Sudarshan’s diagonal representation: the ecstasy and agony of another major discovery in science

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1. Introduction
If the V–A theory represents Professor Sudarshan’s mid-century adventure in Particle Physics, the Diagonal Representation or Optical Equivalence Theorem represents his mid-century adventure in Non-classical Optics. Particle physics already existed as a discipline, but the latter discipline was heralded by this very adventure. The diagonal representation too seems to have brought to its discoverer brought lot more agony than ecstasy.

The ecstasy part is best described by quoting from the abstract of Professor Jeff Kimble to this conference: In 1963 Professor E.C.G. Sudarshan presented the Optical Equivalence Theorem and thereby provided a quantitative, model independent boundary between the classical and manifestly quantum domains of light. More than a decade then passed before nonclassical light first flickered dimly in the laboratory. By now, a zoology of nonclassical sources shine in laboratories around the world, with applications ranging from quantum measurement to quantum computation and communication. The Optical Equivalence Theorem of Sudarshan has been the critical, guiding light in these quests.

While referring to the Born and Wolf classic, the unusual book of J.W. Simmons and M.J. Guttmann [States, Waves and Photons: A Modern Introduction to Light, Addison-Wesley, 1970] takes the liberty to add the subtitle The Bible of Non-quantum Optics, to emphasise the fact that on any issue of optics, which is not specifically quantum, this work has some authoritative insight to offer. The point is that non-quantum subsumes classical. Similarly, nonclassical is more specific than quantum. The question of when exactly nonclassical optics was born is thus a sharp question with a clear answer. Sudarshan’s paper may be called the corner-stone of non-classical optics, notwithstanding the fact that this paper did not occupy anything much more than two printed pages. After all, the poems of Thiruvalluvar occupied not even two full lines each; yet they remain ranked among the greatest works mankind has ever seen!

The agony part is expressed most effectively by Sudarshan himself, with his characteristic brevity in eloquence: Give unto Glauber only what is his. We wanted to understand, perhaps even experience, the nature and magnitude of the agony capsuled in this Biblical quote; and this paper is the result.

The context of the quote is clearly the documents released by the Nobel Foundation to accompany the announcement of Physics Nobel 2005 to Professor Roy Glauber. These
documents assert, in unambiguous terms, that all action took place in the year 1963. Since we are not old enough to have been around doing science in 1963 and have the benefit of first hand experience of what happened, our assessment relies much on the writings of Glauber and Sudarshan during that year. Before we set out in detail our impressions and findings, however, we wish to clarify that understanding the agony is our aim, and not criticising the individuals involved or the authors of the Nobel documents. We believe our description below is how posterity will judge 1963.

2. Before Sudarshan

The semiclassical theory of optical coherence and photoelectric detection was formulated by the Rochester School in the fifties and early sixties. The key element emphasised in these formulations was the central role of the positive frequency part of the real optical field; this is the analytical signal. It was Glauber who showed in his seminal paper (PRL 1963) that quantum correlations could be, and had to be, formulated by simply substituting the positive frequency part of the hermitian field operators in place of the analytic signals in the corresponding classical expressions, leading to normally ordered expressions. He thus brought the coherent states, the eigenstates of these positive frequency operators, into the centre stage. He observed that the density operator of the thermal state is diagonal in the coherent state basis. Finally, Glauber related the photoionization probability of a pair of atoms, atom 1 at $t_1$ and atom 2 at $t_2$, to the quantum correlation function of order four in the field operators (order two in the intensity).

It seems Glauber, at this stage, had little reason to even remotely suspect that this diagonal form for thermal states could be part of a more general result or, equivalently, that the coherent state matrix elements of a density operator [his Eq.(3)] are not all independent of each other. In any case, there is nothing in the text of his paper, or in the list of references cited by him, to suggest that Glauber was, at the time of writing this paper of his, aware of the overcompleteness of the coherent states or its consequences. And the literature he cites reinforces this impression.

Glauber’s PRL paper appeared in print on February 01, 1963. It was received on December 27, 1962.

3. Enter Sudarshan

Sudarshan’s paper appeared in the April 01, 1963 issue of PRL. It was received on March 01, 1963.

Referring to Glauber’s PRL paper, Sudarshan notes: statistical states of a quantized electromagnetic field have been considered recently, and a quantum mechanical definition of coherence function of arbitrary order presented. Sudarshan exhibits little interest in horse-splitting on the fact that Glauber seems to have considered only correlation functions of order four.

Sudarshan refers to the papers of Bargmann, Segal, Klauder, and Schweber on the analytic function representation of operators. None of these had been referred to by Glauber.

Sudarshan draws the reader’s attention to the non-orthogonality and over-completeness of the coherent states, and immediately gets to the heart of the central problem: We can make use of the overcompleteness of the states to represent every density matrix in the “diagonal” form . . .

To complete the diagonal representation

$$\rho = \int d^2 z \phi(z) |z\rangle\langle z|,$$

he presents the inversion formula expressing the phase-space ‘density’, quasi-probability, or diagonal weight function $\phi(z)$ in terms of the density operator $\rho$. Note that the diagonal representation becomes a representation only with the presentation of the inversion formula!
Sudarshan notes that $\phi(z)$, the ‘density’ function in the diagonal representation, is real in view of the hermiticity of $\rho$, but not necessarily positive definite. He shows that, in taking expectation values, every operator should be written in the normally-ordered form to go hand in hand with the diagonal representation, so that the quantum-mechanical expression for all correlation functions will have the same form as their classical counterparts, with the quasi-probability $\phi(\{z_k\})$ now playing the role played in the classical case by the true probability density of the classical ensemble. This is the essence of the Optical Equivalence Theorem. All nonclassicalities, if any, of a given state $\rho$ are fully captured in the departure of the corresponding $\phi(\{z_k\})$ from being a genuine classical probability. Sudarshan notes. This departure continues to remain the very definition of nonclassicality or quantumness, and the guiding light for nonclassical optics. Finally, in Eq. (7) Sudarshan shows how the density operator can be determined from the measured correlation functions, if all the correlation functions are known.

Note: At the time when Sudarshan’s paper is sent to PRL (March 1, 1963), and in fact even at the time when it appeared in print on April 1, 1963, there seems to be no statement or hint of the diagonal representation as a general theorem from anyone else, even as a preprint.

4. Interlude

Glauber’s next paper appeared in the Physical Review of June 15, 1963. It was received on February 11, 1963. It says: The present paper, which is the first of a series of fundamental papers on optics, is devoted largely to defining the concept of coherence. Glauber also gives some hint as to what is to be expected in the next installment of this series. For, after introducing the coherent states in Eqs. (4.9), (4.10) he states: We shall discuss the properties of such states at length in the paper to follow. There is absolutely no indication of anything remotely resembling the diagonal representation as a general theorem! The citation is similar to his PRL paper: no Bargmann, Segal, Klauder, or Schweber.

5. Glauber’s second Phys. Rev. paper, or Sudarshan’s Diagonal representation in Glauber’s own words

This paper appeared in the Physical Review of September 15, 1963. It was received on April 29, 1963, four weeks after Sudarshan’s paper appeared in print. Though Glauber promised a series in his first Phys. Rev. paper, the series seems to have been aborted or abandoned with this one.

Towards the end of the abstract Glauber notes: A particular form is exhibited for the density operator which makes it possible to carry out many quantum mechanical calculations by methods resembling those of classical theory. This representation presents clear insights into the essential distinction between the quantum and classical descriptions of the field. An unsuspecting reader can easily get this wrong, and so a warning could help: this is intended to be part Glauber’s abstract of his own work, and is not to be misread as his eloquent summary of Sudarshan’s work!

Again, while summarizing the contents of the paper, Glauber states: The application of the formalism to physical problems is begun in section VII, where we introduce a particular form for the density operator which seems especially suited to the treatment of radiation by macroscopic sources. Glauber makes here an important point, namely that no useful application of the formalism will emerge without this particular form or representation. At this point Glauber makes no reference to Sudarshan, thereby tempting the reader to approach Section VII with great reverence and even awe, in anticipation of a genuinely new representation which would form the basis for all applications.

Writing Section VII in general, and page 2776 in particular, appears to have been a painful task of great agony, even for Glauber who is one of the all time great expositors. First, Glauber
had to avert the risk of this Section creating an impression in the reader’s mind that the
previous Sections V and VI are, perhaps, just remnants from an earlier draft of Glauber before
Sudarshan’s paper appeared in print. Thus Glauber begins by presenting his view of why the
(unnecessary) discussion in those two Sections had to be gone through: In the preceding Sections
we have demonstrated the generality of the use of coherent states as a basis. Not all fields
require for their description density operators of quite so general a form. Indeed, for a broad
class of radiation fields which includes, as we shall see, virtually all of those studied in optics, it
becomes possible to reduce the density operator to a considerably simpler form. This form is one
which brings to light many similarities between quantum electro-dynamical calculations and the
corresponding classical ones. And he goes on to add: Its use offers deep insights into the reasons
why some of the fundamental laws of optics, such as those for the superposition of fields and
calculation of the resulting intensities, are the same as in classical theory, even when very few
quanta are involved. This is precisely what Sudarshan had asserted in respect of his diagonal
representation, but Glauber does not seem to care, for he has chosen not to cite Sudarshan as
yet!

The promised particular form or representation is presented in Eq. (7.6):

$$\rho = \int P(\alpha) |\alpha\rangle\langle\alpha| d^2\alpha.$$  

Glauber anticipates the risk that at this point the reader could wonder thus: But is this not
simply Sudarshan’s diagonal representation, with $\phi$ replaced by $P$ and $z$ by $\alpha$? So this time he
adds, with manifest reluctance, a footnote: The existence of this form for the density operator
has also been observed by E.C.G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963). His ‘note’ is
discussed briefly at the end of Sec. X. Note that the phrase used is observed, and not even
independently discovered, whereas the normal way of acknowledging an earlier fundamental
discovery forming the very soul of the discipline would have been to say: discovered earlier
by E.C.G. Sudarshan.

So the curious reader moves on to this ‘brief discussion’ in Section X. Indeed there is a
second and last reference to the work of Sudarshan: During the completion of the present paper
a note by Sudarshan has appeared which deals with some of the problems of photon statistics
that have been treated here. Sudarshan has observed the existence of what we have called the $P$
representation of the density operator and has stated its connection with the representation based
on the n-quantum states. To that extent, his work agrees with ours in Section VII and IX.

The above way of citing an earlier published paper, and that too the single most important
discovery in the field, is extraordinary, to say the least. Note that the same phrase ‘observed’
strikes again. How does the poor reader know that Glauber expects ‘observed’ to be read as
‘formulated and proved earlier’, and ‘stated’ to be read as ‘established’? Note also that it is not
‘To that extent, our work of Section VII and IX is along the same lines as implied in Sudarshan’s
work’, but it is ‘To that extent, his work agrees with ours in Section VII and IX”. How does
Glauber know, with certainty, that Sudarshan’s gift of prophecy was so strong as to ensure that
what Sudarshan writes in February would exactly agree with what Glauber will choose to claim
and publish a few months later? Finally, since Sudarshan had observed ‘the existence of’ such
an all important representation much before Glauber, even by Glauber’s own admission, how
did it become lawful for Glauber to name Sudarshan’s baby?

Back to p. 2776. Glauber goes on to specify the class of states for which ‘his’ $P$-representation
exists or, more precisely, states he would be willing to allow. On this one he is categorical:
The function $P(\alpha)$ need not be subject to any regularity conditions, but its singularities must be
integrable ones. It is convenient to allow $P(\alpha)$ to have delta function singularities so that we
may think of a pure coherent state as represented by a special case of equation (7.3). There is
no question of Glauber allowing $P(\alpha)$ to assume negative values, for he says: It is the kind of
operator we might be naturally led to if we were given knowledge that the oscillator is in a coherent state, but one which corresponds to an unknown eigenvalue $\alpha$. That is, only incoherent mixtures or convex sums of coherent states are acceptable for ‘his’ P-representation. [One wonders if Glauber was really aware of the fact that these states will have nothing nonclassical in them. Probably not, as our next consideration shows.]

This lands Glauber on his next difficulty: All the states Glauber has just allowed for ‘his’ representation are already classical according to Sudarshan’s diagonal representation theorem. Being apparently unaware of this fact, and with much zeal, Glauber proceeds to propose towards the end of the page a restricted or fine-tuned definition of classicality/non-classicality divide; this one is original and independent of Sudarshan. He alerts the reader to his observation earlier in the paper that coherent states $|\alpha\rangle$ and $|\alpha'\rangle$ are approximately orthogonal to one another only when $|\alpha - \alpha'| >> 1$, and goes on to add: 

When the function $P(\alpha)$ tends to vary little over such large ranges of the parameter $\alpha$, the non-orthogonality of the coherent states will make little difference, and $P(\alpha)$ will then be interpretable approximately as a probability density. The functions $P(\alpha)$ which vary this slowly will, in general, be associated with strong fields, ones which may be described approximately in classical terms. Glauber does not seem to care that this class which he is willing to allow is nearly empty. Nor does he seem to care that if this decree of his is accepted, even the mixture of a discrete set of coherent states will fail to be classical! In any case, we are not aware of anyone who has taken Glauber’s attempt to modify Sudarshan’s definition of the classicality/nonclassicality divide seriously.

As for his final difficulty in respect of writing this page, Glauber should have been acutely aware of the fact that he has no explicit expression of his own for ‘his’ P-representation. He even goes to the extent of omitting Sudarshan’s formula for the diagonal representation density function, even if it meant that in his entire paper there will be no explicit formula for the function $P(\alpha)$ that plays such a crucial and central role in his theory.

To a beginning student it may appear that Eq.(7.6) is ‘derived’ through Eq.(7.3). This is not true, appearance notwithstanding; indeed, the latter is just the coherent state matrix element of the former.

Glauber knows very well that no integral representation is defined until the inversion formula is established. That is why in the case of his $R(\alpha^*, \beta)$ representation wherein he knows the inversion formula he presents it not just once, but four times: Eqs.(5.4), (5.7), (6.1), and (6.2). That he is not able to define his P-representation in the first place, however, does not seem to deter him from referring in Section IX to some ‘generalized P distribution’, whatever that might mean.

We have always admired Glauber for the clarity of his writings. Thus, we wonder how the half-hearted and self-contradicting incoherent hotchpotch discussions in the second half of this paper, dealing with the diagonal representation, or the P-representation as Glauber would like to name it, could ever have flown out of Glauber’s own pen.

There is little evidence to suggest that Glauber had appreciated in any significant manner the depth, breadth, or universality of Sudarshan’s fundamental theorem. His reluctance to credit Sudarshan fully for the diagonal representation, the soul of nonclassical optics is obvious; may be he was not impressed by the length of Sudarshan’s ‘note’. It is also clear that this representation cannot be perturbatively assimilated into Glauber’s scheme of things, with much conviction.

It should be clear that there was and is only one diagonal representation, and Sudarshan had discovered it in its totality. Glauber’s apparent position that he had a P representation whose details he will not divulge in verifiable terms is bound to be a sure recipe for agony.

6. The Nobel Documents
As noted earlier, the Nobel documents form the immediate provocation for the Biblical ‘Give unto Glauber only what is his’. This part of the agony turns out to be much easier to understand.
Advanced information on the Nobel Prize in Physics 2005 makes several statements which can prove potentially misleading, but we have space for only a few. We begin by quoting from page 4: On the other hand, there lingered an impression that quantum noise only supplied ripples on the field amplitudes of the classical fields. Thus random function theory would account for the observed effects. As an example we quote [8]: “In the conditions under which light fluctuations are usually measured by photoelectric detectors, the semi-classical treatment applies as readily to light of non-thermal origin as to thermal light, and to non-stationary as well as stationary fields.” The subsequent experimental progress rapidly proved the short-comings of the semi-classical approach. The correct theory was published by Roy Glauber in 1963 [9], and this has been the basis for all subsequent theoretical considerations.

To an unsuspecting reader, this quote will seem to clearly identify three phases or time-periods. The first phase is that of die-hard classical or semi-classical physicists of which the authors of [8] are alleged to form a sample. The second is a phase of rapid experimental progress discrediting the alleged die-hards. The third phase is the work of [9], which was to form the final theory for all time to come. And the naive reader will naturally take it that these phases are causally ordered.

Now to the truth: [9] is Glauber’s PRL paper, and we are not aware of Glauber explaining or predicting anything non-classical. [8] is by Mandel, Sudarshan, and Wolf (the Rochester group), the parents of optical coherence at all levels, and this paper was submitted in June 1964, long after Glauber’s PRL and Phys. Rev. papers of 1963. There is a criticism of Glauber’s Nobel winning work in that Glauber had no non-classical aspect to offer, notwithstanding his often repeated anti-classical refrain. Indeed, and as we have seen, Glauber’s view of the diagonal P-representation was too restrictive to allow for anything non-classical, even potentially.

Later part of the page goes on to add: *Glauber introduced the concept of quasiprobabilities into Quantum Optics*. But we have already seen that it was Sudarshan who showed that the diagonal weight function exists as a quasiprobability for all states. Glauber in his later work, renamed it as the P-distribution but claimed that it does not exist for all states. As we will see, this is a position he held even during his Nobel lecture. The real agony is that, to the best of our knowledge, Glauber has never stated clearly and in verifiable terms what ‘exists’ meant in his scheme of things.

Page 5 reads: *E.C.G. Sudarshan [11] drew attention to the use of coherent state representations for the approach of classical physics*. This is an astonishing way of characterizing Sudarshan’s pivotal contribution. Without this work we will be left simply with anti-classical optics with no non-classical component. No wonder, Sudarshan was so provoked!

Another seriously, but more subtly, flawed statement in this document is the one concerning Professor Mandel: *L. Mandel has used the theory to design many ingenious experiments illuminating the quantum nature of light signals*. While seemingly very appreciative of Mandel’s contributions, this statement it terribly patronizing and unjust to the memory of one of the greatest physicists of the second half of the twentieth century. In fact, Mandel was one of the main contributors to the development of coherence theory at all levels: Classical, Semi-classical, and Quantum. It was not Glauber who developed coherence theory to arrive at predictions which could be put to experimental test. It was mainly Mandel who developed quantum coherence theory into a testable theory. Were he alive, it would have been impossible to overlook his contributions while awarding a Nobel prize for the development of quantum theory of coherence.

7. The Nobel Lecture

Commenting on *Information Theory* which came into being, all in one stroke, through the remarkable 1948 work of Shannon, A. I. Khinchin writes in his *Mathematical Foundations of Information Theory* (Dover, 1957): *Rarely does it happen in mathematics that a new discipline achieves the character of a mature and developed scientific theory in the first investigation*
devoted to it. Such in its time was the case with the theory of integral equations, after the fundamental work of Fredholm; so it was with information theory after the work of Shannon.

The Nobel documents had upped the ante by painting as if a similar thing happened with quantum optics in 1963, with Glauber doing a Shannon. Since this was far from being true, the world was keenly looking forward to his Nobel lecture and to his unmatched eloquence to see how he will bridge the gap between these documents and reality. But the lecture proved to be a disappointing anticlimax in this regard. Like floor 13, there is no mention of 1963 in his entire lecture, except for the, by now familiar, negative reference to Sudarshan’s work. None of Glauber’s own three papers of 1963 is cited. Rather, 1951 paper of his, which he himself did not cite in his first two papers of 1963, is cited! Regarding the diagonal weight or ‘his’ P-distribution he says: *It is a member, as we shall see, of a broader class of quasiprobability densities.* But we could not find anything like that in the rest of the lecture, and so his quasiprobability remains to be defined. [English not being our native language, we seem to have misunderstood ‘later’ to mean ‘later in this lecture’]. Then he goes on to add: *The P-representation, unfortunately, is not always available.* And this is where he cites Sudarshan! This is astonishing even by the standards Glauber had heralded in his second Phys. Rev. paper.

8. Final Remarks

Having traced in some detail the source of the agony, it seems appropriate to raise some questions with colleagues in the quantum optics and quantum information community. In raising these questions we refuse to be discouraged by the proverb in Tamil, our mother tongue, which liberally translated reads:*you can wake up the one who is sleeping, but how will you wake up the one who pretends to be sleeping.*

- How long should an individual be subjected to this kind of agony? And for whose fault?
- If Glauber has indeed introduced an integral representation for the density operator, and if it is different from Sudarshan’s diagonal representation, how does one construct this representation for a given density operator?
- If Glauber says his representation does not exist for all density operators, then where does one find the characterization of the division of the set of all density operators into two subsets – the subset for which the representation exists and the one for which it does not?
- Can ‘exist’ in the sense of Glauber be defined in such a way that we can run the test on a given density operator and conclude ‘exists’ or ‘does not exist’, without having to run to Glauber for his impressionistic verdict?
- If Glauber has introduced a quasiprobability distinct from Sudarshan’s diagonal weight, where does one find a characterization of the same?
- Who discovered Glauber-Sudarshan representation? When? As far as we could investigate, they never entered into a joint venture. And when Sudarshan discovered his diagonal representation, he discovered it completely, with no room for anyone to get a piece of the cake.

If 40 plus years of waiting and a Nobel will not encourage Professor Glauber to divulge a precise definition of his P-representation and his ‘broader class of quasiprobability densities’, it should certainly be a matter of agony, not only for Professor Glauber, but also for the entire community.