A geometric property for optimal paths and its applications in first passage percolation

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Abstract
We consider the first passage percolation model in $\mathbb{Z}^d$ with a weight distribution $F$ for $0 < F(0) < p_c$. In this paper, we derive a geometric property for optimal paths to show that all of them have to pass an $M$-exit. By this property, we show that the shape is strictly convex, and we solve the height problem.

1 Introduction of the model and results.
We consider the first passage percolation model on the $L^d = (\mathbb{Z}^d, \mathbb{E}^d)$ lattice, a graph, with the vertices in $\mathbb{Z}^d$ and the edges in $\mathbb{E}^d$ connecting each pair of vertices one unit apart. We assign independently to each edge a non-negative passage time $t(e)$ with a measure $\mu_e$ in a distribution $F(x)$. More formally, we consider the following probability space. As the sample space, we take $\Omega = [0, \infty)^{\mathbb{E}^d}$, whose points are called configurations. If we focus on a particular edge set $E$, we can use $\Omega(E)$ to denote the configurations on $E$. Let $P = \prod_{e \in \mathbb{E}^d} \mu_e$ be the corresponding product measure on $\Omega$. The expectation and variance with respect to $P$ are denoted by $E(\cdot)$ and $\sigma^2(\cdot)$. For any two vertices $u$ and $v$, a path $\gamma$ from $u$ to $v$ is an alternating sequence $(v_0, e_1, v_1, ... , v_i, e_{i+1}, v_{i+1}, ... , v_{n-1}, e_n, v_n)$ of vertices $v_i$ and edges $e_i$ between $v_i$ and $v_{i+1}$ in $\mathbb{E}^d$ with $v_0 = u$ and $v_n = v$. A path is called disjoint if $v_i \neq v_j$ for $i \neq j$. In this paper, we always consider a disjoint path. Given such a path $\gamma$, we define its passage time as

$$T(\gamma) = \sum_{e_i \in \gamma} t(e_i).$$

For any two sets $A$ and $B$, we define the passage time from $A$ to $B$ as

$$T(A, B) = \inf \{ T(\gamma) : \gamma \text{ is a path from } A \text{ to } B \},$$

where the infimum is taken over all possible finite paths from some vertex in $A$ to some vertex in $B$. A path $\gamma$ from $A$ to $B$ with $T(\gamma) = T(A, B)$ is called an optimal path of $T(A, B)$. If we focus

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on the optimal paths in a particular set $J$, we may write

$$T_J(A, B) = \inf \{T(\gamma) : \gamma \subset J\} \text{ for } A, B \subset J.$$  

The existence of such an optimal path has been proven (see Kesten (1986)) if $F(0) \neq p_c$, where $p_c$ is the critical point in percolation. If $t(e) = 0$, the edge is called a zero edge or an open edge; otherwise it is called a closed edge. We also want to point out that the optimal path may not be unique. If all edges in a path are in passage time zero, the path is called a zero path or an open path. If we focus on a special configuration $\omega$, we may write $T(A, B)(\omega)$ instead of $T(A, B)$. When $A = \{u\}$ and $B = \{v\}$ are single vertex sets, $T(u, v)$ is the passage time from $u$ to $v$. We may extend the passage time over $\mathbb{R}^d$. More precisely, if $u$ and $v$ are in $\mathbb{R}^d$, we define $T(u, v) = T(u', v')$, where $u'$ (resp., $v'$) is the nearest neighbor of $u$ (resp., $v$) in $\mathbb{Z}^d$. Possible indetermination can be eliminated by choosing an order on the vertices of $\mathbb{Z}^d$ and taking the smallest nearest neighbor for this order. In this paper, for any $x, y \in \mathbb{R}^d$, $\|x\|$ is denoted by the Euclidean norm and $d(x, y) = \|x - y\|$ is the distance between $x$ and $y$. For any two sets $A$ and $B$ of $\mathbb{R}^d$,

$$d(A, B) = \min \{d(x, y) : x \in A \text{ and } y \in B\}$$

and is denoted as the distance between $A$ and $B$. Sometimes, we need to discuss a norm on $\mathbb{R}^{d-1}$, so we use $\|u - v\|_{d-1}$ for $u, v \in \mathbb{R}^{d-1}$.

### 1.1 Point-point passage time.

Given a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, if $Et(e) < \infty$, by Kingman’s sub-additive theorem, it is well known that

$$\lim_{n \to \infty} \frac{1}{n} T(0, nx) = \inf \frac{1}{n} \text{ET}(0, nx) = \lim_{n \to \infty} \frac{1}{n} \text{ET}(0, nx) = \mu_F(x) \text{ a.s. and in } L_1. \quad (1.1)$$

It is also known (see Kesten (1986)) that

$$\mu_F(x) \text{ is continuous in } x \text{ and } \mu_F(x) = 0 \text{ iff } F(0) \geq p_c. \quad (1.2)$$

In particular, Hammersley and Welsh (1965), in their pioneering paper, investigated

$$a_{0,(n,0,\ldots,0)} = T(0, (n, 0, \ldots, 0)).$$

They showed that

$$\lim_{n \to \infty} a_{0,(n,0,\ldots,0)}/n = \mu_F(1, 0, \ldots, 0) \text{ a.s. and in } L_1. \quad (1.3)$$

For simplicity’s sake, we denote by

$$\mu_F(1, 0, \ldots, 0) = \mu_F.$$
And again, for simplicity’s sake, for any $x$, we let
\[ t_{0,nx} = T(0,nx) \]
and
\[ t_{0,(n,0,\ldots,0)} = a_{0,(n,0,\ldots,0)}. \]
Hammersley and Welsh (1965) also introduced the point-plane passage time. Let
\[ b_{0,n} = \inf \{ T(\gamma) : \gamma \text{ is a path from the origin to } L_n \} \]
and they proved
\[ \lim_{n \to \infty} b_{0,n}/n = \mu_F \text{ a.s. and in } L_1. \]  
(1.4)\]

Since we need the rate of convergence in (1.1), we assume that $t(e)$ is not a constant and satisfies the following:
\[ \int_0^\infty e^{\iota x} dF(dx) < \infty \text{ for some } \iota > 0. \]  
(1.5)\]

When $F(0) < p_c$, the map $x \to \mu(x)$ induces a norm on $\mathbb{R}^d$. The unit radius ball for this norm is denoted by
\[ B := B(F) = \{ u : \mu_F(u) \leq 1 \} \]
and is called the asymptotic shape. The boundary of $B$ is
\[ \partial B := \{ u \in \mathbb{R}^2 : \mu(u) = 1 \}. \]

By (1.2), if $F(0) < p_c$, $B$ is a compact convex deterministic set and $\partial B$ is a continuous convex set (see Kesten (1986)). In particular, let $O$ be the point of the intersection of $B$ and the positive $x_1$-axis. Define for all $t > 0$,
\[ B(t) := \{ u \in \mathbb{R}^d, T(0,u) \leq t \}. \]
The shape theorem (see Cox and Durrett (1981)) is a well-known result stating that for any $\epsilon > 0$,
\[ tB(1-\epsilon) \subset B(t) \subset tB(1+\epsilon) \text{ eventually w.p.1.} \]  
(1.6)\]

One of the most important questions in the first passage percolation model is to ask what the shape of $B$ looks like. It is easy to show (see Fig. 6.1 in Kesten (1986)) that
\[ B \text{ lies between the cube } [-1/\mu_F, 1/\mu_F]^d \text{ and the diamond with the } 2d \text{ corners } (\pm 1/\mu_F, 0, \cdots) \text{ and } (0, \cdots, 0, \pm 1/\mu_F). \]  
(1.7)\]

By using a rate estimate (see Chow and Zhang (2003)), for any $\epsilon > 0$,
\[ P(tB(1-\epsilon) \subset B(t) \subset tB(1+\epsilon)) \geq 1 - \exp(-O(\epsilon t)). \]  
(1.8)
By Zhang’s (2010) Theorem 3

\[ P(b_{0,n} - n\mu_F \ge \epsilon n) \le \exp(-O(\epsilon n)). \] (1.9)

Regarding the length of optimal paths, it is well known (see Prop. (5.8) in Kesten (1986)) that there exists \( \lambda = \lambda(F) \) and \( C_i = C_i(F, \lambda) \) such that for each optimal path \( \gamma_n(x) \),

\[ P(|\gamma_n(x)| \le \lambda n) \le C_1 \exp(-C_2 n). \] (1.10)

In this paper, \( C_i \) denotes a constant with \( 0 < C_i < \infty \) whose precise value is of no important; its value may change from appearance to appearance, but \( C_i \) will always be independent of \( n \) and \( t \), \( k \), and \( m \), although it may depend on \( F \) and \( d \). For simplicity’s sake, we sometimes use \( O(n) \) for \( C_1 n \le O(n) \le C_2 n \) if we do not need the precise value of \( C_i \).

A hyperplane in \( R^d \) always divides \( R^d \) into two parts that do not contain the hyperplane itself. Given \( u = (u_1, \ldots, u_d) \in \partial B \), note that \( B \) is convex, so there exists a hyperplane \( H_u \) containing \( u \) such that one of the divided parts by \( H_u \) does not contain any points of \( B \). The hyperplane \( H_u \) is called a tangent plane of \( u \). Note that the tangent plane of \( u \) is unique. Let

\[ S_u = \partial B \cap H_u. \]

In this paper, we focus on \( S_O \) and will show that \( S_O = O \).

We now show that any optimal path cannot be too tangled. Let \( \mathcal{P}_n(x) \) be the path set containing all the finite paths from the origin to \( nx \) for \( x = (x_1, \ldots, x_d) \). By symmetry, in this paper, we only need to work on \( x \) with \( x_1 > 0 \). An \( M \)-cube is denoted by \( \mathcal{M}(u) = \prod_{i=1}^d [u_i - M, u_i + M] \) for \( u = (u_1, \ldots, u_d) \in Z^d \). We will show that all the optimal paths \( \{\gamma_n(x)\} \) in \( \mathcal{P}_n \) have to cross an \( M \)-cube with a very large probability. We denote by \( L_i \) the hyperplane with \( x_1 = i \). In other words, the plane contains \( (i, 0, \ldots, 0) \), and it uses the \( x_1 \)-axis as the normal vector. We also denote by \( L[i, j] \) (but \( L(i, j) \) ) the space between \( L_i \) and \( L_j \), including \( L_i \) and \( L_j \) (but not including \( L_i \) and \( L_j \)). We also denote by \( L_i \) and \( L[i, j] \) the vertex sets of \( L_i \) and \( L[i, j] \), respectively. Let \( \mathcal{Q}_n(x) \) be the vertex set: the union of all the optimal paths in \( \mathcal{P}_n(x) \) for \( x \) with \( x_1 > 0 \). Let

\[ a_n = \min\{u_1 : u = (u_1, \ldots, u_d) \in \mathcal{Q}_n(x)\} \text{ and } b_n = \max\{u_1 : u = (u_1, \ldots, u_d) \in \mathcal{Q}_n(x)\}. \] (1.11)

For convenience, we denote \( S = \mu_F^{-1}S_O \). For \( \phi > 0 \), let

\[ S_\phi = \{u \in L_1 : d(u, S) \le \phi\}. \]

We know that \( S \subset S_\phi \subset L_1 \). (1.12)

An \( M \)-cube \( \mathcal{M}(u) \) is an \( M \)-exit for \( \mathcal{Q}_n(x) \) if

\[ \gamma_n(x) \cap \mathcal{M}(u) \neq \emptyset \text{ for each optimal path } \gamma_n(x) \in \mathcal{P}_n(x). \]
With this definition, we show the following theorem.

**Theorem 1.** If $F$ is a distribution with an exponential tail in (1.5), $F(0^-) = 0$, and $0 < F(0) < p_c$, then there exist $M$, and $\phi(F, d) > 0$, and $\eta = \eta(F) > 0$, and $C_i = C_i(F, \eta, \phi, M)$ for $i = 1, 2$ such that for any $n$,

$$P \left( \text{there are at least } \eta n \text{ many } M\text{-exits for } Q_n(x) \text{ in } L(a_n, b_n) \right) \geq 1 - C_1 \exp(-C_2 n) \text{ for any } x \in S_{\phi}.$$ 

**Remarks.** 1. We can obtain a more precise value of $\phi$ depending on $d$ in Theorem 1. For example, $\phi$ can be taken to be one when $d \leq 8$.

2. By the construction of the optimal path in Lemma 2.2, we can show that there is an optimal path only crossing a hyperplane $L_i$ once for some $a_n < i < b_n$ with a probability larger than $1 - \exp(-O(n))$. It is possible to show that there exist $\eta n$ many 1-exits for $Q_n(x)$ in Theorem 1 if we make a more precise local surgery in Lemma 2.2.

With Theorem 1, we can now discuss the distribution of these $M$-exits. We divide $L[a_n, b_n]$ into many smaller, equal-sized $\kappa$-cylinders for a small $0 < \kappa < 1$ independent of $n$:

$$L[a_n, a_n + \kappa n], L[a_n + \kappa n, a_n + 2\kappa n], \ldots, L[a_n + i\kappa n, a_n + (i + 1)\kappa n], \ldots, L[b_n - \kappa n, b_n]. \quad (1.13)$$

Here we assume that $i\kappa n$ is an integer for $i = 1, 2, \cdots$; otherwise we can use $\lfloor i\kappa n \rfloor$. We denote the event by $D_{\kappa n}(x)$ that for each $i$, $Q_n(x)$ has at least one $M$-exit $M(v_i)$ in each $\kappa$-cylinder $L[a_n + i\kappa n, a_n + (i + 1)\kappa n]$ defined in (1.13), and

$$\max\{d(v_i, y) : y \in Q_n(x) \cap L[a_n + i\kappa n, a_n + (i + 1)i\kappa n] \} \leq \lambda \kappa n + M$$

for the $\lambda$ defined in (1.9) independent of $n$ and $\kappa$. We have the following corollary from Theorem 1.

**Theorem 2.** If $F$ is a distribution with an exponential tail in (1.5), $F(0^-) = 0$, and $0 < F(0) < p_c$, then for any small $0 < \kappa < 1$, there exist $M$, and $\phi = \phi(k, d) > 0$, and $C_i = C_i(\lambda, \phi, \kappa, F, M)$ for $i = 1, 2$ independent of $n$ such that for any $n$ and any $x \in S_{\phi}$,

$$P(D_{x_n}(\kappa)) \leq C_1 \exp(-C_2 n),$$

where $D_{x_n}(\kappa)$ is the complement of $D_{x_n}(\kappa)$.

**Remark.** 3. From Theorem 2, for any $\kappa > 0$, $\{a_n \leq -\kappa n\}$ or $\{b_n \geq n(1 + \kappa)\}$ has a probability less than $\exp(-O(\kappa n))$. 

5
1.2 Applications.

One of the most important questions in first passage percolation, as we mentioned above, is to understand the shape of $B$. We say that $\partial B$ is strictly convex at $u$ if

$$S_u = u.$$ 

It is widely believed that $B$ is strictly convex (see Kesten (1986)). We show the following theorem to answer this question.

**Theorem 3.** If $F$ has a Bernoulli distribution with $0 < F(0) < p_c$ and $F(1) = 1 - F(0)$, then $\partial B$ is strictly convex at $O$.

We define the height of the optimal paths to be

$$h_n = \max\{d(u, \text{ the } x_1\text{-axis}) : u \in Q_n((1, 0, \ldots, 0))\}.$$ 

It is widely believed (see Hammersley and Welsh (1965); Smythe and Wierman (1978); and Kesten (1986)) that

$$\limsup_{n \to \infty} \frac{h_n}{n} = 0 \text{ in some sense. (1.14)}$$

Equation (1.14) is called the height problem. By Theorem 3 and Kesten’s Lemma 9.10 (1986), we can answer this question affirmatively.

**Corollary.** If $F$ has a Bernoulli distribution with $0 < F(0) < p_c$ and $F(1) = 1 - F(0)$, then

$$\limsup_{n \to \infty} \frac{h_n}{n} = 0 \text{ in probability.}$$

**Remarks.**

4. We show that Theorems 1 and 3 hold with a condition $0 < F(0) < p_c$. The condition that $F(0) > 0$ is crucial. Indeed, there is a counter example (see Durrett and Liggett (1981)) that Theorem 1 cannot hold when $P(t(e) = \alpha) > \bar{p}_c$, where

$$\alpha = \inf\{x : P(t(e) \leq x) > 0\}$$

and $\bar{p}_c$ is the critical point for oriented percolation.

5. In this paper, we only consider the passage times in the paths going to $L_n$ from the origin. On the other hand, we only consider a Bernoulli distribution in Theorem 3 and in the Corollary. The same proof methods in this paper can be adapted to show that Theorems 1 and 2 also hold for the passage times in the paths going to the hyperplane $n\mu_F^{-1}(u)H_u$ from the origin. In addition, the proofs of Theorem 3 and the Corollary can also be adapted to show all the distributions satisfying the requirements in Theorem 1. However, we might need much more complicated notations and many pages to discuss how a path costs extra time when it will not pass an $M$-exit. We would
rather explore them in a separate paper.

6. We may define the height fluctuation exponent to be

$$\xi(d) = \max \left\{ \xi : \limsup_n \frac{Eh_n}{n^\xi} < \infty \right\}.$$ 

It is believed that $\xi(2) = 2/3$. It has been only proved that $0.5 \leq \xi(d) \leq 1$ for all $d$ (see Newman and Piza (1995)). We proved in the Corollary that the height goes to zero, but we are unable to obtain any rate estimate for $\xi(d)$. To show $\xi(d) < 1$, we might need more precise estimates for the distribution of $M$-exits in Theorem 2.

2 Proofs of Theorems 1 and 2.

The proof of Theorem 1 is involved and divided into two steps by the following two separated lemmas. First, for each optimal path, by using a Peierls’ argument, there are $O(n)$ many 2M-cylinders between $L_{i-M}$ and $L_{i+M}$ for large $M$ independent of $n$ such that each cylinder contains only one 2M-cube intersecting the optimal path. Second, by step 1, we construct paths inside these 2M-cubes such that they have $M$-exits for $Q_n(x)$. Each construction takes a positive probability only depending on $M$. We then use a local independence to show the existence of $O(n)$ many $M$-exits with a probability larger than $1 - \exp(-O(n))$.

We now start to work on step 1. In this section, we always consider $x = (x_1, \ldots, x_d)$ with $x \in S_\phi$. Before the proof of Theorem 1, we need to discuss a few properties regarding optimal path $\gamma_n(x) \in P_n(x)$. Let $E_n(1)$ be the event that

$$|\gamma_n(x)| \leq \lambda n,$$ (2.1)

for each optimal path $\gamma_n(x)$ and for a large constant $\lambda$ independent of $n$ and $x$. It follows from (1.9) that

$$P(E_n(1)) \geq 1 - \exp(-O(n)).$$ (2.2)

We also let $E_n(2)$ be the event that

$$(\mu_F(x) - \delta)n \leq t_{0,nx} \leq (\mu_F(x) + \delta)n \text{ for } \delta > 0.$$ (2.3)

By (1.8),

$$P(E_n(2)) \geq 1 - \exp(-O(\delta n)).$$ (2.4)

If $E_n(3) = E_n(1) \cap E_n(2)$,

$$P(E_n(3)) \geq 1 - \exp(-O(\delta n)).$$ (2.5)

Let $\gamma_n(x)$ be an optimal path from the origin to $x$. For each $L_i$ with $a_n < i < b_n$, $\gamma_n(x)$ will cross out $L_i$. Let $u_i$ be the vertex where $\gamma_n(x)$ first meets $L_i$ if we go along $\gamma_n(x)$ from the origin.
to \( n \vec{x} \). Let \( \vec{u}_i \) be the vertex where \( \gamma_n(\vec{x}) \) first meets \( \vec{L}_i \) if we go along \( \gamma_n(\vec{x}) \) from \( n \vec{x} \) to the origin (see Fig. 1). Note that \( \vec{u}_i \) may be equal to \( \vec{u}'_i \). Let \( \gamma_i(1), \gamma_i(2), \gamma_i(3) \) be the sub-paths of \( \gamma_n(\vec{x}) \) from the origin to \( \vec{u}_i \), from \( \vec{u}_i \) to \( \vec{u}'_i \), and from \( \vec{u}'_i \) to \( n \vec{x} \), respectively (see Fig. 1). We have

\[
\gamma_n(\vec{x}) = \gamma_i(1) \cup \gamma_i(2) \cup \gamma_i(3). \tag{2.6}
\]

Let

\[
y_i = \min\{x_1 : (x_1, \ldots, x_l, \ldots, x_d) \in \gamma_i(2)\} \quad \text{and} \quad y'_i = \max\{x_1 : (x_1, \ldots, x_l, \ldots, x_d) \in \gamma_i(2)\}. \tag{2.7}
\]

If \( y'_i - y_i \geq 2M \) for some \( 1 \leq i \leq d \), we say \( \gamma_n(\vec{x}) \) has an \( M \)-arm at \( i \). There might be many vertices with the first coordinate the same as \( y_i \) and \( y'_i \). We simply select two of them: \( y_i \) and \( y'_i \). Let \( \mathcal{F}_M(\vec{x}) \) be the total number of \( M \)-arms of \( \gamma_n(\vec{x}) \).

**Lemma 2.1.** If \( F \) is a distribution with an exponential tail in (1.5), \( F(0^-) = 0 \), and \( F(0^-) < p_c \), then there exist \( \epsilon > 0 \), and \( \phi = \phi(\epsilon, d) > 0 \), and \( M > 0 \), and \( C_i = C_i(\epsilon, \phi, M) \) for \( i = 1, 2 \), such that for any \( n \) and for each optimal path \( \gamma_n(\vec{x}) \in \mathcal{P}_n(\vec{x}) \),

\[
P(\mathcal{F}_M(\vec{x}) \leq n(1 - \epsilon)) \geq 1 - C_1 \exp(-C_2n) \quad \text{for any} \quad \vec{x} \in S_\phi. \tag{2.8}
\]

**Proof.** We first work for \( \vec{x} = (1, 0, \ldots, 0) \). For an optimal path \( \gamma_n(\vec{x}) \in \mathcal{P}(\vec{x}) \), if \( \gamma_n(\vec{x}) \) has an \( M \)-arm at \( j \), then we select \( j' \) (it’s possible \( j = j' \)) such that

\[
y_j \leq j' - M \leq j \leq j' + M \leq y'_j. \tag{2.9}
\]

If \( \gamma_n(\vec{x}) \) does not have an \( M \)-arm at \( j \), then there exists \( j' \) with

\[
y_j, y'_j \in [j' - M, j' + M]. \tag{2.10}
\]

We divide the proof of Theorem 1 into the following two cases. In case (a), we suppose that \( \gamma_j(2) \) will first meet \( \gamma'_j = (y_j', \ldots, y_d') \), and will then meet \( \gamma_j(2) \) for \( y_j' \) and \( y_j \) defined in (2.7) if we go along \( \gamma_j(2) \) from \( \vec{u}_j \) to \( \vec{u}'_j \). In case (b), \( \gamma_j(2) \) will first meet \( \gamma'_j \), and will then meet \( \gamma'_j \) if we go along \( \gamma_j(2) \) from \( \vec{u}_j \) to \( \vec{u}'_j \) (see Fig. 1). We may call them either case (a) \( M \)-arm or case (b) \( M \)-arm at \( j \). Let \( \mathcal{F}_M(a, \vec{x}) \) and \( \mathcal{F}_M(b, \vec{x}) \) be the numbers of case (a) \( M \)-arms and case (b) \( M \)-arms, respectively:

\[
\mathcal{F}_M(\vec{x}) = \mathcal{F}_M(a, \vec{x}) + \mathcal{F}_M(b, \vec{x}). \tag{2.11}
\]

By (2.5),

\[
P(\mathcal{F}_M(a, \vec{x}) \geq n(1 - \epsilon/2)) \leq P(\mathcal{F}_M(a, \vec{x}) \geq n(1 - \epsilon/2), \mathcal{E}_n(3)) + \exp(-O(n)). \tag{2.12}
\]

On \( \mathcal{F}_M(\vec{x}) \geq n(1 - \epsilon) \), we have \( n(1 - \epsilon/2)/(2M) \) many disjoint cylinders \( \{L(j' - M, j' + M)\} \) such that there exist \( y_j \) and \( y'_j \) satisfying (2.10). In case (a) (see Fig. 1), \( \gamma_n(\vec{x}) \) first meets \( \vec{L}_{j' - M} \) at \( \vec{v}_j \).
Figure 1: The figure shows an optimal path from the origin to \( nx \) with an \( M \)-arm at \( i \) and at \( j \) in case (a) and case (b), respectively. Note that \( \gamma_n(x) \) is from the origin to \( nx \), passing through \( X = i \). The sub-path from \( u_i \) to \( u'_i \) is \( \gamma_i(2) \), which does not stay inside an \( M \)-cylinder and is called an \( M \)-arm. We find \( i' \) such that \( y_i \leq i' - M \leq i' + M \leq y'_i \). For simplicity’s sake, we let \( i = i' \).

There are three zigzag paths from \( v_i \) to \( v'_i \), from \( v'_i \) to \( v''_i \), and from \( v''_i \) to \( v'''_i \). Each path has a length larger than \( 2M \) and the sum of the lengths of the three paths is larger than \( 6M \). We call this situation case (a). In case (b), \( \gamma_n(x) \) is from the origin to \( nx \) passing through \( X = j \). The sub-path from \( u_j \) to \( u'_j \) is \( \gamma_j(2) \), which also does not stay inside an \( M \)-cylinder and is called an \( M \)-arm. There are six zigzag paths from \( v_j \) to \( v'_j \), from \( v'_j \) to \( v''_j \), from \( v''_j \) to \( v'''_j \), from \( v'''_j \) to \( z_j \), from \( z_j \) to \( z'_j \), and from \( z'_j \) to \( z''_j \). Each path has a length larger than \( M \) and the sum of the lengths of the six paths is larger than \( 6M \).
After meeting \( v'_j, \gamma_n(x) \) will go back to meet \( L_{j-M} \) at \( v''_j \). After meeting \( v''_j, \gamma_n(x) \) will go to meet \( L_{j+M} \) at \( v'''_j \). Thus, there exist four vertices (see Fig. 1) with

\[
\begin{align*}
v_j &= (x_1(j), \ldots, x_l(j), \ldots, x_d(j)), v'_j = (x_1'(j), \ldots, x_l'(j), \ldots, x_d'(j)), \\
v''_j &= (x_1''(j), \ldots, x_l''(j), \ldots, x_d''(j)), v'''_j = (x_1'''(j), \ldots, x_l'''(j), \ldots, x_d'''(j))
\end{align*}
\]

(2.13)

such that

\[
x_1'(j) - x_1(j) = x_1''(j) - x_1''(j) = x_1''''(j) - x_1''''(j) \geq 2M \text{ with } x_1(j) = x_1''(j), x_1'(j) = x_1''''(j),
\]

(2.15)

where the pieces of \( \gamma'_j, \gamma''_j, \) and \( \gamma''''_j \) are from \( v_j \) to \( v'_j \), from \( v'_j \) to \( v''_j \), and from \( v''_j \) to \( v'''_j \) (see Fig. 1). We call this a zigzag structure. Furthermore, we also have

\( \gamma'_j, \gamma''_j, \gamma''''_j \), which are disjoint in \( j \), and the different triple pieces are in different vertical cylinders.

Moreover, on \( \mathcal{E}_n(2) \), note that \( x = (1, 0, \cdots, 0) \), so

\[
\sum_{j=1}^{n(1-c/2)/(2M)} [T(\gamma'_j) + T(\gamma''_j) + T(\gamma''''_j)] \leq t_{0,1,n,x} \leq (\mu_F + \delta)n.
\]

(2.16)

If \( v_j, v'_j, v''_j, \) and \( v'''_j \) are fixed, then by (1.1) and (2.15) for large \( M \) and small \( \delta \),

\[
6M(\mu_F - \delta) \leq E[T(\gamma_j) + T(\gamma'_j) + T(\gamma''_j)].
\]

(2.17)

Furthermore, by using the concentration estimate (see Theorem 1 in Kesten (1993)), if \( v_j, v'_j, v''_j, \) and \( v'''_j \) are fixed, there are \( C_i \) for \( i = 1, 2 \) independent of \( M \) such that for \( x \geq 1 \)

\[
P \left( \left| T(\gamma'_j) + T(\gamma''_j) + T(\gamma''''_j) - E \left[ T(\gamma'_j) + T(\gamma''_j) + T(\gamma''''_j) \right] \right| \geq x \right) \leq C_1 \exp(-C_2(x^2/M)).
\]

(2.18)

Now we need to fix \( v_j, v'_j, v''_j, \) and \( v'''_j \). We divide \( \mathbb{Z}^d \) into equal \( d \)-dimensional cubes with side-length \( M \), called \( M \)-cubes. More precisely, for \( u = (u_1, \cdots, u_i, \cdots, u_d) \in \mathbb{Z}^d \), an \( M \)-cube is defined to be

\[
B_M(u) = \prod_{i=1}^{d} [Mu_i, Mu_i + M].
\]

On \( \mathcal{E}_n(3) \), \( \gamma_n(x) \) stays on \([-\lambda n, \lambda n]^d \). We then consider the \( M \)-cubes intersecting with \( \gamma_n(x) \). We denote them by \( \gamma_n(x, M) \). Note that if an \( M \)-cube contains a vertex of \( \gamma_n(x) \), then its \( 3^d - 1 \) adjacent neighbor cubes contain at least \( M \) edges of \( \gamma_n(x) \), so on \( \mathcal{E}_n(3) \),

the number of \( M \)-cubes in \( \gamma_n(x, M) \) is less than \((3^d - 1)\lambda n/M \).

(2.19)

Since \( |\gamma_n(x)| \leq \lambda n \), note that these cubes are boundary connected, so by a standard estimate (see (4.24) in Grimmett (1999)), it has at most \( 7^{(3^d-1)Kn/M} \) choices to fix these cubes in \( \gamma_n(x, M) \). We then choose these \( M \)-cubes containing \( v_j, v'_j, v''_j, \) and \( v'''_j \). Note that

\[
\sum_{j=1}^{n} \binom{n}{j} = 2^n,
\]

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Thus, there is a subsequence \( \{v_j\} \) \( j \)'s term to have

\[
(8M^d)(3^d-1)\lambda n/M \leq \exp(d(3^d-1)\lambda n \log(8M)/M)
\]

choices for them. Thus, for a fixed sequence \( \{v_j\} \), \( \{v'_j\} \), \( \{v''_j\} \), if we denote by

\[
T(v_j, v'_j, v''_j, v'''_j) = T(\gamma_j) + T(\gamma''_j) + T(\gamma'''_j)
\]

such that these \( \gamma_j, \gamma''_j, \) and \( \gamma'''_j \) satisfy (2.17)–(2.18), then by (2.19) and the above analysis,

\[
P(\mathcal{F}_M(a, x) \geq n(1 - \epsilon/2), \mathcal{C}_n(3)) \leq \sum_{w_j, w'_j, w''_j, w'''_j} \sum_{j=1}^{n(1-\epsilon/2)/(2M)} T(v_j, v'_j, v''_j, v'''_j) \leq (\mu_F + \delta)n, \mathcal{F}_M(a, x) \geq n(1 - \epsilon/2),
\]

\[
\sum_{w_j, w'_j, w''_j, w'''_j} \sum_{j=1}^{n(1-\epsilon/2)/(2M)} T(v_j, v'_j, v''_j, v'''_j) = (\mu_F + \delta)n, \mathcal{F}_M(a, x) \geq n(1 - \epsilon/2)
\]

(2.20)

On \( \mathcal{F}_M(a, x) \geq n(1 - \epsilon/2) \), note that there are \( n(1 - \epsilon/2)/(2M) \) many disjoint vertical cylinders with these zigzag paths \( \{\gamma_i, \gamma''_i, \gamma'''_i\} \), so if \( \delta \) and \( \epsilon \) are small, by (2.17), there exists a small \( c = c(\delta, \epsilon) < 1 \) such that

\[
P \left( \sum_{j=1}^{n(1-\epsilon/2)/(2M)} T(w_j, w'_j, w''_j, w'''_j) \leq (\mu_F + \delta)n, \mathcal{F}_M(a, x) \geq n(1 - \epsilon/2) \right)
\]

\[
\leq P \left( \sum_{j=1}^{n(1-\epsilon/2)/(2M)} [T(w_j, w'_j, w''_j, w'''_j) - ET(w_j, w'_j, w''_j, w'''_j)] \leq (\mu_F + \delta)n - 3n\mu_F(1 - \epsilon/2)(1 - \delta) \right)
\]

\[
\leq P \left( \sum_{j=1}^{n(1-\epsilon/2)/(2M)} [ET(w_j, w'_j, w''_j, w'''_j) - T(w_j, w'_j, w''_j, w'''_j)] \geq (3 - c)\mu_Fn \right).
\]

(2.21)

By (2.22), if \( \{\sum_{j=1}^{n(1-\epsilon/2)/(2M)} [ET(w_j, w'_j, w''_j, w'''_j) - T(w_j, w'_j, w''_j, w'''_j)] \geq 2\mu_Fn\} \), then there is \( \eta \) independent of \( n \) and \( M \) such that there are at least \( \eta n/M \) many \( j \)'s such that

\[
ET(w_j, w'_j, w''_j, w'''_j) - T(w_j, w'_j, w''_j, w'''_j) \geq \mu_F M.
\]

(2.23)

Thus, there is a subsequence \( \{j_i\} \) such that (2.23) holds for this subsequence. We fix these \( \eta n/M \) many \( j_i \)'s term to have

\[
P \left( \sum_{j=1}^{n(1-\epsilon/2)/(2M)} T(w_j, w'_j, w''_j, w'''_j) \leq (\mu_F + \eta)n, F_M(a, x) \geq n(1 - \epsilon/2) \right)
\]

\[
\leq \left( \frac{n}{2M} \right)^{\eta n/M} P \left( \bigcap_{i=1}^{\eta n/M} \left\{ ET(w_{j_i}, w'_{j_i}, w''_{j_i}, w'''_{j_i}) - T(w_{j_i}, w'_{j_i}, w''_{j_i}, w'''_{j_i}) \geq \mu_F M \right\} \right).
\]

(2.24)
Note that paths \( \gamma_j', \gamma_j'', \text{ and } \gamma_j''' \) are in disjoint cylinders in \( j \), so the events in the intersections of (2.24) are independent. Thus, by (2.18),

\[
\left( \frac{n}{2M} \right) P \left( \bigcap_{i=1}^{\eta n/M} \left( E[T(w_j, w_j', w_j'', w_j''')] - T(w_j, w_j', w_j'', w_j'''] \right) \geq \mu_F M \right) \\
\leq 2^{n/M} \left( P(E[T(w_j, w_j', w_j'', w_j'''] - T(w_j, w_j', w_j'', w_j'''] \geq \mu_F M) \right)^{\eta n/M} \\
\leq 2^{n/M} (C_1 \exp(-C_2 M))^{\eta n/M} = 2^{n/M} C_1^{\eta n/M} \exp(-C_2 \eta n).
\]  

(2.25)

By (2.21)–(2.25), we have

\[
P(F_M(a, x) \geq n(1 - \epsilon/2), \mathcal{E}_n(3)) \\
\leq 7(3^d - 1) \ln^{n/M} 2^{(3^d \ln n/M)} \exp(d(3^d - 1) \ln \log(8 M)/M) 2^{n/M} C_1^{\eta n/M} \exp(-C_2 \eta n).
\]  

(2.26)

Together with (2.13), if \( M \) is large, then for \( x = (1, 0, \ldots, 0) \),

\[
P(F_M(a, x) \geq n(1 - \epsilon/2)) \leq \exp(-O(n)).
\]  

(2.27)

The estimate in (2.27) is called Peierls’ argument.

We now work on any \( x \in S_\phi \) for a small \( \phi > 0 \). If \( x \in S \), then the upper bound in (2.16) is the same. Thus, we can take the same \( c \) in (2.22) to show that (2.27) also holds for \( x \in S \). When \( x \in S_\phi \setminus S \), we have to use \( \mu_F(x) \) as the upper bound in (2.16), but \( \mu_F(x) > \mu_F \). Since \( \mu_F(x) \to \mu_F \) when \( x \) approaches \( S \), we can pick \( \phi > 0 \) small to replace \( \mu_F(x) \) by \( \mu_F + \delta_1(\phi) + \delta \) for a small \( \delta_1(\phi) \) as \( \phi \) small. Thus, we can also have the \( c \) in (2.22) if \( \delta_1(\phi) \), and \( \delta \), and \( \epsilon \) small. Therefore, the same estimate of (2.27) shows that (2.27) also holds for \( x \in S_\phi \).

Now we discuss case (b). \( \gamma_n(x) \) first meets \( L_{j' - M} \) at \( v_j \). After meeting \( L_{j'} \) at \( v_j' \), \( \gamma_n(x) \) will go back to meet \( L_{j' - M} \) at \( v_j'' \). After meeting \( v_j'' \), \( \gamma_n(x) \) will meet \( L_{j'} \) at \( v_j''' \). After meeting \( v_j''' \), \( \gamma_n(x) \) will meet \( L_{j' + M} \) at \( z_j \). Note that \( \gamma_n(x) \) will go back and meet \( L_{j'} \) at \( z_j' \), and go to meet \( L_{j' + M} \) at \( z_j'' \) (see Fig. 1). For fixed vertices \( v_j, v_j', v_j'', v_j''', z_j, z_j', z_j'' \), and \( \delta > 0 \) small, by using a sub-additive property and the same discussion as (2.17), we have

\[
6M(\mu_F - \delta) \leq E[T(v_j, v_j') + T(v_j', v_j'') + T(v_j'', v_j''')] + T(z_j, z_j') + T(z_j, z_j'') + T(z_j', z_j''').
\]  

(2.28)

By using the lower bound of (2.28) and the same proof in case (a),

\[
P(F_M(b, x)) \geq n(1 - \epsilon/2) \leq \exp(-O(n)).
\]  

(2.29)

By (2.12), on \( F_M(x) \geq (1 - \epsilon)n \), if \( F_M(a, x) < (1 - \epsilon/2)n \), then \( F_M(b, x) \geq n(1 - \epsilon/2) \). Therefore, Lemma 2.1 follows from (2.27) and (2.29). \( \Box \)

For any optimal path \( \gamma_n(x) \), on \( F_M(x) \leq (1 - \epsilon)n \), by (2.11), we have at least \( \epsilon n \) many cylinders such that \( \gamma_j(2) \), defined in Lemma 2.1, stays inside the cylinder \( L(j' - M, j' + M) \). By (2.10), \( \gamma_j(2) \)
stays inside the cylinder \( L(j' - M, j' + M) \). We take \( \epsilon' = \epsilon'(M) \leq \epsilon \) small such that there are \( \epsilon'n \) many disjoint cylinders. More precisely, if \( L_{l_j} \) is the center hyperplane of \( L(j' - M, j' + M) \), there are \( \epsilon'n \) many disjoint cylinders \( \{L(l_j - M, l_j + M)\} \) such that \( \gamma_j(2) \), defined above, stays inside the cylinder \( L(l_j - M, l_j + M) \). Let \( v_j \) be the last vertex that \( \gamma_n(x) \) meets at \( L_{l_j-M} \) before meeting \( L_{l_j} \) if we go along \( \gamma_n(x) \) from the origin to \( xn \) (see Fig. 2). Let \( v'_j \) be the last vertex that \( \gamma_n(x) \) meets at \( L_{l_j+M} \) before meeting \( L_{l_j} \) if we go along \( \gamma_n(x) \) from \( xn \) to the origin (see Fig. 2). We denote the sub-paths of \( \gamma_n(x) \) from the origin to \( v_j \), and from \( v_j \) to \( v'_j \), and from \( v'_j \) to \( xn \) by \( \beta_j(1) \), and \( \beta_j(2) \), and \( \beta_j(3) \), respectively. Since \( \gamma_j(2) \) stays inside the cylinder \( L(l_j - M, l_j + M) \),

\[
\beta_j(2) \subset L(l_j - M, l_j + M). \tag{2.30}
\]

We consider all \( \{l_j\} \) such that

\[
\beta_j(2) \subset u_{l_j} + [-M, M] \times [-M_1, M_1]^{d-1} \tag{2.31}
\]

for some \( u_{l_j} \). Let \( D_{M,M_1}(x) \) be the number of all \( \{j\} \) in the cylinders \( \{L(i_j - M, i_j + M)\} \) such that (2.31) holds. Note that if \( D_{M,M_1}(x) < \epsilon'n/2 \) for a large \( M_1 \), then there would be \( \epsilon'n/2 \) many sub-paths \( \beta_j(2) \) of \( \gamma_n(x) \) such that their lengths are larger than \( M_1 \) in disjoint \( \{L(l_j - M, l_j + M)\} \). Thus, if \( M_1 \) is large, then \( |\gamma_n(x)| \geq M_1 \epsilon'n/2 \geq \lambda n \) for the \( \lambda \) defined in (2.1). By using (2.2) and Lemma 2.1, for \( \epsilon' > 0 \) small and \( M, M_1 < \infty \) large, but fixed, if \( x \in S_\phi \) for \( \phi \) in Lemma 2.1,

\[
P(D_{M,M_1}(x) \geq \epsilon'n/2) \geq 1 - \exp(-O(n)). \tag{2.32}
\]

For a simple notation, we take \( M = M_1 \). Thus, \( D_M(x) \) is the number of disjoint cylinders \( \{L(i_j - M, i_j + M)\} \) such that (2.30) and (2.31) hold for \( \beta_j(2) \). We select cubes \( \{u_{l_j} + [-M,M]^d\} \) in a unique way and denote them by \( \{B_j\} \). By (2.32), there exists \( 0 < \epsilon' \) such that

\[
P(D_M(x) \geq \epsilon n) \geq 1 - \exp(-O(n)) \text{ for any } x \in S_\phi. \tag{2.33}
\]

In Lemma 2.1, we finished the first step of the proof of Theorem 1. Now we try to show the second step of the proof of Theorem 1.

**Lemma 2.2.** If \( F \) is a distribution with an exponential tail in (1.5), \( F(0^-) = 0 \), and \( 0 < F(0) < p_c \), then there exist \( M \), and \( \phi > 0 \), and \( C_i = C_i(\phi, M, F) \) for \( i = 1,2 \) such that

\[
P(\text{there is an } M\text{-exit for } Q_n(x) \text{ in } L(a_n,b_n)) \geq 1 - C_1 \exp(-C_2 n) \text{ for any } x \in S_\phi.
\]

**Proof.** We select an optimal path \( \gamma_n(x) \) in \( P_n(x) \) in a unique way. By (2.2), we may assume that \( \gamma_n(x) \subset [-\lambda n, \lambda n]^d \). If a \( 2M \)-cube \( B_j \) contains \( \beta_j(2) \) defined above, we call it is good cube. Furthermore, we call \( B_j \) regularly good if \( B_j \) is good and \( \beta_j(2) \) from \( v_j \) to \( v'_j \) is not open (see Fig. 2). In other words, \( T(\beta_j(2)) > 0 \) in a regularly good cube. By Theorem 5.4 in Grimmett (1999), the probability of existence for an open path from \( v_j \) to \( v'_j \) is less than \( \exp(-O(M)) \) for any two
Figure 2: The figure shows event $\mathcal{G}_j$. Note that $\gamma_n(x)$ is an optimal path from the origin to $x_n$. On $D_M(x) > n\epsilon$, there are at least $\epsilon n$ disjoint cylinders containing good cubes. The figure indicates a $2M$-cylinder from $L_{l_j-M}$ to $L_{l_j+M}$ containing a $2M$ good cube $B_j$, the dotted cube in the figure, between the cylinders. We see that $v_j$ is the last vertex where $\gamma_n(x)$ meets $L_{l_j-M}$ before $L_{l_j}$ if we go along $\gamma_n(x)$ from the origin to $x_n$. And $v'_j$ is the last vertex where $\gamma_n(x)$ meets $L_{l_j+M}$ before $L_{l_j}$ if we go along $\gamma_n(x)$ from $x_n$ to the origin. $\beta_j(1)$, $\beta_j(2)$, and $\beta_j(3)$ are the sub-paths from the origin to $v_j$, from $v_j$ to $v'_j$, and from $v'_j$ to $x_n$. $B_j$ is good since it contains $\beta_j(2)$. $B_j$ is regularly good if $T(\beta_j(2)) > 0$ (not indicated in this figure). $\mathcal{G}_j$ is an event as follows. There is an open path $\zeta_j$ from $v_j$ to $v'_j$, the circle path. $\zeta_j$ is a horizontal segment directly reaching to $L_{l_j}$, which then uses an open edge $e_j$ in $L(l_j, l_j+1)$ to meet $L_{l_j+1}$. After $e_j$, it uses a few line segments in $L(l_j+1, l_j+M)$ to meet $v'_j$. The boundary of $\zeta_j$ is closed on $\mathcal{G}_j$, not indicated in the figure. If $B_j$ is a $G$-cube (not indicated in this figure) and $\mathcal{G}_j$ occurs, then all optimal paths have to use $\zeta_j$ since their passage times are larger than $T(\gamma_j(1) \cup \zeta_j \cup \gamma_j(3))$. 
vertices \( v_j \) and \( v_j' \) in \( B_j \) when \( B_j \) is a fixed cube. By using Peierls’ argument, on \( \{ D_M(x) \geq n \epsilon' \} \), there are more than \( O(n) \) many regularly good \( 2M \)-cubes \( \{ B_i \} \) among these good cubes with a probability larger than \( (1 - \exp(- O(n))). \) In fact, to see this, we simply fix these \( M \)-cubes from the optimal path. Note that by (2.2), we can assume that the length of \( \gamma_n(x) \) is less than \( \lambda n. \) With a large \( M, \) by the same estimate in Lemma 2.1, we have the following argument. If \( D_n'(x) \) is the number of all the regularly good cubes for \( x \in S_\phi, \) then there exists \( 0 < \epsilon < \epsilon'/2 \) such that

\[
P(D'_M(x) \geq \epsilon n) \geq 1 - \exp(- O(n)). \quad (2.34)
\]

For the selected optimal path \( \gamma_n(x) \) above and for each good cube \( B_j, \) if

\[
T(r_n(x)) > T(\gamma_n(x)) - T(\beta_j(2)) \quad (2.35)
\]

for any path \( r_n(x) \in P_n(x) \) not using an edge of any \( \beta_j(2), \) then \( B_j \) is called a \( G \)-cube. Let \( G_n(x) \) be the number of all the \( G \)-cubes. By the definition of a regularly good cube,

\[
G_n(x) \geq D'_n(x). \quad (2.36)
\]

Thus, by (2.33),

\[
P(G_n(x) \geq \epsilon n) \geq P(D'_M(x) \geq \epsilon n) \geq 1 - \exp(- O(n)). \quad (2.37)
\]

Let us consider a special path \( \zeta_j \) from \( v_j \) to \( v_j' \) inside \( B_j. \) \( \zeta_j \) goes from \( v_j \) to meet \( L_{l_j} \) in a strict line segment at \( u_j; \) after \( u_j, \) it goes from \( L_{l_j} \) using edge \( e_j \) to meet \( L_{j+1} \) at \( u_j' \) directly, and then it uses a few line segments in \( L(l_j - M, l_j + M) \cap B_j \) to meet \( v_j' \) (see Fig. 2). Let \( G_j \) be the event that \( \zeta_i \) is open and its boundary edges inside \( L(l_j - M, l_j + M) \cap B_j \) are closed (see Fig. 2). For convenience, let \( E_j \) be the event that there is an optimal path from \( v_j \) to \( v_{j+1}. \) Thus,

\[
G_j \subset E_j = \Omega. \quad (2.38)
\]

On \( G_M(x) \geq \epsilon n, \) we simply list all \( G \)-cubes by

\[
\{ B_1, \ldots, B_i \} \text{ for } i = G_M(x). \quad (2.39)
\]

If there exists \( j \leq i \) such that \( G_j \) occurs in these \( G \)-cubes, we denote the event by \( \{ \exists j, G_j \}; \) otherwise, let its complement be \( \cap_j (E_j \cap G_j^C) \) for all possible \( j \)'s with \( j \leq G_M(x). \) By (2.37),

\[
1 - \exp(- O(n)) \leq P(G_M(x) \geq \epsilon n) \leq P(\{ \exists j, G_j \}, G_M(x) \geq \epsilon n) + P(G_M(x) \geq \epsilon n, \cap_j (E_j \cap G_j^C)). \quad (2.40)
\]

On

\[
\{ \exists j, G_j \}, G_M(x) \geq \epsilon n, \quad (2.41)
\]

note that \( \zeta_j \) is open and \( \beta_j(2) \) is optimal from \( v_j \) to \( v_j'. \) If

\[
\zeta_n(x) = \beta_j(1) \cup \zeta_j \cup \beta_j(3), \quad (2.42)
\]
then on $G_j$, $\zeta_n(x)$ is optimal and crosses $L_j$ once. In addition, on
\[
\{\exists \ j, G_j \}, G_M(x) \geq \epsilon n, \tag{2.43}
\]

note that $T(\zeta_j) = 0$, so for any path $r_n(x) \in P_n(x)$ without using $\beta_j(2)$,
\[
T(r_n(x)) > T(\gamma_n(x)) - T(\beta_j(2)) = T(\gamma_n(x)) - T(\zeta_j) = T(\zeta_n(x)). \tag{2.44}
\]

Thus, for any path $r_n(x) \in P_n(x)$ without using $\beta_j(2)$,
\[
T(r_n(x)) > T(\zeta_n(x)). \tag{2.45}
\]

On the other hand, $\zeta_j$ is the only optimal path from $v_j$ to $v_j'$, so all the optimal paths in $P_n(x)$ have to use an edge of $\zeta_j$. Since $\zeta_j$ is in $B_j$, it implies that $B_j$ is a 2M-exit for $Q_n(x)$. By (2.40),
\[
1 - \exp(-O(n)) \leq P(\exists \ a 2M-exit for $Q_n(x)$ in $L(a_n, b_n)) + P(G_M(x) \geq \epsilon n, \cap \{E_j \cap G_j\}), \tag{2.46}
\]

We now estimate $P(G_M(x) \geq \epsilon n, \cap \{E_j \cap G_j\})$. Let $K_n(x)$ be the event that $E_j$ occurs, but $G_j$ does not occur for all the G-cubes $B_1, \ldots, B_j, \ldots, B_{G_M(x)}$. On $G_M(x) = i$, we fix $B_j = B_j$ for fixed cubes in deterministic 2M-cubes $\{B_j\}$ in the different cylinders for $j \leq i$ such that $E_j \cap G_j$ occurs in each $B_j$. Here $B_1, \ldots, B_j, \ldots, B_i$ are in order if we go along the optimal path $\gamma_n(x)$ from the origin to $nx$. Note that $\{B_j = B_j\}$ are disjoint events for different $B_j$’s. With these observations,
\[
P(G_M(x) \geq \epsilon n, \cap \{E_j \cap G_j\}) = P(K_n(x), G_M(x) \geq \epsilon n) \leq \sum_{i \geq \epsilon n} \sum_{B_j} P(\cap \{E_j \cap G_j, B_j = B_j\}), \tag{2.47}
\]

where the first sum takes all possible $i \geq \epsilon n$, and the second sum takes all possible $B_1, \ldots, B_j, \ldots, B_i$. By (2.38) and (2.47),
\[
P(K_n(x), G_M(x) \geq \epsilon n) \\
\leq \sum_{i \geq \epsilon n} \sum_{B_j} P(\cap \{E_j \cap G_j, B_j = B_j\}, E_i \cap G_j, B_i = B_i) \tag{2.48}
\]

Now we estimate $P(\cap \{E_j \cap G_j, B_j = B_j\}, E_i, B_i = B_i)$. We need to consider all the configurations $\{\omega\}$ in $\{\cap \{E_j \cap G_j, B_j = B_j\}, E_i, B_i = B_i\}$. For simplicity’s sake, we first work on a discrete time $t(e)$ with a distribution $F_m$. For any distribution $F$, we can find discrete distributions $F_m$ such that $F_m \Rightarrow F$. We can start to estimate $P_{F_m}(\cap \{E_j \cap G_j, B_j = B_j\}, E_i, B_i = B_i)$ with distribution $F_m$. We have
\[
P_{F_m}(\cap \{E_j \cap G_j, B_j = B_j\}, E_i, B_i = B_i) \\
= \sum_{\omega} P_{F_m}(\omega \in \{\cap \{E_j \cap G_j, B_j = B_j\}, E_i, B_i = B_i\}), \tag{2.49}
\]
where the sum takes over all possible configurations \( \{\omega\} \). Certainly, we can use distribution \( F \) directly and use the integral to replace the sum in (2.49). However, we would rather use a discrete passage time for simpler notations and let \( m \to \infty \) eventually. For fixed \( B_i \), we focus on the edges inside \( B_i \) between \( L(i - M, i + M) \). We denote the above edge set in \( B_i \) by \( D_i \). For each \( \omega \), we regroup \( \omega \) such that we only account for \( t(e) > 0 \) or \( t(e) = 0 \) for \( e \in D_i \), but ignore the precise value of \( t(e) \). The other edge configurations (not in \( D_i \)) remain the same in \( \omega \). Let \( \{\omega(D_i)\} \) be the configuration after regrouping for each \( \omega \). We can rewrite (2.49) by

\[
P_{F_m}(\cap_{j \leq i-1}\{E_j \cap G_j^C, B_i = B_i\}, E_i, B_i = B_i) = \sum_{\omega(D_i)} P_{F_m}(\omega(D_i)) \in \{\cap_{j \leq i-1}\{E_j \cap G_j^C, B_i = B_i\}, E_i, B_i = B_i\}, \tag{2.50}
\]

where the sum takes in all possible configurations of \( \{\omega(D_i)\} \). For each configuration of

\[
\omega(D_i) = \{\cap_{j \leq i-1}\{G_j^C, B_j = B_j\}, E_i, B_i = B_i\},
\]

we only need to change the closed edges in \( \zeta_i \) to be open to make a configuration of \( G_i \). In other words, we can change at most \((4M)^d \) many edge configurations in \( D_i \) to make a configuration of \( G_i \). Let \( \tau(\omega(D_i)) \) be this change of the configuration. To change each edge from closed to open, it will cost at most \((1 - F(0))/F(0)\) in probability. Since \( F(0) < p_c \leq 0.5 \), it will not cost to change an edge from open to closed. So it will cost at most in total probability

\[
[(1 - F(0))/F(0)]^{(4M)^d} \tag{2.51}
\]

for changing \( \omega(D_i) \) to \( \tau(\omega(D_i)) \) for each \( \omega(D_i) \). Thus,

\[
\sum_{\omega(D_i)} P_{F_m}(\omega) \in \{\cap_{j \leq i-1}\{E_j \cap G_j^C, B_j = B_j\}, E_i, B_i = B_i\}) \leq \sum_{\omega(D_i)} [(1 - F(0))/F(0)]^{(4M)^d} P_{F_m}(\tau(\omega(D_i))). \tag{2.52}
\]

After this change, \( \{\cap_{j \leq i-1}\{E_j \cap G_j^C, B_j = B_j\}, G_i, B_i = B_i\} \) occurs. So

\[
\tau(\omega(D_i)) \in \{\cap_{j \leq i-1}\{E_j \cap G_j^C, B_j = B_j\}, G_i, B_i = B_i\}. \tag{2.53}
\]

However, after changing configurations, two configurations, \( \omega_1(D_i) \) and \( \omega_2(D_i) \), might not be disjoint. In other words, for \( \omega_1(D_i) \neq \omega_2(D_i) \) and

\[
\omega_1(D_i), \omega_2(D_i) \in \{\cap_{j \leq i-1}\{E_j \cap G_j^C, B_j = B_j\}, E_i, B_i = B_i\}, \tag{2.54}
\]

we may have

\[
\tau(\omega_1(D_i)) = \tau(\omega_2(D_i)). \tag{2.55}
\]

Note that the edges in \( D_i \) only have two states, zero or positive, so there are at most \( 2^{(4M)^d} \) different states in \( D_i \). We write \( \tau(\omega(D_i)) \) to be the configuration that \( \zeta_j \) is open, and its boundary edges are
closed, and the other edges remain the same as $\omega$. Thus, there are at most $2^{(4M)^d}$ configurations in $\{\omega(D_i)\}$ that correspond to a configuration in $\tau(\omega(D_i))$. With these observations,

$$
\sum_{\omega(D_i)} P_{F_m}(\tau(\omega(D_i))) \leq \sum_{\tau(\omega(D_i))} 2^{(4M)^d} P_{F_m}(\tau(\omega(D_i))) = 2^{(4M)^d} \sum_{\tau(\omega(D_i))} P_{F_m}(\tau(\omega(D_i))),
$$

(2.56)

where the sum in the right side of (2.56) takes all the possible $\tau(\omega(D_i))$. We say $\tau(\omega_1(D_i))$ and $\tau(\omega_2(D_i))$ are different when $\tau(\omega_1(D_i))$ and $\tau(\omega_2(D_i))$ have different $t(e)$’s in zero or non-zero states in $D_i$.

Together with (2.52) and (2.56),

$$
P_{F_m}(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{E}_i, B_i = B_i)
\leq \left(\frac{1 - F(0)}{F(0)}\right)^{(4M)^d} 2^{(4M)^d} P_{F_m}(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{G}_i, B_i = B_i)
\leq \left(\frac{2(1 - F(0))}{F(0)}\right)^{(4M)^d} P_{F_m}(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{G}_i, B_i = B_i).
$$

(2.57)

By our construction, on $\mathcal{G}_i$, $B_i$ remains the $G$-cube. On the other hand, the other edge configurations of $\tau(\omega(D_i))$ not in $D_i$ remain the same as $\omega(D_i)$. Thus,

$$\{\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{G}_i, B_i = B_i\} \text{ are still disjoint in different } B_j \text{'s.}
$$

(2.58)

Note that (2.57) holds for distribution function $F_m$, and now we need to show (2.57) holds for distribution $F$. We know that if $\xi$ is uniform on $(0,1)$, then for any distribution $U, X = U^{-1}(\xi)$ has a distribution $U$. Let $t(e) = F^{-1}(\xi_e)$ and $t_m(e) = F_m^{-1}(\xi_e)$ for an i.i.d family $\{\xi_e\}$. Since $F_m \Rightarrow F$,

$$
\lim_{m \to \infty} t_m(e) = t(e), \text{ a.s.}
$$

Note that on $\mathcal{E}_n(1)$, $t_{0,nx}$ only uses the edges in $[-\lambda n, \lambda n]^d$, so for finite $n$, by the dominated convergence theorem,

$$
\lim_{m \to \infty} P_{F_m}(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{E}_i, B_i = B_i)
= \lim_{m \to \infty} E_{F_m} I(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{E}_i, B_i = B_i)
= P_F(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{E}_i, B_i = B_i).
$$

(2.59)

If we replace $\mathcal{E}_i$ by $\mathcal{G}_i$, (2.59) also holds. Thus, (2.57) holds for $F$. By (2.57), if we take $m \to \infty$,

$$
\left(\frac{2(1 - F(0))}{F(0)}\right)^{-4M^d} P(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{E}_i, B_i = B_i)
\leq P(\cap_{j\leq i-1}\{\mathcal{E}_j \cap \mathcal{G}_j^C, B_j = B_j\}, \mathcal{G}_i, B_i = B_i).
$$

(2.60)
By (2.60),
\[
P(\cap_{j \leq i-1} \{ E_j \cap G_j^C, B_j = B_j \}, \mathcal{E}_i, B_i = B_i) - P(\cap_{j \leq i-1} \{ E_j \cap G_j^C, B_j = B_j \}, \mathcal{G}_i, B_i = B_i)
\leq P(\cap_{j \leq i-1} \{ E_j \cap G_j^C, B_j = B_j \}, \mathcal{E}_i, B_i = B_i) - \left(2(1 - F(0))\right)^{(-4M)}^{d} \cdot P(\cap_{j \leq i-1} \{ E_j \cap G_j^C, B_j = B_j \}, \mathcal{G}_i, B_i = B_i)
\leq (1 - \exp(-O((4M)^d))) \cdot P(\cap_{j \leq i-1} \{ G_j^C, B_j = B_j \}, B_i = B_i).
\] (2.61)

By using (2.61) in (2.48) together with (2.58),
\[
P(K_n(x), G_M(x) \geq \epsilon n)
\leq \sum_{i \geq \epsilon n} \sum_{B_j} \left[ P(\cap_{j \leq i-1} \{ G_j^C, B_j = B_j \}, \mathcal{E}_i, B_j = B_j) - P(\cap_{j \leq i-1} \{ E_j \cap G_j^C, B_j = B_j \}, \mathcal{G}_i, B_i = B_i) \right]
\leq (1 - \exp(-O((4M)^d))) \sum_{i \geq \epsilon n - 1} \sum_{B_j} P(\cap_{j \leq i-1} \{ G_j^C, B_j = B_j \}).
\] (2.62)

We iterate (2.62) for each $j$ to have
\[
P(K_n(x), G_M(x) \geq \epsilon n) \leq (1 - \exp(-O((4M)^d)))^m \leq \exp(-O(n)).
\] (2.63)

By (2.46) and (2.63), if $x \in \mathbb{S}_p$, then
\[
P(\text{there is a } 2M\text{-exit for } Q_n(x) \text{ in } L(a_n, b_n) \geq \epsilon n) \geq 1 - \exp(-O(n)).
\] (2.64)

Lemma 2.2 follows from (2.64). \qed

**Proof of Theorem 1.** Let $H_M(x)$ be the number of $j$'s such that $E_j \cap G_j^C$ occurs among these $G$-cubes on $G_M(x) \geq \epsilon n$. We estimate $\{ G_M(x) - H_M(x) \leq \eta n, G_M(x) \geq \epsilon n \}$ for $\epsilon > \eta$ defined in Lemma 2.2. We choose these cubes from $B_1, \cdots, B_i$ such that $E_j \cap G_j^C$ occurs in these cubes. Thus, after removing these cubes from $B_1, \cdots, B_i$, $G_j$ occurs in the remaining cubes. By (2.47), if $G_M(x) - H_M(x) \leq \eta n$ is small, on $|\gamma_n(x)| \leq \lambda n$ for all optimal paths of $\mathcal{P}_n(x)$, then
\[
P(G_M(x) - H_M(x) \leq \eta n, G_M(x) \geq \epsilon n) \leq \sum_{i \geq \epsilon - \eta n} \sum_{B_j} \left( \lambda n \right)^{\eta n - \left(\lambda n \right)^{\eta n}} P(\cap_{j \leq i} \{ E_j \cap G_j^C, B_j = B_j \}).
\] (2.65)

By the same estimate of (2.64) together with (1.9), if $x \in \mathbb{S}_p$, then by using the Stirling formula in (2.65),
\[
P(G_M(x) - H_M(x) \leq \eta n, G_M(x) \geq \epsilon n) \leq 2O(\eta n)(1 - \exp(-O(4M)^d))^{(\epsilon - \eta)n}.
\] (2.66)

If we take $\eta > 0$ small, by (2.66), then
\[
P(G_M(x) - H_M(x) \leq \eta n, G_M(x) \geq \epsilon n) \leq \exp(-O(n)).
\] (2.67)
By (2.43) and (2.67), there exists $0 < \epsilon_1 < \epsilon$ such that
\[
P(G_M(x) - H_M(x) \geq \epsilon_1 n, G_M(x) \geq \epsilon n) \geq 1 - \exp(-O(n)). \tag{2.68}
\]
Thus, there are at least $\epsilon_1 n$ many cubes $B_j$ such that $G_j$ occurs in each cube. By using the same proof of (2.45), note that $\{B_j\}$ are in different cylinders, so there are at least $\epsilon_1 n$ many cubes $B_j$, counted as $M$-exits with a probability larger than $1 - \exp(-O(n))$. Therefore, Theorem 1 follows. □

Proof of Theorem 2. Let $\mathcal{D}_{nx}(\kappa)$ be the event that there exists an $M$-exit in each cylinder $L(a_n + ik_n, a_n + (i+1)k_n)$ for each $i$. Let $\gamma_n(x) \in \mathcal{P}_n(x)$, for $x \in S_\phi$, be an optimal path from the origin first meeting $L_{a_n + ik_n}$ at $v_i$. After meeting $v_i$, $\gamma_n(x)$ continues to meet $L_{a_n + (i+1)k_n}$ at $v_{i+1}$, and $\gamma_n(x)$ will not come back to $L_{a_n + (i+1)k_n}$ after $v_{i+1}$. On $|\gamma_n(x)| \leq \lambda n$, $v_i = (x_1(i), \cdots, x_d(i))$ and $v_{i+1} = (x_1(i+1), \cdots, x_d(i+1))$ are two vertices on $L_{a_n + ik_n}$ and on $L_{a_n + (i+1)k_n}$ with
\[
d(x_1(i), x_1(i+1)) \geq \kappa n \text{ and } \| (x_2(i), \cdots, x_d(i)) , (x_2(i), \cdots, x_d(i)) \|_{d-1} \leq \lambda n. \tag{2.69}
\]
We divide $\gamma_n(x)$ into the following three sub-paths:
\[
\gamma_n(x) = \gamma(0, v_i) \cup \gamma(v_i, v_{i+1}) \cup \gamma(v_{i+1}, nx).
\]
Thus, for any $\kappa > \delta_1 > 0$, for fixed $v_i$ and $v_{i+1}$, by (2.69) and (1.8),
\[
P((\mu_F - \delta_1/8)(n\kappa) \geq t_{0,v_i} = T(\gamma(0, v_i)), |\gamma_n(x)| \leq \lambda n) \\
\leq \sum_{v_i} P((\mu_F - \delta_1/8)(n\kappa) \geq t_{0,w_i} = T(\gamma(0, v_i)), |\gamma_n(x)| \leq \lambda n, v_i = w_i) \\
\leq (\lambda n)^d \exp(-O(\delta_1n)) = \exp(-O(\delta_1n)). \tag{2.70}
\]
Thus, by (2.70),
\[
P((\mu_F - \delta_1/8)(n\kappa) \leq t_{0,v_i} = T(\gamma(0, v_i)), |\gamma_n(x)| \leq \lambda n) \geq 1 - \exp(-O(\delta_1n)). \tag{2.71}
\]
Similarly, we have
\[
P((\mu_F - \delta_1/8)(n\kappa) \leq t_{v_i,v_{i+1}} = T(\gamma(v_i, v_{i+1})), |\gamma_n(x)| \leq \lambda n) \geq 1 - \exp(-O(\delta_1n)), \tag{2.72}
\]
and
\[
P((\mu_F - \delta_1/8)(n-(i+1))n\kappa \leq t_{v_{i+1},nx} = T(\gamma(v_{i+1}, nx)), |\gamma_n(x)| \leq \lambda n) \geq 1 - \exp(-O(\delta_1n)). \tag{2.73}
\]
On the other hand, for $v_i$ and $v_{i+1}$ (not fixed), by (1.8) and (1.9), we take $\phi > 0$ small,
\[
1 - \exp(-O(\delta_1n)) \\
\leq P(t_{0,v_i} = t_{0,v_i} + t_{v_i,v_{i+1}} + t_{v_{i+1},nx} \leq n(\mu_F(x) + \delta_1/8), |\gamma_n(x)| \leq \lambda n) \\
\leq P(t_{0,v_i} = t_{0,v_i} + t_{v_i,v_{i+1}} + t_{v_{i+1},nx} \leq n(\mu_F + \delta_1/4), |\gamma_n(x)| \leq \lambda n) \\
\leq \sum_{w_i,w_{i+1}} P(t_{0,w_i} = t_{0,w_i} + t_{w_i,w_{i+1}} + t_{w_{i+1},nx} \leq n(\mu_F + \delta_1/4), v_i = w_i, v_{i+1} = w_{i+1}), \tag{2.74}
\]
where the sum takes all possible $w_i \in L_{\alpha_n+i\kappa n}$ and $w_{i+1} \in L_{\alpha_n+(i+1)\kappa n}$ with
\[
\|w_i\| \leq \lambda n, \|w_{i+1}\| \leq \lambda n.
\] (2.75)

Substituting (2.71)–(2.73) into (2.74), by (2.74) and (1.8),
\[
1 - \exp(-O(\delta_1 n)) \\
\leq \sum_{w_i, w_{i+1}} P(t_{0, nx} = t_{0, w_i} + t_{w_i, w_{i+1}} + t_{w_{i+1}, nx} \leq n(\mu_F + \delta_1), v_i = w_i, v_{i+1} = w_{i+1}) \\
\leq \sum_{w_i, w_{i+1}} P(t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2), v_i = w_i, v_{i+1} = w_{i+1} + 2(\lambda n)^d \exp(-O(\delta_1 n))) \\
\leq \sum_{w_i, w_{i+1}} P(nk(\mu_F - \delta_1/2) \leq t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2), v_i = w_i, v_{i+1} = w_{i+1}) \\
+ 4(\lambda n)^d \exp(-O(\delta_1 n)).
\] (2.76)

With (2.76), on $nk(\mu_F - \delta_1/2) \leq t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2)$ for fixed $i$, $w_i$, and $w_{i+1}$, we can take $\delta_1$ small and use the same proof of Lemma 2.1 for the passage time from $w_i$ to $w_{i+1}$. In fact, by translation invariance, if $\delta_1$ is small, then
\[
P(nk(\mu_F - \delta_1/2) \leq t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2)) = P(nk(\mu_F - \delta_1/2) \leq t_{0, nk\phi} \leq nk(\mu_F + \delta_1/2))
\] (2.77)
for $v \in S_\phi$ for $\phi$ satisfying the condition in Lemma 2.1. Thus, by Lemma 2.1, note that the sub-path $\gamma(w_i, w_{i+1})$ is an optimal path from $w_i$ and $w_{i+1}$,
\[
P(nk(\mu_F - \delta_1/2) \leq t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2))
\geq P(nk(\mu_F - \delta_1/2) \leq t_{w_i, w_{i+1}} \leq nk(\mu_F + \delta_1/2), F_M(w_i, w_{i+1}) \leq nk(1 - \epsilon)) - \exp(-O(\delta_1 n))
\geq P(F_M(w_i, w_{i+1}) \leq nk(1 - \epsilon)) - \exp(-O(\delta_1 n)),
\] (2.78)

where $F_M(w_i, w_{i+1})$ is the number of $M$-arms from $w_i$ to $w_{i+1}$ defined in Lemma 2.1 for $\gamma(w_i, w_{i+1})$. If (2.78) holds, we define $G_M(w_i, w_{i+1})$ to be the number of $G$-cubes in $\gamma(w_i, w_{i+1})$ We also define $K(w_i, w_{i+1})$ to be the number of $G$-cubes in $\gamma(w_i, w_{i+1})$ such that $G_j$ does not occur among these $G$-cubes. Here a $G$-cube in $\gamma(w_i, w_{i+1})$ means that its $\beta_j(2) \subset \gamma(w_i, w_{i+1})$, and
\[
T(r_n) > T(\gamma_n) - T(\beta_j(2))
\]
for any path $r_n \in P_n(x)$. By Lemma 2.2, there exists $\epsilon > 0$ and $\delta_1 > 0$, such that
\[
P(K(w_i, w_{i+1}), G_M(w_i, w_{i+1}) \geq \epsilon\kappa n) \leq \exp(-O(\delta_1 n)).
\] (2.79)

By (2.79), if we define $D_{w_i, w_{i+1}}(\kappa)$ to be the event that there is an $M$-exit of $Q_n(x)$ in $L[a_n + i\kappa n, a_n + (i + 1)\kappa n]$ for fixed $w_i$ and $w_{i+1}$, then
\[
P(D_{w_i, w_{i+1}}(\kappa)) \geq 1 - \exp(-O(\delta_1 n)).
\] (2.80)
Thus, by (2.80),
\[
P(D_n^C(\kappa)) \leq \sum_i \sum_{w_i, w_{i+1}} P(D_{w_i, w_{i+1}}^C(\kappa)) \leq (\lambda n)^{3d} \exp(-O(\delta_1 n)) \leq \exp(-O(\delta_1 n)). \tag{2.81}
\]

Now we estimate the probability of the largest distance of \(v\), the center of \(M(v)\), to vertices of \(Q_n(x)\) in a \(\kappa\)-cylinder. By (2.81),
\[
P(D_n^C(\kappa)) \leq P(D_n^C(\kappa) \cap D_n^C(\kappa)) + \exp(-O(\delta_1 n)). \tag{2.82}
\]
On \(D_n^C(\kappa) \cap D_n^C(\kappa)\), for each \(\kappa\)-cylinder \(L[a_n + i\kappa n, a_n + (i + 1)\kappa n]\), there exists an \(M\)-exit with the center \(v\) for
\[
v \in Q_n(x) \cap L[a_n + i\kappa n, a_n + (i + 1)\kappa n], \tag{2.83}
\]
such that all the optimal paths \(\{\gamma_n(x)\}\) have to pass \(M(v)\). On \(D_n^C(\kappa) \cap D_n^C(\kappa)\), we suppose that there is a vertex \(v' \in \gamma_n(x) \cap L[a_n + i\kappa n, a_n + (i + 1)\kappa n]\) for some \(\gamma_n(x)\) such that
\[
d(v, v') \geq \lambda \kappa n + M \tag{2.84}
\]
for some large \(\lambda\) defined in (1.9). Since \(\gamma_n(x)\) has to pass \(M(v)\), there is \(v'' \in \gamma_n(x) \cap M(v)\) such that
\[
d(v', v'') \geq \lambda \kappa n.
\]
There is an optimal path such that it contains \(v''\) and \(v'\) satisfying (2.84). Note that \(v'\) and \(v''\) are in \(L[a_n + i\kappa n, a_n + (i + 1)\kappa n]\), so
\[
P(D_n^C(\kappa) \cap D_n(\kappa x), d(v'', v') \geq \lambda \kappa n)
\]
\[
\leq P(D_n^C(\kappa) \cap D_n(\kappa x), d(v', v'') \geq \lambda \kappa n, E_n(1)) + \exp(-O(n))
\]
\[
= \sum_{w_i, w_{i+1}} P(D_n^C(\kappa) \cap D_n(\kappa x), d(v', v'') \geq \lambda \kappa n, E_n(1), v_i = w_i, v_{i+1} = w_{i+1}) + \exp(-O(n))
\]
\[
\leq \sum_{w_i, w_{i+1}} P(D_n^C(\kappa) \cap D_n(\kappa x), |\gamma(w_i, w_{i+1})| \geq \lambda \kappa n, v_i = w_i, v_{i+1} = w_{i+1}) + \exp(-O(n)), \tag{2.85}
\]
where the sum in (2.85) takes all \(w_i, w_{i+1}\) in \([-\lambda n, \lambda n]^d\), and \(\gamma(w_i, w_{i+1})\) is the optimal path from \(w_i\) to \(w_{i+1}\). By (2.76), and (2.85), and Kesten’s Proposition 5.8 (1986), if \(\lambda\) is large, but independent of \(n, \kappa,\) and \(x\), then
\[
P(D_n^C(\kappa) \cap D_n(\kappa x), d(v'', v') \geq \lambda \kappa n)
\]
\[
\leq \sum_{w_i, w_{i+1}} P(t_{w_i, w_{i+1}} \leq n(k + \delta_1/2), |\gamma(w_i, w_{i+1})| \geq \lambda \kappa n, v_i = w_i, v_{i+1} = w_{i+1})
\]
\[
+ (3\lambda n)^d \exp(-O(\delta_1 n)) \leq 2(3\lambda n)^d \exp(-O(\delta_1 n)). \tag{2.86}
\]
Theorem 2 follows (2.82) and (2.86). \(\Box\)
3 Proof of Theorem 3.

In this section, \( t(\epsilon) \) takes value 0 or 1 with a probability \( F(0) < p_c \) and \( F(1) = 1 - F(0) \). Suppose that \( S_O \neq O \). Thus, \( S_O \) contains another vertex \( u \neq O \). Let \( \mu_F^{-1}u = u' \in S \subset L_1 \) and \( \mu_F^{-1}O = O' = (1, 0, \cdots, 0) \in S \subset L_1 \). Since \( u' \neq O' \), we suppose that

\[
\|O' - u'\| = \eta \quad \text{for some } \eta > 0.
\]  

For small \( \epsilon > 0 \), we consider

\[
D(n) = \{v \in \mathbb{Z}^d : T(0, v) \leq \mu_F n - \epsilon n\}.
\]  

Let \( \partial D(n) \) be the exterior boundary of \( D(n) \) and \( \Delta D(n) \) be the boundary edges such that each of them has only one of its vertices adjacent to \( \partial D(n) \). It is easy to check that \( D(n) \) only depends on the configurations on

the edges of \( D(n) \) and \( \Delta D(n) \).

Let \( \mathcal{E}_n(\epsilon) \) be the event that

\[
D(n) \subset L(0, n),
\]

and there exist paths \( \gamma(u'n) \) and \( \gamma(O'n) \) parallel to the \( x_1 \)-axis from \( \Delta D(n) \) to \( u'n \) and to \( O'n \), respectively without using edges of \( \partial D(n) \) and \( \Delta D(n) \) such that

\[
|\gamma(u'n)| \leq n\epsilon \quad \text{and} \quad |\gamma(O'n)| \leq n\epsilon.
\]

By (1.8), for \( u' \) and \( O' \) defined above,

\[
P(\mathcal{E}_n(\epsilon)) \geq 1 - \exp(-O(n)) \geq 0.5.
\]  

On \( \mathcal{E}_n(\epsilon) \), if \( D(n) = D \) for a fixed set \( D \), we force the edges in \( \gamma(u'n) \cup \gamma(O'n) \) to be open. It costs at most \( F(0)^{2\epsilon n} \) in probability. Thus, by (3.3)–(3.6),

\[
P(\text{there exist two optimal paths of } b_{0,n} \text{ from the origin to } u'n \text{ and to } O'n \text{ in } L(-\infty, n))
\]

\[
\geq 0.5(F(0))^{2\epsilon n}.
\]  

Let \( b'_{0,n} \) be the passage time from \((2n, 0, \cdots, 0) \) to \( L_n \). By (3.7) and symmetry,

\[
P(\text{there exist two optimal paths of } b'_{0,n} \text{ from } (2n, 0, \cdots, 0) \text{ to } u'n \text{ and to } O'n \text{ in } L[n, \infty))
\]

\[
\geq 0.5(F(0))^{2\epsilon n}.
\]  

Note that if there are optima paths of \( b_{0,n} \) and \( b'_{0,n} \) from the origin and from \((2n, 0, \cdots, 0) \) to \( u'n \), respectively, then there is an optimal from the origin to \((2n, 0, \cdots, 0) \) passing through \( L_n \) once at \( u'n \). Similarly, if there are optima paths of \( b_{0,n} \) and \( b'_{0,n} \) from the origin and from \((2n, 0, \cdots, 0) \) to \( O'n \), respectively, then there is an optimal from the origin to \((2n, 0, \cdots, 0) \) passing through \( L_n \) once
at $O'n$. Note also that the two events in (3.7) and (3.8) are independent, so if $E'_n(\epsilon)$ denotes the event that there are two optimal paths $\gamma'_2n(2nO')$ and $\gamma''_2n(2nO')$ from the origin to $(2n, 0 \cdots, 0)$ such that $\gamma'_2n(2nO')$ only crosses $L_n$ once at $u'n$ and $\gamma''_2n(2nO')$ only crosses $L_n$ once at $O'n$, then

$$P(E'_n(\epsilon)) \geq F(0)^O(\epsilon n).$$

(3.9)

On $E'_n(\epsilon)$, we may assume that

$$L_n \subset L(a_2n + ikn, a_2n + (i + 1)kn)$$

for $k > \epsilon$ and for some $i$. By Theorem 2, for any $\kappa > 0$, we take $\epsilon$ small such that

$$\exp(-O(\epsilon n)) \leq P(E'_n(\epsilon) \cap D_{2nO'}(\kappa)).$$

(3.10)

On $D_{2nO'}(\kappa)$, we may assume that there exists $M$-exit $M(v)$ for $v \in L(a_2n + ikn, a_2n + (i + 1)kn)$. We have

$$d(v, u'n) \leq \lambda kn + M \text{ and } d(v, O'n) \leq \lambda kn + M.$$  

(3.11)

By (3.11) and the triangular inequality,

$$\eta < 2\lambda k + 2M/n.$$  

(3.12)

Since $\eta$ is assumed to be a positive constant, (3.12) will contradict if $\kappa$ is small and $n$ is large. The contradict tells that $S_O = O$. Therefore, Theorem 3 follows. $\square$
References

Chow, Y. and Zhang, Y. (2003). Large deviation in first passage percolation. *Ann. Appl. Probab.* **4** 1601–1614.

Cox, J. T. and Durrett, R. (1981). Some limit theorems for percolation processes with necessary and sufficient condition. *Ann. Probab.* **9** 583–603.

Durrett, R. and Liggett, T. (1981). The shape of the limit set in Richardson’s growth model. *Ann. Probab.* **9** 186–193.

Grimmett, G. (1999). *Percolation.* Springer, Berlin.

Hammersley, J. M. and Welsh, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory. In *Bernoulli, Bayes, Laplace Anniversary Volume* (J. Neyman and L. LeCam, eds.) 61–110. Springer, Berlin.

Kesten, H. (1986). Aspects of first-passage percolation. *Lecture Notes in Math.* **1180** 125–264. Springer, Berlin.

Kesten, H. (1993). On the speed of convergence in first passage percolation. *Ann. Appl. Probab.* **3** 296–338.

Newman, C. and Piza, M. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.* **23** 977–1005.

Smythe, R. T. and Wierman, J. C. (1978). First passage percolation on the square lattice. *Lecture Notes in Math.* **671**. Springer, Berlin.

Zhang, Y. (2010). On the concentration and the convergence rate with a moment condition in first passage percolation. *Stochastic Process Appl.* **120** 1317–1341.

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