Unitary transformation for the system of a particle in a linear potential

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Abstract. – A unitary operator which relates the system of a particle in a linear potential with time-dependent parameters to that of a free particle, has been given. This operator, closely related to the one which is responsible for the existence of coherent states for a harmonic oscillator, is used to find a general wave packet described by an Airy function. The kernel (propagator) and a complete set of Hermite-Gaussian type wave functions are also given.

Introduction. – The existence of coherent and squeezed states for a simple harmonic oscillator \cite{1,2} can be understood from the fact that there exist unitary transformations which leave the time-dependent Schrödinger equation (formally) invariant under the transformations. These transformations have been found as the relations between harmonic oscillators of time-dependent parameters \cite{3,4}, while the transformations between the same simple harmonic oscillators can be applied to a stationary states to give the coherent and squeezed states \cite{4}. On the other hand, Feynman and Hibbs show that the kernel of a general quadratic system is described by the classical solutions of the system \cite{5}. Since the wave function might be derived from the kernel, this suggests that the unitary transformations are described by classical solutions, as explicitly shown in the quadratic systems \cite{6}.

There has been considerable interest for the system of a particle in a linear potential \cite{7,8,9}. This model has eigenfunctions (wave packets) described by the Airy function \cite{10}. The model and the Airy wave functions on a half-line have been used to model the production of high harmonic generation in the laser irradiation of rare gases \cite{11}, and the edge electron gas \cite{12,13}. The model on piecewise domains and the wave functions have been frequently used to model various physical systems \cite{14}. The Schrödinger equation for a free particle has also long been interesting in that the equation is formally identical to the wave equation of a beam of light in the paraxial approximation \cite{15}.

In this article, we will show that there exists a unitary relation between the system of a particle in a linear potential and that of a free particle. Indeed, time-dependent unitary relations have been known to be useful in analyzing quantum systems. Unitary transformations have been extensively used in showing that the Dirac theory goes to the Pauli theory in the non-relativistic limit \cite{16}, and a unitary relation between the system of a charged particle in a
purely time-dependent vector potential and the same system in an accelerated frame without the vector potential has been given in ref. [17]. We will find a unitary transformation which relates the model of a linear potential (with time-dependent parameters) to a free particle system. The transformation resembles the one for a quadratic system which gives coherent states, and could be applied to any wave function of a free particle to give the wave function of a particle in a linear potential. This transformation will be explicitly used to find a general wave packet described by the Airy function, which clearly shows the origin of the Berry-Balazs solution for the wave function of a free particle. Based on the kernel and wave functions of the quadratic system, the kernel (propagator) and a set of Hermite-Gaussian type wave functions for the linear system will also be given, and the completeness of the set will be proved.

The model and a unitary transformation. – The model we will consider is described by the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi(t, x) = -\frac{\hbar^2}{2M(t)} \frac{\partial^2}{\partial x^2} \psi(t, x) - xF(t)\psi(t, x), \]  

(1)

where \( M(t) \) and \( F(t) \) denote the time-dependent mass and external force, respectively. Eq. (1) would describe a charged particle in a linear, scalar potential. The solution \( x_p(t) \) for the classical equation of motion is given as

\[ x_p = \int_0^t 1 \int_0^t F(t') dt' + C \int_0^t 1 M(t') dt' + D, \]  

(2)

which satisfies \( \frac{d}{dt}(M\frac{dx_p}{dt}) = F \), where \( C \) and \( D \) are arbitrary real constants. By defining the operators

\[ O = -i\hbar \frac{\partial}{\partial t} + H = -i\hbar \frac{\partial}{\partial t} + \frac{p^2}{2M} - xF(t), \]  

\[ O_M = -i\hbar \frac{\partial}{\partial t} + H_M = -i\hbar \frac{\partial}{\partial t} + \frac{p^2}{2M}, \]  

(3)

(4)

one may find the unitary relation

\[ U(x_p)O_MU^\dagger(x_p) = O, \]  

(5)

where

\[ U(x_p) = \exp \left[ i \frac{\hbar}{\hbar} [M\dot{x}_p + \xi(x_p)] \right] \exp \left[ -i \frac{\hbar}{\hbar} x_p p \right], \]  

(6)

with the function of time \( \xi \) defined as

\[ \xi(x_p) = -\frac{1}{2} \int_0^t \left[ M(t')\dot{x}_p^2(t') \right] dt'. \]  

(7)

The overdots denote differentiations with respect to the time of the system.

From this unitary relation, one can find that, if a wave function \( \psi_M \) satisfies the Schrödinger equation \( O_M\psi_M = 0 \), the wave function \( U\psi_M \) satisfies the Schrödinger equation \( O(U\psi_M) = 0 \). For a harmonic oscillator system, the center of probability distribution of a (generalized) coherent state, obtained from a (stationary) state through a unitary transformation, moves along the trajectory described by a classical solution \([1,2,3,4]\). The unitary relation between
a linear system and the corresponding free particle system closely resembles the one for (generalized) harmonic oscillators, in that the probability distribution of the unitarily transformed wave function moves globally, according to the classical solution, from the distribution of the original wave function, while the shapes of the two distributions are same. The shapes could evolve under the time-evolution, as in the generalized coherent states. If $F = 0$, the unitary relation becomes a relation between the two physically identical systems, while the relation is still not identically unity if we choose non-zero $C$ and $D$; In this case, if $M(t)$ is a constant, the probability distributions move with the constant speed $C/M$ from each other. As may be clear in the $F = 0$ case, the degrees of freedom of choosing $C$ and $D$ may be interpreted as a manifestation of Galilean symmetry in quantum mechanics.

**Airy wave packets.** – For the linear system with a constant mass $m$, a constant force $f(\equiv \frac{\beta^3}{2m})$, and Hamiltonian $H_m (= \frac{p^2}{2m} - fx)$, there are eigenfunctions described by an Airy function:

$$\phi = \text{Ai}[-\frac{\beta}{\hbar^{2/3}}(x + e)],$$

(8)

with a constant $e$. The Airy function satisfies the equation

$$\frac{d^2}{dz^2}\text{Ai}(z) - z\text{Ai}(z) = 0,$$

(9)

which gives the energy-eigenvalue relation [10]

$$H_m\phi(x) = (\frac{\beta^3}{2m} e)\phi(x).$$

(10)

By choosing a particular solution as

$$x_f = \frac{\beta^3 t^2}{4m^2} + Ct,$$

(11)

and applying the unitary transformation to a stationary wave packet $\psi_f(\equiv \exp[-\frac{\beta^3}{\hbar}(\frac{\beta^3 t}{2m} + mCe)]\phi(x))$, one can find a wave packet $\psi_{free}$ for a free particle system, as

$$\psi_{free} = U^\dagger(x_f)\psi_f$$

$$= \exp\left[\frac{\beta^3}{\hbar} \left\{\frac{m^3}{3\beta^3}(\frac{\beta^3 t}{2m^2} + C)^3 - C^3\right\} - m(\frac{\beta^3 t}{2m^2} + C)(x + \frac{\beta^3 t^2}{4m^2} + Ct + e)\right]\times\text{Ai}[-\frac{\beta}{\hbar^{2/3}}(x + \frac{\beta^3 t^2}{4m^2} + Ct + e)].$$

(12)

If we take $C = 0$ and $e = 0$, $\psi_{free}$ reduces to the wave packet of ref. [7] which propagates in free space without distortion and with constant acceleration. Our derivation of $\psi_{free}$ shows that the solution for a free particle system given in ref. [7] is related to the stationary wave packet of zero energy-eigenvalue in a linear potential, while similar wave packets for the free particle system can also be found from the stationary wave packets of non-zero energy-eigenvalues. By using a unitary operator $U(x_f + D)$, one can obtain another expression for $\psi_{free}$, which is, however, similar to the one given in Eq. (12) with a redefinition of $e$. Even though the Airy wave functions are square-integrable on a half-line ($x < L$) [11] or on a piecewise domain, the wave function (packet) is not square-integrable on the whole line, as have been discussed in detail in ref. [10].
The wave function $\psi_M(\tau, x)$ satisfying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial \tau} \psi_M(\tau, x) = -\frac{\hbar^2}{2M(\tau)} \frac{\partial^2}{\partial x^2} \psi_M(\tau, x)$$

for a free particle with a time-varying mass $M(\tau)$, can be found from $\psi_{\text{free}}$, by replacing $t$ with $\int_0^\tau \frac{m(t')}{M(t')} dt'$. Wave functions satisfying Eq. (1) can then be found by applying the unitary operator to $\psi_M$, as

$$\psi(t, x) = U(x^M_p(t))\psi_M(t, x)$$

$$= \exp\left[i \frac{1}{\hbar} \left( \int_0^t \left( \int_0^{t'} F(t'')dt'' \right) dt' + \frac{1}{3} \left( \frac{\beta^2}{2} \int_0^t dt' \frac{dt''}{M(t')} + \frac{C}{4} \right)^3 + \int_0^t xF(t')dt' \right]$$

$$\times \exp\left[-\frac{\beta}{2} \int_0^t \frac{dt''}{M(t'')} + C \right] \left[ x - x^M_p(t) + e + \frac{\beta^3}{4} \left( \int_0^t \frac{dt''}{M(t'')} \right)^2 + C \left( \int_0^t \frac{dt''}{M(t'')} \right) \right]$$

$$\times \text{Ai} \left[-\frac{\beta}{\hbar^{2/3}} \left( x - x^M_p(t) + e \right) + \frac{\beta^3}{4} \left( \int_0^t \frac{dt''}{M(t'')} \right)^2 + C \left( \int_0^t \frac{dt''}{M(t'')} \right) \right], \quad (13)$$

where

$$x^M_p(t) = \int_0^t \frac{1}{M(t')} \int_0^{t'} F(t'')dt'' dt'.$$  

(14)

With the choice of $C = 0$ and $e = 0$, $\psi(t, x)$ reduces to the one given in ref. [3].

A complete set of wave functions. – The kernel of a harmonic oscillator described by the Lagrangian

$$L^F = \frac{1}{2} M(t) \dot{x}^2 - \frac{1}{2} M(t) u^2(t) x^2 + F(t)x$$

(15)

whose classical equation of motion is given by

$$\frac{d}{dt} (M \frac{dx}{dt} ) + w^2 x = F,$$  

(16)

has been given in ref. [4]. The classical solution is described by two linearly independent homogeneous solutions $u_c(t)$ and $v_s(t)$ and one particular solution $x_{ph}(t)$. By requiring the conditions $v_s(t_a) = 0$, $u_c(t_a) = 1$, $x_{ph}(t_a) = 0$, $\dot{x}_{ph}(t_a) = 0$ on classical solutions, the expression of the kernel given in Eq. (32) of ref. [4] can be simplified to

$$K^F(b, a) = \sqrt{\frac{M(t_b)}{2\pi i \hbar} \frac{\dot{v}_s(t_a)}{v_s(t_b)}}$$

$$\times \exp \left[i \frac{1}{\hbar} \left( \int_0^{t_b} M(t_a) \frac{\dot{v}_s(t_a)}{v_s(t_a)} \right) \right] \left[ x_b - x_{ph}(t_b) \right]^{2} M(t_b) \frac{\dot{v}_s(t_b)}{v_s(t_b)}$$

$$- 2x_b x_b \frac{\dot{v}_s(t_b)}{v_s(t_b)} + 2M(t_b)\dot{x}_{ph}(t_b)x_b$$

$$+ \int_{t_a}^{t_b} \left( M w^2 x_{ph}^2 - M x_{ph}^2 \right) dt],$$  

(17)

without loosing generality. The kernel for the system described by the Hamiltonian $H$ can be found by considering the harmonic oscillator of $w = 0$. By letting $u_c(t_b) = 1$, $v_s(t_b) = \int_{t_a}^{t_b} \frac{dt'}{M(t')}$ and $x_{ph}(t_b) = \int_{t_a}^{t_b} \frac{1}{M(t')} \int_{t_a}^{t'} F(t')dt' dt$, one may find that the kernel (propagator) can be written as

$$K(b, a) = K(t_b, x_b; x_a, t_a)$$
with a positive constant \( b \) then Mehler’s formula \([18]\) and the Schrödinger equation

\[
\frac{\hbar}{\partial t} K(b, a) = - \frac{\hbar^2}{2M(t_b)} \frac{\partial^2}{\partial x_b^2} K(b, a) - x_b F(t_b) K(b, a).
\]

The results for the harmonic oscillator system can also be used to find wave functions for a particle in a linear potential. By defining

\[
v(t) = \int_0^t \frac{dt'}{M(t')} + p, \\
\rho(t) = \sqrt{v^2(t) + b^2},
\]

with a positive constant \( b \) and an arbitrary real constant \( p \), the wave function \( \psi_n \)

\[
\psi_n(t, x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi \hbar}}} \left( \frac{\hbar}{\rho} \right)^{n + \frac{1}{2}} \times \exp \left[ \frac{i}{\hbar} (M \dot{x}_p x + \xi(x_p)) \right] \exp \left[ \frac{(x - x_p)^2}{2\hbar} \left( - \frac{b}{\rho^2} + iM \frac{\dot{\rho}}{\rho} \right) \right] H_n \left( \sqrt{\frac{b}{\hbar}} \frac{x - x_p}{\rho} \right)
\]

may be given, where \( H_n \) is the \( n \)-th order Hermite polynomial and \( x_p \) is given in Eq. (1). After some algebra, one can find the relation

\[
K(b, a) = \sum_{n=0}^{\infty} \psi_n(t_b, x_b) \psi_n^*(t_a, x_a), \quad \text{for} \quad t_b > t_a,
\]

which proves that \( \psi_n \) satisfies the Schrödinger equation of Eq. (1), and the set of \( \{ \psi_n | n = 0, 1, 2, \ldots \} \) is complete. While the relation in Eq. (24) is valid for a general \( M(t) \), by defining

\[
z = \sqrt{\frac{b - i(t_b + p)}{b + i(t_b + p)}} \sqrt{\frac{b + i(t_a + p)}{b - i(t_a + p)}},
\]

then Mehler’s formula \([18]\)

\[
\sum_{n=0}^{\infty} \frac{z^{n+\frac{1}{2}}}{2^n n!} H_n(X) H_n(Y) = \sqrt{\frac{z}{1 - z^2}} \exp \left[ - \frac{z^2}{1 - z^2} (X^2 + Y^2) + 2 \frac{z}{1 - z^2} XY \right],
\]

and the fact that

\[
\frac{z}{1 - z^2} = \frac{\rho(t_b) \rho(t_a)}{2ib(t_b - t_a)},
\]
can be used for the proof of the relation for the case \( M = 1 \). Contrary to the case of the Airy wave function, the shape of the probability distributions of these wave functions evolves as time passes. Following Ref. [15], we define the generalized Gouy phase factor \( \chi(t) \) as

\[
\tan \chi = \frac{v}{b}.
\]

(28)

Expressions for the generalized spot size \( \gamma(t) \) and radius of curvature of the wave front \( s(t) \), which are real, are given through a single complex equality

\[
\frac{1}{\gamma^2(t)} - \frac{i}{\hbar s(t)} = \frac{1}{b + iv(t)}.
\]

(29)

With these definitions, one can find that \( \psi_n \) is written as

\[
\psi_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \times \frac{1}{\sqrt{\gamma}} \exp \left[ \frac{i}{\hbar} \left( M \dot{x}_p x + \xi (x_p) + \frac{(x - x_p)^2}{2 s(t)} \right) - i(n + \frac{1}{2}) \chi - \frac{(x - x_p)^2}{2 \gamma^2} \right] H_n \left( \frac{x - x_p}{\gamma} \right).
\]

(30)

For the case of \( M = 1 \) and \( F = 0 \), if we choose \( x_p = 0 = p \), \( \psi_n \) reduces to the Hermite-Gaussian mode in the paraxial approximation [15]. On the other hand, it is clear that, by applying the unitary transformation given in Eq. (6), the general expression of \( \psi_n(t) \) could be obtained from a Hermite-Gauss mode whose time is \( v (= v(t)) \).

**Summary.** – A unitary transformation which relates the model of a linear potential (with time-dependent parameters) to a free particle system has been given. This transformation closely resembles the one responsible for the existence of coherent states in harmonic oscillators, and the two arbitrary parameters in the transformation have been interpreted as a manifestation of the Galilean symmetry of classical mechanics. While this transformation (with a change of the time-scale) can be used to find a wave function of a particle in a linear potential from any wave function of a free particle, this transformation has been explicitly used to find a general wave packet described by an Airy function. Based on the kernel and wave functions of a generalized harmonic oscillator, the kernel and a set of Hermite-Gaussian type wave functions are also given. The completeness of the set has been proved for a general case, while such a proof is still not available in the harmonic oscillator system.

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