Zeros of tree amplitudes at rest and symmetries of mechanical systems

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Abstract

We consider the tree amplitudes of production of $n_2$ scalar particles by $n_1$ particles of another kind, where both initial and final particles are at rest and on mass shell, in a model of two scalar fields with $O(2)$ symmetric interaction and unequal masses. We find that these amplitudes are zero except for the lowest possible $n_1$ and $n_2$, and that the cancellation of the corresponding Feynman graphs occurs due to a special symmetry of the classical mechanical counterpart of this theory. This feature is rather general and is inherent in various other scalar field theories.
1. Recently, powerful techniques have been developed for calculating tree amplitudes of the production of $n$ scalar particles at $n$-particle threshold by one or two initial particles (virtual or real) \([1, 2, 3, 4, 5, 6, 7, 8]\). The implementation of these techniques revealed an interesting property that some of the amplitudes are equal to zero. One type of the cancellation between the tree graphs occurs for processes involving two on-shell initial particles with non-zero spatial momenta and sufficiently large number of final particles, all of which are at rest. For example, in the theory of one scalar field with $\varphi^4$ self-interaction, the tree amplitudes of the scattering of two particles into $n$ particles at the threshold vanish for $n > 2$ in the case of unbroken reflection symmetry and for $n > 4$ in the case of broken symmetry \([4, 5]\). This property is related to the reflectionlessness of certain potentials in quantum mechanics \([3, 8]\). Different type of the zeros of the tree amplitudes at rest was found in ref.\([8]\).

The processes of the production of $n_1$ particles of the field $\varphi_1$ and $n_2$ particles of the field $\varphi_2$ by one initial particle were considered in the framework of the model with $O(2)$-symmetric interaction and unequal masses, whose lagrangian is

$$L = \frac{1}{2} (\partial_\mu \varphi_1)^2 + \frac{1}{2} (\partial_\mu \varphi_2)^2 - \frac{m_1^2}{2} \varphi_1^2 - \frac{m_2^2}{2} \varphi_2^2 - \lambda (\varphi_1^2 + \varphi_2^2)^2$$ \hspace{1cm} (1)

In the case of broken reflection symmetry $\varphi_1 \rightarrow -\varphi_1$, the production of $n$ particles $\varphi_2$ by one particle $\varphi_1$, where both initial and final particles are at rest and on-shell, is kinematically allowed under certain conditions on $m_1$ and $m_2$. It has been observed that the tree amplitude of this process vanishes at $n > 2$. No explanation of this property was given in ref.\([8]\), although it was suspected that it might be related to a specific symmetry of the interaction $(\varphi_1^2 + \varphi_2^2)^2$.

In this paper we extend this result of ref.\([8]\) and show that the tree amplitudes of the processes of the production of $n_2$ particles $\varphi_2$ by $n_1$ particles $\varphi_1$, all at rest and on-mass-shell, where $n_1$ and $n_2$ are coprime numbers up to one common divisor.
2, vanish in the model (1) except for the cases \( n_1 = n_2 = 2 \) when the reflection symmetry is unbroken and \( n_1 = 1, n_2 = 2 \) when the symmetry is broken. We also relate the cancellation of the corresponding tree diagrams to the integrability of the classical mechanical system with the hamiltonian

\[
H = \frac{1}{2}(\dot{\varphi}_1)^2 + \frac{1}{2}(\dot{\varphi}_2)^2 + \frac{m_1^2}{2}\varphi_1^2 + \frac{m_2^2}{2}\varphi_2^2 + \lambda(\varphi_1^2 + \varphi_2^2)^2
\]

which is obtained from eq.(1) by discarding the space dependence of \( \varphi_1 \) and \( \varphi_2 \). Namely, we show explicitly how the non-trivial symmetry \( \mathfrak{H} \) of the system (2) (which is the simplest case of the Garnier systems \( \mathfrak{H} \)) leads to the nullification of the tree amplitudes at rest.

2. For calculating the tree amplitudes at the threshold, two methods have been employed. One of them is based on recursion relations between diagrams with different numbers of final particles \( \mathfrak{H} \), and the other makes use of classical field equations with special boundary conditions \( \mathfrak{H} \). Let us extend the classical solution method to the case when the initial particles are also on mass shell and at rest.

Let us consider the model of two scalar fields, \( \varphi_1 \) and \( \varphi_2 \), with quartic interaction term,

\[
V(\varphi_1, \varphi_2) = \frac{m_1^2}{2}\varphi_1^2 + \frac{m_2^2}{2}\varphi_2^2 + \sum_{i,k=1,2} \lambda_{ik}\varphi_i^2\varphi_k^2
\]

Our purpose is to calculate the tree amplitudes of the production of \( n_2 \) particles \( \varphi_2 \) by \( n_1 \) particles \( \varphi_1 \), where all initial and final particles are on-shell and their spatial momenta are zero. This process is allowed by energy conservation when \( n_1m_1 = n_2m_2 \). We study the case when no disconnected diagrams exist, so we keep \( n_1 \) and \( n_2 \) coprime up to one common divisor 2.
The LSZ reduction formula for connected amplitudes reads,

\[ < n_2, \varphi_2 | n_1, \varphi_1 > = \prod_{a=1}^{n_2} \prod_{b=1}^{n_1} \int d^4x_a d^4y_b e^{i\epsilon_a x_a - i\epsilon_b y_b} (p_a^2 - m_1^2)(q_b^2 - m_2^2) \frac{\delta}{\delta j_1(x_a)} \frac{\delta}{\delta j_2(y_b)} W[j] |_{j=0} \]  

(3)

In the tree approximation one has

\[ W_{\text{tree}}[j] = S_{cl}[\varphi_1, \varphi_2] \]

where \( \varphi_1[j], \varphi_2[j] \) is the classical solution for the theory with the action

\[ S_{cl}^{(j)} = S_{cl}[\varphi_1, \varphi_2] + \int d^4x j_1 \varphi_1 + \int d^4x j_2 \varphi_2 \]

\( \varphi_1[j] \) and \( \varphi_2[j] \) have to obey the Feynman boundary conditions at \( t \to \pm \infty \). Notice that

\[ \frac{\delta S_{cl}^{(j)}}{\delta j_i} = \varphi_i[j] \]  

(4)

When all particles are at rest, i.e., all \( p_a = 0, \ q_b = 0 \), the space-time dependent sources \( j_i(x) \) and the solutions of the classical field equations \( \varphi_i(x) \) may be replaced by functions \( j_i(t), \ \varphi_i(t) \) depending only on time, and the classical field equations become ordinary differential ones. In this limit, according to eqs.(3) and (4), the mass-shell amplitudes are obtained by substituting

\[ j_1(t) = \rho_1 e^{-i\omega_1 t}, \quad j_2(t) = \rho_2 e^{i\omega_2 t}; \]

and then taking the limit \( \omega_i \to m_i \) in the expression

\[ A_{n_1, \varphi_1 \to n_2, \varphi_2} = (-i)^{n_1 + n_2} (\omega_1^2 - m_1^2)^{n_1} (\omega_2^2 - m_2^2)^{n_2} \left. \frac{\partial^{n_1-1}}{\partial \rho_1^{n_1-1}} \frac{\partial^n}{\partial \rho_2^n} \varphi_1(j, \omega) \right|_{j=0} \]  

(5)

So, we have to consider the following classical equations,

\[ \ddot{\varphi}_1 + m_1^2 \varphi_1 + 2 \sum_k \lambda_{1k} \varphi_1 \varphi_k^2 = \rho_1 e^{-i\omega_1 t} \]

\[ \ddot{\varphi}_2 + m_2^2 \varphi_2 + 2 \sum_k \lambda_{2k} \varphi_2 \varphi_k^2 = \rho_2 e^{i\omega_2 t} \]  

(6)
Let us apply the ordinary perturbation technique to this nonlinear system. To the zeroth order in $\lambda$ (free theory) the solution is

$$\varphi_1^{(0)} = z_1 = \zeta_1 e^{-i\omega_1 t}, \quad \varphi_2^{(0)} = z_2 = \zeta_2 e^{i\omega_2 t}, \quad (7)$$

where

$$\zeta_i = \frac{\rho_i}{m_i^2 - \omega_i^2}$$

At each subsequent step of the iteration procedure, we have to solve the following equations,

$$\ddot{\varphi}_i^{(k)} + m_i^2 \varphi_i^{(k)} = -\sum_j 2\lambda_{ij} (\varphi_i^{(0)} \varphi_j^{(2)})^{(k-1)} \quad (8)$$

where $(\varphi_i \varphi_j^{(2)})^{(k-1)}$ is of order $\lambda^{k-1}$ and therefore is expressed trough $\varphi_i^{(0)}, \varphi_i^{(1)}, \ldots \varphi_i^{(k-1)}$. At $\omega_i \neq m_i$, the perturbative solution is an expansion in $z_1$ and $z_2$, whose coefficients are finite in the limit $\omega_i \to m_i$ until a certain step. At this step (say, $l$-th), the resonance term (i.e. the term oscillating with the frequency $\pm m_i$ in the limit $\omega_i \to m_i$) appears for the first time on the right hand side of eq.(8), so that

$$\ddot{\varphi}_1^{(l)} + m_1^2 \varphi_1^{(l)} = i \frac{A_{n_1n_2}(\omega_1, \omega_2)}{(n_1 - 1)! n_2!} \zeta_1^{n_1-1} \zeta_2^{n_2} e^{i\omega_1 (n_1-1)\omega_1 - n_2 \omega_2} + \text{non-resonance terms} \quad (9)$$

Up to $\lambda^l$, the solution is

$$\varphi(z_1, z_2; \omega_1, \omega_2) = \sum_{(k,r)<(n_1,n_2)} C_{kr}(\omega) z_1^k z_2^r + i \frac{A_{n_1n_2}(\omega_1, \omega_2)}{(n_1 - 1)! n_2!} z_1^{n_1-1} z_2^{n_2} \quad (9)$$

where $C_{kl}(\omega)$ and $A_{n_1n_2}(\omega)$ are finite at $\omega_i \to m_i$. Making use of the definition of $z_i$, eq.(7), we obtain from eq.(9) the following expression for the on-shell amplitudes at rest,

$$A = \lim_{\omega_i \to m_i} \left( -i \right) (\omega_1^2 - m_1^2) \left. \frac{\partial^{n_2-1}}{\partial z_2^{n_2}} \frac{\partial^{n_1}}{\partial z_1^{n_1}} \varphi \right|_{z_i=0} \quad \varphi(z_1, z_2; \omega_1, \omega_2) \quad (10)$$

From eq.(8) we see that

$$A = A_{n_1n_2}(\omega_1, \omega_2) \mid_{\omega_i=m_i}$$

So the amplitudes are determined by the resonance term.
For the actual evaluation of the on-shell amplitudes at rest, it is more convenient to study the *sourceless* field equations, instead of the system (6). To reformulate the procedure, let us compare eq.(9) with the perturbative solution of the sourceless classical equations supplemented by the following conditions

\[ \varphi_i^{(0)} = \hat{z}_i \]  

(11)

where

\[ \hat{z}_1 = \zeta_1 e^{-im_1 t}, \quad \hat{z}_2 = \zeta_2 e^{im_2 t} \]  

(12)

The iteration procedure is again determined by eq.(8). At the \( l \)-th step the resonance term appears for the first time,

\[ \ddot{\varphi}_1^{(l)} + m_1^2 \varphi_1^{(l)} = i \frac{A_{n_1 n_2}(m_1, m_2)}{(n_1 - 1)! n_2!} \zeta_1^{n_1 - 1} \zeta_2^{n_2} e^{im_1 t} + \text{non-resonance terms} \]  

(13)

where \( A_{n_1 n_2}(m_1, m_2) \) is precisely the coefficient \( A_{n_1 n_2}(\omega_1, \omega_2) \) in eq.(9) taken at \( \omega_i = m_i \), i.e. \( A_{n_1 n_2}(m_1, m_2) \) is the amplitude of the process of interest (see eq.(10)). The resonance term in eq.(13) gives rise to the peculiar (growing linearly in time) term in the perturbative solution,

\[ \varphi_1^{(l)} = t e^{im_1 t} \frac{A_{n_1 n_2}(m_1, m_2)}{(n_1 - 1)! n_2!} \frac{1}{2m_1} \zeta_1^{n_1 - 1} \zeta_2^{n_2} + \text{oscillating terms} \]  

(14)

Thus, we obtain the following prescription for calculating the amplitude \( A \) for particles at rest at the tree level. One has to solve the homogeneous ordinary differential equation — the classical equation for space-independent fields — by the perturbation technique; the coefficient of the first peculiar term multiplied by \( 2m_1 (n_1 - 1)! n_2! \) is just equal to the required amplitude.

3. Let us show that there is no peculiar terms in the perturbative solution to classical equations when the corresponding mechanical system possesses a special kind of symmetry.
As an example, let us consider the model with softly broken $O(2)$ symmetry, determined by eq. (1). The corresponding classical system, eq. (2), is the simplest case of the Garnier system and so is integrable [10, 11, 12, 13, 14]. Besides the usual time translation, this system possesses a non-trivial symmetry whose infinitesimal form is

\[
\begin{align*}
\varphi_1 &\mapsto \tilde{\varphi}_1 = \varphi_1 + \epsilon \lambda \varphi_2 (\dot{\varphi}_1 \varphi_2 - \dot{\varphi}_2 \varphi_1) \\
\varphi_2 &\mapsto \tilde{\varphi}_2 = \varphi_2 + \epsilon \left[ \lambda \varphi_1 (\dot{\varphi}_2 \varphi_1 - \dot{\varphi}_1 \varphi_2) + \frac{m_1^2 - m_2^2}{2} \dot{\varphi}_2 \right]
\end{align*}
\]

where $\epsilon$ is a small parameter.

To see how this transformation changes the boundary conditions (11) and (12), we take the limit $\lambda \to 0$ on the right hand side of eq. (15). We obtain for the unbroken symmetry case ($< \varphi_1 > = < \varphi_2 > = 0$)

\[
\begin{align*}
\varphi_1^{(0)} &\mapsto \tilde{\varphi}_1^{(0)} = \varphi_1^{(0)}, \quad \varphi_2^{(0)} &\mapsto \tilde{\varphi}_2^{(0)} = \varphi_2^{(0)} + \epsilon \frac{m_1^2 - m_2^2}{2} \dot{\varphi}_2^{(0)},
\end{align*}
\]

so that

\[
\tilde{\zeta}_1 = \zeta_1, \quad \tilde{\zeta}_2 = \zeta_2 (1 + im_2 \frac{m_1^2 - m_2^2}{2} \epsilon)
\]

The solution $\varphi_{1,2}$ is determined by the parameters $\zeta_1, \zeta_2$; due to the uniqueness of the perturbative solution, we have the following identity

\[
\tilde{\varphi}_{1,2}(\zeta_1, \zeta_2) = \varphi_{1,2}(\tilde{\zeta}_1, \tilde{\zeta}_2)
\]

where the left hand side is given by eq. (14) and $\tilde{\zeta}_i$ on the right hand side are given by eq. (17). Let us compare the peculiar terms in this identity. We obtain from eq. (15) that up to order $\lambda^l$

\[
\tilde{\varphi}_1(\zeta) = \varphi_1(\zeta) + \text{oscillating terms} + O(\lambda^{l+1})
\]

while eqs. (14) and (17) give, again up to order $\lambda^l$,

\[
\varphi_1(\tilde{\zeta}) = \varphi_1(\zeta) + i\epsilon n_2 \frac{m_1^2 - m_2^2}{4} \zeta_1^{n_1-1} \zeta_2^{n_2} A_{n_1 n_2} \frac{A_{n_1 n_2}}{(n_1 - 1)! n_2!} + \text{oscillating terms} + O(\lambda^{l+1})
\]
The identity (18) is satisfied only when

\[(m_1^2 - m_2^2)A_{n_1n_2} = 0,\]

i.e. either \(A_{n_1n_2} = 0\) or \(m_1^2 = m_2^2\). This means that the amplitude of the production of \(n_2\) particles \(\varphi_2\) by \(n_1\) particles \(\varphi_1\), all at rest, does not vanish only for \(n_1 = n_2 = 2\) (recall that we study the case of coprime \(n_1/2\) and \(n_2/2\)).

This analysis can be generalized directly to the broken symmetry case, \(m_1^2 < 0\), \(m_2^2 > m_1^2\), when

\[\langle \varphi_1 \rangle = \frac{|m_1|}{2\sqrt{\lambda}} \quad (21)\]

and instead of eq.(16) one has

\[
\begin{align*}
\varphi_1^{(0)} &\mapsto \tilde{\varphi}_1^{(0)} = \varphi_1^{(0)}, \\
\varphi_2^{(0)} &\mapsto \tilde{\varphi}_2^{(0)} = \varphi_2^{(0)} + \epsilon \left( \frac{|m_1|^2}{4} - \frac{|m_1|^2 + m_2^2}{2} \right) \dot{\varphi}_2^{(0)} \\
&= \varphi_2^{(0)} - \epsilon \frac{|m_1|^2 + 2m_2^2}{4} \dot{\varphi}_2^{(0)}
\end{align*}
\]

So,

\[
\begin{align*}
\tilde{\zeta}_1 &= \zeta_1, \\
\tilde{\zeta}_2 &= \zeta_2 (1 - im_2\epsilon \frac{|m_1|^2 + 2m_2^2}{4})
\end{align*}
\]

Eq.(19) remains valid for the broken symmetry case, while instead of eq.(20) one has

\[
\varphi_1(\tilde{\zeta}) = \varphi_1(\zeta) + \epsilon n_2 \frac{|m_1|^2 + 2m_2^2}{8} \left( \frac{n_1 - 1}{n_1} \right) A_{n_1n_2} + \text{oscillating terms} + O(\lambda^{l+1})
\]

Thus, to satisfy the identity (18) one requires

\[(|m_1|^2 + 2m_2^2)A_{n_1n_2} = 0\]

Therefore, the amplitudes may not vanish only when \(|m_1|^2 + 2m_2^2 = 0\). Since the masses of the excitations around the vacuum (21) are \(m_{\varphi_1} = \sqrt{2}|m_1|, m_{\varphi_2} = \frac{1}{2}|m_1|\).
\[\sqrt{m_1^2 + m_2^2},\] this condition means that \[m_{\varphi_1}^2 = 2m_{\varphi_2}^2,\] i.e. the only non-vanishing tree amplitude at rest is that of the decay of a \(\varphi_1\)-particle into two \(\varphi_2\)-particles.

The absence of the peculiar terms in the perturbation series in this model may be demonstrated by constructing the explicit solution to all orders in \(\lambda\). For example, in the unbroken case, the solution obeying the conditions (11) and (12) is (cf. [8])

\[
\begin{align*}
\varphi_1 &= \hat{z}_1 (1 + \frac{\lambda}{2m_1^2} \hat{z}_2^2) \left( 1 - \frac{\lambda}{2m_1^2} \hat{z}_1^2 - \frac{\lambda}{2m_2^2} \hat{z}_2^2 + \frac{\kappa^2}{4m_1^2 m_2^2} \hat{z}_1^2 \hat{z}_2^2 \right)^{-1} \\
\varphi_2 &= \hat{z}_1 (1 + \frac{\lambda}{2m_1^2} \hat{z}_1^2) \left( 1 - \frac{\lambda}{2m_2^2} \hat{z}_1^2 - \frac{\lambda}{2m_2^2} \hat{z}_2^2 + \frac{\kappa^2}{4m_1^2 m_2^2} \hat{z}_1^2 \hat{z}_2^2 \right)^{-1}
\end{align*}
\]

where

\[
\kappa = \frac{m_1 + m_2}{m_1 - m_2}
\]

Clearly, the expansion of this solution in \(\lambda\) does not contain peculiar terms.

4. So, we find that the nullification of the tree amplitudes for \(\varphi_1\)-particles to create \(\varphi_2\)-particles, all particles being at rest and on mass shell, in the model (1) is directly related to the non-trivial symmetry of the corresponding Hamiltonian system of classical mechanics. The only relevant property of this symmetry is that the expression for the infinitesimal transformation for at least one of the fields \(\varphi_i\) contains a term that is linear in this field or in its derivative. Thus, in any theory of interacting scalar fields possessing the symmetry of the kind described above, the on-shell tree amplitudes at rest must vanish. Another example of such a model is the integrable version of Hénon-Heiles system with arbitrary masses [9, 10], which corresponds to the field theory with the lagrangian

\[L = \frac{1}{2}(\partial_\mu \varphi_1^2) + \frac{1}{2}(\partial_\mu \varphi_2^2) - \frac{m_1^2}{2} \varphi_1^2 - \frac{m_2^2}{2} \varphi_2^2 - \lambda \varphi_1^2 \varphi_2 - 2\lambda \varphi_2^3 (22)\]

The non-trivial symmetry of the corresponding space-independent hamiltonian is [9]

\[
\begin{align*}
\varphi_1 &\mapsto \varphi_1 + 2\lambda (\varphi_1 \varphi_2 - \varphi_2 \varphi_1) + \frac{4m_1^2 - m_2^2}{2} \dot{\varphi}_1, \\
\varphi_2 &\mapsto \varphi_2 + 2\lambda \varphi_1 \varphi_2
\end{align*}
\]
By repeating the above arguments one finds that the only non-vanishing tree amplitude at rest in the model (22) is that of the decay of one $\varphi_1$-particle into two $\varphi_2$-particles.

More examples of the nullification discussed in this paper may be constructed on the basis of wider classes of integrable classical systems, some of which can be found in ref. [10].

Of course, to a given order of the perturbation theory, the nullification of the tree amplitudes at rest for models like (1) or (22) can be seen by explicit evaluation of the Feynman diagrams. In that language, zeros of the amplitudes emerge as the result of cancellations between various diagrams weighted by their symmetry factors. The fact that these zeros are related to the symmetries of classical integrable systems gives the rationale for these, otherwise miraculous, cancellations.

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