Depletion of the Dark Soliton: 
the Anomalous Mode of the Bogoliubov theory

Jacek Dziarmaga\textsuperscript{1,2}, and Krzysztof Sacha\textsuperscript{1}

\textsuperscript{1} Intytut Fizyki Uniwersytetu Jagiellońskiego, Reymonta 4, 30-059 Kraków, Poland
\textsuperscript{2} Los Alamos National Laboratory, Theory Division T-6, Los Alamos, New Mexico 87545, USA

I. INTRODUCTION

Since first experimental realizations of atomic Bose-Einstein condensates (BEC) \cite{1} the condensates have been subject to intensive experimental and theoretical research. The condensates are relatively easy to manipulate and a rich variety of phenomena can be studied in a well controlled way. One example is a dark soliton in analogy to nonlinear optics \cite{2-4}. The dark soliton is a place in a quasi one dimensional condensate where a condensate wave function \( \phi(x) \) changes its sign. Its wave function has a notch where density of the condensate is zero. This notch appears as a dark spot on a bright cigar shaped image of a condensate. Dark solitons were realized in the experiment of the Hannover group \cite{5} and their 3 dimensional analogs were realized in Ref. \cite{6}.

A wave function of the dark soliton is proportional to

\[
\tanh[(x - q)/l], \tag{1}
\]

where \( l \) is a local value of the healing length at the position \( q \) of the soliton. The density of coherent atoms in the notch at \( x = q \) is zero. This creates a hole that might be filled up with incoherent atoms. Given that interactions between atoms are repulsive, depletion of atoms from the condensate to the notch is energetically favourable.

Condensate with a soliton at \( q = 0 \) has an antisymmetric wavefunction of \( x \). This means that all \( N \) atoms are in the same antisymmetric single particle state. The antisymmetry of the condensate wavefunction is preserved by evolution according to the Gross-Pitaevskii equation. However, the full \( N \)-body Hamiltonian depletes pairs of atoms from the antisymmetric condensate wavefunction to symmetric modes of the trap. This two atom process does not change the symmetry of the \( N \)-body state. The symmetric modes have nonzero density in the soliton notch at \( x = 0 \). Simply because the condensate density is zero at \( x = 0 \), the \( x = 0 \) is where we can clearly see the incoherent atoms. In our previous paper \cite{7} we studied a toy model of the dark soliton. We truncated single particle Hilbert space to 3 lowest modes of a one dimensional harmonic trap. This is a self-consistent approximation for small \( N \). We have shown that a state where all atoms are condensed in the same antisymmetric wave function is not a stationary state of a quantum Hamiltonian. Atoms are gradually depleted from the antisymmetric condensate wave function and fill its notch. In this paper we reach similar conclusions in the opposite regime of very large \( N \) which is relevant for the Hannover experiment \cite{5}. We use as a tool the Bogoliubov theory with noninteracting quasiparticles.

In the Bogoliubov theory one considers small fluctuations around a stationary condensate background. In a uniform condensate the soliton (1) has a translational mode \( \partial_q \tanh[(x - q)/l] \) with zero frequency. The same soliton in a nonuniform condensate in a harmonic trap has an anomalous mode with negative frequency equal to \(-1/\sqrt{2}\) times the trap frequency \( \omega_x \) \cite{4}. The nonuniformity breaks the translational invariance, moving the soliton away from the maximal density in the center of the trap lowers its energy. We will see that the anomalous mode is responsible for depletion of atoms from a condensate to the soliton notch. Depletion in the Bogoliubov vacuum has been recently studied in Ref. \cite{8}. However, we argue that depletion in the Bogoliubov vacuum analyzed in Ref. \cite{8} is not experimentally relevant. We construct a quantum state with minimal depletion which approximates the state of the system right after creation of the dark soliton by phase imprinting \cite{5}. Time evolution of such a state is responsible for greying of the dark soliton.

In Ref. \cite{3} it is suggested that deviations from the mean field description of the dark soliton observed experimentally \cite{5} can be explained by dissipation due to collisions of the condensate with a thermal cloud of noncondensed atoms. There is no doubt that dissipation influences dynamics of the soliton in that experiment. However, we show that the dark soliton is greying even in the absence of any thermal cloud or even in 1D where the cloud de-
couples from the soliton [3]. The notch fills up with incoherent atoms quantum depleted from a condensate. Our calculations, which are idealized with respect to the experiment, give a greying time of $O(10\text{ms})$, which is consistent with the experimental observation.

This paper is organized as follows. Section II gives several definitions and approximate formulas valid in the Thomas-Fermi regime of strong interactions. Section III gives Thomas-Fermi approximate analytic expressions for the frequency and the eigenfunctions of the anomalous mode. In Section IV we work out a general expression for a time-dependent number of atoms depleted in the anomalous mode. In Section V the number of depleted atoms is calculated in the Bogoliubov vacuum of the anomalous mode. In Section VI we construct a quantum state with minimal depletion which approximates the state of the system right after creation of the dark soliton. In Section VII we follow time evolution of the state of the system after creation of the dark soliton background are

\begin{align}
\frac{1}{2}u'' + \frac{1}{2}x^2u + 2g\phi_1^2u = -\omega_1u,
\end{align}

\begin{align}
\frac{1}{2}v'' + \frac{1}{2}x^2v + 2g\phi_1^2v = -\omega_2v.
\end{align}

Here $\omega_1 = \frac{1}{2}x^2 + 2g\phi_1^2$ and $\omega_2 = -\omega_1$. For $g \gg 1$ the width of the soliton is very small as compared to the size of the condensate, $l_0 \ll R$.

\section{III. The Anomalous Mode}

Bogoliubov equations [9,10] for small perturbations on the dark soliton background are

\begin{align}
\frac{1}{2}f_+'' + \frac{1}{2}x^2f_+ + 3g\phi_1^2f_+ = -\omega f_+,
\end{align}

\begin{align}
\frac{1}{2}f_-'' + \frac{1}{2}x^2f_- + g\phi_1^2f_- = -\omega f_-.
\end{align}

Eqs.(8,9) have a solution with negative $\omega$ [3,4], the so called anomalous mode. In the Thomas-Fermi regime $\omega$ can be approximated by [3,4]

\begin{align}
\omega \approx -\frac{1}{\sqrt{2}}.
\end{align}

i.e. $-1/\sqrt{2}$ times the frequency of the harmonic trap. This Thomas-Fermi value of $\omega$ follows from effective equations of motion for a position of a dark soliton in Ref. [4], and from a perturbative $(g \gg 1)$ treatment of the Bogoliubov equations [3].

The anomalous mode is a perturbative approximation to the motion of the dark soliton with respect to the background condensate. In the limit of $g \to \infty$ the Thomas-Fermi radius of the condensate (4) is growing to infinity and at the same time the healing length (6) is shrinking to 0. In this limit the condensate is uniform on the lengthscale of the soliton width and the anomalous mode of Eqs.(8,9) becomes

\begin{align}
\lim_{g \to \infty} f_+(x) \propto \frac{1}{\cosh^2(x/l_0)},
\end{align}

\begin{align}
\lim_{g \to \infty} f_-(x) = 0.
\end{align}

This $f_+(x)$ is simply a translational mode of the soliton in an almost uniform condensate. This limit gives us the leading $g \gg 1$ term in the expression for $f_+(x)$.

We use Eq.(9) to find the leading term in $f_-(x)$. To begin with we note that $\phi_0(x)$ satisfies
Comparing Eq. (12) with Eq. (9) one may conclude that $f_-(x) \equiv \alpha \phi_0(x)$ with a certain constant $\alpha$. Substitution of this ansatz and of leading $f_+(x)$ from Eqs. (11) into Eq. (9) and then using Eq. (12) gives an equation

$$-(\mu_1 - \mu_0) \alpha \phi_0(x) + g [\phi_1^2(x) - \phi_0^2(x)] \alpha \phi_0(x) = \frac{\omega}{\cosh^2 (x/l_0)}.$$  

We expect that with a right choice of $\alpha$ this equation is approximately satisfied for $g \gg 1$. As $\mu_1 - \mu_0 = \mathcal{O}(g^0)$ the first term on the left hand side is small as compared to the second term and can be neglected. The second term is localized on a healing length around $x = 0$ and the slow factor $\phi_0(x)$ in this term can be replaced by $\phi_0(0)$. Finally we can use the $g \gg 1$ formulas in Eqs. (3, 5) to get

$$g [\tanh^2 (x/l_0) - 1] \alpha \phi_0^2(0) = \frac{\omega}{\cosh^2 (x/l_0)}.$$  

This equation is satisfied by

$$\alpha = \frac{-\omega}{g \phi_0^2(0)} = \frac{4}{3 \sqrt{g}},$$  

which defines the leading term in $f_-(x)$. After normalization, so that $\int dx (|u|^2 - |v|^2) = \int dx f_+ f_- = 1$, we get

$$f_+(x) \quad g \gg 1 \quad \frac{\sqrt{3g}}{2 \sqrt{2} \cosh^2 (x/l_0)},$$

$$f_-(x) \quad g \gg 1 \quad \sqrt{\frac{2}{3}} \phi_0(x).$$  

We note that $f_+(x)$ is localized within the notch of the dark soliton (5).

**IV. NUMBER OF DEPLETED ATOMS**

We will argue that depletion of atoms from the condensate fills up the notch in the condensate wave function (5) with incoherent atoms. This effect will appear experimentally as greying of the dark soliton. Solution of the equations (7) reveals many different modes corresponding to the Bogoliubov spectrum. However, according to the numerical results in Ref. [8], only the anomalous mode has a wave function localized in the soliton notch [8], compare Eq. (15). From all the modes the anomalous mode will contribute the most to the density of incoherent atoms filling the notch. This expectation is confirmed by numerical calculations of incoherent atom density distribution in the Bogoliubov vacuum state [8], where the density, which is strongly peaked in the notch, is dominated by a contribution from the anomalous mode.

In the following we truncate the Bogoliubov spectrum to the anomalous mode alone.

Having constructed $f_\pm = u \pm v$ we can calculate an operator corresponding to a number of depleted atoms in the anomalous mode

$$d \hat{N}(t) = \int dx \delta \hat{\phi}^\dagger(x, t) \delta \hat{\phi}(x, t),$$

where

$$\delta \hat{\phi}(x, t) = u(x) e^{-i\omega t} \hat{a} + v^*(x) e^{+i\omega t} \hat{a}^\dagger$$

is an annihilation operator of an incoherent atom in the anomalous mode. The operators $\hat{a}, \hat{a}^\dagger$ annihilate/create a Bogoliubov quasiparticle and they fulfill the usual bosonic commutation relation, $[\hat{a}, \hat{a}^\dagger] = 1$. We use Heisenberg picture defined by the Bogoliubov Hamiltonian of the anomalous mode

$$H_B = -|\omega| \hat{a}^\dagger \hat{a}.$$  

The operator of the number of depleted atoms is

$$d \hat{N}(t) = \int dx \left[ |u|^2 \hat{a}^\dagger \hat{a} + |v|^2 \hat{a}^\dagger \hat{a} \right] e^{-2i\omega t} \hat{a}^\dagger \hat{a}^\dagger + |u^*|^2 e^{+2i\omega t} \hat{a} \hat{a} \right].$$

The density of incoherent atoms is the expectation value of the integrand.

**V. DEPLETION IN THE BOGOLIUBOV VACUUM STATE**

For a condensate in a ground state like in Eq. (3) the Bogoliubov spectrum has only positive frequencies (i.e. there are no anomalous modes). The state with no quasi-particles, the Bogoliubov vacuum, is the ground state of the Bogoliubov Hamiltonian. In the absence of any anomalous modes it is natural to calculate the depletion in the Bogoliubov vacuum – the $T = 0$ state of thermodynamic equilibrium [10].

Following the numerical calculation in Ref. [8] we calculate analytically for $g \gg 1$ a number of depleted atoms in the vacuum state of the anomalous mode i.e. the state $|0\rangle_a$ annihilated by $\hat{a}$, $\hat{a}|0\rangle_a = 0$. This vacuum state is an eigenstate of the Bogoliubov Hamiltonian (19) so the number is stationary

$$dN_a = a(0) d\hat{N}(t)|0\rangle_a = \int dx |v(x)|^2,$$

and scales with $g$ as $dN_a \propto g^{2/3}$. We compare the density of depleted atoms in the soliton notch

$$\frac{dN_a}{l_0} \propto g \propto N,$$  

where $N$ is the number of atoms in the condensate. This shows that depletion in the soliton notch is weaker than in the vacuum state.
(where \( N \) is the total number of atoms in the system) to the density of atoms in the condensate
\[
\frac{N}{R} \propto \frac{N}{g^{1/3}} \propto N^{2/3}.
\] (23)

We see that, for a sufficiently large total number of atoms \( N \) the density of incoherent atoms in the notch will exceed the density of the background condensate. The notch will not be visible. When the density of depleted atoms is comparable to the density of the condensate one cannot neglect interactions between quasiparticles. However, because of the potential instability which shows up in the negative frequency of the anomalous mode, we qualitatively expect that, unlike in more common situations where all Bogolubov frequencies are positive, collisions between quasiparticles will not suppress depletion.

In Fig.1 we show condensate and total density plots in the Bogolubov vacuum for the parameters of the Hanover experiment [5] (\( N = 1.5 \times 10^5 \) \(^{87}\)Rb atoms in the harmonic trap with \( \omega_x = 2\pi \times 14 \) s\(^{-1}\) and \( \omega_y = 2\pi \times 425 \) s\(^{-1}\); \( g = 7500 \) [11]). As we can see, even for very reasonable trap parameters and moderate \( N \) the soliton notch is substantially filled with incoherent atoms. For a larger \( N \), the soliton would not be observed at all.

![Graphs showing single particle density function](image)

**FIG. 1.** Single particle density function corresponding to the experimental parameters [5]. The coherent part \(|\phi_1(x)|^2\) [Eq. (5)] is depicted in panel (a), the incoherent part \(|v(x)|^2\) corresponding to the vacuum of the anomalous mode \(|0\rangle_a\) is shown in panel (b). Panel (c) shows the total (i.e. the sum of the coherent and incoherent parts) density function and panel (d) is a zoom on the \( x \approx 0 \) part of panel (c) — note that the density of incoherent atoms at the soliton notch is about 14\% of the maximal density of the surrounding condensate.

The depletion in the Bogolubov vacuum of the anomalous mode is not relevant for dark solitons in current experiments

- The vacuum \(|0\rangle_a\) is not a ground state of the Bogoliubov Hamiltonian (19). Creation of a quasiparticle in this vacuum with the creation operator \( a\dagger \) actually *decreases* the energy. The Bogoliubov vacuum is *not* the state of thermodynamic equilibrium at \( T = 0 \).

- For large enough \( N \) the density of incoherent atoms in the state \(|0\rangle_a\) exceeds the density of the condensate. The linearized Bogoliubov theory breaks down and the calculation of depletion in the state \(|0\rangle_a\) is not self-consistent.

- The Bogoliubov vacuum \(|0\rangle_a\) is *not* the state with minimal possible depletion. When \( v(x) \neq 0 \) quasiparticles cannot be identified with incoherent atoms, compare Eq.(18), and the state with no quasiparticles \(|0\rangle_a\) is not the state with minimal number of incoherent atoms. It is possible to construct a state with a very small depletion as compared to the depletion in the state \(|0\rangle_a\), see the next Section.

**VI. THE MINIMAL DEPLETION STATE**

Solitons are excited experimentally employing a phase imprinting method [5]. We assume that before the phase imprinting the system of \( N \) atoms is in its ground state. The ground state is a condensate (3) with all Bogoliubov modes in a vacuum state. There are no anomalous modes and the density of incoherent atoms is very small as compared to the density of the condensate. The ground state is very close to a perfect condensate with *all* atoms in the condensate wave function (3). We approximate the initial condensate by a perfect condensate without any depletion.

For a perfect initial condensate the phase imprinting can be thought of as a nearly instantaneous operation on the condensate wave function. Right after this operation one gets a condensate with an antisymmetric wave function \( \phi(x) = -\phi(-x) \). To focus on the quantum depletion and not on classical evolution of \( \phi(x) \) we assume that \( \phi(x) = \phi_1(x) \). The initial dark soliton is a (nearly) perfect condensate with the wave function (5). In the Hanover experiment [5] they create moving grey solitons while here, for the sake of simplicity, we assume a stationary dark soliton. We think that in spite of this, qualitative comparison with the experiment is still possible.

The Bogoliubov vacuum \(|0\rangle_a\) is an eigenstate of the \( N \)-body Hamiltonian. For large \( N \) the \(|0\rangle_a\) has substantial depletion. The initial state without depletion is far from \(|0\rangle_a\) or from any other eigenstate. The initial state is not stationary, its depletion may grow from zero to values which far exceed the depletion in the Bogoliubov vacuum. In the following we quantify this effect.

In the framework of the Bogoliubov theory truncated to the anomalous mode (19), the initial depletion-free soliton can be very well approximated by a variational
state $|\psi\rangle$ that minimizes the initial number of atoms depleted from the background condensate $\phi(x)$

$$dN(0) = \langle \psi | d\hat{N}(0) | \psi \rangle .$$ (24)

To find this optimal state it is convenient to introduce bosonic operators $\hat{b}$ and $\hat{b}^\dagger$,

$$\hat{b} = \frac{1}{2} \sqrt{\Omega / \omega} (\hat{a} + \hat{a}^\dagger) + \frac{1}{2} \sqrt{\omega / \Omega} (\hat{a} - \hat{a}^\dagger) ,$$ (25)

where

$$\Omega = 2^{-\frac{13}{2}} (3g)^{1/3} .$$ (26)

In terms of these new bosonic operators the initial number of depleted atoms (20) becomes

$$d\hat{N}(0) = \frac{2\sqrt{\omega \Omega}}{3} \hat{b}^\dagger \hat{b} + \left( \frac{\sqrt{\omega \Omega}}{3} - \frac{1}{2} \right) .$$ (27)

This operator is a sum of a nonnegative operator $\sim \hat{b}^\dagger \hat{b}$ and a constant. $\hat{b}^\dagger \hat{b} \geq 0$ is minimized by the unique vacuum $|0\rangle_b$ of the bosonic annihilation operator $\hat{b}, \hat{b}^\dagger|0\rangle = 0$. The minimal depletion state is $|\psi\rangle = |0\rangle_b$. This vacuum is different from the Bogoliubov vacuum $|0\rangle_a$,

$$|0\rangle_b \neq |0\rangle_a .$$ (28)

In this optimal state $|0\rangle_b$ the initial number of depleted atoms in the soliton notch scales with the number of atoms as $N^{1/3}$. The initial density of incoherent atoms in the notch

$$dN_b(0)/l_0 \propto N^{2/3} ,$$ (29)

where $dN_b(0) = \langle 0|d\hat{N}(0)|0\rangle_b$ grows with $N$ in the same way as the condensate density in Eq. (23).

In Fig. 2 (a,b) we show that for a realistic value of the scattering length and other parameters the minimal density of incoherent atoms in the notch is around 1% of the peak density of the surrounding condensate. Our minimal depletion state is a very accurate approximation to the initial perfect condensate with a dark soliton. As both coherent and incoherent densities grow with $N$ in the same way, compare Eqs. (23,29), the quality of this approximation does not change with increasing $N$. The negligible depletion in the minimal depletion state justifies a posteriori our truncation of the theory to the anomalous mode.

Moreover, close to the initial minimal depletion state $|0\rangle_b$ we can safely use the noninteracting Bogoliubov theory. This theory assumes that the density of incoherent atoms is small as compared to the density of the condensate. In contrast, for large $N$ the Bogoliubov vacuum $|0\rangle_a$ does not satisfy this assumption.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Panel (a): the initial $(t = 0)$ single particle density function for the system initially in the minimal depletion state $|0\rangle_b$ (the values of the parameters correspond to the Hannover experiment [5]). Panel (b) is a zoom on the $x \approx 0$ part of panel (a) — the density of the incoherent atoms at the soliton notch is about 1% of the maximal condensate density. Panels (c)-(d): the same as in the previous panels but for $t_G = 0.42 (\pi/\sqrt{2\omega}) = 10.6$ ms.}
\end{figure}

VII. TIME EVOLUTION OF THE NUMBER OF INCOHERENT ATOMS

The initial density of incoherent atoms in the notch (calculated in the $|0\rangle_b$ state) is negligible, see Fig. 2. The initial dark soliton is very dark. This small depletion is not stationary because the initial minimal depletion state $|0\rangle_b$ is not an eigenstate of the Bogoliubov Hamiltonian (19). From Eqs. (20,25) one can easily deduce time evolution of the number of depleted atoms

$$dN_b(t) = \langle b|d\hat{N}(t)|0\rangle_b =$$

$$dN_b(0) + \frac{(\Omega^2 - \omega^2)^2}{3\sqrt{2}\omega^3\Omega} \sin^2(\omega t) .$$ (30)

The coefficient in front of the $\sin^2(\omega t)$ function scales with the total number of atoms as $N^1$ which implies that the maximal density of depleted atoms in the soliton notch

$$dN_b \left( t = \frac{\pi}{2\omega} \right)/l_0 \propto N^{4/3} .$$ (31)

Hence, for large enough $N$ the density of depleted atoms in the soliton notch will reach the density of the condensate (23) before it reaches the maximum (31) at $t = \pi/2|\omega|$. Eq. (30) might suggest that the system will go through a series of periodic revivals of the initial state $|0\rangle_b$ with a period of $\pi/|\omega|$. This is not a correct conclusion for large $N$. After a greying time

$$t_G < \frac{\pi}{2|\omega|} ,$$ (32)
when the density of incoherent atoms in the notch and the density of the condensate become comparable for the first time, we cannot use the noninteracting Bogoliubov theory any more [10]. After $t_G$, one can ignore neither anharmonicity in the Hamiltonian (19) of the anomalous mode nor interaction of the anomalous mode with other Bogoliubov modes (phonons). We expect that these nonlinearities lead to fast dephasing and prevent any revivals. The number of depleted atoms in the notch initially follows Eq. (30) until at around the greying time $t_G$, their density saturates at a density comparable to the density of the condensate. This will appear as greying of the dark soliton. The nonlinear stage deserves more quantitative analysis. However, we expect that, because of the global instability that shows in the negative frequency of the anomalous mode, the nonlinearities will not suppress depletion.

Eq. (32) gives a crude upper estimate of $t_G$, $t_G < \pi/2|\omega|$. Given Eq. (10) we obtain in real time units

$$t_G < \frac{\pi}{\sqrt{2}\omega_x}.\tag{33}$$

Here $\omega_x$ is an axial trap frequency. In the Hannover experiment of Ref. [5], $\pi/\sqrt{2}\omega_x = 25$ ms while the density of the incoherent atoms at the soliton notch becomes [according to Eq. (30)] comparable to the condensate density after $t_G \approx 10$ ms, see Fig. 2.

**VIII. DISCUSSION AND CONCLUSION**

It is worthwhile to compare solitons to vortices. There are toy model calculations in Ref. [12] for small $N$ that show some incoherent atoms in the vortex core. The mechanism is similar to solitons: 2 atoms in the state with angular momentum $l = 1$ are depleted to states $l = 0$ and $l = 2$. $l = 0$ gives nonzero density in the core. Depletion to an empty core/notch is not limited to solitons.

In contrast to solitons created by phase imprinting, ENS vortices are created by gradual evaporative cooling to a ground state of a rotating trap [13]. This ground state has no negative frequency anomalous mode [10]. This vortex ground state is stationary and has stationary depletion in the core.

In conclusion, we analyze greying of a dark soliton due to depletion in the anomalous mode of the Bogoliubov theory. We give approximate analytic expressions for $u(x)$ and $v(x)$ of the anomalous mode valid in the Thomas-Fermi regime. These expressions are used to work out formulas for a time-dependent number of atoms depleted to the notch. We argue that, contrary to the assumption in Ref. [8], the vacuum of the anomalous mode is far from an optimal approximation to a condensate with a dark soliton. There is a much better state which minimizes depletion from the condensate. We argue that this minimal depletion state can serve as a rough approximation to the state of the system right after creation of the soliton by the phase imprint method. The minimal depletion state is not stationary and the density of incoherent atoms in the notch grows from its negligible initial value until it fills up the notch. For realistic parameters like in the Hannover experiment [5] the notch can be filled up after just 10 ms.

**ACKNOWLEDGMENTS**

We thank James Anglin for discussions and Diego Dalvit for critical reading of the manuscript. J.D. was supported in part by NSA. K.S. acknowledges support of KBN under project 5 P03B 088 21.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995);
K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995); C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. 78, 985 (1997); for a review see *Bose-Einstein Condensation in Atomic Gases*, Proceedings of the International School of Physics “Enrico Fermi”, edited by M. Inguscio, S. Stringari, and C. Wieman (IOS Press, Amsterdam, 1999).

[2] J. K. Taylor, Ed., *Optical Solitons Theory & Experiment* (Cambridge Univ. Press, New York, 1992).

[3] A. E. Muryshev, H. B. van Linden van den Heuvell, and G. V. Shlyapnikov, Phys. Rev. A 60, R2665 (1999); P. O. Fedichev, A. E. Muryshev, G. V. Shlyapnikov, Phys. Rev. A 60, 3220 (1999); A. E. Muryshev, G. V. Shlyapnikov, W. Ertmer, K. Sengstock, and M. Lewenstein, arXiv:cond-mat/011506.

[4] Th. Busch and J. R. Anglin, Phys. Rev. Lett. 84, 2298 (2000).

[5] S. Burger K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).

[6] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Science 287, 97 (2000).

[7] J. Dziarmaga, Z. P. Karkuszewski, and K. Sacha, arXiv:cond-mat/0101008.

[8] C. K. Law, P. T. Leung, and M. C. Chu, arXiv:cond-mat/0110428.

[9] L.P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961); E.P. Gross, Nuovo Cimento 20, 454 (1961); F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

[10] Y. Castin and R. Dum, Phys. Rev. A 57, 3008 (1998);
Y. Castin, in *Les Houches Session LXXII, Coherent atomic matter waves 1999*, edited by R. Kaiser, C. Westbrook and F. David, (Springer-Verlag Berlin Heilderberg New York 2001).

[11] We estimate the one-dimensional $g$ coefficient by $4\pi N a_s L/S$ where $a_s$ is the $s$-wave scattering length, $L = \sqrt{\hbar/m\omega_x}$, and $S$ is an average transverse area calculated within the Thomas-Fermi approximation.

[12] M. A. H. Ahsan et al., Phys. Rev. A 64, 013608 (2001).

[13] K. W. Madison et al., Phys. Rev. Lett. 84, 806 (2000).