Irreducible forms for the metric variations of the action terms of sixth-order gravity and approximated stress–energy tensor

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Abstract

We provide irreducible expressions for the metric variations of the gravitational action terms constructed from the 17 curvature invariants of order six in the derivatives of the metric tensor, i.e., from the geometrical terms appearing in the diagonal heat-kernel or Gilkey–DeWitt coefficient $a_3$. We then express, for a four-dimensional spacetime, the approximated stress–energy tensor constructed from the renormalized DeWitt–Schwinger effective action associated with a massive scalar field. We also construct, for higher dimensional spacetimes, the infinite counterterms of order six in the derivatives of the metric tensor appearing on the left-hand side of Einstein equations as well as the contribution associated with the cubic Lovelock gravitational action. In the appendix, we provide a list of geometrical relations we have used and which are more generally helpful for calculations in two-loop quantum gravity in a four-dimensional background or for calculations in one-loop quantum gravity in higher dimensional background. We also obtain the approximated stress–energy tensors associated with a massive spinor field and a massive vector field propagating in a four-dimensional background.

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1. Introduction

Calculations which must be carried out in field theories defined on curved spacetimes are in general highly non-trivial. This is more particularly true in the context of renormalization of quantum fields and of quantum gravity but, even at the classical level, analogous difficulties can appear, for example, in the context of the radiation reaction problem of gravitational wave
theory. This is mainly due to the systematic occurrence, in these calculations, of Riemann polynomials (i.e., polynomials formed from the Riemann tensor by covariant differentiation, multiplication and contraction) whose complexity, degree and number rapidly increase with the precision of the approximations needed or with the dimension of the gravitational background considered. The results of these calculations may be moreover darkened because of the non-uniqueness of their final forms which moreover complicates the comparison between the works completed by different authors. Indeed, the symmetries of the Riemann tensor as well as Bianchi identities are not used in a uniform manner and monomials formed from the Riemann tensor may be linearly dependent in non-trivial ways. In a beautiful and very useful article [1], Fulling, King, Wybourne and Cummings (FKWC) have proposed to cure this last problem by expanding systematically the Riemann polynomials encountered in calculations on standard bases constructed from group theoretical considerations. They have also displayed such bases for scalar Riemann polynomials of order eight or less in the derivatives of the metric tensor and for tensorial Riemann polynomials of order six or less.

In section 2, we shall use FKWC results to provide irreducible expressions for the metric variations (i.e., for the functional derivatives with respect to the metric tensor) of the action terms associated with the 17 basis elements for the scalar Riemann polynomials of order six in the derivatives of the metric tensor (the so-called curvature invariants of order six). We also provide the irreducible expression for the cubic Lovelock tensor, i.e., the metric variation of the cubic Lovelock gravitational action [2, 3]. The elements of the scalar FKWC-basis previously considered appear in the expression of the diagonal heat-kernel (or Gilkey [4–6]) coefficient \( a_3 \) and therefore in the unrenormalized or renormalized DeWitt–Schwinger effective action [7–12] associated with massive fields propagating in curved spacetime. As a consequence, our results will permit us to unambiguously express, in section 3, for a four-dimensional spacetime, the approximated stress–energy tensor constructed from the renormalized effective action associated with a massive scalar field as well as to construct, for an arbitrary six-dimensional spacetime, the infinite counterterms of order six in the derivatives of the metric tensor which appear on the left-hand side of the bare Einstein equations. In a brief conclusion (section 4), we shall consider some possible immediate prolongations of our present work. Finally, in the appendix, we shall provide a list of geometrical relations we have used (these relations are more generally helpful for calculations in two-loop quantum gravity in a four-dimensional background or for calculations in one-loop quantum gravity in higher dimensional background) and we shall briefly extend the result obtained in section 3 by providing, in the large mass limit, simplified expressions for the approximated stress–energy tensors associated with a massive spinor field and a massive vector field propagating in a four-dimensional background.

All our results are obtained by using the geometrical conventions of Hawking and Ellis [13] as far as the definitions of the scalar curvature \( R \), the Ricci tensor \( R_{pq} \) and the Riemann tensor \( R_{pqrs} \) are concerned and the commutation of covariant derivatives in the form

\[
T^{p...q...rs} - T^{p...q...sr} = + R^{p...}_{tr} T^t..._{q...r} + \cdots - R^{p...}_{qs} T^t..._{r...q} - \cdots.
\]

We use furthermore the FKWC notation \( \mathcal{R}^r_{s,q} \) and \( \mathcal{R}^r_{\lambda_1...\lambda_s} \); \( \mathcal{R}^r_{s,q} \) denotes the space of Riemann polynomials of rank \( r \) (number of free indices), order \( s \) (number of differentiations of the metric tensor) and degree \( q \) (number of factors \( \nabla^p R^r_{...} \)) while \( \mathcal{R}^r_{\lambda_1...\lambda_s} \) denotes the space of Riemann polynomials of rank \( r \) spanned by contractions of products of the type \( \nabla^{\lambda_1} R^r_{...} \). We refer to the FKWC article [1] for more precision on this notation and rigour on the subject.
2. Irreducible forms for the metric variations of the action terms constructed from 17 curvature invariants of order six in the derivatives of the metric tensor and for the cubic Lovelock tensor

In this section, we consider spacetime as an arbitrary $D$-dimensional pseudo-Riemannian manifold $(\mathcal{M}, g_{ab})$ and we assume that its dimension $D$ is sufficiently high in order to avoid any degeneracy of the Riemann tensor as well as any topological constraint associated with Euler–Gauss–Bonnet–Chern–Lovelock densities and numbers. Later, we shall drop this strong hypothesis.

2.1. FKWC-basis for Riemann polynomials of order six and rank zero

The most general expression of a gravitational Lagrangian of order six in the derivatives of the metric tensor is obtained by expanding it on the FKWC-basis for Riemann polynomials of order six and rank zero. This basis consists of the 17 following elements [1]:

$$
\mathcal{R}_6^0: \quad \square \square R
$$

$$
\mathcal{R}_6^{[2,0]}: \quad R \square R \quad R_{pq} R_{p}^{pq} \quad R_{pq} \square R_{pq} \quad R_{pq;rs} R_{pq;rs}^{pq}
$$

$$
\mathcal{R}_6^{[1,1]}: \quad R_{p}^{r} R_{pq}^{r} \quad R_{pq;rs}^{r;qs} \quad R_{pq;rs}^{r} \quad R_{pq;rs}^{r;qs}
$$

$$
\mathcal{R}_6^{[1,3]}: \quad R_{pq}^{r} R_{pq;rs}^{r;qs} \quad R_{pq;rs}^{r} \quad R_{pq;rs}^{r;qs}
$$

(2.1)

This basis is a natural one and is often used in this form in the literature. However, it should be noted that certain authors prefer to use the scalar monomial $R_{pq;rs} \square R_{pq;rs}$ instead of the scalar monomial $R_{pq;rs} R_{pq;rs}^{pq;qs}$. This is the case of Gilkey in [5, 6]. This choice is only a matter of taste because these two terms appear equally in the calculations carried out in field theories defined on curved spacetimes and the elimination of one of them can be achieved by using the identity (A.4). It should also be noted that in these calculations, other Riemann monomials of order six and rank zero, such as $R_{pq} R_{pq;rs}^{r} R_{pq;rs}^{r;qs} R_{pq;rs}^{r} R_{pq;rs}^{r;qs}$ and $R_{pq;rs} R_{pq;rs}^{r} R_{pq;rs}^{r;qs}$, are systematically encountered. They can be eliminated by using the geometrical identities (A.3) and (A.5a)–(A.5c) which permit us to expand them on the FKWC-basis (2.1).

2.2. FKWC-basis for Riemann polynomials of order six and rank two

The functional derivatives with respect to the metric tensor of an action term constructed from a gravitational Lagrangian of order six is obtained by expanding it on the FKWC-basis for Riemann polynomials of order six and rank two. This basis consists of the 42 following elements [1]:

$$
\mathcal{R}_6^{2}: \quad \square R_{ab} \quad \square R_{ab}
$$

$$
\mathcal{R}_6^{[2,0]}: \quad R_{R_{ab}} \quad R_{R_{ab}} \quad R_{R_{pq;rs}^{r} R_{pq;rs}^{r;qs}} \quad R_{R_{pq;rs}^{r} R_{pq;rs}^{r;qs}}
$$

$$
\mathcal{R}_6^{[1,1]}: \quad R_{R_{pq;rs}^{r} R_{pq;rs}^{r;qs}} \quad R_{R_{pq;rs}^{r} R_{pq;rs}^{r;qs}}
$$

(2.2)

This basis is a natural one and is often used in this form in the literature. However, it should be noted that certain authors prefer to use the scalar monomial $R_{pq;rs} \square R_{pq;rs}$ instead of the scalar monomial $R_{pq;rs} R_{pq;rs}^{pq;qs}$. This is the case of Gilkey in [5, 6]. This choice is only a matter of taste because these two terms appear equally in the calculations carried out in field theories defined on curved spacetimes and the elimination of one of them can be achieved by using the identity (A.4). It should also be noted that in these calculations, other Riemann monomials of order six and rank zero, such as $R_{pq} R_{pq;rs}^{r} R_{pq;rs}^{r;qs} R_{pq;rs}^{r} R_{pq;rs}^{r;qs}$ and $R_{pq;rs} R_{pq;rs}^{r} R_{pq;rs}^{r;qs}$, are systematically encountered. They can be eliminated by using the geometrical identities (A.3) and (A.5a)–(A.5c) which permit us to expand them on the FKWC-basis (2.2).
Here, it should be noted that we have slightly modified the FKWC-basis of [1]:

(i) We have replaced the term $R_{pa}^p R_{qb}^b$ proposed in [1] by the term $R_{pa}^p R_{qb}^b$. We think it is more interesting to work with the latter which directly reduces to an element of the scalar basis (2.1) by contraction on the free indices $a$ and $b$. In fact, these two terms are linked by the geometrical identity (A.7) so it is easy to return to the original FKWC-basis.

(ii) We have replaced the terms $R_{pq}^p R_{r}^r a R_{prqb}$, $R_{pq}^p R_{r}^r a R_{prqb}$, $R_{pq}^p R_{r}^r a R_{prqb}$, and $R_{pq}^p R_{r}^r a R_{prqb}$ proposed in [1] by their opposites respectively given by $R_{pq}^p R_{r}^r a R_{prqb}$, $R_{pq}^p R_{r}^r a R_{prqb}$, and $R_{pq}^p R_{r}^r a R_{prqb}$. This choice has been done for obvious mnemotechnic reasons.

In the calculations carried out in field theories defined on curved spacetimes as well as in quantum gravity and more particularly in the calculations we shall achieve in the present section, other Riemann monomials of order six and rank two which are not in the FKWC-basis (2.2) are systematically encountered. They can be eliminated (i.e., expanded on the FKWC-basis (2.2)) more or less trivially from the geometrical identities (A.6)–(A.15e) we have obtained and displayed in the appendix.

2.3. Action terms constructed from the 17 scalar Riemann monomials of order six

From the 17 elements of the FKWC-basis (2.1), we can construct 17 action terms which permit us to express the most general gravitational action constructed from a Lagrangian of order six in the derivatives of the metric tensor. For a general spacetime $(M, g_{ab})$ of an arbitrary dimension $D$ (with $D$ sufficiently high), these 17 action terms are independent but if we assume that the considered spacetime has no boundary (i.e., if $\partial M = \emptyset$), some of these terms are linked together due to Stokes’s theorem. From now on, we shall work under this hypothesis and therefore consider that, for any vector field $V$, we have

$$\int_M d^Dx \sqrt{-g} V^p a \cdot R_p = 0. \quad (2.3)$$

By using integration by parts, the contracted Bianchi identities (A.2b) and (A.2c) as well as the geometrical identities (A.3) and (A.4) and Stokes’s theorem in the form (2.3), it is easy to prove that only the ten action terms

$$\int_M d^Dx \sqrt{-g} R \Box R, \quad (2.4)$$
$$\int_M d^Dx \sqrt{-g} R_{pq} \Box R_{pq}, \quad (2.5)$$
$$\int_M d^Dx \sqrt{-g} R^3, \quad (2.6)$$
$$\int_M d^Dx \sqrt{-g} R_{pq} R_{pq}, \quad (2.7)$$
$$\int_M d^Dx \sqrt{-g} R_{pq} R_{pr}, \quad (2.8)$$
are independent and that one of the remaining action terms vanishes while the other six can be expressed in terms of some of the previous ones:

\[ \int d^{\mathcal{M}} x \sqrt{-g} \Box \Box R = 0, \]  
\[ \int d^{\mathcal{M}} x \sqrt{-g} R_{pq} R^{pq} = \int d^{\mathcal{M}} x \sqrt{-g} \left( \frac{1}{2} R \Box R \right), \]  
\[ \int d^{\mathcal{M}} x \sqrt{-g} R_{pqrs} R^{pqrs} = \int d^{\mathcal{M}} x \sqrt{-g} \left( -\frac{1}{4} R \Box R + R_{pq} R^{pq} - R_{pq} R_{rs} R^{pqrs} \right), \]  
\[ \int d^{\mathcal{M}} x \sqrt{-g} R_{pq}_{rs} R^{pq}_{rs} = \int d^{\mathcal{M}} x \sqrt{-g} \left( -\frac{1}{4} R \Box R - R_{pq} R_{rs} R^{pqrs} + 4R_{pq} R_{rs} R^{pqrs} \right). \]

2.4. Explicit irreducible expressions for the metric variations of the action terms constructed from the 17 scalar Riemann monomials of order six

The functional derivatives with respect to the metric tensor of the ten independent action terms (2.4)–(2.13) can be obtained by using the behaviour of these action terms in the variation

\[ g_{\mu \nu} \rightarrow g_{\mu \nu} + \delta g_{\mu \nu} \]

of the metric tensor. From the corresponding variations of the geometrical tensors \( \sqrt{-g}, R, \Box R, R_{ab}, \Box R_{ab} \) and \( R_{abcd} \) given in appendix A.2 (see (A.18a)–(A.18h)) or in [14] and after tedious calculations, we have expanded the functional derivatives of the ten action
terms (2.4)–(2.13) on the FKWC-basis (2.1) and (2.2) and we have obtained the following results:

\[ H^{(2.0)(1)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R \square R \]
\[ = 2(\square R)_{,ab} - 2(\square R)_{ab} + R_{,ab}R_{,b} + g_{ab}[-2 \square R - (1/2)R_{,p}R^{,p}], \quad \text{(2.22)} \]

\[ H^{(2.0)(3)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R_{pq} \square R^{pq} \]
\[ = (\square R)_{,ab} - \square R_{,ab} + R_{,p(a}R_{,b)}^{,p} - 2R^{pq}R_{pq(,ab)}^{,p} + 6R^{pq}R_{p(a}R_{b)q}^{,p} - 2(\square R^{pq})_{R_{pq}ab} + 4R^{pq}_{,r}R_{r(pq)b}^{,r} \]
\[ + 3R_{,p}R_{,p}^{,p} - R^{pq}_{,p}R_{pq,b} + 2R^{pq}_{,p(a}R_{b)p,q}^{,p} \]
\[ + 4R^{pq}_{,r}R_{r(pq)b}^{,p} - 2R^{pq}_{,r}R_{pq,b} + 4R^{pq}R^{pq}_{,r}R_{pqab} \]
\[ + 2R^{pq}R^{pq}_{,r}R_{r(pq)b}^{,p} + g_{ab}[-(1/2)\square R - 2R_{,pq}R^{pq}] \]
\[ + R_{pq} \square R^{pq} + 2R_{pq,r}R^{pq}_{,r} - (1/2)R_{,p}R^{,p} + (5/2)R_{pq,r}R^{pq}_{,r} - 4R_{pq,r}R^{pq}_{,r} - 2R_{pq}R^{pq}_{,r} + 2R_{pq}R_{rs}R^{pq}_{,rs}], \quad \text{(2.23)} \]

\[ H^{(6.3)(1)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R_{pq}R_{pq} \]
\[ = 6R_{pq,ab} + 6R_{ab}R_{,pq} - 3R^{2}R_{ab} + g_{ab}[-6\square R - 6R_{,p}R^{,p} + (1/2)R^{3}], \quad \text{(2.24)} \]

\[ H^{(6.3)(2)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R_{pq}R_{pq} \]
\[ = RR_{ab} - (\square R)_{ab} + 2R_{,p(a}R_{,b)}^{,p} - R^{pq}_{,ab} \]
\[ + 2R^{pq}R_{pq(,ab)}^{,p} + R_{,ab}R_{,b} + 2R_{,p}R_{,p}^{,p} - 2R_{,p}R_{,p}^{,p} \]
\[ + 2R_{,ab}^{,p}R_{pq,b}^{,p} - 2R^{pq}_{,r}R_{pqab} - R^{pq}_{,r}R_{pq,b} + 2R^{pq}_{,r}R_{pqab} \]
\[ + g_{ab}[-(1/2)\square R - 2R_{,pq}R^{pq} - 2R_{pq}R^{pq} \]
\[ - R_{,p}R^{,p} - 2R_{pq}R^{pq}_{,r} + (1/2)R_{pq}R^{pq}_{,r}], \quad \text{(2.25)} \]

\[ H^{(6.3)(3)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R_{pq}R_{pq} \]
\[ = (3/2)R_{,p(a}R_{b)q}^{,p} - 3R_{,p(a}R_{b)q}^{,p} + 3R^{pq}R_{p(a}R_{b)q}^{,p} \]
\[ + (3/2)R_{pq}R_{pq}^{,p} + 3R^{pq}_{,r}R_{pq,b}^{,p} - 3R^{pq}_{,r}R_{pq}^{,p} \]
\[ + 3R^{pq}_{,r}R_{pq,b}^{,p} + g_{ab}[-(3/2)R_{pq}R^{pq} - (3/2)R_{pq}R^{pq} \]
\[ - (3/2)R_{pq,r}R^{pq}_{,r} - 2R_{pq}R_{pq}^{,r} + (3/2)R_{pq}R_{pq}^{,r}], \quad \text{(2.26)} \]

\[ H^{(6.3)(4)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{g^{ab}} \int_{M} d^{D}x \sqrt{-g} R_{pq}R_{pq}R_{pq} \]
\[ = -(1/2)(\square R)_{ab} + R_{,p(a}R_{b)q}^{,p} + R^{pq}_{,pq(,ab)} \]
\[ - 2R^{pq}_{ab}R_{pq}^{,pq} - (\square R^{pq})_{R_{pq}(,ab)}^{,pq} - 2R^{pq}_{,pq(,ab)}^{,pq} \]
\[ + (1/4)R_{pq}R_{pq,b}^{,p} - R_{,p}R_{pq}^{,pq} + 2R^{pq}_{,pq(,ab)}^{,pq} \]
\[ - 2R^{pq}_{,pq(,ab)}^{,pq} + R^{pq}_{,pq,b}^{,pq} - 2R^{pq}_{,pq(,ab)}^{,pq}. \quad \text{(2.27)} \]
Metric variations in higher-derivative gravity and stress–energy tensor

\[ \begin{align*}
- 2 R^{pqrs} R_{pqab;r} &+ R^{pq} R_{pa} R_{qb} = 2 R^{pq} R^q_{ b} R_{paqb} \\
+ 2 R^{pq} R^r_{pa} R_{pqr(b)} &+ 2 R^{pq} R^s_{pa} R_{sqr(b)} = 2 R_{pq} R^{pqrs} R^q_{rsb} \\
&+ g_{ab} \left( \frac{1}{2} R_{pq} R^{pq} - R_{pq} R^{pq} - R_{pqrs} R^{pqrs} \\
&- 2 R_{pqtr} R^{pqtr} + 2 R_{pqtr} R^{pqtr} + R_{pq} R^p_{qr} R^q_r \\
&- (1/2) R_{pq} R_{rs} R^{pqrs} \right),
\end{align*} \]

(2.27)

\[ \begin{align*}
H^{(6),(5)}_{ab} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4 x \sqrt{-g} R_{pqrs} R^{pqrs} \\
&= 2 R R_{ab} - 4 R \nabla_a \nabla_b - 4 R R^{pq} R_{pqab} + 2 R^{pqrs} R_{pqrs} R_{ab} \end{align*} \]

(2.28)

\[ \begin{align*}
H^{(6),(6)}_{ab} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4 x \sqrt{-g} R_{pqrs} R^{pqrs} \\
&= R_{p(a} R^{p}_{b)} - 2 R_{p(a} \nabla^{p}_{b)} + 2 R^{pq} R_{pq(a} R^{p}_{b)} - 2 R^{pq} R_{pab} R_{pq} \\
&- R_{p(a} R_{pqrb} + 4 R^{pq}_{p(a} R_{pqrb)} + (1/2) R_{pqrs} R^{pqrs}_{(ab)} \\
&+ R_{p(a} R_{qb}^{p} R_{pqrb)} - 4 R^{pq}_{p(a} R_{qb}^{p} R_{pqrb)} - 4 R_{pq} R_{p(a} R^{pq}_{qb)} \\
&+ (1/2) R_{pqrs} R^{pqrs} - 4 R^{pq}_{p(a} R_{qb}^{p} R_{pqrb)} \\
&+ (1/2) R^{pqrs}_{p(a} R_{pqrs} R_{pab} + 4 R^{pq} R_{pab}^{pq} \\
&- 2 R^{pq}_{p(a} R_{pqrs} R_{pab} + 2 R^{pq}_{p(a} R_{pqrs} R_{pab} \\
&- 2 R^{pq}_{p(a} R_{pqrs} R_{pab} + 2 R^{pq}_{p(a} R_{pqrs} R_{pab} \\
&+ 4 R^{pq}_{p(a} R_{pqrs} R_{pab} + 4 R^{pq}_{p(a} R_{pqrs} R_{pab} \\
&+ g_{ab} \left[ -2 R_{pqrs} R^{pqrs} + R_{pqrs} R^{pqrs} + R_{pqrs} R^{pqrs} + R_{pqrs} R^{pqrs} \\
&- (1/4) R_{pqrs} R^{pqrs} + (1/4) R_{pqrs} R^{pqrs} \right].
\end{align*} \]

(2.29)

\[ \begin{align*}
H^{(6),(7)}_{ab} &= \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4 x \sqrt{-g} R_{pqrs} R^{pqrs} R^{rs}_{uv} \\
&= 24 R^{pq}_{(a} R_{pqrb)} - 12 R^{pq}_{a} R_{pqrb} + 12 R^{pq}_{a} R_{pqrb} \\
&+ 3 R^{pqrs}_{a} R_{pqrs} R_{pab} - 6 R^{pqrs}_{a} R_{pqrs} R_{pab} \\
&- 6 R^{pqrs}_{pab} R_{pqrb} + 12 R^{pqrs}_{pab} R_{pqrb} \\
&+ g_{ab} \left[ (1/2) R_{pqrs} R^{pqrs} R^{rs}_{uv} \right].
\end{align*} \]

(2.30)
We obtain seven conserved tensors given by

\[
H_{ab}^{(6,3)(8)} = \frac{1}{\sqrt{-g}} \delta \frac{\delta}{\partial g^{ab}} \int d^Dx \sqrt{-g} R_{prqs} R^p_{\phantom{p}u} \; R^u_{\phantom{u}rvst}
\]

\[
= (3/2) R^{pq} R_{pqab} - 3(\Box R^{pq}) R_{pqab} - 6 R^{pq}_{\phantom{pq}r} R_{rgpb}\]

\[
- (3/4) R^{pqrs} R_{pqrs(ab) + 3 R^p_{\phantom{p}a} R_{pq} - 3 R^{pq}_{\phantom{pq}a} R_{pbq} - 6 R^{pq}_{\phantom{pq}a} R_{b} p q + 3 R^p_{\phantom{p}a} R^q_{\phantom{q}b} R_{b q p} + 6 R^{pq}_{\phantom{pq}r} R_{raqp(a; b)} + (3/4) R^{pqrs}_{\phantom{pqrs}a} R_{pqrs,b} + 3 R^{pq}_{\phantom{pq}r} R_{pra q b} + 3 R^p_{\phantom{p}a} (R^{qr}_{\phantom{qr}a}| p R_{qr s b}) + (3/2) R^{pq} R^{rs}_{\phantom{rs}pa} R_{pqab} - 3 R_{pq} R^{pq}_{\phantom{pq}r} R_{r a b} + 3 R^{pq}_{\phantom{pq}r} R_{pq(a; b)} R_{r b} - (3/2) R^{pqrs} R_{pq(a) R_{r b}} - 9 R^{pq}_{\phantom{pq}r} R_{pq} R_{r b} + (3/2) R^{pq}_{\phantom{pq}r} R_{pq} R_{r b}
\]

\[
+ g_{ab} \left[ (1/2) R_{pqrs} R^p_{\phantom{p}q} R^{rs}_{\phantom{rs}uv} \right].
\]

(2.31)

It should be noted here that the ten geometrical tensors (2.22)–(2.31) are automatically conserved due to the invariance of the actions (2.4)–(2.13) under spacetime diffeomorphisms.

From the relations (2.14)–(2.20) and by using (2.22)–(2.31), we can now directly obtain the functional derivatives with respect to the metric tensor of the seven remaining action terms.

We obtain seven conserved tensors given by

\[
H_{ab}^{(6,1)(1)} = \frac{1}{\sqrt{-g}} \delta \frac{\delta}{\partial g^{ab}} \int d^Dx \sqrt{-g} \Box R = 0,
\]

(2.32)

\[
H_{ab}^{(2,0)(2)} = \frac{1}{\sqrt{-g}} \delta \frac{\delta}{\partial g^{ab}} \int d^Dx \sqrt{-g} R_{pq} R^{pq}
\]

\[
= \frac{1}{2} H_{ab}^{(2,0)(1)}
\]

(2.33a)

\[
= (\Box R)_{ab} - (\Box R)_{ab} + (1/2) R_{a} R_{b} + g_{ab} \left[ - \Box R - (1/2) R_{a} R^{b} \right].
\]

(2.33b)

\[
H_{ab}^{(2,0)(4)} = \frac{1}{\sqrt{-g}} \delta \frac{\delta}{\partial g^{ab}} \int d^Dx \sqrt{-g} R_{pqrs} R^{pqrs}
\]

\[
= - \frac{1}{4} H_{ab}^{(2,0)(1)} + H_{ab}^{(2,0)(3)} + H_{ab}^{(6,3)(3)} + H_{ab}^{(6,3)(4)}
\]

\[
= (1/2) (\Box R)_{ab} - \Box R_{ab} + (1/2) R_{p(a} R^{p}_{b)}
\]

\[
+ 2 R^{p}_{(a} R^{p}_{b)} p q - 3 (\Box R^{pq}) R_{pqab} + 2 R^{pq}_{\phantom{pq}r} R_{pq(a; b)} + 3 R^{pq}_{\phantom{pq}r} R_{p(a} R_{b q) p q} + (3/2) R_{p(a} R^{p}{\phantom{p}b)} + R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq} + 2 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq} + 3 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq} + 2 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq}
\]

\[
+ 2 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq} + 2 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq} + 2 R^{pq}_{\phantom{pq}r} R_{pq} R_{pbq}
\]

\[
+ g_{ab} \left[ R_{pqrs} R^{pqrs} + (1/2) R_{pqrs} R^{pqrs} - (1/2) R_{pqrs} R^{pqrs} \right].
\]

(2.34b)
\[ H_{ab}^{(1,1)(1)}(\delta) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int_M d^Dx \sqrt{-g} R_{;p} R^{;p} \]

\[ = -H_{ab}^{(2,0)(1)}(\delta) \]

\[ = -2(\Box R)_{;ab} + 2(\Box R) R_{ab} - R_{;a} R_{;b} + g_{ab}[2(\Box R) + (1/2) R R_{;p} R^{;p}], \quad (2.35a) \]

\[ H_{ab}^{(1,1)(2)}(\delta) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int_M d^Dx \sqrt{-g} R_{pq;r} R^{pq;r} \]

\[ = -H_{ab}^{(2,0)(3)}(\delta) \]

\[ = -(\Box) R_{;ab} + (\Box) R_{ab} - R_{;p(a} R^{;p}_{b)} + 2 R_{pq} R_{pq;(ab)} \]

\[ - 6 R^{pq} R_{p[a,bq]} + 2(\Box R^{pq}) R_{pqab} - 4 R^{pq,r}_{(a} R_{q;p(b)} \]

\[ - 3 R_{;p} R_{;a} + R_{pq,b} - 2 R_{pq,;a} R_{;b} \]

\[ - 4 R_{pq,;r} R_{pqab} - 2 R_{pq} R_{;a} R_{pq,b} \]

\[ + g_{ab}[(1/2) (\Box R + 2 R_{pq} R^{pq} - R_{pq} (\Box R^{pq}) \]

\[ - 2 R_{pq,rs} R^{pq,rs} + (1/2) R_{pq} R^{pq} - (5/2) R_{pq,r} R^{pq,r} \]

\[ + 4 R_{pq,r} R^{pq,r} - 2 R_{pq} R^{pq} - 2 R_{pq,rs} R^{pq,rs}]. \]  

\[ (2.36b) \]

\[ H_{ab}^{(1,1)(3)}(\delta) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int_M d^Dx \sqrt{-g} R_{pq;r} R^{pq;r} \]

\[ = -\frac{1}{4} H_{ab}^{(2,0)(1)}(\delta) = H_{ab}^{(6,3)(3)} + H_{ab}^{(6,3)(4)} \]

\[ (2.37a) \]

\[ H_{ab}^{(1,1)(4)}(\delta) = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{ab}} \int_M d^Dx \sqrt{-g} R_{pq;r} R^{pq;r} \]

\[ = H_{ab}^{(2,0)(1)}(\delta) - 4 H_{ab}^{(2,0)(3)}(\delta) + 4 H_{ab}^{(6,3)(3)}(\delta) - 4 H_{ab}^{(6,3)(4)}(\delta) \]

\[ - 2 H_{ab}^{(6,3)(6)}(\delta) + H_{ab}^{(6,3)(7)}(\delta) + 4 H_{ab}^{(6,3)(8)}(\delta) \]

\[ = -4 H_{ab}^{(2,0)(4)}(\delta) - 2 H_{ab}^{(6,3)(6)}(\delta) + H_{ab}^{(6,3)(7)}(\delta) + 4 H_{ab}^{(6,3)(8)}(\delta) \]

\[ (2.38b) \]
It should be noted that the last six expressions can also be obtained by considering the variation of the action terms on the left-hand side of (2.15)–(2.20) from (A.20a)–(A.20c). Of course, the corresponding redundant calculations are tedious but we have also achieved them in order to check the validity and the internal coherence of our results.

Finally, it seems to us interesting to also provide the functional derivative of the alternative action term constructed from the gravitational Lagrangian \( R_{pqrs} \). From (A.4) and (2.16) and by using (2.22)–(2.31) or directly by using integration by parts, we obtain for this conserved tensor

\[
H^{(2,0)(alt)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^dx \sqrt{-g} R_{pqrs} \square R_{pqrs}
\]

\[
= -H^{(2,0)(1)}_{ab} + 4H^{(2,0)(3)}_{ab} - 4H_{ab}^{(6,3)(3)} + 4H_{ab}^{(6,3)(4)}
\]

\[
+ 2H_{ab}^{(6,3)(6)} - H_{ab}^{(6,3)(7)} - 4H_{ab}^{(6,3)(8)}
\]

\[
= 4H_{ab}^{(2,0)(4)} + 2H_{ab}^{(6,3)(6)} - H_{ab}^{(6,3)(7)} - 4H_{ab}^{(6,3)(8)}
\]

\[
= -H_{ab}^{(1,1)(4)}.
\]

It appears as the opposite of \( H^{(1,1)(4)}_{ab} \) and its explicit expression can be obtained directly from (2.38c).

2.5. Irreducible form for the cubic Lovelock tensor

In 1971, Lovelock found the most general symmetric and conserved tensor which is quasi-linear in the second derivatives of the metric tensor and does not contain higher derivatives. It therefore generalizes the Einstein tensor \( R_{ab} = (1/2) R g_{ab} \) (see [2, 3] for the original discussion but also the paper by Deruelle and Madore [15] for a historical and physical presentation of this subject). Lovelock found moreover that this tensor can be obtained by functional derivation with respect to the metric tensor of an action constructed from a Lagrangian which is the sum of dimensionally extended Euler densities. The Lovelock gravitational theory is an appealing one, being free of ghosts [16, 17], and is today more particularly considered in the context of string theory and brane cosmology.
The Lovelock Lagrangian $\mathcal{L}_L$ reads

$$\mathcal{L}_L = \sum_{n \geq 0} c_n \mathcal{L}_{(n)},$$  \hfill (2.40)$$

where $c_n$ are the arbitrary coefficients while $\mathcal{L}_{(n)}$ for $n \geq 1$ is the dimensionally extended Euler density of order $n$ in the Riemann tensor (or the Euler–Gauss–Bonnet–Chern–Lovelock invariant of order $n$) given by

$$\mathcal{L}_{(n)} = \frac{1}{2^n} \delta_{\forall x_1 \ldots x_n}^{p_1 \ldots p_n q_1 \ldots q_n} R^{x_1 \ldots x_n}_{p_1 \ldots p_n q_1 \ldots q_n}.$$  \hfill (2.41)$$

Here, $\delta^{p_1 \ldots p_n q_1 \ldots q_n}_{x_1 \ldots x_n}$ denotes the generalized Kronecker symbol which is totally antisymmetric in its upper and lower indices and which can be considered as the dimensionally extended normalized determinant

$$\delta^{p_1 \ldots p_n q_1 \ldots q_n}_{x_1 \ldots x_n} = \sum_{\sigma \in S_n} \text{sign}(\sigma) \delta^{p_1}_{\sigma(x_1)} \delta^{q_1}_{\sigma(x_1)} \cdots \delta^{p_n}_{\sigma(x_n)} \delta^{q_n}_{\sigma(x_n)}.$$  \hfill (2.42)$$

We have $\mathcal{L}_{(0)} = 1$ by convention, $\mathcal{L}_{(1)} = R$ and $\mathcal{L}_{(2)}$ which reduces to the ordinary Gauss–Bonnet density, i.e.,

$$\mathcal{L}_{(2)} = R_{pqrs} R^{pqrs} - 4 R_{pq} R^{pq} + R^2.$$  \hfill (2.43)$$

The part of the Lovelock Lagrangian which is cubic in the Riemann tensor is explicitly given by

$$\mathcal{L}_{(3)} = R^3 - 12 R R_{pq} R^{pq} + 16 R_{pq} R^p R^q + 24 R_{pq} R_{rs} R^{pqrs} + 3 R R_{pqrs} R^{pqrs} - 24 R_{pq} R^p R_{rs} R^{pqrs} + 4 R_{pqrs} R^{pqrs} R^{uv} R_{uv} - 8 R_{pqrs} R^p q R^r u v.$$  \hfill (2.44)$$

and the formalism we have developed in the previous sections permits us to obtain, from the corresponding action

$$S_{(3)} = \int d^D x \sqrt{-g(x)} \mathcal{L}_{(3)}(x),$$  \hfill (2.45)$$

the cubic part of the Lovelock tensor

$$G^{(3)}_{ab} = \frac{1}{\sqrt{-g}} \frac{\delta S_{(3)}}{\delta g^{ab}}.$$  \hfill (2.46)$$

With our previous notation, we have

$$G^{(3)}_{ab} = H^{(6,3)(1)}_{ab} - 12 H^{(6,3)(2)}_{ab} + 16 H^{(6,3)(3)}_{ab} + 24 H^{(6,3)(4)}_{ab} + 3 H^{(6,3)(5)}_{ab} - 24 H^{(6,3)(6)}_{ab} + 4 H^{(6,3)(7)}_{ab} - 8 H^{(6,3)(8)}_{ab}.$$  \hfill (2.47)$$

and more explicitly, by using (2.22)–(2.31), we obtain

$$G^{(3)}_{ab} = -3 R^2 g_{ab} + 12 R R_{pq} R^p g_{ab} + 12 R^p R_{pq} R_{ab} - 24 R^p R_{pq} R_{[ab]q} + 12 R_{pq} R^{pq} R_{[ab]q} + 48 R_{pq} R_{[pq]b} R_{[ab]} - 6 R R_{pq} R_{[pq]} R_{[ab]} - 6 R R_{pq} R_{[pq]} R_{[ab]} + 24 R^p R_{pq} R^{pq} R_{[ab]} - 12 R_{pq} R_{[pq]} R_{[ab]} + 24 R_{pq} R^{pq} R_{[pq]} R_{[ab]} + 12 R_{pq} R_{[pq]} R_{[ab]} + 24 R_{pq} R^{pq} R_{[pq]} R_{[ab]}.$$  \hfill (2.48)$$

This result is not a new one. In fact, it has been obtained by Müller-Hoissen in [18] much more directly. Indeed, in order to functionally derive the Lovelock Lagrangian, it is not necessary to functionally derive independently all the geometrical terms which compound it as we have done. Here, we recover this result mainly in order to check our previous calculations. It should be however noted that the comparison of (2.48) with the result of Müller-Hoissen is not immediate. Indeed, in [18], the FKWC-basis is not used. But, by using (A.15c), it is easy to put the result of Müller-Hoissen in our irreducible form (2.48).
3. Applications: renormalization in the effective action and stress–energy tensor

3.1. Effective action and the Gilkey–DeWitt coefficient $a_3$

In this section, we consider a massive scalar field $\Phi$ propagating on the $D$-dimensional curved spacetime $(\mathcal{M}, g_{ab})$ and obeying the wave equation

$$\Box - m^2 - \xi R) \Phi = 0. \quad (3.1)$$

Here, $m$ is the mass of the scalar field while $\xi$ is a dimensionless factor which accounts for the possible coupling between this field and the gravitational background. The associated DeWitt–Schwinger effective action $W$ [8–12], which contains all the information on the ultraviolet behaviour of the quantum theory, may be represented by the asymptotic series [12]

$$W = \int_{\mathcal{M}} d^D x \sqrt{-g(x)} \left[ \frac{1}{2(4\pi)^{D/2}} \int_0^{+\infty} \frac{d(is)}{(is)^{D/2+1}} e^{-im^2s} \sum_{k=0}^{+\infty} a_k(x)(is)^k \right]. \quad (3.2)$$

Here, $a_k(x)$ are the diagonal DeWitt coefficients and we have for the first four \begin{eqnarray}
a_0 &=& 1, \quad (3.3a) \\
a_1 &=& -(\xi - 1/6) R, \quad (3.3b) \\
a_2 &=& -(1/6)(\xi - 1/5) \Box R + (1/2)(\xi - 1/6)^2 R^2 - (1/180)R_{pq} R^{pq} + (1/180) R_{pqrs} R^{pqrs} \quad (3.3c) \\
\end{eqnarray}

and \begin{eqnarray}
a_3 &=& -(1/60)(\xi - 3/14) \Box \Box R + (1/6)(\xi - 1/6)(\xi - 1/5) R \Box R - (1/90)(\xi - 3/14)R_{pq} R^{pq} - (1/630)R_{pq} \Box R^{pq} + (1/105)R_{pqrs} R^{pqrs} \\
&+& (1/12)(\xi^2 - (2/5)\xi + 17/420)R_{pq} R_{pq} - (1/2520)R_{pqrs} R^{pqrs} \\
&-& (1/1260)R_{pq} R_{pq} R^{pq} + (1/560)R_{pqrs} R^{pqrs} \\
&+& (1/1260)R_{pq} R_{pq} R^{pq} + (1/270)R_{pqrs} R^{pqrs} \\
&-& (1/2835)R_{pqrs} R^{pqrs} - (22/2835)R_{pqrs} R^{pqrs} \quad (3.3d) \\
\end{eqnarray}

The results we have obtained in the previous section are therefore helpful in order to understand those of the physical aspects of the scalar field theory which are more particularly associated with the coefficient $a_3$ since it is of sixth order in the derivatives of the metric tensor. In the following sections, we shall focus our attention on two of them. To be more precise, it is important to recall that the effective action (3.2) is divergent at the lower limit of the integral over $s$ for all positive values of the dimension $D$. By considering the dimensionality $D$ of spacetime as a complex number, the effective action can be regularized by analytic continuation and its divergent part can be extracted coherently. In a four-dimensional background, the divergent part of the effective action is proportional to [8–12]

$$\int_{\mathcal{M}} d^4 x \sqrt{-g(x)} [a_2(x) - m^2 a_1(x) + (m^4/2)a_0(x)] \quad (3.4)$$

while its regularized part is proportional, in the large mass limit, to [10, 11]

$$\int_{\mathcal{M}} d^4 x \sqrt{-g(x)} a_3(x). \quad (3.5)$$
In a six-dimensional background, the divergent part of the effective action is proportional to
\[
\int_M d^6x \sqrt{-g(x)} [a_3(x) - m^2 a_2(x) + (m^4/2) a_1(x) - (m^6/6) a_0(x)].
\]
(3.6)

In both cases, it should be noted that the global (or integrated) Gilkey–DeWitt coefficient
\[
\int_M d^6x \sqrt{-g(x)} a_3(x)
\]
plays a central role. It is therefore necessary to have at our disposal its explicit expression as well as the expressions of the first three global (or integrated) DeWitt coefficients. If we now assume that spacetime has no boundary, by using (3.3d) and (2.14)–(2.20), we can easily obtain them. We have
\[
\int_M d^6x \sqrt{-g} a_0 = \int_M d^6x \sqrt{-g},
\]
(3.8)

\[
\int_M d^6x \sqrt{-g} a_1 = \int_M d^6x \sqrt{-g} [- (\xi - 1/6) R]
\]
(3.9)

\[
\int_M d^6x \sqrt{-g} a_2 = \int_M d^6x \sqrt{-g} [(1/2) (\xi - 1/6)^2 R^2 - (1/180) R_{pq} R^{pq} + (1/180) R_{pqrs} R^{pqrs}]
\]
(3.10)

\[
\int_M d^6x \sqrt{-g} a_3 = \int_M d^6x \sqrt{-g} [(1/12) \xi^2 - (1/30) \xi + 1/336] R \Box R
\]
\[
+ (1/480) R_{pq} \Box R^{pq} - (1/6) (\xi - 1/6)^3 R^3 + (1/180) (\xi - 1/6) R R_{pq} R^{pq}
\]
\[
- (4/2835) R_{pq} R^{pq} R^{rt} + (1/945) R_{pq} R_{rs} R^{pqrs}
\]
\[
- (1/180) (\xi - 1/6) R R_{pqrs} R^{pqrs} + (1/7560) R_{pq} R_{pqrs} R^{pqrs}
\]
\[
+ (1/45360) R_{pqrs} R^{pqrs} R^{rt} - (1/1620) R_{pqrs} R^{pqrs} R^{rt} + (1/270) R_{pqrs} R^{pqrs} R^{rt} + (1/270) R_{pqrs} R^{pqrs} R^{rt}].
\]
(3.11)

3.2. ‘Irreducible’ form for the approximated stress–energy tensor obtained from the DeWitt–Schwinger effective action in four dimensions

In the large mass limit of the quantized scalar field, the renormalized effective action in four dimensions reduces to [10, 11]
\[
W_{\text{ren}} = \frac{1}{32\pi^2 m^2} \int_M d^4x \sqrt{-g(x)} a_3(x)
\]
(3.12)

and from (3.11) we can then write the effective action in the form (see also [10, 11])
\[
W_{\text{ren}} = \frac{1}{192\pi^2 m^2} \int_M d^4x \sqrt{-g} [(1/2) \xi^2 - (1/5) \xi + 1/56] R \Box R
\]
\[
+ (1/140) R_{pq} \Box R^{pq} - (\xi - 1/6)^3 R^3 + (1/30) (\xi - 1/6) R R_{pq} R^{pq}
\]
\[
- (8/945) R_{pq} R^{pq} R^{rt} + (2/315) R_{pq} R_{rs} R^{pqrs}
\]
\[
- (1/30) (\xi - 1/6) R R_{pqrs} R^{pqrs} + (1/1260) R_{pq} R_{pqrs} R^{pqrs}
\]
\[
+ (17/7560) R_{pqrs} R^{pqrs} R^{rt} - (1/270) R_{pqrs} R^{pqrs} R^{rt} + (1/270) R_{pqrs} R^{pqrs} R^{rt}].
\]
(3.13)
By functional derivation of the effective action (3.13), we obtain an approximation for the expectation value of the stress–energy tensor associated with the scalar field. With the notation introduced in section 2, we can write

\[
\langle T_{ab} \rangle_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta g^{ab}}
\]

\[
= \frac{1}{96\pi^2m^2} \left[ (1/2)\xi^2 - (1/5)\xi + 1/56[H_{ab}^{(2,0)(1)} + (1/140)H_{ab}^{(2,0)(3)}
\right.
\]

\[
- (\xi - 1/6)^3 H_{ab}^{(6,3)(1)} + (1/30)(\xi - 1/6)H_{ab}^{(6,3)(2)}
\]

\[
- (8/945)H_{ab}^{(6,3)(3)} + (2/315)H_{ab}^{(6,3)(4)} - (1/30)(\xi - 1/6)H_{ab}^{(6,3)(5)}
\]

\[
+ (1/1260)H_{ab}^{(6,3)(6)} + (17/7560)H_{ab}^{(6,3)(7)} - (1/270)H_{ab}^{(6,3)(8)} \right]
\]

(3.14)

and from the relations (2.22)–(2.31), we have then explicitly

\[
(96\pi^2m^2)(T_{ab})_{\text{ren}} = [\xi^2 - (2/5)\xi + 3/10] - (1/140)\square R_{ab}
\]

\[
- 6(\xi - 1/6)\xi^2 - (1/3)\xi + 1/30RR_{ab}
\]

\[
- (\xi - 1/6)(\xi - 1/5)\square R_{ab} + (1/15)(\xi - 1/7)R_{p(\xi)}R_{,b}^p
\]

\[
+ (1/10)(\xi - 1/6)R \square R_{ab} + (1/42)R_{p(\xi)}R_{ab}^p
\]

\[
+ (1/15)(\xi - 2/7)R_{pq}R_{pq(ab)} + (2/105)R_{pq}R_{pq(ab)}
\]

\[
- (1/70)R_{pq}R_{pq(ab)} + (4/105)R_{pq}\square R_{pq(ab)}
\]

\[
+ (2/35)R_{pq(ab)}R_{pq(ab)} - (1/15)(\xi - 3/14)R_{pq}R_{pq(ab)}
\]

\[
- 6(\xi - 1/4)\xi^2 - (1/2)\xi - (1/5)\xi + 3/4R_{pq}R_{pq(ab)}
\]

\[
+ (1/15)(\xi - 17/84)R_{pq}R_{pq(ab)} + (1/15)(\xi - 1/4)R_{pq}R_{pq(ab)}
\]

\[
- (1/210)R_{pq}R_{pq(ab)} + (1/42)R_{pq}R_{pq(ab)} - (1/105)R_{pq}R_{pq(ab)}
\]

\[
- (1/70)R_{pq}R_{pq(ab)} - (1/15)(\xi - 13/56)R_{pq}R_{pq(ab)}
\]

\[
+ (1/70)R_{pq}R_{pq(ab)} + 3(\xi - 1/6)^3R_{pq}R_{pq(ab)}
\]

\[
- (2/15)(\xi - 1/6)RR_{pq}R_{pq} - (1/15)(\xi - 1/6)R_{pq}R_{pq}R_{pq}
\]

\[
- (2/35)R_{pq}R_{pq}R_{pq} + (1/15)(\xi - 1/6)R_{pq}R_{pq}R_{pq}
\]

\[
+ (1/35)R_{pq}R_{pq}R_{pq} + (1/35)R_{pq}R_{pq}R_{pq} + (1/35)(\xi - 1/6)RR_{pq}R_{pq}R_{pq}
\]

\[
- (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]

\[
+ (2/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]

\[
+ (2/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]

\[
+ (2/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]

\[
+ (2/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]

\[
+ (2/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq} + (4/315)R_{pq}R_{pq}R_{pq}
\]
\[-(1/630)R_{pq}R_{rs}R^{pqrst} - (1/60)(\xi - 1/6)RR_{pqrst}R^{pqrst} + (2/15)(\xi - 1/6)R_{pq}R^p_{rs}R^{qrst} - (1/15)(\xi - 47/252)R_{pqr}R^{pqrs}R_{rsuv} - (4/15)(\xi - 41/252)R_{pqr}R^p_{uv}R_{rsuv} \]. (3.15)

We have therefore expressed the approximated expectation value of the stress–energy tensor associated with the scalar field on the FKWC-basis described in section 2. We have a final expression which is simplified and without any ambiguities. It should be noted that in recent articles [19–21], Matyjasek has calculated this stress–energy tensor but, being only interested in the result in particular spacetimes, he has not obtained a general simplified result valid in an arbitrary background.

To conclude this subsection, we would like to emphasize some possible other simplifications coming from ‘topological’ and geometrical constraints associated with the four-dimensional nature of spacetime. In that special case, it is well-known that the Euler number

\[ \int_M d^4x \sqrt{-g(x)}L(2)(x), \] (3.16)

where \( L(2) \) is given by (2.43) is a topological invariant. As a consequence, its metric variation vanishes identically and we have

\[ H^{(4,2)(1)}_{ab} - 4H^{(4,2)(2)}_{ab} + H^{(4,2)(3)}_{ab} = 0, \] (3.17)

with

\[ H^{(4,2)(1)}_{ab} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4x \sqrt{-g}R^2 \]
\[ = 2R_{ab} - 2RR_{ab} + g_{ab}[-2\Box R + (1/2)R^2], \] (3.18)

\[ H^{(4,2)(2)}_{ab} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4x \sqrt{-g}R_{pq}R^{pq} \]
\[ = R_{ab} - \Box R_{ab} - 2R^{pq}R_{paqb} + g_{ab}[-(1/2)\Box R + (1/2)R_{pq}R^{pq}], \] (3.19)

\[ H^{(4,2)(3)}_{ab} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_M d^4x \sqrt{-g}R_{pqrst}R^{pqrst} \]
\[ = 2R_{ab} - 4\Box R_{ab} + 4R^p_{a}R_{pb} - 4R^{pq}R_{paqb} - 2R^{pq}_{a}R_{pqr} + g_{ab}[1/(2)R_{pqrs}R^{pqrs}], \] (3.20)

or more explicitly,

\[ -2RR_{ab} + 4R^p_{a}R_{pb} + 4R^{pq}R_{paqb} - 2R^{pq}_{a}R_{pqr} + \frac{1}{2}g_{ab}(R_{pqrs}R^{pqrs} - 4R_{pq}R^{pq} + R^2) = 0. \] (3.21)

Moreover, in four dimensions, we have Xu’s geometrical identity (see [22] as well as the introduction of [23] for a simplest derivation and a clear interpretation)

\[ R^3 - 8RR_{pq}R^{pq} + 8R_{pq}R^p_{rs}R^{qr} + 8R_{pq}R_{rs}R^{pqrs} + RR_{pqrs}R^{pqrs} - 4R_{pq}R^p_{rst}R^{qrst} = 0, \] (3.22)

which permits us to write

\[ H^{(6,3)(1)}_{ab} - 8H^{(6,3)(2)}_{ab} + 8H^{(6,3)(3)}_{ab} + 8H^{(6,3)(4)}_{ab} + H^{(6,3)(5)}_{ab} - 4H^{(6,3)(6)}_{ab} = 0. \] (3.23)
Finally, in four dimensions, the cubic Lovelock Lagrangian $\mathcal{L}_3$ given by (2.44) as well as its metric variation, i.e., the cubic Lovelock tensor $G^{(3)}_{ab}$ given by (2.48), vanish identically. As a consequence, we have with (3.17) or (3.21), (3.22) and (3.23), $\mathcal{L}_3 = 0$ and $G^{(3)}_{ab} = 0$ five new geometrical relations which could permit us to fully simplify the expression (3.15) of the approximated stress–energy tensor. We leave this task to the interested reader because the choice of the (scalar and tensorial) Riemann monomials to be eliminated is a matter of taste and depends on the problem treated as well as on the gravitational background considered. We remark however that in expression (3.14) of this stress–energy tensor, it would be certainly interesting to drop two of the ten terms by using (3.23) as well as $G^{(3)}_{ab} = 0$ in the form

$$
H^{(6,3)(1)}_{ab} = 12H^{(6,3)(2)}_{ab} + 16H^{(6,3)(3)}_{ab} + 24H^{(6,3)(4)}_{ab} + 3H^{(6,3)(5)}_{ab}
$$

$$
-24H^{(6,3)(6)}_{ab} + 4H^{(6,3)(7)}_{ab} - 8H^{(6,3)(8)}_{ab} = 0.
$$

3.3. Infinite counterterms appearing on the left-hand side of the bare Einstein equations in six dimensions

As we have noted in section 3.1, the divergent part of the effective action associated with the scalar field is proportional to (3.6) in a six-dimensional background. It can be removed by renormalization of Newton’s gravitational constant and of the cosmological constant and by adding to the Einstein–Hilbert gravitational Lagrangian three counterterms of order four ($R^2$, $R_{pq}R^{pq}$ and $R_{pqrs}R^{pqrs}$) in order to eliminate the divergences associated with the DeWitt coefficient $a_2$ (see (3.10)) as well as ten counterterms of order six ($R \Box R$, $R \Box R_{pq}^r$, $R \Box R_{pq}^q$, $R \Box R_{pq}^{r,s}$, $R_{pq}R^{pq}$, $R_{pq}R^{pq}$, $R_{pq}R_{rs}R^{rs}$, $R_{pq}R_{pqrs}R^{pqrs}$, $R_{pq}R_{pqrs}R^{pqrs}$, $R_{pq}R_{pqrs}R^{pqrs}$) in order to eliminate the divergences associated with the Gilkey–DeWitt coefficient $a_3$ (see (3.11)). These last ten counterterms induce in the bare Einstein equations a correction of sixth order in the derivative of the metric tensor which is of the form

$$
\alpha_1 H^{(2,0)(1)}_{ab} + \alpha_2 H^{(2,0)(3)}_{ab} + \alpha_3 H^{(6,3)(1)}_{ab} + \alpha_4 H^{(6,3)(2)}_{ab} + \alpha_5 H^{(6,3)(3)}_{ab} + \alpha_6 H^{(6,3)(4)}_{ab} + \alpha_7 H^{(6,3)(5)}_{ab} + \alpha_8 H^{(6,3)(6)}_{ab} + \alpha_9 H^{(6,3)(7)}_{ab} + \alpha_{10} H^{(6,3)(8)}_{ab}
$$

with the coefficients $\alpha_i$ containing terms in $1/(D - 6)$ and so diverging in the physical dimension limit. Furthermore, because in six dimensions the Euler number

$$
\int_M d^6x \sqrt{-g(x)} \mathcal{L}_3(x)
$$

is given by (2.44) is a topological invariant, its metric variation $G^{(3)}_{ab}$ given by (2.48) vanishes identically. We then have here again the constraint (3.24) which could permit us to eliminate one of the ten contributions of order six in the bare Einstein equations.

### 4. Concluding remarks

The metric variations of the gravitational action terms constructed from the curvature invariants of order six in the derivatives of the metric tensor have already been considered by numerous authors (see, for example, [19–21, 24–27]). However, in the works achieved until now, all the possible simplifications due to the symmetries of the Riemann tensor as well as to Bianchi identities have not been systematically done. As a consequence, there did not exist until now, in the literature, explicit irreducible formulae for these functional derivatives as was the case for the functional derivatives with respect to the metric tensor of the action terms constructed from the gravitational Lagrangians $\Box R$, $R^2$, $R_{pq}R^{pq}$ and $R_{pqrs}R^{pqrs}$, i.e., from the curvature
invariants of order four in the derivatives of the metric tensor (see, for example, [9]). In the present paper, by using the results obtained by Fulling, King, Wybourne and Cummings [1] based on group theoretical considerations, we have solved this problem unambiguously, filling a void in quantum field theory in curved spacetime. We have then been able to discuss some aspects of quantization linked to the DeWitt–Schwinger effective action (simplified form, in the large mass limit, for the renormalized effective action in four dimensions and simplified forms for the infinite counterterms of order six in the derivatives of the metric tensor which must appear on the left-hand side of the bare Einstein equations in six dimensions). We think that our results could be helpful not only in four dimensions but also in treating some aspects of the quantum physics of extra spatial dimensions which is currently exploding under the impulsion of string theory. In the near future, these results could be more particularly interesting (i) in order to understand the back reaction problem (in the large mass limit of the quantized fields) for a wide class of metrics [28] and (ii) in order to discuss, from a general point of view, the Hadamard renormalization of the stress–energy tensor in an arbitrary $D$-dimensional spacetime [29] and to study precisely its ambiguities.

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Appendix

A.1. Geometrical identities between Riemann polynomials of order six and ranks zero or two

In the present section of the appendix, we provide some geometrical identities which permit us to eliminate ‘alternative’ Riemann monomials of order six by expressing them in terms of elements of the FKWC-basis. These relations are more generally useful for calculations in two-loop quantum gravity in a four-dimensional background or for calculations in one-loop quantum gravity in higher dimensional background. All these relations can be derived more or less trivially from the ‘symmetry’ properties of the Ricci and the Riemann tensors (pair symmetry, antisymmetry, cyclic symmetry)

\[
R_{ab} = R_{ba}, \quad (A.1a)
\]

\[
R_{abcd} = R_{dcab}, \quad (A.1b)
\]

\[
R_{abcd} = -R_{bacd} \quad \text{and} \quad R_{abcd} = -R_{abdc}, \quad (A.1c)
\]

\[
R_{abcd} + R_{adbc} + R_{acdb} = 0, \quad (A.1d)
\]

from the Bianchi identity and its consequences obtained by contraction of index pairs

\[
R_{abcd,e} + R_{abec,d} + R_{abcd,e} = 0, \quad (A.2a)
\]

\[
R_{abcd,d} = R_{ac,b} - R_{bc,a}, \quad (A.2b)
\]

\[
R_{ab}^{;b} = (1/2)R_{;a}, \quad (A.2c)
\]

as well as from the commutation of covariant derivatives in the form (1.1).
It is of course possible to derive numerous scalar relations between scalar Riemann monomials of order six but, in fact, it seems to us that only five of these relations are really important:

\[ R_{pq} R^{pr} - \frac{1}{2} R_{pq} R_{r}^{;r} R^{pr} - R_{pq} R_{r}^{;r} R^{pr} = 0 \quad (A.3) \]

and

\[ R_{pqrs} R_{pqrs} = 4 R_{pqrs} R_{pqrs} + 2 R_{pqrs} R_{pqrs} - R_{pqrs} R_{pqrs} - 4 R_{pqrs} R_{pqrs} = 0 \quad (A.4) \]

Similarly, among all the tensorial relations between Riemann monomials of order six and rank two, we have chosen to retain more particularly the 15 following ones:

\[ \Box (R_{ab}) = \Box R_{ab} + 2 R_{pq;ab} R^{p}_{b} - 2 R^{pq} R_{pqab} - R_{p} R_{ab} + 2 R_{pq} R_{(a,b)} \quad (A.6) \]

and

\[ R_{p}^{q} R_{p}^{ab} = R_{p}^{q} R_{pqab} + R_{q}^{p} R_{pqab} + R_{p}^{q} R_{pqab} \quad (A.7) \]

and

\[ R_{ab}^{p;qr} = R_{ab}^{p;qr} R_{pqab} + R_{ab}^{p;qr} R_{pqab} - \frac{1}{2} R_{ab}^{p;qr} R_{pqab} \quad (A.8) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.10a) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.10b) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.11) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.12) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.13) \]

and

\[ R_{pqrs} R_{pqrs} = \frac{1}{2} R_{pqrs} R_{pqrs} + \frac{1}{2} R_{pqrs} R_{pqrs} \quad (A.14) \]
and
\[
R^{pqrs} R^t_{pq} R_{rsb} = \frac{1}{4} R^{pqrs} R_{pqta} R_{rsb}, \tag{A.15a}
\]
\[
R^{pqrs} R^t_{pq} R^i_{rb} = -\frac{1}{2} R^{pqrs} R_{pqta} R^i_{rsb}, \tag{A.15b}
\]
\[
R^{pqrs} R^t_{pq} R_{rsb} = \frac{1}{2} R^{pqrs} R_{pqta} R_{rsb}, \tag{A.15c}
\]
\[
R^{pqrs} R^t_{pq} R_{rsb} = R^{pqrs} R^t_{pq} R_{rsb} - \frac{1}{4} R^{pqrs} R_{pqta} R_{rsb}, \tag{A.15d}
\]
\[
R^{pqrs} R^t_{pq} R_{rsb} = \frac{1}{4} R^{pqrs} R_{pqta} R_{rsb}. \tag{A.15e}
\]

There exists in particular a lot of other relations involving terms cubic in the Riemann tensor which are useful in calculations but they can be obtained trivially from the five previous ones.

### A.2. Elementary variations

In this short section of the appendix, we provide a list of relations describing the behaviour of some important geometrical tensors in an elementary variation of the metric tensor. These relations are useful to obtain, in section 2, the functional derivatives with respect to the metric tensor of the action terms constructed from the 17 scalar Riemann monomials of order six.

Apart from two of them which we have established, these relations can be found in [14] but we prefer to collect them (i) in order to avoid the reader having to read this reference and (ii) because in certain cases we have adopted a more practical notation.

In the elementary variation
\[
g_{ab} \rightarrow g_{ab} + \delta g_{ab} \tag{A.16}
\]
of the metric tensor, we have

\[
g^{ab} \rightarrow g^{ab} + \delta g^{ab}, \tag{A.17a}
\]
\[
\sqrt{-g} \rightarrow \sqrt{-g} + \delta(\sqrt{-g}), \tag{A.17b}
\]
\[
\Gamma^e_{ab} \rightarrow \Gamma^e_{ab} + \delta \Gamma^e_{ab}, \tag{A.17c}
\]
\[
R \rightarrow R + \delta R, \tag{A.17d}
\]
\[
\Box R \rightarrow \Box R + \delta(\Box R), \tag{A.17e}
\]
\[
R_{ab} \rightarrow R_{ab} + \delta R_{ab}, \tag{A.17f}
\]
\[
\Box R_{ab} \rightarrow \Box R_{ab} + \delta(\Box R_{ab}), \tag{A.17g}
\]
\[
R_{abcd} \rightarrow R_{abcd} + \delta R_{abcd}, \tag{A.17h}
\]

with
\[
\delta g^{ab} = -g^{pa} g^{qb} \delta h_{pq}, \tag{A.18a}
\]
\[
\delta(\sqrt{-g}) = \frac{1}{2} \sqrt{-gh} \quad \text{with} \quad h \equiv g^{pq} h_{pq}, \tag{A.18b}
\]
\[
\delta \Gamma^e_{ab} = \frac{1}{2} \left( -h_{ab} \tau^e + h^e_{a,b} + h^e_{b,a} \right), \tag{A.18c}
\]
\[
\delta R = h^{pq}_{:pq} - \Box h - R^{pq}_{:pq} h_{pq}, \tag{A.18d}
\]
\[
\delta(\Box R) = \Box \left( h^{pq}_{:pq} \right) - \Box \Box h - 2R^{pq}_{:pq} h_{pq} - R^{pq}_{:pq} \Box h_{pq} - (\Box R^{pq}) h_{pq}
- R^{pq}_{:pq} h_{pq} - R_{,p} h^{pq}_{:q} + \frac{1}{2} R_{,p} h^{pp}, \tag{A.18e}
\]
\[ \delta R_{ab} = \frac{1}{2} \left( h^p_{\; ; ap} + h^p_{\; ; cp} - h_{; ab} - h_{; ab} \right), \]  
(A.18f)

\[ \delta (\Box R_{ab}) = \frac{1}{2} \left[ \Box \left( h^p_{\; ; ap} \right) + \Box \left( h^p_{\; ; cp} \right) - \Box (h_{; ab}) - \Box \Box (h_{; ab}) \right] \]

\[ = R_{ab ; p ; q} + \frac{1}{2} R_{ab ; p} - R_{ab ; pq} \]

\[ - \frac{1}{2} R_{ab \; q} \left( \Box h^p_{\; ; b} + h^p_{\; ; b} \right) + \frac{1}{2} R_{pq} \left( \Box h^p_{\; ; a} + h^p_{\; ; a} \right) - \mathcal{R}_{ab} \]

\[ = R_{ab ; q} \left( h^p_{\; ; b} - h^p_{\; ; b} \right) - R_{pq} \left( h_{a}^{q \; ; p} - h_{a}^{q \; ; p} \right), \quad \text{A.18g} \]

\[ \delta R_{abcd} = \frac{1}{2} \left( h_{abcd} - h_{ab, cd} - h_{ad, bc} - h_{ac, bd} - h_{bd, ac} + h_{bc, ad} + R_{bp} h_{ap} \right). \]

(A.18h)

In the elementary variation (A.16) of the metric tensor, we have moreover

\[ R_{a ; b} \rightarrow R_{a ; b} + \delta (R_{a ; b}), \]  
(A.19a)

\[ R_{ab ; c} \rightarrow R_{ab ; c} + \delta (R_{ab ; c}), \]  
(A.19b)

\[ R_{ab ; c, d} \rightarrow R_{ab ; c, d} + \delta (R_{ab ; c, d}), \]  
(A.19c)

\[ R_{ab ; c, d, e} \rightarrow R_{ab ; c, d, e} + \delta (R_{ab ; c, d, e}), \]  
(A.19d)

with

\[ \delta (R_{a ; b}) = (\delta R)_{a ; b}, \]  
(A.20a)

\[ \delta (R_{ab ; c, d}) = (\delta R)_{ab ; c, d} + (\delta R)_{p a} - R_{p a} \right), \]  
(A.20b)

\[ \delta (R_{ab ; c, d, e}) = (\delta R)_{ab ; c, d, e} - R_{p a} \right), \]  
(A.20c)

\[ \delta (R_{ab ; c, d, e}) = (\delta R)_{ab} - (\delta R)_{a} - (\delta R)_{b} + (\delta R)_{p a} \right), \]  
(A.20d)

\[ \delta (R_{ab ; c, d, e}) = (\delta R)_{ab} - (\delta R)_{a} - (\delta R)_{b} + (\delta R)_{p a} \right), \]  
(A.20e)

A.3. ‘Irreducible’ forms for the approximated stress–energy tensors associated with massive spinor and vector fields in four dimensions

In this last subsection of the appendix, we briefly extend the result obtained in section 3.2 by providing, in the large mass limit, simplified expressions for the approximated stress–energy tensors associated with a massive spinor field and a massive vector field propagating in a four-dimensional background. These results have been obtained very easily and quickly from the formalism developed in section 2 and they explicitly prove the power of this formalism.

In the large mass limit, the renormalized effective action of a neutral spinor field is given by [10, 11]

\[ W_{\text{ren}}^{1/2} = \frac{1}{192 \pi^2 m^2} \int_M d^4 x \sqrt{-g} \left( -\frac{3}{280} R \Box R \right) \]

\[ + \left( \frac{1}{28} \right) R_{pq} \Box R_{pq} + \frac{1}{864} R_{pq} \right) R_{pq} \]

\[ - (25/756) R_{pq} R_{pq} R_{pq} + (47/1260) R_{pq} \]

\[ + (7/1440) R_{pqrs} R_{pqrs} + (19/1260) R_{pq} R_{pq} R_{pq} \]

\[ + (29/7560) R_{pqrs} R_{pqrs} R_{pqrs} + (1/108) R_{pqrs} R_{pq} R_{pq} R_{pq} \]. \]  
(A.21)
By functional derivation of this effective action, we obtain for the associated expectation value of the stress–energy tensor the expression (here we use the notation introduced in section 2)

\[
|T_{ab}^{\text{s=1/2}}|_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}^{\text{s=1/2}}}{\delta g^{ab}} \\
= \frac{1}{96\pi^2m^2} \left( -\frac{3}{280}H_{ab}^{(12,0)(1)} + \frac{1}{28}H_{ab}^{(2,0)(3)} + \frac{1}{864}H_{ab}^{(6,3)(1)} - \frac{1}{180}H_{ab}^{(6,3)(2)} \\
- \frac{25}{756}H_{ab}^{(6,3)(3)} + \frac{47}{1260}H_{ab}^{(6,3)(4)} - \frac{7}{1440}H_{ab}^{(6,3)(5)} \\
+ \frac{19}{1260}H_{ab}^{(6,3)(6)} + \frac{29}{7560}H_{ab}^{(6,3)(7)} - \frac{1}{108}H_{ab}^{(6,3)(8)} \right) \tag{A.22}
\]

and from the relations (2.22)–(2.31), we have then explicitly

\[
(96\pi^2m^2)|T_{ab}^{\text{s=1/2}}|_{\text{ren}} = \left( \frac{1}{700} \right) (\Box R)_{ab} - \left( \frac{1}{28} \right) \Box R_{ab} - \left( \frac{1}{120} \right) RR_{ab} \\
+ \left( \frac{1}{120} \right) (\Box R)_{ab} + \left( \frac{23}{840} \right) R_{p(a}R_{b)}^p + \left( \frac{1}{40} \right) R \Box R_{ab} \\
+ \left( \frac{29}{420} \right) R_{pa(b} \Box R_{b)}^p - \left( \frac{19}{420} \right) R^pq R_{pq(a)}^p \tag{A.23}
\]
In the large mass limit, the renormalized effective action of a vector field is given by [10, 11]

\[
W_{\text{ren}}^{\text{s=1}} = \frac{1}{192 \pi^2 m^2} \int d^4 x \sqrt{-g} \left( -(27/280) R \Box R \\
+ (9/28) R_{pq} \Box R^{pq} - (5/72) R^3 + (31/60) R R_{pq} R^{pq} \right.
\]
\[
- (52/63) R_{pq} R^p R^q - (19/105) R_{pq} R_{rs} R^{pqrs} \right.
\]
\[
- (1/10) R R_{pqrs} R^{pqrs} + (61/140) R_{pq} R_{rs} R^{pqrs} \right.
\]
\[
- (67/2520) R_{pqrs} R^{pquv} R^{rs}_{uv} + (1/18) R_{pqrs} R^p_{~q} R^{rs}_{uv} \right). \quad (A.24)
\]

By functional derivation of this effective action, we obtain for the associated expectation value of the stress–energy tensor the expression (here we use again the notation introduced in section 2)

\[
\langle T_{ab} \rangle_{\text{ren}} = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}^{\text{s=1}}}{\delta g_{ab}}
\]
\[
= \frac{1}{96 \pi^2 m^2} \left( -(27/280) H_{ab}^{(2,0)(1)} \right.
\]
\[
+ (9/28) H_{ab}^{(2,0)(3)} - (5/72) H_{ab}^{(6,3)(1)} + (31/60) H_{ab}^{(6,3)(2)}
\]
\[
- (52/63) H_{ab}^{(6,3)(3)} - (19/105) H_{ab}^{(6,3)(4)} - (1/10) H_{ab}^{(6,3)(5)}
\]
\[
+ (61/140) H_{ab}^{(6,3)(6)} - (67/2520) H_{ab}^{(6,3)(7)} + (1/18) H_{ab}^{(6,3)(8)} \right) \quad (A.25)
\]

and from the relations (2.22)–(2.31), we have then explicitly

\[
(96 \pi^2 m^2) \langle T_{ab} \rangle_{\text{ren}} = \left( (9/70) \Box R \right)_{ab} - (9/28) \Box \Box R_{ab} - (1/10) R \Box R_{ab}
\]
\[
- (7/30) \Box R_{ab} + (13/35) R_{p(a} R^p_{b)} - (7/60) R \Box R_{ab}
\]
\[
+ (337/210) R_{p(a} \Box R^p_{b)} + (22/105) R^{pq} R_{pq,(ab)}
\]
\[
+ (34/105) R^{pq} R_{p(a,bq} - (107/210) R^{pq} R_{ab;pq} + (1/21) R^{pq} R_{paqb}
\]
\[
- (22/35) \Box R^{pq} R_{pqab} + (46/35) R^{pq} R_{pq} R_{paqb} \right.
\]
\[
+ (116/105) R_{p(a} \Box R^p_{b)} - (1/42) R^{pq} R_{pq,;ab} \right.
\]
\[
- (1/24) R_{a}R_{b} + (83/210) R_{p} R^{pq}_{(a;b)} - (41/84) R_{p} R^{pq}_{ab} \right.
\]
\[
+ (11/60) R^{pq}_{;a} R_{pq;b} - (14/15) R_{p(a} R_{b;pq} + (221/210) R^{pq}_{;a} R_{pq;b} \right.
\]
\[
+ (113/210) R^{pq}_{;a} R_{pq;b} + (5/21) R^{pq}_{;r} R_{pq;r} \right.
\]
\[
- (107/210) R^{pq}_{;r} R_{pq,;r} - (17/840) R^{pq}_{;r} R_{pq;r}\right.
\]
\[
- (29/105) R^{pq}_{;a} R_{pqr;b} + (5/24) R^2 R_{ab} - (2/5) R_{pq} R^{pq}_{ab}
\]
\[
- (31/60) R^{pq} R_{pq} R_{ab} + (1/21) R^{pq} R_{pq} R_{ab} - (19/30) R^{pq} R_{pqab}
\]
\[
+ (33/35) R^{pq} R_{pqab} - (319/105) R^{pq} R_{pq} R_{pqab} \right.
\]
\[
+ (1/5) R R^{pq}_{;a} R_{pq,;b} + (1/10) R_{a} R_{b} \right.
\]
\[
- (74/105) R_{p(a} R^{pq}_{;rs} R_{rs;b)} - (5/42) R^{pq} R_{pqab} \right.
\]
\[
+ (74/105) R_{p} R^{pq}_{;a} R_{rs;b} - (71/105) R_{pq} R^{pq}_{rs} R^{pq}_{ab} \right.
\]
\[
+ (37/105) R^{pq}_{;a} R_{pq,;b} + (97/105) R^{pq}_{;a} R_{pq,;b} \right.
\]
\[
- (37/105) R^{pq}_{;a} R_{pq,;b} + g_{ab} (9/280) \Box \Box R + (19/120) R \Box R - (1/84) R_{pq} R^{pq} \right). \]
It should be finally noted that it is possible to simplify (A.22) and (A.25) or (A.23) and (A.26) by using the `topological` and geometrical constraints described at the end of section 3.2 which are independent of the quantum field considered.

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\[
- \frac{(223/420)R_{pq} R^{pq} + (79/105)R_{pqrs} R^{pqrs}}{263} \\
+ \frac{(163/1680)R_{p} R^{p} - (17/56)R_{pqrs} R^{pqrs} + (11/420)R_{pqrs} R^{pqrs}}{223} \\
+ \frac{(51/560)R_{pqrs} R^{pqrs} - (5/144)R^{3} + (31/120)RR_{pq} R^{pq}}{17} \\
+ \frac{(1/630)R_{pq} R^{p} R^{pq} - (53/105)R_{pq} R_{rs} R^{pqrs}}{105} \\
- \frac{(1/20)RR_{pqrs} R^{pqrs} + (2/5)R_{pq} R^{p} R^{pqrs}}{20} \\
- \frac{(263/2520)R_{pqrs} R^{pqrs} R^{rs} - (106/315)R_{pqrs} R^{pqrs} R^{rs} - R^{uv}}{223}. \tag{A.26}
\]