Forced oscillations of beams with installation (Part II)

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Abstract. This article (Part II) is a continuation of Part I, published in the same edition, where the spectral problem is considered. Part II examines two types of transverse beam oscillations: forced harmonic and forced random. The beam carries its own distributed mass and discrete mass. The source of oscillations is operating equipment with an element moving during technological operations. The mathematical model of oscillations is presented as a boundary value problem from the main partial differential equation of the hyperbolic type of the fourth order in the spatial coordinate, the second order in time, the boundary conditions and the conditions of the beam sections’ conjugation. The technical theory of the rods’ bending oscillations is used, based on Bernoulli’s hypothesis about the beam plane cross-sections’ invariability. The methods of variables separation and finite differences are applied. The algorithms for solving the problems have been developed, implemented in the Matlab software environment. Verification of the proposed mathematical models is demonstrated using the specific examples. The particular examples have been carried out and the practical conclusions have been outlined.

Introduction
Beams are the widespread elements of buildings and structures of heavy and light industry, petrochemical industries, road transport highways, processing enterprises. Beams are the parts of machines, technological equipment, machine tools, robots and manipulators. The requirements of digital economy, automation and computerization of production required a more accurate and adequate design and construction of beams that experience non-classical man-made and seismic external influences of a dynamic and kinematic nature [1-6]. Beams are subjected to technogenic oscillations, which can be both deterministic and stochastic, are very dangerous and difficult to design. These two types of oscillations are very closely related, since random influences (processes) consist almost always of an infinite set of harmonics [7].

Forced oscillations’ mathematical model
The design diagram of the beam, shown in Figure 1, carries the equipment with discrete mass \( M \) in the middle of the span. The equipment contains mass \( m_0 \), moving plane-parallel in the vertical direction or rotating uniformly around the axis. It is the source of the concentrated force \( F(t) \). In addition to them, the beam itself has its own linear mass \( m \).
There is a wide variety of beams by material, different shapes of cross-sections, constant or variable cross-sections in length, etc. The most suitable in the case, when heavy equipment is installed on the beam and it should be in the same plane with the intermediate floor, is an I-beam paired section.

![Figure 1. Calculation scheme.](image1)

![Figure 2. Sections’ pairing.](image2)

We will use the technical theory of bending oscillations of rods and write out the basic equation of the inhomogeneous differential equation of hyperbolic type in partial derivatives [2].

\[
b u''(x, t) + \mu u(x, t) + \epsilon m u(x, t) = 0, \quad b = EJ, \quad x \in (0, l), \quad t > -\infty. \tag{1}
\]

Traditional notations are used here: \(u(x, t)\) – bending function of the curved longitudinal axis of the beam; \(x, t\) – coordinates in space and time; \(b = EJ\) – constant bending stiffness of a beam along the axis; \(E\) – Young’s modulus of the beam material; \(J\) – axial moment of a beam cross-section inertia; \(m\) – linear density of the beam material; \(\epsilon\) – specific coefficient of internal friction linear viscous forces.

Figures in superscripts correspond to the derivatives with respect to the spatial coordinate, the dots above the letters correspond to the derivatives with respect to time. The number of lines and dots corresponds to the differentiation order.

Boundary conditions and a condition for mating beam sections are added to the main equation. Boundary conditions at the left and right ends of the beam are:

\[
u(0, t) = 0, \quad u'(0, t) = 0, \quad u(l, t) = 0, \quad u''(l, t) = 0. \tag{2}
\]

The condition for mating left and right sections:

\[
b[u''(l/2 - 0, t) - u''(l/2 + 0, t)] - \mu u(l/2, t) = F(t). \quad t > -\infty. \tag{3}
\]

The function \(F(t)\) represents harmonic or random effects of equipment on a beam. Together (1) - (3) form a mathematical model of forced harmonic and random steady-state oscillations. It should be noted that in such problems the initial conditions are not required.

Harmonic oscillations

Harmonic oscillations of the beam are considered in view of the fact that technogenic oscillations are often harmonic or poly-harmonic. In addition, the results obtained for the harmonic oscillations give an approximate idea of random oscillations and greatly facilitate the understanding and solution of the problem about them.

In equation (3), in this case, we assume that the force is described by the function

\[
F(t) = fe^{\lambda t},
\]

where \(f\) – is the harmonic force amplitude, \(\lambda = j\omega, j\) – is an imaginary unit.

In the method of variables separation [8], which will be further applied, the output function has the form of a harmonic oscillation

\[
u(x, t) = X(x) \ e^{\lambda t}.
\]

Here \(X(x)\) – is the oscillation amplitude. Similarly to the case of free oscillations, using the finite difference method [9, 10], it is possible to obtain the inhomogeneous system of linear algebraic equations

\[
A(\lambda) \cdot Y = d,
\]
where $A(\lambda)$ – is a square matrix of order coefficients $n$, $Y = \{y_1, y_2, ..., y_n\}$ – is a vector of discrete arguments replacing a continuous function $X(x)$. The coefficient matrix and the vector on the right side have the form

$$
A(\lambda) = \begin{pmatrix}
1 & -4 & 4 & -1 \\
2 & 5 & 4 & -1 \\
1 & -4 & \mu & -4 & 1 \\
1 & -4 & \mu & -4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -4 & \mu & -4 & 1 \\
\end{pmatrix}
$$

$$
d^T = \{0, ..., d_k, ..., 0\}.
$$

Here zero elements of the matrix and vector are not inscribed, the vector of the right-hand side contains a single element with the number $k = (n+1)/2$, corresponding to the conjugation point of the left and right sections of the beam, the T sign in the superscript corresponds to the operation of vector transposition.

The designations for the matrix elements are introduced:

$$
\mu = b + \Lambda h^4/b, \quad \eta = 10^{-m\lambda^2/b}, \quad \Lambda = m\lambda(\lambda + \varepsilon).
$$

Let us carry out the calculations for a specific beam with the parameters.

**Example 1.** The beam is composed of two I-beams No.30, $l = 8$ m, \( E = 210 \text{ GPa} \), \( J = 14160 \text{ cm}^4 \), \( m = 73 \text{ kg/m} \), \( M = 4000 \text{ kg} \), \( \varepsilon = 0.5 \text{ c}^{-1} \), \( n = 4001 \), \( f = 100 \text{ kN} \), \( \omega = \{1; 15; 50; 100\} \text{ c}^{-1} \), the spatial coordinate step $h = l/(n-1)$. For use in analyzing the results, we indicate the eigenvalues \( \Omega = \{25,51; 393,69\} \text{ c}^{-1} \). It is required to make the calculations and build a graph of the oscillation amplitudes function along the length of the beam $Y(x)$.

The counting results are shown in Figure 3 as graphs $Y(x)$ at the specified frequencies of exposure to equipment $\omega$. Frequency numbers are signed in order.

At low oscillation frequencies, the force $F$, lower than the first natural frequency, the oscillations of the beam have the form of the first form (curves 1, 2). With a further increase in the frequency of disturbances (curves 3, 4), the first resonance oscillations with large amplitudes occur, then they decrease and change the phase of oscillations (curves 3, 4) to the opposite force phase. At high $\omega$ values, the oscillations almost disappear and become imperceptibly small and take place in the second and higher modes of oscillations. These results are of no practical importance and therefore are not shown. At the same time, they may be of theoretical interest.

The results obtained for the harmonic oscillations make it possible to predict the dynamic behavior of a beam under random influences.

**Random oscillations**

Man-made impacts on a beam are often the random processes in time. Moreover, in most cases, the stationary part of such processes lasts longer than the others and poses the greatest danger to building
structures. Therefore, the corresponding oscillations will be further considered as stochastic within the framework of the stationary random processes’ theory. The results on harmonic oscillations obtained above will help to solve such problem in relatively simple ways, without resorting to complex and cumbersome existing methods. For this purpose, in the mathematical model for harmonic oscillations given above, we will make the necessary changes for the random oscillations. They will only concern the input force \( F(t) \). According to the above-mentioned, the output process \( u(x, t) \) will be a centered spatio-temporal random field, stationary in time and inhomogeneous in the spatial coordinate. The task is to find the standard deviations of the output process from the given spectral density of the input random process. Spectral density functions of the input stationary technogenic processes for more frequent cases have the form:

\[
S(\omega) = \frac{4\theta^2 \omega^2}{\pi} \left[ \left( \omega^2 - \theta^2 \right)^2 + 4\omega^2 \theta^2 \right], \quad \omega \in [0, \infty), \quad \theta^2 = \alpha^2 + \beta^2. \tag{4}
\]

Here \( \alpha, \beta \) are the broadband and dominant frequency parameters, \( \sigma \) is the process standard deviation \( \sigma \). Processes with spectral density (4) are called the narrow-band random processes with a characteristic frequency. As it is known, a stationary random process can be represented as a Fourier series [7]

\[
F(t) = \sum_{i=1}^{\infty} \left[ U_i \cos \omega_i t + V_i \sin \omega_i t \right]. \tag{5}
\]

The simple transformations taking into account the relationship between the variance and the correlation function of a stationary random process and the series coefficients correlation (5) give the force variance

\[
D[F(t)] = \sum_{i=1}^{\infty} D_i.
\]

This result is shown in Figure 4 in the form of the so-called discrete line spectrum variances. It turns out that there is a binary correspondence between dispersions and frequencies, that is, each elementary dispersion can be associated with a certain frequency of the spectral density. In this case, the random process variance of the input process \( F(t) \) seems to be an improper integral [10]

\[
D_F = \int_{-\infty}^{\infty} S(\omega) d\omega.
\]

Figure 5 shows that the elementary variance \( S(\omega)d\omega \), shaded in Figure 5 is associated with each frequency \( \omega \). If the elementary variance is taken instead of the driving force amplitude, the output will be the elementary variance of the discrete mass deviations \( dD_x(\omega_k) \). Subsequent summation over \( \omega_k (k = 1, 2, ..., n) \) gives the variance of deviations \( D_x \). A very effective method for obtaining variances for stationary random oscillations at the output, which ultimately will consist in calculating the sum can be obtained from it:
\[ D_z = h_\omega \sum_k S_k. \]

Here \( h_\omega \) is a frequency axis split step. Further calculations are carried out as in Example 1, namely, instead of the amplitude of this harmonic \( F(t) \) the elementary variance of disturbances is inserted into the algorithm and computation program \( S_k h_\omega \). The results of their summation give the output displacement process variance.

Example 2. The beam consists of two steel I-beams No. 30, \( l = 8 \text{ m} \), \( E = 210 \text{ GPa} \), \( J = 14160 \text{ cm}^4 \), \( m = 73 \text{ kg/m} \), \( M = 4000 \text{ kg} \), \( \varepsilon = 0.5 \text{ c}^{-1} \), \( n = 2001 \), \( \beta = \{4; 6; 90; 98\} \text{ c}^{-1} \), \( \sigma_f = 141 \text{ kN} \), the spatial coordinate step \( h = l/(n-1) \). It is required to calculate and plot the function of the standard deviations of the beam oscillations \( \sigma_\omega(x) \).

The calculations shown in the graphs in Figures 3 and 6, specially carried out in the most common frequency range of the equipment operation \( \omega \in [1, 100] \). When considered together, taking into account that standard deviations can only take positive values, they have many quantitative and qualitative similarities.

**Summary**

1. The standard deviations of random oscillations have the values and shapes close to the harmonic oscillations’ amplitudes.
2. Small values of the disturbance frequency (\( \omega = 1, \ldots, 3 \)) and large (\( \omega > 150 \) \text{ c}^{-1} \) are not dangerous for this beam, since the values of the beam dynamic deflections are insignificant (0, ..., 2) mm.

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