Convergence of the solutions of the discounted Hamilton-Jacobi equation: a counterexample

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Abstract

This paper provides a counterexample about the asymptotic behavior of the solutions of a discounted Hamilton-Jacobi equation, as the discount factor vanishes. The Hamiltonian of the equation is a 1-dimensional continuous and coercive Hamiltonian.

1 Introduction and main result

Let $n \geq 1$. Denote by $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ the $n$-dimensional torus. For $c \in \mathbb{R}$, consider the Hamilton-Jacobi equation

$$H(x, Du(x)) = c \quad (E_0)$$

where the Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous and coercive in the momentum. In order to build solutions of the above equation, Lions, Papanicolaou and Varadhan [6] have introduced a technique called ergodic approximation. For $\lambda \in (0, 1]$, consider the discounted Hamilton-Jacobi equation

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = 0 \quad (E_\lambda)$$

(1.1)

By a standard argument, this equation has a unique viscosity solution $v_\lambda : \mathbb{T}^n \to \mathbb{R}$. Moreover, $(-\lambda v_\lambda)$ converges uniformly as $\lambda$ vanishes to a constant $c(H)$ called the critical value. Set $u_\lambda := v_\lambda + c(H) / \lambda$. The family $(u_\lambda)$ is equi-Lipschitz, and converges uniformly along subsequences towards a solution of $(E_0)$, for $c = c(H)$. Note that $(E_0)$ may have several solutions. Recently, under the assumption that $H$ is convex in the momentum, Davini, Fathi, Iturriaga and Zavidovique [2] have proved that $(u_\lambda)$ converges uniformly (towards a solution of $(E_0)$). In addition, they proved that the solution can be characterized using Mather measures and Peierls barriers.

Without the convexity assumption, the question of whether $(u_\lambda)$ converges or not remained open. This paper solves negatively this question and provides a 1-dimensional continuous and coercive Hamiltonian for which $(u_\lambda)$ does not converge*.

Theorem 1.1. There exists a continuous Hamiltonian $H : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{R}$ that is coercive in the momentum, such that $u_\lambda$ does not converge as $\lambda$ tends to 0.

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*Note that for time-dependent Hamilton-Jacobi equations, several counterexamples about the asymptotic behavior of solutions have been pointed out in [1].
The example builds on a class of discrete-time repeated games called stochastic games. The main ingredient is to establish a connection between recent counterexamples to the existence of the limit value in stochastic games (see [8, 9]) and the Hamilton-Jacobi problem†. The remainder of the paper is structured as follows. Section 2 presents the stochastic game example. Section 3 shows that in order to prove Theorem 1.1, it is enough to study the asymptotic behavior of the stochastic game, when the discount factor vanishes. Section 4 determines the asymptotic behavior of the stochastic game.

2 The stochastic game example

Given a finite set $A$, the set of probability measures over $A$ is denoted by $\Delta(A)$. Given $a \in A$, the Dirac measure at $a$ is denoted by $\delta_a$.

2.1 Description of the game

Consider the following stochastic game $\Gamma$, described by:

- A state space $K$ with two elements $\omega_1$ and $\omega_{-1}$: $K = \{\omega_1, \omega_{-1}\}$,
- An action set $I = \{0, 1\}$ for Player 1,
- An action set $J = \{2 - \sqrt{2} + 2^{-2n}, n \geq 1\} \cup \{2 - \sqrt{2}\}$ for Player 2,
- For each $(k, i, j) \in K \times I \times J$, a transition $q(\cdot | k, i, j) \in \Delta(K)$ defined by:
  
  \[
  q(\cdot | \omega_1, i, j) = [ij + (1 - i)(1 - j)]\delta_{\omega_1} + [i(1 - j) + (1 - i)j]\delta_{\omega_{-1}},
  \]
  
  \[
  q(\cdot | \omega_{-1}, i, j) = [i(1 - j) + (1 - i)j]\delta_{\omega_1} + [ij + (1 - i)(1 - j)]\delta_{\omega_{-1}},
  \]
- A payoff function $g : K \times I \times J \to [0, 1]$, defined by
  
  \[
  g(\omega_1, i, j) = ij + 2(1 - i)(1 - j) \quad \text{and} \quad g(\omega_{-1}, i, j) = -ij - 2(1 - i)(1 - j).
  \]

Let $k_1 \in K$. The stochastic game $\Gamma^{k_1}$ starting at $k_1$ proceeds as follows:

- The initial state is $k_1$. At first stage, Player 2 chooses $j_1 \in J$ and announces it to Player 1. Then, Player 1 chooses $i_1 \in I$, and announces it to Player 2. The payoff at stage 1 is $g(k_1, i_1, j_1)$ for Player 1, and $-g(k_1, i_1, j_1)$ for Player 2. A new state $k_2$ is drawn from the probability $q(\cdot | k_1, i_1, j_1)$ and announced to both players. Then, the game moves on to stage 2.
- At each stage $m \geq 2$, Player 2 chooses $j_m \in J$ and announces it to Player 1. Then, Player 1 chooses $i_m \in I$, and announces it to Player 2. The payoff at stage $m$ is $g(k_m, i_m, j_m)$ for Player 1, and $-g(k_m, i_m, j_m)$ for Player 2. A new state $k_{m+1}$ is drawn from the probability $q(\cdot | k_m, i_m, j_m)$ and announced to both players. Then, the game moves on to stage $m + 1$.

†Let us mention the work [4, 5, 3, 10] as other illustrations of the use of repeated games in PDE problems.
Remark 2.1. The action set of Player 2 can be interpreted as a set of randomized actions. Indeed, imagine that Player 2 has only two actions, 1 and 0. These actions are called pure actions. At stage \( m \), if Player 2 chooses \( j_m \in J \), this means that he plays 1 with probability \( j_m \), and 0 with probability \( 1 - j_m \). Denote by \( j_m \in \{0, 1\} \) his realized action. Player 1 knows \( j_m \) before playing, but does not know \( j_m \). If Player 1 chooses \( i_m \in I \) afterwards, then the realized payoff is \( g(k_m, i_m, j_m) \). Thus, the payoff \( g(k_m, i_m, j_m) \) represents the expectation of \( g(k_m, i_m, j_m) \). Likewise, the transition \( q(.|k_m, i_m, j_m) \) represents the law of \( q(k_m, i_m, j_m) \). The transition and payoff in \( \Gamma \) when players play pure actions can be represented by the following matrices:

\[
\begin{array}{c|cc|c|cc}
\omega_1 & 1 & 0 & \omega_1 & 1 & 0 \\
1 & 1 & 0 & 1 & -1 & 0 \\
0 & 0 & 2 & 0 & 0 & -2 \\
\end{array}
\]

The left-hand side matrix stands for state \( \omega_1 \), and the right-hand side matrix stands for state \( \omega_{-1} \). Consider the left-hand side matrix. Player 1 chooses a row (either 1 or 0), and Player 2 chooses a column (either 1 or 0). The payoff is given by the numbers: for instance, \( g(1, 1) = 1 \) and \( g(1, 0) = 0 \). The arrow means that when the corresponding actions are played, the state moves on to state \( \omega_{-1} \); otherwise, it stays in \( \omega_1 \). For instance, \( q(.|\omega_1, 1, 1) = \delta_{\omega_1} \) and \( q(.|\omega_1, 1, 0) = \delta_{\omega_{-1}} \). The interpretation is the same for the right-hand side matrix. In the game \( \Gamma \), Player 1 can play only pure actions (1 or 0), and Player 2 can play 1 with some probability \( j \in J \).

This matrix representation is convenient to understand the strategic aspects of the game.

Let us now define formally strategies. In general, the decision of a player at stage \( m \) may depend on all the information he has: that is, the stage \( m \), and all the states and actions before stage \( m \). In this paper, it is sufficient to consider a restricted class of strategies, called stationary strategies. Formally, a stationary strategy for Player 1 is defined as a mapping \( y : K \times J \to I \). The interpretation is that at stage \( m \), if the current state is \( k \), and Player 2 plays \( j \), then Player 1 plays \( y(k, j) \). Thus, Player 1 only bases his decision on the current state and the current action of Player 2. Denote by \( Y \) the set of stationary strategies for Player 1.

A stationary strategy for Player 2 is defined as a mapping \( z : K \to J \). The interpretation is that at stage \( m \), if the current state is \( k \), then Player 2 plays \( z(k) \). Thus, Player 2 only bases his decision on the current state. Denote by \( Z \) the set of stationary strategies for Player 2.

The sequence \( (k_1, i_1, j_1, k_2, i_2, j_2, ..., k_m, i_m, j_m, ...) \in H_\infty := (K \times I \times J)^\mathbb{N} \) generated along the game is called history of the game. Due to the fact that state transitions are random, this is a random variable. The law of this random variable depends on the initial state \( k_1 \) and the pair of strategies \((y, z)\), and is denoted by \( \mathbb{P}_{y, z}^{k_1} \).

We will call \( g_m \) the \( m \)-stage random payoff \( g(k_m, i_m, j_m) \). Let \( \lambda \in (0, 1] \). The game \( \Gamma^k_\lambda \) is the game where the strategy set of Player 1 (resp. 2) is \( Y \) (resp. \( Z \)), and the payoff is \( \gamma^{k_1}_\lambda \), where

\[
\gamma^{k_1}_\lambda(y, z) = \mathbb{P}_{y, z}^{k_1} \left( \sum_{m \geq 1} (1 - \lambda)^{m-1} g_m \right).
\]
The goal of Player 1 is to maximize this quantity, while the goal of Player 2 is to minimize this quantity. The game $\Gamma^k_\lambda$ has a value, that is:

$$\min_{z \in Z} \max_{y \in Y} \gamma^k_\lambda(y, z) = \max_{y \in Y} \min_{z \in Z} \gamma^k_\lambda(y, z).$$

The value of $\Gamma^k_\lambda$ is then defined as the above quantity, and is denoted by $w_\lambda(k_1)$. A strategy for Player 1 is optimal if it achieves the right-hand side maximum, and a strategy for Player 2 is optimal if it achieves the left-hand side minimum. The interpretation is that if players are rational they should play optimal strategies, and as a result Player 1 should get $w_\lambda(k_1)$, and Player 2 should get $-w_\lambda(k_1)$.

### 2.2 Asymptotic behavior of the discounted value

As we shall see in the next section, for each $\lambda \in (0, 1]$, one can associate a discounted Hamilton-Jacobi equation with $c'(H) = 0$, such that its solution evaluated at $x = 1$ is approximately $w_\lambda(\omega_1)$, for $\lambda$ small enough. Thus, the asymptotic behavior of this quantity needs to be studied.

Define $\lambda_n := 2^{-2n} \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)^{-1}$ and $\mu_n := 2^{-2n-1} \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right)^{-1}$.

**Proposition 2.2.** The following hold:

1. $w_\lambda(\omega_1) = \lim_{\mu \to 0} \lambda w_\lambda(\omega - 1) = 0.$

2. The next section explains how to derive the counterexample and Theorem 1.1 from Proposition 2.2.
3 Link with the PDE problem and proof of Theorem 1.1

The following proposition expresses $w_\lambda$ as the solution of a functional equation called Shapley equation.

**Proposition 3.1.** Let $\lambda \in (0, 1]$ and $u_\lambda := (1 + \lambda)^{-1}w_{\lambda/(1+\lambda)}$. For each $r \in \{-1, 1\}$, the two following equations hold:

(i) $$w_\lambda(\omega_r) = \min_{j \in J} \max_{i \in I} \left\{ g(\omega_r, i, j) + (1 - \lambda) \left[ q(\omega_r|\omega_r, i, j)w_\lambda(\omega_r) + q(\omega_r|\omega_r, i, j)w_\lambda(\omega_{-r}) \right] \right\}$$

(ii) $$\lambda u_\lambda(\omega_r) = \min_{j \in J} \max_{i \in I} \left\{ g(\omega_r, i, j) + q(\omega_r|\omega_r, i, j) \left[ u_\lambda(\omega_{-r}) - u_\lambda(\omega_r) \right] \right\}$$

**Proof.** (a) The intuition is the following. Consider the game $\Gamma^\omega_\lambda$. At stage 1, the state is $\omega_r$. The term $g$ represents the current payoff, and the term $(1 - \lambda)[...]$ represents the future optimal payoff, that is, the payoff that Player 1 should get from stage 2 to infinity. Thus, this equation means that the value of $\Gamma^\omega_\lambda$ coincides with the value of the one-stage game, where the payoff is a combination of the current payoff and the future optimal payoff. For a formal derivation of this type of equation, we refer to [7, VII.1., p. 392].

(b) Evaluating the previous equation at $\lambda/(1 + \lambda)$ yields

$$w_{\lambda/(1+\lambda)}(\omega_r) = \min_{j \in J} \max_{i \in I} \left\{ g(\omega_r, i, j) + \frac{1}{1 + \lambda} \left[ q(\omega_r|\omega_r, i, j)w_{\lambda/(1+\lambda)}(\omega_r) + q(\omega_r|\omega_r, i, j)w_{\lambda/(1+\lambda)}(\omega_{-r}) \right] \right\}$$

Using the fact that $q(\omega_r|\omega_r, i, j) = 1 - q(\omega_{-r}|\omega_r, i, j)$ yields the result.

\[\Box\]

For $p \in \mathbb{R}$, define $H_1 : \mathbb{R} \to \mathbb{R}$ and $H_{-1} : \mathbb{R} \to \mathbb{R}$ by

$$H_1(p) := \begin{cases} - \min_{j \in J} \max_{i \in I} \left\{ g(\omega_1, i, j) - p \cdot ([i(1 - j) + (1 - i)j] \right\}, & \text{if } |p| \leq 2, \\ H_1 \left( \frac{2p}{|p|} \right) + |p| - 2 & \text{if } |p| > 2. \end{cases}$$

$$H_{-1}(p) := \begin{cases} - \min_{j \in J} \max_{i \in I} \left\{ g(\omega_{-1}, i, j) + p \cdot ([i(1 - j) + (1 - i)j] \right\}, & \text{if } |p| \leq 2, \\ H_{-1} \left( \frac{2p}{|p|} \right) + |p| - 2 & \text{if } |p| > 2. \end{cases}$$

For $x \in [-1, 1]$ and $p \in \mathbb{R}$, let

$$H(x, p) := |x|H_1(|p|) + (1 - |x|)H_{-1}(|p|).$$

(3.1)

Note that the definition of $H_1$ and $H_{-1}$ for $|p| > 2$ ensures that $\lim_{|p| \to +\infty} H_1(p) = \lim_{|p| \to +\infty} H_{-1}(p) = +\infty$, thus $\lim_{|p| \to +\infty} H(p) = +\infty$. Note also that for all $x \in [-1, 1]$, $H_1(x, \cdot)$ is increasing on $[-2, 2]$ and $H_{-1}(x, \cdot)$ is decreasing on $[-2, 2]$.

Thanks to Proposition 3.1 (ii) and Proposition 2.2 (i), we have $\lambda u_\lambda(\omega_1) + H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$.
and $\lambda u_\lambda(\omega_1) + H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = 0$.

For $x \in [-1, 1]$, let $u_\lambda(x) = |x|u_\lambda(\omega_1) + (1 - |x|)u_\lambda(\omega_{-1})$. Let $x \in (-1, 1) \setminus \{0\}$. Proposition 2.2 (i) implies that $w_\lambda(\omega_{-1}) \leq w_\lambda(\omega_1)$, thus $u_\lambda(\omega_{-1}) \leq u_\lambda(\omega_1)$ and $|Du_\lambda(x)| = u_\lambda(\omega_1) - u_\lambda(\omega_{-1})$. Consequently, Proposition 3.1 (ii) yields

$$\lambda u_\lambda(x) + H(x, Du_\lambda(x)) = 0. \quad (3.2)$$

Note that the above equation is identical to equation (3.2). The reason why we use the notation $u_\lambda$ and not $v_\lambda$ is that, as we shall see, $c(H) = 0$, thus $u_\lambda$ coincides with $v_\lambda$.

Extend $u_\lambda$ and $H(.,p)$ ($p \in \mathbb{R}$) as 2-periodic functions defined on $\mathbb{R}$. The Hamiltonian $H$ is continuous and coercive in the momentum, and the above equation holds in a classical sense for all $x \in \mathbb{R} \setminus \mathbb{Z}$.

For $x \in \mathbb{R}$, denote by $D^+u_\lambda(x)$ (resp., $D^-u_\lambda(x)$) the super-differential (resp., the sub-differential) of $u_\lambda$ at $x$. Let us show that $u_\lambda$ is a viscosity solution of (3.2) on $\mathbb{R}$. By 2-periodicity, it is enough to show that this is a viscosity solution for $x = 0$ and $x = 1$.

Let us start by $x = 0$. We have $D^+u_\lambda(0) = \emptyset$ and $D^-u_\lambda(0) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$. Let $p \in D^-u_\lambda(0)$. Then $H_{-1}(p) \geq H_{-1}(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_{-1})$, thus $\lambda u_\lambda(0) + H(0, p) \geq 0$. Consequently, $u_\lambda$ is a viscosity solution at $x = 0$.

Consider now the case $x = 1$. We have $D^+u_\lambda(1) = [u_\lambda(\omega_{-1}) - u_\lambda(\omega_1), u_\lambda(\omega_1) - u_\lambda(\omega_{-1})]$ and $D^-u_\lambda(1) = \emptyset$.

Let $p \in D^+u_\lambda(1)$. Then $H_1(p) \leq H_1(u_\lambda(\omega_1) - u_\lambda(\omega_{-1})) = -\lambda u_\lambda(\omega_1)$, thus $\lambda u_\lambda(1) + H(1, p) \geq 0$. Consequently, $u_\lambda$ is a viscosity solution at $x = 1$.

Let us now conclude the proof of Theorem 1.1. Because $H$ is 2-periodic, equation (3.2) can be considered as written on $\mathbb{T}^1$.

As noticed before, equation (3.2) is identical to equation (1.1). Therefore, as stated in the introduction, $-\lambda u_\lambda$ converges to $c(H)$. Proposition 2.2 (ii) implies that $(-\lambda_n, u_{\lambda_n}(1))$ converges to 0, thus $c(H) = 0$. Still by Proposition 2.2 (ii), $(u_{\lambda}(1))$ does not have a limit when $\lambda$ tends to 0: Theorem 1.1 is proved.

4 Proof of Proposition 2.2

4.1 Proof of (i)

Consider Proposition 3.1 (i) for $r = 1$. Take $j = 1/2 \in J$. It yields

$$w_\lambda(\omega_1) \leq \max_{i \in J} \left\{ 1 + (1 - \lambda) \left( \frac{1}{2} w_\lambda(\omega_1) + \frac{1}{2} w_\lambda(\omega_{-1}) \right) \right\} = 1 + \frac{1}{2}(1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})). \quad (4.1)$$

Take $i = 1/2$. This yields

$$w_\lambda(\omega_1) \geq \frac{1}{2} + \frac{1}{2}(1 - \lambda) (w_\lambda(\omega_1) + w_\lambda(\omega_{-1})). \quad (4.2)$$
For $r = -1$, taking $j = 1/2$ and then $i = 1/2$ produce the following inequalities:

\[ w_{\lambda}(\omega-1) \leq -\frac{1}{2} + \frac{1}{2}(1 - \lambda) (w_{\lambda}(\omega_1) + w_{\lambda}(\omega_{-1})) \]  

and

\[ w_{\lambda}(\omega-1) \geq -\frac{1}{2} + \frac{1}{2}(1 - \lambda) (w_{\lambda}(\omega_1) + w_{\lambda}(\omega_{-1})). \]

Combining (4.2) and (4.3) yield $w_{\lambda}(\omega_1) \geq w_{\lambda}(\omega_{-1}) + 1 \geq w_{\lambda}(\omega_{-1})$. Combining (4.1) and (4.4) yield $w_{\lambda}(\omega_{-1}) \geq w_{\lambda}(\omega_1) - 2$, and (i) is proved.

4.2 Proof of (ii)

For $(i, i') \in \{0, 1\}^2$, consider the strategy $y$ of Player 1 that plays $i$ in $\omega_1$ and $i'$ in $\omega_{-1}$ (regardless of Player 2’s actions), and the strategy $z$ of Player 2 that plays $a$ in state $\omega_1$, and $b$ in state $\omega_{-1}$. Denote $\gamma^{i,i'}_{\lambda}(a, b) := \gamma^{\omega_1}_{\lambda}(y, z)$ (resp., $\tilde{\gamma}^{i,i'}_{\lambda}(a, b) := \gamma^{\omega_{-1}}_{\lambda}(y, z)$), the payoff in $\Gamma^{\omega_1}_{\lambda}$ (resp., $\Gamma^{\omega_{-1}}_{\lambda}$), when $(y, z)$ is played.

Proposition 4.1. The following hold:

1. 

\[
\gamma^{0,0}_{\lambda}(a, b) = \frac{-2(a - b - \lambda + b\lambda)}{\lambda(a + b + \lambda - a\lambda - b\lambda)} \]

\[
\gamma^{1,1}_{\lambda}(a, b) = \frac{a - b + \lambda b}{\lambda(a + b + \lambda - a\lambda - b\lambda - 2)} \]

\[
\gamma^{1,0}_{\lambda}(a, b) = \frac{2a + 2b + 2\lambda - ab - a\lambda - 2b\lambda + ab\lambda - 2}{\lambda(b - a + \lambda a - b\lambda + 1)} \]

\[
\gamma^{0,1}_{\lambda}(a, b) = -\frac{2a + 2b - ab - 2b\lambda + ab\lambda - 2}{\lambda(a - b - a\lambda + b\lambda + 1)} \]

2. 

- $\gamma^{0,0}_{\lambda}$ is decreasing with respect to $a$ and increasing with respect to $b$.
- $\gamma^{1,1}_{\lambda}$ is increasing with respect to $a$ and decreasing with respect to $b$.
- $\gamma^{1,0}_{\lambda}$ is increasing with respect to $a$ and $b$.
- $\gamma^{0,1}_{\lambda}$ is decreasing with respect to $a$ and $b$.

Proof. 1. The payoffs $\gamma^{0,0}_{\lambda}(a, b)$ and $\tilde{\gamma}^{0,0}_{\lambda}(a, b)$ satisfy the following recursive equation:

\[
\gamma^{0,0}_{\lambda}(a, b) = a(1 - \lambda)\gamma^{0,0}_{\lambda}(a, b) + (1 - a)(2 + (1 - \lambda)\gamma^{0,0}_{\lambda}(a, b)) \]

\[
\tilde{\gamma}^{0,0}_{\lambda}(a, b) = a(1 - \lambda)\gamma^{0,0}_{\lambda}(a, b) + (1 - a)(-2 + (1 - \lambda)\gamma^{0,0}_{\lambda}(a, b)) \]

Combining these two relations give the first equality. The three other equalities can be derived in a similar fashion.

2. These monotonicity properties are simply obtained by deriving $\gamma^{i,i'}_{\lambda}$ with respect to $a$ and $b$. 

\[ \square \]
For $\lambda \in (0, 1]$, set $p^*(\lambda) := 2 - \sqrt{2} + \left(\frac{3}{4} - \frac{1}{\sqrt{2}}\right) \lambda$. Define a strategy $y$ of Player 1 in the following way:

- in state $\omega_1$, play 0 if $j \leq p^*(\lambda)$, play 1 otherwise,
- in state $\omega_{-1}$, play 1 if $j \leq p^*(\lambda)$, play 0 otherwise.

The rationale behind this strategy can be found in Section 2.2.

For all $n \geq 1$, define

$$
\lambda_n := \frac{2 - 2^n}{\frac{3}{4} - \sqrt{2}} \quad \text{and} \quad \mu_n := \frac{2 - 2^{n-1}}{\frac{3}{4} - \sqrt{2}}.
$$

**Proposition 4.2.** The following hold:

1. 

$$
\lim_{n \to +\infty} \min_{z \in \mathbb{Z}} \gamma_{\lambda_n}(y, z) = \frac{1}{\sqrt{2}}
$$

2. 

$$
\lim_{n \to +\infty} \min_{z \in \mathbb{Z}} \gamma_{\mu_n}(y, z) = \frac{5}{2\sqrt{2}} - 1 > \frac{1}{\sqrt{2}}
$$

**Proof.**

1. For all $(i, i') \in \{0, 1\}$,

$$
\lim_{n \to +\infty} \gamma_{\lambda_n}^{i, i'}(p^*(\lambda_n), p^*(\lambda_n)) = \frac{1}{\sqrt{2}},
$$

and the result follows.

2. Let $z$ be a strategy of Player 2, and $a = z(\omega_1)$ and $b = z(\omega_{-1})$. Note that the interval $(p^*(\mu_n/2), p^*(2\mu_n))$ does not intersect $J$.

The following cases are distinguished:

**Case 1.** $a \leq p^*(\mu_n)$ and $b \leq p^*(\mu_n)$, thus $a \leq p^*(\mu_n/2)$ and $b \leq p^*(\mu_n/2)$

We have $\gamma_{\mu_n}^0(y, z) = \gamma_{\mu_n}^0(a, b) \geq \gamma_{\mu_n}^0(p^*(\mu_n/2), p^*(\mu_n/2)) \xrightarrow{n \to +\infty} \frac{5}{4} \sqrt{2} - 1$

**Case 2.** $a \leq p^*(\mu_n)$ and $b \geq p^*(\mu_n)$, thus $a \leq p^*(\mu_n/2)$ and $b \geq p^*(2\mu_n)$

We have $\gamma_{\mu_n}^0(y, z) = \gamma_{\mu_n}^0(a, b) \geq \gamma_{\mu_n}^0(p^*(\mu_n/2), p^*(2\mu_n)) \xrightarrow{n \to +\infty} \frac{1 + 2\sqrt{2}}{8(-2 + \sqrt{2})}$

**Case 3.** $a \geq p^*(\mu_n)$ and $b \leq p^*(\mu_n)$, thus $a \geq p^*(2\mu_n)$ and $b \leq p^*(\mu_n/2)$

We have $\gamma_{\mu_n}^1(y, z) = \gamma_{\mu_n}^1(a, b) \geq \gamma_{\mu_n}^1(p^*(2\mu_n), p^*(\mu_n/2)) \xrightarrow{n \to +\infty} (-1/16) \frac{-25 + 14\sqrt{2}}{\sqrt{2} - 1}$

**Case 4.** $a \geq p^*(\mu_n)$ and $b \geq p^*(\mu_n)$, thus $a \geq p^*(2\mu_n)$ and $b \geq p^*(2\mu_n)$

We have $\gamma_{\mu_n}^1(y, z) = \gamma_{\mu_n}^1(a, b) \geq \gamma_{\mu_n}^1(p^*(2\mu_n), p^*(2\mu_n)) \xrightarrow{n \to +\infty} -2 + 2\sqrt{2}$

Among these cases, the smallest limit is $\frac{5}{4} (\sqrt{2} - 1)$, and the result follows.
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