Integral Intertwining Operators and Complex Powers of Differential $(q$–Difference) Operators

Boris Feigin
Landau Institute for Theoretical Physics

Feodor Malikov*
Department of Mathematics, Kyoto University , Kyoto 606 Japan

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Abstract

We study a family of modules over Kac-Moody algebras realized in multi-valued functions on a flag manifold and find integral representations for intertwining operators acting on these modules. These intertwiners are related to some expressions involving complex powers of Lie algebra generators. When applied to affine Lie algebras, these expressions give integral formulas for correlation functions with values in not necessarily highest weight modules. We write related formulas out in an explicit form in the case of $\mathfrak{sl}_2$. The latter formulas admit $q$-deformation producing an integral representation of $q$-correlation functions. We also discuss a relation of complex powers of Lie algebra (quantum group) generators and Casimir operators to $(q$–)special functions.

1 Introduction

This paper consists of 2 parts independent within reasonable limits. The 1st one (sect. 2) is devoted to constructing integral intertwining operators acting between Kac-Moody Lie algebra modules realized in multi-valued functions on a flag manifold. The 2nd (sect. 3, 4) is devoted to integral representations

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of solutions to the (quantum) Knizhnik-Zamolodchikov equation. Both have in common, firstly, Kac-Moody Lie algebra (quantum group) singular vector formula as a motivation and, secondly, a new class of Kac-Moody Lie algebra modules as a main object.

1.1 Intertwining Operators

1. Let \( \mathfrak{g} \) be a Kac-Moody Lie algebra on canonical generators \( F_i, H_i, E_i \), \( 0 \leq i \leq n \) and defining relations, associated to a generalized symmetrizable Cartan matrix \( A = (a_{ij}) \):

\[
[h_i, h_j] = 0, \quad [h_i, E_j] = a_{ij} E_j, \quad [h_i, F_j] = -a_{ij} F_j, \\
\text{ad}^{-a_{ij}+1}(E_i)E_j = \text{ad}^{-a_{ij}+1}(F_i)F_j = 0. 
\]

These relations admit the antiinvolution \( \omega \) such that:

\[
\omega(E_i) = F_i, \quad \omega(F_i) = E_i, \quad \omega(H_i) = H_i. 
\]

Denote by \( n_+ \) (\( n_- \) resp.) the subalgebra generated by \( E_0, \ldots, E_n \) (\( F_0, \ldots, F_n \) resp.) and by \( \mathfrak{h} \) - the one generated by \( H_0, \ldots, H_n \). Denote also by \( M(\lambda) \) the Verma module over \( \mathfrak{g} \) with the highest weight \( \lambda \in \mathfrak{h}^* \).

This is a module on 1 generator \( v_\lambda \) and defining relations:

\[
n_+ v_\lambda = 0, \quad H_i v_\lambda = \lambda(H_i). 
\]

We will also be using a contragredient Verma module \( M(\lambda)^c \) which is defined as follows. As a vector space \( M(\lambda)^c \) is dual to \( M(\lambda) \) but the action of \( \mathfrak{g} \) is different from the canonical action on a dual space: if \( f(\cdot) \in M(\lambda)^c \) is a linear functional on \( M(\lambda) \) and \( g \in \mathfrak{g} \) then we set \( g f(\cdot) = f(\omega(g) \cdot) \). Obviously a map dual to a morphism of Verma modules is a morphism of contragredient Verma modules.

A \( \mathfrak{g} \)- morphism of \( M(\lambda) \) is uniquely determined by the image of \( v_\lambda \). It follows from the definition that a non-zero vector \( w \) of a \( \mathfrak{g} \)-module \( W \) may serve as an image of \( v_\lambda \) under a non-zero morphism \( M(\lambda) \to W \) if and only if it satisfies the same conditions as \( v_\lambda \):

\[
n_+ v_\lambda = 0, \quad H_i v_\lambda = \lambda(H_i). 
\] (1)

A non-zero element \( w \) of a \( \mathfrak{g} \)- module \( W \) is said to be a singular vector of the weight \( \lambda \) if it satisfies (1). It follows that the problem of classification of morphisms of Verma modules into a given module is equivalent to that of singular vectors in the latter module. We now recall a singular vector formula obtained in [20].

Let \( (\cdot, \cdot) \) be an invariant inner product on \( \mathfrak{h}^* \), \( \Delta \) be the set of roots of \( \mathfrak{g} \), \( \alpha_0, \ldots, \alpha_n \) be the set of simple roots related to the generators \( E_0, \ldots, E_n \). A root is called positive if it is equal to a non-negative linear combination of simple roots. To each simple root we associate a reflection of the space \( \mathfrak{h}^* \)

\[
r_i \lambda = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i. 
\]
The group generated by these reflections is called the Weyl group of $\mathfrak{g}$. Its action preserves the set of roots and one defines the set of real roots as $\Delta^r = W\{\alpha_0, \ldots, \alpha_n\}$.

We define the shifted action of the Weyl group $W$ on $\mathfrak{h}^*$ by

$$r_\alpha \cdot \lambda = \lambda - \frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \alpha,$$

where $r_\alpha$ stands for the reflection at a real root $\alpha$ and $\rho$ is the fixed element of $\mathfrak{h}^*$ determined by the following conditions $\rho(H_0) = \ldots = \rho(H_n) = 1$.

The Kac-Kazhdan determinant formula [16] implies that for a positive real root $\alpha$ the Verma module $M(\lambda)$ contains a singular vector of the weight $r_\alpha \cdot \lambda$ provided

$$\frac{2(\lambda + \rho, \alpha)}{(\alpha, \alpha)} \subset \{1, 2, \ldots\}.$$  \hspace{1cm} (2)

It also follows from [16] that for a generic $\lambda$ satisfying this condition the singular vector is unique up to proportionality. The formula for this singular vector was found in [20] in the following form.

**Theorem 1.1** Let $r_\alpha = r_{i_k} \cdots r_{i_2} r_{i_1}$ be a reduced decomposition. Set

$$\beta_j = \frac{2(r_{i_{j-1}} \cdots r_{i_1} \cdot \lambda, \alpha_{i_1})}{(\alpha_{i_1}, \alpha_{i_1})} + 1.\hspace{1cm} \text{(3)}$$

Then provided (2) holds the vector

$$F_{i_1}^{\beta_1} \cdots F_{i_2}^{\beta_2} F_{i_1}^{\beta_1} v_\lambda$$

is singular.

This singular vector formula involves complex powers of Lie algebra generators and therefore, its meaning has still to be clarified. Here we restrict ourselves to the following comment. Regard the expression $F_{i_1}^{j_1} \cdots F_{i_2}^{j_2} F_{i_1}^{j_1}$ as a function of integers $(j_1, \ldots, j_k) \in \mathbb{Z}_k$ taking values in the universal enveloping algebra $U(n_-)$. One shows that it admits an analytic continuation to a function defined on $\mathbb{C}^k$ taking values in a certain completion of $U(n_-)$. Then one shows that its value on a sequence $(\beta_1, \ldots, \beta_l)$ defined in Theorem 1.1 actually belongs to $U(n_-)$, provided (3) holds.

This explanation obscures the fact that the intertwiner related to the singular vector (3) is a composition of intertwiners which exist even if (4) does not hold, though may not act between Verma modules. These intertwiners can be easily constructed once one has defined an action of complex powers of the Lie algebra generators in a way that some natural conditions are satisfied. Consider, for example, the expression $F_{i_1}^{\beta_1} v_\lambda$. If one has convinced himself that the easily checked relation

$$[E_i, F_i^{\beta}] = \delta_{i,j} \beta F_j^{\beta - 1}(H_i - \beta + 1), \beta \in \mathbb{N} \hspace{1cm} \text{(4)}$$
makes sense for $\beta \in \mathbb{C}$, one obtains

$$E_j F_{i_1}^{\beta_1} v_\lambda = \delta_{j,i_1} \beta_1 F_{i_1}^{\beta_1, -1} (H_i - \beta_i + 1) v_\lambda = 0.$$  \hspace{1cm} (5)

Therefore, the vector

$$F_{i_1}^{\beta_1} v_\lambda = F_{i_1}^{\lambda(H_{i_1}) + 1} v_\lambda$$  \hspace{1cm} (6)

is a singular vector of a module which we have not yet constructed. The same reasoning for $v_\lambda$ replaced with $F_{i_1}^{\beta_1} v_\lambda$ shows that the vector $F_{i_2}^{\beta_1} F_{i_1}^{\beta_1} v_\lambda$ is singular. This procedure can be iterated arbitrary number of times, always resulting in a singular vector. So provided with a suitable definition of a complex power of a Lie algebra generator one can associate with a Weyl group element $w$ and its decomposition $w = r_{i_1} \cdots r_{i_n} r_{i_1}$ a singular vector $v_\lambda$.

The purpose of this work was to provide this construction with a rigid foundation. We define a family of $\mathfrak{g}$–modules, realized in functions on the big cell of the corresponding flag manifold so that the complex powers of Cartan generators of $\mathfrak{g}$ are well-defined integral operators, acting usually from one module into another.

2. It is known that a contragredient Verma module is realized in sections of a line bundle over the flag manifold with singularities at Schubert cells of codimension $\geq 1$, $\mathfrak{g}$ acting by vector fields. Thus in order to define a complex power of a Cartan generator one has first to define a complex power of a vector field as an operator acting on functions. Linearizing a vector field in a neighborhood of a non-singular point, one reduces the problem to evaluation of a quantity

$$(d \frac{d}{dz})^\mu f(z)$$

for an analytic (probably multi-valued) function $f(z)$ of a complex variable $z$ and an arbitrary $\mu \in \mathbb{C}$. But this is a classical topic. If $\mu \in \mathbb{N}$ then the answer is given by the Cauchy integral

$$(d \frac{d}{dz})^\mu f(z) = \frac{1}{\mu!} \oint_{\sigma} \frac{f(t)}{(t-z)^{\mu+1}} dt,$$

where $f(t)$ is supposed to be regular at $t = z$ and $\sigma$ is a sufficiently small contour enclosing the point $t = z$. In order to get an analytic continuation of a Cauchy integral (over $\mu$) one has to demand that the function $f(t)$ is branching at some point, say at $t = 0$. Then up to some factor (depending only on a branching coefficient of $f(t)$ at $t = 0$) one has

$$(d \frac{d}{dz})^\mu f(z) = \frac{1}{\Gamma(\mu + 1)} \oint_{\sigma_1, \sigma_2} \frac{f(t)}{(t-z)^{\mu+1}} \frac{f(t)}{(t-z)^{\mu+1}} dt,$$  \hspace{1cm} (7)

where $\Gamma(.)$ is the $\Gamma$–function and $\sigma_1, \sigma_2$ are generators of the 1st homotopy group of $\mathbb{C} - \{0, z\}$. The formula (7) is sufficient to describe our construction in the case of $\mathfrak{sl}_2$ which is also one of the cornerstones of the general case.

3. The $\mathfrak{sl}_2$–case. The flag manifold in this case is $\mathbb{C} P^1$, the big cell is $\mathbb{C}$. The Cartan generators are realized as follows:
\[ E = -\frac{d}{dz}, \quad H = -2z\frac{d}{dz}, \quad F = z^2\frac{d}{dz}, \quad (8) \]

where \( z \) is a coordinate on \( \mathbb{C} \). The algebra of all vector fields acts on tensor fields of the form \( f(z)dz^{-\lambda/2}, \ f(z) \in \mathbb{C}[z, z^{-1}]z^\nu, \nu \in \mathbb{C}, \) as follows

\[ p(z)\frac{d}{dz}(f(z)dz^{-\lambda/2}) = (p(z)f'(z) - \lambda/2p'(z)f(z))dz^{-\lambda/2}. \quad (9) \]

Specifying the class of functions \( f(z) \) one obtains the 2-parametric family of \( \mathfrak{sl}_2 \)-modules given by

\[ V(\nu, \lambda) = z^\nu \mathbb{C}[z, z^{-1}]dz^{-\lambda/2}. \quad (10) \]

It is worth mentioning that \( V(0, \lambda) \) contains the submodule \( \mathbb{C}[z]dz^{-\lambda/2} \) isomorphic to a contragredient Verma module with the highest weight \( \lambda \). The latter is irreducible if \( \lambda \) is not a nonnegative integer and contains the \( \lambda + 1 \)-dimensional irreducible \( \mathfrak{sl}_2 \)-submodule spanned by \( \{dz^{-\lambda/2}, zdz^{-\lambda/2}, \ldots, z^\lambda dz^{-\lambda/2}\} \) otherwise.

Now consider a meromorphic tensor field over \( \mathbb{CP}^1 \times \mathbb{CP}^1 \):

\[ s = \frac{dt \ dx}{(t-x)^2}. \]

A direct calculation shows that \( s \) is \( g \)-invariant. Then take a power \( s^\lambda/2+1 \). Whatever the meaning of this expression is for a complex \( \lambda \), the result is also \( g \)-invariant, and one considers the product

\[ f(t) \ dt^{-\lambda/2} s^\lambda/2+1 = \left\{ \frac{f(t)}{(t-z)^{\lambda+2}}dt \right\}dz^{\lambda/2+1}. \]

The result admits integration as in (7), which defines the operator

\[ T: \ V(\nu, \lambda) \rightarrow \ V(\nu - \lambda, -\lambda - 2) \quad (11) \]

\[ T: \ f(x) \ dx^{-\lambda/2} \rightarrow \ \left\{ \frac{1}{\Gamma(\lambda + 2)} \int_{\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1}} \frac{f(t)}{(t-x)^{\lambda+2}}dt \right\}dx^{\lambda/2+1} \]

We have arrived at

**Lemma 1.2** The operator \( T \) as in (11) is \( \mathfrak{sl}_2 \)-linear.

Note that if \( \lambda \) is a nonnegative integer then the restriction of our intertwiner to the submodule \( \mathbb{C}[z]dz^{-\lambda/2} \subset V(0, \lambda) \) is a map dual to the one determined by the singular vector \( F^{\lambda+1}v_\lambda \) of the Theorem and it acts killing the irreducible submodule spanned by \( \{dz^{-\lambda/2}, zdz^{-\lambda/2}, \ldots, z^\lambda dz^{-\lambda/2}\} \) and establishing the isomorphism of the corresponding quotient with the (irreducible) module \( V(0, -\lambda - 2) \).
4. The transparent construction of Lemma 1.2 admits a generalization to the case of a simple finite-dimensional Lie algebra or non-twisted affine Lie algebra. One associates in a way compatible with the action of a group $G$ (related to $g$) to each point $x$ on a flag manifold and a fixed simple root $\alpha$ - a projective line $X_\alpha(x)$ (closed Schubert cell related to this root and attached to this point). Then one constructs $s_i$ (an analogue of $s$ of the $sl_2$-case): a tensor field on the cartesian product of 2 copies of a flag manifold having good properties with respect to the action of $G$ and such that its restriction to each 1-dimensional Schubert cell attached to each point is a section of the line bundle of degree -2. Then one takes an element of an appropriate $g$-module; realizes it, roughly speaking, as a multivalued section of a bundle induced from a character $\lambda$ of the maximal torus; multiplies it by an appropriate power of $s_i$ so that the restriction of the result to each 1-dimensionsal Schubert cell is a differential form and, finally, integrates it over each 1-dimensional Schubert cell. What one gets is a multi-valued section of a line bundle induced from $r_\alpha \cdot \lambda$.

This operator is closely related to the singular vector associated to the simple reflection. One may now associate to an arbitrary element $w$ of the Weyl group (or rather to its reduced decomposition) the corresponding composition of constructed intertwiners. But there are some other possibilities. As suggested by the formula (11) an intertwiner related to an element $w$ of the Weyl group is an integration over a cycle of the Schubert cell $X_w(x)$ with coefficients in a certain local system. Taking composition means a special choice of the cycle. There should be other cycles as well, and there is some evidence suggested by the developments in conformal field theory and geometry (see, for example, [6], [27] and also see below) that these cycles should be related to a certain quantum group.

1.2 Integral Formulas for Correlation Functions

Let $g$ be a finite-dimensional simple Lie algebra, $\hat{g}$ the corresponding non-twisted affine Lie algebra. Let $\lambda_1, \lambda_2$ be weights of $g$, $M(\lambda_1, k)$ ($M(\lambda_2, k)^\vee$) be the Verma (contragredient Verma) module over $\hat{g}$ with the central charge $k$; let also $V$ be a $g$-module and $V((z))$ the module of formal Laurent series in $z$ with coefficients in $V$, regarded as a $\hat{g}$-module with the central charge equal to 0.

Vertex operator is a $\hat{g}$-linear map

$$\Phi(z) : M(\lambda_1, k) \rightarrow M(\lambda_2, k)^\vee \otimes V((z)).$$

One may consider a product of vertex operators $\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)$ and matrix elements $\langle w^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v \rangle$, where $v \in M(\lambda_1, k)$, $w^* \in (M(\lambda_{m+1}, k)^\vee)^*$. One of the central results of conformal field theory is (see [17])

Theorem 1.3 (Knizhnik, Zamolodchikov) The matrix element related to a
pair of vacuum vectors

\[ \Psi(z) = \langle v_{\lambda_{m+1}}^* \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle \]
satisfies a certain system of differential equations.

The explicit form of the above-mentioned system of differential equations called system of Knizhnik-Zamolodchikov equations may be found in sect. 3.1.

One shows that given

\[ \Psi(z) = \langle v_{\lambda_{m+1}}^* \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle \]

the matrix element

\[ \langle v_{\lambda_m}^* \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)F_{j_1}^{\beta_1} \cdots F_{j_l}^{\beta_l}v_{\lambda_1} \rangle \]
is equal to

\[ F_{j_1}^{\beta_1} F_{j_2}^{\beta_2} \cdots F_{j_l}^{\beta_l} \Psi(z). \]

Suppose we are able to define an action of complex powers of \( F_i \)'s (it should seem plausible now). Then if one chooses exponents \( \beta_j \) as prescribed by the singular vector formula (3), the vector \( F_{j_1}^{\beta_1} \cdots F_{j_l}^{\beta_l}v_{\lambda_1} \) is (at least formally) singular and one may hope that the expression

\[ F_{j_1}^{\beta_1} \cdots F_{j_l}^{\beta_l} \Psi(z) \]
also satisfies the Knizhnik-Zamolodchikov equations. We prove that it is true for the usual definition

\[ F_{j_1}^{\beta_1} \cdots F_{j_l}^{\beta_l} \Psi(z) = \prod_{l=1}^l \Gamma(-\beta_j)^{-1} \times \]

\[ \int \{ \exp(-t_1 F_{j_1}) \cdots \exp(-t_l F_{j_l}) \Psi \} \prod_{l=1}^l t_j^{-\beta_j-1} dt_1 \cdots dt_l, \quad (12) \]

where the cycle of integration is defined to be any element of the highest homology group with coefficients in the local system of single-valued branches of the integrand.

1.3 The Quantum Case

The Jimbo - Drinfeld quantum group \( U_q(\mathfrak{g}) \) is a Hopf algebra on generators \( E_i, F_i, K_i^{\pm 1}, 1 \leq i \leq n \) and relations, which represent a deformation of those for the Kac-Moody Lie algebra \( \mathfrak{g} \), \( q \) being a parameter of deformation. Considerable parts of the representation theory of \( U_q(\mathfrak{g}) \) and \( \mathfrak{g} \) are parallel to each other. This, in particular, enabled I. Frenkel and N. Reshetikhin ( see [12] ) to
give definitions of a $q$–vertex operator, correlation function and derive a system of $q$–difference equations ($q$KZ equations) satisfied by the latter.

The same can be said about the representation theory of highest weight modules. (See, for example, [7], [19].) Everything discussed above in the $g$–case admits a direct $q$–deformation. For example, the following simple relation (c.f. formula (4))

$$[E_{i_j}, F_{s_j}] = \delta_{i,j} \frac{q_j^s - q_j^{-s}}{q_j - q_j^{-1}} F_{s_j-1} K_j q_j^{s+1} - K_j^{-1} q_j^{-s-1}$$

together with a calculation analogous to that in (5) implies that the formula, literally coinciding with (3) gives, at least formally, a singular vector in a module over $U_q(g)$. (See [21] for a precise statement analogous to Theorem 1.1.) This makes it plausible that the formula of the type

$$F^{\beta_1}_{i_1} F^{\beta_2}_{i_2} \cdots F^{\beta_l}_{i_l} \Psi(z)$$

gives a solution of $q$KZ equations for a given solution $\Psi(z)$. We prove that it is true provided

$$F^{\beta_1}_{i_1} F^{\beta_2}_{i_2} \cdots F^{\beta_l}_{i_l} \Psi(z)$$
is defined by

$$F^{\beta_1}_{i_1} F^{\beta_2}_{i_2} \cdots F^{\beta_l}_{i_l} \Psi(z) = \prod_{j=1}^{l} \Gamma_q(-\beta_j)^{-1} \times$$

$$\int \{\exp_q(-t_{1} F_{i_1}) \cdots \exp_q(-t_{l} F_{i_l}) \Psi \} \prod_{j=1}^{l} t_{j}^{-\beta_j} d_t d_q t_1 \cdots d_q t_l, \quad (13)$$

Note that (13) is completely analogous to (12) with all the ingredients replaced with their $q$–analogs: Gamma function with $q$–Gamma function, exponential function with $q$–exponential function, and integral with $q$–integral (Jackson integral).

There is also a problem of convergence of the integral in (13). The classical counterpart of this problem is solved by considering the realization of $g$–modules in analytic functions on a flag manifold. We do the same for $U_q(\mathfrak{sl}_2)$ constructing a 2-parametric family of actions of $U_q(\mathfrak{sl}_2)$ by difference operators, which deforms the family of $\mathfrak{sl}_2$–modules $V(\mu, \lambda)$. This also leads to explicit formulas for solutions of $q$KZ equations for $U_q(\mathfrak{sl}_2)$.

**Remarks.** (i) Note that usually KZ equations are considered for finite-dimensional, or at least, highest weight $g$–modules $V_i$’s. This is not the case here. Therefore our integral representations (12, 13) are somewhat complementary to the usual ones (see, for example, the work of Varchenko and Schehtman [20] for the classical case and of Matsuo [22], Reshetikhin [23] for the quantum case). The relation between them is unclear but, hopefully can be established by a special choice of a contour in (12, 13) and sending “branching coefficients” of $V_i$’s to 0;
The appearance of modules realized in multi-valued functions is forced here by the singular vector formula \( (3) \). On the other hand, such modules have naturally emerged in the context of WZW model for \( \widehat{\mathfrak{sl}_2} \). They were first put forward by P.Furlan, A.Ch.Ganchev, R.Paunov and V.B.Petkova in \([11]\) in order to get the Virasoro minimal models correlators from the solutions for KZ equations via a sort of a quantum Drinfeld-Sokolov reduction. The paper \([11]\) also contains some new integral representations for correlation functions. The relation between these integral formulas and ours is not clear but as P.Furlan, A.Ch.Ganchev, R.Paunov and V.B.Petkova kindly informed us at least some of their formulas can be recovered using ours.

As is shown in \([1]\), “the singular vector decoupling condition” implies that in the fractional level case a non-zero vertex operator takes values in a module \( V(\mu, \lambda) \) for some non-integral \( \mu \);

In the main body of the paper we only discuss a \( q \)-deformation of our results on solutions to the classical KZ equations and do not mention a possibility of doing the same with regards to intertwining operators. However this possibility does exist at least in the case of \( U_q(\mathfrak{sl}_2) \). The above mentioned modules \( V(\mu, \lambda) \) admit a deformation \( V_q(\mu, \lambda) \) (see sect.\([4]\)). As a linear space \( V_q(\mu, \lambda) \) is independent of \( \lambda \) and, therefore, the operator \( E^\beta, \beta \in \mathbb{C} \), defined by \([13]\) can be regarded as an operator

\[
E^\beta : V_q(\mu, \lambda) \rightarrow V_q(\mu + \beta, \lambda - 2 \beta).
\]

Direct calculation shows that

\[
E^{\lambda+1} : V_q(\mu, \lambda) \rightarrow V_q(\mu + \lambda + 1, -\lambda - 2)
\]

is \( U_q(\mathfrak{sl}_2) \)-linear, which is, of course absolutely analogous to Lemma \([12]\). We believe that this can also be done in a more general context and hope to discuss it in the next paper.

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2 Construction of Intertwining Operators.

2.1 Flag Manifolds and Line Bundles

In this section we list main geometric notions related to flag manifolds and line bundles. All the material is pretty standard, except the notion of “a linear
bundle related to a complex divisor", which might be new ( and interesting in its own right ). In order not to get bogged down in technicalities we choose the following strategy: firstly, we discuss everything in a finite-dimensional setting and then describe the necessary modifications in the affine case.

1. Let $G$ be a complex simple finite-dimensional Lie group related to a complex simple finite-dimensional Lie algebra $\mathfrak{g}$. ( The class of such algebras is distinguished among all Kac-Moody algebras by the condition that the Cartan matrix $A$ is positive definite, see Introduction.) Denote by $N_-, N_+, T$ the subgroups of $G$ related to the following subalgebras of $\mathfrak{g}$: $n_-, n_+, h$ (resp.).

The flag manifold $F$ is said to be $F = G/B$. It is known that $F$ is a smooth projective variety. Using this we accept the following terminology: by regular function over a given open set we mean a meromorphic function with singularities lying outside this set; by analytic function we mean a (probably multi-valued) function splitting into a power series in a neighborhood of any non-singular point.

$F$ possesses the standard cellular decomposition into Schubert cells $\dot{X}_w$ numerated by elements $w$ of the Weyl group $W$. To describe it recall that an alternative way (to the one chosen in Introduction) to define $W$ is as follows:

$$W = \text{Norm}(T)/T,$$

where $\text{Norm}(T)$ stands for the normalizer of $T$. Therefore $W$ is naturally embedded in $F$ and one sets $\dot{X}_w = N_+w$.

It is known that $\dim \dot{X}_w = l(w)$, where $l(_)\) is the length function on $W$. The Schubert variety $X_w$ is said to be a completion of $\dot{X}_w$ in metric topology. One shows, completing the construction of the cellular decomposition, that

$$F = \bigcup_{w \in W} \dot{X}_w, \quad X_w = \bigcup_{v \leq w} \dot{X}_v,$$

(14)

where $\leq$ stands for the Bruhat ordering on $W$.

The decomposition (14) is in a sense attached to the point $B \in F$. For an arbitrary $x \in F$ set

$$X_w(x) = g(x)X_w,$$

where $g(x) \in G$ satisfies $g(x)B = x \in F$. Though the last property does not determine $g(x)$ uniquely, one verifies that $X_w(x)$ is independent of the choice of $g(x)$. It follows that the map $x \to X_w(x)$ is equivariant: $X_w(gx) = gX_w(x)$.

2. Consider the cartesian product $F \times F$. $G$ naturally acts on $F \times F$, $G$-orbits $\dot{Y}_w$ being numerated by elements of $W$, where $\dot{Y}_w$ is said to be all $(x, y) \in F \times F$ satisfying the following condition: there exists $g \in G$ ( depending on $(x, y)$) such that $(gx, gy) \in (B, \dot{X}_w)$. We denote by $Y_w$ a completion of $\dot{Y}_w$ in metric topology.

There are equivariant projections

$$pr_i : F \times F \to F, \quad i = 1, 2$$

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(on each factor) it follows from the definitions that its restriction to \( Y_w \subset F \times F \) is a fibration

\[ pr_i |_{Y_w} : Y_w \to F \]

with the fiber \( X_w(x) \) over \( x \in F \). The last assertion could have been chosen as a definition of \( X_w(x) \).

3. The Picard group and line bundles. A divisor on a complex variety \( X \) is said to be a complex subvariety of complex codimension 1. One may consider a free abelian group generated by all divisors on \( X \). For example, a meromorphic function \( f \) or, more generally, a meromorphic section of a line bundle determines an element of this group, denoted by \( (f) \) and called divisor of \( f \), where zeros of \( f \) enter \( (f) \) with + and poles - with -.

The Picard group \( \text{Pic}(X) \) is said to be the above free abelian group generated by all divisors on \( X \) modulo subgroup of divisors of meromorphic functions. It is evident that the map assigning to a linear bundle \( L \) over \( X \) a divisor of its (arbitrary) meromorphic section, regarded as an element of \( \text{Pic}(X) \), is a 1-1 correspondence.

In the case \( X = F \) we already have a collection of \( n+1 \) (rank of \( \mathfrak{g} \)) divisors: they are given by the codimension 1 Schubert varieties \( X_i = X_{w_0 r_i}, \ 0 \leq i \leq n \), where \( w_0 \in W \) is an element of the maximal length and \( r_i \) is a reflection at the \( i \)-th simple root. It is well-known that

\[ \text{Pic}(F) = \oplus_{i=0}^{n} \mathbb{Z} X_i. \]

4. Induced line bundles. \( F \) being a homogeneous space, affords another construction of a line bundle. We will be saying that a functional (weight) \( \lambda \in \mathfrak{h}^* \) is integral if \( \lambda(H_i) \in \mathbb{Z}, \ 0 \leq i \leq n \). Each weight \( \lambda \) determines a character of \( \mathfrak{h} \), for the last algebra is commutative and each integral weight can be lifted to a character of the group \( T \), which we also denote by \( \lambda \).

Consider a projection

\[ \pi : G/N_+ \to F = G/B. \]

\( \pi \) is a principal \( T \)-fibration: there is a natural right fiber-wise action of \( T \) on \( G/N_+ \), since \( T \) normalizes \( N_+ \). Denote by \( \Omega^\lambda_+(G/N_+) \) the sheaf of regular \( \lambda \)-homogeneous functions on \( G/N_+ \), i.e. functions satisfying \( f(t x) = \lambda(t) f(x), \ t \in T \).

**Lemma 2.1** For an integral \( \lambda \in \mathfrak{h}^* \) there exists a unique line bundle \( \mathcal{O}(\lambda) \) satisfying:

(i) \( \pi^* \mathcal{O}(\lambda) \) is trivial;

(ii) \( \pi^* \) establishes an isomorphism between the sheaf of regular sections of \( \mathcal{O}(\lambda) \) and \( \Omega^\lambda_+(G/N_+) \). In particular, the spaces of meromorphic sections of \( \mathcal{O}(\lambda) \) and meromorphic \( \lambda \)-homogeneous functions on \( G/N_+ \) are isomorphic;

(iii) the divisor of \( \mathcal{O}(\lambda) \) is equal to \( \sum_i \lambda(H_i) X_i \in \text{Pic}(F) \).
5. **The Borel-Weil theory.** The group $G$ (and, therefore, the algebra $\mathfrak{g}$) acts by left translations on $G/N$. This action preserves $\lambda-$homogeneous functions and therefore, determines a structure of $\mathfrak{g}-$module on meromorphic sections of $\mathcal{O}(\lambda)$.

An integral weight $\lambda$ is called dominant if $\lambda(H_i) \geq 0$, $0 \leq i \leq n$.

**Theorem 2.2 (Borel - Weil).** Let $\lambda$ be an integral weight. Then

(i) $\Gamma(X_{w_0}, \mathcal{O}(\lambda)) \approx M(\lambda)$, where $M(\lambda)$ stands for a contragredient Verma module (see Introduction);

(ii) $\Gamma(F, \mathcal{O}(\lambda)) = 0$ unless $\lambda$ is dominant. If the latter condition is satisfied then $\Gamma(F, \mathcal{O}(\lambda))$ is isomorphic to the irreducible $\mathfrak{g}-$module with highest weight $\lambda$.

6. **Non-integral case.** The item (i) of the Borel-Weil theorem can be understood as a realization of a contragredient Verma module in functions on the “big” Schubert cell. Really, having a meromorphic section of $\mathcal{O}(\lambda)$ fixed and identified with a constant function, one identifies all meromorphic sections of $\mathcal{O}(\lambda)$ with functions on the big cell, elements of $\mathfrak{g}$ being realized by certain order 1 differential operators. (See, for example, explicit formulas for the $\mathfrak{sl}_2-$ case in [9]). Then one observes that the formulas depend on $\lambda$ in a polynomial way and, therefore, make sense for an arbitrary non-integral $\lambda$. This completes the realization of $M(\lambda)$ in functions on the big cell. However in order to construct integral operators we would like to keep track of geometric nature of sections $\mathcal{O}(\lambda)$ for an arbitrary $\lambda$ (see the $\mathfrak{sl}_2-$ case in Introduction).

Observe that the fibration $\pi: G/N \to F = G/B$ is trivial over the big cell $X_{w_0}$ and we fix a trivialization $\phi$. Though $\phi$ is not determined uniquely, one may say that it is determined “almost uniquely”: $\phi$ produces an isomorphism $\phi(X_{w_0}) \approx X_{w_0} \times T$ and, therefore, any other trivialization is different from $\phi$ by an element of $T$.

**Definition.** In view of Lemma 2.1 it is natural to say that for an arbitrary weight $\lambda$ $\mathcal{O}(\lambda, \phi)$ is a sheaf over $X_{w_0}$ defined as follows: the space of its sections $\Gamma(U, \mathcal{O}(\lambda, \phi))$ over an open set $U \subset X_{w_0}$ is said to be the space of functions on an open neighborhood of $\phi(U) \subset G/N$, $\lambda-$ homogeneous and regular on $\phi(U)$.

The above discussion implies that for a pair of trivializations $\phi, \psi$ the corresponding sheaves are canonically isomorphic: $\mathcal{O}(\lambda, \phi) \approx \mathcal{O}(\lambda, \psi)$.

Observe also that if $\lambda$ is integral then each $f \in \Gamma(U, \mathcal{O}(\lambda))$ uniquely determines a meromorphic section of $\mathcal{O}(\lambda)$ understood as a line bundle over $F$ with the divisor $\sum \lambda(H_i)$. In this sense the new definition contains the old one as a particular case.

**Convention.** Using these 2 remarks we will run the risk of reducing the notation $\mathcal{O}(\lambda, \phi)$ to $\mathcal{O}(\lambda)$.

For $f \in \Gamma(U, \mathcal{O}(\lambda))$, $g \in \Gamma(U, \mathcal{O}(\mu))$, $t \in \mathbb{C}$ multiplication and exponentiation gives rise to the following operations:

$$f, g \rightarrow fg \in \Gamma(U, \mathcal{O}(\lambda + \mu)),$$

(16)
\[ f \mapsto f^t \in \Gamma(U, \mathcal{L} \otimes \mathcal{O}(t\lambda)) \text{ if } U \text{ is simply connected.} \quad (17) \]

Another example of dealing with \( \mathcal{O}(\lambda) \) for an arbitrary \( \lambda \) is obtained by considering the \( SL_2 \) case. First of all, for \( SL_2 \) one has: \( N^+ = \mathbb{C}^* \), \( SL_2/N^+ \approx \mathbb{C}^2 - \{ (0,0) \} \) (see [5]), \( F = \mathbb{C}P^1 \), the big cell is \( \mathbb{C} \) and \( \pi \) over \( \mathbb{C} \) is simply the projection \( \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \). A weight is a complex number: \( \lambda \in \mathbb{C} \) and, therefore sections of \( \mathcal{O}(\lambda) \) are naturally determined by expressions \( f(x) \, dx^{-\lambda/2} \), where \( f(x) \) is a meromorphic function. Looking at Introduction, formula (10) one realizes that the integral operator for \( sl_2 \) (see (11)) was obtained by making use of the operations (16,17).

Further for an arbitrary \( G \) and a fixed simple root \( \alpha_i \) of \( G \) one may consider an inclusion \( SL_2 \subset G \), on the level of Lie algebras determined by the \( sl_2 \) --triple: \( \langle E_i, H_i, F_i \rangle \). This gives rise to the inclusion of a flag manifold \( F_i \subset F \). One realizes that the latter identifies \( F_i \) with the Schubert variety \( X_{r_i} \). The following assertion (of course well-known for an integral \( \lambda \in h^* \)) now becomes a tautology:

\[ \mathcal{O}(\lambda) \mid_{X_{r_i}} = \mathcal{O}(\lambda(H_i)). \quad (18) \]

**Remark.** The necessity to understand geometric meaning of a “complex power of a line bundle” appeared several years ago in connection with the developments in the string theory. The matter was treated in [4], [28] in the framework of complex curves rather than homogeneous spaces as it is here.

7. **The affine Lie group case.** Everything we have encountered with so far directly generalizes to the case of an affine Lie group. Making a reference to the book [24] (an exposition of this theory in even more general setting can be found in [18]), we here restrict ourselves to a few comments.

An affine Lie group modulo center is just a loop group \( L(G) \), where an element of the latter is understood as a smooth map \( \gamma : S^1 \rightarrow G \). The corresponding Lie algebra - an affine Lie algebra - is included into the class of Kac-Moody algebras (see sect. 3.1). A Borel subgroup \( B \) and a “maximal nilpotent subgroup” \( \hat{N}^\pm \) are said to consist of boundary values of analytic inside the unit disc maps \( \gamma : \{ z : |z| \leq 1 \} \rightarrow G \), satisfying \( \gamma(0) \in B \) (\( \gamma(0) \in N^\pm \) resp.). The flag manifold is said to be the quotient \( L(G)/B \). It also possesses a cellular decomposition but now one has to distinguish between cells of finite dimension and finite codimension. The former are orbits of elements of an affine Weyl group under \( \hat{N}^+ \), while the latter are also with respect to \( \hat{N}^- \). (This could have been also said with regards to the finite-dimensional case, producing equivalent description of the cells.)

It is crucial that Borel-Weil theory (Theorem 2.2) works in the affine case as well (see the above-mentioned book [24] and [18, 10]). One also realizes that the correspondence among codimension 1 Schubert varieties, line bundles and induced line bundles, our discussion of the fibration \( \pi : G/N^+ \rightarrow G/B \)
together with the subsequent definition of the sheaves $\mathcal{O}(\lambda)$ is independent of the passage to the affine case.

2.2 Construction of an Intertwining Operator related to a simple root.

In this section we use the following unified notations: $\mathfrak{g}, \mathfrak{b}, ..., G, B, F, ...$ stand for either simple finite-dimensional or affine Lie algebra, its Borel subalgebra, ..., the corresponding Lie group, its Borel subgroup, the corresponding flag manifold, ... (resp.). $\mathcal{X}_i = N^{-r_i}, 0 \leq i \leq n$ stands for the Schubert cell of codimension 1. $X_w, X_w, Y_w, Y_w, w \in W$ usually stand for a Schubert cell (variety, $G$-orbit in $F \times F$ resp.) of a finite dimension $l(w)$, except $X_w$, which stands for the big Schubert cell. The last convention is consistent with the previous ones in the finite-dimensional case, while it is not quite so in the affine case, for in the latter case there is no element of the maximal length in the Weyl group.

As is known, for example from the representation theory of complex simple Lie groups, the scheme of constructing intertwiners as integral operators could be as follows. Consider the direct product $F \times F$ and 2 projections $pr_1, pr_2$ on the 1st and the 2nd factors respectively. Take a section $f \in \mathcal{O}(\lambda)$ pull it back on $F \times F$ to get $pr_1^* f \in pr_1^* \mathcal{O}(\lambda)$, “pair” it with “something” defined on $F \times F$ (or on a $G$-invariant subset of $F \times F$), “integrate the result over the fibres of $pr_2$” and so get a section of a bundle defined over $F$. We realize this scheme as follows.

**A. Integral case.** Let $\lambda \in \mathfrak{h}^*$ be integral. Let $D \subset F \times F$ be the diagonal and for a pair of line bundles $\mathcal{H}_1, \mathcal{H}_2$ over $F$ denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$ their exterior product as a line bundle over $F \times F$. One observes that the line bundle $(\mathcal{O}(\alpha_i) \otimes \mathcal{O}(\alpha_i)) |_{Y_{r_i}}$ is related to the divisor $2D$ ($D$ is understood here as a divisor inside $Y_{r_i}$) and we pick up a section $s_i$ of $(\mathcal{O}(\alpha_i) \otimes \mathcal{O}(\alpha_i)) |_{Y_{r_i}}$ so that $(s_i) = 2D$. It is important that $s_i$ is $G$-invariant (since $D$ is also).

Another observation is that if one sets

$$\mathcal{A}_i(\lambda) = ((\mathcal{O}(\lambda) \otimes (\mathcal{O}(\alpha_i)^{-\frac{1}{2} (\lambda + 2\rho)(H_i)}) \otimes (\mathcal{O}(\alpha_i)^{-\frac{1}{2} (\lambda + 2\rho)(H_i)})) |_{Y_{r_i}},$$

then for any $f \in \Gamma(X_{w_0}, \mathcal{O}(\lambda))$, $pr_1^* f \otimes s_i^{-\frac{1}{2} (\lambda + 2\rho)(h_i)}$ is a (meromorphic) section of $\mathcal{A}_i(\lambda)$.

**Lemma 2.3** Restriction of $\mathcal{A}_i$ to $pr_2^{-1}(x) \cap Y_{r_i}$ ($\approx \mathbb{CP}^1$) is a sheaf of differential forms.

**Proof.** The following property of a sheaf $\mathcal{O}(\mu)$ on a flag manifold (see [10])

$$\deg(\mathcal{O}(\mu)|_{X_{r_i}(x)}) = \mu(H_i)$$

implies that for any (meromorphic) section $f$ of $\mathcal{O}(\mu)$ and the above-defined $s_i$ and for an arbitrary coordinate $t$ on $pr_2^{-1}(x) \cap Y_{r_i}$ ($\approx X_{r_i}(x)$) one has

$$pr_1^* f |_{X_{r_i}(x)} = p(t)dt^{-\lambda(H_i)/2},$$

14
\[
\text{s}_i \mid_{X_{r_i}(x)} = C(t-x)^2 dt^{-1},
\]

where \( p(t) \) is a rational function of \( t \) and \( C \) is independent of \( t \). It follows that the restriction of \( p r_1^* f \otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \) to \( X_{r_i}(x) \) is a differential form. □

Making use of this lemma we denote by

\[
\int_{pr_2^{-1}(y)} pr_1^* f \otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)}
\]

the residue of \( pr_1^* f \otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \mid_{pr_2^{-1}(y)} \) at the point \((y,y)\) for any \( f \in \Gamma(\hat{X}_{w_0}, \mathcal{O}(\lambda)) \) and \( y \in F \).

It follows from the definition that \( \int_{pr_2^{-1}(y)} pr_1^* f \otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \) is a section of the bundle \( \mathcal{A}_i \otimes (\mathcal{O}(\alpha_i) \otimes \mathbb{C}) \mid_{Y_{r_i}} \). The last bundle can be restricted to the diagonal and then pushed forward on \( F \) via the isomorphism \( pr_2 \mid_D : D \approx F \).

The result is the bundle \( \mathcal{O}(\lambda) \otimes \mathcal{O}(\alpha_i)^{-\frac{1}{2}(\lambda+\rho)(h_i)} = \mathcal{O}(r_i \cdot \lambda) \). Denote by \( T_i(f) \) the image of \( \int_{pr_2^{-1}(y)} pr_1^* f \otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \) under this composition map. Thus we have constructed a \( \mathbb{C} \)-linear map

\[
T_i : \Gamma(\hat{X}_{w_0}, \mathcal{O}(\lambda)) \to \Gamma(\hat{X}_{w_0}, \mathcal{O}(r_i \cdot \lambda)).
\]

**Theorem 2.4** \( T_i \) is a morphism of \( \mathfrak{g} \)-modules.

**Proof** immediately follows from the construction. \( T_i \) was defined to be a composition of the following operations on sections of \( \mathfrak{g} \)-bundles:

(i) pulling back from \( F \) to \( F \times F \) via \( pr_1 \),

(ii) tensoring with a \( \mathfrak{g} \)-invariant section \( s_i^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \) of the line bundle \( (\mathcal{O}(\alpha_i)^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \otimes (\mathcal{O}(\alpha_i)^{-\frac{1}{2}(\lambda+2\rho)(h_i)} \mid_{Y_{r_i}} \),

(iii) pushing down on \( F \) via \( pr_2 \).

Each of these operations is obviously \( \mathfrak{g} \)-linear. □

**Remarks.** 1. One easily realizes that the constructed morphism \( T_i \) of contragredient Verma modules is dual to the embedding of Verma modules determined by the singular vector \( F_i^{\lambda(H_i)+1}v_\lambda \) (see (3)). On the other hand, the last morphism may not exist, though our \( T_i \) is still defined. In this case \( \lambda(H_i) + 1 < 0 \), the integrand in the definition of \( T_i \) is regular (see also (19)) and, therefore, \( T_i = 0 \) as one should have expected.

2. One may want to define an intertwiner

\[
T_w : \Gamma(\hat{X}_{w_0}, \mathcal{O}(\lambda)) \to \Gamma(\hat{X}_{w_0}, \mathcal{O}(w \cdot \lambda))
\]

related to an arbitrary element of the Weyl group \( w = \cdots r_{i_2} r_{i_1} \) as a composition of the constructed \( T_i \)’s: \( T_w = \cdots T_{i_2} T_{i_1} \). However this does not work since by the above remark one of the factors is usually 0. To obtain the desired formula
one has to “step aside” and define $T_i$’s for a suitable module related to a generic weight $\lambda$.

**B. Nonintegral case.** In order to construct an analogue of $T_i$ for an arbitrary “highest weight” we first define a suitable

1. **Family of $g$—modules.** Fix some group element $g \in G$ and denote by $X_0^{opp}, \ldots, X_n^{opp}$ the “opposite” Schubert cells defined by: $X_i^{opp} = gX_i$. If $\dim g < \infty$, then the natural choice is $g = w_0$, where $w_0 \in W$ is the element of the maximal length, which explains the terminology.

The space $F_0 = F - \bigcup_{0 \leq j \leq n} X_j - \bigcup_{0 \leq j \leq n} X_j^{opp}$ will serve as an analogue of $C^*$ in the $sl_2$—case.

For any local system $\mathcal{L}$ over $F_0$ and any $\lambda \in h^*$ we set $\mathcal{O}(\mathcal{L}, \lambda) = \mathcal{L} \otimes (\mathcal{O}(\lambda) |_{F_0})$. The algebra $g$ naturally acts on sections of $\mathcal{O}(\mathcal{L}, \lambda)$ since it acts on sections of both $\mathcal{L}$ and $\mathcal{O}(\lambda)$. To each local system $\mathcal{L}$ we associate a vector function (collection of exponents)

$$\mu = (\mu_0, \ldots, \mu_n) : F_0 \rightarrow C^{n+1}$$

as follows. One proves that the intersection number of $X_{r_i}(x)$ with $X_j$ as well as with $X_j^{opp}$ is equal to 1 if $i = j$ and - to 0 otherwise for all $x \in F_0$ (this can be derived from the Borel-Weil theorem \[22\] and \[23\]). To each $x \in F_0$ and a number $i$ we associate a projective line $X_{r_i}(x)$ with 2 marked points: $t_0(x) = X_{r_i}(x) \cap X_i$, $t_\infty(x) = X_{r_i}(x) \cap X_i^{opp}$. The projective line punctured at 2 points is homotopically equivalent to a circle. Therefore, the restriction of $\mathcal{L}$ to $X_{r_i}(x)$ determines (and is determined by) a complex number $\mu_i(x)$ - its monodromy coefficient.

**Example 2.5** It follows from the definition that the divisors $(X_j, X_j^{opp})$, $0 \leq j \leq n$ are equivalent (they determine the same line bundle, namely $\mathcal{O}(\lambda^j)$) where $\lambda^j$ is the $j$—th fundamental weight). We fix functions $\phi_0, \ldots, \phi_n$ on $F$ so that $(\phi_j) = (X_j) - (X_j^{opp})$. For a complex vector $\mu = (\mu_0, \ldots, \mu_n)$ define a multivalued function $\phi = \prod_{1 \leq j \leq n} \phi_i^{\mu_j}$. Singlevalued branches of this function determine a local system $\mathcal{L}_\mu$ on $F_0$, the vector function $\mu = (\mu_0, \ldots, \mu_n)$ being constant in this case. The bundle $\mathcal{O}(\mathcal{L}_\mu, \lambda)$ is the simplest example of bundles we will be dealing with.

We will construct an intertwining operator $T_i$ taking a section of $\mathcal{O}(\mathcal{L}, \lambda)$ to a section of $\mathcal{O}(\mathcal{M}, r_i \cdot \lambda)$ for arbitrary $\mathcal{L}$ and $\lambda$, $\mathcal{M}$ being uniquely determined by $\mathcal{L}$ and $\lambda$. Moreover it will be given by essentially the same formula as in the integral case. Set

$$\mathcal{A}_i = ((\mathcal{O}(\mathcal{L}, \lambda) \otimes \mathcal{C}) \otimes (\mathcal{O}(\alpha_i) - \frac{1}{2}(\lambda+2\rho)(H_i)) \otimes (\mathcal{O}(\alpha_i) - \frac{1}{2}(\lambda+2\rho)(H_i))) |_{Y_{r_i}}.$$

(See the previous section, especially \[16\] and \[17\], for the discussion of complex powers of line bundles on flag manifolds.)
As before, the choice of the exponent $-\frac{1}{2}(\lambda + 2\rho)(H_i)$ implies that the restriction of $A_1$ to any fiber of $pr_2 \approx \mathbb{C}P^1$ is “tensorially” a sheaf differential forms, more precisely this is a sheaf of differential forms on $\mathbb{C}P^1$ with coefficients in a certain local system. We now construct this local system.

2. Each projective line $pr^{-1}_2(x) \cong X_{r_1}(x)$ is equipped with three marked points $t_0(x), t_{\infty}(x), x$. Projective line punctured at 3 points is homotopically equivalent to the union of 2 circles. The 1st homotopy group $\pi_1(X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\})$ is a free group on 2 generators $\sigma_1(x), \sigma_2(x)$ which can be thought of as a pair of circles enclosing the points $t_0(x)$ and $x$ respectively. Denote by $L_i(\mathcal{L}, \lambda; x)$ the local system on $X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\}$ determined by the following 1-dimensional representation of $\pi_1(X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\})$:

$$\sigma_1(x) \mapsto \exp(-2\pi \sqrt{-1}\mu_i), \quad \sigma_2(x) \mapsto \exp(2\pi \sqrt{-1}\lambda_i(H_i)).$$

(Recall that $\{\mu_i\}$ is the set of exponents.)

For generic $\mu, \lambda$ the 1st homology group $H_1(X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\})$, $L_i(\mathcal{L}, \lambda; x)$ is 1-dimensional and is generated by the cycle $\sigma_1(x)\sigma_2(x)\sigma_1^{-1}(x)\sigma_2^{-1}(x)$. Now the relation of $A_i$ to the constructed local system is as follows. We again pick out a $G$-invariant section $s_i$ of the bundle $(\mathcal{O}(\alpha_i) \otimes (\mathcal{O}(\alpha_i)))|_{\gamma_{r_1}}$ such that $(s) = 2D$.

Lemma 2.6 For any $f \in \Gamma(\mathcal{O}(\mathcal{L}, \lambda))$ the restriction of $pr^*_1 f \otimes s_i^{-\frac{1}{2}(\lambda + 2\rho)(H_i)}$ to a fiber $pr^{-1}_2(x)$ represents a cohomology class of $X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\}$ with coefficients in $L_i(\mathcal{L}, \lambda; x)$.

Proof. As in Lemma 2.3 the restriction of $pr^*_1 f \otimes s_i^{-\frac{1}{2}(\lambda + 2\rho)(H_i)}$ to $X_{r_1}(x)$ for a fixed $x$ is given by the formula

$$f(t) \over (t-x)^{H_i + 2\rho} dt, \quad f(t) \in t^\mu C[t, t^{-}],$$

where with a certain abuse of notation $x$ stands for the value of $t$ at the point $x$ (c.f. formula 6)). $\Box$

We are almost ready to define an intertwining operator by the formula literally coinciding with that of Theorem 2.4 but in order to understand what its image is we have to introduce one more local system on $F^0$.

3. The assignment $F^0 \ni x \mapsto X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\}$ naturally determines a fibration over $F^0$ with the fiber: projective line punctured at 3 points. This fibration gives rise to a line bundle with a fiber over $x \in F^0$ equal to $H_1(X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\})$, $L_i(\mathcal{L}, \lambda; x)$. Denote this line bundle by $L_i(\mathcal{L}, \lambda)$. A trivialization of the fibration with the fiber projective line punctured at 3 points over a disk induces a trivialization of $L_i(\mathcal{L}, \lambda)$ over the same disk and, therefore an identification of $H_1(X_{r_1}(x) - \{t_0(x), t_{\infty}(x), x\})$, $L_i(\mathcal{L}, \lambda; x)$ for all $x$ from this disk. Obviously the last identification is independent of a trivialization and gives rise to a flat connection called the Gauss-Manin connection.
There arises a local system of horizontal sections of $\mathcal{L}_i(\mathcal{L}, \lambda)$ with respect to this connection.

4. Formula for an intertwiner. We, firstly, have to fix an arbitrary simply connected open subset $\tilde{F}^0 \subset F^0$, for non-trivial sections of our sheaves may not exist over non-simply connected open sets. Now define a linear map

$$\tilde{T}_i : \Gamma(\tilde{F}^0, \mathcal{L}_i(\mathcal{L}, \lambda) \otimes \mathcal{O}(\mathcal{L}, \lambda)) \to \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, r_i \cdot \lambda))$$

to be a composition of the following operations:

(i) $pr_1^* \otimes pr_1^*$: it takes a section $\sigma \otimes f$ of $\mathcal{L}_i(\mu, \lambda) \otimes \mathcal{O}(\mu, \lambda)$ into a section $pr_1^* \sigma \otimes pr_1^* f$ and of the corresponding sheaf over $Y_{r_i}$;

(ii) $\otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(H_i)}$: it tensors the result of the previous operation by $s_i^{-\frac{1}{2}(\lambda+2\rho)(H_i)}$;

(iii) $\int_{pr_2^{-1}}$: by Lemma 2.6 there is a natural pairing of sections of $pr_i^*(\mathcal{L}_i(\mathcal{L}, \lambda))$ with those of $\mathcal{A}_i$ which for a fixed point $(x, y) \in F^0 \times F^0$ is nothing but the value of the cohomology class determined by the restriction of the section of $\mathcal{A}_i$ to the fiber $pr_2^{-1}(y) \cap Y_{r_i}$ on the homology class determined by the value of the section of $pr_i^*(\mathcal{L}_i(\mathcal{L}, \lambda))$ at the point $(x, y)$; the result is a section of a certain bundle which can be easily calculated (see below);

(iv) $pr_2^*$: this is a composition of the restricting to the diagonal $D \in F \times F$ and the consecutive pushing forward on $\tilde{F}^0$ by means of the isomorphism $pr_2 : D \to F$.

We set

$$\tilde{T}_i = pr_2^* \circ (\int_{pr_2^{-1}}) \circ (\otimes s_i^{-\frac{1}{2}(\lambda+2\rho)(H_i)}) \circ (pr_1^* \otimes pr_1^*).$$

It follows from the definition (and can be seen exactly as in the integral case) that the constructed map acts as

$$\tilde{T}_i : \Gamma(\tilde{F}^0, \mathcal{L}_i(\mathcal{L}, \lambda) \otimes \mathcal{O}(\mathcal{L}, \lambda)) \to \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, r_i \cdot \lambda)).$$

Tensoring both sides of the last equality by $\mathcal{L}_i^*(\mathcal{L}, \lambda)$ we finally obtain the desired map

$$T_i : \Gamma(\tilde{F}^0, \mathcal{O}(\mu, \lambda)) \to \Gamma(\tilde{F}^0, \mathcal{L}_i^*(\mu, \lambda) \otimes \mathcal{O}(r_i \cdot \lambda)) = \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}_i^*(\mu, \lambda) \otimes \mathcal{L}_i^*(\mu, \lambda)).$$

**Theorem 2.7** $T_i$ is a morphism of $\mathfrak{g}$–modules.

**Proof** is a literal repetition of that of Theorem 2.4: the map $T_i$ was defined to be a composition of maps each of them being transparently $\mathfrak{g}$–invariant $\Box$.

**Remark.** It is easy (and might be instructive) to examine the $sl_2$– case, considered in Introduction from this general point of view. One has:
\[ V(\mu, \lambda) \approx \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}_\mu, \lambda)) \text{ for an arbitrary simply connected open } \tilde{F}^0 \subset \mathbb{C}^*; \]

\[ s_i = \frac{dx dt}{(t - x)^2}; \]

\[ T_i \text{ from Theorem 2.7 coincides with } T \text{ from Lemma 1.2.} \]

2.3 Construction of Intertwining Operators related to an arbitrary element of the Weyl group.

1. The simplest way to construct an intertwiner \( T_w \) related to an element \( w \in W \) is to consider a reduced decomposition \( w = r_{i_1} \cdots r_{i_l} \) and then set

\[ T_w = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_l}. \]  

(20)

Theorem 2.7 immediately gives

**Lemma 2.8** Operator \( T_w \) as defined in (20) is a \( \mathfrak{g} \)-linear map

\[ T_w : \Gamma^0(\tilde{F}^0, \mathcal{O}(\mathcal{L}, \lambda)) \rightarrow \Gamma^0(\tilde{F}^0, \mathcal{O}(\mathcal{M}, w^{-1} \cdot \lambda)), \]

where \( \mathcal{M} \) is a local system defined as follows: if one sets \( \mathcal{M}_0 = \mathcal{L}, \mathcal{M}_j = \mathcal{L}_{k-j+1}^*(\mathcal{M}_{j-1}, r_{ik-j+1} \cdots r_{ik} \cdot \lambda) \otimes \mathcal{M}_{j-1}, \) then \( \mathcal{M} = \mathcal{M}_k. \)

The construction implies (also see below) that each \( T_i \) basically is an integral of a “1-form” over a “1-cycle” and, therefore, \( T_w \) from (20) is an integral of a “\( k \)-form” over a “\( k \)-cycle”. We are now going to analyze this situation and produce this “\( k \)-form” and the homology group this “\( k \)-cycle” comes from.

2. Consider a \( \mathbb{CP}^1 \)-fibration (a fibration with the fiber \( \mathbb{CP}^1 \))

\[ \pi : A_1 \rightarrow A_0. \]

For any line bundle over \( A_1 \) its degree is said to be the degree of its restriction to (any) fiber of \( \pi \). Let \( \mathcal{M}_1 \) be some line bundle over \( A_1 \) and denote by \( \mathcal{O}(2) \) the line bundle of vector fields over \( A_1 \) tangent to fibers of \( \pi \). Clearly the degree of \( \mathcal{O}(2)_1 \) is equal to 2. Denote the degree of \( \mathcal{M}_1 \) by \( d_1 \). Then the restriction of the bundle \( \mathcal{M}_1 \otimes \mathcal{O}(2)_1^{-d_1/2} \) to the fibers of \( \pi \) is trivial and as is well-known there is a bundle, say \( \mathcal{M}_0 \), such that

\[ \mathcal{M}_1 \otimes \mathcal{O}(2)_1^{-d_1/2} \approx \pi^* \mathcal{M}_0, \]

or

\[ \mathcal{M}_1 \otimes \mathcal{O}(2)^{-d_1/2-1} \approx \pi^* \mathcal{M}_0 \otimes \mathcal{O}(2)_1^{-1}. \]

In other words, there arises a morphism of sections

\[ \phi_1 : \Gamma(\pi^{-1}U, \mathcal{M}_1 \otimes \mathcal{O}(2)_1^{-d_1/2-1}) \rightarrow \Gamma(U, \mathcal{M}_0 \otimes \mathcal{V}_1), \]
where $\mathcal{V}_1$ is a sheaf over $A_0$ is defined to associate to an open set $U \subset A_0$ all holomorphic sections of $\mathcal{O}(2)^{-1}_{1}$ over $\pi^{-1}(U)$ or, equivalently, is a sheaf of volume forms over the fibers of the fibration $\pi$. In particular, for a fixed section $s_1$ of $\mathcal{O}(2)^{-d_1/2-1}_1$ there arises a map

$$\phi(s_1) : \Gamma(\pi^{-1}U, \mathcal{M}_1) \rightarrow \Gamma(U, \mathcal{M}_0 \otimes \mathcal{V}_1). \quad (21)$$

Now consider a sequence of $\mathbb{CP}^1$-fibrations

$$A_k \xrightarrow{\pi_k} A_{k-1} \xrightarrow{\pi_{k-1}^{-1}} \cdots \xrightarrow{\pi_2^{-1}} A_1 \xrightarrow{\pi_1^{-1}} A_0$$

together with the following data:
- $\mathcal{O}(2)_j$ is a line bundle of vector fields tangent to fibers of $\pi_j$;
- $\mathcal{M}_j$ is a line bundle over $A_j$ of degree $d_j$ such that
  $$\mathcal{M}_j \otimes \mathcal{O}(2)^{-d_j/2-1}_j \approx \pi^*\mathcal{M}_{j-1}, \ 1 \leq j \leq k.$$

(The entire sequence $\{\mathcal{M}_j, 1 \leq j \leq k\}$ is uniquely determined by a choice of the bundle $\mathcal{M}_k$.)

Fix sections $s_j$ of the bundles $\mathcal{O}(2)^{-d_j/2-1}_j$. Then the repeated use of the previous construction gives the map

$$\phi(s_1) \circ \cdots \circ \phi(s_k) : \Gamma(\pi^{-1}U, \mathcal{M}_k) \rightarrow \Gamma(U, \mathcal{M}_0 \otimes \mathcal{V}_k), \quad (22)$$

where $\mathcal{V}_k$ is a sheaf over $A_0$ of (fiber-wise) complex-analytic volume forms with respect to the composition fibration

$$\pi(s_1) \circ \cdots \circ \pi(s_k) : A_k \rightarrow A_0.$$

3. The above considerations are related to the intertwiners $T_r$, as follows. First of all, the fibration

$$Y_i \rightarrow F$$

can be regarded as a $\mathbb{CP}^1$-fibration with the fiber over $x \in F$ equal to $X_i(x)$. Then one easily realizes that in the flag manifold case one can easily deal with complex powers of line bundles. In particular, if $\mathcal{O}(\mathcal{L}, \lambda)_1$ is the lift of $\mathcal{O}(\mathcal{L}, \lambda)$ to $Y_i$ then (22) is valid in the form

$$\phi(s_1(\lambda)) : \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, \lambda)_1) \rightarrow \Gamma(\tilde{F}^0, \mathcal{O}(r_i \cdot \lambda)_1 \otimes \mathcal{V}_1), \quad (23)$$

where we denote by $s_i(\lambda)$ the section $s_i^{-\frac{1}{2}(\lambda+2\rho)(H_i)}$ appearing, for example, in Lemma 2.6 and

$\mathcal{V}_1$ is understood to be a sheaf with the fiber over $x \in F^0$: differential forms over $X_i(x)$ with coefficients in $\mathcal{L}_i(\mathcal{L}, \lambda; x)$. Tensoring with $\mathcal{L}_i(\mathcal{L}, \lambda)$ and consecutive integration gives the operator $T_i$. 20
4. We are now going to apply (22) to construct intertwiners related to an arbitrary element of the Weyl group. The possibility of doing this is based on the following Bott-Samelson-Demazure construction (see [8] and also [18] for the Demazure theory in an infinite-dimensional setting).

Obviously $Y_i$ is the set of pairs $\{(x_0, x_1) : x_0 \in F, x_1 \in X_i(x_0)\}$. For a reduced decomposition $w = r_{i_k} \cdots r_{i_1}$ set

$$\tilde{Y}_w = \{(x_0, x_1, \ldots, x_k) : x_0 \in F, x_j \in X_{i_{j-1}}(x_{j-1}), 1 \leq j \leq k\}.$$

Set $w_j = r_{i_j} \cdots r_{i_1}$. There arises the sequence of $\mathbb{C}P^1$-fibrations

$$\sigma_j : \tilde{Y}_{w_j} \to \tilde{Y}_{w_{j-1}},$$

$$\sigma_j : (x_0, \ldots, x_{j-1}, x_j) \mapsto (x_0, \ldots, x_{j-1}).$$

**Theorem 2.9** (see [8], [18]) There exist $G$-invariant rational equivalencies

$$\phi_j : Y_{w_j} \to \tilde{Y}_{w_j}$$

commuting with inclusions $Y_{w_{j-1}} \subset Y_{w_j}$ and with singularities concentrated on $Y_v, v \leq w_{j-1}$.

Denote by $(\mathcal{O}(\alpha_i) \circ \mathcal{O}(\alpha_i))_j$ the lift of $\mathcal{O}(\alpha_i) \circ \mathcal{O}(\alpha_i)$ on $\tilde{Y}_{w_j}$ via the composition map

$$\tilde{Y}_{w_j} \xrightarrow{\sigma_j} \tilde{Y}_{w_{j-1}} \xrightarrow{\sigma_{j-1}} \cdots \xrightarrow{\sigma_1} \tilde{Y}_{w_1} \xrightarrow{\phi_1^{-1}} Y_{i_1}$$

and by $\mathcal{O}(\mathcal{L}, \lambda)_j$ the lift of $\mathcal{O}(\mathcal{L}, \lambda)$ via the composition map

$$\tilde{Y}_{w_j} \xrightarrow{\sigma_j} \tilde{Y}_{w_{j-1}} \xrightarrow{\sigma_{j-1}} \cdots \xrightarrow{\sigma_1} \tilde{Y}_{w_1} \xrightarrow{\phi_1^{-1}} Y_{i_1} \to F.$$

Let $s_{i_j}^{(j)}(\lambda)$ be the lift of the section $s_1(\lambda)$ appearing in (23) on $\tilde{Y}_{w_j}$. The same reasoning as in the “integral” case and the use of (23) shows that there arises a map $\phi(s_{i_j}^{(j)})$ which takes a section of $\mathcal{O}(\mathcal{L}, \lambda)_j$ into a section of $\mathcal{O}(\mathcal{L}, \lambda)_{j-1}$ with coefficients in the sheaf with the fiber over $(x_0, \ldots, x_{j-1})$: differential forms over $\tilde{X}_{i_j}(x_{j-1})$ with coefficients in $L_{i_j}(\mathcal{L}, \lambda; x_{j-1})$. This map is $g$-invariant since it is constructed via the $g$-invariant section $s_{i_j}^{(j)}(\lambda)$.

Now consider the composition

$$\phi(s_{i_1}) \circ \cdots \circ \phi(s_{i_1}).$$

It acts as

$$\phi(s_{i_1}) \circ \cdots \circ \phi(s_{i_1}) : \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, \lambda)) \to \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, r_{i_1} \cdots r_{i_1} \cdot \lambda) \otimes Vol) \quad (24)$$

for a certain sheaf $Vol$. The latter sheaf can be described as follows.
Consider the fibration

$$\sigma_1 \circ \cdots \circ \sigma_k : \tilde{Y}_w \to F.$$ 

Its fiber over \( x \in F \) is \( \tilde{X}_w(x) \). Now recall that each successive application of \( \phi(s_i)^{(j)} \) results in a section of \( \mathcal{O}(r_{ik-j-1} \cdots r_{ik} \cdot \lambda) \) with coefficients in a sheaf of differential forms over fibers with coefficients in the sheaf obtained at the previous step. This means that \( \text{Vol} \) is a sheaf over \( F \) with the fiber over a point \( x \in F^0 \); volume forms over \( \tilde{X}_w(x) \) with coefficients in a certain sheaf over \( \tilde{X}_w(x) \). Denote the latter sheaf by \( \mathcal{V}_x \). Unfortunately we do not possess its direct description. Nevertheless it follows from the definition of \( \phi(s_i)^{(j)} \)'s that what one gets in coordinates is a volume form over \( \tilde{X}_w(x) \) times some multivalued function on \( X_w(x) \). The sheaf of continuous branches of this function is exactly \( \mathcal{V}_x \). It also follows from the construction that the singularities of this function are concentrated on \( \tilde{X}_v(x), v < w \) as well as on the singularities of the sections of \( \mathcal{O}(L, \lambda) \). In other words, the map \( \sigma_j : \tilde{X}_w(x) \to X_w(x) \) makes \( \mathcal{V}_x \) into a local system over \( \tilde{X}_w(x) \cap F^0 \) which we also denote by \( \mathcal{V}_x \).

5. Let \( H_{l(w)}(\mathcal{V}_x) \) be the highest homology group of \( \tilde{X}_w(x) \cap F^0 \) with coefficients in \( \mathcal{V}_x \). There arises a vector bundle over \( F^0 \) with the fiber over \( x \in F^0 \) equal to \( H_{l(w)}(\mathcal{V}_x) \). Denote this bundle by \( \mathcal{H}_{l(w)} \). It is equipped with the canonical Gauss-Manin connection and, therefore, \( \mathfrak{g} \) acts on its sections. Another important point is that there is a map \( \mathcal{H}_{l(w)} \to \text{Vol}^* \), determined by the fiberwise integration. It follows that pushing all sheaves in \([24]\) forward one obtains

$$\Gamma(\tilde{F}^0, \mathcal{O}(L, \lambda) \otimes \mathcal{H}_{l(w)}) \to \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, r_{i_1} \cdots r_{i_k} \cdot \lambda))$$

or, equivalently,

$$T(r_{i_1} \cdots r_{i_k}) : \Gamma(\tilde{F}^0, \mathcal{O}(L, \lambda)) \to \Gamma(\tilde{F}^0, \mathcal{O}(L, r_{i_1} \cdots r_{i_k} \cdot \lambda) \otimes \mathcal{H}_{l(w)}). \quad (25)$$

We have arrived at

**Theorem 2.10** For each reduced decomposition \( w = r_{i_1} \cdots r_{i_k} \) the map \( T(r_{i_1} \cdots r_{i_k}) \) in \([24]\) is \( \mathfrak{g} \)-linear.

6. The \( \mathfrak{g} \)-modules appearing in different sides of \([25]\) are of different size: while \( \Gamma(\tilde{F}^0, \mathcal{O}(L, \lambda)) \) may be regarded as a space of (multi-valued) functions on the flag manifold, it is no longer true for \( \Gamma(\tilde{F}^0, \mathcal{O}(\mathcal{L}, r_{i_1} \cdots r_{i_k} \cdot \lambda) \otimes \mathcal{H}_{l(w)}) \) because there is no reason to think that \( \mathcal{H}_{l(w)} \) is a line bundle. Nevertheless one may look for a line subbundle of the latter bundle such that the space of its sections is closed with respect to the action of \( \mathfrak{g} \). These subbundles are described via the Gauss-Manin connection on \( \mathcal{H}_{l(w)} \). This connection makes the fundamental group \( \pi_1(F^0) \) act on fibers \( H_{l(w)}(\mathcal{V}_x) \). Denote by \( P(H_{l(w)}(\mathcal{V}_x)) \) the projectivization of \( H_{l(w)}(\mathcal{V}_x) \). Each \( \pi_1(F^0) \)-invariant element \( h \) of \( P(H_{l(w)}(\mathcal{V}_x)) \) gives rise to a line subbundle of \( \mathcal{H}_{l(w)} \) which we denote \( \mathcal{H}_{l(w)}(h) \). Obviously \( \mathfrak{g} \)-
acts on the sections of \( \mathcal{H}_{l(w)}(h) \). The inclusion \( \mathcal{H}_{l(w)}(h) \subset \mathcal{H}_{l(w)} \) gives rise to the projection \( \mathcal{H}_{l(w)}^* \subset \mathcal{H}_{l(w)}^* \). We have proved the 1st item of

Lemma 2.11 (i) For each reduced decomposition \( w = r_{i_1} \cdots r_{i_l} \) and a \( \pi_1(F^0) \)-invariant element \( h \) of \( P(H_{l(w)}(V_x)) \) there is a \( g \)-linear map

\[
\Gamma(F^0, \mathcal{O}(L, \lambda)) \to \Gamma(F^0, \mathcal{O}(L, r_{i_1} \cdots r_{i_l} \cdot \lambda) \otimes \mathcal{H}_{l(w)}^*(h));
\]

(ii) There is at least one \( \pi_1(F^0) \)-invariant element of \( P(H_{l(w)}(V_x)) \).

As to the 2nd item, it was actually proved at the beginning of this section. The repeated integration in (20) associates a \( \pi_1(F^0) \)-invariant element of \( P(H_{l(w)}(V_x)) \) to any reduced decomposition \( w = r_{i_1} \cdots r_{i_l} \). The cycle of integration in (20) looks like a product of some simple cycles. As was explained in sect. 2.2, to each 1-dimensional Schubert variety \( X_{r_i}(x) \) one canonically associates a cycle with coefficients in a local system on \( X_{r_i}(x) \) punctured at 3 points. Denote this cycle by \( \sigma_i(x) \). Set \( \sigma_{i_1} \cdots \sigma_{i_l}(x_0) = \{(x_1, \ldots, x_l) : x_j \in \sigma_i(x_{j-1})\} \). Since each \( \sigma_i(x) \) is an eigenvector of the fundamental group of \( F^0 \) (the corresponding homology group is 1-dimensional), the “product” \( \sigma_{i_1} \cdots \sigma_{i_l}(x_0) \) is also. It is not clear whether it depends on the reduced decomposition \( w = r_{i_1} \cdots r_{i_l} \). \( \square \)

3 Solutions to Knizhnik - Zamolodchikov Equations. The classical case.

3.1 Trigonometric Form of Knizhnik-Zamolodchikov Equations and Correlation Functions.

1. We, firstly, prepare notations in order to write down the trigonometric form of Knizhnik-Zamolodchikov equations. (In the exposition we will be following [12].) Let

\[
\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}
\]

be a root decomposition of a simple finite-dimensional Lie algebra \( \mathfrak{g} \). Fix an invariant inner product on \( \mathfrak{g} \) and a basis \( \{h_i \in \mathfrak{h}, g_\alpha \in \mathfrak{g}_\alpha : 1 \leq i \leq n, \alpha \in \Delta\} \) of \( \mathfrak{g} \) so that \( (h_i, h_j) = \delta_{i,j}, (g_\alpha, g_\beta) = \delta_{\alpha, \beta} \). For each \( \mu \in \mathfrak{h}^* \) denote by \( h_\mu \) an element of \( \mathfrak{h} \) satisfying (and uniquely determined) by the condition \( (h_\mu, h) = \mu(h) \).

Set

\[
r = \frac{1}{2} \sum_{i=1}^{n} h_i \otimes h_i + \sum_{\alpha \in \Delta_+} g_\alpha \otimes g_{-\alpha}.
\]

Being an element of \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) \( r \) naturally acts on a tensor product of 2 \( \mathfrak{g} \)-modules. There are \( m^2 \) different ways to make it act on a tensor product of
modules via the following $m^2$ embeddings of $U(g)^{\otimes 2}$ in $U(g)^{\otimes m}$: each of
them is associated to a pair of numbers $1 \leq i, j \leq m$ and sends
\[ U(g)^{\otimes 2} \ni \omega \mapsto \omega_{ij} \in U(g)^{\otimes m}, \]
so that
\[ \omega = \sum a_s \otimes b_s \text{ then } \omega_{ij} = \sum_{s} \prod_{i \neq j}^{j} a_s \otimes 1 \otimes \cdots \otimes 1 \otimes b_s \otimes 1 \otimes \cdots \otimes 1. \]

For a pair $1 \leq i, j \leq m$ introduce the following function in $2$ complex
variables with values in $U(g)^{\otimes m}$:
\[ r(z_i, z_j) = r_{ij} z_i + r_{ji} z_j. \]

The trigonometric form of the Knizhnik-Zamolodchikov equations is the fol-
lowing system of $m$ differential equations on a function $\Psi$ in $m$ complex variables
with values in the tensor product $V_1 \otimes \cdots \otimes V_m$ of $m$ $g$–modules:
\[ (k + h^\vee) \frac{\partial \Psi}{\partial z_i} = \{ \sum_{j \neq i} r_{ij} (z_i, z_j) - \frac{1}{2} (\lambda_1 + \lambda_{m+1} + 2\rho)^{(i)} \} \Psi, \quad (26) \]
where $h^\vee$ is the dual Coxeter number of $g$, $k \in \mathbb{C}$, $\lambda_1, \lambda_{m+1} \in \mathfrak{h}^*$ are regarded as
parameters of the system and, finally, for each $\mu \in \mathfrak{h}^*$ $\mu^{(i)}$ stands for the operator
acting on $V_1 \otimes \cdots \otimes V_m$ as $h_\mu$ applied to the $i$–th factor of $V_1 \otimes \cdots \otimes V_m$. To
keep track of the parameters we will be referring to [21] as $KZ(\lambda_{m+1}, \lambda_1)$.

Let $V_1, \ldots, V_m$ be highest (lowest) weight modules with highest (lowest)
weights $\mu_1, \ldots, \mu_m$ (resp.) and corresponding weight vectors $v_{\mu_1}, \ldots, v_{\mu_m}$. The
simplest non-trivial solution to (26) is found in the case when it is supposed to
take values in the line spanned by $v_{\mu_1} \otimes v_{\mu_2} \otimes \cdots \otimes v_{\mu_m}$.

Lemma 3.1 In the above notations the function
\[ \Psi = \prod_{i < j} (z_i - z_j)^{2(\mu_i, \mu_j)/(k + h^\vee)} (z_i z_j)^{-(\mu_i, \mu_j)/(k + h^\vee)} \times \]
\[ \prod_{i} z_i^{(e\lambda_1 + \lambda_{m+1} + 2\rho, \mu_i)/2(k + h^\vee)} \cdot v_{\mu_1} \otimes v_{\mu_2} \otimes \cdots \otimes v_{\mu_m} \]
is a solution to the system (26).

2. Denote by $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ the non-twisted affine Lie algebra
associated to $g$. The Knizhnik-Zamolodchikov equations were derived in [17] as
equations on matrix elements of certain operators acting between highest weight

modules over $\hat{\mathfrak{g}}$. Let us briefly recall basic definitions, referring to the book [15] for details. Fix the inclusion $\mathfrak{g} \subset \hat{\mathfrak{g}}$, $\mathfrak{g} \mapsto g \otimes 1$. If $F_i, H_i, E_i$, $1 \leq i \leq n$ are canonical generators of $\mathfrak{g}$ then $F_i, H_i, E_i$, $0 \leq i \leq n$ are canonical generators of $\hat{\mathfrak{g}}$ where it is set $E_0 = g_{-\theta} \otimes t$, $F_0 = g_\theta \otimes t^{-1}$ ($\theta$ stands for the maximal root of $\mathfrak{g}$). $H_0 = [E_0, F_0]$. This includes affine Lie algebras into the general theory of Kac-Moody Lie algebras, in particular produces the Cartan and 2 Borel subalgebras $\mathfrak{h} \subset \hat{\mathfrak{h}}$, $\mathfrak{b}^- \subset \hat{\mathfrak{b}}^-$, the triangular decomposition $\hat{\mathfrak{g}} = \hat{\mathfrak{b}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{b}}^+$, root decomposition, the Weyl group etc., as well as the basic objects the of representation theory: Verma module, highest weight modules weight decomposition etc., everything being defined in a natural way.

Let $M(\lambda)$ ($M(\lambda)^c$) be a Verma module (contragredient Verma module) over $\mathfrak{g}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$. We will also denote these modules as $M(\lambda, k)$ (or $M(\lambda, k)^c$), where $\lambda = \hat{\lambda}|_{\mathfrak{h}}$, $k = \hat{k}(c)$. The last equality simply means that the central element $c$ acts as the multiplication by $k \in \mathbb{C}$. The number determined by the last condition is called the level of representation. Observe also that $\hat{\mathfrak{g}}$ is naturally graded by powers of the indeterminate $t$ and this gradation induces the gradation of any highest weight module $N$:

$$N = \oplus_{i \in \mathbb{Z}} N[m],$$

where the degree of the highest weight vector is said to be equal to 0.

An affine Lie algebra also possesses modules which are not included into the general theory of Kac-Moody Lie algebras. For any $\mathfrak{g}$–module $V$ set

$$V[z] = V \otimes \mathbb{C}[z, z^{-1}],$$

$$V((z)) = V \otimes \mathbb{C}((z)),$$

where $\mathbb{C}[z, z^{-1}]$ ($\mathbb{C}((z))$ resp.) is a space of Laurent polynomials (series resp.). Both are naturally equipped with a $\hat{\mathfrak{g}}$–module structure, the level being 0 in both cases and graded by powers of $z$. Note that one can consider the spaces $V[z]z^\nu$, $V((z))z^\nu$ equipped with a natural $\hat{\mathfrak{g}}$–module structure and grading ($\text{deg}(v \otimes z^{n+\nu}) = n$) so that the map

$$v \otimes z^{n+\nu} \mapsto v \otimes z^n$$

provides the isomorphisms

$$V[z] \approx V[z]z^\nu, \ V((z)) \approx V((z))z^\nu,$$

of $\hat{\mathfrak{g}}$–modules.

Consider a ( quadratic ) Casimir operator of $\mathfrak{g}$

$$\Omega = \sum_{\alpha \in \Delta} g_\alpha \otimes g_{-\alpha} + \sum_{i=1}^{n} h^i \otimes h^i.$$
If $\Omega$ acts on a $\mathfrak{g}$–module $V$ as a multiplication by a number, we denote this number by $c_V$. For example, for a highest weight module with a highest weight $\mu$ this number is equal to $(\mu + \rho, \mu)$.

3. **Vertex operator** $\Phi(z)$ is said to be a $\hat{\mathfrak{g}}$–morphism

\[ \Phi(z) : M(\lambda_1, k) \to M^c(\lambda_2, k) \otimes V((z))z^\kappa \]

\[ \kappa = \frac{1}{2(k + h^\vee)}(- (\lambda + \rho, \lambda) + c_V), \]

homogeneous with respect to the above-defined gradation.

We will be often referring to $\Phi(z)$ as a vertex operator acting from $M(\lambda_1)$ to $M(\lambda_2, k)$.

Observe that all $\hat{\mathfrak{g}}$– as well as $\mathfrak{g}$–modules under consideration are semisimple with respect to the action of Cartan subalgebra and, therefore, in addition to the above-defined “exterior” gradation possess the “inner” gradation: the weight decomposition. Now one easily classifies vertex operators.

**Lemma 3.2**

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k); M^c(\lambda_2, k) \otimes V((z))) \approx V(\lambda_1 - \lambda_2), \]

where $V(\lambda_1 - \lambda_2)$ stands for the weight space related to the weight $\lambda_1 - \lambda_2$.

**Proof.** The passage to the dual $V^*$ of $V$ makes the assertion into the following one:

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k) \otimes V^*[z]; M^c(\lambda_2, k)) \approx (V^*)^{(\lambda_1 - \lambda_2)}. \]

Set $\hat{\mathfrak{b}} = \mathfrak{n}_+ \oplus \mathfrak{h}$, $\hat{\mathfrak{b}}^- = \mathfrak{n}_- \oplus \mathfrak{h}$. One proves that

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k) \otimes V^*[z] \approx \text{Ind}_{\mathfrak{b}}^{\hat{\mathfrak{g}}} (\mathbb{C}v_{\lambda_1} \otimes V^*[z]). \]

Frobenius duality implies

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k) \otimes V^*[z]; M^c(\lambda_2, k)) \approx \text{Hom}_{\hat{\mathfrak{b}}} (\mathbb{C}v_{\lambda_1} \otimes V^*[z]; M^c(\lambda_2, k)). \]

Dualizing one obtains

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k) \otimes V^*[z]; M^c(\lambda_2, k)) \approx \text{Hom}_{\hat{\mathfrak{b}}} ((M^c(\lambda_2, k))^*; \mathbb{C}v_{-\lambda_1} \otimes V[z]). \]

Since (by the definition)

\[ (M^c(\lambda_2, k))^* \approx \text{Ind}_{\mathfrak{b}}^{\hat{\mathfrak{g}}} \mathbb{C}v_{-\lambda_2}, \]

one more application of the Frobenius duality implies

\[ \text{Hom}_{\hat{\mathfrak{g}}} (M(\lambda_1, k) \otimes V^*[z]; M^c(\lambda_2, k)) \approx \text{Hom}_{\mathfrak{b}}^{\hat{\mathfrak{g}}} (\mathbb{C}v_{-\lambda_2}; \mathbb{C}v_{-\lambda_1} \otimes V[z]), \]
completing the proof. □

For a generic highest weight a Verma module is isomorphic to the corresponding contragredient Verma module - this is the case we will be interested in - therefore, a composition of vertex operators is well-defined. Namely, if

$$\Phi_i(z_i) : M(\lambda_i, k) \to M(\lambda_{i+1}, k) \otimes V_i(z_i), \ 1 \leq i \leq m$$

are vertex operators, then there arises an operator

$$\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) : M(\lambda_1, k) \to M(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m),$$

where $$(V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m)$$ is said to be the space of Laurent series in $$z_1, \ldots, z_m$$ twisted by certain powers of $$z_i$$ coming from (27) with coefficients in $$V_1 \otimes \cdots \otimes V_m$$.

The composition $$\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)$$ is determined by its matrix elements

$$\langle v_{m+1}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_1 \rangle, \ v_1 \in M(\lambda_1, k), \ v_{m+1} \in M(\lambda_{m+1}, k)^*.$$ 

Each matrix element is obviously a formal series with coefficients in $$V_1 \otimes \cdots \otimes V_m$$.

A correlation function is said to be the matrix element of a composition of vertex operators related to the vacuum vectors:

$$\langle v_{\lambda_{m+1}}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle.$$

Remark that vertex operators can be defined to be acting between not necessarily Verma, or even highest weight modules, though in the latter case existence of a composition of vertex operators becomes more subtle.

**Theorem 3.3 (Knizhnik, Zamolodchikov) (see [13])**

The correlation function

$$\langle v_{m+1}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_{\lambda_1} \rangle$$

satisfies the system KZ($$\lambda_{m+1}, \lambda_1$$) (see (26)).

The proof of Theorem 26, which can be found in [12], implies that the following more precise assertion is valid.

**Lemma 3.4** The statement of Theorem 26 remains valid if

$$\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) : W \to M(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m),$$

is an intertwiner acting from not necessarily Verma module $$W$$, provided the following conditions on $$v_{\lambda_1} \in W$$ hold

$$H v_{\lambda_1} = \lambda_1(H) v_{\lambda_1},$$

$$\langle v_{\lambda_{m+1}}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) | \hat{n}^+ v_{\lambda_1} \rangle = 0.$$
3.2 Main result: Integral Representations of Solutions.

1. As was explained in Introduction it is plausible that a suitable definition of complex powers of generators of \( \hat{g} \) as operators acting on \( (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \) gives rise to solutions of KZ equations. We will give such a definition provided \( V_i \)'s belong to a suitable family of \( g \)-modules. We assume that 
\[
V_i = \Gamma(\tilde{F}^0, \mathcal{O}(L_\nu_i, \mu_i)), \quad 1 \leq i \leq m,
\]
where \( \mu_i, \mu_\nu \) are arbitrary complex numbers, module \( \Gamma(\tilde{F}^0, \mathcal{O}(L, \mu)) \) was defined in sect.2 and the local system \( L_\nu \) was defined in sect.2, Example2.5.

Simply speaking \( \Gamma(\tilde{F}^0, \mathcal{O}(L, \mu)) \) is a module realized in multi-valued functions on the flag manifold. The direct way to this realization is as follows: take \( M(\mu)^c \); it is realized in meromorphic sections of the line bundle \( \mathcal{O}(\lambda) \).

Having a section of this bundle fixed one identifies meromorphic sections with meromorphic functions on the flag manifold, the action of \( g \) being given by 1st order differential operators (see, for example, [9]); then the space \( \Gamma(\tilde{F}^0, \mathcal{O}(L, \mu)) \) is determined by multiplication of meromorphic functions by the multivalued function
\[
\phi = \prod_{1 \leq j \leq n} \phi^j_{\nu j} \text{ defined in the above-mentioned Example2.5} \text{ the action being given by the same 1st order differential operators. Therefore an element of } (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \text{ may (and will ) be regarded as a function of a point on a cartesian product of } m \text{ copies of a flag manifold and complex coordinates } z_1, \ldots, z_m.
\]
The dependence on the latter has been so far formal but it will in reality turn out to determine a multi-valued analytic function.

2. One of the properties of modules \( \Gamma(\tilde{F}^0, \mathcal{O}(L_\nu, \mu)) \) is that a group element \( \exp(tg) \in G, \ g \in g \) may be regarded as an operator sending elements of \( \Gamma(\tilde{F}^0, \mathcal{O}(L_\nu, \mu)) \) to elements of the same space times, probably, another multi-valued function. In order to understand this, first, observe that the group \( G \) acts on meromorphic sections of a line bundle \( \mathcal{O}(\mu) \), corresponding to an integral highest weight \( \mu \). Then one defines an action of \( G \) on complex powers of such sections as follows: if \( A \in G, \ s \) is a section of \( \mathcal{O}(\mu) \) and \( A \cdot s = f_A s \), where \( f_A \) is a function on the flag manifold, then one sets
\[
A \cdot s^\alpha = (f_A)^\alpha \cdot s^\alpha.
\]
Similar formula applies to the product \( \prod s_i^{\alpha_i} \) and since the latter expressions generate the space of sections of \( \mathcal{O}(\mu) \) for an arbitrary \( \mu \) (see the discussion of complex powers of line bundles over a flag manifold in sect.2.1) this completes the definition of the operator \( \exp(tg), \ g \in g \).

Factor-wise application makes \( G \) act on the tensor product \( V_1 \otimes \cdots \otimes V_m \).

3. We are now in a position to define an action of complex powers of elements of \( \hat{g} \) on some elements of \( (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \). First of all, the action of \( \hat{g} \) on \( (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \) is defined by means of a certain Lie algebra homomorphism \( \hat{g} \rightarrow U(g)^{\otimes m} \). For example, on generators of \( \tilde{a}^+ \) it acts as follows:
\[ F_i \mapsto \sum_{j=1}^{m} 1 \otimes \cdots \otimes 1 \otimes F_i \otimes 1 \otimes \cdots \otimes 1, \quad 1 \leq i \leq n, \quad (28) \]

\[ F_0 \mapsto \sum_{j=1}^{m} z_j^{-1} 1 \otimes \cdots \otimes 1 \otimes g_\theta \otimes 1 \otimes \cdots \otimes 1. \quad (29) \]

These formulas combined with the above discussion imply that group elements \( \exp(t_i F_i) \), \( 0 \leq i \leq n \) act on those elements of \((V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m)\) which are analytic (probably multi-valued) functions of \( z_1, \ldots, z_m \). Denote the subspace of \((V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m)\) consisting of analytic (multi-valued) functions of \( z_1, \ldots, z_m \) by \((V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \text{fun}\).

**Definition.** For \( \psi \in (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \text{fun} \), \( A \in \text{End}((V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \text{fun}) \), \( k \in \mathbb{C} \) set

\[ A^\beta \cdot \psi = \Gamma(-\beta)^{-1} \int \{ \exp(-tA) \psi \} t^{-\beta-1} dt, \quad (30) \]

where the contour is an arbitrary element of the local system determined by single-valued branches of the function \( \{ \exp(-tA) \psi \} \) if the latter expression makes sense.

The above discussion shows that the definition makes sense at least for elements of \( \hat{g} \) while the following lemma shows that the definition implies several natural properties.

**Lemma 3.5** The following relations hold provided both sides of them make sense.

(i) \( A^\beta = A^{\beta-n} \cdot A^n \), \( k \in \mathbb{C}, n \in \mathbb{N} \).

(ii) (the binomial theorem) if \([A, B] = 0\), then

\[ (A + B)^\beta = \sum_{j=0}^{\infty} \frac{\beta(\beta-1) \cdots (\beta-j+1)}{j!} A^{\beta-j} B^j; \]

(iii) \([H_i, F_j^\beta] = -\beta a_{ij} F_j^\beta \), \( k \in \mathbb{C} \);

(iv) \([E_i, F_j^\beta] = \delta_{i,j} \beta F_j^{\beta-1}(H_j - \beta + 1), \beta \in \mathbb{C} \)
Proof is a matter of routine calculations.

4. The main result. Let $\Psi \in (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m)^{fun}$ be a correlation function coming from the composition of vertex operators

$$\Phi_m \circ \cdots \circ \Phi_1 : M(\lambda_1, k) \to M(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m).$$

Let $w = r_i \cdots r_2 r_1 \in W$ be a reduced decomposition. Set

$$\beta_j = \frac{2(r_{ij-1} \cdots r_i \cdot \lambda_1, \alpha_i)}{(\alpha_{i_j}, \alpha_i_i)}.$$

Set

$$K_{\Psi,w}(t_1, t_2, \ldots, t_l) = \prod_{j=1}^l \Gamma(-\beta_j)^{-1} \times \{ \exp(-t_1 F_i) \cdots \exp(-t_l F_i) \prod_{j=1}^l t_j^{-\beta_j-1} \Psi \}.$$

Denote by $\mathcal{M}$ the local system (over the domain of $K_{\Psi,w}(t_1, t_2, \ldots, t_l)$) of single-valued branches of $K_{\Psi,w}(t_1, t_2, \ldots, t_l)$. The function $K_{\Psi,w}(t_1, t_2, \ldots, t_l)$ depends on a point on the cartesian product of flag manifolds as well as on $(z_1, \ldots, z_m)$ as on parameters. Therefore also does $\mathcal{M}$. It follows that for every cycle $\sigma \in H_l(\text{domain of } K_{\Psi,w}(t_1, t_2, \ldots, t_l), \mathcal{M})$ the integral

$$\int_{\sigma} K_{\Psi,w}(t_1, t_2, \ldots, t_l) dt_1 dt_2 \ldots dt_l$$

is a function of a point on the cartesian product of flag manifolds and of $(z_1, \ldots, z_m)$.

Theorem 3.6 For any cycle $\sigma \in H_l(\text{domain of } K_{\Psi,w}(t_1, t_2, \ldots, t_l), \mathcal{M})$ the function

$$\int_{\sigma} K_{\Psi,w}(t_1, t_2, \ldots, t_l) dt_1 dt_2 \ldots dt_l$$

satisfies the system $KZ(\lambda_{m+1}, w \cdot \lambda_1)$ (see (26)).

Proof is a formal calculation based on Lemma 3.4 and Lemma 3.5 (iii) and (iv). Firstly observe that the composition of vertex operators $\Phi_m \circ \cdots \circ \Phi_1$ is naturally extended to a $\hat{\mathfrak{g}}$–morphism:

$$\Phi_m \circ \cdots \circ \Phi_1 : M(\lambda_1, k) \otimes \mathcal{C}((t_1, \ldots, t_l)) \to M(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m).$$

Subsequent integration makes it into a $\hat{\mathfrak{g}}$–morphism, denoted symbolically by $\int \circ \Phi_m \circ \cdots \circ \Phi_1$:

$$\int \circ \Phi_m \circ \cdots \circ \Phi_1 : M(\lambda_1, k) \otimes \mathcal{C}((t_1, \ldots, t_l)) \to M(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m).$$
The function
\[ \int K_{\Psi, w}(t_1, t_2, \ldots, t_l) \, dt_1 \, dt_2 \ldots dt_l \]
is equal to the matrix element
\[ \langle v_{\lambda_{m+1}}, \int \Phi_m \circ \cdots \circ \Phi_1 \{ \exp(-t_l F_{i_l}) \cdots \exp(-t_1 F_{i_1}) \prod_{j=1}^l t_j^{-\beta_j} \} \rangle \]
of the morphism \( \int \circ \Phi_m \circ \cdots \circ \Phi_1 \) because of the tautological equality
\[ \int K_{\Psi, w}(t_1, t_2, \ldots, t_l) \, dt_1 \, dt_2 \ldots dt_l = \int \langle v_{\lambda_{m+1}}, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) \{ \exp(-t_l F_{i_l}) \cdots \exp(-t_1 F_{i_1}) \prod_{j=1}^l t_j^{-\beta_j} \} \rangle \, dt_1 \ldots dt_l. \]

Now the choice of the exponents \( \beta_1, \ldots, \beta_l \) and Lemma 3.5 (iii) and (iv) shows that the assumption of Lemma 3.4 is valid for the vector
\[ \{ \exp(-t_l F_{i_l}) \cdots \exp(-t_1 F_{i_1}) \prod_{j=1}^l t_j^{-\beta_j} \} \in M(\lambda_1, k) \otimes \mathbb{C}((t_1, \ldots, t_l)) \]
(c.f. similar reasoning in Introduction, formulas (4,5) ), which completes the proof. \( \square \)

We point out that this proof imitates the following situation:
the function
\[ \int K_{\Psi, w}(t_1, t_2, \ldots, t_l) \, dt_1 \, dt_2 \ldots dt_l \]
is a matrix element of the operator \( \Phi_m \circ \cdots \circ \Phi_1 \) related to a singular vector \( F_{i_l}^{\beta_l} \cdots F_{i_1}^{\beta_1} v_{\lambda_1} \).

### 3.3 Explicit Formulas for \( \hat{sl}_2 \)

We set \( \hat{g} = \hat{sl}_2 \) and suppose that modules \( V_i \) are of the type \( V(\mu, \lambda) \) defined in the Introduction by (10). Under these assumptions one writes out the integral in (13) in an explicit form.

We denote \( E_1, H_1, F_1 \) the Cartan generators of \( sl_2 \) and set \( E_0 = F_1 \otimes t, \) \( F_0 = E_1 \otimes t^{-1} \) so that \( E_i (F_i) \) \( 0 \leq i \leq 1 \) are generators of maximal nilpotent subalgebras of \( \hat{sl}_2 \). It is a matter of direct calculation to show that under the identification
\[ f(x) \, dx^{-\lambda/2} \approx f(x) \]
one has
\[ \exp(-tE_1) f(x) = f(x - t), \quad \exp(-tF_1) f(x) = (-1 - tx)^{\lambda} f \left( \frac{x}{xt + 1} \right). \]
An element of $V(\mu_1, \lambda_1) \otimes \cdots \otimes V(\mu_m, \lambda_m)$ is identified with a function of $(x_1, \ldots, x_m)$ (having the prescribed branching coefficients at coordinate hyper-planes) and a correlation function $\Psi$ takes the form $\Psi = \Psi(x_1, \ldots, x_m; z_1, \ldots, z_m)$. Now formulas (28, 29, 32) imply

$$\exp(-tF_1) \Psi(x_1, \ldots, x_m; z_1, \ldots, z_m) = \prod_{i=1}^{m} (\mu_i - 1 - tx_i) \Psi(x_1/x_1 + 1, \ldots, x_m/x_m + 1; z_1, \ldots, z_m)$$

(33)

$$\exp(-tF_0) \Psi(x_1, \ldots, x_m; z_1, \ldots, z_m) = \Psi(x_1 - tz_1^{-1}, \ldots, x_m - tz_m^{-1}; z_1, \ldots, z_m)$$

(34)

These relations can be easily iterated in order to obtain formulas for, for example

$$\cdots \exp(-t_2F_0) \exp(-t_1F_1) \Psi(x_1, \ldots, x_m; z_1, \ldots, z_m).$$

It is especially so in the case when $\Psi$ is set to be equal to the “simplest” correlation function provided by Lemma 3.1, which is, actually, a monomial as a function of $x$’s. Now observe that it is all we need, since the Weyl group of $\hat{sl}_2$ is a free group generated by 2 reflections $r_0$, $r_1$ and, therefore, each element is uniquely expanded into either

$$\cdots r_0 r_1$$

or

$$\cdots r_1 r_0.$$

Let us now write down the results of direct calculations. Set

$$w = r_1 \cdots r_0 r_1,$$

where $\epsilon \equiv l(mod 2)$;

$$\Psi^{(0)} = \prod_{i<j} (z_i - z_j)^{\mu_i / (k+2)} (z_i z_j)^{\mu_i / (2(k+2))} \times \prod_i z_i^{(\lambda_1 + \lambda_{m+1} + 2) \mu_i / (2(k+2))},$$

$$\Psi^{(l)} = \exp(-t_l F_l) \exp(-t_{l-1} F_1) \cdots \exp(-t_2 F_0) \exp(-t_1 F_1) \Psi^{(0)}.$$

Observe that $\Psi^{(0)}$ is the solution provided by Lemma 3.1 in the case when $v_{\mu_i}$, $1 \leq i \leq m$ are highest weight vectors (of the weight $\mu_i$, $\mu_i \in \mathbb{C}$) and

$$\lambda_1 = \lambda_{m+1} + \mu_1 + \cdots + \mu_m.$$

(Recall that $k$ is a level and a dual Coxeter number for $\hat{sl}_2$ is equal to 2.)

Formulas (33, 34) then give:

$$\Psi^{(l)} = \prod_{i=1}^{m} \{ F_i^{(l)}(x_i; z_i; t_1, \ldots, t_l) \}^{\mu_i} \Psi^{(0)},$$

(35)
where
\[ P^{(l)}_i(x_i; z_i; t_1, \ldots, t_m) = \sum_{j=0}^{l} A_j(x_i; z_j) \sigma_j^{(l)}(t_1, \ldots, t_l), \]
where finally \( A_j(x; z) \) is given by
\[ A_{4i}(x; z) = -z^{-i}, \quad A_{4i+1}(x; z) = -z^{-i}x, \quad A_{4i+2}(x; z) = z^{-i-1}, \quad A_{4i+3}(x; z) = z^{-i-1}x; \]
and
\[ \sigma_j^{(l)} = \sum_{0 \leq i_1 < i_2 < \cdots < i_j < l} t_{2i_1+1} t_{2i_2+1} \cdots t_{2i_j+1}. \]

Example.

\[ \Psi^{(5)} = \prod_{i=1}^{m} \left( -1 - x_i(t_1 + t_3 + t_5) + z_i^{-1}(t_1 t_2 + t_1 t_4 + t_3 t_4) + z_i x_i \right) \]
\[ \prod_{i<j} \left( z_i - z_j \right)^{\mu_i \mu_j / (k+2)} \left( z_i z_j \right)^{\mu_i \mu_j / (k+2)} \times \]
\[ \prod_{l} z_i^{(\lambda_1 + \lambda_{m+1} + 2) \mu_i / (k+2)}; \]

Then in notations of Theorem 3.6 and (35) one gets
\[ K_{\Psi^{(0)},w}(t_1, t_2, \ldots, t_l) = \Psi^{(l)} \times \prod_{i=1}^{l} t_i^{\lambda_1 + (l-i)(k+2)+1}. \] (37)

The function \( K_{\Psi^{(0)},w}(t_1, t_2, \ldots, t_l) \) (as a function of \( t \)'s) is multi-valued and branches at coordinate hyperplanes \( t_i = 0, 1 \leq i \leq m \) and at \( m \) hypersurfaces of the order \( l \) given by
\[ P^{(l)}_i(x_i; z_i; t_1, \ldots, t_l) = 0. \]

Let \( \mathcal{M} \) be the local system of continuous branches of \( K_{\Psi^{(0)},w}(t_1, t_2, \ldots, t_l) \) over the domain of \( K_{\Psi^{(0)},w}(t_1, t_2, \ldots, t_l): D \). Then our main result takes the form:

for any \( \sigma \in H_l(D, \mathcal{M}) \) the integral
\[ \int_{\sigma} K_{\Psi^{(0)},w}(t_1, t_2, \ldots, t_l) dt_1 dt_2 \ldots dt_l \]
satisfies the system \( KZ(\lambda_{m+1}, w, \lambda_1) \) (see (26)), \( K_{\Psi,w}(t_1, t_2, \ldots, t_l) \) being given by (32) (37).

It is worth mentioning that starting with the correlation function given by Lemma 3.1, for \( m = 2 \) at the 1st step ( \( l = 1 \) one obtains a Gauss hypergeometric function. Increasing of \( m \) or of the number of steps gives other remarkable special functions which, besides \( KZ \) equations, satisfy some other differential equations (see sect. 3.3).
4 Solutions to $q$–Knizhnik - Zamolodchikov Equations.

I. Frenkel and N. Reshetikhin have derived (see [12]) a quantum analogue of the trigonometric form of KZ equations. Their approach is based on the observation that the representation theory underlying the classical KZ equations is to a large extent analogous to the representation theory of affine quantum groups.

4.1 Quantum Groups and their Representations

The material of this section is fairly standard. Usually the reference is the work [7].

1. We first recall the definition of the Jimbo-Drinfeld quantum group. For $q \in \mathbb{C}$, $d \in \mathbb{Z}$ set:

$$[n]_d = \frac{1 - q^{2nd}}{1 - q^{2d}},$$

$$[n]_d! = [n]_d \cdots [1]_d,$$

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[n]_d \cdots [n-j+1]_d}{[j]_d!},$$

omitting the subscript if $d = 1$. As usual, $A = (a_{ij})$, $1 \leq i, j \leq n$ stands for a generalized symmetrizable Cartan matrix, symmetrized by non-zero integers $d_1, \ldots, d_n$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j$.

Now if $g$ is a Kac-Moody Lie algebra associated to $A$ the Drinfeld-Jimbo quantum group $U_q(g)$, $q \in \mathbb{C}$ is said to be a Hopf algebra with antipode $S$, comultiplication $\Delta$ and 1 on generators $E_i, F_i, K_i, K_i^{-1}, 0 \leq i \leq n$ and defining relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$  \hfill (38)

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \quad q_i = q^{d_i},$$  \hfill (39)

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i}, \quad q_i = q^{d_i},$$  \hfill (40)

$$\sum_{\nu = 0}^{1-a_{ij}} (-1)^\nu q_i^{(\nu-1+a_{ij})} \begin{bmatrix} 1 - a_{ij} \\ \nu \end{bmatrix}_{d_i} E_i^{1-a_{ij}-\nu} E_i^\nu = 0 \quad (i \neq j),$$  \hfill (41)

the comultiplication being given by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta K_i = K_i \otimes K_i,$$  \hfill (42)

and antipode - by

$$SE_i = -K_i^{-1} E_i, \quad SF_i = -F_i K_i, \quad SK_i = K_i^{-1}.$$  \hfill (43)
The relations admit the $\mathbb{C}$–algebra anti-automorphism $\omega$

$$\omega E_i = F_i, \ \omega F_i = E_i, \ \omega K_i = K_i$$ (44)

Set $U_q^+(\mathfrak{g}) (U_q^{-}(\mathfrak{g}))$ equal to the subalgebra, generated by $E_i$ ($F_i$ resp.) ($1 \leq i \leq n$) and $U_q^0(\mathfrak{g}) = \mathbb{C}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$. One may check that the multiplication induces an isomorphism of linear spaces

$$U_q(\mathfrak{g}) \approx U_q^{-}(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}).$$ (45)

2. Let $\lambda \in \mathfrak{h}^*$. A Verma module $M_q(\lambda)$ is said to be a $U_q(\mathfrak{g})$–module on one generator $v_\lambda$ satisfying the following conditions

$$U^+ v_\lambda = 0, \ K_i v_\lambda = q^i(\lambda, \alpha_i) v_\lambda (i = 1, \ldots, n),$$

$M_q(\lambda)$ is a free $U_q^{-}(\mathfrak{g})$–module on the generator $v_\lambda$.

Such a module exists and unique, which follows almost immediately from (45). We will call $v_\lambda$ a highest weight (or vacuum) vector. An example of a highest weight module is the contragredient Verma module $M_c q(\lambda)$ which is defined as in the classical case: first define the dual to the Verma module by means of the antipode $S$ (see (43)) and then “twist” the action by the antiautomorphism $\omega$ (see (42)). We say that an element $w$ of a $U_q(\mathfrak{g})$–module is a weight vector of the weight $\mu \in P$ if it satisfies $K_i w = q^{(\mu, H_i)}$. A Verma module as well as its submodules and quotients is graded by finite-dimensional weight spaces.

Any quotient $V$ of $M_q(\lambda)$ is called a highest weight module. Theory of highest weight modules over quantum groups for generic $q$ is parallel to that over $\mathfrak{g}$. A couple of examples are:

(i) one establishes a 1-1 correspondence between singular vectors (those annihilated by $U_q^+(\mathfrak{g}))$ and morphisms of Verma modules;

(ii) one proves that $M_q(\lambda)$ has a unique maximal proper submodule $J_q(\lambda)$ and, as was shown in [19], the irreducible quotient $L_q(\lambda) = M_q(\lambda)/J_q(\lambda)$ is a deformation of the irreducible $\mathfrak{g}$–module with highest weight $\lambda$.

Another important (though simple) observation is that the easily checked relation

$$[E_i, F_j^m] = \delta_{i,j} \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}} F_j^{m-1} K_i q_i^{-m+1} - K_i q_i^m - 1$$ (46)

implies that provided $\lambda(H_i) + 1 \in \mathbb{N}$ the vector $F_i^{\lambda(H_i)+1} v_\lambda$ is singular and, therefore, determines a morphism of a certain Verma module in $M_q(\lambda)$. In [21] it was used to derive a quantum analogue of the singular vector formula, here it will serve as a motivation for a $q$–integral representation of $q$–correlation functions (See Introduction.)
3. Recall that the definition of a vertex operator had 2 representation-theoretic ingredients: the highest weight modules and modules of the type \( V[z] \) or \( V((z)) \). The latter do not always admit \( q \)-deformation but they always do so in the case when \( g = sl_n \). The corresponding construction is as follows. Firstly, note that an alternative way to define the module \( V[z] \) for a \( g \)-module \( V \) is to consider the 1-parametric family of \( g \)-modules obtained by the pull-back \( \pi^*(t)V \) of \( V \) with respect to the 1-parametric family \( t \in \mathbb{C} \) of Lie algebra homomorphisms (evaluation maps)

\[
\pi(t) : \hat{g} \to g, \ g \otimes z^m \mapsto t^n g, \ c \mapsto 0.
\]

The above-mentioned theorem of Lusztig (see [19]) implies that (at least finite-dimensional) \( V \) admits a \( q \)-deformation \( V_q \). As was shown, for example in [19], if \( g = sl_n \) then the evaluation map \( \pi(t) : sl_n \to sl_n \subset gl_n \) also admits a \( q \)-deformation to the map \( \pi^*(t) : U_q(sl_n) \to U_q(gl_n) \), which gives a 1-parametric family of \( U_q(sl_n) \)-modules \( \pi^*(t)V_q \). This produces the \( q \)-analogues of \( V[z] \) and \( V((z)) \):

\[
V_q[z] = \pi_q^*(t)V_q \otimes \mathbb{C}[z, z^{-1}], \ V_q((z)) = \pi_q^*(t)V_q \otimes \mathbb{C}((z, z^{-1})).
\]

In general the evaluation map cannot be deformed. In what follows one should think either that \( g = sl_n \) or that the module \( V[z] \) admits a \( q \)-deformation.

### 4.2 \( q \)-Vertex Operators and \( q \)-Correlation Functions

We proceed in complete accordance with the classical case.

*Vertex operator* is said to be a \( U_q(\hat{g}) \)-homomorphism (exactly as in [27])

\[
\Phi(z) : M_q(\lambda_1, k) \to M_q^*(\lambda_2, k) \otimes V_q \otimes \mathbb{C}((z, z^{-1}))z^\kappa.
\]

One also proves that

\[
\text{Hom}_{U_q(\hat{g})}(M_q(\lambda_1, k), M_q^*(\lambda_2, k) \otimes V_q((z))) \approx V_q^{(\lambda_1 - \lambda_2)} \quad (47)
\]

In the same way one defines a composition of vertex operators

\[
\Phi_i(z_i) : M_q(\lambda_i, k) \to M_q(\lambda_{i+1}, k) \otimes (V_q)_i(z_i), \ 1 \leq i \leq m \quad (48)
\]

as an operator

\[
\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) : M_q(\lambda_1, k) \to M_q(\lambda_m, k) \otimes ((V_q)_1 \otimes \cdots \otimes (V_q)_m)(z_1, \ldots, z_m),
\]

where \(((V_q)_1 \otimes \cdots \otimes (V_q)_m)(z_1, \ldots, z_m)\) is said to be the space of Laurent series in \( z_1, \ldots, z_m \) with coefficients in \(((V_q)_1 \otimes \cdots \otimes (V_q)_m)\). It is determined by its matrix elements

\[
\langle v_m^*, \Phi_m(z_m \circ \cdots \circ \Phi_1(z_1))v_1 \rangle,
\]

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where \( v_1 \in M_q(\lambda_1, k) \), \( v_{m+1} \in M_q(\lambda_{m+1}, k) \). The matrix element related to vacuum vectors \( v_1 = v_{\lambda_1}, v_{m+1} = v_{\lambda_{m+1}} \) is said to be a \( q \)-correlation function.

I. Frenkel and N. Reshetikhin have proved that a \( q \)-correlation function satisfies a certain system of \( q \)-difference equations. Let \( \mathcal{R} \) be the universal \( \mathcal{R} \)-matrix of \( U_q(\hat{g}) \). Denote by \( R_{V, W}(z) \) its image as an operator acting from \( V[z] \otimes W[1] \) to \( V((z)) \otimes W[1] \).

**Theorem 4.1** Set \( p = q^{2(k+h^\vee)} \). The \( q \)-correlation function

\[
\Psi(z_1, \ldots, z_m) \equiv \langle v_{m+1}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1)v_1 \rangle
\]
satisfies the following system of \( q \)-difference equations (\( 1 \leq j \leq m \)):

\[
\Psi(z_1, \ldots, z_{j-1},pz_j, z_{j+1}, \ldots, z_m) = R_{j-1,j}(z_{j-1}/pz_j)^{-1} \circ \cdots \circ R_{1,j}(z_1/pz_j)^{-1} \times q^{-(\lambda_1+\lambda_{m+1}+2\rho_j)} \times R_{j,m}(z_j/\lambda_{m+1}) \circ \cdots \circ R_{j,j+1}(z_j/\lambda_{j}) \Psi(z_1, \ldots, z_m),
\]

where \( R_{ij}(z) \) stands for \( R_{(V_q^i)(V_q^j)}(z) \).

One observes that it is not necessary to consider vertex operators only acting between highest weight modules and as in the classical case it follows from the proof of I. Frenkel and N. Reshetikhin that the following more precise assertion is valid.

**Lemma 4.2** The statement of Theorem 4.1 remains valid if

\[
\Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) : W \rightarrow M_q(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m),
\]
is an intertwiner acting from not necessarily Verma module \( W \), provided the following conditions on \( v_{\lambda_1} \in W \) hold

\[
K_i v_{\lambda_1} = q_i^{\lambda_1(H_i)} v_{\lambda_1},
\]

\[
\langle v_{\lambda_{m+1}}^*, \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) U_q^{\vee}(\hat{g}) v_{\lambda_1} \rangle = 0.
\]

In the sequel we will be considering the \( q \)KZ system (49) in the normalized form meaning that \( R \)-matrix is normalized so that

\[
R_{ij}(z) v_i \otimes v_j = v_i \otimes v_j.
\]

Remark that this transformation’s only impact is that a correlation function has to be multiplied by a certain universal factor in order to produce a solution to (49, 50). (We do not want to calculate this factor here.) To keep track of the parameters we will be referring to (49, 50) as \( qKZ(\lambda_{m+1}, \lambda_1) \).
Again in complete accordance with the classical case, one deduces from [17] that if \( \sum_{1 \leq i \leq m} \mu_i = \lambda_1 - \lambda_2 \) then there is only one (up to proportionality) composition of vertex operators [18] and the corresponding correlation function satisfies

\[
\Psi(z_1, \ldots, z_m) = \psi(z_1, \ldots, z_m) \cdot v_1 \otimes \cdots \otimes v_m,
\]

for some \( C \)-valued function \( \psi(z_1, \ldots, z_m) \), where \( v_i \) stands for a highest weight vector of \( (V_q)_{\lambda_i} \). One easily shows that the solution to \( qKZ(\lambda_{m+1}, \lambda_1) \) produced by this composition of vertex operators is the following simple power function

\[
\prod_{j=1}^{m} z_j^{-(\lambda_j + \lambda_{m+1} + 2\rho_j)/2(k + k^\vee)} v_1 \otimes \cdots \otimes v_m.
\]

4.3 \( q \)-Integral Representation of Solutions

1. Here we show that at least formally solutions of \( qKZ \) equations may be written out in a way analogous to that in the classical case with all the ingredients of our integral representations changed for their \( q \)-analogues, in particular with the integrals changed for Jackson integrals. We are unable to verify the convergence of the obtained Jackson integrals because we do not have enough realizations of quantum groups by difference operators. However we will derive the necessary formulas for \( \hat{U}_q(\hat{sl}_2) \) and carry out explicit calculations of \( q \)-correlation functions analogous to those for \( \hat{sl}_2 \).

2. The \( q \)-multinomial theorem. Denote by \( C^*[x_1, \ldots, x_m] \) the algebra of skew polynomials. Recall that the latter is said to by a \( C \)-algebra on \( m \) generators \( x_1, \ldots, x_m \) and relations \( x_j x_i = q^2 x_i x_j \) if \( j > i \). As is well known, the following equality holds in \( C^*[x_1, \ldots, x_m] \):

\[
(x_1 + \cdots + x_m)^n = \sum_{i_1 + \cdots + i_m = n} \frac{[n]}{[i_1]! \cdots [i_m]!} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \quad n \in \mathbb{N}
\]

3. Some \( q \)-calculus. Suppose \( q \in C, |q| < 1 \).

3.1 The \( q \)-analogue of the usual integral is given by the Jackson integral related to a finite or semi-infinite "\( q \)-interval", defined by either

\[
\int_0^c f(t) \, dq \, t = c(1 - q^2) \sum_{n=0}^{\infty} f(cq^{2n})q^{2n}, \quad c \in C
\]

or

\[
\int_0^c f(t) \, dq \, t = c(1 - q^2) \sum_{n=-\infty}^{\infty} f(cq^{2n})q^{2n}, \quad c \in C
\]

the usage of \( q^2 \) being, of course, conventional (see [3]). The Jackson integral enjoys some of the elementary properties of the usual one. Here we point out some of them.
Change of variables. The following is evident

\[ \int_{0}^{\infty} f(at) \, dq t = a^{-1} \int_{0}^{\infty} f(t) \, dq t, \tag{53} \]

where \( \epsilon \) is either 1 or \( \infty \).

Integration by parts. Introduce 2 natural \( q \)-analogues of the derivative:

\[ \partial_q f(x) = \frac{f(x) - f(q^2 x)}{x(1 - q^2)}, \]
\[ \tilde{\partial}_q f(x) = \frac{f(x) - f(q^{-2} x)}{x(1 - q^2)}. \]

One immediately proves the following "\( q \)-integration by parts formula":

\[ \int_{0}^{c} f(x) \tilde{\partial}_q g(x) \, dq t = \int_{0}^{c} \partial_q f(x) g(x) \, dq t - f(c) g(q^{-2} c). \tag{54} \]

One also:
- easily finds an analogue of (54) for a semi-infinite interval;
- defines a Jackson integral over an interval \([a, c] \);
- defines a Jackson integral over a "multi-dimensional region"

\[ \int f(t_1, \ldots, t_m) \, dq t_1 dq t_2 \cdots dq t_m \]

as a repeated Jackson integral and proves that if the function \( f(t_1, \ldots, t_m) \) is naturally restricted then the result is independent of the order of integration. The repeated integration as well as an integral over \([a, c] \) also admits natural analogues of (53, 54).

In what follows the Jackson integrals will be taken over certain "\( q \)-cycles". We are not much concerned about the choice of the region of integration here and restrict ourselves to the following comment. We will usually be dealing with integrals of the form

\[ \int_{0}^{c} f(t) t^n \, dq t, \]

and it will be important that in the integration by parts formula (54) the boundary term vanishes. Therefore, given the integral

\[ \int_{0}^{c} f(t) t^n \, dq t, \]

the interval \([0, c] \) is said to be a \( q \)-cycle if \( f(c) = 0 \). One produces analogous definitions for intervals of other types.
3.2 The following notations are widely used:

\[(a)_i = \prod_{j=0}^{i-1}(1 - aq^{2j}),\]

\[(a)_\infty = \prod_{j=0}^{\infty}(1 - aq^{2j}).\]

The commutative version of the \(q\)-binomial theorem is the following identity (see [13]):

\[\sum_{i=0}^{\infty} \frac{(a)_i}{(q^2)_i} x^i = \frac{(ax)_\infty}{(x)_\infty}.\] (55)

4. Suppose \(A\) is an operator acting on some functional space. We are going to consider expressions of the form

\[\int f(tA) d_q t,\]

which can be understood as operators acting on the same space by

\[\int f(tA) d_q t (\Psi) = \int f(tA)(\Psi) d_q t,\]

provided the right hand side of the last equation converges. In what follows it will be assumed (or checked) that this condition is satisfied and, moreover, this integral possesses all necessary properties, e.g. absolute, uniform convergence, integration by parts etc.

Let \(\Gamma_q(\beta)\) be the usual \(q\)-gamma function. Recall that it is almost uniquely determined by the following functional equation (see [2]):

\[\Gamma_q(\beta + 1) = [\beta]\Gamma_q(\beta).\] (56)

Set

\[\hat{\Gamma}_q(\beta) = q^{-\beta(\beta-1)}\Gamma_q(\beta).\] (57)

It follows that

\[\hat{\Gamma}_q(-\beta + 1) = -[\beta]\hat{\Gamma}_q(-\beta).\] (58)

We will also be using the \(q\)-exponential function

\[\exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.\]

**Definition.** For \(\beta \in \mathbb{C}\) set

\[A^{q,\beta} = \hat{\Gamma}_q(-\beta)^{-1} \int \exp_q(-tA) t^{-\beta-1} d_q t,\] (59)
where the integration is carried out over some $q$–cycle.

This definition is a $q$–deformation of a quite standard definition of a complex power of a linear operator given by (30) and, as the next lemma shows, is necessary to take into account the relations within a quantum group. We also wish to point out that the symbol $A_{q,\beta}$ is slightly ambiguous: actually $A_{q,\beta}$ stands for a collection of operators, each of them being related to the choice of a contour of integration (see, for example, (ii) of the next Lemma.)

**Lemma 4.3** The following relations hold provided both sides of them make sense.

(i) $A_{q,\beta} = A_{q,\beta-n} \cdot A^n$, $k \in \mathbb{C}$, $n \in \mathbb{N}$.

(ii) (the $q$–multinomial theorem) Fix $m$, $r$: $1 \leq r \leq m$; if operators $\{A_j\}$ satisfy $A_jA_i = q^{2}A_iA_j$, $1 \leq i < j \leq m$, then provided a suitable choice of the contour has been made, one has

$$
(A_1 + \cdots + A_m)^{q,\beta} = \sum_{j=0}^{\infty} \sum_{j_1, \cdots, j_{m-1}} \frac{[\beta][\beta - 1] \cdots [\beta - j + 1]}{[j_1]! \cdots [j_{m-1}]!} A_1^{j_1} \cdots A_{r-1}^{j_{r-1}} A_r^{\beta-j} A_{r+1}^{j_r} \cdots A_m^{j_{m-1}}
$$

(iii) $K_iF_{q,\beta}^{\beta}K_i^{-1} = q^{-\alpha_{ij}\beta}F_j^{q,\beta}$;

(iv) $[E_i, F_j^{q,\beta}] = \delta_{i,j} \frac{q_j^{\beta} - q^{-\beta}}{q_j - q^{-1}} F_{q_j^{\beta}} - 1 K_{q_j^{\beta}}^{-1} - K_{q_j^{\beta-1}}^{-1} q_j^{\beta-1}$.

**Proof** is a matter of straightforward calculations. We sketch it briefly.

(i) follows from the relations:

$$
\partial_q(t^{-\mu}) = [\mu]t^{-\mu-1},
$$

$$
\partial_q(\exp_q(-tA)) = -A \exp_q(-tA),
$$

$$
\tilde{\Gamma}_q(-\mu + 1) = -[\mu] \tilde{\Gamma}_q(-\mu),
$$

and the integration by parts formula

$$
\int \exp_q(-tA) \partial_q(t^{-\mu}) d_qt = \int \partial_q \exp_q(-tA) t^{-\mu} d_qt,
$$

(this is the point where the “$q$–cycle condition” is used.)

(ii) follows from the case $m = 2$ of (52) and elementary manipulations with sums and Jackson integrals.
(iii) and (iv) are proved analogously. For example, (iv) follows from the termwise application of its integral analogue, which we have already cited several times:

\[ [E_i, F_j^n] = \delta_{ij} q_j^n - q_j^{-n} q_i - q_j^{-1} F_i^{n-1} K_j q_j^{-n+1} - K_i^{-1} q_j^{n-1} \frac{q_i - q_j^{-1}}{q_j - q_j^{-1}}, \quad n \in \mathbb{N}. \]

5. The main result. We proceed in a complete accordance with the analogous section devoted to classical correlation functions. Let \( \Psi \in (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m) \) be a correlation function coming from the composition of vertex operators

\[ \Phi_m(z_m) \circ \cdots \circ \Phi_1(z_1) : M_q(\lambda_1, k) \to M_q(\lambda_{m+1}, k) \otimes (V_1 \otimes \cdots \otimes V_m)(z_1, \ldots, z_m). \]

Let \( w = r_{i_1} \cdots r_{i_l} r_{i_1} \in W \) be a reduced decomposition. Set

\[ \beta_j = \frac{2(r_{i_{j-1}} \cdots r_{i_1} \cdot \lambda_1, \alpha_{i_j})}{(\alpha_{i_j}, \alpha_{i_j})}. \]

Set

\[ K_{\Psi, w}(t_1, t_2, \ldots, t_l) = \prod_{j=1}^l \tilde{\Gamma}_q(-\beta_j)^{-1} \times \{ \exp_q(-t_1 F_{i_1}) \cdots \exp_q(-t_l F_{i_l}) \prod_{j=1}^l t_j^{-\beta_j} \Psi \}. \]

Theorem 4.4 The integral

\[ \int \sigma K_{\Psi, w}(t_1, t_2, \ldots, t_l) d_q t_1 d_q t_2 \ldots d_q t_l \]

satisfies (if exists) the system \( qKZ(\lambda_{m+1}, w \cdot \lambda_1) \) (see (49,50)).

Proof is a literal repetition of that of Theorem 3.6, where instead of Lemmas 3.4 and 3.5 (iii), (iv), Lemmas 4.2 and 4.3 (iii), (iv) are used. \( \square \)

4.4 Explicit Formulas for \( U_q(\widehat{sl}_2) \)

In this section we carry out an explicit calculation of the integral (60).

1. \( q \)-difference operators realization of the algebra \( U_q(\widehat{sl}_2) \). The algebra \( U_q(\widehat{sl}_2) \) is a Hopf algebra on generators \( E = E_1, K = K_1, F = F_1 \) and relations (or definitions) (58 - 53), where \( A = (2) \). Direct calculations show that the following operators determine a representation of \( U_q(\widehat{sl}_2) \) in the space of functions in 1 variable:

\[ E : f(x) \mapsto -x \frac{q^{-2\lambda+1} f(q^2 x) - q^{2\lambda+1} f(x)}{q^2 - 1}; \]

\[ K : f(x) \mapsto q^{-2\lambda} f(q^2 x); \]

\[ E : f(x) \mapsto \frac{f(x) - f(q^{-2} x)}{(1 - q^{-2}) x}. \]
This action preserves the spaces of polynomials $C[x]$, Laurent polynomials $C[x, x^{-1}]$ and “twisted” Laurent polynomials $x^n C[x, x^{-1}]$. The last space produces a (generically irreducible) $V_q(\nu, \lambda)$-module. The last representation is almost a deformation of the $\mathfrak{sl}_2$-module $V(\nu, \lambda)$ as well as formulas (61 - 63) are almost deformations of the formulas (8 , 9): they would be actual deformations if the latter were composed with the canonical automorphism $E \mapsto F, F \mapsto E, H \mapsto -H$.

One also considers the $U_q(\widehat{\mathfrak{sl}}_2)$-module $V_q(\nu_1, \lambda_1) \otimes \cdots \otimes V_q(\nu_m, \lambda_m)(z_1, \ldots, z_m)$, the action being determined through the associative algebra homomorphism

$$U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)^{\otimes m},$$

$$E \mapsto \Delta^{m-1} E, \ F \mapsto \Delta^{m-1} F,$$

$$F_0 \mapsto E^{(1)} + \cdots + E^{(m)}, \ E_0 \mapsto F^{(1)} + \cdots + F^{(m)},$$

where

$$E^{(i)} = z_i^{-1} K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1, \quad i=1 \ldots m-1,$$

$$F^{(i)} = z_{m-i+1} 1 \otimes \cdots \otimes 1 \otimes E \otimes K \otimes \cdots \otimes K, \quad i=1 \ldots m.$$  

2. $q-$ exponent of a $q-$ difference operator. Recall that the crucial step in getting an explicit form of the classical integral representation (31) was evaluation of an exponent of an order 1 differential operator, which is actually a classical problem of the theory of ordinary differential equations. We are unaware of a general approach to evaluation of a $q-$ exponent of a $q-$ difference operator which arises in (60). However we here report on a straightforward calculation of the quantities

$$\exp_q(-tX) x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$

where $X = F$ or $F_0$ and $x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$ is regarded as an element of the module $V_q(\nu_1, \lambda_1) \otimes \cdots \otimes V_q(\nu_m, \lambda_m)(z_1, \ldots, z_m)$.

**Lemma 4.5** (i) If $x^\alpha \in V(\lambda, \nu)$ then

$$\exp_q(-tF)x^\alpha = x^\alpha \frac{(q^{-2\alpha+2} t/x)_\infty}{(q^2 t/x)_\infty}, \quad (64)$$

$$\exp_q(-tF_0)x^\alpha = x^\alpha \frac{(q^{-2\lambda+2\alpha+1} tx/z)_\infty}{(q^{2\lambda+1} tx/z)_\infty}. \quad (65)$$

(ii)

$$\exp_q(-tF)x_1^{a_1} \cdots x_m^{a_m} = x_1^{a_1} \cdots x_m^{a_m} \prod_{i=0}^{m-1} \frac{(q^{2s_i} t/x_{i+1})_\infty}{(q^{2s_i} t/x_{i+1})_\infty}. \quad (66)$$

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where
\[ s_i = \lambda_i + 2 + \cdots + \lambda_m - \alpha_i - \cdots - \alpha_m + 1, \]
\[ r_i = \lambda_i + 2 + \cdots + \lambda_m - \alpha_i - \cdots - \alpha_m + 1, \]
and
\[ \exp_q(-tF_n)x_1^{\alpha_1} \cdots x_m^{\alpha_m} = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \prod_{i=0}^{m-1} \frac{(q^{2i+1}t/x_i+1)^\infty}{(q^{2i+1}z_i+1)^\infty}, \] (67)
where
\[ \tilde{s}_i = \alpha_i + 1 + \cdots + \alpha_1 - \lambda_i + 1/2, \]
\[ \tilde{r}_i = \alpha_i + \cdots + \alpha_1 + \lambda_i + 1/2, \]

Proof. All assertions are proved by direct calculations. For example, to prove (64) one firstly observes, that for
\[ i \in N \quad F_n x_1^{\alpha_1} \cdots x_m^{\alpha_m} = (-1)^i q^2 \frac{(q^{-2\alpha})_i}{(1 - q^2)^i} x^{n-i}, \]
and then gets
\[ \exp_q(-tF)x^\alpha = x^\alpha \sum_{i=0}^{\infty} \frac{(q^{-2\alpha})_i}{(q^2)_i} (q^2t/x)^i \]
\[ = x^\alpha \frac{(q^{-2\alpha+1})_{2i+1} q t/x}{(q^{2i+1}z_i+1)^\infty}, \] (68)
the last equality following from the commutative version (55) of the $q-$binomial theorem.

(64) is proved in exactly the same way. As to (65), the only difference is that the formula for $F_n x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ or $F_0 x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ is a little bit more complicated. For example, since the elements $E_{ij}$ satisfy the skew polynomial condition:
\[ E^{(j)} E^{(i)} = q^2 E^{(i)} E^{(j)} \text{ if } i < j, \]

$F_n x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ can be expanded by (52). Further application of the commutative version of the $q-$binomial theorem (55) as above gives the required result (67).
\[ \Box \]

As an immediate consequence of Lemma 4.5 and the definition one gets
\[ F^{q,\mu}x^\alpha = x^{\alpha - \mu} q^{-2\mu - 2\alpha} \frac{(q^{-2\alpha})_\infty}{\Gamma_q(-\mu)} \frac{q^2 t/ \Gamma_q(-\mu)}{(q^{2\alpha+2})_\infty} d_q t = \]
\[ q^{2\mu + 2} \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\alpha + 1 - \mu)} x^{\alpha - 2\mu}, \] (69)
\[ F^{q, \mu, x_0^\alpha} = x^{\alpha + \mu} z^{-\mu} q^{-(2\lambda - 2\alpha + 1)} \frac{\Gamma(q^{-\mu - 1})}{\Gamma(q^{-\mu - 1})} \int_0^1 \frac{(q^2 t)^\infty}{(q^{1+2\alpha+2t})^\infty} dq t = q^{2\mu(\lambda - \alpha + 1)} \frac{\Gamma(q(2\lambda - \alpha + 1))}{\Gamma(q(2\lambda - \alpha - \mu + 1))} x^{\alpha + \mu} z^{-\mu}, \tag{70} \]

\[ F^{q, \mu, x_1^\alpha \cdots x_m^\alpha} = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \frac{1}{\Gamma(q(-\mu))} \int t^{-\mu - 1} \prod_{i=0}^{m-1} \frac{(q^{2^i} t/x_{i+1})^\infty}{(q^{2^{i+1}} t/x_{i+1})^\infty} dq t, \tag{71} \]

\[ F^{q, \mu, x_1^\alpha \cdots x_m^\alpha} = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \frac{1}{\Gamma(q(-\mu))} \int t^{-\mu - 1} \prod_{i=0}^{m-1} \frac{(q^{2^i} t/x_{i+1})^\infty}{(q^{2^{i+1}} t/x_{i+1})^\infty} dq t. \tag{72} \]

In integrals (71, 72) the integration is supposed to be carried out over a segment from 0 to a zero of the numerator. One easily sees that for each of the integrals there are \( m \) essentially different choices. (It is, of course, closely related to Lemma 1.3 (ii).) An integral of this type was called by K.Aomoto and K.Mimachi a Jackson integral of the Jordan-Pochhammer type. It is well-known (see, e.g. [13]) that if \( m = 2 \) these integrals produce the Gauss \( q \)-hypergeometric function.

3. Integral representations and series expansions for correlation functions. In view of the formula 51 one gets that the function

\[ \prod_{j=1}^{m} z_j^{-(\lambda_1 + \lambda_{m+1} + 2\rho) j/2(k+h^\vee)} x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m} / 2 \]

is a solution to (49, 50) related to the composition of vertex operators

\[ \Phi_m(z_m) \cdots \Phi_1(z_1) : M_q(\lambda_1) \rightarrow M_q(\lambda_{m+1}, k) \otimes (V(0, \mu_1) \otimes \cdots \otimes V(0, \mu_m)) (z_1, \ldots, z_m) \]

for \( \lambda_1 = \lambda_{m+1} + \mu_1 + \cdots + \mu_m \). Formulas (71, 72) combined with Theorem 4.4 give that the following functions also satisfy the system \( qKZ(\lambda_{m+1}, -\lambda_1 - 2) \) or \( qKZ(\lambda_{m+1}, 2k - \lambda_1 + 2) \) resp.:

\[ F^{q, \lambda_1+1} \prod_{j=1}^{m} z_j^{-(\lambda_1 + \lambda_{m+1} + 2\rho) j/2(k+h^\vee)} x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m} / 2 = \]

\[ \prod_{j=1}^{m} z_j^{-(\lambda_1 + \lambda_{m+1} + 2\rho) j/2(k+h^\vee)} x_1^{\mu_1} x_2^{\mu_2} \cdots x_m^{\mu_m} \frac{1}{\Gamma(q(-\lambda_1 - 1))} \times \]

\[ \int t^{-\lambda_1 - 2} \prod_{i=0}^{m-1} \frac{(q^{2^i} t/x_{i+1})^\infty}{(q^{2^{i+1}} t/x_{i+1})^\infty} dq t, \tag{73} \]

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\[ F_0^{q,k-\lambda_1+1} \prod_{j=1}^{m} z_j^{-(\lambda_1+\lambda_{m+1}+2\rho_j)/2(k+h')} x_1^{\mu_1/2} x_2^{\mu_2/2} \cdots x_m^{\mu_m/2} = \]
\[ \prod_{j=1}^{m} z_j^{-(\lambda_1+\lambda_{m+1}+2\rho_j)/2(k+h')} x_1^{2\mu_1} x_2^{2\mu_2} \cdots x_m^{2\mu_m} \frac{1}{\Gamma_q(-k+\lambda_1-1)} \times \]
\[ \int t^{-k+\lambda_1-2} \prod_{i=0}^{m-1} \frac{\left(q^{2s_i}(t(x_{i+1})/z_{i+1})\right)_\infty}{(q^{2r_i}(t(x_{i+1})/z_{i+1})\right)_\infty} \, dq \, t, \quad (74) \]

where
\[ s_i = \frac{1}{2}(-\mu_{i+1} + \mu_{i+2} + \cdots + \mu_m) + 1, \]
\[ r_i = \frac{1}{2}(\mu_{i+2} + \cdots + \mu_m) + 1, \]
\[ \tilde{s}_i = -\frac{1}{2}(\mu_{i+1} + \cdots + \mu_1 - 1), \]
\[ \tilde{r}_i = -\frac{1}{2}(\mu_i + \cdots + \mu_1 - 1) + \mu_{i+1}. \]

So, we have found solutions to \( q \)-KZ-equations in the form of Jackson integrals of the Jordan-Pochhammer type (see formally similar results in [22]).

In order to get explicitly solutions written as a repeated Jackson integral one has to evaluate a result of a repeated application of a \( q \)-exponent of \( F_1, F_0 \), which we are unfortunately unable to do at present. However our method provides a Laurent series expansion of such solutions. Really, if \( \Psi(z_1, \ldots, z_m) \) is a series expansion of a (not necessarily correlation) function then the functions
\[ F^{\mu,q}\Psi(z_1, \ldots, z_m), \ F_0^{\mu,q}\Psi(z_1, \ldots, z_m) \]
can be evaluated by
(i) expanding the sums
\[ F^{\mu,q} = \sum_{i=1}^{m} F^{(i)}^{\mu,q}, \]
\[ F_0^{\mu,q} = \sum_{i=1}^{m} F_0^{(i)}^{\mu,q}, \]
using the \( q \)-binomial theorem (Lemma 4.3 (ii));
(ii) further term-wise application of these expansions to the series \( \Psi(z_1, \ldots, z_m) \) using formulas \([18, 70]\).

It is easy to see that starting with \( \Psi(z_1, \ldots, z_m) \) polynomial in \( x \)'s this process can be iterated arbitrary number of times, each time giving a converging series.
4.5 Complex Powers of Lie algebra (Quantum Group) generators and Special Functions

1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $G$ be the corresponding group and $\Gamma(\tilde{F}^0, O(L, \lambda_i))$, $1 \leq i \leq m$ be $\mathfrak{g}$–modules of the type we were utilizing in sect. 3.2. Recall that the space $\Gamma(\tilde{F}^0, O(L, \lambda))$ can (and will) be identified with the space of multi-valued functions (depending on the local system $L$), the action of $\mathfrak{g}$ being given by the same 1st order differential operators, which produces the realization of the contragredient Verma module with the highest weight $\lambda$ in (meromorphic) functions on the flag manifold. Also recall that the group elements $g \in G$ as well as complex powers Lie algebra elements can be regarded as operators acting from $\Gamma(\tilde{F}^0, O(L, \lambda))$ to $\Gamma(\tilde{F}^0, O(L', \lambda))$ with some other $L'$ and the same $\lambda$. Since we do not want here to keep track of the local system, instead of $\Gamma(\tilde{F}^0, O(L, \lambda))$ we will be writing $M(\lambda)$ and slightly abusing notation we will be saying that for each $g \in G, X \in \mathfrak{g}, \mu \in \mathbb{C}$ a pair of operators is defined $g, X^\alpha : M(\lambda) \rightarrow M(\lambda)$.

Let $w_0$ be the element of the Weyl group of the maximal length and denote by the same letter its preimage in $G$. Obviously for any $\lambda$ as a constant function on $F$ is the vacuum vector of the module $\otimes_{i=1}^m M(\lambda_i)$ (here the local systems are supposed to be trivial). It follows that the function $w_0$ is the lowest weight vector (i.e. it is annihilated by $n_-$). Therefore $w_0$ is an eigenvector of any Casimir operator of $\mathfrak{g}$. It follows that for any complex number $\alpha$ and any Cartan generator $E_i$ the function $E_i^\alpha w_0$ is also. To be definite, let $\Omega_0, \ldots, \Omega_n$ be a complete set of Casimir operators ($n+1$ is the rank of $\mathfrak{g}$) and let $\Omega_i w_0 = \mu_i w_0$. Then one obtains that the function $E_i^\alpha w_0$ satisfies the following system of differential equations:

$$\Omega_j f = \mu_j f, \ 0 \leq j \leq n.$$ (75)

Observe that the function $E_i^\alpha w_0$ is homogeneous and therefore can be identified with a function of a less number of variables. Under this identification the system (75) is rewritten in the form explicitly depending on $\alpha$. Observe that in all cases, except $\mathfrak{g} = \mathfrak{sl}_2$, the number of equations in (75) is less then the number of variables and one may be interested in finding all other equations which $E_i^\alpha w_0$ satisfies.

2. Example: $\mathfrak{g} = \mathfrak{sl}_2$. In this case $\lambda \in \mathbb{C}$ and $\mathcal{M}(\lambda)$ is understood as $\{ f(x) dx^{-\lambda/2} \}$ for all $f(x)$ from a suitable family of functions on $\mathbb{C}$. The Cartan generators are represented as follows

$$E = -\frac{d}{dx}, \ H = -2x \frac{d}{dx} + \lambda, \ F = x^2 \frac{d}{dx} - \lambda x.$$ 

The above construction of the lowest weight vector in $\otimes_{i=1}^m \mathcal{M}(\lambda_i)$ gives $x_1^{\lambda_1} \cdots x_m^{\lambda_m}$. This is a homogeneous function of $x_1, \ldots, x_m$ of homogeneous degree $\lambda_1 + \ldots + \lambda_m$. 

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\[ \cdots + \lambda_m. \] We identify a homogeneous function \( f(x_1, \ldots, x_m) \) with the non-homogeneous function of \( m - 1 \) variables \( \tilde{f}(t_2, \ldots, t_m) : \tilde{f}(t_2, \ldots, t_m) = f(1, t_2, \ldots, t_m). \) Under this identification differential operators acting on homogeneous functions of a fixed degree are uniquely transformed into differential operators of less by 1 number of variables. If for example, \( m = 2 \) then operators of multiplication \( \text{mult}(x_1), \text{mult}(x_2) \) and differentiation \( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \) as operators acting on functions of degree \( \beta \) are transformed as follows:

\[
\text{mult}(x_2) \rightarrow \text{mult}(t), \quad \frac{\partial}{\partial x_2} \rightarrow \frac{d}{dt}, \quad \text{mult}(x_1) \rightarrow \text{id}, \quad \frac{\partial}{\partial x_1} \rightarrow \beta - t \frac{d}{dt}.
\]

For \( \mathfrak{sl}_2 \) there is essentially 1 Casimir operator \( \Omega = EF + FE + \frac{1}{2}H^2 \) and one can easily write down the equation (75) in this case. Before doing this we remark that the following additional simplification appears here. The element \( \Omega \) may be regarded as a \( \mathfrak{sl}_2 \)-invariant element of the symmetric square \( S^2 \mathfrak{sl}_2 \). The latter algebra acts on a tensor product of a pair of \( \mathfrak{sl}_2 \)-modules by the formula simpler than that for \( U(\mathfrak{sl}_2) \):

\[
X \otimes Xv \otimes w = Xv \otimes Xw.
\]

This gives another action of \( \Omega \) and arguments which led to (75) apply to this action as well. Taking into account all this one finds that the function \( F^{\alpha} x_1^{\lambda_1} x_2^{\lambda_2}(t) \) satisfies the classical hypergeometric equation

\[
t(t - 1)x(t)^{\alpha} + ((1 - \lambda_1 - 2\lambda_2 + \alpha)t - \lambda_1 - \lambda_2 + \alpha - 1)x(t)' + \lambda_2(\lambda_1 + \lambda_2 - \alpha)x(t) = 0,
\]

which is by no means surprise in view of sect. 3.3.

3. Example: \( U_q(\mathfrak{sl}_2) \). Of course all the arguments which led to (75) apply to the case of a “finite-dimensional” quantum group and given its realization by \( q \)-difference operators one obtains a system of \( q \)-difference equations on a certain function. Here we are able to carry out the necessary calculations for \( U_q(\mathfrak{sl}_2) \). In this case we set \( \mathcal{M}_q(\lambda) \) for all \( f(x) \) to be a suitable family of functions on \( \mathbb{C} \), action of \( U_q(\mathfrak{sl}_2) \) being given by (61 - 63). Relations (61 - 63) imply that \( x^{2\lambda} \in \mathcal{M}_q(\lambda) \) is the highest weight vector. The Casimir operator for \( U_q(\mathfrak{sl}_2) \) is given by

\[
\Omega = FE + \frac{Kq + K^{-1}q^{-1}}{(q - q^{-1})^2}.
\]

As above one finds that the function \( F^{\alpha} x_1^{2\lambda_1} x_2^{2\lambda_2} \), which is understood as an element of the module \( \mathcal{M}_q(\lambda_1) \otimes \mathcal{M}_q(\lambda_2) \), satisfies

\[
\Omega f(x_1, x_2) = \frac{q^{2(\lambda_1 + \lambda_2) + 1} + q^{-2(\lambda_1 - \lambda_2) + 1}}{(q - q^{-1})^2} f(x, y).
\]
Again using the fact that the function in question is homogeneous one makes it into a function in 1 variable, the \( q \)- difference operator in 2 variables representing \( \Omega \) being transformed in an obvious way into a \( q \)-difference operator in 1 variable. Finally what one obtains is the usual \( q \)-hypergeometric equation, which is again no surprise in view of the formula \((71)\).

References

[1] Awata H., Yamada Y., Fusion rules for the Fractional Level \( \widehat{\mathfrak{sl}}(2) \) Algebra, KEK-TH-316 KEK Preprint 91-209, January 1992

[2] Askey R., The \( q \)-gamma and \( q \)-beta functions, Applicable Analysis 8 (1988) 125 - 141

[3] Aomoto K., A note on holonomic \( q \)-difference system., Algebraic Analysis I, ed. by M.Kashiwara and T.Kawai Acad. Press (1988) 25 - 28

[4] Beilinson A.A., Schechtman V.V., Determinant bundles and Virasoro algebras., CMP 118 (1988) 651 - 701

[5] Bernstein I.N., Gelfand I.M., Gelfand S.I., Differential operators on the base affine space and a study of \( g \)-modules, Proceedings of the Summer School on Group Representations (I.M.Gelfand, ed.), Bolyai Janos Mathematical Society (Budapest 1971) 39-69. London: Hilger 1975

[6] Bowknegt P., McCarthy J., Pilch K., Free field approach to 2-dimensional conformal field theory., Progress of Theoretical Physics, Supplement No.102 70 (1988) 67-135

[7] De Concini C., Kac V.G., Representations of quantum groups at roots of 1, Progr. in Math. 92 (1990) 471-506

[8] Demazure M., Desingularisation des varietes de Schubert generalisees, Ann.Scient.Ec.Norm.Sup. 4e serie 7 (1974) 53 - 88

[9] Feigin B., Frenkel E., The family of representations of affine Lie algebras, Usp.Math.Nauk (=Russ.Math.Surv.) 43 (1988) 227 - 228 ( in Russian )

[10] Feigin B., Frenkel E., Affine Kac-Moody algebras and semi-infinite flag manifold, CMP. 128 (1990) 2 161 - 189

[11] Furlan P., Ganchev A.Ch., Paunov R., Petkova V.B., Phys.Letters 267 (1991) 63; Solutions of the Knizhnik-Zamolodchikov equations with rational isospins and the reduction to the minimal models, preprint CERN-TH.6289/91, accepted for publication in Nucl.Phys.B
[12] Frenkel I., Reshetikhin N., Quantum affine algebras, commutative systems of difference equations and elliptic solutions to the Yang-Baxter equation, in Proc. of “XX International conference on Differential Geometric Methods in Theoretical Physics” (NY June 6-10, 1991) WS, 1992; Quantum affine algebras and holonomic difference equations, CMP, to appear

[13] Gasper G., Rahman M., Basic hypergeometric series, Encyclopedia of mathematics and its applications 31, Cambridge University Press, 1990

[14] Jimbo M., A $q$–analog of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, Lett. in Math. Phys. 11 (1986) 247-252

[15] Kac V.G. Infinite-dimensional Lie algebras, Boston MA: Birkhauser 1983

[16] Kac V.G., Kazhdan D.A., Structure of representations with highest weight of infinite-dimensional Lie algebras, Adv. Math., 34 (1979), 97-108

[17] Knizhnik V., Zamolodchikov A., Current algebra and Wess-Zumino model in 2 dimensions, Nucl. Phys. B 247 (1984) 83 - 103

[18] Kumar S., Demazure character formula in arbitrary Kac-Moody setting, Inv. Math. 89 (1987) 395 - 423

[19] Lusztig G., Quantum deformations of certain simple modules over enveloping algebras., Adv. Math. 70 (1988) 237-249

[20] Malikov F.G., Feigin B.L., Fuchs D.B., Singular vectors in Verma modules over Kac-Moody algebras, Funkc. Anal. i ego Pril. 20 (1988) 2, 25-37

[21] Malikov F., Quantum groups: singular vectors and BGG resolution, Infinite Analysis - Proceedings of the RIMS Research Project 1991 Part B, 623 - 645, World Scientific Co. Pte. Ltd.

[22] Matsuo A., Quantum algebra structure of certain Jackson integrals, preprint 1992

[23] Mimachi K., Connection problem in holonomic $q$–difference system associated with a Jackson integral of Jordan-Pochhammer type, Nagoya Math. J. 116 (1989) 149 - 161

[24] Pressley A., Segal G., Loop groups, Clarendon Press - Oxford 1988

[25] Reshetikhin N., Jackson-type integrals, Bethe vectors and solutions to a difference analog of the Knizhnik-Zamolodchikov system, preprint, 1992

[26] Schechtman V., Varchenko A., Integral representation of $N$–point correlators in the WZW model, preprint, Max-Planck-Institut fur Mathematik MPI/89-51, 1989
[27] Schechtman V., Varchenko A., *Quantum groups and homology of local systems*, ICM-90, Proceedings of the Satellite Conference “Algebraic Geometry and Analytic Geometry” (Tokyo 1990), Springer, 1991, 182 - 191

[28] Voronov A.A. *A unified approach to string scattering amplitudes*, CMP 131 (1990) 179 - 218; Erratum: 140 (1991) 415 - 416