On methods of estimating cosmological bulk flows

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ABSTRACT

We explore similarities and differences between several estimators of the cosmological bulk flow, \( B \), from the observed radial peculiar velocities of galaxies. A distinction is made between two theoretical definitions of \( B \) as a dipole moment of the velocity field weighted by a radial window function. One definition involves the three dimensional (3D) peculiar velocity, while the other is based on its radial component alone. Different methods attempt at inferring \( B \) for either of these definitions which coincide only for a constant velocity field. We focus on the Wiener Filtering (WF, Hoffman et al. 2015) and the Constrained Minimum Variance (CMV, Feldman et al. 2010) methodologies. Both methodologies require a prior expressed in terms of the radial velocity correlation function. Hoffman et al. (2015) compute \( B \) in Top-Hat windows from a WF realization of the 3D peculiar velocity field. Feldman et al. (2010) infer \( B \) directly from the observed velocities for the second definition of \( B \). The WF methodology could easily be adapted to the second definition, in which case it will be equivalent to the CMV with the exception of the imposed constraint. For a prior with vanishing correlations or very noisy data, CMV reproduces the standard Maximum Likelihood (ML, Kaiser 1988) estimation for \( B \) of the entire sample independent of the radial weighting function. Therefore, this estimator is likely more susceptible to observational biases that could be present in measurements of distant galaxies. Finally, two additional estimators are proposed.

Key words: Cosmology: large-scale structure of the Universe, dark matter

1 INTRODUCTION

On large scales where baryonic processes are dynamically unimportant, peculiar motions of galaxies are likely to be unbiased tracers of the peculiar velocity field of the dominant dark matter. This is in contrast to the spatial distribution of galaxies which is naturally biased with respect to the underlying mass density field. Therefore, catalogs of radial peculiar motions should offer a unique window to any dynamical deviations from standard gravity which may arise in theories for the observed cosmic acceleration (e.g. Hellwing et al. 2014). One important caveat is that the extraction of cosmological information from peculiar velocity catalogs generally involves the biasing relation between galaxies and mass and also observational cuts imposed on the data, e.g. the Zone of Avoidance. An example is a direct calculation of the velocity correlation function where the desired signal is modulated by how galaxies are distributed in the particular velocity catalog (e.g. Davis & Peebles 1983). A large scale moment of the velocity field which has been the subject of vivacious debate is the bulk flow. Although its estimation is affected by the spatial coverage of the data, the bulk flow is fairly insensitive to galaxy-mass biasing Li et al. (2012). Let \( \omega(r) \) be the three dimensional (3D) peculiar velocity as a function of the comoving distance coordinate \( r \) (in \( \text{km s}^{-1} \)) and \( u(r) = \omega \cdot \hat{r} \) be the corresponding radial peculiar velocity field. We consider two theoretical definitions of the bulk flow, \( B \),

\[ B_1 = \int d^3 r g(r) \omega^\alpha(r), \]

(1)

and

\[ B_{11} = 3 \int d^3 r g(r) u(r) \hat{r}^\alpha(r), \]

(2)

where \( \hat{r}^\alpha \) are cosine angles between \( r \) and the Cartesian axes defined by unit vectors \( \hat{x}^\alpha \) (\( \alpha = 1, 2 \& 3 \)). The radial weighting function \( g(r) \) is usually introduced in the definition of \( B \), and it satisfies \( \int d^3 g(r) = 1 \). For a Top-Hat window of radius \( R \) centered on the observer \( g = 3/(4\pi R^3) \) for \( r > R \) and vanishes otherwise. For a Gaussian window \( g \propto \exp(-r^2/2R^2) \). For a constant 3D peculiar velocity, \( \omega = B_0 = \text{const} \), the two definitions yield the same result,
i.e. $B_0$. However, they do not coincide in general (Nusser 2014).

Any moment of the peculiar velocity field can serve as a basis for a quantity that can be estimated from the data for the purpose of constraining cosmological models. The general framework for the various usages of the term bulk flow is the fact that the weighting function $g$ is independent of direction.

We aim at clarifying the connection between methods for inferring bulk flows from sparse and noisy velocity catalogs. Different methods adopt one of the bulk flow definitions above (e.g. Feldman et al. 2010; Hoffman et al. 2015). The lack of equivalence between the two definitions even for a full 3D velocity field, guarantees different results for some of the methods. Further, although not always explicitly stated in the relevant papers, all methods rely on certain assumptions on how the observed radial motions are related to the full 3D velocity field. The bulk flow is defined as integrals over space including regions unexplored by the sparse data. Therefore, even in the ideal case of no observational errors, the data on its own is insufficient to compute $B$. Supplementing the missing information requires extrapolating the observed peculiar velocities to unobserved points in space. This is best done by resorting to the statistical nature of the 3D velocity field within the context of a cosmological model. An unwarranted criticism that is often raised is that the approach is circular in the sense that the inferred $B$ is necessarily consistent with the assumed model. The target quantity (e.g. $B$) is certainly poorly constraint by very noisy data, in which case the prior information dominates. Nevertheless, current catalogs, e.g. the SFI++ (Springob et al. 2007) and Cosmicswells-2 (hereafter CF2, Tully et al. 2013), are sufficiently accurate and dense and they constrain the bulk flow within $\sim 70h^{-1}$ Mpc with very little dependence on the assumed prior (Nusser & Davis 2011). The methods discussed here will be addressed within the context of gaussian random fields. The assumption is justified since motions on large scales obey linear theory for gravitational instability and hence, in the standard paradigm, the velocity and density fields remain gaussian. We further have to specify the statistical nature of the random error on the estimated peculiar velocity. Most common distance indicators are intrinsic scaling relations between galaxy observables involving the distance modulus (log distance) rather than the actual distance. Thus a normal (gaussian) scatter in these relations propagates into a skewed distribution of the measured distance (and peculiar velocity) error (e.g. Lynden-Bell et al. 1988; Springob et al. 2014). The challenge of dealing with non-gaussian errors can be alleviated by following the scheme of Nusser & Davis (1995, 2011). Consider the (inverse) Tully-Fisher relation as a distance indicator: $\eta = sM - \eta_0 + \Delta \eta$ where $s$ and $\eta_0$ are constants, $\eta$ is log the line-width, $M$ and $\Delta \eta$ is a random scatter with a gaussian distribution. For $z \ll 1$, we have $cz = r + u$ and $M = m - 5 \log(r) = M_0 - 5 \log(1 - u/cz)$, where $m$ is the apparent magnitude and $M_0 = m - 5 \log(cz)$ Typically, $u/cz \ll 1$ even for nearby galaxies if $u$ is measured in Local Group frame. Nusser & Davis adopt $\Delta \eta = sM_0 + 2.17su/cz - \eta_0$ as the estimator for the velocity where $u$ is typically expressed in terms of a velocity model such as a bulk flow. This is equivalent to setting $u = 0.46cz(\eta - \eta_0 - sM_0)/s$ as the estimate of the peculiar velocity with a normally distributed random error. A closely related scheme has been advocated more recently by Watkins & Feldman (2015) for obtaining individual peculiar velocities.

We also ignore here any systematic biases in the determination of the peculiar velocities. To mitigate spatial Malquist biases (e.g. Lynden-Bell et al. 1988; Strauss & Willick 1995) we assume that galaxies are placed at their redshift coordinates rather than the observed distance (Aaronson et al. 1982). To linear order this is equivalent to peculiar velocities expressed in terms of actual distance and hence, for simplicity of notation, we shall assume that the velocities are given in terms of the actual distance.

The outline of the paper is as follows. The notation and the statistical tools are given in §2. The “frequentist” approach to inferring $B$ is presented in §3. A new generalized ML estimation is presented here and the standard ML is shown to be recovered as a special case. In §4 we describe the relevant Baysian inference approach. §sec:MV focuses on minimum variance and constrained minimum variance. This section includes new estimator which incorporates constraints in a probabilistic manner. A specific scheme for computing the relevant covariance matrices is given in §6. We end with a general discussion in §7.

2 PRELIMINARIES

2.1 Notation

We are provided with a set of $N$ data points $d_i$ representing observations of the radial peculiar velocities of galaxies,

$$d_i = u_i + e_i,$$  
(3)

where $u_i$ is the underlying true signal and $e_i$ is a random variable representing observational errors. In general $\sigma^2_u = \sigma^2_\nu^2 + \sigma^2_\epsilon^2$ where $\sigma_\nu$ correspond to small scale velocity dispersion and $\sigma_\epsilon$ is proportional to the distance and it arises from the intrinsic scatter in the distance indicator (e.g. the Tully-Fisher relation). The cosine angles, $\hat{n}^\alpha = \hat{r} \cdot \hat{x}^\alpha$, between the radial direction and the Cartesian coordinates, satisfy the orthogonality condition,

$$\int d\Omega \hat{n}_i^\alpha \hat{n}_j^\beta = \frac{4\pi}{3} \delta_K^{\alpha\beta},$$  
(4)

where $\delta_K^{\alpha\beta}$ is the Kronecker delta function which equals to unity for $\alpha = \beta$ and vanishes otherwise. Let $D_{ij} = \langle d_i d_j \rangle$ and $S_{ij} = \langle u_i u_j \rangle$ denote, respectively, the data-data and the underlying signal-signal correlation functions. The angle brackets imply ensemble average over all possible realizations of the quantity inside the brackets. Since the error, $e_i$, and the signal, $u_i$, are uncorrelated, we obtain

$$D_{ij} = \sigma^2_{\epsilon} \delta_{ij} + S_{ij}.$$  
(5)

The specification of the target quantity of interest, i.e. the bulk flow, is done via the cross correlation $P$,

$$P_i^\alpha = \langle d_i B^\alpha \rangle = \langle u_i B^\alpha \rangle.$$  
(6)

\footnote{We shall use the terms “correlation functions” and “covariance matrices” interchangeably.}

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This is the only quantity which involves the definition of the bulk flow. For the definition (2) for example we get

\[ P^{*}_{i} = 3 \int d^{3} r g(r) S_{i}(r) \tilde{n}^{2}(r) , \]

where \( S_{i}(r) = \langle u_{i} u(r) \rangle \). It is also possible to obtain \( P^{*}_{i} \) for the first definition which involves the correlation with transverse component of \( \psi \) (Gorski 1988), but we do not present this here.

We shall use boldface to denote vectors and matrices, e.g. \( \bf{d} \) represents the explicit notation of \( d \), for all \( i \) and \( \bf{d}^{-1} = \sum_{j} D^{-1}_{ij} d_{j} \). We will alternate between these two conventions, as demanded by the facilitation of mathematical manipulations.

2.2 Probabilities

The joint probability distribution function (PDF) \( P(\bf{B} | \bf{d}) \) is our main tool. Specifically, the conditional PDFs \( P(\bf{B} | \bf{d}) \) and \( P(\bf{d} | \bf{B}) \) serve as the basis of the methods described here. We give a brief summary of the properties of conditional normal PDFs (e.g. Bertschinger 1987; Hoffman & Ribak 1991). Let \( \bf{X} \) be an \( n \)-dimensional gaussian random variable divided into two parts \( \bf{X} = (\bf{X}_{1}, \bf{X}_{2}) \). The matrix \( \Sigma = \langle \bf{X} \bf{X}^{T} \rangle \) and its inverse, \( \Sigma^{-1} \) are decomposed as

\[ \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} , \quad \Sigma^{-1} = \begin{bmatrix} \Sigma_{11}^{-1} & \Sigma_{12}^{T} \\ \Sigma_{21} & \Sigma_{22}^{-1} \end{bmatrix} \]

where \( \Sigma_{41} = \langle \bf{X}_{1} \bf{X}_{2}^{T} \rangle \) and

\[ \Sigma_{11}^{-1} = \left( \Sigma_{11} - \Sigma_{12}^{T} \Sigma_{22} \Sigma_{21} \right)^{-1} \]

\[ \Sigma_{22}^{-1} = \left( \Sigma_{22} - \Sigma_{21} \Sigma_{11} \Sigma_{12} \right)^{-1} \]

\[ \Sigma_{12}^{T} = -\Sigma_{11}^{-1} \Sigma_{12} \left( \Sigma_{22} - \Sigma_{21} \Sigma_{11} \Sigma_{12} \right)^{-1} \]

\[ = (\Sigma_{21}^{T})^{-1} \]

The joint PDF \( P(\bf{X}) = P(\bf{X}_{1}, \bf{X}_{2}) \propto \exp(-Q/2) \) with

\[ Q = \bf{X}^{T} \Sigma_{11}^{-1} \bf{X} + \langle \bf{X}_{2} - \mu \rangle^{T} \xi \epsilon^{-1} (\bf{X}_{2} - \mu) \]

where \( \mu = \Sigma_{11}^{-1} \Sigma_{12} \bf{X} _{1} \) and \( \xi = \Sigma_{22} - \Sigma_{21} \Sigma_{11} \Sigma_{12} \).

From this form of \( Q \), the conditional PDF for \( \bf{X}_{2} \) given \( \bf{X}_{1} \) is easily found using Bayes theorem, \( P(\bf{X}_{2} | \bf{X}_{1}) = P(\bf{X}_{2} | \bf{X}_{1}) / P(\bf{X}_{1}) \propto \exp(-\Lambda/2) \) where

\[ \Lambda = (\bf{X}_{2} - \mu)^{T} \xi \epsilon^{-1} (\bf{X}_{2} - \mu) . \]

3 INFEREN CE BASED ON P(\bf{d} | \bf{B});

GENERALIZED ML ESTIMATION

An estimate for \( \bf{B} \) is obtained by means of Maximum Likelihood (ML) estimation, i.e. by maximizing \( P(\bf{d} | \bf{B}) \), the likelihood for observing the data given the model. This is equivalent to finding \( \bf{B} \) which renders a minimum in \( \Lambda \) by solving \( \partial \Lambda / \partial B^\alpha = 0 \). We take \( \bf{X}_{1} = \bf{B} \) to represent the three component of the bulk flow and \( \bf{X}_{2} = \bf{d} \) the \( N \)-dimensional data vector. The specification of the precise definition of the bulk flow is fixed via the covariance matrix \( \Sigma_{12} = \Sigma_{Bd} = \bf{P} = \langle \bf{u} \bf{B} \rangle \). The matrix \( \Sigma_{11} = \Sigma_{BB} \) is a \( 3 \times 3 \) matrix corresponding to \( \langle \bf{B}^{2} \rangle \). Isotropy implies \( \Sigma_{11}^{0}_{0} = \Sigma_{00}^{0}_{0} = \Sigma_{11}^{0}_{1} \). Further, \( \Sigma_{22} = \Sigma_{dd} = \bf{D} \). Therefore, according to (11),

\[ \mu_{i} = \sum_{\alpha} \sigma_{\alpha}^{2} \epsilon_{i\alpha} \]

\[ \xi_{ij} = D_{ij} - \sum_{\alpha} \sigma_{\alpha}^{2} \epsilon_{i\alpha} \epsilon_{j\alpha} \]

and the minimization of the corresponding expression for \( \Lambda \) in (12) yields

\[ B^{\alpha} = B_{0}^{\alpha} = \sum_{\beta} (A^{0}_{\beta}) \sum_{i,j} \sigma_{\alpha}^{2} \xi_{ij} \epsilon_{i\alpha} \epsilon_{j\alpha} \]

where

\[ A^{\alpha\beta} = \sum_{i,j} (\sigma_{\beta}^{2}) \epsilon_{ij} \epsilon_{i\alpha} \epsilon_{j\alpha} \]

and

\[ \Lambda = \sum_{i} (d_{i} - \sum_{\alpha} \langle \bf{u} \rangle \langle \bf{B} \rangle^{\alpha})^{2} / \sigma_{i}^{2} \]

In this expression the residual between the data and the model for \( u_{i} \) is uncorrelated and is entirely due to observational error and possible very small scale velocity dispersion. The solution for \( \partial \Lambda / \partial B^\alpha \) is 0 is

\[ B^{\alpha} = \sum_{\beta} (A^{-1})^{\alpha\beta} \sum_{i=1}^{N} \tilde{d}_{i}^{\beta} \sigma_{i}^{2} \]

where

\[ A^{\alpha\beta} = \sum_{i=1}^{N} \tilde{d}_{i}^{\alpha} \tilde{d}_{i}^{\beta} / \sigma_{i}^{2} \]

The summation is over all particles independent of the form of \( g(r) \). Thus, this special velocity model reproduces the standard ML result (Kaiser 1988). In order to derive estimate for the \( \bf{B} \) within a sphere of radius \( R_{0} \), the actual model has to be modified into \( \psi(r) = \bf{B}_{0} \) for \( r \leq R_{0} \) and \( \psi = 0 \) otherwise.
4 INFEERENCE BASED ON $P(B|d)$: WIENER FILTERING

We now consider the conditional PDF $P(B|d)$. Therefore, we take $X_1 = d$, $X_2 = B$, $\Sigma_1 = D$ and, as before, $\Sigma_{12} = P$. The corresponding expression for $\Lambda$ is $\Lambda = (B - \mu)^T \Sigma^{-1} (B - \mu)$ and its minimum is rendered at

$$B_\alpha = B_{\alpha|d} = \mu_\alpha = \sum_{i,j} D_{ij}^{-1} P_{ij} \delta d_j ,$$

(23)

where the last equality is obtained by substituting the relevant quantities in (13). The covariance matrix $\xi$ does not appear in $B_{\alpha|d}$, but we give it here for later use

$$\xi_{\alpha\beta} = \sigma_B^\alpha \sigma_B^\beta \sum_{i,j} \sigma_B^{-2} D_{ij}^{-1} \delta_{ij} ,$$

(24)

4.1 The relation between $B_{\alpha|d}$ and $B_{\alpha|B}$

Since $P(B|d)P(d) = P(d|B)P(B)$, the expressions (23) and (14) must coincide in the limit of $P(B) = const$, i.e. for $\sigma_B \to \infty$. To see how this happens we resort to the matrix notation and write $A_{\alpha \beta}$ in (15) as

$$A = \Sigma^{-1}_{BB} \Sigma_{dB} \left( \Sigma_{dd} - \Sigma^{-1}_{dB} \Sigma_{BB} \Sigma_{dB} \right)^{-1} \Sigma_{dB}^{T} \Sigma_{dB}^{-1} ,$$

(25)

where the equalities are derived using (9) with $1 \to B$ and $2 \to d$. Further the sum over $i,j$ in the solution (14) transforms into

$$\Sigma_{dB} \Sigma_{dB} \left( \Sigma_{dd} - \Sigma^{-1}_{dB} \Sigma_{BB} \Sigma_{dB} \right)^{-1} d,$$

(26)

$$\Sigma_{dB} \Sigma_{dB} \left( \Sigma_{BB} - \Sigma^{-1}_{dB} \Sigma_{dB} \Sigma_{dB} \right)^{-1} d,$$

$$\Sigma_{dB} \Sigma_{dB} \Sigma_{dB} d,$$

$$\Sigma_{dB} \Sigma_{dB} ,$$

Hence, in matrix notation (14) is

$$A_{\alpha|B} = \Sigma_{BB} B_{\alpha|d} ,$$

(27)

With a little more manipulation we obtain

$$B_{\alpha|B} = \left( \Sigma_{BB} - \left( \Sigma_{BB} \right)^{-1} \right)^{-1} \Sigma_{dB} B_{\alpha|d} ,$$

(28)

where the last equality is obtained from the first two lines in (9). The contribution of $(\Sigma_{BB})^{-1}$ is negligible compared to $\Sigma_{dB} B_{\alpha|d}$ in the limit of $\sigma_B \to \infty$. This is demonstrated by proving,

$$\langle B^T (\Sigma_{BB})^{-1} B \rangle \leq \langle B^T \Sigma_{BB} B \rangle = 3 \sigma_B^4 .$$

(29)

We write $d_i = \mu_i + \Delta_i$, where $\mu_i$ is the mean value of $d_i$ given $B$ (cf. eq. 13) and $\Delta_i$ is a random variable uncorrelated with $B^\alpha$. Hence, in the limit of very large $\sigma_B$, $\Delta_i$.

$$\Delta_i \ll \left( d_i \right) \approx \sum_\alpha \sigma_{\alpha i}^2 \rho_{\alpha i} \Delta_i .$$

(30)

Substituting $(\Sigma_{BB})^{-1} = \Sigma_{BB} - \Sigma_{BB} \Sigma_{dB} \Sigma_{dB} \Sigma_{dB} \Sigma_{dB}$ corresponding to $\sigma_B^\alpha \sigma_B^\beta \sum_{i,j} \Delta_{ij}^{-1} P_{ij} \delta d_j$ the l.h.s. of the inequality becomes $3 \sigma_B^4 \sum_{i,j} \sum_{\alpha} \sigma_{\alpha i}^2 \sigma_{\alpha j}^2 (\Delta^\alpha \Delta^\beta) \approx 0$, for the approximate $\Delta$ given in (30).

4.2 Equivalence with $B$ from a reconstruction of the full 3D peculiar velocity field

Hoffman et al. (2015) use the WF methodology (Zaroubi et al. 1995) to derive the full 3D peculiar velocity field in space, $\mathbf{v}(r)$, from the measured radial peculiar velocities in the CF2 catalog (Tully et al. 2013). They then compute the bulk flow as defined in (1) for a Top-Hat window, i.e. the mean $\mathbf{v}$ within a sphere centered on the observer. They could have also computed the bulk flow according to definition (1). We will show now that $B$ derived from this procedure is (unsurprisingly) entirely equivalent to $B_{\alpha|d}$. The full field is obtained as the one which maximizes the probability $P(\mathbf{v}(r)|d)$. Substituting $X_1 = d$ and $X_2 = \mathbf{v}$ in (11) and minimizing $\Lambda = -2 \ln P + const$ in (12), we obtain

$$v_\alpha(r) = \sum_{i,j} D_{ij}^{-1} (u_i v^\alpha(r)) d_j .$$

(31)

Substituting this expression into either of the bulk flow definitions (1) & (2), leads to the same expression as $B_{\alpha|d}$ in (23).

5 MINIMUM VARIANCE ESTIMATION

So far we have considered estimates which maximize normal conditional PDF. The minimum variance (MV) approach described below yields similar results, but is not restricted to normal PDFs (cf. Zaroubi et al. 1995, for a detailed account). A MV estimate of $B$ is sought as a linear combination of the data,

$$B_{MV} = w d \equiv \sum_{\alpha = 1}^{3} \sum_{n = 1}^{N} \hat{x}^\alpha w^n d_n .$$

(32)

The weights $w$ are found by minimizing the variance, $\Lambda_{\text{MV}}$, of the residual between $B_{\alpha|w}$ and all of its possible realizations,

$$\Lambda_{\text{MV}} = \langle (B - w d)^2 \rangle = \langle B^2 \rangle - 2 \mathcal{P} w + w^T D w .$$

(33)

The minimum in $\Lambda_{\text{MV}}$ is obtained at

$$w_{\text{MV}} = D^{-1} \mathcal{P} .$$

(34)

Implementing this result into (32) yields $B_{\text{MV}}^\alpha = \sum_{i,j} D_{ij}^{-1} P_{ij} d_j$. Therefore, referring to (23), we find $B_{\text{MV}} = B_{\alpha|d}$.

5.1 Constrained Minimum Variance

One may impose constraints to any of the above procedures, as done by Feldman et al. (2010) for the MV estimator. Their constraint is that if the 3D velocity field is $\mathbf{v} = B_\alpha = const$, then (32) must reproduce $B_\alpha$ in the absence of observational errors. Substituting $d_i = u_i = \sum_\alpha B_0^\alpha \xi_\alpha$, in (32), the constraint implies

$$B_0^\alpha = \sum_{i,j} w_i^\alpha B_0^\beta \xi_{ij}^\beta ,$$

(35)

must hold for any $B_0 = const$, yielding

$$w_i^\alpha \xi_{ij}^\beta = \delta_{ij} .$$

(36)
This constraint is incorporated by modifying (33) into
\[
\Lambda_{\text{CMV}} = \langle (B^\alpha - \sum_i w_i^\alpha d_i^2)^2 \rangle + \sum_{\beta} \lambda^{\alpha\beta} \left( \sum_i w_i^\alpha \hat{n}_i^\beta - \delta^{\alpha\beta} \right),
\]
(37)
to be minimized with respect to the Lagrange multipliers, \( \lambda^{\alpha\beta} \) as well as \( w_i^\alpha \). The solution for the minimum point is (c.f. eqs. 8-10 in Agarwal et al. 2012),
\[
w_i^\alpha = \sum_j D^{-1}_{ij} \left( P_i^\alpha - \frac{1}{2} \sum_{\beta=1}^3 \lambda^{\alpha\beta} \hat{n}_j^\beta \right),
\]
(38)
\[
\lambda^{\alpha\beta} = \sum_{\gamma=1}^3 \left( (M^{-1})^{\alpha\gamma} \left( \sum_{i,j} D^{-1}_{ij} P_i^\gamma \hat{n}_j^\beta - \delta^{\alpha\beta} \right) \right),
\]
where
\[
M^{\alpha\beta} = \sum_{i,j} D^{-1}_{ij} \hat{n}_i^\alpha \hat{n}_j^\beta.
\]
(39)
Feldman et al. (2010) adopt the following scheme to approximate \( P_i^\alpha \) (in §6 where we outline an alternative scheme). They write
\[
\langle d_i B^\alpha \rangle = \sum_{j=1}^{N'} w_i^\alpha \langle d_i u_j \rangle
\]
(40)
where
\[
w_i^\alpha = \frac{1}{N'} \sum_{j=1}^{N'} (A^{-1})^{\alpha\beta} \hat{n}_j^\beta,
\]
(41)
are the weights of an isotropic survey of \( N' \) exact radial velocities \( u_j \) measured at random positions \( r_j \), having the radial distribution fixed by \( g(r) \), and
\[
A^{\alpha\beta} = \frac{1}{N'} \sum_{j=1}^{N'} \hat{n}_i^\alpha \hat{n}_j^\beta.
\]
(42)
In the continuum limit of \( N' \to \infty \), \( A^{\alpha\beta} = \int d\Omega r^2 d\Omega g(r) \hat{n}_i^\alpha \hat{n}_j^\beta / \int d\Omega r^2 d\Omega g(r) = 1/3 \delta^{\alpha\beta} \). Hence, \( w_i^\alpha \) is determined by \( d_i B^\alpha = \sum_{j=1}^{N'} w_i^\alpha \langle d_i u_j \rangle \)
(43)
which clearly coincides with (7).

5.2 No velocity correlation & standard ML

We discuss now the case where \( D_{ij} = (\sigma_i^2 + \sigma_j^2) \delta_{ij} \). According to (40), \( \langle d_i B^\alpha \rangle = 0 \) since overlap between the data and the ideal sample is zero. Therefore,
\[
M^{\alpha\beta} = \frac{1}{2} \sum_i \sigma_i^{-2} \hat{n}_i^\alpha \hat{n}_i^\beta
\]
(44)
\[
\lambda^{\alpha\beta} = \frac{1}{2} A^{\alpha\beta}
\]
(45)
\[
w_i^\alpha = \sum_{\beta=1}^3 (A^{-1})^{\alpha\beta} \hat{n}_i^\beta / \sigma_i
\]
(46)
where
\[
A^{\alpha\beta} = \sum_{i=1}^N \hat{n}_i^\alpha \hat{n}_i^\beta / \sigma_i.
\]
(47)
Therefore,
\[
w = w_{\text{ML}},
\]
(48)
i.e. the weights reduce exactly to the ML weights. In contrast, \( B_{\text{mod}} \) approaches zero for uncorrelated velocities, as expected.

5.3 Generalization of the constraint

The constraint imposed above yields weights that recover a constant velocity field in the case of perfect data. The data points are not uniformly distributed and one may argue that this constraint is actually inappropriate since there will always be leakage from other modes of the field.

If desired, other constraints which may be more realistic could be imposed. In particular, the radial velocity field could be decomposed into a functional basis of products of spherical harmonics and spherical Bessel functions (Regos & Szalay 1989; Scharf & Lahav 1993; Fisher et al. 1995). If the velocity field is fully specified by a dipole term then we could write \( u(r) = \sum_{\alpha} a^\alpha \psi(r) \hat{n}^\alpha(r) \) where \( a^\alpha \) are the expansion coefficients and \( \psi \) represent the radial functional basis, e.g. a Bessel function with wavenumber \( k \) (see §6 for details). The constraint is that for the ideal case of this form of the velocity field, the bulk flow is correctly recovered. Adopting the second \( B \) definition in (2), the constraint becomes
\[
3 \int d^3 r g(r) \sum_{\beta} a^\beta \psi(r) \hat{n}^\beta(r) \hat{n}^\alpha(r)
\]
(49)
Performing the angular integration and remembering that the constraint must hold for any \( a^\beta \) we get
\[
\sum_i w_i^\alpha \psi_i \hat{n}_i^\beta = \Psi \delta^{\alpha\beta}
\]
(50)
where \( \Psi = 4 \pi \int d\Omega r^2 g(r) \psi(r) \). Substituting this equality as our constraint in \( \Lambda_{\text{CMV}} \) (instead of 36), the corresponding weights are given by (39) but with the modification, \( \hat{n}_i^\alpha \to \psi_i \hat{n}_i^\alpha \) and \( \delta^{\alpha\beta} \to \delta^{\alpha\beta}(\xi) \).

5.4 Alternative method for imposing constraints

Constraints can be imposed by modifying the conditional PDF \( P(B|d) \propto \exp(-\Lambda/2) \). This is done adding either a quadratic or linear term in \( B \) to \( \Lambda \). Here we present only the quadratic modification since it leads to a particularly elegant result. We write
\[
\Lambda_{\text{mod}} = (B - B_{\text{mod}})^T \xi^{-1} (B - B_{\text{mod}}) + B^T (\xi \Delta)^{-1} B.
\]
(51)
as minus twice the log of the modified PDF, where \( B_{\text{mod}} \) and \( \xi \) are given in (23) and (24), respectively. The \( 3 \times 3 \) matrix, \( \Delta \), is derived by demanding that \( B \) which renders a minimum in \( \Lambda_{\text{mod}} \) also satisfied the desired constraint. For the constraint of Feldman et al. (2010) of \( B = B_0 \) for
6 COMPUTING THE COVARIANCE MATRICES

In a cosmological model with gaussian initial conditions, all statistical properties of the linear density contrast field, δ, are fully specified by the linear density power spectrum, $P(k)$, defined via $\langle \delta(r) \delta(k-r) \rangle = (2\pi)^3 \delta^{(3)}(k-k') P(k)$ where $\delta_k$ is the Fourier transform of $\delta(r)$ and $k$ is the wavenumber. Assuming potential flow, the velocity statistics can readily be related to the density field through the velocity-density relation $\delta = -f(\Omega_m) \nabla \cdot \mathbf{v}$ (Peebles 1980), where $f(\Omega_m)$ is the linear growth factor and $\Omega_m$ is the mass density parameter. All relevant correlation functions can be expressed in terms of definite integrals involving these power spectra. The integrals must be evaluated numerically with lower and upper oscillatory functions (spherical Bessel) appearing in their integrands. Instead of direct numerical integration, we propose evaluating the integrals using the formalism developed by Fisher et al. (1995) and this section heavily relies on that paper. Radial peculiar velocities are conveniently expressed in terms of spherical Bessel functions, $j_l$, and spherical Harmonics, $Y_{lm}$. To do that we imagine a very large spherical volume or radius $R_{\text{max}}$ which entirely encompasses the data and is also sufficiently large that the effects boundary conditions are small. Expressing the density contrast as

$$\delta(r) = \sum_{n=0}^{n_{\text{max}}} \sum_{l,m=n-l}^{n_{\text{max}}} \frac{1}{l} C_{lm} \delta_{lmn} j_l(k_r r) Y_{lm}(\hat{r})$$

(53)

where the wavenumbers $k_n$ are fixed by the boundary conditions at $R_{\text{max}}$ (Fisher et al. 1995). We advocate the boundary conditions which yield gravity potential decaying as $r^{-l}$ for $r > R_{\text{max}}$. Since $j_l Y_{lm}$ are eigenfunctions of the Laplacian, these boundary conditions are equivalent to finding $k_n$ which satisfy $d \ln j_l(k_r r) / d \ln |r|_{\text{max}} = -(l+1)$. The numbers $C_{lm}$ are fixed entirely by the boundary conditions and are given in Table A1 of Fisher et al. (1995). The density coefficients are

$$\delta_{lmn} = \int_{r < R_{\text{max}}} d^3 r j_l(k_r r) Y_{lm}(\hat{r})$$

(54)

Using the linear velocity-density relation one obtains (Regos & Szalay 1989; Scharf & Lahav 1993; Fisher et al. 1995),

$$u(r) = f \sum_{l,m,n} C_{lm} \delta_{lmn} \frac{j_l(k_r r)}{k_r} Y_{lm}(\hat{r})$$

(55)

where $j_l(z) = d j_l(z) / dz |_{z=k_r r}$. For velocities measured with respect to the Local Group frame the $l = 1$ term should be modified by subtracting $1/3$ from $j_1(k_r r)$. Using

$$\langle \delta_{1m1} \delta_{2m2} \rangle = P(k_n) C_{11}^{-1} \delta_{1+1} \delta_{2+1} \sum K_m$$

(56)

this representation for $u(r)$ yields

$$S(r_1, r_2) = f^2 \sum_{l,m,n} \frac{P(k_n)}{k_n} \frac{j_l(k_r r_1) j_l(k_r r_2) Y_{lm}(\hat{r}_1) Y_{lm}(\hat{r}_2)}{2 + 1 + P(k_n) \frac{4\pi}{k_n^2} j_l^2(k_r r_1) j_l^2(k_r r_2) P_l(\hat{r}_1 \cdot \hat{r}_2)}$$

(57)

where $P_l$ is Legendre polynomial of order $l$ and we have used $4\pi/(2l+1) \sum_m Y_{lm}(r_1) Y_{lm}(r_2) = P_l(\hat{r}_1 \cdot \hat{r}_2)$. From $S$, the cross correlation can easily be computed $P$. Using the expression (7) which is appropriate for for definition (2) we get

$$P_{l} = 3 f^2 n_l^* \sum_{n} \frac{P(k_n) j_l^2(k_r r_1)}{k_n^2} \int d r r^2 g(r) j_l^2(k_r r)$$

(58)

7 DISCUSSION

An aim of this contribution is to show that different approaches to the determination of $\mathbf{B}$ are tightly related. All methods yield something which mimics the behavior of the theoretical bulk flow. The non-trivial challenge is to contrast any estimate with predictions of cosmological model. Systematic errors related to observables of galaxy properties, contaminate various methods in different ways. More distant galaxies are more prone to these systematics, therefore, consistency checks must be performed by applying the methods subsamples of the data. Furthermore, spatial coverage of the data depends on galaxy-type and unavoidable cuts on the observables. Thus, methods should be tested on mock galaxy catalogs that mimic the observations in as much as possible, including the observed large scale structure. The constrained simulations (Sorce et al. 2013) designed to match the low redshift large scale structure, can potentially be highly beneficial if combined with galaxy formation models.

We have also emphasized the differences between the methods. Hoffman et al. (2015) provide an estimate for the bulk flow as defined in (1), while Feldman et al. (2010) aim at $\mathbf{B}$ given by (2). These two definitions of $\mathbf{B}$ do not coincide (Nusser 2014), and should not produce identical results. Nonetheless, both methods can easily be adapted to any of the two definitions.

In §3 we derive a new $\mathbf{B}$ estimator which naturally arises from the likelihood $P(\mathbf{d} | \mathbf{B})$. This is a generalization of the standard ML of Kaiser (1988), but incorporates velocity modes beyond the (constant) bulk flow of the entire data catalog. Essentially, this estimator is equivalent to marginalizing $P(\mathbf{d} | \mathbf{B}) / P(\mathbf{B})$ over all modes of a general velocity model, $\mathbf{v}_{\text{model}}$, excluding a mode corresponding a global bulk flow as defined in (2010). Another new estimator is given in §5.4. This estimator incorporates the Feldman et al. (2010) constraint in a new and simple way. But we emphasize that in a Bayesian approach, the prior as expressed in $\mathbf{S}$ and $\mathbf{P}$ correlation functions, already contains all missing information. Additional

2 The following integrals have been used $\int d\Omega \delta^2 Y_{lm} = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{3}{2} l(l+1)}$, $\int d\Omega \delta^2 Y_{l1} = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{3}{2} l}$, $\int d\Omega \delta^2 Y_{l-1} = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{3}{2} l}$, $\int d\Omega \delta^2 Y_{l+1} = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{3}{2} l}$ and $\int d\Omega \delta^2 Y_{l-1} = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{3}{2} l}$.
constraints (not based on extra information) are either statistically redundant or incompatible with the model prior. For example, requiring the estimator to produce \( B_0 = \text{const} \) for the input data \( d_i = \mathbf{r} \cdot B_0 \) (no observational errors), ignores variations in the velocity field on the scales that are not probed by the data.

We have refrained from making any quantitative assessment of the difference between methods and the bulk flow definitions. We hope to do that in the near future as well as comparison applying all methods to observational data, but it is not the point of the paper.

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REFERENCES

Aaronson M., et al., 1982, ApJS, 50, 241
Agarwal S., Feldman H. A., Watkins R., 2012, MNRAS, 424, 2667
Bertschinger E., 1987, ApJ, 323, L103
Davis M., Peebles P. J. E., 1983, The Astrophysical Journal, 267, 465
Feldman H. A., Watkins R., Hudson M. J., 2010, MNRAS, 407, 2328
Fisher K. B., Lahav O., Hoffman Y., Lynden-Bell D., Zaroubi S., 1995, MNRAS, 272, 885
Gorski K., 1988, ApJ, 332, L7
Hellwing W. A., Barreira A., Frenk C. S., Li B., Cole S., 2014, PRL, 112, 221102
Hoffman Y., Ribak E., 1991, ApJL, 380, L5
Hoffman Y., Courtois H. M., Tully R. B., 2015, MNRAS, 449, 4894
Kaiser N., 1988, MNRAS, 231, 149
Li M., et al., 2012, ApJ, 761, 151
Lynden-Bell D., Faber S. M., Burstein D., Davies R. L., Dressler A., Terlevich R. J., Wegner G., 1988, ApJ, 326, 19
Nusser A., 2014, ApJ, 795, 3
Nusser A., Davis M., 1995, MNRAS, 276, 1391
Nusser A., Davis M., 2011, ApJ, 736, 93
Peebles P. J. E., 1980, The large-scale structure of the universe. Princeton University Press, NJ, http://adsabs.harvard.edu/abs/1980lssu.book.....P
Rogos E., Szalay A. S., 1989, ApJ, 345, 627
Scharf C. A., Lahav O., 1993, MNRAS, 264
Sorce J. G., Courtois H. M., Gottlober S., Hoffman Y., Tully R. B., 2013, MNRAS, 437, 3586
Springob C. M., Masters K. L., Haynes M. P., Giovanelli R., Marinoni C., 2007, ApJS, 172, 599
Springob C. M., et al., 2014, MNRAS, 445, 2677
Strauss M. A., Willick J. A., 1995, Phys Rep, 261, 271
Tully R. B., et al., 2013, ApJ, 146, 86
Watkins R., Feldman H. A., 2015, MNRAS, 450, 1868
Zaroubi S., Hoffman Y., Fisher K. B., Lahav O., 1995, ApJ, 449, 446