Verification of the Firoozbakht Conjecture for Primes up to Four Quintillion

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Abstract

If $p_k$ is the $k$th prime, the Firoozbakht conjecture states that the sequence $(p_k)^{1/k}$ is strictly decreasing. We use the table of first-occurrence prime gaps in combination with known bounds for the prime-counting function to verify the Firoozbakht conjecture for primes up to four quintillion ($4 \times 10^{18}$).

Mathematics Subject Classification: 11N05

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1 Introduction

We will examine a conjecture that was first stated in 1982 by the Iranian mathematician Farideh Firoozbakht from the University of Isfahan [8]. It appeared in print in The Little Book of Bigger Primes by Paulo Ribenboim [7, p. 185]. The statement is as follows:

Firoozbakht’s Conjecture. If $p_k$ is the $k$th prime, then the sequence $(p_k)^{1/k}$ is strictly decreasing. Equivalently, for all $k \geq 1$ we have

$$p_{k+1}^k < p_k^{k+1}. \tag{1}$$

The Firoozbakht conjecture is one of the strongest upper bounds for prime gaps. As we will see from Table 1 below, it is somewhat stronger than Cramér’s
conjecture proposed about half a century earlier by the Swedish mathematician Harald Cramér [1]:

**Cramér’s Conjecture.** If \( p_k \) and \( p_{k+1} \) are consecutive primes, then we have
\[
p_{k+1} - p_k = O(\log^2 p_k)
\]
or, more specifically,
\[
\limsup_{k \to \infty} \frac{p_{k+1} - p_k}{\log^2 p_k} = 1.
\]

For the sake of numerical comparison with (1), let us use a modified form of Cramér’s conjecture stated below.

**Modified Cramér Conjecture.** If \( p_k \) and \( p_{k+1} \) are consecutive primes, then
\[
p_{k+1} - p_k < \log^2 p_{k+1}.
\]

This modified form allows us to make predictions of an upper bound for any given prime gap; Table 1 lists a few examples of such upper bounds.

| \( k \) | Consecutive primes \( p_k \) | \( p_{k+1} \) | Upper bounds for \( p_{k+1} \), as predicted by: | Modified Cramér conjecture (solution of \( x = p_k + \log^2 x \)) | Firoozbakht conjecture (solution of \( x^k = p_k^{k+1} \)) |
|---|---|---|---|---|---|
| 5 | 11 | 13 | 19.964 | 17.769 |
| 26 | 101 | 103 | 124.255 | 120.618 |
| 169 | 1009 | 1013 | 1057.493 | 1051.152 |
| 1230 | 10007 | 10009 | 100019999 | 100082220 |
| 9593 | 10000033 | 10000033 | 1000135.579 | 1000123.090 |
| 78499 | 1000003 | 1000033 | 1000135.579 | 1000123.090 |
| 664580 | 10000019 | 1000079 | 10000278.794 | 10000261.534 |

Table 1 shows that, given \( k \) and \( p_k \), the Firoozbakht conjecture (1) yields a tighter bound for \( p_{k+1} \) than the modified Cramér conjecture (2). Indeed, the Firoozbakht upper bound (last column) is below the Cramér upper bound by approximately \( \log p_k \). In *Cramér’s probabilistic model of primes* [1, 3] the parameters of the distribution of maximal prime gaps suggest that inequalities (1) and (2) are both true with probability 1; that is, almost all maximal prime gaps in Cramér’s model satisfy (1) and (2). One may take this as an indication that any violations of (1) and (2) occur exceedingly rarely (if at all).

\footnote{In Cramér’s model with \( n \) urns, the limiting distribution of maximal “prime gaps” is the Gumbel extreme value distribution with scale \( a_n \sim n / \ln n = O(\log n) \) and mode \( \mu_n = n \log(\ln n) / \ln n + O(\log n) = \log^2 n - \log n \log \log n + O(\log n) \) \cite{OEIS:A235402}, here \( \ln n \) denotes the logarithmic integral of \( n \). Hence, for large \( n \), all maximal gap sizes are below \( \log n (\log n - 1) \), except for a vanishing proportion of maximal gaps.}
2 Computational verification for small primes

When primes $p_k$ are not too large, one can directly verify inequality (1) by computation. A simple program that takes a few seconds to perform the verification for $p_k < 10^6$ is available on the author’s website. The program outputs the numeric values of $k$, $p_k$, $p_k^{1/k}$, and an OK if the value of $p_k^{1/k}$ decreases from one prime to the next. It will output FAILURE if the conjectured decrease does not occur. Result: all OKs, no FAILURES. Here is a sample of the output:

| k | p    | $p_k^{1/k}$ | OK/fail |
|---|------|-------------|---------|
| 1 | 2    | 2.000000000 | OK      |
| 2 | 3    | 1.732050808 | OK      |
| 3 | 5    | 1.709975947 | OK      |
| 4 | 7    | 1.626576562 | OK      |
| 5 | 11   | 1.615394266 | OK      |
| 6 | 13   | 1.533406237 | OK      |
| 7 | 17   | 1.498919872 | OK      |
| 8 | 19   | 1.444921323 | OK      |
| 9 | 23   | 1.416782203 | OK      |
|10| 29   | 1.400360331 | OK      |
|11| 31   | 1.366401518 | OK      |
|12| 37   | 1.351087503 | OK      |
| ... | ... | ... | ... |
|78494| 999953| 1.000176022 | OK |
|78495| 999959| 1.000176020 | OK |
|78496| 999961| 1.000176018 | OK |
|78497| 999979| 1.000176016 | OK |
|78498| 999983| 1.000176014 | OK |
|78499| 1000003| 1.000176012 | OK |
|78500| 1000033| 1.000176010 | OK |

Thus primes $p_k < 10^6$ do not violate (1). What about larger primes?

3 What if we do not know $k = \pi(p_k)$?

For large primes $p_k$, the exact values of $k = \pi(p_k)$ are not readily available. Nevertheless, the Firoozbakht conjecture can often be verified in such cases too. When we do not know $\pi(p_k)$ exactly, we can use these bounds for the prime-counting function $\pi(x)$:

$$\pi(x) < \frac{x}{\log x - 1.1} \quad \text{for } x \geq 60184 \quad \text{[2 p. 9, Theorem 6.9]} \tag{3}$$

$$\pi(x) < \frac{x}{\log x - 1.2} \quad \text{for } x \geq 4 \quad \text{(from [3] + computer check for } x < 10^5). \tag{4}$$
Taking the log of both sides of (1) and rearranging, we find that the Firoozbakht conjecture (1) is equivalent to

\[ \pi(p_k) < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \]  
(5)

If we know \( p_k \) and \( p_{k+1} \) (where \( p_k > 60184 \)) but do not know \( \pi(p_k) \), then instead of (5) we may check the stronger condition

\[ \frac{p_k}{\log p_k - 1.1} < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \]  
(6)

For a larger range of applicability \( (p_k > 4) \) we may check another (still stronger) condition:

\[ \frac{p_k}{\log p_k - 1.2} < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \]  
(7)

If (6) or (7) is true, so is (5); and the exact value of \( \pi(p_k) \) is not needed to check (6), (7).

4 Verification for all gaps of a given size \( g \)

We will take (6) and (7) one step further and make \( p_k \) a variable \( (x) \); then \( p_{k+1} = x + g \), where \( g \) is the gap size. We can now solve the resulting simultaneous inequalities

\[ 0 < \frac{x}{\log x - 1.1} < \frac{\log x}{\log(x + g) - \log x} \quad \text{with} \quad x > 60184, \]  
(8)

or, if we are interested in a larger range of applicability,

\[ 0 < \frac{x}{\log x - 1.2} < \frac{\log x}{\log(x + g) - \log x} \quad \text{with} \quad x > 4. \]  
(9)

Here we use the gap size \( g \) as a parameter. In combination with a table of first-occurrence prime gaps \[5\], the solution of (8) and/or (2) will tell us whether a prime gap of size \( g \) may violate the Firoozbakht conjecture for primes \( p_k \approx x \). Consider the following examples.

**Example 1.** Can a prime gap of size 150 violate the Firoozbakht conjecture? To answer this question, we substitute \( g = 150 \) into (8),

\[ 0 < \frac{x}{\log x - 1.1} < \frac{\log x}{\log(x + 150) - \log x} \quad \text{with} \quad x > 60184, \]

\[ \text{regardless of the computation of Sect. 2 which already proves (1) for all} \quad p_k < 10^6 \ldots \]
solve for $x$ and find the “safe bound”

$$x \geq 365323 \quad (\text{or, more precisely, } x > 365322.7038);$$

that is, a gap of 150 does not violate (1) if such a gap occurs between primes above 365323. But there are no prime gaps of size 150 below 365323; in fact, the table of first-occurrence prime gaps [5] indicates that the first such gap follows the prime 13626257. Therefore, a prime gap of size 150 can never violate the Firoozbakht conjecture (1).

**Example 2.** Can a prime gap of size 2 (twin primes) violate the Firoozbakht conjecture? We substitute $g = 2$ into (9),

$$0 < \frac{x}{\log x - 1.2} < \frac{\log x}{\log(x + 2) - \log x} \quad \text{with} \quad x > 4,$$

solve for $x$ and find the “safe bound”

$$x \geq 8 \quad (\text{or, more precisely, } x > 7.8745).$$

So a prime gap of size 2 does not violate (1) if this gap occurs between primes above 8. But we already know that gaps between primes below 8 do not violate the conjecture either (Section 2). Therefore, a prime gap of size 2 (twin primes) can never violate (1).

We have repeated the computation of the above examples for all even values of the gap size $g \in [2, 1476]$ and found that none of these gap sizes could possibly violate (1). A tabulation of “safe bounds” by gap size is available on the author’s website [4]. For $g = 2$ and 4, we had to manually check (1) for a couple of gaps of size $g$ between primes below the respective safe bounds. For $g \in [6, 1476]$, the actual first occurrence of prime gap $g$ is already safe. (“Close calls” occur for record prime gaps in OEIS sequence A005250 [9].) From the prime gaps table [5] we also know that gaps larger than 1476 do not occur below $4 \times 10^{18}$. Thus the validity of Firoozbakht conjecture (1) has been verified for primes up to $4 \times 10^{18}$. We have obtained the following theorem:

**Theorem** Inequality (1) is true for all primes $p_k < 4 \times 10^{18}$.

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Endnotes. All first-occurrence gaps between primes \( p_k < 2^{64} \) have been found by 2019. The results are reported at Mersenneforum.org and included in T.R. Nicely’s tables [5], as well as in OEIS [9, A014320, A335366, A335367]. Using the method of Section 4 these results allow us to confirm that Firoozbakht’s conjecture (1) holds for all primes \( p_k < 2^{64} \).

As in Example 1, the WolframAlpha command

\[
solve 0 < x/(\ln(x)-1.1) < \ln(x)/(\ln(x+1918)-\ln(x))
\]

uses gap \( g = 1918 \) and yields a safe bound a little below \( 2^{64} \). Even lower safe bounds are obtained with the same command for gaps \( g < 1918 \) — and the actual first occurrences of gaps \( g \) are already safe (i.e., the corresponding primes \( p_k \) are larger than their respective safe bounds). Thus, extended computations have shown that

- Firoozbakht’s conjecture (1) is true for all primes \( p_k < 2^{64} \);
- prime gaps of size \( g < 1920 \) cannot violate (1).

(Endnotes added 5 Jan 2023. The URLs [5] updated.)

References

[1] H. Cramér, On the order of magnitude of the difference between consecutive prime numbers. Acta Arith. 2 (1936), 23-46.
[2] P. Dusart, Estimates of some functions over primes without R.H., arXiv:1002.0442 (2010). http://arxiv.org/abs/1002.0442
[3] A. Kourbatov, The distribution of maximal prime gaps in Cramér’s probabilistic model of primes. Int. Journal of Statistics and Probability 3 (2014), No. 2, 18-29. arXiv:1401.6959. http://arxiv.org/abs/1401.6959
[4] A. Kourbatov, Checking the Firoozbakht conjecture: safe bounds, 2015. http://www.javascripter.net/math/primes/firoozbakhtconjecturebounds.htm
[5] T. R. Nicely, First occurrence prime gaps, preprint, 2014 and later years. https://faculty.lynchburg.edu/~nicely/gaps/gaplist.html http://oeis.org/A000101/a000101.pdf
[6] T. Oliveira e Silva, S. Herzog, and S. Pardi, Empirical verification of the even Goldbach conjecture and computation of prime gaps up to \( 4 \cdot 10^{18} \), Math. Comp. 83 (2014), 2033-2060. http://dx.doi.org/10.1090/s0025-5718-2013-02787-1
[7] P. Ribenboim, The Little Book of Bigger Primes, Springer, New York, NY, 2004. http://dx.doi.org/10.1007/b97621
[8] C. Rivera (ed.), Conjecture 30. The Firoozbakht Conjecture, 2002. http://www.primepuzzles.net/conjectures/
[9] N. J. A. Sloane (ed.), The On-Line Encyclopedia of Integer Sequences, 2014. Published electronically at http://oeis.org/ Sequences A014320, A235402, A335366, A335367.