Bounds on Higher Derivative $f(R, \Box R, T)$ Models from Energy Conditions

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This paper studies the viable regions of some cosmic models in a higher derivative $f(R, \Box R, T)$ theory with the help of energy conditions (where $R$, $\Box$ and $T$ are the Ricci scalar, d’Alembert operator and trace of energy momentum tensor, respectively). For this purpose, we assume a flat Friedmann-Lemaître-Robertson-Walker metric which is assumed to be filled with perfect fluid configurations. We take two distinct realistic models, that might be helpful to explore stable regimes of cosmological solutions. After taking some numerical values of cosmic parameters, like crackle, snap, jerk (etc) as well as viable constraints from energy conditions, the viable zones for the under observed $f(R, \Box R, T)$ models are examined.

PACS numbers: 04.50.Kd; 04.20.-q; 98.80.Jk; 98.80.-k

Keywords: Relativistic fluids; Gravitation; Stability

I. INTRODUCTION

It could be helpful to understand the restrictions on general relativity (GR) and improve our theoretical predictions on cosmological scales as modification in GR could provide outstanding results at large scales. Many relativists used modified gravity theories, after the cosmic accelerating picture made by the BICEP2 experiment [1–3], Wilkinson microwave anisotropy probe [4, 5] and the Planck satellite [6–8] which illustrate enigmatic forces behind the cosmic evolution. The dark energy is attributed as an active candidate which influence acceleration in the cosmic expansion. Qadir et al. [9] proposed that GR modification could provide fruitful insights to study issues linked with quantum gravity and dark matter problem. Various modified gravity theories have been proposed by relativists which gained significance due to their additional degrees of freedom (please see modified gravity and DE reviews [10–24]).

The dark side of the universe could be studied by modified theory of gravity that work on the geometrical side of field equations which includes the study of dark energy and dark matter. Different theories have been suggested associated with higher powers of Riemann tensor in Lagrangian, that has been demonstrated useful in cosmology. The most studied framework in this context are $f(R)$ theories, however some suitable modifications are also possible by considering higher derivative terms of the curvature related objects and henceforth called the higher order gravity theories. Starobinsky [25] and Kerner [26] did the pioneering work of introducing the higher derivative terms while exploring solutions avoiding the initial singularity. Further, it is established that these higher derivative gravity models could have significant role in order to study the inflationary expansion of the universe [27].

Bonanno [28] explored exact solutions of electrically charged spherical interiors containing higher derivative terms with null fluid. He also discussed the stability of Cauchy horizon and claimed that in the background of anti de Sitter model, the stable solutions are black holes. Cuzinatto et al. [29] studied the higher derivative theory in which the Lagrangian involves terms of order $n$, e.g., $f(R, \nabla_\mu R, \nabla_\mu_1 \nabla_\mu_2 R, ..., \nabla_\mu_1 ... \nabla_\mu_n R)$, and performed transformation from Jordan-to-Einstein frame in both metric and Palatini formalism. Iihoshi [30] presented a hybrid inflationary scenario in the background of $f(R, \Box R)$ theory, where $\Box$ indicate the d’Alembertian operator. This theory is treated as instinctive generalization of $f(R)$ gravity. Also, he proposed that $f(R, \Box R)$ theory is equivalent to GR coupled with two scalar fields.

By using the dynamical system technique, Tretyakov [31] presented the Minkowski stability issue in the scenario of particular modified theory, i.e. $f(R) + R \Box R$ gravity. He claimed that this method is useful for extracting additional constraints on parameters of various modified gravity theories. There also exists a well-known class of gravity theories in which a more general function of Ricci scalar $R$ replaces the arbitrary function $f(R)$ in the gravitational action

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of GR. One of such theories is $f(R, T)$ gravity, firstly discussed by Harko et al. [32] in which the arbitrary function, incorporates the energy-momentum trace $T$ along with $R$. They also analyzed the self-interacting scalar field models and the Newtonian limit of these modified models. Recently, Abbas et al. [33] have discussed the viability of modified gravity models through gravitational collapse.

Initially, Houndjo et al. [34] discussed the FLRW cosmology in the context of $f(R, □ R, T)$ gravity and found unstable phase of de Sitter model in this scenario. Yousaf et al. [35] discussed the energy conditions (ECs) and the behavior of Friedmann-Lemaitre-Robertson-Walker (FLRW) model in $f(R, □ R, T)$ gravity. They also showed the constraints and the graphical behavior of some of the model parameters. Alvarenga et al. [36] investigated the matter density perturbations in modified $f(R, T)$ models of type $f_1(R) + f_2(T)$, satisfying stress-energy conservation and also compared the results of quasi-static approximations in $f(R, T)$ with that of GR results. They concluded that the unusual behavior of the density contrast constrains the viability of such modified models.

Baffou et al. [37] focused on the cosmological dynamics of low and high red-shift solutions and stability of a modified model, $R + f(T)$, by using the power law and de Sitter solution with linear perturbation and concluded the viability of the considered $f(R, T)$ models. Ilyas et al. [38] explored the formation of compact structures with anisotropic matter content in modified $f(R, T)$ background and concluded the maximum value of pressure and density in the central region. The spherical hydrostatic equilibrium configuration of stellar remnants in the background of $f(R, T) = R + \lambda T$ is analyzed by Moraes et al. [39]. Sahoo et al. [40] also found some interesting results based on $f(R, T)$ gravity. Myrzakulov [41] geometrically constructed few $f(R, T)$ models (where $T$ is the torsion scalar) including models of the form, $f(R, T) = \mu R + \nu T$ and claimed that some of the cosmological outcomes of this theory could explain the phenomenon of accelerated expansion of the universe.

A worth emphasizing aspect in the discussion of singularity theorems and black hole thermodynamics is the ECs which were initially derived in GR by Hawking and Ellis [42]. Several cosmological issues like, expansion history of the universe, phantom fields etc, have been discussed by using the ECs in GR. Also, one can explore additional constraints by exploring EC in the analysis of modified theories to derive general results that hold for a variety of situations. Santos et al. [43] studied the bounds enforced by ECs in the context of $f(R)$ functional form. They explored the null, strong, weak and anisotropic matter condition models from Raychaudhuri equation which differ from those obtained in GR. Bertolami and Sequeira [44] derived the ECs for a particular type of gravity model having non-minimal matter-curvature coupling and discussed their stability via Dolgov-Kawasaki criterion. Zhou et al. [45] after considering FLRW metric proposed two stable $f(G)$ toy models (where $G$ is the Gauss-Bonnet term) for discussing a phantom-like and de Sitter environment. Atazadeh and Darabi [46] assumed two different formulations of $f(R, G)$ gravity and developed viable bounds through ECs.

This paper is aimed to probe the issue of viability of $f(R, □ R, T)$ models in an environment of FLRW perfect fluid metric. Our work is organized as under. The next section will briefly describe $f(R, □ R, T)$ theory along with the corresponding ECs. Section 3 describes viability constraints coming from ECs for two $f(R, □ R, T)$ cosmic models. In the last section, we summarize our main findings.

II. $f(R, □ R, T)$ GRAVITY AND ENERGY CONDITIONS

The usual Einstein Hilbert action (EHA) for $f(R, □ R, T)$ theory can be modified as under [34]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R, □ R, T) + S_M (g^\mu\nu, \psi),$$

where $T$ and $R$ are the traces of the stress-energy and Ricci tensors, respectively, while $\kappa^2 = 8\pi G$ with $G$ as the Newton’s gravitational constant. The operator $\square \equiv \nabla_\mu \nabla^\mu$, in which $\nabla_\mu$ describes covariant derivation. Variations of the above modified EHA with the metric tensor provides

$$\delta S = \frac{1}{2\kappa^2} \int d^4x [f \delta \sqrt{-g} + \sqrt{-g} (f_R \delta R + f_\square R + f_T \delta T) + 2\kappa^2 \delta (\sqrt{-g} L_M)],$$

where $L_M$ indicates Lagrangian for matter field. Equation (2) after substituting $\delta R$, $\delta □ R$, $\delta \sqrt{-g}$ and $\delta T$ with few calculations gives

$$\delta S = \frac{1}{2\kappa^2} \int d^4x \left[ \frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta} f + \sqrt{-g} (T_{\alpha\beta} + \Theta_{\alpha\beta}) f_T \delta g^{\alpha\beta} + f_R \sqrt{-g} (R_{\alpha\beta} + g_{\alpha\beta} □ - □ R_{\alpha\beta} + g_{\alpha\beta} □ + R_{\alpha\beta} □ - □) \right. \nabla_\alpha \nabla_\beta - \nabla_\alpha R_{\beta\gamma} + 2g^\mu\nu \nabla_\mu R_{\alpha\beta\gamma} \nabla_\nu) \delta g^{\alpha\beta} + 2\kappa^2 \frac{\delta (\sqrt{-g} L_M)}{\delta g^{\alpha\beta}} \delta g^{\alpha\beta} \right],$$

(3)
where subscripts $R$, $T$ and $\Box R$ indicate the derivative of the corresponding quantities with respect to $T$, $R$ and $\Box R$, respectively, while $\Theta_{\alpha\beta} = g^{\mu\nu} \delta T_{\mu\nu} / \delta g^{\alpha\beta}$. Equation (3) after simplifications gives rise to

\[
 f_R R_{\alpha\beta} + (g_{\alpha\beta} \Box - \nabla_{\alpha} \nabla_{\beta}) f_R - \frac{1}{2} g_{\alpha\beta} f + (2 f_R (\nabla_{(\alpha} \nabla_{\beta)}) R - \Box R_{\alpha\beta}) - \{R_{\alpha\beta} \Box - \nabla_{\alpha} \nabla_{\beta} + g_{\alpha\beta} \Box - \nabla_{\alpha} R \nabla_{\beta} + 2 g_{\mu\nu} \nabla_{\mu} R_{\alpha\beta} \nabla_{\nu}\} f_R = \kappa^2 T_{\alpha\beta} - f_T (T_{\alpha\beta} + \Theta_{\alpha\beta}),
\]

where $\Theta_{\alpha\beta}$, after selecting $L_m = -p$, turns out to be $\Theta_{\alpha\beta} = 2 T_{\alpha\beta} - p g_{\alpha\beta}$.

We start our analysis by taking a FLRW spacetime which in the background of flat homogeneous state can be given as follows

\[
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
\]

in which $a$ is the scale factor. The stress-energy tensor for the perfect fluid describes the contributions of pressure ($p$) and energy density ($\rho$). This in terms of mathematical expression can be given as

\[
T_{\alpha\beta} = (\rho + p) u_{\alpha} u_{\beta} - p g_{\alpha\beta}.
\]

Here, the fluid four velocity is given by $u_{\mu}$. The corresponding $f(R, \Box R, T)$ equations of motion (4) for the FLRW spacetime (5) and (6) are [34]

\[
2 H f_{\Box R^{''''}} - (2 H^2 + 3 H') f_{\Box R}^{'''} - (5 H^3 + 2 HH' + H''') f_{\Box R'} + 2 \{ -2 H^2 H' + 6 H'^2 + 3 HH'' + 6 H'H'' + 6 H''H'' \}
+ 3 H''H' + H''') f_{\Box R} + H f_{\Box R'} - (H^2 + H') f_R - \frac{f}{6} = \frac{1}{3} [\rho c^2 + (\rho + p) f_T],
\]

\[
f_{\Box R^{''''}} = 5 f_{\Box R}^{''''} + (-8 H^2 + 5 H') f_{\Box R''} + (-23 H^3 + 2 HH' + 4 H'') f_{\Box R'}
+ 2 \left( -2 H^2 H' + 6 H'^2 + 3 HH'' + H''') f_{\Box R} - 2 H f_{\Box R'} - (3 H^2 + H') f_R - f_R'' - \frac{f}{2} = \kappa^2 p.
\]

In terms of the FLRW scale factor, the Hubble ($H$) and deceleration ($q$) are calculated as

\[
H = \frac{\dot{a}}{a}, \quad q = -\frac{1}{H^2} \frac{a''}{a},
\]

whereas jerk ($j$), snap ($s$) and crackle ($l$) are

\[
 j = \frac{a'''}{a H^3}, \quad s = \frac{a''''}{a H^4}, \quad l = -\frac{a'''''}{a H^5}.
\]

The study of the exploring viable bounds on gravity models through ECs has been a source of great interest by many mathematical physicists. The constraints obtained through ECs could lead to analyze the stability of some relativistic systems. The coupling of fluid distributions with the various geometries, like spherical, axial, cylindrical symmetries are often in practice during the modeling of compact objects. In order to have the realistic configurations of these fluids, one must pick the viable formulations of stress-energy tensor. The idea of ECs could provide an effective tool in this regard, in the sense that only those stress-energy tensors are realistic, that satisfy the corresponding ECs. These conditions are observed to be coordinate-invariant (independent of symmetry). In the standpoint of expanding nature, the Raychaudhris equation can be stated as

\[
\frac{d\Theta_1}{d\tau} = -\frac{\Theta_1}{2} + \omega_{\alpha\beta} \omega_{\alpha\beta} - \sigma_{\alpha\beta} - R_{\alpha\beta} k^\alpha k^\beta,
\]

where $\omega_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ are rotation and shear tensor, respectively, whereas $\Theta_1$ is an expansion scalar. These quantities are characterized by the congruences connected with the null vector $k_\mu$. Bamba et al. [47] explored few stable bounds through ECs in $f(G)$ gravitational theories. The following distributions of ECs can be formulated in terms of effective forms of energy density and pressure as

\[
NEC \Leftrightarrow \rho_{\text{eff}} + p_{\text{eff}} \geq 0,
\]
We define a parameter $m$ as follows
\[
m = -\frac{1}{H^6} \frac{a^{(5)}}{a}.
\]

In terms of cosmic parameters, the derivatives of Hubble parameter $H$ are found to be
\[
H' = -H^2 (1 + q),
H'' = H^3 (j + 3q + 2),
H''' = H^4 (s - 4j - 12q - 3q^2 - 6),
H'''' = H^5 (24 + l + 40q + 30q^2 + 10j (2 + q) + 5s),
H^{(5)} = H^6 (-10j^2 - 120j(q + 1) + 6l + m - 30q^3 - 270q^2 + 15qs - 360q + 30s - 120).
\]

In order to solve cumbersome and lengthy $f(R, \Box R, T)$ equations of motion, we consider $f_T = 0$. In this framework, the 00 field equation provides
\[
\rho = -\frac{f}{2} + f_{\Box R}'' + 3(H(H(3q + 1))f_{\Box R}' + H^2 f_{\Box R}((-j + q + 5)) + f_R' - f_R H q) + 2f_{\Box R}(H''' + H^4 (3j + 6q^2 + 23q + 14)),
\]
while the sum of energy density and pressure gives
\[
\rho + p = f_{\Box R}'''' - 2f_R H^2 + 112f_{\Box R} H^4 + f_{\Box R}'' (1 + 5H) + 8f_{\Box R} H''' + 4f_{\Box R} H^4 j - 2f_R H^2 q
+ 184f_{\Box R} H^4 q + 48f_{\Box R} H^4 q^2 - 44H^3 f_{\Box R}' - 7H^3 j f_{\Box R}' - 13H^3 q f_{\Box R}' + H f_R' + 16H^2 f_{\Box R}''
+ 14H^2 q f_{\Box R}'' - f_R'''.
\]

### III. DIFFERENT MODELS

The purpose of this work is to present some viable regions of $f(R, \Box R, T)$ models. We want to analyze the behavior of ECs for the perfect and flat FLRW metric by considering the case of separating $R$ and $\Box R$ formulations. Here, we take models of the forms
\[
f(R, \Box R) = f(R) + f(\Box R).
\]

We proceed our work by considering the following choices of cosmological parameters
\[
H = 0.718, q = -0.64, j = 1.02,
s = -0.39, l = 3.22, m = -11.5.
\]

We shall use above values as well as separable combinations of $f(R)$ and $f(\Box R)$ functions to explore ECs in the following subsections.

#### A. Model 1

First, we choose the tanh Ricci scalar function along with $\beta R \Box R$ term as follows
\[
f(R, \Box R) = R - \alpha \gamma \tanh \left( \frac{R}{\gamma} \right) + \beta R \Box R,
\]
where $\alpha$, $\beta$ and $\gamma$ are constants. After using the above $f(R, \Box R)$ model along with the values of cosmological parameters from Eq. (20), Eq. (18) becomes

$$
\rho = \frac{1}{2\gamma} \left[ 12\beta\gamma H^5 (24 - l + 48q + 18q^2 + 2j (4 + q) - s) + 36\beta\gamma H^6 (4 + j^2 - l + j (13 - 9q) + 55q + 35q^2 + 5q^3 - qs) + 6\gamma H^2 \left[ 1 + q \left\{ -2 + \alpha Sech \left( \frac{6H^2 (-1 + q)}{\gamma} \right) \right\} \right] + \alpha \gamma^2 \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] \right] + 72\alpha H^2 (2 - j + q) Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right],
$$

(22)

wheras Eq. (19) turns out to be

$$
\rho + p = \frac{1}{\gamma^2} H^2 [6\beta\gamma^2 H^3 (24 - l + 48q + 18q^2 + 2j (4 + q) - s) - \gamma^2 \left( 1 - 2\alpha + \cosh \left( \frac{12H^2 (-1 + q)}{\gamma} \right) \right)]
\times (1 + q) Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 - 12\alpha \gamma H^2 \left( -8 + j - 9q - q^2 + s \right) Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] + 6H^4 \left( 124\beta\gamma^2 - 2 - 9\gamma^2 l + 313\beta\gamma^2 q + 178\beta\gamma^2 q^2 + 18\beta\gamma^2 q^3 + 4\beta\gamma^2 s - 2\beta\gamma^2 q s - 48\alpha Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] \right)^4 - 48a q Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^4 - 12\alpha q Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^4 + 96\alpha Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 + 4j^2 (\beta\gamma^2 - 3\alpha Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] \right)^4 + 6\alpha Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 + q (\beta\gamma^2 + 24\alpha)
\times Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^4 - 48\alpha Sech \left[ \frac{6H^2 (-1 + q)}{\gamma} \right] \tanh \left[ \frac{6H^2 (-1 + q)}{\gamma} \right]^2 )])
$$

(23)

For this $f(R, \Box R)$ model, we consider $0 < \gamma < 5$, $-1 < \alpha < 1$ and $-1 < \beta < 1$ ranges of parameters and we will restrict these values by ECs. In order to obtain $\rho > 0$, one requires $\gamma < 3$ along with $\alpha < 0.7$ and $\beta > -0.4$. For $\rho + p > 0$, the viable regions are being shown in Fig. (1) for all possible value of $\gamma$ and $\alpha$ with $\beta > 0$. Furthermore, we analyze the behavior of $\rho$ and $\rho + p$ by fixing one parameter value $\gamma$, as shown in Fig. (2), in which we can see the positivity of both $\rho$ and $\rho + p$ with all ranges of $\gamma$ and positive values of $\beta$ in given ranges. Similar behavior has been observed and shown in Figs. (2) and (3), in which we plot $\rho$ and $\rho + p$ with respect to $\gamma$ along with the positive as well negative values of $\alpha$. Odintsov and Oikonomou [43] presented observational consistent inflationary constraints by providing a comparison between non-singular and singular $R^2$ cosmic model. However, consequences of $R^2$ model consistent with latest Planck data are provided by Odintsov et al. [49]. Bhatti [50] and Yousaf [51] provided some viable models for $\Lambda$-dominated epochs.

### B. Model 2

Next, we take $f(R, \Box R, T)$ model of the form

$$
f(R, \Box R) = R + \alpha \gamma \left[ \left( 1 + \left( \frac{R^2}{\gamma^2} \right)^{-\lambda} \right) - 1 \right] + \beta R \Box R,
$$

(24)

in which $\alpha$, $\lambda$ and $\gamma$ are constants. By making use of this model and Eqs. (9), (10) and (18), we found

$$
\rho = 18\beta H^6 (-2 + j - q) (5 + j + q) + \frac{1}{2} (\alpha \gamma) \left[ 1 - \left( \frac{\gamma^2 + 36H^2 (-1 + q)^2}{\gamma^2} \right)^{-\lambda} \right] - 6H^2 (-1 + q)
$$
FIG. 1: Plot of WEC for model 1 as in Eq. (21), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\alpha$, $\gamma$ and $\beta$.

FIG. 2: Plot of WEC for model 1 as in Eq. (21), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\alpha$ and $\beta$.

FIG. 3: Plot of WEC for model 1 as in Eq. (21), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\gamma$ having $\alpha > 0$. 
\[ + 36\beta H^6 (-1 + q) \left( 12 - 3j + 11q + q^2 - s \right) - 18\beta H^6 (1 + 3q) (6 + 8q + q^2 - s) + 6\beta H^5 (24 - l + 48q + 18q^2 + 2j(4 + q) - s) - 36\beta H^6 (-1 + q) (-8j - 11q - 3q^2 - s) - 3H^2 q[1 \\
- \left\{ 12\alpha\lambda H^2 \left( \gamma^2 + 36H^4(-1 + q)^2 \right)^{\lambda - 1} \right\} (-1 + q) \gamma^{-1} + 6\beta H^4 (-12 + 3j - 11q - q^2 + s) \right] \\
+ 3H \left\{ 12\alpha\lambda H^3 \left( \gamma^2 + 36H^4(-1 + q)^2 \right)^{\lambda - 1} \right\} (-2 + j - q) \gamma^{-1} \{ 864\alpha\gamma\lambda(1 + \lambda)H^7 \\
+ \left( \frac{\gamma^2 + 36H^4(-1 + q)^2}{\gamma^2} \right)^{-\lambda} (-1 + q)^2 (2 - j + q) \} ... \left\{ \left( \frac{\gamma^2 + 36H[t]^4(-1 + q | t|^2)^2}{\gamma^2} \right)^{-1} \right\} \\
- 6\beta H^5 (-48 + l + j (-5 + q) - 81q - 24q^2 + 4s), \right. \]
FIG. 5: Plot of WEC for model 2 as in Eq.(24), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\lambda$, $\gamma$ and $\beta$

FIG. 6: Plot of WEC for model 2 as in Eq.(24), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\alpha$ and $\beta$

[0, 5], [−1, 1], [−1, 1], and [−1, 1], respectively. We shall put constraints on these parameters via ECs. We noticed that $\rho > 0$ requires $\lambda > -0.6$, while $\rho + p > 0$ requires positive values of $\beta$. There are also small regions for which $\rho + p > 0$. The details can be observed from Fig. 5. By fixing one both $\gamma$ and $\lambda$ values, we plotted the viable regions of ECs as Fig. 6. One can observed the positive behavior of $\rho$ for all given ranges of $\alpha$ and $\beta$, while $\rho + p > 0$ requires $\beta$ to be greater than zero. Now, by taking different values of $\beta$ with $\alpha > 0$, we have plotted the diagrams for $\rho$ and $\rho + p$ as shown in Fig. 7. We conclude that for the validity of ECs, $\beta$ should be greater than zero. We check the similar behavior for $\alpha < 0$, which gives the same conclusion e.g. $\beta$ should be non-negative real number. The details of these can be seen from Fig. 8.
FIG. 7: Plot of WEC for model 2 as in Eq. (24), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\gamma$ with $\alpha > 0$.

FIG. 8: Plot of WEC for model 2 as in Eq. (24), left plot show $\rho$ while the right plot show the $\rho + p$ with respect to $\gamma$ with $\alpha < 0$.

IV. SUMMARY

The modified gravity theories have been emerged as among good candidates to study the cosmic acceleration of the expanding universe. The $f(R, \Box R, T)$ gravity theory have gained significance on the basis of curvature matter coupling and can be considered as a generalization of $f(R, T)$ gravity theory. However, there is a crucial difference between both modified theories due to the higher derivative terms of the Ricci scalar in the gravitational Lagrangian and consequently leads to a significant deviation from the geodesic paths. In this paper, we have discussed the ECs in the context of $f(R, \Box R, T)$ gravity theory with two different models which is the best viable method to test the validity of these theories. The WEC has been evaluated using the Raychaudhuri equation which are more general as compared to those obtained in $f(R)$ and $f(R, T)$ gravity theories. We showed that these energy conditions can be satisfied with the modified gravity models. We have found that the obtained inequalities have equivalence with those obtained via $p + \rho \geq 0$ and $\rho \geq 0$ with the limit $p \to p^{\text{eff}}$ and $\rho \to \rho^{\text{eff}}$. We have considered two functional forms of $f$ namely $R - \alpha \gamma \tanh \left( \frac{4}{\alpha} \right) + \beta R \Box R$ and $R + \alpha \gamma \left( \frac{1}{1 + \left( \frac{4}{\alpha} \right)} \right)^{-\lambda} - 1 + \beta R \Box R$. We have used the recent estimated values of the snap, deceleration, jerk and Hubble parameters to show the validity of the different functional forms of $f(R, \Box R, T)$ imposed by the WEC.

For the first functional value of the $f(R, \Box R, T)$ model, we found particular constraints on the parameters $\alpha$, $\beta$ and $\gamma$ to satisfy the energy conditions. We observed the positivity of energy density for the first model when $\alpha < 0.7$, $\beta > -0.4$ and $\gamma < 3$, however, the positivity of $\rho + p > 0$ is observed when $\beta > 0$ with all possible values of $\alpha$ and $\gamma$. These results are indicated in Fig. 1. Similar results have been obtained by fixing the parametric value of $\gamma$ and with different values of $\alpha$ and $\beta$ as shown in Fig. 2. We have also discussed the positivity of energy density and $\rho + p$ for different values of $\beta$ with positive and negative $\alpha$ and presented via plots in Figs. 3 and 4. Similarly, for the second functional value of the $f(R, \Box R, T)$ model, we found particular constraints on the parameters $\alpha$, $\beta$, $\gamma$ and $\lambda$ to satisfy the energy conditions. We have observed that $\rho > 0$ for the second model when $\lambda > -0.6$, however, the positivity of $\rho + p > 0$ is observed when $\beta > 0$ with all possible values of $\alpha$ and $\gamma$. These results are indicated in Fig. 5. Similar results have been obtained by fixing the parametric value of $\lambda$ and $\gamma$ and plotted with respect to $\alpha$ and $\beta$ as shown in Fig. 6. We have also discussed the positivity of energy density and $\rho + p$ for different values of $\beta$ with positive and negative $\alpha$ and presented via plots in Figs. 7-8. One can conclude that the viability of ECs depend on particular values of the model parameters.

It is significant to mention here that regardless of well-motivated physical interpretation of ECs in modified gravity
...our method and interpretation is rather general from the context of higher derivative theory which can be reduced to theories, it is still under discussion due to the confrontation between the observations and the theory. We stress that other modified gravity results under the usual limits.

[1] P. A. R. Ade et al. [BICEP2 Collaboration], Phys. Rev. Lett. 112, 241101 (2014) [arXiv:1403.3985 [astro-ph.CO]].
[2] P. A. R. Ade et al. [BICEP2 and Planck Collaborations], Phys. Rev. Lett. 114, 101301 (2015) [arXiv:1502.00612 [astro-ph.CO]].
[3] P. A. R. Ade et al. [BICEP2 and Keck Array Collaborations], arXiv:1510.09217 [astro-ph.CO].
[4] E. Komatsu et al. [WMAP Collaboration], Astrophys. J. Suppl. 192, 18 (2011) [arXiv:1001.4538 [astro-ph.CO]].
[5] G. Hinshaw et al. [WMAP Collaboration], Astrophys. J. Suppl. 208, 19 (2013) [arXiv:1212.5226 [astro-ph.CO]].
[6] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 571, A1 (2014).
[7] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A13 (2016) [arXiv:1502.01589 [astro-ph.CO]].
[8] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594, A20 (2016) [arXiv:1502.02114 [astro-ph.CO]].
[9] A. Qadir, H. W. Lee, and K. Y. Kim, Int. J. Mod. Phys. D 26, 1741001 (2017).
[10] A. Joyce, B. Jain, J. Khoury and M. Trodden, Phys. Rept. 568, 1 (2015) [arXiv:1407.0059 [astro-ph.CO]].
[11] S. Capozziello and V. Faraoni, Beyond Einstein Gravity (Springer, Dordrecht, 2010).
[12] S. Capozziello and M. De Laurentis, Phys. Rept. 509, 167 (2011) [arXiv:1108.6266 [gr-qc]].
[13] K. Bamba, Capozziello, S., Nojiri, S. and Odintsov, S. D.; Astrophys. Space Sci. 342, 155 (2012) [arXiv:1205.3421 [gr-qc]].
[14] K. Koyama, Rep. Prog. Phys. 79, 046902 (2016) [arXiv:1504.04625 [astro-ph.CO]].
[15] A. de la Cruz-Dombriz and D. Sáez-Gómez, Entropy 14, 1717 (2012) [arXiv:1207.2665 [gr-qc]].
[16] K. Bamba, S. Nojiri and S. D. Odintsov [arXiv:1302.4831] [gr-qc].
[17] K. Bamba, and S. D. Odintsov, [arXiv:1402.7114] [hep-th]. Symmetry 7, 220 (2015) [arXiv:1503.00442 [hep-th]].
[18] Z. Yousaf, K. Bamba and M. Z. Bhatti, Phys. Rev. D 93, 064059 (2016) [arXiv:1603.03175 [gr-qc]].
[19] Z. Yousaf, K. Bamba and M. Z. Bhatti, Phys. Rev. D 93, 124048 (2016) [arXiv:1606.00147 [gr-qc]].
[20] S. Nojiri and S. D. Odintsov eConf C 0602061, 06 (2006); Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007) [hep-th/0601213].
[21] S. Nojiri and S. D. Odintsov [arXiv:0804.4843 [astro-ph]] (2008).
[22] S. Nojiri and S. D. Odintsov [arXiv:0807.0685 [hep-th]] (2008).
[23] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82, 451 (2010) [arXiv:0805.1726 [gr-qc]].
[24] S. Capozziello and M. Francaviglia, Gen. Relativ. Gravit. 40, 357 (2008).
[25] A. A. Starobinsky, Phys. Lett. 91B, 99 (1980).
[26] R. Kerner, Gen. Relativ. Gravit. 14, 453 (1982).
[27] A. A. Starobinsky, Sov. Astron. Lett. 9, 302 (1983); J. D. Barrow and A. C. Ottewill, J. Phys. A 16, 2757 (1983); A. A. Starobinsky and H.-J. Schmidt, Class. Quantum Grav. 4, 695 (1987).
[28] A. Bonanno, arXiv:gr-qc/9801077v1.
[29] R. R. Cuzinatto, C. A. M. de Melo, L. G. Medeiros and P. J. Pompeia, arXiv:1806.0885v2 [gr-qc].
[30] M. Iliosh, J. Cosmol. Astropart. Phys. 02, 022 (2011).
[31] P. V. Tretyakov, Int. J. Geom. Methods Mod. Phys. 12, 1550094 (2015).
[32] T. Harko, F. S. N. Lobo, S. Nojiri, S. D. Odintsov, Phys. Rev. D 84, 024020 (2011).
[33] G. Abbas, M. Tahir, Eur. Phys. J. Plus 133, 476 (2018); G. Abbas, H. Nazar, Eur. Phys. J. C 78, 957 (2018).
[34] M. J. S. Houndjo, M. E. Rodrigues, N. S. Mazhari, D. Momeni and R. Myrzakulov, Int. J. Mod. Phys. D 26, 1750024 (2017).
[35] Z. Yousaf, M. Sharif, M. Ilyas, M. Z. Bhatti, Int. J. Geom. Meth. Mod. Phys. 15, 1850146 (2018) [arXiv:1806.09275 [gr-qc]].
[36] F. G. Alvarenga, A. D. L. Cruz-Dombriz, M. J. S. Houndjo, M. E. Rodrigues, D. Sáez-Gómez, Phys. Rev. D 87, 103526 (2013).
[37] E. H. Baffou, A. V. Kpadonou, M. E. Rodrigues, M. J. S. Houndjo, J. Tossa, Astrophys. Space Sci. 356, 173 (2015).
[38] M. Ilyas, Z. Yousaf, M. Z. Bhatti and B. Masud, Astrophys. Space Sci. 362, 237 (2017); Z. Yousaf, M. Ilyas and M. Z. Bhatti, Eur. Phys. J. C 78, 307 (2018) [arXiv:1804.04953 [physics.gen-ph]]. Eur. Phys. J. C 77, 691 (2017) [arXiv:1710.05717 [gr-qc]]; M. Z. Bhatti, M. Sharif, Z. Yousaf and M. Ilyas, Int. J. Mod. Phys. D 27, 1850044 (2018).
[39] P. H. R. S. Moraes, J. D. V. Arbañil, M. Malheiro, J. Cosmol. Astropart. Phys. 06, 005 (2016).
[40] P. H. R. S. Moraes, P. K. Sahoo, Eur. Phys. J. C 77, 480 (2017); P. H. R. S. Moraes, P. K. Sahoo, P. K. Rev. D 96, 044038 (2017); P. H. R. S. Moraes, P. K. Sahoo, G. Ribeiro, and R. A. C. Correa, arXiv:1712.07569 [gr-qc].
[41] R. Myrzakulov, Eur. Phys. J. C 72, 2203 (2012).
[42] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime (Cambridge University Press, England, 1973).
[43] J. Santos, J. S. Alcaniz, M. J. Rebouc and F. C. Carvalho, Phys. Rev. D 76, 083513 (2007).
[44] O. Bertolami and M. C. Sequeira, Phys. Rev. D 79, 104010 (2009).
[45] S. -Y. Zhou, E. J. Copeland and P. M. Saffin, J. Cosmol. Astropart. Phys. 07, 009 (2009).
[46] K. Atazadeh and F. Darabi, Gen. Relativ. Gravit. 46, 1664 (2014).
[47] K. Bamba, M. Ilyas, M. Z. Bhatti and Z. Yousaf, Gen. Relativ. Gravit. 49, 112 (2017) [arXiv:1707.07380 [gr-qc]].
[48] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D 92, 124024 (2015).
[49] S. D. Odintsov, V. K. Oikonomou and L. Sebastiani, Nucl. Phys. B 923, 608 (2017).
[50] M. Z. Bhatti, Eur. Phys. J. Plus 131, 428 (2016).
[51] Z. Yousaf, Eur. Phys. J. Plus 132, 71 (2017); Eur. Phys. J. Plus 132, 276 (2017); Astrophys. Space Sci. 363, 226 (2018).