A reproducing kernel Hilbert space approach in meshless collocation method

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Abstract
In this paper, we combine the theory of the reproducing kernel Hilbert spaces with the field of collocation methods to solve boundary value problems with a special emphasis on the reproducing property of kernels. Using the reproducing property of the kernels, a new efficient algorithm is proposed to obtain the cardinal functions of a reproducing kernel Hilbert space, which can be applied conveniently for multi-dimensional domains. The differentiation matrices are constructed and also a pointwise error estimate of applying them is derived. In addition, we prove the non-singularity of the collocation matrix. The proposed method is truly meshless, and can be applied conveniently and accurately for high order and also multi-dimensional problems. Numerical results are presented for the several problems such as second- and fifth-order two-point boundary value problems, one- and two-dimensional unsteady Burgers’ equations, and a three-dimensional parabolic partial differential equation. In addition, we compare the numerical results with the best-reported results in the literature to show the high accuracy and efficiency of the proposed method.

Keywords Reproducing kernel Hilbert space · Meshless method · Collocation method · Cardinal functions · Differentiation matrix

Mathematics Subject Classification 65M70 · 65N35 · 65L10 · 34B15 · 65L70

1 Introduction

Meshless methods for numerical solution of the boundary value problems have recently become more and more popular, and several meshless methods were proposed by authors in the literature. Meshless collocation methods based on radial basis functions have been applied successfully in many kinds of differential equations (Zhang et al. 2000; Power et al. 2013).
The meshless collocation radial basis function methods are constructed in a symmetrical and an unsymmetrical form. In both cases, a partial differential equation and its boundary conditions are discretized by point evaluations in certain collocation points. The non-symmetric collocation method used by Kansa is relatively simple to implement; however, it results in an unsymmetric system of equations by forcing the differential equation to be satisfied at the collocation points and the resulting unsymmetric collocation matrix can be singular in exceptional cases (Hon and Schaback 2001). Fasshauer (2005) shows that many of the algorithms and strategies used for solving partial differential equations with polynomial pseudospectral methods can be adapted for the use with radial basis functions. In the most radial basis function methods for solving differential equations, the reproducing property of kernels was not used as a characteristic property in reconstruction scheme. In this paper, we describe how the meshless collocation method based on reproducing kernel Hilbert spaces can be used to solve boundary value problems numerically with an emphasis on reproducing property of kernels and without radial property. In recent years, the reproducing kernel Hilbert space methods have been applied successfully to several nonlinear problems (Abbasbandy et al. 2015; Azarnavid and Parand 2018; Abbasbandy and Azarnavid 2016; Azarnavid et al. 2015, 2018a, b; Emamjome et al. 2017; Arqub 2017; Al-Smadi et al. 2016; Arqub 2016a, b; Akgül and Baleanu 2017; Inc et al. 2012; Akgül et al. 2015; Sakar et al. 2017; Akgül 2015).

The collocation method is usually implemented in the physical space by seeking an approximate solution $u$ of the differential equation in the following form:

$$u_N(x) = \sum_{k=1}^{N} u_N(x_k) h_k(x),$$

where the functions $\{h_k\}_{k=1}^{N}$ are the cardinal functions with respect to the selected collocation points $X = \{x_1, \ldots, x_N\}$; that is, $h_k(x_j) = \delta_{k,j}$ for $1 \leq k, j \leq N$. Here, we consider the collocation method that uses the cardinal functions belong to a reproducing kernel Hilbert space (RKHS). One of the advantages of the radial basis functions collocation method is in dealing with complex geometries and non-uniform discretizations, which has been kept in RKHS collocation method. In addition, in the RKHS collocation method, the boundary points will not appear in discretizations and collocation matrices. Instead of imposing the boundary condition in the collocation matrix, we use the reproducing kernels that satisfy the homogenized boundary conditions of the equation, and then, we can show that the collocation matrix is non-singular if we have an invertible bounded linear operator in main equation. A new and convenient algorithm is proposed based on the reproducing property of kernels, to obtain the cardinal functions, which can be applied for the multi-dimensional case as simple as the one-dimensional case. The operational matrices of differentiation in physical space are constructed and also a pointwise error estimate of applying them is derived. The error estimate of the reconstruction scheme is given as a percentage of the norm of the true solution, which is the only unknown quantity here and the error bound is based on the interpolation error of a known function in reproducing kernel Hilbert space $H$ which can be determined exactly for a given set of collocation point.

The main advantages of the proposed method are as follows; first, it is easy in implementation and programming; second, the proposed method is truly meshless, and in addition, it eliminates the treatment of the boundary conditions, using the reproducing kernels which satisfy the boundary conditions exactly; third, the method can be easily applied for higher order and multi-dimensional boundary value problems with various boundary conditions; fourthly, it does not need to invert any matrix and does not need any linearization process.
for nonlinear time-dependent problems. To demonstrate the efficiency and performance of our method computationally, we apply it on some problems such as high-order differential equations and multi-dimensional nonlinear problems and compare the results with reported results in the literature.

The rest of this paper is organized as follows: In Sect. 2, we describe the construction of the cardinal functions for a subspace of reproducing kernel Hilbert space spanned by some translated kernels. Differentiation matrices and the error estimate of employing them is discussed in Sect. 3. In Sect. 4, we illustrate how the RKHS collocation method can be applied on differential equations by employing two linear differential equations, one- and two-dimensional nonlinear Burgers’ equations and a three-dimensional time-dependent boundary value problem as test problems. Finally, a brief conclusion is given in Sect. 5.

2 Cardinal functions

The discussion of this section is devoted to the obtaining of the multivariate cardinal functions of a finite-dimensional subspace of a reproducing kernel Hilbert space spanned by translated kernels.

Theorem 2.1 (Wendland 2005) Suppose that $H$ is a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \rightarrow \mathbb{R}$. Then, $K$ is positive semi-definite. Moreover, $K$ is positive definite if and only if the point evaluation functionals are linearly independent in $H^*$.

In fact, the RKHS and its reproducing kernel define each other uniquely. For a determined reproducing kernel $K : \Omega \times \Omega \rightarrow \mathbb{R}$ of Hilbert space $H$ and $X = \{x_1, \ldots, x_N\} \subset \Omega$, where $\Omega \subset \mathbb{R}^d$, we define a finite-dimensional linear reconstruction space as follows:

$$S_X = \text{span}\{K(., x_j) : 1 \leq j \leq N\}.$$  \hfill (2.1)

To construct the native space of the kernel from its reproducing kernel $K$, we first define the reconstruction space:

$$S = \{s \in S_X : X \subset \mathbb{R}^d, |X| < \infty\}$$  \hfill (2.2)

containing all potential interpolants of the form

$$s(x) = \sum_{j=1}^{N} c_j K(x, x_j),$$  \hfill (2.3)

for some $c = (c_1, \ldots, c_n)^T \in \mathbb{R}^N$ and $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$. Suppose that we are given a vector

$$f_X = (f(x_1), \ldots, f(x_N))^T \in \mathbb{R}^N$$

of discrete function values, sampled from an unknown function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ at a finite point set $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$. The given data $f_X$ have a unique interpolant of the form (Eq. (2.3)), if the kernel $K$ is positive definite. In this paper, we focus on positive definite kernels. For any $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, from the positive definiteness of the kernel $K$, the matrix $A_X = (K(x_i, x_j))_{1 \leq i, j \leq N} \in \mathbb{R}^{N \times N}$ is positive definite. Let $f = \sum_{j=1}^{N} c_j K(., x_j)$ and $g = \sum_{j=1}^{N} d_j K(., x_j)$ belongs to $S$, and then, using the reproducing property of kernel
for any $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ and all $c, d \in \mathbb{R}^N$. By completion of the inner product space $S$ with respect to its norm $\|\cdot\|_K$, we obtain the RKHS correspond to $K, H = S$. The orthonormal system $\{\psi_i(x)\}_{i=1}^N$ of $S_X$ can be derived from Gram–Schmidt orthogonalization process of $\{K(\cdot, x_i)\}_{i=1}^N$:  

$$\psi_i(x) = \sum_{k=1}^i \beta_{ik} K(\cdot, x_k) \quad (\beta_{ii} > 0, i = 1, 2, \ldots). \quad (2.4)$$

**Proposition 2.1** For any pairwise distinct point set $X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, there exists a unique cardinal basis $\{h_1, \ldots, h_N\} \subset S_X$ satisfying  

$$h_j(x_k) = \begin{cases} 1 & j = k, \\ 0 & j \neq k, \end{cases} \quad (1 \leq j, k \leq N), \quad (2.5)$$

which can be obtain as follows:  

$$h_i(x) = \sum_{k=1}^N \beta_{ki} \psi_k(x), \quad (1 \leq i \leq N), x \in \mathbb{R}^d. \quad (2.6)$$

**Proof** From the uniqueness of interpolant $s: \mathbb{R}^d \to \mathbb{R}$ satisfying $s_X = f_X$, that is  

$$s(x_j) = f(x_j), \quad 1 \leq j \leq N, \quad (2.7)$$

for any $h_i, (1 \leq i \leq N)$, we have a unique interpolant $s_i \in S_X$ that satisfies Eq. (2.5), and can be given as follows:  

$$s_i(x) = \sum_{k=1}^N c_{ik} \psi_k(x) = \sum_{k=1}^N (s_i, \psi_k)_H \psi_k(x)$$

$$= \sum_{k=1}^N \left( s_i, \sum_{j=1}^k \beta_{kj} K(\cdot, x_j) \right)_H \psi_k(x)$$

$$= \sum_{k=1}^N \sum_{j=1}^k \beta_{kj} (s_i, K(\cdot, x_j))_H \psi_k(x)$$

$$= \sum_{k=1}^N \sum_{j=1}^k \beta_{kj} h_k(x_j) \psi_k(x)$$

$$= \sum_{k=1}^N \sum_{j=1}^k \beta_{kj} h_i(x_j) \psi_k(x) = \sum_{k=1}^N \beta_{ki} \psi_k(x), \quad 1 \leq i \leq N,$$

where the functions $s_i |_{1 \leq i \leq N}$ are the involved cardinal functions $h_i |_{1 \leq i \leq N} \in S_X$.  

$\square$
One of the advantages of the above proposition is that the cardinal functions can be easily constructed for the multivariate case, as simple as, one variable case. In Iske (2011), the cardinal functions of $S_X$ are given based on the inversion of the matrix $A_X$. We avoid inverting an ill-conditioned matrix and calculate the cardinal functions using the reproducing property of kernels and a modified Gram Schmidt orthogonalization algorithm, that cost $O(N^2)$ flops versus inversion of matrix $O(N^3)$ flops.

3 Differentiation matrices

One of several ways to implement the collocation method is via the so-called differentiation matrices. Suppose that $H$ is a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \rightarrow \mathbb{R}$. When we just have values of a function $u \in H$ at a scattered point $X = \{x_1, \ldots, x_N\}$ in the domain of $u$, then the value $\mathcal{L}u(z)$ for a fixed point $z \in \Omega$ must be approximated via the following:

$$\mathcal{L}u(z) \simeq \sum_{k=1}^{N} \alpha_k u(x_k),$$

(3.1)

where $\mathcal{L}$ is a linear bounded differential operator. Our aim is to reconstruct operation of $\mathcal{L}$ on an unknown function $u \in H$ from its values $u_X \in \mathbb{R}^N$. Define the linear subspace of $H$:

$$S_X = \text{span}\{K(., x_j) : 1 \leq j \leq N\}.$$ 

Let $S_X^\perp$ be the linear subspace of $H$,

$$S_X^\perp = \{f \in H | f(x_i) = 0, i = 1, \ldots, N\}.$$ 

From the reproducing property of $K$, for any $f \in S_X^\perp$, we have

$$\left( f, \sum_{k=1}^{N} c_k K(., x_k) \right)_H = \sum_{k=1}^{N} c_k (f, K(., x_k))_H = \sum_{k=1}^{N} c_k f(x_k) = 0,$$

so $S_X^\perp$ is the orthogonal complement of $S_X$ and every $u \in H$ can be uniquely decomposed to $u = u_0 + u_0^\perp$, where $u_0 \in S_X$ and $u_0^\perp \in S_X^\perp$. For $1 \leq k \leq N$, we have $u(x_k) = u_0(x_k) + u_0^\perp(x_k) = u_0(x_k)$; then, $u_0(x) = \sum_{k=1}^{N} c_k K(x, x_k) = \sum_{k=1}^{N} u_0(x_k) h_k(x) = \sum_{k=1}^{N} u(x_k) h_k(x)$, is unique interpolant of $u \in S_X$. From Riesz representation theorem, for any bounded linear operator $\mathcal{L}$, there exist $g \in H$, such that $\mathcal{L}u(z) = (g, u)_H$ for a fixed point $z \in \Omega$ and any $u \in H$, then

$$\mathcal{L}u(z) = (g, u)_H \simeq (g, u_0)_H = (g_0 + g_0^\perp, u_0)_H = (g_0, u_0)_H$$

$$= \left( \sum_{k=1}^{N} \alpha_k K(., x_k), u_0 \right)_H = \sum_{k=1}^{N} \alpha_k u_0(x_k) = \sum_{k=1}^{N} \alpha_k u(x_k).$$

(3.2)

We can obtain the coefficients $\alpha_i$, ($1 \leq i \leq N$) as follows:

$$\mathcal{L}h_i(z) = (g, h_i)_H = (g_0 + g_0^\perp, h_i)_H = (g_0, h_i)_H = \left( \sum_{k=1}^{N} \alpha_k K(., x_k), h_i \right)_H$$

$$= \sum_{k=1}^{N} \alpha_k h_i(x_k) = \alpha_i.$$
It is clear that Eq. (3.2) treats the functions $u \in S_X$ exactly. If we evaluate Eq. (3.1) at the grid point $x_i$, ($i = 1, \ldots, N$), for any $u \in S_X$, then we can get

$$\mathcal{L}u(x_i) = \sum_{k=1}^{N} u(x_k) \mathcal{L}h_k(x_i), \quad i = 1, \ldots, N$$

or in matrix-vector notation

$$\mathcal{L}u = Lu,$$

where $\mathcal{L}u = (\mathcal{L}u(x_1), \ldots, \mathcal{L}u(x_N))^T$, $u = (u(x_1), \ldots, u(x_N))^T$ and the entries of the differentiation matrix $L$ are given by $l_{ij} = \mathcal{L}h_j(x_i)$, which can be obtained as follows. From (2.6), we have

$$l_{ij} = \mathcal{L}h_j(x_i) = \sum_{k=1}^{N} \beta_{kj} \mathcal{L}\psi_k(x_i) = (\mathcal{L}\psi_1(x_i), \mathcal{L}\psi_2(x_i), \ldots, \mathcal{L}\psi_N(x_i)).(\beta_{1j}, \beta_{2j}, \ldots, \beta_{Nj})^T.$$ 

In practice, the differentiation matrix can be obtained as follows:

$$L = \Phi^T B, \quad (3.3)$$

where $\Phi_{i,j} = \mathcal{L}\psi_i(x_j)$ and $B$ is a lower triangular matrix with $B_{i,j} = \beta_{ij}$. The differentiation matrix $L$ can now be used to solve partial differential equations. Sometimes, only multiplication by $L$ is required, e.g., for many time-dependent problems, and sometimes, one needs to be able to invert $L$.

**Lemma 3.1** If the linear operator $\mathcal{L}$ is invertible, then the corresponding matrix $L$ is nonsingular.

**Proof** For selected collocation point set $X = \{x_1, \ldots, x_N\}$, suppose that

$$c_1 \mathcal{L}h_1(x) + \ldots + c_N \mathcal{L}h_N(x) = 0,$$

and then, from the linearity of operator $\mathcal{L}$, we have

$$\mathcal{L}(c_1 h_1(x) + \ldots + c_N h_N(x)) = 0.$$ 

It follows that

$$c_1 h_1(x) + \ldots + c_N h_N(x) \equiv 0$$

from the existence of $\mathcal{L}^{-1}$. Then, from the linear independence of cardinal functions, $\mathcal{L}h_j(x)|_{1 \leq j \leq N}$ are linearly independent and the matrix $L$ has full rank. $\square$

**Theorem 3.1** Let $H$ be a reproducing kernel Hilbert space with reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$, and let $\mathcal{L}$ be a linear differential operator. Assume that $X = \{x_1, \ldots, x_N\} \subset \Omega$, where $\Omega \subset \mathbb{R}^d$. Then, for any $z \in \Omega$ and $u \in H$, we have

$$\left| \mathcal{L}u(z) - \sum_{k=1}^{N} u(x_k) \mathcal{L}h_k(z) \right| \leq \|u\|_H \|\varepsilon_X\|_H, \quad (3.4)$$

where $\|\varepsilon_X\|_H = \|\mathcal{L}y K(., z) - \sum_{k=1}^{N} K(., x_k) \mathcal{L}h_k(z)\|_H$ is the norm of interpolation error of a known function $\mathcal{L}y K(., z)$ and $0 \leq \|\varepsilon_X\|^2_H \leq \mathcal{L}_x \mathcal{L}_y K(x, y)|_{x,y=z}.$
Proof From Riesz representation theorem, for any bounded linear operator $L$, there exist $g \in H$, such that $Lu(z) = (g, u)_H$ for any $u \in H$ and $z \in \Omega$, and then
\[
\left| Lu(z) - \sum_{k=1}^{N} u(x_k) Lh_k(z) \right| = \left| (u, g)_H - \left( u, \sum_{k=1}^{N} K(., x_k) Lh_k(z) \right)_H \right| \\
= \left\| (u, g) - \left( u, \sum_{k=1}^{N} K(., x_k) Lh_k(z) \right) \right\|_H \\
\leq \|u\|_H \left\| g - \sum_{k=1}^{N} K(., x_k) Lh_k(z) \right\|_H.
\]

Let $\|e_X\|_H = \|g - \sum_{k=1}^{N} K(., x_k) Lh_k(z)\|_H$. From Eq. (3.2), it is easy to see that $Lh_k(z)|_{1 \leq k \leq N}$ are the interpolation coefficients of $g = L_y K(., z) \in H$, and then, $\|e_X\|_H$ is the interpolation error of the known function $L_y K(., z)$ which can be determined exactly. Then
\[
\|e_X\|_H^2 = \left( \sum_{k=1}^{N} K(., x_k) Lh_k(z), g - \sum_{k=1}^{N} K(., x_k) Lh_k(z) \right)_H \\
= (g, g)_H - 2 \sum_{k=1}^{N} (g, K(., x_k))_H Lh_k(z) + \sum_{k=1}^{N} \sum_{j=1}^{N} K(x_k, x_j) Lh_k(z) Lh_j(z) \\
= (g, g)_H - 2 \sum_{k=1}^{N} \sum_{j=1}^{N} K(x_j, x_k) Lh_j(z) Lh_k(z) + \sum_{k=1}^{N} \sum_{j=1}^{N} K(x_k, x_j) Lh_k(z) Lh_j(z) \\
= (g, g)_H - (Lh(z))^T A_X (Lh(z)),
\]
where $Lh(z) = (Lh_1(z), ..., Lh_N(z))^T$. Therefore, from positive definiteness of matrix $A_X$ and $(g, g)_H = Lg(z) = L(g, K(., y))_H|_{y=z} = L_X L_y K(x, y)|_{x, y=z}$, we have
\[
0 \leq \|e_X\|_H^2 \leq L_X L_y K(x, y)|_{x, y=z},
\]
where the subscript index shows the variable that the linear operator acts on.

The above theorem gives us a pointwise error estimate of the reconstruction scheme as a percentage of the norm of the true solution, which is the only unknown quantity here and the error bound is based on the interpolation error of a known function in reproducing kernel Hilbert space $H$. Suppose that we have a linear differential equation of the form:
\[
Lu = f , \quad (3.5)
\]
for some given $f \in V$ by ignoring boundary condition, where $L : U \to V$ is a linear data map that takes each $u$ in some linear normed space $U$ into the data space $V$. We use the following definition of well-posedness for the above problem given in Schaback (2016). □

**Definition 3.1** An analytic problem as (3.5) is well-posed with respect to a well-posedness norm $\|.\|_W$ on $U$ if there is a constant $C$, such that a well-posedness inequality:
\[
\|u\|_W \leq C \|Lu\|_V , \quad u \in U
\]
holds.

This means that $L^{-1}$ is continuous as a map $L(U) \to U$ in the norms $\|.\|_V$ and $\|.\|_W$. We approximate $u^*$ the true solution of (3.5) by $u_N^*$ defined as (1.1) in finite-dimensional $U_N \subseteq U$, which can be calculate as a finite-dimensional approximation problem from minimization of the residual norm $\|Lu_N - f\|_V$ over all $u_N \in U_N$. □

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Theorem 3.2 Suppose the conditions of Theorem 3.1 hold. Assume a well-posed analytic problem \( Lu = f \) for some given \( f \in V \) in the sense of Definition 3.1 with some constant \( C \), well-posedness norm \( \| \cdot \|_W \) on some RKHS like \( H \) and the data space norm \( \| \cdot \|_V = \max_{z \in \Omega} | \cdot | \). If \( u^* \) is the true solution and \( u^*_N \in S_X \subseteq H \) is the approximated solution, then we have

\[
\| u^* - u^*_N \|_W \leq C \| u \|_H \| e_X \|_H, \tag{3.7}
\]

where \( \| e_X \|_H \) is as defined in Theorem 3.1.

Proof The true solution \( u^* \in H \) can be uniquely decomposed to \( u^* = u_0 + u_0^\perp \), where \( u_0 \in S_X \). From the well-posedness condition (3.6) and (3.4), we have

\[
\| u^* - u^*_N \|_W \leq C \| \mathcal{L}(u^* - u^*_N) \|_V \leq C \| \mathcal{L}(u^* - u_0) \|_V = C \max_{z \in \Omega} | \mathcal{L}(u^* - u_0)(z) | = C \max_{z \in \Omega} | \mathcal{L}u^*(z) - \mathcal{L}u_0(z) |
\]

\[
= C \| \mathcal{L}u^*(z_0) - \mathcal{L}u_0(z_0) \|_V = C \| \mathcal{L}u^*(z_0) - \sum_{k=1}^N u_0(x_k)\mathcal{L}h_k(z_0) \|_V = C \| \mathcal{L}u^*(z_0) - \sum_{k=1}^N u^*(x_k)\mathcal{L}h_k(z_0) \|_V \leq C \| u^* \|_H \| e_X \|_H,
\]

where \( z_0 \) is a point in \( \Omega \).

This theorem shows that, if a problem in mathematical analysis is well-posed, it has a discretization in numerical analysis that is also well-posed (Schaback 2016). Equation (3.7) describes the worst-case error behavior of the approximated solution \( u^*_N \), and the error is given as a percentage of \( \| u^* \|_H \), which is the only unknown quantity in the error bound.

4 Implementation of the method

We now begin with a discussion of some RKHS and their reproducing kernels for construction of differentiation matrices.

Definition 4.1 Let \( H \) denote a Hilbert space of functions \( f : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R} \). Then, a function \( K : \Omega \times \Omega \rightarrow \mathbb{R} \) is said to be the reproducing kernel of \( H \), if and only if \( K(., x) \in H \) for all \( x \in \mathbb{R}^d \), and \( (K(., x), f)_H = f(x) \) for all \( f \in H \) and all \( x \in \mathbb{R}^d \).

Definition 4.2 The inner product space \( W^m_2[a, b] \) is defined as \( W^m_2[a, b] = (u(x)|u^{(m-1)}) \) is absolutely continuous real valued functions, \( u^{(m)} \in L^2[a, b] \). The inner product in \( W^m_2[0, 1] \) is given by the following:

\[
(u(., v(.)))_{W^m_2} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \tag{4.1}
\]

and the norm \( \| u \|_{W^m_2} \) is denoted by \( \| u \|_{W^m_2} = \sqrt{(u, u)_{W^m_2}} \), where \( u, v \in W^m_2[a, b] \).

Theorem 4.1 (Cui and Lin 2009) The space \( W^m_2[a, b] \) is a reproducing kernel space. That is, for any \( u(.,) \in W^m_2[a, b] \) and each fixed \( x \in [a, b] \), there exists \( R_x(.,) \in W^m_2[a, b] \), such that \( (u(.,) , R_x(.,))_{W^m_2} = u(x) \). The reproducing kernel \( R_x(.,) \) can be denoted by the following:

\[
R_y(x) = \begin{cases} 
\sum_{i=1}^{2m} c_i(y)x^{i-1}, & x \leq y, \\
\sum_{i=1}^{2m} d_i(y)x^{i-1}, & x > y.
\end{cases} \tag{4.2}
\]

For more detail about the reproducing kernel Hilbert spaces \( W^m_2[a, b] \) and the method of obtaining their reproducing kernels \( R_x(y) \), refer to Cui and Lin (2009) and references therein.
Proposition 4.1 The reproducing kernel of $W^m_2[a, b]$, for a bounded interval $[a, b]$ is strictly positive definite.

Proof Based on Theorems 4.1 and 2.1, if we can show that the point evaluation functionals are linearly independent, the proof is complete. For pairwise distinct points $x_1, \ldots, x_N \in [a, b]$ and $c \in \mathbb{R}^N$, suppose that $c_1 \delta_{x_1} + \cdots + c_N \delta_{x_N} = 0$; then, for any $f \in W^m_2[a, b]$, we have

$$
(c_1 \delta_{x_1} + \cdots + c_N \delta_{x_N}) f = c_1 f(x_1) + \cdots + c_N f(x_N) = 0.
$$

Let $f(x) = x, x^2, \ldots, x^N \in W^m_2[a, b]$, and then, we have the matrix system $Vc = 0$, where $V$ is the $N \times N$ Vandermonde matrix:

$$
\begin{pmatrix}
  x_1 & \cdots & x_N \\
  x_2 & \cdots & x_N \\
  \vdots & \ddots & \vdots \\
  x_1^N & \cdots & x_N^N
\end{pmatrix} c = 0. \tag{4.3}
$$

From the non-singularity of Vandermonde matrix, we have $c = 0$, so the point evaluation functionals are linearly independent. $\square$

In the following, we illustrate how differentiation matrices can be applied to solve the differential equation. Several examples are discussed in this section which provide samples of how simple differentiation matrices can be applied to the boundary value problems and high-dimensional time-dependent problems. To solve following problems, we may consider linear two-point boundary value problems of second order and fifth order. Then, one- and two-dimensional Burgers’ equations have been considered, as the application of the proposed method on time-dependent problems. Our final example is a three-dimensional problem, to explore the power of the method for solving the multidimensional problems. To show the accuracy and the efficiency of the presented method, the maximum absolute errors, the norm of relative errors, and the root-mean-squared errors of the approximate solutions and the CPU time of the implementation of the method are reported. In the proposed method, first, the nonhomogeneous problem is reduced to a homogeneous one, and then, the functions $R_y(x)$, $j = 1, \ldots, N$ are used as the basis functions to approximate the solution of the homogenized problem, where $R_y(x)$ is the reproducing kernel of $\hat{W}^m_2[a, b]$, and hence, the approximate solution satisfies the boundary conditions exactly. For the method of obtaining the reproducing kernel $R_y(x)$ of $\hat{W}^m_2[a, b]$, refer to Cui and Lin (2009) and references therein. Let

$$
L u(x) = f(x), \quad x \in \Omega \subset R^d, \quad B u(x) = g(x), \quad x \in \partial \Omega,
$$

where $\partial \Omega$ is the boundary of $\Omega$ and $L$ is a differential operator. Then, the boundary conditions can be homogenized using the following:

$$
u(x) = v(x) + h(x), \tag{4.4}
$$

where $h$ satisfies the nonhomogeneous boundary conditions $B u(x) = g(x)$. For more details about homogenization of the one- and multi-dimensional boundary conditions and the method of imposing the boundary conditions in the reproducing kernels, see (Cui and Lin 2009; Azarnavid and Parand 2016). After homogenization of the boundary conditions, the nonhomogeneous problem can be converted to the following form:

$$
L v(x) = F(x), \quad x \in \Omega \subset R^d, \quad B v(x) = 0, \quad x \in \partial \Omega, \tag{4.5}
$$
Table 1 Maximum absolute errors of approximate solutions of Example 4.1 and comparison with radial basis functions collocation method

| N   | \(N = 10\) | \(N = 25\) | \(N = 50\) | \(N = 100\) |
|-----|-------------|-------------|-------------|-------------|
| \(\widehat{W}^5_{[−1, 1]}\) | 9.36088e−5  | 7.80543e−6  | 9.09414e−7  | 1.40113e−7  |
| \(\widehat{W}^5_{[−1, 1]}\) | 1.64341e−6  | 1.96221e−8  | 6.34336e−10 | 2.00868e−11 |
| RBF collocation | 6.32228e−5  | 4.85002e−6  | 9.44819e−7  | 5.60459e−7  |

Table 2 Maximum absolute errors and comparison of results for Example 4.2 with \(N = 13, 26, 52\)

| N   | \(N = 13\) | \(N = 26\) | \(N = 52\) |
|-----|-------------|-------------|-------------|
| Siddiqi et al. (2007), The fifth-order method | 1.3767e−4  | 7.1273e−6  | 4.6950e−7  |
| Siddiqi et al. (2007), The seventh-order method | 1.0024e−4  | 6.8397e−6  | 4.4773e−7  |
| Lv and Cui (2010) | 5.91739e−5 | 3.40705e−7 | 2.03387e−8 |
| Presented method, \(\widehat{W}^6_{[0, 1]}\) | 4.14718e−6 | 3.29059e−7 | 4.60087e−8 |
| Presented method, \(\widehat{W}^8_{[0, 1]}\) | 3.1921e−8  | 9.14844e−10| 3.37252e−11|

where \(F(x) = f(x) - Lh(x)\).

**Example 4.1** Consider the following linear two-point boundary value problem:

\[
\begin{align*}
    u''(x) &= -\frac{\sinh(x)}{(1+cosh(x))^2}, & -1 < x < 1 \\
    u(-1) &= \alpha, & u(1) = \gamma,
\end{align*}
\]

(4.6)

where \(\alpha\) and \(\gamma\) are given, such that the exact solution is \(u(x) = \frac{\sinh(x)}{1+cosh(x)}\).

**Example 4.2** Consider the following fifth-order two-point boundary value problem:

\[
\begin{align*}
    u^{(5)}(x) + u(x) &= g(x), & 0 < x < 1 \\
    u(0) &= 0, & u(1) = 0, \\
    u'(0) &= 1, & u'(1) = -e, \\
    u^{(3)}(0) &= -3,
\end{align*}
\]

(4.7)

where \(g\) is given, such that the exact solution is \(u(x) = x(1 - x)e^x\).

To solve Examples 4.1 and 4.2, first, we construct reproducing kernel spaces \(\widehat{W}^m_{[a, b]} \subset W^m_{[a, b]}\), where \((m \geq 3)\) for Example 4.1 and \((m \geq 5)\) for Example 4.2, such that the kernel of \(\widehat{W}^m_{[a, b]}\) satisfies the homogenized boundary conditions. An approximate solution of the homogenized problem at the grid points \(x_i\) might be obtained by solving the discrete linear system:

\[
Lv = F,
\]

(4.8)

where \(F\) contains the values of the function \(F\) the righthand side function of the differential equation after the homogenization, at the grid points, and \(L\) is the differentiation matrix of
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Table 3  Maximum absolute errors and comparison of results for Example 4.2 with $N = 10, 20, 40$

|                  | $N = 10$   | $N = 20$   | $N = 40$   |
|------------------|------------|------------|------------|
| Caglar et al. (1999) | 0.1570     | 0.0747     | 0.0208     |
| Siddiqi and Akram (2007) | 2.2593e−4 | 1.3300e−5 | 5.2812e−7 |
| Lv and Cui (2010) | 6.29887e−5 | 2.14116e−6 | 7.00280e−8 |
| $\hat{W}_6^2[0, 1]$ | 4.06488e−6 | 6.75653e−7 | 9.77376e−8 |
| $\hat{W}_8^2[0, 1]$ | 6.46414e−8 | 3.078e−9  | 1.19146e−10 |

Fig. 1 Graphs of $\log_{10}|u(x) - u_N(x)|$ with $m = 3, 5$ and $N = 10, 25, 50, 100$, for Example 4.1

Fig. 2 Graphs of $\log_{10}|u(x) - u_N(x)|$ with $m = 6, 8$ and $N = 10, 25, 50, 100$, for Example 4.2

differential operator in (4.6) and (4.7). In other words, the solution at the grid points is given by the following:

$$u = L^{-1}f,$$

and we see that invertibility of $L$ would be required. The presented method is employed for solving the Examples 4.1 and 4.2 which is described in Algorithm 1 step by step. In addition, A Mathematica code of the implementation of the method has been placed in the [http://www.abbasbandy.com/RKHSCOL.cdf](http://www.abbasbandy.com/RKHSCOL.cdf).

The maximum absolute errors of the approximate solutions of Example 4.1 and comparison with radial basis functions collocation method are reported in Table 1. For radial basis functions collocation method, we have used the Gaussian kernel with shape parameter $\epsilon = 1$. The comparison of the maximum absolute errors, of Example 4.2 with the best-reported results in Lv and Cui (2010), Siddiqi et al. (2007), Caglar et al. (1999), and Siddiqi
Algorithm 1 RKHS collocation method for linear differential equations

**Step 1.** Homogenize the boundary conditions of the problem and obtain the homogenized problem (4.5) using the homogenization function $h$ in (4.4).

**Step 2.** Construct the reproducing kernel of $\hat{W}^m_2(a, b)$, which satisfies the homogenized boundary conditions.

**Step 3.** Calculate the coefficients $\beta_{i,k}$, $(1 \leq i, k \leq N)$ in (2.4) using the Gram Schmidt orthogonalization algorithm.

**Step 4.** Construct the operational matrix $L$ related to the linear operator of the problem using (3.3).

**Step 5.** Evaluate the values of $v$, the approximate solution of the homogenized problem (4.5) at the grid points by solving the linear system (4.8).

**Step 6.** Evaluate the values of $u$, the approximate solution of the problem at the grid points using $u = v + h$, where $u$, $v$ and $h$ are the vectors contains the values of $u$, $v$ and $h$ at the grid points.

| Table 4 | Comparison of maximum absolute errors with the existing numerical methods of Example 4.3 for $v = 0.005$, $\sigma = 100$, $\Delta t = 0.01$ at $T = 1.0$ |
|---------|---------------------------------------------------------------|
| $N$     | Rahman et al. (2010) | Mittal and Jain (2012) | Jiwari et al. (2013) | $\hat{W}^3_2[0, 1]$ | $\hat{W}^5_2[0, 1]$ |
|---------|----------------------|------------------------|----------------------|----------------------|----------------------|
| 10      | 1.2458e−7            | 1.215e−7               | 4.708e−8             | 2.18427e−7           | 1.00476e−8           |
| 20      | 3.3944e−8            | 3.062e−8               | 1.091e−8             | 5.0701e−8            | 7.34209e−10          |
| 40      | 1.1249e−8            | 7.644e−9               | 1.980e−9             | 8.45243e−9           | 3.28094e−11          |

and Akram (2007) are shown in Tables 2 and 3. In Figs. 1 and 2, we present the maximum of absolute errors of the approximate solutions in logarithmic scale, for Examples 4.1 and 4.2 in different reproducing kernel spaces and various values of $N$. The reported results show that the accuracy of approximate solutions is closely related to the smoothness order of the reproducing kernels and values of $N$ and as proved in Abbasbandy and Azarnavid (2016) more accurate approximate solutions can be obtained using more mesh points and smoother reproducing kernels.

**Example 4.3** Consider the Burgers’ equation:

$$
\begin{align*}
    \begin{cases}
        u_t + uu_x - vu_{xx} &= 0, & x \in (0, 1), \ t \in (0, T], \\
        u(x, 0) &= f(x), \\
        u(0, t) &= g_1(t), \ u(1, t) = g_2(t),
    \end{cases}
\end{align*}
$$

(4.9)

where $v = \frac{1}{Re}$ and $Re \geq 0$ is the Reynolds number characterizing the size of viscosity and $f, g_1, g_2$ is given, such that the exact solution is $u(x, t) = \frac{2v\pi e^{-\pi^2v \sigma t}\sin(\pi x)}{\pi^2v \sigma + e^{-\pi^2v \sigma t}\cos(\pi x)}$, where $\sigma$ is a parameter. The presented method is employed for solving the Examples 4.3 and 4.4 which is described in Algorithm 2 step by step. For Examples 4.3, the numerical results are presented in Tables 4 and 5 for various values of $N, v$, and they are compared with best-reported results in Rahman et al. (2010), Mittal and Jain (2012), Jiwari et al. (2013).

Graphs of $\log_{10}|u_N(x, t) - u_N(x, t)|$ in $(x, t) \in [0, 1] \times [0, 10]$ with $v = 0.01, 0.005$, $\sigma = 100$, $\Delta t = 0.01$, $N = 40$, and $\hat{W}^5_2[0, 1]$ are given in Fig. 3. The reported results show that more accurate approximate solutions can be obtained using more mesh points and smoother reproducing kernels. The numerical simulations show that the presented method is robust and remain stable as time goes on. For solving time-dependent problems, we used the differentiation matrix for the spatial discretization together with an explicit Euler method with various time steps as Fasshauer (2005).
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Table 5 Comparison of maximum absolute errors with the existing numerical methods of Example 4.3 for $v = 0.01$, $\sigma = 100$, $\Delta t = 0.01$ at $T = 1.0$

| $N$  | Rahman et al. (2010) | Mittal and Jain (2012) | Jiwari et al. (2013) | $\tilde{W}_2^2[0, 1]$ | $\tilde{W}_2^5[0, 1]$ |
|------|----------------------|------------------------|----------------------|------------------------|------------------------|
| 10   | 4.8808e$-$7          | 4.6280e$-$7            | 6.001e$-$11          | 5.70664e$-$7           | 2.81404e$-$8           |
| 20   | 1.4305e$-$7          | 1.1640e$-$7            | 1.010e$-$11          | 1.11397e$-$7           | 1.61939e$-$9           |
| 40   | 5.6677e$-$8          | 2.9068e$-$8            | 1.277e$-$10          | 1.71283e$-$8           | 6.57035e$-$11          |

Table 6 Comparison of maximum absolute errors with the existing numerical methods of Example 4.4 for $v = 0.005$

| $N$  | $\Delta t$ | $T$  | Gao et al. (2013) | Duan et al. (2008) | $\tilde{W}_2^3[0, 1]$ | $\tilde{W}_2^5[0, 1]$ |
|------|------------|------|------------------|------------------|------------------------|------------------------|
| 50   | 0.004      | 2.4  | 1.1e$-$3         | 4.0e$-$3         | 3.11061e$-$5           | 5.00091e$-$6           |
| 100  | 0.001      | 2.4  | 2.8712e$-$4      | 9.9261e$-$4      | 4.35295e$-$5           | 8.59652e$-$7           |

| $N$  | $\Delta t$ | $T$  | Zhu and Wang (2009) | Ramadan et al. (2005) | $\tilde{W}_2^3[0, 1]$ | $\tilde{W}_2^5[0, 1]$ |
|------|------------|------|---------------------|-----------------------|------------------------|------------------------|
| 50   | 0.01       | 2.4  | 6.31491e$-$3       | 2.16784e$-$3         | 2.42535e$-$5           | 3.20613e$-$5           |
| 50   | 0.01       | 1.8  | 5.12020e$-$3       | 2.47189e$-$3         | 9.79958e$-$5           | 6.82184e$-$5           |

**Example 4.4** Consider the Burgers’ equation (4.9) with $f$, $g_1$, $g_2$ is given, such that the exact solution is $u(x, t) = \frac{t}{1 + \left(\frac{t}{t_0}\right)^2 \exp\left(\frac{1}{8t^2}\right)}$, $t \geq 1$, where $t_0 = \exp\left(\frac{1}{8t^2}\right)$. For this example, the numerical results are presented in Table 6 for various values of $N$, $v$ and time $T$, and they are compared with best-reported results in Gao et al. (2013), Duan et al. (2008), Zhu and Wang (2009), and Ramadan et al. (2005). Graphs of $\log_{10}|u(x, t) - u_N(x, t)|$ in $(x, t) \in [0, 1] \times [0, 10]$ with $v = 0.01$, 0.005, $\sigma = 100$, $\Delta t = 0.01$, $N = 40$ and $\tilde{W}_2^5[0, 1]$ are given in Fig. 4. We see that the accuracy increases with increasing the mesh and smoothness of kernels. The numerical simulations show that the presented method is robust and remain stable as time goes on.
Consider the two-dimensional Burger’s equation:

$$u_t + uu_x + uu_y = vu_{xx} + vu_{yy}, \quad 0 \leq x \leq x_N, \quad 0 \leq y \leq y_N, \quad t > 0, \quad (4.10)$$

with the initial condition $u(x, y, 0) = u_0(x, y)$ and the viscous coefficient $v = \frac{1}{Re} > 0$, $Re$ is the Reynolds number. The Dirichlet boundary conditions is given, such that the exact solution is $u(x, y, t) = \frac{1}{1 + e^{x+y-\eta/2Re}}$, and $x_0 = y_0 = 0$, $x_N = y_N = 1$.

**Theorem 4.2** (Arensiaz 1950) Let $W_1$ and $W_2$ be reproducing kernel spaces with reproducing kernels $K_1$ and $K_2$. The direct product $\overline{W} = W_1 \otimes W_2$ is a reproducing kernel space and possesses the reproducing kernel $\overline{K}(x_1, x_2, y_1, y_2) = K_1(x_1, y_1)K_2(x_2, y_2)$.

For solving multi-dimensional problems, we are using the product of reproducing kernels of $W^m [a, b]$ as kernels in multi-dimensional domain. It is easy to see that these kernels are strictly positive definite as proof of Proposition 4.1. An algorithm similar to Algorithm 2 with a few modifications and suitable multi-dimensional kernel can be used for the multi-dimensional problems. For Example 4.5, the comparison of maximum absolute errors $L_\infty$ and relative errors $L_2$ with radial basis functions pseudospectral method and Chebyshev

**Algorithm 2** RKHS collocation method for nonlinear time-dependent problems (Examples 4.3 and 4.4)

**Step 1.** Homogenize the boundary conditions and obtain the homogenized problem (4.5) by $u = U + h$ using the homogenization function $h$ in (4.4).

**Step 2.** Construct the reproducing kernel of $\overline{W}^m [a, b]$, which satisfies the homogenized boundary conditions.

**Step 3.** Calculate the coefficients $\beta_{i,k}$, $(1 \leq i, k \leq N)$ in (2.4) using the Gram Schmidt orthogonalization algorithm.

**Step 4.** Construct the differentiation matrix $D_x$ and $D_y$ related to the differential operator $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

**Step 5.** Evaluate $U^0$, the vectors contains the values of $U(x, 0)$ at the grid points.

**Step 6.** Repeat the following explicit Euler scheme to produce the sequence $U^m$:

$$U^{m+1} = U^m + \Delta t \left( -U^m , * \left( D * U^m \right) - U^m , * h_x - h , * \left( D * U^m \right) - h , * h_x + vD_2 * U^m + v h_{xx} \right),$$

where the symbols $,*$ and $*$ denote the element-wise product and the matrix product, respectively.

**Step 7.** Let $U^m = U^m + h$, where $U^m$, $U^m$ and $h$ are the vectors contains the values of $u(x, m\Delta t)$, $U(x, m\Delta t)$ and $h(x)$ at the grid points.
Table 7  CPU time and comparison of maximum absolute errors \( L_\infty \) and relative errors \( L_2 \) with the existing numerical methods of Example 4.5 using \( N = 25 \) and various \( dt, m \) and \( v \) in \( \tilde{W}_2^5[0, 1] \)

| \( dt \)  | \( T \)  | \( v \)  | RBF PS method | Chebyshev PS method | Presented method | CPU time (s) |
|-------|-------|-------|---------------|----------------------|------------------|------------|
| 0.005 | 1     | 1     | 9.0368e−3 5.7021e−3 | 2.4851e−7 1.9904e−7 | 4.25623e−9 5.65924e−9 | 7.2        |
| 0.001 | 1     | 1     | 9.0366e−3 5.7020e−3 | 9.6413e−8 7.2229e−8 | 4.09451e−9 5.183e−9  | 10.6       |
| 0.005 | 10    | 1     | 8.195e−4 2.9809e−4 | 4.8555e−7 1.6113e−7 | 1.30473e−10 8.08799e−11 | 14.8       |
| 0.001 | 10    | 1     | 8.1993e−4 2.9822e−4 | 9.4330e−8 3.2060e−8 | 3.05613e−11 2.10146e−11 | 48.7       |
| 0.005 | 1     | 0.1   | 2.9854e−1 1.736e−1 | 4.27835e−3 2.12004e−3 | 2.53845e−3 2.572e−3 | 7.6        |
| 0.001 | 1     | 0.1   | 2.9816e−1 1.735e−1 | 3.7823e−3 2.0727e−3 | 2.83507e−3 2.35878e−3 | 11.3       |
| 0.005 | 10    | 0.1   | 2.9024e−3 1.1971e−3 | 3.6637e−15 1.1098e−15 | 8.43362e−20 3.299e−20 | 15.6       |
| 0.001 | 10    | 0.1   | 2.9024e−3 1.1971e−3 | 2.0650e−14 6.4789e−15 | 3.0511e−20 1.58742e−20 | 49.3       |

pseudospectral method using \( N = 25 \) and various \( dt, m \) and \( v \) in \( \tilde{W}_2^5[0, 1] \) is presented in Table 7. For the radial basis functions pseudospectral method, we used the Gaussian kernel with the optimal shape parameter introduced in Fasshauer (2005). The CPU time of the implementation of the proposed method for Example 4.5 is reported in Table 7 to show its computational efficiency.

Example 4.6 Consider the two-dimensional Burger’s equation:

\[
\frac{\partial u}{\partial t} + uu_x + uu_y = vu_{xx} + vu_{yy}, \quad x_0 \leq x \leq x_N, \quad y_0 \leq y \leq y_N, \quad t > 0, \quad (4.11)
\]

with the initial condition \( u(x, y, 0) = u_0(x, y) \) and the viscous coefficient \( v = \frac{1}{Re} > 0 \). \( Re \) is the Reynolds number. The Dirichlet boundary conditions is given, such that the exact solution is \( u(x, y, t) = 0.5 - \tanh(\frac{x+y-t}{2p}) \), and \( x_0 = y_0 = -0.5, x_N = y_N = 0.5 \). For Example 4.6, the CPU time of the implementation of the presented method and the comparison of maximum absolute errors \( L_\infty \) and relative errors \( L_2 \) with radial basis functions pseudospectral method and Chebyshev pseudospectral method using \( N = 25 \) and various \( dt, m \) and \( v \) in \( \tilde{W}_2^5[-0.5, 0.5] \) are presented in Table 8. For the radial basis functions pseudospectral method, we used the Gaussian kernel with the optimal shape parameter introduced in Fasshauer (2005).

Example 4.7 Consider the three-dimensional problem:

\[
\frac{\partial u}{\partial t} = \frac{1}{\pi^2} \nabla^2 u(x, y, z, t) - 2e^{t-\pi(x+y+z)} , \quad (4.12)
\]

where \( 0 \leq x, y, z \leq 1 \) and \( t > 0 \), with the initial condition in \( t = 0 \), and Dirichlet boundary conditions which can be extracted from the analytical solution:

\[
u(x, y, z, t) = e^{t-\pi(x+y+z)} + x + y + z.\]

The relative errors \( L_2 \), of the approximation solutions of Example 4.7 with various \( T, dt, N \) and \( m \) in \( \tilde{W}_2^m[0, 1] \) are reported in Table 9. In Table 10, the numerical results are compared with the best-reported results in Yao and Šarler (2012). Figure 5 shows the graphs of eigenvalues of iteration matrices of forward Euler time stepping, in complex plane,
Euler method which showed almost identical behavior for a smaller time step, so that we can see the implementation of differentiation matrix with an explicit can be obtained using more mesh points and smoother reproducing kernels. From the results show that the presented method remains stable as time goes on, despite the exponential growth of exact solution of the problem in time and more accurate approximate solutions.

| $dt$ | $T$ | $v$ | RBF PS method | Chebyshev PS method | Presented method | CPU time (s) |
|------|-----|-----|----------------|---------------------|-----------------|-------------|
|      |     |     | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
| 0.005 | 1 | 1 | 5.4139e−2 | 1.7775e−2 | 2.84257e−5 | 1.01789e−5 | 6.68948e−6 | 3.63904e−6 | 7.6 |
| 0.001 | 1 | 1 | 5.416e−2 | 1.7781e−2 | 6.53996e−6 | 2.41596e−6 | 1.62154e−6 | 9.35968e−7 | 10.5 |
| 0.005 | 10 | 1 | 6.6292e−4 | 1.8036e−4 | 1.43625e−8 | 3.78170e−9 | 7.56954e−9 | 3.26121e−9 | 14.0 |
| 0.001 | 10 | 1 | 6.6293e−4 | 1.8036e−4 | 5.00597e−9 | 9.10377e−10 | 1.52521e−9 | 6.61975e−10 | 42.3 |
| 0.005 | 1 | 0.1 | 6.1243e−1 | 1.5473e−1 | 1.25696e−2 | 2.29108e−3 | 2.74444e−2 | 7.36095e−3 | 7.7 |
| 0.001 | 1 | 0.1 | 6.1173e−1 | 1.547e−1 | 1.23611e−2 | 2.29390e−3 | 2.52867e−2 | 6.71424e−3 | 10.5 |
| 0.005 | 5 | 0.1 | 6.58091e−3 | 1.79631e−3 | 3.99680e−15 | 6.59355e−16 | 6.66134e−16 | 1.5666e−16 | 10.3 |
| 0.005 | 5 | 0.1 | 6.58245e−3 | 1.79635e−3 | 1.22124e−14 | 3.11285e−15 | 1.7763e−15 | 4.13425e−16 | 24.2 |

Table 9 Relative errors $L_2$, of the approximation solutions of Example 4.7 with various $T$, $dt$, $N$ and $m$ in $\hat{W}_2$ $[0, 1]$

| $T$ | $dt$ | $m$ | $N = 27$ | $N = 64$ | $N = 125$ |
|-----|-----|-----|---------|---------|---------|
| 1   | 0.01 | 3   | 1.68375e−3 | 9.3954e−4 | 5.77729e−4 |
| 1   | 0.01 | 5   | 4.19018e−4 | 1.72518e−4 | 8.92225e−5 |
| 1   | 0.001 | 3   | 1.65172e−3 | 9.1161e−4 | 5.5239e−4 |
| 1   | 0.001 | 5   | 3.93504e−4 | 1.41514e−4 | 6.18902e−5 |
| 5   | 0.01 | 3   | 3.10666e−2 | 1.51374e−2 | 8.47024e−3 |
| 5   | 0.01 | 5   | 7.69788e−3 | 2.78871e−3 | 1.31209e−3 |
| 5   | 0.001 | 3   | 3.04834e−2 | 1.46877e−2 | 8.09662e−3 |
| 5   | 0.001 | 5   | 7.22393e−3 | 2.28496e−3 | 9.06803e−4 |

Table 10 $L_\infty$ and $L_{rms}$, of the approximation solutions of Example 4.7 at $T = 1$ in $\hat{W}_2$ $[0, 1]$ and $\hat{W}_2^5$ $[0, 1]$

| $dt$ | $\hat{W}_2$ $[0, 1], N = 150$ | $\hat{W}_2^5$ $[0, 1], N = 150$ | Yao and Šarler (2012), $N = 160$ |
|------|-----------------|-----------------|-----------------|
|      | $L_\infty$ | $L_{rms}$ | $L_\infty$ | $L_{rms}$ | $L_\infty$ | $L_{rms}$ |
| 0.01 | 1.87124e−3 | 8.1899e−4 | 2.56833e−4 | 1.27696e−4 | 2.98e−3 | 6.39e−4 |
| 0.001 | 1.83218e−3 | 7.79576e−4 | 2.09345e−4 | 8.43759e−5 | 3.88e−3 | 8.32e−4 |
| 0.0001 | 1.82829e−3 | 7.75725e−4 | 2.04977e−4 | 8.06016e−5 | 2.84e−3 | 8.51e−4 |

with $N = 125$, $\hat{W}_2^m$ $[0, 1]$ and various $m$ and $dt$ for Example 4.7. The numerical simulations show that the presented method remains stable as time goes on, despite the exponential growth of exact solution of the problem in time and more accurate approximate solutions can be obtained using more mesh points and smoother reproducing kernels. From the results in Tables 9 and 10, we can see the implementation of differentiation matrix with an explicit Euler method which showed almost identical behavior for a smaller time step, so that we can be assured that the inversion was, indeed, justified for this particular example. To show the computational efficiency of the proposed method, the $L_\infty$ and $L_{rms}$ errors of approximation solutions and the CPU time of the implementation of the method for Example 4.7 with various $N$ are reported in Table 11.
5 Conclusions

In this paper, a new efficient meshless method, a combination of the Reproducing kernel Hilbert space and the meshless collocation method, based on differentiation matrices constructed by cardinal functions of a reproducing kernel Hilbert space is proposed. In comparison with the radial basis functions collocation method (Kansas method), we have the non-singularity in the collocation matrices, and since the boundary conditions are imposed on the trial space instead of the collocation matrices, the implementation of the method is more simple and the method is truly meshless. During the construction process, we have proposed a new and efficient algorithm to obtain the cardinal functions of an RKHS and also we drive the pointwise error estimate of the employing the differentiation matrices. During the construction process, we have proposed a new and efficient algorithm to obtain the cardinal functions of an RKHS and also we drive the pointwise error estimate of the employing the differentiation matrices. To demonstrate the computation efficiency, the mentioned method is
Table 11 \(L_\infty\), \(L_{\text{rms}}\), and CPU time of approximation solutions of Example 4.7 at \(T = 1\) in \(\hat{W}_3^2[0, 1]\) and \(\hat{W}_5^2[0, 1]\) with \(dt = 0.01\)

| \(N\) | \(\hat{W}_3^2[0, 1]\) | \(\hat{W}_5^2[0, 1]\) | CPU time (s) | \(L_\infty\) | \(L_{\text{rms}}\) | CPU time (s) |
|---|---|---|---|---|---|---|
| 27 | 6.2211e−3 | 6.3 | 1.6326e−3 | 6.5990e−4 | 10.2 |
| 64 | 3.3688e−3 | 56.3 | 5.1512e−4 | 2.7348e−4 | 88.3 |
| 100 | 1.1091e−3 | 230.4 | 4.0332e−4 | 1.8535e−4 | 352.8 |

implemented for seven examples and results have been compared with the reported results in the literature which show the validity, efficiency, accuracy, and applicability of the method.

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