Ergodic branching diffusions with immigration: properties of invariant occupation measure, identification of particles under high-frequency observation, and estimation of the diffusion coefficient at nonparametric rates

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Abstract: In branching diffusions with immigration (BDI), particles travel on independent diffusion paths in $\mathbb{R}^d$, branch at position-dependent rates and leave offspring—randomly scattered around the parent’s death position—according to position-dependent laws. We specify a set of conditions which grants ergodicity such that the invariant occupation measure is of finite total mass and admits a continuous Lebesgue density.

Under discrete-time observation, BDI configurations being recorded at discrete times $i\Delta$ only, $i \in \mathbb{N}_0$, we lose information about particle identities between successive observation times. We present a reconstruction algorithm which in a high-frequency setting (asymptotics $\Delta \downarrow 0$) allows to reconstruct correctly a sufficiently large proportion of particle identities, and thus allows to recover $\Delta$-increments of unobserved diffusion paths on which particles are travelling. Picking some few well-chosen observations we fill regression schemes which, on cubes $A$ where the invariant occupation density is strictly positive, allow to estimate the diffusion coefficient of the one-particle motion at nonparametric rates.

MSC classification: primary 60J25, 62M05, 62G05; secondary 60J60, 60J75, 60J80, 62G07, 62G20

Key words: branching diffusions, ergodicity, invariant occupation measure; high frequency observation, reconstruction algorithm, estimation of the diffusion coefficient, kernel estimators, nonparametric rates

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Branching diffusions with immigration (BDI) as strong Markov processes of finite configurations of particles in $\mathbb{R}^d$ have been investigated since Ikeda, Nagasawa and Watanabe (\cite{19}, \cite{20}, \cite{21}, \cite{22}) who study semigroups and their generators, and construct the process by killing and repasting of strong Markov processes. In our BDI model, particles travel on independent diffusion paths, branch at position-dependent rates and leave offspring—randomly scattered around the parent’s death position—according to position-dependent laws; with some variations, this is close to \cite{27}, \cite{28}, \cite{29}, \cite{17}, \cite{18}, \cite{14}. Löcherbach (\cite{27}, \cite{28}) studies likelihood ratios for BDI processes and gives conditions for convergence to limit likelihood ratios, of some type (local asymptotic normality) which is important in parametric statistics, and knowledge of invariant measure or invariant occupation measure is required to check statistical assumptions. Höpfner, Hoffmann and Löcherbach \cite{17} focus on the point process of branching times/positions and estimate nonparametrically the spatial branching rate: again the statistical assumptions require sufficiently explicit knowledge of the invariant occupation measure and its moments. Löcherbach \cite{29} and Höpfner and Löcherbach \cite{18} study the invariant occupation measure on the single-particle space and obtain a continuous Lebesgue density in different settings. Both papers consider the case of local branching, i.e. newborn particles start at their parent’s death position. \cite{29} works with binary reproduction, uniform ellipticity, uniformly bounded $C^\infty$-coefficients and Malliavin calculus; \cite{18} with general position-dependent reproduction laws, finite order of smoothness of coefficients, but restrictive assumptions in view of duality techniques. Assuming that newborn particles scatter randomly around the parent’s death position, Hammer \cite{14} obtains a continuous density for the invariant measure on the configuration space by means of Fourier methods.

Our paper gives a set of relatively general conditions (such as: uniform exponential stability of the expectation semigroup, heat kernel bounds for the single-particle motion subject to position-dependent killing, . . . ; see section 2.1) which grant ergodicity of the BDI process, provide some information on finite ‘moments’ of the invariant measure up to some order (which is explicit from the family of position-dependent reproduction laws, see theorem 2.1.3), and imply that the invariant occupation measure is of finite total mass and admits a continuous Lebesgue density (see theorem 2.1.6). The ‘spatial subcriticality condition’ of \cite{18} reappears as main condition which grants ergodicity of the BDI process with finite invariant occupation measure (see assumption 1.2.2 and lemma 1.2.5 in section 1.2), however, with an important difference: in our model, similar to \cite{14}, we allow for essentially arbitrary non-local branching, i.e. offspring can be scattered randomly around the death position of the parent particle. As a consequence, the process entering our ‘spatial subcriticality condition’ and thus determining the shape of the invariant occupation measure is no longer the single-particle motion itself but a jump diffusion whose jumps represent—in a sense of a ‘many-to-one’-formula—the location
of a ‘typical child’ relative to the parent particle. Our approach allows us to avoid, to a large extent, restrictive smoothness assumptions on the coefficients or ellipticity conditions (of course, uniform ellipticity or smoothness ‘of low order’ may be welcome as a sufficient condition to check e.g. our heat kernel bound assumption 2.1.4 in section 2.1).

We then turn to a setting which has received a lot of attention in statistics of processes: if we observe a continuous-time process at discrete time points \( i \Delta \) only, \( i \in \mathbb{N}_0 \), how do we estimate those quantities which under continuous-time observation would induce mutual singularity of the laws of the process? The main example is volatility in diffusion processes: see e.g. Yoshida [31], Genon-Catalot and Jacod [12], Gobet [13], Podolskij and Vetter [34], Protter and Jacod [23]. In our situation of BDI processes \( (\eta_t)_{t \geq 0} \) which are configuration-valued, observation at discrete times forces us to consider a new type of problem. If we observe at discrete time points \( \{i \Delta : i \in \mathbb{N}_0\} \) not a diffusion path but the trajectory of a BDI process \( (\eta_t)_{t \geq 0} \), we will be left with pairs of configurations \( (\eta_{i \Delta}, \eta_{(i+1)\Delta}) \) without any information on the path history in-between; these are merely pairs of random point measures on the single-particle space. Information on branching or immigration events inside \( (i \Delta, (i+1)\Delta) \) is lost; even in case all particles succeeded to stay alive over the whole time interval \([i \Delta, (i+1)\Delta]\), we do not know which particle of the first configuration did travel to which position of the second configuration. In this context, we propose a reconstruction algorithm (see 3.1.4 and theorems 3.1.6 and 3.1.7 in section 3.1) which in case of high-frequency asymptotics (i.e. \( \Delta \downarrow 0 \)) will be able to recover correctly, to some large extent, particle identities in pairs \( (\eta_{i \Delta}, \eta_{(i+1)\Delta}) \) of successive configurations.

In a next step, we make use of the reconstruction algorithm and of the setting of high-frequency asymptotics \( \Delta \downarrow 0 \) to fill regression schemes for estimation of the diffusion coefficient \( \sigma \sigma^\top \) of the single-particle motion in the BDI process, picking out of the overwhelming amount of discrete-time data \( (\eta_{i \Delta})_{i \in \mathbb{N}_0} \) some few but well-selected pairs \( (\eta_{i \Delta}, \eta_{(i+1)\Delta}) \) of successive configurations for which we are sure –up to exceptional sets of vanishing probability as \( \Delta \downarrow 0 \)– that we reconstruct the particle identities correctly, for all particles involved. Reconstructing in this way particle identities and thus \( \Delta \)-increments for the trajectory of these particles, our regression scheme (see 4.1.1 and theorem 4.1.3 in section 4.1) consists of particles indexed by \( \alpha \) in some index set \( \mathcal{J}(\Delta) \) associated to \( \Delta \), and of pairs

\[
(\mathcal{X}_{\alpha}, Z_{\alpha}) \quad , \quad \alpha \in \mathcal{J}(\Delta)
\]

where \( \mathcal{X}_{\alpha} \) represents the position of particle \( \alpha \) at some random time \( \tau_{\alpha} \Delta \) and \( Z_{\alpha} \) the reconstructed (rescaled) increment \( \frac{\xi_{\alpha}((\tau_{\alpha}+1)\Delta)-\xi_{\alpha}(\tau_{\alpha}\Delta)}{\sqrt{\Delta}} \) for particle \( \alpha \) on \([\tau_{\alpha} \Delta, (\tau_{\alpha}+1)\Delta]\); note that the trajectory itself remains unaccessible from discrete-time data \( (\eta_{i \Delta})_{i \in \mathbb{N}_0} \). For fixed cubes \( A \) in the single-particle space on which the invariant occupation density is strictly positive, we can ensure that the ‘design
variables’ \( \mathcal{X}_\alpha, \alpha \in J(\Delta) \), are approximately regularly spaced over \( A \). If we associate to particles \( \alpha \) their driving Brownian motion \( W_\alpha \), results due to Jacod and Genon-Catalot \cite{12}, see also Podolskij and Vetter \cite{34}, yield good approximations of type

\[
Z_\alpha := \xi_\alpha((\tau_\alpha + 1)\Delta) - \xi_\alpha(\tau_\alpha \Delta) \approx \sigma(\mathcal{X}_\alpha) \frac{W_\alpha((\tau_\alpha + 1)\Delta) - W_\alpha(\tau_\alpha \Delta)}{\sqrt{\Delta}}, \quad \mathcal{X}_\alpha := \xi_\alpha(\tau_\alpha \Delta)
\]

which give

\[
Z_\alpha Z_\alpha^\top \approx (\sigma\sigma^\top)(\mathcal{X}_\alpha) + \text{error terms with some martingale structure}.
\]

All this holds on the ‘good sets’ where our reconstruction is indeed correct, i.e. on the complements of exceptional sets. If the probability of the exceptional sets vanishes as \( \Delta \downarrow 0 \), the contribution of what we believe –falsely on the exceptional set– to be an increment does not vanish: whereas in restriction to the ‘good sets’ density estimation of \( \sigma\sigma^\top(\cdot) \) works as in classical iid regression schemes for nonparametric estimation (Tsybakov \cite{39}), we have to take care of what happens on the exceptional sets in order to reach a balance of both types of contributions to the squared pointwise risk of a nonparametric estimator. We make this explicit in dimension \( d = 1 \) when we discuss kernel estimation of \( \sigma^2(\cdot) \) on intervals \( A \) on which the (continuous) invariant occupation density is strictly positive (theorem 4.3.1 in section 4.3).

The paper is organized as follows. The setting for ergodic BDI processes is exposed in section 1. Continuity of the Lebesgue density of the invariant occupation measure is proved in section 2. The reconstruction algorithm for particle identities in discretely observed BDI processes is the topic of section 3. Section 4 deals with regression schemes, filled from discrete observations, with the aim to estimate the diffusion coefficient of the single-particle motion; the example of kernel estimation of the diffusion coefficient in dimension \( d = 1 \) appears in section 4.3.

1 Ergodic branching diffusions with immigration: our setting

This section introduces branching diffusions with immigration (BDI) and their ergodicity properties. In a first subsection, we introduce BDI as strong Markov processes with life time, close to \cite{27}, \cite{28}, \cite{29}, \cite{18} but more general in that we allow for quite arbitrary spatial scattering of the descendants generated at a branching event (as in \cite{14}). Our method is a construction by killing and repasting of strong Markov processes as in \cite{20}, \cite{21}, \cite{22} or \cite{31}. In a second subsection we state a ‘spatial subcriticality’ condition which grants positive Harris recurrence with finite invariant occupation measure. A third subsection sketches proofs as far as their techniques are relevant for the rest of the paper.
1.1 BDI processes as strong Markov processes with life time

For $d \geq 1$, we write $E := \mathbb{R}^d$, $\mathcal{E} := \mathcal{B}(\mathbb{R}^d)$ and call $(E, \mathcal{E})$ single particle space. We call the space of (ordered) particle configurations $S := \bigcup_{\ell \in \mathbb{N}_0} E^\ell$ configuration space; $\delta$ denoting the void configuration, we have $E^0 = \{\delta\}$. We write $S := \mathcal{B}(S)$ for the Borel-$\sigma$-field on $S$: $(S, S)$ is a Polish space. Lebesgue measure on $S$ is defined layer-wise (for $\ell \geq 1$, its restriction to $E^\ell$ equals Lebesgue measure on $E^\ell$).

The length of a configuration $x \in S$ is denoted by $\ell(x)$, i.e. $\ell(x) = j$ iff $x \in E^j$. Sometimes we write a configuration $x \in S$ as a point measure $x(A) = \sum_{j=1}^{\ell(x)} \epsilon_x(A)$, $A \in \mathcal{E}$ (which equals 0 if $x = \delta$). To measurable functions $f : E \rightarrow \mathbb{R}$ we associate functions $\overline{f} : S \rightarrow \mathbb{R}$ by $\overline{f}(x) := f(x)$, i.e. $\overline{f}(\delta) := 0$.

With these notations, BDI will be a $(S, S)$-valued càdlàg strong Markov process with the following properties (A1)–(A4):

(A1) For $\ell \in \mathbb{N}$, on some random time interval which is specified through (A2) and (A3) below, $\ell$-particle configurations $X^\ell_t$ travel in $E^\ell$ as (strong) solutions to

$$\begin{align*}
X^\ell_t &= (X^1_t, \ldots, X^\ell_t) \quad \text{satisfying} \quad dX^j_t = b(X^j_t)dt + \sigma(X^j_t)dW^j_t, \quad 1 \leq j \leq \ell.
\end{align*}$$

Here $W^1, \ldots, W^\ell$ are independent $d$-dimensional Brownian motions. Drift $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and volatility $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are assumed to be Lipschitz.

(A2) i) Independently of each other, particles living at the same time are killed at position-dependent rate $\kappa : E \rightarrow (0, \infty)$; we assume that $\kappa$ is measurable and locally bounded on $E$.

ii) We have a transition probability $K_1(\cdot, \cdot)$ from $E$ to $\mathbb{N}_0$ such that $p_k(y) := K_1(y, \{k\})$ gives the probability for a particle killed in position $y \in E$ to produce $k$ offspring, $k \in \mathbb{N}_0$.

iii) We have a transition probability $K_2(\cdot, \cdot)$ from $E \times \mathbb{N}_0$ to $S$ with the property

$$K_2((y, k), \cdot) \text{ is concentrated on } E^k \text{ for all } y \in E, \ k \in \mathbb{N}_0$$

which scatters offspring generated at a branching event relative to the parent’s position: $k$-particle offspring of a particle killed in position $y$ will be located in positions

$$y + v_1, \ldots, y + v_k \quad \text{with probability} \quad K_2((y, k), dv_1, \ldots, dv_k), \quad k \geq 1, \ y \in E,$$

and in case $k = 0$ we put $K_2((y, 0), \cdot) = \epsilon_\delta(\cdot)$.
An important special case contained in (A2) iii) is given by product kernels

\[ K_2((y, k), dv_1, \ldots, dv_k) = \prod_{j=1}^{k} K(y, dv_j) \]

for some fixed transition probability \( K(\cdot, \cdot) \) on \((E, \mathcal{E})\). Specializing further, if \( K(y, dv) = q(dv) \) for some probability measure \( q \) on \((E, \mathcal{E})\), independently of \( y \), the distribution of newborn particles relative to their parent’s position is spatially homogeneous. Finally, \( q(dv) := \epsilon_0(dv) \) is the commonly considered case that particles are born exactly at the death position of their parent; we will refer to this special case as \emph{local branching}. In this paper, we shall work under (1) and shall not even assume (2), i.e. we allow for arbitrary non-local branching mechanisms.

\( \text{(A3)} \) For some probability measure \( Q^i \) on \((E, \mathcal{E})\) and some constant \( 0 < c < \infty \), single immigrants arrive at constant rate \( 0 < c < \infty \) and are located in \( E \) according to \( Q^i(dy) \), independently of everything else.

Write \((\Omega, \mathcal{A})\) for the canonical path space of càdlàg functions \([0, \infty) \to S\) with life time \( \zeta \leq \infty \), and \( \eta = (\eta_t)_{t \geq 0} \) for the canonical process on \((\Omega, \mathcal{A})\). Then (A1)–(A3) above determine uniquely a probability measure \( Q \) on \((\Omega, \mathcal{A})\) under which \( \eta \) is a jump diffusion with life time. As long as \( t < \zeta \), jumps (finitely many on finite time intervals) arrive at rate

\[ c + \tau(\eta) = c + \sum_{j=1}^{\ell} \kappa(\eta_t^j) \quad \text{when} \quad \eta_t = (\eta_t^1, \ldots, \eta_t^\ell) \quad \text{belongs to} \quad E^\ell. \]

By convention, the rate is \( c \) when \( \ell = 0 \). Note that by the Lipschitz assumptions on drift and diffusion coefficient in (A1), and by the local boundedness of \( \kappa \) in (A2), the life time \( \zeta \leq \infty \) of the process is the first accumulation point of the sequence of successive jumps times \( T_j \): we have \( T_j < T_{j+1} \) as long as \( T_j \) is finite, and \( \zeta := \sup_j T_j \leq \infty \). On events \( \{T_j < \infty\} \), representing the configuration \( \eta_{T_j^-} \) immediately before the jump by \( x = (x^1, \ldots, x^\ell) \), \( \ell \geq 1 \), the new configuration \( \eta_{T_j} \) at time \( T_j \) is obtained from \( x \) as follows:

\[
\begin{align*}
&\left\{ \begin{array}{l}
(x^1, \ldots, x^{j-1}, x^{j+1}, \ldots, x^\ell) \\
(x^1, \ldots, x^{j-1}, x^j + v_1, \ldots, x^j + v_k, x^{j+1}, \ldots, x^\ell) \\
(x^1, \ldots, x^\ell, y)
\end{array} \right\} \\
&\quad \text{w. pr. } \frac{\kappa(x^j) p_0(x^j)}{c + \kappa(x)} , \\
&\quad \text{w. pr. } \frac{\kappa(x^j) p_k(x^j)}{c + \kappa(x)} K_2((x^j, k), dv_1, \ldots, dv_k) , \\
&\quad \text{w. pr. } \frac{\epsilon_k}{c + \kappa(x)} Q^i(dy) .
\end{align*}
\]

With exception of \( k \) chosen equal to 1, jumps change the length of the configuration. In case \( \ell = 0 \), \( \eta_{T_j^-} \) is the void configuration \( \delta \), thus \( \eta_{T_j} \) a one-particle configuration with \( y \) selected by \( Q^i(dy) \).
When we deal with trajectories of individual particles in the BDI process, we write
\[
d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dW_t
\]
for the single-particle motion on \(E\). Assumption (A1) on drift \(b(\cdot)\) and diffusion coefficient \(\sigma(\cdot)\) grants that the diffusion \(\xi\) has infinite life time.

(A4) i) We have \(\int_0^\infty \kappa(\xi_s)\,ds = \infty\) almost surely, for every choice of a starting point \(y \in E\) for \(\xi\).

ii) Reproduction means \(y \to \rho(y) := \sum_{k \geq 0} k p_k(y)\) are (finite and) locally bounded on \(E\).

Condition (A4) i) grants that all \(T_j\) defined above are almost surely finite stopping times.

Assumptions (A1)–(A4) and all notations of the present subsection will hold throughout the paper. So far, our construction of the BDI process is the canonical one: \((\Omega, \mathcal{A})\) is the canonical path space of càdlàg functions \([0, \infty) \to S\) with life time \(\zeta \leq \infty\), \(\eta = (\eta_t)_{t \geq 0}\) is the canonical process on \((\Omega, \mathcal{A})\), and we have a unique probability law \(Q\) on \((\Omega, \mathcal{A})\) such that \(\eta\) is strongly Markov with the above properties: a jump diffusion with successive jump times \((T_j)_j\) which are finite stopping times and increase towards \(\zeta \leq \infty\).

1.2 Ergodicity

In this subsection, we state a set of sufficient conditions which ensure that the BDI process \(\eta\)

i) is positive Harris recurrent, admitting the void configuration \(\delta\) as a recurrent atom (thus in particular, \(\eta\) will have infinite life time \(\zeta = \infty\));

ii) admits a finite invariant occupation measure on \((E, \mathcal{E})\).

Up to the general form of our kernel \(K_2(\cdot, \cdot)\) in (A2) iii), we follow the same approach as Löcherbach [27], [28], [29], Höpfler and Löcherbach [18] section 1.4; see also Hammer [14] section 3 where the same general form of non-local branching was allowed. Introducing the necessary notation we state the relevant results.

1.2.1 Assumption The functions \(\kappa\) and \(\rho\) are bounded on \(E\).

This assumption guarantees in particular non-explosion of the process \(\eta\): by 1.2.1 \(\eta\) has infinite life time \(\zeta = \infty\) almost surely (and from here on, we will take \(\Omega\) as the usual Skorohod path space of
càdlàg functions \([0, \infty) \to S\). Using the kernel \(K_2(\cdot, \cdot)\) from assumption A2 iii), we define a transition probability \(Q^f(\cdot, \cdot)\) on the single-particle space \((E, \mathcal{E})\) as follows: for \(y \in E\) and \(f : E \to [0, \infty)\) measurable, let

\[
Q^f(y, f) := \frac{1}{\rho(y)} \sum_{k \in \mathbb{N}} p_k(y) \int_{E^k} \left( f(y + v_1) + \cdots + f(y + v_k) \right) K_2 \left((y, k), dv_1, \ldots, dv_k\right);
\]

if \(y \in E\) is such that \(\rho(y) = 0\) we put \(Q^f(y, \cdot) := \nu(\cdot)\), for some fixed probability measure \(\nu\) on \(E\).

In the following we write \(\mathcal{L}\) for the Markov generator of the single-particle-motion \(\xi\) on \(E = \mathbb{R}^d\)

\[
\mathcal{L}f(y) = \sum_{i=1}^d b_i(y) \partial_i f(y) + \frac{1}{2} \sum_{i,j=1}^d a_{i,j}(y) \partial_{i,j} f(y)
\]

where \(a = \sigma \sigma^\top\). Let us introduce a jump diffusion \(\tilde{\xi}\) on \(E\) by defining a generator

\[
\tilde{\mathcal{L}}f(y) := \mathcal{L}f(y) + \kappa(y) \rho(y) \int_E [f(w) - f(y)] Q^f(y, dw);
\]

here, with \(K_2(\cdot, \cdot)\) from (A2) iii), the integral contribution equals

\[
\kappa(y) \sum_{k \in \mathbb{N}} p_k(y) \int_{E^k} \left( f(y + v_1) + \cdots + f(y + v_k) \right) - f(y) \right) K_2 \left((y, k), dv_1, \ldots, dv_k\right).
\]

These generators should be understood in the sense of the corresponding martingale problems. The jump diffusion \(\tilde{\xi}\) can be defined probabilistically in the sense of killing and repasting of strong Markov processes, cf. Ikeda, Nagasawa and Watanabe [19], [20], [21], [22] and Nagasawa [31]: a diffusive motion according to \(\xi\) is killed at position-dependent rate \(\kappa \rho\) and restarted in a position selected by \(Q^f(y, dw)\), independently of everything else. Since \(\xi\) has infinite life time and since \(\kappa \rho\) is bounded in virtue of assumption 1.2.1, also the jump diffusion \(\tilde{\xi}\) has infinite life time.

1.2.2 Assumption We assume

\[
E_y \left( \int_0^\infty e^{-\int_0^t [\kappa(1-\rho)](\xi_s) \, ds} \, dt \right) < \infty \quad \text{for all } y \in E,
\]

\[
y \to E_y \left( \int_0^\infty e^{-\int_0^t [\kappa(1-\rho)](\tilde{\xi}_s) \, ds} \, dt \right) \quad \text{belongs to } L^1(Q^f).
\]

If we think of \(\kappa(1 - \rho)\) as a rate of annihilation/creation of mass, [9] or [10] deal with the total mass of the \(\kappa(1 - \rho)\)-resolvent kernel of the jump diffusion \(\tilde{\xi}\). Assumption 1.2.2 generalizes condition (6) in [18] to kernels satisfying (A2) iii). It implies, see lemma 1.2.4 below, ‘spatial subcriticality’ in the sense of almost certain extinction of families starting from one ancestor located in \(y \in E\).
1.2.3 Definition We shall write \( \eta^r \) for the branching diffusion \( \eta \) without immigration arising as subprocess of all direct descendants of one or several ancestors at some initial time 0. When there is need to specify positions \( y \) for one or \( y_1, \ldots, y_m \) for several ancestors, we write \( \eta^{r,y} \) or \( \eta^{r,y_1, \ldots, y_m} \).

Recalling notation \( \mathcal{T}(x) = \sum_{j=1}^\ell f(x_j) = x(f) \) for \( x = (x_1, \ldots, x_\ell) \in S \) with convention \( \mathcal{T}(\delta) = 0 \), let
\[
H^r(y, f) := E_y \left( \int_0^\infty \mathcal{T}(\eta^r_t) \, dt \right) \leq \infty, \quad y \in E, \ f : E \to [0, \infty) \text{ measurable}
\]
denote the expected occupation measure (finite or not) for \( \eta^r \) starting from one ancestor in \( y \in E \).

1.2.4 Lemma Under assumptions \([1.2.1] \) and \([6] \) of \([1.2.2] \) the total mass of the expected occupation measure for the progeny of an ancestor starting in position \( y \in E \) is finite: We have
\[
H^r(y, 1) = E_y \left( \int_0^\infty \mathcal{T}(\eta^r_t) \, dt \right) = E_y \left( \int_0^\infty e^{-\int_0^t \left( \kappa (1 - \rho) \left( \tilde{\xi}_s \right) \right) \, ds} \, dt \right) < \infty.
\]

1.2.5 Lemma Under assumptions \([1.2.1] \) and \([7] \) of \([1.2.2] \) the BDI process \( \eta \) is positive Harris recurrent, admits the void configuration \( \delta \) as a recurrent atom, and has finite invariant occupation measure
\[
\bar{\mu}(A) = c \left[ Q^1 H^r \right](A) = c \int_E Q^1(dy) H^r(y, 1_A) < \infty, \quad A \in \mathcal{E}
\]
with \( c, Q^1 \) of \([A3] \) and \( H^r(\cdot, \cdot) \) given by \([3] \). The choice of the constant in \([9] \) relates \( \bar{\mu} \) to the invariant probability \( \mu \) of the BDI process through
\[
\bar{\mu}(f) = \mu(f) \quad \text{for all } f : E \to [0, \infty) \text{ measurable}.
\]

1.3 Sketching the proofs, and some further notation

This subsection will sketch proofs for lemmata \([1.2.4] \) and \([1.2.5] \)–assertions which generalize results from \([18] \) to kernels according to \([A2] \) iii– as far as the techniques which appear are of importance for the rest of the paper.

Proof of lemma \([1.2.4] \): Consider the process \( \eta^r \) starting from one ancestor in \( y \in E \). By \([A4] \),
the time \( \tau \) of the first branching event in \( \eta^r \) is a.s. finite, thus \( \eta^r_{\tau-} (E\text{-valued}) \) and \( \eta^r_{\tau-} (S\text{-valued}) \) are well-defined random variables. The strong Markov property yields
\[
H^r(y, f) = E_y \left( \int_{[0, \tau]} f(\eta^r_t) \, dt \right) + E_{\eta^r_{\tau-}} \left( \int_0^\infty \mathcal{T}(\eta^r_t) \, dt \right)
\]
\[
= E_y \left( \int_{[0, \tau]} f(\eta^r_t) \, dt \right) + \mathcal{T}(\eta^r_{\tau-}) \cdot H^r(\cdot, f)(\eta^r_{\tau-})
\]
where we combine definition (5) of \( H^\tau (\cdot , \cdot ) \) with the branching property (i.e. the fact that particles evolve independently). Writing the second contribution on the right hand side conditionally on \( \eta^\tau_{\cdot -} = z \)

\[
\sum_{k=1}^{\infty} p_k(z) \int_{E^k} K_2 ((z,k), dv_1, \ldots, dv_k) [H^\tau (z + v_1, f) + \ldots + H^\tau (z + v_k, f)]
\]

which by definition of \( Q^\tau (\cdot , \cdot ) \) in (3) equals

\[
\rho(z) \int_E Q^\tau (z, dw) H^\tau (w, f),
\]
equation (10) takes the form

\[
H^\tau (y, f) = E_y \left( \int_{[0,\tau]} f(\eta^\tau_t) dt + \rho(\eta^\tau_{\tau -}) \int_E Q^\tau (\eta^\tau_{\tau -}, dw) H^\tau (w, f) \right).
\]

Note that conditionally on \( \eta^\tau_0 = y = \xi_0 \in E \), and up to time \( \tau \) of position-dependent killing at rate \( \kappa \), \( \eta^\tau \) is a single-particle diffusion \( \xi \). Introducing the \( \kappa \)-resolvent kernel of the diffusion \( \xi \)

\[
R_\kappa(y, f) := E_y \left( \int_0^\infty f(\xi_t) e^{-\int_0^t \kappa(\xi_s) ds} dt \right) = E_y \left( \int_0^\tau f(\xi_t) dt \right)
\]

\((y \in E, f : E \to [0,\infty) \) measurable\), the law of \( \eta^\tau_{\tau -} \) starting from \( \eta^\tau_0 = y \in E \) is given by \([R_\kappa \rho](y, \cdot )\), and we can rewrite (11) as

\[
H^\tau (y, f) = R_\kappa(y, f) + [R_\kappa \rho Q^\tau H^\tau](y, f)
\]

which allows for iteration. By (11) and (13), the expected occupation measure (5) has the following interpretation: At rate \( \kappa \), we erase unit mass travelling along the trajectory of \( \xi \), replace it by mass \( \rho \) (generating \( k \) particles with probability \( p_k \), and then merging these \( k \) particles), then shift the location \( \eta^\tau_{\tau -} \) of the merged mass to a random position \( w \) selected according to \( Q^\tau (\eta^\tau_{\tau -}, dw) \). The underlying strongly Markovian system (again defined probabilistically by killing and repasting since the corresponding semigroup is i.g. not contractive) has the generator

\[
\mathcal{L} f(z) - \kappa(z) f(z) + \kappa(z) \rho(z) \int_E Q^\tau(z, dw) f(w)
\]

(14)

\[
= \mathcal{L} f(z) - [\kappa(1-\rho)](z) f(z) + \kappa(z) \rho(z) \int_E Q^\tau(z, dw) [f(w) - f(z)]
\]

\[
= \tilde{\mathcal{L}} f(z) - [\kappa(1-\rho)](z) f(z)
\]

(notations from (4), (3), (3)) and is thus identified as the jump diffusion \( \tilde{\xi} \) on \( E \) ‘killed’ at position-dependent rate \( z \to [\kappa(1-\rho)](z) \) (of course, since we do not assume \( \rho \leq 1 \), speaking of ‘killing’ is abuse of language). Iteration of (13) combined with (14) then provides us with the following explicit solution to (5):

\[
H^\tau (y, f) = \sum_{n \in \mathbb{N}_0} [(R_\kappa \kappa Q^\tau)^n R_\kappa](y, f)
\]

(15)

\[
= E_y \left( \int_0^\infty dt \tilde{\xi}_t e^{-\int_0^t [\kappa(1-\rho)](\tilde{\xi}_s) ds} \right) \leq \infty.
\]
This is the $\kappa(1-\rho)$-resolvent kernel of the jump diffusion $\tilde{\xi}$, well-defined since $f \geq 0$, and finite for bounded $f$ by (1) in assumption 1.2.2. We now take $f \equiv 1$ in (15) and (8). □

Compare the last proof to [18], lemma 1.4 and its proof, and to [14], Prop. 3.2.21 and Cor. 3.2.30, and note the role of the kernel $K_2(\cdot,\cdot)$ from (A2) iii) which scatters offspring produced at branching events: Indeed, the kernel defined in [14], (3.2.23) corresponds exactly to our definition of $Q_r$ in (3).

1.3.1 Remark Following definition 4.10 in Ikeda, Nagasawa and Watanabe [22], the semigroup $(M_t)_{t \geq 0}$ on the single-particle space $(E,\mathcal{E})$

$$(16) \quad M_t(y, f) := E_y \left( \tilde{f}(\eta^t_\cdot) \right), \quad t \geq 0, \ y \in E, \ f : E \to [0, \infty) \text{ measurable}$$

is called expectation semigroup for the branching diffusion without immigration $\eta^x$. In case $f = 1_A$, $M_t(y, A) = E_y (\eta^t_\cdot (A))$ is the expected number of particles visiting $A$ at time $t$ which descend from a single ancestor in $y$ at time 0. This semigroup was implicit in definition 1.2.3, via $H^x(y, f) = \int_0^\infty M_t(y, f) \, dt$. In analogy with the derivation (10)-(15), one can use the strong Markov property and the branching property to obtain by iteration the representation

$$(17) \quad M_t(y, f) = E_y \left( f(\tilde{\xi}_t) \, e^{-\int_0^t \kappa(1-\rho)(\tilde{\xi}_v) \, dv} \right)$$

which identifies the expectation semigroup as the Feynman-Kac semigroup corresponding to the jump diffusion $\tilde{\xi}$ with generator (5) and the ‘potential’ $\kappa(1-\rho)$. We refer to [14], Thm. 3.2.28 for a full proof under our present assumptions.

Identities of the form (16)+(17) expressing the expected number of particles in terms of the dynamics of a single particle have a long history, going back to (at least) Watanabe [40]. They are now commonly called ‘many-to-one’-formulas (see e.g. [15]) and have been generalized in various ways; in particular, in (16) the function $\tilde{f}(\eta^t_\cdot)$ may be replaced by a functional depending on the whole path of the process up to time $t$. However, most of this literature tends to focus on the case of local branching mechanisms where particles reproduce exactly at their death position. See [2] and [30] for versions admitting non-local branching, also employing an auxiliary process as our jump diffusion $\tilde{\xi}$, but still under stronger conditions on the offspring mechanism than our assumption (A2) iii) ([2] assumes in addition constant rates).

Now we can prove lemma 1.2.5.
Proof of lemma 1.2.5:

1) By lemma 1.2.4 (where $\ell(x) \geq 1$ for $x \neq \delta$) and in virtue of (7) in assumption 1.2.2, the expected time to extinction of a subprocess $\eta^{r,j}$ of $\eta$ defined by all descendents of the $j$-th immigrant is finite, the $j$-th immigrant choosing its location according to $Q^i$ by (A3).

Since the process of immigration instants is a Poisson random measure with constant intensity $c$ on $(0,\infty)$, the BDI process $\eta$ will a.s. in the long run return infinitely often to the void configuration $\delta$.

2) By 1), the BDI process $\eta$ can be rewritten in the form of a sum of i.i.d. excursions away from the void configuration $\delta$. Write $R_1, R_2, \ldots$ for the times of successive returns to $\delta$, and define a measure on the configuration space $(S,S)$ by

\( (18) \quad \mu(F) := E_\delta \left( \int_0^{R_1} 1_F(\eta_s) \, ds \right), \quad F \in S. \)

Sets $F \in S$ of positive $\mu$-measure are visited infinitely often in the long run, a.s. for every choice of a starting point in $S$. Thus $\eta$ is a Harris process (we refer to [1] and [35], [32], [33] for Harris recurrence). A Harris process has a unique (up to constant multiples) invariant measure which for the moment we may call $\tilde{\mu}$. We know that $\tilde{\mu}$ is equivalent to $\mu$, and we have ratio limit theorems: for pairs of measurable functions $f, g : S \to [0,\infty)$, with $g$ such that $0 < \tilde{\mu}(g) < \infty$, the limits

\[ \lim_{t \to \infty} \frac{\int_0^t f(\eta_s) \, ds}{\int_0^t g(\eta_s) \, ds} = \frac{\tilde{\mu}(f)}{\tilde{\mu}(g)} \]

exist almost surely, for every choice of a starting point $x \in S$. The structure of $\eta$ as a sum of i.i.d. excursions away from $\delta$ then allows to identify the limits with

\[ \lim_{n \to \infty} \frac{\int_0^{R_n} f(\eta_s) \, ds}{\int_0^{R_n} g(\eta_s) \, ds} = \frac{\mu(f)}{\mu(g)}. \]

Hence invariant measure $\tilde{\mu}$ equals $\mu$ defined in (18), up to constant multiples. This shows that $\mu$ defined in (18) is invariant for the BDI process $\eta$.

3) Associate to the invariant measure $\mu$ on the configuration space $(S,S)$ defined by (18) an invariant occupation measure $\overline{\mu}$ on the single particle space $(E,E)$ via

\[ (19) \quad \overline{\mu}(f) := \mu(\overline{f}) = E_\delta \left( \int_0^{R_1} \overline{f}(\eta_s) \, ds \right), \quad f : E \to [0,\infty) \text{ measurable}. \]

Let $(\tau^1_j)_{j \geq 1}$ denote the sequence of successive immigration times, write $\tau^j_0$ for the time of extinction of the subprocess $(\eta^{r,j}_t)_{t \geq 0}$ of all direct descendents of the ancestor who immigrated at time $\tau^1_j$, then

\[ \lim_{n \to \infty} \frac{\int_0^{R_n} \overline{f}(\eta_s) \, ds}{\int_0^{R_n} \overline{g}(\eta_s) \, ds} = \frac{\mu(\overline{f})}{\mu(\overline{g})} = \frac{\overline{\mu}(f)}{\overline{\mu}(g)}. \]
coincides with $H^r$ from (8) and $Q^i$ from (A3) with

$$
\lim_{j \to \infty} \int_0^{\tau_j} f(\eta_s) ds = \lim_{m \to \infty} \frac{\sum_{j=1}^m \int_{\tau_j}^{\tau_{j+1}} f(\eta_s) ds}{\sum_{j=1}^m \int_{\tau_j}^{\tau_{j+1}} \tilde{g}(\eta_s) ds} = \frac{E_\delta \left( \int_0^\infty \tilde{f}(\eta_s) ds \right)}{E_\delta \left( \int_0^\infty \tilde{g}(\eta_s) ds \right)} = \frac{[Q^i H^r](f)}{[Q^i H^r](g)}
$$

in application of definition 1.2.3. This shows that $\overline{\mu}$ in (19) equals $Q^i H^r$, up to some multiplicative constant. Combining (15) with (1) in assumption 1.2.2 we see that $\mu(1) = \mu(1) = \mu(\ell)$ is finite. This implies $\mu(1) = E_\delta(R_1) < \infty$, and we have the assertion of the lemma up to choice of norming constants: $\mu$ on $(E, \mathcal{E})$ is a finite measure, thus we have positive Harris recurrence of the BDI process $\eta$ with finite invariant occupation measure.

4) It remains to determine the constants. Define $J_n := \max\{j : \tau_j^1 < R_n\}$ with notations of 3). As a consequence of (A3) we have almost surely as $n \to \infty$

$$
J_n \sim c R_n \sim c E_\delta(R_1) n
$$

for every choice of a starting point for the process $\eta$ (where $E_\delta(R_1) > \frac{1}{c}$ shows that the right hand side is necessarily larger than $n$), together with

$$
n \overline{\mu}(f) \sim \int_0^{R_n} \tilde{f}(\eta_s) ds = \sum_{j \leq J_n} \int_{\tau_j}^{\tau_{j+1}} \tilde{f}(\eta_s) ds \sim J_n [Q^i H^r](f) \sim n c E_\delta(R_1) [Q^i H^r](f)
$$

almost surely as $n \to \infty$. This establishes

$$(20) \quad \overline{\mu}(f) = c E_\delta(R_1) [Q^i H^r](f)$$

when invariant measure is defined by (18) and invariant occupation measure by (19). Now, dividing the right hand sides of (18) + (7) and both sides of (20) by $E_\delta(R_1)$ and changing notations correspondingly, we get the assertion of the lemma with respect to the invariant probability. □

2 Some properties of the invariant probability and the invariant occupation measure

We state and prove two theorems on the invariant measure and the invariant occupation measure. Both will be key tools in the statistical context of sections 3 and 4. Theorem 2.1.3 deals with finite ‘moments’ $\mu(\ell^q)$ of the invariant probability $\mu$ of the BDI process of the same order $q$ as the reproduction law in (A2) ii). Theorem 2.1.6 gives conditions which grant existence of a continuous Lebesgue density of the invariant occupation measure $\overline{\mu}$. The proofs are given in sections 2.2 and 2.3.
For the special case of local branching where particles reproduce exactly at their death position, the existence of a continuous invariant occupation density has been considered by Höpfner and Löcherbach [18]; with different methods, Löcherbach [29] and Hammer [14] allow for interactions between particles (see remark 2.1.7 below). In our setting, due to the general form of the kernel in (A2) iii) which scatters offspring generated at a branching event relative to the parent’s position, we take a different approach.

2.1 Two theorems

We introduce further assumptions (not all of these will be in force at the same time) and strengthen preceding ones. From now on, 1.2.1 and 1.2.2 are always assumed, µ is the invariant probability of the BDI process η on the configuration space (S, S), and π the invariant occupation measure on the single particle space (E, E) as specified by (9) in lemma 1.2.5.

2.1.1 Assumption There is some natural number q > 1 such that y → m_q(y) is bounded on E, where

\[ m_q(y) := \sum_{k \in \mathbb{N}_0} k^q p_k(y) \leq \infty \]

denotes q-th moments of the position-dependent reproduction laws \((p_k(y))_k\) at \(y \in E\) in (A2) ii).

Our next assumption strengthens heavily (7) of assumption 1.2.2. Recall the expectation semigroup \((M_t)_{t \geq 0}\) for the branching process without immigration \(\eta^*\) from (16), associated to the expected occupation measure (8), and its representation as a Feynman–Kac semigroup in the ‘many-to-one’-formula (17) in remark 1.3.1.

2.1.2 Assumption With notation \(|||M_t||| := \sup_{y \in E} M_t(y, E) = \sup_{y \in E} E_y \left( e^{-\int_0^t [\kappa (1-\rho) (\tilde{\xi}_v)dv]} \right)\), we have

\[ \limsup_{t \to \infty} \frac{1}{t} \log (|||M_t|||) < 0 . \]

Assumption 2.1.2 implies in particular that the function in (6) is bounded, thus (7) of assumption 1.2.2 holds for any choice of an immigration measure \(Q^1\). Property (21) is known in the general theory of semigroups as uniform exponential stability. We refer to [9], Ch. V, Sec. 1 for a number of
equivalent characterizations which can be used to check our assumption 2.1.2 whenever the semigroup $(M_t)_{t \geq 0}$ is strongly continuous on the Banach space $C_0(E)$ of continuous functions vanishing at infinity.

2.1.3 Theorem Under 1.2.1, 2.1.1 and 2.1.2, we have finite ‘moments’ of the invariant measure

$$\mu(\ell^q) := \int_S \ell^q(x) \mu(dx) = \sum_{\ell \in \mathbb{N}} \ell^q \mu(E^\ell) < \infty$$

where $q > 1$ is specified by assumption 2.1.1.

Theorem 2.1.3 will be proved in section 2.2. Our next assumption concerns the semigroup

$$P^\kappa_t(y, f) := \mathbb{E}_y\left( f(\xi_t) e^{-\int_0^t \kappa(\xi_s) \, ds} \right) \quad t \geq 0, \ y \in E, \ f : E \to [0, \infty) \text{ measurable}$$

of the single-particle motion $\xi$ killed at rate $\kappa$. The semigroup (22) was already implicit in the proof of lemma 1.2.4, see (12). For this semigroup, we shall now require existence of heat kernel bounds (which Hammer [14] used to investigate the invariant measure $\mu$ on $S$, see remark 2.1.7 below).

For sufficient conditions implying such bounds, we refer to Dynkin [7] theorem 0.5 p. 229 appendix paragraph 6, or Friedman [11] theorem 4.5 p. 141.

2.1.4 Assumption The semigroup in (22) admits densities $p^\kappa_t(y, z) \, dz = P^\kappa_t(y, dz)$ with respect to Lebesgue measure which are continuous in $z$ for fixed $y$ and admit bounds

$$p^\kappa_t(y, z) \leq Ct^{-d/2} e^{-\frac{1}{2} \frac{|z-y|^2}{C^2 t}} \quad \text{for all } 0 < t \leq t_0, \ y, z \in E$$

for some $t_0 > 0$ fixed and some positive constant $C$.

Heat kernel bounds 2.1.4 will be a key tool in our proof for the existence of a continuous invariant occupation density, as well as for the results in section 3 below. We stress that 2.1.4 is a strong assumption: even with $d = 1$ and constant killing rate $\kappa \equiv 1$ it does not hold for Ornstein-Uhlenbeck one-particle motion $d\xi_t = -\vartheta \xi_t dt + dW_t$ when the OU parameter $\vartheta$ is different from 0. On the other hand, by Dynkin [7] p. 229, assumption 2.1.4 does hold for all choices of a Hölder continuous and bounded killing rate $\kappa$ whenever the single-particle motion $\xi$ is such that uniform ellipticity holds on $E$ and all $|b^i|, |\sigma^{i,j}|$ in (A1) are bounded. Our final assumption requires that the transition probability $Q^t(\cdot, \cdot)$ of (3) admits bounds of convolution type.
2.1.5 Assumption There exists some finite measure \( \hat{Q}^r \) on the single-particle space \((E, \mathcal{E})\) such that

\[
Q^r(y, A) \leq \hat{Q}^r(A - y), \quad y \in E, \ A \in \mathcal{E}.
\]

Note that (24) is essentially a condition on the transition probability \( K_2(\cdot, \cdot) \) of (A2) iii). Clearly assumption 2.1.5 covers the case of a product structure (2) where \( K(y, dv) = q(dv) \) for some probability measure \( q \) on \( E \): here we take \( \hat{Q}^r := q \) and have equality in (24). It also covers the case of absolutely continuous product structures

\[
K_2((y, k), dv_1, \ldots, dv_k) = \prod_{j=1}^{k} q((y, k), v_j) \nu(dv_j)
\]

for \( \sigma \)-finite measures \( \nu \) on \( E \) when \( \nu \)-densities depend on \( y \) and \( k \) but are uniformly dominated by

\[
q((y, k), v) \leq \hat{q}(v), \quad y \in E, \ k \in \mathbb{N}, \ v \in E
\]

where \( \hat{q} \in L^1(\nu) \); then (24) holds for \( \hat{Q}^r(A) := \int_A \hat{q}(v) \nu(dv) \). Note that we do not require the \( \sigma \)-finite measure \( \nu \) to be Lebesgue-absolutely continuous. Beyond (25) and (26), we see from (3) that assumption 2.1.5 controls in some sense the distance of a ‘typical’ child from its parent’s position.

The following is the second main probabilistic result: heat kernel bounds 2.1.4 for particle motion killed at rate \( \kappa \) and convolution bounds 2.1.5 on the scattering of offspring at branching events allow to obtain a continuous Lebesgue density for the invariant occupation measure. Theorem 2.1.6 will be proved in section 2.3.

2.1.6 Theorem Assume 1.2.1, 2.1.4, 2.1.5, and suppose that the immigration measure \( Q^i \) is such that condition (7) of 1.2.2 is satisfied. If \( d \geq 2 \), suppose in addition that \( Q^i(dx) = q^i(x)dx \) is absolutely continuous with Lebesgue density \( q^i \in L^p(\mathbb{R}^d) \) for some \( p \in \left( \frac{d}{2}, \infty \right] \). Then the invariant occupation measure \( \mu \) on \((E, \mathcal{E})\) (a finite measure by lemma 1.2.5) admits a continuous Lebesgue density \( \gamma \in C_0(E) \).

2.1.7 Remark i) Höpfner and Löcherbach [18] proved existence of a continuous Lebesgue density for \( \mu \) in the special case of local branching, i.e. when \( K_2(\cdot, \cdot) \) of (A2) iii) is of product type (2) with \( K(y, dv) = \epsilon_0(dv) \). Their approach, using stochastic flows of diffeomorphisms, is not directly applicable in our case of non-local branching where we allow for jumps in the distribution of newborn particles, reflected in the jump diffusion \( \tilde{\xi} \) with generator (5). However, it can be adapted to our setting by
using duality theory for (Feller) semigroups. This approach, which will be taken up in another paper, leads to a continuous invariant occupation density under an alternative set of conditions on the single particle motion and the branching and reproduction mechanism. In the present work however, we restrict to the setting of assumptions 2.1.4 and 2.1.5, since the heat kernel bounds \([23]\) will also be used (independently) in the proofs of our results in section 3 below.

ii) For the case of local and binary branching, Löcherbach \([29]\) considered a generalization of the model where coexisting particles move as interacting diffusions. In a \(C^\infty\)-setting, assuming uniform ellipticity, Malliavin calculus establishes the existence of a continuous invariant occupation density (theorem 4.2 in \([29]\)).

iii) Assuming existence of Lebesgue densities \(q^i\) for \(Q^i\) and of transition densities for \(K^2(\cdot, \cdot)\) as in \([25]-[26]\) such that the Fourier transforms of \(q^i\) and \(\hat{q}\) are integrable, Hammer \([14]\) used Fourier methods to deduce existence of a continuous Lebesgue density of the invariant measure \(\mu\) on the configuration space \(S\) from the heat kernel bound assumption 2.1.4 (see assumptions 2.2.1, 2.2.5 and theorem 2.2.8 in \([14]\)), where continuity on \(S\) is understood layer-wise, i.e. for every \(\ell\) the restriction \(\mu(\cdot \cap E^\ell)\) of \(\mu\) to \(E^\ell\) admits a Lebesgue density \(\gamma^\ell\) which belongs to \(C_0(E^\ell)\). We shall not make use of this result in the present paper.

2.2 Proof of theorem 2.1.3

This subsection is devoted to the proof of theorem 2.1.3. Recall that for a measurable function \(f : E \to \mathbb{R}\) we write \(\overline{f} : S \to \mathbb{R}\) for the function \(\overline{f}(x) := \sum_{j=1}^\ell f(x_j), x = (x_1, \ldots, x_\ell) \in S\), with \(\overline{f}(\delta) = 0\). As in definition 1.2.3, \(\eta^\ell\) is the branching diffusion without immigration. Let \((T^\ell_t)_{t \geq 0}\) denote the semigroup of \(\eta^\ell\)

\[
T^\ell_t(x, g) := E_x(g(\eta^\ell_t)), \quad t \geq 0, \ x \in S, \ g : S \to [0, \infty) \text{ measurable}
\]

which is related to the expectation semigroup \((M_t)_{t \geq 0}\) introduced in \([16]\) by

\[
M_t(y, f) = T^\ell_t(y, \overline{f}), \quad t \geq 0, \ y \in E, \ f : E \to [0, \infty) \text{ measurable}.
\]

Moreover, let \((T_t)_{t \geq 0}\) denote the semigroup of the BDI process \(\eta\) on \(S\).

We start with the branching diffusion without immigration \(\eta^\ell\) and study ‘higher moments’ \(T^\ell_t(y, \overline{f'})\) for \(p > 1\) when \(y\) ranges over the single-particle space \(E\). The following is Ikeda, Nagasawa and
2.2.1 Lemma (22) We have a representation

\[ T_t^x (x, \bar{f}^p) = M_t(x, f^p) + \int_0^t ds \int_E M_{t-s}(x, dy) \kappa(y) \sum_{n \geq 2} n p_n(y) \times \]

\[ \times \sum_{(k_1, \ldots, k_n) : 0 \leq k_j < p, \ k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} \int_{E^n} K_n(y, n, dv_1, \ldots, dv_n) \prod_{j=1}^n T_{t-s}^y (y + v_j, \bar{f}^{k_j}) \]

for \( x \in E, f : E \to [0, \infty) \) bounded measurable, \( p \in \mathbb{N} \).

Sketch of Proof:

First, we note that the expectation semigroup (13) has the series representation

\[ M_t(x, f) = P_t^\kappa(x, f) + \sum_{m \in \mathbb{N}} \int_0^t ds_1 \int_E [P_{s_1}^\kappa \kappa \theta Q^\kappa](x, dy_1) \int_0^{s_1} ds_2 \int_E [P_{s_2}^\kappa \kappa \theta Q^\kappa](y_1, dy_2) \cdots \]

\[ \cdots \int_0^{s_{m-1}} ds_m \int_E [P_{s_m}^\kappa \kappa \theta Q^\kappa](y_{m-1}, dy_m) P_{t-s_1-\cdots-s_m}^\kappa (y_m, f) \]

where \((P_t^\kappa)_{t \geq 0}\) and \(Q^\kappa\) are defined in (22) and (33), respectively (see e.g. [13] lemma 3.2.20).

Now proceeding as in [22], we take \( h = 1 \) in their lemma 4.8, eq. (4.75) on p. 139 to obtain

\[ T_t^x (x, \bar{f}^p) = P_t^\kappa(x, f^p) + \int_0^t ds \int_E [P_s^\kappa \kappa](x, dy) \sum_{n \in \mathbb{N}} n p_n(y) \int_{E^n} K_2((y, n), dv_1, \ldots, dv_n) \times \]

\[ \times \sum_{(k_1, \ldots, k_n) : 0 \leq k_j < p, \ k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} \prod_{j=1}^n T_{t-s}^y (y + v_j, \bar{f}^{k_j}) \]

for each \( x \in E \), where we have adjusted their notation to ours. (Essentially, this formula is obtained by conditioning on the first branching event and using the branching property.)

Now we decompose the sums arising in (30)

\[ \sum_{(k_1, \ldots, k_n) : 0 \leq k_j < p, \ k_1 + \cdots + k_n = p} \binom{p}{k_1, \ldots, k_n} \prod_{j=1}^n T_{t-s}^y (y + v_j, \bar{f}^{k_j}) \]

into two terms. The first one collects all indices where \( 0 \leq k_j < p \) for all \( j = 1, \ldots, n \). The remaining second term, collecting indices of type \((k_1, \ldots, k_n) = p e_j\) where \( e_j \) is the \( j \)-th unit vector in \( \mathbb{R}^n \), shrinks to

\[ \sum_{j=1}^n T_{t-s}^y (y + v_j, \bar{f}^p) \]
where the maximal power \( p \) shows up. Both contributions have to be integrated with respect to the kernel

\[
\sum_{n \in \mathbb{N}} p_n(y) \int_{E^n} K_2((y, n), dv_1, \ldots, dv_n).
\]

Using notation (3) and defining

\[
J_p(s; y) := \sum_{n=2}^\infty p_n(y) \int_{E^n} K_2((y, n), dv_1, \ldots, dv_n) \sum_{(k_1, \ldots, k_n): 0 \leq k_j < p,}^p \left( \sum_{k_1 + \cdots + k_n = p} \prod_{j=1}^{n} T_s^p(y + v_j, f^{k_j}) \right)
\]

for \( s \geq 0 \) and \( y \in E \), this gives

\[
T_t^p(x, \bar{f}^p) = P_t^p(x, f^p) + \int_0^t ds \int_E [P_s^p \kappa](x, dy) J_p(t - s; y)
\]

\[
+ \sum_{m=0}^\infty \int_0^t ds_1 \int_E [P_{s_1}^p \kappa \rho Q^p](x, dy_1) \cdots \int_0^{t-s_1-\cdots-s_m-1} ds_m \int_E [P_{s_m}^p \kappa \rho Q^p](y_{m-1}, dy_m) P_t^{s_1+\cdots+s_m}(y_m, f^p)
\]

The structure of the previous display (namely the occurrence of \( T_{t-s}^p(y, \bar{f}^p) \) on the right hand side) allows for iteration: Expanding the last term on the right hand side of (33) leads to

\[
T_t^p(x, \bar{f}^p) = P_t^p(x, f^p) + \int_0^t ds \int_E [P_s^p \kappa](x, dy) J_p(t - s; y)
\]

\[
= \sum_{m=0}^\infty \int_0^t ds_1 \int_E [P_{s_1}^p \kappa \rho Q^p](x, dy_1) \cdots \int_0^{t-s_1-\cdots-s_m-1} ds_m \int_E [P_{s_m}^p \kappa \rho Q^p](y_{m-1}, dy_m) P_t^{s_1+\cdots+s_m}(y_m, f^p)
\]

Using the series representation (29) of the expectation semigroup, we see that the previous display can be transformed into

\[
T_t^p(x, \bar{f}^p) = M_t(x, f^p) + \int_0^t ds \int_E M_s(x, dy) \kappa(y) J_p(t - s; y),
\]

proving the representation (28).

\[\square\]

2.2.2 Lemma Assume \( 1.2.1, 2.1.1 \) and \( 2.2.2 \). Fix a natural number \( q > 1 \) such that \( 2.1.1 \) holds. Then there exist \( \gamma > 0 \) and constants \( C_1, \ldots, C_q \) such that for \( f : E \to [0, \infty) \) bounded and measurable

\[
\| T_t^p(\bar{f}^p) \|_\infty = \sup_{x \in E} |T_t^p(x, \bar{f}^p)| \leq C_p e^{-\gamma t} \| f \|_\infty^p, \quad t \geq 0, \ p = 1, \ldots, q
\]

where \( \ldots |_E \) denotes the restriction of the kernel \( 2.7 \) to the single-particle space \( E \).
Proof: The case \( p = 1 \) is assumption 2.1.2: we know that there exist \( C > 0 \) and \( \gamma > 0 \) such that
\[
|||M_t||| \leq Ce^{-\gamma t}, \quad t \geq 0.
\]
We proceed by induction: let \( 1 < p \leq q \) and assume that (34) already holds for \( k = 1, 2, \ldots, p-1 \), i.e. there are constants \( C_1, C_2, \ldots, C_{p-1} \) such that
\[
|||T^x_j(f^k)|||_\infty \leq C_k e^{-\gamma t} ||f||^k_\infty, \quad t > 0, \ k = 1, \ldots, p-1.
\]
For indices \((k_1, \ldots, k_n)\) appearing in the sum in (28), we have \( 0 \leq k_j \leq p-1 \) and \( k_1 + \cdots + k_n = p \), thus \( k_j \geq 1 \) for at most \( p \) and at least \( 2 \) indices \( j \). Then by induction
\[
\prod_{j=1}^n |||T^x_j(f^{k_j})|||_\infty = \prod_{j:k_j \geq 1} |||T^x_j(f^{k_j})|||_\infty \leq \prod_{j:k_j \geq 1} C_{k_j} e^{-\gamma s} ||f||^{k_j}_\infty \leq C'_p e^{-\gamma s} ||f||^p_\infty
\]
where we define \( C'_p := (\max\{1, C_1, \ldots, C_{p-1}\})^p \). Substituting this into (28) and making use of assumption 2.1.1 and again of 2.1.2 we obtain
\[
|||T^x(f^p)|||_\infty \leq |||M_t||| ||f||^p_\infty + ||\kappa||_\infty \int_0^t ds |||M_{t-s}||| \sum_{n \geq 2} \sum_{\{k_1, \ldots, k_n\}: 0 \leq k_j < p, k_1 + \cdots + k_n = p} \left( \frac{p}{k_1, \ldots, k_n} \right) |||C'_p e^{-\gamma s} ||f||^p_\infty \leq C e^{-\gamma t} ||f||^p_\infty + ||\kappa||_\infty \int_0^t ds e^{-\gamma(t-s)} C'_p e^{-\gamma s} \sum_{n \geq 2} n^p \sum_{\{k_1, \ldots, k_n\}: 0 \leq k_j < p, k_1 + \cdots + k_n = p} ||f||^p_\infty \leq C e^{-\gamma t} ||f||^p_\infty \left( 1 + C'_p ||\kappa||_\infty ||m_p(\cdot)||_\infty \int_0^t e^{-\gamma s} ds \right) \leq C_p e^{-\gamma t} ||f||^p_\infty,
\]
with \( C_p := C \left( 1 + \frac{C'_p ||\kappa||_\infty ||m_p(\cdot)||_\infty}{\gamma} \right) \). Thus (34) is proved.

We turn to the semigroup \((T_t)_{t \geq 0}\) of the branching diffusion \((\eta_t)_{t \geq 0}\) and focus on the void configuration \( \delta \) as starting point at time \( t = 0 \).

2.2.3 Lemma Fix \( p \in \mathbb{N}, t > 0 \) and consider \( f : E \to [0, \infty) \) measurable. Then
\[
T_t(\delta, f^p) = c \sum_{k=0}^{p-1} \binom{p}{k} \int_0^t [Q^i T^x_{s-k}(f^p)] T_s(\delta, f^k) ds. \]

(36)
It is sufficient to prove the assertion in case $p < q$.

**Proof:** Write for short $\nu_t := T_t(\delta, \cdot)$. Since immigration times are distributed according to Poisson random measure with constant intensity $c > 0$, $\nu_t(\mathcal{F}^p)$ has the following explicit form

\[
\nu_t(\mathcal{F}^p) = e^{-ct} \sum_{n=1}^{\infty} c^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n \int_S [Q^i T_{s_1}]^i (d\nu) \cdots \int_S [Q^i T_{s_n}]^i (d\nu) (\mathcal{F}(z_1) + \cdots + \mathcal{F}(z_n))^p
\]

Next, consider $p < q$.

\[
\text{The right hand side in the previous display can be simplified: define}
\]

\[
\tilde{\nu}_t(\mathcal{F}^p) := e^{ct} \nu_t(\mathcal{F}^p).
\]

Differentiating with respect to $t$ (and sorting the terms), we get

\[
\frac{d}{dt} \tilde{\nu}_t(\mathcal{F}^p) = c \tilde{\nu}_t(\mathcal{F}^p) + \frac{p}{k} \int_S [Q^i T_{s_1}]^i (d\nu) \tilde{\nu}_t(\mathcal{F}^p)^{p-k}
\]

Solving this linear inhomogenus ODE by variation of constants yields

\[
\tilde{\nu}_t(\mathcal{F}^p) = e^{ct} \int_0^t e^{-cs} h(s) ds = e^{ct} c \sum_{k=0}^{p-1} \left( \frac{p}{k} \right) \int_0^t [Q^i T_{s_1}]^i (d\nu) e^{-cs} \tilde{\nu}_s(\mathcal{F}^p) ds.
\]

Multiplying by $e^{-ct}$ again, we obtain (36). \qed

**2.2.4 Lemma** Assume 1.2.1, 2.1.1 and 2.1.2. Consider a natural number $q > 1$ for which 2.1.1 holds.

Then for $f : E \to [0, \infty)$ bounded and measurable,

\[
\sup_{0 < t < \infty} T_t(\delta, \mathcal{F}^p) < \infty , \quad 1 \leq p \leq q.
\]

**Proof:** It is sufficient to prove the assertion in case $p = 1$: then $\mathcal{F} = \pi = \ell$. Lemma 2.2.3 allows for recursion. First, in case $p = 1$, we combine $T_s(\delta, \ell^0) = 1$ with (36) and lemma 2.2.2

\[
T_t(\delta, \ell^1) = c \int_0^t [Q^i T_{s_1}]^i (\ell^1) ds \leq c C_1 \int_0^\infty e^{-\gamma s} ds = \frac{c}{\gamma} C_1 =: M_1 < \infty.
\]

Next, consider $p < q$. If our assertion holds for all $1 \leq k \leq p$, with suitable bounds $M_k$, then it holds for $p+1$: by recursion (36),

\[
T_t(\delta, \ell^{p+1}) = c \sum_{k=0}^{p} \left( \frac{p+1}{k} \right) \int_0^t [Q^i T_{s_1}]^i (\ell^{p+1-k}) T_s(\delta, \ell^k) ds
\]

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is bounded (we can apply (34) to every \( k \)-th summand since \( p + 1 \leq q \)) by

\[
c \sum_{k=0}^{p} \binom{p+1}{k} \int_{0}^{\infty} [C_{p+1-k} e^{-\gamma s}][M_k] ds = \frac{c}{\gamma} \sum_{k=0}^{p} \binom{p+1}{k} C_{p+1-k} M_k =: M_{p+1}
\]

and we are done. \( \square \)

Now we have the tools to prove theorem 2.1.3.

**Proof of theorem 2.1.3.** By the ergodic theorem for Harris recurrent processes (see e.g. [1], p. 30) we know that for all \( \mu \)-integrable functions \( g : S \to \mathbb{R} \)

\[
(37) \quad \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} T_{s}(x,g) \, ds = \mu(g) \quad \text{for } \mu\text{-a.e. } x \in S.
\]

By a simple monotone convergence argument, (37) clearly extends to all nonnegative measurable \( g : S \to [0, \infty) \) where the limit is equal to \( +\infty \) if \( g \) is not \( \mu \)-integrable. Moreover, (37) must in particular hold for \( x = \delta \) since \( \mu(\delta) > 0 \). Choosing \( g := f^q \), this gives

\[
(38) \quad \mu(f^q) = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} T_{s}(\delta,f^q) \, ds \leq \infty
\]

for all \( f : E \to [0, \infty) \) measurable and \( q \in \mathbb{N} \). But from (36), we see that \( t \mapsto T_t(\delta,f^q) \) is increasing, thus the limit \( \lim_{t \to \infty} T_t(\delta,f^q) \) exists in \( [0, \infty] \) and must be equal to the Cesàro limit (38) and so

\[
(39) \quad \mu(f^q) = \lim_{t \to \infty} T_t(\delta,f^q) \leq \infty
\]

for all measurable \( f : E \to [0, \infty) \) and \( q \in \mathbb{N} \). Now let \( q \) satisfy assumption 2.1.1. Then we can use lemma 2.2.4 to conclude that the limit (39) is finite for each bounded measurable \( f \). Now the assertion of theorem 2.1.3 follows by choosing \( f \equiv 1 \), i.e. \( f = \ell \). \( \square \)

### 2.2.5 Remark

Note that for the derivation of formula (39), we did not use assumption 2.1.1 nor did we need the full force of assumption 2.1.2 but only positive Harris recurrence of the BDI process \( \eta \) (for which we know from lemma 1.2.5 that e.g. the weaker assumption 1.2.2 is already sufficient). In fact, the recursion (36) can be solved to obtain the following explicit formula

\[
(40) \quad \mu(f^q) = \sum_{n=1}^{q} c^n \sum_{\sum_{j=1}^{n} k_j \geq 1, k_1 + \cdots + k_n = q} \left( \prod_{j=1}^{n} \int_{0}^{\infty} [Q^j T_s^\gamma f^q(j)] ds \right) \leq \infty
\]

for each \( q \in \mathbb{N} \) and \( f : E \to [0, \infty) \) measurable. Our above proof shows that assumptions 2.1.1 and 2.1.2 together are sufficient to ensure finiteness of (40) for bounded measurable \( f \), but they are probably not necessary.
2.3 Proof of theorem 2.1.6

We recall the occupation times kernel $H^F$ for the branching diffusion process without immigration $\eta^F$ defined in (8) with series representation (15)

$$H^F(x, B) = \sum_{n \in \mathbb{N}_0} [(R_\kappa \kappa \varrho Q^F)^n R_\kappa](x, B), \quad x \in E, \; B \subseteq E \text{ Borel},$$

where $Q^F$ is the kernel (3) and

$$R_\kappa(x, dy) = \int_0^\infty P_\kappa^s(x, dy) \, dt, \quad x \in E$$

is the $\kappa$-resolvent of the single-particle diffusion $\xi$ from (12) and (22). Assumption 2.1.4 grants existence of a Lebesgue density

(41) \hspace{1cm} R_\kappa(x, dz) = r_\kappa(x, z) \, dz \quad \text{with} \quad r_\kappa(x, z) := \int_0^\infty p_\kappa^s(x, z) \, dt, \quad x, z \in E.

Thus, by lemma 1.2.5 and (15), the invariant occupation measure

(42) \hspace{1cm} \underline{\tau}(B) = c \, [Q^1 H^F](B) = c \, \sum_{n \in \mathbb{N}_0} [Q^1 (R_\kappa \kappa \varrho Q^F)^n R_\kappa](B) < \infty, \quad B \subseteq E \text{ Borel}

admits a Lebesgue density

(43) \hspace{1cm} z \mapsto c \sum_{n \in \mathbb{N}_0} \underline{\tau}_n(z), \quad \underline{\tau}_n(z) := \int_E [(Q^1 (R_\kappa \kappa \varrho Q^F)^n)(dx) \, r_\kappa(x, z)].

We will show that under the assumptions of theorem 2.1.6 we have $\underline{\tau}_n \in C_0(E)$ for all $n \in \mathbb{N}_0$ and that the series in (43) converges uniformly.

We fix $\varepsilon > 0$ and observe that by the semigroup property of $(P_t^\kappa)_{t \geq 0}$ we can decompose

$$R_\kappa(x, dy) = \int_0^\varepsilon P_t^\kappa(x, dy) \, dt + \int_0^\infty P_{t+\varepsilon}^\kappa(x, dy) \, dt$$

$$=: R_{\kappa, \varepsilon}(x, dy) + R_\kappa P_{\varepsilon}^\kappa(x, dy),$$

where we define $R_{\kappa, \varepsilon}(x, dy) := \int_0^\varepsilon P_t^\kappa(x, dy) \, dt$. For the resolvent density, this means

(44) \hspace{1cm} r_\kappa(x, z) = r_{\kappa, \varepsilon}(x, z) + \int_E R_\kappa(x, dy) p_\varepsilon^\kappa(y, z),

with notation

$$r_{\kappa, \varepsilon}(x, z) := \int_0^\varepsilon p_\varepsilon^\kappa(x, z) \, dt.$$
Now under assumption 2.1.4 we know that if we choose \( \varepsilon \leq t_0 \) and define a density \( \tilde{p}_t(\cdot) \) as the right hand side in the heat kernel bound (23), then we have

\[
p_t^\varepsilon(x,z) \leq \tilde{p}_t(z-x) := Ct^{-d/2} e^{-\frac{1}{4} \frac{|x-z|^2}{t}}, \quad 0 < t \leq \varepsilon, \ x, z \in E
\]
and consequently (by symmetry)

\[
r_{\kappa,\varepsilon}(x,z) \leq \tilde{r}_\varepsilon(x-z) := \int_0^\varepsilon \tilde{p}_t(x-z) \, dt, \quad x, z \in E.
\]

We denote by

\[
\tilde{P}_t(x,dy) := \tilde{p}_t(x-y) \, dy, \quad \tilde{R}_\varepsilon(x,dy) := \tilde{r}_\varepsilon(x-y) \, dy
\]
the corresponding convolution kernels. Regularity of the density \( \tilde{r}_\varepsilon \) depends heavily on the dimension: while \( \tilde{r}_\varepsilon \) is bounded in \( d = 1 \), for \( d \geq 2 \) it has a singularity at the origin. Independently of the dimension we have \( \tilde{r}_\varepsilon \in L^1(E) \) and \( \tilde{p}_t \in L^1(E) \cap C_0^\infty(E) \) for \( 0 < t \leq \varepsilon \), so the kernels \( \tilde{R}_\varepsilon \) and \( \tilde{P}_t \) induce bounded convolution operators on \( L^\infty(E), C_0(E) \) and on \( C_0(E) \), and \( \tilde{P}_t \) induces also a bounded convolution operator \( L^1(E) \to C_0(E) \).

Moreover, by assumption 2.1.5 the ‘jump operator’ corresponding to the kernel \( Q^\varepsilon \) is bounded by

\[
\int_E Q^\varepsilon(x,dy) f(y) \leq \int_E f(x+v) \tilde{Q}^\varepsilon(dv) =: \int_E f(y) \tilde{Q}(x,dy), \quad x \in E, f \geq 0 \text{ measurable},
\]

where \( \tilde{Q}(x,dy) \) denotes the convolution kernel corresponding to the finite measure \( \tilde{Q}^\varepsilon \) in assumption 2.1.5. Since the kernels resp. operators \( \tilde{P}_t \) for \( 0 < t \leq \varepsilon \), \( \tilde{R}_\varepsilon \) and \( \tilde{Q} \) are all convolution kernels resp. operators, they all commute with each other, a fact which we shall exploit heavily below.

### 2.3.1 Lemma

The \( n \)-fold convolution \( \tilde{r}_\varepsilon^{*n} \) of the density \( \tilde{r}_\varepsilon \) with itself has the property

\[
\tilde{r}_\varepsilon^{*n} \in C_0(\mathbb{R}^d) \quad \text{for } n > \frac{d}{2};
\]
in particular \( \tilde{r}_\varepsilon \in C_0(\mathbb{R}) \) for \( d = n = 1 \). The following holds for \( d \geq 2 \): the density \( \tilde{r}_\varepsilon(\cdot) \) is continuous on \( \mathbb{R}^d \setminus \{0\} \) but has a singularity at the origin; we have \( \tilde{r}_\varepsilon(\cdot) \in L^{p^*}(\mathbb{R}^d) \) for all \( 1 \leq p^* < \frac{d}{d-2} \) (where we understand \( \frac{d}{d-2} = \infty \) for \( d = 2 \)).

**Proof:** 1) The fact that \( \tilde{r}_\varepsilon^{*n} \in C_0(\mathbb{R}^d) \) for \( n > \frac{d}{2} \) is most easily seen by Fourier inversion: with \( \mathcal{F} \) denoting the Fourier transform, we have

\[
\mathcal{F}[\tilde{p}_t](\xi) = C^{1+d/2}(2\pi)^{d/2}e^{-\frac{1}{4}Ct\|\xi\|^2}, \quad \xi \in \mathbb{R}^d, t > 0
\]
with the constant \( C \) from assumption 2.1.4. We obtain
\[
\mathcal{F}[\tilde{r}_\varepsilon](\xi) = \int_0^\varepsilon \mathcal{F}[\tilde{p}_t](\xi) \, dt = 2(2\pi C)^{d/2} \frac{1 - e^{-\frac{1}{2}C\varepsilon \|\xi\|^2}}{\|\xi\|^2} \leq \tilde{C} \frac{1 - e^{-\frac{1}{2}\|\xi\|^2}}{\|\xi\|^2} =: h(\|\xi\|), \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]
Consequently,
\[
|\mathcal{F}[\tilde{r}_\varepsilon^n](\xi)| = |(\mathcal{F}[\tilde{r}_\varepsilon](\xi))^n| \leq h(\|\xi\|)^n.
\]
Integration in (hyper-)spherical coordinates shows that \( h(\|\cdot\|)^n \) is integrable on \( \mathbb{R}^d \) if \( n > d/2 \). This gives \( \tilde{r}_\varepsilon^* \in C_0(\mathbb{R}^d) \) by Fourier inversion. The fact that \( \tilde{r}_\varepsilon \) is continuous on \( \mathbb{R}^d \setminus \{0\} \) in any dimension is clear by dominated convergence.

2) The following is from Hammer [14], (3.2.63) on p. 103: for \( d \geq 2 \) and \( \lambda > 0 \), consider the \( \lambda \)-resolvent
\[
\phi_\lambda(x) := \int_0^\infty e^{-\lambda s} p_s(x) \, ds, \quad x \in \mathbb{R}^d
\]
of the heat flow, i.e. \( p_s(x) \) is the density of the normal law \( \mathcal{N}(0, sI_d) \). Then \( x \to \phi_\lambda(x) \) is \( p^* \)-integrable on \( \mathbb{R}^d \) if and only if \( p^* < \frac{d}{d-2} \). This is seen as follows. Sato [36], formulae (30.28)+(30.29) on p. 204, gives an explicit representation of \( \phi_\lambda \)
\[
\phi_\lambda(x) = \text{cst} \left\| x \right\|^{-\frac{d-2}{2}} K_{\frac{d-2}{2}}(\sqrt{2}\lambda \| x \|)
\]
where \( K_{\nu} \) denotes a modified Bessel function (for \( K_{\nu} \), see [36], (4.9) on p. 21, and [10], p. 159) whose asymptotics at 0 and at \( \infty \) are known: when \( r \downarrow 0 \) we have \( K_{\nu}(r) \sim \text{cst} \, r^{-\nu} \) for \( \nu > 0 \) and \( K_0(r) \sim \text{cst} \, \log(r) \); when \( r \uparrow \infty \) we have exponential decay (see Follett [10], p. 160). It follows that \( x \to \phi_\lambda(x) \) is \( p^* \)-integrable on \( \mathbb{R}^d \) if and only if \( p^* < \frac{d}{d-2} \).

3) For the \( L^{p^*} \)-properties of \( \tilde{r}_\varepsilon \), we observe that
\[
\tilde{r}_\varepsilon(x) = \int_0^\varepsilon \hat{p}_t(x) \, dt \leq e^\varepsilon \int_0^\varepsilon e^{-t} \hat{p}_t(x) \, dt \leq Ce^\varepsilon \int_0^\varepsilon e^{-t} t^{-d/2} e^{-\frac{1}{2}C\varepsilon \|x\|^2} \, dt \leq (2\pi C)^{d/2} e^\varepsilon \int_0^\varepsilon e^{-s/C} (2\pi s)^{-d/2} e^{-\frac{1}{2}s \|x\|^2} \, ds \leq \text{cst} \, \phi_{\frac{1}{\varepsilon}}(x)
\]
for all \( x \in \mathbb{R}^d \), with \( \phi_{\frac{1}{\varepsilon}} \) from step 2). So the last assertion of the lemma follows from step 2). \( \square \)

### 2.3.2 Lemma

Under the assumptions of theorem 2.1.6 and with notation \( \eta_n \) from [13], we have \( \eta_n \in C_0(\mathbb{R}^d) \) for all \( n \in \mathbb{N}_0 \).

**Proof:** We use induction on \( n \in \mathbb{N}_0 \).
1) For $n = 0$, definition (43) of $\gamma_0$ combined with decomposition (44) gives

$$z \mapsto \gamma_0(z) = \int_{E} Q^i(dx) r_{\kappa}(x, z) = \int_{E} Q^i(dx) r_{\kappa,e}(x, z) + \int_{E} [Q^i R_{\kappa}](dx) \rho^e_{\kappa}(x, z).$$

Fix $x \in E$. For $t$ sufficiently small, $z \to p^e_{\kappa}(x, z)$ is continuous by assumption 2.1.4. Note first that regardless of the dimension of $E = \mathbb{R}^d$, the function

$$z \mapsto \int_{0}^{\varepsilon} p^e_{\kappa}(x, z) \, dt = r_{\kappa,e}(x, z)$$

is continuous at $z_0 \in E$ whenever $z_0 \neq x$. To see this, fix $z_0 \neq x$ and consider a sequence $z_n \to z_0$; we may assume that there is $\delta > 0$ such that $\|z_n - x\| > \delta$ for all $n$. Then the estimate (23) gives

$$0 \leq p^e_{\kappa}(x, z_n) \leq C t^{-\frac{d}{2}} e^{-\frac{\varepsilon^2}{2\kappa t}} , \quad n \in \mathbb{N} , \ 0 < t < \varepsilon$$

for $\varepsilon$ sufficiently small. Here the right hand side is independent of $n \in \mathbb{N}$ and integrable in $0 < t < \varepsilon$, thus dominated convergence shows

$$\int_{0}^{\varepsilon} p^e_{\kappa}(x, z_n) \, dt \to \int_{0}^{\varepsilon} p^e_{\kappa}(x, z_0) \, dt$$

which establishes (50). Based on this we can check the assertions of the lemma in case $n = 0$. We start with the function

$$z \mapsto \int_{\mathbb{R}} Q^i(dx) r_{\kappa,e}(x, z)$$

on the right hand side of (49).

i) In the special case $d = 1$, the function $z \mapsto r_{\kappa,e}(x, z)$ in (50) is continuous in $z \in \mathbb{R}$ for every $x \in \mathbb{R}$ fixed, and its upper bound $\tilde{r}_{\varepsilon}(\cdot)$ from (40) is in $C_0(\mathbb{R})$ by lemma 2.3.1, thus bounded. Thus dominated convergence shows that the function (51) is continuous and bounded, for any probability measure $Q^i(dx)$ on $\mathbb{R}$. Probability measures on $\mathbb{R}$ being tight, upper bounds $z \mapsto \int_{\mathbb{R}} Q^i(dx) \tilde{r}_{\varepsilon}(z - x)$ with $\tilde{r}_{\varepsilon}(\cdot) \in C_0(\mathbb{R})$ vanish at $\pm \infty$. This implies that the function in (51) belongs to $C_0(\mathbb{R})$.

ii) Let $d \geq 2$ and assume that $Q^i(dx) = q^i(x)dx$ is absolutely continuous with density in $L^p(\mathbb{R}^d)$ for some $p \in \left(\frac{d}{2}, \infty\right)$. Here, without loss of generality we may assume that $p < \infty$, since if $q^i$ is bounded, then (being a probability density) it is in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) = \bigcap_{1 \leq p \leq \infty} L^p(\mathbb{R}^d)$. Then the dual exponent $p^*$ satisfies $1 < p^* < \frac{d}{p - 2}$. By lemma 2.3.1 the function $\tilde{r}_{\varepsilon}(\cdot)$ is in $L^{p^*}(\mathbb{R}^d)$, and is continuous on $\mathbb{R}^d \setminus \{0\}$. We have to consider

$$z \mapsto \int_{\mathbb{R}^d} q^i(x) r_{\kappa,e}(x, z) \, dx \leq \int_{\mathbb{R}^d} q^i(x) \tilde{r}_{\varepsilon}(x - z) \, dx = [q^i * \tilde{r}_{\varepsilon}](z).$$
Since the convolution of two functions from dual $L^p$-spaces is in $C_0(\mathbb{R}^d)$ for $p \in (1, \infty)$ (see e.g. [25], Lemma 2.20, [16], p. 398), the right hand side in (52) is a $C_0$-function of $z$. For convergent sequences $z_n \to z_0$ and $x \neq z_0$ we have
\[ r_{\kappa,\varepsilon}(x, z_n) \to r_{\kappa,\varepsilon}(x, z_0), \quad \hat{r}_\varepsilon(x - z_n) \to \hat{r}_\varepsilon(x - z_0) \]
as $n \to \infty$, using (50) and continuity of $\hat{r}_\varepsilon(\cdot)$ on $\mathbb{R}^d \setminus \{0\}$ by lemma 2.3.1. Integrals on the right hand side of (52) being continuous in $z$, Pratt’s lemma applies (see e.g. [8], theorem VI.5.1, [37], p. 101) and shows
\[ \int_{\mathbb{R}^d} q^\dagger(x) r_{\kappa,\varepsilon}(x, z_n) \, dx \to \int_{\mathbb{R}^d} q^\dagger(x) r_{\kappa,\varepsilon}(x, z_0) \, dx \]
as $n \to \infty$. We have proved that the function (51) is continuous. Since its upper bounds in (52) are in $C_0(\mathbb{R}^d)$, the function (51) is in $C_0(\mathbb{R}^d)$.

iii) So far we have shown that the first term (51) on the right hand side of (49) is a $C_0$-function of $z$. The second term on the right hand side of (49), as a function of $z$, is always in $C_0(\mathbb{R}^d)$: by the regularity properties of $p^\kappa_v$ in assumption 2.1.4 and since $[Q^1 R_\kappa(dx)]$ is a finite measure, the argument is analogous to step i) above, and no regularity of the immigration measure is needed for this term.

We have proved that in case $n = 0$, $z \to \gamma_0(z)$ in (49) has the property stated in the lemma.

2) To prove that all $\gamma_n(\cdot)$ have the property stated in the lemma, we proceed by induction on $n$. Suppose we know already that $\gamma_n \in C_0(E)$. Then by (44) and (43)
\[ \gamma_{n+1}(z) = \int_E [Q^1 R_{\kappa,\epsilon} Q^F]^{n+1}(dx) r_{\kappa}(x, z) \]
\[ = \int_E dx \gamma_n(x) \kappa(x) \rho(x) \int_E Q^F(x, dy) r_{\kappa,\varepsilon}(y, z) + \int_E dx \gamma_{n+1}(x) p^\kappa_v(x, z). \]
Again the second term on the right hand side is a $C_0$-function of $z$ because of the regularity of $p^\kappa_v$ and since $\gamma_{n+1} \in L^1(E)$. We consider the first term and put it using notations (48) in the form
\[ z \to \int_E dx \gamma_n(x) \kappa(x) \rho(x) \int_E Q^F(x, dy) q^\dagger(x, y) r_{\kappa,\varepsilon}(y, z) \]
\[ = \int_E dx \int_E \hat{Q}^F(dv) \gamma_n(x) \kappa(x) \rho(x) q^\dagger(x, x + v) r_{\kappa,\varepsilon}(x + v, z) \]
where $q^\dagger(x, y) \leq 1$ denotes the density of $Q^F(x, dy)$ with respect to $\hat{Q}(x, dy)$, see (43). Here, for each $z_0 \in E$ fixed, the mapping $z \to r_{\kappa,\varepsilon}(x + v, z)$ is continuous at $z_0$ whenever $x + v \neq z_0$, by (50), and $z \to \hat{r}_\varepsilon(x + v, z)$ is continuous at $z_0$ whenever $x + v \neq z_0$, by lemma 2.3.1. Note that the set \{$(x, v) \in E^2 : x + v = z_0$\} is a null set under the measure $dx \otimes \hat{Q}^F(dv)$. The image of the measure $dx \otimes \hat{Q}^F(dv)$ under the mapping $(x, v) \to x + v$ coincides again with Lebesgue measure on $E = \mathbb{R}^d$. 27
Let us write $M_1$ for the image of the measure

$$q^\gamma(x, x + v) \left( \tau_n(x) \kappa(x) \rho(x) \ dx \otimes Q^\gamma(dv) \right) \text{ on } E \times E$$

under the mapping $(x, v) \to x + v$. By assumption \[\text{1.2.1}\] induction assumption which implies $\tau_n \in L^1(E) \cap L^\infty(E)$ and since $q^\gamma(x, y) \leq 1$, the measure $M_1$ on $E$ is finite and admits a Lebesgue density which is bounded: $M_1(du) =: m_1(u)du$ for some $m_1 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Now we rewrite the function on the right hand side in \[\text{63}\] in the form

$$z \to \int_E du \ m_1(u) \ r_{\kappa,\varepsilon}(u, z)$$

where for every $z_0 \in E$ fixed, the mapping $z \to r_{\kappa,\varepsilon}(u, z)$ is continuous at $z_0$ for $M_1$-almost all $u \in E$.

The rest of the argument is analogous to step 1) above with $Q^1(dx)$ replaced by $m_1(u)du$. We only give the details for the case $d \geq 2$: From \[\text{10}\] we have bounds of convolution type

$$z \to \int_E du \ m_1(u) \ \tilde{r}_{\varepsilon}(z - u)$$

where $m_1 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) = \bigcap_{1 \leq p < 2} L^p(\mathbb{R}^d)$ and $\tilde{r}_{\varepsilon}(\cdot) \in L^p(\mathbb{R}^d)$ for $p^* \in (1, \frac{d}{2-2})$ by lemma \[\text{2.3.1}\]. Choosing such a $p^*$ and setting $p := \frac{2}{p^* - 1} \in (\frac{d}{2}, \infty)$ as the dual exponent, we see that the upper bound \[\text{55}\] is in $C_0(\mathbb{R}^d)$, from \[\text{10}\] p. 398. Again Pratt’s lemma applies and shows that the function \[\text{54}\] is in $C_0(\mathbb{R}^d)$. We have proved that $\tau_{n+1}$ is in $C_0(\mathbb{R}^d)$ whenever $\tau_n$ is bounded, for all $n \geq 1$. This concludes the proof of the lemma.

Now we can finish the

**Proof of theorem \[\text{2.1.6}\]**: In view of lemma \[\text{2.3.2}\] it remains only to show that the series \[\text{63}\] converges uniformly. By \[\text{13}\] and the decomposition \[\text{14}\], we have

$$\tau_n(z) = \int_E [Q^1(R_n \kappa \rho Q^\gamma)^n](dx) \ r_{\kappa,\varepsilon}(x, z)$$

$$= \int_E [Q^1(R_n \kappa \rho Q^\gamma)^n](dx) \ r_{\kappa,\varepsilon}(x, z) + \int_E [Q^1(R_n \kappa \rho Q^\gamma)^n R_n](dx) p^\varepsilon(x, z).$$

Moreover, it is easy to show by induction that

$$\kappa \rho Q^\gamma R_n)^n(x, dy) = (\kappa \rho Q^\gamma R_n)^n + \sum_{k=0}^{n-1} (\kappa \rho Q^\gamma R_n)^{n-k} P^\varepsilon(\kappa \rho Q^\gamma R_n, \varepsilon)^k, \quad n \in \mathbb{N}_0.$$

Now we fix $n_0 > \frac{d}{2}$. Then we have for all $n > n_0$

$$\tau_n(z) = \int_E [Q^1(R_n \kappa \rho Q^\gamma)^{n-n_0}(R_n \kappa \rho Q^\gamma)^{n_0}](dx) \ r_{\kappa,\varepsilon}(x, z) + \int_E [Q^1(R_n \kappa \rho Q^\gamma)^n R_n](dx) p^\varepsilon(x, z)$$

$$= \int_E dx \ \tau_{n-n_0}(x) \ \int_E [Q^1(R_n \kappa \rho Q^\gamma)^{n_0-1} \kappa \rho Q^\gamma](x, dy) \ r_{\kappa,\varepsilon}(y, z) + \int_E dx \ \tau_n(x) \ p^\varepsilon(x, z)$$

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where we rewrite the term \((\kappa \rho Q^\varepsilon R_\varepsilon)^{n_0 - 1}\) using \([44]\) with \(n_0 - 1\) in place of \(n\):

\[
\begin{align*}
\int_E dx \tau_{n-n_0}(x) \int_E \left[ (\kappa \rho Q^\varepsilon R_\varepsilon)^{n_0 - 1} \kappa \rho Q^\varepsilon \right] (x, dy) r_{n_\varepsilon}(y, z) \\
+ \int_E dx \tau_{n-n_0}(x) \sum_{k=0}^{n_0-2} \int_E \left[ (\kappa \rho Q^\varepsilon R_\varepsilon)^{n_0 - 1 - k} P_\varepsilon^k (\kappa \rho Q^\varepsilon R_\varepsilon)^k \kappa \rho Q^\varepsilon \right] (x, dy) r_{n_\varepsilon}(y, z) \\
+ \int_E dx \tau_{n}(x) p_\varepsilon^\varepsilon(x, z).
\end{align*}
\]

Rearranging terms, this last equation takes the form

\[
\tau_n(z) = \int_E dx \tau_{n-n_0}(x) \int_E \left[ (\kappa \rho Q^\varepsilon R_\varepsilon)^{n_0 - 1} \kappa \rho Q^\varepsilon \right] (x, dy) r_{n_\varepsilon}(y, z) \\
+ \sum_{k=0}^{n_0-2} \int_E dx \gamma_{n-1-k}(x) P_\varepsilon^k (\kappa \rho Q^\varepsilon R_\varepsilon)^k \kappa \rho Q^\varepsilon \right] (x, dy) r_{n_\varepsilon}(y, z) \\
+ \int_E dx \tau_{n}(x) p_\varepsilon^\varepsilon(x, z).
\]

Using the bounds \([44]-[48]\) and the fact that the operators \(\hat{P}_\varepsilon, \hat{R}_\varepsilon\) and \(\hat{Q}\) induce convolutions and thus all commute, the last display is bounded by

\[
\begin{align*}
\leq & \int_E dx \tau_{n-n_0}(x) \|\kappa \rho\|_\infty^{n_0} \int_E \left[ (\hat{Q} \hat{R}_\varepsilon)^{n_0 - 1} \hat{Q} \right] (x, dy) \hat{r}_\varepsilon(y - z) \\
&+ \sum_{k=0}^{n_0-2} \|\kappa \rho\|_\infty^{k+1} \int_E dx \tau_{n-k-1}(x) \int_E \left[ \hat{P}_\varepsilon^k (\hat{Q} \hat{R}_\varepsilon)^k \hat{Q} \right] (x, dy) \hat{r}_\varepsilon(y - z) \\
&+ \int_E dx \tau_{n}(x) \hat{p}_\varepsilon(x) - z) \\
= & \|\kappa \rho\|_\infty^{n_0} \int_E dx \tau_{n-n_0}(x) \int_E \hat{Q}^{n_0}(x, dy) \hat{r}_\varepsilon^{n_0}(y - z) \\
&+ \sum_{k=0}^{n_0-2} \|\kappa \rho\|_\infty^{k+1} \int_E dx \tau_{n-k-1}(x) \int_E \hat{Q}^{k+1}(x, dy) \left[ \hat{r}_\varepsilon^{k+1} \ast \hat{p}_\varepsilon \right](y - z) \\
&+ \|\tau_n \ast \hat{p}_\varepsilon\|_\infty(z).
\end{align*}
\]

Now \(n_0 > \frac{d}{2}\) implies that the function \(\hat{r}_\varepsilon^{n_0}\) is bounded, by lemma [2.3.1]. Thus we obtain a bound

\[
\begin{align*}
\leq & \hat{Q}^{\varepsilon}(E)^{n_0} \|\kappa \rho\|_\infty^{n_0} \|\hat{r}_\varepsilon^{n_0}\|_\infty \|\tau_{n-n_0}\|_1 \\
&+ \sum_{k=0}^{n_0-2} \hat{Q}^{\varepsilon}(E)^{k+1} \|\kappa \rho\|_\infty^{k+1} \|\hat{r}_\varepsilon^{k+1}\|_1 \|\hat{p}_\varepsilon\|_\infty \|\tau_{n-k-1}\|_1 \\
&+ \|\tau_n\|_1 \|\hat{p}_\varepsilon\|_\infty
\end{align*}
\]

for \(\tau_n(\cdot)\), again using notations \([48]\). Thus we have shown that for \(n > n_0\)

\[
\|\tau_n\|_\infty \leq C \sum_{k=0}^{n_0} \|\tau_{n-k}\|_1 = C \sum_{m=n-n_0}^{n} \|\tau_m\|_1
\]

where \(C > 0\) is some constant that does not depend on \(n\). The total mass \(\bar{\mu}(E) = \sum_{n \in \mathbb{N}_0} \|\tau_n\|_1\) being finite, the last expression is summable in \(n > n_0\).
We have shown that the series converges uniformly in $z$, and since each term is in $C_0(E)$ resp. $C_b(E)$ by lemma the same holds for the limit. This finishes the proof of theorem □

3 Reconstruction of increments for particle trajectories when the BDI process is observed discretely in time

Discretely observed diffusions have received a lot of attention, from Yoshida [41], Genon-Catalot and Jacod [12], Bibby and Sørensen [1], Kessler [24] via Gobet [13] to Podolskij and Vetter [34] or Protter and Jacod [23]. Financial data have been a main motivation, and a main issue is estimation of the unknown volatility or of functionals thereof. If we observe at discrete time points $t_i := i\Delta$, $i \in \mathbb{N}_0$, not a diffusion path but the trajectory of a BDI process $(\eta_t)_{t \geq 0}$, a new type of problem arises: we will be left with pairs of configurations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$, i.e. pairs of random point measures on the single-particle space, without any information on the path history of the continuous-time process in-between. Segments $\eta_{[i\Delta,(i+1)\Delta]}$ of the trajectory of a BDI process will in general contain branching or immigration events, and even for those particles which succeeded to stay alive over the time interval $[i\Delta,(i+1)\Delta]$ –and thus did travel on diffusion paths– any information which particle in $\eta_{i\Delta}$ did travel to which position in the configuration $\eta_{(i+1)\Delta}$ will be lost. In this section we propose an identification algorithm which asymptotically as $\Delta \downarrow 0$ will be able to recover correctly, to some large extent, the particle identities in pairs of successive configurations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$. The algorithm appears in Brandt [5] and was investigated by Berg [3] in dimension $d = 1$. In the present paper, we show that the result holds in arbitrary dimension $d \geq 1$; heat kernel bounds according to assumption play a key role. The reconstruction algorithm is presented in definition the main results are theorems and

3.1 $\varepsilon$-wellspread configurations, identifiable pairs of configurations, the reconstruction algorithm and the problem of correct identification

Recall that we write $x = (x_1, \ldots, x_\ell)$ for configurations $x \in S$ and $x_i = (x_{i,1}, \ldots, x_{i,d})$ for particle positions in $E = \mathbb{R}^d$. In this section, our assumptions will always include for arbitrary choice of an immigration measure, and invariant probability measure $\mu$ on $S$ and invariant occupation measure $\mathcal{P}$ on $E$ are as in lemma [1.2.5]
We call a configuration $x = (x_1, \ldots, x_\ell)$ with $\ell \geq 2$ $\varepsilon$-wellspread if all two-particle subconfigurations $(x_{i_1}, x_{i_2})$, $1 \leq i_1 < i_2 \leq \ell$, are such that

$$\min_{1 \leq j \leq d} |x_{i_1,j} - x_{i_2,j}| \geq \varepsilon \quad \text{for all components } 1 \leq j \leq d.$$  

We extend the definition to $\ell \in \{0, 1\}$ by adopting the convention that one-particle configurations and the void configuration are $\varepsilon$-wellspread.

Write $D(\varepsilon)$ for the set of $\varepsilon$-wellspread configurations in $S$, and $N(\varepsilon) := S \setminus D(\varepsilon)$ for its complement:

$$N(\varepsilon) = \{x = (x_1, \ldots, x_\ell) \in S : \ell \geq 2, \text{there is } i_1 \neq i_2 \text{ and } j \text{ such that } |x_{i_1,j} - x_{i_2,j}| < \varepsilon \}.$$  

Then $N(\varepsilon)$ is the set of all configurations in $S$ for which at least one pair of particles presents $\varepsilon$-close components. The following generalizes [3], theorem 2.11.

3.1.2 Theorem Assuming 1.2.1, 2.1.1 with $q := 3$, 2.1.2 and the heat kernel bounds 2.1.4, we have the following asymptotics for $N(\varepsilon)$ in (57):

$$\mu(N(\varepsilon)) \leq O(\varepsilon) \quad \text{as } \varepsilon \text{ tends to } 0.$$  

The proof will be given in subsection 3.2. We turn to discrete observation of the continuous-time BDI process $\eta = (\eta_t)_{t \geq 0}$. Fix $\Delta > 0$ and let $\{t_i := i\Delta : i \in \mathbb{N}_0\}$ denote a scheme of equidistant observation times. Observing discretely in time, pairs of successive observations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ are merely pairs of finite point measures, possibly of different total mass, without any indication whether or not particles may have died, reproduced or immigrated between times $i\Delta$ and $(i+1)\Delta$, and without any information on particle trajectories in-between. Thus discrete observation raises the problem of ‘particle identification’, to be solved prior to all statistical issues. In the following, we consider configurations $x = (x_1, \ldots, x_\ell(x))$ and $y = (y_1, \ldots, y_\ell(y))$ in $S$, and let $y \circ \pi$ denote the rearrangement $(y_{\pi(1)}, \ldots, y_{\pi(\ell(y))})$ of particles in $y$ by any permutation $\pi$ of $(1, \ldots, \ell(y))$.

3.1.3 Definition For $\Delta > 0$ and $0 < \lambda < \frac{1}{2}$, a pair $(x, y)$ in $S \times S$ is called $(\Delta, \lambda)$-identifiable if

i) $x$ is $4\Delta^\lambda$-wellspread,

ii) $y$ is $2\Delta^\lambda$-wellspread,

iii) $\ell(x) = \ell(y) =: \ell$ for some $\ell \geq 1$,

iv) there is some permutation $\pi$ of $(1, \ldots, \ell)$ (in case $\ell = 1$, $\pi(1) = 1$) which achieves

$$|y_{\pi(i),j} - x_{i,j}| < \Delta^\lambda \quad \text{for all } i = 1, \ldots, \ell, j = 1, \ldots, d.$$  

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Note that by ii), the permutation $\pi$ in iv) –if it exists– is necessarily unique. We write $\text{ID}(\Delta, \lambda)$ for the Borel subset of $(\Delta, \lambda)$-identifiable pairs $(x, y)$ in $S \times S$.

We call a pair $(\eta_i, \eta_{(i+1)\Delta})$ of successive discrete observations $(\Delta, \lambda)$-identifiable when $(\eta_i, \eta_{(i+1)\Delta})$ takes its value in the set $\text{ID}(\Delta, \lambda)$. Based on $3.1.3$ and $3.1.1$ we propose a reconstruction algorithm.

3.1.4 Definition (Reconstruction algorithm): For $(\Delta, \lambda)$-identifiable pairs $(\eta_i, \eta_{(i+1)\Delta})$, for $\pi$ the permutation which achieves $3.1.3$ iv) for $x := \eta_i$ and $y := \eta_{(i+1)\Delta}$, we decide to view $y_{\pi(k)}$ as the position at time $(i + 1)\Delta$ of the particle which was in position $x_k$ at time $i\Delta$ for $k = 1, \ldots, \ell(x)$.

A decision proposed by algorithm 3.1.4 may be correct or incorrect. $(\Delta, \lambda)$-identifiability of a pair of successive observations $(\eta_i, \eta_{(i+1)\Delta})$ is defined in terms of the $\sigma$-field 

$$\mathcal{H}_{i+1}^\Delta \quad \text{with notation} \quad \mathcal{H}_r^\Delta := \sigma(\eta_j : 0 \leq j \leq r), \ r \in \mathbb{N}_0$$

and the algorithm 3.1.4 proposes a decision on the basis of this information. In order to judge whether the proposed decision identifies particles correctly or fails to do so, we need the continuous-time filtration generated by the process $(\eta_t)_{t \geq 0}$, i.e.

$$\mathcal{F}_{(i+1)\Delta} \quad \text{with notation} \quad \mathcal{F}_t = \bigcap_{r > t} \sigma(\eta_s : 0 \leq s \leq r), \ t \geq 0$$

and have to consider path segments

\begin{equation}
\eta_{[i\Delta, (i+1)\Delta]} : [i\Delta, (i+1)\Delta] \ni t \longrightarrow \eta_t \in S
\end{equation}

as $\mathcal{F}_{(i+1)\Delta}$-measurable random variables taking values in $D([i\Delta, (i+1)\Delta], S)$, the path space of càdlàg functions $[i\Delta, (i+1)\Delta] \to S$. The notion introduced now refers to the larger $\sigma$-field $\mathcal{F}_{(i+1)\Delta}$.

3.1.5 Definition For $\Delta > 0$ and $0 < \lambda < \frac{1}{2}$, elements $f$ in $D([i\Delta, (i+1)\Delta], S)$ are $(\Delta, \lambda)$-good if 

i) $f(i\Delta)$ is $4\Delta^\lambda$-wellspread,

ii) there is $\ell \geq 1$ such that $f$ belongs to the space of continuous functions $C([i\Delta, (i+1)\Delta], E^\ell)$,

iii) writing for short $x := f(i\Delta)$ and $y := f((i+1)\Delta)$, we have 

$$|y_{k,j} - x_{k,j}| < \Delta^\lambda \quad \text{for all} \quad k = 1, \ldots, \ell, \ j = 1, \ldots, d.$$
We write \( \mathcal{C}_I(\Delta, \lambda) \) for the Borel set of \((\Delta, \lambda)\)-good elements in the path spaces \( D([i\Delta, (i+1)\Delta], S) \), irrespectively of \( i \in \mathbb{N}_0 \).

Paths segments \( \eta_{[i\Delta, (i+1)\Delta]} \) in \( D \) are called \((\Delta, \lambda)\)-good when \( \eta_{[i\Delta, (i+1)\Delta]} \) takes its value in the set \( \mathcal{C}_I(\Delta, \lambda) \). Then definitions 3.1.5 and 3.1.3 imply the following assertions (59) and (60):

(59) \[ \{ \eta_{[i\Delta, (i+1)\Delta]} \in \mathcal{C}_I(\Delta, \lambda) \} \subset \{ (\eta_{i\Delta}, \eta_{(i+1)\Delta}) \in \mathcal{I}_D(\Delta, \lambda) \} ; \]

(60) on \( \{ \eta_{[i\Delta, (i+1)\Delta]} \in \mathcal{C}_I(\Delta, \lambda) \} \), algorithm 3.1.4 identifies the particles correctly.

The proportion of \((\Delta, \lambda)\)-identifiable pairs observed up to time \( T \) is

(61) \[ \frac{1}{\lfloor T \rfloor} \sum_{i=0}^{\lfloor T \rfloor-1} 1_{\mathcal{I}_D(\Delta, \lambda)}(\eta_{i\Delta}, \eta_{(i+1)\Delta}) ; \]

and (59) allows to lower-bound this by

(62) \[ \frac{1}{\lfloor T \rfloor} \sum_{i=0}^{\lfloor T \rfloor-1} 1_{\mathcal{C}_I(\Delta, \lambda)}(\eta_{[i\Delta, (i+1)\Delta]} ) \]

which is \( \mathcal{F}_T \)-measurable. As a consequence of (61), ratio (62) provides a lower bound for the proportion of pairs of successive discrete observations to which algorithm 3.1.4 first applies and second proposes a correct reconstruction of particle identities. We aim at lower bounds for (62) in stationary regime when \( 0 < \lambda < \frac{1}{2} \) is fixed and \( \Delta > 0 \) is small enough. Below, \( Q_\mu \) is the law on the canonical path space of the BDI process \( \eta = (\eta_t)_{t \geq 0} \) running stationary, i.e. with initial law \( \mu \).

3.1.6 Theorem

Fix \( 0 < \lambda < \frac{1}{2} \).

a) Under the assumptions of theorem 3.1.2, we have as \( \Delta \downarrow 0 \)

(63) \[ Q_\mu \left( \ell(\eta_0) \geq 1, \eta_{[0,\Delta]} \notin \mathcal{C}_I(\Delta, \lambda) \right) \leq O(\Delta^\lambda) , \]

(64) \[ Q_\mu \left( \ell(\eta_0) \geq 1, (\eta_0, \eta_\Delta) \in \mathcal{I}_D(\Delta, \lambda), \eta_{[0,\Delta]} \notin \mathcal{C}_I(\Delta, \lambda) \right) \leq O(\Delta) . \]

b) There is some \( \Delta_0 > 0 \) and some constant \( D < \infty \) such that for all \( 0 < \Delta < \Delta_0 \) the following inequalities hold (note that \( Q_\mu (\ell(\eta_0) \geq 1) \) is strictly smaller than 1):

(65) \[ Q_\mu (\ell(\eta_0) \geq 1) - D \Delta^\lambda < Q_\mu (\eta_{[0,\Delta]} \in \mathcal{C}_I(\Delta, \lambda)) < Q_\mu (\ell(\eta_0) \geq 1) \]

(66) \[ Q_\mu (\eta_{[0,\Delta]} \in \mathcal{C}_I(\Delta, \lambda)) < Q_\mu (\eta_0, \eta_\Delta) \in \mathcal{I}_D(\Delta, \lambda) < Q_\mu (\eta_{[0,\Delta]} \in \mathcal{C}_I(\Delta, \lambda)) + D \Delta . \]
Theorem 3.1.6 will be proved in section 3.2. The main consequence of the theorem is the following: in the language of discretely observed semimartingales, it deals with high frequency observation schemes.

3.1.7 Theorem Write $T$ for deterministic time horizons. Fix $0 < \lambda < \frac{1}{2}$ and let $\Delta$ decrease to 0. Then we have the following convergences in $Q_\mu$-probability:

a) When $T \uparrow \infty$,

$$\frac{1}{|\Delta|} \sum_{i=0}^{\lfloor T/\Delta \rfloor-1} 1_{\mathcal{C}_i(\Delta,\lambda)}(\eta_{i\Delta},(i+1)\Delta)) \quad \rightarrow \quad Q_\mu(\ell(\eta_{0}) \geq 1) = 1 - \mu(E^0).$$

Here the limit is deterministic and strictly between 0 and 1.

b) When $T < \infty$ remains fixed,

$$\frac{1}{|\Delta|} \sum_{i=0}^{\lfloor T/\Delta \rfloor-1} 1_{\mathcal{C}_i(\Delta,\lambda)}(\eta_{i\Delta},(i+1)\Delta)) \quad \rightarrow \quad \frac{1}{T} \int_0^T 1_{\{\ell(s) \geq 1\}} \, ds.$$

Here the limit is a random variable taking values in $[0,1]$.

c) In both cases a) and b) above, we have

$$\frac{1}{|\Delta|} \sum_{i=0}^{\lfloor T/\Delta \rfloor-1} 1_{\{\ell(\eta_{i\Delta}) \geq 1, \eta_{i\Delta},(i+1)\Delta) \notin \mathcal{C}_i(\Delta,\lambda)} = o(Q_\mu)(1) \quad \text{as} \quad \Delta \downarrow 0.$$

On the basis of definitions 3.1.5 and 3.1.3 and of (59), (60), (61) and (62), we can resume theorem 3.1.7 as follows. For high frequency observation schemes, in the sense of asymptotics a) or b), the reconstruction algorithm 3.1.4 first applies to pairs of successive observations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ and second proposes a correct answer to the problem of particle identification

‘for eventually all pairs $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ which have $\ell(\eta_{i\Delta}) > 0$’

asymptotically as $\Delta \downarrow 0$. It is clear that pairs $(\eta_{i\Delta}, \eta_{(i+1)\Delta})$ with $\eta_{i\Delta} = \delta$ are of no use in view of reconstruction of particle identities. The proof of theorem 3.1.7 will be given in section 3.2.3

3.2 Proofs for subsection 3.1

We prove the results of the preceding section.
Proof of theorem 3.1.2: 1) Recall that assumption 2.1.2 implies 2.1.3 for arbitrary choice of an immigration measure. Under 1.2.1, 2.1.1 with \( q = 3 \) and 2.1.2 we know from theorem 2.1.3
\[
\mu(\ell^3) = \sum_{\ell \in \mathbb{N}} \ell^3 \mu(E^{\ell}) < \infty.
\]

2) With respect to \( \varepsilon > 0 \) arbitrary but fixed –which for a while we suppress from notations– write for two-particle configurations \((x_1, x_2) \in E^2\) with \( x_i = (x_{i,1}, \ldots, x_{i,d})\)
\[
h(x_1, x_2) := \sum_{j=1}^{d} 1_{|x_{1,j} - x_{2,j}| < \varepsilon}
\]
and define a function \( g : S \to [0, \infty) \) by
\[
g(x_1, \ldots, x_\ell) := \sum_{1 \leq i_1 < i_2 \leq \ell} h(x_{i_1}, x_{i_2}) \quad \text{for } \ell \geq 2, \quad g \equiv 0 \text{ on } E^0 \cup E^1.
\]
Then we can write for short
\[
1_{\varepsilon(x)}(x) \leq g(x) \quad \text{for all } x \in S
\]
and have for the invariant probability \( \mu \) on \( S \) (use \[18\] in the proof of lemma 1.2.5 plus norming)
\[
\mu(N(\varepsilon)) \leq \int_S g \, d\mu = \frac{1}{E_\delta(R_1)} E_\delta \left( \int_0^{R_1} g(\eta_s) \, ds \right)
\]
where –writing \((T_j)_j\) for the sequence of jump times in the BDI process, and using the Markov property– \[18\] implies that
\[
E_\delta \left( \int_0^{R_1} g(\eta_s) \, ds \right) = \sum_{n=0}^{\infty} E_\delta \left( 1_{\{T_n < R_1\}} \int_{T_n}^{T_{n+1}} g(\eta_s) \, ds \right)
\]
where we can write
\[
E_\delta \left( 1_{\{T_n < R_1\}} \int_{T_n}^{T_{n+1}} g(\eta_s) \, ds \right) = E_\delta \left( 1_{\{T_n < R_1\}} E_{\eta_{T_n}} \left( \int_0^{T_1} g(\eta_s) \, ds \right) \right).
\]

3) We insert an auxiliary step and prove the bound
\[
\int_{E^2} [P_t^\kappa(x_1, dy_1) \otimes P_t^\kappa(x_2, dy_2)] h(y_1, y_2) \leq d C^2(2\pi C)^d 2\varepsilon \frac{1}{\sqrt{2\pi Ct}}
\]
valid for \( 0 < t \leq t_0 \), where we recall that \((P_t^\kappa)_{t \geq 0}\) is the semigroup of the one-particle motion killed at rate \( \kappa \) and \( t_0 \) is from assumption 2.1.4 (note that the right hand side of \[72\] is free from \( x_1 \) and \( x_2 \)). Indeed, for such \( t \) the left hand side of \[72\] is by 2.1.4 smaller than
\[
C^2(2\pi C)^d \int_{E^2} p_{Ct}(y_1 - x_1) p_{Ct}(y_2 - x_2) h(y_1, y_2) \, dy_1 \, dy_2
\]
where \( p_s \) denotes the density (on \( \mathbb{R}^d \)) at time \( s \) for \( d \)-dimensional standard Brownian motion. Since

\[
h(y_1, y_2) = \sum_{j=1}^{d} 1_{\{|y_{1,j} - y_{2,j}| < \varepsilon\}} = \sum_{j=1}^{d} B_\varepsilon(y_{1,j})(y_{2,j})
\]

depends on \( \varepsilon \) (here \( B_\varepsilon \) denotes a ball in \( \mathbb{R}^1 \)), it is sufficient to calculate in (73) the maximum of a one-dimensional normal density to prove (72).

4) Still keeping \( \varepsilon > 0 \) fixed and suppressed from notations, we evaluate typical terms in the expressions (70) and (71) in order to establish a bound

\[
E_{(x_1, \ldots, x_\ell)} \left( \int_0^{T_1} g(\eta_s) \, ds \right) \leq \text{cst} \, d \frac{\ell(\ell - 1)}{2} (t_0^{-1/2} + t_0^{1/2}) ;
\]

here and below, ‘cst’ collects constants which are not of interest (and which may change from line to line). To prove (74), we start from

\[
E_{(x_1, \ldots, x_\ell)} \left( \int_0^{T_1} h(\eta_s^1, \eta_s^2) \, ds \right) = \sum_{1 \leq i_1 < i_2 \leq \ell} E_{(x_1, \ldots, x_\ell)} \left( \int_0^{T_1} h(\eta_s^{i_1}, \eta_s^{i_2}) \, ds \right)
\]

for \( (x_1, \ldots, x_\ell) \in S \) and \( \ell \geq 1 \), with \( g \) and \( h \) as above. We have by definition of the BDI process

\[
E_{(x_1, \ldots, x_\ell)} \left( \int_0^{T_1} h(\eta_s^{i_1}, \eta_s^{i_2}) \, ds \right) = E_{(x_1, \ldots, x_\ell)} \left( \int_0^{\infty} dt \, e^{-ct} \left( \int_0^t \prod_{j=1}^{\ell} P_t^c(x_j, dy_j) \right) (y_{i_1}, y_{i_2}) \right)
\]

\[
= \int_0^{\infty} dt \, e^{-ct} \int_{E^2} \prod_{j=1}^{\ell} P_t^c(x_j, dy_j) \ h(y_{i_1}, y_{i_2})
\]

\[
\leq \int_0^{\infty} dt \, e^{-ct} \int_{E^2} [P_t^c(x_{i_1}, dy_{i_1}) \otimes P_t^c(x_{i_2}, dy_{i_2})] \ h(y_{i_1}, y_{i_2}) .
\]

Let us consider the initial part \( \int_0^{t_0} dt \) of the last integral first: using (72),

\[
\int_0^{t_0} dt \, e^{-ct} \int_{E^2} [P_t^c(x_{i_1}, dy_{i_1}) \otimes P_t^c(x_{i_2}, dy_{i_2})] \ h(y_{i_1}, y_{i_2})
\]

is bounded above by

\[
\int_0^{t_0} dt \, e^{-ct} \, d \, C(2\pi C)^d \, 2\varepsilon \, \frac{1}{\sqrt{2\pi Ct}} \leq \text{cst} \, d \, t_0^{1/2} .
\]

Turning to the remaining part \( \int_{t_0}^{\infty} dt \) of the integral above, we shall prove a bound

\[
\int_{t_0}^{\infty} dt \, e^{-ct} \int_{E^2} [P_t^c(x_{i_1}, dy_{i_1}) \otimes P_t^c(x_{i_2}, dy_{i_2})] \ h(y_{i_1}, y_{i_2}) \leq \text{cst} \, d \, t_0^{-1/2} .
\]

To see this, write the left hand side of (77) as

\[
e^{-ct_0} \int_0^{\infty} dv \, e^{-cv} \ F(v, t_0, x_{i_1}, x_{i_2})
\]

with short notation

\[
F(v, t_0, x_{i_1}, x_{i_2}) := \int_{E^2} [P^c_{t_0} \otimes P^c_{t_0}](x_{i_1}, dy_{i_1}) \otimes [P^c_{t_0} \otimes P^c_{t_0}](x_{i_2}, dy_{i_2}) \, h(y_{i_1}, y_{i_2}) .
\]

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Rearranging terms and applying again (72), this is smaller than
\[
\int_{E^2} [P_{v_1}^{z_1}(x_{i1}, dz_{i1}) \otimes P_{v_2}^{z_2}(x_{i2}, dz_{i2})] \, h(y_{i1}, y_{i2}) \leq \int_{E^2} [P_{v_1}^{z_1}(x_{i1}, dz_{i1}) \otimes P_{v_2}^{z_2}(x_{i2}, dz_{i2})] \, d \, C^2(2\pi C)^d \, 2\varepsilon \frac{1}{\sqrt{2\pi Ct_0}} \leq d \, C^2(2\pi C)^d \, 2\varepsilon \frac{1}{\sqrt{2\pi Ct_0}}
\]
which gives (77). Combining (77), (76) and (75) we have proved (74).

5) Now we insert the bound (74) into the three equations (69), (70) and (71) which gives
\[
\mu(N(\varepsilon)) \leq \text{cst} \, \sum_{n=0}^{\infty} E_\delta \left( 1_{\{T_n < R_1\}} \, E_{\eta_{T_n}} \left( \int_0^{T_1} g(\eta_s) \, ds \right) \right)
\]
\[
\leq \text{cst} \, \varepsilon \, d \, (t_0^{-1/2} + t_0^{1/2}) \, \sum_{n=0}^{\infty} E_\delta \left( 1_{\{T_n < R_1\}} \, (\ell(\eta_{T_n}))^2 \right).
\]
Now from the definition of the BDI process, for all \(x \in S\) with \(\ell(x) \geq 1\),
\[
1 = E_x \left( \int_0^{\infty} dt \, (c + \pi(\eta_t)) e^{-\int_0^t (c + \pi(\eta_s)) \, ds} \right) \leq \text{cst} \, \ell(x) \, E_x(T_1).
\]
Then, absorbing also the dimension \(d\) and the term \((t_0^{-1/2} + t_0^{1/2})\) into the constants, we obtain
\[
\mu(N(\varepsilon)) \leq \text{cst} \, \varepsilon \, \sum_{n=0}^{\infty} E_\delta \left( 1_{\{T_n < R_1\}} \, (\ell(\eta_{T_n})^3 \, E_{\eta_{T_n}}(T_1) \right)
\]
\[
\leq \text{cst} \, \varepsilon \, \sum_{n=0}^{\infty} E_\delta \left( 1_{\{T_n < R_1\}} \, (\ell(\eta_{T_n})^3 \, \int_0^{T_1} (\ell(\eta_s))^3 \, ds \right)
\]
\[
\leq \text{cst} \, \varepsilon \, E_\delta \left( \int_0^{R_1} (\ell(\eta_s))^3 \, ds \right)
\]
which gives
\[
\mu(N(\varepsilon)) \leq \text{cst} \, \varepsilon \, \mu(\ell^3)
\]
and finishes the proof of theorem 3.1.2.

\[ \Box \]

**Proof of theorem 3.1.6:** 1) In a first step, for \(0 < \lambda < \frac{1}{2}\) fixed and \(\Delta > 0\) small enough, define
\[
h_{\Delta, \lambda}(x, y) := \sum_{j=1}^{d} 1_{\{|x_j-y_j| > \Delta \lambda\}} \, , \, x, y \in E = \mathbb{R}^d
\]
for one-particle configurations. We shall show that asymptotically as \(\Delta \downarrow 0\),
\[
(78) \quad \int_E P_{v}^\infty(x, dy) \, h_{\Delta, \lambda}(x, y) = o(\Delta)
\]
where \(o(\Delta)\) denotes bounds which do not depend on \(x \in E\). Indeed, denoting by \(\Phi\) resp. \(\varphi\) the distribution function resp. density of the standard normal law on \(\mathbb{R}\), as in the proof of theorem 3.1.2.
the heat kernel bounds\textsuperscript{2.1.4} yield
\[
\int_E P_\Delta(x, dy) h_{\Delta, \lambda}(x, y) \leq \frac{C}{\sqrt{2\pi C}} \int_E dy p_{\lambda C}(y - x) h_{\Delta, \lambda}(x, y) = d \frac{C}{\sqrt{2\pi C}} \frac{1}{2} \left( 1 - \Phi(\frac{\Delta}{\sqrt{C \Delta}}) \right).
\]
Using \(0 < \lambda < \frac{1}{2}\) and the elementary inequality \(1 - \Phi(v) < \frac{1}{v^2}\Phi(v)\) which is true for all \(v > 0\), this in turn is bounded by
\[
\leq \text{cst} \Delta^{\frac{1}{2} - \lambda} e^{-\frac{x^2}{\Delta}} = \text{cst} \Delta \left( \frac{1}{\Delta} \right)^{\frac{1}{2} + \lambda} e^{-\frac{x^2}{\Delta}} = o(\Delta)
\]
as \(\Delta \downarrow 0\). Here and below, ‘\text{cst}’ denotes constants which may change from line to line.

2) Consider a one-particle motion \(\xi\) killed at rate \(\kappa\), starting at time 0 from \(x \in E\). Over a time interval of length \(\Delta\) we will have three possibilities. Either killing occurs before time \(\Delta\), i.e. \(\xi_{\Delta} = \hat{\delta}\) where \(\hat{\delta}\) represents some cemetery point for processes with life time, or \(\xi_{\Delta}\) takes values in a cube \(U_{\Delta, \lambda}(x) := \bigcup_{j=1}^{d} (x_{j} - \Delta^{\lambda}, x_{j} + \Delta^{\lambda})\) centred at \(x = (x_{1}, \ldots, x_{d})\), or \(\xi_{\Delta}\) takes values in \(E \setminus U_{\Delta, \lambda}(x)\). If we write \(\hat{P}_{\Delta}^{c}(\cdot, \cdot)\) for the semigroup of the one-particle motion including possible jumps to the cemetery point on the extended state space \(\hat{E} := \{\hat{\delta}\} \cup E\), and as before \(P_{\Delta}^{c}(\cdot, \cdot)\) for the killed semigroup on \(E\) in assumption \textsuperscript{2.1.4} this means that asymptotically as \(\Delta \downarrow 0\),
\[
\hat{P}_{\Delta}^{c}\left(x, \left(\{\hat{\delta}\} \cup E \setminus U_{\Delta, \lambda}(x)\right)\right) \leq \|\kappa\|_{\infty} \Delta + P_{\Delta}^{c}(x, E \setminus U_{\Delta, \lambda}(x)) \leq \|\kappa\|_{\infty} \Delta + o(\Delta)
\]
where we have used \textsuperscript{78}. The bounds do not depend on \(x \in E\), for all \(\Delta\) sufficiently small. We turn to \(\ell\)-particle motions, \(\ell \geq 1\), killed and jumping to some cemetery point \(\hat{\delta}\) at rate \(c + \pi(\cdot)\), and write \(\hat{P}_{\Delta}^{c + \pi}(\cdot, \cdot)\) for its semigroup on the extended state space \(\hat{E}^{\ell} := \{\hat{\delta}\} \cup E^{\ell}\). Independence of the killed motions of individual particles shows that
\[
\hat{P}_{\Delta}^{c + \pi}\left(x, \left(\{\hat{\delta}\} \cup E^{\ell} \setminus \bigcup_{i=1}^{\ell} U_{\Delta, \lambda}(x_{i})\right)\right) \leq (c + \|\kappa\|_{\infty} \ell) \Delta + \sum_{i=1}^{\ell} P_{\Delta}^{c}(x_{i}, E \setminus U_{\Delta, \lambda}(x_{i}))
\]
for starting positions \(x = (x_{1}, \ldots, x_{\ell}) \in E^{\ell}\), where again by \textsuperscript{78}
\[
(79) \quad \hat{P}_{\Delta}^{c + \pi}\left(x, \left(\{\hat{\delta}\} \cup E^{\ell} \setminus \bigcup_{i=1}^{\ell} U_{\Delta, \lambda}(x_{i})\right)\right) \leq \text{cst} \ell \Delta \quad \text{as} \quad \Delta \downarrow 0
\]
for all \(\ell \geq 1\), with some global constant not depending on \(\ell \geq 1\) or \(x \in E^{\ell}\).

3) So far we have exploited assumptions\textsuperscript{1.2.1 and 2.1.4} The following argument will exploit \textsuperscript{2.1.1} (with \(q := 3\)) together with \textsuperscript{2.1.2} (which implies \textsuperscript{1.2.2} for arbitrary choice of an immigration measure), and will conclude the proof. Consider a path segment \(\eta_{[\Delta(i+1)\Delta]}\) under \(Q_{\mu}\). By stationarity it is sufficient

to consider $i = 0$, with random initial position $\eta_0 \in S$ selected according to invariant measure $\mu$.

Directly from definitions 3.1.5, 3.1.3 and (57) we have the inclusion

$$\{ \ell(\eta_0) \geq 1, \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda) \} \subset \{ \eta_0 \in \mathbb{N}(4\Delta^3), \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda) \}.$$  

Whenever an initial configuration $x \in S$ with $\ell = \ell(x) \geq 1$ is $4\Delta^3$-wellspread, we consider as in step 2) the $\ell$-particle motion starting from $x$, killed and jumping to a cemetery point $\hat{\delta}$ at rate $c + \overline{p}(-);$ then

$$Q_x(\eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda)) = \mathcal{P}^{\ell+\overline{p}}(x, \{ \hat{\delta} \cup E^\ell \setminus \bigcup_{i=1}^\ell U_{\Delta^3}(x_i) \}) \leq \text{cst} \ell \Delta$$

from (79), with $x = (x_1, \ldots, x_\ell)$. As a consequence, $\overline{p}(1) = \sum_\ell \ell \mu(E^\ell)$ being finite in virtue of lemma 1.2.5, we arrive at

$$Q_\mu(\ell(\eta_0) \geq 1, \eta_0 \in \mathbb{D}(4\Delta^3), \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda)) \leq \text{cst} \Delta \sum_{\ell=1}^\infty \ell \mu(E^\ell) = \mathcal{O}(\Delta)$$

as $\Delta \downarrow 0$. So far, from the above inclusion and the bound (81),

$$Q_\mu(\ell(\eta_0) \geq 1, \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda)) \leq \mu\left(\mathbb{N}(4\Delta^3)\right) + \mathcal{O}(\Delta)$$

as $\Delta \downarrow 0$. Now we make use of assumption 2.1.1 with $q := 3$: applying theorem 3.1.2 to the first term on the right hand side, we have proved assertion (63) in theorem 3.1.6. Note that the rate in (63) comes from the exceptional set $\mathbb{N}(4\Delta^3)$ in the above inclusion.

Assertion (64) follows from the bound (81) since $\{(\eta_0, \eta_\Delta) \in \text{ID}(\Delta, \lambda)\} \subset \{\eta_0 \in \mathbb{D}(4\Delta^3)\}$ by definition in 3.1.3 i). We have proved part a) of theorem 3.1.6

4) We prove part b) of the theorem. By definition in 3.1.5 path segments with $\eta_0 = \delta$ never belong to $\text{CI}(\Delta, \lambda)$, thus

$$\{ \eta_{[0,\Delta]} \in \text{CI}(\Delta, \lambda) \} \cup \{ \ell(\eta_0) \geq 1, \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda) \} = \{ \ell(\eta_0) \geq 1 \}.$$  

By (63) in part a), the $Q_\mu$-probability of the second event on the left hand side is $\mathcal{O}(\Delta^3)$ as $\Delta \downarrow 0$. This establishes (64). Similarly, (59) allows to write

$$\{ \eta_{[0,\Delta]} \in \text{CI}(\Delta, \lambda) \} \cup \{ (\eta_0, \eta_\Delta) \in \text{ID}(\Delta, \lambda), \eta_{[0,\Delta]} \notin \text{CI}(\Delta, \lambda) \} = \{ (\eta_0, \eta_\Delta) \in \text{ID}(\Delta, \lambda) \}$$

where the $Q_\mu$-probability of the second event on the left hand side is $\mathcal{O}(\Delta)$ as $\Delta \downarrow 0$, by (64) in part a). Note that $(\eta_0, \eta_\Delta) \in \text{ID}(\Delta, \lambda)$ implies $\ell(\eta_0) \geq 1$ by definition 3.1.3. This establishes (60).

The proof of theorem 3.1.6 is finished. □

We mention a consequence of theorem 3.1.6 of minor importance.
3.2.1 Proposition  Fix $0 < \lambda < \frac{1}{2}$ and $\Delta \in (0,\Delta_0)$, with $\Delta_0 > 0$ from part b) of theorem 3.1.6. Then for arbitrary choice of a starting point $x \in S$, we have $Q_x$-almost sure convergence as $N \to \infty$

\begin{equation}
\frac{1}{N} \sum_{i=0}^{N-1} 1_{CI(\Delta,\lambda)}(\eta_{[i\Delta,(i+1)\Delta]}) \to Q_\mu\left( (\eta_{[0,\Delta]} \in CI(\Delta,\lambda)) \right)
\end{equation}

(83)

\begin{equation}
\frac{1}{N} \sum_{i=0}^{N-1} 1_{ID(\Delta,\lambda)}(\eta_{i\Delta},\eta_{(i+1)\Delta}) \to Q_\mu\left( (\eta_0,\eta_\Delta) \in ID(\Delta,\lambda)) \right)
\end{equation}

(84)

where the limits (83) and (84) are such that inequalities (63) and (64) hold for $0 < \Delta < \Delta_0$.

Proof: By lemma 1.2.5 the continuous-time process $\eta = (\eta_t)_{t \geq 0}$ is positive Harris. A particular feature of the BDI model is that $\eta$ returning infinitely often to the void configuration $\delta$ will remain there during an independent exponential time with parameter $c > 0$. As a consequence, for $\Delta > 0$ arbitrary but fixed, there will be an infinite number of intervals $[j\Delta,(j+1)\Delta]$ on which $(\eta_t)_{t \geq 0}$ remains visiting the void configuration. Thus the Markov chain of path segments $(\eta_{i\Delta},\eta_{(i+1)\Delta})_{i \in \mathbb{N}_0}$ taking values in $D([0,\Delta],S)$, will admit an infinite number of visits in state $\equiv \delta$ viewed as a path in $D([0,\Delta],S))$. As consequences of this fact, both the path segment chain and the chain of successive pairs $(\eta_{i\Delta},\eta_{(i+1)\Delta})_{i \in \mathbb{N}_0}$ are positive Harris chains. We thus have strong laws of large numbers: the rescaled additive functionals on the left hand side in (83) and (84) converge $Q_x$-almost surely, for every choice of a starting point $x \in S$, to the limits on the right. □

We explain why proposition 3.2.1 is of minor importance. For $0 < \lambda < \frac{1}{2}$ fixed and for $0 < \Delta < \Delta_0$ small but fixed, drawing a large number of discrete-time observations $\eta_{i\Delta} , 0 \leq i \leq N$ (the asymptotics is in $N \to \infty$), the reconstruction algorithm 3.1.4 applies to a proportion of roughly

\[ Q_\mu\left( (\eta_0, \eta_\Delta) \in ID(\Delta,\lambda) \right) > Q_\mu(\ell(\eta_0) \geq 1) - D\Delta^\lambda \]

pairs of successive observations $(\eta_{i\Delta},\eta_{(i+1)\Delta})$, using theorem 3.1.6 it is clear that observed pairs with $\ell(\eta_{i\Delta}) = 0$ are of no use in view of the algorithm. If the decision proposed by algorithm 3.1.4 will be correct for a large amount of the data to which the algorithm applies, there will remain some small proportion of approximately

\[ 0 < Q_\mu\left( (\eta_0, \eta_\Delta) \in ID(\Delta,\lambda) , \eta_{[0,\Delta]} \notin CI(\Delta,\lambda) \right) < D\Delta \]

per cent of the data on which the algorithm 3.1.4 may take a decision which fails to identify the underlying (unobserved) travelling particles correctly. Note that $\Delta$ is small but fixed. In
the language of discretely observed semimartingales \((23, 34)\), proposition 3.2.1 belongs to the framework of ‘low frequency’ asymptotics. A fully satisfactory answer to the problem of particle identities requires a setting of ‘high frequency’ observation, i.e. \(\Delta\) tending to 0. We prove theorem 3.1.7.

**Proof of theorem 3.1.7:**

1) For deterministic \(T\), stationarity allows to write

\[
Q_\mu \left( \frac{1}{\lfloor T \Delta \rfloor} \sum_{i=0}^{\lfloor T \Delta \rfloor - 1} 1\{\ell(\eta_{i+1} \Delta) \geq 1, \eta_{i+1} \Delta \in CI(\Delta, \lambda)\} > \varepsilon \right) 
\]

\[
\leq \frac{1}{\varepsilon} E_\mu \left( \frac{1}{\lfloor T \Delta \rfloor} \sum_{i=0}^{\lfloor T \Delta \rfloor - 1} 1\{\ell(\eta_{i+1} \Delta) \geq 1, \eta_{i+1} \Delta \in CI(\Delta, \lambda)\} \right) 
\]

\[
= \frac{1}{\varepsilon} Q_\mu (\ell(\eta_0) \geq 1, \eta_{0} \Delta \in CI(\Delta, \lambda)) \leq \frac{1}{\varepsilon} O(\Delta^\lambda)
\]

as \(\Delta \downarrow 0\), using theorem 3.1.6. As a consequence, irrespectively of the behaviour of \(T\),

\[
\frac{1}{\lfloor T \Delta \rfloor} \sum_{i=0}^{\lfloor T \Delta \rfloor - 1} 1\{\ell(\eta_{i+1} \Delta) \geq 1, \eta_{i+1} \Delta \in CI(\Delta, \lambda)\} = o(Q_\mu)(1) \quad \text{as} \quad \Delta \downarrow 0.
\]

This is c). Together with a decomposition as in (82) we obtain the \(Q_\mu\)-stochastic equivalence

\[
1\{\ell(\eta_{i+1} \Delta) \geq 1, \eta_{i+1} \Delta \in CI(\Delta, \lambda)\} = 1\{\ell(\eta_0) \geq 1\} + o(Q_\mu)(1) \quad \text{as} \quad \Delta \downarrow 0
\]

which holds in both cases under consideration: the case where time horizon \(T\) is fixed and finite, and the case where \(T\) is increasing to \(\infty\). We underline that high-frequency asymptotics \(\Delta \downarrow 0\) is a necessary condition for (85).

2) When \(T < \infty\) is fixed and \(\Delta \downarrow 0\), the following convergence

\[
\frac{1}{\lfloor T \Delta \rfloor} \sum_{i=0}^{\lfloor T \Delta \rfloor - 1} 1\{\ell(\eta_{i+1} \Delta) \geq 1\} \rightarrow \frac{1}{T} \int_0^T 1\{\ell(\eta_s) \geq 1\} \, ds
\]

holds pathwise since every path of \((\eta_t)_{t \geq 0}\) is a càdlàg function. Here the limit is a random variable taking values in \([0, 1]\). Combining (86) and (85) we have proved b).

3) When \(T \uparrow \infty\) and \(\Delta \downarrow 0\), the following convergence

\[
\lim_{T \uparrow \infty} \lim_{\Delta \downarrow 0} \frac{1}{\lfloor T \Delta \rfloor} \sum_{i=0}^{\lfloor T \Delta \rfloor - 1} 1\{\ell(\eta_{i+1} \Delta) \geq 1\} = \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T 1\{\ell(\eta_s) \geq 1\} \, ds = Q_\mu(\ell(\eta_0) \geq 1)
\]

holds for \(Q_\mu\)-almost all paths of the BDI process \((\eta_t)_{t \geq 0}\), as a consequence of positive Harris recurrence by lemma 1.2.5. Combining (87) and (85) we have proved assertion a) of theorem 3.1.7.

\[\square\]
4 Regression schemes for estimation of the diffusion coefficient from discrete BDI observations

This section needs all assumptions of sections 1 and 2, and in particular relies heavily on theorems 2.1.6 and 3.1.6. For the one-particle motion in (A1), the additional assumption

\[ a(y) := (\sigma \sigma^T)(y) \text{ invertible for all } y \in E = \mathbb{R}^d \]

will be in force. Throughout, the continuous-time BDI process \((\eta_t)_{t \geq 0}\) is stationary, \(Q_\mu\) is the stationary law on the canonical path space as in section 3, and we deal with discrete-time observation at step size \(\Delta\) when \(\Delta\) is small.

4.1 The regression scheme: construction and properties

By theorem 2.1.6, the invariant occupation measure \(\pi\) on the single particle space \(E = \mathbb{R}^d\) admits a Lebesgue density \(\gamma \in C_0(E)\). Fix any cube \(A\) in \(E\) such that

\[ M_1 := \inf\{\gamma(x) : x \in A\} > 0 \]

(in virtue of theorem 2.1.6 such cubes do exist). Fix \(0 < \lambda < \frac{1}{2}\) and consider asymptotics \(\Delta \downarrow 0\) as in theorem 3.1.6; introduce the sequence of integers

\[ n(\Delta) := \lfloor L(A) \Delta^{-\frac{d}{2}} \rfloor \]

increasing to \(\infty\) as \(\Delta \downarrow 0\), with \(L(A)\) the edge length of \(A\) selected in (89).

Writing for short \(n = n(\Delta)\), we partition the cube \(A\) in (89) into \(n^d \sim \text{vol}(A) \Delta^{-\frac{1}{2}}\) cells of equal volume \(\sim \Delta^\frac{d}{2}\) and of equal edge length \(\sim \Delta^\frac{d}{2}\) in every component. In the special case where \(A\) is the unit cube \(A := [0,1]^d\), every such cell is identified through its upper right corner \((\frac{1}{n}, \ldots, \frac{1}{n})\). Then we write \(J(\Delta)\) for the set of all multiindices \(\alpha = (j_1, \ldots, j_d)\) arising in this way, and \(A_\alpha\) for the cell whose upper right corner \((\frac{j_1}{n}, \ldots, \frac{j_d}{n})\) makes appear \(\alpha \in J(\Delta)\). For a general cube \(A\) selected in (89), a linear transformation component by component maps \(A\) to \([0,1]^d\), allows to identify cells \(A_\alpha\) by the image \((\frac{j_1}{n}, \ldots, \frac{j_d}{n})\) of their upper right corner, and thus again yields a representation of \(A\) as a union of cells \(A_\alpha, \alpha \in J(\Delta)\). From (90) we have a one-to-one correspondence between \(J(\Delta)\) and \(\{1, \ldots, n\}^d\) where \(n = n(\Delta)\). We shall make use of the decomposition of \(A\) meeting (89) into cells

\[ A_\alpha, \quad \alpha \in J(\Delta) \]
to fill from discrete BDI observations \((\eta_i \Delta)_{i \in \mathbb{N}_0}\) regression schemes, see 4.1.1 below. Upon careful choice of \(0 < \lambda < \frac{1}{2}\) in definitions 3.1.3 and 3.1.5 and thus in the reconstruction algorithm 3.1.4 such schemes will allow to estimate the diffusion coefficient \((88)\) of the one-particle motion.

4.1.1 Definition (Regression scheme) Fix a cube \(A\) meeting \((89)\). Fix \(0 < \lambda < \frac{1}{2}\) and let \(\Delta\) decrease to 0. For \(\Delta\) small enough, define pairs \((X_\alpha, Z_\alpha) : \alpha \in \mathcal{J}(\Delta)\)

as follows:

i) For \(\alpha \in \mathcal{J}(\Delta)\), define

\[
\tau_\alpha = \tau_\alpha(\Delta) := \inf \{ i \in \mathbb{N}_0 : (\eta_i \Delta, \eta_{i+1}\Delta) \text{ is } (\Delta, \lambda)\text{-identifiable and satisfies } \eta_i \Delta(A_\alpha) \geq 1 \}.
\]

At time \(\tau_\alpha \Delta\), writing for short

\[
x := \eta_{\tau_\alpha} \Delta, \quad y := \eta_{(\tau_\alpha+1)\Delta}, \quad \ell := \ell(x) = \ell(y) \geq 1, \quad x = (x_1, \ldots, x_\ell), \quad y = (y_1, \ldots, y_\ell)
\]

and \(\pi\) for the unique permutation of \((1, 2, \ldots, \ell)\) such that \(y_{\pi(i)}\) is close to \(x_i\) for all \(1 \leq i \leq \ell\)

\[
|y_{\pi(i),j} - x_{i,j}| < \Delta^\lambda \quad \text{for all } 1 \leq i \leq \ell, \quad j = 1, \ldots, d
\]

in the sense of definition 3.1.3 (the permutation is trivial in case \(\ell = 1\)). Then \(\eta_\Delta(A_\alpha) \geq 1\) allows to pick \(m = m(\alpha)\) such that particle \(x_{m(\alpha)}\) is located in the cell \(A_\alpha\) at time \(\tau_\alpha \Delta\). For this \(m(\alpha)\) we define

\[
X_\alpha := x_{m(\alpha)} \quad \text{together with} \quad Z_\alpha := \frac{y_{\pi(m(\alpha))} - x_{m(\alpha)}}{\sqrt{\Delta}}.
\]

ii) In order to do so for all cells \(A_\alpha, \alpha \in \mathcal{J}(\Delta)\), we define

\[
\tau^* = \tau^*(\Delta) = \max \{ \tau_\alpha : \alpha \in \mathcal{J}(\Delta) \}.
\]

We make some comments. First, \(\tau_\alpha\) is a stopping time with respect to the filtration \((\mathcal{H}^\Delta_{i+1})_{i \in \mathbb{N}_0}\) defined after definition 3.1.4 of the reconstruction algorithm; the same holds for \(\tau^*\). We have to show that these stopping times are almost surely finite (at least), then \(X_\alpha\) and \(Z_\alpha\) are well-defined random variables taking values in \(E = \mathbb{R}^d\). Second, it may occur that we fill disjoint cells \(A_\alpha \neq A_{\alpha'}\) simultaneously at the same time. When \(\omega \in \{ \tau_\alpha = \tau_{\alpha'} \}\), pairs \((x, y)\) defined by

\[
x = \eta_{\tau_\alpha} \Delta(\omega) = \eta_{\tau_{\alpha'}} \Delta(\omega), \quad y = \eta_{(\tau_\alpha+1)\Delta}(\omega) = \eta_{(\tau_{\alpha'}+1)\Delta}(\omega)
\]

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being \((\Delta, \lambda)\)-identifiable in the sense of definition 3.1.3, we must have the following: \(m(\alpha) \neq m(\alpha')\) since \(A_\alpha \neq A_{\alpha'}\), thus \(\pi(m(\alpha)) \neq \pi(m(\alpha'))\) since \(\pi\) in 3.1.3 is a permutation, thus \(x_{m(\alpha)} \neq x_{m(\alpha')}\) and \(y_{\pi(m(\alpha))} \neq y_{\pi(m(\alpha'))}\) also in this case.

4.1.2 Proposition In the framework of definition 4.1.1, the stopping times \(\tau^* = \tau^*(\Delta)\) have finite expectation.

The proof given in section 4.2 below will also show that the expected time \(E_\mu(\Delta \tau^*(\Delta))\) which we need to fill the scheme 4.1.1 remains bounded as \(\Delta \downarrow 0\).

4.1.3 Theorem In the framework of 4.1.1, the scheme

\[(X_\alpha, Z_\alpha) : \alpha \in J(\Delta)\]

has the following properties:

i) We have \(X_\alpha \in A_\alpha\) for all \(\alpha \in J(\Delta)\), thus variables \(X_\alpha, \alpha \in J(\Delta)\), are approximately regularly spaced in the cube \(A\).

ii) There are ‘good events’ \(G(\Delta)\) of probability \(\geq 1 - O(\Delta^{\frac{1}{2}})\) on which the full collection \(Z_\alpha, \alpha \in J(\Delta)\), represents true (rescaled) \(\Delta\)-increments of the underlying one-particle motion in \((A1)\).

More precisely, on the ‘good event’ \(G(\Delta)\), there is a collection \(W_\alpha\) of independent \(d\)-dimensional standard Brownian motions and a collection \(\xi_\alpha\) of path segments which are solutions to

\[d\xi_\alpha(s) = b(\xi_\alpha(s))ds + \sigma(\xi_\alpha(s))dW_\alpha(s), \quad 0 \leq s \leq \Delta, \quad \alpha \in J(\Delta)\]

,strong solutions driven by \(W_\alpha\)\) such that all pairs \((X_\alpha, Z_\alpha), \alpha \in J(\Delta)\), admit the representation

\[\left(\frac{\xi_\alpha(\Delta) - \xi_\alpha(0)}{\sqrt{\Delta}}\right) = Z_\alpha, \quad \xi_\alpha(0) = X_\alpha, \quad \alpha \in J(\Delta).

\]

iii) There are exceptional events \(F(\Delta)\) of probability

\[Q_\mu(F(\Delta)) \leq O(\Delta^{\frac{1}{2}})\]

on which at least one entry \(Z_\alpha\) to the scheme 4.1.1, \(\alpha \in J(\Delta)\), fails to represent a true (rescaled) increment of the underlying one-particle motion.

iv) By construction in 4.1.1, we have

\[Z_\alpha \in \mathcal{X}_i \left( -\Delta^{\lambda-1/2}, \Delta^{\lambda-1/2} \right) \text{ for all } \alpha \in J(\Delta).\]
We intend to use the scheme (4.1.1) as a regression scheme for nonparametric estimation of the diffusion coefficient of the one-particle motion in (A1), based on ‘high-frequency’ discrete BDI observations. The motivation is the following (see proposition 6 in Genon-Catalot and Jacod [12], approximation (16) in Podolskij and Vetter [34], or the book Jacod and Protter [23]): on the good events \( G(\Delta) \) in theorem 4.1.3, good approximations

\[
Z_\alpha = \frac{\xi_\alpha(\Delta) - \xi_\alpha(0)}{\sqrt{\Delta}} \approx \sigma(\xi_\alpha(0)) \frac{W_\alpha(\Delta) - W_\alpha(0)}{\sqrt{\Delta}} = \sigma(X_\alpha) U_\alpha(1)
\]

are available for the terms in (91) where by construction

\[
U_\alpha(s) := \frac{W_\alpha(s\Delta) - W_\alpha(0)}{\sqrt{\Delta}}, \ 0 \leq s \leq 1, \ \alpha \in J(\Delta)
\]

are independent \( d \)-dimensional standard Brownian motions on the time interval \([0, 1]\). Clearly \( W_\alpha \) is independent of \( \xi_\alpha(0) = X_\alpha \). Since increments of the same Brownian motion over time intervals \([i\Delta, (i + 1)\Delta]\), \([i'\Delta, (i' + 1)\Delta]\), \( i' \neq i \), are independent, and since different particles have independent driving Brownian motions, the construction in theorem 4.1.3 grants that –in restriction to the good events \( G(\Delta) \)– the collection of Brownian motions \( \{U_\alpha : \alpha \in J(\Delta)\} \) is independent of the collection of design variables \( \{X_\alpha' : \alpha' \in J(\Delta)\} \).

From Itô’s formula, using superscript \( i \) for the components of \( U_\alpha \) and \( \delta^{(i,j)} \) for Kronecker’s \( \delta \),

\[
U_\alpha(1)U_\alpha^T(1) = \left( \delta^{(i,j)} + \int_0^1 U_\alpha^{(i)}(s) dU_\alpha^{(j)}(s) + \int_0^1 U_\alpha^{(j)}(s) dU_\alpha^{(i)}(s) \right)_{1 \leq i,j \leq d}
\]

which means that, always on the ‘good event’ \( G(\Delta) \) in theorem 4.1.3

\[
Z_\alpha Z_\alpha^T = (\sigma \sigma^T)(X_\alpha) + \text{error terms of martingale structure}.
\]

Thanks to theorem 4.1.3 and (94), (4.1.1) provides us –in restriction to the good sets \( G(\Delta) \)– with a regression scheme –in a sense analogous to sections 1.5.1 or 1.6.1 of Tsybakov [39]– for nonparametric estimation of the diffusion coefficient \( a(\cdot) = (\sigma \sigma^T)(\cdot) \) on \( \text{int}(A) \). The design variables \( X_\alpha, \alpha \in J(\Delta) \) are approximately regularly spaced over the cube \( A \) selected in the beginning.

In contrast to the good sets however, the picture is less pleasant on the exceptional sets \( F(\Delta) \): here we have nothing except the trivial bounds from theorem 4.1.3 iv). We shall illustrate the effect of the exceptional sets by an example (kernel estimation in dimension \( d = 1 \)) in section 4.3.
It remains to make precise what ‘good approximation’ in (12) means. We quote a result from Genon-Catalot and Jacod [12]; by theorem 4.1.3, their result applies in restriction to the good set $G(\Delta)$ where all $Z_\alpha$ in (91), $\alpha \in J(\Delta)$, are increments of rescaled one-particle motions. Below, $g$ denotes a polynomial on $\mathbb{R}^d$ of arbitrary finite degree $\gamma$.

4.1.4 Lemma ([12], proposition 6): Assume that the coefficients of the one-particle motion in (A1) are $C^2$ on $E = \mathbb{R}^d$ and satisfy (88) together with (95) $|b_l|, |\sigma_{l,j}|, |\partial_l \sigma_{l,m}|$ are bounded by some constant $L$.

Then, using notations of theorem 4.1.3, (91) and (93), there is some constant $C$ (which depends on $L$ and on the degree $\gamma$ of the polynomial $g$, but does not depend on $\Delta$ as $\Delta \downarrow 0$) such that the following deterministic bound holds:

$$E \left( 1_{G(\Delta)} \left| g(Z_\alpha) - g(\sigma(X_\alpha) U_\alpha(1)) \right|^2 \right| \{X_\alpha' : \alpha' \in J(\Delta)\} \right) \leq C \Delta, \quad \alpha \in J(\Delta).$$

4.2 Proofs for section 4.1

We start with some remarks motivating the construction in 4.1.1. Positive Harris recurrence of the continuous-time process grants that $(\eta_t)_{t \geq 0}$ visits infinitely often the void configuration $\delta$, spending an exponentially distributed time in $\delta$ at every visit. Thus for fixed $\Delta > 0$, discrete observations $(\eta_0, \eta_{i\Delta})_{i \in \mathbb{N}_0}$ form a positive recurrent Markov chain, with recurrent atom {$\delta$} and invariant measure $\mu$ on $S$, and pairs of successive observations $(\eta_{i\Delta}, \eta_{(i+1)\Delta})_{i \in \mathbb{N}_0}$ form a positive recurrent Harris chain, with recurrent atom {$(\delta, \delta)$} and invariant measure $\mu(dx)P_{\Delta}(x, dx')$ on $S \times S$. Positive recurrence ensures that in the long run we will collect an overwhelming amount of data: out of these we pick few but well-selected ones –using the reconstruction algorithm 3.1.4– to fill the scheme 4.1.1.

4.2.1 Proposition For the cube $A$ meeting (89) decomposed into disjoint cells $A_\alpha, \alpha \in J(\Delta)$,

$$\alpha \rightarrow \frac{1}{\overline{\mu}(A)} \int_S \mu(dx) 1_{\{x(A_\alpha) \geq 1\}}$$

defines a probability law on the finite set $J(\Delta)$ which is equivalent to the uniform law on $J(\Delta)$.

Proof: From theorem 2.1.6, the invariant occupation measure $\overline{\mu}$ on the single particle space $E = \mathbb{R}^d$ admits a Lebesgue density $\overline{\gamma} \in C_0(E)$. By choice of the cube $A$ in (89) we have $\overline{\gamma} > 0$ on $A$, hence

$$\alpha \rightarrow \frac{1}{\overline{\mu}(A)} \int_S \mu(dx) x(A_\alpha) = \frac{\overline{\mu}(A_\alpha)}{\overline{\mu}(A)} = \frac{1}{\overline{\mu}(A)} \int_E dy \overline{\gamma}(y) 1_{A_\alpha}(y) > 0$$
defines a probability on \( J(\Delta) \) under which every \( \alpha \in J(\Delta) \) carries strictly positive mass. In the sense of equivalence of measures, this probability is equivalent to the uniform law on \( J(\Delta) \). Also

\[
\alpha \rightarrow \int_S \mu(dx) x(A_\alpha), \quad \alpha \rightarrow \int_S \mu(dx) 1\{x(A_\alpha) \geq 1\}
\]

are equivalent measures on \( J(\Delta) \).

As an application of theorem \[3.1.6\] positive Harris recurrence of \( (\eta_i \Delta)_{i \in \mathbb{N}_0} \) combined with the last proposition yields a proof that the stopping times \( \tau^*(\Delta) \) in proposition \[4.1.2\] have finite expectation:

**Proof of proposition \[4.1.2\]:** For \( \Delta > 0 \) fixed, \[89\] implies that we need at most a geometric number of life cycles to observe the first occurrence of \( \{\eta_i \Delta(\alpha) \geq 1\}, i \in \mathbb{N}_0 \). The expected length of a life cycle is finite. For \( \alpha \in J(\Delta) \) fixed, \[4.2.1\] grants that we need at most a geometric number of occurrences of \( \{\eta_i \Delta(\alpha) \geq 1\} \) to record the first occurrence of \( \{\eta_i' \Delta(\alpha) \geq 1\}, i' \in \mathbb{N}_0 \). Here the success probability of the geometric number of trials is \( O(n^{-d}) = O(\Delta^{\frac{1}{2}}) \), as a consequence of proposition \[4.2.1\]. For \( \Delta \) small enough, theorem \[3.1.6\] ensures that after at most a geometric number of occurrences of \( \{\eta_i' \Delta(\alpha) \geq 1\} \) we will record the first occurrence of \( \{\eta_i'' \Delta(\alpha) \geq 1\}, \eta_i'' \in \mathbb{N}_0 \). By \[89\], this is an occurrence (not necessarily the first one) of the desired event

\[
\{ \eta_i'' \Delta(\alpha) \geq 1, (\eta_i'' \Delta, \eta_{(i''+1) \Delta}) \in \mathcal{ID}(\Delta, \lambda) \}.
\]

Since \( J(\Delta) \) is a finite set, proposition \[4.1.2\] is proved. \( \square \)

We remark that this proof indicates the following: Since \( J(\Delta) \) has \( n^d \) elements, since for every element \( \alpha \) of \( J(\Delta) \) we need in \( Q_{\mu}-\text{expection} \) \( O(n^{d}) \) trials to hit \( A_\alpha \), the expected time \( E_{\mu}(\Delta \tau^*(\Delta)) \) which we need to fill the scheme \[4.1.1\] will be of order

\[
O(n^d n^d \Delta) = O(1)
\]
as \( \Delta \downarrow 0 \). This is the motivation for the choice of \( n = n(\Delta) \) in \[89\].

**Proof of theorem \[4.1.3\]:** For every \( \Delta \) as \( \Delta \downarrow 0 \), the construction in \[4.1.1\] uses a fixed cube \( A \) meeting \[89\], partitioned into \( n^d = O(\Delta^{-\frac{1}{2}}) \) cells \( A_\alpha \) of equal size, \( \alpha \in J(\Delta) \), and pairs

\[
(\eta_{\tau_\alpha \Delta}, \eta_{[(\tau_\alpha+1) \Delta]} \in \mathcal{ID}(\Delta, \lambda) \text{ such that } \eta_{\tau_\alpha \Delta}(A_\alpha) \geq 1, \quad \alpha \in J(\Delta)
\]

(97)
where \( n = n(\Delta) \) is given by \([90]\).

1) We prove assertions iv) and i) of the theorem. By definition in \(3.1.3\) both configurations in \([97]\) have equal length \( \ell \geq 1 \). For every \( \alpha \in J(\Delta) \), a unique permutation \( \pi \) of \( (1, \ldots, \ell) \) maps particles in the configuration \( \eta(\tau_\alpha + 1)\Delta =: y = (y_1, \ldots, y_\ell) \) to particles in the configuration \( \eta(\tau_\alpha)\Delta =: x = (x_1, \ldots, x_\ell) \) via

\[
|y_{\pi(i),j} - x_{i,j}| < \Delta^\lambda \quad \text{for all } 1 \leq i \leq \ell \text{ and all } 1 \leq j \leq d.
\]

Since \( \eta(\tau_\alpha)(A_\alpha) \geq 1 \), some particle \( m = m(\alpha) \) in the configuration \( x \) is visiting \( A_\alpha \) at time \( \tau_\alpha \Delta \). With

\[
X_\alpha := x_m, \quad Z_\alpha := \frac{y_{\pi(m)} - x_m}{\sqrt{\Delta}}
\]

we achieve \( X_\alpha \in A_\alpha \) and \( Z_\alpha \in d X_j = 1 \) \((-\Delta^\lambda, \Delta^\lambda)\).

2) To prove assertions ii) and iii), we define the ‘good events’ in ii) as follows:

\[
G(\Delta) := \{ \text{all pairs } (\eta(\tau_\alpha)\Delta, \eta(\tau_\alpha + 1)\Delta) \text{ in } (97) \text{ are such that } \eta[\tau_\alpha \Delta, (\tau_\alpha + 1)\Delta] \in CI(\Delta, \lambda) \}\.
\]

Then definition \(3.1.5\) and \([59]+[60]\) show the following: in restriction to \( G(\Delta) \), the reconstruction algorithm \(3.1.4\) applied to data \([97]\) will reconstruct all particle identities correctly. In particular it reconstructs correctly –on \( G(\Delta) \), on \( \{\tau_\alpha = i\} \) for \( \alpha \in J(\Delta) \)– the increments \( \xi_\alpha((i+1)\Delta) - \xi_\alpha(i\Delta) \) in the trajectory of the selected particle over the time interval \([i\Delta, (i+1)\Delta]\). Uniquely associated to this particle and this time interval is the increment \( W_\alpha((i+1)\Delta) - W_\alpha(i\Delta) \) of the driving Brownian motion in \((A1)\). Driving Brownian motions for different particles are independent (we might select the same particle twice: then this happens over disjoint time intervals \([i\Delta, (i+1)\Delta] \), \([i'\Delta, (i'+1)\Delta]\), \(i' \neq i\), and we have again independence of the increments of the Brownian motion). This is \([91]\), up to a change of time.

3) It remains to prove the bound in assertion iii). Note that \( G(\Delta) \) belongs to the \( \sigma \)-field \( F_\tau^*(\Delta) \) associated to continuous-time observation up to time \((\tau^* + 1)\Delta\). Define \( F(\Delta) \) as the event that the scheme \([97]\) will involve some pair \((\eta_\Delta, \eta(i+1)\Delta)\) for which the segment \( \eta[i\Delta, (i+1)\Delta] \) lacks to be \((\Delta, \lambda)\)-good. For \( i \) fixed, as a consequence of \(3.1.6\) and stationarity, the probability under \( Q_\mu \) to have

\[
(\eta_\Delta, \eta(i+1)\Delta) \in ID(\Delta, \lambda), \quad \eta[i\Delta, (i+1)\Delta] \notin CI(\Delta, \lambda)
\]

is bounded by \( D\Delta \) for all \( 0 < \Delta < \Delta_0 \). Here \( D \) is some constant \( D \) which does not depend on \( \Delta \). The cube \( A \) being partitioned into \( n^d = \mathcal{O}(\Delta^{-\frac{d}{2}}) \) cells \( A_\alpha, \alpha \in J(\Delta) \), \( n = n(\Delta) \) as in \([91]\), we need \( n^d \) pairs \((\eta_\Delta, \eta(i+1)\Delta)\) to fill the regression scheme \(4.1.1\). This gives

\[
Q_\mu(F(\Delta)) \leq \mathcal{O}(n^d\Delta) = \mathcal{O}(\Delta^{\frac{d}{2}}).
\]
Thus \( F(\Delta) \) is an exceptional event in a sense of vanishing probability as \( \Delta \downarrow 0 \).

4.3 Example: Kernel estimators for the diffusion coefficient in dimension \( d = 1 \)

We restrict the setting to dimension \( d = 1 \). Based on data \( (\eta_i \Delta)_{i \in \mathbb{N}_0} \) from discrete observation at step size \( \Delta \) and on a regression scheme 4.1.1 filled from these data, we wish to construct a kernel estimator for the diffusion coefficient \( \sigma^2(\cdot) \). Calculating its pointwise risk under squared loss, we have to make sure that the influence of the exceptional sets in theorem 4.1.3 iv) does not dominate the (classical) bounds which do hold on the good sets: this will oblige us to select \( \lambda \) quite close to \( \frac{1}{2} \). Recall that \( 0 < \lambda < \frac{1}{2} \) remains fixed in 3.1.3, 3.1.5 and 4.1.1 while asymptotics is in \( \Delta \downarrow 0 \).

We make use of the following notations. For \( \tilde{\beta} \in \mathbb{N} \), a kernel of order \( \tilde{\beta} \) (pp. 5, 10) is a function \( K : \mathbb{R} \rightarrow \mathbb{R} \) supported by \( [-1,1] \) and Lipschitz continuous on \( (-1,1) \) such that

\[
\int K(v) \, dv = 1, \quad \int v^r K(v) \, dv = 0 \quad \text{for} \ r = 1, \ldots, \tilde{\beta}.
\]

For any choice \( h > 0 \) of a bandwidth we write \( K_h(v) := \frac{1}{h} K\left(\frac{v}{h}\right) \).

For \( \beta > 1 \) let \( \beta' \) denote the largest natural number which is strictly smaller than \( \beta \). The Hölder class \( \mathcal{H}(\beta, L) \) of order \( \beta \) (p. 5) is the class of all functions \( f \) in \( \mathcal{C}^{\beta'} \) with the following property: the derivative \( f^{(\beta')} \) of order \( \beta' \) is Hölder with index \( \beta - \beta' \in (0, 1] \) and with Hölder constant \( L \):

\[
\left| f^{(\beta')}(x) - f^{(\beta')}(x') \right| \leq L |x - x'|^{\beta - \beta'}, \quad x, x' \in \mathbb{R}.
\]

An example: whenever in dimension \( d = 1 \) the one-particle motion in (A1) satisfies (88) together with (95), the diffusion coefficient \( \sigma^2 : \mathbb{R} \rightarrow (0, \infty) \) belongs to the Hölder class \( \mathcal{H}(2, L) \). When we model BDI in dimension \( d = 1 \) we might have reasons to work with diffusion coefficients which have ‘more smoothness’: our statistical model below will assume that \( \sigma^2(\cdot) \) belongs to a Hölder class \( \mathcal{H}(\beta, L) \) of order \( \beta \geq 2 \), with \( \beta \) fixed and known. To every \( \beta \geq 2 \) we associate a critical value

\[
\lambda_0(\beta) := \frac{1}{2} - \frac{1}{8(2\beta + 1)} = \frac{8\beta + 3}{16\beta + 8} \in (0, \frac{1}{2}).
\]

In dimension \( d = 1 \), the cube \( A \) meeting (89) in the beginning of section 4.1 is an interval, decomposed according to (90) into

\[
n(\Delta) = \lfloor \text{length}(A) \Delta^{-\frac{1}{2}} \rfloor
\]
disjoint subintervals $A_\alpha$ of equal length $\sim \Delta^{\frac{1}{\beta}}$, $\alpha \in J(\Delta)$. For given order $\beta \geq 2$, it is well known that one needs kernels of order depending on $\beta$ to estimate $\sigma^2(\cdot)$ within class $\mathcal{H}(\beta, L)$ at the optimal nonparametric rate. In our case, exceptional sets being present in theorem 4.1.3, we need more: we also have to relate $\lambda$ in 3.1.3, 3.1.5 and 4.1.1 to the given order $\beta$, and have to work with

$$\lambda_0(\beta) \leq \lambda < \frac{1}{2}$$

with $\lambda_0(\beta)$ from 98. Then, from discrete observation of the BDI process at step size $\Delta$, we fill regression schemes 4.1.1 where $\lambda$ is fixed according to (100) and where $\Delta$ tends to 0: at stage $\Delta$ of the asymptotics, we work with

$$(X_\alpha, Z_\alpha), \alpha \in J(\Delta)$$

(and shall sometimes write $(X_\alpha, Z_\alpha)_{1 \leq \alpha \leq n}$). Next, we fix a kernel $K$ of order $\beta'$, the largest natural number strictly smaller than $\beta$, define the bandwidth $h = h(\Delta)$ by

$$h(\Delta) := (n(\Delta))^{-\frac{1}{2\beta'+1}} = O(\Delta^{\frac{1}{2\beta'+1}})$$

and introduce the estimator

$$\sigma^2(\Delta)(a) := \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \, Y_\alpha \, K_h(X_\alpha - a), \quad Y_\alpha := Z_\alpha^2, \quad a \in \text{int}(A).$$

Recall that all cells $A_\alpha$ are of equal length $\sim \Delta^{\frac{1}{\beta}}$ (which of course is $O(\frac{1}{n})$ as $\Delta \downarrow 0$, but note: length$(A_\alpha)$ asymptotically does not depend on the size of the interval $A$ selected in 89, whereas $\frac{1}{n(\Delta)} = \frac{1}{n}$ does). In this setting, we give a rate as $\Delta \downarrow 0$ for the risk of this estimator, writing $E\sigma^2(\ldots), Q\sigma^2(\ldots)$ instead of $E\mu(\ldots), Q\mu(\ldots)$ in order to stress the dependence on $\sigma^2$.

4.3.1 Theorem For $\beta \geq 2$ fixed, let the diffusion coefficient in (A1) belong to class $\mathcal{H}(\beta, L)$. Choose $0 < \lambda < \frac{1}{2}$ sufficiently close to $\frac{1}{2}$ to satisfy (100). Then asymptotically as $\Delta \downarrow 0$, with $n = n(\Delta)$ from 99, $h = h(\Delta)$ from (101), and $K$ of order $\beta'$, the pointwise risk of the kernel estimator (102) under squared loss satisfies

$$\limsup_{\Delta \downarrow 0} n^{\frac{2\lambda}{2\beta'+1}} E\sigma^2 \left( |\sigma^2(\Delta)(a) - \sigma^2(a)|^2 \right) < \infty$$

at every point $a \in \text{int}(A)$.

The theorem will be proved in subsection 4.4 below. Note that the rate which appears in (103) is the nonparametric rate which is known to be optimal (see Tsybakov [39] section 2.5) under squared loss in standard regression schemes $(X_\alpha', Y_\alpha), Y_\alpha = f(X_\alpha') + \varepsilon_\alpha, 1 \leq \alpha \leq n$ with unknown $f \in \mathcal{H}(\beta, L)$, in the classical setting of i.i.d. square-integrable errors $\varepsilon_\alpha$ and equispaced deterministic design points $X_\alpha'$.
4.4 Proving theorem 4.3.1

We prepare the proof of theorem 4.3.1 by a series of auxiliary steps. The proof will need the diffusion coefficient \( \sigma^2 \) only in restriction to compacts like \( \{ y \in \mathbb{R} : d(y, A) \leq 1 \} \), and we shall write again \( L \) – as before in assumption (99) – to denote a bound on this compact for derivatives of the function \( \sigma^2 \in \mathcal{H}(\beta, L) \) up to order \( \beta' \). Assumptions and notations are those of 4.1.1, of theorem 4.1.3 together with lemma 4.1.4, and of 4.3.1. Often we write for short \( n, h \) instead of \( n(\Delta), h(\Delta) \) etc.

4.4.1 Lemma As \( \Delta \downarrow 0 \), we have deterministic bounds

\[
\sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} n h \left| \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) K_h(X_\alpha - a) - 1 \right| \leq M < \infty
\]

at rate

\[
(104) \quad nh = n^{\frac{2\beta}{2\beta + 1}} = \mathcal{O}(\Delta^{\frac{-\beta}{2\beta + 1}}) \rightarrow \infty.
\]

Recall that all cells \( A_\alpha \) have equal length \( \sim \Delta^{\frac{1}{2}} = \mathcal{O}(\frac{1}{n}) \) for all \( \alpha \in J(\Delta) \).

Proof: Fix \( a \in \text{int}(A) \). By continuity of \( K \) on \((-1,1)\), whenever \( \Delta \) is small enough and a cell \( A_\alpha \) fully contained in \( B_h(a) \), the ball of radius \( h \) around \( a \), we can select a point \( \zeta_\alpha \in \text{cl}(A_\alpha) \) such that

\[
\int_{A_\alpha} K_h(x - a) \, dx = \text{length}(A_\alpha) K_h(X_\alpha - a) \sim \Delta^{\frac{1}{2}} K_h(\zeta_\alpha - a).
\]

Now \( K \) is Lipschitz on \((-1,1)\), thus

\[
\left| K\left( \frac{\zeta_\alpha - a}{h} \right) - K\left( \frac{X_\alpha - a}{h} \right) \right| \leq \mathcal{O} \left( \text{length}(A_\alpha) \frac{1}{h} \right) = \mathcal{O} \left( \frac{1}{nh} \right)
\]

since \( X_\alpha \in A_\alpha \) and by definition of \( n = n(\Delta) \) in (99), thus

\[
\int_{A_\alpha} K_h(x - a) \, dx = \text{length}(A_\alpha) K_h(X_\alpha - a) + \mathcal{O} \left( \frac{1}{(nh)^2} \right).
\]

For \( \mathcal{O}(nh) \) indices \( \alpha \) in \( J(\Delta) \) the cell \( A_\alpha \) will be fully contained in \( B_h(a) \); for at most two additional values of \( \alpha \), a cell \( A_\alpha \) may intersect \( B_h(a) \). Since \( a \in \text{int}(A) \), we have for \( \Delta \) small enough

\[
1 = \int_{B_h(a)} K_h(x - a) \, dx = \sum_{\alpha \in J(\Delta)} \int_{A_\alpha} K_h(x - a) \, dx = \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) K_h(X_\alpha - a) + \mathcal{O} \left( \frac{1}{nh} \right)
\]

which is the assertion. By (99) and (101), \( nh \) tends to \( \infty \) as \( \Delta \downarrow 0 \). □
4.4.2 Lemma  

i) For all powers \( m \in \mathbb{N}_0 \) we have deterministic bounds

\[
\sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \frac{1}{h} |K'|^m(\frac{X_\alpha - a}{h}) \leq M < \infty
\]

with suitable constants \( M = M(K, m) \).

ii) For continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) we have

\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) f(X_\alpha) K_h(X_\alpha - a) \rightarrow f(a) \quad , \quad a \in \text{int}(A)
\]

almost surely and in \( L^q \), \( q \geq 1 \) arbitrary.

iii) For continuous functions \( g : \mathbb{R} \rightarrow \mathbb{R} \) we have

\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) g(\frac{X_\alpha - a}{h}) K_h(X_\alpha - a) \rightarrow \int_{-1}^{1} g(v) K(v) \, dv \quad , \quad a \in \text{int}(A)
\]

almost surely and in \( L^q \), \( q \geq 1 \) arbitrary.

Proof: For (105) it is sufficient to note that \( K \) is bounded, that \( X_\alpha \) belongs to cell \( A_\alpha \) by construction of the regression scheme in 4.1.1 and that the number of cells \( A_\alpha \) which intersect the support of \( K(\frac{X - a}{h}) \) is \( \mathcal{O}(nh) \) which tends to \( \infty \): so the left hand side of (105) is of type

\[
\frac{1}{n} \sum_{a=1}^{n} \frac{1}{h} |K'|^m(\frac{X - a}{h})
\]

and assertion i) is proved. To prove ii) and iii), the same argument shows that for \( \Delta \) small enough, at points \( a \in \text{int}(A) \), the random variables

\[
\frac{X_\alpha - a}{h} \quad \text{where} \quad \alpha \text{ is such that } \frac{X_\alpha - a}{h} \in [-1, 1]
\]

are approximately equispaced over \([-1, 1]\), the spacing of the design variables \( X_\alpha \) being of order \( \text{length}(A_\alpha) \sim \Delta^{\frac{1}{2}} = \mathcal{O}(\frac{1}{\sqrt{n}}) \). As a consequence, left hand sides in ii) and iii) are Riemann sums and converge almost surely as \( \Delta \downarrow 0 \). Since (105) provides constants \( M = M(K, 1) \) such that the convergence is dominated, we have also convergence in \( L^q \) for \( q \geq 1 \) arbitrary. \( \square \)

4.4.3 Lemma  For the kernel \( K \) of order \( \beta' \), we have deterministic bounds

\[
\sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} nh \left| \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) (X_\alpha - a)^j K_h(X_\alpha - a) \right| \leq M < \infty
\]

for every \( 1 \leq j \leq \beta' \). Recall that \( \beta' \) is the greatest integer strictly smaller than \( \beta \).
Proof: Since \( K \) is a kernel of order \( \beta' \), we know from lemma 4.4.2 iii) that for every \( 1 \leq j \leq \beta' \)
\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \left( \frac{X_\alpha - a}{h} \right)^j K_h(X_\alpha - a) \longrightarrow \int_{-1}^1 v^j K(v) dv = 0 \ , \ 1 \leq j \leq \beta'
\]
holds almost surely as \( \Delta \downarrow 0 \), and in \( L^q \) for arbitrary \( q \geq 1 \). We shall combine this with the steps of the proof of lemma 4.4.1 Fix \( 1 \leq j \leq \beta' \) and select \( \zeta_\alpha = \zeta_\alpha(j) \in \text{cl}(A_\alpha) \) such that
\[
\int_{A_\alpha} (x-a)^j K_h(x-a) dx = \text{length}(A_\alpha) (\zeta_\alpha - a)^j K_h(\zeta_\alpha - a) .
\]
Since \( x \to (x-a)^j K_h(x-a) \) is Lipschitz on \( B_h(a) \) and \( X_\alpha \in A_\alpha \), we have
\[
\left| (\zeta_\alpha - a)^j K - (X_\alpha - a)^j K \right| \leq O \left( \text{length}(A_\alpha) \frac{1}{h} \right) = O \left( \frac{1}{nh} \right)
\]
and thus
\[
\int_{A_\alpha} (x-a)^j K_h(x-a) dx = \text{length}(A_\alpha) (X_\alpha - a)^j K_h(X_\alpha - a) + O \left( \frac{1}{(nh)^2} \right) .
\]
For \( O(nh) \) indices \( \alpha \) in \( J(\Delta) \) the cell \( A_\alpha \) will be fully contained in \( B_h(a) \); for at most two additional values of \( \alpha \), a cell \( A_\alpha \) may intersect \( B_h(a) \). Since \( a \in \text{int}(A) \), we have for \( \Delta \) small enough
\[
0 = \int_{B_h(a)} (x-a)^j K_h(x-a) dx = \sum_{\alpha \in J(\Delta)} \int_{A_\alpha} (x-a)^j K_h(x-a) dx = \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) (X_\alpha - a)^j K_h(X_\alpha - a) + O \left( \frac{1}{nh} \right)
\]
for every \( 1 \leq j \leq \beta' \) since \( K \) is a kernel of order \( \beta' \). This is the assertion. \( \square \)

4.4.4 Lemma Our assumptions on the kernel \( K \) combined with the Hölder property for \( \sigma^2 \) grant
\[
\sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} E_{\sigma^2} \left( nh \left[ \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) (\sigma^2(X_\alpha) - \sigma^2(a)) K_h(X_\alpha - a) \right]^2 \right) < \infty .
\]

Proof: Fix \( a \in \text{int}(A) \). We start from a Taylor expansion of \( f := \sigma^2 \in \mathcal{H}(\beta, L) \) at \( a \) as in [39] p. 14
\[
f(a+h) - f(a) = \sum_{j=1}^{\beta'} \frac{f^{(j)}(a)}{j!} h^j + \frac{h^{\beta'}}{\beta'!} \int_0^1 (1 - \tau)^{\beta' - 1} \left[ f^{(\beta')}(a + \tau h) - f^{(\beta')}(a) \right] d\tau
\]
and apply the preceding lemmata. First, derivatives of order \( 1 \leq j \leq \beta' \) in the above square brackets produce terms
\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) (X_\alpha - a)^j K_h(X_\alpha - a)
\]
53
4.4.5 Lemma In restriction to the good events \( G(\Delta) \) of theorem 4.1.3 we have for \( \mathcal{Y}_\alpha = \mathbb{R}_\alpha^2 \)

\[
\sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} E_{\sigma^2} \left( \left( \sum_{a \in \mathcal{J}(\Delta)} \text{length}(A_a) \ 1_{\text{g}(\Delta)}(\mathcal{Y}_\alpha - \sigma^2(\mathcal{X}_\alpha)) \ K_h(\mathcal{X}_\alpha - a) \right)^2 \right) < \infty .
\]

Proof: We use the notations of 4.1.3, 4.1.4 and in particular we have in dimension \( d = 1 \) on the good event \( G(\Delta) \)

\[
\mathcal{Y}_\alpha = \mathbb{R}_\alpha^2 \cong [\sigma(\mathcal{X}_\alpha) \mathcal{U}_a(1)]^2 = \sigma^2(\mathcal{X}_\alpha) \left\{ 1 + 2 \int_0^1 \mathcal{U}_a(s) d\mathcal{U}_a(s) \right\} , \quad \alpha \in \mathcal{J}(\Delta)
\]

with one-dimensional standard Brownian motions \( \mathcal{U}_a, \alpha \in \mathcal{J}(\Delta) \), which by construction in theorem 4.1.3 are independent of each other and independent of the \( \mathcal{X}_\alpha, \alpha \in \mathcal{J}(\Delta) \). Thus as \( \Delta \downarrow 0 \), the expression in square brackets in the assertion is the sum in \( L^2(Q_{\sigma^2}) \) of two terms: first,

\[
S_1(\Delta) := \sum_{a \in \mathcal{J}(\Delta)} \text{length}(A_a) \ 1_{\text{g}(\Delta)}[\mathcal{Y}_\alpha - [\sigma(\mathcal{X}_\alpha) \mathcal{U}_a(1)]^2] \ K_h(\mathcal{X}_\alpha - a)
\]

second, since \( \mathcal{U}_a^2(1) = 1 + 2 \int_0^1 \mathcal{U}_a(s) d\mathcal{U}_a(s) \) on \( G(\Delta) \),

\[
S_2(\Delta) := 2 \sum_{a \in \mathcal{J}(\Delta)} \text{length}(A_a) \ \sigma^2(\mathcal{X}_\alpha) \ 1_{\text{g}(\Delta)} \int_0^1 \mathcal{U}_a(s) d\mathcal{U}_a(s) \ K_h(\mathcal{X}_\alpha - a) .
\]

1) Consider (107) first: the \( \mathcal{U}_a \) being independent of each other and independent of the \( \mathcal{X}_\alpha, \alpha \in \mathcal{J}(\Delta) \),

\[
E_{\sigma^2} \left( [S_2(\Delta)]^2 \right) = 4 E_{\sigma^2} \left( \left( \sum_{a \in \mathcal{J}(\Delta)} \text{length}(A_a) \ \sigma^2(\mathcal{X}_\alpha) \ 1_{\text{g}(\Delta)} \int_0^1 \mathcal{U}_a(s) d\mathcal{U}_a(s) \ K_h(\mathcal{X}_\alpha - a) \right)^2 \right)
\]
reduces to the expectation of the sum of squares of diagonal terms which admits bounds
\[
\leq 4 \frac{L^4}{2} \frac{1}{nh} E_{\sigma^2} \left( \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \frac{1}{h} |K|^2 \left( \frac{X_\alpha - a}{h} \right) \right) \leq 2 L^4 M \frac{1}{nh}
\]
where \( \frac{1}{2} \) is the expectation \( E \left( \left( \int_0^1 U_t dA_t \right)^2 \right) = E \left( \int_0^1 U_t^2 (s) ds \right) \), \( L \) the bound for \( |\sigma| \) from (95), and 
\( M = M(K, 2) \) the deterministic bound from (105). So the sum (107) satisfies
\[
E_{\sigma^2} \left( [S_2(\Delta)]^2 \right) = \mathcal{O} \left( \frac{1}{nh} \right) \quad \text{as } \Delta \downarrow 0.
\]

2) To deal with \( [S_1(\Delta)]^2 \) from (106), we start with the sum of squared diagonal terms
\[
S_3(\Delta) := \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha)^2 1_{g(\Delta)} \left[ Y_\alpha - [\sigma(X_\alpha) U_\alpha(1)]^2 \right] K_h^2(X_\alpha - a)
\]
to which lemma 4.1.4 applies (with \( g(z) = z^2 \) since \( Y_\alpha = Z_\alpha^2 \]): placing conditional expectations (96) inside \( E_{\sigma^2} (\ldots) \) and using bounds \( M = M(K, 2) \) from (105) together with (99) and (101), we obtain
\[
E_{\sigma^2} (S_3(\Delta)) \leq C \Delta \frac{1}{nh} = \mathcal{O} \left( \frac{1}{nh} \right) = \mathcal{O} \left( \frac{1}{n^3 h} \right)
\]
which is negligible as \( \Delta \downarrow 0 \) in comparison to the bound \( \mathcal{O} \left( \frac{1}{nh} \right) \) in step 1).

3) To deal with the sum of non-diagonal contributions
\[
S_4(\Delta) := [S_1(\Delta)]^2 - S_3(\Delta)
\]
to \( [S_1(\Delta)]^2 \) we introduce short notations
\[
V_{\alpha'} := 1_{g(\Delta)} \left[ Y_{\alpha'} - [\sigma(X_{\alpha'}) U_{\alpha'}(1)]^2 \right] , \quad \alpha' \in J(\Delta) , \quad G := \sigma(X_\alpha : \alpha \in J(\Delta))
\]
and write
\[
S_4(\Delta) = \sum_{\alpha' \neq \alpha''} \text{length}(A_{\alpha'}) \text{length}(A_{\alpha''}) V_{\alpha'} V_{\alpha''} K_h(X_{\alpha'} - a) K_h(X_{\alpha''} - a).
\]
Using a regular version \( K(\omega, \cdot) \) of the conditional law
\[
\mathcal{L} \left( \left( 1_{g(\Delta)} G \right)_{\alpha \in J(\Delta)} | G \right)
\]
and Cauchy-Schwarz with respect to \( K(\omega, \cdot) \), lemma 4.1.4 again applies (with \( g(z) = z^2 \)) and yields deterministic bounds
\[
E \left( |V_{\alpha'} V_{\alpha''}| | G \right) \leq \sqrt{E \left( V_{\alpha'}^2 | G \right) E \left( V_{\alpha''}^2 | G \right)} \leq C \Delta
\]
almost surely, whence
\[ E_{\sigma^2}( |S_4(\Delta)| ) \leq C \Delta \ E_{\sigma^2} \left[ \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \frac{1}{h} |K'| \left( \frac{h}{\sigma^2} \right)^2 \right] \leq CM^2 \Delta = O \left( \frac{1}{n^2} \right) \]
using again the bounds \( M = M(K, 1) \) from (105). Thus also \( E_{\sigma^2}( |S_4(\Delta)| ) \) is negligible in comparison to the bound \( O \left( \frac{1}{nh} \right) \) obtained in step 1) as \( \Delta \downarrow 0 \).

4) As a consequence of steps 1) to 3), we have
\[ E_{\sigma^2}( |S_1(\Delta)|^2 + |S_2(\Delta)|^2 ) = O \left( \frac{1}{nh} \right) \]
which finishes the proof of lemma 4.4.5. □

4.4.6 Lemma In general regression schemes \( 4.1.1 \) where \( 0 < \lambda < \frac{1}{2} \), exceptional events \( F(\Delta) \) in theorem 4.1.3 are such that
\[ \sup_{a \in \text{int}(A)} \limsup_{\Delta \downarrow 0} E_{\sigma^2} \left( \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \left[ 1_{F(\Delta)} \left( Y_\alpha - \sigma^2(X_\alpha) \right) K_h(X_\alpha - a) \right] \right)^2 < \infty. \]

Using \( \lambda = \lambda_0(\beta) \), the critical value (98) associated to class \( H(\beta, L) \), we have
\[ E_{\sigma^2} \left( \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \left[ 1_{F(\Delta)} \left( Y_\alpha - \sigma^2(X_\alpha) \right) K_h(X_\alpha - a) \right] \right)^2 = O \left( \frac{1}{nh} \right) \]
as \( \Delta \downarrow 0 \). The left hand side is negligible in comparison to \( O \left( \frac{1}{nh} \right) \) when \( \lambda_0(\beta) < \lambda < \frac{1}{2} \).

Proof: Write \( S_5(\Delta) \) for the sum in square brackets in the assertion. Then \( Y_\alpha = Z_\alpha^2 \) combined with the deterministic bounds on \( Z_\alpha \) from part iv) of theorem 4.1.3 give
\[ |Y_{\alpha'} - \sigma^2(X_{\alpha'})||Y_{\alpha''} - \sigma^2(X_{\alpha''})| \leq O(\Delta^{4(\lambda - \frac{1}{2})}) \quad , \quad \alpha', \alpha'' \in J(\Delta) \]
when \( \Delta \) is small enough (recall that \( \sigma^2 \) is bounded on the interval \( A \)). Using \( \Delta = O(n^{-2}) \) from (99), this bound is \( O(n^{4-8\lambda}) \) as \( \Delta \downarrow 0 \). From part iii) of theorem 4.1.3 and (99) we have
\[ Q_{\sigma^2}(F(\Delta)) \leq O \left( \frac{1}{n} \right) \]
as \( \Delta \downarrow 0 \). Proceeding as in the proof of lemma 4.4.5, using the constants \( M = M(K, 1) \) of (105), we end up with the bound
\[ E_{\sigma^2}( |S_5(\Delta)|^2 ) \leq O \left( n^{3-8\lambda} \right) \quad \text{as} \quad \Delta \downarrow 0 \]
56
which proves the first assertion. The second assertion follows since \( \lambda = \lambda_0(\beta) \) in \([98]\) is such that
\[
\mathcal{O}\left(n^{3-8\lambda_0(\beta)}\right) = \mathcal{O}\left(\frac{1}{n} n^{\frac{4}{n+1}}\right) = \mathcal{O}\left(\frac{1}{nh}\right)
\]
by definition of the bandwidth in \([101]\).

**Proof of theorem 4.3.1** Note first that \( n^{\frac{2\beta}{n+1}} = nh \) by \([99]\) and \([101]\). Then the estimation error
\[
\hat{\sigma}^2(a) - \sigma^2(a)
\]
is decomposed into several terms. First, the difference
\[
\sigma^2(a) - \sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) \sigma^2(a) K_h(X_\alpha - a)
\]
is \(\mathcal{O}(\frac{1}{nh})\) by lemma 4.4.1. Second, by lemma 4.4.4,
\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) (\sigma^2(X_\alpha) - \sigma^2(a)) K_h(X_\alpha - a)
\]
has squared \(L^2(Q_{\sigma^2})\)-norm of order \(\mathcal{O}(\frac{1}{nh})\). Third, on the good events \(G(\Delta)\) of theorem 4.1.3
\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) 1_{G(\Delta)}(Y_\alpha - \sigma^2(X_\alpha)) K_h(X_\alpha - a)
\]
has squared \(L^2(Q_{\sigma^2})\)-norm of order \(\mathcal{O}(\frac{1}{nh})\) by lemma 4.4.5. So far, we could work with arbitrary
\[0 < \lambda < \frac{1}{2}\] fixed. This situation changes drastically with the final contribution
\[
\sum_{\alpha \in J(\Delta)} \text{length}(A_\alpha) 1_{F(\Delta)}(Y_\alpha - \sigma^2(X_\alpha)) K_h(X_\alpha - a)
\]
of the exceptional events \(F(\Delta)\): here we have not more than the trivial bounds from theorem 4.1.3 iv). By lemma 4.4.6 squared \(L^2(Q_{\sigma^2})\)-norms are of order \(\mathcal{O}(n^{3-8\lambda})\) which obliges us to work with
\[
\lambda \geq \lambda_0(\beta) = \frac{1}{2} - \frac{1}{8(2\beta + 1)}
\]
the condition introduced in \([98]\) and \([100]\), to get the contribution from exceptional events balanced under a common \(\leq \mathcal{O}(\frac{1}{nh})\) for all contributions. The proof of theorem 4.3.1 is finished. \(\square\)

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