Decoherence at constant excitation

J M Torres¹, E Sadurní² and T H Seligman¹,³

¹ Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, CP 62210 Cuernavaca, Morelos, México
² Universität Ulm, Institut für Quantenphysik, D-89069 Ulm, Germany
³ Centro Internacional de Ciencias, CP 62210 Cuernavaca, Morelos, México

E-mail: mau@fis.unam.mx (J M Torres)

Received 13 August 2011
Accepted for publication 11 October 2011
Published 17 February 2012
Online at stacks.iop.org/PhysScr/T147/014030

Abstract
We present a simple exactly solvable extension of the Jaynes–Cummings model by adding dissipation. This is done such that the total number of excitations is conserved. The Liouville operator in the resulting master equation can be reduced to blocks of 4 × 4 matrices.

PACS number: 42.50.Pq

1. Introduction

In the field of quantum optics, it is typical to deal with open systems in which some degrees of freedom are treated as an environment. The simplest example is a two-level atom in thermal equilibrium with a continuum of modes [1]. This accounts for an effective spontaneous decay from an excited state to the ground state. To deal with this kind of problem, one usually works under a Markovian assumption (an environment without memory) and as a result one can work with a master equation in the Lindblad form [2], which also follows from the pioneering work of Kossakowski and co-workers [3, 4].

Solutions to this kind of equation, except for a two-level atom, involve complicated expressions due to the extended space in which one is working. The master equation can be written as the action of a super-operator or Liouville operator \( \mathcal{L} \) on a density matrix \( \rho \). If one has an \( n \times n \) density matrix, the super-operator will act on an \( n \times n \) dimensional space.

In this work, we present a new form for a Jaynes–Cummings (JC) model with dissipation. The aim here is simply exact solvability with Kraus operators not considered previously. We show how to construct a master equation whose Liouvillian breaks into blocks not larger than 4 × 4. Here we make an analogue construction for the master equation in the Lindblad form for the JC model in the rotating wave approximation using the Kraus operators that preserve the total number of excitations. This Hamiltonian reduces to 2 × 2 blocks and the corresponding super-operator to 4 × 4 blocks again, thus guaranteeing solvability.

In section 2, we present a review of the JC model, which will allow us to fix the notation to be used in section 3 where we introduce the explicit master equation that describes a system with dissipation, but at a constant number of total excitations. This allows a spectral decomposition of the Liouville operator in terms of 4 × 4 matrices. In section 4, we present explicit solutions to the eigenvalue problem when the atom is in resonance with the cavity and discuss the dynamics of two particular initial states.

2. The Jaynes–Cummings model

Consider the JC Hamiltonian in the rotating wave approximation

\[
H = \delta \sigma_z + g \left( a \sigma_+ + a^\dagger \sigma_- \right).
\]

One can identify an additional conserved quantity, which can be interpreted as the number of excitations

\[
I = a^\dagger a + \frac{1}{2} (\sigma_z + 1).
\]

The basis in which \( I \) is diagonal is given by the states

\[
|n - 1, 1\rangle, \quad |n, 0\rangle.
\]
where \(|n - 1, 1\rangle\) represents a state with \(n - 1\) photons in the field mode and an excited atom. \(|n, 0\rangle\) accounts for \(n\) photons and the atom in the ground state. The action of \(I\) on each of the previous states is given by

\[
I|n - j, j\rangle = n|n - j, j\rangle, \quad j = 0, 1.
\]

In this basis, \(H\) is block-diagonal, with the blocks

\[
H_n = \left( \begin{array}{cc} \delta & g \sqrt{n} \\ \frac{g}{\sqrt{n}} & -\delta \end{array} \right),
\]

with the corresponding eigenenergies

\[
\pm E_n = \pm \sqrt{\delta^2 + g^2 n}.
\]

The eigenstates—the dressed states—of \(H\) are given by

\[
|\phi_n^+\rangle = \cos \theta_n |n - 1, 1\rangle + \sin \theta_n |n, 0\rangle,
\]

\[
|\phi_n^-\rangle = -\sin \theta_n |n - 1, 1\rangle + \cos \theta_n |n, 0\rangle
\]

with

\[
\theta_n = \arctan \left( \frac{E_n - \delta}{E_n + \delta} \right).
\]

### 3. Master equation

The Heisenberg equation for a density matrix in a closed system is given by

\[
\dot{\rho} = -i[H, \rho].
\]

For an open system description, one can consider a master equation in the Lindblad form which describes a non-unitary Markovian evolution [4]. It can be written in terms of an arbitrary set of Kraus operators \(O_j\) as

\[
\dot{\rho} = \mathcal{L}\rho = -i[H, \rho] - \sum_j \frac{\gamma_j}{2} \left( O_j^\dagger \rho O_j + \rho O_j^\dagger O_j - 2 O_j^\dagger \rho O_j \right),
\]

where we have introduced the Liouville super-operator \(\mathcal{L}\) that acts on a density matrix \(\rho\). \(\mathcal{L}\) is a linear operator and the generator of a completely positive dynamical semigroup [4]. Known solvable models [7–9] for this type of equation are, for instance, taking one operator \(O = a\) for a cavity with losses or \(O = \sigma_z\) for an atom with spontaneous decay.

In this work, we explore the possibility of having other types of operators that allow closed solution of the master equation. The condition we impose is that such operators commute with \(I\); this means that this dissipation will conserve the number of excitations. There are many other operators that one could consider, for instance, \(I\) itself, or \(a^dag a\) and \(\sigma_z\). Here we arbitrarily chose the pair \(O_1 = a\sigma_+\) and \(O_2 = a^dag \sigma_-\). With this we can construct the following master equation which describes dissipative dynamics:

\[
\dot{\rho} = \mathcal{L}\rho = -i[H, \rho] - \frac{\gamma_0}{2} (a\sigma_+ a^dag \sigma_- + a^dag \sigma_- a\sigma_+ - 2a^dag \sigma_- a\sigma_+) - \frac{\gamma_1}{2} (a^dag a\sigma_+ \sigma_- a^dag + a^dag \sigma_- a\sigma_+ - 2a\sigma_+ a^dag \sigma_-).
\]

As mentioned above, the operators we have chosen preserve the number of excitations \(I\). Then we find it convenient to work in the basis in which \(I\) is diagonal to represent any density matrix as

\[
\rho = \sum_{n,m=0}^{\infty} \sum_{j,k=0}^{\infty} \rho_{n,m}^{j,k} |n - j, j\rangle \langle n - k, k| = \sum_{n,m} \rho_{n,m}.
\]

Here we have partitioned the density matrix into the 2 \(\times\) 2 matrices \(\rho_{n,m}\) with matrix elements \(\rho_{n,m}^{j,k}\). Note that for \(n = 0\) there is a single state as \(n - j \geq 0\), which in this case implies \(j = 0\). Each \(\rho_{n,m}\) has a definite number of left and right excitations, which can be summarized by its commutation relation with \(I\) as

\[
[I, \rho_{n,m}] = (n - m)\rho_{n,m}.
\]

By construction the Liouvillian preserves the number of excitations and maps any \(\rho_{n,m}\) into another \(\rho_{n,m}\), i.e. it does not couple blocks. Actually one can write an effective Liouville equation for each subspace spanned by a pair of excitations \(n\) and \(m\) as

\[
\dot{\rho}_{n,m} = \mathcal{L}_{n,m}\rho_{n,m},
\]

where each \(\mathcal{L}_{n,m}\) is a super-operator acting on a four-dimensional (4D) space. The full Liouvillian is simply the sum of all these terms and can be expressed as

\[
\mathcal{L} = \sum_{n,m} \mathcal{L}_{n,m}.
\]

If we express each block of the density matrix as a column vector

\[
\rho_{n,m} = \left( \begin{array}{c} \rho_{n,m}^{1,1} \\ \rho_{n,m}^{1,0} \\ \rho_{n,m}^{0,1} \\ \rho_{n,m}^{0,0} \end{array} \right),
\]

then the Liouvillian operator can be written in terms of the 4 \(\times\) 4 matrices

\[
\mathcal{L}_{n,m} = \left( \begin{array}{cccc} -\sqrt{m}\gamma_0 & i\sqrt{m} & -i\sqrt{n} & \sqrt{m}\gamma_1 \\ i\sqrt{m} & -2i\delta - \frac{1}{2}\sqrt{m}\gamma_0 & 0 & -i\sqrt{n} \\ -i\sqrt{n} & 0 & 2i\delta - \frac{1}{2}\sqrt{m}\gamma_0 & i\sqrt{m} \\ \sqrt{m}\gamma_0 & -i\sqrt{n} & i\sqrt{m} & -\sqrt{m}\gamma_1 \end{array} \right),
\]
where we have defined
\[\tilde{\gamma} = \gamma_1 - \gamma_0.\] (18)

For \(m = n\) each block reduces to the Liouville super-operator of a two-level atom in thermal equilibrium with a reservoir and driven by a classical field.

The problem is now reduced to solving the eigenvalue problem for each non-Hermitian \(4 \times 4\) and driven by a classical field. of the Liouville super-operator along the lines of [17]. We leave this for a future investigation.

Now that we have found the formal solution to the problem, we can consider its reduced density matrix, which can be evaluated as
\[\rho(t) = e^{\tilde{\gamma}t} \rho(0).\] (21)

Using the spectral decomposition and biorthonormality, this can be rewritten as
\[\rho(t) = \sum_{n,m=0}^{\infty} \sum_{j=1}^{4} c_{n,m}^{(j)} e^{\lambda_{n,m}^{(j)} t} \tilde{\rho}_{n,m}^{(j)}\] (22)

with the coefficients
\[c_{n,m}^{(j)} = \text{Tr} \{ \tilde{\rho}_{n,m}^{(j)} \rho_0 \}.\] (23)

Now that we have found the formal solution to the problem, we can study the dynamics through the evaluation of relevant quantities. Here, we shall concentrate on the atomic system and consider its reduced density matrix, which can be evaluated as
\[\varrho = \sum_{n} \langle n | \rho | n \rangle\] (24)

to obtain the \(2 \times 2\) matrix
\[\varrho = \begin{pmatrix} \varrho_{11} & \varrho_{10} \\ \varrho_{01} & \varrho_{00} \end{pmatrix} = \sum_{n} \begin{pmatrix} \rho_{n,n}^{1,1} & \rho_{n,n}^{1,0} \\ \rho_{n,n+1}^{0,1} & \rho_{n,n}^{0,0} \end{pmatrix}.\] (25)

One can now evaluate the population inversion of the atomic system
\[W(t) = \varrho_{11}(t) - \varrho_{00}(t) = 2 \varrho_{11}(t) - 1\] (26)
as well as its purity
\[P(t) = \varrho_{11}^2(t) + \varrho_{00}^2(t) + 2 |\varrho_{01}(t)|^2.\] (27)

It is equally interesting to study the behavior of the cavity. The Wigner or Husimi functions should be evaluated to get some insight into the action of the Kraus operators proposed by analyzing the corresponding phase space along the lines of [10]. We leave this for a future investigation.

4. Explicit solutions for zero detuning

We now concentrate on the case when the atom is in resonance with the mode, i.e. zero detuning \(\delta = 0\). Here one is able to find simple explicit solutions for eigenvalues and eigenvectors of the Liouville operator of the master equation (11) by diagonalizing the blocks (17). We shall rescale to unity the coupling between the atom and the cavity, that is, \(g = 1\). The four eigenvalues for each block are given by
\[\lambda_{n,m}^{(j)} = \frac{3}{4} \text{sgn}(n - m) \tilde{\gamma} - \frac{1}{16} \text{sgn}(n - m) \tilde{\gamma}^2 - \frac{1}{4} (m + n)^2.\] (28)

Note that for \(m = n\) the fourth eigenvalue is zero, which corresponds to the stationary state of the dynamics.

It is convenient to introduce
\[\lambda_{n,m}^{(j)} = \frac{3}{4} \tilde{\gamma} - \frac{1}{16} \tilde{\gamma}^2 + \lambda_{n,m}^{(j)}.\] (29)

To simplify the notation, in the following equations we shall write \(\lambda_j = \lambda_{n,m}^{(j)}\) and \(l_j = l_{n,m}^{(j)}\). With this notation, the full set of left and right eigenvectors is written in matrix form:
\[\tilde{\rho}_{n,m} = \begin{pmatrix} (2 + \gamma_1) (\sqrt{m + n}) & -m \gamma_0 + n \gamma_1 - 2 l_1 \\ l_2 (4 + l_1 \tilde{\gamma}) & l_2 (4 + l_1 \tilde{\gamma}) \end{pmatrix},\]
\[\tilde{\rho}_{n,m} = \begin{pmatrix} -2 \gamma_1 l_2 & -m \gamma_0 + n \gamma_1 - 2 l_2 \\ 4 + l_2 \tilde{\gamma} & (\sqrt{m + n}) (4 + l_2 \tilde{\gamma}) \end{pmatrix},\]
\[\tilde{\rho}_{n,m} = \begin{pmatrix} \sqrt{m - n} & \lambda_3 \\ \lambda_3 - \lambda_4 & \sqrt{m - n} \end{pmatrix},\]
\[\tilde{\rho}_{n,m} = \begin{pmatrix} \sqrt{m - n} & \lambda_3 \\ \lambda_3 - \lambda_4 & \sqrt{m - n} \end{pmatrix},\]
\[\tilde{\rho}_{n,m} = \begin{pmatrix} \sqrt{m - n} & \lambda_3 \\ \lambda_3 - \lambda_4 & \sqrt{m - n} \end{pmatrix}.\] (30)
and
\[
\hat{p}_{n,m} = \begin{pmatrix}
\frac{\sqrt{m + \sqrt{n}}}{l_2 - l_1} & -\frac{l_2}{l_2 - l_1} \\
\frac{l_2}{l_2 - l_1} & -\frac{\sqrt{m + \sqrt{n}}}{l_2 - l_1}
\end{pmatrix},
\]
\[
\hat{p}_{n,m}^2 = \begin{pmatrix}
\frac{l_2}{l_2 - l_1} & -i\frac{\sqrt{m + \sqrt{n}}}{l_2 - l_1} \\
is\frac{l_2}{l_2 - l_1} & -\frac{l_2}{l_2 - l_1}
\end{pmatrix},
\]
\[
\hat{p}_{n,m}^3 = \begin{pmatrix}
\frac{(2 - \gamma_1 \lambda_3)(\sqrt{m - \sqrt{n}})}{\lambda_3(4 - \lambda_3 \gamma)} & -\frac{n \gamma_0 + m \gamma_1 + 2 \lambda_4}{\lambda_3(4 - \lambda_3 \gamma)} \\
-\frac{m \gamma_0 + n \gamma_1 + 2 \lambda_4}{\lambda_3(4 - \lambda_3 \gamma)} & \frac{(2 - \gamma_1 \lambda_3)(\sqrt{m - \sqrt{n}})}{\lambda_3(4 - \lambda_3 \gamma)}
\end{pmatrix},
\]
\[
\hat{p}_{n,m}^4 = \begin{pmatrix}
\frac{2 - \gamma_1 \lambda_3}{4 - \lambda_3 \gamma} & \frac{n \gamma_0 + m \gamma_1 + 2 \lambda_3}{4 - \lambda_3 \gamma} \\
\frac{2 \gamma_0 + n \gamma_1 + 2 \gamma_3}{4 - \lambda_3 \gamma} & \frac{2 - \gamma_1 \lambda_3}{4 - \lambda_3 \gamma}
\end{pmatrix}.
\]

For \( m = n \), the proper limit has to be taken in the fourth eigenvector and that is
\[
\hat{p}_{n,n}^4 = \begin{pmatrix}
\frac{4 + n \gamma_0 \gamma_1}{8 + n \gamma^2} & -i\frac{2\sqrt{n}(\gamma_0 - \gamma_1)}{8 + n \gamma^2} \\
i\frac{2\sqrt{n}(\gamma_0 - \gamma_1)}{8 + n \gamma^2} & \frac{4 + n \gamma_0 \gamma_1}{8 + n \gamma^2}
\end{pmatrix}.
\]

Having found explicit solutions for the eigenvalue problem, we are going to use them to investigate the dynamics of two simple initial conditions.

4.1. A dressed state as an initial condition

In this subsection, we investigate the behavior of an eigenvector of \( H \), which is a stationary state if the dissipation is turned off. In this case with zero detuning (\( \delta = 0 \)) such a state has the form
\[
|\phi_n^*\rangle = \frac{1}{\sqrt{2}} (|n - 1, 1\rangle + |n, 0\rangle).
\]

The eigenstates of \( H \) have a definite number of excitations \( n \), and thus the initial state can be expressed as
\[
\rho(0) = \rho_{n,n}(0) = |\phi_n^*\rangle \langle \phi_n^*| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

As different blocks do not couple, we obtain
\[
\rho(t) = \rho_{n,n}(t) = \frac{1}{2} \text{Tr} \left\{ \hat{p}_{n,n}(t) |\phi_n^*\rangle \langle \phi_n^*| \right\} e^{i\gamma_0^* t} \hat{p}_{n,n}^4.
\]

For \( m = n \), the proper limit has to be taken in the fourth eigenvector and that is
\[
\hat{p}_{n,n}^4 = \begin{pmatrix}
\frac{4 + n \gamma_0 \gamma_1}{8 + n \gamma^2} & -i\frac{2\sqrt{n}(\gamma_0 - \gamma_1)}{8 + n \gamma^2} \\
i\frac{2\sqrt{n}(\gamma_0 - \gamma_1)}{8 + n \gamma^2} & \frac{4 + n \gamma_0 \gamma_1}{8 + n \gamma^2}
\end{pmatrix}.
\]

Having found explicit solutions for the eigenvalue problem, we are going to use them to investigate the dynamics of two simple initial conditions.

In this case, we find that the relevant matrix elements are given by
\[
\rho_{n,n}^{1,1}(t) = \frac{2 - \gamma_1 \lambda_3}{4 - \lambda_3 \gamma} \frac{(\gamma_0 - \gamma_1)}{\gamma \lambda_3} + \frac{n \gamma_0 - \gamma_1}{\gamma (4 - \lambda_3 \gamma)} \sum_{j=1}^{A_{j,n}} \frac{\gamma (4 - \lambda_3 \gamma)}{4} e^{i\gamma_0^* t},
\]
\[
\rho_{n,n}^{1,2}(t) = -i\frac{\sqrt{n}(\gamma_0 - \gamma_1)}{\gamma (4 - \lambda_3 \gamma)} \frac{n \gamma_0 - \gamma_1}{\gamma \lambda_3} \sum_{j=1}^{A_{j,n}} \frac{\gamma (4 - \lambda_3 \gamma)}{4} e^{i\gamma_0^* t} (-1)^j.
\]

Once we have the density matrix, we can compute the reduced density matrix for the atomic system from equation (25). The off-diagonal terms vanish and we are left with the diagonal reduced density matrix for the atom
\[
\varrho = \begin{pmatrix} \rho_{n,n}^{1,1} & 0 \\ 0 & 1 - \rho_{n,n}^{1,1} \end{pmatrix}.
\]

To visualize the dynamics we choose to evaluate the population inversion \( W(t) \) of the atomic state, which can be obtained from equation (26) and its purity \( P(t) \) from equation (27). This can be trivially achieved, noting that for the reduced density matrix (37) the matrix element \( \varrho_{11}^{1,1} = \rho_{n,n}^{1,1} \) gives us all the information about the atomic subsystem. In figure 1, we have plotted these quantities for different choices of \( \gamma_1 \) and \( \gamma_0 \).
First we will discuss the case \( \gamma_0 = \gamma_1 \). This is plotted in gray and the results are the constant values \( W = 0 \) and \( P = 1/2 \) in figure 1. Of course this case includes the absence of dissipation, where the dressed state is an eigenvector and therefore a stationary state. For nonzero but equal dissipation constants, we see that the state remains stationary.

For small dissipation constants where we set \( \gamma_0 = 0.08 \) and \( \gamma_1 = 0 \) (black curve), we see small oscillations that eventually decay to the initial values of \( P \) and \( W \).

Increasing in an asymmetric way the dissipation constants \( (\gamma_0 = 0, \gamma_1 = 1.2) \), the oscillations are strongly damped and lead to a greater final population of the excited state (dotted line). We find similar results when \( (\gamma_0 = 1.2, \gamma_1 = 0) \), but here the ground state population dominates for large times (dashed line). For the last two conditions, the behavior of purity is equivalent and leads to a final steady state which is not a complete mixture as was the case for the initial state, i.e. the purity of the atom increases due to the external coupling.

### 4.2. A superposition with two different numbers of excitations

As another example we now consider the time evolution of an initial product state of the field and atom subsystems, that is,

\[
|\Psi_0\rangle = \cos \alpha |n, 0\rangle + \sin \alpha |n, 1\rangle.
\]  

(38)

The first and second terms in the last expression correspond, respectively, to states of \( n \) and \( n + 1 \) excitations. Taking the outer product of this vector, we obtain the initial density matrix, which results in a mixture of terms with these two excitations. The resulting matrix can be expressed as the sum of the four terms

\[
\rho(t) = \rho_{n,n}(t) + \rho_{n,n+1}(t) + \rho_{n+1,n}(t) + \rho_{n+1,n+1}(t),
\]  

(39)

and can be represented as a \( 4 \times 4 \) matrix. Every single term can be computed using the spectral decomposition as in equation (22). Taking a partial trace over the cavity’s degree of freedom to obtain the reduced density matrix of the atomic system, as in equation (25), yields

\[
\varrho(t) = \begin{pmatrix}
\rho_{n,n}(t) + \rho_{n,n+1}(t) & \rho_{n+1,n}(t) \\
\rho_{n,n+1}(t) & \rho_{n,n}(t) + \rho_{n+1,n+1}(t)
\end{pmatrix}.
\]  

(40)

The \( 2 \times 2 \) density matrix is determined by two of its elements as

\[
\varrho_{11}(t) = \cos^2 \alpha \left( \frac{4 + n\tilde{\gamma}_1}{8 + n^2\tilde{\gamma}^2} + \frac{l_{n,n}^{(1)}(2 + l_{n,n}^{(1)}\gamma_1) e^{i l_{n,n}^{(1)} t}}{4 + l_{n,n}^{(1)}(l_{n,n}^{(1)} - l_{n-1,n}^{(1)})} \right)
\]

\[
-\frac{l_{n,n}^{(2)}(l_{n,n}^{(2)} + l_{n,n}^{(2)}\gamma_1) e^{i l_{n,n}^{(2)} t}}{4 + l_{n,n}^{(2)}(l_{n,n}^{(2)} - l_{n-1,n}^{(2)})}
\]

\[
+ \sin^2 \alpha \left( \frac{4 + (n + 1)\tilde{\gamma}_1}{8 + (n + 1)^2\tilde{\gamma}^2} - \frac{l_{n+1,n}^{(1)}(2 + l_{n+1,n}^{(1)}\gamma_1) e^{i l_{n+1,n}^{(1)} t}}{4 + l_{n+1,n}^{(1)}(l_{n+1,n}^{(1)} - l_{n+1,n+1}^{(1)})} \right)
\]

\[
+ \frac{l_{n+1,n+1}^{(2)}(l_{n+1,n+1}^{(2)} + l_{n+1,n+1}^{(2)}\gamma_1) e^{i l_{n+1,n+1}^{(2)} t}}{4 + l_{n+1,n+1}^{(2)}(l_{n+1,n+1}^{(2)} - l_{n+1,n+1}^{(2)})} \right).
\]  

(41)

Figure 2. Population inversion \( W \) (top) and purity \( P \) (bottom) as a function of time for an initial dressed state. We show four curves, all with the same number of excitations \( n = 2 \), and \( \gamma_0 = \gamma_1 = 0 \) for the gray curve, \( \gamma_0 = 0.08, \gamma_1 = 0 \) for the solid black curve, \( \gamma_0 = 1.2, \gamma_1 = 0 \) for the dashed curve and \( \gamma_0 = 0, \gamma_1 = 1.2 \) for the dotted curve.

\[\varrho_{01}(t) = \cos \alpha \sin \alpha \times \left( \frac{(n\tilde{\gamma} + \gamma_1 - 2l_{n,n}^{(1)}) e^{i l_{n,n}^{(1)} t}}{(4 + l_{n,n}^{(1)}\gamma_1)(l_{n,n}^{(1)} - l_{n-1,n}^{(1)})} - \frac{(n\tilde{\gamma} + \gamma_1 - 2l_{n,n}^{(2)}) e^{i l_{n,n}^{(2)} t}}{(4 + l_{n,n}^{(2)}\gamma_1)(l_{n,n}^{(2)} - l_{n-1,n}^{(2)})} \right)
\]

\[+ \frac{(n\tilde{\gamma} + \gamma_1 + 2\lambda_{n,n}^{(3)} e^{i l_{n,n}^{(3)} t}}{(4 - \lambda_{n,n}^{(4)})(\lambda_{n,n}^{(4)} - \lambda_{n,n}^{(3)})} - \frac{(n\tilde{\gamma} + \gamma_1 + 2\lambda_{n,n}^{(4)} e^{i l_{n,n}^{(4)} t}}{(4 - \lambda_{n,n}^{(3)})(\lambda_{n,n}^{(3)} - \lambda_{n,n}^{(2)})} \right).
\]

(41)

In the same way as in section 4.1, from the reduced density matrix of the atom, one can evaluate the inversion of population \( W(t) \) (26) and the purity \( P(t) \) (27). In figure 2, we have plotted the corresponding values for different choices of \( \gamma_1 \) and \( \gamma_0 \).

For the gray curves (\( \gamma_1 = \gamma_0 = 0 \)), we can see that the initial state is not a stationary state of the system without dissipation.

Turning on one of the dissipation constants (\( \gamma_0 = 0.08, \gamma_0 = 0 \)), the black curve shows how the oscillations are damped to end up in a state with almost equal population between excited and ground states and close to the total mixture.

Increasing in an asymmetric way the dissipation constants results in a similar behavior as in the case when we used an initial dressed state. The oscillations are strongly
damped to end up in greater occupation of the excited state for the dotted curve ($\gamma_0 = 0$, $\gamma_1 = 1.2$), and the opposite for the dashed curve ($\gamma_0 = 1.2$, $\gamma_1 = 0$).

5. Conclusions

We have displayed a new open system that can be solved analytically. From the point of view of solvable systems, this is always an interesting step, both formally and as a test ground for approximate or numerical solutions of problems that are not solvable analytically. Solvability here is based on conservation of the excitation number in both the unitary and the non-unitary evolution.

As far as the behavior of the population inversion is concerned, we note that the two terms treat the cavity mode and spin asymmetrically if the two gammas are not equal. Thus, the observed result is not entirely surprising, but it may well deserve a more detailed study because purity will reach its lower limit only with symmetric dissipation.

While this model depends on the exact solvability of fourth-order polynomials, since Galois we know that such a general solution does not exist for higher degrees. We could try to cook up models that correspond to special solvable cases, but we would rather consider the following. If, for example, we have a three or more level atom (or a spin larger than 1/2) or even, say, two two-level systems, according to the discussion in the introduction the dimension of the matrix representation of the super-operator is the square of the dimension of the Hilbert space. Thus for spin 1 we would have a $9 \times 9$ matrix and for two spin 1/2 particles a $16 \times 16$ matrix form of the super-operator. Obviously these are not in general diagonalizable in closed form. However, numerical diagonalization can be considered exact in the sense that arbitrary exactitude can be reached. From that point on, eigenvalues and the dual sets of eigenfunctions can be used to obtain the results required for a study of decoherence.

Other more complicated states, such as coherent states or thermal states can be used as initial states, and will be considered in future work. The question of how to construct a specific experiment that corresponds to these conditions, unfortunately, is still an open problem, but we are striving to find a solution.

Acknowledgments

JMT and THS acknowledge the Alexander von Humboldt Foundation for support during the preparation of the manuscript and Professor W Schleich for discussions and hospitality during their stay at the University of Ulm. The authors acknowledge support from the projects IN114310 of PAPIIT-UNAM and 79613 of CONACyT, Mexico.

References

[1] Carmichael H 1993 An Open System Approach to Quantum Optics (Berlin: Springer)
[2] Lindblad G 1976 On the generators of quantum dynamical semigroups Commun. Math. Phys. 48 119–30
[3] Kossakowski A 1972 On quantum statistical mechanics of non-Hamiltonian systems Rep. Math. Phys. 3 247–74
[4] Gorini V, Kossakowski A and Sudarshan E C G 1976 Completely positive dynamical semigroups of N-level systems J. Math. Phys. 17 821–5
[5] Torres J M, Sadurní E and Seligman T H 2010 Two interacting atoms in a cavity: exact solutions, entanglement and decoherence J. Phys. A: Math. Theor. 43 192002
[6] Sadurní E, Torres J M and Seligman T H 2010 Dynamics of a Dirac oscillator coupled to an external field: a new class of solvable problems J. Phys. A: Math. Theor. 43 285204
[7] Eiselt J and Risken H 1991 Quasiprobability distributions for the Jaynes–Cummings model with cavity damping Phys. Rev. A 43 346–60
[8] Briegel H-J and Englert B-G 1993 Quantum optical master equations: the use of damping bases Phys. Rev. A 47 3311–29
[9] Daebubler B, Risken H and Schoendorff L 1993 Analytic solution for quasiprobability distributions of the Jaynes–Cummings model with cavity damping Phys. Rev. A 48 3955–65
[10] Garraway B M, Knight P L and Plenio M B 1998 Generation and preservation of coherence in dissipative quantum optical environments Phys. Scr. T72 152