On large orientation-reversing finite group-actions on 3-manifolds and equivariant Heegaard decompositions

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Abstract. We consider finite group-actions on closed, orientable and nonorientable 3-manifolds; such a finite group-action leaves invariant the two handlebodies of a Heegaard splitting of $M$ of some genus $g$. The maximal possible order of a finite group-action of an orientable or nonorientable handlebody of genus $g > 1$ is $24(g - 1)$, and in the present paper we characterize the 3-manifolds $M$ and groups $G$ for which the maximal possible order $|G| = 24(g - 1)$ is obtained, for some $G$-invariant Heegaard splitting of genus $g > 1$. If $M$ is reducible then it is obtained by doubling an action of maximal possible order $24(g - 1)$ on a handlebody of genus $g$. If $M$ is irreducible then it is a spherical, Euclidean or hyperbolic manifold obtained as a quotient of one of the three geometries by a normal subgroup of finite index of a Coxeter group associated to a Coxeter tetrahedron, or of a twisted version of such a Coxeter group.

1. Introduction

The maximal possible order of a finite group $G$ of orientation-preserving diffeomorphisms of an orientable handlebody of genus $g > 1$ is $12(g - 1)$ ([Z1], [MMZ]); for orientation-reversing finite group-actions on an orientable handlebody and for actions on a nonorientable handlebody, the maximal possible order is $24(g - 1)$; we will always assume $g > 1$ in the present paper.

Let $G$ be a finite group of diffeomorphisms of a closed, orientable or nonorientable 3-manifold $M$. We define the (equivariant) Heegaard genus of such a $G$-action as the minimal genus $g > 1$ of a Heegaard decomposition of $M$ into two handlebodies of genus $g$ (nonorientable if $M$ is nonorientable) such that both handlebodies are invariant under the $G$-action. Then $|G| \leq 24(g - 1)$, and in the maximal case $|G| = 24(g - 1)$ we call both the $G$-action and the 3-manifold strongly maximally symmetric (the term maximally symmetric is used in various papers for the case of orientation-preserving actions of maximal possible order $12(g - 1)$ on orientable 3-manifolds, see [Z2], [Z7] or the survey [Z4]). In the present paper we characterize the strongly maximally symmetric finite group-actions, using the approach to finite orientation-reversing group-actions on handlebodies in [Z3].
In order to state our results, we introduce some notation (see [T1], [T2] for the following). A **Coxeter tetrahedron** is a tetrahedron all of whose dihedral angles are of the form $\pi/n$ (denoted by a label $n$ of the edge, for some integer $n \geq 2$) and, moreover, such that at each of the four vertices the three angles of the adjacent edges define a spherical triangle (i.e., $1/n_1 + 1/n_2 + 1/n_3 > 1$). Such a Coxeter tetrahedron can be realized as a spherical, Euclidean or hyperbolic tetrahedron in the 3-sphere $S^3$, Euclidean 3-space $\mathbb{R}^3$ or hyperbolic 3-space $\mathbb{H}^3$, and will be denoted by $C(n, m; a, b; c, d)$ where $(n, m)$, $(a, b)$ and $(c, d)$ are the labels of pairs of opposite edges. We denote by $C_{\tau}(n, m; a, b; c, d)$ the Coxeter group generated by the reflections in the faces of $C(n, m; a, b; c, d)$, a properly discontinuous group of isometries of one of these three geometries.

We are interested in particular in the Coxeter tetrahedra $C(n, m) = C(n, m; 2, 2; 2, 3)$ and the corresponding Coxeter groups $C(n, m) = C(n, m; 2, 2; 2, 3)$ which are exactly the following:

- **spherical:** $C(2, 2)$, $C(2, 3)$, $C(2, 4)$, $C(2, 5)$, $C(3, 3)$, $C(3, 4)$, $C(3, 5)$;
- **Euclidean:** $C(4, 4)$;
- **hyperbolic:** $C(4, 5)$, $C(5, 5)$.

We consider also the Coxeter tetrahedra of type $C(n, m; 3, 3; 2, 2)$; such a Coxeter tetrahedron has a rotational symmetry $\tau$ of order two which exchanges the opposite edges with labels 3 and 2 and acts as an inversion on the two edges with labels $n$ and $m$. The involution $\tau$ can be realized by an isometry and hence defines a group of isometries $C_{\tau}(n, m)$ containing the Coxeter group $C(n, m; 3, 3; 2, 2)$ as a subgroup of index two; we call $C_{\tau}(n, m)$ a **twisted Coxeter group**. The twisted Coxeter groups of type $C_{\tau}(n, m)$ are the following:

- **spherical:** $C_{\tau}(2, 2)$, $C_{\tau}(2, 3)$, $C_{\tau}(2, 4)$;
- **Euclidean:** $C_{\tau}(3, 3)$;
- **hyperbolic:** $C_{\tau}(2, 5)$, $C_{\tau}(3, 4)$, $C_{\tau}(3, 5)$, $C_{\tau}(4, 4)$, $C_{\tau}(4, 5)$, $C_{\tau}(5, 5)$.

Our main result is:

**Theorem.** i) A reducible, strongly maximally symmetric $G$-manifold $M$ is obtained by doubling a $G$-action of maximal possible order $24(g-1)$ on an orientable or nonorientable handlebody of genus $g > 1$ (i.e., by taking the double along the boundary of both the handlebody and its $G$-action).

ii) An irreducible, strongly maximally symmetric $G$-manifold $M$ is spherical, Euclidean or hyperbolic and obtained as a quotient of the 3-sphere, Euclidean or hyperbolic 3-space by a normal subgroup of finite index, acting freely, of a Coxeter group $C(n, m)$ or a twisted Coxeter group $C_{\tau}(n, m)$; the $G$-action is obtained as the projection of the Coxeter or twisted Coxeter group to $M$.  

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There are only finitely many possibilities in the spherical case, and most of the finite group-actions are on the 3-sphere. The genera of the strongly maximally symmetric group-actions on $S^3$ can be computed from the orders of the spherical Coxeter groups (see [CM], [Z2, Table 1]), and one obtains:

**Corollary.** The genera of the strongly maximally symmetric finite group actions on the 3-sphere are $g = 2, 3, 5, 11, 6, 17$ and $601$ in the untwisted cases, and $g = 4, 11, 97$ in the twisted cases.

For orientation-preserving actions of maximal possible order $12(g−1)$ on the 3-sphere, the genera are determined in [WZ1, Theorem 3.1] and [WZ2], and there are in addition the values $g = 9, 25, 121$ and $241$.

In section 3 we consider also the case of $G$-actions of maximal possible order $48(g−1)$, allowing $G$-actions which interchange the two handlebodies of a Heegaard splitting of a 3-manifold $M$. We use methods from [Z3], in particular we correct and complete results in [Z3] where the twisted cases are missing. Our results for the spherical case are obtained also in [WWZ] where more general group actions of large orders on the 3-sphere are considered.

2. Proof of the Theorem

Let $G$ be a finite group of maximal possible order $24(g−1)$ which acts on a closed 3-manifold $M$ and leaves invariant the handlebodies $V_g$ and $V'_g$ of a Heegaard decomposition $M = V_g \cup G V'_g$ of genus $g > 1$. By [Z3], each of $V_g/G$ and $V'_g/G$ is a handlebody orbifold obtained by gluing two 3-disk orbifolds $D^3/G_1$ and $D^3/D_2$ (quotients of the closed 3-disk $D^3$ by spherical groups $G_1$ and $G_2$) along a common 2-disk suborbifold $D^2/D_n$ of their boundaries (a quotient of the 2-disk $D^2$ by a dihedral group $D_n$ of order $2n$), with orbifold Euler characteristic $1/|D_n|−1/|G_1|−1/|G_2|$ (see [T1], [T2] for basic facts about orbifolds, and [Z7] for the orientation-preserving case). For the case of maximal possible order $24(g−1)$, there are exactly eight such handlebody orbifolds which are the handlebody orbifolds of orbifold Euler characteristic $-1/24$, the largest possible value smaller than 0; since the orbifold Euler characteristic is multiplicative under finite orbifold coverings, $(-1/24)|G| = 1-g$, $|G| = 24(g−1)$. These minimal handlebody orbifolds can be codified by their orbifold fundamental groups $G_1 \ast_{p_n} G_2$ which, by [Z3, Theorem 1], are exactly the following eight free products with amalgamation:

$$D_2 \ast_{2_2} D_3, \quad D_3 \ast_{2_3} \tilde{A}_4, \quad D_4 \ast_{2_4} \tilde{S}_4, \quad D_5 \ast_{2_5} \tilde{A}_5,$$

$$D_2 \ast_{2_2} D_3, \quad D_3 \ast_{2_3} \tilde{A}_4, \quad D_4 \ast_{2_4} \tilde{S}_4, \quad D_5 \ast_{2_5} \tilde{A}_5;$$

here $D_n = [2, 2, n]$, $\tilde{A}_4 = [2, 3, 3]$, $\tilde{S}_4 = [2, 3, 4]$ and $\tilde{A}_5 = [2, 3, 5]$ denote the extended dihedral, tetrahedral, octahedral and dodecahedral groups (generated by the reflections
in the corresponding spherical triangles with angles $\pi/m$). The group $D_{2n}$ is a spherical group of order $4n$ (isomorphic to the dihedral group $\mathbb{D}_{2n}$), a subgroup of index two in the extended dihedral group $\tilde{\mathbb{D}}_{2n}$, with the standard dihedral action by rotations and reflections on the equatorial section $S^1$ of the 2-sphere $S^2$, the reflections corresponding alternatingly to rotations and reflections in great circles of $S^2$. The extended cyclic group $\mathbb{Z}_n$ of order $2n$ is isomorphic to the dihedral group $D_n$ and acts in the standard way by rotations and reflections on $D^2$, and also on $D^3$.

For example, in the case of $\tilde{\mathbb{D}}_3 * \mathbb{Z}_5 \tilde{\mathbb{A}}_5$ the associated handlebody orbifold $\mathcal{H}(\tilde{\mathbb{D}}_5 * \mathbb{Z}_5 \tilde{\mathbb{A}}_5)$ is obtained as follows. The 3-disk orbifold $D^3/\mathbb{Z}_5$ is a cone over its orbifold boundary, the spherical 2-orbifold $S^2/\tilde{\mathbb{D}}_5$ which is just a triangle with angles $\pi/2, \pi/2$ and $\pi/5$; the singular points of this triangle consists of its three sides which are reflection axes with local groups $\mathbb{Z}_2$, seperated by the three vertices with local groups $\mathbb{D}_2, \mathbb{D}_2$ and $\mathbb{D}_5$. In a similar way, the 3-disk orbifold $D^3/\tilde{\mathbb{A}}_5$ is constructed. In their boundaries, both 3-disk orbifolds $D^3/\tilde{\mathbb{D}}_5$ and $D^3/\tilde{\mathbb{A}}_5$ have a 2-disk suborbifold $D^2/\tilde{\mathbb{Z}}_5$ which is a triangle: two of its sides are reflection axes meeting in a dihedral point $\tilde{\mathbb{D}}_5$, the third side is a nonsingular arc (except for its endpoints) which constitutes the orbifold boundary of $D^2/\tilde{\mathbb{Z}}_5$. The handlebody orbifold $\mathcal{H}_5 = \mathcal{H}(\tilde{\mathbb{D}}_5 * \mathbb{Z}_5 \tilde{\mathbb{A}}_5)$ is then obtained by gluing $D^3/\tilde{\mathbb{D}}_5$ and $D^3/\tilde{\mathbb{A}}_5$ along the 2-suborbifolds $D^2/\tilde{\mathbb{Z}}_5$ in their boundaries (or better, by connecting the two 3-disk orbifolds by a 1-handle orbifold $(D^2/\tilde{\mathbb{Z}}_5) \times [-1, 1]$); note that the orbifold boundary of this handlebody orbifold is a square $D^2([2, 2, 2, 3])$ whose sides are reflection axes meeting in three dihedral points $\mathbb{D}_2$ and one $\mathbb{D}_3$.

Applying the construction to the first four amalgamated free products, one obtains the four handlebody orbifolds

$$
\mathcal{H}_2 = \mathcal{H}(\tilde{\mathbb{D}}_2 *_{\mathbb{Z}_2} \tilde{\mathbb{D}}_3), \quad \mathcal{H}_3 = \mathcal{H}(\tilde{\mathbb{D}}_3 *_{\mathbb{Z}_3} \tilde{\mathbb{A}}_4), \quad \mathcal{H}_4 = \mathcal{H}(\tilde{\mathbb{D}}_4 *_{\mathbb{Z}_4} \tilde{\mathbb{A}}_5), \\
\mathcal{H}_5 = \mathcal{H}(\tilde{\mathbb{D}}_5 *_{\mathbb{Z}_5} \tilde{\mathbb{A}}_5),
$$

and each of these four handlebody orbifolds has the square $D^2([2, 2, 2, 3])$ as its orbifold boundary.

For the remaining four free products with amalgamation one constructs in a similar way the handlebody orbifolds

$$
\tilde{\mathcal{H}}_2 = \mathcal{H}(\tilde{\mathbb{D}}_{2*2} *_{\mathbb{Z}_2} \tilde{\mathbb{D}}_3), \quad \tilde{\mathcal{H}}_3 = \mathcal{H}(\tilde{\mathbb{D}}_{2*3} *_{\mathbb{Z}_3} \tilde{\mathbb{A}}_4), \quad \tilde{\mathcal{H}}_4 = \mathcal{H}(\tilde{\mathbb{D}}_{2*4} *_{\mathbb{Z}_4} \tilde{\mathbb{A}}_5), \\
\tilde{\mathcal{H}}_5 = \mathcal{H}(\tilde{\mathbb{D}}_{2*5} *_{\mathbb{Z}_5} \tilde{\mathbb{A}}_5).
$$

For example, the quotient orbifold $D^3/\tilde{\mathbb{D}}_{2*5}$ is a cone over its orbifold boundary $S^2/\tilde{\mathbb{D}}_{2*5}$ which is a disk with a singular point $\mathbb{Z}_2$ in its interior whose boundary consists of a unique reflection axis starting and finishing in a singular point $\tilde{\mathbb{D}}_5$. Since $S^2/\tilde{\mathbb{D}}_{2*5}$ has again a 2-disk suborbifold $D^2/\tilde{\mathbb{Z}}_5$, one can construct the handlebody orbifold $\mathcal{H}(\tilde{\mathbb{D}}_{2*5} *_{\mathbb{Z}_5} \tilde{\mathbb{A}}_5)$ as before.
Note that the orbifold boundary of each of the four handlebody orbifolds \( \tilde{\mathcal{H}}_n \) is a 2-disk \( \mathcal{D}^2(2, [2, 3]) \) whose boundary consists of two reflection axes intersecting in two dihedral points \( \mathbb{D}_2 \) and \( \mathbb{D}_3 \), and with a singular point \( \mathbb{Z}_2 \) in its interior; note that \( \mathcal{D}^2(2, [2, 3]) \) is a quotient of the square orbifold \( \mathcal{D}^2([2, 2, 2, 3]) \) by a rotational involution \( \tau \).

**Remark.** In the case of orientation-preserving actions on orientable handlebodies and 3-manifolds the maximal order is \( 12(g - 1) \), and the orbifold fundamental groups of the minimal orientable handlebody orbifolds, of Euler characteristic \(-1/12\), are the following four amalgamated free products:

\[
\mathbb{D}_2 \ast_{\mathbb{Z}_2} \mathbb{D}_3, \quad \mathbb{D}_3 \ast_{\mathbb{Z}_3} \mathbb{A}_4, \quad \mathbb{D}_4 \ast_{\mathbb{Z}_4} \mathbb{S}_4, \quad \mathbb{D}_5 \ast_{\mathbb{Z}_5} \mathbb{A}_5
\]

where \( \mathbb{D}_n = (2, 2, n) \), \( \mathbb{A}_4 = (2, 3, 3) \), \( \mathbb{S}_4 = (2, 3, 4) \) and \( \mathbb{A}_5 = (2, 3, 5) \) denote the dihedral, tetrahedral, octahedral and dodecahedral groups, spherical triangle groups which are the orientation-preserving subgroups of index two in the corresponding extended groups (see [Z7]).

Returning to the \( G \)-action on the 3-manifold \( M = V_g \cup_\partial V_g' \), the quotient orbifold \( M/G = (V_g/G) \cup_\partial (V_g'/G) \) is obtained by identifying the minimal handlebody orbifolds \( V_g/G \) and \( V_g'/G \) along their boundaries, and both \( V_g/G \) and \( V_g'/G \) are are one of the eight minimal types described above. Since the boundary of an orbifold of type \( \mathcal{H}_n \) is not homeomorphic to that of an orbifold of type \( \tilde{\mathcal{H}}_m \), both orbifolds \( V_g/G \) and \( V_g'/G \) are of the same type.

Suppose first that \( V_g/G = \mathcal{H}_n \) and \( V_g'/G = \mathcal{H}_m \). The boundary of both \( \mathcal{H}_n \) and \( \mathcal{H}_m \) is the square orbifold \( \mathcal{D}^2([2, 2, 2, 3]) \); up to isotopy, this square has exactly two orbifold homeomorphisms which are the identity map and a reflection in a diagonal of the square (connecting the two opposite vertices of type \( \mathbb{D}_2 \) and \( \mathbb{D}_3 \) and exchanging the other two vertices of type \( \mathbb{D}_2 \)).

Suppose that the boundaries of the handlebody orbifolds \( \mathcal{H}_n \) and \( \mathcal{H}_m \) are identified by the identity map of \( \mathcal{D}^2([2, 2, 2, 3]) \). As explained before, both handlebody orbifolds \( \mathcal{H}_n \) and \( \mathcal{H}_m \) are constructed by identifying two 3-disk orbifolds along a 2-disk suborbifold \( \mathcal{D}^2/\mathbb{Z}_n \) and \( \mathcal{D}^2/\mathbb{Z}_m \) in their boundaries. Identifying the boundaries of \( \mathcal{H}_n \) and \( \mathcal{H}_m \) by the identity map, these 2-disk suborbifolds \( \mathcal{D}^2/\mathbb{Z}_n \) and \( \mathcal{D}^2/\mathbb{Z}_m \) fit together along their boundaries and create a 2-disk suborbifold of \( M/G \) whose boundary consists of two reflection axes meeting in dihedral points \( \mathbb{Z}_n \cong \mathbb{D}_n \) and \( \mathbb{Z}_m \cong \mathbb{D}_m \). If \( n \neq m \), this 2-disk is a bad 2-orbifold, i.e. not covered by a manifold, which is a contradiction since we have the manifold covering \( M \) of \( M/G \). Hence \( n = m \) and \( M/G \) is obtained by doubling \( \mathcal{H}_n \) along the boundary. Then also \( M \) is obtained by doubling the handlebody \( V_g \) and its \( G \)-action along the boundary, so we are in part i) of the Theorem.

Suppose then that the boundaries of \( \mathcal{H}_n \) and \( \mathcal{H}_m \) are identified by a reflection in a diagonal of the square \( \mathcal{D}^2([2, 2, 2, 3]) \). The Coxeter group \( C(n, m) \) acts on the 3-sphere,
Euclidean or hyperbolic 3-space, and the quotient orbifold of this action is the Coxeter tetrahedron \(C(n, m)\): the underlying topological space is the 3-disk, and the singular set consists of the boundary of the tetrahedron (the local group associated to a point is its stabilizer in the Coxeter group). The Coxeter orbifold \(C(n, m)\) has a 2-suborbifold \(D^2([2, 2, 2])\) (a square whose vertices are on the four edges of the tetrahedron with labels 2,2,2 and 3, seperating the two edges with labels \(n\) and \(m\)), and \(D^2([2, 2, 2])\) seperates \(C(n, m)\) into the two handlebody orbifolds \(H_n\) and \(H_m\). So in this case the quotient \(M/G\) is the Coxeter tetrahedral orbifold \(C(n, m)\) and we are in part ii) of the Theorem.

We are left with the cases \(V_g/G = \hat{H}_n\) and \(V_g'/G = \hat{H}_m\). The boundary of each of these minimal handlebody orbifolds is the 2-disk \(D^2([2, 2, 3])\) (seperating the two edges with labels \(n\) and \(m\)) which is invariant under the involution \(\tau\) of \(C(n, m; 3, 3; 2, 2)\). The projection of \(D^2([3, 3, 2, 2])\) to the twisted Coxeter orbifold \(C_\tau(n, m) = C(n, m; 3, 3; 2, 2)/\tau\) which is homeomorphic to the orbifold \(D^2([2, 2, 3])\) and seperates \(C_\tau(n, m)\) into two handlebody orbifolds \(\hat{H}_n\) and \(\hat{H}_m\) (e.g., the quotient of the 1-handle orbifold \(D^2/\mathbb{Z}_5 \times [-1/2, 1/2]\) by the involution \(\tau\) is the 3-disk orbifold \(D^3/\mathbb{D}_{2*5}\), and hence the quotient of \(H(\tilde{A}_5 * \mathbb{Z}_5 \tilde{A}_5)\) by \(\tau\) gives the handlebodly orbifold \(\hat{H}_5 = H((\mathbb{D}_{2*5} * \mathbb{Z}_5 \mathbb{A}_5))\). So by identifying \(\hat{H}_n\) and \(\hat{H}_m\) along their boundaries we obtain the twisted Coxeter orbifold \(C_\tau(n, m)\), and we are in case ii) of the Theorem.

This completes the proof of the Theorem.

3. Examples and comments

The maximal possible order of a \(G\)-action of a closed 3-manifold \(M\) which leaves invariant a Heegaard surface of genus \(g > 1\) is \(48(g - 1)\); in this maximal case, some element of \(G\) has to interchange the two handlebodies of the Heegaard splitting, and the subgroup of index two preserving both handlebodies gives a strongly maximally symmetric \(G\)-action on \(M\).

Suppose that \(M\) is irreducible. Then the subgroup of index two preserving both handlebodies of the Heegaard splitting is obtained from a Coxeter group \(C(n, m)\) or twisted
Coxeter group $C_\tau(n, m)$ as in the Theorem. By the geometrization of finite group actions in dimension 3, we can assume that the whole group $G$ acts by isometries; lifting $G$ to the universal covering, we obtain a group of isometries of $S^3$, $\mathbb{R}^3$ of $\mathbb{H}^3$ containing the Coxeter group $C(n, m)$ or the twisted Coxeter group $C_\tau(n, m)$ as subgroup of index two, and in the second case it contains the Coxeter group $C(n, m; 3, 3; 2, 2) \subset C_\tau(n, m)$ as a subgroup of index 4. Now any 2-fold or 4-fold extension of such a Coxeter group is obtained by adjoining the symmetry group $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ of rotations of the Coxeter tetrahedron to the Coxeter group (see [M] for the hyperbolic case, the other cases are similar), and clearly the presence of such symmetries requires $n = m$.

We denote by $C_\mu(n, n)$ the twisted group generated by $C(n, n) = C(n, n; 2, 2; 2, 3)$ and the involution $\mu$ of the Coxeter tetrahedron $C(n, n; 2, 2; 2, 3)$ which exchanges the two edges with label $n$ and inverts the other four edges, and by $C_{\tau\mu}(n, n)$ the doubly twisted group generated by $C(n, n; 3, 3; 2, 2)$ and the involutions $\tau$ and $\mu$ of the Coxeter tetrahedron $C(n, n; 3, 3; 2, 2)$. The possibilities are then the following:

spherical: $C_\mu(2, 2)$, $C_\mu(3, 3)$, $C_{\tau\mu}(2, 2)$;
Euclidean: $C_\mu(4, 4)$, $C_{\tau\mu}(3, 3)$;
hyberbolic: $C_\mu(5, 5)$, $C_{\tau\mu}(4, 4)$, $C_{\tau\mu}(5, 5)$.

Summarizing, we have:

**Proposition.** Let $M$ be closed, irreducible 3-manifold with a $G$-action of maximal possible order $48(g - 1)$ which leaves invariant the Heegaard surface of a Heegaard splitting of genus $g > 1$ of $M$. Then $M$ is obtained as the quotient of the 3-sphere, Euclidean or hyperbolic 3-space by a normal subgroup of finite index, acting freely, of a twisted Coxeter group $C_\mu(n, n)$ or a doubly twisted Coxeter group $C_{\tau\mu}(n, n)$, and the $G$-action is obtained as the projection of $C_\mu(n, n)$ or $C_{\tau\mu}(n, n)$ to $M$. If $M$ is the 3-sphere, the possible genera are $g = 2, 4$ and 6.

Finally, we discuss an explicit example of a hyperbolic, strongly maximally symmetric 3-manifold $M$ for which, moreover, also the bound $48(g - 1)$ is obtained: this is the hyperbolic 3-manifold $M$ considered in [Z5], [Z7], see also [Z6] for some further properties. The universal covering group of $M$ is a normal torsionfree subgroup of smallest possible index 120 in the Coxeter group $C(5, 5; 3, 3; 2, 2)$, and of index 240 = $24(g - 1)$ in its twisted version $C_\tau(5, 5)$. In particular, $M$ is a strongly maximally symmetric $G$-manifold of genus $g = 11$ (and, by [Z5], also the ordinary Heegaard genus of $M$ is equal to 11). Moreover, it follows from [Z6, Propositions 3.2 and 3.3] that the universal covering group of $M$ is a normal subgroup also of the doubly twisted Coxeter group $C_{\tau\mu}(n, n)$, so $C_{\tau\mu}(n, n)$ projects to an isometry group of order $480 = 48(g - 1)$ of $M$ (which is, in fact, the full isometry group of $M$), and we are in the situation of the Proposition.
We believe that the genus 11 of $M$ is the smallest equivariant genus of a hyperbolic $G$-manifold for which the bound $48(g - 1)$ in the Proposition is obtained and, more generally, also the smallest genus of any strongly maximally symmetric hyperbolic 3-manifold. For this, one has to check the minimal indices of the torsionfree normal subgroups of the other hyperbolic Coxeter and twisted Coxeter groups.

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