Stochastic comparisons, differential entropy and varentropy for distributions induced by probability density functions*

Antonio Di Crescenzo(1), Luca Paolillo(2), Alfonso Suárez-Llorens(3)

(1) Dipartimento di Matematica, Università degli Studi di Salerno
Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy
Email: adicrescenzo@unisa.it – ORCID: 0000-0003-4751-7341

(2) Dipartimento di Matematica, Università degli Studi di Salerno
Via Giovanni Paolo II, 132, 84084 Fisciano (SA), Italy
Email: lpaolillo@unisa.it – ORCID: 0000-0001-7146-4863

(3) Dpto. Estadística e Investigación Operativa, Universidad de Cádiz
Facultad de Ciencias, Campus Universitario
Río San Pedro s/n, 11510 Puerto Real, Cádiz, Spain
Email: alfonso.suarez@uca.es – ORCID: 0000-0002-7679-9328

Abstract

Stimulated by the need of describing useful notions related to information measures, we introduce the ‘pdf-related distributions’. These are defined in terms of transformation of absolutely continuous random variables through their own probability density functions. We investigate their main characteristics, with reference to the general form of the distribution, the quantiles, and some related notions of reliability theory. This allows us to obtain a characterization of the pdf-related distribution being uniform for distributions of exponential and Laplace type as well. We also face the problem of stochastic comparing the pdf-related distributions by resorting to suitable stochastic orders. Finally, the given results are used to analyse properties and to compare some useful information measures, such as the differential entropy and the varentropy.

Keywords: Differential Entropy; Differential Varentropy; Stochastic orders; Information Measures; Decreasing Rearrangement.

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1 Introduction

It is well known that various information measures for absolutely continuous random variables are given by expectations of functions of probability densities. Typical examples in this area

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involves the differential entropy and the related varentropy. These notions have been largely used in several contexts in order to describe the information content and the variability of stochastic systems. However, in some cases the results dealing with these measures are not easily manageable, and require non-trivial efforts in order to appropriately compare the quantities under consideration. Stimulated by the need of dealing with effective probabilistic and statistical tools for assessing the above mentioned information measures, in this paper we introduce the so-called ‘pdf-related distributions’. These are constructed by means of transformation of absolutely continuous random variables through their own probability density functions (pdf’s).

Let us recall that, given an absolutely continuous random variable $X$ having pdf $f(x)$, the interest on transformations of the form $Y = f(X)$ stems in various statistical contexts.

(a) The expectation $E[f(X)]$ is involved in the computation of the relative asymptotic efficiency of the Wilcoxon test relative to the $t$-test (cf. Section 1 of Hodges and Lehmann [19], for instance). Additionally, it is also used in Ahmad and Kochar [1] for testing the classical dispersive ordering between two univariate random variables.

(b) This quantity is also employed in Information Theory for constructing suitable measures of uncertainty. Indeed, the term $\int f(x)^2 \, dx$ is involved in the definition of the ‘Rényi entropy’ of order 2 (see, for instance, Nadarajah and Zografos [21]) and of the ‘differential entropy’ of $X$ as the expectation of the transformation $-\log(f(X))$ (cf. Definition D.2 of Lad et al. [20]).

(c) The analysis of the distribution function of $f(X)$, for $f$ continuous, arises in problems concerning the so-called ‘confidence level’ of $f$, defined as $P[f(X) \leq f(a)]$ and used in Rommel et al. [24] for applications in Functional Data.

Clearly inspired by the previous studied, in this paper we investigate the main properties of the pdf-related distributions, with special reference to (i) the general form of their distribution functions, (ii) the connections with various notions of interest in reliability theory (such as the residual lifetime), (iii) the determination of quantiles. We analyse pdf-related distributions generated by general distributions. However, special attention is devoted to the case when the underlying distributions possess monotone or unimodal pdf’s. Here, unimodality refers to strictly monotone pdf’s or to pdf’s that are first strictly increasing and then strictly decreasing.

The special form that is involved in their definition allows to tackle the problem of comparing the pdf-related distributions by resorting to suitable stochastic orders, such as the usual stochastic order, the dispersive order, the convex trasform order, the star order and the kurtosis order. Hence, in this framework the main results involve both location and variability orderings, and are obtained by techniques based on variability, convexity and rearrangement of pdf’s.

As main application, the pdf-related distributions are finalized to construct a stochastic framework aimed to compare the basic information measures for absolutely continuous random variables. This allows to come to new results on the differential entropy, on the varentropy, and on the varentropy of residual lifetimes.

This is the plan of the paper. In Section 2 we recall some useful notions on random lifetimes and residual lifetimes, also with reference to the related information content, the differential entropy and the differential varentropy. Then, necessary results on unimodal distributions and their residual lifetime distributions are recalled. In Section 3 we introduce the notion of distributions induced by probability density functions, namely pdf-related distributions. Under suitable monotonicity or unimodality assumptions we investigate their distributions and the related residual lifetime distributions, expressed in terms of quantiles. We also obtain a characterization of the pdf-related distribution being $(0, 1)$-uniform for distributions of exponential
and Laplace type. Section 4 focus on results based on stochastic orders, mainly finalized to perform stochastic comparisons between pdf-related distributions, also with reference to re-arrangements of distributions. In Section 5 further stochastic comparisons are performed for the pdf-related distributions, and are also finalized to compare the differential entropy and varentropy of random lifetimes and of residual lifetimes. The new results of the paper are essentially located in Sections 4 and 5.

Throughout the paper we denote by \( X | B \) a random variable having the same distribution of \( X \) conditional on \( B \). Moreover, we write \( Y =_{st} X \) when \( X \) and \( Y \) are identically distributed. Furthermore, \( \phi'(x) \) denotes the derivative of any differentiable function \( \phi(x) \).

2 Background

To improve the readability of the paper, we establish the regularity conditions such as the random variable \( X \) is absolutely continuous having a strictly increasing cumulative distribution function (cdf) \( F(x) = \mathbb{P}(X \leq x) \). We will denote by \( F^{-1}(u) \) the quantile function of \( X \).

Now we provide some key definitions that we will use along the paper. Let \( X \) be a lifetime random variable such that \( F(0) = 0 \). Given a unit which has survived up to time \( t \), its residual life is given by

\[
X_t := \mathbb{X} - t|X > t, \quad t \in S_X.
\]

(1)

By denoting \( F_t(x) = \mathbb{P}(X_t \leq x) \) the cdf of \( X_t \), the survival function and pdf of \( X_t \) are given respectively by:

\[
F_t(x) = 1 - F_t(x) = \frac{F(x + t)}{F(t)},
\]

(2)

\[
f_t(x) = \frac{dF_t(x)}{dx} = \frac{f(x + t)}{F(t)}.
\]

(3)

The conventional approach used to characterize the failure distribution of \( X \) is either by its (instantaneous) hazard rate function

\[
\lambda(x) = \frac{f(x)}{F(x)} = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}[X \leq x + h|X > x], \quad x \in S_X,
\]

(4)

or by the cumulative hazard rate function of \( X \),

\[
\Lambda(x) = - \log F(x) = \int_0^x \lambda(s) \, ds, \quad x \in S_X,
\]

(5)

both functions playing a relevant role in numerous contexts. On the other hand, \( X \) is said to be Increasing Failure Rate (IFR) if \( \lambda(x) \) increases in \( x \). In the same manner we define DFR (Decreasing Failure Rate) if \( \lambda(x) \) decreases in \( x \).

Another key concept is given by the information content of \( X \) defined as the following random variable

\[
IC(X) := - \log f(X).
\]

(6)

This random variable plays a relevant role in Information Theory, since it describes the amount of uncertainty contained in the outcome of \( X \). Indeed, it is well-known that the differential
entropy of $X$ is given by the expectation of $IC(X)$, i.e. (cf. Cover and Thomas \[5\])

$$H(X) := \mathbb{E}[IC(X)] = -\mathbb{E} [\log f(X)] = - \int_{-\infty}^{\infty} f(x) \log f(x) \, dx. \quad (7)$$

In this framework, a relevant notion that measures the variability of the information content is the differential varentropy of $X$, given by (cf. Fradelizi \[12\])

$$V(X) := \text{Var}[IC(X)] = \text{Var}[- \log f(X)] = \mathbb{E}[(IC(X))^2] - [H(X))^2$$

$$= \int_{-\infty}^{\infty} f(x) \log f(x)^2 \, dx - \left[ \int_{-\infty}^{\infty} f(x) \log f(x) \, dx \right]^2. \quad (8)$$

The previous definitions can be also stated for the residual life. It is well known that the residual entropy is given by (cf. Ebrahimi \[8\] and Ebrahimi and Kirmani \[9\])

$$H(X_t) = \mathbb{E}[IC(X_t)] = - \int_{t}^{\infty} f(x) \log \frac{f(x)}{F(t)} \, dx, \quad t \in S_X. \quad (9)$$

The following alternative expressions hold, for $t \in D$:

$$H(X_t) = -\Lambda(t) - \frac{1}{F(t)} \int_{t}^{\infty} f(x) \log f(x) \, dx = 1 - \frac{1}{F(t)} \int_{t}^{\infty} f(x) \log \lambda(x) \, dx.$$

The residual varentropy is (cf. Di Crescenzo and Paolillo \[4\])

$$V(X_t) = \text{Var}[IC(X_t)] = \int_{t}^{\infty} \left( \log \frac{f(x)}{F(t)} \right)^2 \, dx - [H(X_t)]^2$$

$$= \frac{1}{F(t)} \int_{t}^{\infty} f(x) \log f(x)^2 \, dx - [\Lambda(t) + H(X_t)]^2, \quad (10)$$

where $\Lambda(t)$ and $H(X_t)$ are given in \[13\] and \[9\], respectively. See, also, Goodarzi et al. \[13\] and \[14\] for various bounds to the variance of measures in dynamic settings.

We finalize recalling the concept of unimodality which simplifies notably the computation of the distribution function of the transformation $f(X)$ as we will see later on. There are different ways to study the notion of unimodal density on $\mathbb{R}$. We consider one of the most general ways based on the following definition (see Sudhakar and Kumar (1988), page 2).

**Definition 2.1** A random variable $X$ or its distribution function $F$ is called unimodal about a mode (or vertex) $m_F$ if $F$ is convex on $(-\infty, m_F)$ and concave on $(m_F, \infty)$.

A simple consequence of the Definition 2.1 is that if $F$ is unimodal about $m_F$, then apart from a possible mass at $m_F$, $F$ is absolutely continuous and if this is the case then the unimodality of $F$ about $m_F$ is equivalent to the existence of a density $f$, which is non-decreasing on $(-\infty, m_F)$ and non-increasing on $(m_F, \infty)$. Note that this density function $f$ may be constant in a set of positive measure, or not be continuous in a countable number of points on its support. The uniform, Cauchy, Weibull, Gamma, exponential and normal distributions are some examples that satisfies Definition 2.1.

**Remark 2.1** Let $X$ be a random variable under the regularity conditions having a unimodal cdf $F$. If $y \in (0, \infty)$ belongs to the image of the density function $f$, it is easily seen from Definition 2.1 that the set $\{x \in \mathbb{R} : f(x) > y\}$ is always an interval, where we will denote by $l_y$ and $u_y$ the lower and upper limit of that interval, respectively. It is also known that the mode, $m_F$, satisfies that $l_y \leq m_F \leq u_y$. Additionally, if $f$ is symmetric it is apparent that $l_y + u_y = 2m_F$. 

4
Remark 2.2 It is apparent from the Definition 2.1 and the expression (2) that the unimodality of the underlying random variable \( X \) implies the unimodality of the residual life distribution \( X_t \), for all \( t \in S_X \). Additionally, the mode of \( X_t \) is achieved at 0 for all \( t \geq m_F \).

Remark 2.3 Let \( X \) be a random variable under the regularity conditions having a cdf \( F \). Given \( u \in (0, 1) \) and \( t \in S_X \), it is straightforward to show that \( F_t(x) \) also satisfies the regularity conditions and its inverse at \( u \) is given by \( F_t^{-1}(u) = F^{-1}(1 - (1-u)F(t)) - t \).

3 Pdf-related distributions

Let us now define a special type of distribution, named “pdf-related distribution”. This is defined by means of a transformation expressed by the pdf of a given baseline random variable.

Definition 3.1 Let \( X \) be a random variable under the regularity conditions having a pdf \( f(x) \). Then, the random variable \( f(X) \) is named as the pdf-related random variable of \( X \). For any \( y \in \text{Im}^+(f) \) the distribution function of \( f(X) \) is defined as
\[
K(y) := \mathbb{P}(f(X) \leq y).
\]

We first emphasize how \( K(y) \) is affected by location and scale changes. Let \( X \) be a random variable under the regularity conditions; a straightforward computation shows that
\[
K_{aX+b}(y) := \mathbb{P}(f_{aX+b}(aX + b) \leq y) = K_X(|a|y), \tag{12}
\]
where \( a, b \in \mathbb{R}, a \neq 0 \). In such a case it is clear that \([a]f_{aX+b}(aX + b) =_{st} f(X)\). It is also remarkable that \( f_{-X}(-X) =_{st} f(X)\), therefore \( X \) and \(-X\) share the same pdf-related distribution.

Obtaining the distribution function \( K(y) \) is a difficult exercise in practice. However, the assumption of unimodality considerably simplifies its computation. In this case, from Remark 2.1 we obtain that
\[
K(y) := 1 - \mathbb{P}(f(X) > y) = F(l_y) + F(u_y), \quad y \in \text{Im}^+(f). \tag{13}
\]

Example 3.1 From Eq. (13) it is not hard to see that if \( X \) has pdf
\[
f(x) = \left(1 - \left(1 - \frac{1}{x}\right)^{\alpha-1}\right)^{\frac{1}{\alpha-1}}, \quad 0 < x < \frac{\alpha}{\alpha-1},
\]
for \( \alpha > 1 \), then \( K(y) = y^\alpha \), for \( 0 \leq y \leq 1 \). Note that the case \( \alpha \rightarrow 1 \) corresponds to the exponential distribution, \( X \sim \text{Exp}(1) \).

It is worth pointing out that a pdf-related distribution is not necessarily continuous. Indeed, for instance, if \( X \) is uniformly distributed on \((a, b)\), then clearly \( f(X) \) is degenerate in \((b-a)^{-1}\).

Hereafter we obtain a characterization of the pdf-related distribution being \((0, 1)\)-uniform based on exponential and Laplace distributions. It involves the lower and upper limits introduced in Remark 2.1

Proposition 3.1 Let \( X \) be a random variable under the regularity conditions and having a unimodal pdf \( f(x) \) with mode \( m_F \in \mathbb{R} \). Then, \( f(X) \) is uniform on \((0, 1)\) if and only if
\[
l_y - u_y = \ln(y), \quad \forall y \in \text{Im}^+(f).
\]
Additionally, (i) if \( f \) is symmetric with support \( \mathbb{R} \), then \( f(x) = \exp(-2|x - m_F|) \), \( x \in \mathbb{R} \), (ii) if the support of \( f \) is \([m_F, +\infty)\), then \( f(x) = \exp(m_F - x) \), \( x \in [m_F, +\infty) \), and finally, (iii) if the support of \( f \) is \((-\infty, m_F]\), then \( f(x) = \exp(x - m_F) \), \( x \in (-\infty, m_F] \).

**Proof.** From Eq. (13) we know that

\[
\begin{align*}
\text{(i)} & \quad \text{if the support of } f \text{ is } [m_F, +\infty), \text{ then } f(x) = \exp(m_F - x), \quad x \in [m_F, +\infty), \\
\text{(ii)} & \quad \text{if the support of } f \text{ is } (-\infty, m_F], \text{ then } f(x) = \exp(x - m_F), \quad x \in (-\infty, m_F].
\end{align*}
\]

The rest of the proof follows by taking into account that, in the 3 cases, \( l_y - u_y \) is given by \( 2(l_y - m_F) \), \( m_F - u_y \) and \( l_y - m_F \), respectively. \( \square \)

**Example 3.2** An example of non-symmetric pdf leading to the uniform distribution is provided by the following parametric density family

\[
f(x) = \begin{cases} 
\exp \left( \frac{x - m_F}{1 - \alpha} \right), & x \leq m_F \\
\exp \left( \frac{-(x - m_F)}{\alpha} \right), & x \geq m_F.
\end{cases}
\]

Indeed, in this case we have that \( l_y - u_y = \ln(y) \), \( \forall y \in \text{Im}^+(f) \), so that \( f(X) \) is uniform on \((0, 1)\), for all \( \alpha \in (0, 1) \). Note that \( f(x) \) is symmetric only for \( \alpha = 0.5 \).

In the following, when the pdf \( f \) has support \( S_X = (a, b) \), we adopt the following notation: \( f(a) = \lim_{x \to a^+} f(x) \) and \( f(b) = \lim_{x \to b^-} f(x) \).

As application of the construction of pdf-related distributions, hereafter we determine the cdf of the pdf-related random variable of the residual lifetime defined in (1), denoted as

\[
K_t(y) := \mathbb{P}(f_t(X_t) \leq y) = \mathbb{P}(f(X) \leq yF(t) \mid X > t),
\]

with the last identity due to (3).

**Proposition 3.2** Let \( X \) be an absolutely continuous random variable having support \( S_X = (0, b) \).

(a) If the pdf \( f \) is continuous and strictly monotonic in \((t_0, b)\) for a given \( t_0 \in (0, b) \), then for all \( t \in (t_0, b) \) one has

\[
K_t(y) = \begin{cases} 
\frac{F(f^{-1}(yF(t)))}{F(t)}, & y \in \text{Im}^+(f_t) = \left( \frac{f(b)}{F(t)}, \lambda(t) \right), \text{ if } f \text{ is decreasing,} \\
1 - \frac{F(f^{-1}(yF(t)))}{F(t)}, & y \in \text{Im}^+(f_t) = \left( \lambda(t), \frac{f(b)}{F(t)} \right), \text{ if } f \text{ is increasing,}
\end{cases}
\]

where \( f^{-1} \) denotes the inverse of the restriction to \((t_0, b)\) of \( f \).
(b) Let the pdf $f$ be continuous in $S_X$, unimodal and symmetric, strictly increasing for $x \in (0,m]$ and strictly decreasing for $x \in [m,b)$, with $m = b/2$. Then, for all $t \in (0,m]$ one has

$$
K_t(y) = \begin{cases} 
\frac{F(f^{-1}(yF(t)))}{F(t)}, & y \in \left(\frac{f(0)}{F(t)}, \lambda(t)\right) \\
2F(f^{-1}(yF(t)))-F(t), & y \in \left[\lambda(t), \frac{f(m)}{F(t)}\right)
\end{cases}
$$

where $f^{-1}$ is the inverse of the restriction to $(0,m]$ of $f$.

**Proof.** In the case (a), if $f$ is decreasing, from (14) for all $t \in (t_0,b)$ and $y \in \text{Im}^+(f_t)$ we have $K_t(y) = \mathbb{P}(X \geq f^{-1}(yF(t)) | X > t)$. Eq. (15) then follows straightforwardly. When $f$ is increasing the procedure is analogous. In the case (b), when $y \in \left(\frac{f(0)}{F(t)}, \lambda(t)\right)$ the function $K_t(y)$ can be obtained as in case (a), whereas when $y \in \left[\lambda(t), \frac{f(m)}{F(t)}\right)$ we have

$$
K_t(y) = \mathbb{P}(t < X \leq f^{-1}(yF(t)) | X > t) + \mathbb{P}(b - f^{-1}(yF(t)) \leq X \leq b | X > t).
$$

The expression given in (16) thus follows by noting that $F(b-x) = F(x)$ for all $0 \leq x \leq b$. □

Clearly, the cdf given in (16) is continuous in $y = \lambda(t)$, with

$$
K_t(\lambda(t)) = \frac{F(t)}{F(t)}, \quad t \in (0,m],
$$

where the right-hand-side is the odds function of $X$ evaluated at $t \leq m$.

We conclude this section with an example concerning the distribution treated in Proposition 3.2.

**Example 3.3** Let $X$ be a Pareto-type random variable having pdf and cdf given respectively by

$$
f(x) = \frac{1}{(1+x)^2}, \quad x \in (0, +\infty), \quad F(x) = \frac{x}{1+x}, \quad x \in [0, +\infty).
$$

Since $f$ is strictly decreasing for all $x \in (0, +\infty)$, with inverse $f^{-1}(y) = y^{-1/2} - 1$, $y \in (0,1)$, from (15) we have that, for all $t > 0$

$$
K_t(y) = (y(1+t))^{1/2}, \quad y \in \left(0, \frac{1}{1+t}\right).
$$

4 Results based on stochastic orders

With the goal of comparing information measures as the ones given by the differential entropy and the varentropy, we next recall some classical results involving stochastic orders. Roughly speaking, stochastic orders deals about the comparison of two random quantities and can be divided in three kind of comparisons: magnitude, variablity and shape. Let $X$ and $Y$ be two random variables having cdf’s $F(x)$ and $G(x)$ and pdf’s $f(x)$ and $g(x)$, respectively. The following definitions will be relevant.

(i) According to the magnitude, we will say that $X$ is said to be smaller than $Y$ in the usual stochastic order, denoted by $X \leq_{st} Y$, if $F(x) \geq G(x)$, for all $x \in \mathbb{R}$. In some sense, the previous
inequality says that $X$ is less likely than $Y$ to take on large values. The usual stochastic order deals with a characterization in terms of the expectations of increasing transformations, i.e., $X \leq_{st} Y$ if and only if

$$E[g(X)] \leq E[g(Y)]$$

holds for all increasing functions $g$ for which the expectations exist. In particular, just considering $g(x) = x$, the usual stochastic order implies the comparison of the means.

(ii) According to the variability, we will say that $X$ is smaller than $Y$ in the dispersive order, denoted by $X \leq_{disp} Y$, if $F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$, for all $0 < u \leq v < 1$. The dispersive order has a characterization in terms of the density functions evaluated at quantiles, i.e., $X \leq_{disp} Y$, if and only if

$$f(F^{-1}(u)) \geq g(G^{-1}(u)), \quad \text{for all } u \in (0, 1).$$

(iii) According to the shape, we will say that $X$ is smaller than $Y$ in the convex transform order, denoted by $X \leq_{c} Y$, if $G^{-1}(F(x))$ is convex in $x$ on the support of $F(x)$. The convex transform order can be also characterized in terms of the density functions evaluated at quantiles, i.e., $X \leq_{c} Y$, if and only if

$$f(F^{-1}(p)) \geq g(G^{-1}(p)), \quad \text{is increasing in } p \in (0, 1).$$

(iv) According to the shape, if $X$ and $Y$ are non-negative random variables we will say that $X$ is smaller than $Y$ in the star order, denoted by $X \leq_{s} Y$, if

$$\frac{G^{-1}(p)}{F^{-1}(p)}, \quad \text{is increasing in } p \in (0, 1) \iff \log(X) \leq_{disp} \log(Y).$$

(v) According to the shape, if $X$ and $Y$ have symmetric density functions, we will say that $X$ is smaller than $Y$ in the kurtosis order, denoted by $X \leq_{k} Y$, if $G^{-1}(F(x))$ is concave for all $x < Me(X)$, or, equivalently, $G^{-1}(F(x))$ is convex for all $x > Me(X)$, where $Me(X) = F^{-1}(0.5)$ represents the median of $X$. It is pointed out in Oja [22] that the kurtosis order can be characterized in terms of the convex transform order, i.e., $X \leq_{k} Y$ if and only if

$$|X - Me(X)| \leq_{c} |Y - Me(Y)|.$$  

For a comprehensive review on the background of stochastic orders we refer the reader to Shaked and Shantikumar [25].

To conclude the exposition of the stochastic orders we recall the notion of rearrangement of a function which has played a key role in many inequalities in literature. It is described in the seminal book of Hardy et al. [16] and has been studied in the context of entropy, randomness, majorization and dispersion in Hickey [17], [18] and Fernández-Ponce and Suárez-Llorens [11], among other authors. We next recall the decreasing rearrangement of a density function and some of its important properties.

**Definition 4.1** Let $X$ be an absolutely continuous random variable having a pdf $f(x)$. The decreasing rearrangement of $f(x)$, denoted by $f^*(x)$, is given by

$$f^*(x) = \sup\{c : m(c) > x\}, \quad x > 0,$$

where $m(c) = \mu\{t : f(t) > c\}$, $\mu$ denoting the Lebesgue measure.
From Hardy et al. [15], we know that the decreasing rearrangement satisfies the following identity
\[
\int_0^\infty f^*(x) \, dx = \int_{-\infty}^\infty f(x) \, dx = 1.
\]
Then \( f^* \) can be considered as a pdf of a particular random variable that we will denote by \( X^* \).

The following implication is well-known (see Hardy et al. [15], Theorems 9 and 10). Let \( X \) and \( Y \) be absolutely continuous random variables having pdf \( f \) and \( g \), respectively. Then
\[
X^* \leq_{st} Y^* \iff \int_{-\infty}^\infty u(f(x)) \, dx \leq \int_{-\infty}^\infty u(g(x)) \, dx,
\]
for all continuous and concave functions \( u \). It is worth mentioning that \( X^* \leq_{st} Y^* \) was equivalently described as \( \int_0^t f^*(x) \, dx \geq \int_0^t g^*(x) \, dx \) for all \( t > 0 \) in Hickey [17] where it was interpreted as continuous majorisation.

The ordering \( X^* \leq_{st} Y^* \) implies a particular comparison between \( f(X) \) and \( g(Y) \). Rewriting the right side of expression (22), we obtain that \( X^* \leq_{st} Y^* \) is equivalent to
\[
E \left[ \frac{u(f(X))}{f(X)} \right] \leq E \left[ \frac{u(g(Y))}{g(Y)} \right],
\]
for all continuous and concave functions \( u \). The latter relation provides a class of measures of entropy. Indeed, as it is interpreted in Hickey [17], just taking \( u(x) = -x \log(x) \), one has that if \( X^* \leq_{st} Y^* \) then \( H(X) \leq H(Y) \) follows directly from the definition of the differential entropy given in (7). To finalize, the following remarks are straightforward and they will be useful later on.

**Remark 4.1** If \( X \) has a symmetric and unimodal pdf \( f(x) \), then \( X^* =_{st} 2|X - Me(X)| \).

**Remark 4.2** If \( X \) has a strictly decreasing pdf in the interval \((0, +\infty)\), then \( f^*(x) = f(x) \) for all \( x \in (0, +\infty) \), i.e., \( X^* =_{st} X \).

At this point we wonder about what kind of comparisons are we interested in. Keeping in mind that we focus on the comparison of information measures, it seems logical that we are looking for comparisons between two pdf-related random variables \( f(X) \) and \( g(Y) \) that lead to the comparison of the differential entropy and varentropy. For example, from the previous stochastic order definitions we have the following results.

**Lemma 4.1** Let \( X \) and \( Y \) be two random variables under the regularity conditions having pdf’s \( f(x) \) and \( g(x) \), respectively, and having finite differential entropy. If \( f(X) \leq_{st} g(Y) \), then
\[
H(X) \geq H(Y).
\]

**Proof.** From the hypothesis assumption, the expression of the differential entropy given in (7) and considering \( g(x) = \log(x) \), the proof follows easily just using (17). \( \square \)

**Lemma 4.2** Let \( X \) and \( Y \) be two random variables under the regularity conditions having pdf’s \( f(x) \) and \( g(x) \), respectively, and having finite differential varentropy. If \( f(X) \leq_{st} g(Y) \), then
\[
V(X) \leq V(Y).
\]
Proof. From expression (20), \( f(X) \leq_{s} g(Y) \) is equivalent to \( \log f(X) \leq_{disp} \log g(Y) \) which implies that \( \text{Var}[\log f(X)] \leq \text{Var}[\log g(Y)] \) (see, for instance, Section 3.B.2 of [23]). The assertion then follows from the definition of the varentropy given in [8]. \( \square \)

A natural question is whether a certain stochastic order between \( X \) and \( Y \) implies a stochastic comparison between \( f(X) \) and \( g(Y) \), which in turn leads to the comparison of the information measures. The following two examples deserve to be highlighted.

Example 4.1 The usual stochastic ordering between the rearrangements \( X^{*} \) and \( Y^{*} \), \( X^{*} \leq_{st} Y^{*} \), implies the comparison of \( f(X) \) and \( g(Y) \) provided in [24], which in turn implies the comparison of the differential entropy, \( H(X) \leq H(Y) \).

Example 4.2 From [18], if \( X \leq_{disp} Y \) then \( f(F^{-1}(u)) \geq g(G^{-1}(u)) \) for all \( u \in (0, 1) \). Just considering the inverse probability integral transformation and Theorem 1.A.1 in [25], if \( X \leq_{disp} Y \) holds then \( f(X) \geq_{st} g(Y) \) and using Lemma 4.1 we obtain that \( H(X) \leq H(Y) \). For example, it is well known that if \( X \) is IFR, then \( X_{t_{2}} \leq_{disp} X_{t_{1}} \) for all \( t_{1} \leq t_{2} \), see Belzunce et al. [4] and Pellerey and Shaked [23]. From the previous arguments it is apparent that if \( X \) is IFR then \( H(X_{t}) \) is decreasing when \( t \) increases. This result is well known in the literature, see Ebrahimi [8]. A similar result holds for DFR distributions by exchanging all inequalities and replacing decreasing for increasing in the residual differential entropy.

We finalize this section with a result concerning with the comparison of affine transformations.

Proposition 4.1 Let \( X \) be a random variable under the regularity conditions with pdf \( f(x) \), and let \( Y = aX + b \), \( a, b \in \mathbb{R} \), \( a \neq 0 \), having density \( g \). Then, \( f(X) =_{s} g(Y) \) and \( f(X) \geq_{st} (\leq_{st})g(Y) \) if \( |a| \geq (\leq)1 \). Therefore \( V(X) = V(Y) \) and \( H(X) \leq (\geq)H(Y) \) if \( |a| \geq (\leq)1 \).

Proof. By using (12), we easily obtain that \( |a|K_{Y}^{-1}(u) = K_{X}^{-1}(p) \). Then, \( f(X) =_{s} g(Y) \) follows directly by (20). Also using (12), we obtain that \( K_{Y}(y) = K_{X}(|a|y) \geq (\leq)K_{X}(y) \) if \( |a| \geq (\leq)1 \). Therefore, \( f(X) \geq_{st} (\leq_{st})g(Y) \) if \( |a| \geq (\leq)1 \) just using the definition of the stochastic order. The rest of the proof is a direct consequence of Lemma 4.1 and Lemma 4.2 \( \square \)

We would like to emphasize that the relation \( f(X) =_{s} g(Y) \) implies that both pdf-related distribution functions have proportional quantile functions. In such a case, they belong to the same scale family of distributions (see, for instance, Section 4.1 of Di Crescenzo et al. [6]).

5 Differential entropy, varentropy and stochastic orders

As we showed in Example 4.2, the dispersive order, \( \leq_{disp} \), between the underlying random variables \( X \) and \( Y \) implies the usual stochastic order, \( \leq_{st} \), between the pdf-related distributions, \( f(X) \) and \( g(Y) \), which in turn provides the comparison of the differential entropy as discussed in Lemma 4.1. Additionally, we also showed in Example 4.1 that the usual stochastic order, \( \leq_{st} \), between the rearrangements \( X^{*} \) and \( Y^{*} \) of the underlying random variables \( X \) and \( Y \) implies a particular randomness order, see expression (23), between the pdf-related distributions which in turn also provides the comparison of the differential entropy. In this section, we are interested.
in the comparison of the differential varentropy, i.e., we will provide the following type of results:

\[ X \leq_{(1)} Y \implies f(X) \leq_s g(Y) \implies V(X) \leq V(Y) \quad (24) \]

where \( \leq_{(1)} \) is any particular stochastic order and the second implication follows directly from Lemma 4.2. This type of results helps us to provide many possible examples to compare the differential varentropy.

As we mentioned, the rearrangement captures in some sense the degree of randomness in a density function and it has a close connection with the differential entropy. Now we wonder if the rearrangement is also related with the concept of varentropy. From now on, we will assume that the density functions have no flat zones or, equivalently, the rearrangements are strictly decreasing. Next result is of the (24) type.

**Theorem 5.1** Let \( X \) and \( Y \) be two random variables under the regularity conditions having pdf’s \( f \) and \( g \), respectively, with no flat zones. Then

\[ X^* \leq_s Y^* \iff f(X) \leq_s g(Y). \]

**Proof.** From Eq. (2.1) of Hickey [17], the distribution function of \( X^* \) can be expressed as \( F_{X^*}(t) = \mathbb{P}(f(X) > f^*(t)) \). The use of strict inequality in the set \( \{ x \in \mathbb{R} : f(x) > f^*(x) \} \) follows from the assumption of \( f \) having no flat zones. Therefore, one has \( F_{f(X)}(f^*(t)) = 1 - F_{X^*}(t), t \in \mathbb{R} \). Because \( f^* \) is strictly decreasing, we have

\[ F_{f(X)}^{-1}(p) = f^*(F_X^{-1}(1-p)), \quad 0 < p < 1. \quad (25) \]

Then, recalling (20) the relation \( f(X) \leq_s g(Y) \) is satisfied if and only if

\[ \frac{F_{f(X)}^{-1}(p)}{F_{g(Y)}^{-1}(p)} = \frac{g^*(G_Y^{-1}(1-p))}{f^*(F_X^{-1}(1-p))} \]

is increasing in \( p \in (0, 1) \),

which is equivalent to \( X^* \leq_s Y^* \) by virtue of (15).

\[ \square \]

**Remark 5.1** From expression (25) and taking into account the inverse probability integral transform, we obtain that \( f(X) =_{st} f^*(X^*) \). Therefore, it is clear that random variables having the same rearrangements have also the same differential entropy and varentropy. Here we provide an illustrative example. Let \( X \) and \( Y \) be two random variables taking values in \([-1, 1]\), with density functions \( f \) and \( g \) given by

\[ f(x) = 1 - |x|, \quad -1 \leq x \leq 1, \quad g(x) = |x|, \quad -1 \leq x \leq 1, \]

respectively. It is easy to compute that

\[ m_f(c) = m_g(c) = \mu \{ x : g(x) > c \} = 2(1 - c)I_{[0,1]}(c). \]

Then \( f^*(x) = g^*(x) = (1 - x/2)I_{[0,2]}(x) \). Therefore, we have that \( f^*(X^*) =_{st} g^*(Y^*) \) and thus we conclude that \( H(X) = H(Y) \) and \( V(X) = V(Y) \). It is worth mentioning that \( f \) is unimodal but \( g \) is not.

In practice, Theorem 5.1 provides many possible comparisons as we show in the following results.
Corollary 5.1 Let $X$ and $Y$ be two symmetric unimodal random variables under the regularity conditions. Then,

$$X \leq_k Y \quad \text{if and only if} \quad f(X) \leq_* g(Y).$$

Proof. Using jointly expression (21) and Remark 4.1 we obtain that $X \leq_k Y$ if and only if $X^* \leq_c Y^*$, due to the well-known fact that the transform convex order does not depend on scale changes. The result follows easily just applying Theorem 5.1. □

Corollary 5.1 provides many possible comparisons in terms of the differential varentropy. We find in Arriaza et al. [3] a compilation of many unimodal symmetric distributions that are ordered in the kurtosis sense. For example, we know that the classical normal, logistic and Cauchy distributions satisfy that normal $\leq_k$ logistic $\leq_k$ Cauchy. Therefore, just applying Corollary 5.1 and Lemma 4.2 we obtain that the differential varentropy are ordered, i.e.,

$$V(\text{normal}) \leq V(\text{logistic}) \leq V(\text{Cauchy}).$$

Corollary 5.2 Let $X$ and $Y$ be two random variables under the regularity conditions having decreasing density functions with modes $m_X$ and $m_Y$, respectively. Then,

$$X \leq_c Y \quad \text{if and only if} \quad f(X) \leq_* g(Y).$$

Proof. Since changes on location do not affect the transform convex order and using (12) we can consider $m_x = m_y = 0$ without loss of generality. Using Remark 4.2 we obtain that $X \leq_c Y$ if and only if $X^* \leq_c Y^*$. The result follows easily just applying Theorem 5.1. □

Corollary 5.2 also provides many possible comparisons in terms of the differential varentropy as we will see in the following results.

Theorem 5.2 Let $X$ be a nonnegative absolutely continuous random lifetime with support $S_X$ having a cdf $F$ and a pdf $f(x)$. Let $X_t$ be its residual lifetime at time $t$, as defined in (1). Let us suppose that there exists $t_0 \in S_X$ such the pdf of $X_t$, defined in (3), is strictly decreasing $\forall t \geq t_0$. If the ratio

$$\frac{f(F^{-1}(1 - (1 - p)v))}{f(F^{-1}(1 - (1 - p)u))}$$

increases (decreases) in $p \in (0, 1)$, $\forall 0 < v < u < 1$, then $V(X_t)$ increases (decreases) in $t \geq t_0$.

Proof. Given $t_1$ and $t_2$ such that $t_0 < t_1 < t_2$ and using first Corollary 5.2 and then Lemma 4.2 we just need to check that $X_{t_1} \leq_c (\geq_c) X_{t_2}$. From the characterization of the convex transform order given in (19) and from the expressions of the pdf of the residual life given in (23) and the inverse of the distribution function of the residual life given in Remark 2.3 we obtain that

$$X_{t_1} \leq_c (\geq_c) X_{t_2} \iff \frac{f(F^{-1}(1 - (1 - p)\overline{F}(t_1)))}{f(F^{-1}(1 - (1 - p)\overline{F}(t_2)))} \text{ is increasing (decreasing) in } p \in (0, 1).$$

The result follows from the hypothesis just considering $v = \overline{F}(t_2)$ and $u = \overline{F}(t_1)$. □
Example 5.1 Let $X \sim \text{Weibull}(k, \lambda)$ be a Weibull distribution with shape parameter $k$ and scale parameter $\lambda$. From the expression of the cdf of $X$ given by $F(x) = 1 - \exp(-(x/\lambda)^k)$, $x \geq 0$, it is a straightforward matter to compute the ratio

$$\frac{f(F^{-1}(1 - (1 - p)u))}{f(F^{-1}(1 - (1 - p)v))} = \frac{u}{v} \left( \frac{\ln((1 - p)u)}{\ln((1 - p)v)} \right)^{k-1},$$

for all $0 < v < u < 1$. It follows easily that the above ratio is increasing (decreasing) when $k > (\leq) 1$ and it is constant for $k = 1$. It is well-known that Weibull distributions are always unimodal having decreasing density functions after the mode. Then, the pdf’s of the residual lives after the mode satisfy the conditions of Theorem 5.2, see Remark 2.2. Then, $V(X_t)$ increases (decreases) $\forall t \geq \text{mode}$, for $k > (\leq) 1$. On the other hand, the Weibull distribution is IFR (DFR) for $k > (\leq) 1$. Then, just using Example 4.3 we obtain that the differential entropy and the varentropy of the residual lives have opposite directions of growth, namely, if $k > 1$ we obtain that $V(X_t)$ increases and $H(X_t)$ decreases, and the opposite for $k < 1$. The case $k = 1$ is just the exponential distribution where the residual lives have both differential entropy and varentropy constant. Some examples of the varentropy of the residual lives in the Weibull distributions are shown computationally in [7].

We recall that in Theorem 3.4 of [7] it is proved that if a random lifetime $X$ is ILR, i.e. its pdf is log-concave, then the residual varentropy [10] is such that $V(X_t) \leq 1$ for all $t$ in the support of $X$. The condition of log-concave density means that the pdf belongs to the class of strong unimodal densities. Hereafter we show a similar result making use of the above findings.

**Theorem 5.3** Let $X$ be a nonnegative random lifetime under the regular conditions with support $S_X$, and let $X_t$ be its residual lifetime at time $t$ as defined in (1). If, for a given $t \in S_X$, $X_t$ is IFR and its pdf [3] is strictly decreasing, then $V(X_t) \leq 1$.

**Proof.** Since, for a given $t \in S_X$, the residual lifetime $X_t$ is IFR, then for Theorem 4.B.11 of [25] one has that $X_t \leq \text{Exp}(1)$. Moreover, by assumption $X_t$ has a strictly decreasing pdf $f_t(x)$ for all $x \geq 0$. Then, using Corollary 5.2 we know that $X_t \leq \text{Exp}(1)$ if and only if

$$f_{X_t}(X_t) \leq \ast \ g(\text{Exp}(1)),$$

where $g$ is the pdf of Exp(1). Hence, making use of Lemma 4.2 we find

$$V(X_t) \leq V(\text{Exp}).$$

Finally, the assertion follows recalling that $V(\text{Exp}) = 1$. 

**Remark 5.2** From Remark 2.2 we have that the conditions of Theorem 5.3 easily hold for all unimodal IFR distributions when $t$ is greater than the mode of $X$. For example, this is the case of the residual lives of the Weibull distribution for $k > 1$, as it is described in Example 5.1. After the mode the varentropy increases but it has an upper bound equal to 1.

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Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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