INEQUALITIES FOR THE $q$-GAMMA AND RELATED FUNCTIONS

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Abstract. We consider convexity and monotonicity properties for some functions related to the $q$-gamma function. As applications, we give a variety of inequalities for the $q$-gamma function, the $q$-digamma function $\psi_q(x)$, and the $q$-series. Among other consequences, we improve a result of Azler and Grinshpan about the zeros of the function $\psi_q(x)$. We use $q$-analogues for the Gauss multiplication formula to put in closed form members of some of our inequalities.

1. Introduction

Throughout this paper we assume that $0 < q < 1$. The $q$-shifted factorials of a complex number $a$ are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

For convenience we write

$$(a_1, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n, \quad (a_1, \ldots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.$$

For any complex $x$, we let

$$[x]_q = \frac{1 - qx}{1 - q}.$$

for which we have $\lim_{q \to 1} [x]_q = x$. The $q$-gamma function is given by

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad (|q| < 1).$$

It is clear that

$$\Gamma_q(x + 1) = [x]_q \Gamma_q(x)$$

and it is well-known that $\Gamma_q(z)$ is a $q$-analogue for the function $\Gamma(z)$, see Askey [3].

The digamma function is

$$\psi(x) = \left( \log \Gamma(x) \right)' = \frac{\Gamma'(x)}{\Gamma(x)}$$

and its $q$-analogue is given by

$$\psi_q(x) = \left( \log \Gamma_q(x) \right)' = \frac{\Gamma'_q(x)}{\Gamma_q(x)}.$$
The $q$-binomial theorem states that

$$
\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} =: \phi_0(a, -; q, x) \quad (|x| < 1, \ |q| < 1),
$$

where $\phi_0(a, -; q, x)$ is the basic hypergeometric series. For details and historical notes on the $q$-series, the hypergeometric series, and related functions we refer to [4 7]. Our primary goal in this paper is to consider monotonicity and convexity properties of the $q$-gamma function and some of its related functions. As an application, we shall present inequalities involving the functions $\Gamma_q(x)$ and $\psi_q(x)$ along with related functions including the function $\phi_0(a, -; q, x)$. Some of our inequalities involve powers, ratios, and products of these special functions. A crucial tool to achieve some of our inequalities is Jensen’s inequality stating that if $f(x)$ is a convex function on $I$ then for all $x_1, \ldots, x_n \in I$ and all positive $a_1, \ldots, a_n$ one has

$$f\left(\sum_{i=1}^{n} a_i x_i \right) \leq \sum_{i=1}^{n} a_i f(x_i).$$

We mention that refinements of Jensen’s inequality exist in literature and thus any inequality we prove in this paper using Jensen’s inequality can be slightly improved. An example of such a refinement, found by Dragomir et al [6], states that for a real convex function $f(x)$ defined on the interval $I$ one has

$$\frac{1}{k} \sum_{i=1}^{k} f(x_i) - f\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) \geq \max_{1 \leq i \leq j \leq k} \left\{ \frac{f(x_i) + f(x_j)}{2} - f\left(\frac{x_i + x_j}{2}\right) \right\} \geq 0.$$ 

Letting $A_f(x, k)$ denote the middle member of the previous double inequality, we deduce the following refinement of Jensen’s inequality

$$f\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) \leq \frac{1}{k} \sum_{i=1}^{k} f(x_i) - A_f(x, k).$$

For some of other refinements of Jensen’s inequality, see [3 14 15]. Azler and Grinshpan [2 Lemma 4.5] proved that the function $\psi_q(x)$ for $0 < q \neq 1$ has a uniquely determined positive zero $x_0 = x_0(q)$. Among our applications, we shall show that $x_0(q) \in (1, 2)$. We will also provide Ky Fan type inequalities for the $q$-gamma function. Another purpose of our work is to establish a variety of inequalities involving the $q$-series. We note that some of our formulas have been put in closed forms thanks to $q$-analogues of the Gauss multiplication formula for the gamma function which we shall describe now. We recall that the Gaussian multiplication formula for gamma function states that

$$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \cdots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}} \quad (n = 1, 2, \ldots).$$

A famous $q$-analogue for [3] due to Jackson [9 10], (see [7 p. 22]), states that

$$\left(\frac{1 - q^n}{1 - q}\right)^{nz - 1} \Gamma_{q^n}(z) \Gamma_{q^n}\left(z + \frac{1}{n}\right) \cdots \Gamma_{q^n}\left(z + \frac{n-1}{n}\right) = \Gamma_{q^n}(nz) \Gamma_{q^n}\left(\frac{1}{n}\right) \Gamma_{q^n}\left(\frac{2}{n}\right) \cdots \Gamma_{q^n}\left(\frac{n-1}{n}\right) \quad (n = 1, 2, \ldots).$$
Recently, the authors [4] gave the following $q$-analogue for (3)

\[
\prod_{k=1}^{n-1} \Gamma_q \left( \frac{k}{n} \right) = \left( \Gamma_q \left( \frac{1}{2} \right) \right)^{n-1} \frac{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty^{n-1}}{(q; q)_\infty^{n-2}(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty}.
\]

Besides, Sándor and Tóth [19] found

\[
P(n) := \prod_{(k, n) = 1} \Gamma \left( \frac{k}{n} \right) = \frac{(2\pi) \frac{\varphi(n)}{n^2} \prod_{d \mid n} (\frac{q^{\frac{1}{n}}}{q^{\frac{1}{n}}}; q^{\frac{1}{n}})}{e^{\frac{\Lambda(n)}{2}}},
\]

where $\varphi(n)$ is the Euler totient function, $\mu(n)$ is the Möbius mu function, and $\Lambda(n)$ is the Von Mangoldt function. We accordingly let

\[
P_q(n) = \prod_{(k, n) = 1} \Gamma_q \left( \frac{k}{n} \right).
\]

The authors [4] also found the following $q$-analogue (6).

\[
\prod_{k=1}^{n-1} \Gamma_q \left( \frac{k}{n} \right) = \left( \Gamma_q \left( \frac{1}{2} \right) \right)^{n-1} \frac{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty^{n-1}}{(q; q)_\infty^{n-2}(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty}.
\]

2. INEQUALITIES FOR $\psi_q(x)$ AND $\Gamma_q(x)$

Lemma 1. (a) The derivative of the function $\psi_q(x)$ is strictly completely monotonic on $(0, \infty)$, that is,

\[
(-1)^n (\psi_q(x))^{(n)} > 0 \quad x > 0, \quad n = 0, 1, 2, \ldots
\]

(b) For any $x \geq 1$, we have that $x (\psi_q(x))' + 2 \psi_q(x) > 0$.

Proof. Part (a) is an immediate consequence of the series representation

\[
\psi_q(x) = -\log(1 - q) + (\log q) \sum_{n=1}^{\infty} \frac{q^{n+x}}{1 - q^n + x} = -\log(1 - q) + (\log q) \sum_{n=1}^{\infty} \frac{q^{n+x}}{1 - q^n}.
\]

Part (b) is due Azler and Grinshpan [2, Lemma 3.4].

Lemma 2. For all $x > 0$, we have

\[
\frac{q^x \log q}{1 - q^x} + \log[x]_q < \psi_q(x) < \log[x]_q.
\]

Proof. From (11), we deduce that $\log \Gamma_q(x + 1) - \log \Gamma_q(x) = \log[x]_q$. Then by Lagrange mean value theorem, there exists $t \in (0, 1)$ such that

\[
\psi_q(x + t) = \log[x]_q.
\]

As $\psi_q(x)$ is strictly increasing by Lemma 1 the forgoing identity implies that

\[
\psi_q(x) < \psi_q(x + t) < \psi_q(x + 1).
\]

Next, differentiate both sides of (11) to obtain

\[
\psi_q(x) = \frac{q^x \log q}{1 - q^x} + \psi_q(x + 1).
\]

Now, combine (8), (9), and (10) to get the desired inequalities. □
Azler and Grinshpan [2, Lemma 4.5] proved that the function $\psi_q(x)$ for $0 < q \neq 1$ has a uniquely determined positive zero $x_0 = x_0(q)$. For $0 < q < 1$, it turns out that $x_0(q) \in (0, 1)$ as we will see now.

**Theorem 1.** (a) The function $\psi_q(x)$ has a unique zero $x_0$ in the interval $(1, 2)$.

(b) There holds $\Gamma_q(x) \geq \Gamma_q(x_0)$ for all $x \in (0, \infty)$.

**Proof.** First proof of (a) Application of Lemma 2 to $x = 1$ and to $x = 2$ respectively gives

$$\frac{q \log q}{1 - q} < \psi_q(1) < 0 \quad \text{and} \quad \frac{q^2 \log q}{1 - q^2} + \log(1 + q) < \psi_q(2).$$

By the well-known fact that the function $\psi(x)$ is strictly increasing and continuous we will be done if we show that

$$\frac{q^2 \log q}{1 - q^2} + \log(1 + q) > 0.$$  \hspace{1cm} \text{(12)}

Letting $q = \frac{1}{t}$ for $t > 1$ and after simplification (12) becomes

$$\log t < \log(t + 1) - \frac{\log t}{t^2 - 1},$$

or equivalently,

$$(t^2 - 1) \log(t + 1) - t^2 \log t > 0.$$  \hspace{1cm} \text{(12)}

Letting $f(t) = (t^2 - 1) \log(t + 1) - t^2 \log t$, we find that

$$f'(t) = 2t \log(t + 1) - 2t \log t - 1,$$

$$f''(t) = 2 \left( \log \left(1 + \frac{1}{t}\right) - \frac{1}{t + 1} \right).$$

Then by a combination of the previous identity and the well-known inequality $\left(1 + \frac{1}{t}\right)^{t+1} > e$, we deduce that $f''(t) > 0$ from which it follows that $f'(t)$ is strictly increasing. Then from the above, $f'(t) = f'(1) = 2 \log 2 - 1 > 0$, which in turn shows that $f(t)$ is strictly increasing. Therefore $f(t) > f(1) = 0$, establishing the relation (12).

*Second proof of (a)* As $\Gamma_q'(1) = \Gamma_q'(2) = 1$, we have by Rolle’s theorem applied to $\Gamma_q(x)$ on $[1, 2]$ there exists $x_0 \in (1, 2)$ such that $(\Gamma_q(x_0))' = 0$ and hence $\psi_q(x_0) = 0$ as $(\Gamma_q(x))' = \psi_q(x)\Gamma_q(x)$. Since $\Gamma_q(x)$ is strictly convex, its derivative is strictly increasing, and so $x_0$ is unique.

(b) It is well-known that a strict log-convex function is also strict convex and so, $\Gamma_q(x)$ is strict convex on $(0, \infty)$ by Lemma 1(a). That is, $(\Gamma_q(x))'$ is strictly increasing on $(0, \infty)$. Now combine this with the identity $(\Gamma_q(x))' = \psi_q(x)\Gamma_q(x)$ as follows. Then $(\Gamma_q(x))' = 0$ on the left of $x_0$ and $(\Gamma_q(x))' \geq 0$ on the right of $x_0$, showing that the function $\Gamma_q(x)$ is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, \infty)$. This completes the proof.

**Theorem 2.** (a) The function $\log \Gamma_q(x) + x\psi_q(x)$ is strictly increasing on $(1, \infty)$ with a single zero which is in $(1, 2)$.

(b) The function $\log \Gamma_q(x) - x\psi_q(x)$ is strictly decreasing on $(0, \infty)$ with a single zero which is in $(1, 2)$. 

Proof. Let $f(x) = \log \Gamma_q(x) + x\psi_q(x)$. Then by Lemma 3(b), $f'(x) = 2\psi_q(x) + x\psi_q'(x) > 0$ and therefore $f(x)$ is strictly increasing on $(1, \infty)$. We have already seen in the proof of Theorem 3 that $\psi_q(1) < 0 < \psi_q(2)$. It follows that $f(1) = \psi_q(1) < 0 < \psi_q(2) = f(2)$. As the function $f(x)$ is clearly continuous on $(1, \infty)$, the proof is complete for part (a). Part (b) follows in exactly the same way.

Corollary 1. For any $x > 1$ and any positive integer $n$ we have the following double inequality

$$x\psi_q(x) - (x + n)\psi_q(x + n) < \log \frac{\Gamma_q(x)}{\Gamma_q(x + n)} < (x + n)\psi_q(x + n) - x\psi_q(x).$$

Proof. By Theorem 2(a), we have $\log \Gamma_q(x) + x\psi_q(x) < \log \Gamma_q(y) + y\psi_q(y)$ whenever $1 < x < y$. Repeatedly application of this and simplifying yield

$$\log \Gamma_q(x) - \log \Gamma_q(x + 1) < (x + 1)\psi_q(x + 1) - x\psi_q(x)$$

$$\log \Gamma_q(x + 1) - \log \Gamma_q(x + 2) < (x + 2)\psi_q(x + 2) - (x + 1)\psi_q(x + 1)$$

$$\vdots$$

$$\log \Gamma_q(x + n - 1) - \log \Gamma_q(x + n) < (x + n)\psi_q(x + n) - (x + n - 1)\psi_q(x + n - 1).$$

Adding together gives

$$\log \Gamma_q(x) - \log \Gamma_q(x + n) < -x\psi_q(x) + (x + n)\psi_q(x + n),$$

which is equivalent to the first inequality. The second inequality is obtained similarly by considering the function $\log \Gamma_q(x) - x\psi_q(x)$ which is decreasing by Theorem 2(a).

Lemma 3. For any positive integers $k$ and $n$ there holds

(a) $\sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right) = (n-1)\psi_q(1) - n\log \frac{1-q}{1-q^n}$

(b) $\sum_{i=1}^{n-1} \psi_q^{(k)}\left(\frac{i}{n}\right) = (n^{k+1} - 1)\psi_q^{(k)}(1)$.

Proof. In 3 replacing $q^n$ with $q$ and taking logarithms on both sides give

$$\log \frac{1-q}{1-q^n} + \sum_{i=0}^{n-1} \log \Gamma_q(z + \frac{i}{n}) = \log \Gamma_q(nz) + \log \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right).$$

Differentiating with respect to $z$ and then letting $z = \frac{1}{n}$ yield

$$n \log \frac{1-q}{1-q^n} + \sum_{i=1}^{n-1} \psi_q\left(\frac{i}{n}\right) + \psi_q(1) = n\psi_q(1),$$

which is equivalent to the desired identity in part (a). To prove part (b), first differentiate with respect $z$, $k$ times both sides of (13) to obtain

$$\sum_{i=0}^{n-1} \psi_q^{(k)}\left(z + \frac{i}{n}\right) = n^{k+1}\psi_q^{(k)}(nz),$$
then let $z = \frac{1}{n}$ to get
\[\sum_{i=1}^{n-1} \psi_q^{(k)} \left( \frac{i}{n} \right) = (n^{k+1} - 1)\psi_q^{(k)}(1),\]
as desired. □

**Theorem 3.** For any positive integers $k$ and $n$ there holds
\begin{align*}
(a) & \quad (n-1)\left(\psi_q(1) - \psi_q \left( \frac{1}{2} \right) \right) < n \log \frac{1-q}{1-q^n} \\
(b) & \quad (n-1)\psi_q^{(2k-1)} \left( \frac{1}{2} \right) < (n^{2k}-1)\psi_q^{(2k-1)}(1) \\
(c) & \quad (n-1)\psi_q^{(2k)} \left( \frac{1}{2} \right) > (n^{2k}-1)\psi_q^{(2k)}(1).
\end{align*}

**Proof.** (a) By Lemma 1(a), the function $\psi_q(x)$ is strictly concave. Then by an application of Jensen’s inequality to this function with $k = n - 1$ and $x_i = \frac{i}{n}$ for $i = 1, \ldots, n - 1$ we find
\[\psi_q \left( \frac{\sum i}{n-1} \right) > \frac{1}{n-1} \sum \psi_q \left( \frac{i}{n} \right).\]
Then by an appeal to Lemma 3(a) along with simplification we derive
\[\psi_q \left( \frac{\sum i}{n-1} \right) > (n-1)\psi_q \left( \frac{1}{2} \right) - n \log \frac{1-q}{1-q^n},\]
which proves part (a).

(b) By Lemma 1(a), the function $\psi_q^{(2k-1)}(x)$ is strictly convex and therefore by Jensen’s inequality applied to this function with $k = n - 1$ and $x_i = \frac{i}{n}$ for $i = 1, \ldots, n - 1$ one has
\[\psi_q^{(2k-1)} \left( \frac{\sum i}{n-1} \right) < \frac{1}{n-1} \sum \psi_q^{(2k-1)} \left( \frac{i}{n} \right).\]
Now use Lemma 3(b) and simplify to deduce that
\[\psi_q^{(2k-1)} \left( \frac{1}{2} \right) < (n^{2k}-1)\psi_q^{(2k-1)}(1),\]
which is the desired relation in part (b). The similar proof of part (c) is omitted. □

3. Convexity and inequalities for powers, ratios, and products of $\Gamma_q(x)$

**Lemma 4.** Let $f : (0, \infty) \to (0, \infty)$ and let $g(x) = \frac{f(x+1)}{f(x)}$. If $f(x)$ is strictly log-convex on $(0, \infty)$, then for any $x > 0$ and any $a \in (0, 1)$ we have
\[(g(x))^{1-a} < \frac{f(x+1)}{f(x+a)} < (g(x+a))^{1-a}.\]

**Proof.** $f(x)$ is strictly log-convex on $(0, \infty)$ we have for any $u \in [0, 1]$ and any $y \neq z > 0$
\[\log \left( f(uy + (1-u)z) \right) < u \log f(y) + (1-u) \log f(z)\]
or equivalently,

\[(14) \quad f(uy + (1-u)z) < (f(y))^u (f(z))^{1-u}.\]

Let in (14) \( y := x, \ z := x + 1, \) and \( u := 1 - a \) to obtain

\[f(x + a) < (f(x))^{1-a} (f(x + 1))^{a},\]

from which we easily get the first inequality. As to the second inequality, let in (14) \( y := x + a, \ z := x + a + 1, \) and \( u := a \) and proceed as before. □

A classical result by Gautschi [8] states that

\[x^{1-a} < \frac{\Gamma(x+1)}{\Gamma(x+a)} < e^{(1-a)\psi(x+1)} \quad (0 < a < 1).\]

We have the following \(q\)-variant which seems to be new.

**Corollary 2.** Let \( x > 0 \) and \( a \in (0, 1) \). Then

\[
([x]_q)^{1-a} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+a)} < ([x+1]_q)^{1-a}.
\]

**Proof.** Simply apply Lemma 4 to the function \( f(x) = \log \Gamma_q(x) \). □

The following result is well-known.

**Lemma 5.** Let \( I \subseteq \mathbb{R} \) and let \( f : I \to (0, \infty) \).

(a) If \( f(x) \) is concave (strictly concave), then \( \frac{1}{f(x)} \) is convex (strictly convex).

(b) The function \( f(x) \) is log-convex if and only if \( \frac{1}{f(x)} \) is log-concave.

Note that the converse of Lemma 5(a) is not true. For example, the function \( e^x \) is convex but the reciprocal \( e^{-x} \) is not concave.

**Theorem 4.** The function \( (\Gamma_q(x+1))^{\frac{1}{x}} \) is strictly log-concave on \((0, \infty)\).

**Proof.** Letting \( f(x) = (\Gamma_q(x+1))^{\frac{1}{x}} \), we find

\[(15) \quad x^3 \left( \log f(x) \right)^{''} = x^2 \left( \psi_q(x+1) \right)^'' - 2x \psi_q(x+1) + 2 \log \Gamma_q(x+1)
\]

and letting \( h(x) = x^2 \left( \psi_q(x+1) \right)^' - 2x \psi_q(x+1) + 2 \log \Gamma_q(x+1) \), we have \( h'(x) = x^2 \left( \psi_q(x+1) \right)^{''} \) and so \( h'(x) < 0 \) by Lemma 1(a), that is, \( h(x) \) is strictly decreasing on \((0, \infty)\). Then \( h(x) < h(0) = 0 \), which combined with equation (15) implies that \( \log f(x) \) is strictly concave on \((0, \infty)\) and therefore, the desired statement follows. □

**Corollary 3.** The function \( \frac{1}{\Gamma_q(x+1)^{\frac{1}{x}}} \) is strictly log-convex on \((0, \infty)\).

**Proof.** This follows by Theorem 4 and Lemma 5(b). □

A classical gamma version for Corollary 3 is due to Van de Lune [21].

**Corollary 4.** Let

\[g(x) = \left( \frac{\Gamma_q(x+1)}{[x+1]_q} \right)^{\frac{1}{x}}.\]
Then for any \( x > 0 \) and any \( a \in (0, 1) \), we have
\[
(g(x))^{1-a} < \left( \frac{\Gamma_q(x + a + 1)}{\Gamma_q(x + 2)} \right)^{1/2} < (g(x + a))^{1-a}.
\]

**Proof.** This is a direct consequence of Corollary [2] and Lemma [2] and the basic fact that \( \Gamma_q(x + 2) = [x + 1]_q \Gamma_q(x + 1) \).

**Theorem 5.** The function \( \left( \Gamma_q(x) \right)^{\frac{1}{x}} \) is strictly log-convex on \((0, 1]\).

**Proof.** Letting \( f(x) = \left( \Gamma_q(x) \right)^{\frac{1}{x}} \), we get
\[
(\log f(x))'' = x^3 (\log f(x))'' = x^2 (\psi_q(x))' - 2x \psi_q(x) + 2 \log \Gamma_q(x + 1).
\]
Now for the function \( h(x) = x^2 (\psi_q(x))' - 2x \psi_q(x) + 2 \log \Gamma_q(x + 1) \) we find \( h'(x) = x^2 (\psi_q(x))'' < 0 \), that is, the function \( h(x) \) decreases on \((0, 1]\). Then for any \( x \in (0, 1] \) we have with the help of Lemma [1](a) and inequality (11)
\[
h(x) \geq h(1) = \psi_q'(1) - 2 \psi_q(1) > 0.
\]
Thus \( (\log f(x))'' > 0 \) on \((0, 1]\), or equivalently, \( f(x) = \left( \Gamma_q(x) \right)^{\frac{1}{x}} \) is log-convex on \((0, 1]\).

We note that Theorem [4] and Theorem [5] were motivated by results of the second author in Sándor [16] on Euler gamma function. See also [17, 18] for other related results on Euler gamma function.

**Theorem 6.** The function \( \left( \Gamma_q(x) \right)^{x} \) is strictly log-convex on \([1, \infty)\).

**Proof.** We have by a straight computation and Lemma [1](b),
\[
\left( \log (\Gamma_q(x)) \right)^{''} = 2 \psi_q(x) + x (\psi_q(x))' > 0,
\]
showing the desired statement.

**Lemma 6.** Let \( f(x) \) be strictly log-convex on the interval \((0, 1)\). Then we have
\[
\frac{f(1 - 2x(1 - x))}{f(1 - x)} < \frac{f(x)}{f(1 - x)} < \frac{f(2x(1 - x))}{f(1 - x)}.
\]

**Proof.** As \( f(x) \) is strictly log-convex we have for any \( a \in (0, 1) \)
\[
f(a(1 - x) + (a - 1)x) < (f(1 - x))^a (f(x))^{1-a},
\]
which by letting \( a = 1 - x \) means
\[
f(2x^2 - 2x + 1) < (f(1 - x))^{1-x} (f(x))^x.
\]
It follows that
\[
\frac{f(1 - 2x(1 - x))}{f(1 - x)} < \left( \frac{f(x)}{f(1 - x)} \right)^x.
\]
which is the first desired inequality. As to the second inequality, as \( f(x) \) is strictly log-convex we also have for any \( a \in (0, 1) \)

\[
 f(ax + (a - 1)(1 - x)) < (f(x))^a (f(1 - x))^{1 - a},
\]
which by letting \( a = 1 - x \) means

\[
 f(2x(1 - x)) < (f(x))^{1 - x} (f(1 - x))^x,
\]
or equivalently,

\[
 \frac{f(x)}{f(2x(1 - x))} > \left( \frac{f(x)}{f(1 - x)} \right)^x.
\]
This completes the proof. \( \Box \)

**Corollary 5.** For any \( x \in (0, 1) \) we have

\[
\begin{align*}
(a) & \quad \Gamma_q(2x(1 - x)) \Gamma_q(1 - 2x(1 - x)) < \Gamma_q(x) \Gamma_q(1 - x) \\
(b) & \quad \psi_q(2x(1 - x)) \psi_q(1 - 2x(1 - x)) > \psi_q(x) \psi_q(1 - x).
\end{align*}
\]

**Proof.** Let \( f(x) = \Gamma_q(x) \Gamma_q(1 - x) \). Then \( f(x) \) is strictly log-convex being the product of two log-convex functions and we clearly have \( f(x) = f(1 - x) \). Then by virtue of Lemma 6 we get

\[
\frac{\Gamma_q(2x(1 - x)) \Gamma_q(1 - 2x(1 - x))}{\Gamma_q(x) \Gamma_q(1 - x)} < 1 < \frac{\Gamma_q(x) \Gamma_q(1 - x)}{\Gamma_q(2x(1 - x)) \Gamma_q(1 - 2x(1 - x))}.
\]
The simplifying gives part (a). As to part (b), let \( g(x) = \psi_q(x) \psi_q(1 - x) \). Then by Lemma 1(a) we find

\[
(\log g(x))'' = (\psi(x))'' + (\psi(1 - x))'' < 0,
\]
showing by Lemma 5 that the reciprocal \( \frac{1}{g(x)} \) is strictly log-convex. Moreover, it is clear that \( g(x) = g(1 - x) \). Then from Lemma 6 we have

\[
\frac{\psi_q(x) \psi_q(1 - x)}{\psi_q(2x(1 - x)) \psi_q(1 - 2x(1 - x))} < 1.
\]
This completes the proof. \( \Box \)

For our next result, we need the following result of Vasić [20] which is an extension of a famous inequality of Petrović. We refer to [11] for details about Petrović’s inequality.

**Lemma 7.** Let \( f : [0, \infty) \to \mathbb{R} \) be convex. Then for any \( x_1, \ldots, x_n \geq 0 \) and any \( p_1, \ldots, p_n \geq 1, \) we have

\[
\sum_{i=1}^{n} p_i f(x_i) \leq f \left( \sum_{i=1}^{n} \frac{p_i}{p_i - 1} f(x_i) \right).
\]

**Corollary 6.** For any real numbers \( x_1, \ldots, x_n \geq 0 \), we have

\[
\prod_{i=1}^{n} \Gamma_q(x_i) \leq \left[ \frac{\sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i} \right]^q \Gamma_q \left( \sum_{i=1}^{n} x_i \right).
\]

**Proof.** Simply apply Lemma 6 to \( f(x) = \log \Gamma_q(x + 1) \), \( p_1 = \ldots = p_n = 1 \) and note that \( f(0) = 0 \). \( \Box \)
4. Inequalities related to $\frac{\Gamma_q(1-x)}{\Gamma_q(x)}$ and $\Gamma_q\left(\frac{1-x}{x}\right)$

**Theorem 7.** (a) The function $\Gamma_q\left(\frac{1-x}{x}\right)$ is strictly log-convex on $(0, \frac{1}{2}]$.
(b) The function $\frac{\Gamma_q(1-x)}{\Gamma_q(x)}$ is strictly log-concave on $(0, \frac{1}{2}]$.

**Proof.** (a) Let $f(x) = \log \Gamma_q\left(\frac{1-x}{x}\right)$. Then
\[
f_1'(x) = -x^{-1} \psi_q\left(\frac{1-x}{x}\right),
\]
\[
f_1''(x) = \frac{2x}{x^2} \psi_q\left(\frac{1-x}{x}\right) + \frac{1}{x^3} \psi_q'\left(\frac{1-x}{x}\right),
\]
and so, $x^3 f_1''(x) = 2\psi_q(y) + \frac{1}{2} \psi_q'(y)$ where $y = \frac{1-x}{x}$. Noting that $y < \frac{1}{2}$ and that $\frac{1-x}{x} \geq 1$ on $(0, \frac{1}{2}]$, we get with the help of Lemma 1(b) that
\[
x^3 f_1''(x) > 2\psi_q(y) + y \psi_q'(y) > 0.
\]

It follows that $f_1''(x) > 0$ and thus $\Gamma_q\left(\frac{1-x}{x}\right)$ is strictly log-convex on $(0, \frac{1}{2}]$.

(b) Let $f_2(x) = \log \frac{\Gamma_q(1-x)}{\Gamma_q(x)}$. Then $f_2''(x) = \psi_q'(-1-x) - \psi_q'(x)$. As $1-x > x$ on $(0, \frac{1}{2})$ and the function $\psi_q(x)$ is strictly decreasing by Lemma 1(a), we deduce that $f_2''(x) < 0$. This completes the proof. \[\square\]

For our next result we need the following lemma.

**Lemma 8.** There holds
\[\psi_q\left(\frac{1}{2}\right) < 2\psi_q(1).\]

**Proof.** By virtue of Lemma 3 applied to $n = 2$, we have $\psi_q\left(\frac{1}{2}\right) = \psi_q(1) - 2 \log(1 + q\frac{1}{2})$ and so our desired inequality means that $\psi_q(1) > -2 \log(1 + q\frac{1}{2})$. Now, by a combination of Lemma 2 applied to $x = 1$ and the relation (12), we get $\psi_q(1) > \frac{2 \log q}{1-q} - 2 \log(1 + q\frac{1}{2})$, which completes the proof. \[\square\]

**Theorem 8.** For any $x \in (0, \frac{1}{2}]$, one has
\[
\frac{\Gamma_q(1-x)}{\Gamma_q(x)} \leq \Gamma_q\left(\frac{1-x}{x}\right),
\]
with equality only for $x = \frac{1}{2}$.

**Proof.** Let
\[
f(x) = \log \Gamma_q\left(\frac{1-x}{x}\right) + \log \Gamma_q(x) - \log \Gamma_q(1-x).
\]
As $f\left(\frac{1}{2}\right) = 0$, it will be enough to prove that $f(x)$ is decreasing on $(0, \frac{1}{2}]$. Note first that from Theorem 7(a) we have that $\left(\log \Gamma_q\left(\frac{1-x}{x}\right)\right)'$ is increasing on $(0, \frac{1}{2}]$, which implies that
\[
\left(\log \Gamma_q\left(\frac{1-x}{x}\right)\right)' = -\frac{x^{-1}}{x^2} \psi_q\left(\frac{1-x}{x}\right) \leq -4\psi_q(1).
\]
Moreover, since the function $\psi_q(x)$ is concave by Lemma 1(a), we have
\[
\psi_q(x) + \psi_q(1-x) \leq 2\psi_q\left(\frac{x + (1-x)}{2}\right) = 2\psi_q\left(\frac{1}{2}\right).
\]
Now as 

\[ f'(x) = \left( \log \Gamma_q\left( \frac{1-x}{x} \right) \right)' + \psi_q(x) + \psi_q(1-x), \]

we deduce from Lemma 8 and the above facts that 

\[ f'(x) \leq -4\psi_q(1) + 2\psi_q(\frac{1}{2}) = 2\left( \psi_q\left( \frac{1}{2} \right) - 2\psi_q(1) \right) \leq 0. \]

showing that \( f(x) \) is decreasing. This completes the proof. \( \square \)

Recall the notation \( x_0 \) from Theorem 1, which stands for the zero of the function \( \psi_q(x) \) in \((1, 2)\).

**Theorem 9.** There holds

(a) \( \frac{\Gamma_q(1-x)}{\Gamma_q(x)} \leq 1 \) on \((0, \frac{1}{2})\),

(b) \( \Gamma_q\left( \frac{1-x}{x} \right) \geq 1 \) on \((0, \frac{1}{2})\),

(c) \( \Gamma_q\left( \frac{1-x}{x} \right) \leq 1 \) on \([\frac{1}{3}, \frac{1}{2}]\).

**Proof.** Throughout the proof, we let for convenience 

\[ f_1(x) = \log \Gamma_q\left( \frac{1-x}{x} \right) \quad \text{and} \quad f_2(x) = \log \frac{\Gamma_q(1-x)}{\Gamma_q(x)}. \]

As \( \psi_q(x) \) increases by Lemma 2 and \( \psi_q(1) < 0 \) by Lemma 2, we have \( \psi_q(x) < 0 \) on \((0, 1)\) and therefore,

\[ (f_2(x))' = -\psi_q(1-x) - \psi_q(x) > 0, \]

showing that \( f_2(x) \) is strictly increasing on \((0, 1)\). Now as \( f_2\left( \frac{1}{2} \right) = 0 \) we get \( f_2(x) < 0 \) on \((0, \frac{1}{2})\), from which part (a) follows. We now establish the inequalities in the remaining parts by an investigation of the function \( f_1(x) \). We know by Theorem 1 that \( x' \) is a zero for \( \psi_q\left( \frac{1-x}{x} \right) \) if and only if \( x_0 = \frac{1-x'}{x'} \) or \( x' = \frac{1}{1+x_0} \). If \( x > x' \), then it is easy to see that \( \frac{1-x}{x} < x_0 \) and so, \( \psi_q\left( \frac{1-x}{x} \right) < \psi_q(x_0) = 0 \). Similarly, if \( x < x' \), then \( \psi_q\left( \frac{1-x}{x} \right) > 0 \). Noting moreover that \( (f_1(x))' = -\frac{1}{x} \psi_q\left( \frac{1-x}{x} \right) \), we derive that \( f_1(x) \) is strictly decreasing on \((0, x')\), strictly increasing on \((x', \frac{1}{2})\) and it has a minimum at \( x' \). Furthermore, from the basic facts \( f_1\left( \frac{1}{2} \right) = 0 \) and \( \lim_{x \to 0^+} f_1(x) = \infty \), we have that \( f_1(x) \) has a unique zero in \((0, x')\). In fact, since \( f_1\left( \frac{1}{2} \right) = \log \Gamma_q(2) = 0 \), this zero is \( \frac{1}{2} \). To summarize, we have proved so far that \( f_1(x) > 0 \) on \((0, \frac{1}{2})\) and that \( f_1(x) < 0 \) on \([\frac{1}{3}, \frac{1}{2}]\). In other words, parts (b) and (c) have been established. This completes the proof. \( \square \)

We close this section with an inequality of Ky Fan type for the \( q \)-gamma function. For Ky Fan inequalities related to the classical gamma function the reader is referred to Neuman and Sándor [13].

**Theorem 10.** For a positive integer \( k \) and \( i = 1, 2, \ldots, k \), let \( x_i \in (0, \frac{1}{2}] \) and let \( x'_i = 1 - x_i \). Let \( A_k \) denote the arithmetic mean of \( x_i \) and let \( A_k' \) denote the arithmetic mean of \( x'_i \). Then

(a) \[ \frac{\Gamma_q\left( \frac{A_k'}{A_k} \right)}{\Gamma_q\left( \frac{A_k}{A_k} \right)} \leq \left( \prod_{i=1}^{k} \Gamma_q\left( \frac{x_i}{x'_i} \right) \right)^{\frac{1}{k}} \]

(b) \[ \frac{\Gamma_q(A_k)}{\Gamma_q(A_k)} \geq \left( \prod_{i=1}^{k} \Gamma_q(x'_i) \right)^{\frac{1}{k}}. \]
Proof. As the function \( \Gamma_q \left( \frac{1-x}{x} \right) \) is strictly log-convex on \((0, \frac{1}{2}]\) by Theorem 7(a), an application of Jensen’s inequality to this function yields part (a). Moreover, an application of Jensen’s inequality to \( \frac{\Gamma_q(1-x)}{\Gamma_q(x)} \), which is strictly log-concave by Theorem 7(b), gives to part (b). \( \square \)

5. Inequalities related to \( q \)-series

Our first inequalities involves the basic hypergeometric series.

**Theorem 11.** For any positive integer \( n \), any \( x > 0 \), and any \( a \in (0, 1) \), we have

\[
\begin{align*}
(1 - q^x)^{1-a} &< \left( \frac{q^{x+a}; q}{q^{x+1}; q} \right)_n < (1 - q^{x+a})^{1-a} \\
(1 - q^x)^{1-a} &< 1 \phi_0(q^{a-1}, -; q, q^{x+1}) \leq (1 - q^{x+a})^{1-a}.
\end{align*}
\]

Proof. Let \( f(x) = (q^x; q)_n \). Then from \( f(x) = \sum_{i=0}^{n-1} \log(1 - q^{x+i}) \), we get

\[
\left( \log f(x) \right)'' = -(\log q)^2 \sum_{i=0}^{n-1} \frac{q^{x+i}}{(1 - q^{x+i})^2} < 0
\]

which means that the function \( f(x) \) is strictly log-concave and so, \( \frac{1}{f(x)} \) is strictly log-convex by Lemma 3(b). Then by Lemma 4 applied to \( f(x) \),

\[
(1 - q^x)^{1-a} = \left( \frac{(q^x; q)_n}{(q^{x+1}; q)_n} \right)^{1-a} < \left( \frac{q^{x+a}; q}{q^{x+1}; q} \right)_n < \left( \frac{(q^{x+a}; q)_n}{(q^{x+a+1}; q)_n} \right)^{1-a} = (1 - q^{x+a})^{1-a},
\]

which proves part (a). As to part (b), take limits as \( n \to \infty \) in the previous inequalities and use the \( q \)-binomial theorem to obtain

\[
(1 - q^x)^{1-a} \leq \left( \frac{(q^{x+a}; q)_\infty}{(q^{x+1}; q)_\infty} \right) = 1 \phi_0(q^{a-1}, -; q, q^{x+1}) \leq (1 - q^{x+a})^{1-a},
\]

which is the desired double inequality. \( \square \)

**Theorem 12.** For any positive integer \( n \) we have

\[
\frac{1}{(q; q)_\infty} \leq \inf \left\{ \left( \frac{q^{1/2}; q}{q^{1/2} q^{1/2} (1 - q^{1/2})} \right)_{n-1}, \left( \frac{q^{1/2}; q}{q^{1/2} q^{1/2} (1 - q^{1/2})} \right)_{n-1}, \left( \frac{q^{1/2}; q}{q^{1/2} q^{1/2} (1 - q^{1/2})} \right)_{n-1} \right\}
\]

Proof. The function \((1 - q^x) \Gamma_q(x)\) is strictly log-convex by Askey 3. Then by Jensen’s inequality

\[
\log \left( (1 - q^{x_1 + \ldots + x_k})/(k) \Gamma_q(\frac{x_1 + \ldots + x_k}{k}) \right) \leq \frac{1}{k} \left( \log \left( (1 - q^{x_1}) \Gamma_q(x_1) + \ldots + \log \left( (1 - q^{x_k}) \Gamma_q(x_k) \right) \right) \right),
\]

which by taking \( k = n - 1 \) and \( x_i = \frac{1}{n-1} \) and simplifying yield

\[
\left( (1 - q^{1/2}) \Gamma_q(1/2) \right)^{n-1} \leq (q^{1/n}; q^{1/n})_{n-1} \prod_{i=1}^{n-1} \Gamma_q(i/n),
\]

or, by 5,

\[
\left( (1 - q^{1/2}) \Gamma_q(1/2) \right)^{n-1} \leq (q^{1/n}; q^{1/n})_{n-1} (\Gamma_q(1/2))^{n-1} \frac{(q^{1/2}; q^{1/2})_{n-1}}{(q; q)_{n-2} (q^{1/n}; q^{1/n})_{\infty}}.
\]
Simplifying gives

$$\frac{(q^{1/2}; q)_\infty^{n-1}}{(q; q^{1/2})_{\infty}(1 - q^{1/2})_{n-1}} \geq \frac{1}{(q; q)_\infty}.$$  

Now apply Jensen’s inequality to the strictly log-convex function $\Gamma_q(x)$ and proceed as before to obtain

$$\frac{(q^{1/2}; q)_\infty^{n-1}}{(q^{1/2}; q)_\infty q^{1/n}} \geq \frac{1}{(q; q)_\infty}.$$  

Finally, note that $(1 + q^x)\Gamma_q(x)$ is strictly log-convex and use the same sort of argument as before to deduce that

$$\frac{(q^{1/2}; q)_\infty^{n-1}(-q^{1/n}; q^{1/n})_{n-1}}{(q^{1/n}; q^{1/n})_\infty(1 + q^{1/2})_{n-1}} \geq \frac{1}{(q; q)_\infty}.$$  

Combining (17), (18), and (19) yields the desired result. 

**Theorem 13.** For any integer $n > 1$, we have

$$\left( q^{\frac{1}{2}}; q \right)_\infty^{\varphi(n)} \geq \sup \left\{ \prod_{d|n} \left( q^{\frac{1}{2}}; q^{\frac{1}{2}} \right)_\infty^{\mu(d)} \left( 1 - q^{\frac{1}{2}} \right)^{\varphi(n)} \prod_{d|n} \left( q^{\frac{1}{2}}; q^{\frac{1}{2}} \right)_\infty^{\mu(d)} \right\}.$$  

**Proof.** Note first the following well-known facts on the Euler totient function $\varphi(n)$:

$$\sum_{i=1 \atop (i, n)=1}^{n} 1 = \varphi(n) \quad \text{and} \quad \sum_{i=1 \atop (i, n)=1}^{n} i = \sum_{i=1 \atop (i, n)=1}^{n} (n - i) = \frac{n\varphi(n)}{2},$$  

from which it follows that

$$\sum_{i=1 \atop (i, n)=1}^{n} \frac{i}{n} = \frac{\varphi(n)}{2} \quad \text{and} \quad \sum_{i=1 \atop (i, n)=1}^{n} \frac{i}{n} = \frac{1}{2}.$$  

Apply Jensen’s inequality to the function $\Gamma_q(x)$ with $k = \varphi(n)$ and $x_i = \frac{i}{n}$ for $i = 1, \ldots, \varphi(n)$ and use the above to get

$$\log \Gamma_q \left( \frac{1}{2} \right) \leq \frac{1}{\varphi(n)} \log \prod_{i=1 \atop (i, n)=1}^{n} \Gamma_q \left( \frac{i}{n} \right),$$  

which by virtue of (17) means

$$\left( \Gamma_q \left( \frac{1}{2} \right) \right)^{\varphi(n)} \leq P_q(n) = \frac{\left( \Gamma_q \left( \frac{1}{2} \right) \right)^{\varphi(n)}}{\prod_{d|n} \left( (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{\infty} \right)^{\mu(d)}}.$$  

It follows that

$$\left( q^{\frac{1}{2}}; q \right)^{\varphi(n)} \geq \prod_{d|n} \left( q^{\frac{1}{2}}; q^{\frac{1}{2}} \right)^{\mu(d)}.$$  

**Proof of Theorem 12.** Let $x = \frac{1}{1 - q}$ and $y = \frac{1}{1 - q^{1/2}}$. Then

$$\Gamma_q(x) = \frac{\Gamma_q(y)}{y},$$  

and

$$\frac{\Gamma_q(y)}{y} \leq \frac{1}{\varphi(n)} \sum_{i=1 \atop (i, n)=1}^{n} \log \left( \frac{1}{1 - q^{\frac{i}{n}}} \right).$$  

By virtue of (17) we get

$$\frac{\Gamma_q(y)}{y} \leq \frac{1}{(q; q)_\infty}.$$  

Therefore

$$\Gamma_q(x) \leq \frac{1}{(q; q)_\infty}.$$  

This completes the proof of Theorem 12. 

**Example.** Let $q = 2$. Then

$$\Gamma_q(x) \leq \frac{1}{(2; 2)_\infty}.$$  

For $x = \frac{1}{1 - q}$, we have

$$\Gamma_q(x) = \frac{\Gamma_q(y)}{y} \leq \frac{1}{\varphi(n)} \sum_{i=1 \atop (i, n)=1}^{n} \log \left( \frac{1}{1 - q^{\frac{i}{n}}} \right).$$  

By virtue of (17) we get

$$\frac{\Gamma_q(y)}{y} \leq \frac{1}{(q; q)_\infty}.$$  

Therefore

$$\Gamma_q(x) \leq \frac{1}{(q; q)_\infty}.$$  

This completes the proof of Theorem 12.
In the remaining part of the proof we shall need

\[(21) \quad \prod_{\substack{i=1 \\ (i,n)=1}}^{n} \left( 1 - q^\frac{i}{n} \right) = \prod_{d|n} (q^\frac{1}{d}; q^\frac{n}{d}) \mu(\frac{n}{d})\]

which follows by the M"obius inversion formula applied to \(\prod_{i=1}^{n-1} (1 - q^\frac{i}{n}) = (q^\frac{1}{n}; q^\frac{n}{n})_{n-1}\).

Now apply Jensen's inequality to the function \((1 - q^x)\Gamma_q(x)\) with \(k = \varphi(n)\) and \(x_i = \frac{i}{n}\) for \(i = 1, \ldots, \varphi(n)\) to deduce

\[
\log \left( (1 - q^\frac{1}{n})\Gamma_q\left( \frac{1}{2} \right) \right) \leq \frac{1}{\varphi(n)} \log \prod_{\substack{i=1 \\ (i,n)=1}}^{n} (1 - q^\frac{i}{n})\Gamma_q\left( \frac{i}{n} \right),
\]

which by virtue of (17) and (21) means

\[
\left( (1 - q^\frac{1}{n})\Gamma_q\left( \frac{1}{2} \right) \right)^{\varphi(n)} \leq \prod_{d|n} (q^\frac{1}{d}; q^\frac{n}{d}) \mu(\frac{n}{d}) (q^\frac{\varphi(n)}{\varphi(n)}\Gamma_q(\frac{1}{2}))^{\varphi(n)} \prod_{d|n} (q^\frac{\varphi(n)}{\varphi(n)}\Gamma_q(\frac{1}{2}))^{\mu(\frac{n}{d})}\]

Simplifying the foregoing inequality yields

\[(22) \quad (q^\frac{1}{n}; q^\frac{n}{n})^{\varphi(n)} \geq (1 - q^\frac{1}{n})^{\varphi(n)} \prod_{d|n} (q^\frac{1}{d}; q^\frac{n}{d}) \mu(\frac{n}{d})\]

Furthermore, apply Jensen's inequality to the function \((1 + q^x)\Gamma_q(x)\) with \(k = \varphi(n)\) and \(x_i = -\frac{i}{n}\) for \(i = 1, \ldots, \varphi(n)\) and use the same argument as above to obtain

\[(23) \quad (q^\frac{1}{n}; q^\frac{n}{n})^{\varphi(n)} \geq (1 + q^\frac{1}{n})^{\varphi(n)} \prod_{d|n} (q^\frac{1}{d}; q^\frac{n}{d}) \mu(\frac{n}{d})\]

Finally combine (20), (22), and (23) to complete the proof. \(\square\)

For our next result we need the following lemma of Askey [3] which deals with the behaviour of \(\Gamma_q\) as a function of \(q\).

**Lemma 9.** Let \(0 < p < q < 1\). Then

(a) \(\Gamma_p(x) = \Gamma_q(x) \leq \Gamma(x), \quad 0 < x \leq 1\) or \(x \geq 2\)

(b) \(\Gamma_p(x) \geq \Gamma_q(x) \geq \Gamma(x), \quad 1 \leq x \leq 2\).

**Theorem 14.** Let \(0 < p < q < 1\) and let \(n > 1\) be an integer. Then

(a) \(\frac{(p; p)_n}{(p^\frac{1}{n}; p^\frac{n}{n})} (1 - p) \frac{n-1}{n} \leq \frac{(q; q)_n}{(q^\frac{1}{n}; q^\frac{n}{n})} (1 - q) \frac{n-1}{n} \leq \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}\)

(b) \(\frac{(p; p)_n}{(p^\frac{1}{n}; p^\frac{n}{n})} \frac{1}{(1 - p) \frac{n-1}{n}} \geq \frac{(q; q)_n}{(q^\frac{1}{n}; q^\frac{n}{n})} \frac{1}{(1 - q) \frac{n-1}{n}} \geq \frac{(n-1)! (2\pi)^{\frac{n-1}{2}}}{n^{n-\frac{2}{2}}n^{n-\frac{2}{2}}}\)

**Proof.** Note first that

\[
\left( \Gamma_q\left( \frac{1}{2} \right) \right)^{n-1} (q^\frac{1}{n}; q^\frac{n}{n})_{n-1} = (1 - q) \frac{n-1}{n} (q; q)_{\infty}^{n-2},
\]

and therefore the relation (3) boils down to

\[(24) \quad \prod_{k=1}^{n-1} \Gamma_q\left( \frac{k}{n} \right) = \frac{(q; q)_{\infty}^{n}}{(q^\frac{1}{n}; q^\frac{n}{n})} (1 - q) \frac{n-1}{n}.
\]
(a) Let \( x_i = \frac{i}{n} \) for \( i = 1, \ldots, n-1 \) and apply Lemma 9(a) to obtain
\[
\prod_{i=1}^{n-1} \Gamma_p\left(\frac{i}{n}\right) \leq \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right) \leq \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right).
\]
Now use (24) and (3) to deduce
\[
\frac{(p; p)_\infty^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_\infty} (1 - p)^{\frac{n}{n-1}} \leq \frac{(q; q)_\infty^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty} (1 - q)^{\frac{n}{n-1}} \leq \left(\frac{2\pi}{n}\right)^{\frac{n}{2(n-1)}} \frac{n}{n-\frac{1}{2}},
\]
which is the desired inequalities.

(b) As to this part let \( x_i = 1 + \frac{i}{n} \) for \( i = 1, \ldots, n-1 \) and apply Lemma 9(b) to get
\[
\prod_{i=1}^{n-1} \Gamma_p\left(1 + \frac{i}{n}\right) \geq \prod_{i=1}^{n-1} \Gamma_q\left(1 + \frac{i}{n}\right) \geq \prod_{i=1}^{n-1} \Gamma\left(1 + \frac{i}{n}\right).
\]
It follows by combining these inequalities with the basic facts \( \Gamma_q(x+1) = \frac{1 - q^x}{1 - q} \Gamma_q(x) \) and \( \Gamma(x+1) = x\Gamma(x) \) that
\[
\prod_{i=1}^{n-1} \Gamma_p\left(\frac{i}{n}\right) \frac{1 - p^{\frac{1}{n}}}{1 - p} \geq \prod_{i=1}^{n-1} \Gamma_q\left(\frac{i}{n}\right) \frac{1 - q^{\frac{1}{n}}}{1 - q} \geq \prod_{i=1}^{n-1} \Gamma\left(\frac{i}{n}\right).
\]
or equivalently,
\[
\frac{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_{n-1}}{(1 - p)^{n-1}} \frac{(p; p)_\infty^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_\infty} (1 - p)^{\frac{n}{n-1}} \geq \frac{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_{n-1}}{(1 - q)^{n-1}} \frac{(q; q)_\infty^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty} (1 - q)^{\frac{n}{n-1}} \geq \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^{n-1} i^{\frac{i}{n}}.
\]
Finally, an application of (24) and (3) to the foregoing inequalities and simplifying yield
\[
\frac{(p; p)_\infty^n}{(p^{\frac{1}{n}}; p^{\frac{1}{n}})_\infty} \frac{1}{(1 - p)^{\frac{n}{n-1}}} \geq \frac{(q; q)_\infty^n}{(q^{\frac{1}{n}}; q^{\frac{1}{n}})_\infty} \frac{1}{(1 - q)^{\frac{n}{n-1}}} \geq \frac{(n-1)!}{n^{n-\frac{1}{2}}} (2\pi)^{\frac{n}{2(n-1)}}.
\]
This completes the proof. \( \square \)

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