Entanglement criterion based on skew information

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we can establish an entanglement criteria from skew information. our criterion is independent of
Local Uncertainty Relations (LUR) ([3],[4]).

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The following can be found in [1],[3],[4]:
We start with the following observation. Let \( \rho \) be a
density matrix, and let \( M \) be an observable. We denote
the expectation value of \( M \) by \( \langle M \rangle_\rho := \text{Tr}(\rho M) \)
and the variance (or uncertainty) of \( M \) by
\[
\delta^2(M)_\rho := \langle (M - \langle M \rangle_\rho)^2 \rangle_\rho = \langle M^2 \rangle_\rho - \langle M \rangle_\rho^2
\]
(1)

We suppress the dependence on \( \rho \) in our notation, when
there is no risk of confusion. If \( \rho \) is a pure state the variance
is zero iff \( \rho \) is an eigenstate of \( M \). Now we have:

Lemma 1 of [1]. Let \( M_i \) be some observables and
\( \rho = \sum_k p_k \rho_k \) be a convex combination (i.e. \( p_k \geq 0, \sum_k p_k = 1 \))
of some states \( \rho_k \) within some subset \( S \). Then
\[
\sum_i \delta^2(M_i)_\rho \geq \sum_k p_k \sum_i \delta^2(M_i)_{\rho_k}
\]
holds.

In this paper, we will discuss new Relations, and get
new entanglement criteria.

First, note that Wigner and Yanase introduced the fol-
lowing concept in [2], skew information, it was defined as:
\[
I(\rho, M) := \text{Tr}(\rho M^2) - \text{Tr}(\rho^{\frac{1}{2}} M \rho^{\frac{1}{2}} M).
\]
(3)

where \( M \) is an observable.

When \( \rho \) is a pure state, then \( I(\rho, M) \) reduce to the
variance \( \delta^2(M)_\rho \).

\( I(\rho, M) \) has the following celebrated properties:
(a): convex, i.e.,
\[
I(\lambda_1 \rho_1 + \lambda_2 \rho_2, M) \leq \lambda_1 I(\rho_1, M) + \lambda_2 I(\rho_2, M)
\]
for \( \lambda_1 + \lambda_2 = 1, \lambda_1 \geq 0, \lambda_2 \geq 0 \). (4)

(b):
\[
I(\rho_1 \otimes \rho_2, M_1 \otimes I + I \otimes M_2) = I(\rho_1, M_1) + I(\rho_2, M_2)
\]
where \( M_1 \) is an observable of Alice, \( M_2 \) is an observable
of Bob.

We have the following inequality:

Lemma 2. Let \( M_i \) be some observables and
\( \rho = \sum_k p_k \rho_k \) be a convex combination (i.e. \( p_k \geq 0, \sum_k p_k = 1 \))
of some states \( \rho_k \) within some subset \( S \). Then
\[
\sum_i I(\rho, M_i) \leq \sum_k p_k \sum_i I(\rho_k, M_i)
\]
holds. We call a state “violating Lemma 2” iff there are
no states \( \rho_k \) in \( S \) and no \( p_k \) such that Eq. (6) is fulfilled.

Proof. From property (a), we know that the inequality
holds for each \( M_i \): \( I(\rho, M_i) = I(\sum_k p_k \rho_k, M_i) \leq \sum_k p_k I(\rho_k, M_i) \).

\( I(\rho, M) \) and uncertainty \( \delta^2(M)_\rho \) coincide on pure state, i.e. when \( \rho \) is a pure state, then
the inequalities (2) and (6) in lemma 1, 2 both become equality.

let us recall the “Local Uncertainty Relations” (LURs),
introduced by Hofmann and Takeuchi [4]. Let \( A_i \) be ob-
servables on Alice’s space of a bipartite system. If they
do not share a common eigenstate, there is a number
\( C_A > 0 \) such that \( \sum_i \delta^2(A_i)_{\rho_A} \geq C_A \) holds for all states
\( \rho_A \) on Alice’s space. Hofmann and Takeuchi showed:

Proposition 1. [4] Let \( \rho \) be separable and let
\( A_i, B_i, i = 1, ..., n \) be operators on Alice’s (resp. Bob’s) space, fulfilling
\( \sum_{i=1}^n \delta^2(A_i)_{\rho_A} \geq C_A \) and
\( \sum_{i=1}^n \delta^2(B_i)_{\rho_B} \geq C_B \). We define \( M_i := A_i \otimes I + I \otimes B_i \).

Then
\[
\sum_{i=1}^n \delta^2(M_i)_\rho \geq C_A + C_B
\]
holds.

The LURs provide strong criteria which can by con-
bstruction be implemented with local measurements.

We have a dual result of the above Proposition of [3]:

Theorem 1. Let \( \rho \) be separable and let \( A_i, B_i, i = 1, ..., n \) be operators on Alice’s (resp. Bob’s) space, fulfilling
\( \sum_{i=1}^n I(\rho, A_i) \leq C_A \) and \( \sum_{i=1}^n I(\rho, B_i) \leq C_B \).

We define \( M_i := A_i \otimes I + I \otimes B_i \). Then
\[
\sum_{i=1}^n I(\rho, M_i) \leq C_A + C_B
\]
holds.

Proof. From property (b), we know that for product
states, \( I(\rho_A \otimes \rho_B, M_i) = I(\rho_A, A_i) + I(\rho_B, B_i) \), and from
property (a), we know that after mixture, the inequality
(8) holds.

The CCN criterion can be formulated in the following:
see [3]. It makes use of the Schmidt decomposition in
operator space. Due to that, any density matrix \( \rho \) can
be written as
\[ \rho = \sum_k \lambda_k G^A_k \otimes G^B_k. \]  
(9)

where the \( \lambda_k \geq 0 \) and \( G^A_k \) and \( G^B_k \) are orthogonal bases of the observable spaces. Such a basis consists of \( d^2 \) observables which have to fulfill
\[ \text{Tr}(G^A_k G^A_l) = \text{Tr}(G^B_k G^B_l) = \delta_{kl}. \]  
(10)

We refer to such observables as \textit{local orthogonal observables} (LOOs). For instance, for qubits the (appropriately normalized) Pauli matrices together with the identity form a set of LOOs.

**Theorem 2.** To connect our criterion with the LURs, first note that for any LOOs \( G^A_k \) the following relation holds for any states for Alice(Bob).
\[ \sum_{k=1}^{d^2} I(G^A_k) \leq d - 1 \]  
(11)

\textbf{Proof.} Since \( \sum_k \text{Tr}(\rho(G^A_k)^2) = d \mathbb{I} \) and that \( \sum_k \text{Tr}(\rho^{1/2} G^A_k \rho^{1/2} G^A_k) = \text{Tr}(\rho^{1/2})^2 \geq 1 \).

In [3], the authors get the following: for separable states
\[ 1 - \sum_k \langle G^A_k \otimes G^B_k \rangle - \frac{1}{2} \sum_k \langle G^A_k \otimes \mathbb{I} \otimes G^B_k \rangle^2 \geq 0. \]  
(12)

For our criterion, since \( I(\rho, M) := \text{Tr}(\rho M^2) - \text{Tr}(\rho^2 M \rho^2 M) \), so Combining Eq (11) with the method of [3], using the fact that \( \sum_k (G^A_k)^2 = \sum_k (G^B_k)^2 = d \mathbb{I} \) we can repair the above inequality as: for separable states
\[ 1 - \sum_k \langle G^A_k \otimes G^B_k \rangle - \frac{1}{2} \sum_k \text{Tr}(\rho^2 M_k \rho^2 M_k) \leq 0. \]  
(13)

where \( M_k := G^A_k \otimes \mathbb{I} - \mathbb{I} \otimes G^B_k \).

In [3], it was proved that Any state which violates the computable cross norm criterion can be detected by a local uncertainty relation, while the converse is not true.

Numberical experiment show that the our criterion is independent of the LURs.

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