Spreading of relativistic probability densities and Lorentz contraction

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We find the laws for the spreading of the spatial widths (parallel and transverse to the direction of average motion) of the relativistic position probability density for a massive, spinless particle. We find that when the momentum width of the wavepacket is small compared to the average momentum, there is a long time over which spreading is minimal. This result may be useful in particle accelerator design. We also demonstrate the Lorentz contraction of a wavepacket using relativistic probability amplitudes.

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I. INTRODUCTION

In a recent paper [1] this author presented the relativistic transformation properties of momentum/spin-component and position/spin-component probability amplitudes for a massive particle of general spin. The boost transformation properties of the position operator were also derived. In this article, we find the rules for the spreading with time of the spatial widths of the probability distribution (parallel and transverse to the direction of average motion) of a wavepacket Gaussian in momentum space. It will suffice to consider the spinless case.

We also demonstrate the Lorentz contraction of the spatial width of a spinless, massive, wavepacket (in the boost direction) under a boost transformation.

II. DERIVATION OF THE WAVEPACKET SPREADING LAWS

The normalized state vector we consider is

$$|\psi\rangle = \int \frac{d^3k}{\sqrt{\omega}} |k\rangle \Psi(k) = \int \frac{d^3k}{\sqrt{\omega}} |k\rangle e^{-|k-p|^2/4\sigma_p^2} \frac{1}{(2\pi\sigma_p^2)^{3/2}}, \quad (1)$$

where the basis vectors have the covariant normalization

$$\langle k_1 | k_2 \rangle = \omega_1 \delta^3(k_1 - k_2) \quad (2)$$

and each four-momentum is of the form $k^\mu = (\omega, k)^\mu$ with $k^2 = m^2$ and $\omega = \sqrt{k^2 + m^2}$. The factor of $1/\sqrt{\omega}$ in the superposition compensates for the covariant normalization, as discussed in [1].

For calculation convenience, we will only consider the case where the linear momentum distribution is strongly concentrated about the average $p$: $|\sigma_p/p| \ll 1$.

The time-dependent position wavefunction (in the Schrödinger picture) is

$$\psi(t, x) = \int \frac{d^3k}{(2\pi)^3} \Psi(k) e^{i(k \cdot x - \omega(k)t)} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-|k-p|^2/4\sigma_p^2}}{(2\pi\sigma_p^2)^{3/2}} e^{i(k \cdot x - \omega(k)t)}. \quad (3)$$

We expand $\omega(k)$ around $k = p$

$$\omega(k) = \omega + \beta \cdot (k - p) + \frac{|k - p|^2}{2\omega} - \frac{1}{2\omega} (\beta \cdot (k - p))^2 + \ldots \quad (4)$$

where $\omega = \omega(p)$ and $\beta = p/\omega$.

A phase that varies by much less than unity across the wavefunction peak region, $|k - p| \leq \sigma_p$, will contribute negligibly to the integral. We choose to consider only times $t$ with $|t| \ll T$, with

$$\frac{\sigma_p^2 T}{\omega} \sim 1, \quad \text{or} \quad \beta T \sim \frac{p}{\sigma_p} \sigma_x, \quad (5)$$

where $\sigma_x \sigma_p = 1/2$ and $\sigma_x$ is the spatial width at $t = 0$ for this minimal wavepacket. With this choice, the second order terms in Eq. (4) will be relevant, but higher order terms can be ignored.

In the integral, Eq. (3), we change variables to $\rho = k - p$ and take components $\rho_\parallel = \rho \cdot \hat{\beta}$ and $\rho_\perp = \rho - \rho_\parallel \hat{\beta}$. The measure becomes

$$\int d^3k = \int d^3\rho = \int d^2\rho_\perp \int_{-\infty}^{\infty} d\rho_\parallel. \quad (6)$$

We also define components of $x$: $x_\parallel = x \cdot \hat{\beta}$ and $x_\perp = x - x_\parallel \hat{\beta}$. Then the exponent becomes

$$-\frac{|k - p|^2}{4\sigma_p^2} + i(k \cdot x - \omega(k)t) \equiv -\frac{|\rho_\parallel|^2}{4\sigma_p^2} - \frac{\rho_\parallel^2}{4\sigma_p^2} + i\rho_\parallel(x_\parallel - \beta t) - i\rho_\parallel^2 t \gamma^2 2\omega + i\rho_\perp \cdot x_\perp - i\rho_\perp^2 t/2\omega, \quad (7)$$

where $\gamma = 1/\sqrt{1 - \beta^2}$ and we have ignored terms that contribute only to a global phase factor.
We evaluate the integrals using (3.323.2) in the form
\[
\int_{-\infty}^{\infty} dz e^{-z^2/\sigma^2} e^{i\xi z/\sigma} e^{-i\tau z^2/\sigma^2} = \left( \frac{\pi\sigma^2}{1 + i\tau} \right)^{1/2} e^{-\xi^2/4(1+i\tau)}. \tag{8}
\]

The factors we need are
\[
e^{-\xi^2/4(1+i\tau)} = e^{-\xi^2/4(1+\tau^2)} e^{i\xi^2\tau/4(1+\tau^2)}, \tag{9}
\]
with the phase factors not of interest. So we find a factor of the form \(\exp(f(x))\) with
\[
f(x) = \frac{(x|| - \beta t)^2 4\sigma_p^2}{4\{1 + (4\sigma_p^2)^2\gamma^2/4\omega^2\}} + \frac{|x_\perp|^2 4\sigma_p^2}{4\{1 + (4\sigma_p^2)^2\gamma^2/4\omega^2\}}
\]
\[
= \frac{(x|| - \beta t)^2}{4\{\sigma_x^2 + (\frac{\beta}{|p|}\beta t)^2\}} + \frac{|x_\perp|^2}{4\{\sigma_p^2 + (\frac{\beta}{|p|}\beta t)^2\}}. \tag{10}
\]

Completing the remainder of the calculation gives the correct normalization factors for these distributions, and will not be shown here. When we find the position probability density, \(f(x)\) is just multiplied by 2.

So the spreading laws are
\[
\sigma_x(t) = \sqrt{\sigma_x^2 + \left(\frac{1}{\gamma^2} |p| \beta t\right)^2},
\]
\[
\sigma_\perp(t) = \sqrt{\sigma_\perp^2 + \left(\frac{\sigma_p}{|p|} \beta t\right)^2} \quad \text{for} \ \beta t \ll \frac{|p|}{\sigma_p} \sigma_x, \tag{11}
\]
where the variances are defined by
\[
\sigma_x^2 = \langle x_x^2 \rangle - \langle x_x \rangle^2,
\]
\[
\sigma_\perp^2 = \frac{1}{2} \{\langle x_\perp^2 \rangle - \langle x_\perp \rangle^2\}. \tag{12}
\]

We see that the spreading is not, in general, spherically symmetric and that the long time spreading rate in the direction of average motion is suppressed by the \(1/\gamma^2\) factor.

The form of these spreading laws is shown in Figure 1 for \(\sigma_p/|p| = 0.01\) and \(\gamma = 2\). We see that if it was required to have negligible spreading over the course of a scattering experiment, it would suffice to choose the initial and final average positions of the wavepackets so that the propagation time, \(T\), satisfied \(\beta T = \sigma_x/\sqrt{\epsilon}\), with \(\epsilon = \sigma_p/|p|\). This conclusion was reached in [3].

### III. Lorentz Contraction

We consider the state vector, with \(\omega = \sqrt{p^2 + m^2}\),
\[
|\psi\rangle = \int \frac{d^3p}{\sqrt{\omega}} |p\rangle \Psi(p) = \int \frac{d^3p}{\sqrt{\omega}} |p\rangle \frac{e^{-|p|^2/4\sigma_p^2}}{(2\pi\sigma_p^2)^{3/2}}, \tag{13}
\]
representing a spinless particle with vanishing average momentum. The position probability amplitude at \(t = 0\) is (up to an irrelevant phase factor)
\[
\psi(0, x) = \frac{e^{-|x|^2/4\sigma_x^2}}{(2\pi\sigma_x^2)^{1/4}}, \tag{14}
\]
with \(\sigma_x \sigma_p = 1/2\). So the wavepacket is minimal and localized around the origin with a spatial width \(\sigma_x\) at this time.

Using the transformation results from [1], a boost by velocity \(\beta_0\) produces the momentum wavefunction
\[
\Psi'(p) = \sqrt{\gamma_0(1 - \beta_0 \cdot \beta)} \Psi(\Lambda^{-1} p), \tag{15}
\]
with $\gamma_0 = 1/\sqrt{1 - \beta_0^2}$ and $\beta = p/\omega$. With $p' = \Lambda^{-1} p$, we have
\[ p' = p_\perp + \gamma_0 (p_\parallel - \beta_0 \omega), \] (16)
where we have separated the momentum into parts parallel ($p_\parallel$) and perpendicular ($p_\perp$) to the boost velocity. So the exponent contains a factor
\[ |p'|^2 = |p_\perp|^2 + \gamma_0^2 |p_\parallel - \beta_0 \omega|^2. \] (17)
We find that this vanishes where
\[ p_\perp = 0 \quad \text{and} \quad p_\parallel = m\gamma_0 \beta_0, \] (18)
so the modulus-squared of the wavefunction will have its peak there.

We expand Eq. (17) in powers of $p_\parallel$ and $p_\perp = p - m\gamma_0 \beta_0$, to find
\[ |p'|^2 \approx |p_\parallel|^2 + \frac{1}{\gamma_0^2} |p_\parallel - m\gamma_0 \beta_0|^2. \] (19)
The correction terms are of third order in the expansion quantities. Note that the width in momentum in the boost direction will be enlarged by the gamma factor. We choose the momentum width, $\sigma_p$, so that the boosted wavefunction will be narrow in momentum:
\[ \frac{\gamma_0 \sigma_p}{m\gamma_0 \beta_0} = \frac{\sigma_p}{m\beta_0} \ll 1. \] (20)
Then the third order terms we neglected in Eq. (19) will produce higher powers of this small ratio, so are justifiably negligible.

Also we take $\beta \to \beta_0$, its peak value, in the slowly varying factor in Eq. (15), giving
\[ \sqrt{\gamma_0 (1 - \beta_0 \cdot \beta)} \to \frac{1}{\sqrt{\gamma_0}}. \] (21)
The position wavefunction at $t = 0$ is then, defining $x_\parallel$ and $x_\perp$ as components of the position parallel and perpendicular to $\beta_0$, respectively,
\[ \psi'(0, x) = \int \frac{d^3 p}{(2\pi)^{3/2}} e^{i(p_\perp \cdot x_\perp + p_\parallel \cdot x_\parallel)} \frac{1}{\gamma_0} \frac{e^{-|p_\parallel|^2/4\sigma_p^2} e^{-|p_\parallel - m\gamma_0 \beta_0|^2/4\gamma_0^2 \sigma_p^2}}{(2\pi \sigma_p^2)^{3/2}} . \] (22)
Evaluating the integrals using Eq. (8), as we did in the previous section, gives

\[ \psi'(0, x) \propto \exp(-|x_\perp|^2/4\sigma^2_\perp) \exp(-\gamma_0^2|x_\parallel|^2/4\sigma^2_\parallel). \]  \hspace{1cm} (23)

The normalization factors are not shown, but are found to take their correct values. Thus we see the Lorentz contraction of the spatial width in the boost direction,

\[ \sigma_\parallel \rightarrow \frac{\sigma_x}{\gamma_0}, \]  \hspace{1cm} (24)

with \( \sigma_\parallel \) defined as in Eq. (12).

It has been argued \([4]\) that to localize an electron in an arbitrarily small volume would require a large amount of energy, and that pair creation would be the inevitable result. This would be the case if high energy photons were used to perform the localization. We see here that the simplest way to localize a particle in an arbitrarily small volume is to observe it from a boosted frame. The observer in the rest frame could confirm that no pair creation was taking place.

### IV. CONCLUSIONS

Wavepackets that are narrow in momentum have a region large compared to the minimum width, \( \sigma_x \), over which wavepacket spreading is negligible. This is important in modelling scattering experiments, where we want wavepacket spreading to be minimal over the course of the experiment. This issue was discussed by this author, in the nonrelativistic case, for Coulomb scattering \([3]\).

It is anticipated that the results obtained here would be useful in accelerator design.

As a test of the physical relevance of the transformation formulae for probability amplitudes derived in \([1]\), we demonstrated the Lorentz contraction, in the boost direction, of an example wavepacket.

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[1] S. E. Hoffmann, arXiv:1804.00548 (2018).
[2] I. S. Gradsteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, corrected and enlarged ed. (Academic Press, Inc., San Diego, CA, 1980).
[3] S. E. Hoffmann, J. Phys. B: At. Mol. Opt. Phys. 50, 215302 (2017).
[4] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, 1st ed. (McGraw-Hill Inc., 1980).