Long term evolution of CFS-unstable neutron stars and role of differential rotation on short time-scales

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ABSTRACT

I consider differential rotation, associated with radiation-driven Chandrasekhar-Friedman-Schutz (CFS) instability and its possible observational evidences. I focus on the evolution of the apparent spin frequency, which is typically associated with the motion of a specific point on the stellar surface (e.g., polar cap). I start from long-term evolution (on the timescale when instability significantly changes the spin frequency). For this case, I reduce evolution equations to one differential equation and demonstrate that it can be directly derived from energy conservation law. This equation governs evolution rate through sequence of thermally equilibrium states and provides linear coupling for the cooling power and rotation energy losses via gravitational wave emission. In particular, it shows that differential rotation do not affect long-term spin-down. On the contrary, on the short timescales, differential rotation can significantly modify the apparent spin-down, if one examines a strongly unstable star with very small initial amplitude of unstable mode. This statement is confirmed by consideration of Newtonian non-magnetized perfect fluid and dissipative stellar models as well as magnetized stellar model. For example, despite that widely applied evolution equations predict effective spin to be constant in absence of dissipation, the CFS-unstable star should be observed as spinning-down. However, effects of differential rotation on apparent spin-down are negligible for realistic models of neutron star recycling, unless: (1) the neutron star is not magnetized, and (2) r-mode amplitude is modulated faster than the shear viscosity dissipation timescale, being large enough that spin-down can be measured on modulation timescale.

Key words: stars: neutron; instabilities; stars: evolution

1 INTRODUCTION

The radiation-driven instability for rotating stars (Chandrasekhar-Friedman-Schutz, or CFS instability) was suggested by Friedman & Schutz (1978a,b). It is essentially a perturbation enhancement in a rotating star by emission of gravitational waves (GWs). For non-dissipative star models, the instability occurs at arbitrary rotation rate (Andersson 1998; Friedman & Morsink 1998). The dissipation suppresses instability up to a threshold frequency, which depends on temperature (e.g., Lindblom, Owen & Morsink 1998). CFS instability is crucial for neutron stars (NSs) astrophysics, because for most rapidly rotating NSs it can take form of spontaneous excitation of r-modes. R-modes are specific type of oscillating modes, supported by the Coriolis force with predominantly toroidal type of oscillations. R-mode instability can alter NSs evolution (Levin 1999; Ho & Lai 2000; Bondarescu, Teukolsky & Wasserman 2007; Alford & Schwenzer 2014b; Gusakov, Chugunov & Kantor 2014b), so that even a new class of neutron stars may emerge – HOt and Fast Non-Accreting Rotators (HOF-NARs) – which stay hot for billions of years, thanks to r-mode instability (see Chugunov, Gusakov & Kantor 2014 details). Neutron stars with unstable r-modes are considered to be a potential source for a GW astronomy (e.g., Kokkotas & Schwenzer 2016). Even if real NSs are, in fact, stable throughout their lifetime, one can put observational constraints to the critical frequency of the instability and juxtapose them to the theoretical models of instability suppression. Thus, an opportunity to select more viable model would certainly provide new information on neutron stars (see, e.g., Haskell 2015 for recent review).

Up-to-date models of r-mode instability embrace full diversity of neutron star microphysics: they depend on NS core composition (e.g., Jones 2001; Lindblom & Owen 2002; Nayyar & Owen 2006; Andersson, Haskell & Comer 2010; Alford & Schwenzer 2014a), superfluidity (Lindblom & Mendell 2000;
I discuss r-mode instability in magnetized perfect fluid (of Section Kantor & Gusakov Ho & Lai Owen et al. 4 Ho & Lai appeal to the Lagrangian perturbation theory by Haskell, Andersson & Passamonti 2009, 2.2 I discuss several models of differential rotation. I demonstrate that on long timescales the evolution rate through the sequence of thermally equilibrium solutions is presented (in a more detailed form) in Section 4.3. In particular, I apply the results by Friedman et al. (2016), obtained for unstable r-modes in perfect fluid Newtonian stellar models and demonstrate that the unstable star should be observed as spinning down at the rate $\Omega_{\mathrm{app}} \sim (0.5 \div 0.8) \Omega < 0$, dependent on the location of the observed point on the stellar surface (here and below numerical factors correspond to polytropic EOS, $P \propto \rho^{1+1/n}$ with $n = 1$ and r-mode with azimuthal number $m = 2$). In Subsection 4.2 I examine differential rotation in dissipative non-magnetized star. Analysis is performed within generalized toy model by Friedman et al. (2017) and shows that differential rotation can affect apparent spin down rate for growing (unsaturated) and, in some cases, saturated r-modes (corresponding criteria are suggested). In Subsection 4.3 I discuss r-mode instability in magnetized perfect fluid star, applying the results by Friedman et al. (2017) (see Chugunov et al. (2017b) for short summary). They demonstrate that magnetic field suppresses differential motion of fluid elements. Thus, with a good reason I conclude that the apparent spin rate of change $\dot{\Omega}_{\mathrm{app}}$ should be equal to $2\Omega/3 < 0$ for growing modes and to $\Omega$ for saturated r-mode.

In the final Subsection 4.4 I qualitatively assess the role of differential rotation in a realistic scenario of a neutron star recycling by accretion in a low mass X-ray binaries (LMXBs). I argue that in this case the differential rotation can hardly affect the apparent spin down significantly because of the slow evolution of the r-mode amplitude. The latter is associated with a specific feature of neutron star evolution in the instability window: the unstable mode is either saturated or star resides near the instability region’s boundary. However, under specific (and rather unrealistic) conditions, the differential rotation can affect apparent spin down rate. I conclude in Section 5.

I argue that the apparent spin down rate, averaged over long timescale, should not be affected by differential rotation. However, on a short timescale, differential rotation contribution to the apparent spin down can be the same order of magnitude as the decrease rate of the mean rotation frequency.

Following Paper I, in Section 3 I illustrate the results considering r-mode instability in slow rotating Newtonian star models as an example and compare evolution equations with those widely applied (Owen et al. 1998; Ho & Lai 2000). This section also provides a basis for a further discussion of differential rotation in Section 4. As discussed in Paper I, at first glance, my evolution equations differ from those by Owen et al. (1998); Ho & Lai (2000), but this difference is shown to be spurious and associated with mean spin frequency definition. Namely, I define $\Omega$ as derivative of rotation energy over angular momentum, Eq. (1), but definition of $\Omega$ by Owen et al. (1998); Ho & Lai (2000) is corrected for canonical angular momentum of the mode. In particular, this divergence leads to different rate of change of $\Omega$ and $\Omega_{\mathrm{app}}$, which is the most striking in case of r-mode instability in non-dissipative stellar models: equations by Owen et al. (1998); Ho & Lai (2000) predict $\Omega_{\mathrm{app}} = \Omega$, but $\Omega$ is decreasing, and the spin down rate is exponentially growing. Subsequent discussion in Section 4 reveals that the apparent frequency, in fact, decreases for any position of the observed point, even if there is no dissipation, thus favouring $\Omega$ and evolution equations of Paper I. I also discuss the term, suggested by Ho & Lai (2000), which couples evolution of mode amplitude with external (non r-mode) torques.

In Section 4 I discuss several models of differential rotation associated with r-modes as a testbench for general arguments from Subsection 2.3. In particular, I apply the results by Friedman et al. (2016), obtained for unstable r-modes in perfect fluid Newtonian stellar models and demonstrate that the unstable star should be observed as spinning down at the rate $\Omega_{\mathrm{app}} \sim (0.5 \div 0.8) \Omega < 0$, dependent on the location of the observed point on the stellar surface (here and below numerical factors correspond to polytropic EOS, $P \propto \rho^{1+1/n}$ with $n = 1$ and r-mode with azimuthal number $m = 2$). In Subsection 4.2 I examine differential rotation in dissipative non-magnetized star. Analysis is performed within generalized toy model by Friedman et al. (2017) and shows that differential rotation can affect apparent spin down rate for growing (unsaturated) and, in some cases, saturated r-modes (corresponding criteria are suggested). In Subsection 4.3 I discuss r-mode instability in magnetized perfect fluid star, applying the results by Friedman et al. (2017) (see Chugunov et al. (2017b) for short summary). They demonstrate that magnetic field suppresses differential motion of fluid elements. Thus, with a good reason I conclude that the apparent spin rate of change $\dot{\Omega}_{\mathrm{app}}$ should be equal to $2\Omega/3 < 0$ for growing modes and to $\Omega$ for saturated r-mode.

In the final Subsection 4.4 I qualitatively assess the role of differential rotation in a realistic scenario of a neutron star recycling by accretion in a low mass X-ray binaries (LMXBs). I argue that in this case the differential rotation can hardly affect the apparent spin down significantly because of the slow evolution of the r-mode amplitude. The latter is associated with a specific feature of neutron star evolution in the instability window: the unstable mode is either saturated or star resides near the instability region’s boundary. However, under specific (and rather unrealistic) conditions, the differential rotation can affect apparent spin down rate. I conclude in Section 5.

Andersson & Comer 2001; Yoshida & Lee 2003b,a; Lee & Yoshida 2003; Andersson, Glampedakis & Haskell 2009; Haskell, Andersson & Passamonti 2009; Gusakov, Chugunov & Kantor 2014a,b; Kantor & Gusakov 2017; Dommes, Kantor & Gusakov 2018, crust-core coupling (Rientdor 2001; Levin & Ushomisky 2001b; Glampedakis & Anderson 2006a, b), in-medium effects (Kolomeitsev & Voskresensky 2015), etc. To derive this microphysical information from observations properly, one should have an accurate theory, describing evolution of CFS unstable star observational features. This paper deals with one of elements of this theory – role of differential rotation, which can be coupled to CFS instability.

The idea that GWs emission by CFS unstable star can generate differential rotation was suggested by Spruit (1999) within a simple phenomenological model. Inevitable generation of differential rotation by r-mode instability in perfect fluid non-magnetized stellar models was confirmed by Levin & Ushomisky (2001a); Friedman, Lindblom & Lockitch (2016). As discussed by Ho & Lai (2000), differential rotation complicates proper definition of the mean rotation rate $\Omega$, which enters the evolution equations by Owen et al. (1998) and Ho & Lai (2000). In particular, in a recent paper (Chugunov 2017, henceforth Paper I) I pointed that observations are typically sensitive to the secular motion of a specific point on the stellar surface (e.g., polar cap or hot spot), corresponding to an apparent spin frequency $\Omega_{\mathrm{app}}$, which generally does not coincide with $\Omega$ because of differential rotation. Furthermore, apparent spin down rate $\dot{\Omega}_{\mathrm{app}}$ can significantly differ from $\Omega$ rate of change.

Hence, if one naively assumes them to be equal, the conclusions might be misleading. Here I address a question of how one can correctly confront predictions of the r-mode evolution theory with the apparent spin down rate $\dot{\Omega}_{\mathrm{app}}$. Note, that some recent papers suggest (e.g., Kantor, Gusakov & Chugunov 2016) or even make (e.g., Haskell & Patruno 2017; Andersson et al. 2017) such a comparison.

To answer this question, I follow Paper I, where evolution equations were derived via multipolar expansion for GWs by Thorne (1980) (Owen et al. 1998; Ho & Lai 2000 appeal to the Lagrangian perturbation theory by Friedman & Schutz 1978a, b, while deriving evolution equations). This makes it possible to deal directly with Eulerian perturbations of the velocity. For completeness, this derivation is presented (in a more detailed form) in Section 2.1. In the Subsection 2.2 I demonstrate that on long timescales the evolution equations can be reduced to a one, which describes evolution rate through the sequence of thermally equilibrium states. Final Subsection 2.3 of Section 2 is devoted to the discussion, what is the effect of differential rotation on apparent frequency $\Omega_{\mathrm{app}}$, treated as a variable which generally depends on the position of the observed point on NS surface. Real observation should reveal one of the possible values of this variable, corresponding to certain location of the observed point. I neglect other effects on the $\Omega_{\mathrm{app}}$, like the hot spot wandering discussed Patruno & Watts 2012, assuming that they can be treated independently.

I argue that the apparent spin down rate, averaged over long timescale, should not be affected by differential rotation. However, on a short timescale, differential rotation contributes to the apparent spin down can be the same order of magnitude as the decrease rate of the mean rotation frequency.
2 CFS INSTABILITY AND THE EVOLUTION OF AN UNSTABLE STAR

Here I discuss the evolution of CFS unstable relativistic rotating star in its asymptotic rest frame. For completeness, in Subsection 2.1 I present derivation of general evolution equations from paper I in more details. In Subsection 2.2 I discuss evolution on a long timescale. I show that the evolution equations are simplified and can be reduced to one differential equation. The final Subsection 2.3 is devoted to general discussion of differential rotation effects on the apparent spin frequency.

2.1 General evolution equations in asymptotic rest frame

In asymptotic rest frame total mass-energy $E$ and angular momentum $J$ are well defined (see, e.g., Section 19 by Misner, Thorne & Wheeler 1973). Uniform rotation corresponds to the minimal energy at a given angular momentum $J$ (e.g., Boyer & Lindquist 1966; Hartle & Sharp 1967; Stergioulas 2003) and correspondent energy is rotational energy $E_{\text{rot}}$. The rotation frequency can be defined as

$$\Omega = \frac{\partial E_{\text{rot}}}{\partial J}. \quad (1)$$

If total energy $E$ of the star exceeds $E_{\text{rot}}$, the motion of fluid elements is perturbed with respect to uniform rotation. Positively defined excitation energy $E_{\text{ex}}$ can be attributed to the perturbation $E_{\text{ex}} = E - E_{\text{rot}}(J) > 0. \quad (2)$

Emission of the gravitational (or electromagnetic) waves change energy ($E \to E + \delta E$) and angular momentum ($J \to J + \delta J$) of the star (e.g., Misner, Thorne & Wheeler 1973). Change of angular momentum causes variation of rotational energy to $E_{\text{rot}} + \delta E_{\text{rot}}$, where $\delta E_{\text{rot}} = \Omega \delta J$. There are three cases, if we compare $\delta E$ and $\delta E_{\text{rot}}$:

$$\delta E > \delta E_{\text{rot}}: \text{Excitation energy increases} \quad \delta E_{\text{ex}} = \delta E - \delta E_{\text{rot}} > 0. \quad \text{Even if a star initially rotates uniformly, it does not in the final state; perturbations should be excited.}$$

$$\delta E = \delta E_{\text{rot}}: \text{variation of the excitation energy} \quad \delta E_{\text{ex}} = 0. \quad \text{If initial star has uniform rotation (} E_{\text{ex}} = 0\), its final state is also uniform rotation, but spin frequency changes. For example, it happens during magneto-dipolar spin down of pulsar or GW emission by static mountains (see below);

$$\delta E < \delta E_{\text{rot}}: \text{Excitation energy decreases} \quad \delta E_{\text{ex}} < 0. \quad \text{If occurs, only if uniform rotation was perturbed initially and a star’s excitation energy is positive. Gravitational radiation acts as damping mechanism for perturbations. For example, it is the case for GW emission by oscillation modes in non-rotating neutron stars (e.g., Price & Thorne 1969; McDermott et al. 1984).}$$

Axisymmetric rotating star cannot emit GWs, so I consider further the emission associated with oscillation modes. Indeed, if oscillation mode is excited with a frequency $\omega$ (in the asymptotic rest frame) and azimuthal number $m$ [i.e. the perturbations are $\propto e^{(\omega t + m\phi)}$ and cylindrical coordinate system $(z, r, \phi)$ with $e_z = J/J$ is applied], the emission of GWs is permitted.

Multipolar expansion for GWs, formulated by Thorne (1980), gives the rate of energy changes $E_{\text{GW}}^{\text{rot}}$ and angular momentum (in particular, for its projection to the rotation axis, $J_{GW}^{\text{rot}}$) due to GWs emission as sums over multipolar contributions of radiation field in the local wave zone. As it follows from Eqs. (4.16) and (4.23) by Thorne (1980), $E_{GW}^{\text{rot}}$ and $J_{GW}^{\text{rot}}$ are coupled by the equation, which is well-known for emission of electromagnetic waves [see e.g. Section 9.8 by Jackson (1999)]:

$$\frac{\omega}{m} J_{GW}^{\text{rot}} = E_{GW}^{\text{rot}}. \quad (3)$$

Here I discuss emission of GWs, thus the energy is carried away from the star; $E_{GW}^{\text{rot}} < 0$. However, angular momentum of the star can either decrease or increase through emitted GWs, dependent on the sign of $\omega/m$. Definition of $E_{\text{ex}}$ (Eq. 2) permits me to write its rate of change as

$$E_{\text{ex}}^{GW} = \dot{E}_{GW}^{\text{rot}} = \left(1 + \frac{m \Omega}{\omega}\right) \dot{E}_{GW}^{\text{rot}}. \quad (4)$$

This means that excitation energy is increasing (and, consequently, GW emission enhances perturbations) if and only if $(1 + m \Omega/\omega) < 0$. It is easy to check, that this condition is equal to the well-known criteria of CFS instability for eigen-modes: a mode is unstable if it is prograde in the inertial frame, but retrograde in a frame, corotating with the star (e.g., Andersson & Kokkotas 2001; Friedman & Stergioulas 2011). Note, if asymmetry of the uniformly rotating star is static in the frame corotating with the star (e.g., it is associated with elasticity or magnetically supported mountains in crust), the associated GW emission does not enhance excitation energy (at least while the star keeps its shape), because emitted waves correspond to $\omega = -m \Omega$ in asymptotic rest frame and thus $E_{GW}^{\text{rot}} = 0$ according to the equation (4). The same holds true for electromagnetic spin-down by magneto-dipolar emission (see, however, Section 3.1 to specify this statement for a case, which accounts for the effect of differential rotation).

To describe the evolution of CFS unstable star, I parametrize its state by three parameters: total angular momentum $J$, mode energy $E_{\text{ex}}$, and the parameter to characterize thermal state, which I choose to be temperature in the stellar centre $T$. The latter implies an assumption that the star is thermally relaxed, which is applicable for NSs with high accuracy (e.g., Page, Lattimer, Prakash & Steiner 2004; Gusakov, Kazminker, Yakovlev & Gnedin 2005b) because of high thermal conductivity in their depths (see, e.g., Schertin, Baldo & Schulze 2017; Schmitt & Schertin 2017).

1 $E_{\text{ex}}$ should not be confused with canonical energy introduced by Friedman & Schutz (1978a,b), see discussion in Section 3.

2 The pattern speed of the mode is a speed of the surfaces of constant phase of $e^{(\omega t + m\phi)}$, given by $\omega + m \Omega = \text{const}$. In the inertial frame it leads to $\sigma_{\phi}^m = \omega/m = -\nu/m$. For the corotating frame, the azimuthal coordinate is $\phi_{\text{cor}} = \phi - \Omega t$. The pattern speed in this frame is given by $m \Omega + \omega = m \Omega + (\omega + m \Omega)$ and can be written as $d\sigma_{\phi}^{\text{cor}}/dt = \sigma_{\phi}^{\text{cor}} = (m \omega + m \Omega)/m$. Being combined, conditions $\sigma_{\phi}^m > 0$ (prograde mode in inertial frame) and $\sigma_{\phi}^{\text{cor}} < 0$ (retrograde mode in corotating frame) are equal to one condition $1 + m \Omega/\omega < 0$. Note, $\Omega > 0$ due to selection of coordinate system.
for recent results). Let me underline that I do not discuss thermal relaxation of the crust after accretion episodes, which was observed for many of transiently accreting NS (see Wijnands, Degenaar & Page 2017 for recent observational review) and requires detailed modelling of temperature profile inside the crust (e.g., Rutledge et al. 2002; Shuterin et al. 2007; Brown & Cumming 2009; Ootes et al. 2016; Parikh et al. 2018; Meisel et al. 2015). The latter effect is likely negligible for r-modes, because they are primarily localized in the core and temperature profile in the crust should not affect their properties considerably.

The evolution equations, which uses the above parameters, are derived as follows. For angular momentum it follows from Eq. (4):

$$J^G_W = -\frac{m}{\omega + m\Omega} E^G_W. \tag{5}$$

In this equation $\Omega = \Omega(J)$ and determined by Eq. (1). It is worth to note that equation (5) is applicable for any oscillation mode at any spin frequency and even for general relativistic stellar models.

Evolution of the mode energy is associated with GW energy pumping $E^G_W$ and dissipation energy loses $E^G_{\text{dis}}$:

$$E^G_{\text{ex}} = E^G_W + E^G_{\text{dis}} \tag{6}$$

Finally, the thermal evolution of star follows

$$\frac{dT}{dt} = -E^G_{\text{dis}} - L_{\text{cool}}. \tag{7}$$

Here $L_{\text{cool}}$ and $C$ are total cooling power (neutrinos flux from bulk of the star and thermal emission from the surface), and heat capacity of the star, respectively (see, e.g., review by Yakovlev & Pethick (2004) and Ofengeim et al. 2017 for useful analytical approximations). If additional torque (e.g., accretion or magnetic braking) or heating processes (e.g., accretion- or rotation-induced deep crustal heating, Brown et al. 1998; Gusakov et al. 2015) affects to the star, corresponding torque and power $Q_{\text{other}}$ should be added to the right parts of Eqs. (5) and (7), respectively.

To apply equations (5)–(7), one should specify mode properties: $\omega$, $E^G_{\text{ex}}$, and $E^G_{\text{dis}}$ as function of $E_{\text{ex}}$, $J$ and $T$. Of course, this problem is very complicated, especially in the case of general relativity (see, e.g., Ruoff & Kokkotas 2002; Yoshida & Lee 2003a; Lockitch et al. 2003; Krüger et al. 2010; Jasiulek & Chirenti 2017).

### 2.2 Evolution on the long time-scale

On the long timescales, or, to be more precise, on a timescales when excitation and thermal energies can be neglected in comparison to total change of the rotation energy $\Delta E_{\text{rot}}$ over this timescale, evolution equations can be simplified and reduced to one differential equation.

Indeed, on such timescales all energy pumped to the mode by instability is finally converted to heat by dissipation (as a toy intuitive model, unstable mode works as a ‘tube’ of negligible volume to transfer rotation energy to heat).

Thus, time-averaged heating power is equal to $E^G_W$ [one can prove it formally by averaging of Eq. (6) over time, while neglecting the change of the mode energy between initial and final states]. Typically, thermal evolution is much faster than rotational evolution (because the rotation energy exceeds the thermal energy for many orders of magnitude, see, e.g., Alford & Schwemmer 2014b), and the star, affected by CFS instability, rapidly evolves to the state of thermal equilibrium, when the heating by CFS instability and other mechanisms $Q_{\text{other}}$ are compensated by the cooling power $L_{\text{cool}}$. In this state Eq. (5) can be rewritten as follows:

$$\dot{J}^G_W = -\frac{m}{\omega + m\Omega} (Q_{\text{other}} - L_{\text{cool}}). \tag{8}$$

Note, that the right part of this equation depends explicitly only on the $m$ and the mode frequency $\omega$, but not on $E_{\text{ex}}$, $E^G_W$, and $E^G_{\text{dis}}$. The latter quantities, of course, enter this equation, but implicitly: equation (8) determines the evolution rate along thermally equilibrium states on the temperature-angular momentum (spin frequency) plane.

The location of thermally equilibrium states on this plane depends on $E_{\text{ex}}$, $E^G_W$, and $E^G_{\text{dis}}$ and determined by a specific mechanism, which limits CFS-instability. It can be nonlinear saturation (see, e.g., Brink, Teukolsky & Wasserman 2001a, b; Bondarescu, Teukolsky & Wasserman 2009; Haskell, Glampedakis & Andersson 2014) or instability suppression through enhanced dissipation at specific temperature regions ($dE_{\text{ex}}/dT < 0$) (see, e.g., Andersson et al. 2002; Reisenegger & Bonačić 2003b; Haskell et al. 2009; Gusakov et al. 2014b for discussion of particular models).

In the first case, the nonlinear processes lead to rapid increase of $E^G_W$ just as $E_{\text{ex}}$ exceeds saturation energy $E_{\text{sat}}$. It prevents the mode energy growing higher than $E_{\text{ex}}$, which, in turn, determines the maximal value of $J^G_W$.

The latter, via Eq. (8), determines the cooling power (and thus, the temperature), which is required to keep the star in the thermal equilibrium.

In the second case, the star moves along the boundary of the stability region (determined by $E^G_{\text{ex}} + E^G_{\text{dis}} = 0$). Location of this boundary determines stellar temperature and thus the cooling power. $J^G_W$ (and thus the mode energy $E_{\text{ex}}$) is adjusted according to Eq. (8) to provide enough heating to keep star at the required temperature (see, e.g., Gusakov et al. 2014a; Chugunov et al. 2014 for more detailed description).

Implicit dependence of Eq. (8) on $E_{\text{ex}}$, $E^G_{\text{ex}}$, and $E^G_{\text{dis}}$ permits not only to model NS evolution for certain theoretical models, which predict the sequence of thermal equilibrium states (as it was described above), but also to constrain the shape of the instability windows and $J^G_W$ from observational data (e.g. Chugunov et al. 2017).

It is worth noting, that Eq. (8) can be derived directly from conservation laws formulated in Chapter 20 by Misner et al. 1973: ‘the rate of loss of the 4-momentum and angular momentum from the system, as measured gravitationally, is precisely equal to the rate at which matter, fields and GWs carry off 4-momentum and angular momentum’.

The spatial components of the stellar 4-momentum are vanishing, as far as asymptotic rest frame is considered and cooling emission is isotropic. Thus, the change of the total
mass-energy is associated with cooling (photon flux from the surface and neutrino flux from the bulk of the star) and GWs. If thermal and excitation energies are negligible in comparison to the change of rotational energy $\Delta E_{\text{rot}}$, the change of the total mass-energy can be estimated as the change of rotational energy. Combined with Eq. (4), it gives Eq. (8), where power of additional energy sources (e.g., accretion) $Q_{\text{other}}$ is added artificially.

### 2.3 Role of differential rotation

As discussed by Spruit 1999; Rezzolla et al. 2000; Levin & Ushomirsky 2001a; Friedman et al. 2016, the enhancement of CFS unstable mode can lead to differential rotation in the star.\(^4\) As discussed in introduction, the apparent spin frequency $\Omega_{\text{app}}$ is typically associated with the motion of a certain point at the stellar surface (magnetic poles, hot spots, etc.) and can differ from $\Omega(J)$ because of differential rotation. To describe this effect, I treat $\Omega_{\text{app}}$ as a variable, describing the spin frequency, which will be detected, if observation tracks a given point on the surface. Generally, it depends on the location of this point (see Sec. 4). I introduce spin frequency correction for differential rotation

$$\Delta \Omega = \Omega_{\text{app}} - \Omega. \quad (9)$$

This correction is small ($\Delta \Omega \sim \Omega E_{\text{ex}}/E_{\text{rot}} \ll 1$, see e.g. Rezzolla et al. 2000; Levin & Ushomirsky 2001a; Sá 2004; Chugunov 2015; Friedman et al. 2016, obtained for Newtonian stellar models)\(^5\) and clearly can be neglected if one is interested in the change of the spin frequency on a long timescale, as in Section 2.2. However, on a short timescale the differential rotation may affect the apparent spin down rate. Namely,

$$\frac{d \Omega_{\text{app}}}{dt} = \frac{d \Omega}{dt} + \frac{d \Delta \Omega}{dt}. \quad (10)$$

The first term in the right part is given by Eq. (5) and can be estimated as

$$\frac{d \Omega}{dt} \sim \Omega j^{GW} \frac{\dot{E}^{GW}}{J} \sim \dot{E}^{GW} \frac{E_{\text{ex}}}{J}. \quad (11)$$

The second term, by the order of magnitude, is

$$\frac{d \Delta \Omega}{dt} \sim \Omega E_{\text{ex}} \frac{\Delta \Omega}{E_{\text{rot}}} \sim \dot{E}_{\text{ex}} \frac{E_{\text{ex}}}{J}. \quad (12)$$

For $\dot{E}_{\text{ex}} \sim \dot{E}^{GW}$, as it is suggested by Eq. (6), both terms contributing to the apparent spin down are of the same order of magnitude and thus both can affect it. To study this effect accurately, one should not only specify the unstable mode and its properties, but also describe the evolution of differential rotation, which is the second-order effect in perturbation theory. Within full general relativity this task is very complicated, and as for now, has not been performed yet self-consistently (see however Kastaun 2011, who discusses nonlinear decay of large amplitude r-modes to differential rotation).

In the following I limit consideration to r-mode instability in slowly rotating Newtonian perfect fluid stellar models, where several analytical results on the coupling of r-mode instability and differential rotation were recently obtained (Sá 2004; Chugunov 2015; Friedman et al. 2016; Friedman et al. 2017). In Section 4 these results and estimates for evolution of differential rotation in dissipative stellar models are applied as testbenches for the equations, suggested in this paper.

Differential rotation can also result in evolution of the excitation energy through another torques, not associated with r-mode. Similar effect was suggested by Ho & Lai (2000) within different approach (see discussion at the end of Section 3.1). As an example let me discuss a star affected by magnetic braking, estimated within the simplest magneto-dipolar model. As mentioned in Section 2.1, one can derive a counterpart of Eq. (4) for emission of electromagnetic waves. In case of magneto-dipolar emission, associated with rotation of the star, one should assume $m = -1$ and frequency of the emitted waves $\omega^M$ to be equal to the magnetosphere spin velocity. In what follows I assume $\omega^M = \Omega_{\text{app}}$. According to Eq. (4) it gives

$$\dot{E}_{\text{ex}} = \frac{\Delta \Omega}{\Omega} \dot{E}^M, \quad (13)$$

where $\dot{E}^M = -\dot{E}_{\text{ex}}/\tau^M$ and $\tau^M > 0$ are magnetic spin-down power and correspondent time-scale.

It should be noted, that this energy can be pumped not only to the unstable mode, but into differential rotation by itself.\(^6\) Thus one had to be careful when including this term into evolution equations for unstable mode. Note, $\dot{E}_{\text{ex}}^M$ depends on $\Delta \Omega$ and can be competitive with other terms in Eq. (6), if $|\dot{E}^M| > |\dot{E}^{GW}|$ or if differential rotation is strong enough $\Delta \Omega \sim \Omega$. In this paper I suppose that GW emission provides significant contribution to the spin down (e.g. $|\dot{E}^{GW}| \gtrsim |\dot{E}^M|$) and differential rotation is weak (because it is associated with second-order effect in the mode amplitude), thus, I neglect the terms associated with (13) below (except the discussion at the end of the next section).

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\(^4\) Here by ‘differential rotation’ I understand that oscillation-averaged Eulerian velocity differs from uniform rotation. As shown by Chugunov (2015), that differential rotation does not necessarily leads to non-uniform secular motion of fluid elements in oscillating star, but angular velocity of this motion can differ from $\Omega$ (see discussion in Section 4.3).

\(^5\) As discussed in Sections 4.1.1 and 4.2.2 for saturated r-modes in nonmagnetized Newtonian stellar models, differential rotation can exceed this estimate. See these sections for discussion of respective effects.

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\(^6\) Strictly speaking, Eq. (4) does not prove that the energy is pumped directly to the unstable mode, however, this assumption can be supported by general arguments: (a) the pumping is resonant (at the frequency $\omega$ and multipolarity $m$ of the mode, which emits GW), (b) agreement with Lagrangian perturbation theory Friedman & Schutz (1978a,b); Friedman (1978), and (c) analytic results by Friedman et al. (2016).
3 EVOLUTION OF R-MODE UNSTABLE NS WITHIN SLOW ROTATING NEWTONIAN STELLAR MODEL

To illustrate the previous section and to provide basics for subsequent discussion of differential rotation, here I limit my consideration to slow rotating Newtonian stellar models, which are widely applied for studying CFS instability in NSs (see, e.g., Haskell 2015 for recent review). This analysis is close to the one presented in Paper I, but is slightly more detailed. The effects of differential rotation on the apparent frequency are considered in Section 4. For slowly rotating case, Eq. (1) can be written as \( \Omega = \mathcal{J}/I \), where moment of inertia \( I \) does not depend on \( J \). The most unstable mode is r-mode with \( l = m = 2 \) (e.g., Lindblom et al. 1998), which has frequency

\[
\omega = \frac{(m-1)(m+2)}{m+1} \Omega = -\frac{4}{3} \Omega. \tag{14}
\]

For barotropic equation of state (pressure depends only on density), the first order Eulerian perturbations of velocity are purely toroidal and can be written in the following form (e.g., Papaloizou & Pringle 1978)

\[
\delta^{(1)} v = \sqrt{2}\alpha R \Omega \left( \frac{\rho}{R} \right)^m \Im m Y_{m}^{\ell=2} \exp^{\imath \omega t} \tag{15}
\]

Here \( \alpha \) is dimensionless mode amplitude, \( \Im m (Z) \) is imaginary part of complex number \( Z \), and

\[
Y_{m}^{\ell=2} = \frac{1}{l(l+1)} r \nabla \times (r \nabla Y_{m+2}) \tag{16}
\]

is magnetic-type vector spherical harmonic (see e.g., Varshalovich, Moskalev & Khersonskii 1988). Note, that Lindblom et al. (1998); Owen et al. (1998) define \( \delta^{(1)} v \) as a complex value and, as it was noted by Lockitch et al. (2003), calculate energy as a sum of respective values for the real and imaginary parts. In Eq. (15) I define \( \delta^{(1)} v \) as a real value and insert additional multiplier \( \sqrt{2} \) to get the same normalization for the energy in equation (19) as it was suggested by Lindblom et al. (1998); Owen et al. (1998) and widely applied in subsequent papers on r-mode instability. The Eulerian variations of density, pressure, and gravitational potential are of the second order in \( \Omega \), and thus they can be neglected in the leading order of slow rotation approximation (see Lindblom, Mendell & Owen 1999 for r-modes in the second order of slow rotation approximation). As a result, only kinetic energy contributes to the excitation energy of r-mode, which can be written as

\[
E_{ex} = \int \frac{\rho \delta v^2}{2} d^3r + \int \rho \nu_{b} \delta^{(2)} v d^3r + \mathcal{O}(\alpha^3). \tag{17}
\]

Here \( \delta v^2 = \delta v^2(v_i v_j) \) and \( \delta v = \sum_i \delta^{(1)} v_i \) are total perturbation of velocity, presented as a sum over orders in \( \alpha \) [i.e., \( \delta^{(1)} v = \mathcal{O}(\alpha^1) \)]. Integral is taken over stellar volume. The second term in Eq. (17) depends on the second order velocity perturbation \( \delta^{(2)} v \), but only axisymmetric part \( \delta^{(2)} v_{sym} \) can contribute to the integral (due to axisymmetry of \( v^3 \)). The axisymmetric part \( \delta^{(2)} v_{sym} \) is determined up to arbitrary cyclindrically stratified differential rotation (e.g., Sá 2004). However, definition of excitation energy adopted in this paper (Eq. 2) requires that given (perturbed) state of the star should be considered as perturbation of uniformly rotating star with same angular momentum. It imposes constraint:

\[
\delta J = \int \rho (\nu \times \mathbf{r}) d^3r = 0. \tag{18}
\]

So far as unperturbed state is uniform rotation \( \nu_0 = \Omega \times \mathbf{r} \), \( \delta^{(2)} v_{sym} \) contribution to the energy vanishes (in second order in \( \alpha \)). Thus, the second order excitation energy is determined exclusively by the first order perturbations and equals to the kinetic energy in the system corotating with star, which is given by (see, e.g., Lindblom et al. 1998):

\[
E_{ex} = \int \rho \left[ \frac{\delta^{(1)} v_i^2}{2} + \frac{1}{2} \rho \nu^2 \Omega^2 R^{-2m+2} \right] d^3r. \tag{19}
\]

The instability timescale

\[
\tau_{GW}^{-1} = -2 \frac{E_{ex}}{E_{GW}^{\perp}} \tag{20}
\]

can be calculated via multipolar expansion of gravitational radiation for Newtonian sources (see Section V.C in Thorne 1980), as it was done by Lindblom et al. (1998). Namely,

\[
J_{GW} = -m \omega \sum_{l=2}^{\infty} N_l \omega^2 \left( \left| \delta D_{lm} \right|^2 + \left| \delta J_{lm} \right|^2 \right), \tag{21}
\]

where perturbations of the mass \( \delta D_{lm} \) and current \( \delta J_{lm} \) multipole moments are

\[
\delta D_{lm} = \int \rho Y_{lm}^* r^3 d^3r, \tag{22}
\]

\[
\delta J_{lm} = 2 \pi l \| l+1 \right| \left( \rho \nu \delta \nu + \delta \nu \nu \right) \cdot Y_{lm}^* r^3 d^3r \tag{23}
\]

Here

\[
N_l = \frac{4\pi G}{c^{2l+1}} \frac{l(l+1)}{[(l-1)(2l+1)]!!^2} \tag{24}
\]

After r-mode solution (Eq. 15) is substituted into these equations, one obtains that dominant contribution to the instability timescale comes from current multiple (see e.g., Lindblom et al. 1998; Owen et al. 1998) and

\[
\tau_{GW}^{-1} = -32\pi G \frac{(m-1)^2}{c^{2m+3}} \frac{1}{[(2m+1)]!!^2} \left( \frac{m+2}{m+1} \right)^{2l+2} \int_0^R \rho r^{2m+2} dr \tag{25}
\]

\approx \frac{1}{10^5} \left( \frac{R}{10 \, \text{km}} \right)^4 \left( \frac{M}{1.4M_\odot} \right)^{1/2} \left( \frac{\nu}{600 \, \text{Hz}} \right)^6 . \tag{26}
\]

Here \( \nu = \Omega/2\pi \). Friedman et al. (2016) confirms this result by analytic treatment of the r-mode instability up to second order in oscillation amplitude for perfect fluid stellar model. In particular, this paper explicitly reveals the mechanism of energy pumping into the r-modes by GW back-reaction force.

7 To be more specific, as shown by Friedman et al. (2016) for unstable r-modes in perfect fluid stellar models, the exponentially growing part of \( \delta^{(2)} v_{sym} \) is unique, but it does not exclude contribution of time-independent differential rotation to \( \delta^{(2)} v_{sym} \).
Differential rotation and evolution of CFS-unstable stars

The dissipation rate
\[ \tau^{\text{dis}} = -2 \frac{E_{\text{ex}}}{E_{\text{ex}}^2} \]  
(26)
can be calculated for certain model of dissipation as a sum over contributions of relevant dissipation processes (shear viscosity, mutual friction, etc.). Note, any of internal dissipation processes cannot affect total angular momentum of the star, thus the rotational energy is conserved. As a result \( E_{\text{ex}} \) equals to dissipation rate of the total energy. For example, the contribution of the shear viscosity \( \eta \) to the dissipation rate is (Lindblom et al. 1998)
\[ \frac{1}{\tau^3} = (m - 1)(2m + 1) \frac{\int R \eta^2 \rho^2 dr}{\rho^2 + 2 \rho^2 \Omega} \]  
(27)
\[ \approx \frac{1}{2.2 \times 10^5} \left( \frac{R}{10 \text{ km}} \right)^{-5} \left( \frac{M}{1.4 M_{\odot}} \right) \left( \frac{T}{10^8 \text{ K}} \right)^{-2}. \]
The numerical factor corresponds to electron shear viscosity from Shiternin & Yakovlev (2008), fitted by Gusakov et al. (2014b) for Akmal et al. (1998) equation of state parametrized by Heiselberg & Hjorth-Jensen (1999).
Critical temperature of proton superfluidity assumed to be \( T_{\text{cp}} = 2 \times 10^8 \text{ K} \). As noted by Gusakov et al. (2014b), coincidence of the viscosity with results by Flowers & Rho (1979) is accidental, because physics input is essentially different.

Introduction of above timescales permits me to rewrite Eqs. (5) – (7) in the form \( (m = 2 \) is assumed, see Paper I):
\[ \hat{\Omega} = \frac{2Q_{\text{GW}}^2}{\tau_{\text{GW}}} \Omega \]  
(28)
\[ \hat{\alpha} = -\left( \frac{1}{\tau_{\text{GW}}} + \frac{1}{\tau^3} \right) \alpha \]  
(29)
\[ C \hat{T} = \frac{\hat{\alpha}}{\tau_{\text{dis}}^2} \omega^2 + Q_{\text{other}} - L_{\text{cool}} \]  
(30)
Here I, following Owen et al. (1998), introduce dimensionless parameters
\[ \hat{j} = \frac{1}{MR^2} \int_0^R \rho r^2 \text{dr} \approx 1.64 \times 10^{-2}, \]  
(31)
\[ \hat{\gamma} = \frac{I}{MR^2} = \frac{8\pi}{3MR^2} \int_0^R \rho r^4 \text{dr} \approx 0.261, \]  
(32)
\[ \hat{Q} = \frac{m(m + 1) \hat{j}}{4I} \approx 9.38 \times 10^{-2}. \]  
(33)
Equations (28)-(30) can be applied to model evolution of r-mode unstable star within slow rotating Newtonian approximation. As discussed in Section 2, on the long timescales, evolution can be described by one equation (8), and for slow rotating Newtonian stellar models it can be rewritten in the form
\[ L_{\text{cool}} = Q_{\text{other}} + \frac{\hat{\Omega}}{3} \left| \hat{\Omega}_{\text{GW}} \right| = Q_{\text{other}} + \frac{1}{3} \left| \hat{E}_{\text{GW}} \right|. \]  
(34)
The latter equality clearly demonstrates the budget of r-mode instability in slow rotating Newtonian model: 1/3 of the GW spin down power is converted to the heating (and finally emitted from the star by the cooling processes), and the remaining 2/3 of \( \hat{E}_{\text{GW}} \) is directly emitted in form of GWs.

3.1 Comparison with previous works
As discussed in Paper I, the evolution equations (28-30) formally differ from “standard” equations derived by Owen et al. (1998); Ho & Lai (2000), which are widely applied in the papers on the evolution of r-mode unstable NSs. Namely, in the leading order in mode amplitude, the latter equations can be written in form:
\[ \hat{\Omega} = -\frac{2Q_{\text{GW}}^2}{\tau_{\text{GW}}} \Omega, \]  
(35)
\[ \hat{\alpha} = -\left( \frac{1}{\tau_{\text{GW}}} + \frac{1}{\tau_{\text{dis}}} \right) \alpha, \]  
(36)
\[ C \hat{T} = \frac{\hat{\alpha}}{\tau_{\text{dis}}^2} \omega^2 + Q_{\text{other}} - L_{\text{cool}}. \]  
(37)
The difference occurs in spin frequency evolution equations (compare Eqs. 28 and 35). It originates from uncertainties in definition of mean spin frequency, caused by differential rotation, which can be generated in a star as a result of CFS instability (see, e.g., Spruit 1999; Rezzolla et al. 2000; Levin & Ushomirsy 2001a; Friedman et al. 2016 and Section 4). Namely, Owen et al. (1998); Ho & Lai (2000) write total angular momentum as
\[ J = \hat{\Omega} \hat{I} + J_c, \]  
(38)
where canonical angular momentum of r-mode is introduced:
\[ J_c = -(3/2)\hat{\Omega} \hat{M} R^2 \omega^2. \]  
(39)
As a result, their spin frequency parameter is
\[ \hat{\Omega} = (1 + \hat{\Omega}_{\text{GW}}^2) \Omega. \]  
(40)
After this quantity is introduced into standard evolution equations (Eqs. 35-37) they coincide with Eqs. (35-37) in the leading order in \( \alpha \).

Note, for non-dissipative stellar model (\( \tau_{\text{dis}} = \infty \)) \( \hat{\Omega} = 0 \), but \( \hat{\Omega} \) is finite and negative. In Section 4.1 I demonstrate that nondissipative star should be observed as spinning down at rate \( \Omega_{\text{app}} \sim (0.5 \div 0.8) \Omega \), which only numerically differs from \( \hat{\Omega} \).

Equations derived by Owen et al. (1998); Ho & Lai (2000) contain the terms, which are of the next order in \( \alpha^2 \). As discussed in Paper I, these terms are not yet derived accurately (and thus are different for Owen et al. 1998 and Ho & Lai 2000), so it would be more self-consistent to omit them.

Equations suggested by Ho & Lai (2000) contain additional term, which describes evolution of r-mode amplitude governed by the magnetic braking. Namely, \( \alpha/2\tau^3 \) is added to the right-hand side of Eq. (36). This term is supported by the statement that canonical angular momentum is proportional to wave action, being thus adiabatic invariant. As discussed in Section 2.3, similar term can be reproduced by the theory developed in Section 2.1, but it depends on differential rotation \( \Delta \Omega \). In a particular case, when differential motion of fluid elements is suppressed, e.g. by magnetic field (see Section 4.3), \( \Delta \Omega \) is unique (Chugunov 2015) and given by Eq. (64). In this case, assuming that all \( E_{\text{ex}} \) comes to the r-mode, Eq. (13) leads to additional term \( \alpha/2\tau^3 \) in the right hand side of Eq. (29) and agrees with Ho & Lai (2000).

However, in a general case \( \Delta \Omega \) differs from Eq. (65) leading to different additional term in Eq. (29), which contradicts Ho & Lai (2000). It can indicate that part of \( E_{\text{ex}} \)
transfers not to the r-mode, but to the differential rotation. In this case, differential rotation is enhanced by braking and becomes additional degree of freedom, controlled by external torque (as it is in the absence of oscillation modes). So, accurate treatment of CFS instability under action of external torque should deal with this feature. As far as I am concerned, such analysis has not been done yet.

Hopefully, this uncertainty should not affect most applications of r-mode instability, because the magnetic braking terms in mode amplitude evolution can be neglected if \( \tau_{\text{M}} \gg \tau_{\text{GW}} \), which is typically the case.

### 4 ROLE OF DIFFERENTIAL ROTATION

Equations (28-30) can be applied to study evolution of the mode amplitude \( \alpha, \Omega, \) and \( T \). However, as discussed in Section 2.3, the differential rotation, associated with CFS instability, can affect the apparent frequency. In this Section I analyse several simplified models, in which differential rotation can be treated accurately, and use these results to discuss the evolution of NS during recycling in LMXB.

#### 4.1 CFS instability and differential rotation in perfect fluid Newtonian stellar models

Friedman et al. (2016) provides analytical solution for unstable r-modes driven by gravitational radiation reaction force in perfect fluid Newtonian stellar model. In particular, for the model with uniform slow rotation at spin frequency \( \Omega_0 \) at the initial moment of time they demonstrate that:

(a) r-mode grows exponentially and the first order velocity perturbation for r-mode with multipolarity \( m \) can be expressed as [see Eqs. (77) and (93)] by Friedman et al. (2016) and substitute \( l = m \):

\[
\delta^{(1)} \mathbf{v} = 2 \mathbf{m} \left[ \frac{\alpha_{\text{FLL}} \Omega_{\text{r}}^m}{m (m-1) R} \times \nabla \left( \sin^m \theta e^{i(m \phi + \omega t)} \right) \right] e^{-t/R_{\text{GW}}}.
\]

Here \( \omega \) is given by Eq. (14) for \( \Omega = \Omega_0 \). As noted above, the latter agrees with Eq. (25) for \( \Omega = \Omega_0 \). Note, the dimensionless amplitude \( \alpha_{\text{FLL}} \) introduced by Friedman et al. (2016) stays constant, while r-mode is growing. In contrast, here (as in the most of the papers on r-mode instability) the first order perturbation is described by ‘instantaneous’ amplitude \( \alpha = \alpha(t) \), which determines amplitude of the first order velocity perturbations at a given moment \( t \) (see, e.g., Eq. 15). These amplitudes are coupled according to

\[
\alpha_{\text{FLL}}^2 e^{-2t/\tau_{\text{GW}}} = \frac{(2m+1)!}{2^{2m+1} \pi m (m+1) [m(m-1)]^{1/2}} \alpha(t)^2.
\]

For \( m = 2 \) one obtains \( \alpha_{\text{FLL}}^2 e^{-2t/\tau_{\text{GW}}} = 5 \alpha(t)^2 / (8 \pi) \). As far as the results by Friedman et al. (2016) were obtained within perturbation theory and do not include any saturation mechanism, they can be directly applied only during limited time, i.e. as long as \( \alpha(t) \ll 1 \).

(b) the enhancement of r-mode is accompanied by enhancement of oscillation averaged Eulerian perturbation of velocity, which is axisymmetric and can be written in the form [see Eqs. (115), (118) and (119) by Friedman et al. (2016)]:

\[
\delta \alpha^{(2)}_{\text{sym}} \mathbf{v} = -\alpha_{\text{FLL}}^2 \exp(-2t/\tau_{\text{GW}}) \Omega_{\text{r}} \left( \frac{\omega}{R} \right)^{2m-4} \times \left[ \frac{(m+1)^2 \omega^2}{8 R^2} + \frac{m^2 - 1}{2} \left( \frac{\omega}{R} \right)^2 \right] e_{\phi}.
\]

Here \( e_{\phi} \) is a unit vector in the direction of \( \phi \) and auxiliary function

\[
\Upsilon(\omega) = \int_{z_{\text{in}}}^{z_{\text{out}}} \rho \frac{\omega^2}{R^2} dz = \int_{z_{\text{in}}}^{z_{\text{out}}} \rho \frac{\omega^2}{R^2} dz.
\]

The secular motion of fluid elements (with respect to unperturbed uniform rotation at \( \Omega_0 \)), can be described by the drift velocity \( \delta \alpha^{(2)}_{\text{sym}} \mathbf{v} \), which differs from \( \delta \alpha^{(2)}_{\text{sym}} \mathbf{v} \) because of the Stokes drift (see, e.g., Stokes 1847; Longuet-Higgins 1953). For \( m = 2 \) r-mode it can be described by effective velocity (e.g., Rezzolla et al. 2001; Chugunov 2015; Friedman et al. 2016):

\[
\mathbf{v}_s = \frac{3 \Omega_{\text{r}} \omega}{4} \alpha_{\text{FLL}}^2 \exp(-2t/\tau_{\text{GW}}) \left( \frac{\omega}{R} \right)^{2m-2} \left( \frac{z}{R} \right)^2 e_{\phi} - \frac{2}{3} \alpha_{\text{FLL}}^2 \exp(-2t/\tau_{\text{GW}}) \Omega_{\text{r}} \left[ \frac{\omega^2}{4 R^2} + \Upsilon(\omega) \right] e_{\phi}.
\]

As result (Friedman et al. 2016),

\[
\delta \alpha^{(2)}_{\text{sym}} \mathbf{v} = \delta \alpha^{(2)}_{\text{sym}} \mathbf{v} + \mathbf{v}_s
\]

\[
= \frac{3}{2} \alpha_{\text{FLL}}^2 \exp(-2t/\tau_{\text{GW}}) \Omega_{\text{r}} \left[ \frac{\omega^2}{4 R^2} + \Upsilon(\omega) \right] e_{\phi},
\]

where the last equality corresponds to \( m = 2 \). The resulting motion of fluid elements (46) is axisymmetric, cylindrically stratified (i.e., drift velocity depends on \( z \)) only, but differs from uniform rotation.

Let me now map these results to the variables introduced in this paper. To obtain \( \Omega \) as defined by Eq. (1), I need to write down angular momentum as a function of time. As discussed by Levin & Ushomirsky (2001a) (see also Section 3), only axisymmetric part of the velocity perturbations contribute to the total angular momentum, which takes the form

\[
J(t) = \int \rho [\mathbf{v}_0 \times \mathbf{r}] \, d^3r + \int \rho \left[ \delta \alpha^{(2)}_{\text{sym}} \mathbf{v} \times \mathbf{r} \right] \, d^3r
\]

\[
= J_0 - \frac{m (m + 1)}{4} J M R^2 \alpha^2 \Omega_0 e_s,
\]

where Eqs. (31) and (42) were applied.

It is worth to note, that the change of angular momentum \( J = J_0 \) is equal to the canonical angular momentum of r-mode \( J_\delta \) as defined by Friedman & Schutz (1978a,b), as it should be for the eigenmodes evolving under the action of reaction-reaction force in absence of viscosity (Friedman & Schutz 1978a,b). Eq. (47) allows writing down the spin frequency \( \Omega \) as defined in this paper

\[
\Omega(t) = \frac{J(t)}{I} = [1 - Q \alpha^2(t)] \Omega_0
\]

Combined with Eq. (42), it confirms Eqs. (28) and (29) in

\[\text{Eq. (42)}\], it confirms Eqs. (28) and (29) in
Indeed, as shown by Friedman et al. (2017), the thermal evolution equation (Eq. 30) is not affected by instability because dissipation is neglected in this subsection.

As noted above, the apparent spin frequency is typically associated with secular motion of the specific point at the stellar surface, and thus it is given by

$$\Omega_{\text{app}} = \Omega_0 + \frac{Q_0^2}{4\pi^2} \frac{\mathcal{Y}(v)}{\mathcal{V}(v)} \Omega_0. \quad (49)$$

It leads to apparent spin-down rate

$$\frac{d\Omega_{\text{app}}}{dt} = \frac{15}{8\pi^2} \frac{\mathcal{Y}(v)}{\mathcal{V}(v)} \frac{\pi^2}{4R^2} \Omega_0, \quad (50)$$

which depends on $\varpi$ - cylindrical radius of the observed point at the surface.

Corresponding profile of $\Omega_{\text{app}}$ for perfect fluid Newtonian star with polytropic equation of state with $n = 1$ is shown by solid line in Fig. 1. $\Omega_{\text{app}}$ differs from $\Omega(t)$, given by the Eq. (48), even in the leading order in $\alpha(t)$. This results agrees with general arguments presented at Section 2.3 because $E_{\text{ex}} = E_{\text{GW}}$ in the absence of dissipation (see Eq. 6). The profile

$$\Delta \Omega = \Omega_{\text{app}} - \Omega = \left\{ Q - \frac{15}{16\pi} \frac{\pi^2}{4R^2} \right\} \Omega_0. \quad (51)$$

evolves on the gravitational instability timescale.

Summarizing, for non-magnetized Newtonian models of a perfect-fluid star, development of the $r$-mode instability agrees with evolution equations (28)–(30), however the apparent spin down rate is affected by differential rotation (see Eq. 50). It is worth to note, that the GW emission associated with CFS instability results in spin down of all points at stellar surface. In particular, it demonstrates that standard equation (35), predicting $\Omega = \text{const}$ for absence of dissipation ($\tau_{\text{dis}} = 0$), can be rather misleading, if we do not take into account differential rotation effects (i.e. treat $\Omega$ as apparent frequency).

4.1.1 Saturated $r$-modes in non-magnetized perfect fluid stellar models

Previous discussion holds true for nonsaturated modes. For nondissipative star mode coupling mechanism of saturation (Brink et al. 2004b,a; Bondarescu et al. 2009; Bondarescu & Wasserman 2013) can be suppressed because vanishing damping rate of daughter modes require almost perfect tuning in mode triplet (see, e.g., Eq. 4 in Bondarescu & Wasserman 2013). Indeed, as shown by Kastaun (2011), in the absence of dissipation $r$-modes can exist with very large amplitudes $\alpha \sim 1$. However, below I apply traditional simplified model of saturation (e.g. Levin 1999) and assume mode amplitude to be constant after it reaches saturation value $\alpha_{\text{sat}} \ll 1$. In this case the evolution of differential rotation can be qualitatively described by the toy model by Friedman et al. (2017) at magnetic field $B \to 0$ (see also Section 4.2). The model discusses incompressible perfectly conducting rotating fluid with cylindrical symmetry, driven by a force, which mimics the radiation-reaction force associated with CFS instability. The amplitude of this force (per unit volume) is $f_{\text{GW}} \sim f(\varpi)\alpha^2\rho R^3/\tau_{\text{GW}}$. Here $f(\varpi)$ encodes dependence of $f_{\text{GW}}$ on $\varpi$. In absence of viscosity and magnetic field, rotation of cylindrical layers is uncoupled. As a result, constant force $f_{\text{GW}}$ leads to linear increase of differential rotation with time at rate $\Delta \dot{\Omega} \sim f(\varpi)\alpha^2\Omega/\tau_{\text{GW}}$, which depends on the cylindrical radius and contributes to apparent spin down. Thus the differential rotation affects apparent spin-down for saturated $r$-modes in nonmagnetized perfect fluid stellar models.

4.2 Differential rotation in viscous stellar models

Previous discussion in this section neglects viscosity, which can affect profile of differential rotation even it is not enough strong to suppress instability as it is. In this subsection I account for these effects qualitatively within a bit modified toy model by Friedman et al. (2017): I neglect the magnetic field, but account for viscous terms. As described in Subsection 4.1.1, the model discusses incompressible rotating fluid with cylindrical symmetry, driven by a force $f_{\text{GW}}$, which mimics the radiation-reaction force associated with CFS instability. The only nonvanishing component of the velocity is $v_{\phi}$; it is governed by equation (e.g., Landau & Lifshitz 1987)

$$\rho \frac{\partial v_{\phi}}{\partial t} + \eta \left( \frac{1}{r} \frac{\partial v_{\phi}}{\partial r} - \frac{v_{\phi}}{r^2} \right) = f_{\text{GW}}. \quad (52)$$

Note, Fig. 1 in Friedman et al. (2016) represents axisymmetric Eulerian perturbation of the velocity, which indeed can be positive at certain part of the stellar surface, but differs from secular motion of fluid elements due to Stokes drift (see e.g. Stokes 1847; Longuet-Higgins 1953).
In the absence of external force, general solution of this equation can be presented as linear combination of uniform rotation and infinite set of exponentially decaying ($\propto \exp(-\gamma t)$), where $\gamma > 0$) eigenfunctions, with radial dependence $I_1(\sqrt{\gamma^2 - m^2 r^2})$. Here $I_1$ is modified Bessel function of the first kind (solution with modified Bessel function of the second kind are not regular at $r \rightarrow 0$ and thus unphysical). Exact value of time-scales $\gamma$ depends on the condition at the external boundary of the cylinder, but subsequently I will need only order-of-magnitude estimate: $\gamma \sim \gamma = \eta/(\mu R^2)$. Note, that this estimate agrees by order-of-magnitude the with shear viscosity dissipation timescale $\tau^S$, given by Eq. (27). Below I examine the differential rotation before and after saturation of r-modes.

4.2.1 Unsaturated r-modes

According to Eq. (29), the mode amplitude is exponentially increasing at timescale $\tau$ given by:

$$\frac{1}{\tau} = \frac{1}{\tau_{GW}} + \frac{1}{\tau_{dis}}.$$  \hfill (53)

Here I discuss evolution of differential rotation for unstable star ($\tau < 0$) and neglect evolution of $\Omega$ and $T$ in equations (28–30) for simplicity. Thus, I treat $\tau$ as a constant.

The radiation reaction force can be estimated as (Friedman et al. 2016)

$$f_{GW}^{GW} \sim -\alpha^2(t) \rho R \frac{\Omega}{\tau_{GW}} = -\alpha^2(0) \exp\left(-\frac{2t}{\tau}\right) \rho R \frac{\Omega}{\tau_{GW}}.$$ \hfill (54)

The minus sign demonstrates that $f_{GW}^{GW}$ brakes NS rotation. Having expanded this force over eigenfunctions of Eq. (52), I got the following estimate for

$$\Delta \Omega \sim -\frac{\alpha^2(t)}{\gamma \tau} \frac{\tau}{\tau_{GW}} \Omega.$$ \hfill (55)

The contribution of $\Delta \Omega$ to the apparent spin down can be estimated as

$$\frac{d\Delta \Omega}{dt} \sim -2 \frac{\alpha^2(t)}{\gamma \tau^2} \frac{\tau}{2 \tau_{GW}^2} \Omega.$$ \hfill (56)

This value is of the same order of magnitude as $\Omega$, and thus differential rotation can affect apparent spin down rate for viscous stellar models.

4.2.2 Saturated r-modes

To make the discussion more simple, I introduce several evolution timescales

$$\beta^{\Delta \Omega} \equiv \frac{1}{\tau_{\Delta \Omega}} \equiv \frac{1}{\Delta \Omega} \frac{d\Delta \Omega}{dt},$$ \hfill (57)

$$\beta^\Omega \equiv \frac{1}{\tau^\Omega} \equiv \frac{1}{\Omega} \frac{d\Omega}{dt},$$ \hfill (58)

$$\beta^\alpha \equiv \frac{1}{\tau^\alpha} \equiv \frac{1}{\alpha} \frac{d\alpha}{dt}.$$ \hfill (59)

A simplified treatment of saturation (Owen et al. 1998; Ho & Lai 2000; Alford & Schwenzer 2014b; Gusakov et al. 2014b) is substitution of $\tau^{dis}$ with the effective nonlinear dissipation timescale $\tau^{GW} = \frac{\tau^{GW}}{\Omega}$ in the whole equations (35)–(37). As a result, we have $\alpha(t) = \alpha^{sat}$ and, within the toy model, time-independent

$$f_{GW}^{GW} \sim -\alpha^{sat}\rho R \frac{\Omega}{\tau_{GW}}.$$ \hfill (60)

As discussed in Section 4.1.1, for non-magnetized perfect fluid stellar models differential rotation is linearly increasing with time and thus $\Delta \Omega$ can strongly exceed $\alpha^2$ for $t \gg \tau^{GW}$. However, shear viscosity $\eta$ limits the growth of differential rotation. Limiting amplitude of $\Delta \Omega$ can be estimated by equating typical viscous force $f^\eta \sim \eta \Delta \Omega/\tau$ to $f_{GW}^{GW}$. It leads to

$$\Delta \Omega \sim \left(R \alpha^{sat}\right)^2 \frac{\eta}{\gamma \tau_{GW}^2} \sim \left(\alpha^{sat}\right)^2 \frac{\Omega}{\gamma \tau_{GW}^2}.$$ \hfill (61)

This result can be confirmed by accurate expansion of $f_{GW}^{GW}$ over eigenfunctions of Eq. (52). As far as $\gamma^{-1} \sim \tau^S$ and for unstable star $\tau^S > |\tau^{GW}|$, $\Delta \Omega$ can exceed $\alpha^2$ for a factor $\sim 1/|\gamma \tau^{GW}| \sim \tau^S/\tau_{GW}^2$.

As long as $\alpha^{sat}$ depends on $\Omega$ and $T$, $\Delta \Omega$ can vary on the same timescale as $\Omega$ and $T$, $\tau^S \sim \tau^{GW}/\alpha^2$ (see Eq. 28),1 which is very long in comparison with $\tau^{GW}$ and cannot contribute to the apparent spin down as long as $\Delta \Omega < \Omega$.

More accurate treatment of the saturation within the mode coupling mechanism (e.g. Bondarescu et al. 2009) demonstrates that in some cases the r-mode amplitude is modulated and oscillates near saturation value due to complex behaviour of the mode coupling. It can lead to modulation of the $\Delta \Omega$. To model this effect I consider Eq. (52) with oscillating external force $f_{GW}^{GW} \sim \alpha^2 \rho R \Omega \left(1 + \sin(\beta^2 t)\right)/\tau_{GW}$ in the right part. Expanding this force over eigenmodes of Eq. (52), I estimate the typical $\Delta \Omega$ rate of change as

$$\frac{d\Delta \Omega}{dt} \sim \frac{\beta^n}{\sqrt{(\beta^2)^2 + \gamma^2}} \frac{d\Omega}{dt}$$ \hfill (62)

where Eq. (28) was applied. It means, that for $\beta^n \sim \gamma$ differential rotation can contribute to the apparent spin down. This case is not included in realistic models. For example, let us consider a model shown in Fig. 4b by Bondarescu et al. (2009). The modulation timescale (estimated by sight from the plot) is $\tau^n \lesssim 10^5$ s. It is short in comparison with $\gamma \sim \tau^S$ for the temperature of the star discussed on that plot (see Eq. 27 for $T \sim 10^{9–10}$ K). Thus, the apparent spin down can be affected. However, the effect of modulation should be smoothed out on timescales much longer than $\tau^S$. Thus, even in the considered example, the spin down rate, which is estimated from the variation of apparent frequency on the observational timescale $\tau^{dis} \sim 1 \text{ yr} \gg \tau^n$, should not be affected strongly.

The latter argument can be generalized: the modulation of the apparent spin down of a star by differential rotation can be deduced from observations only if one can measure it on the timescale $\tau^{dis} \sim \tau^n$. Note, $\tau^n \sim 1/\gamma$ gives an upper limit for a timescale allowed for measuring the effect of differential rotation on spin down, because for significantly longer $\tau^n \gg 1/\gamma$, contribution of differential rotation to the apparent spin down is suppressed for a factor $1/\gamma \tau^n$ (Eq. 62).

\footnote{As discussed in Section 2.2, a star with saturated unstable mode evolves along thermal equilibrium curve on ($\Omega, T$) plane, thus the timescale of temperature changes is the same as $\tau^S$.}
Is it realistic to measure spin down at such timescale? Indeed, variation of apparent frequency can hardly be measured with high precision, if spin down does not result in the shift of rotation phase $\geq \pi$. This requirement provides a lower bound for $\bar{\Omega} \geq \pi/(r_{\text{GW}}^\text{obs})$, corresponding to lower bound for $\alpha_{\text{sat}}$, which may allow spin down measurement at time-scale $\tau_{\text{obs}}$:

$$\alpha_{\text{sat}} \geq \sqrt{\frac{\tau_{\text{GW}}}{2\pi Q(r_{\text{GW}}^\text{obs})^2}} \approx 1.4 \times 10^{-5} \left( \frac{T}{10^6 \text{ K}} \right)^2 \times \left( \frac{R}{10 \text{ km}} \right)^3 \left( \frac{M}{1.4M_\odot} \right)^{-3/2} \left( \frac{\nu}{600 \text{ Hz}} \right)^{-7/2}. \quad (63)$$

The latter estimate corresponds to $r_{\text{obs}} = r^S$.

Summarizing, in principle differential rotation can affect apparent spin down in viscous non-magnetized star. To do so, the mode amplitude should vary on timescale $\tau^\ast \lesssim 1/\gamma$ and satisfy condition (63) for $\tau_{\text{obs}} \sim \tau^\ast$.

### 4.3 Differential rotation in magnetized star

Now I examine the role of differential rotation in magnetized star. Differential motion of the fluid elements is the reason for magnetic field bending, which produces back reaction: magnetic stresses, generated by the bending. These stresses, in turn, act against subsequent bending and, if the velocity of the fluid motion is below the Alfvén velocity, the magnetic field line has enough time to straighten (e.g. Section 3.14.2 by Alfven & Felthammar 1963), thus preventing the secular motion of the fluid elements (see, e.g., appendix B of Rezzolla et al. 2001 for explicit equation, which couples fluid elements motion and magnetic field evolution). Indeed, as shown by Chugunov et al. (2015); Friedman et al. (2017), in a particular case of r-modes in perfect fluid Newtonian stellar models, the relative displacement of the fluid elements is strongly suppressed, if the magnetic field $B$ is not too weak (namely, if the typical Alfven timescale $\tau_A \sim R/\sqrt{\rho/B}$ does not exceed gravitational instability timescale, corresponding to internal magnetic field $B \gtrsim 3 \times 10^6$ G for non-superconducting NS core;13 subsequently I assume that this condition is true for magnetized star). As a result secular motion of fluid elements becomes close to uniform rotation. The axisymmetric second order Eulerian perturbation of the velocity $\delta_2^{(2)} \nu_{\text{sym}}$ becomes:

$$\delta_2^{(2)} \nu_{\text{sym}} = -\nu_0 + [\delta \Omega \times r] + \frac{15\pi}{32} \alpha^2(t) \left[ \frac{\Omega}{R} \right]^2 \left( \frac{\nu}{\Omega} \right)^2 e_\phi + \delta \Omega \pi \omega e_\phi. \quad (64)$$

12 In is worth to stress that the toy model by Friedman et al. (2017) allows straightforward generalization for r-mode instability in dissipative star, by substitution $\tau^\ast$ instead of $\tau_{\text{GW}}$ as an amplitude growing rate. It leads to conclusion that relative motion of fluid elements is suppressed if $\tau_A \lesssim \tau^\ast$. The latter condition is equal to $B \gtrsim 10^6$ G for $\tau^\ast \sim 1$ yr. If it is satisfied, results of this section hold true.

13 As argued by Chugunov et al. (2015) and proved by Friedman et al. (2017) the enhancement of the magnetic field by differential motion of fluid elements does not lead to instability suppression, at least for low saturation amplitudes. The first order perturbation, given by Eq. (41), is almost unaffected by magnetic field up to $B \lesssim 10^{16}$ G (e.g., Lee 2005; Abbassi et al. 2012; Chirenti & Skákala 2013; Asai et al. 2015).

which differs from uniform rotation due to contribution of the Stokes drift to the secular motion of the fluid elements (see Eq. 46 and Chugunov 2015 for more details). The $\delta \Omega$ term is required to keep $\delta J = 0$ according to definition of perturbations in this paper, which leads to (see Eq. (18)):

$$\delta \Omega = \frac{j}{2\nu} \alpha^2 \Omega = \frac{Q}{3} \alpha^2 \Omega. \quad (65)$$

Note, in spite of the fact that the secular motion of the fluid elements is uniform rotation, spin frequency of this rotation (and the apparent spin frequency) differs from $\Omega$ because the drift velocity, given by Eq. (65), is not vanishing. This effect can be described by $\Delta \Omega = \delta \Omega$, which does not depend on the position (cylindrical radius) of the observed point at the stellar surface (see dashed line in Fig. 1). Correspondent apparent spin down rate is

$$\frac{d\Omega_{\text{app}}}{dt} = \frac{4\tilde{\Omega}^2 \alpha^2}{3\tau_{\text{GW}}(\Omega)}. \quad (66)$$

It is negative ($\tau_{\text{GW}} < 0$), and differs from $\dot{\Omega}$, given by Eq. (28), for a factor of 2/3. Note, (66) provides reasonable estimate for $\Omega_{\text{app}}$ in perfect fluid non-magnetized star (compare solid and dashed lines in Fig. 1).

### 4.4 Differential rotation along recycling scenario

According to recycling scenario of MSP formation, old slowly rotating NS is recycled by accretion from companion star and forms MSP or HOFNAR after accretion ends (Bisnovatyi-Kogan & Komberg 1976; Alpar et al. 1982; Chugunov et al. 2014). Along this evolution, NS can become unstable (e.g., Levin 1999), but the star either evolves along the boundary of the instability window (e.g., Andersson, Jones & Kokkotas 2002; Gusakov et al. 2014a; Chugunov et al. 2017) or plunges deeply into instability window with already saturated r-mode (see, e.g., discussion and analytic formulae in appendix B Gusakov et al. 2014a). Note, as discussed by Kantor et al. (2016), for a neutron star evolving along the boundary of the instability region $\Delta \Omega \sim \alpha^2 \Omega$.

$\Delta \Omega$ may differ from the value, given by Eq. (65) due to Stokes drift with daughter modes, if the saturation occurs, produced by the mode coupling mechanism, as suggested by Arras et al. 2003; Brink et al. 2004a; Bondarescu et al. 2007, 2009.
the r-mode amplitude either is almost constant (and equal to the thermal equilibrium value $\alpha^\text{eq}$) or oscillates, preserving root-mean-square value of amplitude equal to $\alpha^\text{eq}$ (so called $\alpha$-oscillations). In both cases role of differential rotation in apparent spin rate can be described in the same way as it was done here for saturated modes (Sects. 4.2.2, 4.3.1).

Applying the results from these subsections, I come to a conclusion that differential rotation can affect apparent spin down frequencies only for non-magnetized star and only at short timescales $\lesssim \tau_c$. Note, that according to footnote 12, the internal magnetic field should be $B \lesssim 10^9 \text{G} (1 \text{yr}/\tau^\alpha)$ to treat the star as non-magnetized. Typical surface magnetic field of NSs in LMXBs and MSPs is much larger ($\sim 10^8\,\text{--}10^9 \text{G}$), thus if internal magnetic field is not strongly suppressed with respect to surface value, differential rotation does not affect their apparent spin down rate. Even if the field is suppressed, still, relatively high r-mode amplitude (Eq. 63) is required so that one could measure of the spin down rate. I estimate the corresponding intrinsic GW strain amplitude $h_0$ by using Eq. (23) by Owen (2010)

$$h_0 \gtrsim 1.5 \times 10^{-25} \left(\frac{R}{10 \text{ km}}\right) \left(\frac{T}{10^4 \text{ K}}\right)^2 \left(\frac{\nu}{600 \text{ Hz}}\right)^{-1/2} \times \left(\frac{M}{1.4M_\odot}\right)^{-1/2} \left(\frac{r}{1 \text{ kpc}}\right)^{-1},$$

(68)

Here $r$ is distance to the NS. This amplitude is a bit below sensitivity of continuous GW searches (e.g., The LIGO Scientific Collaboration et al. 2018) and thus hypothesis that r-modes with such amplitudes are excited in some of NSs cannot be rejected on the base of direct GW observations. However, indirect observational constraints are possible. Namely, for known neutron stars in LMXBs, amplitudes given by Eq. (63) require enhanced cooling (e.g., direct URCA neutrino emission in the core), so that these NS could to be in thermal equilibrium (e.g. Mahmoodifar & Strohmayer 2013; Kantor et al. 2016), and may be too large to be consistent with apparent spin down rates for some of the sources (see Mahmoodifar & Strohmayer 2013 for details). Hence, it is unlikely that differential rotation affects apparent spin down rate for NSs in LMXBs. The r-mode amplitude in known MSPs is constrained even stronger ($\alpha \lesssim 10^{-7}$, e.g. Alford & Schwerner 2015; Schwener et al. 2017; Chugunov et al. 2017; Bhattacharya et al. 2017) and thus differential rotation cannot affect the apparent spin down rate of known MSPs.

5 SUMMARY, CONCLUSIONS, AND CAVEATS

In this paper I discuss possible imprints of differential rotation on the evolution of the apparent spin frequency $\Omega_{\text{app}}$ of CFS unstable star. Namely, if an unstable star emits GWs, differential rotation and angular momentum evolve. Even though differential rotation is weak, it can evolve much faster than the angular momentum and contribute to the apparent spin-down rate $\Omega_{\text{app}}$ (see Section 2.3). Additional problem, associated with differential rotation, is the uncertainty of the effective spin frequency definition, the results of which is, in particular, different form of the effective spin down rates suggested by Owen et al. (1998); Ho & Lai (2000) and paper I. Two former papers predict that spin down is associated with dissipation, but the latter couples spin down directly with instability timescale. As shown in paper I, this difference is associated only with definition of the effective spin frequency, but still there is a question: which value should corresponds to the apparent spin down rate?

In Section 2.2 I analyse the evolution on the long timescales (total change of the rotational energy due to GW emission strongly exceeds the energies of the unstable mode and the thermal energy). I demonstrate that the evolution equations can be reduced to one differential equation, which describes evolution rate along the sequence of thermally equilibrium states. This equation can be applied not only to study evolution within certain model of r-mode instability suppression, but also to constrains the shape of the instability windows and r-mode amplitude from observational data. For Newtonian stellar models this equation have especially simple form (Eq. (34), which was previously known (e.g., Chugunov et al. 2014). Differential rotation does not affect evolution on the long timescales.

Evolution on the short timescales require more detailed analysis. The apparent frequency is typically associated with motion of certain part of stellar surface (hot spot, magnetic pole, etc.), and thus even for given angular momentum it generally depend on location of the observed part of the star and can be affected by differential rotation. In Sections 4.1 and 4.3 I demonstrate that these features indeed take place for non-dissipative stellar models (except for the saturated r-modes in magnetized star, where $\Omega_{\text{app}} = \Omega$, given by Eq. (28)). In particular, I demonstrate that the star should be observed as spinning down, contrary to the naive expectation, which directly applies equations by Owen et al. (1998); Ho & Lai (2000) and predicts constant spin frequency for absence of dissipation. However, realistic model of NS should be dissipative. I discuss such models qualitatively in Section 4.2 and formulate criteria for the case, when differential rotation can affect observational evolution of the spin frequency: the r-mode amplitude should vary faster than the shear viscosity dissipation timescale and it should be large enough to produce spin down, which can be measured during this timescale.

In Section 4.4, I argue that it is unlikely that differential rotation can have an effect upon observations of NSs evolving along recycling scenario. Namely, detailed evolution scenarios (e.g., Gusakov et al. 2014a,b; Chugunov et al. 2017) predict that even if NS becomes unstable during its evolution, the instability is either saturated or almost suppressed by dissipation.

As discussed in Section 4.2.2, differential rotation, in principle, can affect apparent spin down rate for non-magnetized star. However, even very small internal magnetic field $B \gtrsim 10^9 (1 \text{yr}/\tau^\alpha)$ G suppresses these effects and indirect observational constraints to the r-mode amplitude keep differential rotation effects beyond measuring capabilities. Still, strictly speaking, these effects are not excluded in a newly born NS or even in a recycling neutron star, provided that observation of this particular NS allow of the large r-mode amplitude and negligible internal magnetic field.

The main practical conclusion of the paper is the following: $\Omega$ defined by Eq. (1) is closer to the secular motion of the fluid elements than $\Omega$, suggested by Owen et al. (1998) (see Fig. 1). It favours equations (28)-(30) from Paper I for the
modelling of r-mode unstable NS. However, as discussed in Section 4.4, differential rotation does not affect NS recycling in realistic (dissipative) model. In particular, it means, that the rate of change of effective frequency, \( \Omega \) and \( \hat{\Omega} \), should be almost the same (at the leading order in \( n \))\(^\text{15} \) and can be directly compared with the apparent spin down rate, as it was performed by Kantor et al. (2016); Andersson et al. (2017). Note, however, that this conclusion does not holds true, if we consider strongly unstable star NS with initially small (non-saturated) r-mode amplitude (see Section 4).

The above statements come with some caveats, which, as I expect, do not affect general results. First, the discussion in Section 4 is based on Newtonian models, and, strictly speaking, conclusions’ validity should be checked in a generally relativistic case. I also use very simplified microphysics (barotropic equation of state, with even polytropic approximation for numerical estimates). In particular, I neglect superfluidity in NS core, which can lead to high frequency g-modes (see, e.g., Kantor & Gusakov 2014; Dommes & Gusakov 2016). Although these assumptions are widely used in r-mode literature, their validity should be checked by more refined models. I also neglect that contribution to the differential rotation, which are associated with external (non CFS) torques. Such differential rotation can affect observations (e.g., pulsar glitches are interpreted as effect of relaxation of differential motion of superfluid and normal components of the star, e.g., Anderson & Itoh 1975). However, if the differential rotation induced by external torque is small, it can be (likely) decoupled from the effects discussed here. The strong differential rotation can affect frequencies of unstable oscillations (e.g., Chirenti et al. 2013), but these effects are left beyond scope of this paper. Finally, I do not discuss effects of the solid crust, which can have its own spin frequency, coinciding with the observed one. If crust is magnetically coupled with the core, differential motion of the crust with respect to fluid elements in the core should be suppressed and results of Sec. 4.3 should be generally applicable. In absence of such coupling, an obvious toy model of the crust motion can be obtained by assumption that the crustal spin frequency is given by some average over secular motion of the fluid elements on the top of the core. Within this model all results of the paper are also applicable. However, a strict proof of this statement goes beyond the scope of this paper.

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\(^{15}\) This result can be obtained directly from Eqs. (28) and (35) and the fact that either the unstable NS is located near boundary of instability region or r-mode is saturated (see appendix B of Gusakov et al. 2014a). In the first case, \( \tau_{\text{dis}} \approx -\tau_{\text{GW}} \) by definition of the boundary of instability region, in the second case \( \tau_{\text{dis}} \) in Eq. (35) should be substituted to \( \tau_{\text{eff}} \approx -\tau_{\text{GW}} \). In both cases Eq. (35) becomes equal to (28).

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