ON THE SPECTRUM OF $\bar{X}$-BOUNDED MINIMAL SUBMANIFOLDS

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Abstract

We prove, under a certain boundedness condition at infinity of a $(\bar{X}^T, \bar{X}^\perp)$ component of the second fundamental form, the vanishing of the essential spectrum of a complete minimal $\bar{X}$-bounded and $\bar{X}$-properly immersed submanifold on a Riemannian manifold endowed with a strongly convex vector field $\bar{X}$. The same conclusion also holds for any complete minimal $h$-bounded and $h$-properly immersed submanifold that lies in a open set of a Riemannian manifold $\bar{M}$ supporting a nonnegative strictly convex function $h$. This extends a recent result of Bessa, Jorge and Montenegro on the spectrum of Martin-Morales minimal surfaces. Our proof uses as main tool an extension of Barta’s theorem given in [2].

1 Introduction and main results

Since Calabi in 1965 [4] conjectured that complete minimal hypersurfaces in Euclidean spaces are unbounded, some answers have been given, with a positive answer by Colding and Minicozzi [6] for the case of embedded surfaces, and a negative answer with the counterexamples given by Nadirashvili [14] and by Martin and Morales [12, 13] for the case of immersed nonembedded surfaces.

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Partially supported by FCT through the Plurianual of CFIF.
This conjecture also motivates many other related problems in more general ambient spaces, for instance, on the topological and geometrical properties of minimal submanifolds that are bounded or not, or on the search of conditions for a submanifold to be unbounded. In [3] the structure of the spectrum of the Martin-Morales surfaces is studied, namely it is proved that complete bounded minimal properly immersed submanifolds of the unit open ball of $\mathbb{R}^n$ must have pure point spectrum.

In this note we extend the above result of Bessa, Jorge and Montenegro to an ambient space carrying an almost conformal vector field $\bar{X}$, a concept introduced in ([16, 17]). On a Riemannian $(m+n)$-dimensional manifold $(\bar{M}, \bar{g})$ we say a vector field $\bar{X}$ is almost conformal if

$$2\alpha \bar{g} \leq L_{\bar{X}} \bar{g} \leq 2\beta \bar{g}$$

where $+\infty \geq \beta \geq \alpha > 0$ are constants, and $L_{\bar{X}} \bar{g}(\bar{Y}, \bar{Z}) = \bar{g}(\bar{\nabla}_\bar{Y} \bar{X}, \bar{Z}) + \bar{g}(\bar{\nabla}_\bar{Z} \bar{X}, \bar{Y})$, where $\bar{\nabla}$ is the Levi-Civita connection of $(\bar{M}, \bar{g})$. If we allow $\beta = +\infty$, in this case $\bar{X}$ is named by strongly convex.

An example of almost conformal vector field in a complete Riemannian manifold $M$ is the position vector field $\frac{1}{2} \bar{\nabla} r^2 = \bar{x}$ on a geodesic ball of $M$ of radius $R$ and center $\bar{p}$ that does not intercept the cut locus at $\bar{p}$ and $\sqrt{\kappa}R < \pi/2$ with $\kappa^{\pm} = \kappa^{\pm}(R) = \max\{0, \sup_{B_R(\bar{p})} \bar{K}\}$, where $\bar{K}$ are the sectional curvatures of $\bar{M}$ and $r$ is the distance function on $\bar{M}$ to a given point. In this case $\alpha$ and $\beta$ are well defined functions $\alpha = \alpha_{\kappa^{+}}(R), \beta = \alpha_{\kappa^{-}}(R)$, of $R$, $\kappa^{+}$, and $\kappa^{-} = \kappa^{-}(R) = \min\{0, \inf_{B_R(\bar{p})} \bar{K}\}$ where

$$\alpha_{\kappa}(R) = \begin{cases} R\sqrt{\kappa}\cot(\sqrt{\kappa}R) & \text{for } 0 \leq R < \pi/2\sqrt{\kappa}, \text{ when } \kappa > 0 \\ 1 & \text{for } 0 \leq R < +\infty, \text{ when } \kappa = 0 \\ R\sqrt{-\kappa}\coth(\sqrt{-\kappa}R) & \text{for } 0 \leq R < +\infty, \text{ when } \kappa < 0. \end{cases}$$

A strictly convex function $f$ on $\bar{M}$ with Hess $f \geq \alpha \bar{g}$ defines a strongly convex vector field $\bar{\nabla} f$. Positive homothetic non-Killing vector fields are almost conformal. In $\mathbb{R}^{m+n}$ the position vector field $\bar{X}_x = x$ is such an example. A particular feature of strongly convex vector fields, is that the norm $\|\bar{X}\|$ must take its maximum on the boundary of compact domains (see proposition 1). Therefore $\bar{X}$ cannot be globally defined on a compact manifold $\bar{M}$ without boundary.

Strongly convex vector fields have a role on isoperimetric inequalities for an immersed $m$-dimensional submanifold $F : M \to \bar{M}, m \geq 2$, involving the mean curvature $H$. The Cheeger constant of $M$ is defined by $h(M) = \inf_D A(\partial D)/V(D)$,
where $D$ runs over all compact domains $D$ of $M$ with piecewise smooth boundary $\partial D \subset M$ of respective volume $V(D)$ and area $A(\partial D)$. We recall the following inequality [11]:

$$(\sup_M \| \bar{X}_F \|)^{-1} \leq \frac{1}{\alpha} \left( \frac{1}{m} \mathcal{H}(M) + \sup_M \| H \| \right)$$

(3)

where $\bar{X}_F$ denotes $\bar{X}$ along $F$. Let $\bar{X}^\top$ and $\bar{X}^\perp$ denote the orthogonal projections of $\bar{X}_F$ onto $TM$, and the normal bundle $NM$ respectively. We remark that, following the proof in [11] we see that if $F$ is minimal we have a sharper inequality:

$$(\sup_M \| \bar{X}^\top \|)^{-1} \leq \frac{1}{\alpha} \frac{1}{m} \mathcal{H}(M).$$

(4)

We note that $\bar{X}_F$ (resp.) cannot vanish identically for any minimal immersion $F$ (see lemmas 1 and 2). In the case $M$ is the Euclidean space with the position vector field, $\| \bar{X}^\top \| \leq \| \bar{X}_F \| = \| F \|$. This leads to the following conclusion:

**Theorem 1** ([11]). If $\bar{X}$ is a strongly convex vector field on a neighbourhood of a minimal submanifold $F : M \to \bar{M}$ with zero Cheeger constant, then $\bar{X}^\top$ is unbounded. In the particular case $M = \mathbb{R}^{n+m}$, $F$ is unbounded.

We recall the following inequality due to Cheeger [5],

$$\mathcal{H}^2(D) \leq 4 \mathcal{L}(D)$$

where $\mathcal{L}(D)$ is the fundamental tone of a normal domain $D$ in $M$. For normal bounded domains, $\mathcal{L}(D)$ is the first eigenvalue for the boundary Dirichlet problem. The Rayleigh characterization of the fundamental tone of any open domain $D$ of $M$ is given by

$$\mathcal{L}(D) = \inf \left\{ \frac{\int_D \| \nabla f \|^2}{\int_D f^2} : f \in L^2_{1,0}(D) \right\}$$

where $L^2_{1,0}(D)$ is the completion of $C_0^\infty(D)$ for the norm $\| \phi \|^2 = \int_M \phi^2 + \| \nabla \phi \|^2$. Thus, if $M$ is complete noncompact, $\mathcal{L}(M) = \lim_{R} \mathcal{L}(D_R)$ and $\mathcal{H}(M) \leq \mathcal{H}(D_R)$, where $D_R$ is an exhaustion sequence of bounded domains of $M$ with smooth boundary in $M$. Therefore, from the above inequalities we have the following
estimate for $M$ a bounded domain (possibly with boundary) or a complete Rie-
mannian manifold
\[
(\sup_M \|\bar{\mathbf{X}}_F\|)^{-1} \leq \frac{1}{\alpha} \left( \frac{2}{m} \sqrt{\lambda(M)} + \sup_M \|H\| \right)
\]
\[
(\sup_M \|\bar{\mathbf{X}}^\top\|)^{-1} \leq \frac{1}{\alpha} \frac{2}{m} \sqrt{\lambda(M)}, \quad \text{if } M \text{ is minimal.} \quad (5)
\]

**Definition 1.** Given a vector field $\bar{\mathbf{X}}$ of $\overline{M}$, an immersed submanifold $F : M \rightarrow \overline{M}$ is said $\bar{\mathbf{X}}$-bounded if $\sup_M \|\bar{\mathbf{X}}^\top\| < +\infty$. If $\sup_M \|\bar{\mathbf{X}}^\top\|$ is not achieved, then $F$ is said $\bar{\mathbf{X}}$-proper, if $\|\bar{\mathbf{X}}^\top\| : M \rightarrow [0, \sup_M \|\bar{\mathbf{X}}^\top\|)$ is a proper map.

We will see in proposition 1 that if $M$ is minimal and $\bar{\mathbf{X}}$ is strongly convex, then $\sup_M \|\bar{\mathbf{X}}^\top\|$ is not achieved (in $M$) if condition (6) below holds. Note that if $\bar{\mathbf{X}}$ is the position vector field of $M$, $\|\bar{\mathbf{X}}^\top\| \leq \|\bar{\mathbf{X}}_F(p)\| = r(F(p))$. This implies $\bar{\mathbf{X}}$-boundedness is a weaker concept then the usual boundedness of $M$ in $\overline{M}$. For example, the spiral curve in $\mathbb{R}^2$, $\gamma(t) = ae^{bt}(\cos(e^{abt}), \sin(e^{abt}))$ with $a > 1$ and $b > 0$ constants, is $\bar{\mathbf{X}}$-bounded but unbounded in the usual sense. On the other hand $\bar{\mathbf{X}}$-properness might be a stronger concept than the usual properness of an immersion. We also remark that if $\bar{\mathbf{X}}^\perp = 0$ along all $M$, then $\bar{\nabla}r$ restricted to $M$ is a vector field on $M$. If $r$ is the distance function on $\overline{M}$ from a fixed point $p \in M$, we see that (unit) geodesics of $\overline{M}$ starting at $p$ (that are the integral curves of $\bar{\nabla}r$) lie in $M$. In this case $n = 0$.

Next we state our main theorems:

**Theorem 2.** Let $F : M \rightarrow \overline{M}$ be a complete minimal immersion that is $\bar{\mathbf{X}}$-bounded with $\sup_M \|\bar{\mathbf{X}}^\top\| = R$, where $\bar{\mathbf{X}}$ is a strongly convex vector field of $\overline{M}$ defined on a neighbourhood of $M$, then:

1. $2 \sqrt{\lambda(M)} \geq \delta(M) \geq \frac{ma}{R}$.

2. Furthermore, if the second fundamental form $B$ of $M$ satisfies at points $p \in M$ with $\|\bar{\mathbf{X}}^\top\|$ sufficiently close to $R$,

\[
|\bar{\mathbf{g}}(B(\bar{\mathbf{X}}^\top, \bar{\mathbf{X}}^\top), \bar{\mathbf{X}}^\perp)| \leq \alpha' \|\bar{\mathbf{X}}^\top\|^2,
\]

for some nonnegative constant $\alpha' < \alpha$, and if $M$ is $\bar{\mathbf{X}}$-proper, then the spectrum of $M$ is a pure point spectrum.
The condition (6) does not mean \( \|B\| \) is bounded, even in the case \( \bar{X} \) is the position vector field \( r \frac{\partial}{\partial r} \). In theorem 5 (section 2) we will see that boundedness of the second fundamental form is, in general, not a compatible condition with the boundedness of a complete minimal submanifold, for ambient spaces with sectional curvature bounded from below. Moreover, for the particular case of \( \bar{X} \) being the gradient of a nonnegative convex smooth function \( h : \mathcal{M} \to [0, +\infty) \) we can remove the boundedness condition (6) of theorem 2, if we adapt our definition of boundedness and of properness: \( F \) is \( h \)-bounded if \( \sup_{\mathcal{M}} h \circ F = R < +\infty \), and is \( h \)-proper if \( \sup_{\mathcal{M}} h \circ F \) is not achieved and \( h \circ F : \mathcal{M} \to [0, R) \) is a proper function. We also will see in proposition 1 that \( \sup_{\mathcal{M}} h \circ F \) cannot be achieved for \( F \) a minimal immersion.

**Theorem 3.** Let \( h : \mathcal{M} \to [0, +\infty) \) be a nonnegative convex smooth function and \( F : \mathcal{M} \to \mathcal{M} \) a complete minimal immersion that is \( h \)-bounded. If \( F \) is \( h \)-proper, then the spectrum of \( \mathcal{M} \) is a pure point spectrum.

The above case contains the next example, when \( h = \frac{1}{2} r^2 \), where \( r \) is the distance function to a point \( \bar{p} \) in \( \mathcal{M} \). Note that if \( F \) is \( h \)-bounded, then it is also \( \bar{X} \)-bounded, for \( X \) the position vector field, and the concept of \( h \)-bounded (\( h \)-proper resp.) is equivalent to usual boundedness (properness resp.). Next corollary is a corollary of theorem 2 (1) and theorem 3:

**Corollary 1.** If \( F : \mathcal{M} \to \mathcal{M} \) is a complete bounded minimal submanifold with \( F(\mathcal{M}) \) lying in a open geodesic ball \( B_R(\bar{p}) \) of \( \mathcal{M} \), and \( R \) is in the conditions given in (2), then \( 2\sqrt{\lambda(\mathcal{M})} \geq \eta(\mathcal{M}) \geq \frac{m_R}{R} \), where \( \alpha = \alpha_k(\mathcal{R}) \). Furthermore, if \( F \) is a proper immersion into \( B_R(\bar{p}) \), then the spectrum of \( \mathcal{M} \) is a pure point spectrum.

**Corollary 2.** If \( F : \mathcal{M} \to \mathcal{M} \) is a complete bounded minimal submanifold properly immersed in \( B_R(\bar{p}) \), and \( \mathcal{M} \) is a complete Riemannian manifold with \( \bar{K} \leq 0 \), then \( 2\sqrt{\lambda(\mathcal{M})} \geq \eta(\mathcal{M}) \geq \frac{m}{R} \) and the spectrum of \( \mathcal{M} \) is a pure point spectrum.

The later corollaries are straightforward generalizations of [3]. Donnelly in [7] proved the existence of a non-empty essential spectrum for negatively curved manifolds under certain conditions. This result and corollary 2 gives next corollary:

**Corollary 3.** There is no complete simply connected minimal surface \( F : \mathcal{M}^2 \to \mathcal{M} \) properly immersed into a geodesic ball \( B_R(\bar{p}) \) of a space form \( \mathcal{M} \) of constant sectional curvature \( \bar{K} < 0 \), and satisfying \( \|B\|^2 \to c \) at infinity, for any nonnegative finite constant \( c \).
As we have announced above, in theorem 5 we will see this conclusion can be extended to a considerably more general setting, where we do not need to use spectral theory to prove it, but a generalized Liouville-type result due to Ranjbar-Motlagh [15].

An application of a hessian comparison theorem for the distance function to a totally convex submanifold due to Kasue [10] give us the following theorem:

**Theorem 4.** Let $\overline{M}$ be a connected complete Riemannian manifold with nonnegative sectional curvature and $\Sigma$ a totally convex submanifold of dimension $d \geq n$ that is a closed subset of $\overline{M}$, and let $h = \frac{1}{2}\rho^2$, where $\rho$ is the distance function in $\overline{M}$ to $\Sigma$. If $F : M \to \overline{M}$ is a complete minimal immersed submanifold such that for any $p \in M \setminus F^{-1}(\Sigma)$, \[ \| (\sigma_{F(p)}'(l))^\top \|_2^2 \geq \alpha, \] where $0 < \alpha \leq 1$ is a constant and $\sigma_{F(p)} : [0,l] \to \overline{M}$ is the unique geodesic normal to $\Sigma$ that satisfies $\sigma_{F(p)}(0) \in \Sigma$ and $\sigma_{F(p)}(l) = F(p)$, then:

1. $2\sqrt{\lambda(M)} \geq h(M) \geq \frac{m\alpha}{\sup_M \rho \rho' \rho}$. In particular, if $M$ has zero Cheeger constant, then $\rho \circ F$ is unbounded.

2. If $M$ is $h$-bounded and $h$-properly immersed, then $M$ has pure point spectrum only.

In the last section we apply this general result to submanifolds of a product of Riemannian manifolds.

**2 Some inequalities for minimal submanifolds**

Let $\bar{X}$ be an almost conformal vector field of $\overline{M}$, and $F : M \to \overline{M}$ an immersion of a $m$-dimensional submanifold with second fundamental form $B : \odot^2 TM \to NM$, where $NM$ is the normal bundle of $M$. We give to $M$ the induced Riemannian metric $g = F^\ast \bar{g}$ and the corresponding Levi Civita connection $\nabla$. We denote by $(\cdot)^\top$ and $(\cdot)^\perp$ the orthogonal projections of $T_{F(p)}\overline{M}$ onto $T_pM = dF_p(T_pM)$ and $NM_p$ respectively. We have for $X,Y$ vector fields on $M$, $\nabla_X Y = (\nabla_X Y)^\top$ and $B(X,Y) = (\nabla_X Y)^\perp$. The mean curvature of $M$ is the normal vector given by $H = \frac{1}{m} \text{trace}_g B$. The projection $\bar{X}^\top$ defines a vector field on $M$, and $\bar{X}^\perp$ a section of the normal bundle. Since $\bar{X}_F = \bar{X}^\top + \bar{X}^\perp$, an elementary computation gives

**Lemma 1.** For $Y,Z \in T_pM$, \[ L_{\bar{X}^\top} g(Y,Z) = L_{\bar{X}} \bar{g}(Y,Z) + 2 \bar{g}(B(Y,Z),\bar{X}^\perp). \] In particular, $\bar{X}_F$ cannot vanish everywhere in any open domain of $M$. 


Lemma 2. (1) $m\alpha + mg(H, \vec{X}^\perp) \leq \text{div}_g(\vec{X}^\top) \leq m\beta + mg(H, \vec{X}^\perp)$. If $F$ is minimal then $m\alpha \leq \text{div}_g(\vec{X}^\top) \leq m\beta$, and $\vec{X}^\top$ cannot vanish everywhere in any open domain of $M$.

(2) $g(\nabla ||\vec{X}^\top||, \vec{X}^\top) \geq \alpha ||\vec{X}^\top|| + \frac{1}{||\vec{X}^\top||}g(B(\vec{X}^\top, \vec{X}^\top), \vec{X}^\perp)$.

Proof. Let $e_i$ be an o.n. basis of $T_pM$. At $p$, $\text{div}_g(\vec{X}^\top) = \sum \frac{1}{2}L_{\vec{X}^\top}g(e_i, e_i)$, and an application of previous lemma gives (1) as well (2) since

$$g(\nabla ||\vec{X}^\top||, \vec{X}^\top) = \sum_i \frac{1}{||\vec{X}^\top||}g(\nabla e_i \vec{X}^\top, \vec{X}^\top)g(e_i, \vec{X}^\top) = \frac{1}{2||\vec{X}^\top||}L_{\vec{X}^\top}g(\vec{X}^\top, \vec{X}^\top).$$

\[\square\]

Proposition 1. If $\vec{X}$ is strongly convex, then:

(1) For any bounded domain $D$ of $M$ the norm $||\vec{X}||$ takes its maximum on the boundary $\partial D$.

(2) If $F$ is a minimal immersion and (6) holds, then the supremum of $||\vec{X}^\top||$ cannot be achieved. In particular $M$ cannot be compact without boundary (closed).

(3) If $\vec{X} = \nabla h$ for a smooth nonnegative convex function $h : \overline{M} \rightarrow \mathbb{R}$ and $F : M \rightarrow \overline{M}$ is a minimal submanifold, then the supremum of $h \circ F$ cannot be achieved. In particular $M$ cannot be closed.

Proof. From the inequality $g(\nabla ||\vec{X}||^2, \vec{X}) = 2g(\nabla X \vec{X}, \vec{X}) \geq 2\alpha ||\vec{X}||^2$, all critical points of $||\vec{X}||^2$ are vanishing points. This proves (1). To prove (2) we assume a maximum point $p_0$ of $||\vec{X}^\top||$ exists. Then at $p_0$ we may take $e_1 = \vec{X}^\top / ||\vec{X}^\top||$, and we have by lemma 1 and (6)

$$0 = ||\nabla ||\vec{X}^\top||^2||^2 = 4\sum_i g(\nabla e_i \vec{X}^\top, \vec{X}^\top)^2 \geq 4g(\nabla X \vec{X}, \vec{X})^2 ||\vec{X}^\top||^{-2} \geq (L_{\vec{X}^\top}g(\vec{X}^\top, \vec{X}^\top))^2 ||\vec{X}^\top||^{-2} \geq (2\alpha ||\vec{X}^\top||^2 + 2g(B(\vec{X}^\top, \vec{X}^\top), \vec{X}^\perp))^2 ||\vec{X}^\top||^{-2} \geq C^2$$

where $C = 2(\alpha - \alpha')$, what is impossible. Finally we prove (3). A maximum point $p_0$ of $h \circ F$ satisfies $\Delta(h \circ F)(p_0) \leq 0$, what contradicts

$$\Delta(h \circ F)_p = \sum_i (\text{Hess} h)_F(dF(e_i), dF(e_i)) + m\bar{g}(\nabla h_F, H) \geq m\alpha. \quad (7)$$

\[\square\]
Theorem 5. If $\overline{K}$ is bounded from below, and $F : M \to \overline{M}$ is any complete immersed minimal submanifold with bounded second fundamental form, then for any nonnegative strictly convex function $h : [0, +\infty) \to \overline{M}$ defined in a neighbourhood of $F(M)$, $F$ is $h$-unbounded.

Proof. Let us assume there exists a complete $h$-bounded immersion with $\|B\|^2 \leq b$, $b$ a nonnegative constant. By Gauss equation, the Ricci tensor of $M$ is bounded from below. Indeed, if $Y \in T_p M$ is a unit vector,

$$\text{Ricci}(Y,Y) = \sum_i \overline{K}(Y, e_i) + m \overline{g}(H, B(Y,Y)) - \overline{g}(B(Y,e_i), B(Y,e_i)) \geq m (\inf_{\overline{M}} \overline{K}) - mb - b.$$  

Furthermore, (7) holds for $F$ minimal immersion. Then theorem 2.1 of [15] gives us $\limsup_{r \to +\infty} h \circ F(p) r \geq C$, where $C$ is a positive constant that depends on $m, b, \alpha$ and a lower bound of $\overline{K}$. This contradicts the assumption of $h \circ F$ to be bounded. 

Bessa and Montenegro defined in [2] a quantity on a domain $D$ (bounded or not) of $M$, that here we denote by $c(D)$

$$c(D) = \sup_X \left( \inf_D (\text{div}_g X - \|X\|^2) \right)$$

where $X$ runs over all vector fields on $D$ locally integrable and with a weak divergence. We denote by $c(X) = \text{div}_g X - \|X\|^2$.

Proposition 2 ([2]). $\lambda(D) \geq c(D)$, with equality if $D$ has compact closure with smooth boundary.

Assume $\sup_M \|\vec{X}\|^2 = R < +\infty$ and (6) holds. Set $C = 2(\alpha - \alpha')$. For each $\epsilon > 0$ sufficiently small constant we consider the domain

$$D_\epsilon = \{ p \in M : R^2 > \|\vec{X}\|^2 > R^2 - \epsilon^2 \}.$$

Proposition 3. If $F$ is a minimal submanifold and (6) holds, then for any $0 < \epsilon < R$ sufficiently small,

$$\lambda(D_\epsilon) \geq \frac{mC\alpha}{\epsilon^2}.$$
Proof. We define the function \( f : [\sqrt{R^2 - \varepsilon^2}, R) \to [\varepsilon, +\infty), \) \( f(s) = \frac{c}{\sqrt{R^2 - s^2}} \), and the smooth vector field on \( D_\varepsilon, X = f(t)\bar{X}^\top \), where \( t = \|\bar{X}^\top\| \). Using lemma 2, we have
\[
c(X) = f(t)\text{div}_g(\bar{X}^\top) + g(\nabla(f(t)), \bar{X}^\top) - f(t)^2t^2 \\
\geq f(t)m\alpha + f'(t)(\alpha t + \frac{1}{t} \bar{g}(B(\bar{X}^\top, \bar{X}^\top), \bar{X}^\top)) - f^2(t)t^2.
\]
Note that \( f'(s) \) and \( f^2(s) \) go faster to \( +\infty \) then \( f(s) \), when \( s \to R \). Then we have to require \( f'(t)(\alpha t + \frac{1}{t} \bar{g}(B(\bar{X}^\top, \bar{X}^\top), \bar{X}^\top)) - f^2(t)t^2 \geq 0 \), that holds under condition (6). In this case,
\[
c(X) \geq \frac{Cm\alpha}{R^2 - t^2} \geq \frac{Cm\alpha}{\varepsilon^2}.
\]
Now proposition 2 gives the lower bound for \( \lambda(D_\varepsilon) \).

3 Proof of theorems 2 and 3

Let \( M \) be a complete noncompact \( m \)-dimensional Riemannian manifold, with Laplacian operator \( \Delta \) acting on the domain \( \mathcal{D} \) of \( L^2(M) \), where \( \Delta \phi \in L^2 \) for any \( \phi \in \mathcal{D} \). The spectrum of \( -\Delta \) decomposes as \( \sigma(M) = \sigma_p(M) \cup \sigma_{\text{ess}}(M) \subset [\lambda(M), \infty), \) where \( \sigma_p(M) \) is the pure point spectrum of isolated finite multiplicity eigenvalues, and \( \sigma_{\text{ess}}(M) \) is the essential spectrum. The decomposition principle of [8] states that \( M \) and \( M \setminus K \) have the same essential spectrum, as long as \( K \) is a compact domain of \( M \) with boundary.

Proof of theorem 2. (1) is immediate from (4) and the Cheeger inequality. (2) We can take a sequence \( \varepsilon_k \to 0 \) such that \( \sqrt{R^2 - \varepsilon_k^2} \) are regular values of \( \|\bar{X}^\top\| \). Since \( F \) is \( \bar{X} \)-proper, the sets \( K_{\varepsilon_k} = M \setminus D_{\varepsilon_k} \) are compact with smooth boundary. As in [3] we prove the theorem by showing that \( \lambda(D_{\varepsilon_k}) \to +\infty \) when \( k \to +\infty \), what proves that \( \sigma_{\text{ess}}(M) = \emptyset. \) This is the case by proposition 3. \( \square \)

3. In this case we take the domain of \( M, D_\varepsilon = \{ p \in M : R > h \circ F > R - \varepsilon \} \), and the vector field defined on \( D_\varepsilon \) given by \( X = \nabla(h \circ F)/(R - h \circ F) \). Then using (7),
\[
c(X)_p = \frac{\Delta(h \circ F)(p)}{(R - h(F(p)))} = \frac{\sum_i (\text{Hess} h)_{F(p)}(dF(e_i), dF(e_i))}{(R - h(F(p)))} \geq \frac{m\alpha}{\varepsilon},
\]
and so \( \lambda(D_\varepsilon) \to +\infty \) when \( \varepsilon \to 0. \) \( \square \)
Proof of corollary 3 Assume such immersion exists with \( \|B\|^2 \to c \) at infinity, \( c \geq 0 \) a finite constant. By the Gauss equation the sectional curvature of \( M \) satisfy \( K = \bar{K} - \|B\|^2 \). Then \( M \) has negative sectional curvature and \( K \to \bar{K} - c < 0 \) at infinity. By a result of Donnelly [7] the essential spectrum of \( M \) consists of the half line \([(-\bar{K} + c)/4, +\infty)\) contradicting corollary 2. \( \square \)

4 Ambient space with a totally convex set

**Definition 2.** (1) We say a vector field \( \bar{X} \) of \( \overline{M} \) is almost trace-conformal (strongly trace-convex resp.) along \( M \) if \( 2m\alpha \leq \text{Trace}_{g} F^* L_{g} \bar{X} \leq 2\beta m \) (with \( \beta = +\infty \) resp.), where \( \beta \geq \alpha > 0 \) are constants.

(2) We say that a function \( h : \overline{M} \to [0, +\infty) \) is strictly trace-convex along \( M \) if for some positive constant \( \alpha \), \( \text{Trace}_{g} F^*(\text{Hess} \ h) \geq m\alpha \).

It is elementary to verify next theorem, following the previous proofs:

**Theorem 6.** In the weaker conditions of definitions 2 and 1, the inequality (4) still holds as well the conclusions in theorems 1, 2 and 3.

A subset \( \Sigma \) of \( \overline{M} \) is said to be totally convex if it contains any geodesic connecting two points of \( \Sigma \). If \( \Sigma \) is a submanifold that is a closed subset of \( \overline{M} \), the hessian of the function \( h = \frac{1}{2}\rho^2 \), where \( \rho \) is the distance function in \( \overline{M} \) to \( \Sigma \), satisfies the following comparison theorem:

**Theorem 7 ([10]).** If \( \overline{M} \) is a connected complete Riemannian manifold with non-negative sectional curvature and \( \Sigma \) is a totally convex submanifold of dimension \( d \) that is a closed subset of \( \overline{M} \), then for any \( Y \in T_q\overline{M} \), \( q \notin \Sigma \),

\[
(\text{Hess} \ h)_q(Y, Y) \geq \bar{g}(\sigma'_q(l), Y)^2
\]

where \( \sigma_q : [0, l] \to \overline{M} \) is the unique unit geodesic normal to \( \Sigma \) that satisfies \( \sigma_q(0) \in \Sigma \) and \( \sigma_q(l) = q \).

In [10] the condition on \( \sigma \) is that it must satisfy \( t = \rho(\sigma(t)) \), but this is equivalent to \( \sigma \) meets \( \Sigma \) orthogonally (see [9] chapter 2).

**Proposition 4.** If \( \overline{M} \) is in the conditions of theorem 7 and \( F : M \to \overline{M} \) is a complete minimal immersed submanifold such that for any \( p \in M \setminus F^{-1}(\Sigma) \), \( \|\sigma'_{F(p)}(l)\|^2 \geq \)
\( \alpha \), where \( 0 < \alpha \leq 1 \) is a constant (in particular \( d \geq n \)), then \( h \) is strictly trace-convex along \( M \), \( \sup_M \rho \circ F \) cannot be achieved, and

\[
(\sup_M \rho \circ F)^{-1} \leq \frac{1}{m} \frac{1}{\alpha} h(M).
\]

**Proof.** From theorem 7,

\[
\sum_i (\text{Hess } h)_{F(p)}(dF(e_i), dF(e_i)) \geq \sum_i (\bar{g}(\sigma'_{F(p)}(l), dF(e_i)))^2 = \| (\sigma'_{F(p)}(l))^T \|^2,
\]

what proves \( h \) is strictly trace-convex along \( M \). The last inequality in the proposition is obtained form (4) that holds for \( \bar{X} = \bar{\nu} h \) (see theorem 6), where \( \| \bar{X}^T \| = \| (\bar{\nu} h)^T \| \leq \rho \circ F \| \bar{\nu} \| = \rho \circ F \), by following [11], that we describe now. Since (7) still holds \( \sup_M \rho \circ F \) cannot be achieved, and given a bounded domain \( D \) of \( M \) with boundary \( \partial D \) with unit normal \( \nu \), we have

\[
A(\partial D) \sup_M (\rho \circ F) \geq \left| \int_{\partial D} \bar{g}(\bar{X}^T, \nu) dA \right| \geq \int_D \text{div}_g(\bar{X}^T) dV \geq m \alpha V(D).
\]

\( \square \)

**Proof of theorem 4.** This is an immediate consequence of previous corollary and theorem 6.

\( \square \)

Now we specify for the particular case \( \overline{M} = \Sigma' \times \Sigma \), where \( (\Sigma', g_{\Sigma'}) \) and \( (\Sigma, g_{\Sigma}) \) are Riemannian manifolds of dimension \( d' \leq m \) and \( d \geq n \) respectively where \( d + d' = n + m \). Let us fix a point \( x_0 \in \Sigma' \) and denote by \( r_{\Sigma'} \) the distance function in \( \Sigma' \) to \( x_0 \). We identify \( \Sigma \) with \( x_0 \times \Sigma \), a totally convex set. For \((x, y) \in \overline{M} \), we have

\[
\rho((x, y)) = \bar{d}((x, y), x_0 \times \Sigma) = \bar{d}((x, y), (x_0, y)) = d_{\Sigma'}(x, x_0) = r_{\Sigma'}(x).
\]

Thus, \( h(x, y) = \frac{1}{2} r_{\Sigma'}^2(x) \). If \( F(p) = (x, y) \in \Sigma' \times \Sigma \), and \( l = r_{\Sigma'}(x) \) then \( \sigma_{F(p)}(t) = (\sigma_{\Sigma'}(t), y) \) and \( \sigma'_{F(p)}(l) = ((\sigma_{\Sigma'})(l), 0) \) where \( \sigma_{\Sigma'} \) is a unit geodesic on \( \Sigma' \) with \( \sigma_{\Sigma'}(0) = x_0 \) and \( \sigma_{\Sigma'}(l) = x \). Let \( \pi_{(x, y)} : T_x \Sigma' = T_y \Sigma \rightarrow T_p M \), \( \pi(v) = v^\top \). Therefore, \( F \) is \( h \)-bounded iff \( M \) is immersed in \( B_R(x_0) \times \Sigma \) where \( B_R(x_0) \) is a ball in \( \Sigma' \) of radius \( R < +\infty \), and if \( \pi \) has sup-norm bounded away from zero, then \( h \) is strictly trace-convex on \( M \).

**Proposition 5.** Let \( \Sigma' \) be \( m \)-dimensional and \( \Sigma \) \( n \)-dimensional complete connected Riemannian manifolds with nonpositive sectional curvatures, \( h : \overline{M} \rightarrow [0, +\infty) \),
\[ h(x, y) = \frac{1}{2} r_{\Sigma'}^2(x), \] where \( r_{\Sigma'} \) is the distance function in \( \Sigma' \) to a given point \( x_0 \), and \( B_R(x_0) \) the ball of radius \( R \) of \( \Sigma' \). If \( F : M \rightarrow B_R(x_0) \times \Sigma \) is a complete minimal submanifold \( h \)-properly immersed and there exist a constant \( C > 0 \) such that \( M \) is locally the graph of a local map \( f : B_R(x_0) \rightarrow \Sigma \) with \( f^* g_\Sigma \leq C g_{\Sigma'} \), then \( M \) has pure point spectrum.

**Proof.** First we note that the sectional curvature of \( M \) is also nonnegative. In the particular case \( d' = m \), and if locally \( M \) is the graph of a local map \( f : \Sigma' \rightarrow \Sigma \), then we show that the trace-convexity holds if \( f^* g_\Sigma \leq C g_{\Sigma'} \), for some constant \( C > 0 \). At a given point \( p \in M \), let \( \lambda_1 \geq \ldots \geq \lambda_m \) be the eigenvalues of \( f^* g_\Sigma \) with corresponding \( g_{\Sigma'} \)-o.n. basis \( a_i \) of eigenvectors. Then it follows that at \( F(p) = (x, y) = (x, f(x)) \), \( df_x(a_i) = \lambda_i a_{i+m} \), where \( a_i, a_\alpha, i = 1, \ldots, m, \alpha = m+1 \ldots, m+n \) defines an o.n. basis of \( T_{(x,y)}M \) (note that \( \lambda_i = 0 \) for \( i > \min\{m,n\} \), so we can always find such basis). Then \( e_i = (a_i + \lambda_i a_{i+m})/(1 + \lambda_i^2)^{1/2} \) constitutes an o.n. basis of the graph and

\[
\|((\sigma^{\Sigma'}_F(l))'\|_2 = \|((\sigma^{\Sigma}_F)'(l), 0)\|_2 = \sum_i |g^{\Sigma'}((\sigma^{\Sigma}_F)'(l), a_i)|^2/(1 + \lambda_i^2)^2 \geq \frac{1}{1 + C},
\]

and the proposition is proved. \( \square \)

**Remark.** The previous proposition should be compared with a similar result for the case \( \Sigma \) and \( \Sigma' \) Euclidean spaces in [3]. If \( \Sigma' = \mathbb{R}^m \) and \( \Sigma = \mathbb{R}^n \), according to [1] the immersion in the previous proposition cannot be properly immersed in \( \mathbb{R}^{m+n} \), if \( m \geq n + 1 \).

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