Packing and Covering with Non-Piercing Regions

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Abstract In this paper, we design the first polynomial time approximation schemes for the Set Cover and Dominating Set problems when the underlying sets are non-piercing regions (which include pseudodisks). Earlier, PTASs were known only in the setting where the regions were disks. These techniques relied heavily on the circularity of the disks. We develop new techniques to show that a simple local search algorithm yields a PTAS for the problems on non-piercing regions. We then consider the Capacitated Region Packing problem. Here, the input consists of a set of points with capacities, and a set of regions. The objective is to pick a maximum cardinality subset of regions so that no point is covered by more regions than its capacity. We show that this...
problem admits a PTAS when the regions are $k$-admissible regions (pseudodisks are 2-admissible), and the capacities are bounded by some constant. Our result settles a conjecture of Har-Peled from 2014 in the affirmative. The conjecture was for a weaker version of the problem, namely when the regions are pseudodisks, the capacities are uniform, and the point set consists of all points in the plane. Finally, we consider the Capacitated Point Packing problem. In this setting, the regions have capacities, and our objective is to find a maximum cardinality subset of points such that no region has more points than its capacity. We show that this problem admits a PTAS when the capacity is unity. This extends a result of Ene et al. from 2012.

**Keywords** Local search · Set cover · Dominating set · Capacitated packing · Approximation algorithms

**Mathematics Subject Classification** 52C15 · 05C65 · 05C69 · 68Q25 · 68W25 · 68W05

### 1 Introduction

Geometric packing and covering problems have received wide attention in the last decade, especially in the context of approximation algorithms. Besides the inherent aesthetic appeal, the interest in the geometric setting arises from the fact that in many applications, the packing and covering problems involve geometric objects. For example, see [3,4,14,24,30]. Several tools and techniques have been developed for this purpose, but for many fundamental problems there are still large gaps between the known approximation factors, and the existing hardness results.

Classic techniques for solving packing and covering problems rely on grid-shifting techniques introduced by Hochbaum and Maass [23], and extensions by Erlebach et al. [17] and Chan [10]. All these algorithms are restricted to the setting where the regions are fat. Recent progress has been based mainly on two paradigms. The first is algorithms that use LP rounding [7,9,12,18], and the other is *Local Search* (albeit only in the unweighted setting). Local Search has been used to obtain PTASs\(^1\) for several problems besides packing and covering. For example, see [5,12,19,20,28] and [8,13] for more recent work.

Chan and Har-Peled [12], and Mustafa and Ray [28] obtained PTASs for the Independent Set and Hitting Set problems respectively, via local search when the underlying regions are non-piercing\(^2\). Non-piercing regions are topologically defined and generalize regions like disks, homothets of convex objects, unit height rectangles, arbitrary sized squares, etc. In contrast, for the Set Cover and Dominating Set problems, PTASs exist only when the underlying regions are disks [5,20,28]. Since these are natural and important problems, there have been attempts to extend these results to more general settings. The main difficulty is that the analysis for the case of disks relies heavily on

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\(^1\) A polynomial time \((1 + \epsilon)\)-approximation algorithm for any \(\epsilon > 0\).

\(^2\) A set of simply connected regions is said to be non-piercing if for any pair \(A, B\) of regions, the sets \(A \setminus B\) and \(B \setminus A\) are connected.
the geometry. Durocher and Fraser [15] showed the existence of a PTAS for the Set Cover problem when the regions are pseudodisks satisfying a cover-free condition by dualizing and converting the problem to a Hitting Set problem. Further, they showed that the approach of dualizing the problem cannot be extended to work for a general family of pseudodisks.

In this paper, we develop new techniques to analyze the local search algorithm for problems when the underlying regions are non-piercing. These techniques lead to the first PTAS for the unweighted Set Cover and Dominating Set problems when the underlying regions are non-piercing. In the weighted setting, Chan et al. [11] building on the work of Varadarajan [31] obtained $O(1)$-approximation algorithms for the Set Cover and Dominating Set problems for non-piercing regions with low union complexity. For the weighted Set Cover problem, the current best result is a QPTAS$^3$ of Mustafa et al. [27] that extends the technique of Adamaszek and Wiese [1,2] (which obtains a QPTAS for the Independent Set problem for polygons).

We also develop new techniques for obtaining a PTAS for the Capacitated Region packing problem when the capacities are bounded by a constant and the regions are $k$-admissible$^4$. This result proves a conjecture of Har-Peled [21]. We also consider the dual problem, namely Capacitated Point Packing for non-piercing regions. We show that it admits a PTAS using local search for the special case when the capacities are 1, extending a result of Ene et al. [16], who obtained a PTAS for Capacitated Point Packing for disks with unit capacity in the plane.

2 Preliminaries

Two compact, simply connected regions $A$, $B$ are said to be non-piercing if both $A \setminus B$ and $B \setminus A$ are connected. A set $\mathcal{X}$ of compact, simply connected regions is non-piercing if the regions in $\mathcal{X}$ are pairwise non-piercing. For a region $A$, let $\partial(A)$ denote the boundary of $A$. We assume $\partial(A)$ is oriented counter-clockwise. The boundary divides the plane into two regions the interior of $A$, denoted $\text{int}(A)$, and the exterior of $A$, denoted $\text{ext}(A)$. By the orientation of $\partial(A)$, at any point on it, $\text{int}(A)$ lies to the left of $\partial(A)$. We further assume that any pair of regions in $\mathcal{X}$ intersect properly by which we mean that for any two regions $A$, $B$, $\partial(A) \cap \partial(B)$ consists of a finite set of points, and at each of these points the boundaries of $A$ and $B$ cross (i.e., no tangential intersections are allowed). In this paper, when we use the term region, we implicitly mean that the region is compact and simply connected, and a set of regions is assumed to be properly intersecting.

In some of the applications considered in this paper, we are given a set $\mathcal{R}$ of regions as well as a set $P$ of points in $\mathbb{R}^2$. In this case, we assume that the regions in $\mathcal{R}$ intersect properly, and no point in $P$ lies on the boundary of any of the regions in $\mathcal{R}$, i.e., $\exists \varepsilon > 0$ s.t. each point $p \in P$ is at a distance at least $\varepsilon$ from the boundary of any region $R \in \mathcal{R}$.

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$^3$ A QPTAS is a $(1 + \varepsilon)$-approximation algorithm whose running time is $O(n^{\text{polylog}(n)})$, where $n$ is the input size.

$^4$ $k$-Admissible regions are non-piercing regions whose boundaries intersect at most $k$ times.
3 Our Results

In this section, we define the problems studied in the paper. For each problem, we describe our results, and state related work. For all the problems studied, we show that a standard local search algorithm yields a PTAS. In the rest of the paper, when we say the “local search algorithm”, we implicitly refer to the standard local search algorithm which will be presented in detail in Sect. 5.

Set Cover. Given a finite set of non-piercing regions $X$, and a set of points $P \subset \bigcup X$, compute $Y \subseteq X$ of smallest cardinality such that $P \subset \bigcup Y$.

Note that when the regions are disks, a PTAS for this problem follows from a PTAS for the Hitting-Set problem of halfspaces in $\mathbb{R}^3$ [28] via lifting. However, this technique does not generalize even for pseudodisks. An appealing approach is to try to dualize the problem, and use results for the Hitting Set problem. However, as Durocher and Frazer [15] observed, such a dual does not exist. Currently, the best approximation algorithm is a QPTAS given by Mustafa et al. [27] that works even in the weighted setting. In the unweighted setting, the work of Har-Peled and Quanrud [22] implies a PTAS for the above problem under the assumption that the regions are fat, and that no point is contained in more than a constant number of regions. We obtain a PTAS without these assumptions but with the requirement that the regions are non-piercing. Our work thus complements the work of Har-Peled and Quanrud [22].

**Theorem 3.1** The Local Search algorithm yields a PTAS for the geometric Set Cover problem when the regions are non-piercing.

Dominating Set. Given a finite set of non-piercing regions $X$, find a subset $Y \subseteq X$ of smallest cardinality such that for each $X \in X$, there exists $Y \in Y$ so that $Y \cap X \neq \emptyset$.

Gibson and Pirwani [20] gave a PTAS for this problem, via local search when the regions are disks. However, their proof uses power diagrams [6] which strongly rely on the fact that the regions are circular disks. It is not clear how their result can be generalized to the setting of non-piercing regions, or even pseudodisks. Har-Peled and Quanrud [22] proved that Local Search yields a PTAS for low-density regions, where a set of regions is of low-density if no region is intersected by more than a constant number of regions of size larger than itself. In particular, this implies that the intersection graph has cliques of at most a bounded size. However, even in the case of fat regions, it is not clear how their technique can be applied since the intersection graph of such regions can have cliques of arbitrary size. In case of non-piercing regions too, intersection graphs can have cliques of arbitrary size, and furthermore there is no natural notion of the size of a region. In Sect. 6.1, we prove the following:

**Theorem 3.2** The Local Search algorithm yields a PTAS for the Dominating Set problem in the intersection graph of non-piercing regions.

Capacitated Region Packing. We are given a finite set of non-piercing regions $X$, a set of points $P$ with each point having a capacity at most $\ell$, where $\ell$ is a positive integer constant. The Capacitated Region Packing problem is to compute the largest cardinality set $Y \subseteq X$ such that for each point $p \in P$, the number of regions of $Y$ that contain $p$ is at most its capacity.
Currently, the best algorithm for this problem is by Ene et al. [16] that is an $O(1)$-approximation for arbitrary capacities, extending the work of Chan and Har-Peled [12]. This algorithm works even in the setting where the regions have weights and have linear union complexity. Aschner et al. [5] gave PTAS for fat regions, Har-Peled [21] gave a QPTAS for family of pseudodisks. The results in [5,21] are for the special case where $P = \mathbb{R}^2$. Har-Peled [21] conjectured that a PTAS exists for this problem. Indeed, we obtain a PTAS for the more general problem. In Sect. 6.2 we prove the following theorem.

**Theorem 3.3** The Local Search algorithm yields a PTAS for the Capacitated Region Packing problem when the regions are $k$-admissible for $k = O(1)$, and the capacities of the points are bounded above by a constant $\ell$.

**Capacitated Point Packing.** Given a finite set of non-piercing regions $X$, with a constant capacity $\ell$ and a set of points $P$, compute the largest cardinality set $Q \subseteq P$ such that each region in $X$ contains at most $\ell$ points of $Q$.

Ene et al. [16] gave $O(1)$-approximation algorithms for disks in the plane with arbitrary capacities. For unit capacities, they showed a PTAS for halfspaces in $\mathbb{R}^3$ and disks in the plane. To the best of our knowledge, there is no known $O(1)$-approximation algorithm for pseudodisks, even for unit capacity. We show that the problem admits a PTAS when the regions have unit capacity. In Sect. 6.3 we prove the following theorem.

**Theorem 3.4** The Local Search algorithm yields a PTAS for the Capacitated Point Packing problem for non-piercing regions when the regions have unit capacity.

In the next section, we define the notion of lens-bypassing, and use this to build graphs with a small separator for the Set Cover and the Dominating Set problems thus obtaining a PTAS for these problems. Lens-bypassing is a finer tool than the operation used for constructing a core decomposition in [27], and allows us to simplify one intersection at a time, instead of all intersections with one region at a time. This technique may be of independent interest. For the Capacitated Region Packing problem, we argue that the natural intersection graph relevant to the problem has a small separator by comparing it to a planar graph on the points.

**4 Lens-Bypassing**

In this section, we describe the lens-bypassing operation. Given a set of non-piercing regions, the lens-bypassing operation enables us to simplify the given arrangement so that the resulting simplified arrangement still consists of non-piercing regions, and retains some desired properties (described below) of the initial arrangement.

For two regions $A$ and $B$, each connected component of $A \cap B$ bounded by two arcs, one from the boundary of $A$ and the other from that of $B$ is called a lens. Since the boundary of any region is oriented counter-clockwise, observe that the arcs from $A$ and $B$ forming the boundary of a lens are oriented in opposite directions in the following sense. If $p$ and $q$ are the two intersection points of $\partial(A)$ and $\partial(B)$ on the
boundary of the lens then one arc is oriented from $p$ to $q$ and the other one is oriented from $q$ to $p$.

Let $L_{AB}$ denote the set of lenses formed by the intersection of regions $A$ and $B$. For any $r > 0$, let $D_r$ denote a ball of radius $r$ centered at the origin and let $\oplus$ denote the Minkowski sum. Let $\beta > 0$ be sufficiently small so that for all $\beta' \in (0, \beta]$, and all $\ell_{AB} \in L_{AB}$, the curve $\partial(\ell_{AB} \oplus D_{\beta'})$ does not pass through the intersection of the boundaries of any pair of regions and does not have tangential intersections with the boundary of any region.

We define a lens-bypassing for a lens $\ell_{AB} \in L_{AB}$ formed by $A$ and $B$ in favor of $B$ as follows: leaving $B$ as it is, we modify the boundary of $A$ to follow the boundary of $B$ along the arc of $B$ bounding $\ell_{AB}$, at a sufficiently small distance $\beta > 0$ (chosen as above) away from this arc. More formally, it is the operation of replacing $A$ by $A' = A \setminus (\ell_{AB} \oplus D_{\beta})$. Figures 1 and 2 show the operation of lens-bypassing for a pair of non-piercing regions.

**Remark** Our goal in lens-bypassing is to preserve the union of the regions while simplifying the arrangement. For this, we need $\beta = 0$ in the definition of lens-bypassing. However, this would imply that $A'$ and $B$ share a portion of their boundaries, and thus no longer intersect properly. To avoid this technical complication, we take $\beta$ to be a sufficiently small positive quantity. However, for the rest of the paper, we do not make this distinction and state our results as if $\beta = 0$ for better readability. For instance, we say that lens-bypassing does not change the union of the regions. This is to be understood as: $A' \cup B$ contains all points in $A \cup B$ that are at least a distance $\delta > 0$ away from $\partial(A)$ and $\partial(B)$. Here, we assume that $\beta < \delta$ for an arbitrarily small positive quantity $\delta$.  

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**Fig. 1** The figure shows lenses in the arrangement of non-piercing regions.

**Fig. 2** The figure shows the operation of bypassing lens $\ell_{AB}$ in favor of $B$. 

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Proposition 4.1 Let $A$ and $B$ be two regions in $\mathbb{R}^2$. Let $\ell_{AB}$ be a lens formed by $A$ and $B$. Let $A'$ be the modified version of $A$ obtained after bypassing the lens $\ell_{AB}$ in favor of $B$. If $A$ and $B$ are non-piercing then $A'$ and $B$ are non-piercing.

Proof Assume that $A$ and $B$ are non-piercing. Then $A\setminus B = A'\setminus B$ and so is connected. Now, $B'\setminus A' = (B\setminus A) \cup \ell_{AB}$. Since $B\setminus A$ and $\ell_{AB}$ are connected and share a boundary, their union is also connected. We conclude that $A'$ and $B$ are non-piercing. $\square$

For two regions $A$ and $B$, let $X(A, B)$ denote the intersection points of $\partial(A)$ and $\partial(B)$. Let $\sigma_{AB}$ denote the cyclic sequence of $X(A, B)$ along $\partial(A)$, i.e., walking in counter-clockwise order along $\partial(A)$. Similarly, let $\sigma_{BA}$ denote the cyclic sequence of the intersection points $X(A, B)$ along $\partial(B)$. For two points $x, y$ on $\partial(A)$, we let $\gamma_{xy}(A)$ denote the arc on $\partial(A)$ from $x$ to $y$ in counter-clockwise direction along $\partial(A)$. We simply write $\gamma_{xy}$, when the region $A$ is clear from the context. Two cyclic sequences $\sigma$ and $\sigma'$ on the same set of elements are said to be reverse-cyclic if $\sigma$ can be obtained from $\sigma'$ by reversing the order. For example $x_1, x_2, x_3, x_4, x_1$ and $x_4, x_3, x_2, x_1, x_4$ are reverse-cyclic. For a cyclic sequence $\sigma$, we say $x$ precedes $y$ in $\sigma$, or $x \prec_\sigma y$ if $x$ immediately precedes $y$ in the cyclic sequence $\sigma$.

For a pair of reverse-cyclic sequences $\sigma, \sigma'$, we define a lens formed by the sequences as the pair of elements $\{x, y\}$ that appear consecutively in both $\sigma$ and $\sigma'$ but in reverse order; i.e., $x \prec_\sigma y$ and $y \prec_{\sigma'} x$ or vice versa. Bypassing the lens $\{x, y\}$ is the operation of removing $x$ and $y$ from both $\sigma$ and $\sigma'$, i.e., we obtain modified sequences $\pi = \sigma \setminus \{x, y\}$ and $\pi' = \sigma' \setminus \{x, y\}$.

For regions $A, B$ consider the cyclic sequences $\sigma_{AB}$ and $\sigma_{BA}$. If $x, y$ form a lens in the sequences $\sigma_{AB}$ and $\sigma_{BA}$, it is easy to see that the arcs of $\partial(A)$ and $\partial(B)$ between points $x$ and $y$ in $X(A, B)$ form a lens of the regions $A$ and $B$. If $x \prec_{\sigma_{AB}} y$ and $x \prec_{\sigma_{BA}} y$, then the region bounded by the arcs of $\partial(A)$ and $\partial(B)$ between points $x$ and $y$ forms a region that is contained in either $A\setminus B$ or $B\setminus A$.

We will show that the pair of cyclic sequences $\sigma_{AB}$ and $\sigma_{BA}$ of the intersection points of regions $A$ and $B$ are reverse-cyclic if and only if the regions $A, B$ are non-piercing. Further, a lens formed by the sequences $\sigma_{AB}$ and $\sigma_{BA}$ corresponds to a lens in the intersection of $A$ and $B$, and bypassing a lens in $\sigma_{AB}$ and $\sigma_{BA}$ is the operation of lens-bypassing in $A$ and $B$. The following proposition is intuitively clear and we skip the easy proof.

**Proposition 4.2** If $\sigma, \sigma'$ are two reverse-cyclic sequences on the same set $X$ of elements, then

1. For a lens in the sequences $\sigma, \sigma'$ formed by the pair of elements $\{x, y\}$, bypassing the lens leaves the sequences reverse-cyclic, i.e., the sequences $\pi = \sigma \setminus \{x, y\}$ and $\pi' = \sigma' \setminus \{x, y\}$, are reverse-cyclic.
2. Let $\pi$ and $\pi'$ be obtained from $\sigma$ and $\sigma'$ respectively, by adding elements $x, y \notin X$ between the same pair of consecutive elements in $\sigma$ and $\sigma'$ such that $x \prec_\pi y$ and $y \prec_{\pi'} x$. Then $\pi$ and $\pi'$ are reverse-cyclic.

We now give a combinatorial characterization of non-piercing regions that will be useful for the rest of the paper.
Theorem 4.3 Two regions $A$ and $B$ in $\mathbb{R}^2$ are non-piercing if and only if $\sigma_{AB}$ and $\sigma_{BA}$ are reverse-cyclic.

Proof We prove both directions by induction on $|X(A, B)|$.

First, if $A$ and $B$ are non-piercing, we show that $\sigma_{AB}$ and $\sigma_{BA}$ are reverse-cyclic. The base case is when $|X(A, B)|$ is either 0 or 2 in which case the sequences $\sigma_{AB}$ and $\sigma_{BA}$ are reverse-cyclic. Assume inductively that the theorem holds when $|X(A, B)| < k$, for some $k$. Given two regions $A$, $B$ with $|X(A, B)| = k > 2$, consider $S = \partial(B) \cap A$, the set of chords in $A$ formed by $\partial B$. Since the chords in $S$ are pairwise non-intersecting, there exist two chords so that their end-points are adjacent in both $\partial(A)$, and $\partial(B)$. Suppose that the first chord has end-points $x$ and $y$ and the second chord has end-points $u$ and $v$. Let $R_{xy}$ be the region bounded by $\partial(A)$ and $\partial(B)$ between $x$ and $y$. $R_{uv}$ is defined analogously. Note that if $\partial(A)$ is oriented from $x$ to $y$ and $\partial(B)$ is oriented from $y$ to $x$ (or vice versa) then $R_{xy}$ is a lens formed by $A$ and $B$ and otherwise $R_{xy}$ lies in $A \setminus B$. The same holds for $R_{uv}$. If both $R_{xy}$ and $R_{uv}$ are in $A \setminus B$ then we get a contradiction since they form disconnected components of $A \setminus B$ which we assume is connected. See Fig. 3.

Therefore, at least one of $R_{xy}$ or $R_{uv}$ is a lens formed by $A$ and $B$. Assume without loss of generality that $R_{xy}$ is a lens.

Suppose we bypass the lens $R_{xy}$ in favor of $B$, then by Proposition 4.1 regions $A'$ (obtained by modifying $A$) and $B$ are non-piercing, and $|X(A', B)| = |X(A, B)| - 2$. Figure 4 shows this operation. By the inductive hypothesis, the two sequences $\sigma_{A'B}$ and $\sigma_{BA'}$ are reverse-cyclic. Since $\sigma_{AB}$ and $\sigma_{BA}$ are obtained by adding the pair $\{x, y\}$ in opposite order between the same pair of elements in $\sigma_{A'B}$ and $\sigma_{BA'}$ respectively, we conclude that $\sigma_{AB}$ and $\sigma_{BA}$ are reverse cyclic.

We now show that if $\sigma_{AB}$ and $\sigma_{BA}$ are reverse-cyclic then $A$ and $B$ are non-piercing. We again use induction on $|X(A, B)|$. If $|X(A, B)| = 0$, the regions are disjoint and are therefore non-piercing. Suppose that the theorem is true for $|X(A, B)| < k$. Let $A$, $B$ be two regions with $|X(A, B)| = k$ and such that $\sigma_{AB}$ and $\sigma_{BA}$ are reverse-cyclic. Suppose that we bypass a lens $\ell$ formed by $A$ and $B$ in favor of $B$. Let $x$ and $y$ be the intersection points of $\partial(A)$ and $\partial(B)$ on $\partial(\ell)$. Note that $x$ and $y$ are adjacent in both $\sigma_{AB}$ and $\sigma_{BA}$. Let $A'$ be the region obtained by modifying $A$ as a result of the lens-bypassing.
Then $\sigma_{A'B}$ and $\sigma_{BA'}$ are reverse-cyclic by Proposition 4.2, and $|X(A', B)| = |X(A, B)| - 2$. By the inductive hypothesis, $A'$ and $B$ are non-piercing. Since $A \setminus B = A' \setminus B$ and $B \setminus A = B' \setminus A'$, both $A \setminus B$ and $B \setminus A$ are connected. Thus, the regions $A$ and $B$ are non-piercing.

**Corollary 4.4** For two non-piercing regions $A$ and $B$ if there exist $x$ and $y$ such that $x \prec_{\sigma} y$ and $x \prec_{\sigma} y$, then $|X(A, B)| = 2$.

**Proof** Since the sequences $\sigma_{AB}$ and $\sigma_{BA}$ must be reverse-cyclic, if two of the elements are in the same order it follows that the sequence must be of length 2.

**Corollary 4.5** For two non-piercing regions $A, B$, such that one is not contained in the other, $A \cap B$ is a collection of disjoint lenses.

**Proof** If $A \cap B = \emptyset$, the statement is trivially true. If $|X(A, B)| = 2$ then $A \cap B$ consists of just one lens and we are done. Otherwise, if $|X(A, B)| > 2$, then following the proof of Theorem 4.3, there is at least one lens formed by $A \cap B$. We can now bypass this lens in favor of $B$, thus getting a modified region $A'$ from $A$. We can continue this process, obtaining a sequence of lenses (pairwise disjoint) until the regions do not intersect any more.

The lenses formed in an arrangement of a set $X$ of regions can be ordered as a partial order by inclusion. That is, lenses $\ell_{AB} \prec \ell_{CD}$ if $\ell_{AB} \subseteq \ell_{CD}$. Note that either $C$ or $D$ could be equal to $A$ or $B$. Now we prove the key lemma about lenses.

**Lemma 4.6** Let $X$ be a set of non-piercing regions. Let $\ell_{AB}$ be a minimal lens, defined by regions $A, B \in X$. Let $A'$ be the region obtained from $A$ by bypassing the lens $\ell_{AB}$ in favor of $B$. Then, the regions $X' = (X \setminus \{A\}) \cup \{A'\}$ is a set of non-piercing regions.

**Proof** Let $x, y$ be the two vertices of the lens $\ell_{AB}$ so that the arcs bounding $\ell_{AB}$ are $\gamma_{xy}(A)$, and $\gamma_{yx}(B)$. Let $C$ be any region intersecting $A$. We will show that after bypassing the lens $\ell_{AB}$ in favor of $B$, the modified region $A'$ remains non-piercing with respect to $C$. In particular, we will show that $\sigma_{A'C}$ and $\sigma_{CA'}$ remain reverse-cyclic. Let $S$ be the set of chords in $\ell_{AB}$ formed by $\partial(C)$.

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**Fig. 4** The figure on the left shows the chords of the boundary of $B$ in $A$. The figure on the right shows a lens bypassed in favor of $B$. The part of the boundary of $A$ that is modified is shown as a dotted line.
Suppose there is a chord $\gamma_{pq}(C)$ in $S$, such that both $p$ and $q$ both lie on $\gamma_{yx}(B)$. If $q$ precedes $p$ along $\partial(B)$, i.e., in $\sigma_{BC}$ then $\gamma_{pq}(C)$ and $\gamma_{qp}(B)$ form a lens contained in $\ell_{AB}$, contradicting the minimality of $\ell_{AB}$. Therefore, it follows that $p$ must precede $q$ along $\partial(B)$, i.e., on $\sigma_{BC}$. But, this implies that $p$ precedes $q$ in both reverse-cyclic sequences $\sigma_{BC}$ and $\sigma_{CB}$. By Corollary 4.4 therefore, $\partial(B)$ and $\partial(C)$ do not have any other points of intersection apart from $p$ and $q$. In particular, this implies that $\partial(C)$ does not intersect the boundary of the lens $\ell_{AB}$ at any point other than $p$ or $q$, and so there are no other chords in $S$. After lens-bypassing therefore, $\sigma_{A'C}$ and $\sigma_{CA'}$ are obtained by inserting consecutive points $p, q$ (in fact by points $p', q'$ arbitrarily close to $p$ and $q$ respectively, but we do not make this distinction) between the same two consecutive points in opposite order into $\sigma_{AC}$ and $\sigma_{CA}$. By Proposition 4.2 the sequences remain reverse-cyclic. This case is shown in Fig. 5.

Now, suppose there is a chord $\gamma_{pq}(C)$ in $S$ that joins two points on the boundary of the $\ell_{AB}$ defined by $\partial(A)$, namely $\gamma_{xy}(A)$. If $q$ precedes $p$ on $\gamma_{xy}(A)$, then this forms a lens contained in $\ell_{AB}$ contradicting the minimality of $\ell_{AB}$. Otherwise, $p$ precedes $q$ in $\sigma_{AC}$. But this implies that $p$ precedes $q$ in both reverse-cyclic sequences $\sigma_{AC}$ and $\sigma_{CA}$. Hence, by Corollary 4.4 $\partial(A)$ and $\partial(C)$ have no more points of intersection. In this case, after lens-bypassing, $\partial(A')$ and $\partial(C)$ do not intersect (in fact $A' \subseteq C$). Hence, the regions remain non-piercing. This case is shown in Fig. 6.

If none of the above hold, then all chords in $S$ have one end-point on $\gamma_{xy}(A)$ and the other end-point on $\gamma_{yx}(B)$. In this case, after lens-bypassing $\sigma_{A'C}$ and $\sigma_{CA'}$ are obtained by replacing for each chord of $C$, the end-point of the chord on $\gamma_{xy}(A)$ by its other end-point on $\gamma_{yx}(B)$. The sequences $\sigma_{AC}$ and $\sigma_{A'C}$ are identical except for renaming of the points in $\gamma_{xy}(A)$ by corresponding points in $\gamma_{yx}(B)$. The same
Fig. 7 All chords of $C$ connect a point on $\gamma_x(y)(A)$ and a point $\gamma_y(x)(B)$.

holds for the sequence $\sigma_{CA}$ and $\sigma_{CA'}$. Therefore the sequences $\sigma_{A'C}$ and $\sigma_{CA'}$ remain reverse-cyclic. This case is shown in Fig. 7.

The following corollary follows from the above discussion.

**Corollary 4.7** Let $\ell_{AB}$ be a minimal lens. Let $A'$ be the region obtained after bypassing $\ell_{AB}$ in favor of $B$. Then, for any region $C \neq B$, $A \cap C \neq \emptyset$ implies $A' \cap C \neq \emptyset$.

5 Local Search Framework

The algorithms that yield a PTAS for the problems studied in this paper follow a simple local search framework. While the framework is more general, we describe the local search algorithms in the context of the problems we study in this paper. Since our problems are packing or covering problems on graphs or hypergraphs induced by regions and points in the plane, feasible solutions consist of a subset of points, or a subset of regions. The optimal solution is a feasible solution with either the minimum or the maximum size, depending on the problem. The algorithms take as input a parameter $k$, that depends on the approximation factor we seek. For a $(1 + \varepsilon)$-approximation for a minimization problem, or a $(1 - \varepsilon)$-approximation for a maximization problem, we set $k = O(1/\varepsilon^2)$.

**Local Search Algorithm.** Start with any feasible solution. Let $k$ be the given parameter. Repeatedly attempt to find a better (larger or smaller, depending on whether the problem is a maximization, or minimization problem, respectively) feasible solution by making swaps of at most $k$ elements in the current solution with elements not in the solution. For maximization problems, a swap corresponds to adding at most $k$ elements that are not in the current solution, and removing fewer elements from the current solution such that the resulting solution remains feasible. For minimization problems on the other hand, a swap corresponds to removing at most $k$ elements from the current solution, and adding fewer elements from outside the current solution, so that the resulting solution remains feasible. The algorithm halts and returns the current feasible solution when no such swaps can be made.

For concreteness, we use Set Cover as a running example in the rest of the subsection. Consider the Set Cover problem $(\mathcal{X}, P)$ with a set of points $P$, and a set of non-piercing regions $\mathcal{X}$, and let $n$ be the total number of regions and points in the instance. The goal is to find the smallest subset of the regions that together cover all the input points.
The algorithm starts with a feasible solution, say the trivial solution consisting of all the regions in the input. The algorithm repeatedly attempts to remove at most \( k \) regions from the current solution, and add fewer regions from the regions not in the current solution, so that the resulting collection of non-piercing regions cover all the input points. The algorithm halts when no such improvement is possible. Since each such swap decreases the size of the solution, the number of iterations of the algorithm is at most \( n \). In each iteration, there are at most \( \binom{n}{k} = O(n^k) \) ways to remove \( k \) elements, and at most \( \binom{n}{k-1} = O(n^{k-1}) \) ways to add fewer elements, and \( O(n^2) \) time to check feasibility of the solution. Thus, the total running time is \( O(n^{O(k)}) \).

Let \( \mathcal{R} \) be the set of elements in an optimal solution, and let \( \mathcal{B} \) be the solution returned by the local search algorithm. To show the desired approximation factor of our local search algorithm, we would like to show that \( |\mathcal{B}| \) is close to \( |\mathcal{R}| \). Without loss of generality, we can assume that \( \mathcal{R} \cap \mathcal{B} = \emptyset \) by the following reasoning: Let \( \mathcal{B}' = \mathcal{B} \setminus (\mathcal{R} \cap \mathcal{B}) \), and \( \mathcal{R}' = \mathcal{R} \setminus (\mathcal{R} \cap \mathcal{B}) \). Then, \( \mathcal{R}' \), \( \mathcal{B}' \) are respectively optimal and local search solutions to the set cover problem with points \( P' = P \setminus (\bigcup R \in \mathcal{R} \cap \mathcal{B}) \), and regions \( \mathcal{X}' \setminus (R \in \mathcal{R} \cap \mathcal{B}) \). Now, if we show \( |\mathcal{B}'| \) is close to \( |\mathcal{R}'| \), then it follows that \( |\mathcal{B}| \) is close to \( |\mathcal{R}| \). A similar argument holds for the other problems studied in this paper, and for the analysis we can assume that the local search solution and the optimal solutions are disjoint.

To show that \( |\mathcal{B}| \) is close to \( |\mathcal{R}| \) for an optimization problem, we construct a graph on \( \mathcal{R} \cup \mathcal{B} \) satisfying the following Local Search conditions.

**Definition 5.1 (Local Search conditions)** Given feasible solutions \( \mathcal{R}, \mathcal{B} \) to a problem \( \Pi \), where \( \mathcal{R} \) is an optimal solution, and \( \mathcal{B} \) is a local search solution with \( \mathcal{R} \cap \mathcal{B} = \emptyset \), a bipartite graph \( H \) on \( \mathcal{R} \cup \mathcal{B} \) is said to satisfy the Local Search conditions if it satisfies the following:

1. **Local exchange.** If \( \Pi \) is a minimization problem then we require that for any \( \mathcal{B}' \subseteq \mathcal{B} \), \( (\mathcal{B} \setminus \mathcal{B}') \cup N(\mathcal{B}') \) is a feasible solution. Here, \( N(\mathcal{B}') \) denotes the set of neighbors of \( \mathcal{B}' \) in \( H \). On the other hand, if \( \Pi \) is a maximization problem then we require that for any \( \mathcal{R}' \subseteq \mathcal{R} \), \( (\mathcal{R} \cup \mathcal{R}') \setminus N(\mathcal{R}') \) is a feasible solution.
2. **Sublinear separator.** There exist \( 0 < \alpha, \delta < 1 \) such that for any induced subgraph \( H' \) of \( H \) with vertex set \( V(H') \), there is a vertex separator \( S \subseteq V(H') \) such that \( |S| = O(|V(H')|^\delta) \), and each connected component of \( H' \setminus S \) is of size at most \( \alpha|V(H')| \).

For the Set Cover problem, the local exchange property reduces to the following: Construct a graph on \( \mathcal{B} \cup \mathcal{R} \) such that for each point \( p \in P \), there is an edge between some \( R \in \mathcal{R} \) such that \( R \ni p \) and some \( B \in \mathcal{B} \) such that \( B \ni p \). The graph we construct is a planar graph and therefore satisfies the sublinear separator condition above.

The existence of a graph with the above properties implies a PTAS for corresponding problem. The following result is implicit in [12,28]. See [5, Sect. 2] for a more complete discussion of the Local Search Framework where the following theorem is proved separately for minimization and maximization problems (Thms. 2.4 and 2.7 respectively).
Theorem 5.2 [5,12,28] Consider a minimization or maximization problem $\Pi$. Let $R$ and $B$ be the optimal and local search solutions for the problem. If there exists a bipartite graph $H$ on $R \cup B$ that satisfies the Local Search conditions (Definition 5.1), then the Local Search algorithm is a PTAS for the problem $\Pi$.

Briefly, the argument that Local Search yields a PTAS goes as follows. Since the graph $H$ has a sublinear size separator, we can repeatedly use the separator to split the graph into smaller pieces until we get disjoint pieces of size at most $k$. Since the size of each piece decreases geometrically, and in each piece, the separator size is only sublinear, it follows that the total separator size is at most linear. In fact, the total separator size before we get pieces of size $k$ is at most $|R \cup B|/k^c$, for some $c = c(\alpha, \delta) > 0$.

By the local-exchange property, the Local Search solution is at least as good as the optimal solution within each piece, and since the total separator size is comparatively smaller for large enough $k$, a simple calculation shows that $|B| \sim |R|$, and we obtain a PTAS.

In the rest of the paper, “the Local Search algorithm” refers to the algorithmic framework above, specialized in the natural way to the problem at hand.

6 Applications

In this section, we obtain PTASs for several problems by constructing appropriate graphs satisfying the Local Search conditions (Definition 5.1) for them and applying Theorem 5.2. For the constructions, we primarily use lens-bypassing. In some of the constructions we need the following lemma [29, Lem. 6] which we state here for completeness.

Lemma 6.1 [29] Given a set of non-piercing regions $X$ and a set $P$ of points in the plane, there exists a plane multigraph $H = (P, E)$ such that for any region $X \in X$, the subgraph formed by the vertices in $P \cap X$ and the edges of $H$ lying completely within $X$ is connected.

The lemma as stated is slightly more general than the statement in [29]. Lemma 6 in [29] states the existence of a planar graph. However, the fact that there is a plane multigraph with the properties stated in the above theorem is implicit in the proof (see [29, Lem. 5]).

6.1 Dominating Set and Set Cover

In this subsection, we describe the construction of graphs satisfying the Local Search conditions (Definition 5.1) for the Dominating Set and Set Cover problems, thus obtaining PTASs for these problems using Theorem 5.2.

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5 An embedding of a planar graph in the plane such that the vertices are points and edges are continuous curves between the end-points that are disjoint in their interior. There could be more than one arc joining the same end-points.
For the Set Cover problem, in order to satisfy the local-exchange property, we require a bipartite graph $H = (R \cup B, E)$ (where $\cup$ refers to a disjoint union) with a sublinear sized balanced separator such that for any set $S \subseteq B$, the set $B' = (B \setminus S) \cup N(S)$ is a Set Cover, where $N(S)$ is the set of neighborhoods of the vertices in $H$ corresponding to the regions in $S$. In other words, the required graph $H = (R \cup B, E)$ must satisfy the following property: For each point $p \in P$, there are an $R \in \text{RED}(p)$ and $B \in \text{BLUE}(p)$ that are adjacent in $H$, where $\text{RED}(p)$ is the set of regions in $R$ containing $p$, and $\text{BLUE}(p)$ is the set of regions in $B$ containing $p$.

For the Dominating Set problem, the property that the bipartite graph $H$ is required to satisfy is that for any region $X \in \mathcal{X}$, there is a region $R \in \text{RED}(X)$ and a region $B \in \text{BLUE}(X)$ such that $R$ and $B$ are adjacent in $H$, where $\text{RED}(X)$ is the set of regions in $R$ intersecting $X$, and $\text{BLUE}(X)$ is the set of regions in $B$ intersecting $X$. Let $\mathcal{G}$ denote the regions of $\mathcal{X}$ that are not in $R$ or $B$. Since we assumed that $R \cap B = \emptyset$, the three sets form a partition of the input $\mathcal{X}$.

We give the construction of a planar graph for any set $\mathcal{X} = R \cup B \cup \mathcal{G}$ of non-piercing regions from which the construction of the graphs for the Dominating Set and Set Cover problems follows.

Henceforth we refer to the regions in $R$, $B$, and $\mathcal{G}$ as RED, BLUE, and GREEN regions respectively. We call a region $X \in \mathcal{X}$ rb-intersecting if and only if $X$ intersects a RED as well as a BLUE region. We assume that a region intersects itself. Therefore a RED region is rb-intersecting if it intersects a BLUE region. Similarly, a point in the plane is called rb-intersecting if and only if it is contained in both a RED as well as a BLUE region. As before, we let $\text{RED}(X)$ and $\text{BLUE}(X)$ denote the RED and BLUE regions intersecting $X$, respectively. Similarly, for any point $p$, we let $\text{RED}(p)$ and $\text{BLUE}(p)$ denote the RED and BLUE regions containing $p$ respectively. We now define the properties that our constructed graph satisfies.

**Definition 6.2** Let $\mathcal{X} = R \cup \mathcal{G} \cup B$ be a set of non-piercing regions. A bipartite planar graph $H = (R \cup B, E)$ is called a locality-preserving graph for $\mathcal{X}$ if it satisfies the following properties.

P1 For each rb-intersecting region $X \in \mathcal{X}$, there exist an $R \in \text{RED}(X)$ and a $B \in \text{BLUE}(X)$ such that $R$ and $B$ are adjacent in $H$.

P2 For each rb-intersecting point $p$ in the plane, there exist an $R \in \text{RED}(p)$ and a $B \in \text{BLUE}(p)$ such that $R$ and $B$ are adjacent in $H$.

We can assume that in any instance, all GREEN regions are rb-intersecting as GREEN regions that are not rb-intersecting can be removed, since the property P1 is required to be satisfied only for rb-intersecting regions. (Note that for the Dominating Set problem, since $R$ and $B$ refer to an optimal, and local search solution, respectively, no non-rb-intersecting GREEN regions exist.) We say that a graph $H$ on $R \cup B$ satisfies a region $X \in \mathcal{X}$ if property P1 holds for $X$. We also state this as “region $X$ is satisfied by $H$”. We use a similar terminology for the points. If in a given instance $\mathcal{X}$, there are regions $R \in \mathcal{R}$, $B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $R \cap B \cap G \neq \emptyset$, we say that $\mathcal{X}$ has a RED–BLUE–GREEN intersection. The main theorem we prove in this subsection is the following:
Theorem 6.3  For any set $\mathcal{X} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{B}$ of non-piercing regions, there is a locality-preserving graph $H$.

The broad approach to construct a locality-preserving graph $H$ for $\mathcal{X}$ is as follows. If the instance $\mathcal{X}$ satisfies certain additional conditions, then we can directly describe the construction of such a graph. If the instance $\mathcal{X}$ does not satisfy these additional conditions, we show how we can reduce the instance to one that does.

In order to do this, we describe a sequence of reduction steps that either remove a region, or bypass a minimal lens in the arrangement of the regions, thus getting us closer to an arrangement enjoying the additional conditions alluded to above. These reduction steps have the crucial property that if we are given a locality-preserving graph for the reduced instance, we can obtain a locality-preserving graph for the original instance. We start with a construction of a locality-preserving graph for an instance $\mathcal{X}$ satisfying the additional conditions. Then, in a sequence of lemmas, namely Lemmas 6.5, 6.6 and 6.7 we describe the reduction steps for an instance not enjoying the additional properties. Finally, we can prove Theorem 6.3.

Lemma 6.4  Suppose $\mathcal{X} = \mathcal{R} \sqcup \mathcal{G} \sqcup \mathcal{B}$ is a set of non-piercing regions satisfying the following properties:

1. $R \cap R' = \emptyset$, for all $R, R' \in \mathcal{R}$ and $B \cap B' = \emptyset$, for all $B, B' \in \mathcal{B}$, i.e., the Red regions are pairwise disjoint, and the Blue regions are pairwise disjoint.
2. For each $R \in \mathcal{R}$, $B \in \mathcal{B}$ and $G \in \mathcal{G}$, $R \cap B \cap G = \emptyset$, i.e., there is no Red–Blue–Green intersection.

Then, there is a locality-preserving graph for $\mathcal{X}$.

Proof  In order to construct the graph, we temporarily add Red and Blue points to the arrangement of the regions in the following way: For each intersection $R \cap G$ of a Red region $R$ and a Green region $G$, we place a Red point in $R \cap G$. Similarly, we place a Blue point for each Blue–Green intersection. Since there are no Red–Blue–Green intersections, observe that in the interior of any Green region the Red and Blue regions are disjoint. Therefore, for any Green region, the point we place corresponding to a Red region does not lie in a Blue region, and vice versa.

Now, by Lemma 6.1, applied to the Green regions and the Red and Blue points we place, there is a plane graph $K$ such that for each Green region $G \in \mathcal{G}$, there is an edge in $K$ between a Red point contained in $G$ and a Blue point contained in $G$ lying entirely in $G$.

For each $G \in \mathcal{G}$, we pick one such edge $e_G$ arbitrarily. Let $r$ and $b$ be the Red and Blue end-points of $e_G$, respectively. As observed earlier, $r$ lies in a Red region, and $b$ lies in a Blue region. So, walking from $b$ to $r$ along $e_G$ we encounter a Red region $R$ and a Blue region $B$ that are consecutive along $e_G$. We now extend $B$ along $e_G$ so that $R$ and $B$ now intersect. Note that this can be done in a way that they remain non-piercing. An example is shown in Figs. 8 and 9.

Since the graph $K$ is planar, the extended Blue regions remain disjoint. Extending a Blue region along an edge of $K$ chosen for a Green region may intersect other Green regions in an arbitrary fashion. However, this does not matter as the Green regions will not play any role henceforth. It is possible that the same
A set of BLUE regions (boundary shown in dashed line) in $\mathcal{B}$ and a set of RED regions (boundary shown in solid line) in $\mathcal{R}$ intersect a GREEN region $G \in \mathcal{G}$ (the circle in the figure). The edge $e_G$ connecting a RED point in $G$ and a BLUE point in $G$ lies entirely inside the GREEN disk $G$

Extending the BLUE region $B$ along $e_G$ so that it intersects the adjacent RED region $R$ in a non-piercing manner

BLUE region is extended multiple times to intersect the same RED region. However, the regions remain non-piercing as each extension of the BLUE ensures this property.

By extending the BLUE regions, we have the property that for each $G \in \mathcal{G}$, there are now an $R \in \text{RED}(G)$ and $B \in \text{BLUE}(G)$ such that $R \cap B \neq \emptyset$. Therefore, the intersection graph of the RED and BLUE regions gives us the desired locality-preserving graph. Note that property P2 is trivially satisfied for this graph since we have only extended the regions. Thus, if a point was covered by a RED and a BLUE region, it is still covered by them, and so there is an edge between them in the intersection graph.

Since we ensure that the RED regions are pairwise disjoint, the BLUE regions are pairwise disjoint, and $\mathcal{R} \cup \mathcal{B}$ is non-piercing, the intersection graph of $\mathcal{R} \cup \mathcal{B}$ is planar as observed by Chan and Har-Peled [12].

We now describe the reductions for an instance $\mathcal{X}$ that does not satisfy the conditions of Lemma 6.4. The reduction steps are the following: We first show that we can remove RED–BLUE–GREEN intersections if there are any in our instance. Then, we show that if our instance has two regions such that one is contained in another, then we can remove one of them (this statement is not entirely accurate; we do not get rid of containments where a RED or BLUE region is contained in a GREEN region, but such containments do not affect our construction). Then, we show that we can decrease the number of vertices in the arrangement by bypassing minimal lenses. The latter two reductions are applied repeatedly until none applies. At that point, we can show that the instance satisfies the conditions of Lemma 6.4 and a locality-preserving graph can thus be constructed.

Lemma 6.5 Let $\mathcal{X} = \mathcal{R} \cup \mathcal{B} \cup \mathcal{G}$ be a set of non-piercing regions. Suppose there exist $R \in \mathcal{R}$, $B \in \mathcal{B}$ and $G \in \mathcal{G}$ such that $R \cap B \cap G \neq \emptyset$, i.e., the instance contains

\[ \square \]
a RED–BLUE–GREEN intersection. Then, a locality-preserving graph for the reduced instance $\mathcal{X}' = \mathcal{X} \setminus G$ is a locality-preserving graph for $\mathcal{X}$.

**Proof** Let $H'$ be a locality-preserving graph for $\mathcal{X}'$. By property P2, there is an edge between $R \in \text{RED}(p)$ and $B \in \text{BLUE}(p)$ for $p \in R \cap B \cap G$. This implies that $H'$ is locality-preserving for $\mathcal{X}$ since $\text{RED}(p) \subseteq \text{RED}(G)$ and $\text{BLUE}(p) \subseteq \text{BLUE}(G)$. □

**Lemma 6.6** Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ be a set of non-piercing regions with no RED–BLUE–GREEN intersections. If there are two regions $P$ and $Q$ such that $P \subseteq Q$, and either both are RED or both are BLUE, then the existence of a locality-preserving graph for a suitable reduced instance $\mathcal{X}'$ with one less region than $\mathcal{X}$ implies a locality-preserving graph for $\mathcal{X}$.

**Proof** The reduced instance in this case is obtained by removing $P$, i.e., $\mathcal{X}' = \mathcal{X} \setminus P$. If $H'$ is a locality-preserving graph for $\mathcal{X}'$, then we obtain the locality-preserving graph $H$ for $\mathcal{X}$ by adding $P$ as a vertex, and an edge from $P$ to any RED or BLUE region intersecting $P$ and having a color different from $P$, if such a region exists. If such a region does not exist, then we set $H = H'$. To show that $H$ is locality-preserving for $\mathcal{X}$, note that if $P$ is rb-intersecting, then the edge we added satisfies $P$. Any other region intersecting $P$ also intersects a region of the same color as $P$, namely $Q$. Therefore, such a region is satisfied by $H'$. Similarly, all rb-intersecting points in $P$ remain rb-intersecting in $\mathcal{X}'$ since they are contained in $Q$. Therefore, it follows that the edges in $H'$ satisfy all rb-intersecting points with respect to $\mathcal{X}$. □

**Lemma 6.7** Let $\mathcal{X} = \mathcal{R} \sqcup \mathcal{B} \sqcup \mathcal{G}$ be a set of non-piercing regions such that there are no RED–BLUE–GREEN intersections. Then, a locality-preserving graph $H'$ for a reduced instance $\mathcal{X}'$ is a locality-preserving graph for $\mathcal{X}$. Here, $\mathcal{X}'$ is obtained by bypassing a minimal lens $\ell_{PQ}$ using the following rules:

1. If $P$ and $Q$ have the same color, then bypass $\ell_{PQ}$ in favor of either $P$ or $Q$ chosen arbitrarily.
2. If $\ell_{PQ}$ is contained in a region $R \in \mathcal{R}$, where $R$ is distinct from $P$ and $Q$, which have different colors, then bypass the lens in favor of the region that is not RED.
3. If $\ell_{PQ}$ is contained in a region $B \in \mathcal{B}$, where $B$ is distinct from $P$ and $Q$, which have different colors, then bypass the lens in favor of the region that is not BLUE.

**Proof** We assume without loss of generality that we bypass the lens $\ell_{PQ}$ in favor of $Q$, and let $P'$ be the resulting region corresponding to $P$. Then, $\mathcal{X}' = (\mathcal{X} \setminus P) \cup P'$. By Lemma 4.6, the regions in $\mathcal{X}'$ are non-piercing since we bypass a minimal lens $\ell_{PQ}$.

Suppose $P$ and $Q$ have the same color. By Corollary 4.7, any region $X \neq Q$ intersecting $P$ also intersects $P'$. This ensures that the set of RED or BLUE regions intersecting a region $X \neq Q$ remains unchanged in $\mathcal{X}'$. Thus, $H'$ satisfies all regions except possibly $Q$. Since $P$ and $Q$ have the same color, $Q$ remains rb-intersecting in $\mathcal{X}'$ if it was rb-intersecting in $\mathcal{X}$. Since $P \cup Q$ remains unchanged due to lens-bypassing and the fact that they have the same color, all rb-intersecting points in $\mathcal{X}$ remain rb-intersecting in $\mathcal{X}'$. Since no region gains a new intersection and no point is contained in a new region when going from $\mathcal{X}$ to $\mathcal{X}'$, a locality-preserving graph $H'$ for $\mathcal{X}'$ is also locality-preserving for $\mathcal{X}$.
Now suppose that \( P \) and \( Q \) have distinct colors. Then we bypass \( \ell_{PQ} \) only if it lies in a **Red** or a **Blue** region. Let us assume that \( \ell_{PQ} \) is contained in a **Red** region \( R \). The other case is symmetric. By our assumption that there is no **Red**–**Blue**–**Green** intersection, one of \( P \) or \( Q \) must be **Red**. Since we assume that we bypass \( \ell_{PQ} \) in favor of \( Q \), \( P \) is **Red**. We claim that \( H' \) is locality-preserving with respect to \( X \).

By Corollary 4.7, it follows that for every region \( X \notin \{ P, Q \} \), the set of regions intersecting \( X \) does not change. These regions are therefore satisfied by \( H' \). For the region \( Q \), while it possibly loses its intersection with \( P \), it continues to intersect a **Red** region, namely \( R \). Therefore, \( Q \) is also satisfied by \( H' \). The region \( P' \) may not be **rb**-intersecting even if \( P \) was **rb**-intersecting. This can happen only when \( Q \) is the unique **Blue** region intersecting \( P \), which is **Red** by assumption. However, in that case, consider any point \( p \in \ell_{PQ} \). Such a point \( p \) is **rb**-intersecting in \( X' \) since it lies in \( Q \) and \( R \). Therefore the point \( p \) is satisfied in \( H' \). Now, the fact that \( P \) is satisfied in \( H' \) follows from the fact that any region containing \( p \) also intersects \( P \).

**Proof of Theorem 6.3** Given an instance \( \mathcal{X} = \mathcal{R} \cup \mathcal{B} \cup \mathcal{G} \) of non-piercing regions, we first remove all **Red**–**Blue**–**Green** intersections by applying Lemma 6.5. Then, we repeatedly apply Lemma 6.6 followed by Lemma 6.7 until neither applies. At this point, we claim that the **Red** regions are pairwise disjoint, and the **Blue** regions are pairwise disjoint.

First, note that no **Red** or **Blue** region is contained in another region of the same color by Lemma 6.6. In particular, by Corollary 4.5, any intersection of a pair \( A, B \) of **Red** or **Blue** regions of the same color is a union of lenses formed by them. However, any such lens \( \ell_{AB} \) is removed by Lemma 6.7 so long as it is minimal. We argue that \( \ell_{AB} \) must be minimal. Suppose not. Then, there is a minimal lens \( \ell_{WZ} \) contained in \( \ell_{AB} \). We can check that for all possible colors of \( W \) and \( Z \), Lemma 6.7 applies to \( \ell_{WZ} \) and is thus bypassed.

Thus, when the conditions of Lemmas 6.6 or 6.7 do not apply, the **Red** regions are pairwise disjoint, the **Blue** regions are pairwise disjoint, and there are no **Red**–**Blue**–**Green** intersections. Hence, the conditions of Lemma 6.4 apply and we can obtain a locality-preserving graph \( H' \). This implies a locality-preserving graph \( H \) for the instance \( \mathcal{X} \).

Now, we can prove that the Dominating Set and Set Cover problem for non-piercing regions admits a PTAS.

**Proof of Theorem 3.2** Let \( \mathcal{R} \) denote an optimal solution to the Dominating Set problem. Let \( \mathcal{B} \) denote a solution returned by local search. Let \( \mathcal{G} \) be the remaining regions. Recall that we assume \( \mathcal{R} \cap \mathcal{B} = \emptyset \). Let \( \mathcal{X} = \mathcal{R} \cup \mathcal{B} \cup \mathcal{G} \). Note that since every region in \( \mathcal{X} \) is **rb**-intersecting, by Theorem 6.3 we obtain a planar graph satifying property P1. Note that property P1 is precisely the local exchange property needed in the Local Search conditions (Definition 5.1) for the graph required in the analysis of the Dominating Set problem. Furthermore, being planar, the graph also has balanced vertex separators of size \( O(\sqrt{n}) \) [25], and thus satisfies the sublinear separator condition of the Local Search conditions. By Theorem 5.2, a \((1 + \varepsilon)\)-approximation algorithm follows. 

\( \square \)
Proof of Theorem 3.1 Let \( \mathcal{R} \) denote an optimal solution to the Set Cover problem. Let \( \mathcal{B} \) denote a solution returned by local search. Recall that we assume \( \mathcal{R} \cap \mathcal{B} = \emptyset \). Let \( \mathcal{X} = \mathcal{R} \cup \mathcal{B} \), and let \( \mathcal{G} = \emptyset \).

Theorem 6.3 gives us a planar graph satisfying the required property P2. Since every point \( P \) is in \( R \cap B \), for some \( R \in \mathcal{R} \) and \( B \in \mathcal{B} \), property P2 is precisely the locality exchange property needed for the graph required in the analysis of the Set Cover problem. Furthermore, being planar the graph also has balanced vertex separators of size \( O(\sqrt{n}) \) [25], and thus satisfies the sublinear separator condition of the Local Search conditions. By Theorem 5.2, a \((1 + \epsilon)\)-approximation algorithm follows.

\[ \square \]

6.2 Capacitated Region Packing

Recall that in the Capacitated Region Packing problem, we are given a family of \( r \)-admissible regions \( \mathcal{X} \), a set of points \( P \) with each point having a capacity at most \( \ell \), where \( \ell \) is a positive integer constant. We want to find a maximum cardinality subset \( \mathcal{X}' \subseteq \mathcal{X} \) such that for each point \( p \in P \), the number of regions of \( \mathcal{X}' \) that contain \( p \) is at most its capacity. Unlike the earlier results in this paper, here we require that the regions be \( r \)-admissible for a constant \( r \), i.e., they are non-piercing and their boundaries intersect at most a constant number of times.

Let \( \mathcal{X} = \mathcal{R} \cup \mathcal{B} \) and \( k = 2\ell \). Note that the depth of a point in \( P \) with respect to \( \mathcal{X} \), that is the number of regions in \( \mathcal{X} \) containing it, is at most \( k \). Recall that for the analysis of the Local Search algorithm to go through, we only require to show the existence of a graph satisfying the Local Search conditions (Definition 5.1). Unlike for the Dominating Set, or Set Cover problems, we are unable to construct a planar graph. Instead, we construct a natural graph satisfying the local exchange property of the Local Search conditions, and then show that this graph has a sublinear sized balanced separator.

The graph we construct is as follows: For each point \( p \in P \), add an edge between all regions in \( \mathcal{R} \) containing \( p \) and all regions in \( \mathcal{B} \) containing \( p \). It is easy to check that this graph satisfies the local exchange property of Definition 5.1. We need to show that this graph satisfies the sublinear separator property in Definition 5.1. In fact, we prove that the following super-graph \( H(\mathcal{X}, k) \) satisfies the property: for each point \( p \in \mathbb{R}^2 \) whose depth is at most \( k \), add an edge between all pairs of regions in \( \mathcal{R} \cup \mathcal{B} \) containing \( p \). We will show that \( H(\mathcal{X}, k) \) has a sublinear sized balanced separator. The fact that any subgraph of \( H(\mathcal{X}, k) \) has a sublinear sized balanced separator follows from an identical argument.

We now give an intuitive argument for the existence of a balanced sublinear sized separator in \( H(\mathcal{X}, k) \). Consider the planar graph \( G_p \) on the points in \( P \) such that the subgraph induced by the points in \( R \cap P \) for each region \( R \in \mathcal{R} \cup \mathcal{B} \) is connected. Such a graph is guaranteed by Lemma 6.1. We then conceptually think of the region \( R \) as a spanning tree of this connected subgraph. Then, a vertex separator \( Q \subseteq P \) of \( G_p \) naturally induces a vertex separator on \( H(\mathcal{X}, k) \): replace each point in \( Q \) by the set of regions containing that point. Since each point in \( Q \) is in at most \( k \) regions in \( \mathcal{X} \), the size of the separator obtained is at most \( k \) times the size of \( Q \). Using the fact that
the regions in $X$ have linear union complexity, and their arrangement has low depth ($\leq k$), we conclude that $|P| = O(|X|)$. It follows that $H(X, k)$ has a sublinear sized separator. A standard argument is then used to show that a balanced separator of the same size can be obtained. The formal proof follows.

**Theorem 6.8** Given a set $X$ of $r$-admissible regions for a constant $r$, the graph $H(X, k)$ on $X$ has a balanced separator of size $O(k^{3/2}\sqrt{|X|})$.

**Proof** In order to prove the above statement for $H(X, k)$, we prove it for an isomorphic graph that is the intersection graph of a family of trees $T$ that we obtain in the following way: We put one point in every cell whose depth is at most $k$ in the arrangement of the regions in $X$, and call this point set $P$. By Lemma 6.1 there exists a plane graph $G_P$ on the point set $P$ such that the subgraph induced by $X \cap P$ is connected, for every $X \in X$. For each $X \in X$, consider an arbitrary spanning tree $T_X$ of the subgraph induced by $X \cap P$. Observe that the intersection graph of the family of these spanning trees $T = \{T_X \mid X \in X\}$ is isomorphic to $H(X, k)$. We now claim that such an intersection graph has a small and balanced separator.

**Lemma 6.9** Given a family of trees $T$ as above, its intersection graph has a balanced separator of size $O(k^{3/2}\sqrt{|T|})$.

**Proof** We assign appropriate weights to the points in $P$ to get a balanced weighted separator $S$ in $G_P$. We then use $S$ to obtain a balanced separator for the intersection graph of $T$.

We assign weights to the points in $P$ as follows: we start by assigning a weight of $0$ to each point in $P$. For each region $X \in X$, we add $1/|X \cap P|$ to the weight of each point in $X$. Thus, the weight of a point in $P$ is given by $wt(p) = \sum_{X \ni p} 1/|X \cap P|$.

By the Lipton–Tarjan separator theorem [25], $G_P$ has a separator $S$ of size $O(\sqrt{|P|})$ such that removing $S$ separates the graph into two disjoint sets $A$ and $B$, each of which has at most $2/3$ of the total weight of the points. The separator $S$ for $G_P$ gives a separator for the intersection graph of $T$ by taking $S = \{T \mid S \cap T \neq \emptyset\}$. We claim that $S$ is a small, balanced separator. The fact that removing $S$ separates $T$ into two parts $A, B$ follows since a tree containing a vertex from $A$ and a vertex from $B$ must contain a vertex from $S$.

To show that $|S| \leq O(k^{3/2}\sqrt{|T|})$, we proceed as follows. By our construction, every point in $P$ has depth at most $k$. This implies that $|S| \leq k|S|$. However, $|S| \leq O(\sqrt{|P|})$, as $S$ is a separator in $G_P$. Since $r$-admissible regions have linear union complexity [32] for any constant $r$, the Clarkson–Shor technique [26, p. 141] implies that the number of cells in the arrangement is at most $O(k|T|)$. Thus, $|P| = O(k|T|)$ and hence, $|S| \leq O(k^{3/2}\sqrt{|T|})$.

Now, we need to show that $S$ is balanced. To see this, let $T_A$ be the set of trees whose vertex set is a subset of $A$. $T_B$ is defined similarly. Since, the weight of all trees in $T_A$ were distributed among the points in $A$, $wt(A) \geq |T_A|$. Also, from the planar separator theorem we know that $wt(A) \leq \frac{2}{3}|T|$. Therefore, $|T_A| \leq \frac{2}{3}|T|$. The same holds for $T_B$. Therefore, $S$ is a balanced separator of size $O(k^{3/2}\sqrt{|T|})$ of the intersection graph of $T$.

From Lemma 6.9, it follows that $H(X, k)$ has a balanced separator of size $O(k^{3/2}\sqrt{|X|})$. □
Proof of Theorem 3.3 The graph constructed satisfies the conditions of Theorem 5.2. Therefore a PTAS follows.

Remark Theorem 6.8 seems to allow for a non-constant $k$ since choosing $k = o(|\mathcal{X}|^{1/3})$ is sufficient for obtaining a separator of size sublinear in $|\mathcal{X}|$. However, recall that the ‘sublinear separator’ condition in the Local Search conditions (Definition 5.1) requires a sublinear sized balanced separator for all induced subgraphs and choosing $k$ to be non-constant violates that for subgraphs of size $o(k^3)$.

6.3 Capacitated Point Packing

Recall that in the Capacitated Point Packing problem, we are given a set $P$ of $n$ points, a set $\mathcal{X}$ of non-piercing regions, and a positive integer constant $\ell$. The goal is to obtain the maximum sized subset of points $Q \subseteq P$ such that for every region $X \in \mathcal{X}$, $|X \cap Q| \leq \ell$. We consider the Capacitated Point Packing problem when $\ell = 1$. We show that the Local Search algorithm yields a PTAS for this special case.

Proof of Theorem 3.4 We construct a graph $G$ using Lemma 6.1 on the union of RED and BLUE points. Observe that there are at most one RED point and at most one BLUE point in each region. By Lemma 6.1, for every region that contains a RED point and a BLUE point, there exists an edge between them in $G$. It is easy to check that graph $G$ satisfies the local-exchange property (Definition 5.1). Also, $G$ is planar (by Lemma 6.1) and thus satisfies sublinear separator property (Definition 5.1). By Theorem 5.2, PTAS follows.

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