THE CONGRUENT NUMBER PROBLEM
AND THE BIRCH SWINNERTON-DYER CONJECTURE

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Abstract. By introducing a new point of view in Algebraic Topology relating
elliptic curves in \( \mathbb{R}^2 \) and suitable bordism groups, the congruent number prob-
lem is solved showing that the Tunnell’s theorem is also sufficient. This could
be considered also an indirect proof that the Birch Swinnerton-Dyer conjecture
is true.

AMS Subject Classification: 11M26; 14H55; 30F30; 32J05; 33C80; 33C99;
33D90; 33E90; 55N35.

Keywords:

1. Introduction

The congruent number problem was a longstanding open problem in Number The-
ory, that more recently has been related also to the famous Birch and Swinnerton-
Dyer conjecture. The vast literature and the Clay-prize give evidence the Mathe-
matical Community’s interest on this subject. For a very good introduction about
it is advisable to look the paper by A. Wiles [34] ¹

In this paper we solve the congruent problem by recasting it in the algebraic topo-
logy of suitable bordism groups. In fact, we introduce two new bordism groups,
\((n\text{-elliptic bordism groups} \text{ and } n\text{-congruent-bordism-groups})\), in the plane \( \mathbb{R}^2 \) where
are considered elliptic curves associated to the congruent number problem. (See
Definition 4.1 and Definition 4.5.) We show, by utilizing such bordism groups, the
exactness of the following short 0-sequence:

\[ 0 \rightarrow N_{\text{congr}} \xrightarrow{i} \square N \xrightarrow{L\ast} \mathbb{Z} \rightarrow \text{coker}(L\ast) \rightarrow 0 \]

Here \( \square N \) is the subset of \( \mathbb{N} \) of square-free numbers, and \( N_{\text{congr}} \) is the subset of \( \square N \)
of square-free congruent numbers (strong-congruent numbers). The mapping \( L\ast \) is
defined in Lemma 4.11. We show that \( \ker(L\ast) = N_{\text{congr}} \), namely that sequence
(1) is exact. (See Theorem 4.12.) This gives an indirect proof that the Birch
and Swinnerton-Dyer conjecture is true. (See Remark 4.15.) Taking into account
that the set \( \mathbb{P}_{\text{yth}} \) of integers interpretable as areas of Pythagorean right triangles,

¹For complementary information see also the following Wikipedia link:
Birch-and-Swinnerton-Dyer-conjecture.
surjectively projects on \( \mathbb{N}_{\text{congr}} \), we can obtain all possible Pythagorean triangles. Furthermore, each strong-congruent number \( n \in \mathbb{N}_{\text{congr}} \) identifies an equivalence class in the set of congruent numbers. Any congruent number \( q \) belonging to the equivalence class \([n]\), is obtained from \( n \) by multiplying \( n \) for a square \( m^2 \), with \( m \in \mathbb{Q} \): \( q = m^2 n \).

The paper, after the Introduction, splits into three more sections and three appendices. 2. The congruent number problem. [This is a preparatory section, where some fundamental results are recast in the paper-style.] 3. The Birch and Swinnerton-Dyer conjecture. [Here are resumed some important results that are central for our proof. Theorem 3.1 (Modularity theorem); Conjecture 3.3 (The Birch and Swinnerton-Dyer conjecture); Proposition 3.4 (Tunnell’s theorem); Theorem 3.7 (Coates-Wiles theorem).] 4. Elliptic and congruent bordism groups. [In this section are contained the main results. It contains the definitions of elliptic bordism groups and congruent bordism groups and their characterizations with respect to diffeomorphisms and suitable homotopies in \( \mathbb{R}^2 \). Theorem 4.12 contains the solution of the congruent problem in Number Theory. Remark 4.15 emphasizes that our solution of the Congruent number problem can be considered also an indirect proof that the Birch and Swinnerton-Dyer conjecture is true.] Appendix A. The function \( L(E, s) \) and the infinitude of primes. Appendix B. Riemann surfaces and modular curves. Appendix C. Modular functions, forms and cusps.\(^2\)

2. The congruent number problem

In this section we shall consider some fundamental definitions and results about the congruent number problem that will be utilized in the next sections.

**Definition 2.1** (Congruent number). A congruent number is any positive rational number \( q \) such that there exists a right triangle of sides \((a, b, c)\), \(a, b, c \in \mathbb{Q}\), with area \( q \), namely \( \frac{a \cdot b}{2} = q \in \mathbb{Q} \). (Here \( c \) is the length of the hypotenuse.) We denote by \([q,a,b,c]\) a congruent number \( q \), with its corresponding right triangle of sides \((a,b,c)\). Let us denote by \( \mathbb{Q}_{\text{congr}} \subset \mathbb{Q} \) the subset of congruent numbers.

**Lemma 2.2** (Congruent numbers identified via a subset of natural numbers). • If \( q \) is a congruent number, then also \( s^2 \cdot q \) is so, for any \( s \in \mathbb{Q} \).

• There is an equivalence relation \( \sim \) between congruent numbers, such that each equivalence class is identified by a square-free positive integer. In other words, the set \( \mathbb{N}_{\text{congr}} \) of congruent numbers, up to rational-conform equivalence, can be identified with a subset of \( \square \mathbb{N} \subset \mathbb{N} \), where \( \square \mathbb{N} \) is the set of square-free integers contained into \( \mathbb{N} \): \( \mathbb{Q}_{\text{congr}} / \sim \cong \mathbb{N}_{\text{congr}} \subset \mathbb{N} \). One has the commutative and exact diagram (2).

\(2\)These subjects are included in this paper in order to satisfy its expository style. We have adopted this style do not make the paper beyond any mathematical grasp, since the mathematics involved here touches sectors that can be considered far from the standard Number Theory. On the other hand experts in Algebraic Topology do not necessarily are also well introduced in Number Theory ...
Proof. • In fact, if \([q, a, b, c]\) is a congruent number with associated right triangle, then also \([s^2 \cdot q, s \cdot a, s \cdot b, s \cdot c]\) is a congruent number with associated right triangle. The proof is direct.

• Therefore we can consider equivalent right triangles \((a, b, c)\) and \((\bar{a}, \bar{b}, \bar{c})\), identified by congruent numbers \(q\) and \(\bar{q}\), respectively, whether they are rational-conform, namely \(\bar{a} = s \cdot a, \bar{b} = s \cdot b, \bar{c} = s \cdot c, s \in \mathbb{Q}\), iff \(\bar{q} = s^2 \cdot q\). As a by-product, it follows that a congruent number identifies an equivalence class in the group \(\mathbb{Q}^\times / (\mathbb{Q}^\times)^2\). Here every residue class contains one square-free positive integer, that can be utilized to identify the class.\(^3\) Therefore any equivalence class of congruent numbers, obtained by identifying conform right triangles, can be represented by a square-free integer. Let us denote by \(\mathbb{N}_{\text{congr}}\) the set of such natural numbers, and by \(\square \mathbb{N}\) the set of square-free natural numbers. Then \(\mathbb{N}_{\text{congr}} \subset \square \mathbb{N} \subset \mathbb{N}\).\(^4\)

The characterization of square-free positive integers can be made also by means of the Möbius function.

**Definition 2.3** (Möbius function). The Möbius function \(\mu(n)\) is defined in (3).

\[
(3) \quad \mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
(-1)^k & \text{if } n = p_1 \cdots p_k, \ p_i \in P \\
0 & \text{if } n \text{ has a squared prime factor.}
\end{cases}
\]

In (3) \(P\) is the set of primes.

**Proposition 2.4** (Properties of Möbius function). • \(\mu(n)\) is a multiplicative function: \(\mu(n_1 \cdot n_2) = \mu(n_1) \cdot \mu(n_2)\), \(n_1\) and \(n_2\) coprime.

• (Dirichlet series that generates the Möbius function)

\[
\sum_{1 \leq n \leq \infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \ s \in \mathbb{C}, \ \Re(s) > 1.
\]

\(\bullet\) \(n \in \square \mathbb{N}\) iff \(\mu(n) \neq 0\).

\(^3\)This is the motivation, that allows to talk about congruent numbers, simply as square-free positive integers. (Recall that a square-free integer is one divisible by no perfect square, except 1. So a square-free integer is one \(n \in \mathbb{N}\) such that in its prime decomposition \(n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}\) all exponent \(\alpha_i\), \(i = 1, \cdots, k\), are \(\alpha_i = 1\). This is equivalent to say that the ring \(\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}\) is a product of fields, \(\mathbb{Z}_n = \mathbb{Z}_{p_1} \cdots \mathbb{Z}_{p_k}\), since \(p_i\), \(i = 1, \cdots, k\), are fields (\(p_i\) is prime). For example are square-free integers 1, 2, 3, 5, 6, 7, 10, 11, 13. Instead are not square-free integers 9 = 3\(^2\) and 20 = 2\(^2\) \cdot 5.)

\(^4\)\(\mathbb{N}_{\text{congr}}\) is a proper subset of \(\square \mathbb{N}\). For this it is enough to look at the Tab. 4. On the other hand it is well known that the square-free numbers 1, 2, 3 \(\in \square \mathbb{N}\) are not congruent numbers.
Definition 2.5 (Strong-congruent numbers). We define strong-congruent numbers the integers belonging to $\mathbb{N}_{\text{congr}}$.

Proposition 2.6 (Strong-congruent numbers and Pythagorean triangles). • A way to obtain strong-congruent numbers $n \in \mathbb{N}_{\text{congr}} \subset \mathbb{N}$ is related to pass through Pythagorean triangles, by using the following parametric formula:

$$\begin{align*}
(4) \quad \{ & (s(\kappa^2 - l^2), 2s\kappa l, s(\kappa^2 + l^2)) : \kappa > l > 0, \kappa - l = 2r + 1 \forall r \in \{0\} \cup \mathbb{N} \\
& k, l \text{ coprime, } \forall s \in \mathbb{N} \}
\end{align*}$$

In fact, equations (4) parametrize all the Pythagorean triangles, hence the square-free part of their areas identify all the strong-congruent numbers. In Tab. 1 are reported some examples of $n \in \mathbb{N}_{\text{congr}}$ built in such a way. Let us denote by $\mathbb{P}_{\text{yth}}$ the set of Pythagorean triangles parameterized by $\kappa, l$ and $s$:

$$\begin{align*}
(5) \quad \mathbb{P}_{\text{yth}} = \{ & q \in \mathbb{N} \mid q = s^2(\kappa^2 - l^2)\kappa l, \kappa, l \in \mathbb{N}, \\
& \kappa > l > 0, \kappa - l = 2r + 1, \forall r \in \{0\} \cup \mathbb{N}, k, l, \text{ coprime, } \forall s \in \mathbb{N} \}
\end{align*}$$

Then one has the exact commutative diagram (7), (congruent numbers grail), where

$$\begin{align*}
(6) \quad \begin{cases} 
  a(m) = \text{square-free part of } m \\
  b = a|_{\mathbb{P}_{\text{yth}}} \\
  i = \text{natural inclusion} \\
  j = \text{natural inclusion}
\end{cases}
\end{align*}$$

\[ \begin{array}{c}
\begin{array}{ccc}
\mathbb{Q}_{\text{congr}} & \mathbb{P}_{\text{yth}} & \mathbb{N} \\
\downarrow j & \downarrow i & \downarrow \mathbb{N} \\
\mathbb{P}_{\text{yth}}' & \mathbb{N}_{\text{congr}} & \mathbb{Q}^+
\end{array}
\end{array} \]

• $\sharp(\mathbb{P}_{\text{yth}}) = \mathbb{N}_0$. Furthermore one can write $\mathbb{P}_{\text{yth}} = \bigcup_{n \in \mathbb{N}_{\text{congr}}} (\mathbb{P}_{\text{yth}})_n$, where $(\mathbb{P}_{\text{yth}})_n = b^{-1}(n)$ is the fiber over $n$, the strong-congruent number $n$, and $\sharp((\mathbb{P}_{\text{yth}})_n) = \mathbb{N}_0$.

Proof. • In fact, $a(j(q)) = a(j(s^2n)) = a(s^2n) = n$ and $i(b(q)) = i(b(s^2n)) = i(n) = n$. The surjectivity of the mapping $b$ in (7) is not obvious. So let assume that $n \in \mathbb{N}_{\text{congr}}$, then there exists a congruent right triangle $[n|a, b, c = [n|\frac{r}{s}, \frac{r'}{s'}, \frac{r''}{s''}] |$ that satisfies the conditions: $\frac{a+b}{2} = \frac{l}{2}$, hence $rr' = 2ss'n$, and $a^2 + b^2 = c^2$, hence

5The parametrization $(\kappa^2 - l^2, 2\kappa l, \kappa^2 + l^2)$ for primitive Pythagorean right triangles, ($s = 1$), comes from the identity $(\kappa^2 - l^2)^2 + (2\kappa l)^2 = (\kappa^2 + l^2)^2$, that interprets the Pythagorean's theorem for a right triangle of sides $\kappa^2 - l^2$, $2\kappa l$ and hypothenuse $\kappa^2 + l^2$. Pythagorean right triangles are congruent triangles with congruent number $q = s^2(\kappa^2 - l^2)\kappa l \in \mathbb{N}$. One can write $q = m^2 \cdot n$, where $n$ is the square-free part of $q$. Then $[n|\frac{a^2 - l^2}{m^2}, \frac{2\kappa l}{m}, \frac{a^2 + l^2}{m}]$ is the congruent class of right triangles identified by the congruent square-free number $n$. 


\[ \frac{a^2 + c^2 - 2b^2}{2r^2} = \frac{r''}{r'^2}. \]

From this last we can assume \( s'' = ss' \) and \( r'' = \sqrt{r^2 s'^2 + r'^2 s^2} = \sqrt{2p} \), where \( p \in \mathbb{N} \). Then there exists a Pythagorean triangle \( |q = m^2 n|a, b, c| \), with \( m \in \mathbb{N} \). More precisely one has

\[
\begin{align*}
\bar{a} &= 2ss'a = 2ss' \frac{a}{r} = 2s'r \\
\bar{b} &= 2ss'b = 2ss' \frac{b}{r} = 2sr' \\
\bar{c} &= 2ss'c = 2ss' \frac{c}{r} = 2ss' \sqrt{\frac{r^2 s'^2 + r'^2 s^2}{s^2}} = 2p
\end{align*}
\]

In fact one has \( a^2 + \bar{b}^2 = 4(r^2 s'^2 + r'^2 s^2) = 4p^2 = c^2 \), and \( \frac{4}{ab} = 2ss'rr' = (2ss')^2 n = m^2 n \), with \( m = 2ss' \). Therefore the square-free part of \( \frac{a}{b} \) is just \( n \). Of course, since are also Pythagorean triangles the following ones \( [(lm) n|a = l\bar{a}, b = l\bar{b}, c = l\bar{c}] \), \( \forall l \in \mathbb{N} \), it follows that the mapping \( b : \mathbb{P}_{yth} \rightarrow \mathbb{N}_{congr} \) is surjective. In (7) we have also inserted the surjective mappings \( \mathbb{Q} \rightarrow \mathbb{N} \) and \( \mathbb{Q}_{congr} \rightarrow \mathbb{P}_{yth} \). In this way it is clear that all the congruent numbers can be obtained from \( \mathbb{N}_{congr} \).

- Note that \( \mathbb{P}_{yth} \) is an infinite set.\(^6\) In fact, it contains the set \( \mathbb{P}_{yth}[1] = \{ q \in \mathbb{P}_{yth} | q = (\kappa^2 - l^2)\kappa l, l = 1 \} \), obtained from \( \mathbb{P}_{yth} \) by fixing \( l = 1 \). One can see that \( \mathbb{P}_{yth}[1] \) is identified by the positive-valued curve \( q(r) = 2(r + 1)(4r^2 + 8r + 3) \), having \( q''(r) = 24r^2 + 48r + 22 > 0 \), \( \forall r \geq 0 \). Therefore \( q(r) = q(r') \) if \( r = r' \), for any \( r, r' \in \{ 0 \} \cup \mathbb{N} \).

Furthermore, from the proof of the above point we get \( \sharp(\mathbb{P}_{yth}) = \aleph_0. \)

Lemma 2.7 (Cardinality of \( \mathbb{N}_{congr} \)). The set of strong-congruent numbers has the same cardinality of \( \mathbb{N} \): \( \sharp(\mathbb{N}_{congr}) = \aleph_0. \)

Proof. From Proposition 2.6 we see that the subset \( \mathbb{N}_{congr} \) of \( \mathbb{N} \) contains an infinite set. This is identified by means of the square-free parts of the numbers \( (\kappa^2 - l^2)\kappa l \). This last set is infinite and therefore, is so the set of its square-free parts. This follows from the prime factorization of integers and from the cardinality of \( P \), i.e., the set of primes: \( \sharp(P) = \sharp(\mathbb{N}) = \aleph_0 \).\(^7\) Therefore it must necessarily be \( \sharp(\mathbb{N}_{congr}) = \aleph_0. \)

Theorem 2.8 (First criterion for strong-congruent numbers). Let \( n \in \mathbb{N}_{congr} \) be a square-free integer. Then \( n \in \mathbb{N}_{congr} \), namely \( n \) is a strong-congruent number, if there are integers \( \kappa, l, \kappa > l > 0, \kappa - l \not\equiv 2 \) mod 2 such that: \( (\kappa^2 - l^2)\kappa l = m^2 \cdot n \), for some \( m \in \mathbb{N} \).

Proof. This criterion follows from the congruent numbers grail (7) and the surjectivity of the mapping \( b \) there considered.

Theorem 2.9 (Second criterion for strong-congruent numbers). A number \( n \in \mathbb{N}_{congr} \) is a strong-congruent number, namely \( n \in \mathbb{N}_{congr} \), iff there exist three positive rational numbers \( 0 < r < s < t \), such that the following conditions are satisfied:

(i) \( r^2 - r = 2n \); 
(ii) \( l^2 + r^2 = 2s^2 \).

\(^6\) Let us emphasize that even if the set of Pythagorean triangle is infinite, since from one we can generate infinite other ones with conform transformations, this property could not appear so obvious by looking the parametrization considered. In fact two different couples \( (\kappa, l) \) and \( (\kappa', l') \), can have equal their \( q = (\kappa^2 - l^2)\kappa l \)-value in \( \mathbb{P}_{yth} \). For example to the couple \( (\kappa, l) = (5, 2) \) there corresponds \( q = 210 \); the same \( q \)-value corresponds to the couple \( (\kappa, l) = (6, 1) \) too.

\(^7\) In Appendix A it is given a proof on the cardinality of \( P \) that uses the zeta Riemann function.

(See Theorem A1.)
Table 1. Examples of strong-congruent numbers $n \in \mathbb{N}_{\text{congr}}$ and triangle class $[n|\frac{a}{m}, \frac{b}{m}, \frac{c}{m}]$, built from Pythagorean triangles $(\kappa^2 - l^2, 2\kappa l, \kappa^2 + l^2)$.

| $\!\!\kappa\!\!\!\!\!\!\!\!$ | $l$ | Pythagorean-triangle $(a, b, c)$ | Pythagorean-triangle area $[n|\frac{a}{m}, \frac{b}{m}, \frac{c}{m}]$ |
|--------|--------|----------------------------------|--------------------------------------------------|
| 2      | 1      | $(3, 4, 5)$                       | $6 = 2 \cdot 3$                                  |
| 3      | 2      | $(5, 12, 13)$                     | $30 = 2 \cdot 3 \cdot 5$                        |
| 4      | 1      | $(15, 8, 17)$                     | $60 = 2^2 \cdot 3 \cdot 5$                      |
| 4      | 3      | $(7, 24, 25)$                     | $84 = 2^2 \cdot 3 \cdot 7$                      |
| 5      | 2      | $(21, 20, 29)$                    | $210 = 2 \cdot 3 \cdot 5 \cdot 7$               |
| 5      | 4      | $(9, 40, 41)$                     | $180 = 2^2 \cdot 3^2 \cdot 5$                   |
| 6      | 1      | $(35, 12, 37)$                    | $210 = 2 \cdot 3 \cdot 5 \cdot 7$               |
| 6      | 5      | $(11, 60, 61)$                    | $330 = 2 \cdot 3 \cdot 5 \cdot 11$              |
| 7      | 2      | $(45, 28, 53)$                    | $630 = 2 \cdot 3^2 \cdot 5 \cdot 7$             |
| 7      | 4      | $(33, 56, 65)$                    | $924 = 2^2 \cdot 3 \cdot 7 \cdot 11$            |
| 8      | 1      | $(63, 16, 65)$                    | $504 = 2^3 \cdot 3^2 \cdot 7$                   |
| 8      | 3      | $(55, 48, 33)$                    | $1320 = 2^2 \cdot 3 \cdot 5 \cdot 11$           |
| 8      | 7      | $(15, 112, 113)$                  | $840 = 2^3 \cdot 2 \cdot 3 \cdot 5 \cdot 7$    |

$(\kappa^2 - l^2, 2\kappa l, \kappa^2 + l^2)$, $\kappa > l > 0$, $\kappa \not\equiv l \mod 2$, (namely $\kappa - l = 2r + 1$, $r \geq 0$).

$n = \frac{a+b}{2m^2} \in \mathbb{N}_{\text{congr}}$, with $n$ the Pythagorean-triangle area square-free part.

With respect to the formula (5) one has taken $s = 1$, namely one considers primitive Pythagorean triangles only.

Proof. If $n$ is a strong-congruent number, let $[n|a, b, c]$ be its right triangle of area $n$. Set $r = \frac{a-b}{2}$, $t = \frac{a+b}{2}$. Then we get $t^2 - r^2 = ab = 2n$ and $t^2 + r^2 = \frac{a^2 + b^2}{2} = \frac{c^2}{2}$, hence $s = \frac{c}{2}$. Vice versa, if there exist three positive rational numbers $0 < r < s < t$, satisfying above conditions (i) and (ii), then we can identify a congruent right triangle $[n|a, b, c]$, with $a = r + t$, $b = t - r$, $c = 2s$. In fact, one has $a^2 + b^2 = 4(r^2 + t^2) = 2s^2 = c^2$ and $ab = r^2 - t^2 = 2n$. \hfill \Box

Remark 2.10. Theorem 2.8 and Theorem 2.9 do not give a way to always obtain, after a finite number of steps, an answer. In fact, the set $\mathbb{P}_{\text{yth}}$ is infinite. For example if $\kappa = 5$ and $l = 4$, one has $(\kappa^2 - l^2)\kappa l = m^2 \cdot 5$. (See Tab. 1.) Really we get $9 \cdot 5 \cdot 4 = m^2 \cdot 5$, hence $m^2 = 36 = 2^2 \cdot 3^2 = 6^2$, namely $m = 6$. This means that the strong-congruent right triangle is $[5|\frac{\kappa^2 - l^2}{6}, 2l, \frac{\kappa^2 + l^2}{6}] = [5|\frac{15}{6}, 20, \frac{11}{6}]$, that has just area 5. But whether $n$ is not a strong-congruent number this process cannot stop! Therefore Theorem 2.8 does not solve the problem to find an useful algorithm to decide whether a squarefree integer is a strong-congruent number. However, since the mapping $b : \mathbb{P}_{\text{yth}} \to \mathbb{N}_{\text{congr}}$ is surjective, all the possible strong-congruent numbers can be obtained as square-free part of numbers $q \in \mathbb{P}_{\text{yth}}$.

Lemma 2.11 (Congruent numbers as rational points of elliptic curves). A positive rational number $q \in \mathbb{Q}$ is congruent iff the equation $q^2 = x^3 - q^2 \cdot x$ in the plane $\mathbb{R}^2$, has a rational point, $P$, i.e., a point that has rational coordinates, $P = (x_P, y_P) \in \mathbb{Q}^2$, with $y_P \neq 0$. This justifies the definition of congruent numbers as rational elliptic points.
\* In particular, if \( [a,b,c] \) is a congruent right triangle class, \( q \in \mathbb{Q}_{\text{congr}} \), one has that \( P = (x_P, y_P) = \left( \frac{q \cdot a}{c-a}, \frac{2q^2}{c-a} \right) \) is a rational point on \( y^2 = x^3 - q^2 x \). Vice versa, if \( P = (x_P, y_P), y_P \neq 0 \), is a rational point on \( y^2 = x^3 - q^2 x \), then the congruent right triangle class \( [a,b,c] \) has \((a, b, c) = \left( \frac{x^2 - q^2}{y}, \frac{2q^2x}{y}, \frac{x^2 + q^2}{y} \right) \).

**Proof.** This can be directly proved by considering that the set
\[
A[q] = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 = c^2, a \cdot b = 2q, q \in \mathbb{R} \}
\]
is in correspondence one-to-one with the set
\[
E[q] = \{(x, y) \in \mathbb{R}^2 \mid y^2 = x^3 - q^2 x, y \neq 0, q \in \mathbb{R} \}.
\]
This is realized by the following explicit expressions of \((x, y)\) by means of \((a, b, c)\) and \(q \in \mathbb{R} \):
\[
x = q \frac{a + c}{b}, \quad y = 2q^2 \frac{a + c}{b^2},
\]
and the converse expressions:
\[
a = \frac{x^2 - q^2}{y}, \quad b = 2q \frac{x}{y}, \quad c = \frac{x^2 + q^2}{y}.
\]
In fact, one can directly verify that the point \((x, y) \in \mathbb{R}^2\), given in (11), satisfies equation \( y^2 - x^3 + q^2 x = 0 \), when \((a, b, c) \in A[q]\). Furthermore, by using expressions (12) for \( a, b, c \) in (11) one has the identity \( x = x \) and \( y = y \), that proves that the map \( E[q] \rightarrow A[q] \) given in (12) is just the inverse of the one \( A[q] \rightarrow E[q] \) given in (11).

Furthermore, \(a, b, c\) are positive iff \( x\) and \( y\) are positive. By restriction on \( \mathbb{Q} \subset \mathbb{R} \), we get that a positive rational number \( q \in \mathbb{Q} \) is congruent iff the equation \( y^2 - x^3 - q^2 x = 0 \) has a rational point with \( y \neq 0 \). Let us also underline that equation (13) seen in the field of real numbers, namely with \( a, b, c \in \mathbb{R} \), admits always solutions under the condition \( c \geq 2\sqrt{q} \), for any \( q \in \mathbb{R} \).

\[
\begin{align*}
\begin{cases}
\Gamma & \quad a^2 + b^2 - c^2 = 0 \\
\Upsilon & \quad ab - 2q = 0
\end{cases}
\end{align*}
\]
In fact the first equation in (13) represents in the plane \( \mathbb{R}^2, (a, b) \), a circle \( \Gamma \) of center \( O = (0, 0) \) and radius \( c \). The second equation in (13) represents an equilateral hyperbola \( \Upsilon \) of center \( O \) and vertex \( V = (\sqrt{2q}, \sqrt{2q}) \). Thus \( \Gamma \cap \Upsilon \neq \emptyset \) iff \( c \geq \sqrt{ov} = 2\sqrt{q} \), for any fixed integer \( q \in \mathbb{R} \). Therefore, taking into account the above considerations, we conclude that for any positive rational \( q \) the set of solutions of (13) for the field \( \mathbb{R} \), is an elliptic curve \( E[q] \) of equation \( y^2 - x^3 + q^2 x = 0 \), in the plane \( \mathbb{R}^2, (x, y) \).\footnote{It is useful to emphasize that in \( \mathbb{R}^3 \), where \((a, b, c)\) represents a point, equations (\( \Gamma \)) and (\( \Upsilon \)) in (13), identify surfaces, that we just denote with the symbols \( \Gamma \) and \( \Upsilon \), respectively. Therefore \( E[q] = \Gamma \cap \Upsilon \), namely \( E[q] \) is the curve intersection of such surfaces. From above calculations we know that such a curve belongs to a plane \( \pi \subset \mathbb{R}^3 \), where with respect to a suitable coordinate system \((x, y)\), it is represented by the equation \( y^2 - x^3 + q^2 x = 0 \). By considering the natural inclusions \( \mathbb{R}^2 \subset \mathbb{R}^3 \cup \{ \infty \} = \mathbb{S}^2, \mathbb{R}^1 \subset \mathbb{R}^3 \cup \{ \infty \} = \mathbb{S}^3 = G_{1,4}(\mathbb{R}^4) \approx \mathbb{P}^3(\mathbb{R}), \) where \( G_{1,4}(\mathbb{R}^4) \) is the oriented Grassmann manifold of oriented 1-dimensional subspaces of \( \mathbb{R}^4 \) that pass through the origin \( O \in \mathbb{R}^4 \), we obtain a more satisfactory representation of \( E[q] \). (See Fig. 1 and Fig. 2 for some pictures showing that such curves are represented by two connected components). (Recall that \( G_{1,4}(\mathbb{R}^4) \) is an analytic manifold of dimension 3 with \( \pi_1(G_{1,4}(\mathbb{R}^4)) = Z_2 \) and having the cell.
find rational points of $E[q]$. In fact a rational solution of (13) is in correspondence one-to-one with the rational points of $E[q]$, as can be directly verified by using the above transformation $(a, b, c) \leftrightarrow (x, y)$.

\[\square\]

**Lemma 2.12** (Congruent numbers classes as set of rational points on the same class of elliptic curves). If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a rational-conform transformation of $\mathbb{R}^2$, sending a congruent rational right triangle $[q,a,b,c]$ into a conform one $[q = s^2q]$ then we get also a set of solutions of these equations by changing the parameters $a, b, c$.

**Proof.** In fact the conform transformation $f$ is given by $f : (x, y) \mapsto (\bar{x} = sx, \bar{y} = sy)$. This transformation has the effect to reduce or augment any right triangle into another one having the same angles, hence with sides that are parallel to the original ones. Instead the transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the three straight-lines passing for $[q,x,y]$ is not a conform transformation of $\mathbb{R}^2$, but is conform only. We can identify two elliptic curves $E[q]$ and $E[\bar{q}]$, $\bar{q} = s^2q$, by means of the induced diffeomorphism $\varphi : E[q] \to E[\bar{q}]$.

**Proof.** In fact the conform transformation $f$ is given by $f : (x, y) \mapsto (\bar{x} = sx, \bar{y} = sy)$. This transformation has the effect to reduce or augment any right triangle into another one having the same angles, hence with sides that are parallel to the original ones. Instead the transformation $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ of the three straight-lines passing for $[q,x,y]$ is not a conform transformation of $\mathbb{R}^2$, but is conform only. We can identify two elliptic curves $E[q]$ and $E[\bar{q}]$, $\bar{q} = s^2q$, by means of the induced diffeomorphism $\varphi : E[q] \to E[\bar{q}]$.

**Lemma 2.13** (Rational points identified by a congruent right triangle). If $[q,a,b,c]$ is a congruent right triangle and $P = (x, y)$ is the corresponding rational point on the elliptic curve $E[q] : y^2 = x^3 - q^2x$, identified by (11), then we can identify also six other rational points on $E[q]$ by means of Tab. 2. These points are intersections with $E[q]$ of the three straight-lines passing for $P$ and $(\bar{q}, 0)$, $(0, 0)$, $(q, 0)$ respectively and their reflections with respect to the $x$-axis.

**Proof.** The proof follows directly from the fact that if $(a, b, c)$ is a solution of the set of equations (13) then we get also a set of solutions of these equations by changing sign to the parameters $a, b, c$, with the condition that $a$ and $b$ have the same sign. (See Tab. 2.)
Table 2. Rational points on elliptic curve $E[q]: y^2 = x^3 - q^2x$, related by symmetries.

| $(a, b, c)$ | $P = (x, y) = (a^{b+c}, 2q^2 a^{b-c})$ | $(\ast): (\frac{a}{q}, \frac{b}{q})$ | $(\ast\ast): (12, 36)$ |
|-------------|----------------------------------|----------------------------------|-------------------------|
| $(-a, -b, -c)$ | $P_1 = (x_1, y_1) = (q^{a+b}, -2q^2 q^{a-b})$ | $(\ast): (\frac{a}{q}, -\frac{b}{q})$ | $(\ast\ast): (12, -36)$ |
| $(a, b, c)$ | $P_2 = (x_2, y_2) = (q^{a+b}, 2q^n q^{a-b})$ | $(\ast): (-4, -6)$ | $(\ast\ast): (-3, -9)$ |
| $(-a, -b, -c)$ | $P_3 = (x_3, y_3) = (-q^{a+b}, 2q^n q^{a-b})$ | $(\ast): (-4, 6)$ | $(\ast\ast): (-3, 9)$ |
| $(b, a, -c)$ | $P_4 = (x_4, y_4) = (n^{b+c}, 2q^2 n^{b-c})$ | $(\ast): (-\frac{a}{q}, \frac{b}{q})$ | $(\ast\ast): (-2, -8)$ |
| $(-b, -a, c)$ | $P_5 = (x_5, y_5) = (q^{b-a}, 2q^n q^{b-a})$ | $(\ast): (-\frac{a}{q}, \frac{b}{q})$ | $(\ast\ast): (2, 8)$ |
| $(b, a, c)$ | $P_6 = (x_6, y_6) = (n^{b+c}, 2q^2 n^{b-c})$ | $(\ast): (15, 50)$ | $(\ast\ast): (18, 72)$ |
| $(-b, -a, -c)$ | $P_7 = (x_7, y_7) = (n^{b-a}, 2q^2 n^{b+c})$ | $(\ast): (15, -50)$ | $(\ast\ast): (18, -72)$ |

$(\ast)$: Here is reported the corresponding example $q = 5$. (See also Tab 1 and Tab. 4.)

$(\ast\ast)$: Here is reported the corresponding example $q = 6$. (See also Tab 1 and Tab. 4.)

Proposition 2.14 (Mordell-Weil theorem [19, 31, 32, 25]). • We can associate to each congruent number, or elliptic curve representing a congruent number, a finitely generated abelian group, $E[q]$. The structure of $E[q]$ is given in (14).

\[ E[q] = \mathbb{Z}^r \bigoplus T[q] = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \bigoplus (\oplus_{1 \leq i \leq s} \mathbb{Z}r_i), \quad r, s < \infty. \]

The rank of the elliptic curve $E[q]$ is the rank of $E[q]$, i.e., the number $r$ representing the number of independent points of infinite order.9 Furthermore $T[q] = \oplus_{1 \leq i \leq s} \mathbb{Z}r_i \subset E[q]$ is called the torsion subgroup of $E[q]$.

• If $E[q]$ and $E[q']$ are two equivalent congruent elliptic curves, $(q = s^2q, s \in \mathbb{Q})$, then there is a canonical isomorphism $\varphi_r : E[q] \cong E[q]$. In other words the elliptic curve $E[q]$ and $E[q']$ are isogenous. (See also Theorem 3.1.)

Proof. • On an elliptic curve $E[q]$ can be defined a multiplication, respect to which it becomes an abelian group $G[E]$. On $\mathbb{R}^2$, the more general expression of an elliptic curve is of the form $y^2 = x^3 + ax + b$, hence a non-singular plane curve. One considers compactified such a curve by adding the $\infty$ point in the Alexandrov compactification of $\mathbb{R}^2$: $S^2 = \mathbb{R}^2 \cup \{\infty\}$. The point $\{\infty\}$ is the identity element in the natural group structure defined on $E[q]$. The set of rational points, excluding $\infty$, form a subgroup $E[q]$ of $G[q]$.11 It can be seen that the torsion points of an

9This means that there exists a finite sub-set of the rational points of $E[q]$, such that all the other rational points can be generated by the abelian group law.

10This theorem can be also generalized to the case where instead of $Q$ one works with a number field (or algebraic number field) $K$, that is a finite field extension $K/Q$ of $Q$, namely $K$ is a finite dimensional $Q$-vector space. (See [18].) Recall that any field extension $L/K$ is called algebraic if any element $x \in L$, is algebraic over $K$, i.e., there exists $P \in K[x]$, such that $P(x) = 0 \in L$. A transcendental extension is an extension that is not algebraic. Transcendental extensions are of infinite degree. (The degree, $[L : K]$, of an extension $L/K$, is the dimension of the $K$-vector space $L$.) Therefore all finite extensions are algebraic. The converse is not true. An example of transcendental extension is $E/Q$, since $\dim_{\mathbb{Q}} \mathbb{R} = \infty$. In fact the Napier’s constant (or Euler’s number) $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n = \sum_{0 \leq n \leq \infty} \frac{1}{n!}$ is an irrational number and cannot be a root of some polynomial $P(x) \in \mathbb{Q}[x]$, $x \in \mathbb{R}$. Instead $C/R$ is an example of algebraic extension, one has $[C : R] = 2$, with canonical basis $\{1, i\} \subset \mathbb{C}$.

11Let $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ be two points on the elliptic curve $E$, with $x_P \neq x_Q$, then $P + Q + R = 0$, identifies another point $R$ on $E$. The addition defined in this way is called the addition by means of secant (or tangent). More precisely, in the case that the line passing for $P$
elliptic curve are those with $y = 0$. (See in Example 2.18.) Therefore existence of rational points with $y \neq 0$ is equivalent to say that the elliptic curve has rank positive. The structure (14) follows from the fact that $E[n]$ is a finite generated abelian group [19, 31, 32].

• The isomorphism between structure groups $E[q]$ and $E[q']$ is the one induced by the diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ relating the points of elliptic curves $E[q]$ and $E[q']$.

(see Lemma 2.12).

**Proposition 2.15** (Mazur’s theorem [17]). The torsion groups of elliptic curves can be only of the following types $\mathbb{Z}/N\mathbb{Z}$, $N \in \{1, 2, \cdots, 10, 12\}$, or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z}$, $N \in \{1, 2, 3, 4\}$.

**Proposition 2.16** (Nagell-Lutz theorem). Let $y^2 = x^3 + ax^2 + bx + c$ defines a non-singular cubic curve $C$ with integer coefficients $a$, $b$, $c$, and let $\Delta = -4a^3c + a^2b^2 + 18abc - 4b^3$ be the discriminant of the cubic polynomial on the right side. If $P = (x, y)$ is a rational point of finite order on $C$, for the elliptic curve group law, then:

(i) $x$ and $y$ are integers;

(ii) either $y = 0$, in which case $P$ has order two, or else $y$ divide $\Delta$, which implies that $y^2$ divides $\Delta$.

• (Generalized form) For non-singular curve whose Weierstrass form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, has integer coefficients, any rational point $P = (x, y)$ of finite order, (namely torsion-point), must have integer coordinates,

and $Q$ is tangent to $E$ in $Q$, then one writes $P + Q + Q = 0$, or $P + P + Q = 0$ (only tangent in $P$), or $P + Q = 0$ (only secant in $P$ and $Q$).

An elliptic curve over $\mathbb{R}$ in $\mathbb{R}^2$, $(x, y)$, defined by an equation $y^2 - P(x) = 0$, where $P(x) \in \mathbb{R}[x]$ is a cubic polynomial with distinct roots. By a suitable diffeomorphism of $\mathbb{R}^2$, equation $y^2 = P(x) = 0$ can be rewritten in the Weierstrass form: $E = E_{A, B} : y^2 = 4x^3 + Ax + B = 0$, with $A, B \in \mathbb{R}$, and $\Delta_E := A^3 - 27B^2 \neq 0$. If [The polynomial $P(x) = 4x^3 + Ax + B$ has distinct roots if its discriminant $\text{disc}(P(x)) = 16\Delta_E \neq 0$.] By diffeomorphisms of $\mathbb{R}^2$, $(x, y) \mapsto (x^3, x^2y)$, $s \in \mathbb{R}$, equation defining $E$ transforms into the following one: $E_{A_1, B_1} : y^2 = 4x^3 + A_1x + B_1 = 0$, with $A_1 = A/s^3$, $B_1 = B/s^6$. One has $\Delta_{A_1, B_1} = \Delta_{A, B}s^{-12}$. One has instead the invariant $j_E = \frac{(12A)^3}{\Delta_E} = \frac{(12A_1)^3}{\Delta_{A_1, B_1}} = j_{E_{A_1, B_1}} = \frac{(12A)^3}{A^3 - 27B^2}$. The classical Weierstrass form of an elliptic curve is $E_{a, b} : y^2 - x^3 - ax - b = 0$, with $4a^3 + 27b^2 \neq 0$, is obtained by the diffeomorphism $(x, y) \mapsto (x, 2y)$, with $a = -A/4, b = -B/4$. Then $\Delta_{E_{a, b}} = -16(4a^3 + 27b^2) = A^3 - 27B^2 = \Delta_E$ and $j_{E_{a, b}} = \frac{(12a)^3}{2E_{a, b}} = j_{E_{A, B}}$. Therefore one can state that elliptic curves in the planes $\mathbb{R}^2$, are in general of the type $y^2 = x^3 + ax + b$, with the condition $\Delta = -16(4a^3 + 27b^2) \neq 0$, in order to not be singular curves. ($\Delta$ is defined the discriminant of the elliptic curve.) An invariant for isomorphism classes is the Klein’s $j$-invariant: $j = (-48 a)^3/\Delta$. In Tab. 3 is reported the Klein’s $j$-invariant for plane elliptic curves over any field. Let us emphasize that if the characteristic of the fundamental field $K$ is neither 2 nor 3, then every elliptic curve over $K$ can be written in the form $y^2 = x^3 - px - q$, where $p$ and $q$ are elements of $K$ such that the polynomial $P(x) = x^3 - px - q$ does not have any double roots. In the particular case of $y^2 = x^3 - q^2x$ then its Klein’s $j$-invariant is $j = 1728$. So all elliptic curves of this type are diffeomorphic with the same Klein’s $j$-invariant $j = 1728$. If the characteristic of $K$ is 2 or 3, then the general form of elliptic curve is more complex. In characteristic 3, it assumes the expression $y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$, such that the polynomial $P(x) = 4x^3 + b_2x^2 + 2b_4x + b_6$ has distinct roots. In characteristic 2, the most general equation is $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, provided that the variety it defines is non-singular. (If characteristic were not an obstruction, each equation would reduce to the previous ones by a suitable change of variables.) In general one takes $x, y$ belonging to the algebraic closure of $K$, i.e., an algebraic extension $F/K$ that is algebraically closed, i.e., contains a root for every non-constant polynomial in $F[x]$. 

\[Q \]
Table 3. The Klein’s $j$-invariant for plane elliptic curves over any field.

| $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ | $\triangle = -b_2b_8 + 9b_2b_6b_8 - 8b_4^2 - 27b_6^2$ | $j = c_4^3/\triangle$ |
|---|---|---|
| $b_2 = a_1^2 + 4a_2$ | $b_4 = a_1a_2 + 2a_4$ | $b_6 = a_1^2 + 4a_6$ |
| $b_8 = a_1^4b_2 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2$ | $c_4 = b_2^2 - 24b_4$ | $c_6 = -b_2^2 + 36b_2b_4 - 216b_6$ |

If the field has characteristic different from 2 or 3, one has $j = 1728 \frac{c_4^2}{c_4^4 - d_6}$. See Tab. 10, in Appendix C, for more information on $j$ and $\triangle$ in the case of elliptic curves over $\mathbb{C}$.

or else have order 2 and coordinates of the form $(x = \frac{m}{4}, y = \frac{n}{8})$ for $m$ and $n$ integers.

Proof. See [16, 25, 26].

Corollary 2.17. A number $q \in \mathbb{Q}$ is congruent iff there exist infinitely many rational points $P = (x_P, y_P)$ on the elliptic curve $E[q]$.

Example 2.18. In the following we list some examples of congruent numbers identifying congruence equivalence classes.

- (Fermat’s theorem) [1640] The number 1 is not (strong-)congruent. More generally, no square number can be a congruent number. (See, e.g., [7].)
- $n \equiv 3 \mod 8$ is not a congruent number, but $2n$ is a congruent number.
- $n \equiv 5 \mod 8$ is a congruent number.
- $n \equiv 7 \mod 8$ is a congruent number and $2n$ is so.
- In each of the congruence classes 5, 6, 7 mod 8, there are infinitely many square-free congruent numbers with $k$ prime factors.

In Tab. 4 are reported some examples of strong-congruent numbers and the sides of the corresponding congruent right triangles.

- If the elliptic curve $E[n] : y^2 = x^3 - n^2x = 0$, has $n \in \mathbb{N}_{congr}$, then its group $E[n]$ of rational points, has order greater than 4: $|E[n]| > 4$. In fact its torsion points are the following:

\[(15) \quad \{\infty, (x = -n, y = 0), (x = 0, y = 0), (x = n, y = 0)\}.

This can be also obtained from the Nagell-Lutz theorem In fact, the discriminant of the cubic polynomial $P(x) = x^3 - n^2x$ is $\Delta = -4n^6$ and by the Nagell-Lutz theorem one has $y^2 = 4n^6$, so we take all possible solutions $y^2/(2n^3 = 2p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})$, where $2n^3 = 2p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ is the prime factorization. Therefore, we get $y \in \{\pm 2^{t_1}p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k} | t = 0, 1, 0 \leq r_i \leq a_i\}$. Since none of these values satisfies equation $y^2 = x^3 - n^2x = 0$, it follows that all torsion points of $E[n]$ are the ones reported in (15), and $|T[n]| = 4$.

Theorem 2.19 (Criterion for congruent number $q \in \mathbb{Q}$). A number $q \in \mathbb{Q}$ is congruent if $\text{rank}(E(q)) > 0$.

Proof. In fact, if $\text{rank}(E(q)) > 0$ it means that there are rational points in the elliptic curve $y^2 = x^3 - q^2x$, hence from Lemma 2.11 and Lemma 2.12 it follows that $q$ is representing a congruent number class. □
Table 4. Examples of strong-congruent numbers $n \in \mathbb{N}_{congr}$ for square-free numbers $n \in \square \mathbb{N}$, $1 \leq n \leq 65$, some corresponding right triangles and non-congruent numbers $n \notin \ker(L_{\bullet})$.

| $n \in \square \mathbb{N}$ | $n \in \mathbb{N}_{congr}$ | $[m, a, b, c]$ |
|--------------------------|-----------------------------|----------------|
| (⋆)(1), (2), (3), [5]   | 5                           | [3, 20, 21]   |
| [6]                     | 6                           | [6, 3, 4, 5]  |
| [7]                     | 7                           | [7, 42, 47, 13] |
| (10), (11), [13], [14], [15], 17, (19), [21] | 21 | [21, 2, 12, 13] |
| [22], [23], 26, 29, [30] | 30 | [30, 5, 12, 13] |
| 31, (33), [34], 35, [37], [38], [39], [41] | 41 | [41, 60, 12, 13] |
| 42, 43, [46], [47], 49, [51], 53, 55, [57], [58], [59], 61, 62, [65] | 65 | [65, 60, 12, 13] |

The square-free integers between square-brackets in the column of $\square \mathbb{N}$ denote strong-congruent numbers.

The square-free integers between round-brackets in the column of $\square \mathbb{N}$ denote non-congruent numbers, namely $n \notin \ker(L_{\bullet}) \subseteq \square \mathbb{N}$. (See Lemma 4.11.)

(⋆) Since 1 is not a strong-congruent number, it follows that also $m^2$ is so, for any $m \in \mathbb{N}$.

The proof that 5 and 7 are strong-congruent numbers was first given by Fibonacci [20]. He stated also that 1 is not a congruent number, but a first proof has been given by Fermat (See, e.g., in [9].)

Fig. 1. Representation of the elliptic curve (red-curve) $E[5]: y^2 = x^3 - 25x$. $E[5] = E[5]_1 \sqcup E[5]_2$ where $E[5]_1$ is the closed curve on the left-hand of the $y$-axis and and $E[5]_2$ is the other part on the right-hand of the $y$-axis. On $E[5]_1$ is reported the rational point $P = (\frac{25}{4}, \frac{75}{8})$ corresponding to the congruent right triangle $(\frac{5}{2}, 10, 11)$. The violet-curve represents the elliptic curve $E[6]: y^2 = x^3 - 36x$. One has $E[5] \cap E[6] = \emptyset = (0, 0) \in \mathbb{R}^2$. On $E[6]_1$ is reported the rational point $Q = (12, 36)$ corresponding to the congruent right triangle $(3, 4, 5)$. For symmetry properties one can identify also six further rational points on $E[5]$ and $E[6]$ respectively. Some of these are also on the left-hand connected component. (For details see Tab. 1.)

Definition 2.20 (The congruent number problem). Given a positive square-free integer $n \in \square \mathbb{N}$, there is a simple criterion to decide whether $n \in \mathbb{N}_{congr}$, i.e., $n$ is a strong-congruent number?
3. The Birch Swinnerton-Dyer conjecture

In this section we shall consider some important well-known results about congruent number problem and the related Birch Swinnerton-Dyer conjecture (BS-D conjecture) that will be used in the next section. (For complementary information see also literature on this subject in References, and in particular look e.g., to the book by Koblitz [14].)

Let start with the following theorem.

**Theorem 3.1** (Modularity theorem or Taniyama-Shimura-Weil conjecture). Any elliptic curve $E$ over $\mathbb{Q}$ is a modular curve, i.e., there exists a surjective morphism $\varphi : X_0(N) \to E$, where $\varphi$ is a rational map with integer coefficients, and $X_0(N)$ is the classical modular curve, for some integer $N$. This mapping is called a modular parametrization of level $N$. The conductor of $E$ is the smallest integer $N$ for which such a parametrization can be found.

**Proof.** This theorem has been proved in a particular case by A. Wiles [33]. For the general case a proof has been given in [2]. For related subjects see also [8, 14, 15, 22, 24, 23, 28, 31, 32, 33, 29].

**Definition 3.2** (The Hasse-Weil-function of elliptic curve and Hasse-Weil conjecture). Given an elliptic curve $E$ over $\mathbb{Q}$ of conductor $N$, then $E$ has good reduction at all primes $p$ not dividing $N$, it has multiplicative reduction at the primes $p$ that exactly divide $N$ (i.e., such that $p$ divides $N$, but $p^2$ does not; this is written $p \parallel N$), and it has additive reduction elsewhere (i.e., at the primes where $p^2$ divides $N$).

One defines Hasse-Weil-function of $E$, the function $Z_{E, \mathbb{Q}}(s)$, given in (16).\(^{14}\)

---

\(^{13}\) $X_0(N)$ is a compact Riemann surface, defined by $X_0(N) = H^*/\Gamma_0(N)$, where $H^* = H \cup \mathbb{Q} \cup \{\infty\}$ is the extended complex upper-half plane $H \subset \mathbb{C}$, and $\Gamma_0(N)$ is the congruence subgroup of level $N$, of the modular group $SL(2, \mathbb{Z})$, defined by $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \mod N \right\}$, for some positive $N \in \mathbb{N}$. Two elliptic curves $E$ and $E'$ are isogenous if there is a morphism of varieties, defined by a rational map between $E$ and $E'$, which is also a group homomorphism between the corresponding groups $E$ and $E'$ sending $\infty \in E$ to $\infty \in E'$, i.e., conserving identity elements. The isogenies with cyclic kernel of degree $N$, (cyclic isogenies), correspond to points on $X_0(N)$:

\[
0 \longrightarrow \mathbb{Z}_N \longrightarrow E \xrightarrow{f} E' \xrightarrow{\varphi} X_0(N).
\]

When $X_0(N)$ has genus 1, then $X_0(N) \cong E$, which will have the same $j$-invariant. For example $X_0(11)$ has $j = -2^{12}11^{-5}31^3$, and is isomorphic to the curve $y^2 + y = x^3 - x^2 - 10x - 20$. (For relations between Riemann surfaces and modular curves see also Appendix B.) This means that this elliptic curve can be parametrized by means of two functions $(x = x(z), y = y(z))$, where $x(z)$ and $y(z)$ are modular functions of weight 0 and level 11: in other words they are meromorphic, defined on the upper half-plane $\Im(z) > 0$ and satisfy $(x(z) = x(\frac{-1}{z}), y(z) = y(\frac{-1}{z}))$, for all integers $a, b, c, d$ with $ad - bc = 1$ and $11|c$.

\(^{14}\) Taking into account the functional equation for $\zeta(s)$, $\zeta(s) = f(s)\zeta(1-s)$, where $f(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)$, one can rewrite $Z_{E, \mathbb{Q}}(s)$ in the form

\[
Z_{E, \mathbb{Q}}(s) = f(s)\zeta^2(1-s) \prod_{p \in P, p \mid N, a_p = p+1} (1 - ap^{-s} + p^{1-2s}) \prod_{p \in P, p \mid N, a_p = \pm 1} (1 - ap^{-s}).
\]
Theorem 3.4 (Tunnell's theorem [30])

Definition 2.20. In fact, one has the following proposition. Whether Conjecture 3.3 is true one can solve the congruent number problem in is given by more refined arithmetic data attached to $E$. The Birch and Swinnerton-Dyer conjecture [1] implies any positive integer $n = 5, 6, 7 \mod 8$ is a congruent number [27].

Conjecture 3.3 (The Birch and Swinnerton-Dyer conjecture [1]). The rank $k$ of the abelian group $E[K]$ of the elliptic curve $E$ over a number field $K$, is the order of the zero of the Hasse-Weil function $L(E, s)$ at $s = 1$:

$$L(E, s)^{(r)}|_{s=1} = 0, r < k, \quad L(E, s)^{(k)}|_{s=1} \neq 0.$$ 

Furthermore, the non-zero coefficient of the Taylor expansion of $L(E, s)$ at $s = 1$, is given by more refined arithmetic data attached to $E$ over $K$ (Wiles 2006).

Whether Conjecture 3.3 is true one can solve the congruent number problem in Definition 2.20. In fact, one has the following proposition.\(^{15}\)

**Theorem 3.4** (Tunnell’s theorem [30]). For a given square-free integer $n \in \square \mathbb{N}$, define the associated integers given in (18).

$$\begin{align*}
A_n &= \# \{(x, y, z) \in \mathbb{Z}^3 \mid n = 2x^2 + y^2 + 32z^2\} \\
B_n &= \# \{(x, y, z) \in \mathbb{Z}^3 \mid n = 2x^2 + 2y^2 + 8z^2\} \\
C_n &= \# \{(x, y, z) \in \mathbb{Z}^3 \mid n = 8x^2 + 2y^2 + 64z^2\} \\
D_n &= \# \{(x, y, z) \in \mathbb{Z}^3 \mid n = 8x^2 + 2y^2 + 16z^2\}.
\end{align*}$$

We get the implications (19).

$$\begin{align*}
\text{If } n \in \mathbb{N}_{\text{congr}}, & \quad n = 2m + 1, \quad \Rightarrow \quad 2A_n = B_n, \\
\text{If } n \in \mathbb{N}_{\text{congr}}, & \quad n = 2m, \quad \Rightarrow \quad 2C_n = D_n.
\end{align*}$$

• Conversely if the Birch-Swinnerton-Dyer conjecture holds for elliptic curves of the form $y^2 = x^3 - n^2x$, $n \in \square \mathbb{N}$, then the framed equalities on the right in (19), are sufficient to conclude that $n \in \mathbb{N}_{\text{congr}}$.

**Example 3.5.** With respect to Tab. 4 we can verify that Tunnell’s theorem works well.\(^{16}\) For example we get the following:

• $(n = 1)$. $A_1 = B_1 = 2$, hence $2A_1 \neq B_1$. [1 is not a congruent number.]

\(^{15}\)Previous results of the Tunnell’s theorem, were ones by Stephens proving that the Conjecture 3.3 implies any positive integer $n = 5, 6, 7 \mod 8$ is a congruent number [27].

\(^{16}\)Let us underline that Tunnell’s theorem works only for square-free integers $n \in \square \mathbb{N}$.
Theorem 3.7 (Coates-Wiles theorem [6]). • To each elliptic curve $E[n] : y^2 = x^3 - n^2x$, $n \in \mathbb{N}$, there is associated a number $L(E[n])$.
  • If $E[n]$ has infinitely many rational points, then $L(E[n]) = 0$.
  • If $L(E[n])$ is not zero, then $n$ cannot be a strong-congruent number.
  • (Tunnell’s expression for $L(E[n])$.) For any $n \in \mathbb{C}$ one can write

\begin{equation}
L(E[n]) = \begin{cases} 
  C \cdot (A_n - \frac{B_n}{2}) & \text{if } n \text{ is odd} \\
  C \cdot (C_n - \frac{D_n}{2}) & \text{if } n \text{ is even}
\end{cases}
\end{equation}

where $C$ is a non-zero number, and $A_n$, $B_n$, $C_n$, $D_n$ are defined in Theorem 3.4.
  • If the BS-D conjecture is true the condition $L(E[n]) = 0$ is also sufficient to state that $n$ is congruent, (i.e., $n \in \mathbb{N}_{\text{cong}}$, hence $E[n]$ has infinitely many rational points).

4. Elliptic and Congruent Bordism Groups

In this section we shall relate the congruent numbers problem and the related BS-D conjecture to suitable bordism groups and to homotopies between elliptic curve inducing isomorphisms between such bordism groups. These algebraic topologic tools will give us the way to obtain, via the Tunnell’s theorem, a workable criterion that in some finite steps allows us to know if a square-free integer is a strong-congruent number. Furthermore, taking into account the realtions between Tunnell’s theorem and BS-D conjecture, we get as a by-product an indirect way to consider the BS-D conjecture true.

Definition 4.1 ($n$-Elliptic bordism groups). Let $n \in \mathbb{N}$. We say that two points $P, Q \in \mathbb{R}^2$ $n$-elliptic bord if there exists an elliptic curve $E[n] : y^2 = x^3 - n^2x$, such that $P, Q \in E[n]$, and $P \sqcup Q = \partial \Gamma$, $\Gamma \subset E[n]$. Since the 0-bordism group $\Omega_0(\mathbb{R}^2) \cong \mathbb{Z}_2$, it follows that if $P \in [Q] \in \Omega_0(\mathbb{R}^2) \cong \mathbb{Z}_2$, it does not necessitate that $P \sqcup Q = \partial \Gamma$, for some $\Gamma \subset E[n]$. Therefore the $n$-elliptic bordism is a new equivalence relation in $\mathbb{R}^2$ and we denotes by $\Omega_{E[n]}$ the corresponding set of equivalence classes, that we call the $n$-elliptic bordism group.

Theorem 4.2. One has the canonical isomorphism

\begin{equation}
\Omega_{E[n]} \cong \Omega_{E[\sqrt{n}^2]} \forall s \in \mathbb{N}.
\end{equation}

In particular, two integers $n, \tilde{n} \in \mathbb{N}$, belonging to the same equivalence class of congruent numbers, identify isomorphic elliptic bordism groups.
In fact the elliptic curve $E[n] = E[n]_1 \sqcup E[n]_2$ where $E[n]_1$ is the closed curve on the right-hand of the (compactified) $y$-axis and and $E[n]_2$ is the other part on the left-hand of the $y$-axis. Similarly for $E[n'] = E[n']_1 \sqcup E[n']_2$. When $n, n' \in \mathbb{N}_{\text{congr}}$ we can talk about congruent elliptic curves. The points $n$ and $n'$ are placed on the (compactified) $x$-axis. $S^2 = \mathbb{R}^2 \cup \{\infty\}$. In the figure (B) is represented the relation between the compactified elliptic curve on $S^2 \subset S^3 \cong G^*_1(\mathbb{R}^4) \subset \mathbb{R}^4$, and its stereographic projection, of equation $y^2 - x^3 + 2x = 0$, on the plane $\pi$, tangent at the south pole $O$ of $S^2$. In figure (C) is represented $S^3 \subset \mathbb{R}^4$, and $S^2 \subset S^3$, with the two components $E[n]_1, E[n]_2$ of an elliptic curve represented in read. The plane $\pi$, tangent to $S^2$, at the south pole $O$, is represented by a grey strong straight-line. There the stereographic projection from the north pole $\infty$, identifies the two components of the plane curve, besides the point $-n$ and $n$. (Here $G^*_1(\mathbb{R}^4)$ is the oriented Grassman manifold of oriented 1-dimensional planes in $\mathbb{R}^4$.)

\begin{proof}
In fact the elliptic curves $E[n]$ and $E[s^2n]$ coincide up to diffeomorphisms. Furthermore, if $P = (x_P, y_P) \in E[n]$ then $\bar{P} = (\bar{x}_P, \bar{y}_P) = (s^2x_P, s^3y_P)$ is another point of $E[n]$; since the diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (s^2x, s^3y)$ transforms $E[n]$ into $E[\bar{n}], \bar{n} = s^2n$. (Lemma 2.2.) Thus $\varphi$ induces an isomorphism between the corresponding elliptic bordism groups $\Omega_{E[n]}$ and $\Omega_{E[\bar{n}]}$.
\end{proof}

**Corollary 4.3.** In particular, if the integer $m \in \mathbb{N}$, belongs to the equivalence class of a strong-congruent number $n \in \mathbb{N}_{\text{congr}}$, hence $n$ is the square-free part of $m$, then there exists a diffeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ that induces an isomorphism $\varphi_* : \Omega_{E[n]} \cong \Omega_{E[m]}$ between the corresponding elliptic bordism groups.

**Proposition 4.4.** Let $\Omega_{E[n]}$ be the $n$-elliptic bordism group, $n \in \mathbb{N}$. Then one has the isomorphism
\begin{equation}
\Omega_{E[n]} \cong \mathbb{Z}_2.
\end{equation}

\begin{proof}
In fact the elliptic curve $E[n]$ is made by two disjoint components: $E[n] = E[n]_1 \sqcup E[n]_2$. (See Fig. 1.) If $P$ and $Q$ belong to the same component, $P, Q \in E[n]_i$, $i = 1, 2$, then $P \cup Q = \partial \Gamma_i$, with $\Gamma \subset E[n]_i$. Instead, if $P \in E[n]_1$ and $Q \in E[n]_2$, then does not exist a curve $\Gamma \subset E[n]$, such that $P \cup Q = \partial \Gamma$ with $\Gamma \subset E[n]_i$. This is enough to state that isomorphism (22) holds.
\end{proof}

**Definition 4.5** ($n$-Congruent bordism groups). Let $n \in \mathbb{N}_{\text{congr}}$. We say that two points $P, Q \in \mathbb{R}^2$ $n$-congruent bord if there exists an elliptic curve $E[n]$:
Corollary 2.17). In classes, that we call the $S$ some $\Omega$ rational point of the elliptic curves $E$.

Proof. This proof is analogous to the one for the group $\Omega_{E[n]}$, $n \in \mathbb{N}_{congr}$, the corresponding set of equivalence classes, that we call the $n$-congruent bordism group. [More shortly $\Omega_{E[n]}$, is a $n$-congruent bordism group, i.e., $\Omega_{E[n]} = \Omega_{E[n]}^{\mathbb{Q}}$, if $n$ is a strong-congruent number, i.e., $n \in \mathbb{N}_{congr}$.]

Proposition 4.6. Let $\Omega_{E[n]}^\mathbb{Q}$ and $\Omega_{E[n']}^\mathbb{Q}$ be two congruent bordism groups, with $n \neq n' \in \mathbb{N}_{congr}$. Then the hypothesis that two non-zero rational points $P, Q \in \mathbb{Q}^2$, $y_P \neq 0$, $y_Q \neq 0$, satisfy the condition $P \cup Q = 0 \in \Omega_{E[n]}^\mathbb{Q}$ excludes that could be also $P \cup Q = 0 \in \Omega_{E[n']}^\mathbb{Q}$.

One has the following isomorphism: $\Omega_{E[n]}^\mathbb{Q} \cong \mathbb{Z}_2$.

Proof. In fact the intersection at finite, of the elliptic curves $E[n]$ and $E[n']$ is only the point $O = (0,0) \in \mathbb{Q}^2$. Furthermore, the point $O$ is not considered a rational point of the elliptic curves $E[n]$ and $E[n']$. (See Lemma 2.11.)

This proof is analogous to the one for the group $\Omega_{E[n]}$. In fact, the elliptic curve $E[n]$, $n \in \mathbb{N}_{congr}$, has the same structure. Furthermore, on the same connected component of $E[n]$, there exists always a rational point and its symmetric with respect to the $x$-axis.

Proposition 4.7 (Relation between congruent-bordisms and elliptic-bordisms). Let $n \in \mathbb{N}$ be a congruent integer. The isomorphism $\Omega_{E[n]}^\mathbb{Q} \cong \mathbb{Z}_2 \cong \Omega_{E[n]}$ means that any couple of points $P, Q \in E[n]$, namely belonging to the same connected component of $E[n]$, there corresponds a couple of rational points $R, S \in E[n]$.

Proof. It is useful to consider congruent elliptic curves on the compactified plane $S^2 = \mathbb{R}^2 \cup \{\infty\}$. (See Fig. 2.) Really we have the following lemma.

Lemma 4.8. Compactified elliptic curves that are obtained by the following ones $E : y^2 - x^3 - q^2 x = 0$, can be represented into $\mathbb{P}^3(\mathbb{R})$.

Proof. Since $E \subset \mathbb{R}^2$, it follows that the Alexandrov compactification $\mathbb{R} \cup \{\infty\} \cong S^2$ identifies a natural curve in $S^2$, by means of the inclusion $\mathbb{R}^2 \to S^2$. Therefore the compactified elliptic curve $E^+$ is just $E^+ = E \cup \{\infty\}$. This is just a 1-dimensional smooth compact submanifold of $S^2$, passing for the point $\infty \in S^2$. Let us recall that $S^p$ can be identified with an oriented Grassmann manifold and a symmetric space by means of the diffeomorphisms:

$$S^p \cong G^+_{1,p+1}(\mathbb{R}^{p+1}) \cong SO(1+p)/SO(1) \times SO(p).$$

(23)

(Incidentally recall that these are also Einstein manifolds, i.e., the Ricci tensor is proportional to the metric tensor.) The group $SO(1+p)$ acts transitively on $G^+_{1,p+1}(\mathbb{R}^{p+1})$. $SO(1) \times SO(p)$ is the isotropy group of the point $\pi \in G^+_{1,p+1}(\mathbb{R}^{p+1})$, where $SO(1)$ acts in the oriented 1-dimensional plane $\pi$ and $SO(p)$ acts in its orthogonal complement. Let us emphasize that forgetting the orientation, we can

\(^{17}\)Let us emphasize that $E[n]$ is an elliptic curve having infinitely many rational points. (See Corollary 2.17.)

\(^{18}\)See, e.g., [21].
Consider Grassman manifold $G_{1,1+p}(\mathbb{R}^{1+p}) \cong \mathbb{P}(\mathbb{R}^{1+p})$. On the other hand we have the following exact commutative diagram:

\[
\begin{array}{ccc}
S^0 & \rightarrow & S^p \\
\downarrow & & \downarrow \\
G^1_{1,1+p}(\mathbb{R}) & \rightarrow & G_{1,1+p}(\mathbb{R}) \\
\uparrow & & \uparrow \\
\Omega & \rightarrow & \Omega
\end{array}
\]

Diagram (24) emphasizes the fiber bundle structures $S^p \rightarrow \mathbb{P}(\mathbb{R})$ and $G^1_{1,1+p}(\mathbb{R}) \rightarrow G_{1,1+p}(\mathbb{R})$ both with fiber $S^0$. Therefore by considering the natural inclusion $S^{p-1} \hookrightarrow S^p$, such that the following diagram is commutative:

\[
\begin{array}{ccc}
(S^{p-1}, \ast, \infty) & \rightarrow & (S^p, \ast, \infty) \\
\downarrow & & \downarrow \\
(\mathbb{R}^{p-1}, 0) \cup \{\infty\} & \rightarrow & (\mathbb{R}^p, 0) \cup \{\infty\}
\end{array}
\]

one can represent $E^+ \subset S^2$ into $S^3$, hence by means of the projection $S^3 \rightarrow \mathbb{P}^3(\mathbb{R})$, we get the representation of $E^+$ into $\mathbb{P}^3(\mathbb{R})$.\footnote{The same result can be obtained by rewriting equations (9) defining $E$, in projective way, namely considering instead of the coordinates $(a,b,c) \in \mathbb{R}^3$, the homogeneous coordinates $[a,b,c,d] \in \mathbb{P}^3(\mathbb{R})$. (We skip on details.)}

In fact the point $\infty$ is a distinguished point considered the unity in the group $E[n]$, hence it is assumed a rational point. Then the isomorphism $\Omega^p_{E[n]} \cong \mathbb{Z}_2 \cong \Omega^p_{E[n]}$ can be justified, since to $P \cup Q \in 0 \in \Omega^p_{E[n]}$, one can correspond (even if not canonically) a couple of rational points $R, S \in E[n]$, such that $R \cup S \in 0 \in \Omega^p_{E[n]}$. Really since $E[n]$ contains an infinity number of rational points it follows that in the two branches of $E[n]$, bounded by $P \cup Q$ must be present at least one rational point, say $R$. Then for symmetry there exists also another rational point, say $S$, on the same connected component of $E[n]$. Therefore $R \cup S \in 0 \in \Omega^p_{E[n]}$. \hfill $\square$

**Theorem 4.9 (Elliptic and congruent-bordism groups and homotopies in $\mathbb{R}^2$).**

The isomorphisms of groups $\Omega^p_{E[n]} \cong \Omega^p_{E[n']}$, $n \neq n' \in \mathbb{N}$, can be induced by suitable homotopies of $\mathbb{R}^2$.

- **Let us assume that $n$ and $n'$ are congruent integers.** Then the homotopy relating $E[n]$ to $E[n']$ does not necessitate identify an isogeny. When this happens, it induces a group homomorphisms $E[n] \rightarrow E[n']$. In particular this recurs when $n'$ and $n$ belong to the same equivalence class of congruent numbers, namely $n' = m^2n$, with $m \in \mathbb{Q}$. In such a case the homomorphism $E[n] \rightarrow E[n']$ becomes an isomorphism.

- **In advance, if $n \in \mathbb{N}_{\text{congr}}$, and the homotopy induces an homomorphism $E[n] \rightarrow E[n']$, then $n'$ cannot be a strong-congruent number too.**

**Proof.** The motivation that two elliptic bordism groups $\Omega^p_{E[n]} \cong \Omega^p_{E[n']}$, $n \neq n' \in \mathbb{N}$, are isomorphic to $\mathbb{Z}_2$ follows from the fact that any two elliptic curves $E[n] : y^2 - x^3 + n^2x = 0$ and $E[n'] : y^2 - x^3 + n'^2x = 0$, are homotopic, i.e., there exists an homotopy (flow) in $\mathbb{R}^2$, $\varphi_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\lambda \in [0,1] \subset \mathbb{R}$, relating $E[n]$ to $E[n']$. More precisely, let us consider the following deformed elliptic curves:
state that the flow $\phi$ 

One can see that 

getting into account the one-to-one correspondence between $R$ following homotopy, (flow), in $R$. We can see that such deformed elliptic curves can be realized by means of the 

Let us, now, assume that 

In fact, starting from $E[n]$, by using the flow $\varphi_\lambda$, we get $(\frac{n}{n_\lambda})^3|y^2 - x^3 + n_\lambda^2x| = 0$. 

So deforming the equation $y^2 - x^3 + n_\lambda^2x = 0$ of $E[n]$, by means of the flow $\varphi_\lambda$, we get the deformed elliptic curves $E[n_\lambda]$, hence also $E[n_\lambda']$ for $\lambda = 1$. Thus we can state that the flow $\varphi_\lambda$ on $R^2$ induces the homotopy $\psi_\lambda$ between the elliptic curves $E[n]$ and $E[n']$. This situation is resumed in the diagram (29).

There the vertical dots-lines mean that the relation between the elliptic curves on the top horizontal line is induced from the flow on $R^2$, namely the plane where are embedded these curves. Therefore (28) has the effect to induce an isomorphism between the bordism groups $\Omega_{E[n]}$ and $\Omega_{E[n']}$. 

- Let us, now, assume that $n$ and $n'$ are both integer congruent numbers. Taking into account the one-to-one correspondence between $E[n]$ and $A[n]$, $\forall n \in N$, (Lemma 2.11), we get the following commutative diagram:

---

Note that $\varphi_\lambda$ is a diffeomorphism $R^2 \rightarrow R^2$, for any $\lambda \in [0,1]$. In fact, its jacobian $j(\varphi_\lambda) = (\frac{n}{n_\lambda})^{5/2}$, hence $j(\varphi_\lambda) \in (1, (\frac{n}{n'})^{5/2})$, (whether $n > n'$). Since $\lambda \in [0,1]$, it follows that $j(\varphi_\lambda)$ is an irrational number, except for its boundary value $\lambda = 0$. 

---
\[ (x, y) \quad \varphi \quad (x_1, y_1) \quad \psi \quad (x_2, y_2) \]

If \((x, y) \in E[n]\) is a rational point, the corresponding \((x_1, y_1) \in E'[n']\) does not necessitate to be a rational point too. This should happen when the map \(\psi_1\) is an isogeny. In fact in such a case the corresponding to rational points are rational points too and the diffeomorphisms \(\psi_1\) induce homomorphisms between the groups \(E[n]\) and \(E'[n']\). Therefore, if \((x, y) \in \mathbb{Q}^2\), is a rational point of \(E[n]\), then for the corresponding point on \((x_1, y_1) \in E'[n']\), one has \((x_1, y_1) \in \mathbb{Q}^2\) iff \((x, y) = (\frac{a^n}{m}, (\frac{a^n}{m})^{\frac{1}{2}}y) \in \mathbb{Q}^2\). This condition is satisfied iff \(\frac{a^n}{m} = m^2\), for \(m \in \mathbb{Q}\), or equivalently \(n' = m^2 \cdot n\), with \(m = 1/m\). Therefore, \(\psi_1\) is an isogeny iff the integers \(n'\) and \(n\) belong to the same congruent class.\(^{21}\)

- In particular if \(n \in \mathbb{N}_{\congruent}\), then must be \(n' = m^2 \cdot n\), hence \(n'\) cannot be a strong-congruent number too.

However, from the congruent bordism point of view it is not necessary that above homotopies should be isogenies. In fact, we can use Proposition 4.7 to identify elliptic bordism with congruent-ones. Really one has the isomorphism: \(\Omega_{E[n]}^\mathbb{Q} \cong \mathbb{Z}_2 \cong \Omega_{E'[n']}^\mathbb{Q}\). \(\blacksquare\)

**Corollary 4.10** (Elliptic and congruent-bordism groups and homotopies in \(\mathbb{R}^2\)). Let \(E[n] : y^2 - x^3 + n^2x = 0\), \(n \in \mathbb{N}_{\congruent}\), be a congruent elliptic curve, and \(E'[n'] : y^2 - x^3 + n'^2x = 0\), \(n \in \mathbb{N}\), another elliptic curve. Then there exists an homotopy \(\psi_\lambda : E[n] \to E'[n']\),\(^{22}\) inducing the isomorphism of groups \(\Omega_{E[n]}^\mathbb{Q} \cong \Omega_{E'[n']}^\mathbb{Q}\). Furthermore, if \(E'[n']\) is also a congruent elliptic curve, namely \(\text{rank}(E'[n']) > 0\), then this homotopy induces also the isomorphism \(\Omega_{E[n]}^\mathbb{Q} \cong \Omega_{E'[n']}^\mathbb{Q}\). Furthermore, if \(n' = m^2 \cdot n\), \(m \in \mathbb{Q}\), then the mapping \(\psi_1 : E[n] \to E'[n']\), induces an isogeny \((\psi_1)_* : E[n] \rightarrow E'[n']\) that is a group isomorphism: \(E[n] \cong E'[n']\).

The commutative diagram (32) summarizes above results in this section about the elliptic bordism groups and the congruent bordism groups.

\[ Z_2 \xrightarrow{\varphi} \Omega_{E[\pm 2n]} \xrightarrow{\varphi^*} \Omega_{E[\pm 2n]} \]

\[ Z_2 \xrightarrow{\varphi} \Omega_{E[n]} \xrightarrow{\varphi^*} \Omega_{E[n]} \]

\[ \Omega_{E[n]} \cong \Omega_{E'[n']} \]

\[ \Omega_{E'[n']} \cong \Omega_{E'[n']} \]

---

\(^{21}\) This agree with the property that two alliptic curves \(E\) and \(E'\) are isogenous iff there is a morphism of varieties defined by a rational map between \(E\) and \(E'\), which is also a group homomorphism between the corresponding groups \(\mathbb{E}\) and \(\mathbb{E}'\), sending \(\infty \in E \leftrightarrow \infty \in E'\). In fact, in the actual situation the morphism sending \(E[n]\) to \(E'[n']\) is the diffeomorphism \(\psi_1\), induced from \(\varphi\) given in (28), that in order to be a rational mapping must be \(n' = m^2 \cdot n\), \(m \in \mathbb{Q}\).

\(^{22}\) We denote this circumstance by \(E[n] \simeq E'[n']\), and we say that \(E[n]\) is homotopic to \(E'[n']\).
There $n \in \mathbb{N}$, $n \neq n' \in \mathbb{N}$, $s \in \mathbb{Q}$, the isomorphisms $a$, $b$ and $c$ are induced by homotopies and $\varphi_*$ are induced by diffeomorphisms $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$. The isomorphisms $c$ and $d$ exist if $n'$ is also a congruent integer. Otherwise $\Omega_{E[n']}^Q = \emptyset$.

**Lemma 4.11** (Elliptic and congruent bordism groups vs. congruent number problem). Set

$$\Omega_{E[s]}^Q = \prod_{n \in \mathbb{N}_{\text{congr}}} \Omega_{E[n]}^Q$$

and

$$\Omega_{E[s]} = \prod_{n \in \square \mathbb{N}} \Omega_{E[n]}.$$

Since $\Omega_{E[s]}^Q \cong \mathbb{N}_{\text{congr}} \times \mathbb{Z}_2$ and $\Omega_{E[s]} \cong \square \mathbb{N} \times \mathbb{Z}_2$, we can consider both as trivial fiber bundles over the group $\mathbb{Z}_2$, with discrete fiber $\mathbb{N}_{\text{congr}}$ and $\square \mathbb{N}$ respectively. In other words $\Omega_{E[s]}^Q$ and $\Omega_{E[s]}$ are coverings of $\mathbb{Z}_2$. In the following we shall consider $\mathbb{N}_{\text{congr}}$ and $\square \mathbb{N}$ as topological discrete spaces pointed at $*=5$.

Then one has the following commutative diagram (in the category of coverings):

$$(33)$$

```
0 \quad \Omega_{E[s]}^Q \quad \Omega_{E[s]} \quad \Omega_{E[s]} / \Omega_{E[s]}^Q \quad 0

0 \quad \mathbb{N}_{\text{congr}} \times \mathbb{Z}_2 \quad \square \mathbb{N} \times \mathbb{Z}_2 \quad (\square \mathbb{N} / \mathbb{N}_{\text{congr}}) \times \mathbb{Z}_2 \quad 0

0 \quad \mathbb{N}_{\text{congr}} \quad \square \mathbb{N} \quad \mathbb{Z} \quad \text{coker}(L_\bullet) \quad 0
```

with $L_\bullet$ defined in (34).

$$(34) \quad L_\bullet(n) = \begin{cases} 2A_n - B_n & \text{if } n \text{ is even} \\ 2C_n - D_n & \text{if } n \text{ is odd} \end{cases}$$

where $A_n$, $B_n$, $C_n$ and $D_n$ are defined in (18). The first two vertical lines are exact, and so are the first two horizontal lines in (33). Then in order to solve the congruent problem it is enough to prove that also the bottom horizontal line in (33) is exact. Then, by using the relation between BS-D conjecture and congruent problem, we get an indirect proof that the BS-D conjecture is true for elliptic curves of the type $y^2 - x^3 + n^2 x = 0$, (weak BS-D conjecture).

**Proof.** The first part of the lemma directly follows from above results. Furthermore, note that in general one has that the bottom horizontal line in (33) is a 0-sequence. It is also exact at all points except at $\square \mathbb{N}$ and one has

$$(35) \quad \begin{cases} \text{coim}(L_\bullet) = \square \mathbb{N} / \ker(L_\bullet) \cong \text{im}(L_\bullet) \\ \text{coker}(L_\bullet) = \mathbb{Z} / \text{im}(L_\bullet) \end{cases}$$
Therefore, to state that the bottom horizontal line in (33) is exact is equivalent to state that \( \ker(L_\bullet) = \mathbb{N}_{\text{congr}} \) and, from (35) we get also
\[
\begin{cases}
\text{coker}(L_\bullet) = \mathbb{Z}/(\mathbb{N}/\mathbb{N}_{\text{congr}}) \\
\text{coim}(L_\bullet) = \mathbb{N}/\mathbb{N}_{\text{congr}} \cong \text{im}(L_\bullet)
\end{cases}
\]

(36)

In such a case we can complete commutative diagram (33) with (37).

\[
\begin{array}{ccccccccc}
0 & \to & \Omega_{\mathbb{E}[n]}^\Omega \to & \Omega_{\mathbb{E}[n]}^\Omega / \Omega_{\mathbb{E}[n]}^\Omega & \to & \Omega_{\mathbb{E}[\bullet]}^\Omega / \Omega_{\mathbb{E}[\bullet]}^\Omega & \to & 0 \\
0 & \to & \mathbb{N}_{\text{congr}} \times \mathbb{Z}_2 & \to & \mathbb{N} \times \mathbb{Z}_2 & \to & (\mathbb{N}/\mathbb{N}_{\text{congr}}) \times \mathbb{Z}_2 & \to & 0 \\
0 & \to & \mathbb{N}_{\text{congr}} & \to & \mathbb{N} & \to & \mathbb{Z} & \to & \text{coker}(L_\bullet) & \to & 0
\end{array}
\]

In (37) \( d \) is defined by composition
\[
(\mathbb{N}/\mathbb{N}_{\text{congr}}) \times \mathbb{Z}_2 \to \mathbb{N} \to \mathbb{Z}
\]

Therefore, one has \( \text{im}(d) = \ker(e) \), hence all the commutative diagram (37) is exact too.

This assures that the conditions in (19) are also sufficient to state that \( n \) is a strong-congruent integer, namely \( n \in \mathbb{N}_{\text{congr}} \). Then taking into account Theorem 3.7 we get an indirect proof that the Birch-Swinnerton-Dyer conjecture is true. \( \square \)

We are ready, now, to obtain our main result.

**Theorem 4.12** (The congruent problem solved). The identification of the strong-congruent numbers with the kernel of \( L_\bullet \), \( \mathbb{N}_{\text{congr}} \cong \ker(L_\bullet) \), allows us to decide with a finite number of steps, whether a square-free number \( b \) is a strong-congruent number. In this way we identify all the equivalence classes of congruent numbers.

**Proof.** Let us assume \( n' \in \ker(L_\bullet) \subset \mathbb{N} \), and \( n \in \mathbb{N}_{\text{congr}} \). Then, from Corollary 4.10 we say that there exists a homotopy \( \psi_\lambda : \mathbb{E}[n] \to \mathbb{E}[n'] \) inducing the isomorphism of groups \( \Omega_{\mathbb{E}[n]}^\Omega \cong \Omega_{\mathbb{E}[n']}^\Omega \). Then we can identify a diffeomorphism \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \to (x', y') \), sending the rational point \( (x, y) \in \mathbb{E}[n] \), to a point \( (x_1, y_1) \in \mathbb{E}[n'] \), such that \( (x_1 = \frac{a}{n'}, y_1 = (\frac{n}{n'})^{3/2} y) \). Therefore, to the point \( (x, y) \) on \( \mathbb{E}[n] \) corresponds the strong-congruent right triangle \( (a = \frac{x^2 - n^2}{y}, b = 2n \frac{x}{y}, c = \frac{x^2 + n^2}{y}) \), and to the corresponding point on \( \mathbb{E}[n'] \) it is associated the right triangle \( (a' = \frac{x'^2 - n'^2}{y'}, b' = 2n' \frac{x'}{y'}, c' = \frac{x'^2 + n'^2}{y'}) \). The direct expression of these sides in term of \( (x, y) \), is given in (38).

\[
\begin{aligned}
a' &= \frac{x'^2 - n'^4}{y}(n'^3 n) \frac{1}{2}, & b' &= 2 \frac{x}{y} \left( \frac{n'^3}{n} \right) \frac{1}{2}, & c' &= \frac{x'^2 n'^2 + n'^4}{y}(n'^3 n) \frac{1}{2}
\end{aligned}
\]

(38)
The point \((x_1, y_1)\) is rational iff \(n' = \bar{m}^2 \cdot n\), with \(\bar{m} \in \mathbb{Q}\). In such a case \(a', b'\) and \(c'\) are rational numbers too. But \(n' = m^2 \cdot n\) contradicts the assumption that \(n' \in \mathbb{N}\). Therefore, if the map \(\psi : E[n] \to E[n']\), induces an isogeny \((\psi_1)_* : E[n] \to E[n']\), must necessarily be \(n' \in \mathbb{N}_{\text{congr}}\), hence \(n' = n\). On the other hand, we have also the following lemma.

**Lemma 4.13.** \(\ker(L_\bullet) = \mathbb{N}_{\text{congr}}\).

**Proof.** Set \(\tilde{\Omega}_{E[\bullet]} = \prod_{n \in \ker(L_\bullet)} \Omega_{E[n]}\). Since \(\tilde{\Omega}_{E[\bullet]} \cong \ker(L_\bullet) \times \mathbb{Z}_2\) one can consider \(\tilde{\Omega}_{E[\bullet]}\) as a covering of \(\mathbb{Z}_2\). Then one has the following commutative and exact diagram (in the category of covering spaces):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tilde{\Omega}_{E[\bullet]} & \rightarrow & \Omega_{E[\bullet]}^{\ast} & \rightarrow & \Omega_{E[\bullet]}^{\ast}/\tilde{\Omega}_{E[\bullet]} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \ker(L_\bullet) \times \mathbb{Z}_2 & \rightarrow & \square \mathbb{N} \times \mathbb{Z}_2 & \rightarrow & (\square \mathbb{N}/\ker(L_\bullet)) \times \mathbb{Z}_2 & \rightarrow & 0 \\
\downarrow & & & & \downarrow \tilde{d} & & \downarrow & & \\
0 & \rightarrow & \ker(L_\bullet) & \rightarrow & \square \mathbb{N} & \rightarrow & \mathbb{Z} & \rightarrow & \coker(L_\bullet) & \rightarrow & 0
\end{array}
\]

\(\tilde{d}\) is defined by composition:

\[
(\square \mathbb{N}/\ker(L_\bullet)) \times \mathbb{Z}_2 \rightarrow \square \mathbb{N}/\ker(L_\bullet) \rightarrow \mathbb{Z}
\]

Therefore, one has \(\text{im}(\tilde{d}) = \ker(\bar{\epsilon})\). For a fixed \(n \in \ker(L_\bullet)\) set

\[
\ker(L_\bullet)_n = \{n' \in \ker(L_\bullet) : |n'| \leq n\}.
\]

One has the filtration

\[
\ker(L_\bullet)_5 \subseteq \ker(L_\bullet)_6 \subseteq \ker(L_\bullet)_7 \subseteq \cdots \subseteq \ker(L_\bullet)_n \subseteq \cdots
\]

Similarly, for a fixed \(n \in \mathbb{N}_{\text{congr}}\) set

\[
(\mathbb{N}_{\text{congr}})_n = \{n' \in \mathbb{N}_{\text{congr}} : |n'| \leq n\}.
\]

One has the filtration

\[
(\mathbb{N}_{\text{congr}})_5 \subseteq (\mathbb{N}_{\text{congr}})_6 \subseteq (\mathbb{N}_{\text{congr}})_7 \subseteq \cdots \subseteq (\mathbb{N}_{\text{congr}})_n \subseteq \cdots
\]

Since any strong-congruent number belongs also to \(\ker(L_\bullet)\), it follows that for any \(n \in \mathbb{N}_{\text{congr}}\) one has \((\mathbb{N}_{\text{congr}})_n \subseteq \ker(L_\bullet)_n\). We shall see that the equality holds. This is surely true for \(n = 5, 6, 7\).

\(^{23}\text{This means that for two elliptic curves } E_n \text{ and } E_{n'}, \text{ such that } n, n' \in \mathbb{N}_{\text{congr}}, \text{ namely } n \text{ and } n' \text{ are both strong-congruent numbers, cannot exist an isogeny } E[n] \rightarrow E[n'], \text{ according with Theorem 4.9.}\)

\(^{24}\text{Really one can experimentally verify that it holds also for } n > 7. \text{ For example, from Tab. 4 one can see that } (\mathbb{N}_{\text{congr}}) = \ker(L_\bullet)_n, \text{ for } 5 \leq n \leq 65.\)
Therefore, let us assume that \((N_{congr})_n = \ker(L_\bullet)_n\), for all strong-congruent numbers \(n \leq n_0 \in \mathbb{N}_{congr}\). Let us to prove that it holds also for \(n = \bar{n}_0\), where \(\bar{n}_0\) is the first strong-congruent number greater than \(n_0\). Therefore, let us assume that there exists a square-free number \(m\), contained in \(\ker(L_\bullet)_{\bar{n}_0}\), such that \(m \not\in (N_{congr})_{\bar{n}_0}\).

Let us consider, now, the following lemma.

**Lemma 4.14.**

\[ \sharp(\ker(L_\bullet)) = \sharp(N_{congr}) = \aleph_0. \]

**Proof.** In fact, \(\ker(L_\bullet)\) contains an infinite set, namely \(N_{congr}\), and it is contained in \(\mathbb{N}\). This is essentially a consequence of the well-known Cantor-Schroeder-Bernstein theorem [3, 4, 5, 12] applied to the sequence \(\mathbb{N} \supset \ker(L_\bullet) \supset N_{congr}\), since \(\sharp(\mathbb{N}) = \sharp(N_{congr}) = \aleph_0\). \(\Box\)

From Lemma 4.14 we can assume that there exists a one-to-one map

\[ f : N_{congr} \to \ker(L_\bullet). \]

Since must be \((N_{congr})_{n_0} = \ker(L_\bullet)_{n_0}\), one has the following commutative diagram

\[
\begin{array}{ccc}
(N_{congr})_{\bar{n}_0} & \xrightarrow{f_{\bar{n}_0}} & \ker(L_\bullet)_{\bar{n}_0} \\
\downarrow & & \downarrow \\
(N_{congr})_{n_0} & \xrightarrow{f_{n_0}} & \ker(L_\bullet)_{n_0}
\end{array}
\]

where \(f_n = f|_{(N_{congr})_n}\). On the other hand the mapping \(f_{\bar{n}_0}\) could not be one-to-one. This contradicts the fact that \(f\) realizes the equivalence between \(N_{congr}\) and \(\ker(L_\bullet)\). In fact, the mapping \(f_{\bar{n}_0}\) should be surjective and not only injective. Therefore the assumption that \((N_{congr})_{\bar{n}_0} \neq \ker(L_\bullet)_{\bar{n}_0}\) contradicts the fact that must be \((N_{congr})_{n_0} = \ker(L_\bullet)_{n_0}\). This means that must necessarily be \((N_{congr})_{n} = \ker(L_\bullet)_{n}\), for any strong-congruent number \(n \in N_{congr}\). Thus, we can conclude that \(N_{congr} = \ker(L_\bullet)\). \(\Box\)

From Lemma 4.11 one has the isomorphism \(\Omega_{E[s]}^{\mathcal{Q}} \cong \Omega_{E[s]}\) and the exactness of the commutative diagram (37). In particular the bottom sequence therein, or the sequence (1), is exact. It follows that Problem 2.20 is solved. \(\Box\)

**Remark 4.15** (Theorem 4.12 vs. the Birch Swinnerton-Dyer conjecture). *Our solution of the congruent problem is strongly related to the Tunnell’s theorem, but does not directly refers to the BS-D conjecture. However, since a consequence of the BS-D conjecture is that the framed equalities (19), in the Tunnell’s theorem, are also sufficient to determine strong-congruent numbers, and this conclusion coincides with our Theorem 4.12, we have good chances to argue that the Conjecture 3.3 is true.*

## Appendices

**Appendix A: The function \(L(E,s)\) and the infinitude of primes.**
In this appendix we shall recall some relations between the Riemann zeta function, its generalization \( L(E, s) \) and the proof of the cardinality of the set \( P \) of prime numbers.

**Definition A.1** (Dirichlet character). A Dirichlet character is any function \( \chi : \mathbb{Z} \to \mathbb{C} \) such that the following conditions are satisfied:

1. There exists a positive integer \( \kappa \) (modulus), such that \( \chi(n) = \chi(n + \kappa) \), \( \forall n \in \mathbb{Z} \).
2. \( \chi(n) = \begin{cases} 0 & \text{if } \gcd(n, \kappa) > 1 \\ \neq 0 & \text{if } \gcd(n, \kappa) = 1 \end{cases} \)
3. \( \chi(m \cdot n) = \chi(m) \cdot \chi(n) \), \( \forall m, n \in \mathbb{Z} \).

- A character is called principal if \( \chi(n) = \begin{cases} 1 & \text{if } \gcd(n, \kappa) = 1 \\ 0 & \text{otherwise} \end{cases} \)
- A character is called real if \( \chi : \mathbb{Z} \to \mathbb{R} \), otherwise it is called complex.
- The sign of the character \( \chi \) depends on its value at \(-1\). \( \chi \) is said to be odd if \( \chi(-1) = -1 \) and even if \( \chi(-1) = 1 \).

**Proposition A.1** (Properties of Dirichlet characters). 1) \( \chi(1) = 1 \).
2) \( \chi \) is periodic with period \( \kappa \).
3) If \( a \equiv b \mod \kappa \), then \( \chi(a) = \chi(b) \).
4) If \( a \) is relatively prime to \( \kappa \), \( (\gcd(a, \kappa) = 1) \), then \( \chi(a)^{\varphi(\kappa)} \equiv 1 \mod \kappa \), i.e., \( \chi(a) \) is a \( \varphi(\kappa) \)-th complex root of unity, where \( \varphi(\kappa) \) is the totient function.\(^{25}\)
5) (Trivial character) There exists an unique character, called trivial character, having modulus \( \kappa = 1 \). Then for such a character one has \( \chi(n) = 1 \), \( \forall n \in \mathbb{Z} \).
6) For any character, except the trivial one, one has \( \chi(0) = 0 \).
7) (Relation with character group. Extended residue class characters) If \( \chi : \mathbb{Z} \to \mathbb{C} \) is a Dirichlet character with modulus \( \kappa \), it identifies a character group \( \chi : \mathbb{Z}_\kappa \to \mathbb{C}^* \), with values that are \( \kappa \)-roots of unity. Vice versa if \( \chi : \mathbb{Z}_\kappa \to \mathbb{C}^* \), is a group homomorphism, then it identifies a Dirichlet character \( \chi : \mathbb{Z} \to \mathbb{C} \) with modulus \( \kappa \).
8) (Orthogonality relation) If \( \chi \) is a character modulus \( \kappa \), the \( \sum_{a \mod \kappa} \chi(a) = 0 \), unless \( \chi \) is principal, in which case one has \( \sum_{a \mod \kappa} \chi(a) = \varphi(\kappa) \).

**Theorem A.1** (Cardinality of the primes set \( P \)). The set \( P \) of prime numbers has the same cardinality of \( \mathbb{N} \): \( \sharp(P) = \aleph_0 \).

**Proof.** Since \( P \subset \mathbb{N} \) it is enough to prove that \( P \) is an infinite set. There are different proofs for the infinitude of \( P \), other than first one given by Euclide. Let us give here the one related to the following lemma.

**Lemma A.1** (Euler product). If \( f : \mathbb{N} \to \mathbb{R} \) is a multiplicative function such that \( \sum_{1 \leq n \leq \infty} f(n) \) converges absolutely, then one has the following formula (Euler

\[ \sum_{1 \leq n \leq \infty} f(n) = \prod_{\varphi(\kappa)} \left\{ \cos \left( \frac{2\pi b}{\varphi(\kappa)} \right) + i \sin \left( \frac{2\pi b}{\varphi(\kappa)} \right) \right\}_{b = 0, 1, 2, \ldots, \varphi(\kappa) - 1}. \]

\(^{25}\)This is the Euler’s theorem. In other words \( \chi(a) \) can belong to the following finite set of complex numbers:

\[ \{ \cos \left( \frac{2\pi b}{\varphi(\kappa)} \right) + i \sin \left( \frac{2\pi b}{\varphi(\kappa)} \right) \}_{b = 0, 1, 2, \ldots, \varphi(\kappa) - 1}. \]

\(^{26}\)In Tab. 5, as an example, is reported the multiplication table of \( \mathbb{Z}_1^{10} \) and in Tab. 6 the four Dirichlet characters modulus 10.
Table 5. Multiplication table in \( \mathbb{Z}_{10} = \{1, 3, 7, 9\} = \{m\} \).

| \( m \) \( \backslash \) \( m \) | 1   | 3   | 7   | 9   |
|---------------------|-----|-----|-----|-----|
| 1                   | 1   | 3   | 7   | 9   |
| 3                   | 3   | 9   | 1   | 7   |
| 7                   | 7   | 1   | 9   | 3   |
| 9                   | 9   | 7   | 1   | 3   |

\( 1^{-1} = 1; 3^{-1} = 7; 
7^{-1} = 3; 9^{-1} = 9. \)

Table 6. Dirichlet characters mod 10.

| \( \chi(n) \) \( \backslash \) \( n \) | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \chi_1(n) \)      | 0   | 1   | 0   | 1   | 0   | 1   | 0   | 1   | 0   | 1   |
| \( \chi_2(n) \)      | 0   | 1   | 0   | 0   | 0   | 0   | -i  | 0   | -1  | 0   |
| \( \chi_3(n) \)      | 0   | 1   | 0   | -1  | 0   | 0   | 0   | -1  | 0   | 1   |
| \( \chi_4(n) \)      | 0   | 1   | 0   | -i  | 0   | 0   | 0   | -1  | 0   | -1  |

\( \varphi(10) = 4. \)

For \( \sum_{1 \leq n \leq \infty} f(n) \):  
\[ \sum_{1 \leq n \leq \infty} f(n) = \prod_{p \in P} (1 + f(p) + f(p^2) + \cdots). \]

Proof. The proof is standard. However, it is educationally useful to give an explicit proof also. Let \( q \in \mathbb{N} \) and \( p_1, \cdots, p_k \in P \) be all the primes in \([1, q] \subset \mathbb{N}\). Set \( P_q = \{ p \in P \mid p < q \} \). Then we can write

\[ \prod_{p \in P_q} (1 + f(p) + f(p^2) + \cdots) = \sum_{r_1} f(p_1^{r_1}) \sum_{r_2} f(p_2^{r_2}) \cdots \sum_{r_k} f(p_k^{r_k}) = \sum_{r_1, r_2, \cdots, r_k} f(p_1^{r_1}) f(p_2^{r_2}) \cdots f(p_k^{r_k}) = \sum_{r_1, r_2, \cdots, r_k} f(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}) = P_q[n]. \]

Since for any \( n \in \mathbb{N} \), such that \( n < q \), the corresponding prime decomposition has no factors greater than \( q \), we can write

\[ \left| \sum_{1 \leq n \leq \infty} f(n) - P_q[n] \right| \leq \sum_{q \leq n \leq \infty} |f(n)|. \]

Taking into account that \( \lim_{q \to \infty} \sum_{q \leq n \leq \infty} |f(n)| = 0 \), we get the proof. \( \square \)

Let us recall now that the Riemann zeta function is defined by means of the series \( \zeta(s) = \sum_{1 \leq n \leq \infty} \frac{1}{n^s} \). Since this series converges absolutely for \( \Re(s) > 1 \), we get the Euler product formula for the zeta functions:

\( \zeta(s) = \prod_{p \in P} \frac{1}{1 - p^{-s}}, \Re(s) > 1. \)

In particular for \( s = 2 \) one has

\( \zeta(2) = \sum_{n} \frac{1}{n^2} = \frac{\pi^2}{6}. \)
Thus $\zeta(2)$ is an irrational number that can be represented by $\zeta(2) = \prod_{p \in P} \frac{1}{1-p^{-2}}$. Therefore must necessarily be $P$ an infinite set. \hfill \Box

**Definition A.2.** A Dirichlet series is defined by $\sum_{1 \leq n \leq \infty} \frac{a_n}{n^s}$, $s \in \mathbb{C}$, $a_n \in \mathbb{C}$.

**Example A.1.** The Riemann zeta function $\zeta(s) = \sum_{1 \leq n \leq \infty} \frac{1}{n^s}$ is a Dirichlet series.

**Example A.2.** The Dirichlet $L$-series $L(s, \chi) = \sum_{1 \leq n \leq \infty} \frac{\chi(n)}{n^s}$, $s \in \mathbb{C}$, with $\chi$ a Dirichlet character, in the sense of Definition A.1, is a Dirichlet series. $L(s, \chi)$ converges absolutely and conformally for $\Re(s) \geq 1 + \delta$ (for any positive $\delta$) and admits the Euler product (A.4).

$$L(s, \chi) = \prod_{p \in P} \frac{1}{1 - \chi(p)p^{-s}}.$$  

The Dirichlet $L$-series can be extended by continuation to a meromorphic function on $\mathbb{C}$, yet denoted $L(s, \chi)$. In particular if $\chi = \chi_0$ is the trivial character, hence $\chi(n) = 1$, one has $L(s, \chi_0) = \frac{\zeta(s)}{\prod(1 - p^{-s})}$. So we can write the Riemann zeta function $\zeta(s) = L(s, \chi)/\prod(1 - p^{-s})$. Therefore $L(s, \chi)$ are generalizations of the Riemann zeta function. Similarly, one can define for any elliptic function $E$ a generalized Dirichlet function $L(s, E)$.

**Appendix B: Riemann surfaces and modular curves.**

In this appendix we shall recall some distinguished Riemann surfaces called modular curves that are strictly related to the modular characterization of elliptic curves in $\mathbb{C}$. Let us start with some fundamental definition and results about coverings.

**Definition B.1.** 
- A group $G$ acts (on the left) on a set $X$ if there is a function $\alpha : G \times X \rightarrow X$ such that the following diagrams commute:

$$
\begin{array}{ccc}
G \times X & \xrightarrow{(e, 1)} & G \times X \\
\downarrow \mu \times 1_X & & \downarrow \alpha \\
X & \xrightarrow{\alpha} & X \\
\end{array}
$$

where $\mu : G \times G \rightarrow G$ is the multiplication map and $(e, 1)(x) = (e, x)$, for all $x \in X$, $e$ being the identity of $G$. The orbit for $x \in X$ is the set $Gx = \{ gx \mid g \in G \}$.
- A topological group $G$ acts on a topological space $X$ if there is a continuous action map $\alpha$ respecting above properties (i) and (ii).
- A discrete group $G$ with identity $e$ is said to act properly discontinuously if the following propositions hold:
  (a) For every $x \in X$ there is a neighbourhood $U_x$ such that $gU_x \cap U_x \neq \emptyset \Rightarrow g = e$.
  (b) For every $x, y \in X$, $y \notin Gx$, there are neighbourhoods $V_x, V_y$ of $x$ and $y$ respectively such that $gV_x \cap V_y = \emptyset$, all $g \in G$.

\footnote{With this respect let us recall that $\mathbb{Q}$ is not a complete subspace of $\mathbb{R}$, having empty interior, $\bar{\mathbb{Q}} = \emptyset$, and with closure $\mathbb{R}$, $\overline{\mathbb{Q}} = \mathbb{R}$. Therefore $\partial \mathbb{Q} = \mathbb{R}$ and $\zeta(2) \in \partial \mathbb{Q}$.}

\footnote{The property (b) implies $X/G$ is Hausdorff and the property (a) implies the projection $q : X \rightarrow X/G$ is a covering, i.e., the fibre $F = q^{-1}(p)$ over $p \in X/G$ is discrete.}
Proposition B.1 (Relations between group actions and homotopy groups). • Assume $X$ is a 0-connected topological space. One has a homomorphism
\begin{equation}
\partial : \pi_1(X/G, *) \to \pi_0(F, x_0) = G
\end{equation}
and the following exact sequence
\begin{equation}
0 \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_1(X/G, *) \xrightarrow{\partial} G \longrightarrow 0
\end{equation}
• If $\pi_1(X, x_0) = 0$, then $\pi_1(X/G, *) \cong G$ and we have an action of $G$ on $\pi_n(X/G, *)$:
\begin{equation}
G \times \pi_n(X/G, *) \longrightarrow \pi_{n-1}(F, x_0)
\end{equation}
The induced map $q_* : \pi_n(X, x_0) \to \pi_n(X/G, *)$ is a homomorphism of groups with $G$-action
\begin{equation}
q_*(g\gamma) = g \cdot q_*(\gamma), \quad g \in G, \quad \gamma \in \pi_n(X, x_0)
\end{equation}
such that the following diagram is commutative
\begin{equation}
\begin{array}{ccc}
S^n & \xrightarrow{\gamma} & X \\
\downarrow{g\gamma} & & \downarrow{T_g} \\
X & & X
\end{array}
\end{equation}
where $T_g : X \to X$ is defined by $T_g(x) = gx$.

Definition B.2. A covering $p : \tilde{X} \to X$ is called regular if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$.

Proposition B.2 (Properties of regular coverings). • A discrete group $G$ with identity that acts properly discontinuously on the topological space $X$ induces a regular covering $X \to X/G$. Vice versa every regular covering $X \to X/G$, $G$ acting on $X$ without fixed points (i.e., $gx = x \Rightarrow g = e$), and $X$ Hausdorff, has $G$ acting properly discontinuously. Therefore, $X \to X/G$ is a regular covering.

Example B.1. $\mathbb{Z}$ acts properly discontinuously on $\mathbb{R}$: $n \cdot r = r + n$. Then $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a regular covering. Since $\pi_1(\mathbb{R}, x_0) = 0$, then from exact sequence (B.3) it follows $\pi_1(\mathbb{R}/\mathbb{Z}, *) \cong \mathbb{Z}$. This means that $\mathbb{R}/\mathbb{Z} \cong S^1$, hence $\mathbb{R} \to S^1$ is an universal covering too, since $\mathbb{R}$ is simply connected.

Example B.2 (Riemann surfaces as universal coverings of Riemann surfaces). • From the classification Poincaré-Koebe theorem one knows that every connected Riemann surface $X$ admits a unique complete 2-dimensional real Riemann metric with constant curvature $-1$, $0$, or $1$, inducing the same conformal structure - every metric is conformally equivalent to a constant curvature metric as reported in Tab. 8.
• In Tab. 8 are reported also the corresponding examples with $\pi_1(X, x_0) = 0$. In particular, the uniformization theorem states that any connected Riemann surfaces $Y$ admits as universal covering one of the three fundamental types reported there, i.e., $Y$ is biholomorphic to $X/G$ for some discrete group $G$. 
Table 7. Distinguished congruence groups and properties:
\[ \Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma(1). \]

| Symbol | Quotient isomorphisms | Indexes |
|--------|-----------------------|---------|
| \[ \Gamma_0(N) \] | \( \Gamma_1(N)/\Gamma(N) \cong \mathbb{Z}/N\mathbb{Z} \) | \( [\Gamma(1) : \Gamma(N)] = 2\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^2 \prod_{p\mid N}(1 - \frac{1}{p^2}) \) |
| \[ \Gamma_1(N) \] | \( \Gamma(1)/\Gamma(N) \cong \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \) | \( [\Gamma_0(N) : \Gamma(N)] = \phi(N) = N \prod_{p\mid N}(1 - \frac{1}{p}) \) |
| \( \Gamma(N) \leq \Gamma(1) \) | \( \Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^2 \) | \( [\Gamma(1) : \Gamma_0(N)] = \psi(N) = N \prod_{p\mid N}(1 + \frac{1}{p}) \) |
| \( \Gamma' (N) = \text{S}^{-1} \Gamma_0(N)S \) | | \( \mu(\Gamma_0(N)) = \psi(N) \).
| \( \Gamma''(N) = \text{S}^{-1} \Gamma_1(N)S \) | | \( \mu(\Gamma_1(N)) = \frac{1}{\phi(N)}\psi(N), \text{if } N \geq 3 \).
| \( \Gamma_0(N) \) defined in (B.10); \( \Gamma_1(N) \) defined in (B.11). | | \( \mu(\Gamma(N)) = \frac{1}{\phi(N)}\psi(N), \text{if } N \geq 3 \).

- Modular curves are just Riemann surfaces \( Y \) that can be identified with \( H/\Gamma \) for some congruence group \( \Gamma \).\(^{29}\)

**Example B.3** (Modular curves). • The modular group

\[
\text{(B.7)} \quad \Gamma(1) := \text{SL}_2(\mathbb{Z}) := \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid \det A = 1 \}.
\]

- The principal congruence subgroup of level \( N > 1, N \in \mathbb{N} \), is the subgroup \( \Gamma(N) \subset \Gamma(1) \) defined by the following exact sequence

\[
\text{(B.8)} \quad 0 \longrightarrow \Gamma(N) \longrightarrow \Gamma(1) \overset{\text{mod } N}{\longrightarrow} \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow 0
\]

hence

\[
\text{(B.9)} \quad \Gamma(N) = \{ A \in \Gamma(1) \mid A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } N) \}.
\]

- A congruence subgroup is a subgroup \( \Gamma \leq \Gamma(1) \) that contains \( \Gamma(N) \) for some \( N \); \( \Gamma(N) \leq \Gamma \leq \Gamma(1) \). The smallest such \( N \) is called the level of \( \Gamma \). One has the following properties for congruence subgroups,
  - Each congruence subgroup \( \Gamma \leq \Gamma(1) \) has finite index in \( \Gamma(1) \). The index of \( \pm \Gamma = \Gamma \cup -\Gamma \) is denoted by \( \mu(\Gamma) = [\Gamma : \pm \Gamma] \).
  - Not every subgroup of finite index is a congruence subgroup.\(^{30}\)
  - If \( \Gamma_1 \) and \( \Gamma_2 \) are congruence subgroups, then so is \( \Gamma_1 \cap \Gamma_2 \).\(^{31}\)
  - If \( \Gamma \) is a congruence subgroup, then \( \alpha^{-1}\Gamma \alpha \cap \Gamma(1) \) is also a congruence subgroup, where \( \alpha \in GL^+_2(\mathbb{Q}) := \{ g \in GL_2(\mathbb{Q}) \mid \det(g) > 0 \} \).\(^{32}\)
  - In particular, \( \Gamma \) and \( \Gamma_1 = \alpha^{-1}\Gamma \alpha \) are commensurable subgroups, i.e., \( \Gamma \cap \Gamma_1 \) has finite index in both \( \Gamma \) and \( \Gamma_1 \).
  - In the following we define some distinguished congruence groups corresponding to upper-triangular matrices of level \( N \).

\(^{29}\)The group \( G = GL^+_2(\mathbb{R}) \) acts on \( H \) by fractional linear transformations of \( H \): \( (g, z) \mapsto g(z) = \frac{az + b}{cz + d}, \) if \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \) \( \det(g) > 0. \)

\(^{30}\)For each odd number \( N > 1 \), there is a normal subgroup of index \( 6N^2 \) which is not a congruence subgroup.

\(^{31}\)In fact, \( \Gamma(N) \cap \Gamma(1) = \Gamma(\text{l.c.m.}(N, M)). \)

\(^{32}\)In fact, \( \alpha^{-1}\Gamma \alpha \cap \Gamma(1) \supset \Gamma(ND) \), where \( D = \det(\alpha) \).
\( \Gamma_0(N) := \left\{ A \in \Gamma(1) \mid A \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}. \)

\( \Gamma_1(N) := \left\{ A \in \Gamma(1) \mid A \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \)

In Tab. 7 are reported some properties of such groups.\(^{33}\) \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are respectively preimages of the following Borel subgroups of \( SL_2(\mathbb{Z}/NZ) \):

\( B_0(\mathbb{Z}/NZ) := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad B_1(\mathbb{Z}/NZ) := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}. \)

- (Non-compact modular curves) If \( \Gamma \) is a congruence sub-group, the quotient space \( X_\Gamma^* := H/\Gamma \) can be made into a Riemann surface such that the quotient map \( p_\Gamma : H \to X_\Gamma^* \) is a holomorphic map.

- (Compact modular curves) Let \( H^* := H \cup \mathbb{Q} \cup \{ \infty \} = H \cup \mathbb{P}^1(\mathbb{Q}) \) obtained by \( H \) adding the cusp-points \( \mathbb{P}^1(\mathbb{Q}) \), has a natural structure of compact topological space, on the which acts in natural way the group \( GL_2^+(\mathbb{Q}) \) (hence \( \Gamma \)) by putting \( \gamma(\infty) = \frac{a}{c}. \)

Then \( X(\Gamma) = X_\Gamma := H^*/\Gamma \) has a natural structure of compact Riemann surface that contains \( X_\Gamma^* \) as an open subspace, with a finite complement:

\( \text{cusp}(\Gamma) = \text{cusp}(X_\Gamma) := X_\Gamma \setminus X_\Gamma^* = \mathbb{P}^1(\mathbb{Q})/\Gamma. \)

Furthermore the surjective mapping \( p_\Gamma : H^* \to X(\Gamma) \) is holomorphic.\(^{35}\)

\(^{33}\)An alternative way to write \( \Gamma_0(N) \) is the following:

\[ \Gamma_0(N) = \alpha_N \Gamma(1) \alpha_N^{-1} \cap \Gamma(1) = \beta_N^{-1} \Gamma(1) \beta_N \cap \Gamma(1), \]

with

\[ \alpha_N = \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}, \]

\[ \beta_N = N \alpha_N^{-1} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in GL_2^+(\mathbb{Q}) \]

and

\[ \alpha_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha_N^{-1} = \begin{pmatrix} 0 & cN \\ d/N & 1 \end{pmatrix}. \]

\(^{34}\)\( GL_2^+(\mathbb{Q}) \) acts on \( \mathbb{P}^1(\mathbb{Q}) \) by the rule \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \) if \( [x,y] \in \mathbb{P}^1(\mathbb{Q}) \). In particular one can state that \( \mathbb{P}^1(\mathbb{Q}) = SL_2(\mathbb{Z}) \cdot \infty \), hence \( \mathbb{P}^1(\mathbb{Q})/SL_2(\mathbb{Z}) = \infty \).

\(^{35}\)The topology of \( H^* \) is an open set of the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \). The group \( \Gamma \) acts on the subset \( \mathbb{Q} \cup \{ \infty \} \), breaking it into finitely many orbits called the cusps of \( \Gamma \). If \( \Gamma \) acts transitively on \( \mathbb{Q} \cup \{ \infty \} \), \( X(\Gamma) \) results the Alexandrov compactified of \( X_\Gamma^* \). The topology on \( H^* \) is obtained by taking as basis \( \{ U, U_r, U_{r,m,n} \} \), where \( U \) is any open subset of \( H, U_r := \{ \infty \} \cup \{ z \in H \mid \exists z > r, \ r > 0 \}, U_{r,m,n} \) are transformed of \( U_r \), by means \( \begin{pmatrix} a & -m \\ c & n \end{pmatrix}, \ m, n \in \mathbb{N}, \ an + cm = 1 \). \( X_\Gamma \) is compact since \( H^* \) is compact and \( p_\Gamma : H^* \to X_\Gamma \) is surjective. In general \( X_T \) is the union of finitely many compact sets. For \( \Gamma(1) \) the stabilizer of \( \infty \) is \( \Gamma(1)_{\infty} = \{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}. \) One has

\[ \Gamma_{\infty} = \Gamma \cap \Gamma(1) \supset \left\{ \begin{pmatrix} 1 & NZ \\ 0 & 1 \end{pmatrix} \right\} \text{ for some positive integer } N. \]

Let \( r_{\infty} \) be the minimal positive integer (widt of period) such that \( \begin{pmatrix} 1 & r_{\infty} \\ 0 & 1 \end{pmatrix} \in \Gamma_{\infty}. \) The local coordinate at \( \infty \) is given by

\[ q = e^{2\pi i z/r_{\infty}}. \]
Table 8. Metric classification of Riemann surfaces and their universal coverings

| Name       | Curvature | Simply connected Riemann surfaces | Universal covering classification |
|------------|-----------|-----------------------------------|----------------------------------|
| hyperbolic | $-1$      | $\Delta \cong \mathbb{H}$        | $H \to Y = H/G$, $(Y = H/SL_2(\mathbb{Z}) \cong \mathbb{C}$) |
| parabolic  | $0$       | $\mathbb{C}$                      | $\mathbb{C} \to Y = \mathbb{C}/G$, $(Y = T = \mathbb{C}/L)$ |
| elliptic   | $1$       | $S^1 = \mathbb{C}[\{\infty\}] = \mathbb{P}^1(\mathbb{C})$ | $S^1 \to Y = S^1/G$ |

Upper half-plane: $H := \{ z \in \mathbb{C} | \Im(z) > 0 \}$; Open disc: $\Delta := \{ z \in \mathbb{C} | \Im(z) < 1 \}$.

Lattice: $L := \mathbb{Z} + 2\pi \tau$, $\tau \in \mathbb{C}$, $\tau \neq 0$.

Holomorphic elliptic modular function: $J : H/SL_2(\mathbb{Z}) \cong \mathbb{C}$, $J(z) = g_2^3/(g_2^3 - 27g_3^2)$.

$g_2$ and $g_3$ are the coefficients in the equation $z^3 - g_2z - g_3 = w^3$ of $\mathbb{C}/L$.

Universal coverings $X \to Y$ are called hyperbolic, parabolic, elliptic, according to the type of the Riemann surface $X$.

One has the following properties for congruence subgroups.

**If $\Gamma_1 \leq \Gamma_2$ are two congruence subgroups, then the inclusion map induce a quotient map $p_{\Gamma_1,\Gamma_2} : X_{\Gamma_1} \to X_{\Gamma_2}$ such that the following diagram is commutative.

\[
\begin{array}{ccc}
H^* & \to & H^*/\Gamma_2 \\
p_{\Gamma_1} \downarrow & & \downarrow p_{\Gamma_1,\Gamma_2} \\
X_{\Gamma_1} & \to & X_{\Gamma_2} \\
\end{array}
\]

The mapping $p_{\Gamma_1,\Gamma_2}$ is holomorphic with degree

\[
\deg(p_{\Gamma_1,\Gamma_2}) = \pm \Gamma_2 : \pm \Gamma_1 = \mu(\Gamma_1)/\mu(\Gamma_2).
\]

**If $\alpha \in GL_2^+(\mathbb{Q})$ and $\Gamma_1 \leq \Gamma_2$ are two congruence subgroups related by $\alpha : \alpha\Gamma_1\alpha^{-1} \leq \Gamma_2$, then there exists a unique holomorphic map $p_{\Gamma_1,\Gamma_2} : X_{\Gamma_1} \to X_{\Gamma_2}$ such that $p_{\Gamma_2} \circ \alpha = p_{\Gamma_1,\Gamma_2} \circ p_{\Gamma_1}$.

**Let $\Gamma \subset \Gamma(1)$ be a congruence subgroup containing $\{\pm 1\}$. Then $X_{\Gamma}^*$ is endowed with the quotient topology $p_{\Gamma} \to X_{\Gamma}$. Under this topology $X_{\Gamma}^*$ is Hausdorff. The stabilizer group of $z \in H$ is finite cyclic. Furthermore $\Gamma(z)/\{\pm 1\}$ is isomorphic to one of the following groups: $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$. Up to $\Gamma(1)$-equivalence, the only points with non-trivial stabilizer are $z = i$ and $z = \rho$. There are only finitely many elliptic points (modulo $\Gamma$) of $H$, i.e., $z \in H$ such that $\Gamma(z)/\{\pm 1\}$ is not-trivial.**

**The genus of $X_{\Gamma(N)}$ is given in (B.16).**

\[
\begin{aligned}
g(X_{\Gamma(N)}) = \begin{cases} 
0 & \text{if } N \leq 2 \\
1 + \frac{N^2(N-6)}{24} \prod_{p|N}(1-p^{-2}) & \text{if } N > 2, \; (p = \text{prime}).
\end{cases}
\end{aligned}
\]

**In particular the modular curve $X_0(1)$ can be identified with the Riemann sphere. In fact, the holomorphic map $j : X_0(1) \to \mathbb{P}^1(\mathbb{C})$ sending $SL_2(\mathbb{Z})z \mapsto j(z)$ and $\infty \mapsto \infty$ is a degree 1 map between two compact Riemann surfaces, hence an isomorphism.**

---

36This is true for $\Gamma(1)$, hence true for any subgroup of finite index. Every point of $X_{\Gamma}$ is an orbit $\Gamma.z$ for $z \in H$. If $z$ is not elliptic, then $z$ has $s$ neighborhood $U$ such that $\gamma(U) \cap U \neq \emptyset$ iff $\gamma(z) = z$, i.e., $\Gamma$ acts properly discontinuously on non-elliptic point. If $z$ is elliptic, locally its neighborhood is $D/\mu_2$ or $D/\mu_3$ with local coordinate given by $z \mapsto z^2$ or $z \mapsto z^3$.

37The genus $g(X_{\Gamma(N)})$ is obtained from the ramified covering $X_{\Gamma(N)} \to \Gamma(1)$, $(\Gamma(N)$ is of finite index in $\Gamma(1))$. In this way one obtains also the proof of existence of subgroups of finite index in $\Gamma(1)$ which are not congruence subgroups. [13]
Table 9. Examples of compact modular curves $X(N) = H^*/\Gamma(N)$.

| Name-Symbol | Genus | Cusp number | Galois group $SL_2(\mathbb{Z})/\{\pm 1\}$ of covering $X(N) \to X(1)$ |
|-------------|-------|-------------|-------------------------------------------------|
| Riemann sphere $X_0(1)$ | 0 | 0 | $\{1\}$ |
| Icosahedral $X(5)$ | 0 | 12 | $A_5 \cong PSL_2(5)$ |
| Klein quartic $X(7)$ | 3 | 24 | $PSL_2(7)$ |
| $X(11)$ | 26 | | |
| Classical modular curve $X_0(N)$ | 0, for $N = 1, \cdots, 10, 12$ | | $SL_2(\mathbb{Z})/\{\pm 1\} \cong PSL_2(N)$ for $N$ prime. |

Let $\Gamma$ be a congruence subgroup containing $\{\pm 1\}$. One has the following formula:

(B.17) \[ g(X_\Gamma) = 1 + \frac{d}{12} - \frac{1}{4}e_2 - \frac{1}{3}e_3 - \frac{1}{2}e_\infty \]

with $d = \text{deg}(pr_{\Gamma(1)}), e_\infty$ is the number of cusps on $X_\Gamma$ and $e_r, r = 2, 3$ is the number of elliptic points in the fiber over the order $r$ elliptic point $P_r \in X_{\Gamma(1)}$.

In fact, one has a covering map $pr_{\Gamma(1)} : X_\Gamma \to X_{\Gamma(1)}$ of degree $\text{deg}(pr_{\Gamma(1)}) = [\Gamma(1) : \Gamma], since for non-elliptic point $x$ we have $\sharp(\Gamma x/\Gamma(1)) = [\Gamma(1) : \Gamma]$, and there are only finitely many elliptic points. By means of the Riemann-Hurwitz formula\(^{38}\)

(B.18) \[ 2g(X_\Gamma) - 2 = (2g(X_{\Gamma(1)}) - 2)\text{deg}(pr_{\Gamma(1)}) + b \]

where $b$ is the total ramification degree given by the following formula:

(B.19) \[ b = \sum_{x \in X_{\Gamma}} (e_x - 1) = \sum_{y \in X_{\Gamma(1)}} \text{deg} R_y = \sum_{y \in X_{\Gamma(1)}} \sum_{x \in X_{\Gamma}} \text{deg} p_{\Gamma(1)}^{-1}(y)(e_x - 1), \]

where $e_x$ is the ramification degree at $x \in X_{\Gamma}$. Recall that the ramification points are a subset of the fibers over elliptic points of $X_{\Gamma(1)}$. Then the ramification degree must divide 2 or 3, the only possible orders of elliptic points. Let us recall that the $r$ order elliptic points $P_r \in X_{\Gamma(1)}$ are only $P_2 = i$ and $P_3 = \rho$. Then the number of non-elliptic points in the fiber over $P_r$ is $\frac{d-\rho_s}{r},$ hence $\text{deg} R_r = (r - 1)\frac{d-\rho_s}{r}$.

Furthermore one has $\text{deg} R_\infty = d - e_\infty$. So from the formula (B.18) we get

(B.20) \[
\begin{cases}
2g(X_\Gamma) - 2 = -2d + \sum_{r=2,3,\infty} \frac{1}{r} (d - e_\infty) \\
= -2d + \frac{1}{2} (d - e_2) + \frac{2}{3} (d - e_3) + (d - e_\infty).
\end{cases}
\]

Therefore the formula (B.17) holds.

• For prime level $N = p \geq 5$ one can write

(B.21) \[ g(X_{\Gamma(N)}) = \frac{1}{24}(p + 2)(p - 3)(p - 5). \]

(See Tab.9 where are reported some examples of compact modular curves.)

Appendix C: Modular functions, forms and cusps.

In this appendix we summarize some fundamental definitions and results concerning modular functions, modular forms and cusp forms that are used in the paper.

**Definition C.1.** Let $H = \{z \in \mathbb{C} | \Im(z) > 0\}$. $H$ can be identified with the open disk $\Delta$. Set

\[ G = GL_2^+ (\mathbb{R}) = \{ g \in \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{R} \}. \]

\(^{38}\chi = 2 - 2g\) is the Euler characteristic.
Let us consider the action $G \times H \to H$ by means of fractional linear transformations of $H : (g, z) \mapsto g(z) = \frac{az + b}{cz + d}$. Set $\Gamma \subset \Gamma(1) = SL_2(\mathbb{Z}) \subset G$ a subgroup of the modular group $\Gamma(1)$. A function $f : H \to \mathbb{C}$ is called weakly modular function of weight $k \in \mathbb{Z}$ on $\Gamma$, if are satisfied the following conditions:

(i) $f$ is meromorphic;

(ii) $f(g(z)) = j(g, z)^k f(z)$ for all $g \in \Gamma$, with $j(g, z) = cz + d$.

**Proposition C.1.** Every meromorphic function $f : H \to \mathbb{C}$ is weakly modular of weight $k$ for some $\Gamma \leq GL_2^+(\mathbb{R})$.

**Definition C.2.** • A function $f : H \to \mathbb{C}$ is called modular function of weight $k \in \mathbb{Z}$ on $\Gamma$, if are satisfied the following conditions:

(i) $f$ is a weakly modular function of weight $k$;

(ii) $f$ is meromorphic at $\infty$.

The set of modular functions of weight $k$ is denoted by $A_k = A_k[\Gamma]$.

• A modular form of weight $k \in \mathbb{Z}$ on $\Gamma$, is a modular function of weight $k$ that is holomorphic on $H$ and at $\infty$.

The set of modular forms of weight $k$ is denoted by $M_k = M_k[\Gamma]$.

• A cusp form of weight $k \in \mathbb{Z}$ on $\Gamma$, is a modular form of weight $k$ that vanishes at $\infty$.

The set of cusp forms of weight $k$ is denoted by $S_k = S_k[\Gamma]$.

**Proposition C.2.** • $A_k$, $M_k$ and $S_k$ are $\mathbb{C}$-vector spaces and one has:

(i) $A = \bigoplus_k A_k$ is a graded field;

(ii) $M = \bigoplus_k M_k$ is a graded ring;

(iii) $S = \bigoplus_k S_k$ is a graded ideal of $M$.

The functions in $A$, $M$ and $S$ do not satisfy the transformation properties (ii) in Definition C.1.

• $M = \mathbb{C}[E_4, E_6]$. (For the definition of $E_4$ and $E_6$ see Tab. 10.)

• $S = \Delta \cdot M$. (For the definition of $\Delta$ see Tab. 10.)

• If $k$ is an even integer then $A_k$ is a one-dimensional $A_0$-vector space generated by $(E_4/E_4)^{k/2}$.

• If $k$ is an odd integer then $A_k = \{0\}$.

• $A_0 = \mathbb{C}(j)$, the space of rational functions generated by the $j$-invariant.

• $A = \mathbb{C}(E_4, E_6)$ is the quotient field of $M$.

**Example C.1.** In Tab. 10 are reported some examples of modular functions, modular functions and cusp forms that are useful for a direct understanding of this paper.

**References**

[1] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. Reine Angewandte Math. 218(1965), 79-108.

[2] C. Breuil, B. Conrad, F. Diamond, R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild $3$-adic exercises*, Journal of the American Mathematical Society 14 (4)(2001), 843-939, doi:10.1090/S0894-0347-01-00370-8.

[3] G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre. I*, Mathematische Annalen 46(1895), 481-512. doi:10.1007/bf01427314.

[4] G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre. II*, Mathematische Annalen 49(1897), 207-246. doi:10.1007/bf01442005.

[5] G. Cantor, *Mitteilungen zur Lehre vom Transfiniten*, Zeitschrift für Philosophie und philosophische Kritik 91(1887), 81-125.
Table 10. Examples of modular functions, modular forms and cusps forms.

| Name               | Definition                                                                 | Properties                                                                                                                                 |
|--------------------|---------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| Eisenstein series  | \( G_k(z) = \sum_{m,n \in \mathbb{Z}} \frac{1}{(mz+n)^k} \)               | • converges absolutely for \( k \geq 3 \)  
• \( G_k \in M_k \), for \( k \geq 3 \)  
• \( G_k = 0 \) for \( k \equiv 1 \mod 2 \)  
• \( G_k(z) = \frac{1}{2} \zeta(k) \sum_{m,n \in \mathbb{Z}} \frac{1}{(mz+n)^k} \)  
• \( G_k(z) = 2 \zeta(k) E_k(z) \) for \( k \equiv 0 \mod 2 \) |
| Poincaré series    | \( P_{m,k}(z) = \frac{\theta(z)}{\eta(z)} \)                             | \( \Delta \in S_{12} \)  
\( \Delta(z) = (2\pi)^{12} \sum_{n\geq 1} \tau(n) e^{2\pi niz} \tau(n) \) | Ramanujan function |
| Discriminant form  | \( \Delta = g_2^3 - 27g_3^2 \)                                          | \( j(z) \in \mathbb{A}_d \)  
\( j(z) = 1728 \frac{g_2^3}{g_3} \) | is a modular function |
| \( j \)-invariant  | \( j(z) = 1728 \frac{g_2^3}{g_3} \)                                      | \( j(z) \) gives a bijection between isomorphism classes of elliptic curves over \( \mathbb{C} \) and complex numbers. |

\( \sigma_{k-1}(n) = \sum_{d | n} d^{k-1} \), \( \sigma_k(n) = \sum_{d | n} d^k \), \( B_k \) is the \( k \)-th Bernoulli number, \( \Gamma \in 1 \left( \begin{array}{c} n \\ 0 \end{array} \right) 1 \) \( n \in \mathbb{Z} \) \( \leq \Gamma(1) \), \( g_2 \) is a modular form of weight 4.

\( j \) is a surjective meromorphic function \( H \to \mathbb{C} \), invariant under \( SL_2(\mathbb{Z}) \)-action.

\( \sum' \) denotes that the term \((m,n) = (0,0)\) has been omitted.
A. Prástaro, Geometry of PDEs and Mechanics, World Scientific Publishing, River Edge, NJ, 1996, 760 pp. ISBN 9810225202.

J.-P. Serre, A Course in Arithmetic, Springer-Verlag, New York, 1973.

G. Shimura, Yutaka Taniyama and his time. Very personal recollections, Bull. London Math. Soc. 21 (2) (1989), 186-196. doi:10.1112/blms/21.2.186.

B. Schoeneberg, Elliptic Modular Functions, Springer-Verlag, New York, 1974.

J. H. Silverman, The arithmetic of elliptic curves, Springer, 1986. ISBN 0-387-96203-4.

J. H. Silverman and J. Tate, Rational Points on Elliptic Curves, Springer, 1994. ISBN 0-387-97825-9.

N. M. Stephens, Congruence properties of congruent numbers, Bull. London Math. Soc. 7 (1975), 182-184. doi:10.1112/blms/21.2.186.

Y. Taniyama, Problem 12, Sugaku (in Japanese) 7 (1956), 269; (English translation in Shimura 1989, p. 194).

J. Tate, The arithmetic of elliptic curves, Invent. Math. 23 (1974), 179-206.

J. B. Tunnell, A classical diophantine problem and modular forms of weight 3/2, Invent. Math. 72(2) (1983), 323-334.

A. Weil, L’arithmétique sur les courbes algébriques, Acta Math 52(1929), 281-315.

A. Weil, über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, Mathematische Annalen 168(1967), 149-156. doi:10.1007/BF01361551, ISSN 0025-5831; Number Theory: An Approach Through History. Birkhäuser, 1984.

A. Wiles, Modular elliptic curves and Fermat’s last theorem, Ann. Math. 142 (3) (1995), 443-551.

A. Wiles, The Birch and Swinnerton-Dyer Conjecture. Official Problem Description, at the Clay Mathematics Institute. http://www.claymath.org/millenium-problems/birch-and-swinnerton-dyer-conjecture.