A REPRESENTATION OF GENERALIZED BRAID GROUP IN CLASSICAL BRAID GROUP

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ABSTRACT. We ask if any finite type generalized braid group is a subgroup of some classical Artin braid group. We define a natural map from a given finite type generalized braid group to a classical Artin braid group and ask if this map is an injective homomorphism. We also check that this map is a homomorphism for the generalized braid groups of type $A_n$, $B_n$ and $I_2(k)$.

Key words and phrases. Artin braid group, generalized braid group, Whitehead group, lower $K$-groups, reduced projective class group.

0. INTRODUCTION

The main impetus to this article came from the paper [8] where it was proved that the Whitehead group of any subgroup of the classical (Artin) braid group vanish. A generalization of the Artin braid group is the generalized braid group associated to Coxeter groups. This important class of groups are not very well understood. Naturally one asks if the similar theorem as in [8] is true for generalized braid groups. We consider only finite type generalized braid groups. These groups can be presented in terms of generators and relations, from finite Coxeter groups after removing some of the relations in the Coxeter group presentation. We recall that any finite type generalized braid group is torsion free. In fact these groups are fundamental groups of finite aspherical complexes ([7]).

In fact in [8] it was proved that the Whitehead group of a torsion free subgroup of any finite extension of a strongly poly-free group vanish. (The class of strongly poly-free groups were introduced in [1]). In [1] it was proved that the classical pure braid group is strongly poly-free. On the other hand it is not yet known if the pure braid group associated to a generalized braid group is strongly poly-free. It is not even known if these groups are poly-free (recently this question is also asked by M. Bestvina in [2]). Though for certain class of Coxeter groups it is known that the associated pure braid group is polyfree (see [4]). The pure braid group associated to a generalized braid group is the kernel of the obvious homomorphism
from the generalized braid group onto the associated Coxeter group. A complete classification of finite Coxeter groups is known (see Table 1) ([6]). Here recall that the symmetric groups are Coxeter group of type $A_n$ and the associated generalized braid group is the classical Artin braid group.

As any finite group can be embedded in a symmetric group, we embed a finite Coxeter group in a symmetric group and extend this embedding to a map from the associated generalized braid group to the classical braid group. We ask if we can choose this extension to be an injective homomorphism.

In this article we check that we can choose this extension to be a homomorphism for the generalized braid group of type $A_n$, $B_n$ and $I_2(k)$. Whether this homomorphism is injective is not yet settled.

In Section 4 we give examples of some natural extension in the case of braid group of type $D_n$ which is not a homomorphism.

We prove the following theorem which gives a partial answer to our question:

**Theorem.** Let $W_S$ be a Coxeter group of type $A_n$, $B_n$ or $I_2(k)$, then there is an embedding $e : W_S \to S_m$ for some $m$ and a homomorphism $A_S \to A_m$ making the following diagram commutative:

$$
\begin{array}{ccc}
1 & \longrightarrow & \mathcal{P}A_S \\
| & f_S | & \downarrow \pi_S \\
1 & \longrightarrow & \mathcal{P}A_m \\
| & f_S & \downarrow e \\
& & \mathcal{P}A_m \\
& & \pi_m \\
& & S_m \\
& & 1
\end{array}
$$

Here $A_S$ is the braid group associated to the Coxeter group $W_S$ and $A_m$ is the classical Artin braid group on $(m-1)$-strings.

This paper is an expanded and rewritten version of [11].

1. The question and some consequences

Let $S = \{s_1, s_2, \ldots, s_k\}$ be a finite set and $m : S \times S \to \{1, 2, \ldots, \infty\}$ be a function with the property $m(s, s) = 1$ and $m(s, s') \geq 2$ for $s \neq s'$. The Coxeter group associated to the system $(S, m)$ is by definition the group $W_S = \{s | (ss')^m(s, s') = 1, s, s' \in S\}$ and no relation if $m(s, s') = \infty$. Throughout we always assume that the group $W_S$ is finite. In this special case $m$ is always finite. A complete classification of finite Coxeter groups are known (see [6]). We reproduce the enumeration of all finite Coxeter groups in Table 1. The symmetric groups $S_n$ on $n$ letters are examples of Coxeter groups. These are the Coxeter group of type $A_n$ in the table. The generalized braid group associated to the Coxeter group $W_S$ is $A_S = \{s | ss'ss' \ldots = ss'sss' \ldots , s, s' \in S\}$, here the number of times the factors in $ss'ss' \ldots$ appears is $m(s, s')$, i.e., if $m(s, s') = 3$ then the relation is $ss's = s'ss'$. In such a case we call $A_S$ a braid group of type $W_S$. There is an obvious surjective homomorphism $A_S \to W_S$. Throughout this paper by classical braid group we will mean the generalized braid group associated to the Coxeter group $A_n$. 

Suppose the group $W_S$ is finite. Consider the symmetric group $S_m$ on $m$ letters. It is generated by $N = m - 1$ elements of order 2. We have a surjective homomorphism $A_m \to S_m$, where $A_m$ is the classical Artin braid group on $(m - 1)$-strings. We choose an embedding $e : W_S \to S_m$ (for example let $m \geq |W_S|$). And this embedding can be extended to give a map $f_S : A_S \to A_m$ defined on the generators of $A_S$ in the same way the map $W_S \to S_m$ is given such that the following diagram commutes.

![Diagram](https://via.placeholder.com/150)

We call this diagram of groups as diagram $D$.

Here $PA_S$ and $PA_m$ stand for the pure braid groups of the groups $A_S$ and $A_m$ respectively.

Our question is that:

**Question 1.** Is the map $f_S$ defined above a homomorphism? If yes, then is it injective? If it is not injective then what is the kernel of $f_S$?

Note that injectivity of $e$ implies that $\ker(f_S)$ is same as $\ker(f_S|_{PA_S})$. As the pure braid group is the fundamental group of the complement of reflection hyperplanes in the complex $n$-space corresponding to a faithful representation in $GL(n, \mathbb{R})$ of the associated Coxeter group as a reflection group, some geometric method might be helpful to prove the injectivity of the map $f_S$.

We describe the map $f_S$ in little more details: suppose $S_m$ is generated by the symbols $r_1, r_2, \ldots, r_{m-1}$ and the relations are $r_ir_j = r_jr_i$ whenever $|i - j| \geq 2$, $r_ir_{i+1}r_i = r_{i+1}r_ir_{i+1}$ for $i \leq m - 2$ and $r_i^2 = 1$. Then $A_m = \{r_1, r_2, \ldots, r_N \mid r_ir_j = r_jr_i$ whenever $|i - j| \geq 2, r_ir_{i+1}r_i = r_{i+1}r_ir_{i+1}$ for $i \leq m - 2\}$. The surjective homomorphism $A_m \to S_m$ is the obvious map defined on the generators.

Now define the map $f_S$ by: $f_S(s_i)$ is a coset representative of the coset containing $e(s_i)$ when $e(s_i)$ is considered as an element in $A_m$. Here we note that the answer to the question depends on the choice of the coset representative. In the next section we make a suitable choice of the coset representative, in the cases of the generalized braid groups of type $A_n$, $B_n$ and $I_2(k)$, to show that the map $f_S$ is a homomorphism.

If the answers to the first two questions above are yes then a consequence of the main theorem in [8] will be:

**Consequence 1.1.** The Whitehead group, reduced projective class group and the lower $K$-groups of any subgroup of a finite type generalized braid group vanish.

Also another useful consequence will be:
Consequence 1.2. If the classical Artin braid group has a discrete faithful linear representation then the same is true for any finite type generalized braid group.

Table of finite Coxeter groups

| Notations | Associated graph | Order of the group |
|-----------|-----------------|-------------------|
| $A_n(n>2)$ | ![Graph](image) | $(n+1)!$ |
| $B_n(n>3)$ | ![Graph](image) | $2^n n!$ |
| $D_n(n>5)$ | ![Graph](image) | $n! n$ |
| $I_2(k)$ | ![Graph](image) | $2k$ |
| $H_3$ | ![Graph](image) | 120 |
| $F_4$ | ![Graph](image) | 1152 |
| $H_4$ | ![Graph](image) | 14400 |
| $E_6$ | ![Graph](image) | 51840 |
| $E_7$ | ![Graph](image) | 2903040 |
| $E_8$ | ![Graph](image) | 696729600 |

It is a long standing question if the classical Artin braid group has a discrete faithful linear representation. R. Rouquier has informed the author that recently S. Bigelow has proved that the Artin braid group is linear using Krammer’s representation, a representation that arises from the BMW algebra. The well known Burau representation of the Artin braid groups $A_n$ in the matrix group is known to be not faithful for $n \geq 6$ (for $n \geq 10$ see [10] and for $n \geq 6$ see [9]).

Recently M. Bestvina ([2]) asked the following two questions: (1) Does $A_S$ satisfy
the Tits alternative? (2) Is $A_S$ virtually poly-free?

If the map $f_S$ is an injective homomorphism then the following results will follow from the fact that the classical pure braid group is poly-free:

**Consequence 1.3.** Any finite type generalized braid group is virtually poly-free.

**Consequence 1.4.** Finite type generalized braid groups satisfy Tits alternative. That is if $H \subset A_S$ is a subgroup which is not virtually abelian, then $H$ necessarily contains a nonabelian free group.

Here we recall that the main result in [8] was proved using the fact that the classical pure braid group is strongly poly-free. Hence we ask:

**Question 2.** Is any generalized braid group virtually strongly poly-free?

For curious reader, here we recall the definition of strongly poly-free groups:

**Definition.** A discrete group $\Gamma$ is called strongly poly-free if there exists a finite filtration of $\Gamma$ by subgroups: $1 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$ such that the following conditions are satisfied:

1. $\Gamma_i$ is normal in $\Gamma$ for each $i$
2. $\Gamma_{i+1}/\Gamma_i$ is a finitely generated free group
3. for each $\gamma \in \Gamma$ and $i$ there is a compact surface $F$ and a diffeomorphism $f : F \to F$ such that the induced homomorphism $f_\#$ on $\pi_1(F)$ is equal to $c_\gamma$ in $Out(\pi_1(F))$, where $c_\gamma$ is the action of $\gamma$ on $\Gamma_{i+1}/\Gamma_i$ by conjugation and $\pi_1(F)$ is identified with $\Gamma_{i+1}/\Gamma_i$ via a suitable isomorphism.

**Remark 1.5.** R. Rouquier has informed the author about their paper [5] on finite complex reflection groups. There they have shown, among other results, that except for few cases, for finite complex reflection groups there are presentation of the groups in terms of generators and relations so that the associated generalized braid group can be presented after removing only the finite order relations on the generators in the reflection group presentation. For example this is true in the case of real reflection groups (i.e., Coxeter groups). So the same Questions 1 and 2 may be asked for the generalized braid groups associated to these finite complex reflection groups.

Here recall that a finite reflection subgroup $G \subset GL(n, \mathbb{C})$ has a finite set of generators $\{s \mid s \in S\}$ so that each $s$ fixes (pointwise) a hyperplane $H_s$ in $\mathbb{C}^n$. The group $G$ acts on the space $\mathbb{C}^n - \cup_{s \in S} H_s$ fixed point freely. Consider the quotient space $(\mathbb{C}^n - \cup_{s \in S} H_s)/G$. The fundamental group of the space $(\mathbb{C}^n - \cup_{s \in S} H_s)/G$ is by definition the generalized braid group associated to the reflection group $G$. It is not yet known, if the generalized braid group corresponding to any finite reflection group $G$ can be obtained from $G$ after removing some of the relations in $G$. See [5] for details on this subject.

**Remark 1.6.** As far as I know the results in the Consequences 1.1-1.4 are still open.
2. \( f_S \) is a homomorphism for the braid groups of type \( A_n, I_2(k) \)

The Coxeter groups of type \( A_n \) are the symmetric groups. So there is nothing to prove in this case.

Before we start with the proof of the Theorem in the case \( I_2(k) \) we recall (with a different notation) the presentation of the classical Artin braid group \( A_n \) on \((n-1)\)-strings. It is generated by the symbols \( \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \) with the relations: \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( |i - j| \geq 2 \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for \( i \leq n-2 \). Geometrically the generator \( \sigma_i \) is represented by the following braids (Figure 1):

![Figure 1](image1)

The multiplication is given by juxtaposition of braids. And two braids are said to be same if one can be deformed to the other by an isotopy (fixing the end points 1, 2, \( \cdots \), \( n-1 \)) without any crossing of any two threads of the braids (see Chapter 1 in [3] for more details on this.) The surjective homomorphism \( A_n \to S_n \) is defined by sending \( \sigma_i \) to the transposition \((i, i+1)\).

Now we recall the presentation of the Coxeter groups of type \( I_2(k) \): \( W_S = \{ s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^k = 1 \} \). First we consider the case when \( k \) is even. The case of \( k \) odd will be treated later on.

Note that \( |W_S| = 2k \). We now consider \( k = 2 \). In terms of coset representative we write \( W_S = \{ 1, s_1, s_2, s_1 s_2 \} \). Consider the bijection \( \{ 1, 2, 3, 4 \} \to \{ 1, s_1, s_2, s_1 s_2 \} \) given by \( 1 \mapsto 1, 2 \mapsto s_1, 3 \mapsto s_2, 4 \mapsto s_1 s_2 \). It is easy to check that there is an embedding \( W_S \to S_4 \) sending \( s_1 \mapsto (12)(34) \) and \( s_2 \mapsto (13)(24) = (23)(12)(34)(23) \). Define the map \( f_S : A_S \to A_4 \) by

\[
s_1 \mapsto \sigma_1 \sigma_3 \quad \text{and} \quad s_2 \mapsto \sigma_2 \sigma_1 \sigma_3 \sigma_2
\]

To show that \( f_S \) is a homomorphism we only have to check that the two braids \( \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \) and \( \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3 \) are same. We draw the pictures (Figure 2) of these two braids and see that they are same up to isotopy.

Now we consider the case of \( k = 4 \). In this case the group \( W_S \) has 8 elements. We give an embedding of this group to the symmetric group on 8 letters. We write
in terms of coset representative \( W_S = \{1, s_1, s_2, s_1s_2, s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\} \)
and the bijection between \( W_S \) and \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) preserving the order in the way the elements are written. The embedding \( W_S \to S_8 \) is given by \( s_1 \mapsto (12)(34)(56)(78) \) and
\[
s_2 \mapsto (13)(25)(47)(68) = (23)(12)(45)(34)(23)(67)(56)(45)(78)(67)
\]
Define the map \( f_S : A_S \to A_8 \) by \( s_1 \mapsto \sigma_1 \sigma_3 \sigma_5 \sigma_7 \) and
\[
s_2 \mapsto \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_6 \sigma_5 \sigma_4 \sigma_7 \sigma_6
\]
To check that \( f_S \) is a homomorphism we need to show that \( f_S(s_1s_2s_1s_2) = f_S(s_2s_1s_2s_1) \). Again we actually show by drawing pictures (Figures 3, 4, 5) that the two associated braids are same up to isotopy.
The case of \( k = 6 \) is similar. We just write down the embedding of \( W_S \) in \( S_{12} \):
\[
s_1 \mapsto (12)(34)(56)(78)(910)(1112)
\]
\[
s_2 \mapsto (13)(25)(47)(69)(811)(1012) = (23)(12)(45)(34)(23)(67)(89)(78)(67)(1011)(910)(89)(1112)(1011)
\]
We define \( f_S : A_S \to A_{12} \) by \( f_S(s_1) = \sigma_1 \sigma_3 \sigma_5 \sigma_7 \sigma_9 \sigma_{11} \) and
\[
f_S(s_2) = \sigma_2 \sigma_1 \sigma_4 \sigma_3 \sigma_2 \sigma_6 \sigma_8 \sigma_7 \sigma_6 \sigma_{10} \sigma_9 \sigma_8 \sigma_{11} \sigma_{10}
\]
Again we need to check that \( f_S(s_1s_2s_1s_2s_1s_2s_1s_2s_1s_2) = f_S(s_2s_1s_2s_1s_2s_1s_2s_1s_2) \). We show it in Figures 6 and 7.
In the case of \( k = 2 \) it is easy to see (by drawing the picture of the braid \( f_S(s_1^{m} s_2^{n}) \)) that the map \( f_S \) is in fact injective. See Figure 8.
Now we write down the embedding \( e \) for any even \( k \): \( e : W_S \to S_{2k} \) is defined by
\[
s_1 \mapsto (12)(34)(56)(78) \cdots (2k-1, 2k)
\]
and
\[
s_2 \mapsto (13)(25)(47)(69) \cdots (2k-4, 2k-1)(2k-2, 2k)
\]
Define \( f_S \) by: \( f_S(s_1) = \sigma_1 \sigma_3 \ldots \sigma_{2k-1} \) and to define \( f_S(s_2) \) we do not need to represent it in terms of \( \sigma_i \), but at first we draw the lines from 1 to 3, 2 to 5, 4 to 7, \ldots , 2k \(-\) 2 to 2k then we draw the lines from 3 to 1, 5 to 2, \ldots , 2k \to 2k \(-\) 2 and pass through below if any previous line comes on the way.
It can easily be seen after drawing the pictures of the braids \( f_S(s_1s_2 \ldots s_1s_2) \) and \( f_S(s_2s_1 \ldots s_2s_1) \) that they are same up to isotopy. In fact they are of the type as the second picture in Figure 7.
Finally we consider the case of when \( k \) is odd. In this case we choose the embedding \( e \) in the following way: \( e : W_S \to S_k \) is defined by
\[
s_1 \mapsto (23)(45)(67) \cdots (k-1, k)
\]
and

\[ s_2 \mapsto (12)(34)(56) \cdots (k - 2, k - 1) \]

We define the map \( f_S \) by: \( f_S(s_1) = \sigma_2 \sigma_4 \cdots \sigma_{k-1} \) and \( f_S(s_2) = \sigma_1 \sigma_2 \cdots \sigma_{k-2} \).

We draw the pictures of the braids \( f_S(s_1 s_2 \ldots s_2 s_1) \) and \( f_S(s_2 s_1 \ldots s_2 s_1 s_2) \) for some special cases in Figure 9 and 10 and see that they are same upto isotopy. The general case is easily seen to be true.

Note that \( k = 3 \) is the symmetric groups case. So we consider the special cases \( k = 5, 7 \). In these cases the map \( f_S \) is of the following type: \( k = 5 \):

\[ f_S(s_1) = \sigma_2 \sigma_4 \quad \text{and} \quad f_S(s_2) = \sigma_1 \sigma_3 \]

and \( k = 7 \):

\[ f_S(s_1) = \sigma_2 \sigma_4 \sigma_6 \quad \text{and} \quad f_S(s_2) = \sigma_1 \sigma_3 \sigma_5 \]

Figure 9 shows the case \( k = 5 \) and Figure 10 corresponds to \( k = 7 \).
\[ f_S(s_1) \]

\[ f_S(s_2) \]

\[ f_S(s_1 s_2) \]

\[ f_S(s_2 s_1) \]

Figure 2
Figure 3
$f_{S}(s_1 s_2 s_1 s_2)$

Figure 4
Figure 5

\[ f_S(s_2, s_1, s_2, s_1) \]
\[ f_S(s_1) \]

\[ f_S(s_2) \]

\[ f_S(s_1 s_2) \]

Figure 6
\[ f_S(s_2 s_1) \]

\[ f_S(s_1 s_2 s_1 s_2 s_1 s_2) = f_S(s_2 s_1 s_2 s_1 s_2 s_1) \]

Figure 7
Figure 8
Figure 9
Figure 10
Hence we have proved the following theorem:

**Theorem 2.1.** Let \( \mathcal{W}_S = \{s_1, s_2 \mid s_1^2 = s_2^2 = (s_1s_2)^k = 1 \} \) be the Coxeter group of type \( I_2(k) \). Then there is a homomorphism \( f_S : \mathcal{A}_S \to \mathcal{A}_m \) and an embedding \( e : \mathcal{W}_S \to S_m \), for some \( m \) (depending on \( k \)), making the diagram \( D \) commutative. Also the map \( f_S \) is injective in the case \( k = 2 \).

3. \( f_S \) is a homomorphism for the braid groups of type \( B_n \)

Suppose \( \mathcal{W}_S \) be a Coxeter group of type \( B_n \). It has a representation of the following form:

\[
\mathcal{W}_S = \{s_1, s_2, \ldots, s_n \mid s_1^2 = \cdots = s_n^2 = (s_1s_2)^4 = (s_is_{i+1})^3 \text{ (for } i \geq 2) = (s_is_j)^2 \text{ (for } |i - j| \geq 2) = 1\}.
\]

There is a faithful representation \( e : \mathcal{W}_S \to S_{2n} \) defined on the generators in the following way: \( s_{n-j} \mapsto (j + 1, j + 2)(2n - j - 1, 2n - j) \) for \( 2 \leq n - j \leq n \) and \( s_1 \mapsto (n, n + 1) \).

We define the map \( f_S : \mathcal{A}_S \to \mathcal{A}_{2n} \) by \( f_S(s_{n-j}) = \sigma_{j+1}\sigma_{2n-j-1} \) and \( f_S(s_1) = \sigma_n \).

We check that \( f_S \) is a homomorphism in Figure 11. We show the following equivalence of braids: \( f_S(s_1s_2s_1s_2) = f_S(s_2s_1s_2s_1) \), \( f_S(s_is_{i+1}s_i) = f_S(s_{i+1}s_is_{i+1}) \) for \( i \geq 2 \) and \( f_S(s_is_j) = f_S(s_js_i) \) for \( |i - j| \geq 2 \). The first two equivalence follows from Figure 11 and the last equivalence follows from the braid relation \( \sigma_i\sigma_j = \sigma_j\sigma_i \) for \( |i - j| \geq 2 \).

Thus we have proved the following theorem:

**Theorem 3.1.** Let \( \mathcal{W}_S \) be a Coxeter group of type \( B_n \). Then there is a homomorphism \( f_S : \mathcal{A}_S \to \mathcal{A}_{2n} \) and an embedding \( e : \mathcal{W}_S \to S_{2n} \), making the diagram \( D \) commutative.
Figure 11
4. AN EXAMPLE OF AN EXTENSION $f_S$ WHICH IS NOT A HOMOMORPHISM

Consider the Coxeter group of type $D_n$. It has a representation of the following type:

$$W_S = \{ s_1, \ldots, s_n \mid (s_1 s_2)^3 = (s_2 s_3)^3 = (s_2 s_4)^3 = (s_i s_{i+1})^3 (\text{for } i \geq 4) =$$

$$(s_1 s_2)^2 (\text{for } i \geq 3) = (s_3 s_i)^2 (\text{for } i \geq 4) = (s_i s_j)^2 (\text{for } |i - j| \geq 2 \text{ and } i, j \geq 4) = 1 \}.$$  

We choose the case $n = 4$. In this case there is an embedding of $D_4$ in the symmetric group $S_8$ given by $s_1 \mapsto (34)(56), s_2 \mapsto (23)(67), s_3 \mapsto (35)(46), s_4 \mapsto (12)(78)$. Note that $(35)(46) = (45)(34)(56)(45)$. We define $f_S$ by $f_S(s_1) = \sigma_3 \sigma_5, f_S(s_2) = \sigma_2 \sigma_6, f_S(s_3) = \sigma_4 \sigma_3 \sigma_5 \sigma_4, f_S(s_4) = \sigma_1 \sigma_7$. In Figure 12 we check that the two braids $f_S(s_2 s_3 s_2)$ and $f_S(s_3 s_2 s_3)$ are not same. It is easy to see in Figure 12 that the strings 3 to 3 and 6 to 6 in $f_S(s_2 s_3 s_2)$ are not tangled, whereas the strings 3 to 3 and 6 to 6 in the braid $f_S(s_3 s_2 s_3)$ are tangled which cannot be untangled by an isotopy of braids.

Thus we have showed that the natural map $f_S$ is not a homomorphism, but the diagram $D$ commutes. At the time of writing this paper it was not clear how to choose this extension to be a homomorphism so that the diagram $D$ commute.

Remark 4.1. For the generalized braid groups corresponding to the Coxeter groups of type $H_3, H_4, F_4, E_6, E_7$ and $E_8$ the author found some embedding of the Coxeter group in some symmetric group but either the symmetric group was too large to draw the pictures of the braids for checking if an extension $f_S$ is a homomorphism or a similar situation as in the case of $D_n$ (as above) occurred. Also we note that $H_3$ is a parabolic subgroup of $H_4$, thus it is enough to check if $f_S$ is an injective homomorphism in the case of $H_4$ to prove the same for $H_3$.

Remark 4.2. We used the GAP programme ([12]) to find the embedding $e : W_S \to S_m$ in some concrete cases of the Coxeter group $W_S$ and this was used to find the embedding in general for the Coxeter groups $I_2(k)$ (for $k$ odd) and $B_n$. 


Figure 12
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