SPHERE BUNDLES OVER 4-MANIFOLDS ARE TRIVIAL AFTER LOOPING

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Abstract. We show that except two special cases, the sphere bundle of a vector bundle over a simply connected 4-manifold splits after looping. In particular, this implies that though there are infinitely many inequivalent sphere bundles of a given rank over a 4-manifold, the loop spaces of their total manifolds are all homotopy equivalent.

1. Introduction

Sphere bundles over 4-manifolds are classical and important in topology. There are many famous results especially when the base manifold is the 4-sphere. For instance, in differential topology, Milnor [Mil56] found his exotic 7-sphere as the total manifold of a 3-sphere bundle over the 4-sphere. The total manifolds of general $S^3$-bundles over $S^4$ were eventually classified by Crowley and Escher [C03E] in various categories. In homotopy theory, in their remarkable work [JW54, JW55], James and Whitehead studied deeply the fibrewise homotopy classification of general sphere bundles over spheres. In contrast, there are much fewer studies on sphere bundles over general 4-manifolds, the topology of which should be much harder.

In this paper, we study the homotopy theory of sphere bundles over general simply connected 4-manifolds. The topology of circle bundles over 4-manifolds has been studied by [DL05], from which Beben and Theriault [BT14] have determined their loop homotopy types. The homotopy of 2-sphere bundles was studied by the author recently [Hua22]. Our main theorem below determines the loop homotopy types of sphere bundles over simply connected 4-manifolds for almost all other cases.

Theorem 1.1. Let $(d, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 2}$ and $(d, n) \neq (0, 2), (0, 3)$. Let $N$ be a simply connected closed 4-manifold such that $H^2(N; \mathbb{Z}) \cong \mathbb{Z}^\oplus d$. Let

$$S^n \to M \to N$$

be the sphere bundle of a rank $(n + 1)$ vector bundle over $N$. Then the sphere bundle splits after looping to give a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega N.$$ 

Moreover,

(1). if $d = 0$, $\Omega M \simeq S^3 \times \Omega S^n \times \Omega S^7$; 

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Theorem 2.1 will be proved by the three indicated cases in Section 3, 4, and 5 respectively. As an immediate consequence of Theorem 2.1, we have an isomorphism of homotopy groups
\[ \pi_*(M) \cong \pi_*(S^n) \oplus \pi_*(N), \]
where \( \pi_*(N) \) can be reduced to the homotopy groups of spheres by the further decompositions in the theorem (cf. [BT14, Introduction]). Note that this isomorphism cannot be obtained directly from the long exact sequence of the homotopy groups of the sphere bundle.

Similarly, we have an isomorphism of homology groups
\[ H_*(\Omega M; \mathbb{Z}) \cong H_*(\Omega S^n; \mathbb{Z}) \otimes H_*(\Omega N; \mathbb{Z}), \]
where \( H_*(\Omega S^n; \mathbb{Z}) \) is a rank 1 tensor algebra by the Bott-Samelson theorem [BS53], and \( H_*(\Omega N; \mathbb{Z}) \) was computed by Sa. Basu and So. Basu [BB18, Theorem 4.1] as a quadratic algebra. Further, as Example 5.1 illustrates, in general the homology of the sphere bundle does not split before looping
\[ H_*(M; \mathbb{Z}) \not\cong H_*(S^n; \mathbb{Z}) \otimes H_*(N; \mathbb{Z}). \]

A final remark on the two cases those are not considered in the theorem. The sphere bundle is of the form
\[ S^n \rightarrow M \rightarrow S^4 \]
with \( n = 2 \) or 3. Partial results were obtained by the author [Hua22] for \( n = 2 \) and by Theriault and the author [HT22] for \( n = 3 \). Based on the results there, it can be realized that these two cases are quite different from the other cases when \( n \geq 4 \). In general, when \( n = 2 \) or 3 the sphere bundle does not split after looping, and there are torsions in the loop homology of \( M \). More details can be found in a summary in Section 5.

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2. Rank \((n+1)\) bundles over 4-manifolds

In this section, we review necessary knowledge of rank \((n+1)\) vector bundles over simply connected 4-manifolds with \( n \geq 2 \). From the point of view of manifold topology, it is known that such bundles are classified by their second Stiefel-Whitney classes \( \omega_2 \) and their first Pontryagin classes \( p_1 \), plus an extra Euler class \( e \) when \( n = 3 \). However, for our purpose in this paper, it is convenient to analyze their classifying maps directly from the point of view of homotopy theory. This was already done.
in [Hua22, Section 2] when \( n = 2 \). The general cases are quite similar, and we simply state the necessary results here without proof.

Let \( N \) be a simply connected 4-manifold such that \( H^2(N; \mathbb{Z}) \cong \mathbb{Z}^{\oplus d} \) with \( d \geq 0 \). There is the homotopy cofiber sequence

\[
\begin{array}{ccc}
S^3 & \xrightarrow{\phi} & \bigvee_{i=1}^d S^2 \\
\rho & \xrightarrow{} & S^4 \\
q & \xrightarrow{} & S^3,
\end{array}
\]

where \( \phi \) is the attaching map of the top cell of \( N \), \( \rho \) is the injection of the 2-skeleton, and \( q \) is the pinch map onto the top cell. Since \( \pi_3(BSO(n+1)) = 0 \), it is easy to see that the cofibre sequence (1) implies the short exact sequence of pointed sets

\[
\begin{array}{ccc}
0 & \xrightarrow{} & [S^4, BSO(n+1)] \\
\xrightarrow{q^*} & [N, BSO(n+1)] \\
\xrightarrow{\rho^*} & \bigvee_{i=1}^d S^2, BSO(n+1) & \xrightarrow{} & 0,
\end{array}
\]

in a strong sense that, there is an action of \([S^4, BSO(n+1)]\) on \([N, BSO(n+1)]\) through \( q^* \) such that the sets \( \rho^{-1}(x) \), for \( x \in \bigvee_{i=1}^d S^2, BSO(n+1) \) are precisely the orbits. Further, the short exact sequence is natural for degree 1 maps of 4-manifolds.

A rank \((n + 1)\) vector bundle \( \xi \) over \( N \) is classified by a map \( f : N \rightarrow BSO(n+1) \). The following lemma characterizes the restriction \( \rho^*(f) \) by a circle bundle. Let \( s : S^1 \cong SO(2) \rightarrow SO(n+1) \) be the canonical inclusion of Lie groups. A cohomology class \( z \in H^2(N; \mathbb{Z}) \) is called primitive, if \( z \) is not divisible by any integer \( k \) with \( k \neq \pm 1 \).

**Lemma 2.1.** There exists a class \( \alpha \in H^2(N; \mathbb{Z}) \cong [N, BS^1] \), such that \( \rho^*(f) = (\rho^* \circ (Bs)_*)(\alpha) \) through the composition

\[
[N, BS^1] \xrightarrow{(Bs)_*} [N, BSO(n+1)] \xrightarrow{\rho^*} \bigvee_{i=1}^d S^2, BSO(n+1),
\]

and \( \alpha \) satisfies that

- \( \alpha = 0 \) if \( \omega_2(\xi) = 0 \);
- \( \alpha \) is primitive and \( \omega_2(\xi) \equiv \alpha \mod 2 \) if \( \omega_2(\xi) \neq 0 \);

\( \square \)

The following lemma is a general form of [Hua22, Lemma 2.1]. It decomposes the classifying map \( f \) into two simpler parts through the action \( \cdot \) defined by \( q^* \) in (2).

**Lemma 2.2.** The composition morphism

\[
\Phi : [S^4, BSO(n+1)] \times [N, BS^1] \xrightarrow{(id, (Bs)_*)} [S^4, BSO(n+1)] \times [N, BSO(n+1)] \xrightarrow{} [N, BSO(n+1)]
\]

is surjective. Moreover, for any choice of \( \alpha \) in Lemma 2.1, there exists a class \( f' \in [S^4, BSO(n+1)] \) such that

\[
\Phi(f', \alpha) = q^*(f') \cdot (Bs_*(\alpha)) = f.
\]

\( \square \)
3. The case when \( d \geq 2 \)

Let \( N \) be a simply connected 4-manifold such that \( H^2(N; \mathbb{Z}) \cong \mathbb{Z}^d \) with \( d \geq 2 \). A rank \((n + 1)\) vector bundle \( \xi \) over \( N \) is classified by a map \( f : N \to BSO(n + 1) \). Fix a class \( \alpha \in H^2(N; \mathbb{Z}) \) in Lemma 2.1

**Lemma 3.1.** There exists a simply connected 4-dimensional Poincaré complex \( Q = (S^2 \vee S^2) \cup e^4 \) with a degree 1 map

\[
h : N \to Q,
\]

such that \( h^*(x) = \alpha \) for some class \( x \in H^2(Q; \mathbb{Z}) \).

*Proof.* We consider the case when \( \omega_2(\xi) \neq 0 \) first. By Lemma 2.1, the class \( \alpha \in H^2(N; \mathbb{Z}) \) is primitive. We claim that there exists a class \( \beta \in H^2(N; \mathbb{Z}) \) such that \( \beta \neq \alpha \), and \( \langle \alpha \cup \beta, [N] \rangle = \pm 1 \). Otherwise, by Poincaré duality \( \langle \alpha \cup \beta, [N] \rangle = \pm 1 \). Since \( d \geq 2 \), we can choose another generator \( \gamma \in H^2(N; \mathbb{Z}) \).

By hypothesis, \( \langle \alpha \cup \gamma, [N] \rangle = \pm k \) with \( k \neq 1 \). If \( k = 0 \), let \( \beta = \alpha + \gamma \), and \( \langle \alpha \cup (\alpha + \gamma), [N] \rangle = \pm 1 \). If \( k \geq 2 \), let \( \beta = (1 - k)\alpha + \gamma \), and \( \langle \alpha \cup ((1 - k)\alpha + \gamma), [N] \rangle = \pm 1 \). Hence, the claim is proved.

Then as in [BT14] Proof of proposition 3.2, by abuse of notation the inclusion \( \rho \) of the 2-skeleton of \( N \) can be chosen as

\[
\rho : K \vee (S^2 \vee S^2) \xrightarrow{j \vee (\alpha \vee \beta)} N,
\]

where \( K = \bigvee_{i=1}^{d-2} S^2 \), and the last two \( S^2 \) represent the classes \( \alpha \) and \( \beta \). Let \( Q \) be the homotopy cofibre of the composition \( K \hookrightarrow K \vee (S^2 \vee S^2) \xrightarrow{\rho} N \). Denote by \( h : N \to Q \) the quotient map. It is clear that \( Q = (S^2 \vee S^2) \cup e^4 \) as CW-complex. By the construction \( h^* : H^*(Q; \mathbb{Z}) \to H^*(N; \mathbb{Z}) \) is an injection on \( H^2 \) and an isomorphism on \( H^4 \), and

\[
H^*(Q; \mathbb{Z}) \cong H^*(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).
\]

In particular, \( Q \) is a Poincaré complex. By choosing \( x = h^{-1}(\alpha) \), the lemma is proved when \( \omega_2(\xi) \neq 0 \).

When \( \omega_2(\xi) = 0 \), the class \( \alpha = 0 \) by Lemma 2.1. Choose any primitive class \( \alpha' \in H^2(N; \mathbb{Z}) \). Then we can run through the previous argument for \( \alpha' \) instead of \( \alpha \), except in the last step where we can simply let \( x = 0 \). This proves the lemma when \( \omega_2(\xi) = 0 \).

**Lemma 3.2.** For the Poincaré complex \( Q \) in Lemma 3.1, there is a homotopy equivalence

\[
\Omega Q \simeq \Omega(S^2 \times S^2),
\]

*Proof.* With the construction of \( Q \), the lemma is a special case of [BT14] Lemma 2.3.

**Lemma 3.3.** The classifying map \( f \) of \( \xi \) factors as

\[
f : N \xrightarrow{h} Q \xrightarrow{f_Q} BSO(n + 1),
\]

for some map \( f_Q \).
Proof. It is clear that the degree 1 map $h$ induces a diagram of homotopy cofibrations

$$
\begin{array}{c}
K \vee (S^2 \vee S^2) \xrightarrow{\rho} N \xrightarrow{q} S^4 \\
\downarrow h \downarrow h \quad \quad \downarrow h \\
S^2 \vee S^2 \xrightarrow{e} Q \xrightarrow{q} S^4,
\end{array}
$$

where the maps $q$ and $q$ are the obvious inclusion and quotient maps, and the projection $h'$ is the restriction of $h$ on the 2-skeletons. From (2) it implies the morphism of short exact sequences

$$
\begin{array}{c}
0 \longrightarrow [S^4, BSO(n+1)] \xrightarrow{q^*} [Q, BSO(n+1)] \xrightarrow{e^*} [S^2 \vee S^2, BSO(n+1)] \longrightarrow 0 \\
0 \longrightarrow [S^4, BSO(n+1)] \xrightarrow{q^*} [N, BSO(n+1)] \xrightarrow{\rho^*} [K \vee (S^2 \vee S^2), BSO(n+1)] \longrightarrow 0.
\end{array}
$$

By Lemma 2.2 we know that $f = \Phi(f', \alpha) = q^*(f') \cdot (Bs_x(\alpha))$ for some $f' \in [S^4, BSO(n+1)]$. Let $f_Q = q^*(f') \cdot (Bs_x(x))$ with $x \in [Q, BS^n]$ in Lemma 3.1. Then by the naturality of the actions through the above diagram [Swi02, Proposition 2.48] and Lemma 3.1

$$
h^*(f_Q) = h^*(q^*(f') \cdot (Bs_x(x))) = q^*(f') \cdot (Bs_x(h^*(x))) = q^*(f') \cdot (Bs_x(\alpha)) = f.
$$

This proves the lemma. \[\square\]

We are ready to prove Theorem 1.1 when $d \geq 2$.

Proof of Theorem 1.1 when $d \geq 2$. In this case, by Lemma 3.1 and 3.3 there exists a Poincaré complex $Q = (S^2 \vee S^2) \cup e^4$ such that the classifying map $h$ of $\xi$ factors as $f : N \xrightarrow{h} Q \xrightarrow{f_Q} BSO(n+1)$ for some map $f_Q$. The latter map $f_Q$ determines a vector bundle $\xi_Q$ over $Q$, the pullback of which along $h$ is isomorphic to $\xi$. This implies the morphism of sphere bundles

$$
\begin{array}{c}
S^n \xrightarrow{i} M \xrightarrow{p} N \\
\downarrow \tilde{h} \quad \downarrow h \\
\tilde{Q} \xrightarrow{\tilde{p}} Q
\end{array}
$$

where $i$ and $\tilde{i}$ are fibre inclusions, $p$ and $\tilde{p}$ are bundle projections, $\tilde{h}$ is the induced map, and $\tilde{Q}$ is an $(n + 4)$ dimensional Poincaré complex.

By Lemma 3.2 $\Omega Q \simeq \Omega(S^2 \times S^2)$. Since $n \geq 2$, there is the long exact sequence of homotopy groups

$$
\cdots \longrightarrow \pi_1(\Omega Q) \xrightarrow{\Omega h_1} \pi_1(\Omega Q) \cong \pi_1(\Omega(S^2 \times S^2)) \cong \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(S^n) = 0 \longrightarrow \cdots.
$$
Hence \((\Omega\tilde{p})_*\) is surjective, and there is a map \(i_1 \times i_2 : S^1 \times S^1 \to \Omega \tilde{Q}\) such that the composition

\[
S^1 \times S^1 \xrightarrow{i_1 \times i_2} \Omega \tilde{Q} \xrightarrow{\tilde{p}} \Omega Q \xrightarrow{\simeq} \Omega S^2 \times \Omega S^2
\]

is homotopic to \(E \times E\) with \(E : S^1 \to \Omega S^2\) the suspension map. By the universal property of \(\Omega \Sigma\), there is a unique extension \(I : \Omega S^2 \times \Omega S^2 \to \Omega \tilde{Q}\) of \(i_1 \times i_2\) up to homotopy such that

\[
\Omega S^2 \times \Omega S^2 \xrightarrow{I} \Omega \tilde{Q} \xrightarrow{\tilde{p}} \Omega Q \xrightarrow{\simeq} \Omega S^2 \times \Omega S^2
\]

is homotopic to identity. Therefore, the sphere bundle of \(\xi_Q\) splits after looping to give

\[
\Omega \tilde{Q} \simeq \Omega S^n \times \Omega Q \simeq \Omega S^n \times \Omega S^2 \times \Omega S^2.
\]

In particular, in Diagram (3) \(\Omega \tilde{i}\) has a left homotopy inverse \(\tilde{r}\), which implies that \(\tilde{r} \circ \Omega \tilde{h}\) is a left homotopy inverse of \(\Omega i\). Then the sphere bundle in the top row of Diagram (3) splits after looping, and in particular \(\Omega M \simeq \Omega S^n \times \Omega N\). Further by [BT14, Theorem 1.3] when \(d \geq 2\) there is a homotopy equivalence

\[
\Omega N \simeq S^1 \times \Omega(S^2 \times S^3) \times \Omega(J \vee (J \wedge \Omega(S^2 \times S^3)))
\]

with \(J = \bigvee_{i=1}^{d-2} (S^2 \vee S^3)\). Then in this case the theorem follows by combining the above decompositions.

4. The case when \(d = 1\)

Let \(N\) be a simply connected 4-manifold such that \(H^2(N; \mathbb{Z}) \cong \mathbb{Z}\).

**Lemma 4.1.** There is a homotopy equivalence \(N \simeq \mathbb{C}P^2\).

**Proof.** The generator of \(H^2(N; \mathbb{Z}) \cong \mathbb{Z}\) is represented by a map \(N \to \mathbb{C}P^\infty\). By the cellular approximation theorem it factors through \(\mathbb{C}P^2\) as

\[
N \xrightarrow{\lambda} \mathbb{C}P^2 \to \mathbb{C}P^\infty
\]

for some map \(\lambda\). Then by Poncaré duality, \(\lambda\) induces an isomorphism on cohomology and then is a homotopy equivalence by the Whitehead theorem. \(\square\)

By Lemma 4.1 \(N\) is a homotopy \(\mathbb{C}P^2\). Since we are only interested in homotopy types, we can simply let \(N = \mathbb{C}P^2\).
A rank \((n+1)\) vector bundle \(\xi\) over \(\mathbb{C}P^2\) is classified by a map \(f: \mathbb{C}P^2 \to BSO(n+1)\). Consider the diagram of fibre bundles

\[
\begin{array}{ccc}
S^n & \overset{i}{\longrightarrow} & S^n \\
\downarrow & & \downarrow i \\
S^1 & \overset{j}{\longrightarrow} & X \\
\updownarrow & & \updownarrow \psi \\
S^1 & \overset{j}{\longrightarrow} & S^5 \\
\downarrow & & \downarrow \pi \\
\mathbb{C}P^2 & \overset{p}{\longrightarrow} & M \\
\end{array}
\]

where the bottom bundle is the canonical circle bundle of \(\mathbb{C}P^2\), the rightmost bundle is the sphere bundle of \(\xi\), and \(X\) is a closed \((n+5)\)-manifold.

**Lemma 4.2.** The pullback vector bundle \(\pi^*(\xi)\) is trivial, and in particular \(X \cong S^n \times S^5\).

**Proof.** When \(n \geq 5\), by Bott periodicity \(\pi_5(BSO(n+1)) \cong \pi_4(SO(n+1)) = 0\), and then the lemma follows in this case. When \(2 \leq n \leq 4\), consider the diagram of homotopy cofibrations

\[
\begin{array}{ccc}
* & \overset{\pi}{\longrightarrow} & S^5 \\
\downarrow & & \downarrow \pi' \\
S^2 & \overset{p}{\longrightarrow} & \mathbb{C}P^2 \\
\downarrow & & \downarrow q \\
\mathbb{C}P^2 & \overset{\pi'}{\longrightarrow} & S^4 \\
\end{array}
\]

where \(\pi' = q \circ \pi\). From (2) it implies the morphism of short exact sequence

\[
0 \longrightarrow [S^4, BSO(n+1)] \overset{q^*}{\longrightarrow} [\mathbb{C}P^2, BSO(n+1)] \overset{\rho^*}{\longrightarrow} [S^2, BSO(n+1)] \longrightarrow 0
\]

\[
\pi^* \downarrow \pi^* \\
[S^5, BSO(n+1)] \longrightarrow [S^5, BSO(n+1)].
\]

It is known that the homotopy cofibre of \(\pi\) is \(\mathbb{C}P^3\), and the Steenrod operation \(Sq^2: H^4(\mathbb{C}P^3; \mathbb{Z}/2\mathbb{Z}) \to H^6(\mathbb{C}P^3; \mathbb{Z}/2\mathbb{Z})\) is trivial. This is equivalent to that \(\pi'\) is null homotopic. In particular \(\pi'^* = 0\).

By Lemma [2] we know that \(f = \Phi(f', \alpha) = q^*(f') \cdot (Bs_*(\alpha))\) for some \(f' \in [S^4, BSO(n+1)]\) and \(\alpha \in [\mathbb{C}P^2, BS^1]\). Notice that \(\pi^*(\alpha) = 0\) as \(H^2(S^5; \mathbb{Z}) = 0\). Then by the naturality of the actions through the above diagram [Swi02 Proposition 2.48]

\[
\pi^*(f) = \pi^*(q^*(f') \cdot (Bs_*(\alpha)))
\]

\[
= \pi'^*(f') \cdot (Bs_*(\pi^*(\alpha)))
\]

\[
= 0.
\]

It means that \(\pi^*(\xi)\) is trivial and the lemma is proved. \(\square\)
Remark. Note the above proof is valid for any $G$ such that $\pi_2(G) = 0$, which is the case for any compact semisimple Lie group. Hence, for a such $G$ we have showed for any principal $G$-bundle over $\mathbb{C}P^2$, its pullback along the standard projection $S^5 \to \mathbb{C}P^2$ is trivial.

We are ready to prove Theorem 1.1 when $d = 1$.

**Proof of Theorem 1.1 when $d = 1$.** In this case, consider the circle bundle

$$S^1 \xrightarrow{j} X \xrightarrow{\psi} M$$

in the middle row of Diagram (4). Since $X$ simply connected, $j$ is null homotopic. Therefore, $\Omega \psi$ has a left homotopy inverse $\Omega M \xrightarrow{r} \Omega X$, and then $\Omega M \simeq S^1 \times \Omega X$. Since by Lemma 4.2 $X \simeq S^n \times S^5$, we can define the composition

$$\Omega M \xrightarrow{r} \Omega X \simeq \Omega(S^n \times S^5) \xrightarrow{q_1} \Omega S^n,$$

where $q_1$ is the obvious projection. By Diagram (4) we see that the composition is a left homotopy inverse of $\Omega S^n \xrightarrow{\Omega q_1} \Omega M$. It follows that the sphere bundle in the rightmost column of Diagram (4) splits after looping to give a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega \mathbb{C}P^2 \simeq S^n \times \Omega S^3 \times \Omega S^5.$$

This completes the proof of the theorem when $d = 1$. □

5. **The case when $d = 0$**

Let $N$ be a simply connected 4-manifold such that $H^2(N; \mathbb{Z}) \simeq 0$. Then $N$ is a homotopy $S^4$, and we may let $N = S^4$ as we are only interested in homotopy types.

**Proof of Theorem 1.1 when $d = 0$.** In this case, the sphere bundle is $S^n \xrightarrow{j} M \xrightarrow{p} S^4$ with $n \geq 4$. It induces the long exact sequence of homotopy groups

$$\cdots \to \pi_4(M) \xrightarrow{p_*} \pi_4(S^4) \xrightarrow{\pi_3(S^n)} \pi_3(S^n) = 0 \to \cdots.$$

In particular, $p^*$ is surjective, and then $p$ has a right homotopy inverse. It follows the sphere bundle splits after looping to give a homotopy equivalence

$$\Omega M \simeq \Omega S^n \times \Omega S^3 \simeq S^n \times \Omega S^3 \times \Omega S^5.$$

This completes the proof of the theorem when $d = 0$. □

There are two cases which are excluded in the theorem: $(d, n) = (0, 2)$ or $(0, 3)$. Let us discuss them briefly.
When \((d,n) = (0,3)\), we have the sphere bundle \(S^3 \overset{i}{\to} M \overset{p}{\to} S^4\). Following the notation of Crowley-Escher [C03E], the total manifold \(M\) is denoted by \(M_{\rho,\gamma}\) if it is classified by a pair of integers \((\rho, \gamma) \in \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(SO(4))\). Moreover, as cell complex
\[ M_{\rho,\gamma} \simeq P^4(\gamma) \cup e^7, \]
where the Moore space \(P^4(\gamma)\) is the mapping cone of the degree map \(\gamma : S^3 \to S^3\). When \(\gamma\) is odd, the loop decomposition of the Poincaré complex \(P^4(\gamma) \cup e^7\) was determined by Theriault and the author [HT22]. For any prime \(p\), let \(S^m\{p^r\}\) be the homotopy fibre of the degree \(p^r\) map on \(S^m\). Let \(\gamma = p_1^{r_1} \cdots p_\ell^{r_\ell}\) be the prime decomposition of \(\gamma\). By [HT22, Theorem 1.1], when \(\gamma\) is odd there is a homotopy equivalence
\[ \Omega M_{\rho,\gamma} \simeq \Omega(P^4(\gamma) \cup e^7) \simeq \prod_{j=1}^\ell S^3\{p_j^{r_j}\} \times \Omega S^7. \]

When \(\gamma\) is even, it is a hard problem to determine the loop decomposition of \(M_{\rho,\gamma}\). Partial results are obtained in [HT22, Section 6].

When \((d,n) = (0,2)\), we have the sphere bundle \(S^2 \overset{i}{\to} M \overset{p}{\to} S^4\). This case was studied in [Hua22, Section 5]. Let \(x \in H^2(M;\mathbb{Z})\), \(y \in H^4(M;\mathbb{Z})\) be two generators. Suppose \(x^2 = \pm \gamma y\) for some \(\gamma \in \mathbb{Z}\). Then it can be shown that [Hua22, Lemma 5.1] the circle bundle classified by \(x\) is of the form
\[ S^1 \overset{j}{\to} M_{\rho,\gamma} \to M, \]
where \(j\) is null homotopic. Therefore, there is a homotopy equivalence
\[ \Omega M \simeq S^1 \times \Omega M_{\rho,\gamma}. \]

This case is then reduced to the case when \((d,n) = (0,3)\), and with the results of [HT22] we can prove a nice loop homotopy decomposition when \(\gamma\) is odd, and partial results when \(\gamma\) is even.

In the end of this section, we digress to give an example that is used in the Introduction.

**Example 5.1.** Consider the pullback of the sphere bundle \(M_{\rho,\gamma}\) along the quotient map \(q\)

\[
\begin{array}{ccc}
S^3 & \to & M \\
\downarrow & & \downarrow \\
S^3 & \to & M_{\rho,\gamma} \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
CP^2 & \to & S^4,
\end{array}
\]

which defines the manifold \(M\). By the naturality of the Serre spectral sequence, it is easy to see that the homology of \(M\) satisfies
\[ H_0(M;\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong H_5(M;\mathbb{Z}) \cong H_7(M;\mathbb{Z}) \cong \mathbb{Z}, \quad H_3(M;\mathbb{Z}) \cong \mathbb{Z}/\gamma, \quad H_i(M;\mathbb{Z}) = 0, \text{ other } i. \]

In particular, it implies that as graded groups
\[ H_*(M;\mathbb{Z}) \neq H_*(S^3;\mathbb{Z}) \otimes H_*(CP^2;\mathbb{Z}). \]
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