PROOF OF THE GOLDBACH’S STRONG CONJECTURE BY USING SEMI-CONTINUOUS MODEL OF EVEN NUMBERS

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ABSTRACT. In this paper, we present an explicit and analytic proof for still unproven Goldbach’s strong conjecture. To derive this proof, we first define a heuristic model for representing even numbers called Semi-continuous Model of Even Numbers or briefly S.C.E Model. Then, we employ this model along with using the inequality
\[
\frac{x}{\ln x} \leq \pi(x) \leq 1.2551 \frac{x}{\ln x},
\]
where \(\pi(x)\) denotes the number of all primes smaller than and equal to \(x\), which is presented by Pierre Dusart in his paper [P. Dusart, Explicit estimates of some functions over primes, Ramanujan J. 45 (2016), No. 1, 227–251].

On the one hand, this proof is given for all even numbers \(E \geq 22864\). On the other hand, since the assertion of Goldbach’s strong conjecture is easy to verify for all even numbers \(4 \leq E < 22864\), we turn this conjecture into a theorem.

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1. HISTORY

Basically, the most prominent narration to the content of Goldbach’s strong conjecture refers to the possibility of converting any even number greater than or equal to 4 to sum of two prime numbers; which was expressed by Christian Goldbach in 1742. For years, there have been told many alternatives to this conjecture; but in overall, this conjecture is separated into two branches of weak and strong types, which the latter case refers to the possibility of converting any odd number greater than 5 into the sum of three prime numbers. The proof of weak type conjecture is widely accepted via a publication presented by Harald Helfgott Anderson in 2013\[1\].

However, the strong type of the conjecture is still under examination and investigation of mathematicians in order to find an argument to prove it or to find a counter-example to revoke its generality.

2. INTRODUCTION AND PRELIMINARIES

In the process of studying and working on even numbers in order to find out why or how the process of Goldbach’s conjecture holds true, it can be seen that all even numbers can be considered as a distance from zero with even values. Due to this point of view, in this paper, we will observe a heuristic model in correspondence with the Goldbach’s strong conjecture that we call it Semi-Continuum Model of Even Numbers or briefly S.C.E Model.

By using this model, we can translate Goldbach’s strong conjecture into this model by focusing on \(d_E\) element which will be introduced later. It is necessary to mention that

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we prove this conjecture just for even numbers $E \geq 22864$. On the other hand, since for even numbers $4 \leq E < 22864$ it is easy to verify Goldbach’s strong conjecture, instead of Conjecture, Goldbach’s strong statement can be referred as a Theorem from now on. Since in this model only additive interactions of odd numbers are involved, the number 2, despite being a prime number, will not be considered. However, by expressing Goldbach’s strong conjecture via this model, we can equivalently say that every even number $E \geq 4$ can be viewed as a sum of two primes if and only if the element $d_E$ never vanishes.

In this direction, we employ also the precious inequality presented by Pierre Dusart in [2] as
\[
\frac{x}{\ln x} \leq \pi(x) \leq 1.2551 \frac{x}{\ln x},
\]
where $\pi(x)$ denotes the number of all primes smaller than and equal to $x$. Here, the first inequality holds for all real numbers $x \geq 17$, and the second one holds for all real numbers $x > 1$. The interested reader can observe other inequalities for $\pi(x)$ in that paper, but here the inequality [1] is considered for both simplicity and universality. It will be turned out that by linking or importing this inequality into S.C.E Model, we can come up with the arguments, of which yield that for all even numbers $E \geq 4$, the element $d_E$ never vanishes.

2.1. Basic Semi-continuum Model of Even Numbers or S.C.E Model. In this section, for each positive even number $E$ we uniquely correspond a quadruple representation as
\[
\frac{E}{4} = a_E + b_E + c_E + d_E,
\]
where $a_E, d_E \in \{\frac{n}{2} \mid n \in \mathbb{W}\}$ and $b_E, c_E \in \mathbb{W}$ where $\mathbb{W} = \mathbb{N} \cup \{0\}$. To describe these elements, we first define the notion of additive interaction of odd numbers for a given even number $E$ as follows.

**Definition 2.1.** Let $E$ be a given even number, then for all odd positive numbers $x, y < E$ the notion
\[
x \sim y,
\]
is called an additive interaction for $E$ provided $x \leq y$ and $x + y = E$.

In this setting, for all additive interactions of $E$, we determine the elements of (3) as
\[
\begin{align*}
a_E &= \# \{x \sim y \mid x \neq y \text{ and } x, y \text{ are nonprimes} \} + \frac{1}{2} \# \{x \sim y \mid x \text{ is nonprime} \} \\
b_E &= \# \{x \sim y \mid x \text{ is nonprime but } y \text{ is prime} \} \\
c_E &= \# \{x \sim y \mid x \text{ is prime but } y \text{ is nonprime} \} \\
d_E &= \# \{x \sim y \mid x \neq y \text{ and } x, y \text{ are primes} \} + \frac{1}{2} \# \{x \sim y \mid x \text{ is prime} \},
\end{align*}
\]
where $\#$ denotes cardinality symbol.

**Definition 2.2 (Semi-continuum model of even numbers).** Let $E$ be an even number, then the unique quadruple representation (2) is called the semi-continuum model of $E$, where $a_E, b_E, c_E, d_E$ are as (3)–(6), respectively.

In this case by defining $L_1(E) = a_E + b_E$, $L_2(E) = c_E + d_E$, $R_1(E) = a_E + c_E$ and $R_2(E) = b_E + d_E$, we can also demonstrate the semi-continuum model of even number $E$ as follows.
In this papion shape, for upper vertices, we can see that the values
\[ \lceil L_1(E) \rceil, \lceil R_1(E) \rceil, \]
show respectively the number of odd non-primes in the intervals \([0, \frac{E}{2}]\) and \([\frac{E}{2}, E]\), where \([.\)] is the ceiling function. Similarly, for lower vertices, the same reasonings yield the values
\[ \lceil L_2(E) \rceil, \lceil R_2(E) \rceil, \]
show the number of primes in the intervals \([0, \frac{E}{2}]\) and \([\frac{E}{2}, E]\), respectively. Furthermore, by (2), we can also drive the following relation
\[ E_4 = L_1(E) + L_2(E) = R_1(E) + R_2(E). \] (7)

As it is obvious, since only the odd numbers \(x \leq y < E\) additively interact with each other to represent the even number \(E\), this model is named for semi-continuum model of even numbers.

**Remark 2.3.** We can easily see that
\[ b_E + c_E + 2d_E = L_2(E) + R_2(E) = \pi(E) - 1. \] (8)

Note that, since 2 is a prime but is not an odd number, we subtract 1 from \(\pi(E)\) to get \(L_2(E) + R_2(E)\). On the other hand, we have
\[ b_E + c_E + 2a_E = L_1(E) + R_1(E) = \frac{E}{2} - \pi(E) + 1. \] (9)

**Remark 2.4.** In view of Remark 2.3, the even number 4 is the only exception to our model since \(d_4 = 0\) but we can write \(2 + 2 = 4\).

**Remark 2.5.** For an even number \(E\), if \(\frac{E}{2}\) is also an even number, then we have
\[ L_1 \left( \frac{E}{2} \right) + R_1 \left( \frac{E}{2} \right) = L_1(E) \]
\[ L_2 \left( \frac{E}{2} \right) + R_2 \left( \frac{E}{2} \right) = L_2(E), \]
and if \(\frac{E}{2}\) is an odd number, then we have
\[ L_1 \left( \frac{E}{2} - 1 \right) + R_1 \left( \frac{E}{2} - 1 \right) + \frac{1}{2} = L_1(E) \]
\[ L_2 \left( \frac{E}{2} - 1 \right) + R_2 \left( \frac{E}{2} - 1 \right) = L_2(E). \] (11)

**Example 2.6.** Let \(E = 20\), then by considering the distance of 20 from zero as
we can drive all additive interaction of 20 as follows

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L | R |
| NP | 1 ~ 19 | P |
| P | 3 ~ 17 | P |
| P | 5 ~ 15 | NP |
| P | 7 ~ 13 | P |
| NP | 9 ~ 11 | P |

Now, in view of the representation (2), we can compute quadruple elements of 20 as

\[ a_{20} = 0, \quad b_{20} = 2, \quad c_{20} = 1, \quad d_{20} = 2, \]

using which we can obtain the S.C.E Model of 20 as follows

![S.C.E Model of 20](image)

**Figure 2. S.C.E Model of 20**

2.2. **Linking with Dusart’s Inequality.** Actually, the S.C.E Model by itself is slightly poor to prove Goldbach’s strong conjecture. To enrich it, we utilize Dusart’s inequality \[ 1 \]. To link up with this inequality, we consider the following two main inequalities called *Teeter Inequalities.*

**Lemma 2.7.** Let \( E \geq 17 \) be an even number with quadruple representation

\[
\frac{E}{4} = a_E + b_E + c_E + d_E,
\]

obtained by S.C.E model \[ 3 \]. Then we have

\[
\frac{E}{2} - 1.2551 \frac{E}{\ln E} + 1 < b_E + c_E + 2a_E < \frac{E}{2} - \frac{E}{\ln E} + 1, \tag{12}
\]

\[
\phi(E) = \frac{E}{\ln E} - 1 < b_E + c_E + 2d_E < 1.2551 \frac{E}{\ln E} - 1. \tag{13}
\]

**Proof.** The assertions of the lemma are direct consequences of relations (8), (9) and Dusart’s inequality \[ 1 \], respectively. \( \square \)

**Corollary 2.8.** By subtracting the inequality (13) from (12), we can conclude the following inequality

\[
\frac{E}{\ln E} - \frac{E}{4} - 1 < d_E - a_E < 1.2551 \frac{E}{\ln E} - \frac{E}{4} - 1. \tag{14}
\]

**Remark 2.9.** With the aid of a calculus approach, it turns out that the point \( x = 130.4574578 \) is the root of the following decreasing function
Thus, in view of (14), for even numbers \( E > 130 \) we can follow that
\[
d_E < a_E,
\]
and \( 0 < a_E \).

**Remark 2.10.** For all positive even number \( E \geq 17 \) we have
\[
\frac{E}{4} - 1.2551 \frac{E}{\ln E} + 1 < a_E.
\]

**Proof.** Obviously, since \( \frac{E}{4} - 1.2551 \frac{E}{\ln E} + 1 \notin \{ \frac{n}{2} \mid n \in \mathbb{W} \} \), then \( \frac{E}{4} - 1.2551 \frac{E}{\ln E} + 1 \neq a_E \) for all positive even number \( E \). In the sequel, we continue by contradiction. So, let \( E \) be a positive even number such that \( a_E < \frac{E}{4} - 1.2551 \frac{E}{\ln E} + 1 \) or
\[
2a_E < \frac{E}{2} - 2 \frac{1.2551E}{\ln E} + 2.
\]
If we subtract the inequality (17) from (12) we get
\[
1.2551 \frac{E}{\ln E} - 1 < b_E + c_E,
\]
which contradicts with the inequality (13). \( \Box \)

**Corollary 2.11.** For all positive even number \( E \geq 22864 \) we have
\[
b_E + c_E + d_E < a_E.
\]

**Proof.** First by the inequality (13) we have
\[
b_E + c_E + d_E < 1.2551 \frac{E}{\ln E} - 1,
\]
but by calculus we can see \( \frac{E}{4} - 1.2551 \frac{E}{\ln x} + 1 > 1.2551 \frac{E}{\ln x} - 1 \) for all real number \( x \geq 22864 \), then we can easily obtain (18) by previous remark. \( \Box \)

To obtain a lower and a upper bound for \( L_1(E), L_2(E), R_1(E) \) and \( R_2(E) \) individually with the aid of Dusart’s inequality, we first compute \( L_2(E) \) and the rest will be readily accessible. To this end, we take the Remark (2.5) into account and present the following two lemmas under the assumptions that for a given even number \( E \), firstly \( E/2 \) is also an even number, and secondly \( E/2 \) to be an odd number.

**Lemma 2.12.** Let \( E > 141 \) be an even number, and let \( E/2 \) be an even number, too. Then, the following inequalities hold true
\[
\left(19\right) \quad \frac{E}{\ln \left(\frac{E}{2}\right)} - 1 < L_2(E) < 1.2551 - \frac{E}{\ln \left(\frac{E}{2}\right)} - 1
\]
\[
\left(20\right) \quad \frac{E}{4} - 1.2551 \frac{E}{\ln \left(\frac{E}{2}\right)} + 1 < L_1(E) < \frac{E}{4} - \frac{E}{\ln \left(\frac{E}{2}\right)} + 1
\]
\[
\left(21\right) \quad \frac{E}{\ln E} - 1.2551 \frac{E}{\ln \left(\frac{E}{2}\right)} < R_2(E) < 1.2551 \frac{E}{\ln E} - \frac{E}{\ln \left(\frac{E}{2}\right)}
\]
\[
\left(22\right) \quad \frac{E}{4} - 1.2551 \frac{E}{\ln E} + \frac{E}{\ln \left(\frac{E}{2}\right)} < R_1(E) < \frac{E}{4} - \frac{E}{\ln E} + 1.2551 \frac{E}{\ln \left(\frac{E}{2}\right)}.
\]

**Proof.** First of all, let \( \frac{E}{2} \) be an even number, then in view of the relation (8), and by applying Dusart’s inequality (1) on \( \frac{E}{2} \) we have
\[
\frac{E}{\ln \left(\frac{E}{2}\right)} - 1 < L_2 \left(\frac{E}{2}\right) + R_2 \left(\frac{E}{2}\right) < 1.2551 \frac{E}{\ln \left(\frac{E}{2}\right)} - 1,
\]
and by relation (10) we thus arrive at the following inequality
\[
\frac{E}{\ln(E)} - 1 < L_2(E) < 1.2551 \frac{E}{\ln(E)} - 1.
\]

Since \( \frac{E}{2} - L_2(E) = L_1(E) \), we can compute (20) by multiplying all sides of (19) with \(-1\) and by adding with \( \frac{E}{4} \).

To get the inequality (21), based on the relation (8), it is sufficient to subtract Dusart’s inequality (1) applied on \( E \) from the inequality (19).

To yield the inequality (22), based on the relation (24), it is sufficient to subtract Dusart’s inequality (1) applied on \( \frac{E}{2} \) from the inequality (23).

Now, we present the required second lemma.

**Lemma 2.13.** Let \( E > 141 \) be an even number, and let \( E/2 \) be an odd number. Then, the following inequalities hold true

\[
\begin{align*}
(23) & \quad \frac{E}{\ln(E)} - 1 < L_2(E) < 1.2551 \frac{E}{\ln(E)} - 1 \\
(24) & \quad \frac{E}{4} - 1.2551 \frac{E}{\ln(E)} - 1 < L_1(E) < \frac{E}{4} - 1 \frac{E}{\ln(E)} + 1 \\
(25) & \quad \frac{E}{\ln(E)} - 1.2551 \frac{E}{\ln(E)} - 1 < R_2(E) < 1.2551 \frac{E}{\ln(E)} - 1 \\
(26) & \quad \frac{E}{4} - 1.2551 \frac{E}{\ln(E)} + 1 < R_1(E) < \frac{E}{4} - 1 \frac{E}{\ln(E)} + 1.2551 \frac{E}{\ln(E)} - 1
\end{align*}
\]

**Proof.** First of all, let \( \frac{E}{2} \) be an odd number, then in view of the relation (8), by applying Dusart’s inequality (1) on \( \frac{E}{2} - 1 \) we have
\[
\frac{E}{\ln(E)} - 1 < L_2(E) < 1.2551 \frac{E}{\ln(E)} - 1,
\]
and by relation (11) we thus arrive at the inequality (23).

On the other hand, since \( \frac{E}{2} - L_2(E) = L_1(E) \), we can compute (24) by multiplying all sides of (23) with \(-1\) and by adding with \( \frac{E}{4} \).

To get the inequality (25), based on the relation (8), it is sufficient to subtract Dusart’s inequality (1) applied on \( E \) from the inequality (23).

To yield the inequality (26), based on the relation \( \frac{E}{2} - R_2(E) = R_1(E) \), we follow the same reasoning that of (24). At the end, it is necessary to mention that all inequalities are strict because the parameters \( a_E, b_E, c_E, d_E \) always admit natural or half natural values.

3. **Proof of the Goldbach’s strong Conjecture by Using S.C.E Model**

As we well know, the Goldbach’s strong conjecture asserts that all positive even integers equal or greater than 4 can be expressed as the sum of two primes. It is necessary to mention that we prove this conjecture just for even numbers \( E \geq 22864 \). To prove this conjecture, it is enough to show that for all positive even number \( E \geq 22864 \) the element \( d_E \) in the corresponding S.C.E model never vanishes. To this end, we consider the following theorem.

**Theorem 3.1** (Goldbach). Let \( E \geq 22864 \) be a given even number and let the corresponding \( d_E \) be as in (2), then \( d_E > 0 \).
Proof. Without loss of generality, with the aid of Lemma (2.12), we just prove the theorem for the case when $E/2$ is also an even number. By using Lemma (2.13), the same reasoning yields the theorem for $E$ when $E/2$ to be an odd number.

We proceed the proof by contradiction, so we first let $E \geq 22864$ be a given even number such that $d_E = 0$. By equality (20) we can assume a real number $k > 0$ such that

$$a_E + b_E = \frac{E}{4} - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + 1 + k.$$  

Therefore, we obtain a complementary value of

$$c_E = 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} - 1 - k,$$

so we need to be $d_E = 0$. The above inequality holds if and only if for any real number $\epsilon > 0$ we have

$$(27) \quad 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} - 1 - k - \epsilon < c_E < 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} - 1 - k + \epsilon.$$  

The above inequality together with the inequalities (12), (13), (19) and (22) yields respectively

$$(28) \quad \frac{E}{2} - \frac{E}{\ln E} + 2 - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + k - \epsilon < b_E + 2a_E < \frac{E}{2} - 1.2551 \frac{E}{\ln E} + 2 - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + k + \epsilon$$  

$$(29) \quad 1.2551 \frac{E}{\ln E} - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + k - \epsilon < b_E + 2d_E < \frac{E}{\ln E} - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + k + \epsilon$$  

$$(30) \quad k - \epsilon < d_E < \frac{E}{2} - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + k + \epsilon$$  

$$(31) \quad \frac{E}{4} - \frac{E}{\ln E} + 1 + k - \epsilon < a_E < \frac{E}{4} - 1.2551 \frac{E}{\ln E} + \frac{E}{2 \ln\left(\frac{E}{2}\right)} - 1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} + 1 + k + \epsilon,$$

which all are contradictions because from (28), we deduce

$$1.2551 \frac{E}{\ln E} - \frac{E}{\ln E} < 2\epsilon,$$

and from (29), we obtain

$$1.2551 \frac{E}{\ln E} - \frac{E}{\ln E} < 2\epsilon,$$

and from (30), we arrive at

$$1.2551 \frac{E}{2 \ln\left(\frac{E}{2}\right)} - \frac{E}{2 \ln\left(\frac{E}{2}\right)} < 2\epsilon.$$
and from (31), we see

$$\frac{1.2551}{\ln E} - \frac{E}{\ln E} + \frac{E}{\ln \left(\frac{E}{2}\right)} - \frac{E}{\ln \left(\frac{E}{2}\right)} < 2\epsilon.$$  

□

References

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