KPZ, ASEP and Delta-Bose Gas

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Abstract. We explain the solution of the KPZ equation with narrow wedge initial condition from the point of view of its connections to the ASEP and the δ-Bose gas.

1. Introduction
The KPZ equation is a well known equation which describes surface growth phenomena [1]. Its one dimensional version reads

$$\frac{\partial}{\partial t}h = \frac{1}{2} \lambda \left( \frac{\partial}{\partial x} h \right)^2 + \nu \frac{\partial^2}{\partial x^2} h + \sqrt{D} \eta.$$ (1)

Here $h = h(x,t)$ is the height profile at time $t$, $t \geq 0$, and at position $x \in \mathbb{R}$. $\lambda$ is the strength of the nonlinearity, $\nu$ of the diffusive relaxation and $D$ of the noise. $\eta$ is a Gaussian white noise with covariance

$$\langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t').$$ (2)

Kardar, Parisi and Zhang applied a dynamical renormalization group analysis and showed that the height fluctuations scale as $O(t^{1/3})$. This exponent supports non-Gaussian nature of the phenomena and is shared by many models. The universality class related to these phenomena (in particular associated with this exponent) is called the KPZ universality class [2].

Compared to its fame, not much information is available about the distribution of the height function. A reason may be that the equation itself is not well defined due to the strong irregularity of the noise. One has to first make sense of the equation. Moreover even if one could find a way to define its solutions, it is not clear if and how one can treat the equation to get some explicit information.

From the point of view of statistical mechanics, the most important task is to work out the universal aspects of the model. For this purpose, it is enough to have other models which have no difficulty of being well defined and are accessible by some means. For the KPZ universality class, there are several such models like the asymmetric simple exclusion process (ASEP) and the polynuclear growth (PNG) model. In the last decade, the fluctuation properties of the ASEP have been studied extensively and some universal quantities have been obtained [3–6].

Given the big progress for the ASEP and related models, it was natural to ask if one can treat the KPZ equation itself. In 2010 the first exact solution for the height distribution of the KPZ
equation was obtained in the case of sharp wedge initial conditions [7–11]. The method was to use results for the ASEP and to consider the weakly asymmetric limit. On the other hand it is known that the KPZ equation is related to the $\delta$-Bose gas which is also exactly solvable [12]. It turned out that one can reproduce the result for the KPZ equation by studying the $\delta$-Bose gas [13,14], for further developments see [15,16].

In this article, after explaining the connections between KPZ and ASEP on one side and KPZ and $\delta$-Bose gas on the other side, we discuss the direct relation between the ASEP and the $\delta$-Bose gas based on the duality for the ASEP current fluctuations [17,18].

2. Scaling limit for ASEP

The asymmetric simple exclusion process (ASEP) is a stochastic many particle process. Each particle performs an asymmetric random walk with rate $p$ to the right and $q$ to the left [19–21]. There is an exclusion interaction amongst particles so that a jump is suppressed if the target site is occupied. We assume $q > p \geq 0$, $p + q = 1$. When $p = 0$, all particles hop only to the left, which is then called TASEP (totally ASEP). In this article we only consider the step initial condition, for which all sites on $x < 0$ are empty and $x \geq 0$ are occupied. Let $N(t)$ be the integrated current at the edge between sites 0 and 1, i.e.,

$$N(t) = (\# \text{ hops from 0 to 1 during time } [0,t]) - (\# \text{ hops from 1 to 0 during time } [0,t]).$$

(3)

Let $\eta(x,t)$ denote the particle number (0 or 1) at site $x$ and at time $t$. We define the ASEP height function by

$$h_A(x,t) = \begin{cases} 
2N(t) - \sum_{0 < y \leq x} (2\eta(y,t) - 1), & x > 0, \\
2N(t), & x = 0, \\
2N(t) + \sum_{x < y \leq 0} (2\eta(y,t) - 1), & x < 0.
\end{cases}$$

(4)

We are interested in the scaling behavior of this quantity. For the height at the origin, the basic result is the following. As $t \to \infty$,

$$h_A(0,t/(q-p)) \simeq -\frac{1}{2}t + 2^{-4/3}t^{1/3}\xi_{TW}.$$  

(5)

Here the random amplitude $\xi_{TW}$ is GUE Tracy-Widom distributed,

$$F_{TW}(s) = \mathbb{P}[\xi_{TW} \leq s] = \det(1 - P_s K_{Ai} P_s)_{L^2(\mathbb{R})},$$

(6)

where $\det$ on the right denotes the Fredholm determinant, $P_s$ is the projection onto $[s, \infty)$, and $K_{Ai}$ is the Airy kernel

$$K_{Ai}(x,y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda) \text{Ai}(y + \lambda).$$

(7)

The function $F_{TW}$ describes the distribution of the largest eigenvalue for $N \times N$ GUE random matrices in the large $N$ limit [22]. Such a result was obtained first for the TASEP by Johansson [23] and then generalized to the ASEP by Tracy and Widom [24, 25]. As will be explained in the following section, the above result holds also for the KPZ equation. It is expected that the fluctuations of the height of a curved growing surface in the KPZ universality class are described by the same distribution function, $F_{TW}$. 

2
3. The solution of the KPZ equation for narrow wedge initial conditions
Because of scaling, without loss of generality we can set $\lambda = 1, \nu = 1/2, D = 1$ in the KPZ equation, which then reads
\[
\frac{\partial}{\partial t} h = \frac{1}{2} \left( \frac{\partial}{\partial x} h \right)^2 + \frac{1}{2} \frac{\partial^2}{\partial x^2} h + \eta.
\] (8)
We consider this equation with the narrow wedge initial condition,
\[
h(x, 0) = -|x|/\delta, \quad \delta \ll 1.
\]
The macroscopic shape for finite $t$ is
\[
h(x, t) = \begin{cases} -x^2/(2t) & \text{for } |x| \leq t/\delta, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta. \end{cases}
\]
The exact solution for the distribution of the height at a specified reference point is obtained in [7–11] and given by
\[
h(x, t) = -x^2/2t - \frac{1}{12} \gamma_t^3 + \gamma_t \xi_t,
\] (9)
where
\[
\gamma_t = 2^{-1/3} t^{1/3}.
\] (10)
The distribution of $\xi_t$ is defined by
\[
F_1(s) = P[\xi_t \leq s] = 1 - \int_{-\infty}^{\infty} \exp \left[ -e^{\gamma_t (s-w)} \right] \left( \det(1 - P_u(B_t - P_{Ai})P_u) - \det(1 - P_u B_t P_u) \right) du.
\]
Here $P_{Ai}(x, y) = \text{Ai}(x)\text{Ai}(y)$, $P_u$ is a projection to $[u, \infty)$, and $B_t$ has the kernel
\[
B_t(x, y) = K_{Ai}(x, y) + \int_0^\infty d\lambda (e^{\gamma_t \lambda} - 1)^{-1}(\text{Ai}(x + \lambda)\text{Ai}(y + \lambda) - \text{Ai}(x - \lambda)\text{Ai}(y - \lambda))
\]
In the scaling limit $t \to \infty$, we indeed recover $F_{TW}$. This implies that the KPZ equation is in the KPZ universality class, at the level of the one-point distribution for a curved surface.

4. $\delta$-Bose gas
Using the Cole-Hopf transformation,
\[
Z(x, t) = \exp[h(x, t)],
\] (11)
the KPZ equation becomes
\[
\frac{\partial}{\partial t} Z = \frac{1}{2} \frac{\partial^2}{\partial x^2} Z + \eta Z,
\] (12)
often called stochastic heat equation [26]. Using the Feynman-Kac formula, one obtains
\[
Z(x, t) = E_x \left( \exp \left[ \int_0^t \eta(b(s), t-s) ds \right] Z(b(t), 0) \right),
\] (13)
where $b(t)$ is a standard Brownian motion with $b(0) = x, b(t) = 0$. In this rewriting $b(t)$ can be viewed as a directed polymer in a random potential $\eta$. To make sense of the KPZ equation, one way is to define (13) through the expanded exponential with multiple Ito-integrals and then to recover the KPZ height by inverting (11) [11].
By taking the average over randomness, one obtains
\[
\Xi_{\text{Bose}}(x_1, \ldots, x_N; t) := \langle Z(x_1, t) \cdots Z(x_N, t) \rangle = \langle x_1, \ldots, x_N \rangle e^{-iH_N} |0\rangle ,
\] (14)
where
\[
H_N = -\sum_{j=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{i,j=1,i\neq j}^{N} \delta(x_i - x_j) .
\] (15)

\(H_N\) is the Hamiltonian of the \(\delta\)-Bose gas with attractive \(\delta\)-potential. \(\Xi_{\text{Bose}}(x_1, \ldots, x_N; t)\) satisfies the Schrödinger equation
\[
\frac{\partial}{\partial t} \Xi_{\text{Bose}}(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial x_i^2} \Xi_{\text{Bose}}(x_1, \ldots, x_N; t)
\] (16)
with boundary conditions
\[
\left( \frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i} + 1 \right) \Xi_{\text{Bose}}(x_1, \ldots, x_N; t)|_{x_{i+1} = x_i + 0} = 0 , \quad i = 1, 2, \ldots, N - 1 ,
\] (17)
and the initial condition
\[
\Xi_{\text{Bose}}(x_1, \ldots, x_N; t = 0) = \prod_{i=1}^{N} \delta(x_i) .
\] (18)

The system is known to be exactly solvable. The connection between the KPZ equation and the \(\delta\)-Bose gas was noticed long time ago [12]. Recently summation over excited states was carried out for the \(Z(x, 0) = \delta(x)\) (corresponding to the narrow wedge initial condition for the KPZ equation) by applying the Bethe ansatz. The result (9) in the previous section can be obtained along this route [13,14], but the original derivation was different.

5. KPZ as weakly ASEP limit

The derivation of (9) uses the weakly asymmetric limit of the ASEP [7–11] and thus we briefly recall how the KPZ equation can be obtained from the weakly ASEP in a scaling limit. Let us take a small parameter \(\epsilon > 0\). We consider the scale of space of \(O(\epsilon^{-1})\), time of \(O(\epsilon^{-2})\), and simultaneously set
\[
p = \frac{1}{2} - \frac{1}{2} \sqrt{\epsilon} , \quad q = \frac{1}{2} + \frac{1}{2} \sqrt{\epsilon} .
\] (19)
In the end we take the \(\epsilon \to 0\) limit. (Taking \(\epsilon = 0\) from the outset without scaling space and time, one simply gets the symmetric simple exclusion process (SSEP).) For the ASEP height function \(h_A\), let us define
\[
Z_A(x, t) = \frac{1}{\sqrt{\epsilon}} \exp \left[ \zeta h_A(\epsilon^{-1}x, \epsilon^{-2}t) + \mu \epsilon^{-2}t \right] ,
\] (20)
where
\[
\mu = p + q - 2\sqrt{pq} = \frac{1}{2} \epsilon + \frac{1}{8} \epsilon^2 + O(\epsilon^3) ,
\] (21)
\[
\zeta = \frac{1}{2} \log \frac{q}{p} = \frac{1}{2} \epsilon^{1/2} + \frac{1}{3} \epsilon^{3/2} + O(\epsilon^{5/2}) .
\] (22)
Let us also set
\[
D_{\epsilon} = 2\sqrt{pq} = 1 - \frac{\epsilon}{2} + O(\epsilon^{3/2}) ,
\] (23)
\[
\Delta_{\epsilon} f(x) = \epsilon^{-2} \left( f(x + \epsilon) - 2 f(x) + f(x - \epsilon) \right) .
\] (24)
Then it holds [11, 26]
\[ dZ_A = \frac{1}{2} D_A \Delta_A dt + Z_A dM, \]  
(25)
where \( M \) is a stochastic process, in fact a martingale, in \( t \) with
\[ d\langle M(x)M(y) \rangle = \frac{1}{\epsilon} \delta_{x,y} b_{\epsilon}(\eta, x) dt \]  
(26)
and
\[ b_{\epsilon}(\eta, x) = 1 - (2\eta([\epsilon^{-1}x], \epsilon^{-2}t) - 1)(2\eta([\epsilon^{-1}x], \epsilon^{-2}t) + 1) - O(\sqrt{\epsilon}). \]  
(27)
By comparing (25) with (12) one notices a similarity. In fact it can be shown that, in the limit \( \epsilon \to 0 \), (25) converges to (12). This means that in this weakly asymmetric limit the ASEP tends to the KPZ equation. In other words one can obtain results for the KPZ equation by taking the weakly asymmetric limit for the ASEP.

We start from the following formula for ASEP, due to Tracy and Widom [25],
\[ \mathbb{P}(x_m(t/(q-p)) \leq x) = \int_{C_0} \prod_{k=0}^{\infty} (1 - \mu \tau^k) \det(1 + J(\mu)) \frac{d\mu}{\mu}, \]  
(28)
where \( \tau = p/q \), \( x_m(t) \) is the position of the particle starting from site \( m \) at time \( t = 0 \), \( C_0 \) is a circle around 0 and with radius in \((\tau, 1)\) and
\[ J(\mu; \eta, \eta') = \int_{C_1} \frac{\varphi_{\infty}(\zeta)}{\varphi_{\infty}(\eta')} \frac{\zeta^m}{(\eta')^{m+1}} \frac{\mu f(\mu, \zeta/\eta')}{\zeta - \eta} d\zeta, \]  
(29)
\[ \varphi_{\infty}(\eta) = (1 - \eta)^{-\tau} e^{(\eta/2)/1 - \eta)}, \]  
(30)
\[ f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \mu \tau^k} z^k, \]  
(31)
where \( C_1 \) is a circle around 0 and with radius in \((1, r/\tau)\). By carrying out an intricate asymptotic analysis for (28), one obtains
\[ \lim_{\epsilon \to 0} \mathbb{P} \left[ \frac{2\sqrt{\epsilon} N([\epsilon^{-1}x], \epsilon^{-2}t) - \frac{1}{2} t \epsilon^{-1} - \epsilon^{-1/2}|x| + x^2/2t + \log(2\sqrt{\epsilon})}{\gamma_t} \geq s \right] = F_t(s), \]  
where \( \gamma_t \) is defined by (10). This then translates to the result (9) for the KPZ equation.

6. ASEP and \( \delta \)-Bose
Based on the discussions in the previous two sections, it is natural to ask about the relation between the ASEP and the \( \delta \)-Bose gas. In [17] we explain how the duality of the ASEP found by Schütz [18] is useful to compute exponential moments of the ASEP current. Let us define
\[ N_x = \sum_{j=-\infty}^{x} n_j, \]  
(32)
where \( n_j \) is the number operator at site \( j \) and
\[ Q_x = \tau^{N_x}. \]  
(33)
Set
\[ \Xi_{\text{ASEP}}(x_1, \ldots, x_N; t) = \langle Q_{x_1} \cdots Q_{x_N} \rangle_t. \]  
(34)
Here \( \langle \cdots \rangle \) means the average of the ASEP dynamics for step initial condition. The duality of the ASEP says that this \( N \)-point correlation function in \( Q_x \) satisfies the master equation of the ASEP with \( N \) particles,

\[
\frac{d}{dt} \Xi_{\text{ASEP}}(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} \left( p \Xi_{\text{ASEP}}(\ldots, x_i - 1, \ldots; t) + q \Xi_{\text{ASEP}}(\ldots, x_i + 1, \ldots; t) \right.
\]

\[
\left. - (p + q) \Xi_{\text{ASEP}}(\ldots, x_i, \ldots; t) \right) .
\]

(35)

One has to solve (35) with the boundary condition

\[
p \Xi_{\text{ASEP}}(\ldots, x_i, x_i, \ldots; t) + q \Xi_{\text{ASEP}}(\ldots, x_i + 1, x_i + 1, \ldots; t) = (p + q) \Xi_{\text{ASEP}}(\ldots, x_i, x_i+1, \ldots; t) .
\]

(36)

For step initial condition \( Q_x = \tau^x \Theta(x) \), where \( \Theta(x) = 0(x \leq 0), 1(x \geq 1) \), and hence

\[
\Xi_{\text{ASEP}}(x_1, \ldots, x_N; t = 0) = \tau^{\sum_{i=1}^{N} x_i} \Xi_{\text{ASEP}}(x_1, \ldots, x_N). 
\]

(37)

The ASEP and the XXZ spin chain are related by a simple similarity transformation [18]. Let us set

\[
\Xi_{\text{XXZ}}(x_1, \ldots, x_N) = \tau^{-\frac{1}{2} \sum_{i=1}^{N} x_i} \Xi_{\text{ASEP}}(x_1, \ldots, x_N). 
\]

(38)

Then (38) satisfies

\[
\frac{d}{dt} \Xi_{\text{XXZ}}(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} \left( \Xi_{\text{XXZ}}(\ldots, x_i - 1, \ldots; t) + \Xi_{\text{XXZ}}(\ldots, x_i + 1, \ldots; t) 
\]

\[
- 2 \Delta \Xi_{\text{XXZ}}(\ldots, x_i, \ldots; t) \right),
\]

(39)

where \( \Delta = (\tau^{1/2} + \tau^{-1/2})/2 \) and

\[
\Xi_{\text{XXZ}}(\ldots, x_i, x_i, \ldots; t) + \Xi_{\text{XXZ}}(\ldots, x_i + 1, x_i + 1, \ldots; t) = 2 \Delta \Xi_{\text{XXZ}}(\ldots, x_i, x_i + 1, \ldots; t) .
\]

(40)

The initial condition is

\[
\Xi_{\text{XXZ}}(x_1, \ldots, x_N; t = 0) = \tau^{\frac{1}{2} \sum_{i=1}^{N} |x_i|}.
\]

(41)

Now let us consider the weakly asymmetric limit

\[
x \to \epsilon^{-1} x, \quad t \to \epsilon^{-2} t,
\]

(42)

and (19). We have

\[
\tau = \frac{1 - \sqrt{\epsilon}}{1 + \sqrt{\epsilon}}, \quad \Delta = 1 - \epsilon/2 + O(\epsilon^{3/2}).
\]

(43)

We see that in this limit, the Schrödinger equation (39), (40), (41) tends to the one of the \( \delta \)-Bose gas (16), (17), (18). In this way one understands the interrelation between KPZ, ASEP, and \( \delta \)-Bose gas.

7. Conclusion

In 2010 an important advance was achieved by establishing an analytic solution of the one-dimensional KPZ equation. This will certainly be useful for further studying the equation itself and for comparison with Monte Carlo simulations. Another interesting aspect is the deep connection of the KPZ equation with other models and the resulting similarity in the methods used. In this paper, we explained the connections to the directed polymer problem, \( \delta \)-Bose gas, ASEP, and XXZ spin chain. We discussed only the basic relations. It would be of considerable interest to more deeply elucidate these interrelations.
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