Many 2-Level Polytopes from Matroids

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Abstract. The family of 2-level matroids, that is, matroids whose base polytope is 2-level, has been recently studied and characterized by means of combinatorial properties. 2-level matroids generalize series-parallel graphs, which have been already successfully analyzed from the enumerative perspective.

We bring to light some structural properties of 2-level matroids and exploit them for enumerative purposes. Moreover, the counting results are used to show that the number of combinatorially non-equivalent \((n-1)\)-dimensional 2-level polytopes is bounded from below by \(c \cdot n^{-5/2} \cdot \rho^{-n}\), where \(c \approx 0.03791727\) and \(\rho^{-1} \approx 4.88052854\).

1. Introduction

A hyperplane \(H\) is facet-defining for a polytope \(P\) if it is supporting for \(P\) and \(\dim(P \cap H) = \dim(P) - 1\). A 2-level polytope is a polytope \(P\) such that for each facet-defining hyperplane \(H\), there exists a hyperplane \(H'\) parallel to \(H\) that contains all the vertices of \(P\) not in \(H\). The family of 2-level polytopes appeared in the literature in different areas under different names: in \([DLRS10]\) they are called compressed polytopes and also show up in statistics (see \([Sul06]\)). In the context of combinatorial optimization \([CPT10]\) and \([Lau09]\), 2-level polytopes are related to the so-called exact point configurations: the interest in these configurations is due to the fact that some techniques from polynomial optimization, namely semidefinite programming relaxations, are extremely efficient for these configurations. 2-level polytopes also play a role in the study of extremal centrally-symmetric polytopes \([SWZ09]\).

Two polytopes are combinatorially equivalent if their face lattices are isomorphic. It is known that all 2-level \(n\)-dimensional polytopes are affinely equivalent to \(0/1\)-polytopes (polytopes with vertices in \(\{0,1\}^n\)) and the number of combinatorially non-equivalent \(0/1\)-polytopes is doubly-exponential in the dimension (see \([Zie00]\)). Among the finite number of \(0/1\)-polytopes of fixed dimension, it is natural to ask how many are 2-level, up to combinatorial equivalence.

Though 2-level polytopes are endowed with a very restrictive geometric property, this class is not well-understood and an exact enumeration seems to be complicated. It is easy to see that the 2-levelness is preserved for some polytopal constructions: pyramid, prism, twisted prism and Cartesian product. Moreover some subfamilies of 2-level polytopes are known: two of them are explored in \([PHSZ13]\), the so-called Hansen polytopes \([Han77]\) and Hanner polytopes \([Han56]\), while a third one arises from stable sets of perfect graphs as explained in \([GLS03]\) Ch. 9). Note that the construction of twisted prism over this last family yields the family of Hansen polytopes. Order polytopes of finite posets \([Sta86]\) are also 2-level. Very recently, a new subfamily of 2-level polytopes arising from matroid theory has been characterized in \([GST14]\). More precisely, this subfamily is associated with the base polytopes of the so-called 2-level matroids.

A complete classification of the \(0/1\)-equivalence classes of \(0/1\)-polytopes is only available for dimension 3, 4, 5, and 6 (polytopes up to 12 vertices). Moreover two polytopes that are \(0/1\)-equivalent are also combinatorially equivalent, but the converse is not true. The difficulties in providing a complete list already in dimension 6 suggest that a computational approach to the problem could be unsuccessful. The lack of an exact enumeration in dimension \(\geq 6\) leads to a second natural question, namely the existence of asymptotic bounds for the number of 2-level polytopes.

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By means of the polytopal constructions we mentioned above (pyramid, prism, twisted prism and Cartesian product) exponentially many combinatorially non-equivalent 2-level polytopes can be constructed. In this paper we compute an explicit exponential lower bound for the number of 2-level polytopes via 2-level matroids. More precisely, we prove the following theorem.

**Theorem 1.1.** The number of combinatorially non-equivalent \((n-1)\)-dimensional 2-level polytopes is bounded from below by

\[ c \cdot n^{-5/2} \cdot \rho^{-n}, \]

where \( c \approx 0.03791727 \) and \( \rho^{-1} \) is a computable constant whose value is approximately equal to 4.88052854.

The interest in the subfamily of 2-level matroids is motivated by the fact that it contains more complicated polytopes, namely not obtained by means of elementary polytopal construction. Moreover it allows to determine a large basis for the exponential lower bound.

New combinatorial aspects of 2-level matroids are introduced in Section 3 and give the possibility to increase the control on the enumerative formulas. It is noteworthy that this matroid family generalizes the family of series-parallel graphs, which appears in various areas and has several interesting properties that are likely to have counterparts for the 2-level matroids. In particular, series-parallel graphs have been already successfully studied from an enumerative point of view in [BGK07] and [DFK+11]. To approach the enumeration of 2-level matroids we investigate the so-called matroid tree decomposition associated to these matroids. We analyze the features of the decomposition and we get one of the main results of the paper: we observe that there is an interesting interpretation in terms of acyclic structures. More precisely, we reveal a bijection between 2-level matroids and a family of trees, that we call UMR-trees, whose vertices are labelled by uniform matroids. This last discovery makes 2-level matroids particularly suitable for enumeration. Indeed the family of UMR-trees is exploited in Section 4 to encode all the enumerative information in terms of generating functions and relations (equations) among them by means of the so-called symbolic method in enumerative combinatorics. Finally, powerful analytic techniques are applied to the equations in order to get an asymptotic estimate for the coefficients of the generating functions.

**Structure of the paper.** The paper is structured in the following way: in Section 2 the basics on matroid theory and enumerative combinatorics are stated. In Section 3 we study how to decompose 2-level matroids in terms of tree-like structures (UMR-trees). The structural properties of the UMR-trees are exploited later in Section 4 in order to get counting formulas which can be analyzed by means of analytic techniques, producing the estimate stated in Theorem 1.1.

2. Preliminaries

In this section we introduce the basic notions needed in the rest of the paper. In Subsection 2.1 we focus on definitions and concepts related to matroid theory. In Subsection 2.2 we fix our notation concerning enumeration by means of generating functions and finally in Subsection 2.3 we state the results needed in order to get asymptotic estimates for the coefficients of the generating functions under study.

2.1. Matroids. The basic definition is the following:

**Definition 1.** A matroid of rank \(k\) is an ordered pair \(\mathcal{M} = (E, \mathcal{B})\) consisting of a finite set \(E\) (ground set) and a collection of bases \(\emptyset \neq \mathcal{B} \subseteq \binom{E}{k}\) satisfying the basis exchange axiom: for \(B_1, B_2 \in \mathcal{B}\) and \(x \in B_1 \setminus B_2\) there exists \(y \in B_2 \setminus B_1\) such that \((B_1 \setminus x) \cup y \in \mathcal{B}\).

Matroids are combinatorial objects that generalize graphs and linear dependence: the family of graphic matroids is particularly interesting and useful to visualize examples of matroids. The matroid associated to a graph \(G = (V, E)\) is such that the ground set is given by the set of edges and the collection of bases is given by the set of spanning forests. The rank of a connected graph is clearly \(|V| - 1\). However, it could sometimes be misleading to think in terms of the graph structure, since some information, like the vertex structure, is not retained at matroid level.
A matroid has many equivalent definitions (see Oxley [2011] for more details): we presented the one using the collection of bases. Nevertheless we want to introduce two further collections of sets that can define a matroid. The first one is the collection of independent sets, that is all the sets $X \subseteq E$ such that $X \subseteq B$, for some $B \in \mathcal{B}$. The rank of $X \subseteq E$, denoted by $\text{rank}_M(X)$, is the cardinality of the largest independent subset contained in $X$. The second one is the collection of circuits $\mathcal{C}(M)$. Circuits are minimal dependent sets of $M$. An element $e$ such that $\{e\}$ is a circuit is called loop.

Two matroids $M$ and $\mathcal{N}$ are isomorphic if their collection of circuits are the same up to relabelling of the ground sets $E(M)$ and $E(\mathcal{N})$. More formally $M \cong \mathcal{N}$ if there is a bijection $\phi : E(M) \to E(\mathcal{N})$ such that, for all $X \subseteq E(M)$, $\phi(X) \in \mathcal{C}(\mathcal{N})$ if and only if $X \in \mathcal{C}(M)$.

A fairly simple family of matroids that is of great importance in the rest of the paper are the uniform matroids. The uniform matroid $U_{n,k}$ consists of the ground set $\{1, \ldots, n\}$ and the collection of bases $\binom{[n]}{k}$. The uniform matroids which are also graphic matroids are of the form: $U_{n,0}$, $U_{n,1}$, $U_{n,n-1}$, and $U_{n,n}$. See Figure 1.

**Figure 1.** From left to right, graphical representations of the matroids $U_{n,0}$, $U_{n,1}$, $U_{n,n-1}$, and $U_{n,n}$.

Observe that for $U_{n,0}$ and $U_{n,n}$ we have given one of the many possible graphical representations. Namely, Whitney’s 2-Isomorphism Theorem [Oxl II Thm. 5.3.1] implies that every graph formed by $n$ loops corresponds to $U_{n,0}$ regardless of the vertex structure, while any tree with $n$ edges corresponds to the matroid $U_{n,n}$.

For counting purposes we do not consider the uniform matroids $U_{n,0}$ and $U_{n,n}$, while among the other uniform matroids we need to distinguish the graphic ones from the non-graphic ones. More precisely we write $M_n$ to denote the matroid $U_{n,1}$ (it stands for multiedge) and $R_n$ to denote the matroid $U_{n,n-1}$ (it stands for ring).

The dual matroid $M^*$ of a matroid $M = (E, \mathcal{B})$ is the matroid defined by the pair $(E, \mathcal{B}^*)$ where $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$. For uniform matroids we have $U_{n,k}^* = U_{n,n-k}$ and in particular $R_n^* = M_n$.

An element $e$ is called a coloop of $M$ if it is a loop of $M^*$. A matroid $M$ is self-dual if $M \cong M^*$. For instance all uniform matroids of type $U_{2n,n}$ are self-dual.

**Definition 2.** Let $M = (E, \mathcal{B})$ be a matroid. The base polytope of $M$ is the polytope $P_M := \text{conv}(\{1_B : B \in \mathcal{B}\})$.

It was proved in [GGMSS74] that all the edges of a base polytope are parallel to some difference $e_i - e_j$ of two unit vectors. The base polytopes $P_{R_n}$ and $P_{M_n}$ are $n$-simplices, while the polytopes $P_{U_{n,k}}$ for $2 \leq k \leq n-2$, are called hypersimplices and denoted by $\Delta_{n,k}$. For more background about this family of polytopes we refer to [Zie10].

A 2-level matroid is a matroid such that the corresponding base polytope is 2-level. In [GSL14] an excluded minor characterization for the family of 2-level matroids is provided. The four excluded minors are the following rank 3 matroids on 6 elements: $M(K_4)$, $W^3$, $Q_6$, $P_6$. The first excluded minor of the list is nothing but the graphic matroid of the complete graph on 4 vertices; for more details about these matroids we refer to Oxley’s book [Oxl II] or to the paper where they are used to describe the 2-level matroids [GSL14]. Since $P_6 = ([6],B)$ appears in Example 3, we list here its collection of circuits $\mathcal{C}(P_6) = \{123, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 2456, 3456\}$.
It is important to notice that there is only one circuit with 3 elements. In [BGW03], together with the excluded minor characterization of 2-level matroids, a synthetic description of this class is also provided. Before presenting it, we need to introduce two matroid operations. Let \( M_1 \) and \( M_2 \) be matroids with disjoint ground sets \( E_1 \) and \( E_2 \). The collection
\[ B := \{B_1 \cup B_2 : B_1 \in B(M_1), B_2 \in B(M_2)\} \]
is the set of bases of a matroid on \( E_1 \cup E_2 \), called the **direct sum** of \( M_1 \) and \( M_2 \) and denoted by \( M_1 \oplus M_2 \). On the other hand, if we choose \( e_1 \in E_1 \) and \( e_2 \in E_2 \) such that at least one element is not a coloop and define the collection
\[ B := \{B_1 \cup B_2 \setminus \{e_1, e_2\} : B_1 \in B(M_1), B_2 \in B(M_2), |(B_1 \cup B_2) \cap \{e_1, e_2\}| = 1\}. \]
The pair \((E_1 \cup E_2 \setminus \{e_1, e_2\}, B)\) is a matroid called **2-sum** of \( M_1 \) and \( M_2 \) with base points \( e_1 \) and \( e_2 \). We denote it by \((M_1, e_1) \oplus_2 (M_2, e_2)\). Observe this notation is slightly different from the one used in [Oxl11], but it turns out to be more efficient for the constructive part.

**Theorem 2.1.** Every 2-level matroid can be obtained as a sequence of direct sums and 2-sums of uniform matroids. Moreover every combination of uniform matroids yields a 2-level matroid.

The direct sum and the 2-sum of matroids are closely related to the connectedness of a matroid: a matroid \( M \) is 2-connected (or also connected) if it cannot be written as a proper direct sum of two matroids, and \( M \) is 3-connected if it cannot be written as 2-sum of two matroids each with fewer elements than \( M \).

A **separator** of a matroid \( M \) is a set \( T \subseteq E \) such that \( \text{rank}_M(T) + \text{rank}_M(E \setminus T) = \text{rank}_M(M) \). A matroid \( M \) is 2-connected if and only if there is no separator \( T \) with \( T \) being a proper subset of \( E \). The base polytope \( P_M \) of a matroid \( M = (E, B) \) has dimension \( |E| - c(M) \) where \( c(M) \) is the number of 2-connected components of \( M \). In particular, if \( M \) is 2-connected, then \( \dim(P_M) = |E| - 1 \).

If we try to look at matroid operations from the point of view of base polytopes we have:

- \( P_M^* = 1 - P_M \). This means that the base polytope of the dual matroid \( P_M^* \) is congruent to the base polytope \( P_M \).
- \( P_{M_1 \oplus M_2} = P_{M_1} \times P_{M_2} \), where \( \times \) denotes the Cartesian product of polytopes.
- \( P_{(M_1, e_1) \oplus_2 (M_2, e_2)} \) can be described using the subdirect product construction introduced in [GMC76] as shown in [GS13].

To keep the counting as easy as possible we first deal with 2-connected matroids. This corresponds to consider only sequences of 2-sums of uniform matroids. As a consequence, the polytopes we count cannot be obtained as a Cartesian product of two polytopes (for example no prism is in this family). At the end of Section 4 we show that, asymptotically, the restriction to 2-connected matroids does not alter the exponential growth.

Let us conclude this section with some results for base polytopes that are needed in Section 4 to complete the last step of the asymptotic enumeration of 2-level matroids. The first one appears as part of Exercise 4.9 in [Whi86] Ch. 4.

**Proposition 2.2.** Let \( M \) and \( N \) be 2-connected matroids. \( M \) and \( N \) have isomorphic basis graphs if and only if \( M \cong N \) or \( M \cong N^* \).

The **basis graph** of a matroid \( M \) is the undirected graph with vertex set the collection of all bases of \( M \) such that a basis \( B_1 \) is connected to another basis \( B_2 \) whenever the symmetric difference \( B_1 \Delta B_2 \) has cardinality exactly 2. Note that this is the 1-skeleton of the base polytope.

Since two congruent polytopes have the same 1-skeleton we easily obtain the following corollary, which also appears as an exercise in [BGW03] Ch. 1, Ex. 18.

**Corollary 2.3.** Let \( M \) a 2-connected matroid and \( P_M \) its base polytope. Suppose that \( N \) is a matroid such that the base polytope \( P_N \) is congruent to \( P_M \). Then \( N \cong M \) or \( N \cong M^* \).

It is known that “congruent” \( \Rightarrow \) “combinatorially equivalent”. The converse is in general not true, not even for 0/1-polytopes: for instance we can find full-dimensional 0/1-simplices with different volume [Zic00]. Nevertheless, for the class of base polytopes, we get the following corollary of Proposition 2.2.
Corollary 2.4. Let $\mathcal{M}$ and $\mathcal{N}$ be 2-connected matroids. $P_\mathcal{M}$ is congruent to $P_\mathcal{N}$ if and only if $P_\mathcal{M}$ is combinatorially equivalent to $P_\mathcal{N}$.

Proof. We only need to prove one direction. If $P_\mathcal{M}$ is combinatorially equivalent to $P_\mathcal{N}$, then they have isomorphic face lattices and, in particular, isomorphic 1-skeletons. By Proposition 2.2, we have $\mathcal{M} \cong \mathcal{N}$ or $\mathcal{M} \cong \mathcal{N}^*$ and therefore $P_\mathcal{M}$ is congruent to $P_\mathcal{N}$. \hfill $\Box$

This last corollary allows us to investigate the number of non-congruent 2-level base polytopes, instead of looking at combinatorial equivalence of such polytopes.

2.2. The symbolic method in enumerative combinatorics. Tree-like structures. The reader is referred to [FS09, Ch. 1] to see all the terminology and notation in full detail. Let $(\mathcal{A}, | \cdot |)$ be an admissible combinatorial class, namely a set $\mathcal{A}$ endowed with a size function $| \cdot |$ such that the number of elements in $\mathcal{A}$ of any given size is finite. Then the generating function (GF for short) associated to $\mathcal{A}$ is the formal power series $A(x) = \sum_{a \in \mathcal{A}} x^{|a|} = \sum_{n \geq 0} a_n x^n$. In particular, $a_n$ is the number of elements in $\mathcal{A}$ of size $n$. We also write $[x^n]A(x) = a_n$. For two generating functions $A(x)$ and $B(x)$, we write $A(x) \leq B(x)$ if for each $n$, $[x^n]A(x) \leq [x^n]B(x)$.

The symbolic method in enumerative combinatorics (see [FS09]) gives a direct way to translate combinatorial operations between combinatorial classes into equations between the corresponding GFs. Apart from the usual disjoint union and Cartesian product of combinatorial families (which translate into sum and product of GFs, respectively), we also use the multiset construction: given a combinatorial class $(\mathcal{A}, | \cdot |)$ with GF $A(x)$, the multiset of $\mathcal{A}$ is the combinatorial family obtained by taking multisets of elements in $\mathcal{A}$. The corresponding GF is equal to

$$\text{Mul}(A(x)) = \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} A(x^r) \right).$$

Finally, we also need restricted multiset constructions. Let $\Lambda$ be a subset of positive integers. The multiset operator restricted to $\Lambda$ of $\mathcal{A}$ is the combinatorial family obtained by taking multisets of elements in $\mathcal{A}$ with the restriction that the number of components lies in $\Lambda$. We write this as $\text{Mul}_\Lambda(A(x))$. In particular,

$$\text{Mul}_0(A(x)) = 1, \quad \text{Mul}_1(A(x)) = A(x), \quad \text{Mul}_2(A(x)) = \frac{1}{2} \left( A(x)^2 + A(x^2) \right).$$

The notation $\text{Mul}_{\geq k}$ refers to the multiset operator restricted to $\Lambda = \{k, k+1, ...\}$.

The Dissymmetry Theorem for trees. The Dissymmetry Theorem for trees (see [BLLR97]) provides a general methodology to express a combinatorial class of unrooted trees in terms of related classes of pointed trees. More precisely, let $T$ be a class of unrooted trees. We define the following families of pointed trees: $\mathcal{T}_0$ is built from $T$ by pointing a vertex, $\mathcal{T}_{0 \to 0}$ is the class of trees where an edge of $T$ is pointed and $\mathcal{T}_{0 \to \infty}$ is the class of trees obtained from $T$ by pointing and orienting an edge. The Dissymmetry Theorem for trees asserts that

$$T \cup \mathcal{T}_{0 \to 0} \simeq \mathcal{T}_{0 \to \infty} \cup \mathcal{T}_0,$$

where “$\simeq$” means that there is a bijection between the two combinatorial classes which translates directly into equalities between the counting formulas.

2.3. Asymptotic estimates and analytic combinatorics. By means of analytic methods we can obtain asymptotic estimates for $[x^n]A(x)$ in terms of the singularities of $A(x)$ with minimum complex modulus. Such singularities are called dominant. Since $A(x)$ has non-negative coefficients, one of its dominant singularities (if there are any) is a positive real number by Pringsheim’s Theorem, see [FS09, Thm. IV.6].

With this language, we obtain the asymptotic expansion of $[x^n]A(x)$ by transferring the behaviour of $A(x)$ around its dominant singularities from a simpler function $B(x)$ for which we know the asymptotic behaviour of the coefficients. The first result in this direction is the Transfer Theorem for singularity analysis [FO90, FS09]. For our purposes we present a version of the theorem that covers the case when there is a unique dominant singularity $\rho$. 


with its ground set and its collection of bases. For instance the ring graphic matroids. In particular they are rings and multiedges. Each vertex label must be provided.

Let us consider the matroid-labelled tree in the picture whose vertex labels are all.

\[ \Gamma(\phi) \]

for this topic is the paper \[ Drm97 \]. For convenience, we rephrase it here in a simplified version.

\[ \text{Transfer Theorem for a single singularity} \quad [FO90], \text{ simplified version} \]

\[ \text{Theorem 2.5} \]

\[ \text{Matroid decomposition} \]

This section is devoted to the analysis of the structure of 2-level matroids. Every 2-connected matroid has a tree decomposition which relies on the 2-sum and we refer to \[ Oxl11, \text{Sect. 8.3} \] for a complete overview on this topic. We state here the results which are relevant for the paper and we explore further features of tree decomposition that are specific for the class of 2-level matroids. First let us make precise what we mean by a decomposition.

\[ \text{Definition 3.} \]

A matroid-labelled tree is a tree \( T \) with vertex set \( \{ M_1, \ldots, M_s \} \) for some positive integer \( s \) such that

- each \( M_i \) is a matroid.
- each edge \( t \) of \( T \) is labelled by a subset of \( \cup_i E(M_i) \) of size 2.
- if \( M_{i_1} \) and \( M_{i_2} \) are joined by an edge \( t = \{ e_{i_1}, e_{i_2} \} \) of \( T \), then \( e_{i_1} \in E(M_{i_1}), e_{i_2} \in E(M_{i_2}) \), and either \( e_{i_1} \) is not a separator of \( M_{i_1} \) or \( e_{i_2} \) is not a separator of \( M_{i_2} \).
- the ground sets of the \( M_i \)'s are pairwise disjoint as well as the edges of the tree \( T \).

We call \( M_1, \ldots, M_s \) the vertex labels of \( T \).

\[ \text{Example 1.} \]

Let us consider the matroid-labelled tree in the picture whose vertex labels are all graphic matroids. In particular they are rings and multiedges. Each vertex label must be provided with its ground set and its collection of bases. For instance the ring \( R_4 \) has ground set \( E(R_4) = \)
{1, 2, 3, 4} and collection of bases \( \binom{14}{3} \). For a complete description of the vertex labels we refer to Example 3.

For a matroid-labelled tree \( T \), we can contract an edge \( t \) connecting two vertex labels \( M_{i_1} \) and \( M_{i_2} \). The result is a matroid-labelled tree \( T/t \) with the same edges and vertex labels, except that the vertex labels \( M_{i_1} \) and \( M_{i_2} \) have been gathered into a unique vertex label, namely \( (M_{i_1}, e_{i_1}) \oplus_2 (M_{i_2}, e_{i_2}) \), and the edge \( t \) has been contracted.

**Example 2.** The vertex labels of the matroid-labelled tree introduced in Example 1 are all graphic matroids. Thus, we can represent it as a sequence of 2-sums of graphs. In the picture we specify the ground set for each of the graphs.

In this case, the contraction of an edge in the matroid-labelled tree corresponds to compute the 2-sum of two graphs. If we contract all the edges we get the graph on the right which happens to be a series-parallel graph. **Definition 4** shows that the matroid-labelled tree we are considering is a tree decomposition for the matroid associated to this series-parallel graph.

**Definition 4.** A tree decomposition of a 2-connected matroid \( M \) is a matroid-labelled tree \( T \) such that if \( V(T) = \{M_1, \ldots, M_s\} \) and \( E(T) = \{t_1, \ldots, t_{s-1}\} \), then

- \( E(M) = (E(M_1) \cup E(M_2) \cup \ldots \cup E(M_s)) \setminus (t_1 \cup t_2 \cup \ldots \cup t_{s-1}) \)
- \( |E(M_i)| \geq 3 \) for all \( i \), unless \( |E(M)| < 3 \), in which case \( s = 1 \) and \( M_1 = M \)
- \( M \) is the matroid that labels the single vertex of \( T/t_1, t_2, \ldots, t_{s-1} \).

We now report a theorem appeared in [CE80]. We follow the more recent formulation and notation of [Oxl11, Thm. 8.3.10] except that we replace the words “circuit” and “cocircuit” respectively with “ring” and “multiedge”, according to the names we use later in the paper.
Theorem 3.1. Let $\mathcal{M}$ be a 2-connected matroid. Then $\mathcal{M}$ has a tree decomposition $T$ in which every vertex label is 3-connected, a ring, or a multiedge, and there are no two adjacent vertices that are both labelled by rings or are both labelled by multiedges. Moreover, $T$ is unique up to relabelling of the vertex labels ground sets.

In order to obtain the uniqueness, it is necessary to require that there are no two adjacent vertex labels that are both rings or multiedges, otherwise adjacent rings (or multiedges) could make possible to keep the same tree structure while changing the vertex labels. This additional requirements to get uniqueness justify why we consider separately the labels of type $U$, $R$, and $M$ in Section 4.

The theorem allows us to uniquely represent every matroid by a matroid-labelled tree whose vertex labels are 3-connected matroids (except rings and multiedges). In this paper we want to tackle the problem from a different perspective: instead of starting with a matroid and finding its tree decomposition, the goal is to count how many non-isomorphic matroid-labelled trees can be constructed from a given set of possible vertex labels.

In this constructive process, every time that we establish the adjacency of two vertices, we have to decide one element for each ground set of the two vertex labels to be the base point of the 2-sum. As shown in Example 3, the choice of the elements affects the result of the 2-sum: we can have two non-isomorphic matroids whose tree decomposition have the same tree structure and the same vertex labels, but different labels for the edges of the tree.

Before presenting the example, let us give an explicit description of the collection of circuits of the matroid $M_{i} \circ \mathcal{M}$, namely

$$
\mathcal{C}(\mathcal{M}_{1} \setminus e_{1}) \cup \mathcal{C}(\mathcal{M}_{2} \setminus e_{2}) \cup \{(C_{1} - \{e_{1}\}) \cup (C_{2} - \{e_{2}\}) : e_{1} \in C_{1} \in \mathcal{C}(\mathcal{M}_{1}) \text{ and } e_{2} \in C_{2} \in \mathcal{C}(\mathcal{M}_{2})\}. \tag{3}
$$

Example 3. Consider the 3-connected matroid $P_{6}$ described in Subsection 2.1. Construct a matroid-labelled tree with two adjacent vertex labels, both equal to $P_{6}$ (see Figure 4). Let us label the ground set of the first copy of $P_{6}$ from 1 to 6 and the ground set of the second copy from 7 to 12. Each of the two copies has one circuit of length 3 (we assume $\{1,2,3\}$ and $\{7,8,9\}$) and all the other circuits are of length 4. Consider the matroid $(P_{6},1) \circ \mathcal{M}(P_{6},7)$. It has circuits of length 4, 5, 6. On the other hand, the matroid $(P_{6},4) \circ \mathcal{M}(P_{6},10)$ has circuits of length 4, 5, 6. Thus, the two matroids are not isomorphic.

Nevertheless if we focus our attention on 2-level matroids, the vertex labels are chosen among uniform matroids. In this very particular case the tree structure, together with the vertex labels is enough to determine the matroid uniquely up to matroid isomorphism. The proof of this fact is provided by Lemma 3.3 and Lemma 3.4. The main result of this section is the following theorem, which is required for the enumeration in Section 4.

Theorem 3.2. The family of 2-connected 2-level matroids is in bijection with the family of trees whose vertices are labelled by uniform matroids.

Before presenting the proof of the theorem, we introduce some further definitions. For each vertex label $\mathcal{M}_{i}$ of the tree decomposition of a matroid $\mathcal{M}$, we partition the ground set $E(\mathcal{M}_{i}) = \{e_{1}, \ldots, e_{n}\}$ into two sets: the set $W(\mathcal{M}_{i})$ of elements which are base points for the 2-sum with a vertex label adjacent to $\mathcal{M}_{i}$ and the set $F(\mathcal{M}_{i}) = E(\mathcal{M}_{i}) \setminus W(\mathcal{M}_{i})$. We call $W(\mathcal{M}_{i})$ the set of ideal elements (generalizing the notion of ideal edge in [Tut01 Sect. IV.3]) and $F(\mathcal{M}_{i})$ the set of free elements. Note that the ideal elements do not belong to the ground set of $\mathcal{M}$, while we have $E(\mathcal{M}) = \cup_{i} F(\mathcal{M}_{i})$.

For a matroid $\mathcal{M}$ let us consider the set of its circuits $\mathcal{C}$. We say that $\mathcal{M}$ is transposition invariant with respect to the pair of elements $\{e_{1}, e_{2}\} \subset E(\mathcal{M})$ if we have that $\pi(\mathcal{C}) = \mathcal{C}$, where $\pi$ is the
transposition \((e_1, e_2)\) and
\[
\pi(C) = \{\pi(C) : C \in C\}.
\]
The notation \(\pi(C)\) means that if \(e_1 \in C\) we replace it with \(e_2\) and if \(e_2 \in C\) we replace it with \(e_1\) (note that the circuits containing both \(e_1\) and \(e_2\) or none of them are invariant under such operation).

A matroid is permutation invariant if it is transposition invariant with respect to every couple of elements in the ground set.

**Example 4.** Every uniform matroid \(U_{n,k}\) is permutation invariant, since for every choice of \(e_1, e_2 \in [n] = E(U_{n,k})\), \(\pi(\cdot)\) is a bijection from the set of \((k+1)\)-subsets of \([n]\) to itself. Moreover if a matroid \(\mathcal{M} = ([n], \mathcal{B})\) is permutation invariant, then it is a uniform matroid. Indeed let \(C\) be the circuit with the least number \(s\) of elements, then all the other subsets \(\binom{[n]}{s}\) have to be circuits (by transposition invariance). It also follows that there cannot be other circuits. Thus \(\mathcal{M} = U_{n,s-1}\).

We say that \(\mathcal{M}\) is \(\mathcal{M}_i\)-transposition invariant, for \(\mathcal{M}_i\) vertex label of the tree decomposition if it is transposition invariant with respect to every couple of elements in \(F(\mathcal{M}_i)\).

We say that \(\mathcal{M}\) is node invariant if it is \(\mathcal{M}_i\)-transposition invariant for every vertex label \(\mathcal{M}_i\) of the tree decomposition.

**Lemma 3.3.** Let \(\mathcal{M}\) be a \(\mathcal{M}_i\)-transposition invariant 2-connected matroid and \(\mathcal{U}\) a uniform matroid. For any choice of \(f \in F(\mathcal{M}_i)\), the 2-sum \((\mathcal{M}, f) \oplus_2 (\mathcal{U}, u)\) always yields the same matroid (up to isomorphism).

**Proof.** Consider two elements \(f_1, f_2 \in F(\mathcal{M}_i)\). We want to show that
\[
\mathcal{N}_{f_1} := (\mathcal{M}, f_1) \oplus_2 (\mathcal{U}, u) \cong (\mathcal{M}, f_2) \oplus_2 (\mathcal{U}, u) =: \mathcal{N}_{f_2}.
\]
Notice that \(E(\mathcal{N}_{f_1}) = E(\mathcal{N}_{f_2}) = \{f_2\} \cup \{f_1\}\). We claim that the isomorphism bijection \(\varphi : E(\mathcal{N}_{f_1}) \to E(\mathcal{N}_{f_2})\) is the following.
\[
\varphi(e) = \begin{cases} f_1 & \text{if } e = f_2 \\ e & \text{otherwise.} \end{cases}
\]
We need to show that for every \(X \subset E(\mathcal{N}_{f_1}), X \in C(\mathcal{N}_{f_1})\) if and only if \(\varphi(X) \in C(\mathcal{N}_{f_2})\).

As we have seen in \([4]\) a circuit \(C\) of \(\mathcal{N}_{f_1}\) can be of 3 different types:

- \(C \in C(\mathcal{U} \setminus u)\). In this case \(\varphi(C) = C\) and clearly \(C \in C(\mathcal{N}_{f_2})\).
- \(C \in C(\mathcal{M} \setminus f_1)\). In this case \(C\) is also a circuit of \(\mathcal{M}\), \(f_1 \notin C\) and since \(\mathcal{M}\) is \(\mathcal{M}_i\)-transposition invariant, for \(\pi = (f_1, f_2)\) we have \(\pi(C) \in C(\mathcal{M})\). Notice that \(f_1 \notin C\) implies \(f_2 \notin \pi(C)\), that is \(\pi(C) \in C(\mathcal{M} \setminus f_2)\). Moreover it is easy to observe that \(\varphi(C) = \pi(C)\).
- \(C = (C_1 \setminus \{f_1\}) \cup (C_2 \setminus \{u\})\), \(f_1 \in C_1 \in C(\mathcal{M})\) and \(u \in C_2 \in C(\mathcal{U})\). Since \(\mathcal{M}\) is \(\mathcal{M}_i\)-transposition invariant, we have that \(\pi(C_1) \in C(\mathcal{M})\). Moreover \(f_2 \in \pi(C_1)\). It is also easy to see that \(\varphi(C) = (\pi(C_1) \setminus \{f_2\}) \cup (C_2 \setminus \{u\})\), \(f_2 \in \pi(C_1) \in C(\mathcal{M})\) and \(u \in C_2 \in C(\mathcal{U})\).

The same argument applies to check that all circuits of \(\mathcal{N}_{f_2}\) are circuits of \(\mathcal{N}_{f_1}\) under the map \(\varphi^{-1}\). This concludes the proof. \(\square\)

**Lemma 3.4.** Let \(\mathcal{M}\) be a node invariant 2-connected matroid and \(\mathcal{U}\) a uniform matroid. The 2-sum \((\mathcal{M}, f) \oplus_2 (\mathcal{U}, u)\) is a node invariant matroid for any choice of \(f \in E(\mathcal{M})\) and \(e \in E(\mathcal{U})\).

**Proof.** Choose a vertex label \(\mathcal{M}_i\) of the tree decomposition of \(\mathcal{M}\). Without loss of generality, let us assume \(f \in F(\mathcal{M}_i)\). To prove that \((\mathcal{M}, f) \oplus_2 (\mathcal{U}, u)\) is node invariant, we need to check the transposition invariance for each vertex label. For any vertex label \(\mathcal{M}_j\) and \(f_1, f_2 \in F(\mathcal{M}_j)\), \(f_1, f_2 \neq f\), we have that the set \(C((\mathcal{M}, f) \oplus_2 (\mathcal{U}, u))\) is invariant under \(\pi = (f_1, f_2)\). Indeed \(\mathcal{C}(\mathcal{M} \setminus f)\) and \(\mathcal{C}(\mathcal{U} \setminus u)\) are invariant under \(\pi\) because \(\mathcal{M}\) is node invariant and \(\mathcal{U}\) is permutation invariant. The same holds true for the circuits of the third type, since
\[
\pi((C_1 \setminus f) \cup (C_2 \setminus u)) = (\pi(C_1) \setminus f) \cup (C_2 \setminus u)
\]
and \(f \in \pi(C_1) \in C(\mathcal{M})\) by node invariance of \(\mathcal{M}\).
In the tree decomposition of the 2-sum there is one new node, labelled by the uniform matroid $U$. We still have to check that $(M, f) \oplus_2 (U, u)$ is $U$-transposition invariant. The same argument used above applies to $U$, since it is a permutation invariant matroid.

Proof of Theorem 3.2 Let us start with a uniform matroid $M_1$, $M_2$ is permutation invariant and therefore also node invariant. The 2-sum of $M_1$ with a second uniform matroid $M_2$ yields a node invariant matroid by Lemma 3.3. We can iteratively add by 2-sum new uniform matroids $M_3, M_4, \ldots, M_s$. The matroid we get at every step is clearly node invariant. Moreover, at the $j$-th iteration we have to select which vertex label $M_i$, $i < j$ of $M$ is adjacent to $M_j$. Once we fix $M_i$, the matroid $(M, f) \oplus_2 (M_j, e)$ is independent (up to isomorphism) of the choice of $f \in F(M_i)$ by Lemma 3.3.

We can conclude that the structure of the tree decomposition and the vertex labels are enough to determine uniquely the 2-level matroid. Vice versa Theorem 3.1 together with Theorem 2.4 proves that a 2-connected 2-level matroid uniquely identifies a tree structure with vertex labels chosen among the uniform matroids.

We close the section with a proposition from Ox11 Prop. 7.1.22 which is needed to deal with self-duality in 4.4.

**Proposition 3.5.** Let $M_1$ and $M_2$ be two matroids and $e_i \in E(M_i)$. Then

$$(M_1, e_1) \oplus_2 (M_2, e_2)^* = (M_1^*, e_1) \oplus_2 (M_2^*, e_2).$$

4. Counting UMR-trees

In this section we apply the results in Section 3 to get enumerative formulas for the number of 2-level matroids of fixed size. By means of Theorem 3.2 this is equivalent to a tree enumeration problem, where the labels of vertices are uniform matroids. For technical reasons we distinguish the vertices associated to matroids of type $R_n$ and $M_n$ from the others. From now on we refer to this family of trees as UMR-trees.

More precisely, each UMR-tree has different types of vertices: $R$-vertices, $M$-vertices (respectively associated to uniform matroids $U_{n,n-1}$ and $U_{n,1}$), and $U$-vertices (associated to uniform matroids of type $U_{n,k}$, $2 \leq k \leq n-2$). The degree of a vertex is defined in the traditional way as the number of edges connected to the vertex. We also use an additional type of vertices that we call legs. Legs always have degree 1, and are graphically represented by small black disks. For each free element of a vertex label we draw a leg in the UMR-tree. Observe that legs are always leaves of the tree. Hence, enumerative formulas in terms of the number of legs in our tree model translates directly into counting results in the matroid setting. The combinatorial restrictions we consider in our trees (which naturally arise from the obstructions inherited from the matroid setting) are the following:

1. Trees are unlabelled and not embedded in the plane.
2. Two $R$-vertices (and also two $M$-vertices) are not adjacent.
3. The degree of an $R$-vertex or $M$-vertex is greater or equal than 3, and the degree of an $U$-vertex is greater or equal than 4.

In principle, our goal is to get enumerative formulas for UMR-trees, but due to the Dissymmetry Theorem for trees (Section 2) we need to start encoding rooted families. For this reason we introduce the following technical definition: a UMR-tree is said to be rooted if it has a special leaf of size 0 (namely, it does not contribute to the total amount of legs) that we call virtual leg. Roughly speaking, the virtual leg indicates which is the root of the UMR-tree (namely, the vertex incident with the virtual leg). We use a white triangle to graphically represent the virtual leg. See Figure 3 for an example of a rooted UMR-tree.

If a vertex is incident with the virtual leg, its restricted degree is the total degree minus 1. Notice that $U$-vertices have multiplicity due to the rank of the associated matroid. In other words, once the total degree of a $U$-vertex is fixed (call it $d$), then the possible rank could take a value in $\{2, 3, 4, \ldots, d - 2\}$. This defines $d - 3$ possible different uniform matroids for this vertex.
We start getting relations between these generating functions by means of the root decomposition. Let us start with

to a

We write

Getting formulas for

A similar argument holds changing the root vertex

Auxiliary variable

Easy induction argument we can conclude that for each

A

vertex, respectively. Additionally, we write

functions for rooted trees where the virtual leg is incident with an

A

operator:

Remark: if the root vertex is an

virtual leg followed by a multiset of size greater or equal than 2 of trees rooted

at either an

M

-vertex or an

U

-vertex. This combinatorial description gives us that



Observe that Equations (4) and (5) give that

\( A_R(x) = A_M(x) \). Indeed, from (4) and (5) we obtain that

\[ \sum_{r \geq 1} \frac{1}{r} A_R(x^r) = \sum_{r \geq 1} \frac{1}{r} A_M(x^r). \]

These two formal power series have the same coefficients.

In particular, for each choice of \( n \),

\[ [x^n] \sum_{r \geq 1} \frac{1}{r} A_R(x^r) = [x^n] \sum_{r \leq n} \frac{1}{r} A_R(x^r). \]

Now, applying an easy induction argument we can conclude that for each \( n \),

\[ [x^n] A_R(x) = [x^n] A_M(x). \]

Getting formulas for \( A_U(x) \) is slightly more involved: if the root \( U \)-vertex has total degree \( d \), then it has multiplicity \( d - 3 \). This fact must be encoded in the counting formulas. Let us use an auxiliary variable \( u \) which marks the restricted degree of the rooted \( U \)-vertex (namely, the total degree \( d \) minus 1). Here we emphasize that we do not consider the contribution of the virtual leg to the total number of legs \( n \). This is due to technical reasons that are going to be clear while proceeding with the counting. However, the multiplicity of the \( U \)-vertex must be considered with respect to the total degree of the vertex (thus including the virtual leg) and not with respect to the restricted degree. Indeed, for a \( U \)-vertex rooted tree whose root vertex is of degree \( d \), its restricted degree is equal to \( r = d - 1 \), and its multiplicity is equal to \( d - 3 = r - 2 \).

We write \( a_{n,r} \) for the number of rooted trees with \( n \) non-virtual legs whose virtual leg is attached to a \( U \)-vertex of restricted degree \( r \). The notation \( a_U(x,u) = \sum_{n,r \geq 3} a_{n,r} x^n u^r \) refers to the
corresponding generating function. Then we have

\[ A_U(x) = \sum_{n,r \geq 3} (r - 2)a_{n,r}x^n u^r \bigg|_{u = 1} = \frac{\partial}{\partial u} a_U(x, 1) - 2a_U(x, 1) \]  \hspace{1cm} (6)

Observe now that \( a_U(x, u) \) satisfies the equation \( a_U(x, u) = \text{Mul}_2(u(A_M(x) + A_R(x) + A_U(x) + A_I(x))) \), which arises from the fact that the root \( U \)-vertex has restricted degree \( \geq 3 \) (or equivalently, degree \( \geq 4 \)). Hence we have that

\[ a_U(x, u) = \exp \left( \sum_{r=1}^{\infty} u^r \left( A_R(x^r) + A_M(x^r) + A_U(x^r) + A_I(x^r) \right) \right) - 1 - u \left( A_R(x) + A_M(x) + A_U(x) + A_I(x) \right) - \text{Mul}_2(u(A_M(x) + A_R(x) + A_U(x) + A_I(x))). \]

Now by using Equation (6) we can write \( A_U(x) \) in terms of \( a_U(x, 1) \) and its derivative at \( u = 1 \):

\[ A_U(x) = \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} \left( A_R(x^r) + A_M(x^r) + A_U(x^r) + x^r \right) \right) \left( \sum_{r=1}^{\infty} \left( A_R(x^r) + A_M(x^r) + A_U(x^r) + x^r \right) \right) \]

\[ - (A_R(x) + A_M(x) + A_U(x) + x) - 2 \text{Mul}_2(A_M(x) + A_R(x) + A_U(x) + x) \]

\[ - 2 \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} \left( A_R(x^r) + A_M(x^r) + A_U(x^r) + x^r \right) \right) + \]

\[ + 2 + 2 \left( A_R(x) + A_M(x) + A_U(x) + x \right) + 2 \text{Mul}_2(A_M(x) + A_R(x) + A_U(x) + x). \]

Hence, we have three equations relating \( A_R(x), A_M(x) \) and \( A_U(x) \).

4.2. Application of the Dissymmetry Theorem. We now proceed by applying the Dissymmetry Theorem for trees (see Subsection 2.2) in order to express UMR-trees in terms of rooted ones. Let \( T(x) \) be the generating function of UMR-trees, where \( x \) marks legs. Write \( T_v(x), T_e(x), \) and \( T_d(x) \) the generating functions associated to families of UMR-trees with a pointed vertex, a pointed edge and a pointed (and oriented) edge, respectively. By the Dissymmetry Theorem for trees stated in Equation (1), we have that

\[ T(x) = T_v(x) + T_e(x) - T_d(x). \]  \hspace{1cm} (8)

Let us compute each generating function in terms of the rooted families obtained in Subsection 4.1. Let us start with \( T_v(x) \). This can be written as:

\[ T_v(x) = T_{R-M}(x) + T_{R-U}(x) + T_{R-\cdot}(x) + T_{M-U}(x) + T_{M-\cdot}(x) + T_{U-U}(x) + T_{U-\cdot}(x) \]

where the index of each term shows the type of the end vertices of the pointed edge (for instance, the first term \( R-M \) means that the pointed edge has as end vertices an \( R \)-vertex and an \( M \)-vertex). By cutting the pointed edge and pasting two virtual legs on the ends (see Figure 6), each term in the sum (with the exception of \( T_{U-U}(x) \), which has an additional symmetry) is the product of the corresponding generating functions of rooted families. The single situation where

![Figure 6](image-url)
and, mutatis mutandis, an analogous expression holds for the pointed vertex is not encoded yet and let $t\in T$. Incident with the pointed particular observe that because a pointed leg induces canonically a pointed edge. Let us consider the other situations: generating functions obtained in Subsection 4.1, because now we do not have to consider the symmetry that appeared in the previous family when dealing with an edge linking two $U$-vertices: $T\in T$. The last generating function we want to get is the one of Equation (9) applies for $T_d(x)$. Indeed this generating function can be written as:

$$T_d(x) = T_{R\rightarrow M}(x) + T_{R\rightarrow U}(x) + T_{M\rightarrow}(x)$$

$$+ T_{M\rightarrow R}(x) + T_{M\rightarrow U}(x) + T_{M\rightarrow \bullet}(x)$$

$$+ T_{U\rightarrow M}(x) + T_{U\rightarrow R}(x) + T_{U\rightarrow \bullet}(x) + T_{U\rightarrow U}(x)$$

where the index of each term shows the type of the end vertices for the pointed (directed) edge. In this situation all the computations are similar and easier, because we do not have the extra symmetry that appeared in the previous family when dealing with an edge linking two $U$-vertices:

$$T_d(x) = A_R(x)(A_M(x) + A_U(x) + A_l(x))$$

$$+ A_M(x)(A_R(x) + A_U(x) + A_l(x))$$

$$+ A_U(x)(A_M(x) + A_R(x) + A_U(x) + A_l(x))$$

$$+ A_l(x)(A_R(x) + A_M(x) + A_U(x)).$$

The last generating function we want to get is $T_v(x)$. Observe that $T_v(x)$ is not the sum the generating functions obtained in Subsection 11 because now we do not have to consider the virtual leg. We write

$$T_v(x) = T_{R}(x) + T_{M}(x) + T_{U}(x) + T_{\bullet}(x)$$

where the index of each term indicates the type of the pointed vertex. We want to express now each term by means of the previous rooted families. It is obvious that

$$T_{\bullet}(x) = A_l(x)(A_R(x) + A_M(x) + A_U(x))$$

because a pointed leg induces canonically a pointed edge. Let us consider the other situations: observe that $T_R(x) = \text{Mul}_{\geq 3}(A_M(x) + A_U(x) + A_l(x))$, which is obtained by cutting the edges incident with the pointed $R$-vertex, and pasting a virtual legs on the resulting subtrees. In particular

$$T_R(x) = \text{Mul}_{\geq 3}(A_M(x) + A_U(x) + A_l(x))$$

$$= \text{Mul}_{\geq 2}(A_M(x) + A_U(x) + A_l(x)) - \text{Mul}_2(A_M(x) + A_U(x) + A_l(x))$$

$$= A_R(x) - \text{Mul}_2(A_M(x) + A_U(x) + A_l(x))$$

and, mutatis mutandis, an analogous expression holds for $T_M(x)$. At last, let us study $T_L(x)$: let $t_U(x, u)$ be the generating function of trees with a pointed $U$-vertex, where the multiplicity of the pointed vertex is not encoded yet and $u$ encodes the degree of the pointed $U$-vertex. Then, $t_U(x, u) = \text{Mul}_{\geq 4}(u(A_R(x) + A_M(x) + A_U(x) + A_l(x)))$ and

$$t_U(x, u) = \sum_{n, d \geq 4} t_{n, d} x^n u^d \rightarrow T_U(x) = \sum_{n, d \geq 4} (d - 3)t_{n, d} x^n u^d \bigg|_{u=1}$$

$$= \frac{\partial}{\partial u} t_U(x, 1) - 3t_U(x, 1).$$
Applying the same trick we used for $a_U(x, u)$ in Subsection 3.1 we get that
\[
T_U(x) = \frac{\partial}{\partial u} t_U(x, 1) - 3 t_U(x, 1) = \left( \frac{\partial}{\partial u} - 3 \right) (a_U(x, 1) - u^3 \text{Mul}_3(A_R(x) + A_M(x) + A_U(x) + A_l(x))) \\
= A_U(x) - a_U(x, 1) + (3 - 3) \text{Mul}_3(A_R(x) + A_M(x) + A_U(x) + A_l(x))) \\
= A_U(x) - A(M(x) + A_U(x) + A_l(x)).
\]
Substituting Equations (12), (13) and (14) in (11) we get the expression for $T_u(x)$. Finally, we add (9), (10) and this expression of $T_u(x)$ in Equation (8). All together brings us the generating function $T(x)$, whose first coefficients are $2x^3 + 4x^4 + 10x^5 + 27x^6 + 78x^7 + 246x^8 + 818x^9 + 2871x^{10} + 1046x^{11} + 39358x^{12} + ...$

4.3. Asymptotic analysis. Now we can apply the machinery arising from analytic combinatorics in order to get asymptotic estimates for $[x^n]T(x)$. The main point is based on studying the system of equations which defines $A_R(x)$, $A_M(x)$ and $A_U(x)$, which provides the position and the nature of the dominant singularity of $T(x)$.

In particular, by means of the Drmota-Lalley-Woods methodology (see Subsection 2.2), we obtain the constant growth, which is $\rho^{-1} \approx 4.88052854$ (whose inverse $\rho \approx 0.20489584$ gives the radius of convergence around the origin of the generating function). Possibly more important, we can show that all these generating functions have the same square-root singularity (see the details in the proof). Moreover, the generating function $T(x)$ is an analytic expression of the previous counting formulas, hence the position of the singularity does not change. However, the type of singularity changes due to a combinatorial cancellation arising from the Dissymmetry Theorem for trees applied over UMR-trees. Finally, the asymptotic estimates for the coefficients of $T(x)$ are deduced by means of the Transfer Theorem for singularity analysis (see Theorem 2.6).

Before presenting the proofs, it is worth comparing the growth constant we get with the one for trees applied over UMR-trees. Finally, the asymptotic estimates for the coefficients of $T(x)$ are deduced by means of the Transfer Theorem for singularity analysis (see Theorem 2.6).

\[ X = \sqrt{1 - x/\rho}, \quad A_0 \approx 0.13529174, \quad A_1 \approx -0.23137622, \quad A_2 \approx 0.04653888, \quad A_3 \approx 0.06281332, \quad A_4 \approx 0.06921673, \quad U_0 \approx -0.19340420, \quad U_2 \approx 0.15045323 \quad \text{and} \quad U_3 \approx 0.01018058. \]

**Proof.** As we know that $A_R(x) = A_M(x)$, we just need to analyze the pair of equations (4) and (7). Observe that this pair of equations satisfies the conditions of Theorem 2.6. Assuming that $A_R(x)$ and $A_U(x)$ have a unique singularity $\rho$, then the term
\[
\exp \left( \sum_{x=2}^{\infty} \frac{1}{x} (A_M(x^r) + A_U(x^r) + A_l(x^r)) \right)
\]
in equation (4) is analytic at \( x = \rho \) (similarly in Equation (7)). Hence, we can approximate this term by its Taylor series (which can be computed by an iterative algorithm). Solving now the resulting system of 3 equations by means of Maple computations (namely, the initial pair of equations and the one associated to the jacobian matrix in Equation (2)), gives the solution \( x_0 \approx 0.20489584, A_M(x_0) \approx 0.13529174 \) and \( A_U(x_0) \approx 0.06921673 \). By Theorem 2.6 the position of the singularity of both \( A_M(x) \) and \( A_U(x) \) is located at \( \rho = x_0 \approx 0.20489584 \), and the singular expansion of \( A_M(x) \) and \( A_U(x) \) in a domain denoted at \( \rho \) is of the form

\[
A_R(x) = A_M(x) = A_0 + A_1 X + A_2 X^2 + A_3 X^3 + O(X^4),
\]

\[
A_U(x) = U_0 + U_1 X + U_2 X^2 + U_3 X^3 + O(X^4)
\]

where \( A_i, U_i, i \in \{1, 2, 3, 4\} \) are computable constants. In order to get approximate values of these constants, we substitute the singular expansions of \( A_M(x) \) and \( A_U(x) \) in equations (4) and (7). The terms of the form \( A_M(x^r) \) and \( A_U(x^r) \) \((r \geq 2)\) are also approximated by a truncation of the Taylor series (which can also be computed by an iterative algorithm), because these GFs are analytic at the point \( x = \rho \). At this point we can get a system of equations in the \( A_i \)'s and the \( U_i \)'s by equating the coefficients with same degree of the singular expansions. Solving this system gives the constants claimed in the statement of the theorem. \( \square \)

More precisely, we get the following result for \([x^n]T(x)\):

**Theorem 4.2.** The following asymptotic estimate holds:

\[
[x^n]T(x) = C \cdot n^{-5/2} \cdot \rho^{-n} (1 + o(1))
\]

where \( C \approx 0.07583455 \) and \( \rho \approx 0.20489584 \) are computable constants.

**Proof.** We use the singular expansions for \( A_R(x), A_M(x) \) and \( A_U(x) \) obtained in Lemma 4.1 together with the expressions in equations (5)-(7) in order to get the singular expansion of \( T(x) \):

\[
T(x) = T_0 + T_2 X^2 + T_3 X^3 + O(X^4),
\]

with \( T_0 \approx 0.03457946, T_2 \approx -0.18596384 \) and \( T_3 \approx 0.17921766 \). Observe that the constant multiplying \( X \) in this singular expansion is equal to 0 (due to the unpointing process in the Dissymmetry Theorem for trees). Finally we apply the Transfer Theorem for singularity analysis over this singular expansion. \( \square \)

### 4.4. Dealing with duality. Proof of Theorem 1.1.

The last part is devoted to show that the contribution of self-dual 2-level matroids is exponentially small compared with the estimates we obtained in the previous subsection. Let \( \mathcal{M} \) be a matroid with tree decomposition \( T \), then Proposition 5.2 implies that the tree decomposition of \( \mathcal{M}^* \) has the same tree structure of \( T \). Moreover we replace each vertex label \( \mathcal{M} \) with its dual matroid \( \mathcal{M}^* \).

We are interested in self-dual 2-connected 2-level matroids. The vertex labels are chosen among uniform matroids, and the operation of duality turns labels of type \( R_n \) into labels of type \( R_n \) and vice versa, and \( U_{n,k} \)-labels into \( U_{n,n-k} \)-labels. It is clear that the self-dual labels are of the form \( U_{2n,n} \). Moreover, for technical reasons, we consider also virtual legs and legs to be self-dual.

Our goal is to estimate the contribution of the family of self-dual UMR-trees (namely UMR-trees associated to self-dual matroids) to the total number of UMR-trees. To do that we start analyzing the rooted situation: we write \( A_R(x) = S_R(x) + N_R(x), A_M(x) = S_M(x) + N_M(x) \) and \( A_U(x) = S_U(x) + N_U(x) \), where the generating functions \( S_R(x), S_M(x) \) and \( S_U(x) \) encode self-dual trees whose root vertex is a \( R \)-vertex, a \( M \)-vertex and a \( U \)-vertex, respectively. The generating functions \( N_R(x), N_M(x) \) and \( N_U(x) \) are the ones encoding trees which are not self-dual. Observe that in particular \( S_R(x) = S_M(x) = 0 \), because the dual of each \( R \)-vertex is an \( M \)-vertex, and consequently there are no self-dual trees rooted at either an \( R \)-vertex or an \( M \)-vertex.

We also use a similar notation for unrooted trees. We write \( T(x) = S(x) + N(x) \), where \( S(x) \) is the generating function associated to self-dual (unrooted) trees.

The next lemma tells us that the contribution of self-dual rooted trees is exponentially small:
Lemma 4.3. The following estimate holds:

\[ [x^n] S_U(x) = o([x^n] A_U(x)). \]

Proof. We get and analyze equations for \( S_U(x) \). In this situation, the root vertex is a \( U \)-vertex associated to a uniform matroid of the form \( U_{2,n,n} \). Hence, we notice that the degree of the root vertex determines the rank and, in particular, the multiplicity in the counting is 1. Moreover, he possible restricted degree of the vertex are clearly in the set \( \Lambda = \{3, 5, 7, \ldots \} \).

Now observe that the collection of pending rooted subtrees is a multiset of pairs of rooted trees such that one is the dual of the second, followed by a multiset of odd size of self-dual rooted trees. Hence,

\[
S_U(x) = \text{Mul}_{\{3,5,7,\ldots\}}(S_U(x) + A_U(x)) + \text{Mul}_{\geq 1}(A_R(x^2) + A_M(x^2) + (A_U(x^2) - S_U(x^2))) \text{Mul}_{\{3,5,7,\ldots\}}(S_U(x) + A_U(x)).
\]

Let \( \eta \) be the radius of convergence of \( S_U(x) \). It is obvious that \( \eta \geq \rho \), because the family of self-dual rooted trees are counted in the family of \( U \)-rooted trees. We just need to show that \( \eta \neq \rho \).

Equation (15) can be analyzed in a similar way to the one we find in the proof of Proposition 4.1 However, for our purposes it is enough to bound the coefficients of \( S_U(x) \) by means of crude estimates. Observe that \( \text{Mul}_{\geq 1}(S_U(x) + A_U(x)) \geq \text{Mul}_{\{3,5,7,\ldots\}}(S_U(x) + A_U(x)) \) and

\[
\text{Mul}_{\geq 1}(A_R(x^2) + A_M(x^2) + (A_U(x^2) - S_U(x^2))) \text{Mul}_{\geq 1}(S_U(x) + A_U(x)) \geq \text{Mul}_{\geq 1}(A_U(x^2) + A_M(x^2) + (A_U(x^2) - S_U(x^2))) \text{Mul}_{\{3,5,7,\ldots\}}(S_U(x) + A_U(x)).
\]

Hence, if \( s(x) \) satisfies the equation

\[
s(x) = \text{Mul}_{\geq 1}(s(x) + A_U(x)) + \text{Mul}_{\geq 1}(A_R(x^2) + A_M(x^2) + (A_U(x^2) - s(x^2))) \text{Mul}_{\geq 1}(s(x) + A_U(x))
\]

then \( S_U(x) \leq s(x) \). Observe also that by the combinatorial specification of UMR-trees \( s(x) \leq A_U(x) \). Let \( \gamma \) be the smallest real singularity of \( s(x) \). Observe that this singularity arises either from the square root singularity of the term \( A_R(x^2) + A_M(x^2) + A_U(x^2) \) at \( x \) equals to \( \sqrt{\rho} \approx 0.45265421 \) or from a branch point (smaller than \( \sqrt{\rho} \)) of Equation (15).

We show that the singularity is located at \( \sqrt{\rho} \). Assume the contrary: the smallest singularity \( \gamma \) of \( s(x) \) arises from a branch point of equation (15), such that \( A_R(x^2) + A_M(x^2) + A_U(x^2) \) is analytic at \( \gamma \). Equation (15) can we written as \( s(x) = (\exp(s(x))f(x) - 1 - s(x) - \frac{1}{2}s(x)^2 + g(x)) + h(x)(\exp(s(x))i(x) - 1) \), where the functions \( f(x), g(x), h(x) \) and \( i(x) \) are analytic at \( x = \gamma \). In particular, we can approximate \( A_R(x^2) + A_M(x^2) + A_U(x^2) \) by its Taylor series.

Hence, we have an equation of the form \( s = F(s,x) \). Its hypothetic branch point arises as a coalescence of the solutions of the pair of equations \( s = F(s,x), 1 = F_x(s,x) \) and \( x \leq \sqrt{\rho} \) which do not have solution (these computation has been done with Maple by taking 30 coefficients in the Taylor series of \( A_R(x^2) + A_M(x^2) + A_U(x^2) \)). In particular, this reads that there is no branching point \( \gamma \) strictly smaller than \( \sqrt{\rho} \), and consequently the singularity of \( s(x) \) arises from the singularity of the term \( A_R(x^2) + A_M(x^2) + A_U(x^2) \). To conclude, \( [x^n] s(x) \) has exponential growth of order \( \rho^{-n/2} \), which is exponentially small compared with \( \rho^{-n} \).

Once we know that the number of rooted self-dual trees is exponentially small compared with the total number of rooted trees, we can prove that the number of self-dual unrooted trees is also exponentially small compared with the total number of unrooted trees.

Proposition 4.4. The following estimate hold:

\[ [x^n] S(x) = o([x^n] T(x)). \]

Proof. We obtain a generating function \( D(x) \) such that \( S(x) \leq D(x) \) and that \( [x^n] D(x) = o([x^n] T(x)) \). We split the class of self-dual trees by looking at the type of the center for each self-dual tree. The center of a graph is the set of all vertices such that the greatest distance from
all the vertices is minimal. The center of a tree can be one single vertex or two adjacent vertices (we say it is an edge).

Write \( S(x) = S_o(x) + S_{o-o}(x) \), where \( S_o(x) \) and \( S_{o-o}(x) \) are the generating functions associated to self-dual trees whose center is a vertex and an edge, respectively. We analyze each case separately. We start with self-dual trees whose center is a vertex. In this situation the center is necessarily a \( U \)-vertex labelled by a matroid of type \( U_{2n,n} \). In this case the degree of the rooted vertex determines the rank of the \( U \)-vertex, which has to be counted with multiplicity one. Hence, we have the crude bound \( S_o(x) \leq \text{Mul}(A_R(x^2) + A_M(x^2) + \mathcal{N}_U(x^2)) \text{Mul}(S_U(x) + A_1(x)) \), whose radius of convergence by Lemma 4.3 is strictly bigger than \( \rho \).

Let us study now self-dual trees whose center is an edge. Consider the pair of rooted trees that arise when cutting the edge which plays the role of the center of the tree (and pasting a virtual leg). Two situations may happen:

1. Each tree is self-dual.
2. Each tree is non-self-dual, but one is the dual of the other.

In both cases (1) and (2) we can easily find a bound. Namely, \( S_U(x^2) \) and \( A_R(x^2) + A_M(x^2) + \mathcal{N}_U(x^2) \), respectively. Therefore \( S_{o-o}(x) \leq S_U(x^2) + A_R(x^2) + A_M(x^2) + \mathcal{N}_U(x^2) \). Finally, again by Lemma 4.3, the radius of convergence of \( S_{o-o}(x) \) is strictly bigger than \( \rho \). Hence the result follows. \( \square \)

We can now present the proof that there are exponentially many 2-level polytopes coming from matroid base polytopes:

**Proof of Theorem 1.1** Every 2-connected 2-level matroid \( \mathcal{M} \) on \( n \) elements is, by definition, associated with a 2-level base polytope \( P_{\mathcal{M}} \). The 2-connectedness implies that the dimension of the base polytope is \( n-1 \). By Theorem 2.3 there is only another matroid with the same polytope, namely \( \mathcal{M}^* \).

Denote by \( L_2(n) \) the number of 2-connected 2-level matroids and by \( S_2(n) \) the number of self-dual ones. The number of non-congruent \((n-1)\)-dimensional 2-level polytopes associated with such family is \( \frac{L_2(n) + S_2(n)}{2} \). This yields a lower bound to the number of \((n-1)\)-dimensional 2-level polytopes.

Applying the structural result of Section 3 and using the notation of Subsection 4.2 we easily see that \( L_2(n) = [x^n]T(x) \) and \( S_2(n) = [x^n]S(x) \). We do not have closed formulas for the coefficients of the generating functions, but nevertheless we are able to provide asymptotic estimates: by Theorem 4.2 the number of UMR-trees is asymptotically equal to \( C \cdot n^{-5/2} \cdot \rho^{-n} (1 + o(1)) \), where \( C \approx 0.07583455 \) and \( \rho \approx 0.20489584 \) are computable constants. Due to Proposition 4.3 the contribution of self-dual UMR-trees to this asymptotic is exponentially small. Hence, the number of self-dual UMR-trees up to the duality relation is half of this value plus the number of self-dual UMR-trees. So, Theorem 1.1 holds by dividing the previous bound by 2. \( \square \)

To conclude, observe that we can use the singular expansion of \( T(x) \) in order to get asymptotic estimates for the number of 2-level matroids, including the non-connected ones. This family corresponds with the multiset construction applied over UMR-trees (namely, forests). Hence, the generating function here is \( \text{Mul}(T(x)) = \exp \left( \sum_{r=1}^{\infty} \frac{1}{r} (T(x^r)) \right) \). Observe that

\[
\exp \left( \sum_{r=1}^{\infty} \frac{1}{r} (T(x^r)) \right) = \exp(T(x)) \exp \left( \sum_{r=2}^{\infty} \frac{1}{r} (T(x^r)) \right),
\]

and the second term is analytic at \( x = \rho \). Hence, in a domain dented at \( x = \rho \) the singular expansion of \( \text{Mul}(T(x)) \) is equal to:

\[
\text{Mul}(T(x)) = \exp(T_0 + T_2 X^2 + T_3 X^3 + O(X^4)) \exp \left( \sum_{r=2}^{\infty} \frac{1}{r} (T(\rho^r)) \right),
\]

(see the singular expansion of \( T(x) \) in the proof of Theorem 4.2) which has the expression

\[
\text{Mul}(T(x)) = F_0 + F_2 X^2 + F_3 X^3 + O(X^4),
\]
with $F_0 \approx 1.03526853$, $F_2 \approx -0.19252251$, $F_3 \approx 0.18553841$. Applying now Theorem 2.5 we conclude that

$$[x^n] \text{Mul}(T(x)) = C' \cdot n^{-5/2} \cdot \rho^{-n}(1 + o(1)), $$

with $C' \approx 0.07850913$. Observe that the constant $C'$ is slightly bigger than the constant obtained in the asymptotic estimate for UMR-trees.

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