The Information Flow Problem on Clock Networks

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Abstract. The information flow problem on a network asks whether \( r \) senders, \( v_1, v_2, \ldots, v_r \) can each send messages to \( r \) corresponding receivers \( v_{r+1}, \ldots, v_{n+r} \) via intermediate nodes \( v_{r+1}, \ldots, v_n \). For a given finite \( R \subseteq \mathbb{Z}^+ \), the clock network \( N_s(R) \) has edge \( v_i v_k \) if and only if \( k > r \) and \( k - i \in R \). We show that the information flow problem on \( N_s(\{1, 2, \ldots, r\}) \) can be solved for all \( n \geq r \). We also show that for any finite \( R \) such that \( \gcd(R) = 1 \) and \( r = \max(R) \), we show that the information flow problem can be solved on \( N_s(R) \) for all \( n \geq 3r^3 \). This is an improvement on the bound given in [10] and answers an open question from [9].

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1. The Information Flow Problem

The information flow problem (Definition 1.4) is an important problem for multiuser information theory. This problem was introduced in [1] to formalise the multiple unicast problem. It was shown that the information flow problem is equivalent to the guessing number of a related digraph [9]. The same paper poses an open question regarding the guessing number of a class of digraphs known as clock digraphs (Definition 1.7). Corollary 4.7 answers this question.

Definition 1.1. A network of length \( n \) and width \( r \) is an acyclic digraph \( N \) with vertex set \( \{v_i\}_{i=1}^{n+r} \) such that the input nodes (vertices \( v_1, v_2, \ldots, v_r \)) have no incoming edges. Vertices \( v_{n+1}, v_{n+2}, \ldots, v_{n+r} \) are called the output nodes and vertices \( v_{r+1}, v_{r+2}, \ldots, v_n \) are called intermediate nodes. For any \( r < k \leq n+r \), let \( \Gamma(k) \) denote set of all indices, \( i \), such that \( v_i v_k \) is an edge.

For any positive integer \( m \), let \([m]\) denote the set \( \{1, 2, 3, \ldots, m\} \).

Definition 1.2. For any network \( N \) and any integer \( s \geq 2 \), a circuit on \( N \) over \( \mathbb{Z}_s \), is a \( n \)-tuple of functions \( F = (f_{r+1}, f_{r+2}, \ldots, f_{n+r}) \),

\[
f_k : \mathbb{Z}_s^{\Gamma(k)} \to \mathbb{Z}_s \quad \forall \ r < k \leq n+r,
\]

where \( n \) and \( r \) are the length and width respectively of \( N \). For each input \( c = (c_1, c_2, \ldots, c_r) \in \mathbb{Z}_s^r \), let \( X = (X_1, X_2, \ldots, X_{n+r}) \) denote the unique \((n+r)\)-tuple in \( \mathbb{Z}_s^{n+r} \) such that \( X_i = c_i \) for all \( i \in [r] \) and

\[
X_k = f_k (X_i \mid i \in \Gamma(k)) \quad \forall \ r < k \leq n+r.
\]

\( X \) is called the valuation of \( F \).

Definition 1.3. A circuit, \( F = (f_{r+1}, f_{r+2}, \ldots, f_{n+r}) \), is called linear if and only if each function \( f_k \) is a linear map. For any linear circuit \( F \), let \( M_F \) denote the \( R \)-circuit matrix of \( F \); the linear map \( M_F : \mathbb{Z}_s^r \to \mathbb{Z}_s^{n+r} \) such that \( X = M_F(c) \) for all inputs \( c \in \mathbb{Z}_s^r \), i.e. \( M_F \) is a \((n+r) \times r\) matrix such that

\[
X^T = M_Fc^T
\]

where \( X^T \) and \( c^T \) are the column vectors of the valuation \( X \) and the input \( c \) respectively. The first \( r \) rows of the \( R \)-circuit matrix \( M_F \) are a copy of the \( r \times r \) identity matrix, \( I_r \).
**Definition 1.4.** A network $N$ of width $r$ is **s-solvable** if and only if there exists a circuit on $N$ over $\mathbb{Z}_s$ such that for all inputs $c \in \mathbb{Z}_s^r$, the valuation satisfies

$$(X_1, X_2, \ldots, X_r) = c = (X_{n+1}, X_{n+2}, \ldots, X_{n+r})$$

A network $N$ of width $r$ is **linearly s-solvable** if and only if there exists a linear circuit $F$ on $N$ over $\mathbb{Z}_s$ such that the final $r$ rows of $M_F$ are a copy of $I_r$. For a given network $N$ and an integer $s \geq 2$, the Information Flow Problem asks whether or not $N$ is s-solvable. Similarly, the Linear Information Flow Problem asks whether or not $N$ is linearly s-solvable.

It is natural to consider the information flow problem as an information theory problem in the following way. Each input node, $v_i$, is a sender trying to send a message to its corresponding receiver at node $v_{n+i}$ via the network of internal nodes. The elements of the group $\mathbb{Z}_s$ correspond to the $s$ distinct possible messages that could be sent along each edge. There is a traditional method for solving the information flow problem, called “routing”, in which each intermediate node simply passes on one of the messages it receives. A network can only be solved by routing if and only if there exist vertex disjoint paths from each sender to its corresponding receiver. There are many examples in which a network is solvable, but cannot be solved by routing alone [3,6]. Instead we allow each non-input node, $v_k$, to perform some function, $f_k$, on the messages it receives from nodes $\Gamma(k)$. Each node $v_i$ must send the same message to all nodes $v_k$ such that $i \in \Gamma(k)$. Linear circuits are of interest because they are fast to compute and linear circuits are sufficient to solve a large family of networks (Theorems 2.5 and 4.6).

The information flow problem also has an application to computing the guessing number [4,7] and the information defect [2,5,8] of directed graphs. Specifically, for any network $N$ with input nodes $v_1, v_2, \ldots, v_r$ and output nodes $v_{r+1}, v_{r+2}, \ldots, v_{n+r}$, let $G_N$ denote the digraph obtained by identifying vertex $v_i$ with $v_{n+i}$, for all $1 \leq i \leq r$. The relationship between the s-solvability of a network $N$ and the guessing number (and information defect) of $G_N$ is presented in Theorem 1.5 which originally appears in [9]. Note that for our purposes it does not matter if edge $v_{n+i}v_k$ is replaced with $v_i v_k$ (nor would it make any difference if both edges were included) because, for a circuit which solves the network, the valuation would satisfy $X_i = X_{n+i}$.

**Theorem 1.5.** [2] For any network $N$ of length $n$ and width $r$, if the guessing number of $G_N$ is denoted $\text{gn}(G_N, s)$ and the information defect of $G_N$ is denoted $b(G_N, s)$, then

$$\text{gn}(G_N, s) \leq r \quad \text{and} \quad b(G_N, s) \geq n - r.$$  

We get the equality $\text{gn}(G_N, s) = r$ if and only if $N$ is s-solvable. Moreover, if $N$ is linearly s-solvable then $b(G_N, s) = n - r$.

**Definition 1.6.** For any finite $R \subset \mathbb{Z}^+$ let $r = \max(R)$. For any integer $n > r$ let $N_n(R)$ denote the **clock** network; the network with vertex set $V = \{v_1, v_2, v_3, \ldots, v_{n+r}\}$ and edge set

$$E = \{v_i v_k \mid k > r \text{ and } k - i \in R\}.$$
The network $N_n([r])$ is called the full clock network. To simplify notation, we sometimes write $N_n(r) = N_n([r])$.

**Definition 1.7.** For any finite $R \subset \mathbb{Z}^+$ let $r = \max(R)$. For any integer $n > r$ let $G_{\text{clock}}(n, R)$ denote the clock digraph which has $n$ vertices $\{v_i\}_{i=1}^n$, where $v_i v_j$ is an edge if and only if $j - i$ (modulo $n$) is in $R$. To simplify notation, for any positive integer $r$, we say $G_{\text{clock}}(n, r) = G_{\text{clock}}(n, [r])$.

The clock network inherits its name from the clock digraph, $G_{\text{clock}}(n, R)$, as defined in [9]. When $|R| = 2$, the clock digraph is also known as the Cayley graph Cay$(n, R)$, or the “shift graph” [10]. The clock digraph, $G_{\text{clock}}(n, R)$, can be obtained from the clock network, $N_n(R)$, by identifying nodes $v_i$ and $v_{n+i}$ for all $1 \leq i \leq r$. i.e.

$G_{\text{clock}}(n, r) = G_{N_n(R)}$.

We show in Theorem 2.5 that $N_n(r)$ is always linearly $s$-solvable. By Theorem 1.5 (which originally appears in [9]) this implies that the guessing number and information defect of $G_{\text{clock}}(n, r)$ are $r$ and $n - r$ respectively.

**Proposition 1.8.** For a given finite $R \subset \mathbb{Z}^+$, let $r = \max(R)$, let $M$ be a $(n+r) \times r$ matrix with entries in $\mathbb{Z}_s$ and for $i = 1, 2, \ldots, n+r$, let $\omega(i)$ be the $i^{th}$ row of $M$. If

- the first $r$ rows of $M$ form a copy of the identity matrix $I_r$, and
- for all $r < k \leq n+r$, the row $\omega(k)$ is a linear combination of the rows $\{\omega(i) \mid k - i \in R\}$,

then there exists a circuit, $F$, on $N_n(R)$ such that $M_F = M$.

**Proof.** For $k = r + 1, r + 2, \ldots, n + r$, and $j \in R$, let $\lambda_{kj} \in \mathbb{Z}_s$ be the constants by which $\omega(k)$ is a linear combination of $\{\omega(k - j) \mid j \in R\}$. i.e.

$\omega(k) = \sum_{j \in R} \lambda_{kj} \omega(k - j)$.

![Figure 2. The full clock network $N_8(2)$.](image)
For all pairs \((k, j)\) such that \(k - j \not\in R\) we set \(\lambda_{kj} = 0\). Now let \(F = (f_{r+1}, f_{r+2}, \ldots, f_{n+r})\) be the circuit on \(N_R(n)\) defined by

\[
X_k = f_k \left( X_i \mid i \in \Gamma(k) \right) = \sum_{j \in R} \lambda_{kj} X_{k-j},
\]

and for \(i = 1, 2, \ldots, n + r\), let \(\omega'_i\) be the \(i\)th row of \(M_F\). Since the first \(r\) rows of any \(R\)-circuit matrix form a copy of \(I_r\), we must have \(\omega_i = \omega'_i\) for \(i = 1, 2, \ldots, r\). Then, inductively, for all \(k > r\) we must have \(\omega_k = \sum_{j \in R} \lambda_{kj} \omega_{k-j} = \sum_{j \in R} \lambda_{kj} \omega'_{k-j} = \omega'_k\).

\[\square\]

2. Full Clock Networks

As Theorem 2.5 shows, the full clock network is linearly \(s\)-solvable for all \(s\). This is equivalent to Proposition A in [9], however their proof is incomplete (see Example 2.6). We show that the full clock network is linearly \(s\)-solvable by finding a valid \([r]\)-circuit matrix explicitly.

**Definition 2.1.** For any integers \(a, b > 0\) we can define \(J_{a,b}\) in the following recursive manner. If \(a = b\), then \(J_{a,a} = I_a\) (the \(a \times a\) identity matrix). Otherwise:

- if \(a < b\) then \(J_{a,b} = [J_{a,b-a}, I_a]\),
- if \(a > b\) then \(J_{a,b} = [J_{a-b,b}, I_b]\).

So if \(a > b\) or \(a < b\), then \(J_{a,b}\) is either the horizontal concatenation of \(J_{a,b-a}\) and \(I_a\) or the vertical concatenation of \(J_{a-b,b}\) and \(I_b\) respectively. For example: \(J_{4,3}\) and \(J_{30,43}\) are depicted in Figure 2.

**Proposition 2.2.** If \(A\) is the topleft-most \(a \times a\) sub-matrix of \(J_{n,r}\), then \(|\det(A)| = 1\).

**Proof.** Let \(A\) be the topleft-most \(a \times a\) submatrix of \(J_{n,r}\). We now construct the pair of integers \(p\) and \(q\) in the following way. Initially let \(x = n\) and \(y = r\). Then iteratively perform the following process.

while \(x \geq a\) or \(y \geq a\):

if \(x > y\) replace \(x\) with \(x - y\),

otherwise replace \(y\) with \(y - x\).

Throughout this process (by Definition 2.1) topleft-most \(x \times y\) submatrix of \(A\) is always a copy of \(J_{x,y}\). As soon as both \(x\) and \(y\) are less than or equal to \(a\), we set \(p = x\) and \(q = y\), and terminate this process. Just before the final iteration, we must have had one of \(x\) or \(y\) greater than \(a\), so \(a \leq \max(x, y) = p + q\). Now \(A\) must be in the following form.

\[
A = \begin{bmatrix} J_{p,q} & P \\ Q & S \end{bmatrix}
\]
Figure 4. In the proof of Proposition 2.2, $J_{p,q}$ is the topleft-most $p \times q$ submatrix of $A$ and $A$ is the topleft-most $a \times a$ submatrix of $J_{n,r}$.

where $P$, $Q$ and $S$ are matrices with dimensions $p \times (a-q)$, $(a-p) \times q$ and $(a-p) \times (a-q)$ respectively. Now, there are two cases:

- If $P$ is the left-most $(a-q)$ columns of a copy of $I_p$, then $[Q, S]$ is the topleft-most $(a-p) \times a$ submatrix of a large identity matrix.
- If $Q$ is the top-most $(a-p)$ rows of a copy of $I_q$, then $[P S]$ is the topleft-most $a \times (a-q)$ submatrix of a large identity matrix.

In either case, all the entries of $S$ must be zero because $a-p < q$ and $a-q < p$. Now consider $J_{p,q}$ in which the bottomright-most square submatrix must be a copy of an identity matrix. Explicitly, for any integer $b$ such that $0 \leq b \leq \min(p, q)$, we have

$$J_{p,q} = \begin{bmatrix} \ast & \ast & \ast \\ \ast & I_b & \ast \\ \ast & \ast & I_{a-q} \end{bmatrix},$$

where $\ast$ denotes arbitrary entries. In particular $J_{p,q}$ has this form for $b = p + q - a$. Substituting Equation (2) into Equation (1), we see that $A$ has the form:

$$A = \begin{bmatrix} J_{p,q} & P \\ Q & 0 \end{bmatrix} = \begin{bmatrix} \ast & \ast & I_{a-q} \\ \ast & I_{p+q-a} & 0 \\ I_{a-p} & 0 & 0 \end{bmatrix}$$

where a 0 denotes a submatrix full of zeros, and $\ast$ denotes a submatrix with arbitrary entries. The only non-zero terms in the Leibniz formula for the determinant of $A$ must come exclusively from the submatrices labelled $I_{a-q}$, $I_{p+q-a}$ and $I_{a-p}$. Therefore

$$|\det(A)| = |\det(I_{a-q})| \times |\det(I_{p+q-a})| \times |\det(I_{a-p})| = 1.$$

Definition 2.3. For any positive integers $n$ and $r$ with $n > r$ we define the $(n+r) \times r$ matrix $M_{n,r}$ formed by concatenating a copy of $I_r$ on top of $J_{n,r}$, i.e.

$$M_{n,r} = \begin{bmatrix} I_r \\ J_{n,r} \end{bmatrix}.$$

Proposition 2.4. For any positive integers $n > r$, if $M$ is an $r \times r$ submatrix of $M_{n,r}$ formed by $r$ consecutive rows then $|\det(M)| = 1$.

Proof. Let $M$ be the rows $\omega_{a+1}, \omega_{a+2}, \ldots, \omega_{a+r}$. There are two cases: either $0 \leq a < r$ or $r \leq a \leq n$. 

□
• If \( a < r \) then \( M \) consists of the final \( r - a \) rows of \( I_r \) followed by the initial \( a \) rows of \( \mathcal{I}_{n,r} \). In this case, \( M \) must have the form:

\[
M = \begin{bmatrix}
0 & I_{r-a} \\
A & *
\end{bmatrix}
\]

Where \( A \) is the top-leftmost \( a \times a \) submatrix of \( \mathcal{I}_{n,r} \) and * denotes a submatrix with arbitrary entries. In this case, by Proposition 2.2, we have

\[
|\det(M)| = |\det(I_{r-a})| \times |\det(A)| = 1
\]

• If \( a \geq r \) then \( M \) must have the form:

\[
M = \begin{bmatrix}
* & I_b \\
I_{r-b} & 0
\end{bmatrix}
\]

where * denotes a submatrix with arbitrary entries and \( b \) is the remainder when \( n - a \) is divided by \( r \). In this case we have

\[
|\det(M)| = |\det(I_{r-b})| \times |\det(I_b)| = 1.
\]

\[\square\]

**Theorem 2.5.** For any \( n \geq r > 0 \) and any \( s \geq 2 \), the full clock network, \( N_n(r) \), is linearly \( s \)-solvable.

**Proof.** Consider the matrix \( \mathfrak{M}_{n,r} \) as defined in Definition 2.3. By Proposition 2.4, for any \( s \), the integer span of any \( r \) consecutive rows of \( \mathfrak{M}_{n,r} \) is all \( \mathbb{Z}_r^{s} \). So any row can be expressed as a linear combination of the preceding \( r \) rows. Moreover, the first \( r \) rows of \( \mathfrak{M}_{n,r} \) form a copy of \( I_r \). Therefore \( \mathfrak{M}_{n,r} \) satisfies the conditions of Proposition 1.8, and so there is a circuit \( F \) on \( N_n(r) \) such that \( M_F = \mathfrak{M}_{n,r} \). This circuit linearly solves \( N_n(r) \) because the final \( r \) rows of \( \mathfrak{M}_{n,r} \) form a copy of \( I_r \). \( \square \)

**Example 2.6.**

Let \( F \) be a circuit on \( N_n(r) \) over \( \mathbb{Z}_s \) such that the valuation of \( F \) satisfies

\[
X_i + X_{i+1} + \cdots + X_{i+r} \equiv 0 \pmod{s} \quad \text{for } i = 1, 2, 3, \ldots, n - r
\]

for any input \( c \in \mathbb{Z}_s^r \). To see that this circuit does not solve \( N_n(r) \) in general, observe that for \( n = 7 \) and \( r = 2 \), it does not solve \( N_7(2) \). Explicitly, for any \( c = (c_1, c_2) \in \mathbb{Z}_2^2 \), the valuation must satisfy:

\[
\begin{align*}
X_1 &= c_1 \\
X_2 &= c_2
\end{align*}
\]

because for any valuation of a circuit on \( N_7(2) \), we have \((X_1, X_2) = c\). Moreover, for \( j = 3, 4, 5, 6 \) and \( 7 \), we can deduce:

\[
\begin{align*}
X_3 &= -c_1 - c_2, \\
X_4 &= c_1, \\
X_5 &= c_2, \\
X_6 &= -c_1 - c_2 \\
\text{and } X_7 &= c_1,
\end{align*}
\]

because \( X_{j-2} + X_{j-1} + X_j \equiv 0 \). Finally, if \( F \) solved \( N_7(2) \), then we would have:

\[
\begin{align*}
X_8 &= c_1 \\
\text{and } X_9 &= c_2.
\end{align*}
\]

However, this is not possible; \( X_9 = c_2 \) cannot be determined from only \( X_7 = c_1 \) and \( X_8 = c_1 \).
A modulo 2. So, by Proposition 1.8, \( A \) defined as follows.

\[
\begin{array}{c|cccccccccc}
 n & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
\text{Is } N_n(\{1,3\}) \text{ linearly 2-solvable?} & \times & \times & 
\checkmark & 
\checkmark & 
\checkmark & 
\checkmark & 
\checkmark & 
\checkmark & \checkmark \\
\text{Is } N_n(\{1,3\}) \text{ linearly 3-solvable?} & 
\times & \times & \times & 
\checkmark & 
\checkmark & 
\checkmark & 
\checkmark & 
\checkmark & \checkmark \\
\end{array}
\]

Figure 5. The linear 2-solvability and linear 3-solvability of \( N_n(\{1,3\}) \) for all \( n \geq 4 \).

3. Analysis of a specific case

In this section we investigate the 2-solvability and 3-solvability of the network \( N_n(\{1,3\}) \) for various values \( n \). Firstly, we consider \( n = 7 \) and \( n = 8 \) in the following example.

Example 3.1. Let \( n = 7 \), \( m = 3 \) and \( R = \{1,3\} \) and consider the following two matrices \( A \) and \( B \) defined as follows.

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \quad \text{and} \quad
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
2 & 1 & 0 \\
0 & 2 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

The matrix \( A \) is constructed so that the \( i \)th row is the sum of the \((i - 1)\)th row and the \((i - 3)\)th row modulo 2. So, by Proposition 1.9, \( A \) is a valid \{1,3\}-circuit matrix (over \( \mathbb{Z}_2 \)) and since the bottom 3 rows of \( A \) form an identity matrix, this demonstrates that \( N_7(\{1,3\}) \) is linearly 2-solvable. Similarly the matrix \( B \) demonstrates that \( N_8(\{1,3\}) \) is linearly 3-solvable. It can be verified by a brute force computer search that \( N_7(\{1,3\}) \) is not linearly 3-solvable and \( N_8(\{1,3\}) \) is not linearly 2-solvable.

In general, the \( s \)-solvability of a network depends on \( s \). However, for any \( n \geq 12 \), we can construct a \{1,3\}-circuit matrix of length \( n \) which is valid over \( \mathbb{Z}_s \) for any \( s \geq 2 \) in the following way. For \( n \equiv 0,1,2 \) (mod 3) iteratively concatenate copies of \( I_3 \) to the bottom of \( I_3 \), \( M_{10} \) or \( M_{14} \) respectively, where \( M_{10} \) and \( M_{14} \) are given in Figure 4. We know that no such \{1,3\}-circuit matrix exists for \( n = 7 \) nor \( n = 11 \) because (by brute force computer search) we computed that \( N_7(\{1,3\}) \) is not linearly 3-solvable and \( N_{11}(\{1,3\}) \) is not 2-solvable.

4. General Clock Networks

We saw in the previous section that the network \( N_n(\{1,3\}) \) is linearly \( s \)-solvable for any \( s \), for all \( n \geq 12 \). In this section we generalise this result to arbitrary finite sets of positive integers. Specifically, we determine for which finite \( R \subset \mathbb{Z}^+ \), does there exist a constant \( n_0 \) such that \( N_n(R) \) is \( s \)-solvable for all \( s \) and all \( n \geq n_0 \). We deduce (by Lemma 4.1 and Corollary 4.2) that such an integer \( n_0 \) exists if and only if \( \text{gcd}(R) = 1 \).

Lemma 4.1. If \( n \) is not a multiple of \( \text{gcd}(R) \), then \( N_n(R) \) is not \( s \)-solvable for any \( s \geq 2 \).

Proof. Let \( d = \text{gcd}(R) > 1 \). By definition, each edge \( v_i v_k \) only joins vertices such that \( i \equiv k \) (mod \( d \)). Therefore \( N_n(R) \) is disconnected with at least one component for each residue modulo \( d \). Now consider some index \( a \), and an input \( c = (c_1, c_2, \ldots, c_r) \). If we keep \( c_i \) constant for all \( i \neq a \) and let \( c_a \) vary, then the valuation will only change on vertices in the same component as \( v_a \). Since \( n \) is not a multiple of \( d \),

\[(n + a) - a = n \neq 0 \text{ (mod } d)\].
So the input node, $v_a$, and its corresponding output node, $v_{n+a}$, are in a different components of $N_n(R)$. Therefore $N_n(R)$ is not $s$-solvable for any $s \geq 2$. □

Now consider any $R \subset \mathbb{Z}^+$ such that $\gcd(R) > 1$, there are an infinite number of integers $n$ which are not a multiple of $\gcd(R)$. By Lemma 4.1 this is an infinite number of integers $n$ such that $N_n(R)$ is not $s$-solvable (for any $s$). Therefore there cannot exist any $n_0$ such that $N_n(R)$ is $s$-solvable for all $n \geq n_0$. However, if $n$ is a multiple of $d = \gcd(R) > 1$, then the network $N_n(R)$ is a disjoint union of $d$ copies of

$$N' = N_{n/d}(R') \quad \text{where} \quad R' = \{j/d \mid j \in R\}.$$ 

So $N_n(R)$ is $s$-solvable if and only if $N'$ is $s$-solvable. Since $\gcd(R') = 1$, it suffices now to consider only the cases that $\gcd(R) = 1$. We now make the following definition and propositions, used in Theorem 4.6 and Corollary 4.7.

**Definition 4.2.** Let $s \geq 2$ be an integer, let $R$ be a finite set of positive integers, and let $r = \max(R)$.

- An $R$-atomic matrix is any $r \times r$ matrix, with entries in $\mathbb{Z}_s$, of the form:

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\alpha_r & \alpha_{r-1} & \alpha_{r-2} & \cdots & \alpha_1
\end{bmatrix}
$$

such that $\alpha_j = 0$ for all $j \notin R$.  

**Figure 6.** Matrices $M_{10}$ and $M_{14}$ show that $N_{10}(\{1,3\})$ and $N_{14}(\{1,3\})$ are linearly $s$-solvable for any $s \geq 2$. 

$$
M_{10} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
-1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \text{and} \quad
M_{14} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & -1 \\
0 & 0 & 1 \\
1 & 0 & -1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$
A $R$-step matrix is any $r \times r$ matrix, with entries in $\mathbb{Z}_s$, formed by starting with $I_r$, and then for some $1 \leq t \leq r$, replacing the $t^{th}$ row with $[\beta_1, \beta_2, \ldots, \beta_r],$

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\beta_1 & \beta_2 & \cdots & \beta_t & \cdots & \beta_r \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
$$

where $\beta_i$ is non-zero only if there is some $j \in R$ such that $i + j \equiv t \pmod{r}$. Since $r \in R$, we always allow $\beta_t$ to be non-zero.

For any $1 \leq t \leq r$, the $t$-toggle matrix is the following $r \times r$ matrix, $T(t)$, with entries in $\mathbb{Z}_s$.

$$
T(t) = \begin{bmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 1
\end{bmatrix}
$$

$i.e.$ The $t$-toggle matrix is formed from $I_r$ by replacing row $t$ with a row of $-1$s. A matrix is called a toggle matrix iff it is a $t$-toggle matrix for some $t$.

**Proposition 4.3.** Any $R$-step matrix can be expressed as a product of $r$ $R$-atomic matrices.

**Proof.** Let $P$ denote the only $R$-atomic matrix which is also a permutation matrix; the $R$-atomic matrix for which $\alpha_r = 1$ and $\alpha_i = 0$ for all $i < r$. For any $1 \leq t \leq r$ consider the product

$$A_t A_{r-1} \ldots A_t \ldots A_2 A_1 = S,$$

where $A_t$ is an arbitrary $R$-atomic matrix and $A_i = P$ for all $i \neq t$. This product, $S$, is an arbitrary $R$-step matrix. \(\square\)

**Proposition 4.4.** If $\gcd(R) = 1$ and $r = \max(R) > 1$ then for any $1 \leq t \leq r$ the $t$-toggle matrix can be expressed as a product of $(2r - 3)$ $R$-step matrices.

**Proof.** We inductively define a sequence of subsets,

$$U_2 \subset U_3 \subset U_4 \subset \cdots \subset U_r = [r],$$

in the following manner. Let $U_2 = \{x,t\}$ where $x \in [r]$ is chosen so that $t - x \pmod{r} \in R$. For $k = 3, 4, 5, \ldots, r$, iteratively define $U_k = U_{k-1} \cup \{b\}$ for some $b \in [r] \setminus U_{k-1}$ such that there exists some $a \in U_{k-1}$ such that $a - b \pmod{r} \in R$. We know $a$ and $b$ exist because $\gcd(R) = 1$. Now let $S_k$ be the matrix formed from $I_r$ by replacing the $t^{th}$ row with

$$(x_1, x_2, x_3, \ldots, x_n) \quad \text{where} \quad x_i = \begin{cases} 
-1 & : \text{if } i \in U_k \\
0 & : \text{otherwise.}
\end{cases}
$$

Now we prove that $S_k$ can be expressed as a product of $(2k - 3)$ $R$-step matrices, by induction on $k = 2, 3, 4, \ldots, r$. For the base case ($k = 2$), $S_2$ is a $R$-step matrix. For the inductive step,

$$S_k = E_{ab}(-1)S_{k-1}E_{ab}(1)$$

where $E_{ij}(\lambda)$ is the matrix formed from $I_r$ by replacing the $ij^{th}$ entry with $\lambda$. Note that $E_{ab}(1)$ and $E_{ab}(-1)$ are both $R$-step matrices because $a - b \pmod{r} \in R$, and $S_{k-1}$ can be expressed as a product of $(2(k - 1) - 3)$ $R$-step matrices by the inductive assumption. Therefore $S_k$ can be expressed as a product of

$$1 + (2(k - 1) - 3) + 1 = 2k - 3 \quad \text{R-step matrices.}$$
This completes the induction. For \( k = r \) we have \( U_r = \{1, 2, \ldots, r\} \) and so the \( t \)-toggle matrix is \( T(t) = S_r \), which can be expressed as a product of \( (2r - 3) \) \( R \)-step matrices.

\[ \square \]

**Proposition 4.5.** Any \( r \times r \) permutation matrix can be expressed as the product of at most \( \frac{3r}{2} \) toggle matrices.

**Proof.** First we show that an arbitrary \( k \)-cycle can be expressed as the product of \( k + 1 \) toggle matrices. Explicitly, if \( Q \) is the \( r \times r \) matrix corresponding to the \( k \) cycle, \( (a_1, a_2, \ldots, a_k) \), can be expressed as the product

\[ Q = T(a_1)T(a_2)T(a_3) \cdots T(a_{k-1})T(a_k)T(a_1). \]

Now consider the cyclic decomposition of the permutation; the permutation expressed as the composition of at most \( n/2 \) cycles, such that the sum of the lengths of these cycles is at most \( n \). If each of these cycles are expressed as a product of toggle matrices, then this is

\[ \square \]

**Theorem 4.6.** For any finite \( R \subset \mathbb{Z}^+ \), the network \( N_n(R) \) is linearly \( s \)-solvable if and only if the identity matrix can be expressed as a product of \( n \) \( R \)-atomic matrices with entries in \( \mathbb{Z}_s \).

**Proof.** For any valuation, \( X = (X_1, X_2, \ldots, X_n) \), of a linear circuit on \( N_n(R) \), let \( Y_i \) denote the column vector \( Y_i = (X_{i+1}, X_{i+2}, X_{i+3}, \ldots, X_{i+r})^T \) for \( i = 0, 1, 2, \ldots, n \). Since \( f_k \) is linear,

\[ X_k = f_k (X_{k-j} \mid j \in R) = \sum_{j \in R} \alpha_{kj} X_{k-j}. \]

Therefore we must have \( Y_i = A_i Y_{i-1} \) where \( A_i \) is a \( R \)-atomic matrix. Inductively this implies that \( Y_i = A_i A_{i-1} \ldots A_2 A_1 Y_0 \) for all \( i \geq 0 \) and thus \( Y_n = A Y_0 \), where \( A = A_n A_{n-1} \cdots A_2 A_1 \) is a product of \( n \) \( R \)-atomic matrices. For all inputs \( c \in \mathbb{Z}_s^r \) we have

\[ F(c) = Y_n = A Y_0 = A c. \]

So if \( X \) is the valuation of a circuit which linearly \( s \)-solved \( N_n(R) \), then \( c = Ac \) for all \( c \in \mathbb{Z}_s \). Hence \( A = I_r \), and \( A \) is a product of exactly \( n \) \( R \)-atomic matrices. Conversely the function \( f_k \) can be reconstructed from the \( R \)-atomic matrix \( A_{k-r} \), for each \( k = r + 1, r + 2, \ldots, n + r \), so this construction is reversible.

\[ \square \]

**Corollary 4.7.** Let \( R \) be any finite set of positive integers with \( \gcd(R) = 1 \) and let \( r = \max(R) \). For any \( n \geq 3r^3 \), the network \( N_n(R) \) is linearly \( s \)-solvable for any integer \( s \geq 2 \).

**Proof.** It suffices to show that the identity matrix can be expressed as a product of exactly \( n \) \( R \)-atomic matrices. Let \( P \) denote the only atomic matrix which is also a permutation matrix; the atomic matrix for which \( \alpha_r = 1 \) and \( \alpha_i = 0 \) for all \( i < r \). Note that \( P \) is a \( r \)-cycle and so \( P^r = I_r \). Let \( Q = P^{-n} \) and note that \( Q \) is a permutation matrix. By Proposition 4.5, we can write \( Q \) as a product of \( \leq \frac{3r}{2} \) toggle matrices. By Proposition 4.3, we can write each of these toggle matrices as a product of \( (2r - 3) \) \( R \)-step matrices, and by Proposition 4.3, we can write each of these step matrices as a product of \( r \) \( R \)-atomic matrices. Therefore \( Q \) can be expressed as a product of \( kr(2r - 3) \) \( R \)-atomic matrices. Since \( k \leq \frac{3r}{2} \) and \( n \geq 3r^3 > \frac{3r}{2}r(2r - 3) \), we must have \( n - kr(2r - 3) \geq 0 \) and so

\[ Q \times P^{n-kr(2r-3)} = (Q \times P^{n}) \times (P^{r})^{k(2r-3)} = I_r. \]

Since \( P \) is a \( R \)-atomic matrix and \( Q \) can be expressed as a product of \( kr(2r - 3) \) \( R \)-atomic matrices, we can express \( I_r \) as a product of \( n \) \( R \)-atomic matrices.

\[ \square \]

For finite \( R \subset \mathbb{Z}^+ \) with \( \gcd(R) = 1 \), let \( n_0 = n_0(R) \) be the minimum integer such that \( N_n(R) \) is \( s \)-solvable for all \( s \geq 2 \) for all \( n \geq n_0 \). Corollary 4.7 shows that \( n_0 \) is well defined and that \( n_0 \leq 3r^3 \) (where \( \max(R) = r \)). Theorem 2.5 shows that \( n_0([r]) = r \) and in Section 3, we deduced that \( n_0([1, 3]) = 12 \). The value \( n_0(R) \) for \( R \) in general, remains an open question. We conclude this section with an example which demonstrates that the cubic bound \( n_0 \leq 3r^3 \) in Corollary 4.7 cannot be replaced with any bound less than \( r^2 - r \).
Example 4.8. For any integer \( r \geq 3 \), let \( n = r^2 - r - 1 \) and consider the networks \( N = N_n(\{1, r\}) \) and \( N' = N_n(\{1, r - 1\}) \), and consider the digraphs \( G = G_{\text{clock}}(n, \{1, r\}) \) and \( G' = G_{\text{clock}}(n, \{1, r - 1\}) \). Using Theorem 1.5, the existence of the acyclic network \( N' \) of width \( r - 1 \) such that \( G' = G_{N'} \) implies that
\[
\text{gn}(G', s) \leq r - 1.
\]
Note that \( r(r - 1) \equiv 1 \) modulo \( n \), so \( G \) and \( G' \) are isomorphic, and so \( \text{gn}(G, s) = \text{gn}(G', s) < r \). Now using Theorem 1.5 again (since \( G = G_N \) we can conclude that \( N \) is not \( s \)-solvable. Thus
\[
\text{n}_0(\{1, r\}) \geq r^2 - r,
\]
for all \( r \geq 3 \).

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