POINTWISE BOUNDS FOR THE GREEN'S FUNCTION FOR THE NEUMANN-LAPLACE OPERATOR IN $\mathbb{R}^3$

DAVID HOFF
Indiana University
Department of Mathematics
Bloomington, IN, USA

(Communicated by Toan T. Nguyen)

Bob Glassey and I often discussed the pedagogy of applied analysis, agreeing in particular that elementary facts should have elementary proofs. This work is offered in that spirit and in his memory.

Abstract. We derive pointwise bounds for the Green’s function and its derivatives for the Laplace operator on smooth bounded sets in $\mathbb{R}^3$ subject to Neumann boundary conditions. The proofs require only ordinary calculus, scaling arguments and the most basic facts of $L^2$-Sobolev space theory.

1. Introduction. In this paper we give an elementary derivation of pointwise bounds for the Green’s function and its derivatives for the Laplace operator on a smooth, bounded set in $\mathbb{R}^3$ subject to Neumann boundary conditions. The Green’s function is defined on the cross product of the set with itself and defines integral operators by which a given smooth function can be expressed in terms of its Laplacean in the interior and its normal derivative on the boundary. Our purpose is to derive bounds of arbitrary order using only calculus and the most basic concepts of the theory of $L^2$-Sobolev spaces, in order to make accessible at an elementary level useful results not otherwise easily extracted from the literature (and which incidentally are required for the extension of our own work in [9] to general spatial domains).

The results themselves are by now classical and are included in various ways in the much more sophisticated and extensive treatments in Agmon et. al. [2], [3], so there is no claim here to mathematical originality. (See also [4] and [10] for related work on elliptic equations with Neumann boundary conditions). For the basic construction of the Green’s function we follow very closely the notations and development in Robert [11], whose notes are extracted from a larger work [5]. Robert treats problems in $\mathbb{R}^n$ for general $n \geq 3$ and derives bounds for the Green’s function and its derivatives of order one, but by a method quite different from that presented here. Briefly, our approach applies the construction of an approximation to the Green’s function near the boundary, a simple scaling argument and the application of “interior estimates near the boundary,” the latter being derived in detail in an Appendix. We do, however, include the basic construction in [11], which we regard as expositionally optimal, for the sake of completeness and notational consistency.
We also give a brief discussion of the extension of our own approach to cases \( n > 3 \) at the end of section 4.

The following notations will be in effect throughout the paper:

**Definition 1.1.** The set \( \Omega \) will be bounded, open and connected in \( \mathbb{R}^3 \) with a \( C^{k+1} \) boundary, where \( k \geq 2 \). This means that there is a positive constant \( C_{\Omega} \) and a set \( \mathcal{W}_0 \equiv B_{R_0} \times (-L_0, L_0) \), where \( B_{R_0} \) is the ball of radius \( R_0 \) centered at the origin of \( \mathbb{R}^2 \) and \( R_0 \) and \( L_0 \) are positive constants, such that for every \( y' \in \partial \Omega \) there is a rigid motion \( T \) of \( \mathbb{R}^3 \) (that is, a translation followed by a rotation) and a mapping \( \psi : B_{R_0} \rightarrow (-L_0, L_0) \) such that

- \( T(y') = 0 \)
- \( \psi(0) = 0 \) and \( \nabla \psi(0) = 0 \)
- the derivatives of \( \psi \) up to order \( k + 1 \) are bounded in absolute value by \( C_{\Omega} \) in \( B_{R_0} \)
- \( T(\Omega) \cap \mathcal{W}_0 = \{(z_1, z_2, z_3) : |(z_1, z_2)| < R_0 \text{ and } -L_0 < z_3 < \psi(z_1, z_2)\} \).

With these assumptions there is a continuous unit outer normal vector field \( \nu : \partial \Omega \rightarrow \mathbb{R}^3 \), a nonnegative Radon measure \( dS \) representing surface area on the Borel sets in \( \partial \Omega \) and a bounded linear transformation from \( H^1(\Omega) \) into \( L^2(\partial \Omega) \) whose action on continuous functions in \( H^1(\Omega) \) is their restrictions to \( \partial \Omega \) \( (H^1(\Omega) \subset L^2(\partial \Omega)) \) is the Hilbert space of elements of \( L^2(\Omega) \) having weak derivatives up to order \( j \) which also are in \( L^2 \); and the divergence theorem holds for \( H^1 \) integrands. See [8], sections 3.2 and A.4, for example, for complete statements and proofs. We denote by \( D_\Omega \) the diagonal in \( \Omega \times \Omega \); that is, \( D_\Omega = \{(x, y) \in \Omega \times \Omega \text{ such that } x = y\} \), and similarly for \( D_\Omega^\perp \).

The outward normal derivative of a sufficiently smooth function \( u \) is the directional derivative \( \nabla u \cdot \nu \), where \( \nabla \) is the gradient. This normal derivative will be denoted \( D_\nu u \) and the Laplacian of \( u \) by \( \Delta u \). Also, \( \overline{u} \) will denote the average value of \( u \), that is, the integral of \( u \) divided by the measure of \( \Omega \).

We denote by \( C^j(\Omega) \) the space of real-valued functions on \( \Omega \) having continuous, bounded derivatives on \( \Omega \) up to order \( j \geq 0 \); and for \( j \geq 0 \) and \( \lambda \in (0, 1] \), \( C^{j,\lambda}(\Omega) \) is the Banach space of elements \( \varphi \) in \( C^j(\Omega) \) whose derivatives are Hölder continuous with exponent \( \lambda \); that is, whose \( C^{j,\lambda} \) norm, given by

\[
|\varphi|_{C^{j,\lambda}(\Omega)} \equiv \sum_{|\alpha| \leq j} \sup_{x \in \Omega} |D^\alpha \varphi(x)| + \sup_{|\alpha| \leq j} \sup_{x \neq y \in \Omega} \frac{|D^\alpha \varphi(y) - D^\alpha \varphi(x)|}{|y - x|^\lambda},
\]

is finite. Note that if \( \varphi \in C^{j,\lambda}(\Omega) \), then \( \varphi \) and its derivatives up to order \( j \) have continuous extensions to \( \partial \Omega \).

We recall that the set of restrictions to \( \Omega \) of \( C^\infty \) functions on \( \mathbb{R}^3 \) is dense in \( H^{k+1}(\Omega) \), this fact applying as well to open sets satisfying a “segment condition” (see [1], Theorem 3.22). Also, there is a bounded linear operator mapping \( H^2(\Omega) \) into \( C^{0,1/2}(\Omega) \) whose restriction to smooth functions is the identity, this fact applying to “strong Lipschitz” domains (see [1] Theorem 4.12). The imbedding \( H^2(\Omega) \subset C^{0,1/2}(\Omega) \) will be applied extensively throughout. (The precise meaning here is that a Lebesgue equivalence class in \( H^2 \) includes a function which is in \( C^{0,1/2}(\Omega) \); there will be no need to maintain this distinction, however.) In particular, if \( u \in H^2(\Omega) \), then \( \nabla u \in H^1(\Omega) \) and therefore \( D_\nu u \in L^2(\partial \Omega) \).

We denote by \( B_r(x) \) the ball of positive radius \( r \) centered at a point \( x \), and points \( x \in \mathbb{R}^3 \) will often be denoted \( (x_1, x_2, x_3) = (\tilde{x}, x_3) \) where \( \tilde{x} = (x_1, x_2) \). Finally, we
There is a function \( G \in C(\Omega \times \Omega - \mathcal{D}_\Omega) \) such that for each \( x \in \Omega \), \( G(x, \cdot) \in L^2(\Omega) \). \( G(x, \cdot) \) is the function whose value at \( y \in \Omega \) is \( G(x,y) \), etc., \( \overline{G(x, \cdot)} = 0 \) and if \( u \) is the restriction to \( \Omega \) of a \( C^2 \) function on \( \mathbb{R}^3 \) then
\[
 u(x) - \overline{u} = -\int_{\partial \Omega} G(x,y)(D_\nu u)(y) dS_y + \int_{\Omega} G(x,y)\Delta u(y) dy. \tag{1}
\]

(b) \( G \) is unique in the sense that if \( g \in H^1(\Omega) \) with \( \overline{g} = 0 \) and if for some \( x \in \Omega \) and for all \( u \) as in (a),
\[
 u(x) - \overline{u} = -\int_{\partial \Omega} g(y)(D_\nu u)(y) dS_y + \int_{\Omega} g(y)\Delta u(y) dy \tag{2}
\]
then \( g = G(x, \cdot) \) in the sense of Lebesgue equivalence classes on \( \Omega \).

(c) For all \( x \in \Omega \) and \( \varepsilon > 0 \),
\[
 G(x, \cdot) \in L^2(\Omega) \cap H^{k+1}(\Omega - \mathcal{B}_\varepsilon(x)) \subset C^{k-1,1/2}(\Omega - \mathcal{B}_\varepsilon(x))
\]
and for all \( y \in \Omega \) and \( \varepsilon > 0 \),
\[
 G(\cdot, y) \in L^2(\Omega) \cap H^{k+1}(\Omega - \mathcal{B}_\varepsilon(y)) \subset C^{k-1,1/2}(\Omega - \mathcal{B}_\varepsilon(y)).
\]
The \( L^2(\Omega) \) norms here are bounded by \( C(\Omega, k) \) independently of \( x \) and \( y \) respectively.

(d) For all \( x \in \Omega \) : \( \Delta_y G(x,y) = -|\Omega|^{-1} \) for \( y \in \Omega - \{x\} \) and \( (D_\nu G)(x,y) = 0 \) for \( y \in \partial \Omega \).

(e) \( G(x,y) = G(y,x) \) for \( (x,y) \in \Omega \times \Omega - \mathcal{D}_\Omega \).

(f) For multi-indices \( \alpha \) with \( |\alpha| \leq k-2 \) and for \( (x,y) \in \Omega \times \Omega - \mathcal{D}_\Omega \),
\[
 |D_\alpha y G(x,y)|, \ |D_\alpha x G(x,y)| \leq C(\Omega, k)|x-y|^{-(|\alpha|+1)}.
\]

The overall strategy of the proof is as follows. First, the basic construction of \( G \) is given in section 2 and is taken mostly from Robert \cite{11}. (We give a different proof of the symmetry result (e), however, based on Fourier series, enabled by the fact that \( G(x, \cdot) \in L^2(\Omega) \), which is not true in higher dimensions.) The main point of interest in this paper is the derivation of the pointwise bounds in (f) for \( G \) and its derivatives, especially when both \( x \) and \( y \) are near the boundary and are close. We derive these bounds in section 3, showing first that, near two such points, \( G \) can be represented as the sum of an approximate Green’s function, which is the Green’s function for a half-space whose boundary is a plane tangent to \( \partial \Omega \), and an error term which is an operator acting on \( G \). This error term is shown to be small because \( \partial \Omega \) is nearly flat near the points in question. This construction leads to the bound in (f) but only for \( |\alpha| = 0 \). We then apply this pointwise bound to quantify the statement in (c), giving more precise \( L^2 \) bounds for \( G(x, \cdot) \) on small,
localized \( \overline{\Omega} \)-neighborhoods (sets which are open in \( \overline{\Omega} \)) of boundary points. From this we can then derive \( H^k \) bounds for \( G(x, \cdot) \) on smaller such \( \overline{\Omega} \)-neighborhoods of boundary points (“interior bounds at the boundary”). These \( H^k \) bounds together with a properly scaled version of the imbedding \( H^2 \subset L^\infty \) then combine to prove the pointwise bounds in \( (f) \) for the derivatives of \( G \). (Proofs of the interior bounds at the boundary are elementary but somewhat technical and are given in the Appendix, section 4.)

We remark that while we do show in (c) that \( G(x, \cdot) \in H^{k+1}(\Omega - \overline{B}_\varepsilon(x)) \), for example, so that \( D^2_G(x, \cdot) \in H^{k+1-|\alpha|}(\Omega - \overline{B}_\varepsilon(x)) \subset C^{k-1-|\alpha|}(\Omega - \overline{B}_\varepsilon(x)) \) if \(|\alpha| \leq k - 1\), we do not derive the pointwise bound in \( (f) \) for derivatives of order \( k - 1 \): a loss of one order of regularity occurs in a “flattening of the boundary” argument in section 4.

2. Construction and basic properties. We begin with a review of basic facts about the “fundamental solution” \( \Gamma \) of the Laplace operator, which is defined by
\[
\Gamma(x, y) = -1/(4\pi|x - y|)
\]
for \( x \) and \( y \) in \( \mathbb{R}^3 \) and \( x \neq y \). It is easy to check that \( \Delta \Gamma(x, y) = 0 \) for \( y \neq x \), that \( \Gamma(x, \cdot) \) is locally in \( L^2 \) for every \( x \), and that if \( u \) is defined and continuous in a neighborhood of \( x \in \mathbb{R}^3 \) then
\[
\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} \Gamma(x, y)u(y) dS_y = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} (D_{\nu y} \Gamma)(x, y)u(y) dS_y = u(x). \tag{3}
\]
Applying these facts we arrive at the fundamental representation formula
\[
u(x) = -\int_{\partial \Omega} \left( u(y)(D_{\nu y} \Gamma)(x, y) - \Gamma(x, y)(D_{\nu} u)(y) \right) dS_y + \int_{\Omega} \Gamma(x, y)\Delta u(y) dy \tag{4}
\]
for the restrictions \( u \) of \( C^2 \) functions on \( \mathbb{R}^3 \) to \( \Omega \) and for \( x \in \Omega \). We will construct \( G \) by adding to \( \Gamma \) a function \( h(x, y) \) whose effect is to cancel the first term in the boundary integral above and then normalizing to achieve zero average value.

Before doing this we recall basic facts about weak and strong solutions of the following formal boundary value problem for an unknown function \( \varphi : \Omega \to \mathbb{R} \):
\[
\begin{cases}
\Delta \varphi(x) = f(x) - f, & x \in \Omega, \\
(D_{\nu} \varphi)(x) = 0, & x \in \partial \Omega, \\
\varphi = 0
\end{cases} \tag{5}
\]
for given \( f : \Omega \to \mathbb{R} \). Observe that the first two equations here could be inconsistent without the term \( f \) on the right and that solutions would not be unique without the last condition. To describe the weak form we let \( H^1_{1,\perp} \) be the Hilbert subspace of \( H^1(\Omega) \) whose elements have zero average value. We then formaly multiply the first equation above by \( v \in H^1_{1,\perp} \) and integrate to find that
\[
\int_{\Omega} \nabla \varphi \cdot \nabla v \, dx = -\int_{\Omega} (f - f)v \, dx. \tag{6}
\]
It is easy to check that the bilinear form defined by the left side here is coercive on \( H^1_{1,\perp} \times H^1_{1,\perp} \) and that the right side determines a bounded linear functional on \( H^1_{1,\perp} \) acting on \( v \). The Lax-Milgram theorem ([7] Theorem 5.8 or [6] sect. 6.2.1) therefore applies to show that there is a unique \( \varphi \in H^1_{1,\perp} \) satisfying \( (6) \) for all \( v \in H^1_{1,\perp} \) (and this would be true even if the \( f \) term were omitted in \( (6) \)). Standard elliptic theory (see [9] Theorem 4.2, for example) then applies to show that \( \varphi \in H^{k+1}(\Omega) \subset H^3(\Omega) \).
because \( \partial \Omega \) is \( C^{k+1} \), and therefore that the normal derivative of \( \varphi \) is continuous on \( \partial \Omega \).

We can reverse these steps to recover the original system (5) as follows. Integrating by parts in (6) we find that for all \( v \in H^1_{1,\perp} \),

\[
\int_{\partial \Omega} (D_r \varphi) v \, dS - \int_{\Omega} (\Delta \varphi - (f - \overline{f})) v \, dx = 0. \tag{7}
\]

Now let \( \delta_\varepsilon \) be an approximate \( \delta \)-function at the origin (that is, \( \delta_\varepsilon(x) = \varepsilon^{-3} \delta(x/\varepsilon) \) where \( \delta \) is supported on \( B_1(0) \), is nonnegative and integrates to one) and then take

\[
v(x) = \delta_\varepsilon(x - x_2) - \delta_\varepsilon(x - x_1) \text{ for Lebesgue points } x_1, x_2 \in \Omega \text{ of representatives of } \Delta \varphi \text{ and } f \text{ (observe that } \nabla = 0, \text{ as required).}
\]

Letting \( \varepsilon \to 0 \) we find that \( \Delta \varphi - (f - \overline{f}) \) equals its average value a.e. in \( \Omega \). Then for arbitrary \( v \in H^1_{1,\perp} \) the second integral on the left in (7) is this average value times \( |\Omega| \pi, \text{ which is zero.} \]

The first integral on the left is therefore also zero for all \( v \in H^1_{1,\perp} \), and choosing smooth functions \( v \) to approximate delta functions at a Lebesgue point of \( D_r \varphi \) on \( \partial \Omega \) and adjusted appropriately in the interior, we can deduce that \( D_r \varphi = 0 \) a.e. on \( \partial \Omega \) with respect to the boundary measure. Consequently the average value of \( \Delta \varphi \) is zero, and this together with the aforementioned fact that \( \Delta \varphi - (f - \overline{f}) \) equals its average value a.e. shows that \( \Delta \varphi = f - \overline{f} \) as elements of \( L^2(\Omega) \).

We now construct \( G \), following closely the exposition in [11]. Let \( \eta \) be a smooth, nondecreasing function of \( s \in \mathbb{R} \) such that \( \eta(s) = 0 \) for \( s \leq 1/3 \) and \( \eta(s) = 1 \) for \( s \geq 2/3 \). Fix \( x \) in \( \Omega \), define

\[
f(x,y) = \Delta_y \left( \eta(\tfrac{|x-y|}{\text{dist}(x,\partial \Omega)}) \Gamma(x,y) \right)
\]

then solve

\[
\begin{align*}
\Delta_y w(x,y) &= f(x,y) - \overline{f(x,\cdot)}, & y \in \Omega, \\
D_{y\cdot} w(x,y) &= 0, & y \in \partial \Omega, \\
w(x,\cdot) &= 0
\end{align*}
\tag{8}
\]

for \( w \), as discussed above. Then set

\[
h(x,y) = w(x,y) - \eta(\tfrac{|x-y|}{\text{dist}(x,\partial \Omega)}) \Gamma(x,y)
\]

so that if \( c_x = -\overline{f(x,\cdot)} \) then

\[
\begin{align*}
\Delta_y h(x,y) &= c_x, & y \in \Omega, \\
D_{y\cdot} h(x,y) &= -D_{y\cdot} \Gamma(x,y), & y \in \partial \Omega, \\
|h(x,\cdot)| &\leq |\Gamma(x,\cdot)|_{L^1(\Omega)} \leq C(\Omega)
\end{align*}
\tag{9}
\]

(the proof of Lemma 2.1 below will show that \( c_x = -|\Omega|^{-1} \); and finally define

\[
G(x,y) = \Gamma(x,y) - \overline{\Gamma(x,\cdot)} + h(x,y) - \overline{h(x,\cdot)}.
\]

We then have the following basic facts:

**Lemma 2.1.** Let \( w, h \) and \( G \) be as defined above.

(a) The maps \( x \mapsto w(x,\cdot) \) and \( x \mapsto h(x,\cdot) \) are continuous from \( \Omega \) into \( H^{k+1}(\Omega) \subset C^{k-1,1/2}(\Omega) \).

(b) For \( x \in \Omega \) and \( \varepsilon > 0 \), \( G(x,\cdot) \in L^2(\Omega) \cap H^{k+1}(\Omega - B_\varepsilon(x)) \); and for \( |\alpha| \leq k-1 \), the map \( (x,y) \mapsto D_\alpha G(x,y) \) is continuous on \( \Omega \times \Omega - D_\alpha \Omega \).
(c) If \( K \) is a compact subset of \( \Omega \) then there is a constant \( C(\Omega, k, K) \) such that for 
\( x \in K \),
\[
|w(x, \cdot)|_{H^{k+1}(\Omega)}, \ |h(x, \cdot)|_{H^{k+1}(\Omega)} \leq C(\Omega, k, K)
\]
and for \( |\alpha| \leq k - 1 \) and \( y \in \Omega - \{x\} \),

\[
|D^\alpha_y G(x, y)| \leq C(\Omega, k, K)|x - y|^{-(|\alpha|+1)}.
\]

(d) The representation formula (1) in Theorem 1.2(a) holds and \( G \) satisfies the conclusions in Theorem 1.2(d).

Proof. First observe that the function \( f \) above is zero for \( y \) in a neighborhood of the singular point \( x \), so that \( f(x, \cdot) \in H^{k-1}(\Omega) \) with norm bounded by \( C(\Omega, k, K) \) if \( x \in K \). Standard results for elliptic equations then apply to show that \( w(x, \cdot) \) and therefore \( h(x, \cdot) \) are in \( H^{k+1}(\Omega) \), again with norms bounded by \( C(\Omega, k, K) \) if \( x \in K \). Furthermore, since \( \text{dist}(x, \partial \Omega) \) is a continuous function of \( x \), the map \( x \mapsto f(x, \cdot) \) is continuous from \( \Omega \) into \( H^{k-1}(\Omega) \) so that the maps \( x \mapsto w(x, \cdot) \) and \( x \mapsto h(x, \cdot) \) are continuous from \( \Omega \) into \( H^{k+1} \). The other statements in (a)-(c) then follow by triangulation and by the obvious bounds for \( \Gamma(x, y) \).

To prove (1) we let \( u \) be as in the statement in Theorem 1.2(a) and multiply the first equation in (9) by \( u \) and integrate over \( \Omega \) to obtain
\[
c_x|\Omega| = \int_{\partial \Omega} \left( (Dv_y h)(x, y)u(y) - h(x, y)(Dv_u)(y) \right) dS_y + \int_{\Omega} h(x, y)\Delta u(y) dy.
\]
Taking \( u \equiv 1 \) and applying the boundary condition in (9) we obtain that for small positive \( \varepsilon \),
\[
c_x|\Omega| = -\int_{\partial \Omega} (Dv_y \Gamma)(x, y) dS_y
\]
\[
= -\int_{\partial(\Omega - B_\varepsilon(x))} (Dv_y \Gamma)(x, y) dS_y - \int_{\partial B_\varepsilon(x)} (Dv_y \Gamma)(x, y) dS_y
\]
(outer normals in both integrals). The first integral on the right is zero because \( \Gamma(x, \cdot) \) is harmonic \( \Omega - B_\varepsilon(x) \) and the second integral goes to 1 as \( \varepsilon \to 0 \) by (3). Thus \( c_x = -1/|\Omega| \). Adding (4) and (10) and applying the fact that
\[
\int_{\Omega} \Delta u dy = \int_{\partial \Omega} Dv_u(y) dS_y
\]
to absorb the terms \( \Gamma(x, \cdot) \) and \( h(x, \cdot) \) we then obtain (1). Finally, \( G(x, \cdot) = 0 \) by construction, and \( G(x, \cdot) \in L^2 \) by (a) of the present lemma and the fact that \( \Gamma(x, \cdot) \in L^2 \). The other two properties of \( G \) stated in Theorem 1.2(d), that for \( y \in \partial \Omega, \Delta_y G(x, y) = -|\Omega|^{-1} \) and for \( y \in \partial \Omega, D_{\nu_y} G(x, y) = 0 \), follow directly from (9) and properties of \( \Gamma \).

The next step is to prove the symmetry statement in Theorem 1.2(e). We will apply the Tonelli and Fubini theorems, their use being justified by the bound in (a) of the following lemma. This symmetry will then enable us to extend to the function \( G(\cdot, y) \) the regularity properties in Lemma 2.1(b) and (c) satisfied by the function \( G(x, \cdot) \).

Lemma 2.2. Let \( G \) be the Green’s function constructed in Lemma 2.1.

(a) There is a constant \( C(\Omega, k) \) such that \( |G(x, \cdot)|_{L^2(\Omega)} \leq C(\Omega, k) \) for all \( x \in \Omega \).

(b) \( G(x, y) = G(y, x) \) for all \( x, y \in \Omega \times \Omega - \partial \Omega \).
(c) For \( y \in \Omega \) and \( \varepsilon > 0 \), \( G(\cdot, y) \in L^2(\Omega) \cap H^{k+1}(\Omega - \overline{B}_\varepsilon(x)) \), and for \( |\alpha| \leq k-1 \), the derivatives \( D^\alpha_x G(x, y) \) exist and are continuous on \( \Omega \times \Omega - D_\Omega \) (compare Lemma 2.1(b)).

(d) If \( K \) is a compact subset of \( \Omega \) then there is a constant \( C(\Omega, k, K) \) such that for \( y \in K, x \in \Omega - \{y\} \) and \( |\alpha| \leq k-1 \),

\[
|D^\alpha_x G(x, y)| \leq C(\Omega, k, K)|x-y|^{-(|\alpha|+1)}.
\]

(Compare Lemma 2.1(c)).

Proof. To prove (a) we let \( f \in L^2(\Omega) \) be given and solve the problem (5) for \( \varphi \in H^2 \). Then \( |\varphi|_{L^\infty} \leq C|\varphi|_{H^2} \leq C|f|_{L^2} \) for a constant \( C = C(\Omega, k) \). Therefore by (1),

\[
\left| \int_\Omega G(x, y) f(y) dy \right| = \left| \int_\Omega G(x, y) \Delta \varphi(y) dy \right| = |\varphi(x)| \leq C|f|_{L^2},
\]

since \( \overline{G(\cdot, \cdot)} = 0, \varphi = 0 \) and \( D_\nu \varphi = 0 \) on \( \partial \Omega \). This proves (a).

To prove (b) we first note that there is an orthonormal basis for the subspace of \( L^2(\Omega) \) whose elements have zero average value consisting of (weak) solutions of the system

\[
\begin{align*}
\Delta \varphi_j(x) &= \lambda_j \varphi_j(x), \quad x \in \Omega, \\
(D_\nu \varphi_j)(x) &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

(11)

where \( 0 > \lambda_1 \geq \lambda_2 \geq \ldots \to -\infty \) and each \( \varphi_j \) is in \( H^1_{1+} \cap H^{k+1}(\Omega) \subset C^{1,1/2}(\Omega) \).

(This is a consequence of the spectral theorem, [7] Theorem 5.5, applied to the solution operator \( S : L^2(\Omega) \to H^1_{1+} \) for the problem (5)): that is, the mapping \( f \mapsto Sf = \varphi \) in the notation of (5).) Since \( G(x, \cdot) \in L^2(\Omega) \) and \( \overline{G(\cdot, \cdot)} = 0 \) for fixed \( x \in \Omega \), there is a corresponding Fourier expansion for \( G(x, \cdot) \) in which the coefficients are the \( L^2 \) inner products

\[
\int_\Omega G(x, y) \varphi_j(y) dy = \lambda_j^{-1} \int_\Omega G(x, y) \Delta \varphi_j(y) dy = \lambda_j^{-1} \varphi_j(x)
\]

by (1). Thus for fixed \( x \), \( G(x, \cdot) \) is the \( L^2 \)-limit of the symmetric functions

\[
G^j(x, y) = \sum_{j=1}^J \lambda_j^{-1} \varphi_j(x) \varphi_j(y).
\]

To check that this symmetry persists in the limit we note that \( G \) is measurable on \( \Omega \times \Omega \) by Lemma 2.1(b) and is integrable on \( \Omega \times \Omega \) by Tonelli’s theorem and the \( L^2 \) bound just proved. We now let \( \delta_\varepsilon \) be the approximate delta function at the origin appearing in the discussion preceding (8). Then fixing points \( x_0 \neq y_0 \) in \( \Omega \) and applying the continuity of \( G \) at \( (x_0, y_0) \), Fubini’s theorem, the \( L^2 \) convergence and the dominated convergence theorem we compute

\[
G(x_0, y_0) = \lim_{\varepsilon \to 0} \int \left( \int G(x, y) \delta_\varepsilon(y-y_0) dy \right) \delta_\varepsilon(x-x_0) dx
\]

\[
= \lim_{\varepsilon \to 0} \lim_{J \to \infty} \int \left( \int G^j(x, y) \delta_\varepsilon(y-y_0) dy \right) \delta_\varepsilon(x-x_0) dx
\]

\[
= \lim_{\varepsilon \to 0} \lim_{J \to \infty} \int \left( \int G^j(y, x) \delta_\varepsilon(x-x_0) dx \right) \delta_\varepsilon(y-y_0) dy
\]

\[
= \ldots = G(y_0, x_0).
\]
This proves part (b), and (c) and (d) follow immediately from parts (b) and (c) of Lemma 2.1 and the above symmetry.

The proof of the uniqueness statement in Theorem 1.2(b) can now be given and is nearly identical to that of (a) in the above lemma: Let \( g \) be as in the statement and let \( f \in L^2(\Omega) \) be given. We then solve the system \((5)\) for \( \varphi \in H^2(\Omega) \), apply both \((1)\) and \((2)\) with \( u \) replaced by \( \varphi \) and subtract. The result is that

\[
\int_\Omega (G(x,y) - g(y)) f(y) dy = 0.
\]

It follows that \( G(x, \cdot) = g \) as elements of \( L^2(\Omega) \). Note that this computation requires that both \( G(x, \cdot) \) and \( g \) have zero average value.

We have now proved parts (a)–(e) of Theorem 2.1 with the exception that (a) and (e) are proved so far only with \( \Omega \) replaced by \( \overline{\Omega} \).

3. Pointwise bounds. In this section we complete the proof of Theorem 1.2 by deriving the pointwise bounds in part (f) for \( G \) and its derivatives.

First recall the notation in Definition 1.1 in which \( y' \in \partial \Omega \) and \( T \) is a rigid motion defining a coordinate system in which \( y' = 0 \) and in the notation of that definition, \( \psi(0) = 0 \) and \( \nabla \psi(0) = 0 \). We now define \( U_\delta \subset \Omega \) for \( \delta \leq R_0 \wedge L_0 \) (the smaller of \( R_0 \) and \( L_0 \)) by

\[
U_\delta = \{(\tilde{z},z_3) \in \mathbb{R}^3 : |\tilde{z}| < \delta \text{ and } -\delta + \psi(\tilde{z}) < z_3 < \psi(\tilde{z})\}.
\]

The following lemma gives the “interior estimates at the boundary” referred to earlier.

Lemma 3.1. There is a constant \( C = C(\Omega, k) \) and \( \delta_0 \in (0,R_0 \wedge L_0] \) such that if \( \delta \in (0,\delta_0] \) and if \( y', U_\delta \) and \( \psi \) are as above, then the following hold:

(a) If \( y \) and \( z \) are points in \( U_\delta \) with \( y = (0,y_3) \), then \( |z - y| \leq 2\delta \).

(b) If \( u \in H^k(U_\delta) \) satisfies

\[
\int_{U_\delta} \nabla u(y) \cdot \nabla \varphi(y) dy = c \int_{U_\delta} \varphi(y) dy \tag{12}
\]

for some constant \( c \) and for all \( \varphi \in H^1(U_\delta) \) whose trace on the bottom and lateral surface of \( U_\delta \) is zero, that is, \( \varphi(\tilde{z},z_3) = 0 \) if \( |\tilde{z}| = \delta \) or if \( z_3 = -\delta + \psi(\tilde{z}) \), then for \( j \leq k \)

\[
\sum_{|\alpha| = j} |D^\alpha u|_{L^2(U_{\delta/2})} \leq C(\Omega,k)\delta^{-j} (|u|_{L^2(U_\delta)} + |c|\delta^{7/2}).
\]

Proof. The first statement holds because \( |\psi(\tilde{z})| \leq C_\rho\delta^2 \) and \( \delta \) can be chosen to be small. The proof of the second statement is somewhat lengthy and technical and is therefore deferred to section 4.

We will derive two important consequences of (b): first that the bounds in Theorem 1.2(f) hold when \( x \) and \( y \) are widely separated and second that \( G \) and its derivatives have continuous extensions to \( \partial \Omega \). Before stating these results we examine the embedding \( H^2 \subset L^\infty \), which holds on open sets in \( \mathbb{R}^3 \) satisfying a cone condition ([1] Theorem 4.2). The proof consists in bounding the localized function at the vertex of a cone by the integral of its derivative along a radial ray, integrating by parts once and applying the Cauchy-Schwartz inequality. Let us suppose that an open set \( \mathcal{A} \) satisfies this condition with a cone of unit dimensions. Then for \( C^2 \) functions \( u \) on \( \mathbb{R}^3 \) restricted to \( \mathcal{A} \),

\[
|u|_{L^\infty(\mathcal{A})} \leq C\left( |u|_{L^2(\mathcal{A})} + |Du|_{L^2(\mathcal{A})} + |D^2u|_{L^2(\mathcal{A})} \right)
\]
where $|D^j u|$ denotes the ensemble $\sum_{|\alpha| = j} |D^\alpha u|$ and $C$ depends on the cone parameters, which are assumed to be of order one. Now suppose that $0 \in A$ and that a set $A_s$ is a scaling of $A$; that is, $A_s = \{ y = sx : x \in A \}$ for a positive constant $s > 0$. Then if $v(y) = u(y/s) = u(x)$, we find that

$$|v|_{L^\infty (A_s)} \leq C\left(s^{-3/2}|v|_{L^2(A_s)} + s^{-1/2}|Dv|_{L^2(A_s)} + s^{1/2}|D^2v|_{L^2(A_s)} \right). \quad (13)$$

This result will be applied with important effect in (15) below.

Before proceeding we note that there is a constant $\kappa_0$ depending only on $\Omega$ and on $k$ such that if $y \in \Omega$ with $\text{dist}(y, \partial \Omega) \leq \kappa_0$, then there is a unique $y' \in \partial \Omega$ such that $\text{dist}(y, \partial \Omega) = |y - y'|$.

**Lemma 3.2.** Let $\delta_0$, $U_0$ and $\kappa_0$ be as above.

(a) Assume that $x_0$ and $y_0$ are distinct points in $\Omega$ with $\text{dist}(y_0, \partial \Omega) \leq \kappa_0$ and let $\varepsilon = |x_0 - y_0| \wedge \delta_0$. Define neighborhoods $V_1$ and $V_2$ of $y_0$ as follows (we assume here via Definition 1.1 that, in an appropriate coordinate system, $y_0 = (0, y_0 \delta)$ where $(y_0 \delta) < 0$ and that the point in $\partial \Omega$ closest to $y_0$ is $y_0 = 0$). If $y_0 \in U_0/8$ (case I) take $V_1 = U_0/4$ and $V_2 = U_0/8$; and if $y_0 \notin U_0/8$ (case II) take $V_1 = B_{\varepsilon/8}(y_0)$ and $V_2 = B_{\varepsilon/16}(y_0)$. Then in either case

- $y_0 \in V_2 \subset \subset V_1 \subset \Omega$
- if $y \in V_1$ then $|y - x_0| \geq \varepsilon/2$
- for $j \leq k$

$$\sum_{|\alpha| = j} |D^\alpha y G(x_0, \cdot)|_{L^2(V_2)} \leq C(\Omega, k)\varepsilon^{-j} (|G(x_0, \cdot)|_{L^2(V_1)} + \varepsilon^{7/2}). \quad (14)$$

(b) Given $r > 0$ there is a constant $C(\Omega, k, r)$ such that if $x_0$ and $y_0$ are distinct points in $\Omega$ with $|x_0 - y_0| \geq r$ then

$$\sum_{|\alpha| \leq k - 2} |D^\alpha x G(x_0, y_0)|, \sum_{|\alpha| \leq k - 2} |D^\alpha y G(x_0, y_0)| \leq C(\Omega, k, r).$$

(c) $G(x, y)$ extends continuously to $\overline{\Omega} \times \overline{\Omega} - D_{\overline{\Omega}}$.

**Proof.** The first two items in (a) are easily checked. Concerning the third we first note that in Case I, if $\varphi \in H^1(V_1)$ is zero on the bottom and lateral surface of $V_1$, then

$$\int_{V_1} \nabla y G(x_0, y) \cdot \nabla \varphi(y) \, dy = \int_{\partial V_1} (D_{\nu_{V_1}} G)(x_0, y) \varphi(y) \, dS_y - \int_{V_1} \Delta_y G(x_0, y) \varphi(y) \, dy$$

$$= -|\Omega|^{-1} \int_{V_1} \varphi(y) \, dy$$

by Theorem 1.2(d). The bound in (14) therefore follows from Lemma 3.1(b). The argument is similar but simpler in Case II because there are no boundary effects and standard interior regularity bounds apply (see [9] Theorem 4.1(b) for example).

The result in (b) follows from Lemmas 2.1(c) and 2.2(d) if $\text{dist}(y_0, \partial \Omega) \geq \kappa_0$. Otherwise we can construct the set $V_2$ as in (a) and apply (13) to obtain that for
\[ j \leq k - 2, \]
\[ |D^j_y G(x_0, \cdot)|_{L^\infty(V_2)} \leq C(\Omega, k) \left( \varepsilon^{-3/2} |D^j_y G(x_0, \cdot)|_{L^2(V_2)} + \varepsilon^{-1/2} |D^{j+1}_y G(x_0, \cdot)|_{L^2(V_2)} + \varepsilon^{1/2} |D^{j+2}_y G(x_0, \cdot)|_{L^2(V_2)} \right) \]
\[ \leq C \varepsilon^{-(j+3/2)} \left( |G(x_0, \cdot)|_{L^2(V_1)} + \varepsilon^{7/2} \right) \leq C(\Omega, k, r). \]

Finally to prove (c) we let \( x_0 \) and \( y_0 \) be distinct points in \( \partial \Omega \) and construct sets \( V(x_0) \) and \( V(y_0) \) as in Case I of (a) such that \( x_0 \in \partial \Omega \cap \partial V(x_0), y_0 \in \partial \Omega \cap \partial V(y_0) \) and \( V(x_0) \) and \( V(y_0) \) have disjoint closures. There is therefore a positive number \( r \) such that \( |x - y| \geq r \) for all \( x \in V(x_0) \) and \( y \in V(y_0) \). Since \( k \geq 3 \) the bound in (b) then applies and shows that \( \nabla_x G(x, y) \) and \( \nabla_y G(x, y) \) are uniformly bounded in \( V(x_0) \times V(y_0) \). Thus \( G \) is uniformly Lipschitz and so extends continuously to \( V(x_0) \times V(y_0) \).

We have now proved all of Theorem 1.2 with the exception of the pointwise bounds in Theorem 1.2(f), and these have in fact been proved for points \((x_0, y_0)\) when one of \( x_0 \) or \( y_0 \) is far from \( \partial \Omega \) (Lemmas 2.1(c) and 2.2(d)) or when \( x_0 \) and \( y_0 \) are far apart (Lemma 3.2(b)). For the remaining case that \( x_0 \) and \( y_0 \) are close and both are close to \( \partial \Omega \) we construct a subset \( W_1 \) of \( W_0 \) in Definition 1.1 and make the following claims:

1. **There exist** \( \kappa_1 \) **and** \( R_1 \) **in** \((0, \kappa_0]\) **and** \((0, R_0]\) **respectively (recall Definition 1.1) and a constant** \( C(\Omega, k) \) **such that if** \( W_1 = B_{R_1}(0) \times (-L_0, L_0) \) **then given** \( y \in \Omega \) **with** \( \text{dist}(y, \partial \Omega) \leq \kappa_1 \) **and** \( y' \in \partial \Omega \) **its closest point on** \( \partial \Omega \), **and in a coordinate system in which** \( y' = 0 \) **and** \( y = (0, y_1) \) **where** \( y_1 < 0 \), **the following hold:**

   - \( \psi \) maps \( B_{R_1}(0) \subset \mathbb{R}^2 \) into \((-L_0, L_0)\), \( \psi(0) = 0 \) and \( \nabla \psi(0) = 0 \)
   - \( \Omega \cap W_1 = \{(\tilde{z}, z_3) : |\tilde{z}| < R_1 \text{ and } -L_0 < z_3 < \psi(\tilde{z})\} \)
   - \( y \in W_1 \)
   - If \( \overline{y} \) is the reflected point \( \overline{y} \equiv (0, -y_1) \), which is in the complement of \( \Omega \), then for all \( z \in \partial \Omega \cap W_1 \)
     \[ C^{-1}(|\tilde{z}|^2 + y_1^2) \leq |z - y|^2, \quad |z - \overline{y}|^2 \leq C(|\tilde{z}|^2 + y_1^2) \]
   - If \( z \in \partial \Omega \cap B_{R_1}(y) \) then \( z \in W_1 \).

2. The first two items here are automatic, and we can choose \( \kappa_1 < \kappa_0 \land L_0 \) so that the third holds and \( R_1 \) so that the fourth holds; the last item will then be satisfied if \( R_1 \) and \( \kappa_1 \) are reduced further so that \( R_1 + \kappa_1 < L_0 \). The set \( W_1 \) will be fixed for the remainder of this section.

   Next, if \( y \in \Omega \) with \( \text{dist}(y, \partial \Omega) \leq \kappa_1 \) and if \( \overline{y} \) is the corresponding reflected point defined above, we define an approximate Green’s function

\[ \tilde{\Gamma}(z, y) = \Gamma(y, z) + \Gamma(\overline{y}, z) \]

for \( z \in \Omega \). (Observe that \( \tilde{\Gamma} \) is not symmetric nor is its average value zero.) In the following lemma we derive the important properties of \( \tilde{\Gamma} \) required for the completion of the proof of Theorem 1.2(f). Specifically, (c) below gives a representation for \( G \) at points of interest as the sum of \( \tilde{\Gamma} \) and an error term, the latter being an integral operator acting on \( G \) with kernel \( D_x \tilde{\Gamma} \). Parts (a) and (b) will show that this operator is small in an appropriate sense. For example, whereas the first derivatives of \( \tilde{\Gamma} \) at
Lemma 3.3. There is a constant $C(\Omega,k)$ such that if $y \in \Omega$ with $\text{dist}(y, \partial \Omega) \leq \kappa_1$ then

(a) for all $z \in \partial \Omega \cap W_1$, $|(D_{\nu_z} \hat{\Gamma})(z, y)| \leq C(\Omega,k)|y - z|^{-1}$;

(b) if $R \in (0, R_1]$ and $x \in B_{R/3}(y)$ then

$$\int_{\partial \Omega \cap B_{R/3}(x)} |(D_{\nu_z} \hat{\Gamma})(z, y)||z - x|^{-1} dS_z \leq C(\Omega, k) R|x - y|^{-1};$$

(c) if $\hat{G} \equiv \Gamma + h$ (so that $G(x, \cdot) = \hat{G}(x, \cdot) - \overline{G(x, \cdot)}$), then for $x \in \Omega$

$$\hat{G}(x, y) = \hat{\Gamma}(x, y) - \Gamma(\cdot, y) + \int_{\partial \Omega} (D_{\nu_z} \hat{\Gamma})(z, y) \hat{G}(x, z) dS_z. \quad (17)$$

Proof. To prove (a) we fix $y$ and compute in a coordinate system determined by the rigid motion $T$ in Definition 1.1 in which $y = (0, y_3)$ where $y_3 < 0$ and the closest point to $y$ on $\partial \Omega$ is $y' = 0$ and $\psi$ as in the definition. Then for $z = (\tilde{z}, \psi(\tilde{z})) \in \partial \Omega \cap W_1$, $\nu(z) = (-\nabla \psi(\tilde{z}), 1)/\sqrt{1 + \left|\nabla \psi(\tilde{z})\right|^2}$ and therefore

$$4\pi \sqrt{1 + \left|\nabla \psi(\tilde{z})\right|^2} (D_{\nu_z} \hat{\Gamma})(z, y) = \left(\frac{z - y}{|z - y|^3} + \frac{\tilde{z} - \overline{y}}{|\tilde{z} - \overline{y}|^3}\right) \cdot (-\nabla \psi(\tilde{z}), 1)$$

$$= -\frac{2 \cdot \nabla \psi(\tilde{z})}{|z - y|^3} + \left(\frac{1}{|z - y|^3} - \frac{1}{|\tilde{z} - \overline{y}|^3}\right)(z - \overline{y}) \cdot (-\nabla \psi(\tilde{z}), 1) \quad (18)$$

(observe the crucial cancellation of $y$ and $\overline{y}$ in the first term on the right here).

Elementary estimates based on (16) and the fact that $|\nabla \psi(\tilde{z})| \leq C|\tilde{z}|$ show that each of the two terms on the right is bounded by $C|z - y|^{-1}$, as required.

To prove (b) we first note that if $z \in \partial \Omega \cap B_{R/3}(x)$ then $|z - y| \leq |z - x| + |x - y| < 2R/3 < R_1$ so that $|\tilde{z}| < R_1$ and therefore $z = (\tilde{z}, \psi(\tilde{z})) \in W_1$ by the item following (16) above. The bound in part (a) therefore applies, so if we multiply the integral $I$ in (b) by $|x - y|$ and apply the triangle inequality we obtain

$$|x - y| |z - x| \leq C \int_{\partial \Omega \cap B_{R/3}(x)} \frac{1}{|z - x|} + \frac{1}{|z - y|} dS_z.$$ 

Here $dS_z(\tilde{z}) = \sqrt{1 + |\nabla \psi(\tilde{z})|^2} d\tilde{z} \leq Cd\tilde{z}$, which in polar coordinates in $\mathbb{R}^2$ is $r dr d\theta$; the integral on the right here is then easily seen to be bounded by $C(\Omega, k) R$.

To prove (c) we recall from (9) that $\Delta z h(x, z) = -|\Omega|^{-1}$. Multiplying by $\Gamma(\overline{y}, z)$, which is harmonic in $z \in \Omega$, and integrating, we therefore get

$$\int_{\partial \Omega} \left((D_{\nu_z} \Gamma)(\overline{y}, z) h(x, z) - \Gamma(\overline{y}, z)(D_{\nu_z} h)(x, z)\right) dS_z - \Gamma(\overline{y}, \cdot) = 0;$$

then adding this to the representation (4) with $u = h(x, \cdot)$ we obtain

$$h(x, y) = \int_{\partial \Omega} \left((D_{\nu_z} \hat{\Gamma})(z, y) h(x, z) - \hat{\Gamma}(z, y)(D_{\nu_z} h)(x, z)\right) dS_z - \hat{\Gamma}(\cdot, y). \quad (19)$$

Next we derive a corresponding representation for $\hat{\Gamma}(x, y)$. First, since $\Gamma(\overline{y}, x)$ is harmonic in $x \in \Omega$, we get from (4) that

$$\Gamma(\overline{y}, x) = \int_{\partial \Omega} \left((D_{\nu_z} \Gamma)(x, z) \Gamma(\overline{y}, z) - \Gamma(x, z)(D_{\nu_z} \Gamma)(\overline{y}, z)\right) dS_z.$$
Also, applying the fact that \( \Gamma(y,x) \) is a harmonic function of \( x \in \partial \Omega - B_{\varepsilon}(y) \) for small \( \varepsilon \), we can apply the representation (4) with \( \Omega \) replaced by \( \Omega - B_{\varepsilon}(y) \) then let \( \varepsilon \to 0 \) and apply (3) to get that
\[
\int_{\partial \Omega} \left( (D_{\nu_y} \Gamma)(x,z) \Gamma(y,z) - \Gamma(x,z)(D_{\nu_y} \Gamma)(y,z) \right) dS_z = 0.
\]

Adding the last two equations we obtain
\[
\Gamma(y,x) = \int_{\partial \Omega} \left( (D_{\nu_y} \Gamma)(x,z) \bar{\Gamma}(z,y) - \Gamma(x,z)(D_{\nu_y} \bar{\Gamma})(z,y) \right) dS_z.
\]

We subtract this from (19), recalling that \( (D_{\nu_y} \Gamma)(x,z) = - (D_{\nu_y} \bar{\Gamma})(x,z) \) on \( \partial \Omega \) to obtain finally that
\[
h(x,y) = \Gamma(y,x) - \bar{\Gamma}(x,y) + \int_{\partial \Omega} \left( (D_{\nu_y} \bar{\Gamma})(z,y) \left[ \Gamma(x,z) + h(x,z) \right] \right) dS_z.
\]

The representation in (c) then follows by adding \( \Gamma(x,y) \) to both sides. \( \square \)

We can now complete the proof of Theorem 1.2:

**Proof of Theorem 1.2(f):** We will first prove the bound in (f) for \( |\alpha| = 0 \); bounds for \( |\alpha| > 0 \) will then be shown to follow almost immediately from (15) in Lemma 3.2.

Fix \( y_0 \in \Omega \) with dist\( (y_0, \partial \Omega) \leq \kappa_1 \), let \( R \in (0, R_1) \) be arbitrary for now and fix an \( x_0 \in \Omega \cap B_{R/3}(y_0) \) with \( x_0 \neq y_0 \). Then define
\[
M_{R,x_0} = \sup\{|\tilde{G}(x_0,y)||x_0 - y| : y \in \Omega \cap B_{R/3}(x_0), y \neq x_0 \quad \text{and dist}(y, \partial \Omega) \leq \kappa_1 \},
\]
which is finite by Lemma 2.1 and the obvious bound for \( \Gamma \). We will apply the representation in Lemma 3.3(c) to show that \( M_{R,x_0} \) is bounded by known quantities involving \( \bar{\Gamma} \) plus a multiple of \( M_{R,x_0} \) itself, this multiple being small if \( R \) is small. To do this we fix \( y \) in the set above in the definition of \( M_{R,x_0} \) with \( y \in \Omega \), without loss of generality, and bound the integral on the right side of (17) as follows. If \( z \in \partial \Omega \cap B_{R/3}(x_0) \) then \( |y - z| \geq R/3 \) and we bound both \( |\tilde{G}(x,z)| \) and \( |D_{\nu_z} \bar{\Gamma}(z,y)| \) by \( C(R^{-1}) \), by which we mean a constant that depends on a lower bound for \( R \) away from zero but otherwise depends only on \( \Omega \), \( k \) and the diameter of \( \Omega \). The first of these bounds follows from Lemma 3.2(b) and the second by direct computation. Thus
\[
\left| \int_{\partial \Omega \cap B_{R/3}(x_0)} (D_{\nu_z} \bar{\Gamma})(z,y) \tilde{G}(x_0,z) dS_z \right| \leq C(R^{-1}).
\]

Next, if \( z \in \partial \Omega \) and \( |z - x_0| \in [R/3, 2R/3] \) then again \( |\tilde{G}(x_0,z)| \leq C(R^{-1}) \), and \( |z - y| \leq 2R/3 + R/3 \leq R_1 \), so that the bound in Lemma 3.3(a) holds. Thus
\[
\left| \int_{\partial \Omega \cap [R/3 \leq |z - x_0| < 2R/3]} (D_{\nu_z} \bar{\Gamma})(z,y) \tilde{G}(x_0,z) dS_z \right| \leq C(R^{-1}) \int \frac{dS_z}{|z - y|} \leq C(R^{-1}).
\]

Finally, if \( |z - x_0| < R/3 \) then \( |\tilde{G}(x_0,z)| \leq M_{R,x_0} |z - x_0|^{-1} \) and
\[
\left| \int_{\partial \Omega \cap (B_{R/3}(x_0))} (D_{\nu_z} \bar{\Gamma})(z,y) \tilde{G}(x_0,z) dS_z \right| \leq C(\Omega, k) R M_{R,x_0} |z - x_0|^{-1}
\]
by Lemma 3.3(b). Combining these three bounds we thus have that
\[
\left| \int_{\partial \Omega} (D_{\nu_z} \bar{\Gamma})(z,y) \tilde{G}(x_0,z) dS_z \right| \leq C(R^{-1}) + C(\Omega, k) R M_{R,x_0} |x_0 - y|^{-1}.
\]
It then follows from (17) that
\[ |\tilde{G}(x_0,0)|\leq C(\Omega,k)(1 + C(R^{-1}) + RM_{R,x_0}) \]
for all \( y \) occurring in the definition of \( M_{R,x_0} \). Taking the sup over \( y \) we conclude that \( M_{R,x_0} \) satisfies the same bound. We can therefore choose \( R = R_2 \in (0,R_1) \) sufficiently small depending only on \( \Omega \) and on \( k \) to conclude the following: there are constants \( \kappa_1(\Omega,k), R_2(\Omega,k) \) and \( C(\Omega,k) \) such that if \( y_0 \in \Omega \) with \( \text{dist}(y_0,\partial\Omega) \leq \kappa_1 \) and if \( x_0 \in \Omega \) with \( 0 < |x_0 - y_0| < R_2/3 \), then \( |\tilde{G}(x_0,y_0)| \leq C(\Omega,k)|x_0 - y_0|^{-1} \).

The same bound therefore holds for \( G(x_0,y_0) \) because \( |\tilde{G}(x,\cdot)| \leq C(\Omega,k) \). If \( \text{dist}(y_0,\partial\Omega) \geq \kappa_1 \) or if \( |x_0 - y_0| \geq R_2/3 \) then the same bound holds by Lemma 2.2(d) or Lemma 3.2(b) (but with a different constant \( C(\Omega,k) \)). This proves the bound in Theorem 1.2(f) for \( |\alpha| = 0 \).

Bounds for higher derivatives now follow almost immediately from Lemma 3.2: Let \( x_0 \) and \( y_0 \) be distinct points in \( \Omega \) with \( \text{dist}(y_0,\partial\Omega) \leq \kappa_0 \) and \( |x_0 - y_0| \leq \delta_0 \) and recall the definitions in Lemma 3.2 for \( \varepsilon, V_1 \) and \( V_2 \). The bound proved above for \( G \) shows that
\[ |G(x_0,\cdot)|_{L^2(V_1)} \leq C(\Omega,k)(\int_{V_1} \frac{dy}{|y - x_0|^2})^{1/2} \leq C(\Omega,k)\varepsilon^{1/2} \]
where \( \varepsilon = |x_0 - y_0| \). Revising the last step in the computation in (15) and applying the above bound we then conclude that for \( j \leq 2 \)
\[ |D^j_y G(x_0,y_0)| \leq C(\Omega,k)\left(\varepsilon^{-3/2}|D^j_y G(x_0,\cdot)|_{L^2(V_2)} + \varepsilon^{-1/2}|D^j_y G(x_0,\cdot)|_{L^2(V_2)} \right. \]
\[ \left. + \varepsilon^{1/2}|D^{j+2}_y G(x_0,\cdot)|_{L^2(V_2)} \right) \]
\[ \leq C\varepsilon^{-(j+3/2)}(|G(x_0,\cdot)|_{L^2(V_1)} + \varepsilon^{7/2}) \leq C(\Omega,k)\varepsilon^{-(j+1)} \]
This proves the bound in Theorem 1.2(f) for \( y_0 \) near \( \partial\Omega \) and \( x_0 \) close to \( y_0 \). The same bound for the remaining cases that \( \text{dist}(y_0,\partial\Omega) \geq \kappa_0 \) or that \( |x_0 - y_0| \geq \delta_0 \) again follow from Lemma 2.2(d) and Lemma 3.2(b). This completes the proof of Theorem 1.2.

Appendix: Proof of Lemma 3.1(b). In this section we prove the “interior at the boundary” estimates stated in Lemma 3.1(b) and applied in Lemma 3.2. The first step is to flatten the upper and lower boundaries of \( \mathcal{U}_b \) by making the change of variables \( y \mapsto z(y) \) given by \( \tilde{y} = \bar{y} \) and \( z_3 = y_3 - \psi(y) \). The set \( \mathcal{U}_b \) is then mapped to \( S_\delta = B_3(0) \times (-\delta,0) \). Next we let \( u \) and \( \varphi \) be as in the statement of Lemma 3.1(b) and define \( w(z) = u(y(z)) \) and \( f(z) = \varphi(y(z)) \) where \( y(z) \) is the inverse mapping. Then \( w \in H^k(S_\delta) \), \( f \in H^1(S_\delta) \) and the trace of \( f \) is zero on the bottom and lateral surface of \( S_\delta \). We then compute from (12) that for all such \( f \),
\[ \int_{S_\delta} (A(z)\nabla w(z)) \cdot \nabla f(z) \, dz = c \int_{S_\delta} f(z) \, dz \quad (20) \]
where \( A(z) = I + E(z) \) and \( E \) is a symmetric matrix whose entries are first derivatives of \( \psi \). These entries are therefore \( C^k \) functions of \( \tilde{z} \) and are bounded in \( S_\delta \) by \( C(\Omega,k)\delta \). It will suffice to prove that for \( j \leq k \)
\[ \sum_{|\alpha| = j} |D^\alpha_z w|_{L^2(S_{\delta/2})} \leq C(\Omega,k)\delta^{-j}M \quad (21) \]
where \( M = |w|_{L^2(S_\delta)} + |c|\delta^{7/2} \).
To prove (21) we will construct and apply cut-off functions \( \chi(z) = \zeta(z)\eta(z_3) \) where \( \eta \) will be identically one for \( z_3 \geq -\delta/2 \) and zero in a neighborhood of \( z_3 = -\delta/2 \), and \( \zeta \) is to satisfy the condition

\[
(A^{tr}(z)\nabla\zeta(z)) \cdot \nu(z) = 0
\]  

on the top of \( S_3 \). This condition will be required to eliminate a certain boundary integral and can be achieved as follows. Since \( \nu((\tilde{z}, 0)) \) is the standard basis vector \((0, 0, 1)\) and \( A = I + E \), (22) can be written

\[
a_1(z)\zeta_{z_1} + a_2(z)\zeta_{z_2} + \zeta_{z_3} = 0
\]

where \( a_1 \) and \( a_2 \) are \( C^k \) and are \( O(\delta) \). Thus (22) will hold if \( \zeta \) is constant on the characteristic curves of this first-order pde. Let \( \xi, z \) respect to \( s \) identically one on \( S \) and again the sum on the right is absent if \( \chi \) is constant on these characteristic curves, so that (22) will hold, then inverting the mapping \((\xi, s) \rightarrow (z_1, z_2, z_3)\). Specifically, we let \( \zeta \) for a given \( j \geq 0 \) so that \( \zeta \) is constant on each of these characteristic curves and (22) holds and such that \( \zeta(\tilde{z}, z_3) \) is \( C^k \), is nonincreasing in \( |\tilde{z}| \), is identically one or zero according as \( |\tilde{z}| \leq r_{j+1} \) or \( |\tilde{z}| \geq r_j \), and satisfies \( \nabla\zeta(z) = O(\delta^{-1}) \) and \( D^2\zeta(z) = O(\delta^{-2}) \). In a similar but simpler way we can construct a \( C^\infty \) function \( \chi_2(z_3) \) which is identically one or zero according as \( z_3 \geq -r_{j+1} \) or \( z_3 \leq -r_j \). Finally we let \( \chi = \zeta \eta \) so that \( \chi \) is \( C^k \), \(|D_z\chi(z)| \leq C\delta^{-1} \) and \(|D^2\chi(z)| \leq C\delta^{-2} \), and \( \chi \) is identically one on \( S_{r_{j+1}} \) and identically zero on the complement of \( S_{r_j} \).

Now assume that for some \( j < k \) and for \( i = 0, \ldots, j \)

\[
\sum_{|\alpha|=i} |D^\alpha w|_{L^2(S_\delta)} \leq C(\Omega, k)\delta^{-i} M
\]

which is true for \( j = 0 \). If \( f \in H^{j+1}(S_\delta) \) and \( f = 0 \) on neighborhoods of the bottom and lateral surface of \( S_\delta \), then for \( |\alpha| = j \), \( D^\alpha f \) is an allowable test function in (20), which therefore holds with \( f \) replaced by \( D^\alpha f \). And if each \( \alpha_i \in \{1, 2\} \), that is, if \( D^\alpha \) involves only horizontal derivatives, we can integrate by parts in the horizontal directions (unless \( j = 0 \)).

\[
\int_{S_\delta} (A\nabla D^\alpha w) \cdot \nabla f = \pm \sum_{|\beta|} \binom{\alpha}{\beta} \int_{S_\delta} (D^\beta A)\nabla D^\alpha w \cdot \nabla f + c \int_{W_\delta} D^\alpha f
\]

where the sum is over multi-indices \( \beta \) and \( \gamma \) whose sum is \( \alpha \) and with \(|\gamma| < j \) (the sum is absent if \( j = 0 \) and the last term on the right is present only if \( j = 0 \)). Now let \( \chi \) be the cut-off function constructed above and put \( f(z) = \chi(z)D^\alpha w(z) \). The result is that

\[
\left| \int \chi(A\nabla D^\alpha w) \cdot (\nabla D^\alpha w) dz \right| \leq \left| \int (D^\alpha w)(A\nabla D^\alpha w) \cdot (\nabla \chi) dz \right| + C \sum \int |\nabla D^\alpha w||(\chi||\nabla D^\alpha w| + |\nabla \chi||D^\alpha w|) dz + |c| \int \chi D^\alpha w dz
\]

and again the sum on the right is absent if \( j = 0 \) and the last term on the right is present only if \( j = 0 \). The term on the left dominates \( \int_{S_\delta} \chi |\nabla D^\alpha w|^2 dz \) which in
The right side of (25) with the exception of the term $D_j$ and proving (21).

tatively, we finally conclude that (25) holds for all $|\alpha|=2$, we let $\alpha = \beta + (0,0,2)$ where $D^\beta$ includes no $z_3$-derivatives. Consequently every term on the right side in (26) is known to be bounded by the right side of (25) with the exception of $E^{3,3}D^\beta w_{z_3,z_3}$, and the same is true for the terms on left with the exception of the term $D^\beta w_{z_3,z_3}$. We therefore have that

$$D^\beta w_{z_3,z_3} = -E^{3,3}D^\beta w_{z_3,z_3} + l.o.t.$$ 

Since $E = O(\delta)$ and we can stipulate that $\delta$ is small, we conclude that $D^\alpha w = D^\beta w_{z_3,z_3}$ is bounded as required. This proves that (25) now holds for operators $D^\alpha$ of order $j+1$ involving at most two derivatives with respect to $z_3$. Proceeding inductively, we finally conclude that (25) holds for all $\alpha$ of length $j+1$, completing the induction (23) and proving (21).

**Remark.** The techniques and results of this paper can in fact be extended to dimensions greater than three, *mutatis mutandi*, with one minor change but also one less elementary change. First, since the fundamental solution is not locally in $L^2$ if $n > 3$, the proof of symmetry based on Fourier series given here does not apply.
A different proof, equally elementary, is available however (see [11]). The greater complication is that the interior estimates at the boundary in $L^2$-Sobolev spaces, for which we gave an elementary proof by cutoff functions and self-duality, must be replaced by the corresponding bounds in $L^p$-Sobolev spaces for particular $p \in [1, 2)$ depending on $n$. The proofs are more sophisticated and technical, however, and are beyond the scope and intent of this paper.

REFERENCES

[1] R. A. Adams, Sobolev Spaces, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.
[2] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, Comm. Pure Appl. Math., 12 (1959), 623–727.
[3] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, Comm. Pure Appl. Math., 17 (1964), 35–92.
[4] B. E. J. Dahlberg and C. E. Kenig, Hardy spaces and the Neumann problem in $L^p$ for Laplace’s equation in Lipschitz domains, Ann. of Math., 125 (1987), 437–465.
[5] O. Druet, F. Robert and J. Wei, The Lin-Ni’s Problem for Mean Convex Domains, Mem. Amer. Math. Soc., 2012.
[6] L. C. Evans, Partial Differential Equations, 2nd edition, Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, RI, 2010.
[7] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Berlin, 1983.
[8] D. Hoff, Linear and Quasilinear Parabolic Systems, Mathematical Surveys and Monographs, 251, American Mathematical Society, Providence, RI, 2020.
[9] D. Hoff, Compressible flow in a half-space with Navier boundary conditions, J. Math. Fluid Mech., 7 (2005), 315–338.
[10] C. E. Kenig and J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients, Invent. Math., 113 (1993), 447–509.
[11] F. Robert, Construction and asymptotics for the Green’s function with Neumann boundary conditions, Informal Notes, 2010, available at https://iecl.univ-lorraine.fr/files/2021/04/NotesGreenNeumannRobert.pdf.

Received June 2021; revised October 2021; early access November 2021.

E-mail address: hoff@indiana.edu