A UNIFIED FRAMEWORK FOR CONTINUOUS/DISCRETE
POSITIVE/BOUNDED REAL STATE-SPACE SYSTEMS

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Abstract. There are four variants of passive, linear time-invariant systems, described
by rational functions: Continuous or Discrete time, Positive or Bounded real. By in-
troducing a quadratic matrix inequality formulation, we present a unifying framework
for state-space characterization (a.k.a. Kalman-Yakubovich-Popov Lemma) of the above
four classes of passive systems.

These four families are matrix-convex as rational functions, and a slightly weaker version
holds for the corresponding balanced, state-space realization arrays.

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positive real rational functions, bounded real rational functions, passive linear systems,
state-space realization, K-Y-P Lemma

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1. Introduction

In the study of dynamical systems, passivity is a fundamental property. Thus, it has
been extensively addressed in various frameworks. We here focus on finite-dimensional,
linear, time-invariant, passive systems described by matrix-valued real rational functions
of a complex variable $z$.

We shall use the following notation: $\mathbb{C}_L$ or $\mathbb{C}_R$ is the open Left or Right half
of the complex planes ($\overline{\mathbb{C}_R}$ is the closed right half plane). Let also $\mathbb{D} = \{ z \in \mathbb{C} : 1 > |z| \}$,
$\overline{\mathbb{D}} = \{ z \in \mathbb{C} : 1 \geq |z| \}$, be the open, closed unit disk and $\mathbb{D}^c = \{ z \in \mathbb{C} : |z| > 1 \}$ is
the exterior of the closed unit disk.

For simplicity of exposition we begin with scalar functions terminology:

(a) $\mathcal{P}$, Positive-Real (continuous-time) analytically mapping $\mathbb{C}_R$ to its closure, $\overline{\mathbb{C}_R}$.

See e.g. [4], Chapter 5], [9] Chapter 7], [10] Subsection 2.7.2], and [15].

1The superscript $c$ stands for “complement”.

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(β) $\mathcal{B}$, Bounded-Real, (continuous-time) analytically mapping, $\mathbb{C}_R$ to $\mathbb{D}$. See [3], [4, Section 7.2], [9, Chapter 7], [10, Subsection 2.7.3], and [15].

(γ) $\mathcal{DP}$, Discrete-Time-Positive-Real analytically mapping $\mathbb{D}$ to $\mathbb{C}_R$, the closed right-half plane. See e.g. [21], [33, Lemma 1].

(δ) $\mathcal{DB}$, Discrete-Time-Bounded-Real analytically mapping $\mathbb{D}$ to $\mathbb{D}$. See e.g. [24], [26] and [30].

The above families, to be used in the sequel, are common in Engineering circles. For completeness we point out that in mathematical analysis community there are additional sets:

Herglotz or Carathéodory functions analytically map $\mathbb{D}$ to $\mathbb{C}_R$. See [20]. In other words, if $F(z)$ is a Herglotz function, $F(z^{-1})$ is a $\mathcal{DP}$ function.

Schur functions analytically map $\mathbb{D}$ to its closure $\overline{\mathbb{D}}$. See e.g. [14] and [28]. In other words, if $F(z)$ is a Schur function, $F(z^{-1})$ is a $\mathcal{DB}$ function.

Recall that whenever $F(z)$ is an $p \times m$-valued rational function with no pole at infinity, i.e. $D := \lim_{z \to \infty} F(z)$ is well-defined, one can associate with it a corresponding $(n+m) \times (n+m)$ state-space realization array, $R_F$ i.e.

\[
F(z) = C(zI_n - A)^{-1}B + D \quad R_F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

The $(n+p) \times (n+m)$ realization $R_F$ in Eq. (1.1) is called minimal, if $n$ is the McMillan degree of $F(z)$.

In this work we focus on the case $p = m$ and examine characterizations of the above four families of passive systems through the corresponding state-space realizations. This is also known as the Kalman-Yakobovich-Popov Lemma. For a (modest) account of the vast literature on the subject, beyond those mentioned thus far, see e.g. [1], [8], [13], [19] (a survey), [22], [23], [27], [31] and [32]. For infinite-dimensional versions (all study Schur functions in the above terminology) see e.g. [6], [7], [29].

This work is organized as follows. In Section 2 we give the basic background. Matrix-convex sets are introduced the Section 3. The main result given in Section 4, and in Section 5 it is applied to show matrix-convexity of systems in the framework of state-space realizations.

2. PRELIMINARY BACKGROUND

In the sequel we shall denote by $(\mathbf{H}_n)$ $\overline{\mathbf{H}}_n$, the set of $n \times n$ (non-singular) Hermitian matrices. Skew-Hermitian matrices are denoted by, $i\overline{\mathbf{H}}_n$. It is common to take $\overline{\mathbf{H}}$ and $i\overline{\mathbf{H}}$ as the matricial extension of $\mathbb{R}$ and $i\mathbb{R}$, respectively. Then within Hermitian matrices $(\mathbf{P}_n)$ $\overline{\mathbf{P}}_n$ will be the respective subsets of positive (definite) semi-definite matrices. Recall that $\overline{\mathbf{P}}_n$ may be viewed as the closure of the open set $\mathbf{P}_n$.

For $H \in \mathbf{H}_n$ let us define the following sets satisfying the inclusions of Lyapunov and Stein, respectively

\[
\mathbf{L}_H := \left\{ A \in \mathbb{C}^{n \times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \overline{\mathbf{P}}_n \right\} \quad \mathbf{S}_H := \left\{ A \in \mathbb{C}^{n \times n} : \begin{pmatrix} A \\ I_n \end{pmatrix}^* \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A \\ I_n \end{pmatrix} \in \overline{\mathbf{P}}_n \right\}
\]
\( L_H := \left\{ A \in \mathbb{C}^{n \times n} : \begin{pmatrix} A & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A & 0 \\ I_n & I_n \end{pmatrix} \in \mathbb{P}_n \right\} \quad S_H := \left\{ A \in \mathbb{C}^{n \times n} : \begin{pmatrix} A & 0 \\ I_n & I_n \end{pmatrix} \begin{pmatrix} -H & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} A & 0 \\ I_n & I_n \end{pmatrix} \in \mathbb{P}_n \right\} \).

The above Quadratic Matrix Inclusion formulation is not the common way to describe the families \( L_H \) and \( S_H \). Yet it enables us to present these sets in a common framework. This approach will be taken a step forward in Theorem 4.2 below.

The sets \( L_H \) and \( S_H \) may be viewed as the closure of the open sets \( L_H \) and \( S_H \), respectively. The sets \( L_H \) and \( S_H \) were introduced and studied in [11]. In [5] Tsuyoshi Ando characterized the set \( S_H \).

We now resort to the classical Cayley transform. Recall that \( C(A) \), the Cayley transform of a matrix \( A \in \mathbb{C}^{n \times n} \), is given by

\[
C(A) := (I_n - A)(I_n + A)^{-1} = -I_n + 2(I_n + A)^{-1}, \quad -1 \notin \text{spect}(A).
\]

Recall also that this transform is involutive, i.e. whenever defined, \( C(C(A)) = A \).

It is well known that for a given \( H \in \mathbb{H}_n \),

\[
C(L_H) = \overline{S_H} \quad C(S_H) = L_H.
\]

When \( -H \in \mathbb{P}_n \), the set \( L_H \) is associated with Hurwitz stability of differential equations of the form \( \dot{x} = Ax \). Similarly, when \( H \in \mathbb{P}_n \), the set \( S_H \) is associated with Schur stability of difference equations of the form \( x(k+1) = Ax(k) \).

In the sequel, we shall focus on the special case where in Eq. (2.1) \( H = I_n \), i.e.

\[
\overline{L}_{I_n} := \left\{ A \in \mathbb{C}^{n \times n} : A + A^* \in \overline{\mathbb{P}}_n \right\} \quad \overline{S}_{I_n} := \left\{ A \in \mathbb{C}^{n \times n} : 1 \geq \|A\|_2 \right\}.
\]

One can now extend the above description of four families of scalar real rational functions to matrix-valued set-up.

**Definition 2.1.** Consider the following four families of \( m \times m \)-valued real rational function.

- (\( \alpha \)) \( F \in \mathcal{P} \), means that \( \forall z \in \mathbb{C} \) one has that \( F(z) \in \overline{L}_{I_m} \).
- (\( \beta \)) \( F \in \mathcal{B} \), means that \( \forall z \in \mathbb{C} \) one has that \( F(z) \in \overline{L}_{I_m} \).
- (\( \gamma \)) \( F \in \mathcal{DP} \) means that \( \forall z \in \mathbb{D} \) one has that \( F(z) \in \overline{S}_{I_m} \).
- (\( \delta \)) \( F \in \mathcal{DB} \) means that \( \forall z \in \mathbb{D} \) one has that \( F(z) \in \overline{S}_{I_m} \).

For completeness we recall that combining Definition 2.1 along with Eq. (2.3) reveals that these functions sets are related through the Cayley transform,

\[
\mathcal{B} = C(\mathcal{P}) \quad \mathcal{DB} = C(\mathcal{DP})
\]

\[
F(z) \in \mathcal{P} \iff F \left( \frac{1+z}{1-z} \right) \in \mathcal{DP}
\]

\[
F(z) \in \mathcal{B} \iff F \left( \frac{1+z}{1-z} \right) \in \mathcal{DB}.
\]

(In the Mathematical analysis terminology the formulation is more symmetric, e.g. \( F(z) \) belongs to \( \mathcal{P} \) is equivalent to having \( F(c(z)) \) a Herglotz function).
3. MATRIX-CONVEX SETS

We next resort to the notion of a matrix-convex set, see e.g. [16] and more recently, [17], [18], [25].

Definition 3.1. A family $\mathbf{A}$, of square matrices (of various dimensions) is said to be matrix-convex of level $n$, if for all $\nu = 1, \ldots, n$:

For all natural $k$,

$$\sum_{j=1}^{k} v_j^* v_j = I_\nu \quad \forall v_j \in \mathbb{C}^{\nu \times \nu}$$

having $A_1, \ldots, A_k$ (of various dimensions $1 \times 1$ through $\nu \times \nu$) within $\mathbf{A}$, implies that also

$$\sum_{j=1}^{k} v_j^* A_j v_j ,$$

belongs to $\mathbf{A}$.

If the above holds for all $n$, we say that the set $\mathbf{A}$ is matrix-convex.

In the rest of the section we briefly explore the notion of matrix-convexity.

Lemma 3.2. [22]. The following sets are matrix-convex:

(i) $H$, $iH$, $\mathbb{P}$, $\mathcal{P}$

(ii) $\{ A : \text{Bound} \geq \|A\|_2 \}$ for some Bound $> 0$.

(iii) The (open) closed set $(L_I) \cap \mathcal{T}_I$, see Eq. (2.4).

Recall that the matrix-convexity condition is quite restrictive, so there are not-too-many, non-trivial sets with this property. For example, the sets (i) Toeplitz matrices, (ii) $\{ A : \text{Bound} \geq \|A\|_1 \}$ for some Bound $> 0$, (iii) $\mathcal{L}_P$ with $\alpha I \notin \mathcal{P}$, are convex, but not matrix-convex. Furthermore, matrix-convexity implies both classical convexity and being unitarily-invariant, but the combination of these two properties still falls short of characterizing matrix-convexity. Indeed the set of positively scalar matrices, i.e of the form $\alpha I$, $\alpha > 0$ is unitarily invariant and convex. However, it is not matrix-convex: $A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ belong to the set but not the following combination where

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$\Upsilon_1^\ast \Upsilon_1 + \Upsilon_2^\ast \Upsilon_2 = I_2$.

Nevertheless, the four families of passive rational functions we focus on, do share this property.

Proposition 3.3. Each of the rational functions sets $\mathcal{P}$, $\mathcal{B}$, $\mathcal{DP}$ and $\mathcal{DB}$ is matrix-convex.

Proof: Let $F(z)$ be in $\mathcal{P}$ or in $\mathcal{B}$ and let $F(z_0)$ be the image of a point $z_0$ which lies in the domain of interest $(\mathbb{C}_R$ and $\overline{\mathbb{D}}^c$ for $\mathcal{P}$ and $\mathcal{B}$, respectively) From items $(\alpha)$, $(\beta)$ in Definition 2.1 it follows that as a matrix, $F(z_0)$ is in $L_I$, see Eq. (2.4), which is matrix-convex by item (iii) of Lemma 3.2.

In a similar way, let $F(z)$ be in $\mathcal{DP}$ or in $\mathcal{DB}$, and let $F(z_0)$ be the image of a point $z_0$ which lies in the domain of interest $(\mathbb{C}_R$ and $\overline{\mathbb{D}}^c$ for $\mathcal{DP}$ and $\mathcal{DB}$, respectively). From
In particular the spectrum of each $F(z)$ is in $\mathcal{S}_f$, see Eq. (2.4), which is matrix-convex by item (ii) of Lemma 3.2.

In Proposition 5.3 below, we offer a statement analogous to Proposition 3.3, but in the framework of realization arrays.

We end this section by pointing out that one can go beyond Proposition 3.3.

**Theorem 3.4.** (I) [22] The family $\mathcal{P}$, of $m \times m$-valued positive real rational functions, is a cone, closed under inversion and a maximal matrix-convex family of functions which is analytic in $\mathbb{C}_R$.

Conversely, a maximal matrix-convex cone of $m \times m$-valued rational functions, analytic in $\mathbb{C}_R$, containing the zero degree function $F(s) \equiv I_m$, is the set $\mathcal{P}$.

(II) [23]. A family of $m \times m$-valued real rational functions $F(z)$ which for all $z \in \mathbb{D}^*$ is:
Analytic, matrix-convex and a maximal set closed under multiplication among its elements, is the set $\mathcal{DB}$.

The converse is true as well.

4. Characterization through State-Space: A Unified Framework

Let us construct four $2(n+m) \times 2(n+m)$ matrices, all of compatible four blocks dimensions, with $P \in P_n$:

$$W_{n_1} = \frac{1}{2} \begin{pmatrix} I_n & I_n \\ 0 & 0 \\ I_n - I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -P & 0 \\ 0 & 0 \\ 0 & I_n - I_n \\ 0 & 0 \end{pmatrix}^*$$

$$W_{n_2} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -P & 0 \\ 0 & 0 \\ 0 & I_n \\ 0 & 0 \end{pmatrix}^*$$

$$W_{m_1} = \frac{1}{2} \begin{pmatrix} I_m & I_m \\ 0 & 0 \\ -I_m & I_m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -I_m & 0 \\ 0 & I_m \\ 0 & 0 \\ 0 & I_m \end{pmatrix}^*$$

$$W_{m_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -I_m \\ 0 & I_m \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -I_m & 0 \\ I_m \end{pmatrix}^*$$

and from these “building blocks” we obtain these matrices,

$$(4.1)$$

$$W_\alpha = W_{n_1} + W_{m_1} = \begin{pmatrix} 0 & 0 & -P & 0 \\ -P & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_\beta = W_{n_1} + W_{m_2} = \begin{pmatrix} 0 & 0 & 0 & -P \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_\gamma = W_{n_2} + W_{m_1} = \begin{pmatrix} 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$W_\delta = W_{n_2} + W_{m_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
framework. To this end, we adopt the elegant idea from \cite{15} and \cite{32} to treat the above \((n + m) \times (n + m)\) \(R_F\) as having two faces: (i) of an array and (ii) of a matrix.

**Theorem 4.2.** Let \(R_{F_l}\) be an \((n + m) \times (n + m)\) realization of \(m \times m\)-valued rational function, \(F_l(z)\),

\[
F_l(z) = C_l(zI_n - A_l)^{-1}B_l + D_l \quad \quad R_{F_l} = \begin{pmatrix} A_l & B_l \\ C_l & D_l \end{pmatrix} \quad l = \alpha, \beta, \gamma, \delta.
\]

(I) Consider the relation,

\[
\begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix}^* W_l \begin{pmatrix} R_{F_l} \\ I_{n+m} \end{pmatrix} = Q_l \in P_{n+m} \text{ with } W_{\alpha}, W_{\beta}, W_{\gamma}, W_{\delta} \text{ from Eq. (4.1)}. \tag{4.2}
\]

Then the following is true:

- (\(\alpha\)) If the condition in Eq. (4.2) is satisfied for \(W_{\alpha}\) then \(F_{\alpha}(z)\) is a Positive-Real function.
- (\(\beta\)) If the condition in Eq. (4.2) is satisfied for \(W_{\beta}\) then \(F_{\beta}(z)\) is a Bounded-Real function.
- (\(\gamma\)) If the condition in Eq. (4.2) is satisfied for \(W_{\gamma}\) then \(F_{\gamma}(z)\) is a Discrete-Time-Positive-Real function.
- (\(\delta\)) If the condition in Eq. (4.2) is satisfied for \(W_{\delta}\) then \(F_{\delta}(z)\) is a Discrete-Time-Bounded-Real function.

(II) In each of the four above cases, if the realization \(R_F\) is minimal, then the converse is true as well.

**Proof:** Indeed for \(l = \alpha, \beta, \gamma, \delta\) substituting in Eq. (4.2) \(W_l\) matrices from Eq. (4.1), yields the following explicit right-hand side

\[
\begin{align*}
Q_{\alpha} &= \begin{pmatrix} -PA - A^* P & C^* - PB \\ -B^* P - D^* C & D + D^* \end{pmatrix} \\
Q_{\beta} &= \begin{pmatrix} -PA - A^* P - C^* C & -PB - C^* D \\ -B^* P - D^* C & -D^* D \end{pmatrix} \\
Q_{\gamma} &= \begin{pmatrix} P - A^* PA & -A^* PB + C^* \\ -B^* PA + C & D + D^* - B^* PB \end{pmatrix} \\
Q_{\delta} &= \begin{pmatrix} P - A^* PA - C^* C & -A^* PB - C^* D \\ -B^* PA - D^* C & I_m - D^* D \end{pmatrix}.
\end{align*}
\]

Now for:

- \(Q_{\alpha}\) see e.g. \cite{1}, \cite{4, Chapter 5}, \cite{9, Chapter 7}, \cite{10, Subsection 2.7.2}, \cite{15}, and \cite{31, Theorem 3}.
- \(Q_{\beta}\) see e.g. \cite{3}, \cite{4, Section 7.2}, \cite{9, Chapter 7}, \cite{10, Subsection 2.7.3}, and \cite{15}.
- \(Q_{\gamma}\) see e.g. \cite{21}, \cite{33}.
- \(Q_{\delta}\) see e.g. \cite{24}, \cite{26} and \cite{30}. So the claim is established. \(\square\)

**Remark 4.3.** For completeness we recall in three extensions of Theorem 4.2, which are beyond the scope of this work.

1. If in Eq. (4.1) having \(P \in P_n\) is relaxed to \(H \in H_n\), then Generalized-positivity (boundedness, ...) is obtained. For more details see \cite{1}, \cite{3}, \cite{8, Theorem 10.2} and \cite{15}.
2. If in Eq. (4.2) the right-hand side is restricted to \(P_{n+m}\), then Hyper-positivity (boundedness, ...) is obtained. For further details see \cite{2}.

\(^2\)Like Janus in the Roman mythology
3. In [2] we addressed quantitative subsets of $\mathcal{B}$, i.e. functions where,

$$\sqrt{\frac{1}{\eta + 1}} \geq \sup_{z \in \mathbb{C}_R} \|G(z)\|_2 \quad \eta \in (1, \infty].$$

Note that $\mathcal{B}$ is recovered when $\eta \to \infty$. A state-space characterization of this family is when $W_\beta$ in Eq. (4.2) is substituted by $W_\beta(\eta)$ i.e.

$$\begin{pmatrix} R_{F_\beta} \\ I_{n+m} \end{pmatrix} \ast \begin{pmatrix} 0 & 0 & -P & 0 \\ 0 & \frac{\eta + 1}{\eta - 1} & 0 & 0 \\ -P & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix} \begin{pmatrix} R_{F_\beta} \\ I_{n+m} \end{pmatrix} \in \overline{\mathcal{P}}_{n+m}.$$ \hfill \Box

We next illustrate an application of the unified framework in Theorem 4.2.

**Example 4.4.** Let $F(z)$ and $R_F$ be a rational function and a corresponding $(n+m) \times (m+m)$ realization array, as in Eq. (1.1). Assume that as a matrix $R_F$ is non-singular and let $G(z)$ be defined as $R_G = (R_F)^{-1}$. Multiplying Eq. (4.2) by $R_G$ (recall $= (R_F)^{-1}$) from the left and $R_{G^*}$ from the right, yields

$$(4.3) \quad \begin{pmatrix} I_{n+m} \\ R_G \end{pmatrix}^* W_I \begin{pmatrix} I_{n+m} \\ R_G \end{pmatrix} = \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} U^* \begin{pmatrix} I_{n+m} \\ 0 \end{pmatrix} R_G = R_G Q_I R_{G^*} \in Q_I R_{G^*}.$$ \hfill \Box

From Eq. (4.1) it follows that with $U = \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix}$ one has that $U^* W_\delta U = W_\alpha$ while $U^* W_\delta U = -W_\delta$. In items (a) and (b) below, we examine the system interpretation of this technical observation.

(a) If in Eq. (4.3) $F(z)$ is positive real, i.e. $l = \alpha$, one has that

$$\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} W_\alpha \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} R_G = \tilde{Q}_\alpha \in \overline{\mathcal{P}}_{n+m},$$

namely, $\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* W_\alpha \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} \in \overline{\mathcal{P}}_{n+m}$. One can conclude that also $G(z)$ is positive real.

(b) If in Eq. (4.3) $F \in \mathcal{DB}$, i.e. $l = \delta$, one has that

$$\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} W_\delta \begin{pmatrix} 0 & I_{n+m} \\ I_{n+m} & 0 \end{pmatrix} R_G = \tilde{Q}_\delta,$$

namely, $\begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix}^* (-W_\delta) \begin{pmatrix} R_G \\ I_{n+m} \end{pmatrix} \in \overline{\mathcal{P}}_{n+m}$. Thus, one can say that $G(z)$ is “anti”-$\mathcal{DB}$: More precisely $(G(z))^{-1}$ is a Schur function, i.e. $1 \geq \sup_{z \in \mathbb{D}} \| (G(z))^{-1} \|_2$. \hfill \Box

Consider the four families $\mathcal{R}$, $\mathcal{B}$, $\mathcal{DP}$ and $\mathcal{DB}$. As already mentioned, as rational functions they are related through the Cayley transform, see Eq. (2.5). Theorem 4.2 suggests an additional inter-relations: Through the corresponding state-space realizations, which is next pursued.
5. Sets of Matrix-convex Realization arrays

In this section we address inter-relations within families of realization arrays associated with rational functions. As a preliminary step we recall in the classical notion of transformation of coordinates: Substituting a given state-space realization \( R_F \) by \( \left( \begin{smallmatrix} T^{-1} & 0 \\ 0 & I_m \end{smallmatrix} \right) R_F \left( \begin{smallmatrix} T & 0 \\ 0 & I_m \end{smallmatrix} \right) \), for some non-singular \( n \times n \) matrix \( T \).

Lemma 5.1. Consider the framework of Theorem 4.2 for some \( l \in \{\alpha, \beta, \gamma, \delta\} \).

Up to a change of coordinates, one can take in Eq. (4.2),

\[
\left( \begin{array}{ccc}
R_F^l & I_{n+m} \\
I_{n+m} & I_{n+m}
\end{array} \right)^* \hat{W}_l \left( \begin{array}{ccc}
R_F^l & I_{n+m} \\
I_{n+m} & I_{n+m}
\end{array} \right) \in \mathbb{P}_{n+m},
\]

where the \( \hat{W} \)'s are associated with balanced realization, i.e.

\[
\hat{W}_\alpha = \left( \begin{array}{cc}
0 & 0 \\
0 & -I_n \\
-I_n & 0 \\
0 & 0
\end{array} \right) \quad \hat{W}_\beta = \left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
-I_n & 0 \\
0 & 0
\end{array} \right) \\
\hat{W}_\gamma = \left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & I_n \\
I_n & 0
\end{array} \right) \quad \hat{W}_\delta = \left( \begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & I_n \\
-I_n & 0 \\
0 & 0
\end{array} \right).
\]

A system whose realization satisfies Eq. (5.1) with \( \hat{W}_\alpha \) from Eq. (5.2) is called “internally passive”, see [32, Definition 3].

We also need to resort to following.

Definition 5.2. For all \( k \), let \( v_j \in \mathbb{C}^{(n+m)\times(n+m)} \), \( j = 1, \ldots, k \) be block-diagonal so that

\[
\sum_{j=1}^{k} \begin{bmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{bmatrix}^* \begin{bmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}.
\]

A set \( R \) of \( (n+m) \times (n+m) \) matrices is said to be \( n, m \)-matrix-convex if having \( R_1, \ldots, R_k \) in \( R \), implies that also

\[
R_F := \sum_{j=1}^{k} \begin{bmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{bmatrix}^* \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \begin{bmatrix} v_{j,n} & 0 \\ 0 & v_{j,m} \end{bmatrix},
\]

belongs to \( R \) for all natural \( k \) and all \( v_j \in \mathbb{C}^{(n+m)\times(n+m)} \).

In [22] it was pointed out that the notion of \( n, m \)-matrix-convexity is intermediate between (the more strict) matrix-convexity, and (weaker) classical convexity.

We now pose the following question: For a natural parameter \( k \), let \( F_1(s) , \ldots, F_k(s) \) be a family of \( m \times m \)-valued rational functions all from the same family, admitting \( (n+m) \times (n+m) \) realizations, i.e.

\[
R_{F_j} = \begin{bmatrix} \hat{A}_j & \hat{B}_j \\ \hat{C}_j & \hat{D}_j \end{bmatrix} \quad j = 1, \ldots, k.
\]

Let \( R_F \), a realization of an \( m \times m \)-valued rational function \( F(z) \), be as in Eq. (5.4). We now address the following problem:
Under what conditions having $F_1(z), \ldots, F_k(z)$ in Eq. \eqref{5.5} all in $\mathcal{P}$ (or $\mathcal{B}$ or $\mathcal{D}\mathcal{P}$ or $\mathcal{D}\mathcal{B}$) implies that also the resulting $F(z)$ in Eq. \eqref{5.4} belongs to the same set?

If such a property holds this suggests that out of a small number of “extreme points” of balanced realizations of $\mathcal{P}$ (or $\mathcal{B}$ or $\mathcal{D}\mathcal{P}$ or $\mathcal{D}\mathcal{B}$) rational functions, one can construct a whole “matrix-convex-hull” realizations of functions within the same family. As a sample application, this may enable one to perform a simultaneous balanced truncation model order reduction of a whole family of bounded real functions, in the spirit of \cite{Section 5].

Before addressing this question, a word of caution: For example, $R_1 = \left( \begin{array}{ccc} A & B \\ C & D \end{array} \right)$ and $R_2 = \left( \begin{array}{ccc} -A & B \\ -C & D \end{array} \right)$ are two realizations of the same rational function. Furthermore, $R_1$ is minimal (balanced) if and only if $R_2$ if minimal (balanced). However, $R_3 = \frac{1}{2}(R_1 + R_2) = \left( \begin{array}{ccc} A & 0 \\ 0 & D \end{array} \right)$ is only a non-minimal realization of a zero degree rational function $F(s) \equiv D$.

In a similar way, even when the “extreme points” realizations in Eq. \eqref{5.5} are all balanced, the resulting realization $R_F$ in Eq. \eqref{5.4}, may be not minimal.

We now return to the above question,

**Proposition 5.3.** Consider the framework of Lemma \eqref{5.7} where $l \in \{\alpha, \beta, \gamma, \delta\}$ is prescribed. For a natural parameter $k$, let $F_{1,l}(z), \ldots, F_{k,l}(z)$ be a family of $m \times m$-valued rational functions, admitting $(n+m) \times (n+m)$ realizations as in Eq. \eqref{5.5}, satisfying all Eq. \eqref{5.1} i.e.

\begin{equation}
(5.6) \quad \begin{pmatrix} R_{F_{j,l}} \\ I_{n+m} \end{pmatrix}^* \begin{pmatrix} \tilde{W} \\ I_{n+m} \end{pmatrix} = Q_{j,l} \in \mathcal{P}_{n+m}, \quad l \in \{\alpha, \beta, \gamma, \delta\} \text{ prescribed} \quad j = 1, \ldots, k.
\end{equation}

Then, $R_F$ in Eq. \eqref{5.4} satisfies the same relation, i.e. each of sets $\mathcal{P}, \mathcal{B}, \mathcal{D}\mathcal{P}$ and $\mathcal{D}\mathcal{B}$ is a realization-$m, n$-matrix-convex.

**Proof:** Assume that Eq. \eqref{5.6} holds for $l = \alpha$, i.e.

\[
\begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \\ -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \end{pmatrix} \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} Q_j \in \mathcal{P}_{n+m}, 
\]

and consider matrix-convex combination of realizations as in Eq. \eqref{5.4},

\[
\begin{pmatrix} \sum_{j=1}^{k} v_{j,n}^* A_j v_{j,n} & \sum_{j=1}^{k} v_{j,n}^* B_j v_{j,m} \\ \sum_{j=1}^{k} v_{j,m}^* C_j v_{j,n} & \sum_{j=1}^{k} v_{j,m}^* D_j v_{j,m} \end{pmatrix} \begin{pmatrix} W_\alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{k} v_{j,n}^* A_j v_{j,n} & \sum_{j=1}^{k} v_{j,n}^* B_j v_{j,m} \\ \sum_{j=1}^{k} v_{j,m}^* C_j v_{j,n} & \sum_{j=1}^{k} v_{j,m}^* D_j v_{j,m} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix}.
\]
Thus the case of \( l = \alpha \) is established.

Since \( \hat{W}_{ij}, \hat{W}_j \) and \( \hat{W}_k \) are just permutations of \( \hat{W}_\alpha \), the respective constructions are very similar and thus omitted, and the proof is complete. \( \square \)

Special cases of Proposition 5.3 for \( \mathcal{P} \) and for \( \mathcal{DB} \) were shown in [22] and [23], respectively.

We conclude by illustrating how, by using the above results, one can generate from a single system, a whole collection of them. For simplicity, we address only (a subset of) \( \mathcal{P} \) functions.

**Example 5.4.** Consider the three following rational functions along with the corresponding balanced realizations, where \( a, b \in \mathbb{R} \) are parameters.

\[
F_1(z) = \frac{1}{az} \begin{pmatrix} \frac{b^2}{a^2} + 1 & z - \frac{b}{a} \\ -(z + \frac{b}{a}) & 1 \end{pmatrix} \quad \text{and} \quad R_{F_1} = \begin{pmatrix} 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & -\frac{1}{a} \\ \frac{1}{a} & -\frac{1}{a^2} & 0 & \frac{1}{a^2} \\ 0 & -\frac{1}{a^2} & 0 & 0 \end{pmatrix}
\]

\[(5.7) \quad F_2(z) = \frac{1}{z^2 + 1} \begin{pmatrix} a^2z & a(bz - a) \\ a(bz + a) & (a^2 + b^2)z \end{pmatrix} \quad \text{and} \quad R_{F_2} = \begin{pmatrix} 0 & 1 & a & b \\ -1 & 0 & 0 & -a \\ a & 0 & 0 & 0 \\ b & -a & 0 & 0 \end{pmatrix}
\]

\[
F_3(z) = \frac{z}{1 + z^2} \begin{pmatrix} 1 & \frac{b}{a} - z \\ \frac{b}{a} + z & \frac{b^2}{a^2} + 1 \end{pmatrix} \quad \text{and} \quad R_{F_3} = \begin{pmatrix} 0 & 1 & 1 & \frac{b}{a} \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ \frac{b}{a} & 1 & 1 & 0 \end{pmatrix}
\]

Each of these three functions is positive-real-odd (a.k.a Foster or Lossless), i.e.

\[-F(z) \in \mathbf{L}_{\alpha_1} \quad z \in \mathbb{C}_L \quad \text{and/or} \quad (-I_2 \ 0 \ I_2) R_F + R_F^* (-I_2 \ 0 \ I_2) = 0_{4 \times 4} \]
To employ Proposition 3.3 to generate additional rational functions, let now $\upsilon_j \in \mathbb{C}^{2 \times 2}$ be arbitrary so that $\sum_{j=1}^{3} \upsilon_j^* \upsilon_j = I_2$. Then, with $F_j(s)$ from Eq. (5.7), one has that $\sum_{j=1}^{3} \upsilon_j^* F_j(z) \upsilon_j$ is a $2 \times 2$-valued positive real odd rational function.

Similarly, to generate additional systems by employing Proposition 5.3 let now $\tilde{\upsilon}_j \in \mathbb{C}^{2 \times 2}$ be arbitrary so that $\sum_{j=1}^{3} \tilde{\upsilon}_j^* \tilde{\upsilon}_j = I_2$ and $\sum_{j=1}^{3} \tilde{\upsilon}_j^* \tilde{\upsilon}_j = I_2$. Then, with $R_{F_j}$ from Eq. (5.7), one has that $\sum_{j=1}^{3} \left( \begin{array}{cc} \tilde{\upsilon}_j & 0 \\ 0 & \tilde{\upsilon}_j^* \end{array} \right)^* R_{F_j} \left( \begin{array}{cc} \upsilon_j & 0 \\ 0 & \upsilon_j^* \end{array} \right)$ is a $(2+2) \times (2+2)$ realization (recall, not necessarily minimal) of a positive real odd rational function.

Finally note that we actually started from a single system $F_1$. Indeed, $F_2$ is defined as, $R_{F_2} = (R_{F_1})^{-1}$ (in the sense of inverting a constant $4 \times 4$ matrix). Now, $F_3(z) = (F_1(z))^{-1}$ (in the sense of that the product of pair of rational functions, each of degree two, yields a zero degree rational function, i.e. $F_3(z) F_1(z) \equiv I_2$).

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