Layers and Matroids for the Traveling Salesman’s Paths

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Abstract

Gottschalk and Vygen proved that every solution of the well-known subtour elimination linear program for traveling salesman paths is a convex combination of a set of more and more restrictive “generalized Gao trees” of the underlying graph. In this paper we give a short proof of this, as a \textit{layered} convex combination of bases of a sequence of more and more restrictive matroids. Our proof implies (via the matroid partition theorem) a strongly polynomial, combinatorial algorithm for finding this convex combination. This is a new connection of the TSP to matroids, offering also a new polyhedral insight which was instrumental in the most recent approximation algorithm and bound on the integrality ratio for the $s - t$ path TSP.

\textbf{keywords:} path traveling salesman problem (TSP), matroid partition, approximation algorithm, spanning tree, Christofides’ heuristic, $T$-join, polyhedra

1 Introduction

Gottschalk and Vygen \cite{5} proved that every solution of the well-known subtour elimination linear program for traveling salesman paths is a convex combination of a set of more and more restrictive “generalized Gao trees” of the underlying graph, where a Gao tree is a spanning tree that meets certain cuts in exactly one edge.

In this paper we provide \textit{layered} convex combinations of bases of a sequence of more and more restrictive matroids for a larger set of points, which we call chain-points, generalizing the subtour elimination feasible solutions. This leads to a new connection of the TSP to matroids (observed in \cite{7}), offering also a polyhedral insight, with specific algorithmic consequences, such as a strongly-polynomial combinatorial algorithm for finding this convex combination via the matroid partition theorem. In this paper we show how the technical difficulties for proving the existence of such a particular convex combination and of turning it into an algorithm can be avoided, and a simple proof follows.

The existence of such a convex combination has been used by Sebő and Van Zuylen \cite{7} to prove the so far best approximation ratio for the $s - t$ path traveling salesman problem. This convex combination and the method we exhibit now to prove this result may possibly be adapted

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for proving further results on versions of the traveling salesman problem. The statement itself and its connection to matroids is one more link of the TSP to matroid partition or intersection, a connection interesting for its own sake.

In Section 2 we introduce chain-points, a key notion of our proof, and layered convex combinations, the related matroids, some preliminaries about these and some simple assertions that enable us to “peel off” layers one by one. In Section 3 we execute the induction step of peeling off a layer. We finish this introduction by introducing some notation and terminology.

We assume throughout that we are given a graph $G = (V,E)$, and $s,t \in V$. For $S_1,S_2 \subseteq V$, we denote by $\delta(S_1,S_2) := \{(i,j) \in E : i \in S_1, j \in S_2\}$. If $S_1 = S_2$ we use $E(S_1) = \delta(S_1,S_1)$; if $S_1 = V \setminus S_2$ we use $\delta(S_1) = \delta(S_2) = \delta(S_1,V \setminus S_1)$. For $v \in V$, we let $\delta(v) = \delta(v)$. We denote by $G(S) = (S,E(S))$ the subgraph induced by $S \subseteq V$. For $F \subseteq E$ and $x \in \mathbb{R}^E$, we denote $x(F) := \sum_{e \in F} x_e$. With an abuse of notation and terminology, sets and their incidence vectors will not be distinguished. We denote by $\text{Sp}(G) = \{ x \in \mathbb{R}^E : x(E(U)) \leq |U| - 1 \text{ for all } U \subseteq V, U \neq \emptyset, \text{ and } x(E) = |V| - 1 \}$ the spanning tree polytope (convex hull of spanning trees) of $G$, and by $\text{cone} \text{Sp}(G)$ their cone (vectors that are their non-negative combinations): $\text{Sp}(G) \subseteq \text{cone} \text{Sp}(G)$.

We say that $x \in \mathbb{R}^E$ is a solution to the $(s-t\text{-path TSP})$ subtour elimination LP if $x \in \text{Sp}(G)$, and $x$ satisfies the degree constraints $x(\delta(v)) = 2$ for all $v \in V \setminus \{s,t\}$ and $x(\delta(s)) = x(\delta(t)) = 1$. This is equivalent to the “standard definition” of the subtour elimination linear program: for $\emptyset \subseteq U \subseteq V \setminus \{s,t\}$, the constraints $x(\delta(U)) \geq 2$ and $x(E(U)) \leq |U| - 1$ are equivalent if $x$ satisfies the degree constraints, and for $\{s\} \subseteq U \subseteq V \setminus \{t\}$ similarly, $x(\delta(U)) \geq 1$ and $x(E(U)) \leq |U| - 1$ are equivalent if $x$ satisfies the degree constraints.

2 Chain-points, Layers and Matroids

Let $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k \subseteq V$, and $C := \{ \delta(V_0), \delta(V_1), \ldots, \delta(V_k) \}$. We call spanning trees of $G$ that meet each $C \in C$ in exactly one edge Gao-trees for the chain $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k$. The Gao-edges of a cut $Q \subseteq C$ are those not contained in any other $C \in C$.

We say that $x \in \mathbb{R}^E$ is a chain-point for the chain of sets $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k$, if

(i) $x \in \text{Sp}(G),$

(ii) $x(\delta(V_0)) = x(\delta(V_k)) = 1, x(\delta(V_i)) < 2 \text{ for } i = 1, \ldots, k - 1,$

(iii) $\sum_{\ell \in L_i} x(\delta(v)) = 2|L_i| \text{ for } i = 1, \ldots, k,$

where $L_i := V_i \setminus V_{i-1} \neq \emptyset$ are the level-sets (where $V_{-1} := \emptyset$ and $V_{k+1} := V$). Note that levels $L_0$ and $L_{k+1}$ are not present in (iii).

We may assume without loss of generality that all $V_i$ are distinct. Gao-edges can then be equivalently defined as those joining two consecutive level-sets.

Now, let $E$ be the support of a chain-point $x$. It follows from the proof of Gao [3] that there exists a Gao-tree in $E$ (see below as the $U = V$ case of Lemma 5). Observe that Gao-trees are the set of bases of a matroid on $E$.

As An, Kleinberg and Shmoys [1] noted, if $x$ is a solution to the $s-t$ path TSP subtour elimination LP, then $\{ V_0, V_1, \ldots, V_k \} := \{ A \subseteq V : x(\delta(A)) < 2 \text{ and } s \in A \}$ form a chain, i.e. $\{ V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k \subseteq V \}$, where $V_0 = \{ s \}$ and $V_k = V \setminus \{ t \}$. To check this for the sake of self-containedness, let $A, B \subseteq V, x(\delta(A)) < 2, x(\delta(B)) < 2$, and assume without loss of generality that $s \in A \setminus B, t \in B \setminus A$. Then

$4 > x(\delta(A)) + x(\delta(B)) = x(\delta(A \cap B)) + x(\delta(A \cup B)) + 2x(A \setminus B, B \setminus A)$

by a well-known identity. Suppose for a contradiction that $A \cap B \neq \emptyset$ and $A \cup B \neq V$, and use that $x$ is a solution of the $s-t$ path subtour elimination LP: since in addition $A \cap B$ contains
neither $s$ nor $t$ and $A \cup B$ contains both, $x(\delta(A \cap B)) + x(\delta(A \cup B)) \geq 2 + 2 = 4$, a contradiction. Hence, a solution $x$ to the $s - t$ path TSP subtour elimination LP is a chain-point for the chain $\{A \subseteq V : x(\delta(A)) < 2 \text{ and } s \in A\}$.

Gottschalk and Vyggen [5] showed that it is possible to write a solution $x$ to the $s - t$ path TSP subtour elimination LP as a convex combination of spanning trees such that the coefficients of Gao-trees for the chain $\{A \subseteq V : x(\delta(A)) \leq 2 - \lambda \text{ and } s \in A\}$ sum to at least $\lambda$. We call such a convex combination layered (see below for details).

The advantage of the notion of chain-points rather than subtour elimination LP solutions is that chain-points are closed under “subtracting Gao-trees” (while the degree condition of subtour elimination linear program does not have this property). This allows “peeling off layers” one by one and proving the existence of a layered convex combination by induction (Lemma 2 and Theorem 3).

We now define layered convex combinations somewhat more generally for chain-points, and in terms of the introduced matroids, in more detail, and state our main result, the existence of such a convex combination. Then we state the lemma that allows to peel off layers one by one, that is, to deal with only two layers at a time.

To refer to chain-points and Gao-trees for $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_k$ we use the terms chain-point and Gao-tree without mention of the chain. For a chain-point $x$, we call the values of $x(\delta(V_i))$ for $i = 1, \ldots, k$ the narrow cut sizes. Let the different values of the narrow cut sizes be $2 - \lambda_1 > 2 - \lambda_2 > \ldots > 2 - \lambda_{\ell} = 1$, where $\ell$ is the number of different sizes.

We consider $\ell$ different matroids $(E, B_j)$ whose set of bases $B_j$ are the Gao-trees for the chain of sets $\{V_i : x(\delta(V_i)) \leq 2 - \lambda_1 - \ldots - \lambda_j\}$, $(j = 1, \ldots, \ell)$. We say that $\sum_{j=1}^{\ell} \lambda_j x_j$ is a layered convex combination for $x$ if $x_j$ is in the convex hull of bases in $B_j$ for $j = 1, \ldots, \ell$ and $x = \sum_{j=1}^{\ell} \lambda_j x_j$.

**Theorem 1.** If $x$ is a chain-point, then there exists a layered convex combination for $x$.

The following lemma enables us to “peel off” layers of chain-points one by one, and concentrate on the case when there are only two distinct narrow cut sizes:

Denote the convex hull of Gao-trees of $G$ for the chain $C$ by $\text{Sp}_C(G)$.

**Lemma 2.** Let $x$ be a chain-point, and $x = \varepsilon y + (1 - \varepsilon)x'$ with $y \in \text{Sp}_C(G)$, $x' \in \text{Sp}(G)$. Then $x'$ is a chain point for the chain $(V_i : x(\delta(V_i))) < 2 - \varepsilon$.

**Proof:** Since $y \in \text{Sp}_C(G)$, for all $i = 1, \ldots, k$ we have: $\sum_{v \in L_i} |y(\delta(v))| = 2(|L_i| - 1) + 2 = 2|L_i|$; since $x$ is a chain-point, $\sum_{v \in L_i} |x(\delta(v))| = 2|L_i|$ by (iii). Now (i) holds for $x'$ by assumption; to check (ii) note $x'(\delta(V_i)) = \frac{x(\delta(V_i)) - \varepsilon}{1 - \varepsilon}$, whence $x'(\delta(V_0)) = x'(\delta(V_k)) = 1$ and $x'(\delta(V_i)) < 2$ if $x(\delta(V_i)) < 2 - \varepsilon$ ($i = 1, \ldots, \ell$). Finally, $\sum_{v \in L_i} x'(\delta(v)) = 2\varepsilon|L_i| \geq 2|L_i|$, so (iii) also holds for $x'$.

**Theorem 3.** Let $x$ be a chain-point, and let $\lambda = 2 - \max_{i=0,\ldots,k} x(\delta(V_i))$. Then there exist $y \in \text{Sp}_C(G), x' \in \text{Sp}(G)$ such that

$$x = \lambda y + (1 - \lambda)x'.$$

By Lemma 2 applied to $\varepsilon := \lambda$, the point $x'$ provided by this theorem is a chain-point for $\{V_i : x(\delta(V_i)) < 2 - \lambda\}$, so repeatedly applying Theorem 3 we get Theorem 1. We prove Theorem 3 using the following fractional version of Edmonds’ matroid partition theorem [2], which can be easily stated and proved for rational input from the well-known integer version by multiplying with the denominators of the occurring numbers. We include here a reduction to an explicitly stated version in the literature, a theorem on fractional polymatroids, pointed out to us by András Frank.

**Lemma 4.** Let $M_1 = (E, r_1)$ and $M_2 = (E, r_2)$ be matroids on the same element set, $w \in \mathbb{R}_{\geq 0}^E$, $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ and let $P_i$ be the convex hull of the independent sets of $M_i$ ($i = 1, 2$). There exist $x_1 \in P_1$, $x_2 \in P_2$ so that $\lambda_1 x_1 + \lambda_2 x_2 = w$ if and only if $\lambda_1 r_1(X) + \lambda_2 r_2(X) \geq w(X)$ for all $X \subseteq E$. 

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Theorem 3 follows.

To derive the desired contradiction, we will show that the right hand side is at least 4. First, $f \in w \in P_{f_1} \lambda = \lambda f_1 (i = 1, 2)$, where $P_f$ is the polymatroid associated with $f$.

Our conditions $\lambda_1 r_1 (X) + \lambda_2 r_2 (X) \geq w (X)$ for all $X \subseteq E$ and $w \geq 0$ express exactly that $w \in P_{f_{j_1} + f_2}$. So, by [6, Theorem 44.6] $w = w_1 + w_2$, where $w_i \in P_{f_i} = \lambda_i P_i (i = 1, 2)$. □

3 Peeling off a Layer

Our only debt now is to show that one layer can be peeled off, that is, to prove Theorem 3.

We use Lemma 4 with $w := x$, $M_1$ the matroid whose bases are the Gao-trees and $M_2$ the cycle matroid of $G$, i.e., the matroid on $E$ whose independent sets are forests in $G$. We denote by $p$ the rank function of $M_1$ and by $r$ the rank function of $M_2$. Clearly, $p \leq r$. In the remainder of this section, we show that the condition of Lemma 4 is satisfied for $\lambda_1 = \lambda, \lambda_2 = 1 - \lambda$, i.e. that $\lambda p (x) + (1 - \lambda) r (x) \geq x (X)$ for all $X \subseteq E$. Lemma 4 then implies the existence of $y$ and $x'$ in the convex hull of independent sets of $M_1$ and $M_2$ respectively, and $x (E) = |V| - 1$ implies that they are in fact in the convex hull of bases of $M_1$ and $M_2$, i.e., in $Sp_G (G)$ and $Sp (G)$ respectively, and Theorem 3 follows.

To prove that the condition of Lemma 4 is satisfied we need the following lemma.

Lemma 5. Let $x$ be a chain-point and $U \subseteq V$ such that $x (E (U)) = |U| - 1$. Then

\[ p (E (U)) = r (E (U)) = |U| - 1. \]

Proof: Assume that $x$ and $U$ are as in the condition, and that $E$ is the support of $x$.

Claim 1. The set $I (U) := \{ i \in [0, k + 1] : L_i \cap U \neq \emptyset \}$ is the set of all integers of an interval.

Indeed, suppose for a contradiction that there exists $j \in [0, k], L_j \cap U = \emptyset$ so that $L_{j'} \cap U \neq \emptyset$ for some $j' < j$ and also for some $j' > j$; in other words, $U$ is partitioned by $U \cap V_{j-1}$ and $U \setminus V_j$.

By (ii), and applying (SUBMOD) for $A := V_j, B := V \setminus V_{j-1}$, we get that:

\[ 4 > x (\delta (V_j)) + x (\delta (V \setminus V_{j-1})) = x (\delta (L_j)) + x (\delta (V)) + 2x (\delta (V_{j-1} \setminus V \setminus V_j)). \]

To derive the desired contradiction, we will show that the right hand side is at least 4. First, $x (\delta (V)) = 0$, and (iii) implies that $x (\delta (L_j)) = 2 |L_j| - 2x (E (L_j))$ which is at least 2 since $x (E (L_j)) \leq |L_j| - 1$ for $x \in Sp (G)$. Further, observe that

\[ x (\delta (V_{j-1} \setminus V \setminus V_j)) \geq x (E (U)) - x (E (U \cap V_{j-1})) - x (E (U \cap (V \setminus V_j))) \]

\[ \geq |U| - 1 - (|U \cap V_{j-1}| - 1) - (|U \setminus V_j| - 1) = 1, \]

where the second inequality uses the fact that $x (E (U)) = |U| - 1$ and $x (E (A)) \leq |A| - 1$ for any $A$, and the equality uses the fact that $\{ U \cap V_{j-1}, U \setminus V_j \}$ is a partition of $U$. The claim is proved.

The key claim we need now to prove the lemma is the following.

Claim 2. Let $a, b \in \mathbb{N}, 0 \leq a \leq b \leq k + 1$, and $S := \bigcup_{i=a}^b L_i$. Then $E (S \cap U)$ is connected.

Note that this was shown by Gao [3] for $U := V$ and is the key for the existence of a Gao-tree.

To prove Claim 2, we first show that

\[ (\text{STRICT}) \quad |S| - 2 < x (E (S)) \quad (\leq |S| - 1). \]
If \( a = 0, b = k + 1 \), then \( S = V \) and by (i), \( x(E(V)) = |V| - 1 > |V| - 2 \); if exactly one of \( a = 0 \) or \( b = k + 1 \) holds, then, slightly more generally, subtracting from \( x(E(V)) = |V| - 1 \) the inequality \( x(E(V \setminus S)) \leq |V \setminus S| - 1 \) we get \( x(E(S)) + x(\delta(S)) \geq |S| \) (again from (i)), and then by (ii) \( x(\delta(S)) < 2 \), and we are done again. Finally, suppose \( 1 \leq a \leq b \leq k \), and add up (iii) for \( i = a, a+1, \ldots, b \). We get \( x(E(S)) + \frac{1}{2}x(\delta(S)) = |S| \), where \( \frac{1}{2}x(\delta(S)) \leq \frac{1}{2}x(\delta(V_{a-1})) + \frac{1}{2}x(\delta(V_b)) < 2 \) by (ii), finishing the proof of (STRICT).

Now, by (i), \( x \) is a convex combination of spanning trees, and by (STRICT), at least one of these, denote it \( F \), as all the spanning trees in the convex combination, also contains a spanning tree of \( U \), because of \( x(E(U)) = |U| - 1 \). So \( F \) contains a spanning tree of \( S \cap U \), and the claim is proved.

To finish the proof of the lemma, choose a spanning tree in \( G(L_i \cap U) \) for each \( i \in I(U) \), which is possible by Claim 2 applied to \( S := L_i \); then add a Gao-edge between \( L_i \cap U \) and \( L_{i+1} \cap U \) for each index \( i \) such that \( i, i+1 \in I(U) \), which is possible by applying Claim 2 to \( S := L_i \cup L_{i+1} \). In this way we get a spanning tree \( F \) of \( G(U) \), which is an independent set of \( M_1 \), since it can be completed to a Gao-tree of \( G \) by applying Claim 2 to \( U := V \) and \( S := L_i \) for \( i = 0, \ldots, k + 1 \) and then for \( S := L_i \cup L_{i+1} \) for \( i = 0, \ldots, k \), if \( \{i, i+1\} \cap I(U) \neq \emptyset \). \( \square \)

**Proof of Theorem 3:** First, observe that it suffices to prove the following claim.

**Claim 3.** There exists \( 0 < \varepsilon \leq \lambda \) and \( y \in \text{Sp}_C(G) \), \( x' \in \text{Sp}(G) \), such that \( x = \varepsilon y + (1 - \varepsilon)x' \).

Indeed, if this is true, then \( x - \varepsilon y \in \text{cone.Sp}(G) \), and let \( \varepsilon_{\text{max}} \) be the largest \( \varepsilon \leq \lambda \) such that there exist \( y \in \text{Sp}_C(G) \) such that \( x - \varepsilon y \in \text{cone.Sp}(G) \). Defining \( x' := \frac{x - \varepsilon_{\text{max}}y}{1 - \varepsilon_{\text{max}}} \), we see that \( x' \in \text{Sp}(G) \). So to prove Theorem 3 from Claim 3 we have to prove \( \varepsilon_{\text{max}} = \lambda \).

Suppose for a contradiction that \( \varepsilon_{\text{max}} < \lambda \), then by Lemma 2 applied to \( \varepsilon := \varepsilon_{\text{max}} \), \( x' \) is also a chain point (for the same chain of sets). So then applying Claim 3 to \( x' \), there exists \( \varepsilon' > 0 \) and \( y' \in \text{Sp}_C(G) \) such that \( x' - \varepsilon'y' \in \text{cone.Sp}(G) \). However, then

\[
\text{cone.Sp}(G) \ni x' - \varepsilon'y' = x - \varepsilon_{\text{max}}y - \varepsilon'y' = x - (\varepsilon_{\text{max}} + \varepsilon')z,
\]

where \( z := \frac{\varepsilon_{\text{max}}y + \varepsilon'y'}{\varepsilon_{\text{max}} + \varepsilon'} \in \text{Sp}_C(G) \). This contradicts that \( \varepsilon_{\text{max}} \) is the largest \( \varepsilon \leq \lambda \) such that there exist \( y \in \text{Sp}_C(G) \) such that \( x' = x - \varepsilon y \in \text{cone.Sp}(G) \), finishing the proof of the theorem, provided that Claim 3 is true.

We prove now Claim 3 by checking the condition of the matroid partition theorem in the form of Lemma 4. Let us say that \( \varepsilon \geq 0 \) is suitable for \( X \subseteq E \), if \( \varepsilon > 0 \) and

\[
\varepsilon p(X) + (1 - \varepsilon) r(X) \geq x(X).
\]

By Lemma 4, Claim 3 is equivalent to proving that there exists \( \varepsilon > 0 \) suitable for all \( X \subseteq E \).

If \( p(X) = r(X) \) then by (i) any \( \varepsilon \in [0,1] \) is suitable for \( X \); so assume \( p(X) < r(X) \). If in addition \( x(X) < r(X) \), then \( \varepsilon \) is suitable if and only if \( 0 < \varepsilon \leq \varepsilon_X := \frac{r(X)-x(X)}{r(X)-p(X)} > 0 \). Since the number of such sets is finite, \( \varepsilon_{\text{max}} := \min\{\varepsilon_X : X \subseteq E, x(X) < r(X), p(X) < r(X)\} > 0 \) is suitable for all of them.

It remains to show that there is no other case, that is, \( x(X) = r(X) \) implies \( p(X) = r(X) \). Denote \( V_X \) the set of vertices of (“covered by”) edges in \( X \). We consider the components of \((V_X, X)\). If \((V_X, X)\) has only one component, then \( r(X) = |V_X| - 1 \). Apply Lemma 5 to the graph whose edge-set is the support of \( x \), and \( U := V_X \); then \( X = E(U) \) and by the lemma, \( p(X) = r(X) \) follows.

If \((V_X, X)\) has multiple components, then \( r(X) \) sums up over the components, and the same holds for \( p(X) \) as long as there are not two components that both contain edges in the same cut in
C. Hence, the proof of the claim is completed by showing that it is not possible for two components 
A and B of (V X, X) that both $E(A) \cap \delta(V_i) \neq \emptyset$ and $E(B) \cap \delta(V_i) \neq \emptyset$.
Indeed, if $E(A) \cap \delta(V_i) \neq \emptyset$, then $x(E(A) \cap \delta(V_i)) = x(E(A)) - x(E(A \cap V_i)) - x(E(A \setminus V_i)) \geq 1$, 
since $x(E(A)) = |A| - 1$ because of $x(X) = r(X)$, and $x(E(A')) \leq |A'| - 1$ for any $A' \subseteq A$, because $x \in \text{Sp}(G)$. By applying the same argument to B, if both $E(A) \cap \delta(V_i) \neq \emptyset$ and $E(B) \cap \delta(V_i) \neq \emptyset$, 
then $x(\delta(V_i)) \geq 2$, contradicting (ii).

Our proof implies that a layered convex combination can be found in strongly polynomial time 
with a combinatorial algorithm: Note that Theorem 3 implies that there exists a Gao-tree $T$ and $\lambda' > 0$ such that $x = \lambda'T + (1 - \lambda')x'$ with $x' \in \text{Sp}(G)$ such that (choosing $\lambda'$ to be as large as possible), either $\lambda' = \lambda$ and the number of cuts defining the new chain-point is less than $|C|$ (see Lemma 2), or the dimension of the minimal face of $\text{Sp}(G)$ containing $x'$ is less than the dimension of the minimal face of $\text{Sp}(G)$ containing $x$. Applying this statement iteratively, it follows that a linear number of spanning trees suffices for a layered convex combination. So the number of trees 
we need is asymptotically the same as for an arbitrary decomposition into spanning trees. If $x$ 
is an extreme point solution of the subtour elimination linear program the support of $x$ has size $O(n)$ (see [4]); so only $O(n)$ different trees are needed for a layered convex combination.

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