Note: Effect of localization on mean-field density of state near jamming

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(Dated: 7 August 2018)

PACS numbers: 05.20.-y, 61.43.Fs, 63.20.Pw

The perceptron is a mean-field model of the jamming transition. The model is simple enough to analytically determine the critical exponents of several physical quantities such as the contact number and gap distributions functions. The predicted exponents are the same of those of the hard spheres in the large dimension limit and also reasonably close to the numerical results in finite dimensions. The simplicity of the model also allows us to analytically calculate the density of state \( D(\omega) \), which is the distribution of the eigenvalue spectrum of the Hessian matrix. Near the jamming point, the model predicts for \( \omega \ll \omega_\text{iso} \)

\[
D(\omega) \sim \frac{\omega \sqrt{\omega^2 - \omega_\text{min}^2}}{\omega^2 + \omega_\text{min}^2} \theta(\omega - \omega_\text{min}).
\]

where \( \theta(x) \) is the Heaviside step function and

\[
\omega_\text{min}^2 = c_1 \delta z^2, \quad \omega_\text{iso}^2 = c_2 \delta z^2 - c_3 p.
\]

\( p \) is the pressure and \( \delta z = z - z_\text{iso} \) is the deviation of the contact number \( z \) from the isostatic value \( z_\text{iso} \). \( c_1, c_2 \) and \( c_3 \) are constants. Essentially the same result of eq. (1) is also obtained by the effective medium theory, except the trivial Debye modes. The mean-field perceptron model predicts \( \omega_\text{min} = 0 \) sufficiently near the jamming point, which means \( p = c_2 \delta z^2 / c_3 \). In this case, the scaling behavior of \( D(\omega) \) near the jamming point is

\[
\begin{aligned}
D(\omega) &\sim \begin{cases} 
\text{constant} & (\omega \gg \omega_*) \\
(\omega/\omega_*)^2 & (\omega \ll \omega_*)
\end{cases}, \\
(\omega/\omega_*)^4 & (\omega \ll \omega_\text{iso}),
\end{aligned}
\]

However, it has been revealed that the mean-field prediction of \( D(\omega) \) is inconsistent with the numerical result in finite dimensions. Recent numerical studies in finite dimensions show that, if one carefully removes the phonon mode that follows the Debye low \( \omega^{d-1} \), one obtains

\[
\begin{aligned}
D(\omega) &\sim \begin{cases} 
\text{constant} & (\omega \gg \omega_*) \\
(\omega/\omega_*)^2 & (\omega_\text{iso} \ll \omega \ll \omega_*)
\end{cases}, \\
(\omega/\omega_* + \omega_\text{iso})^4 & (\omega \ll \omega_\text{iso}),
\end{aligned}
\]

where \( \omega_\text{iso} \sim \delta z \) but the proportional constant is much smaller than that of \( \omega_* \). In this note, relying on a bit empirical argument, we reconcile the above discrepancy between the mean-field and finite dimensional results.

The reason of the discrepancy between the mean-field and finite dimensional results is twofold. (i) In finite dimensions, the system is not exactly marginally stable and the pre-stress is smaller than that required by the marginal stability, \( p < p_* \equiv c_2 \delta z^2 / c_3 \). We introduce the distance to the marginal stability as

\[
\varepsilon \equiv (p_* - p) / p_*.
\]

(ii) In finite dimensions, the eigenvectors for \( \omega < \omega_\text{ex0} \) are localized in space, not as in case of the mean-field model where all the modes are extended. This allows us to separate the system into several parts and each of them may have a different value of \( \varepsilon \). To express this fluctuation, we borrow a rather old idea by Gurevich et al. and more recently Ji et al., where they modeled the localized modes by the anharmonic oscillators with different frequencies and \( D(\omega) \) is calculated by summing up the contributions of them. Interestingly, with proper assumptions, this approach correctly reproduces the \( \omega^4 \) scaling for the small value of \( \omega \), though it is not clear how to apply it to the jamming transition. As in case of Ji et al., we consider the distribution function of \( \varepsilon, \mathcal{P}(\varepsilon) \), which is normalized so that \( \int_0^\infty d\varepsilon \mathcal{P}(\varepsilon) = 1 \). We set the small cutoff \( \varepsilon_\text{ex0} \ll 1 \) and assume that \( \mathcal{P}(\varepsilon) = O(\varepsilon_\text{ex0}^{-1}) \) for \( \varepsilon \lesssim \varepsilon_\text{ex0} \) and \( \mathcal{P}(\varepsilon) \sim 0 \) for \( \varepsilon \gg \varepsilon_\text{ex0} \). Then, the mean value of the density of state is calculated as

\[
D(\omega) = \int_0^\infty d\varepsilon \mathcal{P}(\varepsilon) D(\omega, \varepsilon).
\]

Below, using the above assumptions, we show that the scaling behavior of \( D(\omega) \), eq. (4), is correctly reproduced including the scaling factors \( \omega_* \).

We first discuss the scaling behavior in the low frequency limit, \( \omega^2 / p_* \ll \varepsilon_\text{ex0} \). Substituting \( p = (1-\varepsilon)p_* \) into eq. (4), and averaging over \( \varepsilon \), we obtain

\[
\begin{aligned}
D(\omega) &\sim \int_0^\infty d\varepsilon \mathcal{P}(\varepsilon) \frac{\varepsilon \sqrt{\omega^2 - c_3 \varepsilon p_*}}{\omega^2 + \omega_\text{iso}^2} \theta(\omega - \sqrt{c_3 \varepsilon p_*}) \\
&\sim \omega_*^{-3} \mathcal{P}(0) \int_0^\omega d\varepsilon \sqrt{\omega^2 - c_3 \varepsilon p_*} \\
&\sim \varepsilon_\text{ex0}^{-1} (\omega/\omega_*)^4.
\end{aligned}
\]

Defining \( \omega_\text{ex0} \equiv \sqrt{c_3 \varepsilon_\text{ex0} p_*} \), one can see that the above scaling is the same of that of the \( \omega \ll \omega_\text{ex0} \) regime of eq. (4). With the similar calculations, one can confirm that the scaling for \( \omega \gg \omega_\text{ex0} \) is unchanged from the mean-field result, eq. (3). Thus, we recovered the same scaling behaviors of eq. (4). Finally, for concreteness, in

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Fig. 1 we show the numerical result of $D(\omega)$ obtained by assuming $P(\varepsilon) = \varepsilon_{\text{ex0}}^{-1}e^{-\varepsilon/\varepsilon_{\text{ex0}}}$, where $\varepsilon_{\text{ex0}} = 10^{-3}$ and $c_1 = c_2 = c_3 = 1$. If one rescales $\omega$ by $\omega_*$, all the data are collapsed on a single curve as expected from eq. (1).

In summary, we discussed the effects of the localized modes on the density of state $D(\omega)$ by considering the probability distribution function of the proximity to the marginal stability $P(\varepsilon)$. Our calculation reproduces the finite dimensional numerical result near the jamming point, eq. (1). In particular, the theory successfully captures the novel $D(\omega) \sim \omega^4$ scaling including its pressure dependence of the pre-factor, see eq. (6). Note, the derivation of eq. (6) does not depend on the precise form of $P(\varepsilon)$. If $P(\varepsilon)$ is a finite and continuous function at $\varepsilon = 0$, one always gets $D(\omega) \sim \omega^4$ for small $\omega$. This may explain the robustness of the $\omega^4$ scaling against the different interaction potentials and dimension.\textsuperscript{6,8,12,13}

There are still several important points that deserve investigation. A tentative list is the following:

- The cutoff $\varepsilon_{\text{ex0}}$ is related to the average value of $\varepsilon$ as $\varepsilon_{\text{ex0}} \sim \int_0^\infty d\varepsilon P(\varepsilon)\varepsilon \equiv \langle \varepsilon \rangle$. It is reported that in the two dimensional packing near the jamming point, $\langle \varepsilon \rangle \approx 0.045$.\textsuperscript{9,10} We expect $\langle \varepsilon \rangle$ decreases with increasing the dimension since the localized modes are suppressed in high dimension.\textsuperscript{9,12,13} Its dimensional dependence deserves further investigation.

- The scaling of the lowest frequency is changed only if $P(\varepsilon)$ is not finite at $\varepsilon = 0$. For instance, when $P(\varepsilon) \sim A\varepsilon^{-\alpha}$ for small $\varepsilon$, eq. (6) is replaced by $D(\omega) \sim A(\omega/\omega_*)^{4-2\alpha}$. This can correspond to the configuration obtained by quenching from very high temperature. In this case, Lerner and Bouchbinder\textsuperscript{15} observe $D(\omega) \sim \omega^3$ with $\beta < 4$ suggesting that $\alpha > 0$. However, a more recent numerical result shows that $\beta = 4$ for wide range of the initial temperature,\textsuperscript{13} the initial temperature just affects the pre-factor. Further numerical investigations are necessary to determine which of the two scenarios is correct.

- We assume that the system can be divided into several sub-components. Since our theory does not take into account the interactions between the sub-components, the typical length scale of them should be much larger than the correlation length of the system. However, there are many different lengths have been proposed for the jamming transition and it is not very clear which length scale would be relevant to the current argument. The size of the localized excitation recently investigated by Shimada \textit{et al.} might be a promising candidate.

ACKNOWLEDGMENTS

We thank F. Zamponi, P. Urbani, A. Ikeda and H. Mizuno for kind discussions. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement n723955-GlassUniversality).

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