When Are IG-projective Modules Projective? *†

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Abstract

This paper concerns when a finitely generated IG-projective module is projective over commutative Noetherian local rings. We prove that a finitely generated IG-projective module is projective if and only if it is selforthogonal.

1 Introduction

Unless stated otherwise, all rings discussed in this paper are commutative Noetherian local rings, and all modules are finitely generated. Let $R$ be a commutative Noetherian ring. We use $\text{mod} R$ to denote the category of finitely generated $R$-modules. As a common generalization of the notion of projective modules, Auslander and Bridger in [AuB] introduced the notion of finitely generated modules of Gorenstein dimension 0. Such modules are called Gorenstein projective, following Enochs and Jenda’s terminology in [EJ], which are defined as follows:

Definition 1.1 An $R$-module $M$ is said to be Gorenstein projective (G-projective, for short) if there exists an exact sequence of projective modules

$$\text{P} = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \cdots$$

such that $\text{Hom}_R(\text{P}, R)$ is exact and $M \cong \text{Im}(P_0 \rightarrow P_{-1})$.

The exact sequence $\text{P}$ is called a complete projective resolution of $M$.

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We denote $G(R)$ as the full subcategory of $\text{mod} R$ consisting of all Gorenstein projective modules. It is well known that a projective module is Gorenstein projective. It is natural to ask when are the Gorenstein projective modules projective. Our guess is that the Gorenstein projective module is projective if and only if it is self-orthogonal. In [LH], it is proved that this conjecture is true if $R$ is a ring with radical square zero.

**Definition 1.2** An indecomposable $R$-module $M$ is said to be IG-projective if it is $G$-projective and admits either an irreducible epimorphism $P \to M$ or an irreducible monomorphism $M \to P$, with $P$ being a projective module. A (possibly decomposable) module is IG-projective if it is a direct sum of indecomposable IG-projectives.

This notion was introduced by Luo[L], who also prove that if, over such an Artin local algebra $R$ with the simple IG-projective module, then 1-self-orthogonal modules are projective.

In this paper, one sees the isomorphisms as irreducible morphisms. Thus, the projective modules are IG-projective. The main purpose of this paper is to prove that this conjecture is also true for IG-projective modules if $R$ is a commutative Noetherian local ring, which is the following theorem.

**Theorem 1.1** For a commutative Noetherian local ring, a finitely generated IG-projective module is projective if and only if it is selforthogonal.

In the next section, we start by recalling the definitions of Gorenstein dimension and approximation of a module, give several preliminary lemmas involving their properties.

## 2 Preliminaries

In this section, we provide some background material. Throughout this section, let $(R, m, k)$ be a commutative Noetherian local ring with the maximal ideal $m$ and the field $k$. The starting point is a definition of $G$-dimension, introduced by Holm[H]

**Definition 2.1** Let $M$ be an $R$-module. If $n$ is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of $R$-modules with $G_i \in G(R)$ for every $i = 0, 1, \cdots, n$, then we say that $M$ has $G$-dimension at most $n$, and write $G - \dim_R M \leq n$. If such an integer $n$ does not exist, then we say that $M$ has infinite $G$-dimension, and write $G - \dim_R M = \infty$. 
Recall $R$ is called Gorenstein if self-injective dimension of $R$ is finite. The next three lemmas are the properties of $G$-dimension, the proofs are seen in [Ch] and [Ta]

**Lemma 2.1** Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $R$-modules. If two of $L, M, N$ have finite $G$-dimension, then so does the third.

**Lemma 2.2** The following conditions are equivalent:

1. $R$ is Gorenstein;
2. $G - \dim_R M < \infty$ for any $R$-module $M$;
3. $G - \dim_R k < \infty$.

**Lemma 2.3** Suppose that there is a direct sum decomposition $m = I \oplus J$ where $I, J$ are non-zero ideals and $G \dim_R I$ is finite. Then $R$ is a Gorenstein local ring of dimension one.

Next, the notion of a approximation of a module is introduced by Auslander and Reiten [AuR].

**Definition 2.2** Let $\mathcal{X}$ be a full subcategory of $\text{mod}R$ and $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \text{mod}R$. We call $\phi$ a right $\mathcal{X}$-approximation of $M$ if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.

Let $P_1 \to P_0 \to M \to 0$ be a presentation with $P_i$ projective $R$-modules. We write $f^*$ for $\text{Hom}_R(f, R)$, $(-)^*$ for $\text{Hom}_R(-, R)$ and recall that the $R$-module $\text{Coker} f^*$ is called the transpose of $M$, and denote as $\text{Tr} M$; this is well-defined up to projective summands. Here we state an exact sequence and isomorphism of functors for later use. For the proofs, we refer to [VM] and [AF].

**Lemma 2.4** For any $M \in \text{mod}R$, there exists an exact sequence of functors from $\text{mod}R$ to itself:

$$0 \to \text{Ext}_R^1(\text{Tr} M, -) \to M \otimes_R - \xrightarrow{\lambda(\cdot)} \text{Hom}_R(M^*, -) \to \text{Ext}_R^2(\text{Tr} M, -) \to 0.$$ 

**Lemma 2.5** For any $M \in \text{mod}R$, there exist isomorphisms of functors from $\text{mod}R$ to itself:

$$(M \otimes -)^* \cong \text{Hom}_R(M, (-)^*) \cong \text{Hom}_R(-, M^*)$$
3 The main results

In this section, let \((R, m, k)\) be a commutative Noetherian local ring with the maximal ideal \(m\) and the field \(k\), we begin with introducing a proposition, which plays a crucial role in this section. Put \(D(\cdot) = \text{Hom}_R(\cdot, E(R/J))\) where \(J\) is the Jacobson radical of \(R\) and \(E(R/J)\) is the injective envelope of \(R/J\).

**Proposition 3.1** If \((R, m, k)\) is a local ring such that \(G\text{-dim} Dm\) is finite, then there exists an exact sequence

\[0 \to L \to X \to Dk \to 0\]

with \(X\) in \(G(R)\) such that

1. the morphism \(X \to Dk\) is a \(G(R)\)-approximation of \(Dk\) and \(\text{Ext}^i_R(G(R), L) = 0\) for any \(i \geq 1\);

2. the sequence \(0 \to \text{Hom}_R(Dk, G) \to \text{Hom}_R(X, G) \to \text{Hom}_R(L, G) \to 0\) is exact for any Gorenstein projective \(R\)-module \(G\) that is not projective.

**Proof.** Applying the functor \(D(\cdot)\) to the exact sequence \(0 \to m \to R \to k \to 0\), we have \(0 \to Dk \to DR \to Dm \to 0\). Let the morphism \(g : Q \to DR\) be a projective cover of \(DR\) with the projective module \(Q\). Consider a pull-back diagram of the morphisms \(Dk \to DR\) and \(Q \to DR\):

\[
\begin{array}{ccc}
0 & \to & \text{Ker} g \\
\downarrow & & \downarrow \\
0 & \to & Y \\
\downarrow & & \downarrow \\
Dm & \to & Dm
\end{array}
\begin{array}{ccc}
0 & \to & Dk \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\begin{array}{ccc}
0 & \to & Q \\
\downarrow & & \downarrow \\
0 & \to & DR \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

then the sequence \(0 \to \text{Ker} g \to Q \to DR \to 0\) is exact. This induces that \(\text{Ext}^i_R(G(R), \text{Ker} g) = 0\) for \(i > 0\). Since \(G\text{-dim}_RDm\) is finite, by lemma 2.1, so is \(Y\). We consider the strict \(G(R)\)-resolution of \(Y\), say \(0 \to P_s \to P_{s-1} \to \cdots \to P_1 \to X \to Y \to 0\) with all the \(P_i\) being projective and \(X\) belonging to \(G(R)\). Consider the pullback of the morphisms \(X \to Y\) and
Ker $g \to Y$

then the long sequence

$$0 \to P_s \to P_{s-1} \to \cdots \to P_1 \to L \to \text{Ker } g \to 0$$

is exact. Therefore $\text{Ext}^i_R(G(R), \text{Ker } g) = 0$ tells us that $\text{Ext}^i_R(G(R), L) = 0$ for $i > 0$. Thus $X \in G(R)$ implies that the exact $0 \to L \to X \to Dk \to$ is a $G(R)$-approximation of $Dk$. This completes the proof of (1).

Next to prove (2). Let $G$ be an any indecomposable $G$-projective $R$-module that is not projective. We take $\lambda_{G^*}(\cdot)$ to be the morphism $\lambda_{G^*}(\cdot) : G^* \otimes_R - \to \text{Hom}_R(G^{**}, -)$ by $\lambda_{G^*}(\cdot)(a \times -)(f) = f(a) \cdot -$ for any $a \in G^*$, $f \in G^{**}$. Note $\text{Tr} G^* \in G(R)$ and the $G(R)$-approximation of $Dk g : X \to Dk$, we have

$$\text{Ker } \lambda_{G^*}(L) = \text{Ext}^1_R(\text{Tr} R G^*, L) = 0 \quad \text{and} \quad \text{Coker } \lambda_{G^*}(L) = \text{Ext}^2_R(\text{Tr} R G^*, L) = 0$$

By the lemma 2.4, this means that $\lambda_{G^*}(L)$ is an isomorphism. Hence the composite map $\lambda_{G^*}(X) \cdot (G^* \otimes_R \cdot) = \text{Hom}_R(G^{**}, \cdot) \cdot \lambda_{G^*}(L)$ is injective, and so is the map $G^* \otimes_R \cdot$. Thus we have the following commutative diagram

$\begin{array}{cccccccc}
0 & \to & G^* \otimes_R L & \xrightarrow{\lambda_{G^*}(L)} & G^* \otimes_R X & \xrightarrow{\text{Hom}_R(G^{**}, \cdot)} & G^* \otimes_R Dk & \to & 0 \\
0 & \to & \text{Hom}_R(G^{**}, L) & \xrightarrow{\text{Hom}_R(G^{**}, \cdot)} & \text{Hom}_R(G^{**}, X) & \xrightarrow{\text{Hom}_R(G^{**}, \cdot)} & \text{Hom}_R(G^{**}, Dk) & \to & 0
\end{array}$

with exact rows. Since $G \cong G^{**}$ is a non-projective indecomposable module, we have $G^* \otimes Dk \to \text{Hom}_R(G^{**}, Dk)$ is zero. That is, $G^* \otimes \cdot$ is split and we have the exact sequence $0 \to (G \otimes Dk)^* \to (G \otimes X)^* \to (G \otimes L)^* \to 0$. Note from the lemma 2.5, we get the following
That is,

\[
0 \to \text{Hom}_R(D_k, G) \to \text{Hom}_R(X, G) \to \text{Hom}_R(L, G) \to 0
\]

for any non-projective module \( G \) in \( G(R) \).

Let \( M \) be in \( G(R) \). We denote \( \Omega^1(M) \) to be the 1th syzygy module of \( M \). By the definition of Gorenstein projective module, \( \Omega^1(M) \) is in \( G(R) \).

**Proposition 3.2** If \((R, m, k)\) is a local ring such that \( G\text{-dim}Dm \) is finite, then any indecomposable \( IG \)-projective \( R \)-module \( M \) satisfying \( \text{Ext}^i_R(M, M) = 0 \) for \( i \geq 1 \) is projective.

**Proof.** Assume that \( M \) is non-projective. We want to derive a contradiction. Since \( M \) is Irre-Gorenstein projective, there exists the irreducible morphism \( f : P \to M \) or \( h : M \to P \) with a projective module \( P \).

(1) If such an \( f \) exists, then we take a non-split exact sequence \( 0 \to k \to E' \to M \to 0 \). Since \( f \) is irreducible, it follows that \( E' \cong P \oplus E_1 \) and the following diagram is commutative. If \( K = 0 \), then \( M \) is projective. If \( E_1 = 0 \), there is the exact sequence \( 0 \to k \to P \to M \to 0 \). Since \( \text{Ext}^i_R(M, M) = 0 \) for \( i > 0 \), we have \( \text{Ext}^2_R(M, k) = 0 \). That is, \( \text{pd}_R M \) is finite. Hence, \( M \) is projective.
(2) Assume that $h$ exists. Since $G\text{-dim} Dm$ is finite, by proposition 3.1 there exists a short exact sequence:

$$0 \to L \to X \to Dk \to 0$$

of $R$-modules such that $X \to Dk \to 0$ is a $G(R)$-approximation of $Dk$ and $\text{Ext}^i_R(G(R), L) = 0$ for $i > 0$. Take a non-split exact sequence $0 \to M \to E \to Dk \to 0$ in $\text{Ext}^1_R(Dk, M)$, we have the pullback diagram:

\[
\begin{array}{cccc}
0 & 0 & M & M \\
\downarrow & & \downarrow & \\
0 & L & Q & E \\
\downarrow & & \downarrow & \downarrow \\
0 & L & X & Dk \\
\downarrow & & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

with $Q \in G(R)$.

a) If $0 \to M \to Q \to X \to 0$ is split, by the (2) of proposition 3.1 there exists the following commutative diagram

\[
\begin{array}{cccc}
0 & 0 & \text{Hom}_R(Dk, M) & \text{Hom}_R(X, M) \\
\downarrow & & \downarrow & \downarrow \\
0 & \text{Hom}_R(E, M) & \text{Hom}_R(Q, M) & \text{Hom}_R(L, M) \\
\downarrow & & \downarrow & \downarrow \\
\text{Hom}_R(M, M) & \text{Hom}_R(M, M) & \text{Hom}_R(M, M) & \text{Hom}_R(M, M) \\
\downarrow & & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
\]

This induces that $0 \to \text{Hom}_R(Dk, M) \to \text{Hom}_R(E, M) \to \text{Hom}_R(M, M) \to 0$ is exact. So we have the exact sequence $0 \to M \to E \to Dk \to 0$ is split. This is contradicted with it being non-split.
b) The next, let $0 \to M \to Q \to X \to 0$ be non-split. Note from $M \in G(R)$, we take a short exact sequence

$$0 \to M \to P \to M_0 \to 0$$

with $M_0 \in G(R)$. Since the monomorphism $M \to P$ is irreducible and $X$ is in $G(R)$, there is the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & M & \rightarrow & Q & \rightarrow & X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \theta & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & P & \rightarrow & M_0 & \rightarrow & 0
\end{array}
$$

where the morphism $\theta$ is split epimorphic. That is, there is an exact sequence $0 \to Q_0 \to X \to M_0 \to 0$ with $\text{Ker} \theta = Q_0$. Since $L$ is the maximal submodule of $X$, we get the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow 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\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \right]
Consider the push-out diagram of the morphisms $M_0 \to Dk$ and $E \to Dk$:

\[
\begin{array}{c}
0 & \to & 0 \\
\downarrow & & \downarrow \\
M & = & M \\
\downarrow & & \downarrow \\
0 & \to & N \\
\downarrow & & \downarrow \\
Q' & \to & E \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\downarrow & & \downarrow \\
M_0 & \to & Dk \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

we have the sequence $\mathcal{N} : 0 \to M \to Q' \to M_0 \to 0$ is exact. If $\mathcal{N}$ is split, then this induces a contradiction by repeating the proceedings of $a)$. Let $\mathcal{N}$ be a non-split exact sequence. Since $M_0$ is in $G(R)$, there exists the following commutative diagram

\[
\begin{array}{c}
0 & \to & M \\
\downarrow & & \downarrow \\
Q' & \to & M_0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

Since $M \to P$ is irreducible, one have the morphism $Q' \to P$ is split epimorphic. We easily see that $P \cong Q'$. Thus we obtain a commutative diagram

\[
\begin{array}{c}
0 & \to & M \\
\downarrow & & \downarrow \\
P & \to & M_0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

**We claim that** $\text{Ext}^2_R(M, N) = 0$. Since $M$ is selforthogonal, by the exact sequence $0 \to M \to P \to M_0 \to 0$, then $\text{Ext}^1_R(M, M_0) = 0$. Note from our claim, by the exact sequence $0 \to N \to M_0 \to Dk \to 0$, we have $\text{Ext}^1_R(M, Dk) = 0$. This is, $M$ is projective.
Next to prove our claim. Since $\Omega^1(M)$ is the 1th syzygy of $M$, the selforthogonal module $M$ implies that $\text{Ext}^1_R(\Omega^1(M), M) = 0$. Applying the functor $\text{Hom}_R(\Omega^1(M), -)$ to the above diagram, we get a commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
\text{Hom}_R(\Omega^1(M), M) & \to & \text{Hom}_R(\Omega^1(M), M) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_R(\Omega^1(M), N) \\
\downarrow & & \downarrow \\
\text{Hom}_R(\Omega^1(M), P) & \to & \text{Hom}_R(\Omega^1(M), E) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_R(\Omega^1(M), M_0) \\
\downarrow & & \downarrow \\
\delta & \to & \text{Hom}_R(\Omega^1(M), Dk) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}
\]

Note from the exact sequence (*) that $\delta$ is epimorphic. Thus we get an exact sequence

\[0 \to \text{Hom}_R(\Omega^1(M), N) \to \text{Hom}_R(\Omega^1(M), P) \to \text{Hom}_R(\Omega^1(M), E) \to 0\]

This induces $\text{Ext}^1_R(\Omega^1(M), N) = 0$. The 1th syzygy $\Omega^1(M)$ tells us that $\text{Ext}^2_R(M, N) = 0$.

The results of (1) and (2) contrary to the assumption of the proposition. This contradiction completes the proof of the proposition. \(\square\)

Now, let us prove our main theorem.

**Theorem 3.1** Let $(R, m, k)$ be a commutative Noetherian local ring. An IG-projective $R$-module $M$ is projective if and only if $M$ is selforthogonal.

**Proof.** Without loss of generality, let $M$ be an indecomposable module. If $R$ be a Gorenstein ring, then $\text{G-dim}Dm$ is finite. By the proposition 3.2, we have our result.

Let $R$ be a non-Gorenstein ring. Assume that $M$ is an non-projective module. We need to derive a contradiction. Since $M$ is Irre-Gorenstein projective, there exists the irreducible morphism $f: P \to M$ or $h: P \to M$ with a projective module $P$.

(1) If such an $f$ exists, then taking a non-split exact sequence $0 \to k \to E \to M \to 0$ and arguing as in the proof (1) of proposition 3.2, one deduces that $M$ is projective.
(2) If there exists the irreducible monomorphism $h : M \rightarrow P$. Since $M$ is not projective, there exists a maximal submodule $M_1/M$ of $P/M$. Consider the commutative diagram

$$
\begin{array}{c}
0 \\ \\
\downarrow \\
M \\
\downarrow \\
M_1 \\
\downarrow \\
P
\end{array}
\xrightarrow{h}
\begin{array}{c}
0 \\ \\
\downarrow \\
M/MM \\
\downarrow \\
P
\end{array}
\xrightarrow{t}
\begin{array}{c}
0 \\ \\
\downarrow \\
0
\end{array}
$$

the irreducible morphism $h$ implies that $h_1$ is split monomorphic. That is, $M_1 = M \oplus H$ and $M_1$ is a maximal submodule of $P$. Hence, there exists the following commutative diagram

$$
\begin{array}{c}
0 \\ \\
\downarrow \\
M \oplus H \\
\downarrow \\
P \\
\downarrow \\
k \\
\downarrow \\
0
\end{array}
\xrightarrow{m}
\begin{array}{c}
0 \\ \\
\downarrow \\
R \\
\downarrow \\
k \\
\downarrow \\
0
\end{array}
$$

By the Schanuel’s lemma, we have the isomorphism $m \oplus P \cong R \oplus M \oplus H$. Since $M$ is not projective, $M$ is a summand of $m$. Since $R$ is a non-Gorenstein ring, it is contradicted with the Lemma 2.3. Hence, $M$ is projective.

The results of (1) and (2) contrary to the assumption of the theorem. This contradiction completes the proof of the theorem.

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