KAC–MOODY AND VIRASORO ALGEBRAS

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This course develops the representation theory of affine Kac–Moody algebras and the Virasoro algebra. These infinite–dimensional Lie algebras play an important rôle in string theory and conformal field theory; they are the Lie algebras of the loop groups and diffeomorphism group of the circle. We adopt a unitary viewpoint and use supersymmetry as the main technique, as suggested by the supersymmetric coset constructions of Goddard–Kent–Olive and Kazama–Suzuki. Even in the case of finite–dimensional simple Lie algebras, this approach is fruitful. Not only does it give a streamlined route to the classical Weyl character formula (taken from unpublished notes of Peter Goddard), but it also has a natural geometric interpretation in terms of Dirac operators and index theory.

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CHAPTER I. CLIFFORD ALGEBRAS, FERMIONS AND THE SPIN GROUP.

We develop here the theory of fermions in finite-dimensions. This provides the first example of the principle of quantisation: if an algebraic object acts irreducibly on a Hilbert space and a group of automorphisms of the object preserves the equivalence class of the representation, then the automorphism group is implemented by a unique projective representation on the Hilbert space. It is convenient to develop the linear algebra for bosons and fermions in parallel (it is even possible to develop a simultaneous super-theory). Roughly speaking bosons are operators that satisfy the anticommutation relation $aa^* - a^*a = I$ and fermions are operators that satisfy the anticommutation relation $aa^* + a^*a = I$. In this chapter we use fermionic quantisation to construct the spin group, a double covering of the special orthogonal group. In Chapter III, we carry out an analogous treatment for bosons. Bosonic quantisation leads to a construction of the metaplectic group, a double covering of the symplectic group; as applications we will prove the modular transformation properties of theta functions and quadratic reciprocity. Fermionic and bosonic quantisation is also very important in infinite dimensions. The infinite-dimensional theory proceeds very much as in the finite-dimensional case, except that not all automorphisms can be quantised (they must satisfy a ‘Hilbert–Schmidt’ quantisation criterion). In addition there is a remarkable equivalence between the bosonic and fermionic theories in infinite dimensions which has been used to explain the KdV and KP hierarchies in soliton theory.

1. TENSOR, SYMMETRIC AND EXTERIOR ALGEBRAS.

Tensor products. If $V$ and $W$ are finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$, we define their tensor product $V \otimes W$ by taking bases $(v_i)$ and $(w_j)$ in $V$ and $W$ and then decreeing $V \otimes W$ to be the vector space with basis $v_i \otimes w_j$. In general we set $(\sum a_i v_i) \otimes (\sum b_j w_j) = \sum a_i b_j v_i \otimes w_j$, so that $v \otimes w$ is defined for any $v \in V$, $w \in W$. This definition is up to isomorphism independent of the choice of basis. Clearly $\dim(V \otimes W) = \dim(V) \dim(W)$. Iterating we get a similar definition of a $k$-fold tensor product $V_1 \otimes \cdots \otimes V_k$. By definition these is a natural one–one correspondence between the vector space of multilinear maps $V_1 \times \cdots \times V_k \to U$ and $\text{Hom}(V_1 \otimes \cdots \otimes V_k, U)$; this could equally well be used as the universal property characterising the tensor product.

The tensor product has various obvious functorial properties. Thus for example $V_1 \otimes V_2 \cong V_2 \otimes V_1$, $(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$, $V_2 \otimes V_2^* \cong \text{Hom}(V_2, V_1)$, $V_1 \otimes V_2 \cong \text{Hom}(V_2, V_1)$, $\text{Hom}(V_1 \otimes V_2, V_3) \cong \text{Hom}(V_1, V_2^* \otimes V_3)$. Moreover if $f_1 : U_1 \to V_i$ are linear maps, then we have $f_1 \otimes f_2 : U_1 \otimes U_2 \to V_1 \otimes V_2$ sending $u_1 \otimes u_2$ to $f_1(u_1) \otimes f_2(u_2)$.

The tensor algebra. Let $T^n(V) = V^\otimes n = V \otimes \cdots \otimes V$ ($n$ times) and $T(V) = \bigoplus T^n(V)$, the tensor algebra. Multiplication $T^a(V) \cdot T^b(V) \to T^{a+b}(V)$ is defined by concatenation, so that $(v_1 \otimes \cdots \otimes v_a) \times (w_1 \otimes \cdots \otimes w_b) = v_1 \cdots \otimes v_a \otimes w_1 \cdots \otimes w_b$. This makes $T(V)$ into a non–commutative associative algebra.

Action of $S_n$ on $V^\otimes n$. The symmetric group $S_n$ acts on $V^\otimes n$ by permuting the tensor factors. Thus $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in S_n$. Define $\epsilon : S_n \to \{\pm 1\}$ to be the sign homomorphism, assigning $-1$ to an even permutation and $+1$ to an odd permutation. Let

$$S\omega = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \omega, \quad A\omega = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) \sigma \omega$$

be the symmetrising and antisymmetrising operators on $V^\otimes n$.

Symmetric and exterior algebras. Let $S^n(V) = \{\omega \in V^\otimes n : \sigma \omega = \omega \forall \sigma \in S_n\} = SV^\otimes n$ and $\Lambda^k(V) = \{\omega \in V^\otimes n : \sigma \omega = \epsilon(\sigma) \omega \forall \sigma \in S_n\} = AV^\otimes n$. $S(V) = \bigoplus S^n(V)$ and $\Lambda(V) = \bigoplus \Lambda^n(V)$ are called the symmetric and exterior algebras. Their multiplication is defined on homogeneous elements by $\omega_1 \cdot \omega_2 = S(\omega_1 \otimes \omega_2)$ or $\omega_1 \wedge \omega_2 = A(\omega_1 \otimes \omega_2)$ and extended bilinearly to the whole of $S(V)$ or $T(V)$. It is easy to check that $a \cdot (b \cdot c) = S(a \otimes b \otimes c) = (a \cdot b) \cdot c$ and that $a \wedge (b \wedge c) = A(a \otimes b \otimes c) = (a \wedge b) \wedge c$, so that $S(V)$ and $\Lambda(V)$ become associative algebras.

Lemma. $S(V)$ is a commutative ring and $\Lambda(V)$ is a graded commutative ring.

Proof. The first result follows straight from the definitions and is just part of the fact that $S(V)$ coincides with the algebra of polynomial functions on $V^*$ (see below). The algebra $\Lambda(V)$ is $\mathbb{Z}_2$–graded into even or
odd elements, according to degree of homogeneous elements. We set \( \partial a = 0 \) or \( 1 \) according as \( a \) is even or odd. Graded commutativity is just the statement that \( a \wedge b = (-1)^{a \delta b} b \wedge a \), which is immediate from the definitions.

**Concrete realisations of \( S(V) \) and \( \Lambda(V) \).** We map \( S(V) \) into polynomial functions on \( V^* \). Note that \( S^k(V)^* = S^k(V^*) \). We need

**Polarisation Lemma.** The tensors \( \nu^{\otimes m} \) with \( \nu \in V \) span \( S^m V \)

**Proof** Note that if \( X \) is a subspace and \( f(\lambda_1, \ldots, \lambda_m) \) is a polynomial function of \( \lambda_1, \ldots, \lambda_m \) with values in \( X \), then \( \frac{\partial |n|}{\partial \lambda_a} f \) also lies in \( X \) for any multinomial \( \alpha \), since \( X \) is finite–dimensional, so closed. Take \( v_1, \ldots, v_m \in W \) and consider \( f(\lambda) = (\sum \lambda_i v_i)^{\otimes m} \). Up to a constant non–zero factor \( \frac{\partial^{\otimes m}}{\partial \lambda_1 \cdots \partial \lambda_m} \) is the symmetrisation of \( v_1 \otimes \cdots \otimes v_m \). This shows that the symmetrisation of any elementary tensor (and hence any tensor) lies in the subspace \( X \) of \( S^m V \subset V^{\otimes m} \) spanned by the tensors \( \nu^{\otimes m} \).

In particular, \( S^k V^* \) is spanned by tensors \( x^{\otimes n} \) with \( x \in V^* \). Hence the map \( f \mapsto f(x^{\otimes m}) = f(x) \) defines an injection of \( S^k(V) \) into the polynomials of degree \( k \) on \( V^* \). The map is clearly surjective, so we may identify \( f \in S^k(V) \) with the polynomial \( f(x) \). It is easy to see that under this identification \( f : g(x) = f(x)g(x) \), so that as a commutative algebra \( S(V) \) can be identified with the algebra of polynomial functions on \( V^* \).

Note that if \( v_1, \ldots, v_n \) is a basis of \( V \), then a basis of \( \Lambda^k(V) \) is given by \( v_{i_1} \wedge \cdots \wedge v_{i_k} \). Thus \( \dim \Lambda^k(V) = \binom{n}{k} \) and \( \dim \Lambda(V) = 2^n \). In particular \( \Lambda^m(V) = 0 \) for \( m > n \) and \( \Lambda^m(V) \) is one–dimensional. We can also identify \( \Lambda^k(V) \) with alternating multilinear functionals on \( V^* \times \cdots \times V^* \). If \( f \) and \( g \) are homogeneous of degree \( a \) and \( b \) respectively, then exterior multiplication is given by the formula

\[
f \wedge g(x_1, \ldots, x_{a+b}) = \frac{1}{(a+b)!} \sum_{\sigma \in S_{a+b}} \varepsilon(\sigma)f_1(x_{\sigma(1)}, \ldots, x_{\sigma(a)})g(x_{\sigma(a+1)}, \ldots, x_{\sigma(a+b)}).
\]

(Actually the sum can be reduced to a sum over the coset space \( S_{a+b}/S_a \times S_b \) since \( f \otimes g = \varepsilon(\sigma)f \otimes g \) for \( \varepsilon \in S_a \times S_b \)).

Finally note that \( V \to S(V) \) and \( V \to \Lambda(V) \) are functors from the additive category of vector spaces to the multiplicative tensor category of vector spaces. This will not be important for us, although it is the key to quantisation in quantum field theory. As Nelson said, first quantisation is a mystery while second quantisation is a functor. The functoriality appears in the isomorphism \( S(V \oplus W) = S(V) \otimes S(W) \) and \( \Lambda(V \oplus W) = \Lambda(V) \otimes \Lambda(W) \) between (graded) commutative algebras. We need \( (a \otimes b)(c \otimes d) = (-1)^{bdac} ac \otimes bd \) to define the tensor product of graded algebras. The basic rule in discussing graded objects is that if we move a symbol of degree \( \partial_1 \) past a symbol of degree \( \partial_2 \), then a sign \( (-1)^{\partial_1 \partial_2} \) must be introduced. The functor \( S \) corresponds to bosons which satisfy the canonical commutation relations while the functor \( \Lambda \) corresponds to fermions which satisfy the canonical anticommutation relations. The basic idea of supersymmetry is that the bosonic and fermionic theory can be developed in parallel at each stage, so that any concept introduced in one theory has its natural counterpart in the other.

**2. INNER PRODUCTS AND TENSORS.** If \( U \) and \( V \) are real or complex inner product spaces, we can define an inner product on \( U \otimes V \) by taking any positive multiple of the inner product \( (u_1 \otimes v_1, u_2 \otimes v_2) = (u_1, u_2)(v_1, v_2) \). In particular we define the inner product on \( T^k(V) = V^k \) by \( (a_1 \otimes \cdots \otimes a_k, b_1 \otimes \cdots \otimes a_k) = k! \prod (a_i, b_i) \). (The factor of \( k! \) is essential here to guarantee \( \exp(a) \exp(b) = \exp(a, b) \) for \( a, b \in V \).) This inner product extends to \( T(V) \) by declaring the \( T^k(V) \) to be mutually orthogonal. Note that, since \( S(V), \Lambda(V) \subset T(V), \) there are naturally induced inner products on \( S(V) \) and \( T(V) \). The definition immediately give the following explicit formulas in the functional realisations above.

**Lemma.** (a) In \( \Lambda(V) \), we have \( (a_1 \wedge \cdots \wedge a_m, b_1 \wedge \cdots \wedge b_n) = \delta_{mn} \det(a_i, b_j). \)

(b) In \( S(V) \), we have \( (x^m, y^n) = \delta_{mn} n!(x, y)^n \).

We will see that regarded as polynomial functions on \( V^* \subset \mathbb{C}^n \), the inner product in \( S(V) \) agrees with the inner product \( (f, g) = \pi^{-n} \int_{\mathbb{C}^n} f(z) \bar{g}(z) e^{-|z|^2} \), so that \( S(V) \) can be identified with so–called holomorphic Fock space (see below). Part (a) of the lemma shows that if \( (e_i) \) is an orthonormal basis of \( V \), then \( e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \) \( (i_1 < \cdots < i_k) \) is an orthonormal basis for \( \Lambda^k(V) \).
Now both on $\Lambda(V)$ and $S(V)$ we have the operation of multiplication by $v \in V$. We now work out their adjoints.

**Theorem (adjoint derivations).** (a) The adjoint $e(v)^*$ of $e(v)$ is the graded derivation $d_v(v_1 \wedge \cdots \wedge v_k) = \sum (-1)^{i+1} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_i + 1 \wedge \cdots \wedge v_k$ with $d_v(1) = 0$.

(b) The adjoint of multiplication by $v$ is the derivation $\partial_v$ given by $\partial_v(x_1 \cdots x_n) = \sum (x_i, v) \prod_{j \neq i} x_j$ for $x_j \in V$ with $\partial_v(1) = 0$.

**Proof.** (a) We have

\[
(e(w_1)^*v_1 \wedge \cdots \wedge v_{n+1}, w_2 \wedge \cdots \wedge w_{n+1}) = (v_1 \wedge \cdots \wedge v_{n+1}, w_1 \wedge \cdots \wedge w_{n+1})
= \det(v_1, w_1) = \sum (-1)^{i+1} (v_i, w_1) (v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_{n+1}, w_1 \wedge \cdots \wedge w_{n+1}),
\]

expanding the determinant by the first column. This proves the formula for $e(v)^*$. This is usually called “contraction” with $v$ or “interior multiplication”. It is routine to check from the definition of $d_v$ that, if $\omega_1$ and $\omega_2$ are homogeneous, then $d_v(\omega_1 \wedge \omega_2) = d_v(\omega_1) \wedge \omega_2 + (-1)^{\partial_v \omega_1} \omega_1 \wedge d_v \omega_2$. This means that $d_v$ is a graded derivation, the signs being compatible with our previous convention since $d_v$ is odd. Note that $d_v$ is uniquely determined once we declare that it is a graded derivation, $d_v(1) = 0$ and $d_v w = (w, v)$ for $w \in V$.

(b) This can be checked directly using the inner product, as in (a). When $V$ is a complex inner product space, it is also obvious in the functional realisation in terms of polynomials on $\mathbb{C}^n$ with the above inner product, for there clearly $z_1$ has adjoint $\partial/\partial z_1$.

**Theorem (real and complex wave representation).** (a) Let $V$ be an inner product space. Then if $a, b \in V$, the operators $e(a), e(b)$ on $\Lambda(V)$ satisfy the canonical anticommutation relations $e(a)e(b) + e(b)e(a) = 0$, $e(a)^*e(b)^* + e(b)^*e(a)^* = 0$ and $e(a)e(b)^* + e(b)e(a)^* = (a, b)$.

(b) Let $V$ be an inner product space. Then, if $v, w \in V$, the operators $z$ and $\partial_w$ on $S(V)$ satisfy the canonical commutation relations $zw - wz = 0, \partial_z \partial_w - \partial_w \partial_z = 0$ and $\partial_w z - z \partial_w = (z, w)$.

**Proof.** (a) Clearly $e(a)$ and $e(b)$ anticommute, so taking adjoints so too do $e(a)^*$ and $e(b)^*$. Now

\[
(e(a)e(b)^* + e(b)^*e(a))\omega = a \wedge e(b)^*\omega + e(b)^*(a \wedge \omega) = a \wedge e(b)^*\omega + (a, b)\omega - a \wedge (eb)^*\omega = (a, b)\omega.
\]

(b) Clearly $z$ and $w$ commute, hence so do their adjoints $\partial_z$ and $\partial_w$. Now

\[
(\partial_w z - z \partial_w)p = z \partial_w p + (z, w)p - z \partial_w p = (z, w)p.
\]

This proves the last commutation relation.

**Theorem (irreducibility of wave representation).** (a) If $V$ is an complex inner product space, the operators $e(v)$ and $e(v)^*$ act irreducibly on $\Lambda(V)$.

(b) If $V$ is a complex inner product space, the operators $v$ and $\partial_v$ act irreducibly on $S(V)$.

**Proof.** (a) Let $U \neq \{0\}$ be an invariant subspace and take $\omega \neq 0$ in $U$. Then $\omega = \sum a_I v_{i_1} \wedge \cdots \wedge v_{i_k}$ with respect to some orthonormal basis $(v_i)$. Pick a non–zero term of maximal degree, $a_I v_{i_1} \wedge \cdots \wedge v_{i_k}$. Then $e(v_{i_k}) \cdots e(v_{i_1})\omega = a_I$, so that $1 \in U$. Since all of $\Lambda(V)$ can be obtained by applying $e(v)$’s to 1, we see that $U = \Lambda(V)$.

(b) We have to show that the operators $z_1$ and $\partial/\partial z_1$ act irreducibly on the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$. Let $U$ be an invariant subspace and take $p(z) \neq 0$ in $U$. Then $p(z) = \sum a_n z^n$. Pick a non–zero term of maximal degree $a_n z^n$. Then $\partial_n p(z) = a_1$, so that $1 \in U$. Since all polynomials can be obtained by multiplying 1 by $z_i$’s, we see that $U = \mathbb{C}[z_1, \ldots, z_n]$.

3. **THE DOUBLE COMMUTANT THEOREM.** Let $V$ be a finite–dimensional inner product space over $\mathbb{C}$ and let $A \subseteq \text{End}_V$ be a *–subalgebra of End $V$. This means that $I \subseteq A$ and $A$ is a linear subspace closed under multiplication and the adjoint operation $T \mapsto T^*$. For any subset $S \subseteq \text{End}_V$, we define the commutant of $S$ by

\[
S' = \text{End}_S(V) = \{T \in \text{End}_V : Tx = xT \text{ for all } x \in S\}.
\]
**Schur’s Lemma.** (i) A acts irreducibly on V (i.e. has no invariant subspaces) iff \( A' = \mathbb{C} \).

(ii) If A acts on two irreducible subspaces \( V_i \) and \( T \in \text{Hom}_A(V_1, V_2) \) (i.e. commutes with A), then \( T = 0 \) or is an isomorphism.

**Proof.** (Spectral Theorem.) (i) Say A does not act irreducibly and \( U \subset V \) be a proper subspace invariant under A (i.e. U is an A–submodule). Then, if \( P \) is the orthogonal projection onto \( U \), we have \( P \in A' \). So \( A' \neq \mathbb{C} \).

Conversely if \( T \in A' \), then, since \( A' \) is a *–algebra, both Re\( T = T + T^* / 2 \) and Im\( T = T - T^* / 2i \) lie in \( A' \). By the spectral theorem for self–adjoint matrices, so does any projection onto an eigenspace (i.e. a spectral projection). So if \( T \notin \mathbb{C} \), we have produced a projection \( P \in A' \) with \( P \neq 0, I \). The corresponding subspace is invariant.

(ii) If \( v_1 \) and \( v_2 \) are irreducible and \( T \) is an intertwiner, then so is \( T^* \) (simply take adjoints of the intertwining relation and replace \( a \) by \( a^* \)). But then \( TT^* \) and \( T^*T \) are also intertwiners, i.e. \( T^*T \in \pi_1(A)' \) and \( TT^* \in \pi_2(A)' \). They must be scalars by (i), so either both zero or both the same multiple of the identity.

**Double commutant theorem.** If \( A \subset \text{End}(V) \) is a *–algebra, then \( A'' = A \).

**Proof.** (1) If \( U \) is a subspace of \( V \) invariant under \( A \), then so is \( U^\perp \). In particular \( V \) is a direct sum of irreducible \( A \)–submodules.

**Proof.** Say \( \xi \in U^\perp \) and \( a \in A \). Let \( \eta \in U \). Then \( \langle a\xi, \eta \rangle = \langle \xi, a^* \eta \rangle = 0 \) since \( a^* \eta \in U \) and \( \xi \perp U \). So \( a\xi \perp U, \) i.e. \( a\xi \in U^\perp \). So \( V = U \oplus U^\perp \) with \( U \) and \( U^\perp \) \( A \)–modules. We continue this game if \( U \) or \( U^\perp \) fail to be irreducible.

(2) If \( S \in A'' \) and \( v \in V \), there is a \( T \in A \) such that \( Tv = Sv \).

**Proof.** In fact let \( W = Av \subset V \). This is an \( A \)–submodule of \( V \). The orthogonal projection onto \( W \) gives a projection \( E \in \text{End} V \) (\( E^2 = E = E^* \)) which commutes with \( A \), from (1). So \( E \in A' \). But \( S \in A'' \), so \( SE = ES \). This means that \( S \) leaves \( W \) and \( W^* \) invariant. (Note that \( I - E \) is the orthogonal projection onto \( W^\perp \)) But \( v \in W \). So \( Sv \in W = Av \). So \( Sv = Tv \) for some \( T \in A \).

(3) Let \( V' = V \oplus \cdots \oplus V \) (\( m \) times) A acting diagonally, \( a(\xi_1, \ldots, \xi_m) = (a\xi_1, \ldots, a\xi_m) \). This means we can identify \( A \) with a *–subalgebra of \( \text{End}(V') \) (for the initiated, \( V \oplus \cdots \oplus V = V \otimes \mathbb{C}^m \)). It’s easy to check that \( \pi(A)' = A' \otimes M_m(\mathbb{C}) = M_m(A') \), if we write elements of \( \text{End}(V') \) as \( m \times m \) matrices with entries in \( \text{End}(V) \). We go on to check that

\[
\pi(A)'' = (\pi(A)'')' = \pi(A') = \begin{pmatrix} x & x \\ & \ddots \\ & & x \end{pmatrix},
\]

where here (and above) \( \pi \) denotes the embedding \( \text{End}(V) \to \text{End}(V') \) taking operators to diagonal operators.

Take \( m = \dim V \). Set \( v = \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix} \) where \( e_1, \ldots, e_m \) is a basis of \( V \). By step (2), we have \( \pi(A)v = \pi(A)''v \).

But \( \pi(A)'' = \pi(A'') \) from the above. So given \( S \in A'' \) we can find \( T \in A \) such that \( \pi(S)v = \pi(T)v \). Hence

\[
\begin{pmatrix} S \\ S \\ \vdots \\ S \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix} = \begin{pmatrix} T & \cdots \\ \vdots & \ddots \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}.
\]

Thus \( Se_i = Te_i \) for all \( i \) and hence \( S = T \).

**Corollary 1.** A *–algebra \( A \) acts irreducibly iff \( A = \text{End}(V) \).
Corollary 1. All *-representations of End(V) are on direct sums of copies of V.

Proof. Since all representations are sums of irreducibles, it suffices to show that V is the only irreducible representation of End(V). But if W is another inequivalent irreducible, the commutant on V ⊕ W must be End(V) ⊕ End(W) by Schur’s lemma and the double commutant theorem. But the image of End(V) must coincide with its double commutant, a contradiction.

Proof. Choose matrix units in A = End(V) and let W be an A-module. Set W_1 = e_11W and let (w_j) be basis of W_1. Consider the map T : ⊕V ⊗ W_1 → W, ⊕μ_ι j e_ι ⊗ w_j → μ_ι j e_ι w_j. T is surjective since I = ∑ e_ι j e_ι t_j, so that AW_0 = W. T is also injective, for ∑ μ_ι j e_ι w_j = 0 forces μ_ι j w_j = 0 for each i (premultiply by e_(11)) and hence μ_ι j = 0. By construction it commutes with the actions of A. It is even unitary if (w_j) is chosen orthonormal. So W is a direct sum of copies of V.

Remarks. In the exercises this theorem is used to determine the structure of finite-dimensional *-subalgebras of End(V), V a complex inner product space. There is also an infinite-dimensional Hilbert space version of the double commutant theorem due to John von Neumann which is the starting point of the modern theory of operator algebras.

Corollary 2. The vector \( \Lambda = 1 \) is cyclic for a the real*–algebra \( A \) is obvious from the Clifford relations. Hence \( A \) contains all tensors over \( GL(V) \). Thus \( \Lambda = 1 \) is cyclic for \( A \) is a *–algebra since each \( v \) of End(\( V \)) is a *–algebra. So to check that \( B' = A \). The algebra \( A \) is a finite-dimensional subspace, so closed. Now any non-invertible matrix is the limit of invertible matrices: for \( x + εI \) for all \( ε \) sufficiently small. So \( A \) contains all tensors \( w ⊗ \cdots ⊗ w \) even if \( w \) is not invertible. So \( C \) coincides with the fixed points of \( S_m \) in End(\( V \))'s, i.e. the commutant of \( S_m \). (Note that conjugation by \( σ \) gives the permutation action of \( S_m \) on \( W⊗^m \)). Thus \( B' = A \) as required.

4. FERMIONS AND CLIFFORD ALGEBRAS.

Real Clifford algebras. Let \( V \) be a real 2n-dimensional inner product space. Operators \( c(v) \) on a real or complex inner product space \( W \) are said to satisfy the real Clifford algebra relations iff \( v \mapsto c(v) \) is \( ℝ \)-linear, \( c(v)^* = c(v) \) and \( c(a) c(b) + c(b) c(a) = 2(a, b)I \).

Lemma 1. If the operators \( c(v) \) satisfy the real Clifford algebra relations, then the real *–algebra \( A \) they generate is spanned by products \( c(v_{i_1}) c(v_{i_2}) \cdots c(v_{i_k}) \) with \( i_1 < i_2 < \cdots < i_k \) and \( (v_i) \) a basis of \( V \). Moreover \( \dim(A) ≤ 2^{\dim(V)} \).

Proof. Clearly the algebra generated by the \( c(v_i) \)'s is a *–algebra since each \( c(v) \) is self-adjoint. Thus it suffices to prove that \( A_0 = \text{lin}_ℂ(c(v_{i_1}) c(v_{i_2}) \cdots c(v_{i_k})) \) is closed under multiplication by \( c(v_i) \). This, however, is obvious from the Clifford relations. Hence \( A = A_0 \). Clearly \( \dim(A_0) ≤ 2^{\dim(V)} \).

Lemma 2. Let \( c(v) = c(v) + e(v)^* \) acting on \( W = \Lambda_ℝ V \).

(a) The \( c(v)^* \)'s satisfy the real Clifford algebra relations.

(b) The vector \( Ω = 1 \in \Lambda^{0}(V) \) is cyclic for a the real*–algebra \( A \) generated by the \( c(v)^* \)'s, i.e. \( AΩ = Ω(V) \).

(c) The operators \( c(v_{i_1}) c(v_{i_2}) \cdots c(v_{i_k}) \) with \( i_1 < i_2 < \cdots < i_k \) are linearly independent and \( c(v_i) \) is separating for \( A \), i.e. \( _{a}Ω = 0 \) implies \( a = 0 \).

(d) \( c(v_{i_1}) \cdots c(v_{i_k}) ω = v_{i_1} ∧ v_{i_2} ∧ \cdots ∧ v_{i_k} ∧ ω \) lower order terms modulo two.

Proof. (a) By the canonical anticommutation relations,

\[
(c(a) c(b) + c(b) c(a)) = (c(a) + c(a)^*)(c(b) + c(b)^*) = (c(b) + c(b)^*)(c(a) + c(a)^*) = 2(a, b)I.
\]

(b) Let \( W_k = \text{lin}_ℂ\{c(x_1) c(x_2) \cdots c(x_j)Ω : j ≤ k \} \) for \( k ≥ 0 \). We prove by induction that \( W_k = \oplus_{j=0}^{k} Λ^j(V) \).

For \( k = 0 \), this is trivial. For \( k > 0 \), \( c(x_1)x_2 ∧ \cdots ∧ x_k = x_1 ∧ \cdots ∧ x_k \) plus a term in \( Λ^{k-2}(V) \). Thus, by induction, \( x_1 ∧ \cdots ∧ x_k \) lies in \( c(x_1)W_{k-1} + W_{k-2} \) as required.

(c) Since \( 2^{\dim(V)} ≤ \dim(Λ(V)) = \dim AΩ = \dim(A) ≤ 2^{\dim(V)} \), this is obvious from (b) and Lemma 1.

(d) This follows easily by induction on \( k \).
We define the real Clifford algebra \( \text{Cliff}(V) \) to be the real *-algebra generated by the \( c(v) \)'s on \( \Lambda(V) \). We show that \( \text{Cliff}(V) \) has a similar universal property to the group algebra \( \mathbb{C}[G] \). This is defined as the algebra of operators on \( \ell^2(G) \) generated by left translations. Any finite-dimensional unitary representation of \( G \) gives rise to a *-representation of \( \mathbb{C}[G] \) and conversely, so that \( \mathbb{C}[G] \) is the universal algebra for representations of \( G \). We claim that any given Clifford algebra relations \( C(v) \) on \( W \), there is a unique *-representation of \( \text{Cliff}(V) \) sending \( c(v) \) to \( C(v) \).

Uniqueness is clear, since the \( c(v) \)'s generate \( \text{Cliff}(V) \); to prove existence, we take a basis \( (v_i) \) of \( V \) and send the basis element \( c(v_i) \) of \( \text{Cliff}(V) \) to \( C(v_i) \). This is clearly a homomorphism of *-algebras. If \( W = \Lambda(V) \), a real inner product space, we have a natural complexification \( W_C = W \otimes_{\mathbb{R}} \mathbb{C} \). This is just obtained by taking an orthonormal basis for \( V \) and extending the scalars and inner product in the obvious way. The algebra \( A = \text{Cliff}(V) \) and its complexification \( \text{Cliff}(V)_C = A_C = A \oplus iA \) acts on \( W_C \). \( A_C \) is a complex *-algebra and \( \Omega \) is again cyclic and separating for \( A_C \). This means \( A_C \) cannot act irreducibly; for if it did, \( A_C = \text{End}(W_C) \) and \( \Omega \) is not separating for \( \text{End}(W_C) \). Note that \( A \to A\Omega \) gives an isomorphism between \( \text{Cliff}(V) \) and \( \Lambda(V) \) as linear spaces. This allows us to speak about the degree of an element of \( \text{Cliff}(V) \). Note the following immediate consequence of Lemma 2 (d).

**Corollary.** If \( \omega_1 \in \Lambda^a(V) \) and \( \omega_2 \in \Lambda^b(V) \), then \( \omega \cdot \omega_2 = \omega_1 \wedge \omega_2 + \) lower degree terms modulo two.

We now show how introducing a complex structure on \( V \) allows us to produce an irreducible representation of the real Clifford algebra relations. By definition a complex structure on \( V \) is a map \( J \in \text{End}(V) \) such that \( J^2 = -I \) and \( J \) is orthogonal. Since \( \dim(V) = 2n \) is even, such maps always exist. We can then define a complex inner product space \( V_J \) from \( V \) by taking \( J \) to be multiplication by \( i \) and taking the complex inner product on \( V \) as \( (v, w)_C = (v, w)_R - i(Jv, w)_R \), where \( (v, w)_R \) denotes the original real inner product on \( V \).

**Lemma.** \( V_J \) is a complex inner product space with \( (v, v)_R = (v, v)_C \).

**Proof.** Clearly \( (v, w) \in \mathbb{R} \)-bilinear. Moreover \( (v, w)_C = (v, w)_R + i(Jv, w)_R = (v, w)_C + i(Jw, v)_R = (w, v)_C \).

Since \( (Jv, w) = i(v, w) \), it follows that \( (v, w)_C \) is \( \mathbb{C} \)-linear in \( v \) and conjugate linear in \( w \). Now \( (Jv, v)_R = -(v, Jv)_R = (Jv, v)_R \), so that \( (Jv, v)_R = 0 \). Hence \( (v, v)_C = (v, v)_R \) and \( V_J \) is a complex inner product space.

**Theorem.** The formula \( C(v) = e(v) + e(v)^* \) gives a faithful (=injective) irreducible representation of \( \text{Cliff}_C(V) \) on \( S = \Lambda(V_J) \), called the “spin module”. In particular \( \text{Cliff}_C(V) \cong \text{End}(S) \).

**Proof.** Clearly \( v \mapsto C(v) \) is \( \mathbb{R} \)-linear, \( C(v)^* = C(v) \) and \( C(v)C(w) + C(w)C(v) = (e(v) + e(v)^*)(e(w) + e(w)^*) + (e(v) + e(w)^*)(e(v) + e(v)^*) = 2\text{Re}(v, w)_C I = 2(v, w)_R I \).

Hence \( C(v) \) satisfies the real Clifford algebra relations and therefore we get *-homomorphism of \( \text{Cliff}(V) \) into \( \text{End}(\Lambda(V_J)) \). Now the relation \( C(v) = e(v) + e(v)^* \) implies \( C(Jv) = e(v) + e(Jv)^* = e(v) - ie(v)^* \). Hence \( e(v) = \frac{1}{2}(C(v) - iC(Jv)) \) and \( e(v)^* = \frac{1}{2}(C(v) + iC(Jv)) \). But the \( e(v)^* \)'s act irreducibly on \( \Lambda(V) \) (it is the complex wave representation), so the \( C(v)^* \)'s must also act irreducibly. Therefore the \( C(v)^* \)'s generate \( \text{End}(S) \). Thus the image of \( \text{Cliff}_C(V_J) \) has \( \mathbb{C} \)-dimension \( \text{dim}(S)^2 = 2^{2\text{dim}_C(V_J)} = 2^{\dim(V)} \).

But this is the \( \mathbb{C} \)-dimension of \( \text{Cliff}_C(V) \), so the representation of \( \text{Cliff}_C(V) \) is faithful and surjective. Hence \( \text{Cliff}_C(V) \cong \text{End}(S) \). Moreover the representation must a fortiori be faithful on the real subalgebra \( \text{Cliff}(V) \).

5. **QUANTISATION AND THE SPIN GROUP.**

**Bogoliubov automorphisms of \text{Cliff}(V).** Consider the compact group \( SO(V) \).

**Lemma.** \( SO(V) \) is connected.

**Proof.** Any matrix in \( SO(V) \) is conjugate to a block diagonal matrix with \( 2 \times 2 \) diagonal blocks \( D_i = \begin{pmatrix} \cos x_i & \sin x_i \\ -\sin x_i & \cos x_i \end{pmatrix} \), so can be connected by a continuous path to \( I \) by the path of matrices with blocks

\[
D_i = \begin{pmatrix} \cos tx_i & \sin tx_i \\ -\sin tx_i & \cos tx_i \end{pmatrix}.
\]

If \( g \in SO(V), \ v \mapsto c(gv) \) also satisfies the real Clifford algebra relations, so induces an automorphism of \( \text{Cliff}(V) \). In fact \( SO(V) \) acts orthogonally on \( \Lambda(V) \) via \( g(x_1 \wedge \cdots \wedge x_k) = gx_1 \wedge \cdots \wedge gx_k \), so that
ge(v)g^{-1} = e(gv) and hence gc(v)g^{-1} = c(gv) since c(v) = e(v) + e(v)^*. Thus SO(V) normalises Cliff(V) on \(\Lambda(V)\). We write \(a_g\) for the automorphism of Cliff(V) and Cliff_c(V) induced by Ad_g, \(a \mapsto gag^{-1}\). In particular \(g_0 = -I\) acts and gives a period two automorphism \(\gamma = \alpha_{-1}\) of Cliff(V) satisfying \(\alpha c(v) = -c(v)\). This automorphism gives rise to a \(\mathbb{Z}_2\)-grading on \(A = \text{Cliff}(V)\), because we can take the \(+1\) eigenspaces \(A_+\) of \(\gamma\). Clearly \(A_+ \subset A_+\), \(A_+ \subset A_+\) and \(A_+ \subset A_+\). Under the identification \(A = \Lambda(V)\), \(A_+ = \Lambda^{\text{even}}(V)\) and \(A_- = \Lambda^{\text{odd}}(V)\).

Now if \(v \mapsto C(v)\) is the irreducible representation of the Clifford algebra relations on the spin module \(S, v \mapsto C(gv)\) will give another irreducible representation on \(S\). By uniqueness we can find \(U_g \in U(S)\) such that \(C(gv) = U_g C(v)U_g^*\) for all \(v\). Note that \(g \in SO(V)\) commutes with the complex structure \(J\) iff \(g \in SU(V_j)\). In this case \(g\) is canonically implemented on \(S = \Lambda(V_j)\) by \(g(v_1 \cdots \cdots v_k) = gv_1 \cdots \cdots gv_k\). In particular \(g_0 = -I\) commutes with all \(J\)'s, so is canonically implemented on each \(S\): \(g_0\) acts as \(\pm 1\) on \(S^\pm\).

The choice of \(U_g\) is not unique. If \(U_g'\) is another possible choice, then \(U_g'U_g^\prime\) must commute with all \(C(v)\)'s and hence must be a scalar matrix by Schur's lemma. Thus \(U_g\) is uniquely determined up to a phase in \(T\), so that \(U_g\) really gives a homomorphism of \(SO(V)\) into \(U(S)/T = PU(S)\), the projective unitary group. This is what is meant by quantisation. The prequantised action on \(V\) can be implemented on Fock space \(S\) by a unitary; the phase represents the anomaly that usually arises when we quantise. As we shall see, we really get a \(\mathbb{Z}_2\)-valued representation of \(SO(V)\) or equivalently a representation of a double cover, called \(\text{Spin}(V)\), which we now construct. Observe first that \(U_g C(v)U_g^* = C(gv)\), so that \(U_g\) normalises the real subalgebra \(A = \text{Cliff}(V)\) of \(\text{End}(S)\).

**Theorem (Noether–Skolem).** \(g \in \text{End}(S)\) normalises \(A\) iff \(g \in A^* \cdot \mathbb{C}^*\), where \(A^*\) denotes the invertible elements in \(A\).

**Proof.** We know that \(\text{End}(S) = A \oplus \imath A\), a direct sum of real vector spaces. Let \(g = a + ib\) with \(a, b \in A\) and set \(\alpha(a) = gag^{-1}\). Then \((a + ib)x = \alpha(x)(a + ib)\). Hence \(ax = \alpha(x)a\) and \(bx = \alpha(x)b\). Consider the polynomial \(p(t) = \det(a + tb)\). Since \(p(i) \neq 0\), we can find \(t \in \mathbb{R}\) such that \(p(t) \neq 0\). Let \(h = a + tb \in A\) and let \(h^{-1} = u + iv\). Then \(h(u + iv) = I\), so that \(hv = 0\) and hence \(v = 0\). Thus \(h^{-1} = A\). Since \(hx = (a + tb)x = \alpha(x)(a + tb) = \alpha(x)h\), it follows that \(z = h^{-1}g\) commutes with \(A\), so lies in \(\mathbb{C}^*\). Hence \(g = hz\) as claimed.

**Corollary.** For each \(g \in SO(V)\), there is a unitary element \(u_g \in A^*\) uniquely determined up to a sign such that \(u_ge(v)u_g^* = c(gv)\).

**Proof.** Suppose \(u_g = \lambda U_g\). Then \(u_g u_g^* = |\lambda|^2\). Scaling \(u_g\), we may therefore arrange that \(u_g\) is unitary. Since \(A^* \cap \mathbb{C}^* = \mathbb{R}^*\), \(u_g\) is uniquely determined up to sign.

**The spin group.** Let \(\text{Spin}(V) = \{\pm u_g : g \in SO(V)\} \subset \text{Cliff}(V)\), the spin group.

**Lemma.** \(\text{Spin}(V)\) consists of unitaries \(u \in \text{Cliff}(V)\) normalising \(c\) such that the orthogonal transformation \(g\) defined by \(c(gv) = uc(v)u^*\) lies in \(SO(V)\). In particular \(\text{Spin}(V)\) is a closed subgroup of the unitary group of \(A\), so compact.

**Proof.** Clearly any element of \(\text{Spin}(V)\) satisfies these conditions. The converse is obvious from the corollary by uniqueness.

The map \(\text{Spin}(V) \to SO(V)\) is a surjective continuous homorphism, by construction. Its kernel is \(\pm I\), so that \(\text{Spin}(V)\) is a double cover of \(SO(V)\).

**Theorem.** (a) \(\text{Spin}(V)\) is connected.
(b) \(\text{Spin}(V) \subset \text{Cliff}^+(V)\).

**Proof.** (a) Let \(f : \text{Spin}(V) \to \mathbb{Z}\) be a continuous function; we must show it is constant. If we show that \(f(-g) = f(g)\) for all \(g\), then \(f\) will drop to a continuous map of \(SO(V)\) into \(\mathbb{Z}\) and hence be constant, by the connectivity of \(SO(V)\). But \(x(t) = \cos \pi t + c(e_1)c(e_2)\sin \pi t\) \((t \in [0, 1])\) is a continuous path in \(\text{Spin}(V)\) from \(I\) to \(-I\). (To see this either use the representation \(C(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), \(C(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) to write \(x(t) = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}\); or note that \(x(t) = \exp \pi c(e_1)c(e_2)t\) with \(c(e_1)c(e_2)\) skew-adjoint.) Hence \(t \mapsto f(gx(t))\) is continuous so constant. Hence \(f(g) = f(-g)\).
(b) Let $u_0$ be the element of $\text{Cliff}(V)$ implementing the grading automorphism $\gamma$. Then $u_0 c(v) u_0^* = -c(v)$. But $\alpha_g(u_0)$ also implements $\gamma$, so that $\alpha_g(u_0) = \lambda(g) u_0$ with $\lambda(g) = \pm 1$. Thus $\lambda(g)$ is a continuous homomorphism $\text{SO}(V) \to \{\pm 1\}$. Since $\text{SO}(V)$ is connected, $\lambda(g) = 1$. Since $\alpha_g(u_0) = u_g u_0 u_g^*$, this implies that $u_g$ commutes with $u_0$. But then $\gamma(u_g) = u_0 u_g u_0^* = u_g$, so that $u_g \in \text{Cliff}^+(V)$.

**Remark.** There is also an infinitesimal version of the action of $\text{Spin}(V)$ in terms of bilinear combinations of fermions (see the exercises).

### 6. MATRIX GROUPS AND THEIR LIE ALGEBRAS

We start by proving von Neumann’s theorem on closed subgroups of $GL(V)$. We define $\text{gl}(V) = \text{End} V$ with the usual operator norm.

**Lemma (Lie’s formulas).** If $a, b \in \text{End}(V)$ then $(\exp(a/n) \exp(b/n))^n \to \exp(a + b)$ and $(\exp(a/n) \exp(b/n) \exp(-a/n) \exp(-b/n))^n \to \exp[a, b]$.

**Proof.** Recall that $\exp(a) = \sum a^n/n!$ for all $a$ and $\log(1 + x) = \sum (-1)^{n+1} x^n/n$ for $\|x\| < 1$. For $\|a\|$ sufficiently small, we have $\log \exp a = a$ and for $x$ sufficiently small $\exp \log(1 + x) = 1 + x$. Then

$$\log([\exp(a/n) \exp(b/n)]^n) = n \log(1 + (a + b)/n + O(1/n^2)) = a + b + O(1/n) \to a + b,$$

and

$$\log([\exp(a/n) \exp(b/n) \exp(-a/n) \exp(-b/n)]^n) = n^2 \log(1 + [a, b]/n^2 + O(1/n^3)) = [a, b] + O(1/n) \to [a, b].$$

**Theorem (von Neumann).** Let $G$ be a closed subgroup of $GL(V)$ and let

$$\text{Lie}(G) = \{ X \in \text{End} V \mid \exp(tX) \in G \text{ for all } t \}.$$

Then $\text{Lie}(G)$ is a linear subspace of $\text{End} V$ closed under the Lie bracket $[a, b] = ab - ba$ and $\exp(\text{Lie}(G))$ is a neighbourhood of 1 in $G$. In fact if $U$ is a sufficiently small open neighbourhood of 0 then $\exp(U)$ is an open neighbourhood of 1 and $G$ gives a homeomorphism between $U$ and $\exp(U)$.

**Proof.** Lie’s formulas applied to $tX$ and $tY$ immediately show that $\text{Lie}(G)$ is a subspace closed under the bracket $[X, Y] = XY - YX$.

It remains to show that $\exp(\text{Lie}(G))$ is a neighbourhood of 1 in $G$. Let $\text{Lie}(G)^\perp$ be a vector subspace complementing $\text{Lie}(G) \cap \text{gl}(V)$, so that $\text{gl}(V) = \text{Lie}(G) \oplus \text{Lie}(G)^\perp$. By the inverse function theorem, $X \in \text{End} V \implies \exp(X) \text{ gives a homeomorphism between a neighbourhood of 0 in End}(V)$ and 1 in $GL(V)$ (its derivative is $I$). If $\exp(\text{Lie}(G))$ is not a neighbourhood of 1 in $G$, then we can find $g_n \in G$ with $g_n \to 1$ but $g_n \notin \exp(\text{Lie}(G))$. Write $g_n = \exp(X_n) \exp(Y_n)$ with $X_n \in \text{Lie}(G)$, $Y_n \in \text{Lie}(G)^\perp$. By assumption $Y_n \neq 0$ for all $n$. But since $\exp(X_n)$ and $g_n$ are in $G$, it follows that $\exp(Y_n) \in G$ for all $n$. Since $g_n \to 1$, we must have $Y_n \to 0$. By compactness, we may assume by passing to a subsequence if necessary that $Y_n/\|Y_n\| \to Y \in \text{Lie}(G)^\perp$ with $\|Y\| = 1$. Since $\|Y_n\| \to 0$, we can choose integers $m_n$ such that $m_n \|Y_n\| \to t$. Then $\exp(m_n Y_n) = \exp(Y_n)^{m_n} \in G$ has limit $\exp(tY)$. Since $G$ is closed, $\exp(tY) \in G$ for $t > 0$ and hence for all $t$ on taking inverses. So by definition $Y$ lies in $\text{Lie}(G)$, a contradiction.

This result says that matrix groups are Lie groups. If $G$ is a matrix group, we denote its Lie algebra by $\text{Lie}(G)$. We shall be interested in matrix groups that are closed subgroups of $O(n)$. Since $O(n) \subset U(n)$, they are also closed subgroups of $U(n)$.

**Corollary.** Let $G$ and $H$ be matrix groups and $\pi : G \to H$ a continuous homomorphism. Then there is a unique Lie algebra homomorphism $\pi : \text{Lie}(G) \to \text{Lie}(H)$ such that $\pi(\exp(X)) = \exp(\pi(X))$ for $X \in \text{Lie}(G)$.

**Proof.** Uniqueness follows because we may replace $X$ by $tX$ and take the coefficient of $t$. Conversely note that $\pi(\exp(tX))$ is a one parameter subgroup in $H$. Now $H$ is a closed subgroup of $U(n)$; since commuting unitaries can be simultaneously diagonalised, it follows that $\pi(\exp(tX)) = \exp(tA)$ for some matrix skew–adjoint matrix $A$. But then by definition $A$ lies in $\text{Lie}(H)$. We define $\pi(X) = A$. From Lie’s formulas, the map $X \mapsto \pi(X)$ is a Lie algebra homomorphism.

There is also an infinitesimal version of the action of $\text{Spin}(V)$ in terms of bilinear combinations of fermions.
Proposition. (a) \( \text{Lie}(SO(V)) = \{ A \in \text{End}(V) : A^t = -A \} \).
(b) \( \text{Lie}(\text{Spin}(V)) = \{ x \in \text{Cliff}^+(V) | x^t = -x, [x, c(V)] \subset c(V) \} = \text{Im}_R \{ c(a)c(b) - c(b)c(a) : a, b \in V \} \). A basis is given by \( c_i,c(e_j) \) with \( i \neq j \).
(c) If \( \pi : \text{Spin}(V) \to SO(V) \) is the double cover, \( \pi^{-1}(A) = \frac{1}{4} \sum_{i,j} a_{ij}c(e_i)c(e_j) \). In other words, if \( A \in \text{so}(V) \), then \( \pi^{-1}(A) = \frac{1}{4} \sum_{i,j} c(A \cdot e_i)c(e_j) \).
(d) If \( X \in \text{Lie}(\text{Spin}(V)) \), then \( [X, c(v)] = c(\pi(X)v) \).

Proof. (a) is obvious. To prove (b) and (c), note that if \( e^{yt} \) lies in \( \text{Spin}(V) \), then \( y \) is even, \( y^* = -y \) and \( e^{yt}c(v)e^{-yt} = c(e^Atv) \) for some \( A \in \text{Lie}(SO(V)) \). Taking the coefficient of \( t \), we get \( [y, c(v)] = c(Av) \). Let \( (Ae_j, e_i) = a_{ij} \), so that \( a_{ij} \) is antisymmetric and real, and let \( x = \frac{1}{2} \sum_{ij} a_{ij}c(e_i)c(e_j) \). Then

\[
[x, c(v)] = \frac{1}{4} \sum_{ij} a_{ij}c(e_i)c(e_j), c(v)] = \frac{1}{4} \sum_{ij} a_{ij}[-\{c(e_i), c(v)c(e_j) + c(e_i)c(e_j), c(v)]
\]

\[
= \frac{1}{4} \sum_{ij} a_{ij}[(v, e_i)c(e_j) - (v, e_j)c(e_i)] = c(Av),
\]

using the graded commutator \( \{ c(u), c(v) \} = c(u)c(v) + c(v)c(u) = 2(u,v)I \) and the rules for computing graded commutators. Thus \( [y - x, c(v)] = 0 \) for all \( v \in V \) and therefore \( y - x \) must be a real scalar. Since 
\( (y - x)^* = -(y - x) \), we deduce that \( y = x \) as required. The map between Lie algebras is \( \frac{1}{2} \sum a_{ij}c(e_i)c(e_j) \) \( \to (a_{ij}) \) by uniqueness. Finally, if \( A \in \text{Lie}(\text{Spin}(V)) \), then \( e^{At}c(v)e^{-At} = c(\pi(e^{At})v) = c(e^{\pi(A)t}v) \). Taking coefficients of \( t \), we get \( [A, c(v)] = c(\pi(A)v) \), so (d) follows.

7. THE ODD–DIMENSIONAL CASE. The structure of Clifford algebras and spin groups for odd–dimensional inner product spaces can easily be deduced from the even–dimensional case. Let \( V \) be a real inner product space with odd dimension. We may write \( V = V_0 \oplus \mathbb{R}e_0 \), where \( V_0 \) is even–dimensional and \( e_0 \) is a unit vector. Let \( e_1, \ldots, e_m \) be an basis of \( V_0 \), with \( m = 2n \). Suppose that the operators \( c(v) \) satisfy the real Clifford algebra relations \( c(v)c(w) + c(w)c(v) = 2(v, w), c(v)^* = c(v) \).

Lemma. The element \( z = c(e_0)c(e_1)c(e_2) \cdots c(e_m) \) commutes with all \( c(v) \)'s and satisfies \( z^2 = (-1)^n \).

Proof. It is immediate that \( c(e_i)z = zc(e_i) \) for all \( i \), since there are an odd number of \( e_j \)'s. Hence \( c(v)z = zc(v) \) for all \( v \). A simple induction argument shows that \( c(e_0) \cdots c(e_k) = (-1)^{k(k-1)/2}c(e_k)c(e_{k-1}) \cdots c(e_0) \). Thus \( c(e_0)c(e_1) \cdots c(e_m) = (-1)^nc(e_m) \cdots c(e_0) \), so that \( z^2 = (-1)^n \).

Suppose that the \( c(v) \)'s act irreducibly on \( W \), a complex inner product space. By Schur’s lemma, \( z \) must be a scalar, so \( \pm i^n \). Thus

\[
c(e_0) = \mp i^n c(e_1) \cdots c(e_m).
\]

(1)

It follows that \( c(e_1), \ldots, c(e_m) \) already act irreducibly on \( W \). Thus \( \dim(W) = 2^n \) is the standard irreducible representation of \( \text{Cliff}_C(V_0) \). Conversely let \( W \) be an irreducible representation of \( \text{Cliff}_C(V_0) \) and define \( c(e_0) \) by (1). It is easy to check that the \( c(e_i) \)'s satisfy the Clifford relations and hence we get irreducible representations \( \text{Cliff}_C(W) \) of \( \text{Cliff}(V) \). The representations are inequivalent because \( c(e_0) \cdots c(e_m) = z \) with \( z = \pm i^n \). The maximum dimension of \( \text{Cliff}_C(V) \) is \( 2^{n+1} \) and by the double commutant theorem we have a surjection onto \( \text{End}(W_+) \oplus \text{End}(W_-) \). Since this space also has dimension \( 2^{n+1} \), this map is an isomorphism:

\[
\text{Cliff}_C(V) \cong \text{End}(W_+) \oplus \text{End}(W_-).
\]

Lemma. The inclusion \( \text{Cliff}_C^+(V) \subset \text{Cliff}_C(V) \) induces isomorphisms \( \text{Cliff}_C^+(V) \cong \text{End}(W_+) \). The spaces \( W_\pm \) give equivalent irreducible representations of \( \text{Cliff}_C^+(V) \).

Proof. Taking any reordering of the \( e_i \)'s, the previous lemma shows that

\[
c(e_i) = \pm(i^n) c(e_0) \cdots c(e_i) \cdots c(e_m) \in \pi(\text{Cliff}_C^+(V)).
\]
It follows that $\text{Cliff}^+(V)$ acts irreducibly on $W_\pm$. Thus we have a surjection of $\text{Cliff}^+(V)$ onto $\text{End}(W_\pm)$. Since both space have dimension $2^m$, this is an isomorphism, so the first assertion follows. The second follows because a matrix algebra has a unique irreducible representation.

The theory of the spin group for odd-dimensional spaces now proceeds exactly as in the even-dimensional case, but based on $\text{Cliff}^+(V)$ rather than $\text{Cliff}(V)$. The group $SO(V)$ acts by automorphisms on $\text{Cliff}^+(V)$. On the other hand $\text{Cliff}^+(V)$ has a unique irreducible representation $W$. Therefore for each $g \in SO(V)$ there is a unitary $U_g \in \text{U}(W)$, unique up to a scalar multiple, such that $c(gv)c(gw) = U_g c(v)c(w)U_g^*$ for all $v, w \in V$. Since every $c(e_i)$ can be expressed as a product of $n$ elements $c(v)c(w)$, this is equivalent to the condition that $U_g c(v)U_g^* = c(gv)$. Let $A = \pi(\text{Cliff}^+(V)) \subset \text{End}(W)$. Then $A + iA = \text{End}(W)$ and $A \cap iA = \{0\}$. The Noether–Skolem argument then implies that we can find $u_g \in A^+$ with $u_g u_g^* = I$ such that $u_g c(v)u_g^* = c(gv)$. Moreover $u_g$ is unique up to a sign. Let $\text{Spin}(V) = \{\pm u_g : g \in SO(V)\}$. The proofs of the following results are as before.

**Lemma.** $\text{Spin}(V)$ consists of unitaries $u \in \text{Cliff}(V)$ normalising $c(V)$ such that the orthogonal transformation $g$ defined by $c(gv) = uc(v)u^*$ lies in $SO(V)$. In particular $\text{Spin}(V)$ is a closed subgroup of the unitary group of $A$, so compact.

The map $\text{Spin}(V) \to SO(V)$ is a surjective continuous homorphism, by construction. Its kernel is $\pm I$, so that $\text{Spin}(V)$ is a double cover of $SO(V)$.

**Theorem.** (a) $\text{Spin}(V)$ is connected.
(b) $\text{Spin}(V) \subset \text{Cliff}^+(V)$.

There is also an infinitesimal version of the action of $\text{Spin}(V)$ in terms of bilinear combinations of fermions.

**Proposition.** (a) $\text{Lie}(SO(V)) = \{A \in \text{End}(V) : A^t = -A\}$.
(b) $\text{Lie}(\text{Spin}(V)) = \{x \in \text{Cliff}^+(V) : x^* = -x, [x, c(V)] \subset c(V)\} = \text{lin}_\mathbb{R}\{c(a)c(b) - c(b)c(a) : a, b \in V\}$. A basis is given by $c(e_i)c(e_j)$ with $i < j$.
(c) If $\pi : \text{Spin}(V) \to SO(V)$ is the double cover, $\pi^{-1}(A) = \frac{1}{4} \sum c(A \cdot e_i)c(e_i)$.
(d) If $A \in \text{Lie}(\text{Spin}(V))$, then $[\pi^{-1}(A), c(v)] = c(\pi(A)v)$.

8. THE SPIN REPRESENTATIONS OF $\text{Spin}(V)$. We treat the even and odd dimensional cases separately.

**Proposition (irreducibility of spin representations).** If $\text{dim}V$ is even, the spin representations $S^\pm$ of $\text{Spin}(V)$ are irreducible. If $\text{dim}V$ is odd, the spin representations $S$ of $\text{Spin}(V)$ is irreducible.

**Proof.** (1) If $\text{dim}V$ is odd, we have seen that the $c(v)c(w)$’s act irreducibly on $S$. Since the algebra these generate is the double commutant of $\text{Spin}(V)$, $\text{Spin}(V)$ acts irreducibly. In fact $\text{Cliff}(V)$ acts irreducibly on $W^+ \oplus W^-$. There are two ways to see this. (a) We may introduce $\gamma = c(v_1) \cdots c(v_{2m}) \in A$, where $(v_i)$ is an orthonormal basis of $V$. Then $\gamma c(v) = -c(v)\gamma$ for $v \in V$. Thus $\gamma$ commutes with $A$ but is not a scalar. On the other hand $\gamma^2$ is central in $\text{Cliff}(V)$ and unitary, so $\gamma^2 = \pm I$. Thus $\gamma$ must take distinct values on $W^+$ and $W^-$, proving their inequivalence. (b) Take a unit vector $v \in V$ and set $g = c(v)$. Then $g^2 = I$ and $gAg^{-1} = A$. If the two representations $W^\pm$ are equivalent on $A$, then $A$ would be isomorphic to $\text{End}(W^\pm)$ and hence have $1/4$ times the dimension of $\text{Cliff}(V)$. But $\text{Cliff}^+(V) = A$ and $\text{Cliff}^-(V) = Ag$, so that $\text{dim}Cliff(V) = 2 \cdot \text{dim}A$, a contradiction. Finally $\text{Cliff}(V)$ is generated by $c(v)c(w)$’s and therefore by the image spin$(V)$. It follows that the representations $W^\pm$ are irreducible and inequivalent on spin$(V)$ and hence Spin$(V)$.

Every matrix $g \in SO(N)$ is conjugate to a matrix with $2 \times 2$ blocks $\begin{pmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{pmatrix}$ down the diagonal where $j = 1, \ldots, [N/2]$. Note that there is an additional $1$ on the diagonal if $N$ is odd. Thus the complex eigenvalues of $g$ are $e^{\pm \theta_j}$. In any irreducible projective representation $\pi(g)$ of $SO(N)$, the eigenvalues of a
generic block diagonal element $\pi(g)$ are called the weights of the representation (see Chapter II for a more precise definition).

**Lemma (weights of spin representation).** (1) If $\dim(V)$ is even, the weights of the spin representation $W^\pm$ are $\exp i\frac{1}{2} \sum \pm \theta_k$ where the number of plus signs is even for $W^+$ and odd for $W^-$.  
(2) If $\dim(V)$ is odd, the weights of the spin representation $W$ are $\exp \frac{i}{2} \sum \pm \theta_k$.

**Proof.** Let $v_1, \ldots, v_m$ be an orthonormal basis for $V$ and set $v_I = v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $i_1 < \cdots < i_k$. Then $c(v_k) = e(v_k) + e(v_k)^*$ and $c(iv_k) = i(e(v_k) - e(v_k)^*)$. The generators of Lie algebra of the maximal torus are given by $T_j = \frac{1}{4}(c(v_j)c(iv_j) - c(iv_j)c(v_j)) = \frac{1}{2}c(v_j)c(iv_j)$. Thus $T_jv_I = i/2 v_I$ if $j \in I$ and $T_jv_I = -i/2 v_I$ if $j \notin I$. The corresponding self-adjoint operators $S_j$ satisfy $T_j = iS_j$ so that $S_j v_I = \pm v_I$. Since $W^+ = \Lambda^{\text{even}}(V)$ is spanned by $v_I$’s with $I$ even and $W^- = \Lambda^{\text{odd}}(V)$ is spanned by $v_I$’s with $I$ odd, the result follows.

(2) Note that the maximal torus of $SO(V_0)$ coincides with the maximal torus of $SO(V)$. On the other hand the spin representation of $SO(V)$ equals $W^+ \oplus W^-$, where $W^\pm$ are the spin representations of $SO(V_0)$. So the result follows immediately from (1).

**Lemma (grading operator).** If $(e_i)$ is any orthonormal basis of $V$, then $c(e_1) \cdots c(e_n)$ equals $\pm u_0$, the operator implementing the grading. Moreover $u_0^2 = (-1)^{\frac{1}{2} \dim(V)} I$. The grading operator on $S$ is given by $\lambda u_0$ where $\lambda = (i)^{\frac{1}{2} \dim(V)}$.

**Proof.** If $\dim(V) = 2m$, then the elements $a_i = c(e_{2i-1})c(e_{2i})$ commute and satisfy $a_i^2 = -1$. Hence $g = c(e_1) \cdots c(e_n)$ satisfies $g^2 = (-1)^m I$. Moreover $gc(e_i) = -c(e_i)g$. Hence $g = \pm u_0$ and $u_0^2 = g^2 = (-1)^m I$. Now a multiple $\lambda u_0$ of $u_0$ acts as $\pm 1$ on $S^\pm$. Since $u_0^2 = (-1)^m$, we get $\lambda^2 = (-1)^m$.

**Corollary.** $\text{Spin}(V) = \{ u \in \text{Cliff}^+(V) : uu^* = u^*u = I, \ u c(V) u^* = c(V) \}$.

**Proof.** Suppose that $u \in \text{Cliff}^+(V)$ is unitary and that the orthogonal transformation $g$ with $uc(v)u^* = c(gv)$ has determinant $-1$. Define $h \in O(V)$ by $he_1 = -e_1$ and $he_i = e_i$ for $i > 1$. Then $x = g^{-1}h \in SO(V)$, so corresponds to $\gamma \in \text{Spin}(V)$. Hence $h = gx$ corresponds to $w = uw \in \text{Cliff}^+(V)$. But $wc(e_1) \cdots c(e_n)w^* = c(h e_1) \cdots c(h e_n) = -c(e_1) \cdots c(e_n)$, so that $\gamma(w) = -w$, a contradiction.

**Caveat.** If $\dim(V)$ is even and $J$ is a complex structure on $V$, then $U(V_J) \subset SO(V)$ is the subgroup of $SO(V)$ commuting with $J$. Thus $U(V)$ acts canonically on $W_\pm = \Lambda^\pm V_J$ fixing $\Lambda^0 V_J = \mathbb{C}$. Denote this representation by $\pi$. Note however that the representation of $U(V)$ obtained by restricting the spin representations $W_\pm$ of $SO(V)$ is given by $\det(g)^{-1/2} \pi(g)$.
CHAPTER II. COMPACT MATRIX GROUPS AND SIMPLE LIE ALGEBRAS

PART 1. ELEMENTARY STRUCTURE THEORY.

1. COMPACT LIE ALGEBRAS. Let \( G \) be closed subgroup of \( O(V) \) or \( U(V) \), where \( V \) is a real or complex inner product space. Since \( G \) is closed and \( O(V) \) and \( U(V) \) compact, \( G \) must also be compact. We call \( G \) a compact matrix group. Recall that we have defined the Lie algebra of \( G \) as

\[
\text{Lie}(G) = \mathfrak{g} = \{ X \in \text{End} V \mid \exp(tX) \in G \text{ for all } t \}.
\]

By von Neumann's theorem, \( \mathfrak{g} \) is a linear subspace closed under the Lie bracket and the exponential is locally a homeomorphism between neighbourhoods of \( 0 \in \mathfrak{g} \) and \( 1 \in G \). Thus \( \mathfrak{g} \) is indeed a Lie algebra! It is routine to check that the bracket defines the Jacobi identity

\[
[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0
\]

for \( X, Y, Z \in \mathfrak{g} \). An abstract course in Lie algebras might take the antisymmetry of the bracket \( ([X, Y] = -[Y, X]) \) and the Jacobi identity as the defining axioms of a Lie algebra. Since every finite-dimensional Lie algebra can be realised as a Lie subalgebra of matrices, we prefer a more concrete approach. We also proved that every continuous homomorphism between matrix groups \( \pi : G \to H \) gave rise to a unique Lie algebra homomorphism \( \pi : \text{Lie}(G) \to \text{Lie}(H) \) such that \( \pi(\exp(X)) = \exp(\pi(X)) \) for \( X \in \text{Lie}(G) \). The converse of this statement is also true when suitably interpreted. We shall return to this point later.

Note that if \( G \) is a closed subgroup of \( U(V) \) for some \( V \) and hence \( \mathfrak{g} \) carries a real inner product, namely \( \text{Re} \, \text{Tr}(XY^*) = (X, Y) \). This inner product is invariant under \( G \) and therefore under \( \mathfrak{g} \), i.e. \( ([X, Y], Z) + (Y, [X, Z]) = 0 \). We define an involution on \( \mathfrak{g} \) by \( X^* = -X \). This extends to a conjugate linear involution on \( \mathfrak{g}_C = \mathfrak{g} + i\mathfrak{g} \). \( (X + iY)^* = X^* - iY^* = -X + iY \). The inner product extends to a complex inner product on \( \mathfrak{g}_C \) such that \( \text{ad}(X^*) = \text{ad}(X)^* \).

Now suppose that \( \mathfrak{g} \) is a Lie algebra with an invariant inner product \((X, Y)\). We call \( \mathfrak{g} \) a compact Lie algebra. Clearly if \( \mathfrak{b} \) is an ideal in \( \mathfrak{g} \) then so is \( \mathfrak{b}^+ \), and \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}^+ \).

**Lemma.** Every compact Lie algebra is the direct sum of an Abelian algebra (its centre) and simple Lie algebras (of compact type).

**Proof.** \( \mathfrak{g} \) acts as a \(*\)-representation on \( \mathfrak{g} \) via \( \text{ad} \). We can therefore decompose \( \mathfrak{g} \) into a direct sum of irreducibles. Grouping together the copies of the trivial representation into \( \mathfrak{g}_0 \), we have

\[
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,
\]

with \( \mathfrak{g}_0 = \mathfrak{z} \), the centre of \( \mathfrak{g} \). Clearly \([\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_0 \) and \([\mathfrak{g}_j, \mathfrak{g}_i] = 0 \) for \( i \neq j \). By definition of \( \mathfrak{g}_0 \), we must therefore have \([\mathfrak{g}_i, \mathfrak{g}_j] \neq (0) \) for \( i \neq j \). Any \( \text{ad} \)-submodule of \( \mathfrak{g}_i \) is clearly \( \text{ad} \)-invariant, so that \( \mathfrak{g}_i \) is simple and non-Abelian. Indeed \([\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i \), since it is a non-zero ideal; this implies \([\mathfrak{g}, \mathfrak{g}] = \bigoplus_{i>0} \mathfrak{g}_i = \mathfrak{z}^+ \). We call \( \mathfrak{g} \) semisimple if its centre is trivial. Clearly if \( \mathfrak{g} \) is semisimple (and compact), then \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g} \). (Weyl proved that any complex semisimple Lie algebra is the complexification of a compact Lie algebra; see below.)

**Lemma.** If \( \mathfrak{g} \) is semisimple and compact iff the Killing form \( B(X, Y) = \text{Tr}(\text{ad}(X) \text{rad}(Y)) \) is negative definite.

**Proof.** If \( B \) is negative definite, \((X, Y) = -B(X, Y) \) gives an invariant inner product on \( \mathfrak{g} \). If \( X \in \mathfrak{z} \), then \((X, Y) = 0 \) for all \( Y \), so \( X = 0 \) and hence \( \mathfrak{g} \) is semisimple. Conversely if \( \mathfrak{g} \) is semisimple and compact, let \((e_i)\) be an orthonormal basis of \( \mathfrak{g} \) for the invariant inner product \((X, Y)\).

\[
B(X, X) = \text{Tr}(\text{rad}(X)^2) = \sum_i ((\text{ad} X)^2 e_i, e_i) = -\sum_i \|\text{rad}(X)e_i\|^2.
\]

Thus \( B(X, X) \leq 0 \) with equality iff \( \text{ad}(X)e_i = 0 \) for all \( i \) iff \( \text{ad}(X) = 0 \) iff \( X = 0 \).

**Lemma.** If \( \mathfrak{g} \) is compact and semisimple, every derivation of \( \mathfrak{g} \) is inner.
2. EXAMPLES OF SIMPLE COMPACT LIE ALGEBRAS.

Classical compact groups. The classical compact simple Lie groups are $SU(N)$, $SO(2N + 1)$, $U(\mathbb{H}^n)$, $SO(2N)$. It is easy to compute their Lie algebras and verify that they are simple.

Compact Lie algebras constructed from lattices. We shall now give a method of constructing a compact Lie algebra from a lattice, essentially due to Tits. Let $\Lambda$ be a lattice in the real inner product space $V$ such that $\Lambda$ is integral, i.e. $(\alpha, \beta) \in \mathbb{Z}$ for $\alpha, \beta \in \Lambda$, and $\Lambda$ is even, i.e. $(\alpha, \alpha) \in 2\mathbb{Z}$ for $\alpha \in \Lambda$. The lattice has a natural bicharacter $B(\alpha, \beta) = (-1)^{(\alpha, \beta)}$ with values in $\mathbb{Z}_2 = \{\pm 1\}$, i.e. a bilinear form $\Lambda \times \Lambda \to \mathbb{Z}_2$. An $\varepsilon$–factor is a bilinear map $\varepsilon : \Lambda \times \Lambda \to \mathbb{Z}_2$ such that $B(\alpha, \beta) = \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)$ and $\varepsilon(\alpha, \alpha) = (-1)^{\|\alpha\|^2/2}$.

Lemma. Every $\varepsilon$–factor has the form $\varepsilon(\alpha, \beta) = (-1)^{b(\alpha, \beta)}$ where $b : \Lambda \times \Lambda \to \mathbb{Z}$ is a bilinear map such that $b(\alpha, \alpha) \equiv \|\alpha\|^2/2$ modulo 2.

Proof. Let $\alpha_1, \ldots, \alpha_m$ be a $\mathbb{Z}$–basis of $\Lambda$. Define $b : \Lambda \times \Lambda \to \mathbb{Z}$ bilinear by $b(\alpha_i, \alpha_j) = \|\alpha_i\|^2/2$, $b(\alpha_i, \alpha_j) = (\alpha_i, \alpha_j)$ if $i < j$ and $b(\alpha_i, \alpha_j) = 0$ if $i > j$. Clearly $\varepsilon(\alpha, \beta) = (-1)^{b(\alpha, \beta)}$ is an $\varepsilon$–factor. Conversely given an $\varepsilon$–factor, choose $b(\alpha_i, \alpha_j) \in \mathbb{Z}$ such that $\varepsilon(\alpha_i, \alpha_j) = (-1)^{b(\alpha_i, \alpha_j)}$. Extending $b$ bilinearly to $\Lambda \times \Lambda$, we evidently have $\varepsilon = (-1)^b$.

Using the inner product each $\alpha \in \Lambda$ defines a real linear form $\alpha(H) = (H, \alpha)$ on $V$ which extends by complex linearity to $\mathfrak{h} = V \oplus iV$. Let $\Phi = \{\alpha \in \Lambda : \|\alpha\|^2 = 2\}$ and let

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi} CE_{\alpha}.$$ 

Define non–trivial brackets by $[H, E_{\alpha}] = \alpha(H)E_{\alpha}$, $[E_{\alpha}, E_{-\alpha}] = -\alpha$ and $[E_{\alpha}, E_{\beta}] = \varepsilon(\alpha, \beta)E_{\alpha+\beta}$ if $\alpha + \beta$ is a root.

Proposition. The above brackets make $\mathfrak{g}$ into a complex Lie algebra.

Proof. It is easy to verify that the bracket satisfies $[X, Y] = -[Y, X]$ by taking $X$ and $Y$ to be basis elements. We therefore only have to check that the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

is satisfied when $X$ and $Y$ are basis elements. This identity is obvious if at least two of $X, Y, Z$ lie in $\mathfrak{h}$. If $Z$ lies in $\mathfrak{h}$ and $X = E_{\alpha}, Y = E_{\beta}$ then the left hand side is $[X, Y] \{\{\alpha + \beta)(Z) - \alpha(Z) - \beta(Z)\} = 0$. So we may suppose that $X, Y, Z$ all are $E_{\alpha}$’s. Let $\mathfrak{g}_{\alpha} = CE_{\alpha}$. Note that by definition $[h, \mathfrak{g}_{\alpha}] \subseteq \mathfrak{g}_{\alpha}$ and $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Note also that if $\alpha, \beta, \alpha + \beta \in \Phi$, then $(\alpha, \beta) = -1$ and so $\varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha)$.

Suppose then that $X = E_{\alpha}, Y = E_{\beta}$ and $Z = E_{\gamma}$. If $\alpha + \beta + \gamma \notin \Phi \cup \{0\}$, then each term in the Jacobi identity must be zero. So either $\alpha + \beta + \gamma = 0$ or $\alpha + \beta = \gamma \in \Phi$. If $\alpha + \beta + \gamma = 0$, then

$$\varepsilon(\alpha, \beta) = \varepsilon(\beta, \gamma) = \varepsilon(\gamma, \alpha).$$

By symmetry only one of these equalities needs to be proved; the first holds because $\gamma = -\alpha - \beta$ and $\varepsilon(\alpha, \alpha) = -1$, $\varepsilon(\beta, \alpha) = -\varepsilon(\alpha, \beta)$. Hence the left hand side of the Jacobi inequality is proportional to

$$[E_{\alpha+\beta}, E_{\gamma}] + [E_{\beta+\gamma}, E_{\alpha}] + [E_{\gamma+\alpha}, E_{\beta}] = [E_{\gamma+\alpha}, E_{\alpha}] + [E_{\gamma+\beta}, E_{\alpha}] + [E_{\gamma+\alpha}, E_{\beta}] = \gamma + \alpha + \beta = 0.$$ 

Now suppose that $\delta = \alpha + \beta + \gamma \in \Phi$. Expanding $\|\delta\|^2 = 2$, we get

$$(\alpha, \beta) + (\beta, \gamma) + (\gamma, \alpha) = -2.$$ 

Let $a = (\beta, \gamma)$, $b = (\gamma, \alpha)$ and $c = (\alpha, \beta)$. Any permutation of $\alpha, \beta, \gamma$ results in a distinct permutation of $a, b, c$. We have $a + b + c = -2$ and $-2 \leq a, b, c \leq 2$. We may therefore assume that $c \geq 0$. If $c = 0$, we must
have (after permuting) $a = -1$ and $b = -1$ or $a = 0$ and $b = -2$. If $c = 1$, we must have (after permuting) $a = -2$ and $b = -1$. If $c = -2$, we must have $a = -2 = b$. Thus there are four possibilities:

1. $(a, b, c) = (-1, -1, 0)$. Thus $(\alpha, \beta) = 0, \beta + \gamma, \alpha + \gamma \in \Phi$. Thus $\alpha + \beta \notin \Phi \cup \{0\}$. The left hand side of the Jacobi identity becomes

$$0 + \varepsilon(\beta, \gamma)[E_{\beta + \gamma}, E_{\alpha}] - \varepsilon(\gamma, \alpha)[E_{\gamma + \alpha}, E_{\beta}] = [\varepsilon(\beta, \gamma)\varepsilon(\beta + \gamma, \alpha) - \varepsilon(\gamma, \alpha)\varepsilon(\beta, \gamma + \alpha)]E_{\beta} = 0.$$

2. $(a, b, c) = (-1, -2, 1)$. Thus $(\alpha, \beta) = 1$ so that $\alpha + \beta \notin \Phi \cup \{0\}$, $\gamma = -\alpha$ and $\beta + \gamma \in \Phi$. The left hand side of the Jacobi identity is

$$\varepsilon(\beta, \gamma)[E_{\beta + \gamma}, E_{\alpha}] + (\alpha, \beta)E_{\beta} = -\varepsilon(\beta, \gamma)\varepsilon(\alpha, \beta + \gamma)E_{\beta} - E_{\beta} = (-1)^{(\beta, \gamma)}E_{\beta} = 0.$$  

3. $(a, b, c) = (0, -2, 0)$. In this case $\alpha, \gamma \perp \beta$ and $\alpha = -\gamma$. Thus $\beta \pm \alpha \notin \Delta \cup \{0\}$. So two terms vanish and the remaining term vanishes because $[\alpha, E_{\beta}] = (\alpha, \beta)E_{\beta} = 0$.

4. $(a, b, c) = (-2, -2, 2)$. In this case $\beta = \alpha = -\gamma$. The left hand side of the Jacobi identity is trivially zero by skew symmetry of the bracket. This completes the proof.

**Proposition.** Define a complex inner product on $g$ by extending the real inner product on $V$ to a complex inner product on $h = V + iV$ and then decreeing the $E_{\alpha}$’s to be orthonormal and orthogonal to $h$. Define a conjugate-linear map $X \mapsto X^*$ on $g$ by $\alpha^* = -\alpha$ and $E^*_{\alpha} = -E_{\alpha}$. Then $[X, Y]^* = [Y^*, X^*]$ for $X, Y \in g$, $\langle X, Y \rangle = \langle X^*, Y^* \rangle$ and $\text{ad}(X^*) = \text{ad}(X)^*$ for $X \in g$.

**Proof.** This is a routine verification.

**Corollary.** Let $g_0 = \{X \in g : X^* = -X\}$. Then $g_0$ is compact Lie algebra with invariant real inner product $(X, Y)$.

**Proof.** Clearly $g_0$ is closed under bracket and real scalar multiplication. Since $(X, Y) = (X, Y)$ for $X, Y \in g_0$, it follows that the inner product is real on $g_0$. It is invariant since $\text{ad}(X)^* = -\text{ad}(X)$ for $X \in g_0$.

In particular it follows that $g$ is the direct sum of its centre and a set of simple algebras. We now determine the centre and each of the simple summands.

**Proposition.** The centre $\mathfrak{z}$ of $g$ is contained in $h$ and equals $\Phi^\perp \subset h$. Thus $g$ has no centre iff $\Phi$ spans $h$.

**Proof.** The adjoint action of $h$ on $g$ is diagonal: the eigenspace decomposition is $h \oplus \bigoplus g_\alpha$, with $h$ the 0–eigenspace. Thus if $X$ is central, it must lie in the 0–eigenspace of $h$, i.e. $h$. But then we need $0 = [X, E_\alpha] = (\alpha E)(X)E_\alpha$ for all $\alpha \in \Phi$. This happens iff $X \perp \Phi$ as required.

Define $\alpha, \beta, \in \Phi$ to be adjacent if $(\alpha, \beta) \neq 0$. Define $\alpha, \beta \in \Phi$ to be connected if there is a chain of adjacent elements of $\Phi$ linking $\alpha$ and $\beta$. This clearly determines an equivalence relation on $\Phi$. Let $V_0 = \Phi^\perp$. Let these equivalence classes be $\Phi_1, \ldots, \Phi_m$, let $V_\alpha$ be the real–linear span of $\Phi_\alpha$ and let $V_\alpha$ be the $Z$–linear span of $\Phi_\alpha$. Thus $V = V_0 \oplus V_1 \oplus \cdots \oplus V_m$ is an orthogonal direct sum.

**Proposition.** Let $g_\mathfrak{z} = V_1 \oplus iV_1 \oplus \bigoplus_{\alpha \in \Phi} CE_\alpha$. Then $g_\mathfrak{z}$ is a simple non–Abelian Lie algebra and is an ideal in $g$. Moreover $g = \mathfrak{z} \oplus g_\mathfrak{z} \oplus \cdots \oplus g_m$. These ideals are invariant under $*$ and mutually orthogonal.

**Proof.** Clearly each $g_\mathfrak{z}$ is the Lie algebra constructed from the lattice $\Lambda_\mathfrak{z}$ in $V_\mathfrak{z}$. Thus to prove the first part we must show, if $\Phi$ spans $V$ and any two elements of $\Phi$ are connected, that $g$ is simple. Let $a$ be an ideal in $g$. Note the following:

1. If $E_{\alpha} \in a$, then $\alpha \in a$ (since $[E_{\alpha}, E_{-\alpha}] = -\alpha$).
2. If $\alpha \in a$, then $E_{\alpha} \in a$ (since $[h, E_{\alpha}] = CE_{\alpha}$).

We claim that $E_{\alpha} \in a$ for some $\alpha$. Suppose not. Since $a$ is invariant under $\text{ad}(h)$ so can be decomposed into eigenspaces. If no $E_{\alpha}$ lies in $a$, $a$ is the zero eigenspace, so that $a \in h$. But then $\alpha(a) \neq 0$ for some $\alpha$, so that $[a, E_{\alpha}] = CE_{\alpha}$. Hence $E_{\alpha} \in a$ a contradiction.

Since $E_{\alpha} \in a$, so is $\alpha$. We claim that if $E_{\beta}$ is in $a$ and $\beta$ is adjacent to $\alpha$, then $E_{\gamma}$ is in $a$. In fact $E_{\beta} \in a$, so $\beta \in a$. But $[\beta, E_{\gamma}] = (\beta, \gamma)E_{\gamma}$, with $(\beta, \gamma) \neq 0$. Thus $E_{\gamma}$ lies in $a$. Since all elements of $\Phi$ are connected to $\alpha$, it follows that $E_{\beta} \in a$ for all $\beta \in \Phi$ and hence $\beta \in a$ for all $\beta \in \Phi$. Thus $a = g$ and $g$ is therefore simple.
Corollary. \( \mathfrak{g} \) is simple iff \( \Phi \) spans \( V \) and any two elements of \( \Phi \) are connected.

This construction gives all the simple algebras of type \( A, D, E \) (the so-called simply laced algebras). The remaining simple algebras arise as fixed point algebras of lattice automorphisms of these algebras: any automorphism of the lattice \( \Lambda \), preserving the inner product and the \( \varepsilon \)-factor, canonically induces an automorphism of the Lie algebra constructed above. (See the exercises.)

3. MAXIMAL TORI. Let \( G \) be a compact matrix group (not necessarily semisimple). A torus \( T \) in \( G \) is a closed connected Abelian subgroup. Thus if \( t \) is the Lie algebra of \( T \), we have \( T = \exp(t) \) and thus \( T \cong t/\Lambda \) where \( \Lambda = \ker(\exp) \) is a lattice in \( t \). By Kronecker’s theorem, \( T \) is generated topologically by a single element \( t \in T \) (called a topological generator). [In fact if \( x \in T = \mathbb{R}^n/\mathbb{Z}^n \) satisfies \( e_m(x) = e^{2\pi i m \cdot x} \neq 1 \) for all non-zero \( m \in \mathbb{Z}^n \), then \( x \) is a topological generator. Indeed let \( H \) be the closed subgroup generated by \( x \). Then \( H/H^0 \) is finitely generated by \( x \), so cyclic. Taking the appropriate power of \( x \), we do not change the hypotheses but now \( H \) is connected. If \( H \neq T \), then \( \mathfrak{h} \subset \mathfrak{t} = \mathbb{R}^n \) and the kernel of \( \exp \) is \( \Gamma \in \mathfrak{h} \cap \mathbb{Z}^n \), a lattice in \( \mathfrak{h} \). Since \( \mathbb{Z}^n/\Gamma \) is finitely generated and has a free part, it has a (non-trivial) homomorphism onto \( \mathbb{Z} \). Hence there is a homomorphism \( f \) of \( \mathbb{Z}^n \) onto \( \mathbb{Z} \) with \( \Gamma \) in its kernel. Necessarily \( f(x) = x \cdot m \) for some \( m \in \mathbb{Z}^n \). But then \( e_m = 1 \) on \( H \), a contradiction.]

We say that \( T \) is a maximal torus in \( G \) if it not properly contained in any other torus of \( G \). Note that \( T_1 \subset T_2 \) iff \( t_1 \subset t_2 \), so maximal tori always exist.

Lemma. \( T \) is a maximal torus iff \( t \) is a maximal Abelian subalgebra in \( \mathfrak{g} \).

Proof. If \( t \) is maximal Abelian, \( T \) cannot be contained in another torus. If \( t \) is not maximal Abelian, then \( t_1 \subset T_1 \) with \( T_1 \) Abelian. Then \( T' = \exp t_1 \) is connected, closed and Abelian, so a torus, with \( t' \supset t_1 \supset t \), so that \( T' \supset T \), so \( T \) is not maximal Abelian.

Theorem. If \( T \) is a maximal torus in \( G \) with Lie algebra \( \mathfrak{h} \), then \( \mathfrak{g} = \bigcup_{g \in G} ghg^{-1} \).

Proof. Take \( X \in \mathfrak{g} \). Choose \( Y \in \mathfrak{h} \) such that \( \exp Y \) is a topological generator of \( T \). Thus the centraliser of \( Y \) in \( \mathfrak{g} \) is \( \mathfrak{h} \). Next choose \( g \in G \) so that \( \|gXg^{-1} - Y\|^2 \) is minimised, since \( G \) is compact. Replacing \( X \) by \( gXg^{-1} \), we may assume this minimum occurs for \( g = 1 \). Looking at a small variation \( \exp(A)X\exp(-A) \), we must have \( ([X,A],Y) - (Y,[X,A]) = 0 \) for all \( A \). Hence \( (A,[X,Y]) = 0 \) for all \( A \), so that \( [X,Y] = 0 \). Hence \( X \in \mathfrak{h} \), as required.

Theorem. If \( T \) is a maximal torus in \( G \), then \( G = \bigcup_{g \in G} gTg^{-1} \).

Differential geometric remark. If we knew that \( G = \exp(\mathfrak{g}) \), this would follow immediately from the previous theorem. Subjectivity of the exponential map can be proved by a geometric argument (the Hopf–Rinow theorem).

Proof. We show that \( B = \bigcup_{g \in G} gTg^{-1} \) is open and closed in \( G \). Since \( G \) is connected, we must have \( G = B \). Now clearly \( B \) is closed as the continuous image in \( G \) of the compact set \( G \times T \) under the map \( (g,t) \mapsto gtg^{-1} \). So we need only show it is open and for this it is enough to show that each \( t \in T \) is an interior point. Let \( A = C_G(t)^0 \). We consider two extreme cases: \( A = G \) and \( A = T \). In the first case \( t = \exp X \) is central and so, if \( Y \in \mathfrak{g} \), \( \exp(X + Y) \) lies in \( \bigcup g\exp(\mathfrak{h})g^{-1} \), since \( X \in \mathfrak{h} \) and \( \exp(X) \) is central. Thus \( B \) contains an open neighbourhood of \( t \). In the second case, consider the map \( f; h \mapsto X \in \bigcup g\exp(\mathfrak{h})g^{-1} \). We obtain a map \( f \) with \( f(t,Y) = \exp(Y)t\exp(X)\exp(-Y) = t\exp(t^{-1}Yt)\exp(X)\exp(Y) \). The derivative of this map at \( (0,0) \) is \( f'(0,0) = X \). Since \( C_G(t) = \mathfrak{h} \), the map \( Y \mapsto Y - t^{-1}Yt \) is an automorphism of \( \mathfrak{h} \). Thus \( f \) is an isomorphism and \( f \) is locally a diffeomorphism. This provides an open neighbourhood of \( t \) in \( B \).

To handle the general case, we combine these two ideas. Let \( t \in T \) and \( A = C_G(s)^0 \). Thus \( T \subseteq A \subseteq G \). Note that \( t \) is central in the maximal torus \( T \) of \( A \). Thus \( t \) is an interior point of \( \bigcup gA gTg^{-1} \). Thus if \( X \in \mathfrak{a} \) is sufficiently small, \( t\exp(X) \) lies in \( B \). Now take \( Y \in \mathfrak{a} \) and consider the map \( f(X,Y) = \exp(Y)t\exp(X)\exp(-Y) = t\exp(t^{-1}Yt)\exp(X)\exp(-Y) \). Again \( f(0,0)(X,Y) = X \). Thus \( f \) is an isomorphism since \( \mathfrak{a} = C_G(t) \). Thus \( f \) is locally a diffeomorphism at \( (0,0) \) and therefore the image of an open ball around \( (0,0) \) provides an open neighbourhood of \( t \) in \( B \).

Corollary. Every element of \( G \) lies in a maximal torus. In particular \( G = \exp \mathfrak{g} \).
Corollary. Any two maximal tori are conjugate.

Proof. Since a torus is topologically cyclic, one must be contained in a conjugate of the other. Since a conjugate of a maximal torus is a maximal torus, the result follows.

Lemma (centralisers). (1) $x \in C_G(x)^o$ for all $x \in G$.
(2) $C_G(X)$ is connected for all $X \in g$. (More generally if $a$ is an Abelian subalgebra of $g$, then $C_G(a)$ is connected.)

Proof. (1) Let $T$ be a maximal torus containing $x$. Then $x \in T \subseteq C_G(x)^o$.
(2) Let $A = \exp(a)$, a torus. Suppose $x \in C_G(a)$. Then $A \subseteq C_G(x)^o = H$, so $A$ is contained in a maximal torus $T$ in $H$. But $x$ is central in $H$, so $x \in T$. Hence $x \in T \subseteq C_G(a)^o$.

Remark. Note that the more general statement in (2) could also be proved inductively using the single element statement by successively passing to centralisers in centralisers using a basis of $a$.

Corollary. (a) $x \in T$ is contained in exactly one maximal torus iff $C_G(x)^o$ is a maximal torus iff $C_G(x)$ is maximal Abelian.
(b) $C_G(x) = \mathfrak{h}$ iff $C_G(X) = T$.
(c) A maximal torus is maximal Abelian (but not conversely).

Proof. (a) Since $x$ is central in $C_G(x)^o$ and $C_G(x)^o$ is the union of all maximal tori containing $x$ (it is the union of its maximal tori and they all contain $x$), the result follows.
(b) This follows because the Lie algebra of $C_G(X)$ is $C_G(X)$ and $C_G(X)$ is connected.
(c) Say $x$ commutes with $T$. Then $T \subseteq C_G(x)^o$ must contain $x$, since $x$ is central in $C_G(x)^o$.

Corollary. The Weyl group $W = N(T)/T$ is finite.

Proof. By the previous corollary, the continuous map $W \subset \text{Aut}(T) = \text{PGL}_m(\mathbb{Z})$ is injective. Since $W$ is compact, its image is compact and discrete, so finite.

Corollary. $t_1, t_2 \in T$ are conjugate in $G$ iff they are conjugate under the Weyl group $N(T)/T$. (Thus the space of conjugacy classes $G/\text{Ad}G$ is homeomorphic to $T/N(T)$.)

Proof. Let $H = C_G(t_2)^o$ and suppose that $t_2 = gt_1g^{-1}$. Thus $T, gTg^{-1} \subset H$. Since $T$ and $gTg^{-1}$ are maximal tori in $H$, we can find $h \in H$ such that $T = hgTg^{-1}h^{-1}$. Let $x = hg$. Then $t_2 = gt_1 = hgtg^{-1}h^{-1} = xtx^{-1}$ and $x \in N(T)$. The last statement follows because the map $T/N(T) \to G/\text{Ad}G$ is a continuous bijection between compact spaces.

4. REPRESENTATIONS OF SU(2) AND sl(2).

Let $G = SU(2)$, the group of all complex matrices
\[
\begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}
\]
with $|\alpha|^2 + |\beta|^2 = 1$. Thus $G$ is a compact connected group, homeomorphic to $S^3$ [so simply connected]. Since $g \in G$ iff $\det(g) = 1$ and $gg^* = I$, the Lie algebra $\mathfrak{su}(2)$ of $G$ is given by matrices $X$ such that $\text{tr}(X) = 0$ and $X + X^* = 0$, i.e. skew-adjoint matrices with trace zero. We take as a real basis of $\mathfrak{su}(2)$, $X, Y, T$ with $T = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}$. These basis elements are orthogonal with respect to the real inner product $(X, Y) = \text{tr}(XY^*)$ and satisfy the following relations $[X, Y] = T, [T, X] = Y, [Y, T] = X$. Define the complexification of $\mathfrak{su}(2)$ in $M_2(\mathbb{C})$ as $\mathfrak{sl}(2) = \mathfrak{su}(2) + i\mathfrak{su}(2)$. Clearly $\mathfrak{sl}(2) = \{X \in M_2(\mathbb{C}) : \text{tr}(X) = 0\}$, a 3-dimensional complex Lie algebra. It is the Lie algebra of the closed matrix group $SL(2, \mathbb{C})$. The natural complex basis (over $\mathbb{C}$) of $\mathfrak{sl}(2)$ is $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These are related to the real basis by $E = X - iY$, $F = -X - iY$ and $H = -2iT$ (so that $2T = iH$). Thus $H^* = H$, $E^* = F$ and
\[
[H,F] = -2F. \quad \text{(α)}
\]
The important point about the complex basis is that $E$ and $F$ become raising and lowering operators in any finite-dimensional representation. (We will encounter this phenomenon again when we consider
representations of the Heisenberg algebra and pass from the real basis \( P = x, \ Q = id/dx \) to complex basis \( P = x, \ Q = \pm iQ \). The complex operators are again raising and lowering operators, usually called creation and annihilation operators.)

If \( V = \mathbb{C}^2 \), the groups \( SU(2) \) and \( SL(2, \mathbb{C}) \) act on the tensor power \( V^\otimes n \) by \( T(g) = g^\otimes n \). The corresponding action of the Lie algebras is \( X \mapsto T(X) = X \otimes I \otimes \cdots \otimes I \otimes I \otimes \cdots \otimes X \) to be compatible with the exponential map, since \( \exp T(X) = \exp X \otimes \exp X \otimes \cdots \otimes \exp X \). The action of \( SU(2) \) on \( V^\otimes n \) is unitary, so completely reducible. The actions of \( SL(2, \mathbb{C}) \), \( su_2 \) and \( sl_2 \) are also by operators invariant under taking adjoints, so are completely reducible. One of our aims is to classify the irreducible representations of \( SU(2) \) that arise in \( V^\otimes n \). Weyl’s unitarian trick shows that we only need look at representations of \( sl(2) \).

Lemma. \( SU(2), \ SL(2, \mathbb{C}), \ su_2 \) and \( sl_2 \) have the same centraliser algebra on \( V^\otimes n \) and hence the same invariant subspaces.

Proof. Any \( g \in SL(2, \mathbb{C}) \) has a polar decomposition \( g = up \) where \( p = (g^*g)^{1/2} \) and \( u = gp^{-1} \in SU(2) \). The unitary \( u \) can be written as \( \exp(x) \) with \( x \in su_2 \) and \( p \) can be written as \( \exp(iy) \) with \( y \in su_2 \). Note that \( T \) commutes with \( \exp(tA) \) for all \( t \in \mathbb{R} \) iff \( T \) commutes with \( A \) (take the coefficient of \( t \) in \( T \exp(tA) = \exp(tA)T \)). Thus \( T \in End V^\otimes n \) commutes with \( SL(2, \mathbb{C}) \) iff \( T \) commutes with \( T(A) \) for every \( A \in sl_2 \) iff \( T \) commutes with \( SU(2) \).

We now tackle the problem of classifying finite–dimensional irreducible representations of \( sl(2) \). Thus we have operators \( E, \ F \) and \( H \) on \( V \) satisfying (\( * \)). We shall temporarily abandon the adjoint conditions, retaining only the property that \( H \) is diagonalisable. It is easy to check the following commutation relations.

Lemma. (a) \([E^n, F] = nE^{-1}(H + n - 1) = n(H - n + 1)E^{-1} \).
(b) \([F^n, E] = -nF^{-1}(H - n + 1) = -n(H + n - 1)F^{-1} \).
(c) \([H, E^n] = 2nE^n \) and \([H, F^n] = -2nF^n \).

An eigenvector of \( H \) is called a weight vector and the eigenspaces weight spaces. Thus if \( Hv = \lambda v \), \( v \) is a weight vector with weight \( \lambda \). Note that \( HEv = (\lambda + 2)Ev \) and \( HFv = (\lambda - 2)Fv \). Thus \( E \) increases the weight by 2 and \( F \) decreases the weight by 2. For this reason \( E \) and \( F \) are called raising and lowering operators.

Lemma. Let \( V \) be an \( sl(2) \)-module and let \( v \in V \) satisfy \( Hv = \lambda v \) and \( Ev = 0 \). Let \( v_j = (j!)^{-1}F_j v \). Then \( Hv_j = (\lambda - 2j)v \) and \( Ev_j = (\lambda - j + 1)v \).

Proof. Immediate from previous lemma.

Theorem. The irreducible finite–dimensional representations of \( sl(2) \) are classified by their highest weight, a non–negative integer \( d \). The representation \( V_d \) has dimension \( d+1 \) and has a unique highest weight vector \( v \) (up to a scalar multiple). If \( v_0 = v \) and \( v_j = (j!)^{-1}F_j v \) for \( j = 0, \ldots, d \), then the \( v_j \)'s form a basis of \( V_d \) and
\[
H \cdot v_j = (d - 2j)v, \quad F \cdot v_j = (j + 1)v_{j+1}, \quad E \cdot v_j = (d + 1 - j)v_j. \tag{**}
\]

Proof. Let \( v = v_0 \) be a vector of highest weight. Thus \( Hv = \lambda v \) and \( Ev = 0 \). The vector \( v_k = (k!)^{-1}F^k v \) has weight \( \lambda - 2k \), all distinct, so by finite–dimensionality \( F^{d+1}v = 0 \) for some smallest \( d \geq 0 \). Since \( F^{d+1}v_0 = 0 \), we must have \( v_{d+1} = 0 \). But by the lemma, \( Ev_{d+1} = (\lambda - d)v_d \). Since \( v_d \neq 0 \), we get \( \lambda = d \). On the other hand it is easy to verify directly that (\( ** \)) defines a representation of \( sl(2) \) on \( \mathbb{C}^{d+1} = \mathbb{T} \mathbb{C}v_i \). It is irreducible, because if \( U \) is an invariant subspace, it must be a sum of eigenspaces of \( H \) and hence contain some eigenvector. Applying raising and lowering operators we see that all basis vectors lie in \( U \).

Adjoint conditions. If one puts in the self–adjointness conditions \( E^* = F \) and \( H = H^* \), one can give a “no–ghost” argument for \( \lambda = d \):

Lemma. Let \( E, F, H \) be operators on an inner product space \( V \) with \( E^* = F \), \( H^* = H \) satisfying (\( * \)). If \( v \in V \) satisfies \( Ev = 0 \) and \( Hv = \lambda v \), then \( \lambda \) must be a non–negative integer.

Proof. By induction on \( k \), we have \([E, F^k] = (k + 1)F^k(H - kI) \) for \( k \geq 0 \). Hence
\[
(F^{k+1}v, F^k + 1v) = (F^*F^{k+1}v, F^k + 1v) = (EF^{k+1}v, F^{k+1}v) = (k + 1)(\lambda - k)(F^k v, F^k v).
\]
For these norms to be non-negative for all \( k \geq 0 \), \( \lambda \) has to be a non-negative integer.

**Character of a representation.** The representation \( V_d \) coincides with \( S^d V \) since they have the same highest weight and dimension. (It can also be seen directly that \( S^d V \) is irreducible, because this is the linear action on two variable polynomials of degree \( d \).) In particular every irreducible finite-dimensional representation of \( \mathfrak{sl}(2) \) comes from a representation of \( SU(2) \) (and even \( SL(2, \mathbb{C}) \)). The character of a representation \( \pi \) of \( G \) is given by \( \chi(g) = \text{tr}(\pi(g)) \). It is invariant under conjugation, since the trace is.

On the other hand every element of \( SU(2) \) is conjugate to a diagonal matrix \( \left( \begin{array}{cc} \zeta & 0 \\ 0 & \bar{\zeta} \end{array} \right) \), so it is enough to know the character on the diagonal element. From the theorem the character of \( V_d \) on \( \left( \begin{array}{cc} \zeta & 0 \\ 0 & \bar{\zeta} \end{array} \right) \) is \( \chi_d(\zeta) = (\zeta^d - \zeta^{-d})/\zeta - \zeta^{-1} \). It follows that every completely reducible representation is completely specified by its character. In particular this applies to all representations arising as subrepresentations of \( V^{\otimes n} \) and hence tensor products of \( V_d \)'s. By multiplying and expanding the characters we get the celebrated Clebsch–Gordan rules:

\[
V_r \otimes V_s \cong V_{|r-s|} \bigoplus V_{|r-s|+2} \bigoplus \cdots \bigoplus V_{r+s}.
\]

**The Casimir Operator.** The Casimir element is defined as \( C = H^2 + 2(\text{ad} F + \text{ad} E) = H^2 + 2H + 4FE \) on any representation. (Note that \( C = -2(X^2 + Y^2 + T^2) \).) The commutation relations imply that \( C \) commutes with \( H, E, F \) and hence with \( \mathfrak{sl}(2) \). By Schur’s lemma, the Casimir is therefore a constant on \( V_d \) and the constant can be computed by applying \( C \) to \( v_0 \). We get \( CV = (d^2 + 2d)v_0 \). Thus the Casimir distinguishes irreducible representations.

**Complete Reducibility Theorem.** Every finite-dimensional representation \( W \) of \( \mathfrak{sl}(2) \) is completely reducible.

**Remark.** If we considered representations on inner product spaces satisfying \( E^* = F, H^* = H \), this would be immediate; this applies to all the examples above as well as the famous Hodge theory action on hermitian exterior algebras (see Wells, for example). The theorem implies that every finite-dimensional representation extends to \( SU(2) \) and \( SL(2, \mathbb{C}) \) and particular has an invariant inner product.

**Proof (van der Waerden–Casimir).** We may assume \( C \) has only one eigenvalue on \( W \). Let \( W_1 \) be a subspace of \( W \) that is a direct sum of irreducibles of maximal possible dimension. If \( W_1 \neq W \), find an irreducible subspace \( \overline{W} \) of \( W/W_1 \). Because of the eigenvalue assumption on \( C \), \( \overline{W} \) and the irreducible summands of \( W_1 \) are all isomorphic to \( V_d \) for some \( d \). Suppose \( \overline{W} = V/W_1 \). Then \( E^{d+1} \overline{W} = 0 \) so that \( E^{d+1}V \subset W_1 \). On the other hand \( E^{d+1}W_1 = 0 \). Hence \( E \) is nilpotent on \( V \), say \( E^{k+1} = 0, E^k \neq 0 \) on \( V \). Thus \( k \geq d \). Since \( [E^{k+1}, F] = (k+1)(H-k)E^k \), \( k \) must be an eigenvalue of \( H \) so that \( k \leq d \). Hence \( k = d \). Now choose \( v \in V \) such that \( \overline{v} \) is a highest weight vector in \( \overline{W} \). Let \( u = F^d v \) and \( v' = E^d u \). Thus \( \overline{v'} \) is a non-zero multiple of \( \overline{v} \). On the other hand \( Ev' = 0 \) and we have just seen that \( v' = E^d u \) is an eigenvector of \( H \). Thus \( v' \) generates a copy of \( V_d \) not contained in \( W_1 \), a contradiction. Hence \( W \) is completely reducible.

5. **THE ROOT SYSTEM.** Let \( G \) be a compact simple (or semisimple) matrix group with maximal torus \( T \). Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be the corresponding Lie algebras. We may write \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) where \( \mathfrak{m} = \mathfrak{h}^\perp \). Since the inner product is \( \text{Ad} \)-invariant, \( \mathfrak{m} \) is invariant under \( \text{Ad}(T) \). It is a real inner product space on which \( T \) acts without fixed vectors (because \( \mathfrak{h} \) is maximal Abelian). Hence \( \mathfrak{m} \) is even-dimensional the orthogonal matrices \( \text{Ad}(t) \) can simultaneously be put in canonical form. Thus there is an orthonormal basis of \( \mathfrak{m} \) in which \( \text{Ad}(e^t) \) is block diagonal with blocks \( \begin{pmatrix} \cos \alpha_i(T) & \sin \alpha_i(T) \\ -\sin \alpha_i(T) & \cos \alpha_i(T) \end{pmatrix} \) down the diagonal. [Alternatively \( \text{Ad}(T) \) is a torus in \( SO(\mathfrak{m}) \) so contained in a maximal torus.] The eigenvalues of this matrix are \( e^{\pm \alpha_i(T)} \) and the linear function \( \alpha(T) = \pm \alpha_i(T) \) are called roots. Thus each root \( \alpha \) lies in \( \mathfrak{h}^\ast \).

For each root \( \alpha \), we can find orthogonal unit vectors \( x, y \) such that \( [t, x] = \alpha(t)y, [t, y] = -\alpha(t)x \). Consider \( [x, y] \). Then \( \{t, [x, y]\} = \{t, [x, y]\} + \{x, [t, y]\} = 0 \) for \( t \in \mathfrak{h} \). Hence \( [x, y] \) lies in \( \mathfrak{h} \). To calculate which element of \( \mathfrak{h} \) it is, consider \( ([x, y], t) = -g(y, [x, t]) = (y, y)\alpha(t) \). Thus \( [x, y] = T_{\alpha} \), where \( T_{\alpha} \) is the element corresponding to \( \alpha \in \mathfrak{h}^\ast \), i.e. \( (t, T_{\alpha}) = \alpha(t) \). Clearly \( \alpha(T_{\alpha}) = \|\alpha\|^2 \). Define \( X = x/\|\alpha\|, Y = y/\|\alpha\| \) and \( T = T_{\alpha}/\|\alpha\|^2 \). Thus \( [X, Y] = T, [T, X] = Y \) and \( [T, Y] = -X \). We can then form the elements \( E, F \) and
$H$ in the complexification $\mathfrak{g}_C$ as above. Clearly $[t, E] = i\alpha(t)E$ and $[t, F] = -i\alpha(t)F$ for all $t \in \mathfrak{h}$ and the elements $E$ and $F$ are orthogonal. These lie in the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. We call $E, F, H$ the copy of $\mathfrak{sl}(2)$ in $\mathfrak{g}_C$. Thus $H_\alpha = -2iT_\alpha/\|\alpha\|^2$ and $E_\alpha^* = F_\alpha$. Since $\mathfrak{su}(2)$ has a unique invariant norm with $\|H\|^2 = 2$ and $\|H_\alpha\|^2 = 4/\|\alpha\|^2$, we get $\|E^*_\alpha\|^2 = \|F_\alpha\|^2 = \|H_\alpha\|^2/2 = 2/\|\alpha\|^2$.

**Remark.** This construction also makes sense at the level of groups. If $G \subset U(V)$ and $X, Y, T \in \mathfrak{g}$ satisfy $[X, Y] = T, [T, X] = Y, [Y, T] = X$ and span a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$. Since these operators are skew–adjoint, $V$ breaks up as a direct sum of irreducible representations of $\mathfrak{s}$. By the $SU(2)$ theory there is a representation $\pi$ of $SU(2)$ on $V$ such that its generators $X_1, Y_1, T_1$ are sent to $X, Y, T$ under $\pi$. Since $\pi(\exp X) = \exp \pi(X)$, it follows that $\pi(SU(2)) \subseteq G$. The image is a closed connected subgroup of $G$, so a matrix group in its own right. The kernel $Z$ of $\pi$ is a closed subgroup of $SU(2)$ so a matrix group. Since $\pi$ is injective on the Lie algebra of $SU(2)$, $Z$ is discrete so finite. It is normal in $SU(2)$ so $SU(2)$ acts by conjugation on $Z$. But $Z$ is discrete and $SU(2)$ connected. Hence $Z$ is central, so that $Z = \{1\}$ or $\{\pm 1\}$. We call this the copy of $SU(2)$ or $SO(3)$ in $\mathfrak{g}$ corresponding to $\mathfrak{s}$. For this reason we can loosely talk about the “copy of $SU(2)$” in $G$ corresponding to a given root $\alpha$.

The above arguments could also have been carried out directly in the complexification $\mathfrak{g}_C$; in fact we get a useful generalisation which cannot be seen so clearly just working in $\mathfrak{g}$.

**Lemma.** Suppose that $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_{-\alpha}$. Then $[E, F] = -i(F, E^*)T_\alpha = -i(F, E^*)T_\alpha$.

**Proof.** Clearly $[E, F]$ commutes with $\mathfrak{h}$, so lies in $\mathfrak{h}_C$. We have $([E, F], t) = (F, [E^*, t]) = (F, i\alpha(t)E^*) = -i\alpha(t)(F, E^*) = -i(F, E^*)T_\alpha, t$.

**Corollary.** The root $\alpha$ occurs with multiplicity one in $\mathfrak{g}$, so that $\dim \mathfrak{g}_\alpha = 1$.

**Proof.** If not, we can find further $E', F'^\prime$ orthogonal to $E, F$ but with the same relations with $H$. But then $[E, F'] = 0$, since $(F, E^*) = (F', E^*)$ and $[H, F'] = -2F'$. This contradicts the $\mathfrak{sl}(2)$ lemma applied to $V = \mathfrak{g}_C$, with the adjoint representation of $E, F, H$, and $v = F'$.

**Lemma.** If $\alpha \neq \pm \beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta$ is a root and (0) otherwise.

**Proof.** Take the copy of $\mathfrak{sl}(2)$, $E, F, H$, corresponding to the root $\alpha$ and let $V = \oplus_{m \in \mathbb{Z}} \mathfrak{g}_{\beta + \alpha m}$. Then $V$ is invariant under $E, F, H$, so can be written as a direct of irreducible submodules. On the other hand each weight space of $V$ is at most one dimensional, so that $V$ must actually be irreducible. The result follows immediately, because $\text{ad}(E)$ is a raising operator so is an isomorphism between weight spaces of $V$.

**Corollary of proof.** $n(\alpha, \beta) = 2(\alpha, \beta)/(\alpha, \alpha)$ is an integer. The roots of the form $\beta + m\alpha$ are exactly those with $m \in [-p, q]$ where $-p \leq 0 \leq q$ and $p - q = n(\alpha, \beta)$. In particular $\beta - n(\alpha, \beta)\alpha$ is always a root.

**Proof.** The first assertion follows because $n(\alpha, \beta) = \beta(H)$ and $\text{ad} H$ has only integer eigenvalues. The corresponding irreducible representation of $E, F, H$ has lowest $H$–eigenvalue $n(\alpha, \beta) - 2p$ and highest $H$–eigenvalue $n(\alpha, \beta) + 2q$. These must be negatives of each other, so that $n(\alpha, \beta) = p - q$. The last assertion follows because $q - p \in [q, -p]$.

**Proposition.** The root system $\Phi \in \mathfrak{h}^*$, $V$ has the following properties.

R1 $\Phi$ spans $V$.

R2 If $\alpha \in \Phi$, then $\sigma_\alpha \Phi = \Phi$ where $\sigma_\alpha$ is the reflection $\sigma_\alpha (v) = v - (v, \alpha^\vee)/\alpha$ with $\alpha^\vee = 2\alpha/(\alpha, \alpha)$.

R3 $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

R4 If $\alpha \in \Phi$, then the only roots proportional to $\alpha$ are $\pm \alpha$.

**Proof.** (R1) If not, we could find a non–zero $t \in \mathfrak{h}$ such that $\alpha(t) = 0$ for all $\alpha$. But then $t$ would be central, a contradiction. (R2) and (R3) were proved in the preceding lemmas. (R4) If $\beta = \pm \alpha$ is a root, then $2s = n(\alpha, \beta) \in \mathbb{Z}$. So that $2s \in \mathbb{Z}$. Since $\alpha = s^{-1}\beta$, we similarly have $2s^{-1} \in \mathbb{Z}$. Thus we may assume without loss of generality that $s = 2$ and that $3\alpha$ is not a root. Then $V = \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{CH} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-\alpha}$ would give a 5–dimensional representation of $E, F, H$, necessarily irreducible. However $\mathfrak{g}_\alpha \oplus \mathfrak{CH} \oplus \mathfrak{g}_{-\alpha}$ is a subrepresentation, a contradiction.
Lemma. If $\alpha \in \mathfrak{h}^\ast$ is a root and $T_\alpha \in \mathfrak{h}$ the corresponding element of $\mathfrak{h}$, so that $(t, T_\alpha) = \alpha(t)$, then reflection in the ray $\mathbb{R}T_\alpha$ lies in $W = N(T)/T$.

Proof. Take $X, Y, T$ such that $[X, Y] = T$, $[T, X] = Y$ and $[Y, T] = X$. Thus $[t, X] = \alpha(t)Y$ and $[t, Y] = -\alpha(t)X$ for $t \in \mathfrak{h}$. Let $g_s = \exp(sX)$. Then

$$g_s t g_s^{-1} = \text{Ad}(e^{sX}) \cdot t = e^{s\alpha(X)t}.$$ 

If $\alpha(t) = 0$, we therefore have $g_s t g_s^{-1} = t$. On the other hand $g_s T g_s^{-1} = \cos(s)T + \sin(s)X$. Taking $g = g_\pi$, we get $g \in N(T)$, with $gtg^{-1} = t$ if $t \perp T$ and $gtg^{-1} = -T$. Hence the image of $g$ is the reflection in the ray $\mathbb{R}T$.

Example. For $G = SU(N)$, we have $\mathfrak{g}_C = \mathfrak{sl}(N) = \{X \in M_N(\mathbb{C})\}$. We can identify $\mathfrak{h}$ with diagonal matrices \{\$ix : x \in \mathbb{R}^N, \sum x_i = 0\}. The roots vectors are the matrix units $e_{pq}$ with $p \neq q$. Since $e^{ix} e_{pq} e^{-ix} = e^{i(x_p - x_q)} e_{pq}$, the corresponding root is $\alpha_{pq}(x) = x_i - x_q$.

6. THE WEYL GROUP AS A REFLECTION GROUP. Our aim is to obtain a description in terms of roots for a fundamental domain of the Weyl group. For each root $\alpha$, let $\mathcal{H}_\alpha$ be the hyperplane \{\$x : \alpha(X) = 0\} and let $\mathcal{H}_\alpha^+$ be the closed half-space \{\$x : \alpha(X) \geq 0\}. In $\mathfrak{h}$ we define the Weyl chambers to be the connected components of

$$\mathfrak{h}' = \mathfrak{h} \setminus \bigcup_{\alpha} \mathcal{H}_\alpha = \{X : \alpha(X) \neq 0 (\alpha \in \Phi)\}.$$ 

Clearly these are open convex cones. The boundary of each chamber $C$ is contained in $\bigcup \mathcal{H}_\alpha$. Let $W_0$ be the normal subgroup of $W$ generated by reflections in the $\mathcal{H}_\alpha$'s.

Theorem. $W = W_0$ and $W$ permutes the Weyl chambers simply transitively.

Proof. (1) $W$ permutes the Weyl chambers. This is clear because $W$ permutes $\Phi$ and therefore leaves $\mathfrak{h}'$ invariant.

(2) $W_0$ permutes the Weyl chambers transitively. Let $C_1$ and $C_2$ be two Weyl chambers. Fix $x \in C_1$ and consider the boundary sphere $S$ of a small ball in $C$ around $x$. The chamber $C_2$ projects onto an open subset of $S$. Each intersection of distinct hyperplanes $\mathcal{H}_\alpha \cap \mathcal{H}_\beta$ $(\alpha \neq \pm \beta)$ is a subspace of codimension 2 so projects onto a sphere of codimension 1 in $S$. There are only finitely many such spheres so there is a point $y$ in $C_2$ such that the line segment joining $x$ and $y$ misses each double intersection and therefore has only simple (or empty) intersections with each hyperplane $\mathcal{H}_\alpha$. Clearly the composition of the reflection in each of the successive hyperplanes encountered will carry $C_1$ onto $C_2$.

(3) $W = W_0$ and $W$ is simply transitive. To see this, let $W_C = \{\alpha \in W : \sigma C = C\}$ be the stabiliser of $C$. By (2) $W = W_0 \cdot W_C$. Now take $x \in C$ and set $X = |W_C|^{-1} \sum_\sigma \sigma C$. Thus $X \in C$ is fixed by $W_C$. Since $\alpha(X) \neq 0$ for all $\alpha \in \Phi$, $T = C_0(X) = C_G(X)$. Hence $W_C \subset C_G(X)/T = \{1\}$, so that $W_C = \{1\}$ and $W = W_0$.

Corollary. $\overline{C}$ is a fundamental domain for the Weyl group $W$.

Proof. Let $C_1$ and $C_2$ be Weyl chambers with $X \in C_1, Y \in C_2$. If the line segment $[X, Y]$ crosses a hyperplane $\mathfrak{h}_\alpha$, then $\|X - Y\| > \|X - \sigma_\alpha(Y)\|$. Now minimise the distance $X - \sigma(Y)$ over $\sigma \in W$. Any minimum $\sigma Y$ cannot be separated from $X$ by any walls, so that $\sigma Y \in C_1$. Since there is a unique $\sigma$ such that $\sigma Y \in C_1$, it follows that if $X, Y \in C$, then $\|X - \sigma Y\| \geq \|X - Y\|$ for all $\sigma \in W$. This result also holds by continuity for $X, Y \in \overline{C}$; a similar continuity argument shows that $\mathfrak{h} = W \cdot \overline{C}$. Now suppose that $X, \sigma X \in \overline{C}$. Let $Y = \sigma X$ and $\tau = \sigma^{-1}$. Then $0 \geq \|X - \tau Y\| \geq \|X - Y\|$. Hence $X = Y$. Thus the $W$-orbit of any point intersects $\overline{C}$ in just one point, so that $\overline{C}$ is a fundamental domain.

Note that if $C_1, C_2$ are two Weyl chambers, then the number of hyperplanes intersecting the line segment joining $x_1 \in C_1$ and $x_2 \in C_2$ is independent of the choice of $x_1$. If $\Phi_+^+ = \{\alpha : \alpha(x) > 0\}$, it is the number of roots in $\Phi_+^+$ lying in $-\Phi_+^+$; this is because a sign change occurs whenever $x_1$ and $x_2$ lie on opposite sides of $\mathcal{H}_\alpha$. Denote this number by $n(C_1, C_2)$. If we fix a Weyl chamber $C$, we define $n(\sigma) = n(C, C\sigma)$ for $\sigma \in W$. If $x \in C$, we have $\Phi_+^+ = \{\alpha : \alpha(x) > 0\}$, so that $n(\sigma) = |\{\alpha : \alpha(x) > 0, \alpha(\sigma x) > 0\}| = |\{\alpha > 0 : \sigma^{-1} \alpha < 0\}|$. 

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9. GEOMETRIC APPROACH TO WEYL CHAMBERS AND SIMPLE ROOTS. Let $C$ be a fixed Weyl chamber. We call $H_{c}$ a wall of the Weyl chamber $C$ if $H_{c} \cap C$ has non-empty interior in $H_{c}$. We define the simple roots corresponding to $C$ to be those such that $\alpha(x) > 0$ on $C$ and $H_{c}$ is a wall of $C$. The corresponding reflections in the walls are called simple roots.

Lemma. If $\sigma \in W$, then $\sigma$ is a product of $n(\sigma)$ simple reflections. In fact if $\ell(\sigma)$ is the minimum number of simple reflections required for such a product, $\ell(\sigma) = n(\sigma)$. In particular $W$ is generated by simple reflections.

Proof. We prove the result by induction on $n(\sigma)$, the result being trivial for $n = 0$. Take a generic line segment joining $x \in C$ to $y \in \sigma^{-1}C$, crossing the hyperplanes $H_{\beta_{1}}, \ldots H_{\beta_{i}}$ transversely. Then $\sigma^{-1}C = \sigma_{\beta_{1}} \cdots \sigma_{\beta_{i}}C$. By simple transitivity, $\sigma^{-1} = \sigma_{\beta_{i}} \cdots \sigma_{\beta_{1}}$. Thus $\sigma = \sigma_{\beta_{i}} \cdots \sigma_{\beta_{1}}$. Thus $n(C,\sigma C) = \ell$. Let $\tau = \sigma_{\beta_{i}} \cdots \sigma_{\beta_{1}}$. Then $n(C,\tau C) = \ell - 1$, because of the properties of the line segment $[x,y]$. So by induction $\tau$ is the product of $\ell - 1$ simple reflections. Since $\sigma = \sigma_{\beta_{i}} \tau$ and $\beta$ is simple, we see that $\sigma$ is the product of $\ell$ simple reflections as required. Thus $\ell(\sigma) \leq n(\sigma)$.

We now prove that $n(\sigma) = \ell(\sigma)$. Note that if $x \in C$ and $h_{\beta}$ is a wall of $C$, then $x$ and $\sigma_{\beta}x$ are only separated by the hyperplane $h_{\beta}$. Transporting structure by $\tau \in W$, we see that if $\tau x \in \tau C$, $\tau x$ and $\sigma_{\tau}x$ are only separated by the hyperplane $\tau(h_{\beta})$. But $\sigma_{\beta} = \tau_{\beta} \tau^{-1}$, so that $\tau x$ and $\sigma_{\tau}x$ are only separated by $\tau_{\beta}h_{\beta}$. If we write $\sigma = \sigma_{1} \cdots \sigma_{r}$, a product of simple reflections, it follows that there is only one hyperplane separating $x$ and $\sigma_{1}x$, one separating $\sigma_{1}x$ and $\sigma_{1}\sigma_{2}x$, and so on. Thus there is a piecewise linear path from $x$ to $\sigma x$ crossing only $\ell$ hyperplanes. This can only cross more hyperplanes than the straightline joining $x$ and $\sigma x$, so that $\ell(\sigma) \geq n(\sigma)$. Hence $\ell(\sigma) = n(\sigma)$.

Proposition. Each Weyl chamber $C$ is the intersection of the open half spaces corresponding to its walls, i.e. $C = \bigcap_{\alpha \in \Delta} H_{\alpha}$. Moreover $C = \bigcap_{\alpha \in \Delta} H_{\alpha}$. Indeed the inclusion $C \subseteq \bigcap_{\alpha \in \Delta} H_{\alpha}$ is clear. If on the other hand $\alpha(x) \geq 0$ for all $x \in \Delta$ and $z \in C$, then $x_{n} = x + z/n \in C$ and $x_{n} \rightarrow x$. So equality holds.

Now take $\Delta' \supseteq \Delta$ minimal with $C = \bigcap_{\alpha \in \Delta'} H_{\alpha}$. If $\Delta' \neq \Delta$, take $\beta \in \Delta' \setminus \Delta$. Let $\Delta'' = \Delta' \setminus \{\beta\}$. We claim that $H_{\beta}$ does not intersect $\bigcap_{\alpha \in \Delta''} H_{\alpha}$. If not, suppose they meet in $x$. Then $\alpha(x) > 0$ for $\alpha \in \Delta''$ while $\beta(x) = 0$. Thus $x \in \bigcap_{\alpha \in \Delta''} H_{\alpha}$ by our first observations. But then $x \in C \cap H_{\beta}$. Since $\alpha(x) > 0$ for all $\alpha \neq \beta$, this will also be true in a neighbourhood of $x$ in $H_{\beta}$. So $H_{\beta}$ would have to be a wall, a contradiction since by assumption $\beta \notin \Delta$.

Since $C \cap \bigcap_{\alpha \in \Delta''} H_{\alpha}$ and the latter does not meet $H_{\beta}$, they both must lie in $H_{\beta}$. But then $\bigcap_{\alpha \in \Delta''} H_{\alpha} \supset H_{\beta}$, so that $C = \bigcap_{\alpha \in \Delta'} H_{\alpha}$. This contradicts the minimality of $\Delta'$. Hence $\Delta' = \Delta$, so that $C = \bigcap_{\alpha \in \Delta} H_{\alpha}$.

Theorem. The simple roots form a basis of $h^{*}$. Every positive root is a non-negative integral combination of simple roots. The Weyl group orbit of any root contains a simple root; equivalently every hyperplane $H_{\alpha}$ is the wall of some Weyl chamber.

Proof. (1) $\Delta$ spans $h^{*}$. If $x \in h$, then $\alpha(x) > 0$ iff $\alpha_{i}(x) > 0$. Hence $\alpha(x) \leq 0$ for all $\alpha > 0$ iff $\alpha_{i}(x) > 0$ for all $\alpha_{i}$. Similarly $\alpha(x) \leq 0$ for all $\alpha \leq 0$ iff $\alpha_{i}(x) \leq 0$ for all $\alpha_{i}$. Hence $\alpha(x) = 0$ for all $\alpha$ iff $\alpha_{i}(x) = 0$ for all $\alpha_{i}$. Thus the $\alpha_{i}$’s span $h^{*}$.

(2) Each simple root $\alpha$ is non-redundant, i.e. cannot be written as $\alpha = \mu \beta + \nu \gamma$ with $\beta, \gamma \in \Phi^{+}$ non-proportional and $\mu, \nu \geq 0$. If $\alpha = \mu \beta + \nu \gamma$, then $y \in C \cap H_{\alpha}$ implies $\alpha(x) = 0$ and $\beta(x), \gamma(x) \geq 0$. Hence $\beta(x) = \gamma(x) = 0$, so that $C \cap H_{\alpha}$ has at least codimension 1 in $H_{\alpha}$ so cannot have non-empty interior. Thus $H_{\alpha}$ cannot be a wall.

(3) If $\alpha, \beta$ are simple, $\langle \alpha, \beta \rangle \leq 0$. We show that $\langle \alpha, \beta \rangle \leq 0$ for non-redundant roots. In fact we know $\sigma_{\alpha} - \beta = \gamma$ is a root where $\gamma = \sigma_{\alpha} \beta = \beta - 2\alpha, \beta \|^{-1}(\alpha, \beta) \beta$. If $\langle \alpha, \beta \rangle > 0$, then since $\gamma$ or $-\gamma$ is positive, either $\beta$ or $\gamma$ would be non-redundant. Hence $\langle \alpha, \beta \rangle \leq 0$.

(4) The non-redundant roots form a basis of $h^{*}$. If $\Delta$ were not linearly independent, the existence of a linear relation would yield a subset $\Delta_{0} \subseteq \Delta$ and non-negative reals $c_{\alpha}$ such that $\sum_{\alpha \in \Delta_{0}} c_{\alpha} \alpha = \sum_{\beta \notin \Delta_{0}} c_{\beta} \beta$. Then $\gamma(x) > 0$ for $x \in C$ since not all $c$’s are zero. But $\|\gamma\|^{2} = \sum c_{\alpha} c_{\beta}(\alpha, \beta) \leq 0$, so that $\gamma \equiv 0$, a
contradiction.

(5) A positive root is simple iff non–redundant. The simple roots are spanning and contained in the linearly independent set of non–redundant roots, the set of simple roots must coincide with the set of non–redundant roots.

(6) The simple roots form a basis and every positive root is a non–negative combination of simple roots. The non–redundant roots are linearly independent. Since the simple roots are non–redundant by (2) and span \( \mathfrak{h}^* \) by (1), \( \Delta \) forms a basis. Let \( X_i \) be the dual basis in \( \mathfrak{h} \), so that \( \alpha_i(X_j) = \delta_{ij} \). If \( \alpha \geq 0 \), we know that \( \alpha(X_i) \geq 0 \) for all \( i \) implies \( \alpha(X) \geq 0 \). Hence \( \alpha(X_i) \geq 0 \). But \( \alpha = \sum \alpha(X_i) \alpha_i \).

(7) Any \( \mathcal{H}_\alpha \) is the wall of some Weyl chamber. Take \( x \in \mathcal{H}_\alpha \) with \( x \notin \mathcal{H}_\beta \) for \( \beta \neq \pm \alpha \). Take \( y \) in a small ball around \( x \) with \( \alpha(y) > 0 \). Let \( C \) be the Weyl chamber containing \( y \). Then \( \mathcal{H}_\alpha \) is a wall of \( C \) because \( \partial C \) intersects \( \mathcal{H}_\alpha \) in a neighbourhood of \( x \).

(8) \( \Phi = W \cdot \Delta \). By (7) every \( \alpha \in \Phi \) is a simple root for some Weyl chamber \( C' \). But \( C' = \sigma C \) for some \( \sigma \in W \), so that \( \sigma^{-1} \alpha \in \Delta \). Hence \( \Phi = W \cdot \Delta \).

(9) Every positive root is a non–negative integer combination of simple roots. If \( \alpha \in \Phi \), we may have \( \alpha = \alpha \sigma_i \) for \( \sigma \in W \) and \( \sigma_i \in \Delta \). Since \( \sigma \) is a product of simple reflections, it follows that \( \alpha \) is an integer combination of simple roots. By (6) the coefficients must be either all non–negative or non–positive.

Corollary. For each \( x \in \mathfrak{h} \), \( W \cdot x \cap \overline{C} \) is a single point. Thus \( \overline{C} \) is a fundamental domain for \( W \). Moreover if \( x \in \overline{C} \) then \( W_x \) is generated by the simple reflections fixing \( x \), i.e. by the reflections in the walls of \( C \) containing \( x \).

Proof. The result is obvious for \( x \in \mathfrak{h}' \). Otherwise take \( x_n \in \mathfrak{h}' \) with \( x_n \to x \). Since there are only finitely many Weyl chambers, we may assume that \( x_n \in \sigma C \) for a fixed \( \sigma \in W \). Hence \( x \in \overline{C} \).

Now say \( x \in \partial C \) and \( \sigma x \in \overline{C} \) for \( \sigma \neq 1 \). We shall assume the result by induction on \( n(\sigma) = n(C, \sigma C) \), the result being trivial when \( \ell = 0 \). Since \( x \in \partial C \), we have \( x \in \mathcal{H}_\beta \) for some \( \beta \) simple. Since \( \mathcal{H}_\beta \) is a wall, we can find an interior point \( y \in \mathcal{H}_\beta \cap \overline{C} \). Take \( \alpha \in C \) near \( y \) and \( b \in \sigma C \) such the line segment \([a, b] \) is crosses the hyperplanes \( \mathcal{H}_\beta = \mathcal{H}_{\beta_1} \cap \cdots \mathcal{H}_{\beta_n} \) transversely. Let \( \sigma = \sigma_{\beta_1} \cdots \sigma_{\beta_n} \), and \( \tau = \sigma_{\beta_1} \cdots \sigma_{\beta_2} \), so that \( \sigma = \tau \sigma_{\beta_2} \). Let \( C_1 = \sigma_{\beta_2} C \). Then \( n(C, \sigma C) = \ell \) and \( n(C_1, \tau C_1) = \ell - 1 \). Since \( \mathcal{H}_\beta \), \( x = \sigma \beta x \). Thus \( x \in \partial C_1 \) and \( \tau x \in \partial C_1 \). By induction \( \tau x = x \). Since \( \sigma = \tau \sigma_{\beta_2} \), it follows that \( \sigma x = x \), as required. The second assertion follows by induction on \( \ell(\sigma) \), since \( \ell(\tau) = \ell(\sigma) - 1 \).

8. WEYL’S UNIQUENESS THEOREM. It turns out that every simple complex Lie algebra is the complexification of a compact simple Lie algebra, unique up to isomorphism. One proof of this suggested by Cartan minimises the \( \ell^2 \) norm of the structure constants over all choices of orthonormal bases with respect to the Killing form. Weyl’s original proof relied on choosing a basis with real structure constants, similar to the bases in the lattice construction. Our aim here is to show that a compact simple Lie algebra is uniquely determined by its root system.

Theorem A. If \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are compact semisimple Lie algebras with isomorphic complexifications, then \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are isomorphic.

Proof. We may assume that \( \mathfrak{g} \) is the common complexification. let \( J_1 \) and \( J_2 \) be the conjugations corresponding to \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \). Let \( B(x,y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)) \) be the complex Killing form on \( \mathfrak{g} \). Evidently \( B \) restricts to the real Killing forms on both \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \). Take the complex inner product \( (X,Y) = -B(X, J_1 Y) \) on \( \mathfrak{g} \). It is real on \( \mathfrak{g}_1 \). Let \( T = J_2 J_1 \). Then \( T \in \text{Aut}_C(\mathfrak{g}) \) and \( T \) is self–adjoint, since

\[
(TX,Y) = -B(TX, J_1 Y) = -B(X, T^{-1} J_1 Y) = -B(X, J_1 TY) = (X, TY).
\]

Thus \( S = T^2 \in \text{Aut}(\mathfrak{g}) \). We may identify \( \mathfrak{g} \) with its image \( \mathfrak{g} = \text{ad}(\mathfrak{g}) \) in \( \text{End}(\mathfrak{g}) \) with the operator bracket. Since \( S \mathfrak{g} S^{-1} = \mathfrak{g} \), it follows that \( S \mathfrak{g} S^{-i} = \mathfrak{g} \) for all \( t \in \mathbb{R} \). On the other hand it is easily checked that \( J_i S^t = S^{-i} J_i \) for \( i = 1, 2 \). Let \( J_2 = S^t J_2 S^{-t} \). Then \( J_1 J_2 = J_1 S^t J_2 S^{-t} = J_1 J_2 S^{-2t} T^{-1} S^{-2t} \) while \( J_2 J_1 = S^t J_2 S^{-t} J_1 = S^2 T \). These are equal when \( S^t = T^{-2} = S^{-1} \), i.e. when \( t = -1/4 \). Thus \( \theta = S^{-1/4} \) gives an automorphism of \( \mathfrak{g} \) such that \( J_1 \) and \( J_2 = \theta J_2 \theta^{-1} \) commute.

We claim that \( \theta(\mathfrak{g}_1) = \mathfrak{g}_1 \). Let \( \mathfrak{g}_2 = \theta(\mathfrak{g}_1) \). Its conjugation is now \( J_2 \) which commutes with \( J_1 \). Thus \( \mathfrak{g}_1 = \{ X \in \mathfrak{g} : J_1 X = -X \} \) and \( \mathfrak{g}_2 = \{ X \in \mathfrak{g} : J_2 X = -X \} \). Now \( \mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^- \), where \( J_2 = \pm 1 \) on \( \mathfrak{g}_1^+ \). Thus \( \mathfrak{g}_2 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^- \). On the other hand \( -B(X,Y) \) has to be positive definite on \( \mathfrak{g}_2 \) (since \( \mathfrak{g}_2 \) is compact).

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Since it is positive definite on $\mathfrak{g}_1$ and hence $\mathfrak{g}_1^{-}$, it is negative definite on $i\mathfrak{g}_1^{-}$. Therefore $\mathfrak{g}_1 = (0)$ and so $\mathfrak{g}_2 = \mathfrak{g}_1$. Thus $\theta(\mathfrak{g}_2) = \mathfrak{g}_1$ and $\theta$ is an isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$, as required.

**Theorem B.** Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complexifications of compact simple Lie algebras and $f : \mathfrak{h} \to \mathfrak{h}'$ an isometric isomorphism between their maximal abelian subalgebras carrying one root system onto another. Then $f$ extends uniquely to an (complex) isomorphism of $\mathfrak{g}_C$ onto $\mathfrak{g}'_C$ carrying $E_i$ onto $E'_i$.

**Proof (A. Winter).** Uniqueness follows because $f(F_i)$ must be sent onto a multiple of $F'_i$. Since $f(H_i) = H'_i$ and $f(E_i) = E'_i$, the relations $[E_i, F_i] = H_i$ and $[E'_i, F'_i] = H'_i$ force $f(F_i) = F'_i$. Since the $E_i$’s and $F_i$’s generate $\mathfrak{g}$, this uniquely determines $f$.

To prove the existence of the isomorphism, let $\overline{\mathfrak{g}}$ be the subalgebra of $\mathfrak{g} \oplus \mathfrak{g}'$ generated by the elements $\overline{\mathfrak{P}}_i = H_i \oplus H'_i$, $\overline{\mathfrak{E}}_i = E_i \oplus E'_i$ and $\overline{\mathfrak{F}}_i = F_i \oplus F'_i$. The algebra $\overline{\mathfrak{g}}$ has projections $\pi$ and $\pi'$ onto $\mathfrak{g}$ and $\mathfrak{g}'$.

Clearly $\ker(\pi) \subset (0) \oplus \mathfrak{g}'$ and $\ker(\pi') \subset \mathfrak{g} \oplus (0)$. Being invariant under $\text{ad}(\overline{\mathfrak{E}}_i), \text{ad}(\overline{\mathfrak{F}}_i)$ and $\text{ad}(\overline{\mathfrak{P}}_i)$, it follows that $\ker(\pi')$ is invariant under $\text{ad}(E_i), \text{ad}(F_i)$ and $\text{ad}(H_i)$ and hence is an ideal in $\mathfrak{g} \oplus (0)$. Similarly $\ker(\pi)$ is an ideal in $(0) \oplus \mathfrak{g}'$. Since $\mathfrak{g}$ and $\mathfrak{g}'$ are simple, either (and hence both) of these kernels is non–trivial iff $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$.

Suppose therefore that $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}'$. Let $\theta$ be the highest roots for $\mathfrak{h}$ and $\mathfrak{h}'$ with corresponding vectors $E_\theta$ and $E'_\theta$. Let $v = E_\theta \oplus E'_\theta \in \mathfrak{g} \oplus \mathfrak{g}' = \overline{\mathfrak{g}}$. Let $V$ be the $\overline{\mathfrak{g}}$–submodule generated by $v$. Since $v$ is an eigenvector for the $\overline{\mathfrak{P}}_i$, and annihilated by the $\overline{\mathfrak{F}}_i$’s, it is clear that $V$ is just the space obtained by applying monomials in the $\overline{\mathfrak{F}}_i$’s to $v$. As an $\mathfrak{h} \oplus \mathfrak{h}'$–module, the weight $(\beta, \beta)$ occurs with multiplicity one in $V$, since the $\overline{\mathfrak{F}}_i$’s are lowering operators. On the other hand the $\mathfrak{g} \oplus \mathfrak{g}'$ by $E_\theta \oplus E'_\theta$ is just $\mathfrak{g} \oplus \mathfrak{g}'$ by simplicity. This contradiction proves that the kernels are non–trivial and hence that $\mathfrak{g}_C$ and $\mathfrak{g}'_C$ are isomorphic as complex Lie algebras.

**Theorem C (Weyl).** A compact simple Lie algebra is determined up to isomorphism by its root system.

**Proof.** Immediate from Theorems A and B.

9. CLASSIFICATION OF COMPACT SIMPLE LIE ALGEBRAS.

Irreducibility. A root system is said to be irreducible if it cannot be written as the disjoint union of two mutually orthogonal proper subsets.

**Lemma.** A root system is irreducible iff its Weyl group acts irreducibly.

**Proof.** Suppose that $\Phi \subset V$ is the root system. Let $V_1$ be a non–zero $W$–invariant subset of $W$. Since $x - \sigma_\alpha x = (x, \alpha^\vee)\alpha$, either $\alpha \perp x$ for all $x \in V_1$ or $\alpha \in V_1$. Thus $\Phi_1 = \Phi \cap V_1$ and $\Phi_2 = \Phi \cap V_1^\perp$ are orthogonal and have disjoint union $\Phi$. So if $\Phi$ is irreducible, $W$ acts irreducibly. Conversely if $\Phi = \Phi_1 \cup \Phi_2$ is an orthogonal splitting, any reflection $\sigma_\alpha$ fixes pointwise the component in which $\alpha$ does not lie and hence carries the other component into itself. Thus $\Phi_1$ and $\Phi_2$ span orthogonal invariant subspaces, so $W$ does not act irreducibly.

**Cartan matrix.** Let $\Phi$ be a root system with simple roots $\alpha_1, \ldots, \alpha_n$. We define the Cartan matrix $N = (n_{ij})$ by $n_{ij} = 2(\alpha_i, \alpha_j)/\langle \alpha_i, \alpha_i \rangle$. Note that $n_{ii} = 2$ and $n_{ij} \leq 0$ if $i \neq j$; moreover $n_{ij} \neq 0$ iff $n_{ji} \neq 0$.

**Lemma.** An irreducible root system is uniquely determined by its Cartan matrix.

**Proof.** Let $(\alpha_i)$ and $(\alpha'_i)$ be systems of simple roots in $V, V'$ such that $n_{ij} = n'_{ij}$. Define $T : V \to V'$ by $T(\alpha_i) = \alpha'_i$. Then $\sigma_{\alpha_i} \alpha_i = \alpha_i - n_{ij} \alpha_j$. Thus $T \sigma_{\alpha_i} T^{-1} = \sigma'_{\alpha_i}$. Hence $T W T^{-1} = W'$. Since $\Phi = W \cdot \Delta$ and $\Phi' = W' \cdot \Delta'$, we get $T \Phi = \Phi'$. Since $W$ and $W'$ act irreducibly, there is an essentially unique invariant inner product on $V$ and $V'$. Thus $T$ is a scalar multiple of an isometry so the root systems $\Phi$ and $\Phi'$ are equivalent.

Dynkin diagram. Let $m_{ij} = 2\delta_{ij} - n_{ij}$ ($1 \leq i, j \leq n$). Thus $M = (m_{ij})$ is the incidence matrix of a directed graph, called the Dynkin diagram of the root system. Clearly $M$ completely determines the Cartan matrix and hence the root system. Clearly $\Phi$ is irreducible iff the Dynkin diagram is connected.

**The highest root.** Let $\theta$ be the highest weight of the adjoint representation on $\mathfrak{g}$. Since $\theta$ is the highest root and $\sigma_{\alpha_i} \theta = \theta - (\theta, \alpha_i^\vee)\alpha_i$ is also a root, we must have $(\theta, \alpha_i) \geq 0$. Since $\theta$ is a positive root, we may
write $\theta = \sum_{i=1}^{n} d_i \alpha_i$ with $d_i \geq 0$. Since $\alpha_i$ is also a weight of $\mathfrak{g}$, $\theta - \alpha_i \geq 0$. Thus $d_i \geq 1$ for all $i$. Note that since $\mathfrak{g}$ acts irreducibly on $\mathfrak{g}$, there must be a root $\theta - \alpha_j$ for some $j$ since some lowering operator must act non-trivially on $\mathfrak{g}_\theta$. Thus $(\alpha_j, \theta) < 0$.

**Extended Dynkin diagram.** The extended Dynkin diagram arises naturally in the study of affine Lie algebras, but can easily be defined without reference to them. It probably provides the simplest method to classifying Dynkin diagrams. Define $\alpha_0 = -\theta$ extend the definition of $n_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ to include $i$ or $j = 0$. We still have $n_{ii} = 2$ and $n_{ij} \leq 0$. As before define $m_{ij} = 2d_{ij} - n_{ij}$. This is the incidence matrix of a directed graph called the extended Dynkin diagram. The Dynkin diagram is obtained by deleting the node 0 from the extended Dynkin diagram, so it too is connected. As before $m_{ii} = 0$ and $m_{ij} \neq 0$ iff $m_{ji} \neq 0$. Since $\theta = \sum d_i \alpha_i$, we have $\sum d_i \alpha_i = 0$ if we set $d_0 = 1$. Thus we obtain the important equation $\sum m_{ij}d_j = 2d_j$. This equation implies that the extended Dynkin diagram is a directed graph with spectral radius 2. It is easy to classify such graphs.

**Graphs of spectral radius two.** By a graph we shall mean a directed graph where nodes $i$ and $j$ are joined by $m_{ij}$ links. We required $m_{ii} = 0$ for all $i$ (no loops) and $m_{ij} \neq 0$ iff $m_{ji} \neq 0$. The matrix $M = (m_{ij})$ is called the incidence matrix of the graph. We shall suppose that the graph is connected. By Perron–Frobenius theory, the eigenvalue of $M$ of largest modulus is positive and of multiplicity one; it is the unique eigenvalue corresponding to an eigenvector with strictly positive entries. We denote this eigenvalue by $r(M)$ (it is the spectral radius of $M$.) If we take a connected subgraph, Perron–Frobenius theory implies that its spectral radius will be strictly smaller. We use these ideas to classify all connected graphs of spectral radius 2.

**Theorem.** Figure 1 gives a complete list of connected graphs with spectral radius 2.

\[
\begin{align*}
A_1^{(1)} & \\
\begin{array}{ccc}
1 & 1 \\
\circ & \iff & \circ
\end{array} \\
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1
\end{array} \\
\begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array} \\
\begin{array}{ccc}
1 & 1 & 1
\end{array}
\end{align*}
\]

\[
\begin{align*}
A_n^{(1)} & \\
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1
\end{array} \\
\begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array} \\
\begin{array}{ccc}
1 & 1 & 1
\end{array}
\end{align*}
\]

\[
\begin{align*}
B_n^{(1)} & \\
\begin{array}{ccccccc}
1 & 2 & 2 & 2 & 2
\end{array} \\
\begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array} \\
\begin{array}{ccc}
1 & 2 & 1
\end{array}
\end{align*}
\]

\[
\begin{align*}
C_n^{(1)} & \\
\begin{array}{ccccccc}
1 & 2 & 2 & 2 & 2
\end{array} \\
\begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array} \\
\begin{array}{ccc}
1 & 2 & 1
\end{array}
\end{align*}
\]

\[
\begin{align*}
D_n^{(1)} & \\
\begin{array}{ccccccc}
1 & 2 & 2 & 2 & 1
\end{array} \\
\begin{array}{cccccc}
\circ & \circ & \cdots & \circ & \circ & \circ
\end{array} \\
\begin{array}{ccc}
1 & 2 & 1
\end{array}
\end{align*}
\]

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Proof. In these graphs a simple bond means that $m_{ij} = 1 = m_{ji}$. Otherwise the multiplicity of a bond is indicated by the number above it. The numbers in the circles give (a multiple of) the Perron–Frobenius eigenvector corresponding to the eigenvalue 2. It is immediately verified by inspection that all the above graphs have norm 2, so we just have to show that the list is exhaustive. We shall consistently use the fact that a graph of spectral radius 2 cannot have a proper subgraph with spectral radius 2.

If the graph contains a cycle with three or more nodes, it must contain and hence equal a subgraph
Thus we may assume that there are no such cycles. If it contains a node of connected to four or more other nodes, it must contain and hence equal a subgraph $D^{(1)}_4$. Thus we may assume that each node is connected to at most three other nodes. Suppose next that the graph has a bond of multiplicity greater than or equal to 4. If so it has a subgraph of type $A^{(2)}_2$, which it must equal. Now suppose the graph has a bond of multiplicity 3. If the other bond between these nodes had multiplicity greater than or equal to two, then $A^{(1)}_1$ would be a proper subgraph and the graph would have spectral radius greater than two. So it must have multiplicity one, so that the graph would have to be contained and hence equal either $G^{(1)}_2$ or $D^{(2)}_4$. So we may assume all bonds have multiplicity 1 or 2. Suppose it has at least 2 bonds of multiplicity 2. If they are between the same nodes, $A^{(1)}_1$ is a subgraph so the whole graph. If they are between different nodes, then the graph must contain and hence equal one of $C^{(1)}_n$, $A^{(2)}_2$ or $D^{(2)}_{n+1}$. So we may assume that there is only one bond of multiplicity 2. Suppose that there is a node of valency three. It must be connected to one of the nodes in the multiplicity two bond. Thus the graph contains and hence equals one of the graphs $B^{(1)}_n$ or $A^{(2)}_n$. So we may assume it has no nodes of valency three, so that graphs is just one long string. Neither of the nodes in the multiplicity 2 bond can be an endpoint of the string, for the graph would be a proper subgraph of $B^{(1)}_n$ or $A^{(2)}_n$ and hence have spectral radius strictly less than 2. Thus each is connected to a further point; if there no additional points, the graph would be a proper subgraph of $F^{(1)}_4$ (or equally well $E^{(2)}_6$), and hence have spectral radius strictly less than 2. So there must be a fifth point, so the graph contains and hence equals $F^{(1)}_4$ or $E^{(2)}_6$. Thus we may assume the graph has only bonds of multiplicity one and only trivalent vertices. If it has two trivalent vertices, it must contain and hence equal $D^{(1)}_n$. If it had no trivalent vertices, it would be a proper subgraph of $A^{(1)}_n$, which would make its spectral radius less than 2. So we may assume it has exactly one trivalent vertex. If all the branches from that vertex have length greater than or equal to two, it contains and hence equals $E^{(1)}_6$. So some branch has length one. If two had length one, then the graph would be a proper subgraph of $D^{(1)}_n$, so have spectral radius less than 2. Hence one of the branches has length 1 and the two other length greater than or equal to 2. If they both have length greater than or equal to 3, then the graph contains and hence equals $E^{(1)}_7$. So one of branches must have length equal to 2. If the remaining branch has length greater than or equal to 5, the graph contains and hence equals $E^{(1)}_8$. If it had length less than 5, it would be contained in $E^{(1)}_8$, so would have spectral radius less than 2. This completes the proof (which could profitably arranged in a flow chart).

**Classification of Dynkin diagrams.** Figure 2 gives a complete list of possible Dynkin diagrams.

**Proof.** These are the only diagrams that arise from the list of the preceding theorem by removing one node so as not too disconnect the graph. The lattice construction and its fixed point refinement yield root systems corresponding to each Dynkin diagram.
$A_n \quad \circ - \circ - \cdots - \circ - \circ$

$B_n \quad \circ - \circ - \cdots - \circ \Rightarrow \circ$

$C_n \quad \circ - \circ - \cdots - \circ \Leftarrow \circ$

$D_n \quad \circ - \circ - \cdots - \circ - \circ$

$E_6 \quad \circ - \circ - \circ - \circ - \circ$

$E_7 \quad \circ - \circ - \circ - \circ - \circ - \circ$

$E_8 \quad \circ - \circ - \circ - \circ - \circ - \circ - \circ$

$F_4 \quad \circ \Rightarrow \circ - \circ$

$G_2 \quad \circ \Rightarrow \circ$

Figure 2
PART 2. REPRESENTATION THEORY.

In this part we develop the representation theory of an arbitrary compact simple matrix group analogously to that of SU(2). Passing to the Lie algebra, we classify representations by their highest weight and give a description of the representation in terms of lowering and raising operators. We then prove the Weyl character formula by studying the supercharge or Dirac operator. This same procedure will be followed for affine Kac–Moody algebras. The most important example to follow is SU(N), but we use the language of root systems so that the proofs apply in general.

10. ROOT AND WEIGHT LATTICES. Let G be a compact simple matrix group with Lie algebra g. Let T be a maximal torus in G with Lie algebra h and Weyl group $W = N(T)/T$. Let $\Lambda \subset h$ be the kernel of the map $h \to T$ X $e^{2\pi i T}$. Thus $h/\Lambda \cong T$ so that $\Lambda$ is a lattice in h, called the unit or integer lattice. We claim that $T^\vee = \text{Hom}(T, T) \cong \Lambda$. In fact any homomorphism $f : T \to T$ has the form $f(e^{it}) = \exp(itX)$ for a unique $X \in h$ (simply take the infinitesimal representations). Setting $t = 2\pi$, we get $\exp(2\pi X) = 1$, so that $X \in \Lambda$.

The weight lattice. The weight lattice $P(G)$ of T is the group $\hat{T} = \text{Hom}(T, \mathbb{T})$. Looking at the corresponding infinitesimal homomorphism, we see that any $\chi \in \hat{T}$ has the form $\chi(\exp(X)) = e^{\lambda(X)}$ for a unique $\lambda \in \mathfrak{h}^\ast$. Let $P(G)$ be the subgroup of $\mathfrak{h}^\ast$ consisting of weights. Clearly $\chi(\exp(2\pi X)) = 1$ if $X \in \Lambda$, so that $\lambda \in \mathfrak{h}^\ast$ defines a character or weight iff $\lambda(X) \in \mathbb{Z}$ for all $X \in \mathfrak{z}$. Thus $P(G)$ forms a lattice, the weight lattice, and $P(G)$ and $\Lambda$ are dual lattices. We write $P(G) = \Lambda^\ast$ and $\Lambda = P(G)^\ast$. If we restrict the faithful representation of G to V and decompose it in characters, we get a finite set of homomorphisms $\chi_i : T \to \mathbb{T}$. Let $\Gamma_0$ be the subgroup of $\hat{T}$ generated by the $\chi_i$’s. The next result shows that $\Gamma_0 = \hat{T}$.

Lemma. If $\Gamma_0$ is a subgroup of $\Gamma = \text{Hom}(\mathbb{T}^n, \mathbb{T})$ distinguishing the points of $\mathbb{T}^n$, then $\Gamma_0 = \Gamma$.

Proof. $\Gamma/\Gamma_0$ is a finitely generated Abelian group, so there admits a non–trivial homomorphism $\theta$ into $\mathbb{T}$. Now $\Gamma = \mathbb{Z}^n$ with $\mathbb{Z}$-basis $e_1, \ldots, e_n$. Let $t_i = \theta(e_i) \in \mathbb{T}$. Thus $t = (t_i) \in \mathbb{T}^n$. By definition $\theta(\sum m_i e_i) = t^m = e_m(t)$ for $m \in \mathbb{Z}^n$. But then $e_m(t) = 1$ for all $m \in \Gamma_0$, a contradiction.

The root lattice. Let $\Phi$ be the set of non–zero weights appearing in the complexified adjoint representation on $g_{\mathbb{C}}$. Since the adjoint representation is real, if $\alpha \in \Phi$, then $-\alpha \in \Phi$. Let Q be the sublattice of $\mathfrak{h}^\ast$ spanned by $\Phi$. Thus $Q \subseteq P_G$.

The centre of G. The centre Z(G) is a closed subgroup of G, so a Lie group. Its Lie algebra is just the centre of g, so trivial. Hence Z(G) is finite.

Lemma. Z(G) $\cong Q^\ast/P^\ast = (P/Q)^\ast$.

Proof. Note that Z(G) $\subset T$. Thus we may write any $z \in Z(G)$ as $z = e^{2\pi X}$ for $X \in \mathfrak{h}/P^\ast$. Now $e^{2\pi X} \in Z(G)$ iff $\text{ad}(e^{2\pi X}) = 1$ iff $\text{Ad}(e^{2\pi X})$ fixes $g_{\mathfrak{q}}$, for all $\alpha \in \Phi$ iff $e^{2\pi i \alpha(X)} = 1$ for all $\alpha$ iff $\alpha(X) \in \mathbb{Z}$ for all $\alpha \in Q$ iff $X \in Q$. Hence Z(G) $\cong Q^\ast/P^\ast$, as required.

The generalised weight lattice. For each $\alpha$, we have a copy of sl_2 corresponding to $\alpha$, namely $H_\alpha, E_\alpha, F_\alpha$, where $H_\alpha = -2T_\alpha/\|\alpha\|^2$ and $\mu(T_\alpha) = (\alpha, \mu)$. Let V be the defining representation of G (so that G $\subset U(V)$). By the SU(2) theory $\lambda(H_\alpha) \in \mathbb{Z}$ for each weight $\lambda$ of V. Hence $2(\lambda, \alpha)/\|\alpha\|^2 \in \mathbb{Z}$ for each root $\alpha$, i.e. $(\lambda, \alpha^\vee) \in \mathbb{Z}$. This defines a lattice $P(\mathfrak{g}) = \{\lambda \in \mathfrak{h}^\ast : (\lambda, \alpha^\vee) \in \mathbb{Z}\}$, containing $P(G)$. It is called the generalised weight lattice.

The dual root system. The inner product on the real inner product space $\mathfrak{h}$ allows us to identify $\mathfrak{h}$ and $\mathfrak{h}^\ast$. Recall that if $\alpha \in \mathfrak{h}^\ast$, then $\alpha^\vee \in \mathfrak{h}$ is defined by $\langle \alpha^\vee, \lambda \rangle = 2(\alpha, \lambda)/\|\alpha\|^2$.

Proposition. (1) If $\Phi$ is a root system, then $\Phi^\vee$ is also a root system with $W(\Phi^\vee) = W(\Phi)$.
(2) If $\alpha \in \Phi^\vee$, then $\pm \alpha$ is a non–negative integer combination of the $\alpha_i^\vee$’s, $\alpha = \sum n_i \alpha_i$.

Proof. (1) Recall that $\alpha^\vee 2\alpha/(\alpha, \alpha)$. Thus $\alpha^\vee \vee = \alpha$. Also $\alpha \in \Phi$ iff $t^{-\alpha} \in \Phi^\vee$; thus $\Phi^\vee$ is reduced. Since $\Phi$ spans $V$, so too does $\Phi^\vee$. Note also that $\sigma_\beta(\alpha^\vee) = \sigma_{\alpha^\vee}(\sigma_\beta(\alpha)^\vee)$. Finally since $\Phi$ is a root system, $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. Since $\alpha^\vee \vee = \alpha$, the same condition holds for $\Phi^\vee$. Hence $\Phi^\vee$ is an
abstract root system. The Weyl groups coincide because $\sigma_\alpha = \sigma_\alpha$.

(2) Since $\alpha^\vee$ is proportional to $\alpha_i$, the $\alpha^\vee_i$'s are a basis of $V$. So it suffices to show that $\Phi^\vee \subset \bigoplus \mathbb{Z} \alpha^\vee_i = Q_1$ say. Clearly $Q_1$ is invariant under the $\sigma_\alpha$,s, so $W$. If $\alpha \in \Phi^+$, then $\alpha = \sigma_\alpha$ some $i$ (since $h^+_\alpha$ is the wall of some chamber). So $\alpha^\vee = \sigma_\alpha^\vee \in Q_1$, as required.

**The coroot lattice.** This is the lattice in $\mathfrak{h}$ spanned by the coroots $\alpha^\vee$ over $\mathbb{Z}$. The relationship between the lattices is best illustrated in a picture.

$$
\begin{array}{c|c}
Q^* & P(\mathfrak{g}) \\
\cup & \cup \\
\Lambda & P(\mathfrak{g}) \\
\cup & \cup \\
Q^\vee & Q
\end{array}
$$

**Topological remark.** It turns out that the fundamental group of $G$ is just $P(\mathfrak{g})/P(G)$. Thus $G$ is simply connected iff $P(\mathfrak{g}) = P(G)$, i.e. every generalised weight is a weight. This result is due to Hermann Weyl; its proof requires extra analytic or topological tools. In this chapter we shall content ourselves with constructing matrix group $\tilde{G}$ and a homomorphism $f : \tilde{G} \to G$ which is a covering map (ker($f$) is finite and central) such that $P(\tilde{G}) = P(\mathfrak{g})$. (Note that since $f$ is a covering map, the Lie algebra of $\tilde{G}$ is just $\mathfrak{g}$, just like the double cover $\text{Spin}(V) \to SO(V)$.)

**11. POINCARE–BIRKHOFF–WITT THEOREM.** Let $\mathfrak{g}$ be a Lie algebra, possibly infinite dimensional, with basis $X_1, X_2, \ldots$. Then the universal enveloping algebra of $\mathfrak{g}$ has basis $X_{i_1}X_{i_2} \ldots X_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$.

**Proof (Jacobson).** We define $U(\mathfrak{g})$ to be the quotient of the tensor algebra $T(\mathfrak{g})$ by the two-sided ideal generated by $X \otimes Y - Y \otimes X - [X,Y]$ with $X,Y \in \mathfrak{g}$. Let $X_1, X_2, \ldots$ be a basis of $\mathfrak{g}$ (possibly infinite-dimensional). Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$ with basis $X_{i_1} \ldots X_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$. We claim that there is a unique linear 'symbol' map $\sigma : T(\mathfrak{g}) \to S(\mathfrak{g})$ such that

$$
\sigma(a_1 \otimes a_2 \otimes \cdots \otimes (a \otimes b - b \otimes a - [a,b]) \otimes \cdots \otimes a_m) = 0
$$

and

$$
\sigma(X_{i_1} \otimes \cdots \otimes X_{i_n}) = X_{i_1} \cdots X_{i_n}
$$

if $i_1 \leq i_2 \leq \cdots \leq i_n$. Suppose that such a map has been constructed for the linear span of all monomials of degree $\leq n - 1$. We proceed by induction on the length of the permutation required to put a monomial of degree $n$ in 'correct' order. If this length is 0, we simply use (2) to define $\sigma$. Otherwise if the order is wrong we can find a transposition of two adjacent terms which decreases the length of the permutation. We then define

$$
\sigma(X_{i_1} \otimes \cdots \otimes X_{i_n}) = \sigma(X_{i_1} \otimes \cdots \otimes X_{i_{k+1}} \otimes X_{i_k} \otimes \cdots \otimes X_{i_n}) + \sigma(X_{i_1} \otimes \cdots \otimes [X_{i_k},X_{i_{k+1}}] \otimes \cdots \otimes X_{i_n}).
$$

This has to be the case and therefore proves uniqueness of $\sigma$. We must show that $\sigma$ is independent of the choice of $k$. If we used another transposition disjoint from $(k,k+1)$. Then we could apply the same process to the right hand side of the above equation with $(\ell, \ell + 1)$ in place of $(k,k+1)$. We would clearly get the same answer if we did the $(\ell, \ell + 1)$ transposition first followed by $(k,k+1)$, since they are disjoint. If the other transposition was not disjoint, we may suppose that the two transpositions are $(k,k+1)$ and $(k-1,k)$. Thus $i_{k-1} < i_k < i_{k+1}$. Write $Y_j = X_{i_j}$. Then if we use $(k,k+1)$ to define $\sigma$, we get

$$
\sigma(\cdots Y_{k-1} \otimes Y_k \otimes Y_{k+1} \cdots)
= \sigma(\cdots Y_{k-1} \otimes Y_{k+1} \otimes Y_k \cdots) + \sigma(\cdots Y_{k-1} \otimes [Y_k, Y_{k+1}] \cdots)
= \sigma(\cdots Y_{k+1} \otimes Y_{k-1} \otimes Y_k \cdots) + \sigma(\cdots [Y_{k-1}, Y_k] \otimes Y_{k+1} \cdots)
+ \sigma(\cdots [Y_{k-1}, Y_k] \otimes Y_{k+1} \cdots) + \sigma(\cdots Y_{k-1} \otimes [Y_k, Y_{k+1}] \cdots).
$$
If we use \((k - 1, k)\) to define \(\sigma\), we get

\[
\sigma(\cdots Y_{k-1} \otimes Y_k \otimes Y_{k+1} \cdots) = \sigma(\cdots Y_{k+1} \otimes Y_k \otimes Y_{k-1} \cdots) + \sigma(\cdots Y_k \otimes [Y_{k-1}, Y_{k+1}] \cdots) + \sigma(\cdots [Y_{k-1}, Y_k] \otimes Y_{k+1} \cdots).
\]

The difference between these expressions is zero by (1) and the Jacobi identity for \(Y_{k-1}, Y_k\) and \(Y_{k+1}\). The fact that the extension is well-defined means that it has all the stated properties.

Property (2) implies the map \(\sigma\) induces a map \(\sigma : U(\mathfrak{g}) \to S(\mathfrak{g})\). Property (1) implies that \(\sigma\) carries the span of the monomials of degree \(\leq N\) onto \(\bigoplus_{\mu \leq N} S^j(\mathfrak{g})\). Since the monomials of degree \(\leq N\) are spanned by the monomials \(X_1^{\alpha_1} \cdots X_n^{\alpha_n}\) with \(|\alpha| \leq N\) which map into identical linearly independent symbols under \(\sigma\), the theorem follows.

12. HIGHEST WEIGHT VECTORS. Let \(\pi : G \to U(V)\) be a continuous homomorphism. We call \((\pi, V)\) a unitary representation of \(G\) or \(G\)-module. We may decompose \(V\) as a \(T\)-module, \(V = \bigoplus_{\mu \in \hat{T}} V_{\mu}\) where \(e^X \in T\) acts on \(V_{\mu}\) as the scalar \(e^{i\mu(X)}\). We call \(\mu\) a weight of \(V\) and \(V_{\mu}\) the corresponding weight space. Because of the identification of \(\hat{T}\) and the weight lattice \(P(G)\), we have \(\mu \in P(G)\). Notice that the root system is invariant under the Weyl group. Hence \(Q\) and \(Q^\vee\) are invariant under the Weyl group. It follows that \(P(\mathfrak{g})\) and \(Q^\vee\) are also invariant under the Weyl group. Since \(\text{Ad}(g) \exp(2\pi X) = \exp(2\pi \text{Ad}(g) \cdot X)\), for \(X \in \mathfrak{h}\) and \(g \in N(T)\), it follows that the integer lattice is invariant under \(W\). Hence the weight lattice is invariant under \(W\). This also follows by applying the following lemma to a faithful representation.

**Lemma.** The weights of \(V\) are invariant under the Weyl group. In fact if \(\sigma = gT\) for \(g \in N(T)\), then \(gV_{\lambda} = V_{\sigma\lambda}\).

**Proof.** We have

\[
\pi(e^{-X})\pi(g)v = \pi(g)\pi(e^{g^{-1}Xg})v = \pi(g)\pi(e^{i2\pi g^{-1}Xg})v = \pi(g)e^{i\sigma\cdot\lambda(X)}\pi(g)v,
\]

so that \(\pi(g)v\) lies in \(V_{\sigma\lambda}\).

If we pass to the infinitesimal representation of \(\mathfrak{g}\) on \(V\), \(\pi(g)\) and \(\pi(G)\) have the same commutant since \(G\) is connected. As with \(SU(2)\), we shall temporarily drop the skew–adjointness assumption on the matrices \(\pi(X)\). Thus we shall study finite–dimensional representations \(\pi : \mathfrak{g} \to \text{End}(V)\) of the Lie algebra such that the matrices \(\pi(X)\) \((X \in \mathfrak{h})\) are simultaneously diagonalisable. We may therefore decompose \(V\) as a direct sum \(\bigoplus V_{\mu}\) where \(\pi(X)v = i\mu(X)v\) for \(X \in \mathfrak{h}\) with \(\mu \in \text{Hom}(\mathfrak{h}, \mathbb{C})\). We call \(\mu\) a weight of \(V\) and \(V_{\mu}\) the corresponding weight space. On the other hand for each root \(\alpha\) we can construct a copy of \(sl_2\) inside \(\mathfrak{g}\), namely \(E_{\alpha}, F_{\alpha}, H_{\alpha}\) with \(H_{\alpha} = -2iT_{\alpha}/||\alpha||^2\) and \((T_{\alpha}, X) = \alpha(X)\) for \(X \in \mathfrak{h}\). We may consider \(V\) as an \(sl_2\)-module; plainly

\[
\pi(H_{\alpha})v = (\mu, \alpha^\vee)v
\]

for \(v \in V_{\mu}\). This immediately gives the following result.

**Lemma.** (a) Any weight of \(V\) lies in \(P(\mathfrak{g})\), so is a generalised weight.

(b) \(E_{\alpha}V_{\mu} \subseteq V_{\mu+\alpha}\) and \(F_{\alpha}V_{\mu} \subseteq V_{\mu-\alpha}\), so that \(E_{\alpha}\) and \(F_{\alpha}\) act as raising and lowering operators if \(\alpha > 0\).

**Proof.** The first statement follows because \(\pi(H_{\alpha})\) can only have integer eigenvalues by the \(SU(2)\) theory. The second statement follows because \([X, E_{\alpha}] = \alpha(X)E_{\alpha}\) for \(X \in \mathfrak{h}\).

The appearance of lowering and raising operators leads us to define a partial order on \(\mathfrak{h}^*\). We say that \(\lambda \geq \mu\) if \(\lambda - \mu = \sum t_i\alpha_i\) with \(t_i \geq 0\). (In all cases the \(t_i\)'s will be integers.) Clearly every finite–dimensional representation has a highest weight space \(V_{\mu}\), not necessarily unique. Any vector in it, besides being an eigenvector for \(X \in \mathfrak{h}\), will be annihilated by any raising operator \(E_{\alpha}\) for \(\alpha > 0\).

**Theorem.** (1) Every irreducible representation \(V\) has a unique highest weight vector.

(2) Every vector is in linear span of the vectors obtained by successively applying lowering operators to the highest weight vector.

(3) The corresponding weight space is one–dimensional.
(4) The highest weight \( \lambda \) satisfies \((\lambda, \alpha) \geq 0 \) for all \( \alpha > 0 \); we say that \( \lambda \) is dominant.

(5) Two irreducible representations with the same highest weight are isomorphic.

**Proof.** There certainly is at least one highest weight vector, since \( \dim(V) < \infty \). Let \( v \in V_\lambda \) be a non–zero highest weight vector. Thus \( \pi(E_\alpha)v = 0 \) for \( \alpha > 0 \). Every element in the enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) can be written as a sum of monomials \( \mathcal{L}D \) where \( L \) is a product of lowering operators \( \pi(F_\alpha) \) \((\alpha > 0)\), \( D \) is a diagonal operator \((\text{a monomial in } \pi(X)^i \text{ 's for } X \in \mathfrak{h})\) and \( R \) is a product of raising operators \( \pi(E_\alpha) \) \((\alpha > 0)\).

Since \( Rv = 0 \) and \( Dv \) is proportional to \( v \), it follows that \( \mathcal{L}D \mathcal{R}v \) is either 0 or proportional to \( Lv \), so (2) holds. Since applying a lowering operator always lowers the weight, (2) implies uniqueness in (1) as well as (3).

(4) follows from \( \lambda(H_\alpha) = (\lambda, \alpha^\vee \varepsilon) \), \( \pi(E_\alpha)v = 0 \) \((\alpha > 0)\) and the \( SU(2) \) theory. To prove (5), we may suppose we have two irreducible representations \( V_1 \) and \( V_2 \) with highest weight \( \lambda \). Let \( v_1 \) and \( v_2 \) be corresponding non–zero highest weight vectors. Then \( v = v_1 \oplus v_2 \in V_1 \oplus V_2 \) is a highest vector of weight \( \lambda \). Let \( E \) be the \( g \)–submodule generated by \( v \). Since the raising operators annihilate \( v \), \( E \) is spanned by vectors obtained by applying lowering operators to \( v \). Hence \( E_\lambda = \mathbb{C}v \). Let \( f \) be the restriction of the projection \( V_1 \oplus V_2 \to V_1 \) to \( E \). Since \( f(v) = v_1 \) and \( V_1 \) is irreducible, \( f(E) = V_1 \). On the other hand \( \ker(f) \subset V_2 \cap E \) by definition of \( f \). Now \( V_2 \cap E \) is a submodule of \( V_2 \) and \( E \). It does not contain \( v_2 \), because if it did \( v_1 = v - v_2 \) would lie in \( E \) which would contradict \( E_\lambda = \mathbb{C}v \). Thus \( V_2 \cap E \neq V_2 \), so that \( V_2 \cap E = (0) \), by irreducibility of \( V_2 \). It follows that \( f \) is an isomorphism of \( E \) onto \( V_1 \), so that \( E \cong V_1 \) as \( g \)–modules. Similarly \( E \cong V_2 \) and hence \( V_1 \cong V_2 \).

Our next goal will be to prove a converse of this theorem, namely to show that every dominant generalised weight is the highest weight of a finite–dimensional irreducible representation of \( g \) (see the next two sections).

**The fundamental weights.** We know that if \( \alpha_i \) are simple roots, then \( \alpha_i^\vee \) are simple coroots. Thus they form a \( \mathbb{Z} \)–basis for the coroot lattice \( Q^\vee \). Since \( Q^\vee = P(\mathfrak{g})^* \), we get a dual basis \( \lambda_i \in P_\mathbb{Q} \) defined by \( \lambda_i(\alpha_j^\vee) = \delta_{ij} \). Thus \( (\lambda_i, \alpha_j) = \delta_{ij}(\alpha_i, \alpha_i)/2 \). The generalised weights \( \lambda_i \) are called the fundamental weights.

**Simple reflections and positive roots.** If \( \alpha \) is a simple root, then \( \sigma_\alpha \) permutes all the positive roots not equal to \( \alpha \).

**Proof.** Suppose \( \beta \) is a positive root with \( \beta \neq \alpha \). Then \( \beta = \sum_{\gamma \in \Delta} n_\gamma \gamma \). Since \( \beta \neq \alpha \), it is not proportional to \( \alpha \), so \( n_\gamma > 0 \) for some \( \gamma \neq \alpha \). But \( s_\alpha \beta = \beta - \tau \alpha \), so the coefficient of \( \gamma \) in \( s_\alpha \beta \) is also \( n_\gamma \). Thus \( s_\alpha \beta \) must be positive.

**Corollary 1.** If \( \rho \) is half the sum of the positive roots and \( \alpha \) is simple, then \( \rho - \sigma_\alpha \rho = \alpha \).

**Proof.** Immediate since \( \sigma_\alpha \alpha = -\alpha \).

**Corollary 2.** If \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \), then \( \rho = \sum \lambda_i \). Hence \( \rho \in P(\mathfrak{g}) \) and, if \( \sigma \rho = \rho \) for \( \sigma \in W \), then \( \sigma = 1 \).

**Proof.** By Corollary 1, \( (\rho, \alpha_i^\vee) = 1 \). The result follows because \( (\lambda_i) \) is the dual basis to \( (\alpha_i^\vee) \). If \( \sigma \rho = \rho \), then, because \( \rho \) is in the positive Weyl chamber, \( \sigma \) is in the subgroup generated by simple reflections fixing \( \rho \), of which there are none. So \( \sigma = 1 \). Note that this result is obvious for \( SU(N) \). Alternatively, we will see in section 14 that each \( \lambda_i \) is a highest weight vector of a representation whose weights are invariant under \( W \). Thus \( \sigma \lambda_i \leq \lambda_i \). Hence \( \sigma \lambda_i = \lambda_i \) for all \( i \). Hence \( \sigma = 1 \).

**Corollary 3.** \( \rho - \sigma \rho = \sum_{\alpha \in \Phi_+} \alpha \).

**Proof.** Clearly \( \rho - \sigma \rho = \sum \beta \), where the sum is over all \( \beta > 0 \) such that \( \beta = -\sigma \alpha \) with \( \alpha > 0 \). Hence \( \rho - \sigma \rho = -\sum_{\alpha \in \Phi_+} \sigma \alpha \). We get the result by changing \( \sigma \) to \( \sigma^{-1} \) and applying \( \sigma \).

**13. Eigenvalues of the Casimir Operator.** Let \( \mathfrak{g} \) be a Lie algebra and \((x, y)\) an invariant real inner product on \( \mathfrak{g} \). Let \((X_i)\) be an orthonormal basis. Then \( Z = -\sum X_i \otimes X_i \) is an invariant element in \( \mathfrak{g} \otimes \mathfrak{g} \); for clearly \( Z \) is independent of the choice of orthonormal basis and \( \text{ad}(\mathfrak{g}) \cdot X_i \) is also an orthonormal basis. Now let \( V \) be any representation of \( \mathfrak{g} \). Then \( \mathfrak{g} \otimes \mathfrak{g} \to \text{End}(V), X \otimes Y \to XY \) is \( \mathfrak{g} \)-equivariant. Hence the image of \( Z \) commutes with \( \mathfrak{g} \). This image is called the Casimir operator \( \Omega = -\sum \pi(X_i)^2 \). Note that we take the minus sign because, if \( \pi(X_i)^* = -\pi(X_i) \), then \( \Omega = \sum \pi(X_i)^* \pi(X_i) \) is a positive operator.
Recall that if \((x, y)\) is an invariant real inner product on \(\mathfrak{g}\), \((X_i)\) is an orthonormal basis of \(\mathfrak{g}\) and \(\pi : \mathfrak{g} \to \text{End}(V)\) a representation, then the Casimir operator \(C = -\sum \pi(X_i)^2\) commutes with \(\mathfrak{g}\). As for \(\mathfrak{s}(2)\), we can express \(C\) in terms of the elements \(H_i\) and the lowering and raising operators \(E_\alpha, F_\alpha\) \((\alpha > 0)\). Let \((T_i)\) be any orthonormal basis of \(\mathfrak{h}\) and let \(H_\alpha = -2iT_\alpha/\|\alpha\|^2\) with corresponding elements \(E_\alpha, E_{-\alpha} = E_\alpha^*\). Since \(H_\alpha = 4/\|\alpha\|^2\) we have \(\|E_{\pm \alpha}\|^2 = 2/\|\alpha\|^2\). Thus \((T_i)\) and \(\|\alpha\|\|=\sqrt{2}\alpha\) is an orthonormal basis of \(\mathfrak{g}_\mathbb{C}\). If we take any orthonormal basis \((X_i)\) of \(\mathfrak{g}_\mathbb{C}\), we still have \(\Omega = \sum X_i^jX_j\). (This is independent of the choice of orthonormal basis).

**Lemma.** \(\Omega = -\sum T_i^2 - i\sum_{\alpha>0}\|\alpha\|^2E_\alpha^*E_\alpha\). \(\Omega\) acts on the representation \(V_\lambda\) as the scalar \(\|\lambda\|^2 + 2(\lambda, \rho) = \|\lambda + \rho\|^2 - \|\rho\|^2\), where \(\rho = \frac{1}{2}\sum_{\alpha>0}\alpha\).

**Proof.** We have

\[\Omega = \sum T_i^2 + \sum_{\alpha \in \Phi} \frac{\|\alpha\|^2}{2}E_\alpha^*E_\alpha.\]

On the other hand \(E_\alpha F_\alpha = H_\alpha = F_\alpha E_\alpha = -2iT_\alpha/\|\alpha\|^2 + F_\alpha E_\alpha\). Hence

\[\Omega = -\sum T_i^2 + i\sum T_\alpha + \sum_{\alpha>0} \|\alpha\|^2E_\alpha^*E_\alpha.\]

Applying \(\omega\) to the highest weight vector \(v_\lambda \in V_\lambda\) (which is annihilated by \(E_\alpha\) for \(\alpha > 0\)), we get

\[\Omega v_\lambda = ([\lambda, T_i] = \sum_{\alpha>0} (\alpha, \lambda)v_\lambda = [\|\lambda\|^2 + 2(\lambda, \rho)]v_\lambda.\]

**Freudenthal’s Lemma.** If \(\mu\) is a weight of \(V_\lambda\) and \(\nu\) is a weight of \(V_\rho\), then \(|\mu + \nu|^2 \leq |\lambda + \rho|^2\) with equality iff \(\mu = \sigma\lambda\) and \(\nu = \sigma\rho\) for \(\sigma \in W\), necessarily unique.

**Proof.** Take \(\sigma \in W\) such that \(\sigma^{-1}(\mu + \nu) \geq 0\). Since \(\mu\) and \(\nu\) are weights of \(V_\lambda\) and \(V_\rho\) respectively, we have \(\mu_1 = \sigma^{-1}\mu \leq \lambda\) and \(\nu_1 = \sigma^{-1}\nu \leq \rho\), so that \(\lambda + \rho - \mu_1 - \nu_1\) is a sum of positive roots. But then

\[0 = ||\lambda + \rho||^2 - ||\mu_1 + \nu_1||^2 = \lambda + \rho - \mu_1 - \nu_1, \lambda + \rho + \mu_1 + nu_1 \leq (\lambda + \rho - \mu_1 - \nu_1, \rho).

Thus \((\lambda - \mu_1, \rho) = 0 = (\rho - \nu_1, \rho)\) and hence \(\lambda = \mu_1 = \sigma^{-1}\mu, \rho = \nu_1 = \sigma^{-1}\nu\), as required. Uniqueness follows because \(\sigma\rho = \rho\) implies \(\sigma = 1\).

**Corollary.** If \(\mu\) is a weight of \(V_\lambda\), then \(|\mu + \nu|^2 \leq |\lambda + \rho|^2\) with equality iff \(\mu = \lambda\).

**Proof.** In this case \(\nu = \rho\). On the other hand if \(\sigma \rho = \rho\), we must have \(\sigma = 1\). Hence \(\mu = \sigma \lambda = \lambda\).

**14. LIE ALGEBRAL CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS.**

**Generation by simple root vectors.** Let \(\alpha_1, \ldots, \alpha_m\) be the simple positive roots and set \(E_i = E_{\alpha_i}\), \(F_i = F_{\alpha_i}\), and \(H_i = H_{\alpha_i}\).

**Lemma.** \(\mathfrak{g}\) is generated by the \(E_i\)'s and \(F_i\)'s as a Lie algebra.

**Proof.** Let \(\mathfrak{g}_\mathbb{C}\) be the complex Lie algebra generated by all \(E_i, F_i\) and hence \(H_i = [E_i, F_i]\). Clearly \(\mathfrak{g}_\mathbb{C}\) is \(*\)-invariant and hence the complexification of its skew adjoint part \(\mathfrak{g}_0\). Since the \(\alpha_i\)'s are a basis of \(\mathfrak{h}\), the \(H_i\)'s are a basis of \(\mathfrak{h}_\mathbb{C}\). Hence \(\mathfrak{h} \subset \mathfrak{g}_0\). Let \(g_i \in G\) be the Weyl group element in the copy of \(SU(2)\) corresponding to \(\alpha_i\). Thus \(g_i \in \exp \mathfrak{g}_0\) and \(\text{Ad}(g_i)\) is an automorphism of \(\mathfrak{g}_0\). On the other hand \(g_i\) is a representative of the simple reflection \(\sigma_i\) in \(N(T)\) and the \(\sigma_i\)'s generate the Weyl group. Hence \(\mathfrak{g}_0\) and \(\mathfrak{g}_c\) are invariant under \(W\). But if \(g \in N(T)\) corresponds to \(\sigma \in W\), \(\text{Ad}(g) \cdot \mathfrak{g}_0 = \mathfrak{g}_{\sigma \alpha}\). Now by definition every root space \(\mathfrak{g}_{\pm \alpha}\) lies in \(\mathfrak{g}_c\). Since every positive root is in the \(W\)-orbit of a simple root, each root space \(\mathfrak{g}_\alpha\) lies in \(\mathfrak{g}_c\). Hence \(\mathfrak{g}_c = \mathfrak{g}_\mathbb{C}\) and \(\mathfrak{g}_0 = \mathfrak{g}\).

**Corollary.** The \(E_i\)'s generate \(\bigoplus_{\alpha>0} \mathfrak{g}_\alpha\) and the \(F_i\)'s generate \(\bigoplus_{\alpha<0} \mathfrak{g}_\alpha\).
Proof. Let \( g_1 \) and \( g_- \) be the Lie algebras generated by the \( E_i \)'s and \( F_i \)'s respectively. The relation
\[
[E_i,F_j] = \delta_{ij}H_i
\]
shows that \( g_1 \oplus h \oplus g_- \) is a Lie subalgebra of \( g_C \). Since it contains \( E_i, F_i, H_i \), it must be
the whole of \( g_C \) so the result follows.

Lemma (Serre relations). The generators \( E_i, F_i, H_i \) satisfy the following relations:

S1. \([H_i,H_j] = 0\).
S2. \([E_i,F_j] = \delta_{ij}H_i\).
S3. \([H_i,E_j] = n(i,j)E_j \) and \([H_i,F_j] = -n(i,j)F_j \) where \( n(i,j) = 2(\alpha_i,\alpha_j)/||\alpha_i||^2 \).
S_{ij}^+ : ad\( E_i^{-n(i,j)+1}E_j = 0 \) for \( i \neq j \).
S_{ij}^- : ad\( F_i^{-n(i,j)+1}F_j = 0 \) for \( i \neq j \).

Proof. We already know S1, S2 and S3. To prove the \( S_{ij}^- \) (\( i \neq j \)), note that \( \text{ad}(E_i) \cdot F_j = 0 \), \( \text{ad}(H_i) \cdot F_j = -n(i,j)F_j \). Thus the result follows from \( SU(2) \)-theory, because \( F_j \) is a highest weight vector. (In particular \( n(i,j) \leq 0 \).) \( S_{ij}^- \) follows by taking adjoints.

Remark. The affine Kac–Moody algebra is given by similar relations but this time indexed by the extended
Cartan matrix. We have to add an extra triple of generators \( E_0, F_0, H_0 \). The theory of this chapter then
follows almost without change.

Verma module construction (induced modules). Let \( g \) be a Lie algebra, possibly infinite–dimensional,
and \( g_1, g_2 \) subalgebras such that \( g = g_1 \oplus g_2 \). If \( W \) is any finite–dimensional \( g_1 \)–module, then the Verma
module is just the induced module \( U(g) \otimes_{U(g_1)} W \). Since \( U(g) = U(g_1) \otimes U(g_2) \) as vector space by the
Poincaré–Birkhoff–Witt theorem, the Verma module is also isomorphic to \( U(g_1) \otimes W \) as a vector space. This
description makes the action of \( g_1 \) clear, but the action of \( g_2 \) is harder to describe. We therefore give a
more down to earth computational recipe specialised to the case where \( W \) is one–dimensional. (The same
arguments apply in general.) Let \( f : g_2 \rightarrow \mathbb{C} \) be a one–dimensional representation of \( g_2 \). Set \( V = U(g_1) \).
We have to make \( V \) into an \( U(g) \)–module or equivalently a \( g \)–module.

Choose a basis \((b_i)\) of \( g_1 \) consisting of monomials. By the Poincaré–Birkhoff–Witt theorem, any element
in \( U(g) \) can be written uniquely as \( \sum b_ia_i \) with \( a_i \in U(g_2) \). Now take \( x \in U(g) \). Then we have
\( xb_i = \sum b_ia_j(x) \) with \( a_j(x) \in U(g_2) \). Let \( A(x) = a_j(x) \) an infinite matrix with entries in \( U(g_2) \) with finitely
many entries in any row. By uniqueness \( A(xy) = A(x)A(y) \) for \( x, y \in U(g) \). In particular if \( X, Y \in g \), we have
\( [A(X),A(Y)] = A([X,Y]) \). Now define \( x \otimes w = \sum b_j \otimes a_{ij}(x)w \). This is a representation because
\( A(xy) = A(x)A(y) \). By definition the Verma module is a cyclic representation of \( g \) generated by a vector \( w \)
such that \( Xw = f(X)w \) for \( X \in g_2 \). Conversely if \( V' \) is any other such cyclic representation there is clearly
a unique homomorphism of \( V \) onto \( V' \) taking the cyclic vector \( w' \) onto the cyclic vector \( w \). The homomorphism
is given by \( b \otimes w \mapsto bw' \).

Let \( g \) be the Lie algebra of a compact semisimple Lie group \( G \). For each \( \alpha > 0 \) let \( E_\alpha, F_\alpha, H_\alpha \) be the
basis of the Lie algebra \( sl(2)_\alpha \) corresponding to the simple root \( \alpha \). Let \( \lambda \geq 0 \) in \( P(g) \) be a generalised
highest weight. Let \( g_2 = h_\mathbb{C} \oplus \bigoplus_{\alpha>0} g_\alpha \) and \( g_1 = \bigoplus_{\alpha<0} g_\alpha \). These are Lie subalgebras of \( g_\mathbb{C} \) with \( g_\mathbb{C} = g_1 \oplus g_2 \). Consider the 1–dimensional representation sending \( E_\alpha \) to 0 and \( H \in h \) to \( i\lambda(H) \). Let \( M(\lambda) \) be the
the corresponding Verma module. Thus if \( v = v_\lambda \) is the highest weight vector of \( M(\lambda) \), we have \( E_\alpha v_\lambda = 0 \) and
\( Hv_\lambda = \lambda(H)v_\lambda \) for all \( \alpha > 0 \). If \( \alpha_1,\ldots,\alpha_k \) is a numbering of the positive roots, then
a basis of \( M(\lambda) \) is given by \( F_\alpha^{n_1} \cdots F_\alpha^{n_k} v_\lambda \). We know that \( M(\lambda) \) has a unique maximal submodule \( N \) such that
\( L(\lambda) = M(\lambda)/N \) is irreducible as a \( g \)–module. In fact, since \( h \) is diagonalisable, every submodule is the
sum of its weight spaces. Hence if we take \( N \) to be the algebraic sum of all proper submodules, we must
have \( v \notin N \), so that \( N \) is the unique maximal proper submodule. By the \( sl(2) \) theory, if \( \ell_i = (\lambda,\alpha_i') \), then
\( w_i = F_\alpha^{l_i}v_\lambda \) is a singular vector i.e. \( F_iw = 0 \) and \( w \) is an eigenvector for \( h \). It therefore generates a proper
submodule (all weights are strictly less than \( \lambda \)). Hence \( w_i \in N \) for all \( i \). Let \( N_0 \) be the submodule generated
by the \( w_i \)'s.

Theorem (Harish–Chandra). \( L(\lambda) \) is the quotient of \( M(\lambda) \) by the submodule generated by \( F_i^{l_i+1}v_\lambda \) and
is finite–dimensional.

Proof. We have to show that \( N = N_0 \) and \( L(\lambda) \) is finite dimensional. Set \( L = M(\lambda)/N_0 \). Thus \( L \) is a cyclic
module for $\mathfrak{g}$ generated by $v = v_\lambda$ satisfying $Xv = \lambda(X)v$ for $X \in \mathfrak{h}$, $E_\alpha v = 0$ and $F_i^{\ell_i+1}v = 0$. The identity

$$[a^n, b] = \sum_{r=1}^{n} \binom{n}{r} [(ad a)^r b] a^{n-r}$$

implies that the action on $L$ is locally nilpotent, i.e. some power of each $E_i$ or $F_i$ kills any vector. For the $E_i$'s this follows because the $E_i$'s lower energy. For the $F_i$'s it follows because $L$ is spanned by vectors $F_{i_1}\cdots F_{i_k}v$ where $i_1, \ldots, i_k$ are arbitrary (recall that the $F_i$'s generate the $F_\alpha$ subalgebra). Starting from the relation $F_i^{\ell_i+1}v = 0$, successive application of ($\ast$) and the Serre relations show that each $F_i$ is nilpotent on any such monomial vector. This local nilpotence shows that any vector in $L$ lies in a finite dimensional $\mathfrak{sl}(2)$ module for each $i$.

We claim that the weights of $L$ are invariant under the Weyl group $W$. In fact suppose $w \in L$ has weight $\mu$, $H_iw = m_iw$ with $m_i = \mu(H_i) = (\mu, \alpha_i^\vee)$. Then $w$ lies in a sum of $sl(2)$ modules. If $m_i \geq 0$, set $u = f_i^{m_i}w$ and if $m_i < 0$, set $u = e_i^{-m_i}$. Thus $u \neq 0$ by the $sl(2)$ theory and $u$ has weight $\lambda - m_i\alpha_i = \sigma_i\lambda$. Thus the set of $u$ is invariant under each simple reflection $\sigma_i$ and hence the whole of $W$.

To see that $L$ is finite–dimensional, we take the unique $\sigma \in W$ such that $\sigma C = -C$ (note that $-C$ is also a Weyl chamber and the Weyl group acts simply transitively on these). By uniqueness $\sigma^2 = 1$. We claim that $\sigma \lambda$ is the lowest weight of $L$. Since there are plainly only finitely many weights $\mu$ such that $\sigma \lambda \leq \mu \leq \lambda$, each of finite multiplicity, this proves that $L$ is finite–dimensional. To prove the claim, note that $\alpha_i \mapsto -\sigma_i\alpha_i$ must be a permutation of the simple roots because the walls of $C$ and $-C$ correspond to the same simple roots. (In particular $\sigma \Phi^+ = -\Phi^+$, so that $\sigma$ takes the positive roots onto the negative roots.) Let $\mu \leq \lambda$ be a weight of $L$. But then $\sigma \mu$ is also a weight, so that $\mu \sigma = \mu - \sum n_i\alpha_i$ with $n_i \geq 0$. Applying $\sigma$, we get $\mu = \sigma \lambda + \sum \eta_i\alpha_i$, where $\sigma \alpha_i = -\alpha_i$. Thus $\mu \geq \sigma \lambda$, showing that $\sigma \lambda$ is the lowest weight.

Irreducibility of $L$ is now a consequence of the following result.

**Casimir lemma.** Let $V$ be a finite–dimensional cyclic representation of $\mathfrak{g}$ generated by a highest weight vector $v$ of weight $\lambda$. Then $V$ is irreducible.

**Proof.** Note that $V$ must be completely reducible for each $\mathfrak{sl}(2)_i$, so that as above the weights are integrable and invariant under the Weyl group. Suppose that $V$ is not irreducible. Then $V$ must contain a singular vector $w$ of weight $\mu$ strictly lower than $\lambda$: thus $e_iw = 0$ and $h_iw = m_iw$ where $m_i = \mu(h_i) \leq \ell_i$. But then $w$ is a highest weight vector generating an irreducible representation of each $\mathfrak{sl}(2)_i$. On the other hand let $\Omega$ be the Casimir operator of $\mathfrak{g}$. Then $\Omega w = (|\lambda + \rho|^2 - |\rho|^2)v$, so by cyclicity $\Omega = (|\lambda + \rho|^2 - |\rho|^2)I$. Since $\Omega w = (|\mu + \rho|^2 - |\rho|^2)v$, we must have $|\lambda + \rho|^2 = |\mu + \rho|^2$. Choose $\sigma \in W$ such that $\sigma(\mu + \rho) \geq 0$. Then $\sigma \mu < \lambda$ and $\sigma \rho < \rho$. But then by Freudenthal’s lemma $\lambda = \mu$, a contradiction. Hence $V$ is irreducible.

Finally we use the Casimir to argue that $V_\lambda = M(\lambda)$ admits an invariant inner product.

**Theorem.** The representation $V_\lambda$ is unitary.

**Proof (Garland).** The representation $V_\lambda$ is irreducible with highest weight $\lambda$. So there is an isomorphism $T : V_\lambda \to V_\lambda^*$, unique up to a scalar by Schur’s lemma. Hence we get an essentially unique $\mathfrak{g}$-invariant sesquilinear form on $V_\lambda$, $\phi(v, w)$. But $\phi'(v, w) = \phi(w, v)$ is another such form, so that $\phi(v, w) = \phi(v, w)$ for some constant $c \in \mathbb{C}$. Clearly $|c|^2 = 1$. Multiplying $\phi$ by $a$ with $\pi/\alpha = c$, we get $\phi(v, w) = \phi(w, v)$ with $\phi$ non–degenerate. We claim that $\pm \phi$ is positive definite. Clearly all the weight spaces are orthogonal. Since the $\lambda$ weight space is 1–dimensional, we must have $\phi(v_\lambda, v_\lambda) 
eq 0$. Since it is real, we may rescale $\phi$ so that $\phi(v_\lambda, v_\lambda) = 1$. To prove that $\phi$ is positive definite, it suffices to show that $\phi(v, v) \geq 0$ for any weight vector. We prove this by downwards induction on the weights under the usual ordering on weights. In fact if $v$ is a weight vector of weight $\mu \leq \lambda$, then

$$\phi(\Omega v, v) = (|\lambda + \rho|^2 - |\rho|^2)\phi(v, v)$$

while

$$\phi(\Omega v, v) = \sum_{\alpha > 0} \phi(T_1v, T_1v) - \sum_{\alpha > 0} \phi(T_\alpha v, v) + \sum_{\alpha > 0} ||\alpha||^2 \phi(E_\alpha v, E_\alpha v) = (|\mu + \rho|^2 - |\rho|^2)||v||^2 + \sum_{\alpha > 0} \phi(E_\alpha v, E_\alpha v).$$
Comparing (1) and (2) we get

\[(|\lambda + \rho|^2 - |\mu + \rho|^2)\phi(v, v) = \sum \|\alpha\|^2 \phi(E_\alpha v, E_\alpha v).\]

Assume \((v, v) = \phi(v, v) > 0\). Since \(\mu < \lambda\), we have \(\|\mu + \rho\|^2 < |\lambda + \rho|^2\). Since the right hand side is non-negative (by induction), we deduce that \(\phi(v, v) \geq 0\). The induction argument shows that \(\phi(v, w)\) is positive semi-definite. Since \(\phi\) is non-degenerate, \(\phi\) must be positive definite, i.e. a complex inner product.

**Proposition.** Any finite dimensional representation \(W\) of \(\mathfrak{g}\) is completely reducible. If \(V\) is a non-trivial irreducible representation \(C \neq 0\) on \(V\).

**Proof.** We may assume \(C\) has only one eigenvalue on \(W\). Note that \(W\) is completely reducible for each \(\mathfrak{sl}(2)\). This implies that \(h\) is diagonalisable and the weights of \(W\) are in \(P(\mathfrak{g})\). Let \(V_1\) be sum of all the irreducible submodules of \(W\). If \(V_1 \neq W\), find an irreducible subspace \(\overline{V}\) of \(V/W_1\). Suppose \(\overline{V} = V/V_1\) of weight \(\mu\). Let \(\pi\) be a highest weight vector in \(\overline{V}\). We claim that \(\pi\) is an irreducible representation with highest weight \(\rho\). The spin representation. Let \(\mathfrak{su}_2\) be the cyclic module generated by \(v\). It must be irreducible. This contradicts the maximality of \(W_1\). Hence \(W\) is completely reducible.

To prove the claim, observe that if \(E_i v \neq 0\), then \(\mu + \alpha_i\) is a weight of \(W_1\), so is less than some highest weight \(\lambda\). Thus \(\lambda \geq \mu \geq 0 \geq \|\lambda + \rho\|^2 = \|\mu + \rho\|^2\). By Freudenthal’s lemma, \(\lambda = \mu\), a contradiction. The last assertion is obvious from the formula \(\Omega_{V_\lambda} = (\lambda + 2\rho, \lambda)I\).

**15. PROJECTIVE REPRESENTATIONS AND COVERING GROUPS.** For each weight \(\lambda \in P(\mathfrak{g})\) with \(\lambda \geq 0\), we have constructed a representation of \(\mathfrak{g}\) in \(\text{End}(V_\lambda)\) with \(\pi(X)^* = -\pi(X)\). Since \([\mathfrak{g}, \mathfrak{g}] = 0\), each \(\pi(X)\) has trace zero, i.e. \(\pi(X) \in su(V_\lambda)\). Now consider the representation \(\pi_g(X) = \pi(\text{Ad}(g) \cdot X)\). This is also an irreducible representation with highest weight \(\lambda\), so there exists \(U_g \in SU(V_\lambda)\) unique up to an element of \(Z_\lambda\), the finite centre of \(SU(V_\lambda)\), such that \(\pi(gXg^{-1}) = U_g \pi(X) U_g^{-1}\). Let \(G_1 = \{ (u, g) : \pi(gXg^{-1}) = u\pi(X)u^{-1}\}\), a closed subgroup of the compact matrix group \(SU(V_\lambda) \times G\), so a matrix group itself. Let \(\tilde{G}_\lambda = G_1^{\mathfrak{g}}\), the connected component of the identity in \(G_1\). This is also a matrix group. There is a natural homomorphism \(f : G_1 \to G\) with kernel \(G_1 \cap Z\). Since \(G\) is connected and \(f \exp(\mathfrak{g}) = \exp(\mathfrak{g})\), we see that \(f(\tilde{G}_\lambda) = f(G_1^{\mathfrak{g}}) = G\). Thus we have an exact sequence \(1 \to C_\lambda \to \tilde{G}_\lambda \to G \to 1\) with \(C_\lambda = Z_\lambda \cap \tilde{G}_\lambda\), a cyclic central subgroup. The Lie algebras of \(\tilde{G}_\lambda\) and \(G\) can naturally be identified. Moreover \(P(G_\lambda) = \langle P(G), \lambda \rangle\).

In general if \(\pi\) is an irreducible projective unitary representation of \(G\), the same argument shows that we can find \(\tilde{G}\) a connected compact matrix group with a covering homomorphism \(f : \tilde{G} \to G\) and a representation \(\tilde{\pi}\) such that \(\tilde{\pi} = \pi \circ f\) as projective representations.

Let \(\lambda_i\) be a choice of cosets for \(P(\mathfrak{g})/P(G)\) and let \(G_i\) be the central extension (by \(C_\lambda\)) constructed above with projection maps \(f_i : G_i \to G\). Let \(\tilde{G}\) be the connected component of the identity of the closed subgroup of elements \((x_i) \in \prod G_i\) such that \(f_1(x_i) = f_2(x_i)\). Then there is a natural projection \(f : \tilde{G} \to G\) with kernel \(Z = \prod Z_i \cap \tilde{G}\), finite Abelian. By construction \(\lambda_i \in P(\tilde{G})\), so that \(P(\tilde{G}) = P(\mathfrak{g})\). Hence \(Z(\tilde{G}) = P(\tilde{G})/Q\). The group \(\tilde{G}\) has the property that every generalised weight is now a weight.

**Remark.** Weyl’s theorem implies that \(\tilde{G}\) is simply connected.

**16. THE DIRAC OPERATOR AND SUPERSYMMETRY RELATIONS.** Our aim now is to prove Weyl’s character formula for the character of an irreducible representation with highest weight \(\lambda\). The proof is based on supersymmetry and the coset construction of Goddard–Kent–Olive. The supercharge or Dirac operator and its properties lie at the heart of the method which is manifestly unitary. Later we will explain why it also gives a geometric construction of all irreducible representations as twisted harmonic spinors. The Dirac operator or supercharge operator also exists for affine algebras, as shown by Kazama and Suzuki. The same technique can therefore be used for to prove the Weyl–Kac character formula.

**The spin representation.** Let \(V\) be a real inner product space with orthonormal basis \((e_i)\). For \(T \in so(V)\), define \(s(T) = \frac{1}{4} \sum c(T \cdot e_i)c(e_i)\). Recall that \([s(T), c(X)] = c(TX)\).

Let \(G\) be a compact simple matrix group with Lie algebra \(\mathfrak{g}\). Let \(T\) be a maximal torus in \(G\) with Lie algebra \(\mathfrak{h}\). We assume that the invariant inner product on \(\mathfrak{g}\) has been normalised so that the highest root
\( \theta \in \mathfrak{b}^* \) has \( \|\theta\|^2 = 2 \) in the induced norm. Let \( \mathfrak{m} = \mathfrak{h}^* \subset \mathfrak{g} \). Thus \( \mathfrak{m} \) is even dimensional and invariant under \( \text{Ad}(T) \) and \( \text{ad}(h) \). Since it is an inner product space, we may consider the real Clifford algebra \( \text{Cliff} \mathfrak{m} \). It has a unique irreducible representation \( W_\mathfrak{m} \) which is \( \mathbb{Z}_2 \)-graded: \( W_\mathfrak{m} = W_\mathfrak{m}^+ \oplus W_\mathfrak{m}^- \). Let \( V \) be an irreducible representation of \( G \). We define the supercharge or Dirac operator on \( V \otimes W_\mathfrak{m} \) by

\[
Q = \sum (\pi(X_i) + \frac{1}{3}s(X_i))c(X_i).
\]

By definition \( Q \) takes \( V \otimes W_\mathfrak{m}^+ \) into \( V \otimes W_\mathfrak{m}^- \) and commutes with \( T \). As we shall explain below, this operator is really the Dirac operator for a very special connection restricted to an isotypic subspace. However to understand why we take this particular formula we will have to take a supersymmetric path rather than a geometric one.

**A. Computations for** \( \text{Cliff}(\mathfrak{g}) \). Take generators \( c(X) \ (X \in \mathfrak{g}) \) for \( \text{Cliff}(\mathfrak{g}) \). Thus \( \{c(X), c(Y)\} = 2(X, Y) \).

Let \( Q_0 = \sum s(X_i)c(X_i) \) so that \( s(X) = \frac{1}{4} \sum c([X, X_i])c(X_i) \). Evidently \( Q_0 \) is independent of the choice of orthonormal basis, so that \( s(g)Q_0s(g)^{-1} = Q \) since \( s(g)X_is(g)^{-1} \) is another orthonormal basis of \( \mathfrak{g} \). Differentiating this relation, we get

\[
[Q_0, s(X)] = 0.
\]

This is one of the supersymmetry relations; we get the other as follows:

\[
\{Q_0, c(X)\} = \sum s(X_i){c(X_i), c(X)} - [s(X_i), c(X)]c(X_i) = \sum s(X_i)2(X_i, X) + c([X, X_i])c(X_i) = 6s(X).
\]

Thus

\[
\{Q_0, c(X)\} = 6s(X). \tag{2}
\]

**B. Computations in** \( \text{End}(V) \otimes \text{Cliff}(\mathfrak{g}) \). Let \( V \) any unitary \( G \)-module and set

\[
\bar{Q} = \sum (\pi(X_i) + \frac{1}{3}s(X_i))c(X_i).
\]

Then (1) implies that \( [\bar{Q}, \pi(X) + s(X)] = 0 \), while (2) implies

\[
\{\bar{Q}, c(X)\} = \frac{1}{3}Q, c(X) + \sum \pi(X_i){c(X_i), c(X)} = 2(\pi(X) + s(X)).
\]

**C. Coset construction of the Dirac operator.** If \( (X_i) \) is an orthonormal basis of \( \mathfrak{m} \) and \( U \) is an irreducible representation of \( G \), then the Dirac (or supercharge) operator on \( U \otimes W_\mathfrak{m} \) is given by

\[
Q = \sum c(X_i)(\pi(X_i) + \frac{1}{3}s(X_i)).
\]

We can consider the same operator acting on \( U \otimes W_\mathfrak{m} \otimes W_{\mathfrak{h}} = U \otimes W_{\mathfrak{g}} \) and denote it by the same symbol (strictly speaking it should be \( Q \otimes I \)). The supercharge operator \( Q_{\mathfrak{g}} \) also acts on this space as does the supercharge operator for \( \mathfrak{h} \) acting on \( U' \otimes W_{\mathfrak{h}'} \) where \( U' = U \otimes W_{\mathfrak{h}'} \). These operators are given by \( Q_{\mathfrak{g}} = \sum a(\pi(X_a) + \frac{1}{3}s(X_a))c_a \) and \( Q_{\mathfrak{h}} = \sum (\pi(X_A) + s_{\mathfrak{m}}(X_A))c(X_A) \) since \( s_{\mathfrak{h}}(X) = 0 \) for \( X \in \mathfrak{h} \).

**Theorem (coset construction).** \( Q = Q_{\mathfrak{g}} - Q_{\mathfrak{h}} \).

**Proof.** Let \( (X_a) \) be an orthonormal basis of \( \mathfrak{g} \), made up of an orthonormal basis \( (X_i) \) of \( \mathfrak{m} \) and \( (X_A) \) of \( \mathfrak{h} \). Thus we have structure constants \( f_{abc} \) given by \( [X_a, X_b] \propto \sum f_{abc}X_c \). The invariance of the inner product and the orthonormality of \( (X_a) \) imply that \( f_{abc} \) is totally antisymmetric in its three arguments. In particular we have \( s(X_a) = \frac{1}{4} \sum f_{abc}c(X_a)c(X_b) \). Thus

\[
Q_{\mathfrak{g}} - Q_{\mathfrak{h}} = \sum (\pi(X_a) + \frac{1}{3}s_{\mathfrak{g}}(X_a))c(X_a) - \sum (\pi(X_A) + s_{\mathfrak{m}}(X_A))c(X_A)
\]

\[
= \sum \pi(X_a)c(X_a) + \frac{1}{12} \sum_{a,b,c} c(X_a)c(X_b)c(X_c) - \sum \pi(X_A)c(X_A) - \frac{1}{4} \sum_{A,i,j} f_{Aij}c(X_A)c(X_i)c(j)
\]

\[
= \sum \pi(X_i)c(X_i) + \frac{1}{12} \sum_{i,j,k} c(X_i)c(X_j)c(X_k)
\]

\[
= \sum (\pi(X_i) + \frac{1}{3}s_{\mathfrak{m}}(X_i))c(X_i),
\]

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since there are three ways that terms $c(X_A)c(X_i)c(X_j)$ can appear and $f_{abc} = 0$ if two or more coefficients corresponds to basis elements in $\mathfrak{h}$.

17. THE SQUARE OF THE DIRAC OPERATOR.

A. Lichnerowicz’s lemma. Let $c_i = c(X_i)$, so that $\{c_i, c_j\} = 2\delta_{ij}$, and let $R_{ijk\ell} = ([X_i, X_j], [X_k, X_\ell])$.

Apart from the obvious symmetry properties

$$R_{ijk\ell} = R_{kij\ell} = -R_{jik\ell} = -R_{ij\ell k},$$

the tensor $R_{ijk\ell}$ satisfies Bianchi’s first identity, namely

$$R_{ijk\ell} + R_{ik\ell j} + R_{ij\ell k} = 0.$$ 

This follows immediately from the Jacobi identity, since $R_{ijk\ell} = (X_i, [X_j, [X_k, X_\ell]])$.

**Lemma (Lichnerowicz).** $\sum_{i,j,k,\ell} R_{ijk\ell} c_i c_j c_k c_\ell = -2\sum_{i,j} R_{iijj}.$

**Proof.** By Bianchi’s identity we have

$$\sum_{i,j,k,\ell} R_{ijk\ell} c_i c_j c_k c_\ell = -\sum_{i,j,k,\ell} R_{ijk\ell} c_i (c_k c_j c_\ell + c_\ell c_j c_k) = 2\sum_{i,j,k,\ell} R_{ijk\ell} c_i (\delta_{jk} c_\ell + \delta_{k\ell} c_j - 2\delta_{j\ell} c_k - c_j c_\ell).$$

The symmetry properties allow one to delete the $\delta_{i\ell}$ term and, upon rearrangement, we get

$$\sum_{i,j,k,\ell} R_{ijk\ell} c_i c_j c_k c_\ell = \frac{2}{3} \sum_{i,j,k,\ell} R_{ijk\ell} (\delta_{jk} c_\ell - 2\delta_{j\ell} c_k)$$

$$= 2\sum_{ij\ell} R_{iijj\ell} c_i c_\ell = 2\sum_{ij\ell} R_{ijij\ell} (c_i c_\ell + c_\ell c_i)/2 = 2\sum_{ij} R_{ijji}$$

as required.

B. Computations in $\text{Cliff}(g)$. If $(X_i)$ is an orthonormal basis of $g$, let $Q = \sum s(X_i)c(X_i)$ in $\text{Cliff}(g)$.

**Lemma 1.** $Q_0^2$ is a scalar operator.

**Proof.** The supersymmetry relations $\{Q_0, c(X)\} = 6s(X)$ and $[Q_0, s(X)] = 0$ imply

$$[Q_0^2, c(X)] = \{Q_0, c(X)\}Q_0 - Q_0\{Q_0, c(X)\} = 6s(X)Q_0 = 0.$$

So $Q_0^2$ is central in $\text{Cliff}(g)$. Since $Q_0^n$ lies in $\text{Cliff}^+(g)$, it must be a scalar operator.

**Corollary.** $Q_0^2 = 3\sum s(X_i)^2$, so $\sum s(X_i)^2$ is a scalar.

**Proof.** We have

$$\{Q_0, Q_0\} = \{Q_0, \sum s(X_i)c(X_i)\} = \sum [Q_0, s(X_i)]c(X_i) + 3\sum s(X_i)^2 = 6\sum s(X_i)^2,$$

so $\sum s(X_i)^2$ is a scalar, as claimed.

**Lemma 2.** $\sum s(X_i)^2 = -\frac{3}{8} \sum \|[X_i, X_j]\|^2 = -\frac{3}{8} \dim(g) \cdot h_g$, where the dual Coxeter number $h_g$ equals the value of the Casimir in the adjoint representation.

**Remark.** Thus $h_g = -\sum \text{Tr}(\text{ad}(X_i)^2) = \|\theta + \rho\|^2 - \|\rho\|^2$, where $\theta$ is the highest root, i.e the highest weight in the adjoint representation.
Proof. We have

\[ s(X_a) = \frac{1}{4} \sum ([X_a, X_j], X_i) c_i c_j = -\frac{1}{4} \sum (X_a, [X_i, X_j]) c_i c_j, \]

so that

\[ \sum s(X_i)^2 = \frac{1}{16} \sum ([X_i, X_j], [X_k, X_l]) c_j c_k c_l \]

\[ = -\frac{1}{8} \sum \| [X_i, X_j] \|^2 \]

\[ = \frac{1}{8} \sum ([X_i, X_j], [X_i, X_j]) \]

\[ = \frac{1}{8} \sum (\text{ad}(X_i)^2 \cdot X_j, X_j) \]

\[ = \frac{1}{8} \text{Tr}(\sum \text{ad}(X_i)^2) \]

\[ = \frac{1}{8} \text{Tr}(-h_g l) \]

\[ = \frac{1}{8} \dim(g) \cdot h_g. \]

Corollary. \( Q_0^2 = -\frac{1}{2} h_g \cdot \dim(g). \)

Proof. We have \( Q_0^2 = 3 \sum s(X_i)^2 = -\frac{3}{8} h_g \cdot \dim(g). \)

Lemma 3. The weights of \( W_m^\pm \) as a representation of \( h \) are exactly \( \rho - \sum_{\alpha \in S} \alpha \), where \( S \) is an arbitrary subset of positive roots with \( |S| \) even for \( W_m^+ \) and odd for \( W_m^- \). Thus \( \text{ch}_s(W_m) = \text{ch}(W_m^+) - r \text{mch}(W_m^-) = e^\theta \prod_{\alpha > 0} (1 - e^{-\alpha \theta}). \)

Proof. This follows immediately from the formulas for the weights of the spin representations. Recall that every matrix \( g \in \text{SO}(N) \) is conjugate to a matrix with \( 2 \times 2 \) blocks \( \begin{pmatrix} \cos \theta_j & \sin \theta_j \\ -\sin \theta_j & \cos \theta_j \end{pmatrix} \) down the diagonal where \( j = 1, \ldots, [N/2] \). Note that there is an additional 1 on the diagonal if \( N \) is odd. Thus the complex eigenvalues of \( g \) are \( e^{\pm i\theta_j} \). In any irreducible projective representation \( \pi(g) \) of \( \text{SO}(N) \), the eigenvalues of a generic block diagonal element \( \pi(g) \) are called the weights of the representation.

Lemma 3' (weights of spin representation). (1) If \( \dim(V) \) is even, the weights of the spin representation \( W_m^\pm \) are \( \exp i \sum \pm \theta_k \) where the number of plus signs is even for \( W^+ \) and odd for \( W^- \).

(2) If \( \dim(V) \) is odd, the weights of the spin representation \( W^\pm \) are \( \exp \frac{1}{2} \sum \pm i \theta_k \).

Proof. Let \( v_1, \ldots, v_m \) be an orthonormal basis for \( V \) and set \( v_I = v_i_1 \wedge \cdots \wedge v_i_k \) for \( i_1 < \cdots < i_k \). Then \( c(v_k) = e(v_k) + e(v_k)^* \) and \( c(iv_j) = i(e(v_j) - e(v_j)^*) \). The generators of Lie algebra of the maximal torus are given by \( T_j = \frac{i}{4} (c(v_j)c(iv_j) - c(iv_j)c(v_j)) = \frac{i}{4} c(v_j)c(iv_j) \). Thus \( T_j v_I = i/2 v_I \) if \( j \in I \) and \( T_j v_I = -i/2 v_I \) if \( j \notin I \). The corresponding self–adjoint operators \( S_j \) satisfy \( T_j = i S_j \) so that \( S_j v_I = \pm \frac{i}{2} v_I \). Since \( W^+ = \Lambda_{\text{even}}(V) \) is spanned by \( v_I \)'s with \( I \) even and \( W^- = \Lambda_{\text{odd}}(V) \) is spanned by \( v_I \)'s with \( I \) odd, the result follows.

(2) Note that the maximal torus of \( \text{SO}(V_0) \) coincides with the maximal torus of \( \text{SO}(V) \). On the other hand the spin representation of \( \text{SO}(V) \) equals \( W^+ \oplus W^- \), where \( W^\pm \) are the spin representations of \( \text{SO}(V_0) \). So the result follows immediately from (1).

Corollary of Lemma 3. As a representation of \( g \), the irreducible representation \( W \) of \( \text{Cliff}(g) \) contains a highest weight vector of weight \( \rho \); in fact it contains exactly \( N = 2^M \) such vectors where \( M = [m/2] \) with \( m = \dim(h) \).

Proof. We have \( W_h = W_m \otimes W_0 \) with \( h \) acting trivially \( W_0 \) and with highest weight \( \rho \) on \( W_m \); for as we have seen the weights of \( W_m \) are exactly \( \rho - \sum_{\alpha \in S} \alpha \), where \( S \) is a subset of the positive roots. Clearly this means that there are exactly \( N = \dim(W_h) \) vectors of weight \( \rho \) in \( W \).
Lemma 4. \[ \sum \]

Proof. \[ \lambda \] and \[ \nu, \rho \parallel \]

Clearly it suffices to show that all highest weight vectors in \[ \pi \]

proof. \[ \parallel \]

We have already seen that the Casimir of \[ g \] has value \[ 3 \parallel \rho \parallel^2 \] on \[ W \], so any other highest weight \( \lambda = \rho - \sum \alpha \in S \alpha = \rho - \nu \) must satisfy \[ \parallel \lambda + \rho \parallel^2 = 4 \parallel \rho \parallel^2 \]. But

with \( (\nu, \rho) \geq 0 \) and \( (\lambda, \nu) \geq 0 \). Thus \[ \parallel \lambda + \rho \parallel^2 = 4 \parallel \rho \parallel^2 \] only if \( (\nu, \rho) = 0 \) and \( (\lambda, \nu) = 0 \), so that \( \nu = 0 \) and \( \lambda = 0 \).

Lemma 4. \[ \sum s(X_i)^2 = 3 \parallel \rho \parallel^2. \]

Proof. Let \[ W \] be an irreducible representation of \[ \text{Cliff}(g) \]. Now \[ W = W_{\mathfrak{m}} \otimes W_{\mathfrak{h}} \] and \( \mathfrak{h} \) acts trivially on \[ W_{\mathfrak{h}} \]. On the other hand \[ W_{\mathfrak{m}} = \Lambda^*(m_c) \], so that \( \rho \) is a highest weight of \( W \) as a \( g \)-module. In fact although \( \mathfrak{h} \) is canonically quantised, \( \pi(T) = \frac{1}{2} \sum c(T, X_i) c(X_i) \) does not agree with the canonical quantisation. A highest weight vector is given by \( v = \bigwedge_{a > 0} X_a \) and \( TX_a = \alpha(T)X_a \), so that \( \pi(T)v = 2\rho(T)v \). On the other hand \( \pi(T)\Omega = 0 \), so that \( \Omega \) is a lowest weight vector. Since the canonical quantisation has the form \( \pi(T) + \lambda(T) \) and the highest and lowest weight of \( \pi \) have the form \( \pm \mu(T) \) for some \( \mu > 0 \), we must have \( \lambda(T) = \mu(T) = \rho(T) \). The Casimir \( \sum s(X_i)^2 \) therefore acts on the corresponding submodule as \( 3 \parallel \rho \parallel^2 \). Since it acts as a scalar on \( W \), the lemma follows.

Corollary (Freudenthal–de Vries “strange formula”). \[ \parallel \rho \parallel^2 = h_g \cdot \dim g/12. \]

Proof. Immediate from lemmas 2 and 3.

C. Computations in \( \text{End}(V) \otimes \text{Cliff}(g) \). Let \( \pi : g \to \text{so}(V) \) be a representation of \( g \) and set \( Q_g = \sum \pi(X_i) + \frac{1}{3} s(X_i) c(X_i) \).

Theorem. \[ Q_g^2 = \sum \pi(X_i)^2 + \frac{1}{3} \sum s(X_i)^2. \]

Proof. Let \[ Q_0 = \sum s(X_i) c(X_i) \]. Thus \[ \{ Q_0, s(X) \} = 0 \] and \( \{ Q_0, c(X) \} = 6 s(X) \). Moreover \( \{ Q_g, \pi(X) + s(X) \} = 2 \pi(X) + +2 s(X) \). Hence

\[ \{ Q_0, Q_0 \} = \{ Q_g, \sum (\pi(X_i) + s(X_i)) c(X_i) + (\frac{1}{3} - 1) Q_0 \}. \]

The first term gives

\[ \{ Q_g, \sum (\pi(X_i) + s(X_i)) c(X_i) \} = \sum (\pi(X_i) + s(X_i))(2 \pi(X_i) + \frac{2}{3} s(X_i)) \]

while the second term gives

\[ \{ Q_g, Q_0 \} = \sum \pi(X_i) \{ Q_0, c(X_i) \} + \varepsilon \{ Q_0, Q_0 \} = 6 \sum \pi(X_i) s(X_i) + 2 \sum s(X_i)^2. \]

The result follows by substituting in (1) from (2) and (3).

D. Coset computation of \( Q^2 \). By the coset construction \( Q = Q_g - Q_{\mathfrak{h}} \). Thus \( Q_g = Q + Q_{\mathfrak{h}} \), where \( Q \) and \( Q_{\mathfrak{h}} \) anticommute, i.e. \( \{ Q, Q_{\mathfrak{h}} \} = 0 \). Hence

\[ \{ Q_g, Q_g \} = \{ Q + Q_{\mathfrak{h}}, Q + Q_{\mathfrak{h}} \} = \{ Q, Q \} + \{ Q_{\mathfrak{h}}, Q_{\mathfrak{h}} \}. \]
Thus \( Q^2_Q = Q^2 + Q^2_H \), so that \( Q^2 = Q^2_Q - Q^2_H \).

18. WEYL’S CHARACTER AND DENOMINATOR FORMULAS.

**Lemma.** If \( \sigma \in W \), then \( \varepsilon(\sigma) = \det(\sigma) = (-1)^{n(\sigma)} \) where \( n(\sigma) = |\{\alpha > 0 : \sigma\alpha < 0\}| \).

**Proof.** Since \( G \) is connected, \( \text{Ad}(G) \subset SO(g) \). Hence \( N(T) \subset SO(g) \). But \( N(T) \) normalises \( \mathfrak{h} \) and therefore acts on \( m \). If \( g \in N(T) \) it follows that \( \det(m)(g) = \det(\mathfrak{h})(g) \). Since \( \det(m)(t) = \det(\mathfrak{h})(t) = 1 \), we have \( \det(m)(t) = 1 \) for \( t \in T \). Thus if \( \sigma \in W = N(T)/T \), \( \det(m)\sigma \) is well-defined and equals \( \det(\mathfrak{h})(\sigma) \). Let \( X_\alpha, Y_\alpha \) be a basis for \( m \). Clearly \( X_\alpha \wedge Y_\alpha \) is well-defined (independently of the choice of representative in \( N(T) \)). If \( \sigma > 0 \), it equals \( X_\alpha \wedge Y_{\sigma \alpha} \), while if \( \sigma < 0 \) it equals \( -X_{-\sigma} \wedge Y_{-\sigma} \). Thus \( \det(\mathfrak{h})(\sigma) = (-1)^N \) where \( N \) is the number of \( \alpha \) such that \( \sigma\alpha < 0 \), as required.

**Corollary.** \( \pi(\sigma) \in \text{End}(W_m) \) is even or odd according as \( \varepsilon(\sigma) = 1 \) or \(-1\). If \( \xi \in W_\mu \) has weight \( \mu \), then \( \pi(\sigma)\xi \) has weight \( \sigma\mu \).

**Proof.** We already know that if \( T \in O(m) \), then \( \pi(T) \) is even or odd according to the sign of \( \det(T) \), so the first result follows. Thus \( \pi(\sigma)\pi(t)\pi(\sigma)^{-1} = \pm \pi(\sigma)^{-1} \) where we regard \( \pi \) as a representation of \( T \) (recall that \( \rho \) is a weight). Since \( T \) is connected, only the plus sign is possible, so the result follows.

**Lemma (Euler–Poincaré Principle).** Let \( W = W^+ \oplus W^- \) be a vector space \( \mathbb{Z}_2 \)-graded and let \( A \) and \( B \) be even and odd commuting operators on \( W \) with \( B \) diagonalisable. Then \( \text{Tr}_s A = \text{Tr}_s A|_{\ker(B)} \).

**Proof.** Since \( B \) is diagonalisable, so is \( B^2 \). Moreover \( \ker(B) = \ker(B^2) \). Let \( W^\pm = \{ \xi \in W^\pm : B^2\xi = \lambda\xi \} \). If \( \lambda \neq 0 \), then \( B \) gives an isomorphism between \( W^\pm \) and \( W^- \). Since \( A \) commutes with \( B \), \( A \) leaves \( W^\pm \) and the isomorphism given by \( B \) intertwines the two actions of \( A \). Hence \( \text{Tr}_s W^+(A) = \text{Tr}_s W^-(A) \) for \( \lambda \neq 0 \). Hence
\[
\text{Tr}_s A = \text{Tr}_s W^+(A) - \text{Tr}_s W^-(A) = \sum_{\lambda} \text{Tr}_s W^+(A) - \text{Tr}_s W^-(A) = \text{Tr}_s W^+(A) - \text{Tr}_s W^-(A) = \text{Tr}_s A|_{\ker(B)},
\]
as required.

**Theorem (Weyl’s character formula).** \( \text{ch}(V_\lambda) = \Pi^{-1} \sum_{\sigma \in W} \varepsilon(\sigma) e^{\sigma(\lambda+\rho)-\rho}, \) where \( \Pi = \prod_{\alpha>0}(1 - e^{-\alpha}) \).

**Proof.** Consider the \( \mathfrak{h} \) or \( T \) module \( V_\lambda \otimes W_m \). Evidently
\[
\text{ch}_s V_\lambda \otimes W_m = \text{ch}(V) \cdot \text{ch}_s(W_m) = \text{ch}(V_\lambda) \cdot e^{\rho} \prod_{\alpha>0}(1 - e^{-\alpha}).
\]

On the other hand, by the Euler–Poincaré principle,
\[
\text{ch}_s V_\lambda \otimes W_m = \text{ch}_s \ker(Q) \cap (V_\lambda \otimes W_m).
\]

Since \( Q \) commutes with \( \mathfrak{h} \), we can assume \( \xi \in \ker(Q) \cap (V_\lambda \otimes W_m) \) is an \( \mathfrak{h} \)-eigenvector of weight \( \mu + \nu \) say, where \( \mu < \lambda \) and \( \nu < \rho \). Since \( Q \) is skew–adjoint, \( \ker(Q) = \ker(Q) \). But \( Q^2\xi = (||\lambda + \rho||^2 - ||\mu + \nu||^2)\xi \). Therefore we must have \( \mu = \sigma\lambda \) and \( \nu = \sigma\rho \) for some unique \( \sigma \in W \). Note that if \( g \in N(T) \) corresponds to \( \sigma \in W \), then \( g\sigma(\xi) = \sigma(\xi) \). We claim, up to scalar multiples this is the only vector with this weight. In fact suppose that \( \mu + \nu = \mu + \nu \) with \( \lambda \geq \mu_1 \) and \( \rho \geq \nu_1 \). Since \( ||\mu + \nu||^2 = ||\mu + \nu||^2 = ||\lambda + \rho||^2 \), the previous argument implies that \( \mu_1 = \tau\lambda \) and \( \nu_1 = \tau\rho \) for some \( \tau \in W \). But then \( \gamma = \tau^{-1}\sigma \) fixes \( \lambda + \rho \). Since \( \lambda \geq \gamma \lambda \) and \( \rho \geq \gamma \rho \), we get \( \gamma \rho = \rho \) so that \( = \text{id} \). Thus \( \mu = \mu_1 \) and \( \nu = \nu_1 \), so that the vector must lie in the tensor product of the \( \mu \)-weight space of \( V_\lambda \) and the \( \nu \)-weight space of \( W_m \). But each of these weight spaces is obtained by applying \( \sigma \) to \( \lambda \) and \( \rho \) weight spaces; they therefore have multiplicity one. Thus the kernel of \( Q \) (or equivalently \( Q^2 \)) is indexed by elements of \( \sigma \) and has a basis consisting of vectors \( g_\sigma(\nu_1) \otimes \pi(g_\sigma)v_\rho \). The vector \( \pi(g_\sigma)v_\rho \) lies in \( W_m^+ \) according as \( \varepsilon(\sigma) = \pm 1 \). Thus
\[
\text{ch}_s(\ker(Q) \cap (V_\lambda \otimes W_m)) = \sum_{\sigma \in W} \varepsilon(\sigma) e^{\sigma(\lambda+\rho)}.
\]
The character formula follows from (1), (2) and (3).

**Corollary (Weyl’s denominator formula).** \( \sum_{\sigma \in W} \varepsilon(\sigma)e^{\sigma \rho - \rho} = \prod_{\alpha > 0}(1 - e^{-\alpha}). \)

**Proof.** This follows by setting \( \lambda = 0 \), since the character of the trivial representation is identically 1.

**Remark.** Using the denominator formula, we can write \( \text{ch}(V_\lambda) = A(\lambda + \rho)/A(\rho) \) where \( A(\mu) = \sum \varepsilon(\sigma)e^{\sigma \mu} \).

**Corollary (Weyl’s dimension formula).** \( \dim(V_\lambda) = \prod_{\alpha > 0}(\lambda + \rho, \alpha)/\prod_{\alpha > 0}(\rho, \alpha). \)

**Proof.** Let \( X, Y \in \mathfrak{h} \) be the elements such that \( \mu(X) = (\mu, \rho) \) and \( \mu(Y) = (\mu, \rho + \lambda) \) for \( \mu \in \mathfrak{h}^* \). Then by Weyl’s denominator formula

\[
\sum_{\sigma \in W} \varepsilon(\sigma)e^{i(\rho + \lambda)(X_t)} = \sum_{\sigma \in W} \varepsilon(\sigma)e^{i\rho(Y_t)} = \prod_{\alpha > 0} (e^{i\alpha(X)t} - e^{-i\alpha(Y)t}),
\]

and

\[
\sum_{\sigma \in W} \varepsilon(\sigma)e^{i\rho(X_t)} = \prod_{\alpha > 0} (e^{i\alpha(X)t} - e^{-i\alpha(X)t}).
\]

Dividing these we get

\[
\text{Tr}_{V_\lambda}(e^{X_t}) = \prod_{\alpha > 0} \frac{\sin(\lambda + \rho, \alpha)t}{\sin(\rho, \alpha)t}.
\]

The result follows by letting \( t \to 0 \).

**19. REMARKS ON CONNECTIONS AND DIRAC OPERATORS.** Let \( G \) be a with simple compact Lie group with Lie algebra \( \mathfrak{g} \) with invariant inner product \( (X, Y) \). Let \( H \) be a closed subgroup with Lie algebra \( \mathfrak{h} \) and \( \mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m} \). Set \( M = G/H \). Then \( \text{Vect}(M) = (C^\infty(G) \otimes \mathfrak{m})^H \), so that \( X(gh) = \text{ad}(h)^{-1}X(g) \) for \( X \in \text{Vect}(M) \). If \( P = P_\mathfrak{m} \) is the orthogonal projection onto \( \mathfrak{m} \) and \( Y \in \mathfrak{m} \), then \( \bar{Y}(g) = P(g^{-1}Yg) \) defines a vector field with \( \bar{Y}(1) = Y \); translating on the left, we can produce a vector field equal to \( Y \) at a given point. If \( X(g) \) is a vector field and \( f \in C^\infty(M) = C^\infty(G)^H \), we define

\[
Xf(g) = \frac{d}{dt} f(g \exp(X(g)t))|_{t=0}.
\]

It is immediate from the definitions that \( Xf(gh) = Xf(g) \), so that \( Xf \in C^\infty(M) \). The canonical connexion is defined by

\[
\nabla_X \xi(g) = \frac{d}{dt} \xi(g \exp(X(g)t))|_{t=0},
\]

for \( \xi \in \text{Vect}(M) = (C^\infty(G) \otimes \mathfrak{m})^H \). Clearly \( \nabla_X (f \xi) = (Xf) \xi + f\nabla_X \xi \). Any other \( G \)-invariant connexion is given by

\[
\nabla_X \xi = \nabla_X \xi + \text{id} \otimes \alpha_X \xi,
\]

where \( \alpha : \mathfrak{m} \to \text{End}(\mathfrak{m}) \) is an \( H \)-invariant linear mapping; note that if \( X \in (C^\infty(G) \otimes \mathfrak{m})^H \), then \( \text{id} \otimes \alpha(X) \in (C^\infty(G) \otimes \text{End}(\mathfrak{m}))^H \). In our case \( \mathfrak{m} \) has an Ad \( H \)-invariant inner product \( (X, Y) \). This induces a \( G \)-invariant hermitian structure on tangent vectors, \( (X, Y)(g) = (X(g), Y(g)) \in C^\infty(G/H) \). A connexion \( \Delta_X \) is a metric connexion (or compatible with the metric) \( X(\xi, \eta) = (\nabla_X \xi, \eta) + (\xi, \nabla_X \eta) \). It is obvious that \( \nabla_X \) is compatible with the metric; and \( \Delta_X + \alpha_X \) is compatible with the metric iff \( \alpha(m) \subset \text{so}(\mathfrak{m}) \). We call \( \alpha \) the connexion 1–form. We shall take \( \alpha_X(Y) = -\varepsilon[X, Y]|_{\mathfrak{m}} = \varepsilon P_{\mathfrak{m}}[X, Y] \) with \( \varepsilon \in \mathbb{R} \).

Let \( W \) be an irreducible Cliff-\( \mathfrak{m} \)-module, \( \mathbb{Z}_2 \)-graded if \( \text{dim} \) is even. Thus \( W \) is a complex inner product space. Consider \( (C^\infty(G) \otimes W \otimes V)^H \), where \( V \) is any unitary \( H \)-module. Given \( X \in (C^\infty(G) \otimes \mathfrak{m})^H \), a tangent vector, and \( \xi \in (C^\infty(G) \otimes W \otimes V)^H \), define

\[
c(X)\xi = c(X(g)) \otimes \text{id}x(\xi(g)).
\]

Then clearly \( c(X)^* = c(X), c(X)c(Y) + c(Y)c(X) = 2(X, Y)I \) and \( [\nabla_X, c(Y)] = c(\nabla_X Y) \), where the spin connection \( \nabla_X \) is given by \( \nabla_X + s(\alpha_X) \). (Recall that if \( T \in \text{so}(V) \), \( s(T) = \frac{1}{2} \sum c(T \cdot v_i)c(v_i) \), where \( \{v_i\} \) is an orthonormal basis of the real inner product space \( V \).)
The twisted Dirac operator $D_V$ is defined by $D_V = \sum c(X_i)\nabla_{X_i}$, where $(X_i)$ is locally an orthonormal basis of vector fields near $x$. $D_V$ is clearly independent of the local choice of orthonormal basis, so globally defined. By definition $D_V$ is $G$–invariant. We want to find a simpler expression for $D_V$ in terms of an orthonormal basis of $m$.

**Proposition.** If $(X_i)$ is an orthonormal basis of $m$, then on $(C^\infty(G) \otimes W \otimes V)^H$, the Dirac operator is given by

$$D_V = \sum (id \otimes c(X_i) \otimes id)(r(X_i) \otimes id \otimes id - \varepsilon id \otimes s(X_i) \otimes id).$$

**Proof.** Both $D_V$ and the right hand side $D'_V$ are evidently $G$–invariant and act on the correct spaces. Take a section $\xi \in (C^\infty(G) \otimes W \otimes V)^H$. By invariance, it is enough to show that $(D_V \xi)(1) = (D'_V \xi)(1)$. As above we have $X_i$, vector fields on $G/H$, orthonormal at $g = 1$. We then have

$$(D_V \xi)(1) = \sum (c(X_i)\nabla_{X_i} \xi)(1) = \sum c(X_i)(1) d\frac{d}{dt}(\exp X_i(1)t)|_{t=0}$$

$$= \sum c(X_i)([r(X_i)\xi](1) - \varepsilon s(X_i)\xi(1)) = (D'_V \xi)(1),$$

so the result follows.

**20. REMARKS ON DIRAC INDUCTION AND BOTT’S PRINCIPLE.** Consider $\ker(D_V^\pm) \subset (C^\infty(G) \otimes W^\pm \otimes V)$. This is a closed $G$–invariant subspace (in the $C^\infty$ topology). Since $D_V$ is an elliptic operator, we know it is finite–dimensional; in any event it is the closure of the sum of its irreducible subspaces. Let $G$ be a compact matrix group and let $A$ be the $\ast$–algebra generated by the matrix coefficients of a finite–dimensional faithfull representation. Thus $\pi \subset C^\infty(G) \subset C(G) \subset L^2(G)$. By the Stone–Weierstrass theorem, $A$ is uniformly dense in $C(G)$ and hence dense in $L^2(G)$. Let $V \subset L^2(G)$ be a finite–dimensional left invariant subspace. If $f \in C^\infty(G)$ and $\xi \in V$, we have $f \ast \xi \in V$. On the other hand $f \ast \xi \in C^\infty(G)$, so that $V \subset C^\infty(G)$. Now let $W$ be any finite–dimensional irreducible representation of $G$. Then the map of taking matrix coefficients defines a $G \times G$–equivariant embedding $W \otimes W^\ast = C^\infty(G)$. The algebra $A$ is an algebraic direct sum of representations $V \otimes W$ of $G \times G$. If $W \otimes W^\ast$ does not appear in this list, the corresponding elements would have to be orthogonal to $A$, a contradiction. Hence $W \otimes W^\ast \subset A$. On the other hand $\text{Hom}_G(V, C(G)) = V^\ast$ under the map $f \mapsto f^\ast$ with $f^\ast(v) = f(v)(1)$. So $A = \oplus V_1 \otimes V_1^\ast$ (algebraic direct sum) as a $G \times G$–module. Thus $L^2(G) = \oplus V_1 \otimes V_1^\ast$ (Hilbert space direct sum). The multiplicity space of $U$ in $C^\infty(G) \otimes V$ is $\text{Hom}_G(U, C^\infty(G) \otimes V) = \text{Hom}_G(U, V)$ and the multiplicity of $U$ in $(C^\infty(G) \otimes V)^H$ is $\text{Hom}_G(U, (C^\infty(G) \otimes V)^H) = \text{Hom}_H(U, V)$. Moreover $(L^2(G) \otimes V)^H = \oplus V_1 \otimes \text{Hom}_H(V_1, V)$ as a $G$–module.

Thus every representation $U$ of $G$ appears in $(C^\infty(G) \otimes W \otimes V)^H$ with finite multiplicity. The multiplicity space is given by $\text{Hom}_H(U, W \otimes V)$. The operator $D_V$ commutes with $G$ and therefore carries each multiplicity space onto itself. In fact on the space $\text{Hom}_G(U, (C^\infty(G) \otimes W \otimes V)$,

$$D_V = \sum (id \otimes c(X_i) \otimes id)(r(X_i) \otimes id \otimes id - \varepsilon id \otimes s(X_i) \otimes id) = \sum -c(X_i)(\pi(X_i) + \varepsilon s(X_i)).$$

It follows by direct computation (see below) or ellipticity of $D_V$ that $\ker(D_V)$ is finite–dimensional. Note that $D_V$ is skew–adjoint but breaks up as two operators $D_V^{\pm} : (C^\infty(G) \otimes W^\pm \otimes V)^H \to (C^\infty(G) \otimes W^\pm \otimes V)^H$ with finite–dimensional kernels. The index is the formal difference $\text{ind}(D_V^\pm) = [\ker(D_V^\pm)] - [\ker(D_V^\pm)]$ in the representation ring $R(G)$. Let $(C^\infty(G) \otimes W^\pm \otimes V)^H = \oplus V_i \otimes M_i^\pm$, where $M_i^\pm$ is the multiplicity space of $V_i$. Then $D_V^\pm : M_i^\pm \to M_i^\mp$. Since $\ker(D_V^\pm)$ is finite–dimensional, $D_V^\pm$ is almost everywhere an isomorphism of $M_i^\pm$ onto $M_i^\mp$. Thus

$$\text{ind}(D_V^\pm) = \sum (\dim(M_i^+) - \dim(M_i^-)) \cdot [V_i].$$

This proves Bott’s principle: the index depends only on the underlying bundles and not on the elliptic operator between them. one can also compute $\text{ind}(D_V^\pm)$ using supersymmetry and show that if $V$ is positive (in a certain sense), then $\ker(D_V^\pm) = (0)$. This shows that every irreducible representation of $G$ can be realised on a space of twisted harmonic spinors $\ker(D_V^\pm)$ and gives a uniform geometric construction of all irreducible representations.
CHAPTER III. REPRESENTATIONS OF AFFINE KAC–MOODY ALGEBRAS

Background. If $G$ is a compact simply connected group, the corresponding loop group is $LGC^\infty(S^1,G)$ under pointwise multiplication. This group acts by multiplication on $C^\infty(S^1,V)$ whenever $V$ is a finite-dimensional representation of $G$. Let Diff($S^1$) be the group of orientation-preserving diffeomorphisms of $S^1$ and $\text{Rot}(S^1)$ the rotation subgroup. These groups act by automorphisms on $L^2(S^1,V)$. They also act on $C^\infty(S^1,V)$ compatibly with $LG$. The semidirect product acts unitarily on $H = L^2(S^1,V)$ (after correcting the action of $\text{Diff}(S^1)$ by a Radon–Nikodym cocycle). Although $H$ is already a complex Hilbert space, we regard it as a real Hilbert space, taking a new complex structure given by the Hilbert transform $P$. This complex structure defines an irreducible representation of the corresponding real Clifford algebra with generators $P$. $P$ is the Hardy space projection onto the space with vanishing negative Fourier coefficients. This complexification $	ext{Vect}(S^1)$, one can look directly for positive energy projective representations of $C^\infty(S^1,g) × \text{Rot}(S^1)$. We do this below for the Lie subalgebra $Lg \subset C^\infty(S^1,g)$ consisting of (trigonometric) polynomial maps. In view of the fermionic construction, it is not so surprising that every positive energy projective representation of $Lg × \text{Rot}(S^1)$ extends the Witt algebra. This is the Lie algebra of polynomial vector fields on $S^1$, a subalgebra of Lie Diff$(S^1)$. (We establish an infinitesimal version of this fermionic construction in the course of this chapter.)

1. LOOP ALGEBRAS AND THE WITT ALGEBRA. Let $g$ be a simple compact Lie algebra with complexification $g_C$. The (trigonometric) polynomial loops $S^1 \to g$ are spanned by $X\sin m\theta$, $X\cos m\theta$ ($X \in g$, $m \geq 0$) and form a real Lie algebra under pointwise Lie bracket, $[X(\theta), Y(\theta)]$. The complexification has a slightly easier spanning set $X_n = X e^{i n \theta}$. It is called the loop algebra and denoted by $Lg$. Note that

$$[X_m, Y_n] = [X, Y]_{m+n}.$$

(1)

Now there is a natural involution $X \mapsto X^*$ on $g_C$ as well as the usual involution (conjugation) on complex functions. This leads to the (conjugate–linear) involution on $Lg$ given by $(X_n)^* = (X^*)_{-m}$. The Witt algebra corresponds to the complexification of the real Lie algebra of (trigonometric) polynomial vector fields $a(\theta)d/d\theta$ on $S^1$. It has basis $d_n = i e^{i n \theta} d/d\theta$. We can use Leibniz’ rule to compute the Lie brackets:

$$[d_m, d_n] = (m−n)d_{m+n}.$$

(2)

Note that $[d_m, X_n] = i e^{i n \theta} \frac{d}{d\theta}(e^{i n \theta}) X = -n e^{i(m+n)\theta} X = -n X_{m+n}$, so that

$$[d_m, X_n] = -n X_{m+n}.$$

(3)

Note that $d \equiv d_0 = i d/d\theta$ is the vector field corresponding to rotations and $[d, X_m] = -m X_m$. Note that the rotation group $\text{Rot}(S^1) \cong \mathbb{T}$ acts on $Lg$ by $(r_\alpha f)(\theta) = f(\theta - \alpha)$ and $r_\alpha = e^{i d\alpha}$ by Taylor’s theorem. Thus the Witt algebra $\text{ Vect}(S^1)$ acts by Lie algebra derivations on $Lg$. We extend the involution to $\text{ Vect}(S^1)$ by declaring that $d_n^* = d_{-n}$. This picks out the usual real structure on $\text{ Vect}(S^1)$.

2. POSITIVE ENERGY REPRESENTATIONS AND KAC MOODY–ALGEBRAS. We shall be interested in projective, unitary, positive energy representations of $Lg \rtimes C_d$ or $Lg \rtimes \text{Rot}(S^1)$. Thus if $a = Lg \rtimes C_d$ is the semidirect product, we look for inner product spaces $H$ (not complete!!) such that:

(1) Projective: $a$ acts projectively by operators $\pi(A)$ ($A \in a$), i.e. $A \mapsto \pi(A)$ is linear and $[\pi(A), \pi(B)] = \pi([A, B])$ lies in $Cf$ for $A, B \in a$.

(2) Unitary: $\pi(A)^* = \pi(A^*)$.

(3) Positive energy: $H$ admits an orthogonal decomposition $H = \bigoplus_{k \geq 0} H(k)$ such that $D = \pi(d)$ acts on $H(k)$ as multiplication by $k$, $H(0) \neq 0$ and $\dim H(k) < \infty$.
Note that we can take $R_\theta = \pi(r_\theta) = e^{i\theta D}$, so that $R_\theta$ acts on $H(k)$ as multiplication by $e^{i\theta}$. A more general version of the positive energy condition requires only that $H(k) = (0)$ for $k << 0$. Tensoring $H$ be a representation of Rot $S^1$, we may always convert this more general positive energy representation into the normalised form given in (3). The subspace $H(k)$ are called the energy subspaces with energy $k$; the operator $D$ has many names, including the energy operator or hamiltonian operator.

Since the representation is projective, $[\pi(A), \pi(B)] - \pi([A,B]) = b(A,B)I$ where $b(A,B) \in \mathbb{C}$. We call $b$ a 2–cocycle — in fancy language it gives a class in $H^2(a, \mathbb{C})$. The definition immediately implies the antisymmetry condition

$$b(A, B) = -b(B, A)$$

because Lie brackets are antisymmetric; and the Jacobi identity immediately implies that

$$b([A, B], C) + b([B, C], A) + b([C, A], B) = 0$$

for all $A, B, C \in a$. On the other hand we are free to adjust the operators $\pi(A)$ by adding on scalars. Thus to preserve linearity, we change $\pi(A)$ to $\pi(A) + f(A)I$ where $f : a \to \mathbb{C}$ is linear. This changes $b(A, B)$ to $b(A, B) - f([A, B])$. We shall now make such adjustments so that $b$ has a canonical form. We normalise the inner product on $g$ (and hence $h$ and $h^*$) so that $||\theta||^2 = 2$, where $\theta$ is the highest root.

**Theorem.** Representatives of $X(n) = \pi(X_n)$ and $D$ can be chosen so that

$$[X(m), Y(n)] = [X, Y](m + n) - ml(X, Y)\delta_{m+n,0}, \quad [D, X(n)] = -nX(n)$$

and $D = D^*$, $X(n)^* = -X(-n)$ for $X \in g$. Here $\ell$ is a non–negative integer called the level of $H$.

**Remark.** We can extend the inner product $(\cdot, \cdot)$ on $g$ to a complex inner product on $g_C$. If we set $\langle X, Y \rangle = (X, Y)^*$ for $X, Y \in \mathbb{C}$, then we obtain an invariant complex symmetric bilinear form on $g_C$. The commutation relation then becomes $[X(m), Y(n)] = [X, Y](m + n) + m\ell < X, Y > \delta_{m+n,0}$ for $X, Y \in g_C$. When $g = su(n)$, this bilinear form is given by $\langle X, Y \rangle = Tr(XY)$ and is in general a positive multiple of the Killing form.

**Proof.** We start by adjusting the operator $X = X(0)$ for $X \in g$ (the zero modes). Set $b(X(0), Y(0)) = i\ell(X, Y)$. Thus $c([X, Y], Z) + c([Y, Z], X) + c([Z, X], Y) = 0$ and $c(X, Y) = -c(Y, X)$. Since $X^* = -X$ and $\pi(X)^* = -\pi(X)$, it follows that $c(X, Y)$ is real. We show that $c(X, Y) = f([X, Y])$ for $f \in g^*$. Let $(\cdot, \cdot)$ be an invariant inner product on $g$ and write $b(X, Y) = (\delta(X), Y)$ for some linear operator $\delta \in \text{End}(\mathfrak{g})$. The antisymmetry of $b$ implies that $\delta^* = -\delta$, i.e. $\delta$ is skew–adjoint, and the cocycle relation translates into

$$\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)].$$

Thus $\delta$ is a skew–adjoint derivation of $\mathfrak{g}$. We saw in Chapter 2 that any such derivation is inner, i.e. $\delta(X) = [A, X]$ for some $A \in \mathfrak{g}$. Hence $b(X, Y) = ([A, X], Y) = -([X, A], Y) = (A, [X, Y]) = f([X, Y])$ with $f(X) = (A, X)$. Thus we may adjust the $X$’s by pure imaginary scalars so that $[\pi(X), \pi(Y)] = \pi([X, Y])$ with $\pi(X)^* = -\pi(X)$.

Now consider the operators $[D, \pi(X)] = ig(X)I$. Since $D = D^*$ and $\pi(X)^* = -\pi(X)$, $g(X)$ is real. It is also linear and the cocycle relation implies that $g([X, Y]) = 0$. Since $[g, g] = g$, it follows that $g \equiv 0$ and hence that $[D, \pi(X)] = 0$.

This completes the zero mode adjustments. If $n > 0$, we can choose $X(n)$ so that $[D, X(n)] = -nX(n)$, for the left hand side is independent of any choices. Taking $X(-n) = -X(n)^*$ for $X \in \mathfrak{g}$, we still have $[D, X(-n)] = nX(-n)$. These equations imply that the operator $X(n)$ takes the energy space $H(k)$ into $H(k-n)$. Thus the operators $X(-n)$ and $X(n)$ ($n > 0$) raise and lower energy. We thus have $[\pi(X), \pi(Y)] = \pi([X, Y]) + ib(x, y)I$ for some 2–cocycle $b(x, y) \in \mathbb{C}$. We now compute $b(x, y)$.

(1) We have already imposed the condition $[D, X(n)] = -nX(n)$ for $X \in \mathfrak{g}$ which uniquely specifies the choice of $X(n)$’s.

(2) $[X(n), Y(m)] = [X, Y](n + m)$ if $n + m \neq 0$. For these elements lower energy by $n + m$ and scalars preserve energy.
(3) \([X(n), Y(-n)] = [X, Y](0) + \delta_n(X, Y) \cdot I\). For if \([X(n), Y(-n)] - [X, Y](0) = \lambda(X, Y)\), then taking Lie brackets with \(Z(0)\), we find \(\lambda([Z, X], Y) + \lambda(X, [Z, Y]) = 0\). But any \(g\)-invariant bilinear form on \(g\) is a multiple of \((X, Y)\).

(4) \(\delta_n = n\delta_1\) for \(n > 0\). It suffices to show that \(\delta_{n+1} = \delta_n + \delta_1\). But

\[
[A(-1), [X(n + 1), Y(-n)]] = [A(-1), [X, Y](1)] = ([A, [X, Y]](0) - \delta_1(A, [X, Y])].
\]

On the other hand

\[
[A(-1), [X(n + 1), Y(-n)]] = -[X(n + 1), [Y(-n), A(-1)]] - [Y(-n), [A(-1), X(n + 1)]
\]

\[
= -[X(n + 1), [Y, A](-n - 1)] - [Y(-n), [A, X](n)]
\]

\[
= -[X, [Y, A]](0) - [Y, [A, X]](0) - \delta_{n+1}(X, [Y, A]) - \delta_n(Y, [A, X]).
\]

Since \((\cdot, \cdot)\) is \(g\)-invariant, and \([g, g] = g\), the result follows.

(5) \(-\delta_1\) is a non–negative integer \(\ell\). Suppose that \(H(0)\) has a summand \(V_\lambda\) with highest weight \(\lambda\). Let \(v_\lambda\) be a highest weight vector in \(V_\lambda\), so that \(H_i(0)v_\lambda = \lambda(H_i)v_\lambda\) and \(E_i(0)v_\lambda = 0\) for \(i > 0\). Now consider \(E = E_{-\delta}(1), F = E_\delta(-1), H = [E, F] = iT\delta(0) + \delta_1\), where \((T, T) = \theta(T)\). Then \([H, E] = 2E, [H, F] = -2F, H = H^*\) and \(E^* = F\). Moreover \(Ev_\lambda = 0\) and \(Hv_\lambda = (-\theta, \lambda) + \delta_1 v_\lambda\). So by the usual \(sl_2\) lemma, \(-\delta_1 - (\theta, \lambda)\) is a non–negative integer. Hence \(-\delta_1\) must be a non–negative integer \(\ell\).

**Corollary of proof.** Each energy space \(H(k)\) is a \(g\)-module. For level \(\ell \geq 0\), any highest weight \(\lambda\) appearing in \(H(0)\) must satisfy \((\lambda, \theta) \leq \ell\).

In the light of this theorem, we define the affine Kac–Moody algebra \(\widehat{g}\) by

\[
\widehat{g} = Lg \oplus \mathbb{C}d \oplus \mathbb{C}c,
\]

with \([X(m), Y(n)] = [X, Y](m + n) + m < X, Y > \delta_{m+n,0}c\) and \([d, X(n)] = -nX(n)\) (note the minus sign!). All other brackets zero. This contains \(Lg \oplus \mathbb{C}c = \mathcal{L}g\) as an ideal. Note that \(\mathbb{C}c\) is a central subalgebra and \(Lg/\mathbb{C}c = Lg\). Thus \(Lg\) is a central extension of \(Lg\) by \(\mathbb{C}\) and \(\text{Rot}\, S^1\) acts on \(Lg\). \(Lg\) is called an affine Lie algebra.

**Invariant symmetric bilinear form.** We define a bilinear form on \(\widehat{g}\) by

\[
(X_1(m) + \delta_1 d + \gamma_1 c, X_2(n) + \delta_2 d + \gamma_2 c) = (X_1, X_2)\delta_{m+n,0} + \delta_1 \gamma_2 + \delta_2 \gamma_1 = - < X_1, X_2 > \delta_{m+n,0} + \delta_1 \gamma_2 + \delta_2 \gamma_1.
\]

It is straightforward to check that this form is ad–invariant. It is important that it is not positive definite on the ”real part”; for example it is clearly Lorentzian on \(i\mathbb{R}^d + i\mathbb{R}c\).

3. **COMPLETE REDUCIBILITY.**

**Positive energy theorem.** (a) Let \(H\) be a positive energy representation of \(\widehat{g}\) and let \(V = H(0)\) be the lowest energy subspace. If \(\xi \in H(0)\) is cyclic, then some vector in \(H(0)\) generates a lowest energy subspace.

(b) Any positive energy representation is an orthogonal direct sum of irreducible positive energy representations.

**Proof.** (a) Let \(V\) be the subspace of lowest energy. Let \(K\) be any invariant subspace of \(H\). So \(K = \bigoplus K(n)\) and \(H = K \oplus K^\perp\). Note also that \(K^\perp\) is invariant. Now \(H(0) = K(0) \oplus K^\perp(0)\). Moreover the \(\widehat{g}\)-module generated by \(K(0)\) or \(K^\perp(0)\) is contained in \(K\) or \(K^\perp\). But the \(\widehat{g}\)-module generated by \(K(0) \oplus K^\perp(0) = H(0)\) equals \(H\). Hence \(K\) must be the \(\widehat{g}\)-module generated by \(K(0) = K \cap H(0)\). Thus there is a 1–1 correspondence between submodules and certain invariant subspaces of \(H(0)\). Taking \(K\) so that \(KH(t)\) has minimal dimension, we see that \(K\) must be irreducible. Any non–zero vector in \(K \cap H(0)\) must be cyclic, by irreducibility.

(b) Take the cyclic module generated by a vector of lowest energy. This contains an irreducible submodule generated by another vector of lowest energy \(H_1\) say. Now repeat this process for \(H_1^\perp\), to get \(H_2, H_3, \text{ etc.}\). The positive energy assumption shows that \(H = \bigoplus H_i\).
4. CLASSIFICATION OF POSITIVE ENERGY REPRESENTATIONS.

Theorem (Uniqueness). Let \((\pi, H)\) be an irreducible positive energy representation of \(\hat{g}\) of level \(\ell\).

(1) \(H(0)\) is irreducible as an SU\((N)\)–module.

(2) If \(H(0) = V_\lambda\), then \((\theta, \lambda) \leq \ell\).

(3) (Uniqueness) If \(H\) and \(H'\) are irreducible positive energy representations of level \(\ell\) of the above form with \(H(0) \cong H'(0)\) as \(\mathfrak{g}\)–modules, then \(H\) and \(H'\) are unitarily equivalent as representations of \(\hat{g}\).

Proof. (1) Let \(V\) be an irreducible SU\((N)\)–submodule of \(H(0)\). By irreducibility the \(\hat{g}\)–module generated by \(V\) is the whole of \(H^0\). Since \(D\) fixes \(V\), it follows that the \(L^0\mathfrak{g}\)–module generated by \(V\) is the whole of \(H\). The commutation rules show that any monomial in the \(X(n)\)’s can be written as a sum of monomials of the form \(P_+ P_0 P_+\), where \(P_-\) is a monomial in the \(X(n)\)’s for \(n < 0\) (energy raising operators), \(P_0\) is a monomial in the \(X(0)\)’s (constant energy operators) and \(P_+\) is a monomial in the \(X(n)\)’s with \(n > 0\) (energy lowering operators). Hence \(H\) is spanned by products \(P_+ v (v \in V)\). Since the lowest energy subspace of this \(L^0\mathfrak{g}\)–module is \(V\), \(H(0) = V\), so that \(H(0)\) is irreducible as a \(\mathfrak{g}\)–module.

(2) We have already proved this in section 2 by introducing \(E = n + i T_\theta(0), F = n - i T_\theta(0)\) and \(H = [E, F] = \ell + i T_\theta(0)\), where \((T_\theta, T) = \theta(T)\). Thus, if \(v \in H(0)\) has highest weight \(\lambda\), then \(T(0)v = i\lambda(T)v\) and \(Ev = 0\). Thus \(Hv = (\ell + i T_\theta(0))v = (\ell - (\theta, \lambda))v\). Since \(E, F, H\) give a copy of \(\mathfrak{sl}_2\), we get \(\ell \geq (\theta, \lambda)\).

(3) Any monomial \(A\) in operators from \(\mathfrak{g}\) is a sum of monomials \(RDL\) with \(R\) a monomial in energy raising operators, \(D\) a monomial in constant energy operators and \(L\) a monomial in energy lowering operators. Observe that if \(v, w \in H(0)\), the inner products \((A_1 v, A_2 w)\) are uniquely determined by \(v, w\) and the monomials \(A_i\) for \(A_2 A_1\) is a sum of terms \(RDL\) and \((RDLv, w) = (DLv, R^* w)\) with \(R^*\) an energy lowering operator. Hence, if \(H'\) is another irreducible positive energy representation with \(H'(0) \cong H(0)\) by a unitary isomorphism \(v \mapsto v'\), \(U(Av) = Av'\) defines a unitary map of \(H\) onto \(H'\) intertwining \(\hat{g}\).

5. SUGAWARA’S FORMULA FOR \(L_0\).

Sugawara’s formula for \(L_0\). Let \(H\) be a cyclic positive energy representation at level \(\ell\) and let \((X_i)\) be an orthonormal basis of \(\mathfrak{g}\). Let \(L_0\) be the operator defined on \(H^0\) by

\[
L_0 = \frac{1}{N + \ell} \left( -\sum_i \frac{1}{2} X_i(0) X_i(0) - \sum_{n > 0} \sum_i X_i(-n) X_i(n) \right).
\]

Then \(L_0 = D + C/2(N + \ell)\) where the Casimir \(\Delta = -\sum_i X_i(0) X_i(0)\) acts on \(H(0)\) as multiplication by \(C\) and on \(\mathfrak{g}\) as \(2g\) (where \(g\) is the dual Coxeter number, equal to \(N\) for \(\mathfrak{su}(N)\)).

Proof. Since \(\sum_i X_i(a) X_i(b)\) is independent of the orthonormal basis \((X_i)\), it commutes with \(G\) and hence each \(X(0)\) for \(X \in \mathfrak{g}\). Thus \(\sum_i [X, X_i(a)] X_i(b) + X_i(a) [X, X_i](b) = 0\) for all \(a, b\). If \(A = \sum_i \frac{1}{2} X_i(0) X_i(0) + \sum_{n > 0} X_i(-n) X_i(n)\), then using the above relation we get

\[
[X(1), A] = -\ell X(1) + \sum_i \frac{1}{2} ([X, X_i(1)] X_i(0) + X_i)(0) [X, X_i(1)]
\]

\[
+ \sum_n [X, X_i(-n + 1)] X_i(n) + X_i(-n) [X, X_i](n + 1)
\]

\[
= -\ell X(1) + \frac{1}{2} \sum_i [[X, X_i(1)] X_i(0)] - \ell X(1) + \frac{1}{2} \sum_i [[X, X_i]] X_i(1),
\]

since \(\{[X, X_i], X_i\} = 0\) by invariance of \((\cdot, \cdot)\). Hence \([X(1), A] = -(g + \ell) X(1)\), since \(-\sum \text{ad}(X_i)^2 = 2g\). Similarly \([X(-1), A] = (g + \ell) X(-1)\). [Note that if \(H\) were a unitary representation, so that \(X(n)^* = -X(-n)\), then \(A^* = A\) and taking adjoints we get \([X(-1), A] = (g + \ell) X(-1)\). A similar argument could be applied in general using the pairing between \(H\) and its algebraic dual.] Thus \((g + \ell)D + A\) commutes with all \(X(\pm 1)\)'s. Since \([g, g] = g\), these generate \(L^0 g\), and hence \((g + \ell)D + A\) commutes with \(Lg\). Since \((g + \ell)D + A = -\frac{\Delta}{2} \cdot I\) on \(H(0)\) and the cyclic subspace generated by \(H(0)\) is the whole of \(H\), we get \((g + \ell)D + A = -\frac{\Delta}{2} \cdot I\) on \(H\) as required.
Corollary. Let $H$ be a positive energy representation of $\widehat{\mathfrak{g}}$.

(a) If $H$ is irreducible as an $\widehat{\mathfrak{g}}$—module, then it is irreducible as an $\mathcal{L}_\mathfrak{g}$—module.

(b) If $H_1$ and $H_2$ are irreducible $\widehat{\mathfrak{g}}$—modules which are isomorphic as $\mathcal{L}_\mathfrak{g}$—modules, then one is obtained from the other by tensoring with a character of $\text{Rot} S^1$.

Proof. (a) The Sugawara formula show that given $k \geq 0$, there is a finite linear combination $T \xi = D \xi$ for all $\xi \in H(0) \oplus \cdots \oplus H(k)$. Hence the submodule generated by any such $\xi$ also contains the submodules generated by any of the components $\xi_j \in H(j)$ ($j \leq k$). However it is clear that the $\widehat{\mathfrak{g}}$—module generated by any $\xi_j$ is the same as the $\mathcal{L}_\mathfrak{g}$ module generated by $\xi_j$. By irreducibility, it follows that the $\mathcal{L}_\mathfrak{g}$—module generated by $\xi$ is the whole of $H$.

(b) Let $T : H_1 \to H_2$ be a unitary intertwiner for $\widehat{\mathfrak{g}}$. Then $V_\tau TU_\tau$ is also a unitary intertwiner, so must be of the form $\lambda(t)T$ for $\lambda(t) \in T$ by Schur’s lemma. Since $TU_\tau T^* = \lambda(t)V_\tau$, $\lambda(t)$ must be a character of $T$.

Remark. The previous corollary is important because it shows that positive energy representations are classified by up to tensoring with a character of $\text{Rot} S^1$. This will appear as an important feature in our discussion of roots and weights for $\widehat{\mathfrak{g}}$ below.

6. SUGAWARA’S CONSTRUCTION OF THE VIRASORO ALGEBRA.

Theorem. Let $H$ be an irreducible positive energy prepresentation at level $\ell$. If $L - 0$ is defined as above and in addition we set

$$L_m = -\frac{1}{2(\ell + g)} \sum_i \sum_{a+b=m} X_i(a)X_i(b),$$

then

$$[L_m, X(n)] = -nX(n + m), \quad [L_m, L_n] = (m - n)L_{m+n} + \frac{\ell}{12}(m^3 - m)\delta_{m+n,0},$$

where $c = \dim \mathfrak{g} \cdot \ell/(\ell + g)$ and $g$ is the dual Coxeter number.

*Proof.* Since $\sum_i X_i(a)X_i(b)$ is independent of the orthonormal basis $(X_i)$, it commutes with $G$ and hence each $X(0)$ for $X \in \mathfrak{g}$. Thus $\sum_i [X_i, X_i](a)X_i(b) + X_i(a)[X_i, X_i](b) = 0$ for all $a, b$. If $B = \frac{1}{2} \sum_{a+b=m} X_i(a)x_i(b)$, then using the above relation we get $[X(0), B] = 0$. Similarly writing

$$B = \sum_i \frac{1}{2} \sum_{a+b=m, a=b} \sum_{a+b=m, b>a} X_i(a)X_i(b),$$

we get

$$[X(1), B] = \ell X(1) + \sum_i \frac{1}{2} \sum_{a+b=m, a=b} \sum_{a+b=m, b>a} [X_i(a)X_i(b) + X_i(a)[X_i, X_i](b)] = \ell X(1) + \sum_i X_i(\alpha)X_i(\beta) + X_i(\alpha)[X_i, X_i](\beta)$$

In this sum, consider terms $P(\alpha)Q(\beta)$ with $\alpha + \beta = m + 1$: if $\alpha < \beta + 1$, the first term gives a contribution $\sum_i [X_i, X_i](\alpha)X_i(\beta)$, while if $\alpha = \beta + 1$, it gives $\frac{1}{2} \sum_i [X_i, X_i](\alpha)X_i(\beta)$; if $\beta = \alpha + 1$, the second terms gives a contribution $\frac{1}{2} \sum_i X_i(\alpha)$$X_i(\beta)$, while if $\beta > \alpha + 1$, it gives $\sum_i X_i(\alpha)[X_i, X_i](\beta)$. Adding these contributions (when they occur), we get a total of 0 if $\beta > \alpha + 1$: $\sum_i \frac{1}{2} X_i(\alpha)[X_i, X_i](\beta) + X_i(\alpha)[X_i, X_i](\beta)$ if $\beta = \alpha + 1$; $\sum_i [X_i, X_i](\alpha)$ if $\beta = \alpha$; $\frac{1}{2} \sum_i [X_i, X_i](\alpha)X_i(\beta)$ if $\beta = \alpha - 1$; and 0 if $\beta < \alpha - 1$. If $m + 1 = 2k$, then we must have $\alpha = \beta = k$ and

$$[X(1), B] + \ell X(1) = \sum_i [X_i, X_i](k)X_i(k)$$

$$= \frac{1}{2} \sum_i ([X_i, X_i](k)X_i(k) - X_i(k)[X_i, X_i](k))$$

$$= \frac{1}{2} [X_i, X_i](m + 1) - \ell \delta_{k,0}([X_i, X_i](k))$$

$$= gX(m + 1).$$

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If $m = 2k$, then

$$\begin{align*}
[X(1),B] + \ell X(1) &= \sum_i \frac{1}{2} X_i(k) [X,X_i](k+1) + [X,X_i](k)X_i(k+1) + \frac{1}{2} [X,X_i](k+1)X_i(k) \\
&= \frac{1}{2} \sum_i [[X,X_i](k+1),X_i(k)] \\
&= -gX(m+1).
\end{align*}$$

Thus in both cases $[X(1),B] = -(\ell + g)X(m+1)$. Hence $[L_m,X(1)] = -X(m+1)$. Since $L^*_m = L_{-m}$ and $X(1)^* = -X(-1)$, taking adjoints we get $[L_m,X(-1)] = X(m-1)$. Since $[g,g] = g$, $L^0g$ is generated by the $X(1)$’s and $Y(-1)$’s. The relation $[L_m,X(n)] = -nX(m+n)$ then follows easily by induction and the Jacobi identity. Hence

$$[[L_m,L_n],X(p)] = [L_m,[L_n,X(p)]] = -p(m-n)X(p+m+n) = (m-n)[L_{m+n},X(p)].$$

Since the $X(p)$’s act irreducibly, Schur’s lemma implies that $[L_m,L_n] = (m-n)L_{m+n} + \lambda(m,n)I$ for some scalar $\lambda(m,n)$. Note that by definition $L_m$ carries $H(k)$ into $H(k-m)$. Thus both $[L_m,L_n]$ and $L_{m+n}$ carry $H(k)$ into $H(k-m-n)$. Thus $\lambda(m,n) = 0$ if $m + n \neq 0$. Clearly $\lambda(m,-m) = -\lambda(-m,m)$ by antisymmetry of the Lie bracket. Take $\xi \in H(0)$ and $m > 0$. Then $[[L_m,L_{-m}],\xi,\xi] = 2m(L_0\xi,\xi) + \lambda(m,-m)(\xi,\xi)$. On the other hand $L_0\xi = \mu\xi$, where $\mu = \Delta/(\ell + g)$, and $(L_m,L_{-m})\xi = (L_{m-L_{-m}})\xi$. For $m > 0$, we have

$$L_{-m}\xi = \frac{1}{2(\ell + g)} \sum_{i} \sum_{a+b=m:a,b \geq 0} X_i(-a)X_i(-b)\xi.$$

Since $[L_m,X(p)] = -pX(p+m)$ and $L_m\xi = 0$, we get

$$L_m L_{-m} \xi = \frac{1}{2(\ell + g)} \sum_{i} \sum_{a+b=m:a,b \geq 0} aX_i(-a+m)X_i(-b)\xi + bX_i(-a)X_i(-b+m)\xi.$$

Now $S_j = \sum - (X_i(j)X_i(-j)\xi,\xi) = 0$ if $j < 0$ and $= \Delta\|\xi\|^2$ if $j = 0$. Since $X_i(j)X_i(-j) = X_i(-j)X_i(j) + j\ell \cdot I$, $\sum - (X_i(j)X_i(-j)\xi,\xi) = j\ell \cdot \dim g \|\xi\|^2$ for $j > 0$. Hence

$$\begin{align*}
(L_mL_{-m}\xi,\xi) &= \frac{1}{2(\ell + g)} \sum_{b=0}^m (m-b)S_b + \sum_{a=0}^m (m-a)S_a \\
&= \frac{1}{(\ell + g)} [m\Delta\|\xi\|^2 + \sum_{b=0}^m b(m-b)\ell \cdot \dim g \|\xi\|^2] \\
&= \frac{\|\xi\|^2}{\ell + g} (m\Delta + (m^3 - m)/6 \cdot \ell \dim g),
\end{align*}$$

since $\sum_{a=0}^m a(m-a) = (m^3 - m)/6$. Thus $\lambda(m,n) = \delta_{m+n,0}(m^3 - m)/12$ with $c = \ell \dim g/(\ell + g)$, as required.

This result is an example of quantisation. The Witt algebra acts by derivations on $Lg$ preserving the central extension. Thus by Schur’s lemma there is at most one covariant projective representation of it compatible with the group. Thus if $\pi(d_n) = L_n$, we require $[L_n,X(m)] = -mX(m+n)$. By uniqueness, we must have $[L_m,L_n] = (m-n)L_{m+n} + \lambda(m,n)I$. Here $\lambda(n,m)$ is a 2–cocycle. As we show below, by appropriate adjustment of the $L_n$’s by scalars, $\lambda$ can always be normalised so that $\lambda(m,n) = \delta_{m+n,0}c(m^3 - m)/12$ with $c = \ell \dim g/(\ell + g)$, as required.

**Virasoro cocycle lemma.** If $[L_0,L_n] = -nL_n$ for all $n$, then $[L_m,L_n] = (m-n)L_{m+n} + (am^3 + bm)\delta_{m+n,0}I$. If we choose $L_0$ so that $[L_1,L_{-1}] = L_0$, then $a + b = 0$. 

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Proof. Note that \(-n^{-1}[L_0, L_n] \) is independent of adding scalars onto \( L_0 \) or \( L_n \), so we may always choose \( L_n \) so that \([L_0, L_n] = -nL_n\). By the Jacobi identity for \( L_0, L_m \) and \( L_n \), we have \([L_0, [L_m, L_n]] = -(n + m)[L_m, L_n]\). On the other hand \([L_0, L_{m+n}] = -(m + n)L_{m+n}\). Since \([L_m, L_n] = (m - n)L_{m+n} + \lambda(m, n)I\), we must have \( \lambda(m, n) = 0 \) if \( m + n \neq 0 \). Thus
\[
[L_m, L_n] = (m - n) + A(m)\delta_{m+n,0}.
\]
Clearly \( A(m) = -A(-m) \) and \( A(0) = 0 \). Writing out the Jacobi identity for \( L_k, L_n \) and \( L_m \) with \( k + n + m = 0 \), we get
\[
(n - m)A(k) + (m - k)A(n) + (k - n)A(m) = 0.
\]
Setting \( k = 1 \) and \( m = -n - 1 \), we get
\[
(n - 1)A(n + 1) = (n + 2)A(n) - (2n + 1)A(1).
\]
This recurrence relation allows \( A(n) \) to be determined from \( A(1) \) and \( A(2) \). Since \( A(n) = n \) and \( A(n) = n^3 \) give solutions, we see that \( A(m) = am^3 + bm \) for some constants \( a \) and \( b \). Clearly we are free to choose \( L_0 = [L_1, L_{-1}] \) (since we have made no adjustment to \( L_0 \) so far). But then \( A(1) = 0 \) and hence \( a + b = 0 \).

7. WEIGHTS, ROOTS AND THE QUANTUM CASIMIR OPERATOR.

Weights. It is immediately verified that \( \mathfrak{h} = \mathfrak{h} \oplus iRd \oplus iRc \) is a maximal Abelian subalgebra of \( \hat{g} \). If \( H \) is a positive energy representation (in the generalised sense), it first has an energy decomposition \( \bigoplus H(k) \) where \( H(k) \) is the \( k \)-eigenspace of \( d \). If \( H \) has level \( \ell \), then \( c = \ell I \) on \( H \). Each \( H(k) \) breaks up as a sum of \( \mathfrak{h} \)-modules with weights \( \mu \in P(\mathfrak{g}) \). Thus the weights of \( H \) are triples \( \overline{\mathfrak{h}} = (\mu, k, \ell) \in \overline{\mathfrak{h}} \). The dimension of the corresponding weight space in \( H \) is called the multiplicity of the weight.

The Lorentzian inner product on weights. We introduce a real symmetric bilinear form on \( \overline{\mathfrak{h}} \) via
\[
\left< (\mu_1, k_1, \ell_1), (\mu_2, k_2, \ell_2) \right> = (\mu_1, \mu_2) + k_1 \ell_1 + k_2 \ell_2.
\]
This bilinear form is thus obtained by taking the direct sum of the euclidean space \( \mathfrak{h}^* \) with the Lorentzian lattice \( \mathbb{R}^{1,1} \), with indefinite symmetric form \((x_1, y_1) \cdot (x_2, y_2) = (x_1 y_2 + y_1 x_2)\).

Roots and multiplicities. The Lie algebra \( \mathfrak{h} \) acts on \( \hat{g} \) through the adjoint representation preserving the Lorentzian form introduced before (recall \( (X_1, m) + \gamma_1 c + \delta_1 d, X_2(n) + \gamma_2 c + \delta_2 d) = \delta_{m+n,0}(X_1, X_2) + \gamma_1 \delta_2 + \delta_2 \gamma_1 \)). The inner product is non–degenerate on \( \mathfrak{h} \) and the orthogonal complement splits as a direct sum of non–zero eigenspaces of \( \mathfrak{h} \) each of finite multiplicity. Indeed
\[
\mathfrak{h}^\perp = \bigoplus_{n \neq 0} \mathfrak{h}(n) \oplus \bigoplus_{\alpha \in \Phi^+} g_{\alpha}(0) \oplus \bigoplus_{\alpha \in \Phi, n \neq 0} g_{\alpha}(n).
\]
These give weights of \( \overline{\mathfrak{h}} \) which we call affine roots. Since \( c \) is central, they all have the form \((\alpha, *, 0)\). We can list all the roots: \((\alpha, 0, 0) \) with \( \alpha \in \Phi; (\alpha, n, 0) \) with \( \alpha \in \Phi \) and \( n \neq 0 \); and \((0, n, 0) \). We denote the set of affine roots by \( \overline{\Phi} \). We define the positive roots by \( \overline{\Phi}^+ \) to be \((0, n, 0) \) or \((\alpha, n, 0) \) with \( n < 0 \) or, if \( n \neq 0 \), \((\alpha, 0, 0) \) with \( \alpha \in \Phi^+ \). If \( \overline{\Phi} \) is an affine root we denote by \( m_{\overline{\Phi}} \) the multiplicity of the corresponding root space. Thus \((0, n, 0) \) has multiplicity \( m = \dim \mathfrak{h} \) while all other roots have multiplicity one. (These conventions are adopted so that we can use highest weight theory painlessly.) The roots are of two types those of form \( \overline{\alpha} = (\alpha, n, 0) \) with \( \alpha \in \Phi \) and those of the from \( \overline{\alpha} = (0, n, 0) \). The former satisfy \( (\overline{\alpha}, \overline{\alpha}) > 0 \) and are called space–like; the latter satisfy \( (\overline{\alpha}, \overline{\alpha}) = 0 \) and are called time–like.

The simple roots. Let \( \alpha_1, \ldots, \alpha_m \) be the simple roots of \( \mathfrak{g} \) (with respect to a standard Weyl chamber). Let \( \theta \) be the highest root. Set \( \overline{\alpha}_i = (\alpha_i, 0, 0) \) for \( i = 1, \ldots, m \) and \( \overline{\alpha}_0 = (-\theta, -1, 0) \). The \( \overline{\alpha}_i \) are all roots called the simple roots.

Lemma. A space–like root is positive iff it is a non–negative integer combination of simple roots.

Proof. Let \( \overline{\alpha} = (\alpha, n, 0) \) be a space–like root. If \( n = 0 \), the result is known from the finite–dimensional case. If \( n \neq 0 \) and \( \overline{\alpha} = \sum_{i=0}^m n_i \overline{\alpha}_i \), then \( n_0 = -n \) and \( \alpha = \sum_{i=1}^m n_i \alpha_i + n \theta \). If \( n \geq 1 \), then \( n \theta - \alpha = \overline{\alpha}_0 \). If \( n \leq -1 \), then \( -n \theta + \alpha = \overline{\alpha}_0 \). If \( n = 0 \), then \( \alpha = \sum_{i=1}^m n_i \alpha_i + n \theta \), contradicting the fact that \( \alpha \) is space-like. Therefore, every space–like root is a non–negative integer combination of simple roots.
Lemma (Serre relations). The generators \( E_i, F_i, H_i \) satisfy the following relations:

\[ S_1. \quad [H_i, H_j] = 0. \]

\[ S_2. \quad [E_i, F_j] = \delta_{ij} H_i. \]

\[ S_3. \quad [H_i, E_j] = n(i, j) E_j \text{ and } [H_i, F_j] = -n(i, j) F_j \text{ where } n(i, j) = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i). \]

\[ S_{ij}^+. \quad \text{ad} E_i^{-n(i,j)+1} F_j = 0 \text{ for } i \neq j. \]

\[ S_{ij}^- \quad \text{ad} F_i^{-n(i,j)+1} E_j = 0 \text{ for } i \neq j. \]

Proof. We already know S1 and S2. S3 follows from the definition of the Lorentzian inner product and the fact that \( E_0 \) has weight \((-\theta, -1, 0)\) and \( F_0 \) has weight \((\theta, 1, 0)\). To prove the \( S_{ij} \) (\( i \neq j \)), note that \( \text{ad}(E_i) \cdot F_j = 0 \), \( \text{ad}(H_i) \cdot F_j = -n(i, j) F_j \). Thus the result follows from \( SU(2)_1 \)-theory, because \( F_j \) is a highest weight vector. (In particular \( n(i, j) \leq 0 \)). \( S_{ij}^+ \) follows by taking adjoints.

Definition. \( n_{ij} = n(i, j) \) \((i, j \geq 0)\) is called the extended Cartan matrix of \( \mathfrak{g} \). It is the matrix obtained by taking \( \alpha_0 = -\theta \) together with the simple roots of \( \mathfrak{g} \) and therefore coincides with our previous definition of the extended Cartan matrix.

If \( H \) is irreducible, we know that \( H(0) \) is an irreducible \( \mathfrak{g} \)-module. Let \( \lambda \) be its highest weight. The corresponding highest weight vector \( v_\lambda \) has weight \( \lambda = (\lambda, 0, \ell) \). The space \( H \) is spanned by all vectors obtained by applying lowering operators \( F_i \) to \( v_\lambda \). But if \( w \in H \) has weight \( \vec{\pi}, F_i w \) has weight \( \vec{\pi} - \vec{\alpha}_i \). Thus all the weights of \( H \) have the form \( \vec{\pi} = \lambda - \sum_{i=0}^m n_i \vec{\alpha}_i \) with \( n_i \geq 0 \), in analogy to the finite-dimensional case. Again we denote this relation by \( \vec{\lambda} \geq \vec{\pi} \).

The fundamental weights. If \( \theta = \sum m_i \alpha_i^\vee \), we define the fundamental weights by \( \vec{\lambda}_0 = (0, 0, 1) \) (the “vacuum” weight) and \( \vec{\lambda}_i = (\lambda_i, 0, m_i) \) for \( i \geq 1 \). Note that these satisfy \( \langle \vec{\lambda}_i, \vec{\pi}_j \rangle = \delta_{ij} \) with respect to the Lorentzian form where for a space–like vector \( W \) (i.e. one with \( (W, W) > 0 \)) we set \( W^\vee = 2(W, W)^{-1}W \) as in the euclidean case. If we include \( (0, 0, 1) \) with the simple roots and \( \delta = (0, 1, 0) \) with the fundamental weights, we get dual bases of \( \vec{\mathfrak{h}} \). (The inclusion of \( \delta = (0, 1, 0) \) amongst the fundamental weights again reflects the freedom to tensor positive energy representations by characters of \( \text{Rot} S^1 \).)

Dominant weights. A weight \( \vec{\lambda} = (\lambda, k, \ell) \) is said to be dominant if \( \lambda \) is dominant and \( (\lambda, \theta) \leq \ell \). This
is the permissibility condition for the highest weight of \( H(0) \). Plainly \( \overline{\lambda} \) is dominant iff \( (\overline{\lambda}, \pi_i) \geq 0 \) for all \( i \).

We will usually normalise the weight to have \( k = 0 \), using the freedom to tensor by a character of \( \text{Rot} \, S^1 \).

We then have the following analogue of the finite-dimensional result.

**Lemma.** \( \overline{\lambda} = (\lambda, 0, \ell) \) is a dominant weight iff \( \overline{\lambda} = \sum_{i=0}^{m} n_i \overline{\lambda}_i \) with \( n_i \geq 0 \).

**Proof.** This is immediate because \( (\overline{\lambda}_i, \overline{\lambda}_j') = \delta_{ij} \).

**The Weyl weight \( \overline{\rho} \).** We define \( \overline{\rho} = (\rho, 0, g) \). We already know that \( \rho \) is a weight, in fact \( \rho = \sum_{i=1}^{m} \lambda_i \). An analogous statement holds for \( \overline{\rho} \), called the Weyl weight.

**Lemma.** \( \overline{\rho} = \sum_{i=0}^{m} \overline{\lambda}_i \).

**Proof.** We have \( \overline{\lambda}_0 = (0, 0, 1) \) and \( \overline{\lambda}_i = (\lambda_i, 0, m_i) \) for \( i \geq 1 \). Thus we have to prove that \( g = 1 + \sum_{i \geq 1} m_i \).

Recall that \( 2g = (\theta, \theta) + 2(\rho, \theta) \). We have \( \theta = \sum m_i \alpha_i' \) (since \( \theta' = \theta \)), so that

\[
2g = ||\theta||^2 + 2 \sum_{i=1}^{m} (\lambda_i, \theta) = 2 + 2 \sum_{i=1}^{m} m_i,
\]

as required.

**The quantum Casimir operator.** We define the quantum Casimir operator in any positive energy representation to be the operator \( \Omega = L_0 - d \).

**Theorem.** The quantum Casimir operator commutes with \( L_0 \). If \( H \) is a positive energy representation generated by a highest weight vector \( v \) of weight \( \overline{\rho} = (\mu, k, \ell) \), then \( \Omega \) acts as the scalar \( \frac{1}{2} [\overline{\rho} + \overline{\rho} + \overline{\rho} - (\overline{\rho}, \overline{\rho})] / (g + \ell) \) on \( H \).

**Proof.** By construction \( L_0 - d \) commutes with \( g \). Since \( v \) is cyclic for \( L_0 \), it suffices to show that \( \Omega \) acts as the given scalar on \( v \). Since \( v \) is a highest weight vector \( X(n)v = 0 \) for \( n > 0 \), so that

\[
\Omega v = \frac{1}{2(g + \ell)} \left( - \sum X_i(0)^2 v, v \right) = \frac{1}{2(g + \ell)} \left( ||\mu + \rho||^2 - ||\rho||^2 \right) v.
\]

Since \( dv = kv \), we are reduced to showing that

\[
(\overline{\rho} + \overline{\rho} + \overline{\rho}) - (\overline{\rho}, \overline{\rho}) = (\mu + \rho, \mu + \rho) - (\rho, \rho) - 2(\ell + g)k,
\]

which is immediate from the definition of the Lorentzian form.

**8. THE AFFINE WEYL GROUP.**

**Hyperbolic realisation.** Given a space–like vector \( \overline{\alpha} = (\alpha, k, 0) \) with \( \alpha \neq 0 \), we define the hyperbolic reflection by

\[
\sigma_{\overline{\alpha}} \overline{\rho} = \overline{\rho} - 2 \frac{(\overline{\rho}, \overline{\alpha})}{(\overline{\alpha}, \overline{\alpha})} \overline{\alpha}.
\]

Let \( \widehat{W} \) be the group of transformations on \( \overline{\rho} \) generated by hyperbolic reflections in the space–like roots. We call \( \widehat{W} \) the affine Weyl group.

**Proposition 1.** The affine Weyl group permutes the weights and weight multiplicities in an irreducible positive energy presentation.

**Proof.** If \( \alpha \in \Phi \), set \( E = E_\alpha(n) \), \( F = E_{-\alpha}(n) \) and \( H = [E, F] = -2i||\alpha||^{-2} T_\alpha(0) + 2n||\alpha||^{-2} \). Then \( E^* = F \), \( H^* = H \), \( [H, E] = 2E \) and \( [H, F] = -2F \). Thus we have a copy of \( \mathfrak{sl}_2 \). Now suppose that \( w \in H \) has weight \( \overline{\rho} = (\mu, k, \ell) \). Then \( Hw = Mw \) where \( M = ||\alpha||^{-2} (2(\alpha, \mu) + 2n\ell) \). If \( M \geq 0 \), set \( u = E^Mw \). If \( M < 0 \), set \( u = E^{-M}w \). By the \( \mathfrak{sl}_2 \) theory, we know that \( u \neq 0 \). The weight of \( u \) is \( (\mu - \alpha M, k + nM, \ell) = \sigma_{(\alpha, \mu, 0)}(\overline{\rho}) \).

Thus the weights are invariant under the generators of \( \widehat{W} \). Since the inner product is invariant, orthogonal \( u \)'s of weight \( \overline{\rho} \) give rise to orthogonal \( u \)'s. This proves that the multiplicity of \( \sigma(\overline{\rho}) \) is greater than or equal
to the multiplicity of \( \varpi \). Applying \( \sigma^{-1} \), we get the reverse inequality, so the affine Weyl group preserves multiplicities.

We compute \( T = \sigma_{(\alpha,0,0)}^{-1}(\alpha,x,0) \). We have \( T = T_{x,0} \) where \( T = (\mu, k, \ell) = (\mu + \ell \beta, k - (\mu, \beta) - \ell ||\beta||^2/2, \ell) \).

It is easy to verify that \( T \circ T = T_{\alpha+\beta} \) and \( T \sigma T^{-1} = T_{\sigma \alpha} \) for \( \alpha \in h^* \) and \( \sigma \in W \). For \( \alpha \in Q^* \), the \( T_\alpha \)'s generate a copy of \( Q^* \). The group \( W \) normalises this translation group, so the two groups together generate a group isomorphic to \( Q^* \times W \).

**Proposition 2.** \( \hat{W} \cong Q^* \times W \).

**Proof.** We have just seen that \( \hat{W} \) lies in the group generated by \( W \) and the \( T_\alpha \)'s \( (\alpha \in Q^*) \). Clearly \( W \subseteq \hat{W} \) since \( \sigma_\alpha = \sigma_{(\alpha,0,0)} \). In addition we can get \( T_\alpha \in \hat{W} \) by taking \( n = 1 \) in the above discussion. Thus \( \hat{W} \) is generated by \( W \) and the \( T_\alpha \)'s \( (\alpha \in Q^*) \), as required.

**Euclidean realisation.** For fixed level, we consider the first component of the hyperbolic realisation. An affine root \( (\alpha, n, 0) \) gives rise to a transformation \( X \mapsto \sigma_\alpha(X) + n \ell \alpha \). Conjugating by the homothety \( R(X) = \ell X \), we get transformations \( X \mapsto \sigma_\alpha(X) \) which give the usual euclidean generators of the affine Weyl group.Let \( h_{\alpha,n} = \{ X \in h : \alpha(X) = n \} \), an affine hyperplane. Set \( h' = h \cup h_{\alpha,n} \). The connected components of \( h' \) are called Weyl alcoves \( A \). The affine Weyl group \( \hat{W} = \hat{W}(F) \) acting in \( h \) is the group generated by reflections \( \sigma_{\alpha,n} \) in the hyperplane \( h_{\alpha,n} \). Evidently

\[
\sigma_{\alpha,n}(x) = \sigma_\alpha(x) + n \alpha \quad \text{(*)}
\]

Let \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \), identifying \( h \) and \( h^\vee \) using the normalised inner product. Let \( Q \) be the lattice generated by \( F \) in \( h^\vee \) and \( Q^\vee \) the lattice generated by \( F^\vee \) in \( h^\vee \). The Weyl group \( W \) acts in both \( h \) and \( h^\vee \); it is generated by \( \sigma_{\alpha^\vee} = \sigma_\alpha^\vee \) where \( \alpha_i \)'s are walls of a Weyl chamber in \( h \). Under the identification \( h \cong h^\vee \), \( \sigma_\alpha^\vee \) are walls of the corresponding chamber in \( h^\vee \). The Weyl group acts simply transitively on the chambers. Every element of \( \Phi \) is a positive or negative integer combination of the \( \alpha_i \)'s. The following result is an immediate corollary of Proposition 2; it can also easily be proved directly.

**Proposition 2*.** \( \hat{W} = Q^* \times W \), where \( Q^* \) acts on \( h \) by translation.

**Proof.** Formula (*) shows that \( \hat{W} \subseteq Q^* \times W \). Clearly \( W \subseteq \hat{W} \), since \( \sigma_\alpha = \sigma_{(\alpha,0,0)} \). Moreover \( \sigma_{\alpha_1,1} \sigma - \alpha(x) = x + \alpha \). Hence \( Q^* \subseteq \hat{W} \).

**Corollary 1.** Restriction to the first component with \( \ell = 1 \) gives an isomorphism between the hyperbolic and euclidean affine Weyl groups.

**Corollary 2.** \( Q^* \times W \) permutes the \( h_{\alpha,n} \) and hence the Weyl alcoves. Hence the same is true of \( \hat{W} = Q^* \times W \subseteq Q^* \times W \).

**Proof.** Clearly

\[
\sigma(h_{\alpha,n}) = h_{\alpha,n} \quad \text{(1)}
\]

so \( W \) permutes the hyperplanes. Since \( (Q^*, Q) \cong Z \), \( Q^* \) also permutes the hyperplanes; in fact if \( X \in Q^* \) we have

\[
X + h_{\alpha,n} = h_{\alpha,n + \alpha(X)} \quad \text{(2)}
\]

Hence \( Q^* \times W \) permutes the hyperplanes.

**The highest root.** Fix a Weyl chamber \( C \) and and let \( \alpha_1, \ldots, \alpha_\ell \) be the corresponding simple roots. Let \( \theta \) be the highest weight of the adjoint representation on \( g \). Since \( \theta \) is the highest root and \( \sigma_\alpha \theta = \theta - (\theta, \alpha_i) \alpha_i \) is also a root, we must have \( (\theta, \alpha_i) \geq 0 \). Since \( \theta \) is a positive root, we may write \( \theta = \sum_{i=1}^n d_i \alpha_i \) with \( d_i \geq 0 \). Since \( \alpha_i \) is also a weight of \( g \), \( \theta - \alpha_i \geq 0 \). Thus \( d_i \geq 1 \) for all \( i \). Take \( X_1 \in h \) with \( \alpha_i(X_1) = \delta_{ij} \), the dual basis of \( Q^* \). Recall that we have normalised the inner product on \( g \) (and hence \( h \) and \( h^\vee \)) so that \( (\theta, \theta) = 2 \). As usual we identify \( h \) and \( h^\vee \) using this normalised inner product.
Proposition 3. Let $\mathcal{A}$ be the unique Weyl alcove contained in $C$ with $0 \in \overline{\mathcal{A}}$. Then $\mathcal{A} = \{ X \in \mathfrak{h} : (X, \alpha_i) \geq 0, (X, \theta) < 1 \}$ with closure $\overline{\mathcal{A}} = \{ X \in \mathfrak{h} : (X, \alpha_i) \geq 0, (X, \theta) \leq 1 \}$. Moreover $\mathcal{A}$ is a simplex with vertices $m_i^{-1} \lambda_i$, where $\lambda_i$ are the fundamental weights with level $m_i$.

Proof. Clearly $\mathcal{A} = \{ X \in \mathfrak{h} : (X, \alpha) \in (0,1)(\alpha \in \Phi^+) \}$. Let $\mathcal{A}' = \{ X \in \mathfrak{h} : \alpha_i(X) > 0, \theta(X) < 1 \}$. Plainly $\mathcal{A} \subseteq \mathcal{A}'$. Conversely if $X \in \mathcal{A}$, then $\alpha(X) > 0$ for all $\alpha \in \Phi^+$; and, since $\theta - \alpha$ is a non–negative combination of simple roots for any root $\alpha$, $\alpha(X) \leq \theta(X) < 1$ for $\alpha \in \Phi^+$. Thus $X \in \mathcal{A}$, so that $\mathcal{A}' \subseteq \mathcal{A}$. Hence $A = \mathcal{A}'$. Since $\theta = \sum d_i \alpha_i$ with $d_i \geq 0$, this means that $A$ is a simplex with vertices at 0 and $m_i^{-1} \lambda_i$ given by $(Y_i, \alpha_i) = \delta_{ij}$. Since $(\lambda_i, \alpha_i) = m_i \delta_{ij}$ (where $\theta' = \sum m_i \alpha_i'$), we see that $Y_i = m_i^{-1} \lambda_i$ as claimed.

Proposition 4. The affine Weyl group permutes the Weyl alcoves transitively and is generated by the reflections in the walls of $A$ (“simple reflections”). In particular every weight is in the affine Weyl group orbit of a dominant weight.

Proof. Let $W_0$ be the subgroup of $\hat{W}$ generated by the simple reflections. We first prove that $W_0$ permutes the Weyl alcoves transitively. Let $A$ be the standard alcove and let $A'$ be another alcove. Take $X \in A$, $X' \in A'$. Choose $\sigma \in W_0$ minimising $\| X - \sigma X' \|$ and set $Y = \sigma X'$. If the line segment $[X, Y]$ crosses a hyperplane it must also cross a wall $\mathfrak{h}_i$ of $A$; but then $\| X - Y \| \geq \| X - \sigma_i(Y) \|$, contradicting minimality since the simple reflection $\sigma_i$ is in $W_0$. Thus $Y$ must lie in $A$ and hence $A' = \sigma A$ with $\sigma \in W_0$.

Since $W_0$ permutes the Weyl alcoves transitively, facts about $\mathcal{A}$ can be transported to $A\sigma A$. In particular any alcove $A'$ has well–defined walls. Since $\mathcal{A}$ is an alcove for any $\sigma \in \mathcal{Q}^\ast \times W$, it follows that every hyperplane $\mathfrak{h}_{\alpha, n}$ is the wall of some alcove; for (1) and (2) every hyperplane is the image of a wall of $A$ under $Q^\ast \times W$. Let $A'$ be an alcove having $\mathfrak{h}_{\alpha, n}$ as a wall and take $\sigma \in W_0$ such that $A' = \sigma A$. Then $\sigma^{-1} \mathfrak{h}_{\alpha, n}$ is a wall $\mathfrak{h}_i$ of $A$. It follows that $\sigma^{-1} \sigma_{\alpha, n} \sigma = \sigma_i$. Hence $\sigma_{\alpha, n} = \sigma_i \sigma^{-1}$ lies in $\hat{W}$ and so $\hat{W} = W_0$.

Remark. It is also true that the affine Weyl group permutes the Weyl alcoves simply transitively and that the Weyl alcove is a fundamental domain. Although we shall not need these results, we note that they are easy to prove directly for the affine Weyl group of $SU(N)$. Recall that the integer lattice $\Lambda = \mathbb{Z}^N$ acts by translation on $\mathbb{R}^n$. The symmetric group $S_N$ acts on $\mathbb{R}^N$ by permuting the coordinates and normalises $\Lambda$, so we get an action of the semidirect product $\Lambda \times S_N$. The subgroup $\Lambda_0 = \{(N + \ell)(m_i) : \sum m_i = 0 \} \subset \Lambda$ is invariant under $S_N$, so we can consider the semidirect product $W = \Lambda_0 \times S_N$. This is essentially the affine Weyl group of $SU(N)$.

Lemma 1. If $\sigma_i$ is the hyperbolic reflection corresponding to a simple root $\alpha_i$, then $\sigma_i$ permutes $\overline{\Phi} \setminus \{ \alpha_i \}$ (and $\sigma_i \alpha_i = -\alpha_i$). Moreover the $\sigma_i$ preserves the multiplicity of a root. Hence each $\sigma \in \hat{W}$ permutes the roots, preserving their multiplicities.

Proof. We know that $\alpha_i$ is a positive root iff $\alpha = \sum_{i=0}^n n_i \alpha_i$ with $n_i \geq 0$. If $\alpha \neq \alpha_i$, it cannot be a multiple of $\alpha_i$ so $n_j > 0$ for some $j \neq i$. But $\sigma_i \alpha = \alpha - \alpha_i$, so the coefficient of $\alpha_j$ in $\sigma_i \alpha$ is also $n_j$. Thus $\sigma_i \alpha$ must be positive. Note that $\sigma_i(0, n, 0) = (0, n, 0)$, so the $\sigma_i$’s preserve root multiplicities.

Lemma 2. $\sigma_i \overline{p} = \overline{p} - \alpha_i$.

Proof. We have $\sigma_i \overline{\alpha_j} = \overline{\alpha_j} - \delta_{ij} \overline{\alpha_i}$ because $(\overline{\alpha_i}, \overline{\alpha_j'}) = \delta_{ij}$. This implies the result because $\overline{p} = \sum_{i=0}^n \overline{\alpha_i}$.

9. CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS. We now prove an analogue of the Harish–Chandra theorem for an affine Kac–Moody algebra: the proof is almost identical to the finite–dimensional case. Let $\overline{\lambda}$ be a dominant weight. For each simple root $\alpha_i$, let $E_i, F_i, H_i$ be the basis of the Lie algebra $sl(2)$ corresponding to the simple root $\alpha_i$. Let $g_2 = h_{\mathfrak{g}} \oplus \bigoplus_{\alpha_i > 0} \mathfrak{g}_i$ and $g_1 = \bigoplus_{\alpha_i < 0} \mathfrak{g}_i$. These are Lie subalgebras of $\mathfrak{g}$ with $\mathfrak{g} = g_1 \oplus g_2$. We know that $g_2$ is generated by the $E_i$’s and $H_i$ and $g_1$ is generated by the $F_i$’s. Consider the 1–dimensional representation sending $E_i$ to 0 and $H \in \mathfrak{h}$ to $i\overline{H}$. Let $M(\overline{\lambda})$ be the corresponding Verma module. Thus if $v = \overline{\lambda}$ then $M(\overline{\lambda})$ is the highest weight vector of $M(\overline{\lambda})$, we have $E_i v = 0$ and $H(H \lambda) = \overline{\lambda}(H_i)$ where $\overline{\lambda}(H_i) \in \mathbb{Z}$, for all $i$. We know that $M(\overline{\lambda})$ has a unique maximal submodule $N$ such that $L(\overline{\lambda}) = M(\overline{\lambda})/N$ is irreducible as a $\mathfrak{g}$–module. In fact, since $\mathfrak{h}$ is diagonalisable, every submodule is the sum of its weight spaces. Hence if we take $N$ to be the algebraic sum of all proper submodules, we must have $v \not\in N$, so that $N$ is the unique maximal proper submodule. By the $sl(2)$ theory, if $\ell_i = (\overline{\lambda}, \overline{\alpha_i'})$, then $\ell_i$ is an eigenfunction of $\ell_i = \overline{\alpha_i'}$.
then $w_i = F_i^{\ell+1}v_i$ is a singular vector i.e. $E_iw = 0$ and $w$ is an eigenvector for $\hat{g}$. It therefore generates a proper submodule (all weights are strictly less than $\lambda$). Hence $w_i \in N$ for all $i$. Let $N_0$ be the submodule generated by the $w_i$'s.

**Theorem (Harish–Chandra–Kac).** $L(\lambda)$ is the quotient of $M(\lambda)$ by the submodule generated by $F_i^{\ell+1}v_\lambda$.

**Proof.** We have to show that $N = N_0$. Set $L = M(\lambda)/N_0$. Thus $L$ is a cyclic module for $\hat{g}$ generated by $v = \ell\lambda v$ satisfying $Xv = i\lambda(X)v$ for $X \in \mathfrak{g}$, $E_iv = 0$ and $F_i\ell+1v = 0$. The identity

$$[a^n, b] = \sum_{r=1}^{n} \binom{n}{r} [(ad a)^r b] a^{n-r}$$

implies that the action on $L$ is locally nilpotent, i.e. some power of each $E_i$ or $F_i$ kills any vector. For the $E_i$'s this follows because the $E_i$'s lower energy. For the $F_i$'s it follows because $L$ is spanned by vectors $F_{i_1} \cdots F_{i_k} v$ where $i_1, \ldots, i_k$ are arbitrary (recall that the $F_i$'s generate the $\mathfrak{f}_\mathbb{C}$ subalgebra). Starting from the relation $F_i\ell+1v = 0$, successive application of $(*)$ and the Serre relations show that each $F_i$ is nilpotent on any such monomial vector. This local nilpotence shows that any vector in $L$ lies in a finite dimensional $sl(2)$ module for each $i$.

We claim that the weights of $L$ are invariant under the affine Weyl group $\hat{W}$. In fact suppose $w \in L$ has weight $\pi$. Then $H_iw = m_iw$ with $m_i = \mu(H_i) = (\mu, \alpha_i^\vee)$. Then $w$ lies in a sum of $sl(2)$ modules. If $m_i \geq 0$, set $u = F_i^{m_i}w$ and if $m_i < 0$, set $u = E_i^{-m_i}w$. Thus $u \neq 0$ by the $sl(2)$ theory and $u$ has weight $\lambda - m_i\alpha_i = \sigma_i\lambda$. Thus the set of weights is invariant under each simple reflection $\sigma_i$ and hence the whole of $\hat{W}$. As a consequence of this reasoning we have the following result.

**Lemma.** If $\lambda$ is dominant, then $\sigma\lambda \leq \lambda$ for all $\sigma \in \hat{W}$.

**Corollary.** $\tau\pi = \pi$ for $\tau \in \hat{W}$ iff $\tau = 1$.

**Proof.** Let $\lambda_i$ be the fundamental weights of $\hat{g}$. We can apply the lemma to these. Since $\pi = \sum_{\lambda_i \leq \lambda} \lambda_i$, the equality $\tau\pi = \pi$ and the inequalities $\lambda_i \geq \tau\lambda_i$ force $\tau\lambda_i = \lambda_i$ for $i \geq 0$. Hence $\tau = 1$.

Now suppose that $L$ is not irreducible. Then $V$ must contain a singular vector $w$ of weight $\pi$ strictly lower than $\lambda$, thus $E_iw = 0$ and $H_iw = m_iw$ where $m_i = \mu(h_i) \leq \ell_i$. But then $w$ is a highest weight vector generating an irreducible representation of each $sl(2)_i$. On the other hand let $\Omega$ be the quantum Casimir operator of $\hat{g}$ and set $C = 2(\ell + g)\Omega$. Then $Cw = \ell(\lambda + \pi, \lambda + \pi)w$, so by cyclicity $\Omega = ((\lambda + \pi, \lambda + \pi) - (\pi, \pi))\lambda$. Since $\Omega w = ((\pi, \pi) + (\lambda + \pi, \lambda + \pi) - (\pi, \pi)), w$, we must have $((\lambda + \pi, \lambda + \pi) - (\pi, \pi)) = ((\pi, \pi) + (\lambda + \pi, \lambda + \pi) - (\pi, \pi))$. As in the finite–dimensional case, the proof is completed by the contradiction implied by Freudenthal’s lemma.

**Freudenthal’s Lemma.** Let $\lambda = (\lambda_0, 0, 0, \ell)$ be a dominant and let $\pi = (\mu, k, 0, 0)$ be another weight such that $\lambda - \pi = \sum_{i=0}^{m} n_i\alpha_i$ with $n_i \geq 0$. Then $(\lambda + \pi, \lambda + \pi) \geq (\pi, \pi) + (\lambda + \pi, \lambda + \pi)$ with equality iff $\lambda = \pi$.

**Proof.** Take $\tau \in \hat{W}$ such that $\tau(\pi + \pi)$ is dominant. Thus $\tau\pi \leq \lambda$ and $\tau\pi \leq \pi$ by the lemma. But then

$$0 = (\lambda + \pi + \tau\pi, \lambda + \pi + \tau\pi - \tau\pi, \lambda - \tau\pi) \geq (\pi, \pi) + (\lambda, \lambda)$$

Hence $(\pi, \lambda - \tau\pi) = 0$ and $(\tau\pi, \pi - \tau\pi) = 0$, so that $\lambda = \tau\pi$ and $\pi = \tau\pi$. By the corollary above, $\tau = 1$ and hence $\lambda = \pi$, as required.

**11. GARLAND’S ‘NO–GHOST’ THEOREM ON UNITARITY.** Let $H = L(\lambda)$ be the irreducible representation just constructed as a quotient of the Verma module $M(\lambda)$. Consider the algebraic dual $L^*(\lambda) = \bigoplus H(k)^*$; if we take complex multiplication on the dual to be given by $z \cdot \xi = \overline{\xi}$, it is easy to verify that the canonical action of $\hat{g}$ is positive energy of level $\ell$ with highest weight $\lambda$. By Schur’s lemma there is a unique isomorphism of $L(\lambda)$ onto this module, we get a $\hat{g}$–equivariant linear map from $L(\lambda)$ onto its conjugate dual. Since any such map gives and is equivalent to an invariant sesquilinear form on $L(\lambda)$, we deduce that there is an essentially unique invariant sesquilinear form $(v, w)$ on $L(\lambda)$. Since the form is invariant, its kernel is $\hat{g}$–invariant and hence trivial by irreducibility. Thus the form is non–degenerate on
If \( L(\lambda) \) is an irreducible positive energy presentation at level \( \ell \) with \( (\lambda, \theta) \leq \ell \), then the canonical invariant sesquilinear form on \( L(\lambda) \) is positive definite.

**Proof.** Let \( (v, w) \) be the invariant sesquilinear form on \( H = L(\lambda) \). By irreducibility \((\cdot, \cdot)\) is non-degenerate on \( H \). Clearly the spaces \( H(k) \) are orthogonal with respect to \((\cdot, \cdot)\) because \( D = \pi(d) \) is self-adjoint. The form must also be non-degenerate on each \( H(k) \). Each \( H(k) \) is a finite-dimensional \( \mathfrak{g} \)-module and therefore completely reducible. In particular the action of \( \mathfrak{h} \) is diagonalisable on \( H(k) \), so that \( H(k) \) breaks up as a sum of weight spaces for \( \mathfrak{h} \). Since \( \mathfrak{h} \) acts as skew-adjoint operators with respect to \((\cdot, \cdot)\), these eigenspaces must be mutually orthogonal. To prove that \((\cdot, \cdot)\) is positive definite, it therefore suffices to show that \((v, v) \geq 0\) for any vector in \( H(k) \) that is a highest weight vector for \( \mathfrak{g} \). We prove this by induction on \( H(k) \). For \( k = 0 \), this follows from the no-ghost theorem for \( \mathfrak{g} \) proved in Chapter 2. We therefore assume that \((\cdot, \cdot)\) is positive definite on \( \bigoplus_{j \leq k} \) and show that \((v, v) \geq 0\) for \( v \in H(k+1) \) of weight \( \mu \). Now

\[
2(\ell + g)((L_0 - D)v, v) = (\ell + g)[(\lambda + \rho, \lambda + \rho) - (\rho, \rho)](v, v).
\]

But we also have

\[
2(\ell + g)((L_0 - D)v, v) = \frac{1}{2} \sum_i (X_i(0)v, X_i(0)v) - 2(\ell + g)(Dv, v) + \sum_{i, n > 0} (X_i(n)v, X_i(n)v)
\]

\[
\geq (\ell + g)(||\mu + \rho||^2 - 2k)(v, v)
\]

\[
= (\ell + g)[(\rho + \rho, \rho + \rho) - (\rho, \rho)](v, v),
\]

since \((X_i(n)v, X_i(n)v) \geq 0\) for \( n > 0 \) by the induction hypothesis. Combining this equation with (1) we get

\[
[(\lambda + \rho, \lambda + \rho) - (\rho, \rho)](v, v) \geq 0.
\]

By Freudenthal’s lemma, \((\lambda + \rho, \lambda + \rho) - (\rho, \rho) > 0\), so we obtain \((v, v) \geq 0\) as required.

**11. The Character of a Positive Energy Representation.** Our aim now is to determine the character of a unitary irreducible positive energy representation \( H = L(\lambda) \). If \( H \) is a positive energy representation of \( \mathfrak{g} \) or \( \mathfrak{h} \) and \( H = \bigoplus H(n) \), we define the character \( \text{ch} L(\lambda) \) to be the formal power series \( \sum_{n \geq 0} q^n \text{Tr} H(n)(z) \) for \( z = e^T \) with \( T \in \mathfrak{h} \). Although defined as a formal power series in \( q \), the character converges absolutely for \(|q| < 1\) and \( T \in \mathfrak{h}_C \). We can write the character as \( \text{Tr}(q^d z) \). It turns out that \( q^d \) is a trace-class operator for \(|q| < 1\) (for \( 0 < q < 1 \), this means that the positive operator \( q^d \) is diagonalisable with summable eigenvalues). Actually to make the characters invariant under the modular group, it is more natural to take the normalised characters \( \text{Tr}(q^{L_0 - c/24} z) \).

**12. Bosons and Fermions on the Circle.**

**Bosons.** Consider the operators the Lie algebra generated by the operators \( X(n) \) and \( d \), where \( X \in \mathfrak{h} \). We take an orthonormal basis \( (X_i) \) of \( \mathfrak{h} \) and set \( X_i(n) = iA_i(n) \), then we obtain the boson algebra, also often called the oscillator or Heisenberg algebra. If \( \mathfrak{M} = \text{dim} \mathfrak{h} \), then we have \( \mathfrak{M} \) commuting bosonic fields, which physicists usually suggestive write as \( X_i(z) = \sum X_i(n)z^{-n-1} \). The boson algebra is the infinite-dimensional Lie algebra with basis \( \{A_i(n) \} \) satisfying the commutation relations \( [A_i(m), A_j(n)] = m \delta_{ij} \delta_{m+n,0} \). It has a conjugate-linear involution given by \( A_i(n)^* = A_i(-n) \) and a derivation \( d \) given by \( [d, A_i(n)] = -nA_i(n) \) with \( d^* = d \). Note that the zero modes \( A_i(0) \) are central in the semidirect product. Just as with affine algebras, we may consider positive energy unitary representations. As before every positive energy representation is a direct sum of irreducible positive energy representations.

**Theorem.** The boson algebra has a unique irreducible positive energy representation, up to tensoring by characters of \( d \) and the \( A_i(0) \)'s.
Proof. Let \( H \) be a positive energy irreducible representation. Then the \( A_i(0) \)'s leave the lowest energy subspace \( H(0) \) invariant and form a commuting self-adjoint set. Thus we can find \( \Omega \in H(0) \) such that \( A_i(0)\Omega = \mu_i \Omega \). The vector \( \Omega \) is annihilated by the creation operators \( A_i(n) \) for \( n > 0 \), so by the commutation relations (or the Poincaré–Birkhoff–Witt theorem) the submodule generated by \( \Omega \) is spanned by vectors \( \prod A_i(-n)^{k_i} \Omega \) where \( k_{i,n} \geq 0, n \geq 1 \). (Note that the operators \( A_i(-n) \) commute for \( n < 0 \).) It is invariant under \( d \) and the \( A_i(0) \)'s and therefore coincides with \( H \) by irreducibility. Clearly \( H(0) = \mathbb{C} \Omega \). Note that each pair of elements \( A = \sqrt{\mathbb{C}} A_i(n) \) and \( A^* = \sqrt{\mathbb{C}} A_i(-n) \) \( n > 0 \) satisfies the Heisenberg commutation relations \( AA^* - A^*A = I \). This is a copy of the usual Heisenberg Lie algebra. \( A \) is called an annihilation operator and \( A^* \) a creation operator. The following is a standard computation in quantum mechanics.

Lemma. Let \( D = A^* A \) and \( \xi_n = A^n \xi_0 \), where \( \xi_0 \) is a vector satisfying \( A \xi_0 = 0 \).

(a) \( A A^n - A^n A = n A^{n(-1)} \).
(b) \( D \xi_n = n \xi_n \).
(c) \( \langle \xi_n, \xi_m \rangle = \delta_{nm} \langle \xi, \xi \rangle \).

Proof. (a) follows by induction from \( [A_i^n, A_i^n] = I \), since \( A A^n - A^n A = (AA^{n-1} - A^{n-1} A)A^* + A^{n-1}(AA^* - A^*A) = n A^{n-1} \). Next by (a), \( D \xi_n = A^n A \xi_0 = (A^n + n A^n) \xi_0 = n \xi_n \), since \( A \xi_0 = 0 \).

So (b) follows. Since the \( \xi_n \)'s correspond to different eigenvalues of the self-adjoint operator \( D \), they must be pairwise orthogonal. Moreover we have

\[
\langle \xi_n, \xi_m \rangle = (A^n \xi_0, A^m \xi_0) = (A^n A \xi_0, A^m \xi_0) = (A^n A x_0 + n A^{n-1} \xi_0, A^m \xi_0) = n (A^{n-1} \xi_0, A^{m-1} \xi_0).
\]

Thus \( \| \xi_n \|^2 = n \| \xi_n - 1 \|^2 \), so the result follows by induction.

Prophetic remark. Note that \( [D, A] = -A \) and \( [D, A^*] = A^* \). Mathematically this may be viewed as part of the metaplectic action of the symplectic Lie algebra, the bosonic version of spin quantisation for fermions. Below we will use bilinears in bosons to give the Fubini–Veneziano construction of an \( L_0 \) operator implementing \( d \). In fact we will give a construction of the entire Virasoro algebra with central charge 1. Like the Sugawara construction, this is another example of quantisation in infinite dimensions.

The computation above implies that all the vectors \( \prod A_i(-n)^{k_i} \Omega \) are orthogonal with

\[
\| \prod A_i(n)^{k_i} \Omega \|^2 = \prod k_{i,n}! n^{k_{i,n}}.
\]

Conversely the standard Verma module construction produces a similar basis generated from a vector \( \Omega \) with \( A_i(0) \Omega = \mu_i \Omega \). As before, there is unique invariant sesquilinear form on it. By the previous computation it is positive definite and coincides with the formula given above. It may be realised on polynomials \( C[z_{i,n}] \) by \( A_i(n) = n z_{i,n} \), \( A_i(-n) = z_{i,n} \), \( A_i(0) = \mu I \) and \( d z_{i,n} = n z_{i,n} \). To prove the representation is irreducible we proceed as in Chapter 1, acting by annihilation operators \( \partial z_{i,n} \) until we get the vacuum and then acting by creation operators until we get all vectors.

Corollary. The character of the irreducible representation \( H \) of \( \hat{h} \) with \( X_j(0) \) acting as \( i \mu_j \) and \( d \) as \( h \) on \( H(0) \) is \( e^{\mu X_i q^h} \prod_{n \geq 1} (1 - q^n)^{-1} \).

As promised we now produce the quantised action of the Virasoro algebra, which is a simpler case of the Sugawara construction (for an Abelian Lie algebra rather than a simple Lie algebra).

Proposition (Fubini–Veneziano construction). Let

\[
L_0 = \frac{1}{2} A_i(0)^2 + \sum_{j > 0} A_i(-j) A_i(j) = -\sum_{i,j > 0} \frac{1}{2} X_i(0)^2 + \sum_{j > 0} X_i(-j) X_i(j)
\]

and \( L_m = \frac{1}{2} \sum_{i,j \neq 0} A_i(-j) A_i(j + m) \) for \( m \neq 0 \). Then \( [L_m, A_i(n)] = -n A_i(n + m) \) and \( [L_m, L_n] = (m - n) L_{m+n} + M \frac{m(m-1)}{2} \delta_{m,n,0} \). Thus the central charge is \( M \), the number of bosons.

Proof. To check that \( [L_0, A_k(n)] = -n A_k(n) \), we compute

\[
[L_0, A_k(n)] = \sum_{j > 0} (A_i(-j), A_k(n) A_i(j) + A_i(-j) A_i(j), A_k(n)) = -n A_k(n).
\]
A similar computation shows that \([L_m, A_k(n)] = -nA_k(n + m)\). It follows that \([L_m, L_n] - (m - n)L_{m+n}\) commutes with all \(A_k(n)\)'s and hence equals a scalar. Since this operator lowers energy by \(m + n\), this scalar is zero unless \(m + n = 0\). Thus if \(m > 0\), we have \([L_m, L_{-m}] = 2mL_0 + \lambda(m)I\). Now \(L_{-m}\Omega = \frac{1}{2} \sum_{j=0}^{m} A_i(-j)A_i(j - m)\Omega\), so that

\[
[L_m, L_{-m}]\Omega = L_m L_{-m}\Omega = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=0}^{m} L_m A_i(-j)A_i(j - m)\Omega \\
= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=0}^{m} jA_i(-j + m)A_i(j - m)\Omega + (m - j)A_i(-j)A_i(j)\Omega.
\]

Thus

\[
([L_m, L_{-m}]\Omega, \Omega) = m\mu^2/2 + \sum_{j=0}^{m} j(a_{-j}\Omega, a_{-j}\Omega) = m\mu^2/2 + \sum_{j=0}^{m} (m-j) = m\mu^2 + (m^3 - m)/12.
\]

Hence \(\lambda(m) = M(m^3 - m)/12\) as required.

**FERMIONS.** The fermion algebra (or Clifford algebra) is the Lie superalgebra with basis \(\psi_i(n)\) \((n \in \mathbb{Z}, 1 \leq i \leq N)\) satisfying the anticommutation relations \([\psi_i(n), \psi_j(n)] = \delta_{m+n,0}\delta_{ij}\) and the adjoint condition \(\psi_i(n)\ast = \psi_i(-n)\). It has a derivation \(d\) given by \([d, \psi_i(n)] = -m\psi_i(n)\). Note that this is really a collection of \(N\) independent fermi fields on the circle. The zero modes form a subalgebra which can be identified with the real Clifford algebra on an \(N\)-dimensional real inner product space (let \(c_i = \sqrt{2}\psi_i(0)\)). As we have seen in Chapter I, these finite–dimensional algebras behave differently for \(N\) even or odd, so for this reason our account is not entirely parallel to the bosonic case.

We start by defining a cyclic representation of the fermion algebra on the exterior algebra on the unit vectors \(v_{i,j}\) \((j \geq 0)\). \(\psi_i(n)\) acts as \(e(v_{i,n})\ast\) for \(n > 0\) and \(e(v_{i,-n})\) for \(n < 0\); \(\psi_i(0)\) acts as \(\frac{1}{\sqrt{2}}(e(v_{i,0}) + e(v_{i,0})\ast)\). The operator \(d\) acts as the (even) derivation \(dv_{i,n} = n v_{i,n}\). It is clear that we have defined a positive energy representation \(H\) with lowest energy space \(H(0)\), the exterior algebra on the \(v_{i,0}\)'s. This is not an irreducible representation of the zero mode algebra, but the usual creation–annihilation argument shows that any irreducible submodule \(W \subset H(0)\) generates an irreducible representation of the fermion algebra. When \(N\) is odd, this representation will not be graded, because the grading operator does not lie inside the Clifford algebra.

However when \(N\) is even, the grading operator is proportional to \(\psi_1(0) \cdots \psi_N(0)\) and \(W\) is automatically graded. The irreducible module generated by \(W\) is clearly isomorphic to graded tensor product of \(W\) and the exterior algebra on the generators \(v_{i,n}\) with \(n \geq 1\). Conversely any positive energy irreducible representation must have this form: for \(H(0)\) must be irreducible as a module over the zero modes and if \(w_j\) is an orthonormal basis of \(W = H(0)\), then the vectors in the Verma module obtained by applying products of distinct \(\sqrt{2}\psi_i(-n)'s\) to different \(w_j's\) are all orthonormal.

The fermionic version of the Fubini–Venziano construction is defined on the exterior algebra via

\[
L_0 = \frac{N}{16} + \sum_{i,j>0} j\psi_i(-j)\psi_i(j), \quad L_k = \frac{1}{2} \sum_{i,j} j\psi_i(-j)\psi_i(j + k) \quad (k \neq 0).
\]

As before we prove that \([L_0, \psi_i(n)] = -n\psi_i(n)\) and \([L_k, \psi_i(n)] = -(n + \frac{k}{2})\psi_i(n + k)\). Again \([L_m, L_n] - (m - n)L_{m+n}\) commutes with \(\psi_i(n)\) and lowers energy by \(m + n\). If \(m + n > 0\), it therefore acts trivially on the vacuum vector \(\Omega = 1\) and hence everywhere. Thus \([L_m, L_n] = (m-n)L_{m+n}\) for \(m+n>0\). Taking adjoints the same is true for \(m+n<0\). We check directly that \([L_m, L_{-m}]\Omega = \frac{N}{24}(m^3 - m) + \frac{mN}{19}\), by a computation similar to the bosonic one. Hence

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{N}{24}(m^3 - m)\delta_{m+n,0},
\]

so that we get a quantised representation of the Virasoro algebra, with central charge \(c = N/2\).
13. THE KAZAMA–SUZUKI SUPERCHARGE OPERATOR.

A. GKO supercharge operator in Cliff $L_{\mathfrak{g}}$. Let $(X_{a})$ be an orthonormal basis of $\mathfrak{g}$ with $[X_{a}, X_{b}] = \sum f_{abc}X_{c}$. Let $(\psi_{n}(a))$ be Ramond fermions on $L_{\mathfrak{g}}$, so that $\{\psi_{n}(a), \psi_{m}(a)\} = \delta_{nm}$. Let $F_{\mathfrak{g}}$ be the fermionic Fock space giving a cyclic representation of the GKO supercharge operator $Q_{a}$ on $L_{\mathfrak{g}}$. The supercharge operator thus interchanges fermions $\psi_{n}$ with $\pm \psi_{m}$ and bosons $S_{n}$ with $S_{-n}$. Let $S_{a}(n) = -\frac{1}{2} \sum f_{abc}\psi_{b}(m)\psi_{c}(n-m)$ on $F_{\mathfrak{g}}$. Then the anticommutation relations for $\psi$ immediately imply that

$$[S_{a}(n), \psi_{b}(m)] = \sum f_{abc}\psi_{c}(n+m),$$

and $[d, S_{a}(n)] = -nS_{a}(n)$ so that by uniqueness $[S_{a}(m), S_{b}(n)] = \sum f_{abc}S_{c}(n+m)\delta_{n+m,0}C(m)$, where $C(m)$ is a constant. Taking vacuum expectations, i.e., computing $\langle [S_{a}(m), S_{b}(-m)]\Omega, \Omega \rangle$, we find $C(m) = -mg$, with $g$ the dual Coxeter number. Let $Q_{a} = \frac{1}{2} \sum_{a,m} \psi_{a}(m)S_{a}(-m)$ (note that $[\psi_{a}(m), S_{a}(n)] = 0$ so that no normal ordering is required). Clearly $S_{a}(m)^{*} = -S_{a}(m)$ and $Q_{a}^{*} = -Q_{a}$. Then

$$3\{Q_{a}, \psi_{b}(m)\} = \{\psi_{a}(m), \psi_{b}(m)\}S_{a}(-m) + \psi_{b}(m)[S_{a}(-m), \psi_{a}(m)]$$

$$= S_{b}(n) + \sum f_{abc}\psi_{c}(m)\psi_{b}(n-m)$$

$$= S_{b}(n) - \sum f_{bac}\psi_{c}(m)\psi_{b}(n-m)$$

$$= 3S_{b}(n).$$

Thus $\{Q_{a}, \psi_{b}(m)\} = S_{b}(n)$. If $n \neq 0$, let $\xi = -<\psi_{b}(n)\psi_{b}(n)>$. Then $[d, \xi] = -n\xi$ and $[\xi, \psi_{a}(m)] = 0$. So by cyclicity $\xi$ must be a scalar and hence zero. Thus $[Q_{a}, S_{b}^{*}(-m)] = +ng\psi_{b}(n)$; this relation also holds for $n = 0$ because the construction is manifestly $\mathfrak{g}$-invariant. In summary we have obtained the “supersymmetry relations”

$$\{Q_{a}, \psi_{b}(m)\} = S_{b}(n), \quad [Q_{a}, S_{b}(n)] = ng\psi_{b}(n); \quad (*)$$

the supercharge operator thus interchanges fermions $\psi_{b}(n)$ and bosons $S_{b}(n)$.

B. GKO supercharge operator in $\text{End}(H) \otimes \text{Cliff} L_{\mathfrak{g}}$. Let $L(\mathfrak{g})$ be a level $\ell$ positive energy representation of $L_{\mathfrak{g}}$ with corresponding generators $T_{a}(m)$. These satisfy the commutation relations

$$[T_{a}(m), T_{b}(n)] = \sum f_{abc}T_{c}(m+n) - \ell m\delta_{ab}\delta_{m+n,0}.$$ 

We extend the operator $Q_{a}$ to $L(\mathfrak{g}) \otimes F_{\mathfrak{g}}$ as $Q_{a} \equiv I \otimes Q_{a}$. Now consider the operator $Q_{1} = \sum T_{a}(m)\psi_{a}(-m)$ on $L(\mathfrak{g}) \otimes F_{\mathfrak{g}}$. Then

$$\{Q_{1}, \psi_{b}(n)\} = \sum T_{a}(m)\{\psi_{a}(-m), \psi_{b}(n)\} = \sum T_{a}(m)\delta_{ab}\delta_{mn} = T_{b}(n),$$

so that $\{Q_{1}, \psi_{b}(n)\} = T_{b}(n)$. Moreover

$$[Q_{1}, S_{b}(n)] = \sum T_{a}(m)[\psi_{a}(-m), S_{b}(n)] = -\sum f_{bac}T_{a}(m)\psi_{c}(n-m)$$

(1)

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and \([Q_1, T_b(n)] = \sum [T_a(m), T_b(n)] \psi_a(-m)\), so that
\[
[Q_1, T_b(n)] = \sum f_{abc} T_c(m + n) \psi_a(-m) - \ell m \psi_b(n) = \sum f_{bac} T_a(m) \psi_c(n - m) - \ell n \psi_b(n).
\]
(2)

Adding (1) and (2), we get \([Q_1, T_b(n) + S_b(n)] = n \ell \psi_b(n)\). Thus we have
\[
\{Q_1, \psi_b(n)\} = T_b(n), \quad [Q_1, T_b(n) + S_b(n)] = n \ell \psi_b^n.
\]
(\*)

Let \(Q = Q_0 + Q_1\) and \(X_b(n) = T_b(n) + S_b(n)\). The supercharge operator is therefore given by the formula
\[
Q = \sum (T_a(m) + \frac{1}{3} S_a(m)) \psi_a(-m);
\]
the factor 1/3 is very important here and is often given incorrectly in much of the literature. Combining (*)& (**), we get the supersymmetry relations
\[
\{Q, \psi_a(n)\} = X_a(n), \quad \{Q, X_a(n)\} = n(\ell + g) \psi_a(n).
\]

C. Kazama–Suzuki supercharge operator. We keep the above notation, but write \(Q = Q^\mathfrak{h}\) to show the dependence of the construction on \(\mathfrak{g}\). Thus
\[
\{Q^\mathfrak{h}, \psi_a(n)\} = X_a(n), \quad \{Q^\mathfrak{h}, X_a(n)\} = n(\ell + g) \psi_a(n).
\]
(3)

Let \(\mathfrak{h}\) be a maximal torus in \(\mathfrak{g}\) with orthogonal complement \(\mathfrak{m}\). We may choose the orthonormal basis of \(\mathfrak{g}\) to be made up of orthonormal bases \((X_A)\) for \(\mathfrak{h}\) and \((X_i)\) for \(\mathfrak{m}\). We then have fermions \(\psi_a(n)\) and \(\psi_i(n)\). We take the submodule of \(\mathcal{F}_{\mathfrak{g}}\) given by the tensor product \(\mathcal{F}_{\mathfrak{g}} \otimes \mathcal{F}_{\mathfrak{m}}\), where the \(\psi_a(n)\)'s act only on the first factor, irreducibly, and the \(\psi_i(n)\)'s act only on the second factor. Since \(S_A(n)\) commutes with all \(\psi_B(n)\)'s, they act only on the first factor; indeed we have \(S_A(n) = -\frac{1}{2} \sum f_{Aij} \psi_i(m) \psi_j(n - m)\) and \(f_{ABC} = 0\); so that \(S_A(n) = -\sum f_{Aij} \psi_i(m) \psi_j(n - m)\). Consider the representation of \(\mathcal{L}_\mathfrak{h}\) on \(K(\lambda) \otimes \mathcal{F}_{\mathfrak{m}}\) given by \(Y_A(n) = T_A(n) + S_A(n) = X_A(n)\). Thus \([Y_A(m), Y_B(n)] = -m(\ell + g) \delta_{AB} \delta_{m+n,0}\), since \(\mathfrak{h}\) is Abelian. The supercharge operator \(Q^\mathfrak{h}\) corresponding to \(\mathfrak{h}\) on \(K(\lambda) \otimes \mathcal{F}_{\mathfrak{h}}\) is just
\[
Q^\mathfrak{h} = \sum Y_A(m) \psi_A(-m).
\]

It satisfies
\[
\{Q^\mathfrak{h}, \psi_A(n)\} = Y_A(n), \quad \{Q^\mathfrak{h}, Y_A(n)\} = n(\ell + g) \psi_A(n).
\]
(4)

The Kazama–Suzuki supercharge operator is defined by \(Q = Q^\mathfrak{g} - Q^\mathfrak{h}\) (the supersymmetric coset construction). Comparing (3) and (4), we see that
\[
\{Q, \psi_A(n)\} = 0, \quad [Q, X_A(n)] = 0.
\]

The first equation tells us that \(Q\) really acts on \(L(\mathfrak{h}) \otimes \mathcal{F}_{\mathfrak{m}}\), while the second implies that \(Q\) commutes with the natural action of \(\mathcal{L}_\mathfrak{h}\) there. To see why \(Q\) explicitly why \(Q\) acts on \(L(\mathfrak{h}) \otimes \mathcal{F}_{\mathfrak{m}}\), recall that
\[
Q^\mathfrak{g} = \sum (T_a(m) + \frac{1}{3} S_a(m)) \psi_a(-m)\]
and \(Q^\mathfrak{h} = \sum Y_A(m) \psi_A(-m) = \sum (T_A(m) + S_A(m)) \psi_A(-m)\). Thus we get
\[
Q = \sum (T_i(m) + \frac{1}{3} S_i(m)) \psi_i(-m),
\]
where
\[
S_i(n) = -\frac{1}{2} \sum f_{ijk} \psi_i(m) \psi_j(n - m).
\]
Thus
\[
Q = \sum T_i(m) \psi_i(-m) - \frac{1}{6} \sum f_{ijk} \psi_i(m) \psi_j(n) \psi_k(-m - n),
\]
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14. THE SQUARE OF THE SUPERCHARGE OPERATOR.

A. Computations in Cliff $L_0$. We already have proved the supersymmetry relations $\{Q_0, \psi_b(n)\} = S_b(n)$ and $[Q_0, S_b(n)] = n g \psi_b(n)$.

Proposition. $\frac{-1}{g} Q_0^2 = L_0^\psi = \dim g/48$.

Proof. Note that $\frac{-1}{g} Q_0^2 = L_0^\psi$ commutes with $\psi_b^\dagger$. Recall that $Q_0^2 = -Q_0$ and that

$$Q_0 = -\frac{1}{12} \sum f_{abc} \psi_a(m) \psi_b(n) \psi_c(-m - n).$$

The corresponding finite-dimensional supercharge operator is given by $G_0 = -\frac{1}{12} \sum f_{abc} \psi_a(0) \psi_b(0) \psi_c(0)$. Thus $Q_0 \xi = G_0 \xi$ for any $\xi \in H(0)$. Hence $Q_0^2 \Omega = Q_0 G_0 \Omega = G_0^2 \Omega = -\frac{g \dim g}{24} \Omega$. Thus $(-\frac{1}{g} Q_0^2 - L_0^\psi) \Omega = \left(\frac{1}{12} - \frac{1}{12g}\right) \dim g \Omega = \frac{\dim g}{48} \Omega$. Since $\Omega$ is cyclic and $\frac{-1}{g} Q_0^2 - L_0^\psi$ central, the result follows.

Remark. If we have a Sugawara operator $L_0$ with corresponding central charge $c$, we define $L_0 = L_0 - \frac{c}{12}$. If we redefine $L_n = L_n$ for $n \neq 0$, then $[L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}$. We call $L_0$ the normalised Sugawara operator. It is the operator needed to make various characters modular invariant.

B. Computation in End($H$) $\bowtie$ Cliff $L_0$. Let $Q = \sum T_a(m) \psi_a(-m) + \frac{1}{2} S_a(m) \psi_a(-m)$. Set $X_a(m) = T_a(m) + S_a(m)$. Then we already have proved the supersymmetry relations $\{Q, \psi_a(n)\} = X_a(n)$ and $[Q, X_a(n)] = n(\ell + g) \psi_a(n)$.

Proposition. $-(\ell + g)^{-1} Q^2 = L_0^\theta + L_0^\psi$, where $L_0 = L_0 - c/24$ with $c_\theta = \ell \dim g/(\ell + g)$ and $c_\psi = \dim g/16$.

Proof. If we apply the supersymmetry relations, we get $[Q^2, \psi_a(n)] = n(\ell + g) \psi_a(n)$ and $[Q^2, X_a(n)] = n(\ell + g) X_a(n)$. Thus $-(\ell + g)^{-1} Q^2 - L_0^\theta - L_0^\psi$ commutes with the $\psi_a(n)$’s and $X_a(m)$’s. We claim that these operators act cyclically with cyclic vector $\xi \otimes \Omega$, where $\xi \in H(0)$ is a highest weight vector. Since the operator $S_a(n)$ are combinations of biinears in $\psi_b(m)$’s, the cyclic module generated by $\xi \otimes \Omega$ must also be invariant under the $S_a(n)$’s. But then it must also be invariant under $T_a(n) = X_a(n) - S_a(n)$. Since $\xi \otimes \Omega$ is obviously cyclic for the commuting actions of $\psi_a(n)$’s and $T_b(m)$’s, our claim follows. Thus it will suffice to show that $-(\ell + g)^{-1} Q^2 - L_0^\theta - L_0^\psi$ annihilates $\xi \otimes \Omega$. But if $G$ is the finite-dimensional version of $Q$, we have as before that

$$-Q^2(\xi \otimes \Omega) = -G^2(\xi \otimes \Omega) = \left(\frac{g \dim g}{24} + \frac{1}{2}(\|\lambda + \rho\|^2 - \|\rho\|^2)\right)(\xi \otimes \Omega).$$

But $(L_0^\theta + L_0^\psi)(\xi \otimes \Omega) = \left(\frac{\dim g}{16} + \frac{\|\lambda + \rho\|^2 - \|\rho\|^2}{2(\ell + g)} - \frac{\ell \dim g}{24(\ell + g)} - \frac{\dim g}{48}\right)(\xi \otimes \Omega) = \left(\frac{g \dim g}{24} + \frac{1}{2}(\|\lambda + \rho\|^2 - \|\rho\|^2)\right)(\xi \otimes \Omega),$

as required.

C. Coset construction of $Q^2$. By the coset construction $Q = Q_\theta - Q_\hbar$. Thus $Q_\theta = Q + Q_\hbar$, where $Q$ and $Q_\hbar$ anticommute, i.e. $\{Q, Q_\hbar\} = 0$. Hence

$$\{Q_\theta, Q_\hbar\} = \{Q + Q_\hbar, Q + Q_\hbar\} = \{Q, Q\} + \{Q_\hbar, Q_\hbar\}.$$

Thus $Q_\theta^2 = Q^2 + Q_\hbar^2$, so that $Q^2 = Q_\theta^2 - Q_\hbar^2$. Hence we have:

Theorem. $-\frac{1}{\ell + g} Q^2 = L_0^\theta + L_0^\psi - L_0^\hbar$ on $H \otimes F_m$. 

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Proposition. If we use the formula in B for $Q^h$, we get $-\frac{1}{\ell + g}Q^2 = L^0_0 + L^\psi_0$. By definition $L^\psi_0 = L^\psi_0 + Q^0$, so the result follows. (Note that when the tensor product of $H$ and $F_m$ is restricted to $h$, it splits up as a direct sum of positive energy irreducible representations $H_j$. The formula for $Q^2$ is valid on each tensor product $H_j \otimes F_h$)

Corollary. Let $F_m$ be the irreducible representation of Cliff $L_m$. Then on $H \otimes F_m$, we have

$$-\frac{1}{\ell + g}Q^2 = \frac{||\lambda + \rho||^2}{2(\ell + g)} - (L^h_0 - d^\psi - d^\psi),$$

where $d^\psi$ and $d^\psi$ are the energy operators on $H$ and $F_m$.

Proof. Using Freudenthal's strange formula, we get

$$-\frac{1}{\ell + g}Q^2 = L^0_0 + L^\psi_0 - (c_0 + c_\psi, m - c_0)/24$$

$$= (L^\psi_0 - d^\psi) + (L^\psi_0 - d^\psi) - (L^h_0 - d^\psi - d^\psi) - (c_0 + c_\psi, m - c_0)/24$$

$$= \frac{||\lambda + \rho||^2 - ||\rho||^2}{2(\ell + g)} + \frac{\dim m}{16} - \frac{\ell \dim g}{24(\ell + g)} - \frac{\dim m}{24} + \frac{\dim h}{24} - (L^h_0 - d^\psi - d^\psi)$$

$$= \frac{||\lambda + \rho||^2}{2(\ell + g)} - (L^h_0 - d^\psi - d^\psi),$$

as required.

15. KAC'S CHARACTER AND DENOMINATOR FORMULAS.

Proposition. (1) Let $\overrightarrow{\sigma} + \overrightarrow{\tau}$ be a weight of $h$ appearing in $L(\lambda) \otimes F_m$. Then $(\lambda + \overrightarrow{\sigma}) = (\overrightarrow{\sigma} + \overrightarrow{\tau}) = (\overrightarrow{\sigma} \cdot \overrightarrow{\tau})$.

Equality occurs in (1) if $\overrightarrow{\mu} = \sigma \lambda$ and $\overrightarrow{\nu} = \sigma \overrightarrow{\nu}$ for some $\sigma \in W$. In this case $\overrightarrow{\mu} + \overrightarrow{\nu} = \sigma(\lambda + \overrightarrow{\nu})$.

(3) $\sigma \cdot \sigma(\lambda + \overrightarrow{\nu})$ is a bijection from $W$ onto the solutions of (2).

(4) $\sigma(\lambda + \overrightarrow{\nu})$ appears in $L(\lambda) \otimes F_m$ with multiplicity one and corresponds to a tensor $\xi \otimes \eta$, where $\xi$ has weight $\overrightarrow{\tau}$ and $\eta$ has weight $\overrightarrow{\nu}$.

Proof. (1) Let $\overrightarrow{\sigma} + \overrightarrow{\tau}$ be any weight in the tensor product with corresponding vector $\xi$. The corollary in the last section gives the following formula for the square of the supercharge operator

$$-(\ell + g)^{-1}Q^2 = \frac{||\lambda + \rho||^2}{2(\ell + g)} - (L^h_0 - d^\psi - d^\psi).$$

Since $Q^2 = -Q$, the operator $-Q^2$ is positive. Thus

$$2(\ell + 2g)^{-1}(||\lambda + \rho||^2 ||\xi||^2 - ((L^h_0 - d^\psi - d^\psi)\xi, \xi) \geq (\Omega^h_0 - d^\psi - d^\psi)\xi, \xi$$

$$\geq (\Omega^h_0 - d^\psi - d^\psi)\xi, \xi$$

$$= (2(\ell + 2g)^{-1}(\overrightarrow{\sigma} + \overrightarrow{\nu})^2 ||\xi||^2,$$

where $\Omega^h_0$ is the zero–mode Casimir contribution to $L^h_0$. Clearly $\hat{\xi}$ is a highest weight vector for $h$ if and only if $(L^h_0, \xi) = (\Omega^h_0, \xi)$ and only if $L^h_0\xi = \Omega^h_0\xi$. Hence $(\lambda + \overrightarrow{\sigma}, \lambda + \overrightarrow{\tau}) \geq (\overrightarrow{\tau} + \overrightarrow{\nu}, \overrightarrow{\tau} +\overrightarrow{\nu})$: equality is possible only if $\overrightarrow{\mu} + \overrightarrow{\nu}$ is a highest weight of $h$.

(2) Suppose that $\lambda + \overrightarrow{\nu} \geq (\overrightarrow{\sigma} + \overrightarrow{\tau})$ with $\lambda \geq \overrightarrow{\sigma}$ and $\overrightarrow{\nu} \geq \overrightarrow{\tau}$. Take $\tau \in \hat{W}$ such that $\tau(\overrightarrow{\mu} + \overrightarrow{\nu}) \geq 0$. Since $\tau \overrightarrow{\mu}$ is a weight of $L(\lambda)$ and $\tau \overrightarrow{\tau}$ is a weight of $F$, we have $\lambda - \tau \overrightarrow{\mu} \geq 0$ and $\overrightarrow{\nu} - \tau \overrightarrow{\tau} \geq 0$. But then

$$0 = (\lambda + \overrightarrow{\nu} - \tau \overrightarrow{\tau}, \lambda - \tau \overrightarrow{\mu} + \tau \overrightarrow{\nu} - \tau \overrightarrow{\tau}) \geq (\overrightarrow{\nu} + \overrightarrow{\tau}) + (\overrightarrow{\nu} - \tau \overrightarrow{\tau}) \geq 0.$$
The formal power series

\[ D = \varepsilon^\sigma \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^m \]  

satisfies \( D \circ \sigma = \varepsilon(\sigma)D \) for \( \sigma \in \hat{W} \), where \( \varepsilon : \hat{W} \to \{ \pm 1 \} \) is the sign character obtained by the sign character of \( W \) and the projection \( \hat{W} \to W \).

**Proof.** It clearly suffices to show that \( D \circ \sigma_i = \varepsilon(\sigma_i)D \) for each simple reflection \( \sigma_i \). We know that \( \sigma_i \varphi = \varphi - \pi_i \). Moreover, \( \sigma_i \) permutes \( \Phi^+ \setminus \{ \pi_i \} \) and satisfies \( \sigma_i \pi_i = -\alpha_i \). Hence \( D \circ \sigma_i = -D \), as required.

**Theorem (Kac Character Formula).**

\[
\text{ch}_L(\lambda) = \sum_{\sigma \in \hat{W}} \varepsilon(\sigma)e^{\sigma(\lambda + \varphi)}D,
\]

where the denominator \( D \) is given by

\[
D = \prod_{n \geq 1} (1 - q^n)^m \cdot \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \cdot \prod_{n \geq 1, \alpha \in \Phi} (1 - e^{\alpha}q^n) = \prod_{\pi \in \Phi^+} (1 - e^{-\pi})^{\text{mult.}}.
\]

**Proof.** Let \( \text{ch}_aW \) be the supercharacter of a \( \mathbb{Z}_2 \)-graded module. Clearly

\[
\text{ch}_aL(\lambda) \otimes F_m = \text{ch} L(\lambda) \cdot \text{ch}_a F_m.
\]

Moreover, because \( F_m \) can be constructed as the tensor product of an irreducible representation of \( \text{Cliff}(m) \) on \( \Lambda^m \mathfrak{m}_+ \) and the irreducible representation of the non-zero modes \( \psi_i(n) \) on the exterior algebra with generators \( v_i, n \), we easily check that

\[
\text{ch}_a F_m = e^{\varphi} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \cdot \prod_{n \geq 1, \alpha \in \Phi} (1 - e^{\alpha}q^n).
\]

On the other hand \( \text{ch}_a L(\lambda) \otimes F_m \) can be computed by decomposing the tensor product as a direct sum of \( \mathfrak{h} \)-modules, according to the previous proposition. We find

\[
\text{ch}_a L(\lambda) \otimes F_m = \sum_{\sigma \in \hat{W}} \varepsilon'(\sigma)\text{ch} V(\sigma(\lambda + \varphi)),
\]

where \( V(\varphi) \) is the irreducible representation of \( \hat{\mathfrak{h}} \) with highest weight \( \varphi \) and \( \varepsilon'(\sigma) = \pm 1 \) according to whether the weight \( \sigma \varphi \) appears in the even or odd part of \( F_m \). But \text{ch} \( V(\mu) = e^{\mu} \cdot \prod (1 - q^n)^m \), where \( m = \dim \mathfrak{h} \) is the rank of \( \mathfrak{g} \). Hence

\[
\text{ch}_a L(\lambda) \otimes F_m = \sum_{\sigma \in \hat{W}} \varepsilon'(\sigma)e^{\sigma(\lambda + \varphi)} \cdot \prod (1 - q^n)^m.
\]

The character formula follows by combining (1), (2) and (3). It still remains to show that \( \varepsilon'(\sigma) = \varepsilon(\sigma) \). Specialising to \( \lambda = 0 \), we find

\[
e^{\varphi} \prod_{\varphi > 0} (1 - e^{-\varphi}) = \sum_{\sigma \in \hat{W}} \varepsilon'(\sigma)e^{\sigma \varphi}.
\]

We know that all the exponents \( \sigma \varphi \) are distinct and we know that the left hand side is antisymmetric. Since the coefficient of \( e^{\varphi} \) is 1 and \( \varepsilon' \) must coincide with \( \varepsilon \), as required.

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Theorem (Macdonald’s identity/Kac denominator formula).
\[ \sum_{\sigma \in \bar{W}} \varepsilon(\sigma)e^{\sigma(\overline{\mathfrak{p}})} = \prod_{\pi \in \overline{\mathfrak{p}}} (1 - e^{-\pi})^{\text{mult} \alpha}. \]

**Proof.** Since \( \text{ch}(L(0)) = 1 \), the result follows immediately from the character formula with \( \lambda = 0 \).

**Comments on the denominator formula.** Macdonald’s identities for the classical groups were first proved by Dyson independently of root systems and then for general root systems by Macdonald, without representations. In the case of \( \mathfrak{sl}_2 \), we see that the action of \( \bar{W} = \mathbb{Z} \times \mathbb{Z}/2 \) is given by \( \sigma(j, k, \ell) = (-j, k, \ell) \) for \( \sigma \in \mathbb{Z}_2 \) and \( T_n(j, k\ell) = (j + 2n\ell, k + jn + \ell m^2, \ell) \) for \( n \in \mathbb{Z} \). From the denominator formula, we retrieve Jacobi’s celebrated triple product identity:
\[ \sum_{k \in \mathbb{Z}} (-1)^k q^{(k-1)k/2} t^k = \prod_{n}(1 - q^{m-1}t)(1 - q^{m}t^{-1})(1 - q^{m}). \]

We can get another a formula for the characters in terms of theta functions. Let
\[ \Theta_{n,m}(q, z) = \sum_{k \in \mathbb{Z} + \mathbb{Z}} q^{mk^2} e^{2\pi i kmz}. \]

Then if \( 0 \leq j \leq \ell/2 \) is a half–integer, the normalised character of the irreducible positive energy representation of level \( \ell \) with spin \( j \) is given by
\[ \text{ch}(L(\ell, j)) = q^{-c/24} \text{Tr}(q^{L_0} z^{H}) = \frac{\Theta_{2j+1, \ell+2}(q, z) - \Theta_{-2j-1, \ell+2}}{\Theta_{1,2}(q, z) - \Theta_{-1,2}(q, z)}, \]

where \( s = (j + 1/2)/(\ell + 2) - 1/2 \). A similar formula holds for any simple algebra: the sum over the affine Weyl group is first performed as a sum over the coroot lattice followed by an antisymmetrisation over the finite Weyl group. Each sum over the Weyl lattice results in a theta function and we thus get a product formula for an alternating sum of theta functions.

The formula for the level one vacuum character of affine \( SU(2) \) can be further simplified. (This simplification is related to the boson–fermion correspondence and the Frenkel–Kac–Segal construction; an analogue holds for each simply laced simple Lie algebra of types A, D or E.)

**Product formula.** \( \Theta_{n,m}(q, z) \Theta_{n', m'}(q, z) = \sum_{j \in \mathbb{Z}/(m-m')\mathbb{Z}} F_j(q) \Theta_{n+n'+2mj, m+m'}(q, z) \), where \( F_j(q) = \sum_{k \in \mathbb{Z} + x} q^{mm'(m+m')k^2} \) and \( x = (m'n - mn')/(2mn')(m + m') \).

**Proof.** We have
\[ \Theta_{n,m} \Theta_{n', m'} = \sum_{k, k'} q^{mk^2 m' k'^2} e_{mk+k'}(z), \]

where \( k \in \mathbb{Z}/m + \mathbb{Z} \) and \( k' \in \mathbb{Z}/m' + \mathbb{Z} \). Set \( k = j + \frac{n}{m} \) and \( k' = j' + \frac{n'}{m'} \). Define \( s = (k - k')/(m + m') \) and \( s'(mk + m'k')/(m + m') \). Write \( j - j' = (m + m')a + b \) with \( a \in \mathbb{Z} \) and \( 0 \leq b < m + m' \). Then
\[ s \in \frac{nm' - n'm + 2mn'b}{2mm'(m + m')} + \mathbb{Z}, \quad s' \in \frac{n + n' + 2mnj}{2(m + m')} + \mathbb{Z}. \]

This gives a bijection between pairs \((k, k')\) and triples \((s, s', b)\). Since \( mk^2 + m'k'^2 = mm'(m + m')s^2 + (m + m')s'^2 \), we get
\[ \Theta_{n,m} \Theta_{n', m'} = \sum_b (\sum_s q^{mm'(m+m')s^2}) (\sum_{s'} q^{m(m+m')s'^2} e_{m+m'}(s'z)), \]

as required.

**Corollary.** \( \text{ch}(L(1, 0)) = \Theta_{0,1}/\eta(q) \), where \( \eta(q) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \) is Dedekind’s eta function.
Proof. By Jacobi’s triple product identity (the Weyl–Kac denominator formula for $SU(2)$), we have

$$\sum_{k \in \mathbb{Z}} q^{k^2} t^k = \prod_{m \geq 1} (1 + q^{2m-1} t)(1 + q^{2m-1} t^{-1})(1 - q^{2m}).$$

If we specialise $(q,t)$ to $(q^{3/2}, -q^{-1/2})$, we obtain Euler’s pentagonal identity:

$$\varphi(q) = \prod_{m \geq 1} (1 - q^m) = \sum_{m \in \mathbb{Z}} (-1)^k q^{3k^2-k}/2 = \sum_{m \in \mathbb{Z}} (-1)^k q^{3k^2+k}/2.$$

To prove the corollary, we must show that

$$\Theta_{0,1}(\Theta_{1,2} - \Theta_{-1,2}) = q^{\frac{1}{24}} \varphi(q)(\Theta_{1,3} - \Theta_{-1,3}).$$

By the product formula, the left hand side is

$$\Theta_{0,1}(\Theta_{1,2} - \Theta_{-1,2}) = q^{\frac{1}{24}} \varphi(q)(\Theta_{1,3} - \Theta_{-1,3}).$$

as required.
CHAPTER IV. REPRESENTATIONS OF THE VIRASORO ALGEBRA

In this chapter we discuss positive energy unitary representations of the Virasoro algebra. We show that they are classified by a lowest energy \( h \) and a central charge \( c \). Of particular interest are the representations with \( 0 < c < 1 \). A series of these can be constructed by the coset construction of Goddard–Kent–Olive. For these representations \( c = 1 - 6/(m+1) \) with \( m \geq 3 \). The \( h \) values are then \( h_{p,q} = [(p(m+1) - qm)^2 - 1]/4m(m+1) \) where \( 1 \leq q \leq p \leq m-1 \). The coset construction gives representations of the Virasoro algebra on multiplicity spaces and the characters of these multiplicity spaces can be computed explicitly in terms of theta functions and Dedekind’s eta function. Using this information, we present the short proof of Kac’s determinant formula due to Kac and Wakimoto. This is then used to prove the easy part of the unitarity criterion of Friedan–Qiu–Shenker: the only irreducible unitary representations with \( c = 1 - 6/(m+1) \) have \( h = h_{p,q} \) as above. We then use this unitarity criterion to give our own direct proof that the multiplicity spaces are irreducible. (Such a ‘multiplicity one’ theorem seems to hold more generally for \( W \)-algebras.) This gives a very short proof of the Feigin–Fuchs character formula for these values of \( c \) and \( h \). Our method uses unitarity properties rather than a detailed knowledge of null vectors and Verma module resolutions. (These resolutions are not easy; indeed the lengthy derivation of Feigin and Fuchs needs to be supplemented with arguments of Astashkevich on the Jantzen filtration.) Finally we prove the hard part of the unitarity criterion of Friedan–Qiu–Shenker: the only values of \( c \in (0,1) \) yielding unitary representations are \( c = 1 - 6/(m+1) \). Our techniques extend easily to treat the case \( c = 1 \).

1. POSITIVE ENERGY REPRESENTATIONS OF THE VIRASORO ALGEBRA

We shall be interested in **projective**, **unitary**, **positive energy** representations of the Witt algebra \( \mathfrak{d} = \text{Vect} S^1 \). Recall that \( \mathfrak{d} \) is the complexification of the real Lie algebra of (trigonometric) polynomial vector fields \( a(\theta) \, d/d\theta \) on \( S^1 \). It has basis \( d_n = i e^{i n \theta} \, d/d\theta \). We can use Leibniz’ rule to compute the Lie brackets:

\[
[d_m, d_n] = (n-m)d_{m+n}.
\]

We set \( d_n^* = d_{-n} \). This extends to a conjugate-linear involution on \( \mathfrak{d} \). Thus we are looking for inner product spaces \( H \) (not complete!!) such that:

1. **Projective**: \( \mathfrak{d} \) acts projectively by operators \( \pi(A) (A \in \mathfrak{d}) \), i.e. \( A \mapsto \pi(A) \) is linear and \( [\pi(A), \pi(B)] = \pi([A,B]) \) lies in \( C \mathfrak{d} \) for \( A, B \in \mathfrak{d} \).

2. **Unitary**: \( \pi(A)^* = \pi(A^*) \).

3. **Positive energy**: \( H \) admits an orthogonal decomposition \( H = \bigoplus_{k \geq 0} H(k) \) such that some (necessarily unique) representative \( D \) for \( \pi(d_0) \) acts on \( H(k) \) as multiplication by \( k \), \( H(0) \neq 0 \) and \( \dim H(k) < \infty \).

The subspaces \( H(k) \) are called the energy subspaces with energy \( k \); the operator \( D \) has many names, including the energy operator or hamiltonian operator. Since the representation is projective, \( [\pi(A), \pi(B)] = \pi([A,B]) \) and \( \pi([A,B]) = b(A,B)I \) where \( b(A,B) \in \mathbb{C} \). We call \( b \) a 2–cocycle — in fancy language it gives a class in \( H^2(\mathfrak{d}, \mathbb{C}) \). The definition immediately implies the antisymmetry condition

\[
b(A,B) = -b(B,A)
\]

because Lie brackets are antisymmetric; and the Jacobi identity immediately implies that

\[
b([A,B], C) + b([B,C], A) + b([C,A], B) = 0
\]

for all \( A, B, C \in \mathfrak{d} \). On the other hand we are free to adjust the operators \( \pi(A) \) by adding on scalars. Thus to preserve linearity, we change \( \pi(A) \) to \( \pi(A) + f(A)I \) where \( f : \mathfrak{d} \to \mathbb{C} \) is linear. This changes \( b(A,B) \) to \( b(A,B) - f([A,B]) \). We shall now make such adjustments so that \( b \) has a canonical form. We start by choosing the canonical representative \( D \) for \( \pi(d_0) \) as above. By uniqueness, we must have

\[
[L_m, L_n] = (m-n)L_{m+n} + \lambda(m,n)I.
\]

Here \( \lambda(n,m) \) is a 2–cocycle. As we now show, by appropriate adjustment of the \( L_n \)’s by scalars, that \( \lambda \) can always be normalised so that \( \lambda(m,n) = \frac{c}{12}(m^3 - m)\delta_{m+n,0} \), where \( c \) is called the **central charge**. The corresponding central extension of the Witt algebra is usually called the Virasoro algebra.

**Virasoro cocycle lemma.** Representatives \( L_n \) of \( \pi(d_n) \) may be chosen uniquely so that \( [D,L_n] = -nL_n \) for all \( n \). In this case \( [L_m, L_n] = (m-n)L_{m+n} + (am^3 + bm)\delta_{m+n,0}I \). If we choose \( L_0 \) so that \( [L_1, L_{-1}] = L_0 \), then \( a + b = 0 \) and

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.
\]
If instead we take $L_0 = L_0 - c/24$ and $L_n = L_n \ (n \neq 0)$, then

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}. \tag{2}$$

**Proof.** Note that $-n^{-1}[L_0, L_n]$ is independent of adding scalars onto $L_0$ or $L_n$, so we may always choose $L_n$ so that $[L_0, L_n] = -nL_n$. Thus $[D, L_n] = -nL_n$, so that $L_n$ lowers energy by $n$, i.e. takes $H(k)$ into $H(k - n)$. But then $[L_m, L_n]$ and $L_{m+n}$ lower energy by $n + m$. Since $[L_m, L_n] - (m - n)L_{m+n}$ is a scalar, it must be 0 if $n + m \neq 0$. Thus

$$[L_m, L_n] = (m - n) + A(m)\delta_{m+n,0}.$$  

Clearly $A(m) = -A(-m)$ and $A(0) = 0$. Writing out the Jacobi identity for $L_k$, $L_n$ and $L_m$ with $k + n + m = 0$, we get

$$(n - m)A(k) + (m - k)A(n) + (k - n)A(m) = 0.$$  

Setting $k = 1$ and $m = -n - 1$, we get

$$(n - 1)A(n + 1) = (n + 2)A(n) - (2n + 1)A(1).$$

This recurrence relation allows $A(n)$ to be determined from $A(1)$ and $A(2)$. Since $A(n) = n$ and $A(n) = n^3$ give solutions, we see that $A(m) = am^3 + bm$ for some constants $a$ and $b$. Clearly we are free to choose $L_0 = [L_1, L_{-1}]$ (since we have made no adjustment to $L_0$ so far). But then $A(1) = 0$ and hence $a + b = 0$. This gives (1) and (2) follows by an easy manipulation.

**Complete reducibility theorem.** (a) If $H$ is a positive energy unitary representation of $\mathfrak{g}$, then any non-zero vector in the lowest energy subspace $H(0)$ generates an irreducible submodule.

(b) Any positive energy representation is an orthogonal direct sum of irreducible positive energy representations.

**Proof.** (a) Take $v \neq 0$ in $H(0)$ and let $K$ be the $\mathfrak{g}$–invariant subspace it generates. Clearly since $L_nv = 0$ for $n > 0$ and $L_0v = hv$, we see that $K$ is spanned by all products $Rv$ with $R = L_{-n_k} \cdots L_{-n_1}$ with $n_k \geq \cdots \geq n_1 \geq 1$. But then $K(0) = \mathbb{C}v$. We claim that $K$ is irreducible. If not let $K'$ be a submodule and let $P$ be the orthogonal projection onto $K'$. By unitarity $P$ commutes with $\mathfrak{g}$ and hence $D$. Thus $P$ leaves $K(0) = \mathbb{C}v$ invariant, so that $Pv = 0$ or $v$. But $P(Rv) = RPv$. Hence $K' = (0)$ or $K$, so that $K$ is irreducible.

(b) Take the irreducible module generated by a vector of lowest energy $H_1$. Now repeat this process for $H_1^\perp$, to get $H_2$, $H_3$, etc. The positive energy assumption shows that $H = \bigoplus Hi$.

**Uniqueness Theorem.** If $H$ and $H'$ are irreducible positive energy representations of $\mathfrak{g}$ with central charge $c$ and $h = h'$, then $H$ and $H'$ are unitarily equivalent as representations of $\mathfrak{g}$.

**Proof.** Any monomial $A$ in operators from $\mathfrak{g}$ is a sum of monomials $RDL$ with $R$ a monomial in energy raising operators, $D$ a monomial in constant energy operators and $L$ a monomial in energy lowering operators. Observe that if $v, w \in H(0)$, the inner products $(A_1 v, A_2 w)$ are uniquely determined by $v, w$ and the monomials $A_i$: for $A_2^2 A_1$ is a sum of terms $RDL$ and $RDLv, w = (DLv, R^**w)$ with $R^*$ an energy lowering operator. Hence, if $H'$ is another irreducible positive energy representation with $h = h'$, with $H(0) = \mathbb{C}v$ and $H'(0) = \mathbb{C}v'$ for unit vectors $v, v'$, then $U(Av) = Av'$ defines a unitary map of $H$ onto $H'$ intertwining $\mathfrak{g}$.

2. **THE GODDARD–KENT–OLIVE CONSTRUCTION.** We have seen a variety of construction of positive energy representations of the Virasoro algebra in the chapter on affine Kac–Moody algebras: the Segal–Sugawara construction; the Fubini–Veneziano construction using bosons; and the Fubini–Veneziano construction using fermions. We now describe a further “coset” construction of Goddard–Kent–Olive on multiplicity spaces.

**Lemma.** Let $\mathfrak{h}$ be a Lie algebra acting unitarily on the inner product space $H$. Suppose that $H$ is a direct sum of irreducible submodules and that there are only finitely many isomorphism types of irreducible summands
$H_i$. Let $K_i = \text{Hom}_{\mathfrak{h}}(H_i, H)$. Then $K_i$ is naturally an inner product space and the map $\bigoplus K_i \otimes H_i \to H$, 
$\sum \xi_i \otimes \eta_i \mapsto \sum \xi_i \eta_i$ is a unitary map of $\mathfrak{h}$–modules. The operators $A$ on $H$ which commute with $\mathfrak{h}$ act naturally on each $K_i$ by $A_i$. This action is a $*$–homomorphism. Under the unitary isomorphism above, $A$ corresponds to $\oplus A_i \otimes 1$ and $x \in \mathfrak{h}$ to $\oplus 1 \otimes \pi_i(x)$. 

**Proof.** If $S, T \in \text{Hom}(H_i, H)$, then $T^* S \in K_i = \text{End}_{\mathfrak{h}}(H_i) = \mathbb{C}$ by Schur’s lemma. Thus the canonical inner product on $K_i$ is defined by $(S, T) = T^* S$. It is then easy to check the assertions about the map $\oplus K_i \otimes H_i \to H$, since by assumption this map is surjective. The action of $A$ on $K_i = \text{Hom}_{\mathfrak{h}}(H_i, H)$ is defined by $A \xi$. 

**Proposition (coset construction).** Let $\mathfrak{g}$ be a Lie algebra with subalgebra $\mathfrak{h}$. Let $\mathfrak{d}$ be a Lie algebra of derivations action on $\mathfrak{g}$ by $D, X \mapsto [D, X]$ ($D \in \mathfrak{d}, X \in \mathfrak{g}$) such that $\mathfrak{d}$ leaves $\mathfrak{h}$ invariant. Suppose that $\mathfrak{g}$ acts irreducible on the inner product space $H$ and that $(H, \mathfrak{h})$ satisfy the hypotheses of the previous lemma. Suppose in addition that $H$ and the $H_i$’s admit projective unitary actions of $\mathfrak{d}$ compatible with the action of $\mathfrak{g}$ and $\mathfrak{h}$. If $D \in \mathfrak{d}$ acts by $\pi(D)$ on $H$ and $\pi_i(D)$ on $H_i$, then $\pi(D) = \sum I \otimes \pi_i(D) + \sigma_i(D) \otimes 1$, where $\sigma_i$ is a projective unitary representation of $\mathfrak{d}$ on $K_i$; the cocycle of $\mathfrak{d}$ on $K_i$ is the difference of the cocycles of $\mathfrak{d}$ on $H$ and on $H_i$.

**Proof.** Let $\sigma(D) = \pi(D) - \sum I \otimes \pi_i(D)$. By construction this operator commutes with $\mathfrak{h}$. Therefore by the previous lemma $\sigma(D) = \sum \sigma_i(D) \otimes 1$. Now suppose that $c(D_1, D_2)I = [\pi(D_1), \pi(D_2)] - \pi([D_1, D_2])$ and $c_i(D_1, D_2)I = [\pi_i(D_1), \pi_i(D_2)] - \pi_i([D_1, D_2])$ for $D_1, D_2 \in \mathfrak{d}$. If $D_1, D_2 \in \mathfrak{d}$, then

$$
c(D_1, D_2)I = \sum \sigma_i(D_1), \sigma_i(D_2) = \pi([D_1, D_2]) = \sum \sigma_i(D_1), \sigma_i(D_2) - \pi_i([D_1, D_2]) = \sum \sigma_i(D_1), \sigma_i(D_2) - \pi_i([D_1, D_2]) = \sum \sigma_i(D_1), \sigma_i(D_2) - \sigma_i([D_1, D_2]) = \sum \sigma_i(D_1), \sigma_i(D_2).
$$

Looking at this equation on $K_i \otimes H_i$, we get

$$
[\sigma_i(D_1), \sigma_i(D_2)] = \pi([D_1, D_2]) = c(D_1, D_2) - c_i(D_1, D_2),
$$

as required.

We shall apply the coset construction in the following setting. Let $H_0$ be the vacuum representation of $\widehat{\mathfrak{sl}(2)}$ at level one and let $H_{j, \ell}$ be any irreducible representation of $\widehat{\mathfrak{sl}(2)}$ at level $\ell$ with lowest energy space of spin $j \in \frac{1}{2}\mathbb{Z}$. Thus $H_0 \otimes H_{j, \ell}$ gives a positive energy representation of $\widehat{\mathfrak{sl}(2)}$ of level $\ell + 1$. Thus we may write $H_0 \otimes H_{j, \ell} = \bigoplus M_k \otimes H_{k, \ell + 1}$ with the $M_k$ multiplicity spaces. In this case $\mathfrak{g} = \mathcal{Lsl}_2 \oplus \mathcal{Lsl}_2$ and $\mathfrak{h} = \mathcal{Lsl}_2$, embedded diagonally via $X \mapsto X \otimes 1 + 1 \otimes X$. The Witt algebra acts on $\mathfrak{g}$ and $\mathfrak{h}$ and is implemented in both cases by the Sugawara constructions. For $\mathfrak{g}$, it has central charge $3\ell/(\ell + 1)$ while for $\mathfrak{h}$ it has central charge $3(\ell + 1)/(\ell + 3)$. Let $m = \ell + 2$. Subtracting the central charges, we see that there are canonical projective representations of the Virasoro algebra on the multiplicity spaces $M_k$ with central charge

$$
c = 1 - 3[\ell(\ell + 3) - (\ell + 1)(\ell + 2)]/((\ell + 2)(\ell + 3)) = 1 - 6/m(m + 1).
$$

**3. CHARACTER OF THE MULTIPLECTY SPACE.** We recall the formula for the characters of the positive energy representations of $LSU(2)$ (in normalised form). Let

$$
\Theta_{n,m}(q, \zeta) = \sum_{k \in \frac{1}{2m} + \mathbb{Z}} q^{mk^2} \zeta^{2mk}.
$$

Then if $0 \leq j \leq \ell/2$ is a half-integer, the character of the irreducible positive energy representation of level $\ell$ with spin $j$ is given by

$$
\text{ch} L(\ell, j) = \frac{\Theta_{2j+1,\ell+2}(q, \zeta) - \Theta_{-2j-1,\ell+2}(q, \zeta)}{\Theta_{1,2}(q, \zeta) - \Theta_{-1,2}(q, \zeta)}.
$$
Note that the character of a representation of $\widehat{sl}_2$ is $\text{Tr}(q^z)\text{Tr}(q^{z^2})$ where $L_0 = L_0 - c/24$ and $z$ corresponds to the element $\left( \begin{array}{cc} z & 0 \\ 0 & z^{-1} \end{array} \right)$ in $SU(2)$ or $SL(2)$. Similarly the character of a positive energy representation

**Product formula.** $\Theta_{n,m}(q,z)\Theta_{n',m'}(q,z) = \sum_{j\in\mathbb{Z}/(m+m')\mathbb{Z}} F_j(q)\Theta_{n+n'+2m,j,m+m'}(q,z)$, where $F_j(q) = \sum_{k\in\mathbb{Z}} q^{mm'(m+m')k^2}z^{2(km+m'k')}$ and $x = (m'n - mm'+ 2jm'm')/2mm'(m + m')$.

**Proof.** We have

$$\Theta_{n,m}\Theta_{n',m'} = \sum_{k,k'} q^{mk^2m'k^2}z^{2(km+m'k')},$$

where $k \in \frac{m}{m'} + \mathbb{Z}$ and $k' \in \frac{m'}{m} + \mathbb{Z}$. Set $k = j + \frac{m}{m'}$ and $k' = j' + \frac{m'}{m}$. Define $s = (k - k')/(m + m')$ and $s' = (mk + m'k')/(m + m')$. Write $j - j' = (m + m')a + b$ with $a \in \mathbb{Z}$ and $0 \leq b < m + m'$. Then

$$s \in \frac{nm' - n'm + 2mm'b}{2mm'(m + m')} + \mathbb{Z}, \quad s' \in \frac{n + n' + 2mj}{2(m + m')} + \mathbb{Z}.$$

This gives a bijection between pairs $(k, k')$ and triples $(s, s', b)$. Since $mk^2 + m'k'^2 = mm'(m + m')s^2 + (m + m')s'^2$, we get

$$\Theta_{n,m}\Theta_{n',m'} = \sum_b \left( \sum_s q^{mm'(m+m')s^2} \right) \left( \sum_{s'} q^{(m+m')s'^2}z^{2(m+m')s'} \right),$$

as required.

**Corollary.** $\text{ch} L(1,0) = \Theta_{0,1}/\eta(q)$.

**Proof.** By Jacobi’s triple product identity (the Weyl–Kac denominator formula for $SU(2)$), we have

$$\sum_{k\in\mathbb{Z}} q^{k^2}t^k = \prod_{m \geq 1}(1 + q^{2m-1}t)(1 + q^{2m-1}t^{-1})(1 - q^{2m}).$$

If we specialise $(q, t)$ to $(q^{3/2}, -q^{-1/2})$, we obtain Euler’s pentagonal identity:

$$\varphi(q) = \prod_{m \geq 1}(1 - q^m) = \sum_{m\in\mathbb{Z}} (-1)^k q^{3k^2 - k}/2 = \sum_{m\in\mathbb{Z}} (-1)^k q^{3k^2 + k}/2.$$

To prove the corollary, we must show that

$$\Theta_{0,1}(\Theta_{1,2} - \Theta_{-1,2}) = q^{3/2} \varphi(q)(\Theta_{1,3} - \Theta_{-1,3}).$$

By the product formula, the left hand side is

$$(\Theta_{1,3} - \Theta_{-1,3})(\sum_{k\in\frac{1}{m} + \mathbb{Z}} q^{k^2} - \sum_{k'\in\frac{1}{m'} + \mathbb{Z}} q^{k'^2}) = (\Theta_{1,3} - \Theta_{-1,3})q^{3/2} \sum_{k\in\mathbb{Z}} (-1)^k q^{(3k^2 + k)/2} = (\Theta_{1,3} - \Theta_{-1,3})\eta(q),$$

as required.

**Theorem.** $\text{ch} L(0,1) \cdot \text{ch} L(j, \ell) = \sum_{0 \leq k \leq (\ell+1)/2} \psi_k(q) \cdot \text{ch} L(k, \ell + 1)$, where $k - j \in \mathbb{Z}$,

$$\psi_k(q) = \eta(q)^{-1}(\Theta_{a_+,b}(q,1) - \Theta_{a_-,b}(q,1)),$$

$a_\pm = r(m + 1) \mp sm$, $b = m(m + 1)$, $r = 2j + 1$ and $s = 2k + 1$.

**Proof.** By the product formula

$$\Theta_{0,1}(q,z)\Theta_{s,m}(q,z) = \sum_{r \equiv s(2), |r| \leq m} \Theta_{r,m+1}(q,z)f_{rs}(q),$$
where
\[ f_{rs}(q) = \sum_{r' \in r+2m\mathbb{Z}} q^{(r'(m+1)-qm)^2/4m(m+1)}. \]

Note that \( f_{rs}(q) = f_{-r,-s}(q) \). Thus
\[
\Theta_0(q,z)(\Theta_{s,m}(q,z) - \Theta_{-s,m}(q,z)) = \sum_{r \equiv s(2), |r| \leq m} (\Theta_{r,m+1}(q,z) - \Theta_{-r,m+1}(q,z)) f_{rs}(q)
\]
\[
= \sum_{r = 2, 1 \leq r \leq m} (\Theta_{r,m+1}(q,z) - \Theta_{-r,m+1}(q,z))(f_{rs}(q) - f_{-r,-s}(q)),
\]

since there is no contribution for \( r = 0 \). Thus the character of the multiplicity space for \( L(k,\ell+1) \) is
\[
\frac{(f_{rs}(q) - f_{-r,-s}(q))}{\eta(q)} = \frac{(\Theta_{r(m+1)-sm,m(m+1)}(q,1) - \Theta_{r(m+1)+sm,m(m+1)}(q,1))}{\eta(q)},
\]
as required.

4. THE KAC DETERMINANT FORMULA Note that as in Chapter II, section 14, a Verma module \( V(c, h) \) can be constructed which is a representation of the Virasoro algebra with central charge \( c \), generated by a cyclic vector \( \xi_0 \) such that \( L_n \xi = 0 \) for \( n > 0 \) and \( L_0 \xi = h \xi \). It has the universal property that, for any representation generated by a cyclic vector satisfying similar relations, there is unique equivariant mapping sending \( \xi_0 \) to the cyclic vector. A basis of the Verma module is given by monomials in the raising operators \( L_{-i_1} \cdots L_{-i_p}L_{j_1} \cdots L_{j_q} \), where \( n_i \geq 0 \). Clearly the Verma module is a positive energy representation.

We now assume that \( c \) and \( h \) are real. Let \( f : V(c, h) \to \mathbb{C} \) be the linear map picking out the coefficient of \( \xi_0 \) and extend the involution \( L_n^* = L_{-n} \) to a complex involution on the universal enveloping algebra, so that \( (AB)^* = B^* A^* \). We can then define a hermitian form on \( V(c, h) \) by \( (A \xi_0, B \xi_0) = f(B^* A \xi_0) \). By definition it satisfies the invariance condition \((L_n \xi, \eta) = (\xi, L_{-n} \eta)\). Since \( L_n^* = L_0 \), the eigenspaces of \( L_0 \) are orthogonal. Moreover \((\cdot, \cdot)\) is the unique invariant hermitian form on \( V(c, h) \) with \((\xi_0, \xi_0) = 1\): for the orthogonality conditions force \((A \xi_0, B \xi_0) = (B^* A \xi_0, \xi_0) = f(B^* A)\). In particular if \( L(c, h) \) is a unitary irreducible representation and \( T : V(c, h) \to L(c, h) \) is the canonical map, then the invariant hermitian form on \( V(c, h) \) is just the pull back of the inner product on \( L(c, h) \). Let \( K = \{ \xi \in V(c, h) | (\xi, V(c, h)) = 0 \} \). Then \( K \) is invariant under the Virasoro algebra and the hermitian form passes to a non–degenerate invariant hermitian form on \( L = V(h, c)/K \). Since \( K \) is invariant under \( L_0 \), \( L \) is itself a positive energy representation.

We claim that \( L \) is irreducible. In fact let \( L' \) be a submodule of \( L \) and let \( v_0 \) be the image of \( \xi_0 \) in \( L \). \( L' \) can be written as the direct sum of eigenspaces \( L_0 \). Choose \( v \neq 0 \) in \( L' \), the image of \( A \xi_0 \). Thus \((A \xi_0, B \xi_0) \neq 0 \) for some monomial \( B \), by nondegeneracy. Hence \((B^* A v, v_0) \neq 0 \). But then \( L'(0) \neq 0 \), so that \( v_0 \) lies in \( L' \) and thus \( L' = L \).

It follows that the Verma module \( V(c, h) \) is irreducible iff \((\cdot, \cdot)\) is non–degenerate. In particular this happens iff \((\cdot, \cdot)\) is non–degenerate on every energy subspace \( V(N) \). Let \( M_N(c, h) \) be the \( P(N) \times P(N) \) matrix
\[
(L_{-i_1} \cdots L_{-i_p} \xi_0, L_{-j_1} \cdots L_{-j_q} \xi_0)
\]
where \( 1 \leq i_1 \leq \cdots \leq i_p \) and \( 1 \leq j_1 \leq \cdots \leq j_q \) with \( N = \sum i_s = \sum j_t \). The Kac determinant \( \det_N(c, h) \) is the determinant of this matrix. Note that \( L(c, h) \) is unitary iff \( M_N(c, h) \) is positive semi–definite for all \( N \). In this case, it is necessary that \( \det_N(c, h) \geq 0 \) for all \( N \). Using raising and lowering operators to compute the entries of \( M_N(c, h) \), we see that they are all polynomials in \( c \) and \( h \) if \( c, h \in \mathbb{R} \).

Examples. (0) \( \det_0(x, h) = ||\xi_0||^2 = 1 \).
(1) \( \det_1(c, h) = (L_{-1} \xi_0, L_{-1} \xi_0) = 2h \).
(2) \( \det_2(c, h) = \begin{pmatrix} 4h + c/2 & 6h \\ 6h & 8h^2 + 4h \end{pmatrix} = 2h(16h^2 + 2hc - 10h + c) \).

Lemma. If \( L(c, h) \) is unitary, then \( h \geq 0 \) and \( c \geq 0 \).
Proof. The computation of \( \det_1 = 2h \) shows that \( h \geq 0 \). Now for \( n > 0 \) we compute

\[
\|L_{-n} \xi_0 \|^2 = ([L_n, L_{-n}] \xi_0, \xi_0) = 2nh + c(n^3 - n)/12.
\]

For this to be positive for all values of \( n \), we must have \( c \geq 0 \).

**Proposition.** For fixed \( c \), \( \det_N \) is a polynomial in \( h \) of degree \( \sum_{1 \leq rs \leq N} p(N - rs) \). (The coefficient of the highest power of \( h \) is independent of \( c \).)

**Proof.** We prove the result by “degenerating to bosons”. For \( c \) fixed, let \( h = t^{-2} \) and \( a_0 = h^{-1}L_0, a_n = (2h)^{-1/2}L_n \). Thus \( a_0 \xi_0 = \xi_0, [a_m, a_{-m}] = ma_0 + t^2c(m^3 - m)/12 \) and \([a_m, a_n] = (m-n)ta_{m+n} \) if \( m \neq -n, 0 \), \([a_0, a_m] = -mt^2a_m \). Moreover \( A_n = A_{-n} \). If we look at monomials \((a_{-i_1} \cdots a_{-i_r}, \xi_0, a_{-j_s} \cdots a_{-j_0})\), these are polynomials in \( t \). We extend these to \( t = 0 \); this obviously gives the leading order terms in \( h \) in the original problem. In the limit \( t = 0 \), we get the system of oscillators \( a_0 = I, [a_m, a_{-m}] = mI \). For this bosonic system it is immediate that \( x = (a_{-p}^{m_1} \cdots a_{-1}^{m_s}) \xi_0, a_n^{s_1} \cdots a_{-1}^{s_1} \xi_0 \) is zero unless \( m_s = n_s \) for all \( s \), in which case \( x = \prod m_s!s^{m_s} \). This is independent of \( c \). If we substitute these terms into the determinant for \( \det_N \), we see that the off–diagonal terms vanish when \( t = 0 \), so the determinant is given by the product of the diagonal entries, all non–zero. Thus \( \lim_{h \to \infty} h^{-3} \det_N \neq 0 \) is independent of \( c \), where \( M \) is the sum of all \( \sum j_k \)’s with \( \sum j_k k = N \). Let \( m(r,s) \) be the number of partitions of \( N \) in which \( r \) appears exactly \( s \) times. Clearly \( M = \sum_{1 \leq rs \leq N} s \cdot m(r,s) \). Now the number of partitions of \( N \) in which \( r \) appears \( \geq s \) times is \( P(N - rs) \). Thus \( m(r,s) = P(N - rs) - P(N - r(s + 1)) \) (where \( P(0) = 1 \) and \( P(-k) = 0 \) for \( k > 0 \)). Thus

\[
M = \sum s \cdot m(r,s) = \sum \sum s \cdot (P(n - rs) - P(n - r(s + 1))) = \sum_{1 \leq rs \leq N} P(N - rs).
\]

Since \( h^{3M} \) is evidently the highest power of \( h \) with a non–zero coefficient in \( \det_N \), the result follows.

**Definitions.** Let

\[
\varphi_{p,q}(c,h) = h - h_{p,q}(c) = h + (p^2 - 1)(c - 1)/24 \quad \text{and} \quad \varphi_{p,q}(c,h) = (h - h_{p,q}(c))(h - h_{q,p}(c))
\]

\[
= (h - (p - q)^2/4)^2 + h 24(p^2 + q^2 - 2)(c - 1) + \frac{1}{576}(p^2 - 1)(q^2 - 1)(c - 1)^2 + \frac{1}{48}(c - 1)(p - q)^2(pq + 1). (*)
\]

If we parametrise \( c \) as \( c = 1 - 6/m(m+1) \), then

\[
h_{p,q}(c) = \frac{(m + 1)p - mq)^2 - 1}{4m(m + 1)}.
\]

**Kac determinant formula.** \( \det_N(c,h) = C_N \prod_{1 \leq rs \leq N} (h - h_{rs}(c))^{P(N - rs)} \), where \( C_N > 0 \) is independent of \( c \) and \( h \).

**Lemma 1.** If \( t \mapsto A(t) \) is a polynomial mapping into \( N \times N \) matrices and \( \dim \ker A(t) = k \), then \( (t - t_0)^k \) divides \( \det A(t) \).

**Proof.** Take a basis \( v_i \) such that \( A(t_0)v_i = 0 \) for \( i = 1, \ldots, k \). Thus the first \( k \) columns of \( A(t) \) are divisible by \( t - t_0 \) and hence \( (t - t_0)^k \) divides \( \det A(t) \).

**Lemma 2.** Fix \( c \) and regard \( \det_N(c,h) \) as a polynomial in \( h \). If \( \det_k \) vanishes at \( h = h_0 \), then \( (h - h_0)^{P(N - k)} \) divides \( \det_N(c,h) \).

**Proof.** We may take \( k \) minimal subject to \( \det_k(c,h_0) = 0 \). Thus \( V(c,h_0) \) has a singular vector \( v \) at energy level \( k \). By the Poincaré–Birkhoff–Witt theorem, the vectors \( L_{-i_1} \cdots L_{-i_1}v \) are all linearly independent for
\[ i_3 \geq \cdots \geq i_1 \geq 1. \] So at level \( N \), this submodule has dimension \( P(N - k) \). On the other hand this submodule is contained in the kernel of \( \langle \cdot, \cdot \rangle \). Thus \( M_N \) has a kernel of dimension at least \( P(N - k) \) at \( h_0 \). The assertion therefore follows from Lemma 1.

Lemma 3. \( \det_N \) vanishes at \( h_{r,s}(c) \) for \( 1 \leq rs \leq N \).

Proof. By the GKO construction,

\[
\text{ch} L(c^{(m)}, h^{(m)}_{r,s}) \leq \frac{p^h}{\varphi(q)}(1 - q^{rs} - q^{rs'} + \cdots),
\]

where \( r' = m - r, s' = m + 1 - s \) and the inequality is to be understood in terms of coefficients of \( q^i \). It follows that the kernel of \( \langle \cdot, \cdot \rangle \) in \( V(c, h) \) has a non-zero component at each energy level \( N \geq \min(rs, r's') \). Thus \( \det_N \) vanishes at \( h^{(m)}_{r,s} \) for \( m \) sufficiently large. But then \( \det_N \) vanishes at infinitely many points of the curve \( \varphi_{r,s}(c, h) = 0 \) (namely \( (c^{(m)}, h^{(m)}_{r,s}) \)). Since \( \varphi_{r,s}(c, h) \) is irreducible in \( \mathbb{C}[c, h] \), we see that \( \varphi_{r,s} \) divides \( \det_N \) for \( N \geq rs \). Thus \( \det_N \) vanishes at \( h_{r,s}(c) \) for \( 1 \leq rs \leq N \).

Proof of determinant formula (Kac–Wakimoto). By Lemmas 2 and 3, \( \det_N \) is divisible by

\[
\prod_{1 \leq pq \leq N} (h - h_{p,q}(c))^{P(N-pq)},
\]

since the \( h_{p,q}(c) \)'s are distinct for generic \( c \). Since both sides have the same degree in \( h \) and the highest order term in \( h \) is independent of \( c \) (by the Proposition), the result follows.

5. THE FRIEDAN–QUI–SHENKER UNITARITY CRITERION FOR \( h \). We prove the easy part of the FQS criterion for unitarity. (The harder part of their criterion gives the restrictions on the values of \( c \). It depends on a detailed knowledge of the representations \( L(1, m^2/4) \); in this sense, their proof is analogous to the proof of Jones’ index theorem that uses detailed knowledge of the limiting \( SU(2) \) subfactor.)

FQS Theorem. Let \( L(c, h) \) be a unitary representation of the Virasoro algebra with \( c = 1 - 6/m(m+1) \) for \( m \geq 3 \). Then \( h = h_{p,q} \) with \( 1 \leq q \leq p \leq m - 1 \) and \( h_{p,q} = [(p(m+1) - qm)^2 - 1]/4m(m+1) \).

Proof. Let

\[
h_{p,q}(c) = \frac{1}{48}((13 - c)(p^2 + q^2) + \sqrt{(c-1)(c-25)(p^2 - q^2)} - 24pq - 2 + 2c).
\]

Set \( \varphi_{p,p}(c, h) = h - h_{p,p}(c) = h + (p^2 - 1)(c - 1)/24 \) and \( \varphi_{p,q}(c, h) = (h - h_{p,q}(c))(h - h_{q,p}(c)) = (h - (p-q)^2/4)^2 + \frac{h}{24}(p^2 + q^2 - 2)(c - 1) + \frac{1}{576}(p^2 - 1)(q^2 - 1)(c - 1)^2 + \frac{1}{48}(c - 1)(p-q)^2(pq + 1). \)

If we parametrise \( c \) as \( c = 1 - 6/m(m+1) \), then

\[
h_{p,q}(c) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.
\]

It will sometimes be more convenient to use the variable \( x = m + 1/2 \). Thus \( c = 1 - 6/(x^2 - \frac{1}{4}) \) and \( h_{p,q}(c) = [(x(p - q) + \frac{1}{2}(p + q))^2 - 1]/(4x^2 - 1) \). Let \( C_{p,q} \) be the real curve \( \{ (c, h) : \varphi_{p,q}(c, h) = 0 \} \). By the symmetry of \( (*) \) in \( p \) and \( q \), \( C_{p,q} = C_{q,p} \). The form of \( (*) \) also shows that \( \varphi_{p,q}(c, h) > 0 \) for \( c > 1 \) and \( h > 0 \). Hence, for \( h \geq 0 \), the curve \( C_{p,q} \) lies in the region \( c \leq 1 \). We shall only consider it in the domain \( h \geq 0 \) and \( c \geq 0 \). The curve \( C_{p,q} \) is parametrised by \( x \in \mathbb{R} \) with \( x = \pm \infty \) giving its intersection with the line \( c = 1 \) which it touches at \( h = (p-q)^2/4 \), giving two branches \( C_{p,q}^\pm \) according to the sign of \( x \). Clearly \( C_{p,q}^\pm = C_{q,p}^\mp \) and \( C_{p,q}^+ \) is the upper branch if \( p > q \). The curve \( C_{p,q} \) arises at level \( N = pq \). Thus there are infinitely many curves through the point \( (1, M^2/4) \). The figure below shows the possible curves schematically (with a rescaling in the \( h \) direction).
In the degenerate case $p = q$, the curves $C_{p,p}$ become straight lines joining the point $(1, 0)$ to $(0, (p^2 - 1)/24)$, the first case in the figure above. The remaining three cases give all possible shapes for the curve $C_{p,q}$. The curve $C_{p,q}$ cuts the $h$–axis at $h = [(3p - 2q)^2 - 1]/24$ and the $c$–axis at 0 if $|3p - 2q| = 1$, between 0 and 1 if $3p = 2q$ and not at all otherwise.

We next make the important observation that at through the special points $(1, M^2/4)$ there at most one curve of a given level and the upper branch of a higher level curve lies above that of curves of lower level and the lower branch lies below that of the lower level curves the special point. In fact if $C_{p,q}$ occurs at a higher level than $C_{r,s}$ with $(p - q)^2/4 = (r - s)^2/4$ and $p \geq q$, $r \geq s$, then $p > r$, $q > s$, so that $p - q = r - s > 0$ and $p + q > r + s$. Since we evidently have

$$(p - q)x + \frac{1}{2}(p + q))^2 > ((r - s)x + \frac{1}{2}(r + s))^2$$

for $x \geq 0$ and the opposite inequality for $x < 0$, sufficiently large, we see that near $(1, (p - q)^2/4)$, the curve $C_{r,s}$ lies between the upper and lower branches of the curve $C_{p,q}$. Note also that it is not possible for $r - s = p - q$ with $(r, s)$ and $(p, q)$ having the same level. For if $pq = rs$, $p, -q$ and $r, -s$ are roots of the same quadratic. Since $p, r \geq 1$ and $-q, -s \leq -1$, it follows that $p = r$ and $q = s$. Hence at any fixed level, there is at most one curve through $(1, M^2/4)$.

Let $X$ be the strip $[0, 1] \times [0, \infty)$ in the $(c, h)$ plane and let $U_{p,q}$ be the open subset where $\varphi_{p,q}(c, h) < 0$. Its closure $\overline{U_{p,q}}$ is compact and connected with boundary made up of the segment of the curve $C_{p,q}$ in $X$ and parts of the lines $c = 0$ and possibly $h = 0$. In fact if $p = q$, $\overline{U_{p,q}}$ is a rightangle triangle with vertices $(1, 0)$, $(0, 0)$ and $(0, (p^2 - 1)/24)$. While if $p > q$, as may be assumed without loss of generality, $C_{p,q}$ increases as $c$ decreases, cutting the $h$–axis at $(0, [(3p - 2q)^2 - 1]/24)$; similarly the lower branch $C_{p,q}$ cuts the $h$–axis at $(1 - 6/(x^2 - 4), 0)$ with $x = \frac{1}{2}(p + q - 2)/(p - q)$.

Let $X_N = \bigcup_{pq \leq N} U_{p,q}$. We will show that this is the region bounded a explicit sequence of pieces of the curves $C_{p,q}$ as depicted schematically in the diagram below, with the $h$ direction rescaled to make the special points equally spaced on the line $c = 1$. 
Then for \( M = 0,1,2,\ldots,N-1 \) there are curves \( C_{pq} \) with \( p > q, p - q = M \) and \( pq \leq N \). When \( M = N - 1, p = N \) and \( q = 1 \), so the upper branch of the curve cuts the \( x \)-axis at \( c = [3N - 2 - 1]/24 \). The lower branch is decreasing and cuts the \( x \)-axis between 0 and \((N - 1)^2/4\). At the other extreme \( m = 0 \), the curve of highest level through \((1,0)\) is \( C_{p,p} \) where \( p = \lfloor \sqrt{N} \rfloor \), the largest integer less than or equal to \( \sqrt{N} \).

More generally if we take a highest level curve \( C_{p,q} \) through a special point \((1,M^2/4)\) with \( M = p - q > 0 \), then \( pq \leq M < (p+1)(q+1) \). The highest level curve through the special point with \((1,(M+1)^2/4)\) is \( C_{p+1,q} \) if \((p+1)q \leq N \) and \( C_{p,q-1} \) otherwise. The highest level curve \((1,(M-1)^2/4)\) is \( C_{p,q+1} \) if \( p(q+1) \leq N \) and \( C_{p-1,q} \) otherwise. Note that if \( p(q+1) > N \) then necessarily \((p+1)q > N \) since \( p \geq q \), so that for are are at most three possibilities for the highest level curves through the two adjacent special points. In particular if a highest curve through a special point has level \( N \), then the highest level curves through the adjacent points have level \( < N \).

We now work out the points of intersection of these curves: in fact not only do we calculate them but we prove that the key observation of Friedan–Qiu–Shenker that they are these points are characterized as being the first intersections of the highest level curve of level \( N \) through \((M^2/4,1)\) with any other curves of level \( \leq N \).

**Lemma 1.** The curve \( C_{N,1}^+ \) does not intersect any of the other curves at level \( N \).

**Proof.** It suffices to shows that for each value of \( x \) the corresponding point on \( C_{N,1}^+ \) lies above any point on \( C_{p,q}^+ \) for \( p \geq q \) and \( pq \leq N \). But this follows immediately because

\[
x(N-1) + N/2 \geq x(p-q) + (p+q)/2.
\]

Indeed if \( q = 1 \), then \( p < N \) and the result is obvious. If \( q \geq 2 \), then \( N-1 > p-q \). Moreover \( p \leq N/q \leq N/2 \). Since \( q \leq p \), this implies \( p+q \leq 2p \leq N \).

**Lemma 2.** If \( |p-q| \neq |r-s| \) and \( (p,q) \) is not proportional to \( (r,s) \), the curves \( C_{p,q}^+ \) and \( C_{r,s}^+ \) intersect transversely in the \((c,h)\)-plane in the distinct points

\[
(1 - 6/(x^2 - 1/4), [(x(p-q) + 1/2(p+q))^2 - 1]/(4x^2 - 1))
\]

with \( x = \frac{1}{2}(r+s-p-q)/(p-q+s-r) \) and \( x = \frac{1}{2}(r+s+p+q)/(q-p+s-r) \).

**Proof.** The points of intersection are given by the solutions of

\[
x(p-q) + \frac{1}{2}(p+q) = \pm[x(r-s) + \frac{1}{2}(r+s)].
\]

Transversality occurs if and only if the derivatives at \( x \) are equal, i.e.

\[
(p-q)[x(p-q) + \frac{1}{2}(p+q)] = (r-s)[x(r-s) + \frac{1}{2}(r-s)].
\]

Since

\[
|x(p-q) + \frac{1}{2}(p+q)| = |x(r-s) + \frac{1}{2}(r+s)|
\]

while \( |p-q| \neq |r-s| \), this can only happen if both sides above vanish and the two points of intersection coincide. In this case

\[
x = \frac{1}{2}(p+q)/(q-p) = \frac{1}{2}(r+s)/(s-r)
\]

so that \( p/q = r/s \).

**Lemma 3.** If \( C_{p,q}^+ \) is a highest level curve at level \( N \) with \( p \geq q, pq \leq N \) and \( (p+1)(q+1) > N \) then the first intersection with a curve of level \( \leq N \) is with \( C_{p,q-1}^- \) if \( (p+1)q > N \) and with \( C_{p+1,q}^- \) otherwise. The intersection with \( C_{p+k,q+k-1}^- \) takes place at \( x = p+q+k-1/2 \) where \( k = 0 \) or \( 1 \). The first intersection of
Theorem. \( C_{p,q} \) with a curve of level \( \leq N \) with \( C_{p-1,q}^+ \) if \( p(q+1) > N \) and with \( C_{p+1,q}^+ \) otherwise. The intersection with \( C_{p+k-1,q+k-1}^+ \) takes place at \( x = p + q + k - 1/2 \) where \( k = 0 \) or \( 1 \). The intersections are transverse.

Proof. Immediate from Lemma 2.

In the previous diagram we have marked the parts of the highest degree curves lying between these intersections. Let \( X'_N \) be the area bounded by these curves and the \( c \) and \( h \) axes. We now verify the assertion made above about the boundary of the closed region \( X_N \).

**Theorem.** \( X_N = X'_N \).

**Proof.** We assume the result by induction on \( N \), the result being obvious for \( N = 1 \).

At level \( N \), the regions bounded by the curves \( C_{p,q} \) with \( pq = N \) are added. Now when one of these new curves occurs at level \( N \) through \( m^2/4 \), there are no new curves through \( (m \pm 1)^2/4 \). We claim that the new areas added are just the areas between the new curve, the old curve \( C_{p-1,q-1} \) and the places where the new curve cuts the highest level curves to either side. For a curve \( C_{p,q} \) with \( q > 1 \), After the new curve cuts the boundary of the old region it cannot intersect the curved part of the old boundary again by Lemma 3. The same arguments apply to the curve \( C_{N,1} \) using Lemma 1 and Lemma 3.

**Remark.** As observed by Friedan–Qiu–Shenker, an immediate consequence of the theorem is that the new regions added to \( X_{N-1} \) to produce \( X_N \) are exactly the regions bounded by the curves \( C_{p,q} \) of level \( N \), the previous highest level curve \( C_{p-1,q-1} \) through that point and the two adjacent highest level curves \( C_{p-1,q} \) and \( C_{p,q+1} \) through adjacent special points; for the special point \( (1,(N-1)^2/4) \), the new region is the one between \( C_{N,1} \) and \( C_{N-1,1} \). These two types of region are indicated in the diagram above: the first is like a bow–tie; the second like a curved strip.

**Proof of FQS Theorem.** We first prove that for fixed \( c \in (0,1) \) with \( c = 1 - 6/m(m+1) \), the only values of \( h \) for which \( L(c,h) \) can be unitary are the \( h_{p,q}^{(m)} \) with \( p,q \geq 1 \) and \( p + q \leq m \). At level \( N \) define the \( N \)th excluded region \( R_N \) as being the open set where \( \det_N \) is negative for some \( n \leq N \). We claim that

\[
\bigcup R_N = \{(c^{(m)}, h_{p,q}^{(m)}): m > p + q - 1\}.
\]

Note first that every point \((c,h)\) with \( 0 < c < 1 \) and \( h > 0 \) lies in the interior of one of the curves \( C_{p,q} \) for \( p,q \) sufficiently large. Indeed the regions \( U_{p,1} \) sweep out the region since the points of intersection of \( C_{p,1} \) with the \( h \)-axis tend to infinity.

Consider the closure \( F_N \) of all the interiors of the curves \( C_{p,q} \) for \( pq \leq N \). We shall prove that the only parts of \( F_N \) that might not lie in \( R_N \) are the parts of the curves \( C_{p,q}, pq = N \) with \( |m+1/2| > |p+q-1/2| \). In fact the functions \( \det_N \) do not vanish for \( c > 1 \) and therefore have the same sign. Since the Segal–Sugawara construction gives positive values for all \( N \), it follows that \( \det_N \) is positive for all \( c > 1, h > 0 \) and \( N \geq 0 \). Hence if \( p \neq q, \) near \((1,(p-q)^2/4), \) \( \det_N \) is positive for \( c > 1 \). Thus on one side of the curve \( C_{p,q} \) near \((1,(p-q)^2/4), \) \( \det_N \) is positive. Note that near \((1,(p-q)^2/4), \) the function \( \det_N \) changes sign as \( C_{p,q} \) is crossed, because at the \( N \)th stage \( h_{p,q} \) is a simple zero of \( \det_N \). We can therefore exclude all parts of
the interior of $C_{p,q}$ which do not meet $F_{N-1}$ or the closure of the interior of any other $C_{rs}$ with $rs = N$. There are two such regions. Since the first curves of level $\leq N$ met by $C_{p,q}$ are $C_{q-1,p}$ and $C_{q,p-1}$, both of these regions are bounded on one of their sides by a segment of $C_{p,q}$ with $|m + \frac{1}{2}| > p + q - \frac{1}{2}$ (starting at $(1, (p - q)^2/4)$) and on their other sides by segments of curves $C_{rs}$ with $rs < N$. It follows that the new parts of the closure of the interiors of the curves $C_{p,q}$ where det$_N$ vanishes are exactly the boundary parts with $|m + \frac{1}{2}| > p + q - \frac{1}{2}$. Hence the only possible points of unitarity in $h > 0$ and $c \in (0, 1)$ have $h = h^\text{(m)}_{p,q}$ with $p + q < m + 1$.

Finally we specialise to the case when $m \geq 3$ is an integer. We have shown that if $L(c, h)$ is unitary then $h = h^\text{(m)}_{p,q} = [(p(m + 1) - qm)^2 - 1]/4m(m + 1)$ with $p, q \geq 1$ and $p + q \leq m$. We want to show that $h = h_{rs}$ for $1 \leq s \leq r \leq m - 1$. Note that since $p, q \geq 1$ and $p + q \leq m$, we have $p, q \leq m - 1$. If $p \geq q$, we take $r = p, s = q$. Otherwise $q \geq p + 1 \geq 2$. Let $p' = m - p$ and $q' = m + 1 - q$. Then $h^\text{(m)}_{pq} = h^\text{(m)}_{q',p'}$ and $1 \leq q' \leq p' \leq m - 1$. So in this case we may take $r = p'$, $s = q'$. This completes the proof of the FQS proposition.

**Corollary.** For $c = 1 - 6/m(m + 1)$ with $m \geq 3$, the values of $h$ from which $L(c, h)$ is unitary are given by $h = h_{rs,s} = [(r(m + 1) - sm)^2 - 1]/4m(m + 1)$ with $1 \leq s \leq r \leq m - 1$.

6. **THE MULTIPlicity ONE THEOREM.** Each of the multiplicity spaces $K_h$ appearing in the $SU(2) \times SU(2)/SU(2)$ decomposition gives an irreducible representation of the Virasoro algebra, so that $K_h = L(c, h)$.

**Lemma.** Let $h = h_{rs}$ and $M = rs + (m - r)(m + 1 + s) = m(m + 1) - (m + 1)r + ms$. Then

$$\text{ch } L(c, h) = \text{ch } K_h \mod q^{h+M}.$$ 

**Proof.** By the Kac determinant formula we have

$$\det_N(c^{(m)}_{\cdot}, h) = \prod_{1 \leq pq \leq N} (1 - h^\text{(m)}_{pq})^P(N-pq).$$

Note that $h_{rs} = h_{r's'}$. This is the only possible such coincidence, because if $((m + 1)p - qm)^2 = ((m + 1)r - ms)^2$ with $1 \leq p, q \leq m$ and $1 \leq s \leq r \leq m - 1$, we would have $(m + 1)(p \pm r) = m(q \pm s)$. Since $m$ and $m + 1$ are coprime, we would have $p \pm r = am$ and $q \pm s = a(m + 1)$ for some integer $a$. Hence either $p = r$ and $q = s$ or $p = m - r$ and $q = m + 1 - s$. Similar reasoning shows that $rs \neq r's'$. Indeed if $rs = r's'$, we get $rs = (m - r)(m + 1 - s)$. Thus $(m - r)(m + 1 - s) = sm$. Since $m$ and $m + 1$ are coprime, we would have $r$ divisible by $m$ and $s$ by $m + 1$. This contradicts $1 \leq r, s \leq m - 1$. Thus we may assume that $rs < r's'$. It follows that $h = h_{rs}$ is first a zero of det$_N$ when $N = rs$. It has multiplicity one. It has multiplicity $P(N - rs)$ for $rs \leq N < r's'$ and multiplicity $P(N - rs + P(N - r's')$ when $r's' \leq N < M$. By Lemma 1 in section 4, this gives an upper bound for the dimension of the kernel of the form $(\cdot, \cdot)$ and hence a lower for the character:

$$\text{ch } L(c, h_{rs}) \geq q^{h_{rs}}(1 - q^{rs} - q^{r's'}) \mod q^{h_{rs}+M}.$$

Here $\varphi(q) = \prod_{n \geq 1}(1 - q^n)$ and the inequality means that the coefficient of $q^{i+h}$ on the left hand side is greater than or equal to the coefficient on the right hand side for $i \geq M$. On the other hand the right hand side agrees with $\text{ch } K_h \mod q^{h_{rs}+M}$. Since $\text{ch } L(c, h) \leq \text{ch } K_h$, the result follows.

**Proof of Theorem.** By the preceding lemma, we have $\text{ch } L(c, h_{rs})$ and $\text{ch } K_h$ agree for energy levels $< M = rs - (m + 1)r + ms$, where $K_h$ is the $SU(2) \times SU(2)/SU(2)$ multiplicity space. Now suppose that the representation on the multiplicity space $K_h$ is not irreducible. The character computation we have made so far shows that there can be no singular vectors with level $< M$ in $K_h$. For any such would already appear in the cyclic module generated by the lowest energy vector $v_h$ by the equality above. This module is irreducible (by unitarity), so has no singular vectors apart from its lowest energy vector $v_h$. On the other
hand any lowest energy singular vector in $K_h$ would have energy $h' = \left[\left((m+1)p - mq\right)^2 - 1\right]/4m(m+1)$ with $1 \leq q \leq p \leq m - 1$ by the FQS criterion. We will check that $h' < M + h$, i.e.

$$\left[\left((m+1)p - mq\right)^2 - 1\right]/4m(m+1) < rs - (m+1)r + ms + \left[\left((m+1)r - ms\right)^2 - 1\right]/4m(m+1),$$

so that such a vector would have to lie within the energy range discussed above. It follows that $K_h$ is irreducible.

To prove (1), note the left hand side is maximised by taking $p = m - 1$ and $q = 1$, so we must show that

$$\frac{(m^2 - m - 1)^2 - 1}{4m(m+1)} < m(m+1) - (m+1)r + ms + \frac{(m+1)^2 - 1}{4m(m+1)}.$$ 

The quadratic expression in $(r, s)$ is minimised on the triangle $1 \leq s \leq r \leq m - 1$ at its vertices. For fixing $s$, the derivative in $r$ of the right hand side is $-m - 1 + ((m+1)r - ms)/2m < 0$; fixing $r$ the derivative in $s$ is $+m + (ms - (m+1)r)/2m > 0$. Thus in the interior of the triangle we can always decrease the right hand side by moving towards an edge parallel the axes. On interior points of edges parallel to the axes we can decrease the right hand side by moving towards a vertex. On the segment $(r, s) = (t, t)$ with $1 \leq t \leq m - 1$, the derivative in $t$ of the right hand side is $-1 + t/4m(m+1) < 0$. Thus the minimum occurs at a vertex. At the vertex $(r, s) = (m - 1, 1)$, the two extreme terms agree and the middle terms are positive. When $(r, s) = (1, 1)$, the right hand side becomes $m^2 + m - 1$ and the inequality is immediate. When $(r, s) = (m - 1, m - 1)$, the inequality becomes

$$\frac{(m^2 - m - 1)^2 - 1}{4m(m+1)} < (m - 1)(m+1) + \frac{(m-1)^2 - 1}{4m(m+1)}.$$ 

The left hand side equals $(m - 1)(m - 2)/4$ which is less than the first term on the right hand side.

7. **The Feigin–Fuchs Character Formula for the Discrete Series.** The unitary representation $L(c, h)$ with $c = 1 - 6/m(m+1)$ and $h = [(p(m + 1) - qm)^2 - 1]/4(m+1)m$ has normalised character

$$\text{ch} L(c, h) = \eta(q)^{-1}(\Theta_{a_+, b}(q, 1) - \Theta_{a_-, b}(q, 1)), $$

where $\eta(q) = q^{\pm} \prod_{n \geq 1} (1 - q^n)$, $a_\pm = p(m + 1) \mp qm$ and $b = m(m+1)$.

**Proof.** We have just shown that the multiplicity space with lowest energy $h = h_{p,q}$ is irreducible. Thus the character formula for $L(c, h)$ is given by the character of the multiplicity space.

8. **The Friedan–Qiu–Shenker Unitarity Criterion for $c$.** Our aim now is to prove the complete version of the Friedan–Qiu–Shenker unitarity theorem.

**Theorem (Friedan–Qiu–Shenker).** If $0 < c < 1$, then a representation with central charge $c$ is unitary if and only if $c = 1 - 6/m(m+1)$ with $m \geq 3$.

**Remark.** From our previous work, the permitted values of $h$ for a particular $m \geq 3$ are $h_{p,q}(m)$ with $1 \leq q \leq p \leq m - 1$.

We already know that any point not on a curve $C_{p,q}$ cannot be unitary. The points $h_{p,q}(m)$ are precisely the intersections of the different curves $C_{p,q}$. Each intersection $P$ may be described by first choosing a curve $C_{p,q}$ with $P \in C_{p,q}$ and $pq$ minimal and then choosing $C_{p',q'}$ with $P \in C_{p',q'}$ and $p'q'$ minimal. It is easy to check that these intersections are obtained by taking $p' = p + k$, $q' = q + k$ for $k \geq 1$ and $m = p + q + k - 1$. We now rule out all the points between these intersection points. Note that every curve $C_{p,q}$ touches $c = 1$ at $h = (p - q)^2/4$. Thus if $C_{p',q'}$ intersects $C_{p,q}$, then the part of $C_{p,q}$ on the $c < 1$ side of $C_{p',q'}$ is the part with $h$ decreasing if $h' < h$ and $h$ increasing if $h' > h$.

Take a point $P_0$ on $C_{p,q}$ corresponding to $p', q'$ as above. Let $N = pq < N' = p'q'$. Starting from the asymptote at $c = 1$ through $(p' - q')^2/4$, we may follow the $C_{p',q'}$ curve as it travels to $P_0 \in C_{p,q}$. We get a straight line parameterisation of $C_{p',q'}$ by taking $y = m$ as coordinate. Along the way to $P_0$, the curve will cross other curves $C_{p'',q''}$ transversely and simply at points $P_1, \ldots, P_k$. At level $N'$, the dimension
of the null space is 1 on $C_{p',q'}$ away from the intersection points. Near the asymptote on the $c > 1$ side of $C_{p',q'}$, the matrix of inner products $A(c,h)$ is positive–definite. We shall find a open neighbourhood $U$ of the part of the curve $C_{p',q'}$ above $P_0$ (containing $P_1, \ldots, P_k$ and part of the asymptote to $c = 1$) and a rank one spectral projection $P(c,h)$ of $A(c,h)$ in this strip depending continuously on $(c,h)$ such that $P(c,h)A(c,h) = \lambda(c,h)P(c,h)$ with $\lambda(c,h) = 0$ only on $C_{p',q'}$. It will follow that $\lambda(c,h) < 0$ on the $c < 1$ side of $C_{p',q'}$. In particular $A(c,h)$ will have a negative eigenvalue on $C_{p,q}$ in the segment between $P_0$ and the next intersection. This clearly will prove the unitarity theorem.

Using $m$ as parameter, we can replace the curve $C_{p',q'}$ by the $y$–axis. The following result (with $M = 1$) shows the existence and uniqueness of the spectral projection $P(z)$ for $z$ in an open neighbourhood of the $y$–axis with $z \neq P_i$. (This open neighbourhood should of course contain $P_0, \ldots, P_k$.)

**Lemma 1.** Let $A(z)$ be a continuous self–adjoint matrix–valued function on a topological space $Z$ such that ker$(A(z))$ has constant rank 1 (or rank $M$ more generally) for $z \in Z_0$, a closed subset of $Z$, and is invertible otherwise. For each $z \in Z_0$, there is an open neighbourhood $U$ of $z$ such that if $P(z)$ is the orthogonal projection onto $(M \text{th})$ the lowest eigenspace(s) of $A(z)^2$, then $z \mapsto P(z)$ is continuous on $U$. If $Z$ is an open subset of $\mathbb{R}^n$ and $A(z)$ is a smooth (or analytic) function of $z$, then $z \mapsto P(z)$ is also smooth (or analytic) on $U$.

**Proof.** Take $z \in Z_0$. By minimax there is a neighbourhood $U$ of $z$ such that the lowest eigenvalue of $A(t)^2$ is less than $r/2$ and the next eigenvalue is greater than are greater than $2r > 0$. Let $\chi$ be a continuous bump function supported in $(-r,r)$ with $\chi(0) = 1$. Then $P(z) = \chi(A(z)^2)$ for $t \in U$, since the only the lowest eigenvalue of $A(z)$ occurs in $(-r,r)$. Since $\chi$ can uniformly approximated by polynomials on any compact interval, it follows that $z \mapsto A(z)$ is continuous on $U$. The second assertion follows immediately from the contour integral expression for the spectral projection $P(z)$:

$$P(z) = \frac{1}{2\pi i} \int_{|w|=\varepsilon} (wI - A(z))^{-1} dw.$$ 

**Corollary.** There is an open subset $U$ of $Z$ containing $Z_0$ on which $P(z)$ can be defined (uniquely).

**Proof.** Take an open neighbourhood $U_z$ for each point $z \in Z_0$ and set $U = \bigcup_{z \in Z_0} U_z$. By uniqueness the different $P(z)$’s must agree on intersections of these opens.

We next need to use information from the Kac character formula to continue $P(z)$ across the points $P_i$. We simply have to define $P(z)$ in an open neighbourhood of each point $P_i$. Since the intersection at $P_i$ is transverse, we may assume that the transverse curve is the $x$–axis and $P = P_i$ corresponds to the point $(0,0)$.

**Proposition.** In an open neighbourhood of $P = P_i$ there are

(a) a unique continuous determination of a rank one projection $P(z)$ such that $P(z)$ is a spectral projection of $A(z)$ coinciding with the projection onto the kernel of $A(z)$ for $(0,y)$ with $y \neq 0$.

(b) a unique continuous determination of a rank $m$ projection $Q(x)$ on $y = 0$ such that $Q(x)$ is the spectral projection onto the kernel of $A(x,0)$.

Moreover $P$ and $Q$ are orthogonal on $y = 0$ and $P(0) + Q(0)$ is the projection onto the kernel of $A(0,0)$.

**Proof.** Note that $Q(z)$ could be constructed on $y = 0, x \neq 0$ using the method of the previous lemma. We need a variant of this construction. Let $C' = C_{p',q'}$ and let the transverse curve be $C'' = C_{p',q''}$ with $N'' = p''q'' < N'$. The kernel of $M_{N''}$ is rank one on $C''$. We choose parameters such that $P = (0,0), C'$ is the $y$–axis and $C''$ the $x$–axis. As in the previous lemma, let $R(z)$ be a smooth or analytic determination of a spectral subspace of $M_{N''}(z)$ giving the kernel on $y = 0$. Let $u(z) = R(z)u_0/\|R(z)u_0\|$ be a smooth or analytic choice of eigenvector near $z = 0$. Define vectors $u_j(z) = L_{-j_1} \cdots L_{-j_r} u(z)$ at level $N'$ for $\sum j_i = N' - N''$. There are $P(N' - N'')$ such vectors and they form a basis of $A(z)$ for $z = (x,0)$ with $x \neq 0$. The Gram–schmidt orthonormalisation process shows that the orthogonal projection $Q(z)$ onto the subspace spanned by the $u_j(z)$’s is smooth or analytic. Thus $Q(z)$ is a projection of rank $P(N' - N'')$ defined in a neighbourhood of 0. Note that $Q(z)$ is a spectral projection of $A(z)$ for $y = 0, x \neq 0$, but not
necessarily otherwise. Let \( B(z) = Q(z)A(z)Q(z) \) considered as a self-adjoint operator on \( \text{im}(Q(z)) \). Since \( \text{im}(Q(x,0)) \subseteq \ker(A(x,0)) \), we must have \( B(x,0) = 0 \). Hence \( B(x,y) = yB_0(x,y) \) where \( B_0 \) is smooth or analytic. The next lemma shows that \( B_0(0) \) is invertible.

**Lemma 2.** \( \det B_0(0) \neq 0 \).

**Proof.** We claim that \( \det M_N^N(c,h+pq) \neq 0 \) where \( N'' = N' - N \). In fact if the determinant vanished, \( (c,N''+pq) \) would have to lie on some \( C_{rs} \) with \( rs \leq N'' = pq - p'q' \). Thus

\[
(m+1)p + mq = \pm [(m+1)r - ms].
\]

By assumption \( p' = q - 1 + k, q' = p + k \) for some \( k \geq 1 \). Hence

\[
rs \leq p'q' - pq = m(m+1) - (m+1)p - mq.
\]

Combining (1) and (2) yields \( rs \leq (m+1)r - ms \leq m(m+1) \), so that \( (r \pm m)(s \mp (m+1)) \leq 0 \). Since \( 1 \leq r, s \leq m \), it follows that \( r = m \) or \( s = m+1 \) and equality holds in (2). Reducing modulo \( m \) or \( m+1 \), we deduce that \( p = m \) or \( q = m+1 \), neither of which is compatible with \( m = p + q + k - 1 \). Thus the claim holds.

Let \( u(x) \) be the null vector at level \( N'' \) and as above extend \( u \) to \( u(z) \). Since the submodule generated by \( u(z) \) is isomorphic to the Verma module \( M(c,h+pq) \), the corresponding matrix of inner products is \( \psi(z) \cdot M_{N'} \) where \( \psi(z) = (u(z),u(z)) \). But \( \psi(x,0) = 0 \), so that \( y|\psi(x,y) \). From the Kac determinant formula, \( \det M_N^N(x,y) = yf(x,y) \) with \( f(0) \neq 0 \). Hence \( \psi(z) = yg(z) \) with \( g(0) \neq 0 \). But then \( \det B(z) = (u(z),u(z))^p(N''-N')h(z) \) where \( h(0) \neq 0 \). Thus \( y^p(N''-N') \) is the highest power of \( y \) dividing \( \det B(z) \). Since \( B(z) = yB_0(z) \), we must have \( \det B_0(0) \neq 0 \), as required.

By continuity we deduce that \( B_0(z) \) is invertible (possibly by shrinking the neighbourhood of 0). With respect to the orthogonal decomposition corresponding to \( I = Q(z) \oplus (I - Q(z)) \), we may write \( A(z) = \begin{pmatrix} B(z) & C(z) \\ C(z)^* & D(z) \end{pmatrix} \). As above we have \( C(z) = yC_0(z) \), so that \( A(z) = \begin{pmatrix} yB_0(z) & yC_0(z) \\ yC_0(z)^* & D(z) \end{pmatrix} \). We have already seen that \( B_0(z) \) is invertible. If we try to solve \( A(z)v = 0 \) with \( v = \begin{pmatrix} a \\ b \end{pmatrix} \), we find \( a = -B_0^{-1}b \) and \( \langle D - yC^*B^{-1}C \rangle b = 0 \). Looking at the kernel at \( (0,0) \), we see that \( D(0,0) \) has one–dimensional kernel. Likewise \( D(0,0) \) has one–dimensional kernel for \( y \neq 0 \). On the other hand \( D(x,y) \) must have zero kernel for \( x \neq 0 \). Thus we may define \( P(z) \) near \( z = 0 \) as the spectral projection of \( F(z) = D - yC^*B^{-1}C \) corresponding to the lowest eigenvalue, just as in Lemma—1. By definition \( F(z) \) and hence \( P(z) \) is orthogonal to \( Q(z) \) on the \( x \)-axis. By construction \( P(z) \) and \( Q(z) \) have all the required properties.

**Remark.** Recall that if \( P \) and \( Q \) are orthogonal projections with \( \|P - Q\| < 1/2 \), then \( T = PQ + (I - P)(I - Q) \) satisfies check that \( \|T - I\| < 1 \) and \( PT = TP \). Thus \( T \) is an invertible operator conjugating \( P \) into \( Q \). Since \( TT^* \) commutes with \( P, U = (TT^*)^{-1/2}T \) gives a unitary such that \( UQU^* = P \). This means that on a sufficiently small neighbourhood of 0 the projections \( I - Q(z) \) can be identified using a unitary gauge change. This is not true for self-adjoint maps \( A(z) \)!

**Corollary.** The rank one projection–valued function \( P(z) \) can be defined on a neighbourhood of the \( C_{p',q'} \) containing the points \( P_0, \ldots, P_k \). It is continuous and satisfies \( P(z)A(z) = A(z)P(z) = \lambda(z)P(z) \) with \( \lambda(z) = 0 \) iff \( z \in C_{p',q'} \). \( \lambda(z) < 0 \) on the \( c < 1 \) side of \( C_{p',q'} \).

**Proof.** These first part follows taking the open to be a (finite) union of neighbourhoods of the \( P_i \)'s and finitely many other points on \( C_{p',q'} \). For \( c \) near 1, the matrix \( M_{N'}(c,h) \) is positive–definite on the \( c > 1 \) side of \( C_{p',q'} \), invertible off \( C_{p',q'} \) and has one–dimensional kernel on the \( c > 1 \) side of \( C_{p',q'} \). On the other hand \( \det M_N^N < 0 \) on the \( c < 1 \) side of \( C_{p',q'} \). By minimax, at most one eigenvalue of \( A(z) \) can change sign crossing \( C_{p',q'} \). The determinant condition therefore implies that it is the lowest eigenvalue of \( A \) that changes sign. (This evidently corresponds to the lowest eigenvalue of \( A^2 \).) Thus \( \lambda(z) < 0 \) on the \( c < 1 \) side of \( C_{p',q'} \). Since \( \lambda(z) \) is real and non–zero on the \( c < 1 \) side of \( C_{p',q'} \), the last assertion follows.

This last corollary completes the proof of the unitarity theorem.