Busy Beaver Scores and Alphabet Size

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Abstract

We investigate the Busy Beaver Game introduced by Rado (1962) generalized to non-binary alphabets. Harland (2016) conjectured that activity (number of steps) and productivity (number of non-blank symbols) of candidate machines grow as the alphabet size increases. We prove this conjecture for any alphabet size under the condition that the number of states is sufficiently large. For the measure activity we show that increasing the alphabet size from two to three allows an increase. By a classical construction it is even possible to obtain a two-state machine increasing activity and productivity of any machine if we allow an alphabet size depending on the number of states of the original machine. We also show that an increase of the alphabet by a factor of three admits an increase of activity.

1 Introduction

The Busy Beaver Game, as originally defined by Rado [13], is to determine for a given number $n$ of states of deterministic Turing machines over the alphabet $\{0, 1\}$ (0 is the blank symbol) the maximum number of ones produced on an initially blank two-way infinite tape. In each step such a machine reads a tape symbol and depending on the current state writes a symbol, shifts its head one square to the left or to the right, and enters a new state. There is a single halt state (which is traditionally not counted), and on the transition to this state the machine also writes a symbol. What we have just described is sometimes called the quintuple variant of Turing machines in view of the five pieces of information that define a transition. In contrast, the quadruple variant can either move the tape head or write a symbol but not both.

Rado introduced the function $\Sigma(n)$ as the maximum (number of ones produced by machines with $n$ states. The function $S(n)$ denotes the maximum number of steps performed (shift-number) of such machines. He proved that these functions are non-computable and even grow faster than any computable function. Rado also pointed out that these are very simple examples of non-computable functions and that no (explicit) enumeration of computable functions is used in their definition.

The functions are of metamathematical interest as well, since open problems like Goldbach’s conjecture, which can be refuted in a constructive way by a
counterexample, would be settled if \( S(n) \) would be computable for an \( n \) large enough to determine a counterexample by running a Turing machine \[3\] \[4\]. Recently explicit bounds on such an \( n \) have been determined for Goldbach’s conjecture and the Riemann hypothesis along with a Turing machine that cannot be proved to run forever in ZFC \[15\].

Here we consider the generalization of the Busy Beaver Game to alphabets with more than two symbols. As in \[12\] we denote by \( \Sigma(n, m) \) the maximum number of non-blanks produced by any halting deterministic Turing machine with \( n \) states and \( m \) symbols (called productivity) working on an initially blank tape. Similarly, we denote by \( S(n, m) \) the maximum number of steps performed (called activity). Thus the functions defined by Rado are now special cases with \( m = 2 \). For a specific Turing machine \( M \) we denote the two measures by \( \text{productivity}(M) \) and \( \text{activity}(M) \).

A Turing machine \( M \) participating in the generalized Busy Beaver competition can be represented by a table of the form

\[
\begin{array}{cccc}
\text{input symbol} & 0 & 1 & \cdots & m-1 \\
\text{current state} & 1 & & & \\
0 & w_0^1 \delta_0^1 s_0 & w_1^1 \delta_1^1 s_1 & \cdots & w_{m-1}^1 \delta_{m-1}^1 s_{m-1} \\
1 & w_0^2 \delta_0^2 s_0 & w_1^2 \delta_1^2 s_1 & \cdots & w_{m-1}^2 \delta_{m-1}^2 s_{m-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
n & w_0^n \delta_0^n s_0 & w_1^n \delta_1^n s_1 & \cdots & w_{m-1}^n \delta_{m-1}^n s_{m-1} \\
\end{array}
\]

where \( w_i^k \in \{0, 1, \ldots, m-1\} \) indicates the symbol written by \( M \) after reading \( i \) in state \( k \), \( \delta_i^k \in \{L, R\} \) is the direction of the head movement, and \( s_i^k \in \{1, \ldots, n+1\} \) is the new state \( M \) enters. State 1 is the initial state and state \( n+1 \) is the halting state.

As early as 1966, the lower bounds \( \Sigma(3, 3) \geq 12 \) and \( S(3, 3) \geq 57 \) were reported in \[7\] for a non-binary alphabet. Over the following decades, investigations concentrated on computing \( \Sigma(4, 2) \) and on improving lower bounds for larger numbers of states in the classical setting of a binary tape alphabet. The progress in the chase of Busy Beavers is reflected in the following table:

| \( n \) | \( \Sigma(n, 2) \) | \( S(n, 2) \) | references |
|---|---|---|---|
| 1 | 1 | 1 | Rado \[13\] |
| 2 | 4 | 6 | Rado \[13\] |
| 3 | 6 | 21 | Lin, Rado \[8\] |
| 4 | 13 | 107 | Brady \[2\] |
| 5 | \geq 4098 | \geq 47,176,870 | Marxen, Buntrock \[9\] |
| 6 | \geq 3.514 \cdot 10^{18,267} | \geq 7.412 \cdot 10^{69,644} | Kropitz, see \[10\] |

With the exception of lower bounds due to Brady (\( S(2, 3) \geq 38, S(2, 4) \geq 7,195 \)), the search for high scoring machines with more than two symbols did not continue before 2004. As outlined in the survey \[12\], Michel and Brady improved the lower bounds on \( \Sigma(3, 3) \) and \( S(3, 3) \) during that year. Between 2005 and

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1The origin of these bounds communicated to Korfhage by C. Y. Lee of Bell Telephone Lab. is not clear. In \[7\] Lee, Tibor Rado, Shen Lin, Patrick Fischer, Milton Green, and David Jefferson are mentioned in connection with these lower bounds and other early results.
2008 many new machines for non-binary alphabets were found mainly by two teams: Grégory Lafitte and Christophe Papazian, and Terry and Shawn Ligocki (father and son). Lafitte and Papazian could also establish that $\Sigma(2,3) = 9$ and $S(2,3) = 38$, confirming Michel’s conjecture from \[11\] that Brady’s lower bounds dating back almost two decades were tight.

Given the known values and lower bounds for non-binary alphabets, it is natural expect that $\Sigma(n,m)$ and $S(n,m)$ are increasing in both parameters (it is easily shown that they are increasing in their first parameter, see Lemma \[4\] below). An even stronger conjecture was stated by Harland \[5\].

Before presenting our results we cite Harland’s conjecture:

**Conjecture 28 (in [5])** Let $M$ be a $k$-halting Turing machine with $n$ states and $m$ symbols for some $k \geq 1$ with finite activity. Then there is a $k$-halting $n$-state $(m + 1)$-symbol Turing machine $M'$ with finite activity such that

$$\text{activity}(M') > \text{activity}(M) \text{ and productivity}(M') > \text{productivity}(M).$$

Here $k$-halting means that there are $k$ transitions to the halting state.

For $n = 1$, an $n$-state Turing machine has to halt after the first step on a blank in order to have finite activity. As this holds independently of the size of the alphabet, no increase of activity and productivity is possible. We therefore exclude the trivial case $n = 1$.

Notice that the conjecture is stronger than just stating that $\Sigma$ and $S$ are increasing as $m$ grows and $n$ is kept fixed (which it implies by taking highest scoring machines as $M$). The conjecture considers for any specific machine both activity and productivity at the same time. A machine maximizing one of the measures may in fact not maximize the other, as is the case for $n = 3$ where machines with activity 21 produce at most $5 < \Sigma(3)$ ones.

In addition Harland’s conjecture imposes a restriction on the structure of a machine increasing these measures, namely that the number of halting transitions is kept constant for machine $M'$.

Highest scores for small machines still provide evidence in support of the conjecture. We have

$$\Sigma(2,2) = 4 < \Sigma(2,3) = 9 < 2,050 \leq \Sigma(2,4),$$

$$S(2,2) = 6 < S(2,3) = 38 < 3,932,964 \leq S(2,4),$$

$$\Sigma(3,2) = 6 < 374,676,383 \leq \Sigma(3,3),$$

and

$$S(3,2) = 21 < 119,112,334,170,342,540 \leq S(3,3)$$

(results of Rado, Lin, Lafitte, Papazian, T. Ligocki and S. Ligocki, see \[12\] for references).

## 2 Results

It is well-known that activity and productivity grow with the number of states, see the figure on p. 77 of \[9\] or Proposition 27 of \[5\].
Lemma 1 Let $M$ be a Turing machine with $n$ states and $m$ symbols with finite activity. Then there is an $(n+1)$-state $m$-symbol Turing machine $M'$ with finite activity such that $\text{activity}(M') > \text{activity}(M)$ and $\text{productivity}(M') > \text{productivity}(M)$.

The lemma can be proved for any alphabet by redirecting the (unique) halting transition to the new state and having it skip symbols different from the blank while moving the head in one direction. The first blank encountered is replaced with a non-blank and then the machine halts.

An encoding scheme originally developed by Ben-Amram and Petersen [1] and called introspective computing by Luke Schaeffer [15] will be essential in proving Harland’s conjecture for sufficiently large numbers of states.

Theorem 1 For every $m \geq 2$ and $k \geq 1$ there is an $N_{m,k}$ such that for every $k$-halting Turing machine $M$ with $n \geq N_{m,k}$ states and $m$ symbols with finite activity there is an $n$-state, $(m+1)$-symbol $k$-halting Turing machine $M'$ with finite activity such that $\text{activity}(M') > \text{activity}(M)$ and $\text{productivity}(M') > \text{productivity}(M)$.

Proof. Let $M$ be a Turing machine as described in the theorem with $n \geq m$ states. We first notice that w.l.o.g. all $n$ states appear in the unique halting computation of $M$ on the blank tape. For otherwise we omit an unused state $s$ (reducing the number of halting transitions by at most $m$) and redirect all transitions with target $s$ to some remaining state. The resulting Turing machine $\hat{M}$ with $n-1$ states is equivalent to $M$ on a blank tape, since none of the modified transitions is ever reached in the course of the computation. We apply Lemma 1 to $\hat{M}$ resulting in a machine $M'$ with activity($M'$) > activity($M$) and productivity($M'$) > productivity($M$). Since the construction for Lemma 1 preserves the number of halting transitions, it suffices to add at most one halting transition on the new symbol $m$ for each state in order to transform $M'$ into a $k$-halting machine. These transitions will not influence the computation because symbol $m$ is never written onto the tape. In the following we let $N_{m,k} \geq m$.

The next normalization of $M$ is the observation from [1] that in its computation on a blank tape “new” states (states not previously visited) appear in increasing order, i.e., the first state visited and not in the set $\{1, \ldots, s\}$ is $s+1$. This can be achieved by renaming the states appearing in the unique computation of $M$ on a blank tape. A transition followed when a state $s$ is first arrived at is called special, all other transitions are ordinary. Targets of special transitions can be omitted from a description of $M$, as long as there is a flag indicating whether a transition is special. We further note that the number of special transitions is exactly $n$, since by the normalization above all states (including the halt state) are reached.

Finally halting transitions (except the one appearing in the halting computation) are modified, such that they target another state. Obviously this does not influence the computation.

After these transformations, $M$ can be described by the following information:

1. The number $n-1$ in a self-delimiting binary notation, using at most $2 \lceil \log_2 n \rceil$ bits.
2. An array containing \( m(\lceil \log_2 m \rceil + 2) \) bits for every state \( i \in \{1, \ldots, n\} \). These bits correspond to the components (symbol written, head movement, and new state) of a row of the transition table encoding all transitions from a state. The next state is replaced by a flag that is 1 if and only if the transition is special.

3. A list of \( n(m-1) \) destinations of ordinary transitions. The list is sorted according to their first appearance in the computation on a blank tape. A destination can be encoded in \( \lceil \log_2 n \rceil \) bits, since the halting transition is always special and the halting state does not appear in another transition.

In summary, the description of \( M \) requires \( nm[\log_2 n] - n[\log_2 n] + cn \) bits for some constant \( c \) if \( m \) is fixed.

Next we consider the information content of \( n' \) states acting as a ROM in the finite control of a Turing machine with \( m+1 \) symbols. By the technique of introspective computing \([1]\) generalized to \( m+1 \) tape symbols, \( n'm[\log_2 n'] \) bits can be extracted from these states by a fixed extractor machine \( E \) with \( n_E \) states. The extracted bits can be processed by a universal Turing machine \( U \) having \( n_U \) states and simulating machines with \( m \) symbols. As opposed to usual simulators, we let \( U \) write an extra non-blank symbol after it has reached the halting transition of the machine being simulated (notice that this will make sure that activity as well as productivity increase in comparison to \( M \)). A further specific requirement is that \( U \) keeps track of the first appearance of a state and finalizes the transition table according to the flags while simulating a machine. Finally an ordinary universal Turing machine would have exactly one halting transition. In order to satisfy the requirements of Harland’s conjecture we add a sufficient number of (unreachable) states to accommodate \( k-1 \) additional halting transitions.

We let \( d = n_E + n_U \), \( n' = n - d \) and observe that \( n'm[\log_2 n'] = (n-d)m[\log_2 n-d] \geq (n-d)m(\lceil \log_2 n \rceil -1) \geq nm[\log_2 n] - dm[\log_2 n] - nm + dm \geq \text{nm}[\log_2 n] - dm[\log_2 n] - 2nm + dm \geq \text{nm}[\log_2 n] - n[\log_2 n] + cn \) for \( n \geq N_{m,k} \). Therefore \( n' \) states suffice to encode \( M \).

Finally we compose the Turing machine over \( m+1 \) symbols with \( n' \) states encoding machine \( M \), the extractor \( E \), and the universal Turing machine \( U \) to obtain machine \( M' \) with \( n \) states simulating \( M \) and satisfying the theorem. \( \square \)

Next we consider weaker versions of Harland’s conjecture. But first we show some technical Lemmas.

**Lemma 2** For all \( n, m \geq 2 \) we have \( S(n, m) > n \)

Proof. \( S(2,2) = 6 > 2 \), Suppose \( S(n,2) > n \) for some \( n \geq 2 \). By Lemma[1] we get \( S(n+1,2) > n + 1 \) and \( S(n,m) \geq S(n,2) > n \) by adding transitions on \( m-2 \) symbols for a two-symbol champion. \( \square \)

**Lemma 3** If all transitions of Turing machine \( M \) with \( n \) states on the blank move the head in the same direction and \( M \) has finite activity, then we have \( \text{activity}(M) \leq n \).

Proof. If \( M \) makes more than \( n \) steps in one direction, then a state repeats and \( M \) does not stop. \( \square \)
The next result is inspired by the construction in Figure 14 of [5]. In contrast to Theorem 1 it does not preserve the number of halting transitions.

**Theorem 2** For every Turing machine $M$ with $n \geq 2$ states and two symbols having finite activity there is an $n$-state, three-symbol Turing machine $M'$ with finite activity such that $\text{activity}(M') > \text{activity}(M)$.

Proof. Without loss of generality $M$ has maximum activity among all $n$ state, two symbol Turing machines and the first transition of $M$ moves the head to the right.

We let $M'$ have the basic structure of $M$ and add transitions on the new (third) symbol to every state. For a state $s$ to be determined below this transition is halting, while the other transitions are non-halting and can otherwise be arbitrary, since they never will be used.

Consider the tape cell $i$ at the final position of the head in the computation of $M$ on a blank tape. We modify the halting transition taken by $M$ to write the new symbol and move the head depending on the symbols in neighboring cells of $i$.

If cell $i - 1$ contains a blank, we modify the halting transition to move left and go to the initial state. By the normalization of the first transition, $M'$ will move right on the blank (it cannot halt due to Lemma 2 to a state which is chosen as $s$). Then $M$ halts on the new symbol increasing activity by two.

If $i - 1$ contains 1 and there is a state with a transition moving right on 1, we modify the last transition to move left and go to such a state. This will increase activity by one if the transition moving right on 1 is halting, in which case we chosen the current state as $s$. Otherwise activity increases by two as in the previous case if $M'$ returns to cell $i$ in a state chosen as $s$.

If all transitions move left on 1, we consider tape cell $i + 1$. If it contains 1, we modify the halting transition to move right and go to an arbitrary state. Machine $M'$ will either halt immediately or return to cell $i$ in a state chosen as $s$ and halt.

Finally consider a blank in cell $i + 1$. Since for $n \geq 2$ there is a machine with activity exceeding $n$ by Lemma 2, we conclude from Lemma 3 that at least one transition moves the head left on a blank. Go to a state with such a transition and move the head to the right. The resulting Turing machine will halt either when reading cell $i + 1$ or when it returns to cell $i$ in a state chosen as $s$.

In each case $\text{activity}(M') \in \{\text{activity}(M) + 1, \text{activity}(M) + 2\}$. \qed

Next we turn to constructions that increase the alphabet by more than one symbol.

**Theorem 3** For every Turing machine $M$ with $n \geq 2$ states and $m \geq 2$ symbols having finite activity there is a 2-state, $(4nm + 5m)$-symbol Turing machine $M'$ with finite activity such that $\text{activity}(M') > \text{activity}(M)$ and $\text{productivity}(M') > \text{productivity}(M)$.

Proof. Let $M$ be a Turing machine with $n$ states and $m$ symbols. By Lemma 1 there is a machine $M'$ with $n + 1$ states and $m$ symbols increasing activity and productivity. The classical construction from [14] transforms it into an equivalent 2-state machine with $4m(n + 1) + m = 4nm + 5m$ symbols. \qed
Theorem 4 For every Turing machine $M$ with $n \geq 2$ states and $m \geq 3$ symbols having finite activity there is an $n$-state, $3m$-symbol Turing machine $M'$ with finite activity such that $\text{activity}(M') > \text{activity}(M)$.

Proof. If among the Turing machines with $n$ states and $m$ symbols $M$ does not have maximum activity, we choose as $M'$ such a machine and no increase of the tape alphabet is necessary.

Otherwise for every symbol $a$ of $M$ we add new symbols $a_L$ and $a_R$ to the transition table of $M'$. A transition of $M$ on an old symbol writing $a$ is modified to write $a_R$ if it moves the head to the left (indicating that $a_R$ is to the right of the tape head) and similarly $a_L$ if it moves the head to the right. On new symbols $a_R$ and $a_L$ machine $M'$ replaces the new symbol with $a$ and “bounces” back to the right if the symbol was $a_L$ and to the left on $a_R$. Observe that all symbols with subscript $L$ are to the left of the tape head or under it and all symbols with subscript $R$ are to the right of the tape head or under it in the course of the computation of $M'$.

Consider the homomorphism $h$ defined by $h(a) = h(a_L) = h(a_R) = a$ for all symbols of $M$. We claim that for every instantaneous description of $M$ at step $k$ with a tape inscription $w$ of cells visited by $M$ and its head on cell $i$ there is an instantaneous description of $M'$ at step $k' \geq k$ with a tape inscription $w'$ satisfying $h(w) = h(w')$ with its head on cell $i$. This clearly holds for step 0 when there are no modified cells. If $M'$ reads an old symbol $a$ it writes some $b_R$ or $b_L$ while $M$ writes $b$ and both move their heads in the same direction. This clearly maintains the property $h(w) = h(w')$ and that the head positions correspond. If $M'$ reads $a_L$ it has just moved its head left and the neighboring cell contains some symbol $b_R$. Now $M'$ writes $a$, moves right, replaces $b_R$ with $b$, and returns to $a$. In comparison to $M$ two additional steps have been performed while $h(a_L b_R) = h(ab)$. In the same way $M'$ behaves on $a_R$. We conclude that $M'$ halts if and only $M$ does and $\text{activity}(M') \geq \text{activity}(M)$.

To see that $\text{activity}(M') > \text{activity}(M)$ we make use of the assumption that $M$ has maximum activity among the Turing machines with $n$ states and $m$ symbols. By Lemma 2 and Lemma 3 $M'$ has to make at least one turn, which adds at least two steps to the computation of $M'$ in comparison to $M$.  

\square

3 Discussion

We have partially proved Harland’s conjecture. It holds for $n$ sufficiently large and (restricted to the measure activity and without maintaining the number of halting transitions) for $m = 2$. An increase of the alphabet size exceeding one admits similar results for all $n$. In the former construction we have used the technique of interpretation instead of instrumentation (in terms of [1]).

If the Harland’s conjecture is true in general, then it provides further evidence for the symmetry of symbols and states discussed by Shannon in the concluding remarks of [14], since an increase in one of the parameters adds power to the machines.
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