Non-Hermitian Hamiltonians viewed from Heisenberg equation of motion

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Abstract

The Heisenberg picture for non-Hermitian but $\eta$-pseudo-Hermitian Hamiltonian systems is given. Two classes of general non-Hermitian Hamiltonians are proposed, one is non-Hermitian and non-$PT$-symmetric and the other is non-Hermitian but $PT$-symmetric. Their (first order) Heisenberg equations of motion are highly non-trivially complex, however, the corresponding (second order) equations of motion are shown to be real. The real closeness is suggested as a new criterion, based on which the existence of real eigenvalues for the two classes of general non-Hermitian Hamiltonians is determined. The eigenfunctions can be obtained in terms of a similarity transformation. The complementarity and compatibility on the real closeness and the $PT$ symmetry are discussed.

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1 Introduction

It is well known that the operators of physical observables are required to be (Dirac) Hermitian in order to have real eigenvalues in quantum mechanics. However, the Hermiticity can be relaxed to be $\eta$-pseudo Hermiticity or $PT$ symmetry in non-Hermitian quantum mechanics, where $\eta$ is a linear Hermitian or an anti-linear anti-Hermitian operator, and $P$ and $T$ stand for the parity and the time-reversal operators, respectively. Being traced back to the work by Dirac [1] and Pauli [2], the non-Hermitian quantum theories have been developed [3] quickly in recent decades.

Normally, a non-Hermitian Hamiltonian is analyzed in terms of Schrödinger equations in the pseudo-Hermitian and the $PT$-symmetric quantum theories [3]. Because of the non-Hermiticity of Hamiltonians, new concepts are introduced, such as indefinite and positive definite metrics, biorthonormal bases, and modified inner products, etc. Such an analysis has been demonstrated in detail in the review articles [3] for various non-Hermitian quantum systems.

In this paper, instead of solving Schrödinger equations, we analyze eigenvalues of non-Hermitian quantum systems by means of Heisenberg equations of motion. To this end, the Heisenberg picture for non-Hermitian but $\eta$-pseudo-Hermitian Hamiltonians is given as a basis. Then, the first order Heisenberg equations of motion can be derived that are generally complex. If the corresponding second (higher) order equations of motion are real, we use the real closeness as a new criterion to determine the existence of real eigenvalues. In order to clarify this criterion, we then construct two classes of general non-Hermitian Hamiltonians, where one is non-Hermitian and non-$PT$-symmetric and the other is non-Hermitian but $PT$-symmetric, and determine that the two classes of non-Hermitian Hamiltonians have real eigenvalues. The more detailed description is: if a non-Hermitian Hamiltonian has a real second (higher) order equation of motion though the corresponding (first order) Heisenberg equation of motion is complex, we can deduce such a Hermitian Hamiltonian that gives the same second (higher) order equation of motion. Because the two sets of eigenvalues of the Hermitian and the non-Hermitian Hamiltonians are same (up to a real constant), we thus determine the existence of real eigenvalues for the non-Hermitian Hamiltonian. The isospectrum can alternatively be verified by a non-unitary similarity transformation. As a result, it is interesting that the Heisenberg equation of motion is available for the determination of real eigenvalues for non-Hermitian Hamiltonians. In addition, the eigenfunctions of non-Hermitian Hamiltonians are investigated from the point of view of similarity transformations.

This paper is arranged as follows. In the following section, the Heisenberg picture is provided for non-Hermitian but $\eta$-pseudo-Hermitian Hamiltonians. In general, the time evolution of operators, i.e., the Heisenberg equation of motion for non-Hermitian Hamiltonians
is not so obvious as that for Hermitian Hamiltonians. Fortunately, the usual formulation of Heisenberg equations of motion maintains for the non-Hermitian case if a suitably modified inner product is introduced. In section 3, the Swanson model [4] and the Pais-Uhlenbeck oscillator model [5, 6] are chosen as toy models for a simple and initial application of our criterion. It is found that the equations of motion of the two models (second order to the former while fourth order to the latter) are real closed and the real energy spectra are thus obtained. As the main context of the present paper, two classes of general non-Hermitian Hamiltonians are proposed in sections 4 and 5, respectively, where one is non-Hermitian and non-\(PT\)-symmetric and the other is non-Hermitian but \(PT\)-symmetric. Although their (first order) Heisenberg equations of motion are highly non-trivially complex, the corresponding (second order) equations of motion are shown to have the real closeness. Then, the Hermitian (isospectral) Hamiltonians that give the same (second order) equations of motion are deduced, and the existence of real eigenvalues for the two classes of general non-Hermitian Hamiltonians can thus be determined. Finally, a summary is given in section 6; in addition, the complementarity and compatibility on the real closeness and the \(PT\) symmetry are discussed, and the eigenfunctions of the two classes of general non-Hermitian Hamiltonians are briefly investigated in terms of similarity transformations.

2 Heisenberg picture for non-Hermitian Hamiltonians

For a non-Hermitian, \(H^\dagger \neq H\), but \(\eta\)-pseudo-Hermitian Hamiltonian,

\[
H^\dagger \equiv \eta^{-1}H^\dagger \eta = H,
\]

where \(H\) does not depend explicitly on time and \(\eta\), a metric named by Pauli, is a linear Hermitian operator, the modified inner product is defined [2] as

\[
\langle \psi(t)|\psi(t)\rangle_\eta \equiv \langle \psi(t)|\eta|\psi(t)\rangle,
\]

where the wave function at any time \(\psi(t)\) satisfies the Schrödinger equation, \(i\hbar \frac{\partial}{\partial t} \psi(t) = H\psi(t)\), and it can be expressed by the wave function at the initial time as follows:

\[
\psi(t) = \exp(-iHt/\hbar)\psi(0).
\]

For a physical observable \(O\), in accordance with the modified inner product eq. (2), its average value now takes the following form in the Schrödinger picture,

\[
\langle O \rangle^S_{\text{Av}} = \langle \psi(t)|O|\psi(t)\rangle_\eta = \langle \psi(t)|\eta O|\psi(t)\rangle,
\]

which is definitely real because \(O\), as a physical observable, has the same \(\eta\)-pseudo Hermiticity as \(H\), \(O = \eta^{-1}O^\dagger \eta\). Substituting eq. (3) into eq. (1) and using eq. (1), we rewrite the average
value of the operator $O$ to be

$$\langle O \rangle^{S}_{A v} = \langle \psi(0) | \exp(+iH^\dagger t/\hbar) \eta O \exp(-iHt/\hbar) | \psi(0) \rangle$$

$$= \langle \psi(0) | \eta \left\{ \eta^{-1} \exp(+iH^\dagger t/\hbar) \eta \right\} O \exp(-iHt/\hbar) | \psi(0) \rangle$$

$$= \langle \psi(0) | \exp(iHt/\hbar) O \exp(-iHt/\hbar) | \psi(0) \rangle_{\eta}. \quad (5)$$

Alternatively, one can introduce the Heisenberg picture where the average value of the time dependent physical observable $O(t)$ has the form,

$$\langle O(t) \rangle^{H}_{A v} = \langle \psi(0) | O(t) | \psi(0) \rangle_{\eta}. \quad (6)$$

In accordance with the principle that the average value of an arbitrary physical observable is independent of the choice of pictures, that is,

$$\langle O \rangle^{S}_{A v} = \langle O(t) \rangle^{H}_{A v}, \quad (7)$$

we obtain the time dependent observable in terms of the time independent one,

$$O(t) = \exp(iHt/\hbar)O \exp(-iHt/\hbar). \quad (8)$$

The time evolution of $O(t)$, i.e., the Heisenberg equation of motion thus has the usual formulation,

$$\dot{O}(t) = \frac{1}{i\hbar}[O(t), H]. \quad (9)$$

Consequently, eqs. (6) and (9) give the Heisenberg picture under the modified definition of inner products (eq. (5)). In addition, it can be verified that the time dependent operator (sometimes called Heisenberg operator) $O(t)$ maintains the $\eta$-pseudo Hermiticity,

$$O(t) = \eta^{-1}\{O(t)\}^\dagger \eta, \quad (10)$$

due to $H = \eta^{-1}H^\dagger \eta$ and $O = \eta^{-1}O^\dagger \eta$. Further, the relationship of commutators between the two pictures is

$$[O_1(t), O_2(t)] = \exp(iHt/\hbar)[O_1, O_2] \exp(-iHt/\hbar). \quad (11)$$

This leads to the maintenance of Heisenberg commutation relations for canonical time dependent operators. For instance, to the coordinate and the momentum, $x(t)$ and $p(t)$ satisfy

$$[x(t), p(t)] = i\hbar, \quad [x(t), x(t)] = 0 = [p(t), p(t)], \quad (12)$$

if the time independent counterparts obey the (canonical) Heisenberg commutation relations.

In closing this section, we mention that it is a prerequisite that a non-Hermitian Hamiltonian has an $\eta$-pseudo-Hermiticity if the above Heisenberg picture is adopted. In sections 4
and 5 of the present paper, for instance, the two classes of general non-Hermitian Hamiltonians depicted by eq. (29) and eq. (40), respectively, inherently have such pseudo-Hermiticity, see eqs. (38) and (39) for the former, and eqs. (47) and (48) for the latter. Furthermore, the modified inner product defined by eq. (2) ensures that the probability is positive definite for the two classes of non-Hermitian Hamiltonians in both the Schrödinger picture and the Heisenberg picture. For example, for the class expressed by eq. (29), we have

\[ \langle \Phi | \Phi \rangle_\eta \equiv \langle \Phi | \eta \rangle \langle \eta | \Phi \rangle = \langle \Phi | \Omega^\dagger \Omega | \Phi \rangle = \langle \phi | \phi \rangle \geq 0, \]

where eq. (34) and eq. (37) have been utilized. Note that time is hidden because the probability should be independent of the choice of pictures, that is, \( \langle \Phi(t) | \Phi(t) \rangle_\eta = \langle \Phi(0) | \Phi(0) \rangle_\eta \), which is consistent with the preservation of unitarity of time evolution in \( \eta \)-pseudo-Hermitian quantum theory.

3 Two simple models

This section acts as a prelude. We choose two simple models to investigate their real eigenvalues as an initial application of the criterion — the real closeness of equations of motion. One is the Swanson model [4] whose equation of motion is second order, and the other is the Pais-Uhlenbeck oscillator model [5] whose equation of motion is fourth order.

3.1 Swanson model

The Hamiltonian of the Swanson model [4] takes the form,

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 + \frac{ic}{2} (px + xp), \]

where \( m \) and \( \omega \) are the mass and angular frequency of a harmonic oscillator, respectively, and \( c \) is a real constant with the dimension of inverse time. \((x, p)\) is a pair of canonical coordinate and momentum that satisfies the Heisenberg commutation relations,

\[ [x, p] = i\hbar, \quad [x, x] = 0 = [p, p]. \]

The model is constructed by adding the imaginary interacting term \( \frac{ic}{2} (px + xp) \) to the Hamiltonian of the free harmonic oscillator with the angular frequency \( \omega \), where \( c \) may be understood as a coupling constant. Obviously, this Hamiltonian is not Hermitian, \( H^\dagger \neq H \), but \( PT \) symmetric,

\[ H = H^{PT} := (PT)^{-1} H(PT), \]
where the conventional definitions of the parity $P$ and the time-reversal $T$ transformations are as follows:

\[ P : \ x \rightarrow -x, \quad p \rightarrow -p, \quad i \rightarrow +i; \]
\[ T : \ x \rightarrow +x, \quad p \rightarrow -p, \quad i \rightarrow -i. \]  \hspace{1cm} (17)

Instead of solving the Schrödinger equation of the Hamiltonian eq. (14), we derive the Heisenberg equation of motion. Utilizing eq. (9) for coordinate and momentum, we have

\[ \dot{x} = \frac{1}{i\hbar} [x, H], \quad \dot{p} = \frac{1}{i\hbar} [p, H], \]  \hspace{1cm} (18)

and then substituting eqs. (14) and (12) into the above equations, we get the complex Heisenberg equations of motion,

\[ \dot{x} = \frac{p}{m} + icx, \quad \dot{p} = -m\omega^2 x - icp. \]  \hspace{1cm} (19)

Eliminating the momentum operator $p$ in eq. (19), we obtain the quantum equation of motion for the coordinate operator $x$,

\[ \ddot{x} + (\omega^2 + c^2) x = 0. \]  \hspace{1cm} (20)

Note that the second order equation of motion is real closed although its corresponding first order Heisenberg equations of motion (eq. (19)) are complex, and that it describes the harmonic oscillator with the angular frequency $\sqrt{\omega^2 + c^2}$. Therefore, we can deduce a Hermitian Hamiltonian that corresponds to the same second order equation of motion,

\[ h = \frac{p^2}{2m} + \frac{1}{2} m (\omega^2 + c^2) x^2, \]  \hspace{1cm} (21)

which has the same energy spectrum as that of the Swanson model. Consequently, we at once give the real eigenvalues for the Swanson model,

\[ E_n = \hbar \sqrt{\omega^2 + c^2} \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \cdots. \]  \hspace{1cm} (22)

We make several comments on our investigation for the Swanson model. The first is that the $\eta$-pseudo-Hermiticity of the Swanson model is obvious because it is a special case of the two classes of general non-Hermitian Hamiltonians proposed in sections 4 and 5, see eq. (29) and eq. (40). The second comment is that our analysis depends on the real closeness of the second order equation of motion of $x$. If $x$ is eliminated in eq. (19), $p$ satisfies the same real second order equation of motion as eq. (20). This is a special property in the Swanson model.

\footnote{Time is hidden for Heisenberg operators in the following context for the sake of convenience.}
because the Hamiltonian eq. (14) is symmetric under the exchange of $x$ and $p$. This property will be broken in the two classes of general non-Hermitian Hamiltonians, see eqs. (29) and (40). In fact, if and only if the second order equations of motion is real closed to one of the pair of canonical variables $(x, p)$, our criterion works. See the detailed analyses for the two classes of general non-Hermitian Hamiltonians in sections 4 and 5. The last comment is that the effect of the imaginary interaction in the Swanson Hamiltonian, if we focus on eigenvalues, is just to shift the angular frequency of the free harmonic oscillator from $\omega$ to $\sqrt{\omega^2 + c^2}$.

### 3.2 Pais-Uhlenbeck oscillator model

There are several forms of Hamiltonians related to the Pais-Uhlenbeck oscillator model [5], i.e., different Hamiltonians but same equation of motion, among which a $PT$-symmetric Hamiltonian and a $PT$-pseudo-Hermitian one are considered here. The purpose to pick out different forms is to explicitly show that they give the same real fourth order equation of motion, which is independent of the Hamiltonians’ forms and symmetries. Equations of motion act as the basis for us to determine real eigenvalues. In addition, an invariance of parameter exchange existed in the Hamiltonians and their corresponding quantum equation of motion will be emphasized because such an invariance plays a crucial role in determination of the positive definite spectrum of the model.

The $PT$-symmetric Hamiltonian given in ref. [6] has the form,

$$H_1 = \frac{p_1^2}{2\gamma} - ip_2 x_1 + \frac{1}{2} \gamma \left( \omega_1^2 + \omega_2^2 \right) x_1^2 + \frac{1}{2} \gamma \omega_1^2 \omega_2^2 x_2^2,$$

(23)

where $\gamma$, $\omega_1$ and $\omega_2$ are all positive constants, and $(x_j, p_j)$, $j = 1, 2$, are two pairs of canonical coordinates and momenta that satisfy the Heisenberg commutation relations,

$$[x_j, p_k] = i\hbar \delta_{jk}, \quad [x_j, x_k] = 0 = [p_j, p_k], \quad j, k = 1, 2.$$

(24)

$H_1$ possesses a kind of $PT$ symmetry, $H_1 = H_1^{PT} \equiv (PT)^{-1} H_1(PT)$, under the conventional $P$ and $T$ transformations to $(x_1, p_1)$ (see eq. (17)) and the unconventional ones to $(x_2, p_2)$.

It is constructed by determining no classical limits for the quantum Pais-Uhlenbeck oscillator with positive spectra and by performing an (isospectral) operator similarity transformation, see ref. [6] for the details.

The $PT$-pseudo-Hermitian Hamiltonian proposed in ref. [7] takes the form,

$$H_{II} = \frac{p_1^2}{2m} + \frac{1}{2} ma_1^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2} ma_2^2 x_2^2 + i \frac{a_3}{2ma_1a_2} p_1 p_2,$$

(25)

As explained in ref. [6], $x_2$ and $p_2$ transform, in the language of quantum field theory, as parity scalars instead of pseudo-scalars, i.e., $P : x_2 \rightarrow +x_2$, $p_2 \rightarrow +p_2$; $T : x_2 \rightarrow -x_2$, $p_2 \rightarrow +p_2$.  

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which is constructed by adding an imaginary interacting term proportional to $ip_1p_2$ to the Hamiltonian of a free anisotropic planar oscillator, where $m$ is the mass of the two-dimensional oscillator, and $a_1$, $a_2$, and $a_3$ are non-vanishing real constants with the anisotropic condition $a_1 \neq a_2$. $H_{\Pi}$ has $PT$-pseudo-Hermiticity, $H_{\Pi} = H_{\Pi}^\dagger \equiv (PT)^{-1}H_{\Pi}^\dagger(PT)$, under the conventional parity and time-reversal transformations in each dimension (eq. (17)), as shown in ref. [7].

In accordance with the Heisenberg equations of motion eq. (9) and the commutation relations eq. (12), we can derive the first order Heisenberg equations of motion for the two Hamiltonians $H_I$ and $H_{\Pi}$. Although the two groups of Heisenberg equations of motion are complex, after eliminating momentum and coupling terms we deduce the real fourth order ordinary differential equation (ODE)

$$\frac{d^4x_j}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2x_j}{dt^2} + \omega_1^2 \omega_2^2 x_j = 0, \quad j = 1, 2,$$

where for $H_{\Pi}$ the positive and unequal parameters $\omega_1$ and $\omega_2$ are defined under the inequality, $|a_3| < |a_1^2 - a_2^2|$, to be

$$\omega_1 := \sqrt{\frac{a_1^2 + a_2^2 \pm \sqrt{(a_1^2 - a_2^2)^2 - a_3^2}}{2}},$$
$$\omega_2 := \sqrt{\frac{a_1^2 + a_2^2 \mp \sqrt{(a_1^2 - a_2^2)^2 - a_3^2}}{2}}.$$  \hspace{1cm} (27)

Note that $H_I$ and $H_{\Pi}$ have the same real fourth order equations of motion no matter how different they are in the forms and the symmetries.

Now we solve this ODE in the coordinate representation. The characteristic equation has four imaginary roots that constitute two pairs of complex conjugate numbers, i.e., $\pm i\omega_1$ and $\pm i\omega_2$, which implies that the particular solutions proportional to $\exp(\pm i\omega_1t)$ and $\exp(\pm i\omega_2t)$, respectively, describe two decoupled harmonic oscillators with frequencies $\omega_1$ and $\omega_2$. Correspondingly, their energy spectra are already known,

$$E_1 = \hbar \omega_1 \left(n_1 + \frac{1}{2}\right), \quad E_2 = \hbar \omega_2 \left(n_2 + \frac{1}{2}\right), \quad n_1, n_2 = 0, 1, 2, \ldots.$$  \hspace{1cm} (28)

As a result, the solution of the fourth order operator ODE is the linear combination of the particular solutions. Because $\pm H_0$, where $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$ is the Hamiltonian of a harmonic oscillator, lead to the same equation of motion and solution. On the contrary, the particular solution ($\exp(\pm i\omega t)$) consequently corresponds to two energy spectra with the

$^3$H_{I}$ possesses $\eta$-pseudo-Hermiticity, the $\eta$ operator is given in ref. [6].
same absolute value but opposite signs. Thus the total energy spectrum is undetermined yet, that is, there are three possible cases of combination, \( E_1 + E_2, E_1 - E_2, \) and \(-E_1 + E_2\).

Despite the elaborate approaches proposed in literature, for instance, in ref. [6], for getting rid of negative probability or ghost states associated with negative eigenvalues, we simply consider the exchange invariance of the fourth order ODE with respect to the frequencies \( \omega_1 \) and \( \omega_2 \).

In light of the property that the invariance of parameter exchange that exists in an operator maintains in the operator’s eigenvalues, the positive definite spectrum \( E_1 + E_2 \) is thus picked out straightforwardly because it is evident that \( E_1 - E_2 \) or \(-E_1 + E_2\) individually has no invariance under the exchange of \( \omega_1 \) and \( \omega_2 \).

We can make some comments similar to that given in the above subsection. In order to avoid repetition, we just mention that our analysis of real eigenvalues is based on the real closeness of the fourth order ODE. In addition, the action of the invariance of parameter exchange is to fix the positive definite spectrum among real spectra. We note that the invariance of parameter exchange is completely different from the \( PT \)-pseudo-Hermiticity, the former is of internal (parameter) transformation while the latter is of external (spacetime) transformation. Finally, the imaginary interaction in eq. (25) plays the same role as that in the Swanson model (eq. (14)), that is, its effect in the aspect of eigenvalues just shifts the angular frequencies of the free anisotropic two-dimensional oscillator from \(|a_1|\) and \(|a_2|\) to \( \omega_1 \) and \( \omega_2 \) (eq. (27)), respectively.

### 4 A class of general non-Hermitian non-\( PT \)-symmetric models

In the above section we use the real closure of equations of motion as a criterion to analyze two toy models, the Swanson model and the Pais-Uhlenbeck oscillator model, and obtain the same

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\( ^4 \) The negative definite case, \(-E_1 - E_2\), can be omitted directly due to the existence of the major positive definite parts in the eigenvalues of \( H_I \) and \( H_{II} \). In addition, the linear combination of particular solutions does not mean the linear combination of eigenfunctions, it only means that the Pais-Uhlenbeck oscillator model consists of two harmonic oscillators with angular frequencies \( \omega_1 \) and \( \omega_2 \).

\( ^5 \) Although it is not manifest in \( H_{II} \), the invariance presents whatever the upper sign \( (\omega_1 > \omega_2) \) or lower sign \( (\omega_1 < \omega_2) \) in eq. (27) is taken.

\( ^6 \) If an operator \( O(\alpha_1, \alpha_2) \), regardless of its Hermiticity or non-Hermiticity, is invariant under exchange of two parameters \( \alpha_1 \) and \( \alpha_2 \), a linear and invertible operator \( S \) that represents the action of exchange can be introduced and there exists the relation: \( SO(\alpha_1, \alpha_2)S^{-1} = O(\alpha_2, \alpha_1) = O(\alpha_1, \alpha_2) \), where \( S^2 = 1 \).

Suppose the eigenvalue problem: \( O(\alpha_1, \alpha_2)f = \lambda(\alpha_1, \alpha_2)f \), one can prove \( O(\alpha_1, \alpha_2)(Sf) = \lambda(\alpha_2, \alpha_1)(Sf) \).

Due to \([O(\alpha_1, \alpha_2), S] = 0\), the two operators have same eigenfunctions in general, the so-called \( S \) symmetry is unbroken, which leads to \( Sf = \pm f \). Therefore, one obtains \( \lambda(\alpha_1, \alpha_2) = \lambda(\alpha_2, \alpha_1) \), which implies that the invariance of parameter exchange also exists in the eigenvalues of the operator \( O(\alpha_1, \alpha_2) \).

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real eigenvalues as that in literature [4, 6, 7]. This is just a simple and initial application to the criterion because the Swanson model can be diagonalized to be a harmonic oscillator and the Pais-Uhlenbeck oscillator model to be two decoupled harmonic oscillators. Therefore, as the main and important context in the present paper, we have to extend our studies to general non-Hermitian models. To this end, two classes of general non-Hermitian Hamiltonians are proposed, one is non-Hermitian and non-$\text{PT}$-symmetric that is analyzed in this section, and the other is non-Hermitian but $\text{PT}$-symmetric that will be investigated in the section below.

At first, we point out that the real closeness of equations of motion is closely related with the Hermiticity of Hamiltonians. This property provides the basis of our criterion. For a general quantum dynamical system described by a Hermitian Hamiltonian, $H_{\text{Hermitian}} = \frac{\dot{x}^2}{2m} + V(x)$, where $V(x)$ is an arbitrary real potential that is usually required to be differentiable with respect to $x$, the Hamiltonian leads of course to a real second order equation of motion, $m\ddot{x} + V'(x) = 0$. On the contrary, one can deduce a Hermitian Hamiltonian from a real second order equation of motion and fix the Hamiltonian up to a real constant. Consequently, if a non-Hermitian Hamiltonian gives a real second order equation of motion, its Hermitian counterpart can be deduced from the real second order equation of motion. Because they have the same real second order equation of motion, the non-Hermitian Hamiltonian and its Hermitian counterpart are convinced to be isospectral up to a real constant due to an integration of coordinate or momentum. The isospectrum can alternatively be verified through a similarity transformation of Hamiltonians. In this way, one determines the existence of real eigenvalues for such a non-Hermitian Hamiltonian.

Let $\sum_{k=0}^{\infty} c_k x^{k+n}$ be a general series, where the index $n$ can take zero or any of positive integers, and $c_k$’s related to this index are real parameters. The radius of convergence is defined as $R \equiv \lim_{k \to \infty} \left| \frac{c_k}{c_{k+1}} \right|$, and the range of $x$’s average values is required to be less than $R$. Note that $R$ can take infinity, for example, when this series is the Bessel function of the $n$th-degree, $J_n(x)$, where $n = 0, 1, 2, \cdots$. This means that this series can cover special functions. By using the series, we construct the following non-Hermitian Hamiltonian,

$$H = \frac{\dot{p}^2}{2m} + V(x) + \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\}, \quad (29)$$

where $(x, p)$ is a pair of canonical coordinate and momentum that satisfies the Heisenberg commutation relations given by eq. (15). Note that this non-Hermitian Hamiltonian is not $\text{PT}$ symmetric in general under the conventional definitions of $P$ and $T$ given in eq. (17), which can be seen on the one hand because $V(x)$ is generally not an even function, i.e., $(PT)^{-1}V(x)(PT) = V(-x) \neq V(x)$, and on the other hand because the series usually contains powers of even numbers.

According to the Heisenberg picture established in section 2, we derive the Heisenberg equations of motion for the non-Hermitian and non-$\text{PT}$-symmetric Hamiltonian in light of
eqs. (9), (12), and (29),

\[
\dot{x} = \frac{p}{m} + i \sum_{k=0}^{\infty} c_k x^{k+n},
\]

\[
\dot{p} = -V'(x) - \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)' \right\},
\]

where the prime stands for the derivative with respect to \(x\). Although the Heisenberg equations of motion are highly non-trivially complex, surprisingly, we find that the quantum second order equation of motion for the coordinate \(x\) is real closed when eliminating the momentum \(p\),

\[
m\ddot{x} + V'(x) + \frac{m}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2 \right\}' = 0,
\]

Correspondingly, we can deduce a Hermitian Hamiltonian that gives the same real equation of motion,

\[
h = \frac{p^2}{2m} + V(x) + \frac{m}{2} \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2,
\]

which can be fixed up to a real constant that has been set be zero. Because the non-Hermitian non-\(PT\)-symmetric Hamiltonian (eq. (29)) and its Hermitian counterpart (eq. (33)) produce the same equation of motion, they have the same real eigenvalues (up to a constant).

Alternatively, we verify that the non-Hermitian and non-\(PT\)-symmetric Hamiltonian (eq. (29)) can be converted into its Hermitian (isospectral) counterpart (eq. (33)) by a non-unitary similarity transformation. Set

\[
H \Phi = E \Phi, \quad h \phi = E \phi,
\]

\[\text{i.e., } H \text{ and } h \text{ have the same eigenvalues } E, \text{ and } \Phi \text{ and } \phi \text{ are their respective eigenfunctions, by using the Baker-Campbell-Hausdorff formula we can find out such a non-unitary operator } \Omega,\]

\[
\Omega = \exp \left( -\frac{m}{\hbar} \sum_{k=0}^{\infty} c_k \frac{x^{k+n+1}}{k + n + 1} \right),
\]

which is a linear operator, that it connects the two Hamiltonians and their sets of eigenfunctions as follows:

\[
h = \Omega H \Omega^{-1},
\]

\[
\phi = \Omega \Phi.
\]

\[\text{See Appendix A for the detailed derivation.}\]
The detailed derivation is given in Appendix B.

Consequently, we determine the existence of real eigenvalues for the class of non-Hermitian and non-PT-symmetric Hamiltonians by means of the real closure of equations of motion. We emphasize that this criterion is novel and quite straightforward. As to the imaginary interacting potential in eq. (29), \( \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\} \), its effect, when we focus only on eigenvalues, is equivalent to the contribution of the real potential \( \frac{m}{2} \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2 \), see eq. (33).

Before ending this section, we emphasize that the non-Hermitian Hamiltonian eq. (29) inherently possesses the \( \eta \)-pseudo-Hermiticity, which is the prerequisite for us to adopt the Heisenberg picture (see section 2). Taking into account eq. (36) and the Hermiticity of \( h \), we can verify the \( \eta \)-pseudo-Hermiticity of \( H \),

\[
H^\dagger \equiv \eta^{-1}H^\dagger\eta = H,
\]

where \( \eta \) is expressed in terms of \( \Omega \) as follows:

\[
\eta = \Omega^\dagger\Omega. \tag{39}
\]

Note that the metric operator \( \eta \) is both Hermitian and \( \eta \)-pseudo-Hermitian self-adjoint, that is, it is obvious to see \( \eta^\dagger = \eta \), and \( \eta^\dagger \equiv \eta^{-1}\eta^\dagger\eta = \eta \).

5 A class of general non-Hermitian PT-symmetric models

In order to extend the discussion on the real closeness of equations of motion, we now construct a new class of general non-Hermitian but PT-symmetric Hamiltonians. The Hamiltonian is given as follows:

\[
\tilde{H} = \frac{A}{2} x^2 + V(p) + \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right) x + x \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right) \right\}, \tag{40}
\]

where \( A \) is a positive constant with the dimension of \([M][T]^{-2}\), \( V(p) \) is an arbitrary real potential of momentum that is usually required to be differentiable to \( p \), \( \sum_{k=0}^{\infty} a_k p^{k+n} \) is a general series of momentum \( p \), where \( a_k \)'s are real constants, the index \( n \) takes zero or any of positive integers, and \((x,p)\) is a pair of canonical coordinate and momentum that satisfies the Heisenberg commutation relations eq. (15). This Hamiltonian may be understood in the momentum representation of the Hamiltonian (eq. (29)) constructed in the above section, and hence its PT symmetry under the conventional transformations of \( P \) and \( T \) given in eq. (17) is thus restored due to the exchange between coordinate and momentum in eq. (29).
Following the same procedure as in the above section, i.e., using the Heisenberg picture, we at first derive the Heisenberg equations of motion for the coordinate and the momentum from the non-Hermitian PT-symmetric Hamiltonian eq. (40),

\[
\dot{x} = V'(p) + \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)' \right\}, \\
\dot{p} = -Ax - i \sum_{k=0}^{\infty} a_k p^{k+n},
\]

where the prime stands here for the derivative with respect to the momentum \( p \). Similar to eqs. (30) and (31), they are also highly non-trivially complex. However, by using the calculations given in Appendix A and by eliminating the coordinate, we then obtain the real second order equation of motion with respect to the momentum \( p \),

\[
\frac{1}{A} \ddot{p} + V'(p) + \frac{1}{2A} \left\{ \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)^2 \right\}' = 0.
\]

Using this real equation of motion we deduce the corresponding Hermitian Hamiltonian,

\[
\tilde{h} = \frac{A}{2} x^2 + V(p) + \frac{1}{2A} \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)^2.
\]

It is obvious that \( \tilde{h} \) has real eigenvalues that are also the eigenvalues of \( \tilde{H} \). The property of isospectrism can alternatively be verified, similar to Appendix B, from the similarity transformation between \( \tilde{H} \) and \( \tilde{h} \),

\[
\tilde{\Omega} \tilde{H} \tilde{\Omega}^{-1} = \tilde{h},
\]

where the operator \( \tilde{\Omega} \) is found in light of the Baker-Campbell-Hausdorff formula to be,

\[
\tilde{\Omega} = \exp \left( \frac{1}{Ah} \sum_{k=0}^{\infty} \frac{a_k}{k+n+1} p^{k+n+1} \right).
\]

As a result, we determine the existence of real eigenvalues for the class of general non-Hermitian PT-symmetric Hamiltonians (eq. (40)) in terms of the criterion based on the Heisenberg equations of motion. This outcome is also verified by the way based on the similarity transformation (eq. (45)). Similar to the class of models in the above section, the imaginary interacting potential in eq. (40), \( \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)' + x \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)' \right\} \), its effect equals the contribution of the real potential \( \frac{1}{2A} \left( \sum_{k=0}^{\infty} a_k p^{k+n} \right)^2 \) in the aspect of eigenvalues, see eq. (44).

Now it is necessary to emphasize that the non-Hermitian Hamiltonian eq. (40) inherently has the \( \tilde{\eta} \)-pseudo-Hermiticity, which is the prerequisite for us to adopt the Heisenberg picture.
(see section 2). Considering eq. (45) and the Hermiticity of \( \tilde{h} \), we can verify the \( \tilde{\eta} \)-pseudo-Hermiticity of \( \tilde{H} \),

\[
\tilde{H}^\dagger \equiv \tilde{\eta}^{-1} \tilde{H}^\dagger \tilde{\eta} = \tilde{H},
\]

where \( \tilde{\eta} \) is expressed in terms of \( \tilde{\Omega} \) as follows:

\[
\tilde{\eta} = \tilde{\Omega}^\dagger \tilde{\Omega}.
\]

This metric operator \( \tilde{\eta} \) is both Hermitian and \( \tilde{\eta} \)-pseudo-Hermitian self-adjoint, that is, it is obvious to verify \( \tilde{\eta}^\dagger = \tilde{\eta} \), and \( \tilde{\eta}^\dagger \equiv \tilde{\eta}^{-1} \tilde{\eta}^\dagger \tilde{\eta} = \tilde{\eta} \).

In addition, one can deduce the existence of real eigenvalues for \( \tilde{H} \) (eq. (40)) in accordance with the non-Hermitian \( PT \)-symmetric quantum theory \([3]\), which shows that our criterion — the real closeness of equations of motion — is compatible with the \( PT \) symmetry.

6 Summary

In this paper we propose a new criterion — the real closeness of equations of motion — to determine the existence of real eigenvalues for a non-Hermitian Hamiltonian system. The analysis can be fulfilled in the Heisenberg picture we establish for a non-Hermitian but \( \eta \)-pseudo-Hermitian Hamiltonian. We apply the criterion at first to two toy models, the Swanson model and the Pais-Uhlenbeck oscillator model, as an initial trial, and then as the significant context, to two classes of general non-Hermitian Hamiltonians, where one is non-Hermitian non-\( PT \)-symmetric, and the other is non-Hermitian \( PT \)-symmetric. The two classes of general models are shown to have real second order equations of motion, and their Hermitian (isospectral) Hamiltonians are deduced. In fact, our criterion connects the spectra to the Heisenberg equations of motion in non-Hermitian Hamiltonian systems, and such a connection can alternatively be verified by a non-unitary similarity transformation between a non-Hermitian Hamiltonian and a Hermitian one.

The discussion in section 4 shows that the new criterion is complementary to the \( PT \)-symmetric quantum mechanics. The complementarity between the real closeness of equations of motion and the \( PT \) symmetry of non-Hermitian Hamiltonians presents that one can determine the existence of real eigenvalues for a non-Hermitian Hamiltonian by using either the real closeness of equations of motion or the \( PT \) symmetry. In other words, to a model that has real equations of motion but no \( PT \) symmetry, such as eq. (29), one can analyze it as stated in the present paper; on the other hand, to a model that has \( PT \) symmetry but no real equations of motion, such as a class of models with the imaginary cubic potential \( ix^3 \), one can follow the non-Hermitian \( PT \)-symmetric quantum theory \([3]\). Moreover, our discussion in section 5 shows that our criterion is also compatible with the criterion of \( PT \) symmetry.
through investigating the class of non-Hermitian Hamiltonian systems that possesses both
the real equations of motion and the $PT$ symmetry.

In addition, here we simply explain how to determine the eigenfunctions of the two classes
of non-Hermitian Hamiltonians, eq. (29) and eq. (40), both of which lead to real second order
equations of motion, the former is on the coordinate $x$ while the latter on the momentum $p$. For example, to the non-Hermitian Hamiltonian eq. (29), using eq. (33) and eq. (34) we
calculate the set of eigenfunctions of the Hermitian Hamiltonian, $\{\phi\}$, in accordance with
the traditional quantum mechanics. Then, we get the eigenfunctions of the non-Hermitian
Hamiltonian from eq. (37), $\Phi = \Omega^{-1}\phi$, in terms of the non-unitary operator $\Omega$, see eq. (35).

Due to the fact that our discussions in sections 4 and 5 are involved in two representations
of a non-Hamiltonian Hamiltonian system, i.e., the Hermitian and the $\eta$-pseudo-Hermitian
representations, we also give a brief summary on the relationship between them in Appendix C.

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**Appendix A  Derivation of eq. (32)**

Making derivative to eq. (30) with respect to time, we have

$$\ddot{x} = \frac{\dot{p}}{m} + i \sum_{k=0}^{\infty} c_k \left( \dot{x} x^{k+n-1} + \dot{x} \dot{x} x^{k+n-2} + \cdots + \dot{x} x^{k+n-2} \dot{x} x + x^{k+n-1} \dot{x} \right). \quad (49)$$

Substituting eq. (31) into eq. (49), we obtain

$$\ddot{x} = -\frac{1}{m} V'(x) - \frac{i}{2m} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)' p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)' \right\}$$

$$+ i \sum_{k=0}^{\infty} c_k \left( \dot{x} x^{k+n-1} + \dot{x} \dot{x} x^{k+n-2} + \cdots + \dot{x} x^{k+n-2} \dot{x} x + x^{k+n-1} \dot{x} \right). \quad (50)$$
Next, substituting eq. (30) into the third term of eq. (50) and considering the Heisenberg commutation relations (eq. (12)), we reduce this term to be

\[
\frac{i}{2m} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\} - \frac{1}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2 \right\}.
\]

(51)

Combining eq. (50) with eq. (51), we see the cancellation of the imaginary terms and thus derive the real second order equation of motion — eq. (32).

Appendix B Verification of eq. (36)

Using eq. (35) and eq. (12), we make the similarity transformation to the three terms in eq. (29),

\[
\Omega \left\{ \frac{p^2}{2m} \right\} \Omega^{-1} = \frac{p^2}{2m} - \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\} - \frac{m}{2} \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2,
\]

\[
\Omega \left\{ V(x) \right\} \Omega^{-1} = V(x),
\]

\[
\Omega \left\{ \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\} \right\} \Omega^{-1}
\]

\[
= \frac{i}{2} \left\{ \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) p + p \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right) \right\} + m \left( \sum_{k=0}^{\infty} c_k x^{k+n} \right)^2.
\]

By adding the terms in the above three equations, left to left and right to right, we obtain

\[
\Omega H \Omega^{-1} = h,
\]

(52)

where \( h \) is the Hermitian Hamiltonian given in eq. (33).

Appendix C Hermitian and \( \eta \)-pseudo-Hermitian representations

In sections 4 and 5, two classes of general non-Hermitian Hamiltonians are proposed, and their Hermitian (isospectral) counterparts are obtained by two different ways, one is based on the real closeness of equations of motion in the Heisenberg picture, and the other is based on a type of non-unitary similarity transformations. As an example, we summarize below the relationship between the Hermitian and the \( \eta \)-pseudo-Hermitian representations for the non-Hamiltonian non-\( PT \)-symmetric Hamiltonian depicted by eq. (29). As to the
non-Hamiltonian $PT$-symmetric Hamiltonian described by eq. (40), the conclusion is exactly the same.

The Hermiticity or $\eta$-pseudo Hermiticity of a physical variable depends in general on definitions of inner products because the average value of a physical variable is inner-product-dependent and is required to be real. For example, a physical observable $o$, when it presents in the Hermitian representation, is a Hermitian operator under the usual definition of inner products, $\langle \phi_1 | \phi_2 \rangle$, where the set $\{ \phi \}$ is the eigenfunction of the Hermitian Hamiltonian $h$ (eq. (33)), $h\phi = E\phi$, which ensures that its average value is real,

$$\langle o \rangle_\text{Av}^\phi \equiv \langle \phi | o | \phi \rangle = \langle \phi | o^\dagger | \phi \rangle. \quad (53)$$

On the other hand, when this observable, now denoted by $O$, presents in the $\eta$-pseudo Hermitian representation, it should have $\eta$-pseudo Hermiticity,

$$O^\dagger \equiv \eta^{-1}O^\dagger \eta = O, \quad (54)$$

and thus the corresponding average value under the modified inner product \[2\], $\langle \Phi_1 | \Phi_2 \rangle_\eta \equiv \langle \Phi_1 | \eta \Phi_2 \rangle$,

$$\langle O \rangle_\text{Av}^\Phi \equiv \langle \Phi | O | \Phi \rangle_\eta = \langle \Phi | \eta O | \Phi \rangle, \quad (55)$$

is definitely real, where the set $\{ \Phi \}$ is the eigenfunction of the $\eta$-pseudo-Hermitian Hamiltonian $H$ (eq. (29)), $H\Phi = E\Phi$. For an arbitrary physical observable, whatever it is described in the Hermitian representation or in the $\eta$-pseudo-Hermitian representation, its average value must be same,

$$\langle o \rangle_\text{Av}^\phi = \langle O \rangle_\text{Av}^\Phi. \quad (56)$$

In light of eqs. (37), (39), (53), and (55), eq. (56) gives rise to the relation between the two representations for an arbitrary physical observable (including Hamiltonian),

$$O = \Omega^{-1} o \Omega, \quad (57)$$

which is consistent with eq. (36) when $o$ is specified to be $h$ and $O$ to be $H$. We can verify from eq. (57) that $O$ indeed possesses the $\eta$-pseudo-Hermiticity eq. (54). Therefore, we give the consistency of the two descriptions of physical observables in the $\eta$-pseudo-Hermitian representation (cf. eq. (29)) and in the Hermitian representation (cf. eq. (33)). In addition, the commutation relations in the two representations are connected by the same similarity transformation as that satisfied by physical observables:

$$[O_1, O_2] = \Omega^{-1} [o_1, o_2] \Omega. \quad (58)$$

If $[o_1, o_2] = i\hbar$, then $[O_1, O_2] = i\hbar$. That is, the (canonical) Heisenberg commutation relations maintain unchanged under the transformation from the Hermitian representation to the $\eta$-pseudo-Hermitian representation, and vice versa.
At last, it is clearer to see two concrete physical observables we are quite familiar with, the coordinate and the momentum. In the \( h \) (Hermitian) representation, \( x \) and \( p \) are Hermitian, \( i.e., x^\dagger = x \), and \( p^\dagger = p \); on the other hand, if coordinate \( X \) and momentum \( P \) act as physical observables in the \( H \) (\( \eta \)-pseudo-Hermitian) representation, they are related to \( x \) and \( p \), respectively, with the relations: \( X = \Omega^{-1} x \Omega \), and \( P = \Omega^{-1} p \Omega \), and such \( X \) and \( P \) certainly have \( \eta \)-pseudo Hermiticity, \( X^\dagger \equiv \eta^{-1} X^\dagger \eta = X \), and \( P^\dagger \equiv \eta^{-1} P^\dagger \eta = P \), where \( \eta = \Omega^\dagger \Omega \). Because coordinates and momenta are physical observables, their average values are independent of the choice of representations, that is, \( \langle x \rangle^\phi_{Av} = \langle X \rangle^\Phi_{Av} \), and \( \langle p \rangle^\phi_{Av} = \langle P \rangle^\Phi_{Av} \). Moreover, if \( x \) and \( p \) are a pair of canonical variables, \( i.e., [x,p] = i\hbar \), \( [x,x] = 0 = [p,p] \), so are \( X \) and \( P \), \( [X,P] = i\hbar \), \( [X,X] = 0 = [P,P] \).

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