Functional central limit theorems for vicious walkers

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Dedicated to Professor Tokuzo Shiga on his 60th birthday.

Abstract. We consider the diffusion scaling limit of the vicious walker model that is a system of nonintersecting random walks. We prove a functional central limit theorem for the model and derive two types of nonintersecting Brownian motions, in which the nonintersecting condition is imposed in a finite time interval $[0, T]$ for the first type and in an infinite time interval $(0, \infty)$ for the second type, respectively. The limit process of the first type is a temporally inhomogeneous diffusion, and that of the second type is a temporally homogeneous diffusion that is identified with a Dyson’s model of Brownian motions studied in the random matrix theory. We show that these two types of processes are related to each other by a multi-dimensional generalization of Imhof’s relation, whose original form relates the Brownian meander and the three-dimensional Bessel process. We also study the vicious walkers with wall restriction and prove a functional central limit theorem in the diffusion scaling limit.

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1 Introduction

The system of one-dimensional symmetric simple random walks, in which none of walkers have met others in a given time period, is called the vicious walker model. (See Fisher’s paper [8], in which it was introduced as a model of statistical mechanics.) The purpose of this paper is to study the scaling limit of vicious walkers as a stochastic process. Since each random walk tends to a Brownian motion in the diffusion scaling limit, an interacting system of $N$ Brownian motions will be constructed as the scaling limit of vicious walkers with an arbitrary finite number of walkers $N$. We show that a functional central limit theorem for vicious walkers holds and the limit process $X(t) = (X_1(t), X_2(t), \ldots, X_N(t))$ is a temporally inhomogeneous diffusion, that is, its transition probability depends on the time interval $(0, T]$ in which the nonintersecting condition is imposed. We claim that when $N = 2$, the process $(X_2(t) - X_1(t))/\sqrt{T}$ is a one-dimensional Brownian motion conditioned to stay positive during a finite time interval $(0, T]$, which is called a Brownian meander in [35, 40].

We also study the temporally homogeneous diffusion process $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_N(t))$, which is obtained from the previous process by taking $T \to \infty$. The process $Y(t)$ is the Doob h-transform [5] of the absorbing Brownian motion in a Weyl chamber with harmonic function $h_N(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i)$, and can be regarded as a system of Brownian motions with the drift terms acting as the repulsive two-body forces proportional to the inverse of distances between particles [11]. In other words, if we set $T \to \infty$, the scaling limit of vicious walkers can realize a Dyson’s Brownian motion model studied in the random matrix theory [7, 26]. We show the following relation between the processes $X(t)$ and $Y(t)$:

$$P(X(\cdot) \in dw) = \tau_N(T) P(Y(\cdot) \in dw) \frac{1}{h(w(T))},$$

where $\tau_N(T)$ is the normalization constant. This equality is a generalization of Imhof’s formula, which relates the Brownian meander and the three-dimensional Bessel process [13].

The Gaussian ensembles of random matrices can be regarded as the thermodynamical equilibrium of Coulomb gas system and that is the reason why Dyson introduced a one-dimensional model of interacting Brownian particles with (two-dimensional) Coulomb repulsive potentials [7, 26]. Similar relations between our processes and random matrix ensembles can be seen. The distribution of $Y(t)$ is described by using the probability density of eigenvalues of random matrices in the Gaussian unitary ensemble (GUE) with variance $t$. Pandey and Mehta [27, 33] introduced a Gaussian ensemble of Hermitian matrices depending on a parameter $\alpha \in [0, 1]$. When $\alpha = 0$, the ensemble is the Gaussian orthogonal ensemble (GOE), and when
\( \alpha = 1 \), it is the GUE. We will find that the probability density function of \( \sqrt{T/(2T-t)}X(t) \) coincides with that of eigenvalues of matrices in the Pandey-Mehta ensemble with \( \alpha = \sqrt{(T-t)/T} \).

We also study vicious walkers with wall restriction and prove the functional central limit theorem in the diffusion scaling limit. In this case the obtained temporally inhomogeneous diffusion process is a system of nonintersecting Brownian meanders, which is related to the nonstandard classes of random matrices [20, 21].

2 Statement of Results

2.1 Vicious walkers without wall restriction

Let \( (\{S_j\}_{j \geq 0}, P^z) \) be the \( N \)-dimensional Markov chain starting from \( z = (z_1, z_2, \ldots, z_N) \), such that the coordinates \( S_j^k, k = 1, 2, \ldots, N \), are independent simple random walks on \( \mathbb{Z} \). We always take the starting point \( z \) from the set

\[
\mathbb{Z}_N^\prec = \{z = (z_1, z_2, \ldots, z_N) \in (2\mathbb{Z})^N : z_{k+1} - z_k \in 2\mathbb{Z}_+, k = 1, \ldots, N - 1\},
\]

where \( \mathbb{Z}_+ \) is the set of positive integers. Now we consider the condition that any of walkers does not meet other walkers up to time \( m \), i.e.

\[
S_j^1 < S_j^2 < \cdots < S_j^N, \quad 0 \leq j \leq m. \tag{2.2}
\]

We denote by \( Q^z_m \) the conditional probability of \( P^z \) under the event \( \Lambda_m = \{S_j^1 < S_j^2 < \cdots < S_j^N, 0 \leq j \leq m\} \). The process \( (\{S_j\}_{j \geq 0}, Q^z_m) \) is called the vicious walkers (up to time \( m \)) (see Fisher [8]).

![Figure 1: An example of vicious walks without wall restriction.](image)

For \( T > 0 \) and \( z \in \mathbb{Z}_N^\prec \), we consider probability measures \( \mu_{L,T}^z, L \geq 1 \), on the space of continuous paths \( C([0,T] \to \mathbb{R}^N) \) defined by

\[
\mu_{L,T}^z(\cdot) = Q_{L^2T} \left( \frac{1}{L} \mathbf{S}(L^2t) \in \cdot \right),
\]

where \( \mathbf{S}(t), t \geq 0 \), is the interpolation of the random walk \( \mathbf{S}_j, j = 0, 1, 2, \ldots \). We study the limit distribution of the probability \( \mu_{L,T}^z, L \to \infty \).

We put \( \mathbb{R}_N^\prec = \{x \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\} \), which is called the Weyl chamber [9]. By virtue of the Karlin-McGregor formula [15, 16], the transition density function \( f_N(t, y|x) \) of the absorbing Brownian motion in \( \mathbb{R}_N^\prec \) and the probability \( \mathcal{N}_N(t, x) \) that the Brownian motion started at \( x \in \mathbb{R}_N^\prec \) does not hit the boundary of \( \mathbb{R}_N^\prec \) up to time \( t > 0 \) are given by

\[
f_N(t, y|x) = \det_{1 \leq i, j \leq N} \left( (2\pi t)^{-1/2} e^{-(x_i - y_j)^2/2t} \right), \quad x, y \in \mathbb{R}_N^\prec, \tag{2.4}
\]
and

\[ N_N(t, x) = \int_{\mathbb{R}_+^n} dy f_N(t, y|x). \] (2.5)

For an even integer \( n \) and an antisymmetric \( n \times n \) matrix \( A = (a_{ij}) \) we put

\[ \text{Pf}_{1 \leq i < j \leq n}(a_{ij}) = \frac{1}{(n/2)!} \sum_{\sigma} \text{sgn}(\sigma) a_{\sigma(1)\sigma(2)} a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(n-1)\sigma(n)}, \] (2.6)

where the summation is extended over all permutations \( \sigma \) of \( (1, 2, \ldots, n) \) with restriction \( \sigma(2k-1) < \sigma(2k), \) \( k = 1, 2, \ldots, n/2. \) This expression is known as the Pfaffian (see Stembridge [39]). Then we have the following lemma, which is a consequence of the identity given by de Bruijn [4] as shown in Section 4.

**Lemma 2.1** For \( t > 0, x \in \mathbb{R}^N_+ \),

\[ N_N(t, x) = \begin{cases} \text{Pf}_{1 \leq i < j \leq N} F_{ij}(t, x), & \text{if } N \text{ is even} \\ \text{Pf}_{1 \leq i < j \leq N+1} F_{ij}(t, x), & \text{if } N \text{ is odd} \end{cases} \] (2.7)

where

\[ F_{ij}(t, x) = \begin{cases} \Psi \left( \frac{x_j - x_i}{2\sqrt{t}} \right), & \text{if } 1 \leq i, j \leq N \\ 1, & \text{if } 1 \leq i \leq N, j = N + 1 \\ -1, & \text{if } i = N + 1, 1 \leq j \leq N \\ 0, & \text{if } i = N + 1, j = N + 1 \end{cases} \] (2.8)

and \( \Psi(u) = (2/\sqrt{\pi}) \int_0^u e^{-v^2} dv. \)

We put

\[ h_N(x) = \prod_{1 \leq i < j \leq N} (x_j - x_i). \] (2.9)

The first main result is the following theorem.

**Theorem 2.2** (i) For any fixed \( z \in \mathbb{Z}_+^N \) and \( T > 0 \), as \( L \to \infty \), \( \mu^T_{L,T} (\cdot) \) converges weakly to the law of the temporally inhomogeneous diffusion process \( X(t) = (X_1(t), X_2(t), \ldots, X_N(t)), t \in [0, T], \) with transition density \( g^T_N(s, x, t, y) \):

\[ g^T_N(0, 0, t, y) = c_N T^{-N(N-1)/4} \Gamma^{-N^2/2} e^{\left( \frac{|y|^2}{2t} \right)} h_N(y) N_N(T-t, y), \] (2.10)

\[ g^T_N(s, x, t, y) = \frac{f_N(t-s, y|x) N_N(T-t, y)}{N_N(T-s, x)} \] (2.11)

for \( 0 \leq s < t \leq T, x, y \in \mathbb{R}_+^N \), where \( c_N = 2^{-N/2} / \prod_{j=1}^N \Gamma(j/2). \)

(ii) The diffusion process \( X(t) \) solves the following equation:

\[ X_i(t) = B_i(t) + \int_0^t b^T_i(s, X(s)) ds, \quad t \in [0, T], \quad i = 1, 2, \ldots, N, \] (2.12)

where \( B_i(t), i = 1, 2, \ldots, N, \) are independent one-dimensional Brownian motions and

\[ b^T_i(t, x) = \frac{\partial}{\partial x_i} \ln N_N(T-t, x), \quad i = 1, 2, \ldots, N. \]
Then we have the following lemma, which is proved in Section 4 as a consequence of the identity given by

\[ \Lambda \]

the absorbing Brownian motion in \( C \) and \( m \) with wall restriction (up to time \( N \)).

We study the limit distribution of the probability

\[ \hat{\mu}(x) = \int_{0}^{t} \frac{1}{y} ds, \quad t \in [0, \infty), \quad i = 1, 2, \ldots, N. \]  

2.2 Vicious walkers with wall restriction

In this subsection, we impose the condition

\[ S_j^1 \geq 0, \quad 1 \leq j \leq m, \]  

in addition to (2.2) and take the starting point \( z \) from the set

\[ \{ (z_i) \in (2N)^N; z_{k+1} - z_k \in 2Z, k = 1, \ldots, N - 1 \}, \]  

where \( N \) is the set of non-negative integers. That is, there assumed to be a wall at the origin and all walkers can walk only in the region \([0, \infty)\). We denote by \( \tilde{\mu}_{L,T}^z \) the conditional probability of \( I^z \) under the event \( \tilde{\Lambda}_m = \{ 0 \leq S_j^1 < S_j^2 < \cdots < S_j^N, 0 \leq j \leq m \} \). The process \( (\{S_j\}_{j \geq 0}, \tilde{\mu}_{L,T}^z) \) is regarded as the vicious walkers with wall restriction (up to time \( m \)) [22].

For \( T > 0 \) and \( z \in \mathbb{N}^N \), we consider probability measures \( \hat{\mu}_{L,T}^z \) defined by

\[ \hat{\mu}_{L,T}^z(\cdot) = \tilde{\mu}_{L,T}^z \left( \frac{1}{L^2} S(L^2 t) \in \cdot \right). \]  

We study the limit distribution of the probability \( \hat{\mu}_{L,T}^z, \) \( L \to \infty \).

We put \( \mathbf{R}^N_+ = \{ x \in \mathbb{R}^N; 0 \leq x_1 < x_2 < \cdots < x_N \} \). Then the transition density function \( \tilde{f}_N(t, y|x) \) of the absorbing Brownian motion in \( \mathbf{R}^N_+ \) and the probability \( \tilde{N}_N(t, x) \) that the Brownian motion started at \( x \in \mathbf{R}^N_+ \) does not hit the boundary of \( \mathbf{R}^N_+ \) up to time \( t > 0 \) are given by

\[ \tilde{f}_N(t, y|x) = \det_{1 \leq i, j \leq N} \left( (2\pi t)^{-1/2} \left( e^{-\frac{(x_i-y_i)^2}{2t}} - e^{-\frac{(x_j+y_j)^2}{2t}} \right) \right), \quad x, y \in \mathbf{R}^N_+, \]  

and

\[ \tilde{N}_N(t, x) = \int_{\mathbf{R}^N_+} dy \tilde{f}_N(t, y|x). \]  

Then we have the following lemma, which is proved in Section 4 as a consequence of the identity given by de Bruijn [4].
Figure 2: An example of vicious walks with wall restriction.

**Lemma 2.4** For \( t > 0, x \in \mathbb{R}_+^N \),

\[
\hat{N}_N(t, x) = \begin{cases} 
\mathsf{P}_{1 \leq i < j \leq N} \hat{F}_{ij}(t, x), & \text{if } N = \text{even}, \\
\mathsf{P}_{1 \leq i < j \leq N+1} \hat{F}_{ij}(t, x), & \text{if } N = \text{odd},
\end{cases} \tag{2.21}
\]

where

\[
\hat{F}_{ij}(t, x) = \begin{cases} 
\hat{\Psi} \left( \frac{x_i}{\sqrt{2t}}, \frac{x_j}{\sqrt{2t}} \right), & \text{if } 1 \leq i, j \leq N, \\
\Psi \left( \frac{x_i}{\sqrt{2t}} \right), & \text{if } 1 \leq i \leq N, j = N + 1, \\
-\Psi \left( \frac{x_i}{\sqrt{2t}} \right), & \text{if } i = N + 1, 1 \leq j \leq N, \\
0, & \text{if } i = N + 1, j = N + 1,
\end{cases} \tag{2.22}
\]

and

\[
\hat{\Psi}(u_1, u_2) = \frac{2}{\pi} \left[ \int_0^{u_1} dv_1 \int_{u_1-u_2}^{u_2-u_1} dv_2 \exp\left\{ -v_1^2 - (v_1 - v_2)^2 \right\} 
- \int_{u_1}^{u_2} dv_1 \int_{u_2-u_1}^{u_1+u_2} dv_2 \exp\left\{ -v_1^2 - (v_1 - v_2)^2 \right\} \right]. \tag{2.23}
\]

We put

\[
\hat{h}_N(x) = \prod_{1 \leq i < j \leq N} (x_j^2 - x_i^2) \prod_{i=1}^{N} x_i. \tag{2.24}
\]

Then we can obtain the following result.

**Theorem 2.5** (i) For any fixed \( z \in \mathbb{N}_+^N \) and \( T > 0 \), as \( L \to \infty \), \( \hat{\rho}_{L,T}^z(\cdot) \) converges weakly to the law of the temporally inhomogeneous diffusion process \( \hat{X}(t) = (\hat{X}_1(t), \hat{X}_2(t), \ldots, \hat{X}_N(t)) \), \( t \in [0, T] \), with transition density \( \hat{g}_{N}^L(s, x, t, y) \):

\[
\hat{g}_{N}^L(0, 0, t, y) = \hat{c}_N T^{N^2/2} t^{-N(2N+1)/2} \exp \left\{ -\frac{|y|^2}{2t} \right\} \hat{h}_N(y) \hat{N}_N(T - t, y), \tag{2.25}
\]

5
\( \hat{g}_N^T(s, x, t, y) = \hat{N}(t - s, y|x) \hat{N}_N(T - t, y), \) \hspace{1cm} (2.26)

for \( 0 \leq s < t \leq T, x, y \in \mathbb{R}^N_+, \) where \( \hat{c}_N = 1/ \prod_{j=1}^N \Gamma(j). \)

(ii) The diffusion process \( \hat{X}(t) \) solves the following equation:

\[
\hat{X}_i(t) = B_i(t) + \int_0^t \hat{b}_i^T(s, \hat{X}(s))ds, \quad t \in [0, T], \quad i = 1, 2, \ldots, N,
\]

where

\[
\hat{b}_i^T(t, x) = \frac{\partial}{\partial x_i} \ln \hat{N}_N(T - t, x), \quad i = 1, 2, \ldots, N.
\]

Next we consider the case that \( T = T_L \) goes to infinity as \( L \to \infty. \)

**Corollary 2.6**

(i) Let \( T_L \) be an increase function of \( L \) with \( T_L \to \infty \) as \( L \to \infty. \) For any fixed \( z \in \mathbb{N}^N_+, \)

as \( L \to \infty, \) \( \bar{\rho}_L(t, \cdot, \cdot) \) converges weakly to the law of the temporally homogeneous diffusion process \( \hat{Y}(t) = (\hat{Y}_1(t), \hat{Y}_2(t), \ldots, \hat{Y}_N(t)), \) \( t \in [0, \infty), \)

with transition density \( \rho_N(s, x, t, y); \)

\[
\rho_N(0, 0, t, y) = \hat{c}_N t^{-(2N+1)/2} \exp \left\{ -\frac{|y|^2}{2t} \right\} \hat{h}_N(y)^2,
\]

\[
\rho_N(s, x, t, y) = \frac{1}{\hat{h}_N(x)} \hat{f}_N(t - s, y|x) \hat{h}_N(y),
\]

for \( 0 \leq s < t < \infty, x, y \in \mathbb{R}^N_+, \) where \( \hat{c}_N = (2/\pi)^{N/2} / \prod_{j=1}^N \Gamma(2j). \)

(ii) The diffusion process \( \hat{Y}(t) \) solves the following equation:

\[
\hat{Y}_i(t) = B_i(t) + \int_0^t \frac{1}{Y_i(s)} ds + \sum_{1 \leq j \leq N, j \neq i} \left\{ \int_0^t \frac{1}{Y_i(s) - Y_j(s)} ds + \int_0^t \frac{1}{Y_i(s) + Y_j(s)} ds \right\},
\]

\( t \in [0, \infty), i = 1, 2, \ldots, N. \)

### 2.3 Remarks

(i) The process \( X(t) \) ( resp. \( \hat{X}(t) \) ) represents the system of \( N \) Brownian motions ( resp. \( N \) Brownian meanders ) started from the origin conditioned not to collide up to time \( T. \) A limit theorem for one-dimensional random walk conditioned to stay positive was firstly observed by Spitzer [37] and then studied and generalized by many probabilists [1, 12, 3, 6]. For two-dimensional random walk conditioned to stay in a cone, a limit theorem was proved by Shimura [36]. Our theorems are multi-dimensional versions of these limit theorems.

(ii) The process \( Y(t) \) ( resp. \( \hat{Y}(t) \) ) represents the system of \( N \) Brownian motions ( resp. \( N \) three-dimensional Bessel processes ) conditioned never to collide. The function \( h_N(x) \) ( resp. \( \hat{h}_N(x) \) ) is a strictly positive harmonic function for the absorbing Brownian motions in the Weyl chamber \( \mathbb{R}^N_+ \) ( resp. \( \mathbb{R}^N_+ \) ). The process \( Y(t) \) ( resp. \( \hat{Y}(t) \) ) is the corresponding *Doob h-transform* [5, 11]. A functional central limit theorem to the process \( Y(t) \) was also discussed in a recent paper by O’Connell and Yor [31].

(iii) The relation between the Brownian meander and the three-dimensional Bessel process was discussed in Imhof [13]. From our results Imhof’s relation is generalized as follows:

(Without wall restriction) For any \( t_0 = 0 < t_1 < \cdots < t_\ell = T, \) \( \ell \in \mathbb{Z}_+, \)

\[
\prod_{i=1}^\ell g_N^T(t_{i-1}, y_{i-1}, t_i, y_i) = \tau_N T^{N(N-1)/4} \prod_{i=1}^\ell p_N(t_{i-1}, y_{i-1}, t_i, y_i) \frac{1}{h_N(y_\ell)},
\]

(2.31)
for any \( y_i \in \mathbb{R}_+^N, i = 1, 2, \ldots, \ell \), where \( y_0 = 0 \) and
\[
\tau_N = \frac{c_N}{c_N'} = \pi^{N/2} \frac{1}{\prod_{j=1}^N \Gamma(j/2)}.
\]

(With wall restriction) For any \( t_0 = 0 < t_1 < \cdots < t_\ell = T, \ell \in \mathbb{Z}_+ \)
\[
\prod_{i=1}^\ell \tilde{g}_N^T(t_{i-1}, y_{i-1}, t_i, y_i) = \tilde{c}_N T^{N/2} \prod_{i=1}^\ell \tilde{p}_N(t_{i-1}, y_{i-1}, t_i, y_i) \frac{1}{h_N(y_i)},
\]
for any \( y_i \in \mathbb{R}_+^N, i = 1, 2, \ldots, \ell \), where \( y_0 = 0 \) and
\[
\tilde{c}_N = \frac{\tilde{c}_N}{c_N'} = \left( \frac{\pi}{2} \right)^{N/2} \frac{1}{\prod_{j=1}^N \Gamma(2j)}.
\]

(iv) Consider an ensemble of \( N \times N \) complex Hermitian matrices \( \{H\} \). The Gaussian unitary ensemble (GUE) is the ensemble with the probability density function
\[
\mu^{\text{GUE}}(H, \sigma_1^2) = c_1 \exp \left\{ -\frac{1}{2 \sigma_1^2} \text{Tr} H^2 \right\},
\]
where \( \sigma_1^2 \) is variance and \( c_1 = 2^{-N/2}(\pi \sigma_1^2)^{-N^2/2} \). The Gaussian orthogonal ensemble (GOE) is defined as the ensemble of \( N \times N \) real symmetric matrices \( \{A\} \) with the probability density function
\[
\mu^{\text{GOE}}(A, \sigma_2^2) = c_2 \exp \left\{ -\frac{1}{2 \sigma_2^2} \text{Tr} A^2 \right\},
\]
where \( \sigma_2^2 \) is variance and \( c_2 = 2^{-N/2}(\pi \sigma_2^2)^{-N(N+1)/2} \). It is known that the distributions of eigenvalues \( x = (x_1, x_2, \ldots, x_N) \) of these matrix ensembles are given as
\[
g^{\text{GUE}}(x, \sigma_1^2) = \frac{c_N}{N!} \sigma_1^{-N^2} \exp \left\{ \frac{|x|^2}{2 \sigma_1^2} \right\} h_N(x)^2,
\]
and
\[
g^{\text{GOE}}(x, \sigma_2^2) = \frac{c_N}{N!} \sigma_2^{-N(N+1)/2} \exp \left\{ \frac{|x|^2}{2 \sigma_2^2} \right\} h_N(x),
\]
respectively [26]. Theorem 2.2 and Corollary 2.3 give the relation
\[
g_N^T(0, 0, T, y) = N! g^{\text{GOE}}(y, T), \quad y \in \mathbb{R}_+^N,
\]
and
\[
p_N(0, 0, t, y) = N! g^{\text{GUE}}(y, t), \quad y \in \mathbb{R}_+^N, \quad t > 0.
\]

In order to study a Gaussian ensemble of complex Hermitian matrices intermediate between GUE and GOE, Pandey and Mehta considered the following probability density functions with a parameter \( \alpha \in [0, 1] \)
\[
\mu^{PM}(H, \alpha) = \int dA \mu^{\text{GUE}}(H - A, 2\alpha^2 v^2) \mu^{\text{GOE}}(A, 2(1 - \alpha^2) v^2),
\]
where \( v^2 = 1/\{2(1 + \alpha^2)\} \) [27, 33]. They have studied a transition from the GOE to the GUE observed as \( \alpha \) changes from 0 to 1. Let \( g^{PM}(x, \alpha) \) be the probability density function of eigenvalues in this ensemble in Pandey and Mehta. We can show the equality \[18\]
\[
\left( \frac{t(2T-t)}{T} \right)^{N/2} g_N^T\left( 0, 0, t, \sqrt{\frac{t(2T-t)}{T}} x \right) = N! g^{PM}\left( x, \sqrt{\frac{T-t}{T}} \right).
\]
It is shown in [19] that as a consequence of this equality, the Harish-Chandra formula for an integral over the unitary group can be obtained. Similar argument concerning the relation between the process $\tilde{X}(t)$ and the nonstandard classes of random matrices is given in [20].

(v) Spohn [38] constructed nonintersecting Brownian motions on a torus and discussed the infinite volume limit to an infinite system of Dyson-type Brownian motions, which was also constructed by Dirichlet form technique in Osaka [32]. For the present $N$ nonintersecting Brownian motion $X(t)$ in a finite time interval $(0, T)$, two types of temporally inhomogeneous infinite particle systems are obtained by setting $T = T(N)$ and taking $N \to \infty$. If we set $T(N) = 2N$ and observe the bulk configuration of particles at time $t = T(N) + s, -\infty < s \leq 0$, a spatially homogeneous but temporally inhomogeneous system is derived in the infinite particle limit, whose multitime correlation functions have the quaternion determinant expressions with sine-kernel. If we set $T(N) = 2N^{1/3}$ and the particle configuration at time $t = T(N) + s, -\infty < s \leq 0$ around the position $2N^{2/3} - s^2/4$ is observed, a spatially and temporally inhomogeneous system is derived in $N \to \infty$, in which multitime correlation functions are given by the quaternion determinants with Airy-kernel [30, 17]. It is easier to prove the limit theorems for Dyson’s Brownian motion model $Y(t)$ corresponding to the above two kinds of infinite particle limits. The former limit provides a homogeneous infinite system, which coincides with the system studied by Spohn [38], Osaka [32] and Nagao and Forrester [29], and the latter does a temporally homogeneous but spatially inhomogeneous infinite system, which is related with the process recently studied by Prähofer and Spohn [34] and Johansson [14]. See Nagao [28] for $N \to \infty$ limit of the process $\tilde{X}(t)$.

3 Proof of Theorems

3.1 Proof of Theorem 2.2

Let $N_N(m, v|u)$, $u, v \in \mathbb{Z}^N$, be the total number of the vicious walks, in which the $N$ walkers start from $u_i, i = 1, 2, \ldots, N$, and arrive at the positions $v_i, i = 1, 2, \ldots, N$, at time $m$. Then the probability that such vicious walks with fixed end-points are realized in all possible random walks started from the given initial configuration is $N_N(m, v|u)/2^{mN}$, which is denoted by $V_N(m, v|u)$. We also put

$$V_N(m|u) = \sum_{v \in \mathbb{Z}^N} V_N(m, v|u).$$

Define a subset of the square lattice $\mathbb{Z}^2$,

$$L_m = \{(x, y) \in \mathbb{Z}^2 : x + y = \text{even}, \ 0 \leq y \leq m\},$$

and $E_m$ be the set of all oriented edges which connect the nearest-neighbor pairs $((x, y), (x', y'))$ of vertices with $y' = y + 1$ in $L_m$. Then each walk of the $i$-th walker can be represented as a sequence of successive edges connecting vertices $(u_i, 0)$ and $(v_i, m)$ on $(L_m, E_m)$, which we call the lattice path running from $(u_i, 0)$ to $(v_i, m)$. If such lattice paths share a common vertex, they are said to intersect. Under the vicious walk condition, what we consider is a set of all $N$-tuples of nonintersecting lattice paths. Let $\pi_0 \{ (u_i, 0) \}_{i=1}^N \to \{(v_i, m)\}_{i=1}^N$ be the set of all $N$-tuples $(\pi_1, \ldots, \pi_N)$ of nonintersecting lattice paths, in which $\pi_i$ runs from $(u_i, 0)$ to $(v_i, m)$, $i = 1, 2, \ldots, N$. $N_N(m, v|u) = |\pi_0 \{ (u_i, 0) \}_{i=1}^N \to \{(v_i, m)\}_{i=1}^N|$ and the Karlin-McGregor formula [15, 16] gives

$$N_N(m, v|u) = \det_{1 \leq i, j \leq N} \left| |\pi((u_j, 0) \to (v_i, m))| \right|,$$

where $|A|$ denotes the cardinality of a set $A$ and $\pi((u_j, 0) \to (v_i, m))$ the set of lattice paths from $(u_j, 0)$ to $(v_i, m)$. (Such a determinantal formula is also known as the Lindström-Gessel-Viennot formula in the enumerative combinatorics, see [23, 10, 39]). Since $|\pi((u_j, 0) \to (v_i, m))| = \binom{m}{(m + u_j - v_i)/2}$, we have the binomial determinant

$$V_N(m, v|u) = 2^{-mN} \det_{1 \leq i, j \leq N} \left( \binom{m}{(m + u_j - v_i)/2} \right).$$  

(3.1)
For \( L > 0 \) we introduce the following functions:
\[
\phi_L(x) = 2 \left[ \frac{Lx}{2} \right], \quad x \in \mathbb{R}, \text{ and } \phi_L(x) = (\phi_L(x_1), \phi_L(x_2), \ldots, \phi_L(x_N)), \quad x \in \mathbb{R}^N,
\]
where \( [a] \) denotes the largest integer not greater than \( a \). We show the following lemmas.

**Lemma 3.1**

(i) For \( t > 0, x \in \mathbb{Z}_N^N \) and \( y \in \mathbb{R}_N^N \)
\[
\left( \frac{L}{2} \right)^N V_N(\phi_L(t), \phi_L(y)|x) = c_N t^{-N/2} \frac{1}{L} \left( \frac{x}{L} \right)^N \left( 1 + O \left( \frac{|y|}{L} \right) \right), \quad (3.2)
\]
as \( L \to \infty \), where \( c_N = (2\pi)^{-N/2} / \prod_{j=1}^{N} \Gamma(j) \).

(ii) For \( t > 0 \) and \( x \in \mathbb{Z}_N^N \)
\[
V_N(\phi_L(t)|x) = \frac{1}{c_N} h_N \left( \frac{x}{L\sqrt{t}} \right) \left( 1 + O \left( \frac{1}{L} \right) \right), \quad (3.3)
\]
as \( L \to \infty \), where \( \tau_N = \pi^{N/2} \prod_{j=1}^{N} (\Gamma(j)/\Gamma(j/2)) \).

**Proof.** It is enough to consider the case that \( x = 2u, \phi_L(y) = 2v, u, v \in \mathbb{Z}_N^N \) and \( \phi_L(t) = 2\ell, \ell \in \mathbb{Z}_+ \). Then
\[
N_N(\phi_L(t), \phi_L(y)|x) = N_N(2\ell, 2v|2u) = \det_{1 \leq i, j \leq N} \left( \frac{2\ell}{\ell + u_j - v_i} \right),
\]
and
\[
\left( \frac{2\ell}{\ell + u_j - v_i} \right) = \frac{(2\ell)!}{(\ell + u_j - v_i)! (\ell - u_j + v_i)!} = \frac{(2\ell)!}{(\ell - v_i)! (\ell + v_i)!} A_{ij}(\ell, v, u),
\]
with
\[
A_{ij}(\ell, v, u) = \frac{(\ell + v_i - u_j + 1) u_j}{(\ell - v_i + 1) u_j},
\]
where \((a)_0 \equiv 1, (a)_k = a(a+1) \cdots (a+k-1), k \geq 1\). Then
\[
N_N(\phi_L(t), \phi_L(y)|x) = \prod_{i=1}^{N} \left( \frac{(2\ell)!}{(\ell - v_i)! (\ell + v_i)!} \right)^{1 \leq i, j \leq N} \det_{1 \leq i, j \leq N} (A_{ij}(\ell, v, u)). \quad (3.4)
\]
The leading term of \( \det_{1 \leq i, j \leq N} (A_{ij}(\ell, v, u)) \) in \( L \to \infty \) is
\[
D_1(v, u) = \det_{1 \leq i, j \leq N} \left( \frac{\ell + v_i}{\ell - v_i} \right)^{u_j} = (-1)^{N(N-1)/2} \det_{1 \leq i, j \leq N} \left( \frac{\ell + v_i}{\ell - v_i} \right)^{u_{N-j+1}}.
\]
Let \( \xi(u) = (\xi_1(u), \ldots, \xi_N(u)) \) be a partition specified by the starting point \( 2u \) defined by
\[
\xi_j(u) = u_{N-j+1} - (N - j), \quad j = 1, 2, \ldots, N. \quad (3.5)
\]
Noting that the Vandermonde determinant \(\det_{1 \leq i, j \leq N}(z_i^{N-j}) = \prod_{1 \leq i < j \leq N}(z_i - z_j)\), we have

\[
D_1(v, u) = (-1)^{N(N-1)/2} \det_{1 \leq i, j \leq N} \left( \frac{\ell + v_i}{\ell - v_i} \right)^{N-j} s_{\xi(u)}(\ell + v_1, \ell - v_1, \ldots, \ell + v_N, \ell - v_N)
\]

\[
= (-1)^{N(N-1)/2} \prod_{1 \leq i < j \leq N} \left( \frac{\ell + v_i}{\ell - v_i} \right)^{N-j} s_{\xi(u)}(\ell + v_1, \ell - v_1, \ldots, \ell + v_N, \ell - v_N)
\]

\[
= \prod_{1 \leq i < j \leq N} \frac{2(\ell(v_j - v_i))}{(\ell - v_i)(\ell - v_j)} s_{\xi(u)}(\ell + v_1, \ell - v_1, \ldots, \ell + v_N, \ell - v_N),
\]

where \(s_{\lambda}(z_1, \ldots, z_N)\) is the Schur function associated to a partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)\) defined by

\[
s_{\lambda}(z_1, \ldots, z_N) = \frac{\det_{1 \leq i, j \leq N} \left( \frac{z_i^{\lambda_j + N-j}}{z_i^{\lambda_j}} \right)}{\det_{1 \leq i, j \leq N} \left( \frac{z_i^{N-j}}{z_i^j} \right)}.
\]

(See Macdonald [25].) It is a symmetric polynomial of degree \(\sum_{i=1}^{N} \lambda_i\) in \(z_1, \ldots, z_N\) and it is known that (see p.44 in [25])

\[
s_{\lambda}(1, 1, \ldots, 1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

Then the leading term of \(D_1(v, u)\) in \(L \to \infty\) is

\[
D_2(v, u) = \prod_{1 \leq i < j \leq N} \frac{2(\ell(v_j - v_i))}{\ell} s_{\xi(u)}(1, 1, \ldots, 1)
\]

\[
= \ell^{-N(N-1)/2} 2^N(N-1)!/2 h_N(v)h_N(u) \prod_{1 \leq i < j \leq N} \frac{1}{j - i}.
\]

By Stirling’s formula we see that

\[
\prod_{i=1}^{N} \frac{(2\ell)!}{(\ell - v_i)!(\ell + v_i)!} = (\ell \pi)^{-N/2} 2^{2Nt} N(N-1)!/2 \prod_{i=1}^{N} \left( 1 - \frac{v_i^2}{\ell^2} \right)^{-1/2} \left( \frac{1 - v_i/\ell}{1 + v_i/\ell} \right)^{v_i} \left( 1 + \mathcal{O}\left( \frac{1}{\ell} \right) \right).
\]

From (3.4), (3.8) and (3.9)

\[
V_N(\phi_{L^2}(t), \phi_L(y)|x) = 2^{-2Nt} N N_N(\phi_{L^2}(t), \phi_L(y)|x)
\]

\[
= c_N \left( \frac{2}{L} \right)^N h_N \left( \frac{x}{L} \right) h_N \left( \frac{2u}{L} \right) \exp \left\{ -\frac{|y|^2}{2t} \right\} h_N(y) \left( 1 + \mathcal{O}\left( \frac{|y|}{L} \right) \right).
\]

Then we obtain (3.2).

By (3.2) and simple calculation we have

\[
V(\phi_{L^2}(t)|x) = c_N t^{-N^2/2} \frac{1}{\Gamma(N+1)} h_N \left( \frac{x}{L} \right) \int_{\mathbb{R}^N} dy e^{-|y|^2/2t} h_N(y) \left( 1 + \mathcal{O}\left( \frac{1}{L} \right) \right),
\]

(3.10)
as $L \to \infty$. The last integral is the special case ($\gamma = 1/2$ and $a = 1/2t$) of

$$\int_{\mathbb{R}^N} du \, e^{-a|u|^2} \prod_{1 \leq i < j \leq N} |u_j - u_i|^{2\gamma} = (2\pi)^{N/2}(2a)^{-N(\gamma(N-1)+1)/2} \prod_{i=1}^N \frac{\Gamma(1+i\gamma)}{\Gamma(1+\gamma)}$$

(3.11)

found in Mehta (eq.(17.6.7) on page 354 in [26]), whose proof was given in [24]. Then we have (3.3) by elementary calculation. This completes the proof. [1]

**Lemma 3.2** Let $t > 0$ and $x, y \in \mathbb{R}_+^N$. Then

$$\left(\frac{L}{2}\right)^N V_N(\phi_L(t), \phi_L(y)|\phi_L(x)) = f_N(t, y|x) \left(1 + \mathcal{O}\left(\frac{|x-y|}{L}\right)\right),$$

(3.12)
as $L \to \infty$.

**Proof.** From (3.1)

$$\left(\frac{L}{2}\right)^N V_N(\phi_L(t), \phi_L(y)|\phi_L(x))$$

$$= 2^{-N\phi_L(t)} \left(\frac{L}{2}\right)^N \det_{1 \leq i,j \leq N} \left(\frac{\phi_L(x_j) + \phi_L(y_i)}{2}\right)$$

$$= \det_{1 \leq i,j \leq N} \left(2^{-\phi_L(t)-1} L \frac{\phi_L(x_j) + \phi_L(y_i)}{2}\right).$$

Application of Stirling’s formula yields the lemma. [1]

By Donsker’s theorem (see, for instance, Billingsley [2]) we see that $\mathcal{N}_N(t, x)$ is the probability that $N$ Brownian motions do not collide until time $t$. We have the following asymptotic behaviours of the function $\mathcal{N}_N(t, x)$ as $|x|/\sqrt{t} \to 0$.

**Lemma 3.3** Let $t > 0$ and $x \in \mathbb{R}_+^N$. Then

$$\mathcal{N}_N(t, x) = \frac{1}{\pi_N^N} h_N \left(\frac{x}{\sqrt{t}}\right) \left(1 + \mathcal{O}\left(\frac{|x|}{\sqrt{t}}\right)\right), \quad \frac{|x|}{\sqrt{t}} \to 0,$$

(3.13)

where $\pi_N = \pi^{N/2} \prod_{j=1}^N \{\Gamma(j)/\Gamma(j/2)\}$.

**Proof.** First note that

$$f_N(t, y|x) = (2\pi t)^{-N/2} \exp \left\{ -\frac{1}{2t} \sum_{i=1}^N (x_i^2 + y_i^2) \right\} \det_{1 \leq i,j \leq N} \left( e^{x_i y_j/t} \right).$$

We rewrite the determinant as

$$\det_{1 \leq i,j \leq N} \left( e^{x_i y_j/t} \right) = \frac{\det_{1 \leq i,j \leq N} \left( (e^{x_i/t})^{y_N-j+1} \right) \times \det_{1 \leq i,j \leq N} \left( (e^{x_i/t})^{N-j} \right)}{\det_{1 \leq i,j \leq N} \left( (e^{x_i/t})^{N-j} \right)}$$

$$= s_\xi(y) \left( e^{x_1/t}, e^{x_2/t}, \ldots, e^{x_N/t} \right) \prod_{1 \leq i < j \leq N} (e^{x_j/t} - e^{x_i/t}),$$

where $\xi(y) = y_{N-i+1} - (N-i), i = 1, 2, \ldots, N$. Using it

$$f_N(t, y|x) = (2\pi t)^{-N/2} s_\xi(y) \left( e^{x_1/t}, e^{x_2/t}, \ldots, e^{x_N/t} \right)$$

$$\times \exp \left\{ -\frac{|x|^2 + |y|^2}{2t} \right\} \prod_{1 \leq i < j \leq N} (e^{x_j/t} - e^{x_i/t}).$$

(3.14)
Since
\[ \lim_{\delta \to 0} s_{\xi(y)}(e^{x_{1}/t}, \ldots, e^{x_{N}/t}) = s_{\xi(y)}(1, 1, \ldots, 1) = h_N(y) \prod_{j=1}^{N} \frac{1}{\Gamma(j)}, \]
and
\[ \prod_{1 \leq i < j \leq N} (e^{x_{i}/t} - e^{x_{j}/t}) = h_N \left( \frac{x}{t} \right) \left( 1 + \mathcal{O} \left( \frac{|x|}{t} \right) \right), \quad \frac{|x|}{t} \to 0, \]
the function is asymptotically
\[
N_N(t, x) = (2\pi)^{-N/2}t^{-N(N+1)/4}h_N \left( \frac{x}{\sqrt{t}} \right) \prod_{j=1}^{N} \frac{1}{\Gamma(j)} \times \int_{\mathbb{R}^N} dyh_N(y) \exp \left\{ -\frac{|y|^2}{2t} \right\} \left( 1 + \mathcal{O} \left( \frac{|x|}{\sqrt{t}} \right) \right), \quad \frac{|x|}{\sqrt{t}} \to 0.
\]
By (3.11) we have (3.13). \( \blacksquare \)

**Lemma 3.4** For \( z \in \mathbb{Z}_+^N \) and \( T > 0 \) \( \{ \mu_{L,T}^z, L \geq 1 \} \) is tight.

**Proof.** By the Kolmogorov’s tightness criterion it is enough to prove that for any \( \varepsilon > 0 \)
\[
\lim_{\delta \to 0} \sup_{L \geq 1} \mu_{L,T}^z \left( \max_{0 \leq u, v \leq T, |u - v| < \delta} |w(u) - w(v)| \geq \varepsilon \right) = 0. \tag{3.15}
\]
(See, for example, Billingsley [2].) Since \( \mu_{L,T}^z \) is the probability measure of a linearly interpolated random process, (3.15) is derived from the following estimates: as \( \delta \to 0, \)
\[
\lim_{L \to \infty} \sup_{L \geq 1} \mu_{L,T}^z \left( \max_{0 \leq u \leq \delta} |w(u) - w(0)| \geq \varepsilon/2 \right) = o(\delta), \tag{3.16}
\]
\[
\lim_{L \to \infty} \sup_{L \geq 1} \mu_{L,T}^z \left( \max_{0 \leq u \leq \delta} |w(t + u) - w(t)| \geq \varepsilon/2 \right) = o(\delta), \quad t \in [\delta/2, T - \delta]. \tag{3.17}
\]
Under the nonintersecting condition, for any \( i = 1, 2, \ldots, N \)
\[
|w_i(u) - w_i(0)| \leq |w_N(0) - w_i(0)| + (w_i(u) - w_i(0)) - (w_N(u) - w_N(0))_+, \tag{3.18}
\]
where \( a_+ = \max\{a, 0\} \) and \( a_- = \max\{-a, 0\} \). Then the set
\[
\left\{ \max_{0 \leq u \leq \delta} |w(u) - w(0)| \geq \varepsilon/2 \right\}
\]
is included in the set
\[
\left\{ \max_{0 \leq u \leq \delta} (w_1(u) - w_1(0)) - \frac{\varepsilon}{4} - \frac{2N - z_1}{2L} \right\} \cup \left\{ \max_{0 \leq u \leq \delta} (w_N(u) - w_N(0))_+ \geq \frac{\varepsilon}{4} - \frac{2N - z_1}{2L} \right\}.
\]
Noting that \( (w_1(u) - w_1(0))_- \) and \( (w_N(u) - w_N(0))_+ \) are nonnegative submartingales, we can apply Doob’s theorem (see, for instance, Revuz and Yor [35]) to obtain
\[
\mu_{L,T}^z \left( \max_{0 \leq u \leq \delta} |w(u) - w(0)| \geq \varepsilon/2 \right) \leq \left( \frac{8}{\varepsilon} \right)^p E_{L,T}^z \left( |w_1(\delta) - w_1(0)|^p + |w_N(\delta) - w_N(0)|^p \right) \tag{3.19}
\]
for any $p > 1$ and $L > 4(z_N - z_1)/\varepsilon$, where $E_{L,T}^\pi$ represents the expectation with respect to the probability measure $\mu_{L,T}^\pi$.

From Lemmas 3.1, 3.2 and 3.3

$$\limsup_{L \to \infty} E_{L,T}^\pi \left( |w_1(\delta) - w_1(0)|^p + |w_N(\delta) - w_N(0)|^p \right)$$

$$\leq C_1 \limsup_{L \to \infty} \int_{\mathbb{R}^N_+} dy \left( \frac{L}{2} \right)^N V_N(\phi_L(\delta), \phi_L(y)z) V_N(\phi_L(T \delta)|z) (y_1^p + y_N^p)$$

$$\leq C_2 c_N \delta^{-N^2/2} \int_{\mathbb{R}^N_+} dy \exp \left\{ -\frac{|y|^2}{2\delta} \right\} h_N(y)^2 (y_1^p + y_N^p)$$

$$\leq C_3 \delta^{p/2} \int_{\mathbb{R}^N_+} dx \exp \left\{ -\frac{|x|^2}{2} \right\} h_N(x)^2 (x_1^p + x_N^p)$$

$$= O(\delta^{p/2}).$$

Taking $p > 2$, we obtain (3.16).

Fix $t \in [\delta/2, T - \delta]$. By the Markov property

$$\mu_{L,T}^\pi \left( \max_{0 \leq u \leq \delta} |w(t + u) - w(t)| \geq \varepsilon/2 \right)$$

$$= Q_{L^2T}^\pi \left( \max_{0 \leq u \leq \delta} \frac{|S(L^2(t + u)) - S(L^2t)|}{L} \geq \varepsilon/2 \right)$$

$$\leq \frac{1}{P^\pi(\Lambda_{L^2T})} E^\pi \left( \Lambda_{L^2T}, P^{S(L^2t)} \left( \max_{0 \leq u \leq \delta} \frac{|S(L^2u) - S(0)|}{L} \geq \varepsilon/2 \right) \right),$$

(3.20)

where $\Lambda_m = \{S_j^1 < S_j^2 < \cdots < S_j^N, 0 \leq j \leq m\}$ and $E^\pi$ represents the expectation with respect to the probability measure $P^\pi$. By Doob’s inequality for any $x \in \mathbb{Z}^N$ and $p > 1$

$$P^\pi \left( \max_{0 \leq u \leq \delta} \frac{|S(L^2u) - S(0)|}{L} \geq \varepsilon/2 \right) \leq \left( \frac{2}{\varepsilon} \right)^p \left( \frac{S(L^2\delta) - S(0)}{L} \right) \leq C_4 \delta^{p/2}.$$

(3.21)

From Lemma 3.1 (ii)

$$\limsup_{L \to \infty} \mu_{L,T}^\pi \left( \max_{0 \leq u \leq \delta} |w(t + u) - w(t)| \geq \varepsilon/2 \right)$$

$$\leq C_4 \delta^{p/2} \limsup_{L \to \infty} \frac{P^\pi(\Lambda_{L^2T/2})}{P^\pi(\Lambda_{L^2T})} = O(\delta^{p/2 - N(N - 1)/4}).$$

(3.22)

Taking $p > N(N - 1)/2 + 2$, we obtain (3.17). This completes the proof. \[ \square \]

**Lemma 3.5** Let $z \in \mathbb{Z}^N_<$, $0 = t_0 < t_1 < \cdots < t_k = T$ and $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^{nk}$. Then

$$\lim_{L \to \infty} E_{L,T}^\pi \left( \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \theta_j \cdot w(t_j) \right\} \right)$$

$$= \int_{(\mathbb{R}^N_+)^k} dy_1 dy_2 \cdots dy_k \prod_{j=1}^k \phi_N(t_j-1, y_j-1, t_j, y_j) \exp \left\{ -\frac{1}{2} \sum_{j=1}^k \theta_j \cdot y_j \right\},$$

(3.23)

where $y_0 = 0$. \[ 13 \]
Proof. By Lemmas 3.2 and 3.1

\[
\lim_{L \to \infty} \mathbb{E}_{L,T}^T \left( \exp \left\{ -\frac{1}{2} \sum_{j=1}^{k} \theta_j \cdot w(t_j) \right\} \right)
\]

\[
= \lim_{L \to \infty} \frac{1}{P^*_{\alpha}(L^T)} \sum_{x_1 \in \mathbb{Z}^2_+} \cdots \sum_{x_k \in \mathbb{Z}^N_+} E^x \left[ \Lambda_{L^T}, S(\phi_{L^T}(t_1)) = x_1, \right.
\]

\[
\times E^{x_1} \left[ \Lambda_{L^T(t_2-t_1)}, S(\phi_{L^T}(t_2 - t_1)) = x_2, \right.
\]

\[
\times \cdots \times E^{x_{k-1}} \left[ \Lambda_{L^T(t_{k-k-1})}, S(\phi_{L^T}(t_k - t_{k-1})) = x_k, \exp \left\{ \sqrt{-1} \sum_{j=1}^{k} \theta_j \cdot \frac{x_j}{L} \right\} \right] \right]\]

\[
= \lim_{L \to \infty} \frac{1}{P^*_{\alpha}(L^T)} \int_{\mathbb{R}^2_k} dy_1 dy_2 \cdots dy_k \left( \frac{L}{2} \right)^{Nk} V_N(\phi_{L^T}(t_1), \phi_{L^T}(y_1)) \cdots
\]

\[
\times V_N(\phi_{L^T}(t_k - t_{k-1}), \phi_{L^T}(y_k)) \exp \left\{ \sqrt{-1} \sum_{j=1}^{k} \theta_j \cdot y_j \right\}
\]

\[
= \int_{\mathbb{R}^2_k} dy_1 dy_2 \cdots dy_k \prod_{j=1}^{k} g_{L^T}(t_{j-1}, y_{j-1}, t_j, y_j) \exp \left\{ \sqrt{-1} \sum_{j=1}^{k} \theta_j \cdot y_j \right\}.
\]

This completes the proof. \[\square\]

Proof of Theorem 2.2. From Lemmas 3.4 and 3.5 we see that \(\mu^T_{L,T}(\cdot)\) converges weakly to the law of the time inhomogeneous diffusion process \(X(t)\) with transition density \(g_N^T(s, x, t, y)\). Noting that \(N_N(T - t, x)\) is a solution of the heat equation, we see that \(g_N^T(s, x, t, y)\) satisfies the following backward equation:

\[
\frac{\partial}{\partial t} g_N^T(s, x, t, y) = \frac{1}{2} \Delta_x g_N^T(s, x, t, y) + b^T(t, x) \cdot \nabla_x g_N^T(s, x, t, y).
\]  
(3.24)

Then the process \(X(t)\) solves (2.12). This completes the proof of Theorem 2.2.

Proof of Corollary 2.3. By simple observation we see that the estimates concerning the probability \(\mu^T_{L,T}\) in the lemmas are uniform with respects to \(T\). Then (i) is obtained from the properties

\[
\lim_{T \to \infty} g_N^T(0, 0, t, y) = p_N(0, 0, t, y),
\]

\[
\lim_{T \to \infty} g_N^T(s, x, t, y) = p_N(s, x, t, y),
\]

which are derived from Lemma 3.3. Noting that \(h_N(x)\) is a harmonic function, we see that \(p_N(s, x, t, y)\) satisfies the following backward equation:

\[
\frac{\partial}{\partial t} p_N(s, x, t, y) = \frac{1}{2} \Delta_x p_N(s, x, t, y) + \nabla_x \ln h_N(x) \cdot \nabla_x p_N(s, x, t, y).
\]  
(3.25)

Then the process \(Y(t)\) solves (2.15). This completes the proof of Corollary 2.3. \[\square\]

3.2 Proof of Theorem 2.5

Let \(\tilde{N}_N(m, \nu | u)\), \(u, v \in \mathbb{N}_+^N\), be the total number of the vicious walks with wall restriction, in which the \(N\) walkers start form the positions \(u_i, i = 1, 2, \ldots, N\), and arrive at the positions \(v_i, i = 1, 2, \ldots, N\), at time \(m\).
Then the probability that such vicious walks with fixed end-points are realized in all possible random walks started from the given initial configuration is \( \tilde{N}_N(m, v|u)/2^{mN} \), which is denoted by \( \tilde{V}_N(m, v|u) \). We also put
\[
\tilde{V}_N(m|u) = \sum_{v \in \mathbb{N}_+^N} \tilde{V}_N(m, v|u).
\]

By the Karlin-McGregor (Lindström-Gessel-Viennot) formula, we have [22]
\[
\tilde{V}_N(m, v|u) = 2^{-mN} \det_{1 \leq i, j \leq N} \left( \begin{array}{c} m \\ \left( m + u_j - v_i \right)/2 \end{array} \right) - \left( \begin{array}{c} m \\ \left( m + u_j + v_i \right)/2 + 1 \end{array} \right). \tag{3.26}
\]

Let
\[
sp_\lambda(z_1, \ldots, z_N) = \frac{\det(z_i^{\lambda_j + N-j+1} - z_i^{-(\lambda_j + N-j+1)})}{\det(z_i^{N-j+1} - z_i^{-(N-j+1)})}, \tag{3.27}
\]
for a partition \( \lambda = (\lambda_1, \ldots, \lambda_N) \). Remark that \( sp_\lambda(z_1, \ldots, z_N) \) is the character of the irreducible representation corresponding to a partition \( \lambda \) of the symplectic Lie algebra (see, for example, Lectures 6 and 24 in Fulton and Harris [9]). By using the function \( sp_\lambda \) instead of the Schur function, we can show the following lemma by a similar way to the proof of Lemma 3.1.

**Lemma 3.6**

(i) For \( t > 0, x \in \mathbb{N}_+^N \) and \( y \in \mathbb{R}_+^N \)
\[
\left( \frac{\lambda}{2} \right)^N \tilde{V}_N(\phi_{\lambda t}(t), \phi_{\lambda}(y)|x) = \tilde{c}_N t^{-N(2N+1)/2} \tilde{h}_N \left( \frac{x}{L} \right) \exp \left\{ - \frac{|y|^2}{2t} \right\} \tilde{h}_N(y) \left( 1 + O \left( \frac{|y|}{L} \right) \right), \tag{3.28}
\]
as \( L \to \infty \), where \( \tilde{c}_N = (2/\pi)^{N^2/2} \prod_{j=1}^{\infty} \Gamma(2j) \).

(ii) For \( t > 0 \) and \( x \in \mathbb{N}_+^N \)
\[
\tilde{V}_N(\phi_{\lambda t}(t), \phi_{\lambda}(y)|x) = \frac{1}{c_N} \tilde{h}_N \left( \frac{x}{L \sqrt{t}} \right) \left( 1 + O \left( \frac{1}{L} \right) \right), \tag{3.29}
\]
as \( L \to \infty \), where \( \tilde{c}_N = (\pi/2)^{N^2/2} \prod_{j=1}^{\infty} \Gamma(2j)/\Gamma(j) \).

**Proof.** Again we consider the case that \( x = 2u, \phi_{\lambda}(y) = 2v, u, v \in \mathbb{N}_+^N \) and \( \phi_{\lambda t}(t) = 2\ell, \ell \in \mathbb{Z}_+ \). By the equation (3.26)
\[
\tilde{V}_N(\phi_{\lambda t}(t), \phi_{\lambda}(y)|x) = 2^{-2\ell N} \prod_{i=1}^{\ell} \frac{(2\ell)!}{(\ell - v_i)!((\ell + v_i)!) \det_{1 \leq i, j \leq N} \left( \tilde{A}_{ij}(\ell, v, u) \right)}, \tag{3.30}
\]
with
\[
\tilde{A}_{ij}(\ell, v, u) = \left( \frac{\ell + v_i - u_j + 1}{\ell - v_i + 1} \right)_{u_j} - \left( \frac{\ell - v_i - u_j + 1}{\ell + v_i + 1} \right)_{u_j+1}.
\]

Then the leading term of \( \det_{1 \leq i, j \leq N}(\tilde{A}_{ij}(\ell, v, u)) \) as \( L \to \infty \) is
\[
\tilde{D}_1(v, u) = \det_{1 \leq i, j \leq N} \left( \begin{array}{c} \ell + v_i \\ \ell - v_i \end{array} \right)^{u_j} - \left( \begin{array}{c} \ell - v_i \\ \ell + v_i \end{array} \right)^{u_j} = (-1)^{N(N-1)/2} \det_{1 \leq i, j \leq N} \left( \begin{array}{c} \ell + v_i \\ \ell - v_i \end{array} \right)^{u_{N-j+1}} - \left( \begin{array}{c} \ell - v_i \\ \ell + v_i \end{array} \right)^{u_{N-j+1}} = \det_{1 \leq i, j \leq N} \left( \begin{array}{c} \ell - v_i \\ \ell + v_i \end{array} \right)^{N-j+1} - \left( \begin{array}{c} \ell + v_i \\ \ell - v_i \end{array} \right)^{N-j+1} \right)^{sp_\xi(u)} \left( \begin{array}{c} \ell + v_i \\ \ell - v_i \end{array} \right)^{N-j+1},
\]
where $\xi(u) = (\xi_1(u), \ldots, \xi_N(u))$ with $\xi_j(u) = u_{N-j+1} - (N-j+1), j = 1, 2, \ldots, N$. Note that

$$
\det_{1 \leq i, j \leq N} \left( z_i^{N-j+1} - z_i^{-j} \right) = \prod_{j=1}^{N} \left( z_j - \frac{1}{z_j} \right) \prod_{1 \leq i < j \leq N} \left( z_j - z_i \right) \left( \frac{1}{z_i z_j} - 1 \right).
$$

Then by simple calculation we have

$$
\det_{1 \leq i, j \leq N} \left( \left( \frac{\ell - v_i}{\ell + v_i} \right)^{N-j+1} - \left( \frac{\ell + v_i}{\ell - v_i} \right)^{N-j+1} \right)
= \prod_{j=1}^{N} \frac{4\ell v_j}{\ell^2 - v_j^2} \prod_{1 \leq i < j \leq N} \frac{4f^2(v_j^2 - v_i^2)}{(\ell^2 - v_i^2)(\ell^2 - v_j^2)}.
$$

(3.31)

It is known that

$$
sp_a(1, \ldots, 1) = \prod_{1 \leq i < j \leq N} \frac{\ell_j^2 - \ell_j^2}{m_j^2 - m_j^2} \prod_{j=1}^{N} \frac{v_j}{m_j}
$$

(3.32)

with $\ell_j = \lambda_j + N - j + 1, m_j = N - j + 1$ [9]. Then

$$
sp_{\xi(u)}(1, \ldots, 1) = \prod_{1 \leq i < j \leq N} \frac{u_j^2 - u_i^2}{j^2 - i^2} \prod_{j=1}^{N} \frac{u_j}{j} = \hat{h}_N(u) \prod_{j=1}^{N} \frac{1}{\Gamma(2j + 1)}.
$$

(3.33)

Then the leading term of $\hat{D}_2(v, u)$ in $L \to \infty$ is

$$
\hat{D}_2(v, u) = \hat{h}_N(u) \prod_{j=1}^{N} \frac{1}{\Gamma(2j)} \prod_{j=1}^{N} \frac{4v_j}{\ell} \prod_{1 \leq i < j \leq N} \frac{4(v_j^2 - v_i^2)}{\ell^2}
= \prod_{j=1}^{N} \frac{2}{\Gamma(2j)} \hat{h}_N \left( \frac{u}{\ell} \right) \hat{h}_N(2v).
$$

(3.34)

From (3.9), (3.30) and (3.34) we have

$$
\hat{V}_N(\phi_L(t), \phi_L(y)|x)
= \prod_{j=1}^{N} \frac{1}{\Gamma(2j)} \left( \frac{4}{\ell^n} \right)^{N/2} \hat{h}_N \left( \frac{y}{\ell} \right) \hat{h}_N(2u) \exp \left\{ -\frac{|y|^2}{\ell} \right\} \left( 1 + O \left( \frac{|y|}{\ell} \right) \right)
= \left( \frac{2}{L} \right)^N \partial_N t^{-N(2N+1) / 2} \hat{h}_N \left( \frac{X}{L} \right) \hat{h}_N(y) \exp \left\{ -\frac{|y|^2}{2L} \right\} \left( 1 + O \left( \frac{|y|}{L} \right) \right).
$$

Then we obtain (3.28).

By (3.28) and simple calculation we have

$$
\hat{V}(\phi_L^2(t)|x) = \partial_N t^{-N(2N+1) / 2} \hat{h}_N \left( \frac{X}{L} \right) \int_{\mathbb{R}^N} dy \ e^{-|y|^2 / 2L} \hat{h}_N(y) \left( 1 + O \left( \frac{1}{L} \right) \right)
= \frac{\partial_N}{\Gamma(N+1)} \hat{h}_N \left( \frac{X}{L} \right) \left( 1 + O \left( \frac{1}{L} \right) \right),
$$

as $L \to \infty$. The last integral is the special case ($\gamma = 1/2$ and $a = 1$) of

$$
\int_{\mathbb{R}^N} dxe^{-|u|^2 / 2} \prod_{1 \leq i < j \leq N} |u_j|^2 - |u_i|^2 \prod_{j=1}^{N} |u_j|^2 - a
= 2^{aN + \gamma(N-1)} \prod_{j=1}^{N} \frac{\Gamma(1 + j\gamma)\Gamma(a + \gamma(j - 1))}{\Gamma(1 + \gamma)},
$$

(3.35)
Following the same calculation as was done in the proof of Lemma 3.2, we have the following lemma.

**Lemma 3.7** Let $t > 0$ and $x, y \in \mathbb{R}^N_+ <$. Then

\[
\left( \frac{L}{2} \right)^N \tilde{V}_N (\phi_{L^2}(t), \phi_L(y) | \phi_L(x)) = \tilde{f}_N(t, y|x) \left( 1 + O \left( \frac{||x|| + ||y||}{L} \right) \right),
\]

as $L \to \infty$.

We rewrite $\tilde{f}_N(t, y|x)$ as

\[
\tilde{f}_N(t, y|x) = (2\pi t)^{-N/2} s_p \tilde{\xi}_{(y)} \left( e^{x_i/t}, \ldots, e^{x_N/t} \right) \exp \left\{ -\frac{||x||^2 + ||y||^2}{2t} \right\} \times \prod_{j=1}^N \left( e^{x_j/t} - e^{-x_j/t} \right) \left( \prod_{1 \leq i < j \leq N} (e^{x_i/t} - e^{x_j/t}) (e^{(x_i + x_j)/t} - 1) \right) \left\{ \prod_{j=1}^N e^{x_j/t} \right\}^{-N+1}.
\]

Then we can obtain the following lemma by a similar way to prove Lemma 3.3 by virtue of the equations (3.32) and (3.35).

**Lemma 3.8** Let $t > 0$ and $x \in \mathbb{R}^N_+ <$. Then

\[
\tilde{N}_N(t, x) = \frac{1}{\tilde{c}_N} \tilde{h}_N \left( \frac{x}{\sqrt{t}} \right) \left( 1 + O \left( \frac{||x||}{\sqrt{t}} \right) \right), \quad ||x||/\sqrt{t} \to 0,
\]

where $\tilde{c}_N = (\pi/2)^{N/2} \prod_{j=1}^N \{ \Gamma(2j)/\Gamma(j) \}$.

From the above lemmas and the same argument as in the previous subsection we can obtain Theorem 2.5 and Corollary 2.6.

## 4 Proof of Lemmas 2.1 and 2.4

We use the following identity, which is shown in de Bruijn [4]. (See also Appendix in [18].) Lemmas 2.1 and 2.4 are easy consequences of this result as shown below.

**Lemma 4.1** Let $z$ be a square integrable piecewise continuous function on $\mathbb{R}^2$. Then

\[
\int_{\mathbb{R}^2_N} dy \det_{1 \leq i, j \leq N} (z(x_i, y_j)) = Pf_{1 \leq i < j \leq N} (F_{ij}(x)),
\]

where

\[
\hat{N} = \begin{cases} N, & \text{if } N \text{ is even} \\ N + 1, & \text{if } N \text{ is odd} \end{cases}
\]

\[
I_z(x_i) = \int_{-\infty}^{\infty} z(x_i, y) dy,
\]

\[
I_z(x_i, x_j) = \int_{-\infty}^{\infty} \det \begin{pmatrix} z(x_i, y_1) & z(x_i, y_2) \\ z(x_j, y_1) & z(x_j, y_2) \end{pmatrix} dy_1 dy_2,
\]
and
\[
F_{ij}(x) = \begin{cases} 
I_z(x_i, x_j), & \text{if } 1 \leq i < j \leq N, \\
-I_z(x_i, x_j), & \text{if } 1 \leq j < i \leq N, \\
I_z(x_i), & \text{if } 1 \leq i \leq N, j = N + 1, \\
-I_z(x_j), & \text{if } i = N + 1, 1 \leq j \leq N, \\
0, & \text{if } 1 \leq i = j \leq N + 1.
\end{cases}
\]

**Proof of Lemma 2.1.** From the above lemma and integration by substitution, it is enough to show
\[
I_{z_1}(x_i) = 1, \quad I_{z_1}(x_i, x_j) = \Psi \left( \frac{x_j - x_i}{\sqrt{2}} \right),
\]
for \( z_1(x, y) = e^{-(x-y)^2}/\sqrt{\pi} \). The first equation in (4.2) is trivial. Let \( z_0(x) = e^{-x^2}/\sqrt{\pi} \). Then
\[
I_{z_1}(x_i, x_j) = \int_{-\infty}^{\infty} dy_1 \int_{x_i-x_j}^{x_j-x_i} dy_2 \, z_0(y_1) z_0(y_1 + y_2)
= \frac{1}{\pi} \int_{x_i-x_j}^{x_j-x_i} d y_2 \, e^{-y_2^2/2} \int_{-\infty}^{\infty} d y_1 \, e^{-2(y_1+y_2)^2}
= \frac{1}{\sqrt{2\pi}} \int_{x_i-x_j}^{x_j-x_i} d y_2 \, e^{-y_2^2/2} = \Psi \left( \frac{x_j - x_i}{\sqrt{2}} \right).
\]
This completes the proof. ■

**Proof of Lemma 2.4.** It is enough to show
\[
I_{z_2}(x_i) = \Psi(x_i), \quad I_{z_2}(x_i, x_j) = \hat{\Psi}(x_i, x_j),
\]
for \( z_2(x, y) = \left\{ \left( e^{-(x-y)^2} - e^{-(x+y)^2} \right)/\sqrt{\pi} \right\} \chi \{ x \geq 0, y \geq 0 \} \). The first equation in (4.3) is obtained easily. To show the second equation we put
\[
G((a_1, a_2], (b_1, b_2]) = \int_{a_1}^{b_2} \int_{b_1}^{b_2} \, z_0(y_1) z_0(y_1 - y_2),
\]
for \( a_1, a_2, b_1, b_2 \in \mathbb{R} \cup \{ -\infty, \infty \} \). Then
\[
I_{z_2}(x_i, x_j) = G((-\infty, \infty), (-\infty, x_j - x_i]) - G((-\infty, \infty), (-\infty, -x_j + x_i])
- G((x_i, \infty), (-\infty, x_j + x_i]) + G((x_i, \infty), (-\infty, -x_j + x_i])
- G((-\infty, x_j), (-\infty, x_i - x_j]) + G((-\infty, x_j), (-\infty, -x_j - x_i])
+ G((x_j, \infty), (-\infty, x_i + x_j]) - G((x_j, \infty), (-\infty, -x_i + x_j])
= G((-\infty, x_i], (x_i - x_j, x_j - x_i]) - G((x_i, x_j], (x_j - x_i, x_i + x_j])
- G((-\infty, x_i], (-x_i - x_j, x_i - x_j])
= \hat{\Psi}(x_i, x_j).
\]
This completes the proof. ■

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