DISCRETE APPROXIMATIONS OF METRIC MEASURE SPACES
OF CONTROLLED GEOMETRY

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Abstract. We find a necessary and sufficient condition for a doubling metric space to carry a $(1, p)$-Poincaré inequality. The condition involves discretizations of the metric space and Poincaré inequalities on graphs.

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1. Introduction

It is well known that a doubling metric space which also supports some type of Poincaré inequality enjoys many other useful properties (see [12] and the upcoming [13] for examples). However, it is often the case that when presented with an arbitrary metric measure space $(X, d_X, \mu)$, verifying that it satisfies a Poincaré-type inequality is difficult. In this paper we present a method of discretizing a metric measure space that is doubling and supports a Poincaré type inequality (see Section 2 for these definitions). The constructed discretized space will (1) retain the doubling property as well as (2) support its own Poincaré type inequality. We also show that a doubling metric measure space $(X, d_X, \mu)$ only has a Poincaré inequality if some such discretization exists. By discretizing the space, the advantage is that we may verify these properties by checking only a finite number of points for each ball $B \subset X$. With enough symmetry or regularity of a space, this may be simple, as we see in the example in Section 7. This transforms the possibly difficult problem of verifying the doubling and Poincaré properties into a problem that is more computationally feasible. The method of discretization has been well studied in the study of analysis on metric measure spaces. L. Ambrosio, M Colombo, and S. Di Marino used an analog of dyadic cubes, introduced by M. Christ, to study the theory of Sobolev spaces on metric measure spaces (see [1] and [7]). This approach to studying metric measure spaces follows the work of R.R. Coifman and G. Weiss (see [8]). We will use a method of using maximally $\epsilon$-separated subsets to discretize metric measure spaces that also follows their work, but requires different assumptions on our space.

Our method is analogous to that used by P. Herman, R. Peirone, and R. Strichartz to study $p$-energy on the Sierpinski gasket (See [14]). Their work focused on constructing energy forms on the gasket via natural energy forms on discrete approximating graphs. In our paper we are not interested in approximating energies on the metric space since the metric space is already equipped with the energy from the upper gradient structure; we focus instead on Poincaré inequalities (which,

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in turn, are not available in [14]). There has been work in modifying the natural metric on the Sierpinski gasket in order to ensure a Poincaré inequality, and this harmonic Sierpinski gasket discussed in [16] by N. Kajino, and Kusunoki in [20], uses a metric change that may not be bi-Lipschitz, and does not preserve many aspects of the original space. There has also been work studying the limits of Dirichlet forms on post-critically finite fractals following the work of Barlow and Bass (see [2], [11], [21], and [19]). However, these works are done on connected metric graphs, where here we present a discrete condition on highly non-connected spaces. Also, recent notes by J. Cheeger and B. Kleiner [5] and [6] studies Poincaré inequalities on discrete spaces and inverse limits, a different approach than our note here that restrict their scope to metric spaces that are topologically of dimension 1. In their work, they show that a metric space satisfies the (1,1)-Poincaré inequality if it is possible to construct an “inverse limit”, or equivalently a Gromov-Hausdorff limit. In our paper, we show an approach that holds for (1,p)-Poincaré inequalities with $p \geq 1$.

The setting considered in this paper is that of a general metric space $X$, endowed with a metric $d_X$ and a doubling Borel regular measure $\mu$; see Section 2 for precise definitions. We will construct a metric measured graph $(V, d_V, m)$ based on $X$ such that $m$ is a doubling measure, and show that $V$ also supports a Poincaré type inequality when $X$ does. Throughout this note $1 \leq p < \infty$. Our main results are as follows:

**Theorem 1.1.** Let $(X, d_X, \mu)$ be a complete doubling metric measure space that supports a $(1,p)$-Poincaré inequality. Then any discretized space $(V, d_V, m)$, constructed from $(X, d_X, \mu)$ in the manner given in Section 3, is also doubling and supports a $(1,p)$-Poincaré inequality with data quantitatively derived from the data of $(X, d_X, \mu)$.

We then turn our attention to the converse of Theorem 1.1, which requires some preliminary definitions. Let $(V, d_V, m)$ be a graph with metric $d$ and measure $m$. By graph we mean a set of vertices $V$ with an associated edge set $E$, which we suppress in the notation by only referring to the graph as $V$. In this paper for each graph there is a constant $\epsilon_V$ so that if vertices $x$ and $y$ are connected by an edge, the distance $d_V(x,y) = \epsilon_V$. Distances between other vertices $x$ and $y$ are defined via $n \cdot \epsilon_V$ where $n$ is the smallest length of a sequence $x = x_0, x_1, x_2, ..., x_n = y$ where $x_i$ and $x_{i+1}$ are connected by an edge. By $B_V(x, r)$ we mean all vertices of distance strictly less than $r$ from $x$. The measure $m$ is simply an assignment of a positive mass to each vertex and, as it is discrete, it is defined on all subsets of $V$.

For a (discrete) graph $(V, d_V, m)$ and a metric measure space $(X, d, \mu)$ an embedding of $V$ into $X$ is a one-to-one map from the vertex set $V$ into the space $X$. For a sequence of graphs $(V_i, d_{V_i}, m_i)$ with $V_i \subset V_{i+1}$ for $i \geq 0$ by a nested embedding of $(V_i, d_{V_i}, m_i)$ into $X$ we mean a sequence of embeddings $n_i : V_i \rightarrow X$ such that $n_{i+j}|_{V_i} = n_i$ for $i, j \geq 0$. Note this definition of nested embedding does not imply that if $x$ and $y$ are connected by an edge in $V_i$, they are still connected by an edge in $V_{i+1}$. In general, they will not be connected by an edge in $V_{i+1}$. Our second result shows that the discretization from Theorem 1.1 can also yield information about the space $(X, d, \mu)$.

**Theorem 1.2.** Let $(X, d, \mu)$ be a complete doubling metric measure space. Then $(X, d, \mu)$ supports a $(1,p)$-Poincaré inequality if and only if there exists a nested embedded sequence of graphs $(V_i, d_{V_i}, m_i)$ into $X$ such that

1. The Hausdorff distance, $d_H(n_i(V_i), X) = H_i$, is finite for all $i \geq 0$ and $H_i \rightarrow 0$ as $i \rightarrow \infty$.

2. There is a uniform $L > 1$ such that for all $i \geq 0$ and all $x, y \in V_i$,

$$\frac{1}{L} d(n_i(x), n_i(y)) \leq d_{V_i}(x, y) \leq L d(n_i(x), n_i(y))$$
(3) There is a uniform $K > 1$ such that for all $i \geq 0$ all $r > H_i$ and $x \in V_i$

$$\frac{1}{K} \leq \frac{m_i(B_{V_i}(x,r))}{\mu(B_X(x,r))} \leq K$$

(4) $(V_i, d_{V_i}, m_i)$ are all doubling metric measure spaces with uniform doubling constant.

(5) $(V_i, d_{V_i}, m_i)$ all support a $(1,p)$-Poincaré inequality with uniform data.

For the definition of Hausdorff distance see (6.1), for the definition of doubling metric space see Section 2, and for the type of Poincaré inequality assumed see (3.1).

In Section 2, we review applicable definitions for this paper. Section 3 focuses on constructing $V$, a discretization of $X$, and endowing the set with a metric $d_V$ and measure $m$ that are derived from $d_X$ and $\mu$. In Section 4 we verify that $(V, d_V, m)$ also satisfies the doubling property. Section 5 is dedicated to showing that $(V, d_V, m)$ satisfies a discretized version of the Poincaré inequality. Sections 4 and 5 together provide the proof of Theorem 1.1. In Section 6, we review the pertinent definitions of pointed measured Gromov-Hausdorff convergence which will be necessary to our proof of Theorem 1.2. Section 7 discusses an example from Euclidean space showing the necessity of the conditions in Theorem 1.2 as well as showing how one might check for a discrete Poincaré inequality. Finally, Section 8 is dedicated to the proof of Theorem 1.2.

2. Preliminaries

In this section we introduce some necessary definitions. All of this section is standard and may be skipped by the expert on metric measure spaces. A nontrivial locally finite Borel regular measure $\mu$ on a metric space $(X, d_X)$ is called a doubling measure if every metric ball, $B$, has positive and finite measure and there exists a constant, $C \geq 1$, such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r))$$

for each $x \in X$ and $r > 0$. We call the triple $(X, d_X, \mu)$ a doubling metric measure space if $\mu$ is a doubling measure on $X$. The smallest constant $C \geq 1$ such that the above inequality holds is referred to as the doubling constant $C_\mu$ of $\mu$. An $\epsilon$-separated set, $\epsilon > 0$, in a metric space is a set such that every two distinct points in the set are at least $\epsilon$ distance apart. Given a metric space $X$, an $\epsilon$-separated set $A \subset X$ is said to be maximal if for any $x \in X \setminus A$, the distance from $x$ to $A$ is less than $\epsilon$. The metric $d_X$ is said to be doubling metric with constant $N$ if $N \geq 1$ is an integer such that for each ball $B(x,r) \subset X$, every $\frac{r}{2}$-separated set in $B(x,r)$ has at most $N$ points.

It is easy to show that if $(X, d_X, \mu)$ is a doubling metric measure space, then $d_X$ is also a doubling metric with some constant that depends only on $C_\mu$: let $B(x,r)$ be given. Let $A$ be some maximal $r/2$ separated set of $X$. To see that a maximal $r/2$-separated subset of $X$ exists, see Chapter 10 of [12]. If $A \cap B(x,r)$ contains $l$ points, $a_1, a_2, \ldots, a_l$, where $I \subset \mathbb{N}$ is an indexing set, then the set of balls $\{B(a_i, r/2)\}_{i \in I}$ cover $B(x,r)$ by the maximality of $A$. Note that the balls $\{B(a_i, r/4)\}_{i \in I} \subset B(x,2r)$, and are pairwise disjoint. Then for $N \in I$,

$$N\mu(B(x,2r)) \leq \sum_{i=1}^{N} C_\mu^4 \mu(B(a_i, \frac{r}{2})) \leq C_\mu^4 \sum_{i=1}^{N} \mu(B(a_i, \frac{r}{4})) \leq C_\mu^4 \mu(B(x,2r)).$$

Thus, $N \leq C_\mu^4$ when $\mu$ is locally finite. Notice that the assumption that $\mu$ is positive on balls also implies that $N$ must be a finite number, because to be infinite would imply that $\mu(B(x,2r))$ is infinite which contradicts our assumption of $\mu$ being locally finite. Since this holds for all $N \in I$, then the cardinality of $I$ must also be less than or equal to $C_\mu^4$. 


The above metric property was formulated by Coiffman and Weiss in [8], and proves to be of great importance in the study of Sobolev spaces. In particular, doubling spaces can be shown to be separable. If, in addition to being doubling, a metric space is complete, then it is proper. Note that as in the statements of Theorems 1.1 and 1.2 will always assume that $X$ is equipped with a complete metric $d$, and a locally finite Borel regular measure $\mu$.

Let $u$ be a real-valued measurable function on $X$. A non-negative Borel function $\rho : X \to [0, \infty]$ is said to be an upper gradient of $u$ if for all compact rectifiable paths $\gamma : [a, b] \to X$, the following inequality holds:

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} \rho \, ds$$

where $ds$ is the arc-length measure on $\gamma$, induced by the metric $d_X$ on $X$. (see Chapter 7 of [12]). A separable metric measure space $(X, d_X, \mu)$ is said to support a $(1, p)$-Poincaré inequality if every ball $B \subset X$ has positive and finite measure and if there exist constants $C > 0, \lambda \geq 1$ such that

$$(2.1) \quad \int_B |u - u_B| d\mu \leq C r \left( \int_{\lambda B} \rho^p \, d\mu \right)^{\frac{1}{p}}$$

for every measurable function $u : X \to \mathbb{R}$ that is integrable on balls and every upper gradient $\rho$ of $u$. In the above inequality, when the center and radius are clear from context, $B$ is written as the shorthand of $B(x, r)$ and $\lambda B := B(x, \lambda r)$. The notation of $\int_B$ is the average integral over the ball $B$. That is,

$$\int_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu \quad =: \quad u_B$$

for any integrable function $u$ on $B$. The parameters $p, C$, and $\lambda$ are called the data of the Poincaré inequality.

3. Construction of the approximating graphs

Let $(X, d_X, \mu)$ be a doubling metric measure space with doubling constant $C_\mu$, and let $A \subset X$ be a maximal $\epsilon$-separated set for some given $\epsilon > 0$. For each $x \in A$, we associate a vertex $\tilde{x} \in V_\epsilon := V$. We say that $\tilde{x} \sim \tilde{y}$ if and only if $\epsilon \leq d(x, y) \leq 3\epsilon$. We let $\sim$ define an edge set. We will use this relation between points in $V$ to define a metric on $V$. It is worth noting here, that $(V, \sim)$ is a discrete graph, possessing only a discrete topology. In Section 8, this graph will be extended to a connected graph, but for the majority of this paper, all of the calculations involving $V$ will only use points on the vertex set $V$. To highlight this point, we will often only refer to the vertex set $V$ which is in a 1-1 correspondence with the set $A \subset X$, and therefore may be thought of as an embedding in $X$. This canonical embedding can be given by identifying $x \in A$ with $\tilde{x} \in V$. Notice that we require the distance between two points of $A$ to be positive in order for a corresponding edge to be made in $(V, \sim)$. This is to ensure that $(V, \sim)$ has no loops of zero length. We define a distance on $V$, denoted $d_V$, such that $d_V(\tilde{x}, \tilde{y}) = \epsilon$ for all $\tilde{x} \sim \tilde{y}$. We extend this distance function for $\tilde{x}$ and $\tilde{y}$ that do not share an edge the distance between them is the obvious one as stated in the introduction after the statement of Theorem 1.1. It is clear that $d_V$ is a metric. The ball centered at $\tilde{x}$ with radius $r$ is denoted $B_V(\tilde{x}, r) := \{\tilde{y} \in V | d_V(\tilde{x}, \tilde{y}) < r\}$. Note that when $r \leq \epsilon$, we have that $B(\tilde{x}, r) = \{\tilde{x}\}$. We use $B_V$ to denote balls in the graph metric and $B_X$ to denote balls in the original metric space.

It can be shown that a complete and doubling metric measure space that supports a Poincaré inequality is $L$-quasiconvex. That is, for the space $X$, there is a constant $L \geq 1$, depending only on the doubling constant and the data from the Poincaré inequality, such that each pair of points $x, y \in X$ can be joined by a
rectifiable curve $\alpha$ in $X$ such that $\text{length}(\alpha) \leq Ld_X(x,y)$. This is result is due to S. Semmes, but proofs may be found in [18] and [10]. The quasiconvexity of the metric measure spaces allow us to observe a very useful property addressed in the following proposition.

**Proposition 3.1.** For a complete doubling metric space $(X, d_X)$ that is quasiconvex with constant $L$, the canonical embedding of $(V, d_V)$ into $X$ is bi-Lipschitz:

$$\frac{1}{L+1}d_V(\tilde{x}, \tilde{y}) \leq d_X(x, y) \leq 3d_V(\tilde{x}, \tilde{y})$$

**Proof.** We begin by showing the first inequality. Let $\tilde{x}$ and $\tilde{y}$ be two points in the vertex set $V$, and let $x$ and $y$ be their respective corresponding points in $X$. If $d_X(x, y) \leq 3\epsilon$ then $\tilde{x} \sim \tilde{y}$ or $\tilde{x} = \tilde{y}$. Thus, either $d_V(\tilde{x}, \tilde{y}) = \epsilon$ or $d_V(\tilde{x}, \tilde{y}) = 0$. The latter case satisfies the proposition trivially, and the former case follows by seeing that

$$\frac{\epsilon}{L+1} \leq 3\epsilon.$$

Thus, without loss of generality, we assume that $d_X(x, y) > 3\epsilon$. Let $\gamma$ be a rectifiable curve from $x$ to $y$ such that $\text{length}(\gamma) \leq Ld_X(x,y)$. Let $T := \text{length}(\gamma)$, and notice that $T > 3\epsilon$. Since $\gamma$ is rectifiable, we assign it the arc-length parameterization. Choose $K$ as the smallest integer such that $T \leq K\epsilon < Ld_X(x,y) + \epsilon$. Notice that by assumption, $d_X(x,y) > 3\epsilon$, so such $K$ exists. For $i = 0, 1, \ldots, K-1$ we choose $t_i = i\epsilon$, and define $t_K := T$. Then, for $i = 1, \ldots, K$ there are subcurves $\gamma_i := \gamma([t_{i-1}, t_i])$ of $\gamma$ such that $\text{length}(\gamma_i) = \epsilon$ with the exception of $\gamma_K$ which may have length less than or equal to $\epsilon$. Let $x_i := \gamma(t_i)$. By the maximality of $A$, for each $x_i$ there exists a point $z_i \in A$ such that $d_X(x_i, z_i) \leq \epsilon$. It is clear that we can choose $z_0 = x$ and $z_K = y$. We find that for $i = 1, \ldots, K$,

$$d_X(z_{i-1}, z_i) \leq d_X(z_{i-1}, x_{i-1}) + d_X(x_{i-1}, x_i) + d_X(x_i, z_i) \leq 3\epsilon.$$

Since each $z_i$ is in $A$, then it has a corresponding point $\tilde{z_i}$ in $V$. Hence, for each $i$ we have that $\tilde{z_i} \sim z_{i-1}$ or $z_i = z_{i-1}$. So $d_V(\tilde{z_i}, z_{i-1}) \leq \epsilon$, and

$$d_V(\tilde{x}, \tilde{y}) \leq K\epsilon \leq Ld_X(x,y) + \epsilon \leq Ld_X(x,y) + d_X(x,y) = (L + 1)d_X(x,y).$$

The second inequality follows easily from the definition of the distance on $V$ and the triangle inequality on $X$.

We now wish to equip $V$ with a measure $m$ which is related to $\mu$. By the maximality of $A$ (and its one to one correspondence to $V$), $X = \bigcup_{\tilde{x} \in V} B_X(x, \epsilon)$. For any $W \subset V$, we define

$$m(W) := \sum_{\tilde{y} \in W} \mu(B_X(y, \epsilon)).$$

For example, if $r < \epsilon$, then $m(B_V(\tilde{x}, r)) = \mu(B_X(x, \epsilon))$. In particular, for any $\tilde{x} \in V$, we set $m(\tilde{x}) = \mu(B_X(x, \epsilon))$. We see that $m$ is a measure on the $\sigma$-algebra generated by the open balls in $V$. We note that, in general,

$$m(W) \neq \mu \left( \bigcup_{\tilde{y} \in W} B_X(y, \epsilon) \right).$$

Hence, $(V, d_V, m)$ is a metric measure space. For $W \subset V$ and $u : V \to \mathbb{R}$, the definition of $\int_W u(\tilde{x})dm(\tilde{x})$ is given by

$$\int_W u(\tilde{x})dm(\tilde{x}) := \sum_{\tilde{x} \in W} u(\tilde{x})m(\tilde{x}).$$

When the context is clear, we will often use the following notation for a fixed $\tilde{x} \in V$:

$$\int_{\tilde{x} \sim \tilde{y}} u := \sum_{\tilde{x} \sim \tilde{y}} u(\tilde{y})$$
This helps us when we wish to sum only over neighbors, but should not be confused with an integral over the function $u$, as this is in itself a function on $V$ evaluated at the point $\tilde{x}$. This should be made clear from the lack of associated measure in the notation. In a similar manner to (2.1), we may define $u_{B'}$ and $f_{B'}$ for a function $u : V \to \mathbb{R}$:

$$u_{B'} := \int_{B'} u(\tilde{x}) dm(\tilde{x}) := \frac{1}{m(B')} \int_{B'} u(\tilde{x}) dm(\tilde{x})$$

to echo the meaning of $f_{B'} f(x) dx$. That is, $f_{B'} u(\tilde{x}) dm(\tilde{x})$ is an $m$-weighted average value of $u$ over the ball $B'$.

We now describe a discretized version of the $(1,p)$-Poincaré inequality that was introduced by I. Holopainen and P. Soardi in [15].

**Definition 3.2.** We say that $V$ supports a (discrete) $(1,p)$-Poincaré inequality if there exist some constants $C > 0$ and $\lambda \geq 1$ such that for all functions $u : V \to \mathbb{R}$, and each $B' = B(v, r) \subseteq V$,

$$(3.1) \quad \int_{B'} |u(\tilde{x}) - u_{B'}| dm(\tilde{x}) \leq C r \left( \int_{\lambda B'} \left( \int_{\tilde{x} \sim \tilde{y}} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{\epsilon^p} \right) dm(\tilde{x}) \right)^{\frac{1}{p}}.$$ 

Note that the quantity

$$\frac{|u(\tilde{x}) - u(\tilde{y})|}{\epsilon}$$

can be seen as a type of upper gradient when compared to $\rho$ in (2.1) if we consider the edge from $\tilde{x}$ to $\tilde{y}$ as an isometric copy of the interval $[0, \epsilon]$. We wish to rename the quantity on the right hand side of (3.1) for simplicity of exposition. Given a function $u$ on the vertex set $V$, and $\tilde{a} \in V$ we define

$$|\text{grad}(u(\tilde{a}))| := \int_{\tilde{b} \sim \tilde{a}} \frac{|u(\tilde{b}) - u(\tilde{a})|}{\epsilon}.$$ 

This $|\text{grad}(u)|$ function is often referred to as the “$p$-Laplacian”, and is used to define the “graph energy” on $V$ (see [14] for the case that $V$ is a discretization of the Sierpinski gasket). Note that although $\epsilon$ is a fixed number here, later in this note we will be considering a sequence of graphs constructed with different $\epsilon$ values. We suppress the dependency of $|\text{grad}(u)|$ on $\epsilon$ in the notation. Then (3.1) becomes

$$\int_{\tilde{x} \in B'} |u(\tilde{x}) - u_B| dm(\tilde{x}) \leq C' r \left( \int_{\lambda B'} |\text{grad}(u(\tilde{x}))|^p dm(\tilde{x}) \right)^{\frac{1}{p}}$$

where $C'$ depends on $C, p$, and the maximal degree of the graph, as we describe now. Note that the metric doubling constant of $N \leq C'_p$ implies that the maximal degree of our graph is bounded by $C'_p$. That is, any $\tilde{x} \in V$ has at most $C'_p$ neighboring points. Thus, we see that

$$|\text{grad}(u(\tilde{a}))|^p \approx \int_{\tilde{b} \sim \tilde{a}} \left( \frac{|u(\tilde{b}) - u(\tilde{a})|}{\epsilon} \right)^p$$

with $\approx$ meaning that the two differ by a bounded multiplicative constant. Because of this fact, we may substitute the righthand side of (3.1) with the right hand side of (3.2) and absorb this multiplicative constant into the constant from (3.1). This final version (3.2) of the discrete Poincaré inequality is the one we use.

We make a note of a difference between the traditional Poincaré inequality (2.1) and this discretized version. If a space supports some $(1,p)$-Poincaré inequality for any $p \geq 1$, then a simple topological consequence is that the space is connected. Since we will be working with discrete spaces, it is clear that we may not use the traditional Poincaré inequality (2.1) for these. The discrete graphs in this paper are made of isolated points with positive distance between them. That is to say, these spaces are highly disconnected. However, there are still some properties that
we can obtain from our discrete version of the Poincaré inequality that are in line with the traditional version. For example, if \( u \) is a function on a space, and \( u \) has the constant function 0 as an upper gradient, we would like to conclude that \( u \) is a constant function. This is consequence of the space supporting a Poincaré inequality in the traditional sense, but it is also a consequence of a discrete space supporting a discrete Poincaré inequality.

4. Doubling property of the approximating graph

Since the underlying space \((X, d_X, \mu)\) is doubling, it is natural to question whether or not the constructed graph shares this property. The aim of this section is to prove the following lemma:

**Lemma 4.1.** Suppose that \((X, d_X, \mu)\) is doubling with constant \( C_\mu \), and \( X \) is quasiconvex with constant \( L \). Let \( A \) be a maximal \( \epsilon \)-separated subset of \( X \), and let \((V, d_V, m)\) be constructed from \((X, d_X, \mu)\) as before. Then \( m \) is a doubling measure on \( V \).

The space \((V, d_V, m)\) being a doubling space allows us the use of many results of harmonic analysis that extend to doubling spaces. For example, the Lebesgue Differentiation Theorem extends to metric measure spaces under the doubling property. This result is due to the reliance of the Hardy-Littlewood maximal inequality on the doubling property. There are many other extensions that come from having a doubling property. Coifman and Weiss (see [8]) in particular were pioneers of verifying properties of doubling spaces as related to harmonic analysis. J. Luukkainen and E. Saksman showed that every complete doubling metric space carries a doubling measure (see [22]). It is worth noting that the assumption of completeness is essential here. For example, Saksman showed that every metric space without isolated points has a dense subset that does not carry a doubling measure (see [23]). In Section 6 of this paper we will see that having the doubling property is vital to verifying pointed-measured Gromov-Hausdorff convergence of sequences of discretized metric measure spaces.

Before we begin the proof, we note that due to Proposition 3.1,

\[
A \cap B_X \left( \bar{x}, \frac{r}{L+1} \right) \subset B_V(\bar{x}, r) \subset B_X(x, 3r).
\]

**Proof of Lemma 4.1.** We must show that there is some constant \( C_m \geq 1 \) such that for any \( \bar{x} \in V \), and \( r > 0 \), \( m(B_V(\bar{x}, 2r)) \leq C_m m(B_V(\bar{x}, r)) \). Fix \( \bar{x} \in V \). The case where \( 0 < r < \epsilon \) is easily seen due to the uniform bound on the degree of the graph. That is, in this particular case, we have that \( m(B_V(\bar{x}, r)) = m(\{\bar{x}\}) = \mu(B_X(x, \epsilon)) \), and \( m(B_V(\bar{x}, 2r)) \leq m(B_V(\bar{x}, 2\epsilon)) \). We see that

\[
m(B_V(\bar{x}, 2\epsilon)) \leq \sum_{y \sim \bar{x}} \mu(B_X(y, \epsilon))
\]

\[
\leq \deg(\bar{x}) \mu(B_X(x, 2\epsilon))
\]

\[
\leq C^4_\mu \mu(B_X(x, 2\epsilon))
\]

\[
\leq C^5_\mu \mu(B_X(x, \epsilon))
\]

\[
= C^6_\mu m(B_V(\bar{x}, \epsilon)).
\]

Thus, we will consider the case where \( \epsilon \leq r \). By definition,

\[
m(B_V(\bar{x}, 2r)) = \sum_{y \in 2B_V} \mu(B_X(y, \epsilon)).
\]

This may be a problem if the sum is infinite. However, since \( X \) is a doubling metric measure space, there exists an \( N < \infty \) such that there are at most \( N \) points of \( A, y_1, y_2, \ldots, y_N \), in each ball \( B_X(y, 2\epsilon) \). Recall that we verified that \( N \leq C^6_\mu \) in Section 2. This fact, along with the assumption that \( \mu \) is locally finite bypasses such
a problem, and the sum will be finite. For this calculation set  = \lfloor \log_2 (L + 1) \rfloor.

For the first inequality below, we use the fact that  \( B_V(\tilde{x}, 2r) \subset B_X(x, 3(2r + \epsilon)) \) by Proposition (3.1), and the appearance of  in the doubling property. Hence

\[
m(B_V(\tilde{x}, 2r)) = \sum_{y \in 2B_V} \mu(B_X(y, \epsilon)) \leq C^4_\mu \mu(B_X(x, 3(2r + \epsilon)))
\]

\[
\leq C^{6+2\alpha}_\mu \left( B_X \left( x, \frac{2r + \epsilon}{L + 1} \right) \right)
\]

\[
\leq C^{8+2\alpha}_\mu \left( B_X \left( x, \frac{r/2 + \epsilon/4}{L + 1} \right) \right)
\]

\[
\leq C^{8+2\alpha}_\mu \left( B_X \left( x, \frac{r}{L + 1} \right) \right)
\]

\[
\leq C^{8+2\alpha}_\mu \sum_{y \in B_V} \mu(B_X(y, \epsilon))
\]

\[
= C^{8+2\alpha}_\mu m(B_V(\tilde{x}, r))
\]

Hence,  is a doubling measure on  with doubling constant  \( C_m = C^{8+2\alpha}_\mu \).

We also note that the above proof can be easily modified to show the following:

**Lemma 4.2.** Suppose \((X, d_X, \mu)\) is doubling with constant  \( C_\mu \), and is quasisymmetric with constant  \( L \). Let  \( A \) be a maximal \( \epsilon \)-separated subset of  \( X \) and let \((V, d_V, m)\) be constructed as above. Then for  \( x \in A \) and  \( r \geq \epsilon \), there exists a constant  \( K \) such that

\[
\frac{1}{K} m(B_V(\tilde{x}, r)) \leq \mu(B_X(x, r)) \leq K m(B_V(\tilde{x}, r))
\]

where  \( K \) depends only on  \( C_\mu \) and  \( L \).

**Remark 4.3.** In other words, the lemma says that  \( m \) and  \( \mu \) are comparable at scales larger than  \( \epsilon \). Also, the result holds with a multiple  \( a > 1 \) of  \( r \) though repeated use of the doubling property. Hence  \( m(B_V(\tilde{x}, r)) \) and  \( \mu(B_X(x, ar)) \) are comparable with the constant now depending also on  \( a \).

5. **Proof of Theorem 1.1**

**Proof.** In this section we will show that  \( V \) supports a Poincaré inequality in the sense of (3.2). We will do this essentially by transforming a given function  \( \tilde{f} : V \to \mathbb{R} \) into a function  \( f : X \to \mathbb{R} \) by employing a partition of unity, and using the fact that  \( X \) supports a Poincaré inequality in the sense of (2.1) and then reinterpreting this inequality back to the discrete function  \( f \). The rest of the proof lies only in checking the details of this sketch. Let  \( A \) be the maximally \( \epsilon \)-separated subset of  \( X \) that is associated with  \( V \) (as in Section 2). Fix  \( a \in A \), and let  \( \psi_a : X \to \mathbb{R} \) be given by

\[
\psi_a(x) := \min \left\{ 1, \frac{d_X(x, X \setminus B_X(a, 2\epsilon))}{\epsilon} \right\}.
\]

Notice that if  \( x \in B_X(a, \epsilon) \), then  \( \psi_a(x) = 1 \), and if  \( x \notin B_X(a, 2\epsilon) \), then  \( \psi_a(x) = 0 \). Let  \( \varphi_a : X \to \mathbb{R} \) be defined as follows:

\[
\varphi_a(x) := \sum_{b \in A} \psi_b(x).
\]

For any  \( a \in A \),  \( \varphi_a \) is a Lipschitz function with Lipschitz constant equal to  \( C_\mu \), where  \( C \) only depends upon the doubling constant  \( C_\mu \). In fact, we may take  \( C = 5C^3_\mu \). We see that for any  \( x \in X \),

\[
\sum_{a \in A} \varphi_a(x) = 1.
\]
We define \( f : X \to \mathbb{R} \) by:
\[
f(x) := \sum_{a \in A} \tilde{f}(\tilde{a}) \varphi_a(x).
\]

We consider the pointwise upper Lipschitz constant function on \( X \) defined by
\[
\operatorname{Lip} f(x) := \limsup_{r \to 0} \sup_{y \in B_X(x,r)} \frac{|f(x) - f(y)|}{r}
\]

It can be shown that \( \operatorname{Lip} f(x) \) is an upper gradient of \( f \) provided that \( f \) is locally Lipschitz (see Theorem 6.1 in [4]). It is clear from the construction that \( f \) is locally Lipschitz. We will show that for all \( a \in A \) such that \( x \in B_X(a,\epsilon) \),
\[
\operatorname{Lip} f(x) \leq C \sum_{b \sim a} \frac{\tilde{f}(\tilde{a}) - \tilde{f}(\tilde{b})}{\epsilon} = C \left| \operatorname{grad} \tilde{f}(\tilde{a}) \right|
\]
where \( C \) is the Lipschitz constant of the \( \varphi_a \) functions. From this we glean a lower bound for the right half of (3.2).

By the maximality of \( A \), for any \( x \in X \) there is some \( a_0 \in A \) such that \( x \in B_X(a_0,\epsilon) \). Since this ball is open, we assume that \( r \) is small enough such that we may only consider points \( y \in B_X(x,r) \subset B_X(a_0,\epsilon) \), i.e., \( r < \frac{\epsilon - d_X(x,a_0)}{2} \). Let \( D_x = \{ a \in A : d_X(x,a) < 2\epsilon \} \) and \( D_y = \{ a \in A : d_X(y,a) < 2\epsilon \} \). Let \( D = D_x \cup D_y \), which ultimately depends on \( y \), and note that \( D \subset \{ a \in A : d_X(a,a_0) < 3\epsilon \} \). We now show a useful pointwise bound for \( \operatorname{Lip} f(x) \). Observe that if \( a \in A \setminus D_x \) then \( \varphi_a(x) = 0 \) and if \( a \in A \setminus D_y \) then \( \varphi_a(y) = 0 \). Hence,
\[
\frac{|f(x) - f(y)|}{r} = \frac{1}{r} \left| \sum_{a \in D_x} \tilde{f}(\tilde{a}) \varphi_a(x) - \sum_{a \in D_y} \tilde{f}(\tilde{a}) \varphi_a(y) \right|
\]
\[
= \frac{1}{r} \left| \sum_{a \in D} \tilde{f}(\tilde{a}) \varphi_a(x) - \sum_{a \in D} \tilde{f}(\tilde{a}) \varphi_a(y) - \sum_{a \in D} \tilde{f}(\tilde{a}_0) \varphi_a(x) + \sum_{a \in D} \tilde{f}(\tilde{a}_0) \varphi_a(y) \right|
\]

The equality in the second line is due to the fact that \( \sum_{a \in D} \varphi_a(x) = 1 = \sum_{a \in D} \varphi_a(y) \).

After grouping like terms from the above, we continue:
\[
\frac{|f(x) - f(y)|}{r} = \frac{1}{r} \left| \sum_{a \in D} \tilde{f}(\tilde{a})(\varphi_a(x) - \varphi_a(y)) - \sum_{a \in D} \tilde{f}(\tilde{a}_0)(\varphi_a(x) - \varphi_a(y)) \right|
\]
\[
= \frac{1}{r} \left| \sum_{a \in D} (\tilde{f}(\tilde{a}) - \tilde{f}(\tilde{a}_0))(\varphi_a(x) - \varphi_a(y)) \right|
\]
\[
\leq C \frac{1}{r} \sum_{a \in D} |\tilde{f}(\tilde{a}) - \tilde{f}(\tilde{a}_0)| d_X(x,y)
\]
\[
\leq C \frac{1}{\epsilon} \sum_{a \sim a_0} |\tilde{f}(\tilde{a}) - \tilde{f}(\tilde{a}_0)|.
\]

We may now conclude that if \( x \in B(a_0,\epsilon) \) for some \( a_0 \in A \), then there is a constant \( C \) that depends only on \( C_\mu \) such that
\[
(5.1) \quad \operatorname{Lip} f(x) \leq C \left| \operatorname{grad} \tilde{f}(\tilde{a}_0) \right|.
\]

We use this pointwise estimate to compare \( L^p \) estimates of the gradients, in preparation for the Poincaré inequality. Let \( B_X(x,r) \) be a ball in \( X \). Using the results
of Proposition 3.1 and (4.1), we see that
\[
\int_{B_X(x,r)} (\text{Lip } f)^p \, d\mu \leq \sum_{a \in A \cap B_X(x,r+\epsilon)} \left( \int_{B_X(a,\epsilon)} (\text{Lip } f)^p \, d\mu \right)
\]
\[
\leq C^p \sum_{a \in A \cap B_X(x,r+\epsilon)} \left( \int_{B_X(a,\epsilon)} |\text{grad } \tilde{f}(\tilde{a})|^p \, d\mu \right)
\]
\[
= C^p \sum_{a \in A \cap B_X(x,r+\epsilon)} |\text{grad } \tilde{f}(\tilde{a})|^p m(\tilde{a})
\]
\[
\leq C^p \int_{B_{V}(\delta_0,(L+1)(r+2\epsilon))} |\text{grad } \tilde{f}(\tilde{a})|^p \, dm(\tilde{a}).
\]
(5.2)

With the above we now approach (3.2). Let \( a_0 \in A \) be a nearest point to \( x \) in \( A \). Note that if \( r < \epsilon \) the discrete Poincaré inequality (3.2) is trivially valid, so we can now say for all \( r > 0 \) by Lemma 4.2,
\[
\int_{B_X(x,r)} |f - f_{B_X(x,r)}| \, d\mu \leq C r \left( \int_{B_{V}(\delta_0,6\lambda Lr)} |\text{grad } \tilde{f}(\tilde{x})|^p \, dm(\tilde{x}) \right)^{\frac{1}{p}}.
\]
(5.3)

Notice the radius for the average integral on the right hand side is \( 6\lambda Lr = (3-2L)\lambda r \). The 3 appears from the assumption that \( r \geq \epsilon \), and \( 2L \) appears from both the fact that \( L \geq 1 \) and from the constant in Lemma 4.2.

We now wish to verify the remaining part of the Poincaré inequality, i.e. replacing the left hand side of the above inequality with one related to the discrete function \( \tilde{f} \). Instead of looking for the left hand side of (3.2) above, we search for
\[
\int_{\tilde{z} \in B_{V}} \int_{\tilde{w} \in B_{V}} |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{w})| \, dm(\tilde{w}) \, dm(\tilde{z}).
\]
(5.4)

This is a valid substitution since for any metric measure space \((X, d_X, \mu)\), and any measurable function \( u \), the following two properties hold:
\[
\int_B \int_B |u(x) - u(y)| \, d\mu(y) \, d\mu(x) = \int_B \int_B |u(x) - u_B + u_B - u(y)| \, d\mu(y) \, d\mu(x)
\]
\[
\leq 2 \int_B |u(x) - u_B| \, d\mu(x),
\]
and
\[
\int_B |u(x) - u_B| \, d\mu(x) = \int_B \int_B (u(x) - u(y)) \, d\mu(y) \, d\mu(x)
\]
\[
\leq \int_B \int_B |u(x) - u(y)| \, d\mu(y) \, d\mu(x).
\]

Given \( \tilde{a} \in V \) and \( r > 0 \), we look at the ball \( B_{V}(\tilde{a}, r) \subset V \). We fix two points \( \tilde{z}, \tilde{w} \in B_{V}(\tilde{a}, r) \). We note that \( B_X(z, \frac{\epsilon}{2}) \cap A = \{ z \} \subset X \), and \( B_X(w, \frac{\epsilon}{2}) \cap A = \{ w \} \subset X \), by the \( \epsilon \)-seperability of \( A \). Let \( x, y \) be elements of \( B_X(z, \frac{\epsilon}{2}) \), and \( B_X(w, \frac{\epsilon}{2}) \) respectively.

Recalling a useful fact about the \( \varphi \) functions from the partitions of unity, we may write
\[
\tilde{f}(\tilde{z}) = \sum_{b \in A} \tilde{f}(\tilde{z}) \varphi_b(x)
\]
\[
= \sum_{b \in A} \tilde{f}(\tilde{z}) \varphi_b(x) + \left( f(x) - \sum_{b \in A} \tilde{f}(\tilde{b}) \varphi_b(x) \right)
\]
\[
= \sum_{b \in A} (\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{b})) \varphi_b(x) + f(x).
\]
Similarly, we write
\[ \tilde{f}(\tilde{w}) = \sum_{b \in A} (\tilde{f}(\tilde{w}) - \tilde{f}(\tilde{b})) \varphi_b(y) + f(y). \]

Thus,
\[ |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{w})| = \left| \sum_{b \in A} (\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{b})) \varphi_b(x) + f(x) - \sum_{b \in A} (\tilde{f}(\tilde{w}) - \tilde{f}(\tilde{b})) \varphi_b(y) - f(y) \right| \]
\[ \leq |f(x) - f(y)| + \sum_{b \in A} |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{b})| \varphi_b(x) + \sum_{b \in A} |\tilde{f}(\tilde{w}) - \tilde{f}(\tilde{b})| \varphi_b(y). \]

Since \( \varphi_b(x) = 0 \) whenever \( x \notin B_X(b, 2\epsilon) \), and because \( x \in B_X(z, \epsilon/2) \), then the sum from the second term can be taken over all \( b \in A \) such that \( d_X(b, z) < \frac{\epsilon}{2} \), which means that we may instead just sum over neighbors:
\[ |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{w})| \leq |f(x) - f(y)| + \sum_{b \sim \tilde{z}} |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{b})| \varphi_b(x) + \sum_{b \sim \tilde{w}} |\tilde{f}(\tilde{w}) - \tilde{f}(\tilde{b})| \varphi_b(y) \]

We now turn our sights back onto the double sum form of the left hand side of the Poincaré inequality. Using the above comparisons, and recalling that \( x \equiv x_{\tilde{z}} \) and \( y \equiv y_{\tilde{w}} \) depend on \( \tilde{z} \) and \( \tilde{w} \), respectively, we see that
\[
\int_{B_v} \int_{B_v} |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{w})| dm(\tilde{w}) dm(\tilde{z}) \\
\leq \int_{B_v} \int_{B_v} |f(x_{\tilde{z}}) - f(y_{\tilde{w}})| dm(\tilde{w}) dm(\tilde{z}) \\
+ \int_{B_v} \int_{B_v} \sum_{\tilde{z} \sim b} |\tilde{f}(\tilde{z}) - \tilde{f}(\tilde{b})| dm(\tilde{w}) dm(\tilde{z}) \\
+ \int_{B_v} \int_{B_v} \sum_{\tilde{w} \sim b} |\tilde{f}(\tilde{w}) - \tilde{f}(\tilde{b})| dm(\tilde{w}) dm(\tilde{z}).
\]

We work with these terms on the right hand side separately, first with
\[
(5.5) \int_{B_v} \int_{B_v} |f(x_{\tilde{z}}) - f(y_{\tilde{w}})| dm(\tilde{w}) dm(\tilde{z}).
\]

By using the doubling property in the third line and the results of (3.1) in the last line, we see that
\[
\int_{B_v} \int_{B_v} |f(x_{\tilde{z}}) - f(y_{\tilde{w}})| dm(\tilde{w}) dm(\tilde{z}) \\
\leq \sum_{\tilde{z} \in B_v} \sum_{\tilde{w} \in B_v} |f(x_{\tilde{z}}) - f(y_{\tilde{w}})| \mu(B(w, \epsilon)) \mu(B(z, \epsilon)) \\
\leq C_\mu^2 \sum_{\tilde{z} \in B_v} \sum_{\tilde{w} \in B_v} |f(x_{\tilde{z}}) - f(y_{\tilde{w}})| \mu(B(w, \epsilon/2)) \mu(B(z, \epsilon/2)) \\
\leq C_\mu^2 \sum_{\tilde{z}, \tilde{w} \in B_v} \int_{B_X(z, \epsilon)} \int_{B_X(w, \epsilon)} |f(x) - f(y)| \chi_{B(w, \epsilon)}(y) \chi_{B(z, \epsilon)}(x) d\mu(y) d\mu(x) \\
\leq C_\mu^2 \int_{B_X(a, (L+2)r)} \int_{B_X(a, (L+2)r)} |f(x) - f(y)| d\mu(y) d\mu(x),
\]
where \( \chi_A \) as usual stands for the characteristic function of \( A \subset X \). Now for the second term of our inequality from (5.5):

\[
\int_{B_v} \int_{B_v} \sum_{\bar{b} \sim \bar{z}} |\bar{f}(\bar{z}) - \bar{f}(\bar{b})| dm(\bar{w}) dm(\bar{z})
\]

\[
= \int_{B_v} \left[ \int_{B_v} \sum_{\bar{b} \sim \bar{z}} |\bar{f}(\bar{z}) - \bar{f}(\bar{b})| dm(\bar{z}) \right] dm(\bar{w})
\]

\[
= \epsilon \cdot \int_{B_v} \left[ \int_{B_v} |\text{grad } \bar{f}(\bar{z})| dm(\bar{z}) \right] dm(\bar{w})
\]

\[
= \epsilon \cdot m(B_v(\bar{a}, r)) \int_{B_v} |\text{grad } \bar{f}(\bar{z})| dm(\bar{z})
\]

Clearly the same quantity can be used to bound the third term of the summation by transposing \( \bar{z} \) with \( \bar{w} \). Summarizing, from (5.5) we achieve

\[
\int_{B_v} \int_{B_v} |\bar{f}(\bar{z}) - \bar{f}(\bar{w})| dm(\bar{w}) dm(\bar{z}) \leq
\]

\[
C_2^2 \int_{B_x(a,(L+2)r)} \int_{B_x(a,(L+2)r)} |f(x) - f(y)| d\mu(y) d\mu(x)
+ 2\epsilon \cdot m(B_v(\bar{a}, r)) \int_{B_v} |\text{grad } \bar{f}(\bar{z})| dm(\bar{z})
\]

By Lemma 4.2 and Remark 4.3 we are free to average all these integrals to obtain

\[
\int_{B_v} \int_{B_v} |\bar{f}(\bar{z}) - \bar{f}(\bar{w})| dm(\bar{w}) dm(\bar{z}) \leq
\]

\[
C \int_{B_x(a,(L+2)r)} \int_{B_x(a,(L+2)r)} |f(x) - f(y)| d\mu(y) d\mu(x)
+ 2\epsilon \cdot m(B_v(\bar{a}, r)) \int_{B_v} |\text{grad } \bar{f}(\bar{z})| dm(\bar{z}),
\]

where \( C \) is a constant that depends only on \( C_\mu \). We now apply the Poincaré inequality version (5.3) on the first term on the right-hand side of inequality (5.7). Recalling the discussion after (5.4), we achieve

\[
\int_{B_v} \int_{B_v} |\bar{f}(\bar{z}) - \bar{f}(\bar{w})| dm(\bar{w}) dm(\bar{z}) \leq
\]

\[
C_1(L + 2) r \left( \int_{B_v(a,6\lambda L(L+2)r)} |\text{grad } \bar{f}(\bar{z})|^p dm(\bar{z}) \right)^{1/p}
+ 2\epsilon \cdot m(B_v(\bar{a}, r)) \int_{B_v} |\text{grad } \bar{f}(\bar{z})| dm(\bar{z}),
\]

for some constant \( C_1 \) depending on the data of the Poincaré inequality and \( C_\mu \). Now, by employing Hölder’s inequality and the assumption that \( \epsilon < r \) on the second term on the right-hand side we finally conclude:

\[
\int_{B_v} \int_{B_v} |\bar{f}(\bar{z}) - \bar{f}(\bar{w})| dm(\bar{w}) dm(\bar{z}) \leq C_2 r \left( \int_{\Lambda_1 B_v} |\text{grad } \bar{f}(\bar{z})|^p dm \right)^{1/p}.
\]

This is the desired Poincaré inequality. The constants \( C_2 \) and \( \lambda_1 \) ultimately depend only on the data of the Poincaré inequality and the doubling constant of \( X \). Along with Section 4, the above shows that if a metric measure space, \((X,d_X,\mu)\), supports a \((1,p)\)-Poincaré inequality and is doubling, then the discretization, \((V,d_v,m)\), supports a discrete \((1,p)\)-Poincaré inequality and is doubling and Theorem 1.1 is proved. \( \square \)
6. Pointed Measured Gromov-Hausdorff convergence

We would like to answer the converse question to what we have shown: If we have a sequence of graphs that support a Poincaré inequality, have a doubling measure, and “converge” to some base space $X$, does $X$ also support a Poincaré inequality and have a doubling measure $\mu$? To answer this question, we must of course discuss the sense of convergence. In this section we introduce definitions and results about Gromov-Hausdorff convergence that will aid in the answer to the above question. A reader well-versed in the Gromov-Hausdorff topology may safely skip the discussion below and pick up again at Theorems 6.3 and 6.4.

For in-depth discussions of Gromov-Hausdorff convergence from a geometric point of view see the book [3] by D. Burago, Y. Burago, and S. Ivanov. The upcoming book [13] by J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson also discusses the Gromov-Hausdorff topology but from an analytic point of view.

We note that in the context of this paper, each $X_i$ will be separable, and so by the work of M. Fréchet (see [9]), there will always exist some isometric embedding from $X_i$ to $\ell^\infty$.
With these definitions in place, we now present two theorems that will help us answer the guiding questions presented at the beginning of this section.

**Theorem 6.3** ([4], Theorem 9.1). Let \((X_i, d_i, q_i, \mu_i)\) be a sequence of complete spaces which pointed measured Gromov-Hausdorff converge to a complete space \((X, d, q, \mu)\). If each of the measures \(\mu_i\) is doubling with constant \(C_D\), then \(\mu\) is also doubling with constant \(C_D\).

The second theorem that will be of great importance was proved independently by J. Cheeger [4] and S. Keith [17].

**Theorem 6.4** ([4], Theorem 9.6). Let \((X_i, d_i, q_i, \mu_i)\) be a sequence of complete spaces that pointed measured Gromov-Hausdorff converge to a complete space \((X, d, q, \mu)\). Let \(1 \leq p < \infty, C_D, C_p < \infty\) and \(\lambda \geq 1\) be fixed. If each of the measures \(\mu_i\) is doubling with constant \(C_D\), and each space \((X_i, d_i, \mu_i)\) satisfies the \((1, p)\)-Poincaré inequality with constants \(C_p\) and \(\lambda\), then \((X, d, \mu)\) also satisfies the \((1, p)\)-Poincaré inequality with constants \(C_p'\) and \(\lambda'\) depending only on \(p, C_p\) and \(C_D\).

The above two theorems are usually presented with the requirement that \((X_i, d_i)\) be length spaces. However, this requirement is not necessary for the desired results. To see a discussion about the lack of length spaces we have presented, refer to [13] (Chapter 11). We now have the resources necessary to explicitly achieve Theorem 1.2. We first discuss some examples to put the formulation of Theorem 1.2 in context.

7. **Examples from \(\mathbb{R}^2\)**

Here we wish to let \(X = \mathbb{R}^2\), \(d\) be the Euclidean metric, and \(\mu\) be Lebesgue measure. In the next section below we will consider sequences of connected spaces, derived from a sequence of discrete spaces, which converge under the pointed measured Gromov-Hausdorff topology to a space \((\overline{X}, \overline{d})\). Here we wish to show that even when discretized versions of \(\mathbb{R}^2\) are considered, the limit space will not be \(\mathbb{R}^2\) with the Euclidean metric and Lebesgue measure. This is the reason for the flexibility of the conditions in Theorem 1.2.

Consider the discretization sequence generated by the integer grid \(\mathbb{Z} \times \mathbb{Z}\) and dyadic scaling, i.e. \(V_1 = \mathbb{Z} \times \mathbb{Z}\) and \(V_\epsilon = \epsilon_i(\mathbb{Z} \times \mathbb{Z})\) where \(\epsilon_i = 2^{-i}\). Under the scheme for \((V_i, d_i, m_i)\) introduced in Section 3, for large \(n\)

\[
d_n((0,0),(0,1)) = 1/2\quad d_n((0,0),(1,1)) = 1/2.
\]

In fact, one can see that \(d_n\) shrinks distances of neighbors by a factor of \(2^n\) for horizontal and vertical neighbors, leaving the distance between two points along a vertical or horizontal line fixed as \(n \to \infty\). However, the distance between neighbors along a diagonal also shrinks by \(2^n\) under \(d_n\), but the Lebesgue distance between two points shrinks by a factor of \((\sqrt{2}/2^n)\), leaving a distortion of \(\sqrt{2}\). However, verifying the discrete Poincaré inequality on each of the \(V_i\) is a simple task as shown below, and the data is independent of \(i\).

As far as the measures \(m_i\) for this example are concerned, a similar problem occurs. If one pushes these measures forward onto \(\mathbb{R}^2\) we get a sequence of measures, \(\mu_i\), which does not converge to Lebesgue measure. For example

\[
\mu_3(B_{V_3}((0,0),1)) = \frac{43}{16}\pi, \mu_4(B_{V_4}((0,0),1)) = \frac{193}{64}\pi, \text{ and } \mu_5(B_{V_5}((0,0),1)) = \frac{793}{256}\pi.
\]

and \(\lim_{n \to \infty} \mu_n(B_{V_n}((0,0),1)) = \pi^2\). This convergence can be seen geometrically as the vertices in the \(n\)-th discretization cover the unit ball with “squares” which have area \(\pi\) due to the definition of \(m_i\). In fact, \(\mu_i\) will weak-star converge to the Lebesgue measure multiplied by \(\pi\). It should be clear at this point that a less symmetric discretization of \(\mathbb{R}^2\) can lead to measures which are not just multiples of Lebesgue measure. One of the novelties of Theorem 1.2 is that one need not calculate the weak* limit of \(\mu_n\). Instead, it is enough to verify that the collection
of discretezations have the uniform properties (2) through (5). In this case, the doubling constant of $\mu$ is 4, and for all $i \in \mathbb{N}$ the maximum degree of any vertex in $V_i$ is 28. We find that $7128 = 28 \cdot 4^3$ suffices as the doubling constant for all $m_i$. Thus, after verifying a discrete Poincaré inequality, all the conditions of Theorem 1.2 are satisfied, and one can conclude that $(\mathbb{R}^2, d, \mu)$ must also support a Poincaré inequality with the similar data.

Now to verify this discrete Poincaré inequality for our discretization of $\mathbb{R}^2$. For each $i \in \mathbb{N}$ we set $V_i := \mathbb{Z} \times \mathbb{Z}$ where the step size between neighbors is $\frac{1}{2}$. That is, we are taking $\epsilon_i = \frac{1}{2}$. We will consider a ball, $B \subset \mathbb{R}^2$, of radius $n \in N$ centered at the point $(0,0)$. Let $x$ and $y$ be two points in $B$. By the construction of $\mathbb{Z} \times \mathbb{Z}$, there is a path of points $p_0, p_1, \ldots, p_k \in B$ such that $x = p_0 \sim p_1 \sim \ldots \sim p_k = y$. To assign these points, let $\gamma$ be the straight line path in $\mathbb{R}^2$ from $x$ to $y$. We have that $\gamma$ is a rectifiable curve and assume that length$(\gamma) = |x - y|$. We set $t_i$ to be the point on $\gamma$ such that $x = t_0$, $t_k = y$, and for each $i$ we have that $|t_i - t_{i-1}| = \epsilon_i$, with the exception of $|t_k - t_{k-1}|$, which may be less than or equal to $\epsilon_i$. For each $t_i$, along $\gamma$, there is a point in $V_i$ within $\epsilon_i$ distance from $t_i$ on $\gamma$. We let $p_i$ be these points, and see that $|p_i - p_{i-1}| \leq 3\epsilon_i$. Thus, for each $i$ we have that $p_i \sim p_{i-1}$. By the triangle inequality, we see that

$$|f(x) - f(y)| \leq \sum_{i=1}^{k} |f(p_i) - f(p_{i-1})|.$$  

We will integrate both sides of the above inequality. Such integration, along with the observation above yields

$$\sum_{x \in B} |f(x) - f(y)|m_i(x) \leq Cn \sum_{x \in B} \sum_{z \sim x} |f(z) - f(x)|m_i(x),$$

where $C$ is a constant depending on the doubling constant of $m_i$. Since each $m_i$ has a doubling constant that is uniform across all $i \in \mathbb{N}$, then this $C$ is uniform among all $i$ as well. In fact, we may take $C$ to be 256, which is the doubling constant of $\mu$ to the fourth power. Integrating again on both sides we see that

$$\sum_{y \in B} \sum_{x \in B} |f(x) - f(y)|m_i(x)m_i(y) \leq Cn m_i(B) \sum_{x \in B} \sum_{z \sim x} |f(z) - f(x)|m_i(x).$$

By averaging both the two summations on the left hand side of the inequality, we arrive at a $(1, 1)$-Poincaré inequality:

$$\int_B \int_B |f(x) - f(y)|dm_i(x)dm_i(y) \leq Cn \int_B |\operatorname{grad} f(x)|dm_i(x).$$

Through use of Hölder’s inequality we arrive at the desired $(1, p)$-Poincaré inequality:

$$\int_B \int_B |f(x) - f(y)|dm_i(x)dm_i(y) \leq Cn \left( \int_B |\operatorname{grad} f(x)|^p dm_i(x) \right)^{\frac{1}{p}}.$$

Since we have that $C$ is uniform constant independent of $\epsilon_i$, then by Theorem 1.2, $(\mathbb{R}^2, d, \mu)$ will support a $(1, p)$-Poincaré inequality. The ease of the above argument displays the usefulness of Theorem 1.2.

8. Proof of Theorem 1.2

Recall that we begin with a doubling complete metric measure space $(X, d_X, \mu)$. We begin by supposing that $(X, d_X, \mu)$ supports a $(1, p)$-Poincaré inequality. By Theorem (1.1), we can make a nested embedded sequence of graphs $(V_i, d_{V_i}, m_i)$ into $X$ with $\epsilon_i = 1/2^{i-1}$. By construction, we see that

$$d_H(n_i(V_i), X) \leq \epsilon_{V_i} \rightarrow 0.$$
as \( i \to \infty \). Lemma 3.1 guarantees that \( d_{V_i} \) is bi-Lipschitz equivalent to \( d_X \) for all \( x,y \in V_i \). Lemma 4.2 showed that \( \mu \) is comparable to \( m_i \) on balls with radius greater than \( \epsilon_{V_i} \). Section 4 showed that \( m_i \) was doubling, with doubling constant independent of \( \epsilon_{V_i} \), ensuring that the entire family \( (V_i, d_{V_i}, m_i) \) has a uniform doubling constant. Section 5 showed that all \( (V_i, d_{V_i}, m_i) \) support a \((1,p)\)-Poincaré inequality with uniform constants. Thus, one direction of the theorem is proved.

Conversely, assume that there exists a nested embedded sequence of graphs \( (V_i, d_{V_i}, m_i) \) into \( X \) such that

1. The Hausdorff distance, \( d_H(n_i(V_i), X) = H_i \), is finite for all \( i \geq 0 \) and \( H_i \to 0 \) as \( i \to \infty \).
2. There is a uniform \( L > 1 \) such that for all \( i \geq 0 \) and all \( x,y \in V_i \),
   \[
   \frac{1}{L} d(n_i(x), n_i(y)) \leq d_{V_i}(x, y) \leq L d(n_i(x), n_i(y))
   \]
3. There is a uniform \( K > 1 \) such that for all \( i \geq 0 \) all \( r > H_i \) and \( x \in V_i \)
   \[
   \frac{1}{K} \leq \frac{m_i(B_{V_i}(x, r))}{\mu(B_X(x, r))} \leq K
   \]
4. \((V_i, d_{V_i}, m_i)\) are all doubling with uniform doubling constant.
5. \((V_i, d_{V_i}, m_i)\) all support a \((1,p)\)-Poincaré inequality with uniform data.

We must now show that \((X, d_X, \mu)\) supports a \((1,p)\)-Poincaré inequality. Our method of verifying this result requires the transformation of the sequence of graphs, \((V_i)\) into a new sequence of connected topological spaces. This is necessary to use Theorems 6.3 and 6.4 to verify the \((1,p)\)-Poincaré inequality on \( X \).

### 8.1. Extension of \( V_i \)

We may extend each \( V_i \) into a path connected space, \( G_i \), in a similar manner as was done by N. Shanmugalingam in [24] (Section 3), by placing an isomorphic copy of the interval \([0, \epsilon_{V_i}]\) between each set of neighbors. These extensions are not graphs in the traditional sense, but rather, they are a connected space that may be thought of as 1-simplexes. They are connected in the topological sense, and we will often speak of the vertices in \( G_i \), and points on its edges, which are points in an interval \([0, \epsilon_{V_i}]\) with specified associated vertices. We also adjust the metric and measure for \( G_i \). For the metric, we say that two vertices \( \bar{x}, \bar{y} \) in \( V_i \subseteq G_i \) have a distance \( d_{G_i}(\bar{x}, \bar{y}) := d_i(\bar{x}, \bar{y}) \). If \( \bar{x} \) is a vertex and \( \bar{y} \) is on an edge that is connected to \( \bar{x} \), we use the obvious distance inherited from the isomorphic copy of \([0, \epsilon_{V_i}]\). If \( \bar{x} \) is a point on the edge and \( \bar{y} \) is a vertex which is not an endpoint for the edge that \( \bar{x} \) is on, we select the vertex \( \bar{v} \) on the edge that \( \bar{x} \) is on which minimizes the following expression: \( d_{G_i}(\bar{x}, \bar{y}) := d_{V_i}(\bar{y}, \bar{v}) + |\bar{x} - \bar{v}| \). If both \( \bar{x} \) and \( \bar{y} \) are on edges, we just extend as in the previous case in the obvious way. We note that when finding the distance between two points on an edge, it may not be the case that distance is found by selecting the closest vertices to the points, but becomes the minimum distance of 4 different paths that involve the associated vertices. This defines a new metric, \( d_{G_i} \), on our space \( G_i \).

We also build a new measure \( \overline{m}_i \) in terms of \( m_i \) such that \( \overline{m}_i \) is comparable to \( m_i \) on balls with radius greater than \( \epsilon_{V_i} \). This implies that \( \overline{m}_i \) is comparable to \( \mu \) on balls greater than \( \epsilon_{V_i} \) just as \( m_i \) is by assumption (3). If \( U \) is a subset of \( G_i \), then we define

\[
\overline{m}_i(U) = \sum_I \frac{\text{length}(I \cap U)}{\epsilon_{V_i}} \left[ m_i(\bar{x}_I) + m_i(\bar{y}_I) \right],
\]

where the sum is over the intervals \( I = [0, \epsilon_{V_i}] \) such that \( I \cap U \neq \emptyset \), and each \( I \) has associated endpoints \( \bar{x}_I \) and \( \bar{y}_I \). Length is, of course, understood as Lebesgue measure. Because of the uniform bound on degree in the graph \( V_i \) necessitated by the doubling condition property also (3) holds for \( \overline{m}_i \). We now want to verify that
this extended space \((G_i, \hat{d}_i, \bar{m}_i)\) has a doubling measure, and supports a \((1, p)\)-Poincaré inequality in the traditional sense of (2.1). Similar proofs to the next two lemmas can be found in [24] for the case that \(\epsilon_{V_i} = 1\) and \(m_i(\hat{x}) = 1\) for all \(\hat{x} \in V_i \subseteq G_i\). In this paper, we present the more general case that includes more general measures and metrics.

**Lemma 8.1.** The measure \(\bar{m}_i\) is doubling.

**Proof.** We first consider that \(\tilde{x} \in V_i\), and will discuss \(\hat{x}\) being a point on an edge shortly.

**Case 1:** If \(r \leq \epsilon_{V_i}\), then
\[
\frac{r}{\epsilon_{V_i}} m_i(B(\tilde{x}, r)) = \frac{r}{\epsilon_{V_i}} m_i(\tilde{x}) \leq \frac{r}{\epsilon_{V_i}} \sum_I [m_i(\tilde{x}) + m_i(\tilde{y}_I)] = \bar{m}_i(B(\tilde{x}, r)).
\]

Remembering that \(\tilde{x}\) has a bounded degree that only depends on the doubling constant of \(m_i\), we see that
\[
\bar{m}_i(B(\tilde{x}, 2r)) \leq \sum_I \frac{r}{\epsilon_{V_i}} m_i(B(\tilde{x}, 2\epsilon_{V_i})) \leq C_2 r m_i(B(\tilde{x}, \epsilon_{V_i})) = C_2 r \epsilon_{V_i} m_i(B(\tilde{x}, r)),
\]
where \(C\) is a constant that only depends on the doubling constant of \(m_i\). Combining these two facts, we see that
\[
\bar{m}_i(B(\tilde{x}, 2r)) \leq \frac{2 C^2 r}{\epsilon_{V_i}} m_i(B(\tilde{x}, 2r)) \leq 2 C^3 \bar{m}_i(B(\tilde{x}, r)).
\]

**Case 2:** If \(r \geq \epsilon_{V_i}\), then we notice that the largest length of any edge in \(B(\tilde{x}, r)\) is \(\epsilon_{V_i}\). We see that
\[
\bar{m}_i(B(\tilde{x}, r)) = \sum_I \frac{\text{length}(I \cap U)}{\epsilon_{V_i}} [m_i(\tilde{z}_I) + m_i(\tilde{y}_I)]
\]
\[
\leq \sum_I [m_i(\tilde{z}_I) + m_i(\tilde{y}_I)]
\]
\[
\leq 2 C^4 m_i(B(\tilde{x}, r + \epsilon_{V_i}))
\]
\[
\leq 2 C^5 m_i(B(\tilde{x}, r)).
\]
Recall that the max degree of any vertex in \(V_i\) is less than or equal to \(C^4\). It is trivial to see that \(m_i(B(\tilde{x}, r)) \leq \bar{m}_i(B(\tilde{x}, r))\) in this case, and so \(m_i\) and \(\bar{m}_i\) are comparable, and hence \(\bar{m}_i\) is doubling.

Now, we consider the case that \(\tilde{x}\) is not in \(V_i\); this is a little less clear. Let \(\tilde{v}\) be the nearest vertex to \(\tilde{x}\). We have a few cases:

**Case 1:** If \(r/2 \geq \epsilon_{V_i}\), we note that
\[
B(\tilde{v}, r - \epsilon_{V_i}) \subset B(\tilde{x}, r) \subset B(\tilde{v}, r + \epsilon_{V_i})
\]
and so
\[
m_i(B(\tilde{x}, 2r)) \leq m_i(B(\tilde{v}, 2r + \epsilon_{V_i})) \leq C \bar{m}_i(B(\tilde{v}, r/2)) \leq C \bar{m}_i(B(\tilde{v}, r - \epsilon_{V_i})) \leq C \bar{m}_i(B(\tilde{x}, r))
\]
where \(C\) depends on the doubling constant of \(m_i\).

**Case 2:** If \(\epsilon_{V_i} < r < 2\epsilon_{V_i}\), recall \(d_G(\tilde{x}, \tilde{v}) \leq \epsilon_{V_i}/2\), and that
\[
B(\tilde{x}, 2r) \subset B(\tilde{v}, 2r + \epsilon_{V_i}) \subset B(\tilde{v}, 3r).
\]
Hence,
\[
\bar{m}_i(B(\tilde{x}, 2r)) \leq \bar{m}_i(B(\tilde{v}, 2r + \epsilon_{V_i})) \leq C \bar{m}_i(B(\tilde{v}, r/4)),
\]
where again \(C\) depends only on the doubling constant of \(m_i\). By the assumption in this case, \(B(\tilde{v}, r/4) \subset B(\tilde{x}, r/2)\) and so \(\bar{m}_i(B(\tilde{x}, 2r)) \leq C \bar{m}_i(B(\tilde{x}, r))\).

**Case 3:** If \(\epsilon_{V_i}/4 < r < \epsilon_{V_i}\), we note that \(m_i(\tilde{v})\) is comparable to \(m_i(\tilde{w})\) for any \(\tilde{w}\) at distance \(2\epsilon_{V_i}\) or less away from \(\tilde{v}\). This is because of the doubling of \(m_i\). Then \(\bar{m}_i(B(\tilde{x}, 2r))\) is bounded above a constant times the sum of all the \(m_i\) measures of
vertices \( \hat{w} \) at distance \( 2r_{\infty} \) or less away from \( \hat{v} \), which is comparable to \( m_i(\hat{v}) \). But \( \overline{m}_i(B(\hat{x}, r)) \geq m_i(\hat{v})/4 \) because \( r > \epsilon_{\infty}/4 \) and the definition of \( \overline{m}_i \). So doubling follows.

**Case 4:** If \( r < \epsilon_{\infty}/4 \), this case is trivial because \( B(\hat{x}, 2r) \) can only contain edges connected to \( \hat{v} \). \( \square 

**Lemma 8.2.** If \((V_i, d_i, m_i) \) supports a discretized \((1, p)\)-Poincaré inequality in the sense of (3.1), then \((G_i, \hat{d}_{G_i}, \overline{m}_i) \) supports a \((1, p)\)-Poincaré inequality in the sense of (2.1).

**Proof.** Since \( G_i \) is a complete space, and by Lemma 8.1 we have that \( m_i \) is doubling, then it suffices to verify this lemma with Lipschitz functions (see Theorem 8.4.2 in [13]). Let \( u : G_i \to \mathbb{R} \) be a Lipschitz function, and recall that \( \text{Lip} \ u \) is an upper gradient of \( u \). We will assume that our ball, \( B \), is centered at a vertex, \( \hat{x} \). We may do this due to the fact that we may increase the \( \lambda \) value in the data of a Poincaré inequality by a constant that does not depend on \( r \).

**Case 1:** The radius \( r \leq \epsilon_{\infty} \). For \( \tilde{y} \) a neighbor of \( \hat{x} \), let \( r_{\tilde{y}} \) represent a point on the interval connecting \( \hat{x} \) to \( \tilde{y} \) with a distance of \( r \) from \( \hat{x} \). We see that

\[
|u(s_{\tilde{y}}) - u(\hat{x})| \leq \int_{\hat{x}}^{s_{\tilde{y}}} |\text{Lip} \ u(\tau_{\tilde{y}})| \, d\tau,
\]

where \( d\tau \) is Lebesgue measure on \([0, \epsilon_{\infty}] \). We notice the following bound, which proves useful in later calculations. Let \( c \in \mathbb{R} \) and suppose that \( c \leq u_B \). Then we see that

\[
\int_B |u_B - c| \, d\overline{m}_i = u_B - c = \int_B u \, d\overline{m}_i - \int_B c \, d\overline{m}_i = \int_B (u - c) \, d\overline{m}_i \leq \int_B |c - u| \, d\overline{m}_i.
\]

If \( c > u_B \), then we have the similar result that

\[
\int_B |u_B - c| \, d\overline{m}_i = c - u_B \leq \int_B |c - u| \, d\overline{m}_i.
\]

Notice that \( |u(\tilde{y}) - u_B| \leq |u(\tilde{y}) - c| + |c - u_B| \) for each \( \tilde{y} \in G_i \). Hence,

\[
\int_B |u - u_B| \, d\overline{m}_i \leq \int_B |u - c| \, d\overline{m}_i + \int_B |c - u_B| \, d\overline{m}_i \leq 2\int_B |c - u| \, d\overline{m}_i,
\]

for any \( c \in \mathbb{R} \). In particular, we see that

\[
\int_B |u - u_B| \, d\overline{m}_i \approx \inf_{c \in \mathbb{R}} \int_B |c - u| \, d\overline{m}_i.
\]

Notice that \( u(\hat{x}) \) is a constant value, since \( \hat{x} \) is a fixed point in \( B \subset G_i \). Then,

\[
\int_B |u - u_B| \, d\overline{m}_i \leq 2\int_B |u(\hat{x}) - u| \, d\overline{m}_i.
\]

This right hand value is what we will use to show our Poincaré inequality. Recalling (8.1), we see that

\[
\int_B |u(\hat{x}) - u| \, d\overline{m}_i = \frac{1}{\overline{m}_i(B)} \int_B |u(\hat{x}) - u| \, d\overline{m}_i
\]

\[
= \frac{1}{\overline{m}_i(B)} \sum_{y \sim \hat{x}} \int_{\hat{x}}^{s_{\tilde{y}}} |u(\hat{x}) - u(s_{\tilde{y}})| \, d\overline{m}_i(s_{\tilde{y}})
\]

\[
\leq \frac{1}{\overline{m}_i(B)} \sum_{y \sim \hat{x}} \int_{\hat{x}}^{s_{\tilde{y}}} \left( \int_{\hat{x}}^{s_{\tilde{y}}} |\text{Lip} \ u(\tau_{\tilde{y}})| \, d\tau \right) \, d\overline{m}_i(s_{\tilde{y}})
\]

\[
\leq \frac{1}{\overline{m}_i(B)} \sum_{y \sim \hat{x}} \int_{\hat{x}}^{s_{\tilde{y}}} |\text{Lip} \ u(\tau_{\tilde{y}})| \, d\tau \, d\overline{m}_i(s_{\tilde{y}}).
\]

We notice that $d\bar{m}_i(\tau y) = \frac{m_i(\bar{x}) + m_i(\bar{y})}{\epsilon_{\nu_i}} d\tau$, and so continuing

$$
\int_B |u(\bar{x}) - u| d\bar{m}_i = \frac{r}{m_i(B)} \sum_{y \sim \bar{x}} \int_{\bar{x}}^{\bar{y}} |\text{Lip } u(\tau \bar{y})| d\bar{m}_i(\tau \bar{y})
= r \int_B |\text{Lip } u| d\bar{m}_i.
$$

Now, by using Hölder’s inequality, we have a $(1,p)$-Poincaré inequality.

Case 2: The radius $r > \epsilon_{\nu_i}$. We define a new function $\tilde{u}$ as the restriction of $u$ to the vertex set $V_i$. Now, consider the piecewise linear extension of $\tilde{u}$ to $G_i$, which we will denote $\tilde{u}$. Note that $\tilde{u}$ and $u$ agree on $V_i \subset G_i$ and differ on the edge set of $G_i$. Let $f = \tilde{u} - u$, which vanishes on $V_i$ and note that $f_B = \tilde{u}_B - u_B$. Now we consider our Poincaré inequality using the alternate form from (8.2) for the right hand side. Clearly we see that

$$
\int_B | u - u_B |d\bar{m}_i \leq \int_B |\tilde{u} - \tilde{u}_B |d\bar{m}_i + \int_B |f - f_B |d\bar{m}_i.
$$

We will investigate the two terms on the right hand separately. First, we consider

$$
\int_B |\tilde{u} - \tilde{u}_B |d\bar{m}_i.
$$

By the proof of Lemma 8.1, $\bar{m}_i(B(\tilde{y}, \epsilon_{\nu_i}))$ is comparable to $m_i(\tilde{y})$ for all $\tilde{y} \in V_i$. First, we note that

$$
\text{Lip } \tilde{u}(\tilde{z}) = \frac{|\tilde{u}(\tilde{x}) - \tilde{u}(\tilde{y})|}{\epsilon_{\nu_i}}
$$

whenever $\tilde{z}$ is in the edge connecting vertices $\tilde{x}$ and $\tilde{y}$, and by inequality (3.1)

$$
\frac{1}{m_i(B)} \sum_{\tilde{z} \in B} |\tilde{u}(\tilde{z}) - \tilde{u}_B |m_i(\tilde{z}) \leq C r \left( \frac{1}{m_i(\lambda B)} \sum_{\tilde{z} \in \lambda B} |\text{grad } \tilde{u}(\tilde{z})|^{p} m_i(\tilde{z}) \right)^{1/p}
$$

$$
\leq C' r \left( \frac{1}{m_i(2\lambda B)} \int_{2\lambda B} |\text{Lip } \tilde{u}(\tilde{z})|^{p} d\bar{m}_i \right)^{1/p}.
$$

The switch from $C$ to $C'$ is to call attention to the comparability constant that is used to change from $m_i(\tilde{z})$ to $\bar{m}_i(B(\tilde{z}, \epsilon_{\nu_i}))$, and to call attention to the doubling of $\lambda$ in our integral.

We know from one-dimensional calculus that, on edges, linear functions have the smallest $p$-energy integrals amongst all Sobolev functions with the same boundary values. That is, since $\tilde{u}$ is a $p$-harmonic function on each individual edge, those are $\bar{u}$.

$$
\int_I |\text{Lip } \bar{u}|^{p} d\bar{m}_i \leq \int_I |\text{Lip } u|^{p} d\bar{m}_i,
$$

whenever $I$ is an edge connecting two points in $V_i$. To avoid confusion, we will distinguish the average value of $\bar{u}$ on $B$ with respect to $V_i$ and $G_i$ as follows:

$$
\bar{u}_B := \frac{1}{m_i(B)} \sum_{\tilde{x} \in B} \bar{u}(\tilde{x}) m_i(\tilde{x}) \quad \bar{u}_B := \frac{1}{m_i(B)} \int_B \bar{u}(\tilde{x}) d\bar{m}_i(\tilde{x}).
$$

Thus, we have that

$$
\frac{1}{m_i(B)} \sum_{\tilde{z} \in B} |\tilde{u}(\tilde{z}) - \bar{u}_B |m_i(\tilde{z}) \leq C r \left( \int_{2\lambda B} |\text{Lip } u|^{p} d\bar{m}_i \right)^{1/p}.
$$

for some constant $C$. We may assume that $\bar{u}_B = 0$ as subtracting a constant does not change the upper gradient. Let $\tilde{z} \sim \tilde{w}$ in $2B$, and let $\Gamma_{\tilde{z} \tilde{w}}$ be the edge connected
\[ \int_{\Gamma_{\tilde{z} \to \tilde{w}}} |\tilde{u}| |m_i| \leq \frac{|\tilde{u}(\tilde{z})| + |\tilde{u}(\tilde{w})|}{2} m_i(\Gamma_{\tilde{z} \tilde{u}}). \]

Then we see that
\[ \int_B |\tilde{u}| |m_i| \leq \sum_{\tilde{z} \in 2B} |\tilde{u}(\tilde{z})| m_i(B(\tilde{z}, \epsilon_i)). \]

Recalling that \(m_i(B)\) is comparable to \(m_i(B)\) and (8.2), we arrive at
\[ (8.6) \quad \int_B |\tilde{u} - \tilde{u}_B| |m_i| \leq C r \left( \int_{2\lambda B} |\text{Lip } u|^p |m_i| \right)^{1/p}. \]

Now we look at the second term of the right hand side of (8.3):
\[ \int_B |f - f_B| |m_i|. \]

First we see that
\[ \int_B |f - f_B| |m_i| = \sum_{I \cap B \neq \emptyset} \int_I |f - f_B| |m_i|, \]

where \(I\) is an edge in \(G_i\). Recalling that \(f = 0\) on \(V_i\), and using the same argument as in Case 1 on each of these integrals, we easily find that on each edge \(I\) with \(\tilde{z}\) as one of its endpoints,
\[ \int_I |f - f_B| |m_i| \leq 2 \int_I |f| |m_i| = 2 \int_I |f| |m_i| \leq 2 \epsilon_i \int_I |\text{Lip } f| |m_i|. \]

Summing up over all the intervals, taking averages, and noting that \(r > \epsilon_i\), we have
\[ \int_B |f - f_B| |m_i| \leq 2 r \int_B |\text{Lip } f| |m_i|. \]

Hence,
\[ \int_B |f - f_B| |m_i| \leq 2 r \int_B |\text{Lip } f| |m_i| \leq 2 r \left( \int_B |\text{Lip } \tilde{u}| |m_i| + \int_B |\text{Lip } u| |m_i| \right). \]

Recalling the fact stated in (8.4), and applying Hölder’s inequality, we then have that
\[ \int_B |f - f_B| |m_i| \leq 4 r \left( \int_B |\text{Lip } u| |m_i| \right)^{1/p}. \]

Using this as well as the bound from (8.6) in (8.3), we arrive at our (1, p)-Poincaré inequality:
\[ \int_B |u - u_B| |m_i| \leq C r \left( \int_{\lambda_i B} |\text{Lip } u| |m_i| \right)^{1/p}, \]

where \(C\) and \(\lambda_i\) are some constants that depend only on the doubling constant of \(m_i\) and the data for the discretized Poincaré inequality on \(V_i\).

\[ \square \]

8.2. Bi-Lipschitz change in metrics. We have transformed the sequence of discrete spaces, \((V_i, d_i, m_i)\), to a sequence of connected spaces, \((G_i, d_{G_i}, m_i)\), that have a doubling measure, and support a (1, p)-Poincaré inequality with data that depends only on the uniform doubling constant and uniform data of the discrete sequence. We now sketch the rest of our proof. We introduce a change in metric from \((G_i, d_{G_i})\) to a new metric \((G_i, d_{G_i})\), making use of the assumed bi-Lipschitz equivalence of \(d_i\) and \(d_X\) on \(V_i\). For two points \(\tilde{x}\) and \(\tilde{y}\) in \(V_i \subset G_i\), we define
\[ d_{G_i}(\tilde{x}, \tilde{y}) := d_X(x, y), \]

where \(x\) and \(y\) are the associated points of \(\tilde{x}\) and \(\tilde{y}\) in \(X\). We then extend \(d_X\) to all of \(G_i\) in the same manner that we extended \(d_i\) to \(G_i\). We show that this new metric is bi-Lipschitz equivalent to \(d_{G_i}\) in the next paragraph.
This new metric allows us to finish the proof of Theorem 1.2. First, \((G_i, d_{G_i}, \overline{m}_i)\) satisfies a \((1, p)\)-Poincaré inequality with data that only depends upon the data of \((G_i, d_{G_i}, \overline{m}_i)\). This can be seen as the Poincaré inequality is bi-Lipschitz invariant (see chapter 8 of [13]). Second, it is this sequence, \((G_i, d_{G_i}, \overline{m}_i)\) that we will show pointed measured Gromov Hausdorff converges to \((X, d_X, \overline{\mu})\), where \(\overline{\mu}\) is a comparable measure to \(\mu\) on all balls. The relative ease of this conversion is the reason behind the switch to \(d_{G_i}\). Finally, by the result of Cheeger and Keith which we have listed above as Theorem 6.4, \((X, d_X, \overline{\mu})\) will carry a Poincaré inequality. Since measures which are comparable on balls also result in the comparability of integrals of measurable functions, \((X, d_X, \mu)\) will carry the desired Poincaré inequality.

It is not hard to show that \(d_{G_i}\) is a metric, but it is not clear that \(d_{G_i}\) is bi-Lipschitz equivalent to \(d_{G_i}\). It is trivial to see that the the metrics are bi-Lipschitz equivalent when restricted to points on \(V_i \subset G_i\), since \(d_{G_i} = d_X\), and \(d_{G_i} = d_i\) in this case. A more complicated case is when \(\tilde{x}\) and \(\tilde{y}\) are points on edges in \(G_i\). For ease of exposition, we will label the associated vertices of the edges containing these points by: \(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \) and \(\tilde{y}_2\) respectively. First, without loss of generality, we will take \(d_{G_i}(\tilde{x}, \tilde{y}) = d_i(\tilde{x}_1, \tilde{y}_1) + |\tilde{x} - \tilde{x}_1| + |\tilde{y} - \tilde{y}_1|\).

Recall that \(d_{G_i}\) is found by finding a shortest path through \(G_i\) from \(\tilde{x}\) to \(\tilde{y}\). Then we see, by the bi-Lipschitz equivalence of \(d_X\) and \(d_i\), and recalling that \(L \geq 1\),

\[
\begin{align*}
 d_{G_i}(\tilde{x}, \tilde{y}) &\leq d_X(\tilde{x}_1, \tilde{y}_1) + |\tilde{x} - \tilde{x}_1| + |\tilde{y} - \tilde{y}_1| \\
 &\leq Ld_i(\tilde{x}_1, \tilde{y}_1) + L|\tilde{x} - \tilde{x}_1| + L|\tilde{y} - \tilde{y}_1| \\
 &= Ld_{G_i}(\tilde{x}, \tilde{y}).
\end{align*}
\]

Alternatively, if we have that \(d_{G_i}(\tilde{x}, \tilde{y}) = d_X(\tilde{x}_1, \tilde{y}_1) + |\tilde{x} - \tilde{x}_1| + |\tilde{y} - \tilde{y}_1|\), then we see that

\[
\begin{align*}
 d_{G_i}(\tilde{x}, \tilde{y}) &\leq d_i(\tilde{x}_1, \tilde{y}_1) + |\tilde{x} - \tilde{x}_1| + |\tilde{y} - \tilde{y}_1| \\
 &\leq Ld_X(\tilde{x}_1, \tilde{y}_1) + L|\tilde{x} - \tilde{x}_1| + L|\tilde{y} - \tilde{y}_1| \\
 &= Ld_{G_i}(\tilde{x}, \tilde{y}).
\end{align*}
\]

The cases where either \(\tilde{x}\) or \(\tilde{y}\) are vertices are subcases of the above. Indeed, if \(\tilde{x}\) is a vertex, then we may call it \(\tilde{x}_1\) and the term \(|\tilde{x} - \tilde{x}_1|\) is zero, and the above still holds. Thus, we have bi-Lipschitz equivalence of \(d_{G_i}\) and \(d_{G_i}\), with the same bi-Lipschitz constant of \(L\). This implies, by the discussion above, that the family of \((G_i, d_{G_i}, \overline{m}_i)\) supports a \((1, p)\)-Poincaré inequality with uniform data that depends only upon the doubling constant of \(m_i\), \(L\), and the data from \((G_i, d_{G_i}, \overline{m}_i)\).

### 8.3. Pointed measured Gromov Hausdorff convergence of \((G_i, d_{G_i}, \overline{m}_i)\)

This section will be dedicated to proving:

**Lemma 8.3.** A subsequence of \((G_i, d_{G_i}, \overline{m}_i)\) converges in the pointed measured Gromov Hausdorff sense to \((X, d_X, \overline{\mu})\), where \(\overline{\mu}\) is comparable to \(\mu\).

As per the discussion in the first paragraph of the previous subsection, the proof of this lemma will finish the proof of Theorem 1.2.

**Proof of Lemma 8.3.** Let \(q\) be a point in \(V_i\), and \(n_i(q) \in X\) will be called \(q\) by an abuse of notation. Because \((V_i, d_{v_i}, m_i)\) is a nested embedding into \(X\), then there is a representative \(q \in V_i\) for all \(i \in \mathbb{N}\). We begin by showing that \((G_i, d_{G_i}, q) \xrightarrow{GH} (X, d_X, q)\). Let \(r > 0\) and \(0 < \eta < r\) be fixed numbers. For each \(G_i\), we introduce the maps \(f_i : G_i \rightarrow X\) where \(f_i|_{V_i} = n_i\), and \(f_i\) maps points on the edge set to a closest vertex. That is,

\[
f_i(\tilde{x}) = x_1 \in V_i \subset X,
\]
where \( |\tilde{x} - \tilde{x}_1| \leq \frac{c_i}{2} \). Whenever \( \tilde{x} \) is not the midpoint of an edge set, then \( f_i \) is clearly well defined. For a point \( \tilde{x} \in G_i \) on an edge with associated vertices \( \tilde{x}_1, \tilde{x}_2 \in V_i \) such that \( d_{G_i}(\tilde{x}, \tilde{x}_1) = d_{G_i}(\tilde{x}, \tilde{x}_2), \) \( f_i \) may be chosen to take \( \tilde{x} \) to either vertex. It is clear that \( f_i \) are independent of \( r \) and \( \eta \), and may be used for any choice of these numbers. Furthermore, the first requirement of pointed Gromov Hausdorff convergence is trivially satisfied with these maps.

We notice that since we assume that \( V_{i+1} \subset V_i \), and \( H_i \to 0 \), then \( \epsilon_{V_i} \to 0 \) as \( i \to \infty \). Select \( i_0 \) large enough so that \( \epsilon_{V_i} < \frac{\eta}{4} \) for all \( i \geq i_0 \). Let \( \tilde{x}, \tilde{y} \in B_{G_i}(q,r) \) for some \( i \geq i_0 \). If \( \tilde{x} \) and \( \tilde{y} \) are both vertices, then \( d_{X}(x,y) = d_{G_i}(\tilde{x},\tilde{y}) \), and the second requirement of pointed Gromov Hausdorff convergence is guaranteed trivially. However, if \( \tilde{x} \) and \( \tilde{y} \) are not vertices, then we still see that

\[
|d_X(f_i(\tilde{x}), f_i(\tilde{y})) - d_{G_i}(\tilde{x}, \tilde{y})| < 2L\epsilon_{V_i} < \eta.
\]

For the third requirement, we need to verify that \( B_X(q, r - \eta) \subset N_\eta(f_i(B_{G_i}(q, r))) \). This is easily verified since

\[
B_X(q, r - \eta) \subset f_i(B_{G_i}(q, r - \eta + \epsilon_{V_i})) \subset f_i(B_{G_i}(q, r)).
\]

Thus, we see that

\[
(G_i, d_{G_i}, q) \xrightarrow{GH} (X, d_X, q).
\]

Since both \( X \) and \( G_i \) are separable, then there exists isometric embeddings of each into \( \ell^\infty \), since the vertices of \( G_i \) are an embedded subset of \( X \), we can require embeddings that are equal when restricted to \( V_i \subset X \) and \( V_i \subset G_i \). Then we see that

\[
d_{\ell^\infty}^\infty(\iota(\overline{B}_X(q,r)), \iota_i(\overline{B}_{G_i}(q,r))) < H_i + \epsilon_{V_i},
\]

where \( \iota \) and \( \iota_i \) are the embeddings of \( X \) and \( G_i \) into \( \ell^\infty \) respectively. We see that \( H_i + \epsilon_i \) goes to 0 as \( i \to \infty \). Thus, to verify the pointed measured Gromov Hausdorff convergence of \((G_i, d_{G_i}, \overline{\mu}_i)\), we only need to verify that \( \iota(\# \overline{\mu}_i(\overline{B}_{G_i}(q,r))) \) converges in the weak* sense to a measure that is comparable on metric balls to \( \iota(\# \mu(\overline{B}_X(q,r))) \). An application of the Banach-Steinhaus theorem and the Reisz representation theorem guarantee that a subsequence does indeed weak* converge to some measure \( \overline{\mu} \). Thus, we have that

\[
(G_i, d_{G_i}, \overline{\mu}_i) \xrightarrow{\text{GH}} (X, d_X, \overline{\mu}).
\]

By the Theorem 6.3, we know that \((X, d_X, \overline{\mu})\) has \( \overline{\mu} \) as a doubling measure with a constant that ultimately depends on \( L \) and the uniform doubling constant of the family of \( m_i \). We also have, by Theorem 6.4, that \((X, d_X, \overline{\mu})\) supports a \((1,p)\)-Poincaré inequality with data that only depends on \( L \), the uniform doubling constant of the family of \( m_i \), and the uniform data of the family \((V_i, d_i, m_i)\). Since for all \( i \geq i_0 \) we have that \( \overline{\mu}_i \) is comparable to \( \mu \), then \( \overline{\mu} \) is also comparable to \( \mu \) by construction.

We now assert that \((X, d_X, \mu)\) supports a \((1,p)\)-Poincaré inequality. This can be seen since changing (2.1) by a comparable measure only gives a different constant \( C \) which depends upon the comparability constant of the two measures. In this case, the comparability constant depends upon \( L \) and the doubling constant for \( m_i \). Hence, \((X, d_X, \mu)\) supports a \((1,p)\)-Poincaré inequality with data that depends on \( L \), the uniform doubling constant of the family of \( m_i \), and the uniform Poincaré inequality data of the family \((V_i, d_i, m_i)\).

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