THE GLOBAL DERIVED PERIOD MAP

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ABSTRACT. We develop the global period map in the context of derived geometry, generalising Griffiths' classical period map as well as the infinitesimal derived period map. We begin by constructing the derived period domain which classifies Hodge filtrations and enhances the classical period domain. We analyze the monodromy action. Then we associate to any smooth projective map of derived stacks a canonical morphism of derived analytic stacks from the base into the quotient of the derived period domain by monodromy. We conclude the paper by discussing a few examples and a derived Torelli problem. In the appendix we describe how to present derived analytic Artin stacks as hypergroupoids and describe a way of constructing maps into the derived analytification. These are auxiliary results which may be of independent interest.

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1. Introduction

We construct a global derived period map, generalising Griffiths’ period map [21] and the infinitesimal derived period map of deformation theories [13, 16, 17].

We hope that this map will be a useful tool in studying derived moduli of varieties.

Given a polarized smooth projective map \( f : X \to S \) between derived Artin stacks we construct a map \( \mathcal{P} \) from the analytification \( S^{an} \) of the base to a derived analytic stack \( U \) which we call the derived period domain. The underived truncation of \( \mathcal{P} \) is a product of the usual period maps, in particular a closed point \( s \in S \) is sent to the Hodge filtration on the fiber \( X_s \). Moreover, \( \mathcal{P} \) extends the infinitesimal period map.

Our first result constructs the target of the derived period map:

**Theorem 3.11.** There is a derived period domain \( U \) which is a geometric derived analytic Artin stack that extends the classical period domain.
We define $U$ as an open subspace of the analytification of a derived stack $D_n(V, Q)$ which classifies filtrations of a complex $V$ that satisfy the Hodge-Riemann bilinear relations with respect to a shifted bilinear form $Q$. (We use analytification rather than a direct construction in analytic stacks because the theory of derived analytic stacks is not yet as developed as that for derived algebraic stacks.) The main ingredients are Lurie’s derived analytification theorem [33, 34] and Porta’s extensions thereof [41], and some explicit constructions of stacks building on work of Toën and Vezzosi [62].

In order to define maps into this stack we use a universal property for derived analytification, which we deduce from a presentation of derived analytic Artin stacks as hypergroupoids in derived Stein spaces. This theory, which is an application of work by Pridham [48], is developed in the appendix.

We then take the quotient of $U$ by an arithmetic group $\Gamma$ containing the fundamental group of the underlying topological space of $S^\text{an}$. We use Deligne’s work on formality [10] to show that we need not look at the action of the full simplicial loop group of the base but only the fundamental group.

We can now state our main theorem, which is proved using classical Hodge theory, local computations and some homotopy theory:

**Theorem 4.9.** Let $f : X \to S$ be a any polarized smooth projective map between derived Artin stacks where $S$ is of finite presentation. Then there is a derived period map $\mathcal{P} : S^\text{an} \to U/\Gamma$ of derived analytic stacks.

It then follows from the construction that $\mathcal{P}$ enhances the classical period map and the infinitesimal derived period map.

**Remark 1.1.** The study of generalised period maps goes back to work motivated by mirror symmetry [1–3].

1.1. **Outline.** We begin by recalling in Section 2 some basic notions of derived algebraic and derived analytic geometry, in particular the derived analytification functor connecting them, which was defined by Lurie and further described by Porta.

In Section 3 we construct the derived period domain $U$. After recalling the classical period domain in Section 3.1 we review the derived flag variety in Section 3.2. This derived stack classifies filtrations of a complex $V$. For us $V$ will be the cohomology of a fiber of a smooth projective morphism of derived stack. In Section 3.3 we build a geometric derived stack $D_n(V, Q)$ classifying filtrations on a complex $V$ equipped with a bilinear form $Q$, leading to the definition of $U$. The appendix develops the theory of derived analytic Artin stacks as hypergroupoids in derived Stein spaces.
satisfying the Hodge-Riemann orthogonality relation. This is a closure of the derived period domain. We compute the tangent space in Section 3.4. In Section 3.5 we construct the derived period domain $U$ itself using the analytification of $D_n(V, Q)$ and the characterization of open derived substacks in terms of open substacks of the underlying underived stack. We use the universal property of analytification to construct maps into this space. We verify $U$ is a derived enhancement of the usual period domain.

In Section 3.6 we recall Deligne’s result that the derived pushforward of the constant sheaf for a family of smooth projective varieties is formal. We deduce there is no higher monodromy acting on cohomology of the fiber. We then construct the quotient of the derived period domain by the action induced by the monodromy action on $V$.

In Section 4 we construct the derived period map. The first big step is in Section 4.2, where we construct the derived period map locally by pushing forward the relative cotangent complex and showing this gives a map to the derived period domain. We do this by reducing the question to classical Hodge theory. In Section 4.3 we glue the local period maps, using some topological arguments, and the construction is complete. The universal example is given by the moduli stack of smooth polarized schemes. We check that our map is an enhancement of the usual period map in Section 4.4. In Section 4.5 we compute the differential and show that the derived period map extends the infinitesimal derived period map. We briefly talk about examples in Section 4.6.

In Appendix A we develop the theory of hypergroupoids in derived Stein spaces as a model for derived analytic Artin stacks. We use this to construct maps into analytifications.

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We are extremely grateful to Mauro Porta for pointing out a crucial mistake in the first version of this paper and for insightful discussions of possible solutions including the proof of Lemma A.3. These discussions also lead to the joint work [26]. Finally we would like to thank Jon Pridham again for his help in extending his theory of stacks as hypergroupoids to the analytic...
case, which allowed us to fix the error. In particular the proof of Theorem A.6 is based on his ideas.

2. Derived geometry

2.1. Derived algebraic geometry. We will be assuming some familiarity with derived algebraic geometry, which is algebraic geometry that is locally modelled on the model category of simplicial commutative algebras instead of the category of commutative algebras.

There is a vast literature on the subject, developed by Toën-Vezzosi, Lurie and Pridham among others. For an introduction see [57, 59].

Here we just mention a few reminders and conventions.

We will be studying certain derived stacks. Just like a scheme can be represented by a set-valued sheaf on commutative algebras a derived stack can be represented by a simplicial-set-valued hypersheaf on simplicial commutative algebras. (We always take stack to mean higher stack.) In fact derived stacks can be described by a model structure on presheaves of simplicial sets on simplicial commutative algebras. A representable derived stack is also called a derived affine stack, and the image of a simplicial algebra \( A \) in derived stacks is denoted \( R \text{Spec}(A) \).

We will call a derived stack Artin or simply geometric if there exist certain smooth covers. Here we use geometric to mean \( k \)-geometric for some \( k \) rather than 1-geometric. The precise definition is inductive on \( k \), beginning with representable stacks, which are \((-1)\)-geometric. See for example Definition 1.3.3.1 in [62].) Replacing smooth by étale covers one obtains the definition of a Deligne-Mumford stack (we will call them DM stacks for short). A derived stack that is a union of geometric ones is called locally geometric. It is often interesting and consequential to show that certain derived stacks arising as moduli functors, say, are geometric.

Derived stacks come with simplicial algebras of functions \( \mathcal{O} \) and a map \( f : X \to Y \) between derived stacks is called strong if \( \pi_i(\mathcal{O}_X) = \pi_i((f^{-1}\mathcal{O}_Y) \otimes_{\pi_0(f^{-1}\mathcal{O}_Y)} \pi_0(\mathcal{O}_X)) \) for \( i > 0 \).

One fact we will use frequently is that \( k \)-geometric derived stacks are stable under homotopy pullbacks in derived stacks, Corollary 1.3.3.5 in [62].

As we work over \( \mathbb{C} \) the normalization functor \( N \) from simplicial commutative algebras to commutative dg-algebras concentrated in non-positive degrees is a Quillen equivalence and we will change between the two viewpoints.
One can restrict a derived stack to an underived stack along $\text{Alg} \to \text{sAlg}$. We denote this operation by $\pi^0$ (it is often written $t_0$ in the literature).

On the other hand one can truncate a stack to a functor into sets, by applying $\pi_0$ objectwise. We denote this functor by $\pi_0$. (This agrees with the truncation functor $\tau_{\leq 0}$.)

We will often consider the double truncation $\pi_0 \pi^0$ and if $\pi_0 \pi^0(\mathcal{X}) = X$ for some scheme $X$ we will say the derived stack $\mathcal{X}$ is an enhancement of $X$. (Here we deviate a little from the literature, where one typically considers enhancements of higher stacks, i.e. a derived stack $\mathcal{X}$ is an enhancement of the higher stack $X$ if $\pi^0(\mathcal{X}) = X$.)

A derived stack is locally of finite presentation if as a functor on derived rings it commutes with filtered homotopy colimits. It follows from Theorem 8.4.3.18 of [36] that a derived stack $X$ with perfect cotangent complex and such that $\pi^0(X)$ is finitely presented is locally of finite presentation.

We will be considering derived stacks taking values in simplicial sets. In moduli questions these often come from functors valued in $(\infty, 1)$-categories, and several of our constructions are clearest in terms of categories rather than simplicial sets. Our preferred models for $(\infty, 1)$-categories are dg-categories and simplicial categories. We use the right Quillen functors $DK : \text{dgCat} \to \text{sCat}$ obtained by composing truncation, the Dold-Kan construction and the forgetful functor (see [55]), to move from dg-categories to simplicial categories. (We will sometimes use the same name for associated sheaves of dg-categories and simplicial categories). To associate a simplicial set we use the functor $\bar{W} : \text{sCat} \to \text{sSet}$ defined in Definition 1.6 of [47]. It is weakly equivalent to taking the diagonal of the nerve, and the reader is welcome to think of this functor instead. Abusing notation slightly we write $N_W$ for both $\bar{W}$ and $\bar{W} \circ DK$.

$DK$ preserves homotopy limits and $\bar{W}$ preserves homotopy pullbacks of simplicial categories which are homotopy groupoids, see Proposition 1.8 in [47].

**Remark 2.1.** To reassure the reader that the construction $\bar{W}$ is the natural one we can note that if we begin with a simplicial model category $\mathcal{M}$ we can also consider its classifying space, which is just the nerve of the subcategory of weak equivalences. It follows from Corollary 1.10 in [47] and Proposition 8.7 in [50] that this simplicial set is weakly equivalent to $\bar{W} \mathcal{M}$.

**Remark 2.2.** The reader should be advised that there are different definitions of (geometric) derived stacks. We use results by Toën-Vezzosi and Lurie respectively, which a priori live in different frameworks. The framework we
have used above is Toën-Vezzosi’s, where we understand a derived stack as its functor of points.

We will also need to refer to Lurie’s approach using structured topoi. A description can be found in the next section. These are shown to give equivalent models for Deligne-Mumford stacks in [39]. It is worth noting that there is no direct approach to Artin stacks using structured topoi.

The most concrete approach to derived algebraic geometry is in terms of hypergroupoids, this is developed by Pridham in [48], where the equivalence with Toën-Vezzosi’s definitions of Artin stacks is established.

This approach is based on noting that all geometric stacks can be presented as simplicial affine schemes satisfying some technical condition. We will describe this theory in more detail when we extend it to the analytic setting in Appendix A.

There is also some care needed as there are differences in terminology, cf. Remark 2.12 in [47], but the reader can safely ignore these differences unless she or he cares about the precise value of $k$ for which a geometric stack is $k$-geometric.

2.2. Derived analytic geometry following Lurie and Porta. Since the period map is a priori holomorphic and not algebraic we have to construct its derived enhancement in the setting of derived analytic geometry. The theory of derived analytic geometry is still being developed by Lurie, Porta, and Ben-Bassat and Kremnizer.

The main difficulty is that while derived algebraic geometry is modelled on simplicial algebras, derived analytic geometry should be modelled on simplicial analytic algebras, but it is not obvious what those should be. One approach, using Ind-Banach algebras, is developed in [5].

The approach we will use is Lurie’s theory of structured spaces, see [34] and Sections 11 and 12 of [33], in particular as extended by Porta in his work around derived GAGA [40,41]. Porta and Yu have also worked out derived analytic deformation theory [44].

We will use this work largely as a black box, providing us with a good theory of derived analytic DM stacks. In the following we will give some vague explanations while giving references for precise definitions and results. We recommend that the interested reader turn to the introduction of [41] for further explanations.

The crucial object is a category $\mathcal{Top}(\mathcal{F}_{an})$ of $\mathcal{F}_{an}$-structured topoi. We can think of an object as an $\infty$-topos $\mathcal{X}$ together with a sheaf of simplicial commutative rings $\mathcal{O}_{\mathcal{X}}^{alg}$ and some extra structure. (It is fine to think of $\mathcal{X}$ as
a topological space for now, and to not think much about the extra structure at all.)

**Remark 2.3.** Derived analytic spaces have more structure than simply a sheaf of simplicial commutative algebras because one needs to keep track of the action of holomorphic functions on subsets of $\mathbb{C}^n$ by postcomposition. In the differentiable setting Spivak’s definition of simplicial $\mathcal{C}^\infty$-rings is motivated by this issue, see [54].

The objects of interest to us form the subcategory $dAn_C$ of derived analytic spaces, see Definition 1.3 in [41]. They are the enhancements of complex analytic spaces. They also contain a subcategory of derived Stein spaces, which we will denote by $dStein$. (It is called $S_{tn}^{der}$ in loc. cit.)

Together with its analytic topology and the collection of smooth morphisms $dStein$ forms a geometric context in the sense of Porta-Yu [43]. That means that to define derived analytic Artin stacks one can consider the category of simplicial sheaves on $dStein$ and use the notion of geometric stacks in the sense of [43], see Section 8 of [41].

Similarly, on the algebraic side, there is a category $\mathcal{I}op(\mathcal{I}_{\text{et}})$ of $\mathcal{I}_{\text{et}}$-structured topoi, and this category contains a full subcategory of geometric derived stacks equivalent to DM stacks in Toën-Vezzosi’s framework.

There is a forgetful functor $(-)^{\text{alg}} : \mathcal{I}op(\mathcal{I}_{\text{an}}) \rightarrow \mathcal{I}op(\mathcal{I}_{\text{et}})$ which corresponds to forgetting extra structure and considering just the algebraic object $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^{\text{alg}})$.

Then the the following hold:

- There is an analytification functor $(-)^{\text{an}}$ from $\mathcal{I}_{\text{et}}$-structured topoi to $\mathcal{I}_{\text{an}}$-structured topoi, which is right adjoint to the forgetful functor $(-)^{\text{alg}}$. This is an adjunction of functors of $\infty$-categories. See Theorem 2.1.1 in [34].
- The analytification functor sends DM stacks locally of finite presentation to derived analytic spaces, see Remark 12.2.6 in [33].
- The analytification functor can be extended to derived Artin stacks, see Section 8 of [40]. It follows from Lemma 2.35 of [43] applied to the derived context that analytification sends geometric derived algebraic stacks to geometric derived analytic stacks.
- Analytification restricts to the usual analytification on the subcategory of underived schemes, see Proposition 6.5 in [40].
- The natural comparison map $h : (X^{\text{an}})^{\text{alg}} \rightarrow X$ is flat, see Section 6.3 in [40].
• Analytification commutes with the truncations $\pi^0$ and $\pi_0$, see Section 6.6 of [40] and Lemma 2.20 of [43].

It is worth adding some explanation about analytification of Artin stacks. The analytification functor as defined by Lurie only applies to DM stacks as it depends on the notion of a structured topos. It can be turned into a functor on Artin stacks as a left Kan extension, see Section 8 of [40]. But as a left Kan extension the new functor does not have an automatic left adjoint. (This was pointed out to the authors by Mauro Porta.)

Many moduli stacks naturally occur as Artin stacks. This is true in particular for the derived period domain we study here.

In forthcoming work with Mauro Porta [26] we show that analytification is a right adjoint even on the level of Artin stacks. However, for the purposes of this paper we will sidestep this issue by proving the following property:

**Theorem [A.6]** Let $T$ be a derived analytic Artin stack and let $Y$ be a derived algebraic Artin stack locally of finite presentation. Then there is a derived stack $u(T)$ and a natural map $\text{Map}(u(T), Y) \to \text{Map}(T, Y^\text{an})$. This correspondence is natural in $T$ and $Y$. If $T$ is a derived Stein space then $u(T) = R\text{Spec}(\mathcal{O}(T))$.

As the proof involves quite a bit of machinery that is independent of the rest of the paper it has been relegated to the appendix.

**Remark 2.4.** The reader should be warned that even on derived Stein spaces $u$ does not agree with the functor $(-)^{\text{alg}}$, but is rather $T \mapsto R\text{Spec}(\Gamma(T, \mathcal{O}^{\text{alg}}))$. This was pointed out to us by Mauro Porta. Indeed, $T^{\text{alg}}$ has an underlying topological space which is Hausdorff, thus it cannot be an affine scheme.

**Remark 2.5.** We readily admit that Theorem [A.6] is less natural than the analytification adjunction and the reader may prefer to ignore the appendix and refer to the results in [26] instead once they are available.

Next we need to recall some characterizations of sheaves on derived analytic stacks.

**Definition 4.1 in [40]** defines a category of coherent sheaves on a derived analytic space $X$: First one takes the $\infty$-topos $\mathcal{X}$ associated to $X$ and observes there is a category of $\mathcal{O}^{\text{alg}}_{\mathcal{X}}$-modules. The category of coherent sheaves is the subcategory of modules $\mathcal{F}$ whose cohomology sheaves are locally on $\mathcal{X}$ coherent sheaves of $\pi_0(\mathcal{O}^{\text{alg}}_{\mathcal{X}})$-modules. This is entirely analogous to the algebraic case.
Let us unravel this definition. First we make the definition of a derived analytic space a little more precise. Recall the \( \infty \)-topos of a topological space \( T \) can be thought of as the \((\infty, 1)\)-category of sheaves of spaces on \( T \). A model for its hypercompletion is given by the model category of simplicial presheaves on \( T \) (cf. Remark 6.5.0.1 in [35]). But a derived analytic space is always hypercomplete, see Lemma 3.2 of [40]. So locally we are just working with the model category of simplicial sets on a topological space. The space is then equipped with a sheaf of simplicial rings \( \mathcal{O}_X^{\text{alg}} \). (Plus the extra structure that does not affect these definitions).

Many of the derived analytic spaces in this paper will actually have an underlying topological space, so this description applies globally. (This is also true in the algebraic setting for derived schemes.) There will be a brief discussion about sheaves on stacks before Corollary [3.8].

In general, \( \mathcal{O}_X \)-modules are \( \mathcal{O}_X \)-module objects in sheaves on an \( \infty \)-topos \( \mathcal{X} \), cf. Section 2.1 of [33]. As we are working over \( \mathbb{C} \) and assuming we have an underlying topological space we can model this by the model category of sheaves of chain complexes over \( N(\mathcal{O}_X^{\text{alg}}) \) (using the normalization functor \( N \)).

There are cohomology groups defined in this setting, which just come down to the usual cohomology groups, see Definition 7.2.2.14 in [35]. There is a \( t \)-structure on \( \text{Coh}(X) \) and the heart satisfies \( \text{Coh}^\heartsuit(X) \approx \text{Coh}^\heartsuit(\pi^0(X)) \).

**Remark 2.6.** This definition of coherent sheaves may look naive, but Porta in [41] shows that \( \mathcal{O}_X^{\text{alg}} \)-modules are equivalent to a certain more natural category \( S p(\mathcal{S}tr^\text{loc}(X) / \mathcal{O}_X) \).

The cohomology groups can also be used to define bounded below complexes. Note however that \( \mathcal{O}_X^{\text{alg}} \) is not necessarily bounded below. As in the underived case there is an analytification functor on sheaves, provided by pullback along the natural map \( h_X : (X^{\text{an}})^{\text{alg}} \to X \), which sends bounded below coherent sheaves to bounded below coherent sheaves, see [40].

Finally, in order to study tangent spaces in Section [4.5] we will need two results which are as yet unpublished, but will soon appear. Namely, that derived analytic Artin rings are equivalent to derived Artin rings, which will be proven in [42], and that for an Artin stack \( X \) the tangent complex of \( X^{\text{an}} \) is the same as the tangent complex of \( X \), which follows from the above together with the results in [26].

3. THE DERIVED PERIOD DOMAIN
3.1. **The classical period domain.** In this section we construct the target of the period map, the (polarized) derived period domain. The derived period domain is a derived analytic stack enhancing the usual period domain. It will classify filtrations of a complex $V$ equipped with a bilinear form $Q$ which satisfy the Hodge-Riemann bilinear relations.

For background we will recall the construction of the period domain as a subspace of the flag variety in the underived case.

Assume we are given a smooth projective family of varieties $X \rightarrow S$ with fibre $X_s$. The polarized period domain is the moduli of Hodge filtrations on $H^k(X_s, \mathbb{C})$, it is a subspace of the flag variety of $H^k$. To be precise we are given a vector space $H^k = H^k(X_s, \mathbb{Z}) \otimes \mathbb{C}$ with an integral structure and a bilinear form $Q_k(\alpha, \beta) = \langle L^{n-k}\alpha, \beta \rangle$ (here $L$ is the Lefschetz operator and $\langle \cdot, \cdot \rangle$ is the intersection form $\langle \alpha, \beta \rangle = \int \alpha \cup \beta$). Note $Q_k$ is bilinear, symmetric in even degrees and anti-symmetric in odd degrees. These vector spaces are identified for all $s$ as they are diffeomorphism invariants by Ehresmann’s theorem.

**Remark 3.1.** $Q_k$ should not be confused with the hermitian form $Q_k(\alpha, \bar{\beta})$.

Now the polarized period domain is an open subset of a closed subset of the flag, given by the conditions below, which for ease of reference we call the Hodge-Riemann bilinear relations. (This is a slight abuse of language, more correctly the term is only applied to the first and third condition.)

1. $F^p H^k = (F^{k-p+1}H^k)^\perp$ with respect to $Q_k$.
2. $H^k = F^p H^k \oplus F^{k-p+1}H^k$.
3. On $H^p_{prim} = F^p H^k_{prim} \cap F^q H^k_{prim}$ we have $(-1)^{k(k-1)/2} i^{p-q} Q_k(\alpha, \bar{\alpha}) > 0$.

Here $q = k - p$ and $H^k_{prim}$ is the primitive cohomology.

The signature of $Q_k$ on the non-primitive parts of cohomology is easily worked out.

The first condition describes an algebraic subset in the flag variety. We call the closed subset given by condition (1) only the closure of the period domain.

The other two conditions are (analytic) open on the filtrations satisfying the first condition.

**Remark 3.2.** One can view the period domain in two ways: As an open subspace of a closed subspace of a flag, as above, or as a homogeneous space for the group of symmetries, roughly $\text{Aut}(H^k, Q)/\text{Aut}_F(H^k, Q) \subset \text{Aut}(H, Q)/\text{Aut}_F(H, Q)$. Note that this is a quotient of real groups, which
is then shown to be complex as an open subspace of a quotient of complex groups.

This approach seems harder to imitate in the derived setting, which is why our starting point is the flag variety.

Note that the tangent space of the period domain is given by $\text{End}/F^0(\text{End})$, where we write $\text{End}$ for the Lie algebra of endomorphisms of $H^k$ compatible with the pairing, filtered in the usual way, i.e. $F^0(\text{End})$ consists of filtered endomorphisms. See Section 1 of [8]. The image of the period map has tangent space contained in the horizontal tangent space, given by $F^{-1}(\text{End})/F^0(\text{End})$, as a consequence of Griffiths transversality.

To construct the period map for a family $X \to S$ one has to take the quotient of the period domain by the monodromy group $\pi_1(S)$. Note that $\pi_1(S)$ is a subgroup of $\text{Aut}(H^*_\mathbb{Z},Q)$. It acts on integral cohomology, hence the $\mathbb{Z}$-coefficients, and it preserves the bilinear form.

The fundamental group acts properly discontinuously on the period domain.

3.2. **The derived flag variety.** We will now start constructing the derived period domain. We will build it step by step starting from moduli stacks of perfect complexes and filtered complexes.

For simplicity we will define all our stacks on the model category of non-positively graded commutative dg-algebras. We write $A$ for an arbitrary commutative dg-algebra in non-positive degrees. We will consider its model category of dg-modules with the projective model structure (all objects are fibrant).

Before we start with the constructions let us first recall the following useful fact which we will use repeatedly:

**Lemma 3.1.** Suppose we are given an (analytic or algebraic) geometric derived stack $\mathcal{X}$. Then for any open substack $U$ of $\pi^0(\mathcal{X})$ there is a unique geometric derived stack $\mathcal{U}$ with $\pi^0(\mathcal{U}) = U$ that is an open substack of $\mathcal{X}$.

**Proof.** See Proposition 2.1 in [51] for the algebraic case and Proposition 3.16 in [40] for the analytic case. The latter result is for DM stacks, but the proof goes through for Artin stacks.

The heart of both of these propositions is that $\pi^0$ induces an equivalence of sites between $X_\text{et}$ and $(\pi^0X)_\text{et}$, i.e. between étale maps to $X$ and to $\pi^0(X)$. Open immersions are just étale maps which are monic. $\square$
Hence open conditions can be dealt with easily by imposing open conditions on the underlying underived space.

Now we recall our two main building blocks. The following are locally geometric derived stacks locally of finite presentation.

- **Perf**, the moduli stack of perfect complexes. **Perf**(A) is the simplicial set associated to the category of perfect complexes over A. For convenience recall a construction:

  Let **Perf** be the sheaf of dg-categories on dg-algebras that sends A to the dg-category of cofibrant A-modules quasi-isomorphic to perfect modules. We then restrict the morphisms to allow only the ones that become isomorphisms in the homotopy category. Here we call an A-module M perfect if M \otimes_A H^0(A) is homotopy equivalent to a perfect complex of A-modules. Then **Perf**(A) := N_W(\(\mathcal{P}erf(A)\)). (In the literature this is often denoted \(\mathbb{R}Perf\).)

  For a proof that this is a locally geometric derived Artin stack locally of finite presentation, see [14]. Note that this stack is equivalent to the construction of \(\mathcal{M}_1\) in [60], where a perfect object is defined in the more well-known way as a compact object in the homotopy category.

- **Filt\(_n\)**, the stack of filtered perfect complexes of filtration length \(n + 1\). Explicitly, we let **Filt\(_n\)**(A) be the dg-category of sequences \(F^0 \rightarrow \ldots \rightarrow F^1 \rightarrow F^0\) of injections of cofibrant A-modules quasi-isomorphic to perfect modules. Morphism complexes Hom\(_\mathcal{F}\)(\(F^\ast\), \(G^\ast\)) are given by compatible maps \(F^i \rightarrow G^i\) (which are determined on \(F^0\)) and which are invertible in the homotopy category. We will sometimes write this as \(\text{Hom}_\mathcal{F}(F^0, G^0)\).

  That **Filt\(_n\)** = N_W(\(\mathcal{F}ilt\(_n\)\)) is locally geometric and locally of finite presentation can be deduced from Theorem 2.33 and Remark 2.34 in [14], where it is shown that \(\cup_n \mathcal{F}ilt\(_n\)\) and \(\mathcal{F}ilt\(_1\)\) are locally geometry. (Note that the \(n\) we are using here is different from the \(n\) in Di Natale’s \(\mathcal{F}ilt\(_n\)\).)

The next stack also appears in [14], but we will revisit the construction to give an explicit description of the mapping spaces.

**Lemma 3.2.** For a complex V consider \(\mathcal{F}lag^n(V)(A) = \text{holim}(\mathcal{F}ilt\(_n\)(A) \Rightarrow \mathcal{P}erf(A))\), where the two maps are induced by the forgetful functor \(\pi : F^\ast \mapsto F^0\) and the constant functor \(V : F^\ast \mapsto V\). This is the dg-category of pairs \((F^\ast, w_F)\) where \(F^\ast\) is an object of **Filt\(_n\)**(A) and \(w_F : F^0 \mapsto V \otimes A\) is a homotopy equivalence in **Perf**(A).
The morphism complex is given by

$$\text{Hom}((F^*, w_F), (G^*, w_G)) = (\text{Hom}_F(F^0, G^0) \oplus \text{Hom}(F^0, A[1]), \Delta)$$

where $\Delta: (f, h) \mapsto (df, dh + w_G \circ f - w_F)$.

**Proof.** To show that the characterization of the homotopy limit is correct we replace $\mathcal{F}ilt_n$ by $\mathcal{F}ilt^*_n$, which has objects consisting of triples $(F^*, M_F, m_F)$ where $F^*$ is an object in $\mathcal{F}ilt_n$ and $m_F: F^0 \to M_F$ is a homotopy equivalence in $\mathcal{P}erf$. Morphism complexes are given by

$$(\text{Hom}_{\mathcal{F}ilt}(F^0, G^0) \oplus \text{Hom}(M_F, M_G) \oplus \text{Hom}(F^0, M_G)[1], \Delta)$$

where $\Delta: (f, g, h) \mapsto (df, dg, dh + g \circ m_F - m_G \circ f)$.

The natural inclusion is a quasi-equivalence and the map $\mathcal{F}ilt^*_n \to \mathcal{P}erf$ is a fibration. This follows from the standard arguments used in the construction of the path space in dg-categories, see Section 3 of [56].

As dg-categories (with the Dwyer-Kan model structure) are right proper we can compute the homotopy limit as the limit of $\mathcal{F}ilt_n \to \mathcal{P}erf \leftarrow \ast$, which is the category described in the Lemma. \qed

**Lemma 3.3.** The derived flag variety $D\text{Flag}_{n}(V)$ is defined as $N_W$ of the substack $\mathcal{F}\text{lag}_{n}(V)'$ that is given by filtrations such that the maps $H^k(F^{-1} \otimes_A H^0(A)) \to H^k(F^{-i} \otimes_A H^0(A))$ are injections of flat $H^0(A)$-modules. It is geometric.

**Proof.** We first note that $N_W(\mathcal{F}lag'_n(V))$ is the homotopy limit of $\mathcal{F}ilt_n \Rightarrow \mathcal{P}erf$ as $N_W$ commutes with homotopy fibre products of simplicial categories whose homotopy categories are groupoids. As a homotopy pullback of locally geometric stacks it is locally geometric.

To see the substack is locally geometric, use Proposition 2.44 in [14].

We now want to refine this result and show that $D\text{Flag}_{n}(V)$ is geometric. We let $k - 1$ be the amplitude of the complex $V$ and will show that $D\text{Flag}_{n}(V)$ is $(k + 2)$-geometric. We introduce the $k$-geometric stack $\mathcal{F}ilt^*_n(A)$ which classifies filtrations $F^*$ which satisfy that $\text{Ext}^i_{\mathcal{F}ilt_n(H^0(A))}(F^* \otimes_A H^0(A), F^* \otimes_A H^0(A)) = 0$ for $i < -k$. This is a derived geometric $k$-stack by Theorem 2.33 of [14], and thus is $(k + 2)$-geometric. Now we check that all the points in $D\text{Flag}_{n}(V)$ map to $\mathcal{F}ilt^*_n(A) \subset \mathcal{F}ilt_n(A)$. We need to compute Ext-groups for filtered complexes. We do this by considering the Rees construction $@F^i$, cf. [14]. Then $\text{Ext}^i_{\mathcal{F}ilt(H^0(A))}(F^*, F^*) = \text{Ext}^i_{H^0(A)[t]}(\otimes F^i, \otimes F^i)_{G_m}$. As $G_m$ is reductive we can ignore it when checking vanishing of Ext-groups. All $F^i \otimes_A H^0(A)$ have amplitude bounded by the amplitude of $V$. Moreover, they are projective, thus there is a two-term resolution of $\otimes F^i$ by projective...
$R[t]$-modules (see e.g. Theorem 4.3.7 in [65]). Thus the Ext groups vanish in degrees less than $k$.

It follows that we can construct $DFlag_n(V)$ as an open substack of the homotopy fibre of $Filt_n^k \to Perf^k$, where $Perf^k$ is defined analogously to $Filt_n^k$. Thus $DFlag_n(V)$ is $(k + 2)$-geometric. □

Remark 3.3. We choose the condition that the $F_{i-1} \to F_i$ are injective for convenience, it may be dropped without affecting the results.

Moreover one can show that the derived Grassmannian $DFlag_1(V)$ is an enhancement of the usual Grassmannian, i.e. $\pi_0 DGr(V) = Gr(\oplus H^i(V))$, see Theorem 2.42 in [14], and similar for the derived flag varieties.

The tangent complex of $DFlag_n(V)$ at $(F^*, w_F)$ is computed by the homotopy limit of the tangent complexes:

$$T_{F^*, w_F} DFlag(V) = \text{cone}((\chi, 0) : \text{holim}(End_{F^*}(F^0) \Rightarrow End(F^0))[1]$$

Here $End_{F^*}(V)$ is the subcomplex of morphisms that respect the filtration, $\chi$ is the inclusion and 0 is the constant zero map. For this and similar cones we will write $End(F^0)/End_{F^*}(F^0)$

3.3. The derived period domain I: Algebraically. As the next step towards constructing the derived period domain we construct in this section a geometric derived algebraic stack $D$ that classifies filtrations with a bilinear form which satisfy only the Hodge-Riemann orthogonality condition. The derived period domain $U$ will then be the open substack of $D^m$ determined by the second and third Hodge-Riemann condition (positivity).

We fix a non-negative integer $n$ and a complex $V = V_Q \otimes \mathbb{C}$ concentrated in degrees 0 to $2n$ with a symmetric bilinear map $Q : V \otimes V \to \mathbb{Q}[2n]$ that is non-degenerate on cohomology, i.e. $\nu \mapsto Q(\nu, -) : V \to V^\vee[2n]$ is a quasi-isomorphism. Clearly this map can be extended by multiplication to a bilinear map on $V \otimes A$.

Given an $A$-module $W$ we call a map $Q : \text{Sym}^2 W \to A[k]$ for some integer $k$ a shifted bilinear form. $Q$ is called non-degenerate $W$ if the associated map $q : W \to W^\vee[2n]$ is a weak equivalence.

We want to consider the moduli stack $D_n(V, Q)$ of filtrations $F^*$ on the complex $V$ such that $F^i$ and $F^{n-i+1}$ are orthogonal with respect to $Q$.

Theorem 3.4. Let $(V, Q)$ be as above. Then there is a geometric derived stack $D = D_n(V, Q)$ which classifies filtrations of $V$ of length $n+1$ that satisfy
the Hodge-Riemann orthogonality relation with respect to \( Q \). \( D \) enhances the closure of the classical period domain.

Here \( D = N_W(D_n(V, Q)) \) where \( D_n(V, Q)(A) \) is a certain simplicial category whose homotopy category is given as follows: Objects are given by triples \((F^*, w_F, Q_F)\) where

- \( F^* \) is a filtration of perfect \( A \)-modules of length \( n + 1 \),
- \( Q_F \) is a non-degenerate \( 2n \)-shifted bilinear form on \( F^0 \) that vanishes on \( F^i \otimes F^{n+1-i} \) for all \( i \),
- \( w_F : (F_0, Q_F) \simeq (V \otimes A, Q) \) is an isomorphism in the homotopy category of perfect complexes with shifted bilinear form,

such that \( F^* \) gives filtrations on cohomology after tensoring over \( A \) with \( H^0 \) and all \( H^j(F^i \otimes_A H^0(A)) \) are flat.

Morphisms from \((F^*, w_F, Q_F)\) to \((G^*, w_G, Q_G)\) are given by families of morphisms \( F^i \to G^i \) in the homotopy category of \( \text{Filt}_n(A) \) that are compatible with \( w_\cdot \) and \( Q_\cdot \).

**Remark 3.4.** Not that just as in the underived case the real structure on \( V \) only becomes relevant when considering the positivity condition for the period domain.

We will construct \( D_n(V, Q) \) by adding the data of a quadratic form to \( \text{Flag}_n(V) \). As a warm-up and for use in later sections we first prove the following result about a stack constructed by Vezzosi in \([63]\):

**Lemma 3.5.** There is a locally geometric derived stack of perfect complexes with a \( 2n \)-shifted non-degenerate bilinear form, denoted \( Q\text{Perf} \).

Moreover \( Q\text{Perf}(A) = N_W(D\text{Perf}(A)) \) for a simplicial category \( D\text{Perf}(A) \) whose objects are given by perfect complexes \( W \) over \( A \) with a bilinear form \( Q : \text{Sym}^2 W \to A[2n] \) that is non-degenerate. The morphism space \( \text{Map}(D\text{Perf}(A)((W, Q), (W', Q')) \) is given by the homotopy fibre of the map \( f \mapsto Q' \circ \text{Sym}(f) : \text{Map}(D\text{Perf}(A))(W, W') \to \text{Map}(\text{Sym}^2 W, A[2n]) \) over \( Q \).

**Proof.** This is the derived stack that is constructed in Section 3 of \([63]\) and denoted by \( Q\text{Perf}^{\text{nd}}(2n) \) there. We recall the construction and prove the properties we need.

We use simplicial categories as our model for \((\infty, 1)\)-categories. First we define \( Q\text{Perf}' \) using the sheaf of simplicial categories \( D\text{Perf}' \) defined via
the following homotopy pullback diagram.

\[
\begin{array}{ccc}
\mathcal{Q}\mathcal{P}er f'(A) & \longrightarrow & \mathcal{P}er f(A)^\Delta^1 \\
\downarrow & & \downarrow \text{ev}_0 \times \text{ev}_1 \\
\mathcal{P}er f(A) \times \text{Sym}^2_{\Delta^1} & \longrightarrow & \mathcal{P}er f(A) \times \mathcal{P}er f(A)
\end{array}
\]

Here \((-)^\Delta^1\) denotes the functor category from the two object category with one morphism, \(* \to *\). Then \(\text{ev}_0 \times \text{ev}_1\) denotes the product of evaluation maps. \(\text{Sym}^2\) is the functor \(V \mapsto k \otimes_{k[\Sigma_2]} (V \otimes_{\Lambda_2} V)\), where the \(\Sigma_2\)-action is given by swapping factors.

To compute the homotopy limit we need to replace \(\mathcal{P}er f\Delta^1 \to \mathcal{P}er f \times \mathcal{P}er f\) by a fibration. We consider the dg-category \(\mathcal{P}er f\Delta^1\) which replaces morphisms in \(\mathcal{P}er f\Delta^1\), which by definition are commuting squares, by homotopy commutative squares, i.e., in \(\mathcal{P}er f\Delta^1\) we have \(\text{Hom}(f : K \to L, g : M \to N) = (\text{Hom}(K, M) \oplus \text{Hom}(L, N) \oplus \text{Hom}(K, N)[1]), \Delta\) where \(\Delta : (x, y, h) \mapsto (dx, dy, dh + y \circ f - g \circ x)\). As in \(\mathcal{P}er f\) we only consider morphisms which become invertible in the homotopy category. Then we apply the Dold-Kan functor which preserves fibrations.

Unravelling definitions the limit is the category of perfect complexes with a bilinear form. The objects are given by objects \(W \in \mathcal{P}er f(A)\) and \(Q : \text{Sym}^2 W \to A[2n]\) in \(\mathcal{P}er f(A)^\Delta^1\). Morphisms from \((W, Q_W)\) to \((U, Q_U)\) are given by the fibre of \(f \mapsto Q' \circ \text{Sym}^2(f) : \text{Map}(W, U) \to \text{Map}(\text{Sym}^2 W, A[2n])\) over \(Q\).

We apply \(N_W\) to \(\mathcal{Q}Per f'\) and call the resulting derived stack \(QPer f'\). We know that \(N_W\) respects this homotopy limit. Thus by Lemma 3.6 below \(QPer f'\) is a homotopy limit of locally geometric stacks and thus locally geometric.

The condition that \(Q\) is non-degenerate is easily seen to be open. (It is enough to check on \(\pi_0\).) Thus the substack of non-degenerate bilinear forms \(QPer f(2n) \subset QPer f'(2n)\) is also locally geometric.

It is clear from these definitions that \(\pi_0QPer f(2n)(A)\) consists of isomorphism classes of perfect complexes over \(H^0(A)\) with a non-degenerate quadratic form. \(\square\)

**Remark 3.5.** The reason we have to work with simplicial categories is of course that \(f \mapsto \text{Sym}^2(f)\) is not a linear map.

**Lemma 3.6.** \(N_W(\mathcal{P}er f^\Delta^1)\) is a locally geometric stack.
Proof. We can see this by identifying our stack with the linear stack associated to the perfect complex $E = \mathcal{H}om(p_1^* \mathcal{U}, p_2^* \mathcal{U})^*$ on the locally geometric stack $\text{Perf} \times \text{Perf}$. Here $\mathcal{U}$ is the universal complex on $\text{Perf} \times \text{Perf}$. Toën defines linear stacks in Section 3.3 of [59]. For a derived Artin stack $X$ with quasi-coherent complex $E$ we define the stack $V_E$ as a contravariant functor on derived schemes over $X$ by associating $V_E(u) = \text{Map}_{\text{Lqcoh}(S)}(u^*E, O_S)$ to $u : S \to X$. Now $V_E$ is locally geometric whenever $E$ is perfect, the proof follows from Sublemma 3.9 in [60].

To show this agrees with our construction above, we consider the definition of $\text{Perf}^{\Delta^1}$ locally. Over $(V, W) : S \to \text{Perf} \times \text{Perf}$ we have:

$$\text{Map}_{\text{Perf} \times \text{Perf}}(V, W, \text{Perf}^{\Delta^1}) = N_{W}(\text{Perf}^{\Delta^1}(S)) \times \text{Perf}(S) \times \text{Perf}(S) \{ (V, W) \}$$

On the other hand $\mathcal{V}_E((V, W)) = \text{Map}_{S}(\mathcal{H}om(V, W)^*, O_S)$. Both of these compute the mapping space from $V$ to $W$ in $\text{Perf}(S)$. (The characterization in terms of the fibre product can for example be found in Sections 1.2 and 2.2 of [35].) □

Remark 3.6. There is another explicit description of $\mathcal{Q}\text{Perf}$: as the linear stack associated to the perfect complex $\mathcal{H}om(\text{Sym}^2 \mathcal{U}, O[2n])$ on $\text{Perf}$, following the same arguments as Lemma 3.6.

To show this agrees with our construction above, consider the definition of $\mathcal{Q}\text{Perf}$ locally. (Looking at the simplicial set, not the dg-category.) Over $V : S \to \text{Perf}$ we have:

$$\text{Map}_{\text{Perf}}(V, \mathcal{Q}\text{Perf}) = \text{Map}(S, \mathcal{Q}\text{Perf}) \times_{\text{Map}(S, \text{Perf})} \{ V \}$$

$$= \left( \text{Perf}(S) \times_{(\text{Perf} \times \text{Perf})(S)} \text{Perf}^{\Delta^1}(S) \right) \times_{\text{Perf}(S)} \{ V \}$$

$$= \{ (\text{Sym}^2 V, O) \} \times_{\text{Perf} \times \text{Perf}} \text{Perf}(S)^{\Delta^1}$$

And the last line is precisely the mapping space from $\text{Sym}^2 V$ to $O$ in $\text{Perf}(S)$, i.e. it is $\mathcal{V}(\text{Sym}^2(\mathcal{U}))(V)$.

Applying shift and dual does not affect the argument, hence we are done.

We can use the same technique as in Lemma 3.5 to consider filtered complexes with a bilinear form:

Lemma 3.7. There is a locally geometric derived stack $\mathcal{Q}\text{Filt}_n$ of perfect complexes with an $(n + 1)$-term filtration and a non-degenerate bilinear form compatible with the filtration.

To be precise $\mathcal{Q}\text{Filt}_n = N_{W}(\mathcal{Q}\text{Filt}_n)$ for a simplicial category $\mathcal{Q}\text{Filt}_n$ whose objects are filtrations of perfect $A$-modules $F^n \to F^{n-1} \to \ldots \to F^0$.
together with a non-degenerate bilinear form $Q : \text{Sym}^2 F^0 \to A[2n]$ such that $Q$ vanishes on the image of $F^i \otimes F^{n+1-i}$ for all $i$.

The morphism space $\text{Map}_{\mathcal{D}(\text{Perf}(A))}(\{F^*, Q_F\}, \{G^*, Q_G\})$ is given by the fibre of the map $f \mapsto Q_G \circ \text{Sym}(f) : \text{Map}_{\mathcal{Filt}}(F^*, G^*) \to \text{Map}_{\mathcal{Filt}}(\text{Sym}^2 F^*, A[2n])$ over $Q_F$.

**Proof.** We proceed like in the proof of Lemma 3.5 and consider the homotopy pullback diagram

$$
\begin{align*}
\mathcal{D}\mathcal{Filt}_n(A) & \to \mathcal{Filt}(A)^{A^1} \\
\downarrow & \downarrow \text{ev}_0 \times \text{ev}_1 \\
\mathcal{Filt}_n(A) \times \mathcal{Filt}_1(A) \times \mathcal{Filt}_1(A) & \to \mathcal{Filt}_n(A) \times \mathcal{Filt}_1(A)
\end{align*}
$$

Here the map $\mathcal{F}$ sends $F^n \to \ldots \to F^0$ to $\mathcal{F}_F$ as $F_0 \to \ldots \to F^0$.

We replace the right vertical map by a fibration in the same manner as in Lemma 3.5.

Then we see that the objects of $\mathcal{D}\mathcal{Filt}_n(A)$ are filtrations with a bilinear form that vanishes on $\mathcal{F}_F$, i.e. satisfies Hodge Riemann orthogonality. Similarly we write down the mapping spaces.

Again we see that $Q\mathcal{Filt}_n = N_W(\mathcal{D}\mathcal{Filt}_n)$ is a homotopy limit of locally geometric stacks and thus locally geometric. We can show $N_W(\mathcal{Filt}_1^{A^1})$ is locally geometric by an analogue of Lemma 3.6.

Again the substack of non-degenerate forms $Q\mathcal{Filt}_n$ is an open substack. □

**Remark 3.7.** A natural way to understand the orthogonality relation is to consider the map $q : F^0 \to (F^0)^\vee$ induced by $Q$. The filtration $F^*$ induces a filtration $(F^0/F^i)^\vee$ and the orthogonality says precisely that $q$ respects this filtration.

We are now ready to construct the stack $D_n(V, Q)$.

**Proof of Theorem 3.4.** To define $D_n(V, Q)$ we again repeat the construction of Lemma 3.5 this time with $D\text{Flag}$ in place of $\text{Perf}$.

We recall that the homotopy fibre over $V$ of the projection map $\mathcal{Filt}_k \to \mathcal{Perf}$ is given by $\mathcal{Flag}_k(V)$. While it is possible to compute the homotopy limit directly, the homotopy category of the homotopy fibre product can be computed more efficiently using (the proof of) Corollary 1.12 in [47].
Assume we are given a diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ of simplicial categories such that all their homotopy categories $\pi_0(\mathcal{A})$ etc. are groupoids. Then the cited result says we can compute $\pi_0(\mathcal{A} \times_{\mathcal{B}}^{(2)} \mathcal{C})$ as $\pi_0(\mathcal{A}) \times_{\pi_0(\mathcal{B})}^{(2)} \pi_0(\mathcal{C})$.

Here the 2-fibre product of categories is defined to have as objects triples $(a \in \mathcal{A}, c \in \mathcal{C}, \omega : F(a) \cong G(c))$ and morphisms from $(a, c, \omega)$ to $(a', c', \omega')$ are just given by morphisms $f : a \to a'$ and $g : c \to c'$ such that $G(g) \circ \omega = \omega' \circ F(f)$.

We consider first the homotopy fibre of the map of simplicial categories $\text{Filt}^{A^1} \to \text{Perf}^{A^1}$ over $Q : \text{Sym}^2 V \to A[2n]$. We call it $\text{Flag}_1^{A^1}(Q)$, abusing notation and note it is geometric by imitating the proof of Lemma 3.3.

By the above, objects of the homotopy category can be written as diagrams

$$
\begin{array}{ccc}
M^0 & \xrightarrow{w_M} & \text{Sym}^2 V \\
\downarrow & & \downarrow \\
N^0 & \xrightarrow{w_N} & A
\end{array}
$$

where the left hand square commutes strictly and the right hand square is a commutative diagram in the homotopy category.

$\mathcal{D}'(V, Q)$ is defined as the homotopy pullback of the following diagram (suppressing $A$ from the notation):

$$
\begin{array}{ccc}
\mathcal{D}'(V, Q) & \longrightarrow & \text{Flag}_1(Q)^{A^1} \\
\downarrow & & \downarrow \\
\text{Flag}_n(V) & \xrightarrow{\tau \times (0 \to A[2n])} & \text{Flag}_1(\text{Sym}^2 V) \times \text{Flag}_1(A[2n])
\end{array}
$$

We use the same tool again to compute $\mathcal{D}'$ as the 2-fibre product. We find that its homotopy category has objects given by triples $(F^*, w_F, Q_F)$ consisting of a filtration $F^*$, an isomorphism $w_F : F^0 \to V \otimes A$ in the homotopy category and a symmetric bilinear form $Q_F : \text{Sym}^2 F^0 \to A$ compatible with $Q$ that vanishes on $\mathcal{J}_f \subset \text{Sym}^2(F^0)$.

Morphisms are just homotopy commutative collections of objectwise morphisms.

Then $D' := N_W(\mathcal{D}')$ is geometric as a homotopy limit of geometric stacks.
We define $D(A)$ to be the subcategory of $D'(A)$ whose objects satisfy that $H^j(F^i \otimes_A H^0(A)) \to H^j(F^{i-1} \otimes_A H^0(A))$ are injections of flat $H^0(A)$-modules. These are open conditions on $D'$, the proof follows as in Proposition 2.41 of [14].

It remains to check that $D$ enhances the period domain, i.e. that $\pi_0 \pi_0(D)(A)$ is a product of the usual period domains.

To do this we assume $A$ is an algebra (considered as a constant simplicial algebra). We observe that the objects of $D(A)$ are formal, i.e. any $(F^\bullet, w_F, Q_F)$ is equivalent to some filtration on the complex $V \otimes A$ with zero differential. As the complex $F^i$ is perfect and the cohomology groups of the $F^i$ are flat they are projective and the complexes are formal. Thus we get a homotopy equivalence from $F^n \to \ldots \to F^0 \cong V \otimes A$ to $H(F^n) \to \ldots \to H(F^0) = V$, and these are the objects parametrized by the product of closed period domains.

**Remark 3.8.** We note that we can equivalently define $D'(A)$ as the homotopy limit holim((π, V) : $\mathcal{D} Fil_{\mu}(A) \Rightarrow \mathcal{D} Perf(A)$). The arrows are induced by the forgetful functor $\pi : (F^\bullet, Q_F) \leftrightarrow (F^0, Q_F)$ and the constant functor that sends any filtration to $(V, Q)$.

As homotopy limits commute this description as a homotopy fibre at $(V, Q)$ of the projection $\mathcal{D} Fil \to \mathcal{D} Perf$ of homotopy limits is the same as the homotopy pullback of homotopy fibres over $V$ and $Q$ that we used above.

To construct the period map we need to work with perfect complexes not just on derived affines but on general derived stacks. Given a description of the $\infty$-category $\mathcal{D} Coh(U)$ of complexes of quasi-coherent sheaves on any derived scheme $U$ the most natural global definition is that $\mathcal{D} Coh(X) = \text{holim}_i \mathcal{D} Coh(U_i)$ where $U_i$ is a simplicial derived scheme whose realization is the derived stack $X$. As $\mathcal{D} Coh$ is a stack (see for example [62]) this is well-defined.

For our rather explicit approach we will need that sheaves on a stack can be modelled by a (subcategory of a) model category. This can for example be achieved as follows. Given a cover $U_i$ as above consider $QCoh(U_i)$ to be the simplicial model category of complexes of quasi-coherent sheaves on $U_i$, such that $QCoh^{cf}$ is a model for $\mathcal{D} Coh$. Any $U_i \to U_j$ induces a Quillen adjunction. These data form a left Quillen presheaf $i \mapsto QCoh(U_i)$.

We define a category such that an object $G \in QCoh(S)$ is given by the following data: For every $i$ there is $G_i \in QCoh(U_i)$ and for every map $\sigma_{ij} : U_i \to U_j$ there is a comparison map $\phi_{ij} : G_i \to \sigma_{ij}^* G_j$. Morphisms are given by strictly compatibly families of maps. This is the category of
presections with the injective model structure where cofibrations and weak equivalences are defined levelwise. Then we consider those objects $G_\bullet$ such that all $\phi_{ij}$ are weak equivalences. These are called the homotopy cartesian sections and we consider the category of homotopy cartesian sections which are moreover fibrant and cofibrant. This simplicial category is a model for $\mathcal{D}^\text{coh}(S)$.

This result is called strictification, it seems to be part of the folklore. An early reference is [23], see also Appendix B of [62]. A detailed and accessible proof under some finiteness assumptions can be found in [53]. These finiteness assumptions are satisfied if $S$ is of finite type, cf. the proof of Theorem 4.9.

Then $\mathcal{P}erf(S)$ is just the subcategory of fibrant cofibrant homotopy cartesian sections $G_\bullet$ such that each $G_i$ is perfect. The same argument works for $\mathcal{P}erf$, $\mathcal{F}ilt_n$, $\mathcal{QF}ilt_n$ and indeed $\mathcal{D}_n(V,Q)$ instead of $\mathcal{D}^\text{coh}$. There is a slight subtlety in this model: The presection $U \mapsto \mathcal{O}_S(U)$ is not fibrant, so where it appears, for example in the definition of shifted bilinear forms $\mathcal{D}^\text{filt}$, it needs to be replaced fibrantly. Of course this does not affect the homotopy category, so we will not make this explicit if there is no ambiguity.

As an example we state the following global version of Theorem 3.4 to characterize maps from non-affine derived stacks to $D_n(V,Q)$. Here by a perfect complex we mean a fibrant cofibrant homotopy cartesian section as above, and we write $P$ for fibrant cofibrant replacement in the model category of presections.

**Corollary 3.8.** Let $S$ be a derived stack. Then the objects of $\mathcal{D}_n(V,Q)(S)$ may be given by triples $(F^\bullet, w_F, Q_F)$ where

- $F^\bullet$ is a filtration of length $n + 1$ of perfect complexes on $S$
- $Q_F : \text{Sym}^2 F^0 \to P\mathcal{O}_S[2n]$ is a non-degenerate 2n-shifted bilinear form on $F^0$ that vanishes on $F^i \otimes F^{n+1-i}$ for all $i$,
- $w_F : (F_0, Q_F) \simeq (V \otimes \mathcal{O}_S, Q)$ is an isomorphism in the homotopy category of complexes with the shifted bilinear forms,

such that $F^\bullet$ gives filtrations on cohomology after tensoring over $\mathcal{O}_S$ with $H^0(\mathcal{O}_S)$ and all $H^i(F^j \otimes \mathcal{O}_S, H^0(\mathcal{O}_S))$ are flat.

**3.4. The tangent space.**

**Proposition 3.9.** The tangent complex of $D_n(V,Q)$ at the $\mathbb{C}$-point $(F^\bullet, w_F, Q_F)$ is given by $\text{End}_{Q_F}(F^0)[1]$. 
Here $\text{End}_{Q^+_t}(F^0)$ is the complex of endomorphisms of $F^0$ that are antisymmetric with respect to $Q^+_t$, i.e. satisfy $Q^+_t(f, -) + (-1)^{|f|} Q(-, f) = 0$. $\text{End}_{F,Q^+_t}(F^0)$ are those endomorphisms that moreover respect the filtration.

**Proof.** We obtain this result by computing the tangent complexes of $Q\text{Perf}$ and $Q\text{Filt}$, and appealing to the description of $D'_n(V, Q)$ as their homotopy fibre, cf. Remark 3.8.

As $Q\text{Perf} = \text{holim}(\text{Perf} \to \text{Perf} \times \text{Perf} \leftarrow \text{Perf}^{\Delta^1})$ and the tangent space commutes with limits we have the following pullback diagram:

$$
\begin{array}{ccc}
T_{Q\text{Sym}^2V, O}Q\text{Perf} & \longrightarrow & T_Q\text{Perf}^{\Delta^1} \\
\downarrow & & \downarrow \\
T_V\text{Perf} & \longrightarrow & T_{\text{Sym}^2V, O}(\text{Perf} \times \text{Perf})
\end{array}
$$

Let us analyse the ingredients. We recall that all our stacks come from sheaves of dg-categories and then we use Corollary 3.17 from [60] which tells us that to compute the tangent space of the moduli stack of objects of a triangulated dg-category of finite type it suffices to compute the hom-spaces, i.e. $T_{D(M)} = \text{End}(D)[1]$. This applies since we can consider $\text{Perf}$ and $\text{Perf}^{\Delta^1}$ as the moduli stack of objects of a dg-category of finite type.

Thus it follows that

$\begin{align*}
T_V\text{Perf} &= \text{End}(V)[1] \\
T_{\text{Sym}^2V, O}(\text{Perf} \times \text{Perf}) &= (\text{End}(\text{Sym}^2V) \oplus \text{End}(A))[1] \\
T_{f:A \to B}(\text{Perf}^{\Delta^1}) &= (\text{End}(A) \oplus \text{End}(B) \oplus \text{Hom}(A, B)[1], \Delta)[1]
\end{align*}$

Here the last complex has differential $\Delta : (x, y, h) \mapsto (dx, dy, dh + gx - yf)$. This follows from considering hom-spaces in the category $\mathcal{P}\text{erf}^{\Delta^1}$.

Now to consider the map $d\text{Sym}^2$ we have to change to simplicial categories. It is clear that the induced map on derivations sends $f$ to $f \otimes 1 + (-1)^{|f|} 1 \otimes f$.

Putting this together we obtain that the tangent space at $(V, Q)$ is the homotopy kernel of the map $e_Q : \text{End}(V) \to \text{Hom}(\text{Sym}^2V, A[2n])$ defined by $e_Q(f) := Q(f, -) + (-1)^{|f|} Q(-, f)$. We write the kernel as $\text{End}_{Q}(V)$. Over $\mathbb{C}$ the complexes are formal and as $Q$ is non-degenerate $e_Q$ is surjective, so these are just endomorphisms that are anti-symmetric with respect to $Q$. 


We proceed similarly for $QFilt$ to find that $T_{F^*,QF}QFilt$ consists of filtered endomorphisms of $F^0$ that are compatible with $Q_F$.

Then $T_{(F^*,w,F^0,Q_F)}D$ is the homotopy equalizer of the natural inclusion and the zero map. (We use the identification of the tangent spaces to $QPerf$ at $(V \otimes A, Q)$ and $(F^0, Q_F)$.) We conclude by noting that $\text{End}_{F, Q}(F^0) \to \text{End}_Q(F^0)$ is levelwise surjective. □

This is entirely analogous to the tangent complex of the derived flag variety, which is $(\text{End}(V)/\text{End}_c(V))[1]$. We also note that the zeroth cohomology group consists of orthogonal endomorphisms of $V$ modulo filtered orthogonal endomorphisms of $V$. This is of course the tangent space in the underived case.

Remark 3.9. Equivalently we can of course write the tangent complex as the shifted cone on the map

$$q : \text{End}V/\text{End}_c V \to \text{Hom}(\text{Sym}^2V, A[2n])/\text{Hom}_c(\text{Sym}^2V, A[2n])$$

that is induced by $Q$.

As the tangent complex and thus the cotangent complex is perfect and as $\pi_0(D)$ is a scheme of finite presentation we deduce:

**Corollary 3.10.** The derived period stack $D_n(V, Q)$ is locally of finite presentation.

3.5. **The derived period domain II: Analytically.** As the period map and the second Hodge-Riemann bilinear relation are analytic by nature we need to understand the derived analytic period domain.

To add the Hodge-Riemann bilinear relations we need some extra structure, so we recall that $V = H(X_s, \mathbb{C})$ is equipped with a Lefschetz operator $L$ and we define the bilinear forms $Q_k$ on $H^kV$ by $Q_k(\alpha, \beta) = Q(L^{n-k}\alpha, \beta)$.

**Theorem 3.11.** There is a derived analytic period domain $U$, an open analytic substack of $D_n(V, Q)^an$, which is an enhancement of the period domain.

**Proof.** We first apply the Lurie-Porta analytification functor reviewed in Section 2.2 to the geometric stack $D$. This is possible as $D$ is locally of finite presentation by Corollary 3.10.

We need to check that the underived truncation $\pi_0^0(D^an)$ is the closed analytic period domain, but this is immediate from Theorem 3.4 since analytification commutes with truncation.
We note that orthogonality with respect to $Q_k$ follows from orthogonality with respect to $Q$ as in the definition of $D_n(V, Q)$.

Now we construct the open substack $U$ of $D^{an}$. Lemma 3.1 shows that it suffices to construct an open substack $U'$ on the underlying underived stack. We take $U'$ to be the preimage under the natural map $\pi_0^!(D^{an}) \to \pi_0\pi_0^!(D^{an})$ of the classical period domain that is defined inside $\pi_0\pi_0^!(D^{an}) = (\pi_0\pi_0^!(D))^{an}$ by the second and third Hodge-Riemann bilinear relation. Explicitly, it is the open subspace given by the conditions that $F^i H^k \oplus F^{k-i+1} H^k = H^k$ and the form given by $(-1)^{k(k-1)/2} i^{p-q} Q_k(\cdot, \bar{\cdot})$ is positive definite. □

Remark 3.10. As $U$ is open in $D^{an}$ it will have the same tangent complex. Moreover, the tangent complex is unchanged by analytification, see the discussion in Section 4.5.

Let us now consider how we will construct a map into this derived period domain.

The period map is naturally a map into a moduli space, however for technical reasons we have now constructed the derived period domain as an analytification of an algebraic moduli stack, rather than as an analytic moduli stack.

Thus we need to consider the following question: Given the algebraic moduli stack $D$, what can we say about $D^{an}$? It seems too optimistic to expect it to represent some corresponding analytic moduli problem in general, but it is possible to construct maps to $D^{an}$ using Theorem A.6.

Thus, given a derived analytic space $T$, to construct a map $T \to D^{an}$ it suffices to find a map $uT \to D$, and that is what we will do in the next section. (Note that not all maps from $T$ to $D^{an}$ will arise in this way, but the derived period map will.)

For a Stein space we know that $uT$ is $R \text{Spec} \mathcal{O}(T)$. The scheme $uT$ is very large and not expected to be well-behaved (for example $(uT)^{an}$ is certainly not a derived analytic space). Nevertheless, we can study $\text{Map}(uT, D)$ and see that the derived period domain classifies Hodge filtrations of $\mathcal{O}(T)$-modules.

Of course, this is a somewhat unsatisfactory result as we would like to have an analytic moduli interpretation of our derived period domain.

There are several interesting instances where the analytification of a moduli space is the corresponding analytic moduli space. For example, in the underived setting one may observe that the analytification of the Grassmannian is an analytic Grassmannian. One way to see this is that there are matching explicit constructions.
In fact, something very similar holds in this situation and $U$ turns out to be the derived analytic moduli stack of filtrations of $V \otimes O_T$ by perfect complexes that satisfy the Hodge-Riemann bilinear relations.

We prove this result in forthcoming work with Mauro Porta as an application of the main theorem of [26] that in many cases the analytification functor commutes with the mapping space functor.

3.6. **Monodromy and quotient.** The period map only maps to the quotient of the period domain by monodromy, so we examine monodromy for families of algebraic varieties in the derived case, and then construct the quotient of the derived period domain by a group acting on $V$.

Assume we are given a smooth projective map $f : X \to S$ of derived schemes, i.e. a strong map of derived schemes such that $\pi_0\pi(f)$ is smooth and projective. We write $\iota(S)$ for the underlying topological space of $S^{an}$, i.e. $\pi_0S(\mathbb{C})$ considered in the analytic topology.

The fundamental group of $\iota(S)$ acts on cohomology of the fibre as each $R^if_*\Omega_{X/S}$ is a local system on $S$. Looking instead at the complex $Rf_*\mathbb{C}$ we see that it forms a homotopy locally constant sheaf on $\iota(S)$. We will see that $Rf_*\Omega_{X/S} \simeq Rf_*\mathbb{C} \otimes O_S$ in Lemma 4.3.

A homotopy locally constant sheaf is equivalently a representation of the simplicial loop group $\Omega \iota(S)$, see [25], i.e. a priori there is higher monodromy. But the degeneration of the Leray-Serre spectral sequence for families of projective varieties suggests that $Rf_*\mathbb{C}$ is just a direct sum of local systems, i.e. there is no higher monodromy. Indeed, Deligne proves the following as Theorem 1.5 in [10]:

**Theorem 3.12.** Consider the bounded derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$. Let $n \in \mathbb{N}$, $X \in D(\mathcal{A})$ and assume there is $u : X \to X[2]$ inducing isomorphisms:

$$u^i : H^{n-i}(X) \simeq H^{n+i}(X)$$

for $i \geq 0$. Then $X \simeq \sum_i H^i(X)[-i]$.

Moreover, in [12], Deligne constructs a canonical map $\phi : HX \to X$ inducing the identity in cohomology. Now we let $\mathcal{A}$ be the category of sheaves of abelian groups on $\iota(S)$, $X$ be $Rf_*\mathbb{C}$ and $u$ be the map induced by the relative Lefschetz operator. We obtain the following (see [10] 2.6.3): The homotopy locally constant sheaf $Rf_*\mathbb{C}$ is quasi-isomorphic to a direct sum of its cohomology sheaves, which are local systems.

We also need to consider the shifted bilinear forms on $Rf_*\mathbb{C}$ and its cohomology. We may compute $Rf_*(\mathbb{C})$ in such a way (for example using...
a Čech cover of \( t(X) \) over \( t(S) \)) that it comes with a shifted bilinear form \( Q_R \) induced by the pairing \( \langle \alpha, \beta \rangle = \int \alpha \cup \beta \) given by the cup product and the trace map \( \int : R_f^* \mathcal{C}[-2n] \to R^{2n}f_* \mathcal{C} \to \mathbb{C} \) given by Verdier duality. This of course induces the usual pairing \( Q \) on cohomology. \( Q_R \) also pulls back via \( \phi \) to a bilinear form on cohomology \( H(R_f, \mathcal{C}) \) and as \( \phi \) is the identity on cohomology this is again the usual pairing on cohomology. Thus we have a quasi-isomorphism \( \phi : (H(R_f, \mathcal{C}), Q) \to (R_f^* \mathcal{C}, Q_R) \) of complexes of sheaves with shifted bilinear forms.

We deduce:

**Corollary 3.13.** The complex \( R_f^* \mathcal{C} \) with its bilinear form considered as a \( \Omega t(S) \)-representation is equivalent to a representation of \( \pi_1(t(S)) \).

Note that in all this the basis is allowed to be singular, the only assumption is that \( t(S) \) is locally paracompact.

When constructing the global period map we will see (cf. Lemmas 4.3 and 4.4) that this monodromy is the only obstruction to gluing the local derived period maps, i.e. the derived structure of \( S \) (which is infinitesimal) does not interfere.

**Remark 3.11.** On the other hand Voisin shows in [64] that Deligne’s decomposition does not work on the level of algebras. This suggests that extending the period map to keep track of the algebra structure on \( R_f^* \Omega_{X/S} \) poses a significantly harder problem. The bilinear form is defined using the algebra structure, but we have seen that preserving the bilinear form is a much weaker condition than preserving the algebra structure.

**Remark 3.12.** One might wonder what happens if one considers the complex \( R_f^* \Omega_{X/S} \) in the case that \( S \) is underived, i.e. if one considers a classical period map taking values in complexes.

Consider the Hodge filtration \( F^i \) on \( R_f^* \Omega_{X/S} \). Then we note that now all the \( F^i \) are formal, i.e. quasi-isomorphic to \( HF^i \). This follows since they have locally free cohomology. (In the derived setting this argument would not apply.) So the algebraic geometry of \( S \) is not reflected in studying Hodge structures as a complex (as opposed to a collection of cohomology groups), and the topology is not reflected beyond the fundamental group.

This means for example that failures of the Torelli theorem are not resolved by considering a period map taking values in complexes.

We will now connect the monodromy considerations to our construction of \( U \). We think of \( V = V_{\mathbb{Z}} \otimes \mathbb{C} \) as the cohomology complex of a fibre \( X_s \) of our map \( X \to S \) of derived schemes. (Note \( X_s \) is a classical smooth projective...
scheme.) Then we have a strict action of the fundamental group $\pi_1(S)$ on $V$ compatible with the form $Q$. In fact this action factors through the action of the universal arithmetic group $\Gamma = \text{Aut}(V, Q)$ on $V$ with the bilinear form $Q$.

We will now show that the action of $\Gamma$ on $V$ induces an action on the derived period domain.

**Proposition 3.14.** The action of $\Gamma$ on $V$ induces an action on $D_n(V, Q)$ which leaves the derived period domain $U$ invariant and the quotient $U/\Gamma$ exists as a geometric derived analytic stack.

**Proof.** As $V$ comes with a $\Gamma$-action there is a map $B\Gamma \to \text{Perf}$ classifying this representation and $V : * \to \text{Perf}$ factors through it. Note that the simplicial set $B\Gamma$ can be considered as a constant stack (i.e. we stackify the constant functor $B\Gamma$ on derived schemes). We do not expect this to be geometric or locally of finite presentation as $\Gamma$ is an infinite discrete group. As the action is compatible with $Q$ we can consider the map $\gamma : B\Gamma \to Q\text{Perf}$ classifying the pair $(V, Q)$.

Then the pullback of $Q\text{Filt} \to Q\text{Perf}$ along $\gamma$ is a derived stack $D\Gamma$. As the homotopy pullback of $D\Gamma \to B\Gamma$ along the universal cover $* \to B\Gamma$ is equivalent to $D$ (see Remark [3.8]) we see that $D\Gamma$ is the desired quotient. Explicitly, $\Gamma$ acts on $D_h := D\Gamma \times_{B\Gamma} E\Gamma$ via its action on $E\Gamma$, and of course $D_h \cong D$. As analytification commutes with homotopy limits we can consider $D^{an}_\Gamma$ as a quotient of $D^{an}$.

The underlying undervied stack is not necessarily geometric, but we can consider the image of $U$ under $D^{an} \to D^{an}_\Gamma$.

The open substack $U \subset D^{an}$ corresponds to an open substack $U_h$ of $D^{an}_h$, which is preserved by the action of $\Gamma$, as this is true for the underlying undervied spaces where $\Gamma$ is the usual monodromy action.

We write $U/\Gamma$ for the image of $U$ in $D^{an}_\Gamma \approx D^{an}_h/\Gamma$, which is equivalent to the quotient $U_h/\Gamma$.

Now we use the fact that the action on $\pi_0\pi^0(U_h)$ is properly discontinuous to show that $U/\Gamma$ is geometric. To be precise we can consider a cover $\mathcal{U}$ of the underlying undervied space such that the images of any two open sets intersect only finitely many times. (Using once more that open subsets of $\pi_0\pi^0(U)$ give open subsets of $U$.) So $U_h$ provides a cover of $U_h/\Gamma$ that is locally a quotient by a finite group, which is enough to show it is geometric. (Explicitly the cover associated to $\mathcal{U}$ is a cover for $U_h/\Gamma$, as composing a smooth map with a finite map gives a smooth map.) \qed
4. The period map

4.1. The classical period map. In this section, which is only used as background, we recapitulate Griffiths’ period map [21].

For a polarized family of smooth projective varieties $f : X \to S$, where $S$ of finite type, there are holomorphic maps $P_k : S^{an} \to U/\Gamma$, where $U \subset \text{Flag}(F^*, H^k)$ is the polarized period domain and $\Gamma$ is the monodromy group. Here $P_k(s)$ is $F^k H^k \subset \cdots \subset H^k(X_s) \subset H^k(X) \cong H^k(X_0)$.

We define $P(s)$ as the product of the Hodge filtrations on $H^k(X)$ for all $k$.

We give some details: $\Omega^1_{X/S}$ is a vector bundle as $f$ is assumed smooth. Then the relative de Rham complex is given by exterior powers, with a differential which is not $\mathcal{O}$-linear, hence this is not a complex of coherent sheaves. However, it is $f^{-1}\mathcal{O}_S$-linear and thus it is a complex of $f^{-1}\mathcal{O}_S$-modules, which pushes forward along $f$ to a complex of $\mathcal{O}_S$-modules.

Note that Deligne shows that the relative de Rham complex $\Omega^*_{X/S}$ is quasi-isomorphic to $f^{-1}\mathcal{O}_S \otimes \mathbb{C}$ and uses this to show that $Rf_*\Omega^*_{X/S} \cong Rf_*(\mathbb{C}) \otimes \mathcal{O}_S$, see Proposition 2.28 of [11]. This works if $S$ is any analytic space, and if $f$ is just assumed smooth and separated.

By Ehresmann’s theorem we have diffeomorphisms $X_s \to X_0$ for every path from 0 to $s$, showing the cohomology sheaves are locally constant and if $S$ is simply connected $R^if_*\Omega^*_{X/S}$ becomes $H^i(X_s, \mathbb{C}) \otimes \mathcal{O}_S$.

The stupid truncation $\mathcal{F}^p = \Omega^p_{X/S}$ is a subcomplex of sheaves of $f^{-1}\mathcal{O}_S$-modules. One then shows that the $R^if_*\Omega^*_{X/S}$ are vector bundles (and in particular coherent sheaves). This implies that the $R^if_*\mathcal{F}^p = R^if_*\Omega^{zp}_{X/S}$ are complexes of coherent sheaves. This can be done by trivialising the fibration locally and using Grauert’s base change theorem, see [38].

The $R^if_*\mathcal{F}^p$ are moreover subsheaves of the $R^if_*\Omega^*_{X/S}$. This follows from degeneration of the Fröhlicher spectral sequence as the differentials in the $R^if_*$ long exact sequence coming from $\Omega^{zp} \to \Omega^{zp-1} \to \Omega^{p-1}$ vanish. Degeneration of the spectral sequence follows from a dimension count.

Thus the map $P_k$ sending $S$ to the filtration $\{R^k f_*\mathcal{F}^i\}$ of $H^k(\Omega_X)$ is the desired map to the flag variety and it factors through the period domain.

Finally, we can globalize the construction by dividing out by the action of the fundamental group of $S$. We use Ehresmann’s theorem again to pull all our data along any path in the base in a homotopy invariant way, giving an action of the fundamental group of $S$. 
Remark 4.1. These results are often stated for $S$ smooth, but we need not assume this. Deligne’s $\Omega^*_{X/S} \simeq f^{-1}\mathcal{O}_S \otimes \mathbb{C}$ holds over an analytic space, as do Grauert’s theorem (see \textsuperscript{[20]}), and Ehresmann’s theorem (see Demainly’s chapter in \textsuperscript{[6]}).

Grauert’s theorem assumes that $S$ is reduced (there is a version for a non-reduced base). But we may deal with non-reduced bases as part of our proof for the derived case, see Lemma \textsuperscript{4.2} below.

The map $P$ is a priori not algebraic. In fact, to construct the map we needed to locally trivialize $Rf_!\mathbb{C} \otimes \mathcal{O}_S$. In the analytic setting we can do this over any contractible set, so we just need $S$ to be locally contractible. In the algebraic setting this is typically impossible unless $S$ is itself simply connected.

Finally, let us have a look at the differential. Assume for simplicity that $S$ is smooth.

The differential of the period map $dP_{p,k}$ factors through the Kodaira-Spencer map $B \to H^1(X, \mathcal{T}_X)$. In fact $dP$ is the composition of Kodaira-Spencer with the map

$$H^1(X, \mathcal{T}_X) \to \text{Gr}(F^pH^k(X), H^k(X)/F^pH^k)$$

which is given by composing the natural cup and contraction map with the natural quotient and inclusion maps: $F^pH^k \to H^{p,k-p}(X) \to H^{p+1,k-p-1}(X) \to H^k/F^pH^k(X)$.

The period map satisfies Griffiths transversality, i.e. this differential lands inside the subspace $F^{p-1}H^k(X)/H^pH^k(X)$. This can also be expressed as saying the connection on $Rf_*\Omega^*_{X/S}$ maps $F^i$ to $F^{i-1} \otimes \Omega^1_X$.

4.2. The local derived period map. We now define the derived version of the period map. In this section we work with derived schemes modelled on simplicial commutative algebras.

We will consider a polarized smooth projective map $f : X \to S$ between geometric derived stacks. We fix the fibre $X_s$ over some distinguished point $s$. This is a smooth projective variety. We assume that $S$ is of finite presentation.

Remark 4.2. We need our base stack to be locally of finite presentation to use analytification, and we need that it is quasi-compact for some topological arguments in Theorem \textsuperscript{4.9}.

By definition the map $f$ is smooth and projective if $\pi_0\pi^0(f)$ is smooth and projective and $f$ is strong, i.e. $\pi_!(\mathcal{O}_X) \cong \pi_!(\mathcal{O}_S \otimes_{\mathcal{O}_X} \pi_0(\mathcal{O}_X))$. Recall a map of analytic or algebraic spaces is smooth if it is flat and the relative
The cotangent complex $\Omega^{1}_{X/S}$ is locally free. By polarized we just mean that $\pi_{0}\pi^{0}(f)$ is polarized.

We will first restrict ourselves to the case that $S$ is a quasi-separated derived scheme. We note that the definition of a derived scheme $X$ includes that it has an underlying topological space, the same is true for its analytification. In both cases we denote the underlying topological space with the analytic topology by $t(-)$.

We will now construct a map from open pieces of the analytification $S^{an}$ of $S$ into the derived period domain $U \subset D_{n}(V, Q^{an})$, constructed in Theorem 3.11. Explicitly, we take $V$ to be the complex $H^{*}(X_{s}, \Omega^{X}_{s})$ with zero differentials. For $Q$ we take the usual shifted bilinear form defined using product and trace.

As we can only hope to construct the map locally we will restrict to subspaces $T \subset S^{an}$ such that $\pi_{0}\pi^{0}(T)$ is a contractible Stein space. We write $i_{T} : X_{T} \rightarrow X^{an}$ for the pullback of $X^{an}$ to $T$.

Abusing notation we will use $f$ also for $f^{an}$ and $f^{an}|_{X_{T}}$ when there is no ambiguity.

Now by Theorem 3.4 defining a map $T \rightarrow U$ is equivalent to defining a point of $D_{n}(V, Q)(uT)$ that also satisfies the positivity conditions.

In what follows we need to explicitly consider coherent sheaves on derived analytic spaces like $T$. Recall from Section 2.2 that these are just $\mathcal{O}^{alg}_{T}$-modules such that the cohomology sheaves are locally coherent over $\pi_{0}(\mathcal{O}_{T})$. Moreover the underlying $\infty$-topos is the $\infty$-topos of $t(T)$. To compute derived functors we use the local model structure on presheaves.

In particular we can apply the usual direct and inverse image functors between sheaves of complexes, on $t(T)$. There is extra structure on the space of functions on a derived analytic stack, but it will not affect the constructions we need.

The crucial ingredient for Hodge theory is the relative de Rham complex. This is defined as the exterior algebra of the relative cotangent complex (which is a simplicial sheaf).

The relative cotangent complex should be available in every good theory of derived analytic spaces and treatment of the analytic cotangent complex can now be found in [44]. For our purposes we will avoid the analytic theory and just use the analytification of the algebraic cotangent complex. (At least for Deligne-Mumford stacks this agrees with the analytic cotangent complex, see Theorem 5.20 of [44].)
Thus in our set-up we may use the analytification of the relative algebraic cotangent complex. As \( f \) is smooth we think of this as the complex of Kähler differentials. So we are considering \( \Omega^1_{X/S} = \Omega^1_X / f^* \Omega^1_S \) and restrict to \( X^n \), i.e. \( \Omega^1_{X^n/T} : = i^n_T(\Omega^1_{X^n/S})^an = i^n_T h^* \Omega^1_{X/S} \) for the inclusion \( i^n_T : X^n \to X^n \) and the natural map \( h_X : X^n \to X \).

For definitions of the algebraic cotangent complex \( L \), see [62]. We recall some basic local facts: Let \( f^* : A \to B \) be a map of simplicial algebras. The relative cotangent complex \( L_{B/A} \) fits into an exact sequence \( B \otimes_A L_A \to L_B \to L_{B/A} \). For an explicit construction replace \( B \) by an algebra that is free over \( A \), see Section 4 of [19]. In particular then each \( B_n \) is a free algebra over \( A_n \). We know that \( L_A \) is the simplicial \( A \)-module that is given by the Kähler differentials \( \Omega^1_{A_n} \) in degree \( n \). As \( B \) is free over \( A \) we have cofibrations of \( B_n \)-modules \( B_n \otimes \Omega^1_{A_n} \to \Omega^1_{B_n} \) and as the construction of Kähler differentials is functorial in pairs of algebras we find that \( L_{B/A} \) is the simplicial \( B \)-module that is \( \Omega^1_{B_n}/(B_n \otimes \Omega^1_{A/n}) = \Omega^1_{B_n/A_n} \) in degree \( n \). We write \( \Omega^i_{B_n/A_n} \) for \( \wedge^i \Omega^1_{B_n/A_n} \).

We can thus define a chain complex of simplicial modules which is \( \Omega^i_{B_n/A_n} \) in bidegree \((i,n)\). Explicitly we have the following. Here each \( d \) is \( A_i \)-linear.

\[
\begin{array}{cccccccc}
A_0 & \xrightarrow{f^*_0} & B_0 & \xrightarrow{d} & \Omega^1_{B_0/A_0} & \xrightarrow{d} & \Omega^2_{B_0/A_0} & \to \ldots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
A_1 & \to & B_1 & \to & \Omega^1_{B_1/A_1} & \to & \Omega^2_{B_1/A_1} & \to \ldots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
A_2 & \to & B_2 & \to & \Omega^1_{B_2/A_2} & \to & \Omega^2_{B_2/A_2} & \to \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 
\end{array}
\]

We denote the total complex of the associated double complex of \( N(A) \)-modules by \( \Omega_{B/A} \).

**Remark 4.3.** Note that we are indeed taking the exterior algebra in every degree, rather than taking the symmetric algebra in odd degrees, cf. [28].

**Remark 4.4.** A deeper study on differential forms in derived algebraic geometry is done in [37]. However, for a smooth morphism we do not have to worry about the subtleties dealt with in that work, cf. Section 1.2 of [37].

**Lemma 4.1.** *The cotangent complex* \( \Omega^1_{X^n/T} \) *defined above is a coherent sheaf on* \( X^n \). *Restricting to the underived setting we recover the usual cotangent complex for a smooth map, i.e. the sheaf of relative holomorphic differentials.*
Proof. The first claim is true as analytification preserves coherence, see [40].

Next note that holomorphic differentials are the analytification of the sheaf of Kähler differentials, and derived analytification is compatible with truncation, see the proof of Theorem 7.5 in [40]. □

We will now consider the double chain complex of sheaves associated to the simplicial chain complex \((\Omega^*_{X/T})_*\). We will denote its total complex by \(\Omega^*_{X/T}\). This is a complex of \(\mathcal{N}(f^{-1}\mathcal{O}_S)\)-modules.

When \(f\) is smooth we have a derived Poincaré lemma:

Lemma 4.2. Let \(T\) be a derived analytic space that is an open subset of the analytification of a derived algebraic space. Then \(\Omega^*_{X/T}\) as constructed above is a resolution of \(\mathcal{N}(f^{-1}\mathcal{O}_T)\).

Proof. The question is local, so let us first assume \(X \to S\) is a smooth map of derived affine schemes, and let \(T \subset S^{an}\) be an open subspace such that \(\pi_0\pi^0(T)\) is contractible Stein. We consider the map \(A = \mathcal{O}(S) \to B = \mathcal{O}(X)\) of simplicial algebras.

There clearly is a map \(e : \mathcal{N}(A) \to \Omega^*_{B/A}\) of \(\mathcal{N}(A)\)-modules. We claim it is a quasi-isomorphism after base-change to \(\mathcal{O}(T)\), i.e. after analytification and restriction to \(T\). For simplicity we write \(t = h_X \circ i_T\) and note that \(t\) is flat since \(h_X\) is flat, see Section 6.3 of [40]. So we aim to show that \(t^*(e)\) is a quasi-isomorphism.

We consider \(\mathfrak{n}\), the ideal of nilpotents and elements of positive degree in \(\mathcal{N}(A)\). By abuse of notation we will use the same name for the corresponding ideal in \(A\).

This is a simplicial ideal and \(A/\mathfrak{n}\) is reduced and underived. We also consider the ideal \(\mathfrak{n}_B = \mathfrak{n} \otimes_A B\) in \(B\). Note that \(B \otimes_A A/\mathfrak{n}\) is also reduced and constant (as a simplicial algebra) as we assume \(f\) smooth.

The \(\mathfrak{n}_i\) form an exhaustive filtration of \(\mathcal{N}(A)\). As \(\Omega^*_{B/A}\) is an \(\mathcal{N}(A)\)-complex we can filter it by the \(\mathfrak{n}_i\), too. (As \(\Omega^*_{B/A} \otimes \mathcal{N}(A)\mathfrak{n}_i\) are subcomplexes.) We aim to show that the associated graded map of \(t^*(e)\) induces a quasi-isomorphism, which shows that \(t^*(e)\) itself is a quasi-isomorphism. As \(t\) is flat we can equivalently show that the associated graded pieces of \(e\) become quasi-isomorphisms after applying \(t^*\).

Write \(A'\) for \(A/\mathfrak{n}\) and \(B' = B/(\mathfrak{n} \otimes_A B)\). Now we claim the following:

- \(\mathcal{N}(A') = \mathcal{N}(A)/\mathfrak{n}\) and \(\Omega^*_{B'/A'}\) become quasi-isomorphic when applying \(t^*\).
\begin{itemize}
\item $Gr^i_n N(A)$ and $\Omega^i_{B/A} \otimes_{N(A')} Gr^i_n N(A)$ become quasi-isomorphic when applying $\iota^*$.
\item $Gr^i_n (\Omega^i_{B/A}) \simeq \Omega^i_{B'/A'} \otimes_{N(A')} Gr^i_n N(A)$.
\end{itemize}

The first claim follows from the usual underived Poincaré lemma, see for example Deligne [11], observing that $\text{Spec}(B')^{an} \to \text{Spec}(A')^\text{an}$ is a smooth map of analytic spaces. Then we base change to $T_{\text{red}}$ and apply the exact global sections functor.

The second claim follows from the first since $\Omega^m_{B'/A'}$ is flat over $N(A')$ and pullback is compatible with tensor products.

So let us turn to the third claim. Here we will have to work simplicially. We begin with $i = 0$ and show that $\Omega^m_{B/A}/n \simeq \Omega^m_{B'/A'}$. Recall that $\Omega^m_{B/A}$ is $\Omega^m_B/\Omega^m_B \otimes \Omega^1_A$. So we can consider the natural map

$$\Omega^m_B \to \Omega^m_B/\Omega^m_B \otimes \Omega^1_A.$$ 

The map is clearly surjective and we claim the kernel is given by $n \otimes_A \Omega^m_B \otimes \Omega^m_B \otimes \Omega^1_A$.

It follows from the short exact sequence $n/n^2 \to \Omega^1_B \to \Omega^1_B$ that the kernel is given by $d/n + g^{-1}(\Omega^{-1}_{B'} \otimes A', \Omega^1_{A'}) = n \otimes \Omega^m_B + \Omega^1_A \otimes \Omega^m_B$ as $d n \subset \Omega^1_A$.

So we have shown $\Omega^m_{B/A}/n \simeq \Omega^m_{B'/A'}$ as simplicial $A'$-modules. The differential is compatible thus we get a quasi-isomorphism with the associated complexes of $N(A')$-modules, $\Omega^*_B /n \simeq \Omega^*_B /A'$. To consider the other associated graded pieces we observe that for any $A$-modules $M$ we have $(I^n A \otimes M)/(I^{n+1} A \otimes M) \simeq (I^n A/I^{n+1} A) \otimes M/I$. This is valid in the simplicial setting as it holds degree by degree. Again we deduce the quasi-isomorphism of complexes.

Together these claims give the desired quasi-isomorphism on the associated graded modules and we have locally shown the quasi-isomorphism $N(f^{-1} \mathfrak{O}_T) \simeq \Omega^*_{X/T}$ of sheaves. □

We can now consider the derived push-forward $Rf_* \Omega_{X/T}$. Using the product on $\Omega_{X/T}$ and suitable resolutions we may assume this is a sheaf of algebras.

**Lemma 4.3.** Let $f : X \to S$ be a smooth projective morphism of derived schemes and consider $T \subset S^{\text{an}}$ and $\Omega^*_{X/T}$ as above. Assume moreover that $t(T)$ is simply connected. There is a quasi-isomorphism of $\mathfrak{O}_T$-modules $w' : V \otimes \mathfrak{O}_T \to Rf_*(\Omega^*_{X/T})$.

Moreover, we can factor $w' = v \circ \phi : V \otimes \mathfrak{O}_T \to Rf_* \mathbb{C} \otimes \mathfrak{O}_T \to Rf_* (\Omega^*_{X/T})$ where $v$ is an algebra map.
Proof. First we show that \( Rf^*O_T \simeq Rf^*\Omega^\text{an}_{X/T} \).

By Lemma 4.2 we have a weak equivalence \( f^{-1}\mathcal{O}_T \to \Omega X/T \). Now we use the projection formula for complexes of sheaves and the map \( f : t(X_T) \to t(T) \) of topological spaces. As \( f \) is proper \( f_* = f^* \). The projection formula says the natural map 

\[
Rf^* F \otimes^L G \to Rf^*(F \otimes^L f^{-1}(G))
\]

is a quasi-isomorphism. Here \( F = \mathbb{C} \) and \( G = \mathcal{O}_T \) and we call the natural map \( v \).

The tensor product here is over the constant sheaf and can be considered underived. For the projection formula one usually assumes that \( A, B \) are bounded below sheaves (with the cohomological grading) and the structure sheaf is not bounded below in general, but because the fibres of our map of topological spaces are locally finite-dimensional the theorem remains true for unbounded complexes, see §6 of [52].

We may compute the natural map in the category of sheaves of algebras and thus arrange that \( v \) is a homomorphism.

Now \( Rf^*\mathbb{C} \) only depends on the underlying topological space, hence it is a homotopy locally constant sheaf on \( t(T) \) by the classical result. By Corollary 3.13 this is actually formal, i.e. quasi-isomorphic to the graded local system of its cohomology sheaves. But as \( t(T) \) is simply connected this must be a constant sheaf, which is of course quasi-isomorphic to \( V \).

Thus we let \( \phi \) be Deligne’s canonical quasi-isomorphism \( V \to Rf^*\mathbb{C} \), tensored with \( \mathcal{O}_T \).

Next we consider the Hodge filtration. To do this we look at the inclusion of analytic sheaves on \( X \) given by the stupid truncation of the relative de Rham complex, \( \mathcal{F}^i \subset \mathcal{F}^{i-1} \subset \Omega X/T \).

**Lemma 4.4.** Given \( f : X_T \to T \) and \( (V, Q) \) as above we have a diagram of perfect complexes over \( A = N(\mathcal{O}(T)) \)

\[
F^n \to F^{n-1} \to \ldots \to F^0 \leftarrow V \otimes A
\]

where \( F^i \simeq R\Gamma \circ Rf_*(\mathcal{F}^i) \), the \( F^i \) are cofibrant \( A \)-modules and the maps \( F^i \to F^{i-1} \) are injective. Moreover, the conditions of Theorem 3.4 are satisfied, namely:

1. The last map \( w' \) is a quasi-isomorphism.
2. All other maps induce injections on \( H^*(\cdot \otimes_A H^0(A)) \), and the \( H^*(F^i \otimes_A H^0(A)) \) are locally free sheaves.
3. There is a bilinear form \( Q_F \) on \( F^0 \) which is compatible with \( w' \) and \( Q \).
4. The \( F^i \) satisfy orthogonality with respect to \( Q_F \).
In other words we get an object of \( D_n(V, Q)(uT) \), where \( u \) is the forgetful functor to derived algebraic stacks.

**Proof.** We apply the push-forward \( Rf_* \) of sheaves from \( f^{-1} \mathcal{O}_S \)-modules on \( X \) to \( \mathcal{O}_T \)-modules on \( T \) and then global sections and consider the diagram of \( R\Gamma \circ f_* F^i \) for \( i \geq 0 \). These are dg-modules over \( \mathcal{N}(\mathcal{O}(T)) \).

It is clear that we can replace the \( F^i \) by a diagram of inclusions of cofibrant objects. We first deal with some homotopy invariant properties which will not be affected by this.

To show the \( F^i \) are perfect we first consider the pushforward \( Rf_* \Omega^m_{X/S} \) on \( S \) in the algebraic setting. We will show that as \( \Omega^m_{X/S} \) is perfect the pushforward by a smooth proper map is perfect. (A map in derived algebraic geometry is proper if the underlying undervived map is.) This is well-known in the undervived setting.

By our definition of perfect complexes, if the undervived truncation \( F^i \otimes_A H^0(A) \) is perfect so is \( F^i \). So we just need to consider base change for the following diagram:

\[
\begin{array}{ccc}
\pi^0X & \xrightarrow{i_X} & X \\
\downarrow \pi^0(f) & & \downarrow f \\
\pi^0S & \xrightarrow{i_S} & S
\end{array}
\]

This holds by Proposition 1.4 in [58] which asserts that there is a base change theorem for pull-back squares of quasi-compact quasi-separated derived schemes. Thus \( F^i \otimes_A H^0(A) \) is \((\pi^0 f)_*(i_X^* F^i)\), which is perfect by the undervived result.

Next we apply Porta’s GAGA result, see Lemma [4.5] below, to show that pushforward commutes with analytification, thus \( Rf_*(\Omega^m_{X/T}) \) is a pullback (along \( h_S \circ i_T \)) of a perfect complex, and thus perfect. Now \( F^i \) is an iterated extension of perfect complexes and itself perfect.

For the map in the first claim we use the weak equivalence \( w' = v \circ \phi \) from Lemma [4.3].

The second statement follows from classical Hodge theory. As above we start in the algebraic setting and use base change and GAGA to show that \( i_S^* Rf_* \Omega^p_{X/S} \) is locally free because it agrees with \( Rf_! i_X^* \Omega^p_{X/S} = Rf_! \Omega^p_{\pi^0(X)/\pi^0(S)} \), which is well-known to be locally free in classical Hodge theory. Similar for the injections.
To define the shifted bilinear form we will be very explicit about our replacements. Some care is needed since fibrant cofibrant replacements are not strictly monoidal, and the orthogonality condition is strict, rather than up to homotopy.

We choose a cover of $X_T$ and the injective model category of presections as a model for sheaves of $\mathcal{O}_T$-modules on $X_T$, see the discussion before Corollary 3.8.

Objectwise tensor product makes this into a symmetric monoidal model category, the pushout-product axiom holds since cofibrations are defined objectwise. We use fibrant cofibrant replacement using functorial factorization, denote this functor by $P$. We also consider the transfer model category structure on algebra objects, see e.g. [19], and we denote fibrant cofibrant replacement with respect to this structure by $P_{alg}$. Fibrant cofibrant objects are preserved by the forgetful functor.

Next we may replace $F_i$ by a diagram of cofibrations using functorial factorization, call it $Q'(F^i)$, and then by a diagram of cofibrations between fibrant cofibrant objects, write this as $P' F^i$. We observe that by the monoid axiom the diagram $(i, j) \mapsto Q'(F^i) \otimes Q'(F^j)$ is actually cofibrant in the projective model structure, and similarly for $P'$. Thus the weak equivalences of diagrams $Q'(F^i) \otimes Q'(F^j) \rightarrow F^i \otimes F^j \leftarrow Q(F^i \otimes F^j)$ give rise to a lift $Q'(F^i) \otimes Q'(F^j) \rightarrow Q(F^i \otimes F^j)$ as the second map is a trivial fibration. Similarly we have functorial quasi-isomorphisms $P'(F^i) \otimes P'(F^j) \leftarrow Q'(F^i) \otimes Q'(F^j) \rightarrow P(F^i \otimes F^j)$ where the first map is a trivial cofibration, thus there are compatible lifts $\eta_{ij} : P'(F^i) \otimes P'(F^j) \rightarrow P(F^i \otimes F^j)$.

We also choose a quasi-isomorphism $\epsilon : P F^0 \rightarrow P'(F^0)$. Denote the product on $F^0$ by $m$ and then define a multiplication on $P' F^0$ by considering

$$\epsilon \circ Pm \circ \eta_{00} : P' F^0 \otimes P' F^0 \rightarrow P(\mathcal{F}^0 \otimes \mathcal{F}^0) \rightarrow P(\mathcal{F}^0) \rightarrow P'(\mathcal{F}^0)$$

This product vanishes on $P' F^i \otimes P' F^{n+1-i}$, as we can factor it through $\eta_{ij}$ and $m$ vanishes on $F^i \otimes F^{n+1-i}$ for degree reasons.

We push forward both $P' F^i$ and $P_{alg} F^0$ and functorially replace by a diagram of cofibrations again, written as $Q'$. The resulting diagram is our definition of $F^i$.

Now, using the lifting property for cofibrant diagrams as above and adjointness of $f_*$ there are natural and compatible maps

$$\phi_{ij} : Q' f_* P' F^i \otimes Q' f_* P' F^j \rightarrow Q(f_* P' F^i \otimes f_* P' F^j) \rightarrow Q f_* (P' F^i \otimes P' F^j)$$
and we can compose φ₀₀ with Qfᵦ(ε ⊗ Pm ⊗ η₀₀) and a quasi-isomorphism Qfᵦ(Pᵦ(ℱ₀)) → Q'fᵦ(Pᵦ(ℱ₀)) to define a product on ℱ₀ that is zero on ℱᵢ ⊗ ℱᵢ₊₁−ᵢ.

As we are working with fibrant cofibrant objects we also have a quasi-isomorphism F₀ → fᵦP_alg ℱ₀ and we can use Lemma 4.3 to define a second quasi-isomorphism fᵦP_alg ℱ₀ → Ω, where Ω is a fibrant cofibrant replacement of RfᵦC ⊗ ℧₅. Both maps are compatible up to homotopy with the multiplication map. Denote the composition by v'.

Now we know there exists a classical trace map RfᵦC[−2n] → C, which we can tensor with ℧₅ (and replace fibrantly cofibrantly). We use v' to pull back the trace map and define the shifted bilinear form QF on ℱ₀. Orthogonality for the ℱᵢ is clear since the product of sections in ℱᵢ and ℱᵢ₊₁−ᵢ is strictly 0.

The rest of the argument we may consider in the homotopy category. By construction v' is compatible with the product and the trace on Ω, thus with the bilinear form. It is clear that in the homotopy category v is equivalent to an inverse of v'. As φ is also compatible with the bilinear form, so is the map w' = v ◦ φ. □

We needed Porta’s GAGA theorem for potentially unbounded coherent sheaves in this proof. This is a slight strengthening of Theorem 7.5 in [40]. The following proof was communicated to us by Mauro Porta and will appear in [44]:

**Lemma 4.5.** Let f : X → Y be a proper morphism of derived schemes and ℱ a coherent sheaf on X. Then the natural map (Rfᵦℱ)an → Rfᵦan ℱan is an equivalence.

**Proof.** One shows first that if fᵦan : Coh(X) → Coh(Y) has finite cohomological dimension, i.e. there exists n such that for all i > n and all ℱ ∈ Coh(X) we have Rᵢ(fᵦan(ℱ)) = 0, then the result holds for unbounded coherent sheaves.

To see this one follows the proof of Theorem 7.5 in [40]. As in that proof one writes ℱ as τ₋nℱ → ℱ → τ₊nℱ. Now one observes that by the result in the bounded below case (Rfᵦ(τ₋kℱ))an ≃ Rfᵦan(τ₋kℱ)an. On the other hand the assumption of finite cohomological dimension implies that the cohomology groups of (Rfᵦ(τ₋kℱ))an and Rfᵦan(τ₋kℱ)an vanish above degree k + n. Varying n one sees that the natural map (Rfᵦℱ)an → Rfᵦan ℱan is a quasi-isomorphism.

To show that fᵦan has finite coherent cohomological dimension, for every s ∈ S an choose a Stein open neighbourhood s ∈ U ⊊ S such that ˘U is compact in S an. Lemma 6.2 in [43] implies that base change Xₛ'an := U ×ₛ'an X'an can
be covered by finitely many Stein opens. Now \( X^\an_U \) is separated (because \( f \) is proper and thus separated), so the intersection of Stein spaces remains Stein. We can compute cohomology via Čech cohomology attached to this Stein open cover. The open cover is finite, so the complex must be finite as well and \( f^\an \) has finite coherent cohomological dimension.

\[
\square
\]

**Remark 4.5.** We also note that the \( F^i \) are coherent. This is proven by induction, using that \( Rf_* \) is exact and coherent sheaves are a stable subcategory of all sheaves. The crucial ingredient is that the \( Rf_*\Omega^m_{X/T} \) are coherent as pushforwards of coherent sheaves.

It is shown in Proposition 5.5 of [40] that push-forward preserves bounded below coherent complexes. As Porta explained to the authors, the same arguments as in Lemma 4.5 extend the result to the unbounded case if the morphism has bounded cohomological dimension.

**Remark 4.6.** In fact parts of this proof work for every algebraic \( X \to S \). We can associate to \( X \to S \) a filtration on \( V \otimes \mathcal{O}_S \) as long as \( t(S^\an) \) is simply connected.

The lemma says that any \( X \to T \) gives a map from \( u(T) \) to the algebraic moduli stack \( D_n(V,Q) \). To proceed we have to change our target to the analytification of \( D \), and then take the substack \( U \). Note that here we need the polarization in order to define \( U \).

**Corollary 4.6.** In the situation as above there is a derived period map \( T \to U \).

**Proof.** By Theorem A.6 any element in \( D(\Spec \mathcal{O}(T)) \) gives a map \( \mathcal{P} : T \to D^\an \), and by checking the open cohomological conditions (classical Hodge theory) we see that we have \( \mathcal{P} : T \to U \). \( \square \)

### 4.3. The global derived period map.

Having constructed in the previous section a period map for any small patch \( T \) of \( S^\an \), we now glue them together.

We first redo Lemma 4.4 for a map between derived stacks. We consider perfect complexes on \( S^\an \) as fibrant cofibrant homotopy cartesian sections as in the discussion preceding Corollary 3.8.

**Lemma 4.7.** Given \( f : X^\an \to S^\an \) a smooth projective map of derived analytic stacks and \((V,Q)\) as above we have a diagram of perfect complexes on \( S^\an \)

\[
F^n \to F^{n-1} \to \ldots \to F^0 \twoheadrightarrow R
\]
where $\mathcal{R}$ is a fibrant cofibrant model for $Rf_{\ast}\mathcal{C} \otimes O_{S^{an}}$, the $F^i$ are fibrant and cofibrant models for $Rf_{\ast}(\mathcal{F}^i)$, and the maps $F^i \to F^{i-1}$ are injective. Moreover, there is a bilinear form $Q_F$ on $F^0$ and the conditions of Theorem 3.4 are satisfied.

These data look very similar to an object in $\mathcal{D}_n(V, Q)(uS^{an})$, except that we have replaced $V$ by a sheaf.

**Proof.** For lack of a suitable reference we sketch how to define the pushforward map in our setting. (See the proof of Proposition 2.1.1 in [18] for the $\infty$-categorical analogue.)

We choose an affine cover $\{S_i\}_{i \in I}$ of $S$ which induces a cover $\{X^a_{i} = X^{an}_{i} \times_{S^{an}} S^{an}_{i}\}$ of $X^{an}$. All $X_i$ are derived schemes as $f$ is smooth. Now the pushforward maps on the $X^a_{i}$ give a pushforward map of presections. This is well defined since base change holds for quasi-compact and quasi-separated maps of derived schemes, see Proposition 1.4 of [58]. As analytification commutes with pushforward, see Lemma 4.5, this is also true in analytic geometry. The pushforward preserves fibrations (as the adjoint preserves objectwise cofibrations), so it is Quillen.

Thus we may compute $F^i$ as in Lemma 4.4 going through the same yoga of fibrant cofibrant replacement, and obtain a filtration $F^\ast$ of $F^0$. Similarly we compute $\mathcal{R}$ and obtain a map of homotopy cartesian sections $v : \mathcal{R} \to \mathcal{F}^0$. This map is a weak equivalence as it is one locally, so we may find an inverse $v'$ as all objects are fibrant cofibrant. Equipped with $v'$ we can define $Q_F$. All the conditions may be checked locally. \[ \square \]

We now need to combine this with the monodromy action on cohomology. We will write $\mathcal{V}$ for the graded vector space $V = H^\ast(X_{s}, \Omega^{\ast}_{X_{s}})$ with integral structure $V_{Z} = H^\ast(X_{s}, \mathbb{Z})$, equipped with the canonical action of $\Gamma = \text{Aut}(V_{Z}, Q)$. By abuse of notation we will also denote by $\mathcal{V}$ the corresponding sheaf on any space whose fundamental group maps to $\Gamma$.

We first state and prove our main theorem for the case that the base $S$ is a derived scheme.

**Proposition 4.8.** A polarized smooth projective morphism $f : X \to S$ of derived schemes, where $S$ is quasi-separated and of finite presentation, gives rise to a derived period map $\mathcal{P} : S^{an} \to U/\Gamma$, where $U$ is the derived period domain for the pair $(V, Q)$.

**Proof.** We construct a map $u(S^{an}) \to D_{\Gamma}$ and then observe that the associated map to $D_{\Gamma}^{an}$ factors through $U/\Gamma$, using the description in Proposition 3.14.
We recall the map $\gamma : B\Gamma \to Q\text{Perf}$ with $\gamma^*\mathcal{U} = \mathcal{V}$ for the universal perfect complex $\mathcal{U}$. Here the constant stack $B\Gamma$ is just the stackification of the constant presheaf $\mathcal{U}$.

By the proof of Proposition 3.14 it suffices to construct maps $u(S^{an}) \to Q\text{Filt}$ and $u(S^{an}) \to B\Gamma$ together with a homotopy between the two compositions $u(S^{an}) \to Q\text{Perf}$.

We first observe there is a canonical map of derived stacks $\kappa : u(S^{an}) \to B\Gamma$, independently of $f$. To see this we pick a good hypercover of $S^{an}$, i.e. all $t(T_i)$ and their intersections are contractible. Then we have $S^{an} = \text{hocollim}_i T_i$ by Corollary 3.5 in [40]. We also have $N(I) \simeq t(S^{an})$ by the main result of [15]. Thus there is a homomorphism $\pi_1(N(I)) \to \Gamma$. Thus we know there is a map of topological spaces $N(I) \to B\Gamma$ and we can pick a model $\text{hocollim}_J *$ for $B\Gamma$ such that we get a map $I \to J$ which induces (via the maps $uT_i \to *$ of derived algebraic spaces) a map $\text{hocollim}(uT_i) \to B\Gamma$.

By construction $u$ commutes with homotopy colimits (see Appendix A) and we have:

$$\kappa : u(S^{an}) = u(\text{hocollim} T_i) \simeq \text{hocollim} uT_i \to B\Gamma$$

Note that we are comparing homotopy colimits in simplicial sets and derived stacks. But the inclusion of simplicial sets as constant stacks preserves homotopy colimits since it is a left Quillen functor into the local projective model structure on simplicial presheaves, which is a model for derived stacks.

The pullback of sheaves sends the sheaves on $B\Gamma$ to the sheaves of $\mathcal{O}$-modules on $u(S^{an})$ associated to local systems on $t(S^{an})$. (We can check this locally for the map $u(T_i) \to *$, the complex $V$ is sent to $V \otimes \mathcal{O}_{S^{an}}^{\text{alg}}$.) Thus $(\gamma \circ \kappa)^*\mathcal{U} = \mathcal{V} \otimes \mathcal{O}_{S^{an}}^{\text{alg}}$. We can pull back the bilinear form similarly and this determines the composition $\gamma \circ \kappa : u(S^{an}) \to Q\text{Perf}$.

The map to $Q\text{Filt}$ is given by Lemma 4.7, cf. also Corollary 3.8 applied to $uS^{an}$. That lemma also provides the comparison morphism $v'$ from $F^0$ to a fibrant cofibrant replacement of $Rf_*\mathcal{O} \otimes \mathcal{O}_S$. To obtain $w$ we compose with $\phi^{-1}$.

**Theorem 4.9.** Let $f : X \to S$ be a polarized smooth projective morphism of derived geometric stacks where $S$ is of finite type. Then there is a derived period map $\mathcal{P} : S^{an} \to U/\Gamma$, where $U$ is the derived period domain for the pair $(V, Q)$.

**Proof.** We can write $S = \text{hocollim} S_i$ for a simplicial scheme $S_i$, e.g. using [48]. Then $S^{an} = \text{hocollim} S_i^{an}$ since analytification of stacks is defined as a
left Kan extension. Using contractible Stein hypercovers $T_j^{(i)}$ of the $S_i^{an}$ we may also write $S^{an} = \text{hocolim}_{j \in J} T_j$.

The map $f$ induces maps $f_i : X \times_S S_i \to S_i$, and since $f$ is smooth and projective so are the $f_i$. Moreover they all have homeomorphic fibres, so we can fix $V, Q$ and $n$, and thus the derived period domain $U$.

To globalize this construction we first construct a map $\kappa : u(S^{an}) \to B\Gamma$. We will consider $S^{an} = \text{hocolim}_{j \in J} T_j$ and obtain a canonical map

$$\kappa : u(S^{an}) \to \text{hocolim}_j u(T_j) \to \text{hocolim}_j * \simeq \text{hocolim}_j t(T_j) \to B\Gamma$$

just like in the previous proposition.

We know that $Rf_* f^{-1} O_{S^{an}}$ is quasi-isomorphic to $C \otimes O_{S^{an}}$ for a sheaf of abelian groups $C$ (as it is true locally by Lemma 4.3). It remains to show that $C$ is quasi-isomorphic to a locally constant sheaf, i.e. is obtained by pullback from some sheaf on $B\Gamma$. We have to be a bit careful as we pass from the stack $S^{an}$ to the topological space $\text{hocolim}_j t(T_j)$ and, unlike in the case of a derived scheme, sheaves on $S^{an}$ cannot be described as sheaves on $\text{hocolim}_j t(T_j)$ in general. So we first show that $C$ is a pull-back of an infinity local system on $\text{hocolim}_j t(T_j)$, and then that this infinity local system is a plain (graded) local system.

In the following we will abuse notation and denote by $C$ several corresponding objects that live in different categories.

We know from Lemma 4.3 that $C$ is quasi-isomorphic to $V$ on every $T_j$, thus $C \otimes O_{T_j}$ is just the pull back of $V$ along $u(T_j) \to *$. Moreover the transition functions of $C$ are quasi-isomorphisms. Thus we can view $C$ as a homotopy cartesian section of the constant diagram indexed by $J$ that sends every $j$ to the model category of chain complexes. By strictification (e.g. Theorem 1 in [25]) the category of these homotopy cartesian sections is equivalent to the homotopy limit of the constant diagram $J \to \text{dgCat}$ that sends every $j$ to the dg-category $\text{Ch}$ of (fibrant cofibrant) chain complexes. Now we recall that there is a co-action of simplicial sets on any model category, written $(K, \mathcal{D}) \mapsto \mathcal{D}^K$, see Chapter 1 of [22]. This is a Quillen bifunctor, and thus

$$\text{holim}_j \text{Ch} \simeq \text{Ch}^{\text{hocolim}_j *} \simeq \text{Ch}^{\text{hocolim}_j t(T_j)}.$$  

(Here we use that all the $t(T_j)$ are contractible.) Thus $C$ can be considered as an element of $\text{Ch}^{\text{hocolim}_j t(T_j)}$, which is the category of infinity local systems on $\text{hocolim}_j t(T_j)$, see [24]. (To apply the result of [25] we need to consider a diagram indexed by a direct category, without infinite ascending chains of morphisms. By Lemma 3.10 of [48] the simplicial derived scheme $S_\bullet$ is determined by a finite truncation. Moreover, as $S$ is quasi-compact it is enough to consider finitely many $S_i$. For each $i$ we can choose the hypercovers $T_j^{(i)}$ of the Stein spaces
$S^an$ to be bounded. Restricting to non-degenerates we may thus assume that the diagram $J$ is in fact finite.)

Now $\mathcal{C}$ can equivalently be considered as a homotopy locally constant sheaf on $\text{hocolim}(T_j)$, by Theorem 12 of [25]. (The considerations above make sure that $\text{hocolim}(T_j)$ is equivalent to a homotopy colimit of a finite diagram of points, and thus satisfies the conditions of that theorem.)

Thus $\mathcal{C}$ lives in the bounded derived category of an abelian category, namely sheaves of abelian groups on $\text{hocolim}(T_j)$. We can apply Theorem 3.12 since throughout all the equivalences $\mathcal{C}$ kept its Lefschetz operator. We deduce that $\mathcal{C}$ is a direct sum of its cohomology groups. So it is in fact a local system, and given by a representation $\mathcal{V}$ of the fundamental group of $\text{hocolim}(T_j)$. We note that all automorphisms preserve the extra structure on $\mathcal{V}$, so the classifying map of $\mathcal{V}$ still factors through $\Gamma = \text{Aut}(V, \mathcal{O})$. Thus, as in the proof of the previous proposition we have that $\mathcal{C} \otimes \mathcal{O}_{\text{an}}^\text{alg}$ is equivalent to $\kappa^*\mathcal{V} \simeq (\gamma \circ \kappa)^*\mathcal{U}$ in the homotopy category.

To complete the proof we recall that Lemma 4.7 provides a map from $S$ to $Q\text{Filt}$, and there is a natural map $\nu : \mathcal{C} \otimes \mathcal{O}_{\text{an}}^\text{alg} \rightarrow Rf_*(f^{-1}\mathcal{O}_{\text{an}}) \rightarrow Rf_!\mathcal{O}_{\text{an}} = \mathcal{P}^0$. The map is a quasi-isomorphism as we can check locally by Lemma 4.3. We may consider an inverse, and together with the considerations from the previous paragraph we obtain $w : \mathcal{P}^0 \rightarrow \mathcal{C} \otimes \mathcal{O}_{\text{an}}^\text{alg} \rightarrow \mathcal{V} \otimes \mathcal{O}_{\text{an}}^\text{alg}$ in the homotopy category.

Finally, we observe that the map to $\mathcal{D}_\Gamma$ we have constructed factors through $U/\Gamma$ by classical Hodge theory. □

The following example is crucial:

**Example 1.** The universal example is given by taking $S$ to be the moduli stack of polarized schemes described in Example 3.39 of [47], or rather any quasi-compact component $\mathfrak{M}$ of its substack with smooth fibres. This is a derived 1-geometric stack and has a universal family $X \rightarrow \mathfrak{M}$.

It follows from [47] that $\mathfrak{M}$ is locally of finite presentation, so by Theorem 4.9 we have a derived period map $\mathfrak{M} \rightarrow U_\Gamma$.

4.4. **Comparison with the classical period map.** We now need to check that our construction recovers the usual period map in the underived setting. This is covered by the following theorem.

**Theorem 4.10.** Consider $f : X \rightarrow S$ as above and denote the product of the classical period maps associated to $\pi_0\mathcal{P}_0(f)$ by $P : \pi_0\mathcal{P}_0(S^an) \rightarrow \pi_0\mathcal{P}_0(U)$. Then $P = \pi_0\mathcal{P}_0\mathcal{P}$.
Proof. It is enough to check this locally, so let us replace $S^{an}$ by $T$ and assume $T$ is derived Stein. We may replace the target by $D^{an}$, and we know $\pi_0\pi^0 D^{an}$ is the product of the usual closures of the period domains. Then we consider the map $\pi_0\pi^0(\mathcal{P}) : \pi_0\pi^0 T \to \pi_0\pi^0 D^{an}$, where $\mathcal{P}$ is the derived period map obtained via Theorem A.6 from $\mathcal{P}^\# : uT \to D$, which associates to $A$ a certain filtration of $A$-modules.

The theorem breaks down into two claims:

- The partial adjunction $u \dashv an$ is natural with respect to $\pi_0\pi^0$, i.e. $\pi_0\pi^0(\mathcal{P})$ corresponds under the universal property of underived analytification to $\pi_0\pi^0(\mathcal{P}^\#)$.
- The map $\pi_0\pi^0(\mathcal{P}^\#) : \pi_0\pi^0(uT) \to \pi_0\pi^0 D$ corresponds via the universal property to the usual period map.

For the first claim we note that $u$ and $an$ commute with $\pi_0\pi^0$. Then we observe the universal property of derived analytification is constructed from the affine case, see Lemma A.3 which corresponds to the underived case under $\pi_0\pi^0$.

For the second claim we note that by unravelling definitions $\pi_0\pi^0(\mathcal{P}^\#)(T)$ is the filtration on $V \otimes \pi_0\mathcal{O}(uT)$ induced by the Hodge filtration. This is a filtration of projective $\pi_0\mathcal{O}(uT)$-modules. We compare this with the usual period map, which associates to $T$ the Hodge filtration of $\mathcal{T}_T$-modules. Taking global sections we obtain a filtration of modules over $\pi_0\mathcal{O}(T)$ and it is equal to the filtration we constructed above. □

Together with the next section this shows that the map we construct deserves to be called the derived period map. This also shows that the underlying underived map of $\mathcal{P}$ is smooth if $S$ is smooth, but it is clear that $\mathcal{P}$ is not strong, so it is not smooth in the sense of derived analytic geometry.

4.5. Differential of the period map. In this section we will compute the differential of our period map $\mathcal{P}$ by identifying it with the differential of the infinitesimal period map of derived deformation theories considered in [13, 16, 17]. We will not define derived deformation theory here, the reader unfamiliar with it may skip ahead to the concrete description of $d\mathcal{P}$ in Corollary 4.13 and its consequences.

We recall that in [17] for a smooth projective manifold $X$ a period map is defined which goes from the derived deformation functor associated to the Kodaira-Spencer $L_\infty$-algebra $KS_X$ to derived deformations of an $L_\infty$-algebra $E$ associated to the de Rham complex of $X$. To be precise $E$ is $\text{End}(R^\vee(\Omega_X^\vee))/\text{End}_c(R^\vee(\Omega_X^\vee))[-1]$. We will denote this map by $\mathcal{P}_{\text{FMM}} : \mathbb{R}\text{Def}_{KS_X} \to \mathbb{R}\text{Def}_E$. 
Then Definition 3.41 of [13] presents a geometric version $\mathcal{P}_{\text{inf}}$ of this, which sends deformations of $X$ to the derived flag variety. On objects it sends a family $X \to R\text{Spec}(B)$ over a dg-Artin algebra $B$ to the Hodge filtration on the derived de Rham complex.

The following diagram of derived deformation functors is established in [13]:

\[
\begin{array}{ccc}
R\text{Def}_{K_{X}} & \xrightarrow{\mathcal{P}_{\text{MM}}} & R\text{Def}_{E} \\
\downarrow & & \downarrow \\
R\text{Def}_{X} & \xrightarrow{\mathcal{P}_{\text{inf}}} & D\text{Flag}(R\Gamma(X,\Omega_{X}^{*}))
\end{array}
\]

The left vertical map can be considered the inverse Kodaira-Spencer map.

Deformation functors, just like stacks, have tangent complexes, and the tangent of the map $\mathcal{P}_{\text{MM}}$ has a very concrete description, see Corollary 4.13 below. We will now show this agrees (in a reasonable sense) with the differential of our derived period map $\mathcal{P}$.

Before we can make precise what we mean by agreement, let us briefly recall the tangent space in a general setting. The tangent space of a functor $X$ at a point $x : C \to X$ is the functor that associates to any shifted $C$-module $M$ the space $T_{x}(M) = X(C \oplus M) \times_{X(C),x} *$, where $C \oplus M$ is the square-zero extension of $C$ by $M$. If the functor $F$ is homotopy-preserving and homotopy homogeneous then the tangent space is an abelian group and the homotopy groups satisfy $\pi_{i}T_{x}(M) \cong \pi_{i+1}T_{x}(M[-1])$, see for example Section 1 of [45]. The tangent complex as in Section 3.4 has thus $i$-th cohomology given by $\pi_{0}T_{x}(C[-i])$.

The theory of deformations in derived analytic geometry is considered in detail in [44]. Here we will only need that square zero extensions agree in the derived algebraic and the derived analytic setting, in particular that $R\text{Spec}(C \oplus M) = u((R\text{Spec}(C \oplus M))^{an})$. This will be true in any sensible theory, but as the definition of analytic rings is quite subtle we will not provide a proof here, but refer the reader to the forthcoming [42]. There Porta proves that there is no difference between derived Artin rings and (suitably defined) derived analytic Artin rings by establishing an equivalence $B \simeq (B^{\text{anyalg}})$ for derived Artin rings. As the underlying schemes are points ($-^{\text{alg}}$) and $u$ agree in this case.

Now let us prepare a comparison between $d\mathcal{P}$ and $d\mathcal{P}_{\text{inf}}$. We write $B$ for $C \oplus M$. Then to every family $X \to S$ and map $B \to S$ we can associate the
deformation $\tau^*X \to B$ and $\mathcal{P}_{inf}$ then defines an element in $\mathbb{R}Def_E(B)$, i.e. a map $B \to DFlag$. At the same time $\mathcal{P}$ defines an element in $D^m(B^m)$ by postcomposition.

Since $\mathcal{P}$ is a derived analytic map we will compare it with the analytification of $\mathcal{P}_{inf}$.

We note that the map $\mathcal{P}_{inf}$ by definition goes to the deformation space of the derived flag variety, rather than the derived period domain, i.e. it ignores the bilinear form. Thus we replace $\mathcal{P}$ by the composition $\mathcal{P}' : S \to U \to D^m \to DFlag^m$, obtained by composing the period map with the inclusion $U \to D^m$ and the natural forgetful map $D^m \to DFlag^m$.

We note two things: As we work infinitesimally we may ignore monodromy and assume the target of the period map is $U$ rather than $U/\Gamma$. Secondly, we can consider the composition with the map $U \to DFlag^m$, and then we no longer need to refer to the polarization to construct the derived period map.

Thus the following result tells us that the differential of $\mathcal{P}$ is the analytification of the differential of the infinitesimal derived period map.

**Proposition 4.11.** Given $X \to S$ and $\tau : B \to S$ the maps $\mathcal{P}' \circ \tau^m$ and $\mathcal{P}_{inf}^m$ from $B^m$ to $DFlag^m$ agree and the correspondence is natural.

**Proof.** This follows from naturality of the function between mapping spaces in Theorem [A.6]. Denote that map by $\Phi$ and let $B$ be a dg-Artin ring. Note first that $\mathcal{P}^m := \mathcal{P} \circ \tau^m : u(B^m) \to DFlag$ is the map that is sent by $\Phi$ to $\mathcal{P}' \circ \tau^m : B^m \to DFlag^m$. We write $h_B : (B^m)_{alg} \to B$ for the natural map, which corresponds to $1_{B^m}$ under the analytification adjunction. As $B$ has just one point it is clear that $(B^m)_{alg}$ is $u(B^m)$ and $h_B$ also corresponds to the identity under $\Phi$.

We then check $\mathcal{P}^m \circ 1_{B^m} = \mathcal{P}_{inf} \circ h_B$ by unravelling definitions. Both maps send $u(B^m)$ to the filtration $h_B^*(Rf_!\Omega^p_{X/S})$ of $\mathcal{O}_{uB^m}$-modules. Porta’s GAGA theorem shows that pushing forward along $f$, respectively $f^m$, and pulling back along $h_B$ commute.

Now applying $\Phi$ to left hand side of this identity gives $\mathcal{P}' \circ \tau^m$. By naturality the right hand side is sent to $\mathcal{P}_{inf}^m \circ 1_{B^m}$ as $1_{B^m}$ is $\Phi(h_B)$. □

**Proposition 4.12.** The tangent space of $D^m$ at any point agrees with the tangent space of $D$.

**Proof.** This follows from the identification of derived analytic and derived algebraic Artin rings together with the adjointness property of analytification that will be established in [26]. □
In fact, by the same argument analytification does not change any tangent spaces or induced maps between them. We can thus use the computation from [13] to determine the differential of the period map.

**Corollary 4.13.** At a point $s \in S$ the differential of the derived period map $d\mathcal{P} : TS \to TF_{w,Q_r}D_{V,Q}$ is obtained by composing $\tau^* : TS \to T\mathbb{R}Def_X$ with the Kodaira-Spencer map

$$T\mathbb{R}Def_X \to T\mathbb{R}Def_{KS_X} \approx R\Gamma(X, \mathcal{T}_X)[1]$$

and the map

$$R\Gamma(X, \mathcal{T}_X)[1] \to (\underline{\text{End}}_{Q_r}(V)/\underline{\text{End}}_{F,Q_r}(V))[1]$$

induced by the action of $\mathcal{T}_X$ on $\Omega_X$ and the product on derived global sections.

**Proof.** This follows from the theorem as all higher cohomology is determined by evaluating the tangent functor at shifts of $\mathbb{C}$. Other than analytification the only difference to [13] is that the second map goes to $(\text{End}(V)/\text{End}_{r}(V))[1]$. But this map as constructed clearly factors through $(\underline{\text{End}}_{Q_r}(V)/\underline{\text{End}}_{F,Q_r}(V))[1]$.

In other words, the differential of the derived period map can be computed, just like the underived version, by cup and contract.

This characterization of the differential immediately gives the following corollary:

**Corollary 4.14.** The derived period map satisfies Griffiths transversality.

This says that the tangent of the derived period map lies in a subcomplex of the tangent complex of the period domain, namely endomorphisms shifting the filtration by one only. Hence we could call the image of the period map a horizontal derived substack.

**Remark 4.7.** At this stage we would like to say that the infinitesimal derived period map is directly obtained from the global derived period map. Indeed, morally the map $\mathcal{P}_{inf}$ is the restriction of $\mathcal{P}$ to dg-Artin algebras.

However, to make this argument precise we would need to use a moduli stack of all varieties (as opposed to polarized varieties or subschemes of a fixed variety). But the deformation functor $\mathbb{R}Def_{X_0}$ does not extend to an algebraic stack. This can be seen by considering $K3$-surfaces: The deformation functor is unobstructed with a 20-dimensional tangent space, but algebraic families of $K3$-surfaces can only be 19-dimensional.
Remark 4.8. This is a good moment to note that our construction is still in some sense classical, and there should be a generalized (one could also say extended or non-commutative) version of the period map that starts from $R\Gamma(X, \wedge^\ast TX)$, which classifies $A_\infty$-deformations of $D^b_{coh}(X)$. This appears in work by Kontsevich and Barannikov \cite{3, 31}. To take this further, one would need to understand non-commutative Hodge structures, see \cite{30}.

The reader may also consider \cite{16} for an explicit period map for generalized deformations of a Kähler manifold and \cite{29} for a period map for deformations of a non-commutative algebra.

4.6. Examples. We will finish by briefly talking about some examples for the derived period map.

Recall that we already described the universal derived period map defined on substacks of the moduli space $\mathcal{M}$ of polarized schemes in Example 1.

**Example 2.** Another class of examples is given by letting $S$ be a subscheme of the derived Hilbert scheme $D\text{Hilb}_Y$ for some projective variety $Y$. This was constructed originally in \cite{9} and appears in a more modern context in \cite{32}, see also Section 3.3 of \cite{47}. $D\text{Hilb}_Y$ is a geometric stack locally of finite presentation that parametrizes derived families of subschemes of $Y$, to be precise it represents the functor sending $S$ to derived $Y \times S$-schemes which are proper and flat over $S$ and for which the map to $Y \times S$ is a closed immersion almost of finite presentation.

In both of these examples it is useful to observe the following: Fix a smooth scheme $X_0$ or a smooth subscheme $X_0$ of some projective $Y$. Then there is an open substack $S$ of $\mathcal{M}$ respectively $D\text{Hilb}_Y$ containing $X_0$ such that the universal family $f : X \to S$ is smooth and projective. (The underived truncation of $f$ being smooth is an open conditions and the condition that $f$ is strong is again open as it says a certain map of graded modules on $\pi_0(S)$ is an isomorphism.) To apply the main theorem we just need to make sure $S$ is quasi-compact, restricting to some substack if necessary. (Note that this is equivalent to the underlying underived space of $S$ being quasi-compact, see Lemma 3.1.)

The universal families on $\mathcal{M}$ and $D\text{Hilb}_Y$ give smooth projective families of derived schemes $X \to S$ by restriction and thus derived period maps $\mathcal{P} : S^{an} \to U/\Gamma$.

Let us now look infinitesimally at two concrete examples. We note that the tangent complex for the moduli stack $\mathcal{M}$ of polarized schemes at $X$ is an extension of $\mathcal{T}_X[1]$ by $\mathcal{O}_X[1]$, see Example 3.39 in \cite{47}. The tangent
complex of the Hilbert scheme $D \text{Hilb}_Y$ at $X \subset Y$ is $L^*_{X/Y}$, if $X$ is smooth this is just the normal bundle.

Thus the cases where $H^i(\mathcal{J}_X)$ is nonzero for $i > 1$ will often correspond to interesting derived information in $S$.

**Example 3.** If $X_0$ is any Calabi-Yau variety then the infinitesimal derived period map induces an injection on the cohomology groups of tangent complexes, this follows from Theorem B of [27].

We can use this to write down examples where the derived period map is a non-trivial enhancement of the usual period map. Let us for example consider an abelian surface $X$ in a component $\mathcal{M}$ of the moduli stack of polarized varieties. Then $X$ has nontrivial $H^2(T_X)$, and there is a surjection from $\pi_1(T_X|\mathcal{M})$ to $H^2(\mathcal{J}_X)$. Thus the derived period map induces a non-trivial map in degree 1 of the tangent complex at $X$.

**Remark 4.9.** Note that Theorem A of [27] says that whenever the period map induces an injection on the cohomology of tangent spaces then the deformation theory of $X_0$ is unobstructed, as the deformation theory of a filtered complex is governed by a quasi-abelian $L_\infty$-algebra. (In fact generalized deformations of Calabi-Yau varieties are still unobstructed, see [30].)

In particular the derived period map can only see unobstructed deformations of the special fibre, and the kernel of the differential of the derived period map is a reduced obstruction theory.

One question arising immediately from the existence of the period map is when it is injective in a suitable sense, i.e. when a moduli problem can be completely embedded in the period domain (which has a nice description in terms of linear and quadratic data).

One possible way of making this precise would be to ask when $\pi_0 \pi^0(\mathcal{P})$ is an immersion and $d \mathcal{P}$ is injective on homotopy groups.

**Remark 4.10.** We note that this is not the definition of a closed immersion in the sense of [62] (which does not constrain the homotopy groups of the domain), nor is it a strong map (which constrains the homotopy groups of the domain too much). There may be a more natural way of posing this question.

This is a kind of derived Torelli problem. (The reader should note that this term is already used for the problem of determining when the period map determines the derived category of a variety.)
Example 4. Now consider the case where $X_0$ is a hypersurface of degree $d$ in $\mathbb{P}^3$. We may consider it in the moduli stack $\mathcal{M}$ of polarized surfaces. Then derived Torelli is false if $d$ is large enough, i.e. the map on homotopy groups of tangent complexes is not an injection. It follows from our tangent space considerations that derived deformations of $X_0$ in degree 1 surject to $H^2(\mathcal{T}_{X_0})$. One can compute that $H^2(\mathcal{T}_{X_0})$ is nonzero (using Kodaira vanishing, Hirzebruch-Riemann-Roch and the normal exact sequence). But as $H^r(X_0, \Omega^r)$ is concentrated in even degrees the infinitesimal period map must send $H^2(\mathcal{T}_{X_0})$ to zero.

This is particularly interesting since projective hypersurfaces of degree bigger than three satisfy the local Torelli theorem, see [7], thus $\pi_0 \mathcal{M}_0(\mathcal{P})$ is locally an immersion.

Appendix A. Presenting higher analytic stacks

A.1. Introduction. In this appendix we show that Pridham’s framework of presenting higher stacks as hypergroupoids developed in [48] can be used to provide models for derived analytic Artin stacks. For an overview of the general theory see [46].

We then show that this description allows us to define an algebraization construction $u$ and to find a natural map from $\text{Map}(u(T), Y)$ to $\text{Map}(T, Y^{an})$ for a derived Artin stack $Y$ locally of finite presentation and a derived analytic Artin stack $T$.

A.2. Hypergroupoids. In [48] Pridham develops the theory of hypergroupoids as presentations of higher stacks.

To talk about hypergroupoids we need to fix a model category $\mathcal{I}$ with a nice subcategory $\mathcal{A}$ and classes $C$ and $\epsilon$ of morphisms in $\text{Ho}(\mathcal{A})$ and $\text{Ho}(\mathcal{I})$ respectively, satisfying some conditions we will detail below.

Example 5. The motivating example is when $\mathcal{I}$ is the model category of stacks on derived affine schemes over some ground ring $k$ with the étale topology. $\mathcal{A}$ consists of the essential image of the Yoneda embedding, $\epsilon$ is the class of local surjections in $\text{Ho}(\mathcal{I})$ and $C$ the class of smooth maps in $\text{Ho}(\mathcal{A})$ which are moreover in $\epsilon$. We write this quadruple as $(\mathcal{A}_{alg}, \mathcal{A}_{alg}, C_{alg}, \epsilon_{alg})$

We write $s\mathcal{A}$ for $\mathcal{A}^{\Delta^{op}}$. 
Definition. A map \( X \rightarrow Y \) in \( s\mathcal{A} \) is a relative \((n, C)\)-hypergroupoid over \( Y \) if it is a Reedy fibration and the homotopy partial matching maps

\[
X_m \rightarrow M^h_{\Delta^n}(X) \times_{M^h_{\Delta^n}(Y)} Y_m
\]

are in \( C \) for all \( m, k \), and weak equivalences if \( m > n \). A relative hypergroupoid over the final object is simply called a \((n, C)\)-hypergroupoid.

Definition. A map \( X \rightarrow Y \) in \( s\mathcal{A} \) is a trivial relative \((n, C)\)-hypergroupoid over \( Y \) if it is a Reedy fibration and the homotopy matching maps

\[
X_m \rightarrow M^h_{\partial \Delta^n}(X) \times_{M^h_{\partial \Delta^n}(Y)} Y_m
\]

are in \( C \) for all \( m \), and weak equivalences if \( m \geq n \).

We will fix \( n \) in this section and simply speak of hypergroupoids when \( C \) is clear from the context.

Here the matching objects functor \( M^h(X) : s\text{Set}^{op} \rightarrow \mathcal{A} \) is defined by forming the right Kan extension of the functor \( X : \Delta^{op} \rightarrow \mathcal{A} \), see Section 1.1.1 of [48]. It is the derived version of \( \text{Hom}_{\text{Set}}(-, X) \).

Now the geometric realization functor \( | - | \) from \( s\mathcal{A} \) to \( \mathcal{S} \) sends hypergroupoids to geometric stacks, see Proposition 4.1 of [48]. (This is just another name for the homotopy colimit over \( \Delta^{op} \).) For simplicity we will always compose \( | - | \) with stackification (i.e. fibrant replacement in \( \mathcal{S} \)) without mentioning it in the notation.

Theorem 3.3 in [46] says that in the example above the relative category of \((n, \mathcal{C}_\text{alg})\)-hypergroupoids with weak equivalences given by trivial hypergroupoids gives a model for the \( \infty \)-category of strongly quasi-compact \( n \)-geometric derived Artin stacks.

Remark A.1. Recall that a relative category [4] is just a category with a collection of weak equivalences satisfying some very basic conditions. Nevertheless, the model category of relative categories is Quillen equivalent to other models of \( \infty \)-categories (i.e. simplicial categories, quasi-categories, Complete Segal Spaces).

We will denote the levelwise hom-space for \( s\mathcal{A} \) by \( \text{Hom} \), but it is important to note that we cannot compute hom-spaces levelwise. Instead we define

\[
\text{Hom}^\#_{s\mathcal{A}}(X, Y) = \text{Hom}_{\text{pro}(s\mathcal{A})}(\tilde{X}, Y)
\]

where \( \tilde{X} \rightarrow X \) is a \( \mathcal{T} \)-projective relative \( \mathcal{T} \)-cocell where \( \mathcal{T} \) is the class of trivial relative hypergroupoids. A relative \( \mathcal{T} \)-cocell is just a transfinite composition of pullbacks of maps in \( \mathcal{T} \). For detailed definitions see Section 3.2 of [48]. To understand this definition it helps to note that
\[ \pi_0(\text{Hom}^\#(X, Y)) = \lim_{X' \in \mathcal{P}(X)} \pi_0\text{Hom}(X', Y), \]

where the limit is over trivial relative hypergroupoids over \( X \).

Then if \( Y \) is a hypergroupoid we have that \( \text{Hom}^\#_{\mathcal{P}}(X, Y) \cong \text{Map}_\mathcal{P}(|X|, |Y|) \), see Theorem 4.10 in [48].

We want to apply this framework to study analytic stacks. Thus we need a new choice of \((\mathcal{A}, \mathcal{I}, C, \epsilon)\).

Recall that \( d\text{Stein} \) is the full subcategory of \( d\text{An} \) given by those derived analytic spaces whose underlying underived analytic space is Stein. This is constructed as a quasi-category in [40].

We need \( \mathcal{A} \) to be a pseudo-model category. To be precise we need a model category \( \mathcal{I} \), a class of morphisms \( W_{\mathcal{A}} \) in \( \mathcal{A} \) and a fully faithful \( \iota : \mathcal{A} \to \mathcal{I} \) such that

- \( \iota(W_{\mathcal{A}}) = W_{\mathcal{I}} \cap \iota(\mathcal{A}) \). Here \( W_{\mathcal{I}} \) are the weak equivalences in \( \mathcal{I} \).
- \( \mathcal{A} \) closed under weak equivalences in \( \mathcal{I} \), i.e. \( x \simeq \iota(y) \) implies \( x \) is in the image of \( \mathcal{A} \).
- \( \mathcal{A} \) is closed under homotopy pullbacks in \( \mathcal{I} \).

Consider the quasi-category \( d\text{Stein} \) and recall that simplicial categories and quasi-categories are equivalent models of \( \infty \)-categories. We apply \( \mathcal{C} \), the left adjoint of the coherent nerve \( N \), to obtain a simplicial category \( \mathcal{C}[d\text{Stein}] \). We consider \( \text{sSet}^{\mathcal{C}[d\text{Stein}]} \), the simplicial category of functors from \( \mathcal{C}[d\text{Stein}] \) to the simplicial model category of simplicial sets, \( \text{sSet} \). We equip this category with the injective model structure. There is a Grothendieck topology \( \tau \) on \( d\text{Stein} \) induced by étale morphisms of \( \mathcal{T}_{\text{an}} \)-structured topoi (see Definition 2.3.1 of [34]). We localize \( \text{sSet}^{\mathcal{C}[d\text{Stein}]} \) at homotopy \( \tau \)-hypercovers to obtain the model category of stacks, which we call \( \mathcal{I}_{\text{an}} \). As \( \tau \) is subcanonical, see Corollary 3.4 in [40], we have a fully faithful embedding \( y : \mathcal{C}[d\text{Stein}] \to \mathcal{I}_{\text{an}} \).

We let \( \mathcal{A}_{\text{an}} \) be the closure of \( y(\mathcal{C}[d\text{Stein}]) \) under weak equivalences and define \( W_{\mathcal{A}} \) by the first condition above.

Thus the first two conditions are satisfied. For the third condition we need to know that \( d\text{Stein} \) is closed under homotopy pullbacks. By Proposition 1.4 of [40] \( d\text{An} \) has (homotopy) pullbacks. Now note that \( \pi^0 \) commutes with limits and the fibre products of Stein spaces are Stein spaces, see Example 51 (b) in [49].

**Remark A.2.** For the reader’s peace of mind we note that working with simplicial presheaves as simplicial functors is equivalent to working with presheaves defined in the quasi-categorical setting as in [43]. This follows from Proposition 4.2.4.4 of [35]. Setting \( \mathcal{U} = A = \text{sSet} \) we have
\[ N(\mathbf{sSet}^{[d\text{Stein}]})^\circ \cong \text{Fun}(d\text{Stein}, N(\mathbf{sSet})) \] where \( N \) is the coherent nerve. Here \( (-)^\circ \) restricts to fibrant cofibrant objects in a model category. Of course \( N(\mathbf{sSet}) \) is the infinity category of simplicial sets. In other words the simplicial category \( \mathbf{sSet}^{[d\text{Stein}]})^\circ \) models the quasi-category of functors from \( d\text{Stein} \) to simplicial sets. This is true for the injective or projective model structure.

The reason we are working not with quasi-categories but simplicially with \( \mathbf{sSet}^{[d\text{Stein}]})^\circ \) is that we rely on explicit arguments involving simplicial diagrams in Theorem A.1 below.

Having made these observations we will abuse notation and use \( d\text{Stein} \) for \( \mathcal{C}[d\text{Stein}] \) from now on.

Next, given the pseudo-model category \( \mathcal{A}_{an} \subset \mathcal{J}_{an} \) we need a class of \( \epsilon \)-morphisms in \( \mathcal{J}_{an} \), functioning as covers, closed under composition and homotopy pullback. They need to satisfy Properties 1.7 of [48].

We just imitate the algebraic definition and let \( \epsilon_{an} \) be the class of local surjections, i.e. maps such that the associated map of simplicial sheaves is surjective on \( \pi_0 \).

Next we need a class \( \mathcal{C}_{an} \) of morphisms in \( \text{Ho}(\mathcal{A}_{an}) \) containing isomorphisms and closed under composition and homotopy pullback, and satisfying Properties 1.8 of [48]. We define \( \mathcal{C}_{an} \) to consist of smooth maps which are also in \( \epsilon_{an} \).

**Theorem A.1.** \( (\mathcal{A}, \mathcal{J}_{an}, \mathcal{C}_{an}, \epsilon_{an}) \) as above satisfies Properties 1.7 and 1.8 of [48].

**Proof.** We just follow the proof of Proposition 1.19 of [48].

1.8 is true for any simplicial site. For (1) see Proposition 3.1.4 in [61], (2) holds by definition and (3) is clear as smooth maps are defined locally.

For 1.7 we use \( d\text{Stein} \) in place of the category \( \mathcal{T} \) in loc. cit. and then consider simplicial objects in \( \mathcal{J} := \mathcal{J}_{an} = s\text{Pr}_{\text{inj}}(d\text{Stein}) \), which is a category of simplicial presheaves with the local injective model structure on a simplicial site, thus all the computations involved are entirely unchanged.

**Remark A.3.** The last ingredient we need is Assumption 3.20 of [48]. However, we can always satisfy the condition by choosing two universes, see Remark 3.21 in loc. cit.

Note that we may replace objects in \( s\mathcal{A}_{an} \) by objects in \( d\text{Stein}^{\text{op}} \) and talk about genuine simplicial derived Stein spaces. This is shown in Section 4.4.
of [48], the main input being the proof of Lemma 1.3.2.9 of [62], which goes through in the analytic setting.

We can now consider the category of derived analytic stacks as modelled by hypergroupoids. As all the results in [48] hold for analytic stacks we can deduce the following:

**Theorem A.2.** The relative category of \((n, C_{\text{an}})\)-hypergroupoids, with weak equivalences given by relative trivial hypergroupoids, is a model for the \(\infty\)-category of strongly quasi-compact \(n\)-geometric derived analytic Artin stacks.

Here we note that derived Artin stacks are the subcategory of derived analytic stacks \(\mathcal{S}_{\text{an}}\), given by the usual representability conditions, cf. Definition 2.8 of [43].

**Remark A.4.** Note that there is a difference in definition between a geometric context in the sense of [43], like \((\text{Stein}, \tau, P)\), and a HA-context in the sense of [62], which is cited in [48]. But the definition of geometricity is the same in both contexts.

A.3. **Analytification of affines.** The derived analytification functor \(\text{an} : X \mapsto X^{\text{an}}\) between structured \(\infty\)-topoi sends derived affine schemes to \(\mathcal{J}_{\text{an}}\)-affines, see Proposition 2.3.18 of [34]. In particular derived affine schemes locally of finite presentation are sent to derived Stein spaces. (Analytification maps \(\mathcal{I}_{\text{et}}\)-schemes to \(\mathcal{I}_{\text{an}}\)-schemes, and the truncation is sent to a Stein space.) We write \(\text{dAff}_{\text{fp}}\) for derived affine schemes locally of finite presentation. (These are just homotopically finitely presented simplicial algebras.)

We now want to consider a partial left adjoint to this functor and define \(u = R\text{Spec}(O(-))\) on \(\text{dStein}\). This is an affinized forgetful or algebraization functor. It is clear that \(u\) sends affines to affines.

**Lemma A.3.** There is a natural weak equivalence \(\text{Map}_{\text{dStein}}(T, Y^{\text{an}}) \simeq \text{Map}_{\text{dAff}_{\text{fp}}}(uT, Y)\) for a derived Stein spaces \(T\) and a derived affine scheme \(Y\) locally of finite presentation.

**Proof.** We consider the functor \(R\text{Spec}(U(-)) \circ (-)^{\text{alg}}\) between the \(\infty\)-categories \(\text{Top}(\mathcal{I}_{\text{an}})\) and \(\text{dAff}\). This extends the functor \(u\) we are considering. Moreover, as a composition of right adjoints it has a left adjoint: \(\text{dAff} \to \text{Top}(\mathcal{I}_{\text{an}})\) given by inclusion followed by analytification. Now we note that \(\text{an}\) sends \(\text{dAff}_{\text{fp}}\) to \(\text{dStein}\) and conclude. \(\square\)

**Remark A.5.** There is a slightly more concrete way of seeing this weak equivalence: The correspondence is clear if \(Y = \mathbb{k}^1\), as in this case both
sides are just global functions on $T$. Then we can extend to all derived affine schemes by observing that $\text{an}$ preserves limits, and derived affine schemes are generated under limits by $\mathbb{A}^1$.

**Remark A.6.** In fact we can extend $\text{Map}(T, Y^{an}) \simeq \text{Map}(uT, Y)$ to the case where $T$ is a derived analytic space which is a colimit of derived Stein spaces. We can thus extend to simplicial derived Stein spaces, but note that $u$ is not going to preserve hypergroupoids.

### A.4. Analytification of hypergroupoids

We now join the affine adjunction and the construction of hypergroupoids. We will apply the functors $\text{an} : \text{dAff}^{f/p} \to \text{dStein}$ and $u : \text{dStein} \to \text{dAff}$ levelwise to simplicial objects.

Given a derived analytic Artin stack $X$ we construct a hypergroupoid $X$ in derived Stein spaces, with $|X| \simeq \mathcal{X}$, by Theorem 4.7 in [48]. We let $u(\mathcal{X}) := |u(X)|$ and say $u(\mathcal{X})$ is an algebraization of $\mathcal{X}$. Similarly a morphism between Artin stacks has an algebraization by Proposition 4.9 in [48], at the cost of replacing $X$ by a relative trivial hypergroupoid $X' \to X$, which of course still satisfies $|X'| \simeq \mathcal{X}$. We define $\text{an}(\mathcal{Y})$ similarly (we distinguish it from $\mathcal{Y}^{an}$ for the moment).

**Remark A.7.** We note that this is not a functorial definition. It is not even clear that $u(\mathcal{X})$ is well-defined. However, the functor $\text{Hom}^g(u(\mathcal{X}), -)$ on hypergroupoids, which is the object we are interested in, is well-defined, as follows from Theorem [A.6] below.

Thus even if $u$ is not well-defined, we can use it to compute maps into the analytification of an Artin stack.

**Lemma A.4.** **Analytification preserves (trivial) relative hypergroupoids.**

**Proof.** As $\text{an}$ is a right adjoint it preserves homotopy limits in $\text{dAff}$. Moreover it sends smooth maps to smooth maps and preserves local surjections. It follows that it preserves relative hypergroupoids. \qed

Next we check that this analytification agrees with the Lurie-Porta definition that we quoted in Section [2.7]. This also shows it is well-defined.

**Lemma A.5.** Let $Y$ be a hypergroupoid in $\text{dAff}$. Then $|\text{an}(Y)|_{\mathcal{X}^{an}} = (|Y|_{\mathcal{X}^{alg}})^{an}$. Thus $\text{an}(\mathcal{Y}) \simeq \mathcal{Y}^{an}$.

**Proof.** This just says that analytification commutes with geometric realization. This is clear as analytification of Artin stacks is defined as a left Kan extension. \qed
Theorem A.6. Let $\mathcal{X}$ be a derived analytic Artin stack and $\mathcal{Y}$ a derived algebraic Artin stack. With $u$ and $\text{an}$ defined levelwise there exists a map

$$\text{Map}_{\mathcal{S}_{\text{alg}}}(u(\mathcal{X}), \mathcal{Y}) \rightarrow \text{Map}_{\mathcal{S}_{\text{an}}}(\mathcal{X}, \mathcal{Y}_{\text{an}})$$

which is natural in $\mathcal{X}$ and $\mathcal{Y}$.

Proof. By Theorem 4.7 of [48] applied in the analytic and algebraic setting we can find resolutions $\mathcal{X} = |\mathcal{T}|$ and $\mathcal{Y} = |\mathcal{Y}|$. Then we can compute

$$\text{Map}_{\mathcal{S}_{\text{alg}}}(u(\mathcal{T}), \mathcal{Y}) \simeq \text{Hom}_{\mathcal{S}_{\text{alg}}}^\#(uT, Y)$$

by Theorem 4.10 in [48].

Before looking at $\text{Hom}^\#$ in more detail we need some notation. Let $\mathcal{A}$ be $\mathcal{A}_{\text{alg}}$ or $\mathcal{A}_{\text{an}}$. We write $HG$ for the subcategory of $s\mathcal{A}$ formed by hypergroupoids and let $\mathcal{T}$ be the class of relative trivial hypergroupoids. We denote by $\text{Map}_{s\mathcal{A}}$ the mapping space associated to levelwise weak equivalence. We also let $\text{Hom}_{s\mathcal{A}}(X, Z)_n := \text{Hom}_{s\mathcal{A}}(X \times \Delta^n, Z)$ and note there is a natural comparison map $\eta : \text{Hom}_{s\mathcal{A}}(X, Z) \rightarrow \text{Map}_{s\mathcal{A}}(X, Z)$.

We will now adapt the proof of Theorem 3.3 in [46] to show that $\text{Hom}_{s\mathcal{A}}^\#(uT, -)$ is the right derived functor of the mapping space functor $\text{Map}_{s\mathcal{A}}(uT, -)$ on hypergroupoids with respect to $\mathcal{T}$, (Here by a right derived functor we just mean a functor into $\text{Ho}(s\text{Set})$ which sends maps in $\mathcal{T}$ to isomorphisms and is the universal such functor under $\text{Map}$.) We will in fact show that $\text{Hom}^\#$ considered on homotopy categories is the right derived functor of the bifunctor $\text{Map} : pro(s\mathcal{A}) \times HG \rightarrow \text{Ho}(s\text{Set})$ with respect to $\mathcal{T}$ in the second variable and relative $\mathcal{T}$-cocells in the first variable. Here for a pro-object $(X_n)$ and an object $Z$ in $HG$ considered as a constant pro-object we have $\text{Map}_{pro(s\mathcal{A})}(X_n, Z) = \text{hocolim}_n \text{Map}_{s\mathcal{A}}(X_n, Z)$ and similarly for $\text{Hom}$. We recall that $\text{Hom}_{s\mathcal{A}}^\#(X, Z)$ is $\text{Hom}_{pro(s\mathcal{A})}^\#(\bar{X}, Z)$ for a $\mathcal{T}$-projective relative $\mathcal{T}$-cocell $p : \bar{X} \rightarrow X$ which can be chosen functorially, see Proposition 3.24 of [48]. We consider $Z$ as a constant pro-object.

To show that $\text{Hom}^\#$ is the right derived functor we need to check first that it sends maps in $\mathcal{T}$ to isomorphisms, which follows from Lemma 3.19 of [48], and also sends relative $\mathcal{T}$-cocells to isomorphisms, which is Lemma 3.26 in loc. cit. Next, there is a natural transformation $\phi : \text{Map} \rightarrow \text{Hom}^\#$ by the universal property of $\text{Map}$ as $\text{Hom}^\#$ preserves levelwise weak equivalences. To see that $\text{Hom}^\#$ preserves levelwise weak equivalences we write it as $\text{Hom}_{pro(s\mathcal{A})}^\#(\bar{X}, Z)$ where $\bar{X}$ is a diagram of Reedy cofibrant objects and $Z$ is Reedy fibrant, see Proposition 3.24 and Lemma 3.26 of [48].
There is a natural map $\tilde{\eta} : \text{Hom}^\#(X, Z) \to \text{Map}(\tilde{X}, Z)$ and it follows from the universal property of $\text{Map}$ that there is a factorization of $p^* : \text{Map}(X, Y) \to \text{Map}(\tilde{X}, Y)$ as $\tilde{\eta} \circ \phi$.

We can now show $\text{Hom}^\#$ is universal. Let $F$ be some bifunctor $\text{pro}(s\mathcal{A} \times \mathcal{H}) \to \text{Ho}(s\text{Set})$ sending relative $\mathcal{T}$-cocells respectively $\mathcal{T}$ to isomorphisms and equipped with a natural transformation $f : \text{Map} \to F$. We need to factor $f$ through $\phi$. By assumption $F$ sends the relative $\mathcal{T}$-cocell $p : \tilde{X} \to X$ to an isomorphism $p^\#_\phi$. Thus the commutative diagram

$$
\begin{array}{ccc}
\text{Map}(X, Y) & \xrightarrow{f(\cdot, Y)} & F(X, Y) \\
\phi \downarrow & & \downarrow p^\#_\phi \\
\text{Hom}^\#(X, Y) & \xrightarrow{f(\cdot, Y)^\#} & F(\tilde{X}, Y)
\end{array}
$$

gives the desired factorization $(p^\#_\phi)^{-1} \circ f_{X,Y} \circ \tilde{\eta} \circ \phi = f(\cdot, Y)$. We can now fix the first argument for the remainder of the proof.

As we can compute the mapping space in $s\mathcal{A}$ from mapping spaces in $\mathcal{A}$ the weak equivalences $\text{Map}_{s\mathcal{A}}(uT_i, Y_i) \simeq \text{Map}_{s\mathcal{A}^\mathcal{an}}(T_i, Y_i)$ of Lemma A.3 give rise to weak equivalences $\text{Map}_{s\mathcal{A}}(uT, Y) \simeq \text{Map}_{s\mathcal{A}^\mathcal{an}}(T, Y^\mathcal{an})$ natural in $T$ and $Y$.

Thus we have a natural isomorphism $\text{Map}_{s\mathcal{A}}(uT, -) \simeq \text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -) \circ (-)^\mathcal{an}$. To consider the derived functor we have to take some care as $(-)^\mathcal{an}$ does not identify $\mathcal{T}_{\mathcal{alg}}$ and $\mathcal{T}_{\mathcal{an}}$. However, the image of $\mathcal{T}_{\mathcal{alg}}$ under $(-)^\mathcal{an}$ is contained in $\mathcal{T}_{\mathcal{an}}$. Thus we define $R' \text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -)$ as the derived functor of $\text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -)$ with respect to all maps in the image of $\mathcal{T}_{\mathcal{alg}}$ under analytification.

It follows that the derived functor $\text{Hom}^\#_{s\mathcal{A}^\mathcal{an}}(uT, Y)$ is the composition of derived functors $R' \text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -) \circ R(-)^\mathcal{an}$. Then by the universal property of the derived functor there is a natural transformation from $R' \text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -)$ to $R\text{Map}_{s\mathcal{A}^\mathcal{an}}(T, -)$ as the analytifications of elements of $\mathcal{T}_{\mathcal{alg}}$ form a subset of $\mathcal{T}_{\mathcal{an}}$.

Now $(-)^\mathcal{an}$ preserves relative trivial hypergroupoids by Lemma A.4 so it already is its own derived functor. Thus we have constructed a map $\text{Hom}^\#_{s\mathcal{A}^\mathcal{an}}(uT, Y) \to \text{Hom}^\#_{s\mathcal{A}^\mathcal{an}}(T, Y^\mathcal{an})$. Now we apply Theorem 4.10 of [48] to the right hand side to deduce the theorem.

Naturality follows as any morphism $T \to T'$ gives a natural transformation of functors, and by universality a natural transformation between derived functors. We may need to replace our model $T$ for $\mathcal{X}$ by a relative trivial...
hypergroupoid, but this does not affect the argument. (Note that naturality

can be expressed as the fact that the “counit” of our comparison map is

a natural transformation. There is no unit as we do not actually have an

adjunction.)

Remark A.8. Note that the map we construct is not in general a weak
equivalence. (Contrary to a claim in an earlier version of this paper.) A
counterexample is provided by taking $\mathcal{Z}$ a derived Stein space (for example
$\mathbb{C}$) and $\mathcal{Y}$ the moduli stack of perfect complexes. Since $\mathcal{Y}^{an}$ is the analytic
moduli stack of perfect complexes, see [26], we see that the right hand
side consists of perfect complexes on $\mathcal{Z}$ and the left hand side of perfect
complexes over $\mathcal{O}(\mathcal{Z})$. These two categories are not equal as only perfect
complexes on a Stein space that are globally of finite presentation come
from complexes of modules over global sections.

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