A Generalized Theory of Power

Louis L. Scharf, Life Fellow, IEEE and Dongliang Duan, Senior Member, IEEE

Abstract—The complex representation of real-valued instantaneous power may be written as the sum of two complex powers, one Hermitian and the other non-Hermitian, or complementary. A virtue of this representation is that it consists of a power triangle rotating around a fixed phasor, thus clarifying what should be meant by the power triangle. The in-phase and quadrature components of complementary power encode for active and non-active power. When instantaneous power is defined for a Thevenin equivalent circuit, these are time-varying real and reactive power components. These claims hold for sinusoidal voltage and current, and for non-sinusoidal voltage and current. Spectral representations of Hermitian, complementary, and instantaneous power show that, frequency-by-frequency, these powers behave exactly as they behave in the single frequency sinusoidal case. Simple hardware diagrams show how instantaneous active and non-active power may be extracted from metered voltage and current, even in certain non-sinusoidal cases.

I. INTRODUCTION

Definitions and interpretations of active and non-active power in non-sinusoidal systems have engaged the interest of many important and influential investigators [1]-[11]. Nonetheless, there remain issues to be resolved.

The first issue is the resolution of sinusoidal instantaneous power into its constituents, the second is the extension of this resolution to non-sinusoidal systems, and the third is the display of these constituents to the power system engineer who wishes to track power system performance in real time.

In this paper we begin with the standard resolution of real-valued instantaneous power into average, active, and non-active power[1] and offer an easily interpreted complex representation of this real-valued power. In a complex representation of real-valued instantaneous power, a complex power triangle, rotates around a fixed complex phasor. In a Thevenin equivalent circuit, the orthogonal components of this power triangle are in-phase and quadrature components that may be identified with power exchange between active components and resistive components. In any particular circuit this power triangle accounts for the rate at which reactive energy is exchanged between electric fields that sustain voltage, magnetic fields that sustain current, and heat, light, or mass movement that is produced in resistive components. This interpretation provides an alternative view of the power triangle.

We conclude this beginning section with a definition of positive power and negative power. These resolve instantaneous power into positive power delivered from the energy source to the energy destination, and the negative power returned to the source from the destination. During each period of sinusoidal oscillation, the real-valued instantaneous power is negative over short periods of time, during which the rate of exchange of energy from electric and magnetic fields to real elements exceeds the rate at which heat can be dissipated, lights illuminated, or shafts turned. This excess is returned to the source. The averages of each of the positive and negative power components show the role that the power angle plays in determining negative power.

Then arises the question, “how much of this analysis can be extended to non-sinusoidal systems?” In other words, can complex phasors, which are special cases of complex analytic signals, be extended to more general complex analytic signals? And if so, can real-valued instantaneous power still be represented by complex power, in which a slowly reshaped complex power triangle rotates around a slowly varying complex analytic signal? The answer is ‘yes’ in any non-sinusoidal system for which Bedrosian’s theorem applies. In the context of power systems, these are systems in which the non-sinusoidal signal consists of slowly-varying amplitude and phase modulation of a sinusoidal carrier.

Throughout the paper, the line of argument is this: begin with real-valued instantaneous power, give it a complex representation, and use this complex representation to develop intuition, and derive equations and diagrams for extracting the components of real-valued power. It is to be emphasized that power, as we define it, is always real-valued. Nonetheless, complex representations of this real-valued power bring intuition and economies of reasoning. In particular, complex power produces a geometric picture that is not evident in the resolution of this complex power onto the real axis of the Argand plane, where one finds real-valued power. Moreover, it is found that this geometric picture is most illuminating when complex power is resolved into a sum of Hermitian complex power and non-Hermitian complementary complex power. These points will be clarified in due course.

The rest of the paper is organized as follows. In Section II, we will discuss the instantaneous, average, in-phase and quadrature power in sinusoidal systems. Specifically, we will show the interpretation of all the power quantities in a Thevenin equivalent circuit. We will introduce the Hilbert transform and Bedrosian’s Theorem in Section III, which leads to our complex representation of instantaneous power for non-sinusoidal signals discussed in Section IV. Interpretations of our generalized power theory in a Thevenin equivalent circuit with time-varying impedances are presented in Section V. Then, a spectral theory of average power is introduced in Section VI with hardware diagrams for extracting components of instantaneous power given in Section VII. Concluding remarks are given in Section VIII.

Louis L. Scharf is with the Department of Mathematics and the Department of Statistics, Colorado State University, Fort Collins, CO 80521. Email: Louis.Scharf@colostate.edu.

Dongliang Duan is with the Department of Electrical and Computer Engineering, University of Wyoming, Laramie, WY 82071. Email: dduan@uwyo.edu.

1Throughout, we shall follow the terminology of IEEE Standard 1459, 2010.
II. Instantaneous, Average, In-Phase, and Quadrature Power in Sinusoidal Systems

Let us begin with a simple example to motivate our investigations. Define the real voltage \( v(t) = V \cos(\omega_0 t + \phi) \) and the real current \( i(t) = I \cos(\omega_0 t + \phi) \). The corresponding RMS voltage and current are \( V/\sqrt{2} \) and \( I/\sqrt{2} \). The apparent power is defined to be the product \( VI/2 \).

The real-valued instantaneous power is defined to be \( p(t) = v(t)i(t) \), which may be written

\[
p_{vi}(t) = \frac{VI}{2} \cos(\theta - \phi) + \frac{VI}{2} \cos(2\omega_0 t + \theta + \phi),
\]

\[
= \frac{VI}{2} \cos(\theta - \phi) + \frac{VI}{2} \cos(2\omega_0 t + 2\phi)
\] - \( \frac{VI}{2} \sin(\theta - \phi) \sin(2\omega_0 t + 2\phi). \tag{2} \]

Equation (1) shows real-valued instantaneous power to oscillate at frequency \( 2\omega_0 \) and phase delay \( -\frac{(\theta + \phi)}{2\omega_0} \) around the average power \( P_{vi} = \frac{VI}{2} \cos(\theta - \phi) \). This is a scaling of apparent power \( \frac{VI}{2} \) by the power factor \( \cos(\theta - \phi) \). The sum of the first two terms in Eq. (2) is active power. It is always non-negative. The third term is time-varying non-active power. In a Thevenin equivalent circuit, active power is real power and non-active power is reactive power.

A. Complex Representation of Instantaneous Power

There is a complex representation of real-valued instantaneous power, based on a phasor, or complex analytic, representation of real signals:

\[
p_{vi}(t) = \text{Re} \left\{ \frac{VI}{2} e^{j(\theta-\phi)} + \frac{VI}{2} e^{j(\theta+\phi)} e^{j2\omega_0 t} \right\}
\]

\[
= \text{Re} \left\{ \frac{VI}{2} e^{j(\theta-\phi)} + \left( \frac{VI}{2} \cos(\theta - \phi) + j \frac{VI}{2} \sin(\theta - \phi) \right) e^{j2\omega_0 t + 2\phi} \right\}
\]

\[
= \frac{VI}{2} \cos(\theta - \phi) + \frac{VI}{2} \cos(\theta - \phi) \cos(2\omega_0 t + 2\phi)
\] - \( \frac{VI}{2} \sin(\theta - \phi) \sin(2\omega_0 t + 2\phi). \tag{4} \]

The term within the Re operator of Eq. (3) may be termed complex instantaneous power. It consists of two terms: \( (VI/2)e^{j(\theta-\phi)} \) and \( (VI/2)e^{j(\theta+\phi)}e^{j2\omega_0 t} \). The first is a stationary phasor or analytic signal representation of time-invariant average power and the second is a rotating phasor or analytic representation of the time varying component of instantaneous power. In Eq. (4), the rotating phasor is rewritten as a rotation of the stationary phasor \( \frac{VI}{2} e^{j(\theta-\phi)} = \frac{VI}{2} \cos(\theta - \phi) + j \frac{VI}{2} \sin(\theta - \phi) \). In this Cartesian representation, the components \( \frac{VI}{2} \cos(\theta - \phi), \frac{VI}{2} \sin(\theta - \phi) \) are in-phase and quadrature components of the phasor, or adjacent and opposite sides of a right triangle. In this right triangle, the Pythagorean Theorem shows that the hypotenuse has length \( \sqrt{2} V \), which is commonly called apparent power. Figure 1(a) is the complex representation of instantaneous power given in Eq. (3). It consists of a complex phasor, rotating around a fixed phasor. Instantaneous power is read off the diagram as the real part of the complex instantaneous power. Figure 1(b) resolves the rotating phasor into its in-phase and quadrature components, as in Eq. (4). This figure is illuminating because it shows the complex representation of real-valued instantaneous power to consist of a power triangle rotating around the tip of a fixed phasor. The power triangle is a right triangle, defined by in-phase and quadrature components. In this right triangle, the square of apparent power is the sum of squares of active and non-active powers: \( (VI/2)^2 = (VI/2)^2 \cos^2(\theta - \phi) + (VI/2)^2 \sin^2(\theta - \phi) \). In IEEE Standard 1459, this Pythagorean identity is written, \( S^2 = P^2 + Q^2 \), a notation which is justified by the complex representation \( \frac{VI}{2} e^{j(\theta-\phi)} = \frac{VI}{2} \cos(\theta - \phi) + j \frac{VI}{2} \sin(\theta - \phi) \). In a Thevenin equivalent circuit, the square of apparent power is the sum of squares of real and reactive powers. In summary, the power triangle is triangle of Fig. 1(b) that rotates around a fixed phasor.

B. Average Power, Positive Power, and Negative Power

Real-valued instantaneous power is bounded below by \( \frac{VI}{2} [\cos(\theta - \phi) - 1] \) and above by \( \frac{VI}{2} [\cos(\theta - \phi) + 1] \). So it can be negative! This motivates the definition of positive power \( p^+ \) (t) and negative power \( p^- \) (t):

\[
p_{vi}(t) = p^+_vi(t) + p^-_vi(t), \tag{6} \]

\[
p^+_vi(t) = \max(p_{vi}(t), 0) \quad \text{and} \quad p^-_vi(t) = \min(p_{vi}(t), 0). \tag{7} \]

For \( T = \frac{2\pi}{\omega_0} \), the average of negative power over a period is

\[
P_{vi}^- = \frac{1}{T} \int_0^T p^-_vi(t) dt = -\frac{VI}{2} \left[ \frac{\sin(\theta - \phi)}{\pi} - \frac{\theta - \phi \cos(\theta - \phi)}{\pi} \right] \leq 0, \tag{8} \]

and the average of positive power is

\[
p^+_vi(t) = \frac{VI}{2} \cos(\theta - \phi) + \frac{VI}{2} \left[ \frac{\sin(\theta - \phi)}{\pi} - \frac{\theta - \phi \cos(\theta - \phi)}{\pi} \right] \geq 0. \tag{9} \]

These results show that the power angle \( \theta - \phi \) determines not only the fraction of apparent power that is delivered to the load, but it shows that this delivered power is the sum of average positive power and average negative power:

\[
P_{vi} = \frac{1}{T} \int_0^T p_{vi}(t) dt = p^+_vi(t) + p^-_vi(t) = \frac{VI}{2} \cos(\theta - \phi). \tag{10} \]

This result clarifies that the difference between apparent power and delivered power is accounted for by the power returned from the load to the source.

The period of oscillation of instantaneous power is \( T = 2\pi/\omega_0 \), and the fraction of this period during which power is delivered is \( 1 - (\theta - \phi)/\pi \). So even though average power \( \frac{VI}{2} \cos(\theta - \phi) \) decreases trigonometrically with the power angle \( \theta - \phi \), the fraction of time that power is delivered decreases linearly with the power angle. The plot of Figure 2 illustrates typical voltage and currents waveforms for which real-valued instantaneous power is negative over a fraction of a period. The resolution of instantaneous power as a sum of plus power and negative power requires a non-linear decomposition of power.
C. Interpretations in a Thevenin Equivalent Circuit

When can active power be termed real power, and non-active power be termed reactive power?

Consider the Thevenin equivalent circuit in Figure 3 consisting of a complex impedance $Z$ in series with a sinusoidal voltage source $v(t) = \Re\{Ve^{j(\omega t + \theta)}\} = V\cos(\omega t + \theta)$. The sinusoidal current is $i(t) = \Re\{Ie^{j\phi}e^{j(\omega t)}\} = I\cos(\omega t + \theta)$. The circuit constraint is $Ve^{j\theta} = ZIe^{j\phi}$, where the complex impedance is $Z = V/Ie^{j(\phi - \theta)}$. The Cartesian representation of this complex impedance is $Z = R + jX$, where $R = V/I\cos(\theta - \phi)$ is real impedance and $X = V/I\sin(\theta - \phi)$ is reactive impedance. The term $VI/2e^{j(\theta - \phi)}$ may be written $ZI^2/2$, the term $VI/2\cos(\theta - \phi)$ may be written $RI^2/2$ and the term $VI/2\sin(\theta - \phi)$ may be written $XI^2/2$. With these identities, the complex representation of real-valued instantaneous power in Eqs. (3) and (4) may be written

$$p_{vi}(t) = \Re\left[\frac{ZI^2}{2}e^{j(2\omega t + 2\phi)} + \frac{RI^2}{2}e^{j(\omega t + \phi)}\right].$$

Eq. (11) shows that the magnitude of each phasor in the complex representation of real-valued instantaneous power is determined by $ZI^2/2$. Eq. (12) shows the in-phase component of the time varying phasor depends on the resistive component of impedance and the quadrature component depends on the reactive component. The sum of the first two terms in Eq. (13) is active power, and it is determined by the real component of impedance. It may be called the real power that dissipates heat, produces light, and turn shafts. The third term is non-active power, and it is determined by the reactive component of impedance. It may be called the reactive power that accounts for the rate at which energy is exchanged between the electric field that supports voltage and the magnetic field that produces current.

In summary, active power is real power, and non-active power is reactive power, when current and voltage may be identified in a Thevenin equivalent circuit.

Now in Fig 1b, the complex phasor representation of instantaneous power consists of the phasor $ZI^2/2e^{j(2\omega t + 2\phi)}$ rotating around the tip of the stationary phasor $ZI^2/2$. The rotating phasor decomposes orthogonally as $(RI^2/2 + jXI^2/2)e^{j(2\omega t + 2\phi)}$. 

Fig. 3: A Thevenin equivalent circuit.
The phasor representation of the rotating phasor, namely \( R e^{\frac{\theta}{2}} \), has Cartesian coordinates \( (R \cos \frac{\theta}{2}, R \sin \frac{\theta}{2}) \). These are taken to define the sides of a right triangle whose hypotenuse is the apparent power \( Z e^{\frac{\theta}{2}} \). As before, the rotating triangle is the power triangle, the angle \( \theta - \phi \) is the power factor angle, and \( \cos(\theta - \phi) \) is the power factor. The power triangle changes orientation, but it does not change shape. The instantaneous power is the resolution of the complex result onto the real axis of the Argand plane. Power is delivered to the impedance load during \((1-(\theta-\phi)/\pi)T \) seconds of each period, and returned to the voltage source during \(((\theta-\phi)/\pi)T \) seconds of each period. For these \(((\theta-\phi)/\pi)T \) seconds, reactive components are exchanging energy with resistive components at a rate that exceeds the the capacity of the resistive components to dissipate heat, produce light or turn shafts. The excess energy is returned to the voltage source, which means the circuit elements are energizing the source, rather than vice-versa.

III. THE HILBERT TRANSFORM, PHASE-SPLITTER, ANALYTIC SIGNAL, AND BEDROSIAN’S THEOREM

Suppose we had begun with voltage and the current represented as \( v(t) = Re\{\tilde{v}(t)\} \) and \( i(t) = Re\{\tilde{i}(t)\} \), where \( \tilde{v}(t) = Ve^{j\theta}e^{j\omega t} \) and \( \tilde{i}(t) = Ie^{j\phi}e^{j\omega t} \) are the complex analytic representations of real voltage and current. Then we might have defined instantaneous Hermitian power to be

\[ p_{\tilde{v}\tilde{i}}(t) = \tilde{v}^* \cdot \tilde{i} = V I e^{j(\theta-\phi)} \] (14)

and complementary instantaneous power to be

\[ p_{\tilde{v}\tilde{c}}(t) = \tilde{v}^* \cdot \tilde{c} = V I e^{j(\theta+\phi)} e^{j2\omega t} \] (15)

Then the complex analytic representation of Eq. (3) would have been written as

\[ p_{\tilde{v}\tilde{i}}(t) = \frac{1}{2} Re\{p_{\tilde{v}\tilde{i}}(t) + p_{\tilde{v}\tilde{c}}(t)\} \] (16)

In Figure 1(b), the fixed phasor may be re-labeled at \((1/2)p_{\tilde{v}\tilde{i}}(t)\) and the rotating phasor as \((1/2)p_{\tilde{v}\tilde{c}}(t)\). This suggests that the treatment of instantaneous power might be extended from sinusoidal cases to non-sinusoidal, and in fact aperiodic, signals by appealing to the Hilbert transform and complex analytic representations of real signals, as a generalization of the phasor representations of sinusoidal signals.

A. The Hilbert Transform

The Hilbert transform is a linear operator \( H \) that transforms real signals \( x \in L^2(\mathbb{R}) \) into real signals \( \hat{x} \in L^2(\mathbb{R}) \) according to the filtering formula

\[ \hat{x} = Hx : \hat{x}(t) = \int_{-\infty}^{\infty} h(t-u)x(u)du. \] (17)

The real and odd impulse response of the operator \( H \) is \( h(t), -\infty < t < \infty \), with complex Fourier transform \( H(\omega), -\infty < \omega < \infty \):

\[ h(t) = \frac{1}{\pi t} \leftrightarrow \frac{1}{\omega} \text{sgn}(\omega) = H(\omega) \] (18)

The double arrow denotes Fourier transform pair. It is clear that if \( x(t) \to X(\omega) \), then \( \hat{x}(t) \to \text{sgn}(\omega)e^{-j\pi/2}X(\omega) \).

So, frequency-by-frequency, the Hilbert transform leaves the magnitude of the signal spectrum unchanged, but changes its phase by \(-\pi/2\) for positive frequencies and by \(\pi/2\) for negative frequencies. Of course \( X(\omega) \), \( H(\omega) \), and \( \hat{X}(\omega) \) are Hermitian symmetric functions of radian frequency \( \omega \). The filter \( H \) is all-pass, but unrealizable. In [14] a simple first-order rational approximation of \( H \) is proposed, but in our treatment of generalized power, there is no need for such an approximation. A hardware implementation of \( H \) would use a high-order approximation of all-pass \( H \).

The complex signal \( \hat{x}(t) = x(t) + j\hat{x}(t) \) is called the analytic version of \( x(t) \), or the complex analytic version of \( x(t) \), or the complex envelope of \( x(t) \). It may be written as the convolution

\[ \hat{x} = \hat{x}(t) = \int_{-\infty}^{\infty} \psi(t-u)x(u)du \] (19)

where \( \psi(t) = \delta(t) + jh(t) \) is the impulse response of the phase splitter \( \Psi \); the Dirac delta \( \delta(t) \) is the impulse response of the identity operator and \( h(t) \) is the impulse response of the Hilbert transform. The Fourier transform of \( \psi(t) \) is easily shown to be \( \Psi(\omega) = 2\text{step}(\omega) \), where \( \text{step}(\omega) \) is the unit step function. That is, \( \psi(t) = \delta(t) + jh(t) \to 2\text{step}(\omega) = \Psi(\omega) \), and \( \hat{x}(t) \to 2X(\omega)\text{step}(\omega) \). It follows that the spectral representations of real \( x \), \( \hat{x} \) and complex \( \hat{x} \) are

\[ x(t) = \int_{-\infty}^{\infty} 2\text{Re}\{X(\omega)e^{j\omega t}\} \frac{d\omega}{2\pi} \] (20)

\[ \hat{x}(t) = \int_{-\infty}^{\infty} 2\text{Im}\{X(\omega)e^{j\omega t}\} \frac{d\omega}{2\pi} \] (21)

\[ \hat{x}(t) = \int_{-\infty}^{\infty} 2X(\omega)e^{j\omega t} \frac{d\omega}{2\pi} \] (22)

These spectrally-efficient representations exploit the Hermitian symmetry of Fourier transforms for real signals.

It is interesting to note that the celebrated Kramers-Kronig relations are duals of these results. That is, the Kramers-Kronig relations establish that the imaginary part of a spectrum must be the Hilbert transform of the real part for the corresponding signal to be causal, whereas these results establish that the imaginary part of a signal must be the Hilbert transform of the real part for its spectrum to be causal.

Example 1. Consider the real periodic-\( \frac{2\pi}{\omega_0} \) signal

\[ x(t) = \sum_{m=-M}^{M} \frac{A_m}{2} e^{j\theta_m} e^{jm\omega_0 t} = \sum_{m=1}^{M} A_m \cos(m\omega_0 t + \theta_m), \] (23)

\[ A_0 = 0, \quad A_m = A_m, \quad \text{and} \quad \theta_m = -\theta_m. \] (24)

The Hilbert transform of this signal is \( \hat{x}(t) \), and the complex analytic signal is \( \hat{x}(t) \):

\[ \hat{x}(t) = \sum_{m=-M}^{M} \frac{sgn(m)A_m}{2j} e^{j\theta_m} e^{jm\omega_0 t} = \sum_{m=1}^{M} A_m \sin(m\omega_0 t + \theta_m), \] (25)

\[ \hat{x}(t) = \sum_{m=1}^{M} A_m e^{j(m\omega_0 t + \theta_m)}. \] (26)
This may be written \( \tilde{x}(t) = A_1 e^{-j\omega_0 t} + \sum_{n=1}^{M} A_m e^{-j(m\omega_0 t + \theta_m)} \), which is a complex analytic representation of a real sinusoid of frequency \( \omega_0 \), plus harmonic distortion. A special case of this example is \( x(t) = A_1 \cos(\omega_0 t + \theta) \), in which case the Hilbert transform \( \tilde{x}(t) \) and complex analytic signal \( \tilde{x}(t) \) are
\[
\tilde{x}(t) = A_1 \sin(\omega_0 t + \theta) \\
\tilde{x}(t) = x(t) + j\tilde{x}(t) = A_1 e^{j(\omega_0 t + \theta)} = A_1 e^{j\theta} e^{j\omega_0 t} \quad (27)
\]
Of course, in this case the complex analytic signal (or complex envelope) \( \tilde{x}(t) \) is a rotating phasor and \( A_1 e^{j\theta} \) is the corresponding stationary phasor.

Note that, beginning with the complex rotating phasor \( A e^{j\theta_0} e^{j\omega_0 t} \), the real operator \( \text{Re} \) returns the real signal \( A \cos(\omega_0 t + \theta) \), whereas, beginning with the real signal \( A \cos(\omega_0 t + \theta) \), the phase splitter \( \Psi \) returns the complex signal \( A e^{j\omega_0 t} \). For this reason we call the phase splitter the unreal operator.

**B. Bedrosian’s Theorem and Its Implications**

Let \( x(t) = u(t)v(t) \) be a product of real signals, with \( u(t) \leftarrow U(\omega) \) a low-pass signal with \( U(\omega) = 0 \) for \( |\omega| > \Omega \), and \( v(t) \leftarrow V(\omega) \) a high-pass signal with \( V(\omega) = 0 \) for \( |\omega| < \Omega \). Then Bedrosian’s theorem shows \( \tilde{x}(t) = u(t)\tilde{v}(t) \). That is, the low-pass, slowly-varying, factor may be regarded as constant when calculating the Hilbert transform.

**Example 2.** Consider the real aperiodic signal
\[
x(t) = A(t) \cos(\omega_0 t + \theta(t)) \quad (29)
\]
This is a co-sinusoidal carrier of nominal frequency \( \omega_0 \), amplitude and phase modulated by the lowpass amplitude \( A(t) \) and the lowpass phase \( \theta(t) \). Then, by Bedrosian’s theorem, if the bandwidth of \( A(t) \cos(\theta(t)) \) is smaller than \( \omega_0 \), then the Hilbert transform of \( x(t) \) is \( \tilde{x}(t) \) and its complex analytic signal is \( \tilde{x}(t) \):
\[
\tilde{x}(t) = A(t) \sin(\omega_0 t + \theta(t)), \\
\tilde{x}(t) = A(t) \cos(\omega_0 t + \theta(t)) + jA(t) \sin(\omega_0 t + \theta(t)) \\
= A(t) e^{j\theta(t)} e^{j\omega_0 t} \quad (32)
\]
In this complex analytic representation of the real signal \( x(t) \), the complex envelope \( \tilde{x}(t) \) is a rotating phasor with time-varying amplitude and phase. Among its many virtues, this complex analytic representation of the real signal \( x(t) \) provides an unambiguous definition of instantaneous frequency:
\[
\omega(t) = \frac{d}{dt} (\omega_0 t + \theta(t)) = \omega_0 + \frac{d}{dt} \theta(t). \quad (33)
\]
The complex analytic signal may be demodulated with \( e^{-j\omega_0 t} \) to return \( A(t)e^{j\theta(t)} \). In hardware implementations it is common to approximate the phase splitter-complex demodulator by a quadrature demodulator, wherein the real signal is multiplied by \( \cos(\omega_0 t) \) in the real channel and by \( \sin(\omega_0 t) \) in the quadrature channel; each of these channels is then lowpass filtered.

**IV. Complex Representation of Real-Valued Instantaneous Power for Non-Sinusoidal Signals**

Begin with complex analytic voltage and current, \( \tilde{v}(t) \) and \( \tilde{i}(t) \), and define the complex Hermitian power, and the complex complementary power:
\[
\tilde{p}_{\tilde{v}}(t) = \tilde{v}(t)\tilde{\tilde{v}}(t), \quad (34)
\]
\[
\tilde{p}_{\tilde{i}}(t) = \tilde{i}(t)\tilde{\tilde{i}}(t). \quad (35)
\]
It is always the case that the instantaneous power is
\[
p_{vi}(t) = \frac{1}{2} \text{Re} \{ \tilde{p}_{\tilde{v}}(t) + \tilde{p}_{\tilde{i}}(t) \} \quad (36)
\]
Now consider the real voltage \( v(t) = \text{Re}\{V(t)e^{j\theta(t)}e^{j\omega_0 t}\} = V(t) \cos(\omega_0 t + \theta(t)) \) and real current \( i(t) = \text{Re}\{I(t)e^{j\phi(t)}e^{j\omega_0 t}\} = I(t) \cos(\omega_0 t + \phi(t)) \). Assume the functions \( V(t)e^{j\theta(t)} \) and \( I(t)e^{j\phi(t)} \) have bandwidth less than \( \omega_0 \). Then from Bedrosian’s Theorem, it follows that \( \tilde{v}(t) = V(t)e^{j\theta(t)}e^{j\omega_0 t} \) is the complex analytic version of \( v(t) \) and \( \tilde{i}(t) = I(t)e^{j\phi(t)}e^{j\omega_0 t} \) is the complex analytic version of \( i(t) \). These complex analytic signals may be extracted from their real counterparts with the phase-splitter. Then the complex Hermitian power, and the complex complementary power compute as follows:
\[
\tilde{p}_{\tilde{v}}(t) = \tilde{v}(t)\tilde{\tilde{v}}(t) = V(t)I(t)e^{j(\theta(t)-\phi(t))}e^{j(2\omega_0 t+2\phi(t))} \quad (37)
\]
\[
\tilde{p}_{\tilde{i}}(t) = \tilde{i}(t)\tilde{\tilde{i}}(t) = V(t)I(t)e^{j(\phi(t)-\theta(t))}e^{j(2\omega_0 t+2\phi(t))} \quad (38)
\]
The real-valued instantaneous power is
\[
p_{vi}(t) = \text{Re} \left\{ \frac{V(t)I(t)}{2} e^{j(\theta(t)-\phi(t))} + \frac{V(t)I(t)}{2} e^{j(\phi(t)-\theta(t))} e^{j(2\omega_0 t+2\phi(t))} \right\} \quad (39)
\]
\[
= \text{Re} \left\{ \frac{V(t)I(t)}{2} \left[ \cos(\theta(t) - \phi(t)) + j \sin(\theta(t) - \phi(t)) \right] e^{j(2\omega_0 t+2\phi(t))} \right\} \quad (40)
\]
\[
= \frac{V(t)I(t)}{2} \cos(\theta(t) - \phi(t)) + \frac{V(t)I(t)}{2} \cos(\omega_0 t + 2\phi(t)) \quad (41)
\]
Equations (39)-(41) generalize the sinusoidal results of Eqs. (3)-5 to non-sinusoidal signals. These non-sinusoidal signals need not be periodic. The first term in Eqs. (39) and (40) may be called slowly-varying, or short term average power. The second and third terms in Eq. (40) resolve the second term in Eq. (39) into its in-phase and quadrature components. Eq. (41) resolves the real operator to show that instantaneous power is slowly-varying average power plus the in-phase and quadrature components of the component of instantaneous power that oscillates at high frequency \( 2\omega_0 \). Moreover, the sum of the first two terms in Eq. (41) accounts for active power and the third term accounts for non-active power. In a time-varying linear system, active power is real power and non-active power is reactive power.

The geometry generalizes the geometry of Fig. 1(b): The complex representation of instantaneous power consists of the slowly varying phasor \( \frac{V(t)I(t)}{2} e^{j(\theta(t)-\phi(t))} \), around which
rotates the phasor $V(t)I(t) = e^{j(\theta(t) - \phi(t))}$. This rotating phasor decomposes into its in-phase and quadrature components $\frac{1}{2}V(t)\cos(\theta(t) - \phi(t))$ and $\frac{1}{2}V(t)\sin(\theta(t) - \phi(t))$. This decomposition shows that a slowly-varying right triangle, with sides equal to these in-phase and quadrature components, rotates rapidly around the tip of a slowly-varying phasor. This triangle may be called a slowly-varying power triangle.

VI. SPECTRAL THEORY OF AVERAGE POWER

Define average power to be a time average. For sinusoidal voltage and current the results are straightforward:

$$P_{\bar{v}i} = \int p_{\bar{v}i}(t) dt = \frac{VI}{2}e^{j(\theta - \phi)}.$$  

$$P_{\bar{v}i} = \int p_{\bar{v}i}(t) dt = 0,$$ 

$$P_{vi} = \frac{VI}{2} \cos(\theta - \phi).$$

There is no need for a spectral representation of these results. For general time-varying voltage and current, spectral representations bring insight, if correctly interpreted.

Begin with the Fourier transform pairs $v(t) \leftrightarrow V(\omega)$, $i(t) \leftrightarrow I(\omega)$. The current and voltage are real, but their Fourier transforms are complex, with Hermitian symmetry. Give the complex Fourier transforms the polar representations $V(\omega) = A(\omega)e^{j\Theta(\omega)}$ and $I(\omega) = B(\omega)e^{j\Phi(\omega)}$. Recall the corresponding one-sided spectral representations for analytic signals have a factor of 2. Then it is straightforward to write time domain averages as the following frequency-domain averages:

$$P_{\bar{v}i} = \int_{0}^{\infty} 4V(\omega)I^*(\omega) \frac{d\omega}{2\pi} = \int_{0}^{\infty} 4A(\omega)B(\omega)e^{j(\Theta(\omega) - \Phi(\omega))} \frac{d\omega}{2\pi}$$

$$P_{\bar{v}i} = \int_{0}^{\infty} 4V(\omega)I(\omega) \frac{d\omega}{2\pi} = \int_{0}^{\infty} 4A(\omega)B(\omega)e^{j(\Theta(\omega) + \Phi(\omega))} \frac{d\omega}{2\pi}$$

$$P_{vi} = {}^2\int_{0}^{\infty} \left[2A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega))ight. $$

$$+ 2A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega))\cos(2\Phi(\omega))$$

$$- 2A(\omega)B(\omega)\sin(\Theta(\omega) - \Phi(\omega))\sin(2\Phi(\omega)) \frac{d\omega}{2\pi}$$  

The sum of the first two terms is active power and the third term is non-active power.

Frequency-by-frequency, these average powers have the same components as the formulas in the sinusoidal case. The geometry of the complex representation of average power is the geometry of the Hermitian phasor $A(\omega)B(\omega)e^{j(\Theta(\omega) - \Phi(\omega))}$ plus the complementary phasor $A(\omega)B(\omega)e^{j(\Theta(\omega) + \Phi(\omega))}e^{j2\Phi(\omega)}$; the complementary phasor resolves as $4A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega)) + j4A(\omega)B(\omega)\sin(\Theta(\omega) - \Phi(\omega))$ rotated by $e^{j2\Phi(\omega)}$. Frequency-by-frequency, one may speak of a power triangle whose hypotenuse has magnitude equal to the frequency-dependent apparent power, $4A(\omega)B(\omega)$, and whose sides are $2A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega))$ and $2A(\omega)B(\omega)\sin(\Theta(\omega) - \Phi(\omega))$. It cannot be said that there is a broadband rotating

V. INTERPRETATIONS IN A TIME-VARYING THEVENIN EQUIVALENT CIRCUIT

When can active power and non-active power be called real power and reactive power in the case where voltage and current are non-sinusoidal and the impedance is time-varying?

Begin with a Thevenin equivalent circuit for which the impedance $Z(t, \omega)$ is the Fourier transform of the real time-varying impulse response $z(t, \tau): Z(t, \omega) = \int z(t, \tau)e^{-j\omega\tau} d\tau$. Assume the real current is $i(t) = I e^{j(\omega_0 t + \phi)}$, with corresponding complex analytic representation $\tilde{i}(t) = IE^{j(\omega_0 t + \phi)}$. The real voltage is

$$v(t) = \int z(t, \tau)i(t - \tau) d\tau = \text{Re}\{Z(t, \omega_0)IE^{j(\omega_0 t + \phi)}\}.$$  

(42)

where $z(t, \tau) \leftrightarrow Z(t, \omega)$. Assuming that the bandwidth of $Z(t, \omega_0)$ is less than $\omega_0$, Bedrosians theorem shows that the complex analytic voltage is

$$\tilde{v}(t) = Z(t, \omega_0)IE^{j(\omega_0 t + \phi)}.$$  

(43)

This $\tilde{v}(t)$ is a Hermitian, complementary, and real-valued instantaneous power are

$$p_{\tilde{v}i}(t) = \tilde{v}^*\tilde{i}(t) = Z(t, \omega_0)I^2 + JX(t, \omega_0)I^2,$$  

(44)

$$p_{\tilde{v}i}(t) = \tilde{v}\tilde{i}(t) = Z(t, \omega_0)I^2 e^{j(2\omega_0 t + 2\phi)},$$  

(45)

$$p_{vi}(t) = \frac{1}{2}\text{Re}\{p_{\tilde{v}i}(t) + p_{\tilde{v}i}(t)^*\}$$

$$= \frac{R(t)^2 + R(t)^2}{2} \cos(2\omega_0 t + 2\phi) - \frac{X(t, \omega_0)^2}{2} \sin(2\omega_0 t + 2\phi).$$  

(46)

As before, Figure 1(b) may be re-labeled, in this case with time-varying impedance components. The sum of the first two terms in Eq. (46) is active power, and it is real power. The third term is non-active power, and it is reactive power.

Can this analysis be extended to more general currents? It does not seem possible. But this example is useful. It assumes a regulated current source, and a time-varying impedance. And of course by letting $v(t)$ be sinusoidal, and allowing admittance to be time varying, this example applies to a regulated voltage source, and a time-varying admittance.

Frequency-by-frequency, these average powers have the same components as the formulas in the sinusoidal case. The geometry of the complex representation of average power is the geometry of the Hermitian phasor $A(\omega)B(\omega)e^{j(\Theta(\omega) - \Phi(\omega))}$ plus the complementary phasor $A(\omega)B(\omega)e^{j(\Theta(\omega) + \Phi(\omega))}e^{j2\Phi(\omega)}$; the complementary phasor resolves as $4A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega)) + j4A(\omega)B(\omega)\sin(\Theta(\omega) - \Phi(\omega))$ rotated by $e^{j2\Phi(\omega)}$. Frequency-by-frequency, one may speak of a power triangle whose hypotenuse has magnitude equal to the frequency-dependent apparent power, $4A(\omega)B(\omega)$, and whose sides are $2A(\omega)B(\omega)\cos(\Theta(\omega) - \Phi(\omega))$ and $2A(\omega)B(\omega)\sin(\Theta(\omega) - \Phi(\omega))$. It cannot be said that there is a broadband rotating
That is, because the square of a sum is not a sum of squares,

$$\left( \int 2A(\omega)B(\omega) \frac{d\omega}{2\pi} \right)^2 \neq \int (2A(\omega)B(\omega))^2 \left( \cos^2(\Theta(\omega) - \Phi(\omega)) + \sin^2(\Theta(\omega) - \Phi(\omega)) \right) d\omega +$$

$$\int (2A(\omega)B(\omega))^2 \sin^2(\Theta(\omega) - \Phi(\omega)) d\omega$$

This is the basis of Czarnecki’s objection [3] to Budeneu’s definition of reactive power in non-sinusoidal situations [1].

If $V(\omega)$ and $I(\omega)$ are voltage and current in a Thevenin equivalent circuit, then it is straight forward to write $V(\omega) = Z(\omega)I(\omega)$, in which case $V(\omega)I^*(\omega) = Z(\omega)|I(\omega)|^2$ and $V(\omega)I(\omega) = Z(\omega)I^*(\omega)$. If the impedance $Z = R(\omega) + jX(\omega)$ is written as $Z(\omega) = |Z(\omega)|e^{j(\Theta(\omega) - \Phi(\omega))}$, then $R(\omega) = |Z(\omega)|\cos(\Theta(\omega) - \Phi(\omega))$ and $X(\omega) = |Z(\omega)|\sin(\Theta(\omega) - \Phi(\omega))$. Then the frequency domain formulas may be written

$$P_{\vec{v}\vec{i}}(\omega) = 4|Z(\omega)||I(\omega)|^2 e^{j(\Theta(\omega) - \Phi(\omega))}$$

$$P_{\vec{i}\vec{v}}(\omega) = 4|Z(\omega)||I(\omega)|^2 e^{j(\Theta(\omega) - \Phi(\omega))} e^{j2\Phi(\omega)}$$

$$P_{\vec{v}\vec{i}}(\omega) = 2R(\omega)||I(\omega)||^2 + 2R(\omega)|I(\omega)|^2 \cos(2\Phi(\omega))$$

$$- 2X(\omega)||I(\omega)||^2 \sin(2\Phi(\omega))$$

So, frequency-by-frequency, active power depends on real impedance, and non-active power depends on reactive impedance. Active power is real power and non-active power is reactive power. Of course the formula for $P_{\vec{v}\vec{i}}(\omega)$ may be integrated to get average power as average real power plus average reactive power.

This frequency domain analysis goes through essentially unchanged for discrete spectra $\{V_n, n = 0, \pm 1, \ldots\}$, and $\{I_n, n = 0, \pm 1, \ldots\}$, in which case these line spectra would be describing non-sinusoidal periodic signals.

VII. HARDWARE DIAGRAMS FOR EXTRACTING COMPONENTS OF INSTANTANEOUS POWER

The results so far establish the components of Hermitian, complementary, and real-valued instantaneous power. They do not show how these components are to be extracted for monitoring of power system performance.

Begin with the real voltage $v(t)$ and current $i(t)$, measured by volt-meter and current transformer. As illustrated in the hardware diagram of Fig. 4 these are Hilbert transformed to produce the analytic voltage $\tilde{v}(t)$ and analytic current $\tilde{i}(t)$. From these are computed the Hermitian and complementary powers $p_{\vec{v}\vec{i}}$ and $p_{\vec{i}\vec{v}}$. The sum of these may be displayed for the complex representation of real-valued instantaneous power. Or, the complementary power may be demodulated to baseband as $e^{-j2\omega t}p_{\vec{i}\vec{v}}$. This stationary picture is the picture of Fig. 1b at $t = 0$. If the voltage and current are slowly-varying, the picture will be the picture of Eqs. 37 and 38. If the voltage and current are constrained by the impedance of a Thevenin equivalent circuit, the the picture will be the picture of Eqs. 34-36. For computed quantities, the real and imaginary parts of the Hermitian power return $V(t)I(t)\cos(\theta(t) - \phi(t))$ and $V(t)I(t)\sin(\theta(t) - \phi(t))$. These determine the three terms of real-valued power, provided $\omega_0$ and $\phi$ are known; $\omega_0$ is assumed known, or may be extracted with a frequency estimator, and $\phi$ is determined from the identity $e^{-j2\omega t}p_{\vec{i}\vec{v}} = e^{2\phi}p_{\vec{v}\vec{i}}$.

VIII. CONCLUSIONS

Hermitian and complementary complex power may be computed from the Hilbert transforms of real voltage and current. These, in turn, provide an evocative complex representation of real-valued instantaneous power. This representation extends to non-sinusoidal cases, and when there is a Thevenin equivalent circuit, the complex representation may be used to identify active and non-active powers as real and reactive powers. A simple hardware diagram may be used to extract these
components. Frequency domain formulas for average power decompose in a way that is dual, frequency-by-frequency, to the time domain formulas for instantaneous power.

REFERENCES

[1] C. J. Budeanu, Puissances reactives et fictives,” Inst. Raomain de l’energie, Bucharest, Romania, 1927.
[2] W. Shepard and P. Zakikhani, “Suggested definitions of reactive power for nonsinusoidal systems,” Proc. Inst. Electr. Eng., vol 119, pp 1361-1362, Sept, 1972; vol 120, pp 706-798, July 1973.
[3] L. S. Czarnecki, “Considerations on the reactive power in non-sinusoidal situations.” IEEE Trans. Instrum. Meas. vol IM-34, no 3, pp 399-404, Sept. 1985.
[4] C.M. Page, “Reactive power in nonsinusoidal situations,” IEEE Trans Instrum. Meas., vol IM-29, pp 420-423, Dec 1990.
[5] P.S. Filipski, “Apparent power-a misleading quantity in the nonsinusiodal power theory: are all nonsinusoidal power theories doomed to fail?” ETEP, vol 3, no 1, pp 21-26, Feb. 1993.
[6] K.N. Al-Tallaq, “Determination of apparent power components in non-sinusoidal circuits,” IEEE 2006.
[7] S. Psjic, “Modern apparent power definitions: theoretical versus practical approach–the general case,” IEEE Trans Power Delivery, vol 21, no 4, pp 1787-1792, Oct. 2006
[8] Z. Nilolov, Z.Hleba, C. Korsemov, H. Toshev, “Distribution of active and reactive energy in a power line,” Bulgarian Academy of Sciences: Cybernetics and Information Technologies, vol 8, no 2, pp 12-25, 2008.
[9] R.S. Herrera, P. Salmeron, J.R. Vazquez, S.P. Litran, A. Perez, “Generalized instantaneous reactive power theory in polyphase power systems, Escuela Politecnica Superior, University of Huelva, Spain.
[10] G. Todoran, O. Muntean, and A. Buzura, “Determination of active and reactive power in multi-phase systems through analytical signals associated current and voltage signals,” Aca Electrotehnica.
[11] O. Muntean, “Harmonic linear circuits approach using Hilbert transform,”
[12] IEEE, “IEEE Trial-Use Standard Definitions for the Measurement of Electric Power Quantities Under Sinusoidal, Non-sinusoidal, Balanced, or Unbalanced Conditions,” IEEE Std 1459-2000, 2000.
[13] IEEE, “IEEE Standard Definitions for the Measurement of Electric Power Quantities Under Sinusoidal, Non-sinusoidal, Balanced, or Unbalanced Conditions “, IEEE Std 1459-2010, 2010.
[14] M. T. Haque and T. Ise, “Implementation of Single-Phase pq Theory,” , in Proceedings of the Power Conversion Conference-Osaka 2002 (Cat. No.02TH8579), pp. 761-765 vol. 2, Osaka, Japan, April 2-5, 2002.