Abstract. The aim of this short note is to explain how the arguments of the “closing lemma with time control” of F. Abdenur and S. Crovisier [AC12] can be used to answer Question 1 of the article “Instability for the rotation set of homeomorphisms of the torus homotopic to the identity” of S. Addas-Zanata [AZ04].

In this short note, we explain how to get a $C^1$ version of a perturbation result of the rotation set of homeomorphisms of the torus homotopic to the identity, obtained by S. Addas-Zanata in [AZ04]: consider some diffeomorphism $f$ of the torus, isotopic to the identity, and suppose that some extreme point $(t, \omega)$ of the rotation set of $f$ has at least one irrational coordinate. Then there exists a perturbation $g$ of $f$, which is arbitrarily $C^1$-close to $f$, such that the rotation set of $g$ contains some vector that was not in the rotation set of $f$.

We will use the notations of [AZ04]. Let us recall the most useful ones: we will denote $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ the flat torus. The space $D^1(T^2)$ will be the set of $C^1$-diffeomorphism of the torus $T^2$ homotopic to the identity, endowed with the classical $C^1$ topology on compact spaces; $D^1(\mathbb{R}^2)$ will be the set of lifts to the plane of elements of $D^1(T^2)$. Given $\tilde{f} \in D^1(\mathbb{R}^2)$, its rotation set will be defined as

$$\rho(\tilde{f}) = \bigcap_{i=1}^{\infty} \bigcup_{n \geq i} \left\{ \frac{\tilde{f}^n(x) - \tilde{x}}{n} \mid \tilde{x} \in \mathbb{R}^2 \right\}.$$ 

For $\tilde{x} \in \mathbb{R}^2$, we will denote

$$\rho(\tilde{x}, n, \tilde{f}) = \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n}$$

the rotation vector of the segment of orbit $\tilde{x}, \tilde{f}(\tilde{x}), \ldots, \tilde{f}^n(\tilde{x})$, and when it is well defined (for example for a periodic point),

$$\rho(\tilde{x}, \tilde{f}) = \lim_{n \to +\infty} \rho(\tilde{x}, n, \tilde{f}).$$

We will also consider $\omega$ a volume or a symplectic form on $T^2$, whose lift to $\mathbb{R}^2$ will also be denoted by $\omega$.

We will prove the following result.

**Theorem 1.** Let $\tilde{f} \in D^1(\mathbb{R}^2)$ be such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin \mathbb{Q}^2$. Then there exists $\tilde{g} \in D^1(\mathbb{R}^2)$, arbitrarily $C^1$-close to $f$, such that $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$ (and in particular, $\rho(\tilde{g}) \neq \rho(\tilde{f})$).

Moreover, if $\tilde{f}$ preserves $\omega$, then $\tilde{g}$ can be supposed to preserve it too.
Figure 1. If the rotation vector of the initial orbit (in black) is in \{L > 0\}, then the rotation vector of one of the two pseudo orbits (in red and in blue) too.

We will prove this theorem by replacing the \(C^0\) perturbation result of [AZ04] by a closing lemma in topology \(C^1\), obtained by adapting the arguments of Theorem 6 of [AC12].

**Lemma 2** (Closing lemma with rotation control). Let \(\tilde{f} \in D^1(T^2)\), \(L : \mathbb{R}^2 \to \mathbb{R}\) a non-trivial affine form, and \(V\) a \(C^1\)-neighbourhood of \(f\). Then, there exists \(N \in \mathbb{N}\) such that for every non-periodic point \(x\) of \(f\), there exists a neighbourhood \(V\) of \(x\) such that if \(n \geq N\) and \(y \in V\) are such that \(f^n(y) \in V\) and \(L(\rho(\tilde{y}, n, \tilde{f})) > 0\), then there exists \(g \in V\) such that \(y\) is a periodic point of \(g\) satisfying \(L(\rho(\tilde{y}, \tilde{g})) > 0\).

Moreover, if \(f\) preserves \(\omega\), then \(g\) can be supposed to preserve it too.

The idea of the proof of this lemma is identical to that of Theorem 6 of [AC12], by replacing the dichotomy “\(\ell\) divides / does not divide the length of the orbit” by the dichotomy “\(L(\rho(\tilde{x}, n, \tilde{f})) > 0 / L(\rho(\tilde{x}, n, \tilde{f})) \leq 0\)”. More precisely, the proof of the connecting lemma of S. Hayashi [Hay97] builds a “closable” pseudo-orbit from a recurrent orbit of \(f\), by making shortcuts in this orbit; each time such a shortcut is performed there are two possibilities of creating a new pseudo-orbit (see Figure 1). If the initial orbit belongs to the set \{\(L(\rho) > 0\}\), then at least one of these two new pseudo-orbits also belongs to the set \{\(L(\rho) > 0\}\} (as the rotation vector of the initial orbit is a barycentre of the two new ones).

**Proof of Lemma 2.** Simply remark that Proposition 4 of [AC12] still holds when condition

3. The length of the periodic pseudo-orbit \((y_1, \ldots, y_n = y_0)\) is not a multiple of \(\ell\)

is replaced by the condition

3. The periodic pseudo-orbit \((y_1, \ldots, y_n = y_0)\) satisfies \(L(\tilde{y}_n - \tilde{y}_0) > 0\).

The rest of the proof is identical to Section 3.3.1 of [AC12]. \(\square\)

We now explain how this connecting lemma with rotation control can be applied to adapt the proof of Theorem 1 of [AZ04] to the \(C^1\) case. Let us quickly recall the main arguments of the proof in the \(C^0\) case. As the rotation set is convex [MZ89], there exists a supporting line of \(\rho(\tilde{f})\) at \((t, \omega)\), in other words an affine

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1Note that in general, this period is different from \(n\).

2A pseudo-orbit is called closable if Pugh’s algebraic lemma (Lemma 4 of [AC12], see also [Pug67]) can be applied simultaneously to every jump of the pseudo-orbit, to make it become a real orbit.

3This corresponds to the initial argument of [AC12]: “If \(\ell\) does not divide the length of the initial orbit, then it also does not divides the length of at least one of these two new pseudo-orbits”.

4To be rigorous here, pseudo-orbits must be considered in the cover \(\mathbb{R}^2\) and perturbations of diffeomorphisms performed in \(T^2\).
map $L : \mathbb{R}^2 \to \mathbb{R}$ such that $L(t, \omega) = 0$ and $L(v) \leq 0$ for every $v \in \rho(\tilde{f})$. Thus, if we build $g$ close to $f$ such that there exists $v \in \rho(\tilde{g})$ satisfying $L(v) > 0$, then we are done.

The ergodic theorem implies the existence of a point $x_0 \in \mathbb{T}^2$ which is recurrent for $f$ and such that $\rho(\tilde{x}_0, \tilde{f}) = (t, \omega)$. At this point there are two possibilities. Either there exists $n$ arbitrarily large such that $f^n(x_0)$ is close to $x_0$ and $L(\rho(\tilde{x}_0, n, \tilde{f})) > 0$; in this case it suffices to apply a $C^0$ closing lemma to $x_0$ and $f^n(x_0)$ to get the theorem. Or for every $n$ large enough such that $f^n(x_0)$ is close to $x$, we have $L(\rho(\tilde{x}_0, n, \tilde{f})) \leq 0$. This case is a bit more complicated: we begin by proving that in this case, it is possible to suppose that $L(\rho(\tilde{x}_0, n, \tilde{f})) < 0$ (Lemma 3 of [AZ04]). Let $n_0$ be such a number (large enough); a theorem of recurrence of G. Atkinson [Atk76] implies the existence of a time $n_1 \gg n_0$ such that $L_n(\rho(\tilde{x}_0, n_1, \tilde{f}))$ is arbitrarily close to 0. A calculation shows that in this case, $L(\rho(\tilde{f}^{n_0}(\tilde{x}_0), n_1 - n_0, \tilde{f})) > 0$: the rotation vector of the segment of orbit between $\tilde{f}^{n_0}(\tilde{x}_0)$ and $\tilde{f}^{n_1}(\tilde{x}_0)$ belongs to $\{L > 0\}$. It then suffices to apply the $C^0$ closing lemma to $\tilde{f}^{n_0}(\tilde{x}_0)$ and $\tilde{f}^{n_1}(\tilde{x}_0)$.

**Proof of Theorem 1.** Let $\tilde{f} \in D^1(\mathbb{R}^2)$ be such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin Q^2$, and $V$ a $C^1$-neighbourhood of $f$. We fix once for all a lift $f$ of $f$, and choose $L : \mathbb{R}^2 \to \mathbb{R}$ an affine form such that $L(t, \omega) = 0$ and $L(v) \leq 0$ for every $v \in \rho(\tilde{f})$. Let $x_0 \in \mathbb{T}^2$ be a recurrent point of $f$ such that $\rho(\tilde{x}_0, \tilde{f}) = (t, \omega)$. Lemma 2 gives us a number $N \in \mathbb{N}$ and a neighbourhood $V$ of $x_0$. The proof of Theorem 1 of [AZ04] summarized in the previous discussion gives us a point $y = f^{n_0}(x_0)$ (with $n_0$ possibly equal to 0) and a time $n_1 \geq N$ such that $\tilde{f}^{n_1}(\tilde{y}) \in V$ and $L(\rho(\tilde{y}, n_1, \tilde{f})) > 0$. Applying Lemma 2, we get $g \in V$ such that $y$ is a periodic point of $g$ satisfying $L(\rho(\tilde{y}, n_1, \tilde{f})) > 0$. This proves the theorem.

Moreover, if $f$ preserves $\omega$, then $g$ can be supposed to preserve it too. \qed

**Remark 3.** Theorem 1 of [AZ04] is also true in the $C^0$ measure-preserving case. To see it, it suffices to replace the $C^0$ closing lemma by the measure-preserving one (see for example Lemma 13 of [OU41]).

**References**

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