The planar Chain Rule and the Differential Equation for the planar Logarithms

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Abstract

A planar monomial is by definition an isomorphism class of a finite, planar, reduced rooted tree. If $x$ denotes the tree with a single vertex, any planar monomial is a non-associative product in $x$ relative to $m$–array grafting. A planar power series $f(x)$ over a field $K$ in $x$ is an infinite sum of $K$–multiples of planar monomials including the unit monomial represented by the empty tree.

For every planar power series $f(x)$ there is a universal differential $df(x)$ which is a planar power series in $x$ and a planar polynomial in a variable $y$ which is the differential $dx$ of $x$.

We state a planar chain rule and apply it to prove that the derivative $\frac{d}{dx}(\text{Exp}_k(x))$ is the $k$–ary planar exponential series. A special case of the planar chain rule is proved and it derived that the planar universe series $\text{Log}_k(1 + x)$ of $\text{Exp}_k(x)$ satisfies the differential equation

$$(1 + x)\frac{d}{dx}(\text{Log}_k(1 + x)) = 1$$

where $(1 + x)\frac{d}{dx}$ is the derivative which when applied to $x$ results in $1 + x$.

Introduction

For every planar power series $f(x)$ in one variable $x$, there is a universal differential $df(x)$ which is a planar power series in $x$ and a planar polynomial in a variable $y$ which is the differential $dx$ of $x$.

If $g(x)$ is a planar power series of order $\geq 1$, then the substitution of $g(x)$ for $x$ in $f(x)$ gives a power series $f(g(x))$ and the differential $df(g(x))$ of $f(g(x))$ can be computed as

$$d\varphi_g(df)$$

where $\varphi_g$ is the substitution homomorphism induced by $g$ and $d\varphi_g$ is the homomorphism on the algebra of universal differential forms in $x$ and $dx$ extending $\varphi_g$. 

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and mapping $dx$ into $dg$.

This formula is called the planar chain rule.

We note an application to the planar exponential series $Exp_k(x)$, $k \in \mathbb{N}, k \geq 2$, see [G1], section 3.

One obtains a functional equation for the differential $\omega_k$ of $Exp_k(x)$, see section 6. It follows that the derivative $\frac{d}{dx}(Exp_k(x))$ is equal to $Exp_k(x)$.

This gives a conceptual proof of this differential relation, while the proof in [G1], section 3, is quite technical.

We also prove a special chain rule, if $g'(x)$ where $g'(x) = \frac{d}{dx} g(x)$ is the derivative with respect to $x$. From it we find that

$$\left( (1 + x)\frac{d}{dx} \right)(Log_k(1 + x)) = 1,$$

if $Log_k (1 + x)$ is the $k$–ary planar logarithm, see [G2], section 5.

In section 1 notions about $\{x, y\}$- labeled finite, planar, reduced rooted trees are fixed. Some basic facts about the algebra $K\{\{x, y\}$ of power series in $x$ and polynomials in $y$ over a field $K$ are presented in section 2. The universal differential

$$d : K\{\{x\} \rightarrow K\{\{x, y\}\}$$

is introduced in section 4 and the planar chain rule is proved in section 5.

The application to the derivative of $Exp_k(x)$ relative to $x$ is given in section 6.

In section 7 we prove a special rule for the derivative $\frac{d}{dx}$, if the substitution series $g(x)$ satisfies the equation $\frac{d}{dx} g(x) = 1 + g(x)$ and we show that

$$\left( (1 + x)\frac{d}{dx} \right)(Log_k(1 + x)) = 1.$$

If $h_n$ is the homogeneous component of $Log_k(1 + x)$ of degree $n$, then

$$h'_{n+1}(x) = (-n)h_n(x)$$

for $n \geq 1$.

In [G2], section 6, the homogeneous component $h_4$ of $Log_k(1 + x)$ has been computed to be

$$h_4(x) = \left( \frac{k - 3}{3! [2]} - \frac{1}{4! [3]} (k + 1)(k - 2)(k - 3) \right) \cdot x^4 +$$

$$+ \left( \frac{1}{2} \cdot \frac{1}{3! [2]} - \frac{2}{4! [3]} (k - 2) \right) (x \cdot x^3 + x^3 \cdot x)$$
\begin{align*}
+ & \left( \frac{1}{3!} \cdot \frac{3}{2} - \frac{1}{8} - \frac{3}{4!} \right) (x \cdot (x \cdot x^2) + x \cdot (x^2 \cdot x) + (x \cdot x^2) \cdot x + (x^2 \cdot x) \cdot x) \\
+ & \left( \frac{1}{3!} \cdot \frac{3}{2} - \frac{1}{8} - \frac{3}{4!} \right) \cdot x^2 \cdot x^2 \\
+ & \left( \frac{1}{2} \cdot \frac{k-2}{3!} - \frac{2(k+1)(k-2)}{4!} \right) (x \cdot x \cdot x^2 + x \cdot x^2 \cdot x + x^2 \cdot x \cdot x).
\end{align*}

From this it follows that

\begin{align*}
h_3(x) &= (-\frac{1}{3})h'_4(x) = \frac{1}{4} \cdot \frac{k}{[2]} (x \cdot x^2 + x^2 \cdot x) - \frac{1}{3!} \cdot \frac{k-2}{[2]} \cdot x^3, \\
h_2(x) &= (-\frac{1}{2})h'_3(x) = -\frac{1}{2} \cdot x^2, \\
h_1(x) &= -h'_2(x) = x.
\end{align*}

1 Labeled planar rooted trees

Let \(x, y\) be different elements; they will be called variables in the sequel. A \(\{x, y\}\)-labeled finite, planar reduced rooted tree is a pair \(S = (T, \lambda)\) where \(T\) is a finite, planar, reduced rooted tree, see [G1], and \(\lambda\) is a map \(L(T) \to \{x, y\}\) where \(L(T)\) denotes the set of leaves of \(T\).

Then

\[ \text{deg}_x(T, \lambda) := \# \lambda^{-1}(x) \]

is called the \(x\)-degree of \((T, \lambda)\) which is the number of leaves of \(T\) labeled with \(x\) by \(\lambda\).

Also

\[ \text{deg}_y(T, \lambda) := \# \lambda^{-1}(y) \]

is called the \(y\)-degree of \((T, \lambda)\) which is the number of leaves of \(T\) labeled with \(y\) by \(\lambda\). Then \(\text{deg} \ (T, \lambda) = \text{deg} \ T := \# L(T)\) is called the total degree of \((T, \lambda)\).

We denote by \(P(x, y)\) the set of isomorphism classes of \(\{x, y\}\)-labeled finite, planar, reduced rooted trees.

Given \(S_1, \ldots, S_m \in P(x, y)\), \(S_i = (T_i, \lambda_i)\) for \(1 \leq i \leq m\). Then there is a unique \(S \in P(x, y)\) with the following properties:

\(S = (T, \lambda)\) and if \(\rho_T\) is the root of \(T\), then \(T - \rho_T\) is a planar rooted forest whose \(i\)-th component is isomorphic with \(T_i\). The set of leaves of \(L(T)\) is the disjoint union of \(L(T_1), \ldots, L(T_m)\) and the labeling \(\lambda\) induces \(\lambda_i\) on \(T_i\) for all \(1 \leq i \leq m\).
We recall that a planar rooted forest is an ordered disjoint system of planar rooted trees. The construction described above gives a unique map

\[ \bullet_m : (P(x, y))^m \to P(x, y) \]

which is called the \( m \)-ary grafting operation for labeled planar rooted trees. Let \( P'(x, y) = P(x, y) \cup \{1_p\} \) where \( 1_p \) is a symbol representing the empty tree. There is a unique extension of the \( m \)-ary grafting operation to a map

\[ \bullet_m : (P'(x, y))^m \to P'(x, y) \]

satisfying the properties:

- \( \bullet_m(1_p, \ldots, 1_p) = 1_p \)
- \( \bullet_2(S, 1_p) = \bullet_2(1_p, S) = S \) for all \( S \in P'(x, y) \),
- \( \bullet_m(S_1, \ldots, S_m) = \bullet_{m-1}(S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_m) \) if \( S_i = 1_p \) for all \( S_i \in P'(x, y) \).

It is also refered to as the \( m \)-ary grafting operation.

## 2 Polynomials and Power series

Let \( K \) be a field and \( V \) be the \( K \)-vectorspace of all \( K \)-valued functions \( f \) on \( P'(x, y) \) such that for all \( n \in \mathbb{N} \) the set

\[ \{ S \in P'(x, y) : f(S) \neq 0, \text{ord}_x(S) = n \} \]

is finite. For any \( f \in V \) the value \( f(S) \) at \( S \in P'(x, y) \) will be denoted by \( c_S(f) \). It will be called the coefficient of \( f \) relative to \( S \). For any \( m \in \mathbb{N}, m \geq 2 \), we are going to define a \( m \)-ary operation \( \bullet_m \) on \( V \) by putting

\[ f := \bullet_m(f_1, \ldots, f_m) \]

if \( c_S(f) := \sum c_{S_1}(f_1) \cdot \ldots \cdot c_{S_m}(f_m) \) where the summation is extended over all \( S_1, \ldots, S_m \) such that \( S = \bullet_m(S_1, \ldots, S_m) \).

It is easy to check that \( \bullet_m \) is a \( K \)-multilinear map \( V^m \to V \). We will write \( f_1 \times f_2 \times \ldots \times f_m \) for \( \bullet_m(f_1, \ldots, f_m) \) sometimes.

In this note a \( K \)-algebra is by definition a \( K \)-vectorspace \( W \) together with a family \( (\bullet_m)_{m \geq 2} \) of \( K \)-multilinear maps \( \bullet_m : W^m \to W \). In the context of operad theory such an algebra can be defined as an algebra over an operad in the category of \( K \)-linear spaces which is freely generated by \( \{ \mu_m : m \geq 2 \} \) where \( \mu_m \) has degree \( m \), see [MSS], Chap. (1.9). It is a tree operad, see [BV]. Trees are inspired by the attempt to obtain a general composite operation from a collection of indecomposable operations, see [BV].

Other authors, see [SV], use the term hyperalgebra to denote this type of algebras. For the general theory of operads and algebras over operads see also [L], [LR], [H], [F].
Therefore \((V, (\bullet_m)_{m \geq 2})\) is a \(K\)–algebra; it is denoted by \(K\{x, y\}\) and will be referred to as the algebra of planar power series in \(x\) and polynomials in \(y\).

For any \(S \in P'(x, y)\) there is a unique \(f_S \in K\{x, y\}\) such that

\[
 c_T(f_S) = \begin{cases} 
 1 : T = S \\
 0 : T \in P'(x, y), T \neq S.
\end{cases}
\]

We identify \(f_S\) with \(S\) and thus consider \(P'(x, y)\) as a subset of \(K\{x, y\}\). It is easy to check that \(P'(x, y)\) is a \(K\)–linearly independent subset of \(K\{x, y\}\) and that the \(K\)–vectorspace of \(K\{x, y\}\) generated by \(P'(x, y)\) is a subalgebra. It is denoted by \(K\{x, y\}\) and is called the algebra of planar polynomials: in \(x\) and \(y\) over \(K\).

For \(f \in K\{x, y\}\) define

\[
 \text{ord}_x(f) = \infty, \text{if } f = 0
\]

and if \(f \neq 0\), then

\[
 \text{ord}_x(f) = \min\{n \in \mathbb{N} : \text{there is } S \in P(x, y) \text{ with } \deg_x(S) = n \text{ and } c_S(f) \neq 0.\} \]

It is called the \(x\)--order of \(f\).

Let

\[
 |f|_x := \begin{cases} 
 0 : f = 0 \\
 (\frac{1}{2})^{\text{ord}_x(f)} : f \neq 0.
\end{cases}
\]

Then \(|f_1 \cdot f_2 \cdot \ldots \cdot f_m|_x = |f_1|_x \cdot |f_2|_x \cdot \ldots \cdot |f_m|_x\)

as \(\text{ord}(f_1 \cdot f_2 \cdot \ldots \cdot f_m) = \sum_{i=1}^m \text{ord}(f_i)\) and \(\cdot \), \(|\cdot|_x\) defines a distance function on \(K\{x, y\}\) by \(|f, g|_x := |f - g|_x\) which is called the \(x\)--adic distance.

**Proposition 2.1.** Any Cauchysequence in \(K\{x, y\}\) relative to the \(x\)--adic distance is convergent.

**Proof.** By standard arguments. \(\square\)

### 3 Substitution homomorphisms

**Proposition 3.1.** Let \(g, h \in K\{x, y\}\) and let \(\text{ord}_x(g) \geq 1\).
Then there is a unique \(K\)–algebra homomorphism

\[
 \varphi_{(g,h)} = \varphi : K\{x, y\} \to K\{x, y\}
\]

such that
\(\varphi(x) = g\)

\(\varphi(y) = h\)

(iii) \(\varphi\) is continuous with respect to the \(x\)-adic topology.

It is called substitution homomorphism induced by \((g, h)\).

Proof. 1) First we define \(\varphi'(S)\) for any \(S \in P(x,y)\). If \(S = (T, \lambda)\) where \(T\) is a planar rooted tree of degree , there is a \(m\)-ary operation \(\cdot_T\) associated to \(T\), see [G2], Proposition (2.7)

Let \(\varphi'(S) := \cdot_T(Z_1, ..., Z_m)\) where \(Z_i = g\) if \(\lambda(l_i) = x\) and \(Z_i = h\) if \(\lambda(l_i) = y\).

Here \(l_i\) is the \(i\)-th leaf of \(T\).

2) There is a unique \(K\)-linear extension of the map \(\varphi'\) in 1) to a linear map

\[\varphi : K\{\{x, y\} \rightarrow K\{\{x, y\} .\]

3) If \(f \in K\{\{x, y\}, f_n := \sum c_S(f) \cdot S\) where the sum is extended over all \(S \in P(x,y)\) for which \(deg_x(S) = n\), then \(f_n\) is a finite sum and

\[f = \sum_{n=0}^{\infty} f_n.\]

From 1) it follows that \(ord_x(\varphi(f)) \geq ord_x(f)\) for all \(f \in K\{\{x, y\}\) as \(ord_x(g) \geq 1\). Thus \(ord_x(\varphi(f_n)) \geq n\) for all \(n\).

One defines \(\varphi(f) = \sum_{n=0}^{\infty} \varphi(f_n)\). The infinite sum is converging, as the sequence of partial sums is a Cauchy sequence with respect to \(| |_x\). Thus a continuous \(K\)-linear map \(\varphi : K\{\{x, y\} \rightarrow K\{\{x, y\}\) is well defined. One can check that \(\varphi\) is an algebra homomorphism because its restriction to \(K\{\{x, y\}\) is an algebra homomorphism by the definition in 1) and 2).

\[\square\]

Notation: For \(\varphi = \varphi(g, h)\) and \(f \in K\{\{x, y\}\), we denote \(\varphi(f)\) also by \(f(g, h)\). Then \(f = f(x, y)\) as \(\varphi(x, y) = id\). Also \(f(g, h) = f(g(x, y), h(x, y))\). It is obtained by replacing \(x\) by \(g\) and \(y\) by \(h\).

For any \(k \in K\) we get \(f(kx, ky) = k^m f(x, y)\), if \(f\) is homogeneous of degree \(m\).

4 Universal derivation

We denote by \(K\{\{x\}\) the closed unital \(K\)-subalgebra of \(K\{\{x, y\}\) generated by \(x\).
Proposition 4.1. There is a unique continuous $K$-linear map
\[ d : K\{x\} \to K\{x, y\} \]
such that
\begin{enumerate}[(i)]  
  \item $d(x) = y$
  \item $d(f_1 \cdot f_2 \cdot \ldots \cdot f_m) = 
    \sum_{i=1}^{m} (f_1 \cdot \ldots \cdot f_{i-1} \cdot d(f_i) \cdot f_{i+1} \cdot \ldots \cdot f_m) \text{ for all } m \geq 2 \text{ and all } f_1, \ldots, f_m \in K\{x, y\}$
\end{enumerate}

The map $d$ is called the universal derivation on $K\{x\}$. It is customary to denote $d(f)$ by $df$ and call it the differential of $f$. Especially $y = dx$.

Proof. 1) First $d'(S)$ is defined for all $S \in P(x, y)$. If $S = 1_P$, then $d'(1_P) := 0$. If $S \neq 1_P$, $S = (T, \lambda), \lambda(l_i) = x$ for all leaves $l_1, \ldots, l_m$ of $T, m = \deg(T), L(T) = \{l_1, \ldots, l_m\}$, then let $d'(S) := \sum_{i=1}^{m} (T, \lambda^{(i)}), \text{ where } \lambda^{(i)}(l_j) = y \text{ if } j = i \text{ and } \lambda^{(i)}(l_j) = x \text{ if } j \neq i$.

2) As $P'(x)$ is $K$-linearly independent, the map $d'$ in 1) has a unique extension to a $K$-linear map $d_0 : K\{x\} \to K\{x, y\}$.

3) There is a unique continuous extension of $d_0$ to a $K$-linear map $d : K\{x\} \to K\{x, y\}$.

It is easy to check that property (ii) holds. \qed

5 Chain rule

Let $g \in K\{x\}, \ord(g) \geq 1$. As a special case of Proposition (3.1) we get a continuous $K$-algebra homomorphism
\[ \varphi_g = \varphi : K\{x\} \to K\{x\} \]
such that $\varphi(x) = g$.

Also there is a unique continuous $K$-algebra homomorphism
\[ dy : K\{x, y\} \to K\{x, y\} \]
such that
\[ d\varphi(x) = \varphi(x) = g \]
\[ d\varphi(y) = dg \]
Proposition 5.1.

\((d\varphi) \circ d = d \circ \varphi.\)

Proof. 1) Let \(\delta := (d\varphi) \circ d - d \circ \varphi.\) Then \(\delta\) is a continuous \(K-\)linear map from \(K\{\{x\}\}\) into \(K\{\{x,y\}\}\) which satisfies

\[\delta(x) = 0\]

2) It is easy to see that

\[\delta(f_1 \cdot \ldots \cdot f_m) = \sum_{i=1}^{m} \varphi(f_1) \cdot \ldots \cdot \varphi(f_{i-1}) \cdot \delta(f_i) \cdot \varphi(f_{i+1}) \cdot \ldots \cdot \varphi(f_m).\]

Using this formula one can prove by induction on the degree that for any \(S \in P(x)\) one gets \(\delta(S) = 0.\)

If \(S = S_1 \cdot \ldots \cdot S_m,\) then \(deg_x(S_i) < deg_x(S)\) and by induction hypothesis \(\delta(S_i) = 0, \forall i?\) By the formula above then \(\delta(S) = 0\)

3) As \(\delta\) is continuous and \(K\{x\}\) is dense in \(K\{\{x\}\}\) and \(\delta(f) = 0\) by 2) and the linearity of \(\delta,\) we see that \(\delta = 0.\)

Example: Let

\(g = k \cdot x, k \in K.\)

Then

\(\varphi(f(x)) = f(kx)\)

and

\((d\varphi)\omega(x, y) = \omega(kx, ky).\)

If \(f\) (resp. \(\omega\)) is homogeneous of degree \(m,\) then \(f(kx) = k^m f(x), \omega(kx, ky) = k^m \omega(x, y).\)

\[\Box\]

6 The derivative of the exponentials

Let now \(\text{char}(K) = 0, k \in \mathbb{N}k \geq 2,\) and \(f = Exp_k(x)\) be the \(k-\)ary exponential series, see [G1].

Let \(\omega = df\) be the differential of \(f.\) Then \(\omega(kx, ky) = df^k\) by the chain rule.

If \(f = \sum_{n=0}^{\infty} f_n\) and \(f_n\) is the homogeneous part of degree \(n,\) then

\[\omega = \sum_{n=1}^{\infty} df_n\]
and \(df_n\) is the homogeneous component \(\omega_n\) of \(\omega\) of degree \(n\) with \(df_0 = 0\).

More precisely:

\[
\deg_x(df_n) = n - 1, \\
\deg_y(df_n) = 1.
\]

Now by the functional equation satisfied by \(f\), we get

\[
f^k(x) = f(kx) = \sum_{n=0}^{\infty} k^n f_n(x)
\]

\((*)\)

\[
k^n f_n(x) = \sum_{i_1 + \ldots + i_k = n} f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_k}
\]

where the summation is over all

\[(i_1, \ldots, i_k) \in \mathbb{N}^k
\]

with

\[i_1 + \ldots + i_k = n.
\]

**Proposition 6.1.**

\[
k^n \omega_n(x, y) = \sum_{j=1}^{k} \sum_{i_1 + \ldots + i_k = n} f_{i_1} \cdot \ldots \cdot f_{i_j-1} \cdot \omega(f_{i_j}) \cdot f_{i_j+1} \cdot \ldots \cdot f_{i_k}.
\]

**Proof.** Immediate by property (ii) of Proposition 4.1. \(\square\)

There is a unique continuous unital algebra homomorphism

\[
\psi : K\{\{x, y\}\} \rightarrow K\{\{x\}\}
\]

such that \(\psi(x) = x, \psi(y) = 1\).

Recall that \(\psi(f) = f\) for all \(f \in K\{\{x\}\}\).

Then \((f \circ d) : K\{\{x\}\} \rightarrow K\{\{x\}\}\) is the

derivative relative to \(x\), also denoted symbolically by \(\frac{d}{dx}\), see [G1]

**Proposition 6.2.** \(\frac{d}{dx}(\text{Exp}_k(x)) = \text{Exp}_k(x)\).

**Proof.** 1) We have to show that the derivative \(f'_n = \frac{d}{dx}(f_n)\) of the homogeneous component of \(f(x) = \text{Exp}_k(x)\) is equal to \(f_{n-1}\) for \(n \geq 1\).

By applying \(\psi\) to the formula in Proposition (6.1) we get

\[
k^n f'_n(x) = \sum_{j=1}^{k} \sum_{i_1 + \ldots + i_k = n} f_{i_1} \cdot \ldots \cdot f_{i_j-1} \cdot f_{i_j}' \cdot f_{i_j+1} \cdot \ldots \cdot f_{i_k}.
\]
2) We will show that \( f_n'(x) = f_{n-1}(x) \) for \( n \geq 1 \) by induction on \( n \).
If \( n = 1 \), then it is obviously true as \( \text{ord}_x(f - (1 + x)) \geq 2 \).
Let now \( n \geq 2 \). As \( f'_{ij} = 0 \) if \( i_j = 0 \) and \( f'_{ij} = f_{i-1} \) for \( n > 0 \), we obtain

\[
k^n f_n'(x) = \sum_{j=1}^{k} \sum_{i_j \geq 1} f_{i_1} \cdot \ldots \cdot f_{i_{j-1}} \cdot f_{i_j-1} \cdot f_{i_j+1} \cdot \ldots \cdot f_k
\]

Now for any \( 1 \leq i \leq k \) we get

\[
\sum_{i_1 + \ldots + i_k = n-1} f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_k} = k^{n-1} - f_{n-1}(x)
\]

by \((*)\)
It follows that

\[
k^n f_n'(x) = k \cdot k^{n-1} \cdot f_{n-1}(x)
\]

and

\[
f_n'(x) = f_{n-1}(x).
\]

\[\square\]

### 7 Special chain rules and logarithm

Let \( h \in K\{\{x\}\} \). Then there is a unique algebra homomorphism \( \varphi_{(x,h)} = \varphi : K\{x,y\} \rightarrow K\{\{x\}\} \) such that \( \varphi(x) = x \) and \( \varphi(y) = h \). Then

\[
\theta := \varphi \circ d : K\{\{x\}\} \rightarrow K\{\{x\}\}
\]

is a derivation on \( K\{\{x\}\} \) which means that it is \( K \)-linear, continuous and satisfies the general product rule

\[
\theta(f_1 \cdot \ldots \cdot f_m) = \sum_{i=1}^{m} f_1 \cdot \ldots \cdot f_{i-1} \cdot \theta(f_i) \cdot f_{i+1} \cdot \ldots \cdot f_m
\]

for all

\[
f_1, \ldots, f_m \in K\{\{x\}\}
\]

This derivation \( \theta \) will be formally denoted by \( h \frac{d}{dx} \).
Be aware that \( (h \frac{d}{dx})(f) \neq h \cdot \frac{d}{dx}(f) \) in general.
We denote \( \frac{d}{dx}(f) \) also by \( f' \) and call it the derivative of \( f \) with respect to \( x \).

**Proposition 7.1.** Let \( g \in K\{\{x\}\} \) such that \( g' = 1 + g \). Then

\[
\frac{d}{dx}(f(g(x))) = f'(g(x)) + (x \frac{d}{dx})(f)(g(x)) = ((1 + x) \frac{d}{dx}(f))(g(x))
\]
Proof. :1) First we will prove the formula in case $f$ is a tree $T$ of degree $n$
If $n = 1$, then $T = x$ and $\frac{d}{dx}x = 1$ while $(1 + x)\frac{d}{dx}(T) = 1 + x$. Thus $f(g(x)) = g(x)$ and $\frac{d}{dx}(g) = 1 + g$ from which the formula follows. Let now $n > 1$. Then

$$\frac{d}{dx}(T) = \sum_{i=1}^{n} T^{(i)}$$

where $T^{(i)} = \bullet_{T}(z_{1}, \ldots, z_{n})$ with

$$z_{j} = \begin{cases} x: j \neq i \\ 1_{P}: j = i \end{cases}$$

and

$$x \frac{d}{dx}(T) = n \times T$$

Also

$$\frac{d}{dx}(T(g(x))) = \sum_{i=1}^{n} S^{i}$$

with

$$S^{(i)} = \bullet_{T}(W_{1}, \ldots, W_{n})$$

and

$$w_{j} = \begin{cases} g(x): j \neq i \\ g'(x): j = i \end{cases}$$

As $g' = 1 + g$, we obtain from the multilinearity of $\bullet_{T}$, that

$$S(i) = \bullet_{T}(g, \ldots, g) + \bullet_{T(i)}(g, \ldots, g)$$

From these computations it follows that the proposition holds, if $f$ is a tree

2) From 1) one can extend the result to polynomials by multilinearity. As both
sides of the formula are continuous in $f$ the result follows for power series. \hfill \square

Let $Log_{k}(1 + x)$ be the $k$–ary planar logarithm. see [G2], section 5.

**Proposition 7.2.**

$$((1 + x)\frac{d}{dx}(Log_{k}(1 + x)) = 1$$

for all

$$k \in \mathbb{N}, \ k > 2.$$
Proof. Let 
\[ g = \exp_k(x) - 1 \]
Then 
\[ \log_k(\exp_k(x)) = x \]
From the special chain rule above, we can conclude that 
\[ (((1 + x) \frac{d}{dx})(\log_k))\exp_k(x)) = 1 \]
From this the formula follows as \( f(g(x)) = 1 \) only if \( f(x) = 1 \). \( \square \)

**Corollary 7.3.** Let \( h_n \) be the homogeneous component of \( \log_k(1 + x) \) of degree \( n \). Then
\[ h_0 = 0 \]
and
\[ h'_{n+1} = -nh_n \]
for
\[ n \geq 1 \]
Proof. : \( \log_k(1 + x) = \sum_{n=0}^{\infty} h_n =: h \)
and
\[ (1 + x) \frac{d}{dx}(h_n) = h'_n + nh_n, \]
as
\[ (s \frac{d}{dx}(h_n)) = n \times h_n. \]
Moreover \( h'_n \) is homogeneous of degree \( (n - 1) \).
Thus
\[ h'_{n+1} + nh_n = 0 \]
for all
\[ n \geq 1 \]
because
\[ (1 + x) \frac{d}{dx}(h) = 1. \] \( \square \)
Example 7.4. We give the homogeneous terms \( h_n(x) \) of \( \log_k(1 + x) \) of degree \( n \leq 4 \).
Recall that
\[
[n] = \frac{k^n - 1}{k - 1} = \sum_{i=0}^{n-1} k^i
\]
and that
\[
[n]! = \prod_{i=1}^{n} [i].
\]
So especially
\[
[2] = k + 1, \quad [3] = k^2 + k + 1, \quad [3]! = (k + 1)(k^2 + k + 1) = k^3 + 2k^2 + 2k + 1.
\]
For any planar rooted tree \( T \) in \( P \), let \( \{T\} \) be the sum of all planar trees \( S \) in \( P \) whose underlying rooted tree is isomorphic to the underlying rooted tree of \( T \). One can show that \( \exp_k(x) \) and \( \log_k(x) \) are infinite sums over \( \{T\} \) with \( T \in P \).
Then the homogeneous components \( h_n(x) \) for \( n \leq 4 \) can be written as, see [G2]:
\[
h_1(x) = x
\]
\[
h_2(x) = -\frac{1}{2} x^2
\]
\[
h_3(x) = \frac{1}{4} [2] \{x \cdot x^2\} - \frac{1}{3!} \frac{k-2}{[2]} \cdot x^3
\]
\[
h_4(x) = \left(\frac{k-3}{3!} - \frac{1}{3!} (k + 1)(k - 2)(k - 3)\right) \{x^4\} + \frac{1}{2} \frac{1}{3!} \frac{k}{[2]} \{x \cdot x^3\} + \frac{2}{3!} \left(\frac{k}{[2]} - \frac{3}{2} \frac{1}{3} \frac{1}{[3]} (k - 2)\right) \{x \cdot x^2\} + \frac{3}{3!} \frac{1}{2} \frac{1}{3} \frac{1}{[3]} \{x^2 \cdot x^2\} + \frac{3}{2} \frac{1}{3!} \frac{k-2}{[2]} - \frac{2(k+1)(k-2)}{4! [3]} \} \{x \cdot x \cdot x^2\} + \{x \cdot x \cdot x^2\}
\]
Easy computations give
\[
\frac{d}{dx} (\{x \cdot x \cdot x^2\}) = 3 \{x \cdot x^2\} + 6 \{x \cdot x \cdot x\}
\]
\[
\frac{d}{dx} (\{x^2 \cdot x^2\}) = 2 \{x^2\}
\]
\[
\frac{d}{dx} (\{x \cdot (x \cdot x^2)\}) = 6 \{x \cdot x^2\}
\]
\[
\frac{d}{dx} (x \cdot x^3) = 2x^3 + 3 \{x \cdot x^2\}
\]
from which one can derive that

$$h'_4(x) = -3h_3(x)$$

Similarly one gets

$$h'_3(x) = -2h_2(x)$$
$$h'_2(x) = -h_1(x)$$

In [DG] $\text{Log}_2(x)$ was computed to be

$$\text{Log}_2(x) = x - \frac{1}{2} x^2 + \frac{1}{3} (\frac{1}{2} x \cdot x^2 + \frac{1}{2} x^2 \cdot x) -$$

$$-\frac{1}{4} (\frac{4}{21} x \cdot (x \cdot x^2) + \frac{4}{21} x \cdot (x^2 \cdot x) + \frac{5}{21} x^2 \cdot x^2 + \frac{4}{21} (x \cdot x^2) \cdot x + \frac{4}{21} (x^2 \cdot x) \cdot x) +$$

higher terms.

It is obtained by substituting 2 for $k$ in $\text{Log}_k(x)$.

**Open question:** In there a procedure to construct $h_{n+1}(x)$ which is an integral of $(-n) \cdot h_n(x)$ directly from $h_n(x)$?

The problem arises from the fact that the space of homogeneous polynomials of degree $n + 1$ whose derivative is zero is non-trivial for $n \geq 2$

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