Some new function spaces of variable smoothness

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Abstract. A new Besov space of variable smoothness is introduced on which the norm is defined in terms of difference relations. This space is shown to be the trace of a weighted Sobolev space with a weight in the corresponding Muckenhoupt class. Methods of nonlinear spline approximation are applied to derive an atomic decomposition theorem for functions in a Besov space of variable smoothness. A complete description of traces on the hyperplane of a Besov space of variable smoothness and of a weighted Besov space with a weight in the corresponding Muckenhoupt class is given.

Bibliography: 27 titles.

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§1. Introduction

This paper looks at new modifications of Besov-type function spaces of variable smoothness, which are generalizations of the spaces $B^l_{p,q}(\mathbb{R}^n, \{t_k\})$ of [1].

Function spaces of variables smoothness (Besov- and Lizorkin-Triebel-type spaces) and various generalizations of them have been studied extensively. We only mention the papers [2]–[10] (but see the many references given in them).

It is interesting to note that the majority of studies on this subject have been concerned with spaces of variable smoothness consisting of distributions from the space $S'(\mathbb{R}^n)$.

A weight sequence (defining the variable smoothness) $\{s_k\} = \{s_k(\cdot)\}_{k=0}^{\infty}$ will be said to lie in $Y_{\alpha_1,\alpha_2}^{\alpha_3}$ if, for $\alpha_3 > 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$,

\begin{align*}
1) \quad \frac{1}{C_1}2^{\alpha_1(k-l)} \leq \frac{s_k(x)}{s_l(x)} \leq C_12^{\alpha_2(k-l)} \quad &\text{for } l \leq k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n; \\
2) \quad s_k(x) \leq C_2s_k(y)(1 + 2^k|x-y|)^{\alpha_3} \quad &\text{for } k \in \mathbb{N}_0, \quad x, y \in \mathbb{R}^n.
\end{align*}

The constants $C_1, C_2 > 0$ in (1.1) are independent of both the indices $k$ and $l$ and points $x$ and $y$.

In what follows we shall need the standard partition of unity. Let $B^n$ be the unit ball in $\mathbb{R}^n$ and let $\Psi_0 \in S(\mathbb{R}^n)$, $\Psi_0(x) = 1$ for $x \in B^n$, supp $\Psi_0 \subset 2B^n$. Given $j \in \mathbb{N}$, we set $\Psi_j(x) := \Psi_0(2^{-j}x) - \Psi_0(2^{-j+1}x)$ for $x \in \mathbb{R}^n$.

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In [5]–[7] and [9] the Besov spaces of variable smoothness were defined as follows (here we only look at the case when the integration exponents are constant).

**Definition 1.1.** Let \( p, q \in (0, \infty] \), \( \alpha_1, \alpha_2 \in \mathbb{R} \), \( \alpha_3 \geq 0 \), \( \{s_k\} \in Y^{\alpha_3}_{\alpha_1, \alpha_2} \). We let \( B_{p,q}^{\{s_k\}}(\mathbb{R}^n) \) denote the space of all distributions \( f \in S'(\mathbb{R}^n) \) with finite quasi-norm

\[
\|f \| B_{p,q}^{\{s_k\}}(\mathbb{R}^n) := \|s_j F^{-1}(\Psi_j F[f]) \|_q (L_p(\mathbb{R}^n)).
\] (1.2)

In (1.2), \( F \) and \( F^{-1} \) denote the direct and inverse Fourier transform, respectively. Formally replacing the weight sequence \( \{s_k\} \) in Definition 1.1 by the sequence \( \{2^{ks} \gamma\} \) with \( s \in \mathbb{R} \), \( \gamma \in A_{\infty}(\mathbb{R}^n) \), we obtain the definition of a weighted Besov space (see [11], [12]) with Muckenhoupt weight.

We also draw the reader’s attention to [8], [10] and [13], where the axiomatic approach to function spaces (of both constant and variable smoothness) was developed. Instead of the base space \( L_p(\mathbb{R}^n) \) with Muckenhoupt weight.

The spaces examined in [8] and [10] include, as a particular case, the scale of spaces \( Y^{\alpha_3}_{\alpha_1, \alpha_2} \) \( \{s_k\} \) lies in \( Y^{\alpha_3}_{\alpha_1, \alpha_2} \) with the additional assumption that

\[
0 < \alpha_1 \leq \alpha_2 < l.
\] (1.3)

Besov [3], [4] studied spaces of variable smoothness with \( p, q \in (1, \infty) \). It was also assumed that the weight sequence \( \{s_k\} \) lies in \( \text{loc} Y^{\alpha_3}_{\alpha_1, \alpha_2} \) under condition (1.3). The class \( \text{loc} Y^{\alpha_3}_{\alpha_1, \alpha_2} \) differs from the class \( Y^{\alpha_3}_{\alpha_1, \alpha_2} \) in that condition 2) is replaced by the condition

\[
2') \quad s_k(x) \leq 2^{\alpha_3} s_k(y) \quad \text{for } k \in \mathbb{N}_0, \quad |x - y| \leq 2^{-k}.
\] (1.4)

Clearly, \( Y^{\alpha_3}_{\alpha_1, \alpha_2} \subset \text{loc} Y^{\alpha_3}_{\alpha_1, \alpha_2} \), where \( \alpha_3 \) depends only on \( \alpha_3 \) and the constant \( C_2 \) from (1.1). The weighted class \( \text{loc} Y^{\alpha_3}_{\alpha_1, \alpha_2} \) is strictly larger than the class \( Y^{\alpha_3}_{\alpha_1, \alpha_2} \), because the former contains functions with an exponential rate of growth at infinity.

It is also worth mentioning that the methods used in [3]–[10] to prove various assertions about the spaces \( B_{p,q}^{\{s_k\}}(\mathbb{R}^n) \) were based on pointwise estimates of the weight sequence \( \{s_k\} \). This machinery was used in [13], [7], [8] and [10] to establish atomic decomposition theorems (as well as results on molecular and wavelet expansions) provided that the atoms in such a decomposition have zero high order.
moments. The number of zero moments for such atoms is governed by the exponents \(\alpha_1, \alpha_2\) and \(\alpha_3\). Unfortunately, it is not possible to check these conditions in specific problems. For example, if the high-order moments of the atoms from the decomposition of a function \(f : \mathbb{R}^n \to \mathbb{R}\) are zero, then in general we cannot assert that the corresponding moments of the traces of these atoms on the hyperplane \(\mathbb{R}^{n-1}\) are zero. In [9] the trace problem for Besov spaces of various smoothness was solved with the help of the atomic decomposition theorem under certain constraints on the weight sequence \(\{s_k\}\). These constraints allow one to avoid checking that the atoms in the trace decompositions have zero moments. We will show in \(\S\) 6 that these conditions can be relaxed substantially.

An analysis of the definitions of Besov spaces of variable smoothness used in [3]–[5] and [8] shows that in these papers the rather restrictive condition (1.3) was assumed (in the case when these spaces consisted of functions locally integrable to some power). The constraint \(\alpha_1 > 0\) is natural in the case when \(s_k \equiv C_k\) for all \(k \in \mathbb{N}\) (the \(C_k\) are positive constants), otherwise we would need to appeal to the theory of distributions. In the case of variable smoothness this condition is quite crude. The constraint \(l > \alpha_2\) was used to show that the norm in a Besov space is independent of the difference order. This condition is also fairly crude in the setting of variable smoothness. Indeed, when \(p, q \in (1, \infty)\) the author [1] introduced new modifications of Besov spaces of variable smoothness \(\widetilde{B}^l_{p,q} (\mathbb{R}^d, \{t_k\})\) in [1] these spaces were denoted by \(\mathcal{B}^l_{p,q} (\mathbb{R}^d, \{t_k\})\). Furthermore, these new methods should not depend upon the pointwise behaviour of the weight sequence \(\{\gamma_k\}\).

Clearly this calls for a more sophisticated approach to the very concept of variable smoothness. The definition of the weighted class \(\text{loc}Y^\alpha_{\alpha_1,\alpha_2}\) requires revision. There is also a need for new methods which, in particular, are capable of dealing with the space \(\widetilde{B}^l_{p,p} (\mathbb{R}^d, \{\gamma_k\})\), which is the trace of the weighted Sobolev space \(\widetilde{W}^l_p (\mathbb{R}^n, \gamma)\) on a plane of dimension \(1 \leq d < n\), provided that the weight \(\gamma \in A^p_{\text{loc}} (\mathbb{R}^n)\). Note that the weight sequence \(\{\gamma_k\}\) lies in the weighted class \(\text{loc}Y^\alpha_{\alpha_1,\alpha_2}\). However, condition (1.3) may fail to hold for the sequence \(\{\gamma_k\}\) if the weight is ‘sufficiently bad’ (see Remarks 4.2 and 4.4 below).

In this paper, for \(p, q, r \in (0, \infty), l \in \mathbb{N}\), we introduce the Besov space of variable smoothness \(\widetilde{B}^l_{p,q,r} (\mathbb{R}^n, \{t_k\})\), which is a subtle modification of the space \(\widetilde{B}^l_{p,q} (\mathbb{R}^n, \{t_k\})\) in [1]. The norm in this space is defined in terms of the difference relations \(\delta^l_r (Q^n)^f\) (see (2.7)). Here, the weight sequence \(\{t_k\}\) lies in the new weighted class \(X^\alpha_{\alpha_1,\sigma,p}\) (see Definition 2.4). For our purposes the weighted class \(X^\alpha_{\alpha,\sigma,p}\) proves to be more subtle (and altogether more natural!) than the class \(\text{loc}Y^\alpha_{\alpha_1,\alpha_2}\). Necessary and sufficient conditions for a sequence \(\{t_k\}\) to lie in the weighted class \(X^\alpha_{\alpha_1,\alpha_2}\) are expressed in terms of (in a sense) integral estimates, rather than pointwise estimates.

It is worth pointing out that the differences \(\delta^l_r (Q^n)^f\) were used in [13] for the purpose of constructing equivalent norms in Besov- and Lizorkin-Triebel-type spaces. However, our spaces \(\widetilde{B}^l_{p,q,r} (\mathbb{R}^n, \{t_k\})\) do not fit into the axiomatics of [13] because the constraints we place on the weight sequence \(\{t_k\}\) are less restrictive. We also
note that the differences $\delta_l^r(Q^n)f$ were recently employed by Besov [14], [15] to study spaces of functions of zero smoothness.

In dealing with the space $B^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ we shall adjust the methods of nonlinear spline approximation developed in [16] to study classical Besov spaces. We note that the methods of nonlinear spline approximation have never been used to analyze function spaces of variable smoothness and hence may be of independent interest. Using these methods we will be able to state certain theorems on the equivalence of norms in the spaces $B^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ and prove the atomic decomposition theorem for these spaces. We shall also characterize the trace of the space $B^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$. This result extends the results of [9], [11], for any constant $p \in (1, \infty)$.

The paper is organized as follows. In §2 we give auxiliary results and study the simplest properties of the new spaces $B^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$; §3 extends the results in [1] (this section may be looked upon as a rationale for the constructions that follow in §§4–6). In §4 we present the central results of the paper and, in particular, state the atomic decomposition theorem, which will be used in §5 and §6 to derive various embedding and trace theorems.

§2. Auxiliary results

We will adopt the following convention throughout. The symbol $C$ will be used to denote (different) ‘insignificant’ constants in various estimates. Sometimes, if it is required for the purposes of our exposition, we shall indicate the parameters on which one or other constant depends.

By definition, a weight function (a weight) is a measurable function $\gamma: \mathbb{R}^n \rightarrow (0, +\infty)$. Given a measurable set $E \subset \mathbb{R}^n$, we define

$$\gamma(E) := \int_E \gamma(x) \, dx.$$ 

Next, given a measurable set $E$, we denote by $L_p(E)$ the space of all functions $f: E \rightarrow \mathbb{R}$ equipped with the finite quasi-norm

$$||f||_{L_p(E)} := \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}} \quad \text{for } 0 < p < \infty,$$

$$||f||_{L_\infty(E)} := \text{ess sup} |f(x)|.$$

Further, given a measurable set $E$ and a weight $\gamma$, we denote the space of all functions $g: \mathbb{R}^n \rightarrow \mathbb{R}$ with finite quasi-norm $||g||_{L_p(E, \gamma)} := ||\gamma g||_{L_p(E)}$ by $L_p(E, \gamma)$.

In what follows, $Q^n$ will denote an open cube in the space $\mathbb{R}^n$ with sides parallel to the coordinate axes, $r(Q^n)$ will denote the side length of the cube $Q^n$, and $|Q^n|$ will denote its $n$-dimensional Lebesgue measure. For $\delta > 0$, by $\delta Q^n$ we shall mean the cube concentric with $Q^n$ and with side length $r(\delta Q^n) := \delta r(Q^n)$. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, $k \in \mathbb{Z}$, we denote the open dyadic cube of side $2^{-k}$ by

$$Q_{k,m}^n := \prod_{i=1}^n \left( \frac{m_i}{2^k} : \frac{m_i + 1}{2^k} \right)$$
and also define
\[ \tilde{Q}_{k,m}^n := \prod_{i=1}^n \left[ \frac{m_i}{2^k}, \frac{m_i + 1}{2^k} \right]. \]

We also set \[ I^n := \prod_{i=1}^n (-1, 1). \] We shall denote the \( n \)-dimensional ball (sphere) of radius \( \delta \) centred at the origin by \( \delta B^n (\delta S^n) \).

For \( x \in \mathbb{R}^n \) and \( E \subset \mathbb{R}^n \), we put \( x + E := \{ y \in \mathbb{R}^n : y = x + z, z \in E \} \).

Rychkov \[12\] introduced the class of weights \( A_p^{\text{loc}} (\mathbb{R}^n) \), which generalizes the well-known Muckenhoupt class \( A_p (\mathbb{R}^n) \) (for \( 1 < p \leq \infty \)).

**Definition 2.1** (see \[12\]). Let \( p \in (1, \infty), a > 0 \). Given a weight \( \gamma \) we say that \( \gamma \in A_p^{\text{loc}} (\mathbb{R}^n) \) if
\[ C_{\gamma,p,a}^{\text{loc}} := \sup_{Q_n : r(Q_n) \leq a} \frac{1}{|Q^n|} \int_{Q^n} \gamma(x) \, dx \left[ \frac{1}{|Q^n|} \int_{Q^n} \gamma^{-\frac{p'}{p}}(x) \, dx \right]^{\frac{p}{p'}} < +\infty. \]

**Definition 2.2** (see \[17\]). Let \( a > 0 \). We say that a weight \( \gamma \in A_1^{\text{loc}} (\mathbb{R}^n) \) if there exists a constant \( C_{\gamma,1,a}^{\text{loc}} > 0 \) independent of \( Q^n \) such that, for all cubes with side length \( r(Q^n) \leq a \),
\[ \frac{1}{|Q^n|} \int_{Q^n} \gamma(x) \, dx \leq A_\gamma(x) \] for almost all points \( x \in Q^n \).

By \( C_{\gamma,1,a}^{\text{loc}} \) we mean the smallest constant \( A \) satisfying the above inequality.

**Definition 2.3** (see \[12\]). Let \( a > 0 \). We say that a weight \( \gamma \in A_\infty^{\text{loc}} (\mathbb{R}^n) \) if, for some \( \alpha \in (0, 1) \),
\[ \sup_{r(Q^n) \leq a} \left( \sup_{F \subset Q^n, |F| > \alpha |Q^n|} \frac{\gamma(F)}{\gamma(F)} \right) < \infty. \]

**Remark 2.1** (see \[12\]). If a weight \( \gamma \in A_\infty^{\text{loc}} (\mathbb{R}^n) \), then there exists a number \( p_0 \in [1, \infty) \) such that \( \gamma \in A_{p_0}^{\text{loc}} (\mathbb{R}^n) \).

**Remark 2.2**. For \( p \in (1, +\infty] \) the definition of the class \( A_p^{\text{loc}} (\mathbb{R}^n) \) is independent of the choice of the parameter \( a \). For various \( a > 0 \) the constants \( C_{\gamma,p,a}^{\text{loc}} \) are estimated in terms of each other (see \[12\]). One can show that a similar result also holds for \( A_1^{\text{loc}} (\mathbb{R}^n) \).

Given \( f \in L_1^{\text{loc}} (\mathbb{R}^n), a > 0 \), we let \( M_{\leq a} f \) denote the local version of the Hardy-Littlewood maximal function,
\[ M_{\leq a} f(x) := \sup_{x \in Q^n, r(Q^n) \leq a} \frac{1}{|Q^n|} \int_{Q^n} |f(y)| \, dy. \]

The next theorem generalizes Muckenhoupt’s classical result (see, for example, \[18\], Ch. 5, §3, Theorem 1).

**Theorem 2.1** (see \[12\]). Let \( p \in (1, \infty), \gamma \in A_p^{\text{loc}} (\mathbb{R}^n), a > 0 \). Then there exists a constant \( C = C(n, p, a, \gamma) > 0 \) such that
\[ \int_{\mathbb{R}^n} \gamma(x) \{ M_{\leq a} f(x) \}^p \, dx \leq C \int_{\mathbb{R}^n} \gamma(x) |f(x)|^p \, dx \]
for all \( f \in L_p (\mathbb{R}^n, \gamma^{\frac{1}{p'}}). \)
Theorem 2.2 (Hardy’s inequality for sequences). Let $0 < q \leq \infty$, $\mu \leq q$, $\beta \geq 0$, and let $\{a_k\}$ be a sequence of real numbers. Then

$$\left( \sum_{k=0}^{\infty} 2^{qk^\beta} |b_k|^q \right)^{\frac{1}{q}} \leq C \left( \sum_{k=0}^{\infty} 2^{qk^\beta} |a_k|^q \right)^{\frac{1}{q}}$$  \hspace{1cm} (2.1)

(with the obvious modifications for $q = \infty$ or $\mu = \infty$), and the constant $C > 0$ is independent of the sequence $\{a_k\}$ when either

$$|b_k| \leq C \left( \sum_{j=k}^{\infty} |a_j|^\mu \right)^{\frac{1}{\mu}}$$  \hspace{1cm} (2.2)

provided that $\beta > 0$

or

$$|b_k| \leq C 2^{-k\lambda} \left( \sum_{j=0}^{k} 2^{j\mu\lambda} |a_k|^\mu \right)^{\frac{1}{\mu}}$$  \hspace{1cm} (2.3)

provided that $\lambda > \beta$.

In what follows we shall also need the following elementary fact.

Lemma 2.1. Let $r \in (0, \infty]$ and let $f_j \in L^1_{\text{loc}}(\mathbb{R}^n)$ for $j \in \mathbb{N}_0$. Then, for $\mu \leq \min\{1, r\}$,

$$\left\| \sum_{j=0}^{\infty} f_j \mid L^r(\mathbb{R}^n) \right\| \leq \left( \sum_{j=0}^{\infty} \| f_j \mid L^r(\mathbb{R}^n) \|^\mu \right)^{\frac{1}{\mu}}.$$  \hspace{1cm} (2.4)

The proof follows easily since the $l_q$-norm is monotonic with respect to $q$ for $q \geq 1$.

Let $l \in \mathbb{N}$, $r \in (0, \infty]$ and let $\Omega$ be a domain in $\mathbb{R}^n$. For a function $\varphi \in L^1_{\text{loc}}(\Omega)$, $h \in \mathbb{R}^n$, $t > 0$, and a cube $Q^n$ we define the differences of order $l$ as follows (with straightforward modifications for $r = \infty$):

$$\Delta^l(h, \Omega)\varphi(x) := \begin{cases} \sum_{j=0}^{l} C^l_j (-1)^j \varphi(x + jh) & \text{for } [x, x + lh] \subset \Omega, \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (2.5)

$$\bar{\Delta}^l_r(t, \Omega)\varphi(x) := \left( \frac{1}{t^n} \int_{tI^n} |\Delta^l(h, \Omega)\varphi(x)\mid^r \, dh \right)^{\frac{1}{r}} \text{ for } x \in \mathbb{R}^n;$$  \hspace{1cm} (2.6)

$$\delta^l_r(Q^n, \Omega)\varphi := \left( \frac{1}{[r(Q^n)]^{2n}} \int_{r(Q^n)I^n} \int_{Q^n} |\Delta^l(h, \Omega)\varphi(x)\mid^r \, dx \, dh \right)^{\frac{1}{r}}.$$  \hspace{1cm} (2.7)

We set

$$\Delta^l(h)\varphi := \Delta^l(h, \mathbb{R}^n)\varphi, \quad \bar{\Delta}^l_r(t)\varphi := \bar{\Delta}^l_r(t, \mathbb{R}^n)\varphi, \quad \delta^l_r(Q^n)\varphi := \delta^l_r(Q^n, \mathbb{R}^n)\varphi.$$  

Given a cube $Q^n$, for $l \in \mathbb{N}$, $r \in (0, \infty]$, we let $\omega_1(\varphi, Q^n)_r$ denote the modulus of continuity of the function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ on the cube $Q^n$ in the $L^r(Q^n)$-metric; that is,

$$\omega_1(\varphi, Q^n)_r := \sup_{|h| > 0} \| \Delta^l(h, Q^n)\varphi \mid L^r(\mathbb{R}^n) \|.$$
Let \( l \in \mathbb{N}, r \in (0, \infty] \). Given a cube \( Q^n \), we define the local best approximation to the function \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \) in the \( L_r(Q^n) \)-metric by polynomials of degree less than \( l \) by

\[
E_l(\varphi, Q^n)_r := \inf_{\deg(P) < l} \| \varphi - P \|_{L_r(Q^n)}.
\]

Given \( Q^n \) we also define the best local approximation to \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \) in the \( L_r(Q^n) \)-metric by polynomials of coordinate degree less than \( l \) (the total degree of the polynomial is, clearly, at most \( n(l-1) \)) as

\[
\tilde{E}_l(\varphi, Q^n)_r := \inf_{\deg_i(P) < l} \| \varphi - P \|_{L_r(Q^n)},
\]

where the infimum is taken over all polynomials \( P \) whose degree in the variable \( x_i \) is less than \( l \) for each \( i \in \{1, \ldots, n\} \).

The following two-sided estimates are well known (for \( r \geq 1 \), see [19]; for the general setting, see [20]):

\[
C_1 \delta_r(Q^n, Q^n)_r \varphi \leq |Q^n|^{-\frac{1}{r}} \omega_r(Q^n)_r \varphi \leq C_2 \delta_r(Q^n, Q^n)_r \varphi, \tag{2.8}
\]

\[
C_3 \delta_r(Q^n, Q^n)_r \varphi \leq |Q^n|^{-\frac{1}{r}} E_l(\varphi, Q^n)_r \leq C_4 \delta_r(Q^n, Q^n)_r \varphi, \tag{2.9}
\]

where the constants \( C_1, C_2 \) in (2.8) and the constants \( C_3, C_4 \) in (2.9) are independent of both the function \( \varphi \) and the cube \( Q^n \).

Let \( l \in \mathbb{N}, r \in (0, \infty] \) and let \( Q^n \) be a cube. A polynomial \( P_{Q^n} \) will be said to be a polynomial of near best approximation to a function \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \) by polynomials of degree less than \( l \) in the \( L_r(Q^n) \)-metric with constant \( A \geq 1 \) if

\[
\| \varphi - P_{Q^n} \|_{L_r(Q^n)} \leq AE_l(\varphi, Q^n)_r.
\]

The definition a polynomial of almost \( L_r(Q^n) \)-best approximation by polynomials of coordinate degree less than \( l \) to a function \( \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) \) with constant \( A \geq 1 \) is similar.

**Definition 2.4.** Let \( p \in (0, \infty] \). A weight sequence \( \{t_k\} \) is called \( p \)-admissible if \( t_k \in L^1_{\text{loc}}(\mathbb{R}^n) \) for all \( k \in \mathbb{N}_0 \).

**Definition 2.5.** Let \( l \in \mathbb{N}, 0 < p, q, r \leq \infty \), and let \( \{t_k\} \) be a \( p \)-admissible weight sequence. We set

\[
\mathcal{B}_l^{p, q, r}(\mathbb{R}^n, \{t_k\}) := \{ \varphi : \varphi \in L^1_{\text{loc}}(\mathbb{R}^n), \| \varphi \|_{\mathcal{B}_l^{p, q, r}(\mathbb{R}^n, \{t_k\})} < +\infty \}, \tag{2.10}
\]

where

\[
\| \varphi \|_{\mathcal{B}_l^{p, q, r}(\mathbb{R}^n, \{t_k\})} := \left[ \sum_{k=1}^{\infty} \| t_k \Delta_l^k(2^{-k}) \varphi \|_{L_p(\mathbb{R}^n)}^q \right]^{\frac{1}{q}} + \left( \int_{\mathbb{R}^n} t_0^p(x) \| \varphi \|_{L_r(x + I^n)}^p \, dx \right)^{\frac{1}{p}};
\]

\[
\tilde{\mathcal{B}_l^{p, q, r}}(\mathbb{R}^n, \{t_k\}) := \{ \varphi : \varphi \in L^1_{\text{loc}}(\mathbb{R}^n), \| \varphi \|_{\tilde{\mathcal{B}_l^{p, q, r}}(\mathbb{R}^n, \{t_k\})} < +\infty \}, \tag{2.11}
\]
where
\[ \| \varphi \mid \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \| := \left[ \sum_{k=1}^{\infty} \| t_k \delta \varphi (\cdot + 2^{-k}I^n) \|_{L_p(\mathbb{R}^n)}^q \right]^{\frac{1}{q}} + \left( \int_{\mathbb{R}^n} t_0^p(x) \| \varphi \mid L_r(x + I^n) \|_p \, dx \right)^{\frac{1}{p}}, \]
making the obvious modifications for \( p = \infty \) or \( q = \infty \).

Let \( \gamma \) be a weight and let \( l > s > 0 \). We set
\[ \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \gamma) := \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{2^{ks} \gamma \}), \quad \tilde{B}^s_{p,q,r}(\mathbb{R}^n, \gamma) := \tilde{B}^s_{p,q,r}(\mathbb{R}^n, \{2^{ks} \gamma \}). \]
The corresponding spaces will be called weighted Besov spaces with weight \( \gamma \).

**Remark 2.3.** The space \( \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \) (\( \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \)) may turn out to be trivial, containing only the functions that vanish almost everywhere. We give a condition on the parameters \( l, p \) and \( q \) and the \( p \)-admissible sequence \( \{t_k\} \) that guarantees that the corresponding space is nontrivial. Given \( p, q \in (0, \infty] \) and a cube \( Q^n \), assume that
\[ \left( \sum_{k=0}^{\infty} \left( \int_{Q^n} 2^{-klp} t_k^p(x) \, dx \right)^{\frac{2}{q}} \right)^{\frac{1}{q}} < \infty, \]
with the obvious modifications when \( p, q = \infty \).

Under this condition,
\[ C_0^\infty \subset \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \quad (\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})). \]
This follows easily from the Taylor expansion of \( \varphi \), taking the remainder in the Lagrange form.

**Remark 2.4.** The space \( \tilde{B}^l_{p,q,1}(\mathbb{R}^n, \{t_k\}) \) was introduced in [1] for \( p, q \in (1, \infty) \) for weight sequences \( \{t_k\} \in \text{loc} Y_{\alpha_1, \alpha_2}^{\alpha_3} \) without restrictions on the parameters \( \alpha_1 \) and \( \alpha_2 \). The space \( \tilde{B}^l_{p,q,1}(\mathbb{R}^n, \{t_k\}) \) was studied by Kempka and Vybfial [5] for weight sequences \( \{t_k\} \in Y_{\alpha_1, \alpha_2}^{\alpha_3} \) and \( p, q \in (0, \infty] \), under condition (1.3). A space close to the space \( \tilde{B}^l_{p,q,1}(\mathbb{R}^n, \{t_k\}) \) (but different from it!) was studied by Besov in [3] and [4] for \( p, q \in (1, \infty) \) and \( \{t_k\} \in \text{loc} Y_{\alpha_1, \alpha_2}^{\alpha_3} \), under condition (1.3).

**Definition 2.6.** Let \( p \in (0, \infty] \). For a \( p \)-admissible weight sequence \( \{t_k\} \) we set
\[ t_{k,m} := \| t_k \mid L_p(Q_{k,m}^n) \| \quad \text{for} \quad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \] (2.12)
\[ \tilde{t}_k(x) := 2^{kn/p} \sum_{m \in \mathbb{Z}^n} t_{k,m} x Q_{k,m}^n(x) \quad \text{for} \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n \] (2.13)
(if \( p = \infty \) we assume that \( \frac{kn}{p} = 0 \)).

In what follows, the multiple sequence \( \{t_{k,m}\} \) (the weight sequence \( \{\tilde{t}_k\} \)) defined by formula (2.12), (2.13) will be called the multiple sequence (weight sequence) \( p \)-associated with the weight sequence \( \{t_k\} \).
Definition 2.7. Let $\alpha_3 \geq 0, \alpha_1, \alpha_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in (0, +\infty], \alpha = (\alpha_1, \alpha_2)$ and let $
abla = (\sigma_1, \sigma_2)$. We let $X_{\alpha, \sigma, p}^{\alpha_3} = X_{\alpha, \sigma, p}^{\alpha_3}(\mathbb{R}^n)$ denote the set of $p$-admissible weight sequences $\{t_k\}$ satisfying the following conditions:

1) there exist numbers $C_1, C_2 > 0$ such that

$$
\left(2^{kn} \int_{Q_{k,m}^n} \tilde{t}_k^p(x) \right)^{\frac{1}{p}} \left(2^{kn} \int_{Q_{k,m}^n} (\tilde{t}_j)^{-\sigma_1}(x) \right)^{\frac{1}{\sigma_1}} \leq C_1 2^{\alpha_1(k-j)},
$$

(2.14)

$$
0 \leq k \leq j, \quad m \in \mathbb{Z}^n,
$$

$$
\left(2^{kn} \int_{Q_{k,m}^n} \tilde{t}_k^p(x) \right)^{\frac{1}{p}} \left(2^{kn} \int_{Q_{k,m}^n} \tilde{t}_j^{\sigma_2}(x) \right)^{\frac{1}{\sigma_2}} \leq C_2 2^{\alpha_2(j-k)},
$$

(2.15)

(the modifications of (2.6) or (2.7) for $\sigma_1 = \infty$ or $\sigma_2 = \infty$ are clear);

2) for all $k \in \mathbb{N}_0$

$$
0 < t_{k,m} \leq 2^{\alpha_3} t_{k,\tilde{m}} \quad \text{for} \quad m, \tilde{m} \in \mathbb{Z}^n, \quad |m_i - \tilde{m}_i| \leq 1, \quad i = 1, \ldots, n.
$$

(2.16)

Remark 2.5. We denote the subset of $X_{\alpha, \sigma, p}^{\alpha_3}$ consisting of only $p$-admissible weight sequences $\{t_k\}$ by $\tilde{X}_{\alpha, \sigma, p}^{\alpha_3}$. It is clear that $\tilde{X}_{\alpha, \sigma, p}^{\alpha_3} = \text{loc} Y_{\alpha_3, \alpha_1, \alpha_2}^{\alpha_3}$ for $p \in (0, \infty]$,

$$
-\infty < \alpha_1 \leq \alpha_2 < \infty, \quad \alpha_3 \geq 0 \quad \text{and} \quad \sigma_1 = \sigma_2 = \infty.
$$

We fix $-\infty < \alpha_1^i \leq \alpha_2^i < \infty, \alpha_3 \geq 0, \sigma_1^i, \sigma_2^i \in (0, \infty], \quad i = 1, 2$, and set

$$
\alpha^i := (\alpha_1^i, \alpha_2^i), \quad \sigma^i := (\sigma_1^i, \sigma_2^i) \quad \text{for} \quad i = 1, 2.
$$

Elementary arguments based on Hölder’s inequality and the monotonicity of the $l_q$-norm (with respect to $q$) prove the embedding $\tilde{X}_{\alpha, \sigma, p}^{\alpha_3} \subset \tilde{X}_{\alpha, \sigma, p}^{\alpha_3}$ with

$$
\alpha_1^2 = \alpha_1^i + n \min \left\{ \frac{1}{\sigma_1^i} - 1, \frac{1}{\sigma_1^i}, 0 \right\}, \quad \alpha_2^2 = \alpha_2^i + n \max \left\{ \frac{1}{\sigma_2^i} - 1, \frac{1}{\sigma_2^i}, 0 \right\}.
$$

Remark 2.6. Clearly, a multiple sequence $\{t_{k,m}\}$ might be $p$-associated with several weight sequences. However, this will not be an impediment to further constructions if $\{t_k\} \in X_{\alpha, \sigma, p}^{\alpha_3}$. Indeed, using (2.16) we have, by an elementary argument,

$$
\| \varphi | \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) \| \sim \| \varphi | \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{\tilde{t}_k\}) \| \sim \| \varphi | \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \| \quad \text{and} \quad \| \varphi | \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \| \sim \| \varphi | \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \|.
$$

(1)

$$
:= \left\| \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p [\delta_{r}(Q_{k,m}^n) \varphi]^p \right)^{\frac{1}{p}} \right\|_{l_q} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m}^p \| \varphi | L_{r}(Q_{0,m}^n) \| \right)^{\frac{1}{p}}
$$

(2.17)

(with corresponding modifications in the case $p = \infty$).

Here, the constant through which one norm is estimated in terms of the other in (2.17) depends only on $\alpha_3, l, p$ and $n$.

For a fixed $p \in (0, \infty]$, it is clear that there exists a bijection between the multiple sequences $\{t_{k,m}\}$ and the sets of weight sequences $\{t_k\} \in X_{\alpha, \sigma, p}^{\alpha_3}$, for which the multiple sequence $\{t_{k,m}\}$ is $p$-associated with the weight sequence $\{t_k\}$.

Taking (2.17) into account, in what follows the space $\tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\})$ will also be denoted by the symbol $\tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\})$. 

Definition 2.8. Let \( p \in (0, \infty) \), \( d \in \mathbb{N}_0 \), and let a weight \( \gamma^p \in A^\text{loc}_{\infty}(\mathbb{R}^{n + d}) \). We set
\[
\Xi_{k,m} := Q_{k,m}^n \times (2^{-k} B^d \setminus 2^{-k-1} B^d) \quad \text{for} \quad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.
\]
The multiple sequence \( \hat{\gamma}_{k,m} \) defined by
\[
\hat{\gamma}_{k,m} := \|\gamma| L_p(\Xi_{k,m})\| \quad \text{for} \quad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,
\]
will be called the multiple sequence generated by the weight \( \gamma \).

The following important properties of the sequence \( \{\hat{\gamma}_{k,m}\} \) will be required in what follows.

Lemma 2.2. Let \( p \in (0, \infty) \), \( d \in \mathbb{N}_0 \), let the weight \( \gamma^p \in A^\text{loc}_{\infty}(\mathbb{R}^{n + d}) \), and let the multiple sequence \( \hat{\gamma}_{k,m} \) be generated by the weight \( \gamma \). Also let \( m \in \mathbb{Z}^n \), \( k, j \in \mathbb{N}_0 \), \( j \geq k \), and let \( G_{j,k,m} \) be an arbitrary set of cubes \( Q_{j,m}^n \subset Q_{k,m}^n \). Then:

1) the inequality
\[
\sum_{Q_{j,m}^n \subset Q_{k,m}^n} \hat{\gamma}_{p,j,m} \leq C^2 (k-j)^d \hat{\gamma}_{k,m}^p \tag{2.18}
\]
holds, and the constants \( C > 0 \) and \( \delta > 0 \) depend only on \( \gamma \), \( n \) and \( d \);

2) the inequality
\[
\sum_{Q_{j,m}^n \in G_{j,k,m}} \hat{\gamma}_{p,j,m} \leq C \left( \frac{\left| \bigcup_{Q_{j,m}^n \in G_{j,k,m}} Q_{j,m}^n \right|}{\left| Q_{k,m}^n \right|} \right) \delta' \sum_{Q_{j,m}^n \subset Q_{k,m}^n} \hat{\gamma}_{p,j,m} \tag{2.19}
\]
holds, and the constants \( C > 0 \) and \( \delta' > 0 \) depend only on \( \gamma \), \( n \) and \( d \);

3) for \( a > 1 \), \( |m - m'| \leq a, k \geq 0 \),
\[
2^{-\delta_3} \hat{\gamma}_{k,m} \leq \hat{\gamma}_{k,m} \leq 2^{\delta_3} \hat{\gamma}_{k,m},
\]
where the number \( \delta_3 \geq 0 \) depends only on \( \gamma \), \( n \), \( p \), \( d \);

4) for any cube \( Q_{k,m}^n \) and any cube \( Q_{k+1,m}^n \subset Q_{k,m}^n \),
\[
\hat{\gamma}_{k,m} \leq C \hat{\gamma}_{k+1,m},
\]
for \( k \geq 0 \), \( m \in \mathbb{Z}^n \), where the constant \( C > 0 \) depends only on \( \gamma \), \( n \), \( d \), \( p \).

Proof. To prove 3) it suffices to take some cube \( Q^{n + d} \) containing both the sets \( \Xi_{k,m} \) and \( \Xi_{k,m}^d \) and use the fact that \( \gamma^p \) satisfies the doubling condition on the cube \( Q^{n + d} \) with the doubling constant depending only on the constant \( C^\text{loc}_{f,p,r}(Q^{n + d}) \) (the proof of the last fact is similar to that of the corresponding result in [18], Ch. 5). The proof of assertion 4) is similar to that of assertion 3).

We will prove assertion 1); assertion 2) is dealt with in a similar way. It is easy to see that
\[
\frac{\left| \bigcup_{Q_{j,m}^n \subset Q_{k,m}^n} \Xi_{j,m} \right|}{\left| \Xi_{k,m} \right|} \leq C^2 (k-j)^d. \tag{2.20}
\]
Using Definition 2.3 and Remark 2.1 one can easily prove that for some \( \delta(\gamma) > 0 \), for any cube \( Q^n \), \( r(Q^n) \leq a \), and any measurable set \( F \subset Q^n \),

\[
\frac{\gamma^p(F)}{\gamma^p(Q^n)} \leq C \left( \frac{|F|}{|Q^n|} \right)^{\delta(\gamma)},
\]

where the constant \( C > 0 \) is independent of both \( Q^n \) and \( F \).

From (2.20), (2.21) we get (2.18), completing the proof of the lemma.

We let \( \delta_1(\gamma) := \delta_1(\gamma, n, d) \) denote the supremum over all \( \delta \) for which (2.18) holds. Similarly, \( \delta_2(\gamma) := \delta_2(\gamma, n, d) \) will denote the supremum over all \( \delta' \) satisfying (2.19).

Note that in general \( \delta_1 \neq \delta_2 \). Indeed, let \( \gamma^p(x_1, x_2) := |x_1|^\beta \) with \( (x_1, x_2) \in \mathbb{R}^2, \beta > 0 \). Then, clearly \( \gamma^p \in A_\infty(\mathbb{R}^2) \). Also, \( \delta_1(\gamma) = 1 (n = d = 1) \) for any \( \beta > 0 \), whereas \( \delta_2(\gamma) \) depends on \( \beta > 0 \).

**Example 2.1.** We now give an example of a weight sequence \( \{t_k\} \in X_{\alpha_3}^{\alpha_3} \) which will be important in what follows. We note that this example was the main impetus for introducing the classes \( X_{\alpha_3}^{\alpha_3} \).

Let \( d \in \mathbb{N}_0, \ p \in (0, \infty) \), let the weight \( \gamma^p \in A^{loc}_\infty(\mathbb{R}^{n+d}) \), and suppose that a multiple sequence \( \{\tilde{\gamma}_{k,m}\} \) is generated by the weight \( \gamma \). By Remark 2.1 we have \( \gamma^p \in A^{loc}_{pp_0}(\mathbb{R}^{n+d}) \) for some \( p_0 \in [1, \infty) \). Assume that the weight sequence \( \{s_k\} \in \{X_{\alpha_3}^{\alpha_3}\} \). We set

\[
\{s_{k,m}\} = \|s_k \mid L_p(Q^n_{k,m})\|, \quad t_{k,m} := \tilde{\gamma}_{k,m}(2^{kn}p s_k,m),
\]

with \( k \in \mathbb{N}_0, \ m \in \mathbb{Z}^n \). Then the weight sequence \( \{t_k\} = \{\tilde{t}_k\} \) belongs to \( \tilde{X}_{\alpha_3}^{\alpha_3} \) with

\[
\alpha_3 = \alpha_3' + \delta_3(\gamma), \quad \alpha_2 = \alpha_2' - \frac{d(\delta_1(\gamma) - \varepsilon)}{p},
\]

\[
\alpha_1 = \alpha_1' + \frac{n}{\sigma_1} + \frac{n}{p} - \frac{(n + d)p_0}{p} + \frac{d \varepsilon}{p_{p_0}}(\delta_1(\gamma)^{-\frac{pp_0}{p_0}} - n, d - \varepsilon),
\]

\[
\sigma_2 = p, \quad \sigma_1 = \frac{p_0}{p_{p_0}}
\]

for any \( \varepsilon > 0 \). In fact, (2.15) and (2.16) follow easily from assertions 1) and 3) in Lemma 2.2. We will verify (2.14) with \( p_0 > 1 \), the case \( p_0 = 1 \) is dealt with similarly. By Definition 2.2,

\[
\left(2^{kn} \int_{Q^n_{k,m}} \tilde{t}_k^p(x)\right)^{\frac{1}{p}} \leq C_{12} \left(2^{kn} \int_{Q^n_{k,m}} (\tilde{t}_j)^{-\sigma_1}(x)\right)^{\frac{1}{p}}
\]

\[
\leq \left(2^{kn} \int_{Q^n_{k,m}} \tilde{t}_k^p(x)\right)^{\frac{1}{p}} \leq C_{12} 2^{(k-j)(\alpha_1' + \frac{n}{\sigma_1} + \frac{n}{p}) \tilde{\gamma}_{k,m} \left( \sum_{\tilde{m} \in \mathbb{Z}^n} \frac{1}{(\tilde{\gamma}_{j,\tilde{m}})^{p_{p_0}}} \right)^{\frac{pp_0}{pp_0}}}
\]

\[
\leq C_{22} 2^{(k-j)(\alpha_1' + \frac{n}{\sigma_1} + \frac{n}{p}) \tilde{\gamma}_{k,m} 2^j(n+d) \frac{p_0}{p} \left( \sum_{\tilde{m} \in \mathbb{Z}^n} \int_{Q^n_{j,\tilde{m}}} \gamma^{-\frac{pp_0}{p_0}}(x) dx \right)^{\frac{pp_0}{pp_0}}}
\]
\[ \leq C_3 2^{(k-j)(\alpha_1' + \frac{m}{p} + \frac{np}{pp_0}(\delta_1(\gamma - \frac{pp_0}{p_0} , n, d) - \varepsilon))} \gamma_{k,m} 2^{j(n+d) \frac{p_0}{p}} \times \left( \int_{\mathbb{R}^d \cap \Omega_{k,m}} \gamma^{-p \frac{p_0}{p}}(x) \, dx \right) \frac{p_0}{pp_0} \leq C_4 2^{(k-j)\alpha_1}. \]

It is worth pointing out that \( \alpha_i = \alpha_i', i = 1, 2, \) if \( d = 0. \)

Let \( p, q \in (0, \infty], \; r \in (0, p], \; \alpha_1, \alpha_2 \in \mathbb{R}, \; \alpha_3 \geq 0, \; \sigma_1, \sigma_2 \in (0, \infty], \) and let \( \{t_{k,m}\} \) be the multiple sequence \( p \)-associated with a \( p \)-admissible weight sequence \( \{t_k\} \in X_{\alpha, \sigma, p}^{\alpha_3}, \) \( c > 1. \) In the space \( \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}), \) we consider the quasi-norms generated by the multiple sequence \( \{t_{k,m}\} : \)

\[ \| \phi \| \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}, c) \|^{(2)} := \left\| \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p [\delta_r^l(cQ_{k,m}^n, cQ_{k,m}^n) \varphi]^p \right)^{\frac{1}{p}} \right\|_{L_q} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m}^p \| \varphi \|_{L_r(Q_{0,m}^n)} \right)^{\frac{1}{p}}, \]

\[ \| \phi \| \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}, c) \|^{(3)} := \left\| \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p [2^{\frac{kn}{p}} E_l(\varphi, cQ_{k,m}^n) r]^p \right)^{\frac{1}{p}} \right\|_{L_q} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m}^p \| \varphi \|_{L_r(Q_{0,m}^n)} \right)^{\frac{1}{p}}, \]

\[ \| \phi \| \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}, c) \|^{(4)} := \left\| \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p [2^{\frac{kn}{p}} \omega_l(\varphi, cQ_{k,m}^n) r]^p \right)^{\frac{1}{p}} \right\|_{L_q} + \left( \sum_{m \in \mathbb{Z}^n} t_{0,m}^p \| \varphi \|_{L_r(Q_{0,m}^n)} \right)^{\frac{1}{p}}. \]

**Theorem 2.3.** Let \( p, q, r \in (0, \infty], \; \alpha_3 \geq 0, \; \alpha_1, \alpha_2 \in \mathbb{R}, \; \sigma_1, \sigma_2 \in (0, +\infty], \) let \( \{t_k\} \in X_{\alpha, \sigma, p}^{\alpha_3} \) be a \( p \)-admissible sequence, and let \( \{t_{k,m}\} \) be the associated multiple sequence. Then, for \( i = 1, 2, 3, 4, \) the quasi-norms \( \| \cdot \| \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}, c) \|^{(i)} \) are equivalent on the space \( \tilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}). \)

**Proof.** This theorem was proved in \([1]\) for \( r = 1, \) \( p, q \in (1, \infty) \) and \( \{t_k\} \in \text{loc}X_{\alpha_1, \alpha_2}^{\alpha_3}. \) In the general setting the proof is similar if we take \((2.8), (2.9), (2.17)\) and Remark \((2.5)\) into account, and use the estimate

\[ \delta_r^l(cQ_{k,m}^n, cQ_{k,m}^n) \varphi \leq C \sum_{\tilde{m} \in \mathbb{Z}^n} \delta_r^l(Q_{k-j(c), \tilde{m}}^n, cQ_{k,m}^n) \varphi, \]

where \( c > 1 \) is from the hypotheses of the theorem, and \( j(c) \geq 1 \) is the smallest natural number such that \( 2^{-k+j(c)} > c2^{-k}. \) The constant \( C \) in \((2.25)\) depends only on \( n, r, l \) and \( c. \)
Then of (2.26) to the power \( p = 2.7 \).

Remark e space and the corresponding norms are equivalent

Let Theorem 2.5.

Proof Let Lemma 2.3.

Besov space \([21]\), depends on the fact that \( \| \frac{L_r(Q^n)}{Q^n} \| \) is complete for any cube \( Q^n \), and it uses Remark 2.5 and the equivalence of the norm in (2.17). We omit the details, which are quite standard.

Lemma 2.3. Let \( p, q \in (0, \infty] \), \( r \in (0, p], l \in \mathbb{N}, \alpha_3 \geq 0, \alpha_1, \alpha_2 \in \mathbb{R}, \sigma_1, \sigma_2 \in (0, +\infty], \) and let \( \{ t_k \} \in X_{R, \alpha, \sigma}^{\alpha_3} \) be a \( p \)-admissible weight sequence. Then

\[
\bar{B}_{p, q, r}(\mathbb{R}^n, \{ t_k \}) \subset \bar{B}_{p, q, r}(\mathbb{R}^n, \{ t_k \}).
\]

Proof. We shall only consider the case \( p, q \in (0, \infty) \), the arguments when \( p = \infty \) or \( q = \infty \) are similar. We compare the first terms in the quasi-norms (2.10) and (2.11), respectively. Applying Hölder’s inequality to the integral with respect to \( y \) and using (2.13), for \( k \in \mathbb{N} \) we obtain

\[
\sum_{m \in \mathbb{Z}^n} t_{k, m}^p \{ \delta_{I, p}^l(Q_{k, m}^n) \varphi \}^p \leq \sum_{m \in \mathbb{Z}^n} t_{k, m}^p 2^{-n \sigma_1 + n \sigma_2} \int_{Q_{k, m}^n} \left[ \int_{I/2^k} |\Delta^l(h) \varphi(y)|^q dh \right]^p dy
\]

\[
= \int_{\mathbb{R}^n} t_{k}^p(y) [\Delta^l_2(2^{-k}) \varphi(y)]^p dy.
\]  

(2.26)

The proof of the embedding will be complete once we have raised both parts of (2.26) to the power \( \frac{p}{p} \) and summed over all \( k \).

Theorem 2.5. Let \( p, q \in (0, \infty], p \neq \infty, \) let \( p_0 \in [1, \infty), 0 < r_1 \leq r_2 \leq \frac{p}{p_0}, l \in \mathbb{N} \) and let \( \gamma^p \in A_{l, p_0}^{\alpha_3}(\mathbb{R}^n) \). Suppose that \( \alpha_3 \geq 0, 0 < \alpha_1 \leq \alpha_2 < l \), and that the weight sequence \( \{ s_k \} \in \alpha_1, \alpha_2 \). We set

\[
t_k(x) = \gamma(x)s_k(x) \quad \text{for} \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n.
\]

Then

\[
\bar{B}_{p, q, r_1}(\mathbb{R}^n, \{ t_k \}) = \bar{B}_{p, q, r_2}(\mathbb{R}^n, \{ t_k \})
\]

and the corresponding norms are equivalent.

The proof of this theorem depends on the atomic decomposition theorem for the space \( \bar{B}_{p, q, r_2}(\mathbb{R}^n, \{ t_k \}) \), and so we defer it to the end of § 4.

Remark 2.7. If \( \gamma \equiv 1 \) the conclusion of Theorem 2.5 can be extended to the case \( p = \infty \).

The following result, which was proved in [5], will be given in a simplified form for \( p \) and \( q \) constant.
Theorem 2.6. Let \( p, q \in (0, \infty], \alpha_1 > n(\frac{1}{\min\{p,1\}} - 1)[1 + \frac{\alpha_3}{p}], l > \alpha_2 \), and let \( \{s_k\} \in Y^{\alpha_3}_{\alpha_1, \alpha_2} \). Then

\[
B_{p,q}^{\{s_k\}}(\mathbb{R}^n) = \overline{B}_{p,q,1}^l(\mathbb{R}^n, \{s_k\}),
\]

and the corresponding quasi-norms are equivalent.

Combining Theorems 2.5, 2.6 and Remark 2.7 we obtain

Corollary 2.1. Let \( p, q \in [1, \infty], r_1, r_2 \in [1, p], \alpha_3 \geq 0, \alpha_1 > 0, l > \alpha_2 \), and let \( \{s_k\} \in Y^{\alpha_3}_{\alpha_1, \alpha_2} \). Then

\[
B_{p,q}^{\{s_k\}}(\mathbb{R}^n) = \overline{B}_{p,q,r_1}^{l}(\mathbb{R}^n, \{s_k\}) = \overline{B}_{p,q,r_2}^{l}(\mathbb{R}^n, \{s_k\}),
\]

and the corresponding norms are equivalent.

Remark 2.8. The question of whether (or not) the spaces \( \overline{B}_{p,q,r}^l(\mathbb{R}^n, \{s_k\}), \overline{B}_{p,q,r}(\mathbb{R}^n, \{s_k\}) \) and \( B_{p,q}^{\{s_k\}}(\mathbb{R}^n) \) coincide under weaker constraints on the variable smoothness \( \{s_k\} \) (by comparison with Theorems 2.5, 2.6 or Corollary 2.1) is a matter for the future.

Remark 2.9. Combining Theorem 2.5 with Theorem 3.14 from [13] with \( p \in (0, \infty), 0 < r < p, q \in (0, \infty), s > 0, l > s \) and \( \gamma^p \in A_{p/r}(\mathbb{R}^n) \), we obtain

\[
\overline{B}_{p,q,r}^{s}(\mathbb{R}^n, \gamma) = \overline{B}_{p,q,r}^{s}(\mathbb{R}^n, \gamma) = B_{p,q}^{s}(\mathbb{R}^n, \gamma),
\]

and the corresponding norms are equivalent.

§ 3. Trace spaces of weighted Sobolev spaces

As we pointed out in the introduction, the main impetus for studying the spaces \( \overline{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) \) stems from their application to the trace problem for weighted Sobolev spaces.

For a brief overview of the available literature on traces of weighted Sobolev spaces we refer to [1]; we do not dwell on this here.

Let \( p \in [1, \infty], l \in \mathbb{N} \), and let \( \gamma \) be a weight. We fix \( n, d \in \mathbb{N} \). A point in the \((n + d)\)-dimensional Euclidean space \( \mathbb{R}^{n+d} := \mathbb{R}^n \times \mathbb{R}^d \) will be written as a pair \((x, y)\). The plane given in \( \mathbb{R}^{n+d} \) by the equation \( y = 0 \) will be identified with the space \( \mathbb{R}^n \). For \( a > 0 \), we set

\[
^a \mathbb{R}^d := \mathbb{R}^{n+d} \setminus (\mathbb{R}^n \times aB^d)
\]

and put

\[
\Xi_{k,m}^{d,n} := Q_{k,m}^n \times \left( \frac{B^d}{2^k} \setminus \frac{B^d}{2^k+1} \right) \quad \text{for} \quad k \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n.
\]

We let \( W_p^l(\mathbb{R}^{n+d}, \gamma) \) denote the linear space of classes of equivalent functions which have all (Sobolev) generalized derivatives up to order \( l \) inclusive on \( \mathbb{R}^n \). We equip this space with the norm

\[
\| f \|_{W_p^l(\mathbb{R}^{n+d}, \gamma)} = \sum_{|\alpha| \leq l} \| D^\alpha f \|_{L_p(\mathbb{R}^{n+d}, \gamma)}.
\]
In [1] the trace problem was solved for the spaces $\tilde{W}^l_p(\mathbb{R}^n, \gamma)$, which differ slightly from the spaces $W^l_p(\mathbb{R}^n, \gamma)$ in terms of the form of the norm. More precisely, the norm in $\tilde{W}^l_p(\mathbb{R}^n, \gamma)$ does not include certain mixed derivatives.

In what follows we shall require a certain averaging operator, which was constructed in [1]. We shall not give the details of the construction of this operator. For a function $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ we set

$$E_\varepsilon[f](x) := \frac{1}{\varepsilon^{2n}} \sum_{j=1}^{l} \mu_j \int_{\mathbb{R}^n} \Theta\left(\frac{y-x}{\varepsilon}\right) \int_{\mathbb{R}^n} \Theta\left(\frac{z-y}{j\varepsilon}\right) f(z) \, dz \, dy, \quad x \in \mathbb{R}^n. \tag{3.1}$$

The function $\Theta \in C_0^\infty$ in (3.1) is chosen appropriately, $\int_{\mathbb{R}^n} \Theta(x) \, dx = 1$, and the $\mu_j$ are some specially chosen constants (see [1]). Given $k \in \mathbb{N}$, we define $E_k[g] := E_{2^{-k}}[g]$.

**Lemma 3.1.** Suppose that the function $\varphi \in L_1^{\text{loc}}(\mathbb{R}^n)$. Then, for any $\varepsilon > 0$, a multi-index $\alpha$, $|\alpha| = l$, and $x \in \mathbb{R}^n$,

$$|D^\alpha_x E_\varepsilon[\varphi](x)| \leq \frac{1}{\varepsilon^l} \delta^l(x + \varepsilon I^n) \varphi. \tag{3.2}$$

Moreover, for any numbers $0 < \varepsilon_1 < \varepsilon_2$ and a multi-index $\beta$, for $x \in \mathbb{R}^n$,

$$|D^\beta E_{\varepsilon_1} \left[\varphi \right](x) - D^\beta E_{\varepsilon_2} \left[\varphi \right](x)| \leq C \int_{\varepsilon_1}^{\varepsilon_2} \frac{1}{t^{1+|\beta|}} \delta^l(x + t I^n) \varphi \, dt. \tag{3.3}$$

For $\beta = 0$ the proof is given in [1]. The general case is dealt with similarly.

In this section we are not going to give a precise definition of the trace of a function $f \in L_1^{\text{loc}}(\mathbb{R}^{n+d})$ on the plane $y = 0$. This is a standard definition and can be found in Ch. 5 of [22], for example.

Assume that a multiple sequence $\{\tilde{\gamma}_{k,m}\}$ is generated by a weight $\gamma \in A_\infty^{\text{loc}}(\mathbb{R}^{n+d})$. Next, assume that parameters $l \in \mathbb{N}$ and $p \in (1, \infty)$ are fixed. We set

$$\gamma^l_k(x) := \gamma_k(x) := 2^{k(l+\frac{n}{p})} \sum_{m \in \mathbb{Z}^n} \chi_{\tilde{\gamma}^l_{k,m}}(x) \tilde{\gamma}^l_{k,m} \quad \text{for} \quad k \in \mathbb{N}, \quad x \in \mathbb{R}^n,$n)

$$\gamma_{k,m} := 2^{k(l)} \gamma_{k,m} \quad \text{for} \quad k \in \mathbb{N}, \quad m \in \mathbb{Z}^n.$n)

The main result in this section is the following

**Theorem 3.1.** Let $p \in (1, \infty)$, $r \in [1, p)$, $\gamma^p \in A^\text{loc}_{p/r}(\mathbb{R}^{n+d})$, $f \in W^l_p(\mathbb{R}^{n+d}, \gamma)$ and let $l > \frac{d}{r}$. Then the trace $\varphi \in \tilde{B}^{l}_{p,p,r}(\mathbb{R}^n, \{\gamma_k\})$ of the function $f$ exists, and moreover,

$$\|\varphi \mid \tilde{B}^{l}_{p,p,r}(\mathbb{R}^n, \{\gamma_k\})\| \leq C_1 \|f \mid W^l_p(\mathbb{R}^{n+d}, \gamma)\|. \tag{3.4}$$

The constant $C_1$ in (3.4) is independent of $f$.

Conversely, if $\varphi \in \tilde{B}^{l}_{p,p,r}(\mathbb{R}^n, \{\gamma_k\})$, then there exists a function $f \in W^l_p(\mathbb{R}^{n+d}, \gamma)$ such that $\varphi$ is the trace of $f$ on $\mathbb{R}^n$, and moreover,

$$\|f \mid W^l_p(\mathbb{R}^{n+d}, \gamma)\| \leq C_2 \|\varphi \mid \tilde{B}^{l}_{p,p,r}(\mathbb{R}^n, \{\gamma_k\})\|. \tag{3.5}$$

The constant $C_2$ in (3.5) is independent of $\varphi$. 
The next result in an important step in the proof of Theorem 3.1.

**Lemma 3.2.** Let \( p \in (1, \infty) \), \( r \in [1, p) \), \( \gamma^p \in A^\text{loc}_{p/r}(\mathbb{R}^{n+d}) \), \( f \in W^l_p(\mathbb{R}^{n+d}, \gamma) \) and let \( l > \frac{d}{r} \). Then the trace \( \varphi \) of the function \( f \) on \( \mathbb{R}^n \) exists. Moreover, for an arbitrary cube \( Q_{k,m}^n \),

\[
\delta^l_r(Q_{k,m}^n) \varphi \leq \frac{C_3}{2^{k(l-(n+d))/r}} \left\{ \left\| D^\alpha f \right\|_{L^r} \left( C_4 k^m_{k,m} \times \frac{C_5}{2^k} B^d \right) \right\} + \left\| D^\beta f \right\|_{L^r} \left( C_4 k^m_{k,m} \times \frac{C_5}{2^k} B^d \right) \right\}.
\]

(3.6)

The constants \( C_3, C_4 \) and \( C_5 \) in (3.6) are independent of both \( f \) and \( Q_{k,m}^n \).

The proof of this lemma is a straightforward modification of the proof of Lemma 3.1 in [1].

**Proof of Theorem 3.1.** Step 1. Let \( f \in W^l_p(\mathbb{R}^{n+d}, \gamma) \). Then by Lemma 3.2 the trace \( \varphi \) of the function \( f \) on the plane \( \mathbb{R}^n \) exists, and so it suffices to prove estimate (3.4). In turn, this estimate follows if we make an obvious modification to the argument employed in the proof of Theorem 3.1 in [1] (one needs only to apply estimate (3.6) in the appropriate place).

Step 2. We shall explain this part of the proof in detail. As distinct from [1], we need to estimate all the mixed derivatives. So, let \( \{\psi_k\}_{k=0}^\infty \) be a partition of unity for the ball \( B^d \); that is, \( \psi_k(y) \geq 0 \) for \( k \in \mathbb{N}_0 \), \( y \in B^d \) and \( \sum_{k=0}^\infty \psi_k(y) = 1 \) for \( y \in B^d \). In addition,

\[
\psi_0 \in C^\infty \left( B^d \setminus \frac{1}{2} B^d \right), \quad \psi_k \in C^\infty_0 \left( \frac{1}{2^{k-1}} B^d \setminus \frac{1}{2^{k+1}} B^d \right) \text{ for } k \in \mathbb{N},
\]

\[
|D^\beta \psi_k(y)| \leq \frac{C_1}{(\delta_k)^{|\beta|}} \text{ for } y \in B^d, \quad k \in \mathbb{N}_0.
\]

Assume that, for any \( k \in \mathbb{N}_0 \), only two functions \( \psi_k \) and \( \psi_{k+1} \) do not vanish on the set \( 2^{-k} B^d \setminus 2^{-k-1} B^d \). Hence, \( D^\beta \psi_k(y) = -D^\beta \psi_{k+1}(y) \) for \( y \in 2^{-k} B^d \setminus 2^{-k-1} B^d \). The existence of a sequence \( \{\psi_k\}_{k=0}^\infty \) with the above properties may be proved as was done in [22], §4.5 in the proof of the trace theorem for unweighted Sobolev spaces, for example.

We set

\[
f(x, y) := \sum_{k=1}^\infty \psi_k(y) E_2^{-k} [\varphi](x) \text{ for } (x, y) \in \mathbb{R}^n \times B^d,
\]

where, the operator \( E_\varepsilon \) (with \( \varepsilon > 0 \)) is defined in (3.1). We extend the function \( f \) by zero to the set \( n \mathbb{R}^d_1 \).

A multi-index \( \alpha \) will be written as \( (\alpha^1, \alpha^2) = (\alpha^1_1, \ldots, \alpha^1_n, \alpha^2_1, \ldots, \alpha_d^2) \).
Clearly,
\[
\int \int_{\mathbb{R}^n \times \frac{1}{2} B^d} \gamma^p(x, y) \left\{ \sum_{|\alpha|=l, \alpha^2=0} |D^\alpha f(x, y)|^p + \sum_{|\alpha|=l, |\alpha^2| > 0} |D^\alpha f(x, y)|^p \right\} \, dx \, dy \\
= \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) \\
\times \left\{ \sum_{|\alpha|=l, \alpha^2=0} |D^\alpha f(x, y)|^p + \sum_{|\alpha|=l, |\alpha^2| > 0} |D^\alpha f(x, y)|^p \right\} \, dx \, dy.
\]

Taking the properties of the functions \( \psi_k \) into account and applying estimate (3.3), we see that
\[
\sum_{|\alpha|=l, \alpha^2 > 0} \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) |D^\alpha f(x, y)|^p \, dx \, dy \\
= \sum_{|\alpha|=l, \alpha^2 > 0} \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) \\
\times |D^\alpha \psi_k(y) E_{2-k} \varphi(x) + D^\alpha \psi_k(y) E_{2-(k+1)} \varphi(x)|^p \, dx \, dy \\
\leq \sum_{|\alpha|=l} 2^k |\alpha^2|^p \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) |D^\alpha x E_{2-k} \varphi(x) - D^\alpha x E_{2-(k+1)} \varphi(x)|^p \, dx \, dy \\
\leq C 2^{nkp} \gamma_{k,m}^p \left[ \int_{CQ_{k,m}^n} \int_{I_n/2^k} |\Delta^l (h) \varphi(z)| \, dh \, dz \right]^p \, dx \, dy \\
\text{for } k \in \mathbb{N}, m \in \mathbb{Z}^n. \tag{3.7}
\]

The constant \( \tilde{C} \geq 1 \), which is the dilation coefficient of the cubes \( Q_{k,m}^n \), depends only on the diameter of the support of the function \( \Theta \) from (3.1).

Similarly, it follows from (3.2) that
\[
\sum_{|\alpha|=l} \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) |D^\alpha x f(x, y)|^p \, dx \, dy \\
\leq C \sum_{|\alpha|=l} \int \int_{\Xi_{k,m}^n} \gamma^p(x, y) \max\{|D^\alpha x E_{2-k} \varphi(x)|^p, |D^\alpha x E_{2-(k+1)} \varphi(x)|^p\} \, dx \, dy \\
\leq C 2^{nkp} \gamma_{k,m}^p \left[ \int_{CQ_{k,m}^n} \int_{I_n/2^k} |\Delta^l (h) \varphi(z)| \, dh \, dz \right]^p \, dx \, dy \\
\text{for } k \in \mathbb{N}, m \in \mathbb{Z}^n. \tag{3.8}
\]

Using the definition of the function \( f \) and employing Hölder’s inequality with exponents \( r \) and \( r' \), we have, for \( |\alpha|=l \),
\[
\sum_{|\alpha|=l} \int \int_{nR_{l/2}' \mathbb{R}^d} \gamma^p(x, y) |D^\alpha f(x, y)|^p \, dx \, dy \\
\leq C \sum_{m \in \mathbb{Z}^n} \gamma_{0,m}^p \| \varphi \| L_1(CQ_{0,m}^n) \| \|p \\
\leq C \sum_{m \in \mathbb{Z}^n} \gamma_{0,m}^p \| \varphi \| L_r(CQ_{0,m}^n) \| \| \\
\leq C \sum_{m \in \mathbb{Z}^n} \gamma_{0,m}^p \| \varphi \| L_r(Q_{0,m}^n) \| \|, \tag{3.9}
\]
since the cubes $\tilde{C}Q^n_{k,m}$ have finite overlap multiplicity (the constant $\tilde{C}$ is the same as in (3.7)).

Hence, summing (3.7) and (3.8) over $k$ and $m$, taking account of the fact that the cubes $\tilde{C}Q^n_{k,m}$ have finite overlap multiplicity (with fixed $k \in \mathbb{N}$ and variable $m \in \mathbb{Z}^N$), and using (3.9), this gives

$$\sum_{|\alpha|=l} \|D^\alpha f \|_{L_p(\mathbb{R}^{n+d}, \gamma)} \leq C \|\varphi \|_{\tilde{\mathcal{B}}_{p,p,r}(\mathbb{R}^n, \{\gamma_{k,m}\})}. \quad (3.10)$$

To estimate the generalized derivatives $D^\alpha f$ for $|\alpha| < l$, for each $(x, y) \in \mathbb{R}^n \times B^d$, we write the integral representation of the function $D^\alpha f$ in the cone (see §3.4, [22]),

$$V(x, y) = \left\{ (x, y)(1-t) + t(x', y') \mid t \in [0, 1], (x', y') \in \frac{1}{2}B^{n+d}(x, y+3) \right\}$$

(here $\frac{1}{2}B^{n+d}(x, y+3)$ is the ball of radius $\frac{1}{2}$ centred at $(x, y+3)$, and we use Remark 16 from §3.5 in [22].

Let $|\alpha| < l$. Since $f(x, y) = 0$ for $|y| > 1$, we have

$$|D^\alpha f(x, y)| \leq C \sum_{|\beta|=l} \int_{(x,0)+(I^d \times B^d)} |D^\beta f(x, y)| \, dx \, dy \quad \text{for } (x, y) \in \mathbb{R}^n \times B^d.$$

Hence, using the obvious inclusion $A_{p/r}^{\textrm{loc}}(\mathbb{R}^{n+d}) \subset A_p^{\textrm{loc}}(\mathbb{R}^{n+d})$, together with Hölder’s inequality, for $m \in \mathbb{Z}^n$, $|\alpha| < l$, we obtain

$$\int \int_{Q^0_{m} \times B^d} \gamma^p(x, y)|D^\alpha f(x, y)|^p \, dx \, dy$$

$$\leq C \sum_{|\beta|=l} \left[ \frac{\gamma^p(\tilde{C}Q^n_{0,m} \times B^d)}{\gamma^{-p'}(\tilde{C}Q^n_{0,m} \times B^d)} \right]^{\frac{1}{p'}} \times \int \int_{\tilde{C}Q^n_{0,m} \times B^d} \gamma^p(x, y)|D^\beta f(x, y)|^p \, dx \, dy \leq C \sum_{|\beta|=l} \int \int_{\tilde{C}Q^n_{0,m} \times B^d} \gamma^p(x, y)|D^\beta f(x, y)|^p \, dx \, dy. \quad (3.11)$$

Summing estimate (3.11) over $m \in \mathbb{Z}^n$ and taking account of the finite overlap multiplicity of the cubes $\tilde{C}Q^m_{k,m}$ (for fixed $k$ and variable $m$), in view of (3.10) we obtain (3.5).

It remains to show that $\varphi = \text{tr} \mid_{y=0} f$. We fix an arbitrary cube $Q^n$. Almost every point $x \in \mathbb{R}^n$ is a Lebesgue point of $\varphi$ because $\varphi \in L_1^{\textrm{loc}}(\mathbb{R}^n)$. Hence, for almost all $x \in \mathbb{R}^n$,

$$g_\delta(x) := \frac{1}{\delta^n} \int_{x+\delta I^n} |\varphi(x') - \varphi(x)| \, dx' \to 0 \quad \text{as } \delta \to 0.$$

Consequently, by Lebesgue’s convergence theorem,

$$\int_{Q^n} |\varphi(x) - E_\delta[\varphi](x)| \, dx \leq C \int_{Q^n} g_{2\delta}(x) \, dx \to 0 \quad \text{as } \delta \to 0. \quad (3.12)$$
From (3.12) and the definition of $f$ it easily follows that $\varphi$ is the trace of $f$ on the plane $y = 0$.

The proof of Theorem 3.1 is complete.

§4. Atomic decomposition of functions in the spaces $\widetilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{t_{k}\})$

Our aim in this section is to establish the atomic decomposition theorem for functions $\varphi$ in the space $\widetilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{t_{k}\})$. This theorem is one of the principal tools in establishing various embedding and trace theorems (see §§5, 6 below).

We shall also give conditions on a $p$-admissible weight sequence $\{t_{k}\}$ which ensure that the spaces $\widetilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{t_{k}\})$ coincide for various $l \in \mathbb{N}$ and that the corresponding norms are equivalent.

Our arguments depend to a large extent on the methods used in [16]. Throughout this section, we fix numbers $n \in \mathbb{N}$ and $d \in \mathbb{N}_{0}$, and define

$$\Xi_{d,n,k,m} := Q_{k,m}^{n} \times \left( \frac{B^{d}}{2^{k}} \setminus \frac{B^{d}}{2^{k+1}} \right) \quad \text{for} \quad k \in \mathbb{N}_{0}, \quad m \in \mathbb{Z}^{n}.$$  

A point in $(n + d)$-dimensional Euclidean space will be written as a pair $(x, y) := (x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{d})$.

Let $l \in \mathbb{N}$ and let $N_{l-1}$ be the $B$-spline of degree $l - 1$ with knots at the points $t_{i} = i, i \in \{0, 1, \ldots, l\}$. More precisely,

$$N_{l-1}(t) := [0, 1, \ldots, l](t - \cdot)_{+}^{l-1}.$$  

Here, we have used the standard notation for the divided difference (see [23], Ch. 7).

For $k \in \mathbb{N}_{0}$ and $m \in \mathbb{Z}^{n}$, we set

$$N_{k,m}^{l-1}(x) := \prod_{i=1}^{n} N_{l-1}^{l-1}\left(2^{k}\left(x_{i} - \frac{m_{i}}{2^{k}}\right)\right) \quad \text{for} \quad x \in \mathbb{R}^{n}.$$  

The functions $N_{k,m}^{l-1}$ were first introduced by Curry and Schoenberg [24].

We list some properties of the $B$-splines $N_{k,m}^{l-1}$ that will be required in what follows. The corresponding proofs can be found in [24] and [25], for example.

1) The $B$-splines $N_{k,m}^{l-1}$ form a partition of unity on $\mathbb{R}^{n}$ for each fixed $k \in \mathbb{N}_{0}$; that is,

$$\sum_{m \in \mathbb{Z}^{n}} N_{k,m}^{l-1}(x) = 1, \quad x \in \mathbb{R}^{n}. \quad (4.1)$$  

Here, the overlap multiplicity of the supports of splines $N_{k,m}^{l-1}$ is finite and is independent of both $k$ and $m$. We also note that

$$\text{supp} \ N_{k,m}^{l-1} \subset \frac{m}{2^{k}} + \left[0, \frac{l}{2^{k}}\right]^{n} \quad \text{and} \quad N_{k,m}^{l-1}(x) \in (0, 1] \quad \text{for} \quad x \in \frac{m}{2^{k}} + \left(0, \frac{l}{2^{k}}\right]^{n}.$$  

2) On each cube $Q_{k,m}^{n}$ the function $N_{k,m}^{l-1}$ is a polynomial of degree $\leq l - 1$ in each variable.

3) The spline $N_{l-1}^{l-1}$ has a continuous derivative of order $l - 2$ at all points of $\mathbb{R}^{n}$. At the knots $t_{i} = i, i \in \{0, 1, \ldots, l\}$, the spline $N_{l-1}^{l-1}$ has finite one-sided derivatives
of order \( l - 1 \), and on each interval \((i, i + 1), i \in \{0, 1, \ldots, l\}\), the spline \( N^{l-1} \) has continuous derivatives of order \( l - 1 \). Therefore,

\[
\Delta^l(h)N^{l-1}_{k,m}(x) \leq C(2^k|h|)^{l-1} \quad \text{for } x, h \in \mathbb{R}^n. \tag{4.2}
\]

4) Any spline \( S = \sum_{m \in \mathbb{Z}^n} \beta_{k,m} N^{l-1}_{k,m} \) (\( \beta_{k,m} \in \mathbb{R} \) for \( k \in \mathbb{N}_0 \), \( m \in \mathbb{Z}^n \)) can be expanded in a series in splines \( N^{l-1}_{j,m} \) for \( j \geq k \); that is,

\[
S = \sum_{m \in \mathbb{Z}^n} \hat{\beta}_{k,m}(S)N^{l-1}_{j,m}.
\]

We let \( \Sigma^{l-1}_k \) denote the set of all splines \( S \) of the form

\[
S(x) := \sum_{m \in \mathbb{Z}^n} \beta_{k,m} N^{l-1}_{k,m}(x) \quad \text{for } x \in \mathbb{R}^n.
\]

For future purposes we require the concept of a quasi-interpolant, which was first introduced in [25]. Quasi-interpolants were also used in [26] and [16] to construct equivalent norms on unweighted dyadic Besov-type spaces and on classical Besov spaces.

**Definition 4.1** (see [25]). Given \( k \in \mathbb{N}_0 \), \( m \in \mathbb{Z}^n \), we let \( \xi_{k,m} = (\xi_{k,m_1}, \ldots, \xi_{k,m_n}) \) denote the centre of the cube \( Q^{n}_{k,m} \). Assume that all partial derivatives \( D^\nu f \) of \( f \), \( \nu_j \leq l - 1, j \in \{1, \ldots, n\} \), are continuous at each point \( \xi_{k,m} \). For \( k \in \mathbb{N}_0 \), \( m \in \mathbb{Z}^n \), we set

\[
Q^{l-1}_k(f) := \sum_{m \in \mathbb{Z}^n} \alpha_{k,m}(f)N^{l-1}_{k,m},
\]

where

\[
\alpha_{k,m}(f) := \sum_{0 \leq \nu_j \leq 0} a_{k,\nu,m}D^\nu f(\xi_{k,m}),
\]

\[
a_{k,m,\nu} := \prod_{i=1}^n a_{k,m_i,\nu_i}, \quad a_{k,m_i,\nu_i} := \frac{(-1)^{l-1-\nu_i}}{(l-1)!} D^{l-1-\nu_i} \psi_{m_i}(\xi_{k,m_i}),
\]

\[
\psi_{m_i}(t) := \prod_{j=1}^{l-1} \left( \frac{m_i + j}{2^k} - t \right) \quad \text{for } t \in \mathbb{R}.
\]

The operator \( Q^{l-1}_k \) is called a quasi-interpolant.

Throughout this section we fix a constant \( A \geq 1 \).

Let \( P^{n}_{k,m} \) be a polynomial of near best approximation with constant \( A \) in the \( L_r(Q^{n}_{k,m}) \)-metric to the function \( \varphi \in L^r_{\text{loc}}(\mathbb{R}^n) \) (for \( r \in (0, \infty) \)) by polynomials of coordinate degree less than \( l \) on the cube \( Q^{n}_{k,m} \).

We set

\[
g_k(x) := \sum_{m \in \mathbb{Z}^n} P^{n}_{k,m}(x)\chi_{Q^{n}_{k,m}}(x) \quad \text{for } x \in \mathbb{R}^n, \; k \in \mathbb{N}_0.
\]
Finally, following [16], for \( r \in (0, \infty] \) we define
\[
T_{k}^{-1}(\varphi, r)(x) := Q_{k}^{-1}(g_{k})(x) \quad \text{for} \quad x \in \mathbb{R}^{n}, \quad k \in \mathbb{N}_{0},
\]
\[
T_{-1}^{-1}(\varphi, r)(x) := \varphi(x) \quad \text{for} \quad x \in \mathbb{R}^{n}.
\]

We note that in [16] the operator \( T_{k}^{-1} \) acts on functions \( \varphi \) defined on the unit cube.

**Remark 4.1.** The operator \( Q_{k}^{-1} \) is a projection operator from the space of piecewise polynomial functions to the space \( \Sigma_{k}^{-1} \) (we refer the reader to [25] for a proof), and hence
\[
T_{k}^{-1}(\varphi, r)(x) = \sum_{m \in \mathbb{Z}^{n}} \alpha_{k,m}(T_{k}^{-1}(\varphi, r)) N_{k,m}^{-1}(x) \quad \text{for} \quad x \in \mathbb{R}^{n}.
\]

**Lemma 4.1.** Let \( r \in (0, \infty] \). Then for any function \( \varphi \in L_{r}^{\loc}(\mathbb{R}^{n}) \) and \( k \in \mathbb{N}_{0} \) the following estimate holds:
\[
\| \varphi - T_{k}^{-1}(\varphi, r) \|_{L_{r}(Q_{k,m}^{n})} \leq CQ \tilde{E}_l(\varphi, (1 + l)Q_{k,m}^{n}) \leq CE(\varphi, (1 + l)Q_{k,m}^{n}). \tag{4.4}
\]

The constant \( C \) in (4.4) depends only on \( l, n, r, A \).

**Proof.** The first inequality in (4.4) follows from estimate (4.25) in [16], the second inequality is clear.

Let \( p, r \in (0, \infty], \varphi \in L_{r}^{\loc}(\mathbb{R}^{n}), \alpha_{3} \geq 0, \alpha_{i} \in \mathbb{R} \) and let \( \sigma_{i} \in (0, \infty], i = 1, 2 \). For a multiple sequence \( \{t_{k,m}\} \) which is \( p \)-associated with a \( p \)-admissible weight sequence \( \{t_{k}\} \in X_{\alpha, \sigma, p} \) we set
\[
s_{k}^{l} := s_{k}^{l}(\varphi, \{t_{k,m}\})_{r,p} := \inf_{S \in \Sigma_{k}^{l}} \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p} \| \varphi - S \|_{L_{r}(Q_{k,m}^{n})}^{p} \right)^{\frac{1}{p}} \quad \text{for} \quad k \in \mathbb{N}_{0},
\]
\[
s_{l-1}^{l} := s_{l-1}^{l}(\varphi, \{t_{0,m}\})_{r,p} = \left( \sum_{m \in \mathbb{Z}^{n}} t_{0,m}^{p} \| \varphi - L_{r}(Q_{0,m}^{n}) \|^{p} \right)^{\frac{1}{p}}. \tag{4.5}
\]

Note that \( s_{k}^{l}(\varphi, \{t_{k,m}\})_{r,p} < \infty \) if \( \varphi \in \tilde{B}_{p,q,r}^{l}(\mathbb{R}^{n}, \{t_{k}\}) \). Indeed, by (4.4) and in view of Theorem 2.3, we have, for \( k \in \mathbb{N}_{0},
\]
\[
\inf_{S \in \Sigma_{k}^{l}} \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p} \| \varphi - S \|_{L_{r}(Q_{k,m}^{n})}^{p} \right)^{\frac{1}{p}} \leq \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p} \| \varphi - T_{k}^{-1}(\varphi, r) \|_{L_{r}(Q_{k,m}^{n})}^{p} \right)^{\frac{1}{p}} \leq C \| \varphi \|_{\tilde{B}_{p,q,r}^{l}(\mathbb{R}^{n}, \{t_{k}\})} < \infty. \tag{4.6}
\]

**Definition 4.2.** Let \( p, r \in (0, +\infty], s_{k}^{l}(\varphi, \{t_{k,m}\})_{r,p} < \infty \). We say that
\[
U_{k}^{l-1} := U_{k}^{l-1}(\varphi, \{t_{k,m}\}, p) \in \Sigma_{k}^{l-1}
\]
is a spline of near best approximation with constant $A$ to a function $\varphi \in L^p_{\text{loc}}(\mathbb{R}^n)$ if
\[
\left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p \| \varphi - U_{k}^{l-1} | L^p_{r}(Q_{k,m}^n) \right)^{\frac{1}{p}} \leq A_{\beta_{k}}(\varphi, \{t_{k,m}\})_{r,p},
\] (4.7)

**Lemma 4.2** (see [16]). Suppose that a spline $S$ belongs to $\Sigma_{k}^{l-1}$. Then, for any $r \in (0, +\infty]$ and any cube $Q_{k,m}^n$,$
\begin{align*}
C_1 \| S \|_{L^r(Q_{k,m}^n)} & \leq \left( \sum_{m \in \mathbb{Z}^n} | \alpha_{k,m}(S)|^{r} 2^{-kn} \right)^{\frac{1}{r}} \leq C_2 \| S \|_{L^r(C_3Q_{k,m}^n)},
\end{align*}
where the constants $C_1, C_2, C_3 > 0$ are independent of both the cube $Q_{k,m}^n$ and the spline $S$. The corresponding modifications in the case $r = \infty$ are obvious.

The next theorem is an extension of Theorem 4.8 in [16] (which dealt with classical Besov spaces) to the case of Besov spaces of variable smoothness.

We recall that the symbols $\delta_{i}(\gamma, n, d)$ ($i = 1, 2$) were introduced right after the proof of Lemma 2.2.

**Theorem 4.1.** Let $p, r \in (0, \infty)$, $\varphi \in L^p_{\text{loc}}(\mathbb{R}^n)$, $\alpha_3 \geq 0$, $0 < \alpha_1 \leq \alpha_2$, and let $\{s_k\} \in \text{loc}^{\gamma_{\alpha_3}}$ be a weight sequence. Assume that the weight $\gamma^p \in A^\text{loc}_{\alpha_1}(\mathbb{R}^{n+d})$ and that a multiple sequence $\{\tilde{\gamma}_{k,m}\}$ is generated by the weight $\gamma$. Set
\[
t_{k,m} := 2^{kn} s_{k,m} \tilde{\gamma}_{k,m} \quad \text{for} \quad k \in \mathbb{N}_0, \ m \in \mathbb{Z}^n.
\]
Then, for any sufficiently small $\varepsilon > 0$,
\[
\left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p \| \delta_{i}^l(Q_{k,m}^n)\varphi \|^{p} \right)^{\frac{1}{p}} \leq C 2^{-k(\bar{\lambda} + \frac{d(\delta_{1}(\gamma, n, d) - \varepsilon)}{p}) - \alpha_2}
\times \left( \sum_{j=-1}^{k} 2^{jn(\bar{\lambda} + \frac{d(\delta_{2}(\gamma, n, d) - \varepsilon)}{p}) - \alpha_2} \right)^{\mu} (\delta_{j}^l(\varphi, \{t_{k,m}\})_{r,p})^{\mu},
\] (4.9)
where
\[
\bar{\lambda} := \min \left\{ l, l - 1 + \frac{\delta_{2}(\gamma, n, d) - \varepsilon}{p} \right\}, \quad \mu \leq \min \{1, r, p\},
\]
and the constant $C > 0$ depends on $\alpha_1, \alpha_2, \alpha_3, r, l$ and the weight $\gamma$, but is independent of the function $\varphi$.

**Proof.** The main idea of the proof follows that of Theorem 4.8 in [16]. However, we need to modify the proof in [16] to take the properties of the multiple sequence $\{\tilde{\gamma}_{k,m}\}$, which were indicated in Lemma 2.1, into account.

We fix $\varepsilon \in [0, \min\{\delta_{1}(\gamma, n, d), \delta_{2}(\gamma, n, d)\})$ such that (2.18) and (2.19) hold with $\tilde{\delta}_{1}$ and $\tilde{\delta}_{2}$ instead of $\delta$ and $\delta'$ respectively. Here we set
\[
\tilde{\delta}_{1} := \tilde{\delta}_{1}(\gamma, n, d) := \delta_{1}(\gamma, n, d) - \varepsilon, \quad \tilde{\delta}_{2} := \tilde{\delta}_{2}(\gamma, n, d) := \delta_{2}(\gamma, n, d) - \varepsilon.
\]
Let \( U_{j}^{l-1} := U_{j}^{l-1}(\varphi, \{t_{k,m}\}, p) \) be a spline of near best approximation with constant \( A \geq 1 \) to the function \( \varphi \in L^{\text{loc}}(\mathbb{R}^{n}) \). Given \( j \in \mathbb{N}_{0} \), we set \( u_{j}^{l-1} := U_{j}^{l-1} - U_{j-1}^{l-1} \) \( (U_{-1}^{l-1} \equiv 0) \). Then, clearly,

\[
\Delta^{l}(h)\varphi(x) = \Delta^{l}(h)[\varphi - U_{k}^{l-1}](x) + \sum_{j=0}^{k} \Delta^{l}(h)u_{j}^{l-1}(x) \quad \text{for } x, h \in \mathbb{R}^{n}. \tag{4.10}
\]

The inequality

\[
\delta_{r}(Q_{k,m}^{n})\varphi \leq \left( [\delta_{r}(Q_{k,m}^{n})(\varphi - U_{k}^{l-1})]^{\mu} + \sum_{j=0}^{k} [\delta_{r}(Q_{k,m}^{n})u_{j}^{l-1}]^{\mu} \right)^{\frac{1}{\mu}}
\]

follows easily from (4.10) and Lemma 2.1 with \( \mu \leq \min\{1, r, p\} \). Hence, since \( \frac{p}{\mu} \geq 1 \), using Minkowski’s inequality

\[
\left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p}[\delta_{r}(Q_{k,m}^{n})\varphi]^{p} \right)^{\frac{1}{p}} \leq \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p}[\delta_{r}(Q_{k,m}^{n})(\varphi - U_{k}^{l-1})]^{\mu} + \sum_{j=0}^{k} [\delta_{r}(Q_{k,m}^{n})u_{j}^{l-1}]^{\mu} \right)^{\frac{\mu}{\mu + 1}} \frac{1}{\mu + 1}
\]

\[
\leq \left( \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p}[\delta_{r}(Q_{k,m}^{n})(\varphi - U_{k}^{l-1})]^{p} \right)^{\frac{\mu}{p}} + \sum_{j=0}^{k} \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p}[\delta_{r}(Q_{k,m}^{n})u_{j}^{l-1}]^{p} \right)^{\frac{\mu}{p}} \right)^{\frac{1}{p}}
\]

\[
= \left( R_{1}^{1} \right)^{\frac{\mu}{p}} + \sum_{j=0}^{k} \left( R_{2}^{j} \right)^{\frac{\mu}{p}}. \quad (4.11)
\]

Since the cubes \((1 + l)Q_{k,m}^{n}\) have finite overlap multiplicity (which is independent of \( k \) and \( m \)), using Remark 2.2, it is easily seen that

\[
(R_{1}^{1})^{\frac{\mu}{p}} \leq C \left( \sum_{m \in \mathbb{Z}^{n}} t_{k,m}^{p} 2^{kmn} \|\varphi - U_{k}^{l-1} \|_{L_{r}((1 + l)Q_{k,m}^{n})}^{p} \right)^{\frac{\mu}{p}} \leq C(s_{k}(\varphi, \{t_{k,m}\})r, p)^{\mu}. \quad (4.12)
\]

It is worth noting that the differences \( \delta_{r}^{l} \) (rather than \( \bar{\Delta}_{r}^{l} \)) were crucial in obtaining (4.12).

Next, for each \( j \in \mathbb{N}_{0} \) the function \( u_{j}^{l-1} \) can be expanded in a series in terms of the \( B \)-splines \( N_{j,m}^{l-1} \) (by property 4), see above); that is,

\[
u_{j}^{l-1}(x) = \sum_{m \in \mathbb{Z}^{n}} \alpha_{j,m}(u_{j}^{l-1})N_{j,m}^{l-1}(x) \quad \text{for } x \in \mathbb{R}^{n}. \tag{4.13}
\]

Since, given a fixed \( j \), for any point \( x \in \mathbb{R}^{n} \) only a finite number (independent of \( j \) and \( m \)) of splines \( N_{j,m}^{l-1} \) are nonzero, we have

\[
|\Delta^{l}(h)u_{j}^{l-1}(x)|^{r} \leq C \sum_{x \in \text{supp } N_{j,m}^{l-1}} |\alpha_{j,m}|^{r} |\Delta^{l}(h)N_{j,m}^{l-1}(x)|^{r}. \tag{4.14}
\]
For \( k \geq j \in \mathbb{N}_0, \tilde{m} \in \mathbb{Z}^n \) we let \( \Gamma_{j, \tilde{m}} \) denote the set of all cubes \( Q^n_{k, m} \subset Q^n_{j, \tilde{m}} \). Next, let \( \Gamma^1_{j, \tilde{m}} \) denote the set of all cubes \( Q^n_{k, m} \subset Q^n_{j, \tilde{m}} \) for which

\[
(1 + l)Q^n_{k, m} \subset Q^n_{j, \tilde{m}}.
\]

We also define \( \Gamma^2_{j, \tilde{m}} := \Gamma_{j, \tilde{m}} \setminus \Gamma^1_{j, \tilde{m}} \).

In what follows we shall require the following estimate of the measure of the set

\[
F_{j, \tilde{m}} := \bigcup_{Q^n_{k, m} \in \Gamma^2_{j, \tilde{m}}} Q^n_{k, m}
\]

(the proof is similar to that of the corresponding estimate in Theorem 4.8 in [16]). We have

\[
\frac{|F_{j, \tilde{m}}|}{|Q^n_{j, \tilde{m}}|} \leq C 2^{j-k}
\]

for some constant \( C > 0 \) independent of \( j \) and \( \tilde{m} \).

Using (4.15) and (2.19), we obtain

\[
\sum_{Q^n_{k, m} \in \Gamma^2_{j, \tilde{m}}} \gamma^n_{k, m} \leq C 2^{(j-k)\delta(n, \alpha)} \sum_{Q^n_{k, m} \in \Gamma^2_{j, \tilde{m}}} \gamma^n_{k, m}.
\]

The cubes \( Q^n_{k, m} \in \Gamma^1_{j, \tilde{m}} \) satisfy

\[
\delta_r(Q^n_{k, m})N^{l-1}_{j, \tilde{m}} \leq C 2^{(j-k)l},
\]

as \( N^{l-1}_{j, \tilde{m}} \) is a polynomial on the cube \( Q^n_{j, \tilde{m}} \).

Also, by (4.2), the cubes \( Q^n_{k, m} \in \Gamma^2_{j, \tilde{m}} \) satisfy

\[
\delta_r(Q^n_{k, m})N^{l-1}_{j, \tilde{m}} \leq C 2^{(j-k)(-1-1)}.
\]

Next, since \( \{s_k\} \in \text{loc}Y_{\alpha_3}^{\alpha_1, \alpha_2} \), it clearly follows that

\[
s_{k, m} \leq C 2^{(\alpha_2 - \frac{\alpha_1}{p})(k-j)} s_{j, \tilde{m}}
\]

provided that \( Q^n_{k, m} \subset Q^n_{j, \tilde{m}}, k \geq j \in \mathbb{N}_0, m, \tilde{m} \in \mathbb{Z}^n \) (the constant \( C > 0 \) depends only on the weight sequence \( \{s_k\} \)).

Combining estimates (4.8), (4.14), (4.17) and (4.18) and using properties (2.18), (4.16) and (4.19) of the multiple sequences \( \{\gamma^n_{k, m}\} \) and \( \{s_{k, m}\} \), this proves that

\[
R_j^2 \leq C \sum_{m \in \mathbb{Z}^n} \sum_{Q^n_{k, m} \in \Gamma^1_{j, \tilde{m}}} 2^{(k-j)\alpha_2 2 \frac{m}{p} s^n_{j, m} \gamma^n_{k, m} 2^{(j-k)l} p \left[ \sum_{m \in \mathbb{Z}^n} [\alpha_{j, m}]^r \right]}^\frac{p}{r} Q^n_{k, m} \cap \text{supp} N^{l-1}_{j, m} \neq \emptyset
\]

\[
+ C \sum_{m \in \mathbb{Z}^n} \sum_{Q^n_{k, m} \in \Gamma^2_{j, \tilde{m}}} 2^{(k-j)\alpha_2 2 \frac{m}{p} s^n_{j, m} \gamma^n_{k, m} \times 2^{(j-k)(-1-1)p} \left[ \sum_{m \in \mathbb{Z}^n} [\alpha_{j, m}]^r \right]}^\frac{p}{r} Q^n_{j, \tilde{m}} \cap \text{supp} N^{l-1}_{j, m} \neq \emptyset
\]
Further, assume that the conclusion of Theorem 4.2, and \( \Gamma \) the splines shall only indicate the differences. Proof of Theorem 4.2. Let \( \frac{r}{p} \) increase the order of splines approximating the given function. Using this fact, as well as (2.15), (4.16) and (4.19), we follow the proof of Theorem 4.1 by eventually, we obtain
\[
\sum_{m \in \mathbb{Z}^n} t_{j,m}^p [\hat{\delta}_r(Q_{k,m}^n) \varphi]^p \leq C^{-k(l-\alpha_2)} \left( \sum_{j=-1}^{k} 2^{j\mu(l-\alpha_2)} (s_j^{l+1}(\varphi, \{t_{k,m}\})_{r,p})^\mu \right)^\frac{1}{\mu}
\]
(4.21)
(with obvious modifications for \( p = \infty \) or \( \mu = \infty \)). Here, the constant \( C > 0 \) depends on \( \alpha_1, \alpha, \sigma, r, l \), but is independent of \( \varphi \).

**Proof.** To a large extent we shall follow the proof of Theorem 4.1, and hence we shall only indicate the differences.

Clearly, for all \( j \in \mathbb{N}_0, \tilde{m} \in \mathbb{Z}^n \) and all cubes \( Q_{k,m}^n \subset Q_{j,\tilde{m}}^n \),
\[
\delta_r(Q_{k,m}^n) N_{j,\tilde{m}}^{l} \leq C 2^{(j-k)l}
\]
(4.22)
where the constant \( C > 0 \) is independent of \( k, j, m, \tilde{m} \).

Using this fact, as well as (2.15), (4.16) and (4.19), we follow the proof of Theorem 4.1 step by step, replacing all the splines \( U_{k}^{l-1} \) in the proof of Theorem 4.1 by the splines \( U_{k}^{l} \). In view of (4.22), we clearly do not need to deal with the sets \( \Gamma_{j,\tilde{m}}^{1} \) and \( \Gamma_{j,\tilde{m}}^{2} \), and so estimate (4.20) is substantially simplified. Eventually, we obtain the conclusion of Theorem 4.2.

The following result is a corollary to Theorem 4.1.

**Corollary 4.1.** Let \( p, q, \in (0, \infty], \varphi \in L^{\infty} - \text{loc}(\mathbb{R}^n) \), let the weight sequence \( \{s_k\} \in L^{\infty} - \text{loc}(\mathbb{R}^n) \), and let the multiple sequence \( \{s_{k,m}\} \) be \( p \)-associated with \( \{s_k\} \). Further, let \( d \in \mathbb{N}_0 \), let the weight \( \gamma^p \in L^{\infty} - \text{loc}(\mathbb{R}^{n+d}) \), and let the multiple sequence \( \{\gamma_{k,m}\} \) be generated by the weight \( \gamma \). Assume that
\[
\alpha_2 < \lambda + \frac{d \delta_1(\gamma)}{p} \quad \text{for} \quad \lambda := \min \left\{ l, l - 1 + \frac{\delta_2(\gamma)}{p} \right\}.
\]
Set \( t_{k,m} := s_{k,m} \gamma_{k,m} \) for \( k \in \mathbb{N}_0, m \in \mathbb{Z}^n \).
Then a necessary and sufficient condition for \( \varphi \) to lie in \( B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \) is that

\[
N_1(\varphi, l) := \left( \sum_{j=-1}^{\infty} (s^l_j(\varphi, \{t_k\})_{r,p})^q \right)^{\frac{1}{q}} < \infty. \tag{4.23}
\]

Moreover,

\[
N_1(\varphi, l) \sim N_2(\varphi, l) \sim \| \varphi \| B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}), \tag{4.24}
\]

where

\[
N_2(\varphi, l) := \left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t^p_{k,m} \| \varphi - T^{l-1}_k(\varphi, r) \| L^r(Q^n_{k,m}) \|^{\frac{2}{p}} \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} \\
+ \left( \sum_{m \in \mathbb{Z}^n} t^p_{0,m} \| \varphi \| L^r(Q^0_{0,m}) \|^{\frac{2}{p}} \right)^{\frac{1}{p}}.
\]

Proof. We consider the case \( q < \infty \); the case \( q = \infty \) is analogous. Let \( \varphi \in B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \). The estimate

\[
N_1(\varphi, l) \leq N_2(\varphi, l) \leq C \| \varphi \| B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\})
\]

follows from (4.6).

Assume now that \( N_1(\varphi, l) < \infty \). We choose \( \epsilon > 0 \) so small that

\[
\alpha_2 - n < \tilde{\lambda} + \frac{\tilde{d}\tilde{\alpha}}{p}.
\]

Applying Theorem 4.1 with \( \mu \leq \min\{1, q, r\} \) and then using Theorem 2.2, this gives

\[
\| \varphi \| B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\})^q \leq \sum_{k=0}^{\infty} 2^{-kq(\tilde{\lambda} + \tilde{d}\tilde{\alpha} - (\alpha_2 - \frac{n}{p}))} \left( \sum_{j=-1}^{k} 2^{q} \mu^{(\tilde{\lambda} + \tilde{d}\tilde{\alpha} - (\alpha_2 - \frac{n}{p}))}(s^l_j(\varphi, \{t_k\})_{r,p})^\mu \right)^{\frac{q}{\mu}} \leq C[N_1(\varphi, l)]^q, \tag{4.25}
\]

which proves the corollary.

In a similar manner, Theorem 4.2 is instrumental in obtaining the following result.

**Corollary 4.2.** Let \( p, q, r \in (0, \infty], \alpha_1, \alpha_2, \in \mathbb{R}, \alpha_3 \geq 0, \sigma_1 \in (0, \infty], \sigma_2 = p, \) let the multiple sequence \( \{t_{k,m}\} \) be \( p \)-associated with the \( p \)-admissible weight sequence \( \{t_k\} \in X_{\alpha,\sigma,p}^{\alpha_3}. \) If \( l > \alpha_2, \) then the function \( \varphi \in B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}) \) if and only if

\[
N_1(\varphi, l + 1) < \infty. \tag{4.26}
\]

Moreover,

\[
N_1(\varphi, l + 1) \sim N_2(\varphi, l + 1) \sim \| \varphi \| B_{p,q,r}^l(\mathbb{R}^n, \{t_{k,m}\}), \tag{4.27}
\]
Theorem 4.2 can be used to obtain a result on equivalent norms in the space $\tilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{k\})$ for different (sufficiently large) $l$.

**Corollary 4.3.** Let $p,q,r \in (0, \infty]$, $\alpha_{1}, \alpha_{2} \in \mathbb{R}$, $\alpha_{3} \geqslant 0$, $\sigma_{1} \in (0, \infty]$, $\sigma_{2} = p$, $\{k\} \in X_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$, and $l > \alpha_{2}$. Then, for $l' > l$,

$$\|\varphi | \tilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{k\}) \| \sim \| \varphi | \tilde{B}^{l'}_{p,q,r}(\mathbb{R}^{n}, \{k\}) \|.$$ 

**Proof.** We only need to use Corollary 4.2 and Lemma 4.1 to obtain the estimate

$$\|\varphi | \tilde{B}^{l}_{p,q,r}(\mathbb{R}^{n}, \{k\}) \| \leq CN_{2}(\varphi, l') \leq C\| \varphi | \tilde{B}^{l'}_{p,q,r}(\mathbb{R}^{n}, \{k\}) \|.$$ 

The reverse estimate is clear.

**Remark 4.2.** As we pointed out in the introduction, the methods in [4] and [5] enable us to obtain a result similar to Corollary 4.3 for the space $\tilde{B}^{l}_{p,q,1}(\mathbb{R}^{n}, \{k\})$, provided that $\{k\} \in \text{loc}Y_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}} \{\{k\} \in Y_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}} \}$ and $l > \alpha_{2}$.

The following simple example shows that Corollary 4.3 also works under weaker assumptions on the variable smoothness.

Let $p \in (1, \infty)$. We will show that there exists a weight sequence $\{\gamma_{1}^{1}\}$ such that $\{\gamma_{1}^{1}\} \in X_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$, $\alpha_{2} < l$, $\sigma_{2} = p$ and $\{\gamma_{1}^{1}\} \in \text{loc}Y_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$ for $l < \alpha_{2}'$, but $\{\gamma_{1}^{1}\} \notin \text{loc}Y_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$ for any $\alpha_{2}'' < l$. So, let $\varepsilon \in (0, n)$ be a sufficiently small number that will be chosen later. We set

$$(\gamma_{1}^{1})(x, x_{n+1}) := \prod_{i=1}^{n+1} \frac{1}{|x_{i}|^{1-\varepsilon}} \text{ for } (x, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}.$$ 

Note that $(\gamma_{1}^{1}) \in A_{1}(\mathbb{R}^{n+1})$. Consider the multiple sequence

$$(\gamma_{1}^{1}(x, x_{n+1}) := \sum_{m \in \mathbb{Z}^{n}} \chi_{\Gamma_{k,m}^{\gamma_{1}^{1}}}(x) 2^{kn}(\gamma_{1}^{1} x_{n+1}) \text{ for } x \in \mathbb{R}^{n}, k \in \mathbb{N}_{0}.$$ 

Clearly, the inequality

$$\frac{(\gamma_{k+1}^{1})^{p}(0)}{(\gamma_{k}^{1})^{p}(0)} = 2^{pl+n} \frac{\int_{\mathbb{R}^{1+n}} (\gamma_{1}^{1})^{p}(x, x_{n+1}) dx dx_{n+1}}{\int_{0}^{1} (\gamma_{1}^{1})^{p}(x, x_{n+1}) dx dx_{n+1}} \geq 2^{pl+\frac{n}{2}}$$

holds for sufficiently small $\varepsilon \in (0, 1)$. Hence, $\{\gamma_{k}^{1}\} \in Y_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$ only if $\alpha_{2} \geq l + \frac{n}{2p} > l$.

However, by Example 2.1 we have $\{\gamma_{k}^{1}\} \in \tilde{X}_{\alpha_{1, \sigma_{p}}}^{\alpha_{3}}$ for $\sigma_{2} = p$, $\alpha_{2} = l - \frac{\delta_{1}(\gamma)}{p} < l$, and hence all the hypotheses of Corollary 4.3 are satisfied.

We note that by Remark 2.3 the space $\tilde{B}^{l}_{p,p,r}(\mathbb{R}^{n}, \{k\})$ is nontrivial for any $p \in (1, \infty)$ and $r \in [1, p]$. Moreover, by Theorem 3.1 the space $\tilde{B}^{l}_{p,p,r}(\mathbb{R}^{n}, \{k\})$ is the trace of the Sobolev space $W^{l}_{p}(\mathbb{R}^{n+1}, \gamma)$. On some new function spaces 875
The following theorem is an important step in the proof of the atomic decomposition theorem. However, the estimate obtained here may be of independent interest.

Given \( p \in (0, \infty], \theta \in (0, p] \), we set \( p_\theta := \frac{p}{\theta} \), provided that \( p \) and \( \theta \) are not both infinite. If \( \theta = p = \infty \), we set \( p_\theta := 1 \). If \( 0 < \theta < p = \infty \), then we assume that \( p_\theta = \infty \).

**Theorem 4.3.** Let \( p, q, r \in (0, \infty] \), \( \theta \in (0, \min\{p, r\}] \), let \( \{t_k\} \) be a \( p \)-admissible weight sequence in \( X_{\alpha, \sigma, p}^\alpha \) with \( \sigma_1 = \theta r_0' \), \( \alpha_1 > n\left(\frac{1}{p} - \frac{1}{r}\right) \), \( \sigma_2 \in (0, \infty] \), \( \alpha_3 \geq 0 \), and let the multiple sequence \( \{t_{k,m}\} \) be \( p \)-associated with the weight sequence \( \{t_k\} \). Assume that functions \( V_k \in L_r^{\text{loc}}(\mathbb{R}^n) \), \( k \in \mathbb{N}_0 \), and that

\[
\left( \sum_{k=0}^\infty \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p 2^{knp_\theta} \|v_k \mid L_r(Q_{k,m}^n)\|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty,
\]

where \( v_k := V_k - V_{k-1}(V_1 \equiv 0) \) for \( k \in \mathbb{N}_0 \).

Then the series \( \sum_{k=0}^\infty v_k \) converges in \( L_r^{\text{loc}}(\mathbb{R}^n) \) to some function \( \varphi \in L_r^{\text{loc}}(\mathbb{R}^n) \), and moreover,

\[
\left( \sum_{k=0}^\infty \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p 2^{knp_\theta} \|\varphi - V_k \mid L_r(Q_{k,m}^n)\|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} + \left( \sum_{m \in \mathbb{Z}^n} C t_{0,m}^p \|\varphi \mid L_r(Q_{0,m}^n)\|^p \right)^{\frac{1}{p}} \leq C \left( \sum_{k=0}^\infty \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p 2^{knp_\theta} \|v_k \mid L_r(Q_{k,m}^n)\|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}},
\]

(4.28)

where the constant \( C > 0 \) depends on \( \alpha_1, \alpha, \sigma \) and \( r \), but is independent of the function sequence \( \{V_k\} \).

**Proof.** We consider only the case \( p, q \neq \infty \).

We will show that the series \( \sum_{k=0}^\infty v_k \) converges in \( L_r^{\text{loc}}(\mathbb{R}^n) \) to some function \( \varphi \in L_r^{\text{loc}}(\mathbb{R}^n) \). It suffices to show that the series \( \sum_{k=0}^\infty v_k \) converges in \( L_r(Q_{0,m}^n) \) for any cube \( Q_{0,m}^n \).

For any \( j_1 \leq j_2 \in \mathbb{N}_0 \) and \( \mu \leq \min\{1, \theta\} \) it follows from Lemma 2.1 that

\[
\|V_{j_1} - V_{j_2} \mid L_r(Q_{0,m}^n)\| \leq \left( \sum_{j=j_1}^\infty \|v_j \mid L_r(Q_{0,m}^n)\|^{\mu} \right)^{\frac{1}{\mu}} \leq \left( \sum_{j=j_1}^\infty \left( \sum_{m \in \mathbb{Z}^n} \|v_j \mid L_r(Q_{j,m}^n)\|^{\theta} \right)^{\frac{1}{\theta}} \right)^{\frac{1}{1 - \mu}} = K_{j_1,m} ,
\]

(4.29)
Applying Hölder’s inequality to the inner sum (over $\tilde{m}$) with exponents $p_\theta$ and $p'_\theta$ and using (2.14), for any $\mu \leq \min\{1, \theta\}$ we have

$$(K_{j_1,m})^\mu = \sum_{j=j_1}^{\infty} \left( \sum_{m \in \mathbb{Z}^n_{j_1, m}} \frac{t_j^\mu}{t_j^\theta} \frac{\|v_j \| L_r(Q^n_{j,j, m})}{n_p} \right)^{\frac{\mu}{p}}$$

$$\leq C \sum_{j=j_1}^{\infty} \frac{1}{2 \mu r} \frac{t_j^\mu}{t_j^\theta} \left( \sum_{m \in \mathbb{Z}^n_{j_1, m}} \left[ \frac{1}{t_j^\theta} \frac{\|v_j \| L_r(Q^n_{j,j, m})}{n_p} \right]^{\frac{\mu}{\theta p'}} \left( \sum_{\tilde{m} \in \mathbb{Z}^n_{j,j, m}} 2 \frac{t_j^\mu}{t_j^\theta} \frac{\|v_j \| L_r(Q^n_{j,j, \tilde{m}})}{n_p} \right)^{\frac{\mu}{p}} \right)^{\frac{\mu}{p}}$$

$$\leq C \sum_{j=j_1}^{\infty} \frac{2 j \mu n (\frac{n}{\theta} - \frac{r}{p})}{2 j \mu n} \left( \sum_{m \in \mathbb{Z}^n_{j_1, m}} 2 \frac{t_j^\mu}{t_j^\theta} \frac{\|v_j \| L_r(Q^n_{j,j, m})}{n_p} \right)^{\frac{\mu}{p}}. \quad (4.30)$$

We choose $\mu \leq \min\{1, q, \theta\}$ and take $q_\mu := \frac{q}{\mu} \geq 1$. Applying Hölder’s inequality with the exponents $q_\mu$ and $q'_\mu$ to the right-hand side of (4.30) shows that

$$K_{j_1,m} \leq C \left( \sum_{j=j_1}^{\infty} \frac{1}{2 j \mu q_\mu (\alpha_1 - \frac{n}{\theta} - \frac{n}{r})} \right)^{\frac{1}{\mu q_\mu}} \times \left( \sum_{j=j_1}^{\infty} \left( \sum_{m \in \mathbb{Z}^n_{j_1, m}} 2 \frac{t_j^\mu}{t_j^\theta} \frac{\|v_j \| L_r(Q^n_{j,j, m})}{n_p} \right)^{\frac{\mu}{p}} \right)^{\frac{1}{p}} \quad (4.31)$$

Note that for fixed $j_1$ the right-hand side of (4.31) tends to infinity as $\alpha_1$ tends to $\frac{n}{\theta} - \frac{n}{r}$.

From (4.29), (4.31), since the space $L_r(Q^n_{0,m})$ is complete, we find that the series $\sum_{k=0}^{\infty} v_k$ converges in $L_r(Q^n_{0,m})$ to some function $\varphi_m \in L_r(Q^n_{0,m})$, as required. We set $\varphi(x) = \sum_{m \in \mathbb{Z}^n} \chi_{Q^n_{0,m}}(x) \varphi_m(x)$ for $x \in \mathbb{R}^n$.

Now we will prove (4.28). Applying Lemma 2.1 with $f_j = 0$ for $j < k$ and $f_j := v_j \chi_{Q^n_{k,m}}$ for $j \geq k$, and then using Minkowski’s inequality (because $\frac{p}{\mu} \geq 1$), this gives

$$\left( \sum_{m \in \mathbb{Z}^n} 2 \frac{n k p}{t_j^\mu} \frac{\|v_j \| L_r(Q^n_{k,m})}{n_p} \right)^{\frac{1}{p}}$$

$$\leq 2 \frac{n k}{t_k^\mu} \left( \sum_{m \in \mathbb{Z}^n} \frac{t_k^p}{t_k^\mu} \left[ \sum_{j=k}^{\infty} \|v_j \| L_r(Q^n_{k,m}) \right]^{\frac{\mu}{p}} \right)^{\frac{1}{p}}$$

$$\leq 2 \frac{n k}{t_k^\mu} \left( \sum_{j=k}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} \frac{t_k^p}{t_k^\mu} \left[ \sum_{\tilde{m} \in \mathbb{Z}^n_{j,j, m}} \|v_j \| L_r(Q^n_{j,j, \tilde{m}}) \right]^{\frac{\mu}{p}} \right) \right)^{\frac{1}{p}}$$

$$\leq 2 \frac{n k}{t_k^\mu} \left( \sum_{j=k}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} \frac{t_k^p}{t_k^\mu} \left[ \sum_{\tilde{m} \in \mathbb{Z}^n_{j,j, m}} \|v_j \| L_r(Q^n_{j,j, \tilde{m}}) \right]^{\frac{\mu}{p}} \right) \right)^{\frac{1}{p}} \mu =: R_k. \quad (4.32)$$
Arguing as in the proof of (4.30), we obtain for \( j \geq k + 1 \)
\[
2^{-\frac{nkp}{r}} \sum_{m \in \mathbb{Z}^n} t_{k,m}^p \left[ \sum_{\tilde{m} \in \mathbb{Z}^n} \frac{\|v_j | L_r(Q_{j,\tilde{m}}^n)}{L_{r}(Q_{j,\tilde{m}}^n)} \right]^{\frac{p}{q}} \leq 2^{-\frac{nkp}{r}} \sum_{m \in \mathbb{Z}^n} t_{k,m}^p \left( \sum_{\tilde{m} \in \mathbb{Z}^n} \left[ \frac{1}{t_{j,\tilde{m}}} \right]^{\theta p_0} \right)^{\frac{p}{\theta p_0}} \left[ \sum_{\tilde{m} \in \mathbb{Z}^n} t_{j,\tilde{m}}^p \|v_j | L_r(Q_{j,\tilde{m}}^n) \|^p \right] \leq C 2^{(k-j)p(\alpha_1 - (\frac{n}{p} - \frac{n}{r}))} \sum_{m \in \mathbb{Z}^n} t_{j,m}^p 2^{njp} \|v_j | L_r(Q_{j,m}^n) \|^p. \tag{4.33}
\]

We take \( \mu \leq \min\{1, \theta, q\} \). Since \( \alpha_1 > \frac{n}{p} - \frac{n}{r} \) by the hypothesis of the lemma, it follows by Hardy’s inequality, using (4.32) and (4.33), that
\[
\left( \sum_{k=0}^{\infty} R_k^q \right)^{\frac{1}{q}} \leq C \left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{j,m}^p 2^{jnp} \|v_j | L_r(Q_{j,m}^n) \|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{q}}. \tag{4.34}
\]

Now the required estimate for the first term on the left-hand side of (4.27) follows from (4.30) and (4.34).

Next we estimate the second term on the left-hand side of (4.28). Similarly to (4.29), \( \|\varphi | L_r(Q_{0,m}^n) \| \leq CK_{1,m} \). Hence, using (4.31) we easily obtain the estimate
\[
\left( \sum_{m \in \mathbb{Z}^n} t_{0,m}^p \|\varphi | L_r(Q_{0,m}^n) \|^p \right)^{\frac{1}{p}} \leq C \left( \sum_{j=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{j,m}^p 2^{jnp} \|v_j | L_r(Q_{j,m}^n) \|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{q}}. \tag{4.35}
\]

The proof of Theorem 4.3 is complete.

Suppose that for \( \varphi \in L_r^{\text{loc}}(\mathbb{R}^n) \) we have \( \varphi = \sum_{k=0}^{\infty} v_k^l \) in \( L_r^{\text{loc}}(\mathbb{R}^n) \), where
\[
v_k^l(x) := \sum_{m \in \mathbb{Z}^n} \beta_{k,m} N_{k,m}^l(x) \quad \text{for} \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n.
\]

Given \( p, q, r \in (0, \infty] \), we define (with the corresponding modifications for \( p = \infty \) or \( q = \infty \))
\[
N_3(\varphi, l + 1) := \text{inf} \left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\beta_{k,m}|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{q}}, \tag{4.36}
\]
where the infimum in (4.36) is taken over all series \( \sum_{k=0}^{\infty} v_k^l \) converging to the function \( \varphi \) in \( L_r^{\text{loc}}(\mathbb{R}^n) \).

For \( \varphi \in L_r^{\text{loc}}(\mathbb{R}^n) \) we also set (with the corresponding modifications in the case \( p = \infty \) or \( q = \infty \))
\[
N_4(\varphi, l + 1) := \left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\alpha_{k,m}(T_k^l(\varphi, r))|^p \right)^{\frac{2}{p}} \right)^{\frac{1}{q}}. \tag{4.37}
\]

The next result extends Theorem 5.1 in [16] to the case of the Besov spaces of variable smoothness \( \widetilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \).
Corollary 4.4 (atomic decomposition). Let \( p, q, r \in (0, \infty) \), \( \theta \in (0, \min\{r, p\}] \), let \( \{t_k\} \in X_{\alpha, \sigma, p}^{\infty} \) be a \( p \)-admissible weight sequence with \( \alpha_1 > n(\frac{1}{2} - \frac{1}{r}) \), \( l > \alpha_2 \), \( \sigma_1 = \theta p'_0 \), \( \sigma_2 = p \), and let \( \{t_{k,m}\} \) be the multiple sequence \( p \)-associated with the weight sequence \( \{t_k\} \). Then:

1) each function \( \varphi \in \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\}) \) may be expanded in an \( L_r^{\text{loc}}(\mathbb{R}^n) \)-convergent series of splines \( N_{k,m}^l \); that is,

\[
\varphi = \sum_{k=0}^{\infty} v_k^l(\varphi) \quad \text{in the sense of } L_r^{\text{loc}}(\mathbb{R}^n),
\]

(4.37)

where

\[
v_k^l(\varphi)(x) = \sum_{m \in \mathbb{Z}^n} \beta_{k,m}(\varphi) N_{k,m}^l(x) \quad \text{for } x \in \mathbb{R}^n.
\]

Moreover, for some constant \( C > 0 \)

\[
N_3(\varphi, l + 1) \leq N_4(\varphi, l + 1) \leq C \|\varphi\| \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\});
\]

2) if, for some multiple sequence \( \{\beta_{k,m}\} \),

\[
\left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\beta_{k,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty,
\]

then the series \( \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \beta_{k,m} N_{k,m}^l \) converges in \( L_r^{\text{loc}}(\mathbb{R}^n) \) to some function \( \varphi \in \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\}) \) and there exist constants \( C_1, C_2 > 0 \) such that

\[
\|\varphi\| \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\}) \leq C_1 N_3(\varphi, l + 1) \leq C_2 N_4(\varphi, l + 1).
\]

Proof. We argue as in the proofs of Theorem 5.1 and Corollary 5.3 in [16] using Theorem 4.3, Corollary 4.2, Lemma 4.2 and the obvious estimate

\[
\left( \sum_{m \in \mathbb{Z}^n \cap \text{supp } N_{k,m}^{l-1} \neq \emptyset} |\beta_{k,m}|^p \right)^{\frac{1}{p}} \leq \left( \sum_{m \in \mathbb{Z}^n \cap \text{supp } N_{k,m}^{l-1} \neq \emptyset} |\beta_{k,m}|^r \right)^{\frac{1}{r}} \leq C \left( \sum_{m \in \mathbb{Z}^n \cap \text{supp } N_{k,m}^{l-1} \neq \emptyset} |\beta_{k,m}|^p \right)^{\frac{1}{p}} (4.38)
\]

as appropriate, making the obvious modifications in \( (4.38) \) for \( p = \infty \) and \( r = \infty \). The constant \( C \) in \( (4.38) \) depends only on \( n, l, p, r \).

Remark 4.3. Under the hypotheses of Corollary 4.4 the set \( \Sigma_{p,q}^l \) is dense in the space \( \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\}) \), \( p, q \in (0, \infty) \). Indeed, let \( \varphi \in \tilde{B}_{p, q, r}^l(\mathbb{R}^n, \{t_k\}) \). Then, by Corollary 4.4,

\[
\varphi = \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \beta_{k,m}(\varphi) N_{k,m}^l \quad \text{in the sense of } L_r^{\text{loc}}(\mathbb{R}^n),
\]
and so,

$$\left(\sum_{k=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\beta_{k,m}(\varphi)|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq 2N_3(\varphi, l + 1) \leq C\|\varphi \mid \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})\|.$$ 

Hence, taking

$$\varphi_n := \sum_{k=0}^{n} \sum_{m \in \mathbb{Z}^n} \beta_{k,m}(\varphi)N_{k,m},$$

it follows from assertion 2) in Corollary 4.4 that

$$\|\varphi_n - \varphi \mid \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})\| \leq C\left(\sum_{k=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\beta_{k,m}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \rightarrow 0, \quad n \rightarrow \infty.$$

**Remark 4.4.** Let \( p \in (1, \infty). \) We claim that parameters \( r \in (1, p), \alpha_3, \alpha, \sigma \) and a weight sequence \( \{\gamma_k^2\} \in \tilde{X}_{\alpha,\sigma,p} \) can be chosen so as to satisfy all the hypotheses of Corollary 4.4. In addition, \( \{\gamma_k^2\} \in Y_{\alpha_1', \alpha_2'} \) for some \( \alpha_1' < 0, \) but \( \{\gamma_k^2\} \notin Y_{\alpha_1'', \alpha_2''} \) for any \( \alpha_2'' > 0. \)

So, let \( \varepsilon \in (0, p - 1) \) be a sufficiently small number, which will be specified later. We set

$$\gamma^2(p(x_1, \ldots, x_{n+1}) := \prod_{i=1}^{n+1} |x_i|^{p-1-\varepsilon}.$$ \( (\gamma^2)^p \in A_{p/\theta}(\mathbb{R}^{n+1}) \) for some \( \theta \in (1, p). \) For \( l \in \mathbb{N} \) consider the multiple sequence

$$\gamma_{k,m} := 2^{kl^p} \int_{\mathbb{Z}_1^{n}, m} (\gamma^2)^p(x, x_{n+1}) \, dx \, dx_{n+1},$$

where \( k \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n. \) Let

$$\gamma_k^2(x) = 2^{nk} \sum_{m \in \mathbb{Z}^n} \chi_{\tilde{Q}_{k,m}}(x)(\gamma_k^2)^p \quad \text{for} \quad k \in \mathbb{N}_0, \quad x \in \mathbb{R}^n.$$ \n
The inequality

$$\frac{(\gamma_{k+1}^2)^p(0)}{(\gamma_k^2)^p(0)} = 2^{pl+n} \frac{\int_{\mathbb{Z}_0^{1,n}, k+1} (\gamma^2)^p(x, x_{n+1}) \, dx \, dx_{n+1} \, dx_{n+1}}{\int_{\mathbb{Z}_0^{1,n}, k} (\gamma^2)^p(x, x_{n+1}) \, dx \, dx_{n+1}}$$

is obvious.

If \( p > n, \) \( l \leq n, \) then it is easy to check that \( \sup\{\alpha_1 \mid \{\gamma_k^2\} \in Y_{\alpha_1, \alpha_2} \} < 0 \) for sufficiently small \( \varepsilon > 0. \) On the other hand, it follows easily from Example 2.1 that the weight sequence \( \{\gamma_k^2\} \) lies in \( \tilde{X}_{\sigma_1, \sigma_2} = p \frac{p_0}{p_0} (p_0 = \frac{p}{\theta}), \) \( \sigma_2 = p, \) \( \alpha_1 = l + \frac{n}{p} - \frac{(n+1)p_0}{p} = l - \frac{1}{\theta} > 0 \) (because in our setting \( d = 1, p_0 = \frac{p}{\theta}, \theta > 1 \)) and \( \alpha_2 = l - \frac{\delta_1(\gamma^2, n, 1)}{p} < l. \) Hence, all the hypotheses of Corollary 4.4 are satisfied.
Remark 4.5. Let \( p, q \in (0, \infty), p \neq \infty, r \in (0, p] \), \( \gamma^p \in A_{p/r}^{\text{loc}}(\mathbb{R}^n) \) and let \( s > 0 \). Set

\[
t_{k,m} = 2^{ks} \int_{Q_{k,m}^n} \gamma^p(x) \, dx
\]

for \( k \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \). Arguing as in Example 2.1, we conclude that the sequence \( \{t_{k,m}\} \) satisfies the hypotheses of Corollary 4.4. Hence, an appeal to Theorem 2.5 gives the atomic decomposition theorem for the weighted Besov space \( \tilde{B}^{s}_{p,q,r}(\mathbb{R}^n, \gamma) = \tilde{B}_{p,q,r}(\mathbb{R}^n, \gamma) \) as a particular case of Corollary 4.4. Problems on the atomic decomposition of the spaces \( B^{s}_{p,q}(\mathbb{R}^n, \gamma) \) (and their generalizations) were studied in the papers \cite{13} and \cite{17} using different methods (see also the references given in them).

Proof of Theorem 2.5. Step 1. We prove the embedding

\[
\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \subset \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})
\]

for any \( 0 < r \leq \frac{p}{p_0} \). From the definition of the class \( A_{p_0}^{\text{loc}}(\mathbb{R}^n) \), since \( r \leq \frac{p}{p_0} \) we have \( \gamma \in A_{p/r}^{\text{loc}}(\mathbb{R}^n) \). Hence, using Hölder’s inequality and the properties of the sequence \( \{s_k\} \),

\[
\begin{align*}
\int_{\mathbb{R}^n} \gamma^p(x)s_k^p(x) \left[ \delta^l_{r} \left( x + \frac{I^n}{2^k} \right) \varphi \right]^p \, dx & \leq C_1 \sum_{m \in \mathbb{Z}^n} \int_{Q_{k,m}^n} \gamma^p(x)s_k^p(x) \left| \delta^l_{r}(Q_{k,m}^n) \varphi \right|^p \, dx \\
& \leq C_2 \sum_{m \in \mathbb{Z}^n} \int_{Q_{k,m}^n} \gamma^p(x)s_k^p(x) \left( \int_{Q_{k,m}^n} 2^{kn} \frac{\gamma^r(y)}{\gamma^r(y)} \int_{I^n/2^k} |\Delta^l(h)\varphi(y)|^r \, dh \, dy \right) \, dx \\
& \leq C_3 \int_{\mathbb{R}^n} t_k^p(y) [\Delta^l_r(2^{-k})\varphi(y)]^p \, dy.
\end{align*}
\]

The required embedding follows from (4.39).

Step 2. Using the arguments employed in Example 2.1, we conclude that the sequence \( \{t_k\} \) satisfies the hypotheses of Corollary 4.4 with \( 0 < r' \leq \frac{p}{p_0} \). Hence,

\[
\tilde{B}^l_{p,q,r_1}(\mathbb{R}^n, \{t_k\}) = \tilde{B}^l_{p,q,r_2}(\mathbb{R}^n, \{t_k\}),
\]

and the norms are equivalent for \( 0 < r_1 \leq r_2 \leq \frac{p}{p_0} \). So the theorem will be proved if we check that

\[
\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \subset \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})
\]

for any \( 0 < r \leq \frac{p}{p_0} \).

Let

\[
t_k^p(x) := \sum_{m \in \mathbb{Z}^n} \chi_{Q_{k,m}^n}(x) 2^{kn} \|t_k|_{L_p(Q_{k,m}^n)}\|^p := \sum_{m \in \mathbb{Z}^n} \chi_{Q_{k,m}^n}(x) 2^{kn} t_{k,m},
\]

where \( k \in \mathbb{N}_0, x \in \mathbb{R}^n \).

Suppose that \( \varphi \in \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \). Then by Corollary 4.4, \( \varphi = \sum_{j=0}^{\infty} v_j^l(\varphi) \) in the sense of \( L^{\text{loc}}_{r} (\mathbb{R}^n) \), where \( v_j^l(x) = \sum_{m \in \mathbb{Z}^n} \beta_j,m N^l_{j,m}(x) \) for \( x \in \mathbb{R}^n \) and \( j \in \mathbb{N}_0 \).
We set \( \varphi_{1,k} := \sum_{i=0}^{k} v_i \) and \( \varphi_{2,k} := \varphi - \varphi_{1} \). Clearly, for \( k \in \mathbb{N}_0 \)

\[
\frac{1}{C} \| t_k \tilde{\Delta}_r (2^{-k}) \varphi \big| L_p(\mathbb{R}^n) \| \leq \| t_k \tilde{\Delta}_r (2^{-k}) \varphi_{1,k} \big| L_p(\mathbb{R}^n) \| + \| t_k \tilde{\Delta}_r (2^{-k}) \varphi_{2,k} \big| L_p(\mathbb{R}^n) \| =: S_{1,k} + S_{2,k}. \quad (4.40)
\]

Arguing as in the proof of Theorem 4.1, using Example 2.1 and estimate (4.38), for \( \mu \leq \min\{1, r, q\} \) we obtain

\[
S_{1,k} \leq C \left( \sum_{j=0}^{k} 2^{(l-\alpha_2)p(j-k)} \left( \sum_{m \in \mathbb{Z}^n} t_{j,m}^P 2^{j\mu p} \| v_j^I \big| L_r(Q^n_{j,m}) \|^p \right)^{\frac{\mu}{p}} \right)^{\frac{1}{\mu}} \leq C \left( \sum_{j=0}^{k} 2^{(l-\alpha_2)p(j-k)} \left( \sum_{m \in \mathbb{Z}^n} t_{j,m}^P |\beta_{j,m}|^p \right)^{\frac{1}{\mu}} \right). \quad (4.41)
\]

To estimate \( S_{2,k} \) we first note that

\[
[\tilde{\Delta}_r^l (2^{-k}) v_j^I (x)]^p \leq C \left( \| v_j^I (x) \|^r + \sum_{i=1}^{l} \left( 2^{kn} \int_{I^n/2^k} |v_j^I (x+ih)|^r \, dh \right) \right)^{\frac{p}{r}} \leq \left( \| v_j^I (x) \|^r + \int_{2(l+1)Q^n_{k,m}} |v_j^I (y)|^r \, dy \right)^{\frac{p}{r}}. \quad (4.42)
\]

Next, employing property 1) from (1.1), (1.4), using the (local) doubling property of the weight \( \gamma^p \) and the properties of the functions \( N_{k,m}^l \), we have

\[
\int_{\mathbb{R}^n} \gamma^p(x) s_k^P(x) |v_j^I (x)|^p \, dx \leq C 2^{(k-j)p\alpha_1} \int_{\mathbb{R}^n} \gamma^p(x) s_j^P(x) |v_j^I (x)|^p \, dx \leq C 2^{(k-j)p\alpha_1} \sum_{m \in \mathbb{Z}^n} \int_{Q^n_{k,m}} \gamma^p(x) s_j^P(x) \left( \sum_{\tilde{m} \in \mathbb{Z}^n} \beta_{j,\tilde{m}} \right) \leq C 2^{(k-j)p\alpha_1} \sum_{m \in \mathbb{Z}^n} t_{j,m}^P |\beta_{j,m}|^p. \quad (4.43)
\]

Using Example 2.1, estimate (2.16), property 1) from (1.1) and Lemma 4.2, for \( j \geq k \) we have

\[
\sum_{m \in \mathbb{Z}^n} \int_{Q^n_{k,m}} t_{k,m}^P 2^{knp} \| v_j^I \big| L_r(2(1+l)Q^n_{k,m}) \|^p \, dx \leq C \sum_{m \in \mathbb{Z}^n} t_{k,m}^P 2^{knp} \| v_j^I \big| L_r(Q^n_{k,m}) \|^p \leq C 2^{(k-j)p\alpha_1} \sum_{m \in \mathbb{Z}^n} t_{j,m}^P 2^{jnp} \| v_j^I \big| L_r(Q^n_{j,m}) \|^p \leq C 2^{(k-j)p\alpha_1} \sum_{m \in \mathbb{Z}^n} t_{j,m}^P |\beta_{j,m}|^p. \quad (4.44)
\]

Combining estimates (4.41)–(4.44) and arguing as in the proof of (4.32),

\[
S_{2,k} \leq C 2^{\kappa_1} \left( \sum_{j=k}^{\infty} 2^{-j\mu_1} \left( \sum_{m \in \mathbb{Z}^n} t_{j,m}^P |\beta_{j,m}|^p \right)^{\frac{\mu_1}{p}} \right). \quad (4.45)
\]
Substituting the estimates (4.41) and (4.45) into (4.40) and using Hardy’s inequality, we obtain

\[
\left( \sum_{k=1}^{\infty} \| t_k \Delta^k \varphi \| L_p(\mathbb{R}^n) \right)^{1/q} \leq C \left( \sum_{m \in \mathbb{Z}^n} \left( \| \sum_{j,m} t_{j,m}^p \| \beta_{j,m} \| \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.
\]

(4.46)

Finally, taking (4.46) together with Corollary 4.4 completes the proof of Theorem 2.5.

§ 5. Embedding theorems for the spaces \( \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \)

Let \( \beta = \{\beta_j\}_{j=1}^{\infty} \) be a sequence of nonnegative numbers and \( w = \{w_{j,m}\}_{j \in \mathbb{N}, m \in \mathbb{Z}^n} \) be a multiple sequence of nonnegative numbers.

For \( 0 < p, q \leq \infty \), we set (with the corresponding modifications in the case \( p, q = \infty \))

\[
\| a \|_{l_q(\beta l_p(w))} := \{ a = a_{j,m} : a_{j,m} \in \mathbb{R}, \| a \|_{l_q(\beta l_p(w))} < \infty \},
\]

\[
\| a \|_{l_q(\beta l_p(w))} = \left( \beta_j^q \left( \sum_{m \in \mathbb{Z}^n} w_{j,m}^p |a_{j,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \tag{5.1}
\]

**Theorem 5.1** (see [27]). Let \( 0 < p_i, q_i \leq \infty \) for \( i = 1, 2 \).

1) The space \( l_q(\beta l_p(w^1)) \) is continuously embedded into \( l_q(\beta l_p(w^2)) \) if and only if

\[
\sum_{j=1}^{\infty} \left( \frac{\beta_j^2}{\beta_j^1} \right) q^* \left( \sum_{m \in \mathbb{Z}^n} \left( \frac{w_{j,m}^2}{w_{j,m}^1} \right) p^* \right)^{\frac{q^*}{p^*}} < \infty, \tag{5.2}
\]

where

\[
\frac{1}{p^*} := \max \left\{ 0, \frac{1}{p_2} - \frac{1}{p_1} \right\}, \quad \frac{1}{q^*} := \max \left\{ 0, \frac{1}{q_2} - \frac{1}{q_1} \right\}.
\]

2) The space \( l_q(\beta l_p(w^1)) \) is compactly embedded into \( l_q(\beta l_p(w^2)) \) if and only if condition (5.2) is satisfied, and moreover,

\[
\lim_{j \to \infty} \frac{\beta_j^2}{\beta_j^1} \left( \sum_{m \in \mathbb{Z}^n} \left( \frac{w_{j,m}^2}{w_{j,m}^1} \right) p^* \right)^{\frac{1}{p^*}} = 0 \quad \text{if} \quad q^* = \infty, \tag{5.3}
\]

\[
\lim_{|m| \to \infty} \frac{w_{j,m}^1}{w_{j,m}^2} = \infty \quad \text{for all} \quad j \in \mathbb{N} \quad \text{if} \quad p^* = \infty. \tag{5.4}
\]

As a direct corollary of Theorem 5.1 and Corollary 4.4 we obtain

**Corollary 5.1.** Let \( i = 1, 2 \) and let \( 0 < p_i, q_i, r_i \leq \infty \), \( \theta_i \in (0, \min\{p_i, r_i\}] \), \( p_{i\theta_i} = \frac{p_i}{\theta_i} \).

Suppose that for \( i = 1, 2 \)

\[
\{t_k^i\} \in X_{\alpha_i, r_i, p_i}, \quad \alpha_1 > n \left( \frac{1}{\theta_i} - \frac{1}{r_i} \right), \quad \sigma_1^i = r_i(p_{i\theta_i})', \quad \sigma_2^i = p_i', \quad l > \alpha_2^i,
\]

is a \( p \)-admissible weight sequence. Then:
1) the space $\widetilde{B}^l_{p_1,q_1,r_1}(\mathbb{R}^n, \{t^1_k\})$ is continuously embedded into $\widetilde{B}^l_{p_2,q_2,r_2}(\mathbb{R}^n, \{t^2_k\})$ if
\[
\sum_{j=0}^\infty \left( \sum_{m \in \mathbb{Z}^n} \left( \frac{t^2_{j,m}}{t^1_{j,m}} \right)^{p^*_m} \frac{q^*_m}{p^*_m} \right) < \infty,
\]
\[
\frac{1}{p^*_m} := \max\left\{0, \frac{1}{p_2} - \frac{1}{p_1}\right\}, \quad \frac{1}{q^*_m} := \max\left\{0, \frac{1}{q_2} - \frac{1}{q_1}\right\};
\]
\[
\frac{1}{p^*} := \max\left\{0, \frac{1}{p_2} - \frac{1}{p_1}\right\}, \quad \frac{1}{q^*} := \max\left\{0, \frac{1}{q_2} - \frac{1}{q_1}\right\};
\]
\[
(5.5)
\]
\[
2) the space $\widetilde{B}^l_{p_1,q_1,r_1}(\mathbb{R}^n, \{t^1_k\})$ is compactly embedded into $\widetilde{B}^l_{p_2,q_2,r_2}(\mathbb{R}^n, \{t^2_k\})$ if (4.5) is satisfied and, moreover,
\[
\lim_{j \to \infty} \left( \sum_{m \in \mathbb{Z}^n} \left( \frac{t^2_{j,m}}{t^1_{j,m}} \right)^{p^*_m} \frac{q^*_m}{p^*_m} \right) \frac{1}{r^*_m} = 0 \quad \text{if } q^*_m = \infty,
\]
\[
\lim_{|m| \to \infty} \frac{t^2_{j,m}}{t^1_{j,m}} = \infty \quad \text{for all } j \in \mathbb{N}_0 \quad \text{if } p^* = \infty.
\]
\[
(5.6)
\]
\[
(5.7)
\]

§ 6. Traces of the spaces $\widetilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ on planes

In this section we assume that $n \geq 2$. Throughout the section we fix a natural number $n' < n$ and define $n'' := n - n'$. The point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will be denoted by $(x', x'') = (x'_1, \ldots, x'_n, x''_{n+1}, \ldots, x''_n)$ (similarly, we put $m := (m', m'')$ for $m \in \mathbb{Z}^n$). We identify the space $\mathbb{R}^{n'}$ with the plane given in the space $\mathbb{R}^n$ by the equation $x'' = 0$.

Let $\{t^1_k\} \in X^{\alpha_3}_{\alpha, \sigma, p}$ be a $p$-admissible weight sequence and let $\{t^2_{k,m}\}$ be the $p$-associated multiple sequence. For $k \in \mathbb{N}_0$ and $m' \in \mathbb{Z}^{n'}$ we set $t^1_{k,m'} = t^2_{k,(m',0)}$. We set
\[
t^1_k(x') := 2^{kn'} \sum_{m \in \mathbb{Z}^{n'}} X_{q_{k,m'}}(x') t^1_{k,m'} \quad \text{for } k \in \mathbb{N}_0, \quad x' \in \mathbb{R}^{n'}.
\]
Then
\[
t^1_k(x') = 2^{-kn'} p \ t_k(x', 0) \quad \text{for } k \in \mathbb{N}_0, \quad x' \in \mathbb{R}^{n'}.
\]

For $p, r \in (0, \infty]$, $p \neq \infty$, $\theta \in (0, \min\{p, r\}]$ we set $p_\theta := \frac{p}{\theta}$ (as in § 4). In this section it will be convenient to denote the dual exponent to $p_\theta$ by $p_\theta^*$. In other words, $\frac{1}{p_\theta} + \frac{1}{p_\theta^*} = 1$.

In defining the trace of the space $\widetilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ we shall follow the ideas used in [9] (which looked at the trace of the space $B^l_{p(\cdot), q}(\mathbb{R}^n)$).

Given $l \in \mathbb{N}$ we set $\Sigma^l := \bigcup_{k=0}^\infty \Sigma^l_k$ ($\Sigma^l_k$ is defined in § 4). Clearly, $\Sigma^l \subset C(\mathbb{R}^n)$. Hence, it makes sense to talk about the pointwise trace of a function $f \in \Sigma^l \cap \widetilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$. In other words, the function $\text{tr}_{x''=0} f := f(x', 0)$ is well-defined.

To define the trace of an arbitrary function $\varphi \in \widetilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ we shall require the following simple result.
Lemma 6.1. Let
\[ p, q \in (0, \infty), \ r \in (0, \infty), \ \theta \in (0, \min\{r, p\}], \]
\[ \alpha_3 \geq 0, \ \alpha_1 > n\left(\frac{1}{\theta} - \frac{1}{r}\right), \ l > \alpha_2, \ \sigma_1 = \theta^\prime \sigma_0, \ \sigma_2 = p, \]
and let \( \{t_k\} \in X_{\alpha_3,\sigma_0,\sigma_2}^\prime \) be a p-admissible weight sequence. Next, assume that the estimate
\[ \|f(\cdot, 0) \mid B_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\})\| \leq C\|f \mid B_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\})\| \]
holds for some
\[ l' \geq l, \ r' \in (0, \infty), \ \theta' \in (0, \min\{r', p\}], \]
\[ \alpha_3' \geq 0, \ \alpha_1' > n\left(\frac{1}{\theta'} - \frac{1}{r'}\right), \ l' > \alpha_2', \ \sigma_1' = \theta'^\prime \sigma_0, \ \sigma_2' = p, \ \{t_k\} \in X_{\alpha_3',\sigma_0',\sigma_2'}^\prime \]
and any function \( f \in \Sigma_l \cap \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \), where the constant \( C > 0 \) is independent of \( f \).

Then, for any function \( \varphi \in \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \), there exists a unique function \( \varphi' \in \tilde{B}_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\}) \) (up to a nullset with respect to the \( n' \)-dimensional Lebesgue measure) such that if
\[ \|\varphi - \varphi_j \mid \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\})\| \to 0 \quad \text{as} \ j \to \infty \]
for some sequence \( \{\varphi_j\} \in \Sigma_l \cap \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \), then
\[ \|\varphi' - \varphi_j(\cdot, 0) \mid \tilde{B}_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\})\| \to 0 \quad \text{as} \ j \to \infty \]
and moreover,
\[ \|\varphi' \mid \tilde{B}_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\})\| \leq C\|\varphi \mid \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\})\|. \]

The proof of this lemma repeats the corresponding arguments from §2.2 in [9] taking account of Theorem 2.4 and Remark 4.3.

Definition 6.1. Under the hypotheses of Lemma 6.1 let \( \varphi \in \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \). The function \( \varphi' \) constructed in Lemma 6.1 is called the trace of the function \( \varphi \) and denoted by \( \text{tr}_{x'' = 0} \varphi \). The trace of the space \( \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \) on the plane, given in the space \( \mathbb{R}^n \) by the equation \( x'' = 0 \), is the set of equivalence classes of functions \( \varphi' \in \tilde{B}_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\}) \), each of which is the trace of some function \( \varphi \in \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \). The corresponding linear space is denoted by \( \text{Tr}_{x'' = 0} \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \) and the norm on this space is defined to be
\[ \|\varphi' \mid \text{Tr}_{x'' = 0} \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\})\| := \inf_{\varphi' = \text{tr}_{x'' = 0} \varphi} \|\varphi \mid \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\})\| \]
In what follows under the conditions of Lemma 6.1 we shall also use \( \text{Tr} \) to denote the linear operator
\[ \text{Tr}: \tilde{B}_{p,q,r}^{l}(\mathbb{R}^n, \{t_k\}) \to \tilde{B}_{p,q,r}^{l'}(\mathbb{R}^n, \{t_k\}) \]
defined by
\[ \text{Tr}[\varphi](x') = \text{tr}_{x''=0} \varphi(x') \quad \text{for} \quad x' \in \mathbb{R}^{n'}. \]

Recall that in §4, for \( k \in \mathbb{N}_0 \) and \( m = (m', m'') \in \mathbb{Z}^n \), we defined
\[ N_{k,m}^l(x) := \prod_{i=1}^n N_i^l \left( 2^k \left( x_i - \frac{m_i}{2^k} \right) \right) \quad \text{for} \quad x \in \mathbb{R}^n, \]
and hence
\[ N_{k,m}^l(x) := N_{k,m'}^l(x') N_{k,m''}^l(x'') \quad \text{for} \quad x = (x', x'') \in \mathbb{R}^n. \]

We note that, for any \( k \in \mathbb{N}_0 \), \( m' \in \mathbb{Z}^{n'} \),
\[ N_{k,m'}^l(x') = \sum_{m'' \in \mathbb{Z}^{n''}} N_{k,(m',m'')}^l(x',0) \quad \text{for} \quad x' \in \mathbb{R}^{n'}. \quad (6.1) \]

The number of terms on the right-hand side of (6.1) is in fact finite and bounded by a quantity independent of \( m' \) and \( x' \). This follows from the fact that the splines \( N_{k,m}^l \) form a partition of unity and that the multiplicity of the intersection of the supports of the splines \( N_{k,m}^l \) is finite (and independent of \( m \)).

Corollary 4.4 enables us to obtain necessary and sufficient conditions for the trace of the space \( \widetilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) \).

**Theorem 6.1.** Let
\[ p, q \in (0, \infty), \quad r \in (0, \infty), \quad \theta \in (0, \min\{r, p\}], \]
\[ \alpha_3 \geq 0, \quad \alpha_1 > n \left( \frac{1}{\theta} - \frac{1}{r} \right), \quad l > \alpha_2, \quad \sigma_1 = \theta \frac{p}{\sigma_2}, \quad \sigma_2 = p \]
and let \( \{t_k\} \in X_{\alpha_3, \sigma, p}^{\alpha_3} \) be a \( p \)-admissible weight sequence such that the weight sequence \( \{t'_k\} \in X_{\alpha_3, \sigma', p}^{\alpha_3} \) for
\[ l' \geq l, \quad r' \in (0, \infty), \quad \theta' \in (0, \min\{r', p\}], \]
\[ \alpha_3' \geq 0, \quad \alpha_1' > n \left( \frac{1}{\theta'} - \frac{1}{r'} \right), \quad \alpha_2' < l', \quad \sigma_1' = \theta' \frac{p}{\sigma_2'}, \quad \sigma_2' = p. \]

Then the operator
\[ \text{Tr}: \widetilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) \to \widetilde{B}_{p,q,r'}^{l'}(\mathbb{R}^{n'}, \{t'_k\}) \]
is bounded and there exists a (nonlinear) bounded operator
\[ \text{Ext}: \widetilde{B}_{p,q,r'}^{l'}(\mathbb{R}^{n'}, \{t'_k\}) \to \widetilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) \]
such that \( \text{Tr} \circ \text{Ext} = \text{Id} \) on the space \( \widetilde{B}_{p,q,r'}^{l'}(\mathbb{R}^{n'}, \{t'_k\}) \). In particular,
\[ \text{Tr} |_{x''=0} \widetilde{B}_{p,q,r}^l(\mathbb{R}^n, \{t_k\}) = \widetilde{B}_{p,q,r'}^{l'}(\mathbb{R}^{n'}, \{t'_k\}) \]
and the corresponding norms are equivalent.
Proof. Note that under the hypotheses of Theorem 6.1 we can apply Corollaries 4.3 and 4.4 to the spaces $\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})$ and $\tilde{B}^l_{p,q,r'}(\mathbb{R}^{n'}, \{t'_k\})$.

The proof splits naturally into two parts.

1. Let $\varphi \in \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \cap \Sigma^l$. Then $\varphi \in \tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\}) \cap \Sigma^l$ by Corollary 4.3, and the corresponding norms are equivalent. Using Corollary 4.4,

$$\varphi = \sum_{k=0}^{\infty} v_k^l(\varphi)$$ in the sense of $L^1_{\text{loc}}(\mathbb{R}^n)$, \hspace{1cm} (6.2)

where

$$v_k^l(\varphi)(x) = \sum_{m \in \mathbb{Z}^n} \alpha_{k,m}(\varphi) N_{k,m}^l(x) \quad \text{for } x \in \mathbb{R}^n.$$

Moreover,

$$\left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\alpha_{k,m}(\varphi)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \|\varphi\|_{\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})} \|$$

$$\leq C \|\varphi\|_{\tilde{B}^l_{p,q,r}(\mathbb{R}^n, \{t_k\})}. \hspace{1cm} (6.3)$$

We set

$$\alpha'_{k,m'} := \sum_{m'' \in \mathbb{Z}^{n''}} \alpha_{k,(m',m'')(\varphi)} N_{k,m''}^l(0) \quad \text{for } k \in \mathbb{N}_0, \ m' \in \mathbb{Z}^{n'}.$$

Hence,

$$v_k^{l'}(x') := \text{tr} |_{x''=0} v_k^l(x) = \sum_{m \in \mathbb{Z}^n} \alpha_{k,m}(\varphi) N_{k,m}(x',0)$$

$$= \sum_{m' \in \mathbb{Z}^{n'}} \alpha'_{k,m'} N_{k,m'}^l(x') \quad \text{for } x' \in \mathbb{R}^{n'}.$$ \hspace{1cm} (6.4)

In view of (2.16) and (6.2),

$$|\alpha'_{k,m'}| t_{k,m'} \leq C \sum_{m'' \in \mathbb{Z}^{n''}} |\alpha_{k,(m',m'')(\varphi)}| t_{k,(m',m'')} \quad \text{for } k \in \mathbb{N}_0, \ m' \in \mathbb{Z}^{n'}.$$ \hspace{1cm} (6.5)

Next, the estimate

$$\left( \sum_{k=0}^{\infty} \left( \sum_{m' \in \mathbb{Z}^{n'}} t_{k,m'}^p |\alpha'_{k,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq C \left( \sum_{k=0}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} t_{k,m}^p |\alpha_{k,m}(\varphi)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$ \hspace{1cm} (6.6)

is straightforward, and the constant $C > 0$ is independent of the function $\varphi$.

Using (6.3) and (6.6), by Corollary 4.4 we see that

$$\left\| \sum_{k=0}^{N} v_k^l(\cdot, 0) |_{\tilde{B}^l_{p,q,r'}(\mathbb{R}^{n'}, \{t'_{k,m'}\})} \right\| \leq C \left( \sum_{k=0}^{\infty} \left( \sum_{m' \in \mathbb{Z}^{n'}} t_{k,m'}^p |\alpha'_{k,m'}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$\leq C \|\varphi\|_{\tilde{B}^l_{p,q,r'}(\mathbb{R}^{n'}, \{t_{k,m}\})}.$$ \hspace{1cm} (6.7)
From (6.7) and Lemma 6.1 it follows that the trace $\varphi'$ of the function $\varphi$ exists on the plane $x'' = 0$. Moreover,
\[\|\varphi' \, | \, \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})\| \leq C \|\varphi \, | \, \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{t_k\})\|, \tag{6.8}\]
where the constant $C > 0$ is independent of the function $\varphi$. This proves that the trace operator
\[\text{Tr} : \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{t_k\}) \to \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})\]
is bounded.

2. Let $\varphi' \in \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})$. Using the hypotheses of the theorem and Corollary 4.4,
\[\varphi' = \sum_{k=0}^{\infty} v_k^{t'}(\varphi') \text{ in the sense of } L^r_{\text{loc}}(\mathbb{R}^{n'}), \tag{6.9}\]
where
\[v_k^{t'}(\varphi)(x') = \sum_{m' \in \mathbb{Z}^{n'}} \alpha'_{k,m'}(\varphi) N_{k,m'}^{t'}(x') \text{ for } x' \in \mathbb{R}^{n'}.\]

We set
\[\alpha_{k,m} = \alpha'_{k,m'} \text{ for } m' \in \mathbb{Z}^{n'}, \quad m'' \in \mathbb{Z}^{n''}, \quad N_{k,m''}^{t'}(0) \neq 0, \]
\[\alpha_{k,(m',m'')} = 0 \text{ for } m' \in \mathbb{Z}^{n'}, \quad m'' \in \mathbb{Z}^{n''}, \quad N_{k,m''}^{t'}(0) = 0, \tag{6.10}\]
\[v_k^{t}(x) := \sum_{m \in \mathbb{Z}^{n'}} \alpha_{k,m}(\varphi) N_k^{t}(x) \text{ for } x \in \mathbb{R}^n.\]

Hence, using Corollaries 4.3 and 4.4, it is easy to show that the series $\sum_{k=0}^{\infty} v_k^{t'}$ converges in $L^r_{\text{loc}}(\mathbb{R}^n)$ to some function $\varphi \in \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{t_k\})$, and moreover,
\[\|\varphi \, | \, \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{t_k\})\| \leq C_1 \|\varphi' \, | \, \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})\| \leq C_2 \|\varphi' \, | \, \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})\|, \tag{6.11}\]
where the constants $C_1, C_2 > 0$ are independent of the function $\varphi'$.

We set $\text{Ext}[\varphi'] := \varphi$ for $\varphi' \in \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})$. Then by (6.11) the operator
\[\text{Ext} : \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\}) \to \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{t_k\})\]
is bounded. It follows directly from the construction of the function $\varphi$ that
\[\varphi' = \text{tr} \mid_{x''=0} \varphi,\]
and hence $\text{Tr} \circ \text{Ext} = \text{Id}$ on the space $\mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{t'_k\})$.

This proves Theorem 6.1.

We now give some examples to illustrate Theorem 6.1.

**Example 6.1.** Let $p, q \in [1, \infty)$, $r \in [1, p]$, and let the weight sequence $\{s_k\} \in \text{loc} Y_{\alpha_1, \alpha_2} \Xi_{\alpha_3, \infty, p}$ for $\alpha_1 > \frac{n''}{p}$ and $l > \alpha_2$. Then
\[\text{Tr} \mid_{x''=0} \mathcal{B}_{p,q,r}^t(\mathbb{R}^n, \{s_k\}) = \mathcal{B}_{p,q,r}^t(\mathbb{R}^{n'}, \{s'_k\}). \tag{6.12}\]
If now \( \{s_k\} \in Y_{\alpha_1, \alpha_2}^{\alpha_3} \), then for \( \alpha_1 > \frac{2^\sigma}{p}, l > \alpha_2 \), it follows by Corollary 2.1 that

\[
\text{Tr}|_{x''=0}B^{s_k}_{p,q}(\mathbb{R}^n) = B^{s_k}_{p,q}(\mathbb{R}^n').
\] (6.13)

For constant exponents \( p, q \in [1, \infty) \) equality (6.13) coincides with that given in [9].

For \( p, q \in [1, \infty), r = p, s_k = 2^{ks}, \alpha > \frac{2^\sigma}{p}, l > s \), we obtain the classical result due to Besov (a characterization of the trace of the classical Besov space on the plane; see [21], Theorems 1.1, 2.1, 2.2).

**Example 6.2.** Let \( p \in (1, \infty), q \in [1, \infty), r \in [1, p) \) and let the weight \( \gamma \in A^{loc}_{p/r}(\mathbb{R}^n) \). We set

\[
t_k(x') = \gamma_k(x') := 2^{k(s+\frac{\sigma}{p})} \sum_{m' \in \mathbb{Z}^{n'}} \tilde{X}_{\gamma_k, m'}(x') \| \gamma \circ L_p(\Sigma_{k, m'}) \|
\]

for \( k \in \mathbb{N}_0 \) and \( x' \in \mathbb{R}^{n'} \). As a particular case of Theorem 6.1 we obtain a characterization of the trace of the weighted Besov space \( B^{s}_{p,q}(\mathbb{R}^n, \gamma) \) on the hyperplane. In fact, the arguments used in Example 2.1 show that \( \{\gamma_k\} \in X_{\alpha_1, \alpha_2}^{\alpha_3} \), \( \gamma_1 = r\bar{p}, \sigma_2 = p \) and \( \alpha_1 = \alpha_2 = s \). Hence, by Remark 2.9 with \( s > \frac{1}{r}, l > s \), we obtain

\[
\text{Tr}|_{x_n=0}B^{s}_{p,q}(\mathbb{R}^n, \gamma) = \widetilde{B}^{l}_{p,q,r}(\mathbb{R}^{n-1}, \{\gamma_k\}).
\] (6.14)

It is worth noting that this assertion is new and cannot be derived using the available atomic decomposition machinery. Indeed, the number of zero moments for the atoms in the trace decomposition is governed by the exponent \( \alpha_1 \) in the case when \( \{\gamma_k\} \in Y_{\alpha_1, \alpha_2}^{\alpha_3} \). The moment condition need not be tested for \( \alpha_1 > 0 \). A much less general analogue of (6.14) was obtained in [11], where a model weight depending only on the distance to the origin was examined. More precisely, \( \gamma^p(x) = |x|^{\alpha} \) in a small neighbourhood of the origin with \( -n+1 < \alpha < (n-1)(p-1) \). This choice of weight enabled the authors of [11] to avoid testing the zero moment condition for the corresponding atoms from the trace decomposition.

If we consider the principal results obtained in this paper we see that the differences \( \delta^l_r \) can be viewed as the most natural replacements for the differences \( \Delta^l_r \) and \( \Delta^l \) in the definition of weighted Besov spaces (with a fairly complicated weight) and Besov spaces of variable smoothness. The exponent \( r \) proves to be closely related to the exponents \( \alpha_1 \) and \( \sigma_1 \).

Speaking informally, it may be stated that the worse the behaviour of the variable smoothness \( \{t_k\} \) is in the integral sense (which means that the exponents \( \alpha_1, \sigma_1 \) are small) the smaller the exponent \( r \) should be taken in the differences \( \delta^l_r \) in order to reveal the meaningful properties of the corresponding Besov spaces of variable smoothness. It is worth noting that in essence this idea is contained in [13].

However, the methods used in [13] are known to work for weighted Besov and Lizorkin-Triebel spaces with Muckenhoupt weights, but are incapable of dealing with spaces of variable smoothness.
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On some new function spaces

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