Cohomology of Hopf $C^*$-algebras and Hopf von Neumann algebras

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Abstract

We will define two canonical cohomology theories for Hopf $C^*$-algebras and for Hopf von Neumann algebras (with coefficients in their comodules). We will then study the situations when these cohomologies vanish. The cases of locally compact groups and compact quantum groups will be considered in more details.

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The first statement of Remark 1.9(a) in the original published article does not follow directly from Lemma 1.7(a) but it does follow very easily from the argument of Proposition 1.10. Therefore, we change the presentations of Remark 1.9 and Proposition 1.10 (one line was added in the proof of Proposition 1.10). Please find in the following the corrected version of this paper (any further question, comment or correction is welcomed).
0 Introduction

Cohomology theory is an important subject in many branches of Mathematics. In the field of Banach algebras, the vanishing of cohomology defines the interesting notion of amenability which, in the case of group algebras of locally compact groups, generalised the concept of amenable groups. In [28], Ruan studied the operator cohomology of completely contractive Banach algebras and defined the notion of operator amenability. He showed that the Fourier algebra of a locally compact group is operator amenable if and only if the group is amenable. The objective of this paper is to define and study some cohomology theories of Hopf $C^*$-algebras as well as Hopf von Neumann algebras. (Note that the notion of Hopf $C^*$-algebras that we use here is the same as that in [30.1] even though for most of the cases, we will also assume the extra condition mentioned in [30.1] – which is called saturated in this paper; see Definition 1.13). These will be important tools for the study of “locally compact quantum groups”. In particular, we will also investigate the situation when these cohomologies vanish.

In fact, there are four natural approaches to define cohomology theory for Hopf $C^*$-algebras and Hopf von Neumann algebras:

1. analogue of the deformation cohomology for Hopf algebras (see e.g. [31]);
2. analogue of the cyclic cohomology for Hopf algebras (see e.g. [17]);
3. generalisation of the group cohomology (see e.g. [26], [13] and [38] for three different meanings of this);
4. “dual analogue” of the Banach algebra homology (see. e.g. [14]).

For the first approach, we note that the definition of the cochain complex of the deformation cohomology for Hopf algebras requires operations involving both the product and the coproduct in a way which is unnatural for operator algebras (in particular, the maps are not bounded under the “default tensor product”). However, by using some technical results concerning the extended Haagerup tensor product (see [10]), we can still define this sort of analogue for Hopf von Neumann algebras. The same difficulty (together with some other) arise in the case of cyclic cohomology. (We didn’t try to find this analogue but even if this can be done, it is believed that the analogue can only be defined for Hopf von Neumann algebras – with the help of the reshuffle map in [10]). Nevertheless, we will not study these in this paper. Instead, we will study the kind of cohomology theories which are related to those of the third and the fourth approaches.

More precisely, the natural cohomology in this paper is defined according to the fourth approach (a comparison of this cohomology theory with the existing theories of “group cohomology” can be found in Example 2.2) whereas the dual cohomology is conceptually a dual version of the natural cohomology (although not directly related). Note that, even in the case of locally compact groups, the dual cohomology is different from any kind of the cohomology theories known so far (a comparison of the first dual cohomology theory with other cohomology theories can be found in Remark 4.7(a) and a comparison of the dual cohomology theory with a cohomology theory of coalgebras can be found in Remark 2.12(c)). Moreover, we know that for a locally compact group $G$, the cohomology theory for the Fourier algebra $A(G)$ as a Banach algebra is different from the cohomology theory for $A(G)$ as a completely contractive Banach algebra (see [15] and [28] even though they are formally defined in the same way. Hence there is no reason to believe that the dual cohomology of $C_0(G)$ or $C^*_r(G)$ will behave as either the Banach algebra cohomology or the operator cohomology of $A(G)$ (or $L^1(G)$). Therefore, these cohomology theories deserve detail study.

Before we can define the cohomology theories, we need to understand first of all, the comodules of Hopf $C^*$-algebras (and Hopf von Neumann algebras). A comodule of a Hopf $C^*$-algebra $S$ can be thought of as a vector space $X$ together with some topological structures as well as a “scalar-comultiplication” $\beta$ from $X$ to a kind of topological completion of the algebraic tensor product $X \otimes S$. Moreover, this should include the case of coactions on $C^*$-algebras. This means that the range of the coaction should be the “space of multipliers” of the completion of $X \otimes S$. In fact, if we want to define a comodule structure on a Banach space $X$, we will first come across the problem of getting a right topology for $X \otimes S$. Furthermore, the set of
“multipliers” does not behave nicely as required. There are also some other technical difficulties. However, if we consider operator spaces instead of Banach spaces, all these difficulties can be overcome.

In section 1 of this paper, we will recall some basic materials about operator spaces. We will then study multipliers of operator bimodules and give the definition as well as examples of comodules of Hopf C*-algebras. Note that in most of the cases in this paper (except for some results in the final section as well as in the appendix), we will assume that the Hopf C*-algebras are saturated (see Definition 1.13).

In the second section, we will define the natural and the dual cohomology theories mentioned above. We will study some situations when these cohomology theories vanish. If the Hopf C*-algebra is saturated and unital, (i.e. a compact quantum group; see [35] or [39]), we will give some interesting equivalent conditions for the vanishing of the (two-sided) dual cohomology (Theorem 2.10). We will then show (in Corollary 2.13) that in the case when the Hopf C*-algebra is $C^*_r(\Gamma)$ where $\Gamma$ is a discrete group, the vanishing of the dual cohomology is equivalent to the amenability of $\Gamma$. More generally, we will see in the final section that for a general locally compact group $G$, the vanishing of the first dual cohomology of $C^*_r(G)$ is equivalent to the amenability of $G$ (Theorem 4.6(b)). On the other hand, the vanishing of the first dual cohomology of $C_0(G)$ is again equivalent to $G$ being amenable (Theorem 4.6(a)). These are surprising since the dual cohomology theory of Hopf C*-algebras are thought to be not as sharp as theirs von Neumann algebra counterpart (see Remark 4.7(a)).

In section three, we will define comodules of Hopf von Neumann algebras and give the analogues of the above cohomology theories in this case. In particular, we will show that there is a natural Hopf von Neumann algebra comodule structure on the dual space of a comodule and the dual cohomology with coefficients in a given comodule is the same as the natural cohomology with coefficient in the corresponding dual comodule (which is not true in the Hopf C*-algebra case). Furthermore, the dual cohomology also coincides with the operator cohomology (see [28]) of the predual of the underlying Hopf von Neumann algebra.

In the final section, we will give some interesting relations between the vanishing of the (one sided and two sided) dual cohomologies and amenability. In particular, we will give a characterisation of the amenability of discrete semi-groups in terms of the dual cohomology theory and will prove the characterisation of amenable locally compact groups mentioned above. We will also consider the amenability of general Hopf C*-algebras (see [29]).

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## 1 Operator modules and coactions on Operator spaces

In this section, we will recall some properties of operator spaces and define coactions on them.

**Notation:** Throughout this paper, $X$, $Y$ and $Z$ are operator spaces. $\odot$ is the algebraic tensor product while $\otimes$ is the operator spatial tensor product and $\hat{\otimes}$ is the operator projective tensor product (see e.g. [27]).

**Definition 1.1** Let $B$ be a normed $*$-algebra and $N$ be a normed space.

(a) A norm $\| \cdot \|_\alpha$ on the algebraic tensor product $B \odot N$ is said to be a $B$-bimodule cross norm if it is a cross norm (i.e. $\|a \odot x\| = \|a\| \|x\|$) and $\|a \cdot z \cdot b\|_\alpha \leq \|a\|_B \|z\|_\alpha \|b\|_B$ for any $z \in B \odot N$ and $a, b \in B$ (where $a \cdot (c \otimes n) \cdot b = acb \otimes n$).

(b) A $B$-bimodule cross norm on $B \odot N$ is said to be a $L^\infty$ $B$-bimodule cross norm if for any disjoint (self-adjoint) projections $p, q \in B$ (i.e. $pq = 0$), $\|p \cdot z \cdot p + q \cdot z \cdot q\|_\alpha = \max\{\|p \cdot z \cdot p\|_\alpha, \|q \cdot z \cdot q\|_\alpha\}$. 


The following proposition is probably well known and a proof of which can be found in [21] (see also [10]).

**Proposition 1.2** (a) For any Banach space $N$, there is an one to one correspondence between the operator space structures on $N$ and the $L^\infty(K)$-bimodule cross norms on $K \circ N$ (where $K$ is the space of compact operators on the separable Hilbert space $l^2$). In this case, the $L^\infty(K)$-bimodule cross norm on $K \circ N$ is given by the operator spatial tensor product $K \otimes N$.

(b) A linear map $S$ from $X$ to $Y$ is completely bounded if and only if the map $\text{id}_X \otimes S$ extends to a bounded map from $K \otimes X$ to $K \otimes Y$. In this case, $\|S\|_{cb} = \|\text{id}_X \otimes S\|$.

(c) $\mathcal{L}_K(K \otimes X; K \otimes Y) = \{\text{id}_K \otimes S : S \in \text{CB}(X; Y)\}$ (where $\mathcal{L}_K$ means the set of all bounded $K$-bimodule maps). Consequently, $\text{CB}(X; Y) \cong \mathcal{L}_K(K \otimes X; K \otimes Y)$ as normed spaces.

(d) The canonical injection from $K \circ \text{CB}(X; Y)$ to $\text{CB}(X; K \otimes Y)$ gives the natural operator space structure on $\text{CB}(X; Y)$.

The following are some easy facts about projective tensor product. Note that part (a) is easy to check while parts (b) and (c) are the ideas behind the definition of operator projective tensor product (see [5], 5.4) as well as the paragraph following [5, 5.3]). Moreover, part (d) is a direct consequence of part (c).

**Lemma 1.3** Let $X$, $Y$ and $Z$ be operator spaces.

(a) For any $T \in \text{CB}(X; Z)$, the map $T \otimes \text{id}$ on the algebraic tensor product extends to a completely bounded map from $X \hat{\otimes} Y$ to $Z \hat{\otimes} Y$.

(b) $\text{CB}(X \hat{\otimes} Y; Z) \cong \text{CB}(X; \text{CB}(Y; Z)) \cong \text{JCB}(X \times Y; Z)$ (jointly completely bounded bilinear maps from $X \times Y$ to $Z$; see [20]).

(c) $\text{CB}(X; Y^*) = (X \hat{\otimes} Y)^*$.

(d) $\text{CB}(X \hat{\otimes} Y; Z^*) \cong \text{CB}(Y \hat{\otimes} X; Z^*)$.

**Remark 1.4** We call the identification in Lemma (a) (i.e. $\text{CB}(X \hat{\otimes} Y; Z) \cong \text{CB}(X; \text{CB}(Y; Z))$) the standard identification whereas the identification $\text{CB}(Y \hat{\otimes} X; Z) \cong \text{CB}(X; \text{CB}(Y; Z))$ will be called the reverse identification. This distinction is important when $X = Y$ (in which case the two identifications look the same but have different meanings; see the paragraph after Definition 3.4).

The following can be found in [21, 1.5(a)] and is again well known.

**Lemma 1.5** Let $X$ be a closed subspace of an operator space $Y$. If $\psi$ is a completely bounded map from $Y$ to $Z$ and $X \subseteq \text{Ker}(\psi)$, then $\psi$ induces a completely bounded map $\tilde{\psi}$ from $Y/X$ to $Z$ such that $\|\tilde{\psi}\|_{cb} \leq \|\psi\|_{cb}$ and $\tilde{\psi} \circ q = \psi$ (where $q$ is the canonical map from $Y$ to $Y/X$).

The following trivial lemma sets up some notations to be used later.

**Lemma 1.6** Let $W$, $X$, $Y$ and $Z$ be operator spaces.

(a) Any element $F \in \text{CB}(Z; W)$ induces a completely bounded map $\tilde{F}$ from $\text{CB}(Y; Z)$ to $\text{CB}(Y; W)$ such that $\tilde{F}(T) = F \circ T$ and $\|\tilde{F}\|_{cb} \leq \|F\|_{cb}$.

(b) For any $T \in \text{CB}(X; \text{CB}(Y; Z))$, there is a completely bounded map $T^\#: \text{CB}(Z; W) \rightarrow \text{CB}(X; \text{CB}(Y; W))$ (respectively, $T^0 : \text{CB}(W; Y) \rightarrow \text{CB}(X; \text{CB}(W; Z))$) such that $T^\#(F)(x)(y) = F(T(x)(y))$ and $\|T^\#\|_{cb} \leq \|T\|_{cb}$ (respectively, $T^0(F)(x)(w) = T(x)(F(w))$ and $\|T^0\|_{cb} \leq \|T\|_{cb}$).
We would like to study multipliers of operator $A$-bimodules for a $C^*$-algebra $A$. Let us first look at the multipliers of Banach $A$-bimodules. Given an $A$-bimodule $N$, let $M^A_N(N)$ (respectively, $M^A_N$) be the set of all linear maps from $A$ to $N$ that respect the right (respectively, left) $A$-multiplications, i.e., the set of all left (respectively, right) multipliers. Let $M_A(N) = \{(l, r) \in M^A_N(N) \times M^A_N(N) : a \cdot l(b) = r(a) \cdot b \text{ for any } a, b \in A\}$. Elements in $M_A(N)$ are called the multipliers of $N$. A bimodule $N$ is said to be essential if both $A \cdot N$ and $N \cdot A$ are dense in $N$. If $A$ is unital, $N$ is essential simply means that $N$ is a unital $A$-bimodule. Moreover, in this case, $M_A(N) = M^A_N(N) = M^A_A(N) = N$.

**Notation:** From now on, until the end of this section, $A$ is a $C^*$-algebra. Moreover, if $(l, r) \in M_A(N)$ and $a \in A$, we will denote $a \cdot (l, r) = (l(a), r(a))$ and $(l, r) \cdot a = (l \cdot a, r)$.

**Lemma 1.7** Let $A$ be a $C^*$-algebra and $N$ be an essential $A$-bimodule.

(a) Any left or right multiplier on $N$ is automatically bounded.

(b) For any $(l, r) \in M_A(N)$, we have $\|l\|_{M^A_N(N)} = \|r\|_{M^A_N(N)}$.

(c) $M_A(N)$ is a Banach space for the norm defined by $\|(l, r)\| = \|l\|_{M^A_N(N)} = \|r\|_{M^A_N(N)}$.

In fact, part (a) follows from a similar argument as in [25, 3.12.2] and part (b) follows from the fact that $A$ has an approximate unit for $N$ while part (c) is more or less obvious.

Suppose that $Y$ is an operator space with an $A$-bimodule structure. Then $Y$ is called an operator $A$-bimodule if $K \otimes Y$ is a $K \otimes A$-bimodule. In this case, if $Y$ is essential as an $A$-bimodule, then $K \otimes Y$ is an essential $K \otimes A$-bimodule and we call $Y$ an essential operator $A$-bimodule. Moreover, by using [26, 3.3] and some simple arguments concerning the essentialness of the bimodule $Y$ as well as employing the trick of replacing $Y$ with $\mathcal{K} \otimes Y$, we have the following representation lemma (a detail argument can be found in [21]). The triple $(\phi, \pi, \psi)$ satisfying the relation in this lemma is called a **spatial realisation** of $Y$.

**Lemma 1.8** Let $Y$ be an essential operator $A$-bimodule. Then there exist Hilbert spaces $H$ and $K$ as well as a complete isometry $\pi$ from $Y$ to $\mathcal{L}(H; K)$ and faithful non-degenerate representations $\phi$ and $\pi$ of $A$ on $\mathcal{L}(H)$ and $\mathcal{L}(K)$ respectively such that $\phi(b)\pi(y)\psi(a) = \pi(b \cdot y \cdot a)$ for all $a, b \in A$ and $y \in Y$.

This suggests another way to define multipliers: $M^A_N(Y) = \{m \in \mathcal{L}(H; K) : \phi(A)m, m\pi(A) \subseteq \pi(Y)\}$. However, it is not obvious that this definition is independent of the choice of the spatial realisation. Nevertheless, we will see later that it is indeed completely isometrically isomorphic to $M_A(Y)$ (regarded as an operator $A$-bimodule). Let us first give a natural operator space structure on $M_A(Y)$.

**Remark 1.9** (a) Suppose that $M^A_{Y, cb}(Y) = M^A_Y(Y) \cap \text{CB}(A; Y)$ and $M^A_{cb}(Y) = M^A_Y(Y) \cap \text{CB}(A; Y)$ with the induced operator space structures. More precisely, the operator space structure on $M^A_{cb}(Y)$ is given by the canonical injection from $K \otimes M_Y^A(Y)$ to $M^A_{cb}(K \otimes Y) \subseteq \text{CB}(A; K \otimes Y)$ (see Proposition 1.2(d)). We denote by $\|\cdot\|_{\text{usu}}$ and $\|\cdot\|$ the norms on $M^A_Y(Y)$ and $M^A_{cb}(Y)$ induced from $\mathcal{L}(A; Y)$ and $\text{CB}(A; Y)$ respectively. Since $\text{CB}(A; Y)$ can be regarded as the subspace $\mathcal{L}(A; Y)$ of $\mathcal{L}(K \otimes A; K \otimes Y)$ (see Proposition 1.2(b) and (c)), the canonical map from $(M^A_{cb}(Y), \|\cdot\|)$ to $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|_{\text{usu}})$ is an isometry. Therefore, the operator space structure on $M^A_{cb}(Y)$ is given by the canonical embedding from $K \otimes M^A_{cb}(Y)$ to $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|_{\text{usu}})$ (as $(M^A_{cb}(K \otimes Y), \|\cdot\|$) can be regarded as its subspace). The same is true for $M^A_{Y, cb}(Y)$.

(b) Lemma 1.7(b) implies that $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|_{\text{usu}})$ is simultaneously a norm subspace of both $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|)$ and $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|_{\text{usu}})$. Thus, the norms induced on $M^A_{cb}(Y) = M^A_Y(Y) \cap \text{CB}(A; Y)$ from $(M^A_Y(Y), \|\cdot\|)$ and $(M^A_Y(Y), \|\cdot\|_{\text{usu}})$ coincide. Similarly, $M^A_{K \otimes A}(K \otimes Y)$ is simultaneously a norm subspace of both $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|_{\text{usu}})$ and $(M^A_{K \otimes A}(K \otimes Y), \|\cdot\|)$. Therefore, the two embeddings from $K \otimes M^A_{cb}(Y)$ to $\mathcal{L}(K \otimes A; K \otimes Y)$ (induced from $M^A_Y$ and $M^A$) give the same operator space structure on $M^A_{cb}(Y)$ and we use this structure by default.
Proposition 1.10 Let $A$ be a $C^*$-algebra and $Y$ be an essential operator $A$-bimodule. For any spatial realisation $(\phi, \pi, \psi)$ of $Y$ (see the statement before Lemma 1.8), there exists an isometry $\Psi$ from $(M_A(Y), \| \cdot \|_{\text{usus}})$ to $M_A^* (Y)$ such that $\Psi(l, r) \psi(a) = \pi(l(a))$ and $\phi(a) \Psi(l, r) = \pi(r(a))$ $(a \in A; (l, r) \in M_A(Y))$. Moreover, $(M_{Acb}(Y), \| \cdot \|) = (M_A(Y), \| \cdot \|_{\text{usus}})$ and $\Psi$ is a complete isometry.

Proof: Let $\{a_n\}$ be an approximate unit of $A$. Suppose that $(l, r) \in M_A(Y)$. The net $\{\pi(l(a_n))\}$ converges strongly to an element $m \in L(H; K)$ (as $\psi$ is non-degenerate and $l$ is bounded). It is clear that $m \psi(a) = \pi(l(a))$ and $\phi(b)m \psi(a) = \pi(r(b))\psi(a)$ for any $a, b \in A$. Moreover,

$$\|m\| = \sup \{\|m \psi(a)\| : a \in A; \|a\| \leq 1\} = \|l\|_{\text{usus}} = \|(l, r)\|_{\text{usus}}.$$ 

Hence it is not hard to see that the map $\Psi$ that sends $(l, r)$ to $m$ is a surjective isometry from $(M_A(Y), \| \cdot \|_{\text{usus}})$ to $M_A^* (Y)$ (note that any element in $M_A^* (Y)$ defines in the obvious way, an element in $M_A(Y)$) which satisfies the required equalities. As a consequence, $M_{Acb}(Y) = M_A(Y)$ (because both $a \mapsto \pi^{-1}(m \psi(a))$ and $a \mapsto \pi^{-1}(\phi(a)m)$ are completely bounded maps for $m \in M_A^* (Y)$). It remains to show that $\| \cdot \|_{\text{usus}}$ coincides with $\| \cdot \|$ and $\Psi$ is a complete isometry. Observe that by replacing $Y$ with $K \otimes Y$ and $A$ with $K \otimes A$, we have an isometry $\Psi'$ from $(M_{K \otimes A}(K \otimes Y), \| \cdot \|_{\text{usus}})$ to $M_{K \otimes A}^* (K \otimes Y)$. For any $k \in K$ and $a \in A$,

$$\Psi'(id_K \otimes l, id_K \otimes r)(k \otimes a) = (1 \otimes \Psi(l, r))(k \otimes a).$$

Thus, $\Psi$ is an isometry from $(M_A(Y), \| \cdot \|)$ to $M_A^* (Y)$ (recall from Remark 1.9 that $\|ll^*\| = \|(id_K \otimes l, id_K \otimes r)\|_{\text{usus}}$). This also shows that $\| \cdot \|$ and $\| \cdot \|_{\text{usus}}$ agree on $M_A(Y)$. Now, if we replace $Y$ with $K \otimes Y$ only, we have an isometry $\Phi$ from $(M_A(K \otimes Y), \| \cdot \|)$ to $M_A^{id \otimes \pi} (K \otimes Y)$. For any $k, k' \in K$ and $a \in A$, $k \otimes (l, r)$ can be regarded as an element of $M_A(K \otimes Y)$ and

$$\Phi(k \otimes (l, r))(k' \otimes a) = (k \otimes \Psi(l, r))(k' \otimes a).$$

Therefore, the map $id_K \otimes \Psi$ from $K \otimes M_A(Y)$ to $K \otimes M_A^* (Y) \subseteq M_A^{id \otimes \pi} (K \otimes Y)$ is an isometry (note that $(K \otimes M_A(Y), \| \cdot \|)$ is a subspace of $(M_A(K \otimes Y), \| \cdot \|)$ by Remark 1.9 and hence $\Psi$ is a complete isometry by Proposition 1.2(b).

Corollary 1.11 (a) $Y$ is an operator subspace of $M_A(Y)$.
(b) $M_A(Y)$ is a unital operator $M(A)$-bimodule.
(c) If $B$ is another $C^*$-algebra and $Z$ is an essential operator $B$-bimodule, then there exists a complete isometry from $M_B(M_A(Y) \otimes Z)$ to $M_{A \otimes B}(Y \otimes Z)$ that respects both the $A$-bimodule and the $B$-bimodule structures.

Notation: From now on, we may use the identification in Proposition 1.10 implicitly and will regard $M_B(M_A(Y) \otimes Z)$ as subspace of $M_{A \otimes B}(Y \otimes Z)$.

Lemma 1.12 (a) Let $X$ and $Y$ be essential operator $A$-bimodules. Suppose that $\varphi$ is a completely bounded $A$-bimodule map from $X$ to $Y$. Then $\varphi$ induces a completely bounded $M(A)$-bimodule map, again denoted by $\varphi$, from $M_A(X)$ to $M_A(Y)$. If $\varphi$ is completely isometric, then so is the induced map.
(b) Let $A$ and $B$ be $C^*$-algebras and $\psi : A \rightarrow M(B)$ be a non-degenerate $*$-homomorphism. Then $id_Z \otimes \psi$ extends to a complete contraction $id_Z \otimes \psi : M_A(Z \otimes A) \rightarrow M_B(Z \otimes B)$ such that $(id_Z \otimes \psi)(m \cdot a) = (id_Z \otimes \psi)(m) \cdot \psi(a)$ and $(id_Z \otimes \psi)(a \cdot m) = \psi(a) \cdot (id_Z \otimes \psi)(m) \ (m \in M_A(Z \otimes A); a \in M(A))$. If $\psi$ is injective, then $id_Z \otimes \psi$ is a complete isometry. Furthermore, if $\phi$ is a completely bounded map from $Z$ to another operator space $Z'$, then $(\phi \otimes id)(id \otimes \psi) = (id \otimes \psi)(\phi \otimes id)$ on $M_A(Z \otimes A)$.
(c) For any $g \in A^*$ and $T \in CB(X; Y)$ $(X$ and $Y$ are operator spaces), we have $T \circ (id \otimes g) = (id \otimes g)(T \otimes id)$ on $M_A(X \otimes A)$. 

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The map in part (a) is induced by the completely bounded map given in Lemma 16(a). The first two statements of part (b) follow from Proposition 10 while the last statement follows from the fact that \((\varphi \otimes \text{id})(m)(1 \otimes b) = (\varphi \otimes \text{id})(m)(1 \otimes b)\) under the identification in Proposition 10. The map \(\text{id} \otimes g\) in part (c) is defined on \(M_A(X \otimes A) \subseteq M(K(H) \otimes A)\) when \(X \subseteq K(H)\). It is well defined and satisfies the equality in (c) because \(g\) can be decomposed as \(a \cdot g'\) where \(a \in A\) and \(g' \in A^*\).

Next, we will recall from [3, 0.1 & 0.2] the notion of Hopf C*-algebras and their coactions (even though we change some of the terminology in our translation).

**Definition 1.13** (a) Let \(S\) be a C*-algebra with a non-degenerate \(*\)-homomorphism \(\delta\) from \(S\) to \(M(S \otimes S)\). Then \((S, \delta)\) is said to be a Hopf C*-algebra if \(\delta(S)(1 \otimes S), \delta(S)(S \otimes 1) \subseteq S \otimes S\) and \((\delta \otimes \text{id})\delta = (\text{id} \otimes \delta)\delta\). In this case, \(\delta\) is called a coproduct of \(S\). Moreover, a Hopf C*-algebra \((S, \delta)\) is said to be saturated if both of the vector spaces \(\delta(S)(1 \otimes S)\) and \(\delta(S)(S \otimes 1)\) are dense in \(S \otimes S\).

(b) Let \(\mathcal{R}\) be a von Neumann algebra with a weak*-continuous unital \(*\)-homomorphism from \(\mathcal{R}\) to \(\mathcal{R} \hat{\otimes} \mathcal{R}\). Then \((\mathcal{R}, \delta)\) is said to be a Hopf von Neumann algebra if \((\delta \otimes \text{id})\delta = (\text{id} \otimes \delta)\delta\). A Hopf von Neumann algebra \(\mathcal{R}\) is said to be saturated if both \(\delta(\mathcal{R})(1 \otimes \mathcal{R})\) and \(\delta(\mathcal{R})(\mathcal{R} \otimes 1)\) are weak*-dense in \(\mathcal{R} \hat{\otimes} \mathcal{R}\).

(c) Let \(A\) be a C*-algebra and \(M\) be a von Neumann algebra. A non-degenerate \(*\)-homomorphism \(\beta\) from \(A\) to \(M_S(A \otimes S)\) (which is a C*-algebra) is said to be a coaction of \(S\) on \(A\) if \((\beta \otimes \text{id})\beta = (\text{id} \otimes \delta)\beta\). Similarly, a normal \(*\)-homomorphism \(\beta\) from \(M\) to \(M \hat{\otimes} \mathcal{R}\) is said to be a coaction of \(\mathcal{R}\) on \(M\) if \((\beta \otimes \text{id})\beta = (\text{id} \otimes \delta)\beta\).

**Notation:** From now on, unless specified, \((S, \delta)\) is a saturated Hopf C*-algebra and \((\mathcal{R}, \delta)\) is a saturated Hopf von Neumann algebra respectively.

**Definition 1.14** Suppose that \((R, \delta)\) is a (not necessarily saturated) Hopf C*-algebra.

(a) Let \(\beta\) be a completely bounded map from \(X\) to \(M_R(X \otimes R)\). Then \(\beta\) is said to be a right coaction of \(R\) on \(X\) if \((\beta \otimes \text{id})\beta = (\text{id} \otimes \delta)\beta\) is completely bounded on \(X\). Similarly, we can define left coaction as a completely bounded map \(\gamma\) from \(X\) to \(M_R(R \otimes X)\) such that \((\text{id} \otimes \gamma)\gamma = (\delta \otimes \text{id})\gamma\).

(b) A right coaction \(\beta\) is said to be right (respectively, left) non-degenerate if the linear span of \(\{\beta(x) \cdot s : x \in X; s \in R\}\) (respectively, \(\{s \cdot \beta(x) : x \in X; s \in R\}\)) is norm dense in \(X \otimes R\) (we recall that \((l, r) \cdot s = l(s)\) and \(s \cdot (l, r) = r(s)\) for any \((l, r) \in M_R(X \otimes R)\) and \(s \in R\)).

(c) \(X\) is said to be a right \(R\)-comodule (respectively, left \(R\)-comodule) if there exists a right coaction (respectively, left coaction) of \(R\) on \(X\).

The right coaction identity in part (a) actually means that \(\Phi \circ (\beta \otimes \text{id})\beta = (\text{id} \otimes \delta)\beta\) where \(\Phi\) is the forgettable complete isometry from \(M_R(M_R(X \otimes R) \otimes R)\) to \(M_R \hat{\otimes} R(X \otimes R \otimes R)\) given by Corollary 11(c).

**Lemma 1.15** (a) The predual \(\mathcal{R}_\ast\) of \(\mathcal{R}\) has a left (or right) identity if and only if it is unital.

(b) Suppose that \(\beta\) is a right (or left) non-degenerate right coaction of \(S\) in \(X\) and \(\epsilon\) is a counit of \(S\). Then \((\text{id} \otimes \epsilon)\beta = \text{id}\). The same is true for a left coaction.

(c) Let \(\beta\) be a right coaction of \(S\) on \(X\) and let \(Y\) be a closed subspace of \(X\). Suppose that the canonical quotient map \(q\) from \(X\) to \(X/Y\) satisfies the following condition: \((q \otimes \text{id})\beta(Y) = (0)\). Then \(\beta\) induces a right coaction \(\tilde{\beta}\) of \(S\) on \(X/Y\) such that \(\beta \circ q = (q \otimes \text{id}) \circ \tilde{\beta}\).

**Proof:** (a) Suppose that \(\epsilon\) is a left identity of \(\mathcal{R}_\ast\). Then we have \((\nu \otimes \epsilon)\delta((\text{id} \otimes \omega)\delta(s)) = (\nu \otimes (\epsilon \otimes \omega)\delta)\delta(s) = (\nu \otimes \omega)\delta(s)\) (for any \(\omega, \nu \in \mathcal{R}_\ast\) and \(s \in \mathcal{R}\)). Now weak*-density of \(\delta(\mathcal{R})(1 \otimes \mathcal{R})\) in \(\mathcal{R} \hat{\otimes} \mathcal{R}\) implies that \(\epsilon\) is an identity of \(\mathcal{R}_\ast\).

(b) Suppose that \(\beta\) is right non-degenerate. For any \(x \in X\) and \(s \in S\), the tensor \(x \otimes s\) can be approximated by sums of elements of the form \(\beta(y) \cdot t\) \((y \in X; t \in S)\) and hence \(x\) can be approximated by sums of elements of the form \((\text{id} \otimes g)\beta(y)\) \((g \in S^\ast)\); note that \(M_S(X \otimes S)\) can be regarded as a subspace of some...
M(ℋ(H) ⊗ S) by Proposition 1.10. The lemma now follows from the fact that \((\text{id} \otimes e)\beta((\text{id} \otimes g)\beta(y)) = (\text{id} \otimes (e \otimes g)\delta)\beta(y) = (\text{id} \otimes g)\beta(y)\). The other three cases can be proved similarly.

(c) By Lemma 1.16, there exists a map \(\hat{\beta}\) that satisfies the required equality. It remains to check the right coaction identity. In fact, \((\hat{\beta} \otimes \text{id})\beta \circ q = (\hat{\beta} \otimes \text{id})(q \otimes \text{id})\beta = ((q \otimes \text{id})\beta \otimes \text{id})\beta = (q \otimes \text{id} \otimes \text{id})\beta = (\text{id} \otimes \delta)(q \otimes \text{id})\beta\) (note that as we are working with multipliers of an operator bimodule instead of elements in a C\(^∗\)-algebra, caution is needed to be taken for each of the equalities above; in particular, the last equality follows from Lemma 1.16(b)).

**Example 1.16** (a) Let \(\Gamma\) be a discrete group. The reduced group C\(^∗\)-algebra \(C^*_r(\Gamma)\) is a Hopf C\(^∗\)-algebra with coproduct given by \(\delta(\lambda) = \lambda \otimes \lambda\) (where \(\lambda\) is the canonical image of \(r\) in \(C^*_r(\Gamma)\)). If \(\beta\) is a non-degenerate coaction of \(C^*_r(\Gamma)\) on a C\(^∗\)-algebra \(A\) (in the sense of Definition 1.13(c)), then \(A\) can be decomposed as \(A = \bigoplus_{r \in \Gamma} \overline{A_r}\) (see [17, 2.6]). Let \(F\) be any subset of \(\Gamma\) and \(\overline{A_F} = \bigoplus_{r \in F} \overline{A_r}\). Then the restriction \(\beta_F\) of \(\beta\) on \(A_F\) is a right coaction of \(C^*_r(\Gamma)\) on \(A_F\). Moreover, it is not hard to see that this right coaction is also (2-sided) non-degenerate.

(b) Suppose that \(\beta\) is a right (or left) non-degenerate right coaction of \(C^*_r(\Gamma)\) on \(X\). Let \(X_r = \{x \in X : \beta(x) = x \otimes \lambda_r\}\). Then \(X_r = (\text{id} \otimes \varphi_r)\beta(X)\) where \(\varphi_r\) is the functional on \(C^*_r(\Gamma)\) satisfying \(\varphi_r(\lambda_t) = \delta_{r,t}\) (here \(\delta_{r,t}\) means the Kronecker delta) as defined in [17, §2]. Now by the right (respectively, left) non-degeneracy of \(\beta\), we have \(X = \bigoplus_{r \in \Gamma} X_r\).

(c) Let \(G\) be a locally compact group. Then \(C_0(G)\) is a Hopf C\(^∗\)-algebra with a coproduct defined by \(\delta(f)(s,t) = f(st)\) (note that \(M(C_0(G) \otimes C_0(G)) = C_0(G \times G)\)). Right coactions of \(C_0(G)\) on \(X\) are in one to one correspondence with completely bounded actions of \(G\) on \(X\) in the following sense: an action \(\alpha\) of \(G\) on \(X\) is said to be completely bounded if

i. there is \(\lambda > 0\) such that \(\sup\{\|\text{id}_K \otimes \alpha_t(\bar{x})\| : t \in G\} \leq \lambda\|\bar{x}\|\) for any \(\bar{x} \in K \otimes X\);

ii. \(\alpha_\bullet(x)\) is a continuous map from \(G\) to \(X\) for any fixed \(x \in X\);

(or equivalently, \(\alpha\) induces a bounded continuous action of \(G\) on the subspace \(K \otimes X\) of \(K \otimes X\)). Moreover, if the right coaction is a complete isometry, then condition (i) is replaced by the following condition:

i'. \(\sup\{\|\text{id}_K \otimes \alpha_t(\bar{x})\| : t \in G\} = \|\bar{x}\|\) for any \(\bar{x} \in K \otimes X\).

Indeed, by considering \(X\) as a closed subspace of a C\(^∗\)-algebra, we see that \(M(C_0(G) \otimes C_0(G)) = C_0(G;X)\).

Hence as in the case of C\(^∗\)-algebras, a right coaction \(\delta\) induces an action \(\alpha\) of \(G\) on \(X\) such that \(\alpha_t(x) = \delta(x)(t)\). Since \(\delta(X) \subseteq C_0(G;X)\) and \(\delta\) is bounded, there exists \(\lambda > 0\) such that \(\sup\{\|\alpha_t(x)\| : t \in G\} \leq \lambda\|x\|\) and for fixed \(x \in X\), the map \(\alpha_\bullet(x)\) is continuous. As \(\delta\) is completely bounded, we can replace \(X\) with \(K \otimes X\) and show that \(\alpha\) is a completely bounded action. Conversely, let \(\alpha\) be a completely bounded action. If we define \(\delta(x)(t) = \alpha_t(x)\) (for any \(x \in X\) and \(t \in G\)), then condition (ii) implies that \(\delta(x) \in C(G;X)\) (i.e. a continuous map) and so \(k \otimes \delta(x) \in C(G;K \otimes X)\) (for any \(k \in K\)). Furthermore, condition (i) shows that \(id \otimes \delta\) is a bounded map from \(K \otimes X\) to \(C_0(G;K \otimes X) = M(C_0(G) \otimes X) \otimes C_0(G)\)). It is not hard to check that \(\delta\) is a right coaction and the correspondence is established. In this case, \(\delta\) is injective if and only if \(\alpha_t\) is injective for all \(t \in G\) (because \(\alpha_t\alpha_s = \alpha_{st}\)) or equivalently, \(\alpha_t\) is injective for some \(t \in G\). It is the case if and only if \(\alpha_e = I_X\) (note that \(\alpha_e(x) = x\)).

(d) Again when \(S = C_0(G)\), there is an one to one correspondence between right coactions of \(S\) on \(X\) and completely bounded representations of \(G\) on \(X\): a map \(T\) from \(G\) to \(CB(X;X)\) is called a completely bounded representation if

i. \(T_r \circ T_s = T_{rs}\);

ii. \(\sup\{\|T_r\|_{cb} : r \in G\} < \infty\);

iii. the map \(T_e(x)\) from \(G\) to \(X\) is continuous for any \(x \in X\).

In particular, if \(H\) is a Hilbert space and \(H_c\) is the column operator space of \(H\), then right coactions of \(C_0(G)\) on \(H_c\) are exactly bounded continuous representations of \(G\) on \(L(H) = CB(H_c;H_c)\).

(e) By [21, 2.4], a coaction of \(S\) on a Hilbert C\(^∗\)-modules \(E\) (in the sense of [2]) defines a right coaction of \(S\) on the “column space” \(E_c\).
(f) By [21, 2.5 & 2.7], if a corepresentation of $S$ on a Hilbert space $H$ is either unitary or “non-degenerate”, then it gives rise to a coaction of $S$ on $H_c$.

2 Cohomology of Hopf $C^*$-algebras

In this section, we will define and study cohomology theories for Hopf $C^*$-algebras with coefficients in their bicomodules. Let $X$ be an operator space and let $\beta$ and $\gamma$ be respectively a right and a left coactions of $S$ on $X$. Then $(X, \beta, \gamma)$ is said to be a $S$-bicomodule if $(\id \otimes \beta)\gamma = (\gamma \otimes \id)\beta$. Let us first consider a straightforward way to define cohomology (which is a “dual analogue” of Banach algebra homology).

Notation: For simplicity, we may sometimes use $X$ to denote the bicomodule $(X, \beta, \gamma)$. Throughout this section, $S^n$ is the $n$-times spatial tensor product of $S$ (whereas $S^0 = \mathbb{C}$) and $\sigma_{n,k}$ ($1 \leq k \leq n$) is the completely bounded map from $M_{S^n}(S^k \otimes X \otimes S^{n-k})$ to $M_{S^n}(X \otimes S^n)$ defined by $(s_{n-k+1} \otimes \ldots \otimes s_n \otimes x \otimes s_{k-1} \otimes \ldots \otimes s_n)^{\sigma_{n,k}} = x \otimes s_1 \otimes \ldots \otimes s_n$ (see Lemma 1.13(a)).

For $n \geq 1$, we define a completely bounded map $\delta_n$ from $M_{S^n}(X \otimes S^n)$ to $M_{S^{n+1}}(X \otimes S^{n+1})$ by

$$\delta_n(x \otimes s_1 \otimes \ldots \otimes s_n) = \beta(x) \otimes s_1 \otimes \ldots \otimes s_n + \sum_{k=1}^{n} (-1)^k x \otimes s_1 \otimes \ldots \otimes s_{k-1} \otimes \delta(s_k) \otimes s_{k+1} \otimes \ldots \otimes s_n + (-1)^{n+1}(\gamma(x) \otimes s_1 \otimes \ldots \otimes s_n)^{\sigma_{n+1,1}}$$

and $\delta_0(x) = \beta(x) - \gamma(x)^{\sigma_{1,1}}$ ($\delta_n$ is well defined by Lemma 1.12(a) and Corollary 1.12(c)). We need to show that $(M_{S^n}(X \otimes S^n), \delta_n)$ is a cochain complex.

Lemma 2.1 $\delta_n \circ \delta_{n-1} = 0$ for $n = 1, 2, 3, \ldots$

Proof: Note, first of all, that $\delta_1 \circ \delta_0(x) = (\beta \otimes \id)\beta(x) - (\id \otimes \delta)\beta(x) + (\gamma \otimes \id)\beta(x)^{\sigma_{1,1}} - (\id \otimes \beta)\gamma(x)^{\sigma_{2,1}} + (\delta \otimes \id)\gamma(x)^{\sigma_{2,2}} - (\delta \otimes \id)\gamma(x)^{\sigma_{1,1}}\sigma_{1,1} = 0$. This established the equality for $n = 1$. For the case of $n > 1$, the crucial point is to show that $\sum_{k=1}^{n} \sum_{i=1}^{n-1} (-1)^{k+i} (\id x \otimes \id x \otimes \ldots \otimes \id \otimes \id \otimes \id \otimes \delta \otimes \id) = 0$. This can be done by a decomposition (into a sum of summations according to the relative positions of the $i$'s and $k$'s in the original summation) as well as a tedious comparison.

Now we can define a cohomology $H^n(S; X) = \ker(\delta_n)/\text{im}(\delta_{n-1})$ for $n \in \mathbb{N}$ and $H^0(S; X) = \{x \in X : \beta(x) = \gamma(x)^{\sigma_{1,1}}\}$. It is called the natural cohomology of $S$ with coefficient in the bicomodule $(X, \beta, \gamma)$.

Example 2.2 (a) Let $G$ be a locally compact group and $S = C_0(G)$. We have already seen in Example 1.10(c) that $S$ is a Hopf $C^*$-algebra and a right $S$-comodule $X$ is a “completely bounded left $G$-module” and $M_{S}(X \otimes S) = C_b(G; X)$. Hence a $S$-bicomodule is a “completely bounded $G$-bimodule”. In this case, $\psi \in C_0(G; X)$ is in $\ker(\delta_1)$ if and only if it is a derivation in the sense that $\psi(st) = s \cdot \psi(t) + \psi(s) \cdot t$ ($s, t \in G$). Moreover, $\psi \in \im(\delta_0)$ if and only if there exists $x \in X$ such that $\psi(s) = s \cdot x - x \cdot s$ ($s \in G$). Hence $H^1(S; X)$ is formally a kind of group cohomology of $G$.

(b) If $S = c_0(\Gamma)$ for a discrete group $\Gamma$ and the left coaction $\gamma$ is $1 \otimes \id_X$, then $H^1(c_0(\Gamma); X)$ coincides with the first group cohomology $H^1(\Gamma; X)$ (with coefficients in the $c_0(\Gamma)$-bicomodule $X$) that studied in [L3] (see [L3, p.9]).

(c) On the other hand, if $G$ is a profinite group and $\gamma = 1 \otimes \id_X$, the two groups $H^n(C_0(G); X)$ and $H^n(G; X)$ coincide where $H^n(G; X)$ is the group cohomology studied in [38, §9.1] and [67, §2.2].
It is natural to consider the cohomology of a “dual bicomodule” and expect it to relate to the amenability of the Hopf C*-algebra. However, it seems impossible to define dual comodule structure on the dual space of a \(S\)-bicomodule. Nevertheless, we still have a kind of “dual cohomology theory” with the cochain complex starting with the dual space.

Let \(\partial_n\) be the completely bounded map from \(\text{CB}(X; M(S^n))\) to \(\text{CB}(X; M(S^{n+1}))\) defined by

\[
\partial_n(T) = \begin{cases} 
(T \otimes \text{id}) \circ \partial + \sum_{k=1}^n (-1)^k (\text{id}^{n-k} \otimes \partial \otimes \text{id}^{k-1}) \circ T + (-1)^{n+1} (\text{id} \otimes T) \circ \gamma & n = 1, 2, 3, \ldots \\
(T \otimes \text{id}) \circ \partial - (\text{id} \otimes T) \circ \gamma & n = 0
\end{cases}
\]

for any \(T \in \text{CB}(X; M(S^n))\). It is well defined by Lemma 1.12(a) and Corollary 1.11(c). Again, it gives a cochain complex.

**Lemma 2.3** \(\partial_n \circ \partial_{n-1} = 0\) for \(n = 1, 2, 3, \ldots\)

We can now define another cohomology theory by \(H^\beta_0(S; X) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n-1})\) \((n = 1, 2, 3, \ldots)\) and \(H^\beta_j(S; X) = \{ f \in X^* : (f \otimes \text{id}) \circ \partial = (\text{id} \otimes f) \circ \gamma \}\). It is called the dual cohomology of \(S\) with coefficient in \((X, \beta, \gamma)\). The name comes from the fact that it is a cohomology theory “with coefficient in the dual space \(X^\ast\)”. Furthermore, the Hopf von Neumann algebra analogue of this cohomology can actually be regarded as a dual cohomology theory (see Proposition 3.3 and Proposition 3.14 below).

Note that the idea of the dual cohomology is similar to the cohomology theory of coalgebras studied in 9 but although they look alike, they “behave differently” even in the case of discrete groups (see Remark 2.12(c)).

**Example 2.4** Let \(\Gamma\) be a discrete group. Suppose that \((X, \beta, \gamma)\) is a \(C^*_r(\Gamma)\)-bicomodule (see Example 1.10(a)&(b)) such that both \(\beta\) and \(\gamma\) are either left or right non-degenerate. Then by Example 1.10(b), \(X\) can be decomposed into two directed sums \(\oplus_{s \in \Gamma} X^\beta_s\) and \(\oplus_{r \in \Gamma} X^\gamma_r\) corresponding to \(\beta\) and \(\gamma\) respectively (where \(X^\beta_s = (\text{id} \otimes \varphi_s)\beta(X)\) and \(X^\gamma_r = (\text{id} \otimes \varphi_r)\gamma(X)\) is the functional as defined in Example 1.10(b)). Moreover, since \((\text{id} \otimes \beta)\gamma = (\gamma \otimes \text{id})\beta\), for any \(s, t \in \Gamma\), the spaces \(X^\beta_s\) and \(X^\gamma_t\) can be decomposed further into \(\oplus_{r \in \Gamma} (X^\beta_s)_r\) and \(\oplus_{r \in \Gamma} (X^\gamma_t)_r\) corresponding to \(\gamma\) and \(\beta\) respectively such that \((X^\beta_s)_r = (X^\gamma_t)_s\). For any \(x \in (X^\beta_s)_r\), we have \(\beta(x) = x \otimes \lambda_s\) and \(\gamma(x) = \lambda_t \otimes x\). Let \(\alpha \in \text{Ker}(\partial_1)\). Then

\[
\delta(\alpha(x)) = (\alpha(x) \otimes \lambda_s) + (\lambda_t \otimes \alpha(x))
\]

and so \((\varphi_r \otimes \text{id})\delta(\alpha(x)) = \varphi_r(\alpha(x))\lambda_s + \delta_{r,t}(\alpha(x))\lambda_t\) for any \(r \in \Gamma\) (where \(\delta_{r,t}\) is the Kronecker delta). Thus by putting \(r = t\), we obtain \(\alpha(x) = \varphi_t(\alpha(x))(\lambda_t - \lambda_s)\) (note that \((\varphi_t \otimes \text{id})\delta(\alpha(x)) = \varphi_t(\alpha(x))\lambda_t\)). Similarly, we have \(\alpha(x) = \varphi_s(\alpha(x))(\lambda_s - \lambda_t)\). Now for any \(y \in \oplus_{s \in \Gamma} X^\beta_s\), we define

\[
f(y) = \sum \varphi_s(\alpha(y_s))
\]

(where \(y = \sum y_s\) and \(y_s \in X^\beta_s\)). If \(f\) extends to a continuous function on \(X\), then it is not hard to see that \(\partial_0(f) = \alpha\). In fact, for any \(x \in (X^\gamma_t)_s\), we have \((\text{id} \otimes f)\gamma(x) = f(x)\lambda_t = \varphi_s(\alpha(x))\lambda_t\) (by the definition of \(f\) and the fact that \(x \in X^\beta_s\)) and \((f \otimes \text{id})\beta(x) = f(x)\lambda_s = \varphi_s(\alpha(x))\lambda_s\) (as \((X^\gamma_t)_s = (X^\beta_s)_t\)). It is not clear for the moment whether all such functions defined in this way are continuous (this means that \(H^\beta_1(C^*_r(\Gamma; X) = (0))\). However, we will see in Corollary 2.13 that this will imply the amenability of \(\Gamma\). The converse is also true because if \(f \in X^\ast\) such that \(\partial_0(f) = \alpha\), then \(f(x) = \varphi_s(\alpha(x))\) for any \(x \in (X^\beta_s)_t\).

Next, we would like to study the situation when these two cohomology theories vanish. First of all, we will consider the case when the left action \(\gamma\) of the \(S\)-bicomodule \((X, \beta, \gamma)\) is trivial in the sense that \(\gamma = 0\) (there is another meaning for the triviality: \(\gamma = 1 \otimes \text{id}_X\) but we will not consider this situation until Section 4). In this case, the corresponding (one-sided) natural and dual cohomologies will be denoted by \(H^\beta_n(S; X)\) and \(H^{\beta,\ast}_n(S; X)\) respectively. We have the following simple result concerning these one-sided cohomologies.
Proposition 2.5 Let \((S, \delta)\) be a saturated Hopf \(C^*\)-algebra and \((X, \beta)\) be a right \(S\)-comodule.

(a) \(H^0_d(S; X) = (0)\) if and only if \(\beta\) is injective. Moreover, if \(\beta\) is either left or right non-degenerate, then \(H^0_d(S; X) = (0)\).

(b) If \((S, \delta)\) is counital, then \(H^n_\delta(S; X) = (0)\) and \(H^n_d(S; X) = (0)\) \((n \geq 1)\).

(c) If \(H^1_d(S; S) = (0)\), then \((S, \delta)\) is counital.

Proof: (a) The first statement is obvious. To show the second statement, we suppose that \(\beta\) is right non-degenerate and take any \(f \in \text{Ker}(\delta_h)\) (i.e. \((f \otimes \text{id})\beta = 0\)). Now the density of \(\beta(X) \cdot S\) in \(X \otimes S\) will imply that \(f\) is zero. The argument for the case when \(\beta\) is left non-degenerate is similar.

(b) Suppose that \(\epsilon\) is the counit of \(S\). Let \(m \in M_{S^*}(X \otimes S^n)\) be such that \(\delta_n(m) = 0\) and let \(m_\epsilon = (\text{id}_X \otimes \text{id}^{n-1} \otimes \epsilon)(m)\). Then it is clear by Lemma 1.12(b) that

\[
(\beta \otimes \text{id}^{n-1})(m_\epsilon) = (\text{id}_X \otimes \text{id}^n \otimes \epsilon)(\beta \otimes \text{id}^n)(m) = (-1)^{n-1}m - \sum_{k=1}^{n-1} (-1)^k(\text{id}_X \otimes \text{id}^{k-1} \otimes \delta \otimes \text{id}^{n-k-1})(m_\epsilon).
\]

Hence \(m = \delta_{n-1}((-1)^{n-1}m_\epsilon)\). This shows that \(H^n_\delta(S; X) = (0)\). On the other hand, let \(T \in \text{CB}(X; M(S^n))\) such that \(\partial_n(T) = 0\). If \(F = (\epsilon \otimes \text{id}^n) \circ T\), then it is clear from \((\epsilon \otimes \text{id}^n)(\partial_n(T)) = 0\) that

\[
(F \otimes \text{id}) \circ \beta + (-1)^nT + \sum_{k=1}^{n-1} (-1)^k(\text{id}^{n-k-1} \otimes \delta \otimes \text{id}^{k-1}) \circ F = 0.
\]

Hence \(T = (-1)^{n-1}\partial_{n-1}(F)\).

(c) Consider \(\text{id} \in \text{CB}(S; M(S))\). Then clearly \(\partial_1(\text{id}) = 0\) and so there exists \(\epsilon \in S^*\) such that \((\epsilon \otimes \text{id}) \circ \delta = \text{id}\). This shows that \(\epsilon\) is a left identity for \(S^*\) and hence a two-sided identity (by Lemma 1.15(a) and the fact that \(S^*\) is a saturated Hopf von Neumann algebra).

In the following, we will study 2-sided dual cohomology. For the moment, we only have the complete picture for the case when \(S\) is unital (i.e. it represents a compact quantum group) and a partial picture if \(S\) has property (S) (in the sense of [37]). In these cases, the vanishing of the dual cohomology is related to the existence of codiagonals defined as follows.

Definition 2.6 Suppose that \(R\) is a Hopf \(C^*\)-algebra with counit \(\epsilon\). Let \(Y\) be a subspace of \(M(R \otimes R)\) containing \(\delta(R)\) such that \((\text{id} \otimes (\text{id} \otimes f) \circ \delta)(Y) \subseteq Y\) and \(((f \otimes \text{id}) \circ \delta \otimes \text{id})(Y) \subseteq Y\) for any \(f \in R^*\). Then \(F \in Y^*\) is said to be a codiagonal if \(F \circ \delta = \epsilon\) on \(R\) and \(F \circ (\text{id} \otimes (\text{id} \otimes g) \circ \delta) = F \circ ((g \otimes \text{id}) \circ \delta \otimes \text{id})\) on \(Y\) (for any \(g \in R^*\)).

We will defer the illustrations and examples for the codiagonals until Example 2.11 and Remark 2.12(a) & (b).

Note that from this point on, we will need quite a lot of materials from the appendix (unless the readers want to confine themselves in the case of unital Hopf \(C^*\)-algebras – in which case, please see part (b) and (c) of the following Remark). Therefore, perhaps it will be a good idea if the readers can digress to the Appendix at this point (we are sorry that since the materials in the appendix are a bit technical and not in the same favour as the other parts of this paper, we decided to study them in the appendix).

Remark 2.7 Let \(Y = \hat{U}(R \otimes R)\) (see Remark 2.3).

(a) If \(R\) has property (S) (in particular, if \(R\) is a nuclear \(C^*\)-algebra), then by Lemma 1.2, \(Y\) is the biggest unital \(C^*\)-subalgebra of \(M(R \otimes R)\) for which \(\text{id} \otimes \delta\) and \(\delta \otimes \text{id}\) can be extended.

(b) If \(R\) is unital, then \(Y = R \otimes R\) and \(\delta \otimes \text{id}\) and \(\text{id} \otimes \delta\) obviously define on \(Y\) (without \(R\) having the property (S)).

(c) In both of the cases (a) and (b) above, \(Y\) satisfies the condition in Definition 2.6 and \(F \in Y^*\) is a codiagonal if and only if \(F \circ \delta = \epsilon\) and \((F \otimes \text{id}) \circ (\text{id} \otimes \delta) = (\text{id} \otimes F) \circ (\delta \otimes \text{id})\).
Proposition 2.8 Let $S$ be a saturated Hopf $C^*$-algebra. Suppose that $S$ either is unital or has property (S).
(a) If $H^1_d(S; X) = (0)$ for any $S$-bicomodule $X$, then there exist a counit $\epsilon$ on $S$ as well as a codiagonal $F$ on $\hat{U}(S \otimes S)$. Moreover, if $S$ is unital, we obtain the same conclusion even if $H^1_d(S; X)$ vanish only for those $S$-bicomodules $(X, \beta, \gamma)$ such that both $\beta$ and $\gamma$ are 2-sided non-degenerate.
(b) If there exist a counit $\epsilon$ on $S$ and a codiagonal $\hat{F}$ on $M(S \otimes S)$ such that $\hat{F} \circ \delta = \epsilon$ on $M(S)$, then $H^1_d(S; X) = (0)$ for any $S$-bicomodule $(X, \beta, \gamma)$ such that either $\beta$ or $\gamma$ is left (or right) non-degenerate.

Proof: (a) By Proposition 2.4 (c), $S$ has a counit $\epsilon$. Let $U(S)$ be the space $U_{S, S}(S)$ (see Remark 3.3). It is not hard to see that $\delta(U(S)) \subseteq U(S \otimes S)$ (since $(id \otimes \delta)(m)(1 \otimes 1 \otimes s) = (\delta \otimes id)(\delta(m)(1 \otimes s)) \in \delta(U(S)) \otimes S$ and $(\delta \otimes id)(\delta(m)(s \otimes 1 \otimes 1) = (id \otimes \delta)(\delta(m)(s \otimes 1)) \in S \otimes \delta(U(S))$ for any $m \in U(S)$). Let $X$ be the quotient $U(S \otimes S)/\delta(U(S))$ with the canonical quotient map $q$. By Remark 2.4, $id \otimes \delta$ induces a right coaction on $U(S \otimes S)$. Using the first equality above, we have $(q \otimes id)(id \otimes \delta)(U(S)) = (0)$ in $M_S(X \otimes S)$ and by Lemma 1.15 (c), $id \otimes \delta$ induces a right coaction $\beta$ on $X$. Similarly, $\delta \otimes id$ induces a left coaction $\gamma$ on $X$. It is clear that $(X, \beta, \gamma)$ is a $S$-bicomodule. Consider the completely bounded map $T = \epsilon \otimes id - id \otimes \epsilon$ from $U(S \otimes S)$ to $M(S)$. Since $T \circ \delta = 0$, it induces a map $\hat{T} \in CB(X; M(S))$ (by Lemma 1.15). Now for any $m \in U(S \otimes S)$, $\partial_1(\hat{T})(q(m)) = (T \otimes id)(id \otimes \delta)(m) - (\delta \otimes id)(m) + (id \otimes \epsilon)(m)) = 0.$

Hence there exists $G \in X^*$ such that $\partial_0(G) = \hat{T} \ i.e.$ $\epsilon \otimes id - id \otimes \epsilon = (G \otimes q \otimes id)(id \otimes \delta) - (id \otimes G \circ q)(\delta \otimes id).$ Let $F = \epsilon \otimes \epsilon - G \circ q \in U(S \otimes S)^*$. Then

$(F \otimes id)(id \otimes \delta) = \epsilon \otimes id - (G \circ q \otimes id)(id \otimes \delta) = \epsilon \otimes \epsilon - (id \otimes G \circ q)(\delta \otimes id) = (id \otimes F)(\delta \otimes id)$

and $F \circ \delta = (\epsilon \otimes \epsilon) \circ \delta - G \circ q \circ \delta = \epsilon$. It is easy to see that if $S$ is unital, then $\beta$ and $\gamma$ in the above are left as well as right non-degenerate.

(b) Suppose that $T \in CB(X; M(S))$ such that $\partial_1(T) = 0$, i.e. $\delta \circ T = (T \otimes id_S) \circ \beta + (id_S \otimes T) \circ \gamma$. Let $\beta$ be left non-degenerate and $f = \hat{F} \circ (T \otimes id) \circ \beta \in X^*$. Then by the properties of $\beta$ and $\gamma$ as well as the definition of codiagonal, for any $g \in S^*$ and $x \in X$,

$$g(\partial_0(f)(x)) = \hat{F}((id_S^2 \otimes g)((T \otimes id_S)\beta \otimes id_S)\beta(x)) - \hat{F}((g \otimes id_S^2)(id_S \otimes T \otimes id_S)(id_S \otimes \beta)\gamma(x))$$

$$= \hat{F}((id_S^2 \otimes g)(id_S \otimes \delta)(T \otimes id_S)\beta(x)) - \hat{F}((g \otimes id_S^2)(id_S \otimes T \otimes id_S)(id_S \otimes \beta)\gamma(x))$$

$$= \hat{F}((g \otimes id_S^2)(\delta \otimes id_S)(T \otimes id_S)\beta(x)) - \hat{F}((g \otimes id_S^2)(id_S \otimes T \otimes id_S)(\gamma \otimes id_S)\beta(x))$$

$$= \hat{F}((q \otimes id_S^2)((T \otimes id_S^2)\beta \otimes id_S)\beta(x)) + (id_S \otimes T \otimes id_S)(\gamma \otimes id_S)\beta(x)) - \hat{F}((g \otimes id_S^2)(id_S \otimes T \otimes id_S)(\gamma \otimes id_S)\beta(x))$$

$$= \hat{F}((g \otimes id_S^2)(id_S \otimes T \otimes id_S)(\gamma \otimes id_S)\beta(x)) - \hat{F}(\delta((g \otimes id_S)(T \otimes id_S)\beta(x))) = g(T(id_S \otimes \epsilon) \beta(x)) = g(T(x))$$

The final equality follows from Lemma 1.15 (b). The case when $\gamma$ is left non-degenerate can be proved similarly by using $f = \hat{F} \circ (T \otimes id) \circ \gamma$.

Note that the proof of part (a) is similar to that of the dual situation for the existence of diagonals (see e.g. 2.5).

Remark 2.9 This proposition applies in particular to the case when $S$ is the Hopf $C^*$-algebra $S_V$ defined in 3.1.5 (see also 3.3.8) for a coamenable regular multiplicative unitary $V$ (in this case, $S = S_V$ is a nuclear $C^*$-algebra by 1.8). Note that this includes the situation of $C_0(G)$ for a locally compact group $G$ (which need not be amenable).

In the case when the Hopf $C^*$-algebra $S$ is unital, we call a net $\{F_i\}$ in $(S \otimes S)^*$ a bounded approximate codiagonal if $\|(F_i \circ \beta)\|$ is bounded and for any $f \in S^*$, both $\|(F_i \otimes F_i) \circ (\delta \otimes id) - (F_i \otimes f) \circ (id \otimes \delta)\|$ and $\|(F_i \otimes f) \circ (\delta \otimes id) \circ \delta - f\|$ converge to zero.
Theorem 2.10 Suppose that $(S, \delta)$ is a saturated unital Hopf $C^*$-algebra. The following conditions are equivalent.

(i) $S$ has a counit and $S \otimes S$ has a codiagonal;

(ii) $S \otimes S$ has a bounded approximate codiagonal;

(iii) For any $S$-bicomodule $(X, \beta, \gamma)$ such that either $\beta$ or $\gamma$ is right (or left) non-degenerate, $H^2_d(S; X) = (0)$ for $n \geq 1$.

Proof: It is clear that (i) implies (ii). To show that (ii) implies (i), let $\{F_i\}_{i \in I}$ be a bounded approximate codiagonal of $(S \otimes S)^*$ and $F$ be a $\sigma((S \otimes S)^*, S \otimes S)$-limit point of $\{F_i\}_{i \in I}$. By considering a subnet if necessary, we may assume that $\{F_i\}_{i \in I}$ converges to $F$. For any $f \in S^*$ and $s, t \in S$, it is easy to check that $(F \otimes f)\delta(s \otimes t) = (f \otimes F)(\delta \otimes \text{id})(s \otimes t)$ (note that $\delta(s), \delta(t) \in S \otimes S$). Moreover, as $\delta^*$ is $\sigma((S \otimes S)^*, S \otimes S)$-$\sigma(S^*, S)$-continuous, for any $s \in S$ and $f \in S^*$, we have $(\delta^*(F) \otimes f) \delta(s) = \lim_n (\delta^*(F_i))((\delta \otimes f)\delta(s)) = \lim_n (m(F_i) \cdot f)(s) = f(s)$. Thus, $\delta^*(F)$ is a left identity on $S^*$ and hence a two-sided identity (by Lemma 2.11(a)). By Proposition 2.11(b), we have that (iii) implies (i). It remains to show that (i) implies (iii). In fact, the argument is similar to that of Proposition 2.11(b). Let $\epsilon \in I$ be a counit on $S$ and $F$ be a codiagonal on $S \otimes S$. Suppose that $\beta$ is either left or right non-degenerate. For any $T \in Ker(\partial_n) \subseteq CB(X; M(S^n))$, let $R = (id^{|n-1|} \otimes F) \circ (T \otimes \text{id}) \circ \beta$. Then,

$$
\partial_{n-1} R = (id^{|n-1|} \otimes F \otimes \text{id})(T \otimes \text{id})\beta + \sum_{j=1}^{n-1} (-1)^j(id^{n-j-1} \otimes \delta \otimes id^{j-1})(id^{n-1} \otimes F)(T \otimes \text{id})\beta + (-1)^n(id^n \otimes F)(id \otimes T \otimes \text{id})\beta 
$$

In the case when $\gamma$ is left (or right) non-degenerate, we should instead use $R = (F \otimes id^{n-1}) \circ (id \otimes T) \circ \gamma$ in the above argument.

Using this result, we can show that all the dual cohomologies of a Woronowicz AF algebra (see [36, 3.3]) vanish (this will be proved in [22, 3.9]).

Lemma and Example 2.11 Let $\Gamma$ be any discrete amenable group. Then $C^*(\Gamma) \otimes C^*(\Gamma)$ has a codiagonal.

Proof: Note that as $\Gamma$ is amenable, $(C^*(\Gamma) \otimes C^*(\Gamma))^* = B(\Gamma \times \Gamma)$ (the Fourier-Stieltjes algebra of $\Gamma \times \Gamma$). Let $F \in B(\Gamma \times \Gamma)$ and $f \in B(\Gamma)$. Then $F \cdot f = f \cdot F$ if and only if $F(r, s) f(s) = f(r) F(r, s)$ for any $r, s \in \Gamma$. By taking $f = \varphi_r$ (where $\varphi_r$ is as defined in Example 1.1(c)), we see that the above equality is equivalent to $F(r, s) = 0$ if $r \neq s$. Moreover, $F \circ \delta = \epsilon$ if and only if $F(r, r) = 1$. Now consider $F_0 \in l_\infty(\Gamma \times \Gamma)$ defined by $F_0(r, s) = \delta_{r, s}$ (where $\delta_{r, s}$ is the Kronecker delta). It is not hard to see that $F_0$ is positive definite. Indeed, suppose that $\{(r_1, s_1), \ldots, (r_n, s_n)\}$ is any finite set in $\Gamma \times \Gamma$. Then $F_0((r_i, s_i)^{-1}(r_j, s_j)) = 1$ if and only if $r_i^{-1} r_j = s_i^{-1} s_j$. Define an equivalent relation $\sim$ on $\{(r_1, s_1), \ldots, (r_n, s_n)\}$ by $(r, s) \sim (u, v)$ whenever $sr^{-1} = vu^{-1}$. Then $\{F_0((r_i, s_i)^{-1}(r_j, s_j))\}_{i,j=1,\ldots,n}$ is equivalent to a direct sum of square matrices having 1 in all their entries. Hence $F_0 \in B(\Gamma \times \Gamma)_+$ and $C^*(\Gamma) \otimes C^*(\Gamma)$ have a codiagonal.

Remark 2.12 (a) By Corollary 1.12 in Section 4, for any amenable group $G$, the space $\hat{U}(G \times G)$ has a codiagonal. Hence, $C(G) \otimes C(G)$ has a codiagonal for any compact group $G$.

(b) Suppose that $G$ is a locally compact amenable group. The same argument as in the above lemma shows that if $F \in B(G \times G)$ satisfying the second condition of Definition 2.12 then $F(r, s) = 0$ if $r \neq s$. Therefore,
it seems inappropriate to consider codiagonal on $S \otimes S$ (instead of $\hat{U}(S \otimes S)$) for a general Hopf $C^*$-algebras $S$.

(c) After we finished this manuscript, we discovered that an analogue of the dual cohomology for coalgebras has already been studied in [4, §3.1] and a similar equivalence between condition (i) and condition (iii) of Theorem 2.10 (i.e. [3, Thm 3]) was obtained in the purely algebraic setting (but with a different proof). However, by the argument of Lemma 2.11 and [3, Thm 3], for any discrete group $\Gamma$, all the cohomologies of the coalgebra $l^1(\Gamma)$ considered in [3] vanish (note that the functional $F_0$ in the above Lemma is well defined in $B(\Gamma \times \Gamma) = (C^*(\Gamma) \otimes_{\max} C^*(\Gamma))^*$ even if $\Gamma$ is not amenable and the restriction of $F_0$ on $l^1(\Gamma) \otimes l^1(\Gamma)$ is the functional required in [3, Thm3]). (A direct proof for this vanishing statement can also be obtained by using a similar argument as in Example 2.4 in which case we don’t need the function $f$ to be continuous). This, together with the following corollary, shows that the “dual cohomology theory” for Hopf algebras behaves very differently from the one for Hopf $C^*$-algebras.

Corollary 2.13 Let $\Gamma$ be a discrete group. Then $\Gamma$ is amenable if and only if $H^0_d(C^*_r(\Gamma);X) = (0)$ for any $n \geq 1$ and any left (or right) non-degenerate $C^*_r(\Gamma)$-bicomodule $X$ and equivalently, $H^0_d(C^*_r(\Gamma);X) = (0)$ for any left (or right) non-degenerate $C^*_r(\Gamma)$-bicomodule $X$.

Proof: If $\Gamma$ is amenable, then Lemma 2.11 and Theorem 2.10 show that all the dual cohomology of $C^*_r(\Gamma)$ vanish. Now if $H^0_d(C^*_r(\Gamma);X)$ vanishes for any left (or right) non-degenerate $C^*_r(\Gamma)$-bicomodule $(X,\beta,\gamma)$ (and in particular, when $\gamma = 0$), then $\Gamma$ is amenable by Proposition 2.10 (c).

Part of the above corollary (more precisely, the case when $n = 1$) is true for general locally compact groups (see Theorem 3.4(b)). Moreover, by Theorem 3.4(a), the amenability of $G$ is also equivalent to the vanishing of $H^0_d(C_0(G);X)$.

Remark 2.14 Note that if $\Gamma$ is a discrete group such that $C^*(\Gamma) \otimes C^*(\Gamma) = C^*(\Gamma) \otimes_{\max} C^*(\Gamma)$, then the argument in Lemma 2.11 also gives the existence of a codiagonal on $C^*(\Gamma) \otimes C^*(\Gamma)$ and in this case, the dual cohomologies of $C^*(\Gamma)$ vanish. Hence, the vanishing of the dual cohomologies of $C^*(\Gamma)$ seems not strong enough to ensure the amenability of $\Gamma$.

We end this section with the following natural question: in general, is there any relation between the vanishing of the dual cohomology and the amenability or coamenability (see [20]) of the Hopf $C^*$-algebra? Some partial answers will be given in Section 4.

3 Coactions and cohomology of Hopf von Neumann algebras

In this section, we will study coactions and cohomology theories of Hopf von Neumann algebras. We begin with coactions on dual operator spaces (which is a natural generalisation of ordinary coactions on von Neumann algebras).

Notation: Throughout this section, $\mathcal{X}$ is the dual operator space of an operator space $\mathcal{X}$, and we recall from section 1 that $(\mathcal{R},\delta)$ is a Hopf von Neumann algebra.

In order to define a coaction, we need to decide first of all, the range of it (as in the case of Hopf $C^*$-algebras). Note that the range of a coaction on a von Neumann algebra $\mathcal{M}$ by $\mathcal{R}$ is the von Neumann algebra tensor product $\mathcal{M} \otimes \mathcal{R}$. For dual operator spaces, we have the following generalisation. By [4, 2.1], there exists a weak*-homeomorphic complete isometry from $\mathcal{X}$ to some $\mathcal{L}(H)$. Let $\mathcal{R}$ be represented as a von Neumann subalgebra of $\mathcal{L}(K)$ and let $\mathcal{X} \otimes \mathcal{R} = \{\alpha \in \mathcal{L}(H) \otimes \mathcal{L}(K) : (\text{id} \otimes \omega)(\alpha) \in \mathcal{X}; (\nu \otimes \text{id})(\alpha) \in \mathcal{R}\}$
for any $\omega \in \mathcal{R}$ and $\nu \in \mathcal{X}_{r}$ be the Fubini product (see \cite[p.188]{1}). We recall from \cite[3.3]{1} that the Fubini product is independent of the representations of $\mathcal{X}$ and $\mathcal{R}$, and is the dual space of $\mathcal{X} \otimes \mathcal{R}_{r}$. Moreover, if $\mathcal{X}$ is a von Neumann algebra, then $\mathcal{X} \otimes \mathcal{R} = \mathcal{A} \otimes \mathcal{R}$. By Lemma \cite[3.8]{1}, if $\mathcal{Y}$ and $\mathcal{Z}$ are two dual operator spaces and $\varphi$ is a weak*-continuous completely bounded map from $\mathcal{Y}$ to $\mathcal{Z}$, then there exists a weak*-continuous completely bounded map $id \otimes \varphi$ from $\mathcal{X} \otimes \mathcal{Y}$ to $\mathcal{X} \otimes \mathcal{Z}$ such that $(id \otimes \varphi)(t(\omega \otimes \nu)) = t(\omega \otimes \varphi_{*}(\nu))$ ($t \in \mathcal{X} \otimes \mathcal{Y}$; $\omega \in \mathcal{X}_{r}$; $\nu \in \mathcal{Z}_{r}$). This enables us to define the following.

**Definition 3.1** A weak*-continuous completely bounded map $\beta$ from $\mathcal{X}$ to $\mathcal{X} \otimes \mathcal{R}$ (respectively, $\mathcal{R} \otimes \mathcal{X}$) is said to be a normal right coaction (respectively, normal left coaction) if $(\beta \otimes id)\beta = (id \otimes \delta)\beta$ (respectively, $(id \otimes \beta)\beta = (\delta \otimes id)\beta$).

**Remark 3.2** (a) We recall from \cite{2} that a right operator $\mathcal{R}_{s}$-module is an operator space $\mathcal{N}$ with a completely bounded map $m$ from $\mathcal{N} \otimes \mathcal{R}_{s}$ to $\mathcal{N}$ such that $m \circ (m \otimes id) = m \circ (id \otimes \delta_{s})$ (left operator $\mathcal{R}_{s}$-module can be defined similarly). For any normal right coaction $\beta$, the predual map $\beta_{*}$ from $\mathcal{X} \otimes \mathcal{R}_{s}$ to $\mathcal{X}$ gives a right operator $\mathcal{R}_{s}$-module structure on $\mathcal{X}_{r}$.

(b) It is natural to ask whether the dual $\mathcal{R}_{s}$-module structure on $\mathcal{X}$ comes from a normal left coaction (on $\mathcal{X}^{*}$). However, it is not clear why this $\mathcal{R}_{s}$-multiplication can be extended to the operator projective tensor product (or the range of the dual map lies in the Fubini product). Nevertheless, we will see later that it comes from a more general form of coaction (Lemma \cite[5.6]{2}).

**Notation:** Throughout this section, $\mathcal{R}^{n}$ is the $n$-times von Neumann algebra tensor product of $\mathcal{R}$ whereas $\mathcal{R}^{n}_{s}$ is the $n$-times operator projective tensor product of $\mathcal{R}_{s}$ ($n \geq 1$) and we take $\mathcal{R}^{0} = \mathcal{C} = \mathcal{R}_{s}^{1}$.

Suppose that $\mathcal{X}$ is a dual operator space with normal right coaction $\beta$ and normal left coaction $\gamma$ such that $(id \otimes \beta) \circ \gamma = (\gamma \otimes id) \circ \beta$. With the help of Lemma \cite[3.8]{1}, we can define as in the case of Hopf $C^{*}$-algebra, a map $\delta_{n}$ from $\mathcal{X} \otimes \mathcal{R}^{n}$ to $\mathcal{X} \otimes \mathcal{R}^{n+1}$ by $\delta_{n}(x \otimes s_{1} \otimes \ldots \otimes s_{n}) = \beta(x) \otimes s_{1} \otimes \ldots \otimes s_{n} + \sum_{k=1}^{n} (-1)^{k} x \otimes \sigma_{n-k} x_{k} \otimes s_{k+1} \otimes \ldots \otimes s_{n} + (-1)^{n-1} (\gamma(x) \otimes s_{1} \otimes \ldots \otimes s_{n}) \sigma_{n+1,n}$ ($n \geq 1$) and $\delta_{0}(x) = \beta(x) - \gamma(x)\sigma_{1,1}$ (where $\sigma_{n,k}$ is the map from $\mathcal{R}^{n} \otimes \mathcal{X} \otimes \mathcal{R}^{n-k}$ to $\mathcal{X} \otimes \mathcal{R}^{n}$ as defined in Section 2). The same argument as Lemma \cite[2.4]{1} shows that this gives a cochain complex and the cohomology defined is called the normal natural cohomology of $\mathcal{R}$ with coefficient in $(\mathcal{X}, \beta, \gamma)$.

On the other hand, we can also define a map $\partial_{n}$ from $\text{CB}_{n}(\mathcal{X}; \mathcal{R}^{n})$ (the set of all weak*-continuous completely bounded maps from $\mathcal{X}$ to $\mathcal{R}^{n}$) to $\text{CB}_{n}(\mathcal{X}; \mathcal{R}^{n+1})$ by $\partial_{n}(T) = (T \otimes id) \circ \beta + \sum_{k=1}^{n} (-1)^{k}(id^{n-k} \otimes \delta \otimes id^{k-1}) \circ T + (-1)^{n+1}(id \otimes T) \circ \gamma$ ($n \geq 1$) and $\partial_{0}(f) = (f \otimes id) \circ \beta - (id \otimes f) \circ \gamma$. As in the case of Hopf $C^{*}$-algebra, $(\text{CB}_{n}(\mathcal{X}; \mathcal{R}^{n}), \partial_{n})$ is a cochain complex and induces a cohomology $H^{n}_{\sigma,d}(\mathcal{R}; \mathcal{X}) = \text{Ker}(\partial_{n})/\text{Im}(\partial_{n-1})$ which is called the normal dual cohomology of $\mathcal{R}$ with coefficient in $(\mathcal{X}, \beta, \gamma)$. In the case when $\gamma = 0$, we denote it by $H^{n}_{\sigma,d,r}(\mathcal{R}; \mathcal{X})$. Now using a similar argument as that of Proposition \cite[2.4]{1} we have the following.

**Proposition 3.3** Let $\mathcal{R}$ be a saturated Hopf von Neumann algebra as above.

(a) If $\mathcal{R}_{s}$ is unital, then $H^{n}_{\sigma,d,r}(\mathcal{R}; \mathcal{X}) = (0)$ ($n = 1, 2, 3, \ldots$) for any dual operator space $\mathcal{X}$ with a normal right coaction by $\mathcal{R}$.

(b) If $H^{n}_{\sigma,d,r}(\mathcal{R}; \mathcal{R}) = (0)$, then $\mathcal{R}_{s}$ is unital.

However, we are more interested in the existence of a bounded approximate identity of $\mathcal{R}_{s}$ (which is related to amenability). A closer look at the above reveals that this can be achieved if we remove the weak*-continuity. Moreover, by doing so, we can also extend the definition of coactions to general operator spaces. Let us first note that $\mathcal{X} \otimes \mathcal{R} = (\mathcal{X}_{r} \otimes \mathcal{R}_{s})^{*} = \text{CB}(\mathcal{R}_{s}; \mathcal{X})$. Now by translating the coaction identity in terms of $\text{CB}(\mathcal{R}_{s}; \mathcal{X})$, we can define a more general form of coactions.
Definition 3.4 A completely bounded map \( \beta \) from \( X \) to \( \text{CB}(\mathbb{R}_+; X) \) is said to be a right coaction (respectively, a left coaction) if \( \beta(\beta(x)(\omega))(\nu) = \beta(x)(\delta(\nu \circ \omega)) \) (respectively, \( \beta(\beta(x)(\omega))(\nu) = \beta(x)(\delta(\omega \circ \nu)) \)) for any \( x \in X \) and \( \omega, \nu \in \mathbb{R}_+ \). Moreover, a right (or a left) coaction \( \beta \) is said to be non-degenerate if \( \beta(X)(\mathbb{R}_+) \) is dense in \( X \) (which is equivalent to weakly dense). We call \((X, \beta, \gamma)\) a \( \mathbb{R} \)-bicomodule if \( \beta \) is a right coaction and \( \gamma \) is a left coaction on \( X \) by \( \mathbb{R} \) such that \( \beta(\gamma(x)(\omega))(\nu) = \gamma(\beta(x)(\nu))(\omega) \) for any \( x \in X \) and \( \omega, \nu \in \mathbb{R}_+ \).

Note that \( \beta(\beta(x)(\omega))(\nu) = \beta^#(\beta(x)(\omega))(\nu) \) and \( \beta(x)(\delta(\omega \circ \nu)) = \beta^0(\delta(x)(\omega \circ \nu)) \) (where \( \beta^# \) and \( \beta^0 \) are the maps as defined in Lemma 1.1(b)). Hence the right (respectively, left) coaction identity in Definition 3.4 can be simplified to \( \beta^#(\beta) = \beta^0(\delta \circ \sigma) \) (respectively, \( \beta^#(\beta) = \beta^0(\delta_\omega) \)) under the standard identification \( \text{CB}(\mathbb{R}_+; \text{CB}(\mathbb{R}_+; X)) \cong \text{CB}(\mathbb{R}_+; \text{CB}(\mathbb{R}_+; X)) \) (see Remark 1.4). Note that the forms of these simplified coaction identities depend on whether we take the standard identification or the reverse identification (see Remark 1.4).

Example 3.5 (a) Suppose that \( \beta \) is a coaction of \( \mathbb{R} \) on a von Neumann algebra \( \mathcal{M} \). Let \( N \) be any subset of \( \mathcal{M} \) and let \( X_N \) be the closed linear span of the set \( \{(id \otimes \omega)(\beta(x)) : \omega \in \mathbb{R}_+; x \in N\} \). We denote by \( \beta_N \) the composition of the restriction of \( \beta \) on \( X_N \) with the complete isometry from \( \mathcal{M} \otimes \mathbb{R} \) to \( \text{CB}(\mathbb{R}_+; \mathcal{M}) \). Then we have \( \beta_N((id \otimes \omega)(\beta(x)))(\nu) = (id \otimes (\nu \circ \omega))(\beta(x)) \in X_N \) (for any \( \omega, \nu \in \mathbb{R}_+ \) and \( x \in N \)) and it is not hard to see that \( \beta_N \) is a right coaction on \( X_N \).

(b) Suppose that \( \mathbb{R} \) comes from a Kac algebra \( K \) and \( \beta \) is any completely contractive right coaction of \( \mathbb{R} \) on any operator space \( X \). Let \( N \) and \( U \) be the unit balls of \( X \) and \( \mathbb{R}_+ \) respectively. Since \( \beta \in \text{CB}(X; \text{CB}(\mathbb{R}_+; X)) \cong \text{CB}(\mathbb{R}_+; \text{CB}(\mathbb{R}_+; X)) \) is a complete contraction, \( \|\beta(\omega)(x)\| \leq 1 \) for any \( \omega \in U \) and \( x \in N \). It is not hard to see that this defines an action of \( K \) on \( N \) in the sense of [21, 2.2].

(c) For any Hilbert space \( H \), there is an one to one correspondence between right coactions of \( \mathbb{R} \) on the column Hilbert space \( H_\mathbb{C} \) and the representations of \( \mathbb{R} \) on \( H \) (see [21, 2.11]).

(d) Let \( \Gamma \) be a discrete group and \( \beta \) be a coaction of \( C^*_r(\Gamma) \) on a \( C^* \)-algebra \( A \). Then \( \beta \) induces a right coaction \( \hat{\beta} \) of the group von Neumann algebra \( vN(\Gamma) \) on \( A \) (by \( \hat{\beta}(a)(\omega) = (id \otimes \omega)\beta(a) \) for any \( a \in A \) and \( \omega \in A(\Gamma) = vN(\Gamma)_\mathbb{R} \)). Thus, by the next Lemma, we have a left coaction \( \check{\beta} \) of \( vN(\Gamma) \) on \( A^* \). Moreover, as \( \beta \) is injective and \( A^* \otimes A(\Gamma) \) separates points of \( M(A \otimes C^*_r(\Gamma)) \), the subspace \( \check{\beta}(A^*)(A(\Gamma)) \) is weak*-dense in \( A^* \). Notice that the function \( \varphi_s \) defined in Example 1.10(b) is in \( A(\Gamma) \). For any \( t \in \Gamma \), the sets \( \{ f \in A^* : \check{\beta}(f)(\varphi_s) = \delta_s f \} \) (where \( \delta_s \) is the Kronecker delta) and \( \{ \check{\beta}(g)(\varphi_s) : g \in A^* \} \) coincide and we denote this set by \( A_t^* \). If \( r \neq s \in \Gamma \), \( a \in A_r \) and \( f \in A_t^* \), then \( f(a) = \check{\beta}(f)(\varphi_s)(a) = f(id \otimes \varphi_s)\beta(a) = 0 \).

Lemma 3.6 A right coaction \( \beta \) of \( \mathbb{R} \) on \( X \) induces a left coaction \( \check{\beta} \) of \( X^* \) such that \( \check{\beta}(f)(\omega)(x) = f(\beta(x)(\omega)) \) (for \( f \in X^* ; x \in X; \omega \in \mathbb{R}_+ \)). Similarly, a left coaction on \( X \) will induce a right coaction on \( X^* \).

Proof: Let \( \check{\beta} \) be the composition of the completely bounded map \( \beta^# : X^* \rightarrow \text{CB}(X; \mathbb{R}) \) (see Lemma 1.1(b)) with the canonical complete isometry from \( \text{CB}(X; \mathbb{R}) \) to \( \text{CB}(\mathbb{R}_+; X^*) \) (see Lemma 1.14(d)). Thus, \( \beta \) is completely bounded and \( \check{\beta}(f)(\omega)(x) = \beta^#(f)(\omega)(\beta(x))(\omega) = f(\beta(x)(\omega)) \). It remains to show the left coaction identity. Indeed, \( \check{\beta}(\check{\beta}(f)(\omega))(\nu)(x) = f(\check{\beta}(\beta(x)(\nu))(\omega))(x) = f(\beta(x)(\delta(\omega \circ \nu)))(x) = \check{\beta}(f)(\delta(\omega \circ \nu))(x)(\omega) \) for any \( f \in X^* , x \in X \) and \( \omega, \nu \in \mathbb{R}_+ \). The proof of the second statement is the same.

In fact, by a similar argument as in Example 1.10(d), the left coaction \( \check{\beta} \) is normal. It is “weakly non-degenerate” if \( \beta \) is injective. However, we will not need these facts in this paper.

We will again define two cohomology theories for this type of bicomodules. We first consider the analogue of the natural cohomology. Suppose that \( \beta \) is a normal right coaction of \( \mathbb{R} \) on the dual operator space \( X \). As \( \text{CB}(\mathbb{R}_+^n; X) \cong X \otimes \mathbb{R}^n \) under the identification: \( T_n(\omega_1 \otimes \ldots \otimes \omega_n) = (id \otimes \omega_1 \otimes \ldots \otimes \omega_n)(\alpha) \) (for \( \alpha \in X \otimes \mathbb{R}^n \)), the map \( \beta(\omega)(z) = (\beta(z \otimes 1))(z \in X \otimes \mathbb{R}^n) \) can be identified with the map from \( \text{CB}(\mathbb{R}_+^n; X) \) to \( \text{CB}(\mathbb{R}_+^{n+1}; X) \) given by \( \beta(\omega)(T) = \beta(T(\omega_1 \otimes \ldots \otimes \omega_{n+1})) = \beta(\beta(T(\omega_1 \otimes \ldots \otimes \omega_{n+1}))(\omega_1)) \) (\( T \in \text{CB}(\mathbb{R}_+^n; X) \)).
Now for a general $\mathcal{R}$-bicomodule $(X, \beta, \gamma)$, let $\beta(n), \gamma(n)$ and $\delta_{n,k}$ be maps from $\text{CB}(\mathcal{R}^n; X)$ to $\text{CB}(\mathcal{R}^{n+1}; X)$ given by $\beta_n(T)(\omega_1 \otimes \cdots \otimes \omega_{n+1}) = \beta(T(\omega_2 \otimes \cdots \otimes \omega_{n+1}))(\omega_1)$, $\gamma_n(T)(\omega_1 \otimes \cdots \otimes \omega_{n+1}) = \gamma(T(\omega_1 \otimes \cdots \otimes \omega_n))(\omega_{n+1})$ and $\delta_{n,k}(T) = T \circ (\text{id}^{k-1} \otimes \delta \otimes \text{id}^{n-k})$. Then $\beta_n$ is completely bounded since it is the map $\tilde{\beta}$ in Lemma 3.6(a) under the standard identification of Remark 3.4. The same is true for $\gamma_n$. Let

$$
\delta_n = \begin{cases} 
\beta_n + \sum_{k=1}^{n} (-1)^k \delta_{n,k} + (-1)^{n+1} \gamma_n & n \geq 1 \\
\beta(x) - \gamma(x) & n = 0.
\end{cases}
$$

A proof is needed to show that $\delta_n$ gives a cochain complex.

**Lemma 3.7** $\delta_n \circ \delta_{n-1} = 0$ ($n = 1, 2, 3, \ldots$).

**Proof:** For $n = 1$, we have $\delta_1(\delta_0(x)(\omega \otimes \nu)) = \beta(\beta(x)(\nu) - \gamma(x)(\nu))(\omega) - \beta(x)(\delta_0(\omega \otimes \nu)) + \gamma(x)(\delta_0(\omega \otimes \nu)) + \gamma(x)(\omega) = 0$ ($x \in X, \omega, \nu \in \mathcal{R}_*$) (by the left and the right coaction identities). For $n > 1$,

$$
\beta_n \circ \beta_{n-1} = \delta_{n,1} \circ \beta_{n-1}, \quad \beta_n \circ \delta_{n-1,k} = \delta_{n,k+1} \circ \beta_{n-1}, \quad \beta_n \circ \gamma_{n-1} = \gamma_n \circ \beta_{n-1}.
$$

\(\gamma_n \circ \delta_{n-1,k} = \delta_{n,k} \circ \gamma_{n-1}\) and $\gamma_n \circ \gamma_{n-1} = \delta_{n,n} \circ \gamma_{n-1}$.

Thus, in order to show $\delta_n \circ \delta_{n-1} = 0$, we need to check that $\sum_{i=1}^{n} \sum_{k=1}^{n+1} (-1)^{i+k} \delta_{n,i} \circ \delta_{n-1,k} = 0$. This can be shown again by a decomposition and a comparison (similar to Lemma 2.4).

As in section 2, we call the cohomology $H^n(\mathcal{R}; X) = \text{Ker}(\delta_n)/\text{Im}(\delta_{n-1})$ ($n = 1, 2, 3, \ldots$) and $H^0(\mathcal{R}; X) = \langle x \in X : \beta(x) = \gamma(x) \rangle$ the natural cohomology of $S$ with coefficient in $(X, \beta, \gamma)$.

Next, we want to define the dual cohomology analogue for Hopf von Neumann algebras. By Lemma 3.4(b), $\beta$ induces a completely bounded map $\beta_n$ given by $\beta_n(F) = \beta^\#(F) \in \text{CB}(\mathcal{R}^n; \text{CB}(\mathcal{R}^n)) = \text{CB}(\mathcal{R}^n \otimes \mathcal{R}^n)$ (for any $F \in \text{CB}(\mathcal{R}^n)$), i.e., $\beta_n(F)(x_1 \otimes \cdots \otimes x_{n+1}) = F(\beta(x_1) \otimes \cdots \otimes x_{n+1}) (x_1 \otimes \cdots \otimes x_n)$ ($x \in X, x_1, \ldots, x_{n+1} \in \mathcal{R}_*$). Similarly, $\gamma$ induces a completely bounded map $\gamma_n$ such that $\gamma_n(F)(x_1) \otimes \cdots \otimes x_{n+1}) = F(\gamma(x_1)(x_2) \otimes \cdots \otimes x_{n+1})$. Now we let $\partial_n(F) = (\text{id}^{n-1} \otimes \delta \otimes \text{id}^{k-1}) \circ F$ and

$$
\partial_n = \begin{cases} 
\beta_n + \sum_{k=1}^{n} (-1)^k \delta_{n,k} + (-1)^{n+1} \gamma_n & n \geq 1 \\
\beta_0 - \gamma_0 & n = 0.
\end{cases}
$$

By Lemma 3.7 and the proof of Proposition 3.9 below, we have the following.

**Lemma 3.8** $\partial_n \circ \partial_{n-1} = 0$ for $n = 1, 2, \ldots$.

Thus, $\{\partial_n\}$ defines a cohomology $H^n_d(\mathcal{R}; X) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n-1})$ ($n = 1, 2, 3, \ldots$) and $H^0_d(\mathcal{R}; X) = \{f \in X^* : \beta^\#(f) = \gamma^\#(f)\}$ which is called the dual cohomology of $S$ with coefficient in $X$. The use of the term “dual cohomology” can be justified by the following equivalent formulation. This also shows that $H^n_d(\mathcal{R}, \mathcal{R})$ is a more general form of cohomology theory than $H^n_d(\mathcal{R}; \mathcal{R})$.

Suppose that $(X, \beta, \gamma)$ is a $\mathcal{R}$-bicomodule and $\hat{\beta}$ and $\hat{\gamma}$ are respectively the left and the right coactions on $X^*$ given by Lemma 3.6. Then it is easy to see that $(X^*, \hat{\beta}, \hat{\gamma})$ is again a $\mathcal{R}$-bicomodule.

**Proposition 3.9** For any saturated Hopf von Neumann algebra $\mathcal{R}$ and any $\mathcal{R}$-bicomodule $X$, we have $H^n_d(\mathcal{R}; X) \cong H^n(\mathcal{R}; X^*)$ (for $n = 0, 1, 2, 3, \ldots$).

**Proof:** The idea of proof rely on the fact that $\text{CB}(\mathcal{R}^n; X^*) \cong \text{CB}(X; \mathcal{R}^n)$. In this case, the corresponding element $\overline{\beta_n(T)}$ of $T \in \text{CB}(\mathcal{R}^n; X^*)$ is given by $\overline{\beta_n(T)}(\omega_1 \otimes \cdots \otimes \omega_n) = T(\omega_1 \otimes \cdots \otimes \omega_n)(\beta(x_{n+1}))$. Thus,

$$
\overline{\beta_n(T)}(\omega_1 \otimes \cdots \otimes \omega_{n+1}) = \hat{\beta}(T(\omega_1 \otimes \cdots \otimes \omega_n))(\omega_{n+1})(x) = T(\omega_1 \otimes \cdots \otimes \omega_n)(\beta(x)(\omega_{n+1})) = \beta_n(\overline{T})(\omega_1 \otimes \cdots \otimes \omega_{n+1}).
$$
(note that \( \hat{\beta} \) is a left coaction). Similarly, \( \hat{\gamma}_n(T) = \gamma_n(T) \). On the other hand, it is easy to see that \( \hat{\delta}_{n,k}(T) = \partial_{n,n-k+1}(T) \). Therefore, \( \hat{\delta}_n(T) = \gamma_n(T) + \sum_{l=1}^{n} (-1)^{n-l+1} \partial_{n,l}(T) + (-1)^{n+1} \beta_n(T) = (-1)^{n+1} \partial_n(T) \).

Next, we would like to study the vanishing of the dual cohomology of Hopf von Neumann algebras. Again, let us first consider the one-sided case when \( \gamma = 0 \) and use \( H^0_{d,r} \) to denote the dual cohomology defined in this situation.

**Theorem 3.10** Let \( \beta \) be any right coaction of the saturated Hopf von Neumann algebra \( \mathcal{R} \) on an operator space \( X \).

(a) \( H^0_{d,r}(\mathcal{R}; X) = (0) \) if and only if \( \beta \) is non-degenerate.

(b) If \( \mathcal{R}_s \) has a bounded left approximate identity, then \( H^0_{d,r}(\mathcal{R}; X) = (0) \) (\( n \geq 1 \)).

(c) If \( H^1_{d,r}(\mathcal{R}; X) = (0) \), then \( \mathcal{R}_s \) has a bounded left approximate identity.

**Proof:** (a) This part is clear.

(b) Suppose that \( \{\nu_t\} \) is a bounded left approximate identity of \( \mathcal{R}_s \) and \( T \in \text{Ker}(\partial_n) \). Consider the following identification: \( \text{CB}(X; \mathcal{R} \otimes \mathcal{R}^n) \cong \text{CB}(X; \text{CB}(\mathcal{R}_s; \mathcal{R}^n) \cong \text{CB}(\mathcal{R}_s; \text{CB}(X; \mathcal{R}^n)) \) (note that the first isomorphism is different from the one considered in the paragraph preceding Lemma 1.3). Let \( T \in \text{CB}(\mathcal{R}_s; \text{CB}(X; \mathcal{R}^n)) \) be the corresponding element of \( T \) (i.e. \( T(\omega_0)(x)(\omega_1 \otimes \ldots \otimes \omega_{n-1}) = T(x)(\omega_0 \otimes \ldots \otimes \omega_{n-1}) \)). Since \( \partial_n(T) = 0 \), we have, for any \( \omega_0, \ldots, \omega_n \in \mathcal{R}_s \),

\[
0 = T(\omega_0)(\beta(x)(\omega_n))(\omega_1 \otimes \ldots \otimes \omega_{n-1}) + \sum_{k=1}^{n-1} (-1)^k T(\omega_0)(x)(\omega_1 \otimes \ldots \otimes (\omega_{n-k} \cdot \omega_{n-k-1} \otimes \ldots \otimes \omega_n) + (-1)^n T(\omega_0 \cdot \omega_n)(x)(\omega_2 \otimes \ldots \otimes \omega_n).
\]

Moreover, as \( \text{CB}(X; \mathcal{R}^n) \cong (X \otimes \mathcal{R}^n)^* \) (see Lemma 1.3(c)), the bounded net \( \{\hat{T}(\nu_t)\} \) has a subnet \( \{\hat{T}(\nu_{t_j})\} \) that weak*-converges to some \( F \in \text{CB}(X; \mathcal{R}^n) \). Note that \( \nu_{t_j} \cdot \omega \) converges to \( \omega \) for any \( \omega \in \mathcal{R}_s \). Therefore, by putting \( \omega_0 = \nu_{t_j} \), into the above equation and taking limit, we obtain \( 0 = F(\beta(x)(\omega_n))(\omega_1 \otimes \ldots \otimes \omega_{n-1}) + \sum_{k=1}^{n-1} (-1)^k F(x)(\omega_1 \otimes \ldots \otimes (\omega_{n-k} \cdot \omega_{n-k-1} \otimes \ldots \otimes \omega_n) + (-1)^n F(\omega_1)(x)(\omega_2 \otimes \ldots \otimes \omega_n) \) and so \( T = (-1)^{n-1} \partial_{n-1}(F) \) as required.

(c) Recall that the right coaction \( \beta \) of \( \mathcal{R} \) on \( \mathcal{R} \) is given by \( \beta(s)(\omega) = (id \otimes \omega) \delta(s) \) (\( s \in \mathcal{R}_s; \omega \in \mathcal{R}_s \)). As in the proof of Proposition 2.3(b), because id \( \in \text{Ker}(\partial_h) \), there exists \( u \in \mathcal{R}_r \) such that \( \partial_h(u) = id \). Thus, \( (u \otimes x)(s) = u((id \otimes \omega) \delta(s)) = u(\beta(s)(\omega)) = \partial_h(u)(s) = \partial_h(s)(\omega) = s(\omega) \) for any \( \omega \in \mathcal{R}_s \) and \( s \in \mathcal{R} \) (where \( x \) is the second Arens product on \( \mathcal{R}_r = (\mathcal{R}_s)^* \)). Thus \( \mathcal{R}_r \) has a left identity for the second Arens product and so \( \mathcal{R}_s \) has a bounded left approximate identity (see e.g. [18] 5.1.8)).

By Proposition 2.3 \( H^0_{d,r}(\mathcal{R}; X) \cong H^n_{d}(\mathcal{R}; X^*) \) (where \( H^n_d \) is the natural cohomology obtained in the case when the right coaction is zero) for any right \( \mathcal{R} \)-comodule \( X \). The following corollary shows that the vanishing of \( H^n_d \) lies between the existence of a bounded left approximate identity and the existence of an identity in \( \mathcal{R}_s \). In the case when \( \mathcal{R} \) is the bidual of a Hopf C*-algebra, all these three properties coincide (by 2.6).

**Corollary 3.11** If \( H^1_{d,r}(\mathcal{R}; X^*) = (0) \), then there exists a bounded left approximate identity for \( \mathcal{R}_s \). On the other hand, if \( \mathcal{R}_s \) is unital, then \( H^1_{d,r}(\mathcal{R}; X) = (0) \) if and only if \( \mathcal{R}_s \) is a left \( \mathcal{R} \)-comodule.

In fact, the first statement follows clearly from Proposition 2.3 and Theorem 3.10(c). Moreover, suppose that \( u \) is the identity of \( \mathcal{R}_s \) and \( T \in \text{Ker}(\partial_{n+1}) \). If \( F \in \text{CB}(\mathcal{R}^n; X^*) \) is defined by \( F(\omega_1 \otimes \ldots \otimes \omega_{n-1}) = T(u \otimes \omega_1 \otimes \ldots \otimes \omega_{n-1}) \), then it is not hard to see that \( T = \delta_n((-1)^{n}F) \).

It turns out that there is no need to study the vanishing of the 2-sided dual cohomology of a Hopf von Neumann algebra because of its relation with operator cohomology (that studied in [28]). Let us first recall

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Proposition 3.13
For any saturated Hopf von Neumann algebra \( \mathcal{H} \) and \( V \) is an operator \( \mathcal{H} \)-bimodule (see [28] p.1453) or Remark (22). Consider a map \( d_n \) from \( B^n \hat{\otimes} V \) to \( B^{n-1} \hat{\otimes} V \) (where \( B^n \) is the \( n \)-th times operator projective tensor product of \( B \) and \( B^0 = \mathbb{C} \) given by

\[
d_n(a_1 \otimes \ldots \otimes a_n \otimes v) = a_1 \otimes \ldots \otimes a_{n-1} \otimes (a_n \cdot v) + \sum_{i=1}^{n-1} (-1)^{n-i} a_1 \otimes \ldots \otimes a_i \cdot a_{i+1} \otimes \ldots \otimes a_n \otimes v + (-1)^n a_2 \otimes \ldots \otimes a_n \otimes v \cdot a_1.
\]

Note that the \( d_n \)'s that we use here differ from those in [28] by a sign of \((-1)^n\). Now \((B^n \hat{\otimes} V)^*, d^n)\) is a cochain complex and defines the operator cohomology, \( \text{OH}^n(B; V^*) \), from \( B \) to \( V^* \) (i.e. \( \text{OH}^n(B; V^*) = \text{Ker}(d^n) / \text{Im}(d^{n+1}) \) (see [28] p.1455)).

Consider now the completely contractive Banach algebra \( \mathbb{R}_* \). Since \( \text{CB}(\mathbb{R}_* \hat{\otimes} X; X) \cong \text{CB}(X; \text{CB}(\mathbb{R}_*; X)) \), if \( m \in \text{CB}(\mathbb{R}_* \hat{\otimes} X; X) \) is a completely bounded left \( \mathbb{R}_* \)-module on \( X \), then the corresponding map \( \beta_m \in \text{CB}(X; \text{CB}(\mathbb{R}_*; X)) \) given by \( \beta_m(x)(\omega) = m(\omega \otimes x) (x \in X; \omega \in \mathbb{R}_*) \) is a right coaction. Using the same formula, a right coaction defines a left \( \mathbb{R}_* \)-multiplication and these give the following.

**Lemma 3.12** There is an one to one correspondence between completely bounded left (respectively, right) \( \mathbb{R}_* \)-module structures on \( X \) and right (respectively, left) coactions of \( \mathbb{R}_* \) on \( X \).

This implies that there is an one to one correspondence between operator \( \mathbb{R}_* \)-bimodule structures and \( \mathbb{R}_* \)-bicomodule structures on \( X \). Recall that if \( X \) is a \( \mathbb{R}_* \)-bicomodule, the \( \mathbb{R}_* \)-multiplication on \( X \) is given by \( \omega \cdot x = \beta(x)(\omega) \) and \( x \cdot \omega = \gamma(x)(\omega) (\omega \in \mathbb{R}_*; x \in X) \). We can now show that the cochain complex that defines \( H_2^0(\mathbb{R}_*; X) \) is the same as the one that defines the operator cohomology from \( \mathbb{R}_* \) to \( X^* \).

**Proposition 3.13** For any saturated Hopf von Neumann algebra \( \mathcal{R} \) and any \( \mathcal{R} \)-bicomodule \( X \), we have \( H_2^0(\mathcal{R}; X) = \text{OH}^0(\mathcal{R}_*; X^*) \) (for \( n = 1, 2, 3 \ldots \)).

**Proof:** Note that \( \text{CB}(X; \mathcal{R}^n) \cong (\mathcal{R}^n \hat{\otimes} X)^* \) under the identification \( f_T(\omega_1 \otimes \ldots \otimes \omega_n \otimes x) = T(x)(\omega_1 \otimes \ldots \otimes \omega_n) \) \( (T \in \text{CB}(X; \mathcal{R}^n)) \). It is clear that for any \( \omega_1, \ldots, \omega_{n+1} \in \mathbb{R}_* \) and \( x \in X \),

\[
\beta_n(T)(x)(\omega_1 \otimes \ldots \otimes \omega_{n+1}) = f_T(\omega_1 \otimes \ldots \otimes \omega_n \otimes (\omega_{n+1} \cdot x)),
\]

\[
\partial_{n,k}(T)(x)(\omega_1 \otimes \ldots \otimes \omega_{n+1}) = f_T(\omega_1 \otimes \ldots \otimes \omega_{n-k} \otimes \omega_{n-k+1} \cdot \omega_{n-k+2} \otimes \ldots \otimes \omega_{n+k})
\]

and

\[
\gamma_n(T)(x)(\omega_1 \otimes \ldots \otimes \omega_{n+1}) = f_T(\omega_2 \otimes \ldots \otimes \omega_{n+1} \otimes (x \cdot \omega_1)).
\]

Hence \( \partial_n(T)(x)(\omega_1 \otimes \ldots \otimes \omega_{n+1}) = f_T(d_{n+1}(\omega_1 \otimes \ldots \otimes \omega_{n+1} \otimes x)) \) and the complexes \((\mathbb{R}_* \hat{\otimes} X)^*, d_n^* \) and \((\text{CB}(X; \mathcal{R}^n), \partial_n)\) coincide under the above identification.

This, together with [28] 2.1, gives the following characterisation of the vanishing of the dual cohomology of Hopf von Neumann algebras.

**Corollary 3.14** \( H_2^n(\mathbb{R}_*; X) = 0 \) for any \( \mathbb{R} \)-bicomodule \( X \) and any \( n \in \mathbb{N} \) if and only if \( \mathbb{R}_* \) is operator amenable.

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4 Dual cohomology and amenability

In this section, we will give some interesting consequences of the results in the previous sections. If A is a non-zero 2-sided S-invariant closed subalgebra of the dual space $S^*$ (recall that $f \cdot g = (f \otimes g)\delta$ for $f, g \in S^*$), then by [33 III.2.7], there exists a central projection $e \in S^{**}$ such that $A = eS^*$ and by [19] 1.10(b)], $A^*$ is a Hopf von Neumann algebra.

**Notation:** Throughout this section, we will assume that $A$ is a non-zero 2-sided $S$-invariant closed subalgebra of $S^*$ (note the difference between the $A$ in here and in Section 1). Moreover, $(R, \delta)$ is a not necessarily non-degenerate Hopf $C^*$-algebra.

**Lemma 4.1** There is a complete contraction $\Psi_A$ from $MS(X \otimes S)$ to $CB(A; X)$ such that $\Psi_A(m)(\omega) = (id \otimes \omega)(m)$ ($m \in MS(X \otimes S); \omega \in A$). Moreover, if $A$ separates points of $S$, then $\Psi_A$ is a complete isometry.

**Proof:** Let $j$ be the canonical +-homomorphism from $S$ to $A^*$ (given by $j(s)(\omega) = \omega(s)$ for $s \in S$ and $\omega \in A$). Then Lemma 1.12(b) gives a complete contraction $id \otimes j$ from $MS(X \otimes S)$ to $M_{j(S)}(X \otimes j(S))$. Now by Proposition 1.10 and the definition of the Fubini product, we see that $M_{j(S)}(X \otimes j(S))$ can be regarded as an operator subspace of $X^{**} \otimes_R A^* \cong CB(A; X^{**})$ (note that for any $m \in M_{j(S)}(X \otimes j(S))$, $f \in X^*$ and $\omega \in A$, we have $(id \otimes \omega)(m) \in X$ and $(f \otimes id)(m) \in M(j(S))$ by Lemma 1.12(a)). It is not hard to see that the composition $\Psi_A$, of the above two maps satisfies the required conditions (in particular, $\Psi_A(MS(X \otimes S)) \subseteq CB(A; X)$). Furthermore, if $A$ separates points of $S$, then $j$ is a complete isometry and so is $\Psi_A$.

If $\beta$ is a right coaction of $S$ on $X$, then the above lemma shows that $\beta$ induces a right coaction $\tilde{\beta}_A$ of $A^*$ on $X$ such that $\tilde{\beta}_A(x)(\omega) = (id \otimes \omega)(\beta(x))$ for any $\omega \in A$ and $x \in X$ (the coaction identity can be verified easily). Similarly, we can define from a left coaction $\gamma$ of $S$ on $X$, a left coaction $\tilde{\gamma}_A$ of $A^*$ on $X$.

**Proposition 4.2** Suppose that the dual space $S^*$ of a saturated Hopf $C^*$-algebra $S$ contains a non-zero $S$-invariant closed subalgebra $A$ that separates points of $S$. If $A$ is operator amenable (in the sense of [28 2.2]), then $H^1_d(S; X) = (0)$ for any $S$-bicomodule $X$. Consequently, if $S^*$ is operator amenable, then any first dual cohomology of $S$ vanishes.

**Proof:** Let $(X, \beta, \gamma)$ be a $S$-bicomodule. Suppose that $\tilde{\gamma}_A$ and $\tilde{\beta}_A$ are the left and the right coactions of $A^*$ on $X$ as given above. By the definition of operator amenability and Proposition 4.1, $H^1_d(A^*; X) = (0)$. Consider $F \in CB(X; M(S))$ such that $\partial_1(F) = 0$ and let $T = j \circ F \in CB(X; A^*)$ (where $j$ is the map as in the proof of Lemma 1.1). Note that $j$ "preserves the coproducts" (i.e. $\delta(j(s)) = (j \otimes j)\delta(s)$ for all $s \in M(S)$). For any $\omega, \nu \in A$ and $x \in X$, by Lemma 1.12(c),

$$T(\tilde{\beta}_A(x)(\nu))(\omega) - \delta(T(x))(\omega \otimes \nu) + T(\tilde{\gamma}_A(x)(\omega))(\nu) = \omega \circ F((id \otimes \nu)\beta(x)) - (\omega \otimes \nu)\delta(F(x)) + \nu \circ F((\omega \otimes id)\gamma(x))$$

$$= (\omega \otimes \nu) \circ ((F \otimes id) \circ \beta - \delta \circ F + (id \otimes F) \circ \gamma)(x) = 0.$$

Hence $T \in Ker(\partial_1)$ and there exists $f \in X^*$ such that for any $\omega \in A$ and $x \in X$, we have the identities $\omega(F(x)) = T(x)(\omega) = f(\tilde{\beta}_A(x)(\omega)) - f(\tilde{\gamma}_A(x)(\omega)) = \omega((f \otimes id)\beta(x) - (id \otimes f)\gamma(x))$. Since $A$ separates points of $S$ (and hence separates points of $M(S)$ as $A$ is $S$-invariant), the above implies that $F = \partial_0(f)$. Therefore, $H^1_d(S; X) = (0)$.

The proof of the above proposition actually shows that $H^1_d(S; X) \subseteq H^1_d(A^*; X)$ (when $A$ is a $S$-invariant closed subalgebra of $S^*$ that separates points of $S$). Now, we would like to consider a cohomology theory that is even "smaller than" $H^1_d(S; X)$ (in fact, a "restriction" of it) that will help us to study the vanishing of the dual cohomologies of the Hopf $C^*$-algebras associated with locally compact groups. Let $U^n(S) = U_{\omega_{n-1} \otimes A}(S^n)$ for $n \geq 1$ (see Lemma 4.1 and Remark 4.3) and $U^0(S) = \mathbb{C}$. We first show that the cochain complex that defined the dual cohomology can be “restricted to $U^n(S)$” in some cases.
Lemma 4.3  Let \((R, \delta)\) be a (not necessarily saturated) Hopf \(C^*\)-algebra. Suppose that \((X, \beta, \gamma)\) is a \(R\)-bicomodule such that \(\gamma = 1_R \otimes \text{id}_X\). Then \(\partial_n(CB(X; U^n(R))) \subset CB(X; U^{n+1}(R))\).

Proof: Let \(T \in CB(X; U^n(R)), s \in R\) and \(x \in X\). Then
\[
((\text{id}^n \otimes \delta)(T \otimes \text{id})\beta(x))(1 \otimes s) = (T \otimes \text{id} \otimes \delta)(\beta \otimes \text{id})(\beta(x) \cdot s) \in (T \otimes \text{id})(M_R(X \otimes R)) \otimes R \subseteq M(R^{n+1}) \otimes R
\]
(note that \((T \otimes \text{id}) \circ \beta \otimes \text{id}\) is a \(R\)-bimodule map by Lemma 1.12(a)) and
\[
((\text{id}^n \otimes \delta)(\text{id} \otimes T)\gamma(x))(1 \otimes s) = (\text{id}^n \otimes \delta)(\text{id} \otimes T)(x)(1 \otimes s) \in M(R^{n+1}) \otimes R
\]
(by the definition of \(U^n(R)\)). Moreover,
\[
((\text{id}^n \otimes \delta)(\text{id}^{n-1} \otimes \delta)T(x))(1 \otimes s) = (\text{id}^{n-1} \otimes \delta \otimes \text{id})(\text{id}^{n-1} \otimes \delta)T(x)(1 \otimes s) \in M(R^{n+1}) \otimes R
\]
and for \(2 \leq k \leq n,
\[
((\text{id}^n \otimes \delta)(\text{id}^{n-k} \otimes \delta \otimes \text{id}^k)\beta(x))(1 \otimes s) = (\text{id}^{n-k} \otimes \delta \otimes \text{id}^k)(\text{id}^{n-1} \otimes \delta)T(x)(1 \otimes s) \in M(R^{n+1}) \otimes R.
\]
These show that \(((\text{id}^n \otimes \delta)\partial_n(T)(x))(1 \otimes s) \in M(R^{n+1}) \otimes R\). Similarly, we also have \((1 \otimes s)((\text{id}^n \otimes \delta)\partial_n(T)(x)) \in M(R^{n+1}) \otimes R\). Thus \(\partial_n(T)(x) \in U^{n+1}(R)\) as required.

The above lemma says that for any right \(R\)-bicomodule \((X, \beta)\), if we take \(\gamma = 1_R \otimes \text{id}_X\), then \((CB(X; U^n(R)), \partial_n)\) is a cochain subcomplex of \((CB(X; M(R^n)), \partial_n)\). The cohomology \(H^n_{R,d}(R; X)\) defined by this complex is called the restricted left trivial dual cohomology. It turns out that the vanishing of \(H^n_{R,d}(R; X)\) is related to the existence of a left invariant mean on \(U^1(R)\). The idea of the necessity of part (a) in the following proposition comes from [24] p.43.

Proposition 4.4  (a) Suppose that \((R, \delta)\) is a counital (not necessarily saturated) Hopf \(C^*\)-algebra such that \(R\) has property (S) (in particular, if \(R\) is a nuclear \(C^*\)-algebra). Then \(H^n_{R,d}(R; X) = (0)\) for any right \(R\)-comodule \(X\) if and only if there exists a left invariant mean \(\Phi\) on \(U^1(R)\) (see Definition A.3).

(b) If \((S, \delta)\) is a saturated Hopf \(C^*\)-algebra such that \(S\) is unital, then \(H^n_{R,d}(S; X) = (0)\) for all \(n \geq 1\) and for all right \(S\)-comodule \(X\).

Proof: (a) Let \(X = U^1(R)/C \cdot 1\) with the canonical quotient map \(q\) from \(U^1(R)\) to \(X\). As \((q \otimes \text{id})\delta(1) = 0\) (where \(q \otimes \text{id}\) is the map from \(M_R(U^1(R) \otimes R)\) to \(M_R(X \otimes R)\) given by Lemma 1.12(a)), the coproduct \(\delta\) induces a right coaction \(\beta\) from \(X\) to \(M_R(X \otimes R)\) such that \(\beta \circ q = (q \otimes \text{id}) \circ \delta\) (by Lemma 1.15(c)). \(X\) can now be regarded as a \(R\)-bicomodule if we take \(\gamma = 1 \otimes \text{id}_X\). Consider \(T = \text{id}_{U^1(R)} - \epsilon \cdot 1\) (where \(\epsilon\) is the counit of \(R\)). Since \(T(1) = 0\), it induces a map \(T \in CB(X; U^1(R))\) such that \(\bar{T} \circ q = T\) (Lemma 1.5). Now for any \(z \in U^1(R),
\[
\partial_1(T)(q(z)) = (T \otimes \text{id})(\delta(z)) - \delta(T(z)) + 1 \otimes T(z)
\]
\[
\delta(z) - 1 \otimes z - \delta(z) + \epsilon(z) \cdot 1 \otimes 1 + 1 \otimes z - \epsilon(z) \cdot 1 \otimes 1 = 0.
\]
Hence by the hypothesis, there exists \(\phi \in X^*\) such that \(\bar{T} = \partial_1(\phi)\), i.e. \(\phi \circ q \otimes \text{id}) \circ \delta - \phi \circ q \cdot 1 = \text{id}_{U^1(R)} - \epsilon \cdot 1\). Now, let \(\Phi = \epsilon - \phi \circ q\). It is clear that \(\Phi(1) = \epsilon(1) = 1\) and \((\Phi \otimes \text{id}) \circ \delta = \text{id}_{U^1(R)} - (\phi \circ q \otimes \text{id}) \circ \delta = \epsilon(1) - \phi \circ q \cdot 1\). By a similar argument as [24] 2.2 [see also [23] 2.1], the rescaling of either the positive part or the negative part of \(\Phi\) in the Jordan decomposition is a left invariant mean. Conversely, suppose that there exists a left invariant mean \(\Phi\) on \(U^1(R)\). Let \((X, \beta)\) be any right \(R\)-comodule. For any \(T \in CB(X; U^1(R))\) such that \(\partial_1(T) = 0\), take \(f = \Phi \circ T \in X^*\). Then for any \(g \in R^*\) and \(x \in X\), we have by Lemma 1.12(c),
\[
0 = \Phi((\text{id} \otimes g)((T \otimes \text{id})\beta(x))) - \Phi((\text{id} \otimes g)\delta(T(x))) + \Phi(1)g(T(x))
\]
\[
= g((f \otimes \text{id})\beta(x)) - \Phi(T(x))g(1) + g(T(x)).
\]
Hence \( T(x) = (f \otimes \text{id}) \beta(x) - f(x) \cdot 1 = \partial_0(f) \).

(b) It was shown in [39] (see also [34]) that \( S \) has a left Haar state \( \phi \) which is in fact a left invariant mean on \( U^1(S) = S \). Now for any \( T \in \text{Ker}(\partial_n) \), if \( F = (\phi \otimes \text{id}^{n-1}) \circ T \),

\[
0 = (F \otimes \text{id}) \circ \beta + \sum_{k=1}^{n-1} (-1)^k (\text{id}^{n-k-1} \otimes \delta \otimes \text{id}^{k-1}) \circ F + (-1)^n1 \otimes F + (-1)^{n+1}T
\]

which completes the proof.

This proposition, together with Remarks A.3(a) and A.5(b), gives the following corollary. We recall that a discrete semi-group \( \Lambda \) is said to be left amenable if there exists a left invariant mean on \( l^\infty(\Lambda) \) (see Remark A.3(b)). Note that \( U(\Lambda) = U_l(\Lambda) = l^\infty(\Lambda) \) (see Remark A.3(a)) and right coactions of \( c_0(\Lambda) \) on an operator space \( Z \) are bounded homomorphisms from \( \Lambda \) to \( \text{CB}(Z; Z) \) (by a similar argument as in Example 4.10(c)).

**Corollary 4.5** (a) Suppose that \( M \) is a locally compact semi-group with identity. Then \( U_l(M) \) (see Remark A.3(a)) has a left invariant mean if and only if \( H^1_l(c_0(M); Y) = \{0\} \) for any right \( c_0(M) \)-comodule \( Y \).

(b) If \( \Lambda \) is a discrete semi-group with identity, then \( \Lambda \) is left amenable if and only if for any bounded representation \( \pi \) of \( \Lambda \) in \( \text{CB}(Z; Z) \) and any \( T \in \text{CB}(Z, l^\infty(\Lambda)) \) with \( T(z)(r \cdot s) = T(\pi(s)z)(r) + T(z)(s) \) \((r, s \in \Lambda; z \in Z)\), there exists an element \( f \in Z^* \) such that \( T(z)(r) = f(\pi(r)z) - f(z) \).

Propositions 4.2 and 4.4 can also be used to prove the following interesting theorem.

**Theorem 4.6** Let \( G \) be a locally compact group.

(a) \( G \) is amenable if and only if \( H^1_l(c_0(G); X) = \{0\} \) for any \( c_0(G) \)-bicomodule \( X \).

(b) \( G \) is amenable if and only if \( H^1_l(C^*_r(G); X) = \{0\} \) for any \( C^*_r(G) \)-bicomodule \( X \) or equivalently, \( H^1_l(C^*_r(G); X) = \{0\} \) for any 2-sided non-degenerate \( C^*_r(G) \)-bicomodule \( X \).

**Proof:** (a) If \( G \) is amenable, then it is clear that \( OH^1(L^1(G); X^*) = \{0\} \) (since the Banach algebra cohomology \( H^1(L^1(G); X^*) \) vanishes) and thus \( L^1(G) \) is operator amenable. Now by putting \( S = c_0(G) \) and \( A = L^1(G) \) into Proposition 4.2 (see [19] §5), we see that \( H^1_l(c_0(G); X) = \{0\} \). The converse follows directly from Proposition 4.2(a) (note that for any right \( S \)-comodule \( X \), if we take \( \gamma = 1 \otimes \text{id}_X \), then \( H^1_{r,d}(S; X) \subseteq H^1_l(S; X) \)) as well as Remarks A.3(a) (i.e., \( U(G) \subseteq U^1(c_0(G)) \)) and A.3(b).

(b) Suppose that \( G \) is amenable. Then by [28] 3.6, it is easily seen that \( H^1_l(C^*_r(G); X) = \{0\} \) for any 2-sided non-degenerate \( C^*_r(G) \)-bicomodule \( X \), then Proposition 4.2(c) tells us that \( C^*_r(G) \) has a counit and hence \( G \) is amenable.

**Remark 4.7** (a) If the Fourier algebra \( A \) of a saturated Hopf \( C^* \)-algebra \( S \) (recall that the Fourier algebra is the intersection of all non-zero \( S \)-invariant (closed) ideals of \( S^* \); see [12] §5) separates points of \( S \), then by the proof of Proposition 4.2 we have the inclusions: \( H^1_l(S^*; X) \subseteq H^1_{l,A}(A^*; X) \equiv OH^1(A; X^*) \subseteq H^1(A; X^*) \).

In fact, these inclusions, together with the results in [28] and [14], give one of the implications of both parts (a) and (b) of the above Theorem. The other implications tell us that even the vanishing of \( H^1_l(c_0(G); X) \) or \( H^1_l(C^*_r(G); X) \) is strong enough to characterise the amenability of \( G \).

(b) The amenability of \( G \) is also equivalent to the vanishing of the first dual cohomology of \( vN(G) \) (which is exactly [28] 3.6 by Proposition 5.13). The same is true for \( L^\infty(G) \) (by the inclusions in part (a) as well as the results in [14]).
Corollary 4.8 Let $G$ be a locally compact group.

(a) If $G$ is amenable, then there exists a coaction on $\hat{\mathcal{U}}(G \times G) = \{ g \in C_0(G \times G) : r \mapsto r \ast_1 g$ and $r \mapsto r \ast_2 g$ are both continuous maps\}$ (where $r \ast_1 g(s,t) = g(r^{-1}s,t)$ and $r \ast_2 g(s,t) = \Delta(r^{-1})g(s,tr^{-1})$).

(b) If there exists a coaction $F$ on $C_0(G \times G)$ such that $F \circ \delta = \epsilon$ on $C_0(G)$, then $G$ is amenable.

Proof: (a) This follows from Proposition 4.8(a) and Theorem 4.6(a) (together with a similar identification as in Remark 4.3(a)).

(b) For any right $R$-comodule $(X, \beta)$, if we consider $\gamma$ to be the 2-sided non-degenerate left coaction $1 \otimes \text{id}_X$, then Proposition 4.8(b) implies that $(0) = H^1_{d,r}(C_0(G); X) \supseteq H^1_{d,r}(C_0(G); X)$. Hence by Proposition 4.6(a), there exists a left invariant mean on $U^1(C_0(G)) \supseteq U(G)$ and $G$ is amenable.

Next, we will consider the vanishing of the one-sided dual cohomology and relate it to the amenability of Hopf $C^*$-algebras. All the remaining results in this section are obvious and hence no proof will be given.

First of all, we have the following one-sided version of Theorem 4.6 which is a direct consequence of Proposition 4.6.

Corollary 4.9 Let $G$ be a locally compact group.

(a) $H^1_{d,r}(C_0(G); Y) = (0)$ for any right $C_0(G)$-comodule $Y$.

(b) $G$ is amenable if and only if $H^n_{d,r}(C^*_r(G); Y) = (0)$ for any $n \in \mathbb{N}$ and any right $C^*_r(G)$-comodule $Y$. It is the case if and only if $H^1_{d,r}(C^*_r(G); Y) = (0)$ for any 2-sided non-degenerate right $C^*_r(G)$-comodule $Y$.

For the general case, we need to consider the cohomology of Hopf von Neumann algebras instead. We first recall from [20, 2.1] that a Hopf $C^*$-algebra $(S, \delta)$ is said to be left $H_1$-coamenable if the left Fourier algebra $A_S^L$ (i.e. the intersection of all non-zero $S$-invariant left (closed) ideals of $S^*$; see [19, §5]) has a bounded left approximate identity. By Theorem 3.10 we have a characterisation of the amenability of $S$ in terms of the cohomology of the Hopf von Neumann algebra $(A_S^L)^*$.

Corollary 4.10 Suppose that the left Fourier algebra $A$ of $(S, \delta)$ is non-zero. Then $(S, \delta)$ is left $H_1$-coamenable if and only if $H^1_{d,r}(A^*; X) = (0)$ for any right $A^*$-comodule $X$.

Let $(T, V, S)$ be a Kac-Fourier duality in the sense of [19, 5.13]. Then by [20, 3.15], the left (or right) $H_1$-coamenability of $T$ will automatically imply the 2-sided $H_1$-coamenability (i.e. the Fourier algebra of $S$ as defined in Remark 4.7(a) has a bounded 2-sided approximate identity). In this case, we will simply call $(T, V, S)$ amenable (see [20, 3.16]). We have the following characterisation of this amenability in terms of cohomology.

Proposition 4.11 Let $(T, V, S)$ be a Kac-Fourier duality and $(\mu, \nu)$ be any $V$-covariant representation on a Hilbert space $H$. Let $B = \mu^*(L(H)_\nu)$ (note that $B$ will then be both the Fourier algebra and the left Fourier algebra of $S$ by [19, 5.9]). Then $(T, V, S)$ is amenable if and only if $H^1_{d,r}(B^*; X) = (0)$ for any right $B^*$-comodule $X$.

Note that the cohomology considered in this proposition can be regarded as an one-sided version of the operator cohomology (using Proposition 3.13).

Suppose that $V \in \mathcal{L}(H \otimes H)$ and $W \in \mathcal{L}(K \otimes K)$ are regular multiplicative unitaries such that $S_V \cong \hat{S}_W$ (see [3, 1.5]), in particular, if $V$ comes from a Kac system $(H, V, U)$ (see [3, §6]) and $W = \Sigma(1 \otimes U)V(1 \otimes U)\Sigma$ (where $\Sigma \in \mathcal{L}(H \otimes H)$ is defined by $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$). Then $(S_V, V, S_V)$ is a Kac Fourier duality (see [19]). In this case, $V$ is amenable in the sense of Baaj and Skandalis (see [3, A13(c)]) if and only if $(S_V, V, S_V)$ is amenable in the above sense. Moreover, if $A_V = L_V(L(H)_\nu)$ (where $LV$ is the default representation of $S_V$ on $L(H)$), then $A_V^* = S_V^*$ (the weak*-closure of $S_V$ in $\mathcal{L}(H)$). Hence we can express amenability of Kac systems (see [3, 3.3&6.2]) and in particular Kac algebras (see [12]) in terms of cohomology.
Corollary 4.12 Suppose that $V$ is a regular irreducible multiplicative unitary. $V$ is amenable if and only if $H_{a,r}(S''_U; X) = (0)$ for any right $S''_U$-comodule $X$.

In particular, we have a Hopf von Neumann algebra analogue of Corollary 4.11(b) which can be regarded as an one-sided version of [28, 3.6] (in the light of Proposition 3.13).

A Extensions of coactions

Notation: In this appendix, we do not assume the Hopf $C^*$-algebra $(R, \delta)$ to be saturated. Moreover, as usual, all the right and left coactions on $C^*$-algebras in this appendix are $*$-homomorphisms.

The main objective of this appendix is to answer the following natural and interesting question: “If $\beta$ is a coaction of $R$ on a $C^*$-algebra $A$, what is the biggest unital closed subalgebra of $M(A)$ on which $\beta$ can be extended?” This extension is important for some arguments in this paper. We know that in general, it is impossible to extend $\beta$ to a coaction on the whole of $M(A)$ (consider e.g. $A = C_0(G) = R$ and $\beta$ is the coproduct on $C_0(G)$). Inspired by the definition of $U(G)$, we have the following lemma.

Lemma A.1 Suppose that $R$ has property (S) in the sense of [37] or [11] (in particular, if it is a nuclear $C^*$-algebra; see [37, 10]). Let $\beta$ be a (right) coaction of $R$ on a $C^*$-algebra $A$. Then $U\beta(A) = \{ m \in M(A) : \beta(m) \in M_R(M(A) \otimes R) \}$ is a unital $C^*$-subalgebra of $M(A)$ and $\beta$ extends to a coaction (again denoted by $\beta$) of $R$ on $U\beta(A)$.

Proof: It is obvious that $U\beta(A)$ is a unital $C^*$-subalgebra of $M(A)$. We need to show the inclusion: $\beta(U\beta(A)) \subseteq M_R(U\beta(A) \otimes R)$. In fact, since $\beta(U\beta(A))(1 \otimes R) \subseteq M(A) \otimes R$ and $R$ has property (S), it suffices to prove that $(id \otimes f)\beta(m) \in U\beta(A)$ for any $f \in R^*$ and $m \in U\beta(A)$. Note that there exist $f' \in R^*$ and $t \in R$ such that $f = t \cdot f'$. Thus for any $s \in R$, we have $\beta((id \otimes f)\beta(m))(1 \otimes s) = (id \otimes id \otimes f')(id \otimes \delta)\beta(m)(1 \otimes s \otimes t) \in M(A) \otimes R$ (recall that $(id \otimes \delta)\beta(m) \in M_R(M(A) \otimes R \otimes R)$ by Lemma A.1(b)). Finally, the coaction identity of the extension follows from that of $\beta$.

It is clear that $U\beta(A)$ is the biggest closed subalgebra of $M(A)$ on which $\beta$ can be extended. Similarly, we can define $U_r(A)$ for a left coaction $\gamma$. Moreover, we have the following two-sided version of Lemma A.1.

Lemma A.2 Suppose that $R$ has property (S) and there exists a (right) coaction $\beta$ and a left coaction $\gamma$ of $R$ on a $C^*$-algebra $A$ such that $(\gamma \otimes id)\beta = (id \otimes \gamma)\beta$. Then $U_{\beta,\gamma}(A) = \{ m \in M(A) : \beta(m) \in M_R(M(A) \otimes R); \gamma(m) \in M_R(R \otimes M(A)) \}$ is a unital $C^*$-subalgebra of $M(A)$ and $\beta$ (respectively, $\gamma$) extends to a (right) coaction (respectively, left coaction), again denoted by $\beta$ (respectively, $\gamma$), on $U_{\beta,\gamma}(A)$. Moreover, $U_{\beta,\gamma}(A)$ is the biggest unital $C^*$-subalgebra of $M(A)$ for which both $\beta$ and $\gamma$ can be extended.

Proof: Note that $U_{\beta,\gamma}(A) = U\beta(A) \cap U_r(A)$ and so is a unital $C^*$-subalgebra of $M(A)$. We first show that $\beta$ extends to a coaction on $U_{\beta,\gamma}(A)$. By Lemma A.1 and its proof, we need only to show that $(id \otimes f)\beta(m) \in U_r(A)$ for any $m \in U_{\beta,\gamma}(A)$ and $f \in R^*$. Indeed, for any $s \in R$, we have $\gamma((id \otimes f)\beta(m))(s \otimes 1) = (id \otimes id \otimes f)(id \otimes \beta)(\gamma(m)(s \otimes 1)) \in R \otimes M(A)$ and similarly $(s \otimes 1)\gamma((id \otimes f)\beta(m)) \in R \otimes M(A)$. The proof for the extension of $\gamma$ is the same.

Remark A.3 If $A = R$ and $\beta = \delta = \gamma$, we denote $U_{\beta,\gamma}(A)$ by $U(R)$. Moreover, we denote $U_{id \otimes \delta \otimes id}(R \otimes R)$ by $U(R \otimes R)$ and $U_{id \otimes \delta \otimes id}(R^n)$ by $U^n(R)$ (n $\geq$ 1).

(a) Suppose that $M$ be a locally compact semi-group. Then $(C_0(M), \delta_M)$ (where $\delta_M(f)(r, s) = f(rs)$) is a (possibly non-saturated) Hopf $C^*$-algebra. Let $U(M) = \{ g \in C_b(M) : r \mapsto r \cdot g$ and $r \mapsto g \cdot r$ are both norm
continuous} and \( U_t(M) = \{ f \in C_b(M) : r \mapsto r \cdot f \text{ is a norm continuous map} \} \) (where \( r \cdot f(s) = f(sr) \) and \( f \cdot r(s) = f(rs) \)). Then \( U_t(M) = U^1(C_b(M)) \) and \( U(M) = U(C_b(M)) \). In fact, for any \( f \in C_b(M) \), using a similar argument as in Example 1.10(c), \( \delta(f) \in MC_b(M)(C_b(M) \otimes C_b(M)) = C_b(M ; C_b(M)) \) if and only if the map that sends \( r \in M \) to \( r \cdot f \in C_b(M) \) is continuous. Consequently, for a locally compact group \( G \), the space \( U(C_0(G)) \) coincides with \( U(G) \) (note that the inverse and the modular function are not included in the definition of \( U(R) \) but it doesn’t matter).

(b) Let \( R = C^*_r(G) \) and \( u_t \) be the element in \( M(C^*_r(G)) \) corresponding to \( t \in G \). Then it is clear that \( u_t \in U(R) \).

Definition A.4 Suppose that \( \beta \) is a (right) coaction of \( R \) on a \( C^* \)-algebra \( A \).

(a) A closed subspace \( X \) of \( M(A) \) is said to be weakly \( \beta \)-invariant if \( (id \otimes f) \beta(X) \subseteq X \) for any \( f \in R^* \).

(b) Let \( X \) be a weakly \( \beta \)-invariant subspace of \( M(A) \) that contains \( 1_A \). Then \( \Phi \in X^*_r \) is said to be a \( \beta \)-invariant mean if \( \Phi(1_A) = 1 \) and \( \Phi((id \otimes f) \beta(x)) = \Phi(x)f(1_R) \) for any \( f \in R^* \) and \( x \in X \). In the case when \( A = R \) and \( \beta = \delta \), we call such \( \Phi \) a left invariant mean on \( X \).

We can define similarly weakly \( \gamma \)-invariant subspaces and \( \gamma \)-invariant mean for a left coaction \( \gamma \).

Remark A.5 (a) It is clear that if \( \beta \) induces a right coaction on a subspace \( X \) of \( M(A) \) (i.e. \( \beta(X) \subseteq M_R(X \otimes R) \)), then \( X \) is automatically weakly \( \beta \)-invariant. If in addition \( 1_A \in X \), then \( \Phi \in X^*_r \) is a left invariant mean if \( \Phi(1_A) = 1 \) and \( \Phi((id \otimes f) \beta) = \Phi \cdot 1_A \) on \( X \). This applies, in particular, to both \( U^1(R) \) and \( U(R) \) if \( R \) has property (S) (by Lemmas A.3 and A.4).

(b) If \( M \) is a locally compact semi-group, then \( (C_0(M), \delta_M) \) is a (not necessarily saturated) Hopf \( C^* \)-algebra. Let \( X \) be a left invariant subspace of \( C_b(M) \) in the sense that \( r \cdot g \in X \) for any \( g \in X \) and \( r \in M \) where \( r \cdot g(t) = g(tr) \). It is clear that \( \delta_M \) induces a right coaction on \( X \) and so \( X \) is weakly \( \delta_M \)-invariant (by part (a)). If \( X \) contains 1, then \( \Phi \in X^*_r \) is a left invariant mean in the sense of Definition A.4 if and only if \( \Phi(1) = 1 \) and \( \Phi(r \cdot f) = \Phi(f) \) for any \( f \in X \). Hence for a locally compact group \( G \), the left invariant mean on \( U(G) = U(C_0(G)) \) as defined above coincides with the usual definition.

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