of $\gamma$. So we have

$$f_\gamma(p) = f_\gamma(q) - 1.$$  

Combining these equalities, we get the desired equation.

Case II. $\delta = \delta'$. In this case, $\gamma$ does not intersect with $\delta$ close to $q_1$, since by assumption $\gamma$ is in minimal position. Denote by $p = \alpha \cap \sigma$ (resp. $q = \beta \cap \xi$) the first intersection away from $q_1$ in $\alpha \cap \Delta^*_A$ (resp. $\beta \cap \Delta^*_A$) such that the corresponding arc in $\Delta^*_A$ also intersects $\gamma$, see Fig. 3. Then $\gamma$ successively intersects with $\sigma$ and $\xi$, and we denote the intersections by $p'$ and $q'$ respectively. Similar to the first case, we have equalities

$$f_\alpha(s_{q_1}^\alpha) - f_\alpha(p) = f_\gamma(s_{q_3}^\gamma) - f_\gamma(p');$$

$$f_\beta(s_{q_1}^\beta) - f_\beta(q) = f_\gamma(s_{q_1}^\gamma) - f_\gamma(q').$$

Now we denote by $p^0, p^1, \ldots, p^{m+1}$ the ordered intersections in $\alpha \cap \Delta^*_A$ between $s_{q_1}^\alpha$ and $p$, where $p^0 = s_{q_1}^{\alpha}, p^{m+1} = p$, and $p^i = \alpha \cap \sigma^i, 0 \leq i \leq m + 1, \text{for some } \bullet$-arcs $\sigma^i \in \Delta^*_A$. Similarly, we denote by $q^0, q^1, \ldots, q^{n+1}$ the ordered intersections in $\beta \cap \Delta^*_A$ between $s_{q_1}^\beta$ and $q$, where $q^0 = s_{q_1}^{\beta}, q^{n+1} = q$, and $q^i = \beta \cap \xi^i, 0 \leq i \leq n + 1$. Then by the construction of $\sigma$ and $\xi$, we have $m = n$ and $\sigma^i = \xi^i$ for any $0 \leq i \leq m$. We denote these arcs as $\delta_i$, where in particular $\delta = \Delta_A$.

In the following, when we talk about the (left or right) position of a $\sigma$-point with respect to the arcs, we always go from $q_1$ to $q_2$ to $q_3$ to $q_1$. Note that any two arcs $\delta_i, \delta_{i+1}, 0 \leq i \leq m - 1$ are sides of a polygon in $\Delta_A^*$, and if a $\sigma$-point is on the left (resp. right) side of $\alpha$, then it is on
Fig. 3 The second case in Lemma 3.3

the right (resp. left) side of $\beta$, since there exist no $\circ$-points in the interior of a distinguished triangle on the surface. So we have

$$f_\alpha(s^\alpha_{q_1}) - f_\alpha(p'') = f_\beta(s^\beta_{q_1}) - f_\beta(q''),$$

where $p'' = \alpha \cap \delta_m$ and $q'' = \beta \cap \delta_m$.

On the other hand, $\delta_m, \sigma$ and $\varsigma$ are sides of a polygon in $\Delta^*_A$. Now we have three subcases with regards to the position of the $\circ$-point $t$ in this polygon.

Subcase 1. $t$ is on the left side of $\alpha$ and on the right side of $\beta$ and $\gamma$. Then we have the following equalities:

$$f_\alpha(p'') = f_\alpha(p) + 1; \quad f_\beta(q) = f_\beta(q'') - 1; \quad f_\gamma(p') = f_\gamma(q') - 1.$$

Subcase 2. $t$ is on the left side of $\beta$ and on the right side of $\alpha$ and $\gamma$. Then we have:

$$f_\alpha(p'') = f_\alpha(p) - 1; \quad f_\beta(q) = f_\beta(q'') + 1; \quad f_\gamma(p') = f_\gamma(q') - 1.$$

Subcase 3. $t$ is on the left side of $\gamma$, and on the right side of $\alpha$ and $\beta$. Then we have:

$$f_\alpha(p'') = f_\alpha(p) - 1; \quad f_\beta(q) = f_\beta(q'') - 1; \quad f_\gamma(p') = f_\gamma(q') + 1.$$

Finally, the result follows from combining the above equalities in each subcase with the three equalities above. $\square$

The following lemma is a re-statement of Theorem 4.1 in [36] in the context of distinguished triangles on the surface. The key point of Lemma 3.4 is that the compatibility of the gradings in a distinguished triangle on the surface proved in Lemma 3.3 gives rise to three maps in the derived category which form a distinguished triangle in the derived category. This is a key idea underlying many of the proofs in this paper.

**Lemma 3.4** (distinguished triangles on the surface as distinguished triangles) Assume we have a distinguished triangle on the surface as in Fig. 2. Let $f_\alpha$, $f_\beta$ and $f_\gamma$ be gradings of $\alpha, \beta$ and $\gamma$ respectively such that $f_\alpha(s^\alpha_{q_1}) = f_\beta(s^\beta_{q_1})$ and $f_\beta(s^\beta_{q_2}) = f_\gamma(s^\gamma_{q_2})$. Then

1. there are three maps

$$P_{(\alpha, f_\alpha)} \xrightarrow{a} P_{(\beta, f_\beta)}, \quad P_{(\beta, f_\beta)} \xrightarrow{b} P_{(\gamma, f_\gamma)} \quad \text{and} \quad P_{(\gamma, f_\gamma)} \xrightarrow{c} P_{(\alpha, f_\alpha)}[1]$$

in $\mathcal{D}^b(A)$ arising from the intersections of the arcs $\alpha, \beta$ and $\gamma$ at $q_1, q_2$ and $q_3$ respectively;
(2) the maps in (1) give rise to the following distinguished triangle in $\mathcal{D}^b(A)$
\[ P(\alpha, f_\alpha) \xrightarrow{a} P(\beta, f_\beta) \xrightarrow{b} P(\gamma, f_\gamma) \xrightarrow{c} P(\alpha, f_\alpha)[1]. \] (3.1)

Proof The first two maps in (1) follow from the hypotheses on the grading, that is from the fact that $f_\alpha(s^i_{q_1}) = f_\beta(s^i_{q_1})$ and $f_\beta(s^i_{q_2}) = f_\gamma(s^i_{q_2})$. By Lemma 3.3 the equality $f_\gamma(s^i_{q_3}) = f_\alpha(s^i_{q_3}) - 1$ holds and thus the intersection of $\gamma$ and $\alpha$ at $q_3$ induces the third map. It then follows from [36, Theorem 4.1] that the triangle (3.1) is distinguished.

Although distinguished triangles on the surfaces are in general not real triangles on the surface, the relations between the diagonals in a quadrilateral formed by two distinguished triangles on the surfaces are like the ‘flip’ of diagonals in (real) ideal triangulations, see Fig. 4, where $\gamma$ and $\gamma_+$ are the diagonals and all triangles are distinguished. The following proposition shows that this kind of flip on the marked surface corresponds to an instance of the octahedral axiom (and also to a homotopy push-out/pull-back) in the derived category. A similar correspondence is also established in [21] by Dyckerhoff and Kapranov for triangulated categories with 2-periodic dg-enhancements associated with a surface model.

Proposition-Definition 3.5 (flips as homotopy push-outs/pull-backs) Let $\gamma$, $\gamma_i$ and $\alpha_i, i = 1, 2$, be five $\circ$-arcs. Assume that they form two distinguished triangles $\Delta(\gamma_1, \alpha_1, \gamma)$, $i = 1, 2$, on the surfaces as in Fig. 4.

Let $f_\gamma$ be any grading of $\gamma$. Then the following hold.

(1) The arc obtained by smoothing the crossing of $\gamma_1$ and $\alpha_2$ at $q_1$ is homotopic to the arc obtained by smoothing the crossing of $\alpha_1$ and $\gamma_2$ at $q_2$. Then, denoting this arc by $\gamma_+$, the triangles $\Delta(\alpha_1, \gamma_2, \gamma_+)$ and $\Delta(\alpha_2, \gamma_1, \gamma_+)$ are both distinguished triangles on the surfaces.

(2) For $i = 1, 2$, there exist unique gradings $f_{\gamma_+}$, $f_{\gamma_i}$ and $f_{\alpha_i}$ of the arcs $\gamma_+$, $\gamma_i$ and $\alpha_i$ respectively, such that the respective intersections at $q_i$ and $q'_i$ yield the following maps in the category $\mathcal{D}^b(A)$
\[
\begin{align*}
\alpha_i : & P(\gamma, f_\gamma) \rightarrow P(\gamma_i, f_{\gamma_i}) \\
b_i : & P(\gamma_i, f_{\gamma_i}) \rightarrow P(\gamma_+, f_{\gamma+}) \\
c_i : & P(\gamma_+, f_{\gamma+}) \rightarrow P(\alpha_i, f_{\alpha_i}) \\
d_i : & P(\alpha_i, f_{\alpha_i}) \rightarrow P(\gamma, f_\gamma)[1],
\end{align*}
\]
which are parts of the four distinguished triangles in Fig. 5, corresponding to the four distinguished triangles on the surfaces in Fig. 4.

(3) Every square in Fig. 5 is commutative, in particular, $a_1 \circ b_1 = a_2 \circ b_2$ and $(c \circ d := )c_1 \circ d_1 = c_2 \circ d_2$, and the following triangle is a distinguished triangle in $\mathcal{D}^b(A)$,
\[ P(\gamma, f_\gamma) \xrightarrow{(a_1, a_2)} P(\gamma_1, f_{\gamma_1}) \oplus P(\gamma_2, f_{\gamma_2}) \xrightarrow{(b_1, b_2)} P(\gamma_+, f_{\gamma+}) \xrightarrow{\text{cod}} P(\gamma, f_\gamma)[1]. \] (3.2)

(4) The object $P(\gamma_+, f_{\gamma+})$ is the homotopy push-out of the maps
\[ a_i : P(\gamma, f_\gamma) \rightarrow P(\gamma_i, f_{\gamma_i}), i = 1, 2. \]
The object $P(\gamma, f_\gamma)$ is the homotopy pull-back of the maps
\[ b_i : P(\gamma_i, f_{\gamma_i}) \rightarrow P(\gamma_+, f_{\gamma+}), i = 1, 2. \]
**Fig. 4** Flipping the diagonals in a quadrilateral formed by two distinguished triangles \(\triangle(\gamma_1, \alpha_1, \gamma)\) and \(\triangle(\gamma_2, \alpha_2, \gamma)\) on the surface corresponds to a homotopy push-out/pull-back in the derived category.

**Fig. 5** Flipping the diagonals in a quadrilateral formed by two distinguished triangles \(\triangle(\gamma_1, \alpha_1, \gamma)\) and \(\triangle(\gamma_2, \alpha_2, \gamma)\) on the surface corresponds to a homotopy push-out/pull-back in the derived category.

**Proof** (1) Since \(\triangle(\gamma_1, \alpha_1, \gamma)\) is a distinguished triangle on the surface, \(\gamma_1\) is the smoothing of \(\gamma\) and \(\alpha_1\) at \(q_2\). So the smoothing of \(\gamma_1\) and \(\alpha_2\) at \(q_1\) can be obtained in two steps. First by smoothing \(\alpha_1\) and \(\gamma\) at \(q_2\), and then smoothing the resulting arc with \(\alpha_2\) at \(q_1\). Similarly, the smoothing of \(\alpha_1\) and \(\gamma_2\) at \(q_2\) can be obtained in two steps. Namely, by first smoothing \(\gamma\) and \(\alpha_2\) at \(q_1\), and then smoothing the resulting arc with \(\alpha_1\) at \(q_2\). That is, they are both the smoothing of the arcs \(\alpha_1\), \(\gamma\) and \(\alpha_2\) at the two points \(q_2\) and \(q_1\), up to the order of making the smoothing, which does not change the final result. So both successive smoothings of crossings give rise to the arc \(\gamma_+\). Furthermore, by construction and by Proposition-Definition 3.1, the triangles \(\triangle(\alpha_1, \gamma_2, \gamma_+)\) and \(\triangle(\alpha_2, \gamma_1, \gamma_+)\) are both distinguished triangles on the surface.

(2) For \(i = 1, 2\), let \(f_{\gamma_i}\) and \(f_{\alpha_i}\) be the gradings of the arcs \(\gamma_i\) and \(\alpha_i\) respectively such that \(f_{\gamma_i}(s_{\gamma_i}^{\gamma}) = f_{\gamma}(s_{\gamma}^{\gamma})\) and \(f_{\alpha_i}(s_{\alpha_i}^{\gamma}) = f_{\gamma_i}(s_{\gamma_i}^{\gamma})\). Since \(\triangle(\gamma, \gamma_1, \alpha_i), i = 1, 2\) are distinguished triangles on the surface, by lemma 3.4, we have \(f_{\alpha_1}(s_{\alpha_1}^{\gamma}) = f_{\gamma}(s_{\gamma}^{\gamma}) - 1\) and \(f_{\alpha_2}(s_{\alpha_2}^{\gamma}) = f_{\gamma}(s_{\gamma}^{\gamma}) - 1\). Let \(f_{\gamma_+}\) be the grading of \(\gamma_+\) such that \(f_{\gamma_+}(s_{\gamma_1}^{\gamma_+}) = f_{\gamma_1}(s_{\gamma_1}^{\gamma}) = f_{\gamma}(s_{\gamma}^{\gamma}) - 1\).
To sum up, we have the following equalities

\[ f_{\gamma_+}(s_{q_1}^{\alpha_1}) = f_{\alpha_1}(s_{q_1}^{\alpha_1}), \quad f_{\gamma_+}(s_{q_2}^{\alpha_1}) = f_{\alpha_2}(s_{q_2}^{\alpha_1}) - 1; \quad (3.3) \]

\[ f_{\gamma_1}(s_{q_1}^{\gamma_1}) = f_{\gamma_2}(s_{q_2}^{\gamma_1}), \quad f_{\gamma_2}(s_{q_1}^{\gamma_1}) = f_{\gamma_1}(s_{q_1}^{\gamma_1}) - 1. \quad (3.4) \]

Since \( \Delta(\gamma_+, \alpha_1, \gamma_2) \) and \( \Delta(\gamma_1, \gamma_+, \alpha_1) \) are distinguished triangles on the surfaces, then by the above equalities and Lemma 3.3, we have

\[ f_{\gamma_+}(\alpha_1^{\gamma_1}) = f_{\gamma_2}(\alpha_2^{\gamma_2}). \quad (3.5) \]

We then have the following maps in \( D^b(A) \) arising from the intersections \( q_i, q'_i, i = 1, 2, \)

\[
\begin{align*}
\alpha_i : P_{(\gamma, f_\gamma)} &\rightarrow P_{(\gamma_i, f_{\gamma_i})} \\
\beta_i : P_{(\gamma, f_\gamma)} &\rightarrow P_{(\gamma_+, f_{\gamma_+})} \\
\gamma_i : P_{(\gamma_+, f_{\gamma_+})} &\rightarrow P_{(\alpha_i, f_{\alpha_i})} \\
\delta_i : P_{(\alpha_i, f_{\alpha_i})} &\rightarrow P_{(\gamma, f_\gamma)}[1].
\end{align*}
\]

Clearly, these gradings \( f_{\gamma_+}, f_{\gamma_1} \) and \( f_{\alpha_i} \) are unique with respect to the given grading \( f_\gamma \). Then by Lemma 3.4, the four distinguished triangles \( \Delta(\gamma_i, \alpha_i, \gamma), i = 1, 2, \Delta(\alpha_1, \gamma_2, \gamma_+) \) and \( \Delta(\alpha_2, \gamma_1, \gamma_+) \) on the surface give rise to the four distinguished triangles in \( D^b(A) \) in the diagram in Fig. 5.

(3) The fact that the triangle is distinguished follows from [36, Theorem 4.1] (see also [20]). In particular, we have equalities \( a_1 \circ b_1 = a_2 \circ b_2 \) and \( c_1 \circ d_1 = c_2 \circ d_2 \), and every square in Fig. 5 are commutative.

(4) From above, the following square is commutative, and it gives rise to the distinguished triangle \( (3.2) \), so it is a homotopy cartesian square.

\[
\begin{array}{ccc}
P_{(\gamma, f_\gamma)} & \xrightarrow{a_1} & P_{(\gamma_1, f_{\gamma_1})} \\
\downarrow a_2 & & \downarrow b_1 \\
P_{(\gamma_2, f_{\gamma_2})} & \xrightarrow{b_2} & P_{(\gamma_+, f_{\gamma_+})}
\end{array}
\]

Therefore \( P_{(\gamma_+, f_{\gamma_+})} \) is the homotopy push-out of \( a_1 \) and \( a_2 \), and \( P_{(\gamma, f_\gamma)} \) is the homotopy pull-back of \( b_1 \) and \( b_2 \). \( \square \)

**Remark 3.6** From the third part of the above proposition (compare Figs. 4 and 5), we have the following observation, which will be frequently used when we consider approximations of morphisms. Let \( (\alpha, f_\alpha), (\beta, f_\beta) \) and \( (\gamma, f_\gamma) \) be three graded \( \circ \)-arcs sharing a common endpoint \( q \) such that at \( q \) we have in the anti-clockwise order \( \alpha \) followed by \( \beta \) and then \( \gamma \) and \( f_\alpha(s_{q}^{\alpha}) = f_\beta(s_{q}^{\beta}) = f_\gamma(s_{q}^{\gamma}). \) Denote by \( a : P_{(\alpha, f_\alpha)} \rightarrow P_{(\beta, f_\beta)}, b : P_{(\beta, f_\beta)} \rightarrow P_{(\gamma, f_\gamma)} \) and \( c : P_{(\alpha, f_\alpha)} \rightarrow P_{(\gamma, f_\gamma)} \) the morphisms arising from \( q \). Then we have \( c = ab. \) From this we can deduce the geometric form of approximations, however, each case still needs to be considered in detail, since, for example, there might be intersections at both endpoints of the arcs.
4 Silting mutation

In this section, we define the mutation of a silting dissection over a graded marked surface \((S, M, \Delta_A)\), and we show that it gives rise to a geometric interpretation of the mutation of silting objects in \(K^b(\text{proj}A)\).

4.1 Mutation of silting dissections

Let \((\gamma, f_\gamma)\) be a graded arc in a silting dissection \((\Delta, f)\) (c.f. Definition 2.11) over the graded marked surface. Let \(q_1\) and \(q_2\) be the endpoints of \(\gamma\), which may coincide. For \(i = 1, 2\), denote by \(\gamma_i \neq \gamma\) the \(\circ\)-arc in \(\Delta\) with endpoint \(q_i\), which is the first anti-clockwise \(\circ\)-arc in \(\Delta\) following \(\gamma\) based at \(q_i\). Note that \(\gamma_i\) may not exist. Then we have three possible local configurations, see Fig. 6, where we only draw the arcs \(\gamma_i\) such that \(f_{\gamma_i}(s_{\gamma_i}^{q_i}) = f_\gamma(s_{\gamma}^{q_i})\). More precisely, picture (I) means that for both \(i = 1, 2\), \(\gamma_i \neq \gamma\) and \(f_{\gamma_i}(s_{\gamma_i}^{q_i}) = f_\gamma(s_{\gamma}^{q_i})\). Picture (II) means that there exists exactly one \(i\) such that \(\gamma_i \neq \gamma\) and \(f_{\gamma_i}(s_{\gamma_i}^{q_i}) = f_\gamma(s_{\gamma}^{q_i})\). Picture (III) means that there exists no \(i\) such that \(\gamma_i \neq \gamma\) and \(f_{\gamma_i}(s_{\gamma_i}^{q_i}) = f_\gamma(s_{\gamma}^{q_i})\).

We now describe the other arcs appearing in Fig. 6. In picture (I), \(\alpha_i\) is the smoothing of \(\gamma_i\) and \(\gamma\) at the intersection \(q_i\), and \(\gamma_+\) is the smoothing of \(\gamma_1\) and \(\alpha_2\) at \(q_1\), or equivalently, the smoothing of \(\gamma_2\) and \(\alpha_1\) at \(q_2\) (see Proposition 3.5(1)). In picture (II), \(\gamma_+\) is the smoothing of \(\gamma_1\) and \(\gamma\) at \(q_1\). So in both pictures, all the triangles are distinguished triangles on the surface.

Dually, we may consider the \(\circ\)-arcs \(\gamma_i\) in \(\Delta\) based at \(q_i\) which directly follow \(\gamma\) in the clockwise direction. In this case there are also three possible local configurations which we describe in Fig. 7.

Definition 4.1 (Mutation of silting dissections) For the three cases in Fig. 6, we define the left mutation \(\mu_+^{(\gamma, f_\gamma)}((\Delta, f))\) of the silting dissection \((\Delta, f)\) at the graded arc \((\gamma, f_\gamma)\) as follows:

\[
\mu_+^{(\gamma, f_\gamma)}((\Delta, f)) = (\Delta_+^{\gamma}, f_\gamma^+) := \{\mu_+^{(\gamma_1, f_{\gamma_1})}((\alpha, f_\alpha)), (\alpha, f_\alpha) \in (\Delta, f)\},
\]

where

\[
\mu_+^{(\gamma, f_\gamma)}((\alpha, f_\alpha)) = \begin{cases} 
(\gamma^+, f_{\gamma^+}) & \text{if } (\alpha, f_\alpha) = (\gamma, f_\gamma) \\
(\alpha, f_\alpha) & \text{otherwise}
\end{cases}.
\]

with the following conventions:
For $(\gamma, f_\gamma)$ in (I) and (II), the arc $\gamma_+$ is as shown in Fig. 6 (I) and (II) respectively, and the grading $f_{\gamma_+}$ is such that $f_{\gamma_+}(s_{q_1}^{q_1}) = f_{\gamma_1}(s_{q_1}^{q_1})$.

For $(\gamma, f_\gamma)$ in (III), let $\gamma_+ = \gamma$, and set the grading $f_{\gamma_+} = f_\gamma[1]$.

Dually, we define the right mutation $\mu^-_{(\gamma, f_\gamma)}((\Delta, f))$. In particular, for $(\gamma, f_\gamma)$ in (III'), we set $\gamma_- = \gamma$ and $f_{\gamma_-} = f_\gamma[-1]$.

Remark 4.2 Note that the setting in case (I) in the above definition coincides with the setting in Proposition 3.5. So we have maps $a_1$, $b_1$, $c_1$, and $d_1$ as in Proposition 3.5 (2). And we also have a distinguished triangle (3.2) formed by these maps as shown in the third part of the proposition. On the other hand, the setting in case (II) coincides with the setting in Lemma 3.4. So we have maps $a_1$, $b_1$ and $c_1$ and a distinguished triangle formed by them as stated in the lemma.

Proposition-Definition 4.3 The mutations $\mu^+_{(\gamma, f_\gamma)}((\Delta, f))$ and $\mu^-_{(\gamma, f_\gamma)}((\Delta, f))$ of a silting dissection $(\Delta, f)$ are again silting dissections.

Proof We only prove the case for the left mutation $\mu^+_{(\gamma, f_\gamma)}((\Delta, f))$, the proof for the right mutation $\mu^-_{(\gamma, f_\gamma)}((\Delta, f))$ is analogous.

By the construction of $\Delta^+_\gamma$, in all three cases (I), (II), and (III), the only possible intersections between the arcs in $\Delta^+_\gamma$ are at the endpoints. The quadrilateral obtained from smoothing the endpoint crossings of the arcs $\alpha_1, \gamma_2, \alpha_2$, and $\gamma_1$ in case (I) is such that the collection $\Delta^+_\gamma$ does not cut out any subsurface not containing a $\bullet$-marked points in their interior. Thus changing $\gamma$ to $\gamma_+$ in $\Delta$ implies that $\Delta^+_\gamma$ is again an admissible collection. Similarly, in case (II) the collection $\Delta^+_\gamma$ is admissible and in case (III) we have $\Delta^+_\gamma = \Delta$. Furthermore, since $\Delta^+_\gamma$ and $\Delta$ have the same number of arcs, $\Delta^+_\gamma$ is maximal, and therefore it is an admissible dissection.

Now to prove that $\mu^+_{(\gamma, f_\gamma)}((\Delta, f))$ is a silting dissection, by [2, Corollary 3.8 (2)], it is enough to show that for any two graded arcs $(\alpha, f_\alpha)$ and $(\beta, f_\beta)$ in $\mu^+_{(\gamma, f_\gamma)}((\Delta, f))$ such that $\beta$ follows $\alpha$ anticlockwise at a common endpoint $p$, we have $f_\beta(s_\beta^p) \leq f_\alpha(s_\alpha^p)$.

To verify that this condition holds for the arcs in $\mu^+_{(\gamma, f_\gamma)}((\Delta, f))$ is a straightforward check that is left to the reader.
4.2 The compatibility of mutations

Now we have two kinds of mutations, the mutation of silting objects in the derived category \( K^b(\text{proj}A) \) and the mutation of silting dissections in the corresponding geometric model \((S, M, \Delta_A)\). The two mutations fit into the following diagram,

\[
\begin{array}{ccc}
(\Delta, f) & \xrightarrow{\mu_{(\gamma,f_{\gamma})}} & (\Delta, f) \\
\downarrow & & \downarrow \\
P(\Delta, f) & \xrightarrow{\mu_{(\gamma,f_{\gamma})}^{+}} & P_{(\gamma,f_{\gamma})}(\Delta, f)
\end{array}
\]

\[\text{(4.1)}\]

where \((\Delta, f)\) is a silting dissection containing a graded arc \((\gamma, f_{\gamma})\), \(\mu_{(\gamma,f_{\gamma})}^{+}\) is the left mutation at \((\gamma, f_{\gamma})\) of the silting dissection defined in Definition 4.1, \(\mu_{(\gamma,f_{\gamma})}^{+}\) is the left silting mutation of the silting object defined in Definition 2.12, and the vertical correspondence is established in [2, 35] with \(P_{(\gamma,f_{\gamma})}\) the pre-silting object associated to \((\gamma, f_{\gamma})\).

We have a similar diagram with respect to the right mutation of silting dissections and the right mutation of the corresponding silting objects in the derived category.

The following theorem shows that these two mutations are compatible with each other. We only state the compatibility of the left mutations. The statement of the compatibility of the right mutations then is the dual statement of Theorem 4.4.

**Theorem 4.4** [compatibility of mutations] Let \((\Delta, f)\) be a silting dissection, and let \((\gamma, f_{\gamma})\) be a graded arc in it. Denote by \(P_{(\Delta, f)} = \bigoplus_{j=1}^{n} P_{(\gamma_j,f_{\gamma_j})}\) the silting object corresponding to \((\Delta, f)\). Then the left mutation of \((\Delta, f)\) is compatible with the left mutation of \(P_{(\Delta, f)}\), that is, the diagram (4.1) is commutative. Moreover, the exchange triangle of the left silting mutation is given by

1. the distinguished triangle arising from the homotopy push-out of the maps \(a_1\) and \(a_2\) given in Proposition 3.5(3) for case (I):

\[
P_{(\gamma_1,f_{\gamma_1})} \xrightarrow{(a_1,a_2)} P_{(\gamma_1,f_{\gamma_1})} \oplus P_{(\gamma_2,f_{\gamma_2})} \xrightarrow{(b_1,b_2)} P_{(\gamma_1,f_{\gamma_1})} \xrightarrow{\text{cod}} P_{(\gamma,f_{\gamma})}[1];
\]

2. the distinguished triangle arising from the map \(a_1\) for case (II):

\[
P_{(\gamma_1,f_{\gamma_1})} \xrightarrow{a_1} P_{(\gamma_1,f_{\gamma_1})} \xrightarrow{b_1} P_{(\gamma_1,f_{\gamma_1})} \xrightarrow{c_1} P_{(\gamma,f_{\gamma})}[1];
\]

3. the trivial distinguished triangle starting from \(P_{(\gamma,f_{\gamma})}\) for case (III):

\[
P_{(\gamma,f_{\gamma})} \xrightarrow{0} P_{(\gamma,f_{\gamma})}[1] \xrightarrow{id} P_{(\gamma,f_{\gamma})}[1].
\]

**Proof** We begin by proving part (1). From Proposition 3.5 (3), we have the distinguished triangles in the statement. Since in an admissible \(\circ\)-dissection the only intersections of the arcs are at the endpoints, the maps between indecomposable objects in \(P_{(\Delta, f)}\) that arise from these endpoint intersections form a basis of the corresponding morphism spaces. Thus for any graded arc \((\beta, f_{\beta}) \in (\Delta, f)\) with a non-zero morphism space \(\text{Hom}_{D^b(\mathcal{A})}(P_{(\gamma,f_{\gamma})}, P_{(\beta,f_{\beta})})\), \(\beta\) has endpoints \(q_1\) or \(q_2\) in Fig. 6 with \(f_{\beta}(s_{q_1}^{\beta}) = f_{\gamma}(s_{q_1}^{\gamma})\).
or \( f_\beta(s_{q_1}^\beta) = f_\gamma(s_{q_2}^\gamma) \) respectively. Moreover, when \( \beta \neq \gamma \), since \( \gamma_1 \) is the first \( \circ \)-arc in \( \Delta \) anti-clockwise following \( \gamma \) based at \( q_i \) with \( f_{\gamma_i}(s_{q_i}^\gamma) = f_\gamma(s_{q_2}^\gamma) \) for \( i = 1, 2 \), each basis map in \( \text{Hom}_{D^b(A)}(P(\gamma, f_\gamma), P(\beta, f_\beta)) \) (which arises from \( q_i \)) factors through \( a_i \) by Remark 3.6, and thus any map in \( \text{Hom}_{D^b(A)}(P(\gamma, f_\gamma), P(\beta, f_\beta)) \) factors through \( (a_1, a_2) \). On the other hand, since \( \gamma_1 \neq \gamma \), we have \( P(\gamma_1, f_{\gamma_1}) \oplus P(\gamma_2, f_{\gamma_2}) \in \text{add}(P(\Delta, f) \setminus P(\gamma)) \). Therefore, \( (a_1, a_2) \) is the left \( \text{add}(P(\Delta, f) \setminus P(\gamma)) \)-approximation of \( P(\gamma, f_\gamma) \), and the corresponding triangle is the left exchange triangle of the silting mutation of \( P(\Delta, f) \) at \( P(\gamma, f_\gamma) \).

Part (2) follows from a similar argument as in part (1) combined with the argument in the proof of part (3) below. In this case, \( a_1 \) is the left \( \text{add}(P(\Delta, f) \setminus P(\gamma)) \)-approximation of \( P(\gamma, f_\gamma) \), and the triangle in part (2) is the exchange triangle of the left silting mutation.

To prove part (3), we show that the morphism space \( \text{Hom}_{D^b(A)}(P(\gamma, f_\gamma), P(\beta, f_\beta)) \) is zero for any graded arc \( (\beta, f_\beta) \in (\Delta, f) \) different from \( (\gamma, f_\gamma) \), and thus the triangle is the exchange triangle of the left silting mutation.

For contradiction, assume that \( \text{Hom}_{D^b(A)}(P(\gamma, f_\gamma), P(\beta, f_\beta)) \neq 0 \) for some graded arc \( (\beta, f_\beta) \in (\Delta, f) \), then by the same argument as in part (1), \( \beta \) has endpoints \( q_1 \) or \( q_2 \). Without loss of generality, assume that \( \beta \) and \( \gamma \) intersect at \( q_1 \) and that this intersection gives rise to a non-zero map in \( \text{Hom}_{D^b(A)}(P(\gamma, f_\gamma), P(\beta, f_\beta)) \). Then \( f_\gamma(s_{q_1}^\gamma) = f_\beta(s_{q_1}^\beta) \). Note that the existence of \( \beta \) implies the existence of \( \gamma_1 \) (see how to define \( \gamma_1 \) at the beginning of Sect. 4.1). Furthermore at \( q_1 \) we have in anti-clockwise order \( \gamma \) followed by \( \gamma_1 \) and then \( \beta \). Since \( (\Delta, f) \) is a silting dissection, we have \( f_\gamma(s_{q_1}^\gamma) \geq f_{\gamma_1}(s_{q_1}^{\gamma_1}) \) and \( f_{\gamma_1}(s_{q_1}^{\gamma_1}) \geq f_\beta(s_{q_1}^\beta) \). On the other hand, by assumption the grading of \( \gamma_1 \) is such that there is no map from \( P(\gamma, f_\gamma) \) to \( P(\gamma_1, f_{\gamma_1}) \), that is \( f_\gamma(s_{q_1}^\gamma) \neq f_{\gamma_1}(s_{q_1}^{\gamma_1}) \). Therefore \( f_\gamma(s_{q_1}^\gamma) > f_\beta(s_{q_1}^\beta) \). A contradiction.

The following corollary can directly be derived from the above theorem.

**Corollary 4.5** The exchange triangle of a left or a right silting mutation has at most two middle terms.

In general, a silting mutation of a tilting object is not necessarily another tilting object and it is difficult to give a characterisation of when it is and when it is not. However, in the case of gentle algebras, using the surface model, we can characterise precisely when a silting mutation of a tilting object is again a tilting object. We only state the result for the left silting mutation case, the case for right silting mutation is dual. In the below we call a silting dissection \( (\Delta, f) \) such that the corresponding object in \( K^b(\text{proj} A) \) is tilting, a **tilting dissection**.

**Proposition-Definition 4.6** Let \( (\Delta, f) \) be a silting dissection and let \( (\gamma, f_\gamma) \in (\Delta, f) \). Then the left silting mutation \( \mu^+_\gamma(\Delta, f) \) is a tilting dissection if and only if one of the following two conditions holds

1. the graded arc \( (\gamma, f_\gamma) \) is as in of Fig. 6 (I);
2. the graded arc \( (\gamma, f_\gamma) \) is as in of Fig. 6 (II) and no other arc in \( \Delta \) starts or ends at \( q_2 \).

If \( \gamma \) is as in of Fig. 6 (III) then \( \mu^+_\gamma(\Delta, f) \) is a silting dissection which is not tilting.

**Proof** First note that a silting dissection \( (\Delta, f) \) is a tilting dissection if and only if the gradings of any two graded arcs \( (\alpha, f_\alpha) \) and \( (\beta, f_\beta) \) in \( (\Delta, f) \) are compatible at any common endpoint \( q \), that is if \( f_\alpha(s_{q_1}^\alpha) = f_\beta(s_{q_1}^\beta) \).

If \( \gamma \) is as in Fig. 6 (I), then by the proof of Theorem 4.4, we have the equalities \( f_{\gamma_1}(s_{q_1}^{\gamma_1}) = f_{\gamma_1}(s_{q_1}^{\gamma_1}) \) and \( f_{\gamma_2}(s_{q_2}^{\gamma_2}) = f_{\gamma_2}(s_{q_2}^{\gamma_2}) \). On the other hand, the gradings of the remaining arcs in
\(\Delta \setminus \{\gamma\}\) are not changed. So in the silting dissection \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\), the gradings of any two arcs are compatible at the common endpoints. Therefore it is a tilting dissection.

Let \(\gamma\) be an arc as in Fig. 6 (II). If, other than \(\gamma\), there exists no arc in \(\Delta\) starting or ending at \(q_2\) then \(\gamma_+\) is the only arc in \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\) starting or ending at \(q_2\). On the other hand, we have the equality \(f_{\gamma_+}(s_{q_2}^{\gamma_+}) = f_{\gamma_1}(s_{q_2}^{\gamma_1})\). Thus the gradings of the arcs are compatible in \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\). So it is a tilting dissection. If on the other hand, there exists another arc \(\beta\) in \(\Delta\) starting or ending at \(q_2\) then we have \(f_{\gamma_+}(s_{q_2}^{\gamma_+}) = f_{\gamma}(s_{q_2}^{\gamma}) - 1 = f\beta(s_{q_2}^{\beta}) - 1\) in \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\). That is, the gradings of \(\gamma_+\) and \(\beta\) in \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\) are not compatible at \(q_2\).

So the new silting dissection is not a tilting dissection.

If \(\gamma\) is as in Fig. 6 (III), then \((\gamma, f\gamma[1])\) belongs to the new dissection \(\mu^+_{(\gamma, f\gamma)}(\Delta, f)\) and it is clear that this is not a tilting dissection. \(\square\)

We now translate the above geometric statement into a purely algebraic characterisation of when the left silting mutation of a tilting object is again tilting. The case for the right silting mutation of a tilting object can be dually stated.

**Corollary 4.7** Let \(A\) be a gentle algebra. Let

\[
P_{(\Delta, f)} = \bigoplus_{(\gamma, f\gamma) \in (\Delta, f)} P_{(\gamma, f\gamma)}
\]

be a basic tilting object in \(K^b(\text{proj} A)\). Then the left silting mutation

\[
P_{\mu^+_{(\gamma, f\gamma)}(\Delta, f)} = \bigoplus_{(\gamma', f\gamma') \in (\Delta, f), \gamma' \neq \gamma} P_{(\gamma', f\gamma')} \oplus P_{(\gamma_+, f\gamma_+)}
\]

of \(P_{(\Delta, f)}\) at \(P_{(\gamma, f\gamma)}\) is a tilting object if and only if one of the following two conditions hold

1. the exchange triangle giving rise to \(P_{\mu^+_{(\gamma, f\gamma)}(\Delta, f)}\) has two middle terms. That is, it is of the form

\[
P_{(\gamma, f\gamma)} \rightarrow P_{(\gamma_1, f\gamma_1)} \oplus P_{(\gamma_2, f\gamma_2)} \rightarrow P_{(\gamma_+, f\gamma_+)} \rightarrow P_{(\gamma, f\gamma)[1]};
\]

for \(P_{(\gamma_1, f\gamma_1)}\) and \(P_{(\gamma_2, f\gamma_2)}\) indecomposable summands of \(P_{(\Delta, f)}\).

2. the exchange triangle giving rise to \(P_{\mu^+_{(\gamma, f\gamma)}(\Delta, f)}\) has one middle term, that is, it is of the form

\[
P_{(\gamma, f\gamma)} \xrightarrow{a} P_{(\gamma_1, f\gamma_1)} \rightarrow P_{(\gamma_+, f\gamma_+)} \rightarrow P_{(\gamma, f\gamma)[1]},
\]

for some indecomposable summand \(P_{(\gamma_1, f\gamma_1)}\) of \(P_{(\Delta, f)}\) and there exists no non-zero morphism from any indecomposable summand in \(P_{(\Delta, f)} \setminus P_{(\gamma, f\gamma)}\) to \(P_{(\gamma, f\gamma)}\), or if such map \(b\) exists, then \(ba \neq 0\).

If the exchange triangle giving rise to \(P_{\mu^+_{(\gamma, f\gamma)}(\Delta, f)}\) has zero middle term, that is, if it is of the form

\[
P_{(\gamma, f\gamma)} \xrightarrow{0} 0 \xrightarrow{0} P_{(\gamma, f\gamma)[1]} \xrightarrow{id} P_{(\gamma, f\gamma)[1]},
\]

then \(P_{\mu^+_{(\gamma, f\gamma)}(\Delta, f)}\) is not a tilting object.

**Proof** The corollary is just a restatement of Proposition 4.6 using Theorem 4.4. Note that in the second case, the condition in the corollary coincides with the condition that there exists no arc in \(\Delta\) with \(q_2\) as an endpoint except for \(\gamma\) and that this is independent of whether \(\gamma\) is a loop or not. \(\square\)
Fig. 8 By cutting at $\gamma$ in $S$ as in the left picture, in the cut marked surface the two boundary components in $S$ are replaced by one boundary component with two new marked points.

5 Silting reduction

In this section, we give a geometric interpretation of the silting reduction at a pre-silting subcategory arising from an indecomposable pre-silting object in the subcategory of perfect objects of the bounded derived category of a gentle algebra. We do this in terms of cutting the surface associated to the gentle algebra along the arc corresponding to this indecomposable pre-silting object.

For this we fix the following set-up. Let $A$ be a gentle algebra with associated graded marked surface $(S, M, \Delta_A)$. Denote by $\Delta_A^\bullet$ the dual admissible $\bullet$-dissection of $\Delta_A$, and by $\mathcal{K}$ the homotopy category $K^b(\text{proj}A)$. We assume throughout this section that $\gamma$ is a $\circ$-arc without self-intersections, that is, $\gamma$ does not have any self-intersections neither in the interior of $S$ nor at its endpoints. Then it follows from the description of the maps in $\mathcal{K}$, which we recall in Sect. 2.2, that $P_{(\gamma, f_\gamma)}$ is an indecomposable pre-silting object. We set $P = \text{add}(P_{(\gamma, f_\gamma)})$ to be the pre-silting subcategory of $\mathcal{K}$ generated by $P_{(\gamma, f_\gamma)}$.

5.1 Cutting the surface

**Definition 5.1 [the cut marked surface]** Let $(S, M)$ be a marked surface. Let $\gamma$ be an $\circ$-arc in $S$ without self-intersections, that is, $\gamma$ has neither an interior intersection nor an intersection at the endpoints in the boundary.

1. The cut surface $S_\gamma$ is obtained from $S$ by cutting along $\gamma$ and contracting the new boundary segments along $\gamma$.
2. We define the cut marked surface $(S_\gamma, M_\gamma)$ as follows: $P^\bullet_\gamma = P^\bullet, M^\circ_\gamma = M^\circ$, and $M^\circ_\gamma$ is obtained from $M^\circ$ in the following way. Set $M^\circ_\gamma = M^\circ \setminus \{p, q\} \cup \{p_\gamma, q_\gamma\}$, where $p_\gamma$ and $q_\gamma$ are new $\circ$-marked points obtained by cutting along $\gamma$ and then contracting, as described in Figs. 8 and 9. Figure 8 shows the case when the endpoints of $\gamma$ lie on two different boundary components and Fig. 9 shows the case when the endpoints of $\gamma$ lie on the same boundary component.

Then similarly to the discussion in [2, Proposition 1.11], we have several cases (note that in our set-up there are no green punctures in the initial dissection of the marked surface, so some of the cases discussed in [2] do not appear here). As shown in [2, Proposition 1.11], for all cases, there are exactly $|M^\circ| + |P| + b + 2g - 3$ arcs in an admissible $\circ$-dissection of $(S_\gamma, M_\gamma)$, where $g$ is the genus of $S$ and $b$ is the number of connected components of $\partial S$. 

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Case 1: \( \gamma \) is a non-separating curve starting and ending on two different boundary components. In this case, \((S_\gamma, M_\gamma)\) is a marked surface with \(|M^\circ|\) marked points \(\circ\), \(|P|\) punctures, \(b - 1\) boundary components, and genus \(g\).

Case 2: \( \gamma \) is a non-separating curve starting and ending on the same boundary component. In this case, \((S_\gamma, M_\gamma)\) is a marked surface with \(|M^\circ|\) \(\circ\)-marked points, \(|P|\) punctures, \(b + 1\) boundary components, and genus \(g - 1\).

Case 3: \( \gamma \) is a separating curve starting and ending on the same boundary component. In this case, \((S_\gamma, M_\gamma)\) is a disjoint union of two marked surfaces, with a total of \(|M^\circ|\) \(\circ\)-marked points, \(|P|\) punctures, \(b + 1\) boundary components, and such that the sum of their genuses is equal to \(g\).

Note that the assumption that \( \gamma \) is not a loop ensures that the cut marked surface \((S_\gamma, M_\gamma)\) has no \(\circ\)-punctures. When cutting the surface, there may appear components corresponding to disks with only one \(\circ\)-point and one \(\bullet\)-point. Such a surface corresponds to the trivial algebra and as such we can, in general, ignore such components.

### 5.2 Identifying arcs in the cut surface

Recall that we always assume the curves are in minimal position with regards to the initial \(\bullet\)-dissection. All curves which intersect \( \gamma \) in the interior of \( S \) will disappear after cutting. In particular, this holds for any closed curve \( \alpha \). Namely, either \( \alpha \) intersects \( \gamma \) and then there is no corresponding curve in the cut surface or \( \alpha \) does not intersect \( \gamma \) and then there is a unique corresponding closed curve in \((S_\gamma, M_\gamma)\) (up to homotopy). The case of admissible arcs is more complicated, since distinct admissible arcs in \((S, M)\) might be identified in \((S_\gamma, M_\gamma)\).

In order to explicitly describe this, we define an equivalence relation on admissible arcs as follows.

**Definition 5.2** Let \( \alpha \) and \( \beta \) be two admissible arcs intersecting \( \gamma \) at most in their endpoints. We write \( \alpha \xrightarrow{a} \sim_\gamma \beta \), if \( a \) is an oriented intersection from \( \alpha \) to \( \gamma \) and \( \beta \) is obtained from \( \alpha \) by smoothing the crossing with \( \gamma \) at \( a \). We write \( \alpha \xrightarrow{a} \sim_\gamma \beta \) if \( \alpha \) and \( \beta \) are \( \gamma \)-smoothing equivalent, that is, \( \beta \) can be obtained from \( \alpha \) by iterated smoothings with \( \gamma \) at either endpoint of \( \gamma \).

**Remark 5.3** By Proposition-Definition 3.1, it is straightforward to verify that the \( \gamma \)-smoothing equivalence is an equivalence relation on the set of admissible arcs on \((S, M)\) that intersect \( \gamma \) at most in their endpoints.
Fig. 10 Possible positions of an arc \( \alpha_i \) intersecting \( \gamma \) at its endpoints with respect to the oriented intersections arising from the common endpoints, that is the ordering of \( \alpha_i \) and \( \gamma \) at the common endpoints. The black bullet \( q_3 \) in picture (I) corresponds to either a \( \circ \)-marked point or a \( \bullet \)-puncture, while the fat black bullet in picture (III) are used to illustrate that arcs are not contractible.

Let \( \alpha \) be an admissible arc, we denote the \( \gamma \)-smoothing equivalence class of \( \alpha \) by \([\alpha]_\gamma\). Note that \([\alpha]_\gamma\) has only one element if and only if \( \alpha \) does not intersect with \( \gamma \) at the boundary of \( S \). The following lemma is a direct consequence of the definitions.

**Lemma 5.4** Two admissible arcs \( \alpha \) and \( \beta \) in \((S, M)\), which intersect \( \gamma \) at most in their endpoints, are identified in \((S_\gamma, M_\gamma)\) if and only if they are in the same \( \gamma \)-smoothing equivalence class.

We now give a complete list of possible configurations for an admissible arc \( \alpha \) intersecting \( \gamma \) at most in its endpoints.

Case (1): the endpoints of \( \alpha \) are distinct

1. (1.1) \( \alpha \) and \( \gamma \) have only one common endpoint giving rise to an oriented intersection from \( \alpha \) to \( \gamma \) (see Fig. 10 (I) setting \( \alpha = \alpha_1 \));
2. (1.2) \( \alpha \) and \( \gamma \) have only one common endpoint giving rise to an oriented intersection from \( \gamma \) to \( \alpha \) (see Fig. 10 (I) setting \( \alpha = \alpha_2 \));
3. (1.3) \( \alpha \) and \( \gamma \) have two common endpoints giving rise to two oriented intersections from \( \alpha \) to \( \gamma \) (see Fig. 10 (II) setting \( \alpha = \alpha_1 \));
4. (1.4) \( \alpha \) and \( \gamma \) have two common endpoints giving rise to two oriented intersections from \( \gamma \) to \( \alpha \) (see Fig. 10 (II) setting \( \alpha = \alpha_4 \));
5. (1.5) \( \alpha \) and \( \gamma \) have two common endpoints giving rise to an oriented intersection from \( \alpha \) to \( \gamma \) and to an oriented intersection from \( \gamma \) to \( \alpha \) (see Fig. 10 (III) setting \( \alpha = \alpha_2 \) or \( \alpha = \alpha_3 \));

Case (2): the endpoints of \( \alpha \) coincide. Note that in this case there will at most one endpoint of \( \gamma \) coinciding with the endpoint of \( \alpha \) and the following oriented intersections can arise

1. (2.1) one oriented intersection from \( \alpha \) to \( \gamma \) and one oriented intersection from \( \gamma \) to \( \alpha \) (see Fig. 10 (II) setting \( \alpha = \alpha_2 \) or \( \alpha = \alpha_3 \));
2. (2.2) two oriented intersections from \( \alpha \) to \( \gamma \) (see Fig. 10 (III) setting \( \alpha = \alpha_1 \));
3. (2.3) two oriented intersections from \( \gamma \) to \( \alpha \) (see Fig. 10 (III) setting \( \alpha = \alpha_4 \));

**Lemma 5.5** Let \( \alpha \) be an admissible arc in \( S \) which intersects \( \gamma \) only in either one or both of its endpoints. Then \( \alpha \) is one of the admissible arcs described above.

Furthermore, any two admissible arcs intersecting with \( \gamma \) at their endpoints are \( \gamma \)-smoothing equivalent if and only if they both correspond to one of the arcs \( \alpha_i \) in the same picture of Fig. 10. More precisely, \( \alpha_1 \sim \alpha_2 \) in picture (I), \( \alpha_1 \sim \alpha_2 \sim \alpha_3 \sim \alpha_4 \) in pictures (II) and (III).
Proof Note that $\gamma$ always has two distinct endpoints, since in this section we assume that $\gamma$ is not a loop. Then by considering the ordering of $\alpha$ and $\gamma$ at the common endpoints, it is clear that the classification of oriented intersections given before the lemma is complete.

To prove the second part of the lemma, we consider the smoothing equivalence class for each case in Fig. 10. For case (I), we have $\alpha_1 \sim_a \alpha_2$ and $[\alpha_1]_\gamma = \{\alpha_1, \alpha_2\}$. For the other two cases, we redraw pictures (II) and (III) of Fig. 10 as (II) and (III), respectively, in Fig. 11 so that it is easier to label the oriented intersections in the pictures. For case (II), the arcs $\alpha_i, 1 \leq i \leq 4$ are connected by $\gamma$-smoothings as follows:

$$\alpha_1 \sim_{a_1} \alpha_2 \sim_{a_2} \alpha_4, \; \alpha_1 \sim_{a_2} \alpha_3 \sim_{a_4} \alpha_4.$$  

So we have $[\alpha_1]_\gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Case (III) is similar to case (II), namely, the arcs $\alpha_i, 1 \leq i \leq 4$ are connected by $\gamma$-smoothings as follows:

$$\alpha_1 \sim_{b_4} \alpha_2 \sim_{b_2b_3b_4} \alpha_4, \; \alpha_3 \sim_{b_3b_4} \alpha_4,$$

and we have $[\alpha_1]_\gamma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. \qed

Combining Lemmas 5.4 and 5.5, we summarise the above discussion in the following proposition.

Proposition-Definition 5.6 [Identifying arcs in the cut marked surface] Let $(S_\gamma, M_\gamma)$ be the cut marked surface of $(S, M)$ with respect to an $\circ$-arc $\gamma$ without self-intersections. Let $\alpha$ be an $\circ$-arc in $(S, M)$.

1. If $\alpha$ and $\gamma$ intersect in the interior of $S$ then $\alpha$ does not give rise to a curve in $(S_\gamma, M_\gamma)$.
2. If $\alpha$ does not intersect $\gamma$, then $\alpha$ is an arc in $S_\gamma$ and there are no other arcs in $S$ which are identified with $\alpha$ in $(S_\gamma, M_\gamma)$.
3. If $\alpha$ intersects $\gamma$ at an endpoint then $\alpha = \alpha_i$, for $\alpha_i$ as in one of the pictures in Fig. 10, and the other $\alpha_j$ in the same picture are exactly the arcs which are identified with $\alpha_i$ in $(S_\gamma, M_\gamma)$. 

Fig. 11 A redrawing of pictures (II) and (III) in Fig. 10 in the universal cover, where for $i \in \{1, 2\}$, both $q_i$ and $q'_i$ are lifts of $q_i$ in the corresponding pictures in Fig. 10
5.3 Identifying objects in the silting reduction

In the following we will study the silting reduction $\mathcal{K}/\text{thick}_K \mathcal{P}$ and its geometric realization. Let $\mathcal{Z} = \perp \mathcal{P}[>0] \cap \mathcal{P}[<0] \perp$. Then by Theorem 2.13, the composition $\mathcal{Z} \subset \mathcal{K} \xrightarrow{\rho} \mathcal{K}/\text{thick}_K \mathcal{P}$ of natural functors induces an equivalence of triangulated categories:

$$\tilde{\rho} : \mathcal{Z} \mathcal{P} \xrightarrow{\sim} \mathcal{K}/\text{thick}_K \mathcal{P}.$$ (5.1)

So from now we will not distinguish the two categories and also refer to $\mathcal{Z} \mathcal{P}$ as the silting reduction of $\mathcal{K}$ by $\mathcal{P}$. We also recall that by Theorem 2.7 (3), the indecomposable objects in $\mathcal{K}$ only arise from graded $\circ$-arcs or graded closed curves, rather than the infinite arcs, which means the endpoint of a graded arc should not be a $\bullet$-puncture. Recall that for this subsection we have a fixed pre-silting subcategory $\mathcal{P} = \text{add}(\mathcal{P}(\gamma, f_\gamma))$ where $\gamma$ is an $\circ$-arc which is not a loop.

Lemma 5.7 [The objects in $\mathcal{Z}$] Let $\mathcal{P}(\alpha, f_\alpha)$ be an object in $\mathcal{Z}$ corresponding to a graded curve $(\alpha, f_\alpha)$, then $\alpha$ and $\gamma$ do not intersect in the interior of $S$.

Proof For contradiction, suppose that $\alpha$ and $\gamma$ intersect at some point $p$ in the interior of $S$. Without loss of generality we assume that the local configuration of the intersection is as follows. Let $q_i, 1 \leq i \leq 4$, be the intersections nearest to $p$ in $\alpha \cap \Delta^*_A$, respectively in $\gamma \cap \Delta^*_A$.

Then since $\mathcal{P}(\alpha, f_\alpha) \in \perp \mathcal{P}[>0]$, we have

$$f_\alpha(q_1) \geq f_\gamma(q_4), \ f_\alpha(q_3) \geq f_\gamma(q_2),$$

and since $\mathcal{P}(\alpha, f_\alpha) \in \mathcal{P}[<0] \perp$, we have

$$f_\alpha(q_3) \leq f_\gamma(q_4), \ f_\alpha(q_1) \leq f_\gamma(q_2).$$

So we have

$$f_\gamma(q_4) \geq f_\alpha(q_3) \geq f_\gamma(q_2) \geq f_\alpha(q_1) \geq f_\gamma(q_4),$$

and thus they are all equal. A contradiction since we have $f_\alpha(q_1) = f_\alpha(q_3) \pm 1$ and $f_\gamma(q_4) = f_\gamma(q_2) \pm 1$. \qed

Remark 5.8 We mention that the arcs corresponding to indecomposable objects in $\mathcal{Z}$ may intersect with $\gamma$ at their endpoints. On the other hand, the inverse statement of the lemma does not hold, that is, even if $\alpha$ intersects $\gamma$ at the boundary of $S$, rather than in the interior of $S$, $\alpha$ might not give rise to an object in $\mathcal{Z}$ for any grading $f_\alpha$. Examples of this are the arcs $\alpha_2$ and $\alpha_3$ in Lemma 5.11 and Lemma 5.12 below.

The following four lemmas determine when a graded $\circ$-arc corresponds to an object in $\mathcal{Z} \mathcal{P}$, and describe the $(1)$-orbits of this object in $\mathcal{Z} \mathcal{P}$. Each Lemma corresponds to either the case of $\alpha$ and $\gamma$ not intersecting or to one of the three cases in Fig. 10.
Let \( \alpha \) be an \( \circ \)-arc which does not intersect \( \gamma \), then \( P_{(\alpha, f_\alpha)} \) is in \( \mathcal{Z}_P \) for any grading \( f_\alpha \). Moreover, the shift functor \( (1) \) in \( \mathcal{Z}_P \) coincides with \([1] \), and the \([1]\)-orbit of \( P_{(\alpha, f_\alpha)} \) in \( \mathcal{Z}_P \) coincides with the \([1]\)-orbit of \( P_{(\alpha, f_\alpha)} \) in \( \mathcal{K} \).

**Proof** By Theorem 2.9, if there are no intersections between \( \alpha \) and \( \gamma \), then
\[
\text{Hom}_{\mathcal{K}}(P_{(\alpha, f_\alpha)}, P_{(\gamma, f_\gamma)}[i]) = 0
\]
for any \( i \in \mathbb{Z} \) and any grading \( f_\alpha \). So \( P_{(\gamma, f_\gamma)} \in \mathcal{Z}_P \) and \( P_{(\alpha, f_\alpha)}[i] = P_{(\alpha, f_\alpha)}[i] \) by the definition of \( (1) \) in Theorem 2.13. Therefore the \([1]\)-orbit of \( P_{(\alpha, f_\alpha)} \) in \( \mathcal{Z}_P \) coincides with the \([1]\)-orbit of \( P_{(\alpha, f_\alpha)} \) in \( \mathcal{K} \).

**Lemma 5.10** For \( \circ \)-arcs \( \alpha_1 \) and \( \alpha_2 \) as in Fig. 10 (I), let \( f_{\alpha_1} \) and \( f_{\alpha_2} \) be the gradings of \( \alpha_1 \) and \( \alpha_2 \) such that \( f_{\alpha_1}(s_{\alpha_1}^1) = f_\gamma(s_{\alpha_1}^1) \) and \( f_{\alpha_2}(s_{\alpha_2}^1) = f_\gamma(s_{\alpha_2}^1) \) respectively. Then
\[
P_{(\alpha_1, f_{\alpha_1})}[i] \in \mathbb{Z} \iff i \leq 0,
\]
\[
P_{(\alpha_2, f_{\alpha_2})}[i] \in \mathbb{Z} \iff i \geq 0.
\]
Furthermore, the \([1]\)-orbit of \( P_{(\alpha_1, f_{\alpha_1})} \) in \( \mathcal{Z}_P \) is
\[
\cdots \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[-2] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[-1] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})} \cdots
\]

**Proof** Since \( f_{\alpha_1}(s_{\alpha_1}^1) = f_\gamma(s_{\alpha_1}^1) \), the intersection of \( \alpha_1 \) and \( \gamma \) at \( q_1 \) gives rise to a map \( a : P_{(\alpha_1, f_{\alpha_1})} \rightarrow P_{(\gamma, f_\gamma)} \) in \( \mathcal{K} \). Since \( q_1 \) is the unique intersection between \( \alpha_1 \) and \( \gamma \), we have \( P_{(\alpha_1, f_{\alpha_1})}[i] \in \mathcal{P}[<0]^\perp \) for any \( i \in \mathbb{Z} \), and \( P_{(\alpha_1, f_{\alpha_1})}[i] \in \mathcal{P}[>0]^\perp \) if and only if \( i \leq 0 \). Therefore
\[
P_{(\alpha_1, f_{\alpha_1})}[i] \in \mathbb{Z} \iff i \leq 0.
\]
Now since the map \( a \) forms a basis of the space \( \text{Hom}_{\mathcal{K}}(P_{(\alpha_1, f_{\alpha_1})}, P_{(\gamma, f_\gamma)}) \), it is a left \( \mathcal{P} \)-approximation of \( P_{(\alpha_1, f_{\alpha_1})} \), and by Lemma 3.4 (3), it determines a distinguished triangle
\[
P_{(\alpha_1, f_{\alpha_1})} \xrightarrow{a} P_{(\gamma, f_\gamma)} \xrightarrow{\gamma} P_{(\alpha_2, f_{\alpha_2})} \xrightarrow{\alpha} P_{(\alpha_1, f_{\alpha_1})}[1]
\]
for the grading \( f_{\alpha_2} \) of \( \alpha_2 \) such that \( f_{\alpha_2}(s_{\alpha_2}^1) = f_\gamma(s_{\alpha_2}^1) \). So we have \( P_{(\alpha_1, f_{\alpha_1})}[1] = P_{(\alpha_2, f_{\alpha_2})} \) in \( \mathcal{Z}_P \).

By a similar argument, we have
\[
P_{(\alpha_2, f_{\alpha_2})}[i] \in \mathbb{Z} \iff i \geq 0.
\]
Moreover, note that for any \( i < 0 \), we have \( \text{Hom}_{\mathcal{K}}(P_{(\alpha_1, f_{\alpha_1})}, P_{(\gamma, f_\gamma)}[i]) = 0 \), and for any \( i > 0 \), we have \( \text{Hom}_{\mathcal{K}}(P_{(\alpha_2, f_{\alpha_2})}, P_{(\gamma, f_\gamma)}[i]) = 0 \). Therefore \( P_{(\alpha_1, f_{\alpha_1})}[i][1] = P_{(\alpha_1, f_{\alpha_1})}[i+1] \) for any \( i < 0 \), and \( P_{(\alpha_2, f_{\alpha_2})}[i][1] = P_{(\alpha_2, f_{\alpha_2})}[i+1] \) for any \( i > 0 \).

To sum up, these objects are connected by the shift functor \( (1) \) as in the statement. Furthermore the orbit uniquely corresponds to the \( \gamma \)-smoothing equivalence class \( [\alpha_1]_\gamma = [\alpha_2]_\gamma = [\alpha_1, \alpha_2] \) which are identified in \( S_\gamma \) by Proposition 5.6 (3).
(1) If $\alpha_{a_1} = f_\gamma(s_{q_1}^\gamma)$, then $P(\alpha_1, f_{a_1}) \in \mathcal{Z}$ and the $\langle 1 \rangle$-orbit of $P(\alpha_1, f_{a_1})$ is

$$\ldots \xrightarrow{(1)} P(\alpha_1, f_{a_1})[-2] \xrightarrow{(1)} P(\alpha_1, f_{a_1})[-1] \xrightarrow{(1)} P(\alpha_1, f_{a_1}) \xrightarrow{(1)} \ldots$$

for a certain grading $f_{a_4}$ of $\alpha_4$. In particular, $P(\alpha_2, f_{a_2}) \notin \mathcal{Z}$ and $P(\alpha_3, f_{a_3}) \notin \mathcal{Z}$ for any gradings $f_{a_2}$ and $f_{a_3}$ of $\alpha_2$ and $\alpha_3$, respectively.

(2) If $\alpha_{a_1} = f_\gamma(s_{q_1}^\gamma) = m \geq 1$, then $P(\alpha_1, f_{a_1}) \in \mathcal{Z}$ and the $\langle 1 \rangle$-orbit of $P(\alpha_1, f_{a_1})$ is

$$\ldots \xrightarrow{(1)} P(\alpha_1, f_{a_1})[-1] \xrightarrow{(1)} P(\alpha_1, f_{a_2}) \xrightarrow{(1)} P(\alpha_2, f_{a_2})[1] \xrightarrow{(1)} \ldots$$

for certain gradings $f_{a_2}$ and $f_{a_4}$ of $\alpha_2$ and $\alpha_4$, respectively. In particular, $P(\alpha_3, f_{a_3}) \notin \mathcal{Z}$ for any grading $f_{a_3}$ of $\alpha_3$.

(3) If $\alpha_{a_1} = f_\gamma(s_{q_1}^\gamma) = m \leq -1$, then $P(\alpha_1, f_{a_1})[m] \in \mathcal{Z}$ and the $\langle 1 \rangle$-orbit of $P(\alpha_1, f_{a_1})$ is

$$\ldots \xrightarrow{(1)} P(\alpha_1, f_{a_1})[m - 1] \xrightarrow{(1)} P(\alpha_1, f_{a_2}) \xrightarrow{(1)} P(\alpha_2, f_{a_3}) \xrightarrow{(1)} \ldots$$

for certain gradings $f_{a_3}$ and $f_{a_4}$ of $\alpha_3$ and $\alpha_4$, respectively. In particular, $P(\alpha_2, f_{a_2}) \notin \mathcal{Z}$ for any grading $f_{a_2}$ of $\alpha_2$.

**Proof** Since $\alpha_{a_1} = f_\gamma(s_{q_1}^\gamma)$, we have a map $a_1 : P(\alpha_1, f_{a_1}) \rightarrow P(\gamma, f_\gamma)$, which gives rise to the distinguished triangle

$$P(\alpha_1, f_{a_1}) \xrightarrow{a_1} P(\gamma, f_\gamma) \rightarrow P(\alpha_2, f_{a_2}) \rightarrow P(\alpha_1, f_{a_1})[1],$$

where $f_{a_2}$ is the grading of $\alpha_2$ such that $f_\gamma(s_{q_2}^\gamma) = f_{a_2}(s_{q_2}^{a_2})$ and $f_{a_2}(s_{q_2}^{a_2}) = f_{a_1}(s_{q_2}^{a_1}) - 1$.

Now there are three subcases.

Case (1): if $\alpha_{a_1} = f_\gamma(s_{q_1}^\gamma)$, then there is a triangle

$$P(\alpha_1, f_{a_1}) \xrightarrow{a_2} P(\gamma, f_\gamma) \rightarrow P(\alpha_3, f_{a_3}) \rightarrow P(\alpha_1, f_{a_1})[1],$$

where $f_{a_3}$ is the grading of $\alpha_3$ such that $f_\gamma(s_{q_1}^\gamma) = f_{a_3}(s_{q_1}^{a_3})$ and $f_{a_3}(s_{q_1}^{a_3}) = f_{a_1}(s_{q_1}^{a_1}) - 1$.

Suppose $P(\alpha_2, g_{a_2}) \in \mathcal{Z}$ for some grading $g_{a_2}$. Then we have $g_{a_2}(s_{q_2}^{a_2}) \leq f_\gamma(s_{q_2}^\gamma)$ by $P(\alpha_2, g_{a_2}) \in \mathcal{P}[<0]^\perp$, and $g_{a_2}(s_{q_2}^{a_2}) \geq f_\gamma(s_{q_2}^\gamma)$ by $P(\alpha_2, g_{a_2}) \in \mathcal{P}[<0]^\perp$. So

$$g_{a_2}(s_{q_2}^{a_2}) \leq g_{a_2}(s_{q_2}^{a_2}).$$

On the other hand, we have

$$f_{a_2}(s_{q_2}^{a_2}) = f_\gamma(s_{q_2}^\gamma) = f_\gamma(s_{q_2}^\gamma) = f_{a_1}(s_{q_2}^{a_1}) = f_{a_2}(s_{q_2}^{a_2}) + 1.$$

So

$$g_{a_2}(s_{q_2}^{a_2}) = g_{a_2}(s_{q_2}^{a_2}) + 1.$$
Similarly, we may prove that $f_{α_4}$ is a distinguished triangle for some proper grading $f_{α_4}$ of $α_4$. In particular, since $q_1$ and $q_2$ are the only intersections between $α_1$ and $α_2$ and $α_1$ and $α_2$ form a basis of the space Hom($P_{(α_1, f_{α_1})}$, $P_{(γ, f_γ)}$), so $(α_1, α_2)$ is a left $P_{(γ, f_γ)}$-approximation. Thus $P_{(α_1, f_{α_1})}[1] = P_{(α_4, f_{α_4})}$. On the other hand, $P_{(α_1, f_{α_1})}[i] ∈ Z$ if and only if $i ≥ 0$, and $P_{(α_1, f_{α_1})}[i] ∈ Z$ if and only if $i ≤ 0$, giving rise to the wanted (1)-orbit, which uniquely corresponds to the $o$-arc $α_1 = α_4$ in $S_y$.

Case (2): if $f_{α_1}(s_{q_2}^{α_1}) ≥ f_γ(s_{q_2}^{γ}) + 1$, assume that $f_{α_1}(s_{q_2}^{α_1}) - f_γ(s_{q_2}^{γ}) = m ≥ 1$. Now, for $i ∈ Z$,

$$P_{(α_2, f_{α_2})}[i] ∈ P[<0]^⊥ \iff f_{α_2}(s_{q_2}^{α_2}) ≤ f_γ(s_{q_2}^{γ}) + i,$$

$$P_{(α_2, f_{α_2})}[i] ∈ P[>0]^⊥ \iff f_{α_2}(s_{q_2}^{α_2}) ≥ f_γ(s_{q_2}^{γ}) + i.$$ 

On the other hand, we have

$$f_{α_2}(s_{q_2}^{α_2}) = f_γ(s_{q_2}^{γ}),$$

$$f_{α_2}(s_{q_2}^{α_2}) = f_{α_1}(s_{q_2}^{α_1}) - 1 = f_γ(s_{q_2}^{γ}) + m - 1.$$ 

So

$$P_{(α_2, f_{α_2})}[i] ∈ Z \iff 0 ≤ i ≤ m - 1.$$ 

Similarly, we may prove that

$$P_{(α_1, f_{α_1})}[i] ∈ Z \iff i ≤ 0,$$

$$P_{(α_4, f_{α_4})}[i] ∈ Z \iff i ≥ m.$$ 

Now we show that $P_{(α_3, g_{α_3})} \not∈ Z$ for any grading $g_{α_3}$. On the one hand, for $i ∈ Z$,

$$P_{(α_3, g_{α_3})}[i] ∈ P[<0]^⊥ \iff g_{α_3}(s_{q_1}^{α_3}) ≤ f_γ(s_{q_1}^{γ}) + i,$$

$$P_{(α_3, g_{α_3})}[i] ∈ P[>0]^⊥ \iff g_{α_3}(s_{q_1}^{α_3}) ≥ f_γ(s_{q_1}^{γ}) + i.$$ 

So if $P_{(α_3, g_{α_3})}[i] ∈ Z$, then

$$g_{α_3}(s_{q_1}^{α_3}) ≤ g_{α_3}(s_{q_1}^{γ}).$$ 

On the other hand, note that there is a triangle

$$P_{(α_1, f_{α_1})} \xrightarrow{α_2} P_{(γ, f_γ)}[m] \rightarrow P_{(α_3, f_{α_3})} \rightarrow P_{(α_1, f_{α_1})}[1].$$ 

where $f_{α_3}$ is a grading such that $f_{α_3}(s_{q_1}^{α_3}) = f_γ(s_{q_1}^{γ}) + m$ and $f_{α_3}(s_{q_1}^{α_3}) = f_{α_1}(s_{q_1}^{α_1}) - 1$. So

$$f_{α_3}(s_{q_1}^{α_3}) = f_{α_1}(s_{q_1}^{α_1}) - 1 = f_γ(s_{q_1}^{γ}) - 1 = f_{α_3}(s_{q_1}^{α_3}) - m - 1,$$

and $f_{α_3}(s_{q_1}^{α_3}) > f_{α_3}(s_{q_1}^{α_3})$, since $m ≥ 1$. Note that $f_{α_3}$ and $g_{α_3}$ are two gradings on $α_3$ and $f_{α_3}(s_{q_1}^{α_3}) - f_{α_3}(s_{q_1}^{α_3}) = g_{α_3}(s_{q_1}^{α_3}) - g_{α_3}(s_{q_1}^{α_3})$. Thus

$$g_{α_3}(s_{q_1}^{α_3}) > g_{α_3}(s_{q_1}^{α_3}).$$
So we obtain a contradiction and therefore $P_{(\alpha_3, f_{\alpha_3})} \notin Z$ for any grading $g_{\alpha_3}$.

Finally we have the wanted (1)-orbit, which uniquely corresponds to the $\alpha$-arc $\alpha_1 = \alpha_2 = \alpha_4$ over $S_Y$.

Case (3): $f_{\alpha_1}(s_{q_1}^{\alpha_1}) \leq f_{\gamma}(s_{q_2}'^{\gamma}) - 1$, this case is the dual of case (2). $\square$

Lemma 5.12 For $\alpha_1$ as in picture (III) of Fig. 11, let $f_{\alpha_1}$ be the grading of it such that $f_{\alpha_1}(s_{q_1}^{\alpha_1}) = f_{\gamma}(s_{q_2}'^{\gamma})$.

(1) If $f_{\alpha_1}(s_{q_1}^{\alpha_1}) = f_{\alpha_1}(s_{q_1}')$, then $P_{(\alpha_1, f_{\alpha_1})} \in Z$ and the (1)-orbit of $P_{(\alpha_1, f_{\alpha_1})}$ is

$$\cdots \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[-2] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[-1] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})} \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})} \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})}[1] \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})}[2] \xrightarrow{(1)} \cdots$$

for a certain grading $f_{\alpha_4}$ of $\alpha_4$. In particular, $P_{(\alpha_2, f_{\alpha_2})} \notin Z$ and $P_{(\alpha_3, f_{\alpha_3})} \notin Z$ for any gradings $f_{\alpha_2}$ and $f_{\alpha_3}$ of $\alpha_2$ and $\alpha_3$ respectively.

(2) If $f_{\alpha_1}(s_{q_1}^{\alpha_1}) - f_{\alpha_1}(s_{q_1}') = m \geq 1$, then $P_{(\alpha_1, f_{\alpha_1})} \in Z$ and the (1)-orbit of $P_{(\alpha_1, f_{\alpha_1})}$ is

$$\cdots \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[-1] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})} \xrightarrow{(1)} P_{(\alpha_2, f_{\alpha_2})}[1] \xrightarrow{(1)} \cdots$$

$$P_{(\alpha_2, f_{\alpha_2})}[m - 1] \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})}[m] \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})}[m + 1] \xrightarrow{(1)} \cdots$$

for certain gradings $f_{\alpha_2}$ and $f_{\alpha_4}$ of $\alpha_2$ and $\alpha_4$ respectively. In particular, $P_{(\alpha_3, f_{\alpha_3})} \notin Z$ for any grading $f_{\alpha_3}$ of $\alpha_3$.

(3) If $f_{\alpha_1}(s_{q_1}^{\alpha_1}) - f_{\alpha_1}(s_{q_1}') = m \leq -1$, then $P_{(\alpha_1, f_{\alpha_1})}[m] \in Z$ and the (1)-orbit of $P_{(\alpha_1, f_{\alpha_1})}$ is

$$\cdots \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[m - 1] \xrightarrow{(1)} P_{(\alpha_1, f_{\alpha_1})}[m] \xrightarrow{(1)} P_{(\alpha_3, f_{\alpha_3})}[m + 1] \xrightarrow{(1)} \cdots$$

$$\cdots \xrightarrow{(1)} P_{(\alpha_3, f_{\alpha_3})} \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})} \xrightarrow{(1)} P_{(\alpha_4, f_{\alpha_4})}[1] \xrightarrow{(1)} \cdots$$

for certain gradings $f_{\alpha_3}$ and $f_{\alpha_4}$ of $\alpha_3$ and $\alpha_4$ respectively. In particular, $P_{(\alpha_2, f_{\alpha_2})} \notin Z$ for any grading $f_{\alpha_2}$ of $\alpha_2$.

Proof The proof is similar to the proof of Lemma 5.11. $\square$

Combining Lemmas 5.9, 5.10, 5.11, 5.12 and Proposition 5.6, we obtain the following.

Theorem 5.13 Let $P_{(\alpha, f_\alpha)}$ and $P_{(\beta, f_\beta)}$ be two indecomposable objects in $Z_P$ corresponding to $\alpha$-arcs $\alpha$ and $\beta$ in $S$ respectively. Then the following are equivalent

(1) $P_{(\alpha, f_\alpha)}$ and $P_{(\beta, f_\beta)}$ are in the same (1)-orbit in $Z_P$,
(2) $\alpha$ and $\beta$ are in the same $\gamma$-smoothing equivalence class,
(3) $\alpha$ and $\beta$ are identified in $(S_Y, M_Y)$.

5.4 A geometric model of silting reduction

Recall that we denote by $\overline{Z_P}$ the orbit category of the subfactor category $Z_P$ by the shift functor, see Sect. 2.5. We will show in the following that the cut marked surface $(S_Y, M_Y)$ gives a geometric model of $\overline{Z_P}$. 

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Definition 5.14 We call a non-zero indecomposable object $\overline{X}$ in $\overline{\mathcal{Z}}_P$ a string (resp. band) object, if its representative $X$ in $\mathcal{Z}$ is a string (resp. band) object.

Note that the above definition is well-defined and any indecomposable object in $\overline{\mathcal{Z}}_P$ is either a string object or a band object.

We now define three maps $\Phi_1$, $\Phi_2$ and $\Phi_3$ as follows. Let $\text{Ind} \mathcal{Z}$ and $\text{Ind} \overline{\mathcal{Z}}_P$ be the set of indecomposable objects in $\mathcal{Z}$ and $\overline{\mathcal{Z}}_P$, respectively and let $\Phi_1$ be the canonical map $\text{Ind} \mathcal{Z} \to \text{Ind} \overline{\mathcal{Z}}_P$. Denote by $C_\gamma(S, M, \Delta_A)$ the set of graded curves on $(S, M, \Delta_A)$ which have no interior intersection with $\gamma$. Then by Lemma 5.7, there is a canonical map $\Phi_2 : \text{Ind} \mathcal{Z} \to C(S, M, 1_A)$. Finally, denote by $\Phi_3$ the correspondence of curves established in Proposition 5.6 and denote by $C_{\Phi_3}(S_\gamma, M_\gamma)$ the image of $\Phi_3$ in $(S_\gamma, M_\gamma)$. We can summarize the information in the following diagram.

\[
\begin{array}{ccc}
X & \overset{\Phi_1}{\to} & \overline{X} \\
\text{Ind} \mathcal{Z} \downarrow & & \downarrow \Phi \\
C_\gamma(S, M, \Delta_A) & \overset{\Phi_3}{\longrightarrow} & C_{\Phi_3}(S_\gamma, M_\gamma) \\
(\alpha_X, f_\alpha_X) & & \alpha_{\overline{X}} = \alpha_X
\end{array}
\]

Furthermore, we remark that $\Phi_2$ maps any band object in the same one parameter family in $\mathcal{Z}$ to the same graded closed curve in $(S, M, \Delta_A)$. In particular, the parameter associated to a band object simply disappears under this mapping.

Definition 5.15 Define $\Phi : \text{Ind} \overline{\mathcal{Z}}_P \to C_{83}(S, M)$

\[
\overline{X} \mapsto \alpha_{\overline{X}}
\]

as the composition of the three correspondences in the above diagram, given by first lifting a non-zero indecomposable object $\overline{X} \in \text{Ind} \overline{\mathcal{Z}}_P$ to an indecomposable object $X \in \text{Ind} \mathcal{Z}$, then identifying it with the corresponding graded curve $(\alpha_X, f_X)$ on $C_\gamma(S, M, \Delta_A)$, and finally with the associated curve $\alpha_X$ in $C_{\Phi_3}(S_\gamma, M_\gamma)$.

For a non-zero indecomposable object $\overline{X}$ in $\text{Ind} \overline{\mathcal{Z}}_P$, by Lemma 5.7, there is no interior intersection between $\alpha_X$ and $\gamma$, and thus $\alpha_X$ still exists in $S_\gamma$. So $\Phi$ establishes a correspondence from the non-zero indecomposable objects in $\text{Ind} \overline{\mathcal{Z}}_P$ to $\circ$-arcs or closed curves on $S_\gamma$. On the other hand, note that there are many choices for the representative $X$ of $\overline{X}$. However, we now show, that the composition $\Phi$ is independent of the choice of representative and therefore it is a well-defined map.

Lemma 5.16 The map $\Phi$ is a well-defined injection from the set of isomorphism classes of indecomposable objects in $\overline{\mathcal{Z}}_P$ to the set of $\circ$-arcs and pairs consisting of a closed curve and a non-zero element in the base field $k$, which restricts to a bijection from the set of isomorphism classes of indecomposable string objects in $\overline{\mathcal{Z}}_P$ to the set of $\circ$-arcs in $(S_\gamma, M_\gamma)$.

Proof For a non-zero indecomposable band object $\overline{X}$ in $\overline{\mathcal{Z}}_P$, any representative $X$ is a non-zero object in $\mathcal{Z}_P$. So by Lemma 5.7, there is no interior intersection between $\alpha_X$ and $\gamma$, and thus no intersection at all, since $\alpha_X$ is a closed curve. Thus any two distinct lifts of $\overline{X}$ are
associated to the same closed curve $\alpha_X$ in $S_{\gamma}$. Therefore $\Phi$ is well-defined and injective on the band objects.

On the other hand, Theorem 5.13 implies that $\Phi$ is well-defined and injective on the indecomposable string objects. Furthermore, note that any $\alpha$-arc in $S_{\gamma}$ is the image of some $\circ$-arc in $S$, which is always gradable and gives us an indecomposable string object in $\mathbb{Z}_{\mathcal{P}}$. So the correspondence $\Phi$ is surjective, and thus bijective, on the indecomposable string objects.

It follows from the proof above that any closed curve in $(S_{\gamma}, M_{\gamma})$ can be lifted to a unique closed curve in $(S, M)$. We say that a closed curve in $(S_{\gamma}, M_{\gamma})$ is gradable, if the corresponding closed curve in $(S, M, \Delta_A)$ is gradable.

**Theorem 5.17** Let $A$ be a gentle algebra with associated surface model $(S, M, \Delta_A)$. Set $\mathcal{P} = \text{add}(P_{(\gamma, f_{\gamma})})$ to be the pre-silting subcategory in $K^b(\text{proj}A)$ arising from a graded $\circ$-arc $(\gamma, f_{\gamma})$, where $\gamma$ is an $\circ$-arc without self-intersections. Let $\mathbb{Z}_{\mathcal{P}}$ be the silting reduction of $K^b(\text{proj}A)$ by $\mathcal{P}$, and denote by $\mathbb{Z}_{\mathcal{P}}$ the orbit category of $\mathbb{Z}_{\mathcal{P}}$ by the shift functor. Then

1. there is a bijection between the set of isomorphism classes of indecomposable objects in $\mathbb{Z}_{\mathcal{P}}$ and the set of $\circ$-arcs and pairs consisting of a gradable closed curve in $(S_{\gamma}, M_{\gamma})$ and a non-zero element in the base field $k$;

2. under the correspondence in (1), the dimensions of the morphism spaces in $\mathbb{Z}_{\mathcal{P}}$ coincide with the number of oriented intersections of the corresponding $\circ$-arcs and primitive closed curves in $(S_{\gamma}, M_{\gamma})$. More precisely, for any indecomposable objects $\mathbb{X}, \mathbb{Y} \in \mathbb{Z}_{\mathcal{P}}$ associated to $\alpha_X$ and $\alpha_Y$ respectively such that $\alpha_X$ and $\alpha_Y$ are $\circ$-arcs or primitive closed curves, we have

$$\dim \text{Hom}_{\mathbb{Z}_{\mathcal{P}}}^{\bullet}(\mathbb{X}, \mathbb{Y}) = \begin{cases} |\text{Int}_{S_{\gamma}}(\alpha_X, \alpha_Y)| + 2 & \text{if } \alpha_X = \alpha_Y \text{ is a closed curve, and } \mathbb{X} \cong \mathbb{Y}[m] \text{ for some } m \in \mathbb{Z}; \\ |\text{Int}_{S_{\gamma}}(\alpha_X, \alpha_Y)| & \text{otherwise} \end{cases} \quad (5.2)$$

**Proof** The proof of the first part for $\circ$-arcs directly follows from Lemma 5.16. For the case of closed curves, it is enough to remark that any band object in $\mathbb{Z}_{\mathcal{P}}$ uniquely corresponds to a pair consisting of a gradable primitive closed curve and an indecomposable $k[x, x^{-1}]$-module, which in turn by Theorem 2.7 corresponds to a pair consisting of a gradable closed curve and a non-zero element in $k$. We now prove the second part. Let $\mathbb{X}$ and $\mathbb{Y}$ be two indecomposable objects in $\mathbb{Z}_{\mathcal{P}}$.

Assume first that the associated curves $\alpha_X$ and $\alpha_Y$ do not intersect $\gamma$. Then there are no morphisms between $P_{(\gamma, f_{\gamma})}$ and $X[i]$, and between $P_{(\gamma, f_{\gamma})}$ and $Y[i]$, for any $i \in \mathbb{Z}$. Moreover, by Lemma 5.9, on the objects arising from $\alpha_X$ and $\alpha_Y$ the shift functor $(1)$ in $\mathbb{Z}_{\mathcal{P}}$ coincides with $(1)$. Thus

$$\text{Hom}_{\mathbb{Z}_{\mathcal{P}}}(X, Y[i]) = \text{Hom}_{K^b}(X, Y[i])$$

for any $i \in \mathbb{Z}$. So

$$\text{Hom}_{\mathbb{Z}_{\mathcal{P}}}^{\bullet}(\mathbb{X}, \mathbb{Y}) = \text{Hom}_{K^b}^{\bullet}(X, Y)$$

and, in particular,

$$\dim \text{Hom}_{\mathbb{Z}_{\mathcal{P}}}^{\bullet}(\mathbb{X}, \mathbb{Y}) = \dim \text{Hom}_{K^b}^{\bullet}(X, Y).$$

Furthermore, the intersections between $\alpha_X$ and $\alpha_Y$ remain unchanged after cutting the surface, so

$$|\text{Int}_{S_{\gamma}}(\alpha_X, \alpha_Y)| = |\text{Int}_{S_{\gamma}}(\alpha_X, \alpha_Y)|.$$
Fig. 12 o-arcs $\alpha$ and $\alpha'$ which intersect with $\gamma$ at endpoints. The points $q_3$ and $q_4$ may coincide

On the other hand, by Theorem 2.9 (2),

$$
\dim \text{Hom}_K^\bullet (X, Y) = \begin{cases} 
\left| \text{Int}_S(\alpha_X, \alpha_Y) \right| + 2 & \text{if } \alpha_X = \alpha_Y \text{ is a closed curve, and } X \cong Y[m] \text{ for some } m \in \mathbb{Z}; \\
\left| \text{Int}_S(\alpha_X, \alpha_Y) \right| & \text{otherwise.}
\end{cases}
$$

Therefore we have

$$
\dim \text{Hom}_{\mathbb{Z}P}^\bullet (\overline{X}, \overline{Y}) = \begin{cases} 
\left| \text{Int}_{\gamma}(\alpha_X, \alpha_Y) \right| + 2 & \text{if } \alpha_X = \alpha_Y \text{ is a closed curve, and } X \cong Y[m] \text{ for some } m \in \mathbb{Z}; \\
\left| \text{Int}_{\gamma}(\alpha_X, \alpha_Y) \right| & \text{otherwise}
\end{cases}
$$

Now assume that at least one of $\alpha_X$ and $\alpha_Y$ intersects $\gamma$ but that their intersection with $\gamma$ is only at their endpoints. Then $\alpha_X$ and $\alpha_Y$ are both o-arcs, and there are several possibilities depending on the relative positions of $\alpha_X$, $\alpha_Y$ and $\gamma$, see Fig. 10. In the following we give a detailed proof of the case when both $\alpha_X$ and $\alpha_Y$ are arcs such as in case I of Fig. 10, that is, both of them intersect $\gamma$. The other cases are proved in a similar way.

Case I of Fig. 10 splits into two subcases as described in Fig. 12. Indeed, this is the case since by construction both the left side and the right side of equality (5.2) are independent of the representative chosen in the corresponding quotient constructions. For example, if one consider $X$ (resp. $Y$) so that the corresponding $\gamma$-smoothing equivalent class is $\{\alpha, \beta\}$ (resp. $\{\alpha', \beta'\}$), then by Theorem 5.13, one may consider the equation (5.2) up to choosing one representative in each of these.

For convenience, we will denote $\alpha_X$ by $\alpha$ and denote $\alpha_Y$ by $\alpha'$. We also assume that $X = P(\alpha, f_\alpha)$ and $Y = P(\alpha', f_{\alpha'})$, where $f_\alpha$ and $f_{\alpha'}$ are gradings of $\alpha$ and $\alpha'$ respectively such that we have the following maps $a : P(\alpha, f_\alpha) \to P(\alpha', f_{\alpha'})$ and $b : P(\alpha', f_{\alpha'}) \to P(\gamma, f_{\gamma})$ in subcase I, and maps $a : P(\alpha, f_\alpha) \to P(\gamma, f_{\gamma})$ and $b : P(\alpha', f_{\alpha'}) \to P(\gamma, f_{\gamma})$ in subcase II.

Recall from the Lemma 5.10, that the $\langle 1 \rangle$-orbits of $P(\alpha, f_\alpha)$ and $P(\alpha', f_{\alpha'})$ are respectively

$$
\cdots \to P(\alpha, f_\alpha)[-1] \to P(\alpha, f_\alpha) \to P(\beta, f_{\beta}) \to P(\gamma, f_{\gamma}) \to \cdots
$$

and

$$
\cdots \to P(\alpha', f_{\alpha'})[-1] \to P(\alpha', f_{\alpha'}) \to P(\beta', f_{\beta'}) \to P(\gamma, f_{\gamma}) \to \cdots
$$
Subcase I. We have the following
\[
\hom_{\Z_\P}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})) \\
= \bigoplus_{i=-\infty}^{\infty} \hom_{\Z_\P}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i)) \\
= \bigoplus_{i=-\infty}^{\infty} \hom_{\Z_\P}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i)) \oplus \bigoplus_{i=0}^{\infty} \hom_{\Z_\P}(P(\alpha, f_\alpha), P(\beta', f_{\beta'})(i)) \\
= \bigoplus_{i=-\infty}^{\infty} \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i)) \oplus \bigoplus_{i=0}^{\infty} \hom_{\K}(P(\alpha, f_\alpha), P(\beta', f_{\beta'})(i)).
\]

The last equality holds since we know from Fig. 12 that in this case no morphism in the two sums in the last line will factor through an object in \( \P \).

We claim that there is a bijection between the space
\[
\bigoplus_{i=0}^{\infty} \hom_{\K}(P(\alpha, f_\alpha), P(\beta', f_{\beta'})(i))
\]
and the space
\[
\bigoplus_{i=1}^{\infty} \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i)),
\]
and thus there is a bijection between the last sum of spaces in the above equation and the space
\[
\bigoplus_{i=-\infty}^{\infty} \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i)).
\]

Thus finally
\[
\dim \hom_{\Z_\P}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})) = \dim \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})).
\]

To prove this claim it is enough to show that for any \( i \geq 0 \), there is an one-to-one correspondence between \( \hom_{\K}(P(\alpha, f_\alpha), P(\beta', f_{\beta'})(i)) \) and \( \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})(i + 1)) \). However, this directly follows from the following two observations:

1. By assumption \( \beta' \) is the smoothing of \( \alpha' \) and \( \gamma' \). Thus if \( \alpha \) intersects \( \alpha' \) at some point \( \alpha p_{\alpha'} \) different from \( q_1 \) then \( \alpha \) must also intersect \( \beta' \) at some point \( \alpha p_{\beta'} \). Conversely, every intersection of \( \alpha \) with \( \beta' \) implies that there is an intersection of \( \alpha \) with \( \alpha' \) distinct from \( q_1 \).

Thus we have an one-to-one correspondence between the following sets, which maps \( \alpha p_{\alpha'} \) to \( \alpha p_{\beta'} \):

\[
\{ \text{oriented intersections from } \alpha \text{ to } \alpha' \text{ excepting the intersection at } q_1 \} \\
\downarrow \downarrow \\
\{ \text{oriented intersections from } \alpha \text{ to } \beta' \}.
\]

2. Note that, if we denote by \( s_{q_1}^{\beta'} \) respectively \( s_{q_1}^{\alpha'} \), the first intersection of \( \beta' \), respectively of \( \alpha' \), with \( \Delta^* \), then \( f_{\beta'}(s_{q_1}^{\beta'}) = f_{\alpha'}(s_{q_1}^{\alpha'}) - 1 \). This follows from the fact that \( \triangle(\gamma, \beta', \alpha') \) is a distinguished triangle and from the existence of the map \( b \) in the left picture of Fig. 12, so we can apply Lemma 3.3. So we have \( f_{\beta'}(\alpha p_{\beta'}) = f_{\alpha'}(\alpha p_{\alpha'}) - 1 \), and \( \alpha p_{\beta'} \) induces a map from \( P(\alpha, f_\alpha) \) to \( P(\beta', f_{\beta'})(i) \) if and only if \( \alpha p_{\alpha'} \) induces a map from \( P(\alpha, f_\alpha) \) to \( P(\alpha, f_{\alpha'})(i + 1) \). On the other hand, the intersection of \( \alpha \) and \( \alpha' \) at \( q_1 \), gives us a map in \( \hom_{\K}(P(\alpha, f_\alpha), P(\alpha', f_{\alpha'})) \).
However, note that we only consider the space $\text{Hom}_K(Z, P_{(\alpha, f_{\alpha})})$ for $i \geq 0$ in the claim.

To sum up, we have

$$\dim \text{Hom}_{P}(X, Y) = \dim \text{Hom}_{K}^{*}(X, Y) = |\text{Int}_S(\alpha, \alpha')|.$$ 

On the other hand, note that in this case we have

$$|\text{Int}_S(\alpha, \alpha')| = |\text{Int}_{S_{\gamma}}(\alpha, \alpha')|.$$ 

Therefore

$$\dim \text{Hom}_{P}(X, Y) = |\text{Int}_{S_{\gamma}}(\alpha, \alpha')|.$$ 

Subcase II. Similar to the subcase I, we have equalities

$$\text{Hom}_{P}(Z, P_{(\alpha, f_{\alpha})}) = \bigoplus_{i=-\infty}^{\infty} \text{Hom}_{P}(Z, P_{(\alpha', f_{\alpha'})}(i)).$$

$$= \bigoplus_{i=-\infty}^{0} \text{Hom}_{P}(Z, P_{(\alpha, f_{\alpha})}(i)) \oplus \bigoplus_{i=0}^{\infty} \text{Hom}_{P}(Z, P_{(\beta', f_{\beta'})}(i)) \oplus \bigoplus_{i=1}^{\infty} \text{Hom}_{P}(Z, P_{(\beta', f_{\beta'})}(i)).$$

To show the validity of the last equality, one notices that the only intersections between $\alpha$, $\alpha'$, $\beta$ and $\beta'$ with $\gamma$ are the ending points, so the linear extensions of the composition $ac$ are the only maps in the above Hom-space which factor through an object in $P$.

Then similar to the above statement for case I, for any $i \leq 0$, there is an one-to-one correspondence between $\text{Hom}_K(Z, P_{(\alpha, f_{\alpha})})$ and $\text{Hom}_K(Z, P_{(\beta', f_{\beta'})}(i))$. Therefore we have an one-to-one correspondence between $\dim \text{Hom}_{P}(Z, P_{(\alpha, f_{\alpha})})$ and $\dim \text{Hom}_{K}^{*}(P_{(\alpha, f_{\alpha})}, P_{(\beta', f_{\beta'})})/(ac)$. Thus

$$\dim \text{Hom}_{P}(X, Y) = |\text{Int}_S(\alpha, \beta')| - 1 = |\text{Int}_S(\alpha, \alpha')|$$

On the other hand, note that $|\text{Int}_S(\alpha, \alpha')| = |\text{Int}_{S_{\gamma}}(\alpha, \alpha')|$. Therefore

$$\dim \text{Hom}_{P}(X, Y) = |\text{Int}_{S_{\gamma}}(\alpha, \alpha')|.$$

\[ \square \]

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