THE $k_R$-PROPERTY ON FREE TOPOLOGICAL GROUPS

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Abstract. A space $X$ is called a $k_R$-space, if $X$ is Tychonoff and the necessary and sufficient condition for a real-valued function $f$ on $X$ to be continuous is that the restriction of $f$ on each compact subset is continuous. In this paper, we mainly discuss the $k_R$-property on the free topological groups, and generalize some well-known results of K. Yamada’s in the free topological groups.

1. Introduction

Recall that $X$ is called a $k$-space, if the necessary and sufficient condition for a subset $A$ of $X$ to be closed is that $A \cap C$ is closed for every compact subset $C$. It is well-known that the $k$-property generalizing metrizability has been studied intensively by topologists and analysts. A space $X$ is called a $k_R$-space, if $X$ is Tychonoff and the necessary and sufficient condition for a real-valued function $f$ on $X$ to be continuous is that the restriction of $f$ on each compact subset is continuous. Clearly every Tychonoff $k$-space is a $k_R$-space. The converse is false. Indeed, for any uncountable cardinal $\kappa$ the power $\mathbb{R}^\kappa$ is a $k_R$-space but not a $k$-space, see [11, Theorem 5.6] and [10, Problem 7.J(b)]. Now, the $k_R$-property has been widely used in the study of topology, analysis and category, see [3, 4, 5, 6, 13, 16, 17].

Results of our research will be presented in two separate papers. In the paper [14], we mainly extend some well-known results in $k$-spaces to $k_R$-spaces, and then seek some applications in the study of free Abelian topological groups. In the current paper, we shall deter the $k_R$-property on free topological groups and extend some results of K. Yamada’s on free topological groups.

The paper is organized as follows. In Section 2, we introduce the necessary notation and terminologies which are used for the rest of the paper. In Section 3, we investigate the $k_R$-property on free topological groups, and generalize some results of K. Yamada’s. In section 4, we pose some interesting questions about $k_R$-spaces in the class of free topological groups which are still unknown for us.

2. Preliminaries

In this section, we introduce the necessary notation and terminologies. Throughout this paper, all topological spaces are assumed to be Tychonoff, unless otherwise is explicitly stated. First of all, let $\mathbb{N}$ be the set of all positive integers and $\omega$ the first countable order. For a space $X$, we always denote the set of all the non-isolated points by $NI(X)$. For undefined notation and terminologies, the reader may refer to [2], [8], [9] and [15].

Let $X$ be a topological space and $A \subseteq X$ be a subset of $X$. The closure of $A$ in $X$ is denoted by $\overline{A}$. Moreover, $A$ is called bounded if every continuous real-valued function $f$ defined on $A$ is bounded. The space $X$ is called a $k$-space provided that a subset $C \subseteq X$ is closed in $X$ if $C \cap K$ is closed in $K$ for each compact subset $K$ of $X$. A space $X$ is called a $k_R$-space, if $X$ is Tychonoff and the necessary and sufficient condition for a real-valued function $f$ on $X$ to be continuous is that the restriction of $f$ on each compact subset is continuous. Note that every Tychonoff $k$-space is a $k_R$-space. A subset $P$ of $X$ is called a sequential neighborhood of $x \in X$,

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if each sequence converging to $x$ is eventually in $P$. A subset $U$ of $X$ is called \textit{sequentially open} if $U$ is a sequential neighborhood of each of its points. A subset $F$ of $X$ is called \textit{sequentially closed} if $X \setminus F$ is sequentially open. The space $X$ is called a \textit{sequential space} if each sequentially open subset of $X$ is open. The space $X$ is said to be Fréchet-Urysohn if, for each $x \in \overline{A} \subset X$, there exists a sequence $\{x_n\}$ such that $\{x_n\}$ converges to $x$ and $\{x_n : n \in \mathbb{N}\} \subset A$.

\textbf{Definition 2.1.} \[4\] Let $X$ be a topological space.

- A subset $U$ of $X$ is called \textit{$\mathbb{R}$-open} if for each point $x \in U$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$. It is obvious that each $\mathbb{R}$-open is open. The converse is true for the open subsets of Tychonoff spaces.
- A subset $U$ of $X$ is called a \textit{functional neighborhood} of a set $A \subset X$ if there is a continuous function $f : X \to [0, 1]$ such that $f(A) \subset \{1\}$ and $f(X \setminus U) \subset \{0\}$. If $X$ is normal, then each neighborhood of a closed subset $A \subset X$ is functional.

\textbf{Definition 2.2.} Let $\lambda$ be a cardinal. An indexed family $\{X_\alpha\}_{\alpha \in \lambda}$ of subsets of a topological space $X$ is called
- \textit{point-countable} if for any point $x \in X$ the set $\{\alpha \in \lambda : x \in X_\alpha\}$ is countable in $X$;
- \textit{compact-countable} if for any compact subset $K$ in $X$ the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is countable in $X$;
- \textit{locally finite} if any point $x \in X$ has a neighborhood $O_x \subset X$ such that the set $\{\alpha \in \lambda : O_x \cap X_\alpha \neq \emptyset\}$ is finite in $X$;
- \textit{compact-finite} if for each compact subset $K \subset X$ the set $\{\alpha \in \lambda : K \cap X_\alpha \neq \emptyset\}$ is finite in $X$;
- \textit{strongly compact-finite} \[2\] in $X$ if for each set $X_\alpha$ has an $\mathbb{R}$-open neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha \in \lambda}$ is compact-finite in $X$;
- \textit{strictly compact-finite} \[4\] in $X$ if for each set $X_\alpha$ has a functional neighborhood $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha \in \lambda}$ is compact-finite in $X$.

\textbf{Definition 2.3.} \[4\] Let $X$ be a topological space and $\lambda$ be a cardinal. An indexed family $\{F_\alpha\}_{\alpha \in \lambda}$ of subsets of a topological space $X$ is called a \textit{fan} (more precisely, a $\lambda$-fan) in $X$ if this family is compact-finite but not locally finite in $X$. A fan $\{F_\alpha\}_{\alpha \in \lambda}$ is called \textit{strong} (resp. \textit{strict}) if each set $F_\alpha$ has an $\mathbb{R}$-open neighborhood (resp. functional neighborhood) $U_\alpha \subset X$ such that the family $\{U_\alpha\}_{\alpha \in \lambda}$ is compact-finite in $X$.

If all the sets $F_\alpha$ of a $\lambda$-fan $\{F_\alpha\}_{\alpha \in \lambda}$ belong to some fixed family $\mathcal{F}$ of subsets of $X$, then the fan will be called an $\mathcal{F}^\lambda$-fan. In particular, if each $F_\alpha$ is closed in $X$, then the fan will be called a \textit{Clh$^\lambda$-fan}.

Clearly, we have the following implications:

\textit{strict fan} $\Rightarrow$ \textit{strong fan} $\Rightarrow$ \textit{fan}.

Let $\mathcal{P}$ be a family of subsets of a space $X$. Then, $\mathcal{P}$ is called a \textit{k-network} if for every compact subset $K$ of $X$ and an arbitrary open set $U$ containing $K$ in $X$ there is a finite subfamily $\mathcal{P}' \subseteq \mathcal{P}$ such that $K \subseteq \bigcup \mathcal{P}' \subseteq U$. Recall that a space $X$ is an $\mathcal{N}$-space (resp. $\mathcal{K}_0$-space) if $X$ has a $\sigma$-locally finite (resp. countable) $k$-network. Recall that a space $X$ is said to be \textit{Lašnev} if it is the continuous closed image of some metric space.

\textbf{Definition 2.4.} \[7\] A topological space $X$ is a \textit{stratifiable space} if for each open subset $U$ in $X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of $X$ such that
(a) $U_n \subset U$;
(b) $\bigcup_{n=1}^{\infty} U_n = U$;
(c) $U_n \subset V_n$ whenever $U \subset V$.

\textbf{Note:} Each Lašnev space is stratifiable \[9\].

Let $X$ be a non-empty space. Throughout this paper, $X^{-1} = \{x^{-1} : x \in X\}$, which is just a copy of $X$. For every $n \in \mathbb{N}$, the $F_n(X)$ denotes the subspace of $F(X)$ that consists of all
the words of reduced length at most n with respect to the free basis X. Let e be the neutral element of F(X), that is, the empty word. For every n ∈ N, an element x₁x₂⋯xₙ is also called a form for (x₁, x₂, ⋯, xₙ) ∈ (X ⊕ X⁻¹ ⊕ {e})ⁿ. The word g is called reduced if it does not contain e or any pair of consecutive symbol of the form xx⁻¹. It follows that if the word g is reduced and non-empty, then it is different from the neutral element e of F(X). In particular, each element g ∈ F(X) distinct from the neutral element can be uniquely written in the form g = x₁x₂⋯xₙ, where n ≥ 1, ε₁ ∈ \{-1, 1\}, xᵢ ∈ X, and the support of g = x₁x₂⋯xₙ is defined as supp(g) = \{x₁, ⋯, xₙ\}. Given a subset K of F(X), we put supp(K) = ⋃_{g∈K} supp(g).

For every n ∈ N, let
\[ i_n : (X ⊕ X⁻¹ ⊕ {e})ⁿ → F_n(X) \]
be the natural mapping defined by
\[ i_n(x₁, x₂, ⋯, xₙ) = x₁x₂⋯xₙ \]
for each (x₁, x₂, ⋯, xₙ) ∈ (X ⊕ X⁻¹ ⊕ {0})ⁿ.

3. The kᵣ-property on free topological groups

In this section, we investigate the kᵣ-property on free topological groups, and generalize some results of K. Yamada’s. Recently, T. Banakh in [4] proved that F(X) is a k-space if F(X) is a kᵣ-space for a Lašnev space X. Indeed, he obtained this result in the class of more weaker spaces. However, he did not discuss the following question:

**Question 3.1.** Let X be a space. For some n ∈ ω, if Fₙ(X) is a kᵣ-space, is Fₙ(X) a k-space?

First, we give the following Theorem 3.3 which gives a complementary result for the theorem of T. Banakh’s.

**Lemma 3.2.** Let F(X) be a kᵣ-space. If each Fₙ(X) is a normal k-space, then F(X) is a k-space.

**Proof.** It is well-known that each compact subset of F(X) is contained in some Fₙ(X) [2 Corollary 7.4.4]. Hence it follows from [13, Lemma 2] that F(X) is a k-space.

**Theorem 3.3.** Let X be a paracompact σ-space. Then F(X) is a kᵣ-space and each Fₙ(X) is a k-space if and only if F(X) is a k-space.

**Proof.** Since X is a paracompact σ-space, it follows from [2, Theorem 7.6.7] that F(X) is also a paracompact σ-space, hence each Fₙ(X) is normal. Now it is legal to apply Lemma 3.2 and finish the proof.

Next, we shall show that for arbitrary a metrizable space X, the kᵣ-property of Fₙ(X) implies that F(X) is a k-space, see Theorem 3.4. We first prove two technic propositions. To prove them, we need the description of a neighborhood base of e in F(X) obtained in [20].

\[ H₀(X) = \{ h = x₁x₂⋯x₂n ∈ F(X) : \sum_{i=1}^{2n} εᵢ = 0, xᵢ ∈ X \text{ for } i ∈ \{1, 2, ⋯, n\}, n ∈ ℕ \} \]

Obviously, the subset H₀(X) is a clopen normal subgroup of F(X). It is easy to see that each h ∈ H₀(X) can be represented as
\[ h = g₁x₁y₁ε₁g₁⁻¹g₂x₂y₂ε₂g₂⁻¹⋯gₙxₙyₙεₙgₙ⁻¹, \]
for some n ∈ N, where xᵢ, yᵢ ∈ X, εᵢ = ±1, and gᵢ ∈ F(X) for i ∈ \{1, 2, ⋯, n\}. Let P(X) be the set of all continuous pseudometrics on a space X. Then take an arbitrary \( r = \{ ρ_y : g ∈ F(X) \} \in P(X)^{F(X)}. \) Let
\[ p_r(h) = \inf \left( \sum_{i=1}^{n} ρ_y(xᵢ, yᵢ) : h = g₁x₁y₁ε₁g₁⁻¹g₂x₂y₂ε₂g₂⁻¹⋯gₙxₙyₙεₙgₙ⁻¹, n ∈ ℕ \right) \]
for each h ∈ H₀(X). In [20], Uspenskii proved that
(a) \( \rho_\varepsilon \) is a continuous on \( H_0(X) \) and
(b) \( \text{supp}(h) : p_\varepsilon(h) < \delta : r \in P(X)^{F(X)}, \delta > 0 \) is a neighborhood base of \( e \) in \( F(X) \).
Moreover, \( p_\varepsilon(e) = 0 \) for each \( r \in P(X)^{F(X)} \).

Proposition 3.4. For a stratifiable k-space \( X \) if \( F_6(X) \) is a \( k_{R}\)-space, then \( X \) is separable or discrete.

Proof. Assume on the contrary that \( X \) is neither separable nor discrete. Then \( X \) contains a space \( Y = C \bigoplus D \) as a closed subset, where \( C = \{x_n : n \in \omega\} \cup \{x\} \) is a convergent sequence with its limit point \( \{x\} \) and \( D = \{d_\alpha : \alpha \in \omega_1\} \) is a discrete closed subset of \( X \). Since \( D \) is a discrete closed subset of \( X \), we choose a discrete family \( \{O_\alpha\}_{\alpha \in \omega_1} \) of open subsets such that \( d_\alpha \in O_\alpha \) for each \( \alpha \in \omega_1 \). We may assume that \( x_n \neq x_m \) for arbitrary \( n \neq m \) and \( C \cap \bigcup_{\alpha \in \omega_1} O_\alpha = \emptyset \).
Since \( X \) is stratifiable and \( Y \) is closed in \( X \), it follows from [20] that \( F(Y) \) is homeomorphic to a closed subgroup of \( F(X) \). Hence \( F_6(Y) \) is closed subspace of \( F_6(X) \). Next we shall show that \( F_6(X) \) contains a strict \( C^{\omega}-\text{fan} \), a contradiction. Indeed, since \( F_6(X) \) is normal, it suffices to construct a strong \( C^{\omega}-\text{fan} \) in \( F_6(X) \).

For each \( \alpha \in \omega_1 \) choose a function \( f_\alpha : \omega_1 \to \omega \) such that \( f_\alpha|_{\alpha} : \alpha \to \omega \) is a bijection. For each \( n \in \omega \), let
\[
F_n = \{\langle n^{-1} 1 - 1, x_m, d_\beta x_n, n^{-1} d_\alpha^{-1} \rangle : f_\alpha(\beta) = n, \alpha, \beta \in \omega_1, m \leq n\}.
\]
We claim that the family \( \{F_n : n \in \omega\} \) is a strong \( C^{\omega}-\text{fan} \) in \( F_6(X) \). We divide into the proof by the following three statements.

(1) For each \( n \in \omega \), the set \( F_n \) is closed in \( F_6(X) \).

Fix an arbitrary \( n \in \omega \). It suffices to show that the set \( F_n \) is closed in \( F_6(Y) \). Let \( Z = \text{supp}F_n \).
Then it is obvious that \( Z \) is a closed discrete subspace of \( Y \). It follows from [20] that \( F(Z) \) is homeomorphic to a closed subgroup of \( F(Y) \), and thus \( F_6(Z) \) is a closed subspace of \( F_6(Y) \).
Since \( F(Z) \) is discrete and \( F_6 \subset F_6(Z) \), the set \( F_n \) is closed in \( F_6(Y) \) (and thus closed in \( F(X) \)).

(2) The family \( \{F_n : n \in \omega\} \) is strong compact-finite in \( F_6(X) \).

By induction, choose two families of open neighborhoods \( \{W_n\}_{n \in \omega} \) and \( \{V_n\}_{n \in \omega} \) in \( X \) satisfy the following conditions:

(a) for each \( n \in \omega \), \( x_n \in W_n \);
(b) for each \( n \in \omega \), \( x \in V_n \) and \( V_{n+1} \subset V_n \);
(c) for each \( n \in \omega \), we have \( W_n \cap (C \cup D) = \{x_n\} \), \( V_n \cap (D \cup W_n) = \emptyset \) and \( V_n \cap C \subset C_n \), where \( C_n = \{x_m : m > n\} \cup \{x\} \);
(d) \( V_1 \cap \bigcup_{\alpha \in \omega_1} O_\alpha = \emptyset \) and \( W_n \cap \bigcup_{\alpha \in \omega_1} O_\alpha = \emptyset \) for each \( n \in \omega \).

For each \( n \in \omega \), let
\[
U_n = \bigcup \{O_\alpha^{-1} W_n^{-1} O_\beta O_\alpha W_n V_\alpha^{-1} O_\beta^{-1} : f_\alpha(\beta) = n, \alpha, \beta \in \omega_1, m \leq n\}.
\]
Obviously, each \( U_n \) contains in \( F_6(X) \setminus F_7(X) \), and since \( F_6(X) \setminus F_7(X) \) is open in \( F_6(X) \), it follows from [2] Corollary 7.1.19 that each \( U_n \) is open in \( F_6(X) \). We claim that the family \( \{U_n : n \in \omega\} \) is compact-finite in \( F_6(X) \). Suppose not, there exist a compact subset \( K \) in \( F_6(X) \) and an increasing sequence \( \{n_k\} \) such that \( K \cap U_{n_k} \neq \emptyset \) for each \( k \in \omega \). Since \( X \) is stratifiable, \( F(X) \) is paracompact, then the closure of the set \( \text{supp}(K) \) is compact in \( X \). However, for each \( k \in \omega \), since \( K \cap U_{n_k} \neq \emptyset \), there exist \( m_k \in \omega \), \( \alpha_k \in \omega_1 \) and \( \beta_k \in \omega_1 \) such that \( f_{\alpha_k}(\beta_k) = n_k \) and \( K \cap O_\beta^{-1} W_m^{-1} O_{\beta_k} O_{\alpha_k} W_n V_{\alpha_k}^{-1} O_{\beta_k}^{-1} \neq \emptyset \), hence \( \text{supp}(K) \cap O_{\alpha_k} \neq \emptyset \) and \( \text{supp}(K) \cap O_{\beta_k} \neq \emptyset \). Therefore, the set \( \text{supp}(K) \) intersects each element of the family \( \{O_{\alpha_k}, O_{\beta_k} : k \in \omega\} \). Since the set \( \{n_k : k \in \omega\} \) is infinite and \( f_{\alpha_k}(\beta_k) = n_k \) for each \( k \in \omega \), the set \( \{\alpha_k, \beta_k : k \in \omega\} \) is infinite. Then \( \text{supp}(K) \) intersects infinitely many \( O_{\alpha_k}'s \), which is a contradiction with the compactness of \( \text{supp}(K) \). Therefore, the family \( \{U_n : n \in \omega\} \) is compact-finite in \( F_6(X) \). Therefore, the family \( \{F_n : n \in \omega\} \) is strong compact-finite in \( F_6(X) \).

(3) The family \( \{F_n : n \in \omega\} \) is not locally finite at the point \( e \) in \( F_6(Y) \) (and thus not locally finite in \( F_6(X) \)).
Indeed, it suffices to show that \( e \in \bigcup_{n \in \omega} F_n \setminus \bigcup_{n \in \omega} F_n \) in \( F_\delta(Y) \). For any \( \delta > 0 \) and \( r = \{ \rho_g : g \in F(Y) \} \), we shall show that
\[
\{ h \in F_\delta(Y) : p_r(h) < \delta \} \cap \bigcup_{n \in \omega} F_n \neq \emptyset.
\]
Since the sequence \( \{ x_n \} \) converges to \( x \) and \( \rho_{d_n} \) and \( \rho_{d_n^{-1}} \) are continuous pseudometrics on \( Y \) for each \( \alpha \in \omega_1 \), there is \( n(\alpha) \in \omega \) such that \( \rho_{d_n}(x_n, x) < \frac{\delta}{2} \) and \( \rho_{d_n^{-1}}(x_n, x) < \frac{\delta}{2} \) for each \( n \geq n(\alpha) \). Therefore, there are \( n_0 \in \omega \) and uncountable set \( A \subset \omega_1 \) such that \( \rho_{d_n}(x_n, x) < \frac{\delta}{2} \) and \( \rho_{d^{-1}_n}(x_n, x) < \frac{\delta}{2} \) for each \( n \geq n_0 \) and \( \alpha \in A \). Choose \( \alpha \in A \) that has infinitely many predecessors in \( A \). Since \( f_\alpha(\alpha \cap A) \) is an infinite set, there exist \( m > n_0 \) and \( \beta \in \alpha \cap A \) such that \( f_\alpha(\beta) = m \). Then the word
\[
g = d^{-1}_\beta x^{-1} x_{n_0} d_\beta d_\alpha x_{f_\alpha(\beta)} x^{-1} d^{-1}_\alpha \in F_{f_\alpha(\beta)} = F_m.
\]
Furthermore, we have
\[
p_r(g) = p_r(d^{-1}_\beta x^{-1} x_{n_0} d_\beta d_\alpha x_{f_\alpha(\beta)} x^{-1} d^{-1}_\alpha) \leq \rho_{d^{-1}_\beta}(x_{n_0}, x) + \rho_{d_\alpha}(x_m, x) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]
Hence \( e \in \bigcup_{n \in \omega} F_n \). Then the family \( \{ F_n : n \in \omega \} \) is not locally finite at the point \( e \) in \( F_\delta(Y) \).

**Proposition 3.5.** For a metrizable space \( X \) if \( F_\delta(X) \) is a \( k_\delta \)-space, then \( X \) is locally compact.

**Proof.** Assume on the contrary that \( X \) is not locally compact. Then there exists a closed hedgehog subspace
\[
J = \{ x \} \cup \left( \bigcup_{n \in \omega} X_n \right) \cup \{ z_n : n \in \omega \}
\]
such that
1. \( X_n = \{ y_n \} \cup \{ x_{n,j} : j \in \omega \} \) is a closed discrete subset of \( J \) for each \( n \in \omega \);
2. \( \{ z_n : n \in \omega \} \) is a closed discrete subset of \( J \); and
3. \( \{ \{ x \} \cup \bigcup_{n \geq k} X_n : k \in \omega \} \) is a neighborhood base of \( x \) in \( J \).

By Proposition 3.4 the space \( X \) is separable. Next we shall show that \( F_\delta(X) \) contains a strict \( \text{Cld}^\omega \)-fan, which is a contradiction with \( F_\delta(X) \) is a \( k_\delta \)-space. Since \( F_\delta(X) \) is an \( \mathbb{R}_0 \)-space by \([3]\) Theorem 4.1, the subspace \( F_\delta(X) \) is normal, hence it suffices to show that \( F_\delta(X) \) contains a strong \( \text{Cld}^\omega \)-fan. Furthermore, it follows from \([3]\) Proposition 2.9.2 that each compact-finite family of subsets of \( X \) is strongly compact-finite, hence it suffices to show that \( F_\delta(X) \) contains a \( \text{Cld}^\omega \)-fan. For any \( n, j \in \omega \), put
\[
E_{n,j} = \{ z_n^{-1} y_j^{-1} x_{n,j} \}.
\]
Then it is obvious that each \( E_{n,j} \) is closed. Furthermore, it follows from the proof of \([19]\) Proposition 2.1 that \( x \in \bigcap_{n,j \in \omega} E_{n,j} \setminus \bigcap_{n,j \in \omega} E_{n,j} \), and thus the family \( \{ E_{n,j} : n, j \in \omega \} \) is not locally finite at the point \( x \). Next we claim that the family \( \{ E_{n,j} : n, j \in \omega \} \) is compact-finite.

Suppose not, there exist a compact subset \( K \) and two sequences \( \{ n_i \} \) and \( \{ j_i \} \) such that \( K \cap E_{n_i,j_i} \neq \emptyset \). Then the closure of the set \( \text{supp}(K) \) is compact since \( F_\delta(X) \) is paracompact. Since the family \( \{ E_{n,j} : n, j \in \omega \} \) are disjoint each other, one of the sequences \( \{ n_i \} \) and \( \{ j_i \} \) is an infinite set. If \( \{ n_i \} \) is an infinite set, then the closed discrete set \( \{ z_{n_i} : i \in \omega \} \) is contained in \( \text{supp}(K) \), which is a contradiction since \( \text{supp}(K) \) is compact. If \( \{ j_i \} \) is an infinite set and \( \{ n_i \} \) is a finite set, then there exists \( N \in \omega \) such that \( \{ x_{n_i,j_i} : i \in \mathbb{N} \} \subset \bigcup_{j \leq N} X_j \). Obviously, the closed discrete set \( \{ x_{n_i,j_i} : i \in \mathbb{N} \} \) is an infinite set and contains in \( \text{supp}(K) \), which is a contradiction.

Now we can show one of our main theorems in this paper.

**Theorem 3.6.** For a metrizable space \( X \), the following are equivalent:
1. \( F(X) \) is a \( k \)-space;
2. \( F_\delta(X) \) is a \( k_\delta \)-space;
(3) \( F_k(X) \) is a \( k_R \)-space;
(4) the space \( X \) is locally compact separable or discrete.

**Proof.** The equivalence of (1) and (4) was shown in [19]. It is obvious that (2) \( \Rightarrow \) (3). By Propositions 3.4 and 3.5, we have (3) \( \Rightarrow \) (4). \( \square \)

By Theorem 3.6 it is natural to ask the following question:

**Question 3.7.** Let \( X \) be a metrizable space. If \( F_n(X) \) is a \( k_R \)-space for some \( n \in \{4, 5, 6, 7\} \), then is \( F_k(X) \) a \( k_R \)-space?

**Note** For each \( n \in \{2, 3\} \), the answer to the above question is negative. Indeed, for an arbitrary metrizable space \( X \), since \( i_2 \) is a closed mapping, \( F_2(X) \) is a Fréchet-Urysohn space (and thus a \( k \)-space). For \( n = 3 \), we have the following Theorem 3.9. However, for each \( n \in \{4, 5, 6, 7\} \), the above question is still unknown for us.

**Proposition 3.8.** For a metrizable space \( X \) if \( F_3(X) \) is a \( k_R \)-space, then \( X \) is locally compact or \( NI(X) \) is compact.

**Proof.** Assume on the contrary that neither \( X \) is locally compact nor the set of all non-isolated points of \( X \) is compact. Then \( X \) contains a closed subspace

\[
Y = \{x\} \cup \bigcup_{n \in \omega} X_n \oplus \bigoplus_{n \in \omega} C_n,
\]

where for every \( n \in \omega \)

\[
X_n = \{x_{n,i} : i \in \omega\} \text{ is a closed discrete subset of } X,
\]

\[
\{x\} \cup \bigcup_{m \geq n} X_m : n \in \omega\} \text{ is a neighborhood base of } x \text{ in } Y,
\]

\[
C_n = \{c_{n,i} : i \in \omega\} \cup \{c\} \text{ is a convergent sequence with its limit } c_n, \text{ and}
\]

\[
C_n \text{ is contained in the open subset } U_n \text{ of } X \text{ such that the family } \{U_n : n \in \omega\} \text{ is discrete in } X \text{ and } (\{x\} \cup \bigcup_{m \in \omega} X_m) \cap \bigcup_{n \in \omega} U_m = \emptyset.
\]

In order to obtain a contradiction, we shall construct a strict \( \text{Cld}^\omega \)-fan in \( F_3(X) \). For any \( n, i \in \omega \), choose an open neighborhood \( O_n^i \) of the point \( x_{n,i} \) in \( X \) such that the family \( \{O_n^i : i \in \omega\} \) is discrete, \( O_n^i \cap \bigcup_{m \in \omega} U_m = \emptyset \) and \( O_n^i \cap (\{x\} \cup \bigcup_{n \in \omega} X_n) = \{x_{n,i}\} \).

For each \( n, i \in \omega \), let

\[
E(n, i) = \{g_{n,i} = c_n c_{n,i}^{-1} x_{n,i}\}
\]

and

\[
U(n, i) = V_n^i (W_n^i)^{-1} O_n^i,
\]

where \( V_n^i \) and \( W_n^i \) are two arbitrary open neighborhoods of \( c_n \) and \( c_{n,i} \) in \( X \) respectively such that \( V_n^i \cup W_n^i \subset U_n \) and \( V_n^i \cap W_n^i = \emptyset \). Obviously, each \( E(n, i) \) is closed and it follows from [21 Corollary 7.1.19] that \( U(n, i) \) is an open neighborhood of \( E(n, i) \) for each \( n, i \in \omega \). In [19], the author has showed that \( x \in \bigcup_{n,i} E(n, i) \setminus \bigcup_{n,i} E(n, i) \), hence the family \( \{E(n, i) : n, i \in \omega\} \) is not locally finite in \( F_3(X) \). To complete the proof, it suffices to show that the family \( \{U(n, i) : n, i \in \omega\} \) is compact-finite in \( F_3(X) \). Suppose not, there exists a compact subset \( K \) and two sequences \( \{n_j\} \) and \( \{i_j\} \) such that \( K \cap U(n_j, i_j) \neq \emptyset \). Similar to the proof of Proposition 3.5, we can obtain a contradiction. \( \square \)

**Theorem 3.9.** For a metrizable space \( X \) the following are equivalent:

(1) \( F_3(X) \) is a \( k \)-space;
(2) \( F_3(X) \) is a \( k_R \)-space;
(3) the space \( X \) is locally compact or \( NI(X) \) is compact.

**Proof.** The equivalence of (1) and (3) was showed in [19]. The implication of (1) \( \Rightarrow \) (2) is obvious. By Proposition 3.8, we have (2) \( \Rightarrow \) (3). \( \square \)

The following theorem was proved in [14].
Theorem 3.10. Let $X$ be a stratifiable space such that $X^2$ is a $k_R$-space. If $X$ satisfies one of the following conditions, then either $X$ is metrizable or $X$ is the topological sum of $k_\omega$-subspaces.

1. $X$ is a $k$-space with a compact-countable $k$-network;
2. $X$ is a Fréchet-Urysohn space with a point-countable $k$-network.

Since $F_2(X)$ contains a closed copy of $X \times X$, it follows from Theorem 3.10 that we have the following theorem.

Theorem 3.11. Let $X$ be a stratifiable $k$-space with a compact-countable $k$-network. Then the following are equivalent:

1. $F_2(X)$ is a $k$-space;
2. $F_2(X)$ is a $k_R$-space;
3. either $X$ is metrizable or $X$ is the topological sum of $k_\omega$-subspaces.

The following proposition shows that we can not replace “$F_3(X)$” with “$F_4(X)$” in Theorem 3.10. First, we recall a special space. Let

$$M_3 = \bigoplus \{C_\alpha : \alpha < \omega_1\},$$

where $C_\alpha = \{c(\alpha, n) : n \in \mathbb{N}\} \cup \{c_\alpha\}$ with $c(\alpha, n) \to c_\alpha$ as $n \to \infty$ for each $\alpha \in \omega_1$.

Proposition 3.12. The subspace $F_4(M_3)$ is not a $k_R$-space.

Proof. It suffices to show that $F_4(M_3)$ contains a strict Cl$\omega$-fan. It follows from [12] Theorem 20.2 that we can find two families $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ and $\mathcal{B} = \{B_\alpha : \alpha \in \omega_1\}$ of infinite subsets of $\omega$ such that

1. $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta \in \omega_1$;
2. for no $A \subset \omega$, all the sets $A_\alpha \setminus A$ and $B_\alpha \setminus A$, $\alpha \in \omega_1$ are finite.

For each $n \in \omega$, put

$$X_n = \{c(\alpha, n)c_\alpha^{-1}c(\beta, n)c_\beta^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.$$

It suffices to show the following three statements.

1. The family $\{X_n\}$ is strictly compact-finite in $F_4(M_3)$.

Since $M_3$ is a Lašnev space, it follows from [2] Theorem 7.6.7 that $F(M_3)$ is also a paracompact $\sigma$-space, hence $F_4(M_3)$ is paracompact (and thus normal). Hence it suffices to show that the family $\{X_n\}$ is strongly compact-finite in $F_4(M_3)$. For each $\alpha \in \omega_1$ and $n \in \omega$, let $C_\alpha^n = C_\alpha \setminus \{c(\alpha, m) : m \leq n\}$, and put

$$U_n = \{c(\alpha, n)x^{-1}c(\beta, n)y^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1, x \in C_\alpha^n, y \in C_\beta^n\}.$$

Obviously, each $X_n \subset F_4(M_3) \setminus F_3(M_3)$. Since $F_4(M_3) \setminus F_3(M_3)$ is open in $F_4(M_3)$, it follows from [2] Corollary 7.1.19 that each $U_n$ is open in $F_4(M_3)$. We claim that the family $\{U_n\}$ is compact-finite in $F_4(M_3)$. Suppose not, then there exist a compact subset $K$ in $F_4(M_3)$ and a subsequence $\{n_k\}$ of $\omega$ such that $K \cap U_{n_k} = \emptyset$ for each $k \in \omega$. For each $k \in \omega$, choose an arbitrary point

$$z_k = c(\alpha_k, n_k)x_k^{-1}c(\beta_k, n_k)y_k^{-1} \in K \cap U_{n_k},$$

where $x_k \in C_{\alpha_k}^{n_k}$ and $y_k \in C_{\beta_k}^{n_k}$. Since $F_4(M_3)$ is paracompact, it follows from [11] that the closure of the set supp$(K)$ is compact in $M_3$. Therefore, there exists $N \in \omega$ such that

$$\text{supp}(K) \cap \bigcup \{C_\alpha : \alpha \in \omega_1 \setminus \{\alpha_i : \omega_1 : i \leq N\} \} = \emptyset,$$

that is, supp$(K) \subset \bigcup_{\alpha \in \{\gamma_i : \omega_1 : i \leq N\}} C_\alpha$. Since each $z_k \in K$, there exists

$$\alpha_k, \beta_k \in \{\alpha_i \in \omega_1 : i \leq N\}$$

such that $A_{\alpha_k} \cap B_{\beta_k}$ is an infinite set, which is a contradiction with $A_{\alpha_k} \cap B_{\beta_k}$ is finite for all $\alpha, \beta < \omega_1$.

2. Each $X_n$ is closed in $F_4(M_3)$. 

Fix an arbitrary \( n \in \omega \). Next we shall show that \( X_n \) is closed in \( F_4(M_3) \). Let \( Z = \text{supp}(X_n) \). Then \( Z \) is a closed discrete subset of \( M_3 \). Since \( M_3 \) is metrizable, it follows from [20] that \( F(Z) \) is homeomorphic to a closed subgroup of \( F(M_3) \), hence \( F_4(Z) \) is a closed subspace of \( F_4(M_3) \). Since \( F(Z) \) is discrete and \( X_n \subset F_4(Z) \), the set \( X_n \) is closed in \( F_4(Z) \) (and thus closed in \( F_4(M_3) \)).

(3) The family \( \{X_n\} \) is not locally finite at the point \( e \) in \( F_4(M_3) \).

Indeed, it suffices to show that \( e \in \bigcup_{n \in \omega} X_n \setminus \bigcup_{n \in \omega} X_n \). For any \( \delta > 0 \) and \( r = \{\rho_g : g \in F(M_3)\} \), we shall show that

\[
\{h \in F_4(M_3) : p_r(h) < \delta\} \cap \bigcup_{n \in \omega} X_n \neq \emptyset.
\]

since \( \rho_e \) is continuous, we can choose a function \( f : \omega_1 \to \omega \) such that \( \rho_e(c(\alpha, n), c_\alpha) < \delta \) for any \( \alpha \in \omega_1 \) and \( n \geq f(\alpha) \). For each \( \alpha < \omega_1 \), put \( A'_\alpha = \{ n \in A_\alpha : n \geq f(\alpha) \} \) and \( B'_\alpha = \{ n \in B_\alpha : n \geq f(\alpha) \} \). By the condition (b) of the families \( \mathcal{A} \) and \( \mathcal{B} \), it is easy to see that there exist \( \alpha, \beta \in \omega_1 \) such that \( A'_\alpha \cap B'_\beta \neq \emptyset \). So, choose \( n \in A'_\alpha \cap B'_\beta \). Then \( \rho_e(c(\alpha, n), c_\alpha) < \frac{\delta}{2} \) and \( \rho_e(c(\beta, n), c_\beta) < \frac{\delta}{2} \). Let \( z = c(\alpha, n)c_\alpha^{-1}c(\beta, n)c_\beta^{-1} \). Then \( z \in X_n \) and

\[
p_r(z) \leq \rho_e(c(\alpha, n), c_\alpha) + \rho_e(c(\beta, n), c_\beta) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

hence \( z \in \{h \in F_4(M_3) : p_r(h) < \delta\} \cap \bigcup_{n \in \omega} X_n \). \( \square \)

**Theorem 3.13.** Let \( X \) be a metrizable space. If \( F_4(X) \) is a \( k_R \)-space, then \( \text{NI}(X) \) is separable.

**Proof.** Suppose not, then \( X \) contains a closed copy of \( M_3 \). Use the same notation as in Theorem 3.12. Since \( X \) is metrizable, there exists a discrete family \( \{U_\alpha\}_{\alpha \in \omega_1} \) of open subsets in \( X \) such that \( C_\alpha \subset U_\alpha \) for each \( \alpha \in \omega_1 \). For arbitrary \( (\alpha, n) \in \omega_1 \times \omega \), choose open neighborhoods \( V_{\alpha,n} \) and \( W_{\alpha,n} \) of the point \( c(\alpha, n) \) and \( c_\alpha \) in \( X \) respectively such that \( V_{\alpha,n} \cap W_{\alpha,n} = \emptyset \) and \( V_{\alpha,n} \cup W_{\alpha,n} \subset U_\alpha \). For each \( n \in \mathbb{N} \), put and

\[
O_n = \bigcup\{V_{\alpha,n}W_{\alpha,n}^{-1}V_{\beta,n}W_{\beta,n}^{-1} : n \in A_\alpha \cap B_\beta, \alpha, \beta \in \omega_1\}.
\]

Similar to the proof of Theorem 3.12, we can show that the family \( \{O_n\}_{n \in \omega} \) of open subsets is compact-finite in \( F_4(X) \), hence the family \( \{X_n\}_{n \in \omega} \) is a strict \( \text{Cld}^2 \)-fan in \( F_4(X) \), which is a contradiction. \( \square \)

### 4. Open questions

In this section, we pose some interesting questions about \( k_R \)-spaces in the class of free topological groups, which are still unknown for us.

By Theorems 3.9 and 3.11 it is natural to pose the following question:

**Question 4.1.** Let \( X \) be a metrizable space. For each \( n \in \{4, 5, 6, 7\} \), if \( F_n(X) \) is a \( k_R \)-space, is \( F_n(X) \) a \( k \)-space?

In [18], Yamada also made the following conjecture:

**Yamada’s Conjecture:** The subspace \( F_4(X) \) is Fréchet-Urysohn if the set of all non-isolated points of a metrizable space \( X \) is compact.

Indeed, we know no answer for the following question.

**Question 4.2.** Is \( F_4(X) \) a \( k_R \)-space if the set of all non-isolated points of a metrizable space \( X \) is compact?

In particular, we have the following question.

**Question 4.3.** Let \( X = C \bigoplus D \), where \( C \) is a non-trivial convergent sequence with its limit point and \( D \) is an uncountable discrete space. Is \( F_4(X) \) a \( k_R \)-space?

In [3], the authors showed that each closed subspace of a stratifiable \( k_R \)-space is a \( k_R \)-subspace. However, the following two questions are still open.
Question 4.4. Is each closed subgroup of a $k_R$-free topological group $k_R$?

Question 4.5. Is each subspace $F_n(X)$ of a $k_R$-free topological group $k_R$?

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