Neutrality of the color-flavor–locked phase in a Dyson-Schwinger approach

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The role of neutrality constraints for the phase structure of QCD at non-vanishing chemical potentials is studied within a self-consistent truncation scheme for the Dyson-Schwinger equation of the quark propagator in Landau gauge. We find the (approximate) color-flavor–locked phase to be energetically preferred at all potentially relevant densities and for physical values of the quark masses. We furthermore observe the impossibility to define this phase by residual global symmetries and discuss the role of chemical potentials.

I. INTRODUCTION

The phase diagram of quantum chromodynamics (QCD) is expected to contain color-superconducting regimes at sufficiently high densities and low temperatures [1, 2, 3, 4, 5, 6, 7]. Considering three (degenerate) quark flavors, the color-flavor locked (CFL) phase [8] is the ground-state for vanishing temperatures and asymptotically large densities [9, 10]. At densities of potential relevance for compact stellar objects it is, however, not obvious how the splitting of Fermi surfaces due to finite quark masses and neutrality constraints is influencing the ground-state [11, 12, 13]. In Ref. [14] we have elaborated the role of the renormalized strange-quark mass when addressing this question. Including medium modifications of the effective quark interaction decreases the dynamically generated contribution to the strange-quark mass. Therefore, for the physical value of the strange-quark mass the CFL phase has been found to be the ground-state at all potentially relevant densities.

In this paper we will report on the results when incorporating neutrality conditions. As the CFL phase is neutral in the chiral limit due to the residual symmetry, we do not expect it to be influenced strongly. In contrast, the 2SC phase for two flavors is far from being electrically neutral. Therefore, as known from investigations in Nambu-Jona-Lasinio (NJL) type models [13, 15, 16, 17], the region of the 2SC phase is strongly affected by neutrality constraints. When varying the strange-quark mass the corresponding phenomenology can be, in a strongly simplified manner, sketched as follows: In the chiral limit, all quarks have the same Fermi momentum, and due to Luttinger’s theorem, also have the same density. Increasing the strange-quark mass the Fermi momentum of the strange quarks is lowered according to \( p_{F,s} \approx \mu - M_s^2/2\mu \). To first approximation the positive electrical charge of the up-quarks is then compensated by additional down-quarks with Fermi momentum \( p_{F,d} \approx \mu + M_s^2/2\mu \). This mechanism is related to a partial symmetry breaking of the residual symmetry in the CFL phase after the formation of Cooper pairs. Eventually, in some channels ungapped quasiparticle excitations appear, and accordingly these phases are referred to as the gapless CFL (gCFL) phase [18, 19, 20, 21]. At some stage pairing of down- and strange-quarks is no longer favorable, and all remaining Cooper pairs involve an up-quark, which is characterizing the uSC phase. After this, the pairing of up- and strange-quarks might break, and one potentially may find in an intermediate regime the 2SC phase. Finally, all Fermi surfaces are far separated and the ground-state is no longer a spin-zero superfluid. Ungapped uSC and 2SC phases have not been observed in our calculations. It is worth mentioning that gapless phases are expected to be unstable, as some Meissner masses become imaginary [22, 23, 24]. This finding led to a broad discussion of the appearance of inhomogeneous phases [25, 26, 27, 28]. Due to the employed truncations this will not be addressed in this work. As suggested by our previous work [14] we find however those scenarios to be irrelevant for realistic quark masses with gaps as large as we obtain.

This paper is organized as follows: In section II we briefly present the theoretical framework which involves a fully self-consistent treatment of the quark propagator in the CFL-like phase including neutrality constraints. The role of chemical potentials especially in the CFL phase will be discussed in section III. In section IV we discuss the numerical results for the CFL-like phase at non-vanishing strange-quark masses. Finally, we summarize and conclude in section V. The color-flavor basis is given in an Appendix.
II. THE LANDAU-GAUZE QUARK PROPAGATOR AT NON-VANISHING CHEMICAL POTENTIAL

A. The truncated quark Dyson-Schwinger equation at non-vanishing chemical potential

In Refs. [14, 29] a closed truncated Dyson-Schwinger equation (DSE) for the Landau-gauge quark propagator at non-vanishing chemical potential has been solved in the chiral limit and for finite strange-quark masses, respectively. In the present work we follow the same scheme and notations. In the Nambu-Gor’kov basis (see Ref. [4] for a review) the normal quark DSE is coupled to an equation for the gap functions. To describe their structure we define

\[
S_0(p) = \begin{pmatrix} S_0^+ (p) & 0 \\ 0 & S_0^- (p) \end{pmatrix} = -CS_0^T (p) C,
\]

\[
S (p) = \begin{pmatrix} T^+ (p) & T^- (p) \\ T^+ (p) & S^- (p) \end{pmatrix} = -CS^T (p) C,
\]

\[
\Sigma (p) = \begin{pmatrix} \Sigma^+ (p) & \Phi^+ (p) \\ \Phi^+ (p) & \Sigma^- (p) \end{pmatrix},
\]

with \( S_0 \) being the ‘bare’ and \( S \) the full Nambu-Gor’kov propagator. The Nambu-Gor’kov self-energy is denoted by \( \Sigma \). As the Euclidean action is real we furthermore have to impose \( \Phi^- (p) = \gamma_4 \Phi^T (p) \gamma_4 \). The inverse ‘bare’ quark propagator in the presence of a static, isotropic and homogeneous ‘external’ gluon field with non-vanishing time component \( A_4 \) is given by

\[
S_0^{-1} (p) = \begin{pmatrix} -i\vec{p} \cdot \vec{\gamma} - i(p_4 + i\bar{\mu} + \frac{Z_2 F}{Z_2} g A_4) \gamma_4 + m & 0 \\ 0 & -i\vec{p} \cdot \vec{\gamma} - i(p_4 - i\bar{\mu} - \frac{Z_2 F}{Z_2} g A_4) \gamma_4 + m \end{pmatrix}.
\]

Here \( m \) is the mass matrix and \( \bar{\mu} \) the matrix implementing chemical potentials associated to global symmetries. The parameterization and determination of \( \bar{\mu} \) and \( A_4 \) will be discussed below. The renormalization constants \( Z_2 \) and \( Z_{1F} \) also appear in the DSE for the quark propagator

\[
S^{-1} (p) = Z_2 S_0^{-1} (p) + Z_{1F} \Sigma (p)
\]

which explicitly reads

\[
T^\pm = -Z_{1F} \left( Z_2 S_0^{T^{-1}} + Z_{1F} \Sigma^T \right)^{-1} \Phi^\pm S^\pm,
\]

\[
S^{\pm -1} = Z_2 S_0^{T^{-1}} + Z_{1F} \Sigma^\pm - Z_{1F} \Phi^T \left( Z_2 S_0^{T^{-1}} + Z_{1F} \Sigma^T \right)^{-1} \Phi^\pm.
\]

Employing the truncation of Ref. [29], the equations for the self-energy and the gap function then result in

\[
\Sigma^+ (p) = \frac{Z_2^2}{Z_{1F}^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu \lambda_\alpha S^+ (q) \gamma_\nu \lambda_\alpha \left( \frac{\alpha_s (k^2) P_{T\mu}^T}{k^2 + G(k)} + \frac{\alpha_s (k^2) P_{L\mu}^L}{k^2 + F(k)} \right),
\]

\[
\Phi^+ (p) = -\frac{Z_2^2}{Z_{1F}^2} \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu \lambda_\alpha T^+ (q) \gamma_\nu \lambda_\alpha \left( \frac{\alpha_s (k^2) P_{T\mu}^T}{k^2 + G(k)} + \frac{\alpha_s (k^2) P_{L\mu}^L}{k^2 + F(k)} \right),
\]

where \( k = p - q \). Here projectors transverse and longitudinal to the medium have been introduced. The functions \( G \) and \( F \) describe the corresponding medium modifications of the gluon propagator, see Eq. (21) of Ref. [29]. They can be calculated once the coupling is given. Therefore the only input for the quark DSE are the running coupling \( \alpha_s (k^2) \) and the renormalized current quark masses.

As the objective of the presented work is to show that the CFL phase is the physical ground-state for realistic strange-quark masses, we use the most conservative choice for \( \alpha_s (k^2) \) in our approach. This is the running coupling \( \alpha_s (k^2) \) determined in DSE studies of the Yang-Mills sector [30, 31, 32]. As is detailed in Ref. [32] it underestimates chiral symmetry breaking significantly in the Abelian approximation. In Ref. [29] this coupling lead to the smallest critical strange-quark masses. However, we would like to emphasize that in contrast to NJL-type models, our approach is much less sensitive to the choice of the coupling, an effect which can be traced back to the inclusion of the medium polarization.

The renormalization constants are determined in the (chirally broken) vacuum. Due to the vertex construction employed, the quark-gluon vertex renormalization constant, \( Z_{1F} \), cancels in the resulting renormalized equations. For
each flavor, we determine the quark wave-function renormalization constant, $Z_2$, and the renormalization constant $Z_m$, relating the unrenormalized quark mass $m_{0,q} (\Lambda^2)$ at an ultraviolet cutoff $\Lambda$ to the renormalized mass $m_q (\nu)$ via

$$m_{0,q} (\Lambda^2) = Z_m (\nu^2, \Lambda^2) m_q (\nu),$$

(10)

by requiring

$$S_q^+ (p) \big|_{\nu^2 = \nu^2} = -i \not p + m_q (\nu)$$

(11)

at a renormalization scale $\nu$. This corresponds to a momentum-subtraction (MOM) scheme, which results in somewhat smaller numerical values for the quark current masses at a given renormalization scale (usually taken to be 2 GeV). We simply ignore here the difference between $\overline{MS}$ and MOM masses because the effect is of the order of ten percent (when calculated within perturbation theory) and is thus much smaller than the uncertainty in the value of the current masses.

It turns out that, as expected, the mass dependence of the quark wave function renormalization constant, $Z_2$, is negligible as long as the renormalization scale is much larger than the mass. Therefore, we simply drop this dependence and $Z_2$ is determined once and for all in the chiral limit. To keep the number of parameters as small as possible, we set the up- and down-quark masses to zero and vary the strange-quark current mass only.

In the following we will restrict to spatially isotropic phases. In order to solve the DSE of the quark propagator it is advantageous to consider their color-flavor structure first. To get a self-consistent solution we choose suitable sets of matrices $\{P_i\}$ and $\{M_i\}$ in color-flavor space, such that

$$\Sigma^+ (p) = \frac{Z_2}{Z_1} \sum_i \Sigma_i^+ (p) P_i,$$

(12)

$$\Phi^+ (p) = \frac{Z_2}{Z_1} \sum_i \phi_i^+ (p) M_i,$$

(13)

where we have introduced the renormalization-point independent component functions $\Sigma_i^+ (p)$ and $\phi_i^+ (p)$, which are matrix-valued in Dirac space. Full self-consistency is guaranteed in case a basis of all allowed matrices is considered. The dimensionality of this basis in a given phase depends on the residual symmetry in color-flavor space. For the CFL phase this will be detailed below, and the basis used is explicitly given in the Appendix.

The Dirac structure of the self-energies in an even-parity phase can be parameterized by

$$\Sigma_i^+ (p) = \not p \Sigma_{A,i}^+ (p) - i \not p \Sigma_{C,i}^+ (p) + \Sigma_{D,i}^+ (p) - i \gamma_4 \not p \Sigma_{D,i}^+ (p),$$

(14)

$$\phi_i^+ (p) = (\gamma_4 \not p \phi_{A,i}^+ (p) + \gamma_4 \phi_{B,i}^+ (p) + \phi_{C,i}^+ (p) + \not p \phi_{D,i}^+ (p)) \gamma_5,$$

(15)

where $\not p = \not p/|\not p|$, $\not p = \not p \cdot \gamma$, $\not p = \omega_p \gamma_4$ and $\omega_p = ip_\rho + \mu$. Thus we finally need to solve a coupled set of integral equations for the energy- and momentum-dependent functions $\Sigma_{A,B,C,D,i}^+ (p)$ and $\phi_{A,B,C,D,i}^+ (p)$. This is done numerically.

### B. Color neutrality

As has been discussed in $[29, 34, 35, 36]$, we need to allow for constant values of $A_4$, whose DSE reduces to

$$\rho^a (x) \equiv Z_2^2 \frac{1}{2} \int \frac{d^4 p}{(2\pi)^3} \int \frac{d^4 p_4}{2\pi} T_{Y_{D,c.f.NG}} \left( S (p) \Gamma^{(0)a}_{NG4} \right).$$

$$\Gamma^{(0)a}_{NG4}$$

1

$$0.$$

(16)

As $A_4$ is anti-Hermitian within our conventions, we define by

$$\mu_C \equiv \sum_a \mu_a \lambda^a \equiv -i \frac{Z_{1F}}{Z_2} g A_4$$

(17)

the effective color chemical potentials. The color chemical potentials are then adjusted to obtain color neutrality.
C. Electrical neutrality and $\beta$-equilibrium

Electrical neutrality within QCD alone is meaningless. We need to include the electro-weak interaction and its particle content. The conserved electrical charge is then adjusted by its Lagrange multiplier $\mu_Q$ and the chemical potentials for the quarks, including the color chemical potentials, are given by

$$\hat{\mu}_{ab,ij} = \mu \delta_{ab} \delta_{ij} + \sum_d \mu_d \lambda^d_{ab} \delta_{ij} + \mu_Q \delta_{ab} Q_{ij},$$

where $a, b = 1, 2, 3$ denote color and $i, j = 1, 2, 3$ flavor indices. The charges of the quark flavors are encoded by the matrix $Q = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) = \frac{1}{2} T^3 + \frac{1}{2\sqrt{3}} T^8$. Thus the ‘bare’ inverse quark propagator takes the form

$$S_0^{-1}(p) = \begin{pmatrix} -i p_i \gamma_i - i(p_4 + i\bar{\mu}) \gamma_4 + m & 0 \\ 0 & -i p_i \gamma_i - i(p_4 - i\mu^T) \gamma_4 + m \end{pmatrix}.$$  

(19)

Compared to the strong interaction, we can consider leptons as non-interacting particles. Only the electrons will be relevant, as the relevant charge chemical potentials will not or only slightly exceed the mass of other charged leptons. With charge $-1$, their charge chemical potential is $\mu_e = -\mu_Q$. This also means

$$\mu_d = \mu_s = \mu_u + \mu_e$$

(20)

for each color, which is usually referred to as $\beta$-equilibrium. We consider the electrons as massless and their electrical charge density given by $\rho_{el} = \frac{1}{3\pi^2} \mu_Q^3$. Electrical neutrality then enforces

$$\rho_Q(x) = Z_2 \int \frac{d^3p}{(2\pi)^3} \int \frac{dp_4}{2\pi} \text{Tr}_{D,c,f} (QS^+ (p) \gamma_4) + \frac{1}{3\pi^2} \mu_Q^3$$

$$\left\{ \begin{array}{l} \equiv 0. \end{array} \right.$$

(21)

D. Parameterization of the CFL phase

A finite strange-quark mass leads to a partial symmetry breaking of the CFL symmetry in the chiral limit through the mass matrix $m = \frac{1}{3} m_s (1 - \sqrt{3} T^8)$. If in addition $\mu_Q \neq 0$ due to the neutrality constraint, we can directly conclude from its generator $Q = \frac{1}{2} T^3 + \frac{1}{2\sqrt{3}} T^8$ that

$$SU_{c+V}(3) \xrightarrow{m_s \neq 0} SU_{c+V}(2) \otimes U_{c+V}(1)$$

$$\mu_Q \neq 0 \xrightarrow{} U_{c+V}(1) \otimes U_{c+V}(1),$$

(22)

where the residual symmetry is generated by $\tau_3 - \lambda_3^a$ and $\tau_8 - \lambda_8^T$. Those form a Cartan subalgebra of the Lie algebra of $SU_{c+V}(3)$, i.e. a maximum set of commuting matrices. The most general choice of a basis $\{P_i\}$ and $\{M_i\}$, constructed as in Ref. [37] for a less complicated ansatz, is given in the Appendix. It consists of 15 matrices for $\{P_i\}$ and $\{M_i\}$, respectively.

To preserve the residual $U_{c+V}(1) \otimes U_{c+V}(1)$ symmetry, we are only allowed to vary the color chemical potentials $\mu_3$ and $\mu_8$. The question is, whether this is enough to fulfill the requirement $\rho^a(x) = 0$ for $a = 1, \ldots, 8$. To clarify this, we define the color-charge density matrix $\hat{\rho}$ by

$$\hat{\rho} = \int \frac{d^3p}{(2\pi)^3} \int \frac{dp_4}{2\pi} \text{Tr}_{D,c,f} (S^+(p) \gamma_4).$$

(23)

This $3 \times 3$ matrix is symmetric and can be interpreted as the matrix of color-charges in the basis $\{u, d, s\}$. With its help the condition in Eq. (16) is reduced to

$$\text{Tr}_c (\hat{\rho} \lambda^a) = 0.$$  

(24)

As the Gell-Mann matrices form an orthogonal basis, we conclude that we need to adjust $\{\mu_a\}$ to get

$$\hat{\rho} = \frac{\rho}{3} \mathbb{1},$$

(25)
with $\rho$ then being the quark number density.

Varying $\mu_3$ and $\mu_8$, we will surely achieve the requirement in Eq. (24) if the matrix is diagonal. This is the case for all NJL-type investigations \cite{13,15,16,17}. However, this is a further truncation on the self-consistency and enforced by hand. In general the matrix $\hat{\rho}$ is only symmetric, if we also consider the color-flavor structures $P_4, \ldots, P_9$. Therefore all $\mu_a$ for $a = 1, \ldots, 8$ need to be taken into account and the residual symmetry gets completely broken. The same arguments also hold if $\mu_Q = 0$, but $\mu_3 \neq 0$ or $\mu_8 \neq 0$.

We conclude, that a finite strange quark mass induces in general backgrounds fields that break the CFL symmetry completely and the CFL phase can no longer be defined by a residual symmetry. It can only be defined by a continuity argument for the ground-state as a function of $m_8$, that is CFL symmetric for $m_8 = 0$.

Instead of introducing further color chemical potentials aside from $\mu_3$ and $\mu_8$, we will estimate those as described below and choose $\mu_3$ and $\mu_8$ such that $\rho^3 = 0$ and $\rho^8 = 0$. This approach is similar to the truncation being used in NJL-type investigation and we could otherwise not constrain $\{P_i\}$ and $\{M_i\}$. The ansatz becomes self-consistent in the 2SC and uSC phase.

E. Estimating $\mu_a$ not in the Cartan subalgebra

Varying $\mu_3$ and $\mu_8$ only, we adjust the $3 \times 3$-dimensional real and positive charge-density matrix $\hat{\rho}$ to have equal diagonal elements and we define $\text{Tr}(\hat{\rho}) = \frac{1}{2}\rho^3_F$. The off-diagonal elements of $\hat{\rho}$ are strongly suppressed. After diagonalization we obtain $\hat{\rho} = \text{diag}(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3) = D_1\hat{\rho}D$ and, approximating the system by a free gas, we estimate $\hat{\rho}_i = \frac{1}{\pi^2}\rho^3_i$. For a free gas we would therefore need to vary $\mu_3$ and $\mu_8$ in the new basis by $\Delta \hat{\mu}_3 = \frac{1}{2}(\hat{p}_2 - \hat{p}_1)$ and $\Delta \hat{\mu}_8 = \frac{1}{2\sqrt{3}}(2\hat{p}_3 - \hat{p}_1 - \hat{p}_2)$ in order to obtain a neutral phase. This will be used as an estimate of $\mu_a$ for $a \neq 3, 8$ after the transformation into the old color basis.

The eigenvalues of $\hat{\rho}$ can be estimated in an expansion in $\hat{\rho}_{ij} \ll \rho_{11}$ for $i \neq j$ and are given by $\hat{\rho}_{11}$ and $\hat{\rho}_{11} \pm \delta \hat{\rho}$, with $\delta \hat{\rho} = \sqrt{\rho_{12}\rho_{21} + \rho_{23}\rho_{32} + \rho_{13}\rho_{31}}$. The corresponding Fermi momenta $\hat{p}_i$ are approximately given by $\hat{p}_F$ and $\hat{p}_F(1 \pm \delta \hat{\rho}/3\hat{\rho}_{ii})$. Therefore, after ordering the Fermi momenta, we have: $\Delta \hat{\mu}_3 \approx \hat{p}_F\delta \hat{\rho}/3\hat{\rho}_{ii}$ and $\Delta \hat{\mu}_8 \approx 0$. As for a free gas we also suspect the chemical potentials to transform like $\hat{\rho} = D_1\hat{\rho}D$ with $DD_1 = 1$. Thus $\|\Delta \hat{\mu}\| = \|\hat{\mu} - \mu\| = \|\hat{\mu} - \mu\| = \|\Delta \hat{\mu}\|$ for any matrix norm and we obtain

$$\mu_a \lesssim \|\Delta \hat{\mu}\| \approx \frac{\delta \hat{\rho}}{3\hat{\rho}_{ii}} \hat{p}_F, \quad a \neq 3, 8. \quad (26)$$

We will come back to this estimate when discussing our numerical results for the chemical potentials.

III. ELECTRONS IN THE CFL PHASE AND GAPLESS PAIRING

A. Are there electrons in the CFL phase?

It has been argued that the color-neutral CFL phase is automatically electrically neutral \cite{33}. Therefore $\mu_Q = 0$ and no electrons are allowed in the phase, which would have important consequences. The same result is also found in self-consistent NJL-type investigations \cite{13,17}. Since we find deviations, we first describe the reasoning in those models.

Apart from the $SU_c(3) \otimes SU_V(3)$, we can also consider the electrical $U_Q(1)$ generated by $Q$. In the dynamical symmetry breakdown $SU_c(3) \otimes SU_V(3) \rightarrow U_{c+V}(1) \otimes U_{c+V}(1)$. Also $U_Q(1)$ gets broken, however the phase is still symmetric under the so-called $U_Q(1)$ symmetry that is generated by

$$\hat{Q} = Q - \frac{1}{2} \lambda^3 - \frac{1}{2\sqrt{3}} \lambda^8 = \frac{1}{2}(\tau^3 - \lambda^{3T}) + \frac{1}{2\sqrt{3}}(\tau^8 - \lambda^{8T}). \quad (27)$$

In the basis $\{(r, u), (g, d), (b, s), (r, d), (g, u), (r, s), (b, u), (g, s), (b, d)\}$ used in the Appendix, the matrix $\hat{Q}$ is diagonal and the quasiparticles carry the charges $(0, 0, 0, -1, -1, 1, 1, 0, 0)$, respectively. From this it has been concluded that $\rho_Q = 0$ in the fully gapped CFL phase, which then is a $\bar{Q}$-insulator \cite{12}. The quark contribution to the thermodynamical potential $p_q[T, \mu, \mu_3, \mu_8, \mu_Q]$ would then be invariant under

$$p_q[T, \mu, \mu_3, \mu_8, \mu_Q] = p_q[T, \mu, \mu_3 - \frac{1}{2} \mu_Q, \mu_8 - \frac{1}{2\sqrt{3}} \mu_Q, \mu_Q + \mu_Q]. \quad (28)$$
Neglecting the leptons, we would therefore have degenerate ground-states under variation of \( \mu_Q \). Among those the real ground-state among those is then chosen by the minimum of the electronic (or leptonic) contribution to the thermodynamic potential, \( p_{el}[T, \mu, \mu_1, \mu_2, \mu_3, \mu_4] = \frac{1}{12\pi^2}\mu_Q^4 \). We would therefore conclude \( \mu_Q = 0 \), which means that no electrons are allowed in the system.

The whole argument is therefore based on the assumption \( \rho_\tilde{Q} = 0 \). Considering the form of the charge \( \tilde{Q} \), we can concentrate on the separate pairing of \( \{(r, a), (g, u)\} \) and \( \{(r, s), (b, u)\} \), respectively, as only those carry non-vanishing \( \tilde{Q} \)-charge (see Appendix). The requirement \( \rho_\tilde{Q} = 0 \) is therefore equivalent to the statement, that two pairing fermion species with different chemical potentials have the same density in a fully gapped phase. We will now show that this is true for energy-independent gap functions, which are used in NJL-type models and emphasize that energy-dependent gap functions, as in our framework, do not require this.

For the case of two fermion species \( a, b \) we neglect normal self-energies and consider

\[
\begin{pmatrix}
S^+_a \\
S^+_b \\
T^+_a \\
T^+_b
\end{pmatrix}
= \begin{pmatrix}
-i\phi + \mu_a \gamma_4 & -i\phi + \mu_b \gamma_4 & -\gamma_5 \Delta^* & -\gamma_5 \Delta \\
\gamma_5 \Delta & -i\phi - \mu_a \gamma_4 & i\phi - \mu_b \gamma_4 \\
\end{pmatrix}
\]

(29)

giving

\[
S_{a/b}^{-1} = -i\phi + \mu_{a/b} \gamma_4 - |\Delta|^2 \frac{i\phi + \mu_{b/a} \gamma_4}{(p_4 - i\mu_{b/a})^2 + p^2}.
\]

(30)

With the definition of \( D_{a/b} = (p_4 + i\mu_{a/b})^2 + p^2 \) and \( \omega_{a/b} = p_4 + i\mu_{a/b} \), the density \( \rho_{a/b}(x) \) of the fermions species \( a \) and \( b \) turns out to be

\[
\rho_{a/b}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{dp_4}{2\pi} \text{Tr}_D \left(S_{a/b}^+(p)\gamma_4\right)
= \int \frac{d^3p}{(2\pi)^3} \frac{dp_4}{2\pi} \frac{4iD_{b/a}}{D_{b/a}^* + |\Delta|^2} \left(\omega_{a/b}D_{b/a}^* + |\Delta|^2 \omega_{b/a}\right)
\]

(31)

For an energy independent gap function \( \Delta \), we can perform the energy integral and obtain for \( |\mu_a - \mu_b| < 2|\Delta| \)

\[
\rho_{a/b}(x) = \int \frac{d^3p}{(2\pi)^3} \left(\frac{\bar{\mu} - p}{\sqrt{(p - \bar{\mu})^2 + |\Delta|^2}} + \frac{\bar{\mu} + p}{\sqrt{(p + \bar{\mu})^2 + |\Delta|^2}}\right),
\]

(32)

where \( \bar{\mu} = \frac{1}{2}(\mu_a + \mu_b) \). The momentum integral is finite for a gap function that vanishing sufficiently fast and the result shows that for small differences in the chemical potential the densities of both particle species are the same. This has also been found in Ref. [38]. For larger splittings, i.e., \( |\mu_a - \mu_b| > 2|\Delta| \), gapless modes emerge (which will be discussed below).

As we have a non-trivial energy dependence of the gap functions the above argument does no longer hold and we may find qualitative differences from NJL-type investigations. This may be directly verified by using an ansatz for the gap functions. The presence of electrons is therefore also connected to a finite width in the spectral function of the quasiparticles. The latter is included in our investigations and is discussed in Ref. [37] for the SC phase.

We would like to clarify that a finite width of the quasiparticles near the Fermi surface may be an artefact of our current truncation, especially since we have not implemented the Meissner effect and the interaction via magnetic gluons is long-ranged. A closer investigation should be subject of future work.

### B. Gapless pairing

The discussion of the last subsection sheds also light on gapless pairing. For illustration we will again use the simple parameterization in Eq. (29). If the difference in the chemical potentials \( \delta \mu = (\mu_a - \mu_b) \) exceeds the value \( |\Delta| \), the gap in the excitation spectrum vanishes and we find gapless modes even for \( \Delta \neq 0 \). Their dispersion relation are given by \( \det S_{a/b}^{-1} = 0 \), which are the zeros of the denominator of the integrand in Eq. (31). For the CFL phase, gapless
Figure 1: Dependence of certain gap functions $\phi_{C,t}^+$ at the Fermi energy and for vanishing momentum (left) and of the (effective) chemical potentials (right) on the renormalized strange-quark mass. Both dependencies are determined at $\mu = 400$ MeV.

pairing might occur in some channels \[^{[39]}\] and has been found as the preferred homogeneous ground-state in NJL-type investigations \[^{[18, 19, 20, 21]}\] due to neutrality constraints. However, they are expected to be unstable as the Meissner masses become imaginary \[^{[22, 23, 24]}\]. This finding led to a broad discussion of the appearance of inhomogeneous phases \[^{[25, 26, 27, 28]}\]. We will postpone this subject to future investigations and consider here homogeneous phases only. The problem of imaginary Meissner masses in the gap equation has not been addressed yet and does not appear in our truncation either.

For $|\delta \mu| > |\Delta|$ gapless modes create a breached pairing region \[^{[40]}\]. For the occupation numbers implicitly given in Eq. (32) and choosing $\mu_a < \mu_b$, we get

$$n_{a/b}(p) = \begin{cases} \frac{\bar{\mu} + p \mp \sqrt{(p - \bar{\mu})^2 + |\Delta|^2}}{2\sqrt{(p - \bar{\mu})^2 + |\Delta|^2}}, & \text{for } \bar{\mu} - 2\sqrt{\delta \mu^2 - |\Delta|^2} < p < \bar{\mu} + 2\sqrt{\delta \mu^2 - |\Delta|^2}, \\ \frac{(\bar{\mu} - p)}{2\sqrt{(p - \bar{\mu})^2 + |\Delta|^2}} + \frac{(\bar{\mu} + p)}{2\sqrt{(p + \bar{\mu})^2 + |\Delta|^2}}, & \text{else}. \end{cases}$$

(33)

We see that in an interval of the Fermi momentum the occupation number is almost vanishing for one species and is almost unity for the other. The deviation of the occupation numbers from zero and unity in this interval for our simple parameterization is solely coming from the anti-quasiparticles and therefore of order $\Delta^2/\mu^2 \ll 1$. The physical interpretation is that the quasiparticles in this ‘breached pairing region’ do not pair.

IV. NUMERICAL RESULTS

We will now present numerical results for the neutral CFL phase. The amount of data is overwhelming, as we calculate $4 \times 15$ dressing functions depending on energy and momentum for normal self-energy, gap function, normal propagator and anomalous propagator, respectively. However, we refer to Refs. \[^{[14, 29]}\] for details and will only highlight new features coming from the neutrality constraints.

In Fig. 1 we present the dependence of the gap functions on the renormalized strange-quark mass taken at the Fermi energy, i.e., for $p_4 = 0$, and for vanishing momentum. As the color-flavor structure of the pairing is quite involved, we refrain from a determination of the Fermi momenta as we have done in Ref. \[^{[14]}\] and focus on the gap functions at vanishing momentum. This is reasonable as the gap functions $\phi_{C,t}^+$ for moderate chemical potentials are almost constant below the Fermi energy (see Refs. \[^{[14, 29]}\]). The gap functions shown belong to the tensor structures $P_1, P_2, P_3, P_4, P_6$ and $P_7$ given in the Appendix and are labeled according to the quarks involved. As can be seen, the gap functions are weakly dependent on the renormalized strange-quark mass up to values of $\sim 135$ MeV. At this value, the pairing of green strange-quarks with blue down-quarks becomes ungapped as discussed in section III B. This can also be observed in the corresponding occupation numbers (see Fig. 2). From the value $\sim 135$ MeV onwards, the pairing gap of strange- and down-quarks ‘melts’ and above $\sim 180$ MeV we are in the uSC phase. This phase exists up
to a value of $\sim 190$ MeV and in the window of $\sim 190 - 200$ MeV we find the 2SC phase. Above this value no spin-zero pairing takes place anymore.

For numerical reasons we have not been able to compute the pressure difference between the phases. However, the transitions of CFL to gCFL phase, and from there to the uSC phase, are likely to be at most of second order because the self-energy and gap functions change continuously. In addition, if the intrinsic symmetry is not altered, the transitions might even be crossovers. The transitions of uSC to 2SC phase, and of the 2SC to the unpaired phase look like first-order transitions. Here we also refer to our earlier observation, see Ref. [14], that in these phases it is much more difficult to find an energetically disfavored solution. (As we solve the quark propagator DSE, being the variation of the CJT action, by iteration in the high-dimensional space of ‘discretized’ dressing functions, the domain of attraction for the global minimum is as usual strongly dominating.) More important, we want emphasize that those transitions are not in the physically relevant range of the strange-quark mass.

On the rhs of Fig. 1 we present the electron chemical potential $\mu_e$, the color chemical potentials $\mu_3$ and $\mu_8$ as well as an upper bound for the chemical potentials not belonging to the Cartan subalgebra being labeled $\mu_\{3,8\}$. It becomes obvious that electrical neutrality puts the strongest constraint on the phase structure and $\mu_e$ even exceeds the size of the largest gaps in the gCFL, uSC and 2SC phase. It is remarkable to note that $\mu_e$ is non-vanishing in the gapped CFL phase as discussed in section III. This can also be seen in Fig. 2, which shows (in this case for green strange- and blue down-quarks) that the occupation numbers of pairing quarks in the gapped phase need not be identical as concluded for energy independent gap functions in Eq. (33). The bound for the chemical potentials not belonging to the Cartan subalgebra is of the order as the chemical $\mu_3$ and $\mu_8$. This shows that all color chemical potentials are much smaller than the electron chemical potential $\mu_e$. Thus we estimate that the treatment of $\mu_3$ and $\mu_8$ alone is not much better justified than neglecting all color chemical potentials.

In Fig. 2 the occupation numbers for the ungapped pairing of green strange- and blue down-quarks at a chemical potential of $\mu = 400$ MeV and for various renormalized strange-quark masses are presented. Qualitatively those show a breached pairing region as found in Eq. (33) for a simple model study. Due to the interaction, these occupation numbers in the breached pairing region are neither almost vanishing nor close to unity, but more or less close to the occupation numbers in the unpaired phase (see also Ref. [29]).

Last but not least, in Fig. 2 we present the critical values of the renormalized strange-quark mass, i.e., the values of the renormalized strange-quark mass which separate the different phases, as a function of the quark chemical potential. As compared to our analysis given in Ref. [14] the transition from the CFL phase is slightly shifted towards higher values of the strange-quark mass when imposing neutrality constraints. This is indeed the anticipated result: The self-energy and gap functions in the CFL phase are only weakly modified by enforcing neutrality constraints, whereas these functions are strongly affected in the phases at large strange-quark masses. In agreement with the general arguments given in the introduction, towards larger strange-quark masses we first find the uSC phase followed by the 2SC phase. Above some critical value of the renormalized strange-quark mass none of the superfluid phases within the class of our ansatz is energetically favored anymore. This is the case because the Fermi surfaces of the different flavors are to far separated to allow for a spin-zero pairing. In this region one expects pairing of quarks with the same flavor only, see also Ref. [14] for a treatment in the Dyson-Schwinger approach. We also find a gCFL phase, which is supposed to be unstable (cf. the discussion given in the introduction).
strange-quark mass, given as a range of values by the particle data group \cite{41}, this becomes only relevant for the stated upper limit of the strange-quark mass and quark chemical potentials $\lesssim 370\text{MeV}$. For those values NJL-type calculations \cite{13, 15, 16, 17} typically find the chirally broken phase. (In order to achieve a conservative estimate for color-flavor unlocking this does not yet take place in the presently used truncation due to our approximation for the medium polarization.) In addition, it is worth mentioning that solutions of the Bethe-Salpeter equation within the DSE approach in the MOM scheme favor small values for the physical strange quark-current mass \cite{45}, a fact which is also related to the difference in renormalization schemes. Taking into account all these facts we conclude that among all possible color-superconducting phases only the CFL phase is realized for physically relevant strange-quark masses and chemical potentials.

V. CONCLUSIONS

As could have been anticipated from our previous investigation of color-flavor unlocking, the CFL phase stays the ground-state at vanishing temperatures and above the critical chemical potential for the chiral phase transition also when imposing neutrality constraints. The underlying reason for this result, which deviates from the one obtained from NJL-type models, is the inclusion of medium polarization effects. The latter play an important role in presence of a Fermi surface and lead to smaller dynamical mass generation due to damping and screening \cite{14}. As a consequence a stronger explicit symmetry breaking by the bare strange-quark mass as present in nature would be needed to separate the Fermi surfaces and to ‘unlock’ the CFL phase. To reinforce our conclusions we have used a coupling, which is rather weak and not able to give physical values for the chiral condensate and pion decay constant in the vacuum and which thus should favor color-flavor unlocking.

Furthermore, it has been clarified that for a non-vanishing strange-quark mass the CFL phase cannot be defined by a residual global symmetry, and thus should better be named color-flavor-locked-like phase. This can be traced back to the fact that non-vanishing chemical potentials $\mu_3$, $\mu_8$ or $\mu_Q$ induce static gluonic background fields which break the residual symmetry. This effect is, however, estimated to be small.

A surprising feature is the appearance of electrons in the CFL phase which has been expected not to take place. Their presence is allowed due to the energy-dependence of self-energy and gap functions which give the quasiparticles a considerable width but may an artefact of the employed truncation.

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Appendix: THE COLOR-FLAVOR STRUCTURE OF THE CFL PHASE

For the neutral CFL phase with two degenerate quarks and non-vanishing strange-quark mass, we generalize the ansatz of Refs. \cite{11, 14} by choosing the matrices

$$P_i = \begin{pmatrix}
\delta_{i1} & \delta_{i4} & \delta_{i6} \\
\delta_{i5} & \delta_{i2} & \delta_{i7} \\
\delta_{i8} & \delta_{i9} & \delta_{i3} \\
\delta_{i10} & & \\
\delta_{i11} & & \\
\delta_{i12} & & \\
\delta_{i13} & & \\
\delta_{i14} & & \\
\delta_{i15} & & 
\end{pmatrix},$$

\[\text{A.1}\]
This is a complete basis for a residual $U_{c\rightarrow v}(1) \otimes U_{c\rightarrow v}(1)$ symmetry (see Eq. (22)). The basis is defined by

$$\{ (r, u), (g, d), (b, s), (r, d), (g, u), (r, s), (b, u), (g, s), (b, d) \},$$

with $r$, $g$, $b$ denoting the color and $u$, $d$, $s$ the flavor of the quarks.

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