POST-NEWTONIAN TREATMENT OF BAR MODE INSTABILITY IN RIGIDLY ROTATING EQUILIBRIUM CONFIGURATIONS FOR POLYTROPIC STARS

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ABSTRACT

In this paper we determine the onset point of secular instability for the nonaxisymmetric bar mode in rigidly rotating equilibrium configurations in the post-Newtonian approximation, in order to apply it to neutron stars. The treatment is based on a precedent Newtonian analytic energy variational method that we have extended to the post-Newtonian case. This method, based on Landau’s theory of second-order phase transitions, provides the critical value of the ellipsoid polar eccentricity $e$ at the onset of instability, i.e., at the bifurcation point from the axisymmetric Maclaurin to the triaxial Jacobi ellipsoids, and it is valid for any equation of state. The extension of this method to post-Newtonian fluid configurations has been accomplished by combining two earlier orthogonal works, specialized, respectively, to slow rotating configurations but with arbitrary density profile and to constant mass density but arbitrarily fast rotating ellipsoids. We also determine the explicit expressions for the density functionals that allow the generalization of the physical quantities involved in our treatment from the constant mass density to an arbitrary density profile form. We find that, considering homogeneous ellipsoids, the value of the critical eccentricity increases as the stars become more relativistic, in qualitative agreement with previous investigations but with a less marked amount of such an increase. Then we have studied the dependence of this critical value on the configuration equation of state. Considering polytropic matter distributions, we find that the increase in the eccentricity at the onset of instability with the star compactness is confirmed for softer equations of state (with respect to the incompressible case). The amount of this stabilizing effect is nearly independent of the polytropic index.

Subject headings: gravitation — instabilities — relativity — stars: interiors — stars: neutron — stars: rotation

1. INTRODUCTION

The determination of the onset point of instability for nonaxisymmetric modes in rapidly rotating equilibrium configurations is a classic problem. In particular, the $m = 2$, so-called bar mode has long been studied as a result of its relationship with the dissipation mechanisms of viscosity and gravitational radiation. In fact, there are a large number of astrophysical situations in which this instability may appear: the coalescence of a binary neutron star (NS) in a single, rapidly rotating object (Baumgarte et al. 1998); the core collapse in a massive, evolved star or the accretion-induced collapse of a white dwarf (Lai & Shapiro 1995); the coalescence of a white dwarf binary into a progenitor of Type Ia supernovae (Iben & Tutukov 1984; Yungelson et al. 1994) or of isolated millisecond pulsars (Chen & Leonard 1993); the accretion and spin-up of a NS in an X-ray binary system (Chandrasekhar 1970; Friedman & Schutz 1978); the implosion to a black hole of a supramassive neutron star (SMNS) after the spin-up phase (Salgado et al. 1994); and the nonexplosive core contraction of a rapidly rotating massive star (a “fizzler”; Hayashi, Eriguchi, & Hashimoto 1999). This applies not only to situations of collisional systems: the bar formation in self-gravitating collisionless galaxies in purely rotational equilibrium can also be studied as an application of this classic problem (Hohl 1971; Ostriker & Peebles 1973; Binney & Tremaine 1987).

In Newtonian theory a considerable amount of work has been done, and a number of results are well established. Chandrasekhar (1969b), using the tensor virial formalism, found an exact analytic solution for the equilibrium shape and stability of an incompressible, homogeneous, rigidly rotating fluid configuration. In this case, the equilibrium shape in axisymmetry is a Maclaurin spheroid. Nonaxisymmetric instabilities develop in spinning spheroids when the ratio $K/|W|$ of the rotational kinetic to the gravitational potential energy becomes sufficiently large. At the critical value $K/|W| \approx 0.173$, the equilibrium sequence of Maclaurin configurations bifurcates in two different branches of triaxial equilibria, the Jacobi and Dedekind ellipsoids. Since the Maclaurin spheroids are dynamically unstable only for $K/|W| > 0.2738$, the bifurcation point is dynamically stable. However, in the presence of a dissipative mechanism such as viscosity or gravitational radiation, this point becomes secularly unstable to the $m = 2$ bar mode.

After these findings, a number of efforts have been devoted to their extension to more realistic, compressible fluids. Modeling the fluid with a polytropic equation of state (EOS), it was found that for strictly rigid rotation bifurcation to triaxial configurations can exist only when the polytropic index is less than the critical value $n = 0.808$ (Jeans 1919, 1928; James 1964; Tassoul & Ostriker 1970). This is because of the dynamical constraint that the angular velocity at the bifurcation point must be lower than the limiting value at which the centrifugal force balances the gravitational force at the equator ("mass-
shedding’’ limit), and this happens for EOSs sufficiently stiff. Later Ipser & Managan (1985) demonstrated that, while in general the \( m = 2 \) Jacobi and Dedekind bifurcation points do not have the same location along axisymmetric rotating sequences, when considering polytropic axisymmetric sequences with uniform rotation it is found that these two bifurcation points have indeed the same location as in the incompressible case. Lai, Rasio, & Shapiro (1993) obtained the same result constructing triaxial ellipsoid models of rotating polytropic stars in Newtonian gravity and using then an ellipsoidal energy variational method. This approach was originally introduced by Zeldovich & Novikov (1971) for the axisymmetric case and is also illustrated in Shapiro & Teukolsky (1983). Generalizing this approach to the triaxial case, Lai et al. (1993) were able to construct equilibrium sequences for compressible analogs of most classical incompressible objects, such as the Maclaurin, Jacobi, and Dedekind ellipsoids. From these equilibrium and dynamical ellipsoid models, Lai & Shapiro (1995) confirmed that also in the compressible case the Maclaurin configurations bifurcate at \( K / |W| \approx 0.1375 \), independently of the polytropic index \( n \), and that a dynamical bar mode instability sets in at \( K / |W| \approx 0.2738 \). In this paper it is also pointed out that the limit \( n = 0.808 \) imposed by mass shedding in strictly uniformly rotating stars can be bypassed with a slight amount of differential rotation, which can in principle inhibit mass shedding without changing the global structure of the star. On the other hand, a number of authors have constructed numerical models of rotating equilibrium stars, using both polytropic (Bodenheimer & Ostriker 1973; Ostriker & Bodenheimer 1973; Managan 1985; Imamura, Friedman, & Durisen 1985; Ipser & Lindblom 1990) and more realistic EOSs for white dwarfs and NSs (e.g., Ostriker & Tassoul 1969; Durisen 1975; Hachisu 1986; Bonazzola, Frieden, & Gourgoulhon 1996).

An analytic result independent of the EOS has been obtained by Bertin & Radicati (1976, hereafter BR). Using Landau’s theory of second-order phase transitions (Landau & Lifshitz 1967), they found that the transition from the axisymmetric Maclaurin to the triaxial Jacobi sequence always corresponds to \( K / |W| \approx 0.1375 \), if one assumes that the density is constant over ellipsoids with constant eccentricities and that both the internal energy and the enthalpy are independent of the shape of the rotating fluid. A formal treatment of the correspondence between the second-order phase transition and the Maclaurin-Jacobi bifurcation is also presented in Hachisu & Eriguchi (1983).

All these results may be characteristic of Newtonian physics and an \( r^{-2} \) gravitational force. For NSs and relativistic objects, the analysis of equilibrium and stability must be based necessarily on general relativistic models. If the structure of rotating axisymmetric stars in general relativity has been investigated numerically by a number of authors (e.g., Cook, Shapiro, & Teukolsky 1992, 1994a, 1994b, 1996; Salgado et al. 1994), the first fully relativistic perturbation computations of non-axisymmetric instabilities have been published only recently by Stergioulas (1996) and Stergioulas & Friedman (1998), who focused on the instabilities driven by gravitational radiation. An earlier numerical investigation of the effects of relativity on the viscosity-driven bar mode instability was carried out by Bonazzola et al. (1996), and these results have been corroborated by a more detailed analysis (Bonazzola, Frieden, & Gourgoulhon 1998). Stergioulas & Friedman (1998) find that relativistic models are unstable to nonaxisymmetric modes for significantly smaller degrees of rotation than for corresponding Newtonian models. The destabilizing effect of relativity is most striking in the case of the \( m = 2 \) bar mode, which can become unstable even for soft polytropes of index \( n \leq 1.3 \), to be compared with the Newtonian critical value \( n = 0.808 \) reported before. This behavior is in agreement with the result of numerical investigations made by Yoshida & Eriguchi (1997) in the “relativistic Cowling approximation,” in which all metric perturbations are omitted. However, the results of Bonazzola et al. (1996), concerning the effects of general relativity on the viscosity-driven bar mode instability, seem to suggest the opposite effect. The critical polytropic index for the bar mode instability becomes lower, but only slightly, than the Newtonian value as the configuration becomes increasingly relativistic, reaching a value \( n \approx 0.71 \) for very relativistic objects. This behavior suggests that relativistic effects tend to stabilize the configurations. In noting this, Stergioulas & Friedman (1998) concluded that, in general relativity, the onset point of the viscosity-driven and the gravitational radiation-driven \( m = 2 \) modes may no longer coincide as they do in Newtonian theory and that the effect of relativity seems to be very different in the two cases. Yoshida et al. (2002) have recently found, again in the Cowling approximation, that a further destabilizing effect in relativistic configurations is due to differential rotation.

From the analytical point of view, however, the general relativistic treatment of a nonaxisymmetric instability is difficult. Apart from the problem of solving Einstein’s equations without any presupposed symmetry, in the relativistic regime emission of gravitational waves must be taken into account. However, a first step toward the evaluation of general relativistic effects can be made by examining the problem in the so-called post-Newtonian (PN) approximation, where gravitational radiation can be neglected. To obtain this level of approximation, the metric and the stress tensor are expanded as sums of terms of successively higher order in the expansion parameter \( c^{-2} \), where \( c \) is the velocity of light, while each equation is decomposed into a series of equations of successively higher order in \( c^{-2} \). The first PN correction will refer to the terms that are \( \mathcal{O}(c^{-2}) \) (i.e., \( \mathcal{O}(GM/(c^2R)) \), where \( M \) is the configuration mass, \( R \) a length scale of the problem, and \( G \) the gravitational constant) higher than the corresponding Newtonian terms in this expansion. Gravitational radiation enters only at the 2.5 PN level and higher.

PN effects on the equilibria of uniformly rotating, homogeneous objects have been investigated by Chandrasekhar (1965a, 1965b, 1967a, 1967b, 1967c, 1969a) using the tensor virial formalism, and even the post-post-Newtonian (PPN) corrections have been considered with this method (Chandrasekhar & Nutku 1969). In these works integral expressions for global conserved quantities are obtained, but no explicit formulae are given for the PN corrections to the rest mass, angular momentum, and binding energy. A simpler formalism has been proposed only recently by Shapiro & Zane (1998, hereafter SZ), who extended the Newtonian treatment of Lai et al. (1993) to PN gravity. Considering incompressible, rigidly rotating bodies, they were able to construct equilibrium sequences of constant rest mass deriving the analytic functional for the main global parameters characterizing a rotating configuration, and they provided for the first time analytic investigations of the location of the secular instability point in general relativity. Their result is that the value of the ratio \( K / |W| \), defined invariantly, at the onset of bar mode instability increases as the stars become more relativistic, i.e., increases with the compactness parameter \( GM/(c^2R) \), being \( K / |W| \approx 0.1375 \) only in the Newtonian limit \( GM/(c^2R) = 0 \). Since higher
degrees of rotation are required to trigger a viscosity-driven bar mode instability as the stars become more compact, the effect of general relativity is to weaken the instability, at least to PN order. This behavior, consistent with Bonazzola et al. (1996), but contrasting that found by Stergioulas & Friedman (1998), supported the suggestion that in general relativity nonaxisymmetric modes driven unstable by viscosity no longer coincide with those driven unstable by gravitational radiation.

In this paper we wish to investigate analytically the location of bar mode instability points in rotating equilibrium PN configurations for arbitrary EOSs. The paper is organized as follows. We begin with a brief review of the second-order phase transition method of BR in § 2, and then in § 3 we start extending this method to PN configurations, obtaining the expression for the PN total energy (eqs. [54]–[58]). In § 4 we determine the general expressions of the density functionals necessary to model any EOS (eqs. [62], [69], [77], [78], [82], [86], [91], [93], and [96]). In § 5 we give the complete treatment for the analytic determination of the onset point of bar mode instability, and in § 6 we finally evaluate this point for various EOSs. In § 7 we report a discussion of our findings and finally in § 8 the conclusions of our work.

2. THE NEWTONIAN TREATMENT OF BERTIN & RADICATI

As previously reported, the treatment made by BR of the nonaxisymmetric instability that leads from the axisymmetric Maclaurin to the triaxial Jacobi sequence is based on Landau’s theory of second-order phase transitions (Landau & Lifshitz 1967). Second-order phase transitions occur in crystals when, as the temperature decreases, the invariance group of the crystal suddenly reduces to one of its subgroups. In the phase with lower symmetry a new observable, the “order parameter” \( \xi \), which vanishes in the symmetrical phase, is necessary to describe the state of the system together with the thermodynamical variables such as the pressure \( P \) and the temperature \( T \). In general a second-order phase transition occurs along a line in the \((P, T)\)-plane that divides it in two regions corresponding to different symmetries: in region I and on the transition curve the order parameter \( \xi \) vanishes and we have higher symmetry, while in region II \( \xi > 0 \) and there is lower symmetry.

Consider now the total energy \( E \) as a function of the thermodynamical variables: entropy \( S \), volume \( V \), angular momentum \( J \), and of the order parameter \( \xi \). Expanding \( E \) in powers of \( \xi \) in the neighborhood of \( \xi = 0 \), one gets

\[
E = E_0(S, V, J) + \xi E_1(S, V, J) + \xi^2 E_2(S, V, J) + \xi^3 E_3(S, V, J) + \cdots .
\]

(1)

For the equilibrium of a physical configuration with such a total energy, it must be \( \partial E/\partial \xi = 0 \). Now there are two possible solutions: one with \( \xi = 0 \) and the other with \( \xi > 0 \). In the latter case, in which obviously \( E_1 = 0 \), if it is also \( E_3 = 0 \), the change in stability is defined by the condition

\[
E_2(S, V, J) = 0 .
\]

(2)

To discuss the symmetry breaking in the case of a self-gravitating fluid, BR considered that in the symmetrical phase the shape of the fluid is an ellipsoid with polar eccentricity \( 0 < e < 1 \) and that, as a result of the symmetry breaking induced by the instability, the system acquires an equatorial eccentricity \( \xi > 0 \). In this notation the Maclaurin sequence is thus characterized by \( \xi = 0 \), while along the Jacobi sequence we find the solutions with \( \xi > 0 \).

BR make the following general assumptions:

1. The internal energy and the enthalpy are independent of the shape of the rotating fluid.
2. The density is constant over ellipsoids with constant eccentricities (“ellipsoidal approximation”).

Both these assumptions are discussed in Zeldovich & Novikov (1971), and BR show that as a consequence the critical value of the ratio \( K/W \), or equivalently of the eccentricity \( e \), where the Jacobi sequence branches off from the axisymmetric Maclaurin sequence, is given by the transition of \( E_2 \) from positive to negative sign, i.e., by equation (2). This result is not restricted to a particular EOS.

In order to determine this critical value, BR write the total energy as the sum of the gravitational, rotational, and internal energies: \( E = W + K + U \). Assumptions 1 and 2 imply that the internal energy \( U \) is independent of the shape while the gravitational and rotational energies are\(^2\)

\[
W = -\frac{3}{5} \left(\frac{4\pi}{3}\right)^{1/3} \frac{GM^2}{V^{1/3}} \beta [\rho] \phi (e, \xi) ,
\]

(3)

\[
K = \frac{5}{4} \left(\frac{4\pi}{3}\right)^{2/3} \frac{J^2}{MV^{2/3}} \gamma [\rho] f (e, \xi) ,
\]

(4)

thus depending on the fluid shape. Here \( M \) is the mass and \( \gamma [\rho] , \beta [\rho] \) are functionals of the density \( \rho \) that reduce to 1 for

\(^2\) In BR’s original expressions for \( W \) and \( K \) the numerical factors were omitted. Here we restore them.
constant \( \rho \). The functions \( f \) and \( g \) are

\[
\begin{align*}
 f(e, \xi) &= \frac{2(1 - e^2)^{1/3} (1 - \xi)^{1/3}}{2 - \xi}, \\
 g(e, \xi) &= \frac{(1 - e^2)^{1/6}(1 - \xi)^{1/6}}{e} \int_0^{\text{arcsin} e} \left( 1 - \frac{\xi}{e^2} \sin^2 x \right)^{-1/2} dx .
\end{align*}
\]

Minimizing the total energy \( E \) with respect to the volume \( V \) (scalar virial equation), they obtain expressions for the first few terms in equation (1) for the total energy in the neighborhood of \( \xi = 0 \), as a function of the eccentricities \( e, \xi \). By factoring out the part of \( W \) independent of \( e, \xi \) (and calling it \( W_0 \)) and using the notation \( f_\xi = \partial f / \partial x \), the result is

\[
\frac{E_1}{W_0} = \frac{1}{2} (g_\xi + g_\Sigma),
\]

\[
\frac{E_2}{W_0} = \frac{1}{4} \left( g_{\xi\xi} + 2 g_{\xi\xi} + \Sigma^2 g_{\xi\xi} + g_\xi \Sigma + \Sigma g_\Sigma \right),
\]

where

\[
\Sigma = \left( \frac{\partial e}{\partial e} \right)_{S, V, J}
\]

(the variables that appear in the subscript of the latter definition are considered constant). All the derivatives must be calculated at \( \xi = 0 \).

Requiring then the validity of the two equilibrium conditions for the total energy of the ellipsoidal configurations, \( \partial E / \partial e = 0 \) and \( \partial E / \partial \xi = 0 \) at \( e \neq 0, \xi = 0 \), BR first verify that \( E_1 = 0 \) and then calculate the value of \( e \) for which the condition \( E_2 = 0 \) is satisfied. This value \( (e_c = 0.81267) \) corresponds to a critical ratio \( K/|W| = 0.13752 \).

3. THE TOTAL ENERGY IN THE PN APPROXIMATION

One of the goals of this paper is to push to PN order BR’s version of Zeldovich & Novikov’s (1971) energy variational method, which we briefly described in the previous section. Like Zeldovich and Novikov, and every author since then, we shall make the assumption that the density is constant on ellipsoids of fixed eccentricities \( e, \xi \) (we follow BR’s choice of symbols). We thus need to find explicit expressions for the PN corrections to both the kinetic and the potential energy of a fluid configuration.

3.1. Bisnovaty-Kogan & Ruzmaikin’s Work

In their paper, BKR investigate the stability of rotating supermassive stars (SMSs), i.e., those with \( M \geq 10^5 \, M_\odot \), by adding to the usual expressions for the full mass energy, rest mass, and angular momentum the deviations arising from the first- and second-order PN corrections of general relativity in stationary rotating configurations. However, in this particular approximate energy variational method, slowly rotating (Harte 1967) SMSs are considered, and thus the kinetic energy terms due to rotation appear as corrections of the same order of the first PN corrections to the gravitational and internal energies. Similarly, the first PN corrections to the rotational kinetic energy result of the same order as the PPN corrections to the gravitational and internal energies.

BKR start by choosing a metric that allows them to integrate the field equations more easily and to derive from the latter expressions for the total mass energy and angular momentum of a stationary rotating configuration. In a spherical coordinate system \( R, \theta, \phi \) the element of four-space of this metric takes the form

\[
ds^2 = e^\nu (c \, dt - g R^2 \sin^2 \theta \, d\phi)^2 - e^\lambda (dR^2 + R^2 \, d\theta^2 + e^\mu R^2 \sin^2 \theta \, d\phi^2),
\]

where the independent functions \( \nu, \lambda, \mu, g \) are chosen to satisfy Einstein equations. When \( g, \mu \to 0 \), this metric reduces to the spherically symmetric isotropic one (Landau & Lifshitz 1971). In this case \( R \) corresponds to the so-called isotropic coordinate.

Moving to the next point, BKR expand the chosen metric with a power series in \( e^{-2} \) and calculate the total mass energy and angular momentum in the PPN approximation. Then, in order to calculate the first- and second-order corrections to the Newtonian total energy, they move from the coordinate \( R \) to the Newtonian radius \( r \). For this transformation, BKR use the relationship

\[
R = r \left( 1 - \frac{d_1}{c^2} - \frac{d_2}{c^4} \right),
\]
where $d_1$ and $d_2$ are functions of the mass $m$ and radius $r$. In their work, BKR report the following expressions of $d_1$ and $d_2$ correct to the PPN order:

$$d_1 = G \left( \frac{m}{r} + \int_0^r \frac{dm}{r^3} + \frac{1}{r^2} \int_0^r \frac{mrdr}{r} \right) , \quad (12)$$

$$d_2 = G^2 \left[ \frac{m^2}{2r^2} - \frac{m}{r} \int_0^r \frac{dm}{r^2} - \frac{1}{2} \left( \int_0^r \frac{dm}{r} \right)^2 - \frac{m}{r^2} \int_0^r \frac{mrdr}{r^3} + \frac{5}{4r^3} \int_0^r \frac{m^2dr}{r^2} - \frac{3}{2r^2} \int_0^r \frac{mdm}{r^2} + \frac{1}{2} \int_0^r \frac{mdm}{r^2} + \frac{1}{2} \int_0^r \frac{mrdr}{r^3} + \frac{1}{r^4} \int_0^r \frac{mrdr}{r^5} + \frac{1}{r^7} \int_0^r \frac{mrdr}{r^9} \right] + \frac{1}{r^3} \int_0^r \frac{mrdr}{r^5} + G \left( \frac{3}{2r^3} \int_0^r \frac{udm}{r^2} - \frac{1}{2r^3} \int_0^r u \frac{mrdr}{r^4} + \frac{c^2}{3r^5} \int_0^r \frac{\Omega^2 r^4 dr}{r} \right) , \quad (13)$$

where $r_0$ is the Newtonian radius of the sphere, $u$ is the internal energy per unit mass, and $\Omega$ is the angular velocity.

A by-product resulting from the equations given in BKR, which we will use in the following sections of the paper, is the expression of the rest mass of the sphere $M_0$ in terms of coordinate $R$. At PN order, this is

$$M_0 \approx \int_0^{R_0} dm + \frac{3G}{c^2} \left( \int_0^{R_0} \frac{mdm}{R} + \int_0^{R_0} \frac{dm}{R} \right) + \frac{1}{3} \int_0^{R_0} \frac{\Omega^2 R^2 dm}{R} . \quad (14)$$

After the transformation of coordinates, BKR calculate the PN and PPN corrections $E_I$ and $E_{II}$ to the Newtonian energy $E_N$. The results are the following:³

$$E_I = - \frac{G^2}{c^2} \left( \frac{1}{2} \int \frac{m^2 dm}{r^2} - \int \frac{dm}{r} \int m dm + \int \frac{m^2 dm}{r^4} \int mrdr \right) - \frac{G}{c^2} \left( \int \frac{u \frac{dm}{r} + \int \frac{dm}{r} \int u dm}{r} \right) + \frac{1}{3} \int \frac{\Omega^2 r^2 dm}{r} , \quad (15)$$

$$E_{II} = - \frac{G^3}{c^4} \left[ \frac{3}{4} \int \frac{m^3 dm}{r^3} - \frac{3}{2} \int \frac{m^2 dm}{r^2} \int m dm - \frac{1}{2} \int \frac{dm}{r} \int \frac{m^2 dm}{r^4} \int mrdr \right]
+ \frac{5}{4} \int \frac{m^2 dm}{r^4} \int \frac{m^2 dm}{r^2} \int mrdr - \int \frac{dm}{r} \int \frac{m^2 dm}{r^4} \int mrdr
- \int \frac{dm}{r} \int \frac{u \frac{dm}{r} + \int \frac{dm}{r} \int u dm}{r} + \frac{1}{2} \int \frac{u \frac{m^2 dm}{r^2} + \int \frac{dm}{r} \int u dm}{r} - \frac{G}{c^2} \int \frac{u \frac{dm}{r} + \int u dm}{r}$$

$$\frac{1}{2} \int \frac{u \frac{dm}{r} + \int \frac{dm}{r} \int u dm}{r} - \frac{G}{c^2} \left( \frac{4}{9} \frac{1}{r^3} \left( \int \Omega r^2 dm \right)^2 - \frac{8}{9} \frac{\Omega}{r^3} \int \Omega r^2 dm + \frac{1}{3} \int \frac{\Omega^2 mr dm}{R} \right)
+ \int \frac{dm}{r} \int \frac{\Omega^2 r^2 dm}{R} - \frac{2}{3} \int \frac{\Omega^2 \frac{dm}{r} + \int \frac{dm}{r} \int \Omega^2 r^4 dr}{R}
+ \frac{1}{3} \int \frac{dm}{r} \int \frac{\Omega^2 r^2 dm}{R} - \frac{1}{3} \int \frac{\Omega^2 r^2 dm}{R} \right] + \frac{1}{3} \int \frac{u \frac{dm}{r} + \int \frac{dm}{r} \int \Omega^2 r^2 dm}{R} , \quad (16)$$

where $P$ is the pressure and $\rho$ the density profile of the rest mass. In these expressions, integration is carried out over the whole mass of the star, and the limits of the interior integration go from the center to the actual $m$ or $r$.

BKR consider also a correction $E_{ob}$ caused by the stellar oblateness:

$$E_{ob} = \frac{\alpha^2}{5} \frac{G}{c^2} \left( \int_0^{\rho_0} \frac{dm}{r} + \int_0^{\rho_0} \frac{dm}{r} \int \frac{mrdr}{r} \right) + \int_0^{\rho_0} \frac{dm}{r} \int \frac{\rho d\rho}{\rho} \frac{dm}{r} - \frac{\alpha}{15} \int_0^{\rho_0} \frac{\Omega^2 r^2 dm}{R} , \quad (17)$$

where the value of $\alpha$ defines the degree of oblateness. About this value, BKR just note that it is of the same order of the ratio $2GM/(c^2 R)$ between the Schwarzschild radius and the physical radius $R$ of the star. However, since $\alpha$ gives a measure of the stellar oblateness and the oblateness is caused by the rotation of the star, such a parameter must be also related to $\Omega$. Therefore, in the case of slow rotation treated by BKR, the correction $E_{ob}$ is of the PPN order.

³ In BKR’s original expression for $E_{II}$ we have found a few typos. We report here the corrected form.
Finally, BKR find the PN-corrected expression of the angular momentum \( J \) specialized to the case of constant \( \Omega \):

\[
J = \frac{2}{3} \Omega \int_0^r r^2 \, dm + \frac{2}{3} \Omega \int_0^r \left( u + \frac{P}{\rho} \right) r^2 \, dm - \frac{2G}{3} \int_0^r m r \, dm + \frac{4G}{9} \int_0^r \frac{dm}{r} \int_0^r m r \, dr .
\]  
(18)

### 3.2. Shapiro \\& Zane's Work

As we already briefly mentioned in § 1, SZ construct analytic models of incompressible, uniformly rotating stars in PN gravity in order to evaluate their stability against nonaxisymmetric bar modes. For this, an energy variational principle is employed, its equations being exact at PN order. Contrary to BKR's work, this analysis is not restricted to slow rotation, whereby one requires \( R^3/(GM) \ll 1 \), but arbitrarily fast rotation is allowed, so that \( \Omega^2 \) is permitted to reach \( \sim (GM/R^3) \) and stars can suffer considerable rotational distortion. However, this particular energy variational method is valid only for constant density \( \rho_0 \). The latter condition implies that the internal energy vanishes and the Newtonian total energy is just given by

\[
E = W + K .
\]  
(19)

In the choice of the metric, SZ adopt a \( 3 + 1 \) ADM split form (Arnowitt, Deser, \\& Misner 1962) to solve Einstein's equations of general relativity. A subset of these equations is revealed to be well suited to numerical integration in the case of strong-field, three-dimensional configurations in quasi-equilibrium. Moreover, the adopted equations are exact at PN order, where they admit an analytic solution for homogeneous ellipsoids. The most general expression for this kind of metric is

\[
ds^2 = -\alpha^2 \, dt^2 + \gamma_{ij}(dx^i + \beta^i \, dt)(dx^j + \beta^j \, dt) ,
\]  
(20)

where \( \alpha \) and \( \beta^i \) are the lapse and shift functions, respectively. Then SZ choose a “conformally flat” decomposition of the spatial metric:

\[
\gamma_{ij} = \Psi^4 f_{ij} ,
\]  
(21)

where \( \Psi \) is the “conformal factor” and \( f_{ij} \) is the Euclidean metric in the adopted coordinate system. SZ use Cartesian coordinates \( x_i \) (i = 1, 2, 3).

At this point, SZ expand the metric and the stress tensor in terms of \( c^{-2} \) and decompose each ADM equation into a series of equations of different order in \( c^{-2} \). To work in the PN approximation, they retain only the Newtonian terms and those that are \( \mathcal{O}(c^{-2}) \) higher. After such an expansion, they evaluate the conserved quantities of total mass energy \( M \), total rest mass \( M_0 \), and angular momentum \( J \), first in the integral form and then performing the quadratures over the fluid volume, adopting constant density triaxial ellipsoids with semiaxes of the outer surface specified by the values \( a_i (i = 1, 2, 3) \).

Since also the energy of the fluid \( E = (M - M_0)c^2 \) is a conserved quantity, they explicitly report it in the integrated form. For our later applications, we write down here their resulting expressions for \( E \) and \( J \), reintroducing the gravitational constant \( G \) and the velocity of light \( c \) (SZ use geometrized units with \( c = G = 1 \) throughout their paper):

\[
E \approx -\frac{3}{5} \frac{G M_e^2}{R} f + \frac{1}{5} \frac{M_e \Omega^2 R^2}{h} \frac{1}{h} + \frac{G^2 M_e^3}{c^2 R^2} \frac{G M_e^2}{c^2 R} \frac{\Omega^2 R^2 p_1}{h} ,
\]  
(22)

\[
J \approx \Omega M_e R^2 \frac{2}{5h} \left( 1 + \frac{5 GM_e}{2 c^2 R} p_3 h \right) ,
\]  
(23)

where \( f, h, g_i, p_i \) are functions of the ellipsoid axial ratios,

\[
\lambda_1 = \left( \frac{a_2}{a_1} \right)^{2/3} , \quad \lambda_2 = \left( \frac{a_3}{a_2} \right)^{2/3} ,
\]  
(24)

while \( \Omega \) is the angular velocity of the fluid system and

\[
M_e \equiv \int_\nu \rho_0 \, d^3 x = \frac{4\pi}{3} \rho_0 a_1 a_2 a_3 = \frac{4\pi}{3} \rho_0 R^3
\]  
(25)

is a function of the “conformal radial coordinate” \( R \). From the latter definition it is possible to note that \( R \) represents the radius of the spherical configuration with the same volume as the rotating one, and thus it can be considered as a “mean radius.” The PN relationship between the coordinate quantity \( M_e \) and the total baryon rest mass \( M_0 \) is also given:

\[
M_0 \approx M_e + \frac{18}{5} \frac{GM_e^2}{c^2 R} f + \frac{5}{4 c^2 M_e R^2} J^2 .
\]  
(26)

The structure of equation (22) is particularly convenient for performing the required energy variational method, since the full dependence on the two axial ratios is contained in \( f, h \), and, for the PN contributions, \( g_i \) and \( p_i \). They have also checked that
their result agrees with the PN correction to Newtonian energy obtained by Shapiro & Teukolsky (1983) for nonrotating, homogeneous spheres (see § B2 in Appendix B of their paper). For the latter configurations, \( R \) corresponds to the conformal isotropic coordinate.

In order to construct sequences of axisymmetric equilibrium models, SZ first rewrite the energy \( E \) as a function of \( J \), using the relationship
\[
\Omega^2 R^2 \approx \frac{J^2}{M^2 R^2} \frac{25h^2}{4} \left( 1 + \frac{5GM}{2c^2 R^2} \rho_3 h \right) ^2 \approx \frac{J^2}{M^2 R^2} \frac{25h^2}{4} \left( 1 - \frac{5GM}{2c^2 R^2} \rho_3 h \right),
\]
which derives from equation (23), combining it with equation (22). At PN order, they obtain
\[
E \approx - \frac{3}{5} \frac{GM^2}{R} f + \frac{J^2}{4M^2 R^2} h + \frac{G^2 M^4}{c^2 R^2} g_{12} + \frac{25 GM^2}{4c^2 R^2} h^2 p_{123},
\]
where \( p_{123} = p_{12} - p_3 \). Then the equilibrium sequence is determined by minimizing \( E \) with respect to \( \lambda_1 \) and \( \lambda_2 \) holding constant \( M_0 \) and \( J \).

3.3. Energy Functional for Arbitrary Configurations

Now we will combine the orthogonal sets of results presented in the two previous subsections, so as to obtain the general expressions of PN corrections to both the kinetic and the potential energy of a fluid configuration with arbitrary density profile and arbitrary eccentricities \( e, \xi \). We do it as follows.

Considering the kinetic energy, we write it as
\[
K_{\text{TOT}} = K + K_{\text{corr}},
\]
where \( K_{\text{corr}} \) is the correction to the Newtonian kinetic energy, complete to all orders, not just the PN one. For obvious dimensional reasons, we must have
\[
K_{\text{corr}} = \frac{J^2}{MV^{2/3}} p(L[\rho], e, \xi),
\]
where \( J, M, \) and \( V \) are the model angular momentum, mass, and volume, respectively, and \( L[\rho] \) is an adimensional functional (i.e., an application that associates a number to every function) of the density profile \( \rho(r) \). We now rewrite \( L[\rho] \) as
\[
L[\rho] = L[\rho_0] \frac{L[\rho]}{L[\rho_0]},
\]
where \( \rho_0 \) is the density of the constant mass density model with the same total mass, volume, and shape as the stratified model. Obviously, again for dimensional reasons
\[
L[\rho_0] = q \left( \frac{GM}{c^2 V^{1/3}} \right).
\]
The argument of \( q \) is obviously the ratio of the Schwarzschild radius to the physical radius. We now introduce the fact that we are only interested in PN corrections. Clearly, we need to introduce the hypothesis that, as \( GM/(c^2 V^{1/3}) \to 0 \), both functions \( \rho(x) \) and \( q(x) \) are analytic at \( x = 0 \). In other words, we are assuming that it makes sense to expand general relativity terms in powers of \( GM/(c^2 V^{1/3}) \), which seems innocuous enough. In this way, we find
\[
K_{\text{corr}} = \frac{J^2}{MV^{2/3}} \frac{GM}{c^2 V^{1/3}} L[\rho] L[\rho_0] \left( I(e, \xi) + \mathcal{O} \left( \frac{GM}{c^2 V^{1/3}} \right)^2 \right),
\]
where \( I(e, \xi) \) is a function of the model shape only. The first, explicit term is the PN correction to the kinetic energy, for arbitrary shape and density profile. However, by specializing to the case \( \rho = \text{const} \), we now see that \( I(e, \xi) \) must be exactly the function determined by SZ, just rewritten in terms of the polar and equatorial eccentricities \( e \) and \( \xi \) via the relationships
\[
\lambda_1 = (1 - e^2)^{1/3}, \quad \lambda_2 = \left( \frac{1 - e^2 \xi}{1 - \xi} \right)^{1/3}.
\]
In addition, when the model is spherical, \( L[\rho]/L[\rho_0] \) must be the functional determined by BKR’s work. In summary, we take for the PN correction to the kinetic energy
\[
\Delta K = \frac{GJ^2}{c^2 V} \left( \frac{L[\rho]}{L[\rho_0]} \right)_{\text{BKR}} (I(e, \xi))_{\text{SZ}}.
\]
An entirely similar argument yields \( \Delta W \), the PN correction to the potential energy. In this case, we have
\[
W_{\text{TOT}} = W + W_{\text{corr}},
\]
where the correction $W_{\text{corr}}$, complete to all orders, is

$$W_{\text{corr}} = \frac{GM^2}{V^{1/3}} p(L|\rho, e, \xi). \quad (37)$$

Expansion of the general relativity terms in powers of $GM/(c^2 V^{1/3})$ gives

$$W_{\text{corr}} = \frac{GM^2}{V^{1/3}} \frac{GM}{c^2 V^{1/3}} \frac{L|\rho}{L|\rho_0} h(e, \xi) + \mathcal{O}\left(\frac{GM}{c^2 V^{1/3}}\right)^2,$$  

where $h(e, \xi)$ is the eccentricity-transformed shape function $\theta_{12}$ that appears in SZ’s PN correction to the potential energy. In summary, we can write

$$\Delta W = \frac{G^2 M^3}{c^2 V^{2/3}} \left(\frac{L|\rho}{L|\rho_0}\right)_{\text{BKR}} (h(e, \xi))_{\text{SZ}}. \quad (39)$$

However, there is still an obstacle to the straightforward determination of the explicit expressions for the PN corrections $\Delta W$ and $\Delta K$. In fact, from the previous discussion of BKR’s and SZ’s earlier works, it comes out that their PN corrections are expressed in different radial coordinates: Newtonian radius $R_N$ in BKR and conformal radius $R_c$ in SZ. Thus, in combining the results of these two papers, we must be careful with this difference.

We start by rewriting BKR’s PN equation (18) for the angular momentum in the case of rigid rotation as

$$J_{\text{BKR}} = \frac{2}{5} M \Omega R^2 \sigma [\rho] - \frac{34}{315} \frac{GM}{c^2 R_N} M \Omega R^2 \tau [\rho] = \frac{2}{5} M \Omega R^2 \sigma [\rho] \left(1 - \frac{17 GM}{63 c^2 R_N} \frac{\tau [\rho]}{\sigma [\rho]} \right). \quad (40)$$

This form is similar to that adopted by BR for the expressions of the gravitational and rotational energies $W$ and $K$ (see eqs. [3] and [4]), and in a similar way $\sigma [\rho]$ and $\tau [\rho]$ are functionals of the density profile that reduce to 1 for constant $\rho$. In § 4, when we will determine the explicit form of all the density functionals introduced in this section, the origin of the factors that multiply each of them will become clearer.

Then we consider the resulting equations (15) and (16) of BKR for the PN corrections to the Newtonian total energy, omitting those related to the internal energy that vanish in SZ’s case of constant matter density distribution. Keeping in mind that for slow rotation the kinetic energy enters only as a correction, as already noted in § 3.1, we take the terms up to the $\mathcal{O}(c^{-2})$ order and write this total energy in the form, valid for uniform rotation,

$$E_{\text{BKR}} = -\frac{3}{5} \frac{GM^2}{R_N} \beta [\rho] + \frac{5}{1} \frac{GM^3}{MR_N^2 \sigma [\rho]} - \frac{3}{70} \frac{GM^3}{c^2 R_N^2} \delta [\rho] + \frac{23}{175} \frac{GM^2}{c^2 R_N} \Omega^2 R_N^2 \alpha [\rho]. \quad (41)$$

where we have introduced other functionals of the density $\rho$. In particular, this functional $\beta [\rho]$ is exactly the same as that appearing in equation (3).

Now we exploit equation (40) to find

$$\Omega = \frac{5}{2} \frac{J_{\text{BKR}}}{MR_N^2 \sigma [\rho]} \left(1 - \frac{17 GM}{63 c^2 R_N} \frac{\tau [\rho]}{\sigma [\rho]} \right)^{-1} \quad (42)$$

and therefore the PN-approximated relationship

$$\Omega^2 R_N^2 \approx \frac{25}{4} \frac{J_{\text{BKR}}^2}{MR_N^2 \sigma^2 [\rho]} \left(1 + \frac{34 GM}{63 c^2 R_N} \frac{\tau [\rho]}{\sigma [\rho]} \right). \quad (43)$$

Introducing the latter relationship in equation (41), we can rewrite the PN total energy in terms of angular momentum $J$ instead of constant angular velocity $\Omega$. We obtain

$$E_{\text{BKR}} = -\frac{3}{5} \frac{GM^2}{R_N} (\beta [\rho] + \frac{5}{1} \frac{J_{\text{BKR}}^2}{MR_N^2 \sigma^2 [\rho]} - \frac{3}{70} \frac{GM^3}{c^2 R_N^2} \delta [\rho] + \frac{1}{14} \frac{GJ_{\text{BKR}}}{c^2 R_N^3} \frac{1}{\sigma^2 [\rho]} \left(85 \frac{\gamma [\rho]}{\sigma^2 [\rho]} + \frac{23}{2} \alpha [\rho] \right). \quad (44)$$

Since it is possible to write $M = (4\pi/3)\rho_0 R_N^3$, indicating with $\rho_0$ the mean mass density over the whole configuration, we can reduce the latter expression to an equation dependent only on the Newtonian radius $R_N$ and the angular momentum. We get

$$E_{\text{BKR}} = -\frac{3}{5} \frac{(4\pi/3)^2}{G \rho_0^2 R_N^3} \beta [\rho] + \frac{5}{4} \frac{3}{4\pi} \frac{J_{\text{BKR}}^2}{\rho_0 R_N^2 \sigma^2 [\rho]} - \frac{3}{70} \frac{4\pi}{3} \frac{GM^3}{c^2 \rho_0^3 R_N^3} \delta [\rho] + \frac{1}{14} \frac{GJ_{\text{BKR}}}{c^2 R_N^3} \frac{1}{\sigma^2 [\rho]} \left(85 \frac{\gamma [\rho]}{\sigma^2 [\rho]} + \frac{23}{2} \alpha [\rho] \right). \quad (45)$$

At this point, since we are operating in the slow rotation regime, we can use BKR’s equation (11) between the Newtonian and the conformal (quasi-isotropic) radial coordinates. After a proper treatment of equations (12) and (13) for the
“transformation functions” $d_1$ and $d_2$, evaluated at the star boundary, we can write

$$R_e = R_N \left( 1 - \frac{6 GM}{5 c^2 R_N} \mu[\rho] - \frac{1}{15} \frac{\Omega^2 R_N^2}{c^2} \nu[\rho] \right),$$

where the other two density functionals, $\mu[\rho]$ and $\nu[\rho]$, have been introduced. Exploiting again equation (43), we find the same relationship as a function of the angular momentum:

$$R_e \approx R_N \left( 1 - \frac{6 GM}{5 c^2 R_N} \mu[\rho] - \frac{5}{12} \frac{J^2_{BKR}}{c^2 M^2 R_N^2 \sigma^2[\rho]} \nu[\rho] \right).$$

(47)

In order to move to SZ’s radial coordinate $R_e$, we invert the latter relationship, obtaining, at PN order,

$$R_N \approx R_e \left( 1 - \frac{6 GM}{5 c^2 R_e} \mu[\rho] + \frac{5}{12} \frac{J^2_{SZ}}{c^2 M^2 R_e^2 \sigma^2[\rho]} \nu[\rho] \right).$$

(48)

Substituting equation (48) into equation (45), we obtain the total energy in terms of the coordinates adopted by SZ. At PN order,

$$E_{SZ} \approx - \frac{3}{5} \left( \frac{4\pi}{3} \right)^2 \frac{(4\pi/3)^2}{G} \rho_0^2 R_e^2 \beta[\rho] \left( 1 + \frac{6 GM}{5 c^2 R_e} \mu[\rho] + \frac{5}{12} \frac{J^2_{SZ}}{c^2 M^2 R_e^2 \sigma^2[\rho]} \nu[\rho] \right)^5 + \frac{5}{4} \left( \frac{3}{4\pi} \right)^2 \frac{J^2_{SZ}}{\rho_0 R_e^2 \sigma^2[\rho]} \gamma[\rho] \left( 1 + \frac{6 GM}{5 c^2 R_e} \mu[\rho] + \frac{5}{12} \frac{J^2_{SZ}}{c^2 M^2 R_e^2 \sigma^2[\rho]} \nu[\rho] \right)^{-5} - \frac{3}{70} \left( \frac{4\pi}{3} \right)^3 \frac{G^2}{c^2} \rho_0^3 R_e^3 \delta[\rho] + \frac{1}{14} \frac{1}{c^2 R_e^2} \frac{GM}{\sigma^2[\rho]} \left( \frac{85 \gamma[\rho] \tau[\rho]}{9 \sigma[\rho]} + \frac{23}{2} \alpha[\rho] \right),$$

(49)

and after expansion in the PN correction terms, and since now $M = (4\pi/3)\rho_0 R_e^3$ (see eq. [25]), we find

$$E_{SZ} = - \frac{3}{5} \frac{G^2 M^3}{c^2 R_e^2} \beta[\rho] \left( 1 + \frac{6 GM}{5 c^2 R_e} \mu[\rho] + \frac{5}{12} \frac{J^2_{SZ}}{c^2 M^2 R_e^2 \sigma^2[\rho]} \nu[\rho] \right) + \frac{1}{14} \delta[\rho] \left( \frac{6 \beta[\rho] \mu[\rho]}{63} \frac{85 \gamma[\rho] \tau[\rho]}{\sigma[\rho]} - \frac{5}{2} \beta[\rho] \nu[\rho] - 15 \gamma[\rho] \mu[\rho] + \frac{23}{14} \alpha[\rho] \right).$$

(50)

Finally, in order to generalize this expression of the total energy to arbitrary fast rotation, and thus arbitrary eccentricities $e, \xi$, we introduce SZ’s shape functions in such a way that equation (50) is just the limit for $e, \xi \to 0$. The result is

$$E_{SZ} = - \frac{3}{5} \frac{G^2 M^3}{c^2 R_e^2} \beta[\rho] g(e, \xi) + \frac{5}{14} \frac{J^2_{SZ}}{c^2 M^2 R_e^2 \sigma^2[\rho]} f(e, \xi) + \frac{14}{85} \frac{G^2 M^3}{c^2 R_e^2} \left( \frac{6 \beta[\rho] \mu[\rho]}{63} \frac{85 \gamma[\rho] \tau[\rho]}{\sigma[\rho]} - \frac{5}{2} \beta[\rho] \nu[\rho] - 15 \gamma[\rho] \mu[\rho] + \frac{23}{14} \alpha[\rho] \right) h(e, \xi),$$

(51)

where we have rewritten SZ’s functions $f$ and $h$ in terms of $e, \xi$ and named them $g(e, \xi)$ and $f(e, \xi)$ since they are exactly the same as those defined in BR and reported in equations (5) and (6). For what concerns the shape functions $h(e, \xi)$ and $I(e, \xi)$, whose origin we have already discussed above, we present here their full expressions:

$$h(e, \xi) = - \frac{27}{28} \left[ \frac{A_1}{(1 - e^2)^{3/2}} + \frac{A_2}{(1 - e^2)^{1/2}} + A_3 \frac{(1 - e^2)^{3/2}}{(1 - \xi)^{1/3}} \right] + \frac{9}{56} \left[ A_1 A_2 (1 - \xi)^{1/3} + A_1 A_3 (1 - e^2)^{3/2} + A_2 A_3 (1 - \xi)^{1/3} (1 - e^2)^{1/3} \right],$$

(52)

$$I(e, \xi) = - \left[ \frac{39}{28 g} + \frac{3}{40} \frac{(1 - e^2)^{3/2}}{(1 - \xi)^{1/3}} A_1 \right] + f \left[ \left( \frac{33}{280} \frac{(1 - e^2)^{1/3}}{(1 - \xi)^{1/3}} (A_1 + A_2) \right) \right].$$

(53)

The dimensionless coefficients $A_i$ are given in Chandrasekhar (1969b); for their calculation in terms of standard incomplete elliptic integrals involving only the eccentricities $e$ and $\xi$, see Appendix A. The integral $f$ is that reported in Appendix C of SZ. To transform it into a function of $e, \xi$, we used the relationships given by equation (34) between the axial ratios $\lambda_1, \lambda_2$ and the two configuration eccentricities.

The works of BKR and SZ have a common domain of validity, that of slowly rotating, constant density models, for which all functionals reduce to 1, and the shape functions are to be computed for $e, \xi \to 0$. The numerical factor for the
PN correction to the kinetic energy in BKR (eq. [51]) differs from that of SZ (eq. [28]): to wit, \(-457/63\) versus \(-67/7\), respectively. The same disagreement is to be found in the PN corrections to the angular momentum (from eq. [112] for BKR and eq. [23] for SZ we find 1342/1575 and 82/35, respectively). All other terms, instead, coincide. Faced with this dilemma, we remark that the derivation of the coefficients in SZ is buried in heavy computations, most of which are not reported, and which we could thus not check. The computations of BKR, instead, are fully detailed and have allowed us an independent derivation and a step-by-step comparison, from which we deduced that their work is surely correct. At the same time, the computations by SZ satisfy identically equations (106)–(108), which are the obvious PN generalization of the equilibrium conditions in the Newtonian regime, for \(\xi \to 0\). It thus appears that SZ’s equations have the correct low triaxiality limit, except for an overall factor. We have thus decided to adopt SZ’s shape functions, but not the overall factor of the PN terms, which we take instead from BKR’s work. This surely leads to the correct first-order terms, in the limit \(\xi \to 0\), as discussed above. We could not check independently the next term in \(\xi\) of SZ’s shape functions: we simply assumed that they are correct, and we shall check this assumption by comparing our results with numerical investigations (see § 7). Should any further modification be required, it is perfectly clear how one should proceed: in fact, all of the density functionals to be derived here are independent of the overall factors and of the shape functions, and the derivation of the critical eccentricity to the bar mode instability also remains unaffected. At most, marginal numerical differences may result.

3.4. The Final Expression for the PN Total Energy

We can now write the explicit expression of the rotating configuration total energy \(E\) for arbitrary density profiles and eccentricities, which we will use in the PN extension of BR’s treatment of nonaxisymmetric bar mode instability.

The total energy of the fluid will be of the form

\[
E = W + K + U + \Delta W + \Delta K.
\]

(54)

Focusing on each single term of the latter expression, we have that the form of the Newtonian gravitational energy \(W\) is exactly that of equation (3), but with the dimensional quantities referred to conformal coordinates. We repeat it here for completeness:

\[
W = -\frac{3}{5} \left(\frac{4\pi}{3}\right)^{1/3} \frac{GM^2}{V^{1/3}} \beta[\rho(g, \xi)]
\]

(55)

The Newtonian kinetic energy is given by

\[
K = \frac{5}{4} \left(\frac{4\pi}{3}\right)^{2/3} \frac{J^2}{MV^{2/3}} \frac{\gamma[\rho]}{\sigma[\rho]} f(e, \xi)
\]

(56)

while the PN corrections to these two different kinds of energy contributions are

\[
\Delta W = \frac{14}{85} \left(\frac{4\pi}{3}\right)^{2/3} \frac{G^2 M^3}{c^2 V^{2/3}} \left(6\beta[\rho]\mu[\rho] + \frac{1}{14}\delta[\rho]\right) h(e, \xi)
\]

(57)

\[
\Delta K = -\frac{175}{536} \left(\frac{4\pi}{3}\right) \frac{G^2 J^2}{c^2 M^2 V^{2/3}} \frac{1}{\sigma[\rho]} \left(85\gamma[\rho]\frac{\sigma[\rho]}{\sigma[\rho]} - \frac{5}{2}\beta[\rho]\nu[\rho] - 15\gamma[\rho]\mu[\rho] + \frac{23}{14}\alpha[\rho]\right) l(e, \xi)
\]

(58)

Because the internal energy \(U\) is independent of the rotating fluid shape, it is not necessary to write its explicit form.

In order to extend BR’s method to PN configurations, we still must determine the explicit expressions for the density functionals that we have introduced in this chapter. This will be the aim of the next section.

4. THE EXPRESSIONS OF THE DENSITY FUNCTIONALS

When, in their work, BR write the gravitational and rotational energies in the form of equations (3) and (4), they do not give the explicit expression of the newly introduced density functionals \(\beta[\rho]\) and \(\gamma[\rho]\), and we have not found these expressions in the literature. We thus obtained them, together with all the other density functionals introduced in the previous section, by means of the following argument.

The possibility of writing the gravitational and rotational energy terms as in equations (3) and (4) is due to a theorem of dimensional analysis (the so-called \(\Pi\)-theorem; see, e.g., Barenblatt 1996). The density functionals appear separated from the shape functions \(f, g\), their expressions are independent of \(e\) and \(\xi\), and therefore they can be calculated in the simpler spherical case. Moreover, from § 3.3 it is evident that also for the other functionals the determination can be made in this simple case, and all the results found by BKR can be exploited.

The main property of such density functionals is that they generalize the expression of a particular physical quantity from the constant density form to the arbitrary density distribution form, keeping fixed the values of the other physical parameters, and reducing to 1 for constant \(\rho\). Their general form will thus be obtained by the ratio of the physical quantity with which they
are related, written for an arbitrary density distribution, and the expression of the same quantity in the case of constant density.

The determination of these expressions can be made using Newtonian coordinates, since the Newtonian contributions to the total energy take the same form in both coordinate systems, and moreover any PN correction to the density functionals of the PN energy terms would be of PPN order and therefore not interesting in our treatment.

In the rest of this section we will consider each density functional previously introduced and determine its explicit form.

### 4.1. The Density Functional for the Newtonian Gravitational Energy

The functional \( \beta[\rho] \) is given by the ratio of the gravitational energy \( W \) for an arbitrary density distribution \( \rho(r) \) and that for a constant density \( \rho_0 \), at fixed values for \( M \) and \( V \). Indicating with \( R \) the Newtonian radius of the spherical star and with \( m(r) \) the mass contained in a sphere of radius \( r \),

\[
m(r) = 4\pi \int_0^r r^2 \rho(r')dr',
\]

we thus have

\[
W = -\int \frac{Gm(r)}{r} \rho(r)dV = -16\pi^2 G \int_0^R r \rho(r)dr \int_0^r r^2 \rho(r')dr'.
\]

For constant density

\[
W = -\frac{16}{15} \pi^2 G \rho_0^2 R^5 = -\frac{3}{5} \frac{G M^2}{R}.
\]

Note that the latter expression is exactly the factor that multiplies the density functional \( \beta[\rho] \) in the first term on the right-hand side of equation (41). In general, the dimensional factor that appears before the density functional of a physical quantity is given by the expression of that particular physical quantity in the case of constant density matter distribution. This is a consequence of the definition of a density functional.

The ratio between the last two expressions gives our functional

\[
\beta[\rho] = \frac{15}{\rho_0^2 R^5} \int_0^R r \rho(r)dr \int_0^r r^2 \rho(r')dr'.
\]

### 4.2. The Density Functional for the Newtonian Kinetic Energy

In the case of the functional \( \gamma[\rho] \), the ratio between the two rotational energies for different mass density distributions must be evaluated keeping fixed also the value of the angular momentum \( J \). Calling \( \Omega \) the uniform angular velocity of the configuration with arbitrary density distribution and \( \omega \) that of the constant density configuration, we have that the rotational energy can be written as

\[
K = \frac{1}{2} \Omega^2 \int r^2 \sin^2 \theta \rho(r)dV = \frac{4}{15} \pi \Omega^2 \int_0^R r^4 \rho(r)dr,
\]

which for constant density becomes

\[
K = \frac{4}{15} \pi \omega^2 \rho_0 R^5 = \frac{1}{2} M \omega^2 R^2,
\]

and therefore we obtain the functional expression

\[
\gamma[\rho] = \frac{5}{\rho_0^2 R^5} \left( \frac{\Omega}{\omega} \right)^2 \int_0^R r^4 \rho(r)dr.
\]

However, the condition of fixed angular momentum enables us to find the relationship between the two angular velocities \( \Omega \) and \( \omega \). Since the rotational energy can also be written as \( K = \frac{1}{2} I \Omega^2 \), where \( I \) is the momentum of inertia, from equation (63) we obtain the integral expression for the latter physical quantity:

\[
I = \frac{8}{15} \pi \rho_0 R^5 = \frac{2}{5} M R^2.
\]
Thus, the condition that the angular momentum $J = I\Omega$ must be the same in both configurations implies the relationship

$$\frac{\Omega}{\omega} = \frac{\rho_0 R^3}{5 \int_0^r r^4 \rho(r) dr} \quad (68).$$

Introducing this ratio in equation (65) the functional $\gamma[\rho]$ results:

$$\gamma[\rho] = \frac{\rho_0 R^3}{5 \int_0^r r^4 \rho(r) dr}. \quad (69)$$

### 4.3. The Density Functionals for the PN Gravitational and Kinetic Corrections

Considering now the functionals $\delta[\rho]$ and $\sigma[\rho]$, to compute them we can recall, as explained above, BKR’s equations (15) and (16) for the energy corrections up to the PPN order for an arbitrary density distribution. Retaining only the $c(c^{-2})$ terms, we can write the PN gravitational and kinetic corrections, in the case of constant angular velocity, in the form

$$\Delta W = -\frac{G}{c^2} \left[ \frac{1}{2} \int \frac{m^2(r^2 \rho(r) dV)}{r^2} - \int \frac{\rho(r) dV}{r} \int \frac{m(r^2 \rho(r) dV)}{r^4} + \int \frac{m(r \rho(r) dV)}{r^4} \right],$$

$$\Delta K = \frac{G}{c^2} \left[ \frac{4}{3} \int \frac{\rho(r) dV}{r^4} \right] - \frac{G}{c^2} \left[ \frac{1}{3} \int \frac{\rho(r) dV}{r^4} \int \frac{r^2 \rho(r) dV}{r} + \int \frac{m(r \rho(r) dV)}{r^4} - \int \frac{\rho(r) dV}{r} \int m(r) r d r \right]. \quad (70)$$

Substituting equation (59) in both of these last expressions, we obtain

$$\Delta W = 64 \pi^3 \frac{G^2}{c^2} \left\{ -\frac{1}{2} \int_0^R \rho(r) dr \int_0^r r^2 \rho(r') dr' \right\} + \int_0^R \rho(r) dr \int_0^r r' \rho(r') dr' \int_0^r r'' \rho(r'') dr'' \right\} - \int_0^R \rho(r) dr \int_0^r r' \rho(r') dr' \int_0^r r'' \rho(r'') dr'' \right\},$$

$$\Delta K = \frac{16}{3} \pi^2 \Omega^2 \frac{G^2}{c^2} \left[ \frac{4}{3} \int \frac{\rho(r) dV}{r^4} \right] - \frac{5}{3} \int_0^R \rho(r) dr \int_0^r r \rho(r') dr' + \int_0^R \rho(r) dr \int_0^r r^2 \rho(r') dr' \right\} - \int_0^r \rho(r) dr \int_0^r r' \rho(r') dr' \int_0^r r'' \rho(r'') dr'' \right\} + \frac{2}{3} \pi^2 \Omega^2 \int_0^R \rho(r) dr \int_0^\pi u(r) + \frac{2}{\rho} \sin \theta d \theta, \quad (72)$$

which for constant density (and thus $u \equiv 0$) give, after some calculations,

$$\Delta W = -\frac{32}{315} \pi^3 \frac{G^2}{c^2} \rho_0^3, \quad (74)$$

$$\Delta K = \frac{368}{1575} \pi^2 \frac{G^2}{c^2} \rho_0^2 \omega^2 R^2, \quad (75)$$

In the computation of $\Delta K$ (in particular for the last integration) we used the Newtonian result for the pressure $P$ at constant density $\rho_0$ (see, e.g., Chandrasekhar 1965a), which is also reported in SZ’s equation (55) in terms of Cartesian coordinates. We rewrite it here in spherical coordinates:

$$\frac{P}{\rho_0} = \frac{2}{3} \pi G \rho_0 (R^2 - r^2) + \frac{1}{2} \omega^2 r^2 \sin^2 \theta. \quad (76)$$

The last term on the right-hand side of this relationship is negligible in the calculation of $\Delta K$ because of the slow rotation approximation that we are adopting in order to determine the density functionals.
From the above equations, we thus obtain the following expressions for \( \delta[\rho] \) and \( \alpha[\rho] \):

\[
\delta[\rho] = \frac{630}{\rho_0 R^7} \left\{ \frac{1}{2} \int_0^R \rho(r) dr \left[ \int_0^r r'^2 \rho(r') dr' \right]^2 - \int_0^R \rho(r) dr \int_0^r \rho(r') dr' \int_0^{r''} \rho(r''') dr'' \right. \\
+ \left. \int_0^R \rho(r) \frac{dr}{r^2} \int_0^r r'^2 \rho(r') dr' \int_0^{r''} \rho(r'') dr'' \right\} \\
+ \frac{315}{2\pi G \rho_0^2 R^7} \left[ \int_0^R u(r) \rho(r) dr \int_0^r r'^2 \rho(r') dr' + \int_0^R \rho(r) dr \int_0^r u(r') r'^2 \rho(r') dr' \right] .
\] (77)

\[
\alpha[\rho] = \frac{7}{23} \frac{R^3}{\int_0^R r^4 \rho(r) dr} \left\{ \frac{4}{R^3} \left[ \int_0^R r^4 \rho(r) dr \right]^2 - 5 \int_0^R r^2 \rho(r) dr \int_0^r r^4 \rho(r') dr' + \frac{12}{5} \int_0^R r^3 \rho(r) dr \int_0^r r^2 \rho(r') dr' \\
- 6 \int_0^R \rho(r) dr \int_0^r d \theta \int_0^{r'} r^2 \rho(r') dr' + \frac{3}{4\pi G} \int_0^R \left[ u(r) + 2 \frac{P(r)}{\rho(r)} \right] r^4 \rho(r) dr \right\} .
\] (78)

4.4. The Density Functional for the Newtonian Angular Momentum

The density functional \( \sigma[\rho] \) has been introduced in equation (40) in order to generalize the Newtonian constant density angular momentum to the case of arbitrary density distribution. Now, from equation (18) we see that the Newtonian angular momentum in the latter case is

\[
J = \frac{2}{3} \Omega \int_0^R r^2 \rho(r) dV = \frac{8}{3} \pi \Omega \int_0^R r^4 \rho(r) dr ,
\] (79)

which for constant density becomes

\[
J = \frac{8}{15} \pi \omega \rho_0 R^5 = \frac{2}{3} M \omega R^2 .
\] (80)

The ratio between equations (79) and (80) gives for \( \sigma[\rho] \) the result

\[
\frac{\sigma[\rho]}{\rho} = \frac{5}{\rho_0 R^3} \left( \frac{\Omega}{\omega} \right) \int_0^R r^4 \rho(r) dr .
\] (81)

Using equation (68) between the two different angular velocities \( \Omega \) and \( \omega \), we find that in the case of this density functional the simple identity

\[
\sigma[\rho] = 1
\] (82)

is valid for any density profile of the rotating configuration.

4.5. The Density Functional for the PN Correction to the Angular Momentum

BKR’s result reported in equation (18) gives us the PN correction to the angular momentum, which we will call \( \Delta J \), in the form

\[
\Delta J = \frac{2 \Omega}{3c^2} \left\{ \int_0^R \left[ u(r) + \frac{P}{\rho} \right] r^2 \rho(r) dV - \frac{2G}{3} \int_0^R m(r) r \rho(r) dV + \frac{4G}{9} \int_0^R \rho(r) dV \int_0^r \frac{m(r)}{r} dr \right\} .
\] (83)

Introducing the explicit equation (59) for \( m(r) \), this gives

\[
\Delta J = \frac{4}{3} \frac{\Omega}{c^2} \int_0^R r^3 \rho(r) dr \int_0^\pi \left[ u(r) + \frac{P}{\rho} \right] \sin \theta d\theta + \frac{64}{9} \frac{\pi^2 \Omega G}{c^2} \\
\times \left[ - \int_0^R r^3 \rho(r) dr \int_0^\pi r'^2 \rho(r') \sin \theta d\theta + \frac{2}{3} \int_0^R \rho(r) dr \int_0^r r' dr' \int_0^{r''} r''^2 \rho(r''') dr''' \right] .
\] (84)

Considering a constant density, and using equation (76) for the ratio \( P/\rho \), we obtain for slow rotation

\[
\Delta J = -\frac{544}{2835} \frac{\pi^2 G}{c^2} \rho_0 \omega R^7 = -\frac{34}{315} \frac{GM}{c^2 R} M \omega R^2 .
\] (85)
From the last two equations we get, after some calculations during which we exploit again equation (68), the following expression for the density functional $\tau[\rho]$:

$$
\tau[\rho] = \frac{63}{17} \frac{1}{\rho_0 R^2} \int_0^R r^4 \rho(r) dr \left\{ -\frac{3}{4\pi G} \int_0^R \left[ u(r) + \frac{P(r)}{\rho(r)} \right] r^4 \rho(r) dr + 2 \int_0^R r^3 \rho(r) dr \int_0^r r' \rho(r') dr' - \frac{4}{3} \int_0^R \int_0^r r \rho(r) dr \int_0^r r' \rho(r') dr' \right\}.
$$

(86)

4.6. The Density Functionals for the PN Transformation Functions $d_1$ and $d_2$

In equation (46) we have introduced the two density functionals $\mu[\rho]$ and $\nu[\rho]$, when considering the transformation from conformal radial coordinates to Newtonian radial coordinates, which is ruled by equation (11) in the case of slow rotation. Therefore, the expressions for $\mu[\rho]$ and $\nu[\rho]$ have to be derived from those for $d_1$ and $d_2$.

Since we are interested only in PN corrections of order $\mathcal{O}(c^{-2})$, we retain the full equation (12) for the transformation function $d_1$ but only the last term of equation (13), of course evaluated in the case of rigid rotation, for $d_2$. Therefore, on the surface of the spherical star we have

$$
d_1(R) = G \left[ \frac{m(R)}{R} + \frac{1}{R^3} \int_0^R m(r) r dr \right],
$$

(87)

$$
d_2(R) = \frac{c^2 \Omega^2}{3R^2} \int_0^R r^4 dr.
$$

(88)

Considering first the function $d_1$, exploiting equation (59), we can write

$$
d_1(R) = \frac{4\pi G}{R} \left[ \int_0^R r^2 \rho(r) dr + \frac{1}{R^2} \int_0^R r dr \int_0^r r^2 \rho(r') dr' \right],
$$

(89)

which for constant density gives, after some calculations,

$$
d_1(R) = \frac{8}{5} \pi G \rho_0 R^2 = \frac{6}{5} \frac{G M}{R}.
$$

(90)

The expression for the density functional $\mu[\rho]$ thus gives, from the above equations,

$$
\mu[\rho] = \frac{5}{2} \frac{1}{\rho_0 R^2} \left[ \int_0^R r^2 \rho(r) dr + \frac{1}{R^2} \int_0^R r dr \int_0^r r^2 \rho(r') dr' \right].
$$

(91)

On the other hand, moving to the transformation function $d_2$, we can see that equation (88) is independent of the density distribution $\rho(r)$, and thus it is always

$$
d_2(R) = \frac{1}{15} c^2 \Omega^2 R^2.
$$

(92)

This implies the simple identity

$$
\nu[\rho] = 1.
$$

(93)

4.7. The Density Functionals for the Mass Transformation

From equation (14) it is possible to obtain the relationship between the baryon rest mass $M_0$ and the conformal mass of the fluid configuration. In effect, exploiting equation (59), we get

$$
M_0 \approx 4\pi \int_0^R R^2 \rho(R') dR' + 48 \pi^2 \frac{G}{c^2} \left[ \int_0^R R' \rho(R') dR' \int_0^R R' \rho(R') dR' \right] + \frac{4}{3} \frac{\Omega^2}{c^2} \int_0^R R^4 \rho(R') dR' .
$$

(94)
Introducing then three new density functionals and the conformal mass \( M_c = (4\pi/3)\rho_0 R^3 \), in terms of the angular momentum \( J \) we can write

\[
M_0 \approx M_c \theta[\rho] + \frac{18}{5} \frac{G M^2}{c^2 R} \zeta[\rho] g(e, \xi) + \frac{5}{4} \frac{J^2}{c^2 M_c R^5} \eta[\rho] \varphi(e, \xi),
\]

where the shape functions given in equation (26) have been added in order to generalize the fast rotation case, as done in § 3.3 for the PN total energy, with the same criterium in choosing the numerical factor of the PK kinetic correction.

From what we have learned up to now about the calculation of the explicit expressions of the density functionals, it is easy to verify that the two new functionals \( \zeta[\rho] \) and \( \eta[\rho] \) coincide, respectively, with the already treated \( \beta[\rho] \) and \( \gamma[\rho] \) (in the case of \( \zeta[\rho] \) an application of Fubini’s theorem for repeated integrals is required). Considering the other new entry, we obtain

\[
\theta[\rho] = \frac{3}{\rho_0 R^3} \int_0^R R^2 \rho(R')dR'.
\] (96)

5. Analytic Determination of the PN Onset Point of Instability

Now we are ready to extend to PN configurations the treatment of the bar mode instability made by BR in the Newtonian case and briefly described in § 2. We will follow strictly their energy variational method, but adding the PN terms in order to determine analytically the critical value \( e_c \) at which the nonaxisymmetric Jacobi sequence bifurcates from the axisymmetric Maclaurin one in the case of PN arbitrarily fast rotating stars.

First, we make for PN configurations the same assumptions 1 and 2 reported and discussed in § 2. We will follow strictly their energy variational method, but adding the PN terms in order to determine the critical value \( e_c \) at which the nonaxisymmetric Jacobi sequence bifurcates from the axisymmetric Maclaurin one in the case of PN arbitrarily fast rotating stars.

5.1. PN Equilibrium Configurations

In order to construct the equilibrium sequence of relativistic rotating configurations, we must minimize the total energy \( E \) keeping fixed the baryon mass \( M_0 \) and not the conformal mass \( M \). Therefore, we require the validity of both equilibrium conditions

\[
\frac{\partial E}{\partial e} = \frac{\partial E}{\partial R} \frac{\partial R}{\partial e} + \left( \frac{\partial E}{\partial e} \right)_R = 0,
\] (98)

\[
\frac{\partial E}{\partial \xi} = \frac{\partial E}{\partial R} \frac{\partial R}{\partial \xi} + \left( \frac{\partial E}{\partial \xi} \right)_R = 0
\] (99)

at \( e \neq 0, \xi = 0 \), with the constraint \( dM_0 = 0 \). This constraint permits us to obtain the variation of \( R \), since

\[
\frac{\partial M_0}{\partial e} = \frac{\partial M_0}{\partial R} \frac{\partial R}{\partial e} + \left( \frac{\partial M_0}{\partial e} \right)_R = 0
\] (100)

gives for the polar eccentricity

\[
\frac{\partial R}{\partial e} = R_e = - \frac{1}{\partial M_0/\partial R}.
\] (101)

Recalling now equation (95), at PN order we obtain

\[
R_e \approx - \frac{6}{5} \frac{G M}{c^2} \zeta[\rho] g_e - \frac{5}{12} \frac{J^2}{c^2 M^2 R^5} \eta[\rho] \varphi_e
\]

and similarly for the variation with respect to the equatorial eccentricity \( \xi \). By factoring out the parts of the energy contributions independent of \( e, \xi \) (and calling them \( W_0, K_0, \Delta W_0, \Delta K_0 \)), the two equilibrium conditions at PN order give

\[
\frac{5}{R} (W_0 g + K_0 f) R_e + W_0 g_e + K_0 f_e + \Delta W_0 h_e + \Delta K_0 l_e = 0,
\] (103)

\[
\frac{5}{R} (W_0 g + K_0 f) R_e + W_0 g_e + K_0 f_e + \Delta W_0 h_e + \Delta K_0 l_e = 0.
\] (104)
The combination of these last two equations gives, after rearrangement,
\[
\frac{K_0}{W_0} \left( f_{e} - \frac{f_{c}}{g_{c}} \right) + \frac{\Delta W_0}{W_0} \left( \frac{h_{c}}{g_{c}} - \frac{h_{c}^{2}}{g_{c}} \right) + \frac{\Delta K_0}{W_0} \left( \frac{f_{c}}{g_{c}} - \frac{h_{c}}{g_{c}} \right) + \frac{K_0}{W_0} f_{e} - g = 0 .
\] (105)

Now, by recalling equations (5)–(6) and (52)–(53), respectively, for the shape functions \( f, g, h, l \), it is possible to verify the identities
\[
\lim_{\xi \to 0} (f_{e} g_{c} - g_{c} f_{c}) = 0 ,
\] (106)
\[
\lim_{\xi \to 0} (h_{c} g_{c} - g_{c} h_{c}) = 0 ,
\] (107)
\[
\lim_{\xi \to 0} (h_{c} f_{c} - g_{c} l_{c}) = 0 ,
\] (108)

which solve equation (105). It must be pointed out that this solution to equation (105) has an important physical meaning: there is always an equilibrium rotating configuration for any polar eccentricity \( e \neq 0 \) but no equatorial eccentricity \( (\xi = 0) \), independent of the EOS, which in equation (105) is represented by the three ratios \( K_0/W_0, \Delta W_0/W_0, \) and \( \Delta K_0/W_0 \).

Introducing now in equation (103) the explicit expressions of the factors independent of \( e, \xi \) (identified by the subscript 0), we obtain
\[
\frac{5}{R} \left( - \frac{3 GM^2}{5 R} \beta_0 |g| - \frac{5}{4} \frac{J^2}{MR^2} \gamma_0 |\rho| f \right) R_e - \frac{3 GM^2}{5 R} \beta_0 |g| g_{e} + \frac{5}{4} \frac{J^2}{MR^2} \gamma_0 |\rho| \left( \frac{18}{5} \frac{\beta_0 |\rho| g_{e}}{\theta_0 |\rho|} + \gamma_0 |\rho| g_{e} \right) + \left( \frac{15}{2} \frac{J^2}{2 GM^3 R \sigma_0^2 |\rho|} \frac{f_{e}}{\theta_0 |\rho|} \right) g_{e} + \frac{14 G^2 M^3}{85 c^2 R^2} \left( 6 \beta_0 |\rho| \mu_0 |\rho| + \frac{1}{14} \delta_0 |\rho| \right) \right) \left. \frac{1}{\sigma_0 |\rho|} \right| \left. \frac{5}{2} \beta_0 |\rho| |\rho| - 15 \gamma_0 |\rho| \mu_0 |\rho| + \frac{23}{14} \alpha_0 |\rho| \right) l_{e} = 0 .
\] (109)

Combining this expression with equation (102), we get
\[
\frac{5}{3} \beta_0 |\rho| g_{e} + \frac{5}{4} \frac{J^2}{4 GM^3 R \sigma_0^2 |\rho|} \frac{f_{e}}{\theta_0 |\rho|} + \frac{GM}{c^2 R} \left[ \frac{18}{5} \frac{\beta_0 |\rho| g_{e}}{\theta_0 |\rho|} + \frac{5}{4} \frac{J^2}{GM^3 R \sigma_0^2 |\rho|} \frac{f_{e}}{\theta_0 |\rho|} \right] g_{e} + \frac{14 G^2 M^3}{48 c^2 R^2} \left( 6 \beta_0 |\rho| \mu_0 |\rho| + \frac{1}{14} \delta_0 |\rho| \right) \right) \left. \frac{1}{\sigma_0 |\rho|} \right| \left. \frac{5}{2} \beta_0 |\rho| |\rho| - 15 \gamma_0 |\rho| \mu_0 |\rho| + \frac{23}{14} \alpha_0 |\rho| \right) l_{e} + \frac{125}{48} \frac{J^2}{c^2 R^2} \left( \frac{2 GM}{\sigma_0^2 |\rho|} \frac{f_{e}}{\theta_0 |\rho|} \right) = 0 .
\] (110)

In order to reexpress the latter equation in terms of the adimensional parameter \( \Omega^2/(\pi G \rho_0) \), with \( \rho_0 \) the mean density of the configuration, we first combine equations (40) and (46), thus obtaining, for slow rotation,
\[
J = \frac{2}{5} M \Omega^2 R^2 \sigma_0 |\rho| \left[ 1 + \frac{GM}{c^2 R} \left( \frac{12}{5} \mu_0 |\rho| - \frac{17 \tau_0 |\rho|}{63 \sigma_0 |\rho|} \right) p_3(e, \xi) f(e, \xi) \right] .
\] (111)

Generalizing to arbitrary fast rotation with SZ’s shape functions given in equation (23), we can write
\[
J = \frac{2}{5} M \Omega^2 R^2 \sigma_0 |\rho| \left[ 1 + \frac{GM}{c^2 R} \left( \frac{12}{5} \mu_0 |\rho| - \frac{17 \tau_0 |\rho|}{63 \sigma_0 |\rho|} \right) p_3(e, \xi) f(e, \xi) \right] ,
\] (112)
where the expression of the function \( p_3 \) in terms of polar and equatorial eccentricities is
\[
p_3(e, \xi) = \frac{6 g}{5 f} + \frac{24 A_5}{35 f} \left( 1 - e^2 \right)^{2/3} + \frac{18 A_1 + A_2}{35} \frac{1 - \xi}{(1 - e^2)^{2/3}} + \frac{3}{140} (A_1 - A_2)^2 \left( 1 - \xi \right)^{1/3} .
\] (113)

Now from equation (112) we obtain
\[
\frac{J^2}{GM^3 R} = \frac{\Omega^2}{\pi G \rho_0} \frac{3}{2512} \sigma_0^2 |\rho| \left[ 1 + \frac{GM}{41 c^2 R} \left( \frac{12}{5} \mu_0 |\rho| - \frac{17 \tau_0 |\rho|}{63 \sigma_0 |\rho|} \right) p_3 f \right] ,
\] (114)
and introducing this result in the equilibrium condition given by equation (110), we finally get the parameter \( \Omega^2/(\pi G \rho_0) \) as a function of the compactness parameter \( GM/(c^2 R) \) along ellipsoidal equilibrium configurations with polar eccentricity \( e \) for any matter distribution. We fix \( \nu \) at \( GM/(c^2 R) = 0.150 \) the end of validity of our PN approximation. Exploiting the fact that \( \sigma_0 |\rho| \equiv 1 \) independently of the EOS (see § 4.4), at the PN order this function is
\[
\frac{\Omega^2}{\pi G \rho_0} = \frac{4J^2 g_{e}}{f_{e}} \beta_0 |\rho| \left[ \frac{20 f^2}{3 f_{e}} \beta_0 |\rho| \right] GM \left[ \frac{21}{5} g f_{e} \beta_0 |\rho| \right] + \frac{21 f_{e}^{2}}{5 f_{e}} \beta_0 |\rho| + \frac{14}{85} \frac{h_{e}}{\theta_0 |\rho|} \left( 6 \beta_0 |\rho| + \frac{1}{14} \delta_0 |\rho| \right) - \frac{21}{134} \frac{g_{e}}{f_{e}} \left( \frac{85}{63} \tau_0 |\rho| - \frac{5 \beta_0 |\rho|}{2 \gamma_0 |\rho|} - 15 \mu_0 |\rho| + \frac{23}{14} \alpha_0 |\rho| \right) + \frac{21}{41} \frac{p_3 f g_{e}}{\theta_0 |\rho|} \left( \frac{12}{5} \mu_0 |\rho| - \frac{17}{63} \tau_0 |\rho| \right) .
\] (115)
Its graphical representation in the case of homogeneous ellipsoids (i.e., when all the density functionals take the value 1) is reported in Figure 1, where we can see that in the relativistic case the value of $\frac{\Omega^2}{\pi G \rho_0}$ is larger than the Newtonian one at any given polar eccentricity. This confirms the results already found by SZ (but see discussion in § 7) and Chandrasekhar (1965b).

5.2. The PN Secular Instability Point

In order to determine the point of secular instability in the PN case, we must minimize the total energy $E$ with respect to the volume $V$, again keeping fixed the baryonic mass $M_0$ and not the conformal mass $M$. We start introducing the function of $S$, $M_0$, $J$, independent of $e$:

$$\Pi = H - U = PV .$$

The minimization of the total energy $E$ with respect to the volume $V$ gives

$$\frac{\partial E}{\partial V} = \frac{\partial E}{\partial M} \frac{\partial M}{\partial V} + \left( \frac{\partial E}{\partial V} \right)_M = 0 ,$$

while the constraint $dM_0 = 0$ implies

$$\frac{\partial M_0}{\partial V} = \frac{\partial M_0}{\partial M} \frac{\partial M}{\partial V} + \left( \frac{\partial M_0}{\partial V} \right)_M = 0 .$$

The variation of $M$ with respect to $V$ is thus given by

$$\frac{\partial M}{\partial V} = - \frac{(\partial M_0/\partial V)_M}{\partial M_0/\partial M} ,$$
which, recalling equation (95), at PN order gives

\[
\frac{\partial M}{\partial V} \approx \frac{1}{V} \left( \frac{6 G M^2 \zeta[\rho]}{5 c^2 R \theta[\rho]} g + \frac{5}{6 c^2 M R^2} \frac{J^2}{\sigma^2[\rho] \theta[\rho]} f \right). \tag{120}
\]

Therefore, from equation (117) we obtain the scalar virial equation for PN configurations in the following form:

\[
\Pi = \frac{1}{3} \left( W + 2K + 2\Delta W + 3\Delta K \right) + (2W - K) \left( \frac{6 GM \zeta[\rho]}{5 c^2 R \theta[\rho]} g + \frac{5}{6 c^2 M R^2} \frac{J^2}{\sigma^2[\rho] \theta[\rho]} f \right). \tag{121}
\]

If we consider the asymmetrical case, where \( \xi \neq 0 \), the total energy is a function \( E = E(S, M_0, J, \xi) \), which can be expanded in the neighborhood of \( \xi = 0 \) and keeping constant \( S, M_0, J \), obtaining, similarly to equation (1),

\[
E = E_0(S, M_0, J) + \xi E_1(S, M_0, J) + \xi^2 E_2(S, M_0, J) + \xi^3 E_3(S, M_0, J) + \cdots. \tag{122}
\]

Since at constant \( S, M_0, J \) we have \( d\Pi|_{S,M_0,J} = 0 \), the virial equation implies

\[
d \left[ W + 2K + 2\Delta W + 3\Delta K + (K - 2W) \left( \frac{18 GM \zeta[\rho]}{5 c^2 R \theta[\rho]} g + \frac{5}{2 c^2 M R^2} \frac{J^2}{\sigma^2[\rho] \theta[\rho]} f \right) \right]_{S,M_0,J} = 0, \tag{123}
\]

and furthermore

\[
d E|_{S,M_0,J} = \frac{1}{2} dW|_{S,M_0,J} - \frac{1}{2} d\Delta K|_{S,M_0,J} - d \left[ (K - 2W) \left( \frac{9 GM \zeta[\rho]}{5 c^2 R \theta[\rho]} g + \frac{5}{4 c^2 M R^2} \frac{J^2}{\sigma^2[\rho] \theta[\rho]} f \right) \right]_{S,M_0,J}. \tag{124}
\]

We now introduce \( \Sigma \), the derivative of \( e \) with respect to \( \xi \) at constant \( S, M_0, J \). From equation (124) we obtain the following expressions for the expansion terms \( E_1 \) and \( E_2 \), remembering to consider constant the variables that appear as subscripts:

\[
E_1 = \frac{1}{2} \left\{ \frac{\partial^2 W}{\partial \xi^2} \right\}_{S,M_0,J} - \frac{1}{2} \left\{ \frac{\partial \Delta K}{\partial \xi} \right\}_{S,M_0,J} - \frac{1}{2} \left\{ \frac{\partial^2 W}{\partial \xi^2} \right\}_{S,M_0,J} - \frac{1}{2} \left\{ \frac{\partial \Delta K}{\partial \xi} \right\}_{S,M_0,J} = 0, \tag{125}
\]

with the derivatives calculated at \( \xi = 0 \).

To determine analytically the critical eccentricity \( e_c \), we must investigate the condition \( E_2 = 0 \). For this, we insert in the right-hand side of equations (125) and (126) the explicit forms of the shape functions \( f, g, l \). By factoring out the parts independent of \( e, \xi \), and using again the notation \( f_\xi = \partial f/\partial \xi \), we can write the PN expansion coefficients in the form

\[
E_1 \quad \text{and} \quad E_2 \quad \text{in the form}
\]

\[
E_1 = \frac{1}{2} \left( g_c + g_\Sigma \right) + \frac{5}{2 R} \left( R_c + R_\Sigma \right) - \frac{1}{2} K_0 \frac{l_c + l_\Sigma}{W_0} \left( l_c + l_\Sigma \right) - \frac{1}{2} K_0 \left( f_c + f_\Sigma \right) \left( f_c + f_\Sigma \right) - \frac{1}{2} K_0 \left( f_c + f_\Sigma \right) \left( f_c + f_\Sigma \right), \tag{127}
\]
\[ E_{\text{f}} = \frac{1}{4} (g_{\ell e} + 2 \Sigma g_{e e} + \Sigma^2 g_{ee} + g_{e \Sigma} + \Sigma g_{e \Sigma}) \left( 1 + \frac{42}{5} g \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} + \frac{5}{2} \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} - \frac{18 K_0}{5} \frac{f GM \zeta [\rho]}{c^3 R \theta [\rho]} \right) \\
- \frac{1}{4} \frac{\Delta K_0}{W_0} \left( l_{\ell e} + 2 \Sigma l_{e e} + \Sigma^2 l_{ee} + l_{e \Sigma} + \Sigma l_{e \Sigma} \right) + \frac{10}{4} \frac{\Sigma g_{e e}}{R} (R_e + R_c) + \frac{10}{4} \frac{\Sigma g_{e \Sigma}}{R} (R_e + R_c) - \frac{1}{4} (f_{\ell e} + 2 \Sigma f_{e e}) \\
+ \Sigma^2 f_{e e} + f_{e \Sigma} + f_{e \Sigma} \right) \left( \frac{18}{5} \frac{K_0}{W_0} g \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} + \frac{5}{2} \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} - \frac{85}{12} g \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} \right) \\
- \left[ \frac{K_0}{W_0} (f_{\ell e} + f_{e \Sigma}) - 2 (g_{\ell e} + g_{e \Sigma}) \right] \left[ \frac{9}{5} \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} (g_{\ell e} + g_{e \Sigma}) + \frac{5}{4} \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} (f_{\ell e} + f_{e \Sigma}) \right]. \] 

(128)

Now equation (123) allows the determination of \( \Sigma \). In fact, separating the shape functions from the parts independent of the two eccentricities \( e, \zeta \), after some calculations we obtain

\[
\Sigma = -\frac{g_{\ell e}}{g_e} \left[ 1 + \frac{5}{R} g \frac{R_e}{g_{\ell e}} + \frac{10}{5} \frac{K_0}{W_0} g \frac{R_e}{g_{\ell e}} + 2 \frac{\Delta W_0 h_{\ell e}}{W_0 g_{\ell e}} + 3 \frac{\Delta K_0 l_{\ell e}}{W_0 g_{\ell e}} - \frac{18}{5} \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} \left( 4 g - \frac{K_0}{W_0} g \frac{f_{\ell e}}{g_{\ell e}} - \frac{K_0}{W_0} f_{\ell e} \right) \right] \\
- \frac{5}{2} \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} \left( 2 f_{\ell e} - 2 \frac{K_0}{W_0} f_{\ell e} + 2 \frac{g_{f \ell e}}{g_{\ell e}} \right) \left[ 1 + \frac{5}{R} g \frac{R_e}{g_e} + \frac{10}{5} \frac{K_0}{W_0} g \frac{R_e}{g_e} + 2 \frac{\Delta W_0 h_{\ell e}}{W_0 g_{\ell e}} - \frac{18}{5} \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} \left( 4 g - \frac{K_0}{W_0} g \frac{f_{\ell e}}{g_{\ell e}} - \frac{K_0}{W_0} f_{\ell e} \right) \right] \\
+ 3 \left[ \frac{\Delta K_0 l_{\ell e}}{W_0 g_{\ell e}} - \frac{18}{5} \frac{GM \zeta [\rho]}{c^3 R \theta [\rho]} \left( 4 g - \frac{K_0}{W_0} g \frac{f_{\ell e}}{g_{\ell e}} - \frac{K_0}{W_0} f_{\ell e} \right) - \frac{5}{2} \frac{J^2}{c^2 M^2 R^2} \frac{\eta [\rho]}{\theta [\rho]} \left( 2 f_{\ell e} - 2 \frac{K_0}{W_0} f_{\ell e} + 2 \frac{g_{f \ell e}}{g_{\ell e}} \right) \right] \] 

(129)

Now, from equations (106)–(108) we get

\[
\lim_{\xi \to 0} \Sigma = -\frac{g_{\ell e}}{g_e} \bigg|_{\xi = 0} = -\frac{l_{\ell e}}{f_{\ell e}} \bigg|_{\xi = 0} = -\frac{f_{\ell e}}{f_{\ell e}} \bigg|_{\xi = 0} = \frac{1 - e^2}{4e} . \] 

(132)

Inserting these values into equation (127) we can verify that also in the PN treatment it is \( E_1 = 0 \).

If we define

\[
\phi(\xi, e) = -\frac{g_{\ell e}}{g_e} , \] 

(133)

\[
\chi(\xi, e) = -\frac{l_{\ell e}}{f_{\ell e}} , \] 

(134)

\[
\psi(\xi, e) = -\frac{f_{\ell e}}{f_{\ell e}} , \] 

(135)

from equations (129)–(131), we get, at \( \xi = 0 \),

\[
\Sigma_e = \phi_e = \chi_e = \psi_e \] 

(136)
\[
\Sigma_\xi = \phi_\xi + \phi \left\{ \frac{5}{R} \frac{R_g}{g} R_b \frac{R_c}{g_c} + 10 \frac{K_0}{R} \frac{R_g}{g} \frac{R_c}{g_c} + 2 \Delta W_0 \frac{h_c}{g_c} + 3 \frac{K_0}{R} \frac{l_c}{g_c} + 18 \frac{GM}{c^3} \frac{\zeta}{\theta^3} \left( 4g - \frac{K_0}{R} \frac{f_c}{g_c} - \frac{K_0}{R} \frac{f_c}{g_c} \right) \right\} \\
- \frac{5}{2} \frac{J^2}{c^2M^2R^2 \sigma^2[\theta][\theta]} \frac{\eta[\rho]}{2} \left( 2f - \frac{K_0}{R} \frac{f_c}{g_c} + 2 \frac{f_c}{g_c} \right) - \frac{5}{2} \frac{J^2}{c^2M^2R^2 \sigma^2[\theta][\theta]} \frac{\eta[\rho]}{2} \left( 2f - \frac{K_0}{R} \frac{f_c}{g_c} + 2 \frac{f_c}{g_c} \right) \right\} \\
+ \frac{3}{5} \frac{\Delta K_0}{l_c} \frac{\zeta}{\theta^3} \\
\right] \\
+ \frac{3}{5} \frac{GM}{c^3} \frac{\zeta}{\theta^3} \left( 4g - \frac{K_0}{R} \frac{g_c}{g} - \frac{K_0}{R} \frac{g_c}{g} \right) \\
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\]
Equation (128) can thus be rewritten as

\[
\frac{E_2}{W_0} = \left[ \frac{1}{4} g_e (\Sigma_e - \phi_e) + \frac{1}{4} \{g_{e\xi} + \phi g_{e\xi} + \phi^2 g_{e\xi} + \phi \phi_{e\xi} + g_{e\phi_{e\xi}} \} \right] \\
\times \left[ 1 + \frac{42}{5} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 f}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} - \frac{18}{5} \frac{K_0}{W_0} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} \right] \\
- \frac{1}{4} \left[ \frac{f_e (\Sigma_e - \psi_e)}{W_0^4} + \frac{1}{4} \{f_e (\Sigma_e - \psi_e) + \psi^2 f_e + \psi \psi_{e\xi} + \psi_{e\xi} f_e \} \right] \\
\times \left[ \frac{18}{5} \frac{K_0}{W_0} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 K_0}{W_0} \frac{\mathcal{J}^2}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} - \frac{85}{12} \frac{\eta[\rho]}{c^2 M^2 R^2} \right]. \tag{140}
\]

As a consequence of the definitions of \(\phi, \chi, \) and \(\psi,\) the quantities in braces vanish identically, so that in the end we get

\[
\frac{E_2}{W_0} = \frac{1}{4} g_e \{\Sigma_e - \phi_e\} \left[ 1 + \frac{42}{5} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 f}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} - \frac{18}{5} \frac{K_0}{W_0} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} \right] \\
- \frac{1}{4} \left[ \frac{f_e (\Sigma_e - \psi_e)}{W_0^4} \right] \left[ \frac{18}{5} \frac{K_0}{W_0} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 K_0}{W_0} \frac{\mathcal{J}^2}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} - \frac{85}{12} \frac{\eta[\rho]}{c^2 M^2 R^2} \right]. \tag{141}
\]

Inserting in the square brackets the corresponding expressions, and making use of the result

\[
1 + 2 \frac{K_0}{W_0} \frac{f_e}{g_e} + 2 \frac{K_0}{W_0} \frac{f_e}{g_e} + 2 \frac{K_0}{W_0} \frac{f_e}{g_e} = 0
\]

which derives from the equilibrium condition given by equation (103) and exploits the fact that \(\sigma[\rho] = 1\) independent of the EOS, after some rather lengthy calculations we finally obtain the full form of the PN expansion coefficient \(E_2\) in terms of first and second derivatives of the shape functions \(f, g, h, f_e, g_e, h_e\):

\[
\frac{E_2}{W_0} = - \left[ \frac{1}{2} \left( \frac{f_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} - \frac{g_{e\xi}}{g_{e\xi}} - \frac{g_{e\xi}}{g_{e\xi}} \frac{g_{e\xi}}{g_{e\xi}} \right) \left[ \frac{K_0}{W_0} \left( 1 + \frac{18}{5} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 f}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} - \frac{18}{5} \frac{K_0}{W_0} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} \right) \right] \\
+ \left( \frac{K_0}{W_0} \left( \frac{18}{5} \frac{\mathcal{J}^2}{c^2 R_0} \frac{\eta[\rho]}{\vartheta[\rho]} + \frac{5 f}{c^2 M^2 R^2} \frac{\eta[\rho]}{\vartheta[\rho]} \right) \right) \left( \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} - \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} - \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} \frac{g_{e\xi}}{g_{e\xi}} \right) \right] \\
+ \left( 2 + \frac{f_e}{g_e} \frac{K_0}{W_0} \left( \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} - \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} - \frac{h_e}{g_e} \frac{g_{e\xi}}{g_{e\xi}} \frac{g_{e\xi}}{g_{e\xi}} \right) \right) \Delta W_0 \right]. \tag{142}
\]

The full expressions of these derivative functions are reported in Appendix B.

6. EVALUATION OF THE PN CRITICAL POINT

6.1. The Newtonian Limit

We are now able, by using equation (143), to evaluate the critical eccentricity \(e_c\), where the Jacobi non-axisymmetric sequence bifurcates from the Maclaurin axisymmetric sequence, for any PN configuration. But first, in order to check that our PN treatment is consistent with the Newtonian results obtained by BR, we analyze the simplified expression of \(E_2/W_0\) after rejecting all the PN terms. We thus obtain

\[
\frac{E_2}{W_0} = - \frac{1}{2} \frac{K_0}{W_0} \left( \frac{g_{e\xi}}{f_e} \frac{g_{e\xi}}{f_e} - \frac{g_{e\xi}}{f_e} \frac{g_{e\xi}}{f_e} - \frac{g_{e\xi}}{f_e} \frac{g_{e\xi}}{f_e} \right). \tag{144}
\]

At first glance, both equations (144) and (143) are characterized by factors depending on the mass distribution, that is, the ratios \(\Delta W_0/W_0, \Delta W_0/W_0, \Delta K_0/W_0, \) and factors depending on the ellipsoid shape. However, the equilibrium condition \(\partial E/\partial e = 0\), if considered in the Newtonian case, implies that the ratio \(K_0/W_0\) that appears in the Newtonian result for \(E_2\) depends only on the eccentricity \(e\):

\[
\frac{K_0}{W_0} = - \frac{g_e}{f_e}. \tag{145}
\]

Therefore, equation (144) depends only on shape functions, and the condition \(E_2 = 0\) has a unique universal solution \(e_c\), for any internal structure of the rotating configuration, which is \(e_c = 0.81267\), as found in BR.
On the other hand, the addition of the PN terms in the equilibrium condition given by equation (103) makes a separation between physical and geometrical quantities in equation (142) impossible. Thus, the PN result (eq. [143]) for $E_2$ remains a mixed expression of factors depending on the internal structure and factors that are functions of the configuration shape. Therefore, in the case of PN rotating ellipsoids, the critical value $e_c$ that solves the equation $E_2 = 0$ is not universal: to obtain this value we must know the physical quantities $M$, $V$, $J$, together with the distribution of mass expressed by the density functionals.

### 6.2. The Case of Incompressible Fluids

Moving now to PN configurations, we see from equation (143) that the dependence of $e_c$ on the rotating ellipsoid internal structure is via the three ratios $K_0/W_0$, $\Delta W_0/W_0$, and $\Delta K_0/W_0$. Recalling equations (55)–(58) and the definition of zero-subscript quantities as the parts independent of the two eccentricities $e, \xi$, we obtain for these ratios

$$\frac{K_0}{W_0} = \frac{25}{12} \frac{J^2}{GM^3R} \frac{\gamma[\rho]}{\sigma^2[\rho][\beta[\rho]]},$$

(146)

$$\frac{\Delta W_0}{W_0} = - \frac{14}{51} \frac{GM}{c^2R} \left( 6\mu[\rho] + \frac{1}{14} \delta[\beta[\rho]] \right),$$

(147)

$$\frac{\Delta K_0}{W_0} = \frac{875}{1608} \left( \frac{J}{cMR} \right)^2 \frac{1}{\sigma^2[\rho]} \left( \frac{85}{63} \frac{\gamma[\rho] \tau[\rho]}{\sigma[\beta[\rho]]} - \frac{5}{2} \nu[\rho] - 15 \frac{\gamma[\rho] \mu[\rho]}{\beta[\rho]} + \frac{23}{14} \frac{\alpha[\rho]}{\beta[\rho]} \right),$$

(148)

where we have substituted the volume $V$ with the conformal radius $R = (3V/4\pi)^{1/3}$.

By exploiting equation (142), it is then possible to obtain the expression of $J^2$ for any PN rotating configuration at equilibrium:

$$J^2 = \frac{25}{12} \left[ 1 - \frac{14 GM}{51 c^2 R} \left( 6\mu[\rho] + \frac{1}{14} \delta[\beta[\rho]] \right) \frac{h_c}{g_e} - 6 \frac{GM}{c^2 R} \frac{\gamma[\rho]}{\sigma^2[\rho][\beta[\rho]]} g \right] \left[ \frac{1}{GM^3 R} \frac{\gamma[\rho]}{\sigma^2[\rho]} f_c - \frac{1}{134 c^2 M^2 R^2} \frac{\gamma[\rho]}{\sigma^2[\rho]} \frac{1}{\sigma[\beta[\rho]]} \frac{1}{\sigma[\beta[\rho]]} \right] \left[ \frac{85}{63} \frac{\gamma[\rho] \tau[\rho]}{\sigma[\beta[\rho]]} - \frac{5}{2} \nu[\rho] - 15 \frac{\gamma[\rho] \mu[\rho]}{\beta[\rho]} + \frac{23}{14} \frac{\alpha[\rho]}{\beta[\rho]} \right].$$

(149)

By inserting this expression in equations (146) and (148), after some calculations in which we exploit the facts that $\sigma[\rho] = 1$, $\eta[\rho] = \gamma[\rho]$, and $\zeta[\rho] = \alpha[\rho]$ for any EOS, we can obtain also the PN-approximated ratios $K_0/W_0$ and $\Delta K_0/W_0$ in terms of the configuration compactness parameter $GM/(c^2R)$:

$$\frac{K_0}{W_0} = - \frac{1}{f_c/g_e} + \frac{14 GM}{51 c^2 R} \left( 6\mu[\rho] + \frac{1}{14} \delta[\beta[\rho]] \right) \frac{h_c}{g_e} + 6 \frac{GM}{c^2 R} \frac{\gamma[\rho]}{\sigma^2[\rho][\beta[\rho]]} f_c - \frac{35}{134 c^2 R} \left( \frac{85}{63} \frac{\gamma[\rho] \tau[\rho]}{\sigma[\beta[\rho]]} - \frac{5}{2} \nu[\rho] - 15 \mu[\rho] + \frac{23}{14} \frac{\alpha[\rho]}{\beta[\rho]} \right) \frac{h_c}{g_e} + \frac{7}{f_c/g_e} \frac{GM}{c^2 R} \frac{\beta[\rho]}{\theta[\rho]} \frac{g}{f_c/g_e},$$

(150)

$$\frac{\Delta K_0}{W_0} = \frac{35 GM}{134 c^2 R} \left( \frac{85}{63} \frac{\gamma[\rho] \tau[\rho]}{\sigma[\beta[\rho]]} - \frac{5}{2} \nu[\rho] - 15 \mu[\rho] + \frac{23}{14} \frac{\alpha[\rho]}{\beta[\rho]} \right) / f_c / g_e,$$

(151)

In this way, we see that with equation (143) we can evaluate the critical value $e_c$ in the PN approximation for any given mass distribution and any compactness parameter of the configuration.

We start by considering the case of constant density mass distribution, i.e., the case analyzed by SZ. For homogeneous configurations, we already know that all the density functionals discussed in § 4 simply take the constant value 1. Considering different values for the compactness parameter $GM/(c^2R)$, the equation $E_2 = 0$ gives different values of the critical eccentricity $e_c$ at the secular instability onset point, as reported in the first two columns of Table 1 and in Figure 2. The corresponding critical values for the adimensional ratio $\Omega^2/(\pi G\rho_0)$ are given in the first two columns of Table 2.

Another indicator for the onset of instability is the ratio $K/W$ of the kinetic energy to the absolute value of the gravitational potential energy. A relativistic analog of such a ratio can be defined as

$$\frac{K}{|W|} = \frac{(1/2)\Omega J}{(1/2)\Omega J - E},$$

(152)

where the total energy $E$ must be expressed as a function of the angular velocity $\Omega$ by exploiting equations (41) and (46) and following the same procedure of § 3.3. Therefore, in axisymmetry this parameter, which is gauge invariant for rigidly rotating
TABLE 1
Critical Eccentricity $e_c$ at the PN Secular Instability Point for Different Polytropic Indexes $n$ and Different Compactness Parameters $GM/(c^2 R)$

\[
\begin{array}{cccccc}
\hline
GM/(c^2 R) & n = 0 & n = 0.2 & n = 0.4 & n = 0.6 & n = 0.8 \\
\hline
0.000 & 0.8127 & 0.8127 & 0.8127 & 0.8127 & 0.8127 \\
0.010 & 0.8140 & 0.8141 & 0.8141 & 0.8141 & 0.8141 \\
0.020 & 0.8152 & 0.8153 & 0.8153 & 0.8154 & 0.8154 \\
0.030 & 0.8163 & 0.8163 & 0.8164 & 0.8165 & 0.8165 \\
0.040 & 0.8172 & 0.8173 & 0.8174 & 0.8174 & 0.8175 \\
0.050 & 0.8181 & 0.8181 & 0.8182 & 0.8183 & 0.8184 \\
0.060 & 0.8188 & 0.8189 & 0.8190 & 0.8191 & 0.8192 \\
0.070 & 0.8195 & 0.8196 & 0.8197 & 0.8198 & 0.8199 \\
0.080 & 0.8201 & 0.8202 & 0.8203 & 0.8204 & 0.8205 \\
0.090 & 0.8207 & 0.8208 & 0.8209 & 0.8210 & 0.8211 \\
0.100 & 0.8212 & 0.8213 & 0.8215 & 0.8216 & 0.8217 \\
0.110 & 0.8217 & 0.8218 & 0.8219 & 0.8220 & 0.8222 \\
0.120 & 0.8222 & 0.8223 & 0.8224 & 0.8225 & 0.8226 \\
0.130 & 0.8226 & 0.8227 & 0.8228 & 0.8229 & 0.8230 \\
0.140 & 0.8230 & 0.8231 & 0.8232 & 0.8233 & 0.8234 \\
0.150 & 0.8233 & 0.8234 & 0.8236 & 0.8237 & 0.8238 \\
\hline
\end{array}
\]

Note.—The column for $n = 0$ corresponds to the case of constant mass density distribution. We fix at $GM/(c^2 R) = 0.150$ the end of validity of our PN approximation.

Fig. 2.—Critical eccentricity $e_c$ of the PN secular instability point as a function of the compactness parameters $GM/(c^2 R)$, in the case of constant mass density distribution.
objects, at the PN order gives

\[
\frac{K}{|W|} = 1 \left/ \left( 1 + \frac{g f_{\delta_e} \gamma[\rho] - \gamma[\rho]}{2} - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right) - \frac{5 GM}{3 c^2 R} \left[ \frac{9 \beta[\rho]}{2 \theta[\rho]} + \frac{9 f_{\delta_e} \beta[\rho]}{2 f_{\delta_e} \theta[\rho]} \right] + \frac{14 h_{\delta_e}}{85 f_{\delta_e} \gamma[\rho]} \left( \frac{6 \mu[\rho]}{\gamma[\rho]} + \frac{1}{14} \delta[\rho] \right) \right. \\
\left. - \frac{21}{143 f_{\delta_e} \gamma[\rho]} \frac{12}{17} \frac{\tau[\rho] - 5 \beta[\rho] \nu[\rho]}{2 \gamma[\rho]} - 15 \mu[\rho] + \frac{22 \alpha[\rho]}{14 \gamma[\rho]} \right) + \frac{21}{141} p_{12} \left( \frac{12}{5} \frac{\mu[\rho]}{\gamma[\rho]} - \frac{17}{63} \tau[\rho] \right) \left( \frac{g f_{\delta_e} \gamma[\rho]}{f_{\delta_e} \beta[\rho]} - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right) \right) \\
\left/ \left( 1 + \frac{g f_{\delta_e} \gamma[\rho] - \gamma[\rho]}{2} - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right)^2 \right. \\
\left. + \frac{35 GM}{82 c^2 R} \left( \frac{12}{5} \frac{\mu[\rho]}{\gamma[\rho]} - \frac{17}{63} \tau[\rho] \right) \left( \frac{g f_{\delta_e} \gamma[\rho]}{f_{\delta_e} \beta[\rho]} - \gamma[\rho] - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right) \right) \\
\left/ \left( 1 + \frac{g f_{\delta_e} \gamma[\rho] - \gamma[\rho]}{2} - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right)^2 \right. \\
\left. + \frac{35 GM}{82 c^2 R} \left( \frac{12}{5} \frac{\mu[\rho]}{\gamma[\rho]} - \frac{17}{63} \tau[\rho] \right) \left( \frac{g f_{\delta_e} \gamma[\rho]}{f_{\delta_e} \beta[\rho]} - \gamma[\rho] - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right) \right) \\
\left/ \left( 1 + \frac{g f_{\delta_e} \gamma[\rho] - \gamma[\rho]}{2} - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right)^2 \right. \\
\left. + \frac{35 GM}{82 c^2 R} \left( \frac{12}{5} \frac{\mu[\rho]}{\gamma[\rho]} - \frac{17}{63} \tau[\rho] \right) \left( \frac{g f_{\delta_e} \gamma[\rho]}{f_{\delta_e} \beta[\rho]} - \gamma[\rho] - \frac{5}{3} \frac{R f_{\delta_e} \gamma[\rho]}{3 GM^2 f_{\delta_e} \beta[\rho]} U \right) \right) \right).}

In this expression SZ’s shape functions \( p_{12} \) and \( p_3 \) also appear. In terms of polar and equatorial eccentricities, the latter is already given by equation (113), while the former is

\[
p_{12}(e, \xi) = -\frac{27}{140} g + \frac{171}{280} A_3 (1 - e^2)^{2/3} + \frac{111}{280} (A_1 + A_2) \left\{ \frac{(1 - \xi)}{1 - e^2} \right\} + \frac{3}{140} (A_1 - A_2)^2 \left\{ \frac{(1 - \xi)}{1 - e^2} \right\} + \mathcal{J},
\]

where again the integral \( \mathcal{J} \) is that reported in Appendix C of SZ. The critical values of the ratio \( K/|W| \) at the onset point of instability for constant density mass distributions are given in the first two columns of Table 3.

### 6.3. Compressible Fluids: The Polytropic Case

In order to evaluate the critical eccentricity \( e_c \) for a more general mass distribution, we consider now polytropic distributions.

By exploiting the properties of such equilibrium configurations, which are exhaustively described, e.g., in Chandrasekhar (1957), we are able to calculate the values of all the density functionals treated in § 4 for any polytropic mass distribution, i.e., for any polytropic index \( n \), where \( P = k \rho^{1+1/n} \). To do this, we must consider each density functional, rewrite its expression in the case of a polytropic mass distribution, and then evaluate such expression for any index \( n \). We will thus obtain the considered density functional as a function of the polytropic index.

However, in § 1 we pointed out the result of Bonazzola et al. (1996) concerning the critical polytropic index for the onset of the viscosity-driven bar mode instability, reporting that they find a critical value, for very relativistic objects, slightly lower...
than the Newtonian one: \( n \sim 0.71 \) versus \( n = 0.808 \). This means that for intermediate PN configurations the maximum polytropic index for the onset of the viscosity-driven bar mode instability lies between these two values.

Moreover, we must point out here that our PN energy variational method does not provide by itself a critical value for the polytropic index. In fact, no energy variational method can be sensitive to the mass-shedding limit mentioned in § 1, which is a dynamical phenomenon.

The papers that have found critical values of the polytropic index \( n \) for the onset of the bar mode instability are based on dynamical treatments of the rotating configurations, like those used by James (1964) and Bonazzola et al. (1996). On the other hand, the discussion of uniformly rotating equilibrium models beyond the mass-shedding limit is motivated in Lai et al. (1993) with the fact that they are reasonable approximations for the interior of the more realistic, differentially rotating structures, which can probably exist beyond this limit. We also remark that for \( n = 0.5-1.0 \) one obtains models with bulk properties that are comparable to those of observed NSs (Stergioulas 1998\(^4\)).

Because of all the above arguments, we will assume that the maximum polytropic index for the onset of the viscosity-driven bar mode instability in strictly rigidly rotating configurations is \( n = 0.8 \).

The determination, for each density functional, of the function that gives its values in the polytropic index range \( n = 0.8 \) leads to the following expressions (we omit those density functionals whose unitary value is independent of the EOS):

\[
\beta[\rho] = \frac{5}{3} \xi_1 \int_0^{\xi_1} \xi^3 \theta^2(\xi) d\xi \int_0^{\xi_1} \xi^2 \theta^2(\xi) d\xi',
\]

\[
\gamma[\rho] = \frac{3}{5} \xi_1 \int_0^{\xi_1} \xi^3 \theta^2(\xi) d\xi \int_0^{\xi_1} \xi^4 \theta^2(\xi) d\xi',
\]

\[
\delta[\rho] = \frac{70}{3} \xi_1 \int_0^{\xi_1} \xi^3 \theta^2(\xi) d\xi \int_0^{\xi_1} \xi^2 \theta^2(\xi) d\xi',
\]

\[
\left( \frac{1}{2} \int_0^{\xi_1} \theta^2(\xi) d\xi \right)^2 - \int_0^{\xi_1} \xi \theta^2(\xi) d\xi \int_0^{\xi_1} \xi \theta^2(\xi) d\xi' \int_0^{\xi_1} \xi \theta^2(\xi) d\xi''
\]

\[
+ \int_0^{\xi_1} \theta^2(\xi) d\xi \int_0^{\xi_1} \xi \theta^2(\xi) d\xi' \int_0^{\xi_1} \xi \theta^2(\xi) d\xi''.
\]

\[
+ \frac{n}{n+1} \left[ \int_0^{\xi_1} \xi \theta_{n+1}(\xi) d\xi \int_0^{\xi_1} \xi \theta^2(\xi') d\xi' + \int_0^{\xi_1} \xi \theta^2(\xi) d\xi \int_0^{\xi_1} \xi^2 \theta^2(\xi') d\xi' \right],
\]

\[\text{Note:} \text{The column for } n = 0 \text{ corresponds to the case of constant mass density distribution. We fix at } GM/(c^2R) = 0.150 \text{ the end of validity of our PN approximation.}
\]

\[\text{TABLE 3}
\]

| \( GM/(c^2R) \) | \( n = 0 \) | \( n = 0.2 \) | \( n = 0.4 \) | \( n = 0.6 \) | \( n = 0.8 \) |
|-----------------|----------|----------|----------|----------|----------|
| 0.000........... | 0.1375   | 0.1378   | 0.1387   | 0.1404   | 0.1432   |
| 0.010........... | 0.1412   | 0.1409   | 0.1411   | 0.1418   | 0.1431   |
| 0.020........... | 0.1449   | 0.1440   | 0.1434   | 0.1440   | 0.1429   |
| 0.030........... | 0.1486   | 0.1470   | 0.1456   | 0.1442   | 0.1427   |
| 0.040........... | 0.1521   | 0.1500   | 0.1478   | 0.1452   | 0.1423   |
| 0.050........... | 0.1557   | 0.1529   | 0.1499   | 0.1463   | 0.1419   |
| 0.060........... | 0.1592   | 0.1558   | 0.1520   | 0.1473   | 0.1414   |
| 0.070........... | 0.1627   | 0.1587   | 0.1540   | 0.1482   | 0.1408   |
| 0.080........... | 0.1661   | 0.1615   | 0.1560   | 0.1491   | 0.1402   |
| 0.090........... | 0.1696   | 0.1643   | 0.1580   | 0.1500   | 0.1396   |
| 0.100........... | 0.1730   | 0.1671   | 0.1600   | 0.1509   | 0.1390   |
| 0.110........... | 0.1764   | 0.1699   | 0.1619   | 0.1516   | 0.1383   |
| 0.120........... | 0.1796   | 0.1727   | 0.1638   | 0.1525   | 0.1375   |
| 0.130........... | 0.1832   | 0.1754   | 0.1657   | 0.1532   | 0.1367   |
| 0.140........... | 0.1866   | 0.1782   | 0.1676   | 0.1540   | 0.1360   |
| 0.150........... | 0.1899   | 0.1809   | 0.1695   | 0.1548   | 0.1352   |

\[\text{Available at http://www.livingreviews.org/Articles/Volume1/1998-8stergio.}\]
The value of these density functionals for different polytropic indexes \( n \) can be calculated by numerical integrations of these equations. In Figure 3 the six functionals \( \beta [\rho] \), \( \gamma [\rho] \), \( \delta [\rho] \), \( \alpha [\rho] \), \( \tau [\rho] \), and \( \mu [\rho] \) are reported as functions of the index \( n \).
In the case of $\beta[\rho]$ we can compare our result with a formula for the potential energy of polytropic spherical distributions due to Betti and Ritter (Chandrasekhar 1957, chap. 4, eq. [90]):

$$W = -\frac{3}{5-n} \frac{G M^2}{R}.$$  \hspace{1cm} (162)

This equation, together with our equation (55), implies that it must be $\beta[\rho] = 5/(5 - n)$, which is exactly the function that we find.

Now we have all the pieces to evaluate the critical eccentricity $e_c$, where the Jacobi nonaxisymmetric sequence bifurcates from the Maclaurin axisymmetric sequence, marking the PN onset point of the secular bar mode instability, for different polytropic mass distributions and different compactness parameters. Our results are presented in Table 1. For a reproduction in terms of the parameters $\Omega^2/(\sigma G \rho_0)$ and $K/|W|$, we refer to Tables 2 and 3, respectively.

Of course, the density functionals introduced for our PN treatment of bar mode instability and whose expressions have been determined for the first time in the literature in \S 4 can be evaluated for any compressible fluid, not only for those that can be described by a polytropic EOS.

Therefore, in this paper we deposit all the instruments needed for the evaluation of the critical eccentricity $e_c$, where the nonaxisymmetric Jacobi sequence bifurcates from the Maclaurin axisymmetric sequence, once given the fluid configuration EOS.

More realistic EOSs have been considered in the literature, in the form of both schematic analytical models and numerical tables interpolated by means of particular functions. By inserting in the general expressions given in this paper the equations relative to a specific EOS, or their numerical representations, the onset point of bar mode instability for that specific EOS can be evaluated. This may be the object of future work, since we know that new analytical functions are in preparation (P. Haensel 2002, private communication) to describe the internal structure of real NSs.

7. DISCUSSION

We can now discuss the results obtained in the last section, comparing them, when possible, with other works in the literature.

First, in the uniform density case, treated in \S 6.2, we can compare our results with those obtained by SZ, which are given in Table 3 of their work. The comparison of the results is not straightforward, since they adopt a different compactness parameter $GM/(c^2 R_0)$, where $M$ is the total mass energy of the configuration and $R_0$ is the equatorial radius in Schwarzschild coordinates of the spherical, nonrotating configuration. Considering the maximum value for their compactness parameter, SZ point out that it is determined by a limit imposed by the relationship between the conformal and the Schwarzschild compactness parameters in the spherical limit. This function (reported in their eq. [149]) has a maximum for $GM/(c^2 R_0) = 0.28$, corresponding to $GM_c/(c^2 a_1) \approx 0.134$, and therefore SZ state that their PN formalism can be used to investigate relativistic sequences up to a maximum value $[GM/(c^2 R_0)]_{max} \approx 0.28$. Note that the maximum value that we have fixed for our conformal compactness parameter, $GM_c/(c^2 R) = 0.150$, is similar to the value that SZ’s conformal parameter takes in correspondence with $[GM/(c^2 R_0)]_{max}$.

However, if the range of compactness parameters considered is almost the same, the range of critical eccentricities that we present in this work is significantly smaller than that found by SZ. Therefore, we can state that also with our treatment we find that the critical value of the eccentricity for the onset of the bar mode instability increases as the star becomes more relativistic, in the regime in which the PN approximation is valid, but such increase is less marked than in SZ. Thus, the presence of a stabilizing effect due to general relativity on the Jacobi-like bar mode instability is confirmed but weakened in its significance.

Another difference between our PN treatment and SZ’s appears when considering the equilibrium sequences of the rotating configurations. Comparing our Figure 1 with SZ’s analog Figure 1, it is possible to note that we find a much marked increase of the angular velocity in homogeneous ellipsoids with the compactness parameter, at any given value of polar eccentricity. This discrepancy is solved if we adopt in our calculations the overall numerical factors given by SZ (remember the dilemma of \S 3.3). With these values we obtain the PN equilibrium sequences reported in Figure 4, which is identical to SZ’s Figure 1.

If we consider now the results we obtained in the more general compressible polytropic case (see \S 6.3), from Table 1 it is evident that the increase in the critical value of eccentricity for the onset of the bar mode instability with the compactness parameter is confirmed at any polytropic index value, in the regime in which the PN approximation is valid. Thus, the presence of a stabilizing effect due to general relativity on the Jacobi-like bar mode instability is a property also of softer EOSs with respect to the incompressible case.

The only exception to this general trend can be found in the last column of Table 3, which gives the critical values of the ratio $K/|W|$ for a polytropic mass distribution with $n = 0.8$. This column suggests that when increasing the mass concentration toward the center of a rotating configuration, i.e., increasing the value of the polytropic index $n$, there is a value above which the viscosity-driven bar mode instability is strengthened by relativistic effects and no more weakened. We find that such a value lies in the range 0.7–0.8. However, as pointed out in \S 6.3, in the same range also lies the maximum polytropic index for the onset of the instability. Therefore, the column for $n = 0.8$ in Table 3 may be nonrepresentative of the viscosity-driven instability, and in this case the value of the polytropic index where we find the inversion in the instability strength may also define the maximum index for the onset of the instability.

The increase of the critical value of eccentricity with the polytropic index $n$ for a given compactness parameter $GM/(c^2 R)$ is almost negligible; therefore, the onset point of secular instability in our PN treatment may be considered independent of the polytropic mass distribution.
As we have reported in § 1, the same result was reported, but limited to the Newtonian case, in Lai & Shapiro (1995). On the other hand, the numerical investigation of the viscosity-driven bar mode instability carried out by Bonazzola et al. (1996) found a weak dependence of the onset point of instability on the polytropic index $n$ in Newtonian configurations (see in particular their Fig. 3). This discrepancy may be due to the “ellipsoidal approximation” adopted in the analytical works on rotating configurations such as Lai et al. (1993), Lai & Shapiro (1995), BR, and of course this paper. Numerical investigations do not need such an approximation; therefore, they lead to slightly different results on this item.

Finally, Bonazzola et al. (1996) were not able to investigate the secular instability in incompressible fluid configurations, as a result of the problems that their numerical code had in treating the strong discontinuity of the density profile at the surface of the star, which varies suddenly from its constant value to zero. Therefore, up to now no results from relativistic numerical investigations are available on the dependence of the onset point of instability on the configuration compactness. A new numerical code, which solves the discontinuity problem (Gondek-Rosińska & Gourgoulhon 2002), makes the comparison possible between a fully relativistic numerical investigation and SZ’s and our analytical PN treatments.

8. CONCLUSIONS

In this paper we have treated the bar mode secular instability of rigidly rotating equilibrium configurations for neutron stars with an analytic energy variational method in the PN approximation. The method, which is the extension to PN configurations of that used by BR for the Newtonian treatment of bar mode instability, gives results for any (but defined) EOS.

After the full derivation of the equation that gives as its solution the critical eccentricity $e_c$ where the secular bar mode instability sets on, we considered some particular EOSs and evaluated the corresponding critical values. To do this, we introduced density functionals that allowed the generalization of the physical quantities involved in the treatment from the constant mass density to the arbitrary density profile form. The determination of the explicit expressions of such functionals has been made for the first time in the literature.

We started by checking that in the Newtonian case, i.e., when all the PN corrections are null, the critical value for the eccentricity is $e_c = 0.8127$, as found in the precedent literature on this item.

Then we considered PN configurations with a constant density mass distribution. In this case we have found that the critical value $e_c$ depends on the neutron star compactness parameter $GM/(c^2R)$, and it is larger for more relativistic, i.e., more compact, objects. Thus, the effect of general relativity, considered in the PN approximation, is to weaken the bar mode instability, stabilizing the object against secular transition from the axisymmetric Maclaurin to the triaxial Jacobi sequence. This is consistent with the results obtained by SZ in their work on incompressible rotating stars, but we find a less marked stabilizing effect.

Finally, we considered polytropic mass distributions. For these configurations we have calculated numerically the values of all the density functionals involved in our PN energy variational method, obtaining them as functions of the polytropic index $n$ up to the dynamical limit $n \approx 0.8$ imposed by the mass shedding of the rotating configuration. The result is the confirmation
of the increase of the critical eccentricity with the compactness parameter also for \( n > 0 \) (softer) polytropes, of course in the regime in which the PN approximation is valid.

The latter investigation has also shown that for a fixed compactness parameter the increase of the critical value of eccentricity with the polytropic index is negligible, thus extending to PN configurations the independence of the onset point of secular instability on the polytropic mass distribution, at least in the "ellipsoidal approximation" regime.

The formula for the PN total energy (eq. [54]), the expressions of its PN corrections (eqs. [57] and [58]), the explicit general forms of the density functionals introduced in our energy variational method (eqs. [62], [69], [77], [78], [82], [86], [91], [93], and [96]), the expressions specialized to the polytropic case (eqs. [155]–[161]), and the full expressions of the ellipsoidal shape functions together with their derivatives (Appendix B) will all be found useful for future investigations.

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APPENDIX A

THE COEFFICIENTS \( A_i \) IN TERMS OF THE ECCENTRICITIES \( e, \xi \)

The dimensionless coefficients \( A_i \) are given in equations (3.33)–(3.35) of Chandrasekhar (1969b) in terms of standard incomplete elliptic integrals involving only the values \( a_i/i = 1, 2, 3 \) of the three semi-axes of the ellipsoid outer surface. It is possible to calculate them (as in SZ) in terms of the axial ratios \( \lambda_1 = (a_3/a_1)^2/3 \) and \( \lambda_2 = (a_3/a_2)^2/3 \) and therefore as functions of the eccentricities \( e \) and \( \xi \). The standard incomplete elliptic integrals involved in their definition thus become

\[
E(\theta, \phi) = \int_0^\theta (1 - \sin^2 \theta \sin^2 \phi)^{1/2} d\phi ,
\]

\[
F(\theta, \phi) = \int_0^\theta (1 - \sin^2 \theta \sin^2 \phi)^{-1/2} d\phi ,
\]

with

\[
\sin \theta = \frac{\sqrt{\xi}}{e} ,
\]

\[
\sin \phi = e .
\]

After some algebraic manipulation, we obtain the expressions

\[
A_1 = \frac{2(1 - \xi)^{1/2}(1 - e^2)^{1/2}}{e} v(e, \xi) ,
\]

\[
A_2 = \frac{t(e, \xi) i(e, \xi) - 2v(e, \xi) e^2(1 - \xi)^{1/2}(1 - e^2)^{1/2} - 2e(1 - e^2)^{1/2}}{e(e^2 - \xi)} ,
\]

\[
A_3 = \frac{2e(1 - \xi) - 2(1 - \xi)^{1/2}(1 - e^2)^{1/2} j(e, \xi)}{e(e^2 - \xi)} ,
\]

where we have defined the auxiliary functions

\[
v(e, \xi) = \frac{1}{e^2} \int_0^{\arcsin e} \sin^2 x \left( 1 - \frac{\xi}{e^2} \sin^2 x \right)^{-1/2} dx ,
\]

\[
t(e, \xi) = \frac{2e^2(1 - \xi)^{1/2}(1 - e^2)^{1/2} - 2(e^2 - \xi)(1 - \xi)^{1/2}(1 - e^2)^{1/2}}{e^3} ,
\]

\[
i(e, \xi) = \int_0^{\arcsin e} \left( 1 - \frac{\xi}{e^2} \sin^2 x \right)^{-1/2} dx ,
\]

\[
j(e, \xi) = \int_0^{\arcsin e} \left( 1 - \frac{\xi}{e^2} \sin^2 x \right)^{1/2} dx .
\]

It must be noticed, moreover, that the coefficients \( A_i \) are not independent, since the relationship \( A_1 + A_2 + A_3 = 2 \) is valid.
THE FULL EXPRESSIONS OF THE DERIVATIVE FUNCTIONS IN EQUATION (143)

As we have shown in the paper, the critical value of eccentricity for the onset of bar mode instability is given by equating to zero the expression given by equation (143), evaluated in the limit \( \xi \to 0 \). The dependence of this expression on the eccentricity \( e \) is contained in the combination of derivatives of the shape functions \( f, g, h, l \). We report here the full expressions of these derivative functions:

\[
\lim_{\xi \to 0} \frac{g_{\xi}}{g_{e}} = \lim_{\xi \to 0} \frac{l_{e}}{l_{e}} = \lim_{\xi \to 0} \frac{f_{e}}{f_{e}} = \lim_{\xi \to 0} \frac{h_{e}}{h_{e}} = \frac{e^2 - 1}{4e^4}, \tag{B1}
\]

\[
\lim_{\xi \to 0} f_{\xi} = -\frac{e}{9(1 - e^2)^{2/3}}, \tag{B2}
\]

\[
\lim_{\xi \to 0} f_{\xi} = -\frac{(1 - e^2)^{1/3}}{18}, \tag{B3}
\]

\[
\lim_{\xi \to 0} f_{\xi} = -\frac{2e^3(1 - e^2)^{1/6}}{3e\sqrt{1 - e^2} + (2e^2 - 3)\arcsin e}, \tag{B4}
\]

\[
\lim_{\xi \to 0} g_{\xi} = \frac{9 - 4e^2}{12e^3(1 - e^2)^{1/3}} - \frac{27 - 30e^2 + 4e^4}{36e^4(1 - e^2)^{5/6}}\arcsin e, \tag{B5}
\]

\[
\lim_{\xi \to 0} g^{h}_{\xi} = -\frac{27 + 10e^2}{96e^4} - \frac{27 - 30e^2 + 4e^4}{288e^5}, \tag{B6}
\]

\[
\lim_{\xi \to 0} h_{\xi} = \frac{1}{56e^9(1 - e^2)^{2/3}} \left[ -9e^2(-36 + 69e^2 - 49e^4 + 16e^6) + 2\sqrt{1 - e^2}(-324 + 513e^2 - 468e^4 + 152e^6)\arcsin e \\
+ (324 - 729e^2 + 936e^4 - 604e^6 + 96e^8)(\arcsin e)^2 \right], \tag{B7}
\]

\[
\lim_{\xi \to 0} h^{h}_{\xi} = \frac{(1 - e^2)^{1/3}}{1792e^{10}} \left[ 9e^2(-387 + 615e^2 - 304e^4 + 76e^6) + 2\sqrt{1 - e^2}(3483 - 4374e^2 + 3600e^4 + 272e^6)\arcsin e \\
+ (-3483 + 6696e^2 - 8064e^4 + 2432e^6 + 1536e^8)(\arcsin e)^2 \right], \tag{B8}
\]

\[
\lim_{\xi \to 0} h_{\xi} = \left\{ -9(1 - e^2)^{1/6} \left[ -3e^2(9 - 13e^2 + 4e^4) + 2\sqrt{1 - e^2}(27 - 30e^2 + 40e^4)\arcsin e \\
+ (-27 + 48e^2 - 92e^4 + 48e^6)(\arcsin e)^2 \right] \right\} / \left\{ 28e^5 \left[ 3e\sqrt{1 - e^2} + (-3 + 2e^2)\arcsin e \right] \right\}, \tag{B9}
\]

\[
\lim_{\xi \to 0} l_{\xi} = \frac{3}{140e^{15}} \left[ e^2(-1350 + 2295e^2 - 1053e^4 + 77e^6 - 46e^8) \\
- \frac{e}{\sqrt{1 - e^2}}(-2700 + 6390e^2 - 4926e^4 + 1269e^6 - 84e^8 + 32e^{10})\arcsin e \\
+ 9(-150 + 305e^2 - 192e^4 + 36e^6)(\arcsin e)^2 \right], \tag{B10}
\]
\[
\lim_{\xi \to 0} \frac{1}{\xi} = \frac{1}{860, 160 C_0^{24}} \left[ e^4 \left( 123, 525 - 274, 050 e^2 - 652, 320 e^4 + 2, 713, 290 e^6 + 6, 768, 787 e^8 - 25, 404, 672 e^{10} + 25, 029, 960 e^{12} - 9, 597, 704 e^{14} + 1, 941, 024 e^{16} - 475, 104 e^{18} - 172, 736 e^{20} \right) + 6 e^3 \sqrt{1 - e^2} \left( -82, 350 + 155, 250 e^2 + 337, 455 e^4 - 1, 225, 720 e^6 - 2, 115, 507 e^8 + 7, 443, 532 e^{10} - 6, 208, 724 e^{12} + 1, 920, 672 e^{14} - 327, 520 e^{16} + 44, 544 e^{18} + 55, 296 e^{20} \right) \arcsin e - 18 e^2 \left( -41, 175 + 105, 075 e^2 + 51, 540 e^4 - 479, 355 e^6 + 97, 129 e^8 + 1, 327, 914 e^{10} - 1, 798, 488 e^{12} + 920, 144 e^{14} - 203, 264 e^{16} + 20, 480 e^{18} \right) \arcsin e e^2 + 54 e^2 \sqrt{1 - e^2} \left( -9150 + 20, 300 e^2 + 1645 e^4 - 48, 155 e^6 + 56, 168 e^8 - 23, 112 e^{10} + 2304 e^{12} \right) \arcsin e e^3 + 405 \left( -1 + e^2 \right)^{3/2} \left( 305 - 270 e^2 + 36 e^4 \right) \arcsin e e^4 \right], \\
\lim_{\xi \to 0} \frac{1}{\xi} = \frac{1}{g_\xi} = \left\{ \begin{array}{l}
9 e^2 \left( -225 + 474 e^2 + 289 e^4 + 34 e^6 + 30 e^8 + 36 e^{10} \right) + 2 e^2 \sqrt{1 - e^2} \left( 225 - 399 e^2 + 186 e^4 - 9 e^6 + 16 e^8 \right) \arcsin e + 9 \left( -25 + 61 e^2 - 48 e^4 + 12 e^6 \right) \arcsin e e^2 \right\} \left/ \left( 70 e^2 \left( 1 - e^2 \right)^{1/6} \left( 3 e^2 \sqrt{1 - e^2} + 3 e^2 \sqrt{1 - e^2} \right) \arcsin e e^2 \right) \right\} .
\]

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