We investigate a decomposition of a dissipative dynamical map, which is used in quantum dynamics of a finite open quantum system, into two distinct types of mapping on a Hilbert-Schmidt space of quantum states. One type of the maps corresponds to reversible behaviours, while the other to irreversible characteristics. For a finite dimensional system, in which one can equip the state-observable system by a complex matrix space, we employ real vectors or Bloch representations and express a dynamical map on the state space as a real matrix acting on the representation. It is found that rotation and scaling transformations on the real vector space, obtained from the orthogonal-symmetric or the real-polar decomposition, behave as building blocks for a dynamical map. Together with the time parameter, we introduce an additional parameter, which is related to a scaling parameter in the scaling part of the dynamical matrix. Specifically for the Lindblad-type dynamical maps, which form a one-parameter semigroup, we interpret the conditions on the Lindblad map in our framework, where the scaling parameter can be expressed as a function of time, inducing a one-parameter map. As results, we find that the change of the linear entropy or purity, which indicates dissipative behaviours, increases in time and possesses an asymptote expected in thermodynamics. The rate of change of the linear entropy depends on the structure of the scaling part of the dynamical matrix. In addition, the initial state plays an important role in this rate of change. The dissipative behaviours and the partition of eigensubspaces for bit-flipping, phase-flipping, and depolarisation matrices are discussed and illustrated in qubit systems.

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I. INTRODUCTION

A dynamical map on a state space or on an operator space to describe the evolution of a system between two given time-epochs is one of the most versatile mathematical objects used in physics, especially in quantum physics [1, 2]. Since the change in time of many physical quantities in reality are usually explained in the form of differential equations whose solutions will describe physical situations, the dynamical maps governing the equations become the central object in analysis of the problem. Historically, in a typical close quantum system, such a dynamical map is described by a one-parameter strongly continuous unitary group on the operator space or on its dual space, called the state space, inspired by the celebrated work of von Neumann [1]. Despite that this idea sets the standard formulation for dynamical analysis in quantum physics, it has been shown that the formulation is limited by many conditions, which may not be met in practice. In particular, when the closeness property of the system, the time-homogeneity of the dynamics, or reaching equilibrium state within finite time of the system cannot be assumed, the formulation is not well established in physical systems. As consequences, many extensions of the formulation have been proposed in order to explain the behaviours of the quantum systems in unprecedented regimes such as open system, non-equilibrium, or non-stationary regimes. Indeed, this research field is presently very active; see Refs. [2–8] for recent development.

For open quantum systems, any dynamical map can be characterised into three categories: (i) the Liouville-von Neumann type, where the dynamics is described by a unitary group as in the standard quantum formulation [1]; (ii) the Lindblad type, where the complete positivity of the map and the Markovianity are assumed, while decay, decoherence or dephasing are allowed [9, 10], and the dynamical maps can be treated as a one-parameter ultra-weakly continuous semigroup on the operator space; and (iii) the beyond-Lindblad type, where the assumption on complete positivity or Markovianity is dropped [2, 11]. Among these types of the dynamical maps, we observe that they are different in dynamical characteristics, and the most important one is the change in entropy defined by some appropriate functional, for instance, the von Neumann entropy function, along the dynamics [12, 13]. In the Liouville-von Neumann type, the change of the entropy function is zero by the unitary invariance. For the Lindblad-type dynamics, the entropy change behaves mostly monotonic and possesses an asymptote, which is related to a steady state in thermalization and relaxation of the system [5, 10, 14]. Interestingly, the entropy change beyond the Lindblad dynamics remains open. This leads us to investigate using the entropy change as the characterisation parameter of the quantum dynamics.

Assigning another dynamical parameter is not a new idea, but rather it has been developed in many special aspects [15–19]. An obvious example is a path integral formalism or stochastic path integral formulation, in which a random configuration is an additional parameter [15, 16, 19]. Likewise, an assertion of the imaginary time accomplishes the same purpose in non-equilibrium quantum statistics (aka Keldysh or contour integral formulation) [17, 18], where the inverse temperature is identified as the imaginary part of the complex time. We should note important differences between the two aforementioned examples. On one hand, in the path integral formulation, an integration over all possible paths can eliminate additional parameter, thus makes the dynamics of evolution depend on one parameter. On the other hand, in the imaginary time formulation, an additional parameter is not related to the time parameter, thus the dynamics have two separate and independent parameters. In this work, in addition to the usual time parameter to indicate the reversible behaviours, we treat the entropy change as another dynamical parameter to indicate the dissipative or irreversible behaviours.

The key idea of this work lies in introducing a concept of a particular decomposition, which we call the unitary-scaling decomposition, of the Lindblad dynamics in a finite-dimensional open quantum
system. We characterise our interested dynamical matrices into two types: the rotation matrix associated with the unitary evolution, and the scaling matrix describing the dissipative behaviour of the dynamics. In Sections II and III, we give necessary details of the formulation and content concerning the dynamical maps, where the ideas of two-parameter dynamical maps and semigroup properties are presented and discussed in the latter section. In Section IV the consequent results of the formulation are provided. We present the forms of the dynamical maps under a few physical assumptions such as that the rotation part of dynamical map has a time-independent generator, or that the dynamical map is independent of the initial state of the dynamics. As mentioned, we illustrate the dissipative behaviour of the dynamics by using the linear entropy, which retrieves the characteristics of the Lindblad dynamics as expected. We demonstrate our formulation for specific examples of bit-flipping, phase-flipping, and depolarisation matrices in qubit systems in Section V. Conclusion and remarks follow in Section VI.

II. REAL VECTOR REPRESENTATION OF QUANTUM SYSTEM

In our formulation, it is convenient to determine the finite-dimension system or the complex matrix space. We focus the formulation mostly on the real vector representation (i.e. Bloch vector representation, coherent representation) as presented in Refs.[20, 21] with some modification. The framework is provided in the first subsection and the construction of the map is defined in the subsequent subsection. As building blocks of the unitary-scaling decomposition, the unitary and the scaling maps and the idea of real-matrix representations of the map are presented.

A. Real Vector Representation of Self-Adjoint Trace-class Operator

In a finite dimensional quantum system, a Banach space $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on a $d$-dimensional Hilbert space $\mathcal{H}$ is isomorphic to a linear space $M_d(\mathbb{C})$ of $d \times d$-matrices on complex field $\mathbb{C}$. A trace-class $\mathcal{C}_1(\mathcal{H}) := \{ a \in \mathfrak{B}(\mathcal{H}) : \text{Tr}(a) < \infty \}$ of the algebra $\mathfrak{B}(\mathcal{H})$ is identical to the algebra itself and isomorphic to the matrix algebra. Thus, the state-observable description of quantum mechanics can be given by a pair $(\rho, a)$ of $d$-dimensional matrices $\rho \in S_d := \{ \rho \in \mathcal{M}_d(\mathbb{C}) : \rho \geq 0, \text{Tr}(\rho) = 1 \}$ representing a density operator and $a \in \mathbb{H}_d := \{ a \in \mathcal{M}_d(\mathbb{C}) : a = a^* \} \subset \mathcal{M}_d(\mathbb{C})$ representing an observable [22].

First, consider a subspace of Hermitian matrices $\mathbb{H}_d$ which itself forms a Hilbert space with the Hilbert-Schmidt inner product $(a, b)_{HS} := \text{Tr}(a^*b) = \text{Tr}(ab)$ and with norm $||a||^2_{HS} := (a, a)_{HS}$. An element of $\mathbb{H}_d$ is a matrix representation of a self-adjoint operator which also corresponds to an observable in quantum system. An element in $S_d$ or $\mathbb{H}_d$ can be characterised as a real-valued vector, aka Bloch vector, in a compact subspace of a Euclidean real vector space $\mathbb{R}^{d^2-1}$. By using an orthogonal basis set $F = \{ f_\alpha \}, \alpha = 0, 1, \ldots, d^2 - 1$ of $\mathbb{H}_d$ with $f_0 := \frac{1}{\sqrt{d}} \mathbb{1}_d$, such vector is a $(d^2 - 1)$-tuplet $\vec{x}(a) := (x_1(a), x_2(a), \ldots, x_{d^2-1}(a)) \in \mathbb{R}^{d^2-1}$, where the coordinate function $x_\alpha : \mathbb{H}_d \rightarrow \mathbb{R}$ is given by $x_\alpha(a) := (a, f_\alpha)_{HS} = \text{Tr}(af_\alpha)$. Now, define a trace $\text{Tr}$ function by $\text{Tr}(a) = \sum_{\alpha=1}^{d^2-1} x_\alpha(a)$.
where \( a \) is an operator in \( \mathbb{H}_d(\mathbb{C}) \). In this sense, an element \( a \in \mathbb{H}_d \) can be written in the form

\[
a = \frac{\text{Tr}(a)}{d} \mathbb{1}_d + \vec{f} \cdot \vec{x}(a),
\]

where \( \vec{f} = (f_1, f_2, \ldots, f_{d^2-1}) \). Note that \( \vec{f} \) is not a vector in \( \mathbb{R}^{d^2-1} \) but only the \((d^2-1)\)-tuplet of matrix basis \( f_\alpha \). We employ the standard notation of vector algebra for the Euclidean space \( \mathbb{R}^{d^2-1} \).

For a density operator, its real vector representation can be expressed as in Eq. (1) with trace one. Let \( B(r) \subset \mathbb{R}^{d^2-1} \) be a closed ball of radius \( r \) in \( \mathbb{R}^{d^2-1} \) centred at \( \vec{x} \left( \frac{1}{d} \mathbb{1}_d \right) \) (\( \frac{1}{d} \mathbb{1}_d \) is called totally mixed state.) One can observe that

\[
|\vec{x}(a)|^2 = \left\| a - \frac{\text{Tr}(a)}{d} \mathbb{1}_d \right\|_{HS}^2
= \text{Tr} (a^2) - \frac{\text{Tr}(a)^2}{d},
\]

hence the subset of selfadjoint operators \( \{ a : \left\| a - \frac{\text{Tr}(a)}{d} \mathbb{1}_d \right\|_{HS}^2 \leq r^2 \} \) corresponds to the ball \( B(1) \). Eq. (2) arise by the condition that the set \( F \) is orthogonal, and \( f_0 \) is proportional to the identity matrices thus the other matrix: \( f_\alpha, \alpha > 0 \), become traceless. This basis set is related to the tomography basis of quantum process in the field of quantum information. In fact, the choice of this basis is arbitrary, and the most convenient one is preferred in practice. We remark here that

\[
S_d \text{ is interlaced between the ball of radius } r_{in} := \sqrt{\frac{1}{d(d-1)}} \text{ and } r_{out} := \sqrt{\frac{d-1}{d}}, \text{ that is} \]

\[
B(\mathbb{R}^{d^2-1}, r_{in}) \subset S_d \subset B(\mathbb{R}^{d^2-1}, r_{out}).
\]

Generally, this yields to the celebrated Bloch ball for a 2-level system \((d = 2)\) because these three sets are identical as \( r_{in} = r_{out} \), while the structure of state space need not be a ball when the dimension is greater than two.

### B. Real Matrix Representation of a Map on Matrix Space

A Lindblad type dynamical map, which can be treated as one-parameter semi-group either in the Hilbert-Schmidt space \( S_d \) of density operator (Schrödinger picture) or in the Hilbert-Schmidt space \( \mathbb{H}_d \) of self-adjoint operator (Heisenberg picture) can also be represented by a linear transformation on \( S_d \) or \( \mathbb{H}_d \) respectively. In this article, we restrict the consideration to only the transformation among states in the state space \( S_d \). Precisely, we consider the map \( \varphi \) on \( S_d \) and its matrix representation \( T \) in \( M_{d^2-1}(\mathbb{R}) \). \( \varphi \) is written as \( \varphi[T] \) when \( T \) is emphasised. We note that, the translation transformation in \( \mathbb{R}^{d^2-1} \) is not include in our formulation since we consider only linear maps.

In addition, we also assume the following conditions for the dynamical map:

- **C1.** It is positive and trace-preserving, i.e. \( \varphi[T](\rho) \geq 0 \) and \( \text{Tr} (\varphi[T](\rho)) = \text{Tr} (\rho) \) for all density operators \( \rho \) in \( S_d \).

- **C2.** It is contractive, i.e. \( \|\varphi[T](\rho)\|_{HS} \leq \|\rho\|_{HS} \) for all density operators \( \rho \) in \( S_d \).
C3. A totally mixed state $\frac{1}{d}I_d$ is invariant under the map $\varphi[T]$, i.e. $\varphi[T] \left( \frac{1}{d}I_d \right) = \frac{1}{d}I_d$.

One may notice that an operation satisfying these conditions does not effect the first term of the vector representation of the state in Eq. (1). In general case of finite dimensional system [21], the relation between the map $\varphi$ and the matrix $T$ can be given by

$$\varphi[T, \vec{c}](\rho) := \frac{1}{d}I_d + \vec{f} \cdot (T \cdot \vec{x}(\rho) + \vec{c}),$$  \hspace{1cm} (4)

where $\vec{c}$ is a constant vector in $\mathbb{R}^{d^2-1}$ representing the translation transformation in $S_d$. For simplicity, we set the vector $\vec{c} = 0$, thus the translation is not included in our formulation as we mention.

Via this relation, we can see that the positivity and contraction conditions of $T$ in the Euclidean norm $|T \cdot \vec{x}| \leq |\vec{x}|$ correspond to the positivity and contraction conditions of the counterpart map $\varphi[T]$ in the assumptions (1) and (2) while the trace preserving properties of $\varphi[T]$ can be obtained by the traceless property of the basis $f_\alpha$. Eq. (4) fulfils our assumptions and allow us to consider the map on the real vector representation of the state instead. Notice that the mapping $T \mapsto \varphi[T]$ in Eq.(4) is multiplicative, i.e. $\varphi[AB] = \varphi[A] \circ \varphi[B]$. This is a very simple but useful property suggesting that one can consider the decomposition of maps on state space through their counterparts on a real vector space.

Furthermore, we introduce a symbol $P_\alpha$ defined by

$$P_\alpha(\vec{x}(a)) := x_\alpha(a)f_\alpha.$$  \hspace{1cm} (5)

Here, one can also see that this transformation is trace preserving and square-trace preserving, i.e. for a density operator $\rho$, we obtain

$$\text{Tr} (\mathcal{U}(\rho)) = \text{Tr} \left( \frac{1}{d}I_d \right) + \sum_{\alpha=1}^{d^2-1} \text{Tr} \left( P_\alpha (R \cdot \vec{x}(\rho)) \right) = \text{Tr} (\rho) = 1,$$  \hspace{1cm} (6)

$$\text{Tr} (\mathcal{U}(\rho)^2) = \frac{1}{d} + |R \cdot \vec{x}(\rho)|^2 = \text{Tr} (\rho^2),$$  \hspace{1cm} (7)

since $\text{Tr} (f_\alpha) = 0$ for $\alpha = 1, \ldots, d^2 - 1$ and $R$ preserves the Euclidean norm. The latter is related with the degree of mixing of the state. That is, the square-trace $\text{Tr} (\rho^2)$ or purity of the state $\rho$ can be used in the characterisation of mixture, and a unitary process will not change such property.

C. Unitary and Scaling Maps

In quantum mechanics and quantum information, unitary transformations are fundamental objects used to describe an evolution, and their geometric descriptions has been developed in various aspects such as unitary dynamical map, unitary channel in quantum communication, or change of basis in quantum tomography [2, 13, 23]. In this representation, a unitary transformation $U$ on the state space $S_d$ can be represented by a rotation matrix $R$ on vector space $\mathbb{R}^{d^2-1}$ as

$$U(\rho) := \varphi[R](\rho) = \frac{1}{d}I_d + \vec{f} \cdot (R \cdot \vec{x}(\rho)).$$  \hspace{1cm} (5)

Here, one can also see that this transformation is trace preserving and square-trace preserving, i.e. for a density operator $\rho$, we obtain

$$\text{Tr} \left( U(\rho) \right) = \text{Tr} \left( \frac{1}{d}I_d \right) + \sum_{\alpha=1}^{d^2-1} \text{Tr} \left( P_\alpha (R \cdot \vec{x}(\rho)) \right) = \text{Tr} (\rho) = 1,$$  \hspace{1cm} (6)

$$\text{Tr} \left( U(\rho)^2 \right) = \frac{1}{d} + |R \cdot \vec{x}(\rho)|^2 = \text{Tr} (\rho^2),$$  \hspace{1cm} (7)

since $\text{Tr} (f_\alpha) = 0$ for $\alpha = 1, \ldots, d^2 - 1$ and $R$ preserves the Euclidean norm. The latter is related with the degree of mixing of the state. That is, the square-trace $\text{Tr} (\rho^2)$ or purity of the state $\rho$ can be used in the characterisation of mixture, and a unitary process will not change such property.
This also means that a linear entropy defined as
\[ S_L(\rho) := 1 - \operatorname{Tr}(\rho^2) \],
which is also a leading term of the von Neumann entropy \( S_v(\rho) := -\operatorname{Tr}(\rho \ln \rho) \), remains the same in a unitary process.

A scaling map \( S : \mathcal{B}(1) \rightarrow \mathcal{B}(1) \) is another type of the map in our interest. First of all, let \( G = \{ f_0 \} \cup \{ g_\alpha \}_{\alpha=1}^{d^2-1} \) be another orthogonal basis set of \( \mathbb{H}_d \) and let \( J \) be a Jacobian matrix changing the basis sets \( G \) to \( F \). Similar to the set \( F \), we introduce a symbol \( P_{G}^{\alpha} \) by
\[ P_{G}^{\alpha}(\vec{x}) := x_{G}^{\alpha} g_{\alpha}, \]
where \( \vec{x}_{G}^{\alpha} \) is a vector in \( \mathcal{B}(1) \) and \( x_{G}^{\alpha} \) is a norm of \( \vec{x}_{G}^{\alpha}(g_{\alpha}) \). Suppose \( \lambda_{\alpha} \) be a non-negative real number for all \( \alpha = 1, \ldots, d^2 - 1 \), for a given set \( G \), we define the diagonal scaling matrix \( S_D \) by
\[ S_{D} \cdot \vec{x} = \sum_{\alpha=1}^{d^2-1} e^{-\lambda_{\alpha}} P_{G}^{\alpha}(\vec{x}). \] (8)

The entries are written exponential less than 1 meaning that the matrix \( S_D \) can reduce but do not enlarge the magnitude of the vector. It can be understood that the scaling map here is designed to map inside the state space \( S_d \), thus the coefficients in Eq. (8) need to be a positive number less than one. Note also that the phase or the angle of the vector is not changed by this action.

In this sense, the action can be represented in the basis \( F \) as
\[ S = J^{-1} S_{D} J. \] (9)

The scaling map \( P \) on \( S_d \) is defined as a map induced by the scaling matrix \( S \). In particular, we characterise the scaling map into two types. As such, the map \( P \) is called isotropic if all scaling parameters \( \lambda_{\alpha} \) are identical, otherwise it is called anisotropic. For the isotropic case, after setting \( \lambda_{\alpha} = \lambda \) for \( \alpha = 1, 2, \ldots, d^2 - 1 \), one can write
\[ P_{\lambda}(\rho) := e^{-\lambda} \rho + \left( 1 - e^{-\lambda} \right) \frac{1}{d} \mathbb{I}_d. \] (10)

This scaling transformation can represent all positive symmetric contraction matrices on \( \mathbb{R}^{d^2-1} \) since the positive symmetric contraction matrix can be obviously diagonalised with positive entries on such diagonal matrix.

Contrary to the unitary type, this map (either isotropic or anisotropic) does not preserve the purity, while it still preserves the trace of any density operator. That says
\[ \operatorname{Tr}(P(\rho)) = \operatorname{Tr}(\rho) + \sum_{\alpha=1}^{d^2-1} \operatorname{Tr}(P_{\alpha}(S \cdot \vec{x}(\rho))) = \operatorname{Tr}(\rho) = 1, \] (11)
\[ \operatorname{Tr}(P(\rho)^2) = \frac{1}{d} + |S \cdot \vec{x}(\rho)|^2 \leq \operatorname{Tr}(\rho^2). \] (12)

Since it does not maintain the purity of the state, it can be interpreted that the map corresponds to a non-adiabatic change in quantum dynamics. The characteristics of the scaling map together with that of unitary type will be used in the analysis of quantum dynamical map in the next section.

III. DECOMPOSITION OF DYNAMICAL MAP OF LINDBLAD TYPE

To demonstrate the idea of unitary-scaling decomposition in this work, we consider the formulation for Lindblad dynamical map. The original definition of Lindblad map, which is explained in
the Heisenberg picture as a semigroup on a $C^*$-algebra, will be provided in the first subsection. We explore the concept of decomposition of the map in the second subsection where the decomposition is defined for the real vector representation. The suggestion on the two-parameter semigroup properties of the map is given in the final subsection.

A. Lindblad Dynamical Map

In quantum mechanics, it is widely known that an open system is not governed by the Schrödinger type differential equation since the unitary evolution which arise from the equation can exactly not explain irreversible behaviour of the dynamics in actual physical situation. This is one of many frontiers in nowadays statistical physics to understand the more generic class of a physical and mathematical formalism which can describe the dynamics of physical system in general. However, when Markovian property is assumed, i.e. the dynamics is time homogeneous or being completely positive of the dynamical map holds or even near such the case, the descriptions has been successfully given and well explain physical phenomena concerning open system. That are, the Kossakowski-Gorini-Sudarshan formulation \[24\] of the dynamical map of finite dimensional open system, the Lindblad equation \[9\] which is a more general version including the dynamics on infinite dimensional system, and Davies’ \[25\] or Nakajima-Zwanzig \[2, 26\] construction which interprets the Markovian properties of the dynamics as a consequence of technique involving with taking the limit in the composite dynamics. As we mention, in order to exemplify the idea of unitary-dissipative decomposition of dynamical map, in this work, we will focus and brief an overview on the Lindblad type solely.

In general, a dynamical map is given by a continuous mapping $t \mapsto \Phi_t$, where $t \in [0, \infty)$ defines the time parameter of the dynamics and the map $\Phi_t$ is mapping inside either the state space (Schrödinger picture) or operator space (Heisenberg picture). In original work of Ref.\[9\], the map is expressed in Heisenberg picture using $C^*$-algebraic framework that the the dynamical map is acting on the set of quantum operators. Let $\Phi_t$ denote a dynamical map on $S_d$ and $\Phi^*_t$ is its corresponding dual map on the operator space $M_d(\mathbb{C})$ that are related by relation:

$$\text{Tr} \left( \rho \Phi^*_t(a) \right) = \text{Tr} \left( \Phi_t(\rho)a \right),$$

for all $\rho \in S_d$ and all operator $a \in M_d(\mathbb{C})$. We define a Lindblad dynamical map $\Phi^*_t$ is a map satisfying the following conditions.

L1. $\Phi^*_t \in CP_\sigma(\mathcal{H})$,

L2. $\Phi^*_t(\mathbf{1}_B(\mathcal{H})) = \mathbf{1}_B(\mathcal{H})$,

L3. $\Phi^*_t \Phi^*_s = \Phi^*_{t+s}$,

L4. $\lim_{t \searrow 0} \left\| \Phi^*_t - \mathbf{1}_B(\mathcal{H}) \right\| = 0$,

where $CP_\sigma(\mathcal{H})$ is the set of all completely positive maps in $\mathcal{B}(\mathcal{H})$ or equivalently $M_d(\mathbb{C})$. The mapping $\Phi^*$ with its tensor extension $\Phi^* \otimes \mathbb{1}_n : \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C}) \rightarrow \mathcal{B}(\mathcal{H}) \otimes M_n(\mathbb{C})$ is positive for all positive integer $n$. For the details of the mathematical setup of the Lindblad map, readers can consult Refs.\[2, 9, 27\].

Here we turn to the Schrödinger picture and employ the dual-formalism of the map on the state space, since we are primarily concerned with the evolution of a quantum state. In this sense, the
Lindblad type dynamical map $\Phi_t$ on state space $S_d$ can be written in the exponential form $\Phi_t = e^{tL}$ where

$$\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum_{\alpha} D_{h_{\alpha}}(\rho), \quad (13)$$

$$D_{h_{\alpha}}(\rho) = h_{\alpha}\rho h_{\alpha}^\dagger - \frac{1}{2}(h_{\alpha} h_{\alpha}^\dagger \rho + \rho h_{\alpha} h_{\alpha}^\dagger). \quad (14)$$

Here $H = H^\dagger$ is a self-adjoint operator in $\mathfrak{B}(H)$ (typically called Hamiltonian) and $h_{\alpha}$ is an operator in $\mathfrak{B}(H)$ with the condition that $\sum_{\alpha} h_{\alpha}^\dagger h_{\alpha} \in \mathfrak{B}(H)$ where $\alpha$ is in some countable index set.

Furthermore, using the vector representation of the state and owing to the fact that the Lindblad type dynamical map satisfies the assumptions $\mathcal{L}_1$-$\mathcal{L}_3$ for homomorphism $\varphi$ given in the previous section, one can define a matrix $M_t$ (called dynamical matrix) in $M_{d^2-1}(\mathbb{R})$ as a counterpart of $\Phi_t$ as $\Phi_t = \varphi[M_t]$. By this notation, one can consider a vector equation $x_t = M_t \cdot \vec{x}$, where $x_t := \vec{x}(\rho_t)$, $\vec{x} := \vec{x}(\rho)$ and $\rho_t := \Phi_t(\rho)$, rather than the abstract dynamical equation $\rho_t = \Phi_t(\rho)$. Because the map is bounded, which is also continuous, the mapping $t \mapsto \Phi_t$ can be illustrated as a trajectory equipped with $M_t$ on the vector space $\mathbb{R}^{d^2-1}$.

B. Decomposition of Dynamical Matrix

As for the main research problem, it is to analyse the role of decomposition of dynamical map into two types of mapping given in the previous section. We will write the Lindblad map and its real-vector representation in the unitary-scaling decomposition. First, we state the polar decomposition lemma whose proof can be found in Ref. [28].

**Theorem 1 (Polar Decomposition)** Let $A \in M_d$, then it may be written in the form $A = PU$ where $P$ is positive semidefinite and $U$ is unitary. The matrix $P$ is always uniquely determined as $P = (AA^\dagger)^{1/2}$; if $A$ is non-singular, then $U$ is uniquely determined as $U \equiv P^{-1}A$. If $A$ is real, then $P$ and $U$ may be taken to be real.

Here, a matrix $M$ in $M_d(\mathbb{R})$ has a polar decomposition as $M = RS$ for some positive symmetric matrix $S$ and isometric orthogonal matrix $R$ (analogous to $P$ and $U$ in the theorem above). It is also has the form $M = RS$, when $M$ is a normal matrix. Hence, by the construction of a dynamical matrix $M_t$ of the Lindblad map, that is normal, unital and non-singular; thus it has also polar decomposition $M_t = R_tS_t = S_tR_t$, leading to the same decomposition of their induced map $\Phi_t = \varphi[M_t] = \varphi[R_t] \circ \varphi[S_t] = \varphi[S_t] \circ \varphi[R_t]$.

As a transformation, the isometric orthogonal part $R_t$ can be considered as a rotation in $\mathbb{R}^n$, while the positive part $S_t$ becomes a scaling in case $M$ is a contraction as in our case. Hence, $R_t$ and $S_t$, will be called a rotation part and a scaling part, respectively. In this sense, the induced maps will be treated as a unitary evolution and dissipation processes. The interpretation of the unitary mapping on a finite dimensional state space (or operator space) as a rotational transformation on the real vector space is common in the field of information geometry [13]. The latter analogue, which we introduce can reflect the dissipative behaviour, e.g. distilling a heat from the work. This will be clear when we consider the Lindblad map, and its conditions which force a close relationship between the rotation part and the scaling part of the dynamical matrix.
C. Semigroup Properties

Now we will take a brief overview on two-parameter semigroups which we will make connection to the decomposition of a dynamical map in our formulation. From our points of view, the additional parameter is a real non-negative number, and the dynamical map is continuous in this parameter as well as in the time parameter. Consider an ordered pair of parameters \( \alpha = (t, s) \in \mathbb{R}^2_+ := [0, \infty) \times [0, \infty) \), while \( e_t \) and \( e_s \) denote the canonical bases on the set \( \mathbb{R}^2_+ \) corresponding to the parameters \( t \) and \( s \), respectively. Let \( \alpha_i = (r_i, s_i) \), \( i = 1, 2, \ldots \), be any parameter pairs, \( k_t \) and \( k_s \) be positive integers. We define addition, product and norm of ordered pairs by \( \alpha_1 + \alpha_2 := (r_1 + r_2, s_1 + s_2) \), \( \alpha_1 \alpha_2 := (r_1 r_2, s_1 s_2) \), and \( |\alpha_i| := \sqrt{r_i^2 + s_i^2} \), respectively.

**Definition 1** Let \( (\mathcal{E}, \| \cdot \|) \) be any Banach space, a two-parameter semigroup on \( \mathcal{E} \) is a family \( \{ W_\alpha \in \mathcal{B}(\mathcal{E}) : \alpha = (t, s) \in \mathbb{R}^2_+ \} \) with the properties that

1. \( W_{\alpha_1 + \alpha_2} = W_{\alpha_1} \circ W_{\alpha_2} \),
2. \( W_0 = 1_{\mathcal{E}} \).

**Definition 2** A \( t \)-marginal semigroup \( R_t \) is a semigroup given by the relation \( R_t := W_{\alpha_t} \), where \( \alpha \in \mathbb{R}^2_+ \). Let \( S_s \) denote the semigroup corresponding to the parameter \( s \). Let \( A \) and \( B \) be generators of these two marginal semigroups defined as

\[
A\vec{x} := \lim_{t \downarrow 0} \frac{R_t \vec{x} - \vec{x}}{t} = \left( \frac{\partial}{\partial t} R_t \vec{x} \right)_{t=0^+}, \tag{15}
\]
\[
B\vec{x} := \lim_{s \downarrow 0} \frac{S_s \vec{x} - \vec{x}}{s} = \left( \frac{\partial}{\partial s} S_s \vec{x} \right)_{s=0^+}. \tag{16}
\]

In quantum dynamics, almost all known dynamical maps are described by a net of mapping within a given set of states indexed by a time parameter \( t \), \( \Phi_t : \rho \mapsto \rho_t \). One argument we have to concern is that the dynamical map \( \Phi_t \) can be described by two-parameter formulation in the following scenario.

**Definition 3** For a given family of dynamical maps \( \{ W_t : t \geq 0 \} \) on a Banach space \( (\mathcal{E}, \| \cdot \|) \) with a parameter \( t \), and a monotonic function \( s = s(t) \), we say that the map has two parameter version, denoted by \( W_{(t,s)} \), if it can be decomposed into a product \( R_t S_s = S_s R_t = W_t \) where \( R_t \) is a one-parameter isometric semigroup on the Banach space \( (\mathcal{E}, \| \cdot \|) \), i.e. \( \| R_t \| = 1 \), and \( S_s \) is another one-parameter semigroup on the Banach space \( (\mathcal{E}, \| \cdot \|) \) satisfying condition that \( \| S_s \| = |W_t| \). Indeed the map \( W_t \) is two-parameter semigroup with its marginal semigroups \( R_t \) and \( S_s \).

**Remark 1** Remark that the second parameter \( s \) is introduced via the relation \( s = s(t) \), the monotonicity of the relation has to be emphasised to allow validity of its inverse \( t = t(s) \). Here it would be understood that \( W_i(A) = W_{(t,s)}(A) \) for all \( A \) in the Banach space \( (\mathcal{E}, \| \cdot \|) \).

In particular, the Banach space \( \mathcal{E} \) can be either the Hilbert-Schmidt space of trace-class operators or its space of real vector representation containing in \( \mathbb{R}^{d^2-1} \). We consider the latter case first, and we will show that the former case can be obtained in the sense that the dynamical map thereon inherits the properties of dynamical matrices on \( \mathbb{R}^{d^2-1} \). That is, we treat the real vector space \( \mathbb{R}^{d^2-1} \) as the Banach space \( \mathcal{E} \) described above, so the dynamical matrix \( M_i \) is understood as \( W_i \) in the similar fashion. However, there is an interesting question about the second parameter \( s \). From
the definition above, it appears a given relation \( s = s(t) \). Indeed, it is designed to single out the dissipative behaviour of the dynamics from a usual unitary evolution separating the change in the norm of the vector along the dynamics from the change of the angles or phase of the vector (relative to a given reference vector). In the following section, we will interpret the conditions of Lindblad maps in its real vector representation as a special relation between the rotation and scaling parts, which correspond to the change in relative angles and the change in magnitude, respectively.

IV. RESULTS

In the following two subsections, we consider the consequences of the decomposition of the dynamical matrix arisen from the normal property and Markovianity of the Lindblad map. They result in a dynamical matrix and its orthogonal and symmetric parts in the way that they are simultaneously block-diagonalisable and will share the same eigensubspaces. We begin by considering the structure of the rotation matrix as it is an orthogonal transformation leading to the condition of the form of the scaling part. We investigate a special case when the scaling part of the dynamical map is isotropic in the subsequent subsection. The group properties will be revisited again in the final part of this section to intensify the significance of our result.

A. Scaling Part of a Normal Dynamical Matrix

In particular, we consider the normal property of the map that allows isometric part and positive part of the polar form of either the dynamical map or dynamical matrix. Since the orthogonal part of the dynamical matrix can be described by a rotation matrix as well as the symmetric part can be treated as a scaling by the contraction property. In particular, let the composition be written as \( M_t = R_t S_t \), where \( R_t \) and \( S_t \) denote its associate rotation and scaling parts. Here the notation \( S_t \) is not understood as an isotropic scaling with parameter \( t \), i.e. \( t \) denotes a dynamical parameter of the original dynamics not the scaling parameter. We distinguish between the scaling parameter for the isotropic scaling \( S_\lambda \) and a dynamical parameter for the symmetric part \( S_t \) of dynamical matrix \( M_t \) by representing it with a Greek symbol (usually \( \lambda \)) for the former case.

Next, let’s consider a rotation matrix \( R \). The following lemma is from Ref. \[29\].

**Lemma 1** Every element of \( SO(n) \) is conjugate to a block-diagonal matrix,

\[
\text{diag} (r_1, r_2, \ldots, r_m) \text{ if } n = 2m \text{ is even,}
\text{diag} (r_1, r_2, \ldots, r_{m}, 1) \text{ if } n = 2m + 1 \text{ is odd.}
\]

Each \( r_k \) is a \( 2 \times 2 \) block

\[
\begin{pmatrix}
\cos \theta_k & -\sin \theta_k \\
\sin \theta_k & \cos \theta_k
\end{pmatrix},
\]

for some angle \( \theta_k \) and \( 1 = (1) \) block.

**Colloary 1** In the same manner, every element of \( SU(n) \) is conjugate to a diagonal matrix

\[
\text{diag} (\epsilon_1, \epsilon_2, \ldots, \epsilon_n),
\]

where \( \epsilon_k = e^{i\theta_k} \) and \( \prod_{k=1}^{n} \epsilon_k = 1 \). Since the orthogonal group is contained in unitary group, i.e. \( SO(n) \subset SU(n) \), the block \( r_k \) in Lemma 1 can be written in the form of

\[
\begin{pmatrix}
e^{i\theta_k} & 0 \\
0 & e^{-i\theta_k}
\end{pmatrix},
\]
where the corresponding eigenvectors of the block are \( \begin{pmatrix} 1 \\ i \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -i \end{pmatrix} \), respectively.

However, when one considers the field \( \mathbb{R} \), the block \( r_k \) does not correspond to two eigensubspaces given by these two eigenvectors since they are not real eigenvectors. Henceforth, the rotation matrix \( \mathbf{R} \) has only 2-dimensional eigensubspaces unless the overall dimension \( n \) is odd, in which case there is additionally 1 \( \times \) 1 block of \( \mathbf{1} \).

**Collorary 2** In any 2-dimensional eigensubspace of the rotation matrix \( \mathbf{R} \), which is described by the block \( r_k \) a positive symmetric matrix \( s_k \) in \( \mathbb{M}_2(\mathbb{R}) \) commuting with \( r_k \), i.e. \( s_k r_k = r_k s_k \) must be \( a \mathbf{1}_2 \), where \( \mathbf{1}_2 \) is an identity matrix in \( \mathbb{M}_2(\mathbb{R}) \) and \( a \) is a positive real number, unless \( r_k \) is itself identity \( \mathbf{1}_2 \).

This claim can be easily shown by considering a matrix \( s_k \) in this subset in the form of \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \), where \( a, b, \) and \( c \) are real numbers. Solving for the entries from the condition \( s_k r_k = r_k s_k \), one will find that \( b = 0 \) and \( a = c \) is arbitrary, leading to the matrix \( s_k = a \cdot \mathbf{1}_2 \) in which case \( a \) has to be positive real number. Consequently, one may obtain the following proposition.

**Proposition 1** Any scaling matrix \( \mathbf{S} \) in \( \mathbb{M}_n(\mathbb{R}) \) which commutes with a rotation matrix \( \mathbf{R} \) in \( \mathbb{M}_n(\mathbb{R}) \) is conjugate to a block-diagonal matrix,

\[
\text{diag} \left( e^{-\lambda_1} \mathbf{1}_2, e^{-\lambda_2} \mathbf{1}_2, \ldots, e^{-\lambda_m} \mathbf{1}_2 \right)
\]

if \( n = 2m \) is even,

\[
\text{diag} \left( e^{-\lambda_1} \mathbf{1}_2, e^{-\lambda_2} \mathbf{1}_2, \ldots, e^{-\lambda_m} \mathbf{1}_2, e^{-\lambda_{m+1}} \mathbf{1} \right)
\]

if \( n = 2m + 1 \) is odd.

Here \( \lambda_k \) is a real number for \( k = 1, 2, \ldots, m \) and \( \mathbf{1} = (1 \text{ block}) \).

**Proof 1** First, from the condition that matrices \( \mathbf{S} \) and \( \mathbf{R} \) commute, they share the same eigensubspaces. Indeed, both are simultaneously block-diagonisable. Thus, together with Lemma 1, the scaling matrix \( \mathbf{S} \) is conjugate to either one of

\[
\text{diag} \left( s_1, s_2, \ldots, s_m \right) \text{ if } n = 2m \text{ is even},
\]

\[
\text{diag} \left( s_1, s_2, \ldots, s_m, e^{-\lambda_{m+1}} \mathbf{1} \right) \text{ if } n = 2m + 1 \text{ is odd}.
\]

Here, \( s_k \) is a \( 2 \times 2 \) positive symmetric block for \( k = 1, \ldots, m \). Since the commutation will also apply to the eigenspace \( k \), where the blocks \( r_k \) and \( s_k \) commute, and from Collorary 2, therefore, the block \( s_k \) will be proportional to the identity matrix \( \mathbf{1}_2 \).

**Remark 2** 1. In the sense of transformation, one will see that the rotation matrix \( \mathbf{R} \) has an inverse as another rotation matrix, i.e. the block \( r_k \) equipped with the angle of rotation \( \theta_k \) in the diagonal presented in Collorary 2 has an inverse as another \( 2 \times 2 \) rotation block \( r_k^{-1} \), but instead with an angle of rotation \( -\theta_k \). This argument is slightly different when one considers the scaling matrix \( \mathbf{S} \) because it is a contraction by definition. Thus, the inverse of the block \( s_k = e^{-\lambda_k} \mathbf{1}_2 \), which is exactly \( s_k^{-1} = e^{\lambda_k} \mathbf{1}_2 \) is not a contraction matrix since \( \lambda_k > 0 \).
2. In some instance, some blocks $r_k$ and $r_l$ are identical in the way that their angles of rotation are the same, i.e. $\theta_k = \theta_l$. This situation may be called in geometry as an isoclinic rotation, when one considers the overall rotation matrix $r_k \oplus r_l$ on these two eigensubspaces together. This case can occur when the dynamics of the system (or the generator of the dynamics) has a degeneracy in these two eigensubspaces, providing that they share the same eigenvectors (determined on the complex field). However, as long as these two subspaces are treated separately, it does not interfere with the structure of the scaling matrix $S$. In particular, even as the blocks $r_k$ and $r_l$ are degenerate, parameters of the dilations $\lambda_k$ and $\lambda_l$ corresponding to the same pair of eigensubspaces need not be the same. Consequently, the composition of the two matrices $R$ and $S$ is not degenerate in this two eigensubspaces.

Owing to the block diagonal form of the rotation and scaling matrices, the normal dynamical matrix $M_t = R_t S_t$ is conjugate to the same block diagonal such that it can be arranged into the block diagonal form by the same similarity transformation. Therefore, it is convenient to employ this block diagonal form for the dynamical matrix $M_t$. We note here that this block diagonal form arises mostly from the setting of the dynamical matrix, while only the condition of the normal property the Lindblad dynamical map is used. Thus, the same analysis can be adopted to the generalised class of dynamical maps which still possess the normal property. This is an ongoing investigation.

B. Markovianity and Isotropic Scaling

For Markovian case, the dynamical matrix $M_t$ can be written as

$$M_t = \prod_{i=1}^{N} M_{\tau_i},$$

where $\tau_i > 0$ with

$$\sum_{i=1}^{N} \tau_i = t.$$

In the forms of the orthogonal and symmetric parts, if any $S_{\tau_i}$ is isotropic, one obtains that

$$R_t S_t = \prod_{i=1}^{N} R_{\tau_i} S_{\tau_i} = \left( \prod_{i=1}^{N} R_{\tau_i} \right) \left( \prod_{i=1}^{N} S_{\tau_i} \right). \quad (17)$$

In addition, when the matrix $R_t$ is a group with parameter $t$, one can deduce a relation $S_t = \prod_{i=1}^{N} S_{\tau_i}$ obeying the same structure of the addition property of a semigroup. In this sense, we have

$$e^{-\lambda(t)} = \exp \left( -\sum_{i=1}^{N} \lambda(\tau_i) \right), \quad (18)$$

where $\lambda(t)$ denotes the scaling parameter of the map at time $t$ (which is indeed the time duration in the Markovian case). Under this condition where the partition $\{\tau_i\}_{i=1}^{N}$ of the time duration $t$ is
arbitrary, one can observe that the relation $\lambda(t)$ must be a linear function of $t$, so it can be written as $\lambda(t) = \gamma t$, where $\gamma$ is a positive real number [30]. We will discuss the characteristics of the scaling parameter function $\gamma_k(t)$, each block in the non-isotropic case in the next subsection.

The most remarkable consequence of this property concerns the entropy change along the dynamics. Let’s consider again the linear entropy function. First, let

$$S_{L}^{\text{pure}}(t) = \frac{d-1}{d} \left(1 - e^{-2\gamma t}\right)$$

(19)

denote the linear entropy in the case that the initial state is a pure state and $S_{L}^{0}(\rho) := (1 - \text{Tr}(\rho^2))$ denote the linear entropy of the actual initial state. For an initial state $\rho$,

$$S_{L}(\rho_t) = S_{L}^{\text{pure}}(t) + e^{-2\gamma t}S_{L}^{0}(\rho).$$

(20)

The relation above shows that the characteristics of the Lindblad or Markovian dynamics that yields increasing entropy with time and possesses an asymptote or bounded by a certain number $\frac{d-1}{d}$, see Figure 1.

Remark that the situation above occurs when the scaling part is isotropic and the rotation part forms another semigroup. The parameter of scaling is determined up to the entropy function, i.e. the scaling parameter is in one-to-one correspondence with the entropy function. This allows us to consider the scaling parameter as a quantification of non-adiabaticity as we previously introduced.

C. Semigroup Properties Revisited

Now we consider the semigroup properties again in this situation. Suppose we can write $M_t = e^{tL}$ for some matrix $L$ which denote the generator of the semigroup $M_t$. Because the semigroup $M_t$ is a representation of the dynamical semigroup $\Phi_t$, the matrix $L$ will corresponds to the representative of the generator $\Lambda$ of the semigroup $\Phi_t$. The correspondence between the exponential form $M$ in the matrix $L$ and that of $\Phi_t$ follows from multiplicativity of the mapping $\varphi$ and by means of the density of the class of polynomial functions in $H_d$ together with the spectrum theorem.

First, let’s consider the rotation part. According to Collorary [1] it appears that the eigenvector is independent of time $t$ while the eigenvalue is not. The properties are inherited to $KR_K^{-1}$, where the matrix $K$ on $\mathbb{R}^{d^2-1}$ is a similarity transformation matrix from the coordinates in the basis $\tilde{x}(f_{a})$
to the components in the eigensubspace of $R_t$. Note that the matrix $K$ is also independent of time $t$, and it is an orthogonal matrix. Since $S_t$ commutes with $R_t$ by the normal property of $M_t$, we obtain

$$M_t = KR^D S^D_t K^{-1},$$

where $R_t^D$ and $S_t^D$ denote the block-diagonal forms of $R_t$ and $S_t$ respectively. By this notation, it is sufficient to consider the action only in each subspace describing by the block in $R_t^D$ or $S_t^D$. Let $M_t^D := R_t^D S_t^D$ denote the block-diagonal form of $M_t$, which can be obtained by conjugation with $K$, and let $m_k(t)$ be the $k^{th}$ block. In similar fashion, let $r_k(t)$ and $s_k(t)$ be their $k^{th}$ blocks of the matrices $R_t^D$ and $S_t^D$, respectively. For simplicity, we will consider the case that the dimension of the real vector representation $d^2 - 1$ is even, leaving the odd dimensional case as an even dimensional added by the block (1).

Consider the expression of each block $m_k(t) = r_k(t)s_k(t)$ or

$$m_k(t) = e^{\lambda_k(t)} \begin{pmatrix} \cos \theta_k(t) & -\sin \theta_k(t) \\ \sin \theta_k(t) & \cos \theta_k(t) \end{pmatrix},$$

where $\lambda_k(t)$ and $\theta_k(t)$ are functions of $t$ related to the scaling parameter of $S_t$ and the angle of rotation of $R_t$ in the $k^{th}$ eigenblock, respectively. One can see that the scaling matrix $s_k(t)$ in the $2$–dimensional subspace here is clearly isotropic, leading to the following statement.

**Proposition 2** The functions $\lambda_k(t)$ and $\theta_k(t)$ are either simultaneously additive in $t$ or simultaneously non-additive in $t$ for all $k = 1, \ldots, m$ and $2m = d^2 - 1$. Moreover, since the partition of time interval $[0, t]$ into $\{\tau_i\}_{i=1}^N$ is arbitrary, the additivity is identical to the linearity. That is, $\lambda_k(t)$ and $\theta_k(t)$ are both simultaneously linear in $t$ or both are not.

**Proof 2** From the Markovianity, we have $m_k(t_1 + t_2) = m_k(t_1)m_k(t_2)$, which can be written as:

$$m_k(t_1 + t_2) = e^{-\lambda_k(t_1 + t_2)} \begin{pmatrix} \cos \theta_k(t_1 + t_2) & -\sin \theta_k(t_1 + t_2) \\ \sin \theta_k(t_1 + t_2) & \cos \theta_k(t_1 + t_2) \end{pmatrix}.$$  

(22)

$$m_k(t_1)m_k(t_2) = e^{-\lambda_k(t_1) + \lambda_k(t_2)} \begin{pmatrix} \cos \theta_k(t_1) & -\sin \theta_k(t_1) \\ \sin \theta_k(t_1) & \cos \theta_k(t_1) \end{pmatrix} \begin{pmatrix} \cos \theta_k(t_2) & -\sin \theta_k(t_2) \\ \sin \theta_k(t_2) & \cos \theta_k(t_2) \end{pmatrix}$$

(23)

Therefore, if $\theta_k(t)$ is additive in $t$, i.e. $\theta_k(t_1 + t_2) = \theta_k(t_1) + \theta_k(t_2)$, so $\lambda_k(t)$ is. Otherwise, both of them are not additive in $t$.

Unlike the isotropic condition, Proposition 2 implies that

$$S_L(\rho_t) = 1 - \text{Tr} (\rho_t^2)$$

$$= \frac{d - 1}{d} - |K R^D S^D_t K^{-1} \cdot \bar{x}(\rho)|^2$$

$$= \frac{d - 1}{d} - \sum_{k=1}^{m} |r_k(t)s_k(t) \cdot \bar{x}_k(\rho)|^2,$$

(24)
where $\vec{x}_k$ is a 2-dimensional component of the vector $\vec{x}$ in the $k^{th}$ subspace. Since $r_k(t)$ preserves Euclidean norm, it follows that

$$S_L(\rho_t) = \frac{d-1}{d} - \sum_{k=1}^{m} e^{-2\gamma_k t} |\vec{x}_k(\rho)|^2,$$  \hspace{1cm} (25)

where $\lambda_k(t) = \gamma_k t$ as it is a linear function here. From this relation, one can see that the linear entropy is expressed as a weighted sum of the exponential-decay functions of the form $e^{-\gamma_k t}$. It also indicates the composition of the dynamical matrix $M_t$, where the weight in each $k^{th}$ subspace is set by $|\vec{x}_k(\rho)|^2$ of the initial state $\rho$. This expression of the linear entropy reflects the properties of increasing in time and possessing a asymptote from the Markovian dynamics.

From Eq. (25), not only the scaling parameter $\gamma_k$ corresponds to each subspace, but also the mass $|\vec{x}_k(\rho)|^2$ therein the subspace from the initial state affects the dissipative behaviour of the system. As an illustration, let’s consider the case when the initial state $\rho$ has real vector representation lying in only one of the subspaces, namely the $k^{th}$ subspace, while the other components are all null, i.e. $|\vec{x}_k(\rho)|^2 \neq 0$ and $|\vec{x}_l(\rho)|^2 = 0$ for all $l \neq k$. The change of the linear entropy along the dynamics become

$$S_L(\rho_t) = \frac{d-1}{d} - e^{-2\gamma_k t} |\vec{x}_k(\rho)|^2.$$  \hspace{1cm} (26)

This equality is similar to the isotropic case. This is trivial because the scaling matrix is isotropic in each subspace. However, if the real vector representation of the initial state $\vec{x}(\rho)$ has multiple components in more than one subspace, the change in time of the linear entropy is constituted by more than one rates. In summary, the characteristics of map, the initial state (e.g. the state $\rho$) and the relation between them (e.g. the component $\vec{x}_k(\rho)$) affect in the the change of entropy.

**V. EXAMPLES: QUBIT SYSTEMS**

From the result we obtained, in this section, it will be explored them with a typical qubit system. We will not focus in technical details since they are beyond the scope of this article. In the first part, we recall all the formulation again in the form of three dimensional real vector representation of qubit state while the dynamical matrix will be a $3 \times 3$ real matrix. The restrictions on Markovianity and dissipative behaviour are revisited in the second part and physical matrices (processes) in quantum information will be considered in the final subsection.

**A. Real Vector Representation of Qubit**

First qubit system, we can write the state $\rho$ as

$$\rho = \frac{1}{2} \mathbb{1}_2 + \sum_{\alpha=1}^{3} f_{\alpha} x_{\alpha}(\rho),$$

where the set $\{\frac{1}{2} \mathbb{1}_2, f_1, f_2, f_3\}$ denotes the (tomography) basis set. In fact, in the early development in information theory, one of the most famous basis sets is that set of Pauli’s matrices, which is also
FIG. 2. Cross sections of the Bloch ball for the qubit system. On the left hand side, it represents a plane subspace where the component of a given vector is rotated by $R_t$ and scaled by $S_t$. The right figure corresponds to the invariant subspace where the component remains the same via rotation but still has a change in radius by the effects from scaling matrix.

the set of generators of SU(2). This is the usual Bloch representation of the qubit system in which the set of all states can be represented in a unit close ball in $\mathbb{R}^3$; namely the Bloch ball. For the Lindblad map $\Phi_t$, its real matrix representation $M_t$ can be decomposed into the 3–dimensional rotation matrix $R_t$ and the scaling matrix $S_t$, which are real matrix representations of unitary and scaling maps on $S_2$, respectively.

Since the spirit of the partition of the Bloch ball is previously shown, we will begin with this point of view. Consider the rotation part $R_t$; one can see that Lemma 1 is exactly the Euler’s theorem, where the block-diagonal form can be written as

$$R_t^D = \text{diag}(r_1(t), 1).$$

That is any rotation matrix in $\mathbb{R}^3$ rotates components of a real vector representation of an initial state within a certain plane and leaves the other component invariant. We will call the projection onto this plane of any vector a plane component and the remaining orthogonal block an invariant component. In fact, the invariant or orthogonal component is interpreted as an axis of rotation (while this concept is not well defined for $d > 2$). In addition, the two invariant vectors of the rotation in the ball (the pole vectors), are the real vector representation of projections (or pure density matrices in $S_d$) corresponding to the eigenvectors of the Hamiltonian $H$ defined as a generator of $U_t$.

For the scaling matrix $S_t$, it is reduced to the form

$$S_t = \text{diag}(e^{-\gamma_\parallel t}, 1, e^{-\gamma_\perp t}),$$

where the exponents $-\gamma_\parallel t$ and $-\gamma_\perp t$ are the scaling parameters for the plane and invariant subspaces, respectively. The parallel and perpendicular notations are used in the same way for a given vector, the plane component lies in the plane, while the invariant component is perpendicular to the plane. Note also that the linearity arise from the Markovianity of the dynamics. Here, one can see that the scaling of a vector is different between the in-plane and the invariant components. Consequently, the image of the dynamical matrix is reduced from the ball (as the domain) to be a prolate spheroid when $\gamma_\parallel > \gamma_\perp$, an oblate one when $\gamma_\parallel < \gamma_\perp$ or a smaller ball when the scaling is isotropic; as shown in Fig. 2.
B. Entropy Change

Recall the expression of linear entropy in Eq.(25) for \( d = 2 \);

\[
S_L(\rho_t) = \frac{1}{2} - \left( e^{-2\gamma t} |\vec{x}_{||} (\rho)|^2 + e^{-2\gamma t} x_{\perp}^2 (\rho) \right),
\]

which is reduced to \( \frac{1}{2} - e^{-2\gamma t} |\vec{x}(\rho)|^2 \) when the scaling part is isotropic, i.e. \( \gamma_{||} = \gamma_{\perp} = \gamma \). Thus the similar interpretation given in the previous section for this change is also applied to this situation. In addition, for this particular case, the linear entropy can explicitly demonstrate the behaviour of the dynamics by simply considering their geometric interpretation of the state along the dynamics. Although we employ the linear entropy since it is practically convenient to calculate, there is another advantage of this expression. Because the purity or the radius of the Bloch vector is used in result exploration in many experiments, the connection between measurement results to the characterisation of the dynamical map can be made. For instances, one can see that the rate of change in the entropy along the dynamics can be analysed directly from the decomposition of the dynamical matrix or one can extract the invariant component, which should reflect the stationary situation in the dynamics, etc.

Moreover, in a qubit system, the von Neumann entropy can be explicitly expressed as

\[
S_v(\rho) = - p_1 \ln p_1 - p_2 \ln p_2,
\]

\[
= - \frac{1}{2} (1 + r) \ln \frac{1}{2} (1 + r) - \frac{1}{2} (1 - r) \ln \frac{1}{2} (1 - r),
\]

where \( p_1 \) and \( p_2 \) are eigenvalues of state \( \rho \), which are related to the radius \( r := |\vec{x}(\rho)| \) of the vector representation of \( \rho \) as \( p_1 = \frac{1}{2} (1 + r) \) and \( p_2 = \frac{1}{2} (1 - r) \). The radius \( r \) of the state \( \rho_t \) is indeed the same as in the linear entropy, i.e.

\[
r := |\vec{x}(\rho)| = \sqrt{e^{-2\gamma t} |\vec{x}_{||} (\rho)|^2 + e^{-2\gamma t} x_{\perp}^2 (\rho)}.
\]

By this form, the characters of the initial preparation in the selection of the rate of dissipation is not obvious as expressed in the linear entropy, but it is still clear that the change of von Neumann entropy depends on the weights \( |\vec{x}_{||} (\rho)|^2 \) and \( x_{\perp}^2 (\rho) \) via the radius of the vector representation. In particular, when the scaling matrix is isotropic, the von Neumann entropy is simply

\[
S_v(\rho) = - \frac{1}{2} \left( 1 + e^{-\gamma t} r_0 \right) \ln \frac{1}{2} \left( 1 + e^{-\gamma t} r_0 \right) - \frac{1}{2} \left( 1 - e^{-\gamma t} r_0 \right) \ln \frac{1}{2} \left( 1 - e^{-\gamma t} r_0 \right),
\]

where \( r_0 = |\vec{x}| \) and \( \gamma \) is a scaling parameter. Indeed, this also reflects the increasing property and possessing asymptote of the entropy; Fig. [3]

C. Physical Processes

Now we consider some special types of dynamical maps using in quantum information, in which the trace non increasing and completely positive maps may be called channels. They are bit-flipping,
phase-flipping, depolarizing, and amplitude damping \cite{13, 23, 32}.

First we choose the set \{f_1, f_2, f_3\} as the Pauli’s basis set \{\sigma_x, \sigma_y, \sigma_z\}. For a given real number \(0 < p < 1\), bit-flipping and phase-flipping are defined as

\[
BF(\rho) = (1 - p)\rho + p\sigma_x\rho\sigma_x,
\]
\[
PF(\rho) = (1 - p)\rho + p\sigma_z\rho\sigma_z,
\]
respectively. They can be written in the real matrix representation as

\[
M_{BF} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 - 2p & 0 \\
0 & 0 & 1 - 2p
\end{pmatrix},
\]
\[
M_{PF} = \begin{pmatrix}
1 - 2p & 0 & 0 \\
0 & 1 - 2p & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Here one can see that both of them are naturally scaling maps by their forms. In addition, for bit-flipping, one can see that the \(x\)-axis is the invariant subspace, while the \(yz\)-plane is the plane subspace. For the phase-flipping, the \(z\)-axis is the invariant subspace and the \(xy\)-plane are the plane subspace. In both processes, one can see that, the maps does not constitute the rotation as their parts. Furthermore, one can see also that the splitting of the plane and invariant subspaces in both cases depends on the convex combinations in Eqs. (30) and (31) via the convex coefficients for the matrices in Eqs. (32) and (33).

In a different manner, the depolarizing (\(DP\)) is defined by

\[
DP(\rho) = (1 - p)\rho + \frac{p}{2} \mathbb{1}_2
\]
for a given real number \(0 < p < 1\). It does not have the splitting of plane and invariant subspaces. This can be seen clearly from its matrix representation

\[
M_{DP} = \begin{pmatrix}
1 - p & 0 & 0 \\
0 & 1 - p & 0 \\
0 & 0 & 1 - p
\end{pmatrix},
\]

which is an exactly isotropic scaling. Also, the depolarizing process is not composed of any rotation matrix, but only the scaling matrix. In fact, the shape of deformation of the Bloch ball after the
The depolarizing process is still a ball, while those of the bit flipping and phase flipping processes are prolate spheroids with the major axis in the x and z directions, respectively.

Last but not least, we have to remark that since we did not include the translation in our consideration, some dynamical maps or processes may not satisfied the decomposition. A typical example is amplitude damping process (AD), which is defined by

$$AD(\rho) = \left( \begin{array}{cc} 0 & \sqrt{p} \\ 0 & 0 \end{array} \right) \rho \left( \begin{array}{cc} 0 & \sqrt{p} \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{1-p} \end{array} \right) \rho \left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{1-p} \end{array} \right).$$

This mapping is different from the previous examples in that one cannot find the matrix representation of this mapping, i.e. there is no real matrix representation of the map $AD$ on $M_{d^2-1}(\mathbb{R})$ because there is a shift of the center of the Bloch ball via the transformation. This issue may be resolved by mathematical techniques in geometry, e.g. homogenisation or dimension extension of the real vector space. However, the analysis in this direction introduces further questions in both mathematical and physical points of view which are out of the scope of this article, thus we left this issue as further investigation.

VI. CONCLUSIONS

Dynamical maps for open quantum systems are the most versatile objects used to describe the evolution of the systems. Such the maps can be categorized into three groups: the Liouville-von Neuman type, the Lindblad type and the beyond-Lindblad type. We investigate the problem of a decomposition of a dissipative dynamical map of the Lindblad type, which is used in quantum dynamics of a finite open quantum system, into two distinct types of mapping on the space of quantum states represented by density matrices. The formulation and decomposition employ the interplay between the density matrix and the Bloch representations of the state, and the orthogonal-symmetric decomposition of a matrix. One component of the composition map corresponds to reversible behaviours, while the other to irreversible characteristics. It is found that the rotation and the scaling transformations on the real vector space behave as building blocks for a dynamical map. We introduce an additional parameter of the dynamics together with the time parameter. Together with the time parameter, we introduce an additional parameter, which is related to a scaling parameter in the scaling part of the dynamical matrix. Specifically for the Lindblad-type dynamical map, in which the dynamical map forms a one-parameter semigroup satisfying the conditions of complete positivity, normality and Makovianity, we interpret the conditions on the Lindblad map in our framework. As results, we find that the change of linear entropy or purity, as an indicator of dissipative behaviours increases in time and possesses an asymptote expected in thermodynamics. The rate of change of the linear entropy depends on the structure of the scaling part of the dynamical matrix. Moreover, the initial state plays an important role in this rate of change. The issues on initial dependence of dissipative behaviours and the role of eigensubspace partitioning of the dynamical matrix are discussed. Moreover, we have demonstrated our formulation for specific examples of bit-flipping, phase-flipping, and depolarisation matrices in qubit systems.

From the idea of unitary-scaling decomposition, one can see that the dissipative behaviour may be considered in parallel with the periodic behaviour, i.e. via the scaling and the rotation parts. One can set the unitary dynamical map as a centred map and the scaling map as the deviation from the unitary map. In such ways, the use of linear entropy, which is adopted as an indicator of the
dissipative behaviour, as another dynamical parameter becomes a promising tool in understanding quantum open systems in an unprecedented regime.

However, in our formulation, one can see that we add several assumptions, even for the Lindblad dynamics, to make it simple enough to understand the mechanics, but these assumptions may be not met in some instances, e.g. in non-Markovian dynamics, or when the complete positivity fails. It is interesting to remove some of these assumptions and see how the systems behave under the circumstances. Another crucial point is about the entropy function, where the typical entropy function in quantum mechanics is the von-Neumann entropy, while the linear entropy function is only its approximation. Some extensions to use other indicators for dissipative behaviour are also interesting and should be investigated because they may reveal some interesting phenomenon which are eluded by our current formulation. Indeed, these topics are our work in progress.

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[1] J. von Neumann, Mathematical Foundations of Quantum Mechanics (Princeton University Press, 1996).
[2] R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications, Lecture Notes in Physics (Springer-Verlag Berlin Heidelberg, 2007).
[3] J. Fröhlich and B. Schubnel, ArXiv e-prints (2013).
[4] B. Baumgartner, “Characterizing Entropy in Statistical Physics and in Quantum Information Theory,” (2012), arXiv:1206.5727v1 [math-ph].
[5] L. Pucci, M. Esposito, and L. Politi, Journal of Statistical Mechanics: Theory and Experiment 2013, P04005 (2013).
[6] M. Baiesi and C. Maes, New Journal of Physics 15, 013004 (2013).
[7] Modi Kavan, Sci. Rep. 2 (2012), 10.1038/srep00581; 10.1038/srep00581.
[8] E. C. G. Sudarshan, P. M. Mathews, and J. Rau, Phys. Rev. 121, 920 (1961).
[9] G. Lindblad, Communications in Mathematical Physics 48, 119 (1976).
[10] B. Baumgartner, H. Narnhofer, and W. Thirring, Journal of Physics A: Mathematical and Theoretical 41, 065201 (2008).
[11] H. A. Carteret, D. R. Terno, and K. Zyczkowski, Phys. Rev. A 77, 042113 (2008).
[12] A. Wehrl, Rev. Mod. Phys. 50, 221 (1978).
[13] I. Bengtsson and K. Zyczkowski, An Introduction to Quantum Entanglement (Cambridge University Press, 2008).
[14] K. Dietz, Journal of Physics A: Mathematical and General 37, 6143 (2004).
[15] S. Sinha and R. Sorkin, Foundations of Physics Letters 4, 303 (1991).
[16] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets (World Scientific, 2009).
[17] A. Kamenev and A. Levchenko, Advances in Physics 58, 197 (2009).
[18] G. D. Mahan, Many-Particle Physics, Physics of Solids and Liquids (Kluwer Academic Publishers-Plenum Publishers, 2000) Chap. 3.
[19] F. Sakuldee and S. Suwanna, Phys. Rev. E 92, 052118 (2015).
[20] G. Kimura, Physics Letters A 314, 339 (2003).
[21] A. Kossakowski, Open Systems & Information Dynamics 10, 213 (2003).
Indeed, since the domain of this function includes 0, thus the generalized Cauchy’s functional equation in Eq. (13) has also a trivial solution $\lambda(t) = 0$ for all $t = 0$. This associates with the unitary evolution where there is no decaying behaviour in the dynamics. However, we concentrate our analysis on only the non-trivial solution or the linear function.