Spinless Hartree-Fock model of persistent current in rings with single scatterer:
Comparison with correlated models

Radoslav Németh and Martin Možík
Institute of Electrical Engineering, Slovak Academy of Sciences, Dúbravská cesta 9, 841 04 Bratislava, Slovakia

(Dated: March 23, 2022)

Using the self-consistent Hartree-Fock approximation for spinless electrons at zero temperature, we study the persistent current of the interacting electron gas in a one-dimensional continuous ring containing a single \(\delta\) barrier. We calculate the persistent current as a function of the ring circumference, magnetic flux threading the ring, barrier strength, etc. We compare our results with the results of correlated models like the Luttinger liquid model and the Hubbard model solved by means of the renormalization group. A good agreement is found. First, the persistent current decays with the increasing ring circumference \((L)\) faster than \(L^{-1}\) and eventually like \(L^{-\alpha-1}\), where \(\alpha > 0\) depends only on the electron-electron interaction. Second, the persistent current is a sine-shaped function of magnetic flux. This sine-like dependence and in particular the universal power law \(L^{-\alpha-1}\) have so far been believed to arise only in the correlated many-body models. Observation of these features within the Hartree-Fock model is a surprising result.

PACS numbers: 73.23.-b, 73.61.Ey

I. INTRODUCTION

Electron gas in a quantum wire is a realistic one-dimensional (1D) electron system. By tying the wire ends to each other one creates a 1D ring. If the ring circumference is shorter than the electron coherence length, one speaks about the mesoscopic ring. An external magnetic flux applied through the opening of such ring gives rise to an equilibrium current, known as persistent current.

Observations of persistent currents in metallic and semiconducting rings2,3,4 stimulated many theoretical papers focused on the electron-electron (e-e) interaction and disorder in such systems. A quantitative understanding of the observed amplitude of the currents is so far not satisfactory and the role of the e-e interaction and disorder in mesoscopic rings need further research. In this work we focus on the persistent current of interacting spinless electrons in a 1D ring with a single scatterer. Let us review recent results for this simplified problem.

If the e-e interaction is ignored, the leading behavior of the persistent current \((I)\) as a function of the magnetic flux \((\phi)\) and ring circumference \((L)\) can be derived analytically for an arbitrary scatterer.5 It is a periodic function of \(\phi\) with period \(\phi_0 = h/e\), it decays as \(1/L\), and its shape is determined by the absolute value of the transmission amplitude \(t_{kf}\) of the scatterer at the Fermi wave vector \(k_F\). In the limit of small \(t_{kf}\) it reads \(I \propto L^{-1} |t_{kf}| \sin(2\pi \phi/\phi_0)\). These results were derived for a continuous ring and hold also for the ring-shaped tight-binding lattice at half filling.

For a repulsive e-e interaction in the continuous ring, the spinless persistent current can still be derived analytically in the Luttinger liquid model.6 For \(L \to \infty\) and for an arbitrary strength of the scatterer, the current is \(I \propto L^{-\alpha-1} \sin(2\pi \phi/\phi_0)\), where the power \(\alpha > 0\) is universal, depending only on the e-e interaction.

The authors of Ref.6 obtained the spinless persistent current microscopically by solving the Hubbard chain with nearest-neighbor hopping and interaction. Using the renormalization group, they indeed found for a 1D ring with a single scatterer the current which decays faster than \(1/L\) and is a sine-like function of \(\phi\). The shape \(I \propto \sin(2\pi \phi/\phi_0)\) and the universal scaling \(I \propto L^{-\alpha-1}\) were confirmed for long chains and/or strong scatterers.

They also concluded that the sine-like shape and the \(L^{-\alpha-1}\) scaling can only be obtained in the correlated many-body model, not in the Hartree-Fock approximation ignoring the correlation effects. They used a non-self-consistent Hartree-Fock approximation and, indeed, found the qualitatively wrong results. The self-consistent Hartree-Fock approximation, according to these authors, drives the model with a single scatterer into the charge-density-wave groundstate with a finite single-particle gap, which is wrong because a single scatterer cannot change the bulk properties of the system. Hence, no attempt has been done to test the dependence \(I \propto L^{-\alpha-1} \sin(2\pi \phi/\phi_0)\) in the self-consistent Hartree-Fock approximation. However, there is a motivation to do so.

Indeed, recently the self-consistent Hartree-Fock approximation has been used to study tunneling of the weakly-interacting electron gas through a single scatterer in a continuous 1D wire with leads. It has been found that there is no charge-density-wave groundstate in such model, moreover, the tunneling probability has been found to decay with the wire length as \(L^{-2\alpha}\), in accord with correlated models. All this suggests that an attempt to obtain the persistent current \(I \propto L^{-\alpha-1} \sin(2\pi \phi/\phi_0)\) in the self-consistent Hartree-Fock approximation might be meaningful.

We present such attempt in our present work. Using the self-consistent Hartree-Fock approximation at zero temperature, we study the persistent current of the interacting spinless electron gas in a continuous 1D ring with a single \(\delta\) barrier. We compare our results with the results of correlated models, with the Luttinger liquid model and the microscopic Hubbard model. A good...
agreement is found. First, the persistent current decays faster than $L^{-1}$ and eventually like $L^{-\alpha-1}$, where $\alpha > 0$ is universal. Second, the persistent current is a sine-like function of magnetic flux.

The paper is organized as follows. In Sect. II we discuss the Hartree-Fock model of the 1D ring. In Sect. III we present the results for the Hartree-Fock potentials. In Sect. IV the Hartree-Fock results for persistent currents are presented and compared with the results of correlated models. A summary is given in Sect. V.

II. HARTREE-FOCK MODEL OF THE 1D RING

We consider $N$ interacting 1D electrons with free motion along a circular ring threaded by magnetic flux $\phi = BS = AL$, where $S$ is the area of the ring, $B$ is the magnetic field (constant and perpendicular to the ring area), and $A$ is the magnitude of the resulting vector potential (circulating along the ring circumference). Such electron system is described by Hamiltonian

$$
H = \sum_{j}^{N} \left[ \frac{\hbar^2}{2m} \left(-i \frac{\partial}{\partial x_j} + \frac{2\pi \phi}{L} \right)^2 + \gamma \delta(x_j) \right] + \frac{1}{2} \sum_{j \neq i}^{N} V(x_j - x_i),
$$

(1)

where $m$ is the electron effective mass, $x_j$ is the coordinate of the $j$-th electron along the ring, $V(x_j - x_i)$ is the e-e interaction, and $\gamma \delta(x)$ is the potential barrier due to the scatterer. The eigenfunction $\Psi(x_1,x_2,\ldots,x_N)$ and eigenenergy $E$ obey the Schrödinger equation

$$
H\Psi = E\Psi
$$

(2)

with the cyclic boundary condition

$$
\Psi(x_1,\ldots,x_i + L,\ldots,x_N) = \Psi(x_1,\ldots,x_i,\ldots,x_N)
$$

(3)

for $i = 1, 2, \ldots, N$. In the groundstate with the eigenenergy $E_0$ the persistent current reads

$$
I = -\frac{\partial}{\partial \phi} E_0(\phi).
$$

(4)

In the Hartree-Fock approximation, the wave function $\Psi$ is approximated by the Slater determinant

$$
\Psi(x_1,\ldots,x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix}
\psi_{k_1}(x_1) & \cdots & \psi_{k_1}(x_N) \\
\vdots & \ddots & \vdots \\
\psi_{k_N}(x_1) & \cdots & \psi_{k_N}(x_N)
\end{vmatrix},
$$

(5)

where $k_i$ is the quantum number of the $i$-th electron. The wave functions $\psi_k(x)$ obey the Hartree-Fock equation

$$
\left[ \frac{\hbar^2}{2m} \left(-i \frac{\partial}{\partial x} + \frac{2\pi \phi}{L} \right)^2 + \gamma \delta(x) + U_H(x) + U_F(k,x) \right] \psi_k(x) = \varepsilon_k \psi_k(x)
$$

(6)

with the boundary condition

$$
\psi_k(x + L) = \psi_k(x),
$$

(7)

where

$$
U_H(x) = \frac{1}{\psi_k(x)} \sum_{k'} \int dx' V(x - x') |\psi_{k'}(x')|^2
$$

(8)

is the Hartree potential and

$$
U_F(k,x) = -\frac{1}{\psi_k(x)} \sum_{k'} \int dx' V(x - x') \psi_{k'}(x') \psi_{k'}(x)
$$

(9)

is the Fock nonlocal exchange term (expressed as an effective potential for further convenience). In the equations 7 and 9 we sum over all occupied states $k'$.

We introduce the wave functions $\varphi_k(x)$ by substitution

$$
\psi_k(x) = \varphi_k(x) \exp \left(-i \frac{2\pi}{L} \frac{\phi}{\phi_0} x \right).
$$

(10)

Equations 6-9 then give the Hartree-Fock equation

$$
\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \gamma \delta(x) + U_H(x) + U_F(k,x) \right] \varphi_k(x) = \varepsilon_k \varphi_k(x)
$$

(11)

with the boundary condition

$$
\varphi_k(x + L) = \exp \left(i \frac{2\pi}{L} \frac{\phi}{\phi_0} \right) \varphi_k(x),
$$

(12)

where the potentials $U_H(x)$ and $U_F(k,x)$ are still given by equations 7 and 9, but with $\psi$ replaced by $\varphi$.

From equations 11 and 12 the groundstate energy $E_0 = \langle \Psi | H | \Psi \rangle$ can be expressed after simple algebra as

$$
E_0 = \sum_k \left[ \varepsilon_k - \frac{1}{2} \langle \varphi_k | U_H(x) + U_F(k,x) | \varphi_k \rangle \right].
$$

(13)

Setting this expression into 11 we obtain the persistent current in the Hartree-Fock approximation. Once $E_0(\phi)$ is known, $dE_0(\phi)/d\phi$ can be calculated numerically.
As in Refs. 11 and 12 we simplify the equation (9) as
\[ U_F(x) \simeq - \sum_{k'} \int dx' V(x-x') \text{Re} [\psi_{k'}^*(x') \psi_{k'}(x)] \tag{14} \]
by noticing that \( \sum_{k'} \psi_{k'}^*(x') \psi_{k'}(x) \simeq \delta(x-x') \). This 'almost closure relation' was tested in Fig. 2 of Ref. 11, so we do not repeat the same test here. Our comments on the reliability of approximation (14) are given at the end of Sect. IV. Unlike the exact form (11), the Fock potential \( U_F(x) \) is local and independent on \( k \). This saves computer time and memory and allows to study long rings.

In absence of the \( \delta \) barrier, the solution of equation (9) is \( \psi_k(x) = e^{ikx} \) and the potentials \( U_H(x) \) and \( U_F(x) \) become \( x \)-independent constants \( \phi_1 \) and \( \phi_2 \). These constants, \( U_H \) and \( U_F \), only introduce a rigid shift of the energy scale which has no physical effect. Therefore, in the following text we simply consider the potentials \( U_H(x) \) and \( U_F(x) \) reduced by constants \( U_H \) and \( U_F \), respectively. We now discuss the solution of equations (11) and (12) assuming that the potentials \( U_H(x) \) and \( U_F(x) \) are known. Consider first the ring region \( x \in (-L/2, L/2) \) as a straight-line segment of an infinite 1D wire. Inside the segment the potential is \( \epsilon \delta(x) + U_H(x) + U_F(x) \), outside we keep it zero. Therefore, the wave function outside is
\[ \varphi_k(x) = a e^{ikx} + b e^{-ikx}, \quad x < -L/2, \tag{15} \]
\[ \varphi_k(x) = c e^{ikx} + d e^{-ikx}, \quad x > L/2. \tag{16} \]
The amplitudes \( a \) and \( b \) are related to \( c \) and \( d \) by
\[ \begin{pmatrix} c \\ d \end{pmatrix} = T_0 \begin{pmatrix} a \\ b \end{pmatrix}, \tag{17} \]
where \( T_0 \) is the transfer matrix
\[ T_0 = \begin{pmatrix} 1 & 0 \\ \frac{t_k}{r_k} & \frac{1}{t_k} \end{pmatrix} \tag{18} \]
with \( t_k \) and \( r_k \) being the transmission and reflection amplitudes of the electron impinging the region \((-L/2, L/2)\) from the right. To come back to the ring threaded by magnetic flux, we relate \( \varphi_k(-L/2) \) and \( \varphi_k(L/2) \) (eqs. 15 and 16) through the boundary condition (12). Combining this relation with eqs. 17 and 18 we obtain the equation
\[ T \begin{pmatrix} a \\ b \end{pmatrix} = \exp(i2\pi \phi/\phi_0) \begin{pmatrix} a \\ b \end{pmatrix}, \tag{19} \]
where
\[ T = \begin{pmatrix} \frac{1}{t_k} e^{ikL} & -\frac{t_k}{\bar{r}_k} e^{ikL} \\ \frac{1}{t_k} e^{-ikL} & \frac{1}{\bar{r}_k} e^{-ikL} \end{pmatrix} \tag{20} \]
Thus \( \exp(i2\pi \phi/\phi_0) \) is the eigenvalue of matrix \( T \). The product of the eigenvalues of a matrix is its determinant, which is unity in this case. The second eigenvalue is thus \( \exp(-i2\pi \phi/\phi_0) \). Their sum is equal to the matrix trace, which gives the equation for the spectrum:
\[ \cos \left( \frac{2\pi \phi}{\phi_0} \right) = \text{Re} \left[ \frac{\exp(-i\pi L)}{t_k} \right]. \tag{21} \]
Numerical solution of equation (21) has to be combined with computation of the transmission amplitude \( t_k \). We compute \( t_k \) and \( r_k \) in the same way as described for the open wire. The solution of equation (21) gives us the dependence \( k(\phi) \) and finally \( \epsilon_k(\phi) = \hbar^2 k^2(\phi)/2m \).

For each \( k(\phi) \) we calculate the wave function \( \varphi_k(x) \) as follows. From eqs. (15) and (20) we obtain the amplitude
\[ a = \left[ \frac{1}{r_k} - \frac{t_k}{r_k} e^{i(2\pi \phi/\phi_0 + kL)} \right] b \tag{22} \]
and the amplitude \( b \) by normalizing the wave function. Then we express from equation (15) the boundary conditions \( \varphi_k(-L/2) \) and \( d\varphi_k(-L/2)/dx \). We solve the equation (14) with these boundary conditions by using the same finite-difference method as described for open wires.

Once we obtain \( \varphi_k(x) \), we can also perform the self-consistent Hartree-Fock calculation. In the first iteration step we solve equation (11) for the non-interacting gas. The resulting \( \varphi_k(x) \) is used to evaluate the Hartree and Fock potentials. These potentials are used in the second iteration step to obtain new \( \varphi_k(x) \) and new potentials, etc., until the energies \( \epsilon_k \) do not change any more.

In this work we apply the above Hartree-Fock scheme to the GaAs ring with electron density \( n = 5 \times 10^7 \text{ m}^{-1} \), effective mass \( m = 0.067 m_0 \), and e-e interaction
\[ V(x-x') = V_0 e^{-|x-x'|/d}, \tag{23} \]
where \( V_0 = 34 \text{ meV} \) and \( d = 3 \text{ nm} \). We adopt the finite-ranged interaction due to the following reasons. The e-e interaction in electron gas is always screened, so a reasonably chosen screened e-e interaction improves the Hartree-Fock approximation. The e-e interaction has already been used to study the many-body 1D systems, it has been shown to mimic the screened e-e interaction in a realistic 1D system quite well. Finally, we want to compare our study with the correlated models which also rely on the e-e interaction of finite range.

III. HARTREE-FOCK POTENTIAL

In the following we label the transmission and reflection amplitudes of the bare \( \delta \) barrier as \( t_k \) and \( r_k \). It holds that \( \bar{t}_k = k/(k+i\zeta) \) and \( \bar{r}_k = -i\zeta/(k+i\zeta) \), where \( \zeta = \gamma m/h^2 \). Since \( k_F \) and \( m \) are fixed, instead of using \( \gamma \) we parametrize the \( \delta \) barrier by its transmission coefficient \( |\bar{t}_{k_F}|^2 \).
The solid curve in figure 1 shows the self-consistent Hartree-Fock potential in the ring, induced by the scatterer at $x = 0$. The ring is 20 $\mu$m long, we show only a small region around the scatterer. The solid curve in fact involves two overlapping curves for the rings with magnetic flux $\phi \rightarrow 0^+$ and $\phi = 0.25\phi_0$, respectively. Notice that we find the same curve for any $\phi$ as long as we are not far from the $\delta$ barrier (see the next figure). For comparison, the dashed curve shows the result of Ref. 7, the self-consistent Hartree-Fock potential in a GaAs wire connected to leads, with the $\delta$ barrier and parameters as in our ring.

The length of the ring, $L$, is 20 $\mu$m (in the figure only a small region around the scatterer is shown). The solid curve in fact involves two overlapping curves for the rings with magnetic flux $\phi \rightarrow 0^+$ and $\phi = 0.25\phi_0$. Notice that we find the same curve for any $\phi$ as long as we are not far from the $\delta$ barrier (see the next figure). For comparison, the dashed curve shows the result of Ref. 7, the self-consistent Hartree-Fock potential in a GaAs wire connected to leads, with the $\delta$ barrier and parameters as in our ring.

Figure 2 shows all three curves along the entire half length. Here the Friedel oscillations are too dense to be distinguishable, but we can observe their asymptotic decay. We see in the top and middle panel, that the amplitude of the Friedel oscillations in the ring saturates at large distances from the scatterer. This saturation is due to the circular shape and was already observed in the pioneer Hartree-Fock study of the continuous ring.

However, the authors of Ref. 11 discussed also the wire with leads, but without performing any self-consistent Hartree-Fock study. They estimated analytically for extremely strong e-e interaction, that in the infinite wire the amplitude of the Friedel oscillations saturates with distance similarly as in the ring. In other words, at large distances from the scatterer $U_H(x) + U_F(x) \propto x^{-\beta} \cos(2k_F x + \text{const})$, but with $\beta = 0$. This is the charge-density-wave groundstate. In Ref. 6 this groundstate was concluded to hold for any e-e interaction strength, but without providing explicit self-consistent Hartree-Fock results. Nevertheless, due to this charge-density-wave groundstate, the self-consistent Hartree-Fock method was considered to be wrong.

---

**FIG. 1:** Self-consistent Hartree-Fock potential $U_H(x) + U_F(x)$ along a 1D ring, induced by the (not shown) barrier $\gamma \delta(x)$ due to the scatterer at $x = 0$. The $\delta$ barrier is adjusted to have the bare transmission $|\tilde{t}_{\delta, \beta}|^2 = 0.851$ at the Fermi level (14 meV). The length of the ring, $L$, is 20 $\mu$m (in the figure only a small region around the scatterer is shown). The solid curve in fact involves two overlapping curves for the rings with magnetic flux $\phi \rightarrow 0^+$ and $\phi = 0.25\phi_0$. Notice that we find the same curve for any $\phi$ as long as we are not far from the $\delta$ barrier (see the next figure). For comparison, the dashed curve shows the result of Ref. 7, the self-consistent Hartree-Fock potential in a GaAs wire connected to leads, with the $\delta$ barrier and parameters as in our ring.

**FIG. 2:** The same Hartree-Fock potentials as in the preceding figure, but along the entire half length. The top panel and middle panel show the results for the ring with $\phi \rightarrow 0^+$ and $\phi = 0.25\phi_0$, respectively. For comparison, the bottom panel shows the result for the GaAs wire with leads, taken from Ref. 6. Inset to each panel shows the same data in log scale, separately for the positive (top inset) and negative (bottom inset) data. Inset to the bottom panel shows, that the decay of Friedel oscillations in the wire is linear in log scale. It follows the straight (dashed) line with slope $\beta = 0.9$. A detailed analysis shows that $\beta \rightarrow 1$ at extremely large distances.
Recently, the self-consistent Hartree-Fock study has been applied to the continuous 1D wire with a single scatterer, connected to leads. It has been shown, that the charge-density-wave groundstate does not exist in that model at least for weak e-e interaction. In the bottom panel of figure 2 we show the result of Ref. 6 in order to compare with the Hartree-Fock potential in the ring. Inset to the bottom panel shows, that the asymptotic decay of Friedel oscillations in the wire is linear in log scale. This implies the dependence $\propto x^{-\beta}$ with positive $\beta > 0$. A detailed analysis [2] shows that $\beta \to 1$ in the limit of very long wires, i.e., there is no tendency to a non-decaying ($\beta = 0$) periodic wave. Saturation of the oscillation amplitude in the ring is due to the ring geometry and does not mean any charge-density-wave groundstate.

IV. RESULTS FOR PERSISTENT CURRENT

For noninteracting spinless electrons, the persistent current in the continuous 1D ring with a single scatterer can be derived analytically [2]. For even $N$ (considered throughout this paper) the result reads

$$I = \frac{e v_F}{\pi L} \arccos(|\tilde{t}_{k_F}| \cos(\phi' - \pi)) |\tilde{t}_{k_F}| \sin(\phi'),$$

(24)

where $\phi' = 2\pi \phi/\Phi_0$. Equation (24) holds also for the tight-binding model at half-filling [2]. For small $\tilde{t}_{k_F}$

$$I = \frac{e v_F}{2L} |\tilde{t}_{k_F}| \sin(\phi').$$

(25)

For the repulsive e-e interaction, as already mentioned,

$$I \propto L^{-\alpha-1} \sin(\phi')$$

(26)

within the Luttinger-liquid model [2]. Due to the e-e interaction the current vanishes faster than $1/L$.

These analytical predictions were tested in a numerical "experiment" of Ref. 6. The authors of that work studied a ring-shaped Hubbard lattice with nearest-neighbor hopping and nearest-neighbor e-e interaction. They found for the 1D ring with a single scatterer, that the current indeed vanishes faster than $1/L$ and is a sine-like function of $\phi'$. Their calculation was based on the renormalization-group (RG) technique allowing a highly reliable microscopic treatment of correlations.

Let us compare the RG calculation of Ref. 6 with our Hartree-Fock model. In panel $a$ of figure 3 we show the persistent current $LI(\phi')$ calculated by our self-consistent Hartree-Fock approach for $|\tilde{t}_{k_F}|^2 = 0.64$ and various $L$. For qualitative comparison, panels $b$ and $c$ show the data taken from Fig.3 of Ref. 6. These data were obtained for a 1D ring-shaped Hubbard lattice with the same $|\tilde{t}_{k_F}|^2$ and same $N$ as in panel $a$, where $N = 32, 64, \ldots$ for $L = 0.64 \mu m, 1.28 \mu m, \ldots$, respectively. In particular, panel $b$ shows the results of the RG solution while panel $c$ shows the results of the (non-self-consistent) Hartree-Fock solution. The RG solution accounts for the effect of correlations.

The authors of Ref. 6 also performed their own (non-self-consistent) Hartree-Fock calculation. For the purpose of comparison, the results of that calculation are shown in panel $c$ of Fig. 3. One can see that these results exhibit non-physical deviations from the sine-like shape. This non-physical behavior was ascribed to the absence of correlations in the Hartree-Fock model, but we can now ascribe it to the absence of self-consistency (our self-consistent Hartree-Fock results in panel $a$ clearly exhibit the sine-like shape).
In figure 4 we again compare our Hartree-Fock results with the RG results of Ref. [6], but for a stronger \( \delta \) barrier. We see a similar qualitative agreement as in Fig. 3, the peak of the current is now shifted closer to \( \phi' = \pi/2 \).

In figure 5 we present the persistent current \( LI(\phi') \) for \( |\tilde{t}_{k_F}|^2 = 0.03 \) and various \( L \). Panel a shows our self-consistent Hartree-Fock data, panel b the data typical of the Luttinger liquid (see the text).

The right hand side of (27) holds for small \( \tilde{t}_{k_F} \) and/or large \( L \). For small \( \tilde{t}_{k_F} \) one indeed obtains the scaling law

\[
I = \omega L^{-\alpha - 1} \sin(\phi'),
\]

where \( \omega = ev_F |\tilde{t}_{k_F}| d^\alpha / 2 |\tilde{r}_{k_F}| \). The formula (27) was derived by means of the RG method in the limit of weak e-e interaction \( (\alpha \ll 1) \). In that limit \( \alpha \) is given by

\[
\alpha = \frac{V(0) - V(2k_F)}{2\pi \hbar v_F},
\]

where \( V(q) \) is the Fourier transform of the e-e interaction \( V(x - x') \). The Fourier transform of our interaction

\[
\tilde{t}_{k_F} = \frac{\tilde{t}_{k_F}(d/L)\alpha}{\sqrt{|\tilde{r}_{k_F}|^2 + |\tilde{t}_{k_F}|^2 (d/L)^{2\alpha}}} \approx \frac{\tilde{t}_{k_F}(d/L)\alpha}{|\tilde{r}_{k_F}|^2 (d/L)^{2\alpha}}, \quad (27)
\]

where \( d \) is the range of the e-e interaction \( V(x - x') \) and
I that the Hartree-Fock data reproduce the universal law adjusted to fit the open symbols quantitatively. We see from the universal formula (29) and with the factor $\omega$ scaling law (28) with the power $\alpha$. In this calculation the prefactor $\omega$ scaling law (28) with power $\alpha$ was adjusted to fit the Hartree-Fock data quantitatively. The full lines in the right panel show the exact dependence (24) with $\tilde{\alpha}$, $\tilde{\omega}$, $\tilde{V}$ and $\tilde{\mu}$ replaced by the exact amplitude (27).

FIG. 6: Persistent current $LI(\phi' = \pi/2)$ as a function of $L$ for various $\delta$ barriers. Our Hartree-Fock results are shown by open symbols connected by dashed lines. The full lines in the left panel show the scaling law (28) with power $\alpha = 0.0855$. In this calculation the prefactor $\omega$ in eq. (28) was adjusted to fit the Hartree-Fock data quantitatively. The full lines in the right panel show the exact dependence (24) with $\tilde{\alpha}$, $\tilde{\omega}$, $\tilde{V}$ and $\tilde{\mu}$ replaced by the exact amplitude (27).

FIG. 7: Persistent current $LI(\phi' = \pi/2)$ versus $L$ for various $\delta$ barriers and various e-e interaction strength (various $\alpha$). Our Hartree-Fock results are shown by open symbols connected by dashed lines. The full lines show the scaling law (28) with the factor $\omega$ adjusted to fit the Hartree-Fock data quantitatively. In this calculation the powers $\alpha = 0.0277$ and $\alpha = 0.0561$ correspond to the the e-e interaction amplitudes $V_0 = 11$ meV and $V_0 = 22.3$ meV, respectively. Note that we can keep the same value of $\alpha$ by using the e-e interaction (23) with many different combinations of $V_0$ and $d$. However, as long as $\alpha$ remains the same, the choice of $V_0$ and $d$ has no effect on our results (Hartree-Fock potentials, eigenenergies, currents).
could be a purely accidental consequence of wrong approximation in the Hartree-Fock model.

Since it relies on the ‘almost closure relation’ \( \sum_k \psi_k^*(x') \psi_k(x) \simeq \delta(x-x') \), approximation (14) in fact underestimates the exact Fock potential (9) and overestimates the electron transmission and persistent current. Using of the exact Fock potential (9) would just remove a small quantitative difference between the Hartree-Fock and correlated results in the right panel of Fig. 6.

Why the approximation (14) works so well can be qualitatively understood as follows. It can be seen to be a perfect approximation of equation (9) in the extreme case \( V(x-x') \propto \delta(x-x') \). Thus, the shorter the range of the interaction \( V(x-x') \) the better the approximation (14). Our interaction (23) is short-ranged and in this respect more appropriate than the bare Coulomb interaction for which the approximation (14) seems to work well [10,11]. For \( V(x-x') \propto \delta(x-x') \) the Hartree and Fock potentials cancel to zero, the exponentially decaying e-e interaction (23) still produces a weak Hartree-Fock potential.

V. CONCLUDING REMARKS

In conclusion, using the self-consistent Hartree-Fock approximation at zero temperature, we have studied the persistent current of the weakly-interacting spinless electron gas in a continuous 1D ring with a single \( \delta \) barrier. We have compared our results with the results of correlated models, with the Luttinger liquid model and the microscopic Hubbard model. We have found a good agreement. In particular, in the limit of strong \( \delta \) barriers our self-consistent Hartree-Fock approximation reproduces the scaling of the persistent current in the form \( I \propto L^{-\alpha-1} \sin(2\pi\phi/\phi_0) \), where \( \alpha \) is a universal power, depending only on the e-e interaction. We can therefore conclude, that the universal power law is not exclusively the consequence of correlations; it can also be obtained in the self-consistent Hartree-Fock approximation.

We believe that the self-consistent Hartree-Fock approximation would confirm the \( L^{-\alpha-1} \) scaling also in the limit of weak \( \delta \) barriers, where the \( L^{-\alpha-1} \) scaling arises only at extremely large \( L \). Unfortunately, our present computer equipment is too slow for this task.

The self-consistent Hartree-Fock approximation has recently been used to study tunneling of the weakly-interacting electron gas through a single scatterer in a continuous 1D wire with leads. It has been found in the limit of strong \( \delta \) barriers, that the tunneling probability decays with the wire length as \( L^{-2\alpha} \), in accord with correlated models [8,9]. A quite similar agreement with correlated models shows our present Hartree-Fock study by reproducing the \( L^{-\alpha-1} \) scaling of persistent current [5,6].

Acknowledgments

M.M. was supported by the APVT grant APVT-51-021602, R.N. by the VEGA grant 2/3118/23.

* Electronic address: martin.mosko@savba.sk

1 Y. Imry, Introduction to Mesoscopic Physics (Oxford University Press, Oxford, UK, 2002).
2 L. P. Levy, G. Dolan, J. Dumsmuir, and H. Bouchiat, Phys. Rev. Lett. 64, 2074 (1990); V. Chandrasekar, R. A. Webb, M. J. Brady, M. B. Ketchen, W. J. Gallagher, and A. Kleinasser, Phys. Rev. Lett. 67, 3578 (1991); D. Mailly, C. Chapelier, and A. Benoit, Phys. Rev. Lett. 70, 2020 (1993); E. M. Q. Jariwala, P. Mohanty, M. B. Ketchen, and R. A. Webb, Phys. Rev. Lett. 86, 1594 (2001); W. Rabaud, L. Saminadayar, D. Mailly, K. Hasselbach, A. Benoit, and B. Etienne, Phys. Rev. Lett. 86, 3124 (2001).
3 Reviews of this issue can be found in S. Vieffers, P. Koskinen, P. S. Deo, and M. Manninen, Physica E 21, 1 (2004), in U. Eckern, and P. Schwab, J. Low Temp. Phys. 126, 1291 (2002), and in Ref. [1].
4 Y. Imry and R. Landauer, Rev. Mod. Phys. 71, S306 (1999).
5 A. O. Gogolin and N. V. Prokof’ev, Phys. Rev. B 50, 4921 (1994).
6 V. Meden and U. Schollwöck, Phys. Rev. B 67, 035106 (2003).
7 M. Mosko, A. Gendiar, P. Vagner, and Th. Schappers, submitted to Phys. Rev. B.
8 C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 68, 1220 (1992).
9 K. A. Matveev, D. Yue, and L. I. Glazman, Phys. Rev. Lett. 71, 3351 (1993); D. Yue, L. I. Glazman, and K. A. Matveev, Phys. Rev. B 49, 1966 (1994).
10 A. Cohen, K. Richter, and R. Berkovits, Phys. Rev. B 57, 6223 (1998).
11 A. Cohen, R. Berkovits, and A. Heinrich, Int. J. Mod. Phys. B 11, 1845 (1997).
12 Inspired by the Schrödinger/Poisson solver for inversion Si layers [T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. 54, 437 (1982)], we use a similar iterative trick.
13 N. W. Ashcroft, and N. D. Mermin, Solid state physics, Saunders College Publishing, Ed. D. G. Crane, Cornell University, USA 1976.
14 V. Meden, W. Metzner, U. Schollwöck, O. Schneider, T. Stauber, and K. Schönhammer, Europhys. J. B 16, 631 (2000).
15 J. H. Davies, The Physics of Low-Dimensional Semiconductors: An introduction (Cambridge University Press, Cambridge, UK, 1998).