Tilings of non-convex Polygons, skew-Young Tableaux
and determinantal Processes

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1 Introduction and main results

The purpose of this paper is to study random lozenge tilings of non-convex polygonal regions. Non-convex figures are particularly interesting due to the appearance of new statistics for the tiling fluctuations, caused by the non-convexities themselves or by the interaction of these non-convexities. The final goal will be to study the asymptotics of the tiling statistical fluctuations in the neighborhood of these non-convexities, when the polygons tend to an appropriate scaling limit.

The tiling problems of hexagons by lozenges goes back to the celebrated 1911-formula on the enumeration of lozenge tilings of hexagons of sides $a, b, c, a, b, c$ by the Scottish mathematician MacMahon [26]. This result has been extended in the combinatorics community to many different shapes, including non-convex domains, in particular to shapes with cuts and holes; see e.g., Ciucu, Fischer and Krattenthaler [8, 24].

Tiling problems have been linked to Gelfand-Zetlin cones by Cohn, Larsen, Propp [9], and to non-intersecting paths, determinantal processes, kernels and random matrices by Johansson [14, 15, 16]. In [17], Johansson showed that the statistics of the lozenge tilings of hexagons was governed by a kernel consisting of discrete Hahn polynomials; see also Gorin [20]. In [18] and [19], it was shown that, in appropriate limits, the tiles near the boundary between the frozen and
stochastic region (arctic circle) fluctuate according to the Airy process and near
the points of tangency of the arctic circle with the edge as the GUE-minor process.

Tiling of non-convex domains were investigated by Okounkov-Reshetikhin [28]
and Kenyon-Okounkov [22] from a macroscopic point of view. Further important
phenomena for nonconvex domains appear in the work of Borodin, Gorin and
Rains [5], Defosseux [10], Metcalfe [27], Petrov [29, 30], Duse and Metcalfe [12,
13], and Duse, Johansson and Metcalfe [11]. Our present study was motivated
by a new statistics which came up in the context of overlapping Aztec diamonds
[2, 3], when the size of the overlap remains small compared to the size of the
diamonds. It also appears in the context of GUE-matrices with a certain coupling
[3]. This led us to investigate determinantal processes for lozenge tilings of fairly
general non-convex polygons, with non-convexities facing each other. The present
work was motivated by the question whether tuning the sizes of the polygons and
the non-convexities might lead to some new statistics. This is going much beyond
Petrov’s work [30] on the subject, and yet inspired by some of his techniques.

In this paper, we consider a hexagon with several cuts as in Fig.1, and a tiling
with lozenges of the shape as in Fig.2, colored blue, red and green; the cuts can
only be covered with red tiles. Note there is an affine transformation from our
tiles to the usual ones in the literature; see e.g. the simulation of Fig.5. The
right-leaning blue tiles turn into our blue ones, the up-right red ones into our red
ones and the left-leaning green tiles (30°) to our green tiles (45°), all as in Fig.2.

We will be dealing with three different models:

- **Multi-cut model**: many cuts along the upper- and the lower-edge.
- **Lower one-cut model**: many cuts on the upper-edge and one cut on the
  lower-edge.
- **Two-cut model**: one cut on the upper-edge and one on the lower-edge.

Two different determinantal processes, a $K$-process and an $L$-process, will be
considered, depending on the angle at which one looks at the polygons; south
to north for the $K$-process or south-west to north-east for the $L$-process. In this
series of two papers, the first one will focus on the $K$-process, and the second
one on the $L$-process and its asymptotic limit in between the non-convexities.
Nevertheless both processes will be introduced in this paper. The first paper
shows the $K$-process is determinantal and we give its kernel. We believe the $K$-
kernel is a *master-kernel*, from which many new and old kernels can be deduced
in appropriate asymptotic limits.

As shown in Figure 1, we consider a **general non-convex polygonal re-
gion $P$ (multi-cut model)** consisting of taking a hexagon where two opposite
edges have cuts, $u − 1$ cuts $b_1, b_2, \ldots, b_{u−1}$ cut out of the upper-part and $\ell$
cuts $d_1, d_2, \ldots, d_\ell$ cut out of the lower-part. Let $d := \sum_\ell d_i$. Let $b_0$ and $b_u$
be the

\footnote{The $b_i$ and $d_i$’s also denote the size of the cuts.}
“cuts” corresponding to the two triangles added to the left and the right of $\mathcal{P}$ and let $d_0$ be the size of the oblique side. Then $N = b_0 + d_0$ is the distance between the lower and upper edges. The intervals separating the upper-cuts (resp. lower-cuts) are denoted by $n_i$ (resp. $m_i$) and we require them to satisfy $\sum_{i=1}^{\ell+1} m_i = \sum_{i=1}^{d} n_i$, which is equivalent to $\sum_{i=0}^{u} b_i = d + N$. Define $\check{\mathcal{P}}$ to be the quadrilateral (with two parallel sides) obtained by adding the red triangles to $\mathcal{P}$, as in Fig.1.

Introduce the coordinates $(x, n) \in \mathbb{Z}^2$, where $n = 0$ and $n = N$ refer to the lower and upper sides of the polygon, with $x$ being the running variable along the lines $n$. The vertices of $\mathcal{P}$ and $\check{\mathcal{P}}$ all belong to the vertical lines $x = \{\text{half-integers}\}$ of the grid (in Fig.1). The $d = \sum_{i=1}^\ell d_i$ integer points in $\mathcal{P} \setminus \check{\mathcal{P}} \cap \{n = 0\}$ are labeled by $y_1 > \cdots > y_d$. We complete the set that set with the integer points to the left of $\check{\mathcal{P}}$ along $\{n = 0\}$; they are labeled by $y_{d+1} > \cdots > y_{d+N}$ and we set $y_{d+1} = -d - 1$ and $y_{d+N} = -d - N$. This fixes the origin of the $x$-coordinate. Similarly, the integer points $\mathcal{P} \setminus \check{\mathcal{P}} \cap \{n = N\}$ are labeled by $x_1 > \cdots > x_{d+N}$. We assume that $x_i \geq y_i$ for all $1 \leq i \leq d + N$, and that $y_d \notin \{x$-coordinates of an upper-cut$\}$. Define $c := \max\{i \mid x_i > y_d\}$, $b := N-c$ and the set $\mathcal{R} = \{x_1, \ldots, x_c\}$; assume that both $y_1 - x_c$, $y_1 - y_d \leq N - 1$. It follows that $x_{c+1} < y_d < x_c$. We define polynomials$^2$

$$P(z) := (z - y_d + 1)^{N - (y_1 - y_d + 1)} \quad \text{and} \quad Q(z) := \prod_{i=1}^{d+N} (z - x_i).$$

(1)

Although most statements in this paper will remain valid for the general polygons (we will indicate which ones need to be modified), we will often concentrate on the polygonal shape $\mathcal{P}$, as in the two-cut model of Fig. 3, with $\ell = 1$, $u = 2$ and two cuts of same size $d := d_1 = b_1$, one on top and one at the bottom, with equal opposite parallel sides, i.e., $b_u = d_0$ and $N - b_u = N - d_0$. From the definition of $c$ above and the assumptions, we have that, as in the general case,

$$x_{c+1} < y_d < x_c;$$

(2)

$^2$ For any integers $k \in \mathbb{Z}$ and $N \geq 0$ we have $k_0 = 1$ and $k_N = k(k + 1) \ldots (k + N - 1)$.
at \((x,n)\) respectively. The left-most point of the lower cut and of the upper cut are located
implying that the left most and right most points of the hexagon for the two-cut case along \(n=0\) are given by \(x = -d - 1/2\) and \(x = m_1 + m_2 - 1/2\) respectively. The left-most point of the lower cut and of the upper cut are located at \((x,n) = (m_1 - d - \frac{1}{2}, 0)\) and \((x,n) = (n_1 - c - d - \frac{1}{2}, N)\) respectively. Note that the most-right point \(y_1 = m_1 - 1\) in the lower-cut will play an important role!

Besides the \((x,n)\)-coordinates, another set of coordinates \((\xi, \eta)\) will also be convenient (see Fig. 3):

\[
\eta = n + x + \frac{1}{2}, \quad \xi = n - x - \frac{1}{2} \quad \iff \quad x = \frac{1}{2}(\eta - \xi - 1), \quad n = \frac{1}{2}(\eta + \xi). \quad (3)
\]

**The \(K\) and \(L\)-processes.** Given a covering of this polygonal shape with tiles of three shapes, colored in red, blue and green tiles, as in Fig. 2, put a red dot in the middle of the red tiles and a blue dot in the middle of the blue tiles. The red dots belong to the intersections of the vertical lines \(x = \text{integers}\) and the horizontal lines \(n = 0, \ldots, N\); they define a point process \((x,n)\), which we call the \(K\)-process. The initial condition at the bottom \(n=0\) is given by the \(d\) fixed red dots at integer locations in the lower-cut, whereas the final condition at the top \(n=N\) is given by the \(d+N\) fixed red dots in the upper-cut, including the red dots to the left and to the right of the figure, all at integer locations. Notice that the process of red dots on \(\tilde{P}\) form an interlacing set of integers starting from \(d\) fixed dots (contiguous for the two-cut and non-contiguous for the multi-cut model) and growing linearly to end up with a set of \(d+N\) (non-contiguous) fixed dots. This can be viewed as a “truncated” Gel’fand-Zetlin cone!

The blue dots belong to the intersection \(\in P\) of the parallel oblique lines \(x + n = k - \frac{1}{2}\) with the horizontal lines \(n = \ell - \frac{1}{2}\) for \(k, \ell \in \mathbb{Z}\); in terms of the coordinates (3), the blue dots are parametrized by \((\xi, \eta) = (2\ell - k - 1, k) \in \mathbb{Z}^2\), with \((k, \ell)\) as above. It follows that the \((\xi, \eta)\)-coordinates of the blue dots satisfy \(\xi + \eta = 1, 3, \ldots, 2N - 1\). This point process defines the \(L\)-process. The blue dots on the oblique lines also interlace, going from left to right, but their numbers go up, down, up and down again, with a special feature, which is now explained in the two-cut model: their numbers along the oblique lines \(x + n = -\frac{1}{2} + k\) (i.e., \(\eta = k\)) in the strip \(\{\rho\}\) (defined just below) will exactly equal \(r := b - d\), a local minimum along those oblique lines; this will be shown later.

As it turns out, the two kernels \(K\) and \(L\) are highly related, as will be shown in Paper II, where also the asymptotics for the \(L\)-kernel will be carried out. Indeed,
Fig. 2. Three types of tiles, with a height function and with a level line.

Fig. 3: Tiling of a hexagon with two opposite cuts of equal size \((Two-cut\ model)\), with red, blue and green tiles. Here \(d = 2\), \(n_1 = n_2 = 5\), \(m_1 = 4\), \(m_2 = 6\), \(b = 3\), \(c = 7\), and thus \(r = 1\), \(\rho = 2\), \(\sigma = 4\). The \((x, n)\)-coordinates have their origin at the black dot and the \((\xi, \eta)\)-coordinates at the circle given by \((x, n) = (-\frac{1}{2}, 0)\). Red tiles carry red dots on horizontal lines \(n = k\) for \(0 \leq k \leq N\) \((K\text{-process})\) and blue tiles blue dots on oblique lines \(\eta = k\) for \(-d + 1 \leq k \leq m_1 + m_2 + b - 1\) \((L\text{-process})\). The left and right boundaries of the strip \(\{\rho\}\) are given by the dotted oblique lines \(\eta = m_1\) and \(\eta = m_1 + \rho\). Finally, the tilings define \(m_1 + m_2\) non-intersecting paths.
both point processes can be described by dimers on the points of the associated bipartite graph dual to \( \mathbf{P} \); the two kernels are related by the inverse Kasteleyn matrix of the bipartite graph. The aim of this first paper is to find a kernel for the \( \mathcal{K} \)-process, which moreover is in a good form for taking asymptotic limits in the two-cut model.

For future use in the **two-cut model**, two strips within \( \mathbf{P} \) will play a role: (see Fig. 3)

(i) an **oblique strip** \( \{ \rho \} \) extending the oblique segments of the upper- and lower-cuts; that is the region between the lines \( x + n = -\frac{1}{2} + k \) or what is the same \( \eta = k \) for \( m_1 \leq k \leq n_1 + b - d \). The strip \( \{ \rho \} \) has width

\[
\rho := n_1 - m_1 + b - d = m_2 - n_2 + b - d, \tag{4}
\]

and we assume that \( \rho \geq 0 \).

(ii) a **vertical strip** \( \{ \sigma \} \) extending the vertical segments of the upper- and lower-cuts; that is the region between the lines \( x = n_1 - c - \frac{1}{2} \) and \( x = m_1 - d - \frac{1}{2} \). The strip \( \{ \sigma \} \) has width (again same notation for the **name and the width** of the strip!)

\[
\sigma := m_1 - n_1 + c - d = n_2 - m_2 + c - d \geq 0; \tag{5}
\]

this inequality follows from (2). That \( \rho, \sigma \geq 0 \) amounts to the inequalities:

\[
d - b \leq n_1 - m_1 = m_2 - n_2 \leq c - d.
\]

It is natural to assume that the strips \( \{ \rho \} \) (respectively \( \{ \sigma \} \)) have no point in common with the vertical parts (respectively oblique parts) of the boundary \( \partial \mathbf{P} \). This condition for \( \{ \rho \} \) implies \( \{ x + n = -\frac{1}{2} + m_1 \} \cap \{ n = N \} > -c - d - \frac{1}{2} \) and \( \{ x + n = -\frac{1}{2} + n_1 + b - d \} \cap \{ n = 0 \} < m_1 + m_2 - \frac{1}{2} \), implying \( d - b > \max(-m_1, -n_2) \). The condition for the strip \( \{ \sigma \} \) implies \( n_1 - c - \frac{1}{2} > -d - \frac{1}{2} \) and \( m_1 - d - \frac{1}{2} < m_1 + m_2 - c - \frac{1}{2} \), implying \( c - d < \min(n_1, m_2) \). These inequalities combined with condition \( \rho, \sigma \geq 0 \) above imply
Fig. 5. Computer simulation for $n_1 = 105$, $n_2 = 95$, $m_1 = m_2 = 100$, $b = 25$, $c = 30$, $d = 20$ and $n_1 = 50$, $n_2 = 30$, $m_1 = 20$, $m_2 = 60$, $b = 30$, $c = 60$, $d = 20$. Courtesy of Antoine Doeraene.

$$\max(-n_2, -m_1) < d - b \leq m_2 - n_2 = n_1 - m_1 \leq c - d < \min(m_2, n_1). \quad (6)$$

The line $n = N$ consists of three separate sets of integers $\mathbb{L}(eft)$ polygon $P$, a $\mathbb{L}(eft)$ region, an upper-$\mathbb{L}(eft)$ region and the $\mathbb{R}(ight)$ region, containing respectively $b$, $d$ and $c$ integers; in total $N + d$ integers; to wit:

$$
\mathbb{L} := \{x_{d+c+b}, \ldots, x_{d+c+1}\}
\mathbb{C} := \{x_{c+d}, \ldots, x_{c+1}\}
\mathbb{R} := \{x_c, \ldots, x_1\}. \quad (7)
$$

Still for the two-cut model, we define two other sets of $\sigma$ and $\rho$ contiguous integers on the line $y = N$:

$$
\{\bar{\sigma}\} := \{\sigma\} \cap \{n = N\} = \{x_{c+1} + 1, \ldots, y_d - 1\} = \{n_1 - c, \ldots, m_1 - d - 1\}
\{\bar{\rho}\} := \{\rho\} \cap \{n = N\} = \{x_{c+d} - \rho, \ldots, x_{c+d} - 1\}
\quad = \{m_1 - b - c, \ldots, n_1 - c - d - 1\}.
$$
The inequalities \( \mathcal{C} \subset \mathcal{L} \subset \mathcal{R} \) imply that each of the sets \( \mathcal{C} \cup \{ \bar{\sigma} \} \) and \( \{ \bar{\rho} \} \cup \mathcal{C} \) form a contiguous set of integers, such that each of the three sets are completely separated: \( \mathcal{C} < \{ \bar{\sigma} \} < \mathcal{R} \) and \( \mathcal{L} < \{ \bar{\rho} \} < \mathcal{C} \).

Besides the polynomial \( P(z) \) of degree \( N - d \), defined in (8), for which here \( y_1 - y_d + 1 = d \), we define another polynomial \( Q(z) \) of degree \( N + d \), as follows:

\[
P(z) = (z - y_d + 1)_{N-d} =: P_{\rho}(z)Q_{\bar{\sigma}}(z)P_{\sigma}(z)
= (z - x_{d+c} + 1)\rho(z - x_{c+1})\sigma(z - y_d + 1)\sigma
\]

\[
Q(z) := Q_{\mathcal{L}}(z)Q_{\mathcal{C}}(z)Q_{\mathcal{R}}(z) = (z - x_{d+c+1})\sigma(z - x_{c+1})\sigma(z - x_1)c,
\]

where \( P_{\rho}, P_{\sigma}, Q_{\mathcal{L}}, Q_{\mathcal{C}}, Q_{\mathcal{R}} \) are monic polynomials whose roots are given by the sets \( \{ \bar{\rho} \}, \{ \bar{\sigma} \}, \mathcal{L}, \mathcal{C} \) and \( \mathcal{R} \), respectively. Throughout the paper, we will use “the contours \( \Gamma_{\mathcal{R}} \) and \( \Gamma_{\mathcal{L}} \) in \( \mathbb{C} \), encompassing the points of \( \mathcal{R} \) and \( \mathcal{L} \) only”, (9) and no other points of the integrands. The same holds for the notation \( \Gamma(\text{set of points}) \).

The following rational function

\[
h(u) := \frac{Q_{\mathcal{R}}(u)}{P_{\rho}(u)P_{\sigma}(u)Q_{\mathcal{L}}(u)},
\]

appears crucially in the following \( k \)-fold contour integral \(^3\) for \( k \geq 0 \), (see (9))

\[
\Omega_k^\pm(v, z) := \left( \prod_{\alpha=1}^{k} \oint_{\Gamma(\mathcal{L})} \frac{du_{\alpha}h(u_{\alpha})(z - u_{\alpha})^{\pm 1}}{2\pi i (v - u_{\alpha})} \right) \Delta_k^\pm(u).
\]

The oblique lines \( \eta = k + \frac{1}{2} \) for integer \( k \) within \( \mathcal{P} \) contain the red dots (\( \mathcal{K} \)-dots). Among those oblique lines, the left-most oblique one within the strip \( \{ \rho \} \) is given by the line \( y + n = y_1 + 1 \); it divides the polygon \( \mathcal{P} \) into two regions \( y + n \geq y_1 + 1 \text{ and } y + n < y_1 + 1 \); see Fig.3. The distance (measured horizontally)

\[
\tau := (y + n) - (y_1 + 1)
\]

between \((y, n) \in \mathcal{P}\) and the oblique line \((y_1 + 1 \text{ if } \tau < 0 \text{ strictly to the left and } \tau \geq 0 \text{ on the right of that line } \{ \tau = 0 \} \). Similarly the distance

\[
\tau' := (x + m) - (y_1 + \rho)
\]

between \((x, m) \in \mathcal{P}\) and the most-right oblique line \(m + x = y_1 + \rho \text{ if } \tau' = 0 \text{ in the strip } \{ \rho \} \) (also containing red dots) will also play a role. We now state the main Theorems of the paper, valid in the two-cut model:

\(^3\)Set \( \Omega_0^\pm(v, z) = 1 \). A shorthand notation for the Vandermonde is \( \Delta_k(u) := \Delta_k(u_1, \ldots, u_k) = \prod_{1 \leq i < j \leq k} (u_i - u_j) \).
Theorem 1.1  For \((m, x)\) and \((n, y)\) ∈ \(P\), the determinantal process of red dots in the two-cut case is given by the kernel, involving at most \(r + 2\)-fold integrals, where \(r = b - d\) (defined earlier),

\[
\mathbb{K}(m, x; n, y) = -\frac{(y - x + 1)_{n-m-1}}{(n - m - 1)!} \mathbb{1}_{n > m} \mathbb{1}_{y \geq x} + \frac{(N - n)!}{(N - m - 1)!} \oint_{\Gamma(x+N)} \frac{dv}{2\pi i Q(v) Q(v+\delta)} \times \\
\left( \oint_{\Gamma_{\infty}} \frac{dz}{2\pi i (z-v)} \frac{Q(z) Q(z+v) \Omega_{\tau+1}^+(0, 0)}{(z-y)_{n+1}} + \int_{\Gamma_{\tau'}} \frac{dz}{2\pi i (z-y)} \frac{P(z) Q(z) \Omega_{\tau+1}^+(0, 0)}{\Omega_{\tau+1}^+(0, 0)} \right),
\]

where

\[
\Gamma(x+N) := \text{contour containing the set } x + N = \{x, x+1, \ldots\} \\
\Gamma_{\infty} := \text{very large contour containing all the poles of the } z\text{-integrand} \\
\Gamma_{\tau} := \Gamma(y + n - N, \ldots, \min(y_1 - N, y)) 1_{\tau < 0} \\
\quad = \Gamma(x+c-d - \rho + \tau, \ldots, \min(x+c-d - \rho - 1, y)) 1_{\tau < 0}.
\]

The kernel \(\mathbb{K}\) for the multi-cut case will be given in Proposition 7.2 of section 7 and Theorem 7.1' of section 10.

Defining

\[
\theta := m_1 + m_2 - c - d - 1, \quad \delta = n_1 - m_2 = m_1 - n_2,
\]

here is the second main Theorem, due to an involution:

Theorem 1.2  The kernel \(\mathbb{K}\) for the two-cut model has the alternative form:

\[
\mathbb{K}(m, x; n, y) = -\frac{(y - x + 1)_{n-m-1}}{(n - m - 1)!} \mathbb{1}_{n > m} \mathbb{1}_{y \geq x} + \frac{m!}{(n-1)!} \oint_{\Gamma(-y+N)} \frac{dv}{2\pi i Q(v) Q(v+\delta)} \times \\
\left( \oint_{\Gamma_{\infty}} \frac{dz}{2\pi i (z-v)} \frac{Q(z) Q(z+v) \Omega_{\tau+1}^+(0, 0)}{(z-x)_{m+1}} + \int_{\Gamma_{\tau'}} \frac{dz}{2\pi i (z-y)} \frac{P(z) Q(z) \Omega_{\tau+1}^+(0, 0)}{\Omega_{\tau+1}^+(0, 0)} \right),
\]

using the same contours as in previous statement, except for

\[
\Gamma_{\tau'} = \Gamma(\min(n_2 - N - 1, \theta - x), \ldots, \theta - x - m) 1_{\tau' > 0},
\]

with \(\tau'\) as in (13). In this kernel, \(h^{\delta}\) and \(\Omega_{k}^{\delta \pm}\) are given by:

\[
h^{\delta}(u) := \frac{Q_{\rho}(u)}{P_{\rho}(u+\delta) P_{\sigma}(u+\delta) Q_{\sigma}(u)}\]

\[
\Omega_{k}^{\delta \pm}(v, z) := \left( \prod_{\alpha=1}^{k} \oint_{\Gamma(z)} \frac{du_{\alpha} h^{\delta}(u_{\alpha})(z-u_{\alpha})^{\pm 1}}{2\pi i (v-u_{\alpha})} \right) \Delta_{k}^{\pm}(u).
\]

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These formulas can be specialized to known situations: to hexagons with no cuts (Johansson [17]), to non-convex polygons $P$ with cuts at the top only (Petrov [30]), and to the case where the strip $\rho$ reduces to a line (i.e., $\rho = 0$), in which the polygon $P$ (as in Fig.3) can be viewed as two hexagons glued together along one side (Duse-Metcalfe [12]).

Outline. Many steps are necessary to prove this result.

Section 2: Instead of putting the uniform distribution on the red dot-configuration on $P$, it will be more convenient to first consider a non-uniform distribution depending on a parameter $0 < q \leq 1$; this distribution will tend to the uniform one when $q \to 1$. The red dot-configuration will be shown to be equivalent to the set of semi-standard skew-Young Tableaux of a given shape; the latter can be read off the geometry of $P$. The $q$-probability on this set will have a Karlin-McGregor type of formula, which is not surprising due an equivalent formulation in terms of non-intersecting paths. The use of $q$-deformations have been initiated by Okounkov-Reshetikhin [28] and Kenyon-Okounkov [22]. Also it has been used effectively in the work of Petrov [30] for lozenge tilings of hexagons with cuts on the upper-side only.

Section 3 deals with some determinantal identities, but also with a useful, but unusual, integral representation of the elementary symmetric function $h_{y-y_j}(1^n)$. Section 4: Adapting Eynard-Mehta techniques, further developed in [6] and [4], we will construct a Kernel $K_q$, which involves the inverse of a matrix $M$.

Section 5: The matrix $M$ can be transformed so that the inverse is readily computable.

Section 6: The transformed kernel $K_q$ has a multiple integral representation, using integral representations of the many different ingredients.

Section 7: Taking the limit when $q \to 1$ yields a kernel for the multi-cut and the lower one-cut models which involve at worst a $d+2$-fold contour integral, where $d$ is the size of the cuts.

Section 8: This kernel for the two-cut model will then further be reduced to a sum of contour integrations, which are at worst $r+3$-fold integrals.

Section 9: The polygon $P$ has a natural involution, which leads to an alternative form of the kernel. Sections 8 and 9 establish Theorems [3] and [2].

Section 10: This section explains what modifications need to be done for the multi-cut model. It gives another form of the kernel than the one found in Section 7.

2 Interlacing and measures on skew-Young Tableaux

Nonintersecting paths, level lines and the $K$- and $L$-process: The height function on the tiles, given in Fig.2, imply that the heights along the boundary
of the polygon $\mathbf{P}$ are independent of the tiling; for the heights along $\partial \mathbf{P}$, see Fig. 4. The level lines of heights $\frac{1}{2}, \frac{3}{2}, \ldots, m_1 - \frac{1}{2}, m_1 + \frac{1}{2}, \ldots, m_1 + m_2 - \frac{1}{2}$ pass obliquely through green tiles, vertically through blue tiles, and avoid the red tiles. They are the nonintersecting paths going from top to bottom in Fig. 3. It follows that the intersection of the oblique lines $\eta = k$ for integer $-d \leq k \leq n_1 + n_2 + b$ with the level lines determine the blue dots and that the lines $n = k$ for integer $0 \leq k \leq N$ have a red dot at the integers $\notin \text{ level lines}$. In other terms drawing a horizontal line from left to right through the middle of the blue and green tiles in Fig. 3 increases the height by 1 and remains flat along the red tiles. Thus the tilings of the hexagon $\mathbf{P}$ are equivalent to $m_1 + m_2$ non-intersecting level-lines, which in the multi-cut case gets replaced by $\sum m_i = \sum n_i$ non-intersecting level-lines. It follows that

\[
\#\{\text{red dots of the $K$-process on the horizontal line } n = k \text{ within } \tilde{\mathbf{P}}\} = \#\{\text{flat segments of the height function along the line } n = k\} = \left[\text{length of the (line } n = k) \cap \tilde{\mathbf{P}}\right] - \left[\text{increment in height between the two points (line } n = k) \cap \partial \tilde{\mathbf{P}}\right] = (\sum_{i=1}^{\ell+1} m_i + d + k) - (\sum_{i=1}^{\ell+1} m_i) = d + k.
\]

When an oblique line $\eta = k$ traverses a tile, as in Fig. 2, the height increases by 1 for a blue tile and stays flat for a red and green tile. Therefore, we have

\[
\#\{\text{blue dots of the $L$-process on the oblique line } \eta = k \text{ within } \tilde{\mathbf{P}}\} = \left[\text{increment in height along (line } \eta = k) \cap \partial \tilde{\mathbf{P}}\right],
\]

leading to the following pattern for the blue dots for the two-cut model:

\[
\begin{array}{c|c}
\hline
k & \#\{\text{blue dots on } \eta = k\} \\
\hline
n_1 + n_2 + b & 0 \\
\vdots & \vdots \\
n_1 + n_2 & b \\
\vdots & \vdots \\
n_1 + b & b \\
\vdots & \vdots \\
m_1 + \rho & b - d = r \\
\vdots & \vdots \\
m_1 & b - d = r \\
\vdots & \vdots \\
m_1 - d & b \\
\vdots & \vdots \\
b - d & b \\
\vdots & \vdots \\
-d & 0 \\
\hline
\end{array}
\]
This implies that the number of blue dots are $r$ (local minimum) along each of the oblique lines $\eta = k$ within the strip $\{\rho\}$, including its boundary. Outside, that number starts growing linearly in steps of 1 up to $b$, stays fixed for a while and then goes down to 0 linearly in steps of 1.

**The red dot-configurations and skew-Young tableaux.** The red dots on the horizontal lines $\{y = k\} \cap \tilde{P}$ for $0 \leq k \leq N$ are parametrized by $x^{(k)} \in \mathbb{Z}^{d+k}$

$$x^{(k)}_N < \ldots < x^{(k)}_1,$$

subjected to the interlacing pattern below, in short $x^{(k-1)} \prec x^{(k)},$

$$x^{(k)}_{i+1} < x^{(k-1)}_i \leq x^{(k)}_i.$$  \hspace{1cm} (19)

This interlacing follows from an argument similar to [30, 1]. So, we have a truncated Gelfand-Tsetlin cone with prescribed top and bottom:

$$x_{d+N} < \ldots < x_{d+c} < x_{1+c} < \ldots < x_1$$

$$x^{(N-1)}_{d+N-1} < \ldots < x^{(N-1)}_1$$

$$\vdots$$

$$y_d < \ldots < y_2 < \ldots < y_1$$

Note on top we have in the two-cut case the three regions $L, C$ and $R$ of contiguous integers; at the bottom we have one contiguous region, given by the lower-cut. We set $x^{(0)}_k = y_k$ and $x^{(N)}_k = x_k$. The inequalities (19) imply that, given (contiguous) red dots all along $n = N$ in $\tilde{P} \setminus P$, one automatically has the regular pattern of red dots in $\tilde{P} \setminus P$, as illustrated in Figure 1; same remark for $n = 0$.

Moreover, setting for $0 \leq k \leq N$, $\nu^{(k)}_i = x^{(k)}_i + i$, leads to a sequence of partitions

$$\nu^{(0)} \subset \nu^{(1)} \ldots \subset \nu^{(N)},$$  \hspace{1cm} (20)

with prescribed **initial and final condition**,  

$$\mu := \nu^{(0)} \quad \text{and} \quad \lambda := \nu^{(N)}$$

---

4 Define $|x^{(k)}_i| := \sum_{i=1}^{d+k} x^{(k)}_i$ for $0 \leq k \leq N$; in particular for $k = 0$ and $k = N$, we have $|y| = \sum_{i=1}^{d} y_i$ and $|x| = \sum_{i=1}^{d+N} x_i$. That means, the sum is always taken within the parallelogram $\tilde{P}$. Also define $|\nu^{(k)}_i| := \sum_i \nu^{(k)}_i$
and such that each skew diagram $\nu(i) \setminus \nu(i-1)$ is a horizontal strip\(^5\). Note the condition $x_i \geq y_i$ for all $1 \leq i \leq d + N$ on the cuts, mentioned just before (1), guarantees that $\mu \subset \lambda$. In fact the consecutive diagrams $\nu(i) \setminus \nu(i-1)$ are horizontal strips, if and only if the precise inequalities $x_i^{(m)} < x_i^{(m-1)} \leq x_i^{(m)}$ hold.

Putting the integer $i$’s in each skew-diagram $\nu(i) \setminus \nu(i-1)$, for $1 \leq i \leq N$, is equivalent to a skew-Young tableau filled with $N$ numbers $1, \ldots, N$ and exactly $N$. This leads to the following \textbf{semi-standard skew-Young tableau} in the two-cut case, as in Fig.6. An argument, using the height function, which is similar to the one in [1], shows that all configurations are equally likely. So we have \textbf{uniform distribution} on the set of configurations. To conclude, we have:

\[
\{\text{All configuration of red dots}\} \iff \{\text{skew-Young Tableaux of shape } \lambda \setminus \mu \text{ filled with numbers 1 to } N\} \\
\iff \left\{ \begin{array}{l}
\mu = \nu(0) \subset \nu(1) \ldots \subset \nu(N) = \lambda, \text{ with } \mu \subset \lambda \text{ fixed,} \\
\text{ with } \nu(m) \setminus \nu(m-1) = \text{horizontal strip,} 
\end{array} \right\}
\]

with equal probability for each configuration.

---

\[^5\text{An horizontal strip is a Young diagram where each row has at most one box in each column.}\]
Fig. 6: Semi-standard skew-Young Tableau $\lambda \backslash \mu$ filled with numbers $1, \ldots, N$ in one-to-one correspondence with the red dot-process. Semi-standard skew-Young Tableau associated with Fig. 3.

**Uniform and $q$-probability on the set of red dot-configurations.** The last statement above implies that

$$\# \{ \text{configuration of red dots} \} = s_{\lambda \backslash \mu}(1, \ldots, 1, 0, 0 \ldots),$$

and thus we have **uniform probability measure** $P$ on the space of all $N+1$-uples of partitions $\{\nu^{(0)}, \ldots, \nu^{(N)}\}$

$$P(\text{given configuration of red dots}) = P(\nu^{(0)}, \nu^{(2)}, \ldots \nu^{(N)}, \text{ such that } \mu = \nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda)$$

$$= \frac{1}{s_{\lambda \backslash \mu}(1, \ldots, 1, 0, 0 \ldots)} \mathbb{1}_{\nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda}$$

$$= \frac{1}{s_{\lambda \backslash \mu}(1, \ldots, 1, 0, 0 \ldots)} \mathbb{1}_{\nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda}.$$  

(22)

As will be seen, we define a $q$-dependent **probability measure** $P_q$, for $0 < q \leq 1$, on the same space of all $N+1$-uples of partitions, as follows

$$P_q(\nu^{(0)}, \nu^{(2)}, \ldots \nu^{(N)}, \text{ such that } \mu = \nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda)$$

$$= \frac{(q-1)^{\sum_{i=1}^{N-1} |\nu^{(i)}|-(N-1)|\nu^{(0)}|}}{s_{\lambda \backslash \mu}(q^{1-N}, q^{2-N}, \ldots, q^{-1}, q^0)} \mathbb{1}_{\nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda}.$$  

(23)

which for $q \rightarrow 1$ leads to the uniform probability (22). This formula will be explained below.

MacDonald [25] is an excellent reference for the following discussion. Recall skew-Schur polynomials are defined as: for any $\mu \subset \lambda$,

$$s_{\lambda \backslash \mu}(x) := \det(h_{\lambda_i-i+\mu_j+j}(x))_{1 \leq i,j \leq n} = \det(h_{x_i-y_j}(x))_{1 \leq i,j \leq n}, \text{ for } n \geq \ell(\lambda),$$

(24)

where $h_r(x)$ is defined for $r \geq 0$ as

$$\prod_{i \geq 1} (1 - x_iz)^{-1} = \sum_{r \geq 0} h_r(x) z^r,$$

(25)
and \( h_r(x) = 0 \) for \( r < 0 \). Skew Schur polynomials also satisfy the properties

\[
s_{\lambda \setminus \mu}(x, y) = \sum_{\nu} s_{\lambda \setminus \nu}(x) s_{\nu \setminus \mu}(y),
\]

and for \( x = (x_1, \ldots, x_N) \),

\[
s_{\lambda \setminus \mu}(x_1, \ldots, x_N) = \sum_{\mu = \nu^{(0)} \subset \cdots \subset \nu^{(N)}} \prod_{i=0}^{N-1} x_i^{\nu^{(i+1)} - \nu^{(i)}}.
\]

In particular, for \( 0 < q \leq 1 \), from (27)

\[
s_{\lambda \setminus \mu}(q^{1-N}, q^{2-N}, \ldots, q^{-1}, q^0) = \sum_{\mu = \nu^{(0)} \subset \cdots \subset \nu^{(N)}} (q^{-1})^{\sum_{i=1}^{N} (N-i)(|\nu^{(i)}| - |\nu^{(i-1)}|)}
\]

\[
= \sum_{\mu = \nu^{(0)} \subset \cdots \subset \nu^{(N)}} (q^{-1})^{\sum_{i=1}^{N-1} |\nu^{(i)}| - (N-1)|\nu^{(0)}|}.
\]

The expression \( \sum_{i=1}^{N} (N-i)(|\nu^{(i)}| - |\nu^{(i-1)}|) \) in formula (28) is the volume of the three-dimensional figure obtained by putting \( N-i \) cubes of size 1 \( \times \) 1 \( \times \) 1 on top of the squares containing \( i \) in the skew-Young diagram, for \( 1 \leq i \leq N-1 \). This shows why (23) defines a probability measure.

**Some useful formulas:** Notice that

\[
h_r(\alpha^\gamma, \ldots, \alpha^{\gamma-N+1}) = \alpha^{r(\gamma-N+1)} h_r(\alpha^0, \ldots, \alpha^{N-1})
\]

and thus for \( \alpha = q^{-1} \) and \( \gamma = -d \), we have

\[
h_r(q^0, q^{-1}, \ldots, q^{1-N}) = q^{r(1-N-d)} h_r(q^d, \ldots, q^{d+N-1}).
\]

Defining for \( y, z \in \mathbb{R} \) (in terms of the \( q \)-Pochhammer symbol \( (a; q)_k = \prod_{j=0}^{k-1} (1 - aq^j) \) for \( k > 0 \) and = 1 for \( k = 0 \)),

\[
\mathcal{P}_n(z) := \prod_{i=1}^{n} \frac{1 - zq^i}{1 - q^i} =: \frac{(zq; q)_n}{(q; q)_n} =: \sum_{i=0}^{n} c_iz^i \text{ and } \tilde{\mathcal{P}}_n(z) := q^{-dy}\mathcal{P}_n(zq^{-y}),
\]

one checks, using the generating series (25) and the definition of \( \mathcal{P}_n \), that for \( n \geq 0 \),

\[
h_r(q^0, \ldots, q^n) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{du}{u^{r+1}} \prod_{\ell=0}^{n} \frac{1}{1 - uq^\ell} = \mathcal{P}_n(q^r) \text{ for } r \geq 0.
\]
Indeed the left hand side equals 0 for \( r < 0 \), but 0 for \( r < -n \). The integral equals 0 for \( r < 0 \). So, the identity is valid for each of the two indicator functions; both of them will be used. Also by (25),

\[
\begin{align*}
s_{\lambda \mu}(q^0, q^{-1}, \ldots, q^{1-N}) &= \det(h_{x_i-y_j}(q^0, q^{-1}, \ldots, q^{1-N}))_{i \leq j \leq N+d} \\
&= q^{(d+N-1)(\sum_{i=1}^d x_i - \sum_{i=1}^d y_i)} \det(h_{x_i-y_j}(q^d, \ldots, q^{d+N-1}))_{1 \leq i, j \leq N+d} \\
&= q^{-N(d+N-1)(d+(N+1)/2)} q^{(d+N-1) |y|-|x|} \det(H),
\end{align*}
\]

where \( H \) is a square matrix of size \( d + N \):

\[
H = (H_{ij})_{1 \leq i,j \leq N+d} := (h_{x_i-y_j}(q^d, q^{d+1}, \ldots, q^{d+N-1}))_{1 \leq i,j \leq N+d} = (q(x_i-y_j) d h_{x_i-y_j}(q^0, \ldots, q^{N-1}))_{1 \leq i,j \leq N+d},
\]

where formula (30) is used in the second line.

**q-Calculus.** The polynomial \( Q(z) \) has a natural \( q \)-extension, namely (we often write \( q_i := q^x \))

\[
Q_q(z) := \prod_{i=1}^{d+N} (z - q^x_i) = \prod_{i=1}^{d+N} (z - q_i),
\]

and from (31) we have

\[
\begin{align*}
\mathcal{P}_{N-m}(vq^{-x}) &= \prod_{i=1}^{N-m} 1 - \frac{q^i}{1 - q^i} \bigg|_{z=vq^{-x}} = \prod_{i=1}^{N-m} 1 - \frac{vq^{-x}}{1 - q^i} = \frac{(vq^{-x+1}; q)_{N-m}}{(q; q)_{N-m}}.
\end{align*}
\]

Following Petrov [30], we define the \( q \)-hypergeometric function \( \phi(1) \) by means of

\[
\begin{align*}
z \sum_{k=1}^n z^{-k} q^{(k-1)y} \prod_{r=n+1}^{N} (1-q^{r-k}) &=: (q^{N-1}; q^{-1})_{N-n} 2 \phi(1)(q^{-1}, q^{n-1}, q^{N-1} | q^{-1}, q^y) \\
&=: \Phi_q\left(\frac{q^y}{z}\right).
\end{align*}
\]

\( \Delta_k \) stands for the Vandermonde determinant of size \( k \).
Lemma 2.1 For any \( a \in \mathbb{Z} \), remembering the definitions (35) and (31) of \( Q(z) \) and \( \tilde{P}_n^\beta(z) \), the following limits hold

\[
\lim_{q \to 1} P_n(q^x) = \prod_{k=1}^{n} \frac{k + x}{n}, \quad \lim_{q \to 1} \tilde{P}_n^\beta(q^x) = \frac{(x - y + 1)n}{n!},
\]

\[
\lim_{q \to 1 (q-1)^{d+N-1}} Q'(q^x) = Q'(x), \quad \lim_{q \to 1 (q-1)^{d(d-1)/2}} \Delta_d(q^{x_1}, \ldots, q^{x_d}) = \Delta_d(x_1, \ldots, x_d), \quad (37)
\]

and (due to Petrov[30])

\[
\lim_{q \to 1 (q-1)^{N-1}} \frac{1}{g(z)} \int_{\Gamma_\infty} \frac{dz}{2\pi iz} \Phi_q(z^{-1}) \prod_{i=1}^{N-1} (z - q^{\rho_i - y})
\]

\[
= (N - n + 1)! \int_{\Gamma_\infty} \frac{dz}{2\pi i(z - y)_{N-n+2}} \prod_{i=1}^{N-1} (z - \rho_i). \quad (38)
\]

Proof: The limits are straightforward, except for Petrov’s limit formula; its proof can be found in Petrov[30].

3 An alternative integral representation for \( h_{y-y_j}(1^n) \) and some determinantal identities

Proposition 3.1 The following determinantal identity holds, with the constant

\[
C_{N,d} = \prod_{j=1}^{d} \frac{1}{(N-j)!},
\]

\[
\det \left( \frac{(x_\alpha + \beta)_{N-1}}{(N-1)!} \right)_{1 \leq \alpha, \beta \leq d} = C_{N,d} \Delta_d(x_1, \ldots, x_d) \prod_{\alpha=1}^{d} (x_\alpha + d)_{N-d}. \quad (39)
\]

Proof: Each column of the matrix in the determinant \( \det((x_\alpha + \beta)_{N-1})_{1 \leq \alpha, \beta \leq d} \) has the common factor \( (x_\alpha + d)_{N-d} \), which, taken out, leaves us a determinant of the type below; by Lemma 3, Krattenthaler[23], this determinant equals:

\[
\det_{1 \leq i,j \leq n} ((x_i + A_n)(x_i + A_{n-1}) \ldots (x_i + A_{j+1})(x_i + B_j)(x_i + B_{j-1}) \ldots (x_i + B_2))
\]

\[
= \Delta_n(x) \prod_{2 \leq i,j \leq n} (B_i - A_j).
\]

In this expression \( A_j = j - 1, \ B_j = N + j - 2 \) for \( 2 \leq j \leq d \), from which it follows that \( \prod_{2 \leq i,j \leq n} (B_i - A_j) = ((N-1)!^d \prod_{k=1}^{d}((N-k)!)^{-1}. \) This proves identity [39].

\[
^n\text{The explicit form of the constant will never be used.}
\]
Defining $B(y) := \max(y + n, y_1 + 1) - N$, we now define the contours
\[
\gamma_y := \Gamma(y, y - 1, \ldots, B(y)), \text{ if } B(y) \leq y \\
= \emptyset, \text{ if } B(y) > y, \quad (40)
\]
and the contour $\Gamma_\tau$, defined such that
\[
\gamma_y = \Gamma(y, y - 1, \ldots, y + N - N) \setminus \Gamma_\tau, \quad (41)
\]
given that a contour integral will be taken over a rational function with denominator equal to $(z - y)_{N-n+1}$. Then it is easily seen that
\[
\Gamma_\tau := \begin{cases}
\Gamma(\min(y_1 - N, y), \ldots, y + n - N), & \text{if } \tau < 0 \\
\emptyset, & \text{if } \tau \geq 0
\end{cases}, \quad (42)
\]
with $\Gamma_\tau$ containing exactly $-\tau > 0$ points, if $y_1 - N \leq y$ and $N - n + 1$ points, if $y_1 - N \geq y$, and none, when $\tau \geq 0$. This confirms definition (15).

We now state the following general Lemma, which will play a crucial role in the expression of the $K$-kernel.

**Proposition 3.2** Given $0 \leq n \leq N - 1$ and integer points $(y_1 > \cdots > y_j > \cdots)$ satisfying $0 \leq y_1 - y_j \leq N - 1$, the following holds, with the contours $\gamma_y$ and $\Gamma_\tau$ as in (40) and (42)
\[
h_y(y_j) = \oint_{\Gamma_0} \frac{du}{2\pi i u^{y-y_j+1}(1-u)^n} = \frac{(N-n)!}{(N-1)!} \oint_{\gamma_y} \frac{dz(z-y_j+1)_{N-1}}{2\pi i(z-y)_{N-n+1}} \quad (43)
\]

Proof: At first, notice that by Cauchy’s theorem, the integral below, for $n \geq 0$, equals minus the sum of the residues at $z = 1$ and $\infty$, with the residue at $\infty$ vanishing when $n + k - 1 \geq 0$,
\[
\oint_{\Gamma_0} \frac{dz}{2\pi i z^{1+k}(1-z)^n} = \prod_{\ell=1}^{n-1} \frac{k + \ell}{\ell} - \Res_{z=\infty} = \frac{(k+1)_{n-1}}{(n-1)!} - \Res_{z=\infty} \cdot 1_{n+k \leq 0}. \quad (44)
\]

Consider the contour $\gamma_y$ as in (40). If $B(y) > y$, then we have $\max(y + n, y_1 + 1) > y + N$, implying that $y_1 - N > y - 1$, since $0 \leq n \leq N$, and thus
\[ N \geq y_1 - y_j + 1 > y - y_j + N, \] and so \( y - y_j + 1 \leq 0 \), implying that both sides of (43) vanish. In that case, since \( \gamma_y = \emptyset \), we have that, in view of (41),
\[ \Gamma_\tau = \Gamma(y, y - 1, \ldots, y + n - N). \]
So, we now consider \( B(y) \leq y \). We have thus \( \emptyset \neq \{y, y - 1, \ldots, B(y)\} \subset \{y, y - 1, \ldots, y - N + n\} \). Then, since \( y - B(y) = N - n - \max(y_1 + 1 - y - n, 0) \), we have for the right-hand side of (43), upon evaluating by residues and upon using the change of variables \( \ell = y - k \) and \( i = y - j \) in *,
\[
\frac{(N - n)!}{(N - 1)!} \sum_{\ell = B(y)}^{y} \frac{y}{(\ell - y_j + 1) \ldots (\ell - y_j + N - 1)} \prod_{i=y-N+n}^{y} (\ell - i)
\]
\[
= \frac{(N - n)!}{(N - 1)!} \sum_{k=0}^{y-B(y)} \frac{(-1)^k (y - y_j - k + 1)_{N-1}}{k!(N-n-k)!} \prod_{j=0}^{N-n} (j - k)
\]
\[
= \sum_{k=N-n-\max(0,y_1+1-y-n)}^{N-n-\max(0,y_1+1-y-n)} (-1)^k \binom{N-n}{k} \frac{(y - y_j - k + 1)_{N-1}}{(N - 1)!} \tag{45}
\]
provided \( y - y_j - k + N - 1 \geq 0 \) (*) holds for all \( 0 \leq k \leq N - n - \max(0, y_1 + 1 - y - n) \) in the expression on the right-hand side of **. It suffices to check this for the largest value \( k = N - n - \max(0, y_1 + 1 - y - n) \). Indeed, this is so:
(i) If \( y + n \geq y_1 + 1 \), then \( k = N - n \) and \( y - y_j - k + N - 1 = y + n - y_j - 1 \geq y_1 - y_j \geq 0 \).
(ii) If \( y + n < y_1 + 1 \), then \( k = N + y - y_1 - 1 \) and so \( y - y_j - k + N - 1 = y_1 - y_j \geq 0 \); this checks the inequality (*) above. For this same case, namely \( y + n < y_1 + 1 \), and for \( N - n - (y_1 + 1 - y - n) + 1 \leq k \leq N - n \), we also have that the integrand in the last expression (45) has no residue at \( z = 0 \), because the \( z \)-exponent in its denominator equals \( y - y_j - k + 1 \leq y - y_j - (N - n - (y_1 + 1 - y - n) + 1) + 1 = y_1 - y_j - N + 1 \leq 0 \),
using the inequality in the statement. Therefore, in the last expression in (45) the sum can be extended to \(0 \leq k \leq N - n\) and so the final expression reads

\[
\sum_{k=0}^{N-n} (-1)^k \binom{N-n}{k} \int_{\Gamma_0} \frac{z^k \, dz}{2\pi i z - y_i + 1 (1 - z)^N} = \int_{\Gamma_0} \frac{dz}{2\pi i z - y_i + 1 (1 - z)^N} dz
\]

The last formula of (43) follows from formula (41), thus ending the proof of Proposition 3.2.

Both Propositions will be applied to certain determinants, which will come up later, and which depend on the geometry of \(P\). In the corollary below, the \(u_1, \ldots, u_d\) are arbitrary complex variables. Referring to the multi-cut model, remember \(y_1 > \cdots > y_d\) are the integer points in the \(\{\text{lower-cuts}\} \cap \{n = 0\}\) and \(P(z) = (z - y_d + 1)_{N-d}\), as in (1). We define three expressions, which will play an important role in Section 7:

\[
\begin{align*}
\Delta_d(y_{\text{cut}})(u_1, \ldots, u_d) &:= \det \left( \frac{(u_{\alpha} - y_{\beta} + 1)_{N-1}}{(N-1)!} \right)_{1 \leq \alpha, \beta \leq d} \\
\tilde{\Delta}_d(y_{\text{cut}})(w; u_2, \ldots, u_d) &:= \det \left( \begin{array}{ccc}
w^{y_1} & \cdots & w^{y_d} \\
\end{array} \right) \\
\tilde{\Delta}_{d,n}(y; u_2, \ldots, u_d) &:= \int_{\Gamma_0} \frac{dw}{2\pi i w^{y+1}(1 - w)^n} \tilde{\Delta}_d(y_{\text{cut}})(w; u_2, \ldots, u_d) \\
\end{align*}
\]

\[
\begin{align*}
\Delta_d(y_{\text{cut}})(u_1, \ldots, u_d) &= C_{N,d} \prod_{\alpha=1}^d P(u_\alpha), \\
\tilde{\Delta}_{d,n}(y; u_2, \ldots, u_d) &= C_{N,d} \left( \int_{\Gamma(y_{\text{cut}})} \frac{dz}{2\pi i (z - y)^{N-n+1}} - \int_{\Gamma_0} \frac{dz}{2\pi i (z - y)^{N-n+1}} \right) \\
&\times \Delta_d(z, u_2, \ldots, u_d) P(z) \prod_{\alpha=2}^d P(u_\alpha).
\end{align*}
\]

Corollary 3.3 Given the lower one-cut model, the following identities hold for the determinantal expressions in (46), with \(C_{N,d}\) as in (39) and the contour \(\Gamma_\tau\) as in (42):

\[
\Delta_d(y_{\text{cut}})(u_1, \ldots, u_d) = C_{N,d} \Delta_d(u) \prod_{\alpha=1}^d P(u_\alpha),
\]

and

\[
\tilde{\Delta}_{d,n}(y; u_2, \ldots, u_d) = C_{N,d} \left( \int_{\Gamma(y_{\text{cut}})} \frac{dz}{2\pi i (z - y)^{N-n+1}} - \int_{\Gamma_0} \frac{dz}{2\pi i (z - y)^{N-n+1}} \right) \\
\times \Delta_d(z, u_2, \ldots, u_d) P(z) \prod_{\alpha=2}^d P(u_\alpha).
\]
Proof: The first identity (47) follows at once from Proposition 3.1, upon setting $x_\alpha = u_\alpha - y_1$. The second identity (48) follows from moving the $w$-integral to the first row of $\tilde{\Delta}_d(y_{\text{cut}})(w, \ldots, u_d)$, using Proposition 3.2 and finally using the first identity (47), applied to $\Delta_d(y_{\text{cut}})(z, u_2, \ldots, u_d)$:

$$\tilde{\Delta}_d(y_{\text{cut}})(y; u_2, \ldots, u_d) = \oint_{\gamma} (N-n)!dz \frac{d}{2\pi i(z-y)N-n+1} \Delta_d(y_{\text{cut}})(z, u_2, \ldots, u_d)$$

$$= C_{N,d} \oint_{\gamma} (N-n)!dz \frac{d}{2\pi i(z-y)N-n+1} \Delta_d(z, u_2, \ldots, u_d)P(z) \prod_{\alpha=2}^d P(u_\alpha).$$

This ends the proof of Corollary 3.3.

Remark: In the multi-cut case, the same identities hold for the polynomial $P(z)$, as defined in (1), except the right hand side of (47) and the integrand of (48) will involve a symmetric function, as will be discussed in Section 10.

The next lemma deals with identities involving Vandermonde's with certain rows removed. For $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, define the Vandermonde

$$\Delta_m(x) = \det \begin{pmatrix} x_1^{m-1} & \cdots & x_1^0 \\ \vdots & \ddots & \vdots \\ x_m^{m-1} & \cdots & x_m^0 \end{pmatrix} = \prod_{1 \leq i < j \leq m} (x_i - x_j)$$

$$\Delta_{m,k}(x) = \det \begin{pmatrix} x_1^{m-k+1} & \cdots & x_1^0 \\ \vdots & \ddots & \vdots \\ x_m^{m-k+1} & \cdots & x_m^0 \end{pmatrix} = \Delta_m(x)e_{m-k}(x),$$

where $\hat{k}$ refers to removing the column containing the $k$th power. Also

$$e_k(y) = \sum_{i_1 < i_2 < \cdots < i_k} y_{i_1} \cdots y_{i_k}.$$ 

We will need the following Lemma:

Lemma 3.4 Then we have (here $Q(z) := \prod_1^m(z-x_i)\prod_1^n(z-y_i)$)

$$\frac{\Delta_{n+m, \hat{k}}(y)}{\Delta_{n+m}(x,y)} = (-1)^{(m-1)/2} \frac{\Delta_m(x)}{\prod_1^m Q'(x_{\ell})} e_k(y),$$

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and so in particular, setting \( k = 0 \),
\[
\frac{\Delta_n(y)}{\Delta_{n+m}(x,y)} = (-1)^{m(m-1)/2} \frac{\Delta_m(x)}{\prod_{i} Q'(x_i)}.
\] (49)

The following symmetric function, with variables removed, has the integral representation: (with \( Q_q \) defined in (35))
\[
(-1)^{r-1} e_{r-1}(q_1, \ldots, \hat{q}_k, \hat{q}_i, \ldots, \hat{q}_{d}, \ldots, q_{d+N}) = \oint_{\Gamma_{\infty}} \frac{dz}{2\pi i z^{r-1}(z - q_k) \prod_{\ell=1}^{d}(z - q_{i})}.
\] (50)

**Proof:** The first relation is shown by multiplication by \( z^{n-k}(-1)^k \) and summing from \( k = 0 \) to \( n \), which leads to an obvious identity. The second is straightforward and the third is just the residue Theorem.

### 4 From Karlin-McGregor to the \( K_q \)-kernel

Since the \( K \)-process of red dots has a nonintersecting paths description, one expects to have a Karlin-McGregor formula for the \( q \)-dependent probability measure \( P_q \), as in (23) above. This will be shown in Proposition 4.1. Then in Proposition 4.2, we will obtain a kernel by adapting the arguments in Borodin-Ferrari-Prahofer [4] and in Borodin-Rains [6] to these new circumstances. This section refers to the multi-cut model.

We first need some notation. Remember from (19) the interlacing pattern \( 1_{x^{(m-1)}}^{x(m)} \) for \( 1 \leq m \leq N \). To the left of each sequence \( x^{(m-1)} \), we add an extra point, a so-called virtual point \( x^{(m-1)} \) \( = \) virt; it can be thought of as a point at \( -\infty \). Then we define for \( x \in \{ \text{virt} \} \cup \mathbb{Z} \), \( z \in \mathbb{Z} \) and \( k \geq 0 \), the following functions, where the matrix \( H \) was defined in (51):
\[
\chi_i(z) := 1_{z=y_i}, \quad \text{for } 1 \leq i \leq d
\]
\[
\varphi_k(x, z) := q^{(k-1)(z-x)} 1_{x \leq z} \cdot 1_{x=\text{virt}} + q^{(k-1)z} 1_{x=\text{virt}}
\]
\[
= \frac{1}{2\pi i} \oint_{\Gamma_0} du \int_{-\infty}^{\infty} \frac{du}{1 - u q^{k-1}} \left( \frac{1}{u z - z + 1} \right) + q^{(k-1)z} 1_{x=\text{virt}}
\]
\[
= h_{z-x}(q^{k-1}) + q^{(k-1)z} 1_{x=\text{virt}}
\]
\[
\psi_i(z) := \sum_{j=1}^{N+d} (H^{-1})_{ij} 1_{z=x_j}, \quad \text{for } 1 \leq i \leq d + N.
\]

So, in particular we have \( \varphi_k(\text{virt}, z) = q^{(k-1)z} \).
Proposition 4.1  The probability (23) can be written as a product of \( N + 2 \) determinants
\[
P_q(\nu^{(0)}, \nu^{(1)}, \ldots, \nu^{(N)}), \text{ such that } \mu = \nu^{(0)} \subset \nu^{(1)} \subset \ldots \subset \nu^{(N)} = \lambda)
\]
\[
= C_{N,d,q} \det(\chi_{i}(x_{j}^{(0)}))_{1 \leq i,j \leq d}
\times \prod_{m=1}^{N} \det(\varphi_{d+m}(x_{i}^{(m-1)}, x_{j}^{(m)}))_{1 \leq i,j \leq d+m} \det(\psi_{i}(x_{j}^{(N)}))_{1 \leq i,j \leq N+d},
\]
with
\[
C_{N,d,q} = q^{dN(d+N)+\frac{1}{2}N(N^2-1)}.
\]

Define the following \(*\)-operation between two real or complex functions \( f(k) \) and \( g(k) \) defined for \( k \in \mathbb{Z} \):
\[
f * g := \sum_{k \in \mathbb{Z}} f(k)g(k);
\]
e.g., for functions \( \varphi_k \) as above, the operation \(*\) means a convolution \( (\varphi_k * \varphi_{k+1})(x,y) = \varphi_k(x,\cdot) * \varphi_{k+1}(\cdot, y) = \sum_{z \in \mathbb{Z}} \varphi_k(x,z)\varphi_{k+1}(z,y) \). Subsequently, define
\[
\varphi^{n,m+1}(x,y) := (\varphi_n * \cdots * \varphi_m)(x,y)\delta_{n \leq m}.
\]
The functions \( \psi_j \), as in (51), have a natural extension in terms of the geometry of \( \mathcal{P} \): namely, for \( 0 \leq m,n \leq N - 1 \) and \( 1 \leq k \leq d + N \), define:
\[
\psi^{(m+1)}_k(x) := \varphi_{d+m+1}(x,\cdot) * \cdots * \varphi_{d+N}(\cdot, \cdot) * \psi_k(\cdot) = \varphi^{d+m+1,d+N+1}(x,\cdot) * \psi_k(\cdot)
\]
and for \( m = N \) we have \( \psi^{(N+1)}_j = \psi_j \).

Proposition 4.2  The \( \mathbb{K}_q \)-point-process is determinantal with kernel, for \( 0 \leq m,n \leq N \), given by
\[
\mathbb{K}_q(m,x;n,y) = -\varphi_{d+m+1} * \cdots * \varphi_{d+n}(x,y)
+ \sum_{\substack{1 \leq k \leq d+N \\ 1 \leq \ell \leq d}} \psi^{(m+1)}_k(x)(M^{-1})_{k\ell}(\chi_{\ell} * \varphi^{d+1,d+1+n})(y)
+ \sum_{\substack{1 \leq k \leq d+N \\ d+1 \leq \ell \leq d+n}} \psi^{(m+1)}_k(x)(M^{-1})_{k\ell}(\varphi_{\ell}(\text{virt},\cdot) * \varphi^{d+1,d+1+n}(\cdot,y)),
\]
\footnote{The \( \ell \)-summation in the third term of (56) goes up to \( \ell = d + n \); that term contains \( \varphi^{d+n+1,d+n+1} \), which we declare = 1.}
where for $1 \leq i, j \leq N + d$:

$$
M_{k\ell} := \begin{cases} 
\chi_k(\cdot) \ast \psi_{\ell}^{(1)}(\cdot), & 1 \leq k \leq d \\
\varphi_k(virt, \cdot) \ast \psi_{\ell}^{(k+1-d)}(\cdot), & d + 1 \leq k \leq d + N.
\end{cases}
$$

(57)

Before giving the proof of Propositions 4.1 and 4.2, the following Lemma will be needed:

**Lemma 4.3** The following determinantal identities hold: for $1 \leq m \leq N$,

$$
\det(x_i^{(0)})_{1 \leq i,j \leq d} = \prod_{j=1}^{d} \mathbb{1}_{x_j = y_j}.
$$

$$
\det(x_i^{(m-1)}, x_j^{(m)})_{1 \leq i,j \leq d+m} = q^{(k-1)(|x^{(m)}| - |x^{(m-1)}| - d - m)} \mathbb{1}_{x^{(m-1)} \prec x^{(m)}}.
$$

$$
\det(x_i^{(N)})_{1 \leq i,j \leq N+d} = \det(h_{x_i - y_j}(q^d, \ldots, q^{d+N-1}))^{-1} \prod_{j=1}^{d+N} \mathbb{1}_{x_j = x_j}.
$$

(58)

**Remembering the **-operation (53), we have the following convolution properties for $x, y \in \mathbb{Z}$, with the last identity valid for $1 \leq k \leq d + N, 1 \leq m \leq N$:

$$
\varphi^{n+1,m+1}(x, y) = (\varphi_{n+1} \ast \ldots \ast \varphi_m)(x, y) = h_{y-x}(q^n, \ldots, q^{m-1}) \mathbb{1}_{n<m}
$$

$$
= \frac{1}{2\pi i} \oint_{|z| = 1} \frac{dz}{z^{y-x+1}} \prod_{\ell=n}^{m-1} \frac{1}{1 - u q^\ell} \mathbb{1}_{n<m}
$$

$$
\chi_{\ell} \ast (\varphi_{n+1} \ast \ldots \ast \varphi_m)(y) = h_{y-x}(q^n, \ldots, q^{m-1}) \mathbb{1}_{n<m}
$$

$$
\varphi_n(virt, \cdot) \ast (\varphi_{n+1} \ast \ldots \ast \varphi_m)(\cdot, y) = \mathbb{1}_{n<m} \frac{q^{(n-1)y}}{(1-q) \ldots (1-q^{m-n})}
$$

$$
\varphi_n(virt, \cdot) \ast (\varphi_{n+1} \ast \ldots \ast \varphi_m)(\cdot, \cdot) \ast \psi_j(\cdot) =
$$

$$
\frac{1}{(1-q) \ldots (1-q^{m-n})} \sum_{\ell=1}^{N+d} (H^{-1})_{j,\ell} q^{(n-1)x_\ell},
$$

(59)

$$
\varphi^{n+1,m+1}(x, \cdot) \ast \psi_j(\cdot) = \sum_{\ell=1}^{d+N} (H^{-1})_{j,\ell} h_{x_\ell - x}(q^n, \ldots, q^{m-1})
$$

$$
\chi_{\ell}(\cdot) \ast \varphi^{n+1,m+1}(\cdot, \cdot) \ast \psi_j(\cdot) = \sum_{\ell=1}^{d+N} (H^{-1})_{j,\ell} h_{x_\ell - y}(q^n, \ldots, q^{m-1}).
$$

$$
\psi_k^{(m)}(x) = \sum_{\ell=1}^{d+N} (H^{-1})_{k,\ell} h_{x_\ell - x}(q^{d-m-1}, \ldots, q^{d+N-1}).
$$

(60)
Proof: The proof of the first three determinantal identities is straightforward; for instance,

\[
\det(\varphi_k(x_i^{(m-1)}, x_j^{(m)}))_{1 \leq i, j \leq d+m} = q^{(k-1)(\sum_{i=1}^{d+m} x_i^{(m)} - \sum_{j=1}^{d+m} x_j^{(m-1)})} \det(\mathbb{1}_{x_i^{(m-1)} \leq x_j^{(m)}})_{1 \leq i, j \leq d+m}
\]

\[
= q^{(k-1)(|x^{(m)}| - |x^{(m-1)}|)} \mathbb{1}_{x^{(m-1)} \prec x^{(m)}},
\]

taking into account the fact that the last row of the matrix above is given by 
\((q^0 x_1^{(m)}), \ldots, q^{(k-1)} x_d^{(m)})
\]

The convolution properties follow from the identity (26), applied to the partition \(\lambda = (r, 0, 0, \ldots)\) and \(\mu = (s, 0, 0, \ldots)\) with \(r > s\), i.e., \(\mu \subset \lambda\). Indeed one has:

\[
s_\lambda(x) = h_r(x), \quad s_\lambda \backslash \mu(x) = h_r \backslash s(x),
\]

and so, setting \(\nu = (\alpha, 0, 0, \ldots)\), one has

\[
h_{r - \alpha}(x, y) = \sum_{\alpha \in \mathbb{Z}} h_{r - \alpha}(x)h_{\alpha - s}(y).
\]

Another example of proof is:

\[
\varphi_n(virt, \cdot) * (\varphi_{n+1} * \cdots * \varphi_m)(\cdot, \cdot) * \psi_j(\cdot)
\]

\[
= \sum_{\nu \in \mathbb{Z}} q^{(n-1)\nu} \prod_{\ell=1}^{N+d} (1 - q^{m-n})^{H^{-1}_{j\ell}} \mathbb{1}_{z=x_{\ell}}
\]

\[
= \frac{1}{(1 - q) \cdots (1 - q^{m-n})} \prod_{\ell=1}^{N+d} (H^{-1}_{j\ell})^{q^{(n-1)x_{\ell}}},
\]

ending the proof of Lemma 4.3.

Proof of Proposition 4.1: The denominator of the probability (23) can be expressed as a determinant, using formula (33). Also from the second formula (58), it follows that

\[
\prod_{m=1}^{N} \det(\varphi_{d+m}(x_i^{(m-1)}, x_j^{(m)}))_{1 \leq i, j \leq d+m} = q^{(N-1)|x^{(N)}| + d(|x^{(N)}| - |x^{(0)}|)} q^{-\sum_{i=1}^{N-1} |x^{(i)}|} \mathbb{1}_{x^{(0)} \prec x^{(1)} \prec \cdots \prec x^{(N)}}
\]

\[
= q^{(d+N-1)|x^{(0)}| - d|y|} q^{-\sum_{i=1}^{N-1} |x^{(i)}|} \mathbb{1}_{x^{(0)} \prec x^{(1)} \prec \cdots \prec x^{(N)}}
\]
and using the two remaining formulas of (58), one checks:

\[
\mathbb{P}(\mu^{(0)}, \mu^{(2)}, \ldots, \mu^{(N)}, \text{ such that } \mu = \mu^{(0)} \subset \mu^{(1)} \subset \ldots \subset \mu^{(N)} = \lambda) = \frac{\prod_{i,j}^n \mu^{(N)}}{1} \mu^{(0)} \mu^{(1)} \mu^{(N)} = \lambda
\]

\[
= \frac{(q - 1)^{\sum_{\nu} \mu^{(N)} \setminus \mu^{(0)}}}{s \lambda \mu(q^{(q - 1)^{\sum_{\nu} \mu^{(N)} \setminus \mu^{(0)}}}} \mu^{(0)} \mu^{(1)} \mu^{(N)} = \lambda
\]

\[
= q^{N(d + N - 1)(d + N + 1)/2} - \sum_{i,j}^n \mu^{(N)} \setminus \mu^{(0)} \mu^{(1)} \mu^{(N)} = \lambda
\]

\[
= q^{d(d + N - 1)(d + N + 1)/2} - \sum_{i,j}^n \mu^{(N)} \setminus \mu^{(0)} \mu^{(1)} \mu^{(N)} = \lambda
\]

\[
= C_N, d, q \prod_{i,j}^n \det(\chi_i, \chi_j, \mu^{(0)}), \mu^{(1)} \mu^{(N)} = \lambda
\]

establishing the Karlin-MacGregor formula of Proposition 4.1.

Proof of Proposition 4.2. The technology explained in [3, 4] will now be adapted to these new circumstances. Given a set \(\mathcal{X}\), let \(L\) be a positive-definite \(|\mathcal{X}| \times |\mathcal{X}|-\text{matrix with rows and columns parametrized by the points } x \in \mathcal{X}\), with \(L_x\) a \(X \times X\)-submatrix with rows and columns parametrized by \(x \in X\). A random point process (L-ensemble) on \(\mathcal{X}\) will be defined by the probability \(\mathbb{P}(X) = \frac{\det(L_X)}{\det(1 + L)}\) for \(X \in \mathcal{X}\). It is determinantal with kernel \(K = L(1 + L)^{-1}\); that means that the correlation \(\rho(Y) = \mathbb{P}(X \subset \mathcal{X} | Y \subset X) = \det K_Y\). Given a decomposition \(\mathcal{X} = \mathcal{Y} \cup \mathcal{Y}^c\), with \(\mathcal{Y} \neq \emptyset\) and with corresponding block matrix \(1_{\mathcal{Y}}\), the conditional L-ensemble process on \(\mathcal{Y}\) is defined by \(\mathbb{P}(Y) = \frac{\det L_{\mathcal{Y} \cap \mathcal{Y}}}{\det(1_{\mathcal{Y}} + L)}\), and is determinantal with correlation kernel \(K = 1_{\mathcal{Y}} - (1_{\mathcal{Y}} + L)^{-1} \times 1_{\mathcal{Y}}\).

This is now applied to \(\mathcal{X} = \mathcal{Y} \cup \mathcal{Y}^c = (\mathcal{X}^{(0)} \cup \cdots \cup \mathcal{X}^{(N)}) \cup \{1, 2, \ldots, d, x^{(0)}_{d+1}, \ldots, x^{(N-1)}_{d+N}\}\), where \(\mathcal{X}^{(k)} = \mathbb{Z}\) are all the point configurations at level \(0 \leq k \leq N\) and the \(x^{(k)}_{d+k+1} = \text{virt are virtual variables, playing the role of point at } -\infty\) for each set \(\mathcal{X}^{(k)}\) for \(0 \leq k \leq N - 1\).

We now apply this set-up to the specific situation of the paper, although this is completely general. Indeed, we now define (infinite) matrices in terms of the functions appearing in the Karlin-McGregor formula (56), where \(\Phi^{(d)}\) contains the initial condition (points not in \(\mathcal{P}\) at level \(y = 0\), in particular the lower-cut) for the red dots, where the \(T^{k-1,k}\) are the transition functions from one level to

\[One uses \sum_{n=1}^{N} (d + i - 1)(d + i)/2 = \frac{1}{2}dN(d + N) + \frac{1}{2}N(N^2 - 1)\]
another and where $\Psi^{(d+N)}$ contains the final condition for the red dots (points not in $P$ at level $y = N$, in particular the upper-cuts):

- $\Phi^{(d)} = \{1, \ldots, d\} \times \mathcal{X}^{(0)}$-matrix, with $(\Phi^{(d)})(i, x^{(0)}) = \chi_i(x^{(0)})$
- $T_{k-1,k} = (\mathcal{X}^{(k-1)} \cup \{x_{d+k}^{(k-1)}\}) \times \mathcal{X}^{(k)}$-matrix, for $1 \leq k \leq N$
  - with $T_{k-1,k}(x^{(k-1)}, x^{(k)}) = \phi_{d+k}(x^{(k-1)}, x^{(k)})$  
  \begin{equation}
  (62)
  \end{equation}
- $\Psi^{(d+N)} = (\mathcal{X}^{(N)} \times \{1, \ldots, d + N\})$-matrix,
  - with $\Psi^{(d+N)}(x^{(N)}, j) := \psi_j(x^{(N)})$.

The conditional $L$-ensemble on $\mathcal{X}$ with $\mathcal{Y}$ as above is given by the block matrix, where each block has infinite size. The only nonzero determinants of the form $\det L_{Y \cup \mathcal{Y}^c}$ appearing in $\mathbb{P}(Y)$ above are suitable finite minors of this matrix $L$ below. These lead exactly to the Karlin-McGregor formula ($52$), as is easily seen by picking appropriate rows and columns according to the coordinates on top and to the left of the matrix $L$ below:

\[
L = \begin{pmatrix}
1, \ldots, d+N & x_1^{(0)}, \ldots, x_d^{(0)} & x_1^{(1)}, \ldots, x_{d+1}^{(1)} & x_1^{(N)}, \ldots, x_{d+N}^{(N)} \\
0 & \Phi^{(d)} & 0 & 0 & \ldots & 0 \\
0 & 0 & -T^{0,1} & 0 & \ldots & 0 \\
0 & 0 & 0 & -T^{1,2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -T^{N-1,N} \\
\Psi^{(d+N)} & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

We now move rows and columns of $L$ in order to write the matrix as a block $(\mathcal{Y}^c \otimes \mathcal{Y})$-matrix $L_{\mathcal{Y}^c \otimes \mathcal{Y}}$. Indeed, move the rows corresponding to the virtual
variables
\[ x_{d+1}^{(0)}, x_{d+2}^{(1)}, \ldots, x_{d+k}^{(k-1)}, \ldots, x_{d+N}^{(N-1)} \]
respectively to the \( d+1, d+2, \ldots, d+k, \ldots, d+N \)th row, yielding the matrix below, where the \( \mathcal{X}^{(k-1)} \times \mathcal{X}^{(k)} \)-matrices \( W_{k-1,k} \) are the matrices \( T^{k-1,k} \) with the last row removed,

\[
\begin{pmatrix}
0 & E_0 & E_1 & \ldots & E_k & \ldots & E_N \\
0 & 0 & -W_{01} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -W_{k-1,k} & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -W_{N-1,N} \\
\Psi^{d+N} & 0 & 0 & \ldots & 0 & \ldots & 0 \\
\end{pmatrix}
\]

\[
\downarrow d + N
\]

\[
L_{\mathfrak{Y} \otimes \mathfrak{Y}} = \begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}
\]

and

\[
\mathbb{1}_{\mathfrak{Y}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

with

\[
A = 0, \quad B := (E_0, \ldots, E_N), \quad C := (0, \ldots, 0, \Psi^{d+N})^T
\]

\[
D_0 := \begin{pmatrix}
0 & -W_{01} & 0 & \ldots & 0 \\
0 & 0 & -W_{12} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -W_{N-1,N} \\
0 & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

In the matrix \( B \), the \( E_k \) is a \((d+N) \times \mathcal{X}^{(k)}\)-matrix for \( 0 \leq k \leq N \), defined by, for \( k > 0 \),

\[
E_k(i, x^{(k)}) = \begin{cases}
0 & \text{for } i \leq d + k - 1 \\
T^{k-1,k}(x_{d+k}^{(k-1)}, x^{(k)}) = \varphi_{d+k}(x_{d+k}^{(k-1)}, x^{(k)}) & \text{for } i = d + k \\
0 & \text{for } d + k + 1 \leq i \leq d + N
\end{cases}
\]

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and, for \( k = 0 \),

\[
E_0 = \begin{pmatrix}
\Phi^d \\
0 \\
\vdots \\
0
\end{pmatrix} \updownarrow d \updownarrow N
\]

Defining \( D := \mathbb{1}_\mathcal{Y} + D_0 \), the point measure on \( \mathcal{Y} \) is determinantal with correlation kernel, given by

\[
\tilde{K} := \mathbb{1}_\mathcal{Y} - \left( (\mathbb{1}_\mathcal{Y} + L_\mathcal{Y} \otimes \mathcal{Y})^{-1} \right)_{\mathcal{Y} \times \mathcal{Y}} = \mathbb{1}_\mathcal{Y} - \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \right)_{\mathcal{Y} \times \mathcal{Y}} = \mathbb{1}_\mathcal{Y} - D^{-1} - D^{-1}CM^{-1}BD^{-1},
\]

with \( M = BD^{-1}C - A \); it will be shown that this is precisely the matrix \( M \) in (57). To do so, we need to compute the different matrices appearing in the expression (63). It is easily seen that

\[
D^{-1} = \begin{pmatrix}
1 & W_{[0,1]} & \cdots & W_{[0,N]} \\
\vdots & & & \vdots \\
0 & 1 & 0 & 0 \\
0 & \cdots & W_{[N-1,N]} & 1
\end{pmatrix}
\]

with \( W_{[n,m]} := W_{n,n+1} \cdots W_{m-1,m} \mathbb{1}_{m>n} \).

Coordinatewise, the \( \mathcal{X}^{(n)} \times \mathcal{X}^{(m)} \)-matrices \( W_{[n,m]} \) are given by

\[
W_{[n,m]}(x^{(n)}, x^{(m)}) = \varphi_{d+1,n}(x^{(n)}, x^{(n+1)}) \ast \cdots \ast \varphi_{d+m}(x^{(m-1)}, x^{(m)})
\]

Moreover,

\[
BD^{-1} = \begin{pmatrix}
\mathcal{X}^{(0)}_{E_0} & \mathcal{X}^{(1)}_{E_0W_{[0,1]}} & \mathcal{X}^{(m)}_{E_m} + \sum_{k=0}^{m-1} E_kW_{[k,m]} + E_kW_{[k,N]} + \mathcal{X}^{(N)}_{E_N} \\
\end{pmatrix} \updownarrow d + N,
\]

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where

\[
E_k W_{k,m} = \begin{pmatrix} O \\ \varphi_{d+k}(x^{(k-1)}_{d+k}, x^{(k)}) \end{pmatrix} \leftarrow d + k
\times \begin{pmatrix} O \\ \varphi^{d+1+k, d+1+m}(x^{(k)}, x^{(m)}_j) \delta_{m>k} \end{pmatrix}
\]

\[
= \begin{pmatrix} O \\ (\varphi_{d+k} \ast \varphi^{d+1+k, d+1+m})(x^{(k-1)}_{d+k}, x^{(m)}_j) \delta_{m>k} \end{pmatrix} \leftarrow d + k.
\]

Hence \(BD^{-1}\) consists of \(N + 1\) column-blocks, the \(m\)th block being given below, for \(0 \leq m \leq N\), with \(O_{N-m}\) a \((N - m) \times X^{(m)}\) matrix of zeros,

\[
\sum_{k=0}^{m-1} E_k W_{k,m} + E_m = \begin{pmatrix} (\chi_1 \ast \varphi^{d+1,d+1+m})(x^{(m)}) \\
\vdots \\
(\chi_d \ast \varphi^{d+1,d+1+m})(x^{m}) \\
(\varphi_{d+1} \ast \varphi^{d+2,d+1+m})(x^{(0)}_{d+1}, x^{(m)}_j) \\
\vdots \\
(\varphi_{d+k} \ast \varphi^{d+1+k,d+1+m})(x^{(k-1)}_{d+k}, x^{(m)}_j) \\
\vdots \\
(\varphi_{d+m-1} \ast \varphi^{d+m,d+1+m})(x^{(m-2)}_{d+m-1}, x^{(m)}_j) \\
\varphi_{d+m}(x^{(m-1)}_{d+m}, x^{(m)}) \\
O_{N-m} \end{pmatrix} \leftarrow d + 1
\]

\[
\left( \begin{array}{c} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) \left( \begin{array}{c} \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right)
\left( \begin{array}{c} \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right) \left( \begin{array}{c} \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right)
\]

Notice that in the matrix above, we have that the arguments \(x^{(k-1)}_{d+k} = \text{virt.}\).

Moreover

\[
D^{-1} C = \begin{pmatrix} W_{[0,N]} \Psi^{d+N} \\
\vdots \\
W_{[k,N]} \Psi^{d+N} \\
\vdots \\
W_{[N-1,N]} \Psi^{d+N} \\
\Psi^{d+N} \end{pmatrix} = \begin{pmatrix} \psi^{(1)}_j(x^{(0)}) \\
\vdots \\
\psi^{(k+1)}_j(x^{(k)}) \\
\vdots \\
\psi^{(N)}_j(x^{(N-1)}) \\
\psi^{(N+1)}_j(x^{(N)}) \end{pmatrix}_{1 \leq j \leq d+N}, \text{ with } \psi^{(N+1)}_j = \psi_j.
\]
because, for $0 \leq k \leq N - 1$ and $1 \leq j \leq d + N$, we have, using the notation of (55),

\[
(W_{k,N})^d x^{(k)}, j) = W_{k,N}(x^{(k)}, x^{(N)})^d x^{(N)}, j) = \varphi^{d+1+k,d+1+N}(x^{(k)}, x^{(N)}, j) \ast \psi_j(x) = \psi^{(k+1)}(x^{(k)}),
\]

and $\varphi^{d+N}(x^{(N)}, j) = \psi^{(N+1)}(x^{(N)}) = \psi_j(x^{(N)})$. Hence the $(m+1, n+1)^{th}$ block of $(D^{-1}CM^{-1}BD^{-1})$ reads for $0 \leq m, n \leq N$,

\[
\sum_{1 \leq k \leq d, 1 \leq \ell \leq d} \psi_k^{(m+1)}(x^{(m)})(M^{-1})_{k\ell}(\chi_i \ast \varphi^{d+1,d+1+N})(x^{(n)}) + \sum_{1 \leq k \leq d+1, d+1 \leq \ell \leq d+n} \psi_k^{(m+1)}(x^{(m)})(M^{-1})_{k\ell}(\varphi\ell \ast \varphi^{\ell+1,d+1+n})(x^{(d+1-d-1)}, x^{(n)}),
\]

with $x^{(d+1-d-1)} = virt$ and the $(m+1, n+1)^{th}$ block of $(\mathbb{I}_2 - D^{-1})$ reads

\[
-W_{m,n} = \varphi^{d+1+m,d+1+n}(x^{(m)}, x^{(n)}).
\]

Then upon setting $x = x^{(m)}$ and $y = x^{(n)}$ as running variables, and combining the two last equations, we conclude that for $0 \leq m, n \leq N$,

\[
\tilde{K}(m, x; n, y) = (\mathbb{I}_2 - D^{-1} + D^{-1}CM^{-1}BD^{-1})_{m+1,n+1}
\]
gives indeed formula (56), using the notation (59) for $\varphi^{n,m}$. Finally, we have $M = (M_{ij})_{1 \leq i,j \leq d+N} = BD^{-1}C = (BD^{-1})C$, whose entries, upon using (64), are given by

\[
M_{i,j} = \begin{cases}
(\chi_i \ast \varphi^{d+1,d+1+N})(\ast, x) \ast \psi_j(x), & \text{for } 1 \leq i \leq d, \\
(\varphi_i \ast \varphi^{d+1,d+1+N})(\ast, x) \ast \psi_j(x), & \text{for } d+1 \leq i \leq d+N-1, \\
\varphi^{d+N}(\ast, x) \ast \psi_j(x), & \text{for } i = d+N.
\end{cases}
\]

This ends the proof of Proposition 4.2. 

\textbf{Remark:} For each each $0 \leq m \leq N$, consider the linear space $V_m$,

\[
V_m := \text{span}\{\text{functions in brackets of (64)}\} = \text{span}\{\phi_j^{(m)}, 1 \leq j \leq d+m\},
\]

where $\phi_j^{(m)}$, $1 \leq j \leq d+m$ is a new basis of $V_m$, defined by the conditions $\phi_j^{(m)} \ast \psi_j^{(m)} = \mathbb{I}_{i=j}$ for $1 \leq i, j \leq d+m$. Assuming this new basis is such that

\footnote{In the second summation below, we also have $\varphi^{d+1+n,d+1+n} = 1$.}
\( \varphi_{d+m}(\text{virt}, x) = c_m \varphi_{d+m}^{(m)}(x), \ c_m \neq 0, \ 0 \leq m \leq N \) (Assumption A in [6, 4]), then it is shown that the kernel (63) has the following simple form:

\[
K(q)(m, x, n, y) = -\varphi^{d+1+m, d+1+n}(x, y) + \sum_{j=1}^{d+n} \psi_j^{(m)}(x) \phi_j^{(n)}(y).
\]

Assumption A can in our case indeed be achieved for some appropriate choice of basis, which at the end of the day leads to a representation of the kernel in terms of (2, 3)-mops on the circle.

5 Transforming \( M \) into \( \tilde{M} \) and computing its inverse

The purpose of this section is to transform the matrix \( M \), which appears in the kernel (56) for the multi-cut case, into a new matrix \( \tilde{M} \), which is easily invertible. To do so, one needs to define \( D_n := \text{diag}(e^{n-1}, \ldots, e^0) \), \( \Delta_n(q) \) a Vandermonde and \( \Pi_n \) a permutation matrix, (recall \( \mathcal{P}_{N-1}(z) = \sum_{i=1}^{N} e^{i-1}z^i \))

\[
\Delta_n(q) := \begin{pmatrix}
1 & 1 & \ldots & 1 \\
(q^{-1})^{n-1} & (q^{-2})^{n-2} & \ldots & 1 \\
(q^{-2})^{n-1} & (q^{-3})^{n-2} & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
(q^{-(n-1)})^{n-1} & (q^{-(n-2)})^{n-2} & \ldots & 1
\end{pmatrix}, \quad \Pi_n := \begin{pmatrix}
O & 1 \\
1 & O
\end{pmatrix}.
\]

Proposition 5.1 The inverse of the matrix \( \tilde{M} := MT^{-1} \) reads as follows:

\[
\tilde{M}^{-1} = \begin{pmatrix}
\mathbb{T} & O \\
O & \mathbb{T}_N
\end{pmatrix},
\]

where \( T \) is an independent square matrix of size \( d + N \),

\[
T = \begin{pmatrix}
\mathbb{T} & O \\
O & \mathbb{T}_N
\end{pmatrix},
\]

with

\[
T_N := \text{diag}(q^{(N-j)d})_{1 \leq j \leq N} \Delta_n^{(d+N-1)} \Pi_N D_n^{-1} \text{diag}(q^{-(d+N)(d+N-j)})_{1 \leq j \leq N}.
\]

Before giving the proof of Proposition 5.1 we need Lemma 5.2 below. It is useful to modify the matrix \( H \) of size \( d + N \) (as in (34)) to \( \tilde{H} \), by an \( x \) and \( y \) independent transformation \( T \) of block form, so as to be close to the Schur case rather than the skew-Schur case; that is, so that the \( (d + N, N) \)-block of the matrix \( H \) are pure powers of \( q \).
Lemma 5.2 Multiplying the block matrix $H$ with the matrix $T$ above (67), yields ($O_{n,m}$ denotes a $n \times m$-matrix of zeros):

$$\tilde{H} := HT = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \frac{1}{d} & O \\ O & T \end{pmatrix}$$

$$= \begin{pmatrix} \left[h_{x_i-y_j}(q^d, \ldots, q^{d+N-1})\right]_{1 \leq i \leq d, 1 \leq j \leq d} & \left[q^{x_i(N+j)}\right]_{1 \leq i \leq d, 1 \leq j \leq N} \\ \left[q^{d(x_i+y_j)}\left(P_{N-1}(q^{x_i-y_j})\right)\right]_{1 \leq e \leq c \leq \ell} & \left[q^{dx_i q^{x_i(N+j)}}\right]_{1 \leq i \leq d, 1 \leq j \leq N} \end{pmatrix} \quad (68)$$

Proof: At first, we have the following identity (Remember: $P_{n-1}(z) = \sum_{i=1}^{n} e_{i-1} z^{i-1}$; see formula (31))

$$(z^{n-1}, \ldots, z^0) = (P_{n-1}(z(q-1)^{n-1}), \ldots, P_{n-1}(z)) \tilde{T}, \quad \text{with} \quad \tilde{T}_n = A_n^{-1} \Pi_n D_n^{-1}.$$

Indeed, $\tilde{T}$ is such that:

$$z^{\ell-1} = \sum_{k=1}^{n} z^{i-1} e_{i-1} \sum_{k=1}^{n} (q^{-(i-1)})^{(n-k)} \tilde{T}_{k,n-\ell+1},$$

implying, for all $1 \leq \ell, i \leq n$, that

$$e_{i-1} \delta_{i,\ell} = \sum_{k=1}^{n} (q^{-(i-1)})^{(n-k)} \tilde{T}_{k,n-\ell+1},$$

which is statement (69) in matrix notation.

Apply this to the matrix $H$ and $n = N$, where the matrix $A$ of size $(d + N, d)$ consists of the $d$ columns corresponding to the points in the lower-cut $y_1, \ldots, y_d$ and the matrix $B$ of size $(d + N, N)$ to the points $y_{d+1}, \ldots, y_{d+N}$ to the left of $P \cap \{y = 0\}$. One now uses formula (32) and notices that $N - 1 + x_i - y_{d+j} \geq 0$ for $i, j \geq 1$; this enables us to omit the indicator in the formula for the matrix $B = (B_{ij})_{1 \leq i \leq d, 1 \leq j \leq N}$:

$$B_{ij} = \begin{pmatrix} h_{x_i-y_d+j}(q^d, \ldots, q^{d+N-1}) \\ q^{d(x_i+y_j)} P_{N-1}(q^{x_i-y_j}) \end{pmatrix}$$

$$= q^{d(x_i+y_j)} P_{N-1}(q^{x_i-y_j}) \begin{pmatrix} \frac{1}{d} & O \\ O & T \end{pmatrix} = q^{d(x_i+y_j)} P_{N-1}(z(q-1)^{N-j}) \big|_{z=q^{x_i+y_j}}.$$
and thus, in matrix notation,
\[
\text{diag}(q^{-d(x_i+d+N)})B \text{ diag}(q^{(N-j)d}) = \left( P_{N-1}(z(q^{-1})^{N-j}) \right)_{1 \leq i \leq d+N, 1 \leq j \leq N}.
\]

From (69), it follows that for \( \tilde{T}_N = T \), as in (67),
\[
\text{diag}(q^{-d(x_i+d+N)})B \text{ diag}(q^{(N-j)d}) \tilde{T}_N = \left( (z^{N-j})_{z = q^x_i+d+N} \right)_{1 \leq i \leq d+N, 1 \leq j \leq N},
\]

and so
\[
B \text{ diag}(q^{(N-j)d}) \tilde{T}_N = \left( q^{x_i(d+N-j)} \right)_{1 \leq i \leq d+N, 1 \leq j \leq N},
\]

which we rewrite as
\[
BT_N = \left( q^{x_i(d+N-j)} \right)_{1 \leq i \leq d+N, 1 \leq j \leq N},
\]

for \( T_N \) given in terms of \( \tilde{T}_N \), as defined in (69),
\[
T_N := \text{diag}(q^{(N-j)d}) \tilde{T}_N \text{ diag}(q^{-d(x_i+d+N)-j})_{1 \leq j \leq N}.
\]

This corresponds to the matrix \( T_N \) in (67). The multiplication by \( T \) leaves the first \( d \) columns unchanged. Also the last equality in (68) is an immediate consequence of formula (32) which includes the indicator function, since by hypothesis \( N - 1 + x_c - y_1 \geq 0 \). This ends the proof of Lemma 5.2.

Proof of Proposition 5.1: Inserting (55) in the matrix \( M \), as defined in (57), one sees that \( M \) can be written, for appropriate choices of \( \Phi_i' \)s, as:
\[
M_{ij} = \Phi_i(\cdot, \cdot) \ast \psi_j(\cdot) = \Phi_i(\cdot, \cdot) \ast \sum_{\alpha=1}^{d+N} H_{j\alpha}^{-1} \mathbf{1}_{z=x_\alpha} \psi_j(\cdot), \quad 1 \leq i, j \leq N + d.
\]

Since \( \bar{H} = HT \) in Lemma 5.2 this suggests transforming the \( \psi_k \) into a new expression \( \tilde{\psi}_k \), defined by
\[
\tilde{\psi}_j = \sum_{\alpha=1}^{d+N} T_{j\alpha}^{-1} \psi_\alpha = \sum_{\alpha=1}^{d+N} \bar{H}_{j\alpha}^{-1} \mathbf{1}_{z=x_\alpha},
\]

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and consequently $\psi_k^{(m+1)}$ into a new $\tilde{\psi}_k^{(m+1)}$ as defined in \((55)\) and \((60)\),

\[
\tilde{\psi}_k^{(m+1)}(x) := \sum_{\ell=1}^{d+N} (\tilde{H}^{-1})_{k\ell} h_{x\ell-x}(q^{d+m}, \ldots, q^{d+N-1}).
\]

From \((56)\), \((57)\) and \((55)\), this in turn induces a transformation on $M$, preserving the kernel \((56)\):

\[
M_{ij} = (\Phi_i * \sum_{\alpha} T_{j\alpha} \tilde{\psi}_\alpha) = \sum_{\alpha} T_{j\alpha} (\Phi_i * \tilde{\psi}_\alpha) = \sum_{\alpha} (\Phi_i * \tilde{\psi}_\alpha) T_{\alpha j}^T = (\tilde{M} T^T)_{ij},
\]

where $\tilde{M} = MT^{\top-1}$ is the matrix $M$ as in \((70)\), but with $\psi_\alpha$ replaced by $\tilde{\psi}_\alpha$. In view of the form of the kernel \((56)\), we check, using the previous expression, that the kernel is indeed preserved. It suffices by \((56)\) and \((55)\) to check the identity

\[
\sum_k \psi_k M^{-1} = \sum_k \sum_j H_{kj}^{-1} \mathbb{1}_{z=x_j} \sum_\alpha T_{k\alpha}^{\top-1} \tilde{M}_{\alpha j}^{-1}
\]

\[
= \sum_{\alpha,j} (H^T)^{-1}_{\alpha j} \mathbb{1}_{z=x_j} \tilde{M}_{\alpha j}^{-1}
\]

\[
= \sum_{\alpha,j} (\tilde{H})^{-1}_{\alpha j} \mathbb{1}_{z=x_j} \tilde{M}_{\alpha j}^{-1}
\]

\[
= \sum_{\alpha,j} \tilde{\psi}_\alpha \tilde{M}_{\alpha j}^{-1},
\]

inducing a similar relation $\sum_k \psi_k^{(m+1)} M^{-1} = \sum_\alpha \tilde{\psi}_\alpha^{(m+1)} \tilde{M}_{\alpha j}^{-1}$.

We now compute explicitly $\tilde{M}$:

**For $1 \leq i \leq d$ and $1 \leq j \leq d+N$**, setting $n = d$ and $m = d+N$ in \((59)\)

\[
M_{ij} = \chi_i \ast \varphi_{d+1} \ast \ldots \ast \varphi_{d+N} \ast \psi_j = (\varphi_{d+1} \ldots \varphi_{d+N}) (y_i, \cdot) \ast \psi_j (\cdot)
\]

\[
= \sum_{k=1}^{N+d} (H^{-1})_{j,k} h_{x_k-y_i} (q^d, \ldots, q^{N+d-1})
\]

\[
= \sum_{k=1}^{N+d} (H^{-1})_{j,k} H_{k,i} = \mathbb{I}_{ij}
\]

\[
\tilde{M}_{ij} = (MT^{\top-1})_{ij} = \mathbb{I}_{ij},
\]

since the left $(d+N) \times d$-block of $T$ is $\binom{1_d}{0}$. From this discussion it follows that the $\tilde{\psi}_j$, $\tilde{H}$, $\tilde{M}$ are related to each other in the same way as the $\psi_j$, $H$, $M$, via the transformation $\psi_j \mapsto \tilde{\psi}_j$, $H \mapsto \tilde{H}$, $M \mapsto \tilde{M}$.
For \( d + 1 \leq i \leq d + N \) and \( 1 \leq j \leq d + N \), one has, using the same identity for \( \varphi_i(\text{virt}, \cdot) \ast (\varphi_{i+1} \ast \ldots \ast \varphi_{d+N})(\cdot, o) \ast \psi_j(o) \) as in (59), but for \( \psi_j \) replaced by \( \tilde{\psi}_j \), one checks that
\[
\tilde{M}_{ij} = \varphi_i(\text{virt}, \cdot) \ast (\varphi_{i+1} \ast \ldots \ast \varphi_{d+N})(\cdot, o) \ast \tilde{\psi}_j(o)
\]
\[
= \frac{1}{(1 - q) \ldots (1 - q^{N-i+d})} \sum_{\ell=1}^{N+d} (\tilde{H}^{-1})_{j\ell} q^{(i-1)x_{\ell}}
\]
\[
= \frac{1}{(1 - q) \ldots (1 - q^{N-i+d})} \sum_{\ell=1}^{N+d} (\tilde{H}^{-1})_{j\ell} \tilde{H}_{\ell,2d+N-i+1}
\]
\[
= \frac{1}{(1 - q) \ldots (1 - q^{N-i+d})} \delta_{j,2d+N-i+1} = \frac{\delta_{j,2d+N-i+1}}{(q; q)_{d+N-i}}.
\]
So, the matrix \( \tilde{M} \) reads:
\[
\tilde{M} = \left( \begin{array}{c|c}
1_d & O \\
O & (\frac{\delta_{2d+N-i+1}}{(q; q)_{d+N-i}})_{d+1 \leq i,j \leq d+N}
\end{array} \right)
\]
and so
\[
\tilde{M}^{-1} = \left( \begin{array}{c|c|c|c}
1_d & O \\
O & O \\
O & O \\
(q; q)_{N-1}
\end{array} \right)
\]
establishing Proposition 5.1. 

6 The integral representation of the \( K_q \)-kernel

All statements in this section refer to the multi-cut case. The kernel will be given in terms of multiple contour integrals, one of them being a \( d+2 \)-fold integral. But at first, we express the kernel in terms of the basic functions \( h_r \) and \( \tilde{\psi}_{k}^{(m+1)}(x) \), as in (71).

**Proposition 6.1** For \( 0 \leq m, n \leq N \), the \( q \)-kernel reads as follows:
\[
K_q(m,x; n,y) = - h_{y-x}(q^{d+m}, \ldots, q^{d+n-1}) \mathbb{1}_{n>m} \tag{i}
\]
\[
+ q^{dy} \sum_{k=1}^{n} \tilde{\psi}_{d+N-k+1}^{(m+1)}(x) q^{(k-1)y} \prod_{i=n+1}^{N} (1 - q^{i-k}) \tag{ii}
\]
\[
+ \sum_{k=1}^{d} \tilde{\psi}_{k}^{(m+1)}(x) h_{y-y_k}(q^{d}, \ldots, q^{d+n-1}). \tag{iii}
\]
Proof: We use the kernel in Proposition 4.2. Equality $=*$ follows from the inverse (72) of the matrix $M$. Using Lemma 4.3, we then have:

$$\mathbb{K}_q(m, x; n, y) = -\varphi_{d+m+1} \ast \cdots \ast \varphi_d(x, y) + \sum_{1 \leq k \leq d} \tilde{\psi}^{(m+1)}(x)(\tilde{M}^{-1})_{k\ell}(\chi_{k} \ast \varphi_{d+1} \ast \cdots \ast \varphi_{d+n})(y) + \sum_{1 \leq k \leq d+N \atop d+1 \leq \ell \leq d+n} \tilde{\psi}^{(m+1)}(x)(\tilde{M}^{-1})_{k\ell}(\varphi_{\ell}(\text{virt}, \cdot) \ast (\varphi_{\ell+1} \ast \cdots \ast \varphi_{d+n})(\cdot, y))$$

$= -\varphi_{d+m+1} \ast \cdots \ast \varphi_d(x, y) + \sum_{1 \leq k \leq d} \tilde{\psi}^{(m+1)}(x)(\chi_{k} \ast \varphi_{d+1} \ast \cdots \ast \varphi_{d+n})(y) + \sum_{d+1 \leq k \leq d+N \atop d+1 \leq \ell \leq d+n} \tilde{\psi}^{(m+1)}(x) \delta_{2d+N-k+1, \ell}(q; q)_{k-d-1} (\varphi_{\ell}(\text{virt}, \cdot) \ast (\varphi_{\ell+1} \ast \cdots \ast \varphi_{d+n})(\cdot, y))$

$$= -h_{y-x}(q^{d+m}, \ldots, q^{d+n-1}) \mathbb{1}_{n>m} + \sum_{1 \leq k \leq d} \tilde{\psi}^{(m+1)}(x)h_{y-y_k}(q^d, \ldots, q^{d+n-1}) + \sum_{d+N-(n-1) \leq k \leq d+N} \tilde{\psi}^{(m+1)}(x)q^{(2d+N-k)y} (q; q)_{k-d-1} (q; q)_{k-d-(N-n+1)}.$$
and definitions:

\[ h_r(q^d, \ldots, q^{d+n}) = q^r \mathcal{P}_n(q^r) \mathbf{1}_{n+r \geq 0} \quad \text{and} \quad \tilde{\mathcal{P}}_{y-1}(z) := q^{-dy} \mathcal{P}_{n-1}(zq^{-y}). \]

and also that for \(1 \leq r \leq d, 0 \leq m \leq N-1 \) and \(n \geq 1,\)

\[ h_{x_k-x}(q^{d+m}, \ldots, q^{d+N-1}) = q^{(x_k-x)(d+m)} \mathcal{P}_{N-m-1}(q^{x_k-x}) \mathbf{1}_{x_k \geq x} \]

\[ h_{y-y_r}(\bullet) = h_{y-y_r}(q^d, \ldots, q^{d+n-1}) = \frac{1}{2\pi i} \int_0^1 \frac{du}{u^{y-y_r+1}} \prod_{\ell=1}^n \frac{1}{1 - uq^{d-1+\ell}}. \]

The reader is also reminded of the expression, considered in Lemma 2.1

\[ \Phi_q(z^{-1}) := (q^{-N}; q^{-1})_{N-n} 2\phi_1(q^{-1}, q^{n-1}; q^{-N-1}|q^{-1}, z^{-1}). \]

In the next Theorem, one will need \(q\)-analogues of the three determinants (46), depending on the points \(y_{\text{cut}} = (y_1, \ldots, y_d)\) in the lower-cuts. The second will involve the \(q\)-anologue \(h_{y-y_r}(\bullet) := h_{y-y_r}(q^d, \ldots, q^{d+n-1})\) of the function \(h_{y-y_r}(1^n)\) which has the integral representation (75). The definitions are:

\[ \Delta^{(y_{\text{cut}})}_{q,d}(u_1, \ldots, u_d) := \det(\tilde{\mathcal{P}}_{y-1}(u_\alpha))_{1 \leq \alpha, \beta \leq d} \]

\[ \tilde{\Delta}^{(y_{\text{cut}})}_{q,d}(w; u_2, \ldots, u_d) = \det \begin{pmatrix} w^{y_1} & \cdots & w^{yd} \\
^{y_1}_{N-1}(u_2) & \cdots & ^{y_d}_{N-1}(u_2) \\
\vdots & \ddots & \vdots \\
^{y_1}_{N-1}(u_d) & \cdots & ^{y_d}_{N-1}(u_d) \end{pmatrix} \]

\[ \tilde{\Delta}^{(y_{\text{cut}})}_{q,d,n}(y; u_2, \ldots, u_d) := \int_{\Gamma_0} \frac{dw}{2\pi i w^{y_1+1} \prod_{k=1}^n (1 - wq^{d+k-1})} \Delta^{(y_{\text{cut}})}_{q,d}(w, u_2, \ldots, u_d) \]

\[ = \det \begin{pmatrix} h_{y-y_1}(\bullet) & \cdots & h_{y-y_d}(\bullet) \\
^{y_1}_{N-1}(u_2) & \cdots & ^{y_d}_{N-1}(u_2) \\
\vdots & \ddots & \vdots \\
^{y_1}_{N-1}(u_d) & \cdots & ^{y_d}_{N-1}(u_d) \end{pmatrix}. \]

The last expression is obtained by performing the \(w\)-integration on the first row of the matrix in \(\Delta^{(y_{\text{cut}})}_{q,d,n}(y; u_2, \ldots, u_d)\). The expression \(\Delta^{(y_{\text{cut}})}_{q,d,n}(y; u_2, \ldots, u_d)\) clearly vanishes, when \(y \leq y_d - 1\), or what is the same, when \(y\) is strictly to the left of the lower-cuts.
Theorem 6.2  For $0 \leq m, n \leq N$, the kernel for the $K_q$-process reads as follows:

$$q^{(d+m)(x-y)}K_q(m,x;n,y) = \frac{(zq;q)_{n-m-1}}{(q;q)_{n-m-1}} \frac{1}{z=q^{-x}}$$

$$q^{(d+m)(x-y)}K_q(m,x;n,y) = \left( z \frac{zq}{q} \right)^{n-m-1} \frac{d}{q^{(d-1)y}} \oint_{\Gamma(q^{x-y}+N)} \frac{v^m dv}{2\pi i} \cdot \frac{\Phi_q(z^{-1}) dz}{2\pi i} \cdot \frac{P_{N-m-1}(vq^{y-x})}{Q_q(vq^y)} \cdot \frac{Q_q(zq^y)}{Q_q(vq^y)} \cdot \frac{\Omega_q(vq^y, zq^y)}{\Omega_q(v,0)}$$

$$= : (\mathbb{K}_q^{(1)} + \mathbb{K}_q^{(2)} + \mathbb{K}_q^{(3)})(m, x; n, y),$$

where

$$\Omega_q(v, z) := \left( \prod_{j=1}^{d} \oint_{\Gamma(q_{1},...,q_{c})} \frac{du_j}{2\pi i Q_q(u_j)} \right) \prod_{i=1}^{d} \frac{v - u_i}{z - u_i}$$

$$\times \Delta_d(u_1, u_2, \ldots, u_d) \Delta_{q,d}^{(y_{\text{cut}})}(u_1, u_2, \ldots, u_d)$$

$$\bar{\Omega}_q(v, y) := \left( \prod_{j=2}^{d} \oint_{\Gamma(q_{1},...,q_{c})} \frac{du_j}{2\pi i Q_q(u_j)} \right)$$

$$\times \Delta_d(v, u_2, \ldots, u_d) \Delta_{q,d}^{(y_{\text{cut}})}(q, u_2, \ldots, u_d)$$

$$= \sum_{1 \leq r \leq d} (-1)^{r-1} h_{y-y_r}(\bullet) \cdot F_r(v),$$

with $\bullet = (d, \ldots, q^{d+n-1})$ as in (75) and

$$F_r(v) := \left( \prod_{j=1}^{d-1} \oint_{\Gamma(q_{1},...,q_{c})} \frac{du_j}{2\pi i Q_q(u_j)} \right) \det(\frac{P_{N-1}^{y_{\text{cut}}}(u_\alpha)}{1 \leq \alpha \leq d-1}) \Delta_d(v, u_1, \ldots, u_{d-1}).$$

Finally, $\mathbb{K}_q^{(3)}(m, x; n, y) = 0$, when $y \leq y_{d-1}$; when $y$ is to the left of the lower-cuts.

$\Omega_q = 1$ for $d = 0$, $F_r$ and $\bar{\Omega}_q = 1$ for $d=1$. 

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Proof of Theorem 6.2: Expressed as sum of residues, the multiple contour integral 
\( \Omega_q(v, z) \), \( \tilde{\Omega}_q(v, z) \) and \( F_r(v) \) are as follows:

\[
\Omega_q(v, z) := \sum_{1 \leq i_1, \ldots, i_d \leq c} \frac{\Delta_d(q^{x_{i_1}}, \ldots, q^{x_{i_d}}) \Delta_{q,d}(q^{x_{i_1}}, \ldots, q^{x_{i_d}})}{\prod_{\alpha=1}^d Q'_q(q^{x_{i_\alpha}})} \frac{d}{z - q^{x_{i_\alpha}}}
\]

\[
\tilde{\Omega}_q(v, z) := \sum_{1 \leq i_1, \ldots, i_{d-1} \leq c} \frac{\Delta_d(v, q^{x_{i_1}}, \ldots, q^{x_{i_{d-1}}}) \Delta_{q,d,n}(q^y, q^{x_{i_1}}, \ldots, q^{x_{i_{d-1}}})}{\prod_{\alpha=1}^{d-1} Q'_q(q^{x_{i_\alpha}})}
\]

\[
F_r(v) := \sum_{1 \leq i_1, \ldots, i_{d-1} \leq c} \frac{\det(\tilde{H}_N^{-1}(q^{x_{i_\alpha}}))}{\prod_{\alpha=1}^{d-1} Q'_q(q^{x_{i_\alpha}})} \prod_{1 \leq \beta < d, \beta \neq r} \frac{\Delta_d(v, q^{x_{i_1}}, \ldots, q^{x_{i_{d-1}}})}{\prod_{\alpha=1}^{d-1} Q'_q(q^{x_{i_\alpha}})}.
\]

Expressing the determinant of \( \tilde{H} \) of size \( (d+N) \) as a multiple contour integral, using the Laplace expansion for the determinant of a block matrix and
using (68) for $\tilde{H}$ and Lemma 3.4 in the third equality\textsuperscript{11} (remember $q_i := q^{x_i}$)

\[
\begin{align*}
\frac{\det \tilde{H}}{\Delta_{d+N}(q_1, \ldots, q_{d+N})} &= \prod_{i=1}^{d+N} q_i^{d_{xi}} \frac{\det \left( \begin{array}{c|c}
\tilde{p}_{ij} & (q^{x_i}(N-j))_{1 \leq i \leq \ell, 1 \leq j \leq N} \\
0_{d+b,d} & O_{d+b,d}
\end{array} \right)}{\Delta_{d+N}(q_1, \ldots, q_{d+N})} \\
&= (-1)^{d(d-1)/2} \prod_{i=1}^{d+N} q_i^{d_{xi}} \sum_{1 \leq i_1 < \cdots < i_d \leq c} (-1)^{\sigma(i_1, \ldots, i_d)} \det \left( \begin{array}{c}
\tilde{p}_{ij}^{\ell-1}(q_i) \\
\vdots \\
\tilde{p}_{ij}^{\ell-1}(q_d)
\end{array} \right)_{1 \leq j \leq d} \quad (83) \\
&\times \frac{\Delta_N(q_1, \ldots, \widehat{q_{i_1}}, \ldots, \widehat{q_{i_d}}, \ldots, q_{d+N})}{\Delta_{d+N}(q_1, \ldots, q_{d+N})} \\
&= (-1)^{d(d-1)/2} \frac{d^d}{d!} \prod_{i=1}^{d+N} q_i^{d_{xi}} \sum_{1 \leq i_1, \ldots, i_d \leq c} \Delta_{d}(q_{i_1}, \ldots, q_{i_d}) \prod_{j=1}^{d} Q'_q(q_{i_j}) \\
&= (-1)^{d(d-1)/2} \left( \prod_{i=1}^{d+N} q_i^{d_{xi}} \right) \Omega_q(0,0),
\end{align*}
\]

upon using the residue formula (81) for $\Omega_q(0,0)$ in the last equality.

**Expressing the entries of inverse $\tilde{H}^{-1}$ as integrals.** Computing its entries $\tilde{H}^{-1}_{r,k}$ will depend on whether $1 \leq r \leq d$ or $d+1 \leq r \leq d+N$. To do so, the determinant of the adjoint matrix will be expanded as a sum of the determinants of blocks taken from the first part of the matrix times the determinant of a complementary block of the second part. Then using in $\equiv$ the second relation of \textsuperscript{11}Given the set $\{1, \ldots, M\}$, and a subset $(i_1 < \cdots < i_d)$, define

\[
\sigma_M(i_1, \ldots, i_d) := \# \left\{ \text{transpositions needed to map} \right. \\
\left. (1 < \cdots < i_1 < \cdots < i_d < \cdots < M) \right. \\
\left. \text{into} \quad (i_1, \ldots, i_d, 1, 2, \ldots, i_1, \ldots, i_d, \ldots, M) \right\}. \quad (82)
\]
Lemma 3.4 one finds for $1 \leq r \leq d$ and $1 \leq k \leq d + N$:

$$H_{r,k}^{-1} = \frac{d!(-1)^{r+k+d(d-1)/2}}{\Omega_q(0,0)q^{dx_k}} \sum_{1 \leq i_1 < \cdots < i_{d-1} \leq c} (-1)^\sigma \det \tilde{H}$$

$$\times \det \left( \begin{array}{ccc}
\tilde{y}_{N-1}^{(q_1)} & \cdots & \tilde{y}_{N-1}^{(q_1)} \\
\vdots & \ddots & \vdots \\
\tilde{y}_{N-1}^{(q_{d-1})} & \cdots & \tilde{y}_{N-1}^{(q_{d-1})}
\end{array} \right)$$

$$\times (\Delta_N(q_1, \ldots, q_{d-1}, q_{d}, \ldots, q_{d+N})) \times (-1)^{k-1}(\det(\tilde{y}_{N-1}^{(q_1)})_{1 \leq \alpha \leq d-1}^{1 \leq \beta \leq d}) \prod_{\alpha=1}^{d-1} Q'_q(q^{x_\alpha})$$

$$\times (-1)^{\sigma + (d-1)/2} \Delta_d(q_1, q_2, \ldots, q_{d-1})$$

$$\times \frac{d!(-1)^{r-1} q^{-dx_k} F_r(q^{x_k})}{(d-1)! Q'_q(q^{x_k}) \Omega_q(0,0)} = (-1)^{r-1} \int_{\Gamma(q^{x_k})} \frac{v^{-d} dv}{2\pi i Q_q(v) \Omega_q(0,0)} F_r(v).$$

Also the last equality follows from replacing $\sum_{1 \leq i_1 < \cdots < i_{d-1} \leq c}$ by $\sum_{1 \leq i_1, \ldots, i_{d-1} \leq c}$, which brings in the denominator $(d-1)!$. Remember $F_r$ is the $d-1$-fold integral (80).

Using (83) above, we now show that for $1 \leq r \leq N$ the inverse $H_{d+r,k}^{-1}$ has a multiple integral representation, as in the previous computation. Namely one uses in $\hat{=}^*$ the first identity (49) (Lemma 3.4) and the definition (77) of $\Delta(y, q^{x_k})$.

Also notice that the restriction $i_\ell \neq k$ in the sum appearing in $\hat{=}^*$ can be removed, since $\Delta_{N-1}^N = 0$ for $i_\ell = k$. In $\hat{=}^{**}$, one uses (50) (Lemma 3.4) and the terms containing $q^{x_k}$ can be rewritten as a contour integral about $q^{x_k}$, because of the presence of $Q'_q(q^{x_k})$ in the denominator. Finally, in $\hat{=}^{***}$, one uses the definition
of $\Omega_q(v, z)$, thus yielding:

\[
\tilde{H}_{d+r,k}^{-1} = \frac{d!(-1)^{d(d-1)/2}}{\Omega_q(0, 0) q^{d \delta_k}} \text{adj} \left( \begin{array}{c|c} (\mathcal{G}^j_{N-1}(q^{x_i}))_{1 \leq i \leq d, 1 \leq j \leq N} & \left( q^{x_i(N-j)} \right)_{1 \leq i \leq d, 1 \leq j \leq N} \end{array} \right)_{d+r,k},
\]

\[
= \frac{d!(-1)^{d(d-1)/2}}{\Omega_q(0, 0) q^{d \delta_k}} (-1)^{d+r+k} \sum_{1 \leq i_1 < \ldots < i_d \leq c} (-1)^{\sigma} \det \left( \mathcal{G}^j_{N-1}(q^{x_i}) \right)_{1 \leq \alpha, \beta \leq d} \Delta_{d+N}(q_1, \ldots, q_{d+N}) \] \times \Delta \mathcal{N}^{-1}_{N-1}(q_1, q_2, \ldots, q_{d+N})
\]

\[
= \frac{d!(-1)^{d(d-1)/2+r+k}}{\Omega_q(0, 0) q^{d \delta_k}} \sum_{1 \leq i_1 < \ldots < i_d \leq c} (-1)^{\sigma} \Delta_{d+N}(q_1, q_2, \ldots, q_{d+N}) \] \times (-1)^{d(d-1)/2} \sum_{1 \leq i_1 < \ldots < i_d \leq c} \Delta_{d+1}(q_k, q_{i_1}, \ldots, q_{i_d}) \Delta_{q_k}(x, q^x_i), \ldots, q^{x_i}) \Delta_d(q^{x_i}, \ldots, q^{x_i})
\]

\[
= \frac{1}{\Omega_q(0, 0) q^{d \delta_k}} \sum_{1 \leq i_1 < \ldots < i_d \leq c} \Delta_{d+1}(q_k, q_{i_1}, \ldots, q_{i_d}) \Delta_{d}(q^{x_i}, \ldots, q^{x_i})
\]

\[
\times \frac{1}{Q_k^i(q^{x_k})} \prod_{\alpha=1}^d \frac{dz}{Q_k^i(q^{x_i})} = \frac{dz}{4\pi i} \sum_{z=0}^{2\pi i} Q_k^i(q^{x_i}) \Omega_q(0, 0).
\]

**The computation of the $\psi^{(m+1)}$-function**, as defined in (71) of Proposition 6.1 comes in two parts. For $1 \leq r \leq N$, we have, using directly (85) and (75),

\[
\tilde{\psi}^{(m+1)}_{d+r}(x)
\]

\[
\sum_{k=1}^{d+N} (H^{-1})_{d+r,k} h_{x_k-x}^{q^{d+m}, \ldots, q^{d+N-1}} 1_{x_k \geq x}
\]

\[
= q^{-x(d+m)} \sum_{k=1}^{d+N} \tilde{H}_{d+r,k}^{-1} v^{d+m} P_{N-m-1}(v q^{-x}) |_{v=q_k} 1_{x_k \geq x}
\]

\[
= q^{-x(d+m)} \int_{\Gamma(q^x, q^{x+1}, \ldots)} \frac{v^{m} dv}{2\pi i} \int_{\Gamma} \frac{dz}{2\pi i} \frac{P_{N-m-1}(v q^{-x}) Q_k(z) \Omega_q(v, z)}{(z-v) Q_k(v) \Omega_q(0, 0)}.
\]
Similarly for \(1 \leq r \leq d\), using (84), (71) and (80),
\[
\tilde{\psi}_r^{(m+1)}(x) = \sum_{k=1}^{d+N} (H^{-1})_{r,k} h_{x_k - x}(q^{d+m}, \ldots, q^{d+N-1}) \mathbb{1}_{x_k \geq x}
\]
\[
= \frac{d(-1)^{r-1}}{q^{x(d+m)}} \sum_{1 \leq k \leq d+N} q^{x_k (d+m)} P_{N-m-1}(q^{x_k - x}) \int_{\Gamma_{q^{x_k}}} \frac{v^{-d}dv}{2\pi i Q_q(v) \Omega_q(0,0)}
\]
\[
= \frac{d(-1)^{r-1}}{q^{x(d+m)}} \int_{\Gamma(q^{x+d})} \frac{v^{m} P_{N-m-1}(vq^{x})dv}{2\pi i Q_q(v)} \frac{F_v(v)}{\Omega_q(0,0)}.
\]
(87)

Computing each of the terms \(\mathbb{K}_{q}^{(i)}\) in the kernel (78) for \(i = 1, 2, 3\) and for \(0 \leq m, n \leq N\). Indeed, using (75), one finds for (i) of (73):
\[
\mathbb{K}_{q}^{(1)}(m, x; n, y) = h_{y-x}(q^{d+m}, \ldots, q^{d+n-1}) \mathbb{1}_{n > m} \mathbb{1}_{y \geq x}
\]
\[
= z^{d+m} P_{n-m-1}(z) \bigg|_{z = q^{y-x}} \mathbb{1}_{n > m} \mathbb{1}_{y \geq x}
\]
\[
= z^{d+m} \prod_{i=0}^{n-m-2} \frac{1 - z q^{i+1}}{1 - q^{i+1}} \bigg|_{z = q^{y-x}} \mathbb{1}_{n > m} \mathbb{1}_{y \geq x}
\]
\[
= q^{(d+m)(y-x)} \frac{(zq; q)_{n-m-1}}{(q; q)_{n-m-1}} \bigg|_{z = q^{y-x}} \mathbb{1}_{n > m} \mathbb{1}_{y \geq x}.
\]

For \(1 \leq k \leq N, \ m \geq 0\), using formula (86) for \(r = N - k + 1\) and using Petrov’s expression (86), one checks for (ii) of (73):
\[
\mathbb{K}_{q}^{(2)}(m, x; n, y)
\]
\[
= q^{dy} \sum_{k=1}^{n} \psi^{(m+1)}_{d+N-k+1}(x) q^{(k-1)y} \prod_{r=n+1}^{N} (1 - q^{r-k})
\]
\[
= q^{d(y-x) - zm} \int_{\Gamma(q^{x+d})} \frac{v^{m}dv}{2\pi i} \int_{\Gamma_{\infty}} \frac{dz}{2\pi iz} \frac{P_{N-m-1}(vq^{-x}) Q_q(z) \Omega_q(v, z)}{Q_q(v) \Omega_q(0,0)}
\]
\[
\times z \sum_{k=1}^{n} \frac{z^{-k} q^{(k-1)y} \prod_{r=n+1}^{N} (1 - q^{r-k})}{(z - v) v^{-d}}
\]
\[
= q^{d(y-x) - zm} \int_{\Gamma(q^{x+d})} \frac{v^{m}dv}{2\pi i} \int_{\Gamma_{\infty}} \frac{\Phi_q^{(q^{y-z}q^{-1})} dz}{2\pi iz} \frac{P_{N-m-1}(vq^{-x}) Q_q(z) \Omega_q(v, z)}{Q_q(v) \Omega_q(0,0)}
\]
\[
= q^{(y-x)(d+m)} \int_{\Gamma(q^{x+y+d})} \frac{v^{m}dv}{2\pi i} \int_{\Gamma_{\infty}} \frac{\Phi_q^{(z-1)} dz}{2\pi iz} \frac{P_{N-m-1}(vq^{y-z}) Q_q(zq^{y}) \Omega_q(qv^{y}, q^{y}z)}{Q_q(vq^{y}) \Omega_q(0,0)}.
\]

This last equality \(\ast\) is obtained by the substitution \(v \mapsto v q^{y}\) and \(z \mapsto z q^{y}\). Substituting (87) for \(\tilde{\psi}_r^{(m+1)}\) and (75) for \(h_{y-x}\), and using (79), one finds for (iii)

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of \( (73) \):

\[
\mathbb{K}_q^{(3)}(m, x; n, y) = \sum_{1 \leq r \leq d} \psi^{(m+1)}_r(x) h_{y-y_r}(q^d, \ldots, q^{d+n-1})
\]

\[
= \frac{d}{q^{x+d+m}} \int_{\Gamma(q^{x+n})} \frac{v^m \rho_{N-m-1}(vq^{-x}) dv}{2\pi i Q_q(v)} \times \sum_{1 \leq r \leq d} (-1)^{r-1} h_{y-y_r}(\star) \frac{F_r(v)}{\Omega_q(0,0)}
\]

\[
= \frac{d}{q^{x+d+m}} \int_{\Gamma(q^{x+n})} \frac{v^m \rho_{N-m-1}(vq^{-x}) dv}{2\pi i Q_q(v)} \frac{\tilde{\Omega}_q(v, y)}{\Omega_q(0,0)}.
\]

Expression \( \mathbb{K}_q^{(3)} \) as in the kernel \( (78) \) follows by the substitution \( v \rightarrow vq^y \). This ends the proof of Theorem 6.2.

7 The \( \mathbb{K} \)-kernel as a \( d + 2 \)-fold integral (\( d = \text{total size of lower-cuts} \)).

Before stating Theorem 7.1 and Proposition 7.2 we consider the limits for \( q \rightarrow 1 \) of the expressions \( (77) \), yielding by Lemma 2.1 the corresponding expressions \( (46) \):

\[
\lim_{q \rightarrow 1} \Delta^{(\text{ycut})}_{q,d}(\cdot; u_1, \ldots, u_d) = \Delta^{(\text{ycut})}_d(u_1, \ldots, u_d)
\]

\[
\lim_{q \rightarrow 1} \Delta^{(\text{ycut})}_{q,d,n}(\cdot; u_1, \ldots, u_d) = \Delta^{(\text{ycut})}_{d,n}(u_1, \ldots, u_d)
\]

\[
\lim_{q \rightarrow 1} \Delta^{(\text{ycut})}_{q,d}(\cdot; u_1, \ldots, u_d) = \Delta^{(\text{ycut})}_d(u_1, \ldots, u_d)
\]

**Theorem 7.1** The kernel \( \mathbb{K} \) for the red dots in the lower one-cut model has the following form for \( d \geq 0 \), and \( (m, x), (n, y) \in \mathbb{P} \):

\[
\mathbb{K}(m, x; n, y) = -\frac{(y-x+1)(n-m-1)}{(n-m-1)!} \sum_{n \geq m} \sum_{y \geq x} \int_{\Gamma(x+1)} \frac{dv}{2\pi i (N-m) Q(v)} (N-n)!
\]

\[
\times \left( \int_{\Gamma_{\infty}} \frac{dz Q(z)}{2\pi i (z-v)(z-y) N-n+1} + d(z-v) \frac{P(z) \tilde{\Omega}_R(v, z)}{\Omega_R(0,0)} \right),
\]

where \( \tau = (y+n) - (y_1+1) \) and the contour \( \Gamma_{\tau} \) were defined in \( (12) \) and \( (62) \).

Also \( \Gamma_{\infty} = \Gamma(v, y, \ldots, y-N+n) \) is a large circle about all the poles of the
The expressions above contain $\Omega_R$ and $\bar{\Omega}_R$, defined as (for brevity, set $u = (u_1, \ldots, u_d)$ and for contours, see (9)):

$$\Omega_R(v, z) := C_{N,d} \left( \prod_{\alpha=1}^{d} \oint_{\Gamma(x)} \frac{du_\alpha P(u_\alpha)}{2\pi i} \frac{v - u_\alpha}{Q(u_\alpha) z - u_\alpha} \right) \Delta_d(u)^2$$

$$\bar{\Omega}_R(v, z) := C_{N,d} \left( \prod_{\alpha=2}^{d} \oint_{\Gamma(x)} \frac{du_\alpha P(u_\alpha)}{2\pi i} \frac{v - u_\alpha}{Q(u_\alpha) z - u_\alpha} \right) \Delta_d(z, u_2, \ldots, u_d)^2,$$

with $\Omega_R(v, z) = 1$ for $d = 0$ and $\bar{\Omega}_R(v, z) = 1$ for $d = 1$.

**Remark:** The version of Theorem 7.1 for the multi-cut model will be discussed in Section 10.

Before proving Theorem 7.1 we first need the proposition below. Incidentally, the form of the kernel in this proposition is the most convenient one to show that the kernel $K$ is the inverse Kasteleyn matrix, up to some trivial conjugation, this being valid for the multi-cut model as well; see Theorem 9.1 and Paper II.

**Proposition 7.2** The limiting kernel for $q \to 1$ for the multi-cut model has the following form for $d \geq 0$, and $(m, x), (n, y) \in \mathbb{P}$:

$$K(m, x; n, y) := (\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3)(m, x; n, y) = -\frac{(y - x + 1)_{N-m-1}}{(n - m - 1)!} \frac{1_{n>m1_{y \geq x}}}{\Gamma(x+N)} v - u_\alpha z - u_\alpha$$

$$\times \left( \oint_{\Gamma(x)} \frac{dv}{2\pi i} \left( \frac{(N-n)!Q(v)}{(z - y)_{N-n+1}} \frac{\Omega_R(v, z)}{\Omega_R(0, 0)} \right) + d \oint_{\Gamma_0} \frac{dw}{2\pi i w^{y+1}(1 - w)^n} \frac{\bar{\Omega}_R^{(1)}(v, w)}{\Omega_R(0, 0)} \right),$$

where, using the expressions $\Delta^{(ycut)}_d$ and $\bar{\Delta}^{(ycut)}_d$ in (46), (for brevity, set $u = (u_1, \ldots, u_d)$)

$$\Omega_R(v, z) := \left( \prod_{\alpha=1}^{d} \oint_{\Gamma(x)} \frac{du_\alpha}{2\pi i Q(u_\alpha)} \frac{v - u_\alpha}{z - u_\alpha} \right) \Delta_d(u) \Delta^{(ycut)}_d(u)$$

$$\bar{\Omega}_R^{(1)}(v, w) := \left( \prod_{\alpha=2}^{d} \oint_{\Gamma(x)} \frac{du_\alpha}{2\pi i Q(u_\alpha)} \right) \Delta_d(v, u_2, \ldots, u_d) \bar{\Delta}^{(ycut)}_d(w; u_2, \ldots, u_d),$$

In view of the ratios, the precise value of the constant $C_{N,d}$ will actually never appear.
where \( \Omega_R(v, z) = 1 \) for \( d = 0 \), as before, and \( \Omega(1)_{R}(v, w) = w^{y_1} \) for \( d = 1 \). Note the two expressions (97) and (98) for \( \Omega_R(v, z) \) are equivalent in the lower one-cut model.

**Corollary 7.3** Given an hexagon \( P \), with \( \ell - 1 \) cuts on top and none at the bottom, the kernel \( K \) reads

\[
K(m, x; n, y) = -\frac{(y - x + 1)n - m - 1}{(n - m - 1)!} y \geq x
\]

\[
+ \frac{(N - n)!}{(N - m - 1)!} \int_{\Gamma(x + N)} \frac{dv}{2\pi i} \int_{\Gamma_{\infty}} \frac{dz}{2\pi i(z - v)} \left( \frac{(v - x + 1)_{N - m - 1}Q(z)}{(z - y)_{N - n + 1}Q(v)} \right),
\]

where \( \text{setting } a := n_1 + n_2 \)

(i) \( Q(z) \) as in [7] for \( \ell > 1 \). (Petrov [27])

(ii) \( Q(z) = (z - a + 1)b(z + c + 1)b \), for \( \ell = 1 \) (hexagon \( (a, b, c) \)). (Johansson [27])

**Proof of Proposition 7.2** Referring to the kernel (78) in Theorem 6.2, we first have \( \lim_{q \to 1} K_q^{(1)} = K_1 \). The other terms require some argumentation. The term \( K_q^{(2)} \) can be expressed as follows, taking into account the roots \( v = q^{x_j - y} \) of \( Q_q(vq^y) \) and the integration contour \( \Gamma(q^{x_j - y + N}) \),

\[
K_q^{(2)}(m, x; n, y) = \int_{\Gamma_{\infty}} \frac{dz}{2\pi i} \sum_{\substack{1 \leq j \leq d + N \leq x \geq s}} R_j(z),
\]

where

\[
R_j(z) := \text{Res}_{v = q^{x_j - y}} q^{m(x_j - y)} \Phi_q(z^{-1}) P_{N - m - 1}(vq^{-y - x}) Q_q(zq^y) \Omega_q(vq^y, zq^y) \Omega_q(0, 0) \]

\[
= q^{m(x_j - y)} \Phi_q(z^{-1}) P_{N - m - 1}(q^{x_j - x}) \prod_{1 \leq r \leq d + N, r \neq j} \frac{z - q^{x_r - y}}{q^{x_j - y} - q^{x_r - y}} \Omega_q(q^{x_j}, zq^y) \Omega_q(0, 0) \]

\[
= \sum_{1 \leq j_1, \ldots, j_d \leq c \atop j \notin \{j_1, \ldots, j_d\}} \frac{q^{m(x_{j_1} - y)} P_{N - m - 1}(q^{x_{j_1} - x})}{\prod_{1 \leq r \leq d + N, r \neq j} \left( q^{x_j - y} - q^{x_r - y} \right)} (\ast)
\]

\[
\times \Phi_q(z^{-1}) \prod_{1 \leq r \leq d + N, r \neq j, i_1, \ldots, i_d} \left( z - q^{x_r - y} \right) (\ast \ast)
\]

\[
\times \Delta_{q,d}^{(y \text{ cut})}(q^{x_{i_1}}, \ldots, q^{x_{i_d}}) \Delta_d(q^{x_{i_1}}, \ldots, q^{x_{i_d}}) \prod_{a=1}^{d} \frac{q^{x_{j_a} - q^{x_{i_a} - y}}}{\Omega_q(0, 0)} Q_q(q^{x_{i_a}}) (\ast \ast \ast)
\]

\[
= \sum_{1 \leq j_1, \ldots, j_d \leq c \atop j \notin \{j_1, \ldots, j_d\}} ((\ast) \times (\ast \ast) \times (\ast \ast \ast)),
\]

(96)
Equality \( \ast \) is obtained by inserting the expression [81] for \( \Omega_q(v, z) \) into the formula preceding \( \ast \). One also notices that the term \( \prod_{\alpha=1}^{d}(z - q^{x_{i\alpha} - y}) \) in the denominator of \( \Omega_q(q^{x_j}, zq^{y}) \) cancels \( d \) terms in the product \( \prod_{1 \leq r \leq d+N} (z - q^{x_r - y}) \), yielding equality \( (\ast) \). Also \( j \) can be taken different from \( i_1, \ldots, i_d \), because if \( j \) would figure in that set the term would vanish.

Now we let \( q \to 1 \). Since the terms \( (\ast) \) and \( (\ast\ast\ast) \) do not depend on \( z \), the integration in (95) will act on term \( (\ast\ast) \) only. So, at first for given \( x_j \), we have, using Lemma 2.1, the estimate
\[
(\ast) \simeq (q - 1)^{1-N-d} \frac{(x_j - x + 1)N-m-1}{(N-m-1)! \prod_{1 \leq r \leq d+N} (x_j - x_r)}
\]
(97)

Next, again using Lemma 2.1 we have for \( q \sim 1 \),
\[
\oint_{\Gamma_{\infty}} \frac{dz}{2\pi i z} (\ast\ast\ast) = \oint_{\Gamma_{\infty}} \frac{dz}{2\pi i z} \Phi_q(z^{-1}) \prod_{1 \leq r \leq d+N \atop r \neq j, i_1, \ldots, i_d} (z - q^{x_r - y})
\]
\[
\simeq (q - 1)^{N-1(N-n)!} \oint_{\Gamma_{\infty}} \frac{dz}{2\pi i(z - y)N-n+1} \prod_{1 \leq r \leq d+N \atop r \neq j, i_1, \ldots, i_d} (z - x_r)
\]
\[
\simeq (q - 1)^{N-1(N-n)!} \oint_{\Gamma_{\infty}} \frac{dzQ(z)}{2\pi i(z - y)N-n+1(z - x_j)} \prod_{\alpha=1}^{d} (z - x_{i_{\alpha}}).
\]
(98)

Next we need to estimate \( (\ast\ast\ast) \). From (89), combined with some of the limits in Lemma 2.1, one finds the estimate
\[
\Delta_d(q^{x_{i_1}}, \ldots, q^{x_{i_d}}) \Delta_{q,d}^{(Y_{\text{cut}})}(q^{x_{i_1}}, \ldots, q^{x_{i_d}}) \prod_{\alpha=1}^{d} \frac{q^{x_{j} - y} - q^{x_{i_{\alpha}} - y}}{Q'_q(q^{x_{i_{\alpha}}})} = (q - 1)^{d(N+\frac{d-1}{2})-1} \Delta_d(x_{i_1}, \ldots, x_{i_d}) \Delta_{d}^{(Y_{\text{cut}})}(x_{i_1}, \ldots, x_{i_d}) \prod_{\alpha=1}^{d} \frac{x_{j} - x_{i_{\alpha}}}{Q'(x_{i_{\alpha}})}.
\]
(99)
and thus, when \( q \to 1 \), the residue version of expression \( \Omega_R(0,0) \) defined in (81) tends to the residue version of (93), modulo a power of \( (q - 1) \):

\[
\Omega_q(0,0) = \sum_{1 \leq i_1, \ldots, i_d \leq c} \frac{\Delta_d(q^{x_{i_1}}, \ldots, q^{x_{i_d}})\Delta_d^{(y_{\text{cut}})}(q^{x_{i_1}}, \ldots, q^{x_{i_d}})}{\prod_{\alpha=1}^d Q'(q^{x_{i_\alpha}})} \\
\simeq (q - 1)^{-d(N+\frac{d-1}{2})} \sum_{1 \leq i_1, \ldots, i_d \leq c} \frac{\Delta_d(x_{i_1}, \ldots, x_{i_d})\Delta_d^{(y_{\text{cut}})}(x_{i_1}, \ldots, x_{i_d})}{\prod_{\alpha=1}^d Q'(x_{i_\alpha})} \\
= (q - 1)^{-d(N+\frac{d-1}{2})}\Omega_R(0,0).
\]  

(100)

Upon using (89), it follows that the last line (**) of formula (96) is estimated by

\[
** \simeq \left( \frac{q - 1}{\Omega_R(0,0)} \right)^d \frac{\prod_{\alpha=1}^d (x_{j} - x_{i_\alpha})}{Q'(x_{i_\alpha})} \frac{\Delta_d(x_{i_1}, \ldots, x_{i_d})\Delta_d^{(y_{\text{cut}})}(x_{i_1}, \ldots, x_{i_d})}{\Omega_R(0,0)}.
\]  

(101)

Multiply the three contributions (97), (98) and (101) together and do the summation, in which the requirement \( j \notin (i_1, \ldots, i_d) \) can be removed; indeed whenever \( j \in (i_1, \ldots, i_d) \), the sum below would automatically vanish. So we find for each \( x_j \geq x \),

\[
\lim_{q \to 1} \int_{\Gamma_{\infty}} \frac{dz}{2\pi i z} \mathcal{R}_j(z) \\
= \frac{(N - n)!}{(N - m - 1)!} \frac{(x_j - x + 1)_{N-m-1}}{Q'(x_j)} \int_{\Gamma_{\infty}} \frac{dz Q(z)}{2\pi i (z - y)_{N-n+1}(z - x_j)} \\
\times \sum_{1 \leq i_1, \ldots, i_d \leq c} \left( \prod_{\alpha=1}^d \frac{(x_j - x_{i_\alpha})}{(z - x_{i_\alpha})Q'(x_{i_\alpha})} \right) \frac{\Delta_d(x_{i_1}, \ldots, x_{i_d})\Delta_d^{(y_{\text{cut}})}(x_{i_1}, \ldots, x_{i_d})}{\Omega_R(0,0)}. \\
= \frac{(N - n)!}{(N - m - 1)!} \frac{(x_j - x + 1)_{N-m-1}}{Q'(x_j)} \int_{\Gamma_{\infty}} \frac{dz Q(z)}{2\pi i (z - y)_{N-n+1}(z - x_j)} \frac{\Omega_R(x_j, z)}{\Omega_R(0,0)}.
\]

So we conclude, upon summing the \( v \)-residues,

\[
\lim_{q \to 1} \mathcal{K}_q^{(2)}(m, x; n, y) = \frac{(N - n)!}{(N - m - 1)!} \int_{\Gamma(x+N)} \frac{dv}{2\pi i} \int_{\Gamma_{\infty}} \frac{dz}{2\pi i} \frac{(v - x + 1)_{N-m-1}}{(z - v)(z - y)_{N-n+1}} \\
\times \frac{Q(z)\Omega_R(v, z)}{Q(v)\Omega_R(0,0)}.
\]
We now consider the third part $K_q^{(3)}$ of the kernel \[ (78) \], given in residue form as

$$K_q^{(3)}(m,x;n,y) = \frac{d}{q^{(2d+N-1)}} \oint_{\Gamma(q^{x-y+n})} dv \frac{v^m P_{N-m-1}(v q^{x-y}) \tilde{\Omega}_q(v q^y, y)}{2\pi i \prod_{1}^{d+N} (v - q^{x-y}) \Omega_q(0,0)}$$

with $\tilde{\Omega}_q(q^{x_i}, y)$ given in \[ (81) \], from which it follows that, upon using \[ (46) \], \[ (77) \], \[ (75) \], \[ (86) \], Lemma 2.1 and \[ (89) \],

$$\tilde{\Omega}_q(q^{x_j}, y) = \sum_{1 \leq i_2, \ldots, i_d \leq c} \frac{\Delta_d(q^{x_j}, q^{x_{i_2}}, \ldots, q^{x_{i_d}}) \tilde{\Delta}(y_{cut})(q^y, q^{x_{i_2}}, \ldots, q^{x_{i_d}})}{\prod_{\alpha=2}^{d} Q'_q(q^{x_i})}.$$

Inserting this formula into \[ (102) \], and using the limits \[ (37) \], we find

$$\lim_{q \to 1} K_q^{(3)}(m,x;n,y) = \frac{d}{(N - m - 1)!} \oint_{\Gamma(x+N)} dv \frac{\tilde{\Omega}_R^{(1)}(v, z)}{2\pi i \omega^{(1)}(v - x + 1)^{N-m-1}} \frac{\tilde{\Omega}_R'(v, w)}{\Omega_R(0,0)}.$$
in Corollary 3.3 and (46), we find
\[
\oint_{\Gamma_0} dw \bar{\Omega}_{L}^{(1)}(v, w)
\]
\[
= \left( \prod_{\alpha=2}^{d} \oint_{\Gamma(\alpha)} \frac{du_{\alpha}}{2\pi i u_{\alpha}} \right) \Delta_d(v, u_2, \ldots, u_d) \bar{\Delta}_{d, n}(y_{\text{cut}}, y, u_2, \ldots, u_d)
\]
\[
= \left( \oint_{\Gamma(y, y-1, \ldots, y+n-N)} - \oint_{\Gamma, r} \right) \frac{(N-n)! P(z) dz}{2\pi i (z-y) N-n+1} \bar{\Omega}_{R}(v, z),
\]
in terms of \(\bar{\Omega}_{R}(v, z)\), as defined in (91). This establishes formula (90) and thus ends the proof of Theorem 7.1.

Proof of Corollary 7.3: Setting \(d = 0\) in the kernel (92) leads to formulas (94).

Expanding \(1/(z - v) = \sum v^{i}/z^{i+1}\) formula (94) can be written as a finite sum \(\sum p_{i}(x) q_{j}(y)\), with \(p, q\) polynomials. It is unclear in case (ii) how this relates to the extended Hahn kernel in [17].

8 The \(K\)-kernel as a \(r + 3\)-fold integral, with \(r = \) number of particles on the oblique lines of the strip \{\(\rho\}\)

This section will prove Theorem 1.1, which holds for the two-cut model. It shows that the \(K\)-kernel (90) can be reduced from a \(d + 2\)-fold integral to a \(r + 3\)-fold integral (14), where (remember!) \(d\) is the size of the cut, and where \(r := b - d\) is the number of blue dots on the oblique lines \{\(\eta = m + k\}\} for \(0 \leq k \leq \rho\); i.e., within the strip \(\rho\). This reduction is crucial in order to be able to perform the asymptotic analysis, which will occur in Paper II, for the kernels \(K\) and \(L\), when all the sides and cuts of the polygon \(P\) tend to \(\infty\), while \(r\) and \(\rho\) remain finite.

Remember the polynomials \(P = P_{\rho} Q_{C} P_{\sigma}, Q = Q_{L} Q_{C} Q_{R}\) from (8), the rational function \(h\) from (10) and the \(\Omega_{\pm}\)-kernel from (11). In this section we prove the main Theorem 1.1. Before doing that, we need to rewrite the \(\Omega\)-expressions in the \(K\)-kernel (90) in terms of a contour \(\Gamma(L)\) about \(L\), instead of \(R\). This is done in the following proposition:

Proposition 8.1 Considering a hexagon \(P\) subjected to the conditions (6), the kernel for the \(K\)-process of red dots has the following form for \((m, x), (n, y) \in P:\)

\[
K(m, x; n, y) = -\left(\frac{y-x+1}{n-m-1}\right) \sum_{1_{n>m} \leq x} + \oint_{\Gamma(x+N)} \frac{dv(v-x+1)_{N-m-1}}{2\pi i (N-m-1)! Q(v)} \times
\]
\[
\left( \oint_{\Gamma_{\infty}} \frac{dz Q(z) (N-n)! \Omega_{L}(v, z)}{2\pi i (z-v) (z-y) N-n+1} + d \oint_{\Gamma_{s}} \frac{dz P(z) (N-n)! \bar{\Omega}_{L}(v, z)}{2\pi i (z-y) N-n+1} \right),
\]

(104)
where $\Omega_C(v, z)$ and $\tilde{\Omega}_C(v, z)$ are the same expressions as $\Omega_R(v, z)$ and $\tilde{\Omega}_R(v, z)$ in [91], but with the contour $\Gamma(\mathbb{R})$ replaced by $\Gamma(L)$; see [7]. The contour $\Gamma_\tau$ was defined in [15].

To prove Proposition 8.1 we need the following Lemma:

**Lemma 8.2** Given a rational function $R(u)$ with possibly poles within a contour $\Gamma$ and a point $z$ not within $\Gamma$, not a pole of $R(u)$. Then for $0 \leq k - 1 \leq \ell$, we have (the notation $\Gamma \cup z$ refers to the contour $\Gamma$, deformed so as to contain $z \in \mathbb{C}$)

\[
\left(\prod_{\alpha=1}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)}\right) \Delta_\ell^2(u_1, \ldots, u_{\ell+1}) = (k - 1)R(z) \left(\prod_{\alpha=1}^{\ell-k+1} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)}\right) \Delta_{\ell-k+1}^2(u_1, \ldots, u_{\ell+1}) \quad (105)
\]

\[
+ \prod_{\alpha=1}^{\ell-k+1} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \prod_{\alpha=k}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \Delta_\ell^2(u_1, \ldots, u_{\ell+1}).
\]

In particular for $k = \ell + 1$,

\[
\left(\prod_{\alpha=1}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)}\right) \Delta_\ell^2(u_1, \ldots, u_{\ell+1}) = \ell R(z) \left(\prod_{\alpha=1}^{\ell-1} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)}\right) \Delta_{\ell-1}^2(u_1, \ldots, u_{\ell+1}). \quad (106)
\]

**Proof of Lemma 8.2** The identity (105) is obviously true for $k = 1$. We now proceed by induction: let it be true for fixed $k \geq 1$; then we show its truth for $k + 1$. In the second expression on the right hand side of equation (105) one computes the residue at $u_k = z$, yielding

\[
\prod_{\alpha=1}^{k-1} \oint_{\Gamma} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \prod_{\alpha=k}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \Delta_\ell^2(u_1, \ldots, u_{\ell+1})
\]

\[
= \prod_{\alpha=1}^{k-1} \oint_{\Gamma} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \oint_{\Gamma \cup z} \frac{du_k R(u_k)}{2\pi i(u_k - z)} \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \Delta_\ell^2(u_1, \ldots, u_{\ell+1})
\]

\[
= \prod_{\alpha=1}^{k-1} \oint_{\Gamma} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \Delta_\ell^2(u_1, \ldots, u_{\ell+1})
\]

\[
+ \prod_{\alpha=1}^{k-1} \oint_{\Gamma} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \left( R(z) \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup z} \frac{du_\alpha R(u_\alpha)}{2\pi i(u_\alpha - z)} \Delta_\ell^2(u_1, \ldots, u_{\ell+1}) \right)^{\frac{k}{2}}
\]

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\[
\prod_{\alpha=1}^{k} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \, du_\alpha \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \, du_\alpha \Delta^2_{\ell-1}(u_1, \ldots, u_\ell)
\]

\[
+ R(z) \prod_{\alpha=1}^{k-1} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \left( \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \right) \Delta^2_{\ell-1}(u_1, \ldots, \hat{u}_k, \ldots, u_\ell) \prod_{j=1}^{\ell} (u_j - z)
\]

\[
= \prod_{\alpha=1}^{k} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \prod_{\alpha=k+1}^{\ell} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) \Delta^2_{\ell}(u_1, \ldots, u_\ell)
\]

\[
+ R(z) \prod_{\alpha=1}^{k-1} \oint_{\Gamma \cup \mathbb{C}} R(u_\alpha) (u_\alpha - z) \Delta^2_{\ell-1}(u_1, \ldots, u_{\ell-1})
\]

The latter is obtained by a renaming of the variables \((u_1, \ldots, u_{k-1}, u_k, \ldots, u_\ell) \rightarrow (u_1, \ldots, u_{\ell-1})\) and upon noticing that in the second expression the integration over \(\Gamma \cup \mathbb{C}\) can be replaced by \(\Gamma\), since the integrand has no residue at \(z\), ending the proof of Lemma 8.2.

**Proof of Proposition 8.1** The following identities holds:

\[
\frac{\Omega_R(v, z) + d(z - v) \frac{P(z)}{Q(z)} \tilde{\Omega}_R(v, z)}{\Omega_R(0, 0)} = \frac{\Omega_L(v, z)}{\Omega_L(0, 0)} \quad \text{and} \quad \frac{\tilde{\Omega}_R(v, z)}{\tilde{\Omega}_R(0, 0)} = - \frac{\tilde{\Omega}_L(v, z)}{\tilde{\Omega}_L(0, 0)}
\]

It suffices to show that

\[
(-1)^d \Omega_L(v, z) = \Omega_R(v, z) + d(z - v) \frac{P(z)}{Q(z)} \tilde{\Omega}_R(v, z)
\]

\[
(-1)^{d-1} \tilde{\Omega}_L(v, z) = - \tilde{\Omega}_R(v, z)
\]

Indeed, remembering from formula (3), that

\[
\frac{P(u)}{Q(u)} = \frac{P_\rho(u)P_\sigma(u)}{Q_\rho(u)Q_\sigma(u)},
\]

we have that each integrand in the \(d\)-fold integral \(\Omega_L(v, z)\) and in the \(d-1\)-fold integral \(\tilde{\Omega}_L(v, z)\) (as in Proposition 8.1) have respective degree (in a fixed \(u_\alpha\)) equal to

\[
\rho + \sigma - b - c + 2(d-1) = -2, \quad \rho + \sigma - b - c + 2(d-2) + 2 = -2
\]

using the expressions (4) and (5) for \(\rho, \sigma\). So the integrands have no pole at \(\infty\). Thus the contour \(-\Gamma(L)\) in \(\Omega_L\) (as in (108)) can be deformed to a contour
\( \Gamma(\mathcal{R} \cup z) \); so we have \((C_{N,d} \text{ is the constant in Proposition } 3.1)\)

\[
(110) \quad (\frac{1}{d!} \frac{\partial^d}{\partial z^d} \Omega_\mathcal{L}(v, z)) = \left( \prod_{\alpha=1}^d \oint_{\Gamma(\mathcal{R}\cup z)} \frac{du_\alpha P(u_\alpha)}{2\pi i Q(u_\alpha)} u_\alpha - v \right) \Delta_d(u)^2,
\]

which has the form of Lemma \([8.2]\) with \(\ell = d\) and \(R(u) = \frac{P(u)}{Q(u)} (u - v)\), modulo a sign of \((-1)^d\). Then the right hand side of \((110)\) equals, by formula \((106)\) and \((91)\),

\[
\frac{\Omega_\mathcal{R}(v, z)}{C_{N,d}} + d(z - v) \frac{P(z)}{Q(z)} \prod_{\alpha=1}^d \oint_{\Gamma(\mathcal{R})} \frac{du_\alpha P(u_\alpha)}{2\pi i Q(u_\alpha)} (v - u_\alpha)(z - u_\alpha) \Delta_d-1(u_2, \ldots, u_d)^2
\]

\[
= \frac{1}{C_{N,d}} \left( \Omega_\mathcal{R}(v, z) + d(z - v) \frac{P(z)}{Q(z)} \tilde{\Omega}_\mathcal{R}(v, z) \right),
\]

establishing the first identity \((108)\). The second identity \((108)\) is simpler, since the integrands have no pole about \(u_\alpha = z\). Therefore the contour about \(\mathcal{L}\) can be deformed to a contour about \(\mathcal{R}\) in each of the \(d - 1\) integrals. This shows both identities \((108)\) and thus \((110)\), upon using \(\Omega_\mathcal{L}(0, 0) = (-1)^d \Omega_\mathcal{R}(0, 0)\), ending the proof of Proposition 8.1.

\[\text{Proof of Theorem 1.1}\] It suffices to prove:

\[
\frac{\Omega_\mathcal{L}(v, z)}{\Omega_\mathcal{L}(0, 0)} = \frac{Q_\mathcal{L}(v)}{Q_\mathcal{L}(z)} \frac{\Omega_\mathcal{L}^+(v, z)}{\Omega_\mathcal{L}^+(0, 0)}, \quad d \frac{\tilde{\Omega}_\mathcal{L}(v, z)}{\Omega_\mathcal{L}(0, 0)} = \frac{Q_\mathcal{L}(v)}{Q_\mathcal{L}(z)} \frac{\Omega_\mathcal{L}^+(v, z)}{\Omega_\mathcal{L}^+(0, 0)},
\]

\((111)\)

\(\Omega_\mathcal{L}, \tilde{\Omega}_\mathcal{L}\) were defined in \((91)\) with \(\mathcal{R} \rightarrow \mathcal{L}\), and \(\Omega_\mathcal{L}^+\) and \(\Omega_\mathcal{L}^−\) in \((11)\).

The proof will depend on the two identities below \((112)\) and \((113)\). Set \(J := \{i_1 < \cdots < i_k\text{, with } x_{i_\alpha} \in \mathcal{L}\}\), viewed as particles and the corresponding set of holes \(j_1 < \cdots < j_\ell\), also with \(x_{j_\alpha} \in \mathcal{L}\) and \(b = k + \ell\). Let \(\varphi(u)\) be a rational function with no roots or poles along \(\mathcal{L}\). For the ease of notation set momentarily \(\Delta_b(\mathcal{L}) := \Delta_b(x_{d+c+1}, \ldots, x_{d+c+b})\). Then

\[
\prod_{\alpha=1}^k \frac{\varphi(x_{i_\alpha})}{Q_\mathcal{L}(x_{i_\alpha})} = \prod_{x_i \in \mathcal{L}} \frac{\varphi(x_i)}{Q_\mathcal{L}(x_i)} \prod_{\alpha=1}^\ell \frac{Q_\mathcal{L}(x_{j_\alpha})}{\varphi(x_{j_\alpha})}.
\]

\((112)\)

In formula \((49)\) of Lemma \([3.4]\), set \(n = k, m = \ell\) and \(n + m = k + \ell = b\), and \(y = (x_{i_1}, \ldots, x_{i_k})\) and \(x = (x_{j_1}, \ldots, x_{j_\ell})\), and \(Q := Q_\mathcal{L}(z)\), leading to:

\[
\Delta_k(x_{i_1}, \ldots, x_{i_k}) = \pm \frac{\Delta_b(\mathcal{L}) \Delta_\ell(x_{j_1}, \ldots, x_{j_\ell})}{\prod_{\alpha=1}^\ell Q_\mathcal{L}(x_{j_\alpha})}.
\]

\((113)\)
Then, combining the two expressions above, we have for \( k, \ell \geq 0 \) such that \( k + \ell = b \),

\[
\left( \prod_{\alpha=1}^{k} \int_{\Gamma(\xi)} \frac{du_{\alpha} \varphi(u_{\alpha})}{2\pi i Q_{\xi}(u_{\alpha})} \right) \Delta_{k}(u)^{2} = \sum_{i_1 < \cdots < i_k} k! \Delta_{k}^{2}(x_{i_1}, \ldots , x_{i_k}) \left( \prod_{\alpha=1}^{k} \varphi(x_{i_{\alpha}}) \right)
\]

\[
= \frac{k!}{\ell!} \Delta_{b}^{2}(\xi) \prod_{x_{i} \in \xi} \varphi(x_{i}) \ell! \sum_{j_1 < \cdots < j_{\ell}} \left( \prod_{\alpha=1}^{\ell} \frac{Q_{\xi}'(x_{j_{\alpha}})}{Q_{\xi}(x_{j_{\alpha}})} \varphi(x_{j_{\alpha}})(Q_{\xi}'(x_{j_{\alpha}}))^{2} \right) \Delta_{b}^{2}(x_{j_1}, \ldots , x_{j_\ell})
\]

\[
= \frac{k!}{\ell!} \Delta_{b}^{2}(\xi) \left( \prod_{x_{i} \in \xi} \varphi(x_{i}) \right) \left( \prod_{\alpha=1}^{\ell} \int_{\Gamma(\xi)} \frac{du_{\alpha}}{2\pi i \varphi(u_{\alpha})Q_{\xi}(u_{\alpha})} \right) \Delta_{b}^{2}(u).
\]

We now apply the formula above to two different \( k, \ell \) and \( \varphi(u) \) (themselves depending on \( v, z \)):

\[
\varphi(u) := \frac{P(u)(v - u)}{Q_{\xi}(u)Q_{\xi}(u)(z - u)} \quad \text{or} \quad \varphi(u) := \frac{P(u)(v - u)(z - u)}{Q_{\xi}(u)Q_{\xi}(u)}
\]

\[
k = d, \quad \ell = r 
\]

\[
k = d - 1, \quad \ell = r + 1.
\]

Using the expression \((10)\) for \( h(u) \), and remembering the definitions \((9)\) and \((11)\) of \( \Omega_{\xi} \) and \( \Omega_{r} \), the first substitution \((115)\) in \((114)\) leads to the following expression, where \( C_{N,d} \) is the constant in \((39)\) and where \( C_{\xi} \) is a constant\(^{14}\) depending on the geometry of \( P \) only:

\[
\Omega_{\xi}(v, z) = C_{N,d} \left( \prod_{\alpha=1}^{d} \int_{\Gamma(\xi)} \frac{du_{\alpha} P(u_{\alpha}) v - u_{\alpha}}{2\pi i Q(u_{\alpha}) z - u_{\alpha}} \right) \Delta_{d}(u)^{2}
\]

\[
= \frac{dl!}{r!} C_{N,d} C_{\xi} \frac{Q_{\xi}(v)}{Q_{\xi}(z)} \left( \prod_{\alpha=1}^{r} \int_{\Gamma(\xi)} \frac{du_{\alpha} h(u_{\alpha}) z - u_{\alpha}}{2\pi i v - u_{\alpha}} \right) \Delta_{r}(u)
\]

\[
= \frac{dl!}{r!} C_{N,d} C_{\xi} \frac{Q_{\xi}(v)}{Q_{\xi}(z)} \Omega_{r}(v, z),
\]

whereas the second substitution \((115)\) in \((114)\) yields, remembering the definitions \((9)\) and \((11)\) of \( \Omega_{\xi} \) and \( \Omega_{r+1} \),

\[
\tilde{\Omega}_{\xi}(v, z) = C_{N,d} \left( \prod_{\alpha=1}^{d-1} \int_{\Gamma(\xi)} \frac{du_{\alpha} P(u_{\alpha}) (v - u_{\alpha})(z - u_{\alpha})}{2\pi i Q(u_{\alpha})} \right) \Delta_{d-1}(u)^{2}
\]

\[
= \frac{(d - 1)!}{(r + 1)!} C_{N,d} C_{\xi} Q_{\xi}(v)Q_{\xi}(z) \left( \prod_{\alpha=1}^{r+1} \int_{\Gamma(\xi)} \frac{du_{\alpha} h(u_{\alpha})}{2\pi i (v - u_{\alpha})(z - u_{\alpha})} \right) \Delta_{r+1}(u)
\]

\[
= \frac{(d - 1)!}{(r + 1)!} C_{N,d} C_{\xi} Q_{\xi}(v)Q_{\xi}(z) \Omega_{r+1}(v, z).
\]

\(^{13}\) as in Proposition \(8.1\)

\(^{14}\) To be precise, \( C_{\xi} = \Delta(\xi)^{2} \prod_{x_{i} \in \xi} \frac{P(x_{i})}{Q_{\xi}(x_{i})Q_{\xi}(x_{i})Q_{\xi}(x_{i})} = \Delta(\xi)^{2} \prod_{x_{i} \in \xi} \frac{P(x_{i})}{Q(x_{i})}. \)
These two identities lead at once to the ratios (111). Substituting these ratios in the kernel (104) establishes Theorem 1.1.

Corollary 8.3 For \( b = d \), we have that

\[
\frac{\Omega_L(v, z)}{\Omega_L(0, 0)} = \frac{Q_L(v)}{Q_L(z)}.
\]

Proof of Corollary 8.3: Here \( r = 0 \) and thus \( \Omega_r(v, z) = 1 \) and the corollary follows from (111).

9 An alternative form of the \( \mathbb{K} \)-kernel

The aim of this section is to prove Theorem 1.2, which holds for the two-cut model, although many of the arguments discussed below remain valid for the multi-cut case. The lozenge tilings of a polygon is equivalent to dimers on its dual graph, a honeycomb graph (bipartite), consisting of dots, alternating with circles; see Fig. 7. The dots correspond to the shaded triangles and the circles to white triangles. Only one segment emanates out of each vertex, as shown in Figs. 7 and 8. The Kasteleyn matrix for this honeycomb graph is the adjacency matrix and is given by:

\[
\mathbb{K}_{\text{Kast}}(\circ \rightarrow \bullet) = \begin{cases} 1, & \text{if } (m, x) = (n, y), \text{as in (i')} \\ 1, & \text{if } (m, x) = (n - 1, y), \text{as in (ii')} \\ 1, & \text{if } (m, x) = (n - 1, y + 1), \text{as in (iii')} \\ 0, & \text{otherwise} \end{cases}
\]

in terms of the dimers (i), (ii) and (iii) given in Fig. 8. Typically the vertices would belong to the middle of the black and white triangles, as on the left Fig.8. However, the \((x, n)\)-coordinates of our problem correspond to the integer points coinciding with the middle of the lower-edge of the squares, which is the middle of the lower-edge of the shaded triangles on the right Fig.8 above. In order to make the two figures match, one moves the dots in the white triangles to the top-edge and the circles in the shaded triangles to the lower-edge. Thus the Kasteleyn matrix in the new coordinates \((i'), (ii'), (iii')\) reads

\[
\mathbb{K}_{\text{Kast}}(\circ(n, y); \bullet(m, x)) = \begin{cases} 1, & \text{if } (m, x) = (n, y), \text{as in (i')} \\ 1, & \text{if } (m, x) = (n - 1, y), \text{as in (ii')} \\ 1, & \text{if } (m, x) = (n - 1, y + 1), \text{as in (iii')} \\ 0, & \text{otherwise} \end{cases}
\]

with vanishing boundary conditions each time a point lies outside the polygon \( \mathbb{P} \).
Fig. 7. Lozenge tilings and corresponding dimers on the honeycomb lattice

Fig. 8: Three different dimers emanating from a dot and the change of coordinates, by moving the circles to the middle upper-part of the triangle and the dots to the middle of the lower-part of the shaded triangle.

We now state the following Theorem, which will be shown in Paper II and which uses the form of $\mathbb{K}$ stated in Proposition 7.2.

**Theorem 9.1** For the multi-cut case the $\mathbb{K}$-kernel is the inverse of the $\mathbb{K}_{Kast}$; to be precise:

$$\mathbb{K}_{Kast}^{-1}(\bullet(m,x); o(n,y)) = (-1)^{x-y+m-n} \mathbb{K}(m,x;n,y).$$  (118)

Keeping in mind the notation $\theta$ and $\delta$ in (16), we now proceed to:

**Proof of Theorem 1.2** Rotating Fig.3 by $180^\circ$, including the $(x,n)$-axis, we find a new hexagon $P'$, with coordinates $(x',n')$ and parameters $(m'_1, m'_2, n'_1, n'_2, b', c', d') = (n_2, n_1, m_2, m_1, b, c, d)$. Let $'\ $ be this involution. The points $x'_1, \ldots, x'_c, x'_{c+1}, \ldots, x'_{c+d}, x'_{d+c+1}, \ldots, x'_{d+N}$ on the top line of $P'$ have $(x',n')$-coordinates, given by:

$$x' \text{-coordinates} =$$

$$(n_1 + n_2 - 1, \ldots, n_1 + n_2 - c, n'_1 - c - 1, \ldots, n'_1 - c - d, -c - d - 1, \ldots, -d - N) \quad (119)$$

$$n' \text{-coordinates} = (N, \ldots, N).$$
Fig. 9. The lozenges after a rotation of the dimers by 180°.

Denoting the coordinates \((x'(O'), n'(O'))\) = \((x_{d+N} + (d + N), 0)\) of the origin \(O'\) in \(P'\), and using (119), the \((x', m')\)-coordinates of the origin \(O \in P\) are given by (remembering the definition 16 of \(\theta\))

\[
x'(O) = (x'_1 + b) - (d + N) = n_1 + n_2 - 1 + b - (d + N) = \theta
\]

\[
m'(O) = N.
\]

Hence the coordinate transformation \((x, m) \rightarrow (x', m')\) of the points \(p\) is given by

\[
x'(p) = \theta - x(p), \quad m'(p) = N - m(p).
\]

The dimer model, as in Fig. 7, consists of black triangles pointing up and white triangles pointing down, each carrying a black and a white dot respectively; they form together three different tiles, as in Fig. 8. Rotating this figure by 180° has for effect to interchange the two triangles, including the dots representing the black and white dots of the bipartite graph. The rotation does not change the tiles, as can be seen from Fig. 9, but interchanges the role of the black and white triangles, including the dots; in particular the white triangles now point up and the black triangles down. Therefore the Kasteleyn matrices (adjacency matrix of the black and white dots) for the initial and the rotated problems are the transpose of each other; thus we have:

\[
K_{Kast}'(\bullet(n', y') \circ (m', x')) = K_{Kast}^T(\bullet(n', y') \circ (m', x')).
\]  \(120\)

We showed that

\[
(-1)^{x-y+m-n}K(m, x; n, y) = K_{Kast}^{-1}(\bullet(m, x) \circ (n, y)),
\]

and thus for the rotated problem, we have the set of equations for the \(K'\)-kernel for the red dot-process on \(P'\), upon using in = the fact that the black and white dots get interchanged, and using identity (120) in =*

\[
(-1)^{x'-y'+m'-n'}K'(m', x'; n', y') = K_{Kast}^{-1}(\bullet(m', x') \circ (n', y'))^*
\]

\[
= K_{Kast}^{-1}(\circ(m', x') \bullet(n', y'))^*
\]

\[
= K_{Kast}^{-1}(\bullet(n', y') \circ (m', x')).
\]
Then relabeling \((m', x') \leftrightarrow (n', y')\), we have that, since any point \(p \in P\) has coordinates \((m, x)\) or \((m', x')\) in the other coordinates, as long as \((m', x') = (N - m, \theta - x)\)

\[
(-1)^{x'-y'+m'-n'} K'(n', y'; m', x') = K^{-1}_{Kast}(\bullet(m', x') \circ (n', y'))
= K^{-1}_{Kast}(\bullet(m, x) \circ (n, y))
= (-1)^{x-y+m-n} K(m, x; n, y).
\]

So, we have the identity for all \((m, x), (n, y) \in P\),

\[
K(m, x; n, y) = K'(N - n, \theta - y; N - m, \theta - x). \tag{121}
\]

The latter induces the involution

\[
' : (n_1, n_2, m_1, m_2) \rightarrow (m_2, m_1, n_2, n_1).
\]

\[
'. : (m, x; n, y) \rightarrow (N - n, \theta - y; N - M, \theta - x). \tag{122}
\]

Using the involution \((122)\) on the kernel \((14)\), one finds formula \((17)\). For example, the \(\Gamma_{\tau}\)-term in \(K(m, x; n, y)\), as in \((14)\), is present when \(\tau = (y + n) - m < 0\), with \(m_1 = y_1 + 1\). The involuted statement then amounts to the following one, using the values \((16)\) and \((4)\) of \(\theta\) and \(\rho\),

\[
0 > (y' + n') - m_1 = (\theta - x + N - m) - n_2 = -(x + m) + (y_1 + \rho) = -\tau'.
\]

Hence the \(\Gamma_{\tau'}\)-term in \(K(m, x; n, y)\), as in \((17)\), will be present when \(\tau' > 0\). This establishes Theorem 1.2.

10 Multi-cut Polygon

The kernel \((90)\), appearing in Theorem 7.1 can be extended to the multi-cut model; i.e., where the lower-part of the polygon has several cuts, as well. It requires a number of insertions, explained below. Expression \((39)\) in Proposition 3.1 can be rephrased as follows:

**Proposition 3.1’** Given integers \(c_1 < \cdots < c_d\) satisfying \(N - 2 + c_1 - c_d \geq 0\), and given the sum of the gaps \(g := c_d - c_1 - d + 1\) between the integers \(c_1 < \cdots < c_d\), the following holds:

\[
\det \left( \frac{(x_\alpha + c_\beta)_{N-1}}{(N-1)!} \right)_{1 \leq \alpha, \beta \leq d} = C_{N,d} C_{N,d}' \Delta_d(x_1, \ldots, x_d) G_g^{(c)}(x_1, \ldots, x_d) \prod_{\alpha=1}^{d} (x_\alpha + c_d)_{N-1+c_1-c_d}. \tag{123}
\]
where \( E^{(e)}_g \) is a symmetric function of \( x_1, \ldots, x_d \), with coefficients depending on the \( c_i \)’s,
\[
E^{(e)}_g(x_1, \ldots, x_d) = C'_{N,d}(x_1 \ldots x_d)^g + \text{lower order terms},
\]
where the “lower order terms” refer to terms of total degree < \( gd \), with degree in any \( x_i \leq g \), and where
\[
C_{N,d} = \prod_{j=1}^{d} (N-j)!, \quad C'_{N,d} = \Delta_d(c_1, \ldots, c_d) \prod_{j=1}^{d} (d-j)!.
\]

Upon defining the symmetric function
\[
E_g(y_{\text{cut}})(u) := E_g(y_{\text{cut}})(u_1 - y_1, \ldots, u_d - y_1),
\]
with coefficients depending on the points \( y_1 > \cdots > y_d \) in the lower-cuts, Corollary 3.3 gets rephrased as follows:

**Corollary 3.3’** Given the multi-cut model, the following identities hold for the determinantal expressions in (46), with \( C_{N,d} \) as in (39) and the contour \( \Gamma \) as in (42):
\[
\Delta_d(y_{\text{cut}})(u_1, \ldots, u_d) = C_{N,d}C'_{N,d}E_g(y_{\text{cut}})(u)\Delta_d(u) \prod_{\alpha=1}^{d} P(u_{\alpha}),
\]
and
\[
\tilde{\Delta}_{d,n}(y; u_2, \ldots, u_d) = C_{N,d}C'_{N,d} \left( \oint_{\Gamma(y,y-1,\ldots,y-N+n)} - \oint_{\Gamma_{\tau}} \right) \frac{(N-n)!dz}{2\pi i(z-y)N-n+1} \times E_g(y_{\text{cut}})(z, u_2, \ldots, u_d)\Delta_d(z, u_2, \ldots, u_d)P(z) \prod_{\alpha=2}^{d} P(u_{\alpha}).
\]

Finally, Theorem 7.1 remains valid as such, with very small modifications:

**Theorem 7.1’** For \((m, x), (n, y) \in P\), the kernel \( K \) for the red dots in the multi-cut model has the form (90), with
\[
\Omega_{R}(v, z) := C_{N,d}C'_{N,d} \left( \prod_{\alpha=1}^{d} \oint_{\Gamma_{\tau}} \frac{du_{\alpha}P(u_{\alpha})}{2\pi i Q(u_{\alpha})} \frac{v - u_{\alpha}}{z - u_{\alpha}} \right) \Delta_d(u)^2 E_g(y_{\text{cut}})(u)
\]
\[
\overline{\Omega}_{R}(v, z) := C_{N,d}C'_{N,d} \left( \prod_{\alpha=2}^{d} \oint_{\Gamma_{\tau}} \frac{du_{\alpha}P(u_{\alpha})}{2\pi i Q(u_{\alpha})} \frac{v - u_{\alpha}}{z - u_{\alpha}} \right) \times \Delta_d(z, u_2, \ldots, u_d)^2 E_g(y_{\text{cut}})(z, u_2, \ldots, u_d).
\]
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