Polarized triple-collinear splitting functions

at NLO for processes with photons

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We compute the polarized splitting functions in the triple collinear limit at next-to-leading order accuracy (NLO) in the strong coupling $\alpha_s$, for the splitting processes $\gamma \to q\bar{q}\gamma$, $\gamma \to q\bar{q}g$ and $g \to q\bar{q}\gamma$. The divergent structure of each splitting function was compared to the predicted behaviour according to Catani’s formula. The results obtained in this paper are compatible with the unpolarized splitting functions computed in a previous article. Explicit results for NLO corrections are presented up to $O(\epsilon^0)$, in the context of CDR. Also, we analyse the analytical structure of the results, focusing in the implementation of certain checks to verify the validity of the expressions.
I. INTRODUCTION

The multiple-collinear limit of scattering amplitudes in gauge theories is relevant for many reasons. From a phenomenological point of view, higher-order splitting functions are an essential ingredient of subtraction-like algorithms for computing physical cross sections [1]. In particular, multiple collinear splittings at loop level are required to achieve NNLO or even higher perturbative orders. Besides that, parton shower (PS) generators make an extensive use of the collinear behaviour of matrix elements. In order to have a complete description of the collinear splitting, it is important to keep spin correlations from the parent parton. This is the main motivation for computing polarized splitting functions at higher-orders, both increasing the number of collinear particles and loops.

Collinear factorization properties [2, 3] establish that the divergent behaviour of scattering amplitudes is isolated into universal factors called splitting amplitudes [5, 6]. Besides these well-known properties, strict collinear factorization could be broken in certain kinematical configurations [3, 4]. These effects are originated by non-vanishing color correlations among collinear and non-collinear partons, and they could become manifest in the multiple collinear limit at loop-level. So, this constitutes another motivation for exploring higher-order corrections to polarized splittings functions with more than two collinear partons.

For the double-collinear limit at the level of squared matrix-elements, splitting functions are usually called Altarelli-Parisi (AP) kernels [7]. They have been computed at one-loop [8–14] and two-loop level [15–19], both for amplitudes and squared matrix-elements. For the multiple collinear limit, tree-level splitting functions were computed by several authors [20–25], although a full one-loop description is still missing. However, there are some specific results for the triple collinear limit of one-loop amplitudes for the antisymmetric part of $q \to qQ\bar{Q}$ [26] and for processes involving at least one photon [27].

In this article, we compute polarized splitting functions in the triple-collinear limit at NLO in QCD. For the sake of simplicity, we consider only processes involving at least one photon. Quark-started splitting processes are constrained by helicity conservation. So, they turn out to be proportional to the unpolarized splitting functions, which were computed in a previous article [27]. Explicitly, it is possible to write

$$P_{q\to a_1\ldots a_m}(s, s') = \omega_q \delta_{s, s'} \langle \tilde{P}_{q\to a_1\ldots a_m} \rangle,$$

where $\omega_q$ is the number of fermionic degrees of freedom$^1$. For this reason, we only consider the

$^1$ This property is not obvious when working in the context of DREG. The main inconvenient arises from the
polarized splitting functions associated with the processes $\gamma \to q\bar{q}\gamma$, $\gamma \to q\bar{q}g$ and $g \to q\bar{q}\gamma$.

The outline of the paper is the following. In Section II we describe the computational techniques applied to obtain the results. They are mainly based in an extension of Passarino-Veltman procedure at amplitude level, combined with some inversion rules and transcendentality classification. After that, we present results for photon-started processes in Section III. In this section we also include a brief discussion about the structure of these expressions, in order to complement the one exhibited in Ref. [27]. Then we discuss the polarized splitting function for $g \to q\bar{q}\gamma$ and its corresponding NLO corrections, in Section IV. Finally, we present the conclusions in Section V.

II. COLLINEAR LIMITS AND POLARIZED SPLITTING FUNCTIONS

Before centering into the details of the computation of polarized splitting functions, let’s recall some useful definitions to analyse the multiple collinear limit. Let’s consider an $n$-particle process where $m$ particles become collinear at the same time. Collinear momenta are labelled as $p_i$ with $i \in C = \{1, 2, \ldots, m\}$ and these vectors fulfil $p_i^2 = 0$ (massless on-shell partons). The subenergies are defined as $s_{ij} = 2p_ip_j$ and $s_{i,j} = (p_i + p_{i+1} + \ldots + p_j)^2 = p_{i,j}^2$. To avoid (potential) factorization breaking issues [3, 4], we work in the time-like (TL) region, which implies $s_{ij} \geq 0$ for every $i, j \in C$. A proper description of the collinear limit requires the introduction of a pair of light-like vectors ($\tilde{P}^2 = 0$, $n^2 = 0$), such that

$$\tilde{P}^\mu = p_{1,m}^\mu - \frac{s_{1,m}}{2n \cdot \tilde{P}} n^\mu,$$

(2)

corresponds to the collinear direction in the multiparton collinear limit, and $n^\mu$ parametrizes how this limit is approached, with $n \cdot \tilde{P} = n \cdot p_{1,m}$. The longitudinal-momentum fractions $z_i$ are given by

$$z_i = \frac{n \cdot p_i}{n \cdot \tilde{P}}, \quad i \in C,$$

(3)

and they fulfil the constraint $\sum_{i \in C} z_i = 1$.

Factorization properties become explicit when virtual gluons are allowed to have only physical polarizations. For this reason, we work in the light-cone gauge (LCG), which is characterized by the absence of ghosts and

$$d^{\mu\nu}(k, Q) = -\eta^{\mu\nu} + \frac{k^\mu Q^\nu + Q^\mu k^\nu}{Q \cdot k},$$

(4)

extension of $\gamma^5$ to a $D_{ST}$-dimensional space-time, which introduces some ambiguities in the treatment of fermion polarizations. In particular, some interactions can violate helicity-conservation as we described in Ref. [14].
is the physical polarization tensor of a gauge vector boson (gluon or photon) with momentum $k$ and $Q^2 = 0, k \cdot Q \neq 0$. Although the quantization vector $Q$ is arbitrary, we choose $Q = n$ in order to simplify the computation.

Polarized splitting functions are obtained from the tensor product of two amputated splitting matrices. Using collinear factorization properties [2, 3], we know that

$$|A(p_1, \ldots, p_n)\rangle \simeq \sum_{\lambda \in \text{phys.pol.}} S p_\mu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P}) \epsilon_\mu(\vec{P}, \lambda) |A(\vec{P}^{-\lambda}, p_{m+1}, \ldots, p_n)\rangle,$$

where the sum over the physical polarizations of the intermediate parent parton is understood. $a$ is fixed by flavour conservation for processes started by QCD partons, so we drop this label for these configurations. Since we are considering also photon initiated processes, it has to be explicitly specified in our notation to avoid ambiguities. So, in that case we write $\gamma \rightarrow a_1 \ldots a_m$.

In order to make a complete general analysis, in this section we kept the complete flavour labelling to treat simultaneously gluon and photon-started splitting processes.

With the aim of disentangling the different helicity contributions, we remove the polarization vector in the splitting amplitude. Explicitly,

$$|A(p_1, \ldots, p_n)\rangle \simeq \sum_{\lambda, \lambda'} \langle A(p_1, \ldots, p_n) | \mathbf{1} | A(p_1, \ldots, p_n) \rangle$$

thus, after taking the square of this formula, we obtain

$$\langle A(p_1, \ldots, p_n) | \mathbf{1} | A(p_1, \ldots, p_n) \rangle \simeq \sum_{\lambda, \lambda'} \langle A(\vec{P}^{-\lambda}, p_{m+1}, \ldots, p_n)$$

$$\times (\epsilon_\mu(\vec{P}, \lambda))^* S p_\mu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P}) \epsilon_\nu(\vec{P}, \lambda') S p_\nu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P}) \rangle,$$

which allows to define the polarized splitting function according to

$$P_{a \rightarrow a_1 \ldots a_m}^{\mu \nu}(\lambda, \lambda') \equiv \left(\frac{s_{1,m}}{2 \mu^2 \epsilon}\right)^{m-1} (S p_\mu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P}))^\dagger S p_\nu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P})^\dagger,$$

that represents the product of two amputated splitting matrices. This product implies a sum over polarizations (and colors) of all the outgoing collinear partons, but parent parton polarization is not specified. Here, it is crucial to appreciate that the collinear limit is completely described by the object

$$P_{a \rightarrow a_1 \ldots a_m}(\lambda, \lambda') = (\epsilon_\mu(\vec{P}, \lambda))^* S p_\mu^{a \rightarrow a_1 \ldots a_m}(p_1, \ldots, p_m; \vec{P}) \epsilon_\nu(\vec{P}, \lambda'),$$

which implies that we drop terms proportional to $\vec{P}$ and $n$ in the tensorial expansion of $P_{a \rightarrow a_1 \ldots a_m}^{\mu \nu}$.

Moreover, due to Eq. (2), we can make the replacement $p_\mu^m = -p_\mu^{1,m-1}$.
Since the computation of the collinear limit of squared amplitudes can be done using amputated amplitudes, then it is preferable to express our results in terms of $P_{a\rightarrow a_1\ldots a_m}^{\mu\nu}$. Of course, in the helicity formalism, it is more suitable to consider $P_{a\rightarrow a_1\ldots a_m}$. In any case, both expressions can be easily related by contracting with polarization vectors or just by removing them.

Considering collinear factorization at one-loop level,

$$|\mathcal{A}^{(1)}(p_1, \ldots, p_n)\rangle \simeq S P_{a \rightarrow a_1\ldots a_m}^{(1)}(p_1, \ldots, p_m; \tilde{P}) |\mathcal{A}^{(0)}(\tilde{P}, p_{m+1}, \ldots, p_n)\rangle + S P_{a \rightarrow a_1\ldots a_m}^{(0)}(p_1, \ldots, p_m; \tilde{P}) |\mathcal{A}^{(1)}(\tilde{P}, p_{m+1}, \ldots, p_n)\rangle ,$$

then the one-loop correction to the polarized splitting function is given by

$$P_{a \rightarrow a_1\ldots a_m}^{(1),\mu\nu} = \left( \frac{s_{1,m}}{2 \mu^2} \right)^{m-1} \left( S P_{a \rightarrow a_1\ldots a_m}^{(0),\mu}(p_1, \ldots, p_m; \tilde{P}) \right)^\dagger \times S P_{a \rightarrow a_1\ldots a_m}^{(1),\nu}(p_1, \ldots, p_m; \tilde{P}) + \text{h.c.} \ ,$$

We will use this expression as a master formula for all our calculations.

After introducing a general definition for $P_{a \rightarrow a_1\ldots a_m}^{\mu\nu}$, a tensorial basis is required to perform an expansion of this object. When considering an $n$-particle process with $m$-collinear partons, there are $m$ vectors associated with external momenta and a null-vector $n^\mu$ introduced by the quantization procedure. Due to the fact that $P_{a \rightarrow a_1\ldots a_m}^{\mu\nu}$ is a rank-2 tensor and it can depend only on $\{p_i^\mu\}_{i \in C}$ and $n^\mu$, then we define the basis

$$f_{1\mu\nu}^{\mu\nu} = \delta^{\mu\nu} ;$$

$$f_{1+i\sigma_1(i),\sigma_2(i)}^{\mu\nu} = \tilde{p}_{\sigma_1(i),\sigma_2(i)}^{\mu\nu} \quad i \in \{1, \ldots, \Delta_1\} ;$$

$$f_{1+i+\Delta_1\sigma_1(i),\sigma_2(i)}^{\mu\nu} = \tilde{p}_{\mu_1(i),\nu_2(i)}^{\mu\nu} \quad i \in \{1, \ldots, \Delta_2\} ;$$

$$f_{1+j+\Delta_1+\Delta_2\mu_1(m+1)}^{\mu\nu} = \tilde{p}_{j,\mu_1(m+1)}^{\mu\nu} \quad j \in \{1, \ldots, m\} ;$$

$$f_{1+j+\Delta_1+\Delta_2+m}^{\mu\nu} = \tilde{p}_{j,\mu_1(m+1)}^{\mu\nu} \quad j \in \{1, \ldots, m\} ;$$

$$f_{2+\Delta_1+\Delta_2+2m}^{\mu\nu} = \tilde{p}_{m+1,m+1}^{\mu\nu} ;$$

with

$$\tilde{p}_{i,j}^{\mu\nu} = p_i^\mu p_j^\nu + p_j^\mu p_i^\nu ;$$

$$\tilde{p}_{i,j}^{\mu\nu} = p_i^\mu p_j^\nu - p_j^\mu p_i^\nu ;$$

$$\Delta_1 = \frac{m(m+1)}{2} ;$$

$$\Delta_2 = \frac{m(m-1)}{2} ;$$
where we define $p^\mu_{m+1} = n^\mu$ to simplify the notation. In the previous expressions, $\sigma$ is a permutation of pairs of collinear momenta which can also include repeated elements and contributes to the symmetric part; $\rho$ is a permutation that excludes repeated indices. Also it is important to appreciate that $f_1$ is the $D_{ST}$-dimensional metric tensor. This can be chosen freely, since it is associated with the DREG scheme being applied. When working with CDR, it is requested to use $\eta_{D_{ST}}$ to achieve consistency.

It is worth noticing that, in spite of imposing the cancellations induced by the contraction with $\epsilon^\mu(\vec{P})$, we can not completely neglect the remaining elements in the basis. This is related with the computation of tensor-like integrals, which require a complete basis of tensorial structures. Working at the integrand level, we can throw away contributions proportional to $n^\mu$ and $\vec{P}^\mu$ when $\mu$ is the index of the parent parton. After this procedure, we obtain

$$P_{a \to a_1 \ldots a_m}^{\mu \nu} = \sum_{j=1}^{1+\Delta_1+\Delta_2} \left( \int_q A^{(0)}(q) \right) f_j^{\mu \nu} \left| S_\mu \cup S_\nu \right| + \sum_{j=1}^{m+1} \left( \int_q A^{(1)}(q) q^\nu \right) p_j^\mu \left| S_\mu \right| + \sum_{j=1}^{m+1} \left( \int_q A^{(2)}(q) q^\mu \right) p_j^\nu \left| S_\nu \right| + \int_q A^{(3)}(q) q^\mu q^\nu, \quad (22)$$

where $A^{(0)}(q)$ is a scalar function of the loop momenta and $S_a$ implements all the cancellations associated to index $a$. Our approach is different from the usual Passarino-Veltman reduction, since we are not treating integrals as isolated objects. Instead, we are combining them inside the scattering amplitude and, then performing the reduction simultaneously. This method seems to be more efficient because it exploits the symmetries associated with the matrix elements. In both cases, it is mandatory to employ a complete basis to write tensor integrals.

The following step consists in rewriting Eq. (22) projecting over the $\nu = ((m+1)^2 + 1)$ elements of the whole basis. So, we get

$$P_{a \to a_1 \ldots a_m}^{\mu \nu} = \sum_{j=1}^{\nu} A_j f_j^{\mu \nu}, \quad (23)$$

and we define the vector $B_j$ as

$$B_j = \sum_{i=1}^{\nu} A_i f_i^{\mu \nu} (f_j)_{\mu \nu} = (M \cdot A)_j, \quad (24)$$

with the kinematic matrix $(M)_{ij} = f_i^{\mu \nu} (f_j)_{\mu \nu}$. It is important to note that this $\nu$-dimensional matrix contains information about all the possible scalar products among collinear particle momenta.

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2 The validity of this assumption is restricted to TL kinematics. Otherwise, factorization breaking effects described in Refs. [3, 4] could introduce a dependence in the non-collinear partons.
and $n$, together with $(\eta D_{ST})_\mu^\nu = D_{ST}$ (the trace of the $D_{ST}$-dimensional metric tensor). Also, if $D_{ST} = 4$ this matrix becomes singular because momenta are not represented by independent vectors. For this reason, $\text{Det}(M) = \mathcal{O}(\epsilon)$ when $D_{ST} = 4 - 2\epsilon$. Of course, through the computation of $M^{-1}$ we recover the coefficients in the expansion Eq. (23) but this procedure is extremely lengthy due to the size of $M$.

In the special case of the triple collinear limit, we decided to use Cramer’s rule to recover the coefficients inside Eq. (23). First of all, we rewrite the tensorial basis making a distinction according to the symmetry properties. Thus

$$f_1^{\mu\nu} = \eta_{D_{ST}}^{\mu\nu},$$

(25)

$$f_2^{\mu\nu} = 2 \frac{p_1^\mu p_1^n}{s_{123}},$$

(26)

$$f_3^{\mu\nu} = \frac{p_1^\mu p_2^n + p_1^n p_2^\mu}{s_{123}},$$

(27)

$$f_4^{\mu\nu} = 2 \frac{p_2^\mu p_2^n}{s_{123}},$$

(28)

$$f_5^{\mu\nu} = \frac{p_1^\mu p_{123} + p_1^n p_{123}^\mu}{s_{123}},$$

(29)

$$f_6^{\mu\nu} = \frac{p_2^\mu p_{123}^\nu + p_2^n p_{123}^{\mu\nu}}{s_{123}},$$

(30)

$$f_7^{\mu\nu} = 2 \frac{p_{123}^\mu p_{123}^{\nu\mu}}{s_{123}},$$

(31)

$$f_8^{\mu\nu} = \frac{p_1^\mu n^\nu + p_1^n n^\mu}{n \cdot \tilde{P}},$$

(32)

$$f_9^{\mu\nu} = \frac{p_2^\mu n^\nu + p_2^n n^\mu}{n \cdot \tilde{P}},$$

(33)

$$f_{10}^{\mu\nu} = \frac{p_{123}^\mu n^\nu + p_{123}^\nu n^\mu}{n \cdot \tilde{P}},$$

(34)

$$f_{11}^{\mu\nu} = \frac{s_{123} n^\mu n^\nu}{(n \cdot \tilde{P})^2},$$

(35)

are the symmetric structures, while

$$f_{12}^{\mu\nu} = \frac{p_1^\mu p_2^n - p_1^n p_2^\mu}{s_{123}},$$

(36)

$$f_{13}^{\mu\nu} = \frac{p_1^\mu p_{123} - p_1^n p_{123}^\mu}{s_{123}},$$

(37)

$$f_{14}^{\mu\nu} = \frac{p_2^\mu p_{123} - p_2^n p_{123}^\mu}{s_{123}},$$

(38)

$$f_{15}^{\mu\nu} = \frac{p_1^\mu n^\nu - p_1^n n^\mu}{n \cdot \tilde{P}},$$

(39)

$$f_{16}^{\mu\nu} = \frac{p_2^\mu n^\nu - p_2^n n^\mu}{n \cdot \tilde{P}},$$

(40)
\[ f_{17}^{\mu\nu} = \frac{p_{123}^{\mu}\eta^{\nu} - p_{123}^{\nu}\eta^{\mu}}{n \cdot \bar{P}} , \]  

(41)
give rise to the antisymmetric ones. Notice that all the basis elements are dimensionless quantities. Since symmetric and antisymmetric spaces are orthogonal, the matrix \( M \) can be written as

\[ M = \begin{pmatrix} M_{\text{sym}} & 0 \\ 0 & M_{\text{asym}} \end{pmatrix} , \]  

(42)
where \( M_{\text{sym}} \) is a \( 11 \times 11 \) matrix while \( M_{\text{asym}} \) has dimension \( 7 \times 7 \). We are going to treat both contributions independently.

As mentioned before, the determinant of \( M \) vanishes in the limit \( \epsilon \to 0 \). Explicitly,

\[ \det (M) = \det (M_{\text{sym}}) \times \det (M_{\text{asym}}) , \]  

(43)

\[ \det (M_{\text{asym}}) = \Omega^3 , \]  

(44)

\[ \det (M_{\text{sym}}) = -8 \epsilon \Omega^5 , \]  

(45)
and

\[ \Omega = \sum_{i=1}^{3} x_i z_i \left( x_i z_i - \sum_{j \neq i} x_j z_j \right) , \]  

(46)
with the notation

\[ x_i = \frac{-s_{jk} - a0}{-s_{123} - a0} , \]  

(47)
where \((i, j, k)\) is a reordering of the indices set \( \{1, 2, 3\} \) and the special case \( x_0 \equiv 1 \). \( \Omega \) is independent of \( \epsilon \) and cyclically invariant under relabelling of particles. Also, it is important to appreciate that \( M \) becomes singular when working in \( D_{\text{ST}} = 4 \) due to the linear dependence on the momenta.

After specifying the tensor basis, we introduce the vector \( B_j \) following Eq. (24). Due to the cancellations mentioned before, we just need to know 4 coefficients for the symmetric part and only 1 for the antisymmetric one. In other terms, we can expand the polarized splitting function as

\[ P_{a \rightarrow a_1 a_2 a_3}^{\mu\nu} = \sum_{j=1}^{4} A_{j \text{sym}} f_j^{\mu\nu} + A_{j \text{asym}} f_{12}^{\mu\nu} , \]  

(48)
after neglecting contributions proportional to \( n^\mu \) and \( p_{123}^\mu \). To obtain the coefficients \( A_{\text{sym}} \) and \( A_{\text{asym}} \) we use Cramer’s rule by introducing the matrices

\[ (M_{\text{sym}}^{\text{Cramer}})_{ij} = -\frac{\det \bar{M}^{(i,j)}}{8\epsilon \Omega^8} , \quad i \in \{1, \ldots, 4\} , \]  

(49)

\[ (M_{\text{asym}}^{\text{Cramer}})_{j} = -\frac{\det \bar{M}^{(12,j)}}{8\epsilon \Omega^8} , \]  

(50)
where $\tilde{M}^{(i,j)}$ denotes a new matrix formed by replacing the column $i$ of $M$ with the canonical vector $\hat{e}_j$. Thus, $M_{\text{sym}}^{\text{Cramer}}$ is a $4 \times 17$-dimensional matrix while $M_{\text{asym}}^{\text{Cramer}}$ is just a 17-dimensional vector. These matrices allow us to recover only the relevant coefficients, which makes this approach more efficient than inverting the whole system. So,

$$A_{j}^{\text{sym}} = (M_{\text{sym}}^{\text{Cramer}} \cdot B)_{j} \quad j \in \{1, \ldots, 4\},$$

$$A_{\text{asym}} = (M_{\text{asym}}^{\text{Cramer}} \cdot B),$$

lead to the desired expressions.

Finally we would like to make some remarks about the treatment of $B_{j}$. Since each component of this vector is a scalar, all this procedure simplifies the computation of Feynman integrals due to the presence of only scalar ones. We must take into account the existence of different propagators contributing to $B_{j}$. For this reason, we define certain propagator’s basis and we put together all the contributions that can be described inside the same set. Then, integration by parts (IBP) reduction [30, 31] is applied and all the components are expanded using a set of master integrals.

On the other hand, $S_{\text{p}}^{(1)}_{a \to a_1 \ldots a_m}$ can be decomposed as

$$S_{\text{p}}^{(1)}_{a \to a_1 \ldots a_m} = S_{\text{p}}^{(1)\text{div.}}_{a \to a_1 \ldots a_m} + S_{\text{p}}^{(1)\text{fin.}}_{a \to a_1 \ldots a_m},$$

where $S_{\text{p}}^{(1)\text{fin.}}_{a \to a_1 \ldots a_m}$ contains only finite pieces while IR/UV divergences are kept inside $S_{\text{p}}^{(1)\text{div.}}_{a \to a_1 \ldots a_m}$. Moreover, $S_{\text{p}}^{(1)\text{div.}}_{a \to a_1 \ldots a_m}$ can be expressed as

$$S_{\text{p}}^{(1)\text{div.}}_{a \to a_1 \ldots a_m}(p_1, \ldots, p_m; \tilde{P}) = I_{a \to a_1 \ldots a_m}^{(1)}(p_1, \ldots, p_m; \tilde{P}) S_{\text{p}}^{(0)}_{a \to a_1 \ldots a_m}(p_1, \ldots, p_m; \tilde{P}),$$

with the insertion operator

$$I_{a \to a_1 \ldots a_m}^{(1)}(p_1, \ldots, p_m; \tilde{P}) = c_{\text{F}} g_{\text{S}}^2 \left( \frac{-s_{1,m} - i0}{\mu^2} \right)^{-\epsilon} \left\{ \frac{1}{\epsilon^2} \sum_{i,j=1(i\neq j)}^{\tilde{m}} T_i \cdot T_j \left( \frac{-s_{ij} - i0}{-s_{1,m} - i0} \right)^{-\epsilon} \right. \left. + \frac{1}{\epsilon^2} \sum_{i,j=1}^{\tilde{m}} T_i \cdot T_j \left( 2 - (z_i)^{-\epsilon} - (z_j)^{-\epsilon} \right) \right. \left. - \frac{1}{\epsilon} \left( \sum_{i=1}^{\tilde{m}} (\gamma_i - \epsilon \tilde{\gamma}_i^{\text{RS}}) - (\gamma_a - \epsilon \tilde{\gamma}_a^{\text{RS}}) - \frac{\tilde{m} - 2}{2} (\beta_0 - \epsilon \tilde{\beta}_0^{\text{RS}}) \right) \right\},$$

where the color matrix of the collinear particle with momentum $p_i$ is denoted by $T_i$, $\tilde{m}$ counts the number of collinear final state QCD partons and $\tilde{m}$ refers to the total number of QCD partons in the splitting process. The one-loop $d$-dimensional volume factor is given by

$$c_{\text{F}} = \frac{\Gamma (1 + \epsilon) \Gamma (1 - \epsilon)^2}{(4\pi)^{2-\epsilon} \Gamma (1 - 2\epsilon)}. $$

(56)
Final state particles are ordered such that \( \{1, \ldots, \bar{m}\} \) are the colored ones while the remaining ones are singlets under SU\((N_C)\) transformations. Also, it is useful to notice that \( \bar{m} = \bar{m} \) in collinear splittings which are started by non-QCD partons (in this paper, photons).

As can be seen from Eq. (55), all the divergent structure is controlled by the insertion operator \( I_{a \rightarrow a_1 \ldots a_m}^{(1)} \). This object is a matrix in the color space, but for the processes considered it is possible to completely describe its action using a pure \( c \)-number. Let’s explain this point more carefully. Due to color conservation, we have

\[
\sum_i T_i \, S p_{a \rightarrow a_1 \ldots a_m}^{(0)} = S p_{a \rightarrow a_1 \ldots a_m}^{(0)} \, T_a \ .
\]

thus the color charge of the parent parton can be expressed using the color information of the outgoing collinear particles. When \( \bar{m} \leq 3 \), Eq. (57) implies that all the products of color operators inside \( I_{a \rightarrow a_1 \ldots a_m}^{(1)} \) are proportional to the unit matrix. So, we write

\[
I_{a \rightarrow a_1 \ldots a_m}^{(1)} \rightarrow I_{a \rightarrow a_1 \ldots a_m}^{(1)} \, \text{Id} ,
\]

where \( I_{a \rightarrow a_1 \ldots a_m}^{(1)} \) is a pure \( c \)-number.

To conclude the description of \( I_{a \rightarrow a_1 \ldots a_m}^{(1)} \), let’s explain the different coefficients involved in its definition. The flavour coefficients \( \gamma_a \) are given by

\[
\gamma_q = \gamma_{\bar{q}} = 3C_F/2 ,
\]

\[
\gamma_g = \beta_0/2 ,
\]

and \( \beta_0 = (11C_A - 2N_f)/3 \), while \( \gamma_a = 0 \) for non-QCD partons. Besides predicting the \( \epsilon \)-poles, \( I^{(1)} \) also controls the regularization scheme (RS) dependence up to \( \mathcal{O}(\epsilon^0) \) through the coefficients \( \tilde{\gamma}_{i}^{\text{RS}} \) and \( \tilde{\beta}_0^{\text{RS}} \). They are given by

\[
\tilde{\gamma}_{i}^{\text{CDR}} = \tilde{\beta}_0^{\text{CDR}} = 0 ,
\]

in conventional dimensional regularization (CDR), while

\[
\tilde{\gamma}_{q}^{\text{DR}} = \tilde{\gamma}_{\bar{q}}^{\text{DR}} = C_F/2 ,
\]

\[
\tilde{\gamma}_{g}^{\text{DR}} = \tilde{\beta}_0^{\text{DR}} / 2 = C_A/6 ,
\]

in dimensional reduction (DR).

After discussing the divergent structure of splitting functions at NLO, we can exploit this knowledge to write the finite corrections in an advantageous way. If we apply the decomposition
suggested in Eq. (53) and Eq. (54) to the definition given in Eq. (11), we obtain

\[ P^{(1),\mu\nu}_{a\rightarrow a_1...a_m} \equiv \left( \frac{s_{1,m}}{2 \mu^2} \right)^{m-1} \left( S P^{(0),\mu}_{a\rightarrow a_1...a_m} \right)^\dagger \left( S P^{(1)}_{a\rightarrow a_1...a_m} + S P^{(1),\text{fin,}\mu\nu}_{a\rightarrow a_1...a_m} \right) + \text{h.c.} , \]

\[ = 2 \text{Re} \left( P^{(1)}_{a\rightarrow a_1...a_m} (p_1, \ldots, p_m; \vec{P}) \right) P^{(0),\mu\nu}_{a\rightarrow a_1...a_m} + \left( P^{(1),\text{fin,}\mu\nu}_{a\rightarrow a_1...a_m} + \text{c.c.} \right) , \]  

\text{(64)}

with

\[ P^{(1),\text{fin,}\mu\nu}_{a\rightarrow a_1...a_m} = \left( \frac{s_{1,m}}{2 \mu^2} \right)^{m-1} \left( S P^{(0),\mu}_{a\rightarrow a_1...a_m} \right)^\dagger S P^{(1),\text{fin,}\mu\nu}_{a\rightarrow a_1...a_m} , \]  

\text{(65)}

where we must recall that a sum over color and polarization of outgoing collinear particles is always understood. Centering in the triple collinear limit, Eq. (48) can be rewritten as

\[ P^{(1),\text{fin,}\mu\nu}_{a\rightarrow a_1a_2a_3} = c_{a\rightarrow a_1a_2a_3} \left[ \sum_{j=1}^{4} A_j^{(1),\text{fin,}f_{j\mu\nu}} + A_5^{(1),\text{fin,}f_{12}} \right] , \]  

\text{(66)}

with \( c_{a\rightarrow a_1...a_m} \) is a normalization factor which depends on the process and, at this point, we can just take care of the coefficients \( A_j \). Since all the processes studied in this work are of the form \( V \rightarrow q_1\bar{q}_2V_3 \), they turn out to be symmetric under the exchange 1 \( \leftrightarrow \) 2. The tensorial basis has a well-defined behaviour under the symmetry operator \( S_{1\leftrightarrow 2} \), closely related with the symmetry properties in the indices \( \mu \leftrightarrow \nu \); explicitly,

\[ S_{1\leftrightarrow 2} (f_j) = f_j \text{ for } j \in \{1, 3\} , \]  

\text{(67)}

\[ S_{1\leftrightarrow 2} (f_{12}) = -f_{12} , \]  

\text{(68)}

\[ S_{1\leftrightarrow 2} (f_2) = f_4 , \]  

\text{(69)}

so we can infer the behavior of the associated coefficients. Moreover, \( A_4^{(1),\text{fin,}} \) is obtained from \( A_2^{(1),\text{fin,}}. \) Of course, making no assumptions about the symmetry during the computation allows to check for potential errors at the final stage.

The last step in the organization of the finite pieces consists in classifying the different terms according to their transcendental weight. The notion of transcendental weight is related to the number of iterated integrals of rational functions required to express a specific function. In this way, rational functions (including constants) have weight \( 0 \). \( \log(x) \) and \( \pi \) have weight \( 1 \); \( \text{Li}_n(x) \) and \( \zeta_n \) has weight \( n \). Since it is a multiplicative quantity, \( \log(x) \log(y) \) has weight \( 2 \) and so on. It is known that one-loop QCD amplitudes can be expanded using up to weight \( 2 \) functions, when considering only \( \epsilon^0 \) terms. For these reasons and symmetry considerations, the coefficients \( A_j \) can
be written as

\[ A^{(1) \text{fin.}}_j = \sum_{i=0}^{2} C^{(i)}_j + (1 \leftrightarrow 2) \text{ for } j \in \{1, 3\}, \] (70)

\[ A^{(1) \text{fin.}}_2 = \sum_{i=0}^{2} C^{(i)}_2, \] (71)

\[ A^{(1) \text{fin.}}_5 = \sum_{i=0}^{2} C^{(i)}_5 - (1 \leftrightarrow 2), \] (72)

where \( C^{(i)}_j \) includes only functions of transcendental weight \( i \).

As a final comment, let’s notice that unpolarized splitting functions can be recovered by contracting \( P_{\mu \nu}^{a_1 \ldots a_m} \) with \( d_{\mu \nu}(\vec{P}, n) \), i.e.

\[ \langle \hat{P}_{a_1 \ldots a_m} \rangle = \frac{1}{\omega} d_{\mu \nu}(\vec{P}, n) P^{a_1 \ldots a_m}_{\mu \nu}, \] (73)

where \( \omega = 2(1 - \epsilon) \) is the number of physical polarizations associated with the parent vector particle.

### III. PHOTON-STARTED PROCESSES

In this section we present the results associated to the processes \( \gamma \rightarrow q \bar{q} \gamma \) and \( \gamma \rightarrow q \bar{q}g \). In contrast to the path followed in Ref. [27], we start with the simplest splitting with the objective of gaining a better understanding of their structure.

#### A. \( \gamma \rightarrow q \bar{q} \gamma \)

Let’s start with the \( \gamma \rightarrow q \bar{q} \gamma \) splitting amplitude. It is the easiest process in the triple-collinear limit as it involves only Abelian interactions. At LO the splitting amplitude reads

\[ S \gamma \rightarrow q \bar{q} \gamma^{(0)(a_1, a_2)} = e^2 g^2 \mu^2 \text{Id}_{a_1 a_2} \frac{\bar{u}(p_1)}{s_{123}} \left( \frac{\epsilon (p_3) \phi_{13} \epsilon(\vec{P})}{s_{13}} - \frac{\epsilon(\vec{P}) \phi_{23} \epsilon(p_3)}{s_{23}} \right) \nu(p_2), \] (74)

which implies that the LO polarized splitting function can be expressed as

\[ P^{(0), \mu \nu}_{\gamma \rightarrow q \bar{q} \gamma} = e^4 g^4 C_A P^{\mu \nu} \left( p_1, p_2, p_3, \vec{P} \right), \] (75)

where we introduced the pure kinematical function

\[ P^{\mu \nu} \left( p_1, p_2, p_3; \vec{P} \right) = \frac{1}{x_1 x_2} \left( \eta^{\mu \nu} (\epsilon x_1 (1 - x_3) - (1 - x_1)^2) + 2(\epsilon - 1) f_2^{\mu \nu} + 2\epsilon f_3^{\mu \nu} \right) + (1 \leftrightarrow 2). \] (76)
Note that this expression is totally symmetric under the exchange of particles 1 ↔ 2, and that it only involves symmetric elements of the tensorial basis. The function \( P \) describes completely the kinematics of all the splitting processes considered in this article. This is due to the factorization of the color structure at tree-level in the triple collinear limit with photons.

In spite of involving solely symmetric tensorial structures, NLO corrections include non-trivial contributions to \( C_5^{(i)} \). However, as expected, the full splitting function is completely symmetric under 1 ↔ 2. For \( \gamma \to q\bar{q}\gamma \), the normalization factor is given by

\[
c^{\gamma\to q\bar{q}\gamma} = C_A C_F \epsilon_q g_s^4 g_s^2,
\]

while

\[
I^{(1)}_{\gamma\to q_1q_2\gamma_3}(p_1, p_2, p_3; \bar{P}) = \frac{e^2}{\epsilon^2} \left( \frac{-s_{123} - i0}{\mu^2} \right)^{-\epsilon} \left[ -2C_F x_3^{-\epsilon} - 2\epsilon\gamma_q \right],
\]

controls the divergent structure for this process. When comparing \( \epsilon \)-poles in our bare results with the ones predicted by this formula, we found a complete agreement.

Now, let’s show the NLO corrections. The rational terms are described by

\[
C_1^{(0)} = \frac{8(1 - x_1^2)}{x_1 x_2} + \frac{1 - x_1}{x_1},
\]

\[
C_2^{(0)} = \frac{2((x_1^3 - x_1) x_2 + (1 - x_1)^2 + (x_2 + 1)(1 - x_2)^2)}{(1 - x_1 x_2)(1 - x_3)} + \frac{16}{x_1 x_2},
\]

\[
C_3^{(0)} = -\frac{4(x_1(1 - x_2) + 1 - 2}{x_1 x_2(1 - x_3)} - \frac{2}{1 - x_1},
\]

\[
C_5^{(0)} = -\frac{2(x_1 x_2 + x_3^2)}{(1 - x_1)(1 - x_2 x_3(1 - x_3)} - \frac{2}{x_2(1 - x_3)},
\]

while

\[
C_1^{(1)} = \frac{x_2 - 1}{x_2} \left( \frac{\log(x_1)(x_2 - 2x_3)}{1 - x_1} - \frac{2x_3 \log(x_3)}{1 - x_3} \right),
\]

\[
C_2^{(1)} = \frac{(1 - x_2^2)(x_2 - 2x_1 + 2) \log(x_1)}{(1 - x_1) x_2^2 x_3^2} + \frac{(2x_3 - x_1) \log(x_2)}{x_1 x_2^2} + \frac{2 \log(x_3)}{(1 - x_3)^2} \left( (1 - x_2)^2 (2(x_1 - 1)x_2 x_2 + (1 - x_3)^2) + 2(x_3 - 2) \right),
\]

\[
C_3^{(1)} = \frac{2 \log(x_1)((1 - x_1) x_2 + 2x_3 - (x_1 + 1)x_2^2 - x_2(x_1 x_2 + 2x_3))}{(1 - x_1) x_2^2 x_3^2} + \frac{4(x_2(x_3 - 1) + x_1(1 - 6x_2) + 2x_2) \log(x_3)}{x_1 x_3^2(1 - x_3)^2},
\]

\[
C_5^{(1)} = \frac{2((x_1 - 2)x_2 + 2(1 - x_1)^2) \log(x_1)}{(1 - x_1)^2 x_2 x_2^2} - \frac{4 \log(x_3)}{x_2(1 - x_3)^2},
\]
contain the weight 1 functions.

Finally, for the weight 2 contributions we have

\[ C^{(2)}_1 = -2F_1 \left( \frac{1-x_1}{x_2} + \frac{2x_1-5}{x_2} + \frac{x_2-2}{x_1} + \frac{2}{x_1x_2} \right), \]  
\[ (87) \]

\[ C^{(2)}_2 = -2F_1 \left( 2x_2^2 + (1-x_2)^2 \right) \frac{1}{x_1x_2^3} - 2F_2 \left( 2x_1^2 + 2x_1(x_2-1) + (1-x_2)^2 \right) \frac{1}{x_1^2x_2}, \]  
\[ (88) \]

\[ C^{(2)}_3 = -4F_1 \left( (1-x_2)^2 - x_1 \right) \frac{1}{x_1x_2^3}, \]  
\[ (89) \]

\[ C^{(2)}_5 = -4F_1 \left( x_3^2 - x_1 \right) \frac{1}{x_1x_2^2}, \]  
\[ (90) \]

with

\[ F_i = \frac{\pi^2}{6} - \log x_i \log x_3 - \text{Li}_2(1-x_i) - \text{Li}_2(1-x_3) = \mathcal{R}(x_i, x_3), \]  
\[ (91) \]

being originated from the \( \epsilon \)-expansion of standard scalar boxes, after the subtraction of the terms proportional to \( \log^2(x_i) \) included in \( I^{(1)}_{\gamma \to q_1\bar{q}_2\gamma_3}(p_1, p_2, p_3; \tilde{P}) \).

**B. \( \gamma \to q\bar{q}g \)**

The following process is \( \gamma \to q\bar{q}g \), which includes three QCD partons. Since all of them are on-shell final state particles, it is expected that the associated splitting function will be expressed in a very compact form. The corresponding splitting amplitude at tree-level is given by

\[ S\mu_{\gamma \to q_1\bar{q}_2\gamma_3}^{(0)}(a_1, a_2, a_3) = \frac{e_q g_e g_S \epsilon^2}{s_{123}} T^{a_3}_{a_1 a_2} \tilde{u}(p_1) \left( \frac{\ell(p_3) \gamma_{13} \ell(\tilde{P})}{s_{13}} - \frac{\ell(\tilde{P}) \gamma_{23} \ell(p_3)}{s_{23}} \right) u(p_2) \]  
\[ = \frac{g_S}{g_e \epsilon g_q} T^{a_3}_{a_1 a_2} S\mu_{\gamma \to q_1\bar{q}_2\gamma_3}^{(0)}(a_1, a_2), \]  
\[ (92) \]

while the polarized LO splitting function can be written as

\[ P^{(0), \mu\nu}_{\gamma \to q_1\bar{q}_2\gamma_3} = e_q^2 g_e^2 g_S^2 C_A C_F \mathcal{P}^{\mu\nu} (p_1, p_2, p_3; \tilde{P}) \]  
\[ (93) \]

Centering in the NLO corrections, the divergent structure is dictated by

\[ I^{(1)}_{\gamma \to q_1\bar{q}_2\gamma_3}(p_1, p_2, p_3; \tilde{P}) = I^{(1)}_{\gamma \to q_1\bar{q}_2\gamma_3}(p_1, p_2, p_3; \tilde{P}) \]  
\[ + \frac{cT g_S^2 C_A}{\epsilon^2} \left( \frac{-s_{123} - i\epsilon}{\mu^2} \right) \left( x_3^{-\epsilon} - x_1^{-\epsilon} - x_2^{-\epsilon} \right), \]  
\[ (94) \]

and

\[ c^{\gamma \to q\bar{q}g} = C_A C_F e_q^2 g_e^2 g_S^4, \]  
\[ (95) \]
is the global NLO normalization factor. As a usual check, we verified that all the $\epsilon$-poles were equal to those predicted by the expansion of $I^{(1)}_{\gamma \rightarrow q_1 q_2 g_3}$.

Due to the presence of a non-trivial color structure, it is useful to decompose the $C_j^{(i)}$ coefficients according to

$$C_j^{(i)} = C_A C_j^{(i,C_A)} + D_A C_j^{(i,D_A)},$$

(96)

where $D_A = 2C_F - C_A$ is related with the Abelian contributions. Moreover, we find that

$$C_j^{(i,D_A)} = C_j^{(i,\gamma \rightarrow q\bar{q}g)} / 2,$$

(97)

which was expected since the Abelian terms in $\gamma \rightarrow q\bar{q}g$ are the same that those present in the $\gamma \rightarrow q\bar{q}\gamma$ process. So, in order to simplify the presentation of the results, we only write the contributions proportional to $C_A$. The rational terms are given by

$$C_1^{(0,C_A)} = \frac{(1 - x_1)(7x_2 + 8x_3)}{2x_1x_2},$$

(98)

$$C_2^{(0,C_A)} = \frac{8}{x_1x_2} - \frac{(3 - x_2)x_2 - x_1(x_2 + 1)}{2(1 - x_1)x_1x_2},$$

(99)

$$C_3^{(0,C_A)} = -\frac{1}{1 - x_1},$$

(100)

$$C_5^{(0,C_A)} = \frac{1}{x_2(1 - x_3)} - \frac{x_1x_2 + x_3^2}{(1 - x_1)(1 - x_2)x_2(1 - x_3)},$$

(101)

and

$$C_1^{(1,C_A)} = -\frac{3(1 - x_2)\log(x_1)}{2(1 - x_1)},$$

(102)

$$C_2^{(1,C_A)} = \frac{(1 - x_2)^2\log(x_1)}{2(1 - x_1)^2x_1x_2} - \frac{3\log(x_2)}{2x_1x_2},$$

(103)

$$C_3^{(1,C_A)} = -\frac{\log(x_1)(2x_1x_2 + x_3)}{(1 - x_1)^2x_1x_2},$$

(104)

$$C_5^{(1,C_A)} = \frac{(2(1 - x_1)(1 - x_2) - x_1x_2)\log(x_1)}{(1 - x_1)^2x_1x_2},$$

(105)

are the weight 1 contributions. The non-trivial weight 2 terms are given by

$$C_1^{(2,C_A)} = -\frac{F_3(1 - x_1)^2}{x_1x_2},$$

(106)

$$C_2^{(2,C_A)} = -\frac{2F_3}{x_1x_2},$$

(107)

where we used the function

$$F_3 = R(x_1, x_2).$$

(108)
It is interesting to appreciate that this is the last remaining $R$-function involved in the expansion of standard scalar boxes. Also, we obtain the relation

$$
\sum_{j=1}^{2} C_j^{2,A} f_j^{\mu\nu} + (1 \leftrightarrow 2) = F_3 P^{\mu\nu} |_{c_0},
$$

(109)

which tells us that the weight 2 contribution associated with $C_A$ is proportional to the LO splitting function.

Due to the fact that $\gamma \rightarrow q\bar{q}\gamma$ and $\gamma \rightarrow q\bar{q}g$ share some Feynman diagrams in their perturbative expansion, the corresponding NLO corrections are related. This constitutes a cross-check of the results, since they were obtained from independent implementations. Explicitly, we have the relation

$$
P_{\gamma\rightarrow q_1\bar{q}_2\gamma_3}^{\mu\nu} = \frac{e_q^2 g_s^2}{C_F g_s} \left( P_{\gamma\rightarrow q_1\bar{q}_2g_3}^{\mu\nu} |_{C_A \rightarrow 0} \right),
$$

(110)

which is equivalent to cancel all the non-Abelian diagrams from $\gamma \rightarrow q\bar{q}g$ (and adapt the normalization due to the presence of an additional color matrix).

C. Remarks on the structure of the photon-started splittings

In order to make a proper analysis, let’s recall the associated unpolarized results shown in Ref. [27]. Before that, it is useful to introduce the notation

$$
\Delta^{i,j}_{i',j'} \equiv x_i z_j + x_i' (z_j' - 1),
$$

(111)

$$
\bar{\Delta}^{i,j}_{i',j'} \equiv x_i z_j - x_i' (z_j' - 1),
$$

(112)

where the indices correspond to outgoing particles, with $x_0 \equiv 1$ and $z_0 \equiv 1$. For $\gamma \rightarrow q\bar{q}\gamma$ we found

$$
\langle \hat{P}^{(1)\text{fin.}}_{\gamma\rightarrow q_1\bar{q}_2\gamma_3} \rangle = \frac{C_F C_A}{2} e_q^2 g_s^2 \left[ C^{(0,A)} + C^{(1,A)}_1 \log(x_1) + C^{(1,A)}_2 \log(x_3) + C^{(2,A)}_2 R(x_1, x_3) + (1 \leftrightarrow 2) \right],
$$

(113)

with

$$
C^{(0,A)} = \left( x_1 x_2 - z_1 z_2 - \Delta^{1,0}_{0,1} \Delta^{2,0}_{0,2} \right) \left( \frac{2 - 2 x_1 (x_2 + 1)}{x_1 x_2 (1 - x_3)} - \frac{1}{1 - x_1} \right)
$$

$$
- \frac{2 z_1 \Delta^{1,0}_{0,1}}{x_1 x_2} \left( \frac{(x_2 + 1)(1 - x_2)^2}{(1 - x_1) (1 - x_3)} + \frac{x_3 - x_1 x_2}{1 - x_3} + \frac{(3 - x_2) x_2 - x_1 (x_2 + 1)}{2 (1 - x_1)} + 8 \right)
$$

$$
- \frac{8 (1 - x_1)^2}{x_1 x_2} \frac{1 - x_1}{x_1},
$$

(114)
\[ C_{1}^{(1,A)} = \frac{x_1 x_2 - z_1 z_2 - \Delta_{0,1}^{1,0} \Delta_{0,2}^{2,0}}{(1 - x_1) x_1} \left( \frac{x_2 + 2 x_3}{x_2^2} - \frac{x_1 x_2 + 2 x_3}{(1 - x_1) x_2} - \frac{1 + x_1}{1 - x_1} \right) - \frac{z_2 (2 x_3 - x_2) \Delta_{0,2}^{2,0}}{x_1 x_2^2} \]
\[ - \frac{(1 - x_2)^2 z_1 (2 (1 - x_1) + x_2) \Delta_{0,1}^{1,0}}{(1 - x_1)^2 x_2} + \frac{(1 - x_2) (x_2 - 2 x_3)}{(1 - x_1) x_2}, \]  
\[ C_{2}^{(1,A)} = \frac{2 \left( 2 x_1 (z_2 - 1 - \Delta_{0,2}^{2,0} (x_1 x_2 + 1)) - (\Delta_{0,3}^{3,0})^2 - 2 x_2 (z_1 + 2 z_2 - 3) (x_1 x_2 + 2 z_3) - x_2 z_3 \right)}{x_2^2 (1 - x_3)^2} \]
\[ - \frac{2 (x_1^2 (2 z_2 - 3) + (4 z_2 - 13) z_2 + 7 + 2 z_3^2)}{x_1 x_2 (1 - x_3)^2} \]
\[ - \frac{2 (2 x_1 x_2 + (z_1 - 15) z_1 + 7)}{(1 - x_3)^2}, \]  
\[ C^{(2,A)} = \frac{2 \left( x_2 \left( x_3 \Delta_{3,2}^{0,1} + \Delta_{3,0}^{3,0} (\Delta_{2,1}^{0,3} + z_1) + x_2^3 + 2 x_2 x_3 z_1 \right) + (\Delta_{3,2}^{0,1})^2 \right)}{x_1 x_2^3} \]
\[ - \frac{4 \Delta_{0,1}^{1,0} (x_3 - z_1)}{x_1 x_2}, \]  

for the finite NLO corrections, after applying a subtraction procedure analogous to the one described in Eq. (64). On the other hand, the corrections to \( \gamma \to q\bar{q}g \) are given by

\[ \langle \hat{P}_{\gamma \to q\bar{q}g}^{(1) \text{fin.}} \rangle = \frac{D_A g_{\Sigma}^2}{2 e q g_{e}^2} \langle \hat{P}_{\gamma \to q\bar{q}g}^{(1) \text{fin.}} \rangle + \frac{C_A^2 C_F}{2} e q g_{e}^2 g_{S}^4 \left[ C^{(0,B)} + C_{1}^{(1,B)} \log(x_1) \right. 
\[ + \left. (P_{q\bar{q}Z}^{(0)} | \epsilon \rangle \mathcal{R} (x_1, x_2) + (1 \leftrightarrow 2) \right) \], \]  

with

\[ C^{(0,B)} = \frac{16 - 7 x_2 - 2 z_1 z_2 + (1 - z_1)^2 - 15 z_2}{x_1} - \frac{z_1^2}{(1 - x_1) x_2} - \frac{8 z_1^2 + (1 - z_1)^2}{x_1 x_2} \]
\[ + \frac{2 z_1 (1 - z_3) - x_2 (1 - z_1)^2 - (x_2 + 1) z_1}{(1 - x_1) x_1}, \]  
\[ C^{(1,B)} = \frac{z_2 (x_2 (4 x_1 z_1 + x_1 - 1) + 2 x_3 z_1)}{x_1 x_2 (x_1 - 1)^2 x_1 x_2} + \frac{3 x_2^2 + 5 x_2 (z_2 - 1) + 3 z_2^2 - 4 z_2 + 1}{x_1 x_2} - \frac{(1 - x_2)^2 z_1^2}{(1 - x_1)^2 x_1 x_2}, \]  

and

\[ \langle P_{q\bar{q}Z}^{(0)} | \epsilon \rangle = \frac{\langle \hat{P}_{q\bar{q}Z}^{(0)} \rangle}{e q g_{e}^2 g_{S}^2} \bigg|_{\epsilon=0}, \]  

which corresponds to the \( \epsilon^0 \) contribution to the \( g \to q\bar{q}\gamma \) LO unpolarized splitting function.

It is interesting to appreciate that the coefficients \( C_j^{(i)} \) involved in the expansion of the polarized splittings are independent of the longitudinal-momentum fractions \( z_i \), both for \( P_{\gamma \to q\bar{q}\gamma}^{(1) \text{fin.}, \mu \nu} \) and \( P_{\gamma \to q\bar{q}g}^{(1) \text{fin.}, \mu \nu} \). However, the unpolarized version of these splittings depend on \( z_i \) in a non-trivial way.
So, we conclude that these contributions are originated in the contraction of the different tensorial structures with the parent-gluon polarization tensor, $d_{\mu\nu}(\vec{P}, n)$. Explicitly, we have

\begin{align}
  d_{\mu\nu}(\vec{P}, n) f_{1}^{\mu\nu} &= -2(1 - \epsilon), \\
  d_{\mu\nu}(\vec{P}, n) f_{2}^{\mu\nu} &= -2z_1 \Delta_{0,1}^{1,0}, \\
  d_{\mu\nu}(\vec{P}, n) f_{3}^{\mu\nu} &= x_1 x_2 - z_1 z_2 - \Delta_{0,1}^{1,0} \Delta_{0,2}^{2,0}, \\
  d_{\mu\nu}(\vec{P}, n) f_{12}^{\mu\nu} &= 0,
\end{align}

which also justifies the presence of $\Delta_{0,i}^{i,0}$ functions in the final expressions, and $f_{12}^{\mu\nu}$ does not contribute because of its antisymmetry under the exchange $1 \leftrightarrow 2$ (or, equivalently, $\mu \leftrightarrow \nu$). Due to gauge invariance, photon-started splittings at loop-level can be computed using the replacement $d_{\mu\nu} \rightarrow -\eta_{\mu\nu}$ inside gluon propagators.\footnote{3 For further details and a formal proof of this claim, see Ref. \cite{27}, Section IV.} If we remove the polarization vector associated with the parent parton, then it is possible to compute the splitting amplitude without explicitly taking into account the LCG quantization vector $n^\mu$. This property is straightforwardly translated into $P^{\mu\nu}$, because this object is computed using the product of amputated splitting amplitudes. Thus, the coefficients $C_j^{(i)}$ for the collinear processes $\gamma \rightarrow a_1 \ldots a_m$ must be independent of $z_i$ (and, of course, $n \cdot \vec{P}$).

\section{IV. GLUON-STARTED SPLITTING: $g \rightarrow q\bar{q}\gamma$}

Finally, we arrive to the gluon started splitting. Due to the fact that it is originated from cutting an internal gluon, there is a non-trivial color flow through the parent parton. This implies that it is not possible to remove all LCG integrals and trivially cancel all $z_i$ dependence, as happened in $\gamma \rightarrow a_1 \ldots a_m$ processes.

Starting with the tree-level contributions, the splitting amplitude is

\begin{equation}
  S_{p_{q_1 q_2 \gamma 3}}^{(0)(0,a_1,a_2;\alpha)} = \frac{e_q g_s g_s \mu_2^2 T_{a_1 a_2}}{s_{123}} \bar{u}(p_1) \left( \frac{s_{13}}{s} \frac{\not\epsilon(p_3) \not\epsilon(\vec{P})}{s_{13}} - \frac{\not\epsilon(\vec{P}) \not\epsilon(p_3)}{s_{23}} \right) u(p_2),
\end{equation}

and the polarized splitting function is given by

\begin{equation}
  P_{p_{q_1 q_2 \gamma 3}}^{(0),\mu\nu} = \frac{e_q^2 g_s^2}{2} \mathcal{P}^{\mu\nu}(p_1, p_2, p_3; \vec{P}).
\end{equation}

Analysing the NLO corrections to this process, the normalization factor is given by

\begin{equation}
  c_{q\bar{q}\gamma} = \frac{e_q^2 g_s^2}{2},
\end{equation}

for further details and a formal proof of this claim, see Ref. \cite{27}, Section IV.
and the divergent structure is in complete agreement with the one predicted by Catani’s formula, i.e.

\[ I_{q_1 q_2 \gamma_3}^{(1)}(p_1, p_2, p_3; \vec{P}) = \frac{e \Gamma g_s^2}{\epsilon^2} \left( \frac{-s_{123} - t_0}{\mu^2} \right)^{-\epsilon} \left[ C_A \left( 2 - z_1^{-\epsilon} - z_2^{-\epsilon} + x_3^{-\epsilon} \right) - 2 C_F x_3^{-\epsilon} - \epsilon \left( 2 \gamma_q - \gamma_9 - \beta_0 \right) \right]. \]  

(129)

Notice that the \( \epsilon \)-expansion of Eq. (129) involves the presence of \( \log^j(z_i) \) \( (i, j = 1, 2) \), which implies that it could be possible to have some \( z_i \) dependence in \( P_{q_1 q_2 \gamma_3}^{(1), \mu \nu} \).

As we did previously, it is convenient to classify the different color contributions to \( C_j^{(i)} \). So, we use the decomposition

\[ C_j^{(i)} = C_A C_j^{(i, CA)} + D_A C_j^{(i, DA)} + \beta_0 C_j^{(i, \beta_0)}, \]  

(130)

with

\[ C_j^{(i, DA)} = C_j^{(i, \gamma \to q\bar{q} \gamma)}/2, \]  

(131)

because the Abelian component of this splitting function coincides with \( P_{q_1 q_2 \gamma}^{(1), \mu \nu} \). Applying this notation, the terms proportional to \( \beta_0 \) are expressed as

\[ P_{q_1 q_2 \gamma_3}^{(1), \mu \nu} \bigg| \beta_0 = \frac{10}{3} e^{\epsilon \gamma_0} \mathcal{P}^{\mu \nu}, \]  

(132)

because this contribution is originated from the self-energy correction of the parent gluon [14].

After these appreciations, we need to present only \( C_j^{(i, CA)} \) to complete the description of the \( g \to q\bar{q} \gamma \) splitting function. The rational terms are given by

\[ C_1^{(0, CA)} = \frac{5(1 - x_1)^2}{3x_1 x_2} + \frac{1 - x_1}{2x_1}, \]  

(133)

\[ C_2^{(0, CA)} = \frac{(3 - x_2)x_2 - x_1(x_1 + 1)}{2(1 - x_1)x_1 x_2} + \frac{10}{3x_1 x_2}, \]  

(134)

\[ C_3^{(0, CA)} = \frac{1}{1 - x_1}, \]  

(135)

\[ C_5^{(0, CA)} = -\frac{x_3}{(1 - x_1)(1 - x_2)x_2}, \]  

(136)

while the contributions of weight 1 are

\[ C_1^{(1, CA)} = -\frac{3(1 - x_2) \log(x_1)}{2(1 - x_1)}, \]  

(137)

\[ C_2^{(1, CA)} = -\frac{(1 - x_2)^2 \log(x_1)}{2(1 - x_1)^2 x_1 x_2} + \frac{3 \log(x_2)}{2x_1 x_2}, \]  

(138)

\[ C_3^{(1, CA)} = \frac{\log(x_1)(x_1 x_2 + 2x_3)}{(1 - x_1)^2 x_1 x_2}, \]  

(139)

\[ C_5^{(1, CA)} = \frac{\log(x_1)(2x_1 x_2 + x_3)}{(1 - x_1)^2 x_1 x_2}. \]  

(140)
As we could appreciate for the photon-started splittings, all the contributions were independent of \( z_i \) due to the lack of LCG integrals. However, the same behaviour is observed here, at least for weights 0 and 1. In this case, a cancellation among the \( z_i \)-logarithms in \( P_{q_i q' \gamma}^{(1),\mu\nu} \) and those in \( f_{q_i q' \gamma}^{(1)} \) takes place.

The situation changes when studying weight 2 contributions, which are more complicated than in the previous splitting functions. For this reason, a more sophisticated procedure was required for their treatment. The first step consisted in identifying a set of functions to expand these terms. Following the choice shown in Ref. [27] for the unpolarized splitting function \( g \to q\bar{q}\gamma \), we have the basis

\[
F_1 = \frac{\pi^2}{6} - 2\text{Li}_2 \left( 1 - \frac{x_1}{1 - z_1} \right) - 2\text{Li}_2 \left( 1 - \frac{z_2}{1 - z_1} \right) + 2\text{Li}_2 (1 - z_1) + 2 \log (x_2) \log (1 - z_1) + (1 \leftrightarrow 2),
\]

\[
F_2 = \log (x_1) \log (x_2),
\]

\[
F_3 = \frac{\pi^2}{4} - \text{Li}_2 (1 - x_1) - \log (x_1) \log (z_1) + (1 \leftrightarrow 2),
\]

\[
F_4 = \log \left( \frac{x_1}{1 - z_1} \right) \log \left( \frac{1 - z_1}{z_1 z_2} \right) - \log (x_2) \log (1 - z_1),
\]

\[
F_5 = S_{1\leftrightarrow 2} (F_4),
\]

whose associated coefficients are

\[
C^{(2,CA)}_1 = - \frac{F_1}{4 x_1 x_2} + F_2 \left( \frac{x_2^3 (z_1 - 1) + x_3 ((2 - z_1) z_2 + (1 - z_1)^2) - z_2 (2 z_2 + z_3)}{2 x_1 \Delta_{0,1}^{1.0} \Delta_{0,2}^{2.0}} \right)
\]

\[
+ \frac{x_1 (z_1 - z_2) + z_2^2}{\Delta_{0,1}^{1.0} \Delta_{0,2}^{2.0}} + \frac{F_3 (1 - x_1)^2}{x_1 x_2} - \frac{F_4}{2 \Delta_{0,1}^{1.0}} \left( z_3 + \frac{(1 - x_2)^2 + 2 (z_1 - 1)}{x_1 x_2} \right)
\]

\[
- \frac{z_2^2}{x_1 z_3} + \frac{x_1 (z_2 - 1) + x_2 z_2 + z_1 z_2 + (1 - z_3)^2 - 1}{x_2} - \frac{x_1 (1 - z_2) - 6}{2},
\]

\[
C^{(2,CA)}_2 = - \frac{F_1}{2 x_1 x_2} \left( \frac{2 z_2 \Delta_{0,2}^{2.0} ((x_1 - 1) z_1 + (x_2 - 1) z_2 + 1)}{\Omega} + 1 \right) + \frac{F_2}{\Omega} \left( \frac{x_1 ((1 - z_2)^2 - x_2^2 - x_2 z_2)}{x_2} \right)
\]

\[
+ \frac{(x_1 (x_2 + z_2) - z_2)}{x_2} \left( \frac{x_2^2 x_2 - x_1 z_2 \Delta_{0,2}^{2.0} + x_2 z_2^2}{x_1 x_2 \Delta_{0,1}^{1.0}} \right) + \frac{F_3}{x_1 x_2} + \frac{F_4}{\Omega} \left( z_2 ((1 - z_2)^2 - 2 z_3 + 2) \right)
\]

\[
+ \frac{(1 - z_1)(1 - z_2)}{x_2} \left( \frac{z_2 (2 z_2 + z_3) + z_1 - 1)}{x_1} \right) + \frac{2 F_3}{x_1 x_2} + \frac{F_4}{\Omega} \left( z_2 ((1 - z_2)^2 - 2 z_3 + 2) \right)
\]

\[
+ \frac{(x_1 (x_2 + z_2) - z_2)}{x_1 x_2 \Delta_{0,2}^{2.0} + x_2 z_2^2} - \frac{x_1 (x_2^2 + x_2 z_2 + (1 - z_2)^2)}{x_2}
\]

\[
+ \frac{z_2^2 (2 - x_2 z_1 - z_1) + (1 - z_1)^2 (x_2 z_2 - 2 x_2 + 4) + (2 x_2 - 1) z_3^2 - 2 (3 - z_1)(1 - z_1) z_2}{x_1}
\]
\[
\begin{align*}
&+ \frac{(z_2 - 2)z_3^2}{x_1x_2} - (1 - z_1)(2 - z_2^2) + \frac{2(1 - z_2)z_3^2(x_1(1 - z_2) - x_2z_2)}{x_1z_3} \\
&+ \frac{(1 - z_2)(3 - 3z_1(1 - z_2) - z_2(z_2 + 3) - x_2^2z_3)}{x_2} \\
&- F_5z_2 \left( \frac{1}{z_2} + \frac{x_1(z_2^2 + 1) - z_1(1 - z_2) + (1 - z_2)^2z_2}{\Omega z_3} \right) + \frac{x_1(1 - z_2)(z_1 - z_2 + 1) + z_3(2x_2z_1 - x_2z_2 - z_1 + \Delta_{0,2}^{2,0})}{\Omega}, \\
&\mathcal{C}_3^{(2,C_A)} = \frac{F_1}{\Omega x_1x_2} \left( (1 - z_1)z_1(2z_2 - x_1x_2) - x_1z_2(z_2 - 2z_1) + (1 - z_1)^2(2x_1z_2 - x_1 + 1) - \frac{1}{2} \\
&- x_2^2(1 - z_1)z_2 \right) - \frac{F_2}{\Omega} \left( z_1^2z_2^2 + \frac{(1 - z_1)(z_2(1 - 2x_2 - z_1) - z_1 + 1) - 2z_2^2}{x_1} \right) \\
&+ \frac{(1 - z_1)(x_2(1 - z_1) - z_1z_2)z_3^2}{x_1x_2\Delta_{0,1}^{1,0}} - 2(1 - z_1)z_1 \frac{F_4z_2}{z_3} \left( \frac{(1 - z_1)(z_1(\Delta_{0,1}^{1,0} - 1) + 1) - x_1z_2}{x_1x_2\Delta_{0,1}^{1,0}} \right) - z_2(3x_1z_2 - 2) + (x_2 - 2)z_1^2 - x_2) + 2(z_2 - 1)z_1((x_1 - 1)z_1 + 1) + 2(x_1 - 1)z_1^2 + 2z_2 \\
&+ \frac{2x_1z_1^2(z_2 - 1) + (x_1 - 2)(1 - z_2)^2z_2 + 2z_1^3 + z_1(6z_2 - 4z_1^2 - 2)}{\Omega x_2} - \frac{z_3^2(2z_2 - 2z_1)}{\Omega x_1x_2} \right), \\
&\mathcal{C}_5^{(2,C_A)} = \frac{F_2}{x_2\Delta_{0,2}^{2,0}} + \frac{F_4}{z_3x_1x_2} \left( (1 - z_1)^2 + (1 - z_2)^2 \right). 
\end{align*}
\]

For these contributions, there is a non-trivial dependence in both $z_i$ and $\Delta_{0,i}^{1,0}$, not only inside the rational coefficients but also in the definition of the transcendental functions $F_i$. This is a consequence of the presence of Feynman integrals with LCG denominators, which is closely related with non-Abelian interactions.

**A. Comments on cross-checks**

As we did with all the previous processes, the first check consisted in comparing the divergent structure with Catani’s formula. In this particular case, we carefully studied the cancellation of higher weight functions that were multiplying single $\epsilon$-poles. Since we are performing operations with matrices whose elements have $\epsilon$-poles ($M^{-1}$, as defined in Section II), some transcendental weight 2 functions associated with the finite pieces of Feynman integrals could contribute to the divergent IR structure. Of course, Catani’s formula rules out this possibility. However, we explore this issue putting flags in some integrals. Explicitly, triple collinear limit involves the massless
\begin{equation}
I_{\text{LCG}}^{\text{box}} = -i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 (q - p_2)^2 (q - p_{23})^2 (q - p_{123})^2 (n \cdot q)},
\end{equation}

which is known up to order $\epsilon^0$. If we perform a naive general $\epsilon$-expansion, we have

\begin{equation}
I_{\text{LCG}}^{\text{box}} = c_T g_5^2 \left( \frac{-s_{123} - i0}{\mu^2} \right)^{-\epsilon} \left( \frac{B_0}{\epsilon^2} + \frac{B_1}{\epsilon} + B_2 \right),
\end{equation}

where $B_0$ only contains rational functions and $B_i$ incorporates transcendental functions of weight up to $i$. So, we studied the cancellation of single $\epsilon$-poles without writing down the explicit form of $B_2$. Since subtracting $I_{\gamma q_1 q_2 \gamma_3}^{(1)}$ removes all the divergences, we obtained the following equation

\begin{equation}
\frac{1}{\epsilon} \left[ \frac{B_2 + S_{1\rightarrow 2} (B_2)}{2} + D(x_i, z_i) \right] = 0,
\end{equation}

with $D(x_i, z_i)$ only involves rational combinations of weight 2 functions. In consequence, this procedure allowed us to perform a cross-check among our polarized splitting results and the $\epsilon^0$ terms of LCG Feynman integrals, which were computed using other methods.

Following a more conventional path, we also checked that the final result is symmetric under the exchange $1 \leftrightarrow 2$. Another test consisted in taking the limit $C_A \to 0, N_f \to 0$ and compare it with $P_{\gamma \rightarrow q\bar{q}}^{\mu\nu}$. Explicitly, the relation

\begin{equation}
P_{\gamma \rightarrow q_1 q_2 \gamma_3}^{\mu\nu} = \frac{2C_A e^2 g_s^2}{g_5^2} \left( P_{\gamma \rightarrow q_1 q_2 \gamma_3}^{\mu\nu} \big|_{C_A \to 0, N_f \to 0} \right),
\end{equation}

turns out to be successfully verified. On the other hand, we contracted $P_{\gamma \rightarrow q\bar{q}}^{(1)\text{fin..}, \mu\nu}$ with $d_{\mu\nu}(\hat{P}, n)$ to recover the unpolarized splitting (which was computed with an independent implementation). Again, we found a complete agreement.

\section{Conclusions}

In this paper, we computed all the relevant polarized splitting functions in the triple collinear limit, for processes involving at least one photon: $\gamma \rightarrow q\bar{q}\gamma$, $\gamma \rightarrow q\bar{q}g$ and $g \rightarrow q\bar{q}\gamma$. We obtained the NLO corrections to these objects, working in CDR and using TL-kinematics, where strict collinear factorization is fulfilled.

Due to gauge invariance, photon-started polarized splitting functions are completely independent of $n^\mu$, and thus independent of the longitudinal-momentum fractions $z_i$. This reduces the

\footnote{Here we are ignoring the technical details associated with the $i0$ prescription in Feynman integrals because it is not relevant in the kinematical configurations taken into account.}
amount of transcendental functions required to express the results. Moreover, weight 2 components are very simple because they turn out to be proportional to the function \( R(x_i, x_j) \).

The fact that \( P^{\mu
u}_{q_1\bar{q}_2\gamma_3} \) is a gluon-initiated process implies a rather different behaviour of this splitting function compared with the others. In particular, LCG Feynman integrals are required for the computation. Also, the components of transcendental weight 2 depend on \( z_i \) and \( \Delta_{0,i}^{i,0} \). However, all these contributions are isolated in terms proportional to \( C_A \), because the Abelian part is related to \( P^{\mu
u}_{\gamma\gamma_1q_2\gamma_3} \).

All the results shown in this article were compared against their unpolarized version, presented in Ref. [27], and they were consistent. Besides that, we implemented some cross-checks among the polarized splittings, in particular, testing the Abelian limit \( (C_A \to 0, N_f \to 0) \). An alternative test was proposed to check the \( g \to q\bar{q}\gamma \) splitting. Relying in Catani’s formula, it is expected that single \( \epsilon \)-poles do not contain any weight 2 function. On the other hand, the \( \mathcal{O}(\epsilon^0) \) pieces of all the LCG integrals involved only contain this kind of functions. So, we took the LCG massless box and replaced the finite piece with a generic expression. Then, we forced the cancellation of single \( \epsilon \)-poles and obtained an additional constraint which relates Feynman integrals expansions and polarized splittings. Due to the fact that they were computed independently, this comparison provides another check to our results.

Finally, we would like to emphasize that NLO corrections to polarized splitting functions in the triple collinear limit are an essential ingredient for NNNLO computations and beyond. Pure QCD triple-splitting processes, which have a more complicated color structure, will be presented in a forthcoming article.

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