On the Triangle Clique Cover and $K_t$ Clique Cover Problems

Hoang Dau  Olgica Milenkovic  Gregory J. Puleo

Abstract

An edge clique cover of a graph is a set of cliques that covers all edges of the graph. We generalize this concept to $K_t$ clique cover, i.e. a set of cliques that covers all complete subgraphs on $t$ vertices of the graph, for every $t \geq 1$. In particular, we extend a classical result of Erdős, Goodman, and Pósa (1966) on the edge clique cover number ($t = 2$), also known as the intersection number, to the case $t = 3$. The upper bound is tight, with equality holding only for the Turán graph $T(n,3)$. We also extend an algorithm of Scheinerman and Trenk (1999) to solve a weighted version of the $K_t$ clique cover problem on a superclass of chordal graphs. We also prove that the $K_t$ clique cover problem is NP-hard.

1 Introduction

A clique in a graph $G$ is a set of vertices that induces a complete subgraph; all graphs considered in this paper are simple and undirected. A vertex clique cover of a graph $G$ is a set of cliques in $G$ that collectively cover all of its vertices. The vertex clique cover number of $G$, denoted $\theta_v(G)$, is the minimum number of cliques in a vertex clique cover of $G$. An edge clique cover of a graph $G$ is a set of cliques of $G$ that collectively cover all of its edges. The edge clique cover number of $G$, denoted $\theta_e(G)$, is the minimum number of cliques in an edge clique cover. The vertex clique cover number, which is the same as the chromatic number of the complement graph, and the edge clique cover number, also referred to as the intersection number of a graph, have been extensively studied in the literature (see, for instance [5, 6, 12]).

Figure 1: An illustration of a minimum vertex clique cover (left-most), a minimum edge clique cover (middle), and a minimum triangle clique cover (right-most) of the same graph on eight vertices.

We generalize the notions of vertex and edge clique covers by the following definition. Note that we use the word clique to refer to any vertex set inducing a complete subgraph (in contrast to some other authors who require a clique to be a maximal such set).

Definition 1. Let $t$ be a nonnegative integer. A $t$-clique of $G$ is a clique containing exactly $t$ vertices. A $K_t$ clique cover of a graph $G$ is a set of cliques that collectively cover all $t$-cliques in $G$.

Equivalently, a set $\mathcal{C}$ of cliques is a $K_t$ clique cover of $G$ if, for every $t$-clique $S \subset V(G)$, there is a clique $Q \in \mathcal{C}$ with $S \subset Q$. A $K_1$ clique cover is simply a vertex clique cover, while a $K_2$ clique cover is an edge clique cover. We refer to a $K_3$ clique cover as a triangle clique cover. The $K_t$ clique cover number, denoted $\theta_{K_t}$, and the triangle clique cover number, denoted $\theta_{\triangle}$, are also defined accordingly. We illustrate these concepts in Fig. 1.
For positive integers \( n \) and \( k \), the Turán graph \( T(n,k) \) is defined to be the complete \( k \)-partite graph on \( n \) vertices whose part sizes differ by at most 1. Turán graphs frequently arise as extremal graphs for various graph parameters. The following conjecture states that Turán graphs are the unique extremal graphs for the \( K_t \) clique cover problem.

**Conjecture 1.** If \( n \) and \( t \) are positive integers, then for every \( n \)-vertex graph \( G \),

\[
\theta_{K_t}(G) \leq \theta_{K_t}(T(n,t)).
\]  

Equality holds if and only if \( G \cong T(n,t) \).

As motivation for Conjecture 1, we now briefly consider the cases \( t = 1 \) and \( t = 2 \).

When \( t = 1 \), it is obvious that \( \theta_{K_1}(G) = \theta_\emptyset(G) \leq n \) for every graph \( G \) on \( n \) vertices. Equality clearly holds if and only if \( G \) has no edges; a graph with no edges is the trivial Turán graph \( T(n,1) \). Erdős, Goodman, and Pósa [5] proved that \( \theta_{K_1}(G) = \theta_\emptyset(G) \leq \lfloor \frac{n}{t} \rfloor \) for every graph \( G \), with equality holding if and only if \( G \) is the Turán graph \( T(n,2) = K_{\lfloor n/2 \rfloor,\lceil n/2 \rceil} \).

In Sections 2 and 3, we prove the \( t = 3 \) case of Conjecture 1. At the end of Section 3, we also discuss the connections between Conjecture 1 and a theorem of Lehel [8] about covering edges in hypergraphs. The status of Conjecture 1 is summarized in Table 1.

We also consider two natural weighted versions of the \( K_t \) clique cover problem. In Section 5, we consider a variant in which each \( t \)-clique \( S \) is assigned a nonnegative integer weight \( w_S \); we seek a smallest multiset of cliques such that each \( t \)-clique \( S \) is covered at least \( w_S \) times. We give a polynomial-time algorithm to solve this problem on a superclass of chordal graphs, extending a result of Scheinerman and Trenk [15]. The specific class of graphs for which our algorithm works, which we call *semichordal graphs*, is defined and discussed in Section 4.

In Section 6, we consider a weighted version of the \( K_t \) clique cover problem in which each maximal clique \( S \) of \( G \) is assigned an integer cost \( c_S \); the goal is to find a minimum-cost set of maximal cliques such that each \( t \)-clique is covered at least once. In contrast to the polynomial-time algorithm mentioned above, we prove that this variant is NP-hard even when \( k = 2 \) and the graph \( G \) is chordal.

It was shown by Orlin [11] and by Kou, Stockmeyer, and Wong [7] that determining \( \theta_\emptyset(G) \) is an NP-complete problem. Their idea is to reduce the problem of determining \( \theta_\emptyset(G) \), which was known to be NP-complete, to the problem of determining \( \theta_\emptyset(G) \). In Section 7, we generalize the reduction used in [7] to show, by induction, that determining \( \theta_{K_t}(G) \) is an NP-complete problem, for any constant \( t \geq 2 \), by reducing the problem of determining \( \theta_{K_{t-1}}(G) \) to the problem of determining \( \theta_{K_t}(G) \).

## 2 A Minimum-Degree Version of a Theorem of Lovász

In this section, we obtain results similar to the following theorem of Lovász [9]. In the next section, we will apply these results to obtain an upper bound on \( \theta_\Delta(G) \).

**Theorem 1 (Lovász [9]).** Let \( G \) be an \( n \)-vertex graph, and let \( k = \binom{n}{2} - |E(G)| \). If \( t \) is the greatest integer such that \( t^2 - t \leq k \), then \( \theta_\emptyset(G) \leq k + t \).
In this section, we obtain a variant of Theorem 1 by strengthening the hypothesis to include a lower bound on $\delta(G)$, the minimum degree of $G$, rather than just a lower bound on $|E(G)|$. We start with a basic result and then strengthen its weakest nontrivial case.

**Lemma 1.** For any graph $G$, $\theta_e(G) \leq (n - \delta(G)) + \frac{n(n - \delta(G) - 1)}{2}$.

**Proof.** We mimic the proof of Theorem 1. Let $A_1$ be a maximum clique in $G$, and for $i > 1$, let $A_i$ be a maximum clique in $G - (A_1 \cup \cdots \cup A_{i-1})$. Set $a_i = |A_i|$. Let $p$ be the largest index for which $A_i$ is nonempty. As each vertex of $A_p$ has at least one non-neighbor in $A_j$ for $j < p$, we have $p \leq n - \delta(G)$. For each $v \in A_i$ and each $j < i$, let $S_{v,j} = \{v\} \cup (N(v) \cap A_j)$. Each set $S_{v,j}$ is clearly a clique. Let $F$ be the set consisting of all cliques $A_i$ together with all the cliques $S_{v,j}$ where $v \in A_i$ and $j < i$. The set $F$ covers all edges of $G$, and we have

$$\theta_e(G) \leq |F| \leq p + \sum_{i=1}^{p} (i-1)a_i.$$

Subject to the constraints $a_1 + \cdots + a_p = n$ and $a_1 \geq \cdots \geq a_p$, and allowing $a_i$ to take fractional values, the sum $\sum_{i=1}^{p} (i-1)a_i$ is clearly maximized when $a_1 = \cdots = a_p = n/p$. Hence,

$$\theta_e(G) \leq p + n \sum_{i=1}^{p} (i-1) = p + \frac{n(p-1)}{2}.$$

As $p \leq n - \delta(G)$, the conclusion follows. \[\Box\]

When $\delta(G) = n/2$, Lemma 1 gives $\theta_e(G) \leq n^2/4$, which is sharp when $G = K_{n/2,n/2}$. We wish to obtain a sharper bound when $\delta(G)$ is slightly larger than $n/2$.

**Lemma 2.** If $\delta(G) = (n + 1)/2$, then $\theta_e(G) \leq \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4}$.

**Proof.** Define $A_1, \ldots, A_p$ and $F$ as in the proof of Lemma 1. With $a_i = |A_i|$, we have the bound

$$\theta_e(G) \leq |F| \leq p + \sum_{i=1}^{p} (i-1)a_i.$$

If $p < n - \delta(G)$, then repeating the argument in Lemma 1 yields

$$\theta_e(G) \leq p + \frac{n(p-1)}{2} \leq \frac{n^2}{4} - \frac{3n}{4} - \frac{3}{2} < \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4}.$$

If $p = n - \delta(G) = (n - 1)/2$, then we exploit the integrality of $a_i$, which we ignored in the proof of Lemma 1. The sum $\sum_{i=1}^{p} (i-1)a_i$ is maximized, subject to $a_1 + \cdots + a_p = n$ and $a_1 \geq \cdots \geq a_p$, by the sequence with $a_1 = 3$ and $a_i = 2$ for $i \geq 2$. Hence, we obtain the upper bound

$$\theta_e(G) \leq p + 2 \sum_{i=2}^{p} (i-1) = p + p(p-1) = \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4}.$$

Thus, the claimed upper bound on $\theta_e(G)$ holds in both cases. \[\Box\]

**Lemma 3.** If $\delta(G) = n/2 + 1$, then $\theta_e(G) \leq \frac{n^2}{4} - n + 2$. 

Proof. Define $A_1, \ldots, A_p$ and $\mathcal{F}$ as in the proof of Lemma 1. With $a_i = |A_i|$, we have the bound

$$\theta_e(G) \leq |\mathcal{F}| \leq p + \sum_{i=1}^{p} (i-1)a_i.$$ 

If $p < n - \delta(G)$, then repeating the argument in Lemma 1 yields

$$\theta_e(G) \leq p + n(p-1) \leq \frac{n^2}{4} - n - 2 < \frac{n^2}{4} - n + 2.$$ 

If $p = n - \delta(G) = n/2 - 1$, then we again exploit the integrality of $a_i$. The sum $\sum_{i=1}^{p} (i-1)a_i$ is maximized, subject to $a_1 + \ldots + a_p = n$ and $a_1 \geq \cdots \geq a_p$, by the sequence with $a_1 = a_2 = 3$ and $a_i = 2$ for $i \geq 3$. Hence, we obtain the upper bound

$$\theta_e(G) \leq p + 3 + 2 \sum_{i=3}^{p} (i-1) = p + 3 + (p(p-1) - 2) = p + 1 + p(p-1) = \frac{n^2}{4} - n + 2.$$ 

Thus, the claimed upper bound on $\theta_e(G)$ holds in both cases. \qed

3 An Upper Bound on $\theta_\Delta(G)$

In this section, our goal is to generalize the following result of Erdös, Goodman, and Pósa.

**Theorem 2 (Erdös–Goodman–Posa [5]).** If $G$ is an $n$-vertex graph, then $\theta_e(G) \leq \lfloor n^2/4 \rfloor$. Equality holds if and only if $G \cong T(n,2)$.

**Definition 2.** When $G$ is a graph and $r$ is a nonnegative integer, $k_t(G)$ is the number of copies of $K_t$ in $G$.

**Observation 1.** For any nonnegative integer $n$,

$$k_3(T(n,3)) = \begin{cases} \frac{n^3}{27}, & \text{if } n \equiv 0 \pmod{3}, \\ \frac{(n-1)^3}{27} + \frac{(n-1)^2}{9}, & \text{if } n \equiv 1 \pmod{3}, \\ \frac{(n+1)^3}{27} - \frac{(n+1)^2}{9}, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

**Observation 2.** For all $n \geq 3$,

$$k_3(T(n,3)) - k_3(T(n-1,3)) = \left\lfloor \frac{2n/3}{4} \right\rfloor = \begin{cases} \frac{n^2}{9}, & \text{if } n \equiv 0 \pmod{3} \\ \frac{(n-1)^2}{9}, & \text{if } n \equiv 1 \pmod{3} \\ \frac{n^2-n-2}{9}, & \text{if } n \equiv 2 \pmod{3} \end{cases} \geq \frac{(n-1)^2}{9}.$$ 

**Theorem 3.** For any graph $G$, $\theta_\Delta(G) \leq k_3(T(n,3))$. If equality holds, then $G \cong T(n,3)$. 

4
Thus, $\delta(H)$ is the number of triangles that contain $v$. Let $H = G[N(v)]$, i.e., let $H$ be the subgraph of $G$ induced by $N(v)$, and let $F$ be a smallest edge clique cover of $H$. By adding $v$ to each clique in $F$, we obtain a set of cliques $F_1$ covering every triangle that contains $v$. Thus, $F_1 \cup C'$ is a triangle edge cover in $G$. It therefore suffices to show that $\theta_e(H) \leq k_3(T(n, 3)) - k_3(T(n-1, 3))$, which is what we show next. We split the proof into cases according to $d(v)$.

**Case 1:** $d(v) \leq (2n/3)$. In this case, $|V(H)| \leq 2n/3$, so by Theorem 2 and Observation 2 we have

$$\theta_e(H) \leq \left \lfloor \frac{|V(H)|^2}{4} \right \rfloor \leq \left \lfloor \frac{2n}{3} \right \rfloor \leq k_3(T(n, 3)) - k_3(T(n-1, 3)),$$

as desired. If $\theta_e(G) = k_3(T(n, 3))$, then equality must hold throughout the above inequality, and in particular we must have $\theta_e(H) = |V(H)|^2 / 4 = \left \lfloor \frac{2n}{3} \right \rfloor$. By Theorem 2, this implies that $H \cong T(\lfloor 2n/3 \rfloor, 2)$. Furthermore, $\theta_e(G) = k_3(T(n, 3))$ requires that $|C'| = (G - v) = k_3(T(n - 1, 3))$, so by the induction hypothesis, we have $G' \cong T(n-1, 3)$. Thus, $G$ is obtained from $T(n-1, 3)$ by adding a new vertex adjacent to $\lfloor 2n/3 \rfloor$ vertices inducing a complete bipartite graph. This implies that $G \cong T(n, 3)$.

**Case 2:** $d(v) \geq 2n/3 + 1$. Since $v$ was a vertex of minimum degree, every $w \in N(v)$ satisfies $d(w) \geq d(v)$. At most $n - d(v)$ of those neighbors lie outside $N(v)$, so for all $w \in V(H)$, we have

$$d_H(w) \geq d(v) - (n - d(v)) = 2d(v) - n.$$

Thus, $\delta(H) \geq 2d(v) - n$. Lemma 1 yields

$$\theta_e(H) \leq (d(v) - \delta(H)) + \frac{d(v)(d(v) - \delta(H) - 1)}{2} \leq n(d(v) + 2) - d(v)(d(v) + 3).$$

If $d(v) \geq (2n + 4)/3$, then this implies that

$$\theta_e(H) \leq \frac{n(2n+4)}{3} - \frac{2n+4}{3} \left( \frac{2n+4}{3} + 3 \right) = \frac{(n-1)^2 - 27}{9} < \frac{(n-1)^2}{9} \leq k_3(T(n, 3)) - k_3(T(n-1, 3)),$$

and we are done. Similarly, if $d(v) = (2n + 3)/3$, then $n \equiv 0 \pmod{3}$, so we again have

$$\theta_e(H) \leq \frac{n(2n+3)}{3} - \frac{2n+3}{3} \left( \frac{2n+3}{3} + 3 \right) = \frac{n^2 - 3n - 36}{18} < \frac{n^2}{9} = k_3(T(n, 3)) - k_3(T(n-1, 3)).$$

**Case 3:** $d(v) = (2n+1)/3$ or $d(v) = (2n+2)/3$. Since $v$ was a vertex of minimum degree, every $w \in N(v)$ satisfies $d(w) \geq d(v)$. At most $n - d(v)$ of those neighbors lie outside $N(v)$, so for all $w \in V(H)$, we have

$$d_H(w) \geq d(v) - (n - d(v)) = 2d(v) - n.$$

Thus, $\delta(H) \geq 2d(v) - n$. If $\delta(H) \geq 2d(v) - n + 1$, then Lemma 1 yields

$$\theta_e(H) \leq (d(v) - \delta(H)) + \frac{d(v)(d(v) - \delta(H) - 1)}{2} = n - 1 + \frac{d(v)(n - d(v) - 4)}{2} \leq \frac{(n-1)^2}{9},$$

where the last inequality follows from the assumption that $d(v) \geq (2n + 1)/3$.

Hence we may assume that $\delta(H) = 2d(v) - n$. We consider two subcases: either $d(v) = (2n+1)/3$ or $d(v) = (2n+2)/3$. 

5
Theorem 4 by Bollobás [2] and proved by Lehel [8].

Turán graph clique in $C$ we must also have $k$ vertices can be covered by using at most $\theta_e(H) \leq \frac{d(v)^2}{4} - \frac{d(v)}{2} + 1 = \frac{(n-1)^2}{9}$, and so $\theta_e(H) \leq k_3(T(n,3)) - k_3(T(n-1,3))$.

This yields $\theta_\Delta(G) \leq k_3(T(n,3))$ for the case $d(v) = (2n+1)/3$. To obtain the strict inequality $\theta_\Delta(G) < k_3(T(n,3))$, suppose to the contrary that $\theta_\Delta(G) = k_3(T(n,3))$. Since $\theta_e(H) \leq k_3(T(n,3)) - k_3(T(n-1,3))$, we must also have $\theta_\Delta(G - v) = k_3(T(n-1,3))$. By the induction hypothesis, $G - v \cong T(n-1,3)$. In particular, $G - v$ is $K_4$-free. Let $C'$ be a smallest $K_3$ clique cover in $G - v$. Since $G - v$ is $K_4$-free, every clique in $C'$ is a triangle.

Since $\delta(H) = \frac{d(v)+1}{2} > d(v) = \frac{\lfloor V(H) \rfloor}{2}$, every edge in $H$ is contained in some triangle of $H$. Since $G - v$ is $K_4$-free, every triangle of $H$ is contained in $C'$. Let $C$ be the collection of cliques obtained by replacing every triangle $T$ of $H$ contained in $C'$ with the clique $T \cup \{v\}$. Now $C$ covers all triangles in $G$, and

$$|C| = |C'| \leq k_3(T(n-1,3)) < k_3(T(n,3)).$$

This contradicts the hypothesis that $\theta_\Delta(G) = k_3(T(n,3))$. We conclude that $\theta_\Delta(G) < k_3(T(n,3))$ when $d(v) = (2n+1)/3$.

Case 3b: $d(v) = (2n+2)/3$. Here $\delta(H) = 2d(v) - n$ gives $\delta(H) = \frac{n+4}{3} = \frac{d(v)}{2} + 1$. In this case, $n \equiv 2 \pmod{3}$. Since we have assumed also that $n > 3$, we have $n \geq 5$, so Lemma 3 yields

$$\theta_e(H) \leq \frac{d(v)^2}{4} - d(v) + 2 = \frac{n^2}{9} - \frac{4n}{9} + \frac{13}{9} \leq \frac{n^2 - n - 2}{9},$$

where the inequality is strict for $n > 5$. Thus, $\theta_e(H) \leq k_3(T(n,3)) - k_3(T(n-1,3))$, and this inequality is strict for $n > 5$. On the other hand, when $n = 5$, we have $\delta(G) = d(v) = 4$, so that $G$ is a complete graph, which forces $\theta_\Delta(G) = 1 < k_3(T(5,3))$. Thus, when $d(v) = (2n+2)/3$, we have $\theta_\Delta(G) < k_3(T(n,3))$.

Thus, in all cases, $\theta_\Delta(G) \leq k_3(T(n,3))$, and equality holds only in Case 1 when $G \cong T(n,3)$. ■

The results stated in Theorem 3 have a very close connection with their counterparts established for hypergraphs [2, 8]. In the following we discuss the similarity and the differences between our results and those known in the hypergraph literature. A $t$-uniform hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and a hyperedge set $E(H)$, where each hyperedge is a set of some $t$ vertices. A 2-uniform hypergraph is simply a graph. For $p \geq t \geq 2$, let $K_{p,t}$ denote the hyperclique on $p$ vertices, i.e., a set of $p$ vertices of a hypergraph where every subset of $t$ vertices forms a hyperedge. Note that the usual clique $K_t$ is the same as $K_{t,2}$. Let $h_t(n,p)$ denote the maximum number of hyperedges that a $K_{p,t}$-free $t$-uniform hypergraph on $n$ vertices can have. Let $k_t(n,p)$ denote the maximum number of $K_t$ in a $K_{p,t}$-free graph. It was proved by Moon and Moser [10], and by Sauer [14] that $k_t(n,p)$ is precisely the number of $K_t$ in the Turán graph $T(n,p - 1)$. In other words, $k_t(n,p) = k_t(T(n,p - 1))$. The following result was conjectured by Bollobás [2] and proved by Lehel [8].

Theorem 4 (Lehel [8]). The edges of every $t$-uniform hypergraph of order $n$ can be covered by at most $h_t(n,p)$ edges and copies of $K_{p,t}$.

When $t = 2$ and $p = 3$, we have $h_2(n,3) = k_2(n,3) = k_2(T(n,2)) = \left\lfloor \frac{n^2}{4} \right\rfloor$ (by Mantel’s theorem and also by Turán [17]), and hence Theorem 4 reduces to the classical result on edge clique cover by Erdős, Goodman, and Pósa [5] mentioned in the introduction, which states that the edges of every graph on $n$ vertices can be covered by using at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges and triangles.

When $t = 3$, Theorem 4 states that one can use at most $h_3(n,4)$ hyperedges and $K_{4,3}$’s to cover all hyperedges in a 3-uniform hypergraph. However, Theorem 3 does not follow from this statement. Indeed,
our theorem establishes that one can use at most \(k_3(n, 4)\) cliques to cover all triangles in any graph on \(n\) vertices. We emphasize that \(h_3(n, 4)\) is strictly larger than \(k_3(n, 4)\). Recall that \(k_3(n, 4)\), which is the number of triangles in the Turán graph \(T(n, 3)\), can be computed explicitly as \(|n/3|\lfloor(n+1)/3\rfloor\lfloor(n+2)/3\rfloor \approx n^3/27\). By contrast, the determination of \(h_3(n, 4)\), even asymptotically, has remained open since the original work of Turán [17]. Moreover, Turán established that (see also [3])

\[
h_3(n, 4) \geq \begin{cases} m^2(5m - 3)/2, & \text{if } n = 3m, \\ m(5m^2 + 2m - 1)/2, & \text{if } n = 3m + 1, \\ m(m + 1)(5m^2)/2, & \text{if } n = 3m + 2. \\ \end{cases}
\]

The hypergraph that gives rise to this lower bound is a modification of the usual Turán graph \(T(n, 3)\) to the hypergraph setting, which can be constructed as follows. The vertex set is partitioned into three (almost) equal sets \(V_1, V_2,\) and \(V_3\), where \(|V_1| = \lfloor n/3 \rfloor\), \(|V_2| = \lfloor (n+1)/3 \rfloor\), and \(|V_3| = \lfloor (n+2)/3 \rfloor\). The hyperedge set consists of the 3-tuples \(e = \{u, v, w\}\), where either \(u \in V_1, v \in V_2, w \in V_3\), or \(u, v \in V_i\) and \(w \in V_{(i+1) \bmod 3}\). Notice that the first type of hyperedges corresponds to the triangles of the Turán graph \(T(n, 3)\), while the second type of hyperedges corresponds to new triangles that are unique to the hypergraph context. From [2], \(h_3(n, 4)\) is in order of \(5n^3/2 \approx n^3/27\). In fact, Turán conjectured that (2) is actually an equality, which was known to be true for all \(n \leq 13\) [16].

The key point that leads to the difference between \(h_3(n, 4)\) and \(k_3(n, 4)\) is that while a \(K_4\)-free graph (removing all edges that do not belong to any triangles) can be considered as a \(K_{4,3}\)-free 3-uniform hypergraph, the converse is not true. For example, a hypergraph with the vertex set \(\{u, v, w, x\}\) and the hyperedge set \(\{\{u, x, w\}, \{v, x, w\}, \{u, v, w\}\}\) is a \(K_{4,3}\)-free hypergraph, but it corresponds exactly to a \(K_4\) as a graph. Therefore, counting edges in a \(K_{4,3}\)-free 3-uniform hypergraph is not the same as counting triangles in a \(K_4\)-free graph. In fact, the maximum number of hyperedges in such \(K_{4,3}\)-free hypergraphs is larger than the maximum number of triangles in \(K_4\)-free graphs. That is the reason Theorem 4 produces a strictly worse upper bound than the tight upper bound we obtain in Theorem 3. This is also evident from the fact that while according to Theorem 4, the hyperedges of every 3-uniform hypergraph can be covered by using at most \(h_3(n, 4)\) \(K_{4,3}\)'s and hyperedges, Remark 1 states that we cannot cover all triangles in \(K_n\) (\(n \geq 18\)) by \(k_3(n, 4) = k_3(T(n, 3))\) \(K_4\)'s and triangles.

**Remark 1.** It is impossible to cover all triangles in \(K_n\) (\(n \geq 18\)) by only \(k_3(n, 4)\) triangles and copies of \(K_4\). Indeed, since \(k_3(n, 4) \approx n^3/27\) and each \(K_4\) contains four triangles, \(k_3(n, 4)\) triangles and copies of \(K_4\) can cover at most \(\approx 4n^3/27 < n^3/6\) \(\approx (n/3)^3\) triangles, which is the number of triangles in \(K_n\), for \(n\) sufficiently large. It can be easily verified that this conclusion holds for \(n \geq 18\).

Note also that the key lemma (Lemma 4.3) in the proof of Lehel [8] fails if we try to adapt it to the setting of graphs and triangles. The lemma states that for any \(2 \leq t < p\), every \(t\)-uniform hypergraph containing \(m\) edges has a \(K_p\)-free edge subset of cardinality at least \(m/2\). However, its graph-and-triangle version, which would state that every graph containing \(m\) triangles has a \(K_4\)-free subset of triangles of cardinality at least \(m/2\), is no longer correct. A counterexample is the graph \(K_5\), which contains exactly ten triangles, where we cannot find any subset of five triangles that does not include a \(K_4\). Indeed, as established by Moon and Moser [10] and Sauer [14], the Turán graph \(T(5, 3)\) is the \(K_4\)-free graph that contains the largest number of triangles, which is only four. Thus, a direct adaptation of Lehel’s arguments to our setting does not imply our result.

### 4 A Superclass of Chordal Graphs

In this section, we define a class of graphs on which the \(K_t\) clique cover problem can be efficiently solved; our algorithm is given in Section 5.
A graph is said to be chordal if it has no induced cycle of length greater than 3. A well-known characterization of chordal graphs, due to Dirac [4], is that they are the graphs which admit a simplicial elimination ordering: an ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) such that \( N(v_i) \cap \{v_{i+1}, \ldots, v_n\} \) is a clique for each \( i \).

The notion of a perfect elimination order admits a natural generalization, as follows.

**Definition 3** (Aboulker–Charbit–Trotignon–Vušković [1]). Let \( F \) be a set of graphs. An \( F \)-elimination ordering of a graph \( G \) is an ordering \( v_1, \ldots, v_n \) of the vertices of \( G \) such that for each \( i \), the induced subgraph \( G[N(v_i) \cap \{v_{i+1}, \ldots, v_n\}] \) has no induced subgraph isomorphic to a graph in \( F \).

Thus, Dirac’s result states that the chordal graphs are precisely the graphs that admit a \( \{K_2\} \)-elimination ordering. In Section 5, we give an algorithm for computing weighted \( K_t \) clique covers on a superclass of the chordal graphs, defined as follows.

**Definition 4.** A graph \( G \) is semichordal if it admits a \( \{P_3\} \)-elimination ordering, where \( P_3 \) is the path on three vertices.

Equivalently, a graph is semichordal if it admits a vertex ordering \( v_1, \ldots, v_n \) such that for each \( i \), the subgraph induced by \( N(v_i) \cap \{v_{i+1}, \ldots, v_n\} \) is a disjoint union of complete graphs. Since \( K_2 \) is an induced subgraph of \( P_3 \), Dirac’s characterization immediately implies that every chordal graph is semichordal. On the other hand, any cycle \( C_n \) for \( n > 3 \) is a semichordal graph that is not chordal.

As semichordal graphs are defined in terms of the existence of a certain elimination ordering, it would be desirable to have a characterization of these graphs in terms of their forbidden induced subgraphs, analogous to the definition of chordal graphs. Unfortunately, we are aware of no such characterization. The following sufficient (but not necessary) condition for a graph to be semichordal was discovered by Aboulker, Charbit, Trotignon, and Vušković [1].

**Definition 5** ([1]). A graph \( G \) is a wheel if there is a vertex \( v \) of degree at least 3 such that \( G - v \) is isomorphic to a cycle. The vertex \( v \) is the center of the wheel and the subgraph \( G - v \) is the rim of the wheel. A wheel is a 3-wheel if there are three consecutive vertices \( x, y, z \) on the rim such that the center is adjacent to \( x, y, \) and \( z \).

**Theorem 5** ([1]). If \( G \) has no induced subgraph isomorphic to a 3-wheel, then \( G \) is semichordal.

In fact, [1] proves that if \( G \) has no induced subgraph isomorphic to a 3-wheel, then \( G \) satisfies a stronger property guaranteeing that a \( \{P_3\} \)-elimination ordering can be easily found. We refer the reader to [1] for more details.

## 5 Weighted Edges

In this section, we consider a weighted variant of the \( K_t \) clique cover problem. Given a graph \( G \), we assume that each \( t \)-clique \( k \subset V(G) \) is assigned a weight \( w(k) \) representing the number of times that \( S \) must be covered. Our goal is to find a multiset \( C \) of cliques in \( G \) such that each \( t \)-clique \( k \) is covered at least \( w(k) \) times. We formalize these notions as follows.

**Definition 6.** Given a graph \( G \) and an integer \( t \geq 0 \), let \( \mathcal{K}(G) \) be the family of all cliques in \( G \), and let \( \mathcal{K}_t(G) \) be the family of all \( t \)-cliques in \( G \). Let \( w: \mathcal{K}_t \rightarrow \mathbb{Z}_{\geq 0} \) be a weight function on the \( t \)-cliques of \( G \). A \((w, K_t)\)-cover of \( G \) is a function \( f: \mathcal{K}(G) \rightarrow \mathbb{Z}_{\geq 0} \) such that \( \sum_{K \in \mathcal{K}} f(K) \geq w(k) \) for all \( k \in \mathcal{K}_t(G) \), where the sum ranges over all \( K \in \mathcal{K}(G) \) with \( k \subset K \). When \( f \) is a \((w, K_t)\)-cover, we write \( \text{cost}(f) \) for the sum \( \sum_{K \in \mathcal{K}(G)} f(K) \). The \((w, K_t)\)-cover number of \( G \), written \( i_{w,t}(G) \), is the minimum value of \( \text{cost}(f) \) over all \((w, K_t)\)-covers of \( G \).
Observe that when \( w(S) = 1 \) for all \( S \in \mathcal{K}_t \), the \((w, K_t)\)-cover number of \( G \) is just the \( K_t \) clique cover number of \( G \). We also define a corresponding dual problem.

**Definition 7.** The \((w, K_t)\)-clique packing number of \( G \), written \( p_{w,t}(G) \), is the optimum value of the following integer program:

\[
\begin{align*}
\text{maximize } & \sum_{k \in \mathcal{K}_t(G)} w(k)y(k), \\
\text{subject to } & \sum_{k \subseteq K} y(k) \leq 1, \text{ for all } K \in \mathcal{K}(G), \\
& y(k) \geq 0, \text{ for all } k \in \mathcal{K}_t(G) \\
& y(k) \in \mathbb{Z}.
\end{align*}
\]

A feasible solution to this integer program is called a \((w, K_t)\)-packing. When \( y \) is a \((w, K_t)\)-packing write \( \text{val}(y) \) for \( \sum_{k \in \mathcal{K}_t(G)} w(e)y(k) \). (In some circumstances it may be ambiguous which weight function is used to calculate \( \text{val}(y) \), in which case we write \( \text{val}_w(y) \) to specify the weight function being used.)

Let \( i_{w,t}^*(G) \) and \( p_{w,t}^*(G) \) denote the fractional relaxations of \( i_{w,t}(G) \) and \( p_{w,t}(G) \), respectively. Standard LP duality gives

\[
p_{w,t}(G) \leq p_{w,t}^*(G) = i_{w,t}^*(G) \leq i_{w,t}(G).
\]

We wish to show that when \( G \) is semichordal, equality holds.

**Theorem 6.** If \( G \) is semichordal, then for all \( t \geq 1 \) and all \( w : \mathcal{K}_t(G) \to \mathbb{Z}_{\geq 0} \),

\[
i_{w,t}(G) = p_{w,t}(G).
\]

**Proof.** We adapt the argument of Scheinerman and Trenk [15]. We claim that Algorithm 1 produces a \((w, K_t)\)-cover \( f \) and a \((w, K_t)\)-packing \( y \) such that \( \text{cost}(f) = \text{val}(y) \), and that \( y(k) > 0 \) only if \( w(k) > 0 \). Our proof proceeds by induction on \( |E(G)| \). When \( |E(G)| = 0 \) it is clear that the pair of empty functions \((e, e)\) returned by Algorithm 1 has the desired property.

Now suppose that \( |E(G)| > 0 \), let \( v_1, \ldots, v_n \) be the \( \{P_3\}\)-elimination ordering used in Algorithm 1 and let \((f', y') = \text{optpair}(G', w')\). By the induction hypothesis, \( f' \) and \( y' \) are feasible for their respective integer programs, and \( \text{cost}(f') = \text{val}(y') \).

First we argue that \( f \) is feasible. Observe that \( f(k) \geq f'(k) \) for all \( k \in \mathcal{K}_t(G) \). For all \( k \in \mathcal{K}_t(G) - \mathcal{L} \), we have \( w'(k) = w(k) \), and the feasibility of \( f' \) implies that

\[
\sum_{k \subseteq K} f(K) \geq \sum_{k \subseteq K} f'(K) \geq w'(k) = w(k)
\]

for all \( k \in \mathcal{K}_t(G) - \mathcal{L} \). On the other hand, for \( k \in \mathcal{K}_t(Q_i) \), we have \( w(k) \leq w'(k) + t_i \). Since \( f' \) is feasible and \( f(Q_i^*) = t_i \), for these edges we have

\[
\sum_{k \subseteq K} f(K) = f(Q_i^*) + \sum_{k \subseteq K} f'(K) \geq t + w'(e) \geq w(e).
\]

Thus, \( f \) is feasible.

Now we argue that \( y \) is feasible. Consider any clique \( K \). If \( v_1 \notin K \), then \( y(k) = y'(k) \) for all \( t \)-cliques \( k \subseteq K \), so by the induction hypothesis, we have

\[
\sum_{k \subseteq K} y(k) = \sum_{k \subseteq K} y'(k) \leq 1,
\]
Algorithm 1 Recursive algorithm \texttt{optpair} to produce a pair \((f,y)\), where \(f\) is an optimal \((w,K_t)\)-cover and \(y\) is an optimal \((w,K_t)\)-packing.

\begin{algorithm}
\begin{algorithmic}
  \IF {G has no edges}
    \STATE Return \((e,e)\), where \(e\) is the empty function.
  \ELSE
    \STATE Let \(v_1,\ldots,v_n\) be a \(\{P_3\}\)-elimination ordering of \(G\).
    \STATE Let \(Q_1,\ldots,Q_h\) be the components of \(G[N(v_1)]\) of size at least \(t-1\). \{Each \(Q_i\) is a clique.\}
    \STATE Let \(G' = G - v_1\).
    \FORALL {\(i \in \{1,\ldots,h\}\)}
      \STATE Let \(Q_i^* = \{v\} \cup Q_i\).
      \STATE Pick \(Z_i \in K_{t-1}(Q_i)\) to maximize \(w(\{v\} \cup Z_i)\) and let \(t_i = w(\{v\} \cup Z_i)\).
    \ENDFOR
    \STATE Let \(L = K_t(Q_1) \cup \cdots \cup K_t(Q_h)\).
    \STATE Let \(w'(k) = w(k)\) for \(k \in K_t(G) - L\) and let \(w'(k) = \max\{0,w(k) - t_i\}\) for \(k \in K_t(Q_i)\).
    \STATE Let \((f',y') = \texttt{optpair}(G',w')\).
    \STATE Let \(y(k) = y'(k)\) for \(k \in K_t(G')\). \{\(f\) and \(y\) are only partially defined so far\}
    \FORALL {\(i \in \{1,\ldots,h\}\)}
      \STATE Let \(f(Q_i^*) = t_i\).
      \STATE Let \(y(\{v\} \cup Z) = 0\) for all \(Z \in K_{t-1}(Q_i) - Z_i\).
      \IF {\(t_i = 0\) or \(y'(k) > 0\) for some \(k \in K_t(Q_i)\)}
        \STATE Let \(y(\{v\} \cup Z_i) = 0\).
      \ELSE
        \STATE Let \(y(\{v\} \cup Z_i) = 1\).
      \ENDIF
    \ENDFOR
    \STATE Let \(f(Q) = 0\) for all cliques \(Q\) on which \(f\) is not yet defined.
    \STATE Return \((f,y)\).
  \ENDIF
\end{algorithmic}
\end{algorithm}
Thus, we may assume that \( v_1 \in K \). This implies that \( K \subset Q^*_i \) for some \( i \). Observe that

\[
\sum_{k \in K} y(k) \leq \sum_{Z \in K_{i-1}(Q_i)} y\{v\} \cup Z + \sum_{k \in K_i(Q_i)} y(k) = y\{v\} \cup Z_i + \sum_{k \in K_i(Q_i)} y'(k),
\]

where \( \sum_{k \in K_i(Q_i)} y'(k) \leq 1 \) by the feasibility of \( y' \). Thus, if \( y\{v\} \cup Z_i = 0 \), then the constraint for \( K \) is satisfied. The only way the algorithm allows \( y\{v\} \cup Z_i > 0 \) is when \( y'(k) = 0 \) for all \( k \in K_i(Q_i) \), in which case the constraint is again satisfied.

Next we argue that \( y(k) > 0 \) only if \( w(k) > 0 \). Consider any \( k \in K_i(G) \) with \( y(k) > 0 \). If \( v_1 \notin k \), then \( k \in K_i(G') \), so the induction hypothesis implies that \( w'(k) > 0 \). Since \( w'(k) \leq w(k) \), this implies that \( w(k) > 0 \) as well. On the other hand, if \( v_1 \in k \), then \( y(k) = 0 \) is only possible if \( k = \{v_1\} \cup Z_i \) for some \( i \) with \( t_i > 0 \). Since \( t_i = w\{v_1\} \cup Z_i = w(k) \), we again see that \( y(k) > 0 \) implies \( w(k) > 0 \).

Finally we argue that \( \text{cost}(f) = \text{val}(y) \). Let \( R \) be the set the of indices \( i \) such that \( y\{v_1\} \cup Z_i = 1 \). Observe that

\[
\text{cost}(f) = \text{cost}(f') + \sum_{i \in R} f(Q^*_i) = \text{cost}(f') + \sum_{i=1}^h t_i.
\]

By the induction hypothesis, \( \text{cost}(f') = \text{val}_{w'}(y') \). We wish to determine \( \text{val}_{w}(y') \). Observe that

\[
\text{val}_{w}(y') - \text{val}_{w'}(y') = \sum_{y'(k) = 1} [w(k) - w'(k)],
\]

and by the induction hypothesis, \( y'(k) = 1 \) implies that \( w'(k) > 0 \), so that \( w'(k) = w(k) - t_i \). Hence,

\[
\text{val}_{w}(y') - \text{val}_{w'}(y') = \sum_{y'(k) = 1} t_i = \sum_{i \notin R} t_i,
\]

where the last equality holds because \( i \notin R \) implies that either \( t_i = 0 \) or that \( y'(k) > 0 \) for some \( k \in K_i(Q_i) \), in which case feasibility of \( y' \) implies that there is exactly one \( k \) for which this is true. The remaining cliques to count in \( \text{val}(y) \) are the cliques \( \{v_i\} \cup Z_i \) where \( i \in R \). Thus,

\[
\text{val}(y) = \text{val}_{w}(y') + \sum_{i \in R} t_i = \text{val}_{w'}(y') + \sum_{i=1}^h t_i = \text{cost}(y') + \sum_{i=1}^h t_i = \text{cost}(f),
\]

as desired.

\[\blacksquare\]

6 Weighted Maximal Cliques

It is natural to consider the following further generalization of the edge clique cover problem: in addition to weighting the edges of \( G \), we also assign to each maximal clique \( K \) a weight \( w(K) \), and seek to minimize the total weight \( \sum_{K \in \mathcal{K}(G)} w(K)f(K) \) subject to the constraint \( \sum_{e \in K} f(K) \geq w(e) \) for all \( e \in E(G) \). In this section, however, we show that this generalized problem is NP-hard, even when we restrict to instances for which \( G \) is chordal and \( w(e) = 1 \) for all \( e \in E(G) \). For the remainder of this section, we assume that \( w \) weights both the edges and the maximal cliques of \( G \), and we write \( i_{w,t}(G) \) for the optimum value of the above integer program. (In fact, one could also consider a version of the general \( K_t \) clique cover problem in which both the copies of \( K_t \) and the maximal cliques are weighted, but the results of this section show that the \( t = 2 \) case is already hard.)
The basic idea is to reduce the problem of computing $i(G)$ for general graphs to the problem of computing $i_{w,t}(G)$ for chordal graphs with $w(e) = 1$ for all $e \in E(G)$; the former problem has been proved NP-hard by Kou, Stockmeyer, and Wong [7]. In fact, we will require the following stronger NP-hardness result, due to Rosgen and Stewart [13].

**Theorem 7** (Rosgen–Stewart [13]). It is NP-hard to compute $i(G)$ restricted to graphs $G$ with at most $(m + 1)n^3$ maximal cliques, where $m = |E(G)|$ and $n = |V(G)|$.

While our transformation may, on general graphs, result in an exponential increase in the problem size, it only results in a polynomial increase when restricted to the graphs of Theorem 7 with few maximal cliques.

**Lemma 4.** Let $G$ be an $n$-vertex graph with $p$ maximal cliques. There is a chordal graph $H$ with at most $n + (n + 1)p + |E(G)|$ vertices and a weighting $w$ of the edges and cliques of $G$ such that $w(e) = 1$ for all $e \in E(H)$ and $i_{w,t}(H) = i(G)$.

**Proof.** We define a graph $H$ as follows. The vertices of $H$ consist of the vertices of $G$, together with:

- For each $uv \in E(G)$, a vertex $a_{uv}$,
- For each maximal clique $Q$ of $G$, a vertex $b_Q$ and, for each $v \in Q$, a vertex $b_{Q,v}$.

It is clear that $H$ has at most $n + (n + 1)p + |E(G)|$ vertices. The edge set of $H$ is defined as follows:

- For all $u, v \in V(G)$, we have $uv \in E(H)$;
- For each $uv \in E(G)$, we have $ua_{uv}, va_{uv} \in E(H)$;
- For each maximal clique $Q$ of $G$ and each $v \in Q$, we have $vb_Q, vb_{Q,v}, b_{Q,v}b_Q \in E(H)$.

Now $H$ has a simplicial elimination order: we may eliminate all the vertices $a_{uv}$, followed by all vertices $b_{Q,v}$, followed by all vertices $b_Q$, finally followed by all the vertices of $V(G)$. Thus, $H$ is chordal.

Observe that each vertex in $V(H) - V(G)$ belongs to a unique maximal clique of $H$:

- Vertices of the form $a_{uv}$ belong to the unique maximal clique $w.a_{uv}$;
- Vertices of the form $b_Q$ belong to exactly the maximal cliques $b_Q \cup Q$ and $vb_Q. b_{Q,v}$ for $v \in Q$;
- Vertices of the form $b_{Q,v}$ belong to the unique maximal clique $vb_Q. b_{Q,v}$.

Furthermore, these are the only maximal cliques of $H$: if $K$ were a maximal clique of $H$ different from all the cliques above, then we must have $K \subset V(G)$, so that $K$ is a maximal clique in $G$; but then $K \subset K \cup \{b_K\}$, contradicting the maximality of $K$. Thus, the following weighting $w$ gives a weight to every maximal clique of $H$:

$$w(K) = \begin{cases} 
0, & \text{if some } b_{Q,v} \in K \text{ or some } a_{uv} \in K, \\
1, & \text{if } K \text{ is of the form } \{v\} \cup Q \text{ for some maximal } G\text{-clique } Q.
\end{cases}$$

Also let $w(e) = 1$ for all $e \in E(H)$. We claim that $i_{w,t}(H) = i(G)$. Given an optimal $w$-fold covering $f$ of $H$, the set $\{Q : f(Q \cup b_Q) > 0\}$ is an edge-clique covering of $G$, since for each $e \in E(G)$, the only cliques of $H$ that contain $e$ are the cliques of the form $Q \cup b_Q$ where $e \subset Q$. Thus $i_{w,t}(H) \geq i(G)$.

Conversely, given an optimal edge-clique covering $X$ of $G$, we obtain a $w$-fold covering $f$ of $H$ by adding all the cost-0 cliques of $H$; these free cliques cover all edges except the edges of $G$, which are covered by $X$. Therefore, $i_{w,t}(H) = i(G)$. ■

12
Now we deal with the issue of guaranteeing that the transformation in Lemma 4 is polynomial-time. As observed by Rosgen and Stewart [13], for any fixed polynomial $p$, there is a polynomial-time algorithm to enumerate all the maximal cliques of any $n$-vertex graph with at most $p(n)$ maximal cliques; furthermore, there is a polynomial-time algorithm to recognize whether a given graph has at most $n$ maximal cliques.

Thus, when the input instances to the transformation of Lemma 4 are restricted to graphs with at most $(m + 1)n^3$ maximal cliques (as described in Theorem 7), the transformation described in the proof runs in polynomial time. This implies that determining $i_W(G)$ for chordal $G$ is NP-hard when clique weights are permitted, even when $w(e) = 1$ for all $e \in E(G)$.

7 NP-hardness of $K_t$ clique cover problem

The decision version of the $K_t$ clique cover problem is stated below.

---

| KCC($t$) |
|----------|
| **Input:** a graph $G$, and a number $k$;  |
| **Output:** YES if $\theta_{K_t}(G) \leq k$ and NO otherwise; |

---

**Proposition 1.** The decision problem KCC($t$) is NP-complete for any constant $t \geq 1$.

**Proof.** It is obvious that KCC($t$) is in NP. We prove the NP-completeness of this problem by induction on $t$. It is known that KCC(1) is NP-complete [6]. Now suppose that $t \geq 2$ and that KCC($t - 1$) is NP-complete. We aim to show that KCC($t$) is also NP-complete.

Let $G$ be an arbitrary graph of order $n$ and $k \geq 1$. Let $G'$ be the graph obtained from $G$ by introducing

- $s = 1 + \theta_{K_t}(G)$ new vertices $\{u_1, \ldots, u_s\}$, and
- $sn$ new edges that connect the new vertices to all existing vertices of $G$.

As $\theta_{K_t}(G) \leq \binom{n}{t-1}$, the graph $G'$ has order and size polynomial in $n$. Let $k' = sk + \theta_{K_t}(G)$. We demonstrate that $\theta_{K_{t-1}}(G) \leq k$ if and only if $\theta_{K_t}(G') \leq k'$.

Indeed, suppose that $\theta_{K_{t-1}}(G) \leq k$, i.e. there is a set $A$ of at most $k$ cliques in $G$ that collectively cover all $K_{t-1}$ in $G$. Then we can cover all $K_t$ in $G'$ by a set of cliques $B$ obtained from $A$ by adding each vertex in $\{u_1, u_2, \ldots, u_s\}$ to each clique in $A$, together with a minimum set of cliques $C$ of $G$ that can cover all $K_t$ in $G$, which has size $\theta_{K_t}(G)$. In total, this cover has at most

$$s|A| + |C| \leq sk + \theta_{K_t}(G) = k'$$

cliques. Thus, if $\theta_{K_{t-1}}(G) \leq k$ then $\theta_{K_t}(G') \leq k'$. Conversely, suppose that we have a $K_t$ clique cover $D$ of $G'$ of size at most $k'$. Let $D_i$ be the subset of cliques in $D$ that contain the vertex $u_i$, for $1 \leq i \leq s$. As $u_i$ and $u_j$ are not adjacent, for $i \neq j$, $D_i$ and $D_j$ do not have any common cliques. Hence,

$$\sum_{i=1}^{s} |D_i| \leq |D| \leq k'.$$

Therefore, if $i_{\text{min}}$ is an index such that $|D_{i_{\text{min}}}| = \min_{1 \leq i \leq s} |D_i|$, then

$$|D_{i_{\text{min}}}| \leq \left[ \frac{\sum_{i=1}^{s} |D_i|}{s} \right] \leq \left[ \frac{k'}{s} \right] = \left[ \frac{sk + \theta_{K_t}(G)}{s} \right] = k,$$

where the last equality holds because $s = 1 + \theta_{K_t}(G)$. Then, by removing $u_{i_{\text{min}}}$ from all cliques in $D_{i_{\text{min}}}$, we obtain a $K_{t-1}$ clique cover of $G$ of size at most $k$. The proof follows. ■
References

[1] P. Aboulker, P. Charbit, N. Trotignon, and K. Vušković. Vertex elimination orderings for hereditary graph classes. *Discrete Math.*, 338(5):825–834, 2015.

[2] B. Bollobas. Extremal problems in graph theory. *Journal of Graph Theory*, (1):117–123, 1977.

[3] F. Chung and L. Lu. An upper bound for the Turán number $t_3(n,4)$. *Journal of Combinatorial Theory A*, 87:381–389, 1999.

[4] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.

[5] P. Erdős, A. W. Goodman, and L. Pósa. The representation of a graph by set intersections. *Canad. J. Math.*, 18(1):106–112, 1966.

[6] R. M. Karp. Reducibility among combinatorial problems. *Complexity Comput. Comput.*, 40(4):85–103, 1972.

[7] L. T. Kou, L. J. Stockmeyer, and C. K. Wong. Covering edges by cliques with regard to keyword conflicts and intersection graphs. *Commun. ACM*, 21(2):135–139, 1978.

[8] J. Lehel. Covers in hypergraphs. *Combinatorica*, 2(3):305–309, 1982.

[9] L. Lovász. On covering of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 231–236. Academic Press, New York, 1968.

[10] J. W. Moon and L. Moser. On a problem of Turán. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 7:283–286, 1962.

[11] J. Orlin. Contentment in graph theory: Covering graphs with cliques. *Indagationes Mathematicae (Proceedings)*, 80(5):406–424, 1977.

[12] F. S. Roberts. Applications of edge coverings by cliques. *Discrete Appl. Maths.*, 10(1):93–109, 1985.

[13] B. Rosgen and L. Stewart. Complexity results on graphs with few cliques. *Discrete Math. Theor. Comput. Sci.*, 9(1):127–135 (electronic), 2007.

[14] N. Sauer. A generalization of a theorem of Turán. *Res. P. No. 61, Dep. Maths.*, Calgary, 1968.

[15] E. R. Scheinerman and A. N. Trenk. On the fractional intersection number of a graph. *Graphs Combin.*, 15(3):341–351, 1999.

[16] T. H. Spencer. On the size of independent sets in hypergraphs. In *D. Jungnickel and S. A. Vanstone (Eds.), Proceedings of the Marshall Hall Conference, Vermont*, pages 263–273, 1990.

[17] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok (in Hungarian)*, 48:436–452, 1941.