A scale variational principle of Herglotz
By RICARDO ALMEIDA (Aveiro)

Abstract. The Herglotz problem is a generalization of the fundamental problem of the calculus of variations. In this paper, we consider a class of non-differentiable functions, where the dynamics is described by a scale derivative. Necessary conditions are derived to determine the optimal solution for the problem. Some other problems are considered, like transversality conditions, the multi-dimensional case, higher-order derivatives and several independent variables.

1. Introduction

The calculus of variations deals with optimization of a given functional, whose analytic expression is the integral of a given function, that depends on time, space and the velocity of the trajectory:

\[ x \mapsto \int_a^b L(t, x(t), \dot{x}(t)) \, dt. \]

The variational principle of Herglotz can be seen as an extension of such classical theories, but instead of an integral, we have the functional as a solution of a differential equation (see [9], [10]):

\[
\begin{align*}
\dot{z}(t) &= L(t, x(t), \dot{x}(t), z(t)), \quad \text{with } t \in [a, b], \\
z(a) &= z_a.
\end{align*}
\]

Without the dependence of \( z \), we can convert this problem into a calculus of variations problem. In fact, integrating the differential equation

\[ \dot{z}(t) = L(t, x(t), \dot{x}(t)) \]

Mathematics Subject Classification: 49K05, 26A33.
Key words and phrases: calculus of variations, scale derivative.
from \( a \) to \( b \), we obtain

\[ z(b) = \int_a^b \left[ L(t, x(t), \dot{x}(t)) + \frac{z_a}{b - a} \right] dt. \]

Recently, more advances were made namely proving Noether’s type theorems for the variational principle of Herglotz (see e.g. [5], [6], [7], [8], [9], [12]). The aim of this paper is to consider the Herglotz problem, but the trajectories \( x(\cdot) \) may be non-differentiable functions. We believe that this situation may model more efficiently certain physical problems, like fractals.

The organization of the paper is the following. In Section 2 we define what a scale derivative is, following the concept as presented in [2], and we present some of its main properties, like the algebraic rules, integration by parts formula, etc. In Section 3 we prove our new results. After presenting the Herglotz scale problem, we prove a necessary condition that every extremizer must fulfill. Some generalizations of the main result are also presented to complete the study.

2. Scale calculus

We review some definitions and the main results from [2] that we will need. For more on the subject, see references [1], [2], [3].

From now on, let \( \alpha, \beta, h \) be reals in \([0,1]\) with \( \alpha + \beta > 1 \) and \( h \ll 1 \), and consider \( I := [a - h, b + h] \).

**Definition 1.** Let \( f : I \to \mathbb{R} \) be a function. The delta derivative of \( f \) at \( t \) is defined by

\[ \Delta_h[f](t) := \frac{f(t + h) - f(t)}{h}, \quad \text{for} \quad t \in [a - h, b], \]

and the nabla derivative of \( f \) at \( t \) is defined by

\[ \nabla_h[f](t) := \frac{f(t) - f(t - h)}{h}, \quad \text{for} \quad t \in [a, b + h]. \]

If \( f \) is differentiable, then

\[ \lim_{h \to 0} \Delta_h[f](t) = \lim_{h \to 0} \nabla_h[f](t) = f'(t). \]

These two operators can be combined into a single one, where the real part is the mean value of such operators, and the complex part measures the difference between them.
Definition 2. The $h$-scale derivative of $f$ at $t$ is given by

$$\frac{\Box h f}{\Box t} (t) = \frac{1}{2} \left[ (\Delta_h[f](t) + \nabla_h[f](t)) + i (\Delta_h[f](t) - \nabla_h[f](t)) \right],$$

for $t \in [a, b]$. \hfill (1)

For complex valued functions $f$, such definition is extended by

$$\frac{\Box h f}{\Box t} (t) = \frac{\Box h \Re f}{\Box t} (t) + \frac{\Box h \Im f}{\Box t} (t).$$

We now explain how to drop the dependence on the parameter $h$ in the definition of the scale derivative. First, consider the set $C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C})$ of the functions $g \in C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C})$ for which the limit

$$\lim_{h \to 0} g(t,h)$$

exists for all $t \in [a, b]$, and let $E$ be a complementary space of $C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C})$ in $C_0^{0}([a,b] \times [0,1], \mathbb{C})$.

Define $\pi$ the projection of $C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C}) \oplus E$ onto $C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C})$,

$$\pi : C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C}) \oplus E \to C_0^{0,\text{conv}}([a,b] \times [0,1], \mathbb{C})$$

$g := g_{\text{conv}} + g_E \mapsto \pi(g) = g_{\text{conv}}$.

Using these definitions, we arrive at the main concept of [2].

Definition 3. The scale derivative of $f \in C^0(I, \mathbb{C})$, denoted by $\frac{\Box f}{\Box t}$, is defined by

$$\frac{\Box f}{\Box t} (t) := \left\langle \frac{\Box h f}{\Box t} \right\rangle (t), \quad t \in [a, b],$$

where

$$\left\langle \frac{\Box h f}{\Box t} \right\rangle (t) := \lim_{h \to 0} \pi \left( \frac{\Box h f}{\Box t} (t) \right).$$

Definition 4. Given $f : I^n = [a - nh, b + nh] \to \mathbb{C}$, define the higher-order scale derivative of $f$ by

$$\frac{\Box^n f}{\Box t^n} (t) = \left( \frac{\Box^{n-1} f}{\Box t^{n-1}} \right) (t), \quad t \in [a, b],$$

where $\frac{\Box^1 f}{\Box t} := \frac{\Box f}{\Box t}$ and $\frac{\Box^n f}{\Box t^n} := f$. 

We will adopt the notation $\Box^\alpha f(t)$ instead of $\frac{\partial^\alpha f}{\partial t^\alpha}(t)$ when there is no danger of confusion.

Scale partial derivatives are also considered here. They are defined as in the standard case.

**Definition 5.** Let $f : \prod_{i=1}^n [a_i - h, b_i + h] \to \mathbb{R}$ be a function. Define, for each $i \in \{1, \ldots, n\},$

$$\Delta^i_h[f](t_1, \ldots, t_n) := \frac{f(t_1, \ldots, t_{i-1}, t_i + h, t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)}{h},$$

for $t_i \in [a_i - h, b_i]$ and for $t_j \in [a_j - h, b_j + h]$ if $j \neq i,$ and

$$\nabla^i_h[f](t_1, \ldots, t_n) := \frac{f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, t_i - h, t_{i+1}, \ldots, t_n)}{h},$$

for $t_i \in [a_i, b_i + h]$ and for $t_j \in [a_j - h, b_j + h]$, if $j \neq i.$ The $h$-scale partial derivative of $f$ with respect to the $i$-th coordinate is given by

$$\Box^i_h f(t_1, \ldots, t_n) = \frac{1}{2} \left[ (\Delta^i_h[f] + \nabla^i_h[f]) + i (\Delta^i_h[f] - \nabla^i_h[f]) \right],$$

for $t_i \in [a_i, b_i].$

The definition of partial scale derivatives $\Box f/\Box t_i$ is clear.

In what follows, we will denote

$$C^\alpha_\Box([a, b], \mathbb{K}) := \{ f \in C^0(I^n, \mathbb{K}) \mid \Box^k f \in C^0(I^{n-k}, \mathbb{K}) \},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}.$

**Definition 6.** Let $f \in C^0(I, \mathbb{C})$ and $\alpha \in [0, 1].$ We say that $f$ is Hölderian of Hölder exponent $\alpha$ if there exists a constant $C > 0$ such that, for all $s, t \in I,$

$$|f(t) - f(s)| \leq C|t - s|^{\alpha},$$

and we write $f \in H^\alpha(I, \mathbb{C}),$ or simply $f \in H^\alpha$ when there is no danger of mislead.

We say that $f(t_1, \ldots, t_n) \in H^\alpha$ if $f(t_1, \ldots, t_{i-1}, t_i + h, t_{i+1}, \ldots, t_n) \in H^\alpha,$ for all $i \in \{1, \ldots, n\}$ and for all $t_j \in [a_j, b_j], j \neq i.$
Theorem 1. For all $f \in H^\alpha$ and $g \in H^\beta$, we have
\[
\frac{\Box (f \cdot g)}{\Box t}(t) = \frac{\Box f}{\Box t}(t) \cdot g(t) + f(t) \cdot \frac{\Box g}{\Box t}(t), \quad t \in [a, b].
\]

Theorem 2. Let $f \in C^1([a,b], \mathbb{R})$ be such that
\[
\lim_{h \to 0} \int_a^b \left( \frac{\Box h f}{\Box t} \right)_E (t) \, dt = 0,
\]
where
\[
\frac{\Box h f}{\Box t} = \left( \frac{\Box h f}{\Box t} \right)_\text{conv} + \left( \frac{\Box h f}{\Box t} \right)_E.
\]
Then,
\[
\int_a^b \frac{\Box f}{\Box t}(t) \, dt = f(b) - f(a).
\]
As a consequence, we have the following integration by parts formula. If
\[
\lim_{h \to 0} \int_a^b \left( \frac{\Box h (f \cdot g)}{\Box t} \right)_E (t) \, dt = 0,
\]
where $f \in H^\alpha$ and $g \in H^\beta$, then
\[
\int_a^b \frac{\Box f}{\Box t}(t) \cdot g(t) \, dt = [f(t)g(t)]_a^b - \int_a^b f(t) \cdot \frac{\Box g}{\Box t}(t) \, dt.
\]

3. The scale variational principle of Herglotz

The (classical) variational principle of Herglotz is described in the following way. Consider the differential equation
\[
\begin{cases}
\dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), & \text{with } t \in [a, b] \\
z(a) = z_a \\
x(a) = x_a, x(b) = x_b,
\end{cases}
\]
where $x$, $z$ and $L$ are smooth functions. We wish to find $x$ (and the correspondent solution $z$ of the system) such that $z(b)$ attains an extremum. The necessary condition is a second-order differential equation:
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial z} \frac{\partial L}{\partial \dot{x}},
\]
for all $t \in [a, b]$. This can be seen as an extension of the basic problem of calculus of variations. If $L$ does not depend on $z$, then integrating the differential equation along the interval $[a, b]$, we get

$$\int_a^b L(t, x(t), \dot{x}(t)) \, dt \rightarrow \text{extremize}$$

$$x(a) = x_a, \quad x(b) = x_b.$$  

As is well known, many physical phenomena are characterized by non-differentiable functions (e.g., generic trajectories of quantum mechanics [4], scale-relativity without the hypothesis of space-time differentiability [11]). The usual procedure is to replace the classical derivative by a scale derivative, and consider the space of continuous (and non-differentiable) functions. The scale calculus of variations approach was studied in [1], [2], [3] for a certain concept of scale derivative $\Box x(t)$:

$$\int_a^b L(t, x(t), \Box x(t)) \rightarrow \text{extremize}$$

$$x(a) = x_a, \quad x(b) = x_b.$$  

Motivated by this problem, we define the fundamental scale variational principle of Herglotz. First we need to define what extremum is.

**Definition 7.** We say that $z \in C^1([a, b], \mathbb{C})$ attains an extremum at $t = b$ if $z'(b) = 0$.

The problem is then stated in the following way. Consider the system

$$\begin{cases}
\dot{z}(t) = L(t, x(t), \Box x(t), z(t)), & \text{with } t \in [a, b] \\
z(a) = z_a \\
x(a) = x_a, \quad x(b) = x_b.
\end{cases} \quad (4)$$

For simplicity, define

$$[x, z](t) := (t, x(t), \Box x(t), z(t)).$$

We assume that

1. the trajectories $x$ are in $H^\alpha \cap C^1_\Box([a, b], \mathbb{R})$, \( \Box x \in H^\alpha \) and the functional $z$ in $C^2([a, b], \mathbb{C})$,
2. for each $x$, there exists a unique solution $z$ of the system (4),
3. $z_a, x_a, x_b$ are fixed numbers,
(4) the Lagrangian \( L : [a, b] \times \mathbb{R} \times \mathbb{C}^2 \to \mathbb{C} \) is of class \( C^2 \).

Observe that the solution \( z(t) \) actually is a function on three variables, to know \( z = z(t, x(t), \Box x(t)) \). When there is no danger of mislead, we will simply write \( z(t) \).

We are interested in finding a trajectory \( x \) for which the corresponding solution \( z \) is such that \( z(b) \) attains an extremum; and, in particular, what necessary conditions such solutions must fulfill. These equations are called Euler–Lagrange equation types. Again, problem (4) can be reduced to the scale variational problem in case \( L \) is independent of \( z \):

\[
\int_a^b L \left( t, x(t), \Box x(t) + \frac{z_t}{b-a} \right) \, dt \to \text{extremize}.
\]

Theorem 3. If the pair \( (x, z) \) is a solution of problem (4), and \( \frac{\partial L}{\partial \Box x} [x, z] \in H^\alpha(I, \mathbb{C}) \) \( (\alpha \in [0, 1]) \), then \((x, z)\) is a solution of the equation

\[
\frac{\Box}{\partial t} \left( \frac{\partial L}{\partial \Box x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \Box x} [x, z](t), \tag{5}
\]

for all \( t \in [a,b] \).

Proof. Let \( \epsilon \) be an arbitrary real, and consider variation functions of \( x \) of type \( x(t) + \epsilon \eta(t) \), with \( \eta \in H^\beta(I, \mathbb{R}) \cap C^2([a,b], \mathbb{R}) \) \( (\beta \in [0, 1]) \), \( \eta(a) = \eta(b) = \Box \eta(a) = 0 \), and

\[
\lim_{h \to 0} \int_a^b \left( \frac{\Box}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box x} [x, z](t) \right) \right)_{E} \, dt = 0.
\]

The corresponding rate of change of \( z \), caused by the change of \( x \) in the direction of \( \eta \), is given by

\[
\dot{\theta}(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t))|_{\epsilon=0}.
\]

Then

\[
\dot{\theta}(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t))|_{\epsilon=0}
= \frac{d}{d\epsilon} L(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t), z(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t))|_{\epsilon=0}
= \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \frac{\partial L}{\partial \Box x} [x, z](t) \Box \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \theta(t).
\]
We obtain a first order linear differential equation on $\theta$, whose solution is

$$\lambda(b) \theta(b) - \theta(a) = \int_a^b \lambda(t) \left[ \frac{\partial L}{\partial x}[x,z](t) \eta(t) + \frac{\partial L}{\partial \xi x}[x,z](t) \square \eta(t) \right] dt,$$

where

$$\lambda(t) := \exp \left( - \int_a^t \frac{\partial L}{\partial z}[x,z](\tau) d\tau \right).$$

Using the fact that $\theta(a) = \theta(b) = 0$, we get

$$\int_a^b \lambda(t) \left[ \frac{\partial L}{\partial x}[x,z](t) \eta(t) + \frac{\partial L}{\partial \xi x}[x,z](t) \square \eta(t) \right] dt = 0.$$

Integrating by parts the second term, we obtain

$$\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x}[x,z](t) - \frac{\Box}{\Box t} \left( \lambda(t) \frac{\partial L}{\partial \xi x}[x,z](t) \right) \right] \eta(t) dt + \left[ \eta(t) \lambda(t) \frac{\partial L}{\partial \xi x}[x,z](t) \right]_a^b = 0.$$

Since $\eta(a) = \eta(b) = 0$, and $\eta$ is an arbitrary function elsewhere,

$$\lambda(t) \frac{\partial L}{\partial x}[x,z](t) - \frac{\Box}{\Box t} \left( \lambda(t) \frac{\partial L}{\partial \xi x}[x,z](t) \right) = 0,$$

for all $t \in [a,b]$. Since the function $t \mapsto \lambda(t)$ is differentiable, and the function $t \mapsto \frac{\partial L}{\partial \xi x}[x,z](t)$ is in $H^\alpha$, it follows that

$$\lambda(t) \left( \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial \xi x}[x,z](t) \frac{\partial L}{\partial \xi x}[x,z](t) - \frac{\Box}{\Box t} \left( \frac{\partial L}{\partial \xi x}[x,z](t) \right) \right) = 0.$$

Finally, since $\lambda(t) > 0$, for all $t$, we get

$$\frac{\Box}{\Box t} \left( \frac{\partial L}{\partial \xi x}[x,z](t) \right) = \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial \xi x}[x,z](t) \frac{\partial L}{\partial \xi x}[x,z](t),$$

for all $t \in [a,b]$. \hfill \square

**Remark 1.** Assume that the set of state functions $x$ is $C^1([a,b], \mathbb{R})$. Then equation (5) becomes

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \xi x}[x,z](t) \right) = \frac{\partial L}{\partial x}[x,z](t) + \frac{\partial L}{\partial \xi x}[x,z](t) \frac{\partial L}{\partial \xi x}[x,z](t),$$

which is the generalized variational principle of HERGLOTZ as in [10].
Theorem 4. Let the pair \((x, z)\) be a solution of the problem (4), but now \(x(b)\) is free. Then \((x, z)\) is a solution of the equation
\[
\square \frac{\partial L}{\partial \square x}[x, z](t) = \frac{\partial L}{\partial x}[x, z](t) + \frac{\partial L}{\partial z}[x, z](t) \frac{\partial L}{\partial \square x}[x, z](t),
\]
for all \(t \in [a, b]\), and verifies the transversality condition
\[
\frac{\partial L}{\partial \square x}[x, z](b) = 0.
\]

Proof. Following the proof of Theorem 3, the Euler–Lagrange equation is deduced. Then
\[
\eta(t) \lambda(t) \frac{\partial L}{\partial \square x}[x, z](t) \bigg|_a^b = 0.
\]
Since \(\eta(a) = 0\) and \(\eta(b)\) is arbitrary, we obtain the transversality condition. \(\square\)

Multi-dimensional case

For simplicity, we considered so far one state function \(x\) only, but the multi-dimensional case \((x_1, \ldots, x_n)\) is easily studied.

Theorem 5. Let \(\alpha \in ]0, 1[\) and let the vector \((x_1, \ldots, x_n, z)\) be a solution of the problem: find \((x_1, \ldots, x_n)\) that extremizes \(z(b)\), with
\[
\begin{align*}
\dot{z}(t) &= L(t, x_1(t), \ldots, x_n(t), \square x_1(t), \ldots, \square x_n(t), z(t)), \quad \text{with } t \in [a, b] \\
z(a) &= z_a \\
x_i(a) &= x_{ia}, \quad x_i(b) = x_{ib}
\end{align*}
\]
where, for all \(i \in \{1, \ldots, n\}\),
(1) the trajectories \(x_i\) are in \(H^\alpha \cap C^2([a, b], \mathbb{R})\), \(\square x_i \in H^\alpha\) and the functional \(z\) in \(C^2([a, b], \mathbb{C})\),
(2) \(z_a, x_{ia}, x_{ib}\) are fixed numbers,
(3) \(\frac{\partial L}{\partial \square x_i}[x_1, \ldots, x_n, z] \in H^\alpha(I, \mathbb{C})\)
(4) the Lagrangian \(L: [a, b] \times \mathbb{R}^n \times \mathbb{C}^{n+1} \rightarrow \mathbb{C}\) is of class \(C^2\).

Then, for all \(i \in \{1, \ldots, n\}\), \((x_1, \ldots, x_n, z)\) is a solution of the equation
\[
\square \frac{\partial L}{\partial \square x_i}[x_1, \ldots, x_n, z](t) = \frac{\partial L}{\partial x_i}[x_1, \ldots, x_n, z](t) + \frac{\partial L}{\partial z}[x_1, \ldots, x_n, z](t) \frac{\partial L}{\partial \square x_i}[x_1, \ldots, x_n, z](t),
\]
for all \(t \in [a, b]\).
Theorem 6. Let the vector \((x_1, \ldots, x_n, z)\) be a solution of the problem as stated in Theorem 5, but now \(x_i(b)\) is free, for all \(i \in \{1, \ldots, n\}\). Then, for all \(i \in \{1, \ldots, n\}\), \((x_1, \ldots, x_n, z)\) is a solution of the equation
\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}_i}[x_1, \ldots, x_n, z](t) \right) = \frac{\partial L}{\partial x_i}[x_1, \ldots, x_n, z](t) + \frac{\partial L}{\partial z}[x_1, \ldots, x_n, z](t) \frac{\partial L}{\partial \dot{x}_i}[x_1, \ldots, x_n, z](t),
\]
for all \(t \in [a, b]\), and verifies the transversality condition
\[
\frac{\partial L}{\partial \dot{x}_i}[x_1, \ldots, x_n, z](b) = 0.
\]

Higher-order derivatives case

Theorem 7. Let \(\alpha \in [0, 1]\) and let the pair \((x, z)\) be a solution of the problem: find \(x\) that extremizes \(z(b)\), with
\[
\begin{aligned}
\dot{z}(t) &= L(t, x, \square x(t), \ldots, \square^n x(t), z(t)), \quad \text{with } t \in [a, b] \\
z(a) &= z_a \\
\square^i x(a) &= x_{ia}, \, \square^i x(b) = x_{ib}, \quad \text{for all } i \in \{0, \ldots, n-1\},
\end{aligned}
\]
where
1. the trajectories \(x\) are in \(H^\alpha \cap C^\alpha_0([a, b], \mathbb{R})\), \(\square x \in H^\alpha\) and the functional \(z\) in \(C^2([a, b], \mathbb{C})\),
2. \(z_a, x_{ia}, x_{ib}\) are fixed numbers, for all \(i \in \{0, \ldots, n-1\}\),
3. \(\frac{\partial L}{\partial \square^i x}[x, z] \in H^\alpha(I^n, \mathbb{C})\), for all \(i \in \{1, \ldots, n\}\),
4. \([x, z](t) = (t, x, \square x(t), \ldots, \square^n x(t), z(t))\) and \([x](t) = (t, x, \square x(t), \ldots, \square^n x(t))\),
5. the Lagrangian \(L : [a, b] \times \mathbb{R}^{n+1} \to \mathbb{R}\) is of class \(C^2\).

Then, \((x, z)\) is a solution of the equation
\[
\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^n (-1)^{i+1} \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \square^i x}[x, z](t) \right) = 0,
\]
for all \(t \in [a, b]\).

Proof. Let \(x(t) + \epsilon \eta(t)\) be a variation function of \(x\), with \(\epsilon \in \mathbb{R}\) and \(\eta \in H^\beta \cap C^\beta_0([a, b], \mathbb{R})\) (\(\beta \in [0, 1]\)). Also, assume that the variations fulfill the conditions:
1. for all \(i = 0, \ldots, n-1\), \(\square^i \eta(a) = \square^i \eta(b) = 0\), and \(\square^n \eta(a) = 0\),
and rearranging the terms, we get
\[
\lim_{h \to 0} \int_a^b \left( \frac{\Box_h}{\partial t} \left( \lambda(t) \frac{\Box^k}{\partial \Box^i} [x, z][t] \right) \frac{\partial}{\partial \Box^j} \eta(t) \right)_E \, dt = 0.
\]
Define
\[
\theta(t) = \frac{d}{dt} \left|_0^\infty \right. z(t, x(t) + e\eta(t), \Box x(t) + e\Box \eta(t), \ldots, \Box^n x(t) + e\Box^n \eta(t)) \right. |_{e=0}.
\]
Then
\[
\dot{\theta}(t) = \frac{\partial L}{\partial x}[x, z][t] \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial x_i}[x, z][t] \Box^i \eta(t) + \frac{\partial L}{\partial \Box^i}[x, z][t] \theta(t).
\]
Solving this linear ODE, we arrive at
\[
\int_a^b \left( \lambda(t) \frac{\partial L}{\partial x}[x, z][t] \eta(t) + \sum_{i=1}^n \frac{\partial L}{\partial x_i}[x, z][t] \Box^i \eta(t) + \frac{\partial L}{\partial \Box^i}[x, z][t] \theta(t) \right) \, dt = 0,
\]
where
\[
\lambda(t) := \exp \left( - \int_a^t \frac{\partial L}{\partial \Box^i}[x, z][\tau] \, d\tau \right).
\]
Integrating by parts \( n \) times, we obtain the following:
\[
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x}[x, z][t] + \sum_{i=1}^n (-1)^i \frac{\Box^i}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box^j}[x, z][t] \right) \eta(t) \right] \, dt = 0,
\]
and rearranging the terms, we get
\[
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x}[x, z][t] + \sum_{i=1}^n (-1)^i \frac{\Box^i}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box^j}[x, z][t] \right) \eta(t) \right] \, dt = 0,
\]
and
\[
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial \Box^i}[x, z][t] + \sum_{i=1}^n (-1)^i \frac{\Box^i}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box^j}[x, z][t] \right) \eta(t) \right] \, dt = 0.
\]
Since \( \Box^i \eta(a) = \Box^i \eta(b) = 0 \), for all \( i \in \{0, \ldots, n-1\} \) and \( \eta \) is arbitrary elsewhere, we get
\[
\lambda(t) \frac{\partial L}{\partial x}[x, z][t] + \sum_{i=1}^n (-1)^i \frac{\Box^i}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box^i}[x, z][t] \right) = 0,
\]
for all \( t \in [a, b] \).
Theorem 8. Let the pair \((x, z)\) be a solution of the problem as stated in Theorem 7, but now \(\square^i x(b)\) is free, for all \(i \in \{0, \ldots, n-1\}\). Then, \((x, z)\) is a solution of the equation

\[
\lambda(t) \frac{\partial L}{\partial x}[x, z](t) + \sum_{i=1}^{n} (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial x^i} [x, z](t) \right) = 0,
\]

for all \(t \in [a, b]\), and verifies the transversality condition

\[
\sum_{k=i}^{n} (-1)^{k-i} \frac{\partial^k}{\partial t^k} \left( \lambda(t) \frac{\partial L}{\partial x^k} [x, z](t) \right) = 0 \quad \text{at} \quad t = b,
\]

for all \(i \in \{1, \ldots, n\}\).

Several independent variables case

We generalize Theorem 3 for several independent variables. First we fix some notations. The variable time is \(t \in [a, b]\), \(x = (x_1, \ldots, x_n) \in \Omega := \prod_{i=1}^{n} [a_i, b_i]\) are the space coordinates and the state function is \(u := u(t, x)\).

Theorem 9. Let \(\alpha \in \{0, 1\}\) and let the pair \((u, z)\) be a solution of the problem:

find \(u\) that extremizes \(z(b)\), with

\[
\begin{cases}
\dot{z}(t) = \int_{\Omega} L \left( t, x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, z(t) \right) d^n x, & \text{with } t \in [a, b] \\
z(a) = z_a \\
u(t, x) \text{ takes fixed values, } \forall t \in [a, b] \forall x \in \partial \Omega \\
u(t, x) \text{ takes fixed values, } \forall t \in \{a, b\} \forall x \in \Omega,
\end{cases}
\]

(7)

where, for all \(i \in \{1, \ldots, n\}\),

1. the trajectories \(u\) are in \(H^\alpha(I \times \Omega, \mathbb{R}) \cap C_1(I \times \Omega, \mathbb{R})\), \(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \in H^\alpha([a, b] \times \Omega, \mathbb{C})\) and the functional \(z\) in \(C^2([a, b], \mathbb{C})\),
2. \(z_a\) is a fixed number,
3. \(d^n x = dx_1 \ldots dx_n\),
4. \(\frac{\partial L}{\partial t}[u, z], \frac{\partial L}{\partial x}[u, z] \in H^\alpha(I \times \Omega, \mathbb{C})\), where \(\frac{\partial L}{\partial t}[u, z]\) denotes the partial derivative of \(L\) with respect to the variable \(\frac{\partial u}{\partial t}\), and \(\frac{\partial L}{\partial x}[u, z]\) denotes the partial derivative of \(L\) with respect to the variable \(\frac{\partial u}{\partial x_i}\), and \([u, z](t) = (t, x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, z(t))\),
5. \(L : [a, b] \times \Omega \times \mathbb{R} \times C^{n+2} \to \mathbb{C}\) is of class \(C^2\).
Then, \((u, z)\) is a solution of the equation

\[
\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial [u, z]_2}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial [u, z]}[u, z](t) d^n x - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial [u, z]_1}[u, z](t) \right) = 0,
\]

for all \(t \in [a, b]\) and for all \(x \in \Omega\).

**Proof.** Let \(u(t, x) + \epsilon \eta(t, x)\) be a variation function of \(u\), with \(\epsilon \in \mathbb{R}\) and \(\eta \in H^\beta(I \times \Omega, \mathbb{R}) \cap C^1([a, b] \times \Omega, \mathbb{R})\) \((\beta \in ]0, 1[)\). Also, assume that the variations fulfill the conditions:

1. \(\eta(t, x) = 0, \quad \forall t \in [a, b] \forall x \in \partial \Omega\),
2. \(\eta(t, x) = 0, \quad \forall t \in \{a, b\} \forall x \in \Omega\),
3. \(\frac{\partial \eta}{\partial x_i}(a, x) = \frac{\partial \eta}{\partial x_i}(b, x) = 0, \quad \forall x \in \Omega\),
4. for all \(i = 1, 2, \ldots, n\),

\[
\lim_{h \to 0} \int_{a}^{b} \left( \frac{\partial}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial [u, z]_1}[u, z](t) \eta(t) \right) \right) dt = 0.
\]

Proceeding with some calculations, we arrive at the ODE

\[
\dot{\theta}(t) - \int_{\Omega} \frac{\partial L}{\partial [u, z]_2}[u, z](t) d^n x \theta(t) = \int_{\Omega} \frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial [u, z]_1}[u, z](t) \frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \frac{\partial L}{\partial [u, z]_1}[u, z](t) \frac{\partial \eta}{\partial x_i} d^n x.
\]

Solving the ODE, and taking into consideration that \(\theta(a) = \theta(b) = 0\), we get

\[
\int_{a}^{b} \lambda(t) \left[ \frac{\partial L}{\partial u}[u, z](t) \eta + \frac{\partial L}{\partial [u, z]_1}[u, z](t) \frac{\partial \eta}{\partial t} + \sum_{i=1}^{n} \frac{\partial L}{\partial [u, z]_1}[u, z](t) \frac{\partial \eta}{\partial x_i} \right] d^n x dt = 0.
\]
Integrating by parts, and considering the boundary conditions over $\eta$, we get

$$
\int_{a}^{b} \int_{\Omega} \left[ \lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left( \lambda(t) \frac{\partial L}{\partial t}[u, z](t) \right) \right. \\
- \left. \sum_{i=1}^{n} \frac{\square}{\square x_i} \left( \lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) \right] \eta d^n x dt = 0.
$$

By the arbitrariness of $\eta$, it follows that for all $t \in [a, b]$ and for all $x \in \Omega$,

$$
\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\square}{\square t} \left( \lambda(t) \frac{\partial L}{\partial t}[u, z](t) \right) - \sum_{i=1}^{n} \frac{\square}{\square x_i} \left( \lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) = 0.
$$

Since $\lambda(t) > 0$, this condition implies that

$$
\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial t}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial z}[u, z](t) d^n x - \frac{\square}{\square t} \left( \frac{\partial L}{\partial t}[u, z](t) \right) \\
- \sum_{i=1}^{n} \frac{\square}{\square x_i} \left( \frac{\partial L}{\partial x_i}[u, z](t) \right) = 0,
$$

for all $t \in [a, b]$ and for all $x \in \Omega$, and the theorem is proved. \qed

Acknowledgements. We would like to thank the two reviewers for their insightful comments on the paper, as these comments led us to an improvement of the work. This work was supported by Portuguese funds through the CIDMA – Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology (FCT-Fundaç̧ão para a Ciência e a Tecnologia), within project UID/MAT/04106/2013.

References

[1] R. Almeida and D. F. M. Torres, Nondifferentiable variational principles in terms of a quantum operator, *Math. Methods Appl. Sci.* **34** (2011), 2231–2241.

[2] J. Cresson and I. Greff, A non-differentiable Noethers theorem, *J. Math. Phys.* **52** (2011), 023513, 10 pp.

[3] J. Cresson and I. Greff, Non-differentiable embedding of Lagrangian systems and partial differential equations, *J. Math. Anal. Appl.* **384** (2011), 626–646.

[4] R. Feynman and A. Hibbs, Quantum Mechanics and Path Integrals, *MacGraw-Hill, New York*, 1965.

[5] B. Gheorgheva and R. Guenthe, First Noether-type theorem for the generalized variational principle of Herglotz, *Topol. Methods Nonlinear Anal.* **20** (2002), 261–273.
A scale variational principle of Herglotz

[6] B. Georgieva and R. Guenther, Second Noether-type theorem for the generalized variational principle of Herglotz, *Topol. Methods Nonlinear Anal.* 26 (2005), 307–314.
[7] B. Georgieva, R. Guenther and T. Bodurov, Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem, *J. Math. Phys.* 44 (2003), 3911–3927.
[8] R. B. Guenther, J. A. Gottsch and D. B. Kramer, The Herglotz algorithm for constructing canonical transformations, *SIAM Rev.* 38 (1996), 287–293.
[9] R. B. Guenther, C. M. Guenther and J. A. Gottsch, The Herglotz Lectures on Contact Transformations and Hamiltonian Systems, Lecture Notes in Nonlinear Analysis, Vol. 1, *Jabusz Schauder Center for Nonlinear Studies, Nicholas Copernicus University, Torún*, 1996.
[10] G. Herglotz, Berührungstransformationen, Lectures at the University of Göttingen, Göttingen, 1930.
[11] L. Nottale, The theory of scale relativity, *Internat. J. Modern Phys. A* 7 (1992), 4899-4936.
[12] J. C. Orum, R. T. Hudspeth, W. Black and R. B. Guenther, Extension of the Herglotz algorithm to nonautonomous canonical transformations, *SIAM Rev.* 42 (2000), 83–90.
[13] S. Santos, N. Martins and D. F. M. Torres, Higher-order variational problems of Herglotz type, *Vietnam J. of Math.* 42 (2014), 409–419.

RICARDO ALMEIDA
CENTER FOR RESEARCH AND DEVELOPMENT
IN MATHEMATICS AND APPLICATIONS (CIDMA)
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AVEIRO
3810-193 AVEIRO
PORTUGAL
E-mail: ricardo.almeida@ua.pt

(Received August 11, 2015; revised December 22, 2015)