Neuronlike impulses in a travelling wave structure loaded with resonant tunneling diodes and air bridges

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Abstract
In this paper a coplanar waveguide (CPW) periodically loaded with resonant tunneling diodes (RTDs) and air bridges (AB) is presented as a travelling wave (TW) structure for modelling the FitzHugh-Nagumo (FHN) equation and emulating the behaviour of the nerve axon. Based upon an electrical equivalent circuit, the principle of operation is discussed in the context of a lumped-element circuit model being compared to a distributed structure. The phenomena of wave formation and propagation are studied by computer experiments of the underlying nonlinear ordinary difference-differential equations (ODEs) and that of the approximated model nonlinear partial differential equations (PDEs). A key achievement is that this medium supports stable propagating shock waves (kinks and antikinks) when effects of AB are neglected as well as stable traveling pulses only determined by the parameters of the circuit. As a result, compact electronic circuits with features of real neural systems are developed to be an experimental medium mimicking neural activity and to be applied in ultra-fast signal processing.

1. Introduction
Motivated by two pioneering scientific works, there has been a great interest in the study of FitzHugh-Nagumo (FHN) equations, which describe the propagation of an action potential along the nerve axon. In the first one, Hodgkin and Huxley (HH) [1] developed a nerve model of four nonlinear differential equations to describe the ionic and electrical events of the biological membrane occurring during the transmission of an impulse along an axon, which is usually the filament carrying signals from the nerve cell body to other parts of the organism. The second one is provided by FitzHugh [2] who developed a simpler model of HH consisting of a system of two ordinary differential equations (ODEs) which capture key features of the full system. Next, Nagumo et al studied a related equation and added a diffusion term for the conduction process of action potentials along nerve axons [3]. This system, where spatial diffusion in the transmembrane potential is allowed but without any applied external current, is being considered as a basic FHN model [4]. In general, diffusion processes play a very important role in almost any biological phenomenon. Through diffusion many metabolites are exchanged between a cell and its environment or between the blood stream and tissues. In addition, the spatial propagation of a neuron firing depends on the properties of the nerve membrane, which can be affected by drugs or external chemicals. The latter can give rise to cross-diffusive coupling between the components of the system and lead to a modified FHN model with cross-diffusion terms [5]. Hence, the FHN model can be taken as a representative of a wide class of nonlinear—excitable-systems and exhibits wide applications in the modelling of biological processes in cardiac electrophysiology [6]. In this last frame, e.g., basic electrophysiological properties that must be considered as facilitating reentry heartbeat are slow conduction, unidirectional conduction block, and recovery of excitability.

On the other hand, some active transmission lines have the property of shaping signal waveforms during their transmission. As examples, Nagumo et al have described active bistable and monostable transmission lines simulating a nerve axon [3, 7]. In this context, the term ’neuristor’, derived from neuron, is the name suggested
for the whole class of lines that exhibit attenuationless propagation with recovery [8], and from which a variety of neuromorphic electrical circuits can be constructed using available electronic components and the technology of monolithic microwave integrated circuits (MMIC). With respect to the experimental work, on the one hand, several attempts have been done to mimic neuron models using, e.g., analog circuits [9, 10] and field programmable gate arrays [11]. On the other hand, it is well known that resonant tunneling diodes (RTDs) are the fastest switching semiconductor devices currently available in the commercial market. Also, RTDs can be co-integrated leading to a variety of compact and ultra-fast circuits, including resonators, frequency multipliers, multistate memories, analog-to-digital converters, and static random access memory [12, 13]. Furthermore, recent technological advances in III-V semiconductor devices, have led to the development of RTD circuits for millimeter wave generation and processing [14, 15]. In fact, the extremely fast response associated with the negative differential resistance (NDR) of RTDs provided an adequate gain mechanism for the generation of terahertz-frequency (THz) energy.

The main purpose of this paper is to consider coplanar waveguide (CPW) structure based upon RTDs and air bridges (AB) simulating the FHN model of action potential propagation along an axon of a neuron. Our previous works [16, 17] dealing with the monostable RTD line under in consideration, are mainly concerned with ODEs in the wave analysis and less with PDEs that govern the spatial and temporal evolution of the transmembrane voltage. Let us recall that in [17], different behaviors, with tuning of circuit parameters are reported together with physical meaning approaches. Therefore, several technical applications arising from several properties of RTDs; that there is only a foreseeable effect on the line properties; and that there is only a quantitative influence on the wave structures but not a qualitative one, as well as, that there is the possibility for optimizing the structure for technical applications.

Here, we study wavelike phenomena as electrical signaling through numerical solutions of FHN equations, which is an exemplified well-known reaction-diffusion system. Other interesting spatiotemporal processes in electrophysiological properties, including the construction of traveling wave (TW) solutions are also addressed. In section 2 we introduce the basic properties and operating principle of the RTD and AB structure. Next sections concern propagating front waves in section 3 and traveling pulse wave in section 4. In section 5, we conclude the paper.

2. Basic properties and operating principle of the RTD structure

In figure 1, a schematic of a CPW structure is presented where one can see periodically loaded RTDs shunted by air-bridge lines. The cross section and structure of the MMIC in that figure using InP-based technology have been shown in [14]. Kindly note that the air bridge technology has already been incorporated into other designs, such as InP-based RTD [18] and, GaN-based HF planar Schottky barrier diodes [19]. Also a suitable equivalent circuit of the line using discrete elements (’per section n’) is shown. The RTD is represented by its capacitance in parallel with the nonlinear resistance provided by the N-shaped current-voltage characteristic $J(U)$. The CPW losses are considered using the longitudinal resistance $R$. The air bridge line provides a series circuit of the inductance $L$ and its losses $r$. An external dc bias circuit composed of a voltage $V_D$ and a sufficiently high resistance $R_P$ is also inserted into the circuit to emulate a dc current source to bias the RTD as shown in figure 1(c). Let us recall that the relation $S_1/S_2$ between both areas $S_1$ and $S_2$ delimited by the curve $J(U)$, determines the stable nodes of the lattice [16, 17].

Under the above considerations, the nonlinearity of the RTD is determined by the current-voltage relationship approximated by [14]

$$J(U_n) = BU_n(U_n - \alpha)(U_n - \beta)$$

where $\alpha$ and $\beta$ are positive voltage values and $B$ is an amplitude factor.

From figure 1(b), we obtain the following set of nonlinear difference-differential equations describing the voltage wave propagation along the line as

$$\begin{cases} \frac{dU_n}{dt} = \frac{1}{R}(U_{n-1} - 2U_n + U_{n+1}) - BU_n(U_n - \alpha)(U_n - \beta) - I_n \\
L \frac{di_n}{dt} = U_n - rI_n \end{cases} \quad n = 1, 2, ..., NC$$

where $U_n$, $I_n$ are the voltage at element $n$ and the resulting current through $(L, r)$, respectively and, NC is the number of elements considered.

Let us assume that in equation (2), the voltage varies slowly from one unit section to the other. In the framework of long-wavelength approximation, the discrete spatial coordinate $n$ can be replaced by a continuous
one \((n \to x)\) so that \(U_n(t) = U(x, t)\) and \(U_{n \pm 1}(t) = U(x \pm 1, t)\). Therefore, approximate continuous PDEs can be derived by using the Taylor expressions as

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \frac{1}{R} \frac{\partial^2 U}{\partial x^2} - BU(U - \alpha)(U - \beta) - I \\
\frac{L}{\partial t} &= U - rI
\end{align*}
\]

Here, in equation (3), the circuit elements \(R, L, C\) and \(r\) are values per unit length.

However, as it is more usual to work in normalized units, we choose to use the following form of transformed equations

\[
\begin{align*}
\frac{\partial u}{\partial \tau} &= D \frac{\partial^2 u}{\partial x^2} - u(u - \lambda)(u - 1) - w \\
\frac{\partial w}{\partial \tau} &= \varepsilon(u - \gamma w)
\end{align*}
\]

where \(u = \frac{U}{B}, w = \frac{I}{BP}, \tau = \left(\frac{B\beta}{C}\right) t, D = \frac{1}{BP^2}, \varepsilon = \frac{C}{BP\beta}, \gamma = rB/\beta^2\) and \(\lambda = \frac{\alpha}{\beta}\).

The system (4) is the well-known FHN system of equations, modelling action potential propagation along nerve fibers or information processes in neuronal models. The dimensionless constant \(\lambda\) is chosen in such a way that, if \((0 < \alpha < \beta/2)\), then \(0 < \lambda < 1/2\), and hence \((S_1 < S_2)\); or if \((0 < \beta/2 < \alpha < \beta)\), then \(1/2 < \lambda < 1\), and therefore \((S_1 > S_2)\).

We begin our analysis by looking for stationary traveling wave (TW) of system (4) as a function of the single variable \(\xi = x - v\tau\), i.e., \((u(\xi), w(\xi))\) is described as

\[
\begin{align*}
D \frac{d^2 u}{d\xi^2} + v \frac{du}{d\xi} - u(u - \lambda)(u - 1) - w &= 0 \\
v \frac{dw}{d\xi} + \varepsilon(u - \gamma w) &= 0
\end{align*}
\]

where \(v\) is a TW velocity.

**Figure 1.** (a) Schematic of a coplanar waveguide structure periodically loaded with RTDs and air bridges (AB); (b) Electronic equivalent circuit of a section of the CPW structure; (c) The N-shaped current-voltage characteristic of the RTD under current control using an external bias circuit \((V_p, r_p)\).
A traveling pulse is a TW that satisfies \((u, w) \to (0, 0)\) as \(\xi \to \pm \infty\). The existence of a relevant pulse, for some value of \(-v\) has been proved by many authors [20] and Refs. therein, for \(\varepsilon \ll 1\). Mathematically speaking, the system (5) can be rewritten as a system of three ODEs, supplemented by suitable asymptotic boundary conditions and one needs to find a trajectory in the phase plane that is homoclinic to the resting state [21].

Our first goal is to show that the system (4) has solutions which behave qualitatively like propagative pulses and to look for other solutions of possible physical interest. Formally speaking, a TW is a solution of a PDE on an infinite domain that travels at constant velocity and fixed shape. For 1D systems, one can distinguished two types of solitary TW: a traveling front linking a stable resting state to stable excited state and a traveling pulse that begins and ends at the resting state.

3. Traveling front waves in FHN system

It is analytically possible to see what is going on if we consider \(\varepsilon\) to be small in system (4) and this opens interesting technical applications at VHF and technical design in MMIC technologies.

Under this assumption, the FHN system (4) reduces to the so-called scalar bistable equation

\[
D \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \tau} + u(u - \lambda)(u - 1). \tag{6}
\]

Equation (6) has three constant solutions \(u^* = 0, 1\) and \(\lambda\). Of these, the first two correspond to stable equilibrium states, the latter to the unstable equilibrium state. Initially keeping the line in the stable equilibrium state \(u^* = 0\), an appropriated input applied to one end of the line will cause a transition of the state at that end from \(u^* = 0\) to the other stable equilibrium state \(u^* = 1\). The TW analysis converts the PDE (6) into the system of ODE of the first order according to

\[
u(x, \tau) = u(x - \sigma \tau) = \tilde{u}(\zeta), \quad \zeta = x - \sigma \tau, \tag{7}
\]

and the traveling front waves (kink or antikink solutions) correspond to the trajectory connecting the two saddle points on the phase space [16, 17]. This is satisfied by adjusting the TW velocity \(-\sigma\) so that a trajectory leaving one of the saddle point at \(\zeta \to -\infty\) lies on a trajectory approaching the other as \(\zeta \to +\infty\). Thus the TW speed is fixed by the condition that the solution must be physically realistic. Applying the above analysis from equation (6), the TW solution is given by

\[
v(x, \tau) = \frac{1}{1 + \exp \left[ \frac{x}{\sqrt{2D}} + \left( a \frac{1}{2} \right) \tau \right]} \tag{8}
\]

with \(a = \pm \sqrt{2}(\lambda - 1/2)\).

Setting \(D = 1\) in equation (6), the corresponding equation admits other TW solutions (see appendix) and in particularly a 'two phase' solution given as [22]
where $A$, $B$ and $E$ are arbitrary constants.

To be quantitative, we use an input signal given by \( \text{(9)} \) for $\lambda = 0.139$, $D = 0.028; A = B = E = 1$ and consider the sign `'+'` in the expressions of $Z_1$ and $Z_2$ with $\tau = 0$. Within finite differences (FD) method, a numerical solution of (6) is performed using two Dirichlet boundary conditions and the grid in $x$ is then defined over the interval $-60 \leq x \leq 20$ for 101 points. The simulation results are shown in figure 2(a), where one sees the spatial evolution of the kink-type wave and, also indicates good agreement between the analytical and numerical solutions (the successive curves are left to right for $\tau = 0, 20, 40, 60$). The wave is merely displaced along the $x$-axis. Figure 2(b) is a 3D plot of spatio-temporal evolution of the input wave.

4. Traveling pulse wave in FHN system

We recall that the bistable equation cannot support a traveling pulse solution because there is no recovery variable, that is, it does describe an excitable system. Then, we consider the whole FHN equation. This system cannot be solved analytically and nevertheless, we can calculate numerical solutions that demonstrate the

\[
 u(x, \tau) = \frac{A \exp(Z_1) + \lambda B \exp(Z_2)}{A \exp(Z_1) + B \exp(Z_2) + E}
\]
\[
 Z_1 = \pm \frac{\sqrt{2}}{2}x + \left( \frac{1}{2} - \lambda \right)\tau, \quad Z_2 = \pm \frac{\sqrt{2}}{2} \lambda x + \lambda \left( \frac{\lambda}{2} - 1 \right)\tau
\]
general utility of the computer experiments. Two different approaches to the simulation of FHN equation are considered, namely the approximated distributed and lumped-element models. In addition, the lumped version provides a good description of the structure behavior and a deep understanding of both the transmission characteristics and nonlinear properties.

Firstly, in order to model waves that travel along the structure, we consider the system (4) where an external stimulus \(i_p\) is incorporated as

\[
\begin{align*}
\frac{\partial u}{\partial \tau} &= D \frac{\partial^2 u}{\partial x^2} - u(u - \lambda)(u - 1) - w + i_p \\
\frac{\partial w}{\partial \tau} &= \varepsilon (u - \gamma w)
\end{align*}
\]

Here, a single pulse stimulus \(i_p\) at a fixed location on the structure is an externally applied current. Numerical simulations of nonlinear PDEs (4) were carried out by direct integration of PDE using FD method with Dirichlet boundary conditions and the grid in \(x\) is defined on \(0 \leq x \leq 50\) for 300 points. The dimensionless parameters are only slightly different from those used in [23]. Figure 3(a) presents a solution resulting from a single pulse stimulation applied at \(x = 25\) (normalized units). As a result, after a very short time, two stable pulse-like solutions propagating in retrograde (here right to left) and the opposite direction the anterograde one are generated on the structure. Figure 3(b) is the 3D plot of the numerical solution. This wave corresponds to the action potential, i.e., a nerve impulse travelling along the axon.

Secondly, the numerical integration of nonlinear ODEs (2) is performed under the Runge–Kutta method of fourth-order with controlled time step in order to provide the prescribed accuracy. Initial rectangular pulse of

![Figure 4](image-url)
temporal width 6 μs and amplitude 1 V is launched into the network for the study of freely propagating wave. The achievement is that, a stable propagative pulse is generated as shown in figure 4(a), similar as a relaxation oscillation. The corresponding voltage distribution is depicted in figure 4(b). One clearly notices the recovery part of the wave determined by the air bridge together with the RTD current.

5. Conclusion

In conclusion, we have shown that a TW structure based upon RTDs and air bridges in MMIC technology is a good platform emulating the FHN equations simulating the neural system. The study of traveling waves has been undertaken in the context of PDEs and which could be useful to understand several phenomena occurring in electrophysiology properties focus on pattern formation and cardiac rhythm disturbances and for potential technical applications in millimetre and THz ranges, where the RTD is dedicated. Finally, this work could benefit in the implementation of bio-inspired circuits including ultra-fast A/D converters in signal processing.

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Appendix

In this appendix we report other travelling wave front solutions of the bistable equation (6),

\((i)\) \(u(x, \tau) = \frac{A \exp \left[ \pm \frac{\sqrt{2}}{4}(1 - \lambda)x + \frac{1}{2}(1 - \lambda^2)\tau \right] + \lambda}{A \exp \left[ \pm \frac{\sqrt{2}}{4}(1 - \lambda)x + \frac{1}{2}(1 - \lambda^2)\tau \right] + 1}\)

\((ii)\) \(u(x, \tau) = \frac{1}{2} \pm \frac{1}{2} \tanh \left[ \frac{1}{4} \sqrt{2}x + \frac{1}{4}(1 - 2\lambda)\tau + A \right]\)

\((iii)\) \(u(x, \tau) = \frac{1}{2} \lambda + \frac{1}{2} \lambda \tanh \left[ \frac{1}{4} \sqrt{2} \lambda x + \frac{1}{4}(1 - \lambda - 2)\tau + A \right]\)

\((iv)\) \(u(x, \tau) = \frac{1}{2} (1 + \lambda) + \frac{1}{2} (1 - \lambda) \tanh \left[ \frac{1}{4} \sqrt{2} (1 - \lambda)x + \frac{1}{4}(1 - \lambda^2)\tau + A \right]\)

\((v)\) \(u(x, \tau) = \frac{2\lambda}{(1 + \lambda) - (1 - \lambda) \tanh \left[ \frac{1}{4} \sqrt{2} (1 - \lambda)x + \frac{1}{4}(1 - \lambda^2)\tau + A \right]}\)

\((vi)\) \(u(x, \tau) = \frac{1}{2} + \frac{1}{2} \coth \left[ \frac{1}{4} \sqrt{2}x + \frac{1}{4}(1 - 2\lambda)\tau + A \right]\)

\((vii)\) \(u(x, \tau) = \frac{1}{2} \lambda + \frac{1}{2} \lambda \coth \left[ \frac{1}{4} \sqrt{2} \lambda x + \frac{1}{4}(1 - \lambda - 2)\tau + A \right]\)

\((viii)\) \(u(x, \tau) = \frac{1}{2} (1 + \lambda) + \frac{1}{2} (1 - \lambda) \coth \left[ \frac{1}{4} \sqrt{2} (1 - \lambda)x + \frac{1}{4}(1 - \lambda^2)\tau + A \right]\)

\((ix)\) \(u(x, \tau) = \frac{2\lambda}{(1 + \lambda) - (1 - \lambda) \coth \left[ \frac{1}{4} \sqrt{2} (1 - \lambda)x + \frac{1}{4}(1 - \lambda^2)\tau + A \right]}\)

where A is arbitrary constant.

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