

$p$-divisible groups and relative crystalline representations when $e < p - 1$

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Abstract

Let $k$ be a perfect field of characteristic $p > 2$, and let $K$ be a finite totally ramified extension over $W(k)[\frac{1}{p}]$ of ramification degree $e$. Let $R_0$ be a relative base ring over $W(k)[t, t^{-1}, \ldots, t_m]^{\pm}$ satisfying some mild conditions, and let $R = R_0 \otimes W(k) O_K$. We show that if $e < p - 1$, then every crystalline representation of $\pi_1^{\text{et}}(\text{Spec} R[\frac{1}{p}])$ with Hodge-Tate weights in $[0, 1]$ arises from a $p$-divisible group over $R$.

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1 Introduction

Let $k$ be a perfect field of characteristic $p > 2$, and let $W(k)$ be its ring of Witt vectors equipped with the Witt vector Frobenius. Let $K$ be a finite totally ramified extension over $W(k)[\frac{1}{p}]$ with ramification degree $e$, and denote by $O_K$ its ring of integers. If $G$ is a $p$-divisible group over $O_K$, then it is well-known that its Tate module $T_p(G)$ is a crystalline $\text{Gal}(\overline{K}/K)$-representation with Hodge-Tate weights in $[0, 1]$. Conversely, Kisin showed the following result in [Kis06].
Theorem 1.1. (cf. [Kis06 Corollary 2.2.6]) Let $T$ be a crystalline $\text{Gal}(\overline{K}/K)$-representation finite free over $\mathbb{Z}_p$ whose Hodge-Tate weights lie in $[0, 1]$. Then there exists a $p$-divisible group $G$ over $\mathcal{O}_K$ such that $T_p(G) \cong T$ as $\text{Gal}(\overline{K}/K)$-representations.

The goal of this paper is to study the analogous statement in the relative case.

Let $W(k)(t_1^{\pm 1}, \ldots, t_m^{\pm 1})$ be the $p$-adic completion of the polynomial ring $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, and let $R_0$ be a $W(k)(t_1^{\pm 1}, \ldots, t_m^{\pm 1})$-algebra satisfying certain conditions (cf. Section 2.1).

Examples of such $R_0$ include $W(k)(t_1^{\pm 1}, \ldots, t_m^{\pm 1})$ and the formal power series ring $W(k)[s_1, \ldots, s_m]$. Let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ be our base ring, and denote by $\mathcal{G}_R$ the étale fundamental group $\pi_1^\text{et}(\text{Spec}R/(p))$. A relative $p$-adic Hodge theory for such base rings $R$ is developed by Brinon in [Bri08], which is generalized further by Scholze in [Sch13] and by Kedlaya and Liu in [KL15]. In particular, the crystalline period ring $B_{\text{crys}}(R)$ is well-defined, so is crystalline $\mathcal{G}_R$-representations with certain Hodge-Tate weights (cf. [Bri08]).

If $G_\mathcal{R}$ is a $p$-divisible group over $R$, its Tate module $T_p(G_\mathcal{R})$ is a crystalline $\mathcal{G}_R$-representation with Hodge-Tate weights in $[0, 1]$ (cf. [Kim15]). Conversely, we prove the following theorem when the ramification index $e$ is small.

**Theorem 1.2.** Suppose $e < p - 1$. Let $T$ be a crystalline $\mathcal{G}_R$-representation finite free over $\mathbb{Z}_p$ whose Hodge-Tate weights lie in $[0, 1]$. Then there exists a $p$-divisible group $G_\mathcal{R}$ over $R$ such that $T_p(G_\mathcal{R}) \cong T$ as $\mathcal{G}_R$-representations.

There are three major ingredients for the proof of Theorem 1.2. Firstly, Brinon and Trihan proved in [BT08] the generalization of Theorem 1.1 for the case when the base is a complete discrete valuation field whose residue field has a finite $p$-basis. We use this result together with the fact that the $p$-adic completion of $R_0((u))$ is an example of such rings studied loc. cit. Secondly, Kim generalized the Breuil-Kisin classification in the relative setting in [Kis06], and showed that the category of $p$-divisible groups over $R$ is anti-equivalent to the category of Kisin modules of height 1 over $R_0[u]$. Using the classification, we reduce our problem to constructing desired Kisin modules. We remark that our method of constructing the appropriate Kisin modules relies on the assumption that $e < p - 1$. Lastly, to show the statement when $R$ is formal power series ring of dimension 2, we use the purity result for $p$-divisible groups proved in [VZ10] when the ramification index is small.

1.1 Notations

We will reserve $\varphi$ for various Frobenius. To be more precise, let $A$ be an $W(k)$-algebra on which the arithmetic Frobenius $\varphi$ on $W(k)$ extends, and $M$ an $A$-module. We denote $\varphi_A : A \to A$ for such an extension. Let $\varphi_M : M \to M$ be a $\varphi_A$-semi-linear map. This is equivalent to having an $A$-linear map $1 \otimes \varphi_M : \varphi_A^* M \to M$, where $\varphi_A^* M$ denotes $A \otimes \varphi_A M$. We always drop the subscripts $A$ and $M$ from $\varphi$ if no confusion arises. Let $f : A \to B$ be a ring map compatible with Frobenius, that is, $f \circ \varphi_A = \varphi_B \circ f$. Then $\varphi_M$ naturally extends to $\varphi_M^B : M_B \to M_B$ for $M_B := B \otimes_A M$. It is easy to check that $\varphi_B^* M_B = B \otimes_A \varphi_A^* M$ and $1 \otimes \varphi_M^B : \varphi_B^* M_B \to M_B$ is equal to $B \otimes_A (1 \otimes \varphi_M)$. 

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2 Relative $p$-adic Hodge theory and étale $\varphi$-modules

2.1 Base ring and crystalline period ring in the relative case

We follow the same notations as in the Introduction. We recall the assumptions on the base rings and the construction of crystalline period ring in relative $p$-adic Hodge theory in [Bri08] (and in [Kim15] for Breuil-Kisin classification). We also impose some minor additional assumptions which will be needed later. Let $R_0$ be a ring obtained from $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ by a finite number of iterations of the following operations:

- $p$-adic completion of an étale extension;
- $p$-adic completion of a localization;
- completion with respect to an ideal containing $p$.

We assume that either $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \to R_0$ has geometrically regular fibers or $R_0$ has Krull dimension less than 2, and that $k \to R_0/pR_0$ is geometrically integral. In addition, we suppose that $R_0$ is an integral domain containing a Cohen ring $W$ over $W(k)$ such that $R_0$ is formally finite type over $W$, and that $R_0/pR_0$ is a unique factorization domain.

$R_0/pR_0$ has a finite $p$-basis given by $t_1, \ldots, t_m$. The Witt vector Frobenius on $W(k)$ extends (not necessarily uniquely) to $R_0$, and we fix such a Frobenius endomorphism $\varphi : R_0 \to R_0$. Let $\hat{\Omega}_{R_0} := \varprojlim \Omega(R_0/p^n)/W(k)$ be the module of $p$-adically continuous Kähler differentials. By [Bri08] Proposition 2.0.2, $\hat{\Omega}_{R_0} \cong \bigoplus_{i=1}^m R_0 \cdot dt_i$. We work over the base ring $R$ given by $R := R_0 \otimes_{W(k)} O_K$.

Let $\overline{R}$ denote the union of finite $R$-subalgebras $R'$ of a fixed separable closure of $\text{Frac}(R)$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Then $\text{Spec}R[\frac{1}{p}]$ is a pro-universal covering of $\text{Spec}R[\frac{1}{p}]$, and $\overline{R}$ is the integral closure of $R$ in $R[\frac{1}{p}]$. Let $G_R := \text{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1^{\text{ét}}(\text{Spec}R[\frac{1}{p}])$. By a representation of $G_R$, we always mean a finite continuous representation.

The crystalline period ring $B_{\text{cris}}(R)$ is constructed as follows. Let $\overline{R}^\varphi = \varprojlim_{\varphi} \overline{R}/p\overline{R}$. There exists a natural $W(k)$-linear surjective map $\theta : W(\overline{R}) \to \overline{R}$ which lifts the projection onto the first factor. Here, $\overline{R}$ denotes the $p$-adic completion of $\overline{R}$. Let $\theta_{R_0} : R_0 \otimes_{W(k)} W(\overline{R}) \to \overline{R}$ be the $R_0$-linear extension of $\theta$. Define the integral crystalline period ring $A_{\text{cris}}(R)$ to be the $p$-adic completion of the divided power envelope of $R_0 \otimes_{W(k)} W(\overline{R})$ with respect to $\ker(\theta_{R_0})$. Choose compatibly $\epsilon_n \in \overline{R}$ such that $\epsilon_0 = 1$, $\epsilon_n = \epsilon_1^p$ with $\epsilon_1 \neq 1$, and let $\overline{\epsilon} = (\epsilon_n)_{n \geq 0} \in \overline{R}^\varphi$. Then $\tau := \log[\overline{\epsilon}] \in A_{\text{cris}}(R)$. Define $B_{\text{cris}}(R) = A_{\text{cris}}(R)[\frac{1}{p}]$. $B_{\text{cris}}(R)$ is equipped naturally with $G_R$-action and Frobenius endomorphism, and $B_{\text{cris}}(R) \otimes_{R[\frac{1}{p}]} R[\frac{1}{p}]$ is equipped with a natural filtration by $R[\frac{1}{p}]$-submodules. Furthermore, we have a natural integrable connection $\nabla : B_{\text{cris}}(R) \to B_{\text{cris}}(R) \otimes_{R_0} \hat{\Omega}_{R_0}$ such that Frobenius is horizontal and satisfying the Griffiths transversality.
For a $\mathcal{G}_R$-representation $V$ over $\mathbb{Q}_p$, let $D_{\text{cris}}(V) := \text{Hom}_{\mathcal{G}_R}(V, B_{\text{cris}}(R))$. The natural morphism

$$\alpha_{\text{cris}} : D_{\text{cris}}(V) \otimes_{R_0[\frac{1}{p}]} B_{\text{cris}}(R) \to V^\vee \otimes_{\mathbb{Q}_p} B_{\text{cris}}(R)$$

is injective. We say $V$ is crystalline if $\alpha_{\text{cris}}$ is an isomorphism. When $V$ is crystalline, then $D_{\text{cris}}(V)$ is a finite projective $R_0[\frac{1}{p}]$-module, and $D_{\text{cris}}(V) \otimes_{R_0[\frac{1}{p}]} R[\frac{1}{p}]$ has the filtration induced by that on $B_{\text{cris}}(R) \otimes_{R_0[\frac{1}{p}]} R[\frac{1}{p}]$. We define the Hodge-Tate weights similarly as in the classical $p$-adic Hodge theory. Frobenius and connection on $B_{\text{cris}}(R)$ induces those structures on $D_{\text{cris}}(V)$; for the Frobenius endomorphism on $D_{\text{cris}}(V)$, $1 \otimes \varphi : \varphi^* D_{\text{cris}}(V) \to D_{\text{cris}}(V)$ is an isomorphism, and the connection $\nabla : D_{\text{cris}}(V) \to D_{\text{cris}}(V) \otimes_{R_0} \widehat{\Omega}_{R_0}$ is topologically quasi-nilpotent and integrable, satisfying Griffiths transversality and $\varphi$ is horizontal. For a $\mathcal{G}_R$-representation $T$ which is free over $\mathbb{Z}_p$, we say it is crystalline if $T[\frac{1}{p}]$ is crystalline.

Suppose $S_0$ is another relative base ring over $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ satisfying the above conditions and equipped with a choice of Frobenius, and let $b : R_0 \to S_0$ be a $\varphi$-equivariant $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$-algebra map. We also denote $b : R = R_0 \otimes_{W(k)} \mathcal{O}_K \to S := S_0 \otimes_{W(k)} \mathcal{O}_K$ the map induced $\mathcal{O}_K$-linearly. By choosing a common geometric point, this induces a map of Galois groups $\mathcal{G}_S \to \mathcal{G}_R$, and also a map of crystalline period rings $B_{\text{cris}}(R) \to B_{\text{cris}}(S)$ compatible with all structures. If $V$ is a crystalline representation of $\mathcal{G}_R$ with certain Hodge-Tate weights, then via these maps $V$ is also a crystalline representation of $\mathcal{G}_S$ with the same Hodge-Tate weights, and the construction of $D_{\text{cris}}(V)$ is compatible with the base change.

We will consider the following base change maps in later sections. Let $\mathcal{O}_{L_0}$ be the $p$-adic completion of $R_0(p)$, and let $b_L : R_0 \to \mathcal{O}_{L_0}$ be the natural $\varphi$-equivariant map. This induces $b_L : R \to \mathcal{O}_L := \mathcal{O}_{L_0} \otimes_{W(k)} \mathcal{O}_K$. Note that $L = \mathcal{O}_L[\frac{1}{p}]$ is an example of a complete discrete valuation field with a residue field having a finite $p$-basis, studied in [BT08]. On the other hand, for each maximal ideal $q \in \text{mSpec} R_0$, let $\overline{R}_{0,q}$ be the $q$-adic completion of $R_{0,q}$. By the structure theorem of complete regular local rings, we have $\overline{R}_{0,q} \cong \mathcal{O}_q[s_1, \ldots, s_l]$ where $\mathcal{O}_q$ is a complete discrete valuation ring with the maximal ideal $(p)$ and $l \geq 0$ is an integer ($\overline{R}_{0,q}$ is understood to be $\mathcal{O}_q$ when $l = 0$). We consider the natural $\varphi$-equivariant morphism $b_q : R_0 \to \overline{R}_{0,q}$, which induces $b_q : R \to \overline{R}_q := \overline{R}_{0,q} \otimes_{W(k)} \mathcal{O}_K$.

### 2.2 Étale $\varphi$-modules

We study étale $\varphi$-modules and associated Galois representations. Most of the material in this section is a review of [Kim15] Section 7, and the underlying geometry is based on perfectoid spaces as in [Sch12].

Let $R_0$ be a relative base ring over $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ and let $R = R_0 \otimes_{W(k)} \mathcal{O}_K$ as above. Choose a uniformizer $\varpi \in \mathcal{O}_K$. For integers $n \geq 0$, we choose compatibly $\varpi_n \in K$ such that $\varpi_0 = \varpi$ and $\varpi_n^p = \varpi_n$, and let $K_\infty$ be the $p$-adic completion of $\bigcup_{n \geq 0} K(\varpi_n)$. 


Then $K_\infty$ is a perfectoid field and $(\hat{R}_{\wp}^{1/\wp}, \hat{R})$ is a perfectoid affinoid $K_\infty$-algebra. Let $K_\infty^\phi$ denote the tilt of $K_\infty$ as defined in [Sch12], and let $\varpi := (\varpi_n) \in K_\infty^\phi$.

Let $\mathcal{G} := R_0[[u]]$ equipped with the Frobenius extending that on $R_0$ by $\varphi(u) = u^p$. Let $E_{R_\infty}^+ = \mathcal{G}/p\mathcal{G}$, and let $\hat{E}_{R_\infty}^+$ be the $u$-adic completion of $\text{lim}_{\phi} E_{R_\infty}^+$. Let $E_{R_\infty} = E_{R_\infty}^+[\frac{1}{u}]$ and $\hat{E}_{R_\infty} = \hat{E}_{R_\infty}^+[\frac{1}{u}]$. By [Sch12 Proposition 5.9], $(\hat{E}_{R_\infty}, \hat{E}_{R_\infty}^+)$ is a perfectoid affinoid $L^\phi$-algebra, and we have the natural injective map $(\hat{E}_{R_\infty}, \hat{E}_{R_\infty}^+) \hookrightarrow (\hat{R}^+[\frac{1}{u}], \hat{R})$ given by $u \mapsto \varpi$.

Let

$$\hat{R}_{\infty} := W(\hat{E}_{R_\infty}^+) \otimes_{W(K_\infty^\phi)} \vartheta \mathcal{O}_{K_\infty}.$$  

(2.1)

By [Sch12 Remark 5.19], $(\hat{R}_\infty[\frac{1}{u}], \hat{R}_\infty)$ is a perfectoid affinoid $K_\infty$-algebra whose tilt is $(\hat{E}_{R_\infty}, \hat{E}_{R_\infty}^+)$. Furthermore, it is shown in [Kim15] that we have a natural injective map $(\hat{R}_\infty[\frac{1}{u}], \hat{R}_\infty) \hookrightarrow (\hat{R}^+[\frac{1}{u}], \hat{R})$ whose tilt is $(\hat{E}_{R_\infty}, \hat{E}_{R_\infty}^+) \hookrightarrow (\hat{R}^+[\frac{1}{u}], \hat{R})$. For $\mathcal{G}_{R_\infty} := \pi^\phi_!(\text{Spec}\hat{R}_{\infty}[\frac{1}{u}])$, we then have a continuous map of Galois groups $\mathcal{G}_{\hat{R}_\infty} \to \mathcal{G}_R$, which is a closed embedding by [GR03 Proposition 5.4.54]. By the almost purity theorem in [Sch12], $\hat{R}^+[\frac{1}{u}]$ can be canonically identified with the $\varpi$-adic completion of the affine ring of a pro-universal covering of Spec$\hat{E}_{R_\infty}$, and letting $\mathcal{G}_{\hat{E}_{R_\infty}}$ be the Galois group corresponding to the pro-universal covering, there exists a canonical isomorphism $\mathcal{G}_{\hat{E}_{R_\infty}} \cong \mathcal{G}_{\hat{R}_\infty}$.

**Lemma 2.1.** Consider the map of Galois groups $\mathcal{G}_{\mathcal{O}_L} \to \mathcal{G}_R$ induced by choosing a common geometric point for the base change map $b_L : R \to \mathcal{O}_L$ in Section 2.1. Then the images of $\mathcal{G}_{\mathcal{O}_L}$ and $\mathcal{G}_{\hat{R}_\infty}$ inside $\mathcal{G}_R$ generate the group $\mathcal{G}_R$.

**Proof.** $E_{R_\infty}^+$ has a finite $p$-basis given by $t_1, \ldots, t_m, u$. Note that for any element of $g \in \mathcal{G}_R$, there exists an element $h \in \mathcal{G}_{\mathcal{O}_L}$ whose image in $\mathcal{G}_R$ induces the same actions on $t_1^{\varpi}, \ldots, t_m^{\varpi}, \varpi^{1/\wp}$. Since $\hat{R}_\infty = W(\hat{E}_{R_\infty}^+) \otimes_{W(K_\infty^\phi)} \vartheta \mathcal{O}_{K_\infty}$, the actions of $g$ and $h$ are the same on the elements of $\hat{R}_\infty$. Hence, the assertion follows. \hfill $\square$

Now, let $\mathcal{O}_{\mathcal{E}}$ be the $p$-adic completion of $\mathcal{G}[\frac{1}{u}]$. Note that $\varphi$ on $\mathcal{G}$ extends naturally to $\mathcal{O}_{\mathcal{E}}$.

**Definition 2.2.** An étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ where $\mathcal{M}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module and $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear endomorphism such that $1 \otimes \varphi_{\mathcal{M}} : \varphi^*\mathcal{M} \to \mathcal{M}$ is an isomorphism. We say that an étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-module is projective (resp. torsion) if the underlying $\mathcal{O}_{\mathcal{E}}$-module $\mathcal{M}$ is projective (resp. $p$-power torsion).

Let $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}$ denote the category of étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules whose morphisms are $\mathcal{O}_{\mathcal{E}}$-module maps compatible with Frobenius. Let $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{pr}}$ and $\text{Mod}_{\mathcal{O}_{\mathcal{E}}}^{\text{tor}}$ respectively denote the full subcategories of projective and torsion objects. Note that we have a natural notion of a subquotient, direct sum, and tensor product for étale $(\varphi, \mathcal{O}_{\mathcal{E}})$-modules, and duality is defined for projective and torsion objects.
Lemma 2.3. Let $\mathcal{M} \in \text{Mod}^\text{tor}_{\mathcal{O}_E}$ be a torsion étale $\varphi$-module annihilated by $p$. Then $\mathcal{M}$ is a projective $\mathcal{O}_E/p\mathcal{O}_E$-module.

Proof. This follows from essentially the same proof as in [And06, Lemma 7.10].

We consider $W(\overline{R}(\frac{1}{p}))$ as an $\mathcal{O}_E$-algebra via mapping $u$ to the Teichmüller lift $\overline{u}$ of $u$, and let $\mathcal{O}^\ur_{\overline{E}}$ be the integral closure of $\mathcal{O}_E$ in $W(\overline{R}(\frac{1}{p}))$. Let $\widehat{\mathcal{O}}^\ur_{\overline{E}}$ be its $p$-adic completion. Since $\mathcal{O}_E$ is normal, we have $\text{Aut}_{\mathcal{O}_E}(\mathcal{O}^\ur_{\overline{E}}) \cong \mathcal{G}_{E_{R_\infty}} := \pi^\text{et}_1(\text{Spec} E_{R_\infty})$, and by [GR03, Proposition 5.4.54] and the almost purity theorem, we have $\mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{R_\infty}$. This induces $\mathcal{G}_{R_\infty}$-action on $\widehat{\mathcal{O}}^\ur_{\overline{E}}$. The following is proved in [Kim15].

Lemma 2.4. (cf. [Kim15, Lemma 7.2.6 and 7.2.7]) We have $(\widehat{\mathcal{O}}^\ur_{\overline{E}})^{\mathcal{G}_{R_\infty}} = \mathcal{O}_E$ and the same holds modulo $p^n$. Furthermore, there exists a unique $\mathcal{G}_{R_\infty}$-equivariant ring endomorphism $\varphi$ on $\widehat{\mathcal{O}}^\ur_{\overline{E}}$ lifting the $p$-th power map on $\mathcal{O}^\ur_{\overline{E}}/p$ and extending $\varphi$ on $\mathcal{O}_E$. The inclusion $\widehat{\mathcal{O}}^\ur_{\overline{E}} \hookrightarrow W(\overline{R}(\frac{1}{p}))$ is $\varphi$-equivariant where the latter ring is given the Witt vector Frobenius.

Let $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$ be the category of $\mathbb{Z}_p$-representations of $\mathcal{G}_{R_\infty}$, and let $\text{Rep}^\ur_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$ and $\text{Rep}^{\text{tor}}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$ respectively denote the full subcategories of free and torsion objects. For $\mathcal{M} \in \text{Mod}_{\mathcal{O}_E}$ and $T \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$, we define $T(\mathcal{M}) := (\mathcal{M} \otimes_{\mathcal{O}_E} \widehat{\mathcal{O}}^\ur_{\overline{E}})^{\varphi=1}$ and $\mathcal{M}(T) := (T \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}^\ur_{\overline{E}})^{\mathcal{G}_{R_\infty}}$. For a torsion étale $\varphi$-module $\mathcal{M} \in \text{Mod}^\text{tor}_{\mathcal{O}_E}$, we define its \textit{length} to be the length of $\mathcal{M} \otimes_{\mathcal{O}_E} (\mathcal{O}_E/(p))$ as an $(\mathcal{O}_E/(p))$-module.

Proposition 2.5. (cf. [Kim15, Proposition 7.3]) The assignments $T(\cdot)$ and $\mathcal{M}(\cdot)$ are exact equivalences (inverse of each other) of $\otimes$-categories between $\text{Mod}_{\mathcal{O}_E}$ and $\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$. Moreover, $T(\cdot)$ and $\mathcal{M}(\cdot)$ restrict to rank-preserving equivalence of categories between $\text{Mod}^\text{tor}_{\mathcal{O}_E}$ and $\text{Rep}^{\text{tor}}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$ and length-preserving equivalence of categories between $\text{Mod}^\text{tor}_{\mathcal{O}_E}$ and $\text{Rep}^{\text{tor}}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})$. In both cases, $T(\cdot)$ and $\mathcal{M}(\cdot)$ commute with taking duals.

Proof. This is [Kim15, Proposition 7.3]. We remark here for some additional details. Note that $E_{R_\infty}$ is a normal domain and $\pi_1^\text{et}(\text{Spec} E_{R_\infty}) \cong \mathcal{G}_{R_\infty}$. Given Lemma 2.3, the assertion therefore follows from the usual dévissage and [Katz, Lemma 4.1.1]. Note that both functors $T(\cdot)$ and $\mathcal{M}(\cdot)$ are a priori left exact by definition, and exactness can be proved by the same argument as in the proof of [And06, Theorem 7.11].

Suppose $S_0$ is another relative base ring over $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ as in Section 2.1 equipped with a choice of Frobenius, and suppose $b : R_0 \hookrightarrow S_0$ be a $\varphi$-equivariant $W(k)[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$-algebra map which is injective. Let $b : R = R_0 \otimes_{W(k)} \mathcal{O}_K \hookrightarrow S := S_0 \otimes_{W(k)} \mathcal{O}_K$ be the induced injective map. By choosing a common geometric point we have an injective map $\overline{R} \hookrightarrow \overline{S}$, and this induces an injection $\overline{R}_\infty \hookrightarrow \overline{S}_\infty$ by the constructions given at equation (2.1). Hence, the corresponding map of Galois groups $\mathcal{G}_S \to \mathcal{G}_R$ restricts to $\mathcal{G}_{\overline{S}_\infty} \to \mathcal{G}_{R_\infty}$. Let $\overline{\mathcal{G}}_S = S_0[v]$ and let $\mathcal{O}_{\overline{E}, S}$ be the $p$-adic completion of $\mathcal{O}_S(\frac{1}{v})$. Let $\mathcal{M}(\cdot)$
be the functor for the base ring $S$ constructed similarly as above. Let $T \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{\hat{R}_\infty})$. Then $T$ is also a $\mathcal{G}_{\hat{S}_\infty}$-representation via the map $\mathcal{G}_{\hat{S}_\infty} \to \mathcal{G}_{\hat{R}_\infty}$, and we have the natural isomorphism $\mathcal{M}(T) \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E,S} \cong \mathcal{M}_S(T)$ as étale $(\varphi, \mathcal{O}_{E,S})$-modules by the definition of the functors $\mathcal{M}(\cdot)$ and $T(\cdot)$ and by Proposition 2.5.

3 Relative Breuil-Kisin classification

We now explain the classification of $p$-divisible groups over $\text{Spec}R$ via Kisin modules, which is proved in [Kis06] when $R = \mathcal{O}_K$ and generalized in [Kim15] for the relative case. Denote by $E(u)$ the Eisenstein polynomial for the extension $K$ over $W(k)[\frac{1}{p}]$, and let $\mathcal{S} = R_0[u]$ as above.

**Definition 3.1.** Denote by $\text{Kis}^1(\mathcal{S})$ the category of pairs $(\mathcal{M}, \varphi_\mathcal{M})$ where

- $\mathcal{M}$ is a finitely generated projective $\mathcal{S}$-module;
- $\varphi_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear map such that $\text{coker}(1 \otimes \varphi_\mathcal{M})$ is annihilated by $E(u)$.

The morphisms are $\mathcal{S}$-module maps compatible with Frobenius.

Note that for $(\mathcal{M}, \varphi_\mathcal{M}) \in \text{Kis}^1(\mathcal{S})$, $1 \otimes \varphi_\mathcal{M} : \varphi^*\mathcal{M} \to \mathcal{M}$ is injective since $\mathcal{M}$ is finite projective over $\mathcal{S}$ and $\text{coker}(1 \otimes \varphi_\mathcal{M})$ is killed by $E(u)$.

Consider the composite $\mathcal{S} \to \mathcal{S}/u\mathcal{S} = R_0 \xrightarrow{\hat{\phi}} R_0$.

**Definition 3.2.** A Kisin module of height 1 is a tuple $(\mathcal{M}, \varphi_\mathcal{M}, \nabla_\mathcal{M})$ such that

- $(\mathcal{M}, \varphi_\mathcal{M}) \in \text{Kis}^1(\mathcal{S})$;
- Let $\mathcal{N} := R_0 \otimes_{\mathcal{S}, \varphi_\mathcal{M}} \mathcal{M}$ equipped with the induced Frobenius $\varphi_{R_0} \otimes \varphi_\mathcal{M}$. Then $\nabla_\mathcal{M} : \mathcal{N} \to \mathcal{N} \otimes_{R_0} \hat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection commuting with Frobenius.

Denote by $\text{Kis}^1(\mathcal{S}, \nabla)$ the category of Kisin modules of height 1 whose morphisms are $\mathcal{S}$-module maps compatible with Frobenius and connection.

The following theorem classifying the $p$-divisible groups is proved in [Kim15].

**Theorem 3.3.** (cf. [Kim15, Corollary 6.3.1, Remark 6.1.6 and 6.3.5]) There exists an exact anti-equivalence of categories

$$\mathcal{M}^* : \{p\text{-divisible groups over } \text{Spec}R\} \to \text{Kis}^1(\mathcal{S}, \nabla).$$

Let $S_0$ be another base ring satisfying the condition as in Section 2.1 and equipped with a Frobenius, and let $b : R_0 \to S_0$ be a $\varphi$-equivariant map. Then the formation of $\mathcal{M}^*$ commutes with the base change $R \to S := S_0 \otimes_{W(k)} \mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from $b$. 

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Note that if $(\mathcal{M}, \varphi_{\mathfrak{m}}) \in \text{Kis}^1(\mathfrak{G})$, then $(\mathcal{M} \otimes_{\mathfrak{G}} \mathcal{O}_E, \varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{O}_L})$ is a projective étale $(\varphi, \mathcal{O}_E)$-module since $1 \otimes \varphi_{\mathfrak{m}}$ is injective and its cokernel is killed by $E(u)$ which is a unit in $\mathcal{O}_E$. If $G_R$ is a $p$-divisible group over $R$, its Tate module is given by $T_p(G_R) := \text{Hom}_R(\mathbb{Q}_p/\mathbb{Z}_p, G_R \times_R \overline{R})$, which is a finite free $\mathbb{Z}_p$-representation of $G_R$. By [Kim15, Corollary 8.2], we have a natural $G_R^{\infty}$-equivariant isomorphism $T^*(G_R) \otimes_{\mathcal{O}_E} \mathcal{O}_E \cong T_p(G_R)$ where $T^*(G_R) \otimes_{\mathcal{O}_E} \mathcal{O}_E$ denotes the dual of $T(\mathcal{M}^*(G_R) \otimes_{\mathcal{O}_E} \mathcal{O}_E)$.

### 4 Construction of Kisin modules

Throughout this section, we assume $e < p - 1$. We denote $\mathfrak{S}_n := \mathfrak{S}/p^n\mathfrak{S}$ for positive integers $n \geq 1$. Let $T$ be a crystalline $G_R$-representation which is free over $\mathbb{Z}_p$ with Hodge-Tate weights in $[0, 1]$. Let $\mathcal{M} := \mathcal{M}^*(T)$ be the associated étale $(\varphi, \mathcal{O}_E)$-module, where $\mathcal{M}^*(T)$ denotes the dual of $\mathcal{M}(T)$. For each integer $n \geq 1$, denote $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$. Note that $\mathcal{M}_n \cong \mathcal{M}^*(T/p^nT)$. On the other hand, consider the map $b_L : R \to \mathcal{O}_L$ as in Section 2.1. $T$ is also a crystalline $G_{\mathcal{O}_L}$-representation with Hodge-Tate weights in $[0, 1]$, so by [BT08, Theorem 6.10], there exists a $p$-divisible group $G_{\mathcal{O}_L}$ over $\mathcal{O}_L$ such that $T_p(G_{\mathcal{O}_L}) \cong T$ as $G_{\mathcal{O}_L}$-representations. Let $(\mathcal{M}_{\mathcal{O}_L}, \nabla_{\mathfrak{M}_{\mathcal{O}_L}}) := \mathcal{M}^*(G_{\mathcal{O}_L}) \in \text{Kis}^1(\mathfrak{S}_{\mathcal{O}_L}, \nabla)$ be the associated Kisin module over $\mathfrak{S}_{\mathcal{O}_L}$. Denote $\mathcal{M}_{\mathcal{O}_L,n} = \mathcal{M}_{\mathcal{O}_L}/p^n\mathcal{M}_{\mathcal{O}_L}$. The map between the Galois groups $G_{\mathcal{O}_L} \to G_R$ restricts to $G_{\mathcal{O}_L,\infty} \to G_{R,\infty}$. Hence, we have the natural isomorphism $\mathcal{M} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,\mathcal{O}_L} \cong \mathcal{M}_{\mathcal{O}_L} \otimes_{\mathcal{O}_{E,\mathcal{O}_L}} \mathcal{O}_{E,\mathcal{O}_L}$ of étale $(\varphi, \mathcal{O}_{E,\mathcal{O}_L})$-modules. Let $\mathcal{M}_{\mathcal{O}_L} := \mathcal{M} \otimes_{\mathcal{O}_E} \mathcal{O}_{E,\mathcal{O}_L}$ and $\mathcal{M}_{\mathcal{O}_L,n} := \mathcal{M}_{\mathcal{O}_L}/p^n\mathcal{M}_{\mathcal{O}_L}$.

For each $n \geq 1$, we define

$$\mathcal{M}_n := \mathcal{M}_n \cap \mathcal{M}_{\mathcal{O}_L,n}$$

where the intersection is taken as $\mathfrak{S}$-submodules of $\mathcal{M}_{\mathcal{O}_L,n}$. Note that since $\mathcal{M}_n$ is projective over $\mathfrak{S}_n[\frac{1}{u}]$ of rank $d$ and $\mathcal{M}_{\mathcal{O}_L,n}$ is free over $\mathfrak{S}_{\mathcal{O}_L,n}$ of rank $d$, $\mathcal{M}_n$ is a finite $\mathfrak{S}_n$-module. The Frobenius endomorphisms on $\mathcal{M}_n$ and $\mathcal{M}_{\mathcal{O}_L,n}$ induce the Frobenius endomorphism on $\mathcal{M}_n$. Since the Frobenius on $\mathcal{M}_{\mathcal{O}_L,n}$ is injective, we have the injective $\mathfrak{S}$-module morphism

$$1 \otimes \varphi : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathcal{M}_n \to \mathcal{M}_n$$

for each $n$.

**Lemma 4.1.** For each integer $n \geq 1$, we have $\varphi$-equivariant isomorphisms

$$\mathcal{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_E \cong \mathcal{M}_n$$

and

$$\mathcal{M}_n \otimes_{\mathfrak{S}} \mathcal{O}_{E,\mathcal{O}_L} \cong \mathcal{M}_{\mathcal{O}_L,n}.$$
which shows the first isomorphism. On the other hand, since $\mathcal{S} \to \mathcal{S}_{O_L}$ is flat and $\mathcal{M}_{O_L,n}$ is finite free over $\mathcal{S}_{O_L,n}$, we have
\[
\mathcal{M}_n \otimes_{\mathcal{S}} \mathcal{S}_{O_L} \cong (\mathcal{M}_n \otimes_{\mathcal{S}} \mathcal{S}_{O_L}) \cap (\mathcal{M}_{O_L,n} \otimes_{\mathcal{S}} \mathcal{S}_{O_L}) = \mathcal{M}_{O_L,n} \cap (\mathcal{M}_{O_L,n} \otimes_{\mathcal{S}} \mathcal{S}_{O_L}) \cong \mathcal{M}_{O_L,n}
\]
by $\mathcal{S}_n[\frac{1}{u}] \cap \mathcal{S}_{O_L,n} \cong \mathcal{S}_n$.

Lemma 4.2. The cokernel of the $\mathcal{S}$-module map $1 \otimes \varphi : \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_n \to \mathcal{M}_n$ is killed by $E(u)$.

Proof. Let $x \in \mathcal{M}_n$. There exists a unique $y_1 \in \mathcal{O}_E \otimes_{\varphi, \mathcal{O}_E} \mathcal{M}_n \cong \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_n$ such that $(1 \otimes \varphi)(y_1) = E(u)x$. On the other hand, there exists a unique $y_2 \in \mathcal{S}_{O_L} \otimes_{\varphi, \mathcal{S}_{O_L}} \mathcal{M}_{O_L,n}$ such that $(1 \otimes \varphi)(y_2) = E(u)x$. Then we have $y_1 = y_2 \in (\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_n) \cap (\mathcal{S}_{O_L} \otimes_{\varphi, \mathcal{S}_{O_L}} \mathcal{M}_{O_L,n})$.

Since $\mathcal{O}_{L_0}/p\mathcal{O}_{L_0}$ has a finite $p$-basis given by $t_1, \ldots, t_m \in R_0/pR_0$ which also gives a $p$-basis of $R_0/pR_0$, the natural map $\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_{O_L,n} \to \mathcal{S}_{O_L} \otimes_{\varphi, \mathcal{S}_{O_L}} \mathcal{M}_{O_L,n}$ is an isomorphism. Hence,
\[
y_1 \in (\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_n) \cap (\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_{O_L,n}) \cong \mathcal{S} \otimes_{\varphi, \mathcal{S}} (\mathcal{M}_n \cap \mathcal{M}_{O_L,n}) = \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}_n
\]
since $\varphi : \mathcal{S} \to \mathcal{S}$ is flat. This proves the assertion. \qed

For any finite $\mathcal{S}$-module $\mathcal{N}$ equipped with a $\varphi$-semilinear endomorphism $\varphi : \mathcal{N} \to \mathcal{N}$, say $\mathcal{N}$ has $E(u)$-height $\leq 1$ if there exists a $\mathcal{S}$-module map $\psi : \mathcal{N} \to \varphi^*\mathcal{N} = \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{N}$ such that the composite
\[
\varphi^*\mathcal{N} \xrightarrow{1 \otimes \varphi} \mathcal{N} \xrightarrow{\psi} \varphi^*\mathcal{N}
\]
is $E(u) \cdot \text{Id}_{\varphi^*\mathcal{N}}$. By Lemma 4.2, $\mathcal{M}_n$ has $E(u)$-height $\leq 1$.

For each maximal ideal $\mathfrak{q} \in \text{mSpec} R_0$, consider $b_{\mathfrak{q}} : R \to R_{\mathfrak{q}}$ as in Section 2.3. By choosing a common geometric point, we have the induced map of Galois groups $\mathcal{G}_{R_{\mathfrak{q}}} \to \mathcal{G}_R$ which restricts to $\mathcal{G}_{R_{\mathfrak{q}}[\infty]} \to \mathcal{G}_{R[\infty]}$, and $T$ is a crystalline $\mathcal{G}_{R_{\mathfrak{q}}}$-representation with Hodge-Tate weights in $[0, 1]$. Denote $\mathcal{S}_{\mathfrak{q}} := \tilde{R_{\mathfrak{q}}}[u]$.

Proposition 4.3. For each integer $n \geq 1$, $\mathcal{M}_n$ is projective over $\mathcal{S}_n$ of rank $d$.

Proof. Let $\mathfrak{q}$ be a maximal ideal of $R_0$, and let $\mathcal{N}_n := \mathcal{M}_n \otimes_{\mathcal{S}} \mathcal{S}_{\mathfrak{q}}$ equipped with the induced Frobenius endomorphism. Then we have the induced $\mathcal{S}_{\mathfrak{q}}$-linear map $\psi : \mathcal{N}_n \to \mathcal{S}_{\mathfrak{q}} \otimes_{\varphi, \mathcal{S}_{\mathfrak{q}}} \mathcal{N}_n$ such that the composite
\[
\mathcal{S}_{\mathfrak{q}} \otimes_{\varphi, \mathcal{S}_{\mathfrak{q}}} \mathcal{N}_n \xrightarrow{1 \otimes \varphi} \mathcal{N}_n \xrightarrow{\psi} \mathcal{S}_{\mathfrak{q}} \otimes_{\varphi, \mathcal{S}_{\mathfrak{q}}} \mathcal{N}_n
\]
is $E(u) \cdot \text{Id}$. For the isomorphism $\tilde{R_{\mathfrak{q}}}[u] \cong O_{\mathfrak{q}}[s_1, \ldots, s_l]$ as above, consider the projection $\mathcal{S}_{\mathfrak{q}} \to \mathcal{S}_{\mathfrak{q}}/(p, s_1, \ldots, s_l) \cong k_{\mathfrak{q}}[u]$ where $k_{\mathfrak{q}} := O_{\mathfrak{q}}/(p)$. Denote $\overline{\mathcal{N}_n} = \mathcal{N}_n \otimes_{\mathcal{S}_{\mathfrak{q}}} k_{\mathfrak{q}}[u]$
equipped with the induced Frobenius. Then we have the induced $k_q[u]$-linear map $\psi : \overline{M}_n \to k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_n$ such that the composite

$$k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_n \xrightarrow{\varphi} \overline{M}_n \xrightarrow{\psi} k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_n$$

is $u^e \cdot \text{Id}$. Since $k_q[u]$ is a principal ideal domain, $\overline{M}_n$ is a direct sum of its free part and $u$-torsion part $\overline{M}_n \cong \overline{M}_{n,\text{free}} \oplus \overline{M}_{n,\text{tor}}$ as $k_q[u]$-modules. Furthermore, $\varphi$ maps $\overline{M}_{n,\text{tor}}$ into $\overline{M}_{n,\text{tor}}$, and hence the above maps induce

$$k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_{n,\text{tor}} \xrightarrow{\varphi} \overline{M}_{n,\text{tor}} \xrightarrow{\psi} k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_{n,\text{tor}}$$

whose composite is $u^e \cdot \text{Id}$.

We claim that $\overline{M}_{n,\text{tor}} = 0$. Suppose otherwise. Then $\overline{M}_{n,\text{tor}} \cong \bigoplus_{i=1}^b k_q[u]/(u^a_i)$ for some integers $a_i \geq 1$, and $k_q[u] \otimes_{\varphi,k_q[u]} \overline{M}_{n,\text{tor}} \cong \bigoplus_{i=1}^b k_q[u]/(u^{a_i})$. By taking the appropriate wedge product and letting $a = a_1 + \ldots + a_b$, the above maps induce the map of $k_q[u]$-modules

$$k_q[u]/(u^a) \xrightarrow{\varphi} k_q[u]/(u^a) \xrightarrow{\psi} k_q[u]/(u^a)$$

whose composite is equal to $u^e \cdot \text{Id}$. Let $(1 \otimes \varphi)(1) = f(u) \in k_q[u]/(u^a)$, and $\psi(1) = h(u) \in k_q[u]/(u^a)$. Then $u^a | u^a h(u)$, so $u^{(p-1)a} | h(u)$. On the other hand, $f(u) h(u) = u^{eb}$ in $k_q[u]/(u^a)$. This implies $u^{(p-1)a} | u^{eb}$. But $e < p - 1$ and $a \geq b$, so we get a contradiction. Hence, $\overline{M}_{n,\text{tor}} = 0$ and $\overline{M}_n$ is free over $k_q[u]$ of rank $d$, since by Lemma 4.1 $\overline{M}_n[\frac{1}{u}] = (M_n \otimes_{\mathcal{S}} \mathcal{G}_q) \otimes_{\mathcal{S}_q} k_q[u]$ which is projective over $k_q[[u]]$ of rank $d$. Let $b_1, \ldots, b_d \in \mathcal{M}_n$ be a lift of a basis elements of $\overline{M}_n$. By Nakayama’s lemma, we have a surjection of $\mathcal{G}_{q,n}$-modules

$$f : \bigoplus_{i=1}^d \mathcal{G}_{q,n} \cdot e_i \twoheadrightarrow \mathcal{M}_n$$

given by $e_i \mapsto b_i$. Since $\mathcal{M}_n$ is $u$-torsion free and $\mathcal{M}_n[\frac{1}{u}] \cong \mathcal{M}_n \otimes_{\mathcal{S}} \mathcal{G}_q$ is projective over $\mathcal{G}_{q,n}[\frac{1}{u}]$ of rank $d$, $f$ is also injective. Thus, $\mathcal{M}_n = \mathcal{M}_n \otimes_{\mathcal{S}} \mathcal{G}_q$ is projective over $\mathcal{G}_{q,n}$ of rank $d$. Since this holds for every $q \in \text{mSpec} R_0$, it proves the assertion. \(\square\)

**Lemma 4.4.** Let $\mathcal{M}$ and $\mathcal{M}'$ be finite $u$-torsion free $\mathcal{S}$-modules equipped with Frobenius endomorphisms such that $\mathcal{M}[\frac{1}{u}]$ and $\mathcal{M}'[\frac{1}{u}]$ are torsion étale $\varphi$-modules. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ have $E(u)$-height $\leq 1$ and $\mathcal{M}[\frac{1}{u}] = \mathcal{M}'[\frac{1}{u}]$ as étale $\varphi$-modules. Then $\mathcal{M} = \mathcal{M}'$.

**Proof.** Let $\mathcal{L}$ be the cokernel of the embedding $\mathcal{M} \hookrightarrow \mathcal{M} + \mathcal{M}'$ of $\mathcal{S}$-modules. Note that $\mathcal{S} \otimes_{\varphi, \mathcal{S}} (\mathcal{M} + \mathcal{M}') \cong \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M} + \mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}'$ since $\varphi : \mathcal{S} \to \mathcal{S}$ is flat. Thus, $\mathcal{L}$ has $E(u)$-height $\leq 1$, and $\mathcal{L}[\frac{1}{u}] = 0$. Then as in the proof of Proposition 4.3, we have $\mathcal{L} = 0$. So $\mathcal{M} = \mathcal{M} + \mathcal{M}'$. Similarly, $\mathcal{M}' = \mathcal{M} + \mathcal{M}'$. \(\square\)

It is clear that both $p\mathcal{M}_{n+1}$ and $\mathcal{M}_n$ are $u$-torsion free, have $E(u)$-height $\leq 1$ and $p\mathcal{M}_{n+1}[\frac{1}{u}] = p\mathcal{M}_{n+1} = \mathcal{M}_n[\frac{1}{u}]$. We conclude the following Proposition:
**Proposition 4.5.** For each \( n \geq 1 \), we have a \( \varphi \)-equivariant isomorphism

\[
pM_{n+1} \cong M_n.
\]

By Proposition 4.3 and 4.5 if we define the \( \mathcal{G} \)-module

\[
M := \varprojlim_n M_n,
\]

then \( M \in \text{Kis}^1(\mathcal{G}) \). Note that we have a \( \varphi \)-equivariant isomorphism \( M \otimes_{\mathcal{G}} \mathcal{G}_L \cong M_L \) by Lemma 4.1.

**Remark 4.6.** We remark that analogous statements can be proved when \( T \) is a crystalline \( \mathcal{G}_R \)-representation with Hodge-Tate weights in \([0, r]\) for the case \( er < p-1 \), since [BT08] constructs more generally a functor from crystalline representations with Hodge-Tate weights in \([0, r]\) to Kisin modules of height \( r \) when the base is a complete discrete valuation field whose residue field has a finite \( p \)-basis.

To study connections on \( M \), we first consider the following general situation. Let \( A_0 \) be a \( k \)-algebra which is an integral domain. Consider \( n \)-variables \( x_1, \ldots, x_n \), and denote \( \mathbf{x} = (x_1, \ldots, x_n)^t \) and \( x^{[p]} = (x_1^p, \ldots, x_n^p)^t \). An Artin-Schreier system of equations in \( n \) variables over \( A_0 \) is given by

\[
\mathbf{x} = B \mathbf{x}^{[p]} + C
\]

where \( B = (b_{ij})_{1 \leq i, j \leq n} \in M_{n \times n}(A_0) \) is an \( n \times n \) matrix with entries in \( A_0 \) and \( C = (c_i)_{1 \leq i \leq n} \in M_{n \times 1}(A_0) \). Let

\[
A_1 := A_0[x_1, \ldots, x_n]/(x_1 - c_1 - \sum_{i=1}^n b_{1i} x_i^p, \ldots, x_n - c_n - \sum_{i=1}^n b_{ni} x_i^p),
\]

which is the \( A_0 \)-algebra parametrizing the solutions of equation (4.1). \( A_0 \to A_1 \) is étale by [Vas13] Theorem 2.4.1(a)].

**Lemma 4.7.** There exists a non-zero element \( f \in A_0 \) which depends only on \( B \) such that \( A_1[\frac{1}{f}] \) is finite étale over \( A_0[\frac{1}{f}] \).

**Proof.** We induct on \( n \). Suppose \( n = 1 \). If \( \det B \neq 0 \), then equation (4.1) is equivalent to

\[
x_1^p = B^{-1} x_1 - B^{-1} C,
\]

so the assertion holds with \( f = \det B \). If \( \det B = 0 \), then \( B = 0 \) and \( A_1 \cong A_0 \), so the assertion holds trivially.

For \( n \geq 2 \), if \( \det B \neq 0 \), then equation (4.1) is equivalent to

\[
x^{[p]} = B^{-1} \mathbf{x} - B^{-1} C.
\]
Hence, with \( f = \det B, A_1[\frac{1}{f}] \) is finite étale over \( A_0[\frac{1}{f}] \). Suppose \( \det B = 0 \). Denote by \( B^{(i)} \) the \( i \)-th row of \( B \). Then up to renumbering the index for \( x_i \)'s, we have

\[
\sum_{i=1}^{n} e_i B^{(i)} = 0
\]

for some non-zero \( f_1 \in A_0 \) and some \( e_i \in A_0[\frac{1}{f}] \) with \( e_n = 1 \). From equation (4.11), we get

\[
x_n = -\sum_{i=1}^{n-1} e_i x_i + c_n + \sum_{i=1}^{n-1} c_i e_i.
\]

Hence, denoting \( \underline{x}' = (x_1, \ldots, x_{n-1})^t \), equation (4.1) is equivalent to an Artin-Schreier system of equations in \( n - 1 \) variables over \( A_0[\frac{1}{f}] \)

\[
\underline{x}' = B' \underline{x}' + C'
\]

where \( B' \in M_{(n-1) \times (n-1)}(A_0[\frac{1}{f}]) \) and \( C' \in M_{(n-1) \times 1}(A_0[\frac{1}{f}]) \). Note that \( B' \) depends only on \( B \) and not on \( C \). Hence, the assertion follows by induction. \( \square \)

**Proposition 4.8.** There exists \( \tilde{f} \in R_0 \) with \( \tilde{f} \notin pR_0 \) such that the following holds:

Let \( S_0 \) be the \( p \)-adic completion of \( R_0[\frac{1}{f}] \) equipped with the induced Frobenius, and let \( \mathcal{O}_S = S_0[u] \). Let \( M_S = M \otimes_{\mathcal{O}_S} S \) is equipped with the induced Frobenius, so \( M_S \in \text{Kis}^1(\mathcal{O}_S) \). Then there exists a topologically quasi-nilpotent integrable connection

\[
\nabla_{M_S} : (S_0 \otimes_{\mathcal{O}_S} M_S) \to (S_0 \otimes_{\mathcal{O}_S} M_S) \otimes S_0 \bigoplus_{i=1}^m S_0 \cdot dt_i
\]

such that \( \varphi \) is horizontal, so \( (M_S, \nabla_{M_S}) \in \text{Kis}^1(\mathcal{O}_S, \nabla) \). Furthermore, we can choose \( \nabla_{M_S} \) such that \( M_S \otimes_{\mathcal{O}_S} \mathcal{O}_L \) equipped with the induced Frobenius and connection is isomorphic to \( (M_{\mathcal{O}_L}, \nabla_{M_{\mathcal{O}_L}}) \) as Kisin modules over \( \mathcal{O}_L \).

**Proof.** Let \( \mathcal{N} := R_0 \otimes_{\mathcal{O}_S} M \) where the map \( \varphi : \mathcal{O}_S \to R_0 \) denotes the composite \( \mathcal{O}_S \xrightarrow{\text{mod } u} R_0 \xrightarrow{\varphi} R_0 \). Then \( \mathcal{N} \) is a frame as defined in [Lau14, 2.1]. Fix an \( R_0 \)-direct factor \( \mathcal{N}^1 \subset \mathcal{N} \) lifting \( \text{Fil}^1 \mathcal{N}/p\mathcal{N} \subset \mathcal{N}/p\mathcal{N} \). By passing to a Zariski covering of \( \text{Spf}(R_0, p) \), we may assume that \( \mathcal{N}, \mathcal{N}^1 \), and \( \mathcal{N}/\mathcal{N}^1 \) are all free over \( R_0 \). Fix an \( R_0 \)-basis of \( \mathcal{N} \) adapted to the direct factor \( \mathcal{N}^1 \).

Let \( \text{Spf}(A, p) \to \text{Spf}(R_0, p) \) be an étale morphism. Note that \( A \) is equipped with a unique Frobenius lifting of that on \( R_0 \), and \( \widehat{\Omega}_A \cong A \otimes_{R_0} \widehat{\Omega}_{R_0} \cong \bigoplus_{i=1}^m A \cdot dt_i \). We mean by a connection on \( A/p \otimes_{R_0} \mathcal{N} \) an additive morphism

\[
\nabla_{A,1} : A/p \otimes_{R_0} \mathcal{N} \to (A/p \otimes_{R_0} \mathcal{N}) \otimes_{R_0} \widehat{\Omega}_{R_0}
\]
satisfying the Leibnitz rule \((\nabla_{A,1})(ax) = a\nabla_{A,1}(x) + x \otimes da\) for any \(a \in A/p\) and \(x \in \mathcal{N}\).

By [Vas13, Theorem 3.2 Proof], the set of equivalent classes of \(\nabla_{A,1}\) which commutes with the Frobenius \(\varphi(A) \otimes \varphi(\mathcal{N})\) on \(A/p \otimes_{R_0} \mathcal{N}\) can be identified with the set of solutions over \(A/p\) of an Artin-Schreier system of equations

\[ \hat{x} = B \hat{x}^p + C_1 \]

for \(\hat{x} = (x_1, \ldots, x_{dm})^t\), where \(B \in M_{dm \times dm}(R_0/p)\) and \(C_1 \in M_{dm \times 1}(R_0/p)\). When \(A = \mathcal{O}_{L_0}\), it has a solution given by \(\nabla_{\mathfrak{m}_{L_0}}\). Since \(\mathcal{O}_{L_0}/p \cong \text{Frac}(R_0/p)\) and \(R_0/p\) is a unique factorization domain, the solution lies in \((R_0/p)[\frac{1}{f}]\) for some non-zero \(f \in R_0/p\) depending only on \(B\) by Lemma 4.7. Let \(\tilde{f} \in R_0\) be a lift of \(f\), and let \(S_0\) be the \(p\)-adic completion of \(R_0[\frac{1}{f}]\).

Now, for \(n \geq 1\), suppose we are given the connection

\[ \nabla_{S_0,n} : S_0/p^n \otimes_{R_0} \mathcal{N} \rightarrow (S_0/p^n \otimes_{R_0} \mathcal{N}) \otimes_{R_0} \hat{\Omega}_{R_0} \]

such that the Frobenius is horizontal and lifting \(\nabla_{S_0,n} \otimes_{R_0} \hat{\Omega}_{R_0}\) lies in \(S_0/p\) by Lemma 4.7. This proves the assertion.

\[ \nabla_{S_0,n+1} : S_0/p^{n+1} \otimes_{R_0} \mathcal{N} \rightarrow (S_0/p^{n+1} \otimes_{R_0} \mathcal{N}) \otimes_{R_0} \hat{\Omega}_{R_0} \]

such that the Frobenius is horizontal and inducing \(\nabla_{S_0,n} \otimes_{R_0} \hat{\Omega}_{R_0}\) via the natural map \(S_0/p \rightarrow \mathcal{O}_{L_0}\). Then by [Vas13, Theorem 3.2 Proof], the set of equivalent classes of connections

such that the Frobenius is horizontal and lifting \(\nabla_{S_0,n}\) corresponds to the set of solutions over \(S_0/p\) of an Artin-Schreier system of equations

\[ \hat{x} = B \hat{x}^p + C_{n+1}, \]

where \(B\) is the same matrix as above and \(C_{n+1} \in M_{dm \times 1}(S_0/p)\). The solution over \(\mathcal{O}_{L_0}/p\) given by \(\nabla_{\mathfrak{m}_{L_0}}\) lies in \(S_0/p\) by Lemma 4.7. This proves the assertion.

**Proposition 4.9.** Let \(S_0\) be a ring as given in Proposition 4.8, and let \(S = S_0 \otimes_{W(k)} \mathcal{O}_K\). Then there exists a \(p\)-divisible group \(G_S\) over \(S\) such that \(T_p(G_S) \cong T\) as \(\mathcal{G}_S\)-representations.

**Proof.** Let \(G_S\) be the \(p\)-divisible group over \(S\) given by \((\mathfrak{M}_S, \nabla_{\mathfrak{m}_S})\) in Proposition 4.8. Since \(\mathfrak{M}_S \otimes_{\mathcal{O}_S} \hat{\mathcal{G}}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L}\) as Kisin modules, we have \(T_p(G_S) \cong T\) as \(\mathcal{G}_{\mathcal{O}_L}\)-representations. On the other hand, \(\mathfrak{M}_S \otimes_{\mathcal{O}_S} \hat{\mathcal{G}}_{\mathcal{E},S} \cong \mathcal{M} \otimes_{\mathcal{O}_S} \hat{\mathcal{G}}_{\mathcal{E},S}\) as étale \(\varphi\)-modules. Hence, \(T_p(G_S) \cong T\) as \(\mathcal{G}_{S,\infty}\)-representations. Since \(\mathcal{G}_{S,\infty}\) and \(\hat{\mathcal{G}}_{\mathcal{O}_L}\) generate the Galois group \(\mathcal{G}_S\) by Lemma 2.1, we have \(T_p(G_S) \cong T\) as \(\mathcal{G}_S\)-representations.

**5 Proof of the main theorem**

In this section, we finish the proof of Theorem 1.2. We begin by recalling the following well-known lemma about \(p\)-divisible groups.
Lemma 5.1. Let $R_1$ be an integral domain over $W(k)$ such that $\text{Frac}(R_1)$ has characteristic $0$. Then via the Tate module functor $T_p(\cdot)$, the category of $p$-divisible groups over $R_1[\frac{1}{p}]$ is equivalent to the category of finite free $\mathbb{Z}_p$-representations of $G_{R_1} = \pi_1^{\text{´et}}(\text{Spec} R_1[\frac{1}{p}])$. Furthermore, such an equivalence is functorial in the following sense:

Let $R_1 \to R_2$ be a map of integral domains over $W(k)$ such that $\text{Frac}(R_1)$ and $\text{Frac}(R_2)$ have characteristic $0$. Let $G_{R_1}$ be a $p$-divisible group over $R_1$. Then $T_p(G_{R_1}) \cong T_p(G_{R_1} \times_{R_1} R_2)$ as $G_{R_2}$-representations.

We first consider the case when $R$ is a formal power series ring of dimension $2$. Let $T$ be a crystalline $G_R$-representation which is finite free over $\mathbb{Z}_p$ and has Hodge-Tate weights in $[0,1]$.

Proposition 5.2. Suppose $R_0 = O[s]$ for a Cohen ring $O$, and $e \leq p - 1$. Then there exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T$ as $G_R$-representations.

Proof. Let $G$ be a $p$-divisible group over $R[\frac{1}{p}]$ given by Lemma 5.1 such that $T_p(G) \cong T$ as $G_R$-representations. It suffices to show that $G$ extends to a $p$-divisible group $G_R$ over $R$.

By [BT08, Theorem 6.10], there exists a $p$-divisible group $G_{O_L}$ over $O_L$ extending $G \times_{R[\frac{1}{p}]} L$. Let $U = \text{Spec} R \setminus \text{pt}$, where pt denotes the closed point given by the maximal ideal of $R$. Consider the fpqc covering

$$\text{Spec}(R[\frac{1}{p}] \oplus O_L) \to U.$$  

For each positive integer $n \geq 1$, the finite flat group schemes $(G[p^n], G_{O_L}[p^n])$ give a decent datum for the covering by Lemma 5.1. Hence, by considering the corresponding Hopf algebras, we obtain a system of finite flat group schemes $(G_{U,n})_{n \geq 1}$ over $U$ extending $(G[p^n])_{n \geq 1}$. Furthermore, the natural induced sequence of finite flat group schemes

$$0 \to G_{U,1} \to G_{U,n+1} \xrightarrow{x^p} G_{U,n} \to 0$$

is exact again by fpqc descent. So $(G_{U,n})_{n \geq 1}$ is a $p$-divisible group over $U$ extending $G$. Since $e \leq p - 1$, $G_U$ extends to a $p$-divisible group $G_R$ over $R$ by [VZ10, Theorem 3].

Now, let $R_0$ be a general ring satisfying the assumptions in Section 2.1, and let $R = R_0 \otimes_{W(K)} O_K$ with $e < p - 1$. Let $T$ be a crystalline $G_R$-representation free over $\mathbb{Z}_p$ with Hodge-Tate weights in $[0,1]$. Denote by $\mathfrak{M}_G(T)$ the $\mathfrak{S}$-module in $\text{Kis}^1(\mathfrak{S})$ constructed from $T$ as in Section 4. Let $\tilde{f} \in R_0$ be an element as in Proposition 4.8, and let $S_0$ be the $p$-adic completion of $R_0[\frac{1}{f}]$ as in Proposition 4.9. Let $f \in R_0/pR_0$ be the image of $\tilde{f}$ in the projection $R_0 \to R_0/p$. We only need to consider the case when $f$ is not a unit in $R_0/p$. Since $R_0/p$ is a UFD, there exist prime elements $\bar{s}_1, \ldots, \bar{s}_l$ inside $R_0/p$ dividing $f$. Let $s_1, \ldots, s_l \in R_0$ be any preimages of $\bar{s}_1, \ldots, \bar{s}_l$ respectively.
For each \( i = 1, \ldots, l \), consider the prime ideal \( \mathfrak{p}_i = (p, s_i) \subset R_0 \) and let \( R_0^{(i)} := \hat{R}_{0,\mathfrak{p}_i} \) be the \( \mathfrak{p}_i \)-adic completion of \( R_{0,\mathfrak{p}_i} \). Note that \( R_0^{(i)} \) is a formal power series ring over a Cohen ring with dimension 2. We consider the natural \( \varphi \)-equivariant map \( b_i : R_0 \to R_0^{(i)} \), which induces \( b_i : R \to R^{(i)} := R_0^{(i)} \otimes_{W(k)} \mathcal{O}_K \). On the other hand, let \( k_c \) be a field extension of \( \text{Frac}(R_0/pR_0) \) which is a composite of the fields \( \text{Frac}(R_0^{(i)}/p) \) for \( i = 1, \ldots, l \), and let \( k_c^{\text{perf}} = \varprojlim k_c \) be its direct perfection. By the universal property of \( p \)-adic Witt vectors, there exists a unique \( \varphi \)-equivariant map \( b_c : R_0 \to W(k_c^{\text{perf}}) \). Moreover, for each \( i = 1, \ldots, l \), we have a unique \( \varphi \)-equivariant embedding \( R_0^{(i)} \to W(k_c^{\text{perf}}) \) whose composite with \( b_i \) is equal to \( b_c \). Note that the natural embedding \( R_0 \to S_0 \cap \bigcap_{i=1}^l R_0^{(i)} \) as subrings of \( W(k_c^{\text{perf}}) \) is bijective, since \( S_0/p \cap \bigcap_{i=1}^l (R_0^{(i)}/p) = R_0/p \) inside \( k_c^{\text{perf}} \).

By Proposition 5.2 there exists a \( p \)-divisible group \( G_i \) over \( R^{(i)} \) such that \( T_p(G_i) \cong T \) as \( G_{R^{(i)}} \)-representations. We have

\[ (\mathfrak{M}_E(T) \otimes_E \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_{E, R^{(i)}} \cong \mathfrak{M}^*(G_i) \otimes_{E, R^{(i)}} \mathcal{O}_{E, R^{(i)}} \]

as étale \((\varphi, \mathcal{O}_{E, R^{(i)}})\)-modules. Applying Lemma 4.4 we can deduce that \( \mathfrak{M}_E(T) \otimes_E \mathcal{O}_{E, R^{(i)}} \cong \mathfrak{M}^*(G_i) \) compatibly with Frobenius.

Let \( D = D_{\text{cris}}(T[1/p]) \), and denote \( \mathfrak{M} = \mathfrak{M}_E(T) \) and \( \mathcal{N} = R_0 \otimes_{\varphi, \mathcal{O}} \mathfrak{M} \). Let \( \nabla : D \to D \otimes_{R_0} \hat{\Omega}_{R_0} \) be the natural connection.

**Proposition 5.3.** There exists a unique \( \varphi \)-equivariant embedding

\[ h : \mathcal{N} \hookrightarrow D \]

of \( R_0 \)-modules. Furthermore, if we consider \( \mathcal{N} \) as a \( R_0 \)-submodule of \( D \) via \( h \), then \( \nabla \) maps \( \mathcal{N} \) to \( \mathcal{N} \otimes_{R_0} \hat{\Omega}_{R_0} \). Hence, \( \mathfrak{M} \) is a Kisin module of height 1.

**Proof.** By [Kim15 Corollary 6.3.1], there exists a unique \( \varphi \)-equivariant embedding \( h_i : \mathcal{N} \to D \otimes_{R_0} b_i R_0^{(i)} \)

for each \( i = 1, \ldots, l \), and there exists a unique \( \varphi \)-equivariant embedding \( h_c : \mathcal{N} \to D \otimes_{R_0} b_c W(k_c^{\text{perf}}) \). Moreover, by Proposition 4.9 there exists a unique \( \varphi \)-equivariant embedding \( h_S : \mathcal{N} \to D \otimes_{R_0} S_0 \). Note that the maps \( h_1, \ldots, h_l \) and \( h_S \) are compatible with one another, in the sense that their composites with the embedding into \( D \otimes_{R_0} b_c W(k_c^{\text{perf}}) \) are all equal to \( h_c \). Hence, we obtain a \( \varphi \)-equivariant embedding

\[ h : \mathcal{N} \hookrightarrow (D \otimes_{R_0} [1/p]) S_0[1/p] \cap \bigcap_{i=1}^l (D \otimes_{R_0[1/p]} b_i R_0^{(i)}[1/p]) \cong D \otimes_{R_0[1/p]} (S_0[1/p] \cap \bigcap_{i=1}^l R_0^{(i)}[1/p]) = D \]

since \( D \) is projective over \( R_0[1/p] \) and \( S_0[1/p] \cap \bigcap_{i=1}^l R_0^{(i)}[1/p] = R_0[1/p] \). The unicity of \( h \) follows from the unicity of \( h_S \).
Now, note that $\hat{\Omega}_{R_0} \cong \bigoplus_{j=1}^{m} R_0 \cdot dt_j$. So $\nabla$ maps $\mathcal{N}$ to $\mathcal{N} \otimes_{R_0} (\bigoplus_{j=1}^{m} R_0[1] \cdot dt_j)$. On the other hand, by Proposition 4.8, Proposition 5.2, and the compatibility of $D_{\text{cris}}(\cdot)$ with respect to $\varphi$-compatible base changes, we have that $\nabla$ maps $\mathcal{N}$ into $\mathcal{N} \otimes_{R_0} (\bigoplus_{j=1}^{m} S_0 \cdot dt_j)$. Since $\mathcal{N}$ is projective over $R_0$ and $S_0 \cap \bigcap_{i=1}^{l} R_0(i) = R_0$, $\nabla$ maps $\mathcal{N}$ to $\mathcal{N} \otimes_{R_0} (\bigoplus_{j=1}^{m} R_0 \cdot dt_j)$. \hfill \box

**Theorem 5.4.** There exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T$ as $G_R$-representations.

**Proof.** By Proposition 5.3, we have $\mathfrak{M} \in \text{Kis}^1(\varphi, \nabla)$. Furthermore, $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{G}_{\mathcal{O}_L} \cong \mathfrak{M}_{\mathcal{O}_L}$ as Kisin modules over $\mathcal{G}_{\mathcal{O}_L}$, since the Frobenius and connection structure on $\mathfrak{M}$ agree with those on $D$. Thus, if $G_R$ is the $p$-divisible group corresponding to $\mathfrak{M}$, then $T_p(G_R) \cong T$ as $G_{\mathcal{O}_L}$-representations as well as $G_{R_{\infty}}$-representations. The assertion follows from Lemma 2.1. \hfill \box

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