Identical particles are indistinguishable but...

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It is shown that quantum systems of identical particles can be treated as if they were different when they are in well differentiated states. This simplifying assumption allows the consideration of quantum systems isolated from the rest of the universe and justifies many intuitive statements about identical systems. However, it is shown that this simplification may lead to wrong results in the calculation of the entropy.

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I. INTRODUCTION

At the site http://www.sandia.gov/surface_science/stm/ one can see moving pictures, taken with tunnel effect microscope, of Si dimers (two Si atoms stick together) moving in a substrate. The dimers diffuse on the surface in one preferred direction and sometimes get trapped between two fixed chains of Si atoms and move back and forth. Every common sense physicist would agree with the sentence: “the dimer moves back and forth within the trap”. However, a more careful analysis reveals that the sentence is in contradiction with the principles of quantum mechanics because when we say “the dimer” we are assigning an identity to an indistinguishable particle. There are many cases where we use different states in order to distinguish or identify particles that are indistinguishable. This is clearly an error violating a well established principle of quantum mechanics but we may ask how serious this error is “for all practical purposes” (FAPP). After all, when we consider an isolated system, like a hydrogen atom, we are identifying it among myriads of other identical systems and we never question this violation of quantum mechanics. When I think of an individual electron hitting the screen of my PC and producing light as a particle different from another electron hitting the referee’s computer screen, I am in a contradiction with quantum mechanics. The same error I make when I consider one Si atom in a chip of my computer as different from another Si atom in a chip of his computer. How serious is this? Strictly speaking, we are almost always dealing with physical systems build with identical particles and in principle we should treat them as dictated by quantum mechanics, that is, with states symmetric or antisymmetric under the permutation of particles. As far as we know today, the only different particles are the

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three fermion families of quarks and leptons and the gauge bosons of the interactions but in the future we may discover that they are also just different states of some unique particle or system (string, brane, or whatever).

The universe is built of a large number of one or a few identical particles and quantum mechanics teach us how to treat them accordingly. Of course it would be a big nuisance to consider every physical system as an indistinguishable part of the whole universe and therefore it is important to show that, FAPP, the treatment of identical systems, when they are in well differentiated states, can be done as if they were different particles. It has been rigorously proved that the quantum system of two identical particles in different states is equivalent to the corresponding system of two different particles and we are therefore allowed to treat identical particles as different when they are in well differentiated states, for instance, far away or with clearly different properties. The two particle system contains the main ingredients of the problem and the results obtained with it can be generalized to \( N \) particles system. The proof mentioned above is rigourous but its “readability” can be substantially improved. We will therefore see here a simpler presentation of these results in a way accessible to students but without loss of rigour and generality. Complementing the mathematical proofs, we also include many discussions and comments related with the general problem of identical particles in quantum mechanics and in classical physics with more emphasis in the physics of the problems rather than with its formal aspects. One important point emphasized in this work is that the simplifying assumption that sometimes identical particles can be treated as different, may lead to wrong results when calculating the entropy of a system. For the didactic application of this work we suggest several simple exercises.

II. TWO PARTICLE FORMALISM

In order to clarify the notation, we present the Hilbert space formalism for the quantum mechanical treatment of a two particle system, identical or different. Let \( \mathcal{H} \) be the Hilbert space of states for the description of a one particle system. Then, as is well known, the state of a two particle system is an element of \( \mathcal{H} \otimes \mathcal{H} \) (in general, both spaces need not be of the same dimension but for the cases of interest in this work the dimension is the same and we don’t need to take different Hilbert spaces in the tensor product). In some cases, a redundant notation is used \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) in order to associate the first particle to \( \mathcal{H}_1 \) and the second to \( \mathcal{H}_2 \); we will not do this because the right-left order of the factors is sufficient to denote the association.

The permutation of the two particles, or equivalent, the permutation of the states associated to each particle, is an operation of fundamental importance for the definition of the concept of identical particles. We define then the \textit{Permutator} (a unitary transformation) as a linear operator \( \Pi : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) by

\[
\Pi (\Psi \otimes \Phi) = \Phi \otimes \Psi .
\]  

From this definition, it follows as a simple exercise that

\[
\Pi^2 = 1 ,
\]
\( \Pi \chi = \lambda \chi \) with \( \lambda = \pm 1 \) ,

\( \Pi^\dagger = \Pi \) ,

\( \Pi^\dagger (A \otimes B) \Pi = B \otimes A \) ,

where \( \chi \) is an eigenvector of \( \Pi \) and \( A \) and \( B \) are arbitrary operators (the last relation follows easily from the spectral decomposition of the operators). Now we can state the fundamental quantum mechanical postulate for identical particles. The state of a system of two identical particles is an eigenvector of the permutator \( \Pi \) with eigenvalue \( \lambda = 1 \) for identical bosons and \( \lambda = -1 \) for identical fermions. This postulate has an intuitive interpretation: imagine that we try to describe a two identical particle system with a state \( \Psi \otimes \Phi \). Clearly this is inappropriate because we are identifying the first particle with \( \Psi \) and the second with \( \Phi \) in contradiction with the assumption that they are identical. We must therefore correct this error by adding (or better, superposing) the opposite association. That is, \( \Psi \otimes \Phi \pm \Phi \otimes \Psi \) which is an eigenvector of \( \Pi \). This is the strongest form of the identical particle postulate from which other weaker versions can be deduced[3, 4].

From this postulate it follows (simple exercise) that every observable of a two identical particle system, represented by an operator \( O : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \), must be invariant under the permutator. That is, \( \Pi^\dagger O \Pi = O \) or \([O, \Pi] = 0\). The two identical particle observables are therefore of the type \( A \otimes B + B \otimes A \). The fact that we deal with identical particles does not forbid us to consider one particle observables that must take the form \( A \otimes 1 + 1 \otimes A \) and therefore we can not use the measurement of this one particle observables in order to identify a particle.

The form of the state of two identical particles, \( \Psi \otimes \Phi \pm \Phi \otimes \Psi \), reminds us the entangled states where two subsystems share two properties but both properties are shared by both subsystems in a holistic way without individual assignment. For instance, in the singlet state of two spin 1/2 particles (identical or not) \( 1/\sqrt{2}(\varphi_+ \otimes \varphi_- - \varphi_- \otimes \varphi_+ \) -a paradigmatic example for entangled states- we can not tell which particle has spin up and which one has spin down. Both particles share spin up and down simultaneously. A similar situation appears for the state of two identical particles that can be thought of as a state of identity entanglement.

With the permutator operator \( \Pi \) we can define a Symmetrizer \( S \) and an Antisymmetrizer operator \( A \):

\( S = \frac{1}{2}(1 + \Pi) \) \hspace{2cm} (6)

\( A = \frac{1}{2}(1 - \Pi) \) . \hspace{2cm} (7)

These two operators are projectors that project in the subspaces \( \mathcal{H}_S \) and \( \mathcal{H}_A \) that are orthogonal \( \mathcal{H}_S \perp \mathcal{H}_A \) and complete. That is, the Hilbert space is decomposed as an orthogonal sum \( \mathcal{H} \otimes \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_A \). (Simple exercise: prove all this). The state of two identical fermions is an element of \( \mathcal{H}_A \), the state of two identical bosons belongs to \( \mathcal{H}_S \) and the state for two different particles is in \( \mathcal{H} \otimes \mathcal{H} \) and can be decomposed in a symmetrical part plus an antisymmetrical part. For this work, where we are comparing identical particles with different particles, it is irrelevant whether the particles are bosons or fermions an it is therefore convenient to unify both projectors above in
an Identical Particle Projector:

\[ I = \frac{1}{2} (1 + \lambda \Pi) \] with \( \lambda = \pm 1 \). (8)

III. DIFFERENTIATING IDENTICAL PARTICLES

In this section we will show that when two identical particles are in clearly different states they can be treated as different particles. Of course, this is not true for all states and therefore we must clearly state what it means that two states are differentiating. Given two possible one particle states \( \Psi \) and \( \Phi \), we say that they are differentiating if \( \langle \Psi, \Phi \rangle = 0 \). This rigorous criterium can be weakened to FAPP differentiating when \( \langle \Psi, \Phi \rangle \approx 0 \), that is, the scalar product is so small that it will have no measurable consequence. Two eigenstates corresponding to different eigenvalues of an observables are differentiating. Two gaussian states such that their widths are much smaller than their separation are FAPP differentiating (this can be a model for a Si atom in my computer and a Si atom in another computer far away). Sometimes two states can be differentiating due to some different internal property as, for instance, for two electron states with arbitrary location but one with spin “up” and the other with spin “down”, we have \( \langle \Psi(x) \otimes \varphi_+, \Phi(x) \otimes \varphi_- \rangle = \langle \Psi(x), \Phi(x) \rangle \langle \varphi_+, \varphi_- \rangle = 0 \). Two photon states corresponding to orthogonal polarization are differentiating. As with the Si atoms above, all localized states in the classical limit are FAPP differentiating.

However, differentiating states are not sufficient in order to differentiate identical particles. We need more. Besides differentiating states we need to define also differentiating observables, that is, observables that are sensitive to differentiating states. In order to understand this need, consider a two identical particle observable \( A \otimes B + B \otimes A \). The expectation value of this observable in an identical particle state, even if it is a differentiating state, will involve both observables \( A \) and \( B \), both states \( \Psi \) and \( \Phi \) but also both particles. All possible combinations appear. What we need, is to associate one observable, say \( A \), with one state, say \( \Psi \), and the other observable \( B \) with \( \Phi \). For instance, I may be interested in the energy observable of a Si atom in my computer (and not in his) and therefore we want to relate the energy observable with the state \( \Psi \) denoting the localization in my computer. We obtain these state sensitive observables \( A \rightarrow A_{\Psi} \), \( B \rightarrow B_{\Phi} \) using the projectors in the corresponding states,

\[ A_{\Psi} = P_{\Psi}^I A P_{\Psi} \] \hspace{5cm} (9) \\
\[ B_{\Phi} = P_{\Phi}^I B P_{\Phi} \] \hspace{5cm} (10)

where the projector (hermitian and idempotent) in the Hilbert space formalism is given by \( P_{\Psi} = \Psi \langle \Psi, \cdot \rangle \) or, for those addict to the Dirac’s notation, \( P_{\Psi} = |\Psi \rangle \langle \Psi | \). Physically, the operator \( A_{\Psi} \) corresponds to the simultaneous measurement of the observable \( A \) and all properties characteristic of the state \( \Psi \) and therefore we must have \( [A, P_{\Psi}] = 0 \) and also \( [B, P_{\Phi}] = 0 \). With this, we have \( A_{\Psi} = AP_{\Psi} \) and \( B_{\Phi} = BP_{\Phi} \). These state sensitive observables are the quantum mechanical counterpart of the classical observables: the value of an observable in classical physics is always given by the state of the system, that is, they are functions of the coordinates and their
associated canonical momenta that determine the state of the system. Notice that the association of observables with states is perfectly “legal” and it is not a violation of the identity principle: state sensitive observables for two identical particles are built according to the rules of quantum mechanics as

\[ A_\Psi \otimes B_\Phi + B_\Phi \otimes A_\Psi \].

(11)

We can now prove that the treatment of two identical particles in two differentiating states is equivalent to the treatment of two different particles in the corresponding states in the sense that the expectation values of every differentiating observable is the same in both cases. This is the main result of reference[2]. Let us assume that we have two identical particles sharing two differentiating states \( \Psi \) and \( \Phi \); therefore the state of the system is given by

\[ \Xi_{ID} = \sqrt{2} I(\Psi \otimes \Phi) = \frac{1}{\sqrt{2}}(\Psi \otimes \Phi + \lambda \Phi \otimes \Psi) \].

(12)

On the other side if the particles are different the state is

\[ \Xi_{DIF} = \Psi \otimes \Phi \],

(13)

(or the opposite association \( \Phi \otimes \Psi \)). Notice that there is a one to one relation between \( \Xi_{ID} \) and \( \Xi_{DIF} \); in fact, applying \( \sqrt{2} I \) to \( \Xi_{DIF} \) we obtain \( \Xi_{ID} \) and the inverse map is \( \sqrt{2} P_\Psi \otimes P_\Phi \). In spite of this isomorphism, both states denote clearly different physical situations: The state \( \Xi_{ID} \) describes a system of two identical particles where one particle has the properties associated to \( \Psi \) and one particle has the properties of \( \Phi \) but there is no possibility to decide which one is which; whereas the state \( \Xi_{DIF} \) we have two different particles, the first one uniquely identified with the properties of \( \Psi \) and the second with the properties of \( \Phi \). We can now present a mathematical equation that relates these two physically different situations: by direct calculation (exercise) we obtain that for any arbitrary observables \( A \) and \( B \)

\[ \langle \Xi_{ID}, (A_\Psi \otimes B_\Phi + B_\Phi \otimes A_\Psi)\Xi_{ID} \rangle = \langle \Xi_{DIF}, (A \otimes B)\Xi_{DIF} \rangle \].

(14)

(In FAPP differentiating states we neglect terms of order \( |\langle \Psi, \Phi \rangle|^2 \) and get an approximation instead of an equality). Notice that the fact that this equation involves expectation values is not a restriction because every experimentally accessible quantity can be given as the expectation value of a appropriately chosen operator. Therefore, an identical particle system in any differentiating state can be treated as a different particle system with regards to every differentiating observable. With this result, the desired simplification in all physical situations mentioned in the introduction are justified.

IV. SHORT VISIT TO ELEMENTARY PARTICLE PHYSICS

Notice that the equivalence of identical particles in differentiating states with different particles can be applied in both directions. We can treat two identical particles in a simpler way as different particles but also the other way, we can treat two different particles as identical but in
differentiating states. We would first think that nobody would be interested in introducing this complication, however, this possibility has resulted in important discoveries in particle physics. The first application of this idea is due to Heisenberg when he recognized that the two different particles — proton and neutron — could be treated as the same identical particle, the nucleon, but in two different states corresponding to another observable that Wigner called Isospin. In the same way that we don’t consider two electrons with different spin as different particles (although they behave differently, for instance, in an inhomogeneous magnetic field) the proton and neutron are the same identical particle with isospin “up” and “down”. The discovery of isospin had important application in the study of the forces that bind the nucleus: these forces are invariant under rotations in isospin, a fact known as “charge independence” of the nuclear forces. Spin and isospin 1/2 are the smallest representations of a symmetry group denoted by $SU(2)$. Particles with tree charge values, for instance $\pi^+, \pi_0, \pi^-$ correspond to identical particles with isospin one, and in the same way, all hadrons could be assigned an isospin value. The classification of all known hadrons suggested the introduction of the larger group $SU(3)$ that contains isospin and another property (strangeness, initially called “hypercharge”). In this scheme, for instance, the eight particles $p, n, \Sigma^+, \Sigma^0, \Lambda, \Xi^0, \Xi^-$ could be considered as one particle in eight different states of isospin and strangeness. All the then known hadrons could be assigned to a $SU(3)$ multiplet. The search of the smallest representation of the group $SU(3)$ led to the discovery of the tree quarks $u, d, s$. The experimental discovery of new hadrons that did not fit in the scheme, forced the introduction of larger groups and led to the discovery of other quarks. We will not give more details about particle physics in this work. We just wanted to point out that the idea of treating different particles as identical ones in different states, had far reaching consequences in the reductionist study of nature. As mentioned in the introduction, this reduction to more and more fundamental particles is perhaps not finished.

V. IDENTICAL PARTICLES IN CLASSICAL PHYSICS

The arguments presented in previous sections can be used in order to understand the classical limit of statistical mechanics. As a matter of principles, we must start with the idea that all atoms or particles are identical and therefore, in all rigour, should be treated as quantum mechanics dictates, with a state obtained as a superposition of all possible permutations of one particle states. In the same way, all observables should be invariant under the transformation implied by each permutation of particles (as we did for two particles where the state and observables had the form $\Psi \otimes \Phi \pm \Phi \otimes \Psi$ and $A \otimes B + B \otimes A$). Now, from this identical particle quantum statistics we can make a transition to different particles statistic when all $N$ particles in the system can be assigned to $N$ mutually FAPP differentiating states $\Psi_k$, $k = 1 \cdots N$. For instance, let each $\Psi_k$ be gaussian states with widths much smaller than the average separation (small width, $\Delta^2_x \rightarrow 0$, implies large momentum spread, $\Delta^2_p \rightarrow \infty$, that is, large $\langle P^2 \rangle$, or large mean kinetic energy, that is, high temperature). As before, we can now associate observables with states by the definition of the state sensitive observables, for instance, for the one particle hamiltonian, $H \rightarrow H_k = P^\dagger_{\Psi_k} H P_{\Psi_k}$.
with expectation value $\varepsilon_k = \langle \Psi_k, H_k \Psi_k \rangle$. This observable corresponds to the concept of “the energy of the particle in the state $\Psi_k$”. This association of observables with states does not violate the indistinguishability of particles and the $N$ particle observable should be built as the addition of all permutations of the tensor products as was done for two particles in Eq.11. Now the generalization of Eq.14 to $N$ particles tells us that the correct treatment of $N$ identical particles in $N$ FAPP differentiating states is equivalent to the treatment of $N$ different particles, each particle in one different state and with its own values for its observables. But this is just the classical system of $N$ particles.

There is however one case where the classical limit is not defined (even at high temperature) and we must keep the correct quantum mechanical treatment. Assume that we want to count the number of states that can be assumed by $N$ identical particles compatible with some value of the total energy. This is not an observable as before that could be associated to each differentiating state. This number of states is not an expectation value like the ones involved in Eq.14 and it would be therefore wrong to replace it by the number of states obtained in the different particles case. The $N$ identical particles are in one state among a large number of possibilities and if we are interested in this number (and there are very good reasons to be interested in it: the calculation of the entropy), we have no way to relate it with some different particle observable. We just must count the states for identical particles taking care not to count twice the states that differ only by the permutation of particles. The correct counting of states must have the $N!$ introduced by Gibbs in order to obtain the correct value for the entropy, without really knowing that its origin was the quantum mechanical treatment of identical particles.

**VI. EXAMPLES**

In order to consolidate the concepts presented in this work it is convenient to apply them to some simple systems where, in some cases, identical particles can be treated as different and in some other cases not.

Consider the system of two hydrogen atoms located in space at some distance $D$. When the distance is large compared with the extension of the atoms, we can use the location in physical space as FAPP differentiating states and we have the choice of treating the system as composed by two different atoms (differentiated by the location) or, on the contrary, to thereat it as two identical atoms, as quantum mechanics dictates in rigour. When the distance $D$ becomes smaller and smaller we reach the point where we no longer have the choice and we must treat it with the correct quantum mechanic recipe for identical atoms as is done when we deal with the $H_2$ molecule. This treatment of the $H_2$ molecule can be found in many textbooks and will not be repeated here but instead we can sketch how to deal with the two hydrogen atom system when we are allowed to treat it as different atoms. The one atom Hilbert space will involve two factors: one describing the location of the atom in physical space, typically the space of square integrable functions $L^2$, and the other factor involves the Hilbert space spanned by the energy eigenstates of an electron in the Coulomb potential of the atom that we denote by $\mathcal{H}_E$. A typical one atom state is then an element of $\mathcal{H} = L^2 \otimes \mathcal{H}_E$
like $\Psi = \psi(x) \otimes \varphi_n$ (or $\Psi = (\psi, \varphi_n)$ in a simplified notation) where $\psi(x)$ describes the position of the atom in physical space and $\varphi_n$ describes the internal state of the atom, for instance, an energy eigenstate. Similarly, another one atom state could be $\Phi = \phi(x) \otimes \varphi_m = (\phi, \varphi_m)$. Now, the two atom state, considered as identical bosons, is described by a state like $\frac{1}{\sqrt{2}}(\Psi \otimes \Phi + \Phi \otimes \Psi)$. However if the states $\psi(x)$ and $\phi(x)$ are differentiating (or FAPP differentiating) we may make the simplifying assumption that the atoms are different and the state is $\Psi \otimes \Phi$. In this case the differentiating criteria is “the atom that is located around the maximum of $|\psi(x)|^2$ for one of them and around the maximum of $|\phi(x)|^2$ for the other” (exactly the same situation when I say “the Si atom in a chip of my computer” as different from the $Si$ atom in his computer). It is important to emphasize however that the simplifying assumption is not allowed when we want to know the number of states consistent with some value of the energy. Perhaps nobody would be interested in this number for our system of two atoms but for a large number of atoms this number is needed in order to calculate the entropy of the system. If we neglect the kinetic energy associated with the movement of the atoms, the energy of the system is determined by the two indices $(n, m)$ characterizing the internal states of the atoms. Now, if the atoms were distinguishable, then we would have four states compatible with an energy given by $(n, m)$ (assuming for simplicity that $n \neq m$). They are $(\psi, \varphi_n) \otimes (\phi, \varphi_m)$, $(\psi, \varphi_m) \otimes (\phi, \varphi_n)$, $(\phi, \varphi_n) \otimes (\psi, \varphi_m)$ and $(\phi, \varphi_m) \otimes (\psi, \varphi_n)$. However, the atoms are identical and some of these states cannot be considered as different. Therefore we have only two states compatible with a given energy. They are $\frac{1}{\sqrt{2}}[(\psi, \varphi_n) \otimes (\phi, \varphi_m) + (\phi, \varphi_m) \otimes (\psi, \varphi_n)]$ and $\frac{1}{\sqrt{2}}[(\phi, \varphi_n) \otimes (\psi, \varphi_m) + (\psi, \varphi_m) \otimes (\phi, \varphi_n)]$. This factor of two between the identical particle case and the different particle case becomes the factor of $N!$ “discovered” by Gibbs, necessary for the entropy to be an extensive quantity. Identical particles quantum mechanics gives the correct entropy and the different particle approximation fails.

A system similar to the two hydrogen atoms is to consider two identical particles placed in a double square well potential or in two boxes. The different particle approximation when the two square wells are widely separated is an interesting problem left as an exercise. Notice that in this case, the energy eigenstates are differentiating states (they are orthogonal) but are not localized in one or the other well. Adding and subtracting these energy eigenstates we can build FAPP differentiating states corresponding to placing a particle in one well or in the other.

VII. CONCLUSIONS

In principle, the treatment of any particle by quantum mechanics requires the symmetrization or anti-symmetrization of its state with all other identical particles of its sort. Furthermore, every particle or atom that we consider is just one representative of myriads of other identical systems in the universe and apparently a holistic treatment is necessary. However, we have seen in this work that we may use the properties of some states in order to separate out the system of interest from the rest and consider it as a different particle or atom. We may therefore think about “this electron or this Si atom right here” as an individual system differentiated from the rest by some property, for instance, its localization within my computer. In this way, the models that we build,
for instance “one single isolated noninteracting hydrogen atom”, can be consider to be a faithful representation of physical reality. In many cases, the treatment of identical particles as different particles is justified and may be more intuitive. However special care has to be taken when we count the number of states associated with some value of the energy in order to determine the entropy of the system because the number of states in the different particle approximation must be modified in order to get the correct result.

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