AKSZ-BV FORMALISM AND COURANT
ALGEBROID-INDUCED TOPOLOGICAL FIELD THEORIES.

DMITRY ROYTENBERG

ABSTRACT. We give a detailed exposition of the Alexandrov-Kontsevich-Schwarz-Zaboronsky superfield formalism using the language of graded manifolds. As a main illustrative example, to every Courant algebroid structure we associate canonically a three-dimensional topological sigma-model. Using the AKSZ formalism, we construct the Batalin-Vilkovisky master action for the model.

1. Intro and Brief History.

The standard procedure for quantizing classical field theories in the Lagrangian approach is by using the Feynman path integral. From the mathematical standpoint this is somewhat problematic, as it involves “integration” over the infinite-dimensional space of field configurations, on which no sensible measure has been found to exist. Nevertheless, the procedure can be made rigorous in the perturbative approach, provided the classical theory does not have too many symmetries (“too many” means, roughly speaking, an infinite-dimensional space). In the presence of these gauge symmetries, however, the procedure needs to be modified, as one has to integrate over the space of gauge-equivalence classes of field configurations. This can be accomplished by gauge-fixing (choosing a transversal slice to the gauge orbits), and in the late 60s Faddeev and Popov came up with an ingenious method of gauge-fixing the path integral by introducing extra “ghost” fields into the action functional. These ghosts, corresponding to generators of the gauge symmetries but of the opposite Grassman parity, were later incorporated into the cohomological approach of Bechi-Rouet-Stora and Tyutin (BRST), which is now considered the standard approach for quantizing gauge theories.

Unfortunately, the BRST approach fails in more complicated cases involving so-called “open” algebras of symmetries (that is, when the symmetries close under commutator bracket only modulo solutions of the classical field equations). In the early 80s Batalin and Vilkovisky [2] developed a generalization of the BRST procedure which allows, in principle, to handle symmetries of arbitrary complexity. The idea is again to extend the field space by auxiliary fields (known as higher-generation ghosts, antighosts and Lagrange multipliers), as well as their “conjugate antifields”. The field-antifield space has two canonical structures. The first is an odd symplectic form, such that the fields and their respective antifields are conjugate with respect to the corresponding odd Poisson bracket (·, ·) (the “antibracket”). The other is an odd second order differential operator $\Delta$ (the so-called “BV Laplacian”) compatible with the antibracket. The original action functional is extended to a functional $S$ (called the master action) on this odd symplectic supermanifold, obeying the so-called quantum master equation

$$(S, S) - 2i\hbar \Delta S = 0$$
The gauge-fixing is accomplished by choosing a Lagrangian submanifold, and the perturbative expansion is computed by evaluating the path integral over this Lagrangian submanifold. The generalized quantum BRST operator is
\[ Q = -i\hbar \Delta + (S, \cdot). \]

In practice the master action is computed by homological perturbation theory which involves calculating relations among the generators of the Euler-Lagrange ideal as well as the generators of the symmetries, relations among the relations, etc. This is known in homological algebra as the Koszul-Tate resolution, and can be very difficult to carry out in general. But in mid-90s Alexandrov, Kontsevich, Schwarz and Zaboronsky [1] (referred to as AKSZ from now on) found a simple and elegant procedure for constructing solutions to the classical master equation\(^1\):

\[ (S, S) = 0 \]

Their approach uses mapping spaces of supermanifolds equipped with additional structure. Here we use a slightly refined notion of differential graded (dg) manifold, which is a supermanifold equipped with a compatible integer grading and a differential. The grading is needed to keep track of the ghost number symmetry important in some applications. If the source \(N\) is a dg manifold with an invariant measure, and the target \(M\) is a dg symplectic manifold, then the space of superfields \(\text{Maps}(N, M)\) acquires a canonical odd symplectic structure; furthermore, any self-commuting hamiltonian on \(M\) gives rise to a solution of the classical master equation. In case \(N = T[1]N_0\) (corresponding to the algebra of differential forms on a smooth manifold \(N_0\)), one gets the master action for topological field theories on \(N_0\) associated to various structures on the target. In particular, AKSZ show that Witten’s A and B topological sigma-models are special cases of the AKSZ construction.

Cattaneo-Felder [4] and Park [8] further refined the AKSZ procedure by generalizing it to the case of manifolds with boundary, and produced new examples: Cattaneo and Felder studied the Poisson sigma-model [11] on the disk [3][4], while Park considered its higher-dimensional generalizations, the topological open p-branes.

The Poisson sigma model is the most general 2d TFT that can be obtained within the AKSZ framework (at least if the source is \(T[1]N_0\)), the reason being that, for two-dimensional \(N_0\), the symplectic form on the target must have degree 1, if ghost number symmetry is to be preserved. This immediately implies that the target is of the form \(M = T^*[1]M_0\), corresponding to the algebra of multivector fields on a manifold \(M_0\), with the symplectic form corresponding to the Schouten bracket. The differential structure on \(M\) is then necessarily given by a self-commuting bivector field on \(M_0\), i.e. a Poisson structure. The AKSZ procedure gives (the master action for) the Poisson sigma model. Witten’s A and B models can be obtained as special cases of this, when the Poisson tensor is invertible and there is a compatible complex structure.

Now, if we go one step further and consider three-dimensional \(N_0\), the AKSZ formalism requires a symplectic form of degree 2 on the target, and a self-commuting hamiltonian of degree 3. We have shown [9] that such structures are in canonical 1-1 correspondence with what is known as Courant algebroids. A Courant algebroid is given by specifying a bilinear operation on sections of a vector bundle \(E \to M_0\) with an inner product, satisfying certain natural properties. Thus the AKSZ procedure

\(^1\)which, for the cases they consider, also implies the quantum master equation.
yields a canonical 3d TFT associated to any Courant algebroid. Its classical action is given by

\[ S_0[X, A, F] = \int_{N_0} F_i dX^i + \frac{1}{2} A^a g_{ab} dA^b - A^a P^i_a (X) F_i + \frac{1}{6} T_{abc} (X) A^a A^b A^c \]

where the fields are the membrane world volume \( X : N_0 \to M_0 \), an \( X^* E \)-valued 1-form \( A \) and an \( X^* T^* M_0 \)-valued 2-form \( F \); \( g \) is the matrix of the inner product on \( E \), while \( P^i_a \) and \( T_{abc} \) are the structure functions of the Courant algebroid. This action has rather complicated symmetries, requiring the introduction of ghosts for ghosts; the master action contains terms up to degree 3 in antifields, involving up to 3rd derivatives of the structure functions of the Courant algebroid. Known special cases include the Chern-Simons theory (which is an ordinary gauge theory) and Park’s topological membrane.

The paper is organized as follows. Section 2 is a brief introduction to the general theory of differential graded manifolds. Section 3 explains the AKSZ formalism in detail. Section 4 contains a discussion of Courant algebroids and the associated closed membrane sigma model.

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Note. After these notes had been completed, we learned that Ikeda [7] had obtained the Courant algebroid-valued sigma model by examining consistent BV-deformations of the abelian Chern-Simons gauge theory coupled with a zero-dimensional BF theory, rather than by immediate application of AKSZ. Also, Hofman and Park [6][5] considered a generalization of the topological open membrane which takes values in a quasi-Lie bialgebroid [10], which is a Courant algebroid with a choice of a splitting.

2. Differential graded manifolds.

Here we collect the basic notions and fix the notation. The details can be found, for instance, in [12] or [2].

2.1. Graded manifolds.

**Definition 2.1.** A graded manifold \( M \) over base \( M_0 \) is a sheaf of \( \mathbb{Z} \)-graded commutative algebras \( \mathcal{C}(M) \) over a smooth manifold \( M_0 \) locally isomorphic to an algebra of the form \( C^\infty(U) \otimes S(V) \) where \( U \subset M \) is an open set, \( V \) is a graded vector space, and \( S(V) \) is the free graded-commutative algebra on \( V \). Such a local isomorphism is referred to as an affine coordinate chart on \( M \); the sheaf \( \mathcal{C}(M) \) is called the sheaf of polynomial functions on \( M \).

The generators of the algebra \( \mathcal{C}(U) \simeq C^\infty(U) \otimes S(V) \) are viewed as local coordinates on \( M \). Coordinate transformations are isomorphisms of algebras, hence in general non-linear. In what follows we will be mostly concerned with nonnegatively graded graded manifolds. In this case the transformation law for a coordinate of degree
$n$ is of the form $x^n_k = A^i_k (x_0) x^n_i + \text{(terms in coordinates of lower degrees, of total degree } n)$. This explains the word “affine” in the definition.

We have decomposition $\mathcal{C}(M) = \oplus_k \mathcal{C}^k(M)$ according to the degrees. Each $\mathcal{C}^k(M)$ is a sheaf of locally free $C^\infty(M_0)$-modules; in the nonnegative case $\mathcal{C}^\infty(M_0) = \mathcal{C}^0(M)$. We denote by $\mathcal{C}_k(M)$ the subsheaf of algebras generated by $\oplus_{i \leq k} \mathcal{C}^i(M)$. These form a filtration

$$\cdots \subset \mathcal{C}_0(M) \subset \mathcal{C}_1(M) \subset \mathcal{C}_2(M) \subset \cdots$$

which in the nonnegative case is bounded below by $\mathcal{C}_k(M) = \mathcal{C}^0(M) = \mathcal{C}^\infty(M_0)$. In this case there is a corresponding tower of fibrations of graded manifolds

$$(2.1) \quad M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

with $M$ being the projective limit of the $M_n$'s.

**Definition 2.2.** If $M = M_n$ for some $n$, we say $\deg(M) = n$. Otherwise we say $\deg(M) = \infty$.

**Notation 2.3.** Given a vector bundle $A \to M_0$, denote by $A[n]$ the graded manifold obtained by assigning degree $n$ to every fiber variable. The standard choice is $n = 1$. Thus, $\mathcal{C}(A[1])$ is the sheaf of sections of $\wedge^* A^*$ with the standard grading.

**Example 2.4.** For $A = TM_0$ we have graded manifolds $T[1]M_0$ corresponding to the sheaf of differential forms on $M_0$, and $T^*[1]M_0$ corresponding to the sheaf of multi-vector fields. In general, the graded manifold $M_1$ in (2.1) is always of the form $A[1]$ for some $A$, whereas the fibrations $M_{k+1} \to M_k$ for $k > 0$ are in general affine rather than vector bundles.

When working with graded manifolds it is very convenient to use the Euler vector field $\epsilon$, which is defined as the derivation of $\mathcal{C}(M)$ such that $\epsilon(f) = kf$ if $f \in \mathcal{C}^k(M)$. In a local affine chart $\epsilon = \sum \deg(x^a)x^a \frac{\partial}{\partial x^a}$. In particular, $M_0$ is recovered as the set of fixed points of $\epsilon$, which then acts on the normal bundle of $M_0$ in $M$; $\deg(M)$ defined above is simply the highest weight of this action, “the highest degree of a local coordinate”.

Graded manifolds form a category $GrMflds$ with $\text{Hom}(M, N) = \text{Hom}(\mathcal{C}(N), \mathcal{C}(M))$ in graded algebras. Any smooth manifold is a graded manifold with $\epsilon = 0$; this gives a fully faithful embedding into $GrMflds$. Furthermore, the assignment $A \to A[1]$ gives a fully faithful embedding of the category $Vect$ of vector bundles into $GrMflds$.

One also has the forgetful functor into the category $SuperMflds$ of supermanifolds, which only remembers the grading modulo 2. The algebra $\mathcal{C}(M)$ is completed by allowing arbitrary smooth functions of all even variables. The Euler vector field on $M$ is related to the parity operator on the corresponding supermanifold by $P = (-1)^\epsilon$.

**Remark 2.5.** The roles of the $\mathbb{Z}_2$-grading (parity) and the $\mathbb{Z}$-grading are actually quite different. The former is responsible for the signs in formulas and in physics distinguishes bosons from fermions. The latter (the “ghost number” grading in physics) distinguishes physical fields from auxiliary ones (ghosts, antifields, etc.); for us the (nonnegative) integer grading has the added advantage of imposing rigid structure. But in general these two gradings are independent from one another [12]. Our requirement that they be compatible, aside from simplifying the presentation somewhat, is based on the fact that the classical field theories we consider here...
are bosonic. But in principle the AKSZ-BV formalism can handle more general bosonic-fermionic theories.

2.2. Vector bundles. The notion of a vector bundle in \( \text{GrMflds} \) can be defined in two ways. Given a graded manifold \( M \), we can consider a sheaf of locally free \( \mathcal{C}(M) \)-modules. Alternatively, we can consider a graded manifold \( A \) with a surjective submersion \( A \to M \) in \( \text{GrMflds} \) and a linear structure on \( A \) given by an additional Euler vector field \( \epsilon_{\text{vect}} \), which assigns degree 1 to every fiber variable and degree 0 to all functions pulled back from \( M \) (consequently, \( \epsilon_{\text{vect}} \) necessarily commutes with the Euler vector field \( \epsilon_A \) defining the grading on \( A \)). The \( \mathcal{C}(M) \)-module of sections of \( A \) can then be recovered by a variant of the mapping space construction described in the next section. One gets the following generalization of 2.8

Notation 2.6. Given a vector bundle \( A \to M \) in \( \text{GrMflds} \), denote by \( A[n] \) the graded manifold corresponding to the Euler vector field \( \epsilon_A + n\epsilon_{\text{vect}} \). It is again a vector bundle over \( M \) (with the same \( \epsilon_{\text{vect}} \)).

Example 2.7. For a graded manifold \( M \), \( TM \) and \( T^*M \) are vector bundles in \( \text{GrMflds} \), with the Euler vector field \( \epsilon_M \) lifting canonically via Lie derivative (see below). Thus, we also get vector bundles \( T[n]M \) and \( T^*[n]M \) for various integers \( n \). For instance, in \( T^*[n]M \) one has \( \deg(p_\alpha) + \deg(q^\alpha) = n \). Iterating this construction gives rise to many interesting graded manifolds, such as \( T^*[2]T^*[1]M_0 \) and so on.

2.3. Mapping spaces. Because the structure sheaf \( \mathcal{C}(M) \) of a graded manifold can contain nilpotents, care must be taken in defining such notions as points, maps, sections of vector bundles and so on. The most general solution is to use the categorical approach which, as far as we know, goes back to Grothendieck.

We shall assume that for any smooth manifolds \( M_0, N_0 \) the space of smooth maps \( \text{Maps}(M_0, N_0) \) is endowed with a smooth structure making it an infinite-dimensional manifold.

Proposition 2.8. Fix graded manifolds \( M \) and \( N \). Then the functor from \( \text{GrMflds} \to \text{Sets} \) given by \( Z \mapsto \text{Hom}(N \times Z, M) \) is representable. In other words, there exists a graded manifold \( \text{Maps}(N, M) \), unique up to a unique isomorphism, such that \( \text{Hom}(N \times Z, M) = \text{Hom}(Z, \text{Maps}(N, M)) \). Its base \( \text{Maps}(N, M)_0 \) is \( \text{Hom}(N, M) \), viewed as an infinite-dimensional smooth manifold containing \( \text{Maps}(N_0, M_0) \).

Proof. (Sketch) For simplicity assume that \( M \) and \( N \) (but not \( Z \)) are nonnegatively graded, and \( \deg(N) = 1 \) (it will become clear how to handle the general case). Let \( x = \{x_0, x_1\} \) be coordinates on \( N \), \( y = \{y_0, y_1, y_2, \ldots\} \) (subscripts indicate the degree) on \( M \), \( z \) on \( Z \) (the latter are viewed as parameters). Then any morphism from \( N \times Z \) to \( M \) in \( \text{GrMflds} \) has a coordinate expression of the form

\[
\begin{align*}
y_0 &= y_0(x, z) = y_{0,0}(x_0, z) + y_{0,-1}(x_0, z)x_1 + \frac{1}{2} y_{0,-2}(x_0, z)x_1^2 + \cdots \\
y_1 &= y_1(x, z) = y_{1,0}(x_0, z) + y_{1,-1}(x_0, z)x_1 + \frac{1}{2} y_{1,-2}(x_0, z)x_1^2 + \cdots \\
y_2 &= y_2(x, z) = y_{2,0}(x_0, z) + y_{2,-1}(x_0, z)x_1 + \frac{1}{2} y_{2,-2}(x_0, z)x_1^2 + \cdots \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\end{align*}
\]

Here we suppress the running indices, so that for instance \( \frac{1}{2} y_{0,-2}(x_0, z)x_1^2 \) actually means \( \frac{1}{2} y_{0,-2}(0, 0, x_1) \). The coefficients are arbitrary expressions in \( x_0 \) and \( z \) of total degree indicated by the second subscript. Now we let the arbitrary functions of \( x_0, y_{p,q}(x_0) \) (viewed as coordinates of degree \( q \), parametrize \( \text{Maps}(N, M) \). As the parameter
space $Z$ is completely arbitrary, the transformation rules for the $y_{p,q}$’s are determined by those for the $x$’s and $y$’s. The universal property is immediately verified: it’s just a matter of tautologically rewriting $y_{p,q}(x_0, z)$ as $y_{p,q}(x_0)(z)$. The degree 0 coordinates $y_{p,0}(x_0)$ parametrize the “actual” (i.e. degree-preserving) maps from $N$ to $M$.

\begin{remark}
If $N$ is just a point, we recover $M$ as $\text{Maps}(\text{pt}, M)$. This shows the correct way to think of points in graded manifolds. In general, when dealing with points, maps, sections and so on, one must allow them to implicitly depend on arbitrary additional parameters.
\end{remark}

\begin{remark}
In physics, the maps such as $y_{p,0}$ appear as various kinds of physical fields, whereas the non-zero degree $y_{p,q}$’s are referred to as “ghosts”, “antifields” and so on, and considered as non-physical, auxiliary fields. Correspondingly, the degree $q$ in physics is called “ghost number”. Expressions such as $y_p(x) = \sum y_{p-k}(x_0)x_k^k$ are called superfields. In view of the above Proposition, there is a good reason (apart from the physical considerations) to call the non-zero degree $y_{p,q}$’s “ghosts”: they only appear when additional parameters are introduced!
\end{remark}

\begin{remark}
We shall not have a detailed discussion of graded (Lie) groups, but it should be clear how to define them, especially for the reader familiar with super-groups, on which there is by now extensive literature. Graded groups are just group objects in the category $\text{GrMflds}$; the group axioms are expressed by commutative diagrams. In particular, for a graded manifold $M$, $\text{Diff}(M)$ is a graded group, constructed just as in the above Proposition. The direct product $\text{Diff}(N) \times \text{Diff}(M)$ acts on $\text{Maps}(N, M)$ in an obvious way, just like for ordinary manifolds.
\end{remark}

\begin{example}
The following statements are easily verified:

1. $\text{Maps}(S^1, \mathbb{R}[1]) \simeq \mathbb{R}^\infty[1]$ (with Fourier modes as coordinates).
2. $\text{Maps}(\mathbb{R}[-1], M) \simeq T[1]M$ for any graded manifold $M$.
\end{example}

\section{Differentials}

A vector field on a graded manifold $M$ is by definition a derivation of $\mathcal{C}(M)$. Vector fields form a sheaf of graded Lie algebras under the graded commutator.

\begin{definition}
A differential graded manifold is a graded manifold $M$ with a self-commuting vector field $Q$ of degree +1, i.e. $[\epsilon, Q] = Q$ and $[Q, Q] = 2Q^2 = 0$.
\end{definition}

\begin{example}
$\text{Vect}(\mathbb{R}[-1])$ is spanned by the Euler vector field $\epsilon_0 = -\theta \frac{d}{d\theta}$ and the differential $Q_0 = \frac{d}{d\theta}$, with commutation relations as in above definition. It acts on $\text{Maps}(\mathbb{R}[-1], M) \simeq T[1]M$ in an obvious way, giving rise to the grading of differential forms and the de Rham differential $d = dx^i \frac{\partial}{\partial x^i}$. In general, any differential graded manifold is, by definition, acted upon by the graded group $\text{Diff}(\mathbb{R}[-1])$. The differential integrates to an action of the subgroup $\mathbb{R}[1]$ (the “odd time flow”).
\end{example}

\begin{example}
Any vector field $v = \rho^a(x) \frac{\partial}{\partial x}$ on a graded manifold $M$ gives rise to a vector field $\iota_v = (-1)^{\deg(v)} \rho^a(x) \frac{\partial}{\partial x^a}$ on $T[1]M$, and consequently also the Lie derivative $L_v = [\iota_v, d]$. One has $\deg(\iota_v) = \deg(v) - 1$, $\deg(L_v) = \deg(v)$ and $L_{[\iota_v, \iota_w]} = [L_v, L_w]$.

The assignment $M_0 \mapsto (T[1]M_0, d)$ is a full and faithful functor from $M flds$ to $dgM flds$. It is the right adjoint to the forgetful functor which assigns to any dg
manifold $N$ its base $N_0$. In particular, the unit of adjunction gives a canonical dg map $(M, Q) \to (T^*[1]M_0, d)$ for any dg manifold $M$ with base $M_0$, called the anchor.

There are many interesting dg manifolds around. Those that are (as graded manifolds) of the form $A[1]$ for some vector bundle $A \to M_0$ are otherwise known as Lie algebroids. Those that come from Courant algebroids (see next section), however, are not of this form.

2.5. Symplectic and Poisson structures. Recall that the graded manifold structure on $T^*[1]M$ is given by the Euler vector field $\epsilon_{\text{tot}} = L_\epsilon + \epsilon_{\text{vect}}$ where $\epsilon$ gives the grading on $M$ and $L$ denotes the Lie derivative. However, we shall speak of the degree of a differential form on $M$ meaning only the action of the induced Euler vector field $L_\epsilon$, rather than $\epsilon_{\text{tot}}$ or $\epsilon_{\text{vect}}$. Thus, “a $p$-form $\omega$ of degree $q$” means $\epsilon_{\text{vect}} \omega = p \omega$ and $L_\epsilon \omega = q \omega$. In particular, a symplectic structure of degree $n$ is a closed non-degenerate two-form $\omega$ on $M$ with $L_\epsilon \omega = n \omega$.

Given such a symplectic structure, one defines for any function $f \in C(M)$ its Hamiltonian vector field $X_f$ (of degree $\deg(X_f) = \deg(f) - n$) via

$$t_{X_f} \omega = (-1)^{n+1} df$$

and for any two functions $f, g$ their Poisson bracket

$$\{f, g\} = X_f \cdot g = (-1)^{\deg(X_f) \deg(g)} t_{X_f} dg = (-1)^{\deg(f) + 1} t_{X_f} t_{X_g} \omega$$

It’s easily verified that the bracket defines a graded Lie algebra structure on $C(M)[n]$ and $f \mapsto X_f$ is a homomorphism$^3$.

Example 2.16. The basic example of a graded symplectic manifold is $T^*[n]M$ for some graded manifold $M$ and an integer $n$. The symplectic form is canonical $\omega = (-1)^{\hat{\alpha}} dp_a dq^a$ where $\hat{\alpha} = n \mod 2$ and $\hat{\alpha} = \deg(q^a) \mod 2$. The signs are chosen so that we have always $\{p_a, q^b\} = \delta^b_a$. For $n = 1$ and $M = M_0$ an ordinary manifold, the Poisson bracket induced by $\omega$ is just the Schouten bracket of multivector fields; it is easy to show that any nonnegatively graded symplectic manifold of degree 1 is canonically isomorphic to $T^*[1]M_0$ for some ordinary manifold $M_0$.

Example 2.17. If $V$ is a vector space, any nondegenerate symmetric bilinear form on $V$ can be viewed as a symplectic structure on $V[1]$ as follows: $\omega = \frac{1}{2} d \xi^a g_{ab} d \xi^b$ where $g_{ab} = \langle e_a, e_b \rangle$ for some basis $\{e_a\}$ of $V$. Clearly this $\omega$ has degree 2.

The following statements concerning a graded symplectic manifold $(M, \omega)$ are easily verified:

1. If $M$ is nonnegatively graded, $\deg(M) \leq \deg(\omega)$.
2. If $\deg(\omega) = n \neq 0$, then $\omega = d\alpha$, where $\alpha = \frac{1}{n} t_\omega \omega$.
3. If $v$ is a vector field of degree $m \neq -n$, such that $L_v \omega = 0$, then $t_v \omega = \pm d\left(\frac{1}{m+n} t_\omega t_v \omega\right)$.

A corollary of the last statement is that for a graded symplectic manifold of degree $n \neq -1$, any differential preserving $\omega$ is of the form $Q = \{\Theta, \cdot\}$ for some $\Theta \in C^{n+1}(M)$ obeying the Maurer-Cartan equation $\{\Theta, \Theta\} = 0$. A graded manifold equipped with such an $\omega$ and $\Theta$ will be referred to as a differential graded symplectic manifold. For $n = 1$ and $M = T^*[1]M_0$ this is the same as an ordinary Poisson structure on $M_0$.

$^2$Physicists would prefer the term “ghost number”.

$^3$Strictly speaking, one should write $f \mapsto X_f[-n]$.
2.6. Measure and integration. The general integration theory on supermanifolds is quite nontrivial. Fortunately, all we need here is the notion of the integral of a function over the whole graded manifold, i.e. a volume form or a measure. By a measure on $\mathcal{M}$ we shall understand a functional on the space of compactly supported functions (denoted by $f \mapsto \int_{\mathcal{M}} \mu f$) such that locally $\mu$ is the Berezinian measure of the form $F(x)dx$. We call $\mu$ nondegenerate if the bilinear form $\langle f, g \rangle = \int_{\mathcal{M}} \mu fg$ is.

Given a vector field $v$ on $\mathcal{M}$ we say a measure $\mu$ on $\mathcal{M}$ is $v$-invariant if $\int_{\mathcal{M}} \mu (vf) = 0$ for any $f$.

Given two graded manifolds $\mathcal{M}$ and $\mathcal{N}$ and a measure $\mu$ on $\mathcal{N}$ one can define the push-forward or fiber integration of differential forms. This is a chain map $\mu_* : \Omega^k(\mathcal{N} \times \mathcal{M}) \to \Omega^k(\mathcal{M})[\deg \mu]$ defined as follows:

$$\mu_\ast \omega(y)(v_1, \ldots, v_k) = \int_{\mathcal{N}} \mu(x) \omega(x, y)(v_1, \ldots, v_k)$$

If $\mu$ is $v$-invariant for some vector field $v$ on $\mathcal{N}$, one has $\mu_* L_v = 0$, where $v_1$ is the lift to $\mathcal{N} \times \mathcal{M}$ of $v$.

**Example 2.18.** If $N_0$ is a closed oriented smooth $n$-manifold, the graded manifold $T[1]N_0$ has a canonical measure defined as follows:

$$\int_{T[1]N_0} \mu f = \int_{N_0} f^{\top}$$

where $f^{\top}$ denotes the top-degree component of the inhomogeneous differential form $f$. This measure has degree $-n$, is invariant with respect to the de Rham vector field $d$ (Stokes’ Theorem) and also with respect to all vector fields of the form $\iota_v$ (since the top degree component of $\iota_v f$ is always zero); hence, it is invariant with respect to all Lie derivatives $L_v$, i.e. all isotopies of $N_0$ (in fact, all orientation-preserving diffeomorphisms). The induced nondegenerate pairing of differential forms is the Poincare pairing.

3. The AKSZ formalism.

Let us fix the following data.

**The source:** a dg manifold $(\mathcal{N}, D)$ endowed with a nondegenerate $D$-invariant measure $\mu$ of degree $-n-1$ for a positive integer $n$. In practice we will consider $N = T[1]N_0$ for a closed oriented $(n+1)$-manifold $N_0$, with $D = d$, the de Rham vector field, and $\mu$ the canonical measure.

**The target:** a dg symplectic manifold $(\mathcal{M}, \omega, Q)$ with $\deg(\omega) = n$. Then $Q$ is of the form $\{\Theta, \cdot\}$ for a solution $\Theta \in C^{n+1}(\mathcal{M})$ of the Maurer-Cartan equation:

$$\{\Theta, \Theta\} = 0$$

We shall assume here that both $\mathcal{M}$ and $\mathcal{N}$ are nonnegatively graded. Our space of superfields will be $\mathcal{P} = \text{Maps}(\mathcal{N}, \mathcal{M})$. This is a graded manifold, with the degree of a functional referred to as the ghost number. In the algebra of functions on $\mathcal{P}$, generators of non-negative ghost number will be called fields, those of negative ghost number — the antifields. Among the fields we further distinguish between the classical fields (of ghost number zero) and ghosts, ghosts for ghosts, etc. (of positive ghost number). We shall put a $QP$-structure on $\mathcal{P}$, i.e. an odd symplectic form $\Omega$ of degree $-1$ and a homological vector field (the BRST differential) $Q$ of degree $+1$ preserving $\Omega$. It is not guaranteed in general that such a field is Hamiltonian (since
Q and Ω are of opposite degree), but in our case it will be since Ω is exact. Thus
we will obtain a solution S of the classical Batalin-Vilkovisky master equation:

\[(S, S) = 0\]

where \((\cdot, \cdot)\) is the odd Poisson bracket corresponding to ω, otherwise known as the
BV antibracket, while S (of ghost number zero) is the hamiltonian of Q.

Let us first define the Q-structure. As remarked above, the graded group
Diff(N) × Diff(M) acts naturally on P. Therefore, the vector fields D on N and
Q on M induce a pair of commuting homological vector fields on P which we de-
ote by ˆD and Q, respectively. Hence, any linear combination Q = aD + bQ is a
Q-structure. We shall fix the coefficients later to get the master action in the form
we want.

Now for the P-structure. There is a canonical evaluation map ev : N × P → M
given by (x, f) → f(x). This enables us to pull back differential forms on M using
ev, and then push them forward to P using the measure µ. The resulting chain
map \(µ, ev^* : Ω^k(M) → Ω^k(P)\) has ghost number \(deg(µ) = −n − 1\). So let us define
Ω = µ, ev^*ω. If the measure µ is nondegenerate, this defines a P-structure.

It remains to check that the P and Q-structures are compatible and calculate
the master action. We first observe that µ, ev^* applied to functions preserves the
Poisson brackets. That is, we have

\[(\int_N µφ^*(ξ), \int_N µφ^*(η)) = \int_N µφ^*(\{ξ, η\})\]

for ξ, η ∈ C(M) and a superfield φ. Moreover, if Q has hamiltonian Θ, then ˆQ has
hamiltonian \(µ, ev^*Θ\). This gives us the interaction term of our master action:

\[S_{\text{int}}[φ] = \int_N µφ^*(Θ)\]

To see that ˆD also preserves ω, observe that, if µ is D-invariant, then
\(L_Dµ, ev^* = 0\). This is essentially because the evaluation map is invariant under the diagonal action
of Diff(N). Moreover, if ω = da for some 1-form α on M, this implies that up to a
sign, the hamiltonian of ˆD is \(i_Dα\), where \(α = µ, ev^*α\). This gives the kinetic term
\(S_{\text{kin}}\) of the master action.

To see what this all looks like written out explicitly in coordinates, we begin
with the following general remark. In any odd symplectic manifold P with Ω of
degree −1, the ideal generated by functions of negative degree corresponds to a
Lagrangian submanifold L, and in fact P is canonically isomorphic to \(T^*[-1]L\).
In our case this L is the space of fields (including ghosts). It contains the space
of classical fields \(L_0\), corresponding to the ideal generated by functions of nonzero
degree (ghosts and antifields). The restriction of any S of ghost number zero to L
depends only on the classical fields, i.e. is a pullback of a functional S on \(L_0\). This
way we recover the classical action. As the critical points of S are the fixed points
of Q, we see that the solutions of the classical field equations for S are the dg maps
from \((N, D)\) to \((M, Q)\).

Let us assume from now on that \(N = T[1]N_0\) for some closed oriented \((n + 1)\)-
dimensional \(N_0\), with \(D = d\), the de Rham vector field, and µ the canonical measure.
In coordinates \(d = du^i \frac{∂}{∂u^i}\), and we denote the induced differential on superfields
by d instead of ˆD.
As for the target, let $\omega$ be written in Darboux coordinates as $\omega = \frac{1}{2} dx^a \omega_{ab} dx^b$. Here $\omega_{ab}$ are constants, and so the degrees of $x^a$ and $x^b$ must add up to $n$. We shall choose $\alpha = \frac{1}{2} x^a \omega_{ab} dx^b$ as the primitive of $\omega$.

Now, a map $\phi : N \to M$ is parametrized in coordinates by superfields $\phi^*(x^a) = \phi^0 = \phi^0(u, du)$. Then $\Omega$ is given by

$$\Omega = \int_{T[1]N_0} \mu \left( \frac{1}{2} \delta \phi^a \omega_{ab} \delta \phi^b \right) = \int_{N_0} \left( \frac{1}{2} \delta \phi^a \omega_{ab} \delta \phi^b \right)^{\text{top}}$$

To find the field-antifield splitting and compute the normal form for $\Omega$, let us further keep track of the degree of $x^a$ by writing it as a subscript: $x^a_i$ is of degree $0 \leq i \leq n$, then we have

$$\phi_i^a = \sum_{j=0}^{n+1} \phi_{i,j}^a$$

where $\phi_{i,j}^a = \phi_{i,j}^a(u)(du)^j = \frac{1}{j!} \phi_{i,j,\nu_1, \ldots, \nu_j}^a(u) du^{\nu_1} \cdots du^{\nu_j}$ is the $j$-form component of $\phi_i^a$ whose coefficients $\phi_{i,j}^a(u)$ have therefore ghost number $i - j$. It is easy to see then that we must set, for each field $\phi_{i,j}^a$ with $i - j \geq 0$, its conjugate antifield to be $\phi_{i,j}^{\top} = (-1)^{\nu_i} \phi_{n-i, n+1-j}^{\text{top}}$. Then we can rewrite $\Omega$ as

$$\Omega = \int_{N_0} (-1)^i \sum_{i-j \geq 0} \delta \phi_{i,j}^a \delta \phi_{i,j}^{\top}$$

The master action is given by

$$S[\phi] = \int_{T[1]N_0} \mu \left( \frac{1}{2} \phi^a \omega_{ab} d\phi^b + (-1)^{n+1} \phi^a \Theta \right)$$

The classical action $S$ is then recovered by setting all the antifields to zero. The sign in front of the interaction term is chosen so that the solutions of the classical field equations $\delta S = 0$ coincided with dg maps $\phi : (T[1]N_0, d) \to (M, Q = \{ \Theta, \cdot \})$. As we have remarked, $\mu$ is invariant under all orientation-preserving diffeomorphisms of $N_0$, hence $S$ yields a topological field theory.

4. Courant algebroids and the topological closed membrane.

Specializing the above construction to various choices of the target one gets many interesting topological field theories. For instance, in case $n = 1$ the target is necessarily of the form $M = T^*[1]M_0$ for some manifold $M_0$, and the interaction term is given by a Poisson bivector field $\pi$ on $M_0$. The BV quantization of the resulting two-dimensional TFT – the Poisson sigma model – was extensively studied by Cattaneo and Felder \[3\][4].

Here we would like to consider the case $n = 2$. Symplectic nonnegatively graded manifolds $(M, \omega)$ with deg$(\omega) = 2$ were shown in \[9\] to correspond to vector bundles $E \to M_0$ with a fiberwise nondegenerate symmetric inner product $\langle \cdot, \cdot \rangle > 0$ (of arbitrary signature). The construction is as follows. Recall that deg$(M) \leq 2$, hence $M$ fits into a tower of fibrations

$$M = M_2 \to M_1 \to M_0$$

where $M_1$ is of the form $E[1]$ for some vector bundle $E \to M_0$. Restricting the Poisson bracket to $M_1$ gives the inner product. Conversely, given $E, M$ is obtained as the symplectic submanifold of $T^*[2]E[1]$ corresponding to the isometric embedding $E \hookrightarrow E \oplus E^*$ with respect to the canonical inner product on $E \oplus E^*$. If $\{x^i\}$...
are local coordinates on $M_0$ and $\{e_a\}$ is a local basis of sections of $E$ such that $\langle e_a, e_b \rangle = g_{ab} = \text{const.}$, we get Darboux coordinates $\{q^i, p_i, \xi^a\}$ on $M$ (of degrees 0, 2 and 1, respectively), so that
\[
\omega = dp_i dq^i + \frac{1}{2} d\xi^a g_{ab} d\xi^b = d(p_i dq^i + \frac{1}{2} \xi^a g_{ab} d\xi^b)
\]

Notice that the quadratic hamiltonians $C^{2}(M)$ form a Lie algebra under the Poisson bracket, which is isomorphic to the Lie algebra of infinitesimal automorphisms of $E$ preserving $\langle \cdot, \cdot \rangle$.

It was further shown in [9] that solutions $\Theta \in C^{3}(M)$ of the Maurer-Cartan equation $\{\Theta, \Theta\} = 0$ correspond to Courant algebroid structures on $(E, \langle \cdot, \cdot \rangle)$. Such a structure is given by a bilinear operation $\circ$ as well as the operation $\bullet$, i.e. $\circ$ defines a Leibniz algebra on sections of $E$. It follows also that the anchor $\alpha$ induces a homomorphism of Leibniz algebras. The only additional property of $\circ$ concerns its symmetric part, namely
\[
\langle e, e_1 \circ e_2 + e_2 \circ e_1 \rangle = a(e) < e_1, e_2 >
\]

as well as the operation $\circ$ itself:
\[
e \circ (e_1 \circ e_2) = (e \circ e_1) \circ e_2 + e_1 \circ (e \circ e_2)
\]
i.e. $\circ$ defines a Leibniz algebra on sections of $E$. The Maurer-Cartan equation $\{\Theta, \Theta\} = 0$ is equivalent to the defining properties of $\circ$. The corresponding differential $Q$ sends a function $f \in C^{0}(M) = C^{\infty}(M_0)$ to $a^* df$, and a section $e \in \Gamma(E) = C^{1}(M)$ to $e \circ \in C^{2}(M)$.

Now we can write down the sigma-model. Fix a closed oriented 3-manifold $N_0$, with coordinates $\{u^\mu\}$. The classical fields are the degree-preserving maps $T[1]N_0 \to M$, consisting of a smooth map $X : N_0 \to M_0$ (the membrane world-volume), an $X^* E$-valued 1-form $A$ and an $X^* T^* M_0$-valued 2-form $F^4$. The super-fields are written as follows:
\[
\begin{align*}
q^i &= \gamma^i + F^i_\alpha du + \alpha^i_\beta (du)^2 + \gamma^i (du)^3 \\
\xi^a &= \beta^a + A^a du + g_{ab} A^b_\gamma (du)^2 + g^{ab} \beta^a_\beta (du)^3 \\
p_i &= \gamma_i + \alpha_i du + F_i (du)^2 + X^i_\beta (du)^3
\end{align*}
\]

\[\text{Strictly speaking, this is misleading as the transformation law for } p_i \text{ is nonlinear, containing a term quadratic in the } \xi^a \text{s; as a graded manifold, } M \text{ is isomorphic to } E[1] \oplus T^*[2] M_0 \text{ only after a } g \text{-preserving connection has been fixed. This issue complicates a coordinate-free description of the sigma model.}\]
In particular, the $X^*E$-valued scalar $\beta$ and the $X^*T^*M_0$-valued 1-form $\alpha$ are the ghosts, while the $X^*T^*M_0$-valued scalar $\gamma$ is the ghost for ghost. The master action

$$S = \int_{T[1]N_0} \mu(p_1 dq^i + \frac{1}{2} \xi^a g_{ab} d\xi^b - \Theta(q, \xi, p))$$

decomposes as $S = S_0 + S_1 + S_2 + S_3$ where the subscript denotes the number of antifields. Thus, $S_0$ is the classical action:

$$S_0 = \int_{N_0} F_i dX^i + \frac{1}{2} A^a g_{ab} dA^b - A^a P^a(X) F_i + \frac{1}{6} T_{abc}(X) A^a A^b A^c$$

and the rest of the terms are as follows:

$$S_1 = \int_{N_0} \left( -\beta^a P^a X^i \dot{X}_i + (d\beta^c - g^{ac} P^a_\alpha \partial_\alpha) A_1 + \right.$$

$$\left. + (-d\alpha_j - \beta^j \partial_j P^a_\alpha \partial_\alpha + \frac{1}{2} \gamma_j T_{abc} \beta^b \beta^c A^a) F_j + \right.$$

$$\left. + \left( \frac{1}{2} \gamma_j T_{abc} \beta^a \beta^b \beta^c - \beta^a \partial_j P^a_\alpha \gamma_i \right) \right)$$

$$S_2 = \int_{N_0} \left( \frac{1}{2} \gamma_j T_{abc} \beta^a \beta^b \beta^c - \beta^a \partial_j P^a_\alpha \gamma_i \right) F_j F_k F_l$$

$$S_3 = \int_{N_0} \frac{1}{6} \left( \beta^a \partial_j \partial_k \partial_l P^a_\alpha \gamma_i + \frac{1}{6} \partial_j \partial_k \partial_l T_{abc} \beta^a \beta^b \beta^c F_j F_k F_l \right)$$

It appears our sigma model has very complicated 2-algebroid gauge symmetries generated by parameters $\alpha_i, \beta^a$ and $\gamma_i$; writing down the master action without the help of AKSZ would have been extremely difficult. It would be desirable to better understand the structure of the gauge symmetries.

In conclusion, let us point out some special cases.

Example 4.1. Let $M_0 = \{pt\}$. Then $(E, < \cdot, \cdot >)$ is just a vector space with an inner product. A Courant algebroid structure reduces to that of a quadratic Lie algebra with structure constants $T_{abc} = [e_a, e_b, e_c]$. A quick glance reveals that in this case $S_0$ is the classical Chern-Simons functional, for which the master action was written down in [1].

Example 4.2. Let $M = T^*[2]T^*[1]M_0 = T^*[2]T[1]M_0$, with coordinates $\{q^i, \xi^i, \theta_i, p_i\}$ of degree 0.1.1 and 2, respectively (one thinks $\xi^i = dx^i$). Then $\omega = dp_i dq^i + d\xi^i d\theta_i = d(p_i dq^i + \xi^i d\theta_i)$, and we consider $\Theta = \xi^i p_i - \frac{1}{6} c_{ijk}(q) \xi^i \xi^j \xi^k$, where $c = \frac{1}{2} c_{ijk}(q) \xi^i \xi^j \xi^k$ is a 3-form on $M_0$. Clearly $\Theta$ obeys Maurer-Cartan if and only if $dc = 0$. The corresponding Courant algebroid structure on $E = TM \oplus T^*M$ (with the canonical inner product) is given by

$$(X + \xi) \circ (Y + \eta) = [X, Y] + L_X \eta - \iota_Y d\xi + \iota_{X \wedge Y} c$$

The classical fields for the corresponding topological membrane action are comprised of the membrane world-volume $X : N_0 \to M_0$, an $X^* TM_0$-valued 1-form $A$, an $X^* T^* M_0$-valued 1-form $B$ and an $X^* T^* M_0$-valued 2-form $F$. The classical action

$$S_0[X, A, B, F] = \int_{N_0} F_i dX^i + A^j dB_i - A^i F_i + \frac{1}{6} c_{ijk}(X) A^i A^j A^k$$

was considered by Park [S]. We leave it to the reader to write down the master action in this case.
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