MULTIVARIATE DENSITY ESTIMATION VIA ADAPTIVE PARTITIONING (II): POSTERIOR CONCENTRATION

BY LINXI LIU∗,† AND WING HUNG WONG∗,†,‡

Department of Statistics, Stanford University†
Department of Health Research and Policy, Stanford University‡

In this paper, we study a class of non-parametric density estimators under Bayesian settings. The estimators are piecewise constant functions on binary partitions. We analyze the concentration rate of the posterior distribution under a suitable prior, and demonstrate that the rate does not directly depend on the dimension of the problem. This paper can be viewed as an extension of ([12]) where the convergence rate of a related sieve MLE was established. Compared to the sieve MLE, the main advantage of the Bayesian method is that it can adapt to the unknown complexity of the true density function, thus achieving the optimal convergence rate without artificial conditions on the density.

1. Introduction. In this paper, we study the asymptotic behavior of posterior distributions of a class of density estimators based on adaptive partitioning. Density estimation is a fundamental problem in statistics—once an explicit estimate of the density function is obtained, various kinds of statistical inference can follow, including nonparametric testing, clustering, and data compression.

With univariate (or bivariate) data, the most basic non-parametric method for density estimation is the histogram method. In this method, the sample space is partitioned into regular intervals (or rectangles), and the density is estimated by the relative frequency of data points falling into each interval (rectangle). However, this method is of limited utility in higher dimensional spaces because the number of cells in a regular partition of a $p$-dimensional space will grow exponentially with $p$, which makes the relative frequency highly variable unless the sample size is extremely large. In this situation the histogram may be improved by adapting the partition to the data so that larger rectangles are used in the part of the sample space where data is sparse. Motivated by this consideration, researchers have recently developed

∗Supported by NIH grant R01GM109836, and NSF grants DMS1330132 and DMS1407557.

Primary 62G20, secondary 62H10.

Keywords and phrases: density estimation, posterior concentration rate, adaptive partitioning.

1
several multivariate density estimation methods based on adaptive partitioning. For example, by generalizing the classical Polya Tree construction ([3]), [21] developed the Optional Polya Tree (OPT) prior on the space of simple functions. In this prior the partition that supports the simple function is generated by a random recursive partitioning process. As the partition is random a priori, it can be inferred from its posterior distribution once the data is observed. Computational issues related to OPT density estimates were discussed in [14], where efficient algorithms were developed to compute the OPT estimate. In [14], a different way to construct the random partition is introduced where the size of the partition grows linearly instead of geometrically as in OPT. This allows the authors to use sequential importance sampling to sample from the posterior distribution. This Bayesian Sequential Partition (BSP) method is computationally more scalable to higher dimensions than the OPT method. As an application, the methods were used to estimate within-class densities in classification problems, thereby obtaining approximations to the Bayes classifier. When tested on standard data sets with \( p \) ranging from 10-50, the results are competitive to those from leading classification methods such as SVM and boosted tree.

The purpose of the current paper is to address the following questions on such Bayesian density estimates based on partition learning. Question 1: what is the class of density functions that can be well estimated by these methods. Question 2: what is the rate in which the posterior distribution is concentrated around the true density as the sample size increases? For question 1, our analysis will make use of some results from a companion paper [12] on the properties of sieve MLEs where the sieve is constructed by considering simple functions supported by binary partitions of growing sizes. Specifically, [12] showed that if the true density can be approximated in Hellinger distance at a rate of \( I^{-r} \) where \( I \) is the size of the partition, then the convergence rate of the sieve-MLE density estimate is \( O(n^{-r/(2r+1)}) \) up to \( \log n \) terms, where \( n \) is the sample size. We note that the term “well estimated” in question 1 can now be given a more specific meaning, namely that the convergence rate of the estimate should not deteriorate fast when the dimension \( p \) of the sample space is large. [12] gave examples of functions for which approximation rate \( I^{-r} \) is not affected by \( p \) much. These include functions satisfying mixed-Hölder continuity conditions or functions with spatial sparsity as characterized by fast decay of Haar wavelet coefficients. It is well known that sieve MLEs are closely related to penalized estimates which is in turn related to Bayesian methods ([20], [16] and [17]). Thus we expect that the class of density well estimated by the Bayesian methods should be the same class analyzed by [12], i.e. the class of densities that can
be approximated at rate $I^{-r}$ for some $r > 0$. We will see that this is indeed true as a consequence of our main result. Our main result (Theorem 2.1) also provides the answer to the second question: it shows that the posterior probability is concentrated in a shrinking Hellinger ball around the true density, where the radius of the ball is $O(n^{-r/(2r+1)})$ up to log $n$ terms.

Although the convergence rate of the Bayesian method matches that of the sieve MLE, there is an important difference. While this rate is achieved by the Bayesian method without requiring any knowledge of the constant $r$ that characterizes the complexity of the true density function, the sieve MLE can achieve this same rate only if the size of the sieve grows at a rate that depends on $r$, specifically, the size of the partition must be of order $n^{-1/(2r+1)}$. In other words, the Bayesian estimate is adaptive to the complexity of the true density while the sieve MLE is not. This is an important difference in practice.

We now briefly review previous literature on convergence rate of posterior distributions. In breakthrough works [6] and [17], the authors developed general theory on posterior convergence rates and discussed several applications. Following this theory, most results have focused on mixture models ([13] and [4]), because these models allow the study of smooth density functions. Some elegant works include [7] and [8], which studied the concentration rate of the posterior distribution under Dirichlet mixtures of Gaussian priors, and [5] and [15], which examined the posterior concentration rate under the mixtures of Beta priors. Compared to the previous literature, one major improvement of our result is that it can deal with multivariate cases. In particular, the rate attained by our estimate is independent of the dimension $p$, if the true density falls within the support of the prior. When specialized to the univariate case, it still coincides with the previous results. For instance, for one dimensional Hölder space with parameters between 0 and 1, our result is minimax up to a log $n$ term. Another contribution is that our result can adapt to the unknown complexity of the density function. There has been few adaptive rate results for Bayesian density estimates in the literature (see [10] for a more extensive review of recent results on adaptive posterior concentration rates). A notable exception is in [15], where the author obtained adaptive posterior concentration rates for one-dimensional Hölder spaces under mixture Beta priors. Here, our result can adapt to a broader range of density functions, including spatially sparse density functions, Hölder continuous functions, and functions of bounded variation. We gain this advantage at a cost of relatively poor performance for functions with higher order smoothness. It is our belief that in the multivariate case, smoothness is not the best condition to characterize functions that can be
well estimates. The rate under usual smoothness condition is $n^{-\left(\kappa/(2\kappa+p)\right)}$ ([19]), where $\kappa$ is the number of derivatives. Thus high order smoothness cannot guarantee good convergence when $p$ is large.

The article is organized as follows. In Section 2 we define the prior distribution and summarize our main results on posterior concentration rate. We express the posterior measure of the complement of a Hellinger ball as a ratio, where the numerator is the product of prior probability and the likelihood, and the denominator is the normalizing factor. In order to derive the concentration rate, we need to upper bound the numerator and lower bound the denominator. In Section 3 and Section 4, we discuss these upper and lower bounds respectively. Finally, in Section 5, we combine these results to derive the posterior concentration rate.

2. Main results on posterior concentration rate. In this paper, we focus on the density estimation problem in the $p$-dimensional Euclidean space. Let $(\Omega, \mathcal{B})$ be a measurable space and $f_0$ be a compactly supported density function with respect to the Lebesgue measure $\mu$. $Y_1, Y_2, \cdots, Y_n$ is a sequence of independent variables distributed according to $f_0$. After translation and scaling, we can always assume that the support of $f_0$ is contained in the unit cube in $\mathbb{R}^p$. Translating this into notations, we assume that $\Omega = \{(y_1, y_2, \cdots, y_p) : y_i \in [0, 1]\}$. $\mathcal{F} = \{f : f$ is a nonnegative measurable function on $\Omega : \int_\Omega f d\mu = 1\}$ denotes the collection of all the density functions on $(\Omega, \mathcal{B}, \mu)$. Then $\mathcal{F}$ constitutes the parameter space in this problem. Note that $\mathcal{F}$ is an infinite dimensional parameter space.

2.1. Densities on binary partitions. To address the infinite dimensionality of $\mathcal{F}$, we construct a sequence of finite dimensional approximating spaces $\Theta_1, \Theta_2, \cdots, \Theta_I, \cdots$ based on binary partitions. With growing complexity, these spaces provide more and more accurate approximations to the initial parameter space $\mathcal{F}$. Here, we use a recursive procedure to define a binary partition with $I$ subregions of the unit cube in $\mathbb{R}^p$. Let $\Omega = \{(y_1, y_2, \cdots, y_p) : y_i \in [0, 1]\}$ be the unit cube in $\mathbb{R}^p$. In the first step, we choose one of the coordinates $y^l$ and cut $\Omega$ into two subregions along the midpoint of the range of $y^l$. That is, $\Omega = \Omega_0^l \cup \Omega_1^l$, where $\Omega_0^l = \{y \in \Omega : y^l \leq 1/2\}$ and $\Omega_1^l = \Omega \setminus \Omega_0^l$. In this way, we get a partition with two subregions. Note that the total number of possible partitions after the first step is equal to the dimension $p$. Suppose after $I - 1$ steps of the recursion, we have obtained a partition $\{\Omega_{i}\}_{i=1}^I$ with $I$ subregions. In the $I$-th step, further partitioning of the region is defined as follows:

1. Choose a region from $\Omega_1, \cdots, \Omega_I$. Denote it as $\Omega_{i0}$. 

4. L. LIU AND W. H. WONG
2. Choose one coordinate $y^l$ and divide $\Omega_{i_0}$ into two subregions along the midpoint of the range of $y^l$.

Such a partition obtained by $I - 1$ recursive steps is called a binary partition of size $I$. Figure 2.1 displays all possible two dimensional binary partitions when $I$ is 1, 2 and 3.

Now, let

$$\Theta_I = \{ f \in \Theta : f = \sum_{i=1}^{I} \beta_i 1_{\Omega_i}, \sum_{i=1}^{I} \beta_i \mu(\Omega_i) = 1, \{\Omega_i\}_{i=1}^{I} \text{ is a binary partition of } \Omega \text{ of size } I. \}.$$ 

Then, $\Theta_I$ is the collection of the density functions supported by the binary partitions of size $I$. They constitute a sequence of approximating spaces (i.e. a sieve, see [9] and [18] for background on sieve theory). Let $\Theta = \bigcup_{I=1}^{\infty} \Theta_I$ be the space containing all the density functions supported by the binary partitions. Then $\Theta$ is an approximation of the initial parameter space $\mathcal{F}$ to certain approximation error which will be characterized later.

We take the metric on $\mathcal{F}$, $\Theta$ and $\Theta_I$ to be Hellinger distance, which is defined to be

\begin{equation}
\rho(f, g) = \left( \int_{\Omega} (\sqrt{f(y)} - \sqrt{g(y)})^2 dy \right)^{1/2}, \quad f, g \in \Theta. \end{equation}
For $f, g \in \Theta_I$, let $f = \sum_{i=1}^{I} \beta_1^i \mathbf{1}_{\Omega_1^i}$, $g = \sum_{i=1}^{I} \beta_2^i \mathbf{1}_{\Omega_2^i}$, where $\{\Omega_1^i\}_{i=1}^{I}$ and $\{\Omega_2^i\}_{i=1}^{I}$ are binary partitions of $\Omega$. Then the Hellinger distance between $f_I^n$ and $f'_I^n$ can be written as

\[
\rho^2(f, g) = \sum_{i=1}^{I} \sum_{j=1}^{I} \left( \sqrt{\beta_1^i} - \sqrt{\beta_2^j} \right)^2 \mu(\Omega_1^i \cap \Omega_2^j).
\]

We will also use Kullback-Leibler divergence and the variance of the log-likelihood ratio based on a single observation $Y$, which are defined to be

\[
K(f_0, f) = \mathbb{E}_{f_0} \left( \log \frac{f_0(Y)}{f(Y)} \right),
\]

and

\[
V(f_0, f) = \text{Var}_{f_0} \left( \log \frac{f_0(Y)}{f(Y)} \right).
\]

2.2. Approximation error. The accuracy of the approximation to the true density by the elements in $\Theta$ is formulated in the following way. A density function $f \in F$ is said to be well approximated by elements in $\Theta$, if there exists a sequence of $f_I \in \Theta_I$, satisfying that

\[
\rho(f_I, f) = O(I^{-r})(r > 0).
\]

This means that there exists constant $A_1$ and $A_2$, such that $A_1 I^{-r} \leq \min_{g \in \Theta_I} \rho(g, f) \leq \rho(f_I, f) \leq A_2 I^{-r}$. Let $F_0$ be the collection of these density functions. We will first derive posterior concentration rate for the elements in $F_0$ in terms of the parameter $r$. For different function classes, this approximation rate $r$ can be calculated explicitly. This type of results has been discussed in a parallel paper ([12]). In addition to this, we also assume that $f_0$ has finite second moment.

We want to point out that, based on the minimaxity of the Bayes estimator, it is necessary to restrict our attention to a subset of $F$. In [2] and [1], the authors demonstrated that it is impossible to find an estimator which works uniformly well for every $f$ in $F$. This is the case because for any estimator $\hat{f}$, there always exists $f \in F$ for which $\hat{f}$ is inconsistent.

2.3. Prior specification. An ideal prior $\Pi$ on $\Theta = \bigcup_{I=1}^{\infty} \Theta_I$ is supposed to be capable of balancing the approximation error and the complexity of $\Theta$. The prior in this paper penalizes the size of the partition in the sense that the probability mass on each $\Theta_I$ is proportional to $\exp(-\lambda I \log I)$. Given a sample of size $n$, we restrict our attention to $\Theta_n = \bigcup_{I=1}^{n} \Theta_I$, because in practice it is not meaningful to study a partition with the number of
subregions greater than the sample size. This is to say, when \( I \leq n \), \( \Pi(\Theta_I) \propto \exp(-\lambda I \log I) \), otherwise \( \Pi(\Theta_I) = 0 \).

If we use \( T_I \) to denote the total number of possible partitions of size \( I \), then it is not hard to see that \( \log T_I \leq c^* I \log I \), where \( c^* \) is a constant. Within each \( \Theta_I \), the prior is uniform across all binary partitions. In other words, let \( \{\Omega_i\}_{i=1}^I \) be a binary partition of \( \Omega \) of size \( I \), and \( F(\{\Omega_i\}_{i=1}^I) \) is the collection of piecewise constant density functions on this partition (i.e. \( F(\{\Omega_i\}_{i=1}^I) = \{f = \sum_{i=1}^I \frac{\theta_i}{|\Omega_i|} 1_{\Omega_i} : \sum_{i=1}^I \theta_i = 1 \text{ and } \theta_i \geq 0, i = 1, \ldots, I\} \)), then

\[
(2.5) \quad \Pi(F(\{\Omega_i\}_{i=1}^I)) \propto \exp(-\lambda I \log I) / T_I.
\]

Given a partition \( \{\Omega_i\}_{i=1}^I \), the weights \( \theta_i \) on the subregions follow a truncated Dirichlet distribution with parameters all equal to \( \alpha \) (\( \alpha < 1 \)). This is to say, for \( x_1, \ldots, x_I > \tau \) and \( \sum_{i=1}^I x_i = 1 \),

\[
\Pi \left( f = \sum_{i=1}^I \frac{\theta_i}{|\Omega_i|} 1_{\Omega_i} : \theta_1 \in dx_1, \ldots, \theta_I \in dx_I | f \in F(\{\Omega_i\}_{i=1}^I) \right)
\]

(2.6) \( \propto \frac{\Gamma(\alpha I)}{\Gamma(\alpha)^I} \prod_{i=1}^I x_i^{\alpha - 1} \),

otherwise, the prior probability is zero. \( \tau \) is the truncation parameter. In this paper, we set \( \tau \) to be \( DI^{-\kappa} (D, \kappa > 0) \).

2.4. Posterior concentration rate. We are interested in how fast the posterior probability measure concentrates around the true the density \( f_0 \). Under the prior specified above, the posterior probability is the random measure given by

\[
\Pi(B | Y_1, \cdots, Y_n) = \frac{\int_B \prod_{j=1}^n f(Y_j) d\Pi(f)}{\int \prod_{j=1}^n f(Y_j) d\Pi(f)}.
\]

A Bayesian estimator is said to be consistent if the posterior distribution concentrates on arbitrarily small neighborhoods of \( f_0 \), with probability tending to 1 under \( P_0^n \) (\( P_0 \) is the probability measure corresponding to the density function \( f_0 \)). The posterior concentration rate refers to the rate at which these neighborhoods shrink to zero while still possessing most of the posterior mass. More explicitly, we want to find a sequence \( \epsilon_n \to 0 \), such that for sufficiently large \( M \),

\[
\Pi(f : \rho(f, f_0) \geq M \epsilon_n | Y_1, \cdots, Y_n) \to 0 \text{ in } P_0^n - \text{probability}.
\]

The following theorem gives the posterior concentration rate under the prior probability specified in Section 2.3.
Theorem 2.1. $Y_1, \cdots, Y_n$ is a sequence of independent random variables distributed according to $f_0$. $P_0$ is the probability measure corresponding to $f_0$. $\Theta$ is the collection of all the $p$-dimensional density functions supported by the binary partitions as defined in Section 2.1. The prior distribution on $\Theta$ is as specified in Section 2.3. If $f_0 \in F_0$ and $\kappa > \max(2, 4r)$, then $\epsilon_n = n^{-\frac{1}{3p+r}}(\log n)^{2+\frac{1}{r}}$ is posterior concentration rate.

The strategy to show this theorem is to write the posterior probability measure as

$$
\Pi(f : \rho(f, f_0) \geq M \epsilon_n | Y_1, \cdots, Y_n) = \frac{\sum_{I=1}^{\infty} \int_{\{f : \rho(f, f_0) \geq M \epsilon_n\} \cap \Theta} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f)}{\sum_{I=1}^{\infty} \int_{\Theta} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f)}.
$$

(2.7)

The proof still relies on the mechanism developed in the landmark works [6] and [17]. We first derive the upper bounds for the items in the numerator by employing previous results from the study of empirical process in Section 3. Then we lower bound the prior mass of the shrinking ball around the true density in Section 4. In Section 5, these bounds are integrated together, leading to a complete proof of the posterior concentration rate.

2.5. Discussion.

2.5.1. Comparison to the sieve MLE. In the companion work [12], we studied convergence rate of the sieve maximum likelihood estimators. In that paper, the approximating spaces $\Theta_I$ are defined in the same way, and we consider the same subset of density functions $F_0$.

For any $f \in \Theta_I$, the log-likelihood is defined to be

$$
L_n(f) = \sum_{j=1}^{n} \log f(Y_j) = \sum_{i=1}^{I} N_i \log \beta_i,
$$

where $N_i$ is the count of data points in $\Omega_i$, i.e., $N_i = \text{card}\{j : Y_j \in \Omega_i, 1 \leq j \leq n\}$. The maximum likelihood estimator on $\Theta_I$ is defined to be

$$\hat{f}_{n,I} = \arg \max_{f \in \Theta_I} L_n(f).$$

Next theorem presented the result on convergence rate of sieve MLE. It is cited from [12].
Theorem 2.2. For any \( f_0 \in \mathcal{F}_0 \), \( \hat{f}_{n,I} \) is the corresponding maximum likelihood estimator over \( \Theta_I \). \( r \) is the parameter that characterizes the decay rate of the approximation error to \( f_0 \) by the elements in \( \Theta_I \). Assume that \( n \) and \( I \) satisfy

\[
I = \left( \frac{2^8 A_2^2 r}{c_1} \right) \frac{n}{\log n} \left( \frac{1}{2r+1} \right),
\]

where the constant \( c_1 \) can be chosen to be in \((0,1)\), and \( A_2 \) is a constant associated with the decay rate of the approximation error. Then the convergence rate of the sieve MLE is

\[
n^{-\frac{r}{2r+1}} (\log n)^{\frac{1}{2} + \frac{r}{2r+1}}.
\]

Comparing these two rates, we can easily see that they are of the same order up to a logarithmic term. However, for the sieve method, in order to achieve the optimal convergence rate we need to match the size of the partition \( I \) to the sample size \( n \). And this matching depends on some unknown property of the true density function, i.e., the decay rate of the approximation error \( r \). This implies that, in practice it is computationally infeasible to achieve optimal rate under the frequency setting. On the other hand, under Bayesian settings, by imposing a prior on \( \Theta_I \), we are able to achieve the optimal rate without any a priori information. This is one of the major improvements of the Bayesian method.

2.5.2. Computational issues. The total number of binary partitions grows exponentially in \( I \), thus it is urgent to solve the computational issues. In [14], as we mentioned before, the authors imposed a very similar prior distribution. By employing sequential importance sampling, they have designed efficient algorithm to sample from the posterior distribution. Currently, the dimension of the problem can be moderately large, saying around 50.

2.5.3. Applications to different function classes. In the parallel paper, we studied decay rates of the approximation error for different density functions classes, including the densities satisfying a type of sparsity, the space of bounded variation, and mixed-Hölder continuous functions. Since in this paper we use the same approximating spaces, those results still hold. Given this, we can also calculate the corresponding rates of posterior contraction. Based on the minimaxity of Bayesian estimator, these rates are at least upper bounds of minimax convergence rates. In fact, for the one dimensional density functions of bounded variation, the posterior contraction rate is \( n^{-1/3} (\log n)^{5/2} \). If we estimate the density by wavelet thresholding, the convergence rate is \( n^{-1/3} (\log n)^{1/3} \). As a benchmark, the minimax rate of convergence is \( n^{-1/3} \).
2.5.4. Univariate case. In [15], the author investigated rates of convergence for the posterior distribution under the mixture of Beta prior. The true density function is assumed to be Hölder continuous on $[0,1]$. More rigorously, the class of Hölder functions $\mathcal{H}(L, \beta)$ with regularity function $\beta$ is defined as the following: let $\kappa$ be the largest integer smaller than $\beta$, and denote by $f^{(k)}$ its $k$th derivative.

$$\mathcal{H}(L, \beta) = \{ f : [0,1] \to \mathbb{R} : |f^{(k)}(x) - f^{(k)}(y)| \leq L|x - y|^\beta - \kappa \}.$$ 

Then, under a class of location mixtures of Beta models, the concentration rate of the posterior distribution is $n^{-\beta/(2\beta + 1)}$, up to a log $n$ term. It is known that the rate $(n/\log n)^{-\beta/(2\beta + 1)}$ is the minimax rate of convergence for class $\mathcal{H}(\beta, L)$.

Under the prior distribution specified in this paper, we can also study the posterior contraction rate for the Hölder class. However, given the piecewise constant approximations, we will only study the Hölder continuous function on $[0,1]$ with regularity parameter $\beta$ in $(0,1]$. For this class of density functions, we already calculated the decay rate of the approximation error in [12]. Then the convergence rate of the posterior distribution is $n^{-\frac{\beta}{2\beta + 1}} (\log n)^{2 + \frac{1}{2\beta}}$. Up to a log $n$ term, this method still achieves the minimax rate of convergence.

3. Upper bound of the numerator. Briefly speaking, the numerator can be bounded by controlling the complexity of the parameter space $\Theta$. Here, the complexity of the model is measured by the metric entropy. A general discussion of metric entropy can be found in [11]. In this section, we introduce a form of metric entropy with bracketing corresponding to the relevant parameter space, and provide an upper bound for the metric entropy of the approximating spaces defined in Section 2.1. These bounds lead to upper bounds for the items in the numerator of (2.7).

**Definition 3.1.** Let $(\Theta, \rho)$ be a separable pseudo-metric space. $\Theta(\epsilon)$ is a finite set of pairs of functions $\{(f^L_j, f^U_j), j = 1, \cdots, N\}$ satisfying

$$\rho(f^L_j, f^U_j) \leq \epsilon \text{ for } j = 1, \cdots, N,$$

and for any $f \in \Theta$, there is a $j$ such that

$$f^L_j \leq f \leq f^U_j.$$

Let

$$N(\epsilon, \Theta, \rho) = \min \{ \text{card } \Theta(\epsilon) : (3.1) \text{ and } (3.2) \text{ are satisfied} \}.$$
Then, we define the metric entropy with bracketing of $\Theta$ to be

$$H(\epsilon, \Theta, \rho) = \log N(\epsilon, \Theta, \rho).$$

Recall that $\Theta_1, \ldots, \Theta_I, \ldots$ are the approximating spaces defined in section 2.1. The next lemma is devoted to an upper bound for the bracketing metric entropy of $\Theta_I$.

**Lemma 3.1.** Take $\rho$ to be the Hellinger distance. Let $\Theta_I^d = \{f \in \Theta_I : \rho(f, f_0) \leq d\}$. Then,

$$H(\epsilon, \Theta_I^d, \rho) \leq I \log p + (I + 1) \log(I + 1) + I \frac{d}{\epsilon} + c',$$

where $c$ is a constant not dependent on $I$ or $d$.

**Proof.** See [12] proof of Lemma 3.1 and Lemma 3.2.

Our next theorem, which is Theorem 1 in [22], gives a uniform exponential bound for likelihood ratios.

**Theorem 3.1 (Wong and Shen (1995)).** There exist positive constants $a > 0$, $c$, $c_1$ and $c_2$, such that, for any $\epsilon > 0$, if

$$\int_{\epsilon^2/8}^{\sqrt{2}\epsilon} H^{1/2}(u/a, \mathcal{P}, \rho)du \leq cn^{1/2}\epsilon^2,$$

then

$$\mathbb{P}_{f_0}\left(\sup_{\rho(f, f_0) \geq \epsilon, f \in \mathcal{P}} \prod_{i=1}^{n} \frac{f(Y_i)}{f_0(Y_i)} \geq \exp(-c_1n\epsilon^2)\right) \leq 4 \exp(-c_2n\epsilon^2),$$

where $\mathbb{P}_{f_0}$ is understood to be the outer probability measure under $f_0$. The constants $c_1$ and $c_2$ can be chosen in $(0, 1)$ and $c$ can be set as $(2/3)^{5/2}/512$.

Finally, the next lemma provides an upper bound for the items in the numerator in (2.7) when $I$ is sufficiently large.

**Lemma 3.2.** Let $\delta_{n,I} = \left(\frac{I\log I}{n \log n}\right)^{1/2}$. When $n$ and $I$ are sufficiently large, we have

$$\mathbb{P}_{f_0}\left(\sup_{\rho(f, f_0) \geq \delta_{n,I}, f \in \Theta_I} \prod_{i=1}^{n} \frac{f(Y_i)}{f_0(Y_i)} \geq \exp(-c_1n\delta_{n,I}^2)\right) \leq 4 \exp(-c_2n\delta_{n,I}^2).$$
Proof. See [12] proof of Corollary 3.1. \qed

Remark 3.1. Since the metric entropy decreases as $\epsilon$ increases, this lemma also holds for any $\epsilon \geq \delta_{n,I}$. This property is quite useful in the proof of the main theorem.

4. Lower bound of the denominator. In this section, we study how the prior distribution concentrates on the shrinking neighborhoods around the true density function. This is the key to bounding the denominator of (2.7) from below. We develop our results through a series of lemmas. The connection between the lower bounds of the items in the denominator of (2.7) and the concentration rate of the prior distribution is first derived (4.1). By employing a property of Dirichlet distribution (Lemma 4.3) and inequalities bounding Kullback-Leibler divergence by Hellinger distance (Lemma 4.2), we obtain lower bounds of the terms in the denominator of (2.7) in Lemma 4.4.

To begin with, we cite a result from [17]. In this lemma, it is shown that with probability close to 1, the denominator is bounded from below by the prior probability mass concentrating on a ball around $f_0$ multiplied by a coefficient depending on the radius of the ball.

Lemma 4.1 (Shen and Wasserman (2001) Lemma 1). Let $K(\cdot,\cdot)$ and $V(\cdot,\cdot)$ be as defined in (2.3) and (2.4), and let $S(t) = \{f \in \Omega : K(f_0,f) \leq t, V(f_0,f) \leq t\}$. Set $S_n = S(t_n)$. When $t_n$ is a sequence of positive numbers satisfying $nt_n \to \infty$,

$$\mathbb{P}_{f_0} \left( \int_{\Omega} \prod_{j=1}^{n} \frac{f(Y_i)}{f_0(Y_i)} d\Pi(f) \leq \frac{1}{2} \Pi(S_n)e^{-2nt_n} \right) \leq \frac{2}{nt_n}.$$ 

More explicitly, from this lemma we learn that, given the condition $nt_n \to \infty$, $\int_{\Omega} \prod_{j=1}^{n} \frac{f(Y_i)}{f_0(Y_i)} d\Pi(f) \geq \frac{1}{2} \Pi(S_n)e^{-2nt_n}$ with probability close to 1.

It is well known that Hellinger distance can be bounded by the Kullback-Leibler divergence. In [22], they showed that the other direction also holds under an integrability condition. Their results are summarized in the lemma below.

Lemma 4.2 (Wong and Shen (1995) Theorem 5). Let $f$, $f_0$ be two densities, $\rho^2(f,f_0) \leq \epsilon^2$. Suppose that $M_\delta^2 = \int_{\{f_0/f \geq e^{1/\delta}\}} f_0(f_0/f)^\delta < \infty$ for some
\( \delta \in (0, 1] \). Then for all \( \epsilon^2 \leq \frac{1}{2}(1 - e^{-1})^2 \), we have

\[
\int f_0 \log \left( \frac{f_0}{f} \right) \leq \left[ 6 + \frac{2 \log 2}{(1 - e^{-1})^2} + \frac{8}{\delta} \max \{1, \log \left( \frac{M\delta}{\epsilon} \right) \} \right] \epsilon^2,
\]

\[
\int f_0 \left( \log \left( \frac{f_0}{f} \right) \right)^2 \leq 5 \epsilon^2 \left[ \frac{1}{\delta} \max \{1, \log \left( \frac{M\delta}{\epsilon} \right) \} \right]^2.
\]

From the proceeding lemma, we learn that, if \( \rho^2(f, f_0) \leq \epsilon^2 \), then

\[
\max \left( K(f_0, f), \mathbb{E}_{f_0} \left( \left( \log \frac{f}{f_0} \right)^2 \right) \right) = O(\epsilon^2 \log \left( \frac{M\delta}{\epsilon} \right)^2).
\]

This further implies that, there exists a constant \( L \), such that

\[
\left\{ f : \rho(f, f_0) \leq \frac{L\epsilon}{\left( \log \frac{M\delta}{\epsilon} \right)^2} \right\} \subset \left\{ f : K(f_0, f) \leq \epsilon^2, \mathbb{E}_{f_0} \left( \left( \log \frac{f}{f_0} \right)^2 \right) \leq \epsilon^2 \right\}.
\]

This lemma allows us to work on a Hellinger ball instead of a Kullback-Leibler one. The transition is necessary because it is more straightforward to apply a property of the Dirichlet distribution to estimate the probability mass on a Hellinger ball around the true density function. In the lemma below, this particular property of the Dirichlet distribution is stated in terms of \( L_1 \) distance, which is equivalent to the Hellinger distance. We want to point out that this lemma is a variation of Lemma 6.1 in [6] and the proof is adapted from their paper.

**Lemma 4.3.** Let \((X_1, \cdots, X_I)\) be distributed according to the truncated Dirichlet distribution (2.6) with truncation parameter \( \tau \). Let \((x_{10}, \cdots, x_{I0})\) be any point on the \( I \)-simplex. Let \( \epsilon < 1/I \). Assume that \( \tau < \epsilon^2 \).

\[
P \left( \sum_{i=1}^{I} |X_i - x_{i0}| \leq 2\epsilon \right) \geq \frac{\Gamma(\alpha I)}{(\Gamma(\alpha))^I} (\epsilon^2 - \tau)^I.
\]

**Proof.** We can find an index \( i \) such that \( x_{i0} > 1/I \). By relabeling, we can assume that \( i = I \). If \( |x_i - x_{i0}| \leq \epsilon^2 \) for \( i = 1, \cdots, I - 1 \), then

\[
\sum_{i=1}^{I-1} x_i \leq 1 - x_{10} + (I - 1)\epsilon^2 \leq (I - 1)(\epsilon^2 + 1/I) \leq 1 - \epsilon^2 < 1.
\]
Therefore, there exists \( x = (x_1, \cdots, x_I) \) in the simplex with these first \( I - 1 \) coordinates. And
\[
\sum_{i=1}^{I} |x_i - x_{i0}| \leq 2 \sum_{i=1}^{I-1} |x_i - x_{i0}| \leq 2\epsilon^2(I-1) \leq 2\epsilon.
\]
Therefore, the probability on the left hand side of (4.2) is bounded below by
\[
P(|X_i - x_{i0}| \leq \epsilon^2, i = 1, \cdots, I - 1)
\geq \frac{\Gamma(\alpha I)}{(\Gamma(\alpha))^I} \prod_{i=1}^{I-1} \int_{\min((x_{i0}+\epsilon^2),1)}^{\max((x_{i0}-\epsilon^2),\tau)} x_i^{\alpha-1} dx_i (1-\tau).
\]
Since \( \alpha < 1 \), we can lower bound the integrand by 1 and the interval of integration contains at least an interval of length \( \epsilon^2 - \tau \). Therefore, the result above can be further lower bounded by
\[
\frac{\Gamma(\alpha I)}{(\Gamma(\alpha))^I} (\epsilon^2 - \tau)^{I-1}(1-\tau) \geq \frac{\Gamma(\alpha I)}{(\Gamma(\alpha))^I} (\epsilon^2 - \tau)^I.
\]
This finishes the proof.

Now, we are ready to derive lower bounds for the prior probability mass on \( \Theta_I \)'s when \( I \) varies within a certain range. Before stating the result, we want to briefly review the assumptions we made in Section 2.2 and Section 2.3. First, in terms of approximation error, we assume that for any \( f_0 \in \mathcal{F}_0 \), there exists a sequence of \( f_I \in \Theta_I \), such that \( A_1 I^{-r} \leq \min_{g \in \Theta_I} \rho(g, f) \leq \rho(f_I, f) \leq A_2 I^{-r} \) for some positive constants \( A_1 \) and \( A_2 \). Second, we imposed a moment condition on \( \mathcal{F}_0 \). For any \( f \in \mathcal{F}_0 \), we assume that \( \int f^2 < \infty \). At last, given a partition of size \( I \), the weights on the subregions within the partition follow a Dirichlet distribution truncated from below, with the truncation parameter \( \tau = DI^{-\kappa} \) (\( D, \kappa > 0 \)). Under these three assumptions, we will derive the lower bound in the lemma below.

**Lemma 4.4.** Assume that \( f_0 \in \mathcal{F}_0 \). \( \Pi \) is the prior probability specified in Section 2.3, with \( \kappa > \max(2, 4r) \). Let \( t_{n,I} = \epsilon^2 n I = \frac{I \log I}{n \log n} \). When \( I = n^{\frac{1}{2r+1}} \), we have
\[
\frac{\Pi^n_{f_0} \left( \int_{\Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f) \right)}{2nt_{n,I}} \leq \frac{1}{2} \Pi(\Theta_I) \exp(-2nt_{n,I} - c^* I \log I - 4\omega I \log n - I \log \Gamma(\alpha))
\]
\[
\leq \frac{2}{nt_{n,I}},
\]
where $\omega = \max(1, 1/2r)$.

**Proof.** Let $S_{n,I} = \{f \in \Theta_I : K(f_0, f) \leq t_{n,I}, V(f_0, f) \leq t_{n,I}\}$. From lemma 4.1, we have the bound

$$p_{f_0}(\int_{\Theta_I} \prod_{j=1}^{n} f(Y_i) f_0(Y_i) d\Pi(f) \leq \frac{1}{2} \Pi(S_{n,I}) e^{-2nt_{n,I}} \leq \frac{2}{nt_{n,I}}. \tag{4.3}$$

Next step, we will search a lower bound for $\Pi(S_{n,I})$. The way to approach this is to find a subset of $S_{n,I}$ to which we can apply Lemma 4.3. Our argument is as the following.

Define $\tilde{S}_{n,I} = \{f \in \Theta_I : K(f_0, f) \leq t_{n,I}, E_{f_0}(\log f_0(Y) f(Y))^2 \leq t_{n,I}\}$. Note that $E_{f_0}(\log f_0(Y) f(Y))^2 \geq V(f_0, f)$, we have $\tilde{S}_{n,I} \subset S_{n,I}$. From (4.1), we know that $W_{n,I} := \{f \in \Theta_I : \rho(f_0, f) \leq L_{\epsilon_n,I} \frac{1}{\log M} \} \subset \tilde{S}_{n,I}$.

With the truncation parameter $\tau = DI^{-\kappa}$, $M_\delta = O(I^\delta \int_{0}^{(1+\delta)} \frac{1}{n} \log n)$. Furthermore,

$$\frac{\epsilon_n,I}{\log M} = O \left( \frac{I \log I}{n \log n} \frac{1}{\log \left( I^{\delta \kappa} \int_{0}^{(1+\delta)} \frac{1}{n} \log n \right)} \right). \tag{4.4}$$

Under the assumptions that $I = n^{1/2}$, there exists $f_I \in \Theta_I$, such that $\rho(f_0, f_I) < \frac{L_{\epsilon_n,I} M_S}{\log M}$. If we define

$$\tilde{W}_{n,I} := \{f \in \Theta_I : \rho(f, f_I) \leq \frac{L_{\epsilon_n,I} M_S}{\log M} \rho(f_0, f_I)\},$$

by triangle inequality, we know that $\tilde{W}_{n,I} \subset W_{n,I}$. Together with the previous result, we claim that there exists a constant $L'$, such that

$$\tilde{B}_{n,I} := \{f \in \Theta_I : \rho(f, f_I) \leq L' \left( \frac{I \log I}{n \log n} \right)^{1/2} \} \subset \tilde{W}_{n,I},$$

Next, from the fact $\rho^2(f, g) \leq \|f - g\|_{L_1}$, we have

$$B_{n,I} := \{f \in \Theta_I : \|f_I - f\|_{L_1} \leq \frac{L^2 I \log I}{n \log n} \} \subset \tilde{B}_{n,I}.$$
Note that $\Pi(B_{n,I}) = \Pi(\Theta_I)\Pi(B_{n,I}|\Theta_I)$. Assume that $f_I$ is supported by the binary partition $\{\Omega_{i0}\}_{i=1}^I$. Let $F_0 = \{f \in \Theta_I : f = \sum_{i=1}^I \beta_i 1_{\Omega_{i0}}, \beta_i \geq 0, \sum_{i=1}^I \beta_i = 1\}$ be the collection of all the density functions in $\Theta_I$ which are supported by the same binary partition as $f_I$. Then

$$
(4.5) \quad \Pi(B_{n,I}|\Theta_I) \geq \Pi(B_{n,I}|F_0)\Pi(F_0|\Theta_I) \geq \exp(-e^\ast I \log I)\Pi(B_{n,I}|F_0).
$$

Now we apply Lemma 4.3 to bound $\Pi(B_{n,I}|F_0)$ from below. We will works with an $L_1$-ball with radius $\left(\frac{L^2 I \log I}{n \log n}\right)^{\omega}$, where $\omega$ is chosen to be $\max(1, 1/2r)$. We can always assume that $L' < 1$, otherwise we can work with a smaller ball instead. Obviously, this ball is contained in $B_{n,I}$. When $I = n^{\frac{1}{2r+1}}$, we have $\left(\frac{L^2 I \log I}{n \log n}\right)^{\omega} < \frac{1}{I}$. Under the assumptions $\kappa > \max(2, 4r)$, we know that when $I \geq n^\gamma$, $DI^{-\kappa} = o\left(\left(\frac{I \log I}{n \log n}\right)^{2\omega}\right)$. By setting $x_{i0}$ in the lemma to probability mass on $\Omega_{i0}$ under $f_I$, we have

$$
\Pi(B_{n,I}|F_0) \geq \frac{\Gamma(\alpha I)}{(\Gamma(\alpha))^I} \left(\frac{L^2 I \log I}{2n \log n}\right)^{\omega I} - DI^{-\kappa})^I
$$

$$
(4.6) \quad \geq \exp(-I \log \Gamma(\alpha) - 4\omega I \log n).
$$

Combine (4.3), (4.5) and (4.6) together, we get the desired result. \qed

5. Proof of Theorem 2.1. In this section, we will combine the upper bound in Section 3 and the lower bound in Section 4 together to derive the posterior concentration rate.

Proof of Theorem 2.1. Let $\epsilon_n = n^{-\frac{1}{2r+1}}(\log n)^{2+\frac{1}{r}}$ and $\eta_{n,I} = \left(\frac{(I \log I)^{1/r+1}}{n/\log n}\right)^{1/2}$.

First, we divide the items in (2.7) into three blocks. We define

$$
I_{Num} = \sum_{l=1}^{N_1} \int_{\{f : p(f,f_0) \geq M\epsilon_n\} \cap \Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f),
$$

$$
II_{Num} = \sum_{l=N_1+1}^{N_2} \int_{\{f : p(f,f_0) \geq M\epsilon_n\} \cap \Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f),
$$

$$
III_{Num} = \sum_{l=N_2+1}^{n} \int_{\{f : p(f,f_0) \geq M\epsilon_n\} \cap \Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f),
$$

where $N_1 = n^{\frac{1}{2r+1}}(\log n)^{-\frac{1}{2}}$ and $N_2 = Dn^{\frac{1}{2r+1}}(\log n)^2$.

We deal with each block in the numerator separately. Roughly speaking, when $I$ is small, the approximation error to $f_0$ dominates, and these items
can be bounded by the Hellinger distance between \( f \) and \( f_0 \). The items in the middle range can be bounded by controlling the metric entropy of \( \Theta_I \). The items in the last block are negligible because the prior probability decays to zero fast.

We assume that there exists a sequence of \( f_I \in \Theta_I \), such that \( A_1 I^{-r} \leq \min_{g \in \Theta_I} \rho(g, f) \leq \rho(f_I, f) \leq A_2 I^{-r} \) for some positive constants \( A_1 \) and \( A_2 \). When \( I < N_1 \), \( A_1 I^{-r} \) is greater than \( \epsilon_n \). We can apply Lemma 3.2 by setting \( \delta_{n,I} \) to be \( A_1 I^{-r} \). Therefore, as \( n \to \infty \),

\[
I_{Num} \leq \sum_{I=1}^{N_1-1} \Pi(\Theta_I) \exp(-A_1 n I^{-2r})
\]

\[
\leq \left( \sum_{I=1}^{N_1-1} \exp(-2A_1 n I^{-2r}) \right)^{1/2}.
\]

Now, we will estimate the order of the summation in the last line. In order to simplify the notation, we will discuss the order of \( \sum_{I=1}^{N_1-1} \exp(-2A_1 n I^{-2r}) \) in detail.

We know that the mass is centered around \( I = N_1 - 1 \). Power series expansion around that point gives

\[
\sum_{I=1}^{(1-\epsilon)N_1} \leq (1-\epsilon)N_1 \exp \left( -\frac{2A_1 n}{((1-\epsilon)N_1)^{2r}} \right),
\]

which is a lower order term compared to the last term in the summation and thus does not contribute significantly to the summation. Let \( 1 - \delta = \frac{I}{N_1} \), expand

\[
(1-\delta)^{-2r} = 1 + 2r \delta + \left( \frac{-2r}{2} \right) \delta^2 + o(\delta^2).
\]

\[
\sum_{I=(1-\epsilon)N_1}^{N_1-1} \exp\left( -\frac{2A_1 n}{I^{2r}} \right) \leq \int_{(1-\epsilon)N_1}^{N_1} \exp\left( -\frac{2A_1 n}{x^{2r}} \right) dx
\]

\[
\sim \int_{0}^{\epsilon} \exp\left( -2A_1 n \frac{1}{x^{2r}} (\log n)^2 (1-\delta)^{-2r} \right) N_1 d\delta
\]

\[
\sim \int_{0}^{\epsilon} \exp\left( -2A_1 n \frac{1}{x^{2r}} (\log n)^2 (1+2r \delta + o(\delta)) \right) N_1 d\delta
\]

\[
\sim \frac{1}{4r A_1 (\log n)^{1/r+2}} \exp\left( -2A_1 n \frac{1}{x^{2r}} (\log n)^2 \right).
\]
Therefore
\[(5.1) \quad I_{\text{Num}} \leq (\log n)^{-\frac{1}{2}} \exp(-A_1 n^{\frac{1}{2r+1}} (\log n)^2).\]

From Lemma 3.2, we know that if the result applies for $\delta_{n,I}$, then it also applies to $M_{n,I} > \delta_{n,I}$. We have that when $N_1 \leq I \leq N_2$,
\[
II_{\text{Num}} \leq \sum_{I=N_1}^{N_2} \int_{\{f: \rho(f,f_0) \geq M_{n,I}\} \cap \Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f)
\leq \sum_{I=N_1}^{N_2} \exp(-\lambda I \log I) \exp(-M^2 I (\log I)^{1+\frac{1}{r}} \log n)
\leq \left( \sum_{I=N_1}^{N_2} \exp(-2\lambda I \log I) \right)^{1/2} \left( \sum_{I=N_1}^{N_2} \exp\left(-2M^2 I (\log I)^{1+\frac{1}{r}} \log n\right) \right)^{1/2}
\sim \exp\left(-M^2 n^{\frac{1}{2r+1}} (\log n)^2\right),
\]
where the last line is obtained by integration by part.

For $III_{\text{Num}}$, we have
\[(5.2) \quad III_{\text{Num}} \leq \sum_{I=N_2+1}^{n} \int_{\Theta_I} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f)
\sim \exp\left(-n \int f_0 \log(f_0)\right) \sum_{I=N_2+1}^{n} \int_{\Theta_I} \prod_{j=1}^{n} f(Y_j) d\Pi(f).
\]

If we use $x_I$ to represent a partition of size $I$, and $\mathcal{X}_I$ to denote the collection of all binary partitions of size $I$, then the integral in (5.2) can be divided into the integral over each partition as the following:
\[
III_{\text{Num}} \sim \exp\left(-n \int f_0 \log(f_0)\right) \sum_{I=N_2+1}^{n} \int_{\Theta_I} \prod_{j=1}^{n} f(Y_j) d\Pi(f)
\leq \left( \prod_{i=1}^{r} [\alpha_i + n_i] \right) \prod_{i=1}^{r} D(\alpha, \alpha + n_i) \prod_{i=1}^{r} \frac{1}{|\Omega_i|^{n_i}},
\]
where \( \frac{(\Gamma(\alpha))}{\Gamma(\alpha I)} \left( \frac{1}{4I^2} - \tau \right)^{-I} \) is an upper bound for the normalizing constant of the truncated Dirichlet distribution. This inequality can be obtained from Lemma 4.3, because,

\[
\sum_{I=\text{N}_2+1}^{\text{N}_I} \sum_{x_I \in \mathcal{X}_I} \int_{\theta_1, \ldots, \theta_I} \prod_{j=1}^{n} f(Y_j|\theta_1, \ldots, \theta_I, x_I) \Pi(\theta_1, \ldots, \theta_I|x_I) d\theta_1 \ldots d\theta_I
\]

\[
\leq \frac{(\Gamma(\alpha))^I}{\Gamma(\alpha I)} \left( \frac{1}{4I^2} - \tau \right)^{-I} \sum_{I=\text{N}_2+1}^{\text{N}_I} \sum_{x_I \in \mathcal{X}_I} \int_{\theta_1, \ldots, \theta_I} \prod_{j=1}^{n} f(Y_j|\theta_1, \ldots, \theta_I, x_I) \text{Dir}(\theta_1, \ldots, \theta_I - 1; \alpha, \ldots, \alpha|x_I) \Pi(x_I) d\theta_1 \ldots d\theta_I.
\]

Now, we focus on the part inside the summation, and apply Stirling’s approximation to the gamma function,

\[
\frac{D(\alpha + n_1, \ldots, \alpha + n_I)}{D(\alpha, \ldots, \alpha)} \prod_{i=1}^{I} \frac{1}{|\Omega_i|^{n_i}} \]

\[
= \exp \left( \log \Gamma(\alpha I) - I \log \Gamma(\alpha) + \sum_{i=1}^{I} \log \Gamma(\alpha + n_i) 
- \log \Gamma(\alpha I + n) + \sum_{i=1}^{I} n_i \log \frac{1}{|\Omega_i|} \right)
\]

\[
\lesssim \exp \left( \alpha I \log(\alpha I) - \alpha I - I \log(\alpha I + n) \log(\alpha I + n) + \alpha I + n 
+ \sum_{i=1}^{I} [(\alpha + n_i) \log(\alpha + n_i) - (\alpha + n_i) + n_i \log \frac{1}{|\Omega_i|}] \right),
\]

Let \( C(\alpha) = 1/\Gamma(\alpha) - \alpha \), then

\[
(5.3) \quad(5.4) \lesssim \exp \left( \alpha I \log \frac{\alpha I}{\alpha I + n} - n \log(\alpha I + n) + C(\alpha) I + \sum_{i=1}^{I} n_i \log \frac{n_i}{|\Omega_i|} \right).
\]

Given a partition \( \{\Omega_i\}_{i=1}^{I} \), define \( \mu_i = \int_{\Omega_i} f_0, \bar{\mu}_i = n_i/n, \) and \( \nu_i = \mu_i/|\Omega_i| \). Then we have \( \sqrt{n}(\bar{\mu}_i - \mu_i) \to \mathcal{N}(0, \mu_i(1 - \mu_i)) \) in distribution. With this
result, 

\begin{equation}
(5.4) \quad \exp \left( \alpha I \log \frac{\alpha I}{\alpha I + n} - n \log \frac{\alpha I + n}{n} + C(\alpha) I + \sum_{i=1}^{l} n_i \log \frac{\hat{\mu}_i}{\mu_i} + \sum_{i=1}^{l} n_i \log \nu_i \right)
\end{equation}

\begin{align*}
= & \exp \left( \alpha I \log \frac{\alpha I}{\alpha I + n} - n \log \frac{\alpha I + n}{n} + C(\alpha) I + n \int f_0 \log(f_0) - nK(f_0, \hat{f}_x) \\
& + \sum_{i=1}^{l} n_i \log \frac{\hat{\mu}_i}{\mu_i} + n \sum_{i=1}^{l} \log(\nu_i) (\hat{\mu}_i - \mu_i) \right) \\
\sim & \exp \left( \alpha I \log \frac{\alpha I}{\alpha I + n} - n \log \frac{\alpha I + n}{n} + C(\alpha) I + n \int f_0 \log(f_0) - nK(f_0, \hat{f}_x) \right).
\end{align*}

From this result, we know that no matter $I \ll n$ or $I$ is comparable to $n$, the integral over each partition is bounded given that $\lambda$ is large enough. If we plug in this result into the summation, we have

\begin{equation}
III_{num} \lesssim \sum_{I=N_2}^{n} \exp(-I \log I)
\end{equation}

\begin{align*}
\leq & \exp(-Dn^{\frac{1}{2r+1}}(\log n)^2).
\end{align*}

Therefore

\begin{equation}
(2.7) \quad \lesssim \frac{(\log n)^{-\frac{1}{2}} \exp(-A_1 n^{\frac{1}{2r+1}}(\log n)^2) + \exp(-M^2 n^{\frac{1}{2r+1}}(\log n)^2) + \exp(-Dn^{\frac{1}{2r+1}}(\log n)^2))}{\sum_{I=1}^{\infty} \int_{\Theta} \prod_{j=1}^{n} \frac{f(Y_j)}{f_0(Y_j)} d\Pi(f)} \\
\leq \frac{(\log n)^{-\frac{1}{2}} \exp(-A_1 n^{\frac{1}{2r+1}}(\log n)^2) + \exp(-M^2 n^{\frac{1}{2r+1}}(\log n)^2) + \exp(-Dn^{\frac{1}{2r+1}}(\log n)^2)}{\frac{1}{2} \exp \left(- \frac{1}{2r+1} n^{\frac{1}{2r+1}} (\log n)^2 - (\frac{c^*}{2r+1} + 4\omega) n^{\frac{1}{2r+1}} \log n - \frac{1}{2r+1} (\log \Gamma(\alpha) + 1) \right)},
\end{equation}

where the last inequality is obtained by applying Lemma 4.4 to the space $\Theta_f$ with $I = n^{\frac{1}{2r+1}}$. The last line goes to zero when $A_1, M^2$ and $D$ are all greater than $\frac{2}{2r+1}$.

Therefore, we have

\begin{equation}
\Pi(f : \rho(f, f_0) \geq M_{\epsilon_n} | Y_1, \cdots, Y_n) \leq \exp \left(-b n^{\frac{1}{2r+1}}(\log n)^2 \right),
\end{equation}

with probability tending to 1, where $b$ is a positive constant. This concludes the proof.

\hfill \Box
ACKNOWLEDGEMENTS

The authors would like to thank Bai Jiang for helpful discussions.

REFERENCES

[1] Birg, L. and Massart, P. (1998). Minimum contrast estimators on sieves: exponential bounds and rates of convergence. Bernoulli 4 329–375.
[2] Farrell, R. H. (1967). On the Lack of a Uniformly Consistent Sequence of Estimators of a Density Function in Certain Cases. The Annals of Mathematical Statistics 38 471–474.
[3] Ferguson, T. S. (1974). Prior distributions on spaces of probability measures. Ann. Statist. 2 615–629.
[4] Ferguson, T. S. (1983). Bayesian density estimation by mixtures of normal distributions. In Recent advances in statistics 287–302. Academic Press, New York. MR736538 (86a:62057)
[5] Ghosal, S. (2001). Convergence rates for density estimation with Bernstein polynomials. Ann. Statist. 29 1264–1280. MR1873330 (2002i:62079)
[6] Ghosal, S., Ghosh, J. K. and van der Vaart, A. W. (2000). Convergence rates of posterior distributions. The Annals of Statistics 28 500–531.
[7] Ghosal, S. and van der Vaart, A. W. (2001). Entropies and rates of convergence for maximum likelihood and Bayes estimation for mixtures of normal densities. Ann. Statist. 29 1233–1263. MR1873329 (2002i:62078)
[8] Grenander, U. (1981). Abstract Inference. Probability and Statistics Series. John Wiley & Sons.
[9] Hoffmann, M., Rousseau, J. and Schmidt-Hieber, J. (2015). Adaptive posterior concentration rates. The Annals of Statistics.
[10] Kolmogorov, A. N. and Tikhomirov, V. M. (1992). Selected Works of A.N. Kolmogorov. Mathematics and its applications (Kluwer Academic Publishers).: Soviet series v.2. Kluwer Academic Publishers.
[11] Liu, L. and Wong, W. H. (2014). Multivariate density estimation via adaptive partitioning (I): sieve MLE. arXiv preprint arXiv:1401.2597.
[12] Lo, A. Y. (1984). On a class of Bayesian nonparametric estimates. I. Density estimates. Ann. Statist. 12 351–357. MR733519 (85d:62047)
[13] Lu, L., Jiang, H. and Wong, W. H. (2013). Multivariate Density Estimation by Bayesian Sequential Partitioning. Journal of the American Statistical Association 108 1402-1410.
[14] Rousseau, J. (2010). Rates of convergence for the posterior distributions of mixtures of Betas and adaptive nonparametric estimation of the density. The Annals of Statistics 38 146–180.
[15] Shen, X. (1997). On methods of sieves and penalization. Ann. Statist. 25 2555–2591.
[16] Shen, X. and Wasserman, L. (2001). Rates of convergence of posterior distributions. The Annals of Statistics 29 687–714.
[17] Shen, X. and Wong, W. H. (1994). Convergence Rate of Sieve Estimates. The Annals of Statistics 22 pp. 580-615.
[18] Stone, C. J. (1980). Optimal Rates of Convergence for Nonparametric Estimators. The Annals of Statistics 8 1348–1360.
[20] Wahba, G. (1978). Improper Priors, Spline Smoothing and the Problem of Guarding Against Model Errors in Regression. *Journal of the Royal Statistical Society. Series B (Methodological)* 40 pp. 364-372.

[21] Wong, W. H. and Ma, L. (2010). Optional Plya tree and Bayesian inference. *The Annals of Statistics* 38 1433–1459.

[22] Wong, W. H. and Shen, X. (1995). Probability Inequalities for Likelihood Ratios and Convergence Rates of Sieve MLES. *The Annals of Statistics* 23 339–362.

Department of Statistics
Stanford University
390 Serra Mall, Sequoia Hall
Stanford, California 94305
USA