Precise phase measurement is an important research topic in physics and has played a vital role in the famous Michelson-Morley experiment [1] and in the first gravitational wave detection [2]. The topic is related to a fundamental question that is central to quantum metrology: How precisely can the phase be measured? Our understanding of quantum mechanics suggests that it is not possible to measure the phase (\(\phi\)) with infinite precision. When optical interferometry is used for phase measurement, the phase fluctuation or uncertainty (\(\Delta\phi\)) is affected by the photon number fluctuation (\(\Delta n\)). A seminal work of Dirac on the emission and absorption of radiation suggested that the phase uncertainty relation \(\Delta\phi\Delta n \geq 1\) [3,4]. Although the rigor of this relation has been debated (see, for example, [5–7]), there exists a common agreement that the precision of phase measurement depends on the average number (\(n\)) of photons or particles used in the measurement process. Sometimes \(n\) is also understood as the number of interactions between the probe and the phase to be measured, i.e., the number of times the phase is sampled [8]. When the precision of phase measurement is limited by \(\Delta\phi \geq n^{-1/2}\), the corresponding limit is called the standard quantum limit or shot-noise limit. Phase measurements performed with all classical interferometers are bound by the shot-noise limit. However, if quantum effects are used, more precise measurements can be performed. According to the current knowledge (see, for example, [9–11]), the minimum value of the phase uncertainty (\(\Delta\phi\)) is bounded by \(1/n\) from below, i.e., \(\Delta\phi \geq n^{-1}\). This limit is often called the Heisenberg limit. The goal of super-sensitive phase measurement is to achieve the Heisenberg limit or at least to beat the shot-noise limit.

A well-known method of enhancing phase sensitivity beyond the shot-noise limit involves the use of squeezed state of light [12,13]. This method has recently been applied to enhance sensitivity of the LIGO gravitational wave detector [14]. There is another class of methods that uses quantum entanglement involving two or more particles as a key resource to enhance phase sensitivity [15–19]. Entangled states involving two or more particles can also be used to achieve the phase super-resolution [19–23], which refers to a faster oscillation of the output signal of an interferometer with the change in phase. In such a case, it is a two- or many-particle interference pattern that exhibits phase-super resolution.

We introduce a fundamentally different method of super-sensitive and super-resolving phase measurement: we do not use any squeezed or many-particle entangled state to attain the Heisenberg limit; we instead use a many-particle quantum state in which no entanglement exists between two or more particles. The presence of many-particle entanglement is disadvantageous for our scheme because it enhances the phase uncertainty. In fact, when the loss of probing particles is maximum, the quantum state becomes a GreenbergerHorneZeilinger (GHZ) state, and the phase measurement becomes impossible due to very high phase uncertainty. In striking contrast to the super-sensitive phase measurement techniques that use entanglement involving two or more particles as a key resource, our method shows that having many-particle entanglement can be counterproductive in quantum metrology.
The corresponding individual quantum state is given by
\[ p \]  
\[ p \text{ever, for fermions, these modes must be distinct. Paths } p_1, \ldots, p_n \text{ traverses a phase shifter that introduces phase } \phi \text{ into each mode. These modes are then sent through } Q' \text{ and made identical with modes } p_1', \ldots, p_n'. \text{ Particles } 0, 1, \ldots, n \text{ are never detected. } Q \text{ and } Q' \text{ emit in such a way that the resulting quantum state is a superposition of the states that are generated by them individually. Modes } p_0 \text{ and } p_0' \text{ are superposed by a beamsplitter, BS, and phase, } \phi, \text{ is determined by performing measurements at output ports } C \text{ or } D. 

is represented by
\[ |X\rangle = |p_0\rangle |p_1\rangle \ldots |p_n\rangle = \prod_{j=0}^{n} \hat{a}^\dagger (p_j) |\text{vac}\rangle, \quad (1) \]
where \( \hat{a}^\dagger (p_j) \) is the creation operator corresponding to particle \( j \) in path (mode) \( p_j \) and |\text{vac}\rangle represents the vacuum state. The annihilation operator \( \hat{a}_{p_j} \) can be bosonic or fermionic. It obeys the commutation relation \( \hat{a}(p_j)\hat{a}^\dagger (p_k) + \hat{a}^\dagger (p_k)\hat{a}(p_j) = \delta_{jk}, \) where the minus and plus signs are for bosons and fermions, respectively and \( \delta_{jk} \) represents the Kronecker delta. For fermions, the following formula holds in addition: \( \{\hat{a}(p_j)\}^s = \{\hat{a}^\dagger (p_j)\}^s = 0 \) for \( s > 1, \) where \( s \) is a positive integer. Likewise, \( Q' \) can emit the particles in paths (modes) \( p_0', p_1', \ldots, p_n' \) and the corresponding individual quantum state is given by
\[ |X'\rangle = |p_0'\rangle |p_1'\rangle \ldots |p_n'\rangle = \prod_{j=0}^{n} \hat{a}^\dagger (p_j') |\text{vac}\rangle. \quad (2) \]

The phase (\( \phi \)) that we wish to measure is introduced into each of the paths (modes) \( p_1, \ldots, p_n \) by a phase shifter \( Q \) that is placed between \( Q \) and \( Q' \) (Fig. 1). For bosons, these modes do not need to be distinct; however, for fermions, these modes must be distinct. Paths (modes) \( p_1, \ldots, p_n \) are then sent through \( Q' \) and are made identical with paths (modes) \( p_1', \ldots, p_n', \) respectively. \( \gamma \) is the phase difference between paths \( p_0 \) and \( p_0' \), we have chosen the beamsplitter to be lossless and symmetric for simplicity. It follows from Eqs. (1) and (3) that the single-particle detection probability at outputs \( C \) and \( D \) are given by
\[ \langle \hat{n}_C \rangle = \langle \psi | \hat{E}^{(\dagger)}_C \hat{E}_C^{(+)} | \psi \rangle = \frac{1}{2} [1 - \cos (n\phi - \zeta_0)], \quad (7a) \]
\[ \langle \hat{n}_D \rangle = \langle \psi | \hat{E}^{(\dagger)}_D \hat{E}_D^{(+)} | \psi \rangle = \frac{1}{2} [1 + \cos (n\phi - \zeta_0)], \quad (7b) \]
where \( \hat{E}_C^{(+)} = (\hat{E}_C^{(-)})^\dagger, \hat{E}_D^{(+)} = (\hat{E}_D^{(-)})^\dagger, \) and \( \zeta_0 = \xi + \gamma - \pi/2. \) Clearly, single-particle interference patterns with unit visibility are generated at outputs \( C \) and \( D \) when the phase is varied. Making the modes \( p_1, \ldots, p_n \) identical with the modes \( p_1', \ldots, p_n' \), ensures that there is no which-way information and thus the interference occurs.

The presence of \( n\phi \) inside the cosines [Eq. (2)] shows that the detection probabilities oscillate between their maximum and minimum values more frequently for a larger value of \( n \). In Fig. 2(a), we plot the single-particle
Super-sensitivity: phase uncertainty $\Delta \phi$ is plotted against the number of probing particles $n$ (filled circles). The Heisenberg limit (HL) of $1/n$ (solid line) and shot-noise limit (SNL) of $1/\sqrt{n}$ (dashed line) are shown for comparison.

Detection probability at output $C$ against the phase $\phi$ for $n = 2$ and $n = 5$. It is evident that our phase measurement scheme exhibits phase super-resolution in single-particle interference without using two- or many-particle entanglement. This is a striking result because the phase super-resolution is commonly observed in many-particle interference experiments performed with entangled states.

Using Eqs. (5) and (9), we readily find that the phase uncertainty — for both the bosonic and the fermionic cases — is given by

$$\Delta \phi = \frac{1}{n}, \quad (10)$$

for all values of $\phi$. Equation (10) confirms that our scheme allows one to achieve the Heisenberg-limit of phase super-sensitivity.

In Fig. 2(b), we plot $\Delta \phi$ against $n$ (black dots) to illustrate that the phase uncertainty obtained in our scheme attains the Heisenberg limit (solid line). In the same figure, we also show the shot-noise limit (dashed line) for comparison.

We have thus far assumed that there is no loss in the system. We now consider the loss of the probing particles and show why the presence of entanglement between two or more-particles is counterproductive for our phase measurement scheme. In our system, the loss can be effectively represented by the action of an attenuator placed in the paths (modes) of the probing particles between the two sources $Q$ and $Q'$. The mathematical representation of such an attenuator is equivalent to that of a beamsplitter with one unused input port.

Let us denote the amplitude transmission coefficient of the attenuator by $T_j$ for mode $p_j$ ($j = 1, 2, \ldots, n$), where $0 \leq T_j \leq 1$ can be assumed to be real without any loss of generality. Using the quantum mechanical treatment of a beamsplitter, we now replace Eq. (3) by

$$\hat{a}^\dagger (p_i) = e^{-i\phi} \left[ T_i \hat{a}^\dagger (p_i) + \sqrt{1 - T_i^2} \hat{a}^\dagger_0 (p_i) \right], \quad (11)$$

where $l = 1, 2, \ldots, n$ and $\hat{a}^\dagger_0 (p_i)$ corresponds to the vacuum field at the unused port of the beamsplitter that represents the attenuator. Note that $1 - T_i^2$ is the probability of particle $l$ getting lost and $\hat{a}^\dagger_0 (p_i) |\text{vac}\rangle$ can be interpreted as the state representing a lost particle.

Combining Eqs. (4), (6), (7), and (8), we find that the quantum state generated by a lossy system can be expressed as

$$|\psi_{\text{loss}}\rangle = \frac{1}{\sqrt{2^n}} \prod_{l=1}^{n} |p_l\rangle + \frac{1}{\sqrt{2^n}} e^{i(\xi - n\phi)} \prod_{l=1}^{n} \left( T_i \hat{a}^\dagger (p_i) + R_i \hat{a}^\dagger_0 (p_i) \right) |\text{vac}\rangle, \quad (12)$$

where $R_l = \sqrt{1 - T_l^2}$ and $\langle \text{vac} | \hat{a}_0 (p_i) \hat{a}^\dagger_0 (p_i) |\text{vac}\rangle = 1$. Depending on the value of $n$, the state $|\psi_{\text{loss}}\rangle$ is a two- or
many-particle entangled state. It now follows from Eqs. (10), (8), and (12) that for both bosons and fermions [37],

\[ \langle \hat{n}_{\text{diff}} \rangle = \cos (n \phi - \zeta_0) \prod_{l=1}^{n} T_l, \quad (13a) \]

\[ (\Delta n_{\text{diff}})^2 = 1 - \cos^2 (n \phi - \zeta_0) \left( \prod_{l=1}^{n} T_l \right)^2, \quad (13b) \]

where \( \zeta_0 \) is defined below Eq. (7). It is evident from Eq. (13a) that the visibility of the single-particle interference pattern is now given by

\[ V = n \prod_{l=1}^{n} T_l, \quad (14) \]

where \( 0 \leq T_l \leq 1 \). Clearly, the loss of probing particles can be quantified by the visibility.

From Eqs. (14) and (15), we find that the minimum phase uncertainty for a lossy system is

\[ \Delta \phi_{\text{loss}} |_{\min} = \frac{1}{n} \prod_{l=1}^{n} T_l = \frac{1}{nV}. \quad (16) \]

Therefore, when \( V > 1/\sqrt{n} \) and \( n > 1 \), a lossy system implementing our scheme will exhibit better phase sensitivity than the shot-noise limit exhibited by an ideal (lossless) classical system. Let us also note that the phase is usually defined modulo \( 2\pi \) and thus the maximum value of \( \Delta \phi \) must be bounded by \( 2\pi \) from above. It means that when the visibility is less than \( (2n\pi)^{-1} \), no realistic phase measurement is possible. Figure 3 illustrates these results. The Heisenberg limit is achieved when the visibility is unity (i.e., no loss) as suggested by Eqs. (10) and (15).

For achieving the Heisenberg limit, it is absolutely essential that there is no entanglement between two or more particles [Eq. (4)]. The loss of probing particles introduces distinguishability and consequently there is many-particle entanglement [Eq. (12)]. In such a case, the visibility of the interference pattern reduces [Eq. (14)] and the phase uncertainty \( (\Delta \phi) \) increases [Eq. (16)]. In the extreme case of modes \( p_1, \ldots, p_n \) being fully distinguishable from modes \( p_1', \ldots, p_n' \) (i.e., 100% loss), the quantum state becomes a GHZ state (maximally entangled) [35]. Consequently, the visibility becomes zero and the phase measurement can no longer be performed (Fig. 3). Clearly, many-particle entanglement reduces the phase-sensitivity of our measurement scheme: the presence of entanglement is counterproductive. This fact marks a striking difference between our scheme and the super-sensitive phase measurement schemes that use entanglement involving two or more particles (e.g., NOON state) as a key resource.

We conclude by stating that we have introduced a unique method in quantum metrology. A distinct feature of our method is that the probing particles are not detected. Our scheme also does not involve any coincidence measurement or many-particle interference. The detection of single particles is only required and the phase super-resolution is observed in a single-particle interference pattern. Importantly, our scheme allows one to achieve the Heisenberg limit without using two- or many-particle entanglement as a resource. In fact, we have shown that the presence of many-particle entanglement is disadvantageous because it reduces the sensitivity of measurement [38]. The Heisenberg limit is attained only when the probing particles are in a separable state.

An intriguing question that arises in the context of our results concerns the role played by entanglement in important areas of quantum technology. In quantum computing, it is debated whether many-particle entanglement is a key resource for the computational speed-up [29,31]. In quantum communication, measurement-device-independent quantum key distribution can be performed without using entangled photon pairs [32]. Now, our results show that it is possible to achieve the Heisen-
berg limit in quantum metrology without using a many-particle entangled state; a path is thus opened up for more questions about entanglement from a resource theoretical perspective. We, therefore, hope that our results will stimulate further discussion on this topic and help us develop a deeper understanding of quantum physics as a whole.

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[1] A. A. Michelson and E. W. Morley, American Journal of Science 34, 333 (1887).
[2] B. P. Abbott, R. Abbott, T. D. Abbott, M. R. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. X. Adhikari, et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 061102 (2016), URL https://link.aps.org/doi/10.1103/PhysRevLett.116.061102.
[3] P. A. M. Dirac, Proc. R. Soc. London A 114, 243 (1927).
[4] W. Heitler, The Quantum Theory of Radiation (Dover Publications, 1984), 3rd ed.
[5] L. Susskind and J. Glogower, Physics 116, 061102 (2016), URL https://link.aps.org/doi/10.1103/PhysRevLett.116.061102.
[6] S. Barnett and D. Pegg, Journal of Modern Optics 36, 7 (1989).
[7] L. Mandel and E. Wolf, Optical coherence and quantum optics (Cambridge university press, 1995).
[8] V. Giovannetti, S. Lloyd, and L. Maccone, Nature Photonics 5, 222 (2011).
[9] A. S. Lane, S. L. Braunstein, and C. M. Caves, Phys. Rev. A 47, 1667 (1993), URL https://link.aps.org/doi/10.1103/PhysRevA.47.1667.
[10] Z. Y. Ou, Phys. Rev. Lett. 77, 2352 (1996), URL https://link.aps.org/doi/10.1103/PhysRevLett.77.2352.
[11] Z. Y. Ou, Phys. Rev. A 55, 2598 (1997), URL https://link.aps.org/doi/10.1103/PhysRevA.55.2598.
[12] C. M. Caves, Phys. Rev. D 23, 1693 (1981), URL https://link.aps.org/doi/10.1103/PhysRevD.23.1693.
[13] M. Xiao, L.-A. Wu, and H. J. Kimble, Phys. Rev. Lett. 59, 278 (1987), URL https://link.aps.org/doi/10.1103/PhysRevLett.59.278.
[14] M. Tse, H. Yu, N. Kijbunchoo, A. Fernandez-Galiana, P. Dupej, L. Barsotti, C. D. Blair, D. D. Brown, S. E. Dwyer, A. Effler, et al., Phys. Rev. Lett. 123, 231107 (2019), URL https://link.aps.org/doi/10.1103/PhysRevLett.123.231107.
[15] B. Yurke, S. L. McCall, and J. R. Klauder, Phys. Rev. A 33, 4033 (1986), URL https://link.aps.org/doi/10.1103/PhysRevA.33.4033.
[16] B. Yurke, Phys. Rev. Lett. 56, 1515 (1986), URL https://link.aps.org/doi/10.1103/PhysRevLett.56.1515.
[17] J. P. Dowling, Phys. Rev. A 57, 4736 (1998), URL https://link.aps.org/doi/10.1103/PhysRevA.57.4736.
[18] H. Lee, P. Kok, and J. P. Dowling, Journal of Modern Optics 49, 2295 (2002).
[19] A. Kuzmich and L. Mandel, Quantum and Semiclassical Optics: Journal of the European Optical Society Part B 10, 493 (1998).
[20] E. J. S. Fonseca, C. H. Monken, and S. Pádua, Phys. Rev. Lett. 82, 2868 (1999), URL https://link.aps.org/doi/10.1103/PhysRevLett.82.2868.
[21] M. D’Angelo, M. V. Chekhova, and Y. Shih, Phys. Rev. Lett. 87, 013602 (2001), URL https://link.aps.org/doi/10.1103/PhysRevLett.87.013602.
[22] P. Walther, J.-W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, and A. Zeilinger, Nature 429, 158 (2004).
[23] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg, Nature 429, 161 (2004).
[24] M. Lahiri, Phys. Rev. A 98, 033822 (2018), URL https://link.aps.org/doi/10.1103/PhysRevA.98.033822.
[25] M. Lahiri, A. Hochrainer, M. Lahiri, and A. Zeilinger, Physical review letters 116, 080401 (2017).
[26] X. Zou, L. J. Wang, and L. Mandel, Physical review letters 67, 318 (1991).
[27] C. M. Caves, Phys. Rev. D 23, 1693 (1981), URL https://link.aps.org/doi/10.1103/PhysRevD.23.1693.
and \( \hat{n}_{p_0'} = \hat{a}^\dagger(p_0') \hat{a}(p_0') \). However, \( \langle \hat{n}_{n_{\text{diff}}}^2 \rangle = 1 \) for both cases and thus the equations hold for both bosons and fermions.

[37] Once again, \( \langle \hat{n}_{n_{\text{diff}}}^2 \rangle = 1 \) for both bosons and fermions; see [36] in this context.

[38] We ignore the trivial case of \( n = 1 \) because it is not possible to achieve either super-resolution or super-sensitivity in this case.