A NOTE ON THE VALUES OF INDEPENDENCE POLYNOMIALS AT \(-1\)

JONATHAN CUTLER AND NATHAN KAHL

Abstract. The independence polynomial \(I(G;x)\) of a graph \(G\) is \(I(G;x) = \sum_{k=1}^{\alpha(G)} s_k x^k\), where \(s_k\) is the number of independent sets in \(G\) of size \(k\). The decycling number of a graph \(G\), denoted \(\phi(G)\), is the minimum size of a set \(S \subseteq V(G)\) such that \(G - S\) is acyclic. Engström proved that the independence polynomial satisfies \(|I(G;-1)| \leq 2^{\phi(G)}\) for any graph \(G\), and this bound is best possible. Levit and Mandrescu provided an elementary proof of the bound, and in addition conjectured that for every positive integer \(k\) and integer \(q\) with \(|q| \leq 2^k\), there is a connected graph \(G\) with \(\phi(G) = k\) and \(I(G;-1) = q\). In this note, we prove this conjecture.

1. Introduction

Let \(\alpha(G)\) denote the independence number of a graph \(G\), the maximum order of an independent set of vertices in \(G\). The independence polynomial of a graph \(G\) is given by

\[ I(G;x) = \sum_{k=1}^{\alpha(G)} s_k x^k, \]

where \(s_k\) is the number of independent sets of size \(k\) in \(G\). The independence polynomial has been the object of much research (see for instance the survey [7]). One direction of this research, partly motivated by connections with hard-particle models in physics [1, 2, 3, 5, 6], has focused on the evaluation of the independence polynomial at \(x = -1\).

The decycling number of a graph \(G\), denoted \(\phi(G)\), is the minimum size of a set of vertices \(S \subseteq V(G)\) such that \(G - S\) is acyclic. Engström [3] proved the following bound on \(I(G;-1)\), which is best possible.

**Theorem 1.1** (Engström). For any graph \(G\), \(|I(G;-1)| \leq 2^{\phi(G)}\).

Levit and Mandrescu [8] gave an elementary proof of Theorem 1.1 and, in addition, proposed the following conjecture.

**Conjecture 1** (Levit and Mandrescu). Given a positive integer \(k\) and an integer \(q\) with \(|q| \leq 2^k\), there is a connected graph \(G\) with \(\phi(G) = k\) and \(I(G;-1) = q\).

For brevity, in this paper a graph \(G\) with \(\phi(G) = k\) and \(I(G;-1) = q\), with \(|q| \leq 2^k\), will be referred to as a \((k,q)\)-graph. In [9], Levit and Mandrescu provided constructions that gave \((k,q)\)-graphs for all \(k \leq 3\) and \(|q| \leq 2^k\). Also, they gave constructions for every \(k\) provided \(q \in \{2^{\phi(G)}, 2^{\phi(G)} - 1\}\). In this paper, we prove Conjecture 1.
2. The Construction and Proof of Conjecture

The construction proceeds inductively, using particular \((k-1, q)\)-graphs to produce the necessary \((k, q)\)-graphs. First we assemble the tools used in the construction. The most important tool is a recursive formula for \(I(G; x)\) due to Gutman and Harary [1]. We let \(N(v) = \{x \in V(G) : xv \in E(G)\}\) and \(N[v] = \{v\} \cup N(v)\).

**Lemma 2.1.** For any graph \(G\) and any vertex \(v \in V(G)\),

\[
I(G; x) = I(G - v; x) + xI(G - N[v]; x).
\]

Using this, or simply counting independent sets, we can derive the independence polynomial at \(-1\) for small graphs. Some useful examples can be found in Table 2.

| \(G\)  | \(I(G; -1)\) |
|--------|---------------|
| \(K_1\) | 0             |
| \(K_2\) | -1            |
| \(K_3 = C_3\) | -2          |
| \(C_6\) | 2             |

**Table 1.** Some small examples

Since Lemma 2.1 requires a particular vertex \(v \in V(G)\) to be specified, it will often be helpful to root graphs for which we want to compute the independence polynomial at \(-1\). Given a graph \(G\) and a vertex \(v \in V(G)\), the **rooted graph** \(G_v\) is the graph \(G\) with the vertex \(v\) labeled. Of course, \(I(G; -1) = I(G_v; -1)\) for any vertex \(v \in V(G)\).

We now introduce two operations on rooted graphs which will be useful in our proof. The first of these is called **pasting**.

**Definition.** Given two rooted graphs \(G_v\) and \(H_w\), the pasting of \(G_v\) and \(H_w\), denoted \(G_v \& H_w\), is the rooted graph formed by identifying the roots \(v\) and \(w\).

We note two important facts. First, the pasting operation creates no new cycles, and thus \(\phi(G_v \& H_w) \leq \phi(G_v) + \phi(H_w)\). (In our construction the roots will be pendant vertices, and so \(\phi(G_v \& H_w) = \phi(G_v) + \phi(H_w)\).) Second, if for two rooted graphs \(G_v\) and \(H_w\) the quantities \(I(G_v; -1)\) and \(I(H_w; -1)\) have been evaluated using Lemma 2.1, then the value of \(I(G_v \& H_w; -1)\) can be determined in a straightforward way. It is well-known that, letting \(G \cup H\) denote the disjoint union of \(G\) and \(H\), we have

\[
I(G \cup H; x) = I(G; x)I(H; x).
\]

Deleting the pasted vertex in \(G_v \& H_w\) produces a disjoint union of graphs. This fact, and the recurrences

\[
I(G_v; -1) = I(G_v - v; -1) - I(G_v - N[v]; -1)
\]

\[
I(H_w; -1) = I(H_w - w; -1) - I(H_w - N[w]; -1)
\]

then give

\[
I(G_v \& H_w; -1) = I(G_v - v; -1)I(H_w - w; -1) - I(G_v - N[v]; -1)I(H_w - N[w]; -1).
\]

It will be helpful to keep track of the various parts of the above calculation, and in order to do so we introduce the following bookkeeping device. Given a rooted graph \(G_v\), where \(I(G_v; -1) = a\) and \(I(G_v - N[v]; -1) = b\), and hence \(I(G_v; -1) = a - b\), we write \(I(G_v; -1) = (a - b, a, b)\) and say that \(G_v\) has **bracket** \((a - b, a, b)\). An example can be found in Figure 1. Note that for a given rooted
A NOTE ON THE VALUES OF INDEPENDENCE POLYNOMIALS AT $-1$

graph $G_v$ there are unique integers $a$ and $b$, determined by the root, with $I(G_v; -1) = \langle a - b, a, b \rangle$.

Using this notation, the calculations above give the following lemma.

**Lemma 2.2** (Pasting Lemma). If $G_v$ and $H_w$ are rooted graphs on at least two vertices with $I(G_v; -1) = \langle a - b, a, b \rangle$ and $I(H_w; -1) = \langle c - d, c, d \rangle$, then

$$I(G_v \wedge H_w; -1) = ac - bd = \langle ac - bd, ac, bd \rangle$$

and $G_v \wedge H_w$ has bracket $\langle ac - bd, ac, bd \rangle$.

Our second operation is a variation of the pasting operation which, however, is useful enough to merit its own terminology and notation.

**Definition.** Given a rooted graph $G_v$ and an integer $k \geq 0$, the $\ell$-extension of $G_v$, denoted $G_v^\ell$, is the graph formed by identifying the root $v$ with one of the endpoints of a (disjoint) path of length $\ell$ and reassigning the root to the other endpoint of the path.

The length of a path is above measured in edges; for instance for a rooted graph $G_v$, the 0-extension $G_v^0$ is simply $G_v$. As with the pasting operation, no new cycles are created by the extension operation, and so here $\phi(G_v^\ell) = \phi(G)$ for any $\ell$. In addition, the values of the independence polynomial at $-1$ of various extensions of a rooted graph $G_v$ are easy to characterize in terms of the bracket of $G_v$. Indeed, extensions of $G_v$ have the same bracket values, up to sign, but in a different order. The proof of the following lemma follows immediately from the recursion formula and is omitted.

**Lemma 2.3** (Extension Lemma). If $G_v$ is a rooted graph with $I(G_v; -1) = \langle a - b, a, b \rangle$, then

$$I(G_v^1; -1) = \langle -b, a - b, a \rangle$$
$$I(G_v^2; -1) = \langle -a, -b, a - b \rangle$$

and $I(G_v^3; -1) = \langle b - a, -a - b \rangle = -\langle a - b, a, b \rangle = -I(G_v; -1)$.

We illustrate the cycling phenomenon with $C_6$, a graph which will be used in our construction. Obviously we may consider $C_6$ rooted at any given vertex.

(Since $C_3$ has the same set of six brackets, in a different order, when extended, $C_3$ could also have been used in the constructions and proofs to come. We choose $C_6$ solely because $C_6^0$ and $C_6^1$ have positive $I(G; -1)$.)

Using the pasting and extension operations we have our final lemma, which shows that the word “connected” in the conjecture is superfluous. Any disconnected $(k,q)$-graph can be pasted together and extended to produce a connected $(k,q)$-graph.

**Lemma 2.4.** Let $G$ and $H$ be disjoint $(k_1,q_1)$ and $(k_2,q_2)$-graphs, respectively, with $k_1 + k_2 = k$ and $q_1q_2 = q$. Then there is a connected $(k,q)$-graph $F$, i.e., $F$ is connected, $\phi(F) = k_1 + k_2 = k$, and $I(F; -1) = q_1q_2 = I(G \cup H; -1)$. 

![Figure 1. A graph rooted at $r$ with bracket $\langle 5, -3, -8 \rangle$.]
Proof. Root the given graphs as $G_v$ and $H_w$ and let the corresponding brackets be $I(G_v; -1) = \langle q_1, a, b \rangle$ and $I(H_w; -1) = \langle q_2, c, d \rangle$, respectively. Let $F' = (G^2_v \land H^2_w)^1$. By the Extension Lemma, $I(G^2_v; -1) = \langle -a, -b, q_1 \rangle$ and $I(H^2_w; -1) = \langle -c, -d, q_2 \rangle$. Then, by the Pasting Lemma,

$$I(G^2_v \land H^2_w; -1) = \langle bd - q_1 q_2, bd, q_1 q_2 \rangle$$

Therefore, again using the Extension Lemma,

$$I(F'; -1) = I((G^2_v \land H^2_w)^1; -1)$$

$$= \langle -q_1 q_2, bd - q_1 q_2, bd \rangle$$

$$= -q_1 q_2$$

$$= -I(G \cup H; -1).$$

In addition, neither the pasting nor extension operations produce cycles, so $\phi(F) = k_1 + k_2 = k = \phi(G \cup H)$.

Now let $F = (F^2_x \cup K^2_2)^1$, where $F^2_x$ is a rooted version of the graph $F'$ previously. Then by the same analysis as above, we have $I(F; -1) = -I(F' \cup K_2; -1) = -(q_1 q_2)I(F'; -1) = I(G \cup H; -1)$, and $\phi(F) = \phi(F') = \phi(G \cup H)$, as required. \(\square\)

By setting $H = K_2$ and $H = C_6$ in Lemma 2.4 in turn, we obtain the following facts, which will also be useful in the proof. These two facts were also noted by Levit and Mandrescu [9], who used different ad hoc techniques in their constructions of the necessary graphs.

**Corollary 2.5.** If $G$ is a $(k, q)$-graph then there exists (a) a connected $(k + 1, 2q)$-graph and (b) a connected $(k, -q)$-graph.

We now prove Conjecture 1

**Theorem 2.6.** Given a positive integer $k$ and an integer $q$ with $|q| \leq 2^k$, there is a connected graph $G$ with $\phi(G) = k$ and $I(G; -1) = q$.

**Proof.** By Lemma 2.4 we do not need to produce connected $(k, q)$-graphs for all $|q| \leq 2^k$; disconnected $(k, q)$-graphs will suffice. Since $I(G \cup K_1; -1) = 0$ for all $G$, we can consider the case $q = 0$ done for all $k$.

As mentioned previously, our proof proceeds inductively on $k$. When $k = 1$ then $I(C_6; -1) = \langle 2, 1, -1 \rangle$ and, as noted in Table 2, by taking extensions of $C_6$, we rotate through all of $\{2, 1, -1, -2\}$. Thus the theorem is true for $k = 1$.

For the induction step, assume $(k - 1, q)$-graphs are constructible for all $q \leq 2^{k-1}$. By Corollary 2.5(a) we immediately have that $(k, q)$-graphs for even $q \leq 2^k$ are constructible. By Corollary 2.5(b) we also need only construct $(k, q)$-graphs for positive $q \leq 2^k$. It only remains, then, to construct $(k, q)$-graphs for each odd integer in $[0, 2^k]$. To that end, we prove the following claim.
Claim 1. For each odd integer \( q \in [0, 2^k] \), there is a connected \((k, q)\)-graph \( G_v \) such that either \( I(G_v; -1) = \langle q, 2^k, 2^k - q \rangle \) or \( I(G_v; -1) = \langle q, -2^k + q, -2^k \rangle \).

Proof. For \( k = 1 \), we see that the bracket of \( C_0^1 \) has the necessary form, i.e. \( I(C_0^1; -1) = \langle 1, 2, 1 \rangle \). Assume that the hypothesis of the claim is true for \( k - 1 \); we seek to produce \((k, q)\)-graphs for each odd \( q \in [0, 2^k] \) such that \( 2^k \) or \(-2^k\) appears in their bracket. We consider two cases: \( q \in [2^{k-1}, 2^k] \) and \( q \in [0, 2^{k-1}] \).

For the first case, let \( q \) be an odd integer in \([2^{k-1}, 2^k]\). Necessarily then, \( q = 2^k - r \) for some \( r \in [0, 2^{k-1}] \). By the induction assumption, there is some \((k - 1, r)\)-graph \( G_v \) such that either \( I(G_v; -1) = \langle 2^{k-1} - r, 2^{k-1}, r \rangle \) or \( I(G_v; -1) = \langle 2^{k-1} - r, -r, -2^{k-1} \rangle \). By the Pasting Lemma, then, \( I(G_v \land C_0^1; -1) = \langle 2^k - r, 2^k, r \rangle = q \) if the bracket of \( G_v \) is of the first form, or \( I(G_v \land C_0^1; -1) = \langle 2^k - r, -r, -2^k \rangle \) if the bracket of \( G_v \) is of the second form. Thus the claim is true for all \( q \in [2^{k-1}, 2^k] \).

We are left with the second case of the odd \( q \in [0, 2^{k-1}] \). However, because \( 2^k \) appears in all the brackets in the previous case, necessarily these odd \( q \in [0, 2^{k-1}] \) correspond to the \( r \) that appeared in those brackets. Hence extending the constructions for the odd \( q \in [2^{k-1}, 2^k] \) appropriately will produce these \( r \).

The proof of the claim completes the induction, and completes the proof.

3. Acknowledgment

The authors would like to thank Hannah Quense and Tara Wager for helpful discussions.

References

1. Michał Adamaszek, *Special cycles in independence complexes and superfrustration in some lattices*, Topology Appl. 160 (2013), no. 7, 943–950. MR 3037886
2. Mireille Bousquet-Mélou, Svante Linusson, and Eran Nevo, *On the independence complex of square grids*, J. Algebraic Combin. 27 (2008), no. 4, 423–450. MR 2393250 (2009k:05137)
3. Alexander Engström, *Upper bounds on the Witten index for supersymmetric lattice models by discrete Morse theory*, European J. Combin. 30 (2009), no. 2, 429–438.
4. Ivan Gutman and Frank Harary, *Generalizations of the matching polynomial*, Utilitas Math. 24 (1983), 97–106.
5. Liza Huijse and Kareljan Schoutens, *Supersymmetry, lattice fermions, independence complexes and cohomology theory*, Adv. Theor. Math. Phys. 14 (2010), no. 2, 643–694. MR 2721658 (2011i:81152)
6. Jakob Jonsson, *Hard squares with negative activity on cylinders with odd circumference*, Electron. J. Combin. 16 (2009), no. 2, Special volume in honor of Anders Björner, Research Paper 5, 22. MR 2515768 (2010g:05267)
7. Vadim E. Levit and Eugen Mandrescu, *The independence polynomial of a graph—a survey*, Proceedings of the 1st International Conference on Algebraic Informatics, Aristotel Univ. Thessaloniki, Thessaloniki, 2005, pp. 233–254. MR 2186466 (2006k:05163)
8. ________*, A simple proof of an inequality connecting the alternating number of independent sets and the decycling number*, Discrete Math. 311 (2011), no. 13, 1204–1206.
9. ________*, The cyclomatic number of a graph and its independence polynomial at \(-1\)*, Graphs Combin. 29 (2013), no. 2, 259–273.

Department of Mathematical Sciences, Montclair State University, Montclair, NJ

E-mail address: jonathan.cutler@montclair.edu

Department of Mathematics and Computer Science, Seton Hall University, South Orange, NJ

E-mail address: nathan.kahl@shu.edu