Nambu–Goto string with the Gauss–Bonnet term and point–like masses at the ends

Leszek Hadasz† and Tomasz Róg‡

Jagellonian University,
M.Smoluchowski Institute of Physics
Reymonta 4, 30–059 Kraków, Poland

Abstract

In the present paper we investigate classical dynamics of the Nambu–Goto string with Gauss–Bonnet term in the action and point–like masses at the ends in the context of effective QCD string. The configuration of rigidly rotating string is studied and its application to phenomenological description of meson spectroscopy is discussed.

TPJU – 15/96
July 1996

†E–mail: hadasz@thp1.if.uj.edu.pl
‡E–mail: ufrog@thp1.if.uj.edu.pl
The existence of some, at least approximate string representation of QCD, although not rigorously proved, seems to be plausible and useful \[1\]. It is supported by the nature of the $1/N_c$ expansion \[2\], the success of dual models in description of Regge phenomenology, area confinement law found in the strong coupling lattice expansion \[3\] or the existence of flux–line solutions in confining gauge theories \[4, 5\] and the analytical results concerning two–dimensional QCD \[6\]. More arguments can be found in review \[7\].

It is an obvious statement that the role of boundary conditions imposed on the fields can be crucial for the predictions of a theory. The same is also true in string theory — for example, the spectra of states in the case of closed Nambu–Goto \[8\] string theory (with the periodic boundary conditions) and in the case of the open Nambu–Goto string with free ends, satisfying the Neumann boundary conditions, are essentially different. Moreover, in the effective string description of QCD quark’s internal degrees of freedom (colour and electric charges, spin or mass) enter the theory through the boundary conditions, \[9\].

Motivated by such considerations we are led to study the impact of a modifications in the boundary conditions on the string’s dynamics. We shall consider in the present paper the simplest, Nambu–Goto string model, being also the most natural zeroth order approximation to the QCD string.

If we confine ourselves to the string action which is a total derivative (and consequently affects only the boundary conditions for the string), depends only on the first and second order derivatives of the vectors $X^\mu(\tau, \sigma)$ defining the immersion of the string in the four dimensional, Minkowski space–time, and satisfying the natural requirements of being Poincare’ as well as reparametrization (with respect to $\tau$ and $\sigma$) invariant, then we are led \[10\] to the unique possibility,

\[ S_b = -\frac{\alpha}{2} S_{GB} - \beta S_{Ch}. \]  

Here $\alpha$ and $\beta$ are dimensionless parameters ($\alpha > 0$). $S_{GB}$ and $S_{Ch}$ are pseudoeuclidean Gauss–Bonnet and Chern terms which, upon Wick rotation to the Euclidean space, are related to the Euler characteristic of the surface and to the number of its self–intersections, respectively.

If one further wishes to take into account the non–zero masses of quarks, then it is naturally to add to the action the contributions from the massive point–like particles attached to the string ends,

\[ S_p = -m_1 L_1 - m_2 L_2, \]  

where $L_1$ and $L_2$ are invariant lengths of the trajectories of the string ends.

More explicitly, the action of the Nambu–Goto string with modified boundary conditions and point–like masses at the ends is of the form:

\[ S = \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{-g} \mathcal{L} - \sum_{i=1}^{2} m_i \int_{\tau_1}^{\tau_2} dt \sqrt{(d_t X)^2} |_{\sigma = \sigma_i(\tau)}, \]  

with

\[ \mathcal{L} = -\gamma - \frac{1}{2} \alpha R - \beta N. \]
Here $\gamma$ is the string tension, $t$ parametrizes points along the trajectories of the string ends and $g = \det g_{ab}$ with $g_{ab} = \partial_a X^\mu \partial_b X_\mu$ being the induced metrics. The curvature scalar $R$ and the integrand of the Chern term $N$ are defined according to the formulae

$$R = (g^{ab} g^{cd} - g^{ad} g^{bc}) \nabla_a \nabla_b X^\mu \nabla_c \nabla_d X_\mu,$$

$$N = -\frac{1}{2\sqrt{-g}}g^{ac} \epsilon^{bd} \tilde{\epsilon}^{\mu\nu} \nabla_a \nabla_b X^\mu \nabla_c \nabla_d X_\mu,$$

where $\nabla_a$ is the covariant (with respect to the metric $g_{ab}$) derivative and 

$$\tilde{\epsilon}^{\mu\nu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\lambda} \partial_\tau X_\rho \partial_\sigma X_\lambda.$$

In the following we shall assume that the space–like parameter $\sigma$ varies in the interval $[0, \pi]$. It is also convenient to fix the worldsheet parametrization by imposing the conditions [11]:

$$(\dot{X} \pm X')^2 = 0,$$

$$(\ddot{X} \pm \dot{X}')^2 = -\frac{1}{4}q^2,$$

where the dot and the prime mean differentiation with respect to $\tau$ and $\sigma$, respectively, and $q$ is an arbitrary parameter with the dimension of mass. It can be shown then (for details see [10, 11, 12]), that every solution of the string equations of motion and boundary conditions, following from the action (3), corresponds to the solution of the complex Liouville equation [13]:

$$\ddot{\Phi} - \Phi'' = 2q^2 e^\Phi,$$

supplemented with the boundary conditions:

$$\gamma - \alpha q^2 e^{2\Re\Phi} = (-1)^i m_i \frac{\partial}{\partial \sigma} \left( e^{\Re\Phi/2} \right),$$

$$C \frac{\partial}{\partial \tau} \Re\Phi = 0,$$

$$C \cos \left( \Im\Phi/2 \right) = \beta,$$

$$C^2 \frac{\partial}{\partial \sigma} \Im\Phi = 2\beta(-1)^i m_i e^{-\Re\Phi/2},$$

for $\sigma = 0, \pi$,  

where $C = \sqrt{\alpha^2 + \beta^2}$. This correspondence is explicitly established through the relations:

$$e^\Phi = -\frac{1}{q^2 \sin^2 \left[ \frac{F_L(\tau+\sigma)-F_R(\tau-\sigma)}{2} \right]} \frac{F_L'(\tau+\sigma)F_R'(\tau-\sigma)}{F_L(\tau+\sigma)-F_R(\tau-\sigma)},$$

$$X^\mu(\tau, \sigma) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma),$$

$$\frac{\partial}{\partial \tau} X^\mu_{L,R} = \frac{q}{2|F'_{L,R}|} (\cosh \Im F_{L,R}, \cos \Re F_{L,R}, \sin \Re F_{L,R}, \sinh \Im F_{L,R}),$$

$$2$$
where $F_{L,R}$ are arbitrary complex functions which give single valued $\Phi$ satisfying the boundary conditions (10).

We shall also need the formulae for the string’s momentum and angular momentum. The contributions to these quantities from the particles located at the string ends are standard so there is no need to write them down here explicitly while the contributions from the surface terms in (3) can be obtained from the general formulae:

$$P_\mu = \int_0^\pi d\sigma \sqrt{-g} \pi_\mu^0 - \left[ \sqrt{-g} \frac{\partial L}{\partial (\nabla_0 \nabla_1 X^\mu)} \right]_0^\pi,$$

$$M_{\mu\nu} = \int_0^\pi d\sigma \sqrt{-g} \left\{ X_{[\mu} \Pi_{\nu]}^0 - X_{[\mu,a} \frac{\partial L}{\partial (\nabla_a \nabla_0 X^\nu)} \right\} - \left[ \sqrt{-g} X_{[\mu} \frac{\partial L}{\partial (\nabla_0 \nabla_1 X^\nu)} \right]_0^\pi,$$

where

$$\Pi_\mu^0 = -\mathcal{L} \nabla^a X_\mu - \frac{\partial L}{\partial X^\mu_a} + 2 \frac{\partial L}{\partial g^{bc}} g^{ab} \nabla^c X_\mu + \nabla_b \left[ \frac{\partial L}{\partial (\nabla_a \nabla_0 X^\nu)} \right].$$

The formulae (14,15) can be applied in the case of general lagrangian $\mathcal{L}$ being the scalar function of the vector $X^\mu(\tau, \sigma)$ and its covariant derivatives up to the second order and turn out to be quite useful in the practical calculations involving the action (3).

In the following we shall consider only the special case $\beta = 0$ of the general model described by the action (3). The reason for this is that the generic classical solutions of the model with the Chern term correspond to the self–accelerating strings which are hard to interpret from the point of view of the effective QCD string theory and, moreover, this solutions exist only if some (different for different solutions) relations between the parameters of the model ($\gamma, \alpha, \beta$ and $m_i$–s) are satisfied.

A distinguished class of solutions of the Liouville equation (9) is composed of static, i.e. $\tau$–independent fields. They are of the form

$$e^{\Phi} = -\frac{\lambda^2}{q^2} \frac{1}{\cos^2 (\lambda\sigma - d)},$$

where $\lambda$ and $d$ are arbitrary, complex constants.

From the Eqs. (16) and (11) we get

$$F_L = \lambda(\tau + \sigma) - d,$$

$$F_R = \lambda(\tau - \sigma) + d - \pi.$$

To check whether this is actually a solution we have to take into account the boundary conditions (10). They imply that $\lambda$ and $d$ are real numbers satisfying the equations:

$$\frac{\gamma q}{\lambda^2} \cos^4 d - m_1 \sin d \cos^2 d - \frac{\alpha \lambda^2}{q} = 0,$$

$$\frac{\gamma q}{\lambda^2} \cos^4(\pi \lambda - d) - m_2 \sin(\pi \lambda - d) \cos^2(\pi \lambda - d) - \frac{\alpha \lambda^2}{q} = 0.$$
Equations (18) can be easily solved numerically for any specific values of the parameters $\gamma, \alpha$ and $m_i$, giving a family of solutions parametrized by the constant $q$.

In the language of string variables the static Liouville field (16) describes the string which rotates rigidly in the $X^3 = 0$ plane,

$$X^\mu(\tau, \sigma) = \frac{q}{\lambda^2} (\lambda \tau, \cos \lambda \tau \sin(\lambda \sigma - d), \sin \lambda \tau \sin(\lambda \sigma - d), 0). \quad (19)$$

Using the formulae (14,15) we can calculate its energy and the third component of the angular momentum,

$$E = \frac{\pi q \gamma}{\lambda} \left[ 1 + \frac{\sin \pi \lambda \cos(\pi \lambda - 2d)}{\pi \lambda} \right] + m_1 \cos d + m_2 \cos(\pi \lambda - d), \quad (20)$$

$$J = \frac{\pi q^2 \gamma}{2 \lambda^3} \left[ 1 + 2 \frac{\sin \pi \lambda \cos(\pi \lambda - 2d)}{\pi \lambda} + \frac{\sin 2 \pi \lambda \cos(2 \pi \lambda - 4d)}{2 \pi \lambda} \right] -$$

$$- \frac{m_1 q}{\lambda^2} \sin^2 d \cos d - \frac{m_2 q}{\lambda^2} \sin^2 (\pi \lambda - d) \cos(\pi \lambda - d), \quad (21)$$

with the total spatial momentum and the other components of the angular momentum being zero. The values of $\lambda = \lambda(q)$ and $d = d(q)$ which appear in (20) and (21) are determined from the Eqs. (18).

String which rigidly rotates in a plane constitutes a configuration with maximal angular momentum at a given energy. Such configurations compose a classical, leading Regge trajectory. For large values of $q$ one can determine using (18), (20) and (21) the asymptotic form of the relationship between the angular momentum and energy squared to be

$$J = \frac{E^2}{2 \pi \gamma} + \frac{1}{3 \gamma} \sqrt{\frac{2}{\pi}} \left[ \left( \sqrt{m_1^2 + 4 \alpha \gamma} - 2m_1 \right) \sqrt{m_1 + \sqrt{m_1^2 + 4 \alpha \gamma}} + (m_1 \to m_2) \right] E^\frac{1}{2} + \mathcal{O}(E^0). \quad (22)$$

This formula clearly shows that the pertinent Regge trajectory in the discussed model can be raised or lowered with respect to the Regge trajectory in the “pure” Nambu–Goto model,

$$J_{NG} = \frac{1}{2 \pi \gamma} E_{NG}^2,$$

depending on the ratio of the point–like masses $m_i$ squared to the product $\alpha \gamma$.

Although the discussed model is rather simple it is amusing to compare its predictions with the data concerning the spectrum of mesons. One finds that the parameters of the model can be nicely fitted to the energy and angular momentum of the mesonic states lying on the leading Regge trajectories both for the heavy–heavy and heavy–light systems (which is not surprising, as the model of Nambu–Goto string with point–like masses and without Gauss–Bonnet term already reproduces the data quite well, \cite{14}) and for the light–light systems, where the model without Gauss–Bonnet term breaks down. The example is plotted on the Fig.1.
Fig. 1 Regge trajectory of the model fitted to the data from the meson family. The values of the parameters are $m_1 = m_2 = 0, \gamma = 1.96 \times 10^5 (MeV)^2$ and $\alpha = 0.19$.

Let us note a few facts:

- the (approximately) linear asymptotes of the classical Regge trajectories obtained in the discussed model have non–zero intersection with the angular momentum axis, while in the “pure” Nambu–Goto case the quantum corrections are needed to achieve this,

- the “dother” Regge trajectories can be described by considering oscillations of the string around the “rotating rigid rod” configuration (their explicit form can be obtained in a way analogous to the one used in [15]),

- the fitted values of the string tension $\gamma$ and the parameter $\alpha$ are approximately independent of the meson family (i.e. are approximately the same for the heavy–heavy, heavy–light and light–light systems),

- for the light-light systems (excluding the lightest meson in each family — we shall comment on this below) we obtain the quark masses equal approximately to zero, i.e. the point–like masses in the model seems to be connected with the light, current quarks rather then with the heavy constituent quarks of the non–relativistic quark model.
The last point is probably less surprising if one takes into account that addition to the Nambu–Goto string of the Gauss–Bonnet term already results in a non–uniform distribution of the energy density, with the maxima at the string ends, \[13\].

The model seems to have problems with the light mesons of zero spin. Indeed, as illustrated on the Fig.1, for the zero point–like masses the classical solution with zero angular momentum has vanishing energy. But the region of slowly rotating string is precisely the one where the quantum corrections become important. In the paper \[16\] the first quantum correction (the Casimir energy, \[17\]) to the classical energy of the rigidly rotating string configuration in the model of Nambu–Goto string with the Gauss–Bonnet term has been computed using the semiclassical approximation. This correction grows indefinitely for the strings rotating with vanishing frequency, invalidating there the classical picture. The inclusion of point–like masses at the string ends does not qualitatively change this result and, similarly as in the model without point–like masses, the lightest string solution has non–zero energy. It is then naturally to expect that this quasiclassically corrected, lightest string configuration will be a correct candidate for the scalar meson in the discussed effective model.

Acknowledgments

The authors would like to thank dr P. Wegrzyn for inspiration and numerous helpful discussions. One of us (L.H.) was supported by grant KBN 2 P03B 045 10 and by Foundation for Polish Science scholarship.

References

[1] A.M.Polyakov, \textit{Gauge fields and strings,} Harwood Academic Press, 1987.

[2] G.'t Hooft, Nucl.Phys \textbf{B72} (1974) 461.

[3] K.Wilson, Phys.Rev \textbf{D8} (1974) 2445.

[4] H.B.Nielsen and P.Olesen, Nucl.Phys. \textbf{B61} (1973) 45.

[5] M.Baker, J.S.Ball and F.Zachariasen, Phys.Rev. \textbf{D41} (1990) 2612.

[6] David J.Gross, Nucl.Phys. \textbf{B400} (1993) 161. David J.Gross and Washington Taylor IV, Nucl.Phys. \textbf{B400} (1993) 181. David J.Gross and Washington Taylor IV, Nucl.Phys. \textbf{B403} (1993) 395.

[7] J.Polchinski, \textit{Strings and QCD?}, preprint UTTG-92-16 and \texttt{hep-th/9210045}.

[8] Y.Nambu, Lectures on the Copenhagen Summer Symposium (1970), unpublished, O.Hara, Prog.Theor.Phys. \textbf{46} (1971) 1549, T.Goto, Prog.Theor.Phys. \textbf{46} (1971) 1560, L.N.Chang and J.Mansouri, Phys.Rev. \textbf{D5} (1972) 2535, J.Mansouri and Y.Nambu, Phys.Lett. \textbf{39B} (1072) 375.

[9] L.Hadasz, Phys.Lett. \textbf{B324} (1994) 36, L.Hadasz, Acta Phys.Pol. \textbf{B25} (1994) 1419.
[10] P.Węgrzyn, Phys.Rev. D50 (1994) 2769.

[11] B.M.Barbashov, V.V.Nesterenko Introduction to the relativistic string theory, World Sci., Singapore 1990.

[12] P.Węgrzyn, Strings with interacting ends, preprint TPJU–23/94, Mod.Phys.Lett. A, in press; J.Karkowski, Z.Świerczyński and P.Węgrzyn, Nambu–Goto string action with Gauss-Bonnet term, preprint TPJU–24/94, Mod.Phys.Lett. A, in press.

[13] J.Liouville, Math.Phys.P.Appl. 18 (1853) 71, A.R.Forshyt, Theory of Differential Equations, Part 4, vol. 6, New York, Dover, 1959, G.P.Jorjadze, A.K.Pogrebkov, M.C.Polivanov, Doklady Akad. Nauk SSSR 243 (1978) 318.

[14] B.M.Barbashov, Classical dynamics of Rotating Relativistic String with Massive Ends: the Regge Trajectories and Quark Masses, preprint JINR–E2–94–444, submitted to Nucl.Phys. B.

[15] L.Hadasz and P.Węgrzyn, Phys.Rev. D51 (1995) 2891.

[16] L.Hadasz, Ground state energy of the Nambu–Goto string with modified boundary conditions, preprint TPJU 14/96, submitted to Mod.Phys.Lett. A.

[17] H.B.G.Casimir, Proc.Kon.Nederl.Akad.Wetenschap 51 (1948) 739.