RELATIVE EXT GROUPS, RESOLUTIONS, AND SCHANUEL CLASSES

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Abstract. Given a precovering (also called contravariantly finite) class \( F \) there are three natural approaches to a homological dimension with respect to \( F \): One based on Ext functors relative to \( F \), one based on \( F \)-resolutions, and one based on Schanuel classes relative to \( F \). In general these approaches do not give the same result. In this paper we study relations between the three approaches above, and we give necessary and sufficient conditions for them to agree.

0. Introduction

The fact that the category of modules over any ring \( R \) has enough projectives is a cornerstone in classical homological algebra. The existence of enough projective modules has three important consequences:

- To every module \( A \), and integer \( n \) one can define the Ext functor,
  \[ \text{Ext}_R^n(-, A), \]
  with well-known properties, see [1, chap. V].

- Every module \( M \) admits a projective resolution, cf. [1, chap. V]:
  \[ \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0. \]

- Every module \( M \) represents a projective equivalence class \([M]\), and to this one can associate its Schanuel class,
  \[ S([M]) := [\ker \pi], \]
  where \( \pi : P \rightarrow M \) is any epimorphism and \( P \) is projective. One can also consider the iterated Schanuel maps \( S^n(-) \) for \( n \geq 0 \), see Schanuel’s lemma [4, chap. 4, thm. A].

The three fundamental types of objects described above — Ext functors, projective resolutions, and Schanuel classes — are linked together as nicely as one could hope for, in the sense of the following well-known result (see [1, chap. V, prop. 2.1]):

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**Theorem A.** For any $R$–module $M$, and any integer $n \geq 0$ the following conditions are equivalent:

(i) $\text{Ext}_R^{n+1}(M, A) = 0$ for all $R$–modules $A$.

(ii) There exists a projective resolution for $M$ of length $n$,

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0.$$ 

(iii) $S^n([M]) = [0]$.

The equivalent conditions of the theorem above define what it means for a module $M$ to have projective dimension $\leq n$.

In relative homological algebra one substitutes the class of projective modules by any other precovering class $F$, see [1,2]. The fact that $F$ is precovering allows for well-defined constructions of:

- Ext functors $\text{Ext}_F^n(–, A)$ relative to $F$;
- $F$–resolutions, $\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$; and
- Schanuel maps $S^n_F(–)$ relative to $F$.

The constructions of these relative objects are well-known, see for example [2, chap. 8] and [3, lem. 2.2], but for the benefit of the reader we give a short recapitulation in Section 1.

Now, one could hope that there might exist an “$F$–version” of Theorem A, indeed, one would need such a theorem to have a rich and flexible notion of an $F$–dimension. Unfortunately, Theorem A fails for a general precovering class $F$! The aim of this paper is to understand, for a given precovering class $F$, the different kind of obstructions which keep Theorem A from being true.

In Section 2 we investigate how the Ext condition (i) and the resolution condition (ii) in the $F$–version of Theorem A are related:

It is trivial that (ii) $\Rightarrow$ (i) holds always, so we restrict our attention to the converse implication. In Lemma (2.3) we give a necessary condition for (i) $\Rightarrow$ (ii). In Theorem (2.9) we give a sufficient condition for (i) $\Rightarrow$ (ii) in terms of almost epi precovers. We also give concrete examples of precovering classes for which the implication (i) $\Rightarrow$ (ii) fails, and others for which it holds.

In Section 3 we study how the resolution condition (ii) and the Schanuel condition (iii) in the $F$–version of Theorem A are related:

The main results are Theorems (3.4) and (3.8) which give necessary and sufficient conditions for the implication (iii) $\Rightarrow$ (ii), respectively, (ii) $\Rightarrow$ (iii), to hold. We also present concrete examples of precovering classes for which the implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) fail.
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1. Preliminaries

(1.1) Setup. Throughout, \( R \) will be a ring, and all modules will be left \( R \)-modules. We write \( \text{Mod} \, R \) for the category of (left) \( R \)-modules, and \( \text{Ab} \) for the category of abelian groups.

\( \mathcal{F} \) will be any precovering class of modules, cf. (1.2) below, which contains 0 and is closed under isomorphism and finite direct sums.

(1.2) Precovering classes. For definitions and results on precovering classes we generally follow [2, chap. 5 and 8]. We mention here just a few notions which will be important for this paper.

Let \( \mathcal{F} \) be a class of modules. An \( \mathcal{F} \)-precover of a module \( M \) is a homomorphism \( F \to M \) with \( F \in \mathcal{F} \), such that given any other homomorphism \( F' \to M \) with \( F' \in \mathcal{F} \) then there exists a factorization,

\[
\begin{array}{c}
F' \\
\downarrow \\
F \\
\rightarrow \\
M.
\end{array}
\]

If every module admits an \( \mathcal{F} \)-precover then \( \mathcal{F} \) is called precovering. An (augmented) \( \mathcal{F} \)-resolution of a module \( M \) is a complex (which is not necessarily exact),

\[
\cdots \to F_2 \overset{\partial_2}{\to} F_1 \overset{\partial_1}{\to} F_0 \overset{\partial_0}{\to} M \to 0,
\]

with \( F_0, F_1, F_2, \ldots \in \mathcal{F} \), such that

\[
\cdots \to (F, F_2) \overset{(F, \partial_2)}{\to} (F, F_1) \overset{(F, \partial_1)}{\to} (F, F_0) \overset{(F, \partial_0)}{\to} (F, M) \to 0
\]

is exact for all \( F \in \mathcal{F} \). When \( \mathcal{F} \) is precovering, and \( T : \text{Mod} \, R \to \text{Ab} \) is a contravariant additive functor, then one can well-define the \( n \)'th right derived functor of \( T \) relative to \( \mathcal{F} \),

\[
R^n_\mathcal{F} T : \text{Mod} \, R \to \text{Ab}.
\]

One computes \( R^n_\mathcal{F} T(M) \) by taking an non-augmented \( \mathcal{F} \)-resolution of \( M \), applying \( T \) to it, and then taking the \( n \)'th cohomology group of the resulting complex. For a module \( A \) we write:

\[
\text{Ext}_\mathcal{F}^n (\cdot, A) = R^n_\mathcal{F} \text{Hom}_R (\cdot, A).
\]
Note that we underline the Ext for good reasons: There is also a notion of a preenveloping class. If $G$ is preenveloping then one can right derive the Hom functor in the covariant variable with respect to $G$. Thus for each $R$–module $B$ there are functors $\text{Ext}^n_G(B, -)$. However, in general,

$$\text{Ext}^n_F(B, A) \neq \text{Ext}^n_G(B, A)$$

even if $F = G$ is both precovering and preenveloping.

(1.3) $F$–equivalence. Two modules $K$ and $K'$ are called $F$–equivalent, and we write $K \equiv_F K'$, if there exist $F, F' \in F$ with $K \oplus F' \cong K' \oplus F$. We use $[K]$ to denote the $F$–equivalence class containing $K$.

Now let $M$ be any module. By the version of Schanuel’s lemma found in [3, lem. 2.2], the kernels of any two $F$–precovers of $M$ are $F$–equivalent. Thus the class $[\text{Ker } \varphi]$, where $\varphi : F \rightarrow M$ is any $F$–precover of $M$, is a well-defined object depending only on $M$. We write

$$\mathcal{S}_F(M) = [\text{Ker } \varphi].$$

As $F$ is closed under finite direct sums; cf. Setup (1.1), it is not hard to see that $\mathcal{S}_F(M)$ only depends on the $F$–equivalence class of $M$, and hence we get the induced Schanuel map:

$$\text{Mod } R/ \equiv_F \xrightarrow{\mathcal{S}_F} \text{Mod } R/ \equiv_F,$$

For $n > 0$ we write $\mathcal{S}_F^n$ for the $n$–fold composition of $\mathcal{S}_F$ with itself, and we set $\mathcal{S}_F^0 = \text{id}$.

This paper is all about studying relations between the conditions from the following definition.

(1.4) Definition. For any module $M$ and any integer $n \geq 0$ we consider the conditions:

$(E_{M,n})$ $\text{Ext}^{n+1}_F(M, A) = 0$ for all modules $A$.

$(R_{M,n})$ There exists an $F$–resolution of the form

$$0 \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0.$$

$(S_{M,n})$ $\mathcal{S}_F^n([M]) = [0].$

(1.5) Remark. The conditions in Definition (1.4) are labeled according to the following mnemonic rules: “$E$” is for Ext, “$R$” is for Resolution, and “$S$” is for Schanuel.
2. Relative Ext functors and resolutions

In this section we study how the Ext condition and the resolution condition of Definition (1.4) are related. It is straightforward, cf. Proposition (2.1) below, that the resolution condition implies the Ext condition. The converse is, in general, not true, but in Theorem (2.9) we give a sufficient condition on $F$ for this to happen.

(2.1) Proposition. For any precovering class $F$ we have:

$$(R_{M,n}) \implies (E_{M,n}) \text{ for all modules } M \text{ and all integers } n \geq 0.$$  

□

(2.2) Example. There exist precovering classes which are not closed under direct summands:

Let $R$ be a left noetherian ring which is not Quasi–Frobenius, and set $D = R \oplus E$ where $E$ is any non-zero injective $R$–module. Define $F$ to be the class of all modules which are isomorphic to $D^{(\Lambda)}$ for some index set $\Lambda$ (here $D^{\emptyset} = 0$). Note that $F$ is precovering as for example an $F$–precover of a module $M$ is given by the natural map

$$D^{(\text{Hom}_R(D,M))} \longrightarrow M.$$  

To see that $F$ is not closed under direct summands we note that $E$ is a direct summand of $D \in F$. However, there exists no set $\Lambda$ for which $E \cong D^{(\Lambda)}$ (since $R$ is a direct summand of $D^{(\Lambda)}$ for any $\Lambda \neq \emptyset$, and since $R$ is not self-injective).

The example above makes the following lemma relevant:

(2.3) Lemma. A necessary condition for $F$ to satisfy the implication:

$$(E_{M,0}) \implies (R_{M,0}) \text{ for all modules } M,$$

is that $F$ is closed under direct summands.

Proof. Assume that $F$ is not closed under direct summands. Then there exists an $F \in F$ and a direct summand $M$ of $F$ with $M \not\in F$. We claim that $(E_{M,0})$ holds but that $(R_{M,0})$ does not:

As $M$ is a direct summand of $F$, and as $F$ is closed under finite direct sums, cf. Setup (1.1), the abelian group $\text{Ext}_1^F(M, A)$ is a direct summand of $\text{Ext}_1^F(F, A)$ for every module $A$. The latter is zero as $F \in F$, and hence also $\text{Ext}_1^F(M, A) = 0$. Now suppose for contradiction that there do exist an $F$–resolution of $M$ of length zero:

$$0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$
We claim that \( \partial_0 \) must be an isomorphism (contradicting the fact that \( M \not\in \mathcal{F} \)). As \( M \) is a direct summand of \( F \) there is a canonical embedding \( \iota: M \to F \) and a canonical projection \( \pi: F \to M \) with \( \pi \iota = \text{id}_M \). As \( \partial_0 \) is an \( \mathcal{F} \)-precover of \( M \), we get a factorization:

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & M \\
\downarrow{\pi} & & \downarrow{\pi} \\
F_0 & \xrightarrow{\partial_0} & M
\end{array}
\]

It follows that \( \partial_0(\varphi \iota) = \pi \iota = \text{id}_M \), so \( \partial_0 \) is epi and the sequence

\[
0 \to \text{Ker} \partial_0 \xrightarrow{\iota} F_0 \xrightarrow{\varphi} M \to 0
\]

splits. By assumption, \( \text{Hom}_R(G, \partial_0) \) is mono for all \( G \in \mathcal{F} \), so by (†) it follows that \( \text{Hom}_R(G, \text{Ker} \partial_0) = 0 \) for all \( G \in \mathcal{F} \). In particular,

\[
\text{Hom}_R(F_0, \text{Ker} \partial_0) = 0,
\]

and therefore \( \text{Ker} \partial_0 = 0 \) since \( \text{Ker} \partial_0 \) is a direct summand of \( F_0 \). Consequently, \( \partial_0 \) is an isomorphism.

(2.4) **Lemma.** For a homomorphism \( \varphi: F \to M \) the following two conditions are equivalent:

(a) Every endomorphism \( g: M \to M \) with \( g \varphi = \varphi \) is an automorphism.

(b) Every endomorphism \( g: M \to M \) with \( g \varphi = \varphi \) admits a left inverse.

**Proof.** We only need to show that (b) implies (a). Thus assume that \( g \varphi = \varphi \). By assumption (b) there exists a homomorphism \( v: M \to M \) with \( vg = \text{id}_M \). Now

\[
vg = v \varphi = \text{id}_M \varphi = \varphi,
\]

so another application of (b) gives that also \( v \) has a left inverse. As \( v \) has \( g \) as a right inverse, \( v \) must be an automorphisms with \( v^{-1} = g \).  

(2.5) **Definition.** A homomorphism \( \varphi: F \to M \) satisfying the equivalent conditions of Lemma (2.4) is called *almost epi*. The precovering class \( \mathcal{F} \) is called *precovering by almost epimorphisms* if every module has an \( \mathcal{F} \)-precover which is almost epi.
(2.6) **Example.** Clearly, every epimorphism is almost epi, but the converse is in general not true, as for example
\[ \mathbb{Z} \xrightarrow{2} \mathbb{Z} \]
is an almost epimorphism of abelian groups. It follows from Lemma (2.7) below that if a precovering class contains all free modules, then it is precovering by almost epimorphisms.

(2.7) **Lemma.** If there exists an almost epi homomorphism \( \varphi: F \rightarrow M \) with \( F \in \mathcal{F} \) then every \( \mathcal{F} \)-precover of \( M \) is almost epi.

**Proof.** If \( \tilde{\varphi}: \tilde{F} \rightarrow M \) is any \( \mathcal{F} \)-precover of \( M \) then there exists a factorization,
\[ \begin{array}{ccc}
\psi & & \varphi \\
\downarrow & & \downarrow \\
\tilde{F} & \xrightarrow{\tilde{\varphi}} & M.
\end{array} \]
For any endomorphism \( g: M \rightarrow M \) with \( g\tilde{\varphi} = \tilde{\varphi} \) it follows that
\[ g\varphi = g\tilde{\varphi}\psi = \tilde{\varphi}\psi = \varphi, \]
and hence \( g \) must be an automorphism since \( \varphi \) is almost epi.

The next result gives much more information than Example (2.6), namely that there do indeed exist module classes \( \mathcal{F} \) which are precovering by almost epimorphisms, without every \( \mathcal{F} \)-precover being epi. We postpone the proof of Proposition (2.8) to the end of this section.

(2.8) **Proposition.** Consider the local ring \( R = \mathbb{Z}/4\mathbb{Z} \). We denote the generator \( 2 + 4\mathbb{Z} \) of the maximal ideal by \( \xi \), and the residue class field \( R/(\xi) \cong \mathbb{F}_2 \) by \( k \).

Furthermore, let \( \mathcal{F} = \text{Add } k \) be the class of all direct summands of set-indexed coproducts of copies of \( k \). Then the following hold:

(a) \( \mathcal{F} \) is precovering by almost epimorphisms, cf. Definition (2.5).
(b) \( R \) does not admit an epi \( \mathcal{F} \)-precover.

The reason we are interested in classes which are precovering by almost epimorphisms is because of the next result:

(2.9) **Theorem.** Assume that \( \mathcal{F} \) is closed under direct summands and is precovering by almost epimorphisms. Then
\[ (E_{M,n}) \implies (R_{M,n}) \text{ for all modules } M \text{ and all integers } n \geq 0. \]
Proof. First we deal with the case \( n = 0 \): Thus let \( M \) be any module, and assume that \( \text{Ext}_F^1(M, A) = 0 \) for all modules \( A \). We must prove the existence of an \( F \)–resolution of \( M \) of length zero,
\[
0 \longrightarrow G_0 \longrightarrow M \longrightarrow 0.
\]
By assumption on \( F \) we can build an \( F \)–resolution of \( M \) by successively taking almost epi \( F \)–precovers \( \varphi_0, \varphi_1, \varphi_2, \ldots \):
\[
\begin{array}{cccccc}
0 & \rightarrow & K_1 & \rightarrow & 0 & \rightarrow \\
& \varphi_2 & \downarrow i_1 & \rightarrow & \varphi_0 & \rightarrow \\
& \rightarrow & F_2 & \rightarrow & F_1 & \rightarrow \\
\ldots & \rightarrow & F_2 & \rightarrow & F_1 & \rightarrow \\
& \downarrow \partial_2 & \downarrow \partial_1 & \rightarrow & \partial_0 & \rightarrow \\
& & M & \rightarrow & M & \rightarrow \\
& & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
& & K_0 & \rightarrow & 0 & \rightarrow \\
& & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow
\end{array}
\]
We keep in mind that the \( F \)–precovers \( \varphi_n \) are not necessarily epi, and this is the reason why some of the arrows in the diagram above have been dotted. Applying \( \text{Hom}_R(-, A) \), for any module \( A \), to the \( \text{Hom}_R(F, -) \) exact complex,
\[
0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0,
\]
induces by [2, thm. 8.2.3(2)] an exact sequence of relative \( \text{Ext} \) groups,
\[
(*) \quad \text{Ext}_F^0(F_0, A) \xrightarrow{q} \text{Ext}_F^0(K_0, A) \rightarrow \text{Ext}_F^1(M, A) = 0.
\]
As \( F_0 \in F \) we have \( \text{Ext}_F^0(F_0, A) = \text{Hom}_R(F_0, A) \). Furthermore,
\[
\text{Ext}_F^0(K_0, A) = \text{Ker} \text{Hom}_R(\partial_2, A)
\]
\[
= \{ f \in \text{Hom}_R(F_1, A) \mid f \partial_2 = 0 \},
\]
and one verifies that the homomorphism \( q \) is given by \( g \mapsto g\partial_1 \) for \( g \in \text{Hom}_R(F_0, A) \). Applying these considerations to \( A = K_0 \) and considering \( \varphi_1 \in \text{Ext}_F^0(K_0, K_0) \), exactness of (*) implies the existence of a \( g \in \text{Hom}_R(F_0, K_0) \) with \( g\partial_1 = \varphi_1 \), that is, \( gi_0\varphi_1 = \varphi_1 \). As \( \varphi_1 \) is almost epi, \( gi_0 : K_0 \rightarrow K_0 \) must be an automorphism, and hence the sequence
\[
0 \longrightarrow K_0 \xrightarrow{i_0} F_0 \xrightarrow{\pi_0} F_0/K_0 \longrightarrow 0
\]
is split exact. In particular, $F_0/K_0 \in \mathcal{F}$ as $\mathcal{F}$ is closed under direct summands, and we claim that the induced monomorphism,

$$
\begin{array}{ccc}
F_0/K_0 & \xrightarrow{\varphi_0} & M \\
\pi_0 & \downarrow & \varphi_0 \\
F_0 & \xrightarrow{\phi} & M
\end{array}
$$

is an $\mathcal{F}$–precover of $M$. To see this let $\phi: F \to M$ be a homomorphism with $F \in \mathcal{F}$. As $\phi_0: F_0 \to M$ is an $\mathcal{F}$–precover there exists $\psi: F \to F_0$ with $\varphi_0 \psi = \phi$. Consequently, $\pi_0 \psi: F \to F_0/K_0$ satisfies $\varphi_0(\pi_0 \psi) = \phi$.

Now, as $\varphi_0$ is a mono $\mathcal{F}$–precover of $M$,

$$0 \to F_0/K_0 \xrightarrow{\varphi_0} M \to 0$$

is an $\mathcal{F}$–resolution of $M$ of length zero.

Finally we consider the case $n > 0$: We assume that $\text{Ext}^{n+1}_F(M, A) = 0$ for all modules $A$, and we must prove the existence of an $\mathcal{F}$–resolution of $M$ of length $n$. Let $\partial_0: F_0 \to M$ be an $\mathcal{F}$–precover of $M$. By [2, thm. 8.2.3(2)] the Hom$_R(F, -)$ exact complex

$$(\dagger) \quad 0 \to \text{Ker} \partial_0 \to F_0 \xrightarrow{\partial_0} M \to 0$$

induces, for any module $A$, a long exact sequence of relative Ext groups:

$$0 = \text{Ext}^n_F(F_0, A) \to \text{Ext}^n_F(\text{Ker} \partial_0, A) \to \text{Ext}^{n+1}_F(M, A) = 0.$$  

It follows that

$$\text{Ext}^{(n-1)+1}_F(\text{Ker} \partial_0, A) = \text{Ext}^n_F(\text{Ker} \partial_0, A) = 0,$$

so the induction hypothesis implies that $\text{Ker} \partial_0$ admits an $\mathcal{F}$–resolution of length $n - 1$, say,

$$(\ddagger) \quad 0 \to F_n \to \cdots \to F_1 \to \text{Ker} \partial_0 \to 0.$$  

Pasting together $(\dagger)$ and $(\ddagger)$ we get the desired $\mathcal{F}$–resolution of $M$ of length $n$. 

Proof of Proposition (2.8). Note that $R$ is a two-dimensional $k$–vector space with basis $\{1, \xi\}$, so every element of $R$ has a unique representation of the form $a + b \xi$ where $a, b \in k \cong \mathbb{F}_2$. 
Just as in Example (2.2) it follows that $\mathbf{F} = \text{Add} \, k$ is precovering, but we shall also prove this more directly below.

It is useful to observe that a homomorphism $F \to M$ with $F \in \mathbf{F}$ is an $\mathbf{F}$–precover of $M$ if and only if every homomorphism $k \to M$ admits a factorization:

$$(z) \quad \begin{array}{c}
\phi_c \\
\downarrow \\
\varphi_1 \\
\varphi_c \\
k \quad \varphi_1 \\
\downarrow \\
k \\
F \\
\downarrow \\
M.
\end{array}$$

One important consequence of this is that if $F_j \to M_j$ is a family of $\mathbf{F}$–precovers then the coproduct $\coprod_j F_j \to \coprod_j M_j$ is again an $\mathbf{F}$–precover.

For every $c \in k$ there is an $\mathbf{R}$–linear map $\varphi_c : k \to R$, $a \mapsto ac\xi$, and it is not hard to see that, in fact, every $\mathbf{R}$–linear map $k \to R$ has the form $\varphi_c$ for some $c \in k$. Combining this with the commutative diagram

$$(z) \quad \begin{array}{c}
\phi_c \\
\downarrow \\
\varphi_1 \\
\varphi_c \\
k \\
\downarrow \\
k \\
F \\
\downarrow \\
M.
\end{array}$$

observation (z) implies that $\varphi_1 : k \to R$ is an $\mathbf{F}$–precover of $R$. Since $\varphi_1$ is not epi, $R$ cannot be the homomorphic image of any module from $\mathbf{F}$, and this proves (b) from the proposition.

We are now ready to prove part (a) of the proposition, namely that every $\mathbf{R}$–module admits an almost epi $\mathbf{F}$–precover. It is well-known\footnote{The author is convinced that this result and its natural generalizations must be folklore. However, since the author was not able to find a reference, a quick argument is given below.} that every $\mathbf{R}$–module is isomorphic to one of the form

$$k^{(I)} \oplus R^{(J)}$$

Let $M \neq 0$ be any $\mathbf{R}$–module. Since $R = \mathbb{Z}/4\mathbb{Z}$ only has the two proper ideals $(0)$ and $(\xi)$ there are two possibilities:

1. for every $0 \neq x \in M$ we have $\text{Ann}_R(x) = (\xi)$, or
2. there exists $0 \neq x \in M$ with $\text{Ann}_R(x) = (0)$.

In case (1) we can consider $M$ as a module over the field $k = R/(\xi)$, and it follows that $M \cong k^{(I)}$ for some index set $I$. In case (2) there is a monomorphism $R \to M$, and since $R$ is self-injective it follows that $R$ is a direct summand of $M$. Using Zorn’s lemma we find a maximal free (=injective) direct summand $R^{(J)}$ of $M$, and hence we can write $M = M' \oplus R^{(J)}$ where $M'$ satisfies condition (1).
for suitable index sets $I$ and $J$. Hence we only need to show that the module $k^{(I)} \oplus R^{(J)}$ has an almost epi $F$–precover. By the observation (z) it follows that

$$k^{(I)} \oplus k^{(J)} \xrightarrow{\varphi = \begin{pmatrix} \text{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix}} k^{(I)} \oplus R^{(J)}$$

is an $F$–precover. To argue that $\varphi$ is almost epi we let

$$k^{(I)} \oplus R^{(J)} \xrightarrow{g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} k^{(I)} \oplus R^{(J)}$$

be any endomorphism with $\varphi = g \varphi$. We must prove that $g$ is an automorphism. By assumption,

$$\begin{pmatrix} \text{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \text{id}_{k^{(I)}} & 0 \\ 0 & \varphi_1^{(J)} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \varphi_1^{(J)} \\ g_{21} & g_{22} \varphi_1^{(J)} \end{pmatrix}.$$

In particular it follows that $g_{11} = \text{id}_{k^{(I)}}$ and $g_{21} = 0$, so $g$ takes the form

$$g = \begin{pmatrix} \text{id}_{k^{(I)}} & g_{12} \\ 0 & g_{22} \end{pmatrix}.$$

If we can prove that $g_{22} : R^{(J)} \rightarrow R^{(J)}$ is an automorphism, then $g$ must be an automorphism as well with inverse

$$g^{-1} = \begin{pmatrix} \text{id}_{k^{(I)}} & -g_{12} g_{22}^{-1} \\ 0 & g_{22}^{-1} \end{pmatrix}.$$

To see that $g_{22}$ is an automorphism we use another relation from (z), namely that $\varphi_1^{(J)} = g_{22} \varphi_1^{(J)}$. As

$$g_{22} \in \text{Hom}_R(R^{(J)}, R^{(J)}) \cong \left(R^{(J)}\right)^J$$

it follows — if we consider the elements of $R^{(J)}$ as $J$–columns — that $g_{22}$ is given by multiplication from the left by a unique $J \times J$–matrix $(r_{ij})_{i,j \in J}$ with entries from $R$, in which each column $(r_{ij})_{i \in J}$ belongs to $R^{(J)}$. More precisely, $g_{22}$ is given by the formula:

$$R^{(J)} \ni \{s_j\}_{j \in J} \longmapsto (r_{ij})_{i,j \in J} \{s_j\}_{j \in J} = \left\{ \sum_{j \in J} r_{ij} s_j \right\}_{i \in I} \in R^{(J)}.$$

Of course, $\varphi_1^{(J)} : k^{(J)} \rightarrow R^{(J)}$ is given by the $J \times J$–diagonal matrix $\Delta_J \times_J (\varphi_1)$ with $\varphi_1$ in every diagonal entry, and hence $g_{22} \varphi_1^{(J)}$ is given
by the matrix

\[(r_{ij})_{i,j \in J} \Delta_{J \times J}(\varphi_1) = (r_{ij}\varphi_1)_{i,j \in J}.\]

By assumption \(g_{22}(\varphi_1)^{(J)} = \varphi_1^{(J)},\) and consequently we have an equality of \(J \times J\)-matrices:

\[(r_{ij}\varphi_1)_{i,j \in J} = \Delta_{J \times J}(\varphi_1).\]

It follows that:

\[r_{jj}\varphi_1 = \varphi_1 \quad \text{and} \quad r_{ij}\varphi_1 = 0, \quad i \neq j.\]

Now writing \(r_{ij} = a_{ij} + b_{ij}\xi\) with \(a_{ij}, b_{ij} \in k\) and applying the maps above to \(1 \in k\) we get

\[(a_{jj} + b_{jj}\xi)\xi = \xi \quad \text{and} \quad (a_{ij} + b_{ij}\xi)\xi = 0, \quad i \neq j.\]

We see that \(a_{jj} = 1\) and \(a_{ij} = 0\) for \(i \neq j,\) that is,

\[r_{jj} = 1 + b_{jj}\xi \quad \text{and} \quad r_{ij} = b_{ij}\xi, \quad i \neq j.\]

With this information at hand we can see that \(g_{22} = (r_{ij})_{i,j \in J}\) is invertible, in fact, \(g_{22}\) is its own inverse. Let us simply calculate the \((i, j)\)'th entry, \(q_{ij},\) in the product matrix \(g_{22}g_{22}:\) Using that \(\xi^2 = 0\) and that the field \(k \cong \mathbb{F}_2\) has characteristic 2 it follows that:

\[q_{ij} = \sum_{\nu \in J} r_{i\nu}r_{\nu j}\]

\[= \begin{cases} (1 + b_{jj}\xi)^2 & \text{for } i = j \\ (1 + b_{ii}\xi)b_{ij}\xi + b_{ij}(1 + b_{jj}\xi) & \text{for } i \neq j \end{cases} \]

\[= \begin{cases} 1 + 2b_{jj}\xi & \text{for } i = j \\ 2b_{ij}\xi & \text{for } i \neq j \end{cases} \]

\[= \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \]

as desired. \(\square\)

3. Relative resolutions and Schanuel maps

In this section we study how the resolution condition and the Schanuel condition of Definition (1.4) are related. In general, neither of these two conditions imply the other, however, in Theorems (3.4) and (3.8) we give necessary and sufficient conditions for this phenomenon to happen.

(3.1) **Definition.** We say that \(F\) is **weakly closed under direct summands** if for any \(F \in F\) and any direct summand \(M\) in \(F\) with \(F/M \in F,\) the module \(M\) belongs to \(F.\)
Example. There exist precovering classes which are not weakly closed under direct summands:

The precovering class $F$ from Example (2.2) is not closed under direct summands. As $F$ is closed under set-indexed coproducts, it follows from Proposition (3.3) below that $F$ is not weakly closed under direct summands either.

Proposition. A precovering class $F$ is closed under direct summands if and only if $F$ is weakly closed under direct summands and closed under set-indexed (respectively, countable) coproducts in $\text{Mod } R$.

Proof. “If”: Let $M$ be a direct summand of $F \in F$, that is, there exists some module $M'$ with $F = M \oplus M'$. Using Eilenberg’s swindle we consider $F^{(\mathbb{N})}$ and note that

\[(*) \quad M \oplus F^{(\mathbb{N})} \cong F^{(\mathbb{N})}.\]

As $F$ is closed under countable coproducts, $F^{(\mathbb{N})} \in F$, and then $(*)$ implies that $M \in F$ since $F$ is weakly closed under direct summands.

“Only if”: If $F$ is closed under direct summands then obviously $F$ is also weakly closed under direct summands. Since $F$ is precovering and closed under direct summands, the argument in [2, proof of thm. 5.4.1, (2)$\Rightarrow$(1)] shows that $F$ is closed under set-indexed coproducts. \(\square\)

The reason we are interested in classes which are weakly closed under direct summands is because of the next result.

Theorem. A precovering class $F$ satisfies:

\[(z) \quad (S_{M,n}) \Rightarrow (R_{M,n}) \text{ for all modules } M \text{ and all integers } n \geq 0\]

if and only if $F$ is weakly closed under direct summands.

Proof. “Only if”: Under the assumption of $(z)$ we must prove that $F$ is weakly closed under direct summands. Thus, let $M$ be a direct summand of a module $F$ where $F, F/M \in F$. As

\[M \oplus F/M \cong 0 \oplus F\]

we see that $M$ is $F$–equivalent to 0, that is, $S^0_F([M]) = [M] = [0]$. Now the assumption $(z)$ implies the existence of an $F$–resolution of $M$ of length zero,

\[(*) \quad 0 \rightarrow F_0 \xrightarrow{\partial_0} M \rightarrow 0.\]

As in the end of the proof of Lemma (2.3) we see that $\partial_0$ is an isomorphism, and hence $M \cong F_0 \in F$ as desired.
"If": Now assume that $\mathcal{F}$ is weakly closed under direct summands. We will prove (\textcircled{1}) by induction on $n \geq 0$.

We begin with the case $n = 0$: Suppose that $\mathcal{S}_F^n([M]) = [M] = [0]$. By definition there exist $F', F \in \mathcal{F}$ with $M \oplus F' \cong 0 \oplus F \cong F$, and since $\mathcal{F}$ is weakly closed under direct summands it follows that $F_0 := M \in \mathcal{F}$. Thus $M$ admits an $\mathcal{F}$–resolution of $M$ of length zero:

$$0 \to F_0 \overset{\text{id}_M}{\to} M \to 0.$$ 

Next we consider the case $n > 0$: Suppose that $\mathcal{S}_F^n([M]) = [0]$, and take an $\mathcal{F}$–precover $\partial: F_0 \to M$. By definition, $\mathcal{S}_F^{n-1}(\text{Ker } \partial) = \mathcal{S}_F^{n-1} \mathcal{S}_F^n([M]) = \mathcal{S}_F^n([M]) = [0]$, so the induction hypothesis implies the existence of an $\mathcal{F}$–resolution of $\text{Ker } \partial$ of length $n - 1$, say,

$$(b) \quad 0 \to F_n \to \cdots \to F_1 \to \text{Ker } \partial \to 0.$$ 

Pasting together $(b)$ with the complex

$$0 \to \text{Ker } \partial \to F_0 \overset{\partial}{\to} M \to 0$$

gives an $\mathcal{F}$–resolution of $M$ of length $n$, as desired. 

(3.5) Definition. A (precovering) class $\mathcal{F}$ is said to be separating if for every module $M \neq 0$ there exists a non-zero homomorphism $F \to M$ with $F \in \mathcal{F}$.

(3.6) Lemma. For a precovering class $\mathcal{F}$ the following hold:

(a) If every mono $\mathcal{F}$–precover is an isomorphism then $\mathcal{F}$ is separating.

(b) If $\mathcal{F}$ is separating and $\partial: A \to B$ is a homomorphism such that $\text{Hom}_R(F, \partial)$ is mono for all $F \in \mathcal{F}$, then $\partial$ is mono.

Proof. "(a)" Assume that every mono $\mathcal{F}$–precover is an isomorphism, and let $M$ be a module with $\text{Hom}_R(F, M) = 0$ for all $F \in \mathcal{F}$. Thus the map $0 \to M$ is a mono $\mathcal{F}$–precover, and hence an isomorphism by assumption, that is, $M = 0$.

"(b)" Applying the left exact functor $\text{Hom}_R(F, -)$, for any $F \in \mathcal{F}$, to the exact sequence,

$$0 \to \text{Ker } \partial \to A \overset{\partial}{\to} B$$

and using that $\text{Hom}_R(F, \partial)$ is mono, we get that $\text{Hom}_R(F, \text{Ker } \partial) = 0$. As $\mathcal{F}$ is separating it follows that $\text{Ker } \partial = 0$, that is, $\partial$ is mono. \qed
Example. We give two examples of precovering classes \( F \) for which there exist mono \( F \)-precovers which are not isomorphisms:

(a) Let \( R \) be a commutative noetherian ring which is not artinian. As \( R \) is noetherian the class \( F = \text{Inj} R \) of injective \( R \)-modules is precovering by \([2, \text{thm. 5.4.1}]\). However, as \( R \) is not artinian, \( F \) is not separating by \([5, \text{cor. 2.4.11}]\), and hence Lemma \((3.6)\)(a) implies that there must exists mono \( F \)-precovers which are not isomorphisms.

(b) Let \( R \) be a commutative integral domain, and consider for any module \( M \) its torsion submodule,

\[ M_T = \{ x \in M \mid rx = 0 \text{ for some } r \in R \setminus \{0\} \}. \]

A module \( M \) is called torsion if \( M_T = M \), and of course the torsion submodule of any module is torsion.

The torsion modules constitutes a precovering class, in fact, given a module \( M \) it is not hard to see that the inclusion \( M_T \to M \) is a torsion precover of \( M \). In particular, \( 0 = R_T \to R \) is a mono torsion precover of \( R \), but it is not an isomorphism.

The following result shows why we are interested in precovering classes for which every mono precover is an isomorphism.

Theorem. A precovering class \( F \) satisfies:

\((\flat)\) \( (R_{M,n}) \to (S_{M,n}) \) for all modules \( M \) and all integers \( n \geq 0 \)

if and only if every mono \( F \)-precover is an isomorphism.

Proof. “Only if”: Assume \((\flat)\). We must prove that every mono \( F \)-precover is an isomorphism. Any mono \( F \)-precover \( \varphi: F_0 \to M \) gives an \( F \)-resolution of \( M \) of length zero:

\[ 0 \to F_0 \xrightarrow{\varphi} M \to 0, \]

and thus our assumption ensures that \( S_F^0([M]) = [M] = [0] \). This means that \( M \) is a direct summand of some \( F \in F \) with a quotient \( F/M \in F \). In particular, \( M \) is a homomorphic image of \( F \in F \), and this implies that the \( F \)-precover \( \varphi \) must be epi. Consequently, \( \varphi \) is an isomorphism.

“If”: Conversely, assume that every mono \( F \)-precover is an isomorphism. We must show \((\flat)\), which we do by induction on \( n \geq 0 \):

We begin with the case \( n = 0 \). Thus, let \( M \) be any module for which there exists an \( F \)-resolution of length zero:

\[(*)\quad 0 \to F_0 \xrightarrow{\partial} M \to 0.\]
We must argue that $S^n_F([M]) = [0]$. Actually, we prove something even stronger, namely that $M \in \mathcal{F}$. Since $(\ast)$ is an $\mathcal{F}$–resolution we have exactness of

$$
0 \longrightarrow \text{Hom}_R(F, F_0) \xrightarrow{\text{Hom}_R(F, \partial)} \text{Hom}_R(F, M) \longrightarrow 0,
$$

that is, $\text{Hom}_R(F, \partial)$ is an isomorphism for all $F \in \mathcal{F}$. Now our assumption and Lemma (3.6)(a) and (b) gives that $\partial: F_0 \longrightarrow M$ is a mono $\mathcal{F}$–precover. Another application of our assumption then gives that $\partial$ is an isomorphism, and thus $M \cong F_0 \in \mathcal{F}$.

Next we assume that $n > 0$. Let $M$ be a module which has an $\mathcal{F}$–resolution of length $n$,

$$(\dagger) \quad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0.$$

We break up $(\dagger)$ into two complexes,

$$
(1) \quad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\tilde{\partial}_1} \text{Ker} \partial_0 \longrightarrow 0,
$$

$$
(2) \quad 0 \longrightarrow \text{Ker} \partial_0 \xrightarrow{\iota} F_0 \xrightarrow{\partial_0} M \longrightarrow 0,
$$

where $\tilde{\partial}_1$ is the co-restriction of $\partial_1$ to $\text{Ker} \partial_0$. Once we have argued that $\tilde{\partial}_1: F_1 \longrightarrow \text{Ker} \partial_0$ is an $\mathcal{F}$–precover, it will follow that the upper sequence (1) is an $\mathcal{F}$–resolution of $\text{Ker} \partial_0$, and hence the induction hypothesis gives that $S^{n-1}_\mathcal{F}([\text{Ker} \partial_0]) = [0]$. By the lower sequence (2) we have $S^n_\mathcal{F}([M]) = [\text{Ker} \partial_0]$, and thus the desired conclusion follows:

$$
S^n_\mathcal{F}([M]) = S^{n-1}_\mathcal{F}S^n_\mathcal{F}([M]) = S^{n-1}_\mathcal{F}([\text{Ker} \partial_0]) = [0].
$$

To see that $\tilde{\partial}_1: F_1 \longrightarrow \text{Ker} \partial_0$ is an $\mathcal{F}$–precover we let $\varphi: F \longrightarrow \text{Ker} \partial_0$ be any homomorphism with $F \in \mathcal{F}$. As $\partial_0 \varphi = 0$ and as $\text{Hom}_R(F, (\dagger))$ is exact, there exists $\psi: F \longrightarrow F_1$ with $\partial_1 \psi = \iota \varphi$. Since $\iota$ is mono this means that we have a commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\varphi} & \text{Ker} \partial_0, \\
\downarrow{\psi} & & \\
F_1 & \xrightarrow{\partial_1} & \text{Ker} \partial_0,
\end{array}
$$

as desired. \qed

(3.9) Remark. The dual notion of a precover is a preenvelope, see [2, chap.6]. For a preenveloping class $\mathcal{G}$, the reader can imagine how to construct Ext functors, resolutions, and Schanuel maps relative to $\mathcal{G}$, see also [2, chap.8].
Not surprisingly, every result in this paper has an analogue in this “reenveloping context”. We leave it as an exercise for the interested reader to verify this claim.

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