LIPSCHITZ PROPERTY OF HARMONIC MAPPINGS WITH RESPECT TO PSEUDO-HYPERBOLIC METRIC

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Abstract. In this paper, we show that harmonic Bloch mappings are Lipschitz continuous with respect to the pseudo-hyperbolic metric. This result improves the corresponding result of [11, Theorem 1]. Furthermore, we prove the similar property for harmonic quasiregular Bloch-type mappings.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle, and $\overline{\mathbb{D}}$ the closure of $\mathbb{D}$, i.e., $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. For $z \in \mathbb{C}$, the partial derivatives of a complex-valued function $f$ are defined by

\begin{equation}
    f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).
\end{equation}

For $z = re^{i\theta} \in \mathbb{C}$ and $\alpha \in [0, 2\pi]$, the directional derivative of $f$ is defined by

\begin{equation}
    \partial_{\alpha} f(z) = \lim_{r \to 0^+} \frac{f(z + re^{i\alpha}) - f(z)}{r} = e^{i\alpha} f_z(z) + e^{-i\alpha} f_{\overline{z}}(z).
\end{equation}

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Then
\begin{equation}
\Lambda_f(z) := \max_{0 \leq \alpha \leq 2\pi} |\partial_\alpha f(z)| = |f_z(z)| + |f_{\bar{z}}(z)|
\end{equation}

and
\begin{equation}
\lambda_f(z) := \min_{0 \leq \alpha \leq 2\pi} |\partial_\alpha f(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||.
\end{equation}

**Complex dilatation.** A complex-valued function $f$ of the class $C^2$ is said to be harmonic mapping, if it satisfies $\Delta f := 4f_{z\bar{z}} = 0$. Moreover, it was shown in [13] that a function $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its non-vanishing Jacobian $J_f(z) := |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0$, i.e., the dilatation of $f$

$$|\omega_f(z)| = \frac{|f_{\bar{z}}(z)|}{|f_z(z)|} < 1$$

in $\mathbb{D}$.

Assume that $f$ is a harmonic mapping defined in a simply connected domain $\Omega \subseteq \mathbb{C}$. Then $f$ has the canonical decomposition $f = h + g$, where $h$ and $g$ are analytic in $\Omega$. For a sense-preserving harmonic mapping $f$ in $\mathbb{D}$, let

$$\omega(z) = \frac{g'(z)}{h'(z)}$$

be the (second) complex dilatation of $f$. Then $\omega(z)$ is holomorphic mapping of $\mathbb{D}$ and

$$\|\omega\|_\infty := \sup_{z \in \mathbb{D}} |\omega(z)| \leq 1.$$

In this paper, we consider locally univalent and sense-preserving harmonic mappings in $\mathbb{D}$. For basic properties of harmonic mappings, we refer to [9].

It is worth of noting that the composition $f \circ \varphi$ of a harmonic mapping $f$ with a conformal mapping $\varphi$ is a harmonic mapping. Therefore, it is sufficient to consider harmonic mappings defined in the unit disk $\mathbb{D}$. However, $\varphi \circ f$ is not harmonic in general.

**Pseudo-hyperbolic distance.** Fix $w \in \mathbb{D}$ and let $\varphi_w$ be the Möbius transformation of $\mathbb{D}$, that is,

$$\varphi_w(z) = \frac{w - z}{1 - \bar{w}z}, \quad \text{where } z \in \mathbb{D}.$$ 

The pseudo-hyperbolic distance on $\mathbb{D}$ is defined by

$$\rho(z, w) = |\varphi_w(z)|.$$
The pseudo-hyperbolic distance is invariant under Möbius transformations, that is,
\[ \rho(g(z), g(w)) = \rho(z, w), \]
for all \( g \in \text{Aut}(\mathbb{D}) \), the Möbius automorphisms of \( \mathbb{D} \). It has the following useful property (which will be used in proving Theorem 1.4 below):
\[ (1.5) \quad 1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2} = 1 - |w|^2|\varphi'_z(w)|. \]

**Lipschitz continuity with respect to pseudo-hyperbolic metric.** The classical Bloch space for analytic functions are defined as follows.

**Definition 1.1.** We call a function \( h \) a Bloch function (and write \( h \in B \)) if \( h \) is analytic in \( \mathbb{D} \) and
\[ \|h\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|h'(z)| < \infty. \]

The above formula defines a seminorm, and the Bloch functions form a complex Banach space \( B \) with the norm
\[ \|h\|_B = |h(0)| + \|h\|_B. \]

The Bloch space has been considered in many different contexts. For example, very recently, Bohr radius has been established for analytic Bloch spaces in a more general setting (see [14]).

A mapping \( f(z) \) is said to be Lipschitz (resp. co-Lipschitz) in \( \mathbb{D} \) if there exists a constant \( L \) such that the following inequality
\[ \frac{|z_1 - z_2|}{L} \leq |f(z_1) - f(z_2)| \quad (\text{resp. } |f(z_1) - f(z_2)| \leq L|z_1 - z_2|) \]
holds for all \( z_1, z_2 \in \mathbb{D} \), where \( L \geq 1 \) is called the Lipschitz constant. The function \( f \) is said to be bi-Lipschitz if \( f \) is Lipschitz and co-Lipschitz.

It is easy to see that the condition \( h \in B \) does not ensure that \( h \) is a Lipschitz mapping. For example, take \( h(z) = \log(1 - z^2) \), for \( z \in \mathbb{D} \). Then \( h \in B \) since \( \|h\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|h'(z)| \leq 2 \). However, for \( z_1 = x \in (0, 1) \), choose arbitrarily small \( t > 0 \) such that \( z_2 = x + t \in (0, 1) \). Then
\[ \left| \frac{h(z_1) - h(z_2)}{z_1 - z_2} \right| \geq \frac{1}{1 - x} \to \infty \quad \text{as} \quad x \to 1. \]

This shows that \( h \) is not a Lipschitz mapping.

Let
\[ C_{\varphi_w} h(z) = h \circ \varphi_w(z), \]

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where $z, w \in \mathbb{D}$. Then $|C'_{\varphi_w} h(0)| = (1 - |w|^2)|h'(w)|$. In [11], Ghatage, Yan and Zheng showed that $C'_{\varphi_w} h$ is a Lipschitz function with respect to pseudo-hyperbolic metric. They also used this result to study the composition operators $C_{\varphi_w}$ on the Bloch space. In fact, they proved the following theorem.

**Theorem A** [11, Theorem 1]. Let $h$ be in the Bloch space. Then the inequality

$$|(1 - |z|^2)|h'(z)| - (1 - |w|^2)|h'(w)|| \leq 3.31 \rho(z, w)\|h\|_B,$$

holds for all $z, w \in \mathbb{D}$.

Here the constant 3.31 is not sharp, The sharp constant $3\sqrt{3}/2$ was given later by C. Xiong [17].

**Harmonic Bloch space and Bloch-type space.** Analogue to the classical analytic Bloch space, one can define the harmonic Bloch space as follows.

**Definition 1.2.** A harmonic mapping $f$ in $\mathbb{D}$ is called a harmonic Bloch mapping (in sign: $f \in B_h$) if

$$\|f\|_{B_h} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \Lambda_f(z) < \infty.$$

This defines a seminorm, and the space equipped with the norm

$$\|f\|_{B_h} = |f(0)| + \|f\|_{B_h}$$

is called the harmonic Bloch space. It is a Banach space. Clearly, $f = h + \overline{g} \in B_h$ if and only if $h \in B$ and $g \in B$, since

$$\max\{\|h\|_B, \|g\|_B\} \leq \|f\|_{B_h} \leq \|h\|_B + \|g\|_B.$$

The harmonic Bloch space was studied by Colonna [7] as a generalization of the classical Bloch space. We refer to [1,2,6,15,16] and the references therein for more information on $B_h$. More recently, authors in [8,12] investigated extreme points and support points of harmonic and harmonic $\alpha$-Bloch mappings.

Motivated by a number of well-known results on analytic Bloch functions, in [10,14] the authors introduced the harmonic Bloch-type mappings, defined as follows.

**Definition 1.3.** A harmonic mapping $f$ in $\mathbb{D}$ is called a harmonic Bloch-type mapping if

$$\|f\|_{B_h^*} = \sup_{z \in \mathbb{D}} (1 - |z|^2) \sqrt{\left|J_f(z)\right|} < \infty.$$
We denote this class of functions by $B^*_h$ and call the quantity 

$$\|f\|_{B^*_h} = |f(0)| + \|f\|_{B_h}$$

the Bloch-type pseudo-norm of $f$.

It is easy to see that $B_h \subseteq B^*_h$, because $\sqrt{|J_f(z)|} \leq \Lambda_f(z)$, for each $z \in \mathbb{D}$.

**Motivations.** Estimates of the directional derivatives and coefficients, establishing Schwarz lemmas, for harmonic Bloch mappings, or harmonic Bloch-type mappings, and their generalizations have been studied by several authors in, see for example, [3–7,14]. In this paper, our primary goal is to improve the above Theorem A in the case of harmonic Bloch-type mappings. We first improve Theorem A for harmonic Bloch mappings and Bloch-type mappings as follows:

**Theorem 1.4.** Let $f$ be in $B_h$ space. Then the inequality

$$\left|(1-|z|^2)\Lambda_f(z) - (1-|w|^2)\Lambda_f(w)\right| \leq 3\sqrt{3}\rho(z,w)\|f\|_{B_h},$$

holds for all $z, w \in \mathbb{D}$.

Suppose $f(z)$ is a sense-preserving harmonic mapping of $\mathbb{D}$ into a domain $\Omega \subseteq \mathbb{C}$. Then $f(z)$ is a harmonic $K$-quasiregular mapping, if

$$K(f) := \sup_{z \in \mathbb{D}} \frac{|f_z(z)| + |f_\bar{z}(z)|}{|f_z(z)| - |f_\bar{z}(z)|} \leq K,$$

where $K \geq 1$ is a constant.

We show in Lemma 2.2 below that if $f$ is a harmonic $K$-quasiregular mapping in $\mathbb{D}$, then $f \in B^*_h$ if and only if $f \in B_h$. Moreover, by using quasiregularity, we generalize Theorem A as follows.

**Theorem 1.5.** Let $f$ be a harmonic $K$-quasiregular mapping in $\mathbb{D}$ and in $B^*_h$. Then the inequality

$$(1.6) \left|(1-|z|^2)\sqrt{J_f(z)} - (1-|w|^2)\sqrt{J_f(w)}\right| \leq 2.8587(K+1)\rho(z,w)\|f\|_{B^*_h},$$

holds for all $z, w \in \mathbb{D}$.

We conclude this section by stating an open problem.

**Question 1.6.** What are the optimal constants in Theorem 1.4 and Theorem 1.5?
In this section, we prove three lemmas that will be used in proving Theorem 1.5.

**Lemma 2.1.** Let $f$ be in $B_h$ or in $B_h^*$. The respective pseudo-norms of $f$ are Möbius invariant.

**Proof.** For $z, w \in \mathbb{D}$, let 
$$ \lambda = \varphi_w(z) = \frac{w - z}{1 - wz}. $$
Then 
$$ z = \varphi_w(\lambda) = \frac{w - \lambda}{1 - \overline{w}\lambda}. $$
Elementary calculation leads to 
$$ (1 - |z|^2)|\varphi'_w(z)| = 1 - |\lambda|^2. $$
Thus 
$$ \|f \circ \varphi_w\|_{B_h} = \sup_{z \in \mathbb{D}} (1 - |z|^2)(|f_\lambda(\lambda)||\varphi'_w(z)| + |f_{\overline{\lambda}}(\lambda)||\varphi'_w(z)|) $$
$$ = \sup_{z \in \mathbb{D}} (1 - |\lambda|^2)\Lambda_f(\lambda) = \|f\|_{B_h}. $$
Similarly, we have $\|f \circ \varphi_w\|_{B_h^*} = \|f\|_{B_h}$. This completes the proof of Lemma 2.1. \qed

The following lemma shows that if $f$ is a harmonic $K$-quasiregular mapping of $\mathbb{D}$, then $f \in B_h^*$ if and only if $f \in B_h$.

**Lemma 2.2.** Let $f$ be a harmonic $K$-quasiregular mapping in $\mathbb{D}$. Then $f \in B_h$ if and only if $f \in B_h^*$. Moreover, we have 
$$ \|f\|_{B_h^*} \leq \|f\|_{B_h} \leq \sqrt{K} \|f\|_{B_h^*}. $$

**Proof.** Suppose $f \in B_h$. Since $\sqrt{J_f(z)} \leq \Lambda_f(z)$, it is easy to see that 
$$ \sup_{z \in \mathbb{D}} (1 - |z|^2)\sqrt{J_f(z)} \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)\Lambda_f(z) = \|f\|_{B_h}. $$
This implies that $f \in B_h^*$ and $\|f\|_{B_h^*} \leq \|f\|_{B_h}$.

On the other hand, suppose $f \in B_h^*$. The assumption that $f$ is a harmonic $K$-quasiregular mapping of $\mathbb{D}$ ensures that $f$ has the canonical decomposition $f = h + \bar{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$, and 
$$ \frac{|h'| + |g'|}{|h'| - |g'|} = \frac{\Lambda_f^2}{J_f} \leq K. $$
This implies that $\Lambda_f \leq \sqrt{K} \sqrt{J_f}$, and thus, $f \in B_h$. Moreover, we have

$$\|f\|_{B_h} \leq \sqrt{K} \|f\|_{B_h^*}.$$ 

The proof of Lemma 2.2 is complete. □

**Lemma 2.3.** Let $h$ belong to the Bloch space $B$. For $z, w \in \mathbb{D}$, let $\zeta = \varphi_w(z)$ and $g = C_{\varphi_w} h = h \circ \varphi_w$. If $|\zeta| \leq \frac{1}{3}$, then

$$(1 - |\zeta|^2) \left| g'(\zeta) - g'(0) \right| \leq c_1 |\zeta| \|h\|_B,$$

where $c_1 \approx 2.6920$ is the least value of

$$\psi(r) = \frac{1 + r^2/9}{r(1 - r^2)}, \quad 0 < r < 1.$$

**Proof.** For $z, w \in \mathbb{D}$, recall that

$$\zeta = \varphi_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

Then

$$-(1 - |w|^2) h'(w) = (h \circ \varphi_w)'(0).$$

Let $g = h \circ \varphi_w$. We may rewrite the above equation as follows

$$g'(0) = -(1 - |w|^2) h'(w).$$

Note that for any $w \in \mathbb{D}$, by using Cauchy formula for analytic functions, we obtain

$$(1 - |w|^2)|g''(w)| = |(g' \circ \varphi_w)'(0)| = \frac{1}{2\pi r} \left| \int_0^{2\pi} g' \circ \varphi_w(re^{i\theta})e^{-i\theta} d\theta \right|,$$

where $0 < r < 1$. By estimating the integral and noting that $\|g\|_B = \|h\|_B$, one has

$$\left| \int_0^{2\pi} g' \circ \varphi_w(re^{i\theta})e^{-i\theta} d\theta \right| \leq \|g\|_B \int_0^{2\pi} \frac{1}{1 - |\varphi_w(re^{i\theta})|^2} d\theta = \|h\|_B \int_0^{2\pi} \frac{|1 - \overline{w}re^{i\theta}|^2}{(1 - |w|^2)(1 - r^2)} d\theta.$$

It follows from the equality

$$\int_0^{2\pi} \text{Re}(\overline{w}re^{i\theta}) d\theta = 0$$

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and the above discussion that the inequality

\[(1 - |w|^2)|g''(w)| \leq \|h\|_B \frac{1 + r^2|w|^2}{r(1 - r^2)}\]

holds for any \(0 < r < 1\).

Now, consider the function

\[\psi(r) = \frac{1 + r^2/9}{r(1 - r^2)}, \quad 0 < r < 1.\]

Let \(c_1 \approx 2.6920\) denote the least value of \(\psi(r)\). Thus if \(|w| \leq \frac{1}{3}\), then

\[(1 - |w|^2)|g''(w)| \leq c_1\|h\|_B.\]

By using the inequality

\[|g'(\zeta) - g'(0)| \leq \int_0^1 |g''(t\zeta)||\zeta| dt\]

and the assumption that \(|\zeta| \leq \frac{1}{3}\), we have

\[|g'(\zeta) - g'(0)| \leq c_1\|h\|_B \int_0^1 \frac{|\zeta|}{1 - t^2|\zeta|^2} dt\]

\[= c_1\|h\|_B \int_0^{|\zeta|} \frac{ds}{1 - s^2} = c_1\|h\|_B \cdot \frac{1}{2} \ln \frac{1 + |\zeta|}{1 - |\zeta|}.\]

This implies that

\[(1 - |\zeta|^2)|g'(\zeta) - g'(0)| \leq c_1|\zeta|\|h\|_B,\]

because

\[(1 - |\zeta|^2) \ln \frac{1 + |\zeta|}{1 - |\zeta|} \leq 2|\zeta|.\]

The proof of Lemma 2.3 is complete. \(\square\)

3. Proof of main results

Proof of Theorem 1.4. Assume that \(f = h + \bar{g} \in B_h\). First note that \(h \in B\) and \(g \in B\), because

\[\|h\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|h'(z)| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)\Lambda_f(z) = \|f\|_{B_n},\]

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and similarly,
\[ \|g\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| \leq \sup_{z \in \mathbb{D}} (1 - |z|^2)\Lambda_f(z) = \|f\|_{B_h}, \]
where \( \Lambda_f(z) = |h'(z)| + |g'(z)|. \)

By elementary calculations and using Theorem A, we have
\[
\left| (1 - |z|^2)\Lambda_f(z) - (1 - |w|^2)\Lambda_f(w) \right| \\
\leq \left| (1 - |z|^2)|h'(z)| - (1 - |w|^2)|h'(w)| \right| + \left| (1 - |z|^2)|g'(z)| - (1 - |w|^2)|g'(w)| \right| \\
\leq \frac{3\sqrt{3}}{2} \rho(z, w)\|h\|_B + \frac{3\sqrt{3}}{2} \rho(z, w)\|g\|_B \leq 3\sqrt{3} \rho(z, w)\|f\|_{B_h}.
\]
This completes the proof of Theorem 1.4. \( \square \)

**Proof of Theorem 1.5.** Let \( \zeta = \varphi_w(z) \) and \( \psi = f \circ \varphi_w \), where \( z, w \in \mathbb{D} \) and \( f = h + \tilde{g} \) is a harmonic \( K \)-quasiregular mapping in \( \mathbb{D} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). Then \( J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \), and
\[ \sqrt{J_\psi(0)} = (1 - |w|^2)\sqrt{J_f(w)}. \]
Moreover, it follows from (1.5) that
\[ (1 - |z|^2)\sqrt{J_f(z)} = (1 - |\zeta|^2)\sqrt{J_\psi(\zeta)}. \]
Hence,
\[
(1 - |z|^2)\sqrt{J_f(z)} - (1 - |w|^2)\sqrt{J_f(w)} = |(1 - |\zeta|^2)\sqrt{J_\psi(\zeta)} - \sqrt{J_\psi(0)}|.
\]
Following the proof of Theorem 1.4, we now divide our proof into two cases.

**Case 1:** \( |\zeta| \leq \frac{1}{3} \). First, it follows from (3.1) that
\[
\left| (1 - |z|^2)\sqrt{J_f(z)} - (1 - |w|^2)\sqrt{J_f(w)} \right| \\
\leq |\zeta|^2 \sqrt{J_\psi(0)} + (1 - |\zeta|^2) \sqrt{J_\psi(\zeta)} - \sqrt{J_\psi(0)}. \]

By using Definition 1.3 and Lemma 2.1, we have
\[ \sqrt{J_\psi(0)} \leq \|\psi\|_{B_h^*} = \|f \circ \varphi\|_{B_h^*} = \|f\|_{B_h^*}. \]
Next, we estimate \( |\sqrt{J_\psi(\zeta)} - \sqrt{J_\psi(0)}| \) as follows.
By letting $H = h \circ \varphi_w$ and $G = g \circ \varphi_w$, we have
\[
\left| \sqrt{J_\psi(\zeta)} - \sqrt{J_\psi(0)} \right| = \frac{|J_\psi(\zeta) - J_\psi(0)|}{\sqrt{J_\psi(\zeta)} + \sqrt{J_\psi(0)}}
\]
\[
\leq \frac{|H'(\zeta)| + |H'(0)| \cdot |H'(\zeta) - H'(0)|}{|H'(\zeta)|\sqrt{1 - |\omega_\psi(\zeta)|^2} + |H'(0)|\sqrt{1 - |\omega_\psi(0)|^2}} + \frac{|G'(\zeta)| + |G'(0)| \cdot |G'(\zeta) - G'(0)|}{|H'(\zeta)|\sqrt{1 - |\omega_\psi(\zeta)|^2} + |H'(0)|\sqrt{1 - |\omega_\psi(0)|^2}}.
\]
where $\omega_\psi = G'/H'$. Because $f = h + \tilde{g}$ is a harmonic $K$-quasiregular mapping of $\mathbb{D}$, we have
\[
\|\omega_f\|_\infty = \sup_{z \in \mathbb{D}} \frac{|g'(z)|}{|h'(z)|} \leq k,
\]
where $k = \frac{K-1}{K+1} < 1$. A direct calculation leads to
\[
\|\omega_\psi\|_\infty = \sup_{z \in \mathbb{D}} \frac{|G'(z)|}{|H'(z)|} \leq k.
\]
Then
\[
\frac{|H'(\zeta)| + |H'(0)| \cdot |H'(\zeta) - H'(0)|}{|H'(\zeta)|\sqrt{1 - |\omega_\psi(\zeta)|^2} + |H'(0)|\sqrt{1 - |\omega_\psi(0)|^2}} \leq \frac{|H'(\zeta) - H'(0)|}{\sqrt{1 - k^2}},
\]
and
\[
\frac{|G'(\zeta)| + |G'(0)| \cdot |G'(\zeta) - G'(0)|}{|H'(\zeta)|\sqrt{1 - |\omega_\psi(\zeta)|^2} + |H'(0)|\sqrt{1 - |\omega_\psi(0)|^2}} \leq \frac{|G'(\zeta) - G'(0)|}{\sqrt{1 - k^2}}.
\]
These show that
\[
(3.2) \quad \left| \sqrt{J_\psi(\zeta)} - \sqrt{J_\psi(0)} \right| \leq \frac{|H'(\zeta) - H'(0)| + |G'(\zeta) - G'(0)|}{\sqrt{1 - k^2}}.
\]
Moreover, because $f \in B_h^*$, we see from Lemma 2.2 that
\[
\|f\|_{B_h} \leq \sqrt{K}\|f\|_{B_h^*}.
\]
Therefore, it follows from Lemma 2.3 that
\[
(1 - |\zeta|^2)|H'(\zeta) - H'(0)| \leq c_1|\zeta||H|_B \leq c_1|\zeta||f|_{B_h} \leq c_1|\zeta|\sqrt{K}\|f\|_{B_h^*},
\]
and similarly,
\[
(1 - |\zeta|^2)|G'(\zeta) - G'(0)| \leq c_1|\zeta|\sqrt{K}\|f\|_{B_h^*}.
\]
Combining the above inequalities and (3.2) yields

\[(1 - |\zeta|^2) \sqrt{J_\psi(\zeta) - J_\psi(0)} \leq \frac{2c_1 \sqrt{K} \|f\|_{B_h^*}}{\sqrt{1 - k^2}} = c_1(K + 1)|\zeta|\|f\|_{B_h^*}.
\]

Hence, for $|\zeta| \leq \frac{1}{3}$, one has

\[|\zeta|^2 \sqrt{J_\psi(0) + (1 - |\zeta|^2)} \sqrt{J_\psi(\zeta) - J_\psi(0)} \leq c_1(K + 1)|\zeta|\|f\|_{B_h^*} + |\zeta|^2\|f\|_{B_h^*} \leq c_3(K + 1)|\zeta|\|f\|_{B_h^*},
\]

where $c_3 \approx 2.8587$.

**Case 2:** $\frac{1}{3} < |\zeta| < 1$. Since $3|\zeta| > 1$, we have

\[|(1 - |\zeta|^2) \sqrt{J_\psi(\zeta) - J_\psi(0)} \leq \max \left\{((1 - |\zeta|^2) \sqrt{J_\psi(\zeta)}, \sqrt{J_\psi(0)}\right\}
\]

\[\leq \|\psi\|_{B_h^*} < 3|\zeta|\|f\|_{B_h^*}.
\]

Desired inequality (1.6) follows from (3.3) and (3.5). This completes the proof of Theorem 1.5. □

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