POSITIVE DEFINITE FUNCTIONS AND MULTIDIMENSIONAL VERSIONS OF RANDOM VARIABLES

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Abstract. We say that a random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$ is an $n$-dimensional version of a random variable $Y$ if for any $a \in \mathbb{R}^n$ the random variables $\sum a_iX_i$ and $\gamma(a)Y$ are identically distributed, where $\gamma : \mathbb{R}^n \to [0, \infty)$ is called the standard of $X$. An old problem is to characterize those functions $\gamma$ that can appear as the standard of an $n$-dimensional version. In this paper, we prove the conjecture of Lisitsky that every standard must be the norm of a space that embeds in $L_0$. This result is almost optimal, as the norm of any finite dimensional subspace of $L_p$ with $p \in (0, 2]$ is the standard of an $n$-dimensional version ($p$-stable random vector) by the classical result of P. Lévy. An equivalent formulation is that if a function of the form $f(\|\cdot\|_K)$ is positive definite on $\mathbb{R}^n$, where $K$ is an origin symmetric star body in $\mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is an even continuous function, then either the space $(\mathbb{R}^n, \|\cdot\|_K)$ embeds in $L_0$ or $f$ is a constant function. Combined with known facts about embedding in $L_0$, this result leads to several generalizations of the solution of Schoenberg’s problem on positive definite functions.

1. Introduction

Following Eaton [E], we say that a random vector $X = (X_1, ..., X_n)$ is an $n$-dimensional version of a random variable $Y$ if there exists a function $\gamma : \mathbb{R}^n \to \mathbb{R}$, called the standard of $X$, such that $\gamma(a) > 0$ for every $a \in \mathbb{R}^n$, $a \neq 0$, and for every $a \in \mathbb{R}^n$ the random variables

$$\sum_{i=1}^n a_iX_i \quad \text{and} \quad \gamma(a)Y$$

are identically distributed. We assume that $n \geq 2$ and $P\{Y = 0\} < 1$. A problem posed by Eaton is to characterize all $n$-dimensional versions, and, in particular, characterize all functions $\gamma$ that can appear as the standard of an $n$-dimensional version.

It is easily seen [M3], [Ku] that every standard $\gamma$ is an even homogeneous of degree 1 non-negative (and equal to zero only at zero) continuous function on $\mathbb{R}^n$. This means that $\gamma = \|\cdot\|_K$ is the Minkowski
functional of some origin symmetric star body $K$ in $\mathbb{R}^n$. Recall that a closed bounded set $K$ in $\mathbb{R}^n$ is called a star body if every straight line passing through the origin crosses the boundary of $K$ at exactly two points, the origin is an interior point of $K$ and the Minkowski functional of $K$ defined by $\|x\|_K = \min\{s \geq 0 : x \in sK\}$ is a continuous function on $\mathbb{R}^n$. Note that the class of star bodies includes convex bodies containing the origin in their interior.

Eaton [E] proved that a random vector is an $n$-dimensional version with the standard $\| \cdot \|_K$ if and only if its characteristic functional has the form $f(\| \cdot \|_K)$, where $K$ is an origin symmetric star body in $\mathbb{R}^n$ and $f$ is an even continuous non-constant function on $\mathbb{R}$ (see also [K3, Lemma 6.1]). By Bochner’s theorem, this means that the function $f(\| \cdot \|_K)$ is positive definite. Recall that a complex valued function $f$ defined on $\mathbb{R}^n$ is called positive definite on $\mathbb{R}^n$ if, for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$, we have
\[
\sum_{i=1}^m \sum_{j=1}^m c_i \overline{c_j} f(x_i - x_j) \geq 0.
\]

Thus, Eaton’s problem is equivalent to characterizing the classes $\Phi(K)$ consisting of even continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(\| \cdot \|_K)$ is a positive definite function on $\mathbb{R}$. In particular, $\| \cdot \|_K$ appears as the standard of an $n$-dimensional version if and only if the class $\Phi(K)$ is non-trivial, i.e. contains at least one non-constant function. In some places throughout the paper we write $\Phi(E_K)$ instead of $\Phi(K)$, where $E_K = (\mathbb{R}^n, \| \cdot \|_K)$ is the space whose unit ball is $K$.

The problem of characterization of positive definite norm dependent functions has a long history and goes back to the work of Lévy and Schoenberg in the 1930s. Lévy [Le] proved that, for any finite dimensional subspace $(\mathbb{R}^n, \| \cdot \|)$ of $L_q$ with $0 < q \leq 2$, the function $g = \exp(-\| \cdot \|_q)$ is positive definite on $\mathbb{R}^n$, and any random vector $X = (X_1, ..., X_n)$ in $\mathbb{R}^n$, whose characteristic functional is $g$, satisfies the property (1). This result gave a start to the theory of stable processes that has numerous applications to different areas of mathematics. The concept of an $n$-dimensional version is a generalization of stable random vectors.

In 1938, Schoenberg [S1,S2] found a connection between positive definite functions and the embedding theory of metric spaces. In particular, Schoenberg [S1] posed the problem of finding the exponents $0 < p \leq 2$ for which the function $\exp(-\| \cdot \|_p^q)$ is positive definite on $\mathbb{R}^n$, where
\[
\|x\|_q = (|x_1|^q + ... + |x_n|^q)^{1/q}
\]
is the norm the space $\ell^q_n$ with $2 < q \leq \infty$. This problem had been open for more than fifty years. For $q = \infty$, the problem was solved in 1989 by Misiewicz [M2], and for $2 < q < \infty$, the answer was given in [K1] in 1991 (note that, for $1 \leq p \leq 2$, Schoenberg’s question was answered earlier by Dor [D], and the case $n = 2, 0 < p \leq 1$ was established in [F], [H], [L]). The answers turned out to be the same in both cases: the function $\exp(-\| \cdot \|^p_q)$ is not positive definite for any $p \in (0, 2]$ if $n \geq 3$, and for $n = 2$ the function is positive definite if and only if $0 < p \leq 1$. Different and independent proofs of Schoenberg’s problems were given by Lisitsky [Li1] and Zastavnyi [Z1, Z2] shortly after the paper [K1] appeared. For generalizations of the solution of Schoenberg’s problem, see [KL].

The solution of Schoenberg’s problem can be interpreted in terms of isometric embeddings of normed spaces. In fact, the result of Bre-tagnolle, Dacunha-Castelle and Krivine [BDK] shows that a normed space embeds isometrically in $L_p$, $0 < p \leq 2$ if and only if the function $\exp(-\| \cdot \|^p_\infty)$ is positive definite. Hence, the answer to Schoenberg’s problem means that the spaces $\ell^q_n$, $q > 2$, $n \geq 3$ do not embed isometrically in $L_p$ with $0 < p \leq 2$.

The classes $\Phi(K)$ have been studied by a number of authors. Schoenberg [S2] proved that $f \in \Phi(\ell^2_2)$ if and only if

$$f(t) = \int_0^\infty \Omega_n(tr) \, d\lambda(r)$$

where $\Omega_n(| \cdot |_2)$ is the Fourier transform of the uniform probability measure on the sphere $S^{n-1}$, $| \cdot |_2$ is the Euclidean norm in $\mathbb{R}^n$, and $\lambda$ is a finite measure on $[0, \infty)$. In the same paper, Schoenberg proved an infinite dimensional version of this result: $f \in \Phi(\ell^2_2)$ if and only if

$$f(t) = \int_0^\infty \exp(-t^2r^2) \, d\lambda(r).$$

Bretagnolle, Dacunha-Castelle and Krivine [BDK] proved a similar result for the classes $\Phi(\ell_q)$ for all $q \in (0, 2)$ (one just has to replace 2 by $q$ in the formula), and showed that for $q > 2$ the classes $\Phi(\ell_q)$ (corresponding to infinite dimensional $\ell_q$-spaces) are trivial, i.e. contain constant functions only. Cambanis, Keener and Simons [CKS] obtained a similar representation for the classes $\Phi(\ell^1_n)$. Richards [R] and Gneiting [G] partially characterized the classes $\Phi(\ell^q_n)$ for $0 < q < 2$. Aharoni, Maurey and Mityagin [AMM] proved that if $E$ is an infinite dimensional Banach space with a symmetric basis $\{e_n\}_{n=1}^\infty$ such that

$$\lim_{n \to \infty} \frac{\|e_1 + \cdots + e_n\|}{n^{1/2}} = 0,$$
then the class $\Phi(E)$ is trivial. Misiewicz [M2] proved that for $n \geq 3$ the classes $\Phi(\ell^n_\infty)$ are trivial, and Lisitsky [Li1] and Zastavnyi [Z1], [Z2] showed the same for the classes $\Phi(\ell^n_\infty)$, $q > 2$, $n \geq 3$. One can find more related results and references in [M3], [K3].

In all the results mentioned above the classes $\Phi(K)$ appear to be non-trivial only if $K$ is the unit ball of a subspace of $L_q$ with $0 < q \leq 2$. An old conjecture, explicitly formulated for the first time by Misiewicz [M1], is that the class $\Phi(K)$ can be non-trivial only in this case. A slightly weaker conjecture was formulated by Lisitsky [Li2]: if the class $\Phi(K)$ is non-trivial, then the space $(\mathbb{R}^n, \| \cdot \|_K)$ embeds in $L_0$. The concept of embedding in $L_0$ was introduced and studied in [KKYY], the original conjecture of Lisitsky was in terms of the representation (2):

Definition 1. We say that a space $(\mathbb{R}^n, \| \cdot \|_K)$ embeds in $L_0$ if there exist a finite Borel measure $\mu$ on the sphere $S^{n-1}$ and a constant $C \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^n$,

$$\ln \|x\|_K = \int_{S^{n-1}} \ln |(x, \xi)| \, d\mu(\xi) + C. \quad (2)$$

It is quite easy to confirm the conjectures of Misiewicz and Lisitsky under additional assumptions that $f$ or its Fourier transform have finite moments of certain orders; see [Mi1], [Ku], [Li2], [K4].

In this article we prove the conjecture of Lisitsky in its full strength:

Theorem 1. Let $K$ be an origin symmetric star body in $\mathbb{R}^n$, $n \geq 2$ and suppose that there exists an even non-constant continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(\| \cdot \|_K)$ is a positive definite function on $\mathbb{R}^n$. Then the space $(\mathbb{R}^n, \| \cdot \|_K)$ embeds in $L_0$.

Corollary 1. If a function $\gamma$ is the standard of an $n$-dimensional version of a random variable, then there exists an origin symmetric star body $K$ in $\mathbb{R}^n$ such that $\gamma = \| \cdot \|_K$ and the space $(\mathbb{R}^n, \| \cdot \|_K)$ embeds in $L_0$.

In the last section of the paper we use known results about embedding in $L_0$ to point out rather general classes of normed spaces for which the classes $\Phi$ are trivial and whose norms cannot serve as the standard of an $n$-dimensional version.

2. Proof of Theorem 1

As usual, we denote by $S(\mathbb{R}^n)$ the space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^n$ (Schwartz test functions), and by
\[ S'(\mathbb{R}^n) \text{ the space of distributions over } S(\mathbb{R}^n). \text{ If } \phi \in S(\mathbb{R}^n) \text{ and } f \in S'(\mathbb{R}^n) \text{ is a locally integrable function with power growth at infinity, then the action of } f \text{ on } \phi \text{ is defined by} \]
\[ \langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x) \, dx. \]

We say that a distribution is positive (negative) outside of the origin in \( \mathbb{R}^n \) if it assumes non-negative (non-positive) values on non-negative test functions with compact support outside of the origin.

The Fourier transform of a distribution \( f \) is defined by \( \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \) for every test function \( \phi \). A distribution is positive definite if its Fourier transform is a positive distribution.

We use the following Fourier analytic characterization of embedding in \( L_0 \) proved in \([KKYY, \text{Th.3.1}]\):

**Proposition 1.** Let \( K \) be an origin symmetric star body in \( \mathbb{R}^n \). The space \( (\mathbb{R}^n, \| \cdot \|_K) \) embeds in \( L_0 \) if and only if the Fourier transform of \( \ln \|x\|_K \) is a negative distribution outside of the origin in \( \mathbb{R}^n \).

Now we are ready to start the proof of Theorem 1.

**Proof of Theorem 1.** We write \( \| \cdot \| \) instead of \( \| \cdot \|_K \). By Bochner’s theorem, the function \( f(\| \cdot \|) \) is the Fourier transform of a finite measure \( \mu \) on \( \mathbb{R}^n \). We can assume that \( f(0) = 1 \), and, correspondingly, \( \mu \) is a probability measure. The function \( f \) is positive definite on \( \mathbb{R} \), as the restriction of a positive definite function, therefore, \( |f(t)| \leq f(0) = 1 \) for every \( t \in \mathbb{R} \) (see \([VTC, \text{p.188}]\)).

Let \( \phi \) be an even non-negative test function supported outside of the origin in \( \mathbb{R}^n \). For every fixed \( t > 0 \), the function \( f(t\| \cdot \|) \) is positive definite on \( \mathbb{R}^n \), so
\[ \int_{\mathbb{R}^n} f(t\|x\|)\phi(x) \, dx = \langle (f(t\| \cdot \|))^{\wedge}, \phi(x) \rangle \geq 0. \tag{3} \]

For any \( \varepsilon \in (0, 1) \), the integral
\[ g(\varepsilon) = \int_{\mathbb{R}^n} \left( \int_0^\infty t^{-1+\varepsilon} f(t\|x\|) \, dt + \int_1^\infty t^{-1-\varepsilon} f(t\|x\|) \, dt \right) \hat{\phi}(x) \, dx \tag{4} \]
converges absolutely, because \( f \) is bounded by 1 and the function in parentheses is bounded by \( 2/\varepsilon \). By the Fubini theorem,
\[ g(\varepsilon) = \int_0^1 t^{-1+\varepsilon} \left( \int_{\mathbb{R}^n} f(t\|x\|) \hat{\phi}(x) \, dx \right) \, dt \]
\[ + \int_1^\infty t^{-1-\varepsilon} \left( \int_{\mathbb{R}^n} f(t\|x\|) \hat{\phi}(x) dx \right) dt, \]

so by (3) the function \( g \) is non-negative:

\[ g(\varepsilon) \geq 0 \text{ for every } \varepsilon \in (0, 1). \] (5)

Now we study the behavior of the function \( g \), as \( \varepsilon \to 0 \). We have

\[ g(\varepsilon) = \int_{\mathbb{R}^n} \left( \varepsilon \int_0^{\|x\|-\varepsilon} t^{-1+\varepsilon} f(t) \, dt + \|x\| \varepsilon \int_{\|x\|}^\infty t^{-1-\varepsilon} f(t) \, dt \right) \hat{\phi}(x) dx \]

\[ = \int_{\mathbb{R}^n} \frac{\|x\|^{-\varepsilon} - 1}{\varepsilon} \left( \varepsilon \int_0^{\|x\|} t^{-1+\varepsilon} f(t) \, dt \right) \hat{\phi}(x) dx \] (6)

\[ + \int_{\mathbb{R}^n} \frac{\|x\|^\varepsilon - 1}{\varepsilon} \varepsilon \left( \varepsilon \int_{\|x\|}^\infty t^{-1-\varepsilon} f(t) \, dt \right) \hat{\phi}(x) dx \] (7)

\[ + \int_{\mathbb{R}^n} \left( \int_0^{\|x\|} t^{-1+\varepsilon} f(t) \, dt + \int_{\|x\|}^\infty t^{-1-\varepsilon} f(t) \, dt \right) \hat{\phi}(x) dx. \] (8)

We write

\[ g(\varepsilon) = u(\varepsilon) + v(\varepsilon) + w(\varepsilon), \]

where \( u, v, w \) are the functions defined by (6), (7) and (8), respectively.

We start with the function \( w \).

**Lemma 1.**

\[ \lim_{\varepsilon \to 0} w(\varepsilon) = 0. \]

**Proof:** We can assume that \( \varepsilon < 1/2 \). Fix \( a > 0 \). Since \( \hat{\phi} \) is supported outside of the origin, we have \( \int_{\mathbb{R}^n} \hat{\phi}(x) dx = 0 \) and

\[ \int_{\mathbb{R}^n} \left( \int_0^a t^{-1+\varepsilon} f(t) \, dt + \int_a^\infty t^{-1-\varepsilon} f(t) \, dt \right) \hat{\phi}(x) dx = 0, \]

because the expression in parentheses is a constant. Subtracting this from (8) we get

\[ w(\varepsilon) = \int_{\mathbb{R}^n} \left( \int_a^{\|x\|} (t^{-1+\varepsilon} - t^{-1-\varepsilon}) f(t) \, dt \right) \hat{\phi}(x) dx. \]

Now for some \( \theta(t, \varepsilon) \in [0, 2\varepsilon] \),

\[ t^{-1-\varepsilon} |t^{2\varepsilon} - 1| = 2\varepsilon t^{-1-\varepsilon} \theta(t, \varepsilon) |\ln t| \]

\[ \leq 2\varepsilon (1 + a^{-3/2} + \|x\|^{-3/2})(|\ln a| + |\ln \|x\||), \]
so

\[ |w(\varepsilon)| \leq 2\varepsilon \int_{\mathbb{R}^n} ||x|| - a| (1 + a^{-3/2} + ||x||^{-3/2})(| \ln a| + | \ln ||x|||)|\hat{\phi}(x)|dx. \]

By the definition of a star body, \( K \) is bounded and contains a Euclidean ball with center at the origin, so there exist constants \( c, d > 0 \) so that for every \( x \in \mathbb{R}^n \)

\[ c|x|_2 \leq ||x|| \leq d|x|_2, \]

where \( | \cdot |_2 \) is the Euclidean norm in \( \mathbb{R}^n \). Note that \( n \geq 2 \) so \( | \cdot |_2^{-3/2} \)

is a locally integrable function on \( \mathbb{R}^n, \ n \geq 2 \). Also \( \hat{\phi} \) is a test function and decreases at infinity faster than any power of the Euclidean norm. These facts, in conjunction with (10), imply that the integral in the right-hand side of (9) converges, which proves the lemma. \( \square \)

We need the following elementary and well known fact.

**Lemma 2.** Let \( h \) be a bounded integrable continuous at 0 function on \([0, A], \ A > 0\). Then

\[ \lim_{\varepsilon \to 0} \varepsilon \int_0^A t^{-1+\varepsilon} h(t)dt = h(0). \]

**Proof:** We can assume that \( \varepsilon < 1 \). We have

\[ \varepsilon \int_0^A t^{-1+\varepsilon} h(t)dt \]

\[ = \varepsilon \int_0^\varepsilon t^{-1+\varepsilon} (h(t) - h(0))dt + \varepsilon h(0) \int_0^\varepsilon t^{-1+\varepsilon} dt + \varepsilon \int_\varepsilon^A t^{-1+\varepsilon} h(t)dt. \]

The first summand is less or equal to

\[ \varepsilon \max_{t \in [0, \varepsilon]} |h(t) - h(0)| \to 0, \ \text{as} \ \varepsilon \to 0, \]

because \( h \) is continuous at 0. The second summand is equal to

\[ h(0) \varepsilon^\varepsilon \to h(0), \ \text{as} \ \varepsilon \to 0. \]

The third summand is less or equal to

\[ |A^\varepsilon - \varepsilon^\varepsilon| \max_{t \in [0, A]} |h(t)| \to 0, \ \text{as} \ \varepsilon \to 0. \ \square \]
Now we compute the limit at infinity of the function
\[ u(\varepsilon) = \int_{\mathbb{R}^n} \left( \left\| x \right\|^{\varepsilon} - 1 \right) \varepsilon \left( \int_0^{t^{\varepsilon}} f(t) \, dt \right) \hat{\phi}(x) \, dx. \]

**Lemma 3.**
\[ \lim_{\varepsilon \to 0} u(\varepsilon) = -f(0) \int_{\mathbb{R}^n} \ln \left\| x \right\| \hat{\phi}(x) \, dx. \]

**Proof:** Using the estimates
\[ \left| \left\| x \right\|^{\varepsilon} - 1 \right| = \frac{1}{\varepsilon} \int_0^\varepsilon \left\| x \right\|^{-\theta} \ln \left\| x \right\| \, d\theta \leq \ln \left\| x \right\| (1 + \left\| x \right\|^{-1}) \]
and
\[ \varepsilon \int_0^\varepsilon t^{-1+\varepsilon} f(t) \, dt \leq \left\| x \right\|^\varepsilon \leq \left\| x \right\| + 1, \]
we see that the functions under the integral over $\mathbb{R}^n$ in $u(\varepsilon)$ are dominated by an integrable function
\[ \left| \ln \left\| x \right\| (1 + \left\| x \right\|^{-1}) (\left\| x \right\| + 1) \hat{\phi}(x) \right| \]
of the variable $x$ on $\mathbb{R}^n$. Clearly, for $x \neq 0$,
\[ \lim_{\varepsilon \to 0} \frac{\left\| x \right\|^{-\varepsilon} - 1}{\varepsilon} = -\ln \left\| x \right\|. \]
Also, by Lemma 2, for every $x \in \mathbb{R}^n, x \neq 0$
\[ \lim_{\varepsilon \to 0} \varepsilon \int_0^\varepsilon t^{-1+\varepsilon} f(t) \, dt = f(0) = 1, \]
so the functions under the integral by $x$ in $u(\varepsilon)$ converge pointwise to $-\ln \left\| x \right\| \hat{\phi}(x)$. The result follows from the dominated convergence theorem. \qed

Now recall that
\[ v(\varepsilon) = \int_{\mathbb{R}^n} \left( \left\| x \right\|^{\varepsilon} - 1 \right) \varepsilon \left( \int_{\left\| x \right\|}^{\infty} t^{-1-\varepsilon} f(t) \, dt \right) \hat{\phi}(x) \, dx. \]
We have
\[ \varepsilon \int_{\left\| x \right\|}^{\infty} t^{-1-\varepsilon} f(t) \, dt = \varepsilon \int_0^{1/\left\| x \right\|} t^{-1+\varepsilon} f(1/t) \, dt. \]
The difficulty is that we cannot apply Lemma 2 to compute the limit of the right-hand side of the latter equality, because the function $f(1/t)$ may be discontinuous at zero. However, we can avoid this difficulty as follows:
Lemma 4. There exist a sequence \( \varepsilon_k \to 0 \) and a number \( c < 1 \) such that
\[
\lim_{k \to \infty} v(\varepsilon_k) = c \int_{\mathbb{R}^n} \ln \|x\| \phi(x) \, dx.
\]

Proof: By a dominated convergence argument, similar to the one used in the previous lemma, it is enough to prove that there exist a sequence \( \varepsilon_k \to 0 \) and a number \( c < 1 \) such that for every \( x \in \mathbb{R}^n, x \neq 0 \)
\[
\lim_{k \to \infty} \varepsilon_k \int_{\|x\|}^{\infty} t^{-1-\varepsilon_k} f(t) \, dt = c.
\]

For every \( x \neq 0 \) we have
\[
\left| \varepsilon \int_{1/\varepsilon}^{\|x\|} t^{-1-\varepsilon} f(t) \, dt \right| \leq \|x\|^{-\varepsilon} - \varepsilon^\varepsilon \to 0, \quad \text{as} \quad \varepsilon \to 0,
\]
so it is enough to find a sequence \( \varepsilon_k \) and a number \( c < 1 \) such that
\[
\lim_{k \to \infty} \psi(\varepsilon_k) = c < 1,
\]
where
\[
\psi(\varepsilon) = \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t) \, dt.
\]

Since the function \( \psi \) is bounded by 1 on \((0, 1)\), it suffices to prove that \( \psi(\varepsilon) \) cannot converge to 1, as \( \varepsilon \to 0 \).

Suppose that, to the contrary, \( \lim_{\varepsilon \to 0} \psi(\varepsilon) = 1 \). We use the following result from [VTC, p. 205]: if \( \mu \) is a probability measure on \( \mathbb{R}^n \) and \( \gamma \) is the standard Gaussian measure on \( \mathbb{R}^n \), then for every \( t > 0 \)
\[
\mu\{x \in \mathbb{R}^n : |x|_2 > 1/t\} \leq 3 \int_{\mathbb{R}^n} (1 - \hat{\mu}(ty)) \, d\gamma(y), \quad (11)
\]
where \( |\cdot|_2 \) is the Euclidean norm on \( \mathbb{R}^n \). Let \( \mu \) be the measure satisfying \( \hat{\mu} = f(\| \cdot \|) \). For every \( \varepsilon \in (0, 1) \), integrating (11) we get
\[
\varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} \mu\{x \in \mathbb{R}^n : |x|_2 > 1/t\} \, dt
\]
\[
\leq \int_{\mathbb{R}^n} \left( \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t|y|)) \, dt \right) \, d\gamma(y). \quad (12)
\]
Now
\[
\varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} \mu\{x \in \mathbb{R}^n : |x|_2 > 1/t\} \, dt
\]
\[
= \varepsilon \int_{0}^{\varepsilon} t^{-1+\varepsilon} \mu\{x \in \mathbb{R}^n : |x|_2 > t\} \, dt.
\]
and, by Lemma 2, the limit of the left-hand side of (12) as $\varepsilon \to 0$ is equal to $\mu(\mathbb{R}^n \setminus \{0\})$.

On the other hand, the functions

$$h_\varepsilon(y) = \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t\|y\|))dt$$

are uniformly (with respect to $\varepsilon$) bounded by 2. Write these functions as

$$h_\varepsilon(y) = \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon}(1 - f(t\|y\|))dt = \varepsilon^\varepsilon - \|y\|^\varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt$$

$$= \varepsilon^\varepsilon - (\|y\|^\varepsilon - 1)\varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt$$

$$- \varepsilon \int_{\|y\|/\varepsilon}^{1/\varepsilon} t^{-1-\varepsilon} f(t)dt - \varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt.$$ 

For every $y \neq 0$

$$\left| (\|y\|^\varepsilon - 1)\varepsilon \int_{\|y\|/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt \right| \leq \|y\| \varepsilon - 1 \left( \frac{\|y\|}{\varepsilon} \right)^{-\varepsilon} \to 0, \text{ as } \varepsilon \to 0,$$

$$\left| \varepsilon \int_{\|y\|/\varepsilon}^{1/\varepsilon} t^{-1-\varepsilon} f(t)dt \right| \leq |\varepsilon^\varepsilon - (\|y\|/\varepsilon)^{-\varepsilon}| \to 0, \text{ as } \varepsilon \to 0,$$

and by our assumption

$$\varepsilon \int_{1/\varepsilon}^{\infty} t^{-1-\varepsilon} f(t)dt = \psi(\varepsilon) \to 1, \text{ as } \varepsilon \to 0.$$ 

Therefore, the functions $h_\varepsilon$ converge to zero pointwise as $\varepsilon \to 0$ and are uniformly bounded by a constant. By the dominated convergence theorem, the limit of the right-hand side of (12) is equal to 0, as $\varepsilon \to 0$.

Sending $\varepsilon \to 0$ in (12), we get $\mu(\mathbb{R}^n \setminus \{0\}) = 0$, therefore the probability measure $\mu$ is a unit atom at the origin and $f$ is a constant function, which contradicts to the assumption of Theorem 1.

\[\square\]

**End of the proof of Theorem 1:** Let $\varepsilon_k$ be the sequence from Lemma 4. Recall that $g$ is a non-negative function (see (5)). By Lemmas 1, 3, 4,

$$0 \leq \lim_{k \to \infty} g(\varepsilon_k) = \lim_{k \to \infty} (u + v + w)(\varepsilon_k) = (-1 + c) \int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x)dx,$$

where $c < 1$. Therefore,

$$\langle (\ln \| \cdot \|)\hat{\phi}, \phi \rangle = \int_{\mathbb{R}^n} \ln \|x\| \hat{\phi}(x)dx \leq 0$$
for every even non-negative test function $\phi$ supported outside of the origin. By Proposition 1, $(\mathbb{R}^n, \| \cdot \|)$ embeds in $L_0$. □

3. Examples

The concept of embedding of a normed space in $L_0$ was studied in [KKYY]. In particular, it was proved in [KKYY, Th.6.7] that

**Proposition 2.** Every finite dimensional subspace of $L_p$, $0 < p \leq 2$ embeds in $L_0$.

On the other hand, as proved in [KKYY, Th.6.3],

**Proposition 3.** If $(\mathbb{R}^n, \| \cdot \|_K)$ embeds in $L_0$, it also embeds in $L_p$ for every $-n < p < 0$.

The definition and properties of embeddings in $L_p$, $p < 0$ and their connections with geometry can be found [K3, Ch. 6]. Propositions 2 and 3 confirm the place of $L_0$ in the scale of $L_p$-spaces. Speaking informally, the space $L_0$ is larger than every $L_p$, $p \in (0, 2)$, but smaller than every $L_p$, $p < 0$.

There are many examples of normed spaces that embed in $L_0$, but don’t embed in $L_p$, $p \in (0, 2)$ (see [KKYY, Th. 6.5]). In particular, the spaces $\ell^q$, $q > 2$ have this property. In fact, every three dimensional normed space embeds in $L_0$ (see [KKYY, Corollary 4.3]). However, starting from dimension 4, there are many normed spaces that do not embed in $L_0$. The following result from [K3, Th. 4.19] essentially shows that a normed space with dimension greater than 4 does not embed in $L_0$ if the second derivative of its norm at zero in at least one direction is equal to 0.

**Proposition 4.** Let $n \geq 4$, $-n < p < 0$ and let $X = (\mathbb{R}^n, \| \cdot \|)$ be an $n$-dimensional normed space with a normalized basis $e_1, \ldots, e_n$ so that:

(i) For every fixed $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$, the function

$$x_1 \mapsto \|x_1 e_1 + \sum_{i=2}^n x_i e_i\|$$

has a continuous second derivative everywhere on $\mathbb{R}$, and

$$\|x\|_{x_1}''(0, x_2, \ldots, x_n) = \|x\|_{x_1''}''(0, x_2, \ldots, x_n) = 0,$$

where $\|x\|_{x_1}'$ and $\|x\|_{x_1''}$ stand for the first and second partial derivatives by $x_1$ of the norm $\|x_1 e_1 + \cdots + x_n e_n\|$.
(ii) There exists a constant $C$ so that, for every $x_1 \in \mathbb{R}$ and every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ with $\|x_2 e_2 + \cdots + x_n e_n\| = 1$, one has

$$\|x\|''_{x_1^2}(x_1, x_2, \ldots, x_n) \leq C.$$  

(iii) Convergence in the limit

$$\lim_{x_1 \to 0} \|x\|''_{x_1^2}(x_1, x_2, \ldots, x_n) = 0$$

is uniform with respect to $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ satisfying the condition $\|x_2 e_2 + \cdots + x_n e_n\| = 1$.

Then the space $(\mathbb{R}^n, \| \cdot \|)$ does not embed in $L_0$.

Proof: It was proved in [K3, Th. 4.19] that under the assumptions of Proposition 4 the function $\| \cdot \|^{-p}_K$ represents a positive definite distribution if and only if $p \in (n-3, n]$. By [K3, Th. 6.15] the space $(\mathbb{R}^n, \| \cdot \|_K)$ does not embed in $L_p$, $p \in (-1,0)$, so it also does not embed in $L_0$ by Proposition 3. The result follows from Theorem 1. \qed

From Proposition 4 and Theorem 1 we immediately get

**Corollary 2.** If a normed space $(\mathbb{R}^n, \| \cdot \|)$, $n \geq 4$ satisfies the conditions of Proposition 4, then a function of the form $f(\| \cdot \|)$ can be positive definite only if $f$ is a constant function. The norm of such a space cannot appear as the standard of an $n$-dimensional version.

Let us give several examples of spaces satisfying the conditions of Proposition 4. For normed spaces $X$ and $Y$ and $q \in \mathbb{R}$, $q \geq 1$, the $q$-sum $(X \oplus Y)_q$ of $X$ and $Y$ is defined as the space of pairs $\{(x, y) : x \in X, y \in Y\}$ with the norm

$$\|(x, y)\| = (\|x\|_X^q + \|y\|_Y^q)^{1/q}.$$  

It was proved in [K2, Th 2] that such spaces with $q > 2$ satisfy the conditions of Proposition 4 provided that the dimension of $X$ is greater or equal to 3.

Another example is that of Orlicz spaces. Recall that an *Orlicz function* $M$ is a non-decreasing convex function on $[0, \infty)$ such that $M(0) = 0$ and $M(t) > 0$ for every $t > 0$. The norm $\| \cdot \|_M$ of the $n$-dimensional Orlicz space $\ell^n_M$ is defined implicitly by the equality

$$\sum_{k=1}^n M(|x_k|/\|x\|_M) = 1, \ x \in \mathbb{R}^n \setminus \{0\}.$$  

As shown in [K2, Th 3], the spaces $\ell^n_M$, $n \geq 4$ satisfy the conditions of Proposition 4 if the Orlicz function $M \in C^2([0, \infty))$ is such that $M'(0) = M''(0) = 0$. 


Corollary 3. If a normed space \((\mathbb{R}^n, \| \cdot \|)\) contains a subspace isometric to \((X \oplus Y)_q\), where \(q > 2\) and the dimension of \(X\) is at least 3, or contains an Orlicz space \(\ell^4_M\), where \(M\) is an Orlicz function such that \(M \in C^2([0, \infty))\) and \(M'(0) = M''(0) = 0\), then a function of the form \(f(\| \cdot \|)\) can be positive definite only if \(f\) is a constant function.

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