Hairy black holes : from shift symmetry to spontaneous scalarization

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Abstract

The Einstein-Klein-Gordon Lagrangian is supplemented by a non-minimal coupling of the scalar field to specific geometrical invariants : the Gauss-Bonnet and Chern-Simons terms. The coupling is chosen in such a way that large families of hairy black holes exist. These families are characterized, namely, by the number of nodes of the scalar function. The fundamental family encompasses black holes whose scalar hairs appear spontaneously and the solutions presenting shift-symmetric hairs. This feature is very particular to the fundamental (no node) solutions. The model further possesses non-topological solitons : boson stars. The influence of the non-minimal interaction on the spectrum of standard boson stars is examined.

1 Introduction

Attempts to escape the rigidity of the minimal Einstein-Hilbert formulation of gravity and of the limited number of parameters describing its fundamental solutions – the black holes –, lead naturally physicists to emphasize enlarged models of gravity. Besides their purely Academic interests, these attempts are largely motivated nowadays by intriguing problems such as inflation, dark matter and dark energy.

One of the most popular class of extensions of Einstein gravity consists in the inclusion of scalar fields and appeals for natural interactions between the scalar fields and the geometry through higher curvature terms. The general construction of scalar tensor gravity was first obtained in [1]. Recently this theory was revived in the context of Galileon theory [2] and different extensions of it, see e.g. [3].

Apart from their cosmological implications, the extended models of gravity (by scalar or other types of fields) also offer possibilities to escape the limitations of no-hair theorems [4, 5] available in standard gravity. In the last few years, black holes endowed by scalar hairs have attracted a lot of attention and have been constructed in numerous theories. One particularly relevant result is the family of hairy black holes obtained in [6] within the Einstein gravity minimally coupled to a complex scalar field. In this case, the no-hair theorems [4, 5] are bypassed by the rotation of the black hole and the synchronisation of the spin of the black hole with the angular frequency of the scalar field. Recent reviews on the topic of black holes with scalar hairs can be found e.g. in [7, 9, 8]. The general theory of scalar tensor gravity [1, 3] contains a lot of arbitrariness and the construction of compact objects such as black holes, neutron stars or boson stars need to be realized is some particular cases. As an example, the truncation of the Galileon theory to a lagrangian admitting a shift symmetric scalar field was worked out by Sotiriou and Zhou (SZ in the following) [10] and still leads to a large family of models. In the particular case where the scalar field couples linearly to the Gauss-Bonnet invariant [11] hairy black hole were constructed numerically and perturbatively.

Abandoning the hypothesis of shift symmetry, several groups [12, 13, 14] considered during the past year, new types of non-minimal coupling terms between a scalar field and specific geometric invariants (essentially the Gauss-Bonnet term). In these models the occurrence of hairy black holes results from an unstable mode associated to the scalar field equation (a generalized Klein-Gordon equation) in the background of the
underlying metric (the probe limit). The coupling constant characterizing the interaction between the scalar field and the geometric invariant plays a role of a spectral parameter of the linear equation. It is used to say that the hairy black holes appear through a spontaneous scalarization for a sufficiently large value of the coupling constant.

In the present paper we will consider a model of scalar-tensor gravity encompassing the theories presenting a spontaneous scalarization and the shift symmetry property. Families of black holes can be constructed in these models revealing a pattern of extrapolation between shift-symmetric solutions and the phenomenon of spontaneous scalarization. We have shown that this connection hold for coupling the scalar field to the Gauss-Bonnet invariant and to the Einstein-Chern-Simons invariant.

The paper is organized as follow; in Sect 2 we present the model, discuss the ansatz and the general equations. Sect. 3 is devoted to the presentation of the two types of compact objects that exist in the Einstein-Gauss-Bonnet gravity: hairy black holes and boson stars. The hairy black holes are supported by the non-minimal coupling and the associated scalar field can present modes interpreted as excited solutions. The boson stars exist without non-minimal coupling and it is the influence of this new interaction of the spectrum that is examined. All these solutions are obtained within a spherically symmetric ansatz. The last section is devoted to solutions occurring in Einstein-Chern-Simons gravity extended by the scalar field. To activate the Chern-Simons we use a metric of space-time equipped by a NUT charge [18].

2 The model

2.1 The action

We are interested in solutions of the field equations associated with the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \nabla_\mu \phi^* \nabla^\mu \phi - V(\phi) + f(\phi) I(g) \right], \] (2.1)

which extends of the minimal Einstein-Klein-Gordon lagrangian. Here \( R \) is the Ricci scalar and \( \phi \) represents a complex scalar field which – in some circumstances – will be chosen real. The usual Klein-Gordon kinetic term is supplemented by a potential \( V(\phi) \) which will actually be chosen as a function of the combination \( |\phi|^2 \equiv \phi \phi^* \) in order to ensure a \( U(1) \) global symmetry for the scalar sector. In the following \( V \) will be set in the form

\[ V(\phi) = m^2 |\phi|^2 + \lambda_4 |\phi|^4 + \lambda_6 |\phi|^6 \] (2.2)

which is used generically for obtaining Q-balls in the absence of gravity and boson stars when gravity is set in (see e.g. [19], [20] for reviews). The gravity sector is supplemented by a non-minimal coupling between the scalar field and the geometrical invariant \( I(g) \). Two choices of \( I(g) \) will be emphasized:

i) the Gauss-Bonnet-scalar (GB): \( I = \mathcal{L}_{GB} \equiv R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \),

ii) the Chern-Simons-scalar (CS): \( I = \mathcal{L}_{CS} \equiv R \tilde{R} = *R^a_{\ bcd} R^{ab} \).

It is well known that these invariants are total derivatives in four dimensions but they will contribute non trivially to the equations of motion through their non minimal coupling to the scalar field via \( f(\phi) \).

In order to preserve the \( U(1) \) symmetry of the “usual” scalar sector, we will assume that, just like the potential, \( f(\phi) \) is a function of \( |\phi| \). In this paper, we will emphasize the effects of a coupling function of the form

\[ f(\phi) = \gamma_1 |\phi| + \gamma_2 |\phi|^2 \] (2.4)

\[ *R^a_{\ bcd} \] is the Hodge dual of the Riemann-tensor

\[ *R^a_{\ bcd} \equiv \frac{1}{2} \eta^{debf} R^a_{\ bef}, \] (2.3)

where \( \eta^{debf} \) is the 4-dimensional Levi-Civita tensor, \( \eta^{debf} = \epsilon^{debf} / \sqrt{-g} \) and \( \epsilon^{debf} \) the Levi-Civita tensor density.
where \( \gamma_1, \gamma_2 \) are independent coupling constants. Several forms of the function \( f(\phi) \) have been emphasized in the literature where the scalar field is usually chosen real. The EGB theory with \( \gamma_2 = 0 \) and \( V = 0 \) corresponds to a shift-symmetric theory studied by SZ \([10]\), the case \( \gamma_1 = 0 \) is considered in \([12, 13, 14]\). Several choices of the function \( f(\phi) \) have been considered in \([15]\) and very recently in \([16,17]\). Solutions with the form of \( f(\phi) \) above with two independent constants \( \gamma_1, \gamma_2 \) was, to our knowledge, not yet investigated.

Hairy black holes types of solutions in the Einstein-Chern-Simons gravity were emphasized respectively in \([21]\) and \([22]\) for \( \gamma_2 = 0 \) and \( \gamma_1 = 0 \). In these works, the Chern-Simons action is rendered dynamical by endowing the Space-Time with a NUT charge. Finally, solutions in the EGB theory within a Nutty space-time were emphasized in \([23]\).

### 2.2 Equations of motion

The equations of motion (EOM) for the general action \([2.1]\) read

\[
G_{\mu\nu} = 8\pi G \left( T^{(\phi)}_{\mu\nu} + T^{(I)}_{\mu\nu} \right) \tag{2.5}
\]

for the metric function, and

\[
-\Box \phi = -\frac{\partial V}{\partial \phi^*} + \frac{\partial f}{\partial \phi^*} \mathcal{I}(g) \tag{2.6}
\]

for the scalar field. In these equations, \( G_{\mu\nu} \) is the Einstein tensor and \( \Box = \nabla_\mu \nabla^\mu \). The energy momentum \( T^{(\phi)}_{\mu\nu} \) arise from the variation of the standard Klein-Gordon lagrangian:

\[
T^{(\phi)}_{\mu\nu} = \nabla_{(\mu} \phi \nabla_{\nu)} \phi^* - (\nabla_\alpha \phi^* \nabla^\alpha \phi + V(\phi)) g_{\mu\nu} \tag{2.7}
\]

Finally \( T^{(I)}_{\mu\nu} \) is the energy momentum tensor associated with the non-minimal coupling term \(\phi^* \mathcal{I}(g)\).

From Eq.\((2.6)\), one can see that the invariant \(\mathcal{I}(g)\) will act as a source term for the scalar field. Consequently, if one find a space-time solution of the EOM such that \(\mathcal{I}(g) \neq 0\), this solution will automatically present a non-trivial scalar field. This mechanism is known as “curvature induced scalarization”.

### 2.3 The ansatz

#### 2.3.1 Metric

For the scalar-tensor solutions we want to obtain, we used a metric which generalizes the Schwarzschild-NUT solution. In appropriate coordinates, it can be written in the form

\[
ds^2 = -N(r)\sigma^2(r)(dt + 2n \cos \theta d\varphi)^2 + \frac{dr^2}{N(r)} + g(r)(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{2.8}
\]

where \( \theta \) and \( \varphi \) are the standard angles parameterising an \( S^2 \) with the usual range and \( r, t \) are the ‘radial’ and ‘time’ coordinates. The NUT parameter \( n \) appears as a coefficient in the differential form \( dt + 2n \cos \theta d\varphi \) (note that \( n \geq 0 \), without any loss of generality).

The interest of this choice for the metric appears for the choice \( \mathcal{I}(g) = \mathcal{L}_{CS} \). When evaluated with the metric \((2.8)\), it turns out that \( \mathcal{L}_{CS} \) is actually proportional to the NUT charge; so it vanishes for spherically symmetric solutions \( n = 0 \) but becomes non trivial for \( n \neq 0 \), ensuring a non-trivial behaviour of the scalar field via the curvature induced scalarization only for \( n \neq 0 \).

In the decoupling limit \( \gamma_1 = \gamma_2 = 0 \) (implying \( \phi = 0 \)), the functions \( N(r) \), \( \sigma(r) \) and \( g(r) \) are known explicitly:

\[
N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2}, \quad \sigma(r) = 1, \quad g(r) = 0
\]

\(^2\)The expression of \( T^{(I)}_{\mu\nu} \) is generically quite involved and depend of the explicit form of \( \mathcal{I}(g) \). The expression of \( T^{(I)}_{\mu\nu} \) for the two cases considered here can be found in \([22]\) with the same notations.

\(^3\)This behaviour is proper to the CS term. The GB invariant is non-trivial even for \( n = 0 \) and can then source the scalar field in the spherically symmetric case.
and
\[ g(r) = r^2 + n^2. \] (2.10)

This metric possesses a horizon located at
\[ r_h = M + \sqrt{M^2 + n^2} > 0. \] (2.11)

As in the Schwarzschild limit, \( N(r_h) = 0 \) is only a coordinate singularity where all curvature invariants are finite. In fact, a nonsingular extension across this null surface can be found [?].

### 2.3.2 Scalar field

Within the same coordinate system, we choose a scalar field of the form
\[ \phi(x^\mu) = e^{-i\omega t} \phi(r), \] (2.12)

where \( \omega \), the frequency of the scalar field, is a real parameter. This choice is motivated by the study of boson stars. With such an ansatz, it is well known [19] that boson star exist within the minimal coupling of the scalar field to gravity, provided the complex scalar field is supplemented by a mass term (or a more general potential (2.2)).

### 3 Einstein-Gauss-Bonnet case

For the GB case we will be interested only in spherically symmetric solutions (i.e. with \( n = 0 \)). The metric can be set in the form
\[ ds^2 = -N(r)\sigma^2(r)dt^2 + \frac{1}{N(r)}dr^2 + g(r)(d\theta^2 + \sin^2\theta d\phi^2), \] (3.13)

and the usual coordinate choice \( g(r) = r^2 \) will be used throughout this section. With the Ansatz (2.13–2.12), the equations (2.5–2.6) reduces to a system of three coupled differential equations (plus a constraint) for the radial functions \( N, \sigma \) and \( \phi \). Using suitable combinations of the equations, the system is amenable to the form
\[ N' = F_1(N, \sigma, \phi, \phi'), \quad \sigma' = F_2(N, \sigma, \phi, \phi'), \quad \phi'' = F_3(N, \sigma, \phi, \phi') \] (3.14)

where \( F_a, a = 1, 2, 3 \) are involved algebraic expressions whose explicit form is not illuminating enough to be given. In the coming discussion we will set \( c = 1 \) and \( 8\pi G = 1 \); then the equations are invariant under the rescaling
\[ r \to \lambda r, \quad m^2 \to \frac{m^2}{\lambda^2}, \quad \gamma_{1,2} \to \lambda^2\gamma_{1,2}, \] (3.15)

where \( \lambda \) has the dimension of length\(^{-1}\).

In the following we will use implicitly this rescaling to set \( \lambda = 1/r_h \) in case of black holes and \( m = 1 \) in case of boson stars. With this convention, the rescaled quantities \( \gamma_1, \gamma_2 \) are dimensionless.

### 3.1 Hairy black holes

In this section, we restrict (for simplicity) the potential (2.2) to the mass term only (i.e. \( \lambda_4 = \lambda_5 = 0 \)). For black holes, the metric is required to present a regular horizon at \( r = r_h \), i.e. \( N(r_h) = 0 \). This requires \( \omega = 0 \), the scalar field is therefore real. The regularity of the equations at the horizon further implies a non trivial condition for the scalar function and its derivative at \( r = r_h \). The conditions at the horizon can finally summarized as follows
\[ N(r_h) = 0, \quad \phi'(r_h) = \frac{-r_h^2 \pm \sqrt{\Delta}}{8r_h(\gamma_1 + 2\gamma_2\phi(r_h))}, \quad \Delta = r_h^4 - 96\gamma_1^2 - 384(\gamma_2^2\phi(r_h)^2 + \gamma_1\gamma_2\phi(r_h)) \] (3.16)
where we set $m^2 = 0$ (the conditions is more involved for $m > 0$). We note that $\Delta \geq 0$ constitutes a necessary condition for solutions to exist. As we will see in the next section it provides strong limitations on the parameters. The requirement of asymptotically flatness and localized solution demand the conditions

$$\sigma(r \to \infty) = 1, \quad \phi(r \to \infty) = 0.$$  (3.17)

The four conditions (3.16), (3.17) constitute the boundary values of the field equations. No explicit solution of the boundary value problem can be found for generic values of $\gamma_1$ and $\gamma_2$. We relied on a numerical method.

The black holes will be characterized by their mass $M$ and charge $Q$, related respectively to the asymptotic decay of the functions $N(r)$ and $\phi(r)$ :

$$N(r) = 1 - \frac{2M}{r} + O(1/r^2), \quad \phi(r) = \frac{Q}{r} + O(1/r^2).$$   (3.18)

The entropy $S = \pi r^2$ and temperature $T_H = \sigma(r_h)N'(r_h)/(4\pi)$ characterize the solutions at the horizon. Using the equations the temperature can further be specified :

$$N'(r_h) = \frac{1 - m^2 r_h^2 \phi(r_h)^2}{r_h + 4\phi'(r_h)(\gamma_1 + 2\gamma_2 \phi(r_h))}.$$  (3.19)

This result suggests that, in the presence of a mass term, the black holes can eventually be extremal.

### 3.2 Numerical results

We now discuss the pattern of solutions in the $\gamma_1, \gamma_2$ parameter space. Because the equations do not admit closed form solutions, we solved the system by using the numerical routine COLSYS [24] which is well adapted for the problem at hand. It is based on a collocation method for boundary-value differential equations and on damped Newton-Raphson iterations. The equations are solved with a mesh of a few hundred points and relative errors of the order of $10^{-6}$.

To present the results, we find it convenient to start with the shift-symmetric theory (i.e. corresponding to $\gamma_2 = 0$) and the hairy black holes constructed in [10]. A pair of solutions exist for $\gamma_1 \leq \sqrt{1/96}$ (with our normalisation of the non minimal coupling); characterized by the sign $\pm$ in the condition (3.10). We will essentially focus on the solutions corresponding to the 'plus' sign branch which smoothly approaches the Schwarzschild solution in the limit $\gamma_1 \to 0$.

For a fixed value for the parameter $\gamma_1$ with $\gamma_1 \in [0, \sqrt{1/96}]$, the SZ solution can be deformed by increasing (or decreasing) gradually the quadratic coupling constant $\gamma_2$. Let us first discuss the deformation of the SZ solution for large values of $\gamma_1$, i.e. for $\gamma_1 \sim \sqrt{1/96}$. The numerical results show the following features.

(i) Increasing gradually the coupling constant $\gamma_2$ the value $\Delta$ approaches zero at some ($\gamma_1$-depending) critical value, say $\gamma_{2,c}$. Accordingly, no solution exist for $\gamma_2 > \gamma_{2,c}$. This is illustrated on Fig. 1 where a few the parameters are represented as functions of $\gamma_2$ for a few fixed values of $\gamma_1$ (see the purple and red lines). On the left part of the figure the solid and dashed lines represent respectively the quantity $\Delta$ and the mass. The value $\phi(r_h)$ represented by the dot-dashed lines. The corresponding value of $\phi'(r_h)$ is represented on the right side.

(ii) Decreasing $\gamma_2$, a Schwarzschild metric can be approached arbitrarily close, although not exactly. This is due to the fact that the scalar field never reaches $\phi(r) = 0$ due to the presence of the non-homogeneous term in the scalar field equation. Indeed for the Schwarzschild black hole of mass $m$ we have $L_{GB} = 48m^2/r^6$.

The deformation of the SZ solutions corresponding to $0 < \gamma_1 \ll \sqrt{1/96}$ by increasing the coupling $\gamma_2$ leads to a richer pattern. First, the SZ black holes forms a “main branch” of solutions existing up to a maximal value, say $\gamma_2 \leq \gamma_{2,max}$. Then a 'second branch' of solutions exists for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,max}]$. As before, the value $\gamma_{2,c}$ coincide with $\Delta = 0$ and the two branches coincide in the limit $\gamma_2 \to \gamma_{2,max}$. Fig. 1 illustrates
In fact the value $\gamma$ hole. For details about the spectrum of this equation we refer to \[22\].

The two branches evolve in completely different issues: the solutions of the main branch approach uniformly the Schwarzschild black hole while the solutions of the second branch have a non trivial limit and form the two branches.

It is tempting to say that the no-node solutions are 'attracted' by the SZ solutions occurring in the hairy black holes forming two branches on specific intervals of $\gamma$. We discuss above, we observe that these excited hairy black holes exist only for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,m}]$; the lower (resp. upper) bound of this interval corresponds to the $\Delta = 0$ critical value (resp. second eigenvalue of Eq.(3.20).

To summarize, fixing low enough values of $\gamma_1$ and varying $\gamma_2 > 0$, the SZ solution deforms into a family of hairy black holes forming two branches on specific intervals of $\gamma_2$. Taking the limit $\gamma_1 \to 0$, the solutions on the two branches evolve in completely different issues: the solutions of the main branch approach uniformly the Schwarzschild black hole while the solutions of the second branch have a non trivial limit and form the set of so called 'spontaneously scalarized black holes' for $\gamma_2 \in [\gamma_{2,c}, \gamma_{2,max}]$ where $\gamma_{2,c} \approx 0.1734$ and $\gamma_{2,max} \approx 0.1814$. The occurrence of these critical values have different explanations:

1. In the limit $\gamma_2 \to \gamma_{2,c}$, the parameter $\Delta$ (see (3.10)) approaches zero.
2. In the limit $\gamma_2 \to \gamma_{2,max}$, the scalar hairs tends uniformly to zero.

In fact the value $\gamma_{2,max}$ corresponds to an eigenvalue of the scalar field equation

\[
\frac{1}{r^2} \frac{d}{dr} (r^2 N(r) \phi') = \gamma_2 \phi (\frac{48M}{r^6}) , \quad N(r) = 1 - \frac{2M}{r}
\]

considered in the background of Schwarzschild solution. It revealing a tachyonic instability of the trivial solution $\phi = 0$ and, consequently a possibility for the Schwarzschild solution to evolve into an hairy black hole. For details about the spectrum of this equation we refer to \[22\].

\[\text{Figure 1: The value } \phi(x_h) \text{ (black lines), the parameter } \Delta \text{ (red lines) and the mass (blue lines) as functions of } \gamma_2 \text{ for several values of } \gamma_1 \text{ for the EGB with } x_h = 1. \text{ Right : Idem for } \phi'(x_h) \text{ and } \phi'.\]

### 3.2.1 Excited solutions

In the shift-symmetric case, i.e. setting $\gamma_2 = 0$ in the condition \[3.10\] drastically reduces the spectrum of hairy black hole. In fact it allows for a single solution whose scalar field is a monotonic function, excluding the possibility existence of excited solutions. By contrast, the linear equation \[3.20\] possesses a series of critical values of $\gamma_2$ characterized by eigenvectors $\phi$ which present a number of nodes. Any of these solutions leads in principle to a branch of excited hairy black holes of the full system.

Like the fundamental one discusses above, each of them is expected to exist on a specific interval of $\gamma_2$. We constructed numerically the branch corresponding to the first excited (or one-node) solution. A few characteristic parameters are plotted on Fig. 4 (see the $\gamma_1 = 0$, i.e. red lines). Along the case with no node discussed above, we observe that these excited hairy black holes exists for $\gamma_2 \in [\gamma_{2,m}, \gamma_{2,M}]$; the lower (resp. upper) bound of this interval corresponds to the $\Delta = 0$ critical value (resp. second eigenvalue of Eq.(3.20)).

Switching on the parameter $\gamma_1$ leads to a deformation of these exited hairy black holes. As seen from Fig. 2 these black holes exist only for $\gamma_2 \geq \gamma_{2,m}$. This contrast drastically with the case the no-node solutions. It is tempting to say that the no-node solutions are 'attracted' by the SZ solutions occuring in the $\gamma_2 = 0$ limit while the excited solutions, having no equivalent, develop only for large values $\gamma_2$. 
3.2.2 Influence of a mass term

To finish this section, let us discuss the influence of a mass term of the scalar field on the spectrum of hairy black holes formed by the spontaneous phenomenon (i.e. we set $\gamma_1 = 0$ in this section). Including a mass term, the regularity condition reads now

$$\phi'(r_h) = \frac{-B \pm \sqrt{\Delta}}{2A}$$

(3.21)

with

$$A = -\phi_0(12\gamma_2 - m^2r_h^2 + 8\gamma_2\phi_0^2 + 4\gamma_2r_h^4\phi_0^4), \quad B = 8\gamma_2\phi_0(r_h^2 - \phi_0^2(r_h^2 + 8\gamma_2r_h^2 - 64\phi_0^2\gamma_2)),$$

(3.22)

$$\Delta = (1 - m^2\phi_0^2)^2 \left( r_h^2(4r_h^4 - 384\gamma_2^2\phi_0^2) + 256m^2\gamma_2^2\phi_0^4(r_h^4 + 12\gamma_2r_h^2 - 96\gamma_2^2\phi_0^2) + 4096m^4\gamma_2^2\phi_0^8r_h^4 \right)$$

(3.23)

and we posed $\phi(r_h) = \phi_0$. The numerical analysis shows that the inclusion of a mass for the scalar field results in shifting the interval of existence in $\gamma_2$ to larger values. This is demonstrated on Fig. 3. The critical phenomenon limiting the interval of existence is the same as discussed above.

This behaviour can be excepted from the field equation of the scalar field. Indeed for a real scalar field, and a potential restricted to the mass term, (2.6) can be rewritten as

$$-\Box \phi = \gamma_2 \mathcal{I}(g) + 2 \left( \gamma_2 \mathcal{I}(g) - m^2 \right) \phi.$$

In this equation, one can see that the mass act as a “negative shift constant” for the term $\gamma_2 \mathcal{I}(g)$. In the case $\gamma_1 = 0$, scalarized solutions for $m = 0$ appears only when this term become sufficiently important (i.e. when $\gamma_2$ is large enough to ensure the term $\mathcal{I}(g)$ to trigger the scalar field). It is then intuitive to assert that, since the mass just shift down the trigger term of the scalar field, higher values of $\gamma_2$ are needed to allow for spontaneous scalarisation.

3.3 Boson stars

As we mentioned in Sect. 2, it is well known (see e.g. [19]) that regular solutions – the boson stars – exist within the minimal coupling of the scalar field to gravity, provide the – complex – scalar field is of the form
The frequency parameter $\omega$ and the mass $m$ of the scalar field play a crucial role for the existence of these solutions. In place of (3.18), the asymptotic form of the function $\phi(r)$ now reads

$$N(r) = 1 - \frac{2M}{r} + O(1/r^2) \quad , \quad \phi(r) \propto \exp(r\sqrt{m^2 - \omega^2}) ,$$

(3.24)

boson stars then exist for $\omega < m$. Beside the mass $M$, the boson star is characterized by the Noether charge associated to the U(1) symmetry, say $Q$. Details about the Noether current and charge can be found in numerous papers, therefore we just give the final form of the integral to be computed:

$$Q = 8\pi\omega \int \frac{r^2\phi^2}{N\sigma} dr$$

(3.25)

Contrary to monopoles or skyrmions this charge has no topological origin, the boson stars are non-topological solitons.

The numerical construction of these solutions can be realized by solving the field equations discussed above. The boundary conditions now read

$$N(0) = 1 , \quad \phi(0) = F_0 , \quad \phi'(0) = 0 , \quad A(r \to \infty) = 1 , \quad \phi(r \to \infty) = 0 .$$

(3.26)

Practically, the value $F_0$ of the scalar field at the center is used as control parameter in the numerical approach; the frequency $\omega$ has to be fine tuned as a function of $F_0$ for all boundary conditions to be obeyed. The frequency $\omega$, the mass $M$ and the Noether charge $Q$ are then evaluated numerically as functions of $F_0$.

It turns out that these non-topological solitons exist on a finite interval of $\omega/m$, that is to say for $\omega/m \in [\omega_{\text{min}}/m, 1.0]$ with $\omega_{\text{min}}/m \sim 0.76$. In the case of minimal coupling, the plot of the mass $M$ of the soliton versus $\omega$ presents the form of spiral as seen on Fig. 4. It turns out that two or more solutions can exist for specific sub-intervals of the frequency $\omega$. One of the goals of this section will be to investigate the influence of the non-minimal coupling on the pattern of boson stars. Because the potential depends on two independent parameter, we limited the analysis to two particular cases: $V = \phi^2$ and $V = \phi^2(1 - \phi^2)^2$; we refer to these cases respectively as the quadratic and sextic potential.

We can summarize the results as follows: while increasing gradually the non-minimal coupling $\gamma_2$, the numerical results show that:

- (i) For the quadratic potential the $\omega - M$ curve has the tendency to unwind and the minimal frequency decreases;
• (ii) The same feature holds for the sextic potential. In this case the minimal frequency is even lower (although remaining strictly positive) than with the mass term only.

Figure 4: The mass of the boson star as a function $\omega$ for the mass potential. Right: Idem for the sextic potential.

The understanding of the critical phenomenon limiting the boson stars raises naturally. In the **minimally coupled case**, the increase of the control parameter $F_0$ corresponds to a decrease of the value $\sigma(0)$ which tends to zero at some stage. Correspondingly the value $R(0)$ of the Ricci scalar at the center of the star get arbitrarily large, as a consequence the limiting configuration presents a singularity at the center.

Let us now turn to the case of **non minimal coupling** to the Gauss-Bonnet term. Turning on the $\gamma_2$ coupling constant, we observe that the value $\sigma(0)$ decreases when $F_0$ increases; however the limitation in the domain comes from another phenomenon. The source of the problem is related to the denominator of the function $f_3$ in (3.14) which for convenience we denote $D(x)$. The two parameters $\sigma(0), D(0)$ decrease when $F_0$ increases and it appear that for $\gamma_2 \neq 0$, the value $D(0)$ tends to zero much quicker than $\sigma(0)$. This statement is hard to demonstrate because the numerical integration of the equations becomes more and more difficult in this limit. This is due to the fact that -in the coordinate system used- both the numerator and denominator entering in the diagonalized equations become quite large in some domain of the interval of integration. The situation is illustrated on figure 5 where we show the pattern of the solutions in the $\omega - \sigma(0)$ plane (left-part) and in the $D(0) - D(\infty) - \omega$ plane. In this plot, the discriminant $D(x)$ has been normalized with respect to its value at infinity in order to be able to compare the situations for different values of $\gamma_2$. Please note that the scale on the vertical axis of the right plot is logarithmic, illustrating the huge variation of the discriminant while approaching the critical configuration. On this plot, one can appreciate that for $\gamma_2 \neq 0$ the bound in the domain of existence for the solutions is due to the behaviour of $D(0)$, rather than $\sigma(0)$ whose value at the critical point significantly increase with $\gamma_2$.

Let us finally point out that the unwinding phenomenon of the $\omega - M$ relation was observed also for boson stars in five-dimensional gravity [25]. In this case the Gauss-Bonnet term is fully dynamic and no scalar field needs to be added.

Figure 5: The value $\sigma(0)$ as function of $\omega$ for boson stars and three values of $\gamma_2$. Right: Idem for discriminant of the system of equations.
4 Einstein-Chern-Simons case

In the same spirit as in the previous section, we have constructed the black hole solutions in the Einstein-Chern-Simons (ECS) model with the mixed coupling (2.1) and using a NU TTY space-time (2.8) in order to make the Chern-Simons term non trivial.

For generic values of $\gamma_1, \gamma_2$, no explicit solution can be found and, again, we relied on a numerical technique. For the numerical construction, we used the gauge $\sigma(r) = 1$. Then the Einstein-Chern-Simons equations can be transformed into a system of three coupled differential equations of the second order for the functions $N(r), g(r)$ and $\phi(r)$. The desired asymptotic form of the solutions require

$$N(r \to \infty) = 1, \quad \sigma(r \to \infty) = 1, \quad \phi(r \to \infty) = \frac{Q_s}{r}$$

where $Q_s$ is the scalar charge. Imposing an horizon $r = r_h$, i.e. $N(r_h) = 0$, the regularity of the black hole on the horizon requires the following rather complicated conditions

$$g'(r_h) = \frac{1}{N_1}(2 - g_0\phi_0m^2 - 2nN_1\phi_1(\gamma_1 + 2\gamma_2\phi_0))$$

$$0 = 24\gamma_2\phi_0^2\phi_1(N_1)^3 + N_1(2\gamma_2ng_0m^2\phi_0^3 - 12\gamma_2n\phi_0 - g_0^2\phi_1) + g_0^2\phi_0m^2$$

where we posed

$$N(r) = N_1(r - r_h) + O((r - r_h)^2), \quad g(r) = g_0 + g_1(r - r_h) + O((r - r_h)^2), \quad \phi(r_h) = \phi_0 + \phi_1(r - r_h) + O((r - r_h)^2)$$

The pattern of the solutions found in the ECS are very similar to the case of EGB, the results are summarized on Fig. 6 for $n = 0.1$. We have checked that the features of this figure are qualitatively similar for different values of $n$.

Figure 6: The value $\phi'(x_h)$ as function of $\gamma_2$ for several values of $\gamma_1$ for the solutions with $x_h = 1$ and $n = 0.1$. 

10
5 Conclusion

The investigation for hairy black holes in gravity extended by a Gauss-Bonnet term and a scalar field was a source of huge activity over the past year. In particular the stability of such objects was examined in details in [16],[17] and the extension with a cosmological constant in [26]. The coupling function of the scalar field to the Gauss-Bonnet term is, up to now, left as an arbitrary freedom but its form lead to different patterns for the solutions and is crucial for the question of stability.

In this paper we considered as coupling a superposition of the linear and quadratic powers of the scalar field. While spontaneously scalarized black holes – with purely quadratic coupling constant $\gamma_2$ – appear on a very limited interval of the coupling constant $\gamma_2$, we showed that, when adding a linear part (even with small coupling $\gamma_1$), two branches of hairy black holes exist. One of these branches is very close to the spontaneously scalarized black holes while the second extend backward to a solution with shift symmetric scalar field. This feature is specific for the fundamental solutions and is not repeated for excited solution (i.e. with scalar field presenting nodes).

Extending the scalar sector of scalar-tensor gravity to a massive, complex field, we were able to construct boson star solutions in the full theory. The qualitative and quantitative effects of the Gauss-Bonnet term have been reported in details in Sect. 3.3 revealing, for instance, that the presence of the quadratic coupling constant $\gamma_2$ can drastically increase the maximal mass of these objects and the range of $\omega$ (the frequency of the complex scalar field) for which these solutions exist. In this context, we also show that the critical phenomenon limiting the existence of solutions is different in the minimally and non-minimally coupled case.

Finally we studied the solutions for scalar-tensor gravity extended by the same kind of coupling of the scalar field to the Chern-Simons invariant. Here the space-time is endowed with a NUT charge. The pattern of Nutty-Hairy-black holes is qualitatively similar to the case of Gauss-Bonnet.

We hope to address the stability of these new solutions in a near future.
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