Integrable Boundary Interactions for Ruijsenaars’ Difference Toda Chain

J. F. van Diejen¹, E. Emsiz²

¹ Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile.
E-mail: diejen@inst-mat.utalca.cl

² Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile.
E-mail: eemsiz@mat.puc.cl

Received: 13 May 2014 / Accepted: 3 August 2014
Published online: 30 January 2015 – © Springer-Verlag Berlin Heidelberg 2015

Abstract: We endow Ruijsenaars’ open difference Toda chain with a one-sided boundary interaction of Askey–Wilson type and diagonalize the quantum Hamiltonian by means of deformed hyperoctahedral $q$-Whittaker functions that arise as a $t = 0$ degeneration of the Macdonald–Koornwinder multivariate Askey–Wilson polynomials. This immediately entails the quantum integrability, the bispectral dual system, and the $n$-particle scattering operator for the chain in question.

1. Introduction

It is well-known that the open and closed Toda chains may be viewed as limits of the hyperbolic and elliptic Calogero–Moser–Sutherland particle systems, respectively [St,R1, I,R2]. More general integrable open Toda chains with boundary interactions involving potentials of Morse type [Ko,GW,Sk1] and of Pöschl-Teller type [I,KJC] are recovered similarly as degenerations of the Olshanetsky–Perelomov–Inozemtsev generalized Calogero–Moser–Sutherland systems with hyperoctahedral symmetry [I,O,Sh,GLO2]. Moreover, such limiting relations turn out to persist at the level of the Ruijsenaars–Schneider particle systems and Ruijsenaars’ difference (a.k.a. relativistic) Toda chains [R1,R2,R3,E,GLO1,HR,BC], as well as their hyperoctahedral counterparts [D2,C]. Specifically, in the hyperoctahedral case one recovers in this manner generalizations of Ruijsenaars’ open relativistic Toda chain with boundary interactions that were studied at the level of classical mechanics in Refs. [Su1,D1,Su2] and at the level of quantum mechanics in Refs. [KT,D2,E,S,C].

In the present work we consider the Hamiltonian of such an open difference Toda chain endowed with a one-sided four-parameter boundary interaction of Askey–Wilson type. Upon diagonalizing the quantum Hamiltonian in question by means of deformed

This work was supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grants # 1130226 and # 1141114.
hyperoctahedral $q$-Whittaker functions that arise as a $t = 0$ degeneration of the Macdonald–Koornwinder polynomials [K,M], the quantum integrability, the bispectral dual system, and the $n$-particle scattering operator are deduced. For special values of the Askey–Wilson parameters, our chain amounts to a difference counterpart of the $D_n$-type and the $A_{n-1}$-type quantum Toda chains with one-sided boundary potentials of Pöschl-Teller and Morse type, respectively.

The presentation is structured as follows. After introducing our difference Toda chain in Sect. 2 and defining the deformed hyperoctahedral $q$-Whittaker functions in Sect. 3, the diagonalization of the Hamiltonian is carried out in Sect. 4 by identifying the corresponding eigenvalue equation with the $t \to 0$ degeneration of a well-known Pieri formula for the Macdonald–Koornwinder polynomials [D3,M]. The quantum integrals and the bispectral dual system are then discussed in Sects. 5 and 6, respectively. In Sect. 7 analogous results for a difference counterpart of the quantum Toda chain with one-sided boundary potentials of Morse type are obtained by letting one of the boundary parameters tend to zero (which corresponds to a transition from Askey–Wilson polynomials to continuous dual $q$-Hahn polynomials [KLS]). We close in Sect. 8 with an explicit description of the $n$-particle scattering operator that relies on a stationary-phase analysis that was performed in Refs. [R4,D4]. Some useful properties of the Macdonald–Koornwinder multivariate Askey–Wilson polynomials have been collected in a separate appendix at the end.

2. Difference Toda Chain with One-Sided Boundary Interaction of Askey–Wilson Type

Formally, the Hamiltonian of our difference Toda chain is given by the difference operator [D2]:

$$H := T_1 + \sum_{j=2}^{n-1} (1 - q^{x_{j-1} - x_j})T_j$$

$$+ \sum_{j=1}^{n-2} (1 - q^{x_j - x_{j+1}})T_j^{-1} + (1 - q^{x_n-1 - x_n})(1 - q^{x_{n-1} + x_n})T_{n-1}^{-1}$$

$$+ w_+(x_n)(1 - q^{x_{n-1} - x_n})T_n + w_-(x_n)(1 - q^{x_{n-1} + x_n})T_n^{-1} + U(x_{n-1}, x_n),$$

where

$$w_+(x) := \frac{\prod_{0 \leq r \leq 3} (1 - t_1q^x)(1 - q^{2x+1})}{(1 - q^{2x})(1 - q^{2x+1})}, \quad w_-(x) := \frac{\prod_{0 \leq r \leq 3} (1 - t_1^{-1}q^x)}{(1 - q^{2x})(1 - q^{2x-1})},$$

$$U(x, y) := \sum_{\epsilon \in \{1, -1\}} c_\epsilon \frac{(1 - \epsilon q^{x+y})}{(1 - \epsilon q^{x+y})(1 - \epsilon q^{x+y-1})},$$

with

$$c_\epsilon := \frac{1}{2\sqrt{q-1}} \frac{t_0 t_1 t_2 t_3}{\prod_{0 \leq r \leq 3} (1 - \epsilon q^{-1/2} t_r)},$$

and $T_j$ ($j = 1, \ldots, n$) acts on functions $f : \mathbb{R}^n \to \mathbb{C}$ by a unit translation of the $j$th position variable

$$(T_j f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n).$$
Here $q$ denotes a scale parameter and the parameters $t_r$ ($r = 0, \ldots, 3$) play the role of coupling parameters for the boundary interaction of Askey–Wilson type. Upon setting $t_2 = -t_3 = q^{1/2}$, the additive potential term $U(x_{n-1},x_n)$ in $H$ (2.1a)–(2.1d) vanishes. The above Toda chain amounts in this case to a difference analog of the previously studied $D_n$-type quantum Toda chain with Pöschl-Teller boundary potential [I,KJC,O,GLO2]. If we additionally set $t_0 = -t_1 = 1$, then $w_+(x) = w_-(x) = 1$ and we formally recover a $D_n$-type analog of Ruijsenaars’ difference Toda chain [KT,E,S,C] that was introduced at the level of classical mechanics by Suris [Su1].

### 3. Deformed Hyperoctahedral $q$-Whittaker Functions

Let $\Lambda$ denote the cone of integer partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with decreasingly ordered parts $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and let $W$ be the hyperoctahedral group formed by the semi-direct product of the symmetric group $S_n$ and the $n$-fold product of the cyclic group $\mathbb{Z}_2 \cong \{1, -1\}$. Elements $w = (\sigma, \epsilon) \in W$ act naturally on $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ via $w\xi := (\epsilon_1 \xi_{\sigma_1}, \ldots, \epsilon_n \xi_{\sigma_n})$ (with $\sigma \in S_n$ and $\epsilon_j \in \{1, -1\}$ for $j = 1, \ldots, n$). A standard basis for the algebra of $W$-invariant trigonometric polynomials on the torus $\mathbb{T} = \mathbb{R}^n/(2\pi \mathbb{Z}^n)$ is given by the hyperoctahedral monomial symmetric functions

$$m_\lambda(\xi) := \sum_{\mu \in W\lambda} e^{i(\mu, \xi)}, \quad \lambda \in \Lambda,$$

where the summation is meant over the orbit of $\lambda$ with respect to the action of $W$ and the bracket $\langle \cdot, \cdot \rangle$ refers to the usual inner product on $\mathbb{R}^n$ (so $\langle \mu, \xi \rangle = \mu_1 \xi_1 + \cdots + \mu_n \xi_n$). This monomial basis inherits a natural partial order from the hyperoctahedral dominance ordering of the partitions:

$$\forall \mu, \lambda \in \Lambda : \mu \leq \lambda \text{ iff } \sum_{1 \leq j \leq k} \mu_j \leq \sum_{1 \leq j \leq k} \lambda_j \text{ for } k = 1, \ldots, n. \quad (3.2)$$

By definition, the basis of deformed hyperoctahedral $q$-Whittaker functions $p_\lambda(\xi)$, $\lambda \in \Lambda$ is given by the polynomials of the form

$$p_\lambda(\xi) = m_\lambda(\xi) + \sum_{\mu \in \Lambda \text{ with } \mu < \lambda} c_{\lambda,\mu} m_\mu(\xi) \quad (c_{\lambda,\mu} \in \mathbb{C}) \quad (3.3a)$$

such that

$$\langle p_\lambda, m_\mu \rangle_{\hat{A}} = 0 \quad \text{if } \mu < \lambda, \quad (3.3b)$$

where the inner product

$$\langle \hat{f}, \hat{g} \rangle_{\hat{A}} := \int_{\hat{A}} \hat{f}(\xi) \overline{\hat{g}(\xi)} \hat{\Delta}(\xi) d\xi \quad (\hat{f}, \hat{g} \in L^2(\mathbb{A}, \hat{\Delta}(\xi) d\xi)) \quad (3.4a)$$

is determined by the weight function

$$\hat{\Delta}(\xi) := \frac{1}{(2\pi)^n} \prod_{1 \leq j < k \leq n} \left| e^{i(\xi_j + \xi_k)} - e^{i(\xi_j - \xi_k)} \right|^2 \prod_{1 \leq j \leq n} \left| \frac{e^{2i\xi_j}} {\prod_{0 \leq r \leq 3} (\hat{t}_r e^{i\xi_j})} \right|^2 \quad (3.4b)$$
supported on the hyperoctahedral Weyl alcove
\[ \mathcal{A} := \{ (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n \mid \pi > \xi_1 > \xi_2 > \cdots > \xi_n > 0 \}. \] (3.5)
Here \((x)_m := \prod_{l=0}^{m-1} (1 - x q^l)\) and \((x_1, \ldots, x_t)_m := (x_1)_m \cdots (x_t)_m\) refer to standard notations for the \(q\)-Pochhammer symbols, and it is assumed that
\[ q \in (0, 1) \quad \text{and} \quad \hat{t}_r \in (-1, 1) \setminus \{0\} \quad (r = 0, \ldots, 3). \] (3.6)
These deformed hyperoctahedral \(q\)-Whittaker functions \(p_\lambda(\xi), \lambda \in \Lambda\) amount to a \(t \to 0\) degeneration of the more general Macdonald-Koornwinder multivariate Askey–Wilson polynomials introduced in Ref. [K] (cf. Appendix A below).

4. Diagonalization

It is known that the eigenfunctions of Ruijsenaars’ open difference Toda chain consist of \(A_{n-1}\)-type \(q\)-Whittaker functions given by a \(t \to 0\) limit of the Macdonald symmetric functions [GLO1]. In this section our aim is to show that an analogous result holds for the chain with Askey-Wilson type boundary interactions from Sect. 2, upon employing the deformed hyperoctahedral \(q\)-Whittaker functions from Sect. 3. To this end it is convenient to reparametrize the boundary parameters of the Toda chain in terms of the \(q\)-Whittaker deformation parameters (3.6) via
\[ t_0 = \sqrt{q^{-1} t_0 \hat{t}_1 \hat{t}_2 \hat{t}_3}, \quad t_r = \hat{t}_r t_0 / t_0 \quad (r = 1, 2, 3), \] (4.1)
assuming (from now onwards) the additional positivity constraints
\[ \hat{t}_0 > 0 \quad \text{and} \quad \hat{t}_0 t_1 \hat{t}_2 \hat{t}_3 > 0. \] (4.2)
Let \(\rho_0 + \Lambda := \{ \rho_0 + \lambda \mid \lambda \in \Lambda \}\) with
\[ \rho_0 := (\log_q (t_0), \ldots, \log_q (t_0)) \in \mathbb{R}^n. \]
We write \(\ell^2(\rho_0 + \Lambda, \Delta)\) for the Hilbert space of lattice functions \(f : (\rho_0 + \Lambda) \to \mathbb{C}\) determined by the inner product
\[ \langle f, g \rangle_\Delta := \sum_{\lambda \in \Lambda} f(\rho_0 + \lambda)g(\rho_0 + \lambda)\Delta_\lambda \quad (f, g \in \ell^2(\rho_0 + \lambda_n, \Delta)), \] (4.3a)
where
\[ \Delta_\lambda := \frac{\Delta_0}{(qt_0^2_{t_0} 1_{\lambda_n - 1 + \lambda_n})} \prod_{0 \leq r \leq 3} \frac{(t_0 t_r)_{\lambda_n}}{(qt_0 t_r^{-1})_{\lambda_n}} \prod_{1 \leq j < n} \frac{1}{(q)_{\lambda_j - \lambda_{j+1}}} \] (4.3b)
and
\[ \Delta_0 := (q)_{\infty} \prod_{0 \leq r < s \leq 3} (\hat{t}_r \hat{t}_s)_{\infty} = (q)_{\infty} \prod_{1 \leq r \leq 3} (t_0 t_r, qt_0 t_r^{-1})_{\infty}. \] (4.3c)
From the limiting behavior for \(t \to 0\) of the orthogonality relations satisfied by the normalized Macdonald–Koornwinder polynomials \((A.2a)–(A.2c)\), it is immediate that the wave function
\[ \psi_\xi(\rho_0 + \lambda) := \frac{(t_0^2 \lambda_n)}{\prod_{0 \leq r \leq 3} (t_0 t_r)_{\lambda_n}} p_\lambda(\xi) \quad (\lambda \in \Lambda, \xi \in \mathcal{A}). \] (4.4)
satisfies the following orthogonality with respect to the spectral variable $\xi$:

$$
\int_{\Lambda} \psi(\rho_0 + \lambda) \overline{\psi(\rho_0 + \mu)} \hat{\Delta}(\xi) d\xi = \begin{cases} 
\Delta_{\lambda}^{-1} & \text{if } \lambda = \mu, \\
0 & \text{otherwise.}
\end{cases} \quad (4.5)
$$

In other words, the corresponding Fourier transform $F : \ell^2(\rho_0 + \Lambda, \Delta) \to L^2(\Lambda, \hat{\Delta}d\xi)$ given by

$$(F f)(\xi) := \langle f, \psi_\xi \rangle_\Delta = \sum_{\lambda \in \Lambda} f(\rho_0 + \lambda) \overline{\psi_\xi(\rho_0 + \lambda)} \hat{\Delta}(\xi) \quad (4.6a)$$

$(f \in \ell^2(\rho_0 + \Lambda, \Delta))$ constitutes a Hilbert space isomorphism with an inversion formula of the form

$$(F^{-1} \hat{f})(\rho_0 + \lambda) = \langle \hat{f}, \overline{\psi(\rho_0 + \lambda)} \rangle_\Delta = \int_{\Lambda} \hat{f}(\xi) \overline{\psi(\rho_0 + \lambda)} \hat{\Delta}(\xi) d\xi \quad (4.6b)$$

$(\hat{f} \in L^2(\Lambda, \hat{\Delta}d\xi))$. We will refer to $F$ (4.6a), (4.6b) as the deformed hyperoctahedral $q$-Whittaker transform.

The formal Hamiltonian $H$ (2.1a)–(2.1d) restricts to a well-defined discrete difference operator in the space of complex functions on the lattice $\rho_0 + \Lambda$. Indeed, when $t_0 \notin \{1, q^{1/2}\}$ it is manifest that for $x = (x_1, \ldots, x_n)$ at these lattice points we stay away from the poles in the coefficients of $H$ stemming from the denominators of $w_{\pm}(x_n)$ and $U(x_{n-1}, x_n)$ and, moreover, that for any $f : \mathbb{R}^n \to \mathbb{C}$ and any $\lambda \in \Lambda$ the value of $(H f)(\rho_0 + \lambda)$ depends only on evaluations of $f$ at points of $\rho_0 + \Lambda$ (due to the vanishing of $(1 - q^{\epsilon_j - \lambda_j+1})$ at $\lambda_j = \lambda_{j+1}$ $(1 \leq j < n)$ and the vanishing of $w_-(\log_q (t_0) + \lambda_n)$ at $\lambda_n = 0$):

$$(H f)(\rho_0 + \lambda) = \sum_{1 \leq j \leq n} v_j^+(\lambda) f(\rho_0 + \lambda + e_j) + \sum_{1 \leq j \leq n} v_j^-(\lambda) f(\rho_0 + \lambda - e_j) + u(\lambda) f(\rho_0 + \lambda), \quad (4.7)$$

where

$$v_j^+(\lambda) = (1 - q^{\lambda_{j-1} - \lambda_j})(\frac{\prod_{0 \leq r \leq 3} (1 - t_r t_0 q^{\lambda_n})}{(1 - t_0^2 q^{2\lambda_n})(1 - t_0^2 q^{2\lambda_n+1})})^{\delta_{n-j}},$$

$$v_j^-(\lambda) = (1 - q^{\lambda_j - \lambda_{j+1}})(1 - t_0^2 q^{\lambda_{n-1}+\lambda_n})^{\delta_{n-j}} \times (\frac{\prod_{0 \leq r \leq 3} (1 - t_r^{-1} t_0 q^{\lambda_n})}{(1 - t_0^2 q^{2\lambda_n})(1 - t_0^2 q^{2\lambda_n-1})})^{\delta_{n-j}},$$

$$u(\lambda) = \sum_{\epsilon \in \{1, -1\}} \frac{c_\epsilon (1 - \epsilon t_0 q^{\lambda_{n-1}+1/2})}{(1 - \epsilon t_0 q^{\lambda_{n-1}+1/2})(1 - \epsilon t_0^{-1} q^{-\lambda_{n-1}+1/2})},$$

with $c_\epsilon$ taken from (2.1d). Here $\delta_k := 1$ if $k = 0$ and $\delta_k := 0$ otherwise, the vectors $e_1, \ldots, e_n$ denote the standard unit basis of $\mathbb{R}^n$, and $\lambda_0 := +\infty$, $\lambda_{n+1} := -\infty$ by convention (so $(1 - q^{\lambda_0 - \lambda_1}) = (1 - q^{\lambda_n - \lambda_{n+1}}) \equiv 1$). The action of $H$ on lattice functions in Eq. (4.7) extends continuously from $t_0 \notin \{1, q^{1/2}\}$ to the full parameter domain determined by Eqs. (4.1), (4.2) and (3.6).
Our main result implements the Hamiltonian under consideration as a self-adjoint operator in the Hilbert space $\ell^2(\rho_0 + \Lambda, \Delta)$ and provides its spectral decomposition with the aid of the deformed hyperoctahedral $q$-Whittaker transform.

**Theorem 1** (Diagonalization). (i). For boundary parameters $t_r$ (4.1) determined by the $q$-Whittaker deformation parameters $\hat{t}_r$ (3.6), (4.2), the action of the difference Toda Hamiltonian $H$ (2.1a)–(2.1d) given by Eq. (4.7) constitutes a bounded self-adjoint operator in the Hilbert space $\ell^2(\rho_0 + \Lambda, \Delta)$ with purely absolutely continuous spectrum. (ii). The operator in question is diagonalized by the deformed hyperoctahedral $q$-Whittaker transform $F$ (4.6a), (4.6b):

$$H = F^{-1} \circ \hat{E} \circ F,$$

where $\hat{E}$ denotes the bounded real multiplication operator acting on $\hat{f} \in L^2(\hat{\lambda}, \hat{d}\eta)$ via

$$(\hat{E} \hat{f})(\eta) := \hat{E}(\eta) \hat{f}(\eta) \quad \text{with} \quad \hat{E}(\eta) := 2 \sum_{1 \leq j \leq n} \cos(\eta_j).$$

**Proof.** The first part of the theorem is immediate from the second part. To prove the second part it suffices to verify that the deformed hyperoctahedral $q$-Whittaker kernel $\psi_\xi$ satisfies the eigenvalue equation $H \psi_\xi = \hat{E}(\xi) \psi_\xi$, or more explicitly that:

$$\sum_{1 \leq j \leq n} v^+_j(\lambda) \psi_\xi (\rho_0 + \lambda + e_j) + \sum_{1 \leq j \leq n} v^-_j(\lambda) \psi_\xi (\rho_0 + \lambda - e_j) + u(\lambda) \psi_\xi (\rho_0 + \lambda) = \hat{E}(\xi) \psi_\xi (\rho_0 + \lambda).$$

This eigenvalue equation follows from the Pieri formula for the Macdonald–Koornwinder polynomials (A.4) in the limit $t \to 0$. Indeed, it is clear that in the Pieri formula $\lim_{t \to 0} P_\lambda(\xi) = \psi_\lambda(\rho_0 + \lambda)$, $\lim_{t \to 0} \hat{t}_j V^+_j(\lambda) = v^+_j(\lambda)$, $\lim_{t \to 0} \hat{t}_j^{-1} V^-_j(\lambda) = v^-_j(\lambda)$, and one also has that

$$\lim_{t \to 0} \left( \sum_{j=1}^n (\hat{t}_j^{-1} + \hat{t}_j^{-1}) - \sum_{1 \leq j \leq n} V^+_j(\lambda) - \sum_{1 \leq j \leq n} V^-_j(\lambda) \right) = u(\lambda).$$

This last limit formula is not evident but can be deduced from the following rational identity in $q^{x_1}, \ldots, q^{x_n}$:

$$\sum_{j=1}^n \left( \hat{t}_j^{-1} - \hat{t}_j^{-1} w_+(x_j) \prod_{1 \leq k \leq n \atop k \neq j} \frac{1 - t q^{x_j + x_k} - 1 - t q^{x_j - x_k}}{1 - q^{x_j + x_k} - 1 - q^{x_j - x_k}} \right)$$

$$+ \sum_{j=1}^n \left( \hat{t}_j - \hat{t}_j w_-(x_j) \prod_{1 \leq k \leq n \atop k \neq j} \frac{1 - t^{-1} q^{x_j + x_k} - 1 - t^{-1} q^{x_j - x_k}}{1 - q^{x_j + x_k} - 1 - q^{x_j - x_k}} \right)$$

$$= C_t \sum_{\epsilon \in \{1,-1\}} \prod_{0 \leq r \leq 3} (1 - \epsilon t_r q^{-1/2}) \left( 1 - \prod_{j=1}^n \frac{1 - \epsilon t q^{x_j - 1/2} - 1 - \epsilon t^{-1} q^{x_j + 1/2}}{1 - \epsilon q^{x_j - 1/2} - 1 - \epsilon q^{x_j + 1/2}} \right),$$
where \( C_t = -\frac{1}{2} \tau_0^{-1}(1-t)^{-1}(1-q^{-1}t)^{-1} \), upon replacing \( q_t \) by \( \tau_j q^{\lambda_j} (j = 1, \ldots, n) \) and performing the limit \( t \to 0 \). To infer the rational identity itself, one exploits the hyperoctahedral symmetry in the variables \( x_1, \ldots, x_n \) and checks that—as a function of \( x_j \) (with the remaining variables fixed in a generic configuration)—the residues at the (simple) poles on both sides coincide. Hence, the difference of both rational expressions amounts to a \( W \)-invariant Laurent polynomial in \( q^{x_1}, \ldots, q^{x_n} \). The Laurent polynomial in question must actually vanish, as the rational expressions under consideration tend to 0 for \( x_j = (n + 1 - j)c \) in the limit \( c \to +\infty \). □

5. Integrability

The quantum integrability of the difference Toda Hamiltonian \( H \) (2.1a)–(2.1d) is an immediate consequence of the diagonalization in Theorem 1. In effect, a complete system of commuting quantum integrals in the Hilbert space \( L^2(\rho_0 + \Lambda, \Delta) \) is given by the bounded self-adjoint operators

\[
H_l := F^{-1} \circ \hat{E}_l \circ F, \quad l = 1, \ldots, n,
\]

where \( \hat{E}_l : L^2(\Lambda, \Delta) \to L^2(\Lambda, \Delta) \) denotes the real multiplication operator by \( \hat{E}_l(\xi) := m_{\omega_l}(\xi) \) with \( \omega_l := e_1 + \cdots + e_l \) (so \( H_1 = H \)). The operator \( H_l \) (5.1) acts on \( f \in L^2(\rho_0 + \Lambda, \Delta) \) as a difference operator of the form

\[
(H_l f)(\rho_0 + \lambda) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq l, \epsilon_j \in \{1, -1\}, j \in J; \lambda + \epsilon e_j \in \Lambda} C_{\epsilon J}(f)(\rho_0 + \lambda + \epsilon e_J),
\]

where \( e_{\epsilon J} := \sum_{j \in J} \epsilon_j e_j \), \( |J| \) denotes the cardinality of \( J \subset \{1, \ldots, n\} \), and the coefficients

\[
C_{\epsilon J}(f)(\lambda) = \lim_{t \to 0} C_{\epsilon J,t}(f)(\lambda)
\]

arise as \( t \to 0 \) limits of the expansion coefficients in the corresponding Pieri formula for the normalized Macdonald–Koornwinder polynomials \( P_{\lambda}(\xi) \) \((A.1a), (A.1b)\) (cf. \([D3, \text{Sec. 6}]\)):

\[
\hat{E}_l(\xi)P_{\lambda}(\xi) = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq l, \epsilon_j \in \{1, -1\}, j \in J; \lambda + \epsilon e_j \in \Lambda} C_{\epsilon J}(f)(\rho_0 + \lambda + \epsilon e_J)(\xi).
\]

Notice in this connection that the Pieri expansion coefficients

\[
C_{\epsilon J,t}(f)(\lambda) = \Delta_{\lambda + \epsilon e_J} \int_{\Lambda} \hat{E}_l(\xi)P_{\lambda}(\xi)P_{\lambda + \epsilon e_J}(\xi) \Delta(\xi) d\xi
\]

are continuous at \( t = 0 \), because the Macdonald–Koornwinder weight function \( \Delta(\xi) \) and (thus) the polynomials \( P_{\lambda}(\xi) \), \( \lambda \in \Lambda \) are continuous at this parameter value (cf. Appendix A).

In practice it turns out to be very tedious to compute the \( t \to 0 \) limiting coefficients \( C_{\epsilon J}(f)(\lambda) \) explicitly with the aid of the known explicit Pieri formulas for the Macdonald–Koornwinder polynomials in \([D3, \text{Sec. 6}]\) beyond \( l = 1 \). For a particular second quantum integral belonging to the commutative algebra generated by \( H_1, \ldots, H_n \), however, the required computation results to be surprisingly straightforward. More specifically: from
the $t \to 0$ limiting behavior of the $r = n$ (top) Pieri formula for the Macdonald–Koornwinder polynomials in Theorem 6.1 of [D3], one readily deduces that the action on $f \in \ell^2(\rho_0 + \Lambda, \Delta)$ of the operator $H_Q := F^{-1} \circ \hat{Q} \circ F$, where $\hat{Q}$ refers to the self-adjoint multiplication operator in $L^2(\hat{\text{A}}, \hat{\text{d}}\xi)$ by

$$\hat{Q}(\xi) := \prod_{j=1}^{n} (2\cos(\xi_j) - \hat{i}_0 - \hat{i}_0^{-1}),$$

is given explicitly by

$$(H_Q f)(\rho_0 + \lambda) = \sum_{J_+ \cup J_- \cup K_+ \cup K_- = \{1, \ldots, n\}} \sum_{|J_+|+|J_-|+|K_+|+|K_-|=n} u_{K_+,K_-}(\lambda) v_{J_+,J_-}(\lambda) f(\rho_0 + \lambda + e_{J_+} - e_{J_-}), \quad (5.3)$$

with

$$v_{J_+,J_-}(\lambda) = \prod_{j \in J_+} (1 - q^{\lambda_{-j}+\lambda_j}) \prod_{j \in J_-} (1 - q^{\lambda_{-j}+\lambda_j+1}) \times (1 - t_0^2 q^{\lambda_{-1}+\lambda_n}) \delta_{J^c} (n-1) \delta_{J^c} (n) \times (1 - t_0^2 q^{\lambda_{-1}+\lambda_n}) \delta_{J^c} (n-1) \delta_{J^c} (n) w_+ (\lambda_n) \delta_{J^c} (n) w_- (\lambda_n) \delta_{J^c} (n)$

and

$$u_{K_+,K_-}(\lambda) = (-\hat{i}_0)^{|K_-|-|K_+|} \prod_{k \in K_+} (1 - q^{\lambda_{-k}+\lambda_k}) \prod_{k \in K_-} (1 - q^{\lambda_{-k}+\lambda_k+1}) \times (1 - t_0^2 q^{\lambda_{-1}+\lambda_n}) \delta_{K^c} (n) \delta_{K^c} (n) \times (1 - t_0^2 q^{\lambda_{-1}+\lambda_n}) \delta_{K^c} (n) \delta_{K^c} (n) w_+ (\lambda_n) \delta_{K^c} (n) w_- (\lambda_n) \delta_{K^c} (n).$$

Here $\delta_J : \{1, \ldots, n\} \to \{0, 1\}$ denotes the characteristic function of $J \subset \{1, \ldots, n\}$ and $J^c = \{1, \ldots, n\} \setminus J$.

**Corollary 1.** *The difference Toda Hamiltonians $H$ (4.7) and $H_Q$ (5.3) are bounded, self-adjoint, commuting operators in $\ell^2(\rho_0 + \Lambda, \Delta)$ for which the deformed hyperoctahedral $q$-Whittaker functions $\hat{\psi}_\xi$ (4.4) constitute a complete system of (generalized) joint eigenfunctions corresponding to the eigenvalues $\hat{E}(\xi)$ and $\hat{Q}(\xi)$, respectively.*

### 6. Bispectral Dual System

For $t \to 0$ the Macdonald–Koornwinder $q$-difference equation (A.3) amounts to the following eigenvalue equation satisfied by the deformed hyperoctahedral $q$-Whittaker functions:

$$\hat{H} p_\lambda = (q^{-\lambda_1} - 1) p_\lambda \quad (\lambda \in \Lambda), \quad (6.1)$$

with

$$\hat{H} = \sum_{j=1}^{n} \left( \hat{v}_j(\xi)(\hat{T}_{j,1} - 1) + \hat{v}_j(-\xi)(\hat{T}_{j,-1} - 1) \right), \quad (6.2a)$$
and

\[ \hat{v}_j(\xi) = \frac{\prod_{0 \leq r \leq 3}(1 - i_r e^{i\xi_j})}{(1 - e^{2i\xi_j})(1 - qe^{2i\xi_j})} \prod_{1 \leq k \leq n, k \neq j} (1 - e^{i(\xi_j + \xi_k)})^{-1}(1 - e^{i(\xi_j - \xi_k)})^{-1}, \]  

(6.2b)

where \( \hat{T}_{j,q} \) acts on trigonometric (Laurent) polynomials \( \hat{\rho}(e^{i\xi_1}, \ldots, e^{i\xi_n}) \) by a \( q \)-shift of the \( j \)th variable:

\[ (\hat{T}_{j,q} \hat{\rho})(e^{i\xi_1}, \ldots, e^{i\xi_n}) := \hat{\rho}(e^{i\xi_1}, \ldots, e^{i\xi_{j-1}}, qe^{i\xi_j}, e^{i\xi_{j+1}}, \ldots, e^{i\xi_n}). \]

The following proposition is now immediate.

**Proposition 1** (Bispectral Dual Hamiltonian). The \( t = 0 \) Macdonald–Koornwinder q-difference operator \( \hat{H} \) (6.2a), (6.2b) constitutes a nonnegative unbounded self-adjoint operator with purely discrete spectrum in \( L^2(\mathbb{A}, \hat{D}d\xi) \) that is diagonalized by the (inverse) deformed hyperoctahedral q-Whittaker transform \( F \) (4.6a), (4.6b):

\[ \hat{H} = F \circ E \circ F^{-1}, \]  

(6.3a)

where \( E \) denotes the self-adjoint multiplication operator in \( \ell^2(\rho_0 + \Lambda, \Delta) \) of the form

\[ (Ef)(\rho_0 + \lambda) := (q^{-\lambda_1} - 1)f(\rho_0 + \lambda) \]  

(6.3b)

(for \( f \in \ell^2(\rho_0 + \Lambda, \Delta) \) with \( \langle Ef, Ef \rangle_{\Delta} < \infty \)).

One learns from Theorem 1 and Proposition 1 that the eigenfunction transforms diagonalizing the difference Toda Hamiltonian \( \hat{H} \) (4.7) and the \( t = 0 \) Macdonald–Koornwinder difference operator \( \hat{H} \) (6.2a), (6.2b) are inverses of each other. This fact encodes the bispectral duality of the operators under consideration in the sense of Duitsbermaat and Grünbaum [DG, G]: the kernel function \( \hat{\psi}_\xi(\rho_0 + \lambda) \) of the deformed hyperoctahedral q-Whittaker transform \( F \) (4.6a), (4.6b) simultaneously solves the corresponding eigenvalue equations for \( H \) and \( \hat{H} \) in the discrete variable \( \rho_0 + \lambda \) and the spectral variable \( \xi \), respectively.

Explicit commuting quantum integrals for the dual Hamiltonian \( \hat{H} \) (6.2a), (6.2b) are obtained as a \( t \to 0 \) degeneration of the commuting difference operators in [D3, Thm. 5.1]:

\[ \hat{H}_l = \sum_{J \subset \{1, \ldots, n\}, 0 \leq |J| \leq l} \hat{U}_{J', \lgb - J'} \hat{V}_{e_J} \hat{T}_{\epsilon, J, q}, \quad l = 1, \ldots, n, \]  

(6.4)

with \( \hat{T}_{\epsilon, J, q} := \prod_{j \in J} \hat{T}_{\epsilon, j, q} \) and

\[ \hat{V}_{e_J} := \prod_{j \in J} \prod_{0 \leq r \leq 3}(1 - \hat{\tau}_r e^{i\epsilon_j \xi_j}) \prod_{1 \leq k \leq n, k \neq j} (1 - e^{i(\epsilon_j \xi_j + \epsilon_k \xi_k)})^{-1}(1 - e^{i(\epsilon_j \xi_j - \epsilon_k \xi_k)})^{-1} \times \prod_{j, k \in J, j < k} (1 - e^{i(\epsilon_j \xi_j + \epsilon_k \xi_k)})^{-1}(1 - qe^{i(\epsilon_j \xi_j + \epsilon_k \xi_k)})^{-1}, \]
\[ \hat{U}_{K,p} := (-1)^p \sum_{I \subset K, |I| = p} \left( \prod_{j \in I} \frac{\prod_{0 \leq r \leq 3}(1 - \hat{t}_r e^{j\epsilon j})}{(1 - e^{2j\epsilon j})(1 - q e^{2j\epsilon j})} \right) \times \prod_{j \in I} (1 - e^{j(e\xi_j + \xi_k)})^{-1}(1 - e^{j(e\xi_j - \xi_k)})^{-1} \times \prod_{j,k \in I, j < k} (1 - e^{j(e\xi_j + \xi_k)})^{-1}(1 - q^{-1} e^{-i(e\xi_j + \epsilon_k \xi_k)})^{-1} \]  

(\text{so } \hat{H}_1 = \hat{H}). \text{ The diagonalization in Proposition 1 now generalizes to the complete system of commuting quantum integrals } \hat{H}_1, \ldots, \hat{H}_n \text{ as follows.}

**Theorem 2** (Bispectral Dual System). Let \( E_l \) \((1 \leq l \leq n)\) denote the self-adjoint multiplication operator in \( L^2(p + \Delta, \Delta) \) given by

\[ (E_l f)(p + \lambda) := E_{\lambda, l} \cdot f(p + \lambda) \quad (\lambda \in \Delta) \quad (6.5a) \]

(on the domain of \( f \in L^2(p + \Lambda, \Delta) \) for which \( \langle E_l f, E_l f \rangle_\Delta < \infty \)), where

\[ E_{\lambda, l} := q^{-\lambda_1 - \lambda_2 \cdots - \lambda_l - 1}(q^{-\lambda_1} - 1) + 2^{\frac{1}{2}} q^{-\lambda_1 - \lambda_2 \cdots - \lambda_{l-1}}(q^{-\lambda_l} - 1)\delta_{n-l}. \quad (6.5b) \]

The \( q \)-difference operators \( \hat{H}_l \) \((6.4)\) constitute nonnegative unbounded self-adjoint operators with purely discrete spectra in \( L^2(\Lambda, \Delta d\xi) \) that are simultaneously diagonalized by the (inverse) deformed hyperoctahedral \( q \)-Whittaker transform \( F \) \((4.6a), (4.6b)\):

\[ \hat{H}_l = F \circ E_l \circ F^{-1}, \quad l = 1, \ldots, n. \quad (6.5c) \]

**Proof.** It suffices to verify that

\[ \hat{H}_l p_{\lambda} = E_{\lambda, l} p_{\lambda} \quad (\lambda \in \Lambda, \ l = 1, \ldots, n). \]

This is achieved by multiplying the \( l \)th eigenvalue equation in Eq. (5.8) of \([D3]\) by a scaling factor \( t_l^{l(n-l)+(l-1)/2} \) and performing the limit \( t \to 0 \). Indeed, since the Macdonald–Koornwinder polynomial \( p_{\lambda} \) converges to the deformed hyperoctahedral \( q \)-Whittaker function \( p_{\lambda} \), we see from the explicit formulas for the operators in question that the LHS of the cited eigenvalue equation converges in this limit manifestly to \( \hat{H}_l p_{\lambda} \) (up to an overall factor \( t_0^l \)). Hence, the RHS must also have a finite limit for \( t \to 0 \), which confirms that \( p_{\lambda} \) is an eigenfunction of \( \hat{H}_l \) (using again that \( p_{\lambda} \overset{t \to 0}{\longrightarrow} p_{\lambda} \)). For \( l > 1 \) it is not obvious from \([D3, \text{Eq. (5.5)}]\) that the (limiting) eigenvalue is indeed given by \( E_{\lambda, l} \) \((6.5b)\), but this can be deduced quite easily from the asymptotics of \( m_{\lambda} \) and \( \hat{H}_l m_{\lambda} \) at \( \xi = -c \rho, \rho := (n, n-1, \ldots, 2, 1) \) for \( c \to +\infty \). Indeed, one readily computes that for \( c \to +\infty: m_{\lambda} = e^{(\lambda, \rho) c}(1 + o(1)) \) and \( \hat{H}_l m_{\lambda} = E_{\lambda, l} e^{(\lambda, \rho) c}(1 + o(1)) \) (using the explicit formula for \( \hat{H}_l \)) and the asymptotics

\[ \frac{\prod_{0 \leq r \leq 3}(1 - \hat{t}_r e^{i\epsilon j})}{(1 - e^{2i\epsilon j})(1 - q e^{2i\epsilon j})} \overset{c \to +\infty}{\longrightarrow} \begin{cases} t_0^2 & \text{if } \epsilon = 1 \\ 1 & \text{if } \epsilon = -1 \end{cases} \quad (1 \leq j \leq n) \]
and
\[ (1 - q^a e^{i\epsilon (\xi_j \pm \xi_k)})^{-1} \xrightarrow{c \to +\infty} \begin{cases} 0 & \text{if } \epsilon = 1 \\ 1 & \text{if } \epsilon = -1 \end{cases} \ (1 \leq j < k \leq n), \]
where \( a \in \{1, 0, -1\} \). But then also \( p_{\lambda} = e^{(\lambda \cdot \rho)c} (1 + o(1)) \) and \( \hat{H}_l p_{\lambda} = E_{\lambda, l} e^{(\lambda \cdot \rho)c} (1 + o(1)) \) for \( c \to +\infty \) by the triangularity (3.3a) and the property that \( \langle \mu, \rho \rangle < \langle \lambda, \rho \rangle \) if \( \mu < \lambda \). The upshot is that the eigenvalue of \( \hat{H}_l \) on the eigenpolynomial \( p_{\lambda} \) must be equal to \( E_{\lambda, l} \).

The \( q \)-difference operators \( \hat{H}_l \) (6.4) commute in the space of \( W \)-invariant trigonometric polynomials on \( \mathbb{T} \). It is clear from Theorem 2 that this commutativity extends in the Hilbert space in the resolvent sense: for
\[ z_l \notin \sigma(\hat{H}_l) := \{ \lambda, l \mid \lambda \in \Lambda \} \subset [0, +\infty) \ (l = 1, \ldots, n) \]
the resolvents \((\hat{H}_1 - z_1 I)^{-1}, \ldots, (\hat{H}_n - z_n I)^{-1}\) of the unbounded operators \( \hat{H}_1, \ldots, \hat{H}_n \) mutually commute as bounded operators in \( L^2(\Lambda, d\xi) \).

Theorem 2 and Sect. 5 lift the bispectral duality of \( H \) (4.7) and \( \hat{H} \) (6.2a),(6.2b) to the complete systems of commuting quantum integrals. The bispectral dual integrable system \( \hat{H}_1, \ldots, \hat{H}_n \) associated with our difference Toda chain can actually be identified as the strong-coupling limit \(( t = q^g, g \to +\infty) \) of a trigonometric Ruijsenaars-type difference Calogero-Moser system with hyperoctahedral symmetry [D2]. Analogous bispectral dual systems were linked previously to the open quantum Toda chain and Ruijsenaars’ open difference Toda chain. Specifically, the open quantum Toda chain and the strong-coupling limit of Ruijsenaars’ rational difference Calogero-Moser system turn out to be bispectral duals of each other [B,HR,Sk2,Kz], and the same holds true for Ruijsenaars’ open difference Toda chain and the \( t = 0 \) trigonometric/hyperbolic Ruijsenaars-Macdonald operators [GLO1,HR,BC]. Dualities of this type were actually first established for the corresponding particle systems within the realms of classical mechanics: the action-angle transforms linearizing the open Toda chain and the strong-coupling limit of the rational Ruijsenaars-Schneider system are the inverses of each other and the same holds true for the action-angle transforms for Ruijsenaars’ open relativistic Toda chain and the strong-coupling limit of the hyperbolic Ruijsenaars-Schneider system [R1,F].

7. Parameter Reductions

As already anticipated at the end of Sect. 2, for \( \hat{t}_2 = -\hat{t}_3 = q^{1/2} \) and \( \hat{t}_0 = -\hat{t}_1 \to 1 \) (so \( t_0 = -t_1 \to 1 \) and \( t_2 = -t_3 \to q^{1/2} \)) the difference Toda Hamiltonian \( H \) (4.7) and the deformed hyperoctahedral \( q \)-Whittaker functions \( p_{\lambda}(\xi), \lambda \in \Lambda \) degenerate to a difference Toda Hamiltonian and \( q \)-Whittaker functions of type \( D_n \) [Su1,KT,E,S,C]. Even though formally these limiting values of the parameters do not respect our restriction that \( \hat{t}_r \in (-1, 1) \setminus \{0\} \) for \( r = 0, \ldots, 3 \), it is readily inferred from the formulas that the results of Sects. 3–6 nevertheless remain valid at this specialization of the parameters.
In this section we are concerned with the behavior for $\hat{t}_0 \to 0$. In this limit, the difference Toda chain turns out to be governed by a Hamiltonian of the form

$$H = T_1 + \sum_{j=2}^{n} (1 - q^{x_{j-1} - x_j}) T_j + \sum_{j=1}^{n-1} (1 - q^{x_j - x_{j+1}}) T_j^{-1}$$

$$+ \left( \prod_{1 \leq r < s \leq 3} (1 - \hat{t}_r \hat{t}_s q^{x_n - 1}) \right) (1 - q^{x_n}) T_n^{-1}$$

$$+ (\hat{t}_1 + \hat{t}_2 + \hat{t}_3) q^{x_n} + \hat{t}_1 \hat{t}_2 \hat{t}_3 q^{2x_n} (q^{x_{n-1} - x_n} + q^{-x_n} - 1 - q^{-1}).$$ (7.1)

When $\hat{t}_3 = 0$, the Hamiltonian in question constitutes a Ruijsenaars-type difference counterpart of the quantum Toda chain with one-sided boundary potentials of Morse type [Sk1, I]. If in addition $\hat{t}_2 = -1$, then the difference Toda chain under consideration amounts to a quantization of a relativistic Toda chain with boundary potentials introduced by Suris [Su1, KT]. For $\hat{t}_1 = \hat{t}_2 = \hat{t}_3 = 0$ and for $\hat{t}_1 = -\hat{t}_2 = q^{1/2}$ with $\hat{t}_3 = -1$, we recover in turn hyperoctahedral difference Toda chains of type $B_n$ and $C_n$ that are diagonalized by $q$-Whittaker functions of type $C_n$ and $B_n$, respectively [E, S, C]. Again, even though formally none of these specializations respect our restriction that $\hat{t}_r \in (-1, 1) \setminus \{0\}$ (for $r = 1, 2, 3$), it is clear that the formulas below in fact do remain valid.

7.1. Deformed hyperoctahedral $q$-Whittaker function. For $\hat{t}_0 \to 0$, the deformed hyperoctahedral $q$-Whittaker functions $p_\lambda(\xi)$ (3.3a), (3.3b) degenerate into a three-parameter family of orthogonal polynomials $p_\lambda(\xi)$, $\lambda \in \Lambda$ associated with the weight function

$$\hat{\Delta}(\xi) = \frac{1}{(2\pi)^n} \prod_{1 \leq j < k \leq n} \left| (e^{i(\xi_j + i \xi_k)} - q^{-1}) \cdot (e^{i(\xi_j - i \xi_k)} + q^{-1}) \right|_{\infty}^2 \prod_{1 \leq j \leq n} \left| (e^{2i\xi_j})_{\infty} \right|^2.$$

The orthogonality relations for these polynomials read [cf. Eq. (4.5)]

$$\int_{\hat{\Delta}_0} p_\lambda(\xi) p_\mu(\xi) \hat{\Delta}(\xi)d\xi = \begin{cases} \Delta^{-1}_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$ (7.2)

where

$$\Delta_\lambda = \frac{\Delta_0}{(q)^{\lambda_n} \prod_{1 \leq r < s \leq 3} (\hat{t}_r \hat{t}_s)^{\lambda_n}} \prod_{1 \leq j < n} \frac{1}{(q)^{\lambda_j - \lambda_{j+1}}}$$

with

$$\Delta_0 = (q)_{\infty} \prod_{1 \leq r < s \leq 3} (\hat{t}_r \hat{t}_s)_{\infty}.$$  

For $n = 1$, the limit $p_\lambda \xrightarrow{\hat{t}_0 \to 0} p_\lambda$ amounts to a well-known reduction from the Askey-Wilson polynomials to the continuous dual $q$-Hahn polynomials [KLS].
7.2. Hamiltonian. The difference Toda eigenvalue equation \( H \psi = \hat{E}(\lambda) \psi \) becomes in the limit \( \lambda_0 \to 0 \) of the form \( H \phi_\lambda = \hat{E}(\xi) \phi_\lambda \) with \( \phi_\lambda : \Lambda \to \mathbb{C} \) given by \( \phi_\lambda(\lambda) = p_\lambda(\xi) \) (\( \xi \in \Lambda, \lambda \in \Lambda \)), where \( H(7.1) \) acts on \( f : \Lambda \to \mathbb{C} \) via

\[
(Hf)(\lambda) = \sum_{1 \leq j \leq n} v^+_j(\lambda) f(\lambda + e_j) + \sum_{1 \leq j \leq n} v^-_j(\lambda) f(\lambda - e_j) + u(\lambda) f(\lambda),
\]

with

\[
v^+_j(\lambda) = (1 - q^{\lambda_j - 1}),
\]
\[
v^-_j(\lambda) = (1 - q^{\lambda_j - \lambda_{j+1}})(1 - q^{\lambda_n}) \prod_{1 \leq r < s \leq 3} (1 - \hat{\tau}_r \hat{\tau}_s q^{\lambda_{n-s}})^{\delta_{n-s}},
\]
\[
u(\lambda) = (\hat{\tau}_1 + \hat{\tau}_2 + \hat{\tau}_3)q^{\lambda_n} + \hat{\tau}_1 \hat{\tau}_2 \hat{\tau}_3 q^{2 \lambda_n} (q^{-\lambda_n - 1} + q^{-\lambda_n - 1} - 1 - q^{-1})
\]

(subject to the convention that \( \lambda_0 = +\infty \) and \( \lambda_{n+1} = -\infty \)).

7.3. Diagonalization and integrability. Let \( \mathbf{F} : \ell^2(\Lambda, \Delta) \to L^2(\Lambda, \hat{\Delta} d\xi) \) denote the \((\lambda_0 \to 0 \) degenerate) Hilbert space isomorphism determined by the orthogonal basis \( p_\lambda, \lambda \in \Lambda \):

\[
(\mathbf{F} f)(\xi) = \langle f, \phi_\xi \rangle_\Delta = \sum_{\lambda \in \Lambda} f(\lambda) \overline{\phi_\xi(\lambda)} \Delta_\lambda
\]

\((f \in \ell^2(\Lambda, \Delta))\) with

\[
(\mathbf{F}^{-1} \hat{f})(\lambda) = \langle \hat{f}, \phi(\lambda) \rangle_\Lambda = \int_\Lambda \hat{f}(\xi) \phi_\xi(\lambda) \hat{\Delta}(\xi) d\xi
\]

\((\hat{f} \in L^2(\Lambda, \hat{\Delta} d\xi))\), and let \( \hat{E}_l : L^2(\Lambda, \hat{\Delta} d\xi) \to L^2(\Lambda, \hat{\Delta} d\xi) \) \((l = 1, \ldots, n)\) be the multiplication operators defined in accordance with Sect. 5.

The commuting bounded self-adjoint operators \( H_1, \ldots, H_n \) (with absolutely continuous spectra) in \( \ell^2(\Lambda, \Delta) \) given by

\[
H_l = \mathbf{F}^{-1} \circ \hat{E}_l \circ \mathbf{F}, \quad l = 1, \ldots, n,
\]

constitute a complete system of quantum integrals for the difference Toda Hamiltonian \( H_1 = H(7.3) \).

7.4. Bispectral dual system. Let \( \hat{H}_1, \ldots, \hat{H}_n \) denote the commuting \( q \)-difference operators in Eq. (6.4) with \( \hat{\tau}_0 = 0 \) and let \( E_1, \ldots, E_n \) be the self-adjoint multiplication operators in \( \ell^2(\Lambda, \Delta) \) given by [cf. Eqs. (6.5a), (6.5b)]

\[
(E_l f)(\lambda) = E_{\lambda,l} f(\lambda) \quad (\lambda \in \Lambda, l = 1, \ldots, n)
\]

(on the domain of \( f \in \ell^2(\Lambda, \Delta) \) for which \( \langle E_l f, E_l f \rangle_\Delta < \infty \)), with

\[
E_{\lambda,l} = q^{-\lambda - \lambda_2 - \cdots - \lambda_{l-1}} (q^{-\lambda_l - 1}).
\]

Then one has that

\[
\hat{H}_l = \mathbf{F} \circ E_l \circ \mathbf{F}^{-1}, \quad l = 1, \ldots, n,
\]

i.e. the \( q \)-difference operators constitute nonnegative unbounded self-adjoint operators with purely discrete spectra in \( L^2(\Lambda, \hat{\Delta} d\xi) \) that are simultaneously diagonalized by the three-parameter (inverse) deformed hyperoctahedral \( q \)-Whittaker transform \( \mathbf{F} \).
8. Scattering

In Ref. [D4] the scattering operator for a wide class of quantum lattice models was determined by stationary-phase methods originating from Ref. [R4]. It follows from the diagonalization in Theorem 1 that our difference Toda chains fit within this class of lattice models. Indeed, the deformed hyperoctahedral $q$-Whittaker functions $p_{\lambda}, \lambda \in \Lambda$ belong to the family of orthogonal polynomials defined in [D4, Sec. 2], since the orthogonality weight function $\hat{\Delta}(\xi)$ (3.4b) is of the indicated form (with $R = BC_n$) and moreover meets the demanded analyticity requirements. We will close by briefly indicating how the general scattering results from Ref. [D4, Sec. 4.2] specialize in the present difference Toda setting.

Let $H_0$ be the self-adjoint discrete Laplacian in $\ell^2(\Lambda)$ of the form

$$(H_0 f)(\lambda) := \sum_{1 \leq j \leq n} f(\lambda + e_j) + \sum_{1 \leq j \leq n} f(\lambda - e_j) \quad (f \in \ell^2(\Lambda)),$$

and let $\mathcal{H}$ denote the pushforward

$$\mathcal{H} := \Delta^{1/2} H \Delta^{-1/2}$$

of the difference Toda Hamiltonian $H$ (4.7) onto the Hilbert space $\ell^2(\Lambda)$ via the Hilbert space isomorphism $\Delta^{1/2} : \ell^2(\rho_0 + \Lambda, \Delta) \to \ell^2(\Lambda)$ given by

$$(\Delta^{1/2} f)(\lambda) := \Delta^{1/2} f(\rho_0 + \lambda) \quad (f \in \ell^2(\rho_0 + \Lambda, \Delta))$$

(where $\Delta^{-1/2} := (\Delta^{1/2})^{-1}$). Clearly, one has by Theorem 1 that

$$\mathcal{H} = \mathcal{F}^{-1} \hat{\mathcal{E}} \mathcal{F} \quad \text{with} \quad \mathcal{F} := \hat{\Delta}^{1/2} F \Delta^{-1/2},$$

where $\hat{\Delta}^{1/2} : L^2(\hat{\Lambda}, \hat{\Delta}d\xi) \to L^2(\hat{\Lambda})$ denotes the Hilbert space isomorphism given by

$$(\hat{\Delta}^{1/2} \hat{f})(\xi) := \hat{\Delta}^{1/2} \hat{f}(\xi) \quad (\hat{f} \in L^2(\hat{\Lambda}, \hat{\Delta}d\xi))$$

(and $\hat{\mathcal{E}}$ (4.8b) is now regarded as a self-adjoint bounded multiplication operator in $L^2(\hat{\Lambda})$). Moreover, it is elementary that the spectral decomposition of the discrete Laplacian $H_0$ is given by

$$H_0 = \mathcal{F}_0^{-1} \hat{\mathcal{E}} \mathcal{F}_0,$$

where $\mathcal{F}_0 : \ell^2(\Lambda) \to L^2(\hat{\Lambda})$ denotes the Fourier isomorphism

$$(\mathcal{F}_0 f)(\xi) := \sum_{\lambda \in \Lambda} f(\lambda) \chi_{\xi}(\lambda) \quad (f \in \ell^2(\Lambda))$$

with the inversion formula

$$(\mathcal{F}_0^{-1} \hat{f})(\lambda) = \int_{\hat{\Lambda}} \hat{f}(\xi) \chi_{\xi}(\lambda) d\xi$$

(8.5b)
\( \hat{f} \in L^2(\hat{\mathbb{A}}) \). Here we have employed the anti-invariant Fourier kernel
\[
\chi_\xi(\lambda) := \frac{1}{(2\pi)^{n/2} i^n} \sum_{w \in W} \text{sign}(w) e^{i(w(\rho+\lambda),\xi)},
\]
with \( \text{sign}(w) = \epsilon_1 \cdots \epsilon_n \text{sign}(\sigma) \) for \( w = (\sigma, \epsilon) \in W = S_n \ltimes \{1, -1\}^n \) and \( \rho = (n, n-1, \ldots, 2, 1) \). Notice that \( \mathcal{F}_0 \) is recovered from \( \mathcal{F} \) in the limit \( q \to 0, \hat{r} \to 0 \) \((r = 0, \ldots, 3)\).

The scattering operator describing the large-times asymptotics of the difference Toda dynamics \( e^{it\mathcal{H}_0} \) relative to the Laplacian’s reference dynamics \( e^{it\mathcal{H}_0} \) turns out to be governed by an \( n \)-particle scattering matrix \( \hat{S}(\xi) \) that factorizes in two-particle pair matrices and one-particle boundary matrices:
\[
\hat{S}(\xi) := \prod_{1 \leq j < k \leq n} s(\xi_j - \xi_k)s(\xi_j + \xi_k) \prod_{1 \leq j \leq n} s_0(\xi_j), \tag{8.6a}
\]
with
\[
s(x) := \frac{(qe^{ix})_\infty}{(qe^{-ix})_\infty} \quad \text{and} \quad s_0(x) := \frac{(qe^{2ix})_\infty}{(qe^{-2ix})_\infty} \prod_{0 \leq r \leq 3} \frac{(|\hat{r} e^{-ix})_\infty}{(|\hat{r} e^{ix})_\infty}. \tag{8.6b}
\]

To make the latter statement precise, let us denote by \( C_0(\hat{\mathbb{A}}_{\text{reg}}) \) the dense subspace of \( L^2(\hat{\mathbb{A}}) \) consisting of smooth test functions with compact support in the open dense subset \( \hat{\mathbb{A}}_{\text{reg}} \subset \hat{\mathbb{A}} \) on which the components of the gradient
\[
\nabla \hat{E}(\xi) = (-2 \sin(\xi_1), \ldots, -2 \sin(\xi_n)), \quad \xi \in \hat{\mathbb{A}}
\]
do not vanish and are all distinct in absolute value. We now define an unitary multiplication operator \( \hat{S} : L^2(\hat{\mathbb{A}}, d\xi) \to L^2(\hat{\mathbb{A}}, d\xi) \) via its restriction to \( C_0(\hat{\mathbb{A}}_{\text{reg}}) \) as follows:
\[
(\hat{S} \hat{f})(\xi) := \hat{S}(w_\xi \xi) \hat{f}(\xi) \quad (\hat{f} \in C_0(\hat{\mathbb{A}}_{\text{reg}}), \tag{8.7}
\]
where \( w_\xi \in W \) for \( \xi \in \hat{\mathbb{A}}_{\text{reg}} \) is such that the components of \( w_\xi \nabla \hat{E}(\xi) \) are all positive and reordered from large to small.

Theorem 4.2 and Corollary 4.3 of Ref. [D4] then provide the following explicit formulas for the wave operators and scattering operator of our difference Toda chain.

**Theorem 3 (Wave and Scattering Operators).** The operator limits
\[
\Omega^\pm := s - \lim_{t \to \pm\infty} e^{it\mathcal{H}_0} e^{-it\mathcal{H}_0}
\]
converge in the strong \( \ell^2(\hat{\Lambda}) \)-norm topology and the corresponding wave operators \( \Omega^\pm \) intertwining the difference Toda dynamics \( e^{it\mathcal{H}_0} \) with the discrete Laplacian’s dynamics \( e^{it\mathcal{H}_0} \) are given by unitary operators in \( \ell^2(\hat{\Lambda}) \) of the form
\[
\Omega^\pm = \mathcal{F}^{-1} \circ \hat{S}^{1/2} \circ \mathcal{F}_0,
\]
where the branches of the square roots are to be chosen such that
\[
(s(x))^{1/2} = \frac{(qe^{ix})_\infty}{|(qe^{ix})_\infty|} \quad \text{and} \quad s_0(x)^{1/2} = \frac{(qe^{2ix})_\infty}{|(qe^{2ix})_\infty|} \prod_{0 \leq r \leq 3} \frac{|(\hat{r} e^{ix})_\infty|}{|(\hat{r} e^{ix})_\infty|}.\]
Hence, the scattering operator relating the large-times asymptotics of the difference Toda dynamics $e^{i\mathcal{H}t}$ for $t \to -\infty$ and $t \to +\infty$ is given by the unitary operator

$$S := (\Omega^*)^{-1} \Omega = \mathcal{F}_0^{-1} \circ \hat{S} \circ \mathcal{F}_0.$$ 

The degenerate case of the difference Toda chain discussed in Sect. 7 is also covered by Theorem 3, upon setting $\rho_0$ equal to the nulvector in Eq. (8.2), replacing $H$ (4.7) by $\mathcal{H}$ (8.1) and $F$ (4.6a), (4.6b) by $\mathcal{F}$ (7.4a), (7.4b) in $\mathcal{F}$ (8.3), and substituting $\hat{t}_0 = 0$ overall.

Appendix A: Macdonald–Koornwinder Polynomials

This appendix collects some key properties of the Macdonald–Koornwinder multivariate Askey-Wilson polynomials [K,D3,M]. In the case of one variable ($n = 1$), the properties below specialize to well-known formulas for the Askey-Wilson polynomials (see e.g. [KLS]).

The Macdonald–Koornwinder polynomials $p_\lambda(\xi)$ ($\lambda \in \Lambda, \xi \in \mathbb{T}$) are defined as polynomials of the type in Eqs. (3.3a), (3.3b), (3.4a) associated with the weight function [K, Sec. 5], [M, Ch. 5.3]:

$$\hat{\Delta}(\xi) = \frac{1}{(2\pi)^n} \prod_{1 \leq j \leq n} \left| \prod_{0 \leq r \leq 3} (\hat{t}_r e^{i\xi_j})_\infty \right|^2 \prod_{1 \leq j < k \leq n} \left| (\hat{t}_j e^{i(\xi_j + \xi_k)}, \hat{t}_j e^{i(\xi_j - \xi_k)})_\infty \right|^2,$$

with $q \in (0, 1)$ and $t, \hat{t}_r \in (-1, 1) \setminus \{0\}$ ($r = 0, \ldots, 3$). For $t \to 0$ this weight function passes into that of Eq. (3.4b), whence the polynomials in question degenerate in this limit continuously to the deformed hyperoctahedral $q$-Whittaker functions of Sect. 3. Notice in this respect that for $x \in \mathbb{R}$ and $|t| < \varepsilon$ ($\varepsilon < 1$) quotients of the form $(e^{ix})_\infty/(te^{ix})_\infty$ remain bounded in absolute value by $(-1)_\infty/|\varepsilon|_\infty$, so we may interchange limits and integration for $t \to 0$ when integrating trigonometric polynomials against the Macdonald–Koornwinder weight function $\hat{\Delta}(\xi)$ over the bounded alcove $\Lambda$ (by dominated convergence).

The normalized Macdonald–Koornwinder polynomials

$$P_\lambda(\xi) := c_\lambda p_\lambda(\xi) \quad (\lambda \in \Lambda_n),$$

where

$$c_\lambda := \prod_{1 \leq j \leq n} \frac{(\tau_j^2)^{2\lambda_j}}{\prod_{0 \leq r \leq 3} (\hat{t}_r \tau_j)_{\lambda_j}} \prod_{1 \leq j < k \leq n} \frac{(\tau_j \tau_k)^{\lambda_j + \lambda_k}}{(\hat{t}_j \tau_k)_{\lambda_j + \lambda_k}} \frac{(\tau_j \tau_k)^{-1}_{\lambda_j - \lambda_k}}{((\hat{t}_j \tau_k)_{\lambda_j + \lambda_k}((\hat{t}_j \tau_k)^{-1})_{\lambda_j - \lambda_k})}$$

(A.1b)

with $\tau_j := t^{n-j}t_0$ ($j = 1, \ldots, n$) and $t_r$ ($r = 0, \ldots, 3$) given by Eq. (4.1), satisfy the following orthogonality relations [K, Sec. 5], [D3, Sec. 7], [M, Ch. 5.3]:

$$\int_\Lambda P_\lambda(\xi) P_\mu(\xi) \hat{\Delta}(\xi) d\xi = \begin{cases} \Delta_\lambda^{-1} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise}, \end{cases}$$

(A.2a)
with
\[
\Delta_\lambda := \Delta_0 \prod_{1 \leq j \leq n} \left( \frac{1 - \tau_j^2 q^{2\lambda_j}}{1 - \tau_j^2} \prod_{0 \leq r \leq 3} \frac{(t_r \tau_j)_{\lambda_j}}{(q t_r^{-1} \tau_j)_{\lambda_j}} \right) \\
\times \prod_{1 \leq j < k \leq n} \frac{1 - \tau_j \tau_k q^{\lambda_j + \lambda_k}}{1 - \tau_j \tau_k} \frac{(t \tau_j \tau_k)_{\lambda_j + \lambda_k}}{(q t^{-1} \tau_j \tau_k)_{\lambda_j + \lambda_k}} \frac{1 - \tau_j \tau_k^{-1} q^{\lambda_j - \lambda_k}}{1 - \tau_j \tau_k^{-1}} \frac{(t \tau_k^{-1})_{\lambda_j - \lambda_k}}{(q t^{-1} \tau_k^{-1})_{\lambda_j - \lambda_k}}
\]
(A.2b)
and
\[
\Delta_0 := \prod_{1 \leq j \leq n} \frac{(q, t^j)_{\infty}}{(t, \hat{t}_0 \hat{t}_1 \hat{t}_2 \hat{t}_3 \tau^{-n-j}_{\infty})_{\infty}}.
\]
(A.2c)

These orthogonal polynomials satisfy moreover a second-order \(q\)-difference equation [K, Sec. 5], [M, Ch. 5.3, 4.4]:
\[
P_\lambda(\xi) \sum_{j=1}^{n} \left( q^{-1} \hat{t}_0 \hat{t}_1 \hat{t}_2 \hat{t}_3 \tau^{-n-j} q^{\lambda_j} - 1 \right) + t^{j-1} q^{-\lambda_j} - 1
\]
\[
= \sum_{1 \leq j \leq n} \hat{V}_j(\xi) (P_\lambda(\xi - i \log(q)e_j) - P_\lambda(\xi)) + \hat{V}_j(-\xi) (P_\lambda(\xi + i \log(q)e_j) - P_\lambda(\xi)),
\]
(A.3)
with
\[
\hat{V}_j(\xi) := \prod_{0 \leq r \leq 3} \frac{(1 - \hat{t}_r e^{i \xi})}{(1 - e^{2i \xi})} \prod_{1 \leq k \leq n} \frac{1 - t e^{i (\xi + \xi_k)}}{1 - e^{i (\xi + \xi_k)}},
\]
and a Pieri-type recurrence formula [D3, Sec. 6], [M, Ch. 5.3, 4.4]:
\[
P_\lambda(\xi) \sum_{j=1}^{n} (2 \cos(\xi_j) - \hat{t}_j - \hat{t}_j^{-1})
\]
\[
= \sum_{1 \leq j \leq n} \left( \hat{V}_j^+(\lambda) \left( \hat{t}_j P_{\lambda + e_j}(\xi) - P_\lambda(\xi) \right) \right)
\]
\[
+ \sum_{1 \leq j \leq n} \left( V_j^-(\lambda) \left( \hat{t}_j^{-1} P_{\lambda - e_j}(\xi) - P_\lambda(\xi) \right) \right),
\]
(A.4)
with \(\hat{t}_j := t^{n-j} \hat{t}_0 (j = 1, \ldots, n)\) and
\[
V_j^+(\lambda) := \frac{\hat{t}_1}{1 - \hat{t}_0} \prod_{0 \leq r \leq 3} \frac{(1 - \tau_r \tau_j q^{\lambda_j})}{(1 - \tau_j^2 q^{2\lambda_j}) (1 - \tau_j^2 q^{2\lambda_j + 1})} \prod_{1 \leq k \leq n} \frac{1 - t \tau_j \tau_k q^{\lambda_j + \lambda_k}}{1 - \tau_j \tau_k q^{\lambda_j + \lambda_k}} \frac{1 - t \tau_j \tau_k^{-1} q^{\lambda_j - \lambda_k}}{1 - \tau_j \tau_k^{-1} q^{\lambda_j - \lambda_k}}.
\]
\[
V_j^-(\lambda) := \frac{\hat{t}_1}{1 - \hat{t}_0} \prod_{0 \leq r \leq 3} \frac{(1 - \tau_r^{-1} \tau_j q^{\lambda_j})}{(1 - \tau_j^2 q^{2\lambda_j}) (1 - \tau_j^2 q^{2\lambda_j - 1})} \prod_{1 \leq k \leq n} \frac{1 - t^{-1} \tau_j \tau_k q^{\lambda_j + \lambda_k}}{1 - \tau_j \tau_k q^{\lambda_j + \lambda_k}} \frac{1 - t^{-1} \tau_j \tau_k^{-1} q^{\lambda_j - \lambda_k}}{1 - \tau_j \tau_k^{-1} q^{\lambda_j - \lambda_k}}.
\]
(where the vectors \(e_1, \ldots, e_n\) refer to the standard unit basis of \(\mathbb{R}^n\)).
References

[B] Babelon, O.: Equations in dual variables for Whittaker functions. Lett. Math. Phys. 65, 229–240 (2003)

[BC] Borodin, A., Corwin, I.: Macdonald processes. Probab. Theory Relat. Fields 158, 225–400 (2014)

[C] Cherednik, I.: Whittaker limits of difference spherical functions, Int. Math. Res. Not. IMRN 3793–3842 (2009)

[D1] van Diejen, J.F.: Deformations of Calogero-Moser systems and finite toda chains. Theoret. Math. Phys. 99, 549–554 (1994)

[D2] van Diejen, J.F.: Difference Calogero-Moser systems and finite Toda chains. J. Math. Phys. 36, 1299–1323 (1995)

[D3] van Diejen, J.F.: Properties of some families of hypergeometric orthogonal polynomials in several variables. Trans. Am. Math. Soc. 351, 233–270 (1999)

[D4] van Diejen, J.F.: Scattering theory of discrete (pseudo) Laplacians on a Weyl chamber. Am. J. Math. 127, 421–458 (2005)

[DG] Duistermaat, J.J., Grünbaum, F.A.: Differential equations in the spectral parameter. Comm. Math. Phys. 103, 177–240 (1986)

[E] Etingof, P.: Whittaker functions on quantum groups and $\mathcal{q}$-deformed Toda operators. In: Astashkevich, A., Tabachnikov, S. (eds.) Differential Topology, Infinite-Dimensional Lie Algebras, and Applications. Amer. Math. Soc. Transl. Ser. 2, vol. 194., pp. 9–25. Amer. Math. Soc., Providence, RI (1999)

[F] Fehér, L.: Action-angle map and duality for the open Toda lattice in the perspective of Hamiltonian reduction. Phys. Lett. A. 377, 2917–2921 (2013)

[GLO1] Gerasimov, A., Lebedev, D., Oblezin, S.: On $\mathcal{q}$-deformed $\mathfrak{gl}_{l+1}$-Whittaker function III. Lett. Math. Phys. 97, 1–24 (2011)

[GLO2] Gerasimov, A., Lebedev, D., Oblezin, S.: Quantum Toda chains intertwined, St. Petersburg Math. J. 22, 411–431 (2011)

[GW] Goodman, R., Wallach, N.R.: Classical and quantum mechanical systems of Toda-lattice type III. joint eigenfunctions of the quantized systems. Comm. Math. Phys. 105, 473–509 (1986)

[G] Grünbaum, F.A.: The bispectral problem: an overview. In: Bustoz, J., Ismail, M.E.H., Suslov, S.K. (eds.) Special Functions 2000: Current Perspective and Future Directions. NATO Sci. Ser. II Math. Phys. Chem., vol. 30, pp. 129–140. Kluwer Academic Publishers, Dordrecht (2001)

[HR] Hallnäs, M., Ruijsenaars, S.N.M.: Kernel functions and Bäcklund transformations for relativistic Calogero-Moser and Toda systems. J. Math. Phys. 53, 123512 (2012)

[I] Inozemtsev, V.I.: The finite Toda lattices. Comm. Math. Phys. 121, 629–638 (1989)

[KLS] Koekoek, R., Lesky, P., Swarttouw, R.F.: Hypergeometric Orthogonal Polynomials and Their $\mathcal{q}$-Analogues. Springer Monographs in Mathematics. Springer, New York (2010)

[K] Koonwinder, T.H.: Askey-Wilson polynomials for root systems of type $\mathcal{BC}$. In: Richards, D.St.P. (ed.) Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications. Contemporary Mathematics, vol. 138, pp. 189–204. Amer. Math. Soc. Providence RI (1992)

[Ko] Kostant, B.: Quantization and representation theory. In: Luke G.L. (ed.) Representation Theory of Lie Groups. London Mathematical Society Lecture Note Series, vol. 34, pp. 287–316. Cambridge University Press, Cambridge-New York (1979)

[Kz] Kozlowski, K.K.: Aspects of the inverse problem for the Toda chain. J. Math. Phys. 54, 121902 (2013)

[KJC] Kuznetsov, V.B., Jørgensen, M.F., Christiansen, P.L.: New boundary conditions for integrable lattices. J. Phys. A 28, 4639–4654 (1995)

[KT] Kuznetsov, V.B., Tsyganov, A.V.: Quantum relativistic Toda chains. J. Math. Sci. 80, 1802–1810 (1996)

[M] Macdonald, I.G.: Affine Hecke Algebras and Orthogonal Polynomials. Cambridge University Press, Cambridge (2003)

[O] Oshima, T.: Completely integrable systems associated with classical root systems. SIGMA Symmetry Integr. Geom. 3, 061 (2007)

[R1] Ruijsenaars, S.N.M.: Relativistic Toda systems. Comm. Math. Phys. 133, 217–247 (1990)

[R2] Ruijsenaars, S.N.M.: Finite-dimensional Soliton systems. In: Kupershmidt B. (ed.) Integrable and Superintegrable Systems, pp. 165–206. World Scientific Publishing, Teaneck, NJ (1990)

[R3] Ruijsenaars, S.N.M.: Systems of Calogero-Moser type. In: Semenoff, G., Vinet, L. (eds.) Particles and Fields (Banff, 1994). CRM Ser. Math. Phys., pp. 251–352. Springer, New York (1999)

[R4] Ruijsenaars, S.N.M.: Factorized weight functions vs. factorized scattering. Comm. Math. Phys. 228, 467–494 (2002)

[S] Sevostyanov, A.: Quantum deformation of Whittaker modules and the Toda lattice. Duke Math. J. 105, 211–238 (2000)
Shimeno, N.: A limit transition from Heckman-Opdam hypergeometric functions to the Whittaker functions associated with root systems. arXiv:0812.3773

Sklyanin, E.K.: Boundary conditions for integrable quantum systems. J. Phys. A 21, 2375–2389 (1988)

Sklyanin, E.K.: Bispectrality for the quantum open Toda chain. J. Phys. A 46, 382001 (2013)

Suris, Y.B.: Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices. Phys. Lett. A 145, 113–119 (1990)

Suris, Y.B.: The Problem of Integrable Discretization: Hamiltonian Approach. Progress in Mathematics, vol. 219. Birkäuser Verlag, Basel (2003)

Sutherland, B.: An introduction to the Bethe ansatz. In: Shastry, B.S., Jha, S.S., Singh, V. (eds.) Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory (Panchgani, 1985). Lecture Notes in Physics, vol. 242, pp. 1–95. Springer, Berlin (1985)

Communicated by P. Deift