MULTI-PATH MATROIDS

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Abstract. We introduce the minor-closed, dual-closed class of multi-path matroids. We give a polynomial-time algorithm for computing the Tutte polynomial of a multi-path matroid, we describe their basis activities, and we prove some basic structural properties. Key elements of this work are two complementary perspectives we develop for these matroids: on the one hand, multi-path matroids are transversal matroids that have special types of presentations; on the other hand, the bases of multi-path matroids can be viewed as sets of lattice paths in certain planar diagrams.

1. Introduction

In [2] it is shown how to construct, from a pair $P,Q$ of lattice paths that go from $(0,0)$ to $(m,r)$, a transversal matroid $M[P,Q]$ whose bases correspond to the paths from $(0,0)$ to $(m,r)$ that remain in the region bounded by $P$ and $Q$. The basic enumerative and structural properties of these matroids, which are called lattice path matroids, are developed in [2, 3]. This paper introduces multi-path matroids, a generalization of lattice path matroids that share many of their most important properties.

Section 2 starts by reviewing the definition of lattice path matroids as well as an alternative perspective on these matroids that uses collections of incomparable intervals in a linear order. This alternative perspective leads to the starting point for multi-path matroids: the linear order is replaced by a cyclic permutation. In addition to defining and providing examples of multi-path matroids, Section 2 also defines basic concepts that are used in the rest of the paper.

Section 3 shows that the dual and all minors of a multi-path matroid are multi-path matroids (lattice path matroids have the corresponding properties; transversal matroids do not). Proving these properties entails developing several alternative presentations for multi-path matroids. In particular, we show that the bases of a multi-path matroid can be viewed as certain sets of lattice paths in a diagram (such as that in Figure 4 on page 9) that has fixed global bounding paths and one or more pairs of starting and ending points.

The diagrams we develop in Section 3 are crucial tools in the next two sections. Section 4 shows that the Tutte polynomial of a multi-path matroid can be computed in polynomial time in the size of the ground set. This result stands in contrast to the following result of [5]: for any fixed algebraic numbers $x$ and $y$ with $(x-1)(y-1) \neq 1$,
the problem of computing \( t(M; x, y) \) for an arbitrary transversal matroid \( M \) is \( \#P \)-complete. Our work on the Tutte polynomial is cast in the general framework of what we call computation graphs, which allow us to apply the idea of dynamic programming to this computation.

Section 5 shows that, as is true of lattice path matroids, internal and external activities of bases of multi-path matroids have relatively simple lattice-path interpretations. We also sketch a somewhat faster, although more complex, algorithm for computing the Tutte polynomial of a multi-path matroid via basis activities.

The final section addresses several structural properties of multi-path matroids. For instance, we show that multi-path matroids that are not lattice path matroids have spanning circuits and we make some comments about minimal presentations of multi-path matroids.

We close this introduction by recalling several key notions; see [9] for concepts of matroid theory not defined here. A set system is a multiset \( A = (A_j : j \in J) \) of subsets of a finite set \( S \). A transversal of \( A \) is a set \( \{x_j : j \in J\} \) of \( |J| \) distinct elements such that \( x_j \) is in \( A_j \) for all \( j \) in \( J \). A partial transversal of \( A \) is a transversal of a set system of the form \( (A_k : k \in K) \) with \( K \) a subset of \( J \). Edmonds and Fulkerson [6] showed that the partial transversals of a set system \( A \) are the independent sets of a matroid on \( S \). This matroid \( M[A] \) is a transversal matroid and the set system \( A \) is a presentation of \( M[A] \). For a basis \( B \) of \( M[A] \), a matching of \( B \) with \( A \) is a function \( \phi : B \to A \) such that

1. \( b \) is in \( \phi(b) \) for each \( b \) in \( B \) and
2. the number of elements of \( B \) that \( \phi \) maps to any set \( X \) in \( A \) is at most the multiplicity of \( X \) in \( A \).

This terminology is suggested by the interpretation of set systems as bipartite graphs [9, Section 1.6]. In this paper, \( A \) will typically be an antichain, that is, no set in \( A \) contains another set in \( A \). Presentations of transversal matroids are generally not unique. A presentation \( (A_1, A_2, \ldots, A_r) \) of the transversal matroid \( M \) contains the presentation \( (A'_1, A'_2, \ldots, A'_r) \) of \( M \) if \( A'_i \subseteq A_i \) for all \( i \) with \( 1 \leq i \leq r \). We let \( [n] \) denote the set \( \{1, 2, \ldots, n\} \).

2. Basic Definitions

We start by reviewing lattice path matroids and an alternative perspective on these matroids. The majority of this section is devoted to defining multi-path matroids, providing illustrations, and defining notation and concepts that are used in the rest of the paper. Lattice path matroids were introduced in [2]; special classes of lattice path matroids had been studied earlier from other perspectives (see Section 4 of [3]).

A lattice path can be viewed geometrically as a path in the plane made up of unit steps East and North, or, more formally, as a word in the alphabet \( \{E, N\} \), where \( E \) denotes the East step \((1,0)\) and \( N \) denotes the North step \((0,1)\). When viewed as a word in the alphabet \( \{E, N\} \), a lattice path does not have fixed starting and ending points. Thus, one may identify different geometric lattice paths that arise from the same word; whether we identify such paths will depend on, and should be clear from, the context.

Fix lattice paths \( P \) and \( Q \) from \((0,0)\) to \((m,r)\) with \( P \) never going above \( Q \). Let \( \mathcal{P} \) be the set of all lattice paths from \((0,0)\) to \((m,r)\) that go neither above \( Q \) nor
below $P$. For $i$ with $1 \leq i \leq r$, let $N_i$ be the set
\[ N_i = \{ j : \text{step } j \text{ is the } i\text{-th North step of some path in } P \} . \]

The matroid $M[P,Q]$ is the transversal matroid on the ground set $[m+r]$ that has
$(N_1, N_2, \ldots, N_r)$ as a presentation. Note that $M[P,Q]$ has rank $r$ and nullity $m$.

A lattice path matroid is any matroid isomorphic to such a matroid $M[P,Q]$.

Figure 1 shows a lattice path matroid of rank 4 and nullity 5. The sets $N_1$, $N_2$, $N_3$, and $N_4$ are
$\{1, 2, 3, 4, 5\}$, $\{2, 3, 4, 5, 6, 7\}$, $\{5, 6, 7, 8\}$, and $\{7, 8, 9\}$. As this example suggests, the sets
$N_1, N_2, \ldots, N_r$ are intervals in $[m+r]$, and both the left endpoints and the right endpoints form strictly increasing sequences. This motivates the following result from [3].

**Theorem 2.1.** A matroid is a lattice path matroid if and only if it is transversal and some presentation is an antichain of intervals in a linear order on the ground set.

The following result [2, Theorem 3.3] starts to suggest a deeper connection with lattice paths.

**Theorem 2.2.** The map $R \mapsto \{ i : \text{the } i\text{-th step of } R \text{ is North} \}$ is a bijection from $P$ onto the set of bases of $M[P,Q]$.

Multi-path matroids are the generalizations of lattice path matroids that result from using a cyclic permutation in place of the linear order in Theorem 2.1.

Fix a cyclic permutation $\sigma$ of the set $S$. A $\sigma$-interval (or simply an interval) in $S$ is a nonempty subset $I$ of $S$ of the form $\{f_1, \sigma(f_1), \sigma^2(f_1), \ldots, l_I\}$; this $\sigma$-interval is denoted $[f_I, l_I]$ and the elements $f_I$ and $l_I$ are called, respectively, the first and last element of $I$. Note that singleton subsets (which arise if $f_I$ is $l_I$) as well as the entire set $S$ (which arises if $\sigma(l_I)$ is $f_I$) are $\sigma$-intervals. If one views the elements of $S$ placed around a circle in the order given by $\sigma$, then the $\sigma$-intervals are the sets of elements that can be covered by arcs of the circle; in the case of a $\sigma$-interval $[f_I, l_I]$ that is $S$, the arc has a gap between $l_I$ and $f_I$.

We now define our main object of study.

**Definition 2.3.** A multi-path matroid is a transversal matroid that has a presentation by an antichain of $\sigma$-intervals in some cyclic permutation $\sigma$ of the ground set.
The term “multi-path” comes from the alternative perspective on these matroids given in Theorem 3.6. To distinguish the different types of presentations of interest in this paper, presentations of the type in Theorem 2.1 are interval presentations, while those of the type in Definition 2.3 are \(\sigma\)-interval presentations.

Note that the first elements \(f_{I_1}, f_{I_2}, \ldots, f_{I_r}\) that arise from an antichain \(I = (I_1, I_2, \ldots, I_r)\) of \(\sigma\)-intervals are distinct; thus, the rank of \(M[I]\) is \(r\), the number of intervals. Also, for \(I\) to be an antichain of \(\sigma\)-intervals, the set \(S\) can be in \(I\) only if \(r\) is 1. However, Lemma 3.2 shows that the antichain condition in Definition 2.3 can be relaxed without changing the resulting class of matroids; in some cases this relaxation allows \(S\) to be in \(I\).

In the following examples, \(S\) is the set \([n]\) and \(\sigma\) is the cycle \((1, 2, \ldots, n)\). Since a linear order can be “wrapped around” to obtain a cycle, every lattice path matroid is a multi-path matroid. The converse is not true, as the first example shows.

**Example 1.** The 3-whirl \(W^3\) is an excluded-minor for the class of lattice path matroids \([3]\); however, Figure 2 shows that the 3-whirl is a multi-path matroid. The three intervals are \(I_1 = \{1, 2, 3\}\), \(I_2 = \{3, 4, 5\}\), and \(I_3 = \{5, 6, 1\}\). A similar construction shows that all whirls are multi-path matroids.

There is only one interval presentation of a lattice path matroid \(M[P, Q]\) since \(P\) and \(Q\) correspond to, respectively, the greatest and least bases in lexicographic order. (See also \([3]\) Theorem 5.6.) In contrast, even lattice path matroids can have multiple \(\sigma\)-interval presentations, as the next example shows.

**Example 2.** All uniform matroids are lattice path matroids. The following set systems are different \(\sigma\)-interval presentations of the uniform matroid \(U_{3,6}\) of rank 3 on the set \([6]\):

\[
\begin{align*}
&\{(1, 2, 3, 4), \{2, 3, 4, 5\}, \{3, 4, 5, 6\}\}, \\
&\{(1, 2, 3, 4, 5), \{2, 3, 4, 5, 6\}, \{3, 4, 5, 6, 1\}\}.
\end{align*}
\]

A presentation \(A\) of a transversal matroid \(M\) is minimal if no other presentation of \(M\) is contained in \(A\). Interval presentations of lattice path matroids are minimal \([3]\) Theorem 6.1. We next show that multi-path matroids that are not lattice path matroids can have multiple minimal presentations that are \(\sigma\)-interval presentations.
that if some first element \( f \) is in \( I \times <\sigma \) into the proof of closure under duality.

The lattice path interpretations as well as closure under contractions enter multi-path matroids, some of which involve lattice paths and so account for the arbitrary transversal matroids. We also develop several alternative descriptions of and duals. (Analogous properties hold for lattice path matroids but not for arbitrary transversal matroids.) We use \( \Sigma \) to denote the cyclic permutation \( \sigma \) induced in \( X \) by skipping over the elements that are not in \( X \).

There is an induced cycle on the \( \sigma \)-intervals in an antichain \( I = (I_1, I_2, \ldots, I_r) \) of \( \sigma \)-intervals. Indeed, the last elements \( l_{I_1}, l_{I_2}, \ldots, l_{I_r} \) are distinct since \( I \) is an antichain, so a cyclic permutation \( \Sigma \) on \( I \) is given by \( \Sigma(I_j) = I_k \) if \( \sigma_X(I_j) = l_k \) where \( X = \{l_{I_1}, l_{I_2}, \ldots, l_{I_r}\} \). Likewise the cycle \( \sigma_Y \) on \( Y = \{f_{I_1}, f_{I_2}, \ldots, f_{I_r}\} \) induces a cyclic permutation \( \Sigma' \) on \( I \). The assumption that \( I \) is an antichain gives the equality \( \Sigma = \Sigma' \). We use \( \Sigma \) to denote the cyclic permutation of \( I \) induced in this manner from \( \sigma \). For instance, in Example 1, \( \Sigma = (I_1, I_2, I_3) \).

Fix an element \( x \) in a \( \sigma \)-interval \( I \). It will be useful to consider the two parts in which \( I - x \) naturally comes. The first part of \( I - x \) is the empty set if \( x = f_1 \), or the \( \sigma \)-interval \([f_1, \sigma^{-1}(x)]\) if \( x \) is not \( f_1 \). Similarly, the last part of \( I - x \) is the empty set if \( x = l_1 \), or the \( \sigma \)-interval \([\sigma(x), l_1] \) if \( x \) is not \( l_1 \). From the set \( I - x \) alone, references to the first and last parts could be ambiguous (for instance, if \( x = f_1 \) or \( l_1 \)), but it will be clear from the context, so no confusion should result.

Note that an element \( x \) in \( S \) is a loop of \( M[I] \) if and only if \( x \) is in no interval in \( I \). Thus if \( x \) is a loop, then the intervals in \( I \) are intervals in the linear order \( x < \sigma(x) < \sigma^2(x) < \cdots < \sigma^{-1}(x) \), so \( M[I] \) is a lattice path matroid. Likewise, note that if some first element \( f_I \) is not in \( \Sigma^{-1}(I) \), then \( M[I] \) is a lattice path matroid.

3. MINORS, DUALS, AND THE LATTICE PATH INTERPRETATION

This section shows that the class of multi-path matroids is closed under minors and duals. (Analogous properties hold for lattice path matroids but not for arbitrary transversal matroids.) We also develop several alternative descriptions of multi-path matroids, some of which involve lattice paths and so account for the name. The lattice path interpretations as well as closure under contractions enter into the proof of closure under duality.

![Figure 3. A multi-path matroid that has multiple minimal presentations.](image)
We start with a simple lemma that applies to all transversal matroids.

**Lemma 3.1.** Assume $X$ and $Y$ are in a set system $A$ with $X \subseteq Y$. Let $z$ be in $X$ and let $A'$ be obtained from $A$ by replacing one or more occurrences of $Y$ by $Y - z$. Then $M[A] = M[A']$.

**Proof.** Note that it suffices to prove the result in the case that one occurrence of $Y$ is replaced by $Y - z$, and for this it suffices to show that for any basis $B$ of $M[A]$ and matching $\phi : B \to A$, we can find a matching $\phi' : B \to A'$. Clearly there is such a matching $\phi'$ if $z$ is not in $B$, or if $z$ is in $B$ but $\phi(z)$ is not $Y$. Thus assume that $z$ is in $B$ and $\phi(z)$ is $Y$. If $X$ is not in the image of $\phi$, then the map $\phi'$ that agrees with $\phi$ except that $\phi'(z)$ is $X$ is the required matching. Now assume $\phi(x)$ is $X$ for some $x$ in $B$. Since $x$ is in $X$ and therefore in $Y$, the following map $\phi'$ is the required matching:

$$\phi'(w) = \begin{cases} X, & \text{if } w = z; \\ Y - z, & \text{if } w = x; \\ \phi(w), & \text{otherwise}. \end{cases}$$

It is well known and easy to see that if $A = (A_1, A_2, \ldots, A_e)$ is a presentation of a transversal matroid $M$ on $S$, then any single-element deletion $M\setminus x$ is transversal and $A' = (A_1 - x, A_2 - x, \ldots, A_e - x)$ is a presentation of $M\setminus x$. Since deleting $\emptyset$ from any set system in which it appears does not change the associated transversal matroid, we may assume that $\emptyset$ is not in $A'$. Note that if $A$ is an antichain of $\sigma$-intervals, then the sets in $A'$ are $\sigma_{S-x}$-intervals, but there may be containments among these sets. This issue is addressed through the next lemma, which gives a relaxation of the antichain criterion in Definition 2.3.

**Lemma 3.2.** Assume the transversal matroid $M$ has a presentation by a multiset $A$ of $\sigma$-intervals that satisfies the following condition:

(C) if $I \subseteq J$ for $I, J \in A$, then either $f_J$ or $l_J$ is in $I$.

Then $M$ is a multi-path matroid and $A$ contains a $\sigma$-interval presentation of $M$.

**Proof.** If $A$ is an antichain, there is nothing to prove, so assume $I$ and $J$ are in $A$ and $I \subseteq J$. By condition (C) and symmetry, we may assume $f_J$ is in $I$. By replacing $J$ if needed, we may assume no $\sigma$-interval in $A$ whose first element is $f_J$ properly contains $J$. Let $A'$ be the set system obtained from $A$ by replacing $J$ by the $\sigma$-interval $J - f_J$, or eliminating $J$ if $J - f_J$ is empty. Lemma 3.1 gives the equality $M[A] = M[A']$; we will show that $A'$ satisfies condition (C). The presentation of $M$ by $\sigma$-intervals that results from applying this modification as many times as possible must be an antichain, which proves the lemma.

To show that $A'$ satisfies condition (C), first note that the only pairs of intervals that potentially could contradict condition (C) must include $J - f_J$. Let $K$ be another interval in $A'$. If the containment $K \subseteq J - f_J$ holds, then $K$ is a subset of $J$ but does not contain $f_J$; it follows from condition (C) applied to $J$ and $K$ in $A$ that $l_J$ (which is also $l_{J-f_J}$) must be in $K$, as needed. Now assume the containment $J - f_J \subseteq K$ holds. If $l_K$ is in $J - f_J$, there is nothing to show, so assume this is not the case. Since $J$ is the largest set in $A$ that has $f_J$ as its first element, $f_K$ is not $f_J$. If $\sigma^{-1}(f_J)$ were in $K$, then $J$ and $K$ would contradict condition (C) for $A$. Thus, the first element of $K$ must be $\sigma(f_J)$, so $f_K$ is in $J - f_J$, as needed. \qed
It is easy to check that if \((I_1, I_2, \ldots, I_r)\) is an antichain of \(\sigma\)-intervals, then the set system \((I_1 - x, I_2 - x, \ldots, I_r - x)\) satisfies condition \((C)\) of Lemma \ref{lem:contraction}. This observation along with the remarks before that lemma prove the following theorem.

**Theorem 3.3.** The class of multi-path matroids is closed under deletion.

To show that the class of multi-path matroids is closed under contractions, we give presentations of single-element contractions (Lemma \ref{lem:contraction}) that we then show satisfy condition \((C)\) of Lemma \ref{lem:contraction}.

**Lemma 3.4.** Let the antichain \(I\) of \(\sigma\)-intervals be a presentation of \(M\), and let \(\Sigma\) be the cycle \((I_1, \ldots, I_t)\) where \(I_1, \ldots, I_t\) are the \(\sigma\)-intervals that contain a given element \(x\). A presentation of the contraction \(M/x\) is given by:

\[
\begin{align*}
(a) & \; I, \text{ for } t = 0; \\
(b) & \; (I_2, I_3, \ldots, I_r), \text{ for } t = 1; \\
(c) & \; I':= \{(I_1 \cup I_2) - x, (I_2 \cup I_3) - x, \ldots, (I_{t-1} \cup I_t) - x, I_{t+1}, \ldots, I_r\}, \text{ for } t > 1.
\end{align*}
\]

**Proof.** Part \((a)\) holds since \(x\) is a loop of \(M\) if \(t = 0\). If \(t\) is positive, then \(x\) is not a loop, so the bases of \(M/x\) are the subsets \(B\) of \(S - x\) such that \(B \cup x\) is a basis of \(M\). Part \((b)\) follows since matchings \(\phi : B \cup x \rightarrow I\) map \(x\) to \(I_1\). For part \((c)\), we need to show that for subsets \(B\) of \(S - x\), there is a matching \(\phi : B \cup x \rightarrow I\) if and only if there is a matching \(\phi' : B \rightarrow I'\).

Assume first that \(\phi : B \cup x \rightarrow I\) is a matching. Assume \(\phi(x) = I_h\) and \(\phi(b_i)\) is \(I_t\) for all \(i\) with \(i \neq h\) and \(1 \leq i \leq r\). The necessary matching \(\phi'\) is given by:

\[
\phi'(b_i) = \begin{cases} 
(I_t \cup I_{t+1}) - x, & \text{if } 1 \leq i < h; \\
(I_{t-1} \cup I_t) - x, & \text{if } h < i \leq t; \\
I_t, & \text{if } t < i \leq r.
\end{cases}
\]

Now assume \(\phi' : B \rightarrow I'\) is a matching and \(\phi'(b_i) = (I_i \cup I_{i+1}) - x\) for \(1 \leq i < t\). Since \(x\) is in \(I_1, I_2, \ldots, I_t\), to complete the proof it suffices to construct an injection \(\psi : \{b_1, b_2, \ldots, b_{t-1}\} \rightarrow \{I_1, I_2, \ldots, I_t\}\) with each \(b_i\) in \(\psi(b_i)\). Toward this end, classify \(b_1, b_2, \ldots, b_{t-1}\) as follows: \(b_i\) is a *leader* if it is in the first part of \(I_i - x\), otherwise \(b_i\) is a *trailer*. Note that if \(b_i\) is a leader, then \(b_i\) is in the first part of \(I_i - x\) for every \(j\) with \(1 \leq j \leq i\). Similarly, if \(b_i\) is a trailer, then \(b_i\) is in the last part of \(I_i - x\) for every \(j\) with \(i + 1 \leq j \leq t\). Define \(\psi\) as follows: scan \(b_1, b_2, \ldots, b_{t-1}\) in this order and for each leader \(b_i\), let \(\psi(b_i)\) be the first set among \(I_1, I_2, \ldots, I_t\) that is not already in the image of \(\psi\); then scan \(b_{t-1}, b_{t-2}, \ldots, b_1\) in this order and for each trailer \(b_i\), let \(\psi(b_i)\) be the last set among \(I_1, I_2, \ldots, I_t\) not already in the image of \(\psi\). Clearly \(\psi\) is injective and \(b_i\) in \(\psi(b_i)\) for all \(i\).

With this lemma, we can now complete our work on contractions.

**Theorem 3.5.** The class of multi-path matroids is closed under contraction.

**Proof.** We use the notation of Lemma \ref{lem:contraction}. It suffices to show that \(M/x\) is a multi-path matroid. This follows easily from parts \((a)\) and \((b)\) of Lemma \ref{lem:contraction} if \(t\) is at most 1, so assume \(t\) exceeds 1. To show that \(M/x\) is a multi-path matroid, it suffices to show that \(I'\) satisfies condition \((C)\) of Lemma \ref{lem:contraction}. To consider the sets in \(I'\) as \(\sigma_{S-x}\)-intervals, we need only specify the endpoints of any set that is \(S - x\). If \(I_h\) is \(S - x\), where \(t < h \leq r\), we take \(I_h\) to be the \(\sigma_{S-x}\)-interval \([\sigma(x), \sigma^{-1}(x)]\). If \((I_t \cup I_{t+1}) - x\) is \(S - x\), we take this to be the \(\sigma_{S-x}\)-interval \([f_{I_t}, \sigma^{-1}(f_{I_t})]\). Note that there are only three possible containments among the sets in \(I'\):
(i) \((I_i \cup I_{i+1}) - x \subseteq (I_j \cup I_{j+1}) - x\) with \(1 \leq i, j < t\),
(ii) \((I_i \cup I_{i+1}) - x \subseteq I_h\) with \(1 \leq i < t\) and \(t < h \leq r\), and
(iii) \(I_h \subseteq (I_i \cup I_{i+1}) - x\) with \(1 \leq i < t\) and \(t < h \leq r\).

In case (i), note that if \(f_{I_j}\) is not in \((I_i \cup I_{i+1}) - x\), then \(j < i\). It follows that \(I_{j+1}\) is in the \(\sigma\)-interval \([\sigma(x), \sigma^{-1}(I_{j+1})]\), so \(I_{j+1}\) is not in \((I_j \cup I_{j+1}) - x\). This contradicts the assumed containment, so \(f_{I_j}\) is in \((I_i \cup I_{i+1}) - x\) and condition (C) holds in case (i). Note that the containment in case (ii) holds only if \(I_h = [\sigma(x), \sigma^{-1}(x)]\), so condition (C) clearly holds in this case also. Lastly, consider the containment in case (iii). Since \(x\) is not in \(I_h\), if \(f_{I_i}\) were not in \(I_h\), then \(I_h\) would be either contained in or disjoint from \([\sigma(f_{I_i}), \sigma^{-1}(x)]\), so either \(I_h \subseteq I_i\) or \(I_h \subseteq I_{i+1}\) would hold. That both conclusions are contrary to \(I\) being an antichain shows that \(f_{I_t}\) is in \(I_h\), so condition (C) of Lemma 3.2 holds. Thus, \(M/x\) is a multi-path matroid.

We now give an alternative perspective on multi-path matroids that accounts for the name, extends the path interpretation of lattice path matroids, and plays a pivotal role in much of the rest of this paper. Figure 4 illustrates these ideas with the 3-whirl (Example 1 in Section 3). Assume \(M[I]\) has rank \(r\) and nullity \(m\). Fix an element \(x\) of \(M[I]\). (In Figure 4, \(x\) is 1.) Let the cyclic permutation \(\Sigma\) of \(I\) be \((I_1, I_2, \ldots, I_k)\) where the intervals \(I_j\) with \(x \in I_j\) and \(x \neq f_{I_j}\) are \(I_1, I_2, \ldots, I_k\). Note that the linear order \(I_1, I_2, \ldots, I_r\), which plays an important role below, has been specified unless \(k\) is 1 or \(k - 1\) is \(r\). For \(k = 1\), let \(I_1\) be the interval \(I\) in \(I\) that minimizes the size of the interval \([x, f_{I_1}]\). For \(k - 1 = r\), let \(I_1\) be the interval \(I\) in \(I\) that minimizes the size of the interval \([x, I_1]\). Consider the subsets \(\{p_1, p_2, \ldots, p_k\}\) and \(\{p'_1, p'_2, \ldots, p'_k\}\) of \(\mathbb{Z}^2\) where \(p_i = (k - i, i - 1)\) and \(p'_i = p_i + (m, r)\). Let \(L\) and \(L'\) be the lines of slope \(-1\) that contain these sets. Let \(P\) be the lattice path from \(p_1\) to \(p'_1\) formed from the sequence \(x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x)\) by replacing each element \(I_{f_{I_j}}\), for \(I_j \in I\), by a North step and replacing the other elements by East steps. Let \(Q\) be the lattice path from \(p_k\) to \(p'_k\) formed from \(x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x)\) by replacing each element \(I_{f_{I_j}}\), for \(I_j \in I\), by a North step and replacing the other elements by East steps. Note that \(P\) never goes above \(Q\). The lines \(L\) and \(L'\) and the paths \(P\) and \(Q\) bound the region of interest. Label the North and East steps in this region as follows: steps that are adjacent to the points \(p_1, p_2, \ldots, p_k\) are labelled \(x\), those one step away from \(p_1, p_2, \ldots, p_k\) are labelled \(\sigma(x)\), and so on. The resulting diagram, which we denote by \(D(I, x)\), depends on both \(I\) and \(x\). (To simplify the example, the diagram shown in Figure 4 omits the labels on the East steps.) The diagram \(D(I, x)\) captures the set system \(I\): each interval among \(I_k, I_{k+1}, \ldots, I_r\) is the set of labels on the North steps in one row; each interval \(I_i\) among \(I_1, I_2, \ldots, I_{k-1}\) also appears in this way, but split into two parts, with \(x\) and the elements in the last part of \(I_i - x\) appearing among the lowest \(k - 1\) rows and with the elements in the first part of \(I_i - x\) appearing among the highest \(k - 1\) rows. Theorem 3.6, which is a counterpart of Theorem 2.2, shows the significance of \(D(I, x)\).

**Theorem 3.6.** Fix an element \(x\) in a multi-path matroid \(M[I]\). A set \(B\) is a basis of \(M[I]\) if and only if there is a lattice path \(R\) such that

(i) \(R\) goes from a point \(p_i\) to the corresponding point \(p'_i\),
(ii) \(R\) uses East and North steps of the diagram \(D(I, x)\), and
(iii) the labels on the North steps of \(R\) are the elements of \(B\).
Proof. Let \( b_1, b_2, \ldots, b_r \), in this order, be the labels on the North steps of a path \( R \) that satisfies conditions (i) and (ii). Thus, \( b_1, b_2, \ldots, b_r \) are contained, respectively, in \( r \) consecutive intervals in the cycle \((I_1, I_2, \ldots, I_r)\); also, \( b_1, b_2, \ldots, b_r \) are distinct since the North and East steps of \( R \) are labelled, in order, \( x, \sigma(x), \sigma^2(x), \ldots, \sigma^{-1}(x) \). It follows that \( \{b_1, b_2, \ldots, b_r\} \) is a basis of \( M[Z] \).

For the converse, we use the notation established when defining the diagram \( D(\mathcal{I}, x) \). All references to an order on the ground set \( S \) are to the linear order \( x < \sigma(x) < \sigma^2(x) < \cdots < \sigma^{-1}(x) \). Assume \( B \) is a basis of \( M[Z] \) and let \( \phi : B \rightarrow \mathcal{I} \) be a matching. To complete the proof, it suffices prove the following claim.

The elements of \( B \), listed in order as \( b_1, b_2, \ldots, b_r \), are in the sets \( I_{k-t}, I_{k-t+1}, \ldots, I_{k-1}, I_k, \ldots, I_r, I_I, I_2, \ldots, I_{k-t-1} \), respectively, for some \( t \) with \( 0 \leq t \leq k - 1 \).

Indeed, the required path \( R \) takes East steps from \( p_{k-t} \) until a North step labelled \( b_1 \) is reached; after taking that North step, East steps are taken until a North step labelled \( b_2 \) is reached, and so on.

We prove the claim by first constructing a matching for a different set system. For \( i \) with \( 1 \leq i \leq k - 1 \), let \( X_i \) be the first part of \( I_i \) with respect to \( x \) and let \( Y_i \) be \( I_i - X_i \). Note that the set \( \phi^{-1}(\{I_1, I_2, \ldots, I_{k-1}\}) \) is the disjoint union of two subsets whose elements are, in order, say, \( b_1', b_2', \ldots, b_i', b_i'' \) and \( b_1'', b_2'', \ldots, b_i'' \), where each \( b_i' \) is in the subset \( Y_j \) of the set \( I_j = \phi(b_{j}') \) while each \( b_i'' \) is in the subset \( X_j \) of the set \( I_j = \phi(b_{j}'') \). Thus, \( 0 \leq t \leq k - 1 \). Let \( \mathcal{I}' \) be the set system that consists of the intervals

\[ Y_{k-t}, Y_{k-t+1}, \ldots, Y_{k-1}, I_k, \ldots, I_r, X_1, X_2, \ldots, X_{k-t-1}. \]

We also let \( Z_1, Z_2, \ldots, Z_r \), respectively, denote these intervals. Let \( \Phi : B \rightarrow \mathcal{I}' \) be given by

\[
\Phi(b) = \begin{cases} 
Y_{k-1-t+i}, & \text{if } b = b_i' \text{ with } 1 \leq i \leq t; \\
X_i, & \text{if } b = b_i'' \text{ with } 1 \leq i \leq k - 1 - t; \\
\phi(b), & \text{if } b \in \phi^{-1}(\{I_k, I_{k+1}, \ldots, I_r\}).
\end{cases}
\]

The inclusions \( Y_1 \subset Y_2 \subset \cdots \subset Y_{k-1} \) and \( X_{k-1} \subset X_{k-2} \subset \cdots \subset X_1 \) imply that \( \Phi \) is a matching.

Finally, to prove the claim it suffices to show that the \( i \)-th element \( b_i \) of \( B \) is in \( Z_i \). If this statement were false, then either \( b_i < f_{Z_i} \) or \( b_i > l_{Z_i} \). The first option would imply that the \( i \) elements \( b_1, b_2, \ldots, b_i \) can be in only \( i - 1 \) sets, namely \( Z_1, Z_2, \ldots, Z_{i-1} \); the second option would imply that the \( r - i + 1 \) elements

\[ \Phi(\phi^{-1}(\{I_k, I_{k+1}, \ldots, I_r\})) \]

are in only \( i - 1 \) sets.
Figure 5. The dual of the 3-whirl \(W^3\) via flipping diagrams about the line \(y = x\).

\[ b_i, b_{i+1}, \ldots, b_r \] can be in only \(r - i\) sets, namely \(Z_{i+1}, Z_{i+2}, \ldots, Z_r\). Both conclusions are contradicted by the matching \(\Phi\), so the claim and the theorem follow. \(\Box\)

Unlike Theorem 2.2, the correspondence between paths and bases in Theorem 3.6 is not bijective. For example, the two paths in Figure 5 indicated by thick lines correspond to the basis \(\{1, 3, 5\}\). While some bases correspond to a single path, in general each basis corresponds to a family of paths that arise from a single word in the alphabet \(\{E, N\}\) but starting at different points among \(p_1, p_2, \ldots, p_k\).

Note that rotating the diagram \(D(I, x)\) by \(180^\circ\) about the point \((m+k-1/2, r+k-1/2)\) gives the diagram \(D(I, \sigma^{-1}(x))\), using the cycle \(\sigma^{-1}\) in place of \(\sigma\).

Reflecting the diagram \(D(I, x)\) in the line \(y = x\) interchanges the East and North steps. Let \(D^*(I, x)\) denote this reflected diagram. A set \(X\) is the set of labels on the the North steps of a path in \(D^*(I, x)\) that satisfies conditions (i) and (ii) of Theorem 3.6 if and only if \(X\) is the complement of a basis of \(M[I]\). Thus, as illustrated in Figure 5, \(D^*(I, x)\) is a lattice path representation of the dual matroid \(M^*[I]\). Some argument is required, however, to show that \(M^*[I]\) is a multi-path matroid since the set of \(\sigma\)-intervals one obtains from \(D^*(I, x)\) need not be an antichain; in particular, the ground set \(S\) may be among these \(\sigma\)-intervals. For instance, the East steps of a column of \(D(I, x)\) (for example, the column between \(p_1\) and \(p_2\), or that between \(p_2, p_3\) in the first diagram in Figure 6) may be labelled with all elements of \(S\). Also, the first part of an interval that includes \(x\), say between \(p_i\) and \(p_{i+1}\), must be joined with with the corresponding last part between \(p'_i\) and \(p'_{i+1}\), and this union may be \(S\); the second diagram in Figure 6 illustrates this point with the column between \(p_2\) and \(p_3\) (the last of the East steps, labelled 1, 2, 3, 4, is marked) and that between \(p'_2\) and \(p'_3\) (the first of the East steps, labelled 5, 6, 7, 8, 9, is marked). One way to address this problem, in the spirit of the proofs of Theorems 3.3 and 3.5, is to show how to modify the set system that
Figure 6. Two diagrams $D(I, x)$ that, after reflection in the line $y = x$, give set systems that are not antichains.

corresponds to $D^*(I, x)$ to obtain a presentation of $M^*[I]$ by an antichain of $\sigma$-intervals. Instead, we introduce a more general type of diagram (which plays a key role in Section 4) and show that for such a diagram $D$, the sets of labels of the North steps of the paths of $D$ that satisfy conditions (i) and (ii) of Theorem 3.6 are the bases of a multi-path matroid. To avoid excess terminology, we also call these more general objects, which we define below, diagrams; this should create no confusion.

A diagram $D$ is a 5-tuple $(k, m, r, P, Q)$, where $k$ is a positive integer, $m$ and $r$ are non-negative integers, $P$ is a lattice path from $(k - 1, 0)$ to $(k - 1 + m, r)$, and $Q$ is a lattice path from $(0, k - 1)$ to $(m, k - 1 + r)$ that never goes below $P$. For $i$ with $1 \leq i \leq k$, let $p_i$ be $(k - i, i - 1)$ and let $p_i'$ be $p_i + (m, r)$. Let $L$ and $L'$ be the lines of slope $-1$ that contain the points $p_i$ and $p_i'$, respectively. The region $R(D)$ of a diagram $D$ is the set of points in $\mathbb{R}^2$, including the boundary, enclosed by the paths $P$ and $Q$ and the lines $L$ and $L'$. The edges of $D$ are the segments between lattice points in $D$ that are distance 1 apart. Assign label $i$ to an edge in $D$ if it is the $i$-th step in some lattice path that starts at a point on $L$; thus, edges are labelled with the elements of $[m + r]$. A $b$-path is a lattice path contained in the region $R(D)$ that starts at a point $p_i$ and ends in the corresponding point $p_i'$. Thus, any $b$-path contains $r$ North steps and $m$ East steps, and the edges are labelled, in order, $1, 2, \ldots, m + r$. The label-set of a $b$-path $T$ is the set of labels on the North steps of $T$. Let $B(D)$ be the set of all label-sets of $b$-paths in $D$. We now show that $B(D)$ is the set of bases of a multi-path matroid, which we denote by $M[D]$. (To recover multi-path matroids in complete generality, replace the labels $1, 2, \ldots, m + r$ with the elements $x, \sigma(x), \ldots, \sigma^{-1}(x)$, respectively. In much of the rest of the paper, we favor the notational simplicity gained by having $[m + r]$ be the ground set of $M[D]$.)

**Theorem 3.7.** For any diagram $D = (k, m, r, P, Q)$, the collection $B(D)$ of subsets of $[m + r]$ is the set of bases of a multi-path matroid.

**Proof.** By Theorem 3.5, it suffices to prove that $B(D)$ is the set of bases of a contraction of a multi-path matroid $M[I]$. Toward this end, let $D'$ be the diagram $(k, m, r + k + 1, PN^{k+1}, QN^{k+1})$. (See Figure 7.) For $i$ with $1 \leq i \leq k$, let $p_i$
and \( p'_i \) be as above and let \( p''_i \) be \( p'_i + (0, k+1) \). Denote the rows of \( D' \), from the bottom up, by \( R_1, R_2, \ldots, R_{r+2k} \). Let \( \sigma \) be the cycle \((1, 2, \ldots, m + r + k + 1) \). Let \( \mathcal{I} \) consist of the following sets: \( I_j \), for \( j \) with \( 1 \leq j < k \), is the union of the set of labels on the North steps of row \( R_j \) and that of row \( R_{r+k+j+1} \); the set \( I_j \), for \( j \) with \( k \leq j \leq k+r+1 \), consists of the labels on the North steps in row \( R_j \). Each set \( I_j \) is a \( \sigma \)-interval and \( D' \) is the diagram \( D(\mathcal{I}, 1) \) for \( \mathcal{I} \). We claim that \( \mathcal{I} \) is an antichain. Note that each set \( I_j \) has at most \( m+k \) elements and so is a proper subset of \([m+r+k+1]\). The sets \( I_k, I_{k+1}, \ldots, I_{k+r+1} \) form an antichain since we have \( f_{l_k} < f_{l_{k+1}} < \cdots < f_{l_{k+r+1}} \) and \( l_{l_k} < l_{l_{k+1}} < \cdots < l_{l_{k+r+1}} \) for these intervals in the usual linear order on \([m+r+k+1]\). A similar argument shows that \( I_1, I_2, \ldots, I_{k-1} \) form an antichain. Now consider \( I_h \) and \( I_j \) with \( 1 \leq h < k \leq j \leq r+k+1 \). At least one of 1 and \( m+r+k+1 \) is not in \( I_j \), so \( I_h \not\subseteq I_j \). The containment \( I_j \subseteq I_h \) would imply that either \( I_j \subseteq [f_{h_k}, m+r+k+1] \) or \( I_j \subseteq [1, l_{h_k}] \) holds. The first inclusion contradicts the inequality \( 1 \leq f_{l_j} < f_{l_h} \leq m+r+k+1 \) that is evident from the diagram \( D' \); the second containment contradicts the inequality \( 1 \leq l_{l_h} < l_{l_j} \leq m+r+k+1 \) that is also evident from \( D' \). Thus, \( \mathcal{I} \) is an antichain of \( \sigma \)-intervals.

Let \( Z \) consist of the last \( k+1 \) elements of \([m+r+k+1]\). We now show that \( \mathcal{B}(D) \) is the set of bases of the contraction of the multi-path matroid \( M[\mathcal{I}] \) by \( Z \). Since \( Z \) is independent in \( M[\mathcal{I}] \), the bases of \( M[\mathcal{I}]/Z \) are the subsets \( B \) of \([m+r]\) for which \( B \cup Z \) is a basis of \( M[\mathcal{I}] \). Note that the last \( k+1 \) steps in any lattice path whose label set is \( B \cup Z \) are North steps that go from a point \( p'_i \) to the corresponding point \( p''_i \). Thus, \( B \cup Z \) is a basis of \( M[\mathcal{I}] \) if and only if \( B \cup Z \) is the label set of a path in \( D' \) that goes from some point \( p_i \) to the corresponding point \( p_{r+i}'' \) through the point \( p_{r+i}' \). It follows that \( B \) is a basis of \( M[\mathcal{I}]/Z \) if and only if \( B \) is the label set of a \( b \)-path in \( D \), that is, if and only if \( B \) is in \( \mathcal{B}(D) \), as claimed. \( \square \)

That arbitrary diagrams define multi-path matroids allows us to give another perspective on certain minors. (Since the proof of Theorem 3.6 uses Theorem 3.5, this does not replace our earlier work.) Let \( M \) be the multi-path matroid on \([m+r]\) that is represented by a diagram \( D \). Let \( X \) and \( Y \) be disjoint subsets of \([m+r]\) where \( Y \) is independent, \( X \) is coindependent (i.e., the complement of a spanning set), and \( X \cup Y \) consists of the last \( k \) elements of \([m+r]\). From the formulation of

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure7}
\caption{The diagram \( D = (5, 6, 3, E^5 NEN^2, NEN^2 E^5) \) and its extension \( D' = (5, 6, 9, E^5 NEN^8, NEN^2 E^5 N^6) \).}
\end{figure}
minors in terms of bases, it follows that the bases of the minor $M \setminus X/Y$ correspond to the paths in $D$ whose last $k$ steps are determined: these steps are East or North according to whether their labels are in $X$ or $Y$, respectively. The initial segments of paths in $D$ whose last $k$ steps are as specified make up a smaller diagram $D'$. This observation, which is behind the proof of Theorem 3.7, plays an important role in the next section.

The next theorem summarizes the results in this section. The assertion about duality follows from Theorem 3.7 and the remarks before that theorem.

**Theorem 3.8.** The class of multi-path matroids is dual-closed, minor-closed, and properly contains the class of lattice path matroids.

### 4. Tutte Polynomial

The Tutte polynomial has received considerable attention, in part due to its many striking properties (e.g., it is the universal deletion-contraction invariant) and its many important evaluations (e.g., the chromatic and flow polynomials of a graph, the weight enumerator of a linear code, and the Jones polynomial of an alternating knot). (See [4, 12].) In this section, we show that the Tutte polynomial of a multi-path matroid can be computed in polynomial time. This result stands in contrast to the hardness results known for computing the Tutte polynomial of an arbitrary member of many classes of matroids [5, 7, 8, 10, 11]. We cast our work on the Tutte polynomial in a broader framework; we introduce what we call computation graphs, which allow us to apply dynamic programming.

The Tutte polynomial $t(M; x, y)$ of a matroid $M$ on the ground set $S$ can be defined in a variety of ways, perhaps the most basic of which is the following:

$$t(M; x, y) = \sum_{A \subseteq S} (x - 1)^{|r(S) - r(A)|} (y - 1)^{|A| - r(A)}.$$  

The following recurrence relation is more suited to our work. The Tutte polynomial $t(M; x, y)$ is 1 if $M$ is the empty matroid; otherwise, for any element $e$ of $M$,

$$t(M; x, y) = \begin{cases} 
xt(M/e; x, y) & \text{if } e \text{ is an isthmus;} \\
yt(M \setminus e; x, y) & \text{if } e \text{ is a loop;} \\
t(M/e; x, y) + t(M \setminus e; x, y) & \text{otherwise.}
\end{cases}$$

As stated, both of these formulations require roughly $2^{|S|}$ computations. We take advantage of the fact that for a multi-path matroid the recurrence relation can be applied in a manner that involves minors that are easily recognized to be equal; more precisely, the number of different minors that need to be considered turns out to be polynomial in $|S|$, and this allows us to organize the computation in a way that runs in polynomial time. Before turning to multi-path matroids, we establish a general framework for computations of this type.

Let $M$ be a matroid on the set $[n]$. To use the recurrence relation 2, it suffices to consider what we will call the initial minors of $M$, that is, the matroids formed by deleting or contracting, in turn, $n, n - 1, \ldots, h + 2, h + 1$, where at the stage at which an element is deleted, it is not an isthmus, and at the stage at which an element is contracted, it is not a loop. The ground set of an initial minor is an initial segment $[h]$ of $[n]$. Note that if $M \setminus X/Y$ is an initial minor, then $Y$ is independent and $X$ is co-independent.
We define a computation graph $G$ for the matroid $M$ to be an edge-labelled directed graph with label set $\{c,d\}$ that satisfies the following conditions.

1. Each vertex $u$ represents an initial minor $M_u$ of $M$. Every initial minor of $M$ is represented by at least one vertex.
2. Let $u$ be a vertex and let $h$ be the greatest element of the initial minor $M_u$. If there is a $d$-edge from $u$ to $v$, then $M_u \setminus h = M_v$. If there is a $c$-edge from $u$ to $w$, then $M_u / h = M_w$. In addition,
   - (a) if $h$ is an isthmus of $M_u$, then $u$ is the tail of exactly one $c$-edge and no $d$-edge;
   - (b) if $h$ is a loop of $M_u$, then $u$ is the tail of exactly one $d$-edge and no $c$-edge;
   - (c) otherwise $u$ is the tail of exactly one $c$-edge and one $d$-edge.
3. There are two distinguished vertices $v_M$ and $v_\emptyset$; these are the unique vertices that represent the matroid $M$ and the empty matroid, respectively.

By point (1), to construct a computation graph $G$ for a matroid $M$ by using some representation (e.g., a multi-path diagram), apart from the trivial cases in point (3) we are not required to determine whether different representations give the same minor. Note that the restrictions imposed on the edges imply that $u$ is at distance $h$ from $v_\emptyset$ if and only if $M_u$ has $h$ elements; let $V_h$ be the set of such vertices $u$. Then $\{V_0, V_1, \ldots, V_n\}$ is a partition of the vertices of $G$ and any edge that has its tail in $V_h$ has its head in $V_{h-1}$.

Recurrence relation (2) allows us to compute $t(M; x, y)$ from the computation graph $G$. There is a trade-off between several factors that enter into the computation graph: having fewer vertices allows us to compute the Tutte polynomial more quickly, but getting fewer vertices requires recognizing that many initial minors (perhaps with different representations) are equal. A typical application of these ideas would yield a computation graph with polynomially many vertices without determining all instances of equal initial minors. The following lemma helps quantify these observations.

**Lemma 4.1.** We can compute the Tutte polynomial $t(M; x, y)$ from a computation graph $G$ on $\nu$ vertices in $O(\nu rm)$ operations, where $r$ and $m$ are the rank and nullity of $M$.

**Proof.** Partition the vertices of $G$ into blocks $V_0, \ldots, V_{m+r}$, as described before; since $G$ has no oriented cycles this can be done with $O(\nu)$ operations. Assign to every vertex $u$ the Tutte polynomial $t(M_u; x, y)$ in the following manner. First assign 1 to the unique vertex $v_\emptyset$ in $V_0$, then compute the Tutte polynomials for all vertices in $V_1$, then those for all vertices of $V_2$, and so on. To compute the Tutte polynomial $t(M_u; x, y)$ for $u$ in $V_h$, apply recurrence relation (2) in the definition of a computation graph, the edges for which $u$ is the tail indicate which of the three cases of the recurrence to use, and the Tutte polynomials of $M_u \setminus h$ and $M_u / h$ have already been computed because they correspond to vertices of $V_{h-1}$. Thus for every vertex $u$ we just need to add two polynomials or multiply a polynomial by $x$ or $y$, and this can be done in $O(rm)$ operations since $t(M; x, y)$ has at most $rm + r + m$ coefficients. Hence we can compute the Tutte polynomial of every initial minor of $M$, including $M$ itself, in $O(\nu + \nu rm)$, that is, $O(\nu rm)$, operations. \qed
We now focus on the multi-path matroid $M[I]$, or $M$, on $[m + r]$ where $\sigma$ is the cycle $(1, 2, \ldots, m + r)$. Let $D$ be the diagram $D(I, 1)$. We first study the initial minors $M \setminus X/Y$ that arise in constructing a computation graph for $M$, and to do so we work with the diagrams introduced in Section 3. In particular, we show how to obtain a diagram $D'$ for any initial minor $M \setminus X/Y$. The resulting diagrams need not arise from $\sigma$-interval presentations.

It follows from the basis formulation of deletion and contraction that $B$ is a basis of $M \setminus X/Y$ if and only if $B \cup Y$ is a basis of $M$ (recall that $X$ and $Y$ are, respectively, coindependent and independent). These bases, by Theorem 4.2, correspond to b-paths where the last $q = |X \cup Y|$ steps are determined: steps corresponding to elements of $Y$ are North and steps corresponding to elements of $X$ are East. Let $a$ and $b$ be the smallest and largest integers $i$ such that there is a path from $p_i$ to $p_i'$ in $D$ with the last $q$ steps as specified by $X$ and $Y$. (See Figure 8.) For $i$ between $a$ and $b$ let $p_i''$ be the point $p_i' - (|X|, |Y|)$; thus any path from $p_i$ to $p_i'$ whose last $q$ steps are as specified by $X$ and $Y$ goes through the point $p_i''$. Let $P'$ be the lattice path in $D$ from $p_a$ to $p_a''$ that no path in $D$ from $p_a$ to $p_a''$ goes below; similarly, let $Q'$ be the lattice path in $D$ from $p_b$ to $p_b''$ that no path in $D$ from $p_b$ to $p_b''$ goes above. Let $D'$ be the diagram that has $p_a, \ldots, p_b$ as starting points, $p_a'', \ldots, p_b''$ as ending points, and $P'$ and $Q'$ as the bottom and top border. Thus, if $D$ is $(k, m, r, P, Q)$, then $D'$ is $(b - a + 1, m - |X|, r - |Y|, P', Q')$.

**Lemma 4.2.** Let $D = (k, m, r, P, Q)$ be the diagram $D(I, 1)$ of a multi-path matroid $M$ on $[n]$.

1. We can construct from $D$ a diagram $D'$ corresponding to an initial minor $M \setminus X/Y$ in $O(n)$ operations.
2. We can construct from $D$ at most $(n + 1)(\min(r, m) + 1)(k^2 + k)/2$ different diagrams $D'$ corresponding to initial minors of $M$. In particular, $M$ has at most this many initial minors.

**Proof.** The description above for constructing $D'$ from $D$ has two parts: find $a$ and $b$, and then find $P'$ and $Q'$. We sketch how to do these two steps. Since $X$ and $Y$ are coindependent and independent, $a$ and $b$ exist; find them by comparing the last $|X \cup Y|$ steps of $P$ and $Q$ with the steps specified by $X$ and $Y$. (See Figure 8.)

![Figure 8. The shaded region in the second diagram represents the initial minor $M \setminus \{15, 14, 11, 10\}/\{13, 12, 9\}$.


The dotted paths are those specified by $X$ and $Y$.) Specifically: let $N(W, i)$ be the number of North steps among the last $i$ steps of a path $W$ and let $P_{X,Y}$ be the path specified by $X$ and $Y$; then $a$ is

$$\max\{N(P_{X,Y}, i) - N(P, i) : 0 \leq i \leq |X \cup Y|\} + 1.$$  

A similar formula gives $b$, so $a$ and $b$ can be computed in $O(n)$ operations. Construct $P'$ (respectively, $Q'$) by going from $p_a$ to $p''_a$ (respectively, from $p_b$ to $p''_b$), taking East (respectively, North) steps whenever possible. This also takes $O(n)$ operations.

Assertion (2) follows by noting that a diagram is completely determined by (i) the size of $X \cup Y$, (ii) the size of either $X$ and $Y$, (iii) the points $p''_a$ and $p''_b$, and that these two points are determined by the two numbers $a \leq b$ between 1 and $k$. □

We now show how to compute the Tutte polynomial of a multi-path matroid $M$ in polynomial time from its diagram $D = D(I, 1)$. We start by constructing a computation graph for $M$ whose vertices correspond to the diagrams of initial minors of $M$ that are obtained from $D$ as described before Lemma 4.2. Start with a graph that consists of just one vertex $v_M$ that corresponds to the diagram $D$ and iterate the following process.

- Choose a vertex $v$ other than $v_0$ with outdegree 0. Let $M'$, on $[h]$, and $D'$ be the corresponding initial minor and diagram.
- Compute the diagrams $D_d$ and $D_c$ corresponding to $M'\setminus h$, if $h$ is not an isthmus, and $M'/h$, if $h$ is not a loop, as described before Lemma 4.2.
- Find a vertex $d_v$ (respectively, $c_v$) in the computation graph that corresponds to $D_d$ (respectively, $D_c$); if there is no such vertex, add a new vertex to the computation graph. Add a $d$-edge (respectively, a $c$-edge) from $v$ to $d_v$ (respectively, $c_v$).

Stop when the only vertex of outdegree 0 is $v_0$, which corresponds to the empty matroid. The resulting graph $G$ is clearly a computation graph for $M$. The same initial minor $M'$ can be represented more than once in $G$ since different diagrams can represent it, but each diagram $D'$ appears just once and all diagrams have been derived from $D$. By part (2) of Lemma 4.2 the number $ν$ of vertices of $G$ is $O(n \min(r, m)k^2)$. By Lemma 4.1 we can compute $t(M; x, y)$ from $G$ in $O(rmv)$ operations. So now we need only show that this construction of the computation graph can be done in polynomial time.

We show that we can construct $G$ in $O(νn \log ν)$ operations. Consider the operations required for each iteration of the algorithm (each expansion of a vertex $v$ of outdegree 0). First we compute $D_d$ and $D_c$ in $O(n)$ operations and then we check whether they are already in the graph. Comparing two diagrams (i.e., 5-tuples) requires $O(n)$ operations; by using a suitable ordering of the vertices, a binary search using $O(\log ν)$ comparisons suffices to determine whether a given diagram is already in the graph. Thus we need $O(n \log ν)$ operations for any of the $ν$ iterations, so $G$ can be constructed in $O(νn \log ν)$ operations.

Hence the number of operations needed to construct this computation graph and obtain the Tutte polynomial from it is $O(ν(rm + n \log ν))$. To simplify the expression for the number of operations required, note that $r + m$ is $n$ and $k$ is less than $n$; also, $\log ν$ is $O(\log n)$, because $ν$ depends polynomially on $n$. Thus, the work in this section gives the following theorem.
Theorem 4.3. We can compute the Tutte polynomial of a multi-path matroid on $n$ elements in $O(n^6)$ operations.

5. Basis Activities

Another formulation of the Tutte polynomial is given by basis activities, which are also of independent interest. In this section, we describe the internal and external activities of bases of multi-path matroids in terms of lattice paths in diagrams and we sketch an alternative approach to computing the Tutte polynomial of a multi-path matroid through basis activities.

The Tutte polynomial of $M$ can be written as

$$t(M; x, y) = \sum_{B \in \mathcal{B}(M)} x^{i(B)} y^{e(B)},$$

where $\mathcal{B}(M)$ is the collection of bases of $M$ and the exponents $i(B)$ and $e(B)$ are defined as follows. Fix a linear order $<$ on the ground set $S$ of $M$ and let $B$ be a basis of $M$. An element $u$ in $S - B$ is externally active with respect to $B$ if there is no element $v$ in $B$ with $v < u$ for which $(B - v) \cup u$ is a basis. An element $u$ in $B$ is internally active with respect to $B$ if there is no element $v$ in $S - B$ with $v < u$ for which $(B - u) \cup v$ is a basis. The internal activity $i(B)$ of a basis $B$ is the number of elements that are internally active with respect to $B$. The external activity of $B$, denoted $e(B)$, is defined similarly. Note that $i(B)$ and $e(B)$ depend not only on $B$ but also on the order. Equation (3) says that the coefficient of $x^iy^e$ in $t(M; x, y)$ is the number of bases of $M$ with internal activity $i$ and external activity $e$. In particular, the number of such bases is independent of the order.

We will use the following lemma, which is well-known and easy to prove.

Lemma 5.1. Fix a linear order on the ground set $S$ of a matroid $M$ and its dual $M^*$. An element $u$ is internally active with respect to the basis $B$ of $M$ if and only if $u$ is externally active with respect to the basis $S - B$ of $M^*$.

Throughout this section we use the notation and terminology we establish in the next several paragraphs. We assume that the ground set of the multi-path matroid $M[I]$ is $[m + r]$ and that $\sigma$ is the cycle $(1, 2, \ldots, m + r)$. We study the internal and external activities of the bases of $M[I]$ relative to the linear order $1 < 2 < \cdots < m + r$. Let $D$ be the diagram $D(I, 1) = (k, m, r, P, Q)$; recall that $P$ and $Q$ are respectively the bottom and top border of the diagram.

For any subset $X$ of $[m + r]$ the representation $\Pi(X, p)$ of $X$ starting at the lattice point $p$ is the path of $m + r$ steps that starts at $p$ whose $u$-th step is $N$ if $u$ is in $X$, and $E$ otherwise. We say that a path is valid if it is entirely contained in the diagram $D$. Thus, Theorem 3.6 states that the bases of $M[I]$ are the sets $B$ such that, for some $p_i$, the path $\Pi(B, p_i)$ is valid and ends at the corresponding point $p'_i$. Note that if $\Pi(B, p_i)$ and $\Pi(B, p_j)$ are both valid paths for $i < j$, then all paths $\Pi(B, p_i)$ with $i < t < j$ are also valid.

For $v, u$ in $[m + r]$ with $v \leq u + 1$, we use $[v, u]\Pi(X, p_i)$ to denote the path that starts at the beginning of the $v$-th step of $\Pi(X, p_i)$ and follows this path until the end of the $u$-th step. The notation $(v, u)\Pi(X, p_i), [v, u]\Pi(X, p_i)$, and $(v, u)\Pi(X, p_i)$ is defined in the obvious way; for instance $(v, u)\Pi(X, p_i)$ is $[v + 1, u - 1]\Pi(X, p_i)$. In particular, $(v, u)\Pi(X, p_i)$ is defined when $v \leq u - 1$, and $(u - 1, u)\Pi(X, p_i)$ consists of the single point that is common to steps $u - 1$ and $u$ of $\Pi(X, p_i)$.
The following lemma gives the conditions under which an element $u$ of a basis $B$ can be replaced by an element $v$ to yield a basis $B'$. 

**Lemma 5.2.** Let $B$ be a basis of a multi-path matroid $M[I]$ with $u \in B$, $v \notin B$. Let $\Pi(B, p_i)$ be a valid path. Let $B'$ be $(B - u) \cup v$.

1. If $v < u$, then $B'$ is a basis if and only if either
   (a) the path $(v, u)\Pi(B, p_i)$ does not touch the top border $Q$, or
   (b) neither $[1, v]\Pi(B, p_i)$ nor $(u, m + r]\Pi(B, p_i)$ touches the bottom border $P$.

2. If $u < v$, then $B'$ is a basis if and only if either
   (a) the path $(u, v)\Pi(B, p_i)$ does not touch $P$, or
   (b) neither $[1, u]\Pi(B, p_i)$ nor $(v, m + r]\Pi(B, p_i)$ touches $Q$.

**Proof.** By duality it suffices to prove the first claim. By Theorem 3.6, $B'$ is a basis if and only if $B'$ has a valid representation. Compare the paths $\Pi(B', p_i)$ and $\Pi(B', p_{i-1})$ with $\Pi(B, p_i)$. (See Figure 9) Since $\Pi(B', p_i)$ is above $\Pi(B, p_i)$, only $Q$ may prevent $\Pi(B', p_i)$ from being valid, and in that case no $\Pi(B', p_j)$ with $j \geq i$ would be valid; similarly, only $P$ may prevent $\Pi(B', p_{i-1})$ from being valid, and in that case no $\Pi(B', p_{j})$ with $j \leq i - 1$ would be valid. Thus $B'$ is a basis if and only if either $\Pi(B', p_i)$ or $\Pi(B', p_{i-1})$ is valid. These two conditions are equivalent to conditions (1.a) and (1.b), as Figure 9 illustrates. \qed

With the help of this basis exchange lemma we now characterize the internally and externally active elements.

**Theorem 5.3.** Let $B$ be a basis of a multi-path matroid $M[I]$ and let $\Pi(B, p_i)$ be a valid path.

1. An element $u$ in $B$ is internally active if and only if either
   (a) $[u] \subseteq B$, or
   (b) the $u$-th step of $\Pi(B, p_i)$ lies in the top border $Q$ and $(u, m + r]\Pi(B, p_i)$ touches the bottom border $P$.

2. An element $u$ not in $B$ is externally active if and only if either
   (a) $[u] \cap B = \emptyset$, or
   (b) the $u$-th step of $\Pi(B, p_i)$ lies in $P$ and $(u, m + r]\Pi(B, p_i)$ touches $Q$. 

---

Figure 9. Paths $\Pi(B', p_i)$ (dotted line) and $\Pi(B, p_i)$ in part (a), and $\Pi(B', p_{i-1})$ (dotted line) and $\Pi(B, p_i)$ in part (b).
Proof. By duality, we only need to prove part (I).

Note that \( u \) is internally active if \([u] \subseteq B\). Thus, let \( V \) be \([u] - B \) and assume that \( V \) is not empty. Sufficiency follows because if \( u \) satisfies condition (I.b), then it satisfies neither conditions (1.a) nor (1.b) of Lemma 5.2 for any \( v \) in \( V \).

To prove the converse assume that \( u \) is internally active. Let \( v \) be \( \text{max}(V) \). Since \( u \) is internally active, \((B - u) \cup v \) is not a basis, so by condition (1.a) of Lemma 5.2 the path \((v, u)\Pi(B, p_i)\) touches \( Q \). This path has only North steps, by the choice of \( v \), so its ending point has to touch \( Q \). This proves that the \( u \)-th step of \( \Pi(B, p_i) \) lies in \( Q \), so the first part of condition (I.b) holds.

For the second part, let \( v \) be \( \text{min}(V) \). By condition (1.b) of Lemma 5.2 since \((B - u) \cup v \) is not a basis, at least one of the paths \([1, v]\Pi(B, p_i)\) and \((u, m + r)\Pi(B, p_i)\) touches \( P \). We show that if the first path touches \( P \), then so does the second, hence either way \((u, m + r)\Pi(B, p_i)\) touches \( P \), which proves the second part of condition (I.b). Indeed, the minimality of \( v \) implies that \([1, v]\Pi(B, p_i)\) has only North steps. So if \([1, v]\Pi(B, p_i)\) touches \( P \), then it has to touch it from the beginning, that is, \( p_i \) has to be \( p_1 \). Hence \((u, m + r)\Pi(B, p_i)\) touches \( P \) at its ending point \( p_1' \).

Theorem 5.3 shows that we can find the internal and external activities of a basis by just looking at one of its representations in the diagram \( D \). This reduces the problem of counting the number of bases with given internal and external activities to the problem of counting the number of lattice paths of a certain kind in \( D \). In the remainder of this section we sketch a polynomial-time algorithm that computes this number of bases. Note that by Equation (9) this yields a different approach to computing the Tutte polynomial of a multi-path matroid; this approach is slightly quicker than that in the previous section, but it requires keeping track of more details.

The algorithm uses the characterization of activities in Theorem 5.3. Of the conditions in that result, conditions (b) are somewhat more difficult to deal with; we introduce the notion of pseudo-activities to count the steps that are active by conditions (b). Let \( R \) be a valid path in the diagram \( D \) that ends in one of the points \( p_1', \ldots, p_k' \). Let \( s \) be one of its steps and let \( R_s \) be the path that starts at the end of step \( s \) and follows \( R \) until its end. We say that \( s \) is pseudo-internally active in \( R \) if it is a North step that lies in the top border \( Q \) and the path \( R_s \) touches the bottom border \( P \). Similarly we say that \( s \) is pseudo-externally active in \( R \) if it is an East step that lies in \( P \) and \( R_s \) touches \( Q \). Note that, unlike activities, pseudo-activities are not defined for bases, but for paths that end at one of the points \( p_1', \ldots, p_k' \) (e.g., the final segments of paths that correspond to bases).

Let \( p \) be a lattice point of the diagram \( D \) and let \( p_1' \) be one of the ending points \( p_1', \ldots, p_k' \). Let \( a \) and \( b \) be natural numbers with \( a \leq r \) and \( b \leq m \). Let \( \tau_P \) and \( \tau_Q \) be variables that can take on the values \( \text{true} \) and \( \text{false} \). We define \( \Gamma(p, p_1', a, b, \tau_P, \tau_Q) \) to be the number of valid lattice paths starting at \( p \) and ending at \( p_1' \) (consisting of one point if \( p = p_1' \)), with \( a \) pseudo-internally active steps and \( b \) pseudo-externally active steps, and touching \( P \) if and only if \( \tau_P \) is \( \text{true} \), and touching \( Q \) if and only if \( \tau_Q \) is \( \text{true} \). The function \( \Gamma \) satisfies an easily-verified, multi-part recurrence relation of which we mention just two parts. Let \( \gamma \) be \( \Gamma(p, p_1', a, b, \tau_P, \tau_Q) \) and let \( p_E \) and \( p_N \) be, respectively, \( p + (1, 0) \) and \( p + (0, 1) \). If \( p \) is in neither \( P \) nor \( Q \), then

\[
\gamma = \Gamma(p_N, p_1', a, b, \tau_P, \tau_Q) + \Gamma(p_E, p_1', a, b, \tau_P, \tau_Q).
\]
If \( p \) and \( p_N \) are in \( Q \), if \( p \) is not in \( P \), and if \( \tau_Q \) is \textit{true}, then
\[
\gamma = \Gamma(p_N, p'_i, a, b, \tau_P, \tau_Q) + \Gamma(p_E, p'_i, a, b, \tau_P, \text{true}) + \Gamma(p_E, p'_i, a, b, \tau_P, \text{false})
\]
where \( \bar{a} \) is \( a - 1 \) if \( \tau_P \) is \textit{true} and \( a \) if \( \tau_P \) is \textit{false}, and \( \Gamma(p_N, p'_i, a, b, \tau_P, \tau_Q) \) is taken to be 0 if \( \bar{a} < 0 \) (note, for instance, that this term is also 0 if \( \bar{a} > 0 \) and \( \tau_P \) is \textit{false}).

In this way we get a recurrence relation that can be expressed in six parts; in each part, \( \gamma \) is a sum of at most three evaluations of \( \Gamma \), each involving one of the points that \( p \) leads to, namely, \( p_N \) or \( p_E \).

With this multi-part recurrence we can compute all values of \( \Gamma \) by using a dynamic programming algorithm, not unlike in Lemma 4.1. Fix an ending point \( p'_i \).

Consider a point \( a \) to be 0 if \( \bar{a} \) is in \( \Gamma \), that is,
\[
\text{if } \bar{a} \in \Gamma \text{ then } \text{set } a = 0.
\]

In this way we get a recurrence relation that can be expressed in six parts; in each part, \( \gamma \) is a sum of at most three evaluations of \( \Gamma \), each involving one of the points that \( p \) leads to, namely, \( p_N \) or \( p_E \).

We show finally that we can compute the number of bases of internal activity \( i \) and external activity \( e \) from \( \Gamma \). This yields a two-step algorithm for computing the Tutte polynomial of a multi-path matroid: first compute all values of \( \Gamma \), and then obtain the coefficient of each term \( x^i y^e \) in the Tutte polynomial. The algorithm requires \( O(km^2r^2) \) operations, or \( O(n^5) \) where \( n = m + r \) (note that \( k \) is smaller than \( n \)). This algorithm is somewhat faster than that in Section 4.

**Lemma 5.4.** The number of bases of \( M[I] \) with internal activity \( i \) and external activity \( e \) can be found in time \( O(k(i + e)) \) knowing the values of \( \Gamma \).

**Proof.** We give an algorithm that counts the bases with internal activity \( i \) and external activity \( e \) that contain the element 1. Note that the remaining bases are the complements of the bases of the dual with internal activity \( e \) and external activity \( i \) that contain the element 1, so we can compute their number with the same algorithm.

Note that any basis has a unique valid representation that touches the top border \( Q \), so this gives a one-to-one correspondence between bases and certain paths.

For \( t > 0 \) and \( j \) in \([k]\), we define \( \beta(j, t) \) to be the number of bases \( B \) such that
\begin{enumerate}
  \item the internal activity is \( i \) and the external activity is \( e \),
  \item \( \{t\} \subseteq B \) and \( t + 1 \notin B \), and
  \item the path \( \Pi(B, p_j) \) is the unique valid representation that touches \( Q \).
\end{enumerate}

Let \( R \) be the path \( N^t E \) that starts at \( p_j \) and let \( s \) be the last step of \( R \). By conditions (2) and (3), \( R \) coincides with \([1, t + 1] \Pi(B, p_j)\) where \( B \) is any of the bases that \( \beta(j, t) \) is counting; clearly if \( R \) is not valid, then \( \beta(j, t) = 0 \). The first \( t \) North steps of \( R \) are internally active elements in \( B \), but the step \( s \) may or may not be externally active. If \( s \) does not lie in the bottom border \( P \), then by Theorem 5.3 it is not externally active, so
\[
\beta(j, t) = \sum_{(\tau_P, \tau_Q) \in T_P \times T_Q} \Gamma(p, p'_i, i - t, e, \tau_P, \tau_Q),
\]
where \( p \) is the point \( p_j + (1, t) \) where \( R \) ends, \( T_P \) is \{\text{true, false}\} and \( T_Q \) is \{\text{true}\} if \( R \) does not touch \( Q \), and \{\text{true, false}\} if \( R \) touches it. The two possibilities for \( T_Q \) arise from the requirement in condition (3) that the paths touch \( Q \), so if \( R \)
does not touch it, then the remaining part of the path has to. Notice that many terms in this sum may be 0; for instance, if \( i - t \) or \( e \) are greater than 0, then every evaluation of \( \Gamma \) where \( \tau_P \) or \( \tau_Q \) is false is 0.

Now assume that \( s \) lies in \( P \). (This can happen only if \( j \) is 1.) Note that \( s \) is externally active if and only if the remaining part of the path touches \( Q \). Therefore

\[
\beta(j,t) = \sum_{(\tau_P, \tau_Q) \in T_P \times T_Q} \Gamma(p, p'_j, i - t, e - \delta(\tau_Q), \tau_P, \tau_Q),
\]

where \( p \) is the point \( p_j + (1, t) \), the set \( T_Q \) is \{true, false\} or \{true\} according to whether \( R \) does or does not touch \( Q \), the set \( T_P \) is \{true\}, and \( \delta(\tau_Q) \) is 1 if \( \tau_Q \) is true, and 0 otherwise. Note that since \( s \) lies in \( P \) the paths we are counting start on \( P \) so \( \tau_P \) must be true.

To obtain the number of bases with internal activity \( i \) and external activity \( e \) containing 1 we add up all the values of \( \beta \). Since the number \( t \) is bounded by \( i \), we can do the computations in \( O(ki) \) operations. The same algorithm applied to the dual matroid needs \( O(ke) \) operations, hence we can compute the total number of bases of \( M[I] \) of internal activity \( i \) and external activity \( e \) from \( \Gamma \) in \( O(k(i + e)) \) operations.

\[\square\]

6. Further Structural Properties

This final section treats a variety of properties of multi-path matroids and their presentations.

Every connected lattice path matroid with at least two elements has a spanning circuit [3, Theorem 3.3]. The analogous property holds for multi-path matroids, as we now show. By the result just cited, it suffices to focus on multi-path matroids that are not lattice path matroids.

**Theorem 6.1.** A multi-path matroid \( M[I] \) that is not a lattice path matroid has a spanning circuit. Furthermore, every element is in some spanning circuit.

**Proof.** Since multi-path matroids of rank less than 2 are lattice path matroids, we are assuming that the rank is at least 2. The set \( F = \{f_I : I \in I\} \) of first elements is a proper subset of the ground set \( S \). By the comments at the end of Section 2, the first element \( f_I \) of any interval \( I \) in \( I \) is in both \( I \) and \( \Sigma^{-1}(I) \); also, \( M[I] \) has no loops. From these observations, it is immediate to check that \( F \cup x \), for any \( x \) in \( S - F \), is a spanning circuit of \( M[I] \).

\[\square\]

Corollary 6.2 follows from Theorem 6.1 since multi-path matroids that are not lattice path matroids have no loops, and loopless matroids with spanning circuits are connected.

**Corollary 6.2.** Every multi-path matroid that is not a lattice path matroid is connected.

From Corollary 6.2 or directly from Theorem 6.1 it follows that, in contrast to the class of lattice path matroids, the class of multi-path matroids is not closed under direct sums. For example, recall that the 3-whirl \( W^3 \) is a multi-path matroid but not a lattice path matroid. Therefore the direct sum \( W^3 \oplus W^3 \) is neither a lattice path matroid (since \( W^3 \) is a restriction) nor a multi-path matroid. Of course, one could consider the class of matroids whose connected components are multi-path matroids; such matroids can be realized with a simple variation on Definition 2.3.
having the intervals being intervals in the cycles in the cycle decomposition of an
arbitrary permutation of the ground set.

The next theorem gives some indication of how close multi-path matroids are to
lattice path matroids.

**Theorem 6.3.** The restriction of a multi-path matroid to a proper flat is a lattice path matroid.

*Proof.* Let $M$ be the multi-path matroid $M[I]$. The class of lattice path matroids is closed under direct sums [2, Theorem 3.6], so it suffices to prove the assertion for proper flats $F$ for which $M|F$ is connected. The assertion is easily seen to hold for flats of rank 2 or less. Let $F$ be a proper flat of rank 3 or more for which $M|F$ is connected. By Theorem 6.1 and the corresponding result for lattice path matroids [3, Theorem 3.3], the restriction $M|F$ has a spanning circuit $C$. It follows from Hall’s matching theorem that a circuit $C'$ of a transversal matroid has nonempty intersection with exactly $|C'| - 1$ of the sets in any presentation; therefore the inequality $|C| - 1 = r(F) < r(M)$ implies that $C$ is disjoint from at least one interval $I$ in $I$. Thus, $F$ is disjoint from $I$, so $F$ is a flat of the deletion $M \setminus I$. Observe that $M \setminus I$ is a lattice path matroid: by Lemma 3.1, the presentation $(J \setminus I : J \in I, J \neq I)$ of $M \setminus I$ by intervals in $\sigma(I_1), \sigma^2(I_1), \ldots, \sigma^{-1}(f_1)$ contains a presentation of $M \setminus I$ by an antichain of intervals. Since $M \setminus I$ is a lattice path matroid, so is $M|F$. □

Theorem 6.3 allows one to carry over certain results about lattice path matroids
to multi-path matroids. For instance, the description of the circuits of lattice path matroids [3, Theorem 3.9] applies to the nonspanning circuits of multi-path matroids. We mention several other results that are counterparts of results for
lattice path matroids and that may prove useful for the further study of multi-path matroids. Let $M[I]$ be a multi-path matroid of rank $r$ on the set $S$.

1. Let $I_{i_1}, I_{i_2}, \ldots, I_{i_h}$ be the intervals in $I$ that have nonempty intersection with a fixed connected flat $F$ of $M[I]$ of rank greater than 1. Then $\{I_{i_1}, I_{i_2}, \ldots, I_{i_h}\}$ is a $\Sigma$-interval in $I$ and $h$ is $r(F)$.
2. Statement (1) implies that there are at most $r$ connected flats of a fixed rank greater than 1 in $M[I]$. Whirls show that this bound cannot be improved.
3. Statement (1) also implies that any flat of $M[I]$ is covered by at most two connected flats.
4. The elements in any connected flat of $M[I]$ form a $\sigma$-interval in $S$.
5. If $X_1, X_2,$ and $X_3$ are connected flats of $M[I]$, and if no two sets among $X_1, X_2, X_3$ are disjoint, then either one of $X_1, X_2, X_3$ is contained in the union of the other two, or $X_1 \cup X_2 \cup X_3$ is $S$.

(Compare statements (1) and (4) with [3, Theorem 3.11]; compare statements (2) and (3) with [3, Corollary 3.12].)

Our final topic is minimal presentations of multi-path matroids. Example 3 in
Section 2 gives distinct $\sigma$-interval presentations of a multi-path matroid that are also minimal presentations. The next theorem shows that any minimal $\sigma$-interval presentation is also a minimal presentation. Note that the converse is not true: for example, the presentation $(\{1, 4\}, \{2, 4\}, \{3, 4\})$ of $U_{3,4}$ is minimal but these sets are not $\sigma$-intervals for any cycle $\sigma$ on $[4]$. 
Theorem 6.4. The sets in a minimal $\sigma$-interval presentation of a multi-path matroid are cocircuits of the matroid. Any minimal $\sigma$-interval presentation of a multi-path matroid is a minimal presentation.

Proof. Assume that the multi-path matroid $M$ has rank $r$ and that $I$ is a minimal $\sigma$-interval presentation of $M$. Each set in a presentation of a transversal matroid is the complement of a flat of the matroid. Since cocircuits are the least nonempty complements of flats, a presentation by cocircuits is necessarily minimal, so the second assertion of the theorem follows from the first. Let $I$ be in $\mathcal{I}$. Since the complement of $I$ is a flat, the first assertion follows if we show that this complement contains $r-1$ independent elements. In terms of lattice paths, we need to show that there is a lattice path in some diagram $D(\mathcal{I}, x)$ that connects a pair of corresponding points $p_h$ and $p'_h$ and has only one North step that is labelled by an element of $I$.

This statement is trivial if $r$ is 1, so assume $r$ exceeds 1.

Since $\mathcal{I}$ is an antichain and $r$ exceeds 1, some element, say $x$, of $M$ is not in $I$. Let $A$ and $B$ be, respectively, the lower left and upper right points in the row of $D(\mathcal{I}, x)$ that represents $I$. (See Figure 10.) Let $i$ be the least positive integer for which there is a path in $D(\mathcal{I}, x)$ from $p_i$ to $A$. Note that there is a path in $D(\mathcal{I}, x)$ from $p_h$ to $A$ if and only if $h \geq i$. Similarly, let $j$ be the greatest integer for which there is a path from $B$ to $p'_j$. Thus, there is a path in $D(\mathcal{I}, x)$ from $B$ to $p'_h$ if and only if $h \leq j$. It follows that if $i \leq j$, then there is a path in $D(\mathcal{I}, x)$ that connects any pair of corresponding points $p_h$ and $p'_h$ with $i \leq h \leq j$ and that has only one North step labelled by an element of $I$, as desired. We complete the proof by showing that the alternative, the inequality $i > j$, contradicts the assumption that $\mathcal{I}$ is a minimal $\sigma$-interval presentation. The inequality $i > j$ forces $i$ to be greater than 1. If there were a path in $D(\mathcal{I}, x)$ of the form $N^aEQ$ from $p_i$ to $A$, then the path $N^{a+1}Q$ from $p_{i-1}$ to $A$ would also be in $D(\mathcal{I}, x)$, contrary the choice of $i$, so there is only one path from $p_i$ to $A$ and this path consists of all North steps. Similarly, $j < k$ and the unique path from $B$ to $p'_j$ consists of all North steps. From these conclusions, it follows that for any path in $D(\mathcal{I}, x)$, say from $p_h$ to $p'_h$, that uses the North step labelled $f_1$ in the row corresponding to $I$, or any North step immediately above this one, we have $h \geq i$ and the same sequence of steps, but instead going from $p_{h-1}$ to $p'_{h-1}$, remains in $D(\mathcal{I}, x)$. Thus, by deleting $f_1$ from $I$, deleting $\sigma(f_1)$ from $\Sigma(I)$ if $f_{\Sigma(I)} = \sigma(f_1)$, deleting $\sigma^2(f_1)$ from $\Sigma^2(I)$ if $f_{\Sigma^2(I)} = \sigma^2(f_1)$, etc., we obtain a smaller $\sigma$-interval presentation of $M$, that, as desired, contradicts the assumed minimality of $\mathcal{I}$. \hfill \Box

Let $M$ be a matroid of rank $r$ and nullity $m$. Since any hyperplane contains at least $r-1$ of the $r+m$ elements of $M$, any cocircuit has at most $m+1$ elements. From this observation, the following corollary of Theorem 6.4 is evident.

Corollary 6.5. The sets in any minimal $\sigma$-interval presentation of a multi-path matroid of nullity $m$ have at most $m+1$ elements.

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Figure 10. The cases (a) $i \leq j$ and (b) $i > j$ in the proof of Theorem 6.4.

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