Active exploration in adaptive model predictive control*

Anilkumar Parsi, Andrea Iannelli and Roy S. Smith

Abstract—A dual adaptive model predictive control (MPC) algorithm is presented for linear, time-invariant systems subject to bounded disturbances and parametric uncertainty in the state-space matrices. Online set-membership identification is performed to reduce the uncertainty and thus control affects both the informativity of identification and the system's performance. The main contribution of the paper is to include this dual effect in the MPC optimization problem using a predicted worst-case cost in the objective function. This allows the controller to perform active exploration, that is, the control input reduces the uncertainty in the regions of the parameter space that have most influence on the performance. Additionally, the MPC algorithm ensures robust constraint satisfaction of state and input constraints. Advantages of the proposed algorithm are shown by comparing it to a passive adaptive MPC algorithm from the literature.

I. INTRODUCTION

The idea of adaptive control is to make adjustments to the controller using measurement data. The conventional methods of adaptive control like gain scheduling and self tuning regulation are based on certainty equivalence [1]. That is, a model is identified using measurements and is assumed to represent the true system, neglecting the uncertainty of the model parameters. However, these methods cannot handle constraints on states and inputs since the uncertainty is not considered. Model predictive control (MPC) is an optimization based control technique which is popular for its ability to guarantee stability and robust constraint satisfaction under uncertainty [2]. At each time step, the MPC controller solves an optimization problem to compute the control input based on the model of the system. This structure facilitates easy integration of model adaptation into the control algorithm. Thus, a variety of adaptive MPC schemes have been proposed in the recent past using different model structures (state-space, impulse response, etc.) and adaptation techniques (set-membership identification, recursive least-squares, etc.) [3], [4], [5], [6].

Set-membership identification [7] is a technique used when the system is affected by bounded disturbances and noise. In [4], an adaptive MPC algorithm was presented for linear time-varying systems with input and output constraints subject to bounded measurement noise. Robust constraint satisfaction is ensured for all models within a model set which is updated online. An extension of this algorithm applicable to time-invariant systems was proposed in [8], where a worst-case cost is used to improve robustness of the performance. However, the algorithm uses an impulse response model, which results in a large number of parameters and requires identification in a high dimensional space. An alternative method has been proposed in [5] for systems with affine uncertainty in state-space models and subject to bounded additive disturbances. The algorithm uses tube-MPC to ensure robust constraint satisfaction while reducing the uncertainty using set-membership identification. However, in all these methods, the adaptation is passive, that is, the MPC optimizer exploit the fact that identification and control are being simultaneously performed.

These disadvantages can be addressed using dual control [9], a technique which computes control inputs under decision relevant, reducible uncertainty. Since identification is performed online, inputs must have a probing effect to reduce the model uncertainty. An optimal dual control problem can be formulated by explicitly modeling the dependence of uncertainty on the control inputs. However, the optimal solution is given by dynamic programming, whose computational complexity is high for even moderately sized systems [10]. Hence the existing dual control algorithms approximate the optimal control problem to a tractable one, by artificially inducing the probing of parameter space using heuristics. An extension of the adaptive MPC algorithm from [5] was proposed in [11], where an additional constraint is imposed on the control input to ensure persistent excitation. The constraint ensures parameter convergence, but requires tuning to reduce the suboptimality arising from excessive excitation. Moreover, the resulting controller cannot reduce the probing even after the the size of the uncertainty set is decreased. An alternative approach is integrated experiment design, where a metric of model uncertainty is included in the objective function [12]. In [13], linear systems modeled by a linear difference inclusion were controlled using a dual control MPC algorithm. A recursive least squares scheme was used to identify the parameters online, and a functional of the parameter error covariance is included in the MPC cost function. However, a lower covariance of the parameter uncertainty might not always translate into better performance of the system, and the cost function requires tuning of the control performance with the probing effect.

These problems can be mitigated by using an application-oriented approach to dual control [14]. Here, the probing effect is induced by using a measure of the robust performance of the system instead of geometric measures of uncertainty. This ensures that active exploration is performed, that is, the uncertainty is reduced in regions of parameter space to improve the control performance. The idea of using application-oriented exploration has been recently proposed for the linear quadratic regulator problem in [15], while in

* This work is supported by the Swiss National Science Foundation under grant number 200021L78890. The authors are with the Automatic Control Laboratory, ETH Zurich, Switzerland. {aparsi, iannelli, rsmith}@control.ee.ethz.ch
[16] the synthesis approach was modified and the problem setting extended to the finite horizon case. A min-max cost is used to enable the dual effect of the control input, where the maximization is performed over the uncertainty set parameterized as a function of the input.

The main contribution of this paper is to formulate active exploration in a dual adaptive MPC framework using online set-membership identification. For this purpose, the regulation of a linear, time-invariant system with affine uncertainty in the state space matrices is considered. The system is subject to bounded disturbances and must satisfy state and input constraints. Using an approach similar to [17], the problem of feasibility and learning is decoupled. A robust state tube is used to ensure constraint satisfaction. A least mean squares filter is used to estimate the parameter online, and the future constraints on the parameter set are predicted as a function of the control input. A predicted state tube is constructed to be robust against parameter uncertainty in the predicted parameter set, and the cost function is defined as the worst-case cost over the predicted state tube. The algorithm requires a non-convex optimization problem to be solved online. The algorithm has the flexibility to trade-off between the computational complexity arising from the non-convexity and quality of active exploration. The length of the predicted state is a parameter for this trade-off. The performance of the algorithm with varying predicted state tube lengths is compared to a passive adaptive MPC algorithm using numerical simulations.

A. Notation

The sets of real numbers and non-negative real numbers are denoted by \( \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \) respectively. The sequence of integers from \( n_1 \) to \( n_2 \) is represented by \( \mathbb{N}_{n_1}^{n_2} \). For a vector \( \mathbf{b} \), \( \mathbf{b}^\top \) represents its transpose, and \([\mathbf{b}]_i \) refers to the \( i \)th element. The \( i \)th row of a matrix \( A \) is denoted by \([A]_i\), and \( x_{i|t} \) denotes the value of the variable \( x \) at time step \( i+t \) predicted at the time step \( t \). For any scalar-valued function \( J \), \( \max_{h \in \mathbb{H}} J(h) \) refers to the maximum value of \( J \) over the set \( \mathbb{H} \). The Minkowski sum of two sets \( A \) and \( B \) is denoted by \( A \oplus B \), and \( I \) denotes a column vector of appropriate length whose elements are equal to 1. The convex hull of the elements of a set \( S \) is represented by \( \text{co}\{S\} \).

II. PROBLEM CONFIGURATION

A. System description

We consider a discrete time, linear time-invariant system with state \( x_k \in \mathbb{R}^n \), control input \( u_k \in \mathbb{R}^m \) and disturbance \( w_k \in \mathbb{W} \subset \mathbb{R}^n \) at the time step \( k \). The system dynamics can be described according to the parametric equation

\[
x_{k+1} = A(\theta)x_k + B(\theta)u_k + w_k,
\]

where \( \theta \in \mathbb{R}^p \) is an unknown but constant parameter and \( \theta^* \) is its true value. It is assumed that measurements of all the state variables are available at each time step. The parameterization of the state matrices can be described as

\[
A(\theta) = A_0 + \sum_{i=1}^{p} A_i[\theta]_i,
\]

\[
B(\theta) = B_0 + \sum_{i=1}^{p} B_i[\theta]_i,
\]

and the parameter \( \theta \) belongs to a bounded polyhedron given by

\[
\Theta = \{ \theta \in \mathbb{R}^p | H_0 \theta \leq h_0 \},
\]

where \( H_0 \in \mathbb{R}^{n_0 \times p} \), \( h_0 \in \mathbb{R}^{n_0} \), and \( \theta^* \in \Theta \).

Assumption 1: The disturbance set \( \mathbb{W} \) is a bounded set described by \( H_w \in \mathbb{R}^{n_w \times n} \) and \( h_w \in \mathbb{R}^{n_w} \) according to

\[
\mathbb{W} = \{ w \in \mathbb{R}^n | H_w w \leq h_w \}.
\]

The states and inputs of the system must satisfy the constraints

\[
Z = \{ (x_k, u_k) \in \mathbb{R}^n \times \mathbb{R}^m | F x_k + G u_k \leq 1 \},
\]

where \( F \in \mathbb{R}^{n_c \times n} \) and \( G \in \mathbb{R}^{n_c \times m} \). The objective is to regulate the state system from the initial condition \( x_0 \) to the origin, while robustly satisfying the constraints in [5].

B. Online set-membership identification

Set-membership identification is a technique used to identify systems affected by bounded noise with unknown statistical properties [7]. The identification procedure defines a feasible parameter set (FPS), which contains the set of all parameters to be robust against. The FPS is initialized with \( \Theta \) and updated at each time step \( k \) to \( \Theta_k \). To perform the update, a set of non-falsified parameters is constructed using measurement data according to

\[
\delta_{k+1} := \{ \theta \in \mathbb{R}^p | x_{k+1} - A(\theta)x_k - B(\theta)u_k \in \mathbb{W} \} = \{ \theta \in \mathbb{R}^p | H_w x_{k+1} - A(\theta)x_k - B(\theta)u_k \leq h_w \} = \{ \theta \in \mathbb{R}^p | -H_w D_k \theta \leq h_w + H_w d_{k+1} \},
\]

where \( D_k \in \mathbb{R}^{n \times p} \) and \( d_{k+1} \in \mathbb{R}^n \) are

\[
D_k = D(x_k, u_k) := [A_1 x_k + B_1 u_k, \ldots, A_p x_k + B_p u_k],
\]

\[
d_{k+1} := A_0 x_k + B_0 u_k - x_{k+1}.
\]

In (6), \( \delta_{k+1} \) is the set of all parameters \( \theta \) that could have generated the measurement \( x_{k+1} \). This can be used to refine the set \( \Theta_k \). As proposed in [18], using a block of past \( s \) measurements \( [x_{k-s}, \ldots, x_{k-1}] \) improves the identification compared to using a single measurement. Thus, the following non-falsified set is defined,

\[
\Delta_k := \{ \theta \in \mathbb{R}^p | -H_w D_k \theta \leq h_w + H_w d_{t+1} \},
\]

where \( H_\Delta, h_\Delta \) are matrices of appropriate dimensions. The FPS can be updated at each time step using \( \Delta_k \) according to \( \Theta_k = \Theta_{k-1} \cap \Delta_k \). However, this update will increase the number of constraints defining the FPS at every time step. To prevent this increase in parameter set complexity, \( \Theta_k \) is
defined using a finite number of polytopic constraints given in
\[
\Theta_k := \{ \theta \in \mathbb{R}^p | H_\theta \theta \leq h_{\theta_k} \},
\]  
where \( H_\theta \) is chosen offline and \( h_{\theta_k} \in \mathbb{R}^{n_\theta} \) is updated online such that
\[
\Theta_k \supseteq \Theta_{k-1} \cap \Delta_k
\]
is satisfied. This is ensured by calculating \( h_{\theta_k} \) as a solution of the following set of \( n_\theta \) linear programs:
\[
\begin{align*}
[h_{\theta_k}]_i &= \max_{\theta \in \mathbb{R}^p} [H_\theta]_i \theta \\
\text{s.t.} & \quad [H_\Delta] \theta \leq [h_{\Delta}]_i,
\end{align*}
\]
\( i = 1, 2, \ldots, n_\theta. \)

### III. ROBUST STATE TUBE AND CONSTRAINTS

#### A. Tube MPC

To ensure robust constraint satisfaction, the tube MPC approach proposed in [19] is used. The prediction horizon of the MPC problem is \( N \), and the control input is parameterized using a pre-stabilizing feedback gain \( K \in \mathbb{R}^{m \times n} \) as
\[
u_{l|k} = K x_{l|k} + v_{l|k},
\]
where \( \{v_{l|k}\}_{l=0}^{N-1} \in \mathbb{R}^m \) are decision variables in the MPC optimization problem.

**Assumption 2:** The feedback gain \( K \) is chosen such that \( A_\Delta(\theta) = A(\theta) + B(\theta)K \) is asymptotically stable \( \forall \theta \in \Theta \).

The gain \( K \) can be computed using standard robust control techniques, for example, following the approach in [20]. A state tube is designed using the set-based dynamics
\[
\begin{align*}
X_{0|k} &\supseteq \{ x_k \}, \\
X_{l+1|k} &\supseteq A(\theta)X_{l|k} + B(\theta)\nu_{l|k} + \mathbb{W},
\end{align*}
\]
\( \forall \theta \in \Theta_k, \ l = 0, 1, \ldots, N-1, \)

which ensures that \( x_{l|k} \in X_{l|k} \) for all the realizations of uncertainty and disturbance. The tube cross-section at each time step, \( X_{l|k} \), is parameterized by translation and scaling of the set
\[
X_0 := \{ x | H_x x \leq 1 \}
\]
:= \{ x^1, x^2, \ldots, x^v \},

where the vertices \( \{ x^1, x^2, \ldots, x^v \} \) and the matrix \( H_x \in \mathbb{R}^{w \times n} \) are computed offline. The variables \( z_{l|k} \in \mathbb{R}^w \) and \( \alpha_{l|k} \in \mathbb{R}_{\geq 0} \) define the translation and scaling of \( X_0 \) respectively, and are decision variables in the MPC optimization. Then, for \( l = N|k \), the state is parameterized as
\[
\begin{align*}
X_{l|k} &= \{ z_{l|k} \} + \alpha_{l|k}X_0 \\
&= \{ z_{l|k} \} + \alpha_{l|k}\text{co}\{ x^1, x^2, \ldots, x^v \} \\
&= \{ x | H_x (x - z_{l|k}) \leq \alpha_{l|k}1 \}.
\end{align*}
\]

#### B. Reformulation of constraints

The constraints defined in (5) must be satisfied for all values of \( \theta \in \Theta_k \) and disturbances \( \mathbb{W} \). By substituting (11) in (5), one obtains
\[
\begin{align*}
(F + G K) x_{l|k} + G v_{l|k} &\leq 1, \ \forall x_{l|k} \in X_{l|k}, \\
\Leftrightarrow \max_{x \in X_{l|k}} (F + G K) z_{l|k} + G \alpha_{l|k} (F + G K) x &\leq 1, \\
\Leftrightarrow (F + G K) z_{l|k} + G v_{l|k} + \alpha_{l|k} f &\leq 1,
\end{align*}
\]
for all \( l \in N^0 \), \( f_i \) from \( \bar{F} \) parameterized according to (13), and \( \Theta_{l|k} \). To reformulate the set-dynamics in (12), the notation
\[
\begin{align*}
\theta_{l|k} &= z_{l|k} + \alpha_{l|k} x_{l|k}, \\
d_{l|k} &= A_0 x_{l|k} + B_0 v_{l|k} - z_{l+1|k}, \\
u_{l|k} &= K x_{l|k} + \nu_{l|k}, \\
D_{l|k} &= D(x_{l|k}, u_{l|k}),
\end{align*}
\]
is used for \( j \in N^0 \), \( l \in N^N \). The following proposition from [5] formulates the set-dynamics as linear equality and inequality constraints.

**Proposition 1:** Let the state tube \( \{ X_{l|k} \}_{l=0}^{N-1} \) be parameterized according to (13), and \( \bar{w}_i \) is defined as \( \max_{x \in \bar{X}_l} |H_l x|/w_i \) for \( i \in N^0 \).

Then, the set-dynamics (12) are satisfied if and only if for each \( j \in N^0 \) and \( l \in N^N \) there exists \( \Lambda_{l|k}^j \in \mathbb{R}^{w \times n_\theta} \) such that
\[
\begin{align*}
-H_x z_{0|k} - \alpha_{0|k}1 &\leq -H_x x_k, \\
\Lambda_{l|k}^j h_{\theta_k} + H_x d_{l|k} - \alpha_{l|k}1 &\leq -\bar{w}, \\
H_x D_{l|k} &\leq \Lambda_{l|k}^j H_\theta.
\end{align*}
\]

#### C. Terminal set

To obtain an MPC algorithm which ensures recursive feasibility, the state tube is directed to a terminal set. The terminal constraints are imposed on \( x_{N|k} \) and \( \alpha_{N|k} \) since they define the last cross-section of the state tube.

**Assumption 3:** There exists a nonempty terminal set \( \mathcal{X}_T = \{ (z, \alpha) \in \mathbb{R}^w \times \mathbb{R} | z = 0, \alpha \in [0, \bar{\alpha}] \} \), such that for all \( \theta \in \Theta \) it holds that
\[
\alpha \in [0, \bar{\alpha}] \implies \exists \alpha^+ \in [0, \bar{\alpha}] \text{ s.t. } A_\Delta(\alpha^+) \mathbb{W} \subseteq \mathbb{W} \subseteq A_\Delta(\alpha), \alpha \in [0, \bar{\alpha}].
\]

**Assumption 4** implies that the set \( \mathcal{X}_T \) is robustly invariant set for the set-dynamics in \( (z, \alpha) \), with an additional constraint that the set remained centered at origin. If Assumption 3 is satisfied, \( \mathcal{X}_0 \) is chosen to be a robust invariant set, then Assumption 4 is satisfied if \( (x, K x) \in \mathcal{Z} \) \( \forall x \in \mathcal{X}_0 \). Thus, the terminal constraint for the MPC algorithm is
\[
z_{N|k} = 0; \ \alpha_{N|k} \leq \bar{\alpha}.
\]
by the predicted parameter set. The objective function of the MPC optimization problem is defined as a function of the predicted state tube. Finally, the dual adaptive MPC algorithm is described and is properties are briefly discussed.

A. Estimation of θ

An estimate of the parameters is needed to predict the effect of the control action on the identification. For this purpose, a least mean squares (LMS) filter is used to calculate \( \hat{\theta}_k \), which is an estimate of \( \theta \) at the time step \( k \). Using \( \hat{\theta}_k \), the predicted state at the next time step can be written as

\[
\hat{x}_{1|k} = A(\hat{\theta}_k)x_k + B(\hat{\theta}_k)u_k,
\]

and the prediction error at the next time step as

\[
\tilde{x}_{1|k} = A(\theta^*)x_k + B(\theta^*)u_k - \hat{x}_{1|k}.
\]

For a given initial guess \( \hat{\theta}_0 \) and parameter update gain \( \mu \) satisfying \( \frac{1}{\mu} \geq \sup_{x,u \in \mathbb{R}^2} |D(x,u)|^2 \), the estimate \( \hat{\theta}_k \) is recursively updated according to

\[
\hat{\theta}_k = \hat{\theta}_{k-1} + \mu D(x_{k-1}, u_{k-1})^T(x_k - \hat{x}_{1|k-1}),
\]

where \( \hat{\theta}_{k|k} = arg\ min_{\hat{\theta}} ||\theta - \hat{\theta}|| \) denotes the Euclidean projection of the point \( \hat{\theta} \) onto the set \( \Theta_k \). As shown in [11], using the above LMS filter to estimate the parameters \( \theta \) will result in a bounded prediction error.

B. Predicted state tube

At each time step, \( \hat{\theta}_k \) is used to predict the future constraints on the parameter set \( \Theta_k \) as

\[
\delta_k := \{ \theta \in \mathbb{R}^|A(\theta)x_k|B(\theta)u_k \in \mathbb{W} \},
\]

\[
= \{ \theta \in \mathbb{R}^|H_u D_k \theta \leq h_w - H_w D_k \hat{\theta}_k \},
\]

where \( u_{k-1} = Kx_k + v_{0|k} \) is the first control input calculated by the MPC controller. Only \( u_k \) is applied to the system in closed loop, and hence the predicted constraints from other inputs are not considered. A predicted parameter set \( \hat{\Theta}_k \subseteq \Theta_k \) can now be defined as

\[
\hat{\Theta}_k := \hat{\Theta}_k \cap \hat{\delta}_k
\]

\[
= \{ \theta \in \mathbb{R}^|H_u \theta \leq h_{\hat{\theta}_k} \}
\]

\[
= \{ \theta \in \mathbb{R}^|H_u \theta \leq h_{\hat{\theta}_k} \}.
\]

Figure 1a shows the parameter estimate \( \hat{\delta}_k \), the parameter sets \( \Theta_k \) and \( \hat{\Theta}_k \), and the predicted constraints. Using the set \( \hat{\Theta}_k \), a predicted state tube of length \( N \leq N \) is constructed satisfying

\[
\hat{X}_{0|k} \ni \{ x_k \},
\]

\[
\hat{X}_{1|k} \ni A(\theta)\hat{X}_{1|k} + B(\theta)u_{1|k} \ni \mathbb{W}
\]

\[
\forall \theta \in \hat{\Theta}_k, \quad l = N_0^{N-1}.
\]

Note that the control input applied is the same for the set-dynamics (22) and (23), while the parameter sets used are different. The evolution of the predicted state tube and the robust state tube is shown in Figure 1b. Since the parameter sets satisfy \( \hat{\Theta}_k \subseteq \Theta_k \), the predicted state tube lies within the robust state tube. Each set in the predicted state tube is parameterized as \( \hat{X}_{l|k} = \{ \hat{z}_{l|k} \} \oplus \hat{\alpha}_{l|k}X_0 \) where \( \hat{z}_{l|k} \in \mathbb{R}^n, \hat{\alpha}_{l|k} \in \mathbb{R}_{\geq 0}, \) for \( l = N_0 \) are decision variables in the MPC optimization problem. To formulate the predicted set-dynamics in (23) as constraints of the MPC problem, the following definitions are used to simplify notation

\[
\hat{x}_{j|k} = \hat{z}_{l|k} + \hat{\alpha}_{l|k}x_j,
\]

\[
\hat{u}_{j|k} = A_0 \hat{x}_{j|k} + B_0 \hat{u}_{j|k} - \hat{z}_{l+1|k},
\]

\[
\hat{D}_{j|k} = D(\hat{x}_{j|k}, \hat{u}_{j|k}).
\]

The following proposition can be used to represent the dynamics of the predicted state tube as constraints. The proof is similar to Proposition 1 and is omitted.

Proposition 2: The predicted state tube \( \{ \hat{X}_{l|k} \}_{l=0}^{N-1} \) satisfies the set-dynamics (23) if and only if for all \( j \in N_0^N \) and \( l \in N_0^{N-1} \) there exists \( \hat{A}_{l|k} \) such that

\[
-H_{z_{l|k}} \hat{z}_{l|k} \leq -H_{x_{l|k}},
\]

\[
\hat{D}_{l|k} = \hat{D}_{l|k} - H_{u_{l|k}} \hat{z}_{l+1|k} - H_{w_{l|k}} - \hat{z}_{l+1|k},
\]

\[
H_{z_{l|k}} H_{\hat{z}_{l|k}} = \mathbb{I},
\]

\[
H_{z_{l|k}} \hat{D}_{l|k} = \hat{D}_{l|k} H_{\hat{z}_{l|k}}.
\]

C. Predicted worst-case cost

The MPC cost function \( J \) is defined as

\[
J = \sum_{l=0}^{N} J_l,
\]

\[
J_l = \left\{ \begin{array}{ll} \max_{x \in \hat{X}_{l|k}} ||Qx||_{\infty} + ||R\hat{u}_{l|k}||_{\infty}, & \text{if } l \in N_0^{N-1} \\ \max_{x \in \hat{X}_{l|k}} ||Qx||_{\infty} + ||R\hat{u}_{l|k}||_{\infty}, & \text{if } l \in N_0^{N} \end{array} \right.
\]

where \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{m \times m} \) are positive definite cost matrices. A linear cost is chosen so that it can be reformulated using linear inequalities. A quadratic cost would result in second order cone constraints, which increase the computational complexity. The cost function (26) is the sum of the predicted worst-case cost over the horizon \( N \) and the worst-case cost over the remaining prediction horizon. This combination is used because the constraints in (25) are bilinear in the decision variables, unlike (16). This makes the problem non-convex and thus increases the computational complexity of the optimization. The parameter \( N \) offers a trade-off between the computational complexity and active exploration. A higher value of \( N \) increases the effect of a smaller parameter set \( \hat{\Theta}_k \) and thus promotes exploration. However, it also increases the number of bilinear constraints and the computational complexity of the algorithm. This trade-off will be exemplified in the numerical tests shown in Section V.

D. MPC algorithm

The MPC optimization can now be defined using all the elements described above. The decision variables are given
Then, the closed loop system using Algorithm 1 satisfies the 

\[ \hat{\Theta}_k \]

according to (3), and must be computed offline. The value of 

\[ \Theta \]

properties of the algorithm

The following proposition establishes the control theoretic

Adaptive MPC algorithm with active exploration is described

(a) The estimated parameter is \( \hat{\Theta}_k \). The parameter set \( \Theta_k \) is bounded by the blue constraints and the dashed lines represent the predicted constraints. The shaded region shows the predicted parameter set \( \hat{\Theta}_k \).

(b) The state tube \( \{X_{l|k}\}_{l=1}^N \) is shown in blue and predicted state tube \( \{\hat{X}_{l|k}\}_{l=1}^N \) is shown in red (dashed). The values of \( N \) and \( \hat{N} \) are 4 and 2 respectively. The depiction of the terminal set \( X_T \) in \( \mathbb{R}^n \) is shown in black and contains the set \( X_{N|k} \) centered at origin.

Fig. 1: Depiction of parameter set, predicted parameter set, state tube and predicted state tube

Algorithm 1 Adaptive MPC with active exploration

\[
\begin{align*}
\text{Offline} & \quad \text{Choose } K, \mu, \alpha \text{ and } X_0. \text{ Initialize } h_{\theta_k} \text{ and } \hat{\theta}_k. \\
\text{Online} & \quad 1: \ k \leftarrow 1 \\
& \quad 2: \ \text{repeat} \\
& \quad 3: \ \text{Obtain the measurement } x_k \\
& \quad 4: \ \text{Construct } \Delta_k \text{ according to (7)} \\
& \quad 5: \ \text{Update } h_{\theta_k} \text{ using (10) and } \theta_k \text{ according to (20)} \\
& \quad 6: \ \text{Solve optimization problem (23)} \\
& \quad 7: \ \text{Apply the control input } u_k = Kx_k + v_{0|k} \\
& \quad 8: \ k \leftarrow k + 1 \\
& \quad 9: \ \text{until}
\end{align*}
\]

Proof: The properties (i) and (ii) are proven by induction. Let \( \theta^* \in \Theta_k \) for some \( k \geq 0 \) which implies \( \theta^* \in \Delta_k \) according to (9). Since \( u_k \in \mathbb{W} \), the definition of the non-falsified set \( \delta_{k+1} \) in (6) implies \( \theta^* \in \delta_{k+1} \). Since the \( \Delta_{k+1} \) is constructed using \( \delta_{k+1} \) and the last \( s - 1 \) measurements used in \( \Delta_k \), \( \theta^* \in \Delta_{k+1} \). Applying (9) for the time step \( k+1 \) proves \( \theta^* \in \Theta_{k+1} \).

Property (ii) implies recursive feasibility, i.e., if the MPC problem is feasible at the first time step, it remains feasible. For the proof, assume there exists a feasible solution at time step \( k \geq 0 \). It is sufficient to find a feasible solution for the control variables \( \{v_{l|k}\}_{l=1}^{N-1} \) and state tube variables \( \{z_{l|k}, \alpha_{l|k}, \{A_{l|k}^j\}_{j=1}^v\}_{l=0}^N \) at the next time step to prove that the optimization problem is feasible. This is because the state tube satisfies the set-dynamics of the predicted state tube (23). Consider the state tube \( \{X_{l|k}\}_{l=0}^N \) computed at time step \( k \). Assumption 3 implies that the feedback controller \( u = Kx \) maps the set \( X_{N|k} \) to a set \( \alpha^+X_0 \), where \( \alpha^+ \in [0, \alpha] \). Since the relation \( \Theta_k \supseteq \Theta_{k+1} \) holds according to (2), a feasible sequence of control inputs at the next time step is formed by using the time-shifted values of the past inputs as

\[
\begin{align*}
\{v_{l|k+1}\}_{l=0}^{N-2} &= \{v_{l+1|k}\}_{l=0}^{N-2} \\
v_{N-1|k+1} &= 0
\end{align*}
\]

Since the set-dynamics to be satisfied by the state tube are set-inclusions, time-shifted versions of the sets computed at previous time steps are feasible solutions given by

\[
\begin{align*}
\{X_{l|k+1}\}_{l=0}^{N-1} &= \{X_{l+1|k}\}_{l=0}^{N-1} \\
X_{N|k+1} &= \alpha^+X_0
\end{align*}
\]

Property (iii) is a direct result of Proposition 1 and recursive feasibility.

V. Numerical results

In this section, the performance of the dual adaptive MPC (DAMP) algorithm presented in this paper is compared to a passive adaptive MPC (PAMPC) algorithm from [5]. The DAMPC algorithm performs active exploration using a predicted state tube, and the dependence of the performance on the length of the predicted tube \( \hat{N} \) is studied. The PAMPC algorithm does not consider any prediction tube and uses...
the worst-case cost in the MPC cost function. The system matrices used in the simulation are given by

\[
A_0 = \begin{bmatrix} 0.85 & 0.5 \\ 0.2 & 0.6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
B_0 = \begin{bmatrix} 1 & 0.4 \\ 0.2 & 0.4 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0.5 \\ 0.4 & 0 \end{bmatrix}.
\]

The uncertainty in the parameters is described by

\[
\Theta = \{ \theta \in \mathbb{R}^2 \mid \|\theta\|_\infty \leq 1 \},
\]

and the true parameter is \( \hat{\theta} = [0.95, 0.3]^T \). The disturbance set is \( \mathcal{W} = \{ w \in \mathbb{R}^2 \mid \|w\|_\infty \leq 0.1 \} \) and the state and input constraints are described by

\[
Z = \left\{ (x, u) \in \mathbb{R}^{2 \times 2} \right\} \left\{ \begin{array}{l} \|x\|_\infty \leq 10 \\ -0.5 \leq \|u\|_1 \leq 1 \\ -2 \leq \|u\|_2 \leq 2 \end{array} \right\}.
\]

The initial state of the system is \( x_0 = [1, 1.5]^T \). In both PAMPC and DAMPC, the state tube is constructed by translating and scaling the set \( \mathcal{X}_0 = \{ x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1 \} \). The bounded complexity update of \( \Theta_k \) is performed using \( n_0 = 58 \) hyperplanes which are initially chosen as outer bounds of the set \( \Theta \). The cost matrices are given as \( Q = R = I_{2 \times 2} \), the prestabilizing gain used is

\[
K = \begin{bmatrix} -0.5625 & 0 \\ 0 & 0 \end{bmatrix},
\]

and the corresponding terminal set bound \( \bar{\alpha} \) is 0.89. The prediction horizon chosen is \( N = 8 \) time steps for all the algorithms. Two different values of the \( N \) are used, and the corresponding adaptive MPC schemes are referred to as DAMPC\(_2\) and DAMPC\(_3\) for \( N = 2, 5 \) respectively. The DAMPC schemes are initialized with a parameter estimate \( \bar{\theta} = [0.5, 0.5]^T \).

The closed loop trajectories using each of the controllers are given in Figure 2. It can be seen that active exploration improves the performance of regulating the system. The PAMPC scheme achieves a closed loop cost of 6.04, while DAMPC\(_2\) and DAMPC\(_3\) achieve 4.49 (25% lower) and 4.21 (30% lower) respectively. This is because the coefficients of control input \( [u]_2 \) have high uncertainty, and the PAMPC algorithm does not excite this input since the MPC optimizer within does not explicitly include the benefit of online identification. However, the DAMPC algorithms use a higher value of \( [u]_2 \) which improves the identification and reduces the closed loop cost. The updated uncertainty set \( \Theta_0 \) of each scheme after 10 time steps is shown in Figure 3. Even though the uncertainty sets for DAMPC\(_2\) and PAMPC have similar size, the DAMPC\(_2\) algorithm has a lower uncertainty in the parameter \( [\theta]_2 \) which has a stronger influence on the performance. This can be interpreted using Figure 4 which shows the closed loop cost of an MPC controller specifically designed for each plant in the uncertainty set, plotted as a function of the corresponding parameters. Since the goal is to investigate the exploratory actions of the three aforementioned controllers, only the region around the uncertainty sets depicted in Figure 3 is considered. These closed loop trajectories are shown in Figure 2.
costs reveal the relative difficulty in controlling the systems, and thus motivate why some regions of parameter space are removed from the uncertainty set $\Theta_k$ rather than the others. The figure shows that compared to the DAMPC$_2$ controller, the PAMPC controller results in a parameter set associated with worse performance. This is because the predicted worst-case cost function induces exploration so as to remove the systems difficult to control from the future parameter set. Additionally, it can be seen that using a longer predicted state tube horizon $N$ improves exploration. The DAMPC$_5$ algorithm has the smallest uncertainty set, while also having the least closed loop cost. However, the cost-reduction offered by DAMPC schemes is achieved at the price of computational complexity. The PAMPC algorithm solves a series of linear programs, while the DAMPC algorithm needs to solve a non-convex optimization problem for MPC and additionally estimate the parameter. The simulations were performed on a laptop using Intel i7-8550U 1.8 GHz processor, and the optimization problems were setup using YALMIP [21] and solved using IPOPT [22]. The average solver time for the optimization problem in PAMPC was 0.042s, while that of DAMPC$_2$ and DAMPC$_5$ were 0.89s and 1s respectively. A similar trend was observed for the performance and solver times with different values of the predicted state tube length.

VI. CONCLUSION

A dual adaptive MPC scheme was presented for systems with parametric uncertainty in state-space matrices. The algorithm uses online set-membership identification to reduce the uncertainty in the parameters and a tube MPC approach to ensure robust constraint satisfaction. A predicted state-tube is used to capture the effect of the future control inputs on identification, and a predicted worst-case cost is optimized. The resulting optimization problem in the MPC is non-convex, but offers the flexibility to trade-off the computational complexity with performance. The algorithm ensures recursive feasibility and consistency of the parameter set, and performs better compared to a passive adaptive MPC approach from literature while regulating a system.