The Hardy–Littlewood Maximal Operator on Discrete Morrey Spaces

Hendra Gunawan and Christopher Schwanke

Abstract. We discuss the Hardy–Littlewood maximal operator on discrete Morrey spaces of arbitrary dimension. In particular, we obtain its boundedness on the discrete Morrey spaces using a discrete version of the Fefferman–Stein inequality. As a corollary, we also obtain the boundedness of some Riesz potentials on discrete Morrey spaces.

Mathematics Subject Classification. 42B35, 46B45, 46A45.

Keywords. Discrete Morrey spaces, Hardy–Littlewood maximal operator, Riesz potential.

1. Introduction

While the Hardy–Littlewood maximal operator is well known, discrete Morrey spaces were only studied recently in [5] (see also [1] for related works). In this paper, we investigate the boundedness of the (discrete) Hardy–Littlewood maximal operator on discrete Morrey spaces of arbitrary dimension. Some important properties of this operator (and many others) on the $\ell^p(\mathbb{Z}^d)$ spaces were discussed in [9]. See also [6,10–13] for related works on discrete analogues in harmonic analysis.

The boundedness of the (continuous) Hardy–Littlewood maximal operator on the (continuous) Morrey spaces was first studied in [2], whose results were later extended in [7,8] to some generalizations of Morrey spaces. The driving force behind the results in [2,7,8] is the so-called Fefferman–Stein inequality [4, Lemma 1], a result specifically regarding integrable functions defined on $\mathbb{R}^d$. We illustrate in Theorem 2.1 how this inequality, despite its reliance on various tools only available in the continuous setting, can be transformed into a natural discrete analogue. As a consequence of Theorem 2.1, we obtain the boundedness of the discrete Hardy–Littlewood maximal operator on the discrete Morrey spaces in Theorem 3.2.

We begin with some notation and definitions. First, we set $\omega := \mathbb{N} \cup \{0\}$ and use this notation throughout the paper. For $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and $N \in \omega$ define

$$S_{m,N} := \{k \in \mathbb{Z}^d : \|k - m\|_\infty \leq N\},$$
where as usual $\| (x_1, \ldots, x_d) \|_\infty := \max\{|x_i| : 1 \leq i \leq d\}$ for every $(x_1, \ldots, x_d) \in \mathbb{R}^d$. Again following standard conventions, we denote the cardinality of a set $S$ by $|S|$. Then we have $|S_{m,N}| = (2N + 1)^d$ for all $m \in \mathbb{Z}^d$ and each $N \in \omega$. Given $1 \leq p \leq q < \infty$ we define the discrete Morrey space $\ell^p_q(\mathbb{Z}^d)$ to be the space of all functions $x : \mathbb{Z}^d \rightarrow \mathbb{R}$ for which

$$\|x\|_{\ell^p_q(\mathbb{Z}^d)} := \sup_{m \in \mathbb{Z}^d, N \in \omega} |S_{m,N}|^{\frac{1}{q} - \frac{1}{p}} \left( \sum_{k \in S_{m,N}} |x(k)|^p \right)^{1/p} < \infty.$$  

By following the proof of [5, Proposition 2.2], one can readily prove that $\| \cdot \|_{\ell^p_q(\mathbb{Z}^d)}$ defines a norm on $\ell^p_q(\mathbb{Z}^d)$ and that $\ell^p_q(\mathbb{Z}^d)$ is a Banach space with respect to this norm. Indeed, [5, Proposition 2.2] proves the given result for $d = 1$, and its proof is easily adaptable to higher dimensions.

We wish to study the (discrete) Hardy–Littlewood maximal operator on these discrete Morrey spaces of arbitrary dimension. To begin, define the discrete Hardy–Littlewood maximal operator (or for emphasis, the “odd” discrete Hardy–Littlewood maximal operator) $M$ by

$$Mx(m) := \sup_{N \in \omega} \frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x(k)| \quad (x \in \mathbb{R}^{\mathbb{Z}^d}, m \in \mathbb{Z}^d).$$

The operator $M$ is a discrete analogue of the “centered continuous” Hardy–Littlewood maximal operator, which is defined by

$$Mf(y) := \sup_{r > 0} \frac{1}{(2r)^d} \int_{Q_{y,r}} |f(z)|dz \quad (f \in L^1_{\mathrm{loc}}(\mathbb{R}^d), y \in \mathbb{R}^d),$$

where $Q_{y,r} := \{t \in \mathbb{R}^d : \|t - y\|_{\infty} \leq r\}$. While the “odd” discrete Hardy–Littlewood maximal operator will be our primary interest, the following rendition of this function will prove useful in obtaining a discrete analogue of the Fefferman–Stein inequality.

The “even” discrete Hardy–Littlewood maximal operator $\hat{M}$ is defined for $x \in \mathbb{R}^{\mathbb{Z}^d}$ and $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ by

$$\hat{M}x(m) := \sup_{N \in \mathbb{N}} \frac{1}{|R_{m,N}|} \sum_{k \in R_{m,N}} |x(k)|,$$

where

$$R_{m,N} := S_{m,N} \setminus \{(k_1, \ldots, k_d) \in \mathbb{Z}^d : k_i = m_i + N \text{ for some } 1 \leq i \leq d\},$$

so that $|R_{m,N}| = (2N)^d$. Additionally, we define the “uncentered” discrete Hardy–Littlewood maximal operator $\tilde{M}$ by

$$\tilde{M}x(m) := \sup_{S \ni m} \frac{1}{|S|} \sum_{k \in S} |x(k)| \quad (x \in \mathbb{R}^{\mathbb{Z}^d}, m \in \mathbb{Z}^d),$$

where the supremum above is taken over all sets of the form $S = S_{k,N}$, for some $k \in \mathbb{Z}^d$ and $N \in \omega$, that contain $m$. 


We say that two operators $T_1, T_2 : \mathbb{R}^d \to \mathbb{R}^d$ are equivalent if there exist $C_1, C_2 > 0$ such that $C_1 T_1 x(k) \leq T_2 x(k) \leq C_2 T_1 x(k)$ hold for all $x \in \mathbb{R}^d$ and every $k \in \mathbb{Z}^d$. Regarding the operators $M, \hat{M}$, and $\tilde{M}$, we have the following lemma which will be useful in our discussion in the next sections. We leave its proof to the reader.

**Lemma 1.1.** The operators $M, \hat{M}$, and $\tilde{M}$ are pairwise equivalent.

## 2. The Discrete Fefferman–Stein Inequality

In this section, we provide a discrete version of the Fefferman–Stein inequality [4, Lemma 1]. Theorem 2.1 below will be used to obtain the boundedness of the discrete Hardy–Littlewood maximal operator on discrete Morrey spaces in Sect. 3.

We call a function $x \in \mathbb{R}^d$ positive if $x(k) \geq 0$ for each $k \in \mathbb{Z}^d$. Given $A \subseteq \mathbb{R}^d$, we denote the characteristic function of $A$ by $\chi_A$.

**Theorem 2.1.** Let $1 < p < \infty$. There exists $K > 0$ such that for all $x \in \mathbb{R}^d$ and each positive $\phi \in \mathbb{R}^d$,

1. \( \sum_{k \in \mathbb{Z}^d} (M x(k))^p \phi(k) \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p M \phi(k), \)
2. \( \sum_{k \in \mathbb{Z}^d} (\hat{M} x(k))^p \phi(k) \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p \hat{M} \phi(k), \) and
3. \( \sum_{k \in \mathbb{Z}^d} (\tilde{M} x(k))^p \phi(k) \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p \tilde{M} \phi(k). \)

**Proof.** We prove statement (2), from which statements (1) and (3) will follow from Lemma 1.1. For each $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ define the $d$-dimensional cube of volume one by

\[ C_k := \{(y_1, \ldots, y_d) \in \mathbb{R}^d : k_i \leq y_i < k_i + 1 \text{ for all } 1 \leq i \leq d \}. \]

Next note that for every $a \in \mathbb{R}^d$ the function $\bar{a}$ defined by

\[ \bar{a}(t) := \sum_{k \in \mathbb{Z}^d} a(k) \chi_{C_k}(t) \quad (t \in \mathbb{R}^d) \]

is a member of $L^1_{\text{loc}}(\mathbb{R}^d)$ and

\[ \sum_{k \in R_{m,N}} a(k) = \int_{Q_{m,N}} \bar{a}(t)dt \quad (m \in \mathbb{Z}^d, \ N \in \omega), \]

where again $Q_{m,N} = \{t \in \mathbb{R}^d : \|t - m\|_\infty \leq N\}$. Therefore, we have

\[ \sum_{k \in \mathbb{Z}^d} a(k) = \int_{\mathbb{R}^d} \bar{a}(t)dt. \]

Next let $x, \phi \in \mathbb{R}^d$, and suppose $\phi$ is positive. Then, $\bar{\phi}(t) \geq 0$ for every $t \in \mathbb{R}^d$, and in light of the remarks above,

\[ \sum_{k \in \mathbb{Z}^d} (\hat{M} x(k))^p \phi(k) = \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \left( \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{i \in R_{k,N}} |x(i)| \right)^p \phi(k) \chi_{C_k}(t)dt. \]
Furthermore,
\[ \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} \left( \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{i \in R_{k,N}} |x(i)| \right)^p \phi(k) \chi_{C_k}(t) \, dt \]
\[ = \sum_{k \in \mathbb{Z}^d} \int_{C_k} \left( \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \int_{Q_{k,N}} |\bar{x}(s)| \, ds \right)^p \phi(k) \, dt. \]

Note that for each \( k \in \mathbb{Z}^d \) and all \( t \in C_k \) we have
\[ \int_{Q_{k,N}} |\bar{x}(s)| \, ds \leq \int_{Q_{t,N+1}} |\bar{x}(s)| \, ds. \]

Hence
\[ \sum_{k \in \mathbb{Z}^d} \int_{C_k} \left( \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \int_{Q_{k,N}} |\bar{x}(s)| \, ds \right)^p \phi(k) \, dt \]
\[ \leq 2^d \sum_{k \in \mathbb{Z}^d} \int_{C_k} \left( \sup_{N \in \mathbb{N}} \frac{1}{(2(N+1))^d} \int_{Q_{t,N+1}} |\bar{x}(s)| \, ds \right)^p \phi(k) \, dt \]
\[ \leq 2^d \int_{\mathbb{R}^d} (\bar{M}\bar{x}(t))^p \bar{\phi}(t) \, dt. \]

By the Fefferman–Stein inequality \cite[Lemma 1]{4}, there exists \( K > 0 \) such that
\[ \int_{\mathbb{R}^d} (\bar{M}\bar{x}(t))^p \bar{\phi}(t) \, dt \leq K \int_{\mathbb{R}^d} |\bar{x}(t)|^p \bar{M}\phi(t) \, dt. \]

Thus, we obtain
\[ \sum_{k \in \mathbb{Z}^d} (\bar{M}x(k))^p \phi(k) \leq 2^d K \int_{\mathbb{R}^d} |\bar{x}(t)|^p \bar{M}\phi(t) \, dt \]
\[ = 2^d K \sum_{k \in \mathbb{Z}^d} \int_{C_k} |x(k)|^p \left( \sup_{r > 0} \frac{1}{(2r)^d} \int_{Q_{t,r}} \bar{\phi}(s) \, ds \right) \, dt. \]

Next let \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \), and put \( t = (t_1, \ldots, t_d) \in C_k \), so that \( k_i = \lfloor t_i \rfloor \) for each \( i \in \{1, \ldots, d\} \). Suppose that \( 0 < r < \frac{1}{2} \), and let \( D = \{ i \in \mathbb{Z}^d : Q_{t,r} \cap C_i \neq \emptyset \} \).

Note that \( D \neq \emptyset \), as \( k \in D \). Moreover,
\[ \frac{1}{(2r)^d} \int_{Q_{t,r}} \bar{\phi}(s) \, ds = \frac{1}{(2r)^d} \sum_{i \in D} \phi(i) \epsilon_1 \cdots \epsilon_d, \]
where for each \( j \in \{1, \ldots, d\} \) we have either
\[ \epsilon_j = |k_j + 1 - t_j| + r \]
or
\[ \epsilon_j = 2r. \]

Observing that
\[ |k_j + 1 - t_j| \leq r \]
holds for each \( j \in \{1, \ldots, d\} \), we obtain
\[
\frac{1}{(2r)^d} \sum_{i \in D} \phi(i) \epsilon_1 \cdots \epsilon_d \leq \sum_{i \in D} \phi(i) \leq 4^d \frac{1}{(2 \cdot 2)^d} \int_{Q_{k,2}} \tilde{\phi}(s) \, ds.
\]
Also, for \( r \geq \frac{1}{2} \) we get
\[
\frac{1}{(2r)^d} \int_{Q_{k,r}} \tilde{\phi}(s) \, ds \leq \frac{5^d}{(2 \lceil r + 1 \rceil)^d} \int_{Q_{k,\lceil r + 1 \rceil}} \tilde{\phi}(s) \, ds.
\]
Thus
\[
\sum_{k \in \mathbb{Z}^d} \int_{C_k} |x(k)|^p \left( \sup_{r>0} \frac{1}{(2r)^d} \int_{Q_{t,r}} \tilde{\phi}(s) \, ds \right) \, dt
\leq 5^d \sum_{k \in \mathbb{Z}^d} |x(k)|^p \left( \sup_{r>0} \frac{1}{(2 \lceil r + 1 \rceil)^d} \int_{Q_{k,\lceil r + 1 \rceil}} \tilde{\phi}(s) \, ds \right)
\leq 5^d \sum_{k \in \mathbb{Z}^d} |x(k)|^p \left( \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{i \in R_{k,N}} \phi(i) \right)
\leq 5^d \sum_{k \in \mathbb{Z}^d} |x(k)|^p \hat{M}\phi(k).
\]
Therefore,
\[
\sum_{k \in \mathbb{Z}^d} (\hat{M}x(k))^p \phi(k) \leq (10)^d K \sum_{k \in \mathbb{Z}^d} |x(k)|^p \hat{M}\phi(k),
\]
as desired. \( \square \)

3. The Boundedness of the Discrete Maximal Operator

We use the methods of Chiarenza and Frasca [2] in this section, and we additionally require the following lemma.

**Lemma 3.1.** Let \( 1 \leq p \leq q < \infty \). For any \( x \in \ell^p_q(\mathbb{Z}^d) \) we have \( Mx \in \ell^\infty(\mathbb{Z}^d) \) and \( \|Mx\|_{\ell^\infty(\mathbb{Z}^d)} \leq \|x\|_{\ell^p_q(\mathbb{Z}^d)} \).

**Proof.** Let \( x \in \ell^p_q(\mathbb{Z}^d) \), and put \( m^* \in \mathbb{Z}^d \). Then
\[
Mx(m^*) = \sup_{N \in \omega} \frac{1}{(2N + 1)^d} \sum_{k \in S_{m^*,N}} |x(k)| \leq \frac{1}{(2N + 1)^d} \sum_{k \in S_{m,N}} |x(k)|.
\]
In [5, Lemma 2.3], it is shown for \( d = 1 \) that for any \( m \in \mathbb{Z} \) and \( N \in \omega \),
\[
\frac{1}{(2N + 1)^d} \sum_{k \in S_{m,N}} |x(k)| \leq \left( \frac{1}{(2N + 1)^d} \sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}}.
\]
One may readily check that the proof of [5, Lemma 2.3] also holds for general \( d \in \mathbb{N} \). Hence
By Theorem 2.1 (1) there exists $m \geq 24$

[Page 6 of 12] H. Gunawan and C. Schwanke MJOM

It follows from (1) and (2) above that

which proves the lemma. □

Theorem 3.2. Let $1 < p \leq q < \infty$. For all $x \in \ell_q^p(\mathbb{Z}^d)$ we have $Mx \in \ell_q^p(\mathbb{Z}^d)$, and there exists $C > 0$ such that $\|Mx\|_{\ell_q^p(\mathbb{Z}^d)} \leq C\|x\|_{\ell_q^p(\mathbb{Z}^d)}$ holds for all $x \in \ell_q^p(\mathbb{Z}^d)$.

Proof. Let $m \in \mathbb{Z}^d$, and put $N \in \mathbb{N}$ (the case $N = 0$ will be handled later).

By Theorem 2.1 (1) there exists $K > 0$ such that

$$\sum_{k \in \mathbb{Z}^d} (Mx(k))^p \chi_{S_{m,N}}(k) \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p M \chi_{S_{m,N}}(k),$$

and thus

$$\sum_{k \in S_{m,N}} (Mx(k))^p \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p M \chi_{S_{m,N}}(k)$$

$$= K \sum_{k \in S_{m,2N}} |x(k)|^p M \chi_{S_{m,N}}(k)$$

$$+ K \sum_{j=1}^{\infty} \sum_{k \in S_{m,2j+1N \setminus S_{m,2j}N}} |x(k)|^p M \chi_{S_{m,N}}(k).$$

Next note that for every $k \in \mathbb{Z}^d$ we have

$$M \chi_{S_{m,N}}(k) = \sup_{t \in \omega} \frac{1}{(2t + 1)^d} \sum_{i \in S_{k,t}} \chi_{S_{m,N}}(i) = \sup_{t \in \omega} \frac{1}{(2t + 1)^d} |S_{k,t \cap S_{m,N}}|.$$

Let $j \in \mathbb{N}$, and assume $k \in S_{m,2j+1N \setminus S_{m,2j}N}$. Then, $\|k - m\|_{\infty} - N > 0$. Now observe that

1. $S_{k,t \cap S_{m,N}} \neq \emptyset$ if and only if $\|k - m\|_{\infty} \leq t + N$,

that is $t \geq \|k - m\|_{\infty} - N$, and

2. $S_{k,t \cap S_{m,N}} = S_{m,N}$ when $\|k - m\|_{\infty} \leq t - N$,

that is when $t \geq \|k - m\|_{\infty} + N$.

It follows from (1) and (2) above that

$$\sup_{t \in \omega} \frac{1}{(2t + 1)^d} |S_{k,t \cap S_{m,N}}| = \sup_{\|k - m\|_{\infty} - N \leq t \leq \|k - m\|_{\infty} + N} \frac{1}{(2t + 1)^d} |S_{k,t \cap S_{m,N}}|$$

$$\leq \frac{(2N + 1)^d}{2(\|k - m\|_{\infty} - N + 1)^d}$$

$$\leq \left(\frac{3}{2}\right)^d \frac{N^d}{(\|k - m\|_{\infty} - N)^d}.$$
Also observe that for every $k \in \mathbb{Z}^d$ we have $M\chi_{S_m,N}(k) \leq M1(k) = 1$, where $1$ denotes the constant function on $\mathbb{Z}^d$ taking value one. Hence

\[
K \sum_{k \in S_m,2N} |x(k)|^p M\chi_{S_m,N}(k) + K \sum_{j=1}^{\infty} \sum_{k \in S_m,2j+1} |x(k)|^p M\chi_{S_m,N}(k) \\
\leq K \sum_{k \in S_m,2N} |x(k)|^p + \left(\frac{3}{2}\right)^d \sum_{j=1}^{\infty} \sum_{k \in S_m,2j+1} |x(k)|^p \frac{N^d}{(\|k-m\|_\infty - N)^d}.
\]

Now if $k \in S_m,2j+1 \setminus S_m,2j$ then $\|k-m\|_\infty - N > 2^j N - N \geq 2^{j-1} N$. Thus

\[
K \sum_{k \in S_m,2N} |x(k)|^p + \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \sum_{k \in S_m,2j+1} |x(k)|^p \frac{N^d}{(2^j N)^d} \\
= K \sum_{k \in S_m,2N} |x(k)|^p + \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{1}{(2^d)^{j-1}} \sum_{k \in S_m,2j+1} |x(k)|^p.
\]

Next observe that for every $t \in \mathbb{Z}^d$ and all $n \in \omega$ we have

\[
\sum_{k \in S_{t,n}} |x(k)|^p \leq \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2n + 1)^{d - \frac{dp}{q}}.
\]

Hence

\[
K \sum_{k \in S_m,2N} |x(k)|^p + \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{1}{(2^d)^{j-1}} \sum_{k \in S_m,2j+1} |x(k)|^p \\
\leq K \|x\|_{L^q_p(\mathbb{Z}^d)}^p (4N + 1)^{d - \frac{dp}{q}} \\
+ \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{1}{(2^d)^{j-1}} \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2^{j+2} N + 1)^{d - \frac{dp}{q}} \\
\leq 2^{d - \frac{dp}{q}} K \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2N + 1)^{d - \frac{dp}{q}} \\
+ \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{(2^{j+1})^{d - \frac{dp}{q}}}{(2^d)^{j-1}} \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2N + 1)^{d - \frac{dp}{q}} \\
= 2^{d - \frac{dp}{q}} K \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2N + 1)^{d - \frac{dp}{q}} \\
+ 3^d (2^{d - \frac{dp}{q}}) \left(\frac{1}{1 - 2^{-\frac{dp}{q}}} - 1\right) K \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2N + 1)^{d - \frac{dp}{q}} \\
\leq C \|x\|_{L^q_p(\mathbb{Z}^d)}^p (2N + 1)^{d - \frac{dp}{q}},
\]
where \( C_2 = \left( 2^d - \frac{dp}{q} \right) K \vee \left( 3^d \left( 2^d - \frac{dp}{q} \right) \left( \frac{1}{1-2^{\frac{dp}{q}}} - 1 \right) K \right) \). Thus \( m \in \mathbb{Z}^d \) and \( N \in \mathbb{N}, \)

\[
(2N + 1)^{\frac{d}{q} - \frac{d}{p}} \left( \sum_{k \in S_{m,N}} (Mx(k))^p \right)^{\frac{1}{p}} \leq C^{1/p} \|x\|_{\ell_q^p(\mathbb{Z}^d)}.
\]

That this inequality holds for \( N = 0 \) (with \( C = 1 \)) follows from Lemma 3.1. Therefore,

\[
\|Mx\|_{\ell_q^p(\mathbb{Z}^d)} \leq (C^{1/p} \vee 1) \|x\|_{\ell_q^p(\mathbb{Z}^d)},
\]

which completes the proof. \( \square \)

As an application of Theorem 3.2, we obtain the boundedness of some Riesz potentials on discrete Morrey spaces.

**Theorem 3.3.** Let \( 0 < \alpha < d \) and \( 1 < p < q < \frac{d}{\alpha} \). Define

\[
I_\alpha x(k) = \sum_{i \in \mathbb{Z}^d \setminus \{k\}} \frac{x(i)}{\|k - i\|_{\infty}^{\frac{d}{\alpha}}} \quad (x \in \ell_q^p(\mathbb{Z}^d), \ k \in \mathbb{Z}^d).
\]

Set \( s = \frac{dp}{d - \alpha q} \) and \( t = \frac{qs}{p} \). Then \( I_\alpha x \in \ell_t^s(\mathbb{Z}^d) \) for every \( x \in \ell_q^p(\mathbb{Z}^d) \), and there exists a \( C > 0 \) such that

\[
\|I_\alpha x\|_{\ell_t^s(\mathbb{Z}^d)} \leq C \|x\|_{\ell_q^p(\mathbb{Z}^d)} \quad (x \in \ell_q^p(\mathbb{Z}^d)).
\]

**Proof.** Let \( x \in \ell_q^p(\mathbb{Z}^d) \), and let \( m \in \mathbb{Z}^d \). Then

\[
Mx(m) = \sup_{N \in \omega} \frac{1}{(2N + 1)^d} \sum_{k \in S_{m,N}} |x(k)|
\]

\[
\leq \|x(m)\| \vee \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{k \in S_{m,N}} |x(k)|
\]

\[
\leq 2^d \left( \frac{1}{2^d} \sum_{k \in S_{m,1}} |x(m)| \vee \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{k \in S_{m,N}} |x(k)| \right)
\]

\[
\leq 2^d \sup_{r \geq 1} \frac{1}{(2r)^d} \sum_{k \in \mathbb{Z}^d, \|m - k\|_{\infty} \leq r} |x(k)|.
\]

On the other hand,

\[
\sup_{r \geq 1} \frac{1}{(2r)^d} \sum_{k \in \mathbb{Z}^d, \|m - k\|_{\infty} \leq r} |x(k)| \leq \sup_{r \geq 1} \frac{1}{(2|r|)^d} \sum_{k \in \mathbb{Z}^d, \|m - k\|_{\infty} \leq |r|} |x(k)|
\]

\[
= \sup_{N \in \mathbb{N}} \frac{1}{(2N)^d} \sum_{k \in S_{m,N}} |x(k)|
\]

\[
\leq \left( \frac{3}{2} \right)^d Mx(m).
\]
Thus
\[
\left(\frac{2}{3}\right)^d \sup_{r \geq 1} \frac{1}{(2r)^d} \sum_{k \in \mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)| \leq Mx(m) \leq 2^d \sup_{r \geq 1} \frac{1}{(2r)^d} \sum_{k \in \mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)|.
\]

(2)

Next let \( r \geq 1 \), and put \( k \in \mathbb{Z}^d \). Then
\[
I_\alpha x(k) = \sum_{i \in \mathbb{Z}^d \setminus \{k\}} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} = \sum_{0 < \|k-i\|_\infty \leq r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} + \sum_{\|k-i\|_\infty > r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}}.
\]

Define
\[
I_1 := \sum_{0 < \|k-i\|_\infty \leq r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} \quad \text{and} \quad I_2 := \sum_{\|k-i\|_\infty > r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}}.
\]

Then
\[
|I_1| \leq \sum_{j=0}^{\infty} r^{2^{-j}-1} \sum_{r^{-2^{-j}} \leq \|k-i\|_\infty \leq r^{2^{-j}}} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}}.
\]

If \( \|k-i\|_\infty > r^{2^{-j}-1} \) then \( \|k-i\|_\infty^{\alpha-d} < r^{\alpha-d} 2^{-j\alpha} + j d - \alpha + d \). Thus
\[
\sum_{j=0}^{\infty} r^{2^{-j}-1} \sum_{r^{-2^{-j}} \leq \|k-i\|_\infty \leq r^{2^{-j}}} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}}
\leq \sum_{j=0}^{\infty} r^{\alpha-2^{d-\alpha}} \sum_{j=0}^{\infty} \sum_{\|k-i\|_\infty \leq r^{2^{-j}}} 2^{-j\alpha} \frac{1}{(r^{2^{-j}})^d} \sum_{0 < \|k-i\|_\infty \leq r^{2^{-j}}} |x(i)|.
\]

Let \( J = \max\{j \in \omega : r^{2^{-j}} \geq 1\} \). Since \( \sum_{0 < \|k-i\|_\infty \leq r^{2^{-j}}} |x(i)| \) is an empty sum for all \( j > J \), we have
\[
r^{\alpha-2^{d-\alpha}} \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(r^{2^{-j}})^d} \sum_{0 < \|k-i\|_\infty \leq r^{2^{-j}}} |x(i)| = r^{\alpha-2^{d-\alpha}} \sum_{j=0}^{J} \sum_{j=0}^{\infty} \frac{1}{(r^{2^{-j}})^d} \sum_{0 < \|k-i\|_\infty \leq r^{2^{-j}}} |x(i)|.
\]

Using (2), there exists a constant \( C_0 \) (for brevity, we do not record the precise value of this constant) for which
\[
r^{\alpha-2^{d-\alpha}} \sum_{j=0}^{J} \sum_{j=0}^{\infty} \frac{1}{(r^{2^{-j}})^d} \sum_{0 < \|k-i\|_\infty \leq r^{2^{-j}}} |x(i)| \leq C_0 r^{\alpha} Mx(k).
\]

Next note that
\[
|I_2| \leq \sum_{j=0}^{\infty} \sum_{2^jr < \|k-i\|_\infty \leq 2^{j+1}r} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}}.
\]

If \( \|k-i\|_\infty > 2^jr \) then \( \|k-i\|_\infty^{\alpha-d} < (2^jr)^{\alpha-d} \). Hence we obtain
\[
\sum_{j=0}^{\infty} \left( 2^j r \right)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( \sum_{\|k-i\|_{\infty} \leq 2^{j+1} r} |x(i)| \right)^{\frac{1}{q}} \leq \sum_{j=0}^{\infty} (2^j r)^{\alpha - d} \left( \sum_{\|k-i\|_{\infty} \leq 2^{j+1} r} |x(i)| \right)^{\frac{1}{q}}
\]

where we use Hölder’s inequality in the inequality above with reference to the Hölder conjugate \( p' \) of \( p \). Moreover, we have

\[
\sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( \sum_{\|k-i\|_{\infty} \leq 2^{j+1} r} |x(i)| \right)^{\frac{1}{q}} \leq \sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( 2^{j+2} r + 1 \right)^{\frac{1}{p'}} \left( \sum_{\|k-i\|_{\infty} \leq 2^{j+1} |r|} |x(i)| \right)^{\frac{1}{p}}
\]

\[
\leq 8^{\frac{d}{p} - \frac{d}{q}} \sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( 2^{j+2} r + 1 \right)^{\frac{1}{p'}} \left( \sum_{r \in S_{k,2^{j+1} |r|}} |x(i)| \right)^{\frac{1}{p}} \leq 8^{\frac{d}{p} - \frac{d}{q}} \sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( 2^{j+2} r + 1 \right)^{\frac{d}{q}} \|x\|_{\ell_q^n(\mathbb{Z}^d)}.
\]

It is readily checked that there exist constants \( C_1, C_2 > 0 \) such that

\[
8^{\frac{d}{p} - \frac{d}{q}} \sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q}} \left( 2^{j+2} r + 1 \right)^{\frac{d}{q}} \|x\|_{\ell_q^n(\mathbb{Z}^d)} \leq C_1 \sum_{j=0}^{\infty} (2^j r)^{\alpha - d + \frac{d}{p} - \frac{d}{q} + \frac{d}{p}} \|x\|_{\ell_q^n(\mathbb{Z}^d)}
\]

\[
= C_2 \|x\|_{\ell_q^n(\mathbb{Z}^d)} r^{\alpha - \frac{d}{q}}.
\]

Thus for \( C_3 = C_0 \lor C_2 \),

\[
|I_{\alpha} x(k)| \leq C_3 \left( r^\alpha M(x(k)) + r^{\alpha - \frac{d}{q}} \|x\|_{\ell_q^n(\mathbb{Z}^d)} \right).
\tag{3}
\]

Suppose for the moment that \( k \in \mathbb{Z}^d \) satisfies \( M(x(k)) \neq 0 \). By Lemma 3.1 we can take \( r := \left( \frac{\|x\|_{\ell_q^n(\mathbb{Z}^d)}}{M(x(k))} \right)^{\frac{q}{d}} \geq 1 \) in (3) above and obtain

\[
|I_{\alpha} x(k)| \leq C_3 \left[ \left( \frac{\|x\|_{\ell_q^n(\mathbb{Z}^d)}}{M(x(k))} \right)^{\frac{q}{d}} M(x(k)) + \left( \frac{\|x\|_{\ell_q^n(\mathbb{Z}^d)}}{M(x(k))} \right)^{\frac{q}{d}} \|x\|_{\ell_q^n(\mathbb{Z}^d)} \right]^{\alpha - \frac{d}{q}}
\]

\[
= 2C_3 (M(x(k)))^{1 - \frac{\alpha q}{d}} \|x\|_{\ell_q^n(\mathbb{Z}^d)}^{\alpha q}.
\]

On the other hand, if \( M(x(k)) = 0 \) then \( x = 0 \), and so \( I_{\alpha} x(k) = 0 \). Thus, the inequality

\[
|I_{\alpha} x(k)| \leq 2C_3 (M(x(k)))^{1 - \frac{\alpha q}{d}} \|x\|_{\ell_q^n(\mathbb{Z}^d)}^{\alpha q}
\]
holds in this case as well. Hence
\[ \|I_\alpha x\|_{\ell^p(Z^d)} = \sup_{m \in \mathbb{Z}, N \in \omega} \left( \frac{1}{(2N + 1)^{d - \frac{dp}{q}}} \sum_{k \in S_{m,N}} |I_\alpha x(k)|^s \right)^{\frac{1}{s}} \]
\[ \leq \sup_{m \in \mathbb{Z}, N \in \omega} \left( \frac{1}{(2N + 1)^{d - \frac{dp}{q}}} \sum_{k \in S_{m,N}} \left| 2C_3 (Mx(k))^{1 - \frac{aq}{d}} \|x\|_{\ell^q(Z^d)}^{aq/d} \right|^s \right)^{\frac{1}{s}} \]
\[ = 2C_3 \|x\|_{\ell^q(Z^d)}^{aq/d} \sup_{m \in \mathbb{Z}, N \in \omega} \left( \frac{1}{(2N + 1)^{d - \frac{dp}{q}}} \sum_{k \in S_{m,N}} (Mx(k))^p \right)^{\frac{p}{ps}} \]
\[ = 2C_3 \|x\|_{\ell^q(Z^d)}^{aq/d} \|Mx\|_{\ell^p(Z^d)}^p. \]

By Theorem 3.2, there exists \( C > 0 \) such that
\[ 2C_3 \|x\|_{\ell^q(Z^d)}^{aq/d} \|Mx\|_{\ell^p(Z^d)}^p \leq C \|x\|_{\ell^q(Z^d)}^{aq/d} \|x\|_{\ell^p(Z^d)}^p = C \|x\|_{\ell^p(Z^d)}, \]
as desired. \( \square \)

Remark 3.4. The operator defined in (1) may be considered as the discrete fractional integral operator (for the continuous version, see for instance [3]). The proof that we presented above uses an analogue of Hedberg’s inequality, which we obtain right after we have inequality (3). The boundedness of this operator on the \( \ell^p(Z^d) \) spaces can be found in [12].

Acknowledgements

This research was partially supported by the Claude Leon Foundation and by the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) (second author). Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the CoE-MaSS. The first author is supported by ITB Research & Innovation Program 2018.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Berezhnoi, E.I.: A discrete version of local Morrey spaces. Izv. Math. 81, 1 (2017)
[2] Chiarenza, F., Frasca, M.: Morrey spaces and Hardy–Littlewood maximal function. Rend. Mat. Appl. 7(3–4), 273–279 (1987)
[3] Gunawan, H., Eridani, Nakai, E.: On generalized fractional integral operators. Scientiae Math. Japon. Online 10, 307–318 (2014)
[4] Fefferman, C., Stein, E.M.: Some maximal inequalities. Am. J. Math. 93(1), 107–115 (1971)
[5] Gunawan, H., Kikianty, E., Schwanke, C.: Discrete Morrey spaces and their inclusion properties. Math. Nachr. 291(8–9), 1283–1296 (2018)
[6] Magyar, A., Stein, E.M., Wainger, S.: Discrete analogues in harmonic analysis: spherical averages. Ann. Math. 155, 189–208 (2002)
[7] Mizuhara, T.: Boundedness of some classical operators on generalized Morrey spaces. Harmonic Analysis (Sendai, 1990), pp. 183–189, ICM-90 Satell. Conf. Proc., Springer, Tokyo (1991)
[8] Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. Math. Nachr. 166, 95–103 (1994)
[9] Pierce, L.B.: Discrete analogues in harmonic analysis. Ph.D. Dissertation, Princeton University (2009)
[10] Stein, E.M., Wainger, S.: Discrete analogues of singular Radon transform. Bull. Am. Math. Soc. 23, 537–544 (1990)
[11] Stein, E.M., Wainger, S.: Discrete analogues in harmonic analysis I: $\ell^2$ estimates for singular Radon transforms. Am. J. Math. 21, 1291–1336 (1999)
[12] Stein, E.M., Wainger, S.: Discrete analogues in harmonic analysis II: fractional integration. J. d'Analyse Math. 80, 335–355 (2000)
[13] Stein, E.M., Wainger, S.: Two discrete fractional integral operators revisited. J. d’Analyse Math. 87, 451–479 (2002)

Hendra Gunawan
Department of Mathematics
Bandung Institute of Technology
Bandung 40132
Indonesia
e-mail: hgunawan@math.itb.ac.id

Christopher Schwanke
Department of Mathematics
Lyon College
Batesville AR 72501
USA
e-mail: cmschwanke26@gmail.com

Received: January 19, 2018.
Accepted: December 19, 2018.