ADDENDUM TO:
Generically split projective homogeneous varieties

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Abstract

In this addendum we generalize some results of our article [PS10]. More precisely, we remove all restrictions on the characteristic of the base field (in [PS10] we assumed that the characteristic is different from any torsion prime of the group), and complete our classification by the last missing case, namely \( \text{PGO}_{2n}^+ \). We follow our notation from [PS10].

1 Chow rings of reductive groups

1.1. Let \( G_0 \) be a split reductive algebraic group defined over a field \( k \). We fix a split maximal torus \( T \) in \( G_0 \) and a Borel subgroup \( B \) of \( G_0 \) containing \( T \) and defined over \( k \). We denote by \( \Phi \) the root system of \( G_0 \), by \( \Pi \) the set of simple roots of \( \Phi \) with respect to \( B \), and by \( \hat{T} \) the group of characters of \( T \). Enumeration of simple roots follows Bourbaki.

Any projective \( G_0 \)-homogeneous variety \( X \) is isomorphic to \( G_0/P_\Theta \), where \( P_\Theta \) stands for the (standard) parabolic subgroup corresponding to a subset \( \Theta \subset \Pi \). As \( P_i \) we denote the maximal parabolic subgroup \( P_{\Pi\setminus\{\alpha_i\}} \) of type \( i \).

Consider the characteristic map \( c: S(\hat{T}) \rightarrow \text{CH}^*(G_0/B) \) from the symmetric algebra of \( \hat{T} \) to the Chow ring of \( G_0/B \) given in [PS10, 2.7], and denote its image by \( R^* \). According to [Gr58, Rem. 2°], the ring \( \text{CH}^*(G_0) \) can

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be presented as the quotient of $\text{CH}^\ast(G_0/B)$ modulo the ideal generated by the non-constant elements of $R^\ast$.

1.2 Lemma. The pull-back map

$$\text{CH}^\ast(G_0) \to \text{CH}^\ast([G_0, G_0])$$

is an isomorphism.

Proof. Indeed, $B' = B \cap [G_0, G_0]$ is a Borel subgroup of $[G_0, G_0]$, the map

$$[G_0, G_0]/B' \to G_0/B$$

is an isomorphism, and the map $S(\hat{T}) \to \text{CH}^\ast(G_0/B)$ factors through the surjective map $S(\hat{T}) \to S(\hat{T}')$, where $T' = T \cap [G_0, G_0]$.

Let $P$ be a parabolic subgroup of $G_0$. Denote by $L$ the Levi subgroup of $P$ and set $H_0 = [L, L]$. We have

1.3 Lemma. The pull-back map

$$\text{CH}^\ast(P) \to \text{CH}^\ast(H_0)$$

is an isomorphism.

Proof. The quotient map $P \to L$ is Zariski locally trivial affine fibration, therefore the pull-back map $\text{CH}^\ast(L) \to \text{CH}^\ast(P)$ is an isomorphism. Since the composition $L \to P \to L$ is the identity map, the pull-back map $\text{CH}^\ast(P) \to \text{CH}^\ast(L)$ is an isomorphism as well. It remains to apply Lemma 1.2.

1.4 Lemma. The pull-back map

$$\text{CH}^\ast(G_0) \to \text{CH}^\ast(P)$$

is surjective.

Proof. Applying [Gr58, Proposition 3] to the natural map $G_0/B \to G_0/P$ we see that the map $\text{CH}^\ast(G_0/B) \to \text{CH}^\ast(P/B)$ is surjective. But the map $\text{CH}^\ast(P/B) \to \text{CH}^\ast(P)$ is also surjective by Lemma 1.3 and fits into the commutative diagram

$$
\begin{array}{c}
\text{CH}^\ast(G_0/B) \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\text{CH}^\ast(P/B)
\end{array}
\begin{array}{c}
\text{CH}^\ast(G_0) \\
\downarrow
\end{array}
\begin{array}{c}
\text{CH}^\ast(P).
\end{array}
$$

$\square$
1.5 (Definition of $\sigma$). Now we restrict to the situation when $G_0$ is simple. Let $p$ be a prime integer. Denote $\text{Ch}^*(-)$ the Chow ring with $\mathbb{F}_p$-coefficients. Explicit presentations of the Chow rings with $\mathbb{F}_p$-coefficients of split semisimple algebraic groups are given in [Kc85, Theorem 3.5].

For $G_0$ and $H_0$ they look as follows:

$\text{Ch}^*(G_0) = \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p k_1}, \ldots, x_r^{p k_r})$ with $\deg x_i = d_i, 1 \leq d_1 \leq \ldots \leq d_r$;

$\text{Ch}^*(H_0) = \mathbb{F}_p[y_1, \ldots, y_s]/(y_1^{p l_1}, \ldots, y_s^{p l_s})$ with $\deg y_m = e_m, 1 \leq e_1 \leq \ldots \leq e_s$

for some integers $k_i, l_i, d_i,$ and $e_i$ depending on the Dynkin types of $G_0$ and $H_0$.

By the previous lemmas the pull-back $\varphi: \text{Ch}^*(G_0) \to \text{Ch}^*(H_0)$ is surjective. For a graded ring $S^*$ denote by $S^+$ the ideal generated by the non-constant elements of $S^*$. The induced map

$$\text{Ch}^+(G_0)/\text{Ch}^+(G_0)^2 \to \text{Ch}^+(H_0)/\text{Ch}^+(H_0)^2$$

is also surjective. Moreover, for any $m$ with $e_m > 1$ there exists a unique $i$ such that $d_i = e_m$. We denote $i =: \sigma(m)$. The surjectivity implies that

$$\varphi(x_{\sigma(m)}) = cy_m + \text{lower terms}, \quad c \in \mathbb{F}_p^\times.$$

2 Generically split varieties

For a semisimple group $G$ and a prime number $p$ denote by

$$J_\gamma(G) = (j_1(G), \ldots, j_r(G))$$

its $J$-invariant defined in [PSZ08].

2.1 Theorem. Let $G_0$ be a split simple algebraic group over $k$, $G = \gamma G_0$ be the twisted form of $G_0$ given by a 1-cocycle $\gamma \in \text{H}^1(k, G_0)$, $X = \gamma(G_0/P)$ be the twisted form of $G_0/P$, and $Y = \gamma(G_0/B)$ be the twisted form of $G_0/B$. The following conditions are equivalent:

1. $X$ is generically split;

2. The composition map

$$\overline{\text{CH}^*}(Y) \to \text{CH}^*(G_0) \to \text{CH}^*(P)$$

is surjective;
3. For every prime $p$ the composition map

$$\text{Ch}^1(Y) \to \text{Ch}^1(G_0) \to \text{Ch}^1(P)$$

is surjective, and

$$j_{\sigma(m)}(G) = 0 \text{ for all } m \text{ with } d_m > 1.$$ 

Proof. 1⇒2. The same argument as in the proof of Lemma 1.4 (with $Y$ instead of $G_0/B$ and $X$ instead of $G_0/P$).

2⇒3. Clearly, the composition

$$\text{Ch}^* (Y) \to \text{Ch}^*(G_0) \to \text{Ch}^*(P)$$

is surjective for every $p$. In particular, when $d_m > 1$ $\text{Ch}^{d_m}(Y)$ contains an element of the form $x_{\sigma(m)} + a$, where $a$ is decomposable, hence $j_{\sigma(m)}(G) = 0$.

3⇒1. $G_{k(X)}$ has a parabolic subgroup of type $P$; denote the derived group of its Levi subgroup by $H$. We want to prove that $H$ is split. By [PS10, Proposition 3.9(3)] it suffices to show that $J_p(H)$ is trivial for every $p$.

Denote the variety of complete flags of $H$ by $Z$. It follows from the commutative diagram

$$\text{Ch}^*(Y_{k(X)}) \longrightarrow \text{Ch}^*(Z)$$

that $j_m(H) \leq j_{\sigma(m)}(G)$ if $d_m > 1$. Therefore

$$j_m(H) \leq j_{\sigma(m)}(G_{k(X)}) \leq j_{\sigma(m)}(G) = 0$$

when $d_m > 1$. It remains to show that $\text{Ch}^1(Z)$ is rational. But this follows from the commutative diagram

$$\text{Ch}^1(Y) \longrightarrow \text{Ch}^1(Y_{k(X)}) \longrightarrow \text{Ch}^1(Z)$$

$$\text{Ch}^1(G) \longrightarrow \text{Ch}^1(H) \longrightarrow \text{Ch}^1(P).$$

\[\square\]
2.2 Remark.

- If all $e_m > 1$, then the condition on $\text{Ch}^1(Y)$ is void.

- If $G_0$ is different from $\text{PGO}_{2n}^+$ and $e_1 = 1$ (resp. $G_0 = \text{PGO}_{2n}^+$ and $e_1 = e_2 = 1$), then in view of [PS10, Proposition 4.2] it is equivalent to the fact that all Tits algebras of $G$ are split. The latter is also equivalent to the fact that $j_1(G) = 0$ (resp. $j_1(G) = j_2(G) = 0$).

- If $G_0 = \text{PGO}_{2n}^+$ and there is exactly one $m$ with $e_m = 1$, then there are exactly two fundamental weights among $\bar{\omega}_1, \bar{\omega}_{n-1}, \bar{\omega}_n$ whose image with respect to the composition $\text{Ch}^1(Y) \to \text{Ch}^1(G) \to \text{Ch}^1(H)$ equals $y_1$. Then the condition on $\text{Ch}^1(Y)$ is equivalent to the fact that at least one of the Tits algebras corresponding to these fundamental weights in the preimage of $y_1$ is split.

For a simple group $G$ we denote by $A_l$ its Tits algebra corresponding to $\bar{\omega}_l$.

2.3 Theorem. Let $G$ be a group given by a 1-cocycle from $H^1(k, G_0)$, where $G_0$ stands for the split adjoint group of the same type as $G$, and let $X$ be the variety of the parabolic subgroups of $G$ of type $i$.

The variety $X$ is generically split if and only if

| $G_0$     | $i$                          | conditions on $G$                        |
|-----------|------------------------------|------------------------------------------|
| $\text{PGL}_n$ | any $i$                     | $\gcd(\exp A_1, i) = 1$                  |
| $\text{PGSp}_{2n}$ | any $i$                     | $i$ is odd or $G$ is split                |
| $\text{O}_{2n+1}^+$ | any $i$                     | $j_m(G) = 0$ for all $1 \leq m \leq \frac{n+1-i}{2}$ |
| $\text{PGO}_{2n}^+$ | $i$ is odd, $i < n - 1$    | $[A_{n-1}] = 0$ or $[A_n] = 0$, and $j_m(G) = 0$ for all $2 \leq m \leq \frac{n+2-i}{2}$ |
| $\text{PGO}_{2n}^+$ | $i$ is even, $i < n - 1$   | $j_m(G) = 0$ for all $1 \leq m \leq \frac{n+2-i}{2}$ |
| $\text{PGO}_{2n}^+$ | $i = n - 1$ or $i = n$, $n$ is odd | none                                     |
| $\text{PGO}_{2n}^+$ | $i = n - 1$, $n$ is even   | $[A_1] = 0$ or $[A_n] = 0$               |
| $\text{PGO}_{2n}^+$ | $i = n$, $n$ is even       | $[A_1] = 0$ or $[A_{n-1}] = 0$          |
| $\text{E}_6$   | $i = 3, 5$                  | none                                     |
| $\text{E}_6$   | $i = 2, 4$                  | $J_3(G) = (0, *)$                         |
| $\text{E}_6$   | $i = 1, 6$                  | $J_2(G) = (0)$                           |
| $\text{E}_7$   | $i = 2, 5$                  | none                                     |
| $\text{E}_7$   | $i = 3, 4$                  | $J_2(G) = (0, *, *, *)$                  |
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| $\mathbf{E}_7$ | $i = 6$ | $J_2(G) = (0, 0, *, *)$ if char $k \neq 2$ |
| $\mathbf{E}_7$ | $i = 1$ | $J_2(G) = (0, 0, 0, *)$ if char $k \neq 2$ |
| $\mathbf{E}_7$ | $i = 7$ | $J_3(G) = (0)$ and $J_2(G) = (*, 0, *, *)$ if char $k \neq 2$ |

| $\mathbf{E}_8$ | $i = 2, 3, 4, 5$ | none |
| $\mathbf{E}_8$ | $i = 6$ | $J_2(G) = (0, *, *, *, *)$ if char $k \neq 2$ |
| $\mathbf{E}_8$ | $i = 1$ | $J_2(G) = (0, 0, 0, *)$ if char $k \neq 2$ |
| $\mathbf{E}_8$ | $i = 7$ | $J_3(G) = (0, *)$ and $J_2(G) = (0, *, *, *)$ if char $k \neq 3$, $J_2(G) = (0, 0, 0, *)$ if char $k \neq 2$ |
| $\mathbf{E}_8$ | $i = 8$ | $J_3(G) = (0, *)$ and $J_2(G) = (0, 0, 0, *)$ if char $k \neq 3$ |
| $\mathbf{F}_4$ | $i = 1, 2, 3$ | none |
| $\mathbf{F}_4$ | $i = 4$ | $J_2(G) = (0)$ |
| $\mathbf{G}_2$ | any $i$ | none |

("*" means "any value").

Proof. Follows immediately from Theorem 2.1 and [PSZ08, Table 4.13].

This theorem allows to give a shortened proof of the main result of [Ch10]:

2.4 Corollary. Let $G$ be a group of type $\mathbf{E}_8$ over a field $k$ with char $k \neq 3$. If the 3-component of the Rost invariant of $G$ is zero, then $G$ splits over a field extension of degree coprime to 3.

Proof. Let $K/k$ be a field extension of degree coprime to 3 such that the 2-component of the Rost invariant of $G_K$ is zero.

Consider the variety $X$ of parabolic subgroups of $G_K$ of type 7. The Rost invariant of the semisimple anisotropic kernel of $G_{K(X)}$ is zero. Therefore $G_{K(X)}$ splits, and, thus, $X$ is generically split.

By Theorem 2.3, $J_3(G_K) = (0, 0)$, hence by [PS10, Proposition 3.9(3)] $G_K$ splits over a field extension of degree coprime to 3. This implies the corollary.
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