Mild solutions to the Cauchy problem for time-space fractional Keller-Segel-Navier-Stokes system

Ziwen Jiang\textsuperscript{a}, Lizhen Wang\textsuperscript{a,*}

\textsuperscript{a}Center for Nonlinear Studies, School of Mathematics, Northwest University, Xi’an, Shaanxi Province, 710127, China

Abstract

This paper investigates the Cauchy problem of the time-space fractional Keller-Segel-Navier-Stokes model in \( \mathbb{R}^d \) (\( d \geq 2 \)) which can describe both memory effect and Lévy process of the system. The local existence and global existence in Lebesgue space are obtained by means of Banach fixed point theorem and Banach implicit function theorem, respectively. In addition, the regularities of local and global mild solutions are improved in fractional homogeneous Sobolev spaces. Furthermore, some properties of mild solutions including mass conservation, decay estimates, stability and self-similarity are established.

Keywords: Time-space fractional Keller-Segel-Navier-Stokes model; Mild solution; Existence; Higher regularity; Mass conservation

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1. Introduction

In this paper, for \( 0<\beta<1, 1<\alpha\leq 2, \gamma\geq 0 \) and \( d \geq 2 \), we consider the following Cauchy problem for time-space fractional Keller-Segel-Navier-Stokes system:

\[
\begin{cases}
\frac{c_0}{0}D^\beta_t n + (-\Delta)^{\alpha/2}n + u \cdot \nabla n + \nabla \cdot (n \nabla v) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\frac{c_0}{0}D^\beta_t v + (-\Delta)^{\alpha/2}v + u \cdot \nabla v + \gamma v - n = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\frac{c_0}{0}D^\beta_t u + (-\Delta)^{\alpha/2}u + (u \cdot \nabla)u + \nabla \rho + n \nabla \phi = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\
n|_{t=0} = n_0, v|_{t=0} = v_0, u|_{t=0} = u_0 & \text{in } \mathbb{R}^d,
\end{cases}
\]

\textsuperscript{*}Corresponding author

Email addresses: jiangziwensss@163.com (Ziwen Jiang), wanglizhen@nwu.edu.cn (Lizhen Wang)
where \( n = n(x, t) \) denotes the density of cells that diffuses in the manner of Lévy flight and is transported along the velocity field of the incompressible fluid, \( v = v(x, t) \) indicates the concentration of chemical attractant, the vector function \( u = u(x, t) = (u_1(x, t), u_2(x, t), \cdots, u_d(x, t)) \) stands for the fluid velocity field, the scalar function \( \rho = \rho(x, t) \) represents the pressure of the fluid, \( \phi = \phi(x) \) is a given potential function considering the effects of external forces such as the gravitational potential generated by aggregation of cells onto the fluid and \( n_0 = n_0(x), v_0 = v_0(x), u_0 = (u_{1,0}(x), u_{2,0}(x), \cdots, u_{d,0}(x)) \) are prescribed initial data. It can be seen from (1.1) that chemotaxis and fluid are coupled through both the transport of the cells and the chemical \( u \cdot \nabla n, u \cdot \nabla v \) by the fluid and the external force \(-n \nabla \phi \) exerted on the fluid by the cells. \( c_0 D_t^\beta \) is weak Caputo fractional derivative operator of order \( \beta \) introduced in [41, 42] and \( c_0 D_t^\beta w \) models memory effects in time. When function \( w(t) \) is absolutely continuous in time, the definition of weak Caputo derivative is reduced to the following traditional form

\[
c_0 D_t^\beta w(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - s)^{-\beta} \partial_s w(s) \, ds,
\]

where \( \Gamma \) is the Gamma function. According to Chapter V in [52], the nonlocal operator \((-\Delta)^{\alpha/2}\), known as the Laplacian of order \( \alpha/2 \), is given by the Fourier multiplier

\[
(-\Delta)^{\alpha/2} \rho(x) := \mathcal{F}^{-1}\left(|\xi|^\alpha \hat{\rho}(\xi)\right)(x).
\]

Fractional derivatives have been employed to describe the nonlocal effects of the considered equations in references [21, 34, 35]. The motivation of using fractional diffusion instead of classical diffusion comes from the fact that organisms adopt a Lévy process instead of Brownian motion [9, 10, 20, 57, 58] and it is believed that the feeding behavior of some organisms is based on a Lévy process generated by the spatial-fractional diffusion operator \((-\Delta)^{\alpha/2}\) \((0 < \alpha < 2)\). There are some interesting works investigating partial differential equations with fractional Laplacian, see references [7, 11–13, 17, 38, 51]. Chen, Li and Ou [17] solved an open problem posed by Lieb with the method of moving planes in an integral form and classified all the solutions of one type of integral equation equivalent to a family of semi-linear partial differential equations with fractional Laplacian. Caffarelli and Silvestre [11] obtained characterizations for general fractional powers of the Laplacian and other integro-differential operators. Lai, Miao and Zhang [38] constructed a global-time forward self-similar solution to the Navier-Stokes equations with the fractional diffusion \((-\Delta)^{\alpha}\) \((5/6 < \alpha < 1)\) for arbitrarily large self-similar initial data. At the macroscopic level, time fractional operators are associated with anomalous subdiffusion [1, 24, 39, 41, 47, 50, 63]. For example, in [39], Riemann-Liouville fractional
derivative can model anomalous diffusion of continuous time random walks with power law waiting time. Allen, Caffarelli and Vasseur [1] discussed the existence and regularity for weak solutions to porous medium equation with fractional time derivative of Caputo-type and inverse fractional Laplacian operator. Sakamoto and Yamamoto [50] established the unique existence of the weak solution and the asymptotic behavior of fractional diffusion-wave equations.

In the 1970s, Keller and Segel introduced the first mathematical model of chemotaxis describing the aggregation of slime mold amoebae due to attractive chemicals in [32, 33]. There are numerous references investigating the Keller-Segel models such as [6, 14, 18, 25, 37, 49, 59]. Recently, some results related to the fractional Keller-Segel equations were presented in [3, 5, 7, 23, 28, 30, 40, 42, 43, 47, 65, 66]. Escudero [23] constructed the global in time solutions for the fractional diffusion \((-\Delta)^{\alpha/2}\) (1 < \(\alpha\) ≤ 2). Azevedo, Cuevas and Henríquez [3] managed to verify the existence and asymptotic behaviour for the time-fractional Keller-Segel model. Huang and Liu explored the uniqueness and stability of nonlocal Keller-Segel equations with fractional Laplacian in [28]. Li, Liu and Wang [43] considered the existence, integrability, nonnegativity and blow up behaviors of the Cauchy problem of the fractional time-space generalized Keller-Segel equations. Jiang and Wang [30] investigated the global existence and mass conservation of weak solutions to the time-space fractional Keller-Segel equation.

In what follows, we present a brief development of the chemotaxis-fluid system and introduce some related consequences on it. First, the chemotaxis system with the effect of fluid, proposed by Tuval et al. to model the dynamics of populations of aerobic bacteria within the flow of an incompressible Newtonian fluid [56], can be written as

\[
\begin{align*}
\partial_t n + u \cdot \nabla n &= \Delta n - \nabla \cdot (n\chi(c)\nabla c) \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t c + u \cdot \nabla c &= \Delta c - n\kappa(c) \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t u + (u \cdot \nabla) u &= \Delta u - \nabla \rho - n\nabla \phi \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).
\end{align*}
\]

Here, the unknowns \(n, c, u, \rho\) denote the cell density, oxygen concentration, velocity field and pressure of the fluid, respectively. \(\chi(c)\) is the chemotactic sensitivity and \(\kappa(c)\) is the consumption rate of the chemical by the cells. There are a large quantity of literature discussing the chemotaxis-fluid system (1.2), see [15, 16, 22, 44–46, 60, 61, 64] for instance. Duan, Lorz and Markowich [22] obtained the global existence and rates of convergence on classical solutions of (1.2) near constant states. In [45], for the system (1.2) in two dimensional space, Liu and Lorz constructed the global existence of weak solutions for large data and for the chemotaxis-Stokes system.
system with nonlinear diffusion for the cell density in three dimensional space, they proved the global existence of weak solutions. It was shown in [15] that in two dimensional space (1.2) admits a global classical solutions under some suitable assumptions. The global existence, stabilization and convergence rate of solutions to (1.2) on bounded domain \( \Omega \) of global existence of weak solutions. It was shown in [15] that in two dimensional space (1.2) of global mild solutions with small initial data in the scaling invariant space using Banach implicit function theorem in [36]. Jiang et.al [29] established the existence and uniqueness of global mild solution to fractional chemotaxis-Navier-Stokes system. Recently, Azevedo et.al [2] proved the existence of global mild solutions of time fractional Keller-Segel model coupled with the Navier-Stokes fluid with small critical initial data in Besov-Morrey spaces.

For some bioconvection processes, in which the signal substance is produced by the cells themselves, the Keller-Segel-Navier-Stokes model is considered as follows [62]:

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(\nabla c) - n \iota(v) \nabla v) + h(n) \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t c + u \cdot \nabla c &= \Delta c - n\kappa(c) \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t v + u \cdot \nabla v &= \Delta v + g(v)n - \gamma v \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t u + (u \cdot \nabla)u &= \Delta u - \nabla \rho - n\nabla \phi \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),
\end{aligned}
\]  

(1.3)

where the meaning of unknowns \((n, c, u, \rho)\) is same as the one expressed in (1.2), while \(v\) denotes the chemical concentration; \(\chi(c)\) and \(\iota(v)\) are the chemotactic sensitivities, \(h(n)\) is the external source term usually taken as logical source \(\rho n - \mu n^2\), \(\kappa(c)\) stands for the consumption rate of oxygen, \(g(v)\) is the consumption rate of chemical attractant by the cells, and the parameter \(\gamma \geq 0\) presents the decay rate of attractant. Tan and Zhou [53, 54] obtained the global existence and decay estimates of solutions to the double-type chemotaxis system. Taking \(\chi(c) = \iota(v) = g(v) = 1\), \(h(n) = 0\) and \(\kappa(c) = c\) in (1.3), Kozono, Miura and Sugiyama presented the existence of global mild solutions with small initial data in the scaling invariant space using Banach implicit function theorem in [36]. Jiang et.al [29] established the existence and uniqueness of global mild solution to fractional chemotaxis-Navier-Stokes system. Recently, Azevedo et.al [2] proved the existence of global mild solutions of time fractional Keller-Segel model coupled with the Navier-Stokes fluid with small critical initial data in Besov-Morrey spaces.

For some bioconvection processes, in which the signal substance is produced by the cells themselves, the Keller-Segel-Navier-Stokes model is considered as follows [62]:

\[
\begin{aligned}
\partial_t n + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \chi(\nabla c) - n \iota(v) \nabla v) + \rho n - \mu n^2 \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t v + u \cdot \nabla v &= \Delta v + n - \gamma v \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\partial_t u + (u \cdot \nabla)u &= \Delta u - \nabla \rho - n\nabla \phi + f(x, t) \quad \text{in } \mathbb{R}^d \times (0, \infty), \\
\nabla \cdot u &= 0 \quad \text{in } \mathbb{R}^d \times (0, \infty),
\end{aligned}
\]  

(1.4)

where \((n, v, u, \rho)\) is shown as above. Some research related to equation (1.4) have been done.
in references [4, 27, 31, 62]. Winkler [62] constructed the global weak solutions for arbitrarily large initial data to (1.4) and the large time behavior of these solutions under appropriate assumptions. Kang and Kim [31] obtained the generalized solutions for the Keller-Segel system with a degradation source coupled to Navier-Stokes equations in three dimensional space. Hua and Zhang [27] established the global well-posedness for the incompressible Keller-Segel-Navier-Stokes equations in \( \mathbb{R}^3 \).

The work of this present paper is inspired by Kozono, Miura and Sugiyama [36]. We consider the time-space fractional single type of chemotaxis model that the signal substance is produced by the cells themselves under Navier-Stokes fluids as shown in (1.1), covering numerous biologically relevant situations when cells actively use chemotaxis as a means of communication, rather than the double chemotaxis model in [36]. The main mathematical difficulty comes from the fractional diffusion operator and the weak Caputo fractional derivative operator which are nonlocal operators, so the argument of heat semigroup \( e^{t\Delta} \) is infeasible for system (1.1). To overcome this difficulty, we replace the heat semigroup \( e^{t\Delta} \) applied in [36] with the Mittag-Leffler operators \( E_\beta (-t^\beta (-\Delta)^{\frac{\alpha}{2}}) \) and \( E_{\beta,\beta} (-t^\beta (-\Delta)^{\frac{\alpha}{2}}) \). Our goal in this paper is to study the existence and some properties of mild solutions to (1.1).

The rest of this paper is organized as follows. We begin in Section 2 with some basic definitions and properties which will be used in the subsequent proof and the \( L^p - L^q \) estimates of Mittag-Leffler operators are established. Section 3 is dedicated to verifying the existence and uniqueness of mild solution in Lebesgue spaces \( L^p(\mathbb{R}^d) \). On one hand, local existence and uniqueness to (1.1) is verified by Banach fixed point theorem. On the other hand, global existence and uniqueness of mild solution is obtained by Banach implicit function theorem utilized in [36]. Section 4 is devoted to the existence of mild solution in fractional homogeneous Sobolev spaces \( \dot{H}^{\mu,q}(\mathbb{R}^d) \). The mass conservation of local solution and decay estimates, stability of global solution are provided in Section 5 and in addition, the self-similar solution is constructed whenever taking \( \gamma = 0 \) in (1.1).

**Assumption 1.** For \( d \geq 2 \), assume that \( \alpha, \beta \) and exponents \( p, q, r \) satisfy one of the following conditions (I), (II), (III), (IV) and (V):

(I) for any \( 1 < \alpha \leq 2, 0 < \beta < 1, \)

\[
\frac{d}{2\alpha - 2} < q \leq \frac{d}{\alpha - 1}; \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q}; \quad \frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q};
\]
(II) for any $1 < \alpha \leq \frac{3}{2}$, $0 < \beta < 1$,
\[ \frac{d}{\alpha - 1} < q < \infty, \quad \frac{d}{\alpha - 1} < p < \infty, \quad q \leq r < \infty; \]

(III) for any $\frac{3}{2} < \alpha \leq 2$, $0 < \beta \leq \frac{\alpha}{3\alpha - 3}$,
\[ \frac{d}{\alpha - 1} < q < \infty, \quad \frac{d}{\alpha - 1} < p < \infty, \quad q \leq r < \infty; \]

(IV) for any $\frac{3}{2} < \alpha \leq 2$, $\frac{\alpha}{3\alpha - 3} < \beta < 1$,
\[ \frac{d}{\alpha - 1} < q \leq \frac{d\beta}{(3\alpha - 3)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \infty, \quad q \leq r < \infty; \]

(V) for any $\frac{3}{2} < \alpha \leq 2$, $\frac{\alpha}{3\alpha - 3} < \beta < 1$,
\[ \frac{d\beta}{(3\alpha - 3)\beta - \alpha} < q \leq \frac{2d\beta}{(3\alpha - 3)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \frac{d\beta q}{(3\alpha - 3)\beta - \alpha}, \quad q \leq r < \frac{d\beta}{(3\alpha - 3)\beta - \alpha} \frac{d\beta q}{q - d\beta}. \]

**Assumption 2.** For $d \geq 2$, $\alpha, \beta, \mu$ and the exponents $p, q, r$ satisfy either one of the following conditions:

(I) Assume that $\alpha, \beta, \mu$ satisfy either one of the following conditions

(i) $1 < \alpha \leq \frac{5}{4}$, $0 < \beta < 1$, $0 < \mu < \alpha - 1$;

(ii) $\frac{5}{4} < \alpha \leq 2$, $0 < \beta \leq \frac{\alpha}{5\alpha - 5}$, $0 < \mu < \alpha - 1$;

(iii) $\frac{5}{4} < \alpha \leq \frac{4}{3}$, $\frac{\alpha}{5\alpha - 5} < \beta < 1$, $0 < \mu < \alpha - 1$;

(iv) $\frac{4}{3} < \alpha \leq 2$, $\frac{\alpha}{5\alpha - 5} < \beta \leq \frac{\alpha}{4\alpha - 4}$, $0 < \mu < \alpha - 1$;

(v) $\frac{4}{3} < \alpha \leq \frac{3}{2}$, $\frac{\alpha}{4\alpha - 4} < \beta < 1$, $0 < \mu \leq \frac{\alpha}{\beta} - (3\alpha - 3)$;

(vi) $\frac{3}{2} < \alpha \leq 2$, $\frac{\alpha}{4\alpha - 4} < \beta \leq \frac{\alpha}{3\alpha - 3}$, $0 < \mu \leq \frac{\alpha}{\beta} - (3\alpha - 3)$;

and $p, q, r$ satisfy either one of the following inequalities

(1) $\frac{d}{2\alpha - 2 - \mu} < q \leq \frac{d}{\alpha - 1}$, $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$, $\frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q}$;

(2) $\frac{d}{\alpha - 1} < q < \frac{d}{\alpha - 1 - \mu}$, $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$, $q \leq r < \infty$;

(3) $\frac{d}{\alpha - 1 - \mu} \leq q < \infty$, $\frac{d}{\alpha - 1} < p < \infty$, $q \leq r < \infty$.

(II) Assume that $\alpha, \beta, \mu$ satisfy either one of the following conditions
(i) \(1 < \alpha \leq \frac{5}{4}, \quad 0 < \beta < 1, \quad \alpha - 1 \leq \mu < 2\alpha - 2;\)
(ii) \(\frac{5}{4} < \alpha \leq 2, \quad 0 < \beta \leq \frac{\alpha}{5\alpha - 5}, \quad \alpha - 1 \leq \mu < 2\alpha - 2;\)
(iii) \(\frac{5}{4} < \alpha \leq \frac{4}{3}, \quad \frac{\alpha}{5\alpha - 5} < \beta < 1, \quad \alpha - 1 \leq \mu \leq \frac{\alpha}{\beta} - (3\alpha - 3);\)
(iv) \(\frac{4}{3} < \alpha \leq 2, \quad \frac{\alpha}{5\alpha - 5} < \beta \leq \frac{\alpha}{4\alpha - 4}, \quad \alpha - 1 \leq \mu \leq \frac{\alpha}{\beta} - (3\alpha - 3);\)

and \(p, q, r\) satisfy the following inequality

\[(1) \quad \frac{d}{2\alpha - 2 - \mu} < q < \infty, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}, \quad q \leq r < \infty.\]

(III) Assume that \(\alpha, \beta, \mu\) satisfy either one of the following conditions

(i) \(\frac{5}{4} < \alpha \leq \frac{4}{3}, \quad \frac{\alpha}{5\alpha - 5} < \beta < 1, \quad \frac{\alpha}{\beta} - (3\alpha - 3) < \mu < \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2};\)
(ii) \(\frac{4}{3} < \alpha \leq 2, \quad \frac{\alpha}{5\alpha - 5} < \beta \leq \frac{\alpha}{4\alpha - 4}, \quad \frac{\alpha}{\beta} - (3\alpha - 3) < \mu < \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2};\)
(iii) \(\frac{4}{3} < \alpha \leq \frac{3}{2}, \quad \frac{\alpha}{4\alpha - 4} < \beta < 1, \quad \alpha - 1 \leq \mu < \frac{\alpha}{\beta} - \frac{\alpha - 1}{2};\)
(iv) \(\frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{4\alpha - 4} < \beta \leq \frac{\alpha}{3\alpha - 3}, \quad \alpha - 1 \leq \mu < \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2};\)

and \(p, q, r\) satisfy either one of the following inequalities

(1) \(\frac{d}{2\alpha - 2 - \mu} < q \leq \frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}, \quad q \leq r < \infty;\)
(2) \(\frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha} < q < \frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}, \quad q \leq r < \frac{d\beta q}{(3\alpha - 3 + \mu)\beta - \alphaq - d\beta}\).

(IV) Assume that \(\alpha, \beta, \mu\) satisfy either one of the following conditions

(i) \(\frac{4}{3} < \alpha \leq \frac{3}{2}, \quad \frac{\alpha}{4\alpha - 4} < \beta < 1, \quad \frac{\alpha}{\beta} - (3\alpha - 3) \leq \mu \leq \frac{\alpha}{2\beta} - (\alpha - 1);\)
(ii) \(\frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{4\alpha - 4} < \beta \leq \frac{\alpha}{3\alpha - 3}, \quad \frac{\alpha}{\beta} - (3\alpha - 3) \leq \mu \leq \frac{\alpha}{2\beta} - (\alpha - 1);\)
(iii) \(\frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{3\alpha - 3} < \beta < 1, \quad 0 < \mu \leq \frac{\alpha}{2\beta} - (\alpha - 1);\)

and \(p, q, r\) satisfy either one of the following inequalities

(1) \(\frac{d}{2\alpha - 2 - \mu} < q \leq \frac{d}{\alpha - 1}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}, \quad \frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q};\)
(2) \(\frac{d}{\alpha - 1} < q \leq \frac{d}{\alpha - 1 - \mu}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}, \quad q \leq r < \infty;\)
(3) \(\frac{d}{\alpha - 1 - \mu} < q \leq \frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \infty, \quad q \leq r < \infty;\)
(4) \(\frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha} < q < \frac{d\beta}{(3\alpha - 3 + \mu)\beta - \alpha}, \quad \frac{d}{\alpha - 1} < p < \frac{d\beta q}{(3\alpha - 3 + \mu)\beta - \alphaq - d\beta}, \quad q \leq r < \frac{d\beta q}{(3\alpha - 3 + \mu)\beta - \alphaq - d\beta}\).

(V) Assume that \(\alpha, \beta, \mu\) satisfy either one of the following conditions
(i) \( \frac{4}{3} < \alpha \leq \frac{3}{2}, \quad \frac{\alpha}{4\alpha - 4} < \beta < 1, \quad \frac{\alpha}{2\beta} - (\alpha - 1) < \mu < \alpha - 1; \)

(ii) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{4\alpha - 4} < \beta \leq \frac{\alpha}{3\alpha - 3}, \quad \frac{\alpha}{2\beta} - (\alpha - 1) < \mu < \alpha - 1; \)

(iii) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{3\alpha - 3} < \beta < 1, \quad \frac{\alpha}{2\beta} - (\alpha - 1) < \mu \leq \frac{\alpha}{3} - (2\alpha - 2); \)

and \( p, q, r \) satisfy either one of the following inequalities

1. \( \frac{d}{2\alpha - 2 - \mu} < q \leq \frac{d}{\alpha - 1} \), \( \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( \frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q} \),

2. \( \frac{d}{\alpha - 1} < q \leq \frac{d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( q \leq \infty; \)

3. \( \frac{d^3}{(3\alpha - 3\mu)^{3 - \alpha}} < q \leq \frac{2d^3}{(4\alpha - 4)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( q \leq \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

4. \( \frac{2d^3}{(4\alpha - 4)^{3 - \alpha}} < q \leq \frac{2d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

(VI) Assume that \( \alpha, \beta, \mu \) satisfy either one of the following conditions

(i) \( \frac{5}{4} < \alpha \leq \frac{3}{2}, \quad \frac{\alpha}{5\alpha - 5} < \beta < 1, \quad \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2} \leq \mu < \frac{\alpha}{3} + \frac{\alpha - 1}{3}; \)

(ii) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{5\alpha - 5} < \beta \leq \frac{\alpha}{3\alpha - 3}, \quad \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2} \leq \mu < \frac{\alpha}{3} + \frac{\alpha - 1}{3}; \)

(iii) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{3\alpha - 3} < \beta < 1, \quad \alpha - 1 \leq \mu < \frac{\alpha}{3} + \frac{\alpha - 1}{3}; \)

and \( p, q, r \) satisfy the following inequality

1. \( \frac{d}{2\alpha - 2 - \mu} < q \leq \frac{2d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( q \leq \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

(VII) Assume that \( \alpha, \beta, \mu \) satisfy the following condition

(i) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{3\alpha - 3} < \beta < 1, \quad \frac{\alpha}{\beta} - (2\alpha - 2) < \mu < \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2}; \)

and \( p, q, r \) satisfy either one of the following inequalities

1. \( \frac{d}{2\alpha - 2 - \mu} < q \leq \frac{2d^3}{(4\alpha - 4\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( \frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q} \),

2. \( d \frac{d}{\alpha - 1} < q \leq \frac{d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( q \leq \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

3. \( \frac{d}{\alpha - 1} < q \leq \frac{d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q} \), \( q \leq \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

4. \( \frac{2d^3}{(4\alpha - 4)^{3 - \alpha}} < q \leq \frac{2d^3}{(3\alpha - 3\mu)^{3 - \alpha}}, \quad \frac{d}{\alpha - 1} < p < \frac{d^3q}{(3\alpha - 3\mu)^{3} - 3\alpha q - 3d}; \)

(VIII) Assume that \( \alpha, \beta, \mu \) satisfy the following condition

(i) \( \frac{3}{2} < \alpha \leq 2, \quad \frac{\alpha}{3\alpha - 3} < \beta < 1, \quad \frac{\alpha}{2\beta} - \frac{\alpha - 1}{2} \leq \mu < \frac{\alpha}{3}; \)

and \( p, q, r \) satisfy either one of the following inequalities
(1) $\frac{d}{2a - 2 - \mu} < q \leq \frac{2d\beta}{(4a - 4 + \mu)\beta - \alpha}$; \quad $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$; \quad $\frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q}$;

(2) $\frac{2d\beta}{(4a - 4 + \mu)\beta - \alpha} < q \leq \frac{d}{\alpha - 1}$; \quad $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$; \quad $\frac{d}{\alpha - 1} < r < \frac{d\beta q}{(3a - 3 + \mu)\beta - \alpha q - \alpha - \beta}$. 

(IX) Assume that $\alpha, \beta, \mu$ satisfy the following condition

(i) $\frac{3}{2} < \alpha \leq 2$, \quad $\frac{\alpha}{3a - \alpha} < \beta < 1$, \quad $\frac{\alpha}{3a} \leq \mu < \alpha - 1$;

and $p, q, r$ satisfy the following inequality

(1) $\frac{d}{2a - 2 - \mu} < q \leq \frac{d}{\alpha - 1}$; \quad $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$; \quad $\frac{d}{\alpha - 1} < r < \frac{d\beta q}{(3a - 3 + \mu)\beta - \alpha q - \alpha - \beta}$. 

(X) Assume that $\alpha, \beta, \mu$ satisfy the following condition

(i) $\frac{3}{2} < \alpha \leq 2$, \quad $\frac{\alpha}{3a - \alpha} < \beta < 1$, \quad $\frac{\alpha}{23} - \frac{\alpha - 1}{2} \leq \mu < \alpha - 1$;

and $p, q, r$ satisfy either one of the following inequalities

(1) $\frac{d}{\alpha - 1} < q \leq \frac{2d\beta}{(4a - 4)\beta - \alpha}$; \quad $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1 - \mu)q}$; \quad $q \leq r < \frac{d\beta q}{(3a - 3 + \mu)\beta - \alpha q - \alpha - \beta}$;

(2) $\frac{2d\beta}{(4a - 4)\beta - \alpha} < q \leq \frac{2d\beta}{(3a - 3 + \mu)\beta - \alpha}$; \quad $\frac{d}{\alpha - 1} < p < \frac{d\beta q}{(3a - 3 + \mu)\beta - \alpha q - \alpha - \beta}$; \quad $q \leq r < \frac{d\beta q}{(3a - 3 + \mu)\beta - \alpha q - \alpha - \beta}$.

The main results of this paper are presented as follows.

**Theorem 1.1.** Assume that $d \geq 2$, $0 < \beta < 1$, $1 < \alpha \leq 2$, $\gamma \geq 0$, the gravitational potential $\phi$ satisfies $\nabla \phi \in L^d(\mathbb{R}^d)$ and the exponents $p, q$ and $r$ satisfy either one of the following conditions

(1) $\frac{d}{2a - 2} < q \leq \frac{d}{\alpha - 1}$; \quad $\frac{d}{\alpha - 1} < p < \frac{qd}{d - (\alpha - 1)q}$; \quad $\frac{d}{\alpha - 1} < r < \frac{qd}{d - (\alpha - 1)q}$;

(2) $\frac{d}{\alpha - 1} < q < \infty$, \quad $\frac{d}{\alpha - 1} < p < \infty$, \quad $q \leq r < \infty$.

Then for any $n_0 \in L^q(\mathbb{R}^d)$, $\nabla v_0 \in L^r(\mathbb{R}^d)$ and $u_0 \in L^p(\mathbb{R}^d)$, there exists $T > 0$ such that system (I.1) admits a unique mild solution $(n, v, u)$ with

$$ n \in C((0, T]; L^q(\mathbb{R}^d)), \quad \nabla v \in C((0, T]; L^r(\mathbb{R}^d)), \quad u \in C((0, T]; L^p(\mathbb{R}^d)). $$

Furthermore, define the largest time of existence

$$ T_{max} = \sup \{ T > 0 : (I.1) \text{ has a unique solution } (n, v, u) \text{ with } n \in C((0, T]; L^q(\mathbb{R}^d)), \quad \nabla v \in C((0, T]; L^r(\mathbb{R}^d)), \quad u \in C((0, T]; L^p(\mathbb{R}^d)) \}. $$

If $T_{max} < \infty$, then we have

$$ \lim_{t \to T_{max}} \| n(\cdot, t) \|_q = \infty, \quad \lim_{t \to T_{max}} \| \nabla v(\cdot, t) \|_r = \infty, \quad \lim_{t \to T_{max}} \| u(\cdot, t) \|_p = \infty. $$
The gravitational potential \( \phi \) for all \( n \) (3) satisfies \( \alpha \) of Theorem 1.2.

Theorem 1.3. Assume that \( d \geq 2, \gamma \geq 0, \alpha, \beta \) and the exponents \( p, q, r \) satisfy Assumption (1.1). The gravitational potential \( \phi \) satisfies \( \nabla \phi \in L^d(\mathbb{R}^d) \), and there is a constant \( M > 0 \) such that for all \( n_0 \in L^{\frac{d}{2a-\mu}}(\mathbb{R}^d), \nabla v_0 \in L^{\frac{d}{\alpha-1}}(\mathbb{R}^d) \) and \( u_0 \in L^{\frac{d}{\alpha-1}}(\mathbb{R}^d) \) with

\[
\| n_0 \|_{\frac{d}{2a-\mu}} + \| \nabla v_0 \|_{\frac{d}{\alpha-1}} + \| u_0 \|_{\frac{d}{\alpha-1}} + \| \nabla \phi \|_{\frac{d}{\gamma}} < M,
\]

then there exists a unique mild solution \( (n, v, u) \) of (1.1) with the property that

\[
t \frac{d}{\alpha} \left( \frac{2a-\mu}{2} - \frac{1}{\alpha} \right) n \in C((0, \infty); L^q(\mathbb{R}^d)),
\]

\[
t \frac{d}{\alpha} \left( \frac{\alpha-1}{2} - \frac{1}{\alpha} \right) \nabla v \in C((0, \infty); L^r(\mathbb{R}^d)),
\]

\[
t \frac{d}{\alpha} \left( \frac{\alpha-1}{2} - \frac{1}{\alpha} \right) u \in C((0, \infty); L^p(\mathbb{R}^d)).
\]

Theorem 1.3. Assume that \( d \geq 2, 0 < \beta < 1, 1 < \alpha \leq 2, \gamma \geq 0 \), the gravitational potential \( \phi \) satisfies \( \nabla \phi \in L^d(\mathbb{R}^d) \) and the exponents \( p, q, r \) and \( \mu \) satisfy either one of the following conditions

(1) \( 0 < \mu < \alpha - 1, \frac{d}{2a-\mu} < q \leq \frac{d}{\alpha-1}, \frac{d}{\alpha-1} < p < \frac{qd}{d-(\alpha-1)q}, \frac{d}{\alpha-1} < r < \frac{qd}{d-(\alpha-1)q}; \)

(2) \( 0 < \mu < \alpha - 1, \frac{d}{\alpha-1} < q \leq \frac{d}{\alpha-1-\mu}, \frac{d}{\alpha-1} < p < \frac{qd}{d-(\alpha-1-\mu)q}, q \leq r < \infty; \)

(3) \( 0 < \mu < \alpha - 1, \frac{d}{\alpha-1-\mu} < q < \infty, \frac{d}{\alpha-1} < p < \infty, q \leq r < \infty; \)

(4) \( \alpha - 1 \leq \mu < 2\alpha - 2, \frac{d}{2a-\mu} < q < \infty, \frac{d}{\alpha-1} < p < \frac{qd}{d-(\alpha-1-\mu)q}, q \leq r < \infty. \)

Then for any \( n_0 \in H^{\mu,q}(\mathbb{R}^d), \nabla v_0 \in H^{\mu,r}(\mathbb{R}^d) \) and \( u_0 \in H^{\mu,p}(\mathbb{R}^d) \), there exists \( T > 0 \) such that (1.1) admits a unique mild solution \( (n, v, u) \) with

\[
n \in C((0, T]; H^{\mu,q}(\mathbb{R}^d)), \nabla v \in C((0, T]; H^{\mu,r}(\mathbb{R}^d)), u \in C((0, T]; H^{\mu,p}(\mathbb{R}^d)).
\]

Furthermore, define the largest time of existence

\[
T_{max} = \sup\{ T > 0 : (1.1) \ has \ a \ unique \ solution \ (n, v, u) \ with \ n \in C((0, T]; H^{\mu,q}(\mathbb{R}^d)), \nabla v \in C((0, T]; H^{\mu,r}(\mathbb{R}^d)), \ u \in C((0, T]; H^{\mu,p}(\mathbb{R}^d))\}.
\]

If \( T_{max} < \infty \), then we have

\[
\limsup_{t \to T_{max}} \| n(\cdot, t) \|_{H^{\mu,q}} = \infty, \quad \limsup_{t \to T_{max}} \| \nabla v(\cdot, t) \|_{H^{\mu,r}} = \infty, \quad \limsup_{t \to T_{max}} \| u(\cdot, t) \|_{H^{\mu,p}} = \infty.
\]
Theorem 1.4. Assume that $d \geq 2$, $\gamma \geq 0$, $\alpha, \beta, \mu$ and the exponents $p, q, r$ satisfy Assumption 1. Assume that the gravitational potential $\phi$ satisfies $\nabla \phi \in L^d(\mathbb{R}^d)$, and there is a constant $\tilde{M} > 0$ such that for all $n_0 \in L^{\frac{d}{2\alpha - 2}}(\mathbb{R}^d)$, $\nabla v_0 \in L^{\frac{d}{\alpha - 1}}(\mathbb{R}^d)$ and $u_0 \in L^{\frac{d}{\alpha - 1}}(\mathbb{R}^d)$ with

$$\|n_0\|_{\frac{d}{2\alpha - 2}} + \|\nabla v_0\|_{\frac{d}{\alpha - 1}} + \|u_0\|_{\frac{d}{\alpha - 1}} + \|\nabla \phi\|_d < \tilde{M},$$

then there exists a unique mild solution $(n, v, u)$ of (1.1) with the property that

$$t^{\frac{d}{2\alpha - 2} - \frac{1}{\gamma}} \|n\|_{C((0, \infty); L^q(\mathbb{R}^d))}, \quad t^{\frac{d}{(\alpha - 1) - \frac{1}{r}} \|v\|_{C((0, \infty); L^r(\mathbb{R}^d))}, \quad t^{\frac{d}{\alpha - 1 + \frac{1}{r}} \|u\|_{C((0, \infty); \dot{H}^{\mu - p}(\mathbb{R}^d))}}$$

and the other conditions in Theorem 1.1 hold.

Theorem 1.5. Assume that $d \geq \max\{2, 3\alpha - 3\}$ and the other conditions in Theorem 1.4 hold. Suppose $n_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $v_0 \in L^1(\mathbb{R}^d)$, $\nabla v_0 \in L^r(\mathbb{R}^d)$ and $u_0 \in L^p(\mathbb{R}^d)$, then the solution $(n, v, u)$ of (1.1) given by Theorem 1.4 fulfills

$$n \in C((0, T); L^1(\mathbb{R}^d)), \quad v \in C((0, T); L^1(\mathbb{R}^d)).$$

In addition, the following mass conservation is held to be true

$$\int_{\mathbb{R}^d} n(x, t) \, dx = \int_{\mathbb{R}^d} n_0(x) \, dx,$$

$$\int_{\mathbb{R}^d} v(x, t) \, dx = \begin{cases} \int_{\mathbb{R}^d} v_0(x) \, dx + \frac{t^\beta}{\beta \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx, & \gamma = 0, \\ E_\beta(-\gamma t^\beta) \int_{\mathbb{R}^d} v_0(x) \, dx + \frac{1 - E_\beta(-\gamma t^\beta)}{\gamma \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx, & \gamma > 0. \end{cases}$$

Theorem 1.6. Let $d \geq 3$ and $\gamma = 0$. Assume that $(n_0, v_0, u_0)$ and gravitational potential $\phi$ are as in Theorem 1.2. Suppose that $[n_0, v_0, u_0, \phi]$ are homogeneous functions with degree $-(2\alpha - 2), 0, -(\alpha - 1), 0$, respectively, i.e. for any $x \in \mathbb{R}^d, \lambda > 0$,

$$\lambda^{2\alpha - 2}n_0(\lambda x) = n_0(x), \quad v_0(\lambda x) = v_0(x), \quad \lambda^{\alpha - 1}u_0(\lambda x) = u_0(x), \quad \phi(\lambda x) = \phi(x).$$

If $[n_0, v_0, u_0, \phi]$ satisfies (1.5), then the solution $[n, v, u]$ given in Theorem 1.2 is a self-similar one, that is, for all $x \in \mathbb{R}^d$, $t > 0$ and $\lambda > 0$, the following holds,

$$\lambda^{2\alpha - 2}n(\lambda x, \lambda^\beta t) = n(x, t), \quad v(\lambda x, \lambda^\beta t) = v(x, t), \quad \lambda^{\alpha - 1}u(\lambda x, \lambda^\beta t) = u(x, t).$$
2. Notations and Preliminaries

The purpose of this section is to state some notions and functional spaces, to introduce briefly the definition of Caputo derivative (provided in [41, 42]), then to define the mild solution to system (1.1) and to recall some results which will be used later.

For \(1 < p < \infty\), \(\|u\|_p\) denotes the \(L^p\)-norm of the function \(u\) in \(L^p(\mathbb{R}^d)\) space. The space \(C((0, \infty); X)\) stands for the set of continuous functions on \((0, \infty)\) with values in Banach space \(X\). For \(\mu \in \mathbb{R}, 1 < p < \infty\), the fractional nonhomogeneous Sobolev spaces and fractional homogeneous Sobolev spaces are defined as those introduce in [26]:

\[
H^{\mu,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{\mu}{2}} \hat{u}) \in L^p(\mathbb{R}^n) \right\},
\]

\[
\dot{H}^{\mu,p}(\mathbb{R}^n) = \left\{ u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}(|\xi|^\mu \hat{u}) \in L^p(\mathbb{R}^n) \right\},
\]

where \(\mathcal{F}\) is the Fourier transform and the corresponding norms are given by

\[
\|u\|_{H^{\mu,p}} = \|\mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{\mu}{2}} \hat{u})\|_p = \|(I - \Delta)^{\frac{\mu}{2}} u\|_p,
\]

\[
\|u\|_{\dot{H}^{\mu,p}} = \|\mathcal{F}^{-1}(|\xi|^\mu \hat{u})\|_p = \|(-\Delta)^{\frac{\mu}{2}} u\|_p.
\]

Lemma 2.1. [26] Let \(\mu \geq 0, 1 < p < \infty\). Then \(f \in H^{\mu,p}(\mathbb{R}^d)\) if and only if \(f \in L^p(\mathbb{R}^d)\) and \((-\Delta)^{\frac{\mu}{2}} f \in L^p(\mathbb{R}^d)\), that is the norms \(\|f\|_{H^{\mu,p}}\) and \(\|f\|_p + \|f\|_{\dot{H}^{\mu,p}}\) are equivalent.

In order to introduce the generalized definition of Caputo derivative, let us first recall the following definition of limit.

Definition 1. [42] Let \(B\) be a Banach space. For a function \(u \in L^1_{loc}((0,T);B)\), if there exists \(u_0 \in B\) such that

\[
\lim_{t \to 0^+} \frac{1}{t} \int_0^t \|u(s) - u_0\|_B \, ds = 0,
\]

we call \(u_0\) the right limit of \(u\) at \(t = 0\), denoted by \(u(0^+) = u_0\). Similarly, we define \(u(T^-)\), the left limit of \(u\) at \(t = T\), to be the constant \(u_T \in B\) such that

\[
\lim_{t \to T^-} \frac{1}{T - t} \int_t^T \|u(s) - u_T\|_B \, ds = 0.
\]

For \(\beta > -1\), as discussed in [42], we use the following distributions \(\{g_{\beta}\}\) as the convolution kernels

\[
g_{\beta}(t) := \begin{cases} 
\theta(t) \Gamma(\beta) t^{\beta-1}, & \beta > 0, \\
\delta(t), & \beta = 0, \\
\frac{1}{\Gamma(1+\beta)} D \left(\theta(t)t^\beta\right), & \beta \in (-1,0),
\end{cases}
\]

where \(\theta(t) = 1\) if \(0 \leq t < 1\), \(\theta(t) = t\) if \(t \geq 1\), and \(\Gamma(\cdot)\) is the Gamma function.
θ(t) is the standard Heaviside step function and D represents the distributional derivative. 

\[ g_β \] can also be defined for \( \beta \leq -1 \) (see [42]) and consequently

\[ g_{β_1} * g_{β_2} = g_{β_1 + β_2}, \quad ∀ β_1, β_2 \in \mathbb{R}. \quad (2.4) \]

Correspondingly, we can introduce the following time-reflected group

\[ \tilde{C} := \{ \tilde{g}_α : \tilde{g}_α(t) = g_α(-t), α \in \mathbb{R} \}. \]

Clearly, \( \text{supp} \, \tilde{g} \subset (-∞, 0] \) and for \( γ \in (0, 1) \), the following equality is held to be true

\[ \tilde{g}_{-γ}(t) = -\frac{1}{Γ(1-γ)} D(θ(-t)(-t)^{-γ}) = -D\tilde{g}_{1-γ}(t). \quad (2.5) \]

**Definition 2.** [42] Let \( 0 < β < 1 \). Consider \( u \in L^1_{\text{loc}}([0, T); \mathbb{R}) \) such that \( u \) has a right limit \( u(0+) \) at \( t = 0 \) in the sense of Definition [1]. The \( β \)-th order Caputo derivative of \( u \), a distribution in \( \mathcal{D}'(\mathbb{R}) \) with support in \([0, T)\), is defined by

\[ D^β_c u := J_{-β}u - u_0 g_{1-β} = g_{-β} * (θ(t)(u - u_0)), \]

where \( J_α \) denotes the fractional integral operator

\[ J_α u(t) = \frac{1}{Γ(α)} \int_0^t (t-s)^{α-1} u(s)ds. \quad (2.6) \]

Similarly, the \( β \)-th order right Caputo derivative of \( u \) is a distribution in \( \mathcal{D}'(\mathbb{R}) \) with support in \((-∞, T] \), given by

\[ \tilde{D}^β_{c:T} u := \tilde{g}_{-β} * (θ(T-t)(u(t) - u(T))). \]

Now we state the definition of Caputo derivatives for mappings into Banach spaces.

**Definition 3.** [42] Let \( B \) be a Banach space and \( u \in L^1_{\text{loc}}((0, T); B) \). Let \( u_0 \in B \). Define the weak Caputo derivative of \( u \) associated with initial data \( u_0 \) to be \( \overset{c}{D}^β_t u \in \mathcal{D}' \) such that for any test function \( φ \in C^∞_c((-∞, T); \mathbb{R}) \),

\[ \langle \overset{c}{D}^β_t u, φ \rangle := \int_{-∞}^T (u - u_0)θ(t)(\tilde{D}^β_{c:T}φ) dt = \int_0^T (u - u_0)\tilde{D}^β_{c:T}φ dt, \quad (2.7) \]

where \( \mathcal{D}' = \{ v | v : C^∞_c((-∞, T); \mathbb{R}) \to B \text{ is a bounded linear operator} \} \). We call the weak Caputo derivative \( \overset{c}{D}^β_t u \) associated with initial value \( u_0 \) the Caputo derivative of \( u \) if \( u(0+) = u_0 \) in the sense of Definition [1] under the norm of the underlying Banach space \( B \).
In the sequel, we introduce three special functions, Wright function \( W_{\kappa,\lambda} \), Marnardi function \( M_\beta \) and Mittag-Leffler function \( E_{\beta,\gamma} \), respectively. Let \( \kappa > -1 \) and \( \lambda \in \mathbb{C} \). The Wright function \( W_{\kappa,\lambda} \) is defined by the following complex series representation and convergent in the whole complex plane

\[
W_{\kappa,\lambda}(z) := \sum_{j=0}^{\infty} \frac{z^j}{j! \Gamma(\kappa j + \lambda)}.
\]

Mainardi function \( M_\beta : \mathbb{C} \to \mathbb{C} \) is a particular case of the Wright function and is given by

\[
M_\beta(z) := W_{-\beta,1-\beta}(-z), \quad z \in \mathbb{C}.
\]

For real numbers \( \beta \) and \( \gamma \), the Mittag-Leffler function \( E_{\beta,\gamma} : \mathbb{C} \to \mathbb{C} \) is defined as follows

\[
E_{\beta,\gamma}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\beta + \gamma)}, \quad \beta, \gamma \in \mathbb{R}.
\]

The following classical result plays an important role to obtain the structural estimates for the Mittag-Leffler operators \( \{ E_{\beta,\beta}(t^{\beta} \cdot) : t > 0 \} \) and \( \{ E_{\beta,\beta}(t^{\beta} \cdot) : t > 0 \} \).

**Lemma 2.2.** \([19]\) For \( 0 < \beta < 1 \) and \(-1 < r < \infty\), when we restrict \( M_\beta \) to the positive real line, it holds that

\[
M_\beta(t) \geq 0 \text{ for all } t \geq 0, \quad \text{and} \quad \int_0^\infty t^r M_\beta(t) dt = \frac{\Gamma(1 + r)}{\Gamma(1 + \beta r)}.
\]

In what follows, we list some results about the fractional heat semigroup \( \{ e^{-t(-\Delta)^{\frac{\alpha}{2}}} \}_{t>0} \), which is the convolution operator with the fractional heat kernel \( K_t(x) \) and denoted by

\[
e^{-t(-\Delta)^{\frac{\alpha}{2}}} f := K_\alpha(t) f = K_t \ast f, \quad (2.8)
\]

where \( K_t(x) \) is defined via the Fourier transform

\[
K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha}) = t^{-\frac{d}{\alpha}} K(t^{-\frac{1}{\alpha}}x) \quad \text{with} \quad K(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} e^{-|\xi|^\alpha} d\xi. \quad (2.9)
\]

In \([8]\), since \( e^{-t|\xi|^\alpha} \) is a tempered distribution, it holds that \( K_t \in C^\infty((0, \infty) \times \mathbb{R}^d) \). In addition, we define \( K_\gamma(t) := e^{-t\gamma} K_\alpha(t) \) as the damped fractional heat semigroup.

**Lemma 2.3.** \([48]\) For any \( x \in \mathbb{R}^d \), \( 0 < t < \infty \), \( \alpha > 0 \), \( \mu > 0 \) and \( 1 \leq p \leq \infty \), the kernel function \( K(x) \) has the following properties:

(i) \( |K(x)| \leq C(1 + |x|)^{-d-\alpha} \), \( K(x) \in L^p(\mathbb{R}^d) \) and \( K_t(x) \in L^p(\mathbb{R}^d) \).
(ii) \(|(-\Delta)^{\frac{\alpha}{2}}K(x)| \leq C(1 + |x|)^{-d-\mu}, (-\Delta)^{\frac{\alpha}{2}}K(x) \in L^p(\mathbb{R}^d) \text{ and } (-\Delta)^{\frac{\alpha}{2}}K_t(x) \in L^p(\mathbb{R}^d),\)

(iii) \(|\nabla K(x)| \leq C(1 + |x|)^{-d-1}, \nabla K(x) \in L^p(\mathbb{R}^d) \text{ and } \nabla K_t(x) \in L^p(\mathbb{R}^d).\)

As showed in [55], one has the following connections between Mittag-Leffler operators and the Mainardi function:

\[
E_{\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) = \int_0^\infty M_{\beta}(s)K^\alpha(st^\beta) \, ds,
\]

\[
E_{\beta,\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) = \int_0^\infty \beta sM_{\beta}(s)K^\alpha(st^\beta) \, ds,
\]

\[
E_{\beta}(t^\beta(-(\Delta)^{\frac{\alpha}{2}}-\gamma)) = \int_0^\infty M_{\beta}(s)K^\alpha_s(st^\beta) \, ds,
\]

\[
E_{\beta,\beta}(t^\beta(-(\Delta)^{\frac{\alpha}{2}}-\gamma)) = \int_0^\infty \beta sM_{\beta}(s)K^\alpha_s(st^\beta) \, ds.
\]  

(2.10)

In [55], the solution to the following abstract initial value problem

\[
\begin{cases}
\frac{c}{0}D_t^\beta w = Aw + f(t), \\
\phantom{\frac{c}{0}} \quad w|_{t=0} = w_0,
\end{cases}
\]

is given by

\[
w(t) = E_{\beta}(t^\beta A)w_0 + \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}((t-\tau)^\beta A)f(\tau) \, d\tau.
\]

Specialize \(A = -(-\Delta)^{\frac{\alpha}{2}}\) to find that \((n, v, u)\) satisfies the following Duhamel type integral equations:

\[
\begin{cases}
n = E_{\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}})n_0 - \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\frac{\alpha}{2}})(u \cdot \nabla n + \nabla(\nabla n)) \, d\tau, \\
v = E_{\beta}(t^\beta(-(\Delta)^{\frac{\alpha}{2}}-\gamma)v_0 - \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}((t-\tau)^\beta(-(\Delta)^{\frac{\alpha}{2}}-\gamma))(u \cdot \nabla v - n) \, d\tau, \\
u = E_{\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}})u_0 - \int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}(-(t-\tau)^\beta(-\Delta)^{\frac{\alpha}{2}})[P((u \cdot \nabla)u + n\nabla \phi)] \, d\tau,
\end{cases}
\]

(2.11)

where \(P = \{P_{jk} \}_{j,k=1,2,\ldots,d}\) is the projection operator onto the solenoidal vector fields with the expression

\[
P_{jk} = \delta_{jk} + R_jR_k
\]

with Riesz operator \(R_j := \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{\alpha}{2}}\). It is well-known that for all \(1 < q < \infty\), there holds \(PL^q(\mathbb{R}^d) = L^q(\mathbb{R}^d) := \{g \in L^q(\mathbb{R}^d)|\nabla \cdot g = 0\}\).

Now we provide the definition of mild solution as follows.
Definition 4. Let $X$ be a Banach space over space and time. We call that $(n, v, u) \in X$ is a mild solution to system (1.1) if $(n, v, u)$ satisfies the integral equation (2.11) in $X$.

The following lemma regarding the estimates of the gradient of $K^\alpha(t)$ plays an indispensable role in the proof process of $L^p - L^q$ estimates to the Mittag-Leffler operators.

Lemma 2.4. Suppose that $1 \leq q \leq p \leq \infty$, $\mu \geq 0$ and $f \in L^q(\mathbb{R}^d)$. Then the following estimate holds

$$\|(-\Delta)^{\frac{\alpha}{2}} f\|_p \leq C t^{-\frac{\alpha}{2}(\mu + \frac{d}{q} - \frac{d}{p})} \|f\|_q,$$

$$\|(-\Delta)^{\frac{\alpha}{2}} \nabla K^\alpha(t) f\|_p \leq C t^{-\frac{\alpha}{2}(\mu + 1 + \frac{d}{q} - \frac{d}{p})} \|f\|_q,$$

where $C$ is a positive constant and independent of $t$.

Proof. Combining (2.9) and Lemma 2.3, we get

$$\|(-\Delta)^{\frac{\alpha}{2}} K_t(x)\|_r = t^{-\frac{\alpha}{r}} \left( \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{\alpha}{2}} K(t^{-\frac{1}{\alpha}} x) \right|^r dx \right)^{\frac{1}{r}}$$

$$= t^{-\frac{\alpha}{r} - \frac{d}{q} + \frac{d}{p} \frac{1}{\alpha}} \left( \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{\alpha}{2}} K(\xi) \right|^r dx \right)^{\frac{1}{r}}$$

$$\leq C t^{-\frac{\alpha}{r}(\frac{d}{q} + 1 - \frac{1}{r})}.$$

By Young’s inequality, it holds that

$$\|(-\Delta)^{\frac{\alpha}{2}} K^\alpha(t) f\|_p = \|(-\Delta)^{\frac{\alpha}{2}} K_t(x) * f\|_p$$

$$\leq \|(-\Delta)^{\frac{\alpha}{2}} K_t(x)\|_r \|f\|_q$$

$$\leq C t^{-\frac{\alpha}{r}(\frac{d}{q} + 1 - \frac{1}{r})} \|f\|_q,$$

where $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. And the estimates of $\|(-\Delta)^{\frac{\alpha}{2}} \nabla K^\alpha(t) f\|_p$ can be obtained similarly. \hfill \Box

The following proposition gives the $L^p - L^q$ estimates of the Mittag-Leffler operators.

Proposition 2.1. Let $d \geq 2$, $0 < \beta < 1$, $1 < \alpha \leq 2$, $\gamma \geq 0$ and $\mu \geq 0$. Assume that $1 \leq q \leq p \leq \infty$ and $f \in L^q(\mathbb{R}^d)$, then there exists $C > 0$ such that the following estimates hold

(i) If $0 \leq \frac{1}{q} - \frac{1}{p} < \frac{\alpha - \mu}{d}$, we have

$$\|(-\Delta)^{\frac{\alpha}{2}} E_\beta(-t^\beta (-\Delta)^{\frac{\alpha}{2}}) f\|_p \leq C t^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{\alpha}} \|f\|_q,$$

(2.12)

$$\|(-\Delta)^{\frac{\alpha}{2}} E_\beta(t^\beta (-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) f\|_p \leq C t^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{\alpha}} A_{-\frac{d}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\alpha}{\alpha}}(t) \|f\|_q.$$

(2.13)

Particularly, if $p = q$, the constant $C$ can be chosen to be 1.
(ii) If \(0 \leq \frac{1}{q} - \frac{1}{p} < \frac{2\alpha - \mu}{d}\), we have
\[
\|(-\Delta)^{\frac{\alpha}{2}} E_{\beta, \beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}}) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \|f\|_q, \tag{2.14}
\]
\[
\|(-\Delta)^{\frac{\alpha}{2}} E_{\beta, \beta}(t^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} A_{\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\mu}{\alpha}}(t) \|f\|_q. \tag{2.15}
\]

(iii) If \(0 \leq \frac{1}{q} - \frac{1}{p} < \frac{2\alpha - \mu}{d}\), we have
\[
\|(-\Delta)^{\frac{\alpha}{2}} \nabla E_{\beta, \beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}}) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \|f\|_q, \tag{2.16}
\]
\[
\|(-\Delta)^{\frac{\alpha}{2}} \nabla E_{\beta, \beta}(t^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} A_{\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\mu}{\alpha}}(t) \|f\|_q. \tag{2.17}
\]

(iv) If \(0 \leq \frac{1}{q} - \frac{1}{p} < \frac{2\alpha - 1 - \mu}{d}\), we have
\[
\|(-\Delta)^{\frac{\alpha}{2}} \nabla E_{\beta, \beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}}) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \|f\|_q, \tag{2.18}
\]
\[
\|(-\Delta)^{\frac{\alpha}{2}} \nabla E_{\beta, \beta}(t^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma) f\|_p \leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} A_{\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1 + \mu}{\alpha}}(t) \|f\|_q. \tag{2.19}
\]

where \(A_{\sigma}(z) = \int_0^\infty s^\sigma M_{\beta}(s) e^{-sz^\gamma} ds\).

Remark 2.1. For \(-1 < \sigma < \infty\), the boundedness of function \(A_{\sigma}\) has been pointed out in [5]. In fact, by Lemma [2.22] one can easily find that
\[
\sup_{t > 0} A_{\sigma}(t) \leq \frac{\Gamma(1 + \sigma)}{\Gamma(1 + \beta\sigma)}.
\]

Proof. By virtue of the first formula in (2.10) and Lemma [2.4] it follows that
\[
\|(-\Delta)^{\frac{\alpha}{2}} E_{\beta, \beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}}) f\|_p \leq \int_0^\infty M_{\beta}(s) \|(-\Delta)^{\frac{\alpha}{2}} K^\alpha(s^\gamma) f\|_p ds
\]
\[
\leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \|f\|_q \int_0^\infty M_{\beta}(s) s^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\mu}{\alpha}} ds
\]
\[
= C_{\frac{1}{\alpha}} \frac{\Gamma(1 - \frac{\alpha}{\gamma} - \frac{1}{p} - \frac{\mu}{\gamma})}{\Gamma(1 - \frac{d\alpha}{\gamma} - \frac{1}{p} - \frac{\mu}{\gamma})} t^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \|f\|_q,
\]
and the last equality holds due to Lemma [2.22]. Thus (2.12) is verified.

We now come to estimate (2.13). Utilizing the third formula in (2.10), the definition of \(K_{\gamma}^\alpha(t)\) and Lemma [2.4] we see that
\[
\|(-\Delta)^{\frac{\alpha}{2}} E_{\beta, \beta}(t^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma) f\|_p \leq \int_0^\infty M_{\beta}(s) \|(-\Delta)^{\frac{\alpha}{2}} K_\gamma^\alpha(s^\gamma) f\|_p ds
\]
\[
\leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} \int_0^\infty M_{\beta}(s) e^{-st^\gamma} s^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\mu}{\gamma}} ds
\]
\[
\leq Ct^{-\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{d\alpha}{2} - \frac{\mu}{\alpha}} A_{\frac{d\alpha}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{\mu}{\alpha}}(t) \|f\|_q.
\]
Employing (2.10) and Lemma 2.4, we obtain
\[ \|(-\Delta)^{\frac{\mu}{2}} E_{\beta,\beta}(-t^\beta(-\Delta)^{\frac{\mu}{2}}) f\|_p \leq \int_0^\infty \beta s M_\beta(s) \|(-\Delta)^{\frac{\mu}{2}} K^\alpha(st^\beta) f\|_p ds \]
\[ \leq Ct^{-\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} \int_0^\infty M_\beta(s) s^{1 - \frac{d\alpha}{q} - \frac{\mu\beta}{2}} \|f\|_q ds \]
\[ = C \frac{\Gamma(2 - \frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2})}{\Gamma(1 + \beta - \frac{d\alpha}{q} - \frac{\mu\beta}{2})} t^{-\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} \|f\|_q, \]
and
\[ \|(-\Delta)^{\frac{\mu}{2}} \nabla E_{\beta,\beta}(-t^\beta(-\Delta)^{\frac{\mu}{2}}) f\|_p \leq \int_0^\infty \beta s M_\beta(s) \|(-\Delta)^{\frac{\mu}{2}} \nabla K^\alpha(st^\beta) f\|_p ds \]
\[ \leq Ct^{-\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} \int_0^\infty M_\beta(s) s^{1 - \frac{d\alpha}{q} - \frac{\mu\beta}{2}} \|f\|_q ds \]
\[ = C \frac{\Gamma(2 - \frac{1 + \mu}{q} - \frac{d\alpha}{q} - \frac{\mu\beta}{2})}{\Gamma(1 + \beta - \frac{1 + \mu}{q} - \frac{d\alpha}{q} - \frac{\mu\beta}{2})} t^{-\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} \|f\|_q, \]
which yields (2.14) and (2.18). The proofs of the other estimates (2.15)-(2.17) and (2.19) can be verified with the same arguments as the above.

Remark 2.2. Reference [43] listed the $L^p - L^q$ estimates regarding the operators $S_\alpha^\beta(t)$ and $T_\alpha^\beta(t)$, which are defined by
\begin{align*}
S_\alpha^\beta(t)f(x) &:= E_{\beta}(-t^\beta A)f(x) = P(\cdot, t) * f(x), \\
T_\alpha^\beta(t)f(x) &:= t^{\beta-1}E_{\beta,\beta}(-t^\beta A)f(x) = Y(\cdot, t) * f(x),
\end{align*}
and the $L^p - L^q$ estimates in [43] follow from the asymptotic behaviours of $P(x, t)$ and $Y(x, t)$. As a matter of fact, the $L^p - L^q$ estimates of the operators $S_\alpha^\beta(t)$ and $T_\alpha^\beta(t)$ in [43] are equivalent to the estimates listed in Proposition 2.1 whenever $\mu = 0$.

Based on the above estimates, we then establish the time continuity of the Mittag-Leffler operators.

**Proposition 2.2.** For $d \geq 2$, $0 < T \leq \infty$, $0 < \beta < 1$, $1 < \alpha \leq 2$, $\gamma \geq 0$, $\mu \geq 0$ and $p, q$ satisfy the conditions in Proposition 2.1. If $f \in L^q(\mathbb{R}^d)$, then we have
\begin{align*}
t^{\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} (-\Delta)^{\frac{\mu}{2}} E_{\beta}(-t^\beta(-\Delta)^{\frac{\mu}{2}}) f &\in C((0, T); L^p(\mathbb{R}^d)), \\
t^{\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} (-\Delta)^{\frac{\mu}{2}} E_{\beta}(-t^\beta(-\Delta)^{\frac{\mu}{2}} - \gamma) f &\in C((0, T); L^p(\mathbb{R}^d)), \\
t^{\frac{d\alpha}{q} - \frac{1}{p} - \frac{\mu\beta}{2}} (-\Delta)^{\frac{\mu}{2}} \nabla E_{\beta}(-t^\beta(-\Delta)^{\frac{\mu}{2}}) f &\in C((0, T); L^p(\mathbb{R}^d)),
\end{align*}
\[ t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} \nabla E_{\beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)), \tag{2.24} \]
\[ t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} E_{\beta, \beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)), \tag{2.25} \]
\[ t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} E_{\beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)), \tag{2.26} \]
\[ t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} \nabla E_{\beta, \beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)), \tag{2.27} \]
\[ t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} E_{\beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)). \tag{2.28} \]

**Proof.** Fix \( t_0 > 0 \) and consider \( t > t_0 \), the case \( t < t_0 \) follows analogously. Using the definition of Mittag-Leffler operators \((2.10)\), we first write

\[
\| t \frac{d^2}{dt^2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} E_{\beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \|_p
\leq \int_0^\infty s^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha}} M_\beta(s) \| (st^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \left( -\Delta \right)^{\frac{\beta}{2}} K^{\alpha}(st^\beta) f \|_p ds
\leq \int_0^\infty s^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha}} M_\beta(s) \| (st^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} - (st_0^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \| \left( -\Delta \right)^{\frac{\beta}{2}} K^{\alpha}(st_0^\beta) f \|_p ds
\]

\[ + \int_0^\infty s^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha}} M_\beta(s)(st^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \| \left( -\Delta \right)^{\frac{\beta}{2}} K^{\alpha}(st^\beta) f - (st_0^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \| \left( -\Delta \right)^{\frac{\beta}{2}} K^{\alpha}(st_0^\beta) f \|_p ds \]

\[ := I_1(t, t_0) + I_2(t, t_0). \]

Regarding \( I_1(t, t_0) \), in terms of Lemma \( 2.12 \) and \( 2.13 \) in Proposition \( 2.1 \), one obtains that
\[ I_1(t, t_0) \leq \int_0^\infty s^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha}} M_\beta(s) \left( t \frac{d}{dt} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \right) s^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha}} \| f \|_q ds
\leq Ct^{-\frac{d}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \| t \frac{d}{dt} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \| \| f \|_q.
\]

Therefore, \( I_1(t, t_0) \) goes to 0 as \( t \to t_0 \).

As for \( I_2(t, t_0) \), since \( K_t \in C^\infty((0, \infty) \times \mathbb{R}^d) \), we have
\[ I_2(t, t_0) \leq \int_0^\infty M_\beta(s)t_0^\alpha \left( \frac{1}{q} - \frac{1}{p} \right) - \frac{\alpha}{\alpha} \| \left( -\Delta \right)^{\frac{\beta}{2}} K^{\alpha}(st_0^\beta) - (st_0^\beta)^{\frac{1}{2} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{\alpha}{\alpha}} \| f \|_q ds,
\]
\( I_2(t, t_0) \) also goes to 0 as \( t \to t_0 \). Hence this implies that
\[ t \frac{d}{dt} \left( \frac{1}{q} - \frac{1}{p} \right) + \frac{(1 + \mu) \alpha}{\alpha} \left( -\Delta \right)^{\frac{\beta}{2}} E_{\beta}(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) f \in C((0, T); L^p(\mathbb{R}^d)). \]

The other results can be verified in the same way as what we have done above. \( \square \)

**Lemma 2.5.** \( \square \) Let \( \mu > 0, 1 < p < \infty \). If \( f, g \in S(\mathbb{R}^d) \) and \( 1 < p_1, p_2, p_3 < \infty \) satisfies

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},
\]

then the following estimate holds
\[ \| (-\Delta)^{\frac{\beta}{2}} (fg) \|_p \leq C(\| f \|_{p_1} \| g \|_{\dot{H}^{\mu, p_2}} + \| f \|_{\dot{H}^{\mu, p_3}} \| g \|_{p_4}), \]

where \( S(\mathbb{R}^d) \) denotes the Schwartz space.
3. Existence of mild solutions in $L^p$ spaces

In this section, we concentrate on the local existence and global existence of mild solutions to the Cauchy problem of the time-space fractional Keller-Segel-Navier-Stokes system \( (1.1) \). To this end, we take advantage of Banach fixed point theorem and Banach implicit function theorem to obtain the local existence and global existence, respectively.

3.1. Proof of Theorem 1.1

Proof. According to the assumptions in Theorem 1.1, the initial data $n_0 \in L^q(\mathbb{R}^d), \nabla v_0 \in L^r(\mathbb{R}^d)$ and $u_0 \in L^p(\mathbb{R}^d)$, thus there exists a constant $M_0 > 0$ such that the initial data satisfy

$$\|n_0\|_q + \|\nabla v_0\|_r + \|u_0\|_q \leq M_0.$$  

For fixed $T > 0$, we define the Banach space as follows

$$X_T := \{(n, v, u) : n \in C((0, T]; L^q(\mathbb{R}^d)), \nabla v \in C((0, T]; L^r(\mathbb{R}^d)), u \in C((0, T]; L^p(\mathbb{R}^d))\}$$

endowed with the norm

$$\|(n, v, u)\|_{X_T} := \sup_{0 < t \leq T} \|n(t)\|_q + \sup_{0 < t \leq T} \|\nabla v(t)\|_r + \sup_{0 < t \leq T} \|u(t)\|_p,$$

and the closed subset $S$, which is denoted by

$$S := \{(n, v, u) \in X_T : \|(n, v, u)\|_{X_T} \leq 2M_0\}.$$

Then for $0 < t \leq T$, we introduce a mapping $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)(n, v, u)$ on $S$ by defining

\[
\begin{cases}
\mathcal{H}_1(n, v, u) = E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}})n_0 - \int_0^t (t - \tau)^{\beta-1}E_\beta\beta(-(t - \tau)^\beta(-\Delta)^{\frac{\alpha}{2}})(u \cdot \nabla n) \, d\tau \\
\mathcal{H}_2(n, v, u) = E_\beta(t^\beta(-(\Delta)^{\frac{\alpha}{2}} - \gamma))v_0 + \int_0^t (t - \tau)^{\beta-1}E_\beta\beta((t - \tau)^\beta(-(\Delta)^{\frac{\alpha}{2}} - \gamma))n \, d\tau \\
\mathcal{H}_3(n, v, u) = E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}})u_0 - \int_0^t (t - \tau)^{\beta-1}E_\beta\beta(-(t - \tau)^\beta(-\Delta)^{\frac{\alpha}{2}})[P((u \cdot \nabla)u)] \, d\tau \\
- \int_0^t (t - \tau)^{\beta-1}E_\beta\beta(-(t - \tau)^\beta(-\Delta)^{\frac{\alpha}{2}})[P(n \nabla \phi)] \, d\tau.
\end{cases}
\]  

(3.1)
Next, we intend to show that \( \mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) \) defined in (3.1) is a mapping from \( S \) to itself. For any fixed \((n, v, u) \in S\), by (2.13) in Proposition 2.1 and Hölder’s inequality, one has

\[
\| \nabla \cdot E_{\beta, \beta}(-t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}}(un) \|_q \leq C(t - \tau)^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| u \|_p \| n \|_q, \tag{3.2}
\]

and

\[
\| \nabla \cdot E_{\beta, \beta}(-t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}}(n \nabla v) \|_q \leq C(t - \tau)^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| n \nabla v \|_r \| n \|_q, \tag{3.3}
\]

provided \( 0 < \frac{1}{p} < \frac{2 \alpha - 1}{d} \) and \( 0 < \frac{1}{r} < \frac{2 \alpha - 1}{d} \).

For any \( 0 < t \leq T \), we deal with the first formula in (3.1) with the help of (2.12) in Proposition 2.1 (3.2) and (3.3) as follows

\[
\| \mathcal{H}_1(n, v, u) \|_q \leq \| E_{\beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}})n_0 \|_q + \int_0^t (t - \tau)^{-\beta - 1} \| \nabla \cdot E_{\beta, \beta}(-(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(un) \|_q d\tau
\]

\[
+ \int_0^t (t - \tau)^{-\beta - 1} \| \nabla \cdot E_{\beta, \beta}(-t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}}(n \nabla v) \|_q d\tau
\]

\[
\leq \| n_0 \|_q + C \int_0^t (t - \tau)^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| u \|_p \| n \|_q d\tau
\]

\[
+ C \int_0^t (t - \tau)^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| \nabla v \|_r \| n \|_q d\tau.
\]

Since \( p > \frac{d}{\alpha - 1} \) and \( r > \frac{d}{\alpha - 1} \) imply that \(-\frac{\alpha p - \delta}{\alpha p + \beta - 1} > 0 \) and \(-\frac{\alpha p - \delta}{\alpha p + \beta - 1} > 0 \), respectively, we then have

\[
\| \mathcal{H}_1(n, v, u) \|_q \leq \| n_0 \|_q + C T^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| u \|_{C((0,T ]; L^p(\mathbb{R}^d))} \| n \|_{C((0,T ]; L^q(\mathbb{R}^d))}
\]

\[
+ C T^{-\frac{\alpha p - \delta}{\alpha p + \beta - 1}} \| \nabla v \|_{C((0,T ]; L^r(\mathbb{R}^d))} \| n \|_{C((0,T ]; L^q(\mathbb{R}^d))}. \tag{3.4}
\]

Now, we will verify the time continuity of \( \mathcal{H}_1(n, v, u) \). Choose \( 0 < t < t + \delta \leq T \) for some \( \delta > 0 \), then one has

\[
\| \mathcal{H}_1(n, v, u)(t + \delta) - \mathcal{H}_1(n, v, u)(t) \|_q
\]

\[
\leq \| E_{\beta}(-(t + \delta)^\beta (-\Delta)^{\frac{\alpha}{2}})n_0 - E_{\beta}(-t^\beta (-\Delta)^{\frac{\alpha}{2}})n_0 \|_q
\]

\[
+ \int_t^{t+\delta} (t + \delta - \tau)^{\beta - 1} \| \nabla \cdot E_{\beta, \beta}(-(t + \delta - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(un + n \nabla v) \|_q d\tau
\]

\[
+ \int_0^t || ((t + \delta - \tau)^{\beta - 1} \nabla \cdot E_{\beta, \beta}(-(t + \delta - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})
\]

\[
- (t - \tau)^{\beta - 1} \nabla \cdot E_{\beta, \beta}(-(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})) (un + n \nabla v) \|_q d\tau
\]

\[
:= I_1 + I_2 + I_3.
\]
In terms of (2.21) in Proposition 2.2, the first term $I_1$ goes to zero as $\delta \to 0$.

For the second term $I_2$, in a similar fashion as the estimate of (3.4), by Proposition 2.1 and Hölder’s inequality, we see that

$$I_2 \leq C \int^{t+\delta}_t (t + \delta - \tau) \frac{d\alpha}{\alpha} \frac{\beta}{\beta + 1} \|n\|_p \|n\|_q d\tau + C \int^{t+\delta}_t (t + \delta - \tau) \frac{d\alpha}{\alpha} \frac{\beta}{\beta + 1} \|\nabla v\|_r \|n\|_q d\tau$$

$$\leq C \|n\|_{C((0,T];L^p(\mathbb{R}^d))} (\frac{d\alpha}{\alpha} \frac{\beta}{\beta + 1}) \|u\|_{C((0,T];L^p(\mathbb{R}^d))} + \frac{\delta}{\alpha} (\frac{d\alpha}{\alpha} \frac{\beta}{\beta + 1}) \|\nabla v\|_{C((0,T];L^r(\mathbb{R}^d))} ,$$

then it is not difficult to observe that $I_2$ goes to zero as $\delta \to 0$.

Regarding the third term $I_3$, we split it into two parts as follows

$$I_3 \leq \int^{t}_0 ((t + \delta - \tau)^{\beta - 1} - (t - \tau)^{\beta - 1}) \|\nabla \cdot E_{\beta,\beta}(-(t + \delta - \tau)^{\beta}(-\Delta)^{\frac{\beta}{2}})(un + n\nabla v) d\tau d\tau$$

$$+ \int^{t}_0 (t - \tau)^{\beta - 1} \|\nabla \cdot E_{\beta,\beta}(-(t + \delta - \tau)^{\beta}(-\Delta)^{\frac{\beta}{2}}) - \nabla \cdot E_{\beta,\beta}(-(t + \delta - \tau)^{\beta}(-\Delta)^{\frac{\beta}{2}})\| (un + n\nabla v) d\tau d\tau$$

$$:= I_{31} + I_{32},$$

For any $0 < \tau < t$, the fact $(t + \delta)^{-\frac{d\alpha}{\alpha} - \frac{\beta}{\beta}} < (t + \delta - \tau)^{-\frac{d\alpha}{\alpha} - \frac{\beta}{\beta}} < \delta^{-\frac{d\alpha}{\alpha} - \frac{\beta}{\beta}}$ yields that

$$- \int^{t}_0 (t + \delta - \tau)^{-\frac{d\alpha}{\alpha} - \frac{\beta}{\beta}} (t - \tau)^{\beta - 1} d\tau < -C(t + \delta)^{-\frac{d\alpha}{\alpha} - \frac{\beta}{\beta} t^\beta}.$$ Combining the above inequality with (3.2) and (3.3), we find that

$$I_{31} \leq C((t + \delta)^{\beta - \frac{\beta}{\beta} - \frac{d\alpha}{\alpha} - \delta^{\beta - \frac{\beta}{\beta} - \frac{d\alpha}{\alpha} - \delta^{\beta - \frac{\beta}{\beta} + \frac{d\alpha}{\alpha} - \frac{\beta}{\beta} t^\beta}}) \|n\|_{C((0,T];L^p(\mathbb{R}^d))} \|u\|_{C((0,T];L^p(\mathbb{R}^d))}$$

$$+ C((t + \delta)^{\beta - \frac{\beta}{\beta} - \delta^{\beta - \frac{\beta}{\beta} - \delta^{\beta - \frac{\beta}{\beta} + \frac{d\alpha}{\alpha} - \frac{\beta}{\beta} t^\beta}}) \|n\|_{C((0,T];L^p(\mathbb{R}^d))} \|\nabla v\|_{C((0,T];L^r(\mathbb{R}^d))} .$$

Therefore $I_{31}$ goes to zero as $\delta \to 0$.

As regards $I_{32}$, utilizing the estimates listed in (3.2) and (3.3), we derive that

$$I_{32} \leq C \int^{t}_0 (t - \tau)^{\beta - 1} \int^{\infty}_0 \beta s M_{\beta}(s) (\nabla \cdot K^\alpha s(t + \delta)^{\beta} - \nabla \cdot K^\alpha s(t + \delta)^{\beta}) (un + n\nabla v) ds d\tau$$

$$\leq C \int^{t}_0 (t - \tau)^{\beta - 1} \int^{\infty}_0 \beta s M_{\beta}(s) \|\nabla \cdot K_{s(t + \delta)^{\beta} - \nabla \cdot K_{s(t + \delta)^{\beta}}\|_{\alpha} \|u\|_p \|n\|_q d\tau$$

$$+ C \int^{t}_0 (t - \tau)^{\beta - 1} \int^{\infty}_0 \beta s M_{\beta}(s) \|\nabla \cdot K_{s(t + \delta)^{\beta} - \nabla \cdot K_{s(t + \delta)^{\beta}}\|_{\alpha} \|\nabla v\|_r \|n\|_q d\tau.$$

(3.6)

Recall that $K \in C^\infty((0, \infty) \times \mathbb{R}^d)$, $I_3$ goes to zero as $\delta \to 0$. Thus, (3.5)-(3.6) imply that $I_3$ goes to zero as $\delta \to 0$. At this point, this verifies that $H_1(n, v, u) \in C((0,T], L^q(\mathbb{R}^d))$. 

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We next estimate $\|\nabla H_2(n, v, u)\|_r$ and $\|H_3(n, v, u)\|_p$. Once again we employ (3.1), Proposition [2], Hölder’s inequality and the boundedness of the projection operator $P$ to obtain

\[
\|\nabla H_2(n, v, u)\|_r \leq \|\nabla E_\beta(t^\beta(-(-\Delta)^{\frac{\alpha}{2}}) - \gamma)v_0\|_r,
\]

\[
+ \int_0^t (t - \tau)^{\beta - 1}\|\nabla E_\beta,\beta((t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}}) - \gamma)(u \cdot \nabla v)\|_r \, d\tau
\]

\[
+ \int_0^t (t - \tau)^{\beta - 1}\|\nabla E_\beta,\beta((t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}}) - \gamma)n\|_r \, d\tau
\]

(3.7)

\[
\leq \|\nabla v_0\|_r + CT^{-\frac{d\beta}{\alpha}(-\frac{\alpha - 1}{p})}\|u\|_{C((0, T]; L^p(\mathbb{R}^d))}\|\nabla v\|_{C((0, T]; L^r(\mathbb{R}^d))}
\]

\[
+ CT^{-\frac{d\beta}{\alpha}(-\frac{\alpha - 1}{p})\frac{p}{q} + \beta}\|n\|_{C((0, T]; L^q(\mathbb{R}^d))},
\]

and

\[
\|H_3(n, v, u)\|_p \leq \|E_\beta(-t^\beta(-\nabla)^{\frac{\alpha}{2}})u_0\|_p + \int_0^t \int_0^t (t - \tau)^{\beta - 1}\|\nabla E_\beta,\beta((-t - \tau)^\beta(-\nabla)^{\frac{\alpha}{2}})[P(n\nabla \phi)]\|_p \, d\tau
\]

\[
+ \int_0^t (t - \tau)^{\beta - 1}\|\nabla \cdot E_\beta,\beta((-t - \tau)^\beta(-\nabla)^{\frac{\alpha}{2}})[P(u \otimes u)]\|_p \, d\tau
\]

(3.8)

\[
\leq \|u_0\|_p + C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} + \frac{\beta}{q} - 1}\|u\|^2 \, d\tau
\]

\[
+ C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha}(-\frac{1}{p} + \frac{\alpha - 1}{d})\frac{p}{q} + \beta}\|\nabla \phi\|_d \|n\|_q \, d\tau
\]

\[
\leq \|u_0\|_p + CT^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{\alpha - 1}{d})}\|u\|^2_{C((0, T]; L^p(\mathbb{R}^d))} + CT^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{\alpha - 1}{d})\frac{p}{q} + \beta}\|n\|_{C((0, T]; L^q(\mathbb{R}^d))},
\]

provided $p > \frac{d - 1}{\alpha - 1}$, $0 \leq \frac{1}{q} - \frac{1}{r} < \frac{\alpha - 1}{d}$ and $0 \leq \frac{1}{q} - \frac{1}{p} + \frac{1}{d} < \frac{\alpha - 1}{d}$.

On the basis of (3.7) and (3.8), the time continuity can be proved with the same argument as the proof of $H_1(n, v, u) \in C((0, T]; L^q(\mathbb{R}^d))$, which yields that $\nabla H_2(n, v, u) \in C((0, T]; L^r(\mathbb{R}^d))$ and $H_3(n, v, u) \in C((0, T]; L^p(\mathbb{R}^d))$.

Combining the inequalities (3.4), (3.7) and (3.8) leads to

\[
\|H_1(n, v, u)\|_q + \|\nabla H_2(n, v, u)\|_r + \|H_3(n, v, u)\|_p
\]

\[
\leq \|n_0\|_p + \|\nabla v_0\|_r + \|u_0\|_p + C_1 \left(T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})} + T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})}\right)\|n\|_{C((0, T]; L^q(\mathbb{R}^d))}
\]

\[
+ C_2 T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{\alpha - 1}{d})}\|\nabla v\|_{C((0, T]; L^r(\mathbb{R}^d))}\|n\|_{C((0, T]; L^q(\mathbb{R}^d))}
\]

\[
+ C_3 T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{\alpha - 1}{d})}\|u\|_{C((0, T]; L^p(\mathbb{R}^d))}\|n\|_{C((0, T]; L^q(\mathbb{R}^d))} + \|u\|_{C((0, T]; L^p(\mathbb{R}^d))}\|\nabla v\|_{C((0, T]; L^r(\mathbb{R}^d))}
\]

\[
+ \|u\|^2_{C((0, T]; L^p(\mathbb{R}^d))}
\]

\[
\leq M_0 + 2C_1 M_0 \left(T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})} + T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})}\right) + 4C_2 M_0^2 \left(T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})} + T^{-\frac{d\beta}{\alpha}(-\frac{1}{q} + \frac{1}{p} - \frac{\alpha - 1}{d})}\right).
\]
If we choose
\[ T_0 := \min\{8C_1 \|v\|_{L^p(\Omega)}, (8C_1)^{1/p} - 1, 16\tilde{C}_2 M_0 \} \]
it is easily to check that for 0 < t ≤ T_0,
\[ \|\mathcal{H}(n, v, u)\|_q + \|\nabla \mathcal{H}(n, v, u)\|_r + \|\mathcal{H}_3(n, v, u)\|_p \leq 2M_0. \]

Thus, we conclude that \( \mathcal{H}(n, v, u) : S \to S \).

Let \((n_1, v_1, u_1), (n_2, v_2, u_2) \in X_T\). Define
\[
D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2) := \sup_{0 \leq t \leq T} \|(n_1 - n_2)(t)\|_q + \sup_{0 < t \leq T} \|\nabla (v_1 - v_2)(t)\|_r + \sup_{0 < t \leq T} \|(u_1 - u_2)(t)\|_p.
\]
We proceed in the same way as the estimates of (3.4), (3.7) and (3.8) to obtain

\[ \|\mathcal{H}_1(n_1, v_1, u_1) - \mathcal{H}_1(n_2, v_2, u_2)\|_q \leq C T^{-\frac{d \beta}{\alpha}} \|u_1 - u_2\|_{C(0,T); L^p(\Omega)} \|n_1\|_{C(0,T); L^q(\Omega)} \]
\[ + C T^{-\frac{d \beta}{\alpha}} \|n_1 - n_2\|_{C(0,T); L^q(\Omega)} \|u_2\|_{C(0,T); L^p(\Omega)} \]
\[ + C T^{-\frac{d \beta}{\alpha}} \|n_1 - n_2\|_{C(0,T); L^q(\Omega)} \|\nabla v_1\|_{C(0,T); L^r(\Omega)} \]
\[ + C T^{-\frac{d \beta}{\alpha}} \|\nabla (v_1 - v_2)\|_{C(0,T); L^r(\Omega)} \|n_2\|_{C(0,T); L^q(\Omega)} \]
\[ \leq 2C_4 M_0 \left(T^{-\frac{d \beta}{\alpha}} \|u_1 - u_2\|_{C(0,T); L^p(\Omega)} + T^{-\frac{d \beta}{\alpha}} \|n_1 - n_2\|_{C(0,T); L^q(\Omega)} \right) D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2), \]

\[ \|\nabla (\mathcal{H}_2(n_1, v_1, u_1) - \mathcal{H}_2(n_2, v_2, u_2))\|_r \leq C T^{-\frac{d \beta}{\alpha}} \|u_1 - u_2\|_{C(0,T); L^p(\Omega)} \|\nabla v_1\|_{C(0,T); L^r(\Omega)} \]
\[ + C T^{-\frac{d \beta}{\alpha}} \|\nabla (v_1 - v_2)\|_{C(0,T); L^r(\Omega)} \|u_2\|_{C(0,T); L^p(\Omega)} \]
\[ + C T^{-\frac{d \beta}{\alpha}} \|\nabla (v_1 - v_2)\|_{C(0,T); L^r(\Omega)} \|n_2\|_{C(0,T); L^q(\Omega)} \]
\[ \leq C_5 \left(2M_0 T^{-\frac{d \beta}{\alpha}} \|u_1 - u_2\|_{C(0,T); L^p(\Omega)} + T^{-\frac{d \beta}{\alpha}} \|n_1 - n_2\|_{C(0,T); L^q(\Omega)} \right) D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2), \]

and

\[ \|\mathcal{H}_3(n_1, v_1, u_1) - \mathcal{H}_3(n_2, v_2, u_2)\|_p \leq C_6 \left(2M_0 T^{-\frac{d \beta}{\alpha}} \|u_1 - u_2\|_{C(0,T); L^p(\Omega)} + T^{-\frac{d \beta}{\alpha}} \|n_1 - n_2\|_{C(0,T); L^q(\Omega)} \right) D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2). \]

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Combining the above estimates (3.9)-(3.11), it holds that

\[
D_T(\mathcal{H}(n_1, v_1, u_1) - \mathcal{H}(n_2, v_2, u_2)) \leq \left( \tilde{C}_3 M_0 T^{-\frac{d\beta}{\alpha}}(1 - \frac{\alpha}{\alpha - 1}) + 2C_4 M_0 T^{-\frac{d\beta}{\alpha}}(1 - \frac{\alpha}{\alpha - 1}) \right) D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2) + C_5 T^{-\frac{d\beta}{\alpha}}(\frac{1}{\beta} - \frac{1}{\alpha - 1}) + C_6 T^{-\frac{d\beta}{\alpha}}(\frac{1}{\beta} - \frac{1}{\alpha - 1}) \right) D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2).
\]

Then choose \( T_1 \) to satisfy

\[
T_1 := \min\{T_0, \left( \frac{4\tilde{C}_3 M_0}{q} \right)^{\frac{\alpha}{\alpha - 1}}, \left( \frac{8C_4 M_0}{q} \right)^{\frac{\alpha}{\alpha - 1}}, \left( \frac{4C_5}{q} \right)^{\frac{\alpha}{\alpha - 1}}, \left( \frac{4C_6}{q} \right)^{\frac{\alpha}{\alpha - 1}} \},
\]

we conclude that \( \mathcal{H} : S \rightarrow S \) is a strict contraction mapping, that is there exists a constant \( 0 < \varrho < 1 \) such that

\[
D_T(\mathcal{H}(n_1, v_1, u_1) - \mathcal{H}(n_2, v_2, u_2)) \leq \varrho D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2).
\]

Therefore, it follows from Banach fixed point theorem that there exists \( (n, v, u) \in S \) such that \( (n, v, u) = \mathcal{H}(n, v, u) \) is the unique local mild solution to system (1.1).

Finally, we prove the claim regarding \( T_{\max} \) by contradiction. Assume that

\[
\limsup_{t \to T_{\max}} \|n(\cdot, t)\|_q < \infty, \quad \limsup_{t \to T_{\max}} \|\nabla v(\cdot, t)\|_r < \infty, \quad \limsup_{t \to T_{\max}} \|u(\cdot, t)\|_p < \infty.
\]

Following the same approach as we show \( \mathcal{H}_1(n, v, u) \in C((0, T]; L^q(\mathbb{R}^d)), \nabla \mathcal{H}_2(n, v, u) \in C((0, T]; L^r(\mathbb{R}^d)) \) and \( \mathcal{H}_3(n, v, u) \in C((0, T]; L^p(\mathbb{R}^d)) \), for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( T_{\max} - \delta < t_1 < t_2 < T_{\max} \),

\[
\|n(t_1) - n(t_2)\|_q < \epsilon, \quad \|\nabla v(t_1) - \nabla v(t_2)\|_r < \epsilon, \quad \|u(t_1) - u(t_2)\|_p < \epsilon.
\]

Hence, we can define \( n(T_{\max}), v(T_{\max}) \) and \( u(T_{\max}) \) such that

\[
n \in C((0, T_{\max}); L^q(\mathbb{R}^d)), \nabla v \in C((0, T_{\max}); L^r(\mathbb{R}^d)), u \in C((0, T_{\max}); L^p(\mathbb{R}^d)).
\]

Now we consider

\[
\begin{align*}
\tilde{n} &= E_\beta(-(T_{\max} + t)\beta(-\Delta)^\frac{\beta}{2})n_0 \\
&\quad - \int_0^{T_{\max}} (T_{\max} + t - \tau)^{\beta - 1} \nabla \cdot E_{\beta, \beta}(-(T_{\max} + t - \tau)^\beta(-\Delta)^\frac{\beta}{2})(un + n\nabla v) \, d\tau \\
&\quad - \int_0^t (t - \tau)^{\beta - 1} \nabla \cdot E_{\beta, \beta}(-(t - \tau)^\beta(-\Delta)^\frac{\beta}{2})(\tilde{n} + \tilde{n}\nabla \tilde{v}) \, d\tau \\
&:= \tilde{n}_1 + \tilde{n}_2 + \tilde{n}_3,
\end{align*}
\]
For some \( \delta \), then we see that (3.12)-(3.14) has a unique mild solution \((\hat{n}, C)\).

Repeating what has been just done, it holds that

\[
\begin{align*}
\hat{v} &= E_\beta((T_{\max} + t)^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma))v_0 \\
&\quad - \int_0^{T_{\max}} (T_{\max} + t - \tau)^{\beta-1} E_{\beta,\beta}((T_{\max} + t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma))(u \cdot \nabla v - n) \, d\tau \\
&\quad - \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma))(\hat{u} \cdot \nabla \hat{v} - \hat{n}) \, d\tau \\
&= \hat{v}_1 + \hat{v}_2 + \hat{v}_3,
\end{align*}
\]

\[
\hat{u} = E_\beta(-(T_{\max} + t)^\beta(-\Delta)^{\frac{\alpha}{2}})u_0 \\
&\quad - \int_0^{T_{\max}} (T_{\max} + t - \tau)^{\beta-1} E_{\beta,\beta}(-(T_{\max} + t - \tau)^\beta(-\Delta)^{\frac{\alpha}{2}})[P((u \cdot \nabla)u + n\nabla \phi)] \, d\tau \\
&\quad - \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(-(t - \tau)^\beta(-\Delta)^{\frac{\alpha}{2}})[P((\hat{u} \cdot \nabla)\hat{u} + \hat{n}\nabla \phi)] \, d\tau \\
&= \hat{u}_1 + \hat{u}_2 + \hat{u}_3.
\]

For some \( \delta_1 > 0 \), employ (2.21) and (2.22) in Proposition 2.2 to deduce that

\[
\hat{n}_1 \in C((0, \delta_1]; L^q(\mathbb{R}^d)), \quad \nabla \hat{v}_1 \in C((0, \delta_1]; L^r(\mathbb{R}^d)), \quad \hat{u}_1 \in C((0, \delta_1]; L^p(\mathbb{R}^d)).
\]

Following the same approach as we show \( \mathcal{H}_1(n, v, u) \in C((0, T]; L^q(\mathbb{R}^d)), \ \nabla \mathcal{H}_2(n, v, u) \in C((0, T]; L^r(\mathbb{R}^d)) \), \( \mathcal{H}_3(n, v, u) \in C((0, T]; L^p(\mathbb{R}^d)) \), we find that

\[
\hat{n}_2 \in C((0, \delta_1]; L^q(\mathbb{R}^d)), \quad \nabla \hat{v}_2 \in C((0, \delta_1]; L^r(\mathbb{R}^d)), \quad \hat{u}_2 \in C((0, \delta_1]; L^p(\mathbb{R}^d)).
\]

Repeating what has been just done, it holds that

\[
\hat{n}_3 \in C((0, \delta_1]; L^q(\mathbb{R}^d)), \quad \nabla \hat{v}_3 \in C((0, \delta_1]; L^r(\mathbb{R}^d)), \quad \hat{u}_3 \in C((0, \delta_1]; L^p(\mathbb{R}^d)).
\]

Then we see that (3.12)-(3.14) has a unique mild solution \((\hat{n}, \hat{v}, \hat{u})\) with the property

\[
\hat{n} \in C((0, \delta_1]; L^q(\mathbb{R}^d)), \quad \nabla \hat{v} \in C((0, \delta_1]; L^r(\mathbb{R}^d)), \quad \hat{u} \in C((0, \delta_1]; L^p(\mathbb{R}^d)).
\]

For \( t \in (0, \delta_1) \), if we define

\[
n(T_{\max} + t) = \hat{n}(t), \quad v(T_{\max} + t) = \hat{v}(t), \quad u(T_{\max} + t) = \hat{u}(t),
\]

then the fact that \((\hat{n}, \hat{v}, \hat{u})\) is a mild solution on \((0, T_{\max} + \delta_1)\) contradicts with the definition of \( T_{\max} \).

\[
\square
\]
3.2. Proof of Theorem 1.2

Proof. **Step 1.** (Two functional spaces $\mathcal{X}, \mathcal{Y}$ and the map $F$) In this proof, we define

$$
\mathcal{X} := \{ [n_0, v_0, u_0, \phi] \in L^{\frac{d}{\alpha+2}} (\mathbb{R}^d), \nabla v_0 \in L^{\frac{d}{\alpha-1}} (\mathbb{R}^d), u_0 \in L^{\frac{d}{\alpha-1}} (\mathbb{R}^d), \nabla \phi \in L^d (\mathbb{R}^d) \}
$$

with the norm

$$
\| [n_0, v_0, u_0, \phi] \|_{\mathcal{X}} := \| n_0 \|_{L^{\frac{d}{\alpha+2}}} + \| \nabla v_0 \|_{L^{\frac{d}{\alpha-1}}} + \| u_0 \|_{L^{\frac{d}{\alpha-1}}} + \| \nabla \phi \|_d,
$$

and

$$
\mathcal{Y} := \{ [n, v, u] \in C([0, \infty); L^q (\mathbb{R}^d)), \frac{d}{\alpha} \nabla v \in C((0, \infty); L^p (\mathbb{R}^d)),
\frac{d}{\alpha} u \in C((0, \infty); L^p (\mathbb{R}^d)) \}
$$

with the norm

$$
\| [n, v, u] \|_{\mathcal{Y}} := \sup_{0 < t < \infty} \frac{d}{\alpha} \| n(t) \|_q + \sup_{0 < t < \infty} \frac{d}{\alpha} \| \nabla v(t) \|_r + \sup_{0 < t < \infty} \frac{d}{\alpha} \| u(t) \|_r.
$$

It is clear that the space $\mathcal{X}$ and $\mathcal{Y}$ equipped with the norm $\| \cdot \|_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{Y}}$ are Banach spaces.

For $0 < t < \infty$, $[n_0, v_0, u_0, \phi] \in \mathcal{X}$ and $[n, v, u] \in \mathcal{Y}$, we define the map $F$ as follows

$$
F(n_0, v_0, u_0, \phi, n, v, u) := [\tilde{n}, \tilde{v}, \tilde{u}],
$$

\begin{align*}
\tilde{n}(t) &= n(t) - E_\beta (-t^\beta (\Delta)^{\frac{\alpha}{2}}) n_0 + \int_0^t (t - \tau)^{\beta - 1} E_\beta,\beta \left( -(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} \right) (u \cdot \nabla n) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta - 1} \nabla \cdot E_\beta,\beta \left( -(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} \right) (n \nabla v) \, d\tau,
\end{align*}

\begin{align*}
\tilde{v}(t) &= v(t) - E_\beta (t^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma) v_0 - \int_0^t (t - \tau)^{\beta - 1} E_\beta,\beta \left( (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma \right) n \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta - 1} E_\beta,\beta \left( (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma \right) (u \cdot \nabla v) \, d\tau,
\end{align*}

\begin{align*}
\tilde{u}(t) &= u(t) - E_\beta (-t^\beta (\Delta)^{\frac{\alpha}{2}}) u_0 + \int_0^t (t - \tau)^{\beta - 1} E_\beta,\beta \left( -(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} \right) \\
&\quad \times \left[ P((u \cdot \nabla) u + n \nabla \phi) \right] \, d\tau.
\end{align*}

(3.15)

**Step 2.** ($F : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ is a continuous map) To begin with, we show that $\frac{d}{\alpha} \tilde{n} \in$
Based on (3.15), we estimate $\|\tilde{n}\|_q$ as follows.

$$
\|\tilde{n}(t)\|_q \leq \|n(t)\|_q + \|E_\beta(-t^\beta(-\Delta)^{\frac{q}{2}})n_0\|_q \\
+ \int_0^t (t-\tau)^{\beta-1}\|\nabla \cdot E_{\beta, \beta}(-(t-\tau)^\beta(-\Delta)^\frac{q}{2})(un)\|_q d\tau \\
+ \int_0^t (t-\tau)^{\beta-1}\|\nabla \cdot E_{\beta, \beta}(-(t-\tau)^\beta(-\Delta)^\frac{q}{2}) (n\nabla v)\|_q d\tau \\
:= A_0 + A_1 + A_2 + A_3.
$$

(3.16)

Now, applying the $L^p - L^q$ estimates of Mittag-Leffler operators in Proposition 2.1, we estimate $A_1, A_2$ and $A_3$, respectively. For the term $A_1$, recalling (2.12), if $q > \frac{d}{2\alpha - 2}$, we arrive at

$$
A_1 \leq C t^{-\frac{d\alpha}{\beta}(\frac{2\alpha - 2}{d} - \frac{1}{q})}\|n_0\|_{\frac{q}{2\alpha - 2}}.
$$

(3.17)

As for $A_2$, based on (2.18) and Hölder’s inequality, we obtain

$$
A_2 \leq C \int_0^t (t-\tau)^{-\frac{d\alpha}{\beta}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) - \frac{\beta}{\alpha} + \beta - 1}\|un(\tau)\|_q d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{d\alpha}{\beta} + \frac{1}{\alpha} + \frac{\beta - 1}{\alpha}} \frac{d\alpha}{\beta}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) \frac{d\alpha}{\beta}\left(\frac{\alpha - 1}{2} - \frac{1}{q}\right)\|u(\tau)\|_p \tau^{\frac{d\alpha}{\beta}\left(\frac{\alpha - 1}{2} - \frac{1}{q}\right)}\|n(\tau)\|_q d\tau \\
\leq C \sup_{\tau > 0} \tau^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|u(\tau)\|_p \sup_{\tau > 0} \tau^{\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|n(\tau)\|_q \\
\times t^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}B\left(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta, \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha q} - \frac{3\beta(\alpha - 1)}{\alpha} + 1\right),
$$

(3.18)

where conditions (I), (II), (III), (IV) and (V) in Assumption 1 ensure that

$$
-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta > 0, \quad \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha q} - \frac{3\beta(\alpha - 1)}{\alpha} + 1 > 0.
$$

Hence, we deduce that the Beta function $B\left(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta, \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha q} - \frac{3\beta(\alpha - 1)}{\alpha} + 1\right) \leq C$, then (3.18) satisfies the following estimates

$$
A_2 \leq Ct^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)} \sup_{\tau > 0} \tau^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|u(\tau)\|_p \sup_{\tau > 0} \tau^{\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|n(\tau)\|_q.
$$

(3.19)

As regards $A_3$, utilizing (2.18) and Hölder’s inequality again, we derive

$$
A_3 \leq C \int_0^t (t-\tau)^{-\frac{d\alpha}{\beta}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) - \frac{\beta}{\alpha} + \beta - 1}\|(n\nabla v)(\tau)\|_q d\tau \\
\leq C \int_0^t (t-\tau)^{-\frac{d\alpha}{\beta} + \frac{1}{\alpha} + \frac{\beta - 1}{\alpha}} \frac{d\alpha}{\beta}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) \frac{d\alpha}{\beta}\left(\frac{\alpha - 1}{2} - \frac{1}{q}\right)\|n(\tau)\|_q \tau^{\frac{d\alpha}{\beta}\left(\frac{\alpha - 1}{2} - \frac{1}{q}\right)}\|\nabla v(\tau)\|_r d\tau \\
\leq C \sup_{\tau > 0} \tau^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|\nabla v(\tau)\|_r \sup_{\tau > 0} \tau^{\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}\|n(\tau)\|_q \\
\times t^{-\frac{d\alpha}{\beta}\left(\frac{2\alpha - 2}{d} - \frac{1}{q}\right)}B\left(-\frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \beta, \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{3\beta(\alpha - 1)}{\alpha} + 1\right).
$$

(3.20)
Since conditions (I), (II), (III), (IV) and (V) in Assumption 1 imply that
\[ -\frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \beta > 0, \quad \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{3\beta(\alpha - 1)}{\alpha} + 1 > 0, \]
we have
\[ A_3 \leq Ct\frac{d\beta}{\alpha} \left( \frac{2\alpha - 2}{d - 1} \right) \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| \nabla v(\tau) \|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| n(\tau) \|_q. \tag{3.21} \]

Combining the above estimates (3.17), (3.19) and (3.21), one has
\[ \sup_{t > 0} t^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| \tilde{n}(t) \|_q \leq C \| n_0 \|_p \sup_{t > 0} t^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| n(t) \|_q \]
\[ \times \left( 1 + \sup_{t > 0} t^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| u(t) \|_p + \sup_{t > 0} t^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \| \nabla v(t) \|_r \right). \tag{3.22} \]

The time continuity can be proved with the same argument as the proof of Theorem 1.1 then we claim that
\[ t^{\frac{d\beta}{\alpha}} \left( \frac{2\alpha - 2}{d - 1} \right) \tilde{n} \in C((0, \infty); L^q(\mathbb{R}^d)). \]

In views of (3.15), \( \| \nabla \tilde{v}(t) \|_r \) can be written as
\[ \| \nabla \tilde{v}(t) \|_r \leq \| \nabla v(t) \|_r + \| E_\beta(t^\beta (- \Delta)^{\frac{\alpha}{2}} - \gamma)) \nabla v_0 \|_r \]
\[ + \int_0^t (t - \tau)^{\beta - 1} \| \nabla E_{\beta, \beta} \left( (t - \tau)^\beta (- \Delta)^{\frac{\alpha}{2}} - \gamma \right) \left( u \cdot \nabla v \right)(\tau) \|_r d\tau \]
\[ + \int_0^t (t - \tau)^{\beta - 1} \| \nabla E_{\beta, \beta} \left( (t - \tau)^\beta (- \Delta)^{\frac{\alpha}{2}} - \gamma \right) n(\tau) \|_r d\tau \]
\[ := Q_0 + Q_1 + Q_2 + Q_3. \tag{3.23} \]

For the term \( Q_1 \), by virtue of (2.12) in Proposition 2.1 it holds that for \( r > \frac{d}{\alpha - 1} \),
\[ Q_1 \leq Ct^{-\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} \| \nabla v_0 \|_\frac{d}{\alpha - 1} \tag{3.24} \]

As for \( Q_2 \), based on (2.19) and Hölder’s inequality, we calculate that
\[ Q_2 \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right) - \frac{\beta}{\alpha}} \| u \nabla v \|_\frac{d}{\alpha - 1} d\tau \]
\[ \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha}} \| u(\tau) \|_p \| \nabla v(\tau) \|_r d\tau \]
\[ \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha}} \tau^{\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right) - \frac{\beta}{\alpha}} \| u(\tau) \|_p \| \nabla v(\tau) \|_r d\tau \]
\[ \leq C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} \| u(\tau) \|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} \| \nabla v(\tau) \|_r \]
\[ \times t^{-\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} B \left( -\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha p} + \frac{d\beta}{\alpha r} - \frac{2\beta(\alpha - 1)}{\alpha} + 1 \right) \]
\[ \leq Ct^{-\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} \| u \|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left( \frac{\alpha - 1}{d - 1} \right)} \| \nabla v \|_r, \tag{3.25} \]
where we have used the fact
\[ B\left(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{2\beta(\alpha - 1)}{\alpha}\right) + 1 \leq C \]
due to conditions (I), (II), (III), (IV) and (V) in Assumption 1 to derive the last inequality of (3.25). Regarding \( Q_3 \), using (2.19) and Hölder’s inequality, we get
\[
Q_3 \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|n(\tau)\|_q d\tau
\]
\[
\leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} + \frac{d\beta}{\alpha q} + \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} - \frac{2\beta(\alpha - 1)}{\alpha}} \|n(\tau)\|_q d\tau
\]
\[
\leq C t^{-\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|n(\tau)\|_q
\]
\[ \leq C t^{-\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|n(\tau)\|_q, \tag{3.26} \]

where Assumption 1 ensure that \( B\left(-\frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} - \frac{2\beta(\alpha - 1)}{\alpha}\right) + 1 \leq C \).

Thus, (3.24), (3.25) and (3.26) lead to the following estimate,
\[
\sup_{t > 0} t^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|\nabla \tilde{v}\|_r \leq C \|\nabla v_0\|_{\frac{d}{\alpha - 1}} + \sup_{t > 0} \tau^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|\nabla v(\tau)\|_r
\]
\[ + C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|u(\tau)\|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|\nabla v(\tau)\|_r
\]
\[ + C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|n(\tau)\|_q. \tag{3.27} \]

The time continuity of \( t^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \nabla \tilde{v} \) can be verified similarly as the proof of Theorem 1.1.

Therefore, we derive that
\[ t^{\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \nabla \tilde{v} \in C((0, \infty); L'(\mathbb{R}^d)). \]

Next, we shall estimate \( \|\tilde{u}\|_p \) with the aid of (3.15),
\[
\|\tilde{u}(t)\|_p \leq \|u(t)\|_p + \|E_\beta(-t^{\beta}(-\Delta)^{\frac{\beta}{2}})u_0\|_p
\]
\[ + \int_0^t (t - \tau)^{\beta - 1} \|\nabla \cdot E_\beta(\sigma (-\Delta)^{\frac{\sigma}{2}})(u \otimes u)(\tau)\|_p d\tau
\]
\[ + \int_0^t (t - \tau)^{\beta - 1} \|E_\beta(\sigma (-\Delta)^{\frac{\sigma}{2}})(n \nabla \phi)(\tau)\|_p d\tau \tag{3.28} \]
\[ := B_0 + B_1 + B_2 + B_3. \]

To estimate the term \( B_1 \), for \( p > \frac{d}{\alpha - 1} \), Proposition 2.1 leads to
\[ B_1 \leq C t^{-\frac{d\beta}{\alpha} + \frac{\beta}{\alpha} - \frac{2\beta(\alpha - 1)}{\alpha}} \|u_0\|_{\frac{d}{\alpha - 1}}. \tag{3.29} \]
For the term $B_2$, due to conditions (I), (II), (III), (IV) and (V) in Assumption 1, it holds

$$B(\beta - \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha}; 2d\beta \frac{\alpha}{\alpha p} - \frac{2\beta(\alpha - 1)}{\alpha} + 1) \leq C.$$  

Combining the boundedness of the projection operator $P$, (2.18) in Proposition 2.1 and Hölder’s inequality, we obtain

$$B_2 \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} \frac{2}{p} \frac{1}{2} + \frac{\beta - 1}{\alpha}} \|P(u \otimes u)(\tau)\|_p \ d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} + \beta - 1} \|u(\tau)\|_p \ d\tau$$

$$\leq C t^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a}} \|P(u \otimes u)(\tau)\|_p \ d\tau$$

$$\leq C t^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a}} \|P(u \otimes u)(\tau)\|_p \ d\tau$$

(3.30)

As regards $B_3$, the boundedness of the projection operator $P$, (2.14) and Hölder’s inequality indicate that

$$B_3 \leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a}} \|P(n\nabla \phi)(\tau)\|_q \ d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a}} \|n(\tau)\|_q \ |\nabla \phi|_d \ d\tau$$

$$\leq C t^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a}} \|P(n\nabla \phi)(\tau)\|_q \ |\nabla \phi|_d$$

(3.31)

Since conditions (I), (II), (III), (IV) and (V) in Assumption 1 imply that

$$-\frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta > 0, \quad \frac{d\beta}{\alpha q} = \frac{2\beta(\alpha - 1)}{\alpha} + 1 > 0,$$

then we have

$$B_3 \leq C t^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a} \frac{1}{a}} \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \frac{(2\alpha - 2)}{a} \frac{1}{a}} \|n(\tau)\|_q \ |\nabla \phi|_d.$$  

(3.32)

Combine (3.29), (3.30) and (3.32) to yield

$$\sup_{t > 0} \frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a} \frac{1}{a} \|u\|_p \leq C \|u_0\|_d + C \|\nabla \phi\|_d \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \frac{(2\alpha - 2)}{a} \frac{1}{a}} \|n(\tau)\|_q$$

$$+ C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a} \frac{1}{a}} \|u(\tau)\|_p (1 + \sup_{t > 0} \tau^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a} \frac{1}{a}} \|u(\tau)\|_p).$$  

(3.33)

Hence it follows that

$$t^{\frac{d\beta}{\alpha} \frac{(\alpha - 1)}{a} \frac{1}{a}} \bar{u} \in C((0, \infty); L^p(\mathbb{R}^d)).$$
Then, from (3.22), (3.27) and (3.33), we conclude that $F(n_0, v_0, u_0, \phi, n, v, u) \in \mathcal{Y}$ with
\[\|F(n_0, v_0, u_0, \phi, n, v, u)\|_\mathcal{Y} \leq C\|[n_0, v_0, u_0, \phi]\|_\mathcal{X} + C\|[n, v, u]\|_\mathcal{Y}(1 + \|[n, v, u]\|_\mathcal{Y} + \|\nabla \phi\|_d).

**Step 3.** (The map $F(n_0, v_0, u_0, \phi, \cdot, \cdot, \cdot)$ is of class $C^1$ from $\mathcal{Y}$ into itself) For each $[n, v, u] \in \mathcal{Y}$, we define a linear map $L_{[n,v,u]}(\bar{n}, \bar{v}, \bar{u}) = [\tilde{N}, \tilde{V}, \tilde{U}]$ on $\mathcal{Y}$ by
\[
\begin{aligned}
\tilde{N}(t) &= \bar{n}(t) + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}}(u \cdot \nabla \bar{n} + \bar{u} \cdot \nabla n) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} \nabla \cdot E_{\beta,\beta}(-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}}((\bar{n} \nabla v + n \nabla \bar{v})) \, d\tau,
\end{aligned}
\]
\[
\begin{aligned}
\tilde{V}(t) &= \bar{v}(t) + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}} - \gamma)((\bar{u} \cdot \nabla v + u \cdot \nabla \bar{v})) \, d\tau \\
&\quad - \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}} - \gamma)) \bar{n} \, d\tau,
\end{aligned}
\]
\[
\begin{aligned}
\tilde{U}(t) &= \bar{u}(t) + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})\beta \cdot \nabla \cdot P((\bar{u} \cdot \nabla) u + (u \cdot \nabla) \bar{u}) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})P(\bar{n} \nabla \phi) \, d\tau.
\end{aligned}
\]
We intend to show that for each fixed $[n_0, v_0, u_0, \phi] \in \mathcal{X}$, $L_{[n,v,u]}$ is the Fréchet derivative of $F(n_0, v_0, u_0, \phi, n, v, u)$ at $[n, v, u] \in \mathcal{Y}$. We define $[N, V, U]$ by
\[
[N, V, U] := F(n_0, v_0, u_0, \phi, n + \bar{n}, v + \bar{v}, u + \bar{u}) - F(n_0, v_0, u_0, \phi, n, v, u) - L_{[n,v,u]}(\bar{n}, \bar{v}, \bar{u}).
\]
In view of (3.15) and (3.34), it is not difficult to find that
\[
\begin{aligned}
N(t) &= n(t) + \bar{n}(t) - E_{\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}})n_0 \\
&\quad + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})((u + \bar{u}) \cdot \nabla (n + \bar{n})) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} \nabla \cdot E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})((n + \bar{n}) \nabla (v + \bar{v})) \, d\tau \\
&\quad - (n(t) - E_{\beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}})n_0 + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})(u \cdot \nabla n) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} \nabla \cdot E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})(u \nabla v) \, d\tau \\
&\quad - (\bar{n}(t) + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})(u \cdot \nabla \bar{n} + \bar{u} \cdot \nabla n) \, d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} \nabla \cdot E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})(\bar{n} \nabla v + n \nabla \bar{v}) \, d\tau \\
&\quad = \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}})(\bar{u} \cdot \nabla \bar{n} + n \nabla \bar{v}) \, d\tau.
\end{aligned}
\]
Due to the similar estimates in (3.18) and (3.20), one obtains

\[
\|N(t)\|_q = \| \int_0^t (t - \tau)^{3 - 1} E_{\beta, \beta}(- (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(\bar{u} \cdot \nabla \bar{n}) \, d\tau \|_q \\
+ \| \int_0^t (t - \tau)^{3 - 1} \nabla \cdot E_{\beta, \beta}(- (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(\bar{n} \nabla \bar{v}) \, d\tau \|_q \\
\leq C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{p}\right)} \|\bar{u}(\tau)\|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} \|\bar{n}(\tau)\|_q \\
\times t^{-\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} B \left(- \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} - \frac{3\beta(\alpha - 1)}{\alpha} + 1\right) \\
+ C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{p}\right)} \|\nabla \bar{v}(\tau)\|_r \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} \|\bar{n}(\tau)\|_q \\
\times t^{-\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} B \left(- \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{3\beta(\alpha - 1)}{\alpha} + 1\right).
\]

By (3.15) and (3.34), it holds that

\[
\nabla V(t) = \int_0^t (t - \tau)^{3 - 1} \nabla E_{\beta, \beta}((t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}} - \gamma)(\bar{u} \cdot \nabla \bar{v})(\tau) \, d\tau
\]

with

\[
\|\nabla V(t)\|_r \leq C \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{p}\right)} \|\bar{u}(\tau)\|_p \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}(\tau)\|_r \\
\times t^{-\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{q}\right)} B \left(- \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{2\beta(\alpha - 1)}{\alpha} + 1\right).
\]

In the same way as above, we see from (3.15) and (3.34) that

\[
U(t) = \int_0^t (t - \tau)^{3 - 1} E_{\beta, \beta}(-(t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}}) P((\bar{u} \cdot \nabla \bar{n})) \,(\tau) \, d\tau,
\]

and it holds

\[
\|U(t)\|_p \leq C t^{-\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{p}\right)} B \left(- \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{2d\beta}{\alpha p} - \frac{2\beta(\alpha - 1)}{\alpha} + 1\right) \\
\times \left( \sup_{\tau > 0} \tau^{\frac{d\beta}{\alpha} \left(\frac{\alpha - 1}{\alpha} - \frac{1}{p}\right)} \|\bar{u}(\tau)\|_p \right)^2.
\]

Therefore, (3.36), (3.38), (3.40) yield that for each \([n_0, v_0, u_0, \phi] \in \mathcal{X}\) and each \([n, v, u] \in \mathcal{Y}\)

\[
\lim_{\|[n, v, u]\|_\mathcal{Y} \to 0} \frac{\|[N, V, U]\|_\mathcal{Y}}{\|[\bar{n}, \bar{v}, \bar{u}]\|_\mathcal{Y}} = \lim_{\|[n, v, u]\|_\mathcal{Y} \to 0} \frac{\|[F(n_0, v_0, u_0, \phi, n + \bar{n}, v + \bar{v}, u + \bar{u}) - F(n_0, v_0, u_0, \phi, n, v, u) - \mathcal{L}[n, v, u](\bar{n}, \bar{v}, \bar{u})]\|_\mathcal{Y}}{\|[\bar{n}, \bar{v}, \bar{u}]\|_\mathcal{Y}} = 0.
\]
This implies that the Fréchet derivative of $F$ at point $[n_0, v_0, u_0, \phi, n, v + \bar{v}, u + \bar{u}] \in X \times Y$ in the direction to $[n, v, u]$ is equal to $L_{[n,v,u]}(\bar{n}, \bar{v}, \bar{u})$.

**Step 4.** (Fréchet derivative $L_{[n,v,u]}(\bar{n}, \bar{v}, \bar{u})$ at $[n, v, u] = [0, 0, 0]$ is a bijective mapping) From Step 3, for $[\bar{n}, \bar{v}, \bar{u}] \in Y$, we have an expression $L_{[0,0,0]}(\bar{n}, \bar{v}, \bar{u}) = [\bar{N}_0, \bar{V}_0, \bar{U}_0]$ as

$$
\bar{N}_0(t) = \bar{n}(t), \quad \bar{V}_0(t) = \bar{v}(t) - \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-1)^{\frac{1}{2}} - \gamma)) \bar{\alpha}(\tau) d\tau, \quad \bar{U}_0(t) = \bar{u}(t).
$$

Hence it is easy to see that $\bar{N}_0 = \bar{V}_0 = \bar{U}_0 = 0$ implies that $\bar{n} = \bar{v} = \bar{u} = 0$, which suggests that $L_{[0,0,0]}$ is injective.

On the other hand, for every $[\bar{N}_0, \bar{V}_0, \bar{U}_0] \in Y$, we may take $[\bar{n}, \bar{v}, \bar{u}] \in Y$ as

$$
\bar{n}(t) = \bar{N}_0(t), \quad \bar{v}(t) = \bar{V}_0(t) + \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((-1)^{\frac{1}{2}} - \gamma)) \bar{\alpha}(\tau) d\tau, \quad \bar{u}(t) = \bar{U}_0(t),
$$

so that

$$
L_{[0,0,0]}(\bar{n}, \bar{v}, \bar{u}) = [\bar{N}_0, \bar{V}_0, \bar{U}_0].
$$

Therefore, we prove that $L_{[0,0,0]}$ is surjective from $Y$ onto itself.

**Step 5.** (Existence and uniqueness) Now, it follows from Banach implicit function theorem that there is a $C^1$-map $g$ defined as

$$
g : X_M \to Y_M,
$$

and there exists some $M > 0$ such that

$$
g(0, 0, 0, 0) = [0, 0, 0],
$$

$$
F(n_0, v_0, u_0, \phi, g(n_0, v_0, u_0, \phi)) = [0, 0, 0], \quad \forall [n_0, v_0, u_0, \phi] \in X_M,
$$

where

$$
X_M := \{[n_0, v_0, u_0, \phi] \in X : \|[n_0, v_0, u_0, \phi]\|_X < M\},
$$

$$
Y_M := \{[n, v, u] \in Y : \|[n, v, u]\|_Y < M\}.
$$

Thus, the existence and uniqueness of mild solutions to system (1.1) are the consequences of the existence of the continuously differential map $g$ from $X_M$ to $Y_M$. 

**4. Existence of mild solutions in fractional homogeneous Sobolev spaces**

In this section, we investigate the existence of the solution to (1.1) in fractional homogeneous Sobolev spaces. The methods of proving the local and global existence result are still Banach fixed point theorem and Banach implicit function theorem. The proofs in this section are similar to Section 3.1 and Section 3.2.
4.1. Proof of Theorem 1.3

Proof. In this proof, we proceed as the same argument with the proof of Theorem 1.1 using Banach fixed point theorem. According to the assumptions in Theorem 1.3, the initial data \( n_0 \in H^{\mu,q}(\mathbb{R}^d), \nabla v_0 \in H^{\mu,r}(\mathbb{R}^d) \) and \( u_0 \in H^{\mu,p}(\mathbb{R}^d) \), there exists a constant \( \mathcal{M} > 0 \) such that the initial data satisfy

\[
\|n_0\|_{H^{\mu,q}} + \|\nabla v_0\|_{H^{\mu,r}} + \|u_0\|_{H^{\mu,p}} \leq \mathcal{M}.
\]

Define the Banach space

\[
X_{\mu,T} := \{(n, v, u) : n \in C((0, T]; H^{\mu,q}(\mathbb{R}^d)), \nabla v \in C((0, T]; H^{\mu,r}(\mathbb{R}^d)), u \in C((0, T]; H^{\mu,p}(\mathbb{R}^d))\}
\]

with the norm given by

\[
\|(n, v, u)\|_{X_{\mu,T}} := \sup_{0 < t \leq T} \|n(t)\|_{H^{\mu,q}} + \sup_{0 < t \leq T} \|\nabla v(t)\|_{H^{\mu,r}} + \sup_{0 < t \leq T} \|u(t)\|_{H^{\mu,p}},
\]

and consider the closed subset \( \tilde{S} \), which is defined by

\[
\tilde{S} := \{(n, v, u) \in X_{\mu,T} : \|(n, v, u)\|_{X_{\mu,T}} \leq 2\mathcal{M}\}.
\]

The mapping \( \mathcal{H} \) is the one given in (3.1).

In light of Lemma 2.3, it is obvious that

\[
\|(-\Delta)^{\frac{\beta}{2}}(un)\|_{\frac{\mu+q}{\mu+r}} \leq C \left( \|(-\Delta)^{\frac{\beta}{2}}u\|_{\mu} \|n\|_{q} + \|(-\Delta)^{\frac{\beta}{2}}n\|_{q} \|u\|_{\rho} \right), \tag{4.1}
\]

\[
\|(-\Delta)^{\frac{\beta}{2}}(n\nabla v)\|_{\frac{\mu+q}{\mu+r}} \leq C \left( \|(-\Delta)^{\frac{\beta}{2}}\nabla v\|_{\mu} \|n\|_{q} + \|(-\Delta)^{\frac{\beta}{2}}n\|_{q} \|\nabla v\|_{r} \right).
\]

In views of the definition of \( \mathcal{H}_1(n, v, u) \) listed in (3.1), Proposition 2.1, the above inequality (4.1) and Lemma 2.1, we evaluate that for any \( 0 < t \leq T \):

\[
\|\mathcal{H}_1(n, v, u)\|_{H^{\mu,q}} \leq \left\{ \begin{array}{ll}
C \|E_{\beta}(-t^\beta(-\Delta)^{\frac{\beta}{2}})(un)\|_q \\
+ \int_0^t (t - \tau)^{\frac{\beta}{2} - 1} \|\nabla \cdot E_{\beta,\beta}(-t - \tau)^{\beta}(-\Delta)^{\frac{\beta}{2}}(-\Delta)^{\frac{\beta}{2}}(un)\|_q d\tau \\
+ \int_0^t (t - \tau)^{\frac{\beta}{2} - 1} \|\nabla \cdot E_{\beta,\beta}(-t - \tau)^{\beta}(-\Delta)^{\frac{\beta}{2}}(-\Delta)^{\frac{\beta}{2}}(n\nabla v)\|_q d\tau \end{array} \right. \tag{4.2}
\]

\[
\leq \|n_0\|_{H^{\mu,q}} + CT \frac{d\beta}{\alpha} \left( \frac{1}{p} - \frac{\alpha - 1}{d} \right) \|u\|_{C([0,T]; H^{\mu,p}(\mathbb{R}^d))} \|n\|_{C([0,T]; H^{\mu,q}(\mathbb{R}^d))} \|\nabla v\|_{C([0,T]; H^{\mu,r}(\mathbb{R}^d))} + CT \frac{d\beta}{\alpha} \left( \frac{1}{r} - \frac{\alpha - 1}{d} \right) \|n\|_{C([0,T]; H^{\mu,q}(\mathbb{R}^d))},
\]

here the last inequality holds since the conditions (1)-(4) in Theorem 1.3 ensure that

\[
- \frac{d\beta}{\alpha} \left( \frac{1}{p} - \frac{\alpha - 1}{d} \right) > 0, \quad - \frac{d\beta}{\alpha} \left( \frac{1}{r} - \frac{\alpha - 1}{d} \right) > 0.
\]

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The inequality (4.2) together with (3.4) in Section 3.1 yields that

\[
\| \mathcal{H}_1(n, v, u) \|_{H^{\mu,q}} \leq \| n_0 \|_{H^{\mu,q}} + C T^{-\frac{d\alpha}{\alpha - 1}} \| u \|_{C((0,T];H^{\mu,p}(\mathbb{R}^d))} \| n \|_{C((0,T];H^{\mu,q}(\mathbb{R}^d))}
\]
\[
+ C T^{-\frac{d\alpha}{\alpha - 1}} \| \nabla v \|_{C((0,T];H^{\mu,r}(\mathbb{R}^d))} \| n \|_{C((0,T];H^{\mu,q}(\mathbb{R}^d))}.
\]

Similarly, by virtue of (3.1), Proposition 2.1, Lemma 2.5 and Lemma 2.1 together with (3.7)
in Section 3.1, we obtain that

\[
\| \mathcal{H}_2(n, v, u) \|_{H^{\mu,r}} \leq \| \nabla v_0 \|_{H^{\mu,r}} + C T^{-\frac{d\alpha}{\alpha - 1}} \| n \|_{C((0,T];H^{\mu,q}(\mathbb{R}^d))}
\]
\[
+ C T^{-\frac{d\alpha}{\alpha - 1}} \| u \|_{C((0,T];H^{\mu,p}(\mathbb{R}^d))} \| \nabla v \|_{C((0,T];H^{\mu,r}(\mathbb{R}^d))}.
\]

In terms of the boundedness of the operator \( P \), (2.18) in Proposition 2.1 and Hölder’s inequality, we have

\[
\int_0^t (t - \tau)^{\beta - 1} \| (-\Delta)^{\frac{\beta}{2}} E_{\beta,\beta}(- (t - \tau)^{\beta} (-\Delta)^{\frac{\beta}{2}}) [P(n \nabla \phi)] \|_p \, d\tau
\]
\[
\leq C \int_0^t (t - \tau)^{-\frac{d\alpha}{\alpha + 1} + \beta} \| \nabla \phi \|_d \| n \|_q \, d\tau
\]
\[
\leq C T^{-\frac{d\alpha}{\alpha + 1} + \beta} \| n \|_{C((0,T];L^q(\mathbb{R}^d))},
\]
provided \( 0 \leq \frac{1}{q} - \frac{1}{p} + \frac{1 + \mu}{d} < \frac{\alpha - 1}{\alpha} \). Therefore, it holds that

\[
\| \mathcal{H}_3(n, v, u) \|_{\dot{H}^{\mu,p}} \leq \| (-\Delta)^{\frac{\beta}{2}} E_{\beta,\beta}(- (t - \tau)^{\beta} (-\Delta)^{\frac{\beta}{2}}) u_0 \|_p
\]
\[
+ \int_0^t (t - \tau)^{\beta - 1} \| (-\Delta)^{\frac{\beta}{2}} \nabla \cdot E_{\beta,\beta}(- (t - \tau)^{\beta} (-\Delta)^{\frac{\beta}{2}}) [P(u \otimes u)] \|_p \, d\tau
\]
\[
+ \int_0^t (t - \tau)^{\beta - 1} \| (-\Delta)^{\frac{\beta}{2}} E_{\beta,\beta}(- (t - \tau)^{\beta} (-\Delta)^{\frac{\beta}{2}}) [P(n \nabla \phi)] \|_p \, d\tau
\]
\[
\leq \| u_0 \|_{\dot{H}^{\mu,p}} + C T^{-\frac{d\alpha}{\alpha + 1} + \frac{1 + \mu}{d}} \| u \|_{C((0,T];H^{\mu,p}(\mathbb{R}^d))}^2
\]
\[
+ C T^{-\frac{d\alpha}{\alpha + 1} + \frac{1 + \mu}{d}} \| n \|_{C((0,T];L^q(\mathbb{R}^d))}^2.
\]

Hence combining the above inequality (4.5) and (3.8) in Section 3.1, we find

\[
\| \mathcal{H}_3(n, v, u) \|_{H^{\mu,p}} \leq \| u_0 \|_{H^{\mu,p}} + C T^{-\frac{d\alpha}{\alpha - 1}} \| u \|_{C((0,T];H^{\mu,p}(\mathbb{R}^d))}^2
\]
\[
+ C(T^{-\frac{d\alpha}{\alpha + 1} + \frac{1 + \mu}{d}} + T^{-\frac{d\alpha}{\alpha - 1} - \frac{2 + \alpha}{\alpha} + \beta} + T^{-\frac{d\alpha}{\alpha - 1} - \frac{2 + \alpha}{\alpha} + \beta}) \| n \|_{C((0,T];H^{\mu,q}(\mathbb{R}^d))}.
\]

In light of (4.3), (4.4) and (4.6), if we choose \( \tilde{T}_0 \) satisfy

\[
\tilde{T}_0 := \min \{(8\tilde{C}_1)^{\frac{\alpha_{qr}}{\alpha_{qr} - (\alpha - 1)\mu q}}, (16\tilde{C}_1)^{\frac{\alpha_{qp}}{\alpha_{qp} - (\alpha - 1)\mu pq}}, (16\tilde{C}_1)^{\frac{\alpha_{rq}}{\alpha_{rq} - (\alpha - 1)\mu r}}, (16\tilde{C}_2\mathcal{M})^{\frac{\alpha_{pq}}{\alpha_{pq} - (\alpha - 1)\mu q}}, (16\tilde{C}_2\mathcal{M})^{\frac{\alpha_{pq}}{\alpha_{pq} - (\alpha - 1)\mu r}}\},
\]

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it is not difficult to find that \( \mathcal{H}(n, v, u) : \tilde{S} \to \tilde{S} \), that is
\[
\|\mathcal{H}_1(n, v, u)\|_{H^{\mu,q}} + \|\nabla\mathcal{H}_2(n, v, u)\|_{H^{\mu,r}} + \|\mathcal{H}_3(n, v, u)\|_{H^{\mu,p}} \leq 2M, \quad 0 < t \leq \tilde{T}_0.
\]

In the sequel, we shall calculate the distance estimates. For \((n_1, v_1, u_1), (n_2, v_2, u_2) \in X_{\mu,T}\), using similar estimates as (4.3), (4.4) and (4.6), it is clear that
\[
\|\mathcal{H}_1(n_1, v_1, u_1) - \mathcal{H}_1(n_2, v_2, u_2)\|_{H^{\mu,q}} \\
\leq CT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d})\|u_1 - u_2\|_{C((0,T]; H^{\mu,p}(R^d))} \|n_1\|_{C((0,T]; H^{\mu,q}(R^d))} \\
+ CT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d})\|n_1 - n_2\|_{C((0,T]; H^{\mu,q}(R^d))} \|u_2\|_{C((0,T]; H^{\mu,p}(R^d))} \\
+ CT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d})\|n_1 - n_2\|_{C((0,T]; H^{\mu,q}(R^d))} \|\nabla v_1\|_{C((0,T]; H^{\mu,r}(R^d))} \\
+ CT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d})\|\nabla v_1 - \nabla v_2\|_{C((0,T]; H^{\mu,r}(R^d))} \|n_2\|_{C((0,T]; H^{\mu,q}(R^d))} \\
\leq 2C_4M(T^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d}) + T^{-\frac{d\beta}{\alpha}}(\frac{1}{q} - \frac{a-1}{d}))D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2),
\]
and
\[
\|\mathcal{H}_3(n_1, v_1, u_1) - \mathcal{H}_3(n_2, v_2, u_2)\|_{H^{\mu,p}} \\
\leq CT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d})\|u_1 - u_2\|_{C((0,T]; H^{\mu,p}(R^d))} \|n_1\|_{C((0,T]; H^{\mu,q}(R^d))} + \|u_2\|_{C((0,T]; H^{\mu,p}(R^d))} \\
+ C\left(T^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d}) - \frac{a}{d} + \beta\right)\left(T^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d}) - \frac{a(q\beta)}{\alpha} + \beta\right)\|n_1 - n_2\|_{C((0,T]; H^{\mu,q}(R^d))} \\
\leq C_6(2MT^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d}) + T^{-\frac{d\beta}{\alpha}}(\frac{1}{q} - \frac{a-1}{d}) + T^{-\frac{d\beta}{\alpha}}(\frac{1}{q} - \frac{a-1}{d}))D_T(n_1 - n_2, v_1 - v_2, u_1 - u_2).
\]

Define
\[
D_{\mu,T}(n_1 - n_2, v_1 - v_2, u_1 - u_2) := \sup_{0 \leq t \leq T} \|n_1 - n_2(t)\|_{H^{\mu,q}} \sup_{0 < t \leq T} \|v_1 - v_2(t)\|_{H^{\mu,r}} \sup_{0 < t \leq T} \|u_1 - u_2(t)\|_{H^{\mu,p}},
\]
therefore,
\[
D_{\mu,T}(\mathcal{H}(n_1, v_1, u_1) - \mathcal{H}(n_2, v_2, u_2)) \leq \left(\tilde{C}_3M T^{-\frac{d\beta}{\alpha}}(\frac{1}{p} - \frac{a-1}{d}) + 2C_4MT^{-\frac{d\beta}{\alpha}}(\frac{1}{q} - \frac{a-1}{d})\right)D_{\mu,T}(n_1 - n_2, v_1 - v_2, u_1 - u_2).
\]
If we choose $\tilde{T}_1$ satisfy
\[
\tilde{T}_1 := \min\{T, \left(\frac{4C_3 \mathcal{M}}{\varrho}\right)^{\frac{1}{\alpha-d-(\alpha-1)p}}, \left(\frac{SC_4 \mathcal{M}}{\varrho}\right)^{\frac{1}{\alpha-(\alpha-1)r}}, \left(\frac{4C_5}{\varrho}\right)^{\frac{1}{\alpha-(\alpha-1)\varrho}}, \left(\frac{SC_6}{\varrho}\right)^{\frac{1}{\alpha-d-(\alpha-1)\varrho}}\},
\]
then for $0 < \varrho < 1$, we have
\[
D_{\mu,T}(\mathcal{H}(n_1, v_1, u_1) - \mathcal{H}(n_2, v_2, u_2)) \leq \varrho D_{\mu,T}(n_1 - n_2, v_1 - v_2, u_1 - u_2).
\]
The claim that $\mathcal{H} : \tilde{S} \to \tilde{S}$ is a strict contraction map can be verified as the proof of Theorem 1.1. The existence and uniqueness of mild solution to (1.1) in $\tilde{S}$ then follow from Banach fixed point theorem.

Following the approach of the proof of Theorem 1.1 the statement as regards $T_{\text{max}}$ can be obtained similarly, and we omit the details. \qed

4.2. Proof of Theorem 1.4

**Proof.** The proof of Theorem 1.4 is similar to that of Theorem 1.2. We consider Banach spaces $\mathcal{Y}_\mu$,
\[
\mathcal{Y}_\mu := \{[n, v, u] \mid \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) n \in C((0, \infty); L^q(\mathbb{R}^d)), \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) v \in C((0, \infty); L^r(\mathbb{R}^d)), \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) u \in C((0, \infty); H^{\mu,q}(\mathbb{R}^d)), \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) \nabla v \in C((0, \infty); H^{\mu,r}(\mathbb{R}^d)), \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) \nabla u \in C((0, \infty); H^{\mu,p}(\mathbb{R}^d))\}
\]
endowed with the norm
\[
\|[n, v, u]\|_{\mathcal{Y}_\mu} := \sup_{0 < t < \infty} \left\{ \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) ||n(t)||_q + \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) ||v(t)||_r + \frac{d\alpha}{d}(\frac{2\alpha-2-1}{d}) ||u(t)||_p \right\}
\]
\[
\leq \sup_{0 < t < \infty} \left\{ ||n(t)||_{H^{\mu,q}} + \left[ E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n_0 \right]_{H^{\mu,q}} + \int_0^t (t-\tau)^{-\beta-1} \left\| \nabla \cdot E_{\beta,\beta} \left(- (t-\tau)^\beta (-\Delta)^{\frac{\alpha}{2}} \right) (-\Delta)^{\frac{\alpha}{2}} (\nabla v) \right\|_q d\tau \right\} + \int_0^t (t-\tau)^{-\beta-1} \left\| \nabla \cdot E_{\beta,\beta} \left(- (t-\tau)^\beta (-\Delta)^{\frac{\alpha}{2}} \right) (-\Delta)^{\frac{\alpha}{2}} (n \nabla v) \right\|_q d\tau.
\]
By virtue of (2.12) in Proposition 2.1, it is obvious that if \( q > \frac{d}{2\alpha - 2 + \mu} \), it holds
\[
\| E_\beta(-t^\beta(-\Delta)^{\frac{\beta}{2}})n_0\|_{H^{\mu,q}} \leq Ct^{-\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)}} \| n_0 \| \cdot \frac{d}{2\alpha - 2 + \mu}.
\]

Proposition 2.1 and Lemma 2.5 yield that
\[
\int_0^t (t - \tau)^{-\frac{\beta}{\alpha}} \| \nabla \cdot E_{\beta,\beta} \left( - (t - \tau)^\beta (-\Delta)^{\frac{\beta}{2}} (un) \right) \|_q d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha} - \frac{\beta}{\alpha} - 1} \left( \| (\nabla \cdot u)\|_q \| n \|_q + \| (\nabla \cdot n)\|_q \| u \|_p \right) d\tau \\
\leq Ct^{-\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)}} B \left( \frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta, \frac{d\beta}{\alpha q} + \frac{1}{p}, \frac{1}{q}, \frac{1}{p} \right) - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1 \\
\times \left( \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{1}{q} - 1 \right)} \| u \|_{H^{\mu,p}} \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)} \| n \|_q \\
+ \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{1}{q} - 1 \right)} \| u \|_{H^{\mu,q}} \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)} \| n \|_q \right),
\]

where the exponents \( \alpha, \beta, \mu, p, q, r \) satisfy the conditions in Assumption 2, which imply that
\[
-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \beta > 0, \quad \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha p} - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1 > 0.
\]

Similarly, Proposition 2.1 and Lemma 2.5 lead to
\[
\int_0^t (t - \tau)^{-\frac{\beta}{\alpha}} \| \nabla \cdot E_{\beta,\beta} \left( - (t - \tau)^\beta (-\Delta)^{\frac{\beta}{2}} (n \nabla v) \right) \|_q d\tau \\
\leq C \int_0^t (t - \tau)^{-\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)}} B \left( \frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \beta, \frac{d\beta}{\alpha q} + \frac{1}{c}, \frac{1}{q}, \frac{1}{p} \right) - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1 \\
\times \left( \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{1}{q} - 1 \right)} \| \nabla v \|_{H^{\mu,r}} \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)} \| n \|_q \\
+ \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{1}{q} - 1 \right)} \| \nabla v \|_{H^{\mu,q}} \sup_{\tau > 0} \tau^\frac{d\beta}{\alpha \cdot \left( \frac{2\alpha - 2 + \mu}{d} - \frac{1}{q} \right)} \| n \|_q \right),
\]

where Assumption 2 ensure that
\[
-\frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \beta > 0, \quad \frac{d\beta}{\alpha q} + \frac{d\beta}{\alpha r} - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1 > 0.
\]
Hence we obtain that
\[ \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \tilde{n}(t) \|_{\dot{H}^{\mu,q}} \leq C \| u_0 \|_{\dot{H}^{\mu,p}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| n(t) \|_{\dot{H}^{\mu,q}} \]
\[ + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| u \|_{\dot{H}^{\mu,p}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| n \|_{q} \]
\[ + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| n \|_{\dot{H}^{\mu,q}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| n \|_{q} \]
(4.8)

Based on (3.15), we utilize Proposition 2.2 and Lemma 2.5 again to calculate that
\[ \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \nabla v(t) \|_{\dot{H}^{\mu,r}} \leq C \| \nabla v_0 \|_{\dot{H}^{\mu,r}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \nabla v \|_{\dot{H}^{\mu,r}} \]
\[ + C \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \nabla v \|_{\dot{H}^{\mu,r}} + C \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \nabla v \|_{\dot{H}^{\mu,r}} \]
(4.9)

and
\[ \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| \tilde{u}(t) \|_{\dot{H}^{\mu,p}} \leq C \| u_0 \|_{\dot{H}^{\mu,p}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| u(t) \|_{\dot{H}^{\mu,p}} \]
\[ + C \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| u \|_{\dot{H}^{\mu,p}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| u \|_{\dot{H}^{\mu,p}} \]
\[ + C \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| u \|_{\dot{H}^{\mu,p}} + \sup_{t > 0} t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \| n \|_{q} \| \nabla \phi \|_{d} \]
(4.10)

where \( p, q, r, \alpha, \beta, \mu \) in Assumption 2 satisfy
\[ \beta - \frac{d\beta}{\alpha} > 0, \quad \frac{d\beta}{\alpha} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1 > 0, \quad \beta - \frac{d\beta}{\alpha} \left( \frac{1}{q} - \frac{1}{r} \right) - \frac{\beta}{\alpha} > 0, \]
From (3.22), (3.27), (3.33) and (4.8)-(4.10), we conclude that

\[
\frac{d\beta}{\alpha p} + \frac{1}{p} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1 > 0,
\quad \frac{d\beta}{\alpha q} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1 > 0,
\quad 2\frac{d\beta}{\alpha p} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1 > 0,
\quad \frac{\beta - \frac{d\beta}{\alpha} \frac{1}{p}}{\alpha} - \frac{(1 + \mu)\beta}{\alpha} > 0.
\]

The time continuity follows in a similar approach as the argument in the proof of Theorem 1.1, then we have

\[
t^{\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \bar{n} \in C((0, \infty); \dot{H}^{\mu, q}(\mathbb{R}^d)),
\]

\[
t^{\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \nabla \bar{v} \in C((0, \infty); \dot{H}^{\mu, r}(\mathbb{R}^d)),
\]

\[
t^{\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \bar{u} \in C((0, \infty); \dot{H}^{\mu, p}(\mathbb{R}^d)).
\]

From (3.22), (3.27), (3.33) and (4.8)-(4.10), we conclude that \( F(n_0, v_0, u_0, \phi, n, v, u) \in \mathcal{Y}_\mu \) with

\[
\|F(n_0, v_0, u_0, \phi, n, v, u)\|_{\mathcal{Y}_\mu} \leq C \|[n_0, v_0, u_0, \phi]\|_X + C\|[n, v, u]\|_{\mathcal{Y}_\mu}(1 + \|[n, v, u]\|_{\mathcal{Y}_\mu} + \|\nabla \phi\|_d).
\]

Following what we have done before in the proof of Step 3 of Theorem 1.2, recall the definition of \( N(t), \nabla V(t) \) and \( U(t) \) listed in (3.35), (3.37) and (3.39), respectively, we arrive at

\[
\|N(t)\|_{\dot{H}^{\mu, q}} \leq C t^{-\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} B(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha} \frac{1}{q} \frac{1}{p} \frac{1}{r} - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1)
\times \left(\sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{n}\|_{\dot{H}^{\mu, q}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{n}\|_{q}
\right.
\]

\[
+ \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{n}\|_{\dot{H}^{\mu, q}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{n}\|_{p}
\]

\[
\left. + C t^{-\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} B(-\frac{d\beta}{\alpha r} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha} \frac{1}{q} \frac{1}{p} - \frac{(3\alpha - 3 + \mu)\beta}{\alpha} + 1)
\times \left(\sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{\dot{H}^{\mu, p}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{q}
\right.
\]

\[
\left. + \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{\dot{H}^{\mu, q}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{r}
\right)
\]

\[
\|\nabla V(t)\|_{\dot{H}^{\mu, r}} \leq C t^{-\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} B(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{d\beta}{\alpha} \frac{1}{q} \frac{1}{p} \frac{1}{r} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1)
\times \left(\sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{\dot{H}^{\mu, p}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{r}
\right.
\]

\[
\left. + C \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{\dot{H}^{\mu, q}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\nabla \bar{v}\|_{p}
\right),
\]

\[
\|U(t)\|_{\dot{H}^{\mu, p}} \leq C t^{-\frac{d\beta}{\alpha p} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} B(-\frac{d\beta}{\alpha p} - \frac{\beta}{\alpha} + \frac{2d\beta}{\alpha p} - \frac{(2\alpha - 2 + \mu)\beta}{\alpha} + 1)
\times \left(\sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{u}\|_{\dot{H}^{\mu, p}} \sup_{t > 0} t^{\frac{d\beta}{\alpha} \left(\frac{2\alpha - 2 + \mu}{\alpha} - \frac{1}{q}\right)} \|\bar{u}\|_{p}
\right).
\]
Therefore, (3.36), (3.38), (3.40) and (4.11)-(4.13) indicate that for each \([n_0, v_0, u_0, \phi] \in \mathcal{X}\) and each \([n, v, u] \in \mathcal{Y}_\mu\), it holds
\[
\lim_{\|\bar{n}, \bar{v}, \bar{u}\|_{\mathcal{Y}_\mu} \to 0} \|\bar{N}, \bar{V}, \bar{U}\|_{\mathcal{Y}_\mu} = 0,
\]
which implies that the Fréchet derivative of \(F\) at point \([n_0, v_0, \phi, n + \bar{n}, v + \bar{v}, u + \bar{u}] \in \mathcal{X} \times \mathcal{Y}_\mu\) in the direction to \([n, v, u]\) is equal to \(L_{[n,v,u]}(\bar{n}, \bar{v}, \bar{u})\).

Applying Banach implicit function theorem, we observe that, for some \(\bar{M} > 0\), there is a \(C^1\)-map \(\bar{g} : \mathcal{X}_{\bar{M}} \to \mathcal{Y}_{\mu, \bar{M}}\) such that
\[
\bar{g}(0, 0, 0, 0) = [0, 0, 0],
F(n_0, v_0, u_0, \phi, \bar{g}(n_0, v_0, u_0, \phi)) = [0, 0, 0], \quad \forall[n_0, v_0, u_0, \phi] \in \mathcal{X}_{\bar{M}},
\]
where
\[
\mathcal{X}_{\bar{M}} := \{[n_0, v_0, u_0, \phi] \in \mathcal{X} : \|[n_0, v_0, u_0, \phi]\|_{\mathcal{X}} < \bar{M}\},
\mathcal{Y}_{\mu, \bar{M}} := \{[n, v, u] \in \mathcal{Y}_{\mu} : \|[n, v, u]\|_{\mathcal{Y}_{\mu}} < \bar{M}\}.
\]
Thus, the existence and uniqueness of mild solution to system (1.1) are the consequence of the existence of the continuously differential map \(\bar{g}\) from \(\mathcal{X}_{\bar{M}}\) to \(\mathcal{Y}_{\mu, \bar{M}}\).

5. Properties of mild solutions

In this section, we explore the integrability of the local mild solution if \(n_0 \in L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d), v_0 \in L^1(\mathbb{R}^d), \nabla v_0 \in L^r(\mathbb{R}^d)\) and \(u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)\), and then obtain the mass conservation. In addition, we prove the decay estimates and stability of global mild solution as a byproduct of Theorem 1.2. At the end of this paper, we investigate the self-similar solution whenever taking \(\gamma = 0\) in system (1.1).

**Lemma 5.1.** Assume \(P(x, t), Q(x, t)\) to be those defined in (2.20), then \(P(x, t), Q(x, t)\) are nonnegative and integrable. In particular, it holds
\[
\int_{\mathbb{R}^d} P(x, t) \, dx = 1, \quad \int_{\mathbb{R}^d} Q(x, t) \, dx = \frac{1}{\Gamma(\beta)}.
\]

**5.1. Proof of Theorem 1.3**

**Proof.** In this proof, we consider the Banach space
\[
X^1_{T} := \{(n, v, u) : n \in C((0, T]; L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)), v \in C((0, T]; L^1(\mathbb{R}^d)), \nabla v \in C((0, T]; L^r(\mathbb{R}^d)), u \in C((0, T]; L^p(\mathbb{R}^d))\}
\]
with the norm given by
\[ \| (n, v, u) \|_{X_T^1} := \sup_{0 \leq t \leq T} \| n(t) \|_1 + \sup_{0 \leq t \leq T} \| n(t) \|_q + \sup_{0 \leq t \leq T} \| v(t) \|_1 + \sup_{0 \leq t \leq T} \| \nabla v(t) \|_r + \sup_{0 \leq t \leq T} \| u(t) \|_p, \]
and the closed subset \( S_1 := \{ (n, v, u) \in X_T^1 : \| (n, v, u) \|_{X_T^1} \leq 2M_1 \} \). The mapping \( \mathcal{H} \) is the one defined in (3.1).

Next for \( 0 < t \leq T \), we will estimate \( \| \mathcal{H}_1(n, v, u) \|_1 \), \( \| \mathcal{H}_2(n, v, u) \|_1 \) and \( \| \mathcal{H}_3(n, v, u) \|_1 \), respectively. Proposition 2.1 Hölder’s inequality and interpolation inequality imply that
\[
\| \mathcal{H}_1(n, v, u) \|_1 \leq \| E_{\beta}(-t^{\beta}(-\Delta)^{\frac{\beta}{2}}) n_0 \|_1 + \int_0^t (t-\tau)^{-\beta-1} \| \nabla \cdot E_{\beta,\beta}(-\tau)^{\beta}(-\Delta)^{\frac{\beta}{2}}(un) \|_1 \, d\tau \\
+ \int_0^t (t-\tau)^{-\beta-1} \| \nabla \cdot E_{\beta,\beta}(-\tau)^{\beta}(-\Delta)^{\frac{\beta}{2}}(n\nabla v) \|_1 \, d\tau \\
\leq \| n_0 \|_1 + C \int_0^t (t-\tau)^{-\alpha+\beta-1} \left( \| u \|_p \| n \|_q \right) \, d\tau \\
\leq \| n_0 \|_1 + C \int_0^t (t-\tau)^{-\alpha+\beta-1} \left( \| u \|_p \| n \|_q \right) \, d\tau \\
\leq \| n_0 \|_1 + CT^{-\beta+\beta} \| n \|_{C((0,T];L^q(\mathbb{R}^d))} \| n \|_{C((0,T];L^q(\mathbb{R}^d))} \| \nabla v \|_{C((0,T];L^q(\mathbb{R}^d))} \| \nabla v \|_{C((0,T];L^q(\mathbb{R}^d))},
\]
(5.1)
where \( 0 < \theta_1, \theta_2 < 1 \) satisfy \( 1 - \frac{1}{p} = 1 - \theta_1 + \frac{\theta_1}{q} \) and \( 1 - \frac{1}{r} = 1 - \theta_2 + \frac{\theta_2}{q} \).

With a similar fashion as (5.1), we have the following estimates
\[
\| \mathcal{H}_2(n, v, u) \|_1 \leq \| v_0 \|_1 + CT^{\beta} \| n \|_{C((0,T];L^q(\mathbb{R}^d))} \| n \|_{C((0,T];L^q(\mathbb{R}^d))} \\
+ CT^{-\alpha+\beta} \| u \|_{C((0,T];L^p(\mathbb{R}^d))} \| v \|_{C((0,T];L^1(\mathbb{R}^d))} \| \nabla v \|_{C((0,T];L^q(\mathbb{R}^d))},
\]
(5.2)
where \( 0 < \theta_3 = \frac{rd}{p(rd-d+r)} < 1 \).

Combining (3.3), (3.7), (3.8) and (5.1), (5.2), the claim that \( \mathcal{H} : S_1 \rightarrow S_1 \) is a strict contract mapping can be proved as the same argument with the proof of Theorem 4.1. Then the existence
\[
n \in C((0, T); L^1(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)), \quad v \in C((0, T); L^1(\mathbb{R}^d)), \\
\nabla v \in C((0, T); L^r(\mathbb{R}^d)), \quad u \in C((0, T]; L^p(\mathbb{R}^d)),
\]
can be performed similarly as we did in the proof of Theorem 4.1 in Section 3.1.

As soon as we have the integrability, that is
\[
\int_{\mathbb{R}^d} n(x, t) \, dx = \int_{\mathbb{R}^d} E_{\beta}(-t^{\beta}(-\Delta)^{\frac{\beta}{2}}) n_0(x) \, dx \\
- \int_{\mathbb{R}^d} \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(-\tau)^{\beta}(-\Delta)^{\frac{\beta}{2}} (\nabla \cdot (un + n\nabla v)) \, d\tau \, dx,
\]
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by Lemma 5.1, we then get
\[ \int_{\mathbb{R}^d} E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n_0(x) \, dx = \int_{\mathbb{R}^d} n_0(x) \, dx. \]

For any \( t > 0 \), we have \( un + n\nabla v \in C((0, t]; L^1(\mathbb{R}^d)) \) since \( (n, v, u) \in X^1_T \). By approximating \( un + n\nabla v \) with \( C^\infty_c((0, t] \times \mathbb{R}^d) \), we find
\[
\int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) (\nabla \cdot (un + n\nabla v)) \, d\tau \, dx = 0.
\]

Thus, we obtain the following mass conservation
\[
\int_{\mathbb{R}^d} n(x, t) \, dx = \int_{\mathbb{R}^d} n_0(x) \, dx.
\]

In addition, on the one hand, whenever \( \gamma = 0 \), we derive that
\[
\int_{\mathbb{R}^d} v(x, t) \, dx = \int_{\mathbb{R}^d} E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) v_0(x) \, dx
- \int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) (\nabla \cdot (uv)) \, d\tau \, dx \tag{5.3}
+ \int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n \, d\tau \, dx.
\]

In light of Lemma 5.1, we get
\[
\begin{align*}
\int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n(x, \tau) \, d\tau \, dx & = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} \int_{\mathbb{R}^d} n(x, \tau) \, dx \, d\tau \\
& = \frac{t^\beta}{\beta \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx.
\end{align*}
\]

Then (5.3) becomes
\[
\int_{\mathbb{R}^d} v(x, t) \, dx = \int_{\mathbb{R}^d} v_0(x) \, dx + \frac{t^\beta}{\beta \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx.
\]

On the other hand, whenever \( \gamma > 0 \), we obtain
\[
\begin{align*}
\int_{\mathbb{R}^d} v(x, t) \, dx & = \int_{\mathbb{R}^d} E_\beta(t^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) v_0(x) \, dx \\
& - \int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}((t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) (\nabla \cdot (uv)) \, d\tau \, dx \tag{5.4} \\
& + \int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}((t - \tau)^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) n \, d\tau \, dx.
\end{align*}
\]
Since the semigroup property of Mittag-Leffler functions and Lemma 5.1, one has
\[
\int_{\mathbb{R}^d} E_\beta(t^\beta(-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) v_0(x) \, dx = \int_{\mathbb{R}^d} E_\beta(-t^\beta E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma)) v_0(x) \, dx
\]
\[= E_\beta(-t^\beta) \int_{\mathbb{R}^d} v_0(x) \, dx. \]

Notice that \( E_{\beta,\beta}(z) = \beta E_\beta'(z) \) and \( E_\beta(0) = 1 \), then the following identities hold to be true.

\[
\int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}((t - \tau)^{\beta}(-(-\Delta)^{\frac{\alpha}{2}} - \gamma)) n(t) \, d\tau \, dx
\]
\[= \int_{\mathbb{R}^d} \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(-\gamma(t - \tau)^{\beta}) E_{\beta,\beta}(-t - \tau)^{\beta}(-\Delta)^{\frac{\alpha}{2}}) n(x, \tau) \, d\tau \, dx
\]
\[= \frac{1}{\gamma \Gamma(\beta)} \int_0^t -\gamma(t - \tau)^{\beta-1} E_\beta'(-\gamma(t - \tau)^{\beta}) d\tau \int_{\mathbb{R}^d} n_0(x) \, dx
\]
\[= \frac{1 - E_\beta(-\gamma t^\beta)}{\gamma \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx. \]

Thus, (5.4) becomes
\[
\int_{\mathbb{R}^d} v(x, t) \, dx = E_\beta(-\gamma t^\beta) \int_{\mathbb{R}^d} v_0(x) \, dx + \frac{1 - E_\beta(-\gamma t^\beta)}{\gamma \Gamma(\beta)} \int_{\mathbb{R}^d} n_0(x) \, dx.
\]

**Remark 5.1.** Due to technical reasons that the limitation of the relationship between \( p, q \) in \( L^p - L^q \) estimates of Mittag-Leffler operators and the nonlocal effect of fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\), we cannot obtain the integrability and mass conservation of global mild solution to system (1.1) in this paper.

5.2. Some properties to the global mild solutions

On the basis of Theorem 1.2, we can easily have the following corollaries.

**Corollary 5.1.** Suppose that \((n, v, u)\) is the mild solution to system (1.1) given by Theorem 1.2, which exhibits the following decay behaviors, that is, there exists \( C > 0 \) such that for any \( t > 0 \),
\[
t^{\frac{d\beta}{\alpha} - \frac{1}{q}} \|n(t) - E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n_0\|_q \leq C, \quad (5.5)
\]
\[
t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \|\nabla v(t) - \nabla E_\beta(t^\beta(-\Delta)^{\frac{\alpha}{2}} - \gamma) v_0\|_r \leq C, \quad (5.6)
\]
\[
t^{\frac{d\beta}{\alpha} - \frac{1}{p}} \|u(t) - E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) u_0\|_p \leq C. \quad (5.7)
\]
Proof. Obviously, on the basis of (2.11), the decay estimate (5.5), (5.6) and (5.7) can be obtained strictly by (3.18)-(3.20), (3.25)-(3.26) and (3.30)-(3.32) in the proof of Theorem 1.2, respectively.

Next, we are going to prove the global stability of mild solution in Theorem 1.2, under the initial disturbance and the perturbation of external forces.

**Corollary 5.2.** Let the exponents \( p, q, r \) be as in Assumption 1 and \( M \) be as in Theorem 1.2. Assume that two initial data \((n_0, v_0, u_0)\) and \((n'_0, v'_0, u'_0)\) and two gravitational potential \( \phi \) and \( \phi' \) satisfy that

\[
\|n_0\|_{\frac{d}{2\alpha-2}} + \|\nabla v_0\|_{\frac{d}{\alpha-1}} + \|u_0\|_{\frac{d}{\alpha-1}} + \|\nabla \phi\|_d < M. \tag{5.8}
\]

\[
\|n'_0\|_{\frac{d}{2\alpha-2}} + \|\nabla v'_0\|_{\frac{d}{\alpha-1}} + \|u'_0\|_{\frac{d}{\alpha-1}} + \|\nabla \phi'\|_d < M. \tag{5.9}
\]

Suppose that \((n, v, u)\) and \((n', v', u')\) are mild solutions of (1.1) given by Theorem 1.2 with \( (n_0, v_0, u_0)|_{t=0} = (n'_0, v'_0, u'_0)|_{t=0} \). For any \( \eta > 0 \), there is a constant \( \delta = \delta(d, p, q, r, \eta) \) such that

\[
\|n_0 - n'_0\|_{\frac{d}{2\alpha-2}} + \|\nabla v_0 - \nabla v'_0\|_{\frac{d}{\alpha-1}} + \|u_0 - u'_0\|_{\frac{d}{\alpha-1}} + \|\nabla \phi - \nabla \phi'\|_d < \delta, \tag{5.10}
\]

then we have

\[
\sup_{0 < t < \infty} t^{\frac{d}{\alpha} \left(\frac{2\alpha-2}{d} - \frac{1}{q}\right)} \|n(t) - n'(t)\|_q + \sup_{0 < t < \infty} t^{\frac{d}{\alpha} \left(\frac{\alpha-1}{d} - \frac{1}{r}\right)} \|\nabla v(t) - \nabla v'(t)\|_r
\]

\[
+ \sup_{0 < t < \infty} t^{\frac{d}{\alpha} \left(\frac{\alpha-1}{d} - \frac{1}{p}\right)} \|u(t) - u'(t)\|_p < \eta. \tag{5.11}
\]

Proof. Due to the proof of Theorem 1.2, we observe that \( g \) is a \( C^1 \)-map. Suppose that the initial data and gravitational potential \( n_0, v_0, u_0, \phi, n'_0, v'_0, u'_0, \phi' \) satisfy (5.8) and (5.9), respectively. By the continuity of \( g \), it holds that for any \( \eta > 0 \), there exists \( \delta = \delta(d, p, q, r, \eta) \) such that if

\[
\|[n_0, v_0, u_0, \phi] - [n'_0, v'_0, u'_0, \phi']\|_{X_M} \leq \delta,
\]

then

\[
\|[n, v, u] - [n', v', u']\|_{Y_M} \leq \eta.
\]

That completes the proof.

At the end of this section, for \( \gamma = 0 \), we prove the result in Theorem 1.6 with respect to the self-similar solution.
Proof. Based on (2.11), we derive the mild solution of (1.1) with $\gamma = 0$ as follows.

\[
\begin{aligned}
    \mathbf{v}(x, t) &= E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) v_0 - \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}( - (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(u \cdot \nabla n + \nabla \cdot (n \nabla v)) \, d\tau, \\
    \mathbf{u}(x, t) &= E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) u_0 - \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}( - (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})[P((u \cdot \nabla)u + n \nabla \phi)] \, d\tau.
\end{aligned}
\]

For initial data $(n_0, v_0, u_0)$ given in Theorem 1.2, we construct the following iterative sequence

\[
\begin{aligned}
    n_1 &= E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) n_0, \\
    v_1 &= E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) v_0, \\
    u_1 &= E_\beta(-t^\beta(-\Delta)^{\frac{\alpha}{2}}) u_0, \\
    n_{j+1} &= n_j - \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}( - (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(u_j \cdot \nabla n_j + \nabla \cdot (n_j \nabla v_j)) \, d\tau, \\
    v_{j+1} &= \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}( - (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})(u_j \cdot \nabla v_j - n_j) \, d\tau, \\
    u_{j+1} &= \int_0^t (t - \tau)^{\beta - 1} E_{\beta, \beta}( - (t - \tau)^\beta (-\Delta)^{\frac{\alpha}{2}})[P((u_j \cdot \nabla)u_j + n_j \nabla \phi)] \, d\tau.
\end{aligned}
\]

Recall that $K^\alpha f(x) = (K^\alpha f)(x) = t^{-\frac{d}{2}}(K(t^{-\frac{1}{2}} \cdot) * f)(x)$ in (2.8), by the assumption on homogeneity of initial data (1.8), it is easy to check that the following identities holds,

\[
\begin{aligned}
    \lambda^{\alpha-2}(K^\alpha(\lambda^2 t_0))(\lambda x) &= (K^\alpha(t_0))(x), \\
    (K^\alpha(\lambda^\alpha t_0))(\lambda x) &= (K^\alpha(t_0))(x), \\
    \lambda^{\alpha-1}(K^\alpha(\lambda^\alpha t_0))(\lambda x) &= (K^\alpha(t_0))(x).
\end{aligned}
\]

(5.13)

We shall infer that $n_j(x, t)$ are self-similar. First, for $j = 1$, with the help of (2.10) and (5.13), one has

\[
\begin{aligned}
    \lambda^{\alpha-2}n_1(\lambda x, \lambda^\frac{\alpha}{2} t) &= \lambda^{\alpha-2} \int_0^\infty M_\beta(s)(K^\alpha(\lambda^\alpha s^\beta) n_0)(\lambda x) \, ds \\
    &= \int_0^\infty M_\beta(s)(K^\alpha(s^\beta) n_0)(x) \, ds \\
    &= n_1(x, t).
\end{aligned}
\]

(5.14)
Proceeding by induction, for $j = 2, 3, \cdots$, we verify that $\lambda^{2\alpha-2}n_j(\lambda x, \lambda^{\frac{2}{\beta}} t) = n_j(x, t)$. Indeed,

$$
\begin{align*}
\lambda^{2\alpha-2}n_j(\lambda x, \lambda^{\frac{2}{\beta}} t) &= \lambda^{2\alpha-2} \int_0^t (t - \tau)^{\beta-1} \left( E_{\beta, \beta} \left( - \lambda^\alpha (t - \tau)^{\beta} (-\Delta)^{\frac{\alpha}{\beta}} \right)(u_{j-1} \cdot \nabla n_{j-1})(\lambda^{\frac{2}{\beta}} \tau) \right)(\lambda x) d\tau \\
&\quad + \lambda^{2\alpha-2} \int_0^t (t - \tau)^{\beta-1} \left( E_{\beta, \beta} \left( - \lambda^\alpha (t - \tau)^{\beta} (-\Delta)^{\frac{\alpha}{\beta}} \right) \right) \\
&\quad \times \left( \nabla \cdot (n_{j-1} \nabla v_{j-1}) \right)(\lambda^{\frac{2}{\beta}} \tau))(\lambda x) d\tau \\
&= \int_0^t (t - \tau)^{\beta-1} \left( E_{\beta, \beta} \left( - (t - \tau)^{\beta} (-\Delta)^{\frac{\alpha}{\beta}} \right)(u_{j-1} \cdot \nabla n_{j-1})(\tau) \right)(x) d\tau \\
&\quad + \int_0^t (t - \tau)^{\beta-1} \left( E_{\beta, \beta} \left( - (t - \tau)^{\beta} (-\Delta)^{\frac{\alpha}{\beta}} \right) \right) \left( \nabla \cdot (n_{j-1} \nabla v_{j-1}) \right)(\tau))(x) d\tau \\
&= n_j(x, t).
\end{align*}
$$

In a similar way, we obtain that for $j \in \mathbb{N}$,

$$
v_j(\lambda x, \lambda^{\frac{2}{\beta}} t) = v_j(x, t), \quad \lambda^{\alpha-1}u_j(\lambda x, \lambda^{\frac{2}{\beta}} t) = u_j(x, t).
$$

Since $[n, v, u]$ is the limit in $\mathcal{Y}$ of the sequence $[n_j, v_j, u_j]$, it follows that $[n, v, u]$ is the self-similar solution to system (1.1) with $\gamma = 0$.

\[\square\]

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