Baxters’s $Q$-operators for the simplest $q$-deformed model.

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Abstract

In the present paper we describe the procedure of the $Q$-operators construction for the $q$-deformed model, described by the Lax operator, which is important to formulate the Bethe ansatz for the Sin-Gordon model. This Lax operator can also be considered as some massless limit of the Lax operator of SG model. We constructed two $R$ operators which are the universal intertwiners for the Lax operators. The traces of its monodromies over the auxiliary space are Baxter operators i.e. the operator solutions of $T-Q$ equation. We also found the intertwining relations which imply the mutual commutativity of the corresponding $Q$-operators.

1 Introduction

This article continues the series of articles devoted to the investigation of Baxter $Q(x)$ operators for various quantum integrable systems. Long ago, in his famous papers [2] Baxter has introduced this object for the solution of the eigenvalue problem of XYZ spin chain. In the frameworks of the Quantum Inverse Scattering Method or Algebraic Bethe Ansatz (see, for example, [1]) $Q$ operator could be defined as follows. Let $T(x) = L_N(x)\ldots L_1(x)$ be the monodromy matrix of the Lax operator $L_k(x)$. The trace of the monodromy is the transfer matrix $t(x)$. In the cases, we are going to consider the Lax operators, as well, as its monodromy $T(x)$ are intertwined by rational or trigonometric $R$-matrices. Then $Q(x)$ operator is defined by the following equations:

$$t(x)Q(x) = a(x)Q(x+i) + b(x)Q(x-i)$$

in the rational case and

$$t(x)Q(x) = a(x)Q(qx) + b(x)Q(q^{-1}x)$$

in the trigonometric case. As it follows from the general approach [1], the roots of the eigenvalues of $Q$ are the solutions of Bethe equation and consequently it defines the eigenvectors and eigenvalues of $t(x)$. Functions $a(x)$ and $b(x)$ are in our case polynomials (usual or trigonometric) and determined by the factorization of the quantum determinant of the Lax operator. These equations are discrete analogues of the differential equations of the second order (see for example [5], where was shown how Baxter equation for inhomogeneous XXX spin chain turns to the differential equation of the
second order, which allows to find spectrum of the Gaudin model). So we expect two independent solution \( Q \) of Baxter equation with the same \( t \). Indeed, in the paper [4], where authors considered the case of XXX and XXZ spin chains the second polynomial solution was found (this fact was first pointed out in [3]). In the present paper we consider the problem of construction of operator \( Q \) i.e. the operator solution of (2) with given transfer matrix \( t(x) = tr \, T(x) \). Actually the method presented in our paper is similar to the method which Baxter exploited for quantization of XXZ and XYZ spin chains [2]. In the previous papers [6, 7, 9] (see also [8] where was noticed the relation of \( Q \)-operators with quantum Backlund transformation) the expressions for two basic operators \( M \) acting in quantum spaces and auxiliary infinite-dimensional space were derived. In the paper [6] also was discovered the relationship between \( Q \)-operator and the Bloch solutions of quantum linear problem. The traces of the monodromies of \( M \)-operators are the \( Q \)-operators. In the paper [10] the one-parametric family of \( Q \)-operators for the XXX spin chain was constructed, however there is the problem to construct two independent solution in this case. In the present paper we have constructed the basic \( M \)-operators for the simplest Lax operator connected with the trigonometric \( R \)-matrix.

2 The intertwining relations with the Lax operator

2.1 Universal \( R \)-matrix as \( Q \)-operator

Let us consider the problem of construction of Baxter operators for the integrable model connected with the Lax operator (\( x \) is the spectral parameter):

\[
L(x) = \begin{pmatrix} u & xv^{-1} \\ -xv & u^{-1} \end{pmatrix}
\]

(3)

Operators \( u \) and \( v \) form the Weyl pair:

\[
wv = qvu.
\]

(4)

Here we consider the case of \(|q| < 1\). We will use not only operators generated by the integer powers of \( u \) and \( v \) but also the noninteger powers of \( u \) and \( v \). So the operators \( u \) and \( v \) have to be considered as the exponents of Heisenberg operators \( p \) and \( q \) with the commutation relation \([p, q] = -i\). It is well known that the operators \( L \) are intertwined by the trigonometric \( R \)-matrix:

\[
R(z) = \begin{pmatrix} qz - q^{-1}z^{-1} & z - z^{-1} & q - q^{-1} \\ q - q^{-1} & z - z^{-1} & \end{pmatrix}
\]

(5)

\( R \)-matrix acts in the tensor product of the two dimensional auxiliary spaces 1 and 2, \( L_1 \) and \( L_2 \) act in local quantum space and in the spaces 1 and 2 respectively. The intertwining equations are

\[
R_{12}(x/y)L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}(x/y)
\]

(6)
Since the operators \( u \) and \( v \) are invertible there is no usual Bethe anzats for the monodromy matrix of Lax operators \( (3) \). However it is possible to construct Bethe ansatz for the product of two Lax operators and to arrive to the formulation of the Sin-Gordon model within the QISM approach (see, for example \( [1] \)).

First we construct the universal \( R \) - matrix. The defining equations are:

\[
R_{12}L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}. \tag{7}
\]

In contrast to \( (6) \), \( R_{12} \) intertwines \( L_1, L_2 \) in "quantum" space. Here indices 1 and 2 denote different Weyl spaces (with the same \( q \)) and Lax operators act in the common auxiliary two-dimensional space. The corresponding four equations are:

\[
\begin{align*}
R &\cdot (u_1u_2 - xy \cdot v_2v_1^{-1}) = (u_1u_2 - xy \cdot v_1v_2^{-1}) \cdot R \\
R &\cdot (u_1^{-1}u_2 - xy \cdot v_1v_2^{-1}) = (u_1^{-1}u_2 - xy \cdot v_2v_1^{-1}) \cdot R \\
R &\cdot (yu_1v_2^{-1} + xu_2v_1) = (xu_1v_2^{-1} + yu_2v_1) \cdot R \\
R &\cdot (yu_1^{-1}v_2 + xu_2v_1) = (yu_1v_2 + xu_2^{-1}v_1) \cdot R \tag{8}
\end{align*}
\]

It is clear that \( R = R(x/y) \). Now let us use the following ansatz:

\[
R_{12}(z) = P_{12}r_{12} \tag{9}
\]

where \( P_{12} \) is the permutation operator i.e.:

\[
u(v)_{1,2} \cdot P_{12} = P_{12} \cdot u(v)_{2,1}
\]

This ansatz leads to the following single equation (along with \([r, u_1u_2] = 0\)):

\[
r(z) \{qw_{12} + z\} u_2^{-1}v_1^{-1} = u_2^{-1}v_1^{-1} \{1 + q^{-1}w_{12}\} r(z) \tag{11}
\]

We choose the following solution of this equation:

\[
r_{12}(z) = f(u_1u_2, z)g(w_{12}, z) \tag{12}
\]

where \( f(z) = (u_1u_2)^{inz/lnq} \), \( w_{12} = u_1v_1u_2v_2^{-1} \) and \( g \) satisfies the following recursion:

\[
\frac{g(q^2w_{12})}{g(w_{12})} = \frac{1 + qz^{-1}w_{12}}{1 + zqw_{12}} \tag{13}
\]

We will use the following the solution of this recursion

\[
g(w_{12}) = \frac{(-zqw_{12}; q^2)_{\infty}}{(-z^{-1}qw_{12}; q^2)_{\infty}} \tag{14}
\]

Constructed \( R \) - matrix differs from the Faddeev-Volkov \( R \) - matrix (see \([1]\)), by factor \( f \). As it will be seen, this \( R \) - matrix is connected to the \( Q \) - operator of model under consideration. To see it let us consider some multiplication rules between the \( R \) - matrix and Lax operator. Lax operator is degenerate at the point \( x_0 = iq^{-1/2} \):

\[
L(x_0) = \left( \begin{array}{cc} iq^{-1/2}v_1^{-1} & \cdot \left( -ivuq^{1/2}, 1 \right) = \tau^{-1}u^{-1}\rho^- \end{array} \right) \tag{15}
\]
Introduce also
\[ \tau^+ = \left( 1, -iq^{1/2}v^{-1}u \right), \quad \rho^+ = \left( \frac{1}{iq^{1/2}vu} \right) \] (16)

(it is clear that \( \rho^- \rho^+ = \tau^+ \tau^- = 0 \)) to construct the projectors:
\[ \Pi^- = \tau^-(\rho^- \tau^-)^{-1} \rho^-, \quad \Pi^+ = \rho^+ (\tau^+ \rho^+)^{-1} \tau^+ \] (17)

Their properties as the projectors are:
\[ (\Pi^+)^2 = \Pi^+, \quad \Pi^+ \Pi^- = 0, \quad \Pi^+ + \Pi^- = 1 \] (18)

Let us also write down the expression for the operator that is inverse to the Lax operator:
\[ \tilde{L}(x) = \begin{pmatrix} u^{-1} & -qxv^{-1} \\ qxv & u \end{pmatrix}, \quad L(x) \tilde{L}(x) = 1 + x^2q \] (19)

It turns out that our \( R \) - matrix satisfies the following triangle conditions:
\[ \Pi^- L(x) R_{12}(-iq^{1/2}x) \Pi^+ = \Pi^+ R_{12}(-iq^{1/2}x) L(x) \Pi^- = 0 \] (20)

Here the projectors act in the two dimensional auxiliary space and in the Weyl space number 2. Moreover, the following multiplication rules hold true:
\[ \rho^- L(x) R(-iq^{1/2}x) = R(-iq^{-1/2}x) u_1 \rho^- \]
\[ L(x) R(-iq^{1/2}x) \rho^+ = \rho^+ R(-iq^{3/2}x) u_1^{-1} (1 + qx^2) \]
\[ R(-iq^{1/2}x) L(x) \tau^- = \tau^- u_1 R(-iq^{3/2}x) \]
\[ \tau^+ R(-iq^{1/2}x) L(x) = u_1^{-1} (1 + x^2q) R(-iq^{-1/2}x) \tau^+ \] (21)

For the monodromy of \( R(-iq^{1/2}x) \)
\[ \hat{Q}^{(1)}(x) = R_{N,a}(-iq^{1/2}x) R_{N-1,a}(-iq^{1/2}x) \ldots R_{1,a}(-iq^{1/2}x), \] (22)
\[ tr_a \hat{Q}^{(1)}(x) = Q^{(1)}(x) \] (23)

These rules guarantee the validity of Baxter equation:
\[ t(x) Q^{(1)}(x) = u_1^{-1} \ldots u_N^{-1} (1 + x^2q)^N \cdot Q^{(1)}(q^{-1}x) + u_N \ldots u_1 \cdot Q^{(1)}(qx) \] (24)

provided the trace of \( \hat{Q}^{(1)}(x) \) exists. In our case it is sufficient to demand that the trace of permutation is equal to the identity operator
\[ tr_2 P_{12} = id_1. \] (25)

As an example let us consider the simplest case of one degree of freedom. In this case using the property (25) we immediately arrive at the following expression for \( Q \) - operator:
\[ Q(iq^{-1/2}z) = tr_2 R(z) = u_1^{2mnz/lnq} \sum_{k=0} \left( -qu_1^2 \right)^k q^{k^2} \frac{(z^2; q^2)_k}{(q^2; q^2)_k} \] (26)
2.2 Universal $R$-matrix for $q$-DST model

Let us construct also the universal $R$-matrix for the $q$-deformed integrable DST model \[9\]. The Hamiltonian of this model is

$$H = \sum_{n=1}^{N} \left[ u_n^{-2} + a_n^+ u_n^{-1} a_{n-1} u_{n-1}^{-1} \right]$$  \(27\)

where $a_n, a_n^+, u_n$ is the generators of $q$-oscillator algebra satisfying:

$$au = qua, \quad ua^+ = qa^+u, \quad a+a = q^{-1}(1-u^2), \quad aa^+ = q^{-1}(1-q^2 u^2)$$  \(28\)

This Hamiltonian belongs to the family of commuting operators generated by the transfer matrix

$$t(x) = TrT(x) = TrL_N(x) \ldots L_1(x)$$  \(29\)

where the Lax operator for this model is given by

$$L(x) = \begin{pmatrix} u x^{-1} + x u^{-1} & a^+ \\ a & x u \end{pmatrix}$$  \(30\)

and is intertwined by the $R$-matrix \[5\]. In contrast to the model, considered in the previous section, there exists the Algebraic Bethe ansatz for $q$-DST model (the reference state satisfies $a \omega_0 = 0$). The defining equations for the universal $R$-matrix are \[7\]. Again, it is suitable to extract the permutation operator. The final result is

$$R_{12}(z) = P_{12} \cdot (u_1 u_2)^{ln z/ln q} \sum_{k=0}^{\infty} (a_2^+ a_1 u_1^{-1} u_2^{-1})^k (-zq^2)^k/z^2 q^2)$$  \(31\)

It can be written down in the formal form similar to the expression for universal $R$-matrix from previous subsection

$$R_{12}(z) = P_{12} \cdot (u_1 u_2)^{ln z/ln q} (-g^2 z^{-1} w_{12}; q^2)^\infty / (-g^2 z w_{12}; q^2)^\infty$$  \(32\)

where $w_{12} = u_1^{-1} u_2^{-1} a_2^+ a_1$. Again, constructing projectors from the Lax operator at the degeneration point (in this case $z = 1$), considering the multiplication rules (they are similar to \[21\]) and taking the trace we arrive to the Baxter equation for the transfer-matrix of the model and the $Q$ - operator, which is the trace of monodromy of $R$-matrices

$$t(x)Q(x) = (x-x^{-1})^N Q(q^{-1} x) u_1 \ldots u_N + x u_1^{-1} \ldots u_N^{-1} Q(qx)$$  \(33\)

$$Q(x) = Tr_a R_{N,a} \ldots R_{1,a}$$  \(34\)

For the $q$-oscillator algebra the trace is well defined object, namely if

$$A = \sum_{k=0}^{\infty} \left\{ (a^+)^k A_k^+(u) + a^k A_k^-(u) \right\}$$  \(35\)
then
\[ tr \, A = \sum_{n}^{\infty} A_{0}(q^{n}) \]  \hspace{1cm} (36)

but in our case it is sufficient to use the property \(25\) to obtain the expressions for \(Q\)-
operators. However, in contrast to the case of Weyl algebra, we are able to write down
the permutation in the following form
\[ P_{12} = C_{0}(u_{1}, u_{2}) + \sum_{n=1}^{\infty} \left\{ (a_{1}^{+}a_{2})^{k} \cdot C_{n}^{+}(u_{1}, u_{2}) + (a_{1}^{-}a_{2})^{k} \cdot C_{n}^{-}(u_{1}, u_{2}) \right\} , \]  \hspace{1cm} (37)

where
\[ C_{0} = \Delta(u_{1} - u_{2}), \]
\[ C_{n}^{+} = \Delta(u_{1} - q^{n}u_{2}) \cdot g_{2}^{-1}(u_{1}) \cdots g_{2}^{-1}(q^{n-1}u_{1}), \]
\[ C_{n}^{-} = \Delta(u_{2} - q^{n}u_{1}) \cdot g_{2}^{-1}(u_{2}) \cdots g_{2}^{-1}(q^{n-1}u_{2}) \]  \hspace{1cm} (38)

where \(g_{2}(u) = q^{-1}(1 - q^{2}u^{2})\) and \(\Delta(n_{1} - n_{2})\) is the Kroneker symbol. It is clear that \(25\) in the case of \(q\)-oscillator algebra is in agreement with the rule \(36\).

3 The second \(Q\)-operator

Let us also write down the expression for the operator which is defined by the following
equations:
\[ \tilde{L}(x) = \begin{pmatrix} u^{-1} & -q xv^{-1} \\ q x v & u \end{pmatrix}, \quad L(x)\tilde{L}(x) = 1 + x^{2}q \]  \hspace{1cm} (39)

Let us consider now the following intertwining equation:
\[ \tilde{L}(y)L(x)\tilde{R} = \tilde{R}L_{1}(x)\tilde{L}_{2}(y) \]  \hspace{1cm} (40)

The corresponding equations are
\[ (u_{1}u_{2}^{-1} + q xyv_{2}^{-1}v_{1}) \cdot \tilde{R} = \tilde{R} \cdot (u_{1}u_{2}^{-1} + q xyv_{2}v_{1}^{-1}) \]
\[ (xu_{2}^{-1}v_{1}^{-1} - q yv_{2}^{-1}u_{1}^{-1}) \cdot \tilde{R} = \tilde{R} \cdot (xv_{1}^{-1}u_{2} - q yu_{1}v_{2}^{-1}) \]
\[ (q yv_{2}u_{1} - xv_{1}u_{2}) \cdot \tilde{R} = \tilde{R} \cdot (q yv_{2}u_{1}^{-1} - xv_{1}v_{2}^{-1}) \]
\[ (u_{1}^{-1}u_{2} + q xyv_{2}v_{1}^{-1}) \cdot \tilde{R} = \tilde{R} \cdot (u_{1}^{-1}u_{2} + q xyv_{2}^{-1}v_{1}) \]  \hspace{1cm} (41)

It is clear that in this case \(\tilde{R} = \tilde{R}(xy)\). From the first and the last equations we see that
\[ u_{2}^{-1}v_{1}^{-1} \cdot \tilde{R} = \tilde{R} \cdot v_{1}^{-1}u_{2}, \quad u_{1}^{-1}v_{2}^{-1} \cdot \tilde{R} = \tilde{R} \cdot u_{1}v_{2}^{-1} \]
\[ u_{1}v_{2} \cdot \tilde{R} = \tilde{R} \cdot u_{1}^{-1}v_{2}, \quad u_{2}v_{1} \cdot \tilde{R} = \tilde{R} \cdot u_{2}^{-1}v_{1} \]  \hspace{1cm} (42)

Let us look for \(\tilde{R}\) in the form
\[ \tilde{R}(z) = u_{2}^{2 lin_{1}/lin} \cdot \tilde{r}(\tilde{w}_{12}, z) \]  \hspace{1cm} (43)
Here we denote $\tilde{w}_{12} = u_1^{-1}v_1^{-1}u_2v_2$. Then for $f$ we obtain:

$$f(q^2\tilde{w}_{12}, z) = \frac{1 + qz\tilde{w}_{12}^{-1}}{1 + qz\tilde{w}_{12}} = qz\tilde{w}_{12}^{-1} \cdot \frac{1 + q^{-1}z^{-1}\tilde{w}_{12}}{1 + qz\tilde{w}_{12}}$$

The following operator function satisfies this recursion:

$$f(\tilde{w}_{12}, z) = \theta(z^{-1}q^{-2}\tilde{w}_{12}) \cdot \frac{(-qz\tilde{w}_{12}; q^2)^\infty}{(-q^{-1}z^{-1}\tilde{w}_{12}; q^2)^\infty} \cdot (1 + q^{-1}z^{-1}\tilde{w}_{12})_{\infty}$$

Here and in the previous subsection we use standard notations for theta function and $q$-exponent:

$$\theta(\alpha) = \sum z^\alpha, \quad \theta(q^2\alpha) = q^{-1}\alpha^{-1}\theta(\alpha)$$

$$(x; q)_n = (1 - x)(1 - qx) \ldots (1 - q^{n-1}x)$$

$$(x; q)_{\infty} = \prod_{k=0}^{\infty} (1 - q^kx), \quad (1 - x)(qx; q)_{\infty} = (x; q)_{\infty}$$

It turns out that this intertwiner has similar to (21) rules of multiplication with the Lax operator:

$$\tau^+ L(x)\tilde{R}(iq^{-1/2}x) = \tilde{R}(iq^{-3/2}x)u_1\tau^+$$

$$L(x)\tilde{R}(iq^{-1/2}x)\tau^- = \tau^- \tilde{R}(iq^{1/2}x)u_1^{-1}(1 + qx^2)$$

$$\tilde{R}(iq^{-1/2}x)L(x)\rho^+ = \rho^+ u_1\tilde{R}(iq^{-3/2}x)$$

$$\rho^- \tilde{R}(iq^{-1/2}x)L(x) = u_1^{-1}(1 + x^2q)\tilde{R}(iq^{1/2}x)\rho^-$$

Again, taking the trace of monodromy

$$\tilde{Q}^{(2)}(x) = \tilde{R}_{N,a}(iq^{-1/2}x)\tilde{R}_{N-1,a}(iq^{-1/2}x)\ldots \tilde{R}_{1,a}(iq^{-1/2}x),$$

$$tr_a\tilde{Q}^{(2)}(x) = Q^{(2)}(x)$$

we arrive to the following Baxter equation:

$$t(x)Q^{(2)}(x) = u_1^{-1} \ldots u_N^{-1}(1 + x^2q)^N \cdot Q^{(2)}(q^{-1}x) + u_N \ldots u_1 \cdot Q^{(2)}(qx)$$

However, we do not known the suitable definition of trace operation in this case (mainly due the presence of the factor $u_2^{2n_{u_1}lnq}$ in $\tilde{R}$).

### 4 Intertwining relations between $R$ and $\tilde{R}$ matrices

The intertwiners introduced in the previous section are constructed in particular to satisfy the following commutative relations:

$$[t(x), Q^{(1,2)}(y)] = 0$$
Here we get the intertwining equations between $R$ and $\tilde{R}$ matrices, that imply the mutual commutativity the corresponding $Q$-operators:

$$[Q^{(1)}(x), Q^{(1)}(y)] = 0, \ [Q^{(1)}(x), Q^{(2)}(y)] = 0, \ [Q^{(2)}(x), Q^{(2)}(y)] = 0 \quad (50)$$

Let us algebraically prove the first intertwining relation (our proof is similar to the proof presented in [11]):

$$R_{12}(x)R_{13}(y)R_{23}(y/x) = R_{23}(y/x)R_{13}(y)R_{12}(x) \quad (51)$$

It is the Yang-Baxter equation and implies the mutual commutativity of $Q^{(1)}$ operators. Let us write down $R$ in the following form:

$$R_{ab}(x) = P_{ab} \cdot (u_a u_b)^{-lnx/lnq} \cdot \frac{(-zq w_{ab}; q^2)_\infty}{(-z^{-1} q w_{ab}; q^2)_\infty} = P_{ab} \cdot f_{ab}(x) \cdot K(w_{ab}, x) \quad (52)$$

The proof consists of four parts. First we move permutations to the left (the products of permutations in left and right sides are equal) and get:

$$f_{23}(x)K(w_{23}, x)f_{12}(y)K(w_{12}, y)f_{23}(y/x)K(w_{23}, y/x) = f_{12}(y/x)K(w_{12}, y/x)f_{23}(y)K(w_{23}, y)f_{12}(x)K(w_{12}, x) \quad (53)$$

Second we cancel the operator-factors $f$. We notice that:

$$w_{12}u_2^\alpha = q^{-\alpha}u_2^\alpha w_{12}, \ w_{23}u_2^\alpha = q^{\alpha}u_2^\alpha w_{23} \quad (54)$$

Using this properties let us transform (53) to the following form:

$$f_{23}(x)f_{12}(y)f_{23}(y/x)K(w_{23}y^{-1}, x)K(w_{12}y/x, y)K(w_{23}, y/x) = f_{12}(y/x)f_{23}(y)f_{12}(x)K(w_{12}y, y/x)K(w_{23}y^{-1}, y)K(w_{12}, x) \quad (55)$$

It is easy to verify that

$$f_{23}(x)f_{12}(y)f_{23}(y/x) = f_{12}(y/x)f_{23}(y)f_{12}(x) \quad (56)$$

Finally it remains to prove the following:

$$K(w_{23}y^{-1}, x)K(w_{12}y/x, y)K(w_{23}, y/x) = K(w_{12}y, y/x)K(w_{23}y^{-1}, y)K(w_{12}, x) \quad (57)$$

Introducing the new variables: $U = w_{23}, \ V = w_{12}$ we can rewrite (57) in the following form:

$$K(U y^{-1}, x)K(V y/x, y)K(U, y/x) = K(V y, y/x)K(U x^{-1}, y)K(V, x) \quad (58)$$

Introduce the notation:

$$S(U) = (-q U; q^2)_\infty \quad (59)$$

Making use of the threefold Jacobi product:

$$(y; q^2)_\infty(y^{-1}; q^2)_\infty(q^2; q^2)_\infty = \theta_{q^2}(y)$$
Equation (58) transforms to the following form
\[
S((xy)^{-1}U)S(y/xU^{-1})\theta^{-1}(x/yU)S(Vx^{-1})S(xy^{-2}V^{-1}) \\
\theta^{-1}(Vx^{-1}y)S(Ux/y)S(x/yU^{-1})\theta^{-1}(y/xU) = \\
S(xV)\theta^{-1}(y^2/xV)S(xy^{-2}V^{-1})S((xy)^{-1}U)S(xy^{-1}U^{-1}) \\
\theta^{-1}(xy^{-1}U)S(x^{-1}V)S(V^{-1}x^{-1})\theta^{-1}(xV)
\]
(60)

The third step is the cancellation of all theta functions. The main formulae in use are
\[
U\theta(V) = \theta(V)q^{-1}V^{-1}U \Rightarrow f(U)\theta(V) = \theta(V)f(q^{-1}V^{-1}U) \\
U^{-1}\theta(V) = \theta(V)VU^{-1}q^{-1} \Rightarrow f(U^{-1})\theta(V)f(q^{-1}VU^{-1})
\]
(61)

After proper pulling through of theta functions we get the cancellations of all theta functions and
\[
S(q^{-1}yx^{-2}UV)S(q^{-1}y^{-1}U^{-1}V^{-1})S(UV^2q^{-2}yx^{-1}) \\
S(q^2xy^{-3}V^{-2}U^{-1})S(xy^{-2}V^{-1})S(xV) = \\
S(xV)S(y^{-2}xV^{-1})S((xy)^{-1}U) \\
S(xy^{-1}U^{-1})S(q^{-1}yx^{-2}UV)S(q^{-1}y^{-1}U^{-1}V^{-1})
\]
(62)

On the last step we make use of the "pentagon" identity:
\[
S(V)S(U) = S(U)S(q^{-1}UV)S(V).
\]
(63)

Multiple (eight times) use of this identity leads to the fact that (62) is true. The other intertwining relation may be checked analogous way
\[
R_{23}(x/y)\tilde{R}_{12}(x)\tilde{R}_{13}(y) = \tilde{R}_{13}(y)\tilde{R}_{12}(x)R_{23}(x/y)
\]
(64)

5 Conclusion

The results obtained in the present paper are quite formal (except the result for the universal $R$ - matrix of $q$ - DST model). To derive all formulae we use only Weyl-type commutation relations (for generic $\alpha$ and $\beta$)
\[
u^\alpha v^\beta = q^{\alpha\beta}v^\beta u^\alpha.
\]

Nothing have been said about the representation of these operators. Due to the existence of the trace operation in Weyl algebra (actually, we need only $Tr_2 P_{12} = id_1$), it is possible to construct at least the first $Q$ - operator connected with the universal $R$ - matrix. Formally, the trace of monodromy of the second intertwiner $- \tilde{R}$ - matrix, which intertwines the Lax operator and the inverse Lax operator, satisfies the Baxter equation, but suitable trace procedure in this case is unknown. It would be interesting to see the relationship between the $M$ - operators considered in our paper and the similar objects for the Lax operator of Sin-Gordon model. Using $R$ and $\tilde{R}$ matrices,
constructed in our paper one can construct $R$ and $\tilde{R}$ matrices for the product of two Lax operators [3]

$$L_{SG} = L_{2n,a}(kx)L_{2n-1,a}(k^{-1}x).$$  \hfill (65)

However to arrive to the Lax operator of SG model one should impose some constrain on the Weyl operators

$$u_{2n}u_{2n-1}v_{2n}v_{2n-1}^{-1} = 1. \hfill (66)$$

This constrain does not commutate with $R$ and $\tilde{R}$ matrices for the product (65).

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