Maximal Perimeters of Polytope Sections and Origin-Symmetry

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Abstract
Let $P \subset \mathbb{R}^n$ ($n \geq 3$) be a convex polytope containing the origin in its interior. Let $\text{vol}_{n-2}(\text{relbd}(P \cap \{t\xi + \xi^\perp\}))$ denote the $(n-2)$-dimensional volume of the relative boundary of $P \cap \{t\xi + \xi^\perp\}$ for $t \in \mathbb{R}$, $\xi \in S^{n-1}$. We prove the following: if $\text{vol}_{n-2}(\text{relbd}(P \cap \{t\xi + \xi^\perp\})) = \max_{t \in \mathbb{R}} \text{vol}_{n-2}(\text{relbd}(P \cap \{t\xi + \xi^\perp\}))$ for all $\xi \in S^{n-1}$, then $P$ is origin-symmetric, i.e., $P = -P$. Our result gives a partial affirmative answer to a conjecture by Makai et al. We also characterize the origin-symmetry of $C^2$ star bodies in terms of the dual quermassintegrals of their sections; this can be seen as a dual version of the conjecture of Makai et al.

Keywords Convex bodies · Convex polytopes · Origin-symmetry · Sections

Mathematics Subject Classification 52B15 · 52A20 · 52A38

1 Introduction

A convex body $K \subset \mathbb{R}^n$ is a convex and compact subset of $\mathbb{R}^n$ with non-empty interior. Convex bodies are extensively studied objects in convex geometry; see the standard references [3, 12] for known results and open problems. We say $K$ is origin-symmetric if it is equal to its reflection through the origin, i.e., $K = -K$. Many results for convex
bodies depend on the presence of origin-symmetry. For example, the well-known Funk
Section Theorem (e.g. [3, Thm. 7.2.6]) states that whenever $K_1, K_2 \subset \mathbb{R}^n$ are origin-
symmetric convex bodies such that

$$\text{vol}_{n-1}(K_1 \cap \xi^\perp) = \text{vol}_{n-1}(K_2 \cap \xi^\perp) \quad \text{for all } \xi \in S^{n-1},$$

then necessarily $K_1 = K_2$. Here, $\xi^\perp := \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$.

It is both interesting and useful to find properties which are equivalent to origin-
symmetry. Several such characterizations are already known. For example, a convex
body $K \subset \mathbb{R}^n$ is origin-symmetric if and only if every hyperplane through the origin
splits $K$ into two halves of equal $n$-dimensional volume. This result began with Funk
[2] for $n = 3$; see [1, 11] for more information. Schneider [11] established the corre-
sponding statement when the $n$-dimensional volume is replaced by $(n-1)$-dimensional
surface area.

Brunn’s Theorem implies that if $K$ is origin-symmetric, then

$$\text{vol}_{n-1}(K \cap \xi^\perp) = \max_{t \in \mathbb{R}} \text{vol}_{n-1}(K \cap \{t\xi + \xi^\perp\}) \quad \text{for all } \xi \in S^{n-1}. \quad (1)$$

Here, $\{t\xi + \xi^\perp\}$ denotes the translate of $\xi^\perp$ containing $t\xi$. Using a particular integro-
differential transform, Makai et al. [8] proved the converse statement: a convex body $K$ which contains the origin in its interior and satisfies (1) must be origin-symmetric. In fact, they proved a more general statement for star bodies. Ryabogin and Yaskin [10] gave an alternate proof using Fourier analysis, as well as a new characterization of origin-symmetry via conical sections. A stability version of the result of Makai et al. was established in [13].

Makai et al. [8] conjectured a further characterization of origin-symmetry in terms
of the quermassintegrals of sections. Recall that the quermassintegrals $W_l(K)$ of a
convex body $K \subset \mathbb{R}^n$ arise as coefficients in the expansion

$$\text{vol}_n(K + tB^n_2(o, 1)) = \sum_{l=0}^{n} \binom{n}{l} W_l(K)t^l, \quad t \geq 0.$$ 

The addition of sets here is the well-known Minkowski addition

$$K + tB^n_2(o, 1) := \{x + ty : x \in K, \ y \in B^n_2(o, 1)\}.$$ 

Refer to [12] for a thorough overview of mixed volumes and quermassintegrals. For
any $0 \leq l \leq n-2$ and $\xi \in S^{n-1}$, consider the quermassintegral $W_l((K - t\xi) \cap \xi^\perp)$ of
the $(n-1)$-dimensional convex body $(K - t\xi) \cap \xi^\perp$ in $\xi^\perp$. If $K$ is origin-symmetric,
then the monotonicity and positive multilinearity of mixed volumes together with the
Alexandroff–Fenchel inequality imply

$$W_l(K \cap \xi^\perp) = \max_{t \in \mathbb{R}} W_l((K - t\xi) \cap \xi^\perp) \quad \text{for all } \xi \in S^{n-1}. \quad (2)$$
For $l = 0$, (2) is equivalent to (1), as $W_0((K - t\xi) \cap \xi^\perp)$ is the $(n - 1)$-dimensional volume of $(K - t\xi) \cap \xi^\perp$. Makai et al. conjectured that if $K$ contains the origin in its interior and satisfies (2) for any $1 \leq l \leq n - 2$, it must be origin-symmetric. Makai and Martini [7] proved a local variant of the conjecture for smooth perturbations of the Euclidean ball; otherwise, the problem is completely open.

In this paper, we consider the case of convex polytopes which satisfy (2) for $l = 1$. A convex polytope $P \subset \mathbb{R}^n$ is a convex body which is the convex hull of finitely many points. By a polytope we will always mean a convex polytope, unless explicitly stated otherwise. It is a common practice to restrict unsolved problems for general convex bodies to the class of polytopes (e.g. [9, 14, 15, 17, 18]) because polytopes have additional structure. Up to a constant depending on the dimension, $W_1((P - t\xi) \cap \xi^\perp)$ is the $(n - 2)$-dimensional surface area of the $(n - 1)$-dimensional polytope $(P - t\xi) \cap \xi^\perp$ in $\xi^\perp$. Letting $\text{vol}_{n-2}(\text{relbd}(P \cap \{t\xi + \xi^\perp\}))$ denote the $(n - 2)$-dimensional volume of the relative boundary of $P \cap \{t\xi + \xi^\perp\}$, we prove the following:

**Theorem 1.1** Let $P \subset \mathbb{R}^n$, $n \geq 3$, be a convex polytope containing the origin in its interior, and such that

$$\text{vol}_{n-2}(\text{relbd}(P \cap \xi^\perp)) = \max_{t \in \mathbb{R}} \text{vol}_{n-2}(\text{relbd}(P \cap \{t\xi + \xi^\perp\})) \quad (3)$$

for all $\xi \in S^{n-1}$. Then $P = -P$.

We introduce some notation and simple lemmas in Sect. 2. The proof of Theorem 1.1 is presented in Sect. 3. Finally, in Sect. 4, we briefly explain how to characterize the origin-symmetry of $C^2$ star bodies using the dual quermassintegrals of sections; this is a dual version of the conjecture of Makai et al. [8].

### 2 Some Notation & Auxiliary Lemmas

The origin in $\mathbb{R}^n$ is denoted by $0$. The affine hull and linear span of a set $A \subset \mathbb{R}^n$ are respectively denoted by $\text{aff}(A)$ and $\text{span}(A)$. We let $\mathbb{R}x := \text{span}(x)$ be the line through $x \in \mathbb{R}^n \setminus \{0\}$ and the origin. We use $|\cdot|_2$ for the Euclidean norm, $B^n_2(x, r)$ for the Euclidean ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$, and $S^{n-1}$ for the unit sphere. For $\xi \in S^{n-1}$, we define $\xi^+ := \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq 0\}$, $\xi^- := \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$, and $S^{n-1}(\xi, \epsilon) := S^{n-1} \cap B^n_2(\xi, \epsilon)$ for small $\epsilon > 0$. Finally, $[\xi_1, \xi_2] := S^{n-1} \cap \{\alpha\xi_1 + \beta\xi_2 : \alpha, \beta \geq 0\}$ gives the shorter geodesic segment connecting linearly independent $\xi_1$, $\xi_2 \in S^{n-1}$.

For any $(n - 2)$-dimensional polytope $G \subset \mathbb{R}^n$ whose affine hull does not contain the origin, define $\eta_G \in S^{n-1}$ to be the unique unit vector for which

- the line $\mathbb{R}\eta_G$ and $\text{aff}(G)$ intersect orthogonally;
- $G \subset \eta_G^- := \{x \in \mathbb{R}^n : \langle x, \eta_G \rangle \leq 0\}$.

For each $t > 0$,

$$\text{reflec}(G, t) := \{x \in \mathbb{R}^n : \langle x, \eta_G \rangle = t \text{ and the line } \mathbb{R}x \text{ intersects } G\}$$
is an \((n - 2)\)-dimensional polytope in \(\mathbb{R}^n\); see Fig. 1. In words, \(\text{reflec}(G, t)\) is the positive homothetic copy of \(-G\) lying in \(\{t\eta_G + \eta_G^\perp\}\) so that every line connecting a vertex of \(\text{reflec}(G, t)\) to the corresponding vertex of \(G\) passes through the origin.

**Lemma 2.1** Let \(Q \subset \mathbb{R}^n\) be a polytope for which the origin is not a vertex. Let \(S^{n-1}(\theta_0, \varepsilon)\) be a spherical cap of radius \(\varepsilon > 0\) centred at \(\theta_0 \in S^{n-1}\). There exists \(\theta \in S^{n-1}(\theta_0, \varepsilon)\) such that \(\theta^\perp\) does not contain any vertices of \(Q\).

**Proof** If \(u_1, \ldots, u_d\) are the vertices of \(Q\), choose any \(\theta\) from the non-empty set \(S^{n-1}(\theta_0, \varepsilon) \setminus (u_1^+, \ldots, u_d^+)\). \(\square\)

The proof of the following lemma is trivial.

**Lemma 2.2** Let \(I \subset \mathbb{R}\) be an open interval. Let \(\{f_j\}_{j=1}^n\) be a collection of differentiable \(\mathbb{R}^n\)-valued functions on \(I\). Define \(F(t) := \det(f_1(t), \ldots, f_n(t))\). Then \(F\) is differentiable on \(I\) with

\[
F'(t) = \sum_{j=1}^n \det(f_1(t), \ldots, f_{j-1}(t), f_j'(t), f_{j+1}(t), \ldots, f_n(t)).
\]

### 3 Proof of Theorem 1.1

Let \(P\) be a convex polytope containing the origin in its interior and satisfying (3) for all \(\xi \in S^{n-1}\). Our proof has two distinct parts. We first need to prove that

\[
\text{reflec}(G, t) \text{ is an } (n - 2)\text{-dimensional face of } P \text{ for some } t > 0 \quad (4)
\]

whenever \(G\) is an \((n - 2)\)-dimensional face of \(P\). To the contrary, we suppose \(G_0\) is an \((n - 2)\)-dimensional face of \(P\) for which (4) is false. We find a special spherical
cap $S^{n-1}(\xi_0, \varepsilon)$. For every $\xi \in S^{n-1}(\xi_0, \varepsilon)$, $\xi \perp$ misses all the vertices of $P$, while intersecting $G_0$ and no other $(n - 2)$-dimensional face $G$ of $P$ which is parallel to $G_0$. The face $G$ is parallel to $G_0$ if $\text{aff}(G) = \text{aff}(G_0) + x$ for some $x \in \mathbb{R}^n$. We derive a “nice” equation from (3) which is valid for all $\xi \in S^{n-1}(\xi_0, \varepsilon)$. Forgetting the geometric meaning, we analytically extend this nice equation to all $\xi \in S^{n-1}$, excluding a finite number of great subspheres. Studying the behaviour near one of these subspheres, we arrive at our contradiction.

We conclude that for every vertex $v$ of $P$, the line $\mathbb{R}v$ contains another vertex $\tilde{v}$ of $P$. In the second part of our proof, we prove that $\tilde{v} = -v$. Hence, $P$ is origin-symmetric.

### 3.1 First Part

Assume that our polytope $P$ has an $(n - 2)$-dimensional face $G_0$ for which (4) is false. That is, reflec$(G_0, t)$ is not identically equal to an $(n - 2)$-dimensional face of $P$ for any $t > 0$. We still allow that, for any given $t > 0$, reflec$(G_0, t)$ may have a non-empty intersection with another $(n - 2)$-dimensional face of $P$ which may or may not be parallel to $G_0$.

By the convexity of $P$, the hyperplane $\text{aff}(o, G_0)$ contains at most one other $(n - 2)$-dimensional face of $P$ (besides $G_0$) which is parallel to $G_0$. If such a face exists, it must lie in $\text{aff}(o, G_0) \cap \{t\eta_{G_0} + \eta \perp_{G_0}\}$ for some $t > 0$ because $P$ contains the origin in its interior. We additionally assume without loss of generality that reflec$(G_0, t)$ is not contained within an $(n - 2)$-dimensional face $G$ of $P$ for any $t > 0$ (else we replace $G_0$ with $G$).

**Lemma 3.1** There is a $\xi_0 \in S^{n-1}$ such that

(i) the hyperplane $\xi_0 \perp$ does not contain any vertices of $P$;
(ii) $\xi_0 \perp$ intersects $G_0$ but no other $(n - 2)$-dimensional faces of $P$ parallel to $G_0$;
(iii) there is exactly one vertex $v$ of $G_0$ contained in $\xi_0 := \{x \in \mathbb{R}^n : \langle x, \xi_0 \rangle \geq 0\}$.

**Proof** Choose $\theta \in S^{n-1}$ so that $\theta \perp = \text{aff}(o, G_0)$. Let $\eta := \eta_{G_0}$ be the unit vector defined as before. There are only two possibilities: either $\theta \perp$ does not contain any $(n - 2)$-dimensional face of $P$ parallel to $G_0$ (besides $G_0$), or it contains exactly one.

If the first case is true, we can of course choose an affine $(n - 3)$-dimensional subspace $L$ lying within $\text{aff}(G_0)$ which does not pass through any vertices of $G_0$, and strictly separates exactly one vertex $v \in G_0$ from the others.

Suppose the second case is true, i.e., there is a unique $t_0 > 0$ for which $H := \{x \in \mathbb{R}^n : \langle x, \eta \rangle = t_0\} \cap \theta \perp$ contains an $(n - 2)$-dimensional face $G$ of $P$. By definition and assumption, reflec$(G_0, t_0)$ lies in $H$ and is not contained in $G$. We can choose $\tilde{L} \subset H$ to be an $(n - 3)$-dimensional affine subspace which, within $H$, strictly separates exactly one vertex $\tilde{v} \in \text{reflec}(G_0, t_0)$ from both $G$ and the remaining vertices of reflec$(G_0, t_0)$. Let $v$ be the vertex of $G_0$ lying on the line $\mathbb{R}\tilde{v}$.

Regardless of which case was true, set $L := \text{aff}(o, \tilde{L}) \subset \theta \perp$. The $(n - 2)$-dimensional subspace $L$ intersects $G_0$ but no other $(n - 2)$-dimensional faces of $P$ parallel to $G_0$, and separates $v$ in $\theta \perp$ from the remaining vertices of $G_0$. Perturbing $L$ within $\theta \perp$ if necessary (see Lemma 2.1), $L$ also does not contain any vertices of $P$. Choose $\phi \in S^{n-1} \cap \theta \perp \cap L \perp$. 

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Define the slab $\theta_{\alpha}^\perp := \{ x \in \mathbb{R}^n : |\langle x, \theta \rangle| \leq \alpha \}$, with $\alpha > 0$ small enough so that $\theta_{\alpha}^\perp$ only contains vertices of $P$ lying in $\theta^\perp$. Necessarily, $\theta_{\alpha}^\perp$ also only contains the $(n-2)$-dimensional faces of $P$ parallel to $G_0$ which lie entirely in $\theta^\perp$. Choose $\beta > 0$ large enough so that the slab $\{ x \in \mathbb{R}^n : |\langle x, \phi \rangle| \leq \beta \}$ contains $P$. Let $\xi_0 \in S^{n-1}$ be such that $\xi_0^\perp = \operatorname{aff}(o, \alpha \theta + \beta \phi + L)$ and $\langle v, \xi_0 \rangle \geq 0$. We then have $\xi_0^\perp \cap P \subset \theta_{\alpha}^\perp$ and $\xi_0^\perp \cap \theta^\perp = L$. It follows from the construction of $\theta_{\alpha}^\perp$ and $L$ that $\xi_0$ has the desired properties.

Let $\{ E_i \}_{i \in I}$ and $\{ F_j \}_{j \in J}$ respectively be the edges and facets (i.e., $(n-1)$-dimensional faces) of $P$ intersecting $\xi_0^\perp$. Consider a spherical cap $S^{n-1}(\xi_0, \varepsilon)$ of radius $\varepsilon > 0$ centred at $\xi_0$. For $\varepsilon > 0$ small enough, the set

$$\{ x \in \mathbb{R}^n : |\langle x, \xi \rangle| \leq \varepsilon \text{ for some } \xi \in S(\xi_0, \varepsilon) \}$$

does not contain any vertices of $P$. Consequently, the map

$$t \mapsto \operatorname{vol}_{n-2}(\operatorname{relbd}(P \cap \{ t\xi + \xi^\perp \})) = \sum_{j \in J} \operatorname{vol}_{n-2}(F_j \cap \{ t\xi + \xi^\perp \})$$

is differentiable in a neighbourhood of $t = 0$ for each $\xi \in S^{n-1}(\xi_0, \varepsilon)$. Therefore, (3) implies

$$\sum_{j \in J} \frac{d}{dt} \operatorname{vol}_{n-2}(F_j \cap \{ t\xi + \xi^\perp \}) \bigg|_{t=0} = 0 \quad (5)$$

for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$. We need to find an expression for this derivative.

For each $i \in I$, let $u_i + l_i s$ be the line in $\mathbb{R}^n$ containing $E_i$; $u_i$ is a point on the line, $l_i$ is a unit vector parallel to the line, and $s$ is the parameter. Clearly, $\{ t\xi + \xi^\perp \}$ intersects the same edges and facets as $\xi_0^\perp$ for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$ and $|t| \leq \varepsilon$. The intersection point of $\{ t\xi + \xi^\perp \}$ with the edge $E_i$ is given by

$$p_i(\xi, t) := u_i + l_i \left( \frac{t - \langle u_i, \xi \rangle}{\langle l_i, \xi \rangle} \right).$$

Note that $\xi_0^\perp$ intersects exactly those edges of $G_0$ which are adjacent to the vertex $v$. Whenever $E_i$ is an edge of $G_0$ adjacent to $v$, we put $u_i := v$ and choose $l_i$ so that it gives the direction from another vertex of $G_0$ to $v$; this ensures $\langle l_i, \xi_0 \rangle > 0$.

For each $j \in J$, there is a pair of vertices from the facet $F_j$ such that the line through them does not lie in a translate of $\operatorname{aff}(G_0)$, and with one of the vertices on either open side of $\xi_0^\perp$. Translating this line if necessary, we obtain an auxiliary line $w_j + m_j s$ which

- lies within $\operatorname{aff}(F_j)$;
- does not lie within an $(n-2)$-dimensional affine subspace parallel to $\operatorname{aff}(G_0)$;
- is transversal to $\xi^\perp$ for every $\xi \in S^{n-1}(\xi_0, \varepsilon)$;
intersects \{ t\xi + \xi^\perp \} at a point in the relative interior of \( F_j \) for every \( \xi \in S^{n-1}(\xi_0, \varepsilon) \) and \( |t| \leq \varepsilon \).

Again, \( w_j \) is a point on the line, \( m_j \) is a unit vector parallel to the line, and \( s \) is the parameter. The intersection point of \{ t\xi + \xi^\perp \} with \( w_j + m_j s \) is given by

\[
q_j(\xi, t) := w_j + m_j \left( \frac{t - \langle w_j, \xi \rangle}{\langle m_j, \xi \rangle} \right).
\]

Note that we necessarily have \( \xi_0 \not\in m_j \) for all \( j \in J \). Let us reiterate that \( q_j(\xi, t) \) lies in the relative interior of \( F_j \).

Consider a facet \( F_j \), and an \((n-2)\)-dimensional face \( G \) of \( P \) which intersects \( \xi_0^\perp \) and is adjacent to \( F_j \). Observe that \( G \cap \{ t\xi + \xi^\perp \} \) is an \((n-3)\)-dimensional face of the \((n-2)\)-dimensional polytope \( F_j \cap \{ t\xi + \xi^\perp \} \), for each \( \xi \in S^{n-1}(\xi_0, \varepsilon) \) and \( |t| \leq \varepsilon \). Express \( G \cap \{ t\xi + \xi^\perp \} \) as an up-to-boundaries disjoint union of \((n-3)\)-dimensional simplices whose vertices correspond to the vertices of \( G \cap \{ t\xi + \xi^\perp \} \); that is, each simplex has vertices \( p_{i_1}(\xi, t), \ldots, p_{i_{n-2}}(\xi, t) \) for some \( i_1, \ldots, i_{n-2} \in I \). Triangulating every such \((n-3)\)-dimensional face \( G \cap \{ t\xi + \xi^\perp \} \) in this way, we get a triangulation of \( F_j \cap \{ t\xi + \xi^\perp \} \) by taking the convex hull of the simplices in its relative boundary with \( q_j(\xi, t) \).

**Remark** The description and orientation of a simplex \( \Delta \) in the triangulation of \( F_j \cap \{ t\xi + \xi^\perp \} \) in terms of the ordered vertices \( \{ p_{i_1}(\xi, t), \ldots, p_{i_{n-2}}(\xi, t), q_j(\xi, t) \} \) is independent of \( \xi \in S(\xi_0, \varepsilon) \) and \( |t| \leq \varepsilon \).

Setting \( n_j \in S^{n-1} \) to be the outer unit normal to \( F_j \), \( \text{vol}_{n-2}(F_j \cap \{ t\xi + \xi^\perp \}) \) is then a sum of terms of the form

\[
\text{vol}_{n-2}(\Delta) = \frac{\det (p_{i_1}(\xi, t) - q_j(\xi, t), \ldots, p_{i_{n-2}}(\xi, t) - q_j(\xi, t), n_j, \xi)}{(n-2)! \sqrt{1 - \langle n_j, \xi \rangle^2}}; \tag{6}
\]

e.g. see [3, p. 14] for the volume formula for a simplex. We assume the column vectors in the determinant are ordered so that the determinant is positive. Differentiating (6) at \( t = 0 \) with the help of Lemma 2.2 gives

\[
\frac{1}{(n-2)! \sqrt{1 - \langle n_j, \xi \rangle^2}} \sum_{\gamma=1}^{n-2} \det (X_{i_1}(\xi), \ldots, \tilde{X}_{i_{\gamma}}(\xi), \ldots, X_{i_{n-2}}(\xi), n_j, \xi), \tag{7}
\]

where for \( 1 \leq \gamma \leq n-2, \)

\[
X_{i_\gamma}(\xi) := p_{i_\gamma}(\xi, 0) - q_j(\xi, 0) = u_{i_\gamma} - \langle u_{i_\gamma}, \xi \rangle l_{i_\gamma} - w_j + \langle w_j, \xi \rangle m_j,
\]

\[
\tilde{X}_{i_\gamma}(\xi) := \left. \frac{d}{dt} (p_{i_\gamma}(\xi, t) - q_j(\xi, t)) \right|_{t=0} = \frac{l_{i_\gamma}}{\langle l_{i_\gamma}, \xi \rangle} - \frac{m_j}{\langle m_j, \xi \rangle}.
\]
The left hand side of (5) is a sum of expressions having the form (7). That is, (5) is equivalent to
\[
\sum_{\Delta} \left( \frac{\sum_{\gamma=1}^{n-2} \det (X_i(\xi), \ldots, \tilde{X}_{i_j}(\xi), \ldots, X_{i_{n-2}}(\xi), n_j, \xi)}{(n-2)! \sqrt{1 - \langle n_j, \xi \rangle^2}} \right) = 0 \tag{8}
\]
for every \( \xi \in S^{n-1}(\xi_0, \varepsilon) \), where the first summation is over all appropriately ordered indices \( \{i_1, \ldots, i_{n-2}, j\} \) corresponding to vertices of simplices \( \Delta \) in our triangulation of the relative boundary of \( P \cap \xi_0 \perp \). Forget the geometric meaning of (8). Clearing denominators on the left-hand side of (8) gives a function of \( \xi \) which we denote by \( \Phi(\xi) \). Because \( \Phi \) is a sum of products of scalar products of \( \xi \) and terms \( \sqrt{1 - \langle n_j, \xi \rangle^2} \), we are able to consider \( \Phi \) as a function on all of \( S^{n-1} \) such that \( \Phi \equiv 0 \) on \( S^{n-1}(\xi_0, \varepsilon) \).

**Lemma 3.2** \( \Phi(\xi) = 0 \) for all \( \xi \in S^{n-1} \).

**Proof** Suppose \( \zeta \in S^{n-1} \) is such that \( \Phi(\zeta) \neq 0 \). We have \( \Phi(n_j) = 0 \) for all \( j \in J \), so \( \xi \neq n_j \). There is a \( \zeta_1 \) from the relative interior of \( S^{n-1}(\xi_0, \varepsilon) \) which is not parallel to \( \xi \), and is such that the geodesic segment \( [\zeta_1, \xi] \) connecting \( \zeta_1 \) to \( \xi \) contains none of the \( n_j \). Choose \( \zeta_2 \in S^{n-1} \) which is perpendicular to \( \zeta_1 \), lies in \( \text{aff}(\{\zeta_1, \xi\}) \), and is such that \( \langle \zeta_2, \xi \rangle > 0 \). Let \( \tilde{\Phi} \) be the restriction of \( \Phi \) to \( \xi \in [\zeta_1, \xi] \), and adopt polar coordinates \( \xi = \zeta_1 \cos \phi + \zeta_2 \sin \phi \). As a function of \( \phi \in [0, \arccos \langle \zeta_1, \xi \rangle] \), \( \tilde{\Phi} \) is a sum of products of \( \cos \phi \), \( \sin \phi \), and \( \sqrt{1 - \langle n_j, \zeta_1 \cos \phi + \zeta_2 \sin \phi \rangle^2} \). The radicals in the expression for \( \tilde{\Phi} \) are never zero because \( [\zeta_1, \xi] \) misses all of the \( n_j \), so \( \tilde{\Phi} \) is analytic. Consequently, \( \tilde{\Phi} \) must be identically zero, as it vanishes in a neighbourhood of \( \phi = 0 \). This is a contradiction. \( \square \)

Lemma 3.2 implies that the equality in (8) holds for all \( \xi \in S^{n-1} \setminus A \), where \( A \) is the union over \( i \in I \) and \( j \in J \) of the unit spheres in \( l_i^\perp \) and \( m_j^\perp \). Of course, we have \( \pm n_j \in S^{n-1} \cap m_j^\perp \) for each \( j \in J \), so \( \{\pm n_j\}_{j \in J} \subset A \). We will consider the limit of the left-hand side of (8) along a certain path in \( S^{n-1} \setminus A \) which terminates at a point in \( A \).

The \((n-2)\)-dimensional face \( G_0 \) is the intersection of two facets of \( P \) belonging to \( \{F_j\}_{j \in J} \), say \( F_1 \) and \( F_2 \). The normal space \( G_0^\perp := \langle \text{aff}(G_0) - \text{aff}(G_0) \rangle^\perp \) of \( G_0 \) is two-dimensional and spanned by \( n_1 \) and \( n_2 \), the outer unit normals of \( F_1 \) and \( F_2 \). For \( s \in \mathbb{R} \), define
\[
\tilde{n}_s := \frac{(1-s)n_1 + sn_2}{\| (1-s)n_1 + sn_2 \|_2}.
\]
Choose \( \tilde{\varepsilon} > 0 \) small enough so that \( \langle v, \tilde{n}_s \rangle > 0 \) for all \( s \in [1 - \tilde{\varepsilon}, 1 + \tilde{\varepsilon}] \); this is possible because we have \( \langle v, n_1 \rangle > 0 \) and \( \langle v, n_2 \rangle > 0 \). Consider the non-degenerate geodesic segments
\[
A_+ := \{ \tilde{n}_s : 1 - \tilde{\varepsilon} \leq s \leq 1 \} \quad \text{and} \quad A_- := \{ \tilde{n}_s : 1 \leq s \leq 1 + \tilde{\varepsilon} \}
\]
in $S^{n-1} \cap G_0^+ \cong S^1$. The arc $A_+$ is not contained in the normal space of any other $(n-2)$-dimensional face of $P$ intersected by $\xi_0^\perp$, because $\xi_0^\perp$ does not intersect any other $(n-2)$-dimensional faces parallel to $G_0$; neither is $A_+$ contained in $m_j^\perp$ for any $j \in J$, because $m_j$ is not contained in a translate of $\text{aff}(G_0)$. The same conclusions also hold for $A_-$. Therefore, we can choose $s_+ \in (1 - \tilde{\varepsilon}, 1)$ and $s_- \in (1, 1 + \tilde{\varepsilon})$ so that for $\tilde{n} \in \{\tilde{n}_+, \tilde{n}_-\}$

- $\tilde{n}$ is not a unit normal for any $(n-2)$-dimensional face of $P$ intersected by $\xi_0^\perp$, besides $G_0$;
- $\tilde{n} \not\perp m_j$, hence $\tilde{n} \neq \pm n_j$, for all $j \in J$;
- $\langle v, \tilde{n} \rangle > 0$.

For the time being, arbitrarily fix $s_0 \in \{s_+, s_-\}$ and set $\tilde{n} := \tilde{n}_{s_0}$. For small $\delta > 0$, define the unit vector

$$\xi_\delta := \frac{\tilde{n} + \delta \xi_0}{|\tilde{n} + \delta \xi_0|_2}.$$

Clearly,

$$\lim_{\delta \to 0^+} \xi_\delta = \tilde{n} \in S^{n-1} \cap G_0^+ \subset \bigcup_{i \in I} (S^{n-1} \cap l_i^\perp) \subset A.$$

We have $\langle \xi_\delta, l_i \rangle, \langle \xi_\delta, m_j \rangle \neq 0$ for all $i \in I, j \in J$ whenever

$$0 < \delta < \min \left\{ \frac{|\langle \tilde{n}, l_i \rangle|}{|\langle \xi_0, l_i \rangle|}, \frac{|\langle \tilde{n}, m_j \rangle|}{|\langle \xi_0, m_j \rangle|} : i \in I \text{ such that } \langle \tilde{n}, l_i \rangle \neq 0, \ j \in J \right\}.$$

The previous minimum is well defined and positive, because $\xi_0 \not\perp l_i, m_j$ and $\tilde{n} \not\perp m_j$ for all $i \in I, j \in J$. So $\xi_\delta \in S^{n-1} \setminus A$ for small enough $\delta > 0$.

Now, replace $\xi$ with $\xi_\delta$ in (8), multiply both sides of the resulting equation by $\delta^{n-2}$, and take the limit as $\delta$ goes to zero. Consider what happens to the expressions (7) multiplied by $\delta^{n-2}$ in this limit. We have

$$\lim_{\delta \to 0^+} \delta X_{i_y}(\xi_\delta) = \lim_{\delta \to 0^+} \left( \delta u_{i_y} - \delta \frac{\langle u_{i_y}, \tilde{n} + \delta \xi_0 \rangle}{\langle l_{i_y}, \tilde{n} + \delta \xi_0 \rangle} l_{i_y} - \delta w_j + \delta \frac{\langle w_j, \tilde{n} + \delta \xi_0 \rangle}{\langle m_j, \tilde{n} + \delta \xi_0 \rangle} m_j \right)$$

$$= \begin{cases} 0 & \text{if } \tilde{n} \not\perp l_{i_y}, \\ -\frac{\langle u_{i_y}, \tilde{n} \rangle}{\langle l_{i_y}, \xi_0 \rangle} l_{i_y} & \text{if } \tilde{n} \perp l_{i_y}, \end{cases}$$

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and

\[
\lim_{\delta \to 0^+} \delta \tilde{X}_{i_\nu}(\xi_\delta)
= \lim_{\delta \to 0^+} \left( \frac{\delta |\tilde{n} + \delta \xi_0|^2}{\langle l_{i_\nu}, \tilde{n} + \delta \xi_0 \rangle} - \frac{\delta |\tilde{n} + \delta \xi_0|^2}{\langle m_j, \tilde{n} + \delta \xi_0 \rangle} \right)
= \begin{cases} 
 0 & \text{if } \tilde{n} \not\perp l_{i_\nu}, \\
 l_{i_\nu} & \text{if } \tilde{n} \perp l_{i_\nu},
\end{cases}
\]

because \( \xi_0 \not\perp l_i \) for all \( i \in I \) and \( \tilde{n} \not\perp m_j \) for all \( j \in J \). Therefore, \( \delta^{n-2} \) times expression (7) vanishes in the limit if at least one index \( i_\nu \) in (7) corresponds to an edge direction \( l_{i_\nu} \) which is not perpendicular to \( \tilde{n} \). If \( l_{i_1}, \ldots, l_{i_{n-2}} \) are all perpendicular to \( \tilde{n} \), then the limit of \( \delta^{n-2} \) times expression (7) becomes

\[
\frac{(-1)^{n-3} \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_j, \tilde{n})}{(n - 2)! \sqrt{1 - \langle n_j, \tilde{n} \rangle^2}} \sum_{\omega=1}^{n-2} \left( \frac{\langle u_{i_\omega}, \tilde{n} \rangle - \langle u_{i_\nu}, \tilde{n} \rangle}{\langle l_{i_\nu}, \xi_0 \rangle} \right).
\]

If the determinant in (9) is non-zero, then \( l_{i_1}, \ldots, l_{i_{n-2}} \) are linearly independent. Therefore, \( l_{i_1}, \ldots, l_{i_{n-2}} \) span an \( (n-2) \)-dimensional subspace which is parallel to the \( (n-2) \)-dimensional face \( G \) of \( P \) to which the edges \( E_{i_1}, \ldots, E_{i_{n-2}} \) belong. Necessarily, \( \tilde{n} \) will be a unit normal for \( G \), so \( G = G_0 \) by our choice of \( \tilde{n} \). We conclude that the limit of \( \delta^{n-2} \) times (7) only has a chance of being non-zero if (7) corresponds to an \( (n-2) \)-dimensional simplex \( \Delta_j \) in our triangulation of \( G_1 \cap \xi_0^\perp \), \( j = 1 \) or \( j = 2 \), with the base of \( \Delta_j \) being an \( (n-3) \)-dimensional simplex in the triangulation of \( G_0 \cap \xi_0^\perp \).

If (7) comes from such a \( \Delta_1 \) in the triangulation of \( F_1 \cap \xi_0^\perp \), then the limit of \( \delta^{n-2} \) times (7) is given by (9), and simplifies further to the non-zero term

\[
\frac{(-1)^{n-3} n^{-3} s_0 \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2)}{(n - 3)! (l_{i_1}, \xi_0) \cdots (l_{i_{n-2}}, \xi_0) |(1 - s_0) n_1 + s_0 n_2| 2 \sqrt{1 - \langle n_1, \tilde{n} \rangle^2} \neq 0. \tag{10}
\]

The distinct indices \( i_1, \ldots, i_{n-2} \) correspond to the vertices of a simplex in the triangulation of \( G_0 \cap \xi_0^\perp \), ordered so that the expression in (6) for the facet \( F_1 \) is positive. The important fact that (10) is non-zero is clear once we observe that the determinant is non-zero. Indeed, the unit vectors \( l_{i_1}, \ldots, l_{i_{n-2}} \) are necessarily linearly independent and perpendicular to both \( n_1 \) and \( n_2 \), because they give the directions for distinct edges of \( G_0 \) with the common vertex \( v \). Similarly, when (7) comes from such a \( \Delta_2 \) in the triangulation of \( F_2 \cap \xi_0^\perp \), its product with \( \delta^{n-2} \) has the non-zero limit

\[
\frac{(-1)^{n-3} \det(l_{k_1}, \ldots, l_{k_{n-2}}, n_2, n_1)}{(n - 3)! (l_{k_1}, \xi_0) \cdots (l_{k_{n-2}}, \xi_0) |(1 - s_0) n_1 + s_0 n_2| 2 \sqrt{1 - \langle n_2, \tilde{n} \rangle^2} \neq 0. \tag{11}
\]

The distinct indices \( k_1, \ldots, k_{n-2} \) correspond to the vertices of a simplex in the triangulation of \( G_0 \cap \xi_0^\perp \), ordered so that the expression in (6) for the facet \( F_2 \) is positive.
We will now consider the signs of the determinants in (10) and (11).

**Lemma 3.3** The determinants \( \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2) \) in (10) have the same sign for any collection of indices \( i_1, \ldots, i_{n-2} \) with the previously described properties. The determinants in (11) also all have the same sign. However, the signs of the determinants in (10) and (11) may differ.

**Proof** We will transform the determinant \( \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2) \) into a determinant having the same form as in (6). We will do this with the aid of two linear transformations. Before we define these transformations, we first describe several subspaces on which they act.

Let \( y \in G_0 \cap \xi_0^\perp \). Note that

\[
o \in ((G_0 - y) \cap \xi_0^\perp) = ((G_0 \cap \xi_0^\perp) - y) \subset ((F_1 - y) \cap \xi_0^\perp) = ((F_1 \cap \xi_0^\perp) - y).
\]

Consider the \((n - 3)\)-dimensional subspace

\[
L := \text{span} ( (G_0 \cap \xi_0^\perp) - y ),
\]

which is orthogonal to the three-dimensional subspace

\[
L^\perp = \text{span} (n_1, n_2, \xi_0).
\]

Define the \((n - 2)\)-dimensional subspace

\[
L_{n_1} := \text{span}(n_1, L).
\]

The projections \( n_2|n_1^\perp \) and \( \xi_0|n_1^\perp \) are non-zero, linearly independent, and orthogonal to \( L_{n_1} \). So, the orthogonal complement of \( L_{n_1} \) is the two-dimensional plane

\[
L_{n_1}^\perp = \text{span} (n_2|n_1^\perp, \xi_0|n_1^\perp).
\]

Also consider the \((n - 2)\)-dimensional subspace

\[
M := \text{span} ( (F_1 \cap \xi_0^\perp) - y )
\]

for which

\[
M^\perp = \text{span}(n_1, \xi_0) = \text{span}(n_1, \xi_0|n_1^\perp).
\]

Observe that \( M \cap L_{n_1} = L \). Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be the special orthogonal matrix which acts as the identity on \( L_{n_1} \), and rotates \( n_2|n_1^\perp \) through the plane \( L_{n_1}^\perp \) to a vector parallel to, and with the same direction as, \( \xi_0|n_1^\perp \). Of course,

\[
Tn_1 = n_1.
\]
Recall that $v$ denotes the unique vertex of $G_0$ which lies in the interior of the half-space $\xi_0^+$. Note that $v - y \in G_0 - y$, but $v - y \notin L$. We have $(n_2|n_1^\perp) \perp (v - y)$, because $n_1, n_2 \perp (v - y)$. Since orthogonal transformations preserve inner products,

$$\langle \xi_0|n_1^\perp, T(v - y) \rangle = |\xi_0|n_1^\perp|_2|T(n_2|n_1^\perp)|_2^{-1} \langle T(n_2|n_1^\perp), T(v - y) \rangle$$

$$= |\xi_0|n_1^\perp|_2|T(n_2|n_1^\perp)|_2^{-1} \langle n_2|n_1^\perp, v - y \rangle = 0 \quad \text{and} \quad \langle n_1, T(v - y) \rangle = (T(n_1), T(v - y)) = \langle n_1, v - y \rangle = 0.$$

It follows that

$$T(v - y) \in (n_1^\perp \cap (\xi_0|n_1^\perp)^\perp) = \text{span}(n_1, \xi_0|n_1^\perp)^\perp = M.$$  

Furthermore, we know that

$$T(v - y) \notin L,$$

because $v - y \notin L$ and the invertible map $T$ acts as the identity on $L \subset L_{n_1}$.

The definition of our second linear transformation depends on the relative position of $T(v - y)$ to $q_1(\xi_0, 0) - y$. Recall that the vector $q_1(\xi_0, 0) \in F_1 \cap \xi_0^\perp$ comes from the intersection of $\xi_0^\perp$ with the auxiliary line $w_1 + m_1s$ corresponding to the facet $F_1$. Therefore,

$$q_1(\xi_0, 0) - y \in M.$$

Furthermore, we have

$$q_1(\xi_0, 0) - y \notin L,$$

because $q_1(\xi_0, 0)$ lies in the relative interior of $F_1$.

The subspace $L$ splits $M$ into two halves, relatively open within $M$, which together must contain the vectors $T(v - y)$ and $q_1(\xi_0, 0) - y$. If these points lie in the same half, let $\widetilde{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the identity. If $T(v - y)$ and $q_1(\xi_0, 0) - y$ lie in opposite halves, let $\widetilde{T}$ be the orthogonal transformation which reflects points across the hyperplane

$$\text{span}(M^\perp, L) = \text{span}(n_1, \xi_0|n_1^\perp, L).$$

In either case, observe that

$$\widetilde{T}n_1 = n_1 \quad \text{and} \quad \widetilde{T}(\xi_0|n_1^\perp) = \xi_0|n_1^\perp,$$

and set $u := \widetilde{T}T(v - y) + y$. We then have

$$u, q_1(\xi_0, 0) \in M + y = \text{aff}(F_1 \cap \xi_0^\perp)$$

and

$$u, q_1(\xi_0, 0) \notin L + y = \text{aff}(G_0 \cap \xi_0^\perp).$$
The affine subspace \( \text{aff}(G_0 \cap \xi_0^+) \) splits \( \text{aff}(F_1 \cap \xi_0^+) \) into two halves, with \( u \) and \( q_1(\xi_0, 0) \) contained within the relative interior of the same half.

Fix a set of ordered indices \( i_1, \ldots, i_{n-2} \) from (10). Each unit vector \( l_{i_\gamma} \) is parallel to an edge \( E_{i_\gamma} \) of \( G_0 \). Indeed, we have

\[
l_{i_\gamma} = \frac{v - p_{i_\gamma}(\xi_0, 0)}{|v - p_{i_\gamma}(\xi_0, 0)|_2},
\]

where \( p_{i_\gamma}(\xi_0, 0) \) is the previously defined intersection of \( E_{i_\gamma} \) with \( \xi_0^+ \). Note that \( p_{i_\gamma}(\xi_0, 0) - y \in L \). As both \( T \) and \( \tilde{T} \) act as the identity on \( L \), we see

\[
\tilde{T}T(p_{i_\gamma}(\xi_0, 0) - y) = p_{i_\gamma}(\xi_0, 0) - y.
\]

We then find that

\[
\text{det}(l_1, \ldots, l_{n-2}, n_1, n_2) = \text{det}
\begin{pmatrix}
v - p_{i_1}(\xi_0, 0) & \cdots & v - p_{i_{n-2}}(\xi_0, 0) \\
v - p_{i_{n-2}}(\xi_0, 0) & \cdots & v - p_{i_{n-2}}(\xi_0, 0) \\
n_1, n_2 - (n_1, n_2)n_1 & \cdots & n_1, n_2 - (n_1, n_2)n_1
\end{pmatrix}
= C \text{det}(\tilde{T}T(v - y) - \tilde{T}T(p_{i_1}(\xi_0, 0) - y), \ldots, \tilde{T}T(v - y) - \tilde{T}T(p_{i_{n-2}}(\xi_0, 0) - y), \tilde{T}Tn_1, \tilde{T}T(n_2|n_1^+))
\]

\[
\begin{array}{c}
\frac{C(-1)^{n-2}|T(n_2|n_1^+)|_2}{|\xi_0|n_1^+|_2} \text{det}(p_{i_1}(\xi_0, 0) - u, \ldots, p_{i_{n-2}}(\xi_0, 0) - u, n_1, \xi_0|n_1^+)) \\
\frac{C(-1)^{n-2}|T(n_2|n_1^+)|_2}{|\xi_0|n_1^+|_2} \text{det}(p_{i_1}(\xi_0, 0) - u, \ldots, p_{i_{n-2}}(\xi_0, 0) - u, n_1, \xi_0),
\end{array}
\]

where

\[
C = \pm \left( \prod_{\gamma=1}^{n-2} |v - p_{i_\gamma}(\xi_0, 0)|_2 \right)^{-1}.
\]

The sign of \( C \) depends on the definition of \( \tilde{T} \). Importantly, the sign of

\[
\frac{C(-1)^{n-2}|T(n_2|n_1^+)|_2}{|\xi_0|n_1^+|_2}
\]

is independent of any particular choice of appropriate indices in (10). The function

\[
t \mapsto \text{det}
\begin{pmatrix}
p_{i_1}(\xi_0, 0) - ((1 - t)u + tq_1(\xi_0, 0)), \ldots
\end{pmatrix}
\]

\[
\begin{array}{c}
p_{i_{n-2}}(\xi_0, 0) - ((1 - t)u + tq_1(\xi_0, 0)), n_1, \xi_0
\end{array}
\]

is independent of any particular choice of appropriate indices in (10).
is continuous for \( t \in [0, 1] \); it is also non-vanishing for such \( t \) because the line segment connecting \( u \) to \( q_1(\xi_0, 0) \) lies in \( \text{aff}(F_1 \cap \xi_0^+) \) and does not intersect \( \text{aff}(G_0 \cap \xi_0^-) \). By the Intermediate Value Theorem, the determinant in (12) must have the same sign as

\[
\det(p_{i_1}(\xi_0, 0) - q_1(\xi_0, 0), \ldots, p_{i_{n-2}}(\xi_0, 0) - q_1(\xi_0, 0), n_1, \xi_0).
\]

Recalling formula (6), we recognize that the previous determinant is positive. We conclude that the sign of \( \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2) \) is independent of the choice of appropriate indices in (10). A similar argument shows that the sign of the determinant in (11) is also independent of the choice of appropriate indices \( k_1, \ldots, k_{n-2} \). \qed

We return to the altered (8), obtained by multiplying both sides of (8) by \( \delta^{n-2} \) and replacing \( \xi \) with \( \xi_\delta \). In view of Lemma 3.3 and expressions (10) and (11), the limit of the left-hand side of the altered (8) becomes

\[
\lim_{\delta \to 0^+} \delta^{n-2} \sum_{\Delta} \frac{(-1)^{n-3}(v, \tilde{\mu})^{n-3}s_0 \det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2)}{(n-3)!|l_{i_1}, \xi_0| \cdots |l_{i_{n-2}}, \xi_0| |1 - s_0|n_1 + s_0n_2|2 \sqrt{1 - |n_1, \tilde{\mu}|^2}} = \sum \frac{(-1)^{n-3}(v, \tilde{\mu})^{n-3}(1 - s_0) \det(l_{k_1}, \ldots, l_{k_{n-2}}, n_2, n_1)}{(n-3)!|l_{k_1}, \xi_0| \cdots |l_{k_{n-2}}, \xi_0| |1 - s_0|n_1 + s_0n_2|2 \sqrt{1 - |n_2, \tilde{\mu}|^2}} \times \sum \frac{\det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2)}{|l_{i_1}, \xi_0| \cdots |l_{i_{n-2}}, \xi_0|}. \tag{13}
\]

The third and fourth summations are taken over indices \( i_1, \ldots, i_{n-2} \in I \) corresponding to the vertices \( \{p_{i_1}(\xi_0, 0), \ldots, p_{i_{n-2}}(\xi_0, 0)\} \) of simplices in the triangulation of \( G_0 \cap \xi_0^+ \), ordered so that the expression in (6) is positive for \( F_1 \). For each set of indices \( i_1, \ldots, i_{n-2} \), we wrote \( k_1, \ldots, k_{n-2} \) for the suitable rearrangement which makes (6) positive for \( F_2 \). The \( \pm \) in (13) depends on whether or not the determinants

\[
\det(l_{i_1}, \ldots, l_{i_{n-2}}, n_1, n_2)
\]

have the same sign as the determinants

\[
\det(l_{k_1}, \ldots, l_{k_{n-2}}, n_2, n_1).
\]

Whether this \( \pm \) is a plus or a minus does not depend on our previous arbitrary choice of \( s_0 \in \{s_+, s_\} \). Recalling that \( 0 < s_+ < 1 < s_- \), we also observe that

\[
\frac{s_+}{\sqrt{1 - |n_1, \tilde{\mu}|^2}} + \frac{1 - s_+}{\sqrt{1 - |n_2, \tilde{\mu}|^2}} > 0 \quad \text{and} \quad \frac{s_-}{\sqrt{1 - |n_1, \tilde{\mu}|^2}} - \frac{1 - s_-}{\sqrt{1 - |n_2, \tilde{\mu}|^2}} > 0.
\]
So, if the ± in (13) is a plus, let us have chosen $s_0 = s_+$; if the ± is a minus, let us have chosen $s_0 = s_-$.

It is now clear that (13) is non-zero for the appropriately selected $s_0$. Indeed, we know that $\langle v, \widetilde{n} \rangle > 0$, $\langle l_i, \xi_0 \rangle > 0$ for all edges $E_i$ of $G_0$ intersected by $\xi_0 \perp 0$, and the determinants in (13) are all non-zero with the same sign. However, (13) being non-zero contradicts the fact that the limit of the right-hand side of the altered (8) is zero. This contradiction implies that our starting assumption, “there exists an $(n−2)$-dimensional face $G_0$ of $P$ which does not satisfy (4)”, is false. It then follows that every $(n−2)$-dimensional face of $P$ satisfies (4).

### 3.2 Second Part

For every $(n−2)$-dimensional face $G$ of $P$, $\text{reflec}(G, t)$ is also an $(n−2)$-dimensional face of $P$ for some $t > 0$. From this fact, we can immediately conclude the following:

- If $v$ is a vertex of $P$, then the line $Rv$ contains exactly one other vertex of $P$. This second vertex, which we will denote by $\widetilde{v}$, necessarily lies on the opposite side of the origin as $v$.
- If $u$ and $v$ are vertices of $P$ connected by an edge $E(u, v)$, then $\widetilde{u}$ and $\widetilde{v}$ are connected by an edge $E(\widetilde{u}, \widetilde{v})$ parallel to $E(u, v)$.

We prove $P = −P$ by showing $\widetilde{v} = −v$ for every vertex $v$. To the contrary, suppose there is a vertex $v$ for which $|v|_2 < |\widetilde{v}|_2$. Let $\{v_i\}_{i=0}^k$ be a sequence of vertices of $P$ such that $v_0 = v$, $v_k = \widetilde{v}$, and the vertices $v_i$ and $v_{i+1}$ are connected by an edge $E(v_i, v_{i+1})$ for each $0 \leq i \leq k − 1$. It follows from the previous itemized observations that the triangle $T(o, v, v_1)$ with vertices $\{o, v, v_1\}$ is similar to the triangle $T(o, \widetilde{v}, \widetilde{v}_1)$; see Fig. 2. Given that $|v|_2 < |\widetilde{v}|_2$, we must also have $|v_1|_2 < |\widetilde{v}_1|_2$. Continuing this argument recursively, the similarity of the triangle $T(o, v_i, v_{i+1})$ to the triangle $T(o, \widetilde{v}_i, \widetilde{v}_{i+1})$ implies $|v_{i+1}|_2 < |\widetilde{v}_{i+1}|_2$ for $1 \leq i \leq k − 1$. But then $|\widetilde{v}|_2 = |v_k|_2 < |\widetilde{v}_k|_2 = |v|_2$, which is a contradiction.
4 Dual Quermassintegrals of Sections

We say a set \( L \subset \mathbb{R}^n \) is star-shaped if every one-dimensional subspace intersects \( L \) in a (possibly degenerate) line segment which includes the origin. The radial function of such a star-shaped \( L \) is defined by

\[
\rho_L(\xi) := \max \{ a \geq 0 : a\xi \in L \} \quad \text{for} \quad \xi \in S^{n-1},
\]

and it completely determines \( L \):

\[
L = \{ o \} \cup \{ x \in \mathbb{R}^n \setminus \{ o \} : |x|_2 \leq \rho_L(x/|x|_2) \}.
\]

The radial function naturally extends to a positively homogeneous function on \( \mathbb{R}^n \setminus \{ 0 \} \) with homogeneity \(-1\). A star body is a star-shaped \( L \subset \mathbb{R}^n \) whose radial function is positive and continuous. Note that a convex body which contains the origin in its interior is also a star body. Finally, note that a star body is \( C^2 \) if its radial function is in \( C^2(\mathbb{R}^n \setminus \{ 0 \}) \).

Dual quermassintegrals (and, more generally, dual mixed volumes) were introduced by Lutwak [6]. The dual quermassintegrals \( \tilde{W}_l(L) \) of a star body \( L \subset \mathbb{R}^n \) arise as coefficients in the expansion

\[
\text{vol}_n(L + tB^2_n(o, 1)) = \sum_{l=0}^{n} \binom{n}{l} \tilde{W}_l(L) t^l, \quad t \geq 0.
\]

Here, \( L + tB^2_n(o, 1) \) denotes the radial sum of the star bodies \( L \) and \( tB^2_n(o, 1) \). That is, \( L + tB^2_n(o, 1) \) is the star body whose radial function is the sum of the radial functions for \( L \) and \( tB^2_n(o, 1) \):

\[
\max \{ a \geq 0 : a\xi \in L + tB^2_n(o, 1) \} := \rho_L(\xi) + t\rho_{B^2_n(o, 1)}(\xi), \quad \xi \in S^{n-1}.
\]

There are also explicit expressions for the dual quermassintegrals in terms of the radial function (e.g. [3, formula (A.57)]):

\[
\tilde{W}_l(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L^{n-l}(\xi) \, d\xi.
\]

See [3, 12] for further details.

There are many parallels between quermassintegrals and dual quermassintegrals, so it is natural to consider the conjecture of Makai et al. [8] in the dual setting. Given a star body \( L \subset \mathbb{R}^n \) and any \( \xi \in S^{n-1} \), \( (L - t\xi) \cap \xi^\perp \) is an \((n-1)\)-dimensional star body in \( \xi^\perp \) for all \( t \in \mathbb{R} \) which are in a small enough neighborhood about zero. Therefore, for each integer \( 0 \leq l \leq n-2 \), we can define the function

\[
\tilde{W}_{l, \xi}(t) := \tilde{W}_l((L - t\xi) \cap \xi^\perp)
\]
on a small neighborhood about zero, where \( \tilde{W}_l((L - t\xi) \cap \xi^\perp) \) is the dual quermass-integral of the \((n-1)\)-dimensional star body \((L - t\xi) \cap \xi^\perp \) in \(\xi^\perp \). For all \( t \in \mathbb{R} \) at which \( \tilde{W}_{l,\xi} \) is well defined, we have

\[
\tilde{W}_{l,\xi}(t) = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \rho_{L - t\xi}^{n-1-l}(\eta) \, d\eta = \kappa_{n-1-l} \int_{G(\xi^\perp, n-1-l)} \text{vol}_{n-1-l}((L - t\xi) \cap H) \, dH, \tag{14}
\]

where the second equality is given by the dual Kubota formula (e.g. [3, Thm. A.7.2]). Here, \( G(\xi^\perp, n-1-l) \) is the Grassmannian of \((n-1-l)\)-dimensional subspaces in \(\xi^\perp \), and \( dH \) is the \( O(n-1) \)-invariant regular Borel probability measure on \( G(\xi^\perp, n-1-l) \). Using (14), we extend the definition of \( \tilde{W}_{l,\xi}(t) \) to all \( t \in \mathbb{R} \).

Note that \( \tilde{W}_{l,\xi} \) is twice continuously differentiable in a neighbourhood of zero when \( L = C^2 \). We prove the following theorem.

**Theorem 4.1** Let \( L \subset \mathbb{R}^n \) be a \( C^2 \) star body and \( 1 \leq l \leq n - 2 \). Then

\[
\frac{d}{dt} \tilde{W}_{l,\xi}(t) \bigg|_{t=0} = 0 \quad \forall \, \xi \in S^{n-1} \tag{15}
\]

if and only if \( L = -L \).

The proof of Theorem 4.1 follows from formulas derived in [16]. These formulas involve spherical harmonics, and the fractional derivatives of \( \tilde{W}_{l,\xi} \) at \( t = 0 \). Recall that a spherical harmonic of dimension \( n \) and degree \( m \) is the restriction to \( S^{n-1} \) of a real-valued harmonic and homogeneous polynomial of degree \( m \) in \( n \) variables. Importantly, there is an orthogonal basis of \( L^2(S^{n-1}) \) consisting of spherical harmonics. Furthermore, any two spherical harmonics of the same dimension and different degrees are orthogonal. The standard reference for spherical harmonics in convex geometry is [4].

Let \( h \) be a complex-valued \( L^1 \) function on \( \mathbb{R} \) which is \( m \) times continuously differentiable in a neighbourhood of zero. Let \( q \in \mathbb{C} \setminus \{0, 1, \ldots, m - 1\} \) with real part \(-1 < \text{Re}(q) < m \). The fractional derivative of \( h \) of order \( q \) at zero is given by

\[
h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left( h(t) - \sum_{k=0}^{m-1} \frac{d^k}{ds^k} h(s) \bigg|_{s=0}^t \right) \, dt
\]

\[
+ \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t) \, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{1}{k!(k-q)} \frac{d^k}{dt^k} h(t) \bigg|_{t=0};
\]

see, for example, [5]. Defining \( h^{(k)}(0) \) by the limit for \( k = 0, 1, \ldots, m - 1 \), we get an analytic function \( q \mapsto h^{(q)}(0) \) for \( q \in \mathbb{C} \) with \(-1 < \text{Re}(q) < m \). Fractional derivatives extend the notion of classical differentiation:

\[
h^{(k)}(0) = (-1)^k \frac{q^k}{dt^k} h(t) \bigg|_{t=0} \quad \text{for} \quad k = 0, 1, \ldots, m.
\]
When $L$ is $C^2$ we can consider the fractional derivatives of $\tilde{W}_{l,\xi}$ at zero of order $q$, for $q \in \mathbb{C}$ with $-1 < \text{Re}(q) < 2$.

The next lemma was essentially proven in [16], but we reproduce a detailed proof here for completeness. Recall that the Legendre polynomials $P_m^n : \mathbb{R} \to \mathbb{R}$, of degree $m \geq 0$ and fixed dimension $n \geq 2$, are an orthogonal system of polynomials on $[-1, 1]$ with respect to the weight function $(1 - t^2)^{(n-3)/2}$, and normalized by the condition $P_m^n(1) = 1$.

**Lemma 4.1** Let $L \subset \mathbb{R}^n$ be a $C^2$ star body. Let $0 \leq l \leq n - 2$ be an integer. Let $H_m^n \in L^2(S^{n-1})$ be a spherical harmonic of dimension $n$ and degree $m \in \mathbb{N} \cup \{0\}$. Define $f_{l,m} \in L^1(\mathbb{R})$ by

$$f_{l,m}(t) := (n - 1)\kappa_{n-1} P_m^n(t)(1 - t^2)^{(n-3-l)/2} \chi_{[-1,1]}(t).$$

Then

$$\int_{S^{n-1}} \tilde{W}_{l,\xi}(0) H_m^n(\xi) d\xi = \frac{n - 1 - l}{(n - 1 - l - q)(n - 1)} f_{l,m}(0) \int_{S^{n-1}} \rho_L^{n-1-l-q}(\xi) H_m^n(\xi) d\xi$$

for all $q \in \mathbb{C} \setminus \{n - 1 - l\}$ with $-1 < \text{Re}(q) < 2$.

**Proof** Fix $q \in \mathbb{R}$ with $-1 < q < 0$, for the time being. For any integer $1 \leq k \leq n - 1$ and any $\xi \in S^{n-1}$, we have

$$\tilde{W}_{n-1-k,\xi}(0) = \int_{0}^{\infty} \frac{|t|^{-l-q}(1 + \text{sgn}(t)) \tilde{W}_{n-1-k,\xi}(t)}{2\Gamma(-q)} dt = \int_{\mathbb{R}} \frac{|\rho_L^{n-k-q}(1 + \text{sgn}(t)) \tilde{W}_{n-1-k,\xi}(t)|}{2\Gamma(-q)k_{k-1}^{n-1}} dt dH,$$

where the last equality is obtained using [3, formula (A.61), p. 412] and Fubini’s Theorem. Continuing the computation, we find

$$\int_{G(\xi^+,k)} \int_{L(\text{span}(H,\xi))} \frac{|\langle x, \xi \rangle|^{-l-q}(1 + \text{sgn}(\langle x, \xi \rangle))}{2\Gamma(-q)k_{k-1}^{n-1}} dx dH$$

$$= \int_{G(\xi^+,k)} \int_{S^{n-1}(\text{span}(H,\xi))} \frac{|\langle \theta, \xi \rangle|^{-l-q}(1 + \text{sgn}(\langle \theta, \xi \rangle))\rho_L^{k-q}(\theta)}{2(k - q)\Gamma(-q)k_{k-1}^{n-1}} d\theta dH$$

$$= \int_{G(\xi^+,k)} \int_{S^{n-1}(\text{span}(H,\xi))} \int_{-1}^{1} \frac{|\rho_L^{k-q}(t\xi + \eta\sqrt{1 - t^2})|}{2(k - q)\Gamma(-q)k_{k-1}^{n-1}} dt d\eta dH$$

$$\frac{1}{2} \int_{S^{n-1}(\text{span}(H,\xi))} \int_{-1}^{1} \frac{|\rho_L^{k-q}(t\xi + \eta\sqrt{1 - t^2})|}{2(k - q)(n - 1)\Gamma(-q)} dt d\eta.$$
where the last equality is obtained using [5, formula (2.22)]. Finalizing the calculation gives

$$
= \int_{S^{n-1}} |\langle \theta, \xi \rangle|^{-q} (1 + \text{sgn} \langle \theta, \xi \rangle) (1 - |\langle \theta, \xi \rangle|^2)^{-(n-1-k)/2} \rho_L^{k-q} (\theta) \frac{d\theta}{2k^{-1}(k-q)(n-1)\Gamma(-q)}.
$$

That is, for any integer $0 \leq l \leq n - 2$ and any $\xi \in S^{n-1}$, we have

$$
\tilde{W}_{l,\xi}^{(q)}(0) = \frac{n - 1 - l}{(n - 1 - l - q)(n - 1)} \times \int_{S^{n-1}} |\langle \theta, \xi \rangle|^{-q} (1 + \text{sgn} \langle \theta, \xi \rangle) \rho_L^{n-1-l-q} (\theta) \frac{d\theta}{2\Gamma(-q)(1 - |\langle \theta, \xi \rangle|^2)^{l/2}}.
$$

Now, define the integral operator $I_{l,q} : C(S^{n-1}) \rightarrow C(S^{n-1})$ by

$$(I_{l,q} f)(\xi) := \int_{S^{n-1}} F_{l,q}(\langle \theta, \xi \rangle) f(\theta) \, d\theta \quad \text{for } f \in C(S^{n-1}),$$

where

$$F_{l,q}(t) := \frac{|t|^{-1-q} (1 + \text{sgn} t)}{2\Gamma(-q)(1 - t^2)^{l/2}} \quad \text{for } t \in [-1, 1].$$

Using the previous definition,

$$\tilde{W}_{l,\xi}^{(q)}(0) = \frac{n - 1 - l}{(n - 1 - l - q)(n - 1)} (I_{l,q} \rho_L^{n-1-l-q})(\xi). \quad (17)$$

Let $H^n_m \in L^2(S^{n-1})$ be any spherical harmonic of dimension $n$ and degree $m \in \mathbb{N} \cup \{0\}$. By the Funk–Hecke Theorem (e.g. [4, Thm. 3.4.1]),

$$I_{l,q} H^n_m = \lambda_{l,m}(q) H^n_m, \quad (18)$$

where

$$\lambda_{l,m}(q) := (n - 1)\kappa_{n-1} \int_{-1}^{1} F_{l,q}(t) P^n_m(t)(1 - t^2)^{(n-3)/2} \, dt$$

$$= \frac{(n - 1)\kappa_{n-1}}{2\Gamma(-q)} \int_{-1}^{1} |t|^{-1-q} (1 + \text{sgn} t) P^n_m(t)(1 - t^2)^{(n-3-l)/2} \, dt$$

$$= \frac{(n - 1)\kappa_{n-1}}{\Gamma(-q)} \int_{0}^{1} t^{-1-q} P^n_m(t)(1 - t^2)^{(n-3-l)/2} \, dt.$$

Observe that $\lambda_{l,m}(q)$ is the fractional derivative of

$$f_{l,m}(t) = (n - 1)\kappa_{n-1} P^n_m(t)(1 - t^2)^{(n-3-l)/2} \chi_{[-1,1]}(t)$$
of order $q$ at $t = 0$. This $f_{l,m}$ is infinitely smooth in a neighborhood of the origin.

**Remark** The Funk–Hecke Theorem as stated in Groemer’s book requires $F_{l,q}$ to be bounded on the interval $[-1,1]$. Our $F_{l,q}$ is not bounded, and it is not integrable over $[-1,1]$ for $l > 1$. However, both integrals in (18) are finite for all integers $1 \leq l \leq n - 2$. We obtain (18) by applying the Funk–Hecke Theorem to the bounded approximating functions $F_{\delta}(t) := \chi_{[\delta,1-\delta]}(t)F_{l,q}(t)$ for small $\delta > 0$, and taking the limit.

Use (17), Fubini’s Theorem, and (18) to obtain

\[
\int_{S^{n-1}} \tilde{W}_{l,\xi}^{(q)}(0) H_m^{n}(\xi) \, d\xi = \frac{n-1-l}{(n-1-l-q)(n-1)} \int_{S^{n-1}} (I_{l,q} \rho_L^{n-1-l-q}(\xi) H_m^{n}(\xi)) \, d\xi
\]

\[
= \frac{n-1-l}{(n-1-l-q)(n-1)} \int_{S^{n-1}} \rho_L^{n-1-l-q}(\xi) (I_{l,q} H_m^{n}(\xi)) \, d\xi
\]

\[
= \frac{(n-1-l)f_{l,m}(q)}{(n-1-l-q)(n-1)} \int_{S^{n-1}} \rho_L^{n-1-l-q}(\xi) H_m^{n}(\xi) \, d\xi.
\]

Therefore, we have

\[
\int_{S^{n-1}} \tilde{W}_{l,\xi}^{(q)}(0) H_m^{n}(\xi) \, d\xi = \frac{(n-1-l)f_{l,m}(q)}{(n-1-l-q)(n-1)} \int_{S^{n-1}} \rho_L^{n-1-l-q}(\xi) H_m^{n}(\xi) \, d\xi
\]

for all $q \in \mathbb{R}$ with $-1 < q < 0$, for all integers $0 \leq l \leq n - 2$, and for every spherical harmonic $H_m^{n}$ of order $m \in \mathbb{N} \cup \{0\}$. We want to extend the validity of (19) to other values of $q$ using analytic continuation. It is clear that the right-hand side of (19) is an analytic function of $q \in \mathbb{C}$ for $\text{Re}(q) > -1$ and $q \neq n - 1 - l$; indeed, $f_{l,m}$ is infinitely smooth in a neighborhood of the origin, and $\rho_L$ is bounded from above and away from zero.

To see that the left-hand side of (19) is also analytic, consider the following. Let $\delta > 0$ be small enough so that:

- The star body $L$ contains the closed ball of radius $\delta$ centered at the origin.
- For all $\xi \in S^{n-1}$ and all $-\delta < t < \delta$, $(L - t\xi) \cap \xi^\perp$ is an $(n-1)$-dimensional star body within $\xi^\perp$.

Then

\[
\tilde{W}_{l,\xi}(t) = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \rho_L^{n-1-l}(\eta) \, d\eta
\]

for all $-\delta < t < \delta$ and $\xi \in S^{n-1}$. It is not difficult to prove the following facts:
• For each $\xi \in S^{n-1}$ and $\eta \in S^{n-1} \cap \xi^\perp$, the function $g_{\xi,\eta}(t) := \rho_{L-t\xi}(\eta)$ is continuously differentiable on the open interval $(-\delta, \delta)$ (recall that $L$ is $C^1$ smooth).

The function $g_{\xi,\eta} : (-\delta, \delta) \to [0, \infty)$ and its derivative are bounded, uniformly with respect to $\xi \in S^{n-1}$ and $\eta \in S^{n-1} \cap \xi^\perp$.

These facts then imply:

• For every $\xi \in S^{n-1}$, $\widetilde{W}_l,\xi(t)$ is continuously differentiable on $(-\delta, \delta)$.

• The function $\widetilde{W}_l,\xi : (-\delta, \delta) \to [0, \infty)$ and its derivative are bounded, uniformly with respect to $\xi \in S^{n-1}$.

Therefore, it follows that

$$q \mapsto \int_{S^{n-1}} \widetilde{W}_l(0) H_m^n(\xi) \, d\xi$$

is analytic on the domain

$$\{q \in \mathbb{C} : -1 < \text{Re}(q) < 2, \ q \neq 0, 1\},$$

with derivatives passing through the integral. By analytic continuation, (19) is valid for $q \in \mathbb{C}$ with $-1 < \text{Re}(q) < 2$, $q \neq n - 1 - l$, and $q \neq 0, 1$. Taking limits using Lebesgue’s Dominated Convergence Theorem, we then see (19) is also true for $q = 0, 1$. \hfill \Box

**Proof of Theorem 4.1** Let $H_{m}^{n}$ be any spherical harmonic of dimension $n$ and odd degree $m$. For $q = 1$, Lemma 4.1 gives us

$$- \int_{S^{n-1}} H_{m}^{n}(\xi) \widetilde{W}_l(0) \, d\xi = \int_{S^{n-1}} H_{m}^{n}(\xi) \widetilde{W}_l(1) \, d\xi = \frac{(n - 1 - l) f_{l,m}(0)}{(n - 2 - l)(n - 1) \int_{S^{n-1}} H_{m}^{n}(\xi) \rho_{L}^{n-2-l}(\xi) \, d\xi}$$

when $l \neq n - 2$. The denominator of the right hand side of (16) is 0 when $l = n - 2$, and so is the numerator:

$$\lim_{q \to 1} \int_{S^{n-1}} H_{m}^{n}(\xi) \rho_{L}^{n-1-2-2}(\xi) \, d\xi = \int_{S^{n-1}} H_{m}^{n}(\xi) \, d\xi = 0$$

because of the orthogonality of spherical harmonics with different degrees. Applying de l’Hospital’s rule we find that

$$- \int_{S^{n-1}} H_{m}^{n}(\xi) \widetilde{W}_l(0) \, d\xi = \int_{S^{n-1}} H_{m}^{n}(\xi) \widetilde{W}_l(1) \, d\xi = \frac{f_{l,m}(0)}{n - 1} \int_{S^{n-1}} H_{m}^{n}(\xi) \log(\rho_{L}(\xi)) \, d\xi$$

$\square$
when \( l = n - 2 \). Calculating

\[
\begin{align*}
f_{l,m}^{(1)}(0) &= -(n-1)\kappa_{n-1} \frac{d}{dt} P_m^n(t)(1-t^2)^{(n-3-l)/2} \bigg|_{t=0} \\
&= -(n-1)\kappa_{n-1} \frac{d}{dt} P_m^n(t) \bigg|_{t=0},
\end{align*}
\]

it then follows from \([4, \text{Lem. 3.3.9 and 3.3.8}]\) that \( f_{l,m}^{(1)} \neq 0 \) for odd \( m \).

If (15) is true, then (20) and (21) respectively show that the spherical harmonic expansions of \( \rho_{L_{n-2-l}} \) (when \( l \neq n - 2 \)) and \( \log \rho_L \) (when \( l = n - 2 \)) do not have any harmonics of odd degree; consequently, \( \rho_L \) must be an even function and \( L = -L \). If \( L = -L \), then it is easy to see that (15) is valid. \( \square \)

Restricting \( L \) to the class of \( C^2 \) convex bodies containing the origin in their interiors, we see via Brunn’s Theorem that Theorem 4.1 is equivalent to the statement that

\[
\tilde{W}_{l,\xi}(0) = \max_{t \in \mathbb{R}} \tilde{W}_{l,\xi}(t) \quad \text{for all} \quad \xi \in S^{n-1}
\]

if and only if \( L \) is origin symmetric.

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