EMERGENCE OF BENFORD’S LAW IN CLASSICAL MUSIC

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ABSTRACT. We analyzed a large selection of classical musical pieces composed by Bach, Beethoven, Mozart, Schubert and Tchaikovsky, and found a surprising connection with mathematics. For each composer, we extracted the time intervals each note was played in each piece and found that the corresponding data sets are Benford distributed. Remarkably, the logarithmic distribution is not only present for the leading digits, but for all digits.

1. Introduction

What does the Moonlight Sonata by Beethoven have in common with the Swan Lake ballet by Tchaikovsky? They both exhibit Benford distributed time intervals for their constituent musical notes. This is not a singular result. We analyzed hundreds of musical pieces composed by Bach, Beethoven, Mozart, Schubert, and Tchaikovsky and found that in each case, the note durations are Benford distributed.

Our data consists of a selection of MIDI files downloaded from the music archive http://www.kunstderfuge.com, which is a major resource housing thousands of music files. We chose a collection of sonatas, concertos, etc., for a total of 521 files. It should be noted that depending of the structure of each musical piece, that piece may be spread over several files. For instance, Tchaikovsky’s Swan Lake has 4 acts with each act broken to parts, each part corresponding to a single file, for a total 43 files. We used Mathematica to obtain the time duration each note was played in a given file. In our analysis we ignored the dynamics, thus the quieter parts were given the same weight as the louder ones. These data were compiled into tables, which were analyzed for their digit distributions. With no exception, we observed the emergence of Benford’s law across the works of each of the composers we studied.

This paper is structured as follows. First, we present a short introduction to Benford’s law. Then, we analyze the digit distribution in the time tables and find it to be Benford in all digits. For our analysis we used a quantile-quantile representation in which the experimental data sets were plotted against the theoretical Benford distribution and found a remarkable close agreement.

2. Benford’s Law

Benford’s law comes from the empirical observation that in many data sets the leading digits of numbers are more likely to be small than large, for instance 1 is more likely to occur as the leading digit than 2, which in turn is more likely the first digit than 3, etc. This observation was first published by Newcomb in 1881 [1], and given experimental support in 1938 by Benford [2] who analyzed over 20,000 numbers collected from naturally occurring data sets such as the area
of the riverbeds, atomic weights of elements, etc. Explicitly, he showed that the probability of \( d \) being the first digit is

\[
P_d = \log_{10} \left(1 + \frac{1}{d}\right), \quad d = 1, 2, \ldots, 9
\]

which came to be known as Benford’s law. The first digit probabilities are illustrated in Fig. (1). Similarly, there are logarithmic expressions for the probabilities of the second, third and other digits. For instance, the probability of a number having its first digit \( d_1 \) and second digit \( d_2 \) is

\[
P_{d_1d_2} = \log_{10} \left(1 + \frac{1}{d_1d_2}\right).
\]

Thus, in a Benford distributed data set, the probability of a number having its first digit equal to 2 and the second digit equal to 6 is \( P_{26} = \log_{10} \left(1 + \frac{1}{26}\right) \approx 0.0164 \).

In general, a set \( \{x_n\} \) of real positive numbers is Benford if

\[
\lim_{N \to \infty} \frac{\#\{1 \leq n \leq N : S(x_n) \leq t\}}{N} = \log t, \quad \text{for all } t \in [1, 10)
\]

where

\[
S(x) = 10^{\log_{10} x - \log_{10} \lfloor x \rfloor}, \quad x > 0
\]

is the significand function \( S : \mathbb{R}^+ \to [1, 10) \). In the above definition \( \lfloor x \rfloor \) denotes the floor function. In other words, the significand function simply gives the first part of the scientific notation of any number. For example, the significand of 143 is \( S(143) = 1.43 \).

The current research reaffirms the ubiquity of Benford’s law in many collections of numerical data. For a large list of applications ranging from identifying fraud in financial data [4] to the set of distances from Earth to stars [5], see [6].

3. Data Extraction and Analysis

We chose a collection of sonatas, concertos, etc., for a total of 521 files and, using Mathematica, we extracted the time duration each note was played in a given musical piece. For example, Table 1 illustrates part of our data for the Moonlight Sonata (Opus 27, No. 2) by Beethoven. It contains nine notes and their corresponding cumulative times in seconds each note was played during the entire musical piece. For each of the 32 Beethoven’s piano sonatas we constructed similar
data sets, and performed a digit distribution analysis on the collected data on the
notes’ duration. In this case the data set was comprised of 2043 values which is
roughly 32 (sonatas) × 88 (number of piano keys) minus the total number of notes
in all sonatas that were not played. We made a histogram of frequencies for digits
1 through 9 as the first digits. The result was compared with Benford’s law eq.

![Figure 2](image2.png)

**Figure 2.** The histograms represents the digit distribution of time
intervals for each piano key played for the 32 piano sonatas by
Beethoven vs. the theoretical (Benford) distribution.

Similarly we made a histogram of frequencies of numbers starting with 10,
11, 12, through 99 and compared with theoretical two-digit Benford law eq. (2).
The results are illustrated in Figure 2. Note the good fit of the experimental data
with the theoretical prediction. But the goodness of fit extends beyond the first
two digits, as evidenced by the linearity of the points in the Q-Q plot in Figure
![Figure 3](image3.png)

**Figure 3.** Quantile-Quantile Plot comparing the theoretical Ben-
ford and experimental for the 32 Beethoven sonatas.

two digits, as evidenced by the linearity of the points in the Q-Q plot in Figure

Because the Q-Q plot compares the experimental CDF with Benford’s Law
theoretical definition in Eq. (3), we conclude that the analyzed times are Benford distributed for all digits.

Repeating the same analysis for Tchaikovsky’s Swan Lake ballet, we obtained the Q-Q plot shown in Figure (4).

Comparing Figure (3) with Figure (4) we see that indeed the time durations in these musical pieces are Benford distributed.

Similarly, analyzing a large selection of music files by J. S. Bach, Mozart, and Schubert we obtained Benford distributed time durations. The complete selection contains 72 pieces by Bach, 32 pieces by Beethoven, 41 pieces by Mozart, 271 pieces by Schubert, and 105 pieces by Tchaikovsky. Finally, we did the union of all data sets, and as expected, the fit is extremely good. The results are presented in Figure (5).
In conclusion, based our extended analysis, we would like to advance the conjecture that the time durations in classical pieces are Benford distributed.

**References**

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