REFINED ENERGY INEQUALITY WITH APPLICATION TO WELLPPOSEDNESS FOR THE FOURTH ORDER NONLINEAR SCHRÖDINGER TYPE EQUATION ON TORUS

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Abstract. We consider the time local and global well-posedness for the fourth order nonlinear Schrödinger type equation (4NLS) on the torus. The nonlinear term of (4NLS) contains the derivatives of unknown function and this prevents us to apply the classical energy method. To overcome this difficulty, we introduce the modified energy and derive an a priori estimate for the solution to (4NLS).

1. Introduction

We consider the fourth order nonlinear Schrödinger type equation (4NLS) on the torus

$$\begin{cases}
i\partial_t \psi + \partial_x^2 \psi + \nu \partial_x^4 \psi = \mathcal{N}(\psi, \overline{\psi}, \partial_x \psi, \partial_x^2 \psi, \partial_x^3 \psi, \partial_x^4 \psi), \\
\psi(0, x) = \phi(x), \quad x \in \mathbb{T},
\end{cases}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\psi : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ is an unknown function, and $\phi : \mathbb{T} \to \mathbb{C}$ is a given function. The nonlinear term $\mathcal{N}$ is given by

$$\mathcal{N}(\psi, \overline{\psi}, \ldots, \partial_x^2 \psi, \partial_x^4 \psi) = \lambda_1|\psi|^2 \psi + \lambda_2|\psi|^4 \psi + \lambda_3(\partial_x \psi)^2 \overline{\psi} + \lambda_4|\partial_x \psi|^2 \psi + \lambda_5 \psi^2 \overline{\partial_x \psi} + \lambda_6|\psi|^2 \partial_x^2 \psi,$$

where $\nu \neq 0$ and $\lambda_j$, $j = 1, \cdots, 6$ are real constants. The equation (1.1) arises in the context of a motion of vortex filament. More precisely, using the localized induction approximation, Da Rios [5] proposed some equation which approximates the three dimensional motion of an isolated vortex filament embedded in an inviscid incompressible fluid filling an infinite region. The Da Rios equation is reduced to the cubic nonlinear Schrödinger equation

$$i\partial_t \psi + \partial^2_x \psi = -\frac{1}{2}|\psi|^2 \psi, \quad (t, x) \in \mathbb{R} \times \mathbb{T}$$

via the Hasimoto transform [10]. To describe the motion of actual vortex filament more precisely, some detailed models taking into account the effect from higher order corrections of equation have been introduced by Fukumoto-Moffatt [7]. The Fukumoto-Moffatt equation is rewritten as (1.1) by using the Hasimoto transform. For the physical background of (1.1), see Fukumoto-Moffatt [7].

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In this paper we consider the time local well-posedness for (1.1) on the Sobolev spaces $H^m(\mathbb{T})$. Our notion of well-posedness contains the existence and uniqueness of the solution and the continuity of the data-to-solution map. We also consider the persistent property of the solution, that is, the solution describes a continuous curve in $H^m(\mathbb{T})$ whenever $\phi \in H^m(\mathbb{T})$. Our motivation to consider the time local well-posedness for (1.1) is that we are interested in the stability of the standing wave solution $\psi(t, x) = e^{i\omega t} \varphi_{\omega}(x)$ to (1.1). When (1.1) is completely integrable (see the later half of this section below for the detail), (1.1) has the sech-type standing wave solution. The orbital stability in $H^m(\mathbb{R})$ of the sech-type standing wave solution is proved by [20]. On the other hand, we easily see that (1.1) has a exact periodic standing wave solution of the form $\psi(t, x) = \kappa e^{i\tau x + i\omega t}$ for some real constants $\kappa, \tau$ and $\omega$. It is interesting that the sech-type standing wave and the periodic standing wave correspond to the tornado like curve and the helicoid curve in the motion of the vortex filament, see Kida [16].

As the first step to show the orbital stability of the sech-type and the periodic standing wave, we need to prove the global well-posedness for (1.1) in the Sobolev spaces on the real line $\mathbb{R}$ and on the torus $\mathbb{T}$, respectively. Concerning the local well-posedness of (1.1) on real line $\mathbb{R}$, Segata [23, 24, 25] and Huo-Jia [11, 12] proved that the initial value problem of (1.1) is locally well-posed in Sobolev space $H^s(\mathbb{R})$ with $s > 1/2$ by using the Fourier restriction method introduced by Bourgain [3] and Kenig-Ponce-Vega [14, 15]. As far as we know, there is no result on the well-posedness of (1.1) under the periodic boundary condition.

In this paper we focus on the well-posedness of (1.1) on the torus. There is a large literature on the well-posedness for the dispersive equations in the torus. See for instance [6, 13, 19, 21] for the linear dispersive equations and [11, 3, 4, 8, 10, 22, 26, 27] for the non-linear dispersive equations. We summarize the well-posedness on the derivative nonlinear Schrödinger equation with the periodic boundary condition. Tsutsumi-Fukuda [26, 27] proved the local and global well-posedness for the Schrödinger equation with some nonlinearity on the torus by using the classical energy method. Grünrock-Herr [8] and Herr [10] obtained sharp well-posedness results for some derivative nonlinear Schrödinger equation on the torus by using the Fourier restriction method. The well-posedness of the Schrödinger equation for more general derivative nonlinearity in the $n$-dimensional torus was given by Chihara [4]. We notice that the classical energy method does not works for his setting. In [4] he conquered this problem by using the pseudo-differential operators with non-smooth coefficients on the torus.

As we shall see below, the dispersive equations on the torus do not have fine properties compared to the real line case. Therefore the proof of the well-posedness on the torus become increasingly harder than the real line case. To state our results more precisely, we introduce several notations. Given a function $\psi$ on $\mathbb{T}$, we define the Fourier coefficient of $\psi$, by

$$\hat{\psi}(n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(x)e^{-inx}dx, \quad n \in \mathbb{Z}. $$
Let $m$ be a non-negative integer. $H^m(\mathbb{T})$ denotes the all tempered distributions on $\mathbb{T}$ satisfying
\[
\|\psi\|_{H^m} = \left(\sum_{n \in \mathbb{Z}} |\langle n \rangle^{2m}|\hat{\psi}(n)|^2\right)^{1/2} < +\infty,
\]
where $\langle n \rangle = \sqrt{1 + n^2}$.

The main result in this paper is the following:

**Theorem 1.1.** Let $m \geq 4$ be an integer. Then (1.1) is locally well-posed in the following sense: For any $\phi \in H^m(\mathbb{T})$, there exists a time $T = T(\|\phi\|_{H^m}) > 0$ and a unique solution $\psi$ of (1.1) satisfying
\[
\psi \in C([0, T]; H^m(\mathbb{R})).
\]
Moreover, the data-to-solution map $H^m(\mathbb{T}) \to C([0, T]; H^m(\mathbb{T}))(\phi \mapsto \psi(t))$ is continuous.

The difficulty in the proof of time local well-posedness of (1.1) arises in so called “loss of a derivatives”. More precisely, the standard energy estimate gives only the following:
\[
\frac{d}{dt}\|\partial_x^m \psi(t)\|_{L^2_x}^2 = 2\{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6\} \text{Im} \int_{\mathbb{T}} \overline{\psi} \partial_x \psi \cdot \partial_x^m \overline{\psi} \partial_x^{m+1} \psi dx
\]
\[-2\lambda_5 \text{Im} \int_{\mathbb{T}} \psi^2 (\partial_x^{m+1} \psi)^2 dx + \ell.o.t.
\]

Since the first and second terms in the right hand side of (1.3) contain the $(m + 1)$-st derivatives of $\psi$, we cannot control those factors in terms of $H^m$ norm of $\psi$. Therefore this estimate does not give an a priori estimate for the solution.

For the real line case, the unitary group $\{e^{it(\partial_x^2 + i\nu \partial_x^4)}\}_{t \in \mathbb{R}}$ generated by the linear operator $i\partial_x^2 + i\nu \partial_x^4$ gains extra smoothness in space variable, see Kenig-Ponce-Vega [13]. Thanks to this smoothing property for $\{e^{it(\partial_x^2 + i\nu \partial_x^4)}\}_{t \in \mathbb{R}}$, in [23, 11, 24, 12] they could overcome a loss of derivatives and guarantee the well-posedness of (1.1) on $\mathbb{R}$. However, for the periodic case the corresponding unitary group does not have such a fine properties (see e.g., [9]) and it is not likely that the contraction mapping principle guarantees the well-posedness for (1.1) on $\mathbb{T}$. Since this is the case we abandon making use of the property of the unitary group $\{e^{it(\partial_x^2 + i\nu \partial_x^4)}\}_{t \in \mathbb{R}}$ and try to this issue by a different approach.

Let us return the estimate (1.3). If we contrive to eliminate the worst terms, we can obtain an a priori estimate of solution. In this paper we take a hint from Kwon [17] which is concerned with the well-posedness for the fifth-order KdV equation on $\mathbb{R}$, we introduce the “modified” energy:
\[
[E_m(\psi)](t) = \|\partial_x^m \psi(t)\|_{L^2_x}^2 + \|\psi(t)\|_{L^2_x}^2 + C_m\|\psi(t)\|_{L^2_x}^{4m+2}
\]
\[+ \frac{\lambda_5}{\nu} \text{Re} \int_{\mathbb{T}} (\partial_x^{m-1} \psi)^2 \overline{\psi}^2 dx + \frac{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6}{4\nu} \int_{\mathbb{T}} |\partial_x^{m-1} \psi|^2 |\psi|^2 dx,
\]
where $C_m$ is a sufficiently large constant depending only on $m$ so that $E_m(\psi)$ is positive. Thanks to the correction terms we can eliminate the worst factors in (1.3) and evaluate
the $H^m$ norm of the solution $\psi$ to (1.1) in terms of the $H^m$ norm of the initial data $\phi$. This is a crucial point in the proof of Theorem 1.1.

It is known that (1.1) is completely integrable if and only if $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8$, $\lambda_3 = -3\nu/2$, $\lambda_4 = -\nu$, $\lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$. In this case (1.1) has infinitely many conservation quantities, see Langer and Perline [18]. The first three conservation quantities for (1.1) are given by

$$I_0(\psi) = \frac{1}{2} \int_T |\psi|^2 dx,$$

$$I_1(\psi) = \frac{1}{2} \int_T |\partial_x \psi|^2 dx - \frac{1}{8} \int_T |\psi|^4 dx,$$

$$I_2(\psi) = \frac{1}{2} \int_T |\partial_x^2 \psi|^2 dx + \frac{3}{4} \int_T |\psi|^2 \overline{\partial_x \psi} \partial_x \psi dx + \frac{1}{8} \int_T |\psi|^2 \overline{\partial_x^2 \psi} dx,$$

$$+ \frac{5}{8} \int_T (\partial_x \psi)^2 \overline{\psi}^2 dx + \frac{3}{4} \int_T |\partial_x \psi|^2 |\psi|^2 dx + \frac{1}{16} \int_T |\psi|^6 dx.$$

In general, the conservation quantities for (1.1) are expressed as

$$I_m(\psi) = \frac{1}{2} \int_T |\partial_x^m \psi|^2 + \int_T Q_m(\psi, \overline{\psi}, \ldots, \partial_x^{m-1} \psi, \overline{\partial_x^{m-1} \psi}) dx,$$

where $Q_m$ are some polynomials in $(\psi, \overline{\psi}, \ldots, \partial_x^{m-1} \psi, \overline{\partial_x^{m-1} \psi})$ satisfying the inequalities $|Q_m| \leq C_m \|\psi\|^\alpha_{L^2_T} \|\overline{\partial_x^m \psi}\|_{L^2_T}^{\beta_m}$ for some $\alpha_m > 0$ and $0 < \beta_m < 2$. Therefore combining Theorem 1.1 the conservation laws $I_m(\psi(t)) = I_m(\psi(0))$ and Young’s inequality, we obtain the global existence theorem for (1.1) in $H^m(T)$:

**Theorem 1.2.** Assume $\lambda_1 = -1/2$, $\lambda_2 = -3\nu/8$, $\lambda_3 = -3\nu/2$, $\lambda_4 = -\nu$, $\lambda_5 = -\nu/2$ and $\lambda_6 = -2\nu$. Then (1.1) is globally well-posed in $H^m(T)$ with an integer $m$ greater than 3.

Finally we point out that by combining our proof with the estimates for the fractional derivatives we may well be able to extend Theorem 1.1 to the case where $m$ is not an integer. In this paper we do not touch on this issue.

The plan of this paper is as follows. Section 2 is devoted to the parabolic regularization associated to (1.1). In Section 3, we introduce the modified energy and give an a priori estimate for the solution to (1.1). Then we shall prove the existence of solution to (1.1). In Section 4 we give the proofs of the uniqueness and the persistent properties of solution to (1.1), and the continuous dependence of the solution to (1.1) on the initial data.

2. PARABOLIC REGULARIZATION

In this section, we consider the parabolic regularization of (1.1) in $H^m(T)$. We first give the Gagliardo-Nirenberg inequality for the periodic functions.

**Lemma 2.1.** Let $l$ and $m$ be integers satisfying $0 \leq l \leq m - 1$ and let $2 \leq p \leq \infty$. Then there exists a constant $C$ depending only on $l, m$ and $p$ such that for any $\psi \in H^m(T)$,

$$\|\partial_x^l \psi\|_{L^p_T} \leq C \times \left\{ \begin{array}{ll} \|\psi\|_{L^2_T}^{1-\alpha} \|\partial_x^m \psi\|_{L^2_T}^\alpha & (1 \leq l \leq m - 1), \\ \|\psi\|_{L^2_T}^{1-\alpha} \|\partial_x^m \psi\|_{L^2_T}^\alpha + \|\psi\|_{L^2_T}^2 & (l = 0), \end{array} \right.$$
where \( \alpha = (l + 1/2 - 1/p)/m \). Especially, we have \( \|\partial_x^l \psi\|_{L^p} \leq C\|\psi\|_{L^2}^{1-\alpha} \|\psi\|_{H^m}^\alpha \).

**Proof.** See for instance, [22, Section 2]. \( \square \)

Next we consider the parabolic regularization of (1.1). Let us introduce the regularizing sequence used in Bona-Smith [2]. Let \( \varphi \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \varphi(\xi) \leq 1 \) for \( \xi \in \mathbb{R} \), \( \varphi^{(k)}(0) = 0 \) for \( k \in \mathbb{N} \) and \( \varphi(\xi) \) tends exponentially to 0 as \( |\xi| \to \infty \). We define for \( \epsilon \in (0,1] \),

\[
\phi_\epsilon(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \varphi(\epsilon n) \hat{\phi}(n)e^{inx}.
\]

Then, \( \{\phi_\epsilon\}_{\epsilon>0} \in L^\infty(\mathbb{T}) \) and \( \|\phi - \phi_\epsilon\|_{H^m} \to 0 \) as \( \epsilon \to 0 \). Furthermore, for any \( l \geq 0 \),

\[
\|\phi_\epsilon\|_{H^{n+l}} \leq C\epsilon^{-l}\|\phi\|_{H^m},
\]

\[
\|\phi - \phi_\epsilon\|_{H^m} \leq C\epsilon^l\|\phi\|_{H^m},
\]

\[
\|\phi - \phi_\epsilon\|_{H^{n+l}} \leq C\epsilon^l\|\phi\|_{H^m}.
\]

We consider the regularized problem of (1.1):

\[
\left\{ \begin{array}{l}
 i\partial_t \psi_\epsilon + \partial_x^2 \psi_\epsilon + (\nu + i\epsilon)\partial_x^4 \psi_\epsilon = \mathcal{N}(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^4 \overline{\psi_\epsilon}), \\
 \psi_\epsilon(0, x) = \phi_\epsilon(x),
\end{array} \right. \tag{2.1}
\]

where \( \psi_\epsilon : \mathbb{R} \times \mathbb{T} \to \mathbb{C} \) is an unknown function, and \( \phi_\epsilon : \mathbb{T} \to \mathbb{C} \) is a Bona-Smith approximation of \( \phi \). Concerning the solvability of (2.1), we have the following lemma.

**Lemma 2.2.** Let \( m \geq 3 \) be an integer. For any \( \phi \in H^m(\mathbb{T}) \), there exists a time \( T_\epsilon = T(\epsilon, \|\phi\|_{H^m}) > 0 \) and a unique solution \( \psi_\epsilon \) of (2.1) satisfying

\[
\psi_\epsilon \in C([0, T_\epsilon), H^m(\mathbb{T})).
\]

**Proof.** We shall prove (2.1) by using the Banach fixed point theorem. Let \( \{W_\epsilon(t)\}_{t \geq 0} \) be the contraction semi-group generated by the linear operator \( i\partial_x^2 + i\nu \partial_x^4 - \epsilon\partial_x^4 \):

\[
[W_\epsilon(t)\phi](x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \hat{\phi}(n)e^{inx+(-in^2+i\nu n^4-\epsilon n^4)t}.
\]

Then, the initial value problem (2.1) is rewritten as the integral equation

\[
\psi_\epsilon(t) = W_\epsilon(t)\phi_\epsilon - i \int_0^t W_\epsilon(t - \tau)\mathcal{N}(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^4 \overline{\psi_\epsilon})(\tau)d\tau.
\]

We put \( r = \|\phi\|_{H^m} \). For \( T > 0 \), we define

\[
X^T_T = \{\psi \in C([0, T]; H^m(\mathbb{T})) | \sup_{t \in [0, T]} \|\psi(t)\|_{H^m} \leq 2r\}.
\]

We shall show that the map

\[
\Phi(\psi_\epsilon) = W_\epsilon(t)\phi_\epsilon - i \int_0^t W_\epsilon(t - \tau)\mathcal{N}(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^4 \overline{\psi_\epsilon})(\tau)d\tau
\]

is a contraction on \( X^T_T \) for choosing \( T \) suitably.
We easily see that
\[
\| \Phi(\psi)(t) \|_{H_x^m} \leq \| \phi \|_{H_x^m} + \int_0^t \| W_\epsilon(t - \tau) \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau) \|_{H_x^m} d\tau. \quad (2.2)
\]

By Plancherel’s identity, we obtain
\[
\int_0^t \| W_\epsilon(t - \tau) \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau) \|_{H_x^m} d\tau
\]
\[
= \int_0^t \left\{ \sum_{n \in \mathbb{Z}} \langle n \rangle^{2m} |\hat{\mathcal{N}}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau, n) e^{(\mu n^2 + \nu n^4 - \epsilon n^2)(t - \tau)} |^2 \right\}^{1/2} d\tau
\]
\[
= \int_0^t \left\{ \sum_{n \in \mathbb{Z}} \langle n \rangle^{2m-4} |\hat{\mathcal{N}}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau, n) |^2 \langle n \rangle^4 e^{-2\epsilon n^2(t - \tau)} \right\}^{1/2} d\tau
\]
\[
\leq \int_0^t \sup_{n \in \mathbb{Z}} \{ \langle n \rangle^2 e^{-\epsilon n^2(t - \tau)} \} \| \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau) \|_{H_x^m} d\tau.
\]

Since \( \sup_{n \in \mathbb{Z}} \{ \langle n \rangle^2 e^{-\epsilon n^2(t - \tau)} \} \leq 1 + \epsilon^{-1/2} (t - \tau)^{-1/2} \),
\[
\int_0^t \| W_\epsilon(t - \tau) \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau) \|_{H_x^m} d\tau \leq C \int_0^t \left\{ 1 + \epsilon^{-1/2} (t - \tau)^{-1/2} \right\} \| \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(\tau) \|_{H_x^m} d\tau
\]
\[
\leq C (t + \epsilon^{-1/2} T^{1/2}) \sup_{t \in [0, T]} \| \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(t) \|_{H_x^m}.
\]

Collecting \((2.2)\) and \((2.3)\), we have
\[
\sup_{t \in [0, T]} \| \Phi(\psi)(t) \|_{H_x^m} \leq \| \phi \|_{H_x^m} + C(T + \epsilon^{-1/2} T^{1/2}) \sup_{t \in [0, T]} \| \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(t) \|_{H_x^m}.
\]

By Sobolev’s embedding, we have
\[
\sup_{t \in [0, T]} \| \mathcal{N}(\psi, \psi, \ldots, \partial_x^2 \psi, \partial_x^2 \bar{\psi})(t) \|_{H_x^{m-2}} \leq C(1 + \sup_{t \in [0, T]} \| \psi(t) \|_{H_x^m}^2) \sup_{t \in [0, T]} \| \psi(t) \|_{H_x^m}^3.
\]

Therefore,
\[
\sup_{t \in [0, T]} \| \Phi(\psi)(t) \|_{H_x^m} \leq r + C(T + \epsilon^{-1/2} T^{1/2})(1 + r^2)^3.
\]

We can easily check that \( \Phi(\psi_e) \in C([0, T]; H^m(\mathbb{T})) \). Therefore, by choosing \( T_e > 0 \) sufficiently small so that \( C(T_e + \epsilon^{-1/2} T_e^{1/2})(1 + r^2)^3 < 1 \) we have \( \psi_e \in X_T^r \). By a similar
way, for $\psi^1, \psi^2 \in X_T$, we have
\[
\sup_{t \in [0,T]} \|\Phi(\psi^1)(t) - \Phi(\psi^2)(t)\|_{H^m_T} \\
\leq C(T + \epsilon^{-1/2}T^{1/2})(1 + r^2)\nu^2 \sup_{t \in [0,T]} \|\psi^1(t) - \psi^2(t)\|_{H^m_T} \\
< \sup_{t \in [0,T]} \|\psi^1(t) - \psi^2(t)\|_{H^m_T}.
\]

Consequently, we have that $\Phi$ is a contraction on $X_T$. The Banach fixed point theorem implies the unique existence of solution to (2.1) in $X_T$ which completes the proof of Lemma 2.2.

□

3. Modified Energy

In this section, by using the modified energy, we give an a priori estimates for the solution to (2.1) obtained by Lemma 2.2.

Let $m \geq 1$ be an integer. We introduce the modified energy:
\[
[E_m(\psi)](t) = \||\partial^m_x \psi(t)\|^2_{L^2} + \|\psi(t)\|^2_{L^2} + C_m\|\psi(t)\|_{L^2}^{4m+2} + \frac{\lambda_3}{\nu} \text{Re} \int_T (\partial^{m-1}_x \psi)^2 \psi^2 dx + \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{4\nu} \int_T |\partial^{m-1}_x \psi|^2 |\psi|^2 dx,
\]
where $C_m$ is a sufficiently large constant depending only on $m$ so that $E_m(\psi)$ is positive. This is possible because of the following reason. The Gagliano-Nirenberg inequality (Lemma 2.1) implies
\[
\frac{\lambda_3}{\nu} \text{Re} \int_T (\partial^{m-1}_x \psi)^2 \psi^2 dx + \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{4\nu} \int_T |\partial^{m-1}_x \psi|^2 |\psi|^2 dx \\
\geq -\frac{1}{2} \||\partial^m_x \psi(t)\|^2_{L^2} - \frac{1}{2} \|\psi(t)\|^2_{L^2} - D_m \|\psi(t)\|_{L^2}^{4m+2}
\]
with some positive constant $D_m$ depending only on $\nu, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ and $m$. Hence we obtain
\[
[E_m(\psi)](t) \geq \frac{1}{2} \||\partial^m_x \psi(t)\|^2_{L^2} + \frac{1}{2} \|\psi(t)\|^2_{L^2} + (C_m - D_m)\|\psi(t)\|_{L^2}^{4m+2}.
\]

Choosing $C_m$ so large that $C_m > D_m$, we have $[E_m(\psi)](t) > 0$. We notice that
\[
\frac{1}{2} \|\psi(t)\|^2_{H^m_T} \leq [E_m(\psi)](t) \leq C(\|\psi(t)\|^m_{L^2_T} + 1)\|\psi(t)\|^2_{H^m_T}. \tag{3.1}
\]

Lemma 3.1. Let $\psi_\epsilon \in C([0,T_\epsilon), H^m(T))$ be a solution to (2.1). Then, there exists positive constants $C$ and $T = T(\|\phi\|_{H^m_T})$ which are independent of $\epsilon$ such that
\[
\|\psi_\epsilon(t)\|^2_{H^m_T} \leq C(T, \|\phi\|_{L^2_T})\|\phi\|_{H^m_T},
\]
for any $t \in [0,T)$. 

Proof. We first evaluate \([E_m(\psi)](t)\). Applying the \(m\)-th derivative the both sides of (2.1), taking the inner product of the resultant equation with \(\partial^m_x \psi\), and adding the complex conjugation of the produce, we obtain

\[
\frac{d}{dt} \|\partial^m_x \psi(t)\|_{L^2_x}^2 + 2\epsilon \|\partial^{m+2}_x \psi(t)\|_{L^2_x}^2 = 2\text{Im} \int_T \partial^m_x N(\psi, \bar{\psi}, \partial^2_x \psi, \partial^2_x \bar{\psi}) \partial^m_x \bar{\psi} dx.
\]

Using the Leibniz rule, we obtain

\[
\partial^m_x N(\psi, \bar{\psi}, \partial^2_x \psi, \partial^2_x \bar{\psi}) = (2\lambda_3 + m\lambda_6)\bar{\psi} \partial_x \psi + (\lambda_4 + m\lambda_6)\psi \partial_x \bar{\psi}) \partial^{m+1}_x \psi \\
+ (\lambda_4 + 2m\lambda_5)\bar{\psi} \partial_x \psi \partial_x \bar{\psi} \partial^{m+2}_x \psi + \lambda_6 |\psi|^{2}\partial^{m+2}_x \psi + \lambda_5 \psi^2 \partial^{m+2}_x \bar{\psi} \\
+ P_1(\psi, \bar{\psi}, \partial^2_x \psi, \partial^2_x \bar{\psi}),
\]

where \(P_1\) is a linear combination of the cubic terms \(\partial^{j_1}_x \psi \partial^{j_2}_x \bar{\psi} \partial^{j_3}_x \psi\) with \(j_1 + j_2 + j_3 = m\), the cubic terms \(\partial^{j_1}_x \psi \partial^{j_2}_x \bar{\psi} \partial^{j_3}_x \bar{\psi}\) with \(j_1 + j_2 + j_3 = m + 2\) and \(j_1 + j_2 + j_3 \leq m\), and the quintic terms \(\partial^{j_1}_x \psi \partial^{j_2}_x \bar{\psi} \partial^{j_3}_x \psi \partial^{j_4}_x \bar{\psi} \partial^{j_5}_x \bar{\psi}\) with \(j_1 + j_2 + j_3 + j_4 + j_5 = m\). Hence the Hölder and Gagliardo-Nirenberg (Lemma 2.1) inequalities imply

\[
\|P_1\|_{L^2_x} \leq C \bigg( \|\psi\|_{L^2_x}^{(2m-1)/m} \|\bar{\psi}\|_{H^m_x}^{(m+1)/m} + \|\psi\|_{L^2_x}^{(4m-2)/m} \|\bar{\psi}\|_{H^m_x}^{(m+2)/m} \\
+ \|\psi\|_{L^2_x}^{(2m-3)/m} \|\bar{\psi}\|_{H^m_x}^{(m+3)/m} \bigg)
\]

\[
\leq C [E_m(\psi_e)](t)^{3/2}.
\]

In the last inequality we used the inequalities

\[
\|\psi\|_{L^2_x} \leq [E_m(\psi)](t)^{\alpha} \quad \text{for any} \quad \frac{1}{4m+2} \leq \alpha \leq \frac{1}{2},
\]

\[
\|\psi\|_{H^m_x} \leq [E_m(\psi)](t)^{1/2}.
\]
Substituting (3.3) and (3.4) into (3.2), we have
\[
\frac{d}{dt} \| \partial_x^m \psi_e(t) \|_{L^2}^2 + 2\epsilon \| \partial_x^{m+2} \psi(t) \|_{L^2}^2 \quad (3.5)
\]
\[
= 2(2\lambda_3 + m\lambda_6) \text{Im} \int \overline{\psi} \partial_x \psi \partial_x^m \psi \partial_x^{m+1} \psi dx
+ 2(\lambda_4 + m\lambda_6) \text{Im} \int \overline{\psi} \partial_x \psi \partial_x^m \psi \partial_x^{m+1} \psi dx
+ 2(\lambda_4 + 2m\lambda_5) \text{Im} \int \overline{\psi} \partial_x \psi \partial_x^m \psi \partial_x^{m+1} \psi dx

\]
\[+ 2\lambda_6 \text{Im} \int |\psi|^{2} \partial_x^m \psi \partial_x^{m+2} \psi dx + 2\lambda_5 \text{Im} \int \overline{\psi} \partial_x^m \psi \partial_x^{m+2} \psi dx
+ 2\text{Im} \int \psi_e \partial_x^m \psi_e \partial_x^{m+2} \psi_e dx \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\]
The inequality (3.4) and the Schwarz inequality imply
\[|I_6| \leq \| P_1 \|_{L^2} \| \partial_x^m \psi \|_{L^2} \leq C|E_m(\psi_e)| (t)^2.
\]
An integration by parts yields
\[|I_3| \leq C|E_m(\psi_e)| (t)^2.
\]
We can express \(I_2, I_4\) and \(I_5\) in terms of \(I_1\) by using an integration by parts:
\[
I_2 = 2(\lambda_4 + m\lambda_6) \text{Im} \int \overline{\psi} \partial_x \psi \partial_x^m \psi \partial_x^{m+1} \psi dx
+ R_1(\psi_e, \overline{\psi}_e, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi}_e),
\]
\[
I_4 = -4\lambda_6 \text{Im} \int \overline{\psi} \partial_x \psi \partial_x^m \psi \partial_x^{m+1} \psi dx
+ R_2(\psi_e, \overline{\psi}_e, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi}_e),
\]
\[
I_5 = -2\lambda_5 \text{Im} \int \overline{\psi} \partial_x^{m+1} \psi dx
+ R_3(\psi_e, \overline{\psi}_e, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi}_e),
\]
where \(R_1, R_2\) and \(R_3\) satisfy
\[|R_1| + |R_2| + |R_3| \leq C|E_m(\psi_e)| (t)^2.
\]
Substituting above equations into (3.5), we have
\[
\frac{d}{dt} \| \partial_x^m \psi_e(t) \|_{L^2}^2 + 2\epsilon \| \partial_x^{m+2} \psi(t) \|_{L^2}^2
\]
\[= 2\{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6\} \text{Im} \int \overline{\psi} \partial_x \psi_e \cdot \partial_x^m \psi_e \partial_x^{m+1} \psi_e dx
- 2\lambda_5 \text{Im} \int \overline{\psi} \partial_x^{m+1} \psi_e dx + R_4(\psi_e, \overline{\psi}_e, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi}_e),
\]
where $R_4$ satisfies
\[ |R_4| \leq C[E_m(\psi_e)](t)^2. \]

On the other hand, from the equation (1.1), we have
\[
\frac{d}{dt} \text{Re} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e}^2 \, dx
\]
\[ = 2 \text{Re} \int_T \partial_x^{m-1} \psi_e \partial_x \partial_x^{m-1} \psi_e \cdot \overline{\psi_e}^2 \, dx + 2 \text{Re} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e} \partial_t \overline{\psi_e} \, dx \]
\[ = -2\epsilon \text{Re} \int_T \partial_x^{m-1} \psi_e \partial_x^{m+3} \psi_e \cdot \overline{\psi_e}^2 \, dx - 2 \text{Im} \int_T \partial_x^{m-1} \psi_e \partial_x^{m+1} \psi_e \cdot \overline{\psi_e}^2 \, dx \]
\[ -2\nu \text{Im} \int_T \partial_x^{m-1} \psi_e \partial_x^{m+3} \psi_e \cdot \overline{\psi_e}^2 \, dx \]
\[ + 2 \text{Im} \int_T \partial_x^{m-1} \psi_e \partial_x^{m-1} \mathcal{N}(\psi_e, \overline{\psi_e}, \ldots, \partial_x^2 \psi_e, \partial_x^2 \overline{\psi_e}) \cdot \overline{\psi_e}^2 \, dx \]
\[ -2\epsilon \text{Re} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e} \partial_x^{m} \overline{\psi_e} \, dx + 2 \text{Im} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e} \cdot \partial_x^{m+2} \overline{\psi_e} \, dx \]
\[ + 2\nu \text{Im} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e} \cdot \partial_x^{m} \overline{\psi_e} \, dx \]
\[ -2 \text{Im} \int_T (\partial_x^{m-1} \psi_e)^2 \overline{\psi_e} \cdot \mathcal{N}(\psi_e, \overline{\psi_e}, \ldots, \partial_x^2 \psi_e, \partial_x^2 \overline{\psi_e}) \, dx \]
\[ \equiv I_7 + I_8 + I_9 + I_{10} + I_{11} + I_{12} + I_{13} + I_{14}. \]

An integration by parts yields
\[ I_7 = -2\epsilon \text{Re} \int_T (\partial_x^{m+1} \psi_e)^2 \overline{\psi_e}^2 \, dx \]
\[ + R_5(\psi_e, \overline{\psi_e}, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi_e}), \]
\[ I_9 = -2\nu \text{Im} \int_T (\partial_x^{m+1} \psi_e)^2 \overline{\psi_e} \, dx \]
\[ + R_6(\psi_e, \overline{\psi_e}, \ldots, \partial_x^m \psi_e, \partial_x^m \overline{\psi_e}), \]

where $R_5$ and $R_6$ satisfy
\[ |R_5| + |R_6| \leq C[|\psi_e|]^{(2m-3)/m} |\psi_e|^{(2m+3)/m} \]
\[ \leq C[E_m(\psi_e)](t)^2. \]
Integrating by parts, we also obtain
\[
|I_8| + |I_{12}| \leq C \| \psi_\epsilon \|_{L_x^2}^{(2m-1)/m} \| \psi_\epsilon \|_{H_x^m}^{(2m+1)/m}
\]
\[
\leq C [E_m(\psi_\epsilon)](t)^2,
\]
\[
|I_{11}| + |I_{13}| \leq C \| \psi_\epsilon \|_{L_x^2}^{(2m-3)/m} \| \psi_\epsilon \|_{H_x^m}^{(2m+3)/m}
\]
\[
\leq C [E_m(\psi_\epsilon)](t)^2,
\]
\[
|I_{10}| + |I_{14}| \leq C \| \psi_\epsilon \|_{L_x^2}^{4} \| \psi_\epsilon \|_{H_x^m}^{2} + \| \psi_\epsilon \|_{L_x^2}^{(6m-1)/m} \| \psi_\epsilon \|_{H_x^m}^{(2m+1)/m}
\]
\[
+ \| \psi_\epsilon \|_{L_x^2}^{(4m-2)/m} \| \psi_\epsilon \|_{H_x^m}^{(2m+2)/m}
\]
\[
\leq C [E_m(\psi_\epsilon)](t)^2.
\]

Substituting above equations into (3.7), we have
\[
\frac{d}{dt} \text{Re} \int_T (\partial_x^{m-1} \psi_\epsilon)^2 \overline{\psi_\epsilon^2} dx = -2\nu \text{Im} \int_T (\partial_x^{m+1} \psi_\epsilon)^2 \overline{\psi_\epsilon^2} dx + 2\epsilon \text{Re} \int_T (\partial_x^{m+1} \psi_\epsilon)^2 \overline{\psi_\epsilon^2} dx
\]
\[
+ R_7(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^m \psi_\epsilon, \partial_x^m \overline{\psi_\epsilon}),
\]
where $R_7$ satisfies
\[
|R_7| \leq C [E_m(\psi_\epsilon)](t)^2.
\]

By an argument similar to (3.8), we obtain
\[
\frac{d}{dt} \int_T |\partial_x^{m-1} \psi_\epsilon|^2 |\psi_\epsilon|^2 dx
\]
\[
= -8\nu \text{Im} \int_T \overline{\psi_\epsilon} \partial_x \psi_\epsilon \cdot \partial_x \overline{\psi_\epsilon} \partial_x^{m+1} \psi_\epsilon dx - 2\epsilon \int_T |\partial_x^{m+1} \psi_\epsilon|^2 |\psi_\epsilon|^2 dx
\]
\[
+ R_8(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^m \psi_\epsilon, \partial_x^m \overline{\psi_\epsilon}),
\]
where $R_8$ satisfies
\[
|R_8| \leq C [E_m(\psi_\epsilon)](t)^2.
\]

Finally, we obtain
\[
\frac{d}{dt} \| \psi_\epsilon(t) \|_{L_x^2}^{4m+2} + 2(2m + 1)\epsilon \| \psi_\epsilon(t) \|_{L_x^2}^{4m} \| \partial_x^2 \psi_\epsilon(t) \|_{L_x^2}^2
\]
\[
\leq C [E_m(\psi_\epsilon)](t)^2.
\]
Collecting (3.7), (3.8), (3.9), and (3.10), we have
\[
\frac{d}{dt} E_m(t) + 2\epsilon\|\partial_x^{m+2} \psi_\epsilon(t)\|^2_{L^2} + 2(2m+1)\epsilon\|\psi_\epsilon(t)\|^2_{L^2} \|\partial_x^2 \psi_\epsilon(t)\|^2_{L^2} \leq \frac{2\lambda_5}{\nu} \epsilon \text{Re} \int_T (\partial_x^{m+1} \psi_\epsilon(t))^{2m} \psi_\epsilon^2 \, dx \\
- \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{2\nu} \int_T |\partial_x^{m+1} \psi_\epsilon|^2 |\psi_\epsilon|^2 \, dx \\
+ R_9(\psi_\epsilon, \overline{\psi_\epsilon}, \ldots, \partial_x^m \psi_\epsilon, \overline{\partial_x^m \psi_\epsilon}),
\]
where \( R_9 \) satisfies
\[
|R_9| \leq C E_m(\psi_\epsilon)^2(t)^2.
\]
Since the sum of the first and second terms in the right hand side of (3.11) are bounded by \( \epsilon \|\partial_x^{m+2} \psi_\epsilon\|^2_{L^2} + C E_m(\psi_\epsilon)^2(t)^2 \), we obtain
\[
\frac{d}{dt} [E_m(\psi_\epsilon)](t) = \epsilon\|\partial_x^{m+2} \psi_\epsilon(t)\|^2_{L^2} + 2(2m+1)\epsilon\|\psi_\epsilon(t)\|^2_{L^2} \|\partial_x^2 \psi_\epsilon(t)\|^2_{L^2} \leq C E_m(\psi_\epsilon)^2(t)^2.
\]
Therefore
\[
\frac{d}{dt} [E_m(\psi_\epsilon)](t) \leq C E_m(\psi_\epsilon)^2(t)^2.
\]
We note that the constant \( C \) is independent of \( \epsilon \in (0, 1] \). From the above inequality we have
\[
[E_m(\psi_\epsilon)](t) \leq \frac{[E_m(\psi_\epsilon)](0)}{1 - Ct[E_m(\psi_\epsilon)](0)},
\]
for \( 0 \leq t < \min\{T_\epsilon, C^{-1}[E_m(\psi_\epsilon)](0)^{-1}\} \). Combing this inequality, \( \|\phi_e\|_{H^m} \leq \|\phi\|_{H^m} \) for any \( \epsilon \in (0, 1] \) and (3.11), we see
\[
\|\psi_\epsilon\|^2_{H^m} \leq C (\|\phi\|^2_{L^2} + 1) \|\phi\|^2_{H^m}.
\]
for \( 0 \leq t < \min\{T_\epsilon, C^{-1}(\|\phi\|^2_{L^2} + 1)^{-1}\|\phi\|^2_{H^m} \} \). Let \( T \equiv (2C)^{-1}(\|\phi\|^2_{L^2} + 1)^{-1}\|\phi\|^2_{H^m} \). Then for any \( 0 < t < \min\{T_\epsilon, T\} \), we have
\[
\|\psi_\epsilon\|^2_{H^m} \leq 2C (\|\phi\|^2_{L^2} + 1) \|\phi\|^2_{H^m}.
\]
If \( T_\epsilon < T \), we can apply Lemma 2.2 to extend the solution in the same class to the interval \([0, T]\). Therefore we obtain the desired result. \( \square \)

Using Lemma 3.1 we obtain the existence of the solution to (1.1):

**Lemma 3.2.** Let \( m \geq 4 \) be an integer. For any \( \phi \in H^m(\mathbb{T}) \), there exists a time \( T = T(\|\phi\|_{H^m}) > 0 \) and a solution \( \psi \) of (1.1) satisfying
\[
\psi \in L^\infty([0, T]; H^m(\mathbb{R})).
\]
Proof. Let $\phi \in H^m(\mathbb{T})$ and let $\{\phi_\epsilon\}_\epsilon \subset H^\infty(\mathbb{T})$ be a Bona-Smith approximation of $\phi$. Then by Lemma 2.2 there exists a unique solution $\psi_\epsilon \in C([0, T_\epsilon); H^m(\mathbb{T}))$ to (2.1). Lemma 3.1 yields that there exists $T = T(\|\phi\|_{H^m}) > 0$ which is independent of $\epsilon$ such that $\{\psi_\epsilon\}_\epsilon$ is uniformly bounded in $L^\infty(0, T; H^m(\mathbb{T}))$ with respect to $\epsilon \in (0, 1]$. By a standard limiting argument, it is inferred that a subsequence of $\psi_\epsilon$ convergence in $L^\infty(0, T; H^m(\mathbb{T}))$ weak* to a solution $\psi$ of (1.1) such that $\psi_\epsilon \in L^\infty(0, T; H^m(\mathbb{T}))$. We omit the detail. \hfill $\square$

4. Proof of Theorem 1.1

In the preceding sections, we proved the existence of the solution to (1.1). In this section, we complete the proof of Theorem 1.1 by showing the following three assertions

(i) uniqueness of the solution

(ii) persistent properties of the solution

(iii) continuous dependence of the solution upon initial data

4.1. Uniqueness. Let $\psi_1$ and $\psi_2$ be two solutions to (1.1) with same initial data satisfying $\sup_{t \in [0, T]} \|\psi_j(t)\|_{H^2_T} < \infty$, $j = 1, 2$. We shall show that $\psi_1 \equiv \psi_2$ for $t \in [0, T)$. To prove this, it suffices to show that $\psi = \psi_2 - \psi_1$ satisfies $\|\psi(t)\|_{H^2_T} \equiv 0$ because this identity and $\psi(0) \equiv 0$ implies $\psi \equiv 0$. The reason we prove $\|\psi(t)\|_{H^2_T} \equiv 0$ instead of driving $\|\psi(t)\|_{L^2_T} \equiv 0$ is that the corresponding modified energy for $L^2$ involves the anti-derivatives of $\psi$.

The standard energy estimate yields

$$
\frac{d}{dt} \|\psi(t)\|_{L^2_T}^2 = 2\text{Im} \int_T \{\mathcal{N}(\psi + \psi_1, \bar{\psi} + \bar{\psi}_1, \ldots, \partial^2_x \psi + \partial^2_x \psi_1, \partial^2_x \bar{\psi} + \partial^2_x \bar{\psi}_1)
- \mathcal{N}(\psi_1, \bar{\psi}_1, \ldots, \partial^2_x \psi_1, \partial^2_x \bar{\psi}_1)\} \psi \bar{\psi} dx.
$$

(4.1)

$$
\frac{d}{dt} \|\partial_x \psi(t)\|_{L^2_T}^2 = 2\text{Im} \int_T \partial_x \{\mathcal{N}(\psi + \psi_1, \bar{\psi} + \bar{\psi}_1, \ldots, \partial^2_x \psi + \partial^2_x \psi_1, \partial^2_x \bar{\psi} + \partial^2_x \bar{\psi}_1)
- \mathcal{N}(\psi_1, \bar{\psi}_1, \ldots, \partial^2_x \psi_1, \partial^2_x \bar{\psi}_1)\} \partial_x \psi \bar{\psi} dx
- 2(2\lambda_3 + \lambda_4) \text{Im} \int_T \partial_x \psi_1 \bar{\psi}_1 \cdot \partial^2_x \psi \partial_x \bar{\psi} dx
- 2\lambda_5 \text{Im} \int_T \partial^2_x (\partial^2_x \psi)^2 dx + R_{10}(\psi_1, \bar{\psi}_1, \ldots, \partial^2_x \psi_1, \partial^2_x \bar{\psi}_1, \psi_2, \bar{\psi}_2, \ldots, \partial^2_x \psi_2, \partial^2_x \bar{\psi}_2),
$$

(4.2)

where $R_{10}$ satisfies

$$
|R_{10}| \leq C(\|\psi_1\|_{H^2_T}^2 + \|\psi_1\|_{H^2_T}^4 + \|\psi_2\|_{H^2_T}^2 + \|\psi_2\|_{H^2_T}^4) \|\psi\|_{H^2_T}^2.
$$
On the other hand, a direct calculation yields

\[
\frac{2\lambda_3 + \lambda_4}{4\nu} \frac{d}{dt} \int_{\mathbb{T}} |\psi_1|^2 |\psi|^2 \, dx
\]

(4.3)

\[
= -2(2\lambda_3 + \lambda_4) \text{Im} \int_{\mathbb{T}} \partial_x \psi_1 \overline{\psi_1} \cdot \partial_x^2 \psi \partial_x \overline{\psi} \, dx
\]

\[
+ R_{11}(\psi_1, \overline{\psi_1}, \ldots, \partial_x^3 \psi_1, \partial_x^2 \overline{\psi_1}, \psi_2, \overline{\psi_2}, \ldots, \partial_x^3 \psi_2, \partial_x^2 \overline{\psi_2}),
\]

\[
\frac{\lambda_5}{\nu} d \frac{d}{dt} \text{Re} \int_{\mathbb{T}} \psi_1^2 (\overline{\psi})^2 \, dx
\]

(4.4)

\[
= 2\lambda_5 \text{Im} \int_{\mathbb{T}} \psi_1^2 (\partial_x^2 \overline{\psi})^2 \, dx
\]

\[
+ R_{12}(\psi_1, \overline{\psi_1}, \ldots, \partial_x^3 \psi_1, \partial_x^2 \overline{\psi_1}, \psi_2, \overline{\psi_2}, \ldots, \partial_x^3 \psi_2, \partial_x^2 \overline{\psi_2}),
\]

where $R_{11}$ and $R_{12}$ satisfy

\[
|R_{11}| + |R_{12}| \leq C(\|\psi_1\|^2_{H^2} + \|\psi_1\|_{H^2}^{\alpha} + \|\psi_2\|^2_{H^2} + \|\psi_2\|_{H^2}^{\alpha})\|\psi\|^2_{H^2}.
\]

Here we set

\[
[\hat{E}_1(\psi)](t) = \|\partial_x \psi(t)\|^2_{L^2} + \hat{C}_1\|\psi(t)\|^2_{L^2}
\]

\[
+ \frac{2\lambda_3 + \lambda_4}{4\nu} \int_{\mathbb{T}} |\psi|^2 |\psi|^2 \, dx + \frac{\lambda_5}{\nu} \text{Re} \int_{\mathbb{T}} \psi_1^2 (\overline{\psi})^2 \, dx,
\]

where $\hat{C}_1$ is a sufficiently large constant depending only on $m$, $\sup_{t \in [0, T]} \|\psi_1(t)\|_{H^2}$ and $\sup_{t \in [0, T]} \|\psi_2(t)\|_{H^2}$ so that $\hat{E}_1(\psi)$ is positive. Then, from (4.1), (4.2), (4.3) and (4.4), we obtain

\[
\frac{d}{dt} [\hat{E}_1(\psi)](t)
\]

\[
\leq C(\|\psi_1\|^2_{H^2} + \|\psi_1\|_{H^2}^{\alpha} + \|\psi_2\|^2_{H^2} + \|\psi_2\|_{H^2}^{\alpha})\|\psi\|^2_{L^2}
\]

\[
\leq C[\hat{E}_1(\psi)](t).
\]

Hence Gronwall’s lemma yields

\[
[\hat{E}_1(\psi)](t) \leq [\hat{E}_1(\psi)](0) e^{ct}.
\]

(4.5)

Since $[\hat{E}_m(\psi)](0) = 0$, Gronwall’s lemma yields $[\hat{E}_1(\psi)](t) \equiv 0$. Combination of this identity and the equality $0 \leq \|\psi(t)\|_{H^2} \leq [\hat{E}_1(\psi)](t)$ implies $\psi \equiv 0$, which completes the proof of the uniqueness.

4.2. Persistence of solution. To prove the persistent property of the solution to (1.1) which is obtained by Lemma 3.2, we employ the Bona-Smith approximation. We denote $\phi$, the Bona-Smith approximation of $\phi$.

**Lemma 4.1.** Let $\phi \in H^m(\mathbb{T})$ with $m \geq 3$ and let $\psi_\alpha$, $\psi_\epsilon$ denote the solution to (1.7) corresponding to the initial data $\phi_\alpha$ and $\phi_\epsilon$, respectively. Then there exists $C = C(T, ||\phi||_{H^m}) >$
0 such that for $0 \leq \alpha < \epsilon \leq 1$,
\[
\sup_{t \in [0,T)} \| \psi_\alpha(t) - \psi_\epsilon(t) \|_{H^m_x}
\leq C(e^{m-3} + \| \phi - \phi_\alpha \|_{H^m_x} + \| \phi - \phi_\epsilon \|_{H^m_x}).
\] (4.6)

**Proof.** We put $\psi = \psi_\alpha - \psi_\epsilon$. We first evaluate $\| \psi(t) \|_{H^1_x}$. Replacing $\psi = \psi_1 - \psi_2$ by $\psi = \psi_\alpha - \psi_\epsilon$ in (4.5), we have
\[
[E_1(\psi)](t) \leq [E_1(\psi)](0)e^{Ct},
\] (4.7)
for $t \in [0,T)$, where
\[
[E_1(\psi)](t) = \| \partial_x \psi(t) \|_{L^2_x}^2 + \tilde{C}_1 \| \psi(t) \|_{L^2_x}^2 + \frac{2\lambda_3 + \lambda_4}{4\nu} \int_T |\psi_\alpha|^2|\psi|^2 dx + \frac{\lambda_5}{\nu} \text{Re} \int_T \psi_\alpha(\overline{\psi})^2 dx.
\]
$\tilde{C}_1$ is a sufficiently large constant depending only on $m$, $\sup_{t \in [0,T)} \| \psi(t) \|_{H^m_x}$ so that $E_1(\psi)$ is positive and $C$ in (4.7) depends only on $\sup_{t \in [0,T)} \| \psi(t) \|_{H^2_x}$. Since
\[
\| \psi(t) \|_{H^1_x}^2 \leq [E_1(\psi)](t),
\] (4.8)
\[\begin{align*}
[E_1(\psi)](0) &\leq C\| \phi - \phi_\epsilon \|_{H^1_x}^2 \leq C(\| \phi - \phi_\alpha \|_{H^1_x}^2 + \| \phi - \phi_\epsilon \|_{H^1_x}^2) \leq C(\alpha^{2(m-1)} + e^{2(m-1)}) \leq C e^{2(m-1)},
\end{align*}
\] (4.9)
the inequalities (4.7), (4.8) and (4.13) lead to the inequality
\[
\| \psi(t) \|_{H^1_x} \leq C e^{m-1}.
\] (4.10)

Next, we evaluate $\| \psi(t) \|_{H^m_x}$. By an argument similar to (3.2),
\[
\frac{d}{dt} \| \partial_x^m \psi(t) \|_{L^2_x}^2
= 2\text{Im} \int_T \partial_x^m \{ \mathcal{N}(\psi + \bar{\psi}, \bar{\psi} + \psi, \ldots, \partial_x^2 \psi + \partial_x^2 \bar{\psi}, \partial_x^2 \bar{\psi} + \partial_x^2 \psi) - \mathcal{N}(\psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^2 \bar{\psi}_\epsilon) \} \partial_x^m \bar{\psi} dx.
\] (4.11)

Using the Leibniz rule, we have
\[
\partial_x^m \{ \mathcal{N}(\psi + \bar{\psi}, \bar{\psi} + \psi, \ldots, \partial_x^2 \psi + \partial_x^2 \bar{\psi}, \partial_x^2 \bar{\psi} + \partial_x^2 \psi) - \mathcal{N}(\psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^2 \psi_\epsilon, \partial_x^2 \bar{\psi}_\epsilon) \}
= \{ (2\lambda_3 + m\lambda_6)\partial_x^m \psi_\alpha \partial_x \psi_\alpha + (\lambda_4 + m\lambda_6)\psi_\alpha \partial_x^m \bar{\psi}_\alpha \} \partial_x^{m+1} \psi
\]
\[\begin{align*}
&+ (\lambda_4 + 2m\lambda_5)\psi_\alpha \partial_x^m \psi_\alpha \partial_x^{m+1} \bar{\psi}_\alpha \\
&+ \lambda_6 \psi_\alpha^2 \partial_x^{m+2} \psi + \lambda_6 \bar{\psi}_\alpha^2 \partial_x^{m+2} \bar{\psi}
\end{align*}
\] (4.12)
\[\begin{align*}
&+ P_2(\psi_\alpha, \bar{\psi}_\alpha, \ldots, \partial_x^m \psi_\alpha, \partial_x^m \bar{\psi}_\alpha, \psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^{m+2} \psi_\epsilon, \partial_x^{m+2} \bar{\psi}_\epsilon).
\end{align*}
\]

1By the inequality $\| \phi_\epsilon \|_{H^m_x} \leq \| \phi \|_{H^m_x}$ and Lemma 32, we can choose $C_1$ independently of $\alpha$. 

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and
\[ P_2(\psi_\alpha, \bar{\psi}_\alpha, \ldots, \partial_x^m \psi_\alpha, \partial_x^m \bar{\psi}_\alpha, \psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^{m+2} \psi_\epsilon, \partial_x^{m+2} \bar{\psi}_\epsilon) \]
\[ = \{(2\lambda_3 + m\lambda_6)(\bar{\psi}_\alpha \partial_x \psi + \partial_x \bar{\psi}_\psi) + (\lambda_4 + m\lambda_6)\psi_\alpha \partial_x \bar{\psi} + \partial_x \bar{\psi}_\psi\}\partial_x^{m+1} \psi_\epsilon \]
\[ + (\lambda_4 + 2m\lambda_5)(\psi_\alpha \partial_x \psi + \partial_x \psi_\psi)\partial_x^{m+1} \bar{\psi}_\epsilon \]
\[ + \lambda_6(\bar{\psi}_\alpha \psi + \psi_\epsilon \bar{\psi}_\psi)\partial_x^{m+2} \psi_\epsilon + \lambda_5(\psi_\alpha \psi + \psi_\psi)\partial_x^{m+2} \bar{\psi}_\epsilon \]
\[ + P_3(\psi_\alpha, \ldots, \partial_x^m \psi_\alpha, \psi_\epsilon, \ldots, \partial_x^m \bar{\psi}_\epsilon), \]
where \( P_3 \) satisfies
\[ \| P_3 \|_{L^2} \leq C(\| \psi_\alpha \|_{H^m}^2 + \| \psi_\epsilon \|_{H^m}^4 + \| \psi_\epsilon \|_{H^m}^2 + \| \psi_\epsilon \|_{H^m}^4)\| \psi \|_{H^m} \]
\[ \leq C\| \psi \|_{H^m}. \]
Combining the inequalities (4.10) and (4.13) and \( \| \psi_\epsilon \|_{H^{m+2}} \leq C\epsilon^{-2} \| \psi \|_{H^m} \), we have
\[ \| P_2 \|_{L^2} \leq C(\| \psi_\alpha \|_{H^1}^2 + \| \psi_\epsilon \|_{H^1}^4)\| \psi_\epsilon \|_{H^m+2} \| \psi \|_{H^1} + C\| \psi \|_{H^m} \]
\[ \leq C\epsilon^{m-3} + C\| \psi \|_{H^m}. \]
The identity (1.12) and a standard energy estimate yield
\[ \frac{d}{dt}\| \partial_x^m \psi(t) \|_{L^2}^2 \]
\[ = 2\{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6\}\Im \int_T \partial_x \psi_\alpha \psi_\epsilon \partial_x^{m+1} \psi \partial_x^{m} \bar{\psi} dx \]
\[-2\lambda_5 \Im \int_T \psi_\alpha^2 (\partial_x^{m+1} \psi)^2 dx \]
\[ + R_{13}(\psi_\alpha, \bar{\psi}_\alpha, \ldots, \partial_x^m \psi_\alpha, \partial_x^m \bar{\psi}_\alpha, \psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^{m+2} \psi_\epsilon, \partial_x^{m+2} \bar{\psi}_\epsilon). \]
where \( R_{13} \) satisfies
\[ |R_{13}| \leq C\epsilon^{m-3} + C\| \psi \|_{H^m}^2. \]
On the other hand, a direct calculation yields
\[ \frac{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6}{4\nu} \frac{d}{dt} \int_T |\psi_\alpha|^2 |\partial_x^{m-1} \psi|^2 dx \]
\[ = -2\{2\lambda_3 + \lambda_4 + 2(m - 1)\lambda_6\}\Im \int_T \partial_x \psi_\alpha \psi_\epsilon \partial_x^{m+1} \psi \partial_x^{m} \bar{\psi} dx \]
\[ + R_{14}(\psi_\alpha, \bar{\psi}_\alpha, \ldots, \partial_x^m \psi_\alpha, \partial_x^m \bar{\psi}_\alpha, \psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^{m+2} \psi_\epsilon, \partial_x^{m+2} \bar{\psi}_\epsilon), \]
and
\[ \frac{\lambda_5}{\nu} \frac{d}{dt} \Re \int_T \psi_\alpha^2 (\partial_x^{m-1} \psi)^2 dx \]
\[ = 2\lambda_5 \Im \int_T \psi_\alpha^2 (\partial_x^{m+1} \psi)^2 dx \]
\[ + R_{15}(\psi_\alpha, \bar{\psi}_\alpha, \ldots, \partial_x^m \psi_\alpha, \partial_x^m \bar{\psi}_\alpha, \psi_\epsilon, \bar{\psi}_\epsilon, \ldots, \partial_x^{m+2} \psi_\epsilon, \partial_x^{m+2} \bar{\psi}_\epsilon), \]
where \( R_{14} \) and \( R_{15} \) satisfy
\[ |R_{14}| + |R_{15}| \leq C\| \psi \|_{H^m}^2. \]
From (4.14), (4.15) and (4.16), we obtain
\[
\frac{d}{dt}\left\{ \|\partial_x^m \psi(t)\|^2_{L^2_x} + \frac{2\lambda_3 + \lambda_4 + 2(m-1)\lambda_6}{4\nu} \int_T |\psi|_2^2 |\partial_x^{m-1}\psi|^2 \, dx \right\} + \frac{\lambda_5}{\nu} \operatorname{Re} \int_T \psi_\alpha^2 (\partial_x^{m-1}\psi)^2 \, dx \leq C\|\psi\|_{H^m_x}^2 + C\epsilon^{m-3}.
\] (4.17)

Here we set
\[
[\tilde{E}_m(\psi)](t) = \int_T |\partial_x^m \psi|^2 \, dx + \tilde{C}_m \int_T |\psi|^2 \, dx + \int_T |\psi|^2 |\partial_x^{m-1}\psi|^2 \, dx + \operatorname{Re} \int_T \psi_\alpha^2 (\partial_x^{m-1}\psi)^2 \, dx,
\]
where \(\tilde{C}_m\) is a sufficiently large constant depending only on \(m\), \(\sup_{t \in [0,T]} \|\psi(t)\|_{H^m_x}\) so that \(\tilde{E}_m(\psi)\) is positive. Then the inequality (4.17) is expressed in terms of \(\tilde{E}_m\):
\[
\frac{d}{dt}[\tilde{E}_m(\psi)](t) \leq C[\tilde{E}_m(\psi)](t) + C\epsilon^{m-3}.
\]

Therefore Gronwall’s lemma leads to the inequality
\[
[\tilde{E}_m(\psi)](t) \leq C([\tilde{E}_m(\psi)](0) + \epsilon^{m-3})e^{CT}.
\]

Therefore we have (4.6) which completes Lemma 4.1.

Let us prove the persistent property of the solution to (1.1). Let \(\phi \in H^m(T)\) and \(\{\phi_\epsilon\}_{\epsilon > 0} \subset H^\infty(T)\) be a Bona-Smith approximation of \(\phi\). Lemma 3.2 yields there exists \(T = T(\|\phi_\epsilon\|_{H^m_x})\) and a unique solution \(\psi(t) \in L^\infty(0,T;H^\infty(T))\) to (1.1). Since \(\|\phi_\epsilon\|_{H^m_x} \leq \|\phi\|_{H^m_x}\), we can choose \(T\) independently of \(\epsilon\). By Lemma 4.1 \(\{\psi(t)\}_\epsilon\) is Cauchy sequence in \(C(0,T;H^m(T))\). Consequently we see that \(\phi \in C(0,T;H^m(T))\). This guarantees the persistent property of the solution in Theorem 1.1.

4.3. **Continuity of data-to-solution map.** As the final step of the proof of Theorem 1.1, we prove that the data-to-solution map \(S_t : H^m(T) \to C([0,T);H^m(T)) (\phi \mapsto \psi(t))\) associated to (1.1) is continuous. To this end, we shall prove the following: Let \(\phi \in H^m(T)\). For any \(\eta > 0\) there exists \(\delta > 0\) such that if \(\tilde{\phi} \in H^m(T)\) satisfies
\[
\|\phi - \tilde{\phi}\|_{H^m_x} < \delta,
\]
then
\[
\sup_{t \in [0,T]} \|S_t(\phi) - S_t(\tilde{\phi})\|_{H^m_x} < \eta.
\]
Let \( \{\phi_\epsilon\}_{\epsilon>0} \) and \( \{\tilde{\phi}_\epsilon\}_{\epsilon>0} \) be the Bona-Smith approximations of \( \phi \) and \( \tilde{\phi} \), respectively. By the triangle inequality, we have

\[
\sup_{t \in [0,T]} \|S_t(\phi) - S_t(\tilde{\phi})\|_{H^m_T} \leq \sup_{t \in [0,T]} \|S_t(\phi) - S_t(\phi_\epsilon)\|_{H^m_T} + \sup_{t \in [0,T]} \|S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\|_{H^m_T} + \sup_{t \in [0,T]} \|S_t(\tilde{\phi}_\epsilon) - S_t(\tilde{\phi})\|_{H^m_T}.
\]

(4.18)

Letting \( \alpha \) tend to 0 in (4.6), we have

\[
\sup_{t \in [0,T]} \|S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\|_{H^m_T} \leq \ T(\epsilon^{m-3} + \|\phi - \phi_\epsilon\|_{H^m_T}).
\]

(4.19)

By a similar argument as the derivation of (4.6), we obtain

\[
\sup_{t \in [0,T]} \|S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\|_{H^m_T} \leq C(\epsilon^{m-3} + \|\phi_\epsilon - \tilde{\phi}_\epsilon\|_{H^m_T}).
\]

(4.20)

Combining the above inequality with the triangle inequality

\[
\|\phi_\epsilon - \tilde{\phi}_\epsilon\|_{H^m_T} \leq \|\phi_\epsilon - \phi\|_{H^m_T} + \|\phi - \tilde{\phi}\|_{H^m_T} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H^m_T},
\]

we have

\[
\sup_{t \in [0,T]} \|S_t(\phi_\epsilon) - S_t(\tilde{\phi}_\epsilon)\|_{H^m_T} \leq C(\epsilon^{m-3} + \|\phi - \tilde{\phi}\|_{H^m_T} + \|\phi - \phi_\epsilon\|_{H^m_T} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H^m_T}).
\]

Substituting (4.19), (4.20) and (4.21) into (4.18), we obtain

\[
\sup_{t \in [0,T]} \|S_t(\phi) - S_t(\tilde{\phi})\|_{H^m_T} \leq C(\epsilon^{m-3} + \|\phi_\epsilon - \tilde{\phi}_\epsilon\|_{H^m_T} + \|\phi - \tilde{\phi}\|_{H^m_T} + \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H^m_T}).
\]

(4.22)

We first choose \( \delta > 0 \) so that \( C\delta < \eta/4 \). Since \( \phi_\epsilon \to \phi \) and \( \tilde{\phi}_\epsilon \to \tilde{\phi} \) in \( H^m \) as \( \epsilon \to 0 \), there exists \( \epsilon_0 > 0 \) such that for \( 0 < \epsilon \leq \epsilon_0 \),

\[
\|\phi - \phi_\epsilon\|_{H^m_T} < \frac{\eta}{4}, \quad \|\tilde{\phi} - \tilde{\phi}_\epsilon\|_{H^m_T} < \frac{\eta}{4}.
\]

Further choosing \( \epsilon_0 \) sufficiently small so that \( C\epsilon_0^{m-3} < \eta/4 \), we have that if \( \tilde{\phi} \in H^m(T) \) satisfies \( \|\phi - \tilde{\phi}\|_{H^m_T} < \delta \), then taking \( 0 < \epsilon \leq \epsilon_0 \) in (4.22), we have

\[
\sup_{t \in [0,T]} \|S_t(\phi) - S_t(\tilde{\phi})\|_{H^m_T} < \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} = \eta.
\]

The proof of Theorem 1.1 is now complete.

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