Supercuspidal support of irreducible modulo ℓ-representations of SLₙ(F)

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Abstract
Let k be an algebraically closed field with characteristic ℓ ≠ p. We show that the supercuspidal support of irreducible smooth k-representations of Levi subgroups M' of SLₙ(F) is unique up to M'-conjugation, where F is either a finite field of characteristic p or a non-archimedean locally compact field of residual characteristic p.

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1 Introduction
Let F be a non-archimedean locally compact field with residual characteristic p, and k be an algebraically closed field with characteristic ℓ ≠ p. Let G be a connected reductive group defined over F or a finite field F_q, where q is a power of p. Denote by G the group of F (resp. F_q) rational points G(F) (resp. G(F_q)).

The supercuspidal support of an irreducible smooth k-representation π of G is important during the study of the theory of representations of G. When ℓ is equal to 0, supercuspidal representations are all cuspidal, and there is a quick proof that the cuspidal support of π is unique up to G-conjugation. When ℓ is positive, an example of cuspidal but not supercuspidal k-representation has been found in [V1] when G = GL₂. The cuspidal support of

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an irreducible $k$-representation of $G$ is always unique up to $G$-conjugation, while the supercuspidal support is not, and an example of non-uniqueness has been found when $G = \text{Sp}_8$ in [Da]. However, the uniqueness of supercuspidal support is true when $G = \text{GL}_n$ and a proof was given in [V2]. In this article, the same result is proved when $G = \text{SL}_n$ and its Levi subgroups. The uniqueness of supercuspidal support is the base stone of the Bernstein decomposition of the category $\text{Rep}_k(\text{GL}_n(F))$ of smooth $k$-representations of $\text{GL}_n(F)$ in [Helm]. Since we obtain the same result for $\text{SL}_n(F)$, it shows a substantial possibility that the category $\text{Rep}_k(\text{SL}_n(F))$ can be decomposed relative to inertially equivalent classes of supercuspidal pairs of $\text{SL}_n(F)$ (see Definition 2.4 for supercuspidal pair and two supercuspidal pairs are said to be inertially equivalent if they are $\text{SL}_n(F)$-conjugate to each other up to an unramified character).

This article is the first step of generalising the result in [Helm] to $\text{SL}_n(F)$. Let $W(k)$ be the ring of Witt vectors of $k$. In [Helm], Helm gave a Bernstein decomposition of $\text{Rep}_{W(k)}(\text{GL}_n(F))$, from which one can deduce the Bernstein decomposition of $\text{Rep}_k(\text{GL}_n(F))$. It is worth noting that firstly the coefficient $W(k)$ is essentially needed for his later work with Emerton of the local Langlands correspondence for $\text{GL}_n$ in families, secondly his proof relies on a family of injective objects in $\text{Rep}_{W(k)}(\text{GL}_n(F))$, which is constructed from the $W(k)$-projective covers of cuspidal $k$-representations of $\text{GL}_n(\mathbb{F}_q)$. In this article, we consider $k$-representations of Levi subgroups $M'$ of $\text{SL}_n(\mathbb{F}_q)$ in Section 3 where we also study $W(k)[M']$-modules and the $W(k)$-projective covers of cuspidal $k$-representations. Since the fractional field $K$ of $W(k)$ must not be sufficient large for finite group $\text{GL}_n(\mathbb{F}_q)$, we need more discussion about this coefficient here. In Section 4 we consider $k$-representations of Levi subgroups of $p$-adic groups $\text{SL}_n(F)$.

To be more precisely, in Section 3 for an irreducible cuspidal $k$-representation $\nu$ of a Levi subgroup $M'$, the $W(k)[M']$-projective cover $P_\nu$ of $\nu$ can be constructed from Gelfand-Graev $W(k)$-lattice. A computation of $r_{L'}^{M'}P_\nu$ gives the uniqueness of supercuspidal support of $\nu$, where $L'$ denotes a Levi subgroup of $M'$ and $r_{L'}^{M'}$ denotes the normalised parabolic restriction relative to $L'$.

In Section 4 a basic fact is that for an irreducible cuspidal $k$-representation $\pi'$ of $M'$, there is an irreducible cuspidal $k$-representation $\pi$ of $M$ such that $\pi|_{M'}$ contains $\pi'$ as a sub-representation, where $M$ is a Levi subgroup of $G = \text{GL}_n(F)$. Let $(L, \tau)$ belongs to the supercuspidal support of $\pi$, and $\tau = \otimes I \tau_i \tau_i|_{M'}$ where $I$ is finite. We first prove that the supercuspidal support of $\pi'$ is contained in $\bigcup_{i \in I} (L', \tau_i|_{M'})$, where $(L', \tau_i|_{M'})$ denotes the $M'$-conjugacy class of $(L', \tau_i)$. Then we generalise the operator of derivative defined in [BEZE] for $\text{GL}_n$ to $M'$. Since $M'$ can not be written as a direct product of special linear groups in lower rank, the author can not find a way to avoid the complication of notations in Section 4. At the end, we deduce that there exists one unique $i_0 \in I$ such that $(L', \tau_{i_0})$ belongs to the supercuspidal support of $\pi'$ from the unicity of Whittaker model of $\pi$.

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## 2 Cuspidal and supercuspidal representations

### Basic notations

Let $F$ be a non-archimedean locally compact field with residual characteristic $p$, and $k$ be an algebraically closed field with characteristic $\ell \neq p$. Let $G$ be a
connected reductive group defined over $F$ or $F_q$, where $q$ is a power of $p$, and $G$ be the group of $F$ (resp. $F_q$) rational points of $G$. In this article, a $k$-representation of $G$ is always assumed to be smooth.

Fix a Borel subgroup of $G$. Let $M$ be a standard Levi subgroup of $G$. Denote by $i^G_M$ and $r^G_M$ the normalised parabolic induction and normalised parabolic restriction. Let $K$ be a closed subgroup of $G$. Denote by $\text{ind}^G_K$ the compact induction from $K$ to $G$, and $\text{res}^G_K$ the restriction from $G$ to $K$.

**Definition 2.1.** Let $\pi$ be an irreducible $k$-representation of $G$, we say

- $\pi$ is cuspidal, if for any proper Levi subgroup $M$ and irreducible $k$-representation $\sigma$ of $M$, $\pi$ does not appear as a sub nor a quotient-representation of $i^G_M \sigma$;
- $\pi$ is supercuspidal, if for any proper Levi subgroup $M$ and irreducible $k$-representation $\sigma$ of $M$, $\pi$ does not appear as a subquotient of $i^G_M \sigma$.

We say a pair $(M, \sigma)$ consisting with a Levi subgroup $M$ and an irreducible $k$-representation $\sigma$ is a cuspidal (resp. supercuspidal) pair, if $\sigma$ is cuspidal (resp. supercuspidal). We say

- a cuspidal pair $(M, \sigma)$ belongs to the cuspidal support of $\pi$, if $\pi$ is a sub or a quotient-representation of $i^G_M \pi$;
- a supercuspidal pair $(M, \sigma)$ belongs to the supercuspidal support of $\pi$, if $\pi$ is a subquotient of $i^G_M \sigma$.

**Remark 2.2.** In the above definition, $\pi$ being cuspidal is equivalent with $r^G_M \pi$ being zero for any proper Levi $M$ of $G$.

**Reduction to cuspidal cases** Let $\pi$ be an irreducible $k$-representation of $G$. The cuspidal support of $\pi$ is unique up to $G$-conjugation as proved in [V1]. To prove the uniqueness of supercuspidal support of $\pi$, it is enough to prove the same result for each irreducible cuspidal $k$-representations of Levi subgroups of $G$. In fact, let $(M, \sigma)$ be a cuspidal pair inside the cuspidal support of $\pi$, if $\pi$ is a sub or a quotient-representation of $i^G_M \pi$.

We deduce that $\sigma$ is a sub-quotient of $r^G_M i^G_L \tau$, to which apply the filtration given in [V1, §II.2.18], then we obtain that up to a conjugation of $w$, which is an element of the Weyl group of $G$, $(w(L), w(\tau))$ belongs to the supercuspidal support of $\sigma$.

3 $k$-representations of finite groups $\text{SL}_n(F)$

In this section, let $G' = \text{SL}_n$ and $G = \text{GL}_n$ be defined over $F_q$, where $q$ is a power of a prime number $p$. Denote by $G' = G'(F_q)$ and $G = G(F_q)$. Recall that $k$ is an algebraically closed field with characteristic $\ell \neq p$. Let $W(k)$ be the ring of Witt vectors of $k$ and $\mathcal{K}$ the fractional field of $W(k)$, and $\overline{\mathcal{K}}$ an algebraic closure of $\mathcal{K}$. We have two main purposes in this section, one is to prove Theorem 3.11. The other one is to construct the $W(k)[M']$-projective cover of an irreducible cuspidal $k$-representation of $M'$, where $M'$ denotes a Levi subgroup of $G'$.

Notice that the center of $G'$ is disconnected but the center of $G$ is connected, so we follow the method of [DelLiu] (page 132), which is also applied in [Bon]: consider the regular inclusion $i : G' \rightarrow G$, then we want to use functor $\text{Res}^G_{G'}$ to deduce properties from $G$-representations to $G'$-representations.
3.1 Projective modules

Regular inclusion \( i \) We summarize the context we will need in Section 2 of [Bon]:

Let \( \mathcal{F} \) be the Frobenius morphism of the Galois group \( \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \), where \( \mathbb{F}_q \) is an algebraic closure of \( \mathbb{F}_q \). \( \mathcal{F} \) induces an isogeny of \( G \), which we also denote by \( \mathcal{F} \). In particular, the invariant group \( G^{\mathcal{F}} = G \). The canonical inclusion \( i \) from \( G' \) to \( G \) commutes with \( \mathcal{F} \) and maps \( \mathcal{F} \)-stable maximal torus to \( \mathcal{F} \)-stable maximal torus. If we fix one \( \mathcal{F} \)-stable maximal torus \( T \) of \( G \) and denote by \( T' = i^{-1}(T) \), then \( i \) induces a bijection between the root systems of \( G \) and \( G' \) relative to \( T \) and \( T' \). Furthermore, \( i \) gives a bijection between standard \( \mathcal{F} \)-stable parabolic subgroups of \( G \) and \( G' \) with inverse \( \cap G' \), which respects subsets of simple roots contained by parabolic subgroups. Besides, restricting \( i \) to a \( \mathcal{F} \)-stable Levi subgroup \( L \) of a \( \mathcal{F} \)-stable parabolic subgroup of \( G \) is the canonical inclusion from \( L' \) to \( L \).

From now on, we fix a \( \mathcal{F} \)-stable maximal torus \( T_0 \) of \( G \), and fix \( T_0' = i(T_0) \) of \( G' \) as well. For any \( \mathcal{F} \)-stable standard Levi subgroup \( L \), we always denote by \( L' = i(L) \), and denote by \( L \) and \( L' \) the corresponding split Levi subgroups \( L^{\mathcal{F}} \) and \( L'^{\mathcal{F}} \) respectively.

Now we consider the dual groups. Let \( (G^*, T^0, \mathcal{F}^*) \) and \( (G'^*, T'^0, \mathcal{F}^*) \) be triples dual to \( (G, T_0, \mathcal{F}) \) and \( (G', T'_0, \mathcal{F}) \), where \( G^* \) is dual to \( G \) and \( F^* \) is the dual isogeny of \( F \). We deduce a canonical surjective morphism \( \iota^*: G^* \rightarrow G'^* \), which commutes with \( F^* \) and maps \( T_0^* \) to \( T'^0 \). For any \( \mathcal{F} \)-stable standard parabolic subgroup \( P \) and its \( \mathcal{F} \)-stable Levi subgroup \( L \), we use \( P' \) and \( L' \) to denote the \( \mathcal{F} \)-stable standard parabolic subgroups \( P \cap G' \) and Levi subgroups \( L \cap G' \), respectively, then we have:

\[
\iota^*(L^*) = L'^*.
\]

After denoting by \( L'^{*\cap} = L'^* \) and by \( L^{\cap *\prime} = L^* \), we have:

\[
\iota^*(L^*) = L'^*.
\]

Lusztig series and \( \ell \)-blocks From now on, if we consider a semisimple element \( \tilde{s} \in L^* \) for any split Levi subgroup \( L^* \) of \( G^* \), we always denote by \( s \) the image \( i^*(\tilde{s}) \) and by \( [\tilde{s}] \) the \( L^* \)-conjugacy class of \( s \), and a similar definition for \( [s] \). We say a semisimple element is \( \ell \)-regular if \( \ell \) does not divide its order. Since the order of \( \tilde{s} \) is divisible by the order of \( s \), we have that \( s \) is \( \ell \)-regular if \( \tilde{s} \) is \( \ell \)-regular. By the theory of Delign-Lusztig, an irreducible \( k \)-representation \( \pi \) of \( L \) corresponds to a semisimple conjugacy class \( [\tilde{s}] \), where \( \tilde{s} \) is \( \ell \)-regular.

Let \( G(\mathbb{F}_q) \) be a finite group of Lie type, where \( G \) is a connected reductive group defined over \( \mathbb{F}_q \). For any irreducible representation \( \chi \) of \( G(\mathbb{F}_q) \), let \( e_\chi \) denote the central idempotent of \( \mathbb{K}(G(\mathbb{F}_q)) \) associated to \( \chi \) (see definition in the beginning of [BrMi §2]). Fixing a semisimple element \( s \in G^*(\mathbb{F}_q) \), let \( \mathbb{E}(G(\mathbb{F}_q), (s)) \) be the Lusztig serie of \( G(\mathbb{F}_q) \) corresponding to \( [s] \). If \( s \) is \( \ell \)-regular, define

\[
\mathcal{E}_{\ell}(G(\mathbb{F}_q), s) := \bigcup_{t \in (C_{G^*}(s)^{F^*})_{\ell}} \mathbb{E}(G(\mathbb{F}_q), (ts)).
\]

Here \( (C_{G^*}(s)^{F^*})_{\ell} \) denotes the group consisting with all \( \ell \)-elements of \( C_{G^*}(s)^{F^*} \), where \( C_{G^*}(s) \) is the centraliser group of \( s \) in \( G \), and \( ts \) is semisimple as well. Now define:

\[
b_s = \sum_{\chi \in \mathcal{E}_{\ell}(G(\mathbb{F}_q), s)} e_\chi,
\]

which obviously belongs to \( \mathbb{K}(G(\mathbb{F}_q)) \).

For the convenience reason, we state a theorem in [BrMi] below.
Theorem 3.1 (Broué, Michel). Let $s \in G^*$ be a semisimple $\ell$-regular element, and $L'$ be the set of prime numbers except $\ell$. Define $\mathbb{Z}_\ell = \mathbb{Z}[1/r]_{r \in L'}$, where $\mathbb{Z}$ denotes the ring of integers.

Remark 3.2. We view $\mathbb{Z}_{\ell}$ as a subring of $\mathbb{K}$. Let $K^\text{unr}$ be an unramified closure in $\mathbb{K}$ of $\mathbb{Q}_\ell$ and $O^\text{unr}$ be the ring of integers of $K^\text{unr}$, in fact we have $b_s \in O^\text{unr}[G]$. By the proof of Theorem 9.12 of [CE], we know that the support of $b_s$ is contained in $O^\text{unr}[G]$. Theorem 3.1 of [CE] implies that $O^\text{unr}[G] \cap \mathbb{Z}_\ell[G] = O^\text{unr}[G]$. In particular, $O^\text{unr} \subset W(\mathbb{F}_\ell) \subset W(k)$, since $W(\mathbb{F}_\ell)$ is the completion of the $\ell$-adic topology of an unramified closure of $\mathbb{Q}_\ell$. It is worth noting that $K^\text{unr}$ and $\mathbb{K}$ must not be sufficiently large for $G$. In particular, there exists cuspidal $\mathbb{K}$-representation of $\text{GL}_2(\mathbb{F}_\ell)$ which is not defined over $\mathbb{K}$.

Proposition 3.3. For any split Levi subgroup $L$ (resp. $L'$) and any semisimple $\ell$-regular element $\bar{s} \in L^*$ (resp. $s \in L'^*$), we have: $b_s \in O^\text{unr}[L]$ (resp. $b_s \in O^\text{unr}[L']$).

Proof. We deduce from the analysis above and the definition that $e_\chi \in K[L]$. Combining this with Theorem 3.1 we conclude that $b_s \in O^\text{unr}[L]$. The same for $b_s$.

Gelfand-Graev lattices and its projective direct summands In this section, we construct the projective cover of an irreducible cuspidal $k$-representation of $L'$ by using Gelfand-Graev lattice, and prove that it is a direct summand of the projective cover of an irreducible cuspidal $k$-representations of $L$ after restricted to $L'$ (see Proposition 3.10).

For a split Levi subgroup $L'$ of $G'$, fix a rational split Borel subgroup $B'$ with unipotent radical $U_{L'}$. Denote by $O_{U_{L'}}$ the set of non-degenerate $\mathbb{K}$-characters of $U_{L'}$. Let $L$ be the Levi subgroup of $G$ such that $L \cap G' = L'$, the group $U_{L'} = U_L$ is also the unipotent radical of $L$. Notice that $O_{U_L}(L) = O_{U_{L'}}(L')$ consists only one unique $L$-conjugacy class, but multiple $L'$-conjugacy classes.

Let $(K, O, k)$ be a splitting $\ell$-modular system of $G$, where $K$ is a finite extension of $\mathbb{K}$ sufficiently large for $G$ and $O$ is its ring of integers. For an $\mu \in O_{U_{L'}}$, it contains an $O[U_{L'}]$-lattice, and we denote it by $O_{\mu}$. Define $Y_{L', \mu} = \text{ind}_{U_{L'}}^{L'} O_{\mu}$, the Gelfand-Graev lattice associated to $\mu$. In fact, we have that $Y_{L', \mu}$ is defined up to the $T'$-conjugacy class of $\mu$. Take any $\ell$-regular semisimple element $s \in L'^*$, define:

$$Y_{L', \mu, s} = b_s \cdot Y_{L', \mu}.$$  

Meanwhile, from the definition we have directly that

$$\sum_{[s]} b_s = 1,$$

where the sum runs over all the $\ell$-regular semisimple $L'^*$-conjugacy class $[s]$. So:

$$Y_{L', \mu} = \sum_{[s]} Y_{L', \mu, s}.$$  

Since $O_{\mu}$ is projective and the compact induction respects projectivity, we know that $Y_{L', \mu}$ is a projective $O[L']$-module. Proposition 3.3 implies that $Y_{L', \mu, s}$ are $O[L']$-modules, which are direct components of projective $O[L']$-module $Y_{L', \mu}$, hence $Y_{L', \mu, s}$ are also projective $O[L']$-modules.

We define

$$\mathcal{E}_{\ell}(G) := \bigcup_{z \text{ semi-simple, } \ell \text{-regular}} \mathcal{E}(G, z).$$  

5
Definition 3.4 (Gruber, Hiss). Let G be the group of \( \mathbb{F}_q \)-points of an algebraic group defined over \( \mathbb{F}_q \), and \((K,O,k)\) be a splitting \( \ell \)-modular system. Let \( Y \) be an \( O[G] \)-lattice with ordinary character \( \psi \). Write \( \psi = \psi_V + \psi_L \), such that all constituents of \( \psi_V \) and non of \( \psi_L \) belong to \( \mathcal{E}_V(G) \). Then there exists a unique pure sublattice \( V \leq Y \), such that \( Y/V \) is an \( O[G] \)-lattice whose character is equal to \( \psi_V \). The quotient \( Y/V \) is called the \( \ell \)-regular quotient of \( Y \) and denoted by \( \pi_{\ell}(Y) \).

Corollary 3.5. Let \( L' \) be a split Levi subgroup of \( G' \), and \( s \) be an \( \ell \)-regular semisimple element in \( L' \). For any \( \mu \in O_U(L') \), the module \( Y_{L',\mu,s} \) is indecomposable.

Proof. Since \( Y_{L',\mu,s} \) is a projective \( O[L'] \)-module, the section \S 4.1 of [GrHi] or Lemma 5.11(Hiss) in [Geck] tells us that it is indecomposable if and only if its \( \ell \)-regular quotient \( \pi_{\ell}(Y_{L',\mu,s}) \) is indecomposable. Inspired by section 5.13. of [Geck], we consider \( K \otimes \pi_{\ell}(Y_{L',\mu,s}) \), which is the unique irreducible sub-representation of \( K \otimes Y_{L',\mu} \) lying in Lusztig serie \( \mathcal{E}(L',(s)) \). The module \( \pi_{\ell}(Y_{L',\mu,s}) \) is torsion-free, so we deduce that \( \pi_{\ell}(Y_{L',\mu,s}) \) is indecomposable.

Proposition 3.6. Let \( L' \) be a split Levi subgroup of \( G' \), and \( \mu \in O_U(L') \). All the \( \ell \)-regular indecomposable direct summands \( Y_{L',\mu,s} \) of Gelfand-Graev lattice \( Y_{L',\mu} \) are defined over \( O^{unr} \). In particular, there exist indecomposable projective \( W(k)[L'] \)-modules \( \mathcal{Y}_{L',\mu,s} \) such that \( \mathcal{Y}_{L',\mu,s} \otimes W(k) O \cong Y_{L',\mu,s} \).

Proof. Since \( U_{L'} \) are \( p \)-groups, \( \mu \) is defined over \( O^{unr} \), which is equivalent to say that there is a \( O^{unr}[U_{L'}] \)-module \( O_{\mu} \) such that \( O_{\mu} \cong O \otimes_{O^{unr}} O \). Define a projective \( O^{unr}[L'] \)-module \( \text{Ind}_{U_{L'}}(O_{\mu}) \). By Remark 3.2, we define \( \mathcal{Y}_{L',\mu,s} = b_{s} Y_{L',\mu,s} \), which is indecomposable from the fact that \( \mathcal{Y}_{L',\mu,s} \otimes W(k) k = Y_{L',\mu,s} \otimes_{O} k \) is indecomposable.

Remark 3.7. Let \( B_L \) be a split Borel subgroup of \( L \), such that \( B_L \cap L' = B_{L'} \). Since \( U_{L'} \) is also the unipotent radical of \( B_{L'} \). We can repeat the proof for \( Y_{L',\bar{s}} \) and see that they are also defined over \( O^{unr} \).

After the above discussion, we consider the \( \ell \)-modular system \((K,W(k),k)\) instead of a splitting \((K,O,k)\). For a split Levi subgroup \( L \) of \( G \), since the set \( O_U(L) \) consists with only one orbit under conjugation of a split maximal torus of \( L \), the Gelfand-Graev \( W(k) \)-lattice is unique, and we denote it by \( Y_L \). All the discussion above work for \( Y_L \). In particular, for an \( \ell \)-regular semisimple element \( \bar{s} \in L^* \), we denote by \( Y_{L,\bar{s}} \) the indecomposable projective direct summand \( b_{\bar{s}} \cdot Y_L \). Now we study the relation between \( Y_{L,\bar{s}} \) and \( Y_{L',\mu,s} \).

Corollary 3.8. Let \( \bar{s} \in L^* \) be a semisimple \( \ell \)-regular element, then:

\[
\text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) \cong b_{\bar{s}} \cdot \text{res}_{L'}^{L}(Y_L).
\]

Proof. We know directly from definition that for any semisimple \( \ell \)-regular \( s' \in G^* \):

\[
b_{s'} \cdot \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) \cong b_{s'} \cdot \text{res}_{L'}^{L}(Y_L),
\]

Meanwhile \( b_{s'} \cdot \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) \) is a projective \( W(k)[G'] \)-module, so it is free over \( W(k) \). Proposition 11.7 in [Brou] told us that \( b_{s'} \cdot \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) \otimes K = 0 \) if \( [s'] \neq [s] \) with \( s = i^{*}(\bar{s}) \), which means \( b_{s'} \cdot \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) = 0 \). Combine this with

\[
\bigoplus_{[s]} b_{s'} \cdot \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L) = \text{res}_{L'}^{L}(b_{\bar{s}} \cdot Y_L),
\]
where $[s']$ run over the semisimple conjugacy classes of $L^*$. We obtain the result.

**Proposition 3.9.** For a split Levi subgroup $L$ of $G$, let $L'$ be the split Levi subgroup $L \cap G'$ of $G'$. Denote by $Z(L)$ and $Z(L')$ the center of $L$ and $L'$ respectively. We have an equation:

$$\text{res}^L_{L'} Y_L = |Z(L) : Z(L')| \bigoplus_{[\mu] \in O_U(L')} Y_{L', \mu},$$

where $[\mu]$ denote the $T'$-orbit of $\mu$.

**Proof.** Let $B$ be a split Borel subgroup of $L$ and $B = B \cap L'$ the corresponding split Borel of $L'$, and $U'$ denotes the unipotent radical of $B'$, observing that $U'$ is also the unipotent radical of $B$. Fixing one non-degenerate character $\mu$ of $U'$, let $O_\mu$ be its $W(k[U'])$-lattice. By the transitivity of induction, we have:

$$\text{Stab}([\mu]) \subset \text{Stab}(Y_{L', \mu}) \subset \text{Stab}(Y_{L', \mu} \otimes \overline{\tau}),$$

where $\text{Stab}$ denotes the group of stabiliser. On the other hand, the proof of lemma 2.3 a) in [DiF] tells that

$$\text{Stab}(Y_{L', \mu} \otimes \overline{\tau}) \subset \text{Stab}([\mu]).$$

So the inclusion above is in fact a bijection. Combine this with the statement of lemma 2.3 a) in [DiF], we finish our proof.

**Proposition 3.10.** Fix a semisimple $\ell$-regular $s \in G^*$, define $S_{[s]}$ to be the set of semisimple $\ell$-regular $\tilde{G}^*$-conjugacy classes $[\tilde{s}] \subset \tilde{G}^*$ such that $i^*[\tilde{s}] = [s]$. Then

$$\bigoplus_{[\tilde{s}] \in S_{[s]}} \text{res}_{L'} Y_{L, \tilde{s}} = |Z(L) : Z(L')| \bigoplus_{\mu \in O_U(L')} Y_{L', \mu, s}$$

**Proof.** By definition that $Y_{L, \tilde{s}} = b_{\tilde{s}} \cdot Y_L$. Multiplying $b_s$ on both sides of the equation in Proposition $3.9$ and considering Corollary $3.8$ we conclude that for any $\ell$-regular semisimple $G^*$-conjugacy class $[s]$, $\bigoplus_{[\tilde{s}] \in S_{[s]}} \text{res}_{L'} Y_{L, \tilde{s}}$ is a projective direct summand of $|Z(L) : Z(L')| \bigoplus_{\mu \in O_U(L')} Y_{L', \mu, s}$. Meanwhile, let $S = \{S_{[s]} | s \in G^*, s \text{ semisimple } \ell \text{-regular}\}$, then Proposition $3.9$ can be written as:

$$\bigoplus_{S_{[s]} \in S} \bigoplus_{[\tilde{s}] \in S_{[s]}} \text{Res}_{L'} Y_{L, \tilde{s}} = |Z(L) : Z(L')| \bigoplus_{[s] \in O_U(L')} Y_{L', \mu, s}$$

Since there is a natural bijection between $S$ and the set of semisimple conjugacy classes $\{[s]\}$ in $G^*$, we deduce the equation desired.
3.2 Uniqueness of supercuspidal support

In this part, we will prove the main theorem 3.11 for this section.

**Theorem 3.11.** Let $L'$ be a standard split Levi subgroup of $G'$ and $\nu$ be a cuspidal $k$-representation of $L'$. Then the supercuspidal support of $\nu$ is unique up to $L'$-conjugation.

Let $P_\nu$ denote the $W(k)[L']$-projective cover of $\nu$. To prove the theorem above, we will follow the strategy below:

1. For any standard Levi subgroup $M'$ of $L'$, prove that $r_{M'}^L P_\nu$ is either equal to 0 or indecomposable.

2. Prove that there is only one unique standard split Levi subgroup $M'$ of $L'$, such that $r_{M'}^L P_\nu$ is cuspidal.

Let $(M', \theta)$ be a supercuspidal $k$-pair of $L'$. From the proof of Proposition 3.2 of [Hiss], we have that $(M', \theta)$ belongs to the supercuspidal support of $(L', \nu)$, if and only if $\text{Hom}(r_{M'}^L P_\nu, \theta) \neq 0$. Combining this fact with step 1 as above, we obtain that $r_{M'}^L P_\nu$ is the $W(k)[M']$-projective cover of $\theta$. Proposition 2.3 of [Hiss] states that an irreducible $k$-representation of $M'$ is supercuspidal if and only if its projective cover is cuspidal, hence Theorem 3.11 is equivalent to step 2.

**Remark 3.12.**

- The discussion above is true as well for Levi subgroups $L$ of $G$.

- Proposition 3.2 of [Hiss] concerns $k[L']$-projective cover, but from Proposition 42 of [Ser] we know that there is a surjective morphism of $k[L']$-modules from the $W(k)[L']$-projective cover to the $k[L']$-projective cover, and hence obtain the same result for $W(k)[L']$-projective cover.

**Proposition 3.13.** Let $\nu$ be an irreducible cuspidal $k$-representation of $L'$. There exists a simple $k[L']$-module $\tilde{\nu}$, and a surjective morphism $\text{res}^k_{L'} \tilde{\nu} \to \nu$. Furthermore, let $Y_{L, \tilde{s}}$ be the $W(k)[L']$-projective cover of $\tilde{\nu}$, where $\tilde{s} \in G^*$ is an $\ell$-regular semisimple element, then there exists $\mu \in O_{L'}(L')$ such that $Y_{L', \mu, s}$ is the $W(k)[L']$-projective cover of $\nu$.

**Proof.** By using Mackey formula to $\text{ind}^L_{L'} \nu$, we can find such $\tilde{\nu}$.

Since the restriction functor respects projectivity, we deduce the second part from Corollary 3.5 and Proposition 3.10.

Let $M'$ be a standard split Levi subgroup of $L'$. It is clear that $\mu|_{M'}$ belongs to $O_{L'}(M')$. Now consider the intersection $[s] \cap M^*$. As in the paragraph above Proposition 5.10 of [Helm], $[s] \cap M^*$ consists of one $M'$-conjugacy class or is empty, so does $[s] \cap M^*$. For the first case, $Y_{M', \mu|_{M'}, [s] \cap M^*}$ is well defined, and for the second case, we define it to be 0. From now on, we will always use $Y_{M', \mu, s}$ to simplify $Y_{M', \mu|_{M'}, [s] \cap M^*}$. We use the same manner to define $Y_{M, s}$.

**Proposition 3.14.** Let $\nu$ be an irreducible cuspidal $k[L']$-representation, and $\tilde{\nu}$, $Y_{L', \mu, s}$, $Y_{L, \tilde{s}}$ be as in Proposition 3.13. Then $r_{M'}^L Y_{L', \mu, s}$ is equal to 0 or indecomposable and isomorphic to $Y_{M, \mu, s}$ as $W(k)[M']$-module.

**Proof.** In the proof of Proposition 3.10 we know that $Y_{L', \mu, s}$ is a direct summand of $\text{Res}^L_{L'} (Y_{L, \tilde{s}})$. Observing that the unipotent radical of $M'$ is also the unipotent radical of $M$, we deduce directly from the definition that $r_{M'}^L (\text{res}^L_{L'} (Y_{L, \tilde{s}})) = \text{res}^M_{M'} (r_{M}^L Y_{L, \tilde{s}})$, and Proposition 5.10
in [Helm] states that $r^L_M(\pi_{\tilde{s},\tilde{\tilde{s}}}) = \mathcal{Y}_{M,\tilde{s}}$. The statements above, combining with the fact that parabolic restriction is exact and respects projectivity, implies that $r^L_M\mathcal{Y}_{L',\mu,s}$ is a projective direct summand of $\text{res}^M_M\mathcal{Y}_{M,\tilde{s}}$. Suppose $[\tilde{s}] \cap M^*$ is empty, then the same for $[\tilde{s}] \cap M^*$, hence we have $\mathcal{Y}_{M,\tilde{s}} = r^L_M\mathcal{Y}_{L',\mu,s} = 0$.

Now consider the second case. Suppose that $[\tilde{s}] \cap M^*$ is non-empty, which is a $M^*$-conjugacy class $[\tilde{s}']$ for $\tilde{s}' \in M^*$. Let $\mu'$ be the non-degenerate character $\text{res}^L_U\mu$, where $U_L$ and $U_M$ denote the unipotent radical of $L$ and $M$ respectively. Corollary 15.15 in [Bon] gives an equation:

$$r^L_M\mathcal{Y}_{L',\mu,s} \otimes \overline{\mathcal{K}} = \mathcal{Y}_{M',\mu',s'} \otimes \overline{\mathcal{K}}.$$ 

which means the $\ell$-regular quotient (see Definition 3.3) of $r^L_M\mathcal{Y}_{L',\mu,s}$ is indecomposable. Notice that Corollary 15.11 in [Bon] tells us that the sub-representation of $\text{Res}_L^M\mathcal{Y}_{M,\tilde{s}} \otimes \overline{\mathcal{K}}$ corresponding to $[\tilde{s}']$ has multiplicity one, and the equation above says that the irreducible sub-representation corresponding to $[\tilde{s}']$ of $r^L_M\mathcal{Y}_{L',\mu,s} \otimes \overline{\mathcal{K}}$ and $\mathcal{Y}_{M',\mu',s'} \otimes \overline{\mathcal{K}}$ coincide, hence these two projective direct summands of $\text{Res}_L^M\mathcal{Y}_{M,\tilde{s}}$ coincide each other.

We move on to the second step of Theorem 3.11. The statement of step 2 is true for $L$, hence there only left the proposition below to finish our proof:

**Proposition 3.15.** Let $\mathcal{Y}_{L',\mu,s}$, $\mathcal{Y}_{L,\tilde{s}}$, $\tilde{\mathcal{Y}}$ be as in Proposition 3.13, then for any standard split Levi $L'$ of $L$, we have $r^L_M\mathcal{Y}_{L',\mu,s} = \mathcal{Y}_{M',\mu,s}$ is cuspidal if and only if $r^L_M\mathcal{Y}_{L,\tilde{s}} = \mathcal{Y}_{M,\tilde{s}}$ is cuspidal.

**Proof.** The regular inclusion $i$ induces a bijection preserving partial order between standard Levi subgroups of $G$ and $G'$, the statement in the proposition is equivalent to say that for any split Levi $L'$ of $L'$,

$$r^L_M\mathcal{Y}_{L',\mu,s} = 0 \iff r^L_M\mathcal{Y}_{L,\tilde{s}} = 0.$$ 

The proof of Proposition 3.14 tells us

$$r^L_M\mathcal{Y}_{L',\mu,s} \leftrightarrow \text{res}^M_M\mathcal{Y}_{M,\tilde{s}},$$ 

hence the direction $\Rightarrow$ is clear.

Now consider the other direction. Notice that $r^L_M\mathcal{Y}_{L',\mu,s}$ is an $O[M']$-lattice, and definition 5.9 in [Geck] tells us that $r^L_M\mathcal{Y}_{L',\mu,s} = 0$ if and only if its $\ell$-regular quotient $\pi_{\ell}(r^L_M\mathcal{Y}_{L',\mu,s}) = 0$. By [Bon] Corollary 15.15], the $\overline{\mathcal{K}}[M']$-module $(\pi_{\ell}(r^L_M\mathcal{Y}_{L',\mu,s})) \otimes \overline{\mathcal{K}}$ is the sum of irreducible $\overline{\mathcal{K}}[M']$-submodules of $\text{ind}_U^L\mu$ corresponding to $[\tilde{s}] \cap M^*$, where $[\tilde{s}]$ denotes the $L^*$-conjugacy class. Hence $(\pi_{\ell}(r^L_M\mathcal{Y}_{L',\mu,s})) \otimes \overline{\mathcal{K}} = 0$ implies $[\tilde{s}] \cap M^* = 0$, which means $[\tilde{s}] \cap M^* = 0$, and $\mathcal{Y}_{M,\tilde{s}} = 0$.

\[\square\]

4 \textit{k-representations of p-adic groups \text{SL}_n(F)}

Let $F$ be a non-archimedean locally compact field of residual characteristic $p$, which is different from $\ell$. Let $M$ be a Levi subgroup of $G = \text{GL}_n(F)$, we always denote by $M'$ the Levi subgroup of $G'$ such that $M'M' = M'$. Let $\pi$ be an irreducible cuspidal $k$-representation of $M'$, by [Im] Proposition 2.2 there exists an irreducible $k$-representation $\pi'$ of $M$ such that $\pi'$ appears as a direct component of $\pi|_{M'}$, which is semisimple with finite length (a same proof
as in [Ta] can be generalised to the case when $\ell$ is positive as explained in [C]). Furthermore, any such $\pi$ is cuspidal, which follows from the fact that the unipotent radical of a Parabolic subgroup of $G$ lies in the kernel of the determinant function. The supercuspidal support of $\pi$ is unique up to $M$-conjugation ([V2]).

We prove in Section 4.1 that the supercuspidal support of $\pi'$ is also unique up to $M$-conjugation (Proposition 4.4), which is the first description of supercuspidal support. We will first generalise the definition of $n$-th derivative given by Bernstein and Zelevinsky in [BeZe] for complex representations of $GL_n(F)$ to $k$-representations of Levi subgroups $M'$ of $SL_n(F)$, which gives a link between the highest derivative of an irreducible representation and that of its supercuspidal support. Then we deduce the uniqueness of supercuspidal support of irreducible $k$-representations of $M'$ in Theorem 4.14., based on the result of Proposition 4.4, by considering its Whittaker model of a fixed non-degenerate character.

4.1 First description of supercuspidal support

**Lemma 4.1.** Let $\pi$ be an irreducible $k$-representation. If $\pi \otimes \chi \circ \det$ is supercuspidal for a $k$-quasicharacter $\chi$ of $F^\times$, then $\pi$ is supercuspidal.

**Proof.** Assume that there is a supercuspidal representation $\tau$ of a proper Levi $L$ of $M$ such that $\pi$ is an irreducible subquotient of $i_M^L \tau$. Then $\pi \otimes \chi \circ \det$ is a subquotient of $i_M^L \tau \otimes \chi \circ \det$, which follows from the equivalence

$$i_M^L \tau \otimes \chi \circ \det \cong (i_M^L \tau) \otimes \chi \circ \det.$$

The above equivalence is obtained from [§1,5.2,d][V1], by noticing that for any parabolic subgroup containing $L$, its unipotent radical is a subset of the kernel of the determinant function.

**Lemma 4.2.** Let $\pi'$ be an irreducible cuspidal $k$-representation of $M'$, and $\pi$ an irreducible $k$-representation of $M$ containing $\pi'$. Then $\pi'$ is supercuspidal if and only if $\pi$ is supercuspidal.

**Proof.** Let $L$ be a Levi subgroup of $M$, and $L' = L \cap M'$. We have $\text{res}_M^L r_M^L \pi \cong r_{L'}^M \text{res}_M^M \pi$, while $\text{res}_M^M \pi$ is a direct sum of $M$-conjugations of $\pi'$, hence the later one is zero, and we obtain that $\pi$ is cuspidal.

We assume that $\pi$ is non-supercuspidal, which means there exists a supercuspidal representation $\tau$ of a proper Levi subgroup $L$ of $M$, the representation $\pi$ is a subquotient of $i^L_M \tau$. Now by §5.2 [BeZe], we obtain:

$$\text{res}_M^M i^M_L \tau \cong i_{L'}^M \text{res}_L^L \tau.$$

There must be a direct component $\tau'$ of $i^L_M \tau$, $\pi'$ be an irreducible subquotient of $i^L_M \tau'$. Hence $\pi'$ is not supercuspidal, which contradicts with the assumption.

[Ta, Corollary 2.5] can be generalised to $k$-representations. We write this proposition here for convinient

**Proposition 4.3.** Let $\pi'$ be an irreducible cuspidal $k$-representation of $M'$. If $\pi_1, \pi_2$ two irreducible cuspidal $k$-representations of $M$, such that $\pi'$ appears as a direct component of $\text{res}_M^M \pi_1$ and $\text{res}_M^M \pi_2$ in common, then there exists a $k$-quasicharacter of $F^\times$ verifying that $\pi_1 \cong \pi_2 \otimes \chi \circ \det$.  

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Proposition 4.4. Let \( \pi' \) be an irreducible cuspidal k-representation of \( M' \), and \( \pi \) an irreducible cuspidal k-representation of \( M \) such that \( \pi \) contains \( \pi' \). Let \([L, \tau]\) be the supercuspidal support of \( \pi \), where \( L \) is a Levi subgroup of \( M \) and \( \tau \) an irreducible supercuspidal k-representation of \( L \). Let \( \tau' \) be an arbitrary direct component of \( \res_M^L \tau \). A supercuspidal pair belonging to the supercuspidal support of \( \pi' \) is M-conjugated to \( (L', \tau') \).

Proof. Let \( L_0' \) be a Levi subgroup of \( M' \) and \( \tau_0' \) an irreducible supercuspidal k-representation of \( L_0' \). Let \( \tau_0 \) be an irreducible k-representation of \( L_0 \) containing \( \tau_0' \), hence \( \tau_0 \) is supercuspidal as in Lemma 4.2.

Now suppose that \( \pi' \) is an irreducible subquotient of \( \res_{L_0}^{M'} \tau_0' \). By the same reason as in the proof of Lemma 4.2, we know that there must be an irreducible subquotient of \( \res_{L_0}^{M'} \tau_0' \), noted as \( \pi_0 \), such that \( \pi' \) is a direct component of \( \res_M^L \pi_0 \). From [13] Corollary 2.5, there exists a k-quasicharacter \( \chi \) of \( F^\times \) such that \( \pi_0 \cong \pi \otimes \chi \circ \det \). On the other hand, the supercuspidal support of \( \pi \otimes \chi \circ \det \) is the M-conjugacy class of \((L, \tau \otimes \chi \circ \det)\). Hence we may assume that \( L_0 = L \) and \( \tau_0 \cong \tau \otimes \chi \circ \det \), and deduce that \( \tau_0' \) is a direct component of \( \res_L^L \tau \otimes \chi \circ \det \cong \res_L^L \tau \). \( \square \)

4.2 The \( n \)-th derivative and parabolic induction

This section is a direct generalisation of the part of derivatives given in [BeZe] for \( \GL_n \). Nevertheless, except the convenience reason, we write this section because the author believe the notation system for \( G' \) is worthy to be introduced, which is different and complicated compared to that of \( \GL_n \). The complexity arises from the fact that the Levi subgroups \( M' \) cannot be written in the form of a product of \( \SL_n \) groups in lower rank, so a method of recursion can not be applied here.

Let \( n_1, \ldots, n_m \) be a family of integers, and let \( M_{n_1, \ldots, n_m} \) be the product \( \GL_{n_1} \times \cdots \times \GL_{n_m} \), which can be canonically embedded into \( \GL_{n_1 + \cdots + n_m} \). Let \( M'_{n_1, \ldots, n_m} \) be \( M_{n_1, \ldots, n_m} \cap \SL_{n_1 + \cdots + n_m} \), and \( P_{n_1} \) the mirabolic subgroup of \( \GL_{n_1} \). For any \( i \in \{1, \ldots, m\} \), let \( U_{n_i} \) be the subset of \( \GL_{n_i} \), consisted with upper-triangular matrix with 1 on the diagonal. We denote \( U_{n_1, \ldots, n_m} = U_{n_1} \times \cdots \times U_{n_m} \) by \( U_{n_1, \ldots, n_m} \).

Definition 4.5. Let \( n_1, \ldots, n_m \) be a family of positive integers, and \( s \in \{1, \ldots, m\} \). We define:

- the mirabolic subgroup at place \( s \) of \( M_{n_1, \ldots, n_m} \), as \( P_{(n_1, \ldots, n_m), s} = \GL_{n_1} \times \cdots \times \GL_{n_{s-1}} \times \GL_{n_{s+1}} \times \cdots \times \GL_{n_m} \);
- the mirabolic subgroup at place \( s \) of \( M'_{n_1, \ldots, n_m} \), as \( P'_{(n_1, \ldots, n_m), s} = \GL_{n_1} \times \cdots \times \GL_{n_{s-1}} \times \GL_{n_{s+1}} \times \cdots \times \GL_{n_m} \cap M'_{n_1, \ldots, n_m} \).

We fix \( \theta_i \) a non-degenerate character of \( U_{n_i} \). It is clear that \( U_{n_1, \ldots, n_m} \) is a subgroup of \( P_{(n_1, \ldots, n_m), s} \) and \( P'_{(n_1, \ldots, n_m), s} \) for any \( s \in \{1, \ldots, m\} \). Let \( V_{n_s-1} \) be the additive group of k-vector space with dimension \( n_s-1 \), which can be embedded canonically as a normal subgroup in \( U_{n_1} \times \cdots \times U_{n_m} \). The subgroup \( V_{n_s-1} \) is normal both in \( P_{(n_1, \ldots, n_m), s} \) and \( P'_{(n_1, \ldots, n_m), s} \), furthermore, we have \( P_{(n_1, \ldots, n_m), s} = M_{n_1, \ldots, n_{s-1}, n_s-1, \ldots, n_m} V_{n_s-1} \) and \( P'_{(n_1, \ldots, n_m), s} = M'_{n_1, \ldots, n_{s-1}, n_s-1, \ldots, n_m} V_{n_s-1} \). Under the notation system as explained above, the notation \( M_0 \) is meaningful in this section, which denotes a subgroup of \( P_{(1,1)} \), hence is the trivial group.

Denote \( \theta \) a k-character of \( U_{n_1} \times \cdots \times U_{n_m} \). For any \( k \)-representation \((E, \rho)\) inside \( \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}) \), where \( E \) is the representation space of \( \rho \), let \( E_{\theta} \) denote the subspace of \( E \) generated by elements in form of \( \rho(g)a - \theta(g)a \), where \( g \in V_{n_s-1}, a \in E \). We define the
Definition 4.6. Fix a non-degenerate character $\theta$ of $U_{n_1} \times \cdots \times U_{n_m}$.

- Let $(E, \rho) \in \text{Rep}_k(P'_{(n_1, \ldots, n_m), s})$, define
  $$\Psi_s^- : \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}) \to \text{Rep}_k(M'_{n_1, \ldots, n_s-1, \ldots, n_m}),$$
  the canonical projection from $E$ to $E(1, s)$;
- Let $(E, \rho) \in \text{Rep}_k(M'_{n_1, \ldots, n_s-1, \ldots, n_m})$, and write $P'_{(n_1, \ldots, n_m), s} = M'_{n_1, \ldots, n_s-1, \ldots, n_m} \cdot V_{n_s-1}$. Define
  $$\Psi_s^+ : \text{Rep}_k(M'_{n_1, \ldots, n_s-1, \ldots, n_m}) \to \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}),$$
  which maps $(E, \rho)$ to $(E, \Psi_s^+(\rho))$ by $\Psi_s^+(\rho)(mg)(a) = \rho(m)(a)$, for any $m \in M'_{n_1, \ldots, n_s-1, \ldots, n_m}, g \in V_{n_s-1}$ and $a \in E$;
- Let $(E, \rho) \in \text{Rep}_k(P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s})$, define
  $$\Phi_{\theta,s}^- : \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}) \to \text{Rep}_k(P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}),$$
  which maps $E$ to $E(\theta, s)$, and consider the restricted representation from $M'_{n_1, \ldots, n_s-1, \ldots, n_m}$ to $P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}$;
- Let $(E, \rho) \in \text{Rep}_k(P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s})$. Consider the composed canonical embedding
  $$P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s} \hookrightarrow M_{n_1, \ldots, n_s-1, \ldots, n_m} \hookrightarrow P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s},$$
  and denote $(E, \rho_{\theta})$ the $k$-representation of $P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s} \cdot V_{n_s-1}$ under the above embedding, such that $\rho_{\theta}(pg) = \theta(g)\rho(p)$, for any $p \in P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}, g \in V_{n_s-1}$.
  Define
  $$\Phi_{\theta, s}^+ : \text{Rep}_k(P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}) \to \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}),$$
  by taking $\Phi_{\theta, s}^+(\rho) = \text{ind}_{P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}}^{P'_{(n_1, \ldots, n_s-1, \ldots, n_m), s}} V_{n_s-1} \rho_{\theta}$.

Remark 4.7. By the reason that for any $m \in \mathbb{Z}$ the group $V_m$ is a limite of pro-$p$ open compact subgroups, the four functors defined above are exact.

The notion of derivatives is well defined for $k$-representations of $G$, now we consider the parallel operator of derivatives for Levi subgroups of $G'$.

Definition 4.8. Fix a non-degenerate character $\theta$ of $U_{n_1} \times \cdots \times U_{n_m}$. Let $(E, \rho) \in \text{Rep}_k(P'_{(n_1, \ldots, n_m), s})$, for any integer $s \in \{1, \ldots, m\}$ and $1 \leq d \leq n_1 + \ldots + n_s$, we define the derivative $\rho_{\theta,s}^{(d)}$:

- when $1 \leq d \leq n_s$, define
  $$\rho_{\theta,s}^{(d)} = \Psi_s^- \circ (\Phi_{\theta,s}^-)^{d-1} : \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}) \to \text{Rep}_k(M'_{n_1, \ldots, n_s-d, \ldots, n_m}, s);$$
- when $n_s + 1 \leq d = n_s + \ldots + n_{s-l} + n'$, where $0 \leq l \leq s-1$ and $1 \leq n' \leq n_{s-l-1}$, then
  $$\rho_{\theta,s}^{(d)} = \Psi_{s-l}^- \circ (\Phi_{\theta,s-l}^-)^{n'-1} \circ \Phi_{\theta,s-l}^- \circ \ldots \circ \Phi_{\theta,s-l}^- \circ \Phi_{\theta,s-l}^- \circ \ldots \circ \Phi_{\theta,s-l}^- \circ \rho,$$
  and
  $$\rho_{\theta,s}^{(d)} : \text{Rep}_k(P'_{(n_1, \ldots, n_m), s}) \to \text{Rep}_k(M'_{n_1, \ldots, n_{s-l}-1-n', n_{s-l}+1, \ldots, n_m}).$$
Definition 4.9. Suppose $m$ is bigger than 2. To simplify our notations, we introduce $\text{Ind}^{m-1}_m : \text{Rep}_k(G_1) \to \text{Rep}_k(G_2)$ according to different cases:

- When $G_1 = M'_m$ and $G_2 = M'_{m-1} + M_m$, we embed $G_1$ into $G_2$ as in the figure case I, and $\text{Ind}^{m-1}_m$ is defined as $i_{U,1}$, and the later one is defined as in §1.8 of [BeZe];

- When $G_1 = P'_{(n_1,\ldots,n_m),m}$ and $G_2 = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}$, we embed $G_1$ into $G_2$ as in the figure case II, and $\text{Ind}^{m-1}_m$ is defined as $i_{U,1}$;

- When $G_1 = P'_{(n_1,\ldots,n_{m-1}),m-1}$ and $G_2 = P'_{(n_1,\ldots,n_{m-1}+n_m),m-1}$, we embed $G_1$ into $G_2$ as in the figure case III, and $\text{Ind}^{m-1}_m$ is defined as $i_{U,1} \circ \varepsilon$. Here $\varepsilon$ is a character of $P'_{(n_1,\ldots,n_m),m-1}$. Write $g \in P'_{(n_1,\ldots,n_m),m-1} \subset M'_{n_1,\ldots,n_m}$ as $(g_1,\ldots,g_m)$, define $\varepsilon(g) = |\det(g_m)|$, the absolute value of $\det(g_m)$, which is a power of $p$. This $k$-character is well defined since $p \neq l$.

Proposition 4.10. Assume that $\rho_1 \in \text{Rep}_k(M'_{n_1,\ldots,n_m})$, $\rho_2 \in \text{Rep}_k(P'_{(n_1,\ldots,n_m),m})$, and $\rho_3 \in \text{Rep}_k(P'_{(n_1,\ldots,n_{m-1}),m-1})$. The functor $\text{Ind}^{m-1}_n$ is defined as in [4.7] according to different cases, and we have the following properties:

1. In $\text{Rep}_k(P'_{(n_1,\ldots,n_{m-1}+n_m),m-1})$, there exists an exact sequence:
   $$0 \to \text{Ind}^{m-1}_m(\rho_1|_{P'_{m-1,m-1}}) \to (\text{Ind}^{m-1}_m(\rho_1)|_{P'_{m-1,m-1}} \to \text{Ind}^{m-1}_m(\rho_1|_{P'_{m,m}}) \to 0,$$
where $P'_{m,m-1}$ denotes $P'_{(n_1,...,n_m),m-1}$, $P'_{m-1,m-1}$ denotes $P'_{(n_1,...,n_{m-1}+n_m),m-1}$, and $P'_{m,m}$ denotes $P'_{(n_1,...,n_m),m}$.

2. When $2 \leq m$, let $\hat{\theta}$ be a non-degenerate character of $U_{n_1} \times ... \times U_{n_{m-2}} \times U_{n_{m-1}+n_m}$, such that $\hat{\theta}|_{U_{n_1} \times ... \times U_{n_m}} \cong \theta$. We have equivalences:

- $\text{ind}_{m-1} \circ \Psi_m^{-1}(\rho_2) \cong \Psi_{m-1}^{-1}(\rho_2)$;
- $\text{ind}_{m-1} \circ \Phi_{\theta,m}^{-1}(\rho_3) \cong \Phi_{\theta,m-1}^{-1}(\rho_3)$.

3. We have an equivalence:

$$\Psi_{m-1}^{-1} \circ \text{ind}_{m-1}(\rho_3) \cong \text{ind}_{m-1} \circ \Psi_{m-1}^{-1}(\rho_3),$$

and an exact sequence:

$$0 \rightarrow \text{ind}_{m-1} \circ \Phi_{\theta,m}^{-1}(\rho_3) \rightarrow \Phi_{\theta,m-1}^{-1}(\rho_3) \rightarrow \text{ind}_{m-1}(\Psi_{m-1}^{-1}(\rho_3)|_{P'}) \rightarrow 0,$$

where $P' = P'_{(n_1,...,n_{m-1}+n_m),m}$.

Proof. As proved in the Appendix of [C], Theorem 5.2 in [BeZe] holds for $k$-representations of $M'$. Now let $n = n_1 + ... + n_m$.

For (1): Let $M' = M'_1,...,n_m$ be embedded into $G' = M'_1,...,n_{m-1}+n_m$ as in definition 4.9, figure 1. Define functor $F$ as $F(\rho_1) = \rho_1|_{P'_{(n_1,...,n_{m-1}+n_m),m-1}}$, where the functor $F$ is equivalently defined as in §5.1 [BeZe] under the following setting (the notation $M$ in [BeZe] corresponds $M'$ here):

$$U \text{ as in figure 1, } \hat{\theta} = 1, P = M'U;$$

$$N = P'_{(n_1,...,n_{m-1}+n_m),m-1}, V = \{e\}, Q = NV.$$
and \( sgn(\sigma) \) denotes the signal of \( \sigma \), and \( sgn(\sigma) L_{m} \) denotes an element in \( M_{n_{1}, \ldots, n_{m}} \), which is equal to identity on the first \( m - 1 \) blocs, and a scalar matrix with value of \( sgn(\sigma) \) on the last bloc. Now we consider condition (4) from 5.1 [BeZe]:

- Since \( V = \{ e \} \), it is clear that \( \omega(P), \omega(M') \) and \( \omega(U) \) are decomposable with respect to \( (N, V) \);
- Let us consider \( \omega^{-1}(Q) = \omega^{-1}(N) \).

To study the intersection \( \omega^{-1}(N) \cap (M \cdot U) \), first we consider a Levi subgroup \( M'_{n_{1}, \ldots, n_{m-1}+n_{m}-1,1} \) of \( G' \) and the corresponding standard parabolic subgroup

\[
P' = M'_{n_{1}, \ldots, n_{m-1}+n_{m}-1,1} \cdot V_{n_{m-1}+n_{m}-1},
\]

where \( V_{n_{m-1}+n_{m}-1} \) denotes the unipotent radical of \( P' \). We have \( N \subset P' \), hence \( \omega^{-1}(N) \subset \omega^{-1}(P') \). As in 6.1 of [BeZe], after fix a system \( \Omega \) of roots, and denote \( \Omega' \) the set of positive roots. Then by Proposition in 6.2 [BeZe], we can write \( \omega^{-1}(P') = G(S) \) and \( P = G(P), U = U(M) \) in the manner as in 6.1 [BeZe], where \( S, P \) and \( M \) are convex subset of \( \Omega \). So by a same computation as in Proposition in 6.1 [BeZe], we have:

\[
\omega^{-1}(P') \cap P = (\omega^{-1}(P') \cap M') \cdot (\omega^{-1}(P') \cap U).
\]

Notice that \( \omega^{-1}(P') \cap U = \omega^{-1}(N) \cap U \), we deduce that:

\[
\omega^{-1}(N) \cap P = (\omega^{-1}(N) \cap M') \cdot (\omega^{-1}(N) \cap U).
\]

In the formula of \( \Phi_{\varepsilon} \) in 5.2 [BeZe], since \( U \cap \omega^{-1}(N) = U \), the characters \( \varepsilon_{1} = \varepsilon_{2} = 1 \). Hence we obtain the exact sequence desired.

For (2). In this part, the functor \( ind_{m}^{m-1} \) was defined differently according to different cases of Definition 4.9. First we consider the first part concerning about \( \Psi_{m} \). Define functor \( F \) as \( \Psi_{m-1} \circ ind_{m}^{m-1} \). We write \( F \) as in §5.1 [BeZe] in the situation:

\[
G = P'_{(n_{1}, \ldots, n_{m-1}+n_{m}), m-1}, M = P'_{(n_{1}, \ldots, n_{m}), m}, U \text{ are defined in Definition 4.9 case II},
\]

\[
N = M'_{n_{1}, \ldots, n_{m-1}+n_{m}-1} \cdot V = V_{n_{m-1}+n_{m}-1}.
\]

Condition (1) and (2) of §5.1 [BeZe] is clear. Since \( Q = G \), and there is only one \( Q \)-orbit on \( X = P \setminus G \), conditions (3), (4) hold trivially. Thus we obtain the equivalence:

\[
ind_{m}^{m-1} \circ \Psi_{m-1}^{\rho_{2}} = \Psi_{m-1}^{\rho_{2}} \circ ind_{m}^{m-1}.
\]

For the second part concerning about \( \Phi_{\varepsilon, m-1} \). Define functor \( F \) as \( \Phi_{\varepsilon, m-1} \circ ind_{m}^{m-1} \). We write \( F \) as in §5.1 [BeZe] in the situation:

\[
G = P'_{(n_{1}, \ldots, n_{m-1}+n_{m}), m-1},
\]

\[
N = P'_{(n_{1}, \ldots, n_{m-1}+n_{m}-1), m-1} \cdot V = V_{n_{m-1}+n_{m}-1},
\]

and \( M, U \) are defined as the case II of 4.9 Same orbits and same computation as in Proposition 4.13 (b) of [BeZe] can be applied here.

For part (3). Define functor \( F = \Psi_{m-1}^{\rho_{2}} \circ ind_{m}^{m-1} \), we have (in the manner of §5.1 [BeZe]):

\[
G = P'_{(n_{1}, \ldots, n_{m}), m}.
\]
N = M_{n1,\ldots,nm-1}, V = V_{nm-1},
and M, U as in [BeZa] case II. There is only one Q-orbit on P\setminus G, and condition (1) − (4) and (+) in §5.1 [BeZa] hold. Notice that \( \varepsilon \circ \Psi_{m-1} \simeq \Psi_{m-1} \) (\( \varepsilon \) is defined in Definition 4.9). After applying theorem 5.2 of [BeZa], we obtain the equivalence:

\[
\Psi_{m-1} \circ \text{ind}_m^{m-1} \rho_3 \simeq \text{ind}_m^{m-1} \circ \Psi_{m-1} \rho_3.
\]

Define functor \( F = \Phi^{-}_{\theta,m-1} \circ \text{ind}_m^{m-1} \). We have (in the manner of §5.1 [BeZa]):

\[
G = P'_{(n1,\ldots,nm),m},
\]

\[
N = P'_{(n1,\ldots,nm-1+n_{m-1}),m-1}, V = V_{nm-1+n_{m-1}},
\]

and M, U are defined as the case II of Definition 4.9. As in the proof of part (2), the group Q has two orbits on P\setminus G: the closed one P \cdot e and the open one P^{-1} \cdot \omega_0. The condition (4) can be justified as part (2), and condition (+) is clear since \( \omega_0(U) \cap V = 1 \). Now we apply theorem 5.2 [BeZa]. The functor corresponds to the orbit P \cdot e is \( \text{ind}_m^{m-1} \circ \Phi^{-}_{\theta,m-1} \) by noticing \( \varepsilon \circ \Phi^{-}_{\theta,m-1} \simeq \Phi_{\theta,m-1} \circ \varepsilon \). Now we consider the functor corresponds to the orbit P \cdot \omega_0^{-1}. Following the notation as §5.1 [BeZa], the character \( \psi' = \omega_0^{-1}(\psi|_{M\omega_0^{-1}(V)}) \) is trivial. The character \( \varepsilon_1 \) is trivial, and \( \varepsilon_2 \simeq \varepsilon^{-1} \). Hence the functor corresponded to this orbit is

\[
\text{ind}_m^{m-1} \circ \text{res}_{P_1}^{M_{n1,\ldots,nm-1+n_{m-1}}} \circ \Psi^{-}_{m-1},
\]

from which we deduce the exact sequence desired. □

**Corollary 4.11.** The functor \( \text{ind}_m^{m-1} \) is defined respectively to the corresponding cases as in Definition 4.9.

1. Let \( \rho \in \text{Rep}_k(P'_{(n1,\ldots,nm),m}) \), and \( \theta, \dot{\theta} \) be as in Proposition 4.10 part (2). Assume that \( 1 \leq i \leq n_m \), then we have an equivalence about taking i-th derivative

\[
(\text{ind}_m^{m-1} \rho)^{(i)}_{\theta,m-1} \simeq \text{ind}_m^{m-1} \rho_{\theta,m}^{(i)},
\]

2. Let \( \rho \in \text{Rep}_k(P'_{(n1,\ldots,nm),m-1}) \). Assume that \( 1 \leq i \leq n_{m-1} + n_m \), then \( (\text{ind}_m^{m-1} \rho)^{(i)}_{\theta,m-1} \) is filtrated by \( \text{ind}_m^{m-1}((\rho_{\theta,m}^{(i-j)})_{\theta,m-1}) \), where \( [1, i - n_m] \leq j \leq i \) and \( [1, i - n_m] \) denotes the bigger integer;

3. Let \( \rho \in \text{Rep}_k(M'_{n1,\ldots,nm}) \). Assume that \( i \geq 0 \), then \( (\text{ind}_m^{m-1} \rho)^{(i)}_{\theta,m-1} \) is filtrated by \( \text{ind}_m^{m-1}((\rho_{\theta,m}^{(i-j)})_{\theta,m-1}) \), where \( [1, i - n_m] \leq j \leq i \);

4. Let \( \rho \in \text{Rep}_k(M'_{n1,\ldots,nm}) \), there is an equivalence:

\[
(\text{ind}_1 \circ \cdots \circ \text{ind}_m^{m-2} \circ \text{ind}_m^{m-1} \rho)^{(n1,\ldots,nm)}_{(1)} \simeq (\cdots ((\rho_{(n1,\ldots,nm)})_{(1)})_{(1)} \cdots)_{(1)}.
\]

**Proof.** Part (1) follows from the exactness of \( \Phi_{\theta,m}, \Psi_{m}^- \) and 4.10 (2); (2) from (1) and 4.10 (3), (3) from (1), (2), and 4.10 (1). Part (4) follows from (3), by noticing that

\[
\text{ind}_1 \circ \cdots \circ \text{ind}_m^{m-2} \circ \text{ind}_m^{m-1} \rho \simeq i_{M_{n1,\ldots,nm}'} \rho.
\]

In fact, this is the transitivity of parabolic induction. □
4.3 Uniqueness of supercuspidal support

We take the same notations as in Section 4.2.

Proposition 4.12. Let $\tau \in \text{Rep}_k(\text{M}_n^{', \ldots , , n_m})$, and $\theta$ a non-degenerate character of $U_{n_1, \ldots , n_m}$. Then $\tau_{\theta,m}^{(n_1 + \cdots + n_m)} \neq 0$ is equivalent to say that $\text{Hom}_k[\text{U}_{n_1, \ldots , n_m}](\tau, \theta) \neq 0$. In particular, this is equivalent to say that $(\text{U}_{n_1, \ldots , n_m}, \theta)$-coinvariants of $\tau$ is non-trivial.

Proof. In this proof, we use $U$ to denote $U_{n_1, \ldots , n_m}$. For the first equivalence, notice that $\Phi_{\theta,m}^\tau(\tau) \neq 0$ is equivalent to say that $(V_{n_{m-1}}, \theta)$-coinvariants of $\tau$ is non-trivial. For $1 \leq s \leq n_m - 1$, let $V_s$ be the subgroup of $U$ consisting with the matrices with non-zero coefficients only on the $(s + 1)$-th line and the diagonal. Let $W$ denote the representation space of $\tau$. The space of $\tau_{\theta,m}^{(n_1 + \cdots + n_m)}$ is isomorphic to the quotient of $W$ by the subspace $W_\theta$ generated by $g_s(w) - \theta(g_s)w$, for every $s$ and $g_s \in V_s$, $w \in W$. Meanwhile, since the subgroups $V_s$'s generate $U$ and $\theta$ is determined by $\theta|_{V_s}$ while considering every $s$, the subspace $W_\theta$ of $W$ is isomorphic to the subspace generated by $g(w) - \theta(g)w$, where $g \in U$. Hence $\tau_{\theta,m}^{(n_1 + \cdots + n_m)} \neq 0$ is equivalent to say that $(U_{n_1, \ldots , n_m}, \theta)$-coinvariants of $\tau$ is non-trivial. The second equivalence is clear, since the $(U, \theta)$-coinvariants of $\tau$ is the largest quotient of $\tau$ such that $U$ acts as a multiple of $\theta$. \hfill $\Box$

Proposition 4.13. Let $\tau \in \text{Rep}_k(\text{M}_n^{', \ldots , , n_m})$, and $\rho$ be a subquotient of $\tau$. Let $\theta$ be a non-degenerate character of $U_{n_1, \ldots , n_m}$, and $\rho_{\theta,m}^{(n_1 + \cdots + n_m)}$ is non-trivial, then $\tau_{\theta,m}^{(n_1 + \cdots + n_m)}$ is non-trivial.

Proof. We consider the $(n_1 + \cdots + n_m)$-th derivative functor corresponding to the non-degenerate character $\theta$, from the category $\text{Rep}_k(\text{M}_n^{', \ldots , , n_m})$ to the category of $k$-vector spaces, which maps $\tau$ to $\tau_{\theta,m}^{(n_1 + \cdots + n_m)}$. By Definition 4.13 and Remark 4.17, this functor is a composition of functors $\Psi_\tau$ and $\Phi_{\theta,m}^\tau$, hence is exact, from which we deduce that $\tau_{\theta,m}^{(n_1 + \cdots + n_m)}$ is non-trivial. \hfill $\Box$

Theorem 4.14. Let $M'$ be a Levi subgroup of $G'$, and $\rho$ an irreducible $k$-representation of $M'$. The supercuspidal support of $\rho$ is a $M'$-conjugacy class of one unique supercuspidal pair.

Proof. Since the cuspidal support of irreducible $k$-representation is unique, to prove the uniqueness of supercuspidal support, it is sufficient to assume that $\rho$ is cuspidal. Let $\pi$ be an irreducible cuspidal $k$-representation of $M$, such that $\rho$ is a sub-representation of $\text{res}_M^M \pi$. Let $(L, \pi)$ be a supercuspidal pair of $M$, and $[L, \pi]$ consists the supercuspidal support of $\pi$. We have $\text{res}_M^M \pi \cong \oplus_{i \in I} \pi_i$, where $I$ is a finite index set. According to Proposition 4.4, the supercuspidal support of $(M', \rho)$ is contained in the union with respect to $i \in I$ of $M'$-conjugacy class of $(L', \pi_i)$. To finish the proof of our theorem, it remains to prove that there exists one unique $i_0 \in I$ such that $(L', \pi_{i_0})$ is contained in the supercuspidal support of $(M', \rho)$.

After conjugation by $G'$, we could assume that $M' = M_{n_1, \ldots , n_m}$ and $L' = M_{k_1, \ldots , k_l}$ for a family of integers $m, l, n_1, \ldots , n_m, k_1, \ldots , k_l \in N^*$. There exists a non-degenerate character $\theta$ of $U = U_{n_1, \ldots , n_m}$, such that $\rho|_{\theta,m}^{(n_1 + \cdots + n_m)} \neq 0$. In fact, let $\theta$ be any non-degenerate character of $U$ and write $\text{res}_M^M \pi \cong \oplus_{s \in S} \pi_s$, where $S$ is a finite index set. We have:

$$\pi^{(n_1 + \cdots + n_m)} \cong (\pi|_M|_{\theta,m}^{(n_1 + \cdots + n_m)}) \cong \oplus_{s \in S} \pi_s|_{\theta,m}^{(n_1 + \cdots + n_m)},$$

where $\pi_{\theta,m}^{(n_1 + \cdots + n_m)}$ indicates the $(n_1 + \cdots + n_m)$-th derivative of $\pi$. As in Section [III, 5.10, 3] of [Y], we have $\dim(\rho|_{\theta,m}^{(n_1 + \cdots + n_m)}) = 1$, hence there exists one element $s_0 \in S$ such that

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\( (\pi_{s_0})^{(n_1+\cdots+n_m)}_{(t,\theta),m} \neq 0 \). Notice that \( \rho \) are isomorphic to some \( \pi_{s_0} \), which means there exists a diagonal element \( t \in M \), such that the \( t \)-conjugation \( t(\pi_{s_0}) \cong \rho \). The character \( t(\theta) \) is also non-degenerate of \( U \), and we have \( (t(\pi_{s_0}))^{(n_1+\cdots+n_m)}_{(t,\theta),m} \cong (\pi_{s_0})^{(n_1+\cdots+n_m)}_{(t,\theta),m} \) as \( k \)-vector spaces.

We conclude that \( \dim \rho^{(n_1+\cdots+n_m)}_{(t,\theta),m} = 1 \). To simplify the notations, we assume \( t = 1 \).

If \( \rho \) is a subquotient of \( L_i^\ell \tau_i \) for some \( i \in I \). By (4) of Corollary 4.11 and Proposition 4.13, the derivative \( \tau^{(n_1+\cdots+n_i)}_{i,\theta,m} \neq 0 \). By section [§III, 5.10, 3)] of [V1], the derivative \( \tau^{(n_1+\cdots+n_i)}_{\theta,m} \) = 1, which means the dimension of \( (U_{L_i},\theta) \)-coinvariants of \( \tau \) is 1 (by 4.12). Notice that the \( (U_{L_i},\theta) \)-coinvariants of \( \tau \) is the direct sum of \( (U_{L_i},\theta) \)-coinvariants of \( \tau_i \) for every \( i \in I \), which implies that there exists one unique \( i_0 \in I \) whose \( (U_{L_i},\theta) \)-coinvariants is non-zero with dimension 1. By Proposition 4.12 and Proposition 4.13, this is equivalent to say that there exists one unique \( i_0 \in I \), such that the derivative \( \tau^{(n_1+\cdots+n_m)}_{i_0,\theta,m} \neq 0 \). \( \square \)

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