NONTRAPPING SURFACES OF REVOLUTION WITH LONG LIVING RESONANCES

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Abstract. We study resonances of surfaces of revolution obtained by removing a disk from a cone and attaching a hyperbolic cusp in its place. These surfaces include ones with non-trapping geodesic flow (every maximally extended non-reflected geodesic is unbounded) and yet infinitely many long living resonances (resonances with uniformly bounded imaginary part, i.e. decay rate).

Let $\lambda_k$ be the sequence of resonances satisfying

$$\Re \lambda_k = \pi b \frac{k}{\log k} \left( 1 + O \left( \frac{\log \log k}{\log k} \right) \right),$$

$$\Im \lambda_k = -bj \frac{1}{2} + O \left( \frac{1}{\log k} \right),$$

where $j = 1$ if $a + b \neq 0$, and $j = 2$ if $a + b = 0$.

The most interesting case is $a + b = 0$. Then $f \in C^1$, so the geodesic flow on $(X, g)$ is well-defined and nontrapping (see Proposition 2.1) and yet there exists a sequence of long-living resonances, that is a sequence $\lambda_k$ with $|\Im \lambda_k|$ bounded. This seems to be a new phenomenon.

Figure 1. The surface of revolution $(X, g)$ for $a + b = 0$, $a + b > 0$, and $a + b < 0$. Let $a < 0 < b$, and let $(X, g)$ be the surface of revolution

$$X = \mathbb{R} \times S^1, \quad g = dr^2 + f(r)^2 d\theta^2, \quad f(r) = \begin{cases} 1 + ar & \text{if } r \leq 0, \\ e^{-br} & \text{if } r > 0. \end{cases}$$
Figure 2. This figure plots the resonances $\lambda_k$ in the complex plane, with $k$ ranging from 10 to 1000 in steps of 10, for $(a, b) = (-1, 1), (-2, 1), (-1, 2),$ and $(-2, 2)$. They were computed by solving equation (2.7) numerically in Mathematica using FindRoot, initialized with the leading term of (1.1).

For many scattering problems it is known that sequences of long-living resonances are impossible when the geodesic or bicharacteristic flow is nontrapping: such results go back to Lax and Phillips [LaPh], and Vainberg [Va1] for asymptotically Euclidean scattering, and have been recently extended to asymptotically hyperbolic scattering by Vasy [Va2]. The result closest to the setting of the Theorem is that of [Da], where it is shown that smooth nontrapping manifolds with cusp and funnel have no sequences of long living resonances (many more references can be found in that paper).

Sequences of long-living resonances have been found for a variety of scattering problems, going back to work of Selberg [Se] on finite volume hyperbolic quotients and Ikawa [Ik] on Euclidean obstacle scattering, but in each case the geodesic or bicharacteristic flow has been trapping; see e.g. Dyatlov [Dy] for some recent results and many references.

It is interesting to compare the asymptotic formula (1.1) to analogous asymptotics for one dimensional potential scattering. By work of Regge [Re], Zworski [Zw1] (see also Stepin and Tarasov [StTa]), if $V: \mathbb{R} \to \mathbb{R}$ is supported in $[-L/2, L/2]$, is smooth away from $\pm L/2$ and vanishes to order $j \geq 0$ at $\pm L/2$, then the resonances of $-\partial_r^2 + V(r)$ satisfy

$$\lambda_k = \frac{\pi}{L} k - i \frac{j + 2}{L} \log k + O(1) \quad (1.2)$$

in the right half plane. Note that, as in our result, the decay rates (values of $|\text{Im} \lambda_k|$) are related to the regularity of the potential, with more regularity giving faster decay.

Sequences of resonances asymptotic to logarithmic curves (as in (1.2)) have been found in other situations where the coefficients of the differential operator are not $C^\infty$: see work of
Zworski [Zw2] for scattering by radial potentials and Burq [Bu] for scattering by two convex obstacles, one of which has a corner. In §3 we prove that if we take $b < 0 < a$, and

$$X_1 = (-1/a, \infty) \times S^1, \quad g_1 = dr^2 + f_1(r)^2d\theta^2, \quad f_1(r) = \begin{cases} 1 + ar & \text{if } r \leq 0, \\ e^{-br} & \text{if } r > 0, \end{cases}$$

then for a fixed Fourier mode the resonances closest to the real axis obey an asymptotic more similar to (1.2), namely

$$\lambda_k = \pi ak - \frac{ija}{2} \log k + O(1).$$

This suggests that the reflected geodesics (which, unlike the transmitted ones, are trapped) do not by themselves seem to be enough to produce sequences of long living resonances.

Recently, Baskin and Wunsch [BaWu] study another scattering problem with nonsmooth coefficients (and in particular trapping of non-transmitted geodesics): they show that there are no long-living resonances for nontrapping Euclidean scattering by a metric perturbation with cone points. For this they use the result of Melrose and Wunsch [MeWu] that at cone points diffracted singularities for the wave equation are weaker than transmitted ones, as well as the method of Vainberg [Va1] and Tang and Zworski [TaZw] which relates resolvent continuation and propagation of singularities. For a general metric $g \in C^{1,\alpha}$, $\alpha > 0$, with only conormal singularities (as in the case of a jump such as we have) de Hoop, Uhlmann, and Vasy [DUV] show that reflected singularities are weaker than transmitted ones: these results lead one to conjecture that there are no long-living resonances for nontrapping Euclidean scattering by a metric perturbation with $C^{1,\alpha}$ jump singularities. When the discontinuities are more severe, as in the case of the transmission obstacle problem (where $g$ itself is discontinuous), long living resonances and even resonances with $|\text{Im } \lambda_k| \to 0$ have been observed (see Cardoso, Popov, and Vodev [CPV] and references therein): note however that in that case, unlike in our $C^{1,1}$ case, the geodesic equation does not have unique solutions.

To prove the Theorem we use the fact that, on each Fourier mode in the angular variable, $\Delta_g$ is an ordinary differential operator and the integral kernel of $(\Delta_g - \lambda^2)^{-1}$ can be written in terms of Bessel functions. Resonances occur at values of $\lambda$ satisfying a transcendental equation (see (2.7) below), and we use Bessel function asymptotics to analyze these solutions for a fixed Fourier mode $m$.

It would be interesting to find out if increasing the regularity of $(X, g)$ further leads to further increases in $|\text{Im } \lambda_k|$. The results of [Da], that smooth nontrapping manifolds with cusp and funnel have no sequences of long living resonances, suggests that the answer is yes. To study this, one could modify $f$ on $(-R, 0)$ for some $R > 0$ to make more derivatives of $f$ continuous at 0. Then one could replace the Hankel function asymptotics used in (2.13) by WKB asymptotics (as in e.g. [Ol, Chapter 10, §3.1]), and compute further terms of the asymptotic expansion of the right hand side of (2.15).

Another interesting problem is to find all the long living resonances of $(X, g)$. This would require asymptotics uniform in the Fourier mode $m$. Such asymptotics for Bessel functions in terms of Airy functions are known (see e.g. [AbSt, §9.3.37]) but the resulting transcendental equation seems difficult to solve.

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2. Proof of Theorem

Proposition 2.1. When $a + b = 0$ the manifold $(X, g)$ has a nontrapping geodesic flow.

Proof. The geodesic equations of motion for this flow are

\[ \ddot{r} - f' f(\dot{\theta})^2 = 0, \quad (2.1) \]
\[ \ddot{\theta} = 0. \quad (2.2) \]

If $f \in C^{4,1}$, there is a unique solution. Additionally, if $\dot{\theta} = 0$, then $\dot{r} = 0$ is disallowed because that would describe a stationary solution. In this case $(2.1)$ reduces to $\dot{r} = 0$ supplemented with the condition $\dot{r} \neq 0$, and this forces $r \to \pm \infty$ linearly.

We take $a$ negative and $b$ positive, so $f'$ is always negative and thereby from the first equation of motion $(2.1)$ we find $\ddot{r} \leq 0$. Thus as $t$ increases, $r$ must tend toward either a constant or negative infinity. In the former case, we must also have that $\dot{r} \neq 0$, and this forces $r \to \pm \infty$ linearly.

If $\dot{f} f(\dot{\theta})^2 \to 0$ which is impossible if $\dot{\theta}$ does not vanish, so $r$ cannot tend to a constant. Hence $r$ tends to negative infinity and describes a nontrapping geodesic. \(\square\)

In $(r, \theta)$ coordinates the Laplacian on $(X, g)$ is given by

\[ \Delta_g = -f^{-1} \partial_r f \partial_r - f^{-1} \partial_\theta f^{-1} \partial_\theta = -\partial^2_r - f' f^{-1} \partial_r - f^{-2} \partial^2_\theta. \]

Let

\[ P(m) = -\partial^2_r - f^{-1} f' \partial_r + f^{-2} m^2. \]

On functions of the form $u(r)e^{im\theta}$ the Laplacian acts as

\[ \Delta_g u(r)e^{im\theta} = P(m)u(r)e^{im\theta}. \]

Proposition 2.2. The outgoing resolvent, defined by

\[ (P(m) - \lambda^2)^{-1} : L^2(\mathbb{R}, f(r)dr) \to L^2(\mathbb{R}, f(r)dr), \quad \text{Im} \lambda > 0, \]

is holomorphic on the half plane $\text{Im} \lambda > 0$. Its integral kernel continues meromorphically to a covering space of $\{ \lambda \in \mathbb{C} : \lambda \neq 0, \lambda \neq \pm b/2 \}$.

Proof. For $\lambda \in \mathbb{C} \setminus \{0\}$ and $m > 0$, the Helmholtz equation

\[ (P(m) - \lambda^2)u = 0, \quad (2.3) \]

is solved by

\[ u(r) = \begin{cases} 
  c_1 H^{(1)}_{m/a}(\lambda(r + 1/a)) + c_2 H^{(2)}_{m/a}(\lambda(r + 1/a)) & \text{if } r \leq 0, \\
  c_3 e^{br} I_{m/2}(\frac{m}{b} e^{br}) + c_4 e^{br}/2 K_{m/2}(\frac{m}{b} e^{br}) & \text{if } r > 0,
\end{cases} \]

where $H^{(1)}$ and $H^{(2)}$ are the Hankel functions (as in [AbSt, §9.1]), $I$ and $K$ are the modified Bessel functions (as in [AbSt, §9.6]), $\nu$ is given by

\[ \nu = \sqrt{\frac{1}{4} - \frac{\lambda^2}{b^2}} = -\frac{i \lambda}{b} \left(1 + O(\lambda^{-2})\right), \quad (2.4) \]

and $c_1, c_2, c_3, c_4 \in \mathbb{C}$ are taken such that $u$ and $u'$ are continuous at 0.
By the method of variation of parameters, the inhomogeneous Helmholtz equation
\[ (P(m) - \lambda^2)u = v \]  
(2.5)
is solved by
\[ u(r) = \int_{-\infty}^{\infty} R(r, r')v(r')dr', \]  
(2.6)
where
\[ R(r, r') = -\psi_1(\max\{r, r'\})\psi_2(\min\{r, r'\})/W(r'), \]
where \( \psi_1 \) and \( \psi_2 \) are linearly independent solutions of (2.3) and \( W = \psi_1\psi_2' - \psi_2\psi_1' \) is their Wronskian.

This is the integral kernel of an operator bounded on \( L^2(\mathbb{R}, f(r)dr) \) for \( \text{Im}\ \lambda > 0 \) if and only if
\[ \psi_1(r) = e^{br/2}K_\nu\left(\frac{m}{b}e^{br}\right), \quad r > 0, \]
\[ \psi_2(r) = H^{(2)}_{m/a}(\lambda(r + 1/a)), \quad r < 0 \]
up to an overall constant factor. The condition on \( \psi_1 \) is justified by the asymptotics [AbSt, §9.2.3, §9.2.4] (that \( H^{(1)} \) grows exponentially and \( H^{(2)} \) decays exponentially as \( r \to -\infty \)). The condition on \( \psi_2 \) is justified by [AbSt, §9.7.1, §9.7.2] (that \( I \) grows double-exponentially and \( K \) decays double-exponentially as \( r \to \infty \)).

Meromorphic continuation now follows from the fact that the Hankel function terms are entire in \( \log \lambda \) and the modified Bessel function terms are entire in \( \nu \). Poles of \( R(r, r') \) occur at the values of \( \lambda \) for which the Wronskian \( W \) vanishes, or equivalently for which there is \( c \in \mathbb{C} \) such that
\[ u(r) = \begin{cases} H^{(2)}_{m/a}(\lambda(r + 1/a)) & \text{if } r \leq 0 \\ ce^{br/2}K_\nu\left(\frac{m}{b}e^{br}\right) & \text{if } r > 0, \end{cases} \]
is continuous along with its first derivative at 0. That is,
\[ H^{(2)}_{m/a}\left(\frac{\lambda}{a}\right) = cK_\nu\left(\frac{m}{b}\right) \]
\[ \lambda H^{(2)'}_{m/a}\left(\frac{\lambda}{a}\right) = cmK_\nu'\left(\frac{m}{b}\right) + \frac{cb}{2}K_\nu\left(\frac{m}{b}\right). \]

Dividing the equations to eliminate \( c \) gives
\[ \frac{K_\nu'(\frac{m}{b})}{K_\nu(\frac{m}{b})} = \frac{\lambda H^{(2)'}_{m/a}(\frac{\lambda}{a})}{mH^{(2)}_{m/a}(\frac{\lambda}{a})} - \frac{b}{2m}. \]  
(2.7)

It is important to keep track of the branch of \( H^{(2)} \): as the range of \( \lambda \) extends from \( \{\text{Im}\ \lambda > 0\} \) to \( \{\text{Im}\ \lambda > 0\} \cup \{\text{Re}\ \lambda > b/2\} \), the range of \( \text{arg}(\lambda/a) \) extends from \((-\pi, 0)\) to \((-3\pi/2, \pi/2)\).

Note that no poles (i.e. solutions to (2.7)) can occur when \( \text{Im}\ \lambda > 0 \). This follows from the self-adjointness of \( \Delta_g \) on a suitable domain, but can also be checked directly from the fact that at such a pole we would have \( \langle P(m)u, u \rangle_{L^2} = \lambda^2\|u\|^2_{L^2} \) for the corresponding \( u \), and the left hand side of this equation is nonnegative \( \langle P(m)u, u \rangle_{L^2} \geq 0 \) via integration by parts, but this is a contradiction since \( \lambda^2 \not\in [0, \infty) \). This also shows that (2.6) is the unique \( L^2 \) solution to (2.5) when \( v \in L^2, \text{Im}\ \lambda > 0 \).
For \( m = 0 \), the analysis is similar and slightly simpler: the only change is that \( I_\nu\left(\frac{m}{b}e^{br}\right) \) is replaced by \( e^{br} \) and \( K_\nu\left(\frac{m}{b}e^{br}\right) \) is replaced by \( e^{-br} \).

Poles of the meromorphic continuation of the integral kernel \( R(r, r') \) are called resonances of \( P(m) \).

**Proposition 2.3.** For any \( m > 0, \varepsilon > 0 \) there are \( \lambda_0 > 0, k_0 \in \mathbb{N} \) such that the resonances of \( P(m) \) in the region \( \{ |\lambda| > \lambda_0, |\arg \lambda| < \pi/2 - \varepsilon \} \) form a sequence \( (\lambda_k)_{k \geq k_0} \) satisfying

\[
\text{Re } \lambda_k = \pi b \frac{k}{\log k} \left( 1 + O\left( \frac{\log \log k}{\log k} \right) \right),
\]

\[
\text{Im } \lambda_k = -\frac{b j}{2} + O\left( \frac{1}{\log k} \right),
\]

where \( j = 1 \) if \( a + b \neq 0 \), and \( j = 2 \) if \( a + b = 0 \).

**Proof.** Since resonances are solutions to (2.7), we will deduce this from Bessel function asymptotics. Recall the definitions [AbSt, §9.6.10, §9.6.2]:

\[
I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(1 + \nu)} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! \Gamma(\nu + k + 1)},
\]

\[
K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} \left[ I_{-\nu}(z) - I_\nu(z) \right].
\]

Define the remainders \( R_1, R_2, R_3, R_4 \) by the equations

\[
I_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(1 + \nu)} (1 + R_1), \quad I_{-\nu}(z) = \frac{z^{-\nu}}{2^{-\nu} \Gamma(1 - \nu)} (1 + R_2)
\]

\[
I'_\nu(z) = \frac{\nu}{z} \frac{z'^\nu}{2^\nu \Gamma(1 + \nu)} (1 + R_3), \quad I'_{-\nu}(z) = -\frac{\nu}{z} \frac{z'^{-\nu}}{2^{-\nu} \Gamma(1 - \nu)} (1 + R_4),
\]

and observe that \( R_j = O(\nu^{-1}) \) for each fixed \( j \) and \( z \). We simplify the resulting formula for \( K \) using the Gamma reflection formula [AbSt, §6.1.17]:

\[
K_\nu(z) = \frac{\pi}{2 \sin \pi \nu} \left[ \left( \frac{z}{2} \right)^{-\nu} \frac{1 + R_2}{\Gamma(1 - \nu)} - \left( \frac{z}{2} \right)^{\nu} \frac{1 + R_1}{\Gamma(1 + \nu)} \right]
\]

\[
= \frac{\Gamma(1 - \nu) \Gamma(\nu)}{2} \left[ \left( \frac{z}{2} \right)^{-\nu} \frac{1 + R_2}{\Gamma(1 - \nu)} - \left( \frac{z}{2} \right)^{\nu} \frac{1 + R_1}{\nu \Gamma(\nu)} \right]
\]

\[
= \frac{1}{2} \left[ \left( \frac{z}{2} \right)^{-\nu} \Gamma(\nu)(1 + R_2) + \left( \frac{z}{2} \right)^{\nu} \Gamma(-\nu)(1 + R_1) \right].
\]

Similarly

\[
K'_\nu(z) = \frac{\nu}{2z} \left[ \left( \frac{z}{2} \right)^{\nu} \Gamma(-\nu)(1 + R_3) - \left( \frac{z}{2} \right)^{-\nu} \Gamma(\nu)(1 + R_4) \right].
\]

The quotient is

\[
\frac{K'_\nu(z)}{K_\nu(z)} = \frac{\nu}{z} \left[ \frac{1 + R_3 - g(\nu)(1 + R_4)}{1 + R_1 + g(\nu)(1 + R_2)} \right],
\]

where \( g(\nu) = (z/2)^{-2\nu} \Gamma(\nu)/\Gamma(-\nu) \).
Consequently (2.16) becomes so that we then cross multiply:

\[
\frac{\lambda H_{m/a}(\frac{\lambda}{a})}{mH_{m/a}(\frac{\lambda}{a})} - \frac{b}{2m} = -i\lambda(1 + R_5),
\]

where \( R_5 = O(\nu^{-1}). \)

Plugging (2.11) and (2.13) into (2.7) gives

\[
\frac{1 + R_3 - g(\nu)(1 + R_1)}{1 + R_1 + g(\nu)(1 + R_2)} = -i\lambda(1 + R_5).
\]

We then cross multiply:

\[
(1 + R_3 - g(\nu)(1 + R_4))b\nu = (-i\lambda(1 + R_5))(1 + R_1 + g(\nu)(1 + R_2)).
\]

We further substitute

\[
i\lambda = -b\nu(1 + R_6)
\]

(see (2.4)), and divide the entire expression by \( b\nu. \) This gives

\[
1 + R_3 - g(\nu)(1 + R_4) = (1 + R_5)(1 + R_6)(1 + R_1 + g(\nu)(1 + R_2)).
\]

Solving for \( g(\nu) \) we find

\[
g(\nu)(1 + R_4 + (1 + R_2)(1 + R_5)(1 + R_6)) = -R_3 + R_6 + R_5(1 + R_6) + R_1(1 + R_5)(1 + R_6), \quad (2.15)
\]

\[
2g(\nu) = R_1 - R_3 + R_5 + R_6 + O\left(\frac{1}{\nu^2}\right). \quad (2.16)
\]

The leading terms of these four remainders are as follows (see (2.9) for \( R_1, \) (2.10) for \( R_3, \) (2.12) and (2.13) for \( R_5, \) and (2.14) for \( R_6):\)

\[
R_1 = \frac{z^2}{4\nu} + O\left(\frac{1}{\nu^2}\right), \quad R_5 = -\frac{a + b}{2b\nu} - \frac{a^2 - 4m^2}{8b^2\nu^2} + O\left(\frac{1}{\nu^3}\right), \quad (2.17)
\]

\[
R_3 = \frac{z^2}{4\nu} + O\left(\frac{1}{\nu^2}\right), \quad R_6 = -\frac{1}{8\nu^2} + O\left(\frac{1}{\nu^4}\right). \quad (2.18)
\]

(See also \texttt{http://math.mit.edu/~datchev/dkk.nb} for Mathematica code which computes \( R_5. \)) Note that by (2.9) and (2.10), and the recurrence relation \([\text{AbSt, §}9.6.26],\) we have

\[
\frac{z^{\nu}}{2^{\nu+1}(1 + \nu)}(R_1 - R_3) = I_\nu(z) - \frac{z}{\nu}I_\nu'(z) = -\frac{z}{\nu}I_{\nu+1}(z) = -\frac{z^{\nu+2}}{2^{\nu+1}\nu\Gamma(2 + \nu)} \left(1 + O\left(\frac{1}{\nu}\right)\right),
\]

so that

\[
R_1 - R_3 = -\frac{z^2}{2\nu^2} + O\left(\frac{1}{\nu^3}\right). \quad (2.19)
\]

Consequently (2.16) becomes

\[
g(\nu) = -\frac{a + b}{4b\nu} + O\left(\frac{1}{\nu^2}\right).
\]
In the case where \(a + b = 0\), (2.15) becomes
\[
geq = -\frac{1}{4\nu^2} + O \left( \frac{1}{\nu^3} \right),
\]
Summing up, we have found that
\[
geq = \frac{c_j}{\nu^j} + O \left( \frac{1}{\nu^{j+1}} \right),
\]
where \(c_j \neq 0\), \(j = 1\) if \(a + b \neq 0\) and \(j = 2\) if \(a + b = 0\).

We now analyze \(\tilde{g}(\nu)\) using Stirling’s approximation: \(\Gamma(1+\xi) = \sqrt{2\pi\xi^{\frac{1}{2}}}e^{-\xi}(1 + O(\xi^{-1}))\) (see e.g. [AbSt, §6.1.37]):
\[
\tilde{g}(\nu) = \left( \frac{\nu}{2} \right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(\nu)} = \left( \frac{\nu}{2} \right)^{-2\nu} \frac{\sqrt{2\pi\nu^{\nu + 1/2}e^{-\nu}} (1 + O(\frac{1}{\nu}))}{\sqrt{2\pi}(-\nu)^{-\nu + 1/2}e^{\nu} (1 + O(\frac{1}{\nu}))}
\]
\[
= \left( \frac{\nu}{2} \right)^{-2\nu} \nu^{\nu + 1/2}(-\nu)^{-\nu - 1/2} \left( 1 + O \left( \frac{1}{\nu} \right) \right).
\]
Note to apply Stirling’s approximation we use our assumption that \(|\arg \lambda| < \pi/2 - \epsilon\).

Using log to denote the principal branch of the logarithm, we have
\[
\log \tilde{g}(\nu) = \left( \nu + \frac{1}{2} \right) \log \nu + \left( \nu - \frac{1}{2} \right) \log(-\nu) - 2\nu \log \frac{\nu}{2} + O \left( \frac{1}{\nu} \right).
\]
Plugging this into (2.20), we obtain
\[
\left( \nu + \frac{1}{2} \right) \log \nu + \left( \nu - \frac{1}{2} \right) \log(-\nu) - 2\nu \log \frac{\nu}{2} = c_j - j \log \nu - 2\pi ik + O \left( \frac{1}{\nu} \right),
\]
where \(c_j\) is a suitable branch of \(\log c_j\), and \(k \in \mathbb{Z}\).

Since \(\text{Im} \nu < 0\) (because \(\text{Re} \lambda > b/2\)), we have \(\log(-\nu) = \log \nu + \pi i\) and hence
\[
c_j - 2\pi ik = 2\nu \log \nu + j \log \nu + \left( \nu - \frac{1}{2} \right) i\pi - 2\nu \log \frac{\nu}{2} + O \left( \frac{1}{\nu} \right)
\]
\[
= (2\nu + j) \left( \log \nu + \frac{i\pi}{2} - \log \frac{\nu}{2} \right) + j \log \frac{\nu}{2} - i\pi \frac{j + 1}{2} + O \left( \frac{1}{\nu} \right)
\]
\[
= (2\nu + j) \left( \log (2\nu + j) - \frac{1}{2\nu + j} + \frac{i\pi}{2} - \log \nu \right) - i\pi \frac{j + 1}{2} + j \log \frac{\nu}{2} + O \left( \frac{1}{\nu} \right)
\]
\[
= (2\nu + j) \left( \log \frac{i(2\nu + j)}{\nu} \right) - i\pi \frac{j + 1}{2} + j \log \frac{\nu}{2} - 1 + O \left( \frac{1}{\nu} \right)
\]
\[
= (2\nu + j) \left( \log \frac{i(2\nu + j)}{\nu} \right) - i\pi \frac{j + 1}{2} + j \log \frac{\nu}{2} - 1 + O \left( \frac{1}{\nu} \right).
\]

With \(\tilde{\nu} = i(2\nu + j)/\nu\) this becomes
\[
\frac{2\pi k}{\nu} = \tilde{\nu} \log \tilde{\nu} + r(\tilde{\nu}),
\]
where the remainder \(r(\tilde{\nu})\) is holomorphic, independent of \(k\), and bounded as \(\text{Re} \tilde{\nu} \to +\infty\) (since \(\text{Re} \lambda \to +\infty\) implies \(\text{Im} \nu \to -\infty\) and hence \(\text{Re} \tilde{\nu} \to +\infty\)). We will show that if \(\lambda_0\)
and \( k \) are sufficiently large, then (2.21) has a unique solution \( \tilde{\nu}_k \) corresponding to a \( \lambda \) with \( \Re \lambda \geq \lambda_0 \), and we will compute its asymptotics. Changing variables back to \( \lambda \) will give (2.8).

For this we use the Lambert W function. Recall that for \( \Re \zeta > 0 \), the equation
\[
\tilde{\nu} \log \tilde{\nu} = \zeta \tag{2.22}
\]
has a unique solution with \( \Re \log \tilde{\nu} > 1 \) and \( |\Im \log \tilde{\nu}| < \pi/2 \) (and hence a unique solution with \( \Re \tilde{\nu} > 1 \)), given by the principal branch of the Lambert W function (see [CGHJK, Figure 4 and Figure 5]). As \( \Re \zeta \to \infty \), it obeys (see [CGHJK, (4.20)])
\[
\tilde{\nu}(\zeta) = \frac{\zeta}{\log \zeta} \left( 1 + O \left( \frac{\log \log \zeta}{\log \zeta} \right) \right). \tag{2.23}
\]
For \( k \) sufficiently large, this reduces solving (2.21) with \( \Re \tilde{\nu} > 1 \) to solving
\[
\frac{2\pi k}{ez} = \zeta + r(\tilde{\nu}(\zeta))
\]
for \( \zeta \) with \( \Re \zeta > 0 \). But e.g. [Ol, Chapter 1, Theorem 5.1] guarantees that such a solution exists and is unique, provided \( k \) is sufficiently large. We denote the solution by \( \zeta_k \), and the corresponding \( \tilde{\nu} \) by \( \tilde{\nu}_k \).

Then (2.23) becomes
\[
\log \tilde{\nu}_k = \log \left( \frac{2\pi k}{ez} + O(1) \right) - \log \left( \frac{2\pi k}{ez} + O(1) \right) + O \left( \frac{\log \log \left( \frac{2\pi k}{ez} + O(1) \right)}{\log \left( \frac{2\pi k}{ez} + O(1) \right)} \right),
\]
that is to say
\[
\tilde{\nu}_k = \frac{2\pi k}{ez \log k} \left( 1 + O \left( \frac{\log \log k}{\log k} \right) \right). \tag{2.24}
\]

To obtain more precise asymptotics for \( \Im \tilde{\nu} \), we take the imaginary part of (2.21):
\[
\Re \tilde{\nu}_k \arg \tilde{\nu}_k + \Im \tilde{\nu}_k \log |\tilde{\nu}_k| = O(1)
\]
Note that, since \( \Re \tilde{\nu}_k > 0 \), both terms on the left have the same sign. Consequently \( \Im \tilde{\nu}_k \log |\tilde{\nu}_k| = O(1) \) and so
\[
\Im \tilde{\nu}_k = O \left( \frac{1}{\log k} \right).
\]
Recalling that in terms of \( \lambda \) we have
\[
\lambda_k = \frac{b \tilde{\nu}_k ez - ibj}{2} \left( 1 + O \left( \frac{1}{\tilde{\nu}_k^2} \right) \right),
\]
we conclude that
\[
\Re \lambda_k = \frac{bez}{2} \Re \tilde{\nu}_k \left( 1 + O \left( \frac{1}{\tilde{\nu}_k^2} \right) \right) = \pi b \frac{k}{\log k} \left( 1 + O \left( \frac{\log \log k}{\log k} \right) \right),
\]
\[
\Im \lambda_k = \left( \frac{bez}{2} \Im \tilde{\nu}_k - \frac{bj}{2} \right) \left( 1 + O \left( \frac{1}{\tilde{\nu}_k^2} \right) \right) = -\frac{bj}{2} + O \left( \frac{1}{\log k} \right).
\]
\( \square \)
From (2.8) we can deduce a Weyl asymptotic for the resonances up to a given real part. Put

\[ N(\lambda) = \# \{ k : \lambda_0 \leq \text{Re} \lambda_k \leq \lambda \} . \]

Then, if \( W = -\log k \),

\[ \lambda_k = \pi b \frac{k}{\log k} (1 + r(k)) = -\frac{\pi b}{We^W} (1 + r) . \]

We obtain from this that \( We^W = -\frac{\pi b}{\lambda} (1 + r) \) where \( W = -\log k \), and thus

\[
W = \log \left( \frac{\pi b}{\lambda} (1 + r) \right) - \log \left( -\log \left( \frac{\pi b}{\lambda} (1 + r) \right) \right) + O \left( \frac{\log (-\log \left( \frac{\pi b}{\lambda} (1 + r) \right))}{\log \left( \frac{\pi b}{\lambda} (1 + r) \right)} \right) \\
= -\log \lambda + \log \pi b + O(r) - \log \log \lambda + O \left( \frac{1}{\log \lambda} \right) + O \left( \frac{\log \log \lambda}{\log \lambda} \right) .
\]

From this we get the result

\[ k = e^{-W} = \frac{\lambda \log \lambda}{\pi b} \left( 1 + O \left( \frac{\log \log \lambda}{\log \lambda} \right) \right) , \]

and hence

\[ N(\lambda) = \frac{\lambda \log \lambda}{\pi b} \left( 1 + O \left( \frac{\log \log \lambda}{\log \lambda} \right) \right) . \]

On the other hand

\[
\frac{1}{2\pi} \text{Vol}\{ \rho \in \mathbb{R}, r \geq 0 : \rho^2 + m^2 e^{2br} \leq \lambda^2 \} = \frac{\lambda \log \lambda}{\pi b} + O(\lambda) ,
\]

allowing us to interpret our result as a Weyl asymptotic for \( N(\lambda) \): note this is the same as the asymptotic obeyed by the eigenvalues of \(-\partial_r^2 + m^2 e^{2br}\) on \((0, \infty)\): see e.g. [Ti, (7.4.2)].

3. Nontrapping surfaces with funnel

If instead \( b < 0 < a \) and \((X_1, g_1)\) is given by

\[ X_1 = (-1/a, \infty) \times S^1 , \quad g_1 = dr^2 + f_1(r)^2 d\theta^2 , \quad f_1(r) = \begin{cases} 1 + ar & \text{if } r \leq 0 , \\ e^{-br} & \text{if } r > 0 , \end{cases} \]

then \((X_1, g_1)\) is a surface of revolution with a cone point and a funnel. In the case \( a = 1 \) the cone point disappears and we have a surface of revolution with a disk and a funnel.

Let

\[ P_1(m) = -\partial_r^2 - f_1^{-1} f_1' \partial_r + f_1^{-2} m^2 . \]

On functions of the form \( u(r)e^{im\theta} \) the Laplacian acts as

\[ \Delta_{g_1} u(r)e^{im\theta} = P_1(m) u(r)e^{im\theta} . \]

**Proposition 3.1.** For any \( m > 0 \) there are \( \lambda_0 > 0, k_0 \in \mathbb{N} \) such that the resonances of \( P_1(m) \) in the region \( \{ \text{Re} \lambda \geq \lambda_0 \} \) form a sequence \( (\lambda_k)_{k \geq k_0} \) satisfying

\[ \lambda_k = \pi a k - \frac{ija}{2} \log k + O(1) \]

where \( j = 1 \) if \( a + b \neq 0 \), and \( j = 2 \) if \( a + b = 0 \).
Proof. We replace $H^{(2)}$ by $J$ and $K$ by $I$ in (2.7), so that resonances are given by solutions to

$$
\frac{I'_\nu(m/b)}{I_\nu(m/b)} = \frac{\lambda J'_m/a(\frac{\lambda}{a})}{m J_m/a(\frac{\lambda}{a})} - \frac{b}{2m},
$$

(3.1)

where now (2.4) is replaced by

$$\nu = \sqrt{\frac{1}{4} - \frac{\lambda^2}{b^2}} = \frac{i}{b} \left( 1 - \frac{b^2}{8\lambda^2} + O\left( \frac{1}{\lambda^4} \right) \right),$$

so that again $\text{Re } \nu > 0$ when $\text{Im } \lambda > 0$. This follows by the same proof as Proposition 2.2, with the difference that the asymptotics [AbSt, §9.2.3, §9.2.4] for the Hankel functions must now be replaced by [AbSt, §9.1.7, §9.1.8, §9.1.9]. (We are using here the Friedrichs extension of the Laplacian at the cone point — see e.g. [MeWu, (3.3)]).

Using (2.9), (2.10), and (2.19), the left hand side of (3.1) becomes

$$\frac{I'_\nu(z)}{I_\nu(z)} = \frac{\nu}{z} + \frac{z}{2\nu} + O\left( \frac{1}{\nu^2} \right)$$

Substituting, we get

$$\frac{I'_\nu(m/b)}{I_\nu(m/b)} = \frac{i\lambda}{m} - \frac{i(b^2 + 4m^2)}{8m\lambda} + O\left( \frac{1}{\lambda^2} \right),$$

so (3.1) becomes

$$\frac{J'_m/a(\frac{\lambda}{a})}{J_m/a(\frac{\lambda}{a})} = i + \frac{b}{2\lambda} - \frac{i(b^2 + 4m^2)}{8\lambda^2} + O\left( \frac{1}{\lambda^3} \right).$$

(3.2)

The large argument asymptotics for $J'$ and $J$ [AbSt, §9.2.5, 9.2.11] say that

$$\frac{J'_m/a(\frac{\lambda}{a})}{J_m/a(\frac{\lambda}{a})} = - \frac{R \sin \chi - S \cos \chi}{P \cos \chi - Q \sin \chi},$$

(3.3)

where $\chi = \frac{\lambda}{a} - \frac{\pi m}{2a} - \frac{\pi}{4}$ and $P, Q, R, S$ each have an asymptotic expansion in $\lambda^{-1}$. Combining (3.2) and (3.3), plugging in the expansions for $P, Q, R, S$, and solving for $\exp(-2i\chi)$ gives

$$e^{-2i\chi} = -a + b + \frac{a^3 - a^2b - 4am^2 - 4bm^2}{16a^2\lambda^2} + O\left( \frac{1}{\lambda^3} \right).$$

(3.4)

(See also http://math.mit.edu/~datchev/dkk.nb for Mathematica code which computes $e^{-2i\chi}$.) If $a \neq -b$, then the first term is the leading term. Otherwise, the first term vanishes and the second term simplifies to $a^2/8\lambda^2$ and becomes the leading term. As in the previous section, we write

$$e^{-2i\chi} = c_j \lambda^{-j}(1 + O(\lambda^{-1})).$$

We solve for $\lambda$ by taking the log of both sides:

$$-2i\chi - 2\pi ki = \log c_j - j \log \lambda + O(\lambda^{-1})$$

$$-2i \left( \frac{\lambda}{a} - \frac{\pi m}{2a} - \frac{\pi}{4} \right) = -2\pi ki = \log c_j - j \log \lambda + O(\lambda^{-1})$$

$$\frac{2j}{a} \lambda - j \log \lambda = -2\pi ki + O(1).$$
Substituting $\tilde{\lambda} = \frac{-2i}{ja} \lambda$, we get

$$\tilde{\lambda} + \log \frac{\tilde{\lambda} j a i}{2} = \frac{2\pi k i}{j} + O(1)$$

$$\tilde{\lambda} + \log \tilde{\lambda} = \frac{2\pi i k}{j} + O(1).$$

Using the Lambert W function to solve this (as we solved (2.21) above), we see that

$$\tilde{\lambda}_k = -\frac{2\pi i k}{j} - \log k + O(1)$$

or that

$$\lambda_k = \pi a k - \frac{ija}{2} \log k + O(1).$$

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