Radiation reaction in quantum mechanics

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Abstract

The Lorentz-Dirac radiation reaction formula predicts that the position shift of a charged particle due to the radiation reaction is of first order in acceleration if it undergoes a small acceleration. A semi-classical calculation shows that this is impossible at least if the acceleration is due to a time-independent potential. Thus, the Lorentz-Dirac formula gives an incorrect classical limit in this situation. The correct classical limit of the position shift at the lowest order in acceleration is obtained by assuming that the energy loss at each time is given by the Larmor formula.

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A classical charge $e$ with four-velocity $u^\mu$ emits electromagnetic radiation with energy-momentum $P^\mu$ according to the Larmor formula:

$$\frac{dP^\mu}{d\tau} = \frac{2\alpha}{3} a^2 u^\mu,$$

(1)

where $\tau$ is the proper time along the world line of the charge $e$ and where $a = \sqrt{-\dot{u}^\mu \dot{u}_\mu}$ (with $\dot{u}^\mu = du^\mu/d\tau$) is the acceleration of the charge. (See, e.g., Ref. [1].) We have defined $\alpha = e^2/4\pi$. It is widely believed that the radiation reaction is given by the Lorentz-Dirac formula [2,3]. The corresponding four-force $K^\mu$ is given by

$$K^\mu = \frac{2\alpha}{3} (-a^2 u^\mu + \frac{d^2 u^\mu}{d\tau^2}).$$

(2)

One can readily verify that $u^\mu K_\mu = 0$. (This is necessary for consistency of Newton’s equation $m \ddot{u}^\mu = K^\mu$, where $m$ is the mass of the charge.) Moreover, if the acceleration lasts only for a finite time, the total energy-momentum lost due to this four-force agrees with the total energy-momentum radiated in the form of electromagnetic radiation according to the Larmor formula (1).

Although the Lorentz-Dirac formula (2) is widely accepted, it is known to have some unphysical features. (See, e.g., Refs. [1,4].) Let us consider one-dimensional motion and let $u^\mu = (u^0, u^1) = (\cosh \beta, \sinh \beta)$. Then the Lorentz-Dirac force is $K^\mu = \frac{2\alpha}{3} (\sinh \beta, \cosh \beta) \dddot{\beta}$. Newton’s equation reads

$$\frac{d\beta}{d\tau} = \frac{2\alpha}{3m} \frac{d^2 \beta}{d\tau^2} + F_{\text{ext}}(\tau),$$

(3)

where $F_{\text{ext}}(\tau)$ represents an external force. This equation allows runaway solutions. For example, if $F_{\text{ext}}(\tau) = 0$, then $\beta(\tau) = \beta_0 \exp(3m\tau/2\alpha)$ is a solution. Runaway solutions can be excluded by rewriting Eq. (3) as follows:

$$m \frac{d\beta}{d\tau} = \int_{\tau_0}^\infty e^{-s} F_{\text{ext}}(\tau + \tau_0 s) ds,$$

where $\tau_0 = 2\alpha/3m$. However, this equation violates causality. This causality violation is usually dismissed as unimportant [1] because its time scale $\tau_0$ is very small. (It is of order $10^{-24}$ sec for electrons.) This nevertheless casts some doubts on the validity of the Lorentz-Dirac theory. In this Letter we will show that in fact the Lorentz-Dirac theory does not describe the classical limit of the quantum theory if the acceleration is due to a static potential varying in one dimension, say in the $z$-direction. (Moniz and Sharp concluded that the Lorentz-Dirac formula is reproduced in the classical limit by studying the Heisenberg equations [3]. It is difficult to find the reason why we disagree since our method is quite different from theirs.)

Once we start considering a quantum system with a static potential, a natural alternative classical theory emerges. The $z$-component of the momentum is not conserved in the rest frame of the potential. However, the energy is conserved. Hence, it is natural to expect that the correct classical limit is obtained by assuming that the energy is lost at each time according to the Larmor formula (1), disregarding the momentum conservation in the $z$-direction. We call this theory the Larmor theory in this Letter. We show that the Larmor
theory is likely to be the correct classical theory. We use natural units $\hbar = c = 1$ unless otherwise stated throughout this Letter.

We consider a charged particle moving in the $z$-direction under the influence of a potential $V(z)$ which is constant if $|z|$ is sufficiently large. (Thus, the acceleration lasts only for a finite time.) Let us first clarify how the Lorentz-Dirac and Larmor theories differ.

We work in the non-relativistic approximation. Let us assume that the kinetic energy is dominant so that the particle always moves in the positive $z$-direction. The energy is defined as $E = \frac{1}{2}mv^2 + V(z)$, where $v$ is the velocity of the particle. The rate of energy change in the Lorentz-Dirac theory reads

$$\frac{dE_D}{dt} = \frac{2}{3}\alpha \ddot{v}v,$$  \hspace{1cm} (4)

where $\ddot{v} = \frac{d^2v}{dt^2}$. On the other hand, according to the Larmor theory this quantity is

$$\frac{dE_L}{dt} = -\frac{2}{3}\alpha \dot{v}^2.$$  \hspace{1cm} (5)

To first order in $\alpha$, one can let $v$ on the right-hand sides of Eqs. (4) and (5) be the velocity without any radiation reaction. By imposing the condition that $E_D = E_L$ for $t \to -\infty$ one has

$$E_D - E_L = \frac{\alpha}{3} \frac{d}{dt}v^2.$$  \hspace{1cm} (6)

Note here that this quantity vanishes for $t \to +\infty$ because $V(z)$ is assumed to become constant for large $|z|$. Thus, the two theories give the same final velocity.

Now, let $z$ denote the position without the radiation reaction. Let $\Delta_D v$ and $\Delta_D z$ be the corrections to the velocity and position at the lowest order in $\alpha$ according to the Lorentz-Dirac theory and $\Delta_L v$ and $\Delta_L z$ be the corresponding quantities for the Larmor theory. Then, using $V'(z) = -m\dot{v}$, $\Delta_D v = (d/dt)(\Delta_D z)$ and $\Delta_L v = (d/dt)(\Delta_L z)$, one finds from Eq. (6)

$$m\frac{d}{dt}\left(\frac{\Delta_D z - \Delta_L z}{v}\right) = \frac{2\alpha}{3} \frac{d}{dt} \ln v.$$  \hspace{1cm} (6)

Let the initial and final velocities without the radiation reaction be $v_i$ and $v_f$, respectively. Then, for $t \to +\infty$, one has

$$\Delta_D z - \Delta_L z = \frac{2\alpha}{3} \frac{v_f}{v_i} \ln \frac{v_f}{v_i}.$$  \hspace{1cm} (6)

Now, let $a(t)$ be the acceleration. Then, to lowest order in $a(t)$ one finds

$$\Delta_D z - \Delta_L z = \frac{2\alpha}{3m}(v_f - v_i) = \frac{2\alpha}{3m} \int_{-\infty}^{+\infty} a(t) \, dt.$$  \hspace{1cm} (7)

Thus, the two theories differ in the position shift, and the difference is of first order in acceleration.

Next let us derive the formula for the position shift at $t = 0$ for the Larmor theory, assuming that $a(t) = 0$ for $t > 0$ for simplicity. The energy formula (4) gives

$$\frac{d}{dt}(v\Delta_L v - \dot{v}\Delta_L z) = -\frac{2\alpha}{3m}a(t)^2.$$  \hspace{1cm} (4)
From this we find the position shift at \( t = 0 \) to lowest order in \( a(t) \) as
\[
(\Delta L_z)_0 = -\frac{2\alpha}{3\bar{p}} \int_{-\infty}^{t} \left( \int_{-\infty}^{t'} a(t')^2 dt' \right) dt
= \frac{2\alpha}{3\bar{p}} \int_{-\infty}^{+\infty} t a(t)^2 dt,
\]
where we have replaced \( mv \) by the average momentum \( \bar{p} \). Define \( \hat{a}(k) \) to be the Fourier transform of \( a(t) \), i.e.,
\[
\hat{a}(k) = \int_{-\infty}^{+\infty} dt a(t)e^{ikt}.
\]
Then we find
\[
(\Delta L_z)_0 = -\frac{2i\alpha}{3\bar{p}} \int_{-\infty}^{+\infty} dk \frac{d}{dk} \hat{a}(k) \hat{a}(k).
\] (8)

Note that this position shift is of second order in acceleration. Since the difference in the position shift for \( t \to +\infty \) given by Eq. (7), which equals that at \( t = 0 \) if \( a(t) = 0 \) for \( t > 0 \), is of first order, the position shift \( (\Delta D_z)_0 \) at \( t = 0 \) according to the Lorentz-Dirac theory is of first order in acceleration. It will be shown next that the position shift is of second order in acceleration in quantum theory and, therefore, that the Lorentz-Dirac theory gives an incorrect classical limit.

The quantum system corresponding to the classical one considered above is given, to lowest non-trivial order in \( e \), by the following Hamiltonian:
\[
H = \frac{\hat{p}^2}{2m} + V(z) + H_{em} - \frac{e}{m} \mathbf{A} \cdot \hat{\mathbf{p}},
\] (9)
where \( H_{em} \) is the Hamiltonian describing the free electromagnetic field and where \( \hat{\mathbf{p}} = -i\partial/\partial \mathbf{x} \). The relevant Hilbert space is the tensor product of the space of non-relativistic one-particle Schrödinger wave functions and the Fock space of photons. The underlying theory is the scalar QED with an external potential for the charged scalar field. (See, e.g., Ref. [6] for the non-relativistic version of this theory.) One arrives at the Hamiltonian (9) by adopting the Coulomb gauge and dropping the terms of second order in \( e \).

First let us consider wave functions evolved in time by the Hamiltonian \( \hat{H}_0 = \hat{p}^2/2m + V(z) \). They represent accelerated particles without radiation reaction. Let \( V(z) = V_0 = \text{const.} \) for sufficiently large and negative values of \( z \) and \( V(z) = 0 \) for sufficiently large and positive values of \( z \). Writing the solution to the time-independent Schrödinger equation as \( \phi_P(z) \exp(iP_x x + iP_y y) \), where the energy is \( E = (P^2 + P_x^2 + P_y^2)/2m \), one finds
\[
\left[ -\frac{1}{2m} \frac{d^2}{dz^2} + V(z) \right] \phi_P(z) = \frac{P^2}{2m} \phi_P(z).
\]

1The term proportional to \( e^2 A^\mu A_\mu \phi \phi \) contributes only to the mass renormalization at the lowest order. The term representing the Coulomb interaction produces a contribution corresponding to Coulomb scattering of the charged particle off a vacuum loop under the influence of \( V(z) \). This contribution vanishes if one assumes that the potential \( V(z) \) is the same (including the sign) for both the positively and negatively charged particles. We will make this assumption here.
Let us assume that \(\frac{P^2}{2m} \gg |V(z)|\) and that \(V(z)\) is smooth enough so that the WKB approximation is applicable. The right-moving solutions \(\phi_P(z)\) are given by

\[
\phi_P(z) = \kappa(z)^{-1/2} \exp \left[ i \int_0^z \kappa(z) \, dz \right],
\]

where \(\kappa(z) = [P^2 - 2mV(z)]^{1/2}\).

Let the initial state be \(\Psi_0 = \psi_0 \otimes |0\rangle\), where \(|0\rangle\) is the vacuum state of the electromagnetic field and \(\psi_0\) is a wave packet of the charged particle given by

\[
\psi_0 = \int \frac{dPdP_\perp}{\sqrt{(2\pi)^3}} f(P, P_\perp)\sqrt{P} \phi_P(z) e^{i\mathbf{P}_\perp \cdot \mathbf{x}_\perp - iE(P, P_\perp)t}.
\]

Here, \(P_\perp = (P_x, P_y)\), \(E(P, P_\perp) = (P^2 + P_\perp^2)/2m\), \(\mathbf{x}_\perp = (x, y)\) and \(\int dPd^2P_\perp |f(P, P_\perp)|^2 = 1\).

The function \(f(P, P_\perp)\) is assumed to be non-zero only for the values of \(P\) much larger than \(|2mV(z)|^{1/2}\). The wave packet is assumed to be sharply peaked about a classical trajectory near the region with \(V'(z) \neq 0\). The electromagnetic field \(\mathbf{A}(x)\) in the Schrödinger picture can be expanded as

\[
\mathbf{A}(x) = \sum_{j=1}^2 \int \frac{d^3k}{(2\pi)^3} \sqrt{2k} \left[ \epsilon^{(j)} a_{kj} e^{ik \cdot \mathbf{x}} + \epsilon^{(j)} a_{kj}^\dagger e^{-ik \cdot \mathbf{x}} \right],
\]

where \(\epsilon^{(j)}\) are the (real) polarizations of the photon satisfying \(\epsilon^{(j)} \cdot k = 0\) and where \(k = |\mathbf{k}|\).

The annihilation and creation operators \(a_{kj}\) and \(a_{kj}^\dagger\) satisfy the usual commutation relations

\[
[a_{kj}, a_{k'j'}] = \delta_{j,j'} \delta(k' - k) \quad \text{and} \quad [a_{kj}, a_{kj}^\dagger] = [a_{kj}^\dagger, a_{kj}] = 0.
\]

By straightforward application of the time-dependent perturbation theory with the interaction Hamiltonian \(H_I = -(e/m)\mathbf{A} \cdot \mathbf{p}\), one finds the first-order component in the final state as

\[
\psi_1 = e^2 \sum_{j=1}^2 \int d^2p_\perp dp \int \frac{d^3k}{(2\pi)^3} \sqrt{P} \phi_P(z) \langle p|p_\perp|kj\rangle 
\times \int dz' \delta_P(z') \left[ i \epsilon^{(j)} \cdot \mathbf{P}_\perp \phi_P(z') + \epsilon^{(j)} \cdot \nf \phi_P(z') \right] e^{ikz'},
\]

where

\[
|p| = |p| + k_\perp \quad \text{and} \quad P = \sqrt{p^2 + p_\perp^2 - P_\perp^2 + 2mk}.
\]

At the lowest order in the WKB approximation, one has \(\phi_P'(z) \approx \sqrt{P^2 - 2mV(z)}^{1/2} \phi_P(z)\).

By choosing the wave packet such that \(|P| \ll P\) if \(f(P, P_\perp) \neq 0\), the term proportional to \(\epsilon^{(j)} \cdot \mathbf{P}_\perp\) can be neglected. One can then choose the two polarizations so that \(\epsilon_z^{(2)} = 0\). Thus, one has

\[
\psi_1 \approx ie \int d^2p_\perp dp \int \frac{d^3k}{(2\pi)^3} \sqrt{P} \phi_P(z) \langle p|p_\perp|kj\rangle,
\]

where
\[
I_{pk} = \sqrt{\frac{P}{p}} \int_{-\infty}^{+\infty} dz \sqrt{P^2 - 2mV(z)} \phi_p(z) \phi_P(z)e^{-ikz} \\
\approx \int_{-\infty}^{+\infty} \exp \left[ i \int_{0}^{z} g(z') dz' - ikz \right].
\]

We have defined
\[
g(z) = [P^2 - 2mV(z)]^{1/2} - [p^2 - 2mV(z)]^{1/2}.
\tag{10}
\]

It is assumed here that the WKB approximation is applicable to the final-state wave functions as well. This assumption is valid unless the photon carries away most of the kinetic energy of the charged particle. The phase space for such final states can be made arbitrarily insignificant by increasing \(P\). Hence, it is reasonable to assume that the contribution from such final states can be neglected.

The integral \(I_{pk}\) can be rewritten by integrating by parts and dropping the surface terms as
\[
I_{pk} = -i \int_{-\infty}^{+\infty} dz \frac{g'(z)}{[g(z) - k_z]^2} \exp \left[ i \int_{0}^{z} g(z') dz' \right] e^{-ikz}.
\]

(The surface terms can be dropped because we are dealing with a wave packet here, and the final result would be the same if we introduced a damping factor which made the surface terms vanish.) Now, the function \(g(z)\) in Eq. (10) is multiplied by \(\hbar^{-1}\) if one restores \(\hbar\). Hence, in the classical limit \(\hbar \to 0\), the integral \(I_{pk}\) tends to zero due to rapid oscillations if \(P \neq p\). Since we are interested in the classical limit, we make the approximation \(P \approx p\). Now, by using conservation of transverse momentum, \(P_\perp = p_\perp + k_\perp\), one finds \(|P_\perp^2 - p_\perp^2| \leq k(|P_\perp| + |p_\perp|)\). Note that \(m \gg |p_\perp| + |p_\perp|\) because the charged particle is assumed to be non-relativistic. Therefore, we conclude that \(|P_\perp^2 - p_\perp^2| \ll 2mk\). From \(P = \sqrt{p^2 + p_\perp^2 - P_\perp^2 + 2mk}\) we have \(P \approx [p^2 + 2mk]^{1/2}\). Thus, in the non-relativistic and classical regime, one can make the following approximation:
\[
g(z) = [P^2 - 2mV(z)]^{1/2} - [p^2 - 2mV(z)]^{1/2} \approx \frac{k}{v_p(z)},
\]

where \(v_p(z) = [p^2 - 2mV(z)]^{1/2}/m\) is the velocity of the classical particle with final momentum \(p\) in the \(z\)-direction. By using this approximation one finds
\[
I_{pk} = i \int_{-\infty}^{+\infty} dz \frac{k v_p'(z)e^{-ikz}}{[k - k_zv_p(z)]^2} \exp \left[ i \int_{0}^{z} \frac{k}{v_p(z')} dz' \right].
\]

Noting that \(k \gg k_zv_p(z)\) because the particle is non-relativistic, one finds
\[
I_{pk} \approx \frac{i}{k} \int_{-\infty}^{+\infty} dt a_p(t)e^{ikt} = \frac{i}{k} \hat{a}_p(k),
\]

where \(a_p(t)\) is the acceleration of the classical particle with the final momentum \(p\), with \(t = 0\) at \(z = 0\). Thus, the first-order part of the final-state wave function in the momentum representation is
\[
\hat{\psi}_1 \approx -e \epsilon_k^{(1)} \frac{\hat{a}_p(k)}{\sqrt{(2\pi)^32k^3}} f(p, \mathbf{p}_\perp).
\]

6
We have let $P \approx p$ and $P_{\perp} \approx p_{\perp}$. The probability of emission is obtained by integrating $|\hat{\psi}_1|^2$ over $p$, $p_{\perp}$ and $k$. Here, we note that the amplitude of emission is proportional to the acceleration. Hence, the position shift must be at least of second order in acceleration because the probability is proportional to the amplitude squared. Thus, the Lorentz-Dirac theory cannot give the correct classical limit.

By recalling that $\int dp d^2p_{\perp} |f(p, p_{\perp})|^2 = 1$, the emission probability is found to be

$$
\mathcal{P} = \frac{2e^2}{3} \int \frac{d^3k}{(2\pi)^3 2k^3} |\hat{a}_p(k)|^2 = \frac{4\alpha}{3} \int_0^\infty \frac{dk}{2\pi k} |\hat{a}_p(k)|^2,
$$

where $p$ is taken to be the average value for the wave packet. We have used the fact that the average of $\langle \epsilon_z^{(1)} \rangle^2$ is $2/3$. The probability $\mathcal{P}$ is infrared divergent if $\hat{a}_p(0) \neq 0$, i.e., if the initial and final velocities are different. The expected energy of the emitted photon is

$$
E = \frac{e^2}{3\pi} \int_0^\infty \frac{dk}{2\pi} |\hat{a}_p(k)|^2 = \frac{2\alpha}{3} \int_{-\infty}^{+\infty} dt a(t)^2.
$$

Thus, the classical Larmor formula (1) is reproduced.

Now, the position operator (in the momentum space) of the final-state wave packet evolved back to $t = 0$ can be approximated by $i\partial / \partial p$. (The time $t = 0$ is automatically picked out because the phase factor describing the time dependence of the wave function is chosen to be one at $t = 0$.) Let us concentrate on the sector with $p_{\perp}$ fixed and suppress $p_{\perp}$. The contribution to the expectation value of $i\partial / \partial p$ from the tree diagram of first order in $\epsilon$ is given by

$$
(z)_0 = \int d^3k \langle \hat{\psi}_1, \hat{\psi}_1 \rangle
= \frac{4e^2}{3} \int \frac{d^3k}{(2\pi)^3 2k^3} \langle \hat{a}_p(k)f(p), \hat{a}_p(k)f(p) \rangle.
$$

where

$$
\langle A, B \rangle = \frac{i}{2} \int dp \left[ A \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} B \right].
$$

The acceleration is given by $-V'(z)/m$ as a function of $z$, and its $p$ and $t$ dependence is only through $z = pt/m$. Thus, one has $[t(\partial / \partial t) - p(\partial / \partial p)]a_p(t) = 0$. Hence,

$$
\frac{\partial}{\partial p} \hat{a}_p(k) = -\frac{1}{p} \hat{a}_p(k) - \frac{k}{p} \frac{\partial}{\partial k} \hat{a}_p(k).
$$

By using this formula and the fact that $\hat{a}_p(-k) = \overline{\hat{a}_p(k)}$ (because $a_p(t)$ is real), one obtains

$$
(z)_0 = \frac{4\alpha}{3} \langle f(p), f(p) \rangle \int_0^\infty \frac{dk}{2\pi k} |\hat{a}_p(k)|^2
- \frac{2i\alpha}{3} \int_{-\infty}^{+\infty} \frac{dp}{p} |f(p)|^2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \overline{a_p(k)} \frac{\partial}{\partial k} \hat{a}_p(k).
$$

The first term is the expectation value of the operator $i\partial / \partial p$ (without radiation reaction) times the probability of emission. This term will be cancelled by the “one-loop” correction.
without emission at this order due to unitarity. This means that the expectation value of the final-state position shift evolved back to \( t = 0 \) is given by

\[
(\Delta z)_0 = \frac{2i\alpha}{3} \int_{-\infty}^{+\infty} \frac{dp}{p} |f(p)|^2 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \hat{a}_p(k) \frac{\partial}{\partial k} \hat{a}_p(k).
\]

It is clear that one obtains Eq. (8) in the limit where the wave packet becomes concentrated at a point in the momentum space. Thus, the Larmor theory gives the correct position shift.

In summary, it was shown in this Letter that the Lorentz-Dirac radiation reaction formula does not give the correct classical limit of the quantum mechanical system with a static potential, by demonstrating that the position shift due to that formula is incorrect. It was also shown that the correct classical limit of the position shift is obtained by assuming that the energy is lost according to the Larmor formula, disregarding momentum conservation.

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REFERENCES

[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975)
[2] H. A. Lorentz, *Theory of Electrons*, (Dover, New York, 1952).
[3] P. A. M. Dirac, Proc. R. Soc. London **A167**, 148 (1938).
[4] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[5] E. J. Moniz and D. H. Sharp, Phys. Rev. D **10**, 1133, (1974); *ibid.* **15**, 2850, (1977).
[6] L. I. Schiff, *Quantum Mechanics*, (McGraw-Hill, New York, 1968).