Jet Fractions in $e^+e^-$ Annihilation

Garth Leder

Cavendish Laboratory, Cambridge University,
Madingley Road, Cambridge CB3 0HE. ENGLAND

email: leder@hep.phy.cam.ac.uk

Abstract

The jet fractions expected in $e^+e^-$ annihilation are calculated analytically in the leading log approximation up to 6 jet order for both the JADE and Durham algorithms.

---

1 This research was supported in part by the UK Particle Physics and Astronomy Research Council.
Contents

1 Introduction 3
   1.1 QCD and Jet Fractions ............................. 3
   1.2 The Leading Logarithmic Approximation .............. 5
   1.3 Soft and Collinear Divergences ...................... 6

2 Analytic Calculation of Jet Fractions 7

3 Moment Generating Function Method 9

4 Results 13

5 Conclusion 15

A Program Details 16
   A.1 Step Functions and Recursive Branching ............. 16
   A.2 Linked Lists ........................................ 17
   A.3 Fractions and Euclid’s Algorithm ................... 17
1 Introduction

1.1 QCD and Jet Fractions

QCD is now well accepted as the theory of strong interactions. It has a running coupling constant $\bar{\alpha}_S = \alpha_S(Q^2)/2\pi$, which is only low enough for perturbation theory to be valid at high energies: $Q^2$ is the relevant momentum scale. In the LEP and SLC $e^+e^-$ colliders, the centre-of-mass energy $Q \gtrsim 90$ GeV is high enough for perturbation theory to be valid in the early stages of the strong interaction \cite{1}; it can therefore be used to calculate the production of quarks and gluons (partons). These energies also allow the determination of the QCD colour factors $C_F$ and $C_A$ \cite{2,3}. The partons then combine at lower momentum scales where perturbation theory is not valid into a large number of hadrons in the process known as hadronisation. It is these hadrons that are actually measured in the detectors.

The distribution of experimentally measured hadrons closely follows the theoretically calculated partons in its flow of momentum and quantum numbers \cite{4,5}. The uncertainty principle relates the distance over which an interaction takes place to the momentum transfer involved. Interactions with a momentum scale greater than about 1 GeV are in the perturbative regime. In the non-perturbative regime there are large distances between partons, and the momentum transfer at this stage is therefore small. This phenomenon, known as Local Parton-Hadron Duality, allows hadronisation to be modelled by Monte Carlo event generators such as JETSET \cite{6,7} and HERWIG \cite{8,9,10}. The former uses a string model in which the gluon field energy is converted into hadrons, whilst the latter uses a low-mass cluster model. These two event generators yield similar results, for example in the determination of $\alpha_S$ \cite{11,12,13,14}, giving some confidence in their validity.

One of the common observables used in the study of hadronic final states is the number of jets as defined by a given jet algorithm. The $n$th jet fraction $R_n$ is then defined as

$$R_n = \frac{\sigma_n}{\sigma_{total}}$$

where $\sigma_n$ is the cross section for $n$ jets and $\sigma_{total}$ is the total cross section. Jet fractions may be used to calculate $\alpha_S$ by comparing parton-level theoretical calculations with measured hadron-level rates, provided the same jet algorithm is used in the two stages of the analysis.

Jet algorithms use a measure to determine the closeness of pairs of tracks. The JADE algorithm \cite{15,16} uses the invariant mass

$$y_{ij} = \frac{(p_i + p_j)^2}{Q^2} = \frac{2p_ip_j}{Q^2}$$

where $p_i$ and $p_j$ are the four-momenta of the two particles (assumed massless) and $Q^2$ is the centre-of-mass energy squared of the complete event as defined above. Let the partons
have energies $E_i$, $E_j$ and relative angle $\theta$ in the centre-of-mass frame. The measure is then

$$y_{ij}^J = \frac{2E_iE_j(1 - \cos \theta)}{Q^2}.$$  \hspace{1cm} (3)

At each stage, the pair of tracks with the smallest measure are combined if this measure is smaller than a fixed cut-off, ie if $y_{ij} < y_{cut}$. The algorithm stops once all pairs have a measure greater than $y_{cut}$, at which point the remaining tracks are termed jets. In experimental analyses the tracks are hadrons in the detector; in theoretical calculations they are partons. The validity of comparing these two situations is at least suggested by Local Parton-Hadron Duality, and might provide information about non-perturbative effects.

The Durham algorithm [17] is identical to the JADE algorithm except that it uses as its measure

$$y_{ij}^D = \frac{2 \min\{E_i^2, E_j^2\}(1 - \cos \theta)}{Q^2}. \hspace{1cm} (4)$$

A further variant on this theme is the Geneva algorithm [18], again identical except for its measure:

$$y_{ij}^G = \frac{8}{9} \frac{E_iE_j(1 - \cos \theta)}{(E_i + E_j)^2}. \hspace{1cm} (5)$$

This algorithm has not been widely used in experimental analyses. For each measure there are a number of different possible procedures for combining pairs of tracks, the simplest of which is to add their four-momenta. The calculations presented in this paper are of an accuracy not to be affected by the procedure used.

The mass of a current quark ($\sim$ few MeV) [19] is much less than the characteristic QCD scale ($\Lambda_{QCD} \simeq 200$ MeV). For the above algorithms the effect of the masses is swamped by non-perturbative effects [20], and the quarks may be considered massless.

At small values of $y_{cut}$ the jet fractions are dominated by soft gluons, which have a useful property known as angular ordering. Consider the emission of such a soft gluon from either of a pair of partons separated by an angle $\theta$. The emitting partons may be treated as classical currents: the emission from each of these currents interferes, but is coherent due to the colour structure. The resulting cross section [21], written in terms of the energies and angles of the partons, may be split into two parts, each containing a collinear singularity. When averaged over the azimuthal direction the contribution from each of these parts is non-zero only within a cone of angle $\theta$: the emission may be separated into two parts, each of which describes emission from a single parton, but restricted to the cone described above. Emission at larger angles adds up to be equivalent to emission from the parent parton [22]. This argument holds to next-to-leading log order, and was discovered independently by Mueller [23] and Ermolaev and Fadin [24]. Coherence effects of this type have been observed at a variety of energies [25, 26, 27].
1.2 The Leading Logarithmic Approximation

Hadronisation corrections set in at $y_{\text{cut}} \sim m_{\text{Hadrons}}^2/Q^2 \sim (1 \text{ GeV})^2/Q^2$: at LEP energies this gives a lower limit on $y_{\text{cut}}$ of $\sim 0.0001$. At values of $y_{\text{cut}} < 0.1$, such that

$$L = \ln \left( \frac{1}{y_{\text{cut}}} \right) > 2.5,$$

(6)

the divergent regions of phase space due to soft, collinear gluons dominate the integrand. Perturbation theory then gives a series in two variables, in increasing powers of $\bar{\alpha}_S$ and decreasing powers of $L$. For jet fractions the leading term in $L$ has two powers of $L$ for every power of $\bar{\alpha}_S$, one from each of the soft and collinear divergences. At each order of $\bar{\alpha}_S$ the series in $L$ can be truncated to give a valid approximation, provided that $\bar{\alpha}_S L^2 < 1$. I shall work to leading log order, in which only the highest power of $L$ is kept. At LEP energies $\bar{\alpha}_S \simeq 0.117 \pm 0.005$ [29], giving a lower limit on $y_{\text{cut}}$ of $\sim 0.05$. At higher energies this limit will be lowered, widening the valid range of $y_{\text{cut}}$.

At lower values of $y_{\text{cut}}$ at LEP energies it is necessary to resum the series to all orders in $\bar{\alpha}_S$. This is possible if the coefficients follow, or are at least close to, the pattern of a known function; for example the leading log coefficients for the two jet fraction in the Durham algorithm follow the pattern of the coefficients of the exponential function [30]. This phenomenon is known as exponentiation [31], and depends on the factorisation of the phase space of the relevant observable: the QCD matrix elements are guaranteed to factorise for soft, collinear gluons [32]. These resummed predictions compare well with
The same properties do not hold for the JADE algorithm, which has more complicated phase space boundaries due to extra gluon-gluon correlations: the coefficients follow no discernable pattern for the most commonly used recombination prescriptions [34]. Calculations for jet fractions exist at leading order in $\bar{\alpha}_S$ and leading log order up to six jet order for the JADE algorithm [35], and up to four jet order for the Durham algorithm [17, 36]. The JADE calculations only include the abelian terms, that is only the diagrams with no triple gluon vertices. The existing results are here extended to include the non-abelian terms. The existence of the triple gluon vertex has been confirmed at LEP [2]. Some of the coefficients have also been investigated numerically [37].

Jet clustering has also been used as a first stage in experimental analyses to reduce the theoretical errors in other observables [38]. The calculation of jet fractions can also be used as a test of the consistency of QCD between different experimental situations, for example between $e^+e^-$ annihilation and deep inelastic scattering.

1.3 Soft and Collinear Divergences

In the perturbative calculation of parton branching there are two types of divergence that occur even aside from renormalisation. In the case of branching from an outgoing parton (time-like branching), and after integrating over the azimuthal direction, the cross section can be expressed as [33]

$$\frac{d\sigma_{n+1}}{d\sigma_n} = \bar{\alpha}_S V(z) dz \frac{\sin \theta d\theta}{1 - \cos \theta}$$

(7)

where $\theta$ is the angle between the partons. For $q \rightarrow qg$ branching $z$ is the energy fraction of the emitted gluon, whilst for $g \rightarrow gg$ branching it is the energy fraction of either gluon. The formulation in terms of parton energies and angles is used in order to take into account both types of divergence. $V(z)$ is the splitting function which gives the probability of parton branching after averaging and summing over initial and final spin states respectively. In the soft (small $z$) limit the two splitting functions relevant to this analysis [21] are

$$V_F^C(z)[q \rightarrow qg] = C_F \frac{1 + (1 - z)^2}{z} \simeq \frac{2C_F}{z} + \mathcal{O}(1)$$

(8)

and

$$V_G^C(z)[g \rightarrow gg] = 2C_A \left[ z (1 - z) + \frac{1 - z}{z} + \frac{z}{1 - z} \right] \simeq \frac{2C_A}{z} + \mathcal{O}(1).$$

(9)

These splitting functions diverge at $z = 0$. The evolution equation (7) is also divergent in the collinear (small $\theta$) limit:

$$\frac{d\sigma_{n+1}}{d\sigma_n} \simeq 2\bar{\alpha}_S V(z)dz \frac{d\theta}{\theta} \simeq 2\bar{\alpha}_S C \frac{dz}{z} \frac{d\theta^2}{\theta^2},$$

(10)
where $C = C_F$ or $C_A$ according to the type of branching. This divergence is a consequence of the partons’ masslessness, but the expression would be logarithmically enhanced even with small parton masses. The splitting function for gluon to quark-antiquark branching is

$$V_F^G(z|g \to q\bar{q}) = \frac{1}{2}N_f \left[z^2 + (1-z)^2\right], \quad (11)$$

where $N_f$ is the number of active quark flavours at the relevant energy. This function contains no soft divergence, and therefore this type of branching does not contribute at leading log order. Similarly the four gluon vertex is suppressed by a factor of $\bar{\alpha}_s$.

For an observable to be physical it must not be affected by any of the divergences in the theory. It must therefore take the same value if any parton is replaced by an arbitrary number of partons such that the total momentum and colour flow of the group remains the same. Such an observable is called collinear-safe. For the same reason it must also remain unchanged if a parton emits an arbitrary number of gluons of zero energy, in which case it is called infrared-safe. This suggests that a parton is significant for its momentum and colour flow, rather than as a directly measurable particle.

2 Analytic Calculation of Jet Fractions

For the leading log order result all the gluons are soft and collinear. The soft limit enables the energy fraction of each parton to be approximated as $\epsilon_i \simeq E_i/Q$. Note that the original quark and antiquark both have $\epsilon = 1$ to lowest order in $\epsilon$. For two partons branching off the same initial parton the collinear limit ($\theta$ small) allows the approximation $2(1 - \cos \theta) \simeq \theta^2 + O(\theta^4)$ to be made. The measures (3) and (4) then simplify to

$$y^J_{ij} = \epsilon_i \epsilon_j \theta^2, \quad (12)$$
$$y^D_{ij} = \min\{\epsilon^2_i, \epsilon^2_j\} \theta^2. \quad (13)$$

For two gluons originating one from the quark and one from the antiquark, the collinear limit is $\theta \simeq \pi$ and the measures become

$$y^J_{ij} = \epsilon_i \epsilon_j, \quad (14)$$
$$y^D_{ij} = \min\{\epsilon^2_i, \epsilon^2_j\}. \quad (15)$$

These approximations are sufficient for the leading log order calculation. In the small $\theta$ limit the perpendicular momentum of the less energetic particle with respect to the more energetic particle is $k_\perp = Q\epsilon \sin \theta \simeq Q\epsilon \theta$ where $Q\epsilon$ is the energy of the less energetic particle as defined above. The Durham measure is thus seen to be equal to the perpendicular momentum squared divided by $Q^2$.

The phase space for $n$ jets (10) is

$$\frac{d\epsilon_1}{\epsilon_1} ... \frac{d\epsilon_n}{\epsilon_n} \frac{d\theta^2_1}{\theta^2_1} ... \frac{d\theta^2_n}{\theta^2_n}. \quad (16)$$
The substitutions
\begin{align}
\ln \left( \frac{1}{\epsilon_i} \right) &= x_i L, \quad \ln \left( \frac{1}{\theta_i^2} \right) = y_i L, \quad \ln \left( \frac{1}{y_{\text{cut}}} \right) = L, \quad (17)
\end{align}

transform this to
\begin{align}
dx_1...dx_n dy_1...dy_n. \quad (18)
\end{align}

The formal integration range for the original variables \( \epsilon_i, \theta_i^2 \) is taken as \( 0 \rightarrow 1 \), remembering that only small values of \( \epsilon \) and \( \theta \) contribute at leading log order. This translates into \( x_i, y_i \) space as \( 0 \rightarrow \infty \). In the actual calculation it is simplest to put in these limits as step functions, ie \( \Theta(x_1)...\Theta(x_n)\Theta(y_1)...\Theta(y_n) \), where
\begin{align}
\Theta(x) &= \begin{cases} 
1 & x \geq 0, \\
0 & x < 0. 
\end{cases} \quad (19)
\end{align}

I will work for the moment with the JADE algorithm. Consider two gluons radiated successively by a quark with energy fractions \( \epsilon_i \) and \( \epsilon_j \), and at angles \( \theta_i \) and \( \theta_j \) respectively to the quark. In order to generate the maximum number of logarithms \( L \) the angles must be strongly ordered, ie \( \theta_j \ll \theta_i \). The smaller of the angles \( \theta_j \) may therefore be ignored, and the angle between the two partons taken as \( \theta_i \). The condition that these two gluons are resolved as separate jets is \( y_{ij} > y_{\text{cut}} \), which appears in the integrand as
\begin{align}
\Theta(\epsilon_i \epsilon_j \theta_i^2 - y_{\text{cut}}); \quad (20)
\end{align}

the substitutions \( (17) \) transform this to
\begin{align}
\Theta(1 - x_i - x_j - y_i). \quad (21)
\end{align}

This divides the phase space into two halves separated by a plane. Taken together with the appropriate conditions for other pairs of partons the overall integration region is a ‘polygon’ in a space with two dimensions for every emitted gluon.

For the specific diagram

\begin{align}
\text{Diagram Image}
\end{align}
the conditions that no two partons will be combined by the JADE algorithm are

| Condition                | Expression |
|--------------------------|------------|
| Angular ordering         | $\Theta(\theta_2 - \theta_3)$ |
| Quark-gluon              | $\Theta(\epsilon_1 \theta_1^2 - y_{cut})$ |
| Gluon-gluon              | $\Theta(\epsilon_1 \epsilon_2 - y_{cut})$  
|                         | $\Theta(\epsilon_1 \epsilon_3 - y_{cut})$  
|                         | $\Theta(\epsilon_2 \epsilon_3 \theta_3^2 - y_{cut})$ |

This gives as the integration

$$R = \int_{-\infty}^{\infty} \Theta(y_3 - y_2) \Theta(1 - x_1 - y_1) \Theta(1 - x_1 - x_2) \Theta(1 - x_1 - x_3) \Theta(1 - x_2 - x_3 - y_2) \Theta(x_1) \Theta(x_2) \Theta(x_3) \Theta(y_1) \Theta(y_2) \Theta(y_3) dx_1 dx_2 dx_3 dy_1 dy_2 dy_3.$$  

(22)

This calculation may be done using the C program outlined in appendix (A): the answer is $53/360$, which yields a contribution of

$$\frac{1}{2} 2^3 C_F^2 C_A \frac{53}{360} \tilde{\alpha}_S^3 L^6 = \frac{424}{135} (\tilde{\alpha}_S L^2)^3 \quad \text{(23)}$$

to the final answer. The factor of $1/2$ is statistical, and results from gluons 2 and 3 being identical. In this context a pair of gluons are identical if the swapping of their coordinates leaves the integrand invariant: such gluons must either contribute a statistical factor of $1/2$ or have their energies ordered. To reach physically meaningful results it is necessary to add up the results from each of the diagrams contributing at the relevant order. At leading log order for $n$ jets this consists of all the tree level diagrams involving the original quark-antiquark pair and $n - 2$ emitted gluons; the creation of further quark-antiquark pairs is suppressed by one factor of $L$ as there is no associated soft divergence.

It is of course possible to integrate over the variables in any order. The accuracy of the program had been tested for a number of integrands: it gives the same answer for several different integration orders. The results presented in this paper have been found using at least two different integration orders, always with consistent results. All the results have also been checked numerically, using the VEGAS Monte Carlo routine in Fortran 77. Here the integration was performed directly in $\epsilon, \theta$ space. Again the results are consistent within the errors of the Monte Carlo.

## 3 Moment Generating Function Method

The moment generating function for a probability distribution $p(n)$ is defined as

$$\phi(u) = \sum_{n=0}^{\infty} u^n p(n). \quad \text{(24)}$$
The probabilities themselves are recovered using the formula
\[
p(n) = \frac{1}{n!} \left[ \frac{d^n}{du^n} (\phi(u)) \right]_{u=0}.
\] (25)

For the case of a $q\bar{q}$ pair produced in $e^+e^-$ annihilation, the moment generating function for the Durham algorithm may be calculated to next-to-leading order using the coherent branching formalism [40]: it is
\[
\phi^2_q(Q, u) = u^2 e^{2F_0(Q,u)},
\]
where
\[
F_i(Q, u) = \frac{d^i}{du^i} \int_{Q_0}^{Q} \Gamma_q(Q, q) \left[ \phi_g(q, u) - 1 \right] dq
= \int_{Q_0}^{Q} \Gamma_q(Q, q) \left[ \frac{d^i}{du^i} \phi_g(q, u) \right] dq, \quad i \geq 1.
\] (26)

Here
\[
\Gamma_q(Q, q) = \frac{4C_F \tilde{\alpha}_s}{q} \left[ \ln \left( \frac{Q}{q} \right) - \frac{3}{4} \right]
\]
is the $q \to qg$ branching probability, and $\phi_g(q, u)$ is the moment generating function for a single gluon. At leading log order $\tilde{\alpha}_s$ may be taken as a constant, evaluated at the momentum scale of the interaction. Defining
\[
G_i(Q, u) = \frac{d^i}{du^i} \left[ \phi_q^2(Q, u) \right],
\]
(27)

it may easily be calculated that
\[
\begin{align*}
G_0 & = \left[ u^2 \right] e^{2F_0} \\
G_1 & = \left[ 2u + 2u^2 F_1 \right] e^{2F_0} \\
G_2 & = \left[ 2 + 8u F_1 + u^2 \left( 2F_2 + 4F_1^2 \right) \right] e^{2F_0} \\
G_3 & = \left[ 12F_1 + u \left( 12F_2 + 4F_1^2 \right) + u^2 \left( 2F_3 + 24F_1^2 + 48F_2F_1^2 + 8F_2^2 \right) \right] e^{2F_0} \\
G_4 & = \left[ 24F_2 + 48F_1^2 + u \left( 48F_2F_1 + 96F_3^2 + 64F_1^3 \right) + u^2 \left( 2F_4 + 16F_1F_3^2 + 12F_4F_1^2 + 48F_2F_1^2 + 8F_2^2 \right) \right] e^{2F_0} \\
G_5 & = \left[ 40F_3 + 240F_1F_2 + 160F_3^2 + u \left( 20F_4 + 160F_1F_3 + 120F_2^2 + 480F_1^2F_2 + 16F_4^2 \right) + 320F_2^2F_1^2 + 320F_4^2 + O(u) \right] e^{2F_0} \\
G_6 & = \left[ 60F_4 + 440F_1F_3 + 360F_2^2 + 1200F_1F_2^2 + 320F_4^2 + O(u) \right] e^{2F_0},
\end{align*}
\]
(28)

where the $G_i$ and $F_i$ are evaluated at $(Q, u)$. The quark Sudakov form factor is calculated as
\[
\Delta_q(Q) = e^{F_0(Q,0)} = e^{-\int_{Q_0}^{Q} \Gamma_q(Q, q) dq}
\]
(30)

using $\phi_q(q, 0) = 0$ as calculated in the next section [41]. The jet rates themselves are found using equation (25) to be
\[
\begin{align*}
R_2(Q) & = \Delta_q^2(Q) \\
R_3(Q) & = (2F_1) \Delta_q^2(Q)
\end{align*}
\]
\[
\begin{align*}
R_4(Q) & = \left( F_2 + 2F_1^2 \right) \Delta_4^2(Q) \\
R_5(Q) & = \left( \frac{1}{3}F_3 + 2F_1F_2 + \frac{4}{3}F_1^3 \right) \Delta_2^2(Q) \\
R_6(Q) & = \left( \frac{1}{12}F_4 + \frac{11}{18}F_1F_3 + \frac{1}{2}F_2^2 + \frac{5}{3}F_1^2F_2 + \frac{4}{9}F_1^4 \right) \Delta_3^2(Q),
\end{align*}
\]
where the \( F_i \) are evaluated at \((Q, 0)\).

The gluon moment generating function is \[\phi_g(Q, u) = [u + u^2D_0(Q, u)] e^{C_0(Q, u)},\] \(32\)

where
\[
D_i(Q, u) = \frac{d^i}{du^i} \int_{Q_0}^Q \Gamma_f(q)e^{E_0(q, u)}dq = \int_{Q_0}^Q \Gamma_f(q) \left[ \frac{d^i}{du^i}e^{E_0(q, u)} \right] dq,
\]
\(33\)

\[
E_i(Q, u) = \frac{d^i}{du^i} \int_{Q_0}^Q \{ 2\Gamma_g(Q, q) - \Gamma_g(Q, q) \} \left[ \phi_g(q, u) - 1 \right] + \Gamma_f(q) \} dq
\]
\[
E_i(Q, u) = \int_{Q_0}^Q \{ 2\Gamma_g(Q, q) - \Gamma_g(Q, q) \} \left[ \frac{d^i}{du^i}\phi_g(q, u) \right] dq, \quad i \geq 1,
\]
\(34\)

and
\[
C_i(Q, u) = \frac{d^i}{du^i} \int_{Q_0}^Q \{ \Gamma_g(Q, q) \left[ \phi_g(q, u) - 1 \right] - \Gamma_f(q) \} dq
\]
\[
C_i(Q, u) = \int_{Q_0}^Q \Gamma_g(Q, q) \left[ \frac{d^i}{du^i}\phi_g(q, u) \right] dq, \quad i \geq 1.
\]
\(35\)

Here
\[
\Gamma_g(Q, q) = \frac{4C_A\tilde{\alpha}_S}{q} \left[ \ln \left( \frac{Q}{q} \right) - \frac{11}{12} \right]
\]
\(36\)

and
\[
\Gamma_f(q) = \frac{2N_f\tilde{\alpha}_S}{3q}
\]
\(37\)
are \( g \to gg \) and \( g \to q\bar{q} \) branching probabilities respectively. Defining
\[
A_i(Q, u) = \frac{d^i}{du^i} \left[ \phi_g(Q, u) \right],
\]
\(38\)

it may easily be calculated that
\[
\begin{align*}
A_0(Q, u) & = [u + u^2D_0] e^{C_0} \\
A_1(Q, u) & = [1 + u (C_1 + 2D_0) + u^2 (D_1 + D_0C_1)] e^{C_0} \\
A_2(Q, u) & = [2C_1 + 2D_0 + u (C_2 + C_1^2 + 4D_1 + 4D_0C_1) + u^2 (D_2 + 2D_1C_1 + D_0C_2 + D_0C_1^2)] e^{C_0} \\
A_3(Q, u) & = [3C_2 + 3C_1^2 + 6D_1 + 6D_0C_1 + u (C_3 + 3C_1C_2 + C_1^3 + 6D_2 + 12D_1C_1 + 6D_0C_2 + 6D_0C_1^2) + O (u^2)] e^{C_0}
\end{align*}
\]

\(11\)
Figure 2: Quantity dependencies in the moment generating function method

| Quantity | Depends on |
|----------|------------|
| $A_n$    | $C_{0..n-1}, D_{0..n-2}$ |
| $E_n$    | $A_n$      |
| $B_n$    | $E_{0..n-1}$ |
| $C_n$    | $A_n$      |
| $D_n$    | $B_n$      |

Figure 2 shows the dependencies of each of the quantities $A_i$ to $E_i$. The quantities may be calculated cyclically in the order in which they are given. The key point is that $A_n$ only depends (explicitly or implicitly) on $A_{0..n-1}$ at $u = 0$, and so the recursive nature of equations (32) to (35) is avoided. It is interesting to note how the colour factors are introduced by the different quantities: for example each factor of $C_i(Q,0)$ introduces a colour factor of $C_A$ through its dependency on $\Gamma_g(Q,q)$ (equation (36)).

\[
A_4(Q, u) = \left[4C_3 + 12C_1C_2 + 4C_1^3 + 12D_2 + 24D_1C_1 + 12D_0C_2 + 12D_0C_1^2 + O(u)\right] e^{C_0},
\]  

where the $A_i$, $C_i$ and $D_i$ are evaluated at $(Q, u)$. The gluon Sudakov form factor is calculated as

\[
\Delta_g(Q) = e^{C_0(Q,0)} = e^{-\int_{Q_0}^{Q} \Gamma_g(Q,q) + \Gamma_f(q) dq}
\]  

again using $\phi_g(q,0) = 0$. The $A_i$ are evaluated at $(Q,0)$ to be

\[
A_0(Q,0) = 0
\]

\[
A_1(Q,0) = \Delta_g(Q)
\]

\[
A_2(Q,0) = 2(C_1 + D_0) \Delta_g(Q)
\]

\[
A_3(Q,0) = 3(C_2 + C_1^2 + 2D_1 + 2D_0C_1) \Delta_g(Q)
\]

\[
A_4(Q,0) = 4(C_3 + 3C_1C_2 + C_1^3 + 3D_2 + 6D_1C_1 + 3D_0C_2 + 3D_0C_1^2) \Delta_g(Q),
\]  

where the $C_i$ and $D_i$ are also evaluated at $(Q,0)$. Defining a further form factor as

\[
\Delta_f(Q) = e^{E_0(Q,0)} = e^{-\int_{Q_0}^{Q} 2\Gamma_g(Q,q) - \Gamma_f(q) dq - \Gamma_g(q) dq} = \frac{\Delta_2^2(Q)}{\Delta_g(Q)},
\]  

and defining

\[
B_i(Q, u) = \frac{d^n}{du^n} [e^{E_0(Q,u)}],
\]  

it may similarly be calculated that

\[
B_0(Q,0) = \Delta_f(Q)
\]

\[
B_1(Q,0) = (E_1) \Delta_f(Q)
\]

\[
B_2(Q,0) = (E_2 + E_1^2) \Delta_f(Q),
\]  

where the $E_i$ are also evaluated at $(Q,0)$.
4 Results

$C_F$ and $C_A$ are the QCD colour factors, $\bar{\alpha}_S = \alpha_S/2\pi$ and $L = \ln(1/y_{\text{cut}})$. $R_n^J$ and $R_n^D$ are the leading log order $n$ jet rates for the JADE and Durham algorithms respectively. The JADE results were found by direct integration as described in section (2), and the Durham results using both that and the moment generating function method described in section (3): the two methods agree at all calculated orders. In addition, as mentioned earlier, the results for both algorithms have been checked numerically using the Monte Carlo Integrator VEGAS, again with agreement within the errors of the Monte Carlo.

\begin{align*}
    R_2^J &\sim \left[ \frac{1}{2} \right] (C_F\bar{\alpha}_S L^2) \\
    R_3^J &\sim \left[ \frac{3}{2} + \frac{1}{6} \left( \frac{C_A}{C_F} \right) \right] (C_F\bar{\alpha}_S L^2)^2 \\
    R_4^J &\sim \left[ \frac{31}{45} + \frac{5}{18} \left( \frac{C_A}{C_F} \right) + \frac{1}{36} \left( \frac{C_A}{C_F} \right)^2 \right] (C_F\bar{\alpha}_S L^2)^3 \\
    R_5^J &\sim \left[ \frac{571}{2520} + \frac{4091}{20160} \left( \frac{C_A}{C_F} \right) + \frac{643}{11520} \left( \frac{C_A}{C_F} \right)^2 + \frac{107}{20160} \left( \frac{C_A}{C_F} \right)^3 \right] (C_F\bar{\alpha}_S L^2)^4 \\
    R_2^D &\sim \left[ \frac{1}{2} \right] (C_F\bar{\alpha}_S L^2) \\
    R_3^D &\sim \left[ \frac{1}{2} + \frac{1}{12} \left( \frac{C_A}{C_F} \right) \right] (C_F\bar{\alpha}_S L^2)^2 \\
    R_4^D &\sim \left[ \frac{1}{6} + \frac{1}{12} \left( \frac{C_A}{C_F} \right) + \frac{1}{90} \left( \frac{C_A}{C_F} \right)^2 \right] (C_F\bar{\alpha}_S L^2)^3 \\
    R_5^D &\sim \left[ \frac{1}{24} + \frac{1}{24} \left( \frac{C_A}{C_F} \right) + \frac{7}{480} \left( \frac{C_A}{C_F} \right)^2 + \frac{17}{10080} \left( \frac{C_A}{C_F} \right)^3 \right] (C_F\bar{\alpha}_S L^2)^4. 
\end{align*}

An additional result is the abelian contribution to the 7 jet fraction for the two algorithms:

\begin{align*}
    R_7^J &\sim \left[ \frac{1093}{18900} \right] (C_F\bar{\alpha}_S L^2)^5, \\
    R_7^D &\sim \left[ \frac{1}{120} \right] (C_F\bar{\alpha}_S L^2)^5. 
\end{align*}
Whilst these abelian contributions cannot be used on their own, they provide further evidence that the JADE coefficients do not follow an obvious pattern, while the Durham abelian coefficients exponentiate.

One qualitative aspect of the Durham algorithm is that soft, collinear gluons are combined only such that at least one of the pair has no other partons closer to it in angle than its partner. This suggests that the phase space boundaries given by comparing other pairs of gluons should have no effect on the result: this was confirmed for all the results quoted above.

These results are in complete agreement with the results of Brown and Stirling as presented in [35] and [36] for the JADE and Durham algorithms respectively, and also with the results of [17] for the Durham algorithm, there calculated using moment generating functions as in section (3). The results presented here include, in addition to these previous results, the non-abelian ($C_A$) terms for the JADE algorithm, the five and six jet terms for the Durham algorithm, and the abelian seven jet terms for both algorithms. The abelian results, the integrands for which are generated automatically, are limited by the memory available to the computer. The integrands for the non-abelian results are put in by hand, and the results are therefore limited by the number of relevant diagrams: the seven jet calculation involves 32 such diagrams, compared with only 13 for six jets.

For the specific case of SU(3), $C_F = 4/3$ and $C_A = 3$. The results of equation (15) then become

\[
\begin{align*}
R_2^J & \sim 1.00 \\
R_3^J & \sim 2.67 (\bar{\alpha}_S L^2) \\
R_4^J & \sim 3.33 (\bar{\alpha}_S L^2)^2 \\
R_5^J & \sim 3.45 (\bar{\alpha}_S L^2)^3 \\
R_6^J & \sim 3.24 (\bar{\alpha}_S L^2)^4 \\
R_2^D & \sim 1.00 \\
R_3^D & \sim 1.33 (\bar{\alpha}_S L^2) \\
R_4^D & \sim 1.22 (\bar{\alpha}_S L^2)^2 \\
R_5^D & \sim 0.97 (\bar{\alpha}_S L^2)^3 \\
R_6^D & \sim 0.72 (\bar{\alpha}_S L^2)^4. 
\end{align*}
\]

Again there is no clear pattern in the JADE results. Although the numerical coefficients do appear to be under control, it is difficult to see how they might be resummed. The root of the problem lies in the necessity to compare all gluon pairings in the definition of the phase space. Here the Durham algorithm is an improvement, in that fewer pairs of gluons need to be compared.

One further new result is the sequence of coefficients for a cascade generated by a single
The coefficients decrease less rapidly than the exponential series owing to the increasing number of triple gluon vertices.

5 Conclusion

The jet fractions for the JADE and Durham algorithms have been calculated up to six jets in the leading order in $\bar{\alpha}_S$ and leading log order approximation. The coefficients for the JADE algorithm exhibit no easily discernable pattern, suggesting that they may be difficult to resum.

I am most grateful to G. P. Salam and B. R. Webber for many useful discussions. I would also like to thank the CERN Theory Division for hospitality while part of this work was carried out.
A Program Details

A.1 Step Functions and Recursive Branching

Where it appears as part of an integrand, a step function is defined mathematically as a change in the limits of the integration, i.e.

\[ \int_{-\infty}^{\infty} \Theta(x - a)f(x)dx = \int_{a}^{\infty} f(x)dx. \]  

(49)

A more complicated situation arises when there are two step functions, either of which may give the lower limit:

\[ \int_{-\infty}^{\infty} \Theta(x - a)\Theta(x - b)f(x)dx = \left\{ \begin{array}{l} \int_{b}^{\infty} f(x)dx & a \geq b, \\ \int_{a}^{\infty} f(x)dx & b \geq a. \end{array} \right. \]  

(50)

The above result can be written in a combined form as

\[ \int_{-\infty}^{\infty} \Theta(x - a)\Theta(x - b)f(x)dx = \Theta(a - b) \int_{a}^{\infty} f(x)dx + \Theta(b - a) \int_{b}^{\infty} f(x)dx. \]  

(51)

In an integration over more than one dimension the constants \( a \) and \( b \) can become functions of the remaining variables of integration. The information already assumed about \( a \) and \( b \) is written in terms of step functions and the integration over the next dimension is therefore of the same form. An integration in \( n \) dimensions has been branched recursively to two \( n - 1 \) dimensional integrations.

The scheme works analogously for upper limits:

\[ \int_{-\infty}^{\infty} \Theta(a - x)\Theta(b - x)f(x)dx = \left\{ \begin{array}{l} \int_{-\infty}^{a} f(x)dx & a \leq b, \\ \int_{b}^{\infty} f(x)dx & b \leq a. \end{array} \right. \]  

(52)

If both limits are given by step functions, the step function giving the lower limit must switch on before the step function giving the upper limit switches off:

\[ \int_{-\infty}^{\infty} \Theta(x - a)\Theta(b - x)f(x)dx = \Theta(b - a) \int_{a}^{b} f(x)dx. \]  

(53)

The most general case involves an unspecified number of both upper and lower limits. The answer contains one branch for every possible pairing of an individual upper limit and an individual lower limit. For each branch there are three sources of step functions in addition to any step functions contained implicitly in \( f(x) \):

- the lower limit must be greater than other possible lower limits,
- the upper limit must be smaller than other possible upper limits,
- the upper limit must be greater than the lower limit.
A.2 Linked Lists

At each stage of the integration it is not practical to predict the number of terms that the polynomial component of the integrand will contain, nor the number of step functions needed. Therefore it is essential to have a data structure that allows for an arbitrary number of terms at each stage, for example a linked list. The individual terms are stored separately, each one with pointers to the next and previous terms. C [11, 12] supports dynamic memory allocation: functions are included which can insert or remove terms at any desired location, memory being taken from or returned to the pool of free memory as required. The recursive design of the integration ensures that only one integrand need be stored for each dimension at a time, thus minimising the memory use.

A.3 Fractions and Euclid’s Algorithm

The initial integrand for each calculation consists numerically purely of integers; real numbers do not occur. This is the case for both the JADE and Durham algorithms. The integrand can therefore be expressed purely using rational numbers at all stages of the calculation, including the final answer. The handling of fractions is done by a separate program module; fractions are cancelled using Euclid’s algorithm for the highest common factor (HCF) of two integers [13, 14], as described below:

Let the two numbers whose HCF is to be found be $x_1$ and $x_2$, such that $x_1 > x_2$. The procedure is best described schematically:

\[
\begin{align*}
    x_1 &= m_1 x_2 + x_3 \\
    x_2 &= m_2 x_3 + x_4 \\
        &\vdots \\
    x_n &= m_n x_{n+1}.
\end{align*}
\]

(54)

At each stage the first number is divided by the second number, the remainder being given by the third number. Any common factor of $x_1$ and $x_2$ must divide $x_1 - m_1 x_2 = x_3$, and is hence a common factor of $x_2$ and $x_3$. Any common factor of $x_2$ and $x_3$ must also be a factor of $x_1$, and hence the HCF of $x_1$ and $x_2$ is the same as the HCF of $x_2$ and $x_3$. This argument is repeated for all equations in the above series to show that the original HCF required is also the HCF of $x_n$ and $x_{n+1}$: this is clearly $x_{n+1}$ itself.
References

[1] N. Magnoli, P. Nason, and R. Rattazzi, Phys. Lett. B 252, 271 (1990).

[2] ALEPH Collaboration, PLB 284, 151 (1992).

[3] OPAL Collaboration, Zeit. Phys. C 65, 367 (1995).

[4] Y. L. Dokshitzer, V. A. Khoze, A. H. Mueller, and S. I. Troyan, Perturbative QCD, Editions Frontieres, 1991.

[5] Y. I. Azimov, Y. L. Dokshitzer, V. A. Khoze, and S. I. Troyan, Zeit. Phys. C 27, 65 (1985).

[6] T. Sjöstrand, Computer Phys. Commun. 39, 347 (1984).

[7] M. Bengtsson and T. Sjöstrand, Computer Phys. Commun. 43, 367 (1987).

[8] G. Marchesini et al., Computer Phys. Commun. 67, 465 (1992).

[9] G. Marchesini and B. R. Webber, Nucl. Phys. B 238, 1 (1984).

[10] G. Marchesini and B. R. Webber, Nucl. Phys. B 310, 461 (1988).

[11] OPAL Collaboration, Zeit. Phys. C 59, 1 (1993).

[12] ALEPH Collaboration, Phys. Lett. B , 163 (1992).

[13] L3 Collaboration, Phys. Lett. B 284, 471 (1992).

[14] DELPHI Collaboration, Zeit. Phys. C 59, 21 (1993).

[15] JADE Collaboration, Zeit. Phys. C 33, 23 (1986).

[16] JADE Collaboration, Phys. Lett. B 213, 235 (1988).

[17] S. Catani, Y. L. Dokshitzer, M. Olsson, G. Turnock, and B. R. Webber, Phys. Lett. B 269, 432 (1991).

[18] S. Bethke, Z. Kunszt, D. E. Soper, and W. J. Stirling, NPB 370, 310 (1992).

[19] H. Leutwyler, hep-ph/9602253 (1996).

[20] G. Rodrigo, hep-ph/9609213 (1996).

[21] B. R. Webber, Ann. Rev. Nucl. Part. Sci. 36, 253 (1986).

[22] Y. L. Dokshitzer, V. A. Khoze, S. I. Troyan, and A. H. Mueller, Rev. Mod. Phys. 60, 373 (1988).
[23] A. H. Mueller, Phys. Lett. B 104, 161 (1981).
[24] B. I. Ermolaev and V. S. Fadin, Sov. Phys.-JETP 33, 269 (1981).
[25] Two-Gamma Collaboration, Phys. Rev. Lett. 57, 945 (1986).
[26] P. D. Sheldon et al., Phys. Rev. Lett. 57, 1398 (1986).
[27] OPAL Collaboration, Phys. Lett. B 247, 617 (1990).
[28] OPAL Collaboration, Zeit. Phys. C 49, 375 (1991).
[29] B. R. Webber, QCD and jet physics, in 27th Int. Conf. on High Energy Physics, Glasgow, 1994, edited by P. Bussey and I. Knowles, IOP, 1994.
[30] B. R. Webber, Multijets in $e^+e^-$ annihilation at 500 GeV, in Proc. 1991 ECFA Workshop, 1991.
[31] N. Brown, J. Phys. G: Nucl. Part. Phys. 17, 1561 (1991).
[32] F. A. Berends and W. T. Giele, Nucl. Phys. B 306, 759 (1988).
[33] S. Catani, Jet physics at LEP and SLC, in 18th Johns Hopkins Workshop on Current Problems in Particle Theory, edited by R. Casalbouni, G. Domokos, S. Kovesi-Domokos, and B. Monteleoni, World Scientific, 1994.
[34] B. R. Webber, Jets in perturbation theory, in QCD - 20 Years Later, World Scientific, 1992.
[35] N. Brown and W. J. Stirling, Phys. Lett. B 252, 657 (1990).
[36] N. Brown and W. J. Stirling, Zeit. Phys. C 53, 629 (1992).
[37] G. Kramer and H. Spiesberger, Zeit. Phys. C 71, 111 (1996).
[38] ALEPH Collaboration, Phys. Lett. B 257, 479.
[39] R. K. Ellis, W. J. Stirling, and B. R. Webber, QCD and Collider Physics, CUP, to be published.
[40] S. Catani, L. Trentadue, G. Turnock, and B. R. Webber, Nucl. Phys. B 407, 3 (1993).
[41] M. Banahan, D. Brady, and M. Doran, The C Book, Addison-Wesley, second edition, 1991.
[42] B. W. Kernighan and D. M. Ritchie, The C Programming Language, Prentice Hall, second edition, 1988.
[43] T. L. Heath, The Thirteen Books of Euclid’s Elements, volume 2, CUP, 1908.
[44] H. Davenport, The Higher Arithmetic, CUP, 1982.