On the theory of kinetic equations for interacting particle systems with long range interactions

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Abstract

In this paper we review the formal derivation of different classes of kinetic equations for long range potentials. We consider suitable scaling limits for Lorentz and Rayleigh gases as well as interacting particle systems whose dynamics can be approximated by means of kinetic equations. The resulting kinetic equations are the Landau and the Balescu-Lenard equations. In the derivation of the kinetic equations particular emphasis is made in the fact that all the kinetic regimes can be obtained approximating the dynamics of the interacting particle systems by the evolution of a single particle in a random force field with a friction term which is due to the interaction with the surrounding particles. The case of particles interacting by means of Coulombian potentials as well as the cutoffs which yield the so-called Coulombian logarithm are discussed in detail.

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1 Introduction

It is well known that a large class of many particle systems which evolve by means of Newton equations can be described, under suitable assumptions on the potentials describing the particle interactions, by means of a function $f(x, v, t)$ which yields the density of particles in the phase space $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$. The evolution of the function $f(x, v, t)$, which is usually termed as one-particle distribution function, in these cases is given by a kinetic equation. Some examples of kinetic equations are the Boltzmann equation, the Landau equation and the Balescu-Lenard equation.

The specific kinetic equation which describes a given scaling limit of interacting particles systems depends on the properties of the potentials which describe the interaction between the particles as well as on the scalings assumed for the magnitudes describing the interaction potential, like the strength and range of the potential as well as other properties of the potential which will be described in detail in this paper.

One of the main goals of kinetic theory is to describe a system of particles whose evolution is given by the equations:

$$
\frac{dX_j}{d\tau} = V_j, \quad \frac{dV_j}{d\tau} = -\sum_k \nabla \Phi_\varepsilon (X_j - X_k), \quad j \in S
$$

(1.1)

where $S$ is a (countable) set of indexes. The potentials $\Phi_\varepsilon (X) = \Phi (X; \varepsilon)$ depend on a parameter $\varepsilon$ which eventually will be sent to zero. The problem is to characterize the families of potentials $\Phi_\varepsilon$ for which it is possible to describe the evolution of the system by means of an equation for the one-particle distribution function $f(x, v, t)$. A precise definition of this function will be given later. We will assume in all the following that the velocities of the particles are of order one. This can always be assumed by means of a suitable change in the unit of time. Notice that the microscopic variables are denoted as $X, V, \tau$ while the macroscopic variables, which will be defined in detail later, will be denoted as $x, v, t$. We assume that for a typical particle, $|V_k|$ and $|v_k|$ are of order one and also that the typical microscopic distance between two particles is of order one, i.e. $|X_j - X_k| \sim 1$.

We now describe some families of problems which are simpler than (1.1), but that will allow to approximate the dynamics of (1.1) in a suitable asymptotic limit. The common feature of these models is that they describe the dynamics of a tagged particle that moves in a random medium. More precisely, we are interested in the evolution of a particle characterized by its position and velocity $(X(\tau), V(\tau))$ whose dynamics is given by the system of equations:

$$
\frac{dX}{d\tau} = V, \quad \frac{dV}{d\tau} = -\Lambda_\varepsilon (V) + F_\varepsilon (X, \tau; \omega)
$$

(1.2)

where $F_\varepsilon$ is a random force field defined for $\omega$ in a suitable probability space $\Omega$ and $\Lambda_\varepsilon (V)$ is a function of the velocity which can be thought of as a friction term which depends only on the particle velocity $V$. Notice that we will not assume that $\Lambda_\varepsilon (V)$ and $V$ are necessarily parallel.

We will assume that the initial positions and velocities $\{(X_j, V_j) : j \in S\}$ in (1.1) are chosen according to some probability distribution which is spatially homogeneous and with a distribution of velocities $g = g(v)$. In order to make precise the connection between (1.1) and (1.2) we must choose the random force field $F_\varepsilon$ and the friction term $\Lambda_\varepsilon (V)$ as functionals of $g$, i.e.:

$$
\Lambda_\varepsilon (V) = \Lambda_\varepsilon (V; g), \quad F_\varepsilon (X, \tau; \omega) = F_\varepsilon (X, \tau; \omega; g).
$$

(1.3)
The precise form of this functionals will be described in Section 5.2. In most of the paper we will restrict ourselves to the study of spatially homogeneous particle distributions. Therefore, we will assume that the random field $F_{\varepsilon}(X, \tau; \omega)$ is invariant under space translations. Some examples of non-homogeneous particle distributions will be discussed in Section 8.3. In this case, in order to obtain consistent kinetic limits we need to assume that the length scale of the inhomogeneities is comparable to the mean free path of the system.

A second type of dynamical systems which we will consider in this paper and that might be used to approximate the dynamics of the system (1.1) are the so-called Rayleigh gases. These systems give the evolution of a tagged particle in the force field generated by a countable set $S$ of infinitely many particles (scatterers), each of them is the center of a potential field. Moreover, it is assumed that the particles in the tagged particle and the scatterers interact by means of the usual Newton’s laws, but that the scatterers do not interact among themselves. Suppose that we denote the position and velocity of the tagged particle as $(X, V)$ and the positions and velocities of the scatterers as $\{(Y_k, W_k)\}_{k \in S}$. We will take a system of units in which the mass of the tagged particle is 1 and we will restrict for the moment to the case in which the mass of all the scatterers and the tagged particle is the same. Then, the set of equations describing the dynamics of a Rayleigh gas is:

$$\frac{dX}{d\tau} = V, \quad \frac{dV}{d\tau} = -\sum_{j \in S} \nabla \phi_{\varepsilon}(X - Y_j)$$

$$\frac{dY_k}{d\tau} = W_k, \quad \frac{dW_k}{d\tau} = -\nabla \phi_{\varepsilon}(Y_k - X), \quad k \in S$$

(1.4)

where we assume that $\phi_{\varepsilon}$ is the interaction potential which gives the interaction between the tagged particle and each of the scatterers.

Note that from the physical point of view, this Rayleigh gas system is a good approximation for the dynamics of a tracer particle moving in a background of particles for which their mean free path is much larger than the mean free path of the tracer particles. Rayleigh gases have been extensively studied (cf. [6], [48], [49] and [10]). We remark that we call Rayleigh gas the system that is denoted as ideal Rayleigh gas in [49] (see also [39]).

We can characterize a Rayleigh gas by means of a random point process in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$. A class of measures which are extensively used in kinetic theory are the generalized Poisson measures. This measures are uniquely characterized by a (typically nonfinite) measure $g \in \mathcal{M}_+(\mathbb{R}^3 \times \mathbb{R}^3)$ defined in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$. Then, the corresponding probability measure is determined uniquely assuming that the particles are independently distributed and that the average number of particles in a Borel set $A \subset \mathbb{R}^3 \times \mathbb{R}^3$ is given by:

$$\int_A g \,(dX, dV).$$

(1.5)

It will be convenient to consider measures $g$ yielding bounded particles densities in the space of particle positions. These measures satisfy:

$$\rho \in L^\infty (\mathbb{R}^3),$$

(1.6a)

where $\rho$ is defined by

$$\rho(X) = \int_{\mathbb{R}^3} g \,(X, dV).$$

(1.7)
We can define a random evolution for the tagged particle \((X, V)\) assuming that the initial distribution of scatterers in the phase space is given by a probability distribution. We will restrict ourselves to the case in which the initial distribution of scatterers \(\{(V_k(0), W_k(0))\}_{k \in S}\) is determined using a generalized Poisson probability distribution associated to a measure \(g\) in the phase space.

Notice that we can assume that the scatterers generate a random force field \(F_\varepsilon(X, \tau) = -\sum_{j \in S} \nabla \phi_\varepsilon(X - Y_j)\). However, the problem \((1.4)\) is different from \((1.2)\) because the force field \(F_\varepsilon(X, \tau)\) is modified by the tagged particle \((X, V)\), while in the dynamics \((1.2)\) the time-dependent force field \(F_\varepsilon(X, \tau; \omega)\) is unaffected by the tagged particle. Nevertheless, it turns out that in some rescaling limits which will be made precise in Subsection 5.2, it is possible to approximate the dynamics given by \((1.4)\) using the much simpler dynamics \((1.2)\).

In this paper we are mostly concerned with the kinetic limits associated to the many particle system \((1.1)\) for long range potentials. Precise definitions will be in Section 5, but we can say here by definiteness that these are weak interaction potentials with a range much larger than the typical distance between particles. In this situation, under suitable assumptions, there exists a macroscopic time scale \(T_\varepsilon\) which is much larger than time required for a particle to travel the typical distance between particles. The time \(T_\varepsilon\) is the typical time in which the velocity of one particle changes by an amount of order one. We can define a macroscopic time variable \(t = \frac{\tau}{T_\varepsilon}\). It turns out that in some scaling limits (see Section 5) we can approximate for small but macroscopic times, the dynamics \((1.1)\) by means of at least one of the dynamics \((1.2), (1.4)\) depending on the form of the interaction potentials.

More precisely, in order to describe in which sense the solutions of the system \((1.1)\) can be approximated by those of the systems \((1.2)\) or \((1.4)\), we need to introduce the concept of one-particle distribution, which we will define in the general situation in which the particles are not distributed in a homogeneous manner in space. We will assume that the particle configurations are chosen according to a generalized Poisson distribution \(\hat{P}_0\) which is completely determined by a measure \(f_0 \in \mathcal{M}_+(\mathbb{R}^3 \times \mathbb{R}^3)\) (cf. \((1.5)\)), which is usually referred to as intensity measure. We will assume also that the density \(\rho_0\) defined by means of \(\rho_0(X) = \int_{\mathbb{R}^3} f_0(X, dV)\) satisfies \(\rho_0 \in L^\infty(\mathbb{R}^3)\) (cf. \((1.6a), (1.7)\)). Suppose that we choose a particle configuration \(\{(\hat{X}_j, \hat{V}_j)\}\) according to the probability measure \(\hat{P}_0\) and we make evolve each of the particles this configuration by means of anyone of the dynamics \((1.2), (1.4)\). The evolution of each of the particles of this configuration is assumed to be independent of the others. Notice that in particular in the case of the evolution \((1.4)\) this requires to assume that the distribution of scatterers \(\{(X_k, V_k) : k \in S\}\) appearing in \((1.2)\) is chosen in an independent manner for each of the particles where the evolution of each of the particles in \(\{(\hat{X}_j, \hat{V}_j)\}\) is independent from the others. This yields an evolution of the original probability measure \(\hat{P}_0^\tau\) to a new probability measure \(\hat{P}_\tau^\varepsilon\). For the particular evolution just described the probability measure \(\hat{P}_\tau^\varepsilon\) is a new generalized Poisson measure for any \(\tau > 0\) due to the fact that the particles \(\{(\hat{X}_j, \hat{V}_j)\}\) evolve independently from each other. Therefore the probability measure \(\hat{P}_\tau^\varepsilon\) can be completely characterized by means of a density \(f_\varepsilon(X, V; \tau)\) in the phase space.

It turns out that for suitable choices of the random force fields \(F_\varepsilon\) and friction coefficients \(\Lambda_\varepsilon\) in \((1.2)\) or the interaction potentials \(\phi_\varepsilon\) in \((1.4)\) we can approximate the evolution of the functions \(f_\varepsilon(X, V; \tau)\) as \(\varepsilon \to 0\) by means of a function \(f\) which satisfies a Markovian integro-
differential equation. More precisely, the following limit exists in the sense of measures:

\[
f(x, v, t) = \lim_{\varepsilon \to 0} f_\varepsilon (T_\varepsilon x, v; T_\varepsilon t) \tag{1.8}
\]

where \(T_\varepsilon\) is the macroscopic time scale defined above and it satisfies \(\lim_{\varepsilon \to 0} T_\varepsilon = \infty\). Moreover, the measure \(f\) solves a linear equation with the form:

\[
\partial_t f + v \cdot \partial_x f = K[f(t, x, \cdot)](v) + f(\cdot, \cdot, 0) = f_0(\cdot) \tag{1.9}
\]

where \(K[\cdot]\) is an integro-differential operator, depending on the type of interactions acting only on the velocity variable of \(f\) for each fixed \(x\). Their precise form for different types of particle interactions will be described in Section 5. We will denote the variables \(x\) and \(t\) in (1.8) as macroscopic space and macroscopic time respectively.

Equation (1.9) contains all the information about the kinetic regime which gives the dynamics as \(\varepsilon \to 0\) for particles evolving according to (1.2) or (1.4). If we consider random force fields depending on a distribution \(g\) of particle velocities as in (1.3) we would obtain a collision operator \(K[\cdot]\) depending also on \(g\). Then, the corresponding kinetic equation would take the form:

\[
\partial_t f + v \cdot \nabla_x f = K[f; g] \tag{1.10}
\]

where the operator \(f \to K[f; g]\) acts only on the variables \(v\) of \(f\) at each point \(x \in \mathbb{R}^3\). Then, in the homogeneous case (1.10) reduces to:

\[
\partial_t f = K[f; g]. \tag{1.11}
\]

We can now state the relation between the kinetic limit for the dynamic of the many particle system given by (1.1) and the kinetic limits for (1.2) and (1.4) described above. Suppose that we define a generalized Poisson measure \(P_0\) given by a density \(f_0 \in \mathcal{M}_+ (\mathbb{R}^3 \times \mathbb{R}^3)\) as above. We consider then the evolution of the particle configurations by means of (1.1). Assuming the dynamics (1.1) is well-defined with \(P_0\)—probability one, we can define the associated evolution mapping \(U^\tau\):

\[
U^\tau_\varepsilon((X_k, V_k)_{k \in S}) = (X_k(\tau), V_k(\tau)),
\]

with associated inverse \(U^{-\tau}\). This defined then define a new measure \(P_\tau^\varepsilon\) for each positive time by means of

\[
P_\tau^\varepsilon = P_0 \circ U^{-\tau}_\varepsilon.
\]

In the case of the evolution given by (1.1), the new probability measure is not a generalized Poisson measure for positive times anymore. The difference between the probability measure \(P_\tau^\varepsilon\) and a generalized Poisson measure is usually measured using the correlation functions appearing in the BBGKY hierarchies ([11], [14], [47]), although we will not use this approach in most of this paper (except by a short discussion in Section 6). However, in the kinetic regime, the probability measures \(P_\tau^\varepsilon\) converge as \(\varepsilon \to 0\) to a generalized Poisson measure which can be characterized by a measure \(f\) in the phase space. We claim that in the kinetic regime the evolution of the measure can be computed by means of the nonlinear equation:

\[
\partial_t f + v \cdot \partial_x f = Q[f], \tag{1.12}
\]
where $Q$ is given by

$$Q[f] = K[f; f]$$  \hspace{1cm} (1.13) $$

and $K[f; g]$ is the collision operator for the reduced model in (1.10), associated to the density $g$. In the case of homogeneous particle distributions (1.12) reduces to:

$$\partial_t f = Q[f].$$  \hspace{1cm} (1.14) $$

The rationale behind the closure assumption is the following. Suppose that at a given time $t$ the distribution of particles in the phase space is characterized by the measure $f(\cdot, t)$. We claim that the evolution of an individual particle can be approximated, at least for small macroscopic times for which $f$ is approximately constant, by means of one of the dynamics (1.2) or (1.4) with a distribution of particles given by $g = f(\cdot, t)$. Therefore, due to (1.10), the distribution of particles at time $t + h$ can be expected to be given by:

$$f(x, v, t + h) = f(x, v, t) + h[-v \cdot \partial_x f(x, v, t) + K[f; f(x, v, t)]](x, v, t)$$  \hspace{1cm} (1.15) $$

and taking the limit $h \to 0$ we obtain (1.12), (1.13).

The operator $K$ in (1.15) is the corresponding one to the kinetic limit associated to (1.2) or (1.4), depending on the type of interaction under consideration. On the other hand, as we indicated above, the dynamics of the Rayleigh gases described by the equations (1.4), can be approximated in the case of long range interactions and in the kinetic limit by means of the dynamics of a tagged particle with friction moving in a random force field (cf. (1.2)). This will allow to derive in all the cases the kinetic equations for long range potentials, deriving the corresponding kinetic limits for (1.2), a task much simpler than to derive kinetic limits for the original system of equations (1.1). One of the main questions that needs to be answered in order to fulfill this program is to obtain the formulas for the random force field $F_\varepsilon$ and the friction coefficient $\Lambda_\varepsilon$ appearing in (1.2) from the interactions $\phi_\varepsilon$ for the Rayleigh gas (1.4). This will be made in Subsection 5.2.

It is worth to remark that the approximation of the dynamics of a particle moving in a "medium" which consists in large number of particles by means of one equation with the form (1.2) is extensively used in Statistical Physics, not only in situations in which the dynamics of the system can be approximated by a kinetic equation. A well known example is the description of the Brownian motion of a particle in a viscous fluid. The dynamics of the Brownian particle can be approximated using a Langevin equation, which contains a friction term acting on the Brownian particle, and a random term (white noise). The main difference between this problem and the problems considered in this paper is that to find the connection between the microscopic dynamics and the macroscopic coefficient characterizing the properties of the medium (in this case the friction coefficient), cannot be made in a manner so explicit in the case of a viscous fluid (described by Stokes equations) as in the cases in which the dynamics of the medium can be approximated by a kinetic equation. It is well known that there is general formula connecting the properties of the noise term in (1.2) with the friction coefficient known as the fluctuation-dissipation Theorem (cf. [28]). The fluctuation-dissipation theorem can be expected to hold for systems in which there exists a mechanism driving the distribution of the scatterers to an equilibrium (i.e. not for Rayleigh gases). This connection between noise and friction is due to the fact that the equilibrium distribution at a given temperature is the Gibbs distribution, which in the kinetic regime reduces to Maxwellian distributions for both
the tagged particle and the scatterers. The variances of the velocities are related through the principle of equipartition of energy which holds at equilibrium. The equation describing the evolution of the tagged particle is a Fokker-Planck equation containing a friction- and a diffusive term in the space of velocities, which must be related in order to yield the decided value of the variance.

Three classes of kinetic equations with the form \( \text{(1.12), (1.13)} \) have been used extensively in the physics literature to approximate the dynamics of systems described by means of \( \text{(1.1)} \), namely the Boltzmann equation, the Landau equation and the Balescu-Lenard equation. The kinetic approximations are valid if the characteristic potential energy of each particle does not exceed its kinetic energy. This means that the particle trajectories are nearly rectilinear over lengths of the order of the typical distance between particles due to the weakness of the interactions. However, this might happen in two different ways. First we can have strong interactions, with a range much shorter than the typical particle distance. In this case, the particles of the system interact rarely, but when they do, they experience velocity deflections of order one. The resulting kinetic equation is the Boltzmann equation. The second possibility is to have weak interaction potentials between particles with a range comparable larger or equal than the typical particle distance. In this case the deviations of the trajectory from a rectilinear path are due to the accumulation of many small random deflections which are due to the interaction with many different particles. In this case the resulting kinetic equation is the Landau or the Balescu-Lenard equation. The difference between Landau and Balescu-Lenard stems from the different forms of the random force field \( F_{\varepsilon} \) and the friction coefficient \( \Lambda_{\varepsilon} \) in \( \text{(1.2)} \). Details about the computation of \( F_{\varepsilon} \) and \( \Lambda_{\varepsilon} \) from the particle interactions will be given in Subsection \( \text{5.2} \). Here we just indicate that in the cases in which the Landau equation is applicable, \( F_{\varepsilon} \) and \( \Lambda_{\varepsilon} \) are computed assuming that scatterers move in straight lines. In the case in which the limit kinetic equation is Balescu-Lenard, we must take into account in the computation of \( F_{\varepsilon} \) and \( \Lambda_{\varepsilon} \), the interactions between the scatterers among themselves. The alternative depends on the type of interactions taking place between the particles in \( \text{(1.1)} \). In the cases in which the Balescu-Lenard equation is applicable, the mutual interactions between the scatterers can be described using a single function which is usually termed as dielectric constant. This function and its main properties will be discussed in Subsection \( \text{5.2.1} \). Notice that the Landau equation is just a particular case of the Balescu-Lenard equation in which the dielectric function is just one.

It is worth to mention that the Landau equation can arise as a kinetic limit of \( \text{(1.1)} \) for some short range potentials. The typical situation in which this can happen is the so-called grazing collisions limit which is characterized by weak interaction potentials with a range much shorter or comparable to the typical particle distance. In such a systems the changes in the velocity of one particle are due to the sum of the effects of many pairwise independent weak collisions, each of them yielding small deflections. We refer for instance to \( \text{[2, 15, 16, 22]} \).

In principle, it is not possible to approximate \( \text{(1.1)} \) by means of the dynamics \( \text{(1.2)} \) in the case in which the resulting kinetic equation is a Boltzmann equation. This is due to the fact that the friction term \( \Lambda_{\varepsilon} \) in \( \text{(1.2)} \) is due to the long range interaction of many scatterers acting on the tagged particle, while in the cases in which the kinetic regime is the Boltzmann equation the interactions between particles are due only to pairwise collisions. Therefore, the Boltzmann kinetic regime can be obtained only approximating \( \text{(1.1)} \) by means of a Rayleigh
gas dynamics as in (1.4). However, it will be seen in Subsection 5.6 that the dynamics of a tagged particle in such a Rayleigh gas can be replaced in the case of hard sphere potentials by a new system with inelastic collisions which in the limit $\varepsilon \to 0$ yields the same kinetic dynamics as the original Rayleigh gas. The inelastic collisions can be thought as the analogous of the friction coefficient $\Lambda_\varepsilon$ in the Boltzmann kinetic regime.

The derivation of the three types of kinetic equations discussed above has been extensively considered in the physical literature. The Landau and the Balescu-Lenard equations have been considered mostly in the plasma physics literature, in the particular case in which the particles interact by means of Coulomb potentials. We will consider this case in Subsection 5.2.2 as well as more general interaction potentials.

The connection between the Balescu-Lenard equation and the dynamics of tagged particles in effective media has been pointed out by several authors (cf. [29, 30, 33, 40]). The roots of some of the ideas explained above in order to obtain the kinetic limit of systems of particles interacting by means of long range potentials can be found in the seminal paper by Bogoliubov ([11]), in which the fluctuations of the particle distributions are described using BBGKY hierarchies (see also [4, 5, 35]).

The kinetic limits which describe the dynamics of a tagged particle moving among a set of fixed scatterers has been considered in [38]. Systems in which a tracer particle interacts with scatterers which are not affected by the tracer particle are usually called Lorentz gases. The Lorentz gas can be obtained as a formal limit of a Rayleigh gas in which the mass of the scatterers is much larger than the mass of the tagged particle.

In this case the distribution function $f(x, v, t)$ yields the probability of finding the tagged particle at a position of the phase space. The evolution of this function is described in the resulting kinetic limits studied in [38] either by a linear Boltzmann equation or by a linear Landau equation.

The main conclusion obtained in [38] for Lorentz gases are the following. The description of $f(x, v, t)$ by means of a kinetic equation is possible only if the kinetic energy of a typical particle is much larger than its potential energy. As indicated above this can happen in two different ways, namely because the range of the potentials is much shorter than the average particle distance, or because the interaction potentials are very weak and the tagged particle deflections are due to the addition of many small independent deflections due to different particles. It has been see in [38] that in order to decide if the limit kinetic equation is a Boltzmann or a Landau equation it is convenient to compute two different time scales $T_{BG}$ and $T_L$ which are denoted as Boltzmann-Grad and Landau time scales respectively. The time $T_{BG}$ is the typical time in which the tagged particle arrives sufficiently close to one scatterer to experience a deflection of its velocity comparable to the velocity itself. The time scale $T_L$ is the characteristic time in which the velocity of the tagged particle experiences a change comparable to it due to the accumulation of small random deflections.

It was seen in [38] that the dynamics of a tagged particle can be described by means of a linear Boltzmann equation if $1 \ll T_{BG} \ll T_L$. On the other hand, the description by means of a Landau equation is possible if $1 \ll T_L \ll T_{BG}$. The time scale in which the kinetic evolution takes place is $T_\varepsilon = \min \{T_{BG}, T_L\}$. A second condition which the interactions must satisfy is that they must become independent on distances of order $T_\varepsilon$ (assuming that the characteristic velocity of the tagged particle is of order one). Actually, it has been seen in [38] that for some choices of interaction potentials the deflections of the tagged particles at times of the same
order as $T_\varepsilon = T_L$ have correlations of order one and the resulting limit dynamics cannot be described by a kinetic equation.

It turns out that in interacting particle systems (cf. (1.1)), as well as in Rayleigh gases, it is possible to define in an analogous manner the characteristic time scales $T_{BG}$ and $T_L$. As in the Lorentz gases, these time scales allow us to determine whether the particle system can be described as a kinetic limit, and whether the resulting equation is a Boltzmann equation or a Landau/Balescu-Lenard equation. If we assume that the particle velocities are of order one the mean free path is $T_\varepsilon$. The two conditions required to obtain a kinetic equation in the limit $\varepsilon \to 0$ are the same as those obtained in the case of fixed scatterers, namely that the mean free path is much larger than the typical particle distance (i.e. $T_\varepsilon \gg 1$) and that the deflections of the particle velocities for particles at distances larger than the mean free path become uncorrelated as $\varepsilon \to 0$.

1.1 Notation and structure of the paper.

We will use the following notation in this paper:

\[ \mathbb{N}_* = \{0, 1, 2, 3, \ldots\}, \quad \mathbb{N} = \{1, 2, 3, \ldots\} \quad (1.16) \]

For a function or measure $g(x, v)$ on the phase space, we denote the associated spatial density by $\rho[g]$ by:

\[ \rho[g] (x) = \int_{\mathbb{R}^3} g(x, w) \, dw. \quad (1.17) \]

The plan of the paper is the following. In Section 2 we summarize the different kinetic equations that can be obtained to approximate the dynamics of systems with the form (1.1) in suitable asymptotic limits. Section 3 summarizes the main properties of random force fields generated by systems of moving particles. Section 4 introduces the Boltzmann time scale and the Landau time scale in a manner analogous to [38]. Section 5, which is the main of the paper shows how to approximate the dynamics of both a tagged particle in a Rayleigh gas (cf. (1.4)) or a system of interacting particles (cf. (1.1)) by means of kinetic equations, in cases in which the interactions take place by means of long range potentials. Section 6 compares the approach used in Section 5 to derive kinetic equations with the classical approach based in BBKGY hierarchies. Moreover, this approach allows to obtain probability measures yielding the distribution of particles for times $t > 0$. Section 7 contains a description of some phenomena associated to the Vlasov equation which play an important role in the theory of this equation in the whole space $\mathbb{R}^3$. In Section 8 we discuss Rayleigh gases described by equations with the form (1.4) in the case in which the mass of the scatterers is very large. It turns out that in that case the dynamics of the tagged particle can be approximated as the motion of a tagged particle moving in a static random force field, i.e. a Lorentz gas. Finally in Section 9 we discuss how to adapt the previous ideas to the case of spatially nonhomogeneous distributions.

2 Kinetic descriptions of many particle systems.

We summarize in this Section the different kinetic equations which approximate the dynamics of systems of particles described by the equations (1.1) as well as the form of the interaction potentials and the asymptotic limits in which such kinetic approximation is valid.
We recall briefly the features that characterize each of the models at the level of particles:

| Model                  | Scatterers influence tagged particle | Tagged particle influences scatterers | Scatterers influence other scatterers |
|------------------------|--------------------------------------|---------------------------------------|---------------------------------------|
| Lorentz gas            | Yes                                  | No                                    | No                                    |
| Rayleigh gas           | Yes                                  | Yes                                   | No                                    |
| Interacting particle system | Yes                              | Yes                                   | Yes                                    |

| Model                  | Momentum conserved in the collisions | Energy conserved in the collisions |
|------------------------|--------------------------------------|-----------------------------------|
| Lorentz gas            | No                                   | No (moving scatterers)            |
|                        |                                      | Yes (fixed scatterers)            |
| Rayleigh gas           | Yes                                  | Yes                               |
| Interacting particle system | Yes                              | Yes                               |

2.1 Evolution of a tagged particle in a Lorentz gas.

In the next three Subsections we describe the kinetic equations which describe the behaviour of Lorentz and Rayleigh gases as well as interacting particle systems in suitable kinetic limits and for suitable classes of potentials. We restrict ourselves here for simplicity to the case of spatially homogeneous systems.

The dynamics of a tagged particle in a Lorentz gas is described by the following set of evolution equations:

\[
\begin{align*}
\frac{dX}{d\tau} &= V, \\
\frac{dV}{d\tau} &= -\sum_{j \in S} \nabla \phi_e (X - Y_j), \\
\frac{dY_k}{d\tau} &= W_k, \quad \frac{dW_k}{d\tau} = 0, \quad k \in S
\end{align*}
\]

We observe that the Lorentz gas can be thought as the interaction of tagged particles with a random force field (static or moving).

As it has been proved in [38] it is possible to show that different linear kinetic evolution equation arise, for a given scaling, depending on the decay $s$ as well as on the singularities of the interaction potential, assuming that all the scatterers are at rest, i.e. $W_k = 0$ for all $k \in S$. Using similar methods, it is possible to obtain the following equations describing the evolution of the distribution of velocities $f(v, t)$ which characterizes the tagged particle. We assume in all the following that the typical interparticle distance between scatterers is $d = 1$.

We will assume also that the mean free path is much larger than the interparticle distance, i.e. $\ell_e >> d$. We then have the following possibilities.
(1) For potentials with the form $\Phi_\varepsilon(Y) = V\left(\frac{|Y|}{\varepsilon}\right)$, with $\varepsilon \to 0$ and $V(s)$ decreasing faster than $\frac{1}{s^2}$, with $\alpha > 1$, the kinetic time scale is $T_\varepsilon = \frac{1}{\varepsilon^2}$ and the resulting equation is the linear Boltzmann (with an additional averaging due to the possible distribution of velocities of the scatterers):

$$
\partial_t f(v,t) = \mathcal{L}_B(f)(v,t)
$$

$$
\mathcal{L}_B(f)(v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega \; \tilde{B} (\omega \cdot (v - v_*), |v - v_*|) g(v_*) \left[ f(v') - f(v) \right]
$$

where

$$
v' = v - 2 (\omega \cdot (v - v_*)) \omega
$$

The cross section $\tilde{B}$ is different from the one computed for the usual collisions between two particles because we assume that the scatterer is not affected by the tagged particle. The simplest way of studying this scattering process is to use a coordinate system moving at the scatterer speed $v_*$. Notice that the solutions of (2.2) do not preserve the energy or momentum for the distribution of tagged particles, something that could be expected since these quantities are not preserved in the individual collisions.

(2) For potentials with the form $\Phi_\varepsilon(Y) = \varepsilon V(e^\alpha |Y|)$, with $\varepsilon \to 0$, $\alpha < \frac{5}{4}$, $V(s)$ decreasing faster than $\frac{1}{s^d}$ as $s \to \infty$, with $\alpha > 1$, assuming that the average particle distance is $d = 1$, and choosing the time scale $T_\varepsilon = \varepsilon^{4\alpha-2}$ which satisfies $T_\varepsilon \gg 1$ as $\varepsilon \to 0$ we obtain that the distribution of particle velocities of the tagged particle $f(v,t)$ solves the kinetic equation

$$
\partial_t f(v,t) = \mathcal{L}_L(f)(v,t)
$$

$$
\mathcal{L}_L(f)(v) = \sum_{j,k} \int_{\mathbb{R}^3} dv_* g(v_*) \partial_j \left( \kappa \left( \delta_{j,k} - \frac{(v - v_*)_j (v - v_*)_k}{|v - v_*|^2} \right) \right) \partial_k f(v,t)
$$

where $\kappa$ is the diffusion coefficient in the space of particle velocities, which is given by:

$$
\kappa = \frac{\pi}{2} \frac{1}{|v - v_*|} \int_0^\infty dk |\tilde{V}(|k|)|^2 |k|
$$

and $\tilde{V}$ is the Fourier transform of $V$. Similar equations can be obtained in the case of potentials $V(s)$ behaving as $\frac{1}{s}$ as $s \to \infty$. Nevertheless the formula (2.4) has to be modified, due to the presence of logarithmic divergences. These divergences can be compensated by means of a suitable change in the time scale. We refer to Subsection 4.2 for details about this case. A similar situation can be seen in the case of fixed scatterers in Subsection 3.2 of 354.

The solutions of (2.3) do not preserve neither the energy or the momentum of the particle distribution $f(v,t)$. It is worth to remark that the equations obtained in 354 in the case of scatterers at rest can be obtained as special cases of the equations (2.2), (2.3) just taking the distribution $g(v_*) = \delta(v_*)$. Actually, more general classes of potentials than those described in the point (2), including potentials $V(s)$ behaving as Coulombian potentials for large $s$, have been considered in 38. In such a cases a logarithmic correction, termed as Coulombian logarithm, must be included in the time scale $T_\varepsilon$. The Coulombian logarithm will be extensively discussed later.
2.2 Evolution of a tagged particle in a Rayleigh gas.

We summarize now the classes of kinetic equations that can be obtained for Rayleigh gases. We recall that the set of equations describing the dynamics of a Rayleigh gas is given by (1.4), namely

\[
\frac{dX}{d\tau} = V, \quad \frac{dV}{d\tau} = -\sum_{j \in S} \nabla \phi_\varepsilon (X - Y_j)
\]

\[
\frac{dY_k}{d\tau} = W_k, \quad \frac{dW_k}{d\tau} = -\nabla \phi_\varepsilon (Y_k - X), \quad k \in S
\]

All the results included in this paper are in space dimension three and that the average distance between the scatterers is \(d = 1\). In all the cases the mean free path \(\ell_\varepsilon\) is much larger than the average distance between particles. In the case of Landau kinetic equations we mention here some examples of potentials yielding these equations. Other scaling limits yielding this behaviour will be discussed in Subsection 5.1.

1. For potentials: \(\Phi_\varepsilon (Y) = V \left( \frac{|Y|}{\varepsilon} \right)\), with \(\varepsilon \to 0\) and \(V (s)\) decreasing faster than \(\frac{1}{s^\alpha}\), with \(\alpha > 1\) we can approximate the dynamics of the Rayleigh gas (1.4) using the Boltzmann-Grad time scale \(\tau = T_\varepsilon t\) with \(T_\varepsilon = \frac{1}{\varepsilon^2}\) using the Boltzmann equation (cf. [6], [39], [48]):

\[
\frac{\partial_tf (v,t)}{dt} = Q_B (g, f)(v,t)
\]

\[
Q_B (g, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} B ((v - v_s) \cdot \omega, |(v - v_s)|) \left( g(v'_s)f(v') - g(v_s)f(v) \right) \ dv_s \ dw
\]

where \(v'_s, v'\) are now given by the standard formulas for the collisions, namely

\[
v' = v - \omega \cdot (v - v_s) \omega, \quad v'_s = v_s + \omega \cdot (v - v_s) \omega.
\]

2. We can consider potentials with the form \(\Phi_\varepsilon (Y) = \varepsilon V \left( \frac{|Y|}{\varepsilon L_\varepsilon} \right)\), with \(V\) smooth, \(1 \ll L_\varepsilon \ll \frac{1}{\varepsilon}\) as \(\varepsilon \to 0\) and with \(V (s)\) decreasing faster as \(\frac{1}{s^\alpha}\) or potentials with the form \(\Phi_\varepsilon (Y) = V \left( \frac{|Y|}{\varepsilon} \right)\) with \(V\) smooth behaving as \(V (s) \sim \frac{1}{s}\) as \(s \to \infty\). The kinetic regime is obtained in the time scale \(t = \frac{T_\varepsilon}{\varepsilon}\) with \(T_\varepsilon = \frac{1}{\varepsilon^2 L_\varepsilon^2}\) in the first case and \(T_\varepsilon = \frac{1}{\varepsilon^2 \log (\frac{1}{\varepsilon})}\) in the second case. In this case, the resulting kinetic equation, up to some trivial rescaling of constants in the time variable is the Landau equation for a Rayleigh Gas (see for instance [48]) which reads as

\[
\frac{\partial_tf (v,t)}{dt} = Q_L (g, f)(v,t)
\]

\[
Q_L (g, f)(v) = \sum_{j,k} \int_{\mathbb{R}^3} dv_s g(v_s) \partial_j \left( \kappa \left( \delta_{j,k} - \frac{(v - v_s)_j (v - v_s)_k}{|v - v_s|^2} \right) (v_{s,k} + \partial_k)f(v,t) \right)
\]

(2.7)
with \( \kappa \) given as in (2.4). We remark that the main difference between the kinetic equations (2.3) and (2.7) is the presence of the friction term \( v_{*,k} f(v,t) \). This term yields a change in the energy of the tagged particle, which is due to the fact that the tagged particle can exchange energy with the scatterers, differently from the case of the Lorentz gases.

2.3 Evolution of interacting particle systems.

As indicated in the Introduction we can obtain kinetic equations which approximate the dynamics of the system (1.1) in suitable scaling limits using as an intermediate step the kinetic limit of the dynamics of either (1.2) or (1.4) during small macroscopic times. The derivation of the kinetic equations is made by means of the closure equation (1.13) where \( K[f,g] \) is the kinetic kernel which corresponds to the kinetic limit associated to (1.2) or (1.4). The corresponding kinetic equation is then (restricting ourselves to spatially homogeneous situations) is given by (1.14).

Some of the cases of interaction potentials and the approximations of the form (1.2) or (1.4) used in the kinetic limit are the following ones. We write also the resulting kinetic equations which describe the evolution of the particle system (1.1) as well as the asymptotic limit in which this approximation is valid. In the case of Landau and Balescu-Lenard equations, the specific scaling limits yielding these kinetic equations will be made precise in Subsection 5.2.

1. For potentials: \( \Phi_\varepsilon(Y) = V \left( \frac{|Y|}{\varepsilon} \right) \), with \( \varepsilon \to 0 \) and \( V(s) \) decreasing faster than \( \frac{1}{s^\alpha} \), with \( \alpha > 1 \) we can approximate the evolution of the distribution of velocities by means of the Boltzmann equation.

\[
\partial_t f(v,t) = Q_B(f,f)(v,t),
\]
\[
Q_B(f,f)(v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega B(n \cdot \omega, |v-v_*|) \left[ f(v')f(v'_*) - f(v)f(v_*) \right],
\]  
(2.8)

where \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \) and \( n = n(v,v_*) = \frac{(v-v_*)}{|v-v_*|} \). Here \( (v,v_*) \) is a pair of velocities in incoming collision configuration and \( (v',v'_*) \) is the corresponding pair of outgoing velocities defined by the collision rule

\[
v' = v + ((v_* - v) \cdot \omega) \omega, \quad (2.9)
\]
\[
v'_* = v_* - ((v_* - v) \cdot \omega) \omega. \quad (2.10)
\]

The collision kernel \( B(n \cdot \omega, |v-v_*|) \) is proportional to the cross section for the scattering problem associated to the collision between two particles.

2. Landau equation.

\[
\partial_t f(v,t) = Q_L(f,f)(v,t)
\]
\[
Q_L(f,f)(v) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} dv_* a(v-v_*)(\nabla_v - \nabla_{v_*}) f(v)f_*(v) \right),
\]  
(2.11)

where \( a = a(V) = a(v-v_*) \) denotes the matrix with components
\[ a_{i,j}(V) = \frac{\pi^2}{4} \int_{\mathbb{R}^3} dk \ k_i k_j \delta(k \cdot V) |\hat{\Phi}(k)|^2 = \frac{A}{|V|} \left( \delta_{i,j} - \hat{V}_i \hat{V}_j \right) \] for some \( A > 0 \) \( (2.12) \)

which is determined by the pair interaction potential \( \Phi \).

3. Balescu-Lenard equation.

\[
\partial_t f(v, t) = Q_{BL}(f, f)(v, t) \tag{2.13}
\]

\[
Q_{BL}(f, f)(v) = \nabla_v \cdot \left( \int dv^* a(v - v^*, v)(\nabla_v - \nabla_{v^*})f(v)f^*(v) \right),
\]

where the matrix \( a \) is given by

\[
a_{i,j}(V, v) = \frac{\pi^2}{4} \int_{\mathbb{R}^3} dk \ k_i k_j \delta(k \cdot V) \frac{|\hat{\Phi}(k)|^2}{|\varepsilon(k, v \cdot v)|^2} \tag{2.14}
\]

Here \( \varepsilon \) is the so-called dielectric function, which will be discussed in detail in Subsection 5.2.1.

We summarize the kinetic limits for the scaling limits of interacting particle systems. As before, we assume the initial configurations of particles to be random. Particle velocities are assumed to be of order one, and we choose the length scale such there are 1 particles per unit volume. We consider weak, radially symmetric potentials \( \phi_{ps} \) of the following form:

\[
\phi_{\varepsilon}(x) = \varepsilon \phi(x/L_{\varepsilon}).
\]

Then we obtain the following table for the kinetic equations of the associated scaling limits and the kinetic timescale \( T_{\varepsilon} \), depending on the choice of potential and the scaling:

| \( \phi(x) \in \mathcal{S}(\mathbb{R}^3) \) , or \( \phi \sim \frac{1}{|x|^s} \), \( |x| \geq 1 \), \( s > 1 \) | \( \phi \sim \frac{1}{|x|^s} \), \( |x| \geq 1 \) |
|---|---|
| \( L_{\varepsilon} = 1 \) | Landau, \( T_{\varepsilon} = \varepsilon^{-2} \) | Landau, \( T_{\varepsilon} = \varepsilon^{-2} |\log \varepsilon| \) |
| \( L_{\varepsilon} = \varepsilon^{-\alpha}, \alpha \in (0, \frac{1}{3}] \) | Landau \( T_{\varepsilon} = \varepsilon^{4s-2} \) | Landau, \( T_{\varepsilon} = \varepsilon^{4s-2} |\log \varepsilon| \) |
| \( L_{\varepsilon} = \varepsilon^{-\frac{1}{3}} \) | Balescu-Lenard \( T_{\varepsilon} = \varepsilon^{-\frac{5}{2}} \) | Landau, \( T_{\varepsilon} = \varepsilon^{-\frac{5}{2} |\log \varepsilon|} \) |

3 Random force fields generated by moving particles.

3.1 Generalities about random force fields.

As indicated in the Introduction one of our goals is to approximate the dynamics of (1.1) in some scaling limits by means of the dynamics of a tagged particle in some families of random force fields. In this Section, we describe some general properties of random force fields as well as some specific random force fields generated by sets of moving particles. Other random force fields will not be directly related to fields generated by point particles, but they will be useful in order to approximate the dynamics (1.1) by (1.2). Those force fields will be described in Subsection 5.2 as well as their relation with the fields constructed in this Section.
Since we want to consider random force fields having singularities at the particle centers, we will introduce some notation to deal with this case. We will denote as $\Lambda$ the space of particle configurations in $\mathbb{R}^3$. More precisely, the elements of $\Lambda$ are locally finite subsets of $\mathbb{R}^3$, or more precisely, sequences with the form $\{x_k\}_{k \in \mathbb{N}}$ such that $\# \left( \{x_k\}_{k \in \mathbb{N}} \cap B_R(0) \right) < \infty$ for any $R < \infty$. Notice that we do not need to assume that $x_k \neq x_j$ for $k \neq j$. In order to allow force fields which diverge at some points we define $\mathbb{R}^3_*$ as the compactification of $\mathbb{R}^3$ using a single infinity point $\infty$. For a function $F \in C(\mathbb{R}^3; \mathbb{R}_3^3)$ we write $F(x_0) = \infty$ for some $x_0 \in \mathbb{R}^3$, if the function satisfies $\lim_{x \to x_0} |F(x)| = \infty$.

We then introduce the following notation:

$$C_* (\mathbb{R}^3) = C (\mathbb{R}^3; \mathbb{R}_3^3).$$

Since $\mathbb{R}^3$ is a metric space, we can endow $C_* (\mathbb{R}^3)$ with a metric topology in the usual manner. Most of the random force fields used in this paper will be more regular than just continuous. We then define:

$$C^k_* (\mathbb{R}^3) = \left\{ F \in C_* (\mathbb{R}^3) : F \in C^k (\mathbb{R}^3 \setminus F^{-1} (\infty)) \right\} \text{ where } k = 1, 2, ...$$

Therefore, the elements of $C^k_* (\mathbb{R}^3)$ are just $C^k$ functions at the points where they are bounded. We will not need to define any topology on the spaces $C^k_* (\mathbb{R}^3)$.

We are interested in time dependent random force fields. We then define the metric space $C (\mathbb{R} : C_* (\mathbb{R}^3))$. We could define similarly $C ((0, T] : C_* (\mathbb{R}^3))$, but we will use in this paper only time dependent random force fields defined globally in time.

Therefore we define:

$$C^k (\mathbb{R} : C^k_* (\mathbb{R}^3)) = \left\{ F \in C (\mathbb{R} : C_* (\mathbb{R}^3)) : F \in C^k ((\mathbb{R} \times \mathbb{R}^3) \setminus F^{-1} (\infty)) \right\}$$

for $k = 1, 2, ...$. Notice that $C^k (\mathbb{R} : C^k_* (\mathbb{R}^3))$ is the subset of the set of functions of $C (\mathbb{R} : C_* (\mathbb{R}^3))$ which have $k$ continuous derivatives at the points where $F$ is bounded. We will use the shorthand notation $F \in C^k$ for $F \in C^k (\mathbb{R} : C^k_* (\mathbb{R}^3))$.

We introduce a $\sigma$–algebra on the space $C (\mathbb{R} : C_* (\mathbb{R}^3))$ generated by the cylindrical sets, i.e. the $\sigma$–algebra generated by the sets $\{ Y(t_0, x_0) \in B \}$ where $B$ is a Borel set of $\mathbb{R}_3^3$ and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^3$. We will denote this $\sigma$–algebra as $\mathcal{B}$.

All the random force fields in which we are interested in this paper are contained in the following definition.

**Definition 3.1** Let $(\Omega, \mathcal{F}, \mu)$ a measure space where $\mathcal{F}$ is a $\sigma$–algebra of subsets of $\Omega$ and $\mu$ is a probability measure. A random force field is a measurable mapping $F$ from $\Omega$ to the set of functions $C (\mathbb{R} : C_* (\mathbb{R}^3))$ with respect to the $\sigma$–algebra $\mathcal{B}$.

Notice that a random force field defines a probability $P$ on the $\sigma$–algebra $\mathcal{B}$, which consists of subsets of $C (\mathbb{R} : C_* (\mathbb{R}^3))$, by means of:

$$P (A) = \mu (\{ \omega \in \Omega : F (\omega) \in A \}) \ , \ A \in \mathcal{B}.$$ 

A random force field can be characterized by the family of random variables

$$\{ \omega \to F (X, \tau; \omega) , \ X \in \mathbb{R}^3 , \ \tau \in [0, T] \}.$$
We can define the action of the group of spatial translations on $C^k (\mathbb{R} : C^k_\ast (\mathbb{R}^3))$ by means of:

$$T_a F (X, \tau) = F (X + a, \tau)$$
for each $a \in \mathbb{R}^3$.

and the group of time translations by means of:

$$U_b F (X, \tau) = F (X, \tau + b)$$
for each $b \in \mathbb{R}$.

We will say that a random force field $F$ is invariant under spatial translations (or just invariant under translations) if we have:

$$\mu (\{ \omega \in \Omega : F (\cdot ; \omega) \in A \}) = \mu (\{ \omega \in \Omega : T_a F (\cdot ; \omega) \in A \})$$
for each $A \in \mathcal{B}$ and any $a \in \mathbb{R}^3$. We will say that a random force field $F$ is invariant under time translations (or stationary) if:

$$\mu (\{ \omega \in \Omega : F (\cdot ; \omega) \in A \}) = \mu (\{ \omega \in \Omega : U_b F (\cdot ; \omega) \in A \})$$
for each $A \in \mathcal{B}$ and any $b \in \mathbb{R}$.

Actually, all the random force fields considered in this paper will be also invariant under rotations. Given $M \in SO (3)$ we define the action of the group $SO (3)$ on $C^k (\mathbb{R} : C^k_\ast (\mathbb{R}^3))$ by means of:

$$R_M F (X, \tau) = F (MX, \tau)$$
for each $M \in SO (3)$.

Then, the random force field $F$ is invariant under the group $SO (3)$ if:

$$\mu (\{ \omega \in \Omega : F (\cdot ; \omega) \in A \}) = \mu (\{ \omega \in \Omega : R_M F (\cdot ; \omega) \in A \})$$
for each $A \in \mathcal{B}$ and any $M \in SO (3)$.

3.2 Time dependent particle configurations and random force fields.

3.2.1 Random particle configurations

In this Section we describe a family of random force fields that are generated by particles distributed randomly in the phase space according to the Poisson distribution at time $t = 0$ which move at constant velocity for positive times. We will take as unit of length the typical distance between particles $d$, i.e. $d = 1$. The particle velocities are distributed according to a finite nonnegative measure $g = g (dv)$, independently from the particle positions. Note that $\int_{\mathbb{R}^3} g (dv)$ is the spatial particle density. We will choose the unit of time $\tau$ in such a way that the average particle velocity is of order one. Each particle is the center of a radial potential $\Phi = \Phi (|y|) \in C^2 (\mathbb{R}^3 \setminus \{0\})$. We will consider two different types of potentials. First we will consider potentials of order one with short ranges, i.e. smaller range than the particle distance $d$. Second we will consider weak potentials with arbitrary range, but typically larger or equal than the particle distance $d$.

More precisely, we will denote as $\Lambda_p$ the space of locally finite particle configurations in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$. Each of these particle configurations can be represented by a sequence $\{(x_k, v_k)\}_{k \in \mathbb{N}}$ with $x_k \in \mathbb{R}^3$ and $v_k \in \mathbb{R}^3$ where all the sequences which can be obtained from
another by means of a permutation of the particles are equivalent and represent the same particle configuration. We consider the \( \sigma \)-algebra \( \Sigma_p \) generated by the sets:

\[
U_{B,n} = \{(x_k, v_k)\}_{k \in \mathbb{N}} \in \Lambda_p : \# \{(x_k, v_k) \cap B\} = n
\]

for each \( n \in \mathbb{N} \), any Borel set \( B \subset \mathbb{R}^3 \times \mathbb{R}^3 \). We define a measure in \( \mathbb{R}^d \times \mathbb{R}^d \) by means of the product measure \( dxg \ (dv) \). We then define a probability measure \( \nu_g \) on \( \Sigma_p \) by means of:

\[
\nu_g \left( \bigcap_{j=1}^J U_{B_j,n_j} \right) = \prod_{j=1}^J \left[ \frac{\int_{B_j} dxg \ (dv)}{(n_j)!} \right] - \int_{B_j} dxg (dv)
\]

(3.1)

where \( B_j \) is a Borel set of \( \mathbb{R}^3 \times \mathbb{R}^3 \) and \( n_j \in \mathbb{N} \) for each \( j \in \{1, 2, \ldots, J\} \) with \( B_j \cap B_k = \emptyset \) if \( j \neq k \).

We define the free flow evolution group \( T(\tau) \), \( \tau \in \mathbb{R} \), on the space of particle configurations \( \Lambda_p \) as follows. Suppose that we represent a particle configuration \( \xi \in \Lambda_p \) by the sequence \( \{(x_k, v_k)\}_{k \in \mathbb{N}} \). Then we define:

\[
T(\tau) \xi = \{(x_k + v_k \tau, v_k)\}_{k \in \mathbb{N}} , \ \tau \in \mathbb{R} \quad (3.2)
\]

This definition yields a mapping \( T(\tau) : \Lambda_p \rightarrow \Lambda_p \) which is independent of the specific sequence used to label \( \xi \in \Lambda_p \). It is not hard to exhibit examples of particle configurations \( \xi \in \Lambda_p \) for which the configuration defined by means of (3.2) is not locally finite for \( \tau \neq 0 \). This can be achieved giving to some particles placed very far away from the origin large velocities which transport infinitely many particles to a bounded region for some times \( t_0 \in \mathbb{R} \).

However, this does not happen with probability one if the particles are chosen according to the probability measure \( \nu_g \) defined in (3.1). More precisely, we have the following result:

**Proposition 3.2** Let \( (\Lambda_p, \Sigma_p, \nu_g) \) the measure space of particle configurations, with \( \nu_p \) as in (3.1). Then, the evolution group \( T(\tau) \) defined by means of (3.2) is well defined (i.e. \( T(\tau) \xi \) is locally finite) for any \( \tau \in \mathbb{R} \), for a.e. \( \xi \in \Lambda_p \).

The pushforward measure \( \nu_g \circ T(-\tau) \) satisfies \( \nu_g \circ T(-\tau) = \nu_g \) for each \( \tau \in \mathbb{R} \).

**Proof.** First we note that \( T(\tau) \xi \in \Lambda_p \) holds \( \nu_g \) a.e. for every \( \tau \in \mathbb{R} \). We have

\[
\nu_g(\xi \in \Lambda_p : T(\tau) \xi \text{ not loc. finite}) \leq \sum_{n=1}^{\infty} \nu_g(\xi \in \Lambda_p : T(\tau) \xi \cap B_n \text{ infinite}),
\]

therefore it suffices to show \( \nu_g(\xi \in \Lambda_p : T(\tau) \xi \cap B_R \times \mathbb{R}^3 \text{ infinite}) = 0 \) for every \( R > 0 \). To this end we observe that

\[
\sum_{n=1}^{\infty} \nu_g(\xi \in \Lambda_p : \exists k \in S, n \leq |v_k| \leq n + 1, x_k \in B_R - \tau v_k) < \infty.
\]

Hence by Borel-Cantelli we have \( \nu_g(\xi \in \Lambda_p : \exists N > 0 x_k + \tau v_k \in B_R \Rightarrow x_k \in B_N) = 1 \), so \( \nu_g(\xi \in \Lambda_p : T(\tau) \xi \cap B_R \times \mathbb{R}^3 \text{ infinite}) = 0 \) as claimed.

Furthermore, for any Borel sets \( A, B \subset \mathbb{R}^3 \) and \( n \in \mathbb{N} \) we have

\[
\nu_g(\xi \in \Lambda_p : |\{T(\tau) \xi \cap A \times B\}| = n) = \nu_g(\xi \in \Lambda_p : |\{k \in S : v_k \in B, x_k \in A - \tau v_k\}| = n)
\]

\[
= \nu_g(\xi \in \Lambda_p : |\{k \in S : v_k \in B, x_k \in A\}| = n)
\]

\[
= \nu_g(\xi \in \Lambda_p : |\xi \cap A \times B\}| = n).
\]

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Repeating the same computation for the cylinder sets shows \( \nu_g = \nu_g \circ T(-\tau) \).

Actually several of the random force fields considered in this paper will contain at least two different types of particles having different types of charges. This is due to the fact that in order to define some of the long range potentials, in particular those behaving for large values as Coulombian potentials, a electroneutrality condition is required in order to be able to define spatially homogeneous random force fields (cf. [38], Theorem 2.13). On the other hand, there is not reason to assume that in multicomponent systems all the particles have the same velocity distribution. Suppose that we consider systems with \( L \) different types of particles having respectively the charges \( \{q_\ell \} \), \( \ell \in \{1, 2, ..., L\} \) and velocity distributions \( \{g_\ell \} \).

\( \nu \) being the smallest Radon measure on \( \mathbb{R}^d \), we define:

\[
\nu_{g, \ell}(dv) = \frac{1}{N} \int_{\mathbb{R}^d} g_\ell(x, dv) dv,
\]

where \( N = \# B_j \) and \( \ell \in \{1, 2, ..., L\} \).

This way we can define randomly moving particles in \( \mathbb{R}^d \) with different types of particles and different velocities. This can be done by means of a permutation of the particles within a single species. Now, given a family of radially symmetric interaction potentials \( \{\phi_\ell\} \), \( \ell \in \{1, 2, ..., L\} \), we can define a family of random force fields taking as starting point the random particle configurations defined in the previous section. Given a family of radially symmetric interaction potentials \( \phi = \phi(|x|) \) such that \( \nabla \phi \in C^2_b(\mathbb{R}^d) \), we want to give a meaning to the following expressions in order to define suitable random force fields. In the case of particle configurations in \( \Lambda_p \):

\[
F(x, \tau; \omega; g) = -\sum_{k \in \mathbb{N}} \nabla \phi(x - x_k - v_k \tau), \quad \omega = \{(x_k, v_k)\}_{k \in \mathbb{N}} \in \Lambda_p.
\]
and in the case of particle configurations with different types of charges \( \{Q_\ell\}_{\ell=1}^L \) and velocity distributions \( \{g_\ell (dv)\}_{\ell=1}^L \) the goal is to give a meaning to expressions like:

\[
F \left( x, \tau; \omega; \{g_\ell\}_{\ell=1}^L \right) = - \sum_{\ell=1}^L \sum_{k \in \mathbb{N}} Q_\ell \nabla \phi \left( x - x_{k,\ell} - v_{k,\ell} \tau \right), \quad \omega \in \Lambda_p^L
\]

(3.7)

with \( \omega \) as in (3.3). Similarly, we define truncations of the expressions above, defined by

\[
F^R \left( x, \tau; \omega; g \right) = - \sum_{k \in \mathbb{N} \mid x_k \leq R} \nabla \phi \left( x - x_k - v_k \tau \right), \quad \omega = \{(x_k, v_k)\}_{k \in \mathbb{N}} \in \Lambda_p
\]

(3.8)

\[
F^R \left( x, \tau; \omega; \{g_\ell\}_{\ell=1}^L \right) = - \sum_{\ell=1}^L \sum_{k \in \mathbb{N} \mid x_{k,\ell} \leq R} Q_\ell \nabla \phi \left( x - x_{k,\ell} - v_{k,\ell} \tau \right), \quad \omega \in \Lambda_p^L.
\]

(3.9)

The convergence of the series on the right-hand side of (3.6), (3.7) for \( \tau = 0 \) and a large class of interaction potentials \( \phi \) has been considered in [38]. Given that the distribution of points \( \{x_k + v_k \tau\}_{k \in \mathbb{N}} \) is given by a Poisson distribution (cf. 3.2) in \( \mathbb{R}^3 \) these results hold for any \( \tau \in \mathbb{R} \). The convergence of the series in (3.6), (3.7) is not immediate for potentials \( \phi (|x|) \) decreasing like nonintegrable power laws for large values of \( |x| \). Using the methods in [38] we might then see that the right-hand side of (3.6) can be given a meaning for any \( \tau \in \mathbb{R} \) with probability one and \( \phi (|x|) \sim \frac{C}{|x|^s}, \quad s > 1 \) defining the right-hand side of (3.6) as the limit as \( R \to \infty \) of the sum over points contained in a sphere \( B_R(0) \). In the case \( s = 1 \) it has been proved in [38] that such a limits exist and they define a random force field invariant under translations if the electroneutrality condition (3.3) holds.

The type of arguments used in [38] can be adapted to prove that the random force fields in (3.6), (3.7) are defined for all \( \tau \in \mathbb{R} \) with probability one. The following set of conditions for the function \( \phi \) will be used in the definition of the random force fields \( F \).

\[
|\phi (x)| + |x| |\nabla \phi (x)| \leq \frac{C}{|x|^s} \quad \text{for } |x| \geq 1 \text{ with } s > 2
\]

(3.10)

\[
\left| \phi (x) - \frac{A}{|x|^s} \right| + |x| \left| \nabla \phi (x) + \frac{Ax}{|x|^{s+2}} \right| \leq \frac{C}{|x|^{s+1}} \quad \text{for } |x| \geq 1 \text{ with } s \in (1, 2), \quad A \in \mathbb{R}
\]

(3.11)

\[
\left| \phi (x) - \frac{A}{|x|^s} \right| + |x| \left| \nabla \phi (x) + \frac{Ax}{|x|^3} \right| \leq \frac{C}{|x|^{2+\delta}} \quad \text{for } |x| \geq 1 \text{ with } A \in \mathbb{R}, \quad \delta > 0.
\]

(3.12)

In the three formulas (3.10)-(3.12) we assume that \( C > 0 \).

We will further assume that the functions \( g, \, g_\ell \) satisfy:

\[
\int g(v) |v|^{4+\kappa} \, dv < \infty,
\]

(3.13)

\[
\int g_\ell (v) |v|^{4+\kappa} \, dv < \infty,
\]

(3.14)

for some \( \kappa > 0 \). We then have the following result:

**Proposition 3.3** Let \( \phi : \mathbb{R}^3 \to \mathbb{R} \) be a radially symmetric interaction potential such that \( \phi, \nabla \phi, \nabla^2 \phi \in C_s (\mathbb{R}^3) \). Let \( B_1(0) \) be the unit ball in \( \mathbb{R}^3 \). We have the following.
(i) Suppose that $\phi$ satisfies (3.11) and that $\nabla^2 \phi$ is bounded in $L^p (B_1 (0))$ for some $p > 1$. Let $\nu_g$ be the probability measure in the measure space $(\Lambda_p, \Sigma_p, \nu_g)$ defined by means of (3.12) where $g$ satisfies (3.13). Then for any $x \in \mathbb{R}^3$ the series in (3.6) converges absolutely for all $\tau \in \mathbb{R}$ for $\nu_g$–almost $\omega \in \Lambda_p$. Moreover, the series (3.6) defines a random force field in $C (\mathbb{R} : C_\ast (\mathbb{R}^3))$.

(ii) Suppose that $\phi$ satisfies (3.11) and that $\nabla^2 \phi$ is bounded in $L^p (B_1 (0))$ for some $p > 1$. Let $\nu_g$ be the probability measure in the measure space $(\Lambda_p, \Sigma_p, \nu_g)$ defined by means of (3.12) where $g$ satisfies (3.13). Then, for each $x \in \mathbb{R}^3$ the following limit exists for all $\tau \in \mathbb{R}$ for $\nu_g$–almost $\omega \in \Lambda_p$:

$$F (x, \tau; \omega; g) = - \lim_{R \to \infty} \sum_{|x| \leq R} \nabla \phi (x - x_k - v_k \tau)$$

Moreover, the series in (3.12) defines a random force field in $C (\mathbb{R} : C_\ast (\mathbb{R}^3))$.

(iii) Suppose that $\phi$ satisfies (3.12) and that $\nabla \phi$ is in the Sobolev space $H^s (B_1 (0))$ with $s > \frac{1}{2}$. Let $(\Lambda_p^L, \mathcal{F}^L, \nu^{(g)}_{(\theta)L=1})$ be the measure space defined by means of (3.12), (3.4) where the functions $\{g^L_{(\theta)L=1}\}$ satisfy (3.14) and the electroneutrality condition (3.5) holds. Then, for each $x \in \mathbb{R}^3$ the following limit exists for all $\tau \in \mathbb{R}$ for $\nu^{(g)}_{(\theta)L=1}$–almost $\omega \in \Lambda_p^L$:

$$F (x, \tau; \omega; \{g^L_{(\theta)L=1}\}) = - \lim_{N \to \infty} \sum_{L=1}^L \sum_{|x| \leq 2^N} Q \nabla \phi (x - x_k - v_k \tau)$$

Moreover, the series in (3.12) defines a random force field in $C (\mathbb{R} : C_\ast (\mathbb{R}^3))$.

**Proof.** We will only sketch the proof of Proposition 3.3 in the case (iii) since it is the most involved. The generalization to (ii) and (iii) can be made along similar lines using ideas analogous to the proof of Theorem 2.6 in [38].

We first split $F$ into the contribution due to the close particles and the long range contribution. To this end we split $\phi$ as $\phi = \phi_1 + \phi_2$ where $\phi_1$ is supported in the unit ball and it is smooth away from the origin. The function $\phi_2$ is smooth in the whole space $\mathbb{R}^3$. More precisely we introduce a cutoff $\eta \in C^\infty (\mathbb{R}^3)$ such that $\eta (x) = \eta (|x|), 0 \leq \eta \leq 1, \eta (x) = \frac{1}{2}$ if $|x| \leq 1, \eta (x) = 0$ if $|x| \geq 1$. We set

$$\phi_1 (x) := \phi (x) \eta (|x|), \quad \phi_2 (x) := \phi (x) \{1 - \eta (|x|)\} .$$

We then define the random force fields $F_1, F_2$, as in (3.16), using the potentials $\phi_1, \phi_2$, so that $F = F_1 + F_2$. We estimate first the contribution $F_2$. To this end we define the random variables

$$f_0^{(2)} (x, \tau; \omega) = - \sum_{\ell=1}^L \sum_{|x_k| \leq 1} Q \nabla \phi_2 (x - x_k - v_k \tau) ,$$

$$f_j^{(2)} (x, \tau; \omega) = - \sum_{\ell=1}^L \sum_{2^{j-1} < |x_k| \leq 2^j} Q \nabla \phi_2 (x - x_k - v_k \tau) , \quad j = 1, 2, 3, \ldots$$
Then

\[ - \sum_{\ell=1}^{L} \sum_{\{ |x_{k,\ell}| \leq 2^{N} \}} Q \ell \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau) = \sum_{j=0}^{N} f_j (x, \tau; \omega) \]  

(3.18)

Notice that using the fact that the Poisson distributions for the different charges have rate \( \nu \) and that they are invariant under translations. Then:

We assume that \( \phi \) in independent of \( \omega \). Hence, from (3.20) using the estimates above, we obtain

\[ \mathbb{E} \left[ f_j^{(2)} (x, \tau; \omega) \right] = 0. \]  

(3.19)

We now observe that

\[ \mathbb{E} \left[ \int_{0}^{T} f_j^{(2)} (x, \tau; \omega) \otimes f_m^{(2)} (x, \tau; \omega) \, d\tau \right] = \delta_{j,m} \int_{0}^{T} d\tau \mathbb{E} [f_j (x, \tau; \omega) \otimes f_m (x, \tau; \omega)]. \]  

(3.20)

Then, using that the distributions of particles in the sets \( \{ 2^{j-1} < |x_{k,\ell}| \leq 2^j \} \) (and \( \{ |x_{k,\ell}| \leq 2 \} \) are mutually independent and then the random variables \( f_j^{(2)} (x, \tau; \omega) \) are mutually independent, we obtain

\[ \mathbb{E} \left[ f_j^{(2)} (x, \tau; \omega) \otimes f_m^{(2)} (x, \tau; \omega) \right] = \sum_{\ell=1}^{L} \sum_{\{ 2^{j-1} < |x_{k,\ell}| \leq 2^j \}} (Q \ell)^2 \mathbb{E} \left[ \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau) \otimes \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau) \right]. \]

We now use the fact that the Poisson distributions for the different charges have rate \( \int_{\mathbb{R}^3} g_\ell (dv) \) and that they are invariant under translations. Then:

\[ \mathbb{E} \left[ \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau) \otimes \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau) \right] = \int_{\mathbb{R}^3} dy \int_{\{ 2^{j-1} < |y| \leq 2^j \}} [\nabla \phi_2 (x - y - \nu \tau) \otimes \nabla \phi_2 (x - y - \nu \tau) \] \[ \times g_\ell (dv) \]. \]

Since \( \phi_2 \) is smooth near the origin and \( \nabla \phi_2(|y|) \sim \frac{1}{|y|^2} \) for large distances, we obtain

\[ |\mathbb{E} \left[ \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_1) \otimes \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_2) \right]| \leq \int_{\mathbb{R}^3} dy \int_{\{ 2^{j-1} < |y| \leq 2^j \}} \frac{g_\ell (dv)}{\left( 1 + |x - y| \right)^2 \left( 1 + |x - y| \right) \left( 1 + |x - y| \right)^2}. \]

We assume that \( |x| \) is bounded, and that \( \tau_1, \tau_2 \) are bounded. We estimate separately the contributions due to the region where \( |v| \leq 2^{\frac{j}{2}} \) and the region where \( |v| > 2^{\frac{j}{2}} \). For \( |v| \leq 2^{\frac{j}{2}} \) we have

\[ |\mathbb{E} \left[ \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_1) \otimes \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_2) \right]| \leq \frac{C}{2^{j}} \]

and for \( |v| > 2^{\frac{j}{2}} \)

\[ |\mathbb{E} \left[ \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_1) \otimes \nabla \phi_2 (x - x_{k,\ell} - v_{k,\ell} \tau_2) \right]| \leq C 2^{-\frac{j}{2}} 2^{3j} \]

for some constant \( C \) independent of \( j \). Hence, from (3.21) using the estimates above, we obtain

\[ \mathbb{E} \left[ \int_{0}^{T} f_j^{(2)} (x, \tau; \omega) \otimes f_\ell^{(2)} (x, \tau; \omega) \, d\tau \right] = \left[ \int_{0}^{T} K_j (\tau) \, d\tau \right] \delta_{j,\ell} \]

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for \( j = 0, 1, 2, \ldots \), where:

\[
|K_j(\tau)| \leq C 2^{-(\frac{3}{2}-3)j} \quad \text{for } \tau \in [0, T]
\]

for some \( C > 0 \), which depends only on \( T \).

Arguing similarly, we can obtain estimates for the derivatives of the functions \( f^{(2)}_j \). More precisely, we have the estimate

\[
E \left[ \int_0^T \left| \frac{\partial_s f^{(2)}_j(x, \tau; \omega)}{\partial \tau} \right|^2 d\tau \right] \leq \tilde{C}_T 2^{-(\frac{3}{2}-3)j}, \quad j = 0, 1, 2, \ldots
\]

whence, using Morrey’s Theorem, we obtain that

\[
E \left[ \sup_{0 \leq \tau \leq T} \left| f^{(2)}_j(x, \tau; \omega) \right| \right] \leq C_T 2^{-(\frac{3}{2}-3)j}, \quad j = 0, 1, 2, \ldots
\]

We can then prove, using Borel-Cantelli Theorem as in [38], that the limit as \( N \to \infty \) of the right-hand side of (3.18) exists for all \( \tau \in [0, T] \).

We now prove the existence of the random force field \( F_1 \) associated to the localized part of the potential \( \phi_1 \). To this end we define random functions \( f^{(1)}_0, f^{(1)}_1 \) as in (3.17) replacing \( \phi_2 \) by \( \phi_1 \). Arguing as before and using the fact that \( \phi_2 \in H^s(\mathbb{R}^3) \) we obtain the estimate

\[
E \left[ \int_0^T \left| \frac{\partial_s f^{(1)}_j(x, \tau; \omega)}{\partial \tau} \right|^2 d\tau \right] \leq C_T 2^{-(\frac{3}{2}-3)j}, \quad j = 0, 1, 2, \ldots
\]

with \( p > 1 \), where \( \partial_s^\tau \) is the fractional derivative of order \( s \) defined by means of incremental quotiens (see for instance [1, 13]). Then, using that \( s > \frac{1}{2} \) we obtain

\[
E \left[ \sup_{0 \leq \tau \leq T} \left| f^{(1)}_j(x, \tau; \omega) \right| \right] \leq C_T 2^{-(\frac{3}{2}-3)j}, \quad j = 0, 1, 2, \ldots
\]

We can then argue as in the estimate of the random force field \( F_2 \) due to \( \phi_2 \) to prove the convergence of the corresponding series uniformly for all \( \tau \in [0, T] \). Then the convergence of the right hand side of (3.16) follows.

**Remark 3.4** We could also define random force fields as in Proposition 3.3 for interaction potentials \( \phi \) satisfying \( \phi(x) \sim \frac{A}{|x|^s} \) as \( |x| \to \infty \) for \( \frac{1}{2} < s < 1 \), as it was made in [38] for stationary particle distributions. The kinetic equations that can arise for this class of potentials for interacting particle systems will be shortly discussed in Subsection 8.1.

**Remark 3.5** We can define also singular potentials \( \phi \) taking only the values 0 and \( \infty \) in the case of hard-sphere interactions with simple modifications of the previous arguments.

**Remark 3.6** Notice that if we assume additional regularity for \( \phi \), say \( \phi \in C^k \) we obtain a similar regularity for the random force field \( F \).

We now introduce some classes of random force fields which can be obtained as in Proposition 3.3 by means of suitable choices of potentials \( \phi = \phi_\varepsilon \) with \( \varepsilon \to 0 \). The resulting random force fields will be of two different types which will be denoted as Boltzmann and Landau random force fields respectively.
Boltzmann random force fields. We will say that a family of random force fields defined by means of (3.6) or (3.7) in the sense of Proposition 3.3 is a family of Boltzmann random force fields if the functions $\phi_{\varepsilon}$ have the form:

$$
\phi_{\varepsilon}(x) = \Phi \left( \frac{|x|}{\varepsilon} \right), \quad x \in \mathbb{R}^3 \setminus \{0\}, \quad \varepsilon > 0
$$

(3.21)

where $\Phi(s)$ is compactly supported.

The force fields with the form (3.21) will yield a kinetic limit as $\varepsilon \to 0$. Their key feature is that they yield particle deflections of order one if the interacting particles approach at distances of order $\varepsilon$.

Remark 3.7 It would be possible to make less restrictive assumptions on $\Phi$, for instance that $\Phi$ decreasing exponentially or $\Phi(s) \sim \frac{C}{s^a}$ as $s \to \infty$ with $a > 1$ (cf. [38]). (The critical exponents depend on the space dimension).

Landau random force fields. The families of Landau random force fields are characterized by the fact that the interactions generated by each individual particle are small and tend to zero as $\varepsilon \to 0$. We will say that a family of random force fields with anyone of the forms (3.6) or (3.7) is a family of Landau random force fields if the function $\phi_{\varepsilon}$ has the form:

$$
\phi_{\varepsilon}(x) = \varepsilon \Phi \left( \frac{|x|}{L_{\varepsilon}} \right), \quad x \in \mathbb{R}^3, \quad \varepsilon > 0
$$

(3.22)

where $\Phi \in C^2(\mathbb{R}^3)$. We can choose the characteristic length $L_{\varepsilon}$ in one of the two possible ways. Either:

$$
L_{\varepsilon} \gg 1
$$

(3.23)

or

$$
L_{\varepsilon} \sim 1 \quad \text{or} \quad L_{\varepsilon} \ll 1
$$

(3.24)

In order to be able to define the random force fields $F_{\varepsilon}$ by means of (3.6) or (3.7) (or in the more precise forms (3.15), (3.16) in Proposition 3.3) we need to assume a sufficiently fast decay of $\Phi(s)$ as $s \to \infty$ or to impose suitable electroneutrality conditions if, say (3.12) holds.

We have seen in [38] how to obtain a kinetic limit for tagged particles moving in stationary random force fields. Using analogous arguments it is possible to derive kinetic equations describing the evolution of a tagged particle moving in time dependent random force fields with the form (3.6) or (3.7) and (3.22) in the limit $\varepsilon \to 0$. Some additional assumptions on $L_{\varepsilon}$ are also required, in order to obtain uncorrelated velocity deflections over distances of the order of the mean free path. The case (3.24) corresponds to the so-called grazing collisions limit. In this case, although the limit kinetic equation is a Landau equation, we can interpret the dynamics as a sequence of weak, Boltzmann-like, binary collisions. In the case in which $L_{\varepsilon} \gg 1$ the resulting limit kinetic equation is a linear Landau equation.

3.3 Scatterer distributions in Rayleigh gases.

In the case of the dynamics (1.4) the evolution of the tagged particle cannot be described only by means of the action of the random force field generated by the scatterers. This can
be seen most clearly is the case in which the mass of the tagged particle is the same as the mass of the scatterers (or comparable) and the interaction between the tagged particle and the scatterers are as in the case of the Boltzmann random force fields (cf. (3.21)). It will be seen in Subsection 5.2 that also in the case of Landau interaction potentials (cf. 3.22) the dynamics of the tagged particle depends on whether the scatterers are affected by the tagged particle or not. This may be seen by studying the binary interaction (cf. Subsection 5.6).

The information that we need to keep about the system of scatterers in order to compute the evolution of a tagged particle which evolves according to (1.4) is the whole configuration of positions and velocities of the scatterers at any time (i.e. \( \{(X_k(t), V_k(t))\}_{k \in \mathbb{N}} \)) as well as their charges, if there are different types of particles, and the form of the potentials yielding the interactions. If the tagged particle and the scatterers have a different mass, the ratio between these masses would also play a role in the determination of the dynamics. We could also consider different types of scatterers having different masses. The distribution of velocities of the different scatterers then plays also a role in the dynamics of the tagged particle \((X(t), V(t))\).

4 Computation of the kinetic timescale and limit equation

As indicated in the Introduction the paper [38] describes how to obtain kinetic limits for the dynamics of tagged particles moving in a field of fixed scatterers. There are basically two different types of kinetic limits, namely Boltzmann and Landau. It has been seen in [38] that it is possible to define two time scales \( T_{BG} \) and \( T_L \) which are the characteristic times for the velocity of the tagged particle to experience a deflection of order one due to a binary collision and to the accumulation of many small random deflections due to weak interactions with many particles respectively.

In the case of more complicated particle dynamics, like (1.1) or (1.4) we can also define kinetic time scales. We consider first the case of a tagged particle moving in a Rayleigh gas, i.e. the dynamics of the tagged particle is described by means of (1.4). In the case of systems described by means of (1.1) we will compute the kinetic time scales approximating the dynamics of a tagged particle by means of (1.2). This approximation will be obtained in Section 5.2. The kinetic time scale associated to equations with the form (1.2) can be readily obtained and this will provide a method to obtain the kinetic time scale for systems described by (1.1). The kinetic time scale for weakly interacting particle systems will be computed in Section 5.2.

In the case of tagged particles moving in a Rayleigh gas (cf. (1.4)) we can define a characteristic kinetic time and also the mean free path for a particle moving in a Rayleigh gas as follows. Suppose that \((X,V)\) is the position and velocity of a tagged particle whose dynamics is given by (1.4) where the initial velocity distribution of the scatterers is given by the measure \(g\) in the case of particles of a single type, or a set of measures \(\{g_\ell\}_{\ell=1}^L\) in the case of particles of different types as discussed in Subsection 3.3. We define the mean free path associated to the family of potentials \(\phi_{\varepsilon}\) and to the set of particle distributions as the value \(T_{\varepsilon}\) such that:

\[
\mathbb{E}\left[|V_{\varepsilon}(T_{\varepsilon}) - V_{\varepsilon}(0)|^2\right] = \frac{1}{4} |V_{\varepsilon}(0)|^2
\]  

(4.1)
where we assume that $|V_0(0)|$ is the characteristic speed of the tagged particle.

It has been seen in [38] how to estimate $T_\varepsilon$ in the case of static scatterers, splitting the random force field in the sum of a Boltzmann part and a Landau part. We can then define a Boltzmann-Grad limit time scale $T_{BG}$ by means of $(4.1)$ for the Rayleigh gas associated to the Boltzmann part of the potential and the Landau time scale $T_L$ which is computed applying $(4.1)$ to the Landau part of the potential. In most of the interaction potentials considered in [38] one of the two time scales is much larger than the other as $\varepsilon \to 0$. Then, given that in the kinetic limit the deflections are additive we obtain:

$$T_\varepsilon \sim \min\{T_{BG}, T_L\} \text{ as } \varepsilon \to 0.$$ 

A similar decomposition in Boltzmann part and Landau part can be made in the case of arbitrary Rayleigh gases. We can use then the same approach to estimate $T_\varepsilon$ as in [38] in the kinetic limit $\varepsilon \to 0$. We then recall the decomposition of the interaction potentials obtained in [38].

Suppose that $\phi_\varepsilon$ is as in $(1.4)$. The potentials for which there is a Boltzmann part are characterized by the existence of a collision length $\lambda_\varepsilon$ satisfying $\lim_{\varepsilon \to 0} \lambda_\varepsilon = 0$ and such that:

$$\lim_{\varepsilon \to 0} \phi_\varepsilon (\lambda_\varepsilon y) = \Psi (y)$$

uniformly in compact sets of $\mathbb{R}^3 \setminus \{0\}$, where $\Psi (y) \neq 0$. The distance $\lambda_\varepsilon$ is the characteristic distance at which the tagged particle experiences deflections of order one due to the interaction with one of the scatterers.

We then split $\phi_\varepsilon$ as:

$$\phi_{B,\varepsilon} (x) = \phi_\varepsilon (x) \eta \left( \frac{|x|}{M \lambda_\varepsilon} \right), \quad \phi_{L,\varepsilon} (x) = \phi_\varepsilon (x) \left[ 1 - \eta \left( \frac{|x|}{M \lambda_\varepsilon} \right) \right] \quad (4.2)$$

where $\eta \in C^\infty (\mathbb{R}^3)$ is a radially symmetric cutoff function satisfying $\eta (y) = 1$ if $|y| \leq 1$, $\eta (y) = 0$ if $|y| \geq 2$, $0 \leq \eta \leq 1$. We assume that $M$ is a large number, which eventually might be sent to infinity at the end of the argument. It would be also possible to take $M = M_\varepsilon$ with $M_\varepsilon \to \infty$ as $\varepsilon \to 0$ at a sufficiently slow rate to control the transition region between Boltzmann and Landau collisions. In $(4.2)$, $\phi_{B,\varepsilon}$ stands for Boltzmann and $\phi_{L,\varepsilon}$ for Landau.

We now compute the characteristic time scales $T_{BG}$ and $T_L$ which yield the expected time to have velocity deflections comparable to the tagged particle velocity itself for the Boltzmann part of the interaction $\phi_{B,\varepsilon}$ and the Landau part of the interaction $\phi_{L,\varepsilon}$ for a tagged particle moving in a Rayleigh gas.

### 4.1 Computation of $T_{BG}$ for a tagged particle in a Lorentz gas with moving scatterers.

We first compute the characteristic time scale to obtain relevant deflections for a tagged particle moving in a Rayleigh gas generated by potentials with the form $\phi_{B,\varepsilon}$ in $(1.2)$. To compute the deflection time using the definition $(4.1)$ would require some involved computations, since it would require to compute the contributions to the deflection of having multiple collisions,
something that would require some tedious computations. Therefore, instead of using the definition (4.1) we will compute a simpler quantity, namely the length of a tube in which the probability of finding one particle is of order one.

We will restrict ourselves to the case in which the velocity of the tagged particle experiences deflections of order one at distances of order $\varepsilon$ of one scatterer. Suppose that a tagged particle moves at speed $V$. We assume that the initial positions of the scatterers $x_j$, as well as their velocities $v_j$, denoted as $\omega = \{(x_k, v_k)\}_{k \in J}$ are determined by means of the Poisson measure defined in (3.1). Suppose without loss of generality that the initial position of the tagged particle is the origin. Since its initial velocity is $V$ we have that the position of the tagged particle at time $t$ is $Vt$. We then define $T_{BG}$ in the following way:

$$T_{BG} = \inf \left\{ \tau : P \left( \exists k : \min_{0 \leq s \leq \tau} |Vs - (x_k + v_k s)| \leq \varepsilon \right) \geq \frac{1}{2} \right\}.$$  

To compute $T_{BG}$, we compute the probability above for each $\tau > 0$. To this end, we decompose the velocity space $\mathbb{R}^3$ into disjoint cubes $Q_k$:

$$\mathbb{R}^3 = \bigcup_{\ell \in \mathbb{N}} Q_\ell, \quad Q_\ell = \{ v \in \mathbb{R}^3 : x = c_\ell + p, p \in [0, \varepsilon/\tau)^3 \},$$

for some appropriately chosen centers $c_\ell \in \mathbb{R}^3$. Then for $\tau > 0$ we have

$$P \left( \exists k : \min_{0 \leq s \leq \tau} |Vs - (x_k + v_k s)| \leq \varepsilon \right) = \sum_{\ell \in \mathbb{N}} P \left( \exists k : \min_{0 \leq s \leq \tau} |Vs - (x_k + v_k s)| \leq \varepsilon, v_k \in Q_\ell \right) \sim \sum_{\ell \in \mathbb{N}} P \left( \exists k, s \in [0, \tau] : |x_k - s(V - c_\ell)| \leq \varepsilon, v_k \in Q_\ell \right) \sim \tau \lambda_\varepsilon^2,$$

where $\lambda_\varepsilon$ is the collision length. Therefore, the Boltzmann-Grad timescale $T_{BG}$ is given by

$$T_{BG} \sim \lambda_\varepsilon^{-2},$$

as in the Lorentz gas with fixed obstacles, see [38].

### 4.2 Computation of $T_L$ for a tagged particle in a Lorentz gas with moving scatterers.

We now compute the characteristic time in which the deflections of the velocity are of order one for one tagged particle moving in a time dependent Landau random force field with the form (3.6) (or (3.7)) generated by potentials with the form (3.22) with $\Phi (s) \sim \frac{1}{s}$ as $s \rightarrow \infty$. To this end we compute the variance of the velocity deflections experienced by a particle moving in a straight line during the time $T$. This deflection is given by:

$$D_\varepsilon (T; \omega) = \int_0^T F_\varepsilon (Vt, t; \omega) \, dt$$

where we assume that the initial position of the tagged particle is $X = 0$ and the velocity is $V$. We have $\mathbb{E} [D_\varepsilon (T)] = 0$. We compute:

$$\mathbb{E} [D_\varepsilon (T) \otimes D_\varepsilon (T)] = \int_0^T dt_1 \int_0^T dt_2 \mathbb{E} [F_\varepsilon (Vt_1, t_1; \omega) \otimes F_\varepsilon (Vt_2, t_2; \omega)] \quad (4.3)$$

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In order to indicate how to compute (4.3) we consider the case of random force fields (3.6) with \( \phi = \phi_\varepsilon \) given by:
\[
\phi_\varepsilon (x) = \varepsilon \Phi \left( \frac{x}{L_\varepsilon} \right)
\]
where \( \Phi (\xi) \) decreases sufficiently fast as \( |\xi| \to \infty \), for instance as in (3.10). In order to compute (4.3) we approximate it assuming that we have \( N \) particles independently distributed in \( B_{R_N} (0) \times \mathbb{R}^3 \subset \mathbb{R}^6 \) with \( N = \frac{4\pi (R_N)^3}{3} \) with the probability density \( \frac{1}{|B_{R_N} (0)|} g (v) \). (It would be also possible to use a macrocanonical distribution, but the result is equivalent in the limit \( N \to \infty \)). We will denote as \( D_\varepsilon^N (T) \) the corresponding deflections and:
\[
F (x, \tau; \omega; g) = - \sum_{k=1}^N \nabla \phi_\varepsilon (x - x_k - v_k \tau)
\]

Then:
\[
\mathbb{E}[D_\varepsilon^N (T) \otimes D_\varepsilon^N (T)] = \sum_{k=1}^N \sum_{j=1}^N \int_0^T \int_0^T dt_1 \int_0^T dt_2 \mathbb{E} \left[ \nabla \phi_\varepsilon (V t_1 - x_k - v_k t_1) \otimes \nabla \phi_\varepsilon (V t_2 - x_j - v_j t_2) \right]
\]
\[
= \sum_{k=1}^N \int_0^T dt_1 \int_0^T dt_2 \mathbb{E} \left[ \nabla \phi_\varepsilon (V t_1 - x_k - v_k t_1) \otimes \nabla \phi_\varepsilon (V t_2 - x_k - v_k t_2) \right]
\]
\[
= \frac{N}{|B_{R_N}|} \int_0^T dt_1 \int_0^T dt_2 \int_{B_{R_N}} dy \int_{\mathbb{R}^3} dv g (v) \nabla \phi_\varepsilon ((V - v) t_1 - y) \otimes \nabla \phi_\varepsilon ((V - v) t_2 - y)
\]
\[
= \int_0^T dt_1 \int_0^T dt_2 \int_{B_{R_N} (0)} dy \int_{\mathbb{R}^3} dv g (v) \nabla \phi_\varepsilon ((V - v) t_1 - y) \otimes \nabla \phi_\varepsilon ((V - v) t_2 - y)
\]

Taking the limit \( N \to \infty \) we obtain:
\[
\mathbb{E}[D_\varepsilon (T) \otimes D_\varepsilon (T)] = \int_0^T dt_1 \int_0^T dt_2 \int_{\mathbb{R}^3} dv g (v) \nabla \phi_\varepsilon ((V - v) t_1 - y) \otimes \nabla \phi_\varepsilon ((V - v) t_2 - y)
\]
\[
= \varepsilon^2 \int_0^T dt_1 \int_0^T dt_2 \int_{\mathbb{R}^3} dv g (v) \nabla \phi_\varepsilon ((V - v) t_1 - y) \otimes \nabla \phi_\varepsilon \left( \frac{y - (V - v) (t_1 - t_2)}{L_\varepsilon} \right) \otimes \nabla \phi_\varepsilon \left( \frac{y}{L_\varepsilon} \right)
\]
\[
= \varepsilon^2 (L_\varepsilon)^2 \int_0^T dt_1 \int_{-\frac{T-t_1}{L_\varepsilon}}^{\frac{T-t_1}{L_\varepsilon}} K (V; \tau) d\tau
\]

where:
\[
K (V; \tau) = \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} dv g (v) \nabla \phi_\varepsilon (\xi + (V - v) \tau) \otimes \nabla \phi_\varepsilon (\xi)
\]

We will now assume that \( T \gg L_\varepsilon \). Then, for most of the values of \( t_1 \) in the integral we have that \( \frac{T-t_1}{L_\varepsilon} \gg 1 \) and \( \frac{t_1}{L_\varepsilon} \gg 1 \). On the other hand, the function \( K (V; \tau) \) is integrable in \( \tau \) under the assumption (3.10). There are some boundary effects in the integrals if \( t_1 \) or \( (T-t_1) \) are of order \( L_\varepsilon \). We then obtain the approximation:
\[
\mathbb{E} \left[ D_\varepsilon (T) \otimes D_\varepsilon (T) \right] \sim \varepsilon^2 (L_\varepsilon)^2 T \int_{-\infty}^{\infty} K (V; \tau) d\tau
\]
We then obtain that the characteristic time scale is:

\[ T = \frac{1}{\varepsilon^2 (L_\varepsilon)^2}, \]

and the diffusion coefficient for the tagged particle is the matrix:

\[ d(V) = \int_{-\infty}^{\infty} K(V; \tau) \, d\tau \]

which is diagonal if \( g \) is isotropic. Notice that in order to obtain a dynamics without correlations (and also in order to have self-consistency of the previous argument), we need to have \( T \gg L_\varepsilon \), i.e.:

\[ \varepsilon^2 (L_\varepsilon)^3 \ll 1. \]

It is interesting to compute the characteristic time in the case of potentials decreasing as Coulombian potentials. In this case we need to take at least two types of charges in order to have electroneutrality. We consider random force fields that are the limit as \( N \to \infty \) of fields with the form:

\[ F(x, \tau; \omega; g) = -\varepsilon \sum_{k=1}^{N} \nabla \Phi (x - x_k - v_k \tau) + \varepsilon \sum_{k=1}^{N} \nabla \Phi (x - y_k - w_k \tau) \]

where the particles \( \{(x_k, v_k)\} \) and \( \{(y_k, w_k)\} \) are chosen independently and \( \Phi \) is a smooth function satisfying:

\[ \Phi(x) \sim \frac{1}{|x|} \text{ as } |x| \to \infty \text{ and } \nabla \Phi(x) \sim -\frac{x}{|x|^3} \text{ as } |x| \to \infty. \]

We can then compute the variance of the deflections arguing as above. We then obtain:

\[ \mathbb{E} [D_\varepsilon (T) \otimes D_\varepsilon (T)] = 2\varepsilon^2 \int_{0}^{T} dt_1 \int_{0}^{T} dt_2 \int_{(\mathbb{R}^3)^2} d(y, v) g(v) \nabla \Phi (y - (V - v) (t_1 - t_2)) \otimes \nabla \Phi (y) \]

\[ = 2\varepsilon^2 \int_{0}^{T} dt_1 \int_{-t_1}^{T-t_1} K(V; s) \, ds, \]

where:

\[ K(V; s) = \int_{(\mathbb{R}^3)^2} d(y, v) g(v) \nabla \Phi (y + (v - V) s) \otimes \nabla \Phi (y). \]

Notice that the function \( K(V; s) \) is well defined for each \( s \in \mathbb{R} \). We can compute the asymptotics of \( K(V; s) \) as \( |s| \to \infty \). We can assume without loss of generality that \( s > 0 \) since the case \( s < 0 \) would be similar. We rescale \( y = s \xi \), then:

\[ K(V; s) = s^3 \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} dvg(v) \nabla_y \Phi (s (\xi + (v - V))) \otimes \nabla \Phi (s \xi). \]

Taking the limit as \( s \to \infty \) and using the asymptotics of \( \nabla \Phi (y) \) we obtain:

\[ K(V; s) \sim \frac{1}{s} \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} dvg(v) \frac{(\xi + (v - V))}{|\xi + (v - V)|^3} \otimes \frac{\xi}{|\xi|^3} \quad (4.4) \]
This integral exists. Indeed, the integrals on $\xi$ are well defined if $v \neq V$. On the other hand, if $v \to V$ we obtain a divergence like $\frac{1}{|v-V|}$. We will assume that $g(v)$ is smooth enough to have integrability of $\int_{\mathbb{R}^3} dv \frac{g(v)}{|v-V|}$. Then the integral is well defined and we obtain the asymptotics:

$$K(V; s) \sim \frac{\Gamma(V, \text{sgn}(s))}{s} \text{ as } |s| \to \infty$$

(4.5)

for some function $\Gamma : \mathbb{R}^3 \times \{-1, +1\} \to M_3(\mathbb{R}^3)$. We then obtain:

$$\mathbb{E} [D_\varepsilon(T) \otimes D_\varepsilon(T)] \sim 2\varepsilon^2 \int_0^T dt \int_{t_1}^{T-t_1} \frac{\Gamma(V, \text{sgn}(s))}{|s|} ds \sim d(V) \varepsilon^2 T \log(T)$$

(4.6)

where $d(V) \in M_3(\mathbb{R}^3)$ is a nonnegative matrix.

This yields the characteristic time scale:

$$T = T_\varepsilon \sim \frac{1}{\varepsilon^2 \log \left(\frac{1}{\varepsilon}\right)}$$

(4.7)

which is exactly the same time scale obtained in [38] for random force fields generated by static distributions of particles (Lorentz gases). We obtain a different type of diffusion coefficient. In particular in the case of Lorentz gases the diffusion takes place only on the sphere of constant velocity. This is not the case for Rayleigh gases.

The logarithmic term $\log \left(\frac{1}{\varepsilon}\right)$ is a well known correction appearing in systems with Coulombian interactions. It is usually termed as Coulombian logarithm (cf. [33]).

We can now compute the evolution of the distribution function $f(v, t)$ yielding the distribution of particle velocities for the tagged particle evolving according to (2.1) with random force fields $F$ as in (3.6), (3.7), (3.22) and a distribution of scatterer velocities $g$. We introduce a new time scale $t$ by means of $\tau = T_\varepsilon t$. Then $f$ satisfies:

$$\partial_t f(v, t) = \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial}{\partial v_j} \left(d_{j,k}(v) \frac{\partial f}{\partial v_j}\right)(v, t)$$

where $d(v) = (d_{j,k}(v))_{j,k=1,2,3}$ is as in (4.6) with the functions $K$ and $\Gamma$ as in (4.4), (4.5).

As in the case of fixed scatterers, for potentials of the form $\frac{1}{|x|^s}$ with $s < 1$ as $|x| \to \infty$, there is no Markovian evolution dynamics for the tagged particle. This is due to strong correlations of the systems that prevail for times comparable to the mean free time. The computation is analogous to the one given for fixed scatterers in [38].

5 Approximation by the dynamics of a tagged particle in a random force field with friction.

In this Section we show how to approximate the dynamics of the systems (1.1), (1.4) by means of an equation of the form (1.2), at least during small macroscopic times in suitable scaling limits. The method can be applied in systems in which the range of the potential is much
larger than the average distance between particles. The rationale behind the method is to decompose the force made by the scatterers on the tagged particle in a friction term, which is the reaction to the force made by the tagged particle on the scatterers and yields their deflection, plus a random force term which is the sum of the forces produced by the scatterers in the tagged particle assuming that they are not affected by it. The friction term is due to the fact that the tagged particle is moving against the mean velocity of the surrounding medium.

We begin approximating the dynamics of a tagged particle given by (1.4) which yields a simpler problem, since the mutual interactions between the scatterers do not need to be taken into account. Later, we will consider the full interacting particle system (1.2). We will restrict our analysis to three types of potentials, namely weak potentials with a finite range much larger than the distance between particles, potentials behaving for large distances as Coulomb potentials and finally the case of so-called "grazing collisions" in which the interactions between particles are very weak and have a range smaller or similar to the particle distance.

5.1 Approximating the dynamics of a particle in a Rayleigh gas as a tagged particle in a random force field plus friction.

In this subsection we study the dynamics of a tagged particle in a Rayleigh gas, which we show can be approximated by the equation (cf. (1.2)). The main novelty of these systems in comparisons to the Lorentz gases is the onset of the friction term $\Lambda_p$. The physical reason for the onset of this term is the reaction force of the scatterers on the tagged particle, which is due to the fact that the scatterers are affected by the tagged particle.

5.1.1 Particle interactions with finite range much larger than the particle distance.

Suppose that the position and velocity of a tagged particle $(X, V)$ evolves according to (1.4) where the interaction potential $\phi_\varepsilon (x)$ is as in (3.22), (3.23). We will assume that $\Phi (s)$ decreases fast enough as $s \to \infty$, say exponentially, although in space dimension three, a decay like $\frac{1}{s^a}$ with $a > 1$ would be enough. As a first step we approximate the distribution of scatterers as the sum of a constant density plus some gaussian fluctuations in some suitable topology. To this end we introduce a new variable $y_k = \frac{Y_k}{L_\varepsilon}$ which will be useful to describe the system on a scale where this Gaussian approximation is valid, but that is smaller than the mean free path. In order to keep the particle velocities of order one we introduce a new time scale by means of $\tilde{t} = \frac{t}{L_\varepsilon}$. We write also $\xi = \frac{X}{L_\varepsilon}$. Then (1.4) becomes:

\begin{align}
\frac{d\xi}{dt} &= V, \quad \frac{dV}{dt} = -\varepsilon \sum_{j \in S} \nabla_\xi \Phi (\xi - y_j) \\
\frac{dy_k}{dt} &= W_k, \quad \frac{dW_k}{dt} = -\varepsilon \nabla_y \Phi (y_k - \xi), \quad k \in S. \tag{5.1}
\end{align}

The goal is to approximate the dynamics of the scatterers by means of a continuous density. To this end we introduce the following particle density in the phase space:

\begin{equation}
f_\varepsilon (y, w, \tilde{t}) = \frac{1}{(L_\varepsilon)^3} \sum_k \delta (y - y_k) \delta (w - W_k). \tag{5.2}
\end{equation}
We can then rewrite the first two equations of (5.1) as:

$$\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^d} \nabla \varphi (\xi - \eta) \rho_\varepsilon (\eta, \tilde{t}) \, d\eta$$

(5.3)

where $\rho_\varepsilon (y, \tilde{t}) = \rho [f_\varepsilon (\cdot, \tilde{t})] (y)$ is the spatial density introduced in (1.14). On the other hand, the second set of equations of (5.1) implies that:

$$\partial_t f_\varepsilon (y, w, \tilde{t}) + w \cdot \nabla_y f_\varepsilon (y, w, \tilde{t}) - \varepsilon \nabla_y \varphi (y - \xi) \cdot \nabla_w f_\varepsilon (y, w, \tilde{t}) = 0.$$  

(5.4)

We have then reformulated (5.1) as (5.3), (5.4).

We can now take formally the limit $\varepsilon \rightarrow 0$. To this end, notice that $f_\varepsilon (y, w, 0)$ is of order one and it converges in the weak topology to $g (w)$. In order to obtain the evolution for different rescalings of $L_\varepsilon$ with $\varepsilon$ we compute the asymptotic behaviour in law of $f_\varepsilon (y, w, 0)$ as $\varepsilon \rightarrow 0$. The following Gaussian approximation (5.5) will be used repeatedly in the following:

$$\mathbb{E} \left[ (f_\varepsilon (y_a, w_a, 0) - g (w_a)) (f_\varepsilon (y_b, w_b, 0) - g (w_b)) \right] = \frac{g (w_a)}{(L_\varepsilon)^2} \delta (y_a - y_b) \delta (w_a - w_b).$$

(5.5)

This approximation will be justified in Appendix A where it will be seen how to derive it from the empirical densities associated to particle distributions given by the Poisson measure.

Assuming (5.5), it is natural to look for solutions of (5.4) with the form:

$$f_\varepsilon (y, w, \tilde{t}) = g (w) + \frac{1}{(L_\varepsilon)^2} \zeta_\varepsilon (y, w, \tilde{t}).$$

(5.6)

Then, using the fact that the contribution to the integral $\int_{\mathbb{R}^d} \nabla \varphi (\xi - \eta) \rho_\varepsilon (\eta, \tilde{t}) \, d\eta$ due to the term $g (w)$ vanishes, we obtain that $\zeta_\varepsilon (y, w, \tilde{t})$ solves the following problem:

$$\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon (L_\varepsilon)^2 \int_{\mathbb{R}^d} \nabla \varphi (\xi - \eta) \tilde{\rho}_\varepsilon (\eta, \tilde{t}) \, d\eta,$$

(5.7)

$$\partial_t \zeta_\varepsilon (y, w, \tilde{t}) + w \cdot \nabla_y \zeta_\varepsilon (y, w, \tilde{t}) - \varepsilon (L_\varepsilon)^2 \nabla_y \varphi (y - \xi) \cdot \nabla_w \left( g (w) + \frac{\zeta_\varepsilon (y, w, \tilde{t})}{(L_\varepsilon)^2} \right) = 0,$$

(5.8)

where $\tilde{\rho}_\varepsilon (y, \tilde{t}) = \rho [\zeta_\varepsilon (\cdot, \tilde{t})] (y)$ is the associated spatial density (cf. (1.17)).

Notice that we assume that the potential $\varphi$ decreases sufficiently fast to guarantee that the integral $\int_{\mathbb{R}^d} \nabla \varphi (\xi - \eta) \tilde{\rho}_\varepsilon (\eta, \tilde{t}) \, d\eta$ is well defined for a random particle distribution $\zeta_\varepsilon (y, w, \tilde{t})$ given by (5.5), (5.6). In the case of potentials $\varphi$ decreasing as some power laws it is possible to give a meaning to this integral by means of a limit procedure.

Then, using the fact that $L_\varepsilon \rightarrow \infty$ we obtain the following limit problem formally. We write

$$\theta_\varepsilon = \varepsilon (L_\varepsilon)^2.$$  

(5.9)

Given that the range of the interaction potentials is of order $|y| \approx 1$ we need to have $\theta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ in order to obtain a kinetic limit. On the other hand, since $L_\varepsilon \rightarrow \infty$, making the Gaussian approximation $\zeta_\varepsilon \rightarrow \zeta$, we can approximate the problem (5.7), (5.8) as:

$$\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\theta_\varepsilon \int_{\mathbb{R}^d} \nabla \varphi (\xi - \eta) \tilde{\rho} (\eta, \tilde{t}) \, d\eta.$$  

(5.10)
where \( \tilde{\rho}(y, \tilde{t}) = \rho(\tilde{\xi}(\cdot, \tilde{t}))(y) \) and

\[
\left( \partial_{\tilde{t}} + w \cdot \nabla \right) \zeta(y, w, \tilde{t}) - \theta_{2} \nabla g \Phi(y - \xi) \cdot \nabla_{w} g(w) = 0, \quad \zeta(y, w, 0) = N(y, w). \tag{5.11}
\]

Using (5.8) we obtain:

\[
\mathbb{E}[N(y, w)] = 0, \quad \mathbb{E}[(N(y_{a}, w_{a})) N(y_{b}, w_{b})] = g(w_{a}) \delta(y_{a} - y_{b}) \delta(w_{a} - w_{b}). \tag{5.12}
\]

To prove rigorously that the term \( \frac{1}{(L_{s})^{2}} \zeta \) can be neglected in (5.8) is a challenging mathematical problem, that will not be considered in this paper.

Due to the linearity of (5.11)-(5.12) we can write \( \zeta(y, w, \tilde{t}) \) as \( \zeta_{1}(y, w, \tilde{t}) + \zeta_{2}(y, w, \tilde{t}) \), where:

\[
\partial_{\tilde{t}} \zeta_{1}(y, w, \tilde{t}) + w \cdot \nabla \zeta_{1}(y, w, \tilde{t}) = 0, \quad \zeta_{1}(y, w, 0) = N(y, w) \tag{5.13}
\]

\[
\partial_{\tilde{t}} \zeta_{2}(y, w, \tilde{t}) + w \cdot \nabla \zeta_{2}(y, w, \tilde{t}) - \theta_{2} \nabla \Phi(y - \xi) \cdot \nabla_{w} g(w) = 0, \quad \zeta_{2}(y, w, 0) = 0. \tag{5.14}
\]

Equation (5.13) can be solved explicitly using characteristics:

\[
\zeta_{1}(y, w, \tilde{t}) = N(y - w\tilde{t}, w). \tag{5.15}
\]

The stochastic process on the right-hand side of (5.13) has non trivial correlations in time. On the other hand, it is easily seen using (5.12) that for any \( \tilde{t} \in \mathbb{R} \) the stochastic process \( \zeta_{1}(y, w, \tilde{t}) \) is the same in law as \( N(y, w) \).

We can approximate the solution of (5.14) as follows. As long as the time \( \tilde{t} \) is smaller than the mean free path we can approximate \( V \) as a constant and then \( \xi(\tilde{t}) = \xi(0) + V\tilde{t} \) (cf. (5.10)). Assuming that \( \xi(0) = 0 \) without loss of generality, we can approximate (5.14) as:

\[
\partial_{\tilde{t}} \zeta_{2}(y, w, \tilde{t}) + w \cdot \nabla \zeta_{2}(y, w, \tilde{t}) - \theta_{2} \nabla \Phi(y - V\tilde{t}) \cdot \nabla_{w} g(w) = 0, \quad \zeta_{2}(y, w, 0) = 0.
\]

The solution of this equation is given by:

\[
\zeta_{2}(y, w, \tilde{t}) = \theta_{2} \nabla_{w} g(w) \cdot \int_{0}^{\tilde{t}} \nabla \Phi(y - w(\tilde{t} - s) - Vs) \, ds. \tag{5.16}
\]

This function takes a simpler form in a coordinate system moving with speed \( V \), i.e. \( \Xi(\tilde{y}, \tilde{t}) = \zeta_{2}(y + V\tilde{t}, w, \tilde{t}) \). Indeed, in this coordinate system, the asymptotics can be easily computed as:

\[
\Xi(\tilde{y}, \tilde{t}) \rightarrow \Xi_{\infty}(\tilde{y}, w) = \theta_{2} \nabla_{w} g(w) \cdot \int_{-\infty}^{0} \nabla \Phi(\tilde{y} + (w - V) s) \, ds \quad \text{as} \quad \tilde{t} \rightarrow \infty. \tag{5.17}
\]

Indeed, as it will be seen below the characteristic time scale for the change of \( V \) is \( (\theta_{2})^{-2} \) and therefore the effect of the difference \( (\xi - V\tilde{t}) \) in the term containing \( \tilde{\rho}_{2} \) would be a lower order term during these time scales.

The limit function \( \Xi_{\infty} \) is singular for \( w = V \) for general values of \( \tilde{y} \). Notice that the stabilization of \( \Xi \) to \( \Xi_{\infty} \) takes place in times \( \tilde{t} \) of order one, i.e. in the microscopic time scale. The stabilization of the particle density around the tagged particle to a stationary distribution (in a coordinate system moving with the tagged particle) in the microscopic time scale, which is much shorter than the macroscopic time scale, was seemingly noticed first by Bogoliubov. Notice that in this regime the function \( \zeta_{2} \) approaches asymptotically a travelling wave solution.
which moves at the same speed as the tagged particle. This solution is sometimes referred as cloud. Physically, the cloud represents the distortion in the distribution of scatterers due to the presence of the tagged particle.

We can now compute the evolution of the tagged particle using (5.11). Using the decomposition into \( \zeta_1, \zeta_2 \) (cf. (5.13)-(5.14)) we can decompose the force acting on the tagged particle in two pieces. The one associated to \( \zeta_1 \) is a time dependent random force field which is not affected by the tagged particle. The term associated to \( \zeta_2 \) yields a deterministic term which depends only on the velocity of the tagged particle. Let \( \tilde{\rho}_{1,2}(y, \tilde{t}) = \rho(\zeta_{1,2}(\cdot, \tilde{t}))(y) \), then \( \tilde{\rho}_1(y, \tilde{t}) \) is a Gaussian random variable which can be characterized by means of the following set of expectations and correlations (cf. Appendix A):

\[
\mathbb{E} \left[ \tilde{\rho}_1 (y, \tilde{t}) \right] = 0
\]

\[
\mathbb{E} \left[ \tilde{\rho}_1 (y_1, \tilde{t}_1) \tilde{\rho}_1 (y_2, \tilde{t}_2) \right] = \frac{1}{(\tilde{t}_1 - \tilde{t}_2)^3} g \left( \frac{y_1 - y_2}{\tilde{t}_1 - \tilde{t}_2} \right), \quad \tilde{t}_1 > \tilde{t}_2 \tag{5.18}
\]

We can then approximate in the second equation in (5.10) the terms containing \( \tilde{\rho}_2 \) using that \( \xi \approx V \tilde{t} \) for \( \tau \) smaller than the mean free path. However, it is relevant to notice that the approximation \( \xi \approx V \tilde{t} \) cannot be made in the term containing the random density \( \tilde{\rho}_1 \) unless \( \tilde{t} \ll (\theta_\varepsilon)^{-2} \) because for \( \tilde{t} \gtrsim (\theta_\varepsilon)^{-2} \) the resulting correction would be of order \( \theta_\varepsilon \frac{1}{|\xi - V \tilde{t}|} \) which would be of order \( (\theta_\varepsilon)^2 \) for times \( \tilde{t} \) of order one and therefore it would give a contribution of order one in times of order \( (\theta_\varepsilon)^{-2} \).

We then have the following approximation:

\[
\frac{dV}{dt} \approx -\theta_\varepsilon \int_{\mathbb{R}^3} \nabla_\xi \Phi (\xi - \eta) \tilde{\rho}_1 (\eta, \tilde{t}) \, d\eta + \theta_\varepsilon \int_{\mathbb{R}^3} \nabla_\xi \Phi (\eta) \tilde{\rho}_2 (\eta + V \tilde{t}, \tilde{t}) \, d\eta.
\]

For the second integral, we use the approximation (5.17):

\[
\theta_\varepsilon \int_{\mathbb{R}^3} \nabla_\xi \Phi (\eta) \tilde{\rho}_2 (\eta + V \tilde{t}, \tilde{t}) \, d\eta \sim - (\theta_\varepsilon)^2 \Lambda_g (V), \quad \text{with}
\]

\[
\Lambda_g (V) = -\int_{\mathbb{R}^3} d\eta \int_{\mathbb{R}^3} dw \nabla_\xi \Phi (\eta) \left[ \nabla_w g(w) \cdot \int_{-\infty}^{0} \nabla_\eta \Phi (\eta + (w - V) s) \, ds \right]. \tag{5.19}
\]

Additionally, define the random force field given by:

\[
F_g (\xi, \tilde{t}) = -\int_{\mathbb{R}^3} \nabla_\xi \Phi (\xi - \eta) \tilde{\rho}_1 (\eta, \tilde{t}) \, d\eta = \int_{\mathbb{R}^3} \nabla_\eta \Phi (\eta) \tilde{\rho}_1 (\eta + \xi, \tilde{t}) \, d\eta. \tag{5.20}
\]

We can then write the evolution equation of \( V \) as:

\[
\frac{dV}{dt} = \theta_\varepsilon F_g (\xi, \tilde{t}) - (\theta_\varepsilon)^2 \Lambda_g (V), \quad \frac{d\xi}{dt} = V. \tag{5.21}
\]

The random force field \( F_g (\xi) \) is a Gaussian field with zero average. The correlation between the values of the field at different points can be computed using (5.18) and (5.20). We have:

\[
\mathbb{E} \left[ F_g (\xi_1, \tilde{t}_1) \otimes F_g (\xi_2, \tilde{t}_2) \right] = \int_{\mathbb{R}^3} d\eta_1 \int_{\mathbb{R}^3} d\eta_2 \frac{\nabla \Phi (\eta_1) \otimes \nabla \Phi (\eta_2)}{(\tilde{t}_1 - \tilde{t}_2)^3} g \left( \frac{\eta_1 + \xi_1 - \eta_2 - \xi_2}{\tilde{t}_1 - \tilde{t}_2} \right).
\]
Notice that (5.21) yields an approximation for the dynamics of the tagged particle by means of the dynamics of a particle which moves with a friction coefficient which is due to the fact that the tagged particle moves against the medium. A time dependent random gaussian field acts also in the particle.

Actually, in order to write a differential equation for the probability density which describes the position and velocity of the tagged particle, it is convenient to define a random variable \( B_g(t; V) = F_g(V, t) \) where the times \( t \) are much shorter than the mean free time for deflections. Then:

\[
\mathbb{E} \left[ B_g(t; V) \right] = 0
\]

\[
\mathbb{E} \left[ B_g(t_1; V) B_g(t_2; V) \right] = \int_{\mathbb{R}^3} d(\eta_1, \eta_2) \frac{\nabla_\eta \Phi(\eta_1) \otimes \nabla_\eta \Phi(\eta_2)}{(t_1 - t_2)^3} g \left( V + \frac{\eta_1 - \eta_2}{t_1 - t_2} \right), \quad \tilde{t}_1 > \tilde{t}_2.
\]

Then, (5.21) can be rewritten, for times in which the change of \( V \) is small, as:

\[
\frac{dV}{dt} = \frac{1}{\theta_c} B_g(\tilde{t}; V) - (\theta_c)^2 \Lambda_g(V), \quad \frac{dx}{dt} = V
\]

Equation (5.23) can be thought as an approximated Langevin equation. The noise term \( B_g(\tilde{t}; V) \) decorrelates in times \( \tilde{t} \) of order one. Therefore, in a suitable scaling limit we have that the term \( B_g(\tilde{t}; V) \) behaves as a ”white noise” term. Notice that we can expect the zero average ”noise” term \( \theta_c B_g(\tilde{t}; V) \) and the deterministic term \( (\theta_c)^2 \Lambda_g(V) \) to yield contributions of the same order of magnitude, comparable to the velocity of the tagged particle, in a time scale of order \( t \approx \frac{1}{(\theta_c)^2} \). This suggests to introduce a macroscopic time scale \( T_c = \frac{L}{(\theta_c)^2} \). Then, the macroscopic time scale \( t \) and the macroscopic spatial variable \( x \) can be defined in terms of the microscopic time scale \( \tau \) and the microscopic spatial variable \( X \). We then have:

\[
t = \frac{\tau}{T_c} = (\theta_c)^2 \tilde{t}, \quad x = \frac{X}{T_c} = (\theta_c)^2 X, \quad w = V
\]

where we use the fact that the microscopic and macroscopic velocity are the same. Then (5.23) becomes:

\[
\frac{dx}{dt} = w, \quad \frac{dw}{dt} = \frac{1}{\theta_c} B_g \left( \frac{t}{(\theta_c)^2}; w \right) - \Lambda_g(w)
\]

for times \( t \) small. We now notice that, since \( \theta_c \to 0 \) we can approximate \( \frac{1}{\theta_c} B_g \left( \frac{t}{(\theta_c)^2}; w \right) \) by means of a white noise term. More precisely, we will write \( \frac{1}{\theta_c} B_g \left( \frac{t}{(\theta_c)^2}; w \right) \to \eta(t) = \eta(t; w) \) where (cf. (5.22)):

\[
\mathbb{E}[\eta(t_1; w)] = 0, \quad \mathbb{E}[\eta(t_1; w) \otimes \eta(t_2; w)] = D_g(w) \delta(t_1 - t_2).
\]

where:

\[
D_g(w) = \int_0^\infty \frac{ds}{s} \int_{\mathbb{R}^3} d\eta_1 \int_{\mathbb{R}^3} d\eta_2 \nabla_\eta \Phi(\eta_1) \otimes \nabla_\eta \Phi(\eta_2) g \left( w + \frac{\eta_1 - \eta_2}{s} \right)
\]

Notice that the integral is well defined if \( g(v) \) decreases sufficiently fast as \( |v| \to \infty \). The matrix \( D_g(w) \) is nonnegative. We then obtain the following approximation of (5.21) for small \( t \):

\[
\frac{dx}{dt} = w, \quad \frac{dw}{dt} = \eta(t; w) - \Lambda_g(w)
\]
Therefore, if we denote the probability density describing the distribution of the tagged particle as $f(x, w, t)$ we obtain the following evolution equation for it
\[
\partial_t f + w \cdot \partial_x f = \partial_w \left( \frac{1}{2} D_g(w) \partial_w f + \Lambda_g(w) f \right).
\]
(5.26)

**Remark 5.1** The earliest study of systems with long range interactions can be found in (Bogoliubov). The approach introduced there is based on the BBGKY hierarchy, arguing that the truncated correlation function $g_2$ and truncated correlations of higher order stabilize on a much shorter timescale than the one-particle distribution function $f_1$. In our approach this corresponds to the stabilization of the function $H_g$, as well as of the noise $B_g$ to a stationary process on the short time scale introduced by Bogoliubov.

### 5.1.2 Coulomb point particles.

We now consider the evolution of a tagged particle $(X, V)$ evolving according to the Rayleigh gas dynamics (cf. (1.4)) with interaction potentials decreasing as the Coulomb potential at large distances. We recall that in this case, in order to be able to define the random force field we need to impose suitable electroneutrality conditions. To be concise, we restrict our analysis to the case in which we have only two types of charges, having opposite signs, but it would be possible to adapt the arguments to more complicated charge distributions. We will consider the following types of interaction potentials:
\[
\phi_\varepsilon(X) = \Phi \left( \frac{|X|}{\varepsilon} \right) \quad \text{or} \quad \phi_\varepsilon(X) = \varepsilon \Phi \left( \frac{|X|}{\varepsilon} \right),
\]
(5.27)

where $\Phi$ is a smooth function which behaves for large values as $\Phi(s) \sim \frac{1}{s}$ as $s \to \infty$, as well as the corresponding asymptotic formulas for the derivatives of $\Phi$. The first type of potential in (5.27) includes for instance the Landau part associated to Coulomb potentials (cf. (4.2)).

We will assume that there are two types of scatterers, having charges $+1$ and $-1$ respectively. We will denote them as $\{Y^\pm_k\}_{k \in S}$ respectively. Then, the evolution equation for the tagged particle is described by the following set of equations:
\[
\begin{align*}
\frac{dX}{d\tau} &= V, \\
\frac{dV}{d\tau} &= -\sum_{j \in S} \nabla_X \phi_\varepsilon \left( X - Y^+_j \right) + \sum_{j \in \tilde{S}} \nabla_X \phi_\varepsilon \left( X - Y^-_j \right) \\
\frac{dY^\pm_k}{d\tau} &= W_k^\pm, \\
\frac{dW^\pm_k}{d\tau} &= \mp \nabla_{Y^\pm_k} Y^\pm_k - X, \quad k \in S.
\end{align*}
\]
(5.28)

Arguing as in Subsection 5.1.1 we can approximate these equations replacing the particle distributions by particle densities in the phase space. We have two different types of charges and therefore we need to introduce two different densities. We define new variables by means of:
\[
y^\pm_k = \frac{Y^\pm_k}{L_\varepsilon}, \quad \xi = \frac{X}{L_\varepsilon}, \quad \tilde{\tau} = \frac{\tau}{L_\varepsilon}
\]
(5.29)

In this case there is no canonical choice of the characteristic length $L_\varepsilon$. The first type of potentials in (5.27) has a characteristic length $\varepsilon$ and the second one has a characteristic length of order one. However, on these length scales we cannot approximate the distributions by continuous densities. In the case of interacting particle systems which will be consider
in Subsection 5.2 there will be a natural choice of \( L_\varepsilon \), namely the so-called Debye screening length. However, in the case of the Rayleigh gas under consideration here such a natural choice does not exist since the power law does not have any characteristic length. We will take any length \( L_\varepsilon \) much larger than one:

\[ L_\varepsilon \gg 1. \]

Additional conditions on \( L_\varepsilon \) will be made precise later. Let \( f_\varepsilon^\pm (y, w, t) \) be the empirical densities defined as in (5.2).

We define also a rescaled potential \( \tilde{\phi} \) by means of:

\[
\phi_\varepsilon (X) = \tilde{\phi}_\varepsilon \left( \frac{X}{L_\varepsilon} \right) = \tilde{\phi}_\varepsilon (\xi). \tag{5.30}
\]

Then, the equations for the tagged particle in (5.28) can be rewritten as:

\[
\frac{d\tilde{\xi}}{dt} = V, \quad \frac{dV}{dt} = (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_\varepsilon (\xi - \eta) \left[ \rho^\varepsilon_\varepsilon (\eta, \tilde{t}) - \rho^\varepsilon_\varepsilon (\eta, \tilde{t}) \right] d\eta, \tag{5.31}
\]

where \( \rho^\varepsilon_\varepsilon (y, \tilde{t}) = \rho[f^\varepsilon_\varepsilon^\pm (\cdot, \tilde{t})] \). On the other hand the equations for the scatterers in (5.28) imply:

\[
\partial_t f^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) + w \cdot \nabla_y f^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) \mp \nabla_y \tilde{\phi}_\varepsilon (y - \xi) \cdot \nabla_w f^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) = 0. \tag{5.32}
\]

Suppose that the distribution of velocities of the scatterers \( \{ Y^\varepsilon_k \}_{k \in S} \) are given by functions \( g^\varepsilon^\pm (w) \). Then arguing as in Subsection 6.1.1 we can approximate the initial data for (5.32) by means of Gaussian stochastic processes with average \( g^\varepsilon^\pm (w) \) and \( g^\varepsilon^- (w) \) respectively which satisfy:

\[
\mathbb{E} \left[ (f^\varepsilon_\varepsilon^\pm (y, w, 0) - g^\varepsilon^\varepsilon (w)) (f^\varepsilon_\varepsilon^\pm (y', w', 0) - g^\varepsilon^\varepsilon (w')) \right] = \frac{g^\varepsilon^\varepsilon (w)}{(L_\varepsilon)^3} \delta (y - y') \delta (w - w') \tag{5.33}
\]

\[
\mathbb{E} \left[ (f^\varepsilon_\varepsilon^\pm (y, w, 0) - g (w)) (f^\varepsilon_\varepsilon^- (y', w', 0) - g^- (w')) \right] = 0. \tag{5.34}
\]

The last equation ensures that the distributions of scatterers are mutually independent.

It is then natural, arguing as in Subsection 6.1.1, to look for solutions with the form:

\[
f^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) = g^\varepsilon^\varepsilon (w) + \frac{1}{(L_\varepsilon)^2} \zeta^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}). \tag{5.35}
\]

Then, neglecting higher order terms, we obtain the the particle fluctuations satisfy approximately the following problems:

\[
\partial_t \zeta^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) + w \cdot \nabla_y \zeta^\varepsilon_\varepsilon^\pm (y, w, \tilde{t}) \mp (L_\varepsilon)^3 \nabla_y \tilde{\phi}_\varepsilon (y - \xi) \cdot \nabla_w g^\varepsilon^\varepsilon (w) = 0. \tag{5.36}
\]

with random initial data \( \zeta^\varepsilon_\varepsilon^\pm (y, w, 0) = N^\varepsilon^\pm (y, w) \) satisfying:

\[
\mathbb{E} \left[ N^\varepsilon^\pm (y, w) \right] = 0, \quad \mathbb{E} \left[ N^\varepsilon_\varepsilon^\pm (y, w-a) N^- (y, w-b) \right] = 0 \tag{5.37}
\]

\[
\mathbb{E} \left[ N^\varepsilon^\pm (y, w-a) N^\varepsilon^- (y, w-b) \right] = g^\varepsilon^\varepsilon (w-a) \delta (y-a-y_b) \delta (w-a-w_b). \tag{5.38}
\]

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The electroneutrality condition requires that:
\[
\int_{\mathbb{R}^3} g^+ (w) \, dw = \int_{\mathbb{R}^3} g^- (w) \, dw.
\]

Then (5.31) can be rewritten as:
\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -(L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_\varepsilon (\xi - \eta) \left[ \zeta^+_\varepsilon (\eta, w, \tilde{t}) - \zeta^-_\varepsilon (\eta, w, \tilde{t}) \right] \, d\eta. \tag{5.39}
\]

Summarizing, we have reduced the original problem to (5.36) - (5.39). It is natural, in order to derive a kinetic limit, to consider the previous model as \(\varepsilon \to 0\). Moreover, we will assume that \(\tilde{t}\) is sufficiently small so that we can assume that \(V\) is constant. Assuming that \(\xi (0) = 0\) then we have the approximation \(\xi \simeq V \tilde{t}\). We can then approximate (5.36) as:
\[
\partial_t \zeta^\pm (y, w, \tilde{t}) + w \cdot \nabla_y \zeta^\pm (y, w, \tilde{t}) = (L_\varepsilon)^2 \nabla_y \tilde{\phi}_\varepsilon (y - V \tilde{t}) \cdot \nabla_w g^\pm (w) = 0. \tag{5.40}
\]

Using (5.27), (5.30) and the asymptotic behaviour of the function \(\Phi\) we obtain the approximation:
\[
\nabla_y \tilde{\phi}_\varepsilon (y - V \tilde{t}) \simeq -\varepsilon \frac{(y - V \tilde{t})}{|y - V \tilde{t}|^3}.
\]

In any case, due to the presence of logarithmic divergences we will keep in the equation the whole gradient \(\nabla_y \tilde{\phi}_\varepsilon (y - V t)\).

We can then solve (5.40) with initial data satisfying (5.37), (5.38) arguing as in Subsection 5.1.1.

\[
\zeta^\pm (y, w, \tilde{t}) = N^\pm (y - w \tilde{t}, w) \pm (L_\varepsilon)^2 \nabla_w g^\pm (w) \cdot \int_0^{\tilde{t}} \nabla_y \tilde{\phi}_\varepsilon (y - w (\tilde{t} - s) - V s) \, ds.
\]

We can then approximate (5.39) as:
\[
\frac{d\xi}{dt} = V
\]
\[
\frac{dV}{dt} = \tilde{F}_g (\xi, t) - 2 (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_\varepsilon (\xi - \eta) \left[ \nabla_w g^\pm (w) \cdot \int_0^{\tilde{t}} \nabla_y \tilde{\phi}_\varepsilon (\eta - w (\tilde{t} - s) - V s) \, ds \right] \, d(\eta, w) \tag{5.41}
\]

where \(F_g (\xi, t)\) is the time dependent random force field:
\[
F_g (\xi, t) = -2 (L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_\varepsilon (\xi - \eta) \left[ N^+ (\eta - w \tilde{t}, w) - N^- (\eta - w \tilde{t}, w) \right] \, d\eta dw.
\]

On the other hand we can write the last integral term in (5.41) in a more suitable form. We approximate \(\xi\) as \(V \tilde{t}\) during the time scale in which the velocity is approximately constant. Then, the last integral term in (5.41) becomes:
\[
-2 (L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_\varepsilon (V \tilde{t} - \eta) \left[ \nabla_w g^\pm (w) \cdot \int_0^{\tilde{t}} \nabla_\eta \tilde{\phi}_\varepsilon (\eta - w (\tilde{t} - s) - V s) \, ds \right] \, d\eta dw
\]
\[
= 2 (L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_\eta \tilde{\phi}_\varepsilon (\eta) \left[ \nabla_w g^\pm (w) \cdot \int_{-\tilde{t}}^0 \nabla_\eta \tilde{\phi}_\varepsilon (\eta + (V - w) s) \, ds \right] \, d\eta dw.
\]
We can rewrite these equations in the original variables $X, V, \tau$.

$$\frac{dX}{d\tau} = V, \quad \frac{dV}{d\tau} = F_g(X, \tau) - \Lambda_g(V).$$

(5.42)

Here $F_g(X, \tau)$ is the random force field:

$$F_g(X, \tau) = -\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla Z \phi_\varepsilon(Z - X) \left[ N^+(Z - \omega \tau, w) - N^-(Z - \omega \tau, w) \right] dZ dw$$

and $\Lambda_g(V)$ is the friction coefficient:

$$\Lambda_g(V) = -2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla Z \phi_\varepsilon(Z) \left[ \nabla w g(w) \cdot \int_{-\tau}^0 \nabla Z \phi_\varepsilon(Z + (V - \omega) s) ds \right] dZ dw.$$

(5.43)

It turns out that in this case we cannot replace in the last term $\int_{-\tau}^0$ by $\int_{-\infty}$ as it was made in the case of potentials with a shorter range (cf. Subsection 5.1.1). This fact is related to the so-called Coulombian logarithm, that has been already seen in Section 4.2. Using (5.12) we obtain:

$$\frac{1}{(L_\varepsilon)^2} N^\pm \left( \frac{1}{L_\varepsilon}, Z \right) = N^\pm (Z, w).$$

We can now estimate the mean free path using the form of $\phi_\varepsilon$ in (5.27). To this end we remark that:

$$\mathbb{E}[F_g(X)] = 0$$

$$\mathbb{E}[F_g(X_1, \tau) \otimes F_g(X_2, \tau')] = 2 \int_{(\mathbb{R}^3)^2} d(w, Z) \nabla \phi_\varepsilon(Z - X_1) \otimes \nabla \phi_\varepsilon(Z - (V - (\tau - \tau') - X_2)) g(w).$$

We need to estimate the time scale in which these random forces yield deflection velocities of order one. To this end, we can approximate the path by means of a rectilinear path during lengths shorter than the mean free path. We can assume that the particle begins to move at $X = 0$. Then, the deflections due to the interaction with the random force field $F_g$ up to time $T_\varepsilon$ are given by:

$$D_\varepsilon(T_\varepsilon) = \int_0^{T_\varepsilon} F_g(Vs, s) ds.$$

Therefore $\mathbb{E}[D_\varepsilon(T_\varepsilon)] = 0$ and:

$$\mathbb{E}[D_\varepsilon(T_\varepsilon) \otimes D_\varepsilon(T_\varepsilon)] = \int_0^{T_\varepsilon} ds_1 \int_0^{T_\varepsilon} ds_2 \mathbb{E}[F_\varepsilon(Vs_1, s_1) \otimes F_\varepsilon(Vs_2, s_2)]$$

$$= 2 \int_0^{T_\varepsilon} ds_1 \int_0^{T_\varepsilon} ds_2 \int_{\mathbb{R}^3} dZ \int_{\mathbb{R}^3} g(w_1) dw_1 \nabla Z \phi_\varepsilon(Z - (V - w) s_1) \otimes \nabla Z \phi_\varepsilon(Z - (V - w) s_2).$$

We can approximate this covariance matrix using the form of $\phi_\varepsilon$ in (5.27). Then:

$$\mathbb{E}[D_\varepsilon(T_\varepsilon) \otimes D_\varepsilon(T_\varepsilon)] = 2\varepsilon^2 \int_0^{T_\varepsilon} ds_1 \int_0^{T_\varepsilon} ds_2 \int_{\mathbb{R}^3} dZ \cdot \int_{\mathbb{R}^3} g(w) dw \Phi'(\|Z - (V - w) s_1\|) \Phi'(\|Z - (V - w) s_2\|) \frac{(Z - (V - w) s_1)}{|Z - (V - w) s_1|} \otimes \frac{(Z - (V - w) s_2)}{|Z - (V - w) s_2|}.\]
Then, using the change of variables $Z = (s_2 - s_1)Y$ we obtain:

$$
\mathbb{E} [D_\varepsilon (T_\varepsilon) \otimes D_\varepsilon (T_\varepsilon)] = 2\varepsilon^2 \int_0^{T_\varepsilon} ds_1 \int_0^{T_\varepsilon} |s_2 - s_1|^3 ds_2 \int_{\mathbb{R}^3} dY \cdot \int_{\mathbb{R}^3} g(w + V) dw \Phi'(|s_2 - s_1|Y) \Phi'(|s_2 - s_1|Y + w) Y \frac{Y}{|Y|} \otimes \frac{(Y + w)}{|Y + w|}.
$$

If $|s_2 - s_1|$ is of order one the function of $|s_2 - s_1|$ obtained by means of $\int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw (\cdots)$ is bounded. On the other hand, using the asymptotic behaviour of $\Phi$ we obtain the following asymptotic formula:

$$
|s_2 - s_1|^3 \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} g(w + V) dw \Phi'(|s_2 - s_1|Y) \Phi'(|s_2 - s_1|Y + w) Y \frac{Y}{|Y|} \otimes \frac{(Y + w)}{|Y + w|}
$$

$$
\sim \frac{1}{|s_2 - s_1|} \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} g(w + V) dw Y \frac{Y}{|Y|^2} \otimes \frac{(Y + w)}{|Y + w|^2} \text{ as } |s_2 - s_1| \to \infty
$$
whence:

$$
\mathbb{E} [D_\varepsilon (T_\varepsilon) \otimes D_\varepsilon (T_\varepsilon)] \sim 2\varepsilon^2 T_\varepsilon \log (T_\varepsilon) \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} g(w + V) dw Y \frac{Y}{|Y|^2} \otimes \frac{(Y + w)}{|Y + w|^2},
$$

(5.44)
as $T_\varepsilon \to \infty$. This gives the characteristic time scale for the evolution of the velocity is:

$$
T_\varepsilon \sim \frac{1}{2\varepsilon^2 \log (1/\varepsilon)} \text{ as } \varepsilon \to 0,
$$

(5.45)
which is the expected time scale for the diffusive part of the Coulombian interactions. In particular the time scale contains the Coulombian logarithm.

We can compute also the characteristic time scale for the friction term $\Lambda_\varepsilon (V)$. To this end we compute the size of the contribution of this term. More precisely, the deflection produced by this term in times $T_\varepsilon$ is:

$$
d_\varepsilon (T_\varepsilon) = 2 \int_0^{T_\varepsilon} ds_1 \int_{\mathbb{R}^3} \nabla Z \Phi_\varepsilon (Z) \left[ \nabla_w g(w) \cdot \int_{-s_1}^0 \nabla Z \Phi_\varepsilon (Z + (V - w) s_2) ds_2 \right] dZ dw
$$

Using (5.27) we obtain, using the change of variables $s_2 \to -s_2$ and applying Fubini:

$$
d_\varepsilon (T_\varepsilon) = 2\varepsilon^2 \int_0^{T_\varepsilon} (T_\varepsilon - s_2) \Psi (s_2; V) ds_2
$$

(5.46)
where:

$$
\Psi (s_2; V) = \int_{\mathbb{R}^3} dZ \int_{\mathbb{R}^3} dw \Phi'(|Z|) \Phi'(|Z + (V - w) s_2|) \left( \nabla_w g(w) \cdot \frac{Z + (w - V) s_2}{|Z + (w - V) s_2|} \right) \frac{Z}{|Z|}.
$$

Using our assumptions on $\Phi$ we obtain that $\Psi (s_2; V)$ is bounded for $s_2$ bounded and it behaves asymptotically as $s_2 \to \infty$ as:

$$
\Psi (s_2; V) \sim \frac{1}{s_2} \int_{\mathbb{R}^3} dZ \int_{\mathbb{R}^3} dw \frac{1}{|Z|^2 |Z + (w - V)|^2} \left( \nabla_w g(w) \cdot \frac{Z + (w - V)}{|Z + (w - V)|} \right) \frac{Z}{|Z|}
$$

40
where we have used the change of variables $Z \to s_2 Z$. Then:

$$d_\varepsilon (T_\varepsilon) \sim 2\varepsilon^2 T_\varepsilon \log (T_\varepsilon) \int_{\mathbb{R}^3} dZ \int_{\mathbb{R}^3} dw \frac{1}{|Z|^2 |Z + (w - V)|^2} \left( \nabla_w g (w) \cdot \frac{Z + (w - V)}{|Z + (w - V)|} \right) \frac{Z}{|Z|} \tag{5.47}$$

In order to derive the evolution equation for the probability density which describes the position of the tagged particle we rescale the time $\tau$ and introduce a macroscopic time scale $t = \frac{1}{T_\varepsilon}$ with $T_\varepsilon = \frac{1}{2\varepsilon^2 \log (\varepsilon)}$ and the space variable $x = \frac{X}{T_\varepsilon}$. Then combining (5.42), (5.44), (5.47) we obtain the following stochastic differential equation for the tagged particle:

\[
\frac{dx}{dt} = V, \quad \frac{dV}{dt} = W_g (t; V) - \lambda_g (V)
\]

where:

\[
\lambda_g (V) = - \int_{(\mathbb{R}^3)^2} d(w, Z) \frac{1}{|Z|^2 |Z + (w - V)|^2} \left( \nabla_w g (w) \cdot \frac{Z + (w - V)}{|Z + (w - V)|} \right) \frac{Z}{|Z|}, \tag{5.48}
\]

and

\[
\mathbb{E} [W_g (t; V)] = 0, \tag{5.49}
\]

\[
\mathbb{E} [W_g (t; V) \otimes W_g (t'; V)] = \delta (t - t') \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} g (w + V) dw \frac{Y}{|Y|^2} \otimes \frac{(Y + w)}{|Y + w|^2} = D_g (V). \tag{5.50}
\]

**Remark 5.2** Contrarily to the case studied in Subsection 5.1.1, the friction term $H_g$ and the noise $B_g$ do not stabilize on a short Bogoliubov time scale, as can be seen by the logarithmic divergence of the integral in (5.43) as $\tau \to \infty$ and the divergence of $\frac{1}{T_\varepsilon} \int_{0}^{T_\varepsilon} (T_\varepsilon - s_2) \Psi (s_2; V) ds_2$ (cf. (5.46)) as $T_\varepsilon \to \infty$. This logarithmic divergence yields the logarithmic correction of the time scale characteristic of Coulombian interactions.

### 5.1.3 The threshold between short and long range potentials in the decay $\frac{1}{|x|}$

Something that was seen in [38] is that in the case of Lorentz gases and interaction potentials with the form $\frac{\varepsilon}{|x|}$ the threshold which separates between short range and long range potentials corresponds to $s = 1$, i.e. to potentials decreasing like Coulombian potentials. Actually, the same separation between long range and short range potentials holds in the case of Rayleigh gases.

The rationale behind this difference between short and long range potentials can be easily understood at the physical level (cf. [7] for an explanation of the onset of the Coulombian logarithm in the case of Coulombian potentials). Suppose that we consider a tagged particle moving along a rectilinear path which will be denoted as $\ell$ during a length $L$. We consider the
deflections of the tagged particle induced by a set of scatterers distributed according to the Poisson distribution with an average distance 1. More precisely, we distinguish the deflections produced by the scatterers at dyadic distances, i.e. at a distance $2^n$ and $2^{n+1}$ from $\ell$ for each $n \in \mathbb{Z}$, where we assume that $2^n$ is smaller than the range of the potential. We will denote the distances between $2^n$ and $2^{n+1}$ for each $n$ as a "dyadic". The deflection experienced by the tagged particle is a random variable with zero average, due to the fact that the scatterers are symmetrically distributed with respect to the line $\ell$. On the other hand if we estimate the variance of the deflections produced by the particles at distances between $2^n$ and $2^{n+1}$ it can be readily seen that the magnitude of this variance decreases exponentially with $n$ if $s > 1$ and increases exponentially with $n$ (as long as $2^n \lesssim L$) for $s < 1$. The contribution to the deflections of the scatterers at distances larger than $L$ is negligible if $s > \frac{1}{2}$ (cf. [38] for a detailed proof of this in the case of static scatterers). Notice that for these "far-away" scatterers the change of angle subtended by the tagged particle is very small, i.e. we can say that no collision is taking place. In the case of $s = 1$ the magnitude of the deflections is the same for all the values of $n$ as long as $2^n \lesssim L$. Actually, this is the reason for the onset of the Coulombian logarithm, namely the fact that we need to add the deflections produced by the different dyadic cylinders and the number of these cylinders is of order $\log (L)$, if we assume that the potential $\frac{1}{|x|}$ is cut at distances of order $|x| \simeq 1$.

It turns out that a similar picture takes place in the case of moving scatterers, i.e. for Rayleigh gases. The onset of the logarithmic correction for potentials behaving like Coulomb for large distances has been seen above. Seemingly nonkinetic models, with long range correlations due to the long range of the potentials arise for $s \in \left(\frac{1}{2}, 1\right)$ as it happens in the Lorentz gases considered in [38], in spite of the fact that for Rayleigh gases the scatterers move. However, we will not continue the study of this case in this paper. (Nevertheless, the case $s \in \left(\frac{1}{2}, 1\right)$ will be discussed in Subsection 8.1 in the case of interacting particle systems).

5.1.4 The case of grazing collisions.

Another example of Rayleigh gas dynamics (cf. [14]) which can be approximated in the same manner as above is the case which is usually termed with the name of grazing collisions. This corresponds to the case of particles which interact weakly with a tagged particle, with an interaction range smaller or similar to the average particle distance and do not interact between themselves. We will assume that there is only one type of scatterers. Therefore the system under consideration has the form:

$$
\frac{dX}{d\tau} = V , \quad \frac{dV}{d\tau} = -\sum_{j \in S} \nabla X \phi_{\varepsilon} (X - Y_j) \tag{5.51}
$$

$$
\frac{dY_k}{d\tau} = W_k , \quad \frac{dW_k}{d\tau} = -\nabla Y \phi_{\varepsilon} (Y_k - X) , \quad k \in S
$$

where we assume that the interaction potential is:

$$
\phi_{\varepsilon} (X) = \varepsilon \Phi \left( \frac{|X|}{\ell_{\varepsilon}} \right) \tag{5.52}
$$

where $\Phi = \Phi (s)$ is a smooth function which decreases sufficiently fast as $s \to \infty$, as well as the corresponding asymptotic formulas for the derivatives of $\Phi$. We can assume for instance
that $\Phi(s) \sim \frac{1}{s^\alpha}$ as $s \to \infty$ with $\alpha > 1$, or that $\Phi(s)$ decreases exponentially. We assume that $\ell \lesssim 1$. In particular we might assume that $\ell \to 0$. We then argue as in Subsections 5.1.1 and introduce a new set of variables by means of:

$$y_k = \frac{Y_k}{L}, \quad X = \frac{X}{L}, \quad \tilde{t} = \frac{\tau}{L}$$

The length $L$ is only an auxiliary length $L \gg 1$ that enables us to approximate the background distribution by a Guassian. Notice that in this case it is not convenient to take $L = \ell$ because since $\ell \lesssim 1$ we would not obtain an approximation of the distribution of particles by means of a random distribution. Consider again the empirical density defined by (5.2). We also define a rescaled potential by means of:

$$\phi_e(X) = \tilde{\phi}_e \left( \frac{X}{L} \right) = \tilde{\phi}_e(\xi)$$

Using the change of values $\tilde{t} = \frac{\tau}{L}, \quad \xi = \frac{X}{L}$ we can rewrite (5.51), (5.52) as:

$$\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = \sum_{j \in S} \nabla_\xi \tilde{\phi}_e(\xi - y_j) = -(L\ell)^3 \int_{\mathbb{R}^3} \nabla_\xi \tilde{\phi}_e(\xi - \eta) \rho_e(\eta, \tilde{t}) \, d\eta$$

where $\rho_e(\eta, \tilde{t}) = \rho[f_\epsilon(\cdot, \tilde{t})](y)$ as defined in (1.17).

On the other hand, using the equations for the scatterers in (5.51) we obtain:

$$\partial_\tilde{t} f_\epsilon(y, w, \tilde{t}) + w \cdot \nabla_y f_\epsilon(y, w, \tilde{t}) - \frac{1}{(L\ell)^3} \sum_k (y - y_k) \left[ \nabla_y \tilde{\phi}_e(y_k - \xi) \cdot \nabla_w \delta(w - W_k) \right] = 0$$

Arguing as Subsections 5.1.1, 5.1.2 it is natural to approach the initial value for (5.53) as:

$$\mathbb{E}[(f_\epsilon(y_a, w_a, 0) - g(w_a))(f_\epsilon(y_b, w_b, 0) - g(w_b))] = \frac{g(w_a)}{(L\ell)^3} \delta(y_a - y_b) \delta(w_a - w_b)$$

where $g(w)$ is the distribution of velocities of the scatterers $\{Y_k\}_{k \in S}$. We then look for a solution (5.53), (5.54) with the form:

$$f_\epsilon(y, w, \tilde{t}) = g(w) + \frac{1}{(L\ell)^2} \zeta_\epsilon(y, w, \tilde{t})$$

Then, approximating (5.53) by the leading order terms we obtain:

$$\left( \partial_\tilde{t} + w \cdot \nabla_y \right) \zeta_\epsilon(y, w, \tilde{t}) - (L\ell)^2 \nabla_y \tilde{\phi}_e(y - \xi) \cdot \nabla_w g(w) = 0, \quad \zeta_\epsilon(y, w, 0) = N_\epsilon(y, w)$$

where the (random) initial data is characterized by:

$$\mathbb{E}[N_\epsilon(y, w)] = 0$$

$$\mathbb{E}[N_\epsilon(y_a, w_a) N_\epsilon(y_b, w_b)] = \frac{g(w_a)}{(L\ell)^3} \delta(y_a - y_b) \delta(w_a - w_b).$$
On the other hand, the evolution of the tagged particle (cf. (5.51)) can be approximated as:

\[
\frac{d\xi}{dt} = V , \quad \frac{dV}{dt} = -(L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \tilde{\phi}_\varepsilon (\xi - \eta) \zeta_\varepsilon (\eta, w, \tilde{t}) \, d\eta d\eta.
\] (5.58)

We can solve these equations as in Subsection 5.1.1. We then obtain:

\[
\zeta_\varepsilon (y, w, \tilde{t}) = N_\varepsilon (y - w\tilde{t}, w) + (L_\varepsilon)^3 \nabla_w g (w) \cdot \int_{0}^{\tilde{t}} \nabla_y \tilde{\phi}_\varepsilon (y - w (\tilde{t} - s) - Vs) \, ds
\]

where we use the approximation \( \xi \simeq V \tilde{t} \) which is valid for times in which the tagged particle travels over distances shorter than the mean free path. In order to obtain the evolution of the tagged particle we approximate also \( \xi \) as \( V \tilde{t} \). Then, the velocity of the tagged particle for times shorter than the mean free path is given by:

\[
\frac{dV}{dt} = (L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \tilde{\phi}_\varepsilon (\eta) N_\varepsilon (\eta - w\tilde{t}, w) \, d\eta d\eta +
\]

\[
+ (L_\varepsilon)^3 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla \tilde{\phi}_\varepsilon (\eta) \nabla_w g (w) \cdot \int_{-\infty}^{0} \nabla \tilde{\phi}_\varepsilon (\eta + (V - w) s) \, ds d\eta d\eta
\]

We change to the original variables:

\[
\frac{dV}{d\tau} = \int_{\mathbb{R}^3} dX \int_{\mathbb{R}^3} dw \nabla_X \phi_\varepsilon (X + w\tau) \tilde{N} (X, w) +
\]

\[
+ \int_{\mathbb{R}^3} dX \int_{\mathbb{R}^3} dw \nabla_X \phi_\varepsilon (X) \nabla_w g (w) \cdot \int_{-\infty}^{0} \nabla_X \phi_\varepsilon (X + (V - w)) \, ds.
\]

where is the following Gaussian noise, which is independent of \( \varepsilon \)

\[
\mathbb{E} [\tilde{N} (X, w)] = 0
\]

\[
\mathbb{E} [\tilde{N} (X_a, w_a) \tilde{N} (X_b, w_b)] = g (w_a) \delta (X_a - X_b) \delta (w_a - w_b).
\]

We then write the evolution equation of the tagged particle as

\[
\frac{dX}{d\tau} = V , \quad \frac{dV}{d\tau} = \frac{\theta_\varepsilon}{\sqrt{\ell_\varepsilon}} B_g \left( \frac{\tau}{\ell_\varepsilon} \right) - (\theta_\varepsilon)^2 A_g (V)
\] (5.59)

Here \( \theta_\varepsilon \) is given by:

\[
\theta_\varepsilon = \varepsilon \ell_\varepsilon,
\] (5.60)

the friction term \( A_g (V) \) is given by

\[
A_g (V) = - \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla_Y \Phi (|Y|) \nabla_w g (w) \cdot \int_{-\infty}^{0} \nabla_Y \Phi (|Y + (V - w) s|) \, ds
\] (5.61)

and \( B_g (\cdot) \) is a gaussian stationary stochastic process defined in the time variable such that

\[
\mathbb{E} [B_g (s)] = 0 , \quad \mathbb{E} [B_g (0) B_g (s)] = \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla_Y \Phi (|Y + ws|) \otimes \nabla_Y \Phi (|Y|) g (w)
\] (5.62)

We observe that the noise \( B_g \) decorrelates in times of order \( \ell_\varepsilon \) as it might be expected.
We can now rewrite (5.59) using a macroscopic time scale \( t = (\theta \varepsilon)^2 \tau \) and the new space variable \( x = (\theta \varepsilon)^2 X \). We keep the velocity \( v = V \). Then, taking the limit \( \varepsilon \to 0 \) we obtain the following approximation

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{1}{\theta \varepsilon \sqrt{\varepsilon}} B_g \left( \frac{t}{(\theta \varepsilon)^2 \ell} \right) - \Lambda_g (v) = \tilde{B}_g (t) - \Lambda_g (v)
\]

(5.63)

where \( \Lambda_g \) and \( B_g \) are as in (5.61), (5.62) and \( \tilde{B}_g \) satisfies:

\[
\mathbb{E} \left[ \tilde{B}_g (s) \right] = 0, \quad \mathbb{E} \left[ \tilde{B}_g (s_1) \tilde{B}_g (s_2) \right] = D \delta (s_1 - s_2)
\]

\[
D = \int_0^\infty ds \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla Y \Phi (|Y + ws|) \otimes \nabla Y \Phi (|Y|) g (w).
\]

(5.64)

We then obtain the following evolution equation for the distribution function \( f (x, v, t) \).

\[
\partial_t f (x, V, t) + V \cdot \nabla_x f (x, V, t) = \partial_V \left( \frac{D}{2} \partial_V f + \Lambda_g (V) f \right) (x, V, t).
\]

(5.65)

5.2 Approximating the dynamics of an interacting particle system using the dynamics of a tagged particle with friction in a random medium.

We now describe how to approximate the dynamics of the many particle system (1.1) by means of (1.2) in the case of weak particle interactions. The main idea is to apply to each particle of the system a method analogous to the one used in the previous Subsection for Rayleigh gases, namely to approximate the dynamics of a particle as the motion in a random force field with a friction coefficient. The only difference is that in the computation of the random force field and the friction coefficient acting on each particle we must take into account the interaction between the background particles, which is not present in the case of Rayleigh gases.

Actually, the decomposition the action in a tagged particle of a medium composed of many of particles as a friction term and a noise term has been used since the earliest times of Statistical Physics. Indeed, Einstein’s analysis of Brownian motion (cf. [19]), assumes that the forces acting on one particle moving in a viscous fluid can be decomposed in the sum two parts, namely the friction force, which can be computed solving Stokes equations, and a random force field, which is assumed to have the form of a white noise in time. The determination of the amplitude of the white noise was made using a version of the fluctuation-dissipation theorem. This theorem, which is a very general result in Statistical Physics provides a relation between the friction acting on a tagged particle and the noise part, which is due to the fact that at equilibrium equipartition of the energy holds.

The approach of approximating the dynamics of a single particle in the system (1.1) is similar from the conceptual point of view. The only difference is that in the case of kinetic limits of (1.1), we can obtain explicit formulas for the friction coefficient and the random force field acting on each particle in terms of the interaction potential describing the microscopic dynamics. In the case of a Brownian particle moving in a viscous flow, the direct computation of the friction and the random force field in terms of the microscopic interactions between the particles is a rather difficult task.

We remark that the computations in this Subsection yield a method to obtain the characteristic time scale for the evolution of the distribution of velocities of a particle system described by (1.1) in the particular scaling limits under consideration.

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5.2.1 Particle interactions with finite range much larger than the particle distance.

Computation of the friction coefficient and a random force field. We begin considering the case of interaction potentials with the form \( \xi \). We consider the dynamics of a distinguished particle \((X, V)\) in an interacting particle system. Denoting as \((Y_k, W_k)\), \(k \in S\) the position and velocity of the remaining particles of the system, using the change of variables (5.29) as well as (1.1) we obtain:

\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon \sum_{j \in S} \nabla \xi \Phi (\xi - y_j) \tag{5.66}
\]

\[
\frac{dy_k}{dt} = W_k, \quad \frac{dW_k}{dt} = -\varepsilon \nabla y \Phi (y_k - \xi) - \varepsilon \sum_{j \in S} \nabla y \Phi (y_k - y_j), \quad k \in S
\]

In order to approximate this system by an evolution equation of the form (1.2), first recall the empirical measure \( f_{\varepsilon} \) defined in (5.2). Then, the first equation of (5.66) can be rewritten as:

\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon \int_{\mathbb{R}^3} \nabla \xi \Phi (\xi - \eta) \rho_{\varepsilon} (\eta, t) \, d\eta \tag{5.67}
\]

with \( \rho_{\varepsilon} (y, t) = \rho (f_{\varepsilon} (\cdot, t))(y) \) (cf. (1.17)). The second set of equations in (5.66) becomes:

\[
\frac{dy_k}{dt} = W_k, \quad \frac{dW_k}{dt} = -\varepsilon \nabla y \Phi (y - \xi) - \varepsilon (L_\varepsilon)^3 (\nabla y \Phi * \rho_{\varepsilon})(y_k, t), \quad k \in S. \tag{5.68}
\]

Using (5.68) we can derive, arguing as in Subsection 5.1.1, the following evolution equation for the particle density \( f_{\varepsilon} \):

\[
\partial_t f_{\varepsilon} (y, w, t) + w \cdot \nabla_y f_{\varepsilon} (y, w, t) - \varepsilon \left[ \nabla_y \Phi (y - \xi) + (L_\varepsilon)^3 (\nabla y \Phi * \rho_{\varepsilon})(y, t) \right] \cdot \nabla_w f_{\varepsilon} (y, w, t) = 0 \tag{5.69}
\]

Using the change of variables (5.6), we can rewrite (5.69) as:

\[
(\partial_t + w \cdot \nabla_y) \zeta_{\varepsilon} (y, w, t) - \varepsilon (L_\varepsilon)^2 \nabla_y \Phi (y - \xi) + \varepsilon (L_\varepsilon)^3 (\nabla y \Phi * \tilde{\rho}_{\varepsilon})(y, t) \cdot \nabla_w \left( g (w) + \frac{1}{(L_\varepsilon)^2} \zeta_{\varepsilon} (y, w, t) \right) = 0.
\]

where \( \tilde{\rho} (y, t) = \rho (\zeta (\cdot, t))(y) \). Neglecting the term \( \frac{1}{(L_\varepsilon)^2} \zeta_{\varepsilon} \) which might be expected to be small compared with \( g \) we obtain the following approximation for the fluctuations of the density of scatterers:

\[
((\partial_t + w \cdot \nabla_y) \zeta_{\varepsilon} (y, w, t) - \varepsilon (L_\varepsilon)^2 \nabla_y \Phi (y - \xi) + \varepsilon (L_\varepsilon)^3 (\nabla y \Phi * \tilde{\rho}_{\varepsilon})(y, t) \cdot \nabla_w g (w) = 0. \tag{5.70}
\]

The term \( \tilde{\rho}_{\varepsilon} \) has the same order of magnitude as \( \zeta_{\varepsilon} \). Then, the terms \( w \cdot \nabla_y \zeta_{\varepsilon} (y, w, t) \) and \( \varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla y \Phi (y - \eta) \tilde{\rho}_{\varepsilon} (\eta, t) \, d\eta \cdot \nabla_w g (w) \) yield a comparable contribution if \( \varepsilon (L_\varepsilon)^3 \) is of order one. To consider the general case, we will assume that \( \varepsilon (L_\varepsilon)^3 \rightarrow \sigma \) as \( \varepsilon \rightarrow 0 \), where \( \sigma \) can be zero, a positive number or infinity. (In this last case the dependence of \( \sigma \) on \( \varepsilon \) must
be preserved, but it will not be explicitly written for the sake of simplicity). We can then expect that \( \zeta \sim \varepsilon \) as \( \varepsilon \to 0 \) where \( \zeta \) solves the problem:

\[
(\partial_t + w \cdot \nabla_y \zeta) (y, w, \tilde{t}) = \left[ \varepsilon (L_\varepsilon)^{\frac{3}{2}} \nabla_y \psi (y - V \tilde{t}) + \sigma (\nabla_y \Phi * \tilde{\rho}) (y, \tilde{t}) \right] \cdot \nabla_w g (w) = 0 \quad (5.71)
\]

\[
\zeta (y, w, 0) = N (y, w)
\]

where \( \tilde{\rho} (y, \tilde{t}) = \rho (\zeta (\cdot, \tilde{t})) (y) \) and \( N \) is as in (5.12). We have also made use of the approximation \( \xi \approx V \tilde{t} \) for times shorter than the mean free time, and a particle that starts at the origin \( \xi (0) = 0 \) without loss of generality. We can then decompose \( \zeta \) as:

\[
\zeta (y, w, \tilde{t}) = \zeta_1 (y, w, \tilde{t}) + \zeta_2 (y, w, \tilde{t}). \quad (5.72)
\]

Here \( \zeta_1 \) is given by

\[
(\partial_t + w \cdot \nabla_y \zeta_1) (y, w, \tilde{t}) - \sigma \nabla_w g (w) \cdot (\nabla_y \Phi * \tilde{\rho}_1) (y, \tilde{t}) \, d\eta = 0 \quad (5.73)
\]

\[
\zeta_1 (y, w, 0) = N (y, w) \quad (5.74)
\]

with \( \tilde{\rho}_1 (\eta, \tilde{t}) = \rho (\zeta_1 (\cdot, \tilde{t})) (y) \) (cf. (1.17)), and \( \zeta_2 \) is given by:

\[
(\partial_t + w \cdot \nabla_y) \zeta_2 (y, w, \tilde{t}) - \sigma \nabla_w g (w) \cdot (\nabla_y \Phi * \tilde{\rho}_2) (y, \tilde{t}) = \varepsilon (L_\varepsilon)^{\frac{3}{2}} \nabla_y \psi (y - V \tilde{t}) \cdot \nabla_w g (w) \quad (5.75)
\]

\[
\zeta_2 (y, w, 0) = 0 \quad (5.76)
\]

with \( \tilde{\rho}_2 (\eta, \tilde{t}) = \rho (\zeta_2 (\cdot, \tilde{t})) (y) \). Notice that the function \( \zeta_2 \) still depends on \( \varepsilon \).

The set of equations (5.72)-(5.76) yields the fluctuations of the scatterer density in the phase space. The contribution \( \zeta_1 \) contains the "noisy" part of the fluctuations. The contribution \( \zeta_2 \) yields the perturbation to the scatterers density induced by the presence of the distinguished particle \( (X, V) \). It is crucial to notice that if \( \sigma \) is of order one, the resulting densities \( \zeta_1, \zeta_2 \) would be different from those obtained for Rayleigh gases (cf. (5.15), (5.16)). The terms proportional to \( \sigma \) give the contribution due to the interactions of the scatterers with themselves. The problems (5.73)-(5.74) and (5.75)-(5.76) are linear and can be solved using Fourier and Laplace transforms (cf. (31), (33)).

**Remark 5.3 (On the Balescu-Lenard rescaling)** We have seen that for \( \sigma \) of order one, the equation which describes the evolution of the density fluctuations (5.71) contains a term

\[
\sigma \left( \int_{\mathbb{R}^3} \nabla_y \psi (y - \eta) \tilde{\rho} (\eta, \tilde{t}) \, d\eta \right) \cdot \nabla_w g (w)
\]

yielding contributions of the same order of magnitude as the transport term \( w \cdot \nabla_y \zeta (y, w, \tilde{t}) \). This is the property characterizing the so-called Balescu-Lenard limit. The possible onset of this term is the main difference between the dynamics of particles moving in a Rayleigh gases (where the term accounting by self-interactions between the scatterers does not appear) and the dynamics of a particle in an interacting particle system. It is important to remark that the scaling limit obtained above yielding the Balescu-Lenard limit is only valid for interaction potentials having a large but finite range. In the case of Coulombian potentials, which will be discussed in Subsection 5.2.2, a characteristic length in which the free transport of particles becomes of the same order of magnitude as the self-interactions term arises naturally. This characteristic length, known as Debye screening length, will be discussed in detail in Subsection 5.2.2.
Remark 5.4 It is interesting to remark that in the case of solutions of the Vlasov equation with integrable initial data there are rigorous results in [12] which show that the density fluctuations can be approximated by the linearized Vlasov equation. Unfortunately those results cannot be applied for the problem considered above since the particles are distributed homogeneously in the whole space.

Remark 5.5 The fact that the noise term in the Balescu-Lenard equation can be obtained by means of the evolution of the white noise by means of a linearized Vlasov-Equation has been formulated in the physical literature with different degrees of generality (cf. [33, 40, 46]). The most general formulation of this idea so far is the one in [33] where an arbitrary background of particle velocities is considered. The analysis in [40] is restricted to perturbations near the Maxwellian distribution. There is a physically very clear interpretation of Balescu-Lenard in terms of force acting on a cloud of particles generated by a tagged particle moving in a background of scatterers in [43, 44, 46].

In order to solve the problems (5.73)-(5.74) and (5.75)-(5.76) we define a fundamental solution associated to the operator on the left hand side of (5.73), (5.75). We define the background of scatterers in [43, 44, 46].

\[ \text{Remark 5.5} \]

The fact that the noise term in the Balescu-Lenard equation can be obtained by means of the evolution of the white noise by means of a linearized Vlasov-Equation has been formulated in the physical literature with different degrees of generality (cf. [33, 40, 46]). The most general formulation of this idea so far is the one in [33] where an arbitrary background of particle velocities is considered. The analysis in [40] is restricted to perturbations near the Maxwellian distribution. There is a physically very clear interpretation of Balescu-Lenard in terms of force acting on a cloud of particles generated by a tagged particle moving in a background of scatterers in [43, 44, 46].

\[ \text{Remark 5.4} \]

It is interesting to remark that in the case of solutions of the Vlasov equation with integrable initial data there are rigorous results in [12] which show that the density fluctuations can be approximated by the linearized Vlasov equation. Unfortunately those results cannot be applied for the problem considered above since the particles are distributed homogeneously in the whole space.

\[ \text{Remark 5.5} \]

The fact that the noise term in the Balescu-Lenard equation can be obtained by means of the evolution of the white noise by means of a linearized Vlasov-Equation has been formulated in the physical literature with different degrees of generality (cf. [33, 40, 46]). The most general formulation of this idea so far is the one in [33] where an arbitrary background of particle velocities is considered. The analysis in [40] is restricted to perturbations near the Maxwellian distribution. There is a physically very clear interpretation of Balescu-Lenard in terms of force acting on a cloud of particles generated by a tagged particle moving in a background of scatterers in [43, 44, 46].
Notice that in this way we approximate the dynamics of the tagged particle using a friction term and a random force field. Notice also that we are using the notation $H_g$ to denote the quantity that was $-\Lambda_g$ in the previous Subsection.

Actually we are interested in the dynamics given by (5.81) in times of the order of the mean free time between collisions. In those times we have $\tilde{t} \gg 1$. Therefore it is natural to study the asymptotic behaviour of $\tilde{F}_g(\xi, \tilde{t})$ and the friction coefficient $H_g(\tilde{t}; V)$ as $\tilde{t} \to \infty$.

We have:

$$\tilde{H}_g(\infty; V) = \int_{(\mathbb{R}^3)_r} \nabla_y\Phi(y) \int_0^\infty ds G_\sigma(y - \eta, w, w_0, s) \left[ \nabla_\eta\Phi(\eta + Vs) \cdot \nabla w g(w_0) \right] d\eta dw_0 dydw$$

and we can then approximate the evolution of the tagged particle in time scales larger than $\tilde{t}$ as:

$$\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = \varepsilon (L_\varepsilon)^2 \tilde{F}_g(\xi, \tilde{t}) + \varepsilon^2 (L_\varepsilon)^3 H_g(\infty; V)$$

Concerning the random force field, we need to keep the dependence of the field on $\tilde{t}$, since the random force field is time dependent, but we want to consider the asymptotic behaviour of the field for large times. To this end we define the random force field:

$$F_g(\xi, \tilde{t}) = \lim_{T \to \infty} \tilde{F}_g(\xi, \tilde{t} + T)$$

It is possible to give a physical interpretation to both terms $H_g(\tilde{t}; V)$ and $F_g(\xi, \tilde{t})$ as follows. The term $H_g(\tilde{t}; V)$ is due to the fact that the presence of the tagged particle induces a force in the surrounding distribution of scatterers. These scatterers rearrange their positions as a consequence of their mutual interactions and the forces induced by the tagged particle and this results in a reaction force acting on the tagged particle. On the other hand the fluctuations of the particle density yield a random force field. These fluctuations of the particle density are rearranged due to the effect of the mutual interactions between the particles. The resulting force field after these fluctuations reach a steady state is $F_g(\xi, \tilde{t})$. Notice that:

$$F_g(\xi, \tilde{t}) = \lim_{T \to \infty} \left[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_y\Phi(y) \int_{\mathbb{R}^3} dw_0 G_\sigma(y - \eta, w, w_0, \tilde{t} + T) N(\eta + \xi, w_0) dydw \right]$$

In order to have a well defined quantity by means of (5.82) and a noise with integrable time correlations by means of (5.84) we need to make some assumptions in $g(w)$ and the interaction potential $\Phi$ in order to have suitable decay properties for $G_\sigma(y, w, w_0, \tilde{t})$ as $\tilde{t} \to \infty$. The conditions that must be imposed on $g$ in order to obtain such a decay will be discussed in the next Subsection. We just remark here that this decay condition for $G_\sigma$ means, from the physical point of view that the ”medium” formed by the set of particles is stable under its own interactions. We will just assume in the following that the decay of $G_\sigma$ in $\tilde{t}$ is sufficiently fast to ensure the convergence of all the integrals appearing in the rest of this Subsection converge.

We then assume that the friction term is well defined by means of (5.82). We compute now the statistical properties of the noise term defined by means of (5.84). Notice that $F_g(\xi, \tilde{t})$ is a Gaussian noise with zero average. Therefore, in order to characterize $F_g(\xi, \tilde{t})$ we just need
to compute the covariance function: which is given by:

$$
\mathbb{E} \left[ F_g (\xi_1, \tilde{t}_1) F_g (\xi_2, \tilde{t}_2) \right] = \lim_{T \to \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_y \Phi (y_1) \nabla_y \Phi (y_1) dy_1 dy_2 dw_1 dw_2 \\
\cdot \int_{\mathbb{R}^3} dn_1 \int_{\mathbb{R}^3} dw_{0,1} \int_{\mathbb{R}^3} dn_2 \int_{\mathbb{R}^3} dw_{0,2} G_\sigma (y_1 - \eta_1, w_1, w_{0,1}, \tilde{t}_1 + T) G_\sigma (y_2 - \eta_2, w_2, w_{0,2}, \tilde{t}_2 + T) \cdot
\cdot \mathbb{E} \left[ N (\eta_1 + \xi_1, w_{0,1}) N (\eta_2 + \xi_2, w_{0,2}) \right].
$$

Using (5.12) and evaluating the integrals using the dirac masses, we obtain

$$
\mathbb{E} \left[ F_g (\xi_1, \tilde{t}_1) \otimes F_g (\xi_2, \tilde{t}_2) \right] = \lim_{T \to \infty} \int_{(\mathbb{R}^3)^2} \nabla_y \Phi (y_1) \otimes \nabla_y \Phi (y_2) dy_1 dy_2 \\
\cdot \int_{(\mathbb{R}^3)^2} d\eta_1 dw_{0,1} \Xi_\sigma (y_1 - \eta_1, w_0, \tilde{t}_1 + T) \Xi_\sigma (y_2 - \eta_2 + \xi_2, w_0, \tilde{t}_2 + T) q (w_0),
$$

where \( \Xi_{\sigma} \) is the velocity marginal of \( G_\sigma \) defined by

$$
\Xi_\sigma (y, w, t) = \int_{\mathbb{R}^3} G_\sigma (y, w, w, t) dw.
$$

Notice that we have reduced the dynamics of the tagged particle to the stochastic differential equation (5.83) with the friction term \( H_g (V) \) given by (5.82) and the random force field \( F_g (\xi, \tilde{t}) \) having zero average and variance (5.85).

In order to compute the limit in (5.83) and to check that the friction term \( H_g (V) \) in (5.82) is well defined we need to study the properties of the function \( G_\sigma (y, w, w_0, \tilde{t}) \) and in particular its long time asymptotics as \( \tilde{t} \to \infty \).

**Stability properties of a Vlasov medium.** As indicated above, in order to obtain a well defined noise term \( F_g (\xi, \tilde{t}) \) and friction coefficient \( H_g (\tilde{t}; V) \) the function \( G_\sigma \) that solves (5.77)-(5.78) has to decay sufficiently fast as \( \tilde{t} \to \infty \). This decay is closely related to the stability properties of the system of particles described by equations with the form (1.11) with interaction potentials as in (3.22), (3.23). For these potentials makes sense to approximate (1.11) by means of the Vlasov equation. The stability of homogeneous distributions of particles with a distribution of velocities given by \( q (w) \) which can be approximated using (5.69) was first considered in [31, 33] where the linearized Vlasov equation around a homogeneous distribution of particles was considered. It was found in that paper the possibility of damping of perturbations in a homogeneous equation in spite of the fact that the Vlasov equation does not include the effect of time irreversible effects like particle collisions.

A general condition on the distribution of velocities \( q (w) \) yielding stability of a homogeneous medium was given in [41]. A rigorous proof of stability of the homogeneous state for a large class of velocity distributions under the nonlinear Vlasov equation with Coulombian potentials in the torus has been obtained in [37]. In this paper we restrict ourselves to the analysis of the linearized problem (5.77)-(5.78). We discuss conditions on the potentials \( \Phi \) and the velocity distributions \( q \) yielding a behaviour on \( G_\sigma \) for large \( \tilde{t} \) allowing to define the friction coefficient \( H_g (V) \) and the random force field \( F_g \) by means of (5.82) and (5.85) respectively.

From the physical point of view the decay of \( G_\sigma \) for long times means that the homogeneous distribution of scatterers is stable under the combined effect of their dispersion of velocities.
and their long range mutual interactions. We notice in particular that the interaction between the scatterers does not yield an exponential growth of the density inhomogeneities in the phase space.

The simplest way of solving (5.77)-(5.78) is applying Fourier in the $x$ variable and Laplace in the time variable $\tilde{t}$, as it was made in [31, 33]. We define the following Fourier-Laplace transform:

$$
\tilde{F}(k, v, z) = \frac{1}{(2\pi)^2} \int_0^\infty \!\! dz \int_{\mathbb{R}^3} \!\! dx e^{-i x \cdot k - z \tilde{t}} F(x, v, \tau)
$$

Taking the transform of (5.77)-(5.78) we obtain:

$$(z + i(k \cdot w)) \tilde{G}_\sigma (k, w, w_0, z) - \sigma (2\pi)^{\frac{3}{2}} (\nabla_w g(w) \cdot i k) \tilde{\Phi}(k) \tilde{\Xi}_\sigma (k, w_0, z) = \frac{\delta (w - w_0)}{(2\pi)^{\frac{3}{2}}},$$  

(5.86)

We can rewrite (5.86) as:

$$
\tilde{G}_\sigma (k, w, w_0, z) = \sigma (2\pi)^{\frac{3}{2}} (\nabla_w g(w) \cdot i k) \tilde{\Phi}(k) \tilde{\Xi}_\sigma (k, w_0, z) + \frac{\delta (w - w_0)}{(2\pi)^{\frac{3}{2}} (z + i(k \cdot w))},
$$

(5.87)

Integrating in $w$ we obtain:

$$
\tilde{\Xi}_\sigma (k, w_0, z) = \left( \sigma (2\pi)^{\frac{3}{2}} \tilde{\Xi}_\sigma (k, w_0, z) \tilde{\Phi}(k) \int_{\mathbb{R}^3} \frac{i k \cdot \nabla_w g(w)}{z + i k \cdot w} dw + \frac{1}{(2\pi)^{\frac{3}{2}} (z + i k w_0)} \right).
$$

(5.88)

Then we can represent $\tilde{\Xi}_\sigma$ explicitly as:

$$
\tilde{\Xi}_\sigma (k, w_0, z) = \frac{1}{(2\pi)^{\frac{3}{2}} (z + i(k \cdot w_0)) \Delta_\sigma (k, z)}
$$

(5.89)

where:

$$
\Delta_\sigma (k, z) = 1 - \sigma (2\pi)^{\frac{3}{2}} \tilde{\Phi}(k) \int_{\mathbb{R}^3} \frac{(i k \cdot \nabla_w g(w))}{z + i(k \cdot w)} dw.
$$

(5.90)

Using (5.87) we obtain also:

$$
\tilde{G}_\sigma (k, w, w_0, z) = \frac{\sigma (\nabla_w g(w) \cdot i k) \tilde{\Phi}(k)}{(z + i k \cdot w_0)(z + i k \cdot w)} \Delta_\sigma (k, z) + \frac{\delta (w - w_0)}{(2\pi)^{\frac{3}{2}} (z + i(k \cdot w_0))},
$$

(5.91)

In order to avoid the exponential growth of the disturbances we need the following stability condition:

$$
\Delta_\sigma (k, z) \neq 0 \text{ for } \Re(z) \geq 0 \text{ and any } k \in \mathbb{R}^3.
$$

(5.92)

We remark that in case of a smooth decaying potential $\phi$, this criterion can for example be satisfied choosing $0 < \sigma \ll 1$ small.

Notice that in the formulas of $\tilde{F}_g(\xi, \tilde{t})$ and $H_g(\tilde{t}; V)$ we only make use of $\tilde{\Xi}_\sigma$.

Inverting the Fourier-Laplace transform in , we obtain:

$$
\Xi_\sigma (x, w_0, \tilde{t}) = \frac{1}{2\pi i} \int_{\mathbb{R}^3} \frac{e^{ik \cdot x} dk}{(2\pi)^{\frac{3}{2}}} \int_{\gamma} \frac{e^{iz} dz}{(z + i(k \cdot w_0)) \Delta_\sigma (k, z)}.
$$

(5.93)
We can consider possible analyticity properties of the function $\Delta_\sigma (k, z)$ \[5.90\] in the variable $z$, in order to obtain a possible decay of the integral above in time.

We rewrite the function in terms of the Radon transform $H (s, \theta)$ of $g$, which for $\theta \in S^2$ and $s \in \mathbb{R}^3$ is defined as

$$
H (s; \theta) = \int_{\{ \theta \cdot w = s \}} g (w) dS (w).
$$

With this definition, the integral appearing in \[5.90\] can be rewritten as

$$
\int_{\mathbb{R}^3} \frac{(\theta \cdot \nabla_w g (w))}{\zeta + i (\theta \cdot w)} \, dw = \int_{-\infty}^{\infty} \frac{ds}{\zeta + is} \frac{\partial H (s; \theta)}{\partial s}.
$$

We will assume that $H (s; \theta)$ is analytic in a strip $\{ |\text{Im} (s)| < \delta_0 \}$ for all the values of $\theta \in S^2$. Then $\Delta_\sigma (k, z)$ would be analytic in a strip $\{ |\text{Im} (\zeta)| < \delta_0 \}$. Initially we assume that $\text{Re} (\zeta) > 0$. We can move the line of integration from $\mathbb{R}$ to $\mathbb{R} - \frac{i\pi}{2}$. It then follows that

$$
\Psi (\zeta; \theta) := \int_{\mathbb{R}^3} \frac{(\theta \cdot \nabla_w g (w))}{\zeta + i (\theta \cdot w)} \, dw = \int_{\mathbb{R} - \frac{i\pi}{2} i} \frac{ds}{\zeta + is} \frac{\partial H (s; \theta)}{\partial s} \tag{5.94}
$$

is analytic for $\text{Re} (\zeta) > - \frac{\delta_0}{2}$.

Now suppose that for (possibly smaller value) $\delta_0 > 0$ we can further ensure that $\Delta_\sigma (k, z)$ does not vanish for $\text{Re} (\zeta) > - \frac{\delta_0}{2}$. Then the function

$$
\Delta_\sigma (k, z) = 1 - \sigma \hat{\Phi} (k) \Psi \left( \frac{z}{|k|}; \frac{k}{|k|} \right)
$$

is analytic in $z$ on a region of the form $\{ \text{Re} (z) \geq - \delta_1 |k| \}$.

With this we return to the friction term $H_g$ given by:

$$
H_g (\tilde{t}; V) = \int_{(\mathbb{R}^3)^4} \nabla_y \Phi (y) \int_0^\tilde{t} ds \Xi_\sigma (y - \eta, w_0, s) \left[ \nabla_\eta \Phi (\eta + V s) \cdot \nabla_w g (w_0) \right] d\eta dw_0 dy
$$

Inserting the Fourier-Laplace representation of $\Xi_\sigma$ we obtain

$$
H_g (\tilde{t}; V) = \int_{(\mathbb{R}^3)^4} \int_0^\tilde{t} ds \left[ \frac{1}{2\pi i} \frac{e^{ik \cdot (y - \eta)}}{(2\pi)^3} \int_{\gamma} dz \frac{e^{iz \cdot \nabla_y \Phi (y)} \left[ \nabla_\eta \Phi (\eta + V s) \cdot \nabla_w g (w_0) \right]}{(z + i (k \cdot w_0)) \Delta_\sigma (k, z)} \right] d\eta dw_0 dy
$$

The integrals in $y, \eta$ are explicit and yield

$$
H_g (\tilde{t}; V) = \int_0^\tilde{t} e^{ik \cdot V s} ds \int_{\mathbb{R}^3} k \left| \Phi (k) \right|^2 \frac{1}{2\pi i} \int_{\gamma} dz \frac{e^{iz \cdot \nabla_w g (w_0)}}{(z + i (k \cdot w_0))} \Delta_\sigma (k, z) \int_{\mathbb{R}^3} \nabla_\eta \Phi (\eta + V s) \, d\eta
$$

By definition of $\Psi \ \ \ \ \ [5.94]$ this is equal to

$$
H_g (\tilde{t}; V) = \int_0^\tilde{t} e^{ik \cdot V s} ds \int_{\mathbb{R}^3} \frac{k}{|k|} \left| \Phi (k) \right|^2 \frac{1}{2\pi i} \int_{\gamma} dz \frac{e^{iz \cdot \nabla_w g (w_0)}}{(z + i (k \cdot w_0))} \Delta_\sigma (k, z) \Psi \left( \frac{z}{|k|}; \theta \right)
$$

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The analyticity properties of $\Delta_\sigma (k, z)$ and $\Psi \left( \frac{z}{|k|}; \theta \right)$ allow to deform $\gamma$ to a contour contained in $\text{Re} \left( z \right) < 0$. We then obtain exponential decay of the integral as $s \to \infty$. This gives

$$H_g (V) = H_g (\infty; V) = - \frac{k}{|k|} \frac{1}{2\pi i} \int_{\gamma} \frac{1}{(ik \cdot V + z)} \Delta_\sigma (k, z) \Psi \left( \frac{z}{|k|}; \theta \right) dz.$$  

(5.95)

This is the friction coefficient associated to Balescu-Lenard. Notice that the function $H_g$ depends on the velocity distribution $g$ through the dependence of $\Psi \left( \frac{z}{\lambda}; \theta \right)$ on $g$ (cf. (5.94)).

We now use the Fourier-Laplace representation of $\Xi_\sigma$ to compute the time correlation of the forces.

$$\mathbb{E} \left[ F_g (\xi_1, t_1) F_g (\xi_2, t_2) \right] = \lim_{T \to \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_y \Phi (y_1) \nabla_y \Phi (y_2) \frac{dy_1 dy_2}{(2\pi)^3} \int_{\mathbb{R}^3} g (w_0) dw_0 \cdot \int_{\mathbb{R}^3} \frac{e^{ik \cdot (y_1 - y_2)}}{(2\pi)^3} \frac{1}{\Delta_\sigma (k, z_1)} e^{-ik \cdot (\xi_2 - \xi_1)} dw_2$$

(5.96)

The integral in $\eta_1$ gives a Dirac mass in $k = k_1 = -k_2$, so:

$$\mathbb{E} \left[ F_g (\xi_1, t_1) \otimes F_g (\xi_2, t_2) \right] = \lim_{T \to \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_y \Phi (y_1) \otimes \nabla_y \Phi (y_2) \frac{dy_1 dy_2}{(2\pi)^3} \int_{\mathbb{R}^3} g (w_0) dw_0 \cdot \int_{\mathbb{R}^3} \frac{e^{ik \cdot y_1}}{(2\pi)^3} \frac{1}{\Delta_\sigma (k, z_1)} e^{-ik \cdot (\xi_2 - \xi_1)} dw_2$$

(5.97)

We will assume that $\Delta_\sigma (k, z_1)$ is analytic in the regions indicated above. We can then compute the integrals along the circuits $\gamma$ using residues. The contribution to the integral in the region $\text{Re} \left( z_j \right) < 0$ converges exponentially to zero as $T \to \infty$. The exponent depends on $|k|$ but due to the integrability in $k$ we can take the limit of those terms. Therefore, we are left only with the contributions due to residues at $z_1 = -i (k \cdot w_0), \ z_2 = i (k \cdot w_0)$. We then obtain:

$$\mathbb{E} \left[ F_g (\xi_1, t_1) \otimes F_g (\xi_2, t_2) \right] = \int_{\mathbb{R}^3} g (w_0) dw_0 \int_{\mathbb{R}^3} (k \otimes k) \Phi (k) \frac{2}{|\Delta_\sigma (k, -i (k \cdot w_0)) \Delta_\sigma (-k, i (k \cdot w_0))|} e^{ik \cdot w_0 (\xi_2 - \xi_1)} dk.$$  

Using the identity $\Delta_\sigma (-k, z^*) = (\Delta_\sigma (k, z))^*$ we obtain:

$$\mathbb{E} \left[ F_g (\xi_1, t_1) \otimes F_g (\xi_2, t_2) \right] = \int_{\mathbb{R}^3} g (w_0) dw_0 \int_{\mathbb{R}^3} (k \otimes k) \Phi (k) \frac{2}{|\Delta_\sigma (k, -i (k \cdot w_0)) \Delta_\sigma (-k, i (k \cdot w_0))|^2} e^{ik \cdot w_0 (\xi_2 - \xi_1)} dk.$$  

(5.98)
Therefore, the main contribution to the function $\Delta_{\sigma}(k, z)$ is due to the integral term which yields the effect of the self-consistent interactions between the particles. A problem which deserves more detailed analysis is the study of the stability properties of the corresponding Vlasov medium for this particular scaling limit. We plan to address this issue in the future.

Remark 5.6 It is interesting to notice that we could consider scaling limits in the potentials yielding $\sigma = \infty$, i.e. interaction potentials with the form (5.22), (5.23) with $\varepsilon (L_\varepsilon)^3 \to \infty$. Thus, the main contribution to the function $\Delta_{\sigma}(k, z)$ is due to the integral term which yields the effect of the self-consistent interactions between the particles. A problem which deserves more detailed analysis is the study of the stability properties of the corresponding Vlasov medium for this particular scaling limit. We plan to address this issue in the future.

Derivation of the kinetic equation yielding the evolution of the tagged particle.

We can now write the evolution equation for the distribution function $f(x, V, t)$. We will restrict ourselves to the case in which $\sigma = 1$, but the argument should be adapted if $\sigma \gg 1$ as $\varepsilon \to 0$. In this case we can use the same rescalings as in Subsection 5.1.1. Assuming that $\tilde{t} \gg 1$ we can approximate (5.83) as

$$
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = \varepsilon (L_\varepsilon)^{\frac{3}{2}} F_{\varepsilon}(\xi, \tilde{t}) + \varepsilon^2 (L_\varepsilon)^{\frac{3}{2}} H_{\varepsilon}(V)
$$

It is then natural to introduce a time scale $t = \frac{\tilde{t}}{\varepsilon (L_\varepsilon)^{\frac{3}{2}}} = \frac{\tau}{\varepsilon (L_\varepsilon)^{\frac{3}{2}}}$, and a macroscopic length scale $x = \frac{\xi}{\varepsilon (L_\varepsilon)^{\frac{3}{2}}} = \frac{X}{\varepsilon (L_\varepsilon)^{\frac{3}{2}}}$, whence

$$
\frac{dx}{dt} = V, \quad \frac{dV}{dt} = \frac{1}{\varepsilon (L_\varepsilon)^{\frac{3}{2}}} F_{\varepsilon} \left( x_0 + V \varepsilon^2 (L_\varepsilon)^{3} t , \varepsilon^2 (L_\varepsilon)^{3} t \right) + H_{\varepsilon}(V)
$$

(5.99)

where we approximate $\xi$ in $F_{\varepsilon}$ by means of a linear function. We assume that $\varepsilon (L_\varepsilon)^{\frac{3}{2}} \to 0$ as $\varepsilon \to 0$. Then (5.99) can be approximated by the stochastic differential equation:

$$
\frac{dx}{dt} = V, \quad \frac{dV}{dt} = \eta(t; V) + H_{\varepsilon}(V)
$$

where, using (5.98) we obtain that

$$
\mathbb{E} \left[ \eta(t_1; V) \otimes \eta(t_2; V) \right] = \delta(t_1 - t_2) \cdot D(V)
$$

where

$$
D_{\varepsilon}(V) = \int_0^\infty dt \int_{\mathbb{R}^3} g(w_0)dw_0 \int_{\mathbb{R}^3} [k \otimes k] \frac{\Phi(k)}{\Delta_{\sigma}(k, -i(k \cdot w_0))} dk
$$

$$
= \int_{\mathbb{R}^3} g(w_0) dw_0 \int_{\mathbb{R}^3} [k \otimes k] \frac{\Phi(k)}{[k \cdot (V - w_0) + 0^+] \Delta_{\sigma}(k, -i(k \cdot w_0))} dk
$$

(5.100)

We can then obtain the equation yielding the evolution of the distribution of particles $f(x, v, t)$ which is given by:

$$
\partial_t f(x, v, t) + v \cdot \partial_x f(x, v, t) = \partial_v \left( \frac{D_f(v)}{2} \partial_v f - H_f(v) f \right)(x, v, t)
$$

(5.101)

where notice that we assume that the diffusion matrix in the velocity space is chosen assuming that the distribution of velocities at each point $x$ is given by $f(x, \cdot, t)$.
Dielectric function. It is customary in the physical literature to assume that the set of particles described by means of Vlasov equations, can be interpreted as an effective medium in which the particle density rearranges due to the action of an external field. The effective properties of the medium are usually described by means of the so-called dielectric function. 

In order to define the dielectric function we consider the following generalization of the Vlasov equation (cf. (5.101)) in which we replace the force due to a tagged particle in the Vlasov medium by an arbitrary force term $F_{\text{ext}}(y, w) :$

$$\partial_t f(y, w, \tilde{t}) + w \cdot \nabla_y f(y, w, \tilde{t}) + \left[ F_{\text{ext}}(y, w) - \sigma \int_{\mathbb{R}^3} \nabla_y \Phi(y - \eta) \rho(\eta, \tilde{t}) \, d\eta \right] \cdot \nabla_w f(y, w, \tilde{t}) = 0$$

where $\sigma = \varepsilon (L_v)^3$. We consider solutions of this equation close to the spatially homogeneous solution $g(w)$. We define $h(y, w, \tilde{t}) = f(y, w, \tilde{t}) - g(w)$. Then, assuming that $F_{\text{ext}}$ and $h$ are small and linearizing we obtain:

$$\partial_t h(y, w, \tilde{t}) + w \cdot \nabla_y h(y, w, \tilde{t}) + \left[ F_{\text{ext}}(y, \tilde{t}) - \sigma \int_{\mathbb{R}^3} \nabla_y \Phi(y - \eta) \tilde{\rho}(\eta, \tilde{t}) \, d\eta \right] \cdot \nabla_w g(w) = 0$$

(5.102)

with:

$$\tilde{\rho}(\eta, \tilde{t}) = \int_{\mathbb{R}^3} h(\eta, w, \tilde{t}) \, dw$$

(5.103)

Notice that the total force exerted by the combination of external forces and the forces due to the particles of the Vlasov medium is:

$$F(y, \tilde{t}) = F_{\text{ext}}(y, \tilde{t}) - \sigma \int_{\mathbb{R}^3} \nabla_y \Phi(y - \eta) \tilde{\rho}(\eta, \tilde{t}) \, d\eta$$

(5.104)

In order to define the dielectric function we consider external forces with the form $F_{\text{ext}}(y, \tilde{t}) = e^{i(\omega t + k \cdot y)}F_0$ with $\omega \in \mathbb{R}$, $k \in \mathbb{R}^3$, $F_0 \in \mathbb{R}^3$. In that case the force $F(y, \tilde{t})$ is proportional to $e^{i(\omega t + k \cdot y)}$. We define the dielectric function $\varepsilon = \varepsilon(k, \omega) \in C(\mathbb{R}^3 \times \mathbb{R}; M_{3 \times 3}(\mathbb{C}))$, where we denote as $M_{3 \times 3}(\mathbb{C})$ the set of $3 \times 3$ matrices with complex coefficients, by means of:

$$F(y, \tilde{t}) = [\varepsilon(k, \omega) F_0] e^{i(\omega t + k \cdot y)}$$

(5.105)

In order to compute the function $h$ for $F_{\text{ext}}(y, \tilde{t}) = e^{i(\omega t + k \cdot y)}F_0$ we look for solutions of (5.102), (5.103) with the form $h(x, w, t) = e^{i(\omega t + k \cdot y)}H(w)$. Then $H(w)$ satisfies:

$$i(\omega t + k \cdot w) H(w) - (2\pi)^3 2\varepsilon_l \, \nabla_w g(w) \cdot k \, \Phi(k) \tilde{\rho_0} = -F_0 \cdot \nabla_w g(w)$$

(5.106)

where:

$$\tilde{\rho_0} = \int_{\mathbb{R}^3} H(w) \, dw$$

$$\int_{\mathbb{R}^3} \nabla_y \Phi(y - \eta) e^{-ik \cdot (y - \eta)} \, d\eta = (2\pi)^3 / 2 \, \delta(k - k)$$

Therefore:

$$H(w) = \frac{(2\pi)^3 2\varepsilon_l \, \nabla_w g(w) \cdot k \, \Phi(k) \tilde{\rho_0}}{(\omega t + k \cdot w + i0^+)} - \frac{F_0 \cdot \nabla_w g(w)}{i(\omega t + k \cdot w + i0^+)}$$

55
Then, integrating in \( w \):

\[
\tilde{\rho}_0 - (2\pi)^{\frac{3}{2}} \sigma \tilde{\rho}_0 \hat{\Phi} (k) \int_{\mathbb{R}^3} \frac{\nabla_w g (w) \cdot k}{(\omega t + k \cdot w + i0^+)} dw = - \int_{\mathbb{R}^3} \frac{F_0 \cdot \nabla_w g (w)}{i (\omega t + k \cdot w + i0^+)} dw
\]

whence

\[
\tilde{\rho}_0 = - \frac{1}{\Delta_\sigma (k, i\omega)} \int_{\mathbb{R}^3} \frac{F_0 \cdot \nabla_w g (w)}{i (\omega t + k \cdot w + i0^+)} dw
\]

Then, using that \( F_{\text{ext}} (y, \tilde{t}) = e^{i(\omega t + k \cdot y)} F_0 \) it follows that:

\[
F (y, \tilde{t}) = F_{\text{ext}} (y, \tilde{t}) - \sigma \int_{\mathbb{R}^3} \nabla_y \Phi (y - \eta) \tilde{\rho} (\eta, \tilde{t}) d\eta
\]

\[
= e^{i(\omega t + k \cdot y)} F_0 + \int_{\mathbb{R}^3} \nabla_y \Phi (y - \eta) e^{i(\omega t + k \cdot \eta)} d\eta \int_{\mathbb{R}^3} \frac{\nabla_w g (w) \cdot F_0}{i (\omega t + k \cdot w + i0^+)} dw
\]

\[
= e^{i(\omega t + k \cdot y)} \left( F_0 + \frac{(2\pi)^{\frac{3}{2}} \sigma}{\Delta_\sigma (k, i\omega)} \hat{\Phi} (k) k \int_{\mathbb{R}^3} \frac{\nabla_w g (w) \cdot F_0}{(\omega t + k \cdot w + i0^+)} dw \right)
\]

whence:

\[
\epsilon (k, \omega) = I + \frac{(2\pi)^{\frac{3}{2}} \sigma}{\Delta_\sigma (k, i\omega)} \hat{\Phi} (k) k \int_{\mathbb{R}^3} \frac{k \otimes \nabla_w g (w)}{(\omega t + k \cdot w + i0^+)} dw
\]

### 5.2.2 Coulomb-like interaction potentials.

We now consider interacting particle systems with the form (1.1) in which the interaction potentials are as in (5.27) (where \( \Phi (s) \sim \frac{1}{s} \) as \( s \to \infty \)). This question has been extensively studied in the physical literature due to his relevance in astrophysics and the in the theory of plasmas (cf. for instance ([42]), ([50])). In this case we cannot assume as in Subsection 5.2.1 that the interaction potential has a large, but finite range. On the contrary, in this case, the range of the potential is infinity. However, it turns out that, assuming some stability conditions for the Vlasov medium similar to the ones discussed above, there exists a characteristic length, namely the so-called Debye length which yields the effective interaction length between the particles of the system. This length is characterized by the fact that the changes in the particle density in the phase space due to the dispersion of the velocities is comparable to the ones due to the forces due to the particle deflections (or also by the presence of a tagged particle). A precise definition will be given in (5.11).

Another difference between the case of Coulomb potentials and the problem considered in Subsection 5.2.1 is that in this second case, there is a well defined friction coefficient acting over a tagged particle moving at speed \( V \) (cf. (5.81)) and a well defined noise acting on the tagged particle (cf. (5.81), (5.79)). Both the friction term and the noise stabilize to their asymptotic behaviour in large microscopic times. In the case of Coulombian interactions such stabilization of the friction and noise term in macroscopic times does not take place. On the contrary, they increase logarithmically. It is well known that there is not a large difference in practice between a magnitude converging to a value or increasing logarithmically. Nevertheless, the main consequence of this logarithmic behaviour will be the onset in the macroscopic time scale of the Coulombian logarithm as it might be expected in a system with Coulombian interactions.
Finally, it is interesting to remark that there is a difference between the interacting particle systems (1.1) and the case of Rayleigh gases with Coulombian interactions considered in Subsection 5.1.2. In this second case, some of the logarithmic factors are due to the fact that all the dyadic scales between the cutoff length for the Coulombian potential and the macroscopic scale yield contributions of a similar size in the computation of the friction term and the noise term. (See the discussion in Subsection 5.1.3 about the contributions due to dyadics for Coulombian potentials). In the case of interacting particle systems the "dyadic" scales contributing to the friction and noise term are those between the particle size and the Debye length. Due to this, some of the numerical factors appearing in the formula of the macroscopic scale are different for Rayleigh gases and interacting particle systems. (Compare (5.45) and (5.137)).

We now consider the evolution of a system of particles in which the particle interactions behave for large distances as Coulombian potentials. We will assume in most of the following that the potentials are smooth and in particular, that the deflection experienced by two colliding particles which interact by means of this potential is small. In the case of point charges, it is possible to cut the potential as it was made in [38] in order to separate the Boltzmann collisions (due to close binary encounters between particles) and weak deflections due to the effect of many random collisions. Given that the Boltzmann collisions take place in a larger time scale than the small deflections (due to the presence of the Coulombian logarithm) we will ignore that part of the potential in the following.

Assume that there are two types of scatterers with opposite charges, namely \( \{ Y_k \}_{k \in S} \) and \( \{ \tilde{Y}_k \}_{k \in \tilde{S}} \) having opposite charges. The main difference between the case considered in Subsection 5.2.1 and the case considered in this Subsection, besides the fact that we need to include in the dynamics at least two types of particles in order to have electroneutrality, is the fact that due to the power law behaviour of the potential, there is not any intrinsic length scale that we can call the range of the potential.

More precisely, we will assume that the interaction potential between the particles has the form in (5.27). We can assume in particular that \( \Phi (\xi) \simeq \frac{1}{|\xi|} \) for large values, introducing a cutoff near the origin in order to avoid large deflections as indicated above. We rescale the variables as in (5.29), where \( L_\varepsilon \) will be chosen now as the so-called Debye length which will be chosen shortly. We will just assume for the moment that \( L_\varepsilon \gg 1 \), i.e. that it is larger than the average particle distance. Then, the evolution of the rescaled system of particles (in which we distinguish a tagged particle) is given by:

\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\sum_{j \in S} \nabla \xi \tilde{\phi}_\varepsilon (\xi - y_j^+) + \sum_{j \in S} \nabla \xi \tilde{\phi}_\varepsilon (\xi - y_j^-), \quad \frac{dy_k^\pm}{dt} = W_k^\pm \tag{5.107}
\]

\[
\frac{dW_k^\pm}{dt} = \mp \nabla \tilde{\phi}_\varepsilon (y_k^\pm - \xi) - \sum_{j \in S, j \neq k} \left( \nabla \tilde{\phi}_\varepsilon (y_k^\pm - y_j^+) - \nabla \tilde{\phi}_\varepsilon (y_k^\pm - y_j^-) \right), \quad k \in S.
\]

Notice that we assume that all the particles have the same mass. It would be possible to consider more general situations.

We approximate the system of equations (5.107) by a Vlasov equation. We then introduce particle densities as it was made in (5.31). Then, arguing as in Subsection 5.1.2 we obtain
the following evolution equation for the tagged particle
\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = - (L_\epsilon)^2 \int_{\mathbb{R}^3} \nabla_x \tilde{\phi}_{\epsilon} (\xi - \eta) \left[ \rho_{\epsilon}^+(\eta, \tilde{\epsilon}) - \rho_{\epsilon}^- (\eta, \tilde{\epsilon}) \right] d\eta, \tag{5.108}
\]
and for the scatterers distributions:
\[
\partial_t f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) + w \cdot \nabla_y f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) - \left[ (L_\epsilon)^2 \int_{\mathbb{R}^3} \nabla_y \tilde{\phi}_{\epsilon} (y - \eta) (\rho_{\epsilon}^+ - \rho_{\epsilon}^-) (\eta, \tilde{\epsilon}) d\eta \pm \nabla_y \tilde{\phi}_{\epsilon} (y - \xi) \right] \cdot \nabla_w f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) = 0. \tag{5.109}
\]
where \( \rho_{\epsilon}^\pm = \rho[f_{\epsilon}^\pm] \).

We can now define the Debye screening length \( L_\epsilon \). This length will be chosen in order to make the terms associated to particle transport (i.e. \( w \cdot \nabla_y f_{\epsilon}, w \cdot \nabla_y f_{\epsilon} \)) and the terms describing the self-interaction between scatterers (i.e. \( \nabla_y \tilde{\phi}_{\epsilon} * \rho_{\epsilon} \cdot \nabla_w f_{\epsilon} \) and similar ones) comparable. To find the correct scale, we use that \( \tilde{\phi}_{\epsilon} (\xi) = \phi_{\epsilon} (L_\epsilon \xi) \sim \frac{1}{\xi} \) if \( |\xi| \) is of order one we obtain the following approximation:
\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = - \varepsilon (L_\epsilon)^2 \int_{\mathbb{R}^3} \nabla_x \left( \frac{1}{|\xi - \eta|} \right) \left[ \rho_{\epsilon}^+(\eta, \tilde{\epsilon}) - \rho_{\epsilon}^- (\eta, \tilde{\epsilon}) \right] d\eta \tag{5.110}
\]
\[
\partial_t f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) + w \cdot \nabla_y f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) - \left[ \varepsilon (L_\epsilon)^2 \int_{\mathbb{R}^3} \nabla_y \left( \frac{1}{|y - \eta|} \right) (\rho_{\epsilon}^+ - \rho_{\epsilon}^-) (\eta, \tilde{\epsilon}) d\eta \pm \frac{\varepsilon}{L_\epsilon} \nabla_y \left( \frac{1}{|y - \xi|} \right) \right] \cdot \nabla_w f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) = 0. \tag{5.111}
\]
Then, choosing
\[
\varepsilon (L_\epsilon)^2 = 1 \tag{5.111}
\]
the self consistent force term is of the same order of magnitude as the convection.

With this choice of scaling, the system \((5.108)-(5.109)\) reads:
\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = - (L_\epsilon)^2 \int_{\mathbb{R}^3} \nabla_x \Phi (L_\epsilon |\xi - \eta|) \left[ \rho_{\epsilon}^+(\eta, \tilde{\epsilon}) - \rho_{\epsilon}^- (\eta, \tilde{\epsilon}) \right] d\eta \\
0 = \partial_t f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) + w \cdot \nabla_y f_{\epsilon}^\pm (y, w, \tilde{\epsilon}) - (L_\epsilon)^2 \left[ \int_{\mathbb{R}^3} \nabla_Y \Phi (L_\epsilon |y - \eta|) (\rho_{\epsilon}^+ - \rho_{\epsilon}^-) (\eta, \tilde{\epsilon}) d\eta \pm \frac{\varepsilon}{L_\epsilon} \nabla_Y \Phi (L_\epsilon |y - \xi|) \right] \cdot \nabla_w f_{\epsilon}^\pm (y, w, \tilde{\epsilon}). \tag{5.112}
\]
where we use the fact that \( \Phi = \Phi (X) \) and we denote as \( \nabla_X \Phi \) the usual gradient with respect to the variable \( X \).

The system \((5.112)\) must be solved with the initial conditions \((5.38)-(5.39)\), where the average \( \int_{\mathbb{R}^3} f_{\epsilon}^+ (y, w, 0) - f_{\epsilon}^- (y, w, 0) dW \) vanishes.

We can now linearize as in \((5.39)\) to obtain:
\[
\partial_t \xi_{\epsilon}^\pm (y, w, \tilde{\epsilon}) + w \cdot \nabla_y \xi_{\epsilon}^\pm (y, w, \tilde{\epsilon}) - (L_\epsilon)^2 \left[ \int_{\mathbb{R}^3} \nabla_X \Phi (L_\epsilon |y - \eta|) (\rho_{\epsilon}^+ - \rho_{\epsilon}^-) (\eta, \tilde{\epsilon}) d\eta \pm \varepsilon L_\epsilon \nabla_X \Phi (L_\epsilon |y - \xi|) \right] \cdot \nabla_w g (w) = 0. \tag{5.113}
\]
with initial conditions (5.37), (5.38). We can formulate a problem for the difference \( \lambda_\varepsilon (y, w, \tilde{t}) = \zeta^+ (y, w, \tilde{t}) - \zeta^- (y, w, t) \). Due to the electroneutrality condition we have:

\[
\left( \tilde{\rho}_\varepsilon^+ - \tilde{\rho}_\varepsilon^- \right) (\eta, \tilde{t}) = \frac{1}{(L_\varepsilon)^2} \int_{\mathbb{R}^3} \left( \left( \zeta_\varepsilon^+ - \zeta_\varepsilon^- \right) (\eta, w, \tilde{t}) \right) dw = \frac{1}{(L_\varepsilon)^2} \int_{\mathbb{R}^3} \lambda_\varepsilon (\eta, v, \tilde{t}) dv. \tag{5.114}
\]

Then from (5.113) we obtain:

\[
0 = \partial_t \lambda_\varepsilon (y, w, \tilde{t}) + w \cdot \nabla_y \lambda_\varepsilon (y, w, \tilde{t}) - 2 \left( (L_\varepsilon)^2 \int_{\mathbb{R}^3} \nabla_X \Phi (L_\varepsilon | y - \eta |) \lambda_\varepsilon (\eta, v, \tilde{t}) dv + \varepsilon (L_\varepsilon)^2 \nabla_X \Phi (L_\varepsilon | y - \xi |) \right) \cdot \nabla_w g (w).
\]

and where using (5.37), (5.38) we obtain:

\[
\mathbb{E} [\lambda_\varepsilon (y, w, 0)] = 0
\]

\[
\mathbb{E} [\lambda_\varepsilon (y_a, w_a, 0) \lambda_\varepsilon (y_b, w_b, 0)] = 2 g (w_a) \delta (y_a - y_b) \delta (w_a - w_b). \tag{5.115}
\]

Using the linearity of the problem we can decompose \( \lambda_\varepsilon \) as:

\[
\lambda_\varepsilon = \lambda_1 + \lambda_2 \tag{5.116}
\]

where we do not write explicitly the dependence of \( \lambda_1, \lambda_2 \) in \( \varepsilon \) and \( \lambda_1, \lambda_2 \) solve the following problems:

\[
\partial_t \lambda_1 (y, w, \tilde{t}) + w \cdot \nabla_y \lambda_1 (y, w, \tilde{t}) - 2 (L_\varepsilon)^2 \nabla_w g (w) \int_{\mathbb{R}^3} \nabla_X \Phi (L_\varepsilon | y - \eta |) d\eta \int_{\mathbb{R}^3} \lambda_1 (\eta, v, \tilde{t}) dv = 0 \tag{5.117}
\]

\[
\mathbb{E} [\lambda_1 (y, w, 0)] = 0, \quad \mathbb{E} [\lambda_1 (y_a, w_a, 0) \lambda_1 (y_b, w_b, 0)] = 2 g (w_a) \delta (y_a - y_b) \delta (w_a - w_b) \tag{5.118}
\]

and:

\[
\partial_t \lambda_2 (y, w, \tilde{t}) + w \cdot \nabla_y \lambda_2 (y, w, \tilde{t}) = 2 \left[ (L_\varepsilon)^2 \int_{\mathbb{R}^3} \nabla_X \Phi (L_\varepsilon | y - \eta |) d\eta \int_{\mathbb{R}^3} \lambda_2 (\eta, v, \tilde{t}) dv + \varepsilon (L_\varepsilon)^2 \nabla_X \Phi (L_\varepsilon | y - \xi |) \right] \cdot \nabla_w g (w) \tag{5.119}
\]

\[
\lambda_2 (y, w, 0) = 0 \tag{5.120}
\]

In order to obtain \( \lambda_1 \) and \( \lambda_2 \) we need to study the fundamental solution for the linearized Vlasov system given by the solution of:

\[
\partial_t G (y, w; w_0, \tilde{t}) + w \cdot \nabla_y G (y, w; w_0, \tilde{t}) - 2 (L_\varepsilon)^2 \nabla_w g (w) \cdot \int_{\mathbb{R}^3} \nabla_X \Phi (L_\varepsilon | y - \eta |) d\eta \int_{\mathbb{R}^3} G (\eta, v; w_0, \tilde{t}) dv = 0 \tag{5.121}
\]

\[
G (y, w; w_0, 0) = \delta (y) \delta (w - w_0) \tag{5.122}
\]

We can solve (5.121), (5.122) applying Fourier in the variable \( y \) and Laplace in \( \tilde{t} \). If we denote this transform as \( \hat{G} = \hat{G} (k, w; w_0, z) \). Then:

\[
(z + i (k \cdot w)) \hat{G} (k, w; w_0, z) - \frac{2 (2\pi)^\frac{3}{2}}{(L_\varepsilon)^2} \hat{\Phi} \left( \frac{k}{L_\varepsilon} \right) (ik \cdot \nabla_w g (w)) \Xi (k; w_0, z) = \frac{\delta (w - w_0)}{(2\pi)^\frac{3}{2}},
\]
where \( \tilde{\Xi}(k; w_0, z) = \rho[\tilde{G}(:, w_0, z)](k) \). Then we have

\[
\tilde{G}(k, w; w_0, z) = \frac{\delta (w - w_0)}{(2\pi)^{\frac{3}{2}} (z + i (k \cdot w_0))} + \frac{2 (2\pi)^{\frac{3}{2}} \hat{\Phi} \left( \frac{k}{L_c} \right) (ik \cdot \nabla_w g(w))}{(z + i (k \cdot w))} \tilde{\Xi}(k; w_0, z) \tag{5.123}
\]

Therefore, after integrating in \( w \) we find an explicit representation of \( \tilde{\Xi} \):

\[
\tilde{\Xi}(k; w_0, z) = \frac{1}{(2\pi)^{\frac{3}{2}} (z + i (k \cdot w_0))} \Delta_e(k, z) \tag{5.124}
\]

where:

\[
\Delta_e(k, z) = 1 - \frac{2 (2\pi)^{\frac{3}{2}} \hat{\Phi} \left( \frac{k}{L_c} \right) \int \frac{(ik \cdot \nabla_w g(w))}{(z + i (k \cdot w))} dw}{(z + i (k \cdot w))} \tag{5.125}
\]

Inserting (5.124) back into (5.123) we obtain the representation:

\[
\tilde{G}(k, w; w_0, z) = \frac{\delta (w - w_0)}{(2\pi)^{\frac{3}{2}} (z + i (k \cdot w_0))} + \frac{\hat{\Phi} \left( \frac{k}{L_c} \right) (ik \cdot \nabla_w g(w))}{(z + i (k \cdot w))} \Delta_e(k, z) \tag{5.123}
\]

We will assume the usual stability condition of the medium, i.e. that the function \( \Delta_e(k, \cdot) \) is analytic in \( \{\text{Re}(z) > 0\} \) for each \( k \in \mathbb{R}^3 \).

We can now obtain \( \lambda_e \) using (5.116) as well as the fundamental solution. We approximate in (5.116) the tagged particle position \( \xi \) as \( Vt^\epsilon \), something that it is admissible for trajectory lengths shorter than the mean free path.

\[
\lambda_e(y, w, t^\epsilon) = \int_{\mathbb{R}^3} d\eta \int_{\mathbb{R}^3} dw_0 G(y - \eta, w, w_0, t^\epsilon) \lambda_1(\eta, w_0, 0) + \frac{2 \varepsilon (L_c)^{\frac{3}{2}}}{L_c} \int_0^t ds \int_{\mathbb{R}^3} dw_0 G(y - \eta, w, w_0, t - s) \nabla_X \Phi(L_c | \eta - V s|) \cdot \nabla_w g(w_0)
\]

and, using the first equation in (5.112), we obtain:

\[
\frac{d\xi}{dt} = V , \quad \frac{dV}{dt} = \frac{1}{(L_c)^2} \tilde{F}_g(\xi, t) - \frac{1}{(L_c)^3} \tilde{H}_g(\tilde{t}; V) \tag{5.126}
\]

where:

\[
\tilde{F}_g(\xi, \tilde{t}) = - \int_{\mathbb{R}^3} (L_c)^2 \nabla_Y \Phi(L_c | y - \xi|) \Xi(y - \eta, w_0, \tilde{t}) \lambda_1(\eta, w_0, 0) d(\eta, w_0, y)
\]

\[
\tilde{H}_g(\tilde{t}; V) = 2 \int_{\mathbb{R}^3} (L_c)^2 \nabla_Y \Phi(L_c | y - \tilde{V}\tilde{t}|) \int_0^\tilde{t} ds \Xi(y - \eta, w_0, \tilde{t} - s) d(\eta, w_0, y)
\]

\[
(L_c)^2 \nabla_Y \Phi(L_c | \eta - V s|) \cdot \nabla_w g(w_0)
\]

We observe that the scalings introduced in (5.126), (5.127) have been chosen to obtain quantities of order one.

Notice that \( \tilde{F}_g(\xi, \tilde{t}) \) is a Gaussian random force field. Using (5.118) we obtain:

\[
\mathbb{E} \left[ \tilde{F}_g(\xi, \tilde{t}) \right] = 0 \tag{5.128}
\]
and a covariance
\[ \mathbb{E} \left[ \tilde{F}_g (\xi_1, \tilde{t}_1) \otimes \tilde{F}_g (\xi_2, \tilde{t}_2) \right] = 2 \int_{\mathbb{R}^3} dy_1 \int_{\mathbb{R}^3} dy_2 (L_\varepsilon)^2 \nabla_X \Phi (L_\varepsilon | y_1 - \xi_1 |) \otimes (L_\varepsilon)^2 \nabla_X \Phi (L_\varepsilon | y_2 - \xi_2 |) \cdot \int_{\mathbb{R}^3} g (w_{0,1}) dw_{0,1} \int_{\mathbb{R}^3} d\eta \int_{\mathbb{R}^3} dw_1 \int_{\mathbb{R}^3} dw_2 G (y_1 - \eta, w_1, w_{0,1}, \tilde{t}_1) G (y_2 - \eta, w_2, w_{0,1}, \tilde{t}_2). \]

It is convenient to reformulate (5.126) using the set of macroscopic variables which is defined as the set of time and space scales in which the tagged particle experiences deflections of order one. This scale is unknown at this point, and it will be determined later. We denote as \( T_\varepsilon \) the characteristic microscopic time in which the deflections of the velocities become comparable to the velocity itself. Therefore, it is natural to introduce the following macroscopic variable:

\[ t = \frac{\tau}{T_\varepsilon} = \frac{\tilde{t}}{T_\varepsilon} \quad (5.129) \]

where we use that \( \tilde{t} = \frac{\tau}{\varepsilon} \) with \( L_\varepsilon \) as in (5.111) and we define \( \tilde{T}_\varepsilon = \frac{T_\varepsilon}{\varepsilon} \).

We introduce a macroscopic spatial variable:

\[ x = \frac{X}{T_\varepsilon} = \frac{\xi}{T_\varepsilon}, \quad v = V, \quad x(t) = \xi (\tilde{t}) \]

Then (5.126) becomes:

\[ \frac{dx}{dt} (t) = v(t), \quad \frac{dv}{dt} (t) = \frac{T_\varepsilon}{(L_\varepsilon)^2} \tilde{F}_g (\xi (\tilde{T}_\varepsilon t), \tilde{T}_\varepsilon t) - \frac{T_\varepsilon}{(L_\varepsilon)^2} \tilde{H}_g \left( \tilde{T}_\varepsilon \tilde{t}; V (\tilde{T}_\varepsilon t) \right) =: F_g (x(t), t) - H_g (t; v(t)) \quad (5.130) \]

We now compute the asymptotics of the friction term \( \tilde{H}_g (\tilde{t}; V) \) given in (5.127) as \( \tilde{t} \to \infty \), i.e., for times for which the tagged particle moves at distances much larger than the Debye screening length. Using (5.124) we obtain:

\[ \tilde{H}_g (\tilde{t}; V) = \frac{1}{(2\pi)^3} \int_{(\mathbb{R}^3)^3} dy d\eta dw_0 (L_\varepsilon)^2 \nabla_X \Phi (L_\varepsilon (y - V \tilde{t})) \int_0^{\tilde{t}} ds \int_{\mathbb{R}^3} dk e^{i k (y - \eta)} \cdot \int_{\mathbb{R}^3} dz e^{i (\tilde{t} - \tilde{s})} \frac{1}{(L_\varepsilon)^2} \nabla_X \Phi (L_\varepsilon (\eta - V s)) \cdot \nabla_w g (w_0). \]

Then, rewriting the equation above in terms of the Fourier transform of \( \Phi \), the friction term becomes

\[ \tilde{H}_g (\tilde{t}; V) = \frac{1}{i \pi} \int_0^{\tilde{t}} ds \int_{\mathbb{R}^3} dk \frac{e^{i k V s}}{(L_\varepsilon)^4} \left| \Phi \left( \frac{k}{L_\varepsilon} \right) \right|^2 \int_{\mathbb{R}^3} dz \frac{e^{i z s}}{\Delta (k, z)} \int_{\mathbb{R}^3} dw_0 \left[ \frac{k \cdot \nabla_w g (w_0)}{z + i (k \cdot w_0)} \right]. \]

We define, analogously as in the case considered in the previous Subsections,

\[ \Psi (z, k) = \Psi \left( \frac{z}{|k|}, \frac{k}{|k|} \right) = \int_{\mathbb{R}^3} dw_0 \left[ \frac{k \cdot \nabla_w g (w_0)}{z + i (k \cdot w_0)} \right] \quad (5.131) \]
whence, using (5.126),
\[
\Delta_{\varepsilon} (k, z) = 1 - \frac{2 (2\pi)^2}{(L_\varepsilon)^2} \Phi \left( \frac{k}{L_\varepsilon} \right) \Phi \left( \frac{z}{|k|}, \frac{k}{|k|} \right).
\] (5.132)

Thus
\[
\tilde{H}_g (\hat{t}; V) = \frac{1}{\pi i} \int_{\mathbb{R}^3} \frac{k dk}{(L_\varepsilon)^2} \Phi \left( \frac{k}{L_\varepsilon} \right) \left[ 2 \int_0^\hat{t} d\hat{t} \left[ \int_\gamma \left( \int_{L_\varepsilon p, L_\varepsilon z} e^{i(p \cdot V + \zeta) \tau} d\zeta \right) \Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right) \right] \right].
\] (5.133)

We will assume that the function $g$ has analyticity properties analogous to the ones assumed in Subsection 5.2.1. Moreover, we will also assume that the Penrose stability condition holds, i.e. $\Delta_{\varepsilon} (k, z) \neq 0$ for $\text{Re} \ (z) \geq 0$ and $k \in \mathbb{R}^3 \setminus \{0\}$.

Using the changes of variables $k = L_\varepsilon p$, $z = L_\varepsilon \zeta$, $s = \tau$, which allows to return to the microscopic variables in Fourier, we can rewrite (5.133) as:
\[
\tilde{H}_g (\hat{t}; V) = \frac{1}{\pi i} \int_{\mathbb{R}^3} \frac{p dp}{(L_\varepsilon)^2} \Phi \left( \frac{p}{L_\varepsilon} \right) \left[ 2 \int_0^\hat{t} d\hat{t} \left[ \int_\gamma \left( \int_{L_\varepsilon p, L_\varepsilon \zeta} e^{i(p \cdot V + \zeta) \tau} d\zeta \right) \Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right) \right] \right].
\] (5.134)

Using also (5.132) we obtain:
\[
\tilde{H}_g (\hat{t}; V) = \frac{1}{\pi i} \int_0^\hat{t} d\hat{t} \int_{\mathbb{R}^3} \frac{p dp e^{i p \cdot V \tau}}{p} \Phi \left( \frac{p}{L_\varepsilon} \right) \left[ 2 \int_\gamma \left( \frac{\Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right) e^{i \zeta \tau} d\zeta}{1 - \frac{2 (2\pi)^2}{(L_\varepsilon)^2} \Phi \left( \frac{p}{L_\varepsilon} \right) \Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right)} \right] \right].
\]

The, due to the Penrose stability condition we can perform a contour deformation for the integration in $\zeta$ to bring the contour to the region $\{\text{Re} \ (z) < 0\}$. Therefore, we can take the limit $\hat{t} \to \infty$ and we obtain
\[
\tilde{H}_g (\infty; V) = \frac{1}{\pi i} \int_{\mathbb{R}^3} p dp \Phi (p) \left[ 2 \int_\gamma \frac{\Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right) d\zeta}{1 - \frac{2 (2\pi)^2}{(L_\varepsilon)^2} \Phi \left( \frac{p}{L_\varepsilon} \right) \Psi \left( \frac{\zeta}{|p|}, \frac{p}{|p|} \right)} \right].
\]

Using now residues to compute the integral along the contour $\gamma$ we obtain:
\[
\tilde{H}_g (\infty; V) = 2 \int_{\mathbb{R}^3} p dp \Phi (p) \left[ \frac{\Psi \left( -i p \cdot V, \frac{p}{|p|} \right)}{1 - \frac{2 (2\pi)^2}{(L_\varepsilon)^2} \Phi \left( \frac{p}{L_\varepsilon} \right) \Psi \left( -i p \cdot V, \frac{p}{|p|} \right)} \right].
\] (5.135)

This formula yields the asymptotic friction coefficient acting on a particle which moves at speed $V$. We first notice that the integral in the right hand side of (5.135) is convergent. Indeed, if $|p| \to \infty$ we can assume that $|\Phi (p)|$ decays sufficiently fast, say exponentially, due to the cutoff for small distances we made for the potential. On the other hand, since the potential $\Phi$ is behaving at large distances as Coulombian potential $\Phi (p) \sim \frac{c}{|p|^2}$, $c > 0$ as $|p| \to 0$ then at a first glance the terms $p |\Phi (p)|^2$ would yield a logarithm divergence as $|p| \to 0$. Nevertheless, this divergence does not take place due to the presence of the term
\[ \frac{1}{L^2} \hat{\Phi}(p) \text{ in the denominator of (5.135). This provides a suitable cutoff of the singularities for } |p| \text{ of order } \frac{1}{L^2}. \]

We remark that some care is needed in the deformation of the contour \( \gamma \) appearing in the integral in (5.134). This is due to the fact that the region of analyticity of the function \( (\Delta_x(k, z))^{-1} \) in the \( z \) variable, becomes very small as \( |k| \to 0 \). This is very closely related to the so-called Langmuir waves, which will be discussed in Subsection 7.1 and therefore, we refer there for more details about this issue.

It is relevant to remark that the relevant contributions in the integral (5.135) are those with \( |p| \approx 1 \). Using Plancherel’s formula it follows that these contributions are those between the region where we cut the potential (i.e. \( \Phi \)) with \( \Xi \cdot \nabla \Phi \), refer there for more details about this issue.

Remark 5.7 It is interesting to notice that in the case of interacting particle systems with Coulombian interactions there exists a limit friction coefficient \( H_g(\infty; \nu) \) differently from the case of Rayleigh gases with Coulombian interactions considered in Subsection 5.1.2 where the friction coefficient diverges logarithmically as \( \tau \to \infty \). The reason for this different behaviour is the fact that in the case of interacting particle systems we have the cutoff effects introduced by the Debye screening length.

In order to obtain the evolution equation for the probability distribution associated to the tagged particle \((x, v)\) we approximate the function \( x \) by \( vt \) for small \( t \). We then define a random variable:

\[ D_g(t) = \int_0^t F_g(vs, s) \, ds \]

The random variable \( D_g(t) \) measures the particle deflections for small \( t \). We have:

\[ \mathbb{E}[D_g(t)] = 0 \]  

and:

\[ \mathbb{E}[D_g(t_1) \otimes D_g(t_2)] = 2L_c^2 \hat{\Theta}^2 \int_0^{t_1} \int_0^{t_2} d(s_1, s_2) \int_{\mathbb{R}^3} d\eta \int_{\mathbb{R}^3} dw_0 g(w_0) \Xi(y_1 - \eta, w_0, \hat{T}_c, s_1) \cdot \hat{\Xi}(y_2 - \eta, w_0, \hat{T}_c, s_2) \cdot \int_{(\mathbb{R}^2)^2} d(y_1, y_2) \nabla_x \hat{\Phi}(L_c | y_1 - \hat{T}_c vs_1) \otimes \nabla_x \hat{\Phi}(L_c | y_2 - \hat{T}_c vs_2). \]
We insert the Fourier-Laplace representation of $\Xi$ (5.124) and obtain:

$$\mathbb{E}[D_g(t_1) \otimes D_g(t_2)] = \frac{2L^2_T\pi^3}{L^2_T\pi^3} \int_0^{t_1} \int_0^{t_2} d(s_1, s_2) \int_{(\mathbb{R})^3} d(w_0, k) \int_{\gamma} d\tilde{z}_1 \int_{\gamma} d\tilde{z}_2 e^{z_1 T_x s_1 + i k w_0} \cdot g(w_0) \int_{(\mathbb{R})^2} d(y_1, y_2) \frac{\nabla \Phi \left( L_x \left| y_1 - T_x v s_1 \right| \right) \otimes \nabla \Phi \left( L_x \left| y_2 - T_x v s_2 \right| \right)}{\Delta_x(k, z_1) \Delta_x(-k, z_2)}.$$ 

We perform the contour integrals in $\gamma$:

$$\mathbb{E}[D_g(t_1) \otimes D_g(t_2)] = \frac{2L^2_T\pi^3}{L^2_T\pi^3} \int_0^{t_1} \int_0^{t_2} d(s_1, s_2) \int_{(\mathbb{R})^3} d(w_0, k) d\tilde{z}_2 e^{-i k w_0 T_x (s_1 - s_2)} \cdot e^{i k L_x k (y_1 - y_2)} g(w_0) \int_{(\mathbb{R})^2} d(x_1, x_2) \frac{\nabla \Phi \left( L_x \left| y_1 - T_x v s_1 \right| \right) \otimes \nabla \Phi \left( L_x \left| y_2 - T_x v s_2 \right| \right)}{\Delta_x(k, i k w_0) \Delta_x(-k, -i k w_0)}.$$ 

Using the identity $\Delta_x(a, ib) = \Delta_x^*(-a, -ib)$ for $b \in \mathbb{R}$ and changing timescale as in (5.129) we obtain:

$$\mathbb{E}[D_g(t_1) \otimes D_g(t_2)] = \frac{2L^2_T\pi^3}{L^2_T\pi^3} \int_0^{T_{t_1} t_1} \int_0^{T_{t_2} t_2} d(\tau_1, \tau_2) \int_{(\mathbb{R})^3} d(w_0, k) d\tilde{z}_2 e^{i k L_x k (v - w_0)(\tau_1 - \tau_2)} \cdot e^{i k L_x k (x_1 - x_2)} g(w_0) \int_{(\mathbb{R})^2} d(x_1, x_2) \frac{\nabla \Phi(x_1) \otimes \nabla \Phi(x_2)}{|\Delta_x(k, i k w_0)|^2}.$$ 

Performing the integral in $x_1, x_2$ leads to the Fourier representation:

$$\mathbb{E}[D_g(t_1) \otimes D_g(t_2)] = \frac{2}{L^2_T} \int_0^{T_{t_1} t_1} \int_0^{T_{t_2} t_2} d(\tau_1, \tau_2) \int_{(\mathbb{R})^3} d(w_0, k) g(w_0) \frac{\langle k \otimes k \rangle \hat{\phi}(k)^2 e^{-i k (v - w_0)(\tau_1 - \tau_2)}}{|\Delta_x(L_x k, i L_x k w_0)|^2}.$$ 

To determine the asymptotics of this expression for $\varepsilon \to 0$ we observe that the integral in $k$ behaves like:

$$\int_{\mathbb{R}^3} dk \frac{\langle k \otimes k \rangle \hat{\phi}(k)^2 e^{-i k (v - w_0)(\tau_1 - \tau_2)}}{|\Delta_x(L_x k, i L_x k w_0)|^2} \approx \frac{\epsilon^2 c_1}{|v - w_0| |\tau_1 - \tau_2| \sqrt{|v - w_0|^2}} \left( I - \frac{(v - w_0) \otimes (v - w_0)}{|v - w_0|^2} \right) \mathbb{1}_{|v - w_0| |\tau_1 - \tau_2| \leq L_x},$$

since the dielectric function (5.125) behaves as:

$$|\Delta_x(L_x k, i L_x k w_0)|^{-1} \approx \begin{cases} 1, & \text{for } k \gtrsim 1/L_x \\ \frac{k}{L_x}, & \text{for } k \lesssim 1/L_x \end{cases}$$

Then the left hand side of (5.140) can be approximated as $\int_{|k| \leq 1} \frac{(k \otimes k)}{|k|^4} e^{-i k (v - w_0)(\tau_1 - \tau_2)} dk$. This integral can be approximated by the right hand side of (5.140) with $c_1 = \frac{\epsilon^2}{2}$. A simple way to see this is to use the tensor structure of the integral and to compute this tensor in the case in which $(v - w_0)$ is parallel to the $x_1$ coordinate axis.
Using the asymptotic formula (5.140) we finally obtain

\[ \mathbb{E} [D_g (t_1) \otimes D_g (t_2)] = \frac{2c_1^2}{L^2} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 \int_{\mathbb{R}^3} dv_0 \int_{\mathbb{R}^3} dw_0 \frac{g(w_0)}{|v - w_0|} \left| \tau_1 - \tau_2 \right| I - \frac{(v - w_0) \otimes (v - w_0)}{|v - w_0|^2} \mathbb{1}_{|v - w_0| \leq L} \]

\[
\sim 4c_1^2 \varepsilon^2 \log(L_e) T_e (t_2 - t_1) \int_{\mathbb{R}^3} dv_0 \frac{g(w_0)}{|v - w_0|} \left( I - \frac{(v - w_0) \otimes (v - w_0)}{|v - w_0|^2} \right) \]

\[
\sim 2c_1^2 \varepsilon^2 T_e \log \left( \frac{1}{\varepsilon} \right) (t_2 - t_1) \int_{\mathbb{R}^3} dv_0 \frac{g(w_0)}{|v - w_0|} \left( I - \frac{(v - w_0) \otimes (v - w_0)}{|v - w_0|^2} \right) \]

where we use (5.111) in the last step. Notice that we assume that \( \int_{\mathbb{R}^3} dv_0 \frac{g(w_0)}{|v - w_0|} < \infty \) Using (5.137) we then obtain

\[
\mathbb{E} [D_g (t_1) \otimes D_g (t_2)] \sim 2c_1^2 (t_2 - t_1) \int_{\mathbb{R}^3} dv_0 \frac{g(w_0)}{|v - w_0|} \left( I - \frac{(v - w_0) \otimes (v - w_0)}{|v - w_0|^2} \right) = 2 (t_2 - t_1) A_g (v). \tag{5.141} \]

It is possible to interpret the asymptotics of the inverse Fourier transforms in (5.140) in terms of the correlations of a random force field evaluated at two different points \( x_1 \) and \( x_2 \). The correlations between two points \( x_1, x_2 \) decrease then as \( \frac{1}{|x_1 - x_2|} \) for distances smaller than the Debye length. For distances larger than the Debye length \( L_e \) the decay of the correlations is much faster and therefore, the Coulombian logarithm is due only to the range of distances between the particle size \( \varepsilon \) and the Debye length \( L_e \). This is different in the case of Rayleigh gases with Coulombian interactions, where the range of distances contributing to the kinetic regime goes from the particle size \( \varepsilon \) to the mean free path.

Using (5.130), (5.136), (5.141) we can now derive the evolution equation for the distribution function \( f (x, v, t) \) where \( t = \frac{T}{T_e} \) and \( T_e \) is as in (5.137). We have:

\[
\partial_t f (x, v, t) + v \cdot \partial_x f (x, v, t) = \partial_v \left[ A_f (v) \partial_v f + h_f (v) f \right] (x, v, t) \tag{5.142} \]

where \( h_f (v), A_f (v) \) are computed as in (5.130), (5.141).

**Remark 5.8** It is interesting to notice that (5.142) is a Landau equation, instead of a Balescu-Lenard equation as in principle could be expected for long range interactions. The reason for this is the presence of the Coulombian logarithm as well as the fact that the contribution to the coefficients \( A_f, A_f (v) \) is due to the dyadics smaller than the Debye screening length and since all these contributions are comparable the total contribution is larger than the one due to the Debye screening length that would be the dominant one if the kinetic equation had been the Balescu-Lenard equation.

**Remark 5.9** Notice that, differently from the case of Rayleigh gases with Coulombian interactions, after rescaling out the logarithmic term both the friction coefficient and the deflections \( A_g (v) \) yield a well defined limit as \( \tilde{t} \to \infty \), where \( \tilde{t} \) is the mesoscopic limit. In the case of Rayleigh gases both quantities diverge logarithmically as \( \tilde{t} \to \infty \).

**Remark 5.10** Notice that a heuristic explanation of (5.141) is that the Vlasov evolution of the white noise which describes the fluctuations of the particle density yields a "coloured noise" which decorrelates on distances of the order of the Debye length.
5.3 Weak coupling case.

We now apply analogous methods to derive the kinetic equation which yields the evolution of a system of particles in the weak coupling limit. In this case the interaction potential is small and its range is of the same order as the average distance between the particles. In this case the evolution equations are:

\[
\frac{dX}{d\tau} = V, \quad \frac{dV}{d\tau} = -\sum_{j \in S} \nabla X \phi_\varepsilon (X - Y_j) \tag{5.143}
\]

\[
\frac{dY_k}{d\tau} = W_k, \quad \frac{dW_k}{d\tau} = -\nabla X \phi_\varepsilon (Y_k - X) - \sum_{j \in S, j \neq k} \nabla X \phi_\varepsilon (Y_k - Y_j), \quad k \in S
\]

where we assume that the average distance between the particles is of order one and the interaction potential is:

\[
\phi_\varepsilon (X) = \varepsilon \Phi (|X|) \tag{5.144}
\]

where \(\Phi (|X|)\) decreases sufficiently fast as \(|X| \to \infty\).

We can derive a kinetic equation describing the behaviour of this system using methods analogous to those in Subsections 5.1.4 and 5.2.1. We introduce an auxiliary length \(L_\varepsilon \gg 1\) which will allow to approximate the particle distributions by means of continuous densities.

We then define new variables:

\[
y_k = \frac{Y_k}{L_\varepsilon}, \quad \xi = \frac{X}{L_\varepsilon}, \quad \tilde{t} = \frac{\tau}{L_\varepsilon}
\]

Then:

\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon \sum_{j \in S} \nabla \xi \Phi (L_\varepsilon (\xi - y_j))
\]

\[
\frac{dy_k}{dt} = W_k, \quad \frac{dW_k}{dt} = -\varepsilon \nabla y_k \Phi (L_\varepsilon (y_k - \xi)) - \varepsilon \sum_{j \in S, j \neq k} \nabla y_k \Phi (L_\varepsilon (y_k - y_j)), \quad k \in S
\]

We define:

\[
f_\varepsilon (y, w, \tilde{t}) = \frac{1}{(L_\varepsilon)^3} \sum_k \delta (y - y_k) \delta (w - W_k)
\]

Then:

\[
\frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla \xi \Phi (L_\varepsilon (\xi - \eta)) \rho_\varepsilon (\eta, \tilde{t}) \, d\eta
\]

\[
\frac{dy_k}{dt} = W_k, \quad \frac{dW_k}{dt} = -\varepsilon \nabla y \Phi (L_\varepsilon (y - \xi)) - \varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla y \Phi (L_\varepsilon (y - \eta)) \rho_\varepsilon (\eta, \tilde{t}) \, d\eta, \quad k \in S
\]

where:

\[
\rho_\varepsilon (y, \tilde{t}) = \int_{\mathbb{R}^3} f_\varepsilon (y, w, \tilde{t}) \, dw
\]

Therefore:

\[
\partial_t f_\varepsilon (y, w, \tilde{t}) + w \cdot \nabla_y f_\varepsilon (y, w, \tilde{t})
\]

\[
- \left[ \varepsilon \nabla y \Phi (L_\varepsilon (y - \xi)) + \varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla y \Phi (L_\varepsilon (y - \eta)) \rho_\varepsilon (\eta, \tilde{t}) \, d\eta \right] \cdot \nabla_y f_\varepsilon (y, w, \tilde{t}) = 0
\]
It turns out that in this case the term containing \( \varepsilon (L_{\varepsilon})^3 \int_{\mathbb{R}^3} \nabla y \Phi (L_{\varepsilon} (y - \eta)) \rho_{\varepsilon} (\eta, \tilde{t}) \, d\eta \) is negligible compared with the transport term \( w \cdot \nabla f_{\varepsilon} (y, w, \tilde{t}) \). We can then approximate this equation as:

\[
\partial_{\tilde{t}} f_{\varepsilon} (y, w, \tilde{t}) + w \cdot \nabla y f_{\varepsilon} (y, w, \tilde{t}) - \varepsilon \nabla y \Phi (L_{\varepsilon} (y - \xi)) \cdot \nabla w f_{\varepsilon} (y, w, \tilde{t}) = 0
\]

We are then exactly in the same situation as in Subsection 5.4.4 with \( \ell_{\varepsilon} = 1 \). We then obtain the following approximated equation for each of the particles of the system:

\[
\frac{dx}{dt} = v, \quad \frac{dv}{dt} = \frac{1}{\varepsilon} B_g \left( \frac{t}{\varepsilon^2} \right) + H_g (v)
\]

where \( x = \varepsilon^2 X, \; v = V, \; t = \varepsilon^2 \tau \) and:

\[
H_g (V) = \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla X \Phi (|Y|) \nabla w g (w) \cdot \int_{-\infty}^0 \nabla X \Phi (|Y + (V - w) s|) \, ds \tag{5.145}
\]

\[
E [B_g (s)] = 0, \quad E [B_g (0) B_g (s)] = \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla X \Phi (|Y + ws|) \otimes \nabla X \Phi (|Y|) g (w)
\]

We then define:

\[
D_g = \int_0^\infty ds \int_{\mathbb{R}^3} dY \int_{\mathbb{R}^3} dw \nabla X \Phi (|Y + ws|) \otimes \nabla X \Phi (|Y|) g (w) \tag{5.146}
\]

Then, the distribution function \( f (x, v, t) \) is determined by means of the equation

\[
\partial_t f (x, v, t) + v \cdot \partial_x f (x, v, t) = \partial_v \left[ \frac{D}{2} \partial_v f + H_f (v) f \right] (x, v, t) \tag{5.147}
\]

**Remark 5.11** The methods used in the whole Section 5.2 can be applied with minor modifications also to derive kinetic approximations for a tracer particle moving in a medium of interacting particle systems, in the case in which the interaction between the tracer particle and the scatterer is different from the interactions between the scatterers themselves.

### 5.4 Kinetic equations for Rayleigh gases.

We now summarize the kinetic equations and the corresponding rescaling limits yielding them that we have obtained in the previous Sections for a tagged particles evolving according to the equations \( \{1.4\} \) (i.e. Rayleigh gases).

- In the case of potentials with the form

\[
\phi_{\varepsilon} (x) = \varepsilon \Phi \left( \frac{|x|}{L_{\varepsilon}} \right) \text{ with } L_{\varepsilon} \gg 1
\]
and \( \Phi (s) \) decreasing faster than \( \frac{1}{s^\alpha} \) with \( \alpha > 1 \) (cf. (3.22), (3.23)) we have seen that the distribution of particle velocities \( f(x, w, t) \) evolves according to the equation (cf. (5.26))

\[
\partial_t f + w \cdot \partial_x f = D_w \cdot \left( \frac{1}{2} D_g(w) \partial_w f + \Lambda_g(w) f \right)
\]

where \( D_g(w) \) is as in (5.25) and \( \Lambda_g(w) \) is as in (5.19). The formula relating the microscopic \( \tilde{t} \) and macroscopic variable \( t \) is \( t = (\theta \epsilon)^2 \tilde{t} \) with \( \theta \epsilon = \epsilon \left( L \epsilon \right)^3 \).

- In the case of Coulombian potentials, i.e. potentials with one of the forms

\[
\phi_\epsilon (x) = \Phi \left( \frac{|x|}{\epsilon} \right) \quad \text{or} \quad \phi_\epsilon (x) = \epsilon \Phi (|x|)
\]

with \( \Phi (s) \sim \frac{1}{s} \) as \( s \to \infty \) (cf. (5.27)) we obtain the following kinetic equation (cf. (5.50))

\[
\partial_t f + V \cdot \partial_x f = \partial_V \cdot \left( \frac{1}{2} D_V(V) \partial_V f + \lambda_g(V) f \right)
\]

where \( \lambda_g(V) \) is as in (5.48) and \( D_g(V) \) is as in (5.49). The relation between the microscopic time scale \( \tau \) and the macroscopic time scale is given by \( \tau = \frac{t}{2 \epsilon^2 \log\left( \frac{1}{\epsilon} \right)} \).

- In the case of potentials with the form (weak coupling case)

\[
\phi_\epsilon (x) = \epsilon \Phi \left( \frac{|x|}{L \epsilon} \right), \quad L \epsilon \lesssim 1
\]

where \( \Phi (s) \) decreases faster than \( \frac{1}{s^\alpha} \) with \( \alpha > 1 \) (cf. (5.52)) we can approximate the distribution of particle velocities \( f \) by means of the solutions of the kinetic equation (cf. (5.64a))

\[
\partial_t f + V \cdot \partial_x f = \partial_V \cdot \left( \frac{1}{2} D \partial_V f + \Lambda_g(V) f \right)
\]

where \( D \) is as in (5.64a) and the friction coefficient \( \Lambda_g(V) \) is as in (5.61).

### 5.5 Kinetic equations for interacting particle systems.

We now summarize the precise scaling limit in which a system of particles described by the system of equations (1.11) can be approximated by means of Landau and Balescu Lenard equations.

- In the case of interaction potentials with the form

\[
\phi_\epsilon (x) = \epsilon \Phi \left( \frac{|x|}{L \epsilon} \right) \quad \text{with} \quad \epsilon (L \epsilon)^3 \to 1
\]
and $\Phi(s)$ decreasing faster than $\frac{1}{s^\alpha}$ with $\alpha > 1$ (cf. (3.22), (3.23)) the kinetic equation yielding the evolution of the distribution function $f$ is given by the Balescu-Lenard equation (cf. (5.101))

$$\partial_t f + v \cdot \partial_x f = \partial_v \left( \frac{D_f(v)}{2} \partial_v f - H_f(v) f \right)$$

where $H_g(V)$ is defined in (5.95) and $D_g(V)$ in (5.100). The formula relating the microscopic time scale $\tilde{t}$ with the macroscopic time scale $t$ is $\tilde{t} = \varepsilon^2 (L\varepsilon)^3 t$.

- In the case of Coulombian potentials as in (5.148) (cf. also (5.27)) the evolution of the distribution function $f$ is given by the Landau equation (cf. (5.142))

$$\partial_t f + v \cdot \partial_x f = \partial_v \left( A_f(v) \partial_v f + h_f(v) f \right)$$

where $A_f(v)$ is defined in (5.141) and $h_f(v)$ has been defined in (5.136). The microscopic time scale $\tau$ and the macroscopic time scale are related by means of the formula $\tau = \frac{t}{\varepsilon^2 \log(\frac{1}{\varepsilon})}$. The fact that the kinetic equation for Coulombian potentials is the Landau equation instead of the Balescu-Lenard equation, due to the presence of the Coulombian logarithm, has been originally noticed by Landau and Lenard (cf. [33, 35]).

- We considered one version of the weak coupling case with potentials of the form (5.149) with $\ell = 1$. The kinetic equation describing the evolution of the particle distribution $f$ is the following version of the Landau equation (cf. (5.147))

$$\partial_t f + v \cdot \partial_x f = \partial_v \left( \frac{D_f}{2} \partial_v f + H_f(v) f \right)$$

where $H_g(V)$ has been defined in (5.145) and $D_g$ in (5.146). The relation between the microscopic time scale $\tilde{t}$ and the macroscopic time scale $t$ is $t = \varepsilon^2 L\varepsilon \tilde{t}$.

**Remark 5.12** A few rigorous results deriving kinetic equations of Landau or Balescu-Lenard form for interacting particle systems have been recently obtained. In these results the particles interact by means of weak potentials with a range comparable or larger than the average particle distance. See for instance [9, 17, 18] for results for the full particle system and the time of validity of the kinetic equation is shorter than the macroscopic time scale. In [51, 55] the kinetic equation is derived for macroscopic times of order one but the particle system is approximated by a truncated BBGKY hierarchy.
5.6 Reformulation of the dynamics of a particle in a Rayleigh-Boltzmann gas as the dynamics of a particle in a Boltzmann random force field with nonelastic collisions.

We have seen in Subsection 5.1 that the dynamics of a tagged particle in a Rayleigh gas under some smallness assumptions on the interaction potentials can be approximated by means of the dynamics of a tagged particle in a random force field with a friction coefficient (cf. for instance (5.19), (5.24)). Both the friction coefficient and the random force field are due to the combined effect of many scatterers. Due to this it is remarkable that, as we will see in this Subsection, a similar decomposition of the forces acting on a tagged particle moving in a Rayleigh-Boltzmann gas, can be made at least in the case in which the interaction between the tagged particle and the scatterers takes place by means of hard-sphere potentials. More precisely, we will show that the dynamics of a tagged particle in such a situation is equivalent to the dynamics of a particle in the random force field generated by a set of moving scatterers, whose dynamics is not affected by the tagged particle, but where the collisions between the tagged particle and the scatterers are non elastic. Such non elastic collisions play a role analogous to the friction term in (5.24).

To prove our claim we must study first the elastic collisions between a tagged particle and one scatterer. We will assume that the tagged particle and the scatterer have the same mass and the same radius $a$. We will denote the velocities of the tagged particle before and after the collision as $w_1, w'_1 \in \mathbb{R}^3$ respectively and the velocities of the scatterer before and after the collision as $w_2, w'_2 \in \mathbb{R}^3$. We denote as $x_1, x_2$ the position of the tagged particle and scatterer at the collision time respectively. Then $|x_1 - x_2| = 2a$. We write:

$$\zeta = \frac{(x_1 - x_2)}{|x_1 - x_2|} \quad (5.150)$$

The conservation of momentum and energy yields:

$$w_1 = w_1 + w_2 = w'_1 + w'_2, \quad (w_1)^2 + (w_2)^2 = (w'_1)^2 + (w'_2)^2$$

Then the exchange of momentum between the particles will be assumed to be proportional to $\zeta$. Moreover, we can analyze the collision in the coordinate system in which $w_2 = 0$. Then:

$$w'_1 - w_1 = -w'_2 = \lambda \zeta \quad (5.151)$$

for some $\lambda \in \mathbb{R}$. Then, the conservation of energy yields:

$$(w_1)^2 + (w_2)^2 = (w_1 + \lambda \zeta)^2 + \lambda^2 |\zeta|^2 = (w_1)^2 + 2\lambda (w_1 \cdot \zeta) + 2\lambda^2$$

and using that $|\zeta| = 1$ we obtain:

$$\lambda = -(w_1 \cdot \zeta) \quad (5.152)$$

whence:

$$w'_1 = w_1 - (w_1 \cdot \zeta) \zeta, \quad w'_2 = (w_1 \cdot \zeta) \zeta \quad (5.153)$$

We now prove that we can obtain an equivalent dynamics for the tagged particle assuming that the scatterer does not modify its trajectory in the collision (i.e. the random force field is not affected by the tagged particle), but including in the collision an additional term which makes the collision inelastic.
We will denote the velocities of the tagged particle and the scatterer before the collision as \(v_1, v_2\) respectively and after the collision as \(v'_1, v'_2\). Then \(v_2 = v'_2 = 0\) and \(v_1 = w_1, v'_1 = w'_1\). We assume that during the collision an impulse \(I = w'_1 - w_1\) is transmitted to the tagged particle. Then, taking into account (5.151), (5.152) we obtain:

\[
I = -(w_1 \cdot \zeta) \zeta
\]

Notice that this impulse can be thought as a friction term because the energy of the tagged particle is reduced. Indeed, we have:

\[
(v'_1)^2 - (v_1)^2 = -(w_1 \cdot \zeta)^2 \leq 0
\]

Moreover, the collision is kinematically possible, in spite of the fact that the velocity of the scatterer is not modified. Indeed, the collision is kinematically possible if we have \((w'_1 \cdot \zeta) \leq 0\). Actually we have:

\[
w'_1 \cdot \zeta = w_1 \cdot \zeta - (w_1 \cdot \zeta)|\zeta|^2 = 0
\]

which confirms that the collision is possible. Notice that this formula implies in addition that in the collision in a coordinate system in which \(w_2 = 0\) we obtain that the tagged particle 1 moves perpendicularly to the vector \(\zeta\) connecting both particles immediately after the collision.

It is natural to ask if it is possible to reformulate the problem of the collision between two particles which interact by means of a potential that depends only on their distance. The goal would be to check if the deflection experience by one of the particles (the tagged particle) is the same as the one experienced by a tagged particle which interacts with a scatterer which affects the dynamics of the tagged particle but is not affected by it, including in addition a friction term depending only on the velocity and the distance between the particles. More precisely the problem is the following. We have two interacting particles with will be assumed to have the same mass that can be chosen to be one. The interaction potential is \(V(|X|)\). Then, the elastic collision problem between the two particles is:

\[
\frac{dX_1}{dt} = V_1, \quad \frac{dX_2}{dt} = V_2
\]

\[
\frac{dV_1}{dt} = -\nabla V_{X_1}(|X_1 - X_2|), \quad \frac{dV_2}{dt} = -\nabla V_{X_2}(|X_2 - X_1|)
\]

Suppose that we solve a collision problem for the previous system of equations imposing \(V_1 (\infty) = V_{1,in}\) and assuming that the collision between two particles is characterized by an impact parameter \(b \in \mathbb{R}^3\). The problem is the following one. Is it possible to find a friction force \(\Phi (V_1 - V_2; |X_1 - X_2|)\) such that the solution of the following problem

\[
\frac{dY_1}{dt} = W_1, \quad \frac{dW_1}{dt} = -\nabla V_{X_1}(|Y_1|) + \Phi (W_1; |Y_1|), \quad W_1 (\infty) = V_{1,in}
\]

with the same impact parameter \(b\) yields \(W_2 (\infty) = V_{1,out}\)?

Notice that, differently from the problem considered in Subsections 5.1, 5.2 where we obtain an approximation of all the forces acting on a tagged particle by means of a friction term plus the evolution in a random force field, the description of the collisions given above is just a mathematical construction without any real physical significance. Nevertheless, the possibility of this reformulation of the collision problem has some independent interest.
6 Evolution of the probability distribution of particles to a correlated measure.

We have seen in Subsections 5.2 that for interacting particle systems of the form (1.1) we can derive a kinetic approximation in the case of small interaction potentials. This was achieved by replacing the particle distribution by a Gaussian random field evolving by means of a Vlasov equation, and a friction term. Then the dynamics of a tagged particle in this random background approximates the evolution of a particle in the interacting particle system (1.1) for short but macroscopic times.

The resulting kinetic equations are the Landau or the Balescu-Lenard equation. There is an alternative way of deriving these equations which was introduced by Bogoliubov (cf. [11]). Suppose that we have a system of particles that solves (1.1). We assume also that the initial distribution of particles in the phase space is given by a generalized Poisson distribution \( \nu_g \) as in (3.1) which is characterized by a distribution of velocities \( g \). Our goal is to define a family of evolutions of the probability measure \( \tau \rightarrow P^\varepsilon_\tau \), \( t \geq 0 \), with \( P^\varepsilon_0 = \nu_g \). The measure \( P^\varepsilon_\tau \) is defined as follows. Suppose that we denote as \( \omega \in \Omega \) the random empirical measure associated to the initial particle configuration \( (X_i, V_i)_{i \in I} \) in the phase space, i.e.

\[
\omega = \sum_{i \in I} \delta(x - X_i)\delta(v - V_i).
\]  

We denote as \( U^\varepsilon_\tau \) the evolution operator given by the evolution of the particle system (1.1). We will assume that \( U^\varepsilon_\tau \) is well defined (and measurable) for each \( \tau > 0 \) and define the measure \( P^\varepsilon_\tau \) as:

\[
P^\varepsilon_\tau = P^\varepsilon_0 \circ (U^\varepsilon_\tau)^{-1} = \nu_g \circ (U^\varepsilon_\tau)^{-1}.
\]

Due to the particle interaction, the measures \( P^\varepsilon_\tau \) are in general no Poisson measures. The usual way of characterizing these measures is by means of the many-particle correlation functions \( g_n \) which are obtained solving the so-called BBGKY hierarchy:

\[
\partial_t g_n(\tau, \alpha_n) + \sum_{k=1}^n v_k \nabla y_k g_n(\tau, \alpha_n) - \sum_{k=1}^n \int_{\mathbb{R}^6} d\eta_{n+1} \nabla y_k \phi_\varepsilon(y_k - y_{n+1}) \nabla v_k g_{n+1}(\tau, \alpha_n, \eta_{n+1}) = \sum_{k=1}^n \sum_{\ell=1}^n \nabla y_k \phi_\varepsilon(y_k - y_{\ell}) \nabla v_k g_n(\tau, \alpha_n).
\]

Assuming that the BBGKY hierarchy allows to characterize uniquely the probability measures \( P^\varepsilon_\tau \) we would have:

\[
P^\varepsilon_\tau = K^\varepsilon [g (\cdot, t) ; \{g_k (\cdot, \tau)\}_{k=2}^\infty]
\]

where \( K \) is the operator that yields a random measure on the phase space \( \mathbb{R} \times \mathbb{R} \) for a given set of correlation functions \( g_k \). The measure is uniquely determined by the correlation functions \( g_k \) under relatively weak conditions, e.g. if there exists \( C > 0 \) such that

\[
\|g_k (\cdot, \tau)\|_{L^\infty} \leq C^n.
\]

See for instance [11], [15] for more details on this so-called moment problem.
It was noticed by Bogoliubov that the time scale for the evolution of the correlation functions for two or more particles \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \) is much shorter if \( \varepsilon \to 0 \) than the one for the one-particle distribution \( g(\cdot, \tau) \). Further, they are expected to stabilize to time-independent functions \( G_k \) which only depend on \( g \). Approximating the correlation functions for two or more particles by the functions \( G_k \) yields an approximation for the evolution of the one-particle distribution \( g \) on a much longer time scale.

Bogoliubov’s approach is particularly useful in the derivation of Landau and Balescu-Lenard equations. We will illustrate the use of the method in the case of interaction potentials with the form (3.22), (3.23). The analysis of this problem through an approximation by the dynamics of a tagged particle with an effective friction term moving in a random force field has been made in Subsection 5.2.1. In this particular setting we can linearize the evolution equations yielding the functions \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \). It turns out that these linearized equations can be explicitly solved in terms of the solutions of the corresponding linearized Vlasov equation (cf. [52]). In particular, the characteristic time scale for the evolution of the functions \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \) is the same as the one of the linearized Vlasov equation (cf. for instance (5.70)).

In the case of potentials with the form (3.22), (3.23) and since the particle velocities are of order one we have that the characteristic time for the linearized Vlasov equation (and the equations for the functions \( \{g_k(\cdot, t)\}_{k=2}^\infty \)) is of order \( L_\varepsilon \). This time scale is much shorter than the timescale \( t \), in which the particles move along lengths of the order of the mean free path. If the stability conditions discussed in Subsection 5.2.1 hold, we have similar stability properties for the linearized operator which yields the evolution of the functions \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \). It then follows that the functions \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \) approach an equilibrium

\[
G_k(\cdot; g), \quad k \geq 2,
\]

which depends on the function \( g \). The equations yielding the evolution of \( \{g_k(\cdot, \tau)\}_{k=2}^\infty \) contains the one-particle distribution \( g \). Since this evolution takes place on a longer timescale, we can approximate the dynamics of \( g(\cdot, t) \) using \( g_k(\cdot, t) \approx G_k(\cdot; g(\cdot, t)) \) for \( k \geq 2 \). By (6.4), we obtain the following approximation for the random measure \( P_\varepsilon \)

\[
\mu^\varepsilon \simeq \mathbb{K}_\varepsilon [g(\cdot, \tau) ; \{G_k(\cdot; g(\cdot, \tau))\}_{k=2}^\infty] := \mathbb{H}_\varepsilon [g(\cdot, \tau)]
\]

as \( \varepsilon \to 0 \).

**Remark 6.1** The measure \( \mu \) given by the functions \( G_k \) through the relation (6.3) is a random measure on \( \mathbb{R}^3 \times \mathbb{R}^3 \), but not necessarily related to a point process on this space. Hence, in contrast to the measure in (5.1), \( \mu \) is in general not of the form (6.1). The problem whether such a representation holds is called full K-moment problem. For necessary and sufficient conditions, see for example [24].

Finally, an approximation for the evolution of \( g(\cdot, t) \) follows by replacing \( g_2 \) by \( G_2 \) in (6.3). In the regime described in Subsection 5.2.1 this yields either the Landau or the Balescu-Lenard equation.

The previous argument, besides providing a different derivation of the Landau and Balescu-Lenard equations (as well as a blueprint for a possible rigorous derivation of these kinetic equations taking as starting point a Hamiltonian mechanical system), provides also an interesting description of the probability distributions which characterize the particle distributions in macroscopic times (cf. (3.5)).

An exposition on possible applications of the BBGKY-method to various kinetic limits and their fluctuations can be found in [27].
Some relevant physical properties of the Vlasov-Poisson equation.

In this Section we discuss a few properties of particle systems which are due to their collective behaviour and can be described, if the interactions between the particles are weak but have long range, using the Vlasov equation. We will discuss two specific phenomena. The first is the existence of some oscillations of the particle density for particles interacting by means of Coulomb potentials which damp very slowly. These oscillations are usually termed as Langmuir waves. The second issue that we discuss here is the phenomenon of screening. The two phenomena are well known in the physical literature (cf. for instance [25, 33]) but both of them are relevant in the rigorous mathematical study of some problems related to the Vlasov equations and for this reason we will describe them here.

Both phenomena can be related to the asymptotic behaviour of the dielectric function \( \epsilon (k, \omega) \) defined in Subsection 5.2.1. More precisely, Langmuir waves are related to the asymptotic behaviour of \( \epsilon (k, \omega) \) as \( k \to 0 \). On the other hand, screening properties act for arbitrary, dynamic situations. However, they become particularly simple to describe in static situations, i.e. for \( \omega \to 0 \).

7.1 Langmuir waves.

Langmuir waves are some oscillations with wavelength much larger than the Debye screening length which take place in plasmas (cf. [25], [26]). We will use also the term Langmuir waves to refer to some oscillations with very large wavelength for particle systems interacting by means of Coulomb-like potentials like the ones in (5.27).

Langmuir waves can be described using the linearized Vlasov equation. We will illustrate the meaning of these waves using the linearized Vlasov equation for a one-component plasma (assuming then than in the original problem there is a background of charge to ensure electroneutrality) or a two component plasma with two opposite signs. We then assume that the linearized problem around a constant distribution of particles has the following form:

\[
\partial_t h(x, v, t) + v \cdot \nabla_x h(x, v, t) + F \cdot \nabla_v f_0(v) = 0
\]

\[
F(x, t) = -\int dy \int dv h(y, v, t) \left[ (L \epsilon)^2 \nabla_y \Phi(L \epsilon |x - y|) \right]
\]

where we have rescaled the length in order to make the range of the potential or the screening length of order one. We look for solutions of this equation with the form:

\[
h(x, v, t) = e^{i(\omega t + k \cdot x)} H(v), \quad k \in \mathbb{R}^3, \quad \omega \in \mathbb{C}
\]

whence \( H \) solves:

\[
i \omega H(v) + (v \cdot k) H(v) + F_0 \cdot \nabla_v f_0(v) = 0
\]

\[
F_0 = -\int dy \int dv H(v) e^{-ik \cdot y} \left[ (L \epsilon)^2 \nabla_y \Phi(L \epsilon |y|) \right]
\]
whence:
\[ H(v) = -\frac{F_0 \cdot \nabla_v f_0(v)}{i(\omega + (v \cdot k))} \]
and:
\[ F_0 = \int dy \int dv \frac{(F_0 \cdot \nabla_v f_0(v))}{i(\omega + (v \cdot k))} e^{-\imath k \cdot y} \left[ (L_\epsilon)^2 \nabla_y \Psi(L_\epsilon |y|) \right] \]

We can write this in matrix form:
\[
\left( I - \int dy \int dv \frac{e^{-\imath k \cdot y}}{i(\omega + (v \cdot k))} (L_\epsilon)^2 \nabla_y \Psi(|y|) \otimes \nabla_v f_0(v) \right) F_0 = 0
\]
for some \( F_0 \in \mathbb{C}^3 \). This yields the following eigenvalue problem:
\[
\det \left( I - \int dy \int dv \frac{e^{-\imath k \cdot y}}{i(\omega + (v \cdot k))} \nabla_y \Psi(|y|) \otimes \nabla_v f_0(v) \right) = 0 \tag{7.2}
\]
where \( k \in \mathbb{R}^3 \) is given. The stability condition \( \text{(7.2)} \) implies that the solutions of the equation \( \text{(7.2)} \) are contained in the half-plane \( \{ \text{Im}(\omega) > 0 \} \). However, in the very relevant case of Coulombian potentials \( \Psi(|y|) = \frac{1}{|y|} \) it turns out that there is a root \( \omega = \omega(k) \) which converges to a real value \( \Omega_0 \neq 0 \) as \( |k| \to 0 \). We have that \( \text{Im}(\omega(k)) > 0 \) for all \( k \in \mathbb{R}^3 \). However, the following asymptotic formula holds if we assume that the distribution of particle velocities \( g \) is Maxwellian
\[
\text{Im}(\omega(k)) \sim -c_0 \omega_p \left( \frac{k_D}{k} \right)^3 \exp \left( -\frac{k_D^2}{2|k|^2} \right) \quad \text{as } |k| \to 0 \tag{7.3}
\]
where \( k_D = \frac{\omega_p}{\langle v^2 \rangle} \) is the variance of the velocity, \( \omega_p = \sqrt{4\pi} \) and \( c_0 = \sqrt{\frac{4}{3}} \) (cf. \[ \text{[26], Chapter 10} \). Formula \( \text{(7.3)} \) is valid for the Coulomb potential \( \Psi(s) = \frac{1}{|s|} \), but similar formulas hold for potentials \( \Psi(s) \) behaving asymptotically as \( \frac{1}{|s|} \) as \( |s| \to \infty \). The main consequence of \( \text{(7.3)} \) is the existence of solutions of \( \text{(7.1)} \) with very large wavelength (compared with the Debye length) and damping in time extremely slowly. A consequence of this is the existence of solutions of the linear problem \( \text{(7.1)} \) which converge to equilibrium very slowly in suitable Sobolev spaces. This is a result that has been rigorously proved in \[ \text{[20, 21]} \] for the one-dimensional and radial version of \( \text{(7.1)} \) with Coulombian interactions. The extremely slow damping of very large wavelengths plays a crucial role in the stabilization of the correlation function for Coulombian potentials, as has been discussed in \[ \text{[52]} \] both in the case of Maxwellian and non-Maxwellian distributions of particle velocities. The asymptotics of slowly damping, long wavelength waves arising in the Vlasov-Poisson system has been rigorously studied also in \[ \text{[8]} \].

For general non Coulombian potentials, Langmuir waves do not exist. More precisely, for general interaction potentials we have \( \omega(k) \to 0 \) as \( |k| \to 0 \).

It is worth to mention that the mathematical properties of the function \( \Delta_\epsilon(k, z) \) (cf. \[ \text{[51, 131, 61, 124]} \) yielding the existence of Langmuir waves, are relevant in the derivation of \[ \text{(5.135) and (5.139)} \) using contour deformation arguments. Notice that due the function \( (\Delta_\epsilon(k, z))^{-1} \) is analytic in the variable \( z \) as long as \( \Delta_\epsilon(k, \cdot) \) does not have a zero. On the other hand, the existence of Langmuir waves implies the existence of one zero of \( \Delta_\epsilon(k, z) \) with \( z = \omega i \) and \( \omega \) satisfying \( \text{(7.3)} \). Therefore, in the case of Coulombian potentials the function \( (\Delta_\epsilon(k, z))^{-1} \) is analytic in the \( z \) variable in a region with the form \( \{ \text{Re}(z) \geq -C \exp \left( -\frac{\pi}{|k|^2} \right) \} \)
for some positive constants $C$ and $a$. Moreover, the size of this region of analiticity is rather optimal. In particular, it is not possible to obtain analiticity of the function $(\Delta_{\epsilon}(k, \cdot))^{-1}$ in a region with the form $\{\text{Re}(z) \geq -C |k|\}$ as in the case of long range, fast decaying potentials considered in Section 5.2.1. The main consequence of having an analiticity region for $(\Delta_{\epsilon}(k, z))^{-1}$ so small is that the contour deformations yielding to (5.135) must be made with some care. In particular, the cutoffs introduced by the function $1 - \frac{2(2\pi)^3}{(L_{\epsilon})^2}\Phi(p)\Psi\left(-\frac{ip\cdot V}{|p|}, \frac{p}{|p|}\right)$ as $|p| \to 0$, must be used in order to estimate the contributions of the region $\{|p| \ll 1\}$.

An important consequence of the existence of the Langmuir waves is the fact that the perturbations of the initial particle distribution with a wavelength much larger than the Debye length tend to dissappear very slowly. In particular it would be relevant to understand what is the effect of such long wave perturbations in the solutions of the initial value problems for the linearized Vlasov equation yielding the friction term (5.135) and the noise term (5.139).

We further observe that in the case of Coulombian interaction potentials if the initial particle distribution has correlations at distances larger than the Debye screening length we can expect these correlations to vanish on a very long time scale. In particular, the existence of a kinetic limit in this case is not clear a priori. Another interesting issue is to determine if there are Langmuir waves for potentials with the form (3.22)-(3.23), or if Langmuir waves are restricted to the case of Coulombian potentials.

7.2 Screening effects.

It is interesting to remark that screening effects (i.e. charge rearrangement which tends to damp the effects of charges unbalances in a system) does not require irreversible effects, namely collisions. Actually the screening properties can be described using just the (collisionless) Vlasov equation. Screening is closely related to Landau damping and to the stability of a medium discussed in Subsection 5.2.1. This fact is well understood in the physical literature (cf. [53]). We will just discuss here some simple mathematical results for the linearized Vlasov equation which illustrate the build-up of screening in time.

Suppose that we consider a system of particles which can be approximated by means of the Vlasov equation (5.113), (5.114). We assume that the interactions between the particles are Coulombian, and that the distribution of particles is spatially homogeneous and characterized by a distribution of velocities $f_0(v)$. We will write $h = \zeta^+ - \zeta^-$ to denote the perturbation of the particle density in the phase space and replace $\tilde{t}$ by $t$ in order to discharge the notation. This perturbation will be assumed to be sufficiently small to ensure that $h$ can be described using the linearized Vlasov equation. We will assume that the initial distribution of particles is $f_0(v)$ and that we place a small charge at the origin at rest. We can then approximate the evolution of the perturbation of the density of particles in the phase space $h$ by means of the following set of equations:

\[
\begin{align*}
\partial_t h(x, v, t) + v \cdot \nabla_x h(x, v, t) + \tilde{F} \cdot \nabla_v f_0(v) & = 0 \quad (7.4) \\
\tilde{F}(x, t) & = -\nabla_x \Psi(|x|) - \int dy \int dv h(y, v, t) \nabla_x \Psi(|x - y|) \\
h(x, v, 0) & = 0
\end{align*}
\]
where $\Psi(|x|) = \frac{1}{4\pi|x|}$. We assume, without loss of generality, that $\int f_0(v)\,dv = 1$. We will assume also that $f_0(v) = f_0(|v|)$. Notice that we can write $\tilde{F}(x, t) = -\nabla_x\varphi(x, t)$, with

$$\varphi(x, t) = \Psi(|x|) + \int dy \int dv \, h(y, v, t) \Psi(|x - y|)$$

Then

$$-\Delta \varphi(x, t) = \delta(x) + \rho(x, t) \quad \rho(x, t) = \int h(x, v, t)\,dv$$

(7.5)

Applying Duhamel’s formula to (7.4) we obtain

$$h(x, v, t) = \int_0^t [\nabla_x \varphi(x - v(t - s), s) \cdot \nabla_v f_0(v)]\,ds$$

and

$$\rho(x, t) = \int_0^t ds \int dv [\nabla_x \varphi(x - v(t - s), s) \cdot \nabla_v f_0(v)]$$

$$= \int_0^t (t - s)\,ds \int f_0(v) \Delta_x \varphi(x - v(t - s), s)\,dv$$

and, using the change of variables $y = v(t - s)$ we obtain

$$\rho(x, t) = \int_0^t \frac{ds}{(t - s)^2} \int f_0\left(\frac{y}{t - s}\right) \Delta_x \varphi(x - y, s)\,dy$$

$$= \Delta_x \left(\int_0^t \frac{ds}{(t - s)^2} \int f_0\left(\frac{y}{t - s}\right) \varphi(x - y, s)\,dy\right)$$

Using (7.5) we obtain

$$-\Delta_x \varphi(x, t) = \delta(x) + \Delta_x \left(\int_0^t \frac{ds}{(t - s)^2} \int f_0\left(\frac{y}{t - s}\right) \varphi(x - y, s)\,dy\right)$$

$$= \delta(x) + \Delta_x \left(\int_0^t \frac{ds}{s^2} \int f_0\left(\frac{y}{s}\right) \varphi(x - y, t - s)\,dy\right)$$

(7.6)

The equation (7.6) describes the onset of screening effects. This has been studied in detail in §4. We just describe here the steady states of (7.6) which describe the long time asymptotics of this equation if $f_0$ satisfies the stability conditions described in Subsection 5.2.1. The stationary solutions of (7.6) satisfy

$$-\Delta_x \varphi_\infty(x) = \delta(x) - \Delta_x \left(\int_0^\infty \frac{ds}{s^2} \int f_0\left(\frac{y}{s}\right) \varphi_\infty(x - y)\,dy\right)$$

(7.7)

We can rewrite (7.7) computing the integral $\int_0^\infty \frac{ds}{s^2} f_0\left(\frac{x}{s}\right)$. To this end we write $y$ in spherical coordinates $y = rw$, $r = |y| > 0$, $\omega \in S^2$. We then have $f_0\left(\frac{w}{s}\right) = f_0\left(\frac{\omega}{s}\right)$, since we assumed that $f_0$ is invariant under rotations. Then

$$\int_0^\infty \frac{ds}{s^2} f_0\left(\frac{y}{s}\right) = \int_0^\infty \frac{ds}{s^2} f_0\left(\frac{r}{s}\right) = \frac{1}{r} \int_0^\infty f_0(\xi) \xi^2 d\xi = \frac{1}{4\pi r}$$

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Thus
\[
\int_0^\infty \frac{ds}{s^2} \int f_0 \left( \frac{y}{s} \right) \varphi_\infty (x + y) \, dy = \int \frac{\varphi_\infty (x - y) \, dy}{4\pi |y|} = \int \frac{\varphi_\infty (y) \, dy}{4\pi |x - y|}
\]
Therefore (7.7) implies
\[
-\Delta_x \varphi_\infty (x) = \delta (x) + \int \Delta_x \left( \frac{1}{4\pi |x - y|} \right) \varphi_\infty (y) \, dy = \delta (x) - \varphi_\infty (x)
\]
whence
\[
-\Delta_x \varphi_\infty (x) + \varphi_\infty (x) = \delta (x)
\]

The solution of this equation is \( \varphi_\infty (x) = e^{-|x|/4\pi |x|} \). This solution exhibits the expected screening property in distances of the order of the screening length.

An alternative derivation of the screening properties taking as starting point the Vlasov equation can be found in [27] and [33]. The approach in those books is based in solving (7.4) using Fourier and Laplace transforms in order to compute the dielectric constant \( \varepsilon (k, \omega) \) which, for \( \omega \to 0 \) (i.e. in the stationary regime) is proportional \( \frac{1}{|k|^2 + 1} \), which is precisely the Fourier tranform of the potential \( e^{-|x|/4\pi |x|} \). It is relevant to emphasize the link between screening properties and Landau damping (or stability of the Vlasov medium) which plays a crucial role in the analysis of the long time asymptotics of the solutions to (7.6).

Some interesting problems suggested by the previous computations are the following ones

(i) To study the stationary solution and its stability for the solution of the Vlasov equation which describes the distribution of charges around a Dirac charge moving at constant speed.

(ii) To check if the screening properties considered above take place also for smooth potentials behaving asymptotically as \( \frac{1}{|x|^s} \) as \( |x| \to \infty \). Notice that in this case the potential generated by a set of charges cannot be described using a PDE, but the screening properties very likely depend only on the asymptotics of the potential and not in the details of it.

8 Other scaling limits yielding kinetic equations.

8.1 The case of very slowly decreasing potentials.

In the case of Lorentz gases, it has been seen in [28] that for interaction potentials with the form \( V_\varepsilon (x) = \varepsilon V (x) \) with \( V (x) \sim \frac{1}{|x|^s} \) as \( |x| \to \infty \) with \( \frac{1}{2} < s < 1 \) it is not possible to derive a kinetic equation describing the evolution of a tagged particle as \( \varepsilon \to 0 \). The reason for this is that the deflections experienced by the tagged particle have correlations of order one in distances of the order of the mean free path. Note that in order to obtain a well defined random force field in which the tagged particle moves, we need to assume an electroneutrality
condition. Nevertheless, the long range correlations of the deflections occur in spite of this assumption.

In the case of interacting particle systems the situation is different due to the presence of the Debye screening. The screening allows to consider that the interaction between the particles takes place by means of a potential with finite range in the form that has been seen in the derivation of the friction coefficient \( H_g(\infty; v) \) and the noise term \( D_g(t) \) (cf. (5.136), (5.139)). Differently from the case of Coulombian potentials, the Coulombian logarithm does not appear because not all the dyadic regions contribute in the same way. In the case \( \frac{1}{2} < s < 1 \), the largest contribution to the deflections can be expected to be due to the interaction between particles placed at distances of the order of the Debye length. This gives a picture rather different from the one taking place in the case of Lorentz gases, but it is also rather different from the picture in the usual Landau equation for Coulombian potential. Seemingly the resulting kinetic equation described the dynamics of the interacting particle system in this case is a Balescu-Lenard equation in which the main interaction take place at distances of the order of the Debye length. We plan to address this problem in the future.

8.2 The case of very heavy scatterers

We now study the motion of a tagged particle in a background of much heavier scatterers. It turns out that in some cases it is possible to approximate the dynamics of the tagged particle as a Lorentz gas (with stationary or moving scatterers). This is possible in the cases in which the interactions of the tagged particle with the scatterers do not modify almost the energy of the tagged particle (although they modify in a significant manner its direction of motion). The equations of motion for a Rayleigh gas in the case in which the mass of the tagged particle is different from the mass of the scatterers are

\[
\begin{align*}
\frac{dX}{d\tau} &= V, & m\frac{dV}{d\tau} &= -\sum_{j \in S} \nabla \phi_\varepsilon(X - Y_j) \\
\frac{dY_k}{d\tau} &= W_k, & \frac{dW_k}{d\tau} &= -\nabla \phi_\varepsilon(Y_k - X), & k \in S
\end{align*}
\]  

(8.1) (8.2)

where \((X, V)\) is the position and velocity of the tagged particle and \(\{(Y_k, W_k)\}_{k \in S}\) are the positions and velocities of the scatterers. We assume that the mass of the scatterers is one and the mass of the tagged particle is \(m\). We will assume in this Section that \(m \ll 1\). Moreover, we will assume also that the initial velocity of the tagged particle is of order one, as well as the average speed of the scatterers.

We examine first in which cases it is possible to approximate the dynamics of (8.1), (8.2) by means of a Lorentz gas yielding a linear Boltzmann equation with the form (2.2) instead of the linear Boltzmann equation (2.5). Notice that the key difference between both types of equations is that in the second one the change of velocity of the scatterer must be taken into account. Therefore, the dynamics of the tagged particle cannot be assumed to be the motion in a given random force field as in the Lorentz gas case. Suppose that the velocity of the tagged particle before the collision is \(v\) and the velocity of the scatterer before the collision is
v_s. Then, the velocities of the tagged particle and the scatterer after the collision become

\[ v - 2 \left( \frac{(v - v_s)}{m + 1} \cdot \omega \right) \omega, \quad v_s + 2 \left( \frac{m(v - v_s)}{m + 1} \cdot \omega \right) \omega \quad (8.3) \]

We can then obtain several distributions of velocities in which the velocity of the scatterers is almost constant in the collisions and we can approximate \([2.3]\) by \([2.2]\) (or equivalently the Rayleigh gas dynamics by a Lorentz dynamics). We will assume that the time scale is the characteristic time between collisions. Therefore, we can assume that the number of collisions in this time scale is of order one, and it is enough to examine under which conditions the change of velocity of the scatterers is negligible in a single collision.

Suppose first that the velocity of both the scatterers and the tagged particle is of the same order of magnitude, say one. Then \((8.3)\) becomes approximately, as \(m \to 0\)

\[ v - 2 ((v - v_s) \cdot \omega) \omega, \quad v_s \]

Therefore, in this regime, the velocity of the scatterers is near constant at the collisions and we can approximate \([2.3]\) by \([2.2]\).

Another situation in which a Rayleigh gas with very heavy scatterers can be approximated by means of Lorentz gas corresponds to the case in which the velocities of the particles and scatterers are determined by the equipartition of energy principle. This would imply that \(m |v|^2 \approx |v_s|^2\). Therefore, if we assume that the velocity of the tagged particle is of order one, it follows that \(|v_s|\) is of order \(\sqrt{m}\). Then \(v_s + 2 \left( \frac{m(v - v_s)}{m + 1} \cdot \omega \right) \omega \sim v_s\). Moreover in this case, the velocity of the tagged particle after each collision is approximately

\[ v - 2 (v \cdot \omega) \omega \]

Therefore, in this case, the dynamics of the Rayleigh gas can be approximated as a Lorentz gas with scatterers at rest. More generally, the dynamics of a tagged particle with mass \(m \ll 1\) and a velocity of order one moving in a Rayleigh gas can be approximated by means of the dynamics of a Lorentz gas (with scatterers at rest) if the velocity of the scatterers satisfies \(m \ll |v_s| \ll 1\).

We now consider the dynamics of a tagged particle in the Rayleigh gas \((8.1), (8.2)\) in the case of potentials with the form \((3.22), (3.23)\). We assume that the both the velocity of the tagged particle and the scatterers is of order one. We rescale the variables as

\[ y_k = \frac{Y_k}{mL_e}, \quad \bar{t} = \frac{\tau}{mL_e}, \quad \xi = \frac{X}{mL_e} \]

Notice that we use a time scale suitable to describe the evolution of the tagged particle. Then, the evolution equations are:

\[ \frac{d\xi}{dt} = V, \quad \frac{dV}{dt} = -\varepsilon \sum_{j \in S} \nabla_\xi \Phi (\xi - y_j) \]

\[ \frac{dy_k}{dt} = W_k, \quad \frac{dW_k}{dt} = -m\varepsilon \nabla_y \Phi (y_k - \xi), \quad k \in S \quad (8.4) \]
We assume that the potential $\Phi (s)$ decreases fast enough as $s \to \infty$ (say exponentially). We assume also that $m \ll 1$.

Arguing as in Subsection 5.1.1, we can approximate these equations as:

$$\partial_t f_\varepsilon (y, w, t) + w \cdot \nabla_y f_\varepsilon (y, w, t) - m \varepsilon \nabla_y \Phi (y - \xi) \cdot \nabla_w f_\varepsilon (y, w, t) = 0$$  \hspace{1cm} (8.5)

$$\frac{d\xi}{dt} = V , \quad \frac{dV}{dt} = -\varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \Phi (\xi - \eta) \rho_\varepsilon (\eta, t) \, d\eta$$  \hspace{1cm} (8.6)

where:

$$f_\varepsilon (y, w, t) = g (w) + \frac{1}{(L_\varepsilon)^2} \zeta_\varepsilon (y, w, t)$$  \hspace{1cm} (8.7)

and $\zeta_\varepsilon (y, w, 0)$ will be fixed shortly. We can linearize the equation of $f_\varepsilon$. Then:

$$\partial_t \zeta_\varepsilon (y, w, t) + w \cdot \nabla_y \zeta_\varepsilon (y, w, t) - m \theta_\varepsilon \nabla_y \Phi (y - \xi) \cdot \nabla_w g (w) = 0$$  \hspace{1cm} (8.8)

$$\zeta_\varepsilon (y, w, 0) = N (y, w)$$  \hspace{1cm} (8.9)

where $\theta_\varepsilon = \varepsilon (L_\varepsilon)^2$ and where:

$$\mathbb{E} [N (y, w)] = 0 , \quad \mathbb{E} [(N (y_a, w_a)) N (y_b, w_b)] = g (w_a) \delta (y_a - y_b) \delta (w_a - w_b)$$  \hspace{1cm} (8.10)

We can compute now the term $\zeta_\varepsilon (y, w, t)$ as in Subsection 5.1.1. We obtain the following approximation for $\zeta_\varepsilon (y, w, t)$:

$$\zeta_\varepsilon (y, w, t) = \zeta_1 (y, w, t) + \zeta_2 (y, w, t)$$

$$\zeta_1 (y, w, t) = N (y - w t, w)$$

$$\zeta_2 (y, w, t) = m \theta_\varepsilon \nabla_w g (w) \cdot \int_0^t \nabla_y \Phi (y - w (t - s) - V s) \, ds$$

Using that $m \to 0$ we obtain the following approximation for times $t$ of order one:

$$\zeta_\varepsilon (y, w, t) = N (y - w t, w)$$

We can now write the evolution equation for the tagged particle. Arguing as in Subsection 5.1.1, we obtain:

$$\frac{d\xi}{dt} = V , \quad \frac{dV}{dt} = -\varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \Phi (\xi - \eta) \tilde{\rho}_\varepsilon (\eta, t) \, d\eta$$  \hspace{1cm} (8.11)

where:

$$\tilde{\rho}_\varepsilon (y, t) = \int_{\mathbb{R}^3} \zeta_\varepsilon (y, w, t) \, dw = \int_{\mathbb{R}^3} N (y - w t, w) \, dw$$  \hspace{1cm} (8.12)

Notice that the equation for the tagged particle (5.11) becomes in this limit the dynamics of a tagged particle in a given time dependent random force field, i.e. a Lorentz gas. Therefore, in this particular problem we obtain the same picture as in the case of Boltzmann collisions. The properties of the random force field can be obtained easily. We have

$$f (\xi, t) = -\varepsilon (L_\varepsilon)^3 \int_{\mathbb{R}^3} \nabla_\xi \Phi (\xi - \eta) \int_{\mathbb{R}^3} N (\eta - w t, w) \, dw \, d\eta$$
whence

\[ \mathbb{E} \left[ f (\xi, \tilde{t}) \right] = 0 \]

\[ \mathbb{E} \left[ f (\xi_1, \tilde{t}_1) \otimes f (\xi_2, \tilde{t}_2) \right] = \varepsilon^2 (L_\varepsilon)^6 \int_{\mathbb{R}^3} d\eta \int_{\mathbb{R}^3} dw \nabla_\xi \Phi (\xi_1 - \eta) \otimes \nabla_\xi \Phi (\xi_2 - \eta + w (\tilde{t}_1 - \tilde{t}_2)) g(w) \]

The dynamics of the tagged particle in this random force field can then be studied using the type of methods developed in [38] (adapted to the time dependent case). Notice that the random force field \( f (\xi, \tilde{t}) \) is not a field generated by a Poisson distribution of scatterers as discussed in Subsection 3.2. The Gaussian approximation of the random force field above is sufficient to compute the particle deflections which are the only relevant quantity required to obtain the evolution of the tagged particle. Nevertheless, it would be possible to use similar arguments to approximate the dynamics of the tagged particle by means of the dynamics in a Lorentz gas with interaction potentials obtained by means of the Poisson probability measure.

Notice that if the velocities of the scatterers and the tagged particle are chosen by means of the equipartition of energy principle, i.e. \( |v| \) of order one (say) and \( |v_*| \) of order \( \sqrt{m_a} \) a similar approximation would imply that the random force field \( f \) would be independent on \( \tilde{t} \).

We will not continue the discussion here any further, but we remark that it is possible to apply similar ideas to describe the dynamics of a tagged particle with mass \( m \ll 1 \) moving in a background of much heavier interacting particles (with mass 1). Similar arguments imply that the friction term, which is associated to the deflections experience by the scatterers would be negligible. However, an interesting feature of this problem is that the resulting random force field in which the tagged particle moves would not be just due to the free motion of the scatterers, but also by their self-consistent interactions, as it has been in the derivation of Balescu-Lenard equations. Notice that the resulting kinetic equation is a linear kinetic equation without friction term, but in which the diffusion coefficient is affected by the interactions taking place between the scatterers.

8.3 Nonhomogeneous situations.

We have considered in the previous Sections only spatially homogeneous particle distributions. We can apply similar methods to study some classes of nonspatially homogeneous situations. We will not consider the details in this paper, but we just indicate under which assumptions we can obtain kinetic equations analogous to the ones obtained in the homogeneous case.

The specific scaling limit in which we can obtain a nonspatially homogeneous kinetic equation is the following. Suppose that we consider a set of particles in the phase space \( \mathbb{R}^3 \times \mathbb{R}^3 \) by means of a nonhomogeneous Poisson distribution characterized by the rate:

\[ f_0 \left( \frac{x}{L_h}, v \right), \quad L_h \gg 1. \]

We assume that the average distance between particles is one and that the characteristic velocity of the particles is of order one too. Suppose that the particle configuration evolves according to the system of equations (1.1). Choosing \( L_h \) as the mean free path we obtain the
version of the kinetic equation containing the transport term $v \cdot \nabla x$ which does not vanish for nonhomogeneous solutions.

More precisely, in the case of Rayleigh Gases we obtain the following family of linear kinetic equations

$$\partial_t f + v \cdot \nabla_x f = Q(g, f), \quad f = f(x, v, t) \quad (8.13)$$

where the linear collision operator $Q = Q_B(g, f)$ with $Q_B$ given as in (2.5) in the case of the Rayleigh-Boltzmann equation, and $Q = Q_L$ with $Q_L$ given as in (2.7) in the case of the Rayleigh-Landau equation.

In the case of interacting particle systems we obtain the following family of nonlinear kinetic equations

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad f = f(x, v, t) \quad (8.14)$$

where the collision operator $Q = Q_B$ with $Q_B$ given as in (2.8) in the case of the Boltzmann equation, $Q = Q_L$ with $Q_L$ given as in (2.11) in the case of the Landau equation and $Q = Q_{BL}$ with $Q = Q_{BL}$ given as in (2.13) in the case of the Balescu-Lenard equation.

The nonhomogeneous kinetic equations are meaningful only if $L_h$ is comparable or much larger than the mean free path. If $L_h$ is much larger than the mean free path we would obtain that the kinetic equations would be given approximately by homogeneous equations, whose solutions converge to Maxwellian distributions characterized by local temperatures, density and momentum which would change in the characteristic scale $L_h$. This situation corresponds to the so-called hydrodynamic limit that we will not be discussed in this paper.

9 Conclusions.

In this paper we have examined the precise conditions under which we can approximate the dynamics of many particle systems or tracer particles in Rayleigh gases by means of kinetic equations. In particular we have focused mostly in the case in which the particles interact by means of weak, but long range potentials, including potentials behaving for long distances as the Coulomb potential. In this type of problems the resulting kinetic equations are the classical Landau and Balescu-Lenard equations. In the case of system with Coulombian interactions a distinct logarithmic correction known as Coulombian logarithm appears in the formula of the mean free path and in the formula of the macroscopic time scale.

We have examined in detail how to derive formally both types of kinetic equations (Landau and Balescu-Lenard), approximating the dynamics of the tracer particles or the particles of the system by means of the dynamics of particles in random force fields. In the kinetic limit under consideration such a dynamics can be approximated as the dynamics of a particle with a friction coefficient which is affected by a random force field, which is not affected by the tagged particle itself. These results allow to obtain nonlinear evolution equations describing the evolution of many interacting particle systems in suitable asymptotic limits.

A general idea that we have used repeatedly to obtain kinetic approximations of the particle systems which interact by means of weak interactions is to approximate the dynamics of each particle by means of the dynamics of a particle interacting with a random force field. It then turns out that the dynamics of the tagged particle can be reformulated as the dynamic
of a particle in which two forces act, namely a friction force and a random force field which is not affected by the tagged particle itself.

We have discussed also the relation between the derivations of Landau and Balescu-Lenard equations given in this paper and the traditional derivation obtained using the methods of \[3, 11, 23\] and Lenard based in the BBGKY hierarchies. We have discussed also several phenomena taking place in some many interacting particle systems due to their collective behaviour, like screening as well as the slow decay of some long range oscillations known as Langmuir waves.

A Approximation of particle distributions by means of gaussian densities.

The approximation of Poisson point processes by means of gaussian fields in distances much larger than the average particle distance is a well known consequence of the Central Limit Theorem. We summarize here the main properties that we will use of the resulting gaussian fields for particle distributions distributed in the phase space \((x, v)\).

For each particle configuration \(\{(x_k, v_k)\} \in S\in \mathbb{R}^3\times\mathbb{R}^3\) chosen according to the rate \(dxg\) we define \(y_k = \frac{x_k}{L_\varepsilon}, W_k = v_k, y = \frac{x}{L_\varepsilon}, w = v\), with \(L_\varepsilon \gg 1\) and then the empirical distribution:

\[
f_\varepsilon (y, w) = \frac{1}{(L_\varepsilon)^3} \sum_k \delta (y - y_k) \delta (w - W_k), \quad f_\varepsilon \in \mathcal{M}_+ (\mathbb{R}^3 \times \mathbb{R}^3)
\]

Then, for any compactly supported test function \(\varphi \in C_0 (\mathbb{R}^3 \times \mathbb{R}^3)\) we have:

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon (y, w) \varphi (y, w) \, dydw = \frac{1}{(L_\varepsilon)^3} \sum_k \varphi (y_k, W_k)
\]

Then, taking the expectation with respect to the Poisson measure we obtain:

\[
E \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon (y, w) \varphi (y, w) \, dydw \right] = \frac{1}{(L_\varepsilon)^3} E \left[ \sum_k \varphi (y_k, W_k) \right]
\]

The right-hand side of this identity can be computed approximating \(\varphi\) by piecewise constant functions and using the properties of the Poisson distribution. Then:

\[
E \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon (y, w) \varphi (y, w) \, dydw \right] = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi (y, w) g (w) \, dydw
\]

We estimate now the variance of \(\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon \varphi dydw\). We have:

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon (y_a, w_a, 0) f_\varepsilon (y_b, w_b, 0) \varphi (y_a, w_a) \varphi (y_b, w_b) \, dy_adw_adyw_bw_b
= \frac{1}{(L_\varepsilon)^6} \sum_k \sum_\ell \varphi(y_k, W_k) \varphi(y_\ell, W_\ell)
\]
In order to compute this integral we approximate \( \varphi \) by means of piecewise functions. Therefore we need to compute the following expectations (where we denote as \( \chi_A \) the characteristic function of the set \( A \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \)):

\[
\mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_A (y', W_{y'}) \right], \quad \mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_B (y', W_{y'}) \right]
\]

where \( A \) and \( B \) are disjoint sets. We have:

\[
\mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_A (y', W_{y'}) \right] = \mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \chi_A (y, W_k) \chi_A (y', W_{y'}) \right] + \mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \chi_A (y, W_k) \chi_A (y', W_{y'}) \right] = \frac{1}{(L_c)^6} \mathbb{E} \left[ (n(A))^2 \right] + \mathbb{E} \left[ \frac{n(A)}{(L_c)^6} \right] = \frac{1}{(L_c)^6} \mathbb{E} \left[ (n(A))^2 \right]
\]

On the other hand we have:

\[
\mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_B (y', W_{y'}) \right] = \mathbb{E} \left[ \frac{n(A) n(B)}{(L_c)^6} \right] = \frac{1}{(L_c)^6} \mathbb{E} \left[ n(A) \right] \mathbb{E} \left[ n(B) \right]
\]

We write:

\[
c_A = (L_c)^3 \int_A g(v) dy dv = (L_c)^3 J_A
\]

whence:

\[
\mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_B (y', W_{y'}) \right] = \frac{1}{(L_c)^6} \mathbb{E} \left[ n(A) \right] \mathbb{E} \left[ n(B) \right] = J_A J_B
\]

We compute

\[
\mathbb{E} \left[ (n(A))^2 \right] = \sum_{N=0}^{\infty} N^2 \frac{(c_A)^N}{N!} e^{-c_A} = (c_A)^2 + c_A = (L_c)^6 (J_A)^2 + (L_c)^3 J_A,
\]

since \( \sum_{N=0}^{\infty} N^2 \frac{(x)^N}{N!} = e^x (x^2 + x) \). Then:

\[
\mathbb{E} \left[ \frac{1}{(L_c)^6} \sum_k \sum_{y, y'} \chi_A (y, W_k) \chi_A (y', W_{y'}) \right] = \frac{1}{(L_c)^6} \mathbb{E} \left[ (n(A))^2 \right] = (J_A)^2 + \frac{1}{(L_c)^3} J_A
\]

By decomposing \( \varphi \) in linear combinations of characteristic functions of disjoint sets we get:

\[
\frac{1}{(L_c)^6} \mathbb{E} \left[ \sum_{k, l} \varphi (y, W_k) \varphi (y_l, W_{y_l}) \right] = \int_{(\mathbb{R}^3)^2} \varphi (y, w) g(v) dv dy + \frac{1}{(L_c)^3} \int_{(\mathbb{R}^3)^2} (\varphi (y, w))^2 g(v) dv dy
\]

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whence:

\[
\mathbb{E} \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_\varepsilon (y_a, w_a, 0) f_\varepsilon (y_b, w_b, 0) \varphi (y_a, w_a) \varphi (y_b, w_b) \, dy_a dw_a dy_b dw_b \right] \\
= \frac{1}{(L\varepsilon)^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (\varphi (y, w))^2 \, g (v) \, dy \, dv
\]

or in a formal manner:

\[
\mathbb{E} [f_\varepsilon (y_a, w_a, 0) f_\varepsilon (y_b, w_b, 0)] = \frac{g (w_a)}{(L\varepsilon)^3} \delta (y_a - y_b) \delta (w_a - w_b)
\]

We now compute the correlations associated to the density \( \tilde{\rho}_1 \). We have:

\[
\begin{align*}
\tilde{\rho}_1 (y, \tilde{t}) &= \int_{\mathbb{R}^3} \zeta_1 (y, w, \tilde{t}) \, dw \\
\zeta_1 (y, w, \tilde{t}) &= N (y - w \tilde{t}, w) \\
\mathbb{E} [(N (y_a, w_a)) N (y_b, w_b)] &= g (w_a) \delta (y_a - y_b) \delta (w_a - w_b)
\end{align*}
\]

Then:

\[
\begin{align*}
\mathbb{E} [\tilde{\rho}_1 (y_1, \tilde{t}_1) \tilde{\rho}_1 (y_2, \tilde{t}_2)] &= \int_{\mathbb{R}^3} \, dw_1 \int_{\mathbb{R}^3} \, dw_2 \mathbb{E} [\zeta_1 (y_1, w_1, \tilde{t}_1) \zeta_1 (y_2, w_2, \tilde{t}_2)] \\
&= \int_{\mathbb{R}^3} \, dw_1 \int_{\mathbb{R}^3} \, dw_2 \mathbb{E} [N (y_1 - w_1 \tilde{t}_1, w_1) N (y_2 - w_2 \tilde{t}_2, w_2)] \\
&= \int_{\mathbb{R}^3} \, dw_1 g (w_1) \delta (y_1 - y_2 - w_1 (\tilde{t}_1 - \tilde{t}_2))
\end{align*}
\]

Suppose now that \( \tilde{t}_1 \geq \tilde{t}_2 \). Then:

\[
\begin{align*}
\mathbb{E} [\tilde{\rho}_1 (y_1, \tilde{t}_1) \tilde{\rho}_1 (y_2, \tilde{t}_2)] &= \int_{\mathbb{R}^3} \, dw_1 g (w_1) \delta (y_1 - y_2 - w_1 (\tilde{t}_1 - \tilde{t}_2)) \\
&= \frac{1}{(\tilde{t}_1 - \tilde{t}_2)^2} \int_{\mathbb{R}^3} \, dw_1 g (w_1) \delta \left( \frac{y_1 - y_2}{\tilde{t}_1 - \tilde{t}_2} - w_1 \right) = \frac{1}{(\tilde{t}_1 - \tilde{t}_2)^3} g \left( \frac{y_1 - y_2}{\tilde{t}_1 - \tilde{t}_2} \right)
\end{align*}
\]

Henceforth:

\[
\begin{align*}
\mathbb{E} [\tilde{\rho}_1 (y_1, \tilde{t}_1) \tilde{\rho}_1 (y_2, \tilde{t}_2)] &= \frac{1}{(\tilde{t}_1 - \tilde{t}_2)^3} g \left( \frac{y_1 - y_2}{\tilde{t}_1 - \tilde{t}_2} \right) , \quad \tilde{t}_1 > \tilde{t}_2.
\end{align*}
\]

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