Consistent truncations around half-maximal AdS$_5$ vacua of 11-dimensional supergravity

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Abstract
We use exceptional field theory to systematically study all possible consistent truncations around any $M$-theory half-maximal vacua of the warped product form AdS$_5 \times M_3 \times S^2 \times S^1$, with $M_3$ a three-dimensional manifold. We show that, for any of these vacua, only truncations with at most three vector multiplets are consistent. Furthermore, the possible gaugings are restricted to be either $SU(2) \times U(1)$ or $ISO(3) \times U(1)$ where, in the first case, the $U(1)$ factor can have different embeddings inside the global symmetry group $SO(5, n)$, where $n \leq 3$ equals the number of vector multiplets. This rules out the possibility of any other gauging arising as a consistent truncation around the aforementioned $M$-theory vacua. Our analysis shows that of the many flows from half-maximal to quarter-maximal AdS$_5$ vacua constructed in five-dimensional supergravity in Bobev (2018 J. High Energy Phys. 6 86), only those corresponding to an adjoint mass deformation in the dual SCFT can be uplifted to 11-dimensional supergravity. The other flows are five-dimensional artefacts without a higher-dimensional origin. Furthermore, consistent truncations with vector multiplets exist only if the vacuum satisfies certain conditions, which we derive.

Keywords: AdS-CFT correspondence, flux compactifications, supergravity models, superstring vacua

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1. Introduction

A common problem in the study of string and supergravity compactifications is how to construct a lower-dimensional theory which captures relevant aspects of their dynamics. If there is a separation of scales, we can integrate out all massive modes above the compactification scale to obtain a lower-dimensional effective theory. However, without such a separation of scales, as for example in all well-understood AdS vacua\(^3\) of string theory and supergravity, choosing a basis of modes to keep in the truncation is a difficult affair. As a minimum, we should require that the lower-dimensional theory obtained after truncation reproduces solutions to the original higher-dimensional supergravity. Such consistent truncations used to be rare and difficult to construct \([3]\), with the only systematic construction arising for group manifolds, and isolated examples existing such as the \(S^7\) \([4]\) and \(S^4\) \([5, 6]\) truncations of 11-dimensional supergravity. Despite these difficulties, consistent truncations have long played an important role in the AdS/CFT correspondence \([7]\) and as a solution-generating tool in supergravity.

Recently, the number of examples of consistent truncations and their systematic understanding has grown rapidly \([8–32]\). A key tool has been the development of exceptional field theory (ExFT) \([33–36]\) and exceptional generalised geometry (EGG) \([37–39]\). ExFT and EGG are reformulations of supergravity theories where metric and flux degrees of freedom are treated on the same footing in a way that makes an \(E_{d(d)}\) symmetry manifest. One of the benefits of this reformulation is that consistent truncations are now captured naturally and systematically. For example, consistent truncations preserving all supersymmetries are described by generalised Scherk–Schwarz truncations \([8, 9, 40–45]\) on ‘generalised Leibniz parallelisable’ manifolds \([9]\). This has led to construction of the consistent truncation of IIB supergravity on \(S^5\) \([46]\), as well as a number of new consistent truncations on products of spheres and hyperboloids \([8, 9, 14–18, 21, 31]\). On the other hand, consistent truncations preserving half the supersymmetries \([19, 22]\) in \(11 - d\) dimensions are described by a generalised \(\text{Spin}(d - 1 - n)\) structure, where \(n\) labels the number of vector multiplets, and consistent truncations preserving a general number of supersymmetries were most recently described in \([29]\). Moreover, these tools led to a proof—\([22]\) for the half-maximal and subsequently \([29]\) for the general supersymmetric case—of the conjecture that every supersymmetric AdS vacuum of 10-/11-dimensional supergravity admits a consistent truncation keeping only the gravitational supermultiplets \([47]\).\(^4\)

The difference between ExFT and EGG is that the first uses a formally enlarged set of coordinates, forming a representation of \(E_{d(d)}\). However, all fields are constrained by the ‘section condition’ to depend only on a subset of physical coordinates. Nonetheless, choosing appropriate embeddings of the physical coordinates within the full enlarged set can be a powerful tool in ExFT, for example for constructing consistent truncations to dyonically gauged supergravities \([18]\) or in establishing relations between IIA and IIB truncations \([14, 21]\). For the purposes of this paper, we will always solve the section condition of ExFT in a canonical way, so that there is no difference between ExFT and EGG. Thus, we will simply be referring to ExFT from now onwards but the analysis could be performed identically in EGG. While our analysis could have been performed in EGG, this has not been done before. In particular, our paper differs substantially from the analysis, performed in EGG, of consistent truncation of

\(^3\)It is conjectured that no AdS vacua of string theory admit a separation of scales \([2]\).

\(^4\)This conjecture was first motivated by explicitly constructing the consistent truncations around the most general minimally supersymmetric AdS\(_5\) backgrounds of IIB and 11-dimensional supergravity in \([47, 48]\).
AdS$_5$ vacua in [29], which constructs one new truncation but does not analyse any other possible truncations, other than the very general considerations already reported in [22], which show that any consistent truncation can have at most five vector multiplets, and in the case of AdS$_5$ the gauging must include SU(2) x U(1)$_R$.

In the context of holography, it is especially useful to obtain consistent truncations around AdS vacua. These can then be used to study deformations of the AdS vacuum, such as those holographically modelling RG flows or constructing asymptotically AdS black holes. Therefore, it would be particularly useful to have a classification of possible consistent truncations around known AdS vacua. In [25, 26], ExFT was used to systematically classify all possible consistent truncations around supersymmetric AdS$_6$ and AdS$_7$ vacua of 10-dimensional type IIB and IIA supergravity, respectively. In the present paper, we will extend this analysis to the case of AdS$_5$ vacua preserving 16 supercharges by classifying all their possible consistent truncations to half-maximal five-dimensional gauged supergravity with arbitrary number of vector multiplets. Therefore, our work can be seen as a higher-dimensional completion of [49].

Recently, a consistent truncation with three vector multiplets was constructed in [28] around the Maldacena–Nuñez vacua obtained by wrapping M5 branes around a Riemann surface [50]. This was later also recovered using generalised geometry in [29]. Here we put these results into a systematic context, by showing that this truncation with three vector multiplets is the largest possible truncation around general LLM configurations [51], and only possible for the Maldacena–Nuñez solution [50]. However, we also show that other consistent truncations with up to three vector multiplets may be possible and take first steps in constructing them. In particular, we derive precise conditions that need to be satisfied for these truncations. We once again emphasise that the novelty in our analysis is not the use of ExFT vs EGG, which for the purposes of the analysis presented here is just a matter of nomenclature. Rather, our analysis goes beyond the EGG result of [29] by studying general consistent truncations with vector multiplets around half-maximal AdS$_5$ vacua of M-theory. By contrast [29] investigates but one possible consistent truncation.

Our analysis already leads to some interesting results. For example, we are able to show that of the flows from half-maximal to quarter-maximal AdS$_5$ vacua studied in [1], only those corresponding to a mass deformation of an adjoint scalar in the dual SCFT can be uplifted to 11-dimensional supergravity. The other flows are five-dimensional artefacts without a higher-dimensional completion.

This paper is organised as follows. In section 2, we review $E_{6(6)}$ ExFT and how half-maximal AdS$_5$ vacua and consistent truncations around them are described within it. In section 3, we use this method to construct the generalised $G$-structures for all half-maximal AdS$_5$ vacua of 11-dimensional supergravity. This formulation is used in section 3.2 to construct consistent truncations keeping only the gravitational supermultiplet and in section 4 to systematically study all possible consistent truncations with vector multiplets around them. Finally, in section 5, we analyse the particular case where the internal space is locally a $S^4$ fibration. In the remaining part of this section we give a summary of our results.

### 1.1. Summary of results and comparison with earlier work in the literature

The main goal of this paper is to systematically study and classify all possible half-maximal consistent truncations around vacua of the form

$$\text{AdS}_5 \times M_3 \times S^2 \times S^1, \quad (1.1)$$
which are classified in terms of a function $D(x_1, x_2, y)$ on the three-dimensional manifold $M_3$ satisfying the TodA equation \[51\]
\[
(\partial_{x_1}^2 + \partial_{x_2}^2) D + \partial_y^2 e^D = 0. 
\]

Our method is based on describing all these geometries in terms of generalised $G$-structures, as we do in section 3, which gives us a natural framework for our classification. In this description it is convenient to define the following function and forms

\[
p = 2 \times 5^{1/6} \sqrt{2} R^2 y, \quad \nu_1 = 2 \times 5^{1/6} \sqrt{2} R^2 \, dx_1, \quad \nu_2 = 2 \times 5^{1/6} \sqrt{2} R^2 \, dx_2,
\]
\[
d\chi = -5^{1/6} \sqrt{2} R^2 \left( \epsilon^{ij} \partial_i D \, dx_j \wedge dy + \partial_y e^D \, dx_1 \wedge dx_2 \right),
\]

where $R$ is the AdS$_5$ radius and the exactness of $d\chi$ requires (1.2).

In half-maximal supergravity two kinds of consistent truncations can be considered: a truncation keeping only the gravity multiplet, which we call minimal, and truncations keeping also extra massless vector multiplets. The first type exists for any solution of the TodA equation (1.2), as follows from the general conjecture of [47], and they were explicitly constructed in [52]. In section 3.2 we show how these results can be re-derived in our language with a simple ansatz at the level of the generalised $G$-structures.

The case of consistent truncations keeping extra vector multiplets deserves more attention. To the best of our knowledge, the only known example in the literature is a consistent truncation with three vector multiplets around the Maldacena–Núñez vacua, resulting in a five-dimensional gauged supergravity with gauge group ISO(3) × U(1). This was only constructed recently, first in [28] and then recovered using EGG in [29].

Unlike this previous work in the literature, the purpose of the current paper is to obtain a full classification of all possible consistent truncations around any half-maximal AdS$_5$ vacua. Therefore we do not construct consistent truncations in a case-by-case approach but instead analyse the conditions for existence of such truncations around any vacua described by a solution of (1.2), with any number of vector multiplets and any gauge group. As a result of our analysis in section 4, we prove that there exist no consistent truncations with more than three vector multiplets and we rule out any possible resulting gauge group except for the aforementioned ISO(3) × U(1) as well as SU(2) × U(1), where the U(1) factor of the later can have different embeddings into the global symmetry group SO(5, $n$), with $n$ the number of vector multiplets. Furthermore, for those cases that are not ruled out by our analysis, the consistent truncation only exists if the function $D$ satisfy some extra conditions on top of (1.2), which we derive. Our results are expressed in terms of pairs of forms ($\tilde{\nu}$, $\Phi$) defined by

\[
\tilde{\nu} = \nu_1 \nu_2 + \nu_2 \nu_3 + \nu_3 \, dp,
\]
\[
\Phi = \frac{1}{2 \times 5^{1/6} \sqrt{2} R^2} \left[ \left( \nu_1 \partial_y D - e^{-\frac{1}{2} D} \nu_3 \partial_1 D \right) \, dp \wedge \nu_2 
\right.
\]
\[
\left. + \left( -\nu_2 \partial_y D + e^{-\frac{1}{2} D} \nu_3 \partial_2 D \right) \, dp \wedge \nu_1 \right] + \nu_2 \, d\nu_1 - \nu_1 \, d\nu_2 + \nu_3 \, d\chi,
\]

where the different objects involved are defined in (1.3) and $\nu_1$, $\nu_2$ and $\nu_3$ are three functions of $M_3$ satisfying

\[ v_1^2 + v_2^2 + v_3^2 = 1. \]
Table 1. Summary of the possible consistent truncations with \(n\) vector multiplets around half-maximal AdS\(_5\) vacua and their gaugings. The prime on U(1) denotes a non-standard embedding of the U(1) \(\subset\) SO(5, \(n\)), whereas U(1) indicates a linear combination of U(1) \(\subset\) SO(5) and U(1) \(\subset\) SO(\(n\)) which can be read off from (2.9). The \(n = 1\) case is characterised by the forms \(\bar{\nu}\) and \(\Phi\) given in (1.4) satisfying (1.5). The \(n = 2\) case transforming in the \(1 \oplus q \oplus 1 - q\) representation is characterised by two pairs of forms, \((\bar{\nu}_1, \Phi_1)\) and \((\bar{\nu}_2, \Phi_2)\), both of the form (1.4) and satisfying individually (1.5) together with (1.6). Consistent truncations with \(n > 3\) do not exist.

| \(n\) | \((\text{SU}(2) \times \text{U}(1))_R\) rep | Conditions to be satisfied | Gauging |
|---|---|---|---|
| 1 | \(I_0\) | \(d\bar{\nu} = d\Phi = 0\) | \(\text{SU}(2) \times \text{U}(1)\) |
| 2 | \(I_0 \oplus I_q\) | Two different \(I_0\)'s satisfying (1.6) | \(\text{SU}(2) \times \text{U}(1)'\) |
| 3 | \(I_0 \oplus I_0 \oplus I_0\) | Three different \(I_0\)'s satisfying (1.6) pairwise | \(\text{SU}(2) \times \text{U}(1)'\) |
| 3 | \(I_0 \oplus I_q \oplus I_{-q}\) | \(I_0\) and \(I_q \oplus I_{-q}\) satisfying (1.6) pairwise | \(\text{SU}(2) \times \text{U}(1)'\) |
| 3 | \(3_0\) | \(S^4\) fibred over Riemann surface | ISO(3) \(\times\) U(1) |

For truncations with more than one vector multiplet, multiple pairs \((\bar{\nu}_1, \Phi_1), (\bar{\nu}_2, \Phi_2), \ldots\) (characterised by triplets of functions \((v_1, v_2, v_3), (w_1, w_2, w_3), \ldots\) respectively) are involved. In this case, apart from satisfying (1.5) independently, they also need to satisfy

\[
v_1 w_1 + v_2 w_2 + v_3 w_3 = 0,
\]

pairwise. The possible truncations with vector multiplets, as well as the extra conditions for their existence and the resulting gaugings are summarised in table 1, where the mentioned ISO(3) \(\times\) U(1) truncation is recovered as one instance in our classification. For any background satisfying the conditions in 1, our setup immediately gives a consistent truncation ansatz.

## 2. Half-supersymmetric AdS\(_5\) vacua and consistent truncations from \(E_{6(6)}\) ExFT

In this section we review how consistent truncations can be systematically studied in ExFT using the language of generalised \(G\)-structures, which will be used in the upcoming sections. For details about this formalism we refer to [22, 19, 20]. We summarise the main aspects of \(E_{6(6)}\) ExFT and our notation in appendix A.

## 2.1. Half-supersymmetric AdS\(_5\) vacua in \(E_{6(6)}\) ExFT

Supersymmetric vacua are naturally described in ExFT in terms of generalised \(G\)-structures. In order for \(M_5 \times M_\text{int}\) to be a half-supersymmetric vacuum, the internal space \(M_\text{int}\) has to admit 16 no-where vanishing real Killing spinor fields. As shown in [19, 20, 22], this is equivalent to an algebraic and differential condition on \(M_\text{int}\).

The algebraic condition is that \(M_\text{int}\) must admit a generalised USp(4) structure, defined by six nowhere vanishing generalised vectors \(\{J_u, \bar{K}\}\), with \(u = 1, \ldots, 5\), transforming in the
27 representation of $E_{6(6)}$ and satisfying the algebraic constraints

\[ J_u \wedge J_v = \frac{1}{5} \delta_{uv} J_w \wedge J^w, \]
\[ K \wedge \hat{K} = 0, \]
\[ \hat{K} \wedge J_u \wedge J^w > 0, \]  
(2.1)

where $u$ indices are raised/lowered by $\delta_{uv}$ and the wedge products between generalised objects are defined in (A.4) and (A.5). The maximal commutant of $\text{USp}(4) \subset E_{6(6)}$ is the $R$-symmetry group $\text{USp}(4)_R$, that rotates the structures $J_u$ among themselves and leaves $\hat{K}$ invariant. For the upcoming discussion, it is useful to define the following fields

\[ K = \frac{1}{5} J_u \wedge J^u, \quad \kappa^3 = K \wedge \hat{K}, \]  
(2.2)

where $K$ is an object in the $27$ representation of $E_{6(6)}$ and $\kappa$ a density of weight $1/3$. Note that the generalised vector fields $J_u$ and $\hat{K}$ in (2.1) have weight $1/3$ and thus $K$ has weight $2/3$.

In order to describe an AdS vacuum, the structure defined by (2.1) needs to be weakly integrable [22], which is equivalent to demanding that the internal geometry satisfies the corresponding BPS conditions. The presence of a non-zero cosmological constant breaks the $R$-symmetry group $\text{USp}(4)_R$ into $(\text{SU}(2) \times \text{U}(1))_R$, which is the $R$-symmetry group preserving the AdS$_5$ vacuum, or equivalently the $R$-symmetry group of the holographically dual superconformal field theory. By splitting the index $u$ into $(A, i)$ with $A = 1, 2, 3$ being the SU(2) index and $i = 1, 2$ the U(1) one, the weak integrability condition implies the following differential constraints

\[ \mathcal{L}_{J_A} J_B = -\frac{5^{1/6}}{\sqrt{2}} \epsilon_{ABC} J^C, \]
\[ \mathcal{L}_{\hat{K}} J_i = -\frac{5^{1/6}}{\sqrt{2}} \epsilon_{ij} J^j, \]
\[ \mathcal{L}_{\hat{K}} J_A = \mathcal{L}_{J_A} J_i = \mathcal{L}_{J_J} J_A = \mathcal{L}_{J_J} J_j = \mathcal{L}_{J_K} \hat{K} = 0, \]  
(2.3)

where we have fixed conventions such that $R$ is the radius of the AdS$_5$ vacua. The differential operator $\mathcal{L}$ is the $E_{6(6)}$ generalised Lie derivative defined in (A.2).

Finally, let us stress that the generalised USp(4) structure (2.1) encodes all the information about the background. In fact, the generalised USp(4) structure also defines a generalised metric $\mathcal{M}$ [29], given explicitly in (A.6). From this object, the supergravity background fields can be read off using the dictionaries described in appendix A.2.

2.2. Consistent truncations around half-maximal AdS$_5$ vacua

As shown in [22], a consistent truncation of 11-dimensional/10-dimensional maximal supergravity to a half-maximally supersymmetric supergravity in five dimensions is described in a similar way to the AdS$_5$ vacua reviewed in section 2.1. Here we focus on consistent truncations around half-maximal AdS$_5$ vacua, whose construction we review in the following.

2.2.1. Minimal consistent truncation. Given any half-maximal AdS$_5$ vacuum, we can immediately construct a minimal consistent truncation to half-maximal gauged supergravity with
SU(2) × U(1) gauging [53] and containing only the gravitational supermultiplet using the following truncation ansatz

\[ J^u_M(x, Y) = X(x) J^u_M(Y), \]
\[ \hat{K}^M(x, Y) = X^{-2}(x) \hat{K}^M(Y), \] (2.4)

for the scalar sector and

\[ A^\mu_M(x, Y) = A^\mu_u(x) J^u_M(Y) + A^\mu_0(x) \hat{K}^M(Y), \]
\[ B_{\mu\nu}M(x, Y) = -B_{\mu\nu}^u(x) \hat{J}^u_{M}(Y) - B_{\mu\nu}^0(x) K^M(Y), \]
\[ g_{\mu\nu}(x, Y) = \bar{g}_{\mu\nu}(x) \kappa^{5/3}(Y), \] (2.5)

for the other fields to the \( E_6(6) \) ExFT. Here, \( Y^M \) denote the internal coordinates on \( M_{\text{int}} \) and \( x^\mu \) the external coordinates on \( M_5 \), \( X \) is the scalar field, \( (A^\mu_u, A^\mu_0) \) are the vector fields, \( (B_{\mu\nu}^u, B_{\mu\nu}^0) \) the tensor fields and \( \bar{g}_{\mu\nu} \) the metric of the five-dimensional gravitational supermultiplet. The ExFT generalised metric is then computed using \( J^u \) and \( \hat{K} \) instead of \( J^u \) and \( \hat{K} \) in (A.6).

2.2.2. Consistent truncation with vector multiplets. Moreover, for special half-maximal AdS vacua, we can construct a consistent truncation with vector multiplets. This arises when the AdS vacuum admits a generalised \( \text{Spin}(5-n) \subset \text{USp}(4) \) structure, where \( n \) labels the number of vector multiplets obtained in the five-dimensional gauged supergravity, and we define \( \text{Spin}(1) = \text{Spin}(0) = \mathbb{Z}_2 \). A generalised \( \text{Spin}(5-n) \) structure is equivalent to \( M_{\text{int}} \) admitting a further \( n \) generalised vector fields, \( \bar{J}^u, \bar{J}^\mu \) satisfying the algebraic conditions

\[ \bar{J}^u \wedge \bar{J}^\nu = -\delta^{u\nu} K, \]
\[ \bar{J}^\mu \wedge \bar{J}^\nu = 0. \] (2.6)

Note that this immediately implies that we can only construct truncations with at most \( n \leq 5 \) vector multiplets.

Moreover, for the truncation around the AdS vacuum to be consistent, the following differential conditions must be satisfied

\[ \mathcal{L}_{\bar{J}^u} \bar{J}^\nu = -f_{\mu\nu}^u \bar{J}^\mu, \]
\[ \mathcal{L}_{\bar{J}^\mu} \bar{J}^\nu = -f_{\mu\nu}^u \bar{J}^\mu - f_{\rho\nu}^u \bar{J}^\rho, \]
\[ \mathcal{L}_{\bar{J}^u} \hat{K} = 0, \]
\[ \mathcal{L}_{\hat{K}} \bar{J}^u = -\xi^\nu_{\mu} \bar{J}^\nu, \] (2.7)

where \( \xi^\nu_{\mu}, f_{\mu\nu}^u, f_{\rho\nu}^u \) and \( f_{\mu\nu}^u \) are constants. In particular, the first and last equations of (2.7) imply that the extra generalised vector fields \( \bar{J}^u \) corresponding to the vector multiplets must form a representation of the \((\text{SU}(2) \times \text{U}(1))_R\) symmetry group of the AdS vacuum. This results in a gauged supergravity with embedding tensor [54] whose components can be read off as explained in [22]. Thus, we find
are defined by the vacuum (2.3) and the remaining non-vanishing components are defined by (2.7). The embedding tensor defines the gauging of the supergravity via the gauge-covariant derivative

\[ D_\mu = \nabla_\mu - A_\mu^a f_{ab} t_{bc} - A_\mu^0 \xi_{bc} t_{bc}, \]  

(2.9)

where \( a, b = 1, \ldots, 5 + n \) are raised/lowered by the SO(5, n) metric

\[ \eta_{ab} = \begin{pmatrix} \delta_{uv} & 0 \\ 0 & -\delta_{u\bar{v}} \end{pmatrix}, \]  

(2.10)

and \( t_{ab} = t_{\bar{a}b} \) are the SO(5, n) generators.

For the consistent truncation ansatz, it is useful to define the \( 5 + n \) generalised vector fields

\[ J^M_a = (J^M_0, J_a^M), \]  

(2.11)

as well as

\[ \bar{J}_a = J_a \wedge \bar{K}. \]  

(2.12)

Now the consistent truncation ansatz is given by

\[ \bar{J}_a^M(x,Y) = X(x) b_a^\mu(x) J_\mu^M(Y), \]

\[ \bar{K}^M(x,Y) = X^{-1}(x) \bar{K}^M(Y), \]

\[ A_\mu^M(x,Y) = A_\mu^\gamma(x) J_\gamma^M(Y) + A_\mu^0(x) \bar{K}^M(Y), \]

\[ B_{\mu \nu}^M(x,Y) = -B_{\mu \nu}^\mu(x) J_{\mu, \nu}^M(Y) - B_{\mu \nu}^0(x) \bar{K}_{\mu, \nu}(Y), \]

\[ g_{\mu \nu}(x,Y) = \delta_{\mu \nu}(x) \kappa^{2/3}(Y), \]  

(2.13)

where the \( x \)-dependent fields are the fields of the five-dimensional gauged supergravity. As explained in [22], the \( b_a^\mu \) are constrained by

\[ b_a^\mu b^\nu b^\lambda \eta_{ab} = \delta_{\mu \nu}, \]  

(2.14)

and are identified by SO(5) rotations on the \( u, v \) indices. Thus, the \( b_a^\mu \in \text{SO}(5, n)/\text{SO}(5) \times \text{SO}(n) \), together with \( X \in \mathbb{R}^+ \) parameterise the scalar coset space of the five-dimensional half-maximal gauged supergravity.

3. Half maximal AdS5 vacua from 11d supergravity

The general half-maximal AdS5 vacua of \( M \)-theory take the local form \( \text{AdS}_5 \times S^2 \times S^1 \times M_3 \), where \( M_3 \) is a three-dimensional compact manifold with boundaries. In order to study their consistent truncations, we must first describe these backgrounds in terms of generalised USp(4)
structures. Using the decomposition of ExFT fields into 11-dimensional supergravity discussed in appendix A.2, the most general USp(4) structures on $S^3 \times S^1 \times M_3$, defined by (2.1), satisfying the differential conditions (2.3) are

\[
J_A = \frac{5^{1/6} \sqrt{2}}{R} v_A + \left( p dY_A \wedge \text{vol}_{S^3} + \chi \wedge dY_A + Y_A dp \wedge \text{vol}_{S^3} - Y_A d\chi \right)
\]
\[
+ \frac{R}{51/6 \sqrt{2}} \left( -Y_A (p d\chi + \chi \wedge dp) \wedge \text{vol}_{S^2} \wedge \text{vol}_{S^3} + Y_A \chi \wedge d\chi \wedge \text{vol}_{S^2} - dp \wedge d\chi \wedge \theta_A \wedge \text{vol}_{S^1} \right),
\]
\[
J_i = d(w_i \nu_i - \epsilon w_i \nu_2)
\]
\[
+ \frac{R}{51/6 \sqrt{2}} d\left( \epsilon w_i (p dv_1 - \chi \wedge \nu_2) \wedge \text{vol}_{S^2} + w_i (p dv_2 + \chi \wedge \nu_1) \wedge \text{vol}_{S^2} \wedge \text{vol}_{S^3} \right),
\]
\[
\hat{K} = \frac{5^{1/6} \sqrt{2}}{R} v_{S^3},
\]
where the forms $\text{vol}_{S^3}$, $\text{vol}_{S^2}$ and $\text{vol}_{S^1}$ are the volume forms on $M_3$, $S^3$ and $S^1$ and $Y_A$, $w_i$ are the embedding coordinates of $S^2 \hookrightarrow \mathbb{R}^3$ and $S^1 \hookrightarrow \mathbb{R}^2$, respectively, satisfying

\[
Y_A Y_B \delta^{AB} = w_i w_j \delta^{ij} = 1,
\]
and we use $\delta^{AB}$ and $\delta_{ij}$ to raise/lower the $A, B = 1, \ldots, 3$ and $i, j = 1, 2$ indices respectively. We also defined the Killing vectors on the round $S^2$ and $S^1$ as $Y_A$ and $w_i$, respectively, as well as $\theta_A \equiv *dY_A$, the Hodge dual with respect to the round $S^2$, and $\epsilon w_i \equiv \epsilon_{ij} w^j$. The details of all these objects can be found in appendix B. Finally the objects $\nu_1, \nu_2$ are one-forms on $M_3$ and $p$ a function on $M_3$ satisfying the constraints

\[
dv_2 \wedge dp = -\nu_1 \wedge d\chi, \quad dv_1 \wedge dp = \nu_2 \wedge d\chi, \quad \nu_1 \wedge dv_2 = dp \wedge d\chi \geq 0.
\]
\[
\nu_2 \wedge dv_2 = \nu_1 \wedge dv_1, \quad \nu_1 \wedge dv_2 = -\nu_2 \wedge dv_1,
\]

(3.3)

The field $K = \frac{1}{2} J_a \wedge J^a$ is

\[
K = \sqrt{\frac{7}{5}} \left( -dp \wedge d\chi \wedge \text{vol}_{S^3} + \frac{R}{51/6 \sqrt{2}} (p \text{vol}_{S^3} - \chi) \otimes dp \wedge d\chi \wedge \text{vol}_{S^2} \wedge \text{vol}_{S^3} \right),
\]

(3.4)

and satisfies

\[
K \wedge \hat{K} = \sqrt{\frac{7}{5}} p dp \wedge d\chi \wedge \text{vol}_{S^3} \wedge \text{vol}_{S^2} \geq 0.
\]

(3.5)

Up to diffeomorphisms and field redefinitions, the constraints (3.3) are generically solved by

\[
dp = 2 \times 5^{1/6} \sqrt{2} R^2 dy,
\]
\[
\nu_1 = 2 \times 5^{1/6} \sqrt{2} R^2 e^{D/2} (\cos \Theta dx_1 + \sin \Theta dx_2),
\]
\[
\nu_2 = 2 \times 5^{1/6} \sqrt{2} R^2 e^{D/2} (-\sin \Theta dx_1 + \cos \Theta dx_2),
\]

(3.6)
together with
\[ d\chi = -5^{1/6} \sqrt{2} R^2 \left( \epsilon^{ij} \partial_i d x_j \wedge d y + \partial_y e^D d x_1 \wedge d x_2 + 2 d(\Theta d y) \right), \] (3.7)
where \((x_1, x_2, y)\) is a local coordinate frame on \(M_3\) and \(D\) and \(\Theta\) arbitrary functions of \(M_3\). The consistency condition \(d^2 \chi = 0\) requires
\[ \partial_D^2 D + \partial_D^2 D + \partial_D^2 e^D = 0, \] (3.8)
which is the Toda equation (1.2). Note also that the forms (3.6) are linearly independent at each point of the bulk, which implies that
\[ \nu_1 \wedge \nu_2 \wedge d p \neq 0, \] (3.9)
everywhere except on the boundaries. Using (3.6) and (3.7), the condition \(K \wedge \hat{K} > 0\) becomes
\[ -16 e^D R^6 y \partial_D y \wedge \text{vol}_3 \wedge \text{vol}_{S^2} \wedge \text{vol}_{S^1} > 0, \] (3.10)
which we solve by taking
\[ y > 0, \quad \partial_D y < 0. \] (3.11)

### 3.1. AdS\(_5\) vacua fields

Given the half-maximal structures (3.1), we can construct the corresponding generalised metric \(\mathcal{M}_{5N}\) using (A.6). From this, we obtain the corresponding 11-dimensional fields making use of the dictionary between supergravity objects and ExFT described in (A.18).

For the metric one obtains
\[ d s^2 = f_1 d s^2_{\text{AdS}_5} + f_4 (e^D d x_1 d x_2 + d y^2) + f_3 (d \beta + A_i d x_i)^2 + f_2 d s^2_{S^2}, \] (3.12)
with
\[ f_4 = \frac{R^2 e^{-4 \lambda}}{1 - y^2 e^{-6 \lambda}}, \quad f_3 = 4 R^2 e^{2 \lambda} (1 - y^2 e^{-6 \lambda}), \quad f_2 = R^2 y^2 e^{-4 \lambda}, \]
\[ A_i = \frac{1}{2} \epsilon_{ij} \partial_j D, \] (3.13)
where
\[ e^{-6 \lambda} = - \frac{\partial_D D}{y (1 - y \partial_D D)}. \] (3.14)
The AdS\(_5\) warp factor \(f_1\) can also be read off from the structures as in (A.8) and in our case is
\[ f_1 = | \det g |^{-1/3} \kappa^2 = 4 e^{2 \lambda}. \] (3.15)
We observe that the function \(\Theta\) can be completely removed from the geometry through a local shift of the coordinate \(\beta\), and we will therefore set from now on \(\Theta = 0\).
Finally, the gauge field \( C^{(3)} \) can also be read off from (A.18). Using the forms (3.6) we obtain
\[
C^{(3)} = R^3 \left( 4y^3 e^{-6\lambda}(d\beta + A_i dx^i - d\Theta) - 4 \left( \frac{1}{2 \times 5^{1/6} \sqrt{2R^2} \chi} - y d\Theta + y A_i dx^i \right) \right) \\
\wedge \text{vol}_2, \tag{3.16}
\]
with field strength
\[
F^{(4)} = dC^{(3)} = R^3 \left( d \left( 4y^3 e^{-6\lambda}(d\beta + A_i dx^i - d\Theta) \right) + \hat{d}B \right) \wedge \text{vol}_2, \tag{3.17}
\]
with
\[
d\hat{B} = 2 \left( y^2 \partial_i \left( \frac{1}{y} \partial_x e^\beta \right) dx_1 \wedge dx_2 + y \partial_1 \partial_2 D dy \wedge dx_1 \wedge dy \right), \tag{3.18}
\]
and we observe that the function \( \Theta \) can be removed by the same local shift of \( \beta \) we used to remove it from the metric. These geometries are those obtained in [51].

3.2. Minimal consistent truncation

As shown in [22] and reviewed in section 2.2.1, we can use the generalised USp(4) structure of the AdS_5 vacua to immediately construct a consistent truncation to five-dimensional SU(2) \times U(1) gauged supergravity keeping only the gravitational supermultiplet. The truncation ansatz is given by (2.4) and (2.5) and all that is left to do is to use the dictionary between ExFT and 11-dimensional supergravity to compute the supergravity fields. Here we restrict ourselves for simplicity to the scalar sector, for which we find the uplift formulae
\[
\begin{align*}
\tilde{d}x^2 &= \tilde{f}_4 (e^D dx_i dx^i + dy^2) + \tilde{f}_3 (d\beta + A_i dx^i)^2 + \tilde{f}_2 dx_2^2 \\
\tilde{F}^{(4)} &= \left( d \left( 4y^3 e^{-6\lambda}(d\beta + A_i dx^i) \right) + \hat{d}B \right) \wedge \text{vol}_2,
\end{align*}
\]
where \( A_i dx^i \) and \( \hat{d}B \) are the same forms as in the vacuum and
\[
\begin{align*}
\tilde{f}_4 &= \frac{R^2 X^{-2} e^{-4\lambda}}{1 - y^2 e^{-6\lambda}}, & \tilde{f}_3 &= 4 R^2 X^4 e^{2\lambda} (1 - y^2 e^{-6\lambda}), & \tilde{f}_2 &= R^2 X^{-2} y^2 e^{-4\lambda},
\end{align*}
\]
with
\[
e^{-6\lambda} = - \frac{\partial_i D}{y (X^{-3} - y \partial_i D)}. \tag{3.21}
\]
This agrees with the consistent truncation constructed in [52].

4. Consistent truncations with vector multiplets

In section 2.2.2, we reviewed that for an AdS vacuum to admit a consistent truncation with vector multiplets around it, its internal manifold should admit \( n \) extra vector multiplets \( J_g \) satisfying the constraints (2.6) and (2.7). We will now use these conditions to classify all possible
consistent truncations with vector multiplets around the half-maximal AdS vacuum. Using our results, it is straightforward to construct the consistent truncations with vector multiplets from the formulae (2.13), although we will not do so here.

In particular, (2.7) implies that the additional $J_a$ have to organise into representations of $(SU(2) \times U(1))^R$. Furthermore, the explicit expressions of $J_i$ in (3.1) show that these are ‘trivial’ generalised vector fields whose action via the generalised Lie derivative vanishes automatically

$$\mathcal{L}_{J_i} = 0,$$

for any tensor it acts on. This immediately implies that $f_{\mu}^a = 0$, which is consistent with the outcome of the analysis of the embedding tensor constraints of the five-dimensional gauged supergravity [49]. Moreover, the $SU(2)$ representations of $J_a$ must be of odd dimensions, i.e. be representations of SO(3).

Together with the fact that $n \leq 5$, we are then left with the following possibilities for the vector multiplets, which we discuss in the subsequent sections:

- Up to five $SU(2) \times U(1)$ singlets (sections 4.1 and 4.2),
- One $SU(2)$ triplet (section 4.3),
- One $SU(2)$ triplet and up to two singlets (section 4.4),
- One $SU(2)$ quintuplet, i.e. the symmetric traceless representation of SO(3) (section 4.5),
- Up to two $U(1)$ doublets (sections 4.6 and 4.7),
- One $SU(2)$ triplet and one $U(1)$ doublet (section 4.8),
- One $U(1)$ doublet and up to three singlets or two $U(1)$ doublets and one singlet (section 4.9).

By ‘$U(1)$ doublet’ we mean the complex $U(1)$ representation with general charge $q$ under $U(1)$. We will now analyse these various possibilities in turn and show that, at most, only the following possibilities are allowed:

- Up to three $SU(2) \times U(1)$ singlets,
- One $SU(2)$ triplet,
- One $U(1)$ doublet,
- One $U(1)$ doublet and one singlet.

### 4.1. One singlet under $(SU(2) \times U(1))^R$

To construct a consistent truncation with one extra vector multiplet, the internal manifold should admit an extra generalised vector $J$. The most general generalised vector satisfying (2.6) and compatible with the symmetries is

$$\bar{J} = (\bar{\Phi} + \bar{\nu} \wedge \text{vol}_3) + \frac{R}{5 \sqrt{2}} \left( (p \bar{\Phi} - \chi \wedge \bar{\nu}) \wedge \text{vol}_3 \wedge \text{vol}_3 - \bar{\Phi} \wedge \chi \wedge \text{vol}_3 \right),$$

(4.2)

where $\bar{\nu}$ and $\bar{\Phi}$ are a one- and a two-form satisfying the constraints

$$\nu_2 \wedge \Phi = -\bar{\nu} \wedge d\nu_1, \quad \nu_1 \wedge \Phi = \bar{\nu} \wedge d\nu_2,$$

$$dp \wedge \Phi = \bar{\nu} \wedge d\chi, \quad \bar{\nu} \wedge \Phi = dp \wedge d\chi,$$

(4.3)
and $\nu_1$, $\nu_2$ and $dp$ are the objects appearing in the structures (3.1). Constraints (4.3) can be completely solved as follows. Given that $\nu_1$, $\nu_2$ and $\nu_3$ form a basis of one-forms at each point of the manifold, one can write

$$\bar{\nu} = \nu_1 \nu_1 + \nu_2 \nu_2 + \nu_3 \nu_3$$  \hspace{1cm} (4.4)

with $\nu_1$, $\nu_2$ and $\nu_3$ three functions of $M_3$. Then, using (3.6) (with $\Theta = 0$), the first three equations of (4.3) fix

$$\Phi = \frac{1}{2 \times 5^{1/6} \sqrt{2} R^2} \left[ \left( \nu_1 \partial_i D - e^{-\frac{2}{3} D} \nu_3 \partial_i D \right) dp \wedge \nu_2 + \left( -\nu_2 \partial_i D + e^{-\frac{2}{3} D} \nu_3 \partial_i D \right) dp \wedge \nu_1 \right] + v_2 dv_1 - v_1 dv_2 + v_3 dv_3$$ \hspace{1cm} (4.5)

and the last condition of (4.3) becomes

$$(\nu_1^2 + \nu_2^2 + \nu_3^2) dp \wedge d\chi = dp \wedge d\chi.$$ \hspace{1cm} (4.6)

Therefore, given the fact that $dp \wedge d\chi \neq 0$, a general solution to (4.3) given the fields (3.6) is

$$\bar{\nu} = \nu_1 \nu_1 + \nu_2 \nu_2 + \nu_3 \nu_3 \quad \text{with} \quad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1,$$

$$\Phi = \frac{1}{2 \times 5^{1/6} \sqrt{2} R^2} \left[ \left( \nu_1 \partial_i D - e^{-\frac{2}{3} D} \nu_3 \partial_i D \right) dp \wedge \nu_2 + \left( -\nu_2 \partial_i D + e^{-\frac{2}{3} D} \nu_3 \partial_i D \right) dp \wedge \nu_1 \right] + v_2 dv_1 - v_1 dv_2 + v_3 dv_3$$ \hspace{1cm} (4.7)

Finally, the differential conditions (2.7) imply that

$$d\bar{\nu} = d\Phi = 0.$$ \hspace{1cm} (4.8)

In terms of the solution (4.7), the condition $d\bar{\nu} = 0$ becomes

$$\partial_2 \left( e^{\frac{2}{3} D} \nu_1 \right) = \partial_1 \left( e^{\frac{2}{3} D} \nu_2 \right),$$

$$\partial_1 \nu_3 = \partial_2 \left( e^{\frac{2}{3} D} \nu_1 \right),$$

$$\partial_2 \nu_3 = \partial_3 \left( e^{\frac{2}{3} D} \nu_2 \right),$$ \hspace{1cm} (4.9)

and using these, the condition $d\Phi = 0$ can be written as

$$\partial_1 \left( e^{\frac{2}{3} D} (\partial_i D)^2 \nu_1 \right) + \partial_2 \left( e^{\frac{2}{3} D} (\partial_i D)^2 \nu_2 \right) + e^{-D} \partial_3 \left( e^{2 D} (\partial_i D)^2 \nu_3 \right) = 0.$$ \hspace{1cm} (4.10)

Thus, a consistent truncation with a single vector multiplet requires the existence of a one-form on $M_3$ such that its components in the $\nu_1$, $\nu_2$, $dp$ basis form a unit-norm triplet ($\nu_1$, $\nu_2$, $\nu_3$) such that (4.9) and (4.10) are satisfied. The resulting five-dimensional gauged supergravity has embedding tensor $f_{abc}$, $\xi_{ab}$ with $a = (u, 6)$ and whose only non-zero components are

$$f_{ABC} = \frac{5^{1/6} \sqrt{3}}{R} \epsilon_{ABC}, \quad \xi_{ij} = \frac{5^{1/6} \sqrt{3}}{R} \epsilon_{ij}.$$ \hspace{1cm} (11.1)

Hence the five-dimensional supergravity has SU(2) $\times$ U(1) gauging.
4.2. Multiple \((SU(2) \times U(1))_R\) singlets

To analyse the more general case of \((SU(2) \times U(1))_R\) singlets, we first consider the case of two \((SU(2) \times U(1))_R\) singlets. Following the discussion in the previous section this is described by generalised vectors of the form (4.2) characterised by \(n\) pairs \((\bar{\nu}_i, \bar{u}_i),\) each of them satisfying individually (4.7), (4.9) and (4.10). Furthermore, the vectors formed with the components of the different \(\bar{\nu}_i\) in the \([\nu_1, \nu_2, d\]) basis will have to be perpendicular pairwise. This condition implies that we can keep at most three \((SU(2) \times U(1))_R\) singlets. The resulting gauged supergravity has embedding tensor \(f_{ABC}, \xi_{ij}\) with \(a = (u, \bar{A})\), with \(\bar{u} = 1, \ldots, n\) with \(n \leq 3\) labelling the number of vector multiplets. The only non-zero components are

\[
\begin{align*}
    f_{ABC} &= \frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{ABC}, \\
    \xi_{ij} &= \frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{ij}.
\end{align*}
\]

Hence the five-dimensional supergravity has \(SU(2) \times U(1)\) gauging.

4.3. Triplet under \(SU(2)\)

We next analyse the possibility of having three vectors organising into a triplet under \(SU(2)\). The most general extra generalised vectors compatible with the symmetries and satisfying the algebraic constraints (4.3) are

\[
J_A = \rho y_A + (p dy_A \wedge vol_3 + \chi \wedge dy_A + y_A \Phi + y_A \bar{\nu} \wedge vol_3) + \frac{1}{\rho} \left( y_A (p \Phi - \chi \wedge \bar{\nu}) \wedge vol_3 \wedge vol_3 - y_A \Phi \wedge \chi \wedge vol_3 + dp \wedge \theta_A \wedge vol_3 \right),
\]

where \(\bar{\nu}\) and \(\Phi\) must satisfy the same constraints (4.3) and are therefore again solved by (4.7). However, the differential conditions (2.7) now imply the equations

\[
\begin{align*}
    \bar{\nu} &= dp, \\
    -p d\Phi + 2 \bar{\nu} \wedge \Phi &= 0,
\end{align*}
\]

which, with some work, one can show that they can only be solved if the function \(D\) characterizing the vacuum is of the form

\[
e^D = e^{\sigma(x_1, x_2)}(-y^2 + \text{constant}),
\]

with \(\sigma\) any function of \((x_1, x_2)\) satisfying

\[
\partial_1^2 \sigma + \partial_2^2 \sigma - 2e^\sigma = 0.
\]

This form implies that the internal space is \(S^4\) fibred over a Riemann surface as in [50], as we will discuss in more detail in section 5.

As a result of this, the embedding tensor \(f_{ABC}, \xi_{ij}\) with \(a = (u, \bar{A})\) is given by

\[
\begin{align*}
    f_{ABC} &= \frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{ABC}, \\
    f_{A\bar{B}C} &= -2 \frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{A\bar{B}C}, \\
    f_{A\bar{B}C} &= -\frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{A\bar{B}C}, \\
    \xi_{ij} &= \frac{5^{1/6}}{6} \sqrt{2} \bar{R} \epsilon_{ij},
\end{align*}
\]
and all other components vanishing. Therefore, the gauge group of the resulting supergravity is $\text{ISO}(3) \times U(1)$. Explicit uplift expressions for this consistent truncation have been constructed using $\text{SO}(5)$ maximal gauged supergravity in [28], while this consistent truncation has also been studied using generalised geometry in [29]. As we have shown here, this consistent truncation exists if and only if the AdS$_5$ vacuum is of the type of $S^4$ fibred over a Riemann surface.

### 4.4. One SU(2)$_R$ triplet and multiple (SU(2) × U(1))$_R$ singlets

For a background to allow a truncation keeping one triplet of SU(2) and multiple singlets, it has to allow both of them independently, satisfying the conditions described in the previous sections, together with extra compatibility conditions. However, even the case with one singlet is not possible. In order to have a truncation with the triplet, the internal space must be $S^4$ fibred over a Riemann surface. In this case, it is possible to show that the differential condition (4.13) for the singlet cannot be solved. Therefore, a truncation keeping an SU(2)$_R$ triplet and a singlet is not possible, nor is a truncation keeping a triplet and multiple singlets.

### 4.5. Quintuplet of SU(2)$_R$

Next we investigate the possibility of adding five extra $\bar{J}_{(AB)}$ transforming in the symmetric traceless representation of SU(2). We consider the most general ansatz compatible with the symmetries, namely

$$\bar{J}_{(AB)} = \bar{\rho} y_{(AB)} + \bar{\varrho} y_{(A} dy_{B)} \wedge \text{vol}_3 + \bar{\varphi} y_{(A} \theta_{B)} \wedge \text{vol}_3,$$

$$+ y_{(A} \bar{x}_1 \wedge dy_{B)} + y_{(A} \bar{x}_2 \wedge \theta_{B)} + y_{(AB)} \Phi + y_{(AB)} \bar{\nu} \wedge \text{vol}_3,$$

$$+ (\bar{\rho} y_{(AB)} ) \Pi \wedge \text{vol}_3 + \bar{\eta} y_{(AB)} \text{vol}_3 \wedge \text{vol}_2,$$

$$+ m y_{(A)} \text{vol}_3 \wedge dy_{(B)} \wedge \text{vol}_3 + n y_{(A)} \theta_{(B)} \wedge \text{vol}_3),$$

where ( . . . ) indicate traceless symmetrisation. It is easy to show, however, that it is impossible for (4.18) to satisfy the algebraic conditions (2.6). Therefore, there are no consistent truncations with five vector multiplets transforming as a quintuplet of SU(2)$_R$.

### 4.6. Doublet of U(1)$_R$

We next investigate the possibility of constructing a truncation with vector multiplets transforming as a U(1)$_R$ doublet of general charge $q$, which is not necessarily the same as that of the U(1) vacuum structures. In particular, we look for extra generalised vectors transforming as

$$L^K_J = -\frac{S^{1/6} \sqrt{2}}{R} q \epsilon_{ij} J^i,$$

with $q \in \mathbb{Z}$ the U(1) charge. Using the functions $w_{(q)}$ and differential forms $dw_{(q)}$ defined in appendix B, the most general generalised vectors one can construct satisfying the algebraic constraints (2.6) and compatible with the symmetries are
\[ \tilde{q} = \left( w_{q_i} \Phi_2 + \omega w_{q_i} \Phi_1 + w_1 \bar{\nu}_2 \wedge \text{vol}_{S^1} + \omega w_{q_i} \bar{\nu}_1 \wedge \text{vol}_{S^1} \right) \\
+ \frac{1}{\rho} \left( w_{q_i} (p \Phi_2 - \chi \wedge \bar{\nu}_2) \wedge \text{vol}_{S^2} + \omega w_{q_i} (p \Phi_1 - \chi \wedge \bar{\nu}_1) \wedge \text{vol}_{S^2} \wedge \text{vol}_{S^1} \\
- w_{q_i} \Phi_2 \wedge \chi \wedge \text{vol}_{S^2} - \omega w_{q_i} \Phi_1 \wedge \chi \wedge \text{vol}_{S^2} \right), \]  

(4.20)

where \( \nu_i \) and \( \Phi \) satisfy (4.7). Furthermore, writing \( \bar{\nu}_i = v_i^0 \nu_1 + v_i^2 \nu_2 + v_i^3 d\rho \), they also need to satisfy

\[ \sum_{k=1}^{3} v_k^{(i)} v_k^{(j)} = \delta^{ij}. \]  

(4.21)

The differential conditions imply

\[ q \, d\Phi_i = -\epsilon_{ij} d\bar{\nu}_j, \]  

(4.22)

which in terms of \( v_k^{(i)} \) become

\[ q \left( v_i^0 \partial_i D + v_i^0 \partial_2 D + 2 v_i^0 \partial_2 v_i^0 \right) = 2 e_i^j e^{-2D} \left( \partial_i \left( e^{2D} v_j^0 \right) - \partial_j \left( e^{2D} v_i^0 \right) \right), \]

\[ q \left( 2 v_i^0 \partial_i e^{2D} - v_i^0 \partial_2 D \right) = 2 e^j_i \left( \partial_j \left( e^{2D} w_i^0 \right) - \partial_i w_j^0 \right), \]

\[ q \left( 2 v_i^0 \partial_i e^{2D} - v_i^0 \partial_2 D \right) = -2 e_j^i \left( \partial_j \left( e^{2D} w_i^0 \right) - \partial_i w_j^0 \right). \]

(4.23)

With these conditions satisfied, we find the embedding tensor \( f_{abc}, \xi_{a}, \), with \( a = (u, i) \), whose only non-zero components are given by

\[ f_{abc} = \frac{5^{1/6} \sqrt{2}}{R} \epsilon_{ABC}, \quad \xi_j = \frac{5^{1/6} \sqrt{2}}{R} \epsilon_{ij}, \quad \xi_{ij} = -\frac{5^{1/6} \sqrt{2}}{R} q \epsilon_{ij}. \]  

(4.24)

The resulting gauge group is still given by SU(2) × U(1) but the \( \xi_{ij} \) term changes the embedding of the U(1) inside SO(5, 2) so that it is a linear combination of U(1) ⊂ SO(5) ⊂ SO(5, 2) and U(1) ∼ SO(2) ⊂ SO(5, 2), see (2.9).

This consistent truncation can be used to uplift the RG flows of [1] between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) AdS5 vacua, which require \( |q| < 2 \). Our results show that the parameter \( q \in \mathbb{Z} \) must be quantised for the uplift to be possible. Therefore, the only gauged supergravity with a flow between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) AdS5 vacua which could be uplifted to 11-dimensional supergravity have the value \( |q| = 1 \). [1] also considers flows between supergravities with multiple U(1)R doublets, but we will show these cannot arise from consistent truncations of 11-dimensional supergravity in section 4.7. Interestingly, this critical value, \( |q| = 1 \), corresponds to precisely the only case where the ratio of central charges of the dual CFT’s is given by the well-known value \( \frac{2}{3} \) [55, 56]. As shown in [56], this ratio arises whenever the flow from a \( \mathcal{N} = 2 \) to \( \mathcal{N} = 1 \) superconformal field theory is triggered by a mass deformation of an adjoint scalar. Since our results show that the other flows between \( \mathcal{N} = 4 \) and \( \mathcal{N} = 2 \) AdS5 vaca found in [1] with \( |q| \neq 1 \) cannot be uplifted, this strongly suggests that only flows from \( \mathcal{N} = 4 \) to \( \mathcal{N} = 2 \) AdS5 vaca corresponding to adjoint mass deformations can be realised in a consistent truncation. Thus other flows from half-maximal to quarter-maximal AdS5 vaca can only arise directly in 11-dimensional supergravity, if they exist at all.
It would be particularly interesting to find solutions of the conditions (4.21) and (4.23), which would provide an explicit uplift of the flow of [1].

4.6.1. Uplift of RG flows to type IIB. One may ask if the flows of [1] can alternatively be uplifted by a consistent truncation to IIB string theory. However, it is easy to see that this is only possible when $\rho = 2$ in the notation of [1] or $q = 1$ in the notation of section 4.6. In this case, the uplift corresponds to the FGPW flow [55] between $AdS_5 \times S^5$ and the $N = 2$ $AdS_5$ Pilch–Warner solution, or its quotients.

To see this, note that all $N = 4$ $AdS_5$ vacua of IIB are of the type $AdS_3 \times S^5/Z_3$. In particular, they have a $Z_3 \subset SU(2) \subset Spin(5)$ structure, which implies that one can keep at most two vector multiplets in a truncation, see (2.6). However, these vector multiplet structures are exactly those arising from $AdS_3 \times S^5$ which survive the $Z_3$ truncation. In particular, the $N = 4$ gauged supergravity obtained this way is just the truncation of the five-dimensional $N = 8$ $SO(6)$ supergravity that is invariant under the $Z_3$ action. This supergravity corresponds to the desired gauging of [1] for the value $\rho = 2$ in the notation of [1] (corresponding to $q = 1$ in our conventions of (4.6)). The corresponding flow is nothing but the FGPW flow [55] between the $N = 8$ $AdS_5 \times S^5$ vacuum and the Pilch–Warner $N = 2$ $AdS_5$ vacuum [57] of IIB string theory, or analogous flows starting from $AdS_5 \times S^5/Z_3$.

4.7. Two $U(1)_R$ doublets

Following the results of the previous section, we consider two doublets of generalised vectors $\tilde{j}^{(1)}_i$ and $\tilde{j}^{(2)}_i$ characterised by $(\bar{\nu}^{(1)}_i, \bar{\phi}^{(1)}_i)$ and $(\bar{\nu}^{(2)}_i, \bar{\phi}^{(2)}_i)$, with $i = 1, 2$, respectively. Analogous to other cases discussed above, the four vectors formed by the components of $\bar{\nu}^{(1)}_i$ and $\bar{\nu}^{(2)}_i$, $i = 1, 2$, in the $(\nu_1, \nu_2, dp)$ basis have to be perpendicular pairwise. However, given that the space is only three-dimensional, this condition cannot be satisfied, ruling out the existence of these consistent truncations. This result excludes the uplift of various flow solutions of [1].

4.8. One $SU(2)_R$ triplet and one $U(1)_R$ doublet

For this consistent truncation to exist, one needs to require that the function $D$ is of the form (4.15) (ensuring the existence of the triplet) as well as that the two one-forms $\tilde{\nu}^{(doub.)}_i$ are perpendicular to $\tilde{\nu}^{(trip.)}_i = dp$ in the $(\nu_1, \nu_2, dp)$ basis. This implies that $\tilde{\nu}^{(doub.)}_1 = \nu_1 \nu_1 + \nu_2 \nu_2$, with $\nu_1^2 + \nu_2^2 = 1$, and either $\tilde{\nu}^{(doub.)}_2 = -\nu_2 \nu_1 + \nu_1 \nu_2$ or $\tilde{\nu}^{(doub.)}_2 = -\nu_2 \nu_1 - \nu_1 \nu_2$. Using these ansätze and $D$ of the form (4.15), one can show that the equation (4.23) have no solutions and we therefore conclude that having a truncation with a $U(1)_R$ doublet and a $SU(2)_R$ triplet is not possible.

4.9. Multiple $U(1)_R$ doublets with multiple ($SU(2) \times U(1))_R$ singlets

We have already seen in section 4.7, that it is impossible to keep multiple $U(1)_R$ doublets. Therefore, the case of multiple $U(1)_R$ doublets with some ($SU(2) \times U(1))_R$ singlets is also not possible.

This leaves us with the case of one $U(1)_R$ doublet together with multiple ($SU(2) \times U(1))_R$ singlets. Analogous to previous cases, these truncations exist if one can find a set of multiple $\tilde{\nu}^{(trip.)}_i$ which satisfy (4.9) and (4.10) and the vectors formed by their components in the $(\nu_1, \nu_2, dp)$ basis are perpendicular pairwise as well as perpendicular to the ones coming from the two $\tilde{\nu}^{(doub.)}_i$, which satisfy (4.23). This implies that we can keep at most one singlet together with one $U(1)_R$ doublet. The resulting embedding tensor $f_{abc}$, $\xi_{ab}$, with $a = (u, \bar{u}, 8)$, now has
as its only non-zero components

\[ f_{ABC} = \frac{5^{1/6}}{R} \sqrt{2} \epsilon_{ABC}, \quad \xi_{ij} = \frac{5^{1/6}}{R} \sqrt{2} \epsilon_{ij}, \quad \bar{\xi}_{\bar{i} \bar{j}} = -\frac{5^{1/6}}{R} \sqrt{2} \epsilon_{\bar{i} \bar{j}}. \]  

(4.25)

The gauging is again given by SU(2) \(\times\) U(1) but due to \(\xi_{ij} \neq 0\) the embedding of the U(1) inside SO(5, 3) is again a linear combination of U(1) \(\subset\) SO(5) \(\subset\) SO(5, 3) and U(1) \(\subset\) SO(3) \(\subset\) SO(5, 3), see (2.9).

5. Consistent truncations of SO(5) gauged supergravity around half-maximal AdS\(_5\) vacua via SL(5) ExFT

An important class of half-maximal AdS\(_5\) vacua arise as near horizon limit of M5 branes wrapped on Riemann surfaces [50]. As expected, these form a subclass of the geometries (3.12), but they are also vacua of the seven-dimensional maximally supersymmetric SO(5) gauged supergravity obtained by truncating 11-dimensional supergravity on a \(S^4\) [5].

In this section, we analyse consistent truncations of the seven-dimensional theory around these vacua in terms of the generalised parallelisation of \(S^4\) used to construct SO(5) gauged supergravity as a generalised Scherk–Schwarz reduction. A particular example of such a consistent truncation keeping three vector multiplets was analysed in [29] using generalised geometry and constructed using different methods in [28]. Here, as in the previous section, we want to analyse all possible consistent truncations around these vacua. The outcome of our analysis is that the truncation of [28, 29] is the only possible one keeping vector multiplets. This implies that all other truncations of section 4 do not have a seven-dimensional origin and can only be uplifted to 11-dimensional supergravity.

5.1. \(S^4\) reduction and its half-maximal AdS\(_5\) vacua

As shown in [8, 9], the \(S^4\) is generalised parallelisable in SL(5) ExFT. This means that its generalised tangent bundle admits a globally defined frame, which can be used to define a consistent truncation preserving all supersymmetries. This generalised parallelisation is given by the following objects in the \(10\)

\[ E_{IJ(10)} = R^{-1} v_{IJ} + R^2 \sigma_{IJ} + R^{-1} \epsilon_{IJ} A_{(3)}, \]  

(5.1)

where \(I, J = 1, \ldots, 5\), \(R\) is the radius of the four-sphere, and the objects \(v_{IJ}\) and \(\sigma_{IJ}\) are defined in appendix B. The three form \(A_{(3)}\) is defined such that

\[ dA_{(3)} = 3 R^3 \text{vol}_{S^4}. \]  

(5.2)

Furthermore, one can also define global frames for other generalised bundles of the sphere in other representations of SL(5). For instance, the parallelisation of the bundles transforming in the \(5\) and the \(\bar{5}\) are defined by

\[ E_{I(5)} = R \bar{Y}_I - R^4 Y_I \text{vol}_{S^4} + R \bar{Y}_I A_{(3)}, \]

\[ E_{\bar{I}(\bar{5})} = -R^4 \beta_I + \bar{Y}_I A_{(3)} + \bar{Y}_I, \]  

(5.3)

where we refer again to appendix B for the definitions of the objects appearing. The generalised parallelisation (5.1) generates a SO(5) algebra that can be broken to SU(2) \(\times\) U(1) by choosing
the embedding coordinates for the $S^4$ as (B.16), as discussed in appendix B. Splitting the index $I = (A, i)$, we define the following generalised objects

$$E_{A_{(10)}} = \frac{1}{2} \epsilon_{ABC} E_{B_{(10)}}, \quad E_{A_{(5)}} = E_{I = A_{(5)}}, \quad E_{A_{(\bar{5})}} = E_{I = A_{(5)}},$$

$$E_{S_{(10)}} = \frac{1}{2} \epsilon_{ij} E_{ij_{(10)}}, \quad E_{i_{(5)}} = E_{I = i_{(5)}}, \quad E_{i_{(\bar{5})}} = E_{I = i_{(5)}},$$

(5.4)

where now the objects $E_{A_{(10)}}$ and $E_{S_{(10)}}$ generate the SU(2) and the U(1) algebras respectively.

In terms of the objects on the two- and one-spheres, the gauge field in (5.2) can be written as

$$A = r^3 \text{vol}_{S^2} \wedge \text{vol}_{S^1}.$$  

(5.5)

With the parallelisation of the $S^4$ and the decomposition of the $E_6(6)$ covariant objects into $SL(5) \times GL(2)$ objects described in appendix C, one can now construct the most general geometries with 16 supercharges of the local form AdS$_5 \times \Sigma \times S^4$, with $\Sigma$ a Riemann surface. The most general structures satisfying the conditions (2.1) and (2.3) are

$$J_A = 5^{1/6} \sqrt{2} E_{A_{(10)}} + R^2 E_{A_{(5)}} \otimes (\xi \wedge \tilde{\xi}) + R E_{A_{(\bar{5})}} \otimes \varphi,$$

$$J_i = 5^{1/6} \sqrt{2} R^2 \left[ E_{i_{(5)}} \otimes (\xi \wedge \varphi) + \epsilon E_{i_{(5)}} \otimes (\xi \wedge \varphi) \right] - 5^{1/6} \sqrt{2} R \left[ E_{i_{(\bar{5})}} \otimes \xi - \epsilon E_{i_{(\bar{5})}} \otimes \tilde{\xi} \right],$$

$$\hat{K} = 5^{1/6} \sqrt{2} E_{S_{(10)}},$$

(5.6)

where $\varphi, \xi$ and $\tilde{\xi}$ are one-forms on $\Sigma$ satisfying

$$\xi \wedge \tilde{\xi} > 0,$$

(5.7)

as well as

$$d \varphi = -\xi \wedge \tilde{\xi}, \quad d \xi = -\varphi \wedge \tilde{\xi}, \quad d \tilde{\xi} = \varphi \wedge \xi.$$  

(5.8)

This result can be embedded into the general one (3.1) obtained in section 3 by taking

$$p = 5^{1/6} \sqrt{2} R^2 r, \quad \chi = -5^{1/6} \sqrt{2} R^2 r \varphi,$$

$$\nu_1 = 5^{1/6} \sqrt{2} R^2 \sqrt{1 - r^2} \xi, \quad \nu_2 = 5^{1/6} \sqrt{2} R^2 \sqrt{1 - r^2} \tilde{\xi},$$

(5.9)

where the coordinate $r$, which is here one of the coordinates on the $S^4$, plays the role of the coordinate $y$ in section 3. Comparing these fields with (3.6), we conclude that for all these cases the function $D$ is of the form

$$e^{D(x_1, x_2, r)} = (1 - r^2) e^{\sigma(x_1, x_2)},$$

(5.10)

with $\sigma(x_1, x_2)$ a function on the Riemann surface, which is the form of the Maldacena–Nuñez configuration [50].

5.2. Consistent truncations with vector multiplets

In section 4, we argued that geometries described by a function $D$ of the form (5.10) allowed for a consistent truncation with three vector multiplets around it, with gauge group ISO(3) $\times$ U(1). This truncation can also be embedded into maximal seven-dimensional SO(5) gauged supergravity, as explicitly constructed in [28], and also analysed in [29]. In this section, we will argue
that this truncation and the minimal one are the only half-maximal consistent truncations one can construct around half-maximal AdS5 vacua of seven-dimensional SO(5) gauged supergravity. We stress, however, that this does not necessarily exclude truncations around these vacua that cannot be embedded into the seven-dimensional theory. To analyse the existence of such truncations, we refer to the general discussion of section 4, where one should take $D$ of the form (5.10).

As reviewed in 2.2.2, in order to have consistent truncations the internal geometry should admit extra generalised vectors forming a generalised Spin($5 - n$) $\subset$ USp(4) structure. Since we want the consistent truncation to come from seven-dimensional gauged supergravity, the generalised Spin($5 - n$) $\subset$ USp(4) structure must not deform the generalised parallelisation of the $S^4$. Therefore, the SL(5) part of the generalised structures must be written in terms of (5.4).

We start by analysing the case of a single vector multiplet, transforming as a singlet of the $R$-symmetry group. From (5.4), we observe that the only object transforming as a singlet is $E_{S^1,1} \omega_{(q)}^i$ in appendix B. The forms $\chi$ and $\bar{\chi}$ satisfy

$$\bar{\chi} \wedge \xi = \chi \wedge \bar{\xi}, \quad \chi \wedge \xi = -\bar{\chi} \wedge \bar{\xi}, \quad \chi \wedge \bar{\chi} = -\xi \wedge \bar{\xi}, \quad (5.12)$$

which can be solved in general by taking

$$\chi = V_1 \xi + V_2 \bar{\xi}, \quad \bar{\chi} = \pm(V_2 \xi - V_1 \bar{\xi}), \quad (5.13)$$

with $V_1$ and $V_2$ two functions of the Riemann surface satisfying $V_1^2 + V_2^2 = 1$. Given this solution, and unlike in the general case discussed in section 4.6, the differential conditions (2.7) can only be solved if the charge is $q = 1$. In this situation the forms $\chi$ and $\bar{\chi}$ need to satisfy

$$d \bar{\chi} = -\chi \wedge \varphi, \quad d \chi = \bar{\chi} \wedge \varphi, \quad (5.14)$$

and one can show, in a very similar way to the discussion of section 4.8, that these equations cannot be solved given that the function $D$ characterising the vacuum satisfies the Toda equation. This concludes our proof that the only possible consistent truncations around half-maximal AdS5 vacua in seven-dimensional SO(5) gauged supergravity are the minimal truncation keeping only the gravitational supermultiplet or the consistent truncation with three vector multiplets and ISO(3) × U(1) gauging constructed in [28, 29].

6. Conclusions

In this paper, we studied half-maximal AdS5 vacua of $M$-theory and their consistent truncations. We showed that ExFT allows us to fully classify all possible consistent truncations
to half-maximal gauged supergravity with arbitrary number of vector multiplets. By working systematically through all possible cases, we showed that the largest possible consistent truncation of the half-maximal AdS5 vacua contain three vector multiplets, with the possible gaugings ISO(3) × U(1) or SU(2) × U(1), where in the latter the U(1) is embedded either as a subgroup of SO(5) ⊂ SO(5, 3) or as a linear combination of U(1) ⊂ SO(5) ⊂ SO(5, 3) and U(1) ⊂ SO(3) ⊂ SO(5, 3). We were able to show that the ISO(3) × U(1) gauging arises if and only if the internal space is a $S^4$ warped over a Riemann surface. This consistent truncation has previously been constructed in [28] and studied from a generalised geometry perspective in [29].

We also derived differential conditions which must be satisfied by the AdS5 vacua in order to admit a consistent truncation with one or two vector multiplets. The resulting gaugings are SU(2) × U(1) with the U(1) again possibly embedded in two different ways for the case of two vector multiplets. Any other five-dimensional half-maximal gauged supergravity with AdS5 vacua preserving 16 supercharges cannot be uplifted to M-theory. Therefore, our results can be seen as a higher-dimensional completion of the five-dimensional analysis of the possible gaugings giving rise to half-maximal AdS5 vacua in [49].

In section 5, we analysed in more detail truncations around vacua whose internal space is locally a $S^4$ fibration over a Riemann surface. As mentioned above and previously shown in [28, 29], these admit an ISO(3) × U(1) truncation around them, which can be constructed via the SO(5) gauged supergravity obtained by reducing 11-dimensional supergravity on $S^4$. We argue that this is the only consistent truncation with vector multiplets that can be obtained in this way.

A natural follow-up question to our work is to better understand the differential equations that arise in the classification of consistent truncations with vector multiplets and to find examples of AdS5 vacua satisfying these conditions, or alternatively, proving that no such vacua can exist. This would finally settle the question of which five-dimensional half-maximal gauged supergravities with half-maximal AdS5 vacua can be uplifted to 11-dimensional supergravity. A particularly interesting example would be the half-maximal theory analysed in [1] which contains RG flows between half-maximal and $N = 2$ AdS5 vacua. This theory belongs to the class we studied in section 4.6. Therefore, if the conditions (4.21) and (4.23) can be solved, the flow constructed in [1] can be immediately uplifted to 11-dimensional supergravity using our results.

Already we were able to find an important result regarding these flows. By giving a higher-dimensional origin to coupling constants appearing in the five-dimensional supergravity theories considered in [1], we showed that these coupling constants have to satisfy a particular quantisation condition. This implies that many of the RG flows constructed in [1] cannot be uplifted to 11-dimensional supergravity with a consistent truncation. Indeed, the only RG flow from an $\mathcal{N} = 4$ to an $\mathcal{N} = 2$ AdS5 vacuum considered in [1] that could arise from a consistent truncation of 11-dimensional supergravity has the famous ratio of central charges given by $\frac{27}{32}$ [55, 56]. The other flows only exist as five-dimensional artefacts without a higher-dimensional completion. This strongly suggests that only those flows dual to adjoint mass deformations in the SCFT can studied holographically.

Finally, ExFT has recently been developed as a powerful tool to study the full Kaluza–Klein spectrum around vacua of gauged supergravities [58, 59]. In these methods, consistent truncations play an important role, since they allow us to study all vacua lying in the same truncation, even those with less or no (super-)symmetry remaining. So far, the method has been restricted to consistent truncations preserving all supercharges [60–65]. However, a systematic construction
of half-maximal truncations can open the door to adapting these methods to consistent truncations preserving fewer supersymmetries. This might give access to the Kaluza–Klein spectra around vacua of half-maximal gauged supergravities which arise as truncations of maximal 10- or 11-dimensional supergravity.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. $E_{6(6)}$ exceptional field theory

In this appendix, we summarise relevant aspects of $E_{6(6)}$ ExFT [66]. This theory is a reformulation of 11-dimensional supergravity based on a split into 5 external and 6 internal directions. The internal degrees of freedom are organised into representations of the exceptional group $E_{6(6)}$. The internal coordinates themselves are embedded into a larger set of generalised coordinates transforming in the $27$ representation of $E_{6(6)}$, but subject to the ‘section condition’

$$d^{MNP} \partial_N \otimes \partial_P = 0,$$

where $M = 1, \ldots, 27$, $d^{MNP}$ is the totally symmetric cubic invariant of $E_{6(6)}$ and the derivatives act on any pair of fields or gauge parameters in the theory. This constraint is solved by restricting the dependence of all fields to a subset of the generalised coordinates, breaking the $E_{6(6)}$ symmetry to some smaller subgroup. Upon solving the section condition in this way, ExFT reduces to 11-dimensional or type II supergravity.

The internal diffeomorphisms and gauge transformations are unified into the notion of generalised diffeomorphism along the generalised internal coordinates. Its action, parameterised by a gauge parameter $\Lambda \in 27$ acting on any generalised vector $V \in 27$, is given by the generalised Lie derivative

$$\mathcal{L}_\Lambda V^M = \Lambda^N \partial_N V^M - V^N \partial_N \Lambda^M + 10 d_{NPQ} V^M \partial_N \Lambda^P \partial_Q \Lambda^N.$$  

(A.2)

Using this definition, the algebra of generalised diffeomorphism closes only upon solving the section condition (A.1).

While the analysis of [63] studies the Kaluza–Klein spectra of vacua of half-maximal gauged supergravity in 3 dimensions, these are assumed to come from consistent truncation of higher-dimensional half-maximal supergravity. Therefore, these consistent truncations preserve all supersymmetries.

Throughout this paper we will take our generalised vector fields to have weight 1/3.

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The field content of $E_{6(6)}$ ExFT is given by [66]

$$\{g_{\mu\nu}, A_\mu^N, B_{\mu\nu}N, \mathcal{M}_{MN}\}, \quad \mu = 0, \ldots, 4,$$  \hspace{1cm} (A.3)

where the external metric $g_{\mu\nu}$ is an $E_{6(6)}$ singlet, the one-form $A_\mu^N$ transforms in the 27 representation and the two-form $B_{\mu\nu}N$ in the $\overline{27}$ representation of $E_{6(6)}$. All purely internal degrees of freedom are encoded into the generalised metric $\mathcal{M}_{MN}$, that parameterise the coset space $E_{6(6)}/USp(8)$.

Finally, throughout this paper, we make use of the following $E_{6(6)}$-invariant algebraic operations: given two vectors in the 27 representation, we define a wedge product $\wedge$ as the algebraic map [22] (the same operation is denoted by $\cdot$ in [67])

$$27 \otimes 27 \rightarrow \overline{27},$$

$$(V_1, V_2) \rightarrow (V_1 \wedge V_2)_M = d_{MNP}V_1^NV_2^P,$$  \hspace{1cm} (A.4)

where $d_{MNP}$ is the $E_{6(6)}$ invariant. Furthermore, we define the $\wedge$ product between a 27 and a $\overline{27}$ representations as

$$27 \otimes \overline{27} \rightarrow 1,$$

$$(V, W) \rightarrow V \wedge W = V^NW_N.$$  \hspace{1cm} (A.5)

By choosing a solution to the section constraint, these wedge products decompose into standard wedge products between differential forms, as we review in appendix A.2.

A.1. Generalised metric from half-maximal structures

In the main part of the paper, we have described half-maximal AdS$_5$ vacua in terms of weakly integrable generalised USp(4) structures. However, these structures also define a generalised metric $\mathcal{M}$, from which the supergravity background fields can be read off using the dictionaries described in appendix A.2. These expression can be obtain constructed analogously to the SO(5, 5) and SL(5) cases constructed in [25]. For the $E_{6(6)}$ case we obtain

$$\mathcal{M}_{MN} = 2 \times 5^{1/3} \left( \frac{1}{\kappa^4} \tilde{J}_{uM}\tilde{J}^u_N + \frac{1}{\kappa^4} K_MK_N - \frac{1}{2\kappa} d_{MNP}\tilde{K}^P \right. \right.$$  

$$\left. - \frac{10}{3\kappa^4} e^{u_1\ldots u_5}(d \cdot \hat{J}_{u_1})_{MP}(d \cdot \hat{J}_{u_2})_{PQ}(d \cdot \hat{J}_{u_3})_{QR}(d \cdot \hat{J}_{u_4})_{RS}(d \cdot \hat{J}_{u_5})_{SN} \right),$$  \hspace{1cm} (A.6)

where we defined

$$\hat{J}_{uM} = d_{MNP}\tilde{J}^N_u,$$

$$(d \cdot \hat{J}_u)_{MN} = d_{MNP}\hat{J}^P_u,$$  \hspace{1cm} (A.7)

Finally, the warp factor of the five-dimensional metric is given by

$$f_1 = |\det g|^{-1/3}\kappa^2,$$  \hspace{1cm} (A.8)

where $g$ is the internal metric that can be read off from (A.6) and the dictionary to 11-dimensional supergravity given in appendix A.2.
Note that the coefficients in (A.6) and (A.8) are not unique since \(J_u\) and \(\hat{K}\) can be rescaled, whilst still satisfying (2.1). Indeed, in [29], the generalised metric was given with different coefficients to our (A.6). As a result of such a rescaling, the coefficients in the differential conditions (2.3) would change. However, the coefficients in (A.6) become unique once we impose the differential conditions (2.3).

A.2. \(E_{6(6)}\) ExFT and 11-dimensional supergravity

By taking the 11-dimensional solution to the section constraint, the \(E_{6(6)}\) symmetry is broken to GL(6). Under such a breaking, a generalised vectors \(J \in 27\) with weight 1/3 and a generalised vector \(K \in 27\) with weight 2/3 decompose as

\[
J = v + \lambda_{(2)} + \lambda_{(5)},
\]
\[
K = \omega_{(1,6)} + \omega_{(3)} + \omega_{(1)},
\]

where \(v\) is a vector and \(\lambda_{(n)}\) and \(\omega_{(n)}\) are forms of degree \(n\). The object \(\omega_{(1,6)}\) transforms as a section of \(T^*M \otimes \Lambda^6 T^*M\). In components we write

\[
J^N = (J^n, J_{n12}, J_{n1...n6}),
\]
\[
K^N = (K^n, K_{n12}, K_{n1...n6}).
\]

The group \(E_{6(6)}\) has a totally symmetric invariant tensor \(d_{MNP}\), whose non-zero components are

\[
d^M_{npqrst} = \sqrt{\frac{2}{5}} \delta^n_p \epsilon_{rs}^{pqst},
\]
\[
d_{npqrst} = \frac{1}{\sqrt{10}} \epsilon_{npqrst},
\]

where we use the conventions where the tensor \(d^{MNP}\) has the same numerical factors as \(d_{MNP}\), and it is normalised as\(^7\)

\[
d_{NPQ} d^{MPQ} = \delta^M_N.
\]

Furthermore, it satisfies the cubic identity

\[
d_{SMN} d_{PQ} d^{STR} = \frac{2}{15} \delta^R_{(M} \delta^N_{P)} q_{QR}.
\]

\(^7\) Note that the numerical factors of \(d_{MNP}\) given in (A.11) differ from those in [66]. The reason is a different summation convention: in ours, we add a factorial factor every time we sum over anti-symmetrised indices, namely,

\[
V^N_{12} V^s_2 = V^N_{12} V^s_2 + \frac{1}{2!} V^N_{1n12} V^s_{n12} + \frac{1}{3!} V^N_{12...ns} V^s_{2...ns},
\]

whereas their summation convention is without the factorial factors.
With the splitting (A.9), the wedge products (A.4) and (A.5) decompose as
\[
(J_u \wedge J_v)_{(1,6)} = \sqrt{\frac{2}{5}} d x^m \otimes \lambda_{m(1)} \wedge \lambda_{m(5)},
\]
\[
(J_u \wedge J_v)_{(4)} = \sqrt{\frac{2}{5}} \left( \frac{1}{2} \lambda_{u(2)} \wedge \lambda_{v(2)} - \iota_{u(\lambda_v(5))} \right),
\]
\[
(J_u \wedge J_v)_{(1)} = -\sqrt{\frac{2}{5}} \iota_{v(\lambda_u(2))},
\]
\[
K \wedge \hat{K} = (\iota_v \omega_{(1,6)})_{(6)} + \lambda_{(2)} \wedge \omega_{(4)} + \lambda_{(5)} \wedge \omega_{(1)},
\]
where $J_u$, $J_v$, $\hat{K} \in \mathbb{C}^7$ and $K \in \mathbb{C}^7$. The object $\lambda_{m(1)}$ in the first line is defined as
\[
\lambda_{m(1)} = \lambda_{mp} d x^p,
\]
where $\lambda_{mp}$ are the components of $\lambda_{(2)}$, i.e. $\lambda_{(2)} = \frac{1}{2} \lambda_{mp} dx^m \wedge dx^p$. Also, the object $(\iota_v \omega_{(1,6)})_{(6)}$ is a contraction of the vector $v$ with the one-form part of $\omega_{(1,6)}$, thus obtaining a six-form:
\[
(\iota_v \omega_{(1,6)})_{(6)} = V^m \omega_{m(6)},
\]
The generalised Lie derivative (A.2) with parameter $J_u \in \mathbb{C}^7$ acting on a vector $J_v \in \mathbb{C}^7$ decomposes under the split (A.9) as
\[
(L_{J_u} J_v)_{(e)} = L_{e\nu} v_{\nu},
\]
\[
(L_{J_u} J_v)_{(2)} = L_{\nu} \lambda_{v(2)} - \iota_{v(\nu)} d \lambda_{u(2)},
\]
\[
(L_{J_u} J_v)_{(5)} = L_{\nu} \rho_{v(5)} - \iota_{v(\nu)} d \rho_{u(5)} - \lambda_{v(2)} \wedge d \lambda_{u(2)}.
\]

A.2.1. 11-dimensional fields from the ExFT generalised metric. As mentioned above, all purely internal degrees of freedom of the 11-dimensional supergravity are encoded into the generalised metric $\mathcal{M}_{MN}$. For instance, the components $\mathcal{M}_{m\bar{n}}$ and $\mathcal{M}_{m\bar{n}}$ are [66]
\[
\mathcal{M}_{m\bar{n}} = (\text{det } g)^{-2/3} g_{m\bar{n}},
\]
\[
\mathcal{M}_{m\bar{n}} = \frac{1}{2} (\text{det } g)^{-2/3} g_{m\bar{n}} C_{m\bar{n}pqr},
\]
where the barred indices in the left-hand side indicate that they correspond to the (dualised) five-form components.

Appendix B. Spheres conventions

We summarise here the notation and conventions for the spheres of different dimensions appearing in the paper. In general, one defines a unit radius sphere $S^d \subset \mathbb{R}^{d+1}$ in terms of $d + 1$ functions $\mathcal{Y}_I$, $I = 1, \ldots, d + 1$ satisfying
\[
\mathcal{Y}_I \mathcal{Y}^I = 1.
\]
\[\text{Note that the numerical factors are different from those appearing in [66]. The reason is the different summation convention as explained in footnote 5.}\]
The round metric on the sphere and its volume form are
\[ ds^2 = dY_I \otimes dY_I, \quad \text{vol}_{S^d} = \frac{1}{d!} \epsilon_{I_1 \ldots I_{d+1}} Y^{I_1} dY^{I_2} \wedge \ldots \wedge dY^{I_{d+1}}. \] (B.2)

This metric space has \( d(d+1)/2 \) Killing vectors which can be labelled by an antisymmetric index \([I,J]\) and, given a local coordinate base \( x^i \), with \( i = 1 \ldots d \), can be written as
\[ V_{[I,J]}^i = 2 \epsilon^{ij} Y_I^j dY^I, \] (B.3)
where \( g^{ij} \) is the inverse metric on the sphere.

### B.1. The two-sphere

The two-sphere \( S^2 \) is defined by three functions \( Y^A \) with \( A = 1, 2, 3 \) and satisfying relation (B.1). The three Killing vectors (B.3) can be organised into
\[ \pounds_V Y^A = \frac{1}{2} \epsilon_{ABC} Y^C. \] (B.4)

Furthermore, we also introduce the one-forms which are Hodge-dual to \( dY_A \), given by
\[ \theta_A = \epsilon_{ABC} Y^B dY^C. \] (B.5)

Together with the other vector and forms on the two-sphere these satisfy the following relations
\[ \iota_V \theta_B = \delta_{AB} - y_A \theta_B, \quad Y_A dY^A = Y_A \theta^A = y_A v^A = 0. \] (B.6)

Finally, all objects transform in a natural way under the SO(3) action generated by the Killing vectors, namely
\[ \pounds_L v_A \theta_B = -\epsilon_{ABC} \theta^C, \quad \pounds_L v_A Y_B = -\epsilon_{ABC} Y^C, \quad \pounds_L \theta_A \theta_B = -\epsilon_{ABC} \partial_C \theta^C. \] (B.7)

### B.2. The one-sphere

The circle \( S^1 \) can be described in terms of two functions \( w^i, i = 1, 2 \) satisfying (B.1). There is a single Killing vector,
\[ v_{S^1} = \frac{1}{2} \epsilon^{ij} w_j. \] (B.8)

The functions \( w^i \) and their external derivatives transform naturally under its action, namely
\[ \pounds_v w^i = -\epsilon^{ij} w^j, \quad \pounds_v \epsilon \epsilon^{ij} w^j, \] (B.9)

Given the set of functions \( w^i \), one can construct another set of functions
\[ \epsilon w^i = \epsilon^{ij} w_j, \] (B.10)

which under the action of \( v_{S^1} \) transform in the same way as \( w^i \). Some useful relations are
\[ dw^i = -\epsilon w^i \text{vol}_{S^1}, \quad d(\epsilon w^i) = w^i \text{vol}_{S^1}. \] (B.11)
Finally, one can also construct functions transforming as \( U(1) \) representation of arbitrary integer weight under the action of \( v_{S^1} \). For instance, if one takes the defining functions of the circle to be

\[
  w_i = (\cos \beta, \sin \beta),
\]

where \( \beta \in [0, 2\pi) \) is the angle on the circle, one can construct the functions

\[
  w_{(q)i} = (\sin(q\beta), \cos(q\beta)),
\]

with \( q \in \mathbb{Z} \), which transform under the action of \( v_{S^1} \) as

\[
  L_{v_{S^1}} w_{(q)i} = -q \epsilon_{ij} w_{(q)j},
  L_{v_{S^1}} d w_{(q)i} = -q \epsilon_{ij} d w_{(q)j},
\]

and analogous with \( \epsilon w_{(q)i} \).

### B.3. The four-sphere

We describe the four-sphere \( S^4 \) in terms of five functions \( Y^I \) satisfying (B.1). Other forms that are relevant for our discussion are

\[
  \sigma_{IJ} = \frac{1}{2} \epsilon_{IJKL} Y^K Y^L, \\
  \beta_I = \frac{1}{3!} \epsilon_{IJKL} Y^{JK} Y^L.
\]

The four-sphere can be described in terms of the defining functions two- and one-spheres by taking

\[
  Y^i = (-r Y^4, \sqrt{1 - r^2} w^i),
\]

where \( Y^4 \) and \( w^i \) are the coordinates for the two- and the one-spheres defined above, namely \( Y_A Y^A = w_i w^i = 1 \), and \( r \in [0, 1) \) is an extra coordinate. The minus sign in front of the \( S^2 \) part ensures that the four-, two- and one-spheres have all positive-definite volume forms.

### Appendix C. SL(5) ExFT and its embedding in \( E_{6(6)} \)

The relevant group for constructing an ExFT adapted to a \( 7 + 4 \) splitting of 11-dimensional supergravity is \( SL(5) \). In this ExFT [33], the generalised internal coordinates transform in the \( 10 \) of \( SL(5) \). Other relevant representations appearing in its construction are the \( 5 \) and \( 5 \).

By taking the 11-dimensional supergravity solution of the section constraint, the \( SL(5) \) group is broken to \( GL(4) \). The objects transforming in the \( 10 \), \( 5 \) and \( 5 \) with weights \( \frac{1}{5}, \frac{1}{5} \) and \( \frac{2}{5} \), respectively, under the generalised Lie derivative decompose as

\[
  V_{10} = v + \lambda_{(2)}, \\
  V_{5} = \lambda_{(3)} + \lambda_{(0)}, \\
  V_{5} = \lambda_{(1)} + \lambda_{(4)},
\]

where \( v \) is a vector and \( \lambda_{(n)} \) an \( n \)-form.
In appendix A.2 we discussed that taking the 11-dimensional supergravity solution to the section constraint breaks the group $E_{6(6)}$ to GL(6). Here we solve the section conditions in a different way, adapted to the case where the internal space has itself a fibration structure with a 4 + 2 split. In particular, this structure breaks $E_{6(6)}$ to SL(5) × GL(2). Then, the usual 11-dimensional solution to the section constraint is recovered by taking the SL(5) solution to the section constraint and letting fields depend also on the GL(2) coordinates. However, for certain calculations it is useful to keep the SL(5) covariance, as we will see now. Under the splitting $E_{6(6)} \rightarrow$ SL(5) × GL(2), an element $V^N$ in the 27 representation of $E_{6(6)}$ decomposes as

$$V = V_{10,(0)} + V_{5,(2)} + V_{5,(1)} + V_6(v),$$

where the bold subscripts indicate the SL(5) representation and the subscript $(n)$ that the object transforms like a GL(2) $n$-form (or vector for $(v)$). In components,

$$V^N = (V^{ab}, V^a_{\alpha\beta}, V_{a\alpha\nu}, V^\nu),$$

where $a, b, = 1, \ldots, 5$ label the fundamental SL(5) indices and $\alpha, \beta = 1, 2$ label the fundamental GL(2) indices. The $E_{6(6)}$ invariant $d_{MN}$ is given by

$$d_{ab\,cd\,e\,f} = \frac{1}{\sqrt{10}}\epsilon_{abcde} \left( \frac{1}{2} \epsilon^{\alpha\beta} \right), \quad d_{ab\,cd\,e\,f} = -\frac{1}{\sqrt{5}} \delta^e_{ab},$$

where the factors match the same conventions as in (A.11). The wedge product (A.4) decomposes as

$$(J_a \wedge J_b)_{ab,\alpha\beta} = \frac{1}{\sqrt{10}} \left( \epsilon_{abcdef} J_a^{\, cd} J_b^{\, de} - 4 J_a^{[a] (v]} J_b^{([b] \alpha)} \right),$$

$$(J_a \wedge J_b)_a = \frac{\sqrt{2}}{5} \left( \epsilon_{abcdef} J_a^{\, be} J_b^{\, de} - J_a^{(a\, \nu)} J_b^{(\nu\, a\, \alpha)} \right),$$

$$(J_a \wedge J_b)^a_a = -\frac{\sqrt{2}}{5} \left( J_a^{\, ab} J_b^{\, b\alpha} + J_a^{(a\, \nu)} J_b^{(\nu\, a\, \alpha)} \right),$$

$$(J_a \wedge J_b)_a (\nu\, \gamma) = \frac{\sqrt{2}}{5} \left( J_a^{\, a\, (a\nu \gamma)} J_b^{(\nu\, a\, \gamma)} \right),$$

and the generalised Lie derivatives (A.2) as

$$(\mathcal{L}_{J_a} J_b)^a_{\alpha\beta} = \mathcal{L}_{(J_a\, b\nu\gamma)} (J_b)^{a\gamma} + L_{(J_a\, b\nu\gamma)} (J_b)_{\gamma\alpha} - L_{(J_a\, b\nu\gamma)} (J_b)_{\gamma\beta} + J_a^{(a\, \nu)} (\partial (J_a\, 5, (1)))^a_{\alpha\beta} + \frac{1}{2} \epsilon_{abcdef} J_a^{\, cd} J_b^{\, de} J_{a\, \nu\, \alpha\beta},$$

$$(\mathcal{L}_{J_a} J_b)^a_{\alpha\beta} = \mathcal{L}_{(J_a\, b\nu\gamma)} (J_b)^{a\gamma} + L_{(J_a\, b\nu\gamma)} (J_b)_{\gamma\alpha} + 2 J_a^{(a\, \nu)} (\partial (J_a\, 5, (2)))^{(\nu\, a\, \beta)} + 2 (\partial (J_a\, 5, (1)))^{(a\, \nu\, (1)) (a\, \nu\, (1))} J_b^{(b\, \beta\, \gamma)},$$

$$+ 2 \partial (J_a^{(a\, \nu\, (1)) (a\, \nu\, (1))} J_b^{(b\, \beta\, \gamma)} - (J_b)^{\gamma\alpha} (\partial (J_a\, 5, (2)))^{(\nu\, a\, \beta)} + 2 (\partial (J_a\, 5, (1)))^{(a\, \nu\, (1)) (a\, \nu\, (1))} J_b^{(b\, \beta\, \gamma)}.$$
\[ (\mathcal{L}_J J_v)_{a_0} = \mathcal{L}_{(J_v)_{10}(J_v)}_{a_0} + \mathcal{L}_{(J_v)_{c_0}(J_v)}_{a_0} - \frac{1}{4} \epsilon_{abcde} J_v^b \delta_c \left( \partial (J_{a_5(1)}) \right)^{d e} - \partial_{ab} J_v^a J_v^b_{a_0} \]
\[ + J_v^a \left( \partial (J_{a_5(2)}) \right)_{a_0} - \partial_{ab} J_v^a J_v^b_{a_0} + J_v^a \partial_{ab} J_v^b_{a_0} - \frac{1}{4} \epsilon_{abcde} \partial_a J_v^b \delta_c \left( J_v^d \right) \]
\[ = \mathcal{L}_{(J_v)_{10}(J_v)}^a + \mathcal{L}_{(J_v)_{c_0}(J_v)}^a - \mathcal{L}_{(J_v)_{10}(J_v)}^a, \tag{C.6} \]

where $\mathcal{L}$ in the right-hand side is the SL(5) generalised Lie derivatives and $L$ is the usual Lie derivative along the GL(2) directions. The operator $\partial$ is the SL(5) ExFT nilpotent differential operator [22, 67] and $d$ is the usual exterior derivative with respect to the GL(2) coordinates.

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