A GENERALIZED CATONI’S M-ESTIMATOR UNDER FINITE α-TH MOMENT ASSUMPTION WITH α ∈ (1, 2)

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ABSTRACT. We generalize the M-estimator put forward by Catoni in his seminal paper [4] to the case in which samples can have finite α-th moment with α ∈ (1, 2) rather than finite variance, our approach is by slightly modifying the influence function φ therein. The choice of the new influence function is inspired by the Taylor-like expansion developed in [5]. We obtain a deviation bound of the estimator, as α → 2, this bound is the same as that in [4]. Experiment shows that our generalized M-estimator performs better than the empirical mean estimator, the smaller the α is, the better the performance will be. As an application, we study an ℓ1 regression considered by Zhang et al. [22] who assumed that samples have finite variance, and relax their assumption to be finite α-th moment with α ∈ (1, 2).

1. INTRODUCTION

Let $X_1, ..., X_n$ be a sequence of samples drawn from a distribution, its empirical mean estimator is defined by

$$\bar{X} = \frac{X_1 + ... + X_n}{n}.$$ 

$\bar{X}$ has an optimal minimax mean square error among all mean estimators in all models including Gaussian distributions, but its deviation is suboptimal for heavy tail distribution [4].

Catoni put forward in his seminal paper [4] a new M-estimator for heavy-tailed samples with finite variance by solving the following equation about $\theta$

$$\sum_{i=1}^{n} \varphi(\beta(X_i - \theta)) = 0 \quad \text{with} \quad -\log \left(1 - x + \frac{|x|^2}{2}\right) \leq \varphi(x) \leq \log \left(1 + x + \frac{|x|^2}{2}\right),$$

where $\beta > 0$ is a parameter to be tuned and $\varphi$ is non-decreasing and called influence function. The deviation performance of this estimator is much better than $\bar{X}$. Catoni’s idea has been broadly applied to many research problems, see for instance [1, 18, 6, 7, 8, 13, 15, 20]. The finite variance assumption plays an important role in Catoni’s analysis, but it rules out many interesting distributions such as Pareto law [12, 19, 5, 10], which describes the distributions of wealth and social networks.

We generalize Catoni’s M-estimator to the case in which samples can have finite α-th moment with α ∈ (1, 2). Our approach is by replacing Catoni’s influence function with the one satisfying $-\log \left(1 - x + \frac{|x|^2}{2}\right) \leq \varphi(x) \leq \log \left(1 + x + \frac{|x|^2}{2}\right)$. The choice of the new influence function is inspired by the Taylor-like expansion developed in [5]. By an argument very similar to Catnoi’s, we obtain a deviation upper bound which coincides with that in [4] as α ↑ 2 (see Theorem 2.6 and Remark 2.7 below). Experiment shows that our generalized M-estimator performs better than the empirical mean estimator, the smaller the α is, the better the performance will be.

Catoni’s argument for finding M-estimator can be split into two steps. The first step is to find two deterministic values $\theta_-$ and $\theta_+$, both depending on a parameter $\beta$ to be tuned later, such that the M-estimator $\hat{\theta}$ falls between $\theta_-$ and $\theta_+$ with a high probability. The $\theta_-$
and \( \theta_+ \) were obtained explicitly by solving two quadratic algebraic equations \( B_-(\theta) = 0 \) and \( B_+(\theta) = 0 \) respectively in [4], whereas in our setting the corresponding equations are not quadratic and the solutions do not have explicit forms. Alternatively, we first prove that \( B_-(\theta) = 0 \) has a largest solution, while \( B_+(\theta) = 0 \) has a smallest one, and then use them as a replacement of \( \theta_- \) and \( \theta_+ \) in our analysis. The second step is to show that as one chooses \( \beta > 0 \) sufficiently small, the difference between \( \theta_- \) and \( \theta_+ \) can be as small as we wish, whence the estimator can be localized in a small interval with a high probability. In our analysis, we also choose a sufficiently small \( \beta \) (depending on \( \alpha \)) to make our estimator fall in a small interval with the above special solutions as its two end points. As \( \alpha \uparrow 2 \), our result coincides with that in [4].

As an application of our generalized estimator, we consider the \( \ell_1 \)-regression with heavy-tailed samples studied by Zhang et al. [22] who assumed the samples have finite variance. The linear regression considered in [22] aims to find the minimizer \( \theta^* \) of the minimization problem as follows

\[
\min_{\theta \in \Theta} R_{\ell_1}(\theta) \quad \text{with} \quad R_{\ell_1}(\theta) = \mathbb{E}_{(x,y) \sim \Pi} [ |x^T \theta - y | ],
\]

where \( \Pi \) is a probability distribution, and \( \Theta \subseteq \mathbb{R}^d \) is the set in which \( \theta^* \) is located. In practice, \( \Pi \) is not known, one usually draws a data set \( T = (x_1, y_1), \ldots, (x_n, y_n) \) from \( \Pi \) and considers the following empirical optimization problem:

\[
\min_{\theta \in \Theta} \hat{R}_{\ell_1}(\theta) \quad \text{with} \quad \hat{R}_{\ell_1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} |x_i^T \theta - y_i |.
\]

The theoretical guarantees for bounded or sub-Gaussian distributed \( \Pi \) have been discussed in many papers, see for instance [3, 11, 21].

Inspired by Catoni's work, Zhang et al. considered the case that \( \Pi \) is heavy-tailed with finite variance and proposed a new minimization problem

\[
\min_{\theta \in \Theta} \hat{R}_{\varphi, \ell_1}(\theta) \quad \text{with} \quad \hat{R}_{\varphi, \ell_1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} \varphi(\beta |y_i - x_i^T \theta | ),
\]

where \( \varphi \) is the same as that in [4] and \( \beta > 0 \) is a parameter to be tuned. A new estimator was established from this minimization problem and an error bound was obtained. When the sample size \( n \) tends to infinity, this error bound tends to zero.

Thanks to the analysis of Section 2, we extend the results in [22] to the case in which samples can have finite \( \alpha \)-th moment with \( \alpha \in (1, 2) \), our approach is by replacing the original \( \varphi \) with the one in Section 2 and solving the corresponding minimization problem. We establish a similar error bound for our estimator and prove that it tends to zero as \( n \rightarrow \infty \).

The organization of this paper is as follows. In Section 2, we give the deviation analysis of M-estimator for the case that only \( \alpha \)-th central moment is known and show that the M-estimator has a better performance compared with the empirical mean. In Section 3, we state the upper bounds and the corresponding lower bounds on the empirical mean. In the last section, we discuss the \( \ell_1 \)-regression with heavy-tailed distributions (variance equals infinity).

2. A GENERALIZED CATONI’S M-ESTIMATOR AND ITS DEVIATION ANALYSIS

Let \( (X_i)_{i=1}^{\infty} \) be a sequence of i.i.d. samples drawn from some unknown probability distribution \( \Pi \) on \( \mathbb{R} \). We assume that there exists some \( \alpha \in (1, 2) \) such that

\[
\mathbb{E}|X_1|^\alpha < \infty.
\]
Further denote

\[
m = \mathbb{E}[X_1], \quad v = \mathbb{E}|X_1 - m|^\alpha.
\]

Inspired from Catoni’s idea in [4] and the Taylor-like expansion develop in [5], we consider a non-decreasing function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that

\[
-\log \left( 1 - x + \frac{|x|^\alpha}{\alpha} \right) \leq \varphi(x) \leq \log \left( 1 + x + \frac{|x|^\alpha}{\alpha} \right).
\]

The widest possible choice of \( \varphi \) (see Figure 1) compatible with these inequalities is

\[
\varphi(x) = \begin{cases} 
\log \left( 1 + x + \frac{x^\alpha}{\alpha} \right), & x \geq 0, \\
-\log \left( 1 - x + \frac{|x|^\alpha}{\alpha} \right), & x < 0.
\end{cases}
\]

Let \( \beta \) be some strictly positive real parameter that will be chosen later and denote the estimator of the mean \( m \) by \( \hat{\theta} \), which is the solution to the equation

\[
\sum_{i=1}^n \varphi\left( \beta(X_i - \hat{\theta}) \right) = 0.
\]

For further use, we denote

\[
r(\theta) = \frac{1}{\beta n} \sum_{i=1}^n \varphi\left( \beta(X_i - \theta) \right), \quad \theta \in \mathbb{R}.
\]

It is easy to see \( r(\theta) \) is a non-increasing random variable since \( \varphi \) is non-decreasing.

Let us briefly explain how to find the estimator \( \hat{\theta} \) into the following two steps: the first step is to find two deterministic values \( \theta_- \) and \( \theta_+ \), both depending on \( \beta \), such that \( r(\theta_-) > 0 > r(\theta_+) \) with a high probability, from the non-decreasing property of \( r \), we know that \( \theta_- < \hat{\theta} < \theta_+ \) holds with a high probability; the second is to show that as we choose \( \beta > 0 \) sufficiently small, the difference between \( \theta_- \) and \( \theta_+ \) can be as small as we wish, whence the estimator can be localized in a small interval with a high probability.
Lemma 2.1. Keep the same notations and assumptions as above. Then, for any \( \theta \in \mathbb{R} \), we have
\[
\mathbb{E}\left[ \exp (\beta nr(\theta)) \right] \leq \exp \left( n\beta (m - \theta) + \frac{2^{\alpha - 1}n\beta^\alpha}{\alpha} \left( v + |m - \theta|^\alpha \right) \right)
\]
and
\[
\mathbb{E}\left[ \exp (-\beta nr(\theta)) \right] \leq \exp \left( -n\beta (m - \theta) + \frac{2^{\alpha - 1}n\beta^\alpha}{\alpha} \left( v + |m - \theta|^\alpha \right) \right).
\]

Proof. We first recall the \( C_r \)-inequality, that is, for any \( a, b \in \mathbb{R} \) and \( p > 0 \), we have
\[
|a + b|^p \leq \max \{2^{p-1}, 1\}(|a|^p + |b|^p),
\]
and if \( p > 1 \), it is easy to verify
\[
|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).
\]
Then, noting that \( X_i, i = 1, \cdots, n \) are i.i.d., by (2.1), we have
\[
\mathbb{E}\left[ \exp (\beta nr(\theta)) \right] = \mathbb{E}\left[ \exp \left( \sum_{i=1}^{n} \varphi(\beta(X_i - \theta)) \right) \right]
\[
= \left( \mathbb{E}\left[ \exp \left( \varphi(\beta(X_1 - \theta)) \right) \right] \right)^n
\[
\leq \left( 1 + \beta(X_1 - \theta) + \frac{\beta^\alpha}{\alpha}|X_1 - \theta|^\alpha \right)^n,
\]
and noting that \( \alpha \in (1, 2) \), by (2.5), we further have
\[
\mathbb{E}\left[ \exp (\beta nr(\theta)) \right] \leq \left[ 1 + \beta(m - \theta) + \frac{\beta^\alpha}{\alpha}\mathbb{E}|X_1 - m + m - \theta|^\alpha \right]^n
\[
\leq \left[ 1 + \beta(m - \theta) + \frac{2^{\alpha - 1}\beta^\alpha}{\alpha} \left( v + |m - \theta|^\alpha \right) \right]^n
\[
\leq \exp \left( n\beta(m - \theta) + \frac{2^{\alpha - 1}n\beta^\alpha}{\alpha} \left( v + |m - \theta|^\alpha \right) \right),
\]
where the last inequality is by the inequality \( 1 + x \leq e^x \) for any \( x \in \mathbb{R} \), (2.3) is proved and the inequality (2.4) can be proved in the same way. The proof is complete. \( \square \)

According to (2.3) and (2.4), for any \( \epsilon \in (0, \frac{1}{2}) \), we denote
\[
B_+(\theta) = m - \theta + \frac{(2\beta)^{\alpha - 1}}{\alpha} (v + |m - \theta|^\alpha) + \frac{\log(e^{-1})}{n\beta},
\]
\[
B_-(\theta) = m - \theta - \frac{(2\beta)^{\alpha - 1}}{\alpha} (v + |m - \theta|^\alpha) - \frac{\log(e^{-1})}{n\beta}.
\]

Lemma 2.2. Keep the same notations and assumptions as above. Then, for any \( \theta \in \mathbb{R} \), we have
\[
\mathbb{P}(r(\theta) < B_+(\theta)) \geq 1 - \epsilon
\]
and
\[
\mathbb{P}(r(\theta) > B_-(\theta)) \geq 1 - \epsilon.
\]
In particular, for any \( \theta \in \mathbb{R} \), we have
\[
\mathbb{P}(B_-(\theta) < r(\theta) < B_+(\theta)) \geq 1 - 2\epsilon.
\]
It is easy to verify that

\( B \) when

\[
\lim_{\theta \to \infty}
\]

then the equation

\( \theta \) immediately follows from (2.6) and (2.7).

Let us now consider the following two equations:

\( \theta \) is

\[
\text{Further, when } \theta \geq m. \text{ The derivative of function } B_+(\theta) \text{ is}
\]

\[
\frac{2^{\alpha-1}}{\alpha} \beta^\alpha v - \frac{\alpha - 1}{2^\alpha} + \frac{\alpha \log(\epsilon^{-1})}{2^{\alpha - 1} n} \leq 0.
\]

When \( B_+(m + \frac{1}{2^\beta}) = 0 \), that is, \( \beta^\alpha v = \frac{\alpha - 1}{2^\alpha} - \frac{\alpha \log(\epsilon^{-1})}{2^{\alpha - 1} n} \), then the equation \( B_+(\theta) = 0 \) has the unique solution \( \theta = m + \frac{1}{2^\beta} \). When \( B_+(m + \frac{1}{2^\beta}) < 0 \), that is, \( \beta^\alpha v < \frac{\alpha - 1}{2^\alpha} - \frac{\alpha \log(\epsilon^{-1})}{2^{\alpha - 1} n} \), then the equation \( B_+(\theta) = 0 \) has two solutions.

Furthermore, when \( \theta \leq m \),

\[
B_+(\theta) > 0.
\]
which implies that the smallest solution $\theta_+ > m$. By (2.12), we have

$$B_+(m + 1) = -1 + \frac{(2\beta)^{\alpha-1}}{\alpha}(v + 1) + \frac{\log(\epsilon^{-1})}{n\beta} \leq 0.$$ 

Hence, there exists $\tilde{\theta} \in (m, m + 1]$ such that $B_+(\tilde{\theta}) = 0$. Since $\theta_+$ is the smallest solution to the equation $B_+(\theta) = 0$, we have $\theta_+ \in (m, m + 1]$.

By the same argument as above, we can also prove the equation $B_-(\theta) = 0$ has at least one solution and $\theta_- \in [m - 1, m)$. The proof is complete. \qed

Lemma 2.4. Let $\beta, \theta_-$ and $\theta_+$ be specified in Lemma 2.3. We have

$$\Pr\left(\theta_- \leq \tilde{\theta} \leq \theta_+\right) \geq 1 - 2\epsilon.\quad (2.13)$$

Proof. By (2.8), we know that the following event holds with a probability at least $1 - 2\epsilon$:

$$r(\theta_-) > 0 \quad \text{and} \quad r(\theta_+) < 0.$$ 

Since $r(\theta)$ is a continuous function and non-increasing, $r(\theta) = 0$ has a solution $\tilde{\theta}$ between $\theta_-$ and $\theta_+$ such that

$$\theta_- \leq \tilde{\theta} \leq \theta_+$$

holds with a probability at least $1 - 2\epsilon$. \qed

Remark 2.5. From the above proof, it is easy to deduce (2.11) from (2.12), that is, (2.12) is a stronger condition.

Now, we can give the main result in this section, which can give a deviation upper bound for the M-estimator $\hat{\theta}$.

Theorem 2.6. For any $\epsilon \in (0, \frac{1}{2})$ and positive integer $n$, we assume

$$n \geq \left(\frac{2v + 1}{\alpha}\right) \alpha 2\alpha \log(\epsilon^{-1}) \frac{n}{v}.\quad (2.14)$$

Then, the inequality

$$|m - \hat{\theta}| < \frac{2\left(2\alpha \log(\epsilon^{-1})\right)^{\alpha-1}}{n} \left(\frac{n}{\alpha - \left(2\alpha \log(\epsilon^{-1})\right)^{\alpha-1}}\right)^{\frac{1}{\alpha}} = O\left(\left(\log(\epsilon^{-1})\right)^{\frac{\alpha-1}{\alpha}} n^{-\frac{\alpha-1}{\alpha}}\right)$$

holds with probability at least $1 - 2\epsilon$.

Remark 2.7. In Theorem 2.6, if we take $\alpha = 2$, then the right hand side is $O\left(\left(\log(\epsilon^{-1})\right)^{\frac{1}{2}} n^{-\frac{1}{2}}\right)$, which has the same order as [4, Proposition 2.4].

Proof. By (2.14), (2.12) is fulfilled and thus Lemma 2.3 tells us that the equation

$$B_+(\theta) = m - \theta + \frac{(2\beta)^{\alpha-1}}{\alpha}(v + |m - \theta|^{\alpha}) + \frac{\log(\epsilon^{-1})}{n\beta} = 0$$

has the smallest solution $\theta_+ \in (m, m + 1]$. Denote $x = m - \theta$, $x_+ = m - \theta_+$ and

$$\hat{B}_+(x) = x + \frac{(2\beta)^{\alpha-1}}{\alpha}(v + |x|^{\alpha}) + \frac{\log(\epsilon^{-1})}{n\beta},$$

then $\hat{B}_+(x_+) = 0$ with $x_+ \in [-1, 0)$. Since $\alpha \in (1, 2)$, we further have

$$x_+ + \frac{(2\beta)^{\alpha-1}}{\alpha}(v - x_+) + \frac{\log(\epsilon^{-1})}{n\beta} \geq 0.\quad (2.15)$$
By (2.12), we have \( 1 - \frac{(2\beta)^{\alpha - 1}}{\alpha} > 0 \), solving (2.15) gives
\[
x_+ \geq \frac{(2\beta)^{\alpha - 1}v + \frac{\alpha \log(e^{-1})}{n\beta}}{\alpha - (2\beta)^{\alpha - 1}}.
\]
Let \((2\beta)^{\alpha - 1}v = \frac{\alpha \log(e^{-1})}{n\beta}\), i.e., \(\beta = \frac{1}{2}\left(\frac{2\alpha \log(e^{-1})}{n\beta}\right)^\frac{1}{\alpha}\), we have
\[
m - \theta_+ \geq -\frac{2\left(\frac{2\alpha \log(e^{-1})}{n\beta}\right)^\frac{\alpha - 1}{\alpha} v^{\frac{1}{\alpha}}}{\alpha - (2\beta)^{\alpha - 1}}.
\]
Similarly, we get
\[
m - \theta_- \leq \frac{2\left(\frac{2\alpha \log(e^{-1})}{n\beta}\right)^\frac{\alpha - 1}{\alpha} v^{\frac{1}{\alpha}}}{\alpha - (2\beta)^{\alpha - 1}}.
\]
Then, (2.13) implies that the inequality
\[
|\hat{\theta} - m| \leq \frac{2\left(\frac{2\alpha \log(e^{-1})}{n\beta}\right)^\frac{\alpha - 1}{\alpha} v^{\frac{1}{\alpha}}}{\alpha - (2\beta)^{\alpha - 1}}
\]
holds with probability at least \(1 - 2\epsilon\). \(\square\)

The empirical mean estimator is defined by
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,
\]
we postpone to study deviation bounds for the empirical mean \(\bar{X}\) in Section 3 below.

In Figures 2-5, we compare the bound on the deviations of the M-estimator \(\hat{\theta}\) with the deviations of the empirical mean \(\bar{X}\), when the sample distribution is a Pareto distribution with shape parameter \(\frac{2+\alpha}{2}\) and scale parameter \(\left(\frac{2+\alpha}{2}\right)^{-\frac{1}{\alpha}}\) (see, e.g., [12, Chapter 23]), that is,

\[
\mathbb{P}(X_1 > x) = 2^{-1}(\frac{2 + \alpha}{2 - \alpha})^{-\frac{2+\alpha}{2\alpha}}x^{-\frac{2+2\alpha}{2\alpha}}, \quad x \geq \left(\frac{2 + \alpha}{2 - \alpha}\right)^{-\frac{1}{\alpha}},
\]
\[
\mathbb{P}(X_1 \leq x) = 2^{-1}(\frac{2 + \alpha}{2 - \alpha})^{-\frac{2+\alpha}{2\alpha}}(-x)^{-\frac{2+2\alpha}{2\alpha}}, \quad x \leq -\left(\frac{2 + \alpha}{2 - \alpha}\right)^{-\frac{1}{\alpha}}.
\]

By the definition, it is easy to verify that \(m = \mathbb{E}X_1 = 0\) and \(v = \mathbb{E}|X_1 - m|^\alpha = 1\). We can get figures for the upper bound of \(\hat{\theta}\), the upper bound and lower bound of \(\bar{X}\). It is obvious from Figures 2-5 that the \(\hat{\theta}\) has a better performance when \(\epsilon\) is small enough. We can also see that the smaller the \(\alpha\) is, the better the performance of \(\hat{\theta}\) will be comparing with that of \(\bar{X}\). The parameters for Figures 2-5 are in Table 1, where 0.001 : 0.001 : 0.08 means the range of \(\epsilon\) is from 0.001 to 0.08 with step-size 0.001. The ranges of \(\epsilon\) in Table 1 satisfy (3.2) and the values of \(n\) in Table 1 satisfy (2.14) and (3.2).

| Table 1. Parameters in Figures 2-5 |
|----------|----------------|-------|
| Figure 2 | 1.9 | 0.001:0.001:0.08 | 500 |
| Figure 3 | 1.8 | 0.001:0.001:0.08 | 500 |
| Figure 4 | 1.5 | 0.001:0.001:0.08 | 500 |
| Figure 5 | 1.2 | 0.01:0.001:0.08 | 3000 |
3. THE DEVIATION UPPER AND LOWER BOUNDS OF THE EMPIRICAL MEAN ESTIMATOR

At the end of the previous section, we compare our M-estimator $\hat{\theta}$ with the empirical mean $\bar{X}$. In this section, we will prove the upper bounds and the corresponding lower bounds on the empirical mean used in these comparisons.

3.1. Upper bounds.

Lemma 3.1. Let $(X_i)_{i=1}^n$ be a sequence of random variables independently drawn from some distribution $\mathcal{D}$ with mean $m$ and $\alpha$-th central moment $v$. Then, denote the empirical mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, we have

$$\mathbb{P}\left(|\bar{X} - m| \geq \left(\frac{v}{\epsilon n^{\alpha-1}}\right)^{\frac{1}{\alpha}}\right) \leq 2\epsilon.$$
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**Figure 4.** Deviations of $\hat{\theta}$ from the sample mean, compared with those of empirical mean

**Figure 5.** Deviations of $\hat{\theta}$ from the sample mean, compared with those of empirical mean

**Proof.** Noticing that $(X_i - m)_{i=1}^n$ are i.i.d. random variables with mean zero, by [16, Theorem 2], we have

$$\mathbb{E}\left| \sum_{i=1}^n (X_i - m) \right|^\alpha \leq 2 \sum_{i=1}^n \mathbb{E}|X_i - m|^\alpha = 2nv,$$

which implies

$$\mathbb{P}\left( |\bar{X} - m| \geq \left( \frac{v}{\epsilon n^{\alpha-1}} \right)^\alpha \right) \leq \frac{1}{\epsilon n^{\alpha-1}} \frac{\mathbb{E}|\bar{X} - m|^\alpha}{\mathbb{E}|X_i - m|^\alpha} \leq \frac{1}{\epsilon n^{\alpha-1}} \frac{\mathbb{E}\sum_{i=1}^n |X_i - m|^\alpha}{\mathbb{E}|X_i - m|^\alpha} \leq 2\epsilon,$$

the desired result follows. □
3.2. Lower bounds. Comparing with Lemma 3.1, the following lemma gives a lower bound for the deviations of the empirical mean for some specific distributions.

**Lemma 3.2.** For any value of the $\alpha$-th central moment $v$, any deviation $\eta > 0$, there is some distribution $\Pi$ with mean zero and $\alpha$-th central moment $v$ such that

\[
(3.1) \quad \mathbb{P}(\bar{X} \geq \eta) = \mathbb{P}(\bar{X} < -\eta) \geq \frac{v}{3n^{\alpha-1}\eta^{\alpha}} (1 - \frac{v}{n^{\alpha}\eta^{\alpha}})^{n-1},
\]

where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ with $(X_i)_{i=1}^{n}$ independently drawn from the distribution $\Pi$. Furthermore, if

\[
(3.2) \quad \epsilon < (3e)^{-1} \text{ and } n \geq 2,
\]

the inequality

\[
|\bar{X} - m| \geq \left( \frac{v}{3n^{\alpha-1}\epsilon} \right)^{\frac{1}{\alpha}} (1 - \frac{3e\epsilon}{n})^{n-1}
\]

holds with probability at least $2\epsilon$.

**Proof.** Let us consider the random variable $X$, which has the following distribution:

\[
\mathbb{P}(X = 0) = 1 - \frac{v}{n^\alpha \eta^\alpha}, \quad \mathbb{P}(X = n\eta) = \mathbb{P}(X = -n\eta) = \frac{v}{3n^\alpha \eta^\alpha}
\]

and

\[
\mathbb{P}(X \in (x, \infty) \setminus \{n\eta\}) = \frac{q}{2\gamma} x^{-\gamma}, \quad x \in (p, \infty) \setminus \{n\eta\}
\]

\[
\mathbb{P}(X \in (-\infty, x) \setminus \{-n\eta\}) = \frac{q}{2\gamma} |x|^{-\gamma}, \quad x \in (-\infty, -p) \setminus \{-n\eta\},
\]

where $\gamma \in (\alpha, 2)$, $p = \left( \frac{v}{3n^{\alpha-1}} \right)^{\frac{1}{\alpha}} n\eta$ and $q = \frac{v}{2\gamma} \left( \frac{v}{3n^{\alpha-1}} \right)^{\frac{1}{\alpha}} (n\eta)^{\gamma - \alpha}$. It is easy to check that $\mathbb{E}X = 0$ and

\[
\mathbb{E}|X|^\alpha = (n\eta)^\alpha \frac{v}{3n^\alpha \eta^\alpha} + (n\eta)^\alpha \frac{v}{3n^\alpha \eta^\alpha} + \frac{q}{\gamma - \alpha} p^{\alpha - \gamma} = \frac{v}{3} + \frac{v}{3} + \frac{v}{3} = v.
\]

Let $(X_i)_{i=1}^{n}$ be i.i.d., which have the same distribution as $X$. Then,

\[
\mathbb{P}(\bar{X} \geq \eta) = \mathbb{P}(\bar{X} < -\eta) \geq \mathbb{P}(\bar{X} = \eta) \geq \frac{v}{3n^{\alpha-1}\eta^\alpha} (1 - \frac{v}{n^\alpha \eta^\alpha})^{n-1},
\]

so (3.1) is proved.

Take $\eta = \frac{v}{3n^{\alpha-1}\epsilon^\prime}$, we have

\[
\frac{v}{3n^{\alpha-1}\eta^\alpha} (1 - \frac{v}{n^\alpha \eta^\alpha})^{n-1} = \epsilon(1 - \frac{3e\epsilon}{n})^{-n(1 - \frac{3e\epsilon}{n})^{n-1}} (1 - \frac{3e\epsilon}{n})^{-n-1} (1 - \frac{3e\epsilon}{n})^{-n-1}.
\]

If $\epsilon < (3e)^{-1}$, then $(1 - \frac{3e\epsilon}{x})^{x-1} \geq (1 - \frac{1}{x})^{x-1}$. For any $x \geq 1$, we denote $f(x) = (1 - \frac{1}{x})^{x-1}$, then

\[
f'(x) = (1 - \frac{1}{x})^{x-1} \left( \log(1 - \frac{1}{x}) + \frac{(x - 1)}{x^2(1 - \frac{1}{x})^2} \right)
\]

\[
= (1 - \frac{1}{x})^{x-1} \left( \log(1 - \frac{1}{x}) + \frac{1}{x} \right).
\]

Noting that $(1 - \frac{1}{x})^{x-1} > 0$, let $g(x) = \log(1 - \frac{1}{x}) + \frac{1}{x}$ for $x \geq 1$, then we have

\[
g'(x) = \frac{1}{x^2} - \frac{1}{x} = \frac{1}{x^2(x - 1)} > 0
\]
and
\[ \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \left( \log(1 - \frac{1}{x}) + \frac{1}{x} \right) = 0, \]
which imply \( g(x) \leq 0 \), so we have \( f'(x) \leq 0 \) for \( x \geq 1 \). What’s more, we have
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} (1 - \frac{1}{x})^{-1} = e^{-1}, \]
which implies \( (1 - \frac{3\epsilon e}{n})^{n-1} \geq e^{-1} \). Therefore, we have
\[ \frac{v}{3n^{\alpha-1} \eta^\alpha} (1 - \frac{v}{n^{\alpha} \eta^\alpha})^{n-1} \geq e(1 - \frac{3\epsilon e}{n})^{-(n-1)} (1 - \frac{3\epsilon e}{n})^{n-1} = \epsilon. \]
The proof is complete. \( \square \)

4. \( \ell_1 \)-REGRESSION FOR HEAVY-TAILED SAMPLES HAVING FINITE \( \alpha \)-TH MOMENT WITH \( \alpha \in (1, 2) \)

The linear regression considered in [22] aims to find the unknown minimizer \( \theta^* \) of the following minimization problem:
\[ \min_{\theta \in \Theta} R_{\ell_1}(\theta) \quad \text{with} \quad R_{\ell_1}(\theta) = \mathbb{E}_{(x,y) \sim \Pi} \left[ |x^T \theta - y| \right], \]
where \( \Pi \) is the population’s distribution, and \( \Theta \subseteq \mathbb{R}^d \) is the set in which \( \theta^* \) is located. In practice, \( \Pi \) is not known, one usually draws a data set \( \mathcal{T} = (x_1, y_1), \ldots, (x_n, y_n) \) from \( \Pi \) and consider the following empirical optimization problem:
\[ \min_{\theta \in \Theta} \hat{R}_{\ell_1}(\theta) \quad \text{with} \quad \hat{R}_{\ell_1}(\theta) = \frac{1}{n} \sum_{i=1}^{n} |x_i^T \theta - y_i|. \]
Inspired by Catoni’s work, Zhang et al. [22] considered the case that \( \Pi \) is heavy tailed with finite variance and proposed a new minimization problem
\[ \min_{\theta \in \Theta} \hat{R}_{\varphi, \ell_1}(\theta) \quad \text{with} \quad \hat{R}_{\varphi, \ell_1}(\theta) = \frac{1}{n\beta} \sum_{i=1}^{n} \varphi(\beta |y_i - x_i^T \theta|), \]
where \( \varphi \) is the same as that in [4] and \( \beta > 0 \) is to be determined later.

Thanks to the analysis of Section 2, we extend the results in [22] to the case in which samples can have finite \( \alpha \)-th moment with \( \alpha \in (1, 2) \), the approach is by replacing the original \( \varphi \) with (2.1).

4.1. Main results of this section. Before stating the main results, we first give some definitions and assumptions.

Definition 4.1. Let \( (\Theta, d) \) be a metric space, and \( K \) be a subset of \( \Theta \). Then a subset \( \mathcal{N} \subseteq K \) is called an \( \varepsilon \)-net of \( K \) if for every \( \theta \in K \), we can find a \( \hat{\theta} \in \mathcal{N} \) such that \( d(\theta, \hat{\theta}) \leq \varepsilon \). The covering number is the minimal cardinality of the \( \varepsilon \)-net of \( \Theta \) and denoted by \( N(\Theta, \varepsilon) \).

We shall assume:

Assumption A1 (i) The domain \( \Theta \) is totally bounded, that is, for any \( \varepsilon > 0 \), there exists a finite \( \varepsilon \)-net of \( \Theta \).

(ii) The expectation of the \( \alpha \)-th moment of \( x \) is bounded, that is,
\[ \mathbb{E}_{(x,y) \sim \Pi} [|x|^\alpha] < \infty. \]

(iii) The \( \ell_\alpha \)-risk of all \( \theta \in \Theta \) is bounded, that is,
\[ \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) = \sup_{\theta \in \Theta} \mathbb{E}_{(x,y) \sim \Pi} [|y - x^T \theta|^\alpha] < \infty. \]
Then, we can state the second theorem, which will be proved in subsection 4.2.

**Theorem 4.2.** Let \( \theta^* \) and \( \hat{\theta} \) be the minimizers of (4.1) and (4.2), respectively. Under Assumption **A1**, for any \( \delta \in (0, \frac{1}{2}) \), with probability at least \( 1 - 2\delta \), we have

\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\theta^*) \\
\leq 2E|x_1| + \left( \frac{2^{\alpha-1}\varepsilon}{\alpha}E|x_1|^{\alpha} + \frac{2^{\alpha-1} + 1}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) \right) \beta^{\alpha-1} + \frac{1}{n\beta} \log \frac{N(\Theta, \varepsilon)}{\delta^2} \\
\text{for any } \varepsilon > 0. \quad \text{Furthermore, let} \quad \\
\beta = \left( \frac{1}{n} \log \frac{N(\Theta, \varepsilon)}{\delta^2} \right)^{\frac{1}{\alpha}},
\]

we have

\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\theta^*) \\
\leq 2E|x_1| + \left( \frac{2^{\alpha-1}\varepsilon}{\alpha}E|x_1|^{\alpha} + \frac{2^{\alpha-1} + 1}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + 1 \right) \left( \frac{1}{n} (d \log (6nr) + \log \frac{1}{\delta^2}) \right)^{\frac{\alpha-1}{\alpha}}.
\]

In order to compute the covering number, we further assume:

**Assumption **A2** The domain \( \Theta \subseteq \mathbb{R}^d \), and its radius is bounded by a constant \( r \), that is, \( \|\theta\| \leq r, \quad \forall \theta \in \Theta \).

Then, we have the following corollary, which will be proved in subsection 4.2.

**Corollary 4.3.** Keep the same notations and assumptions in Theorem 4.2. In addition, we suppose the **Assumption **A2** holds. Then, for any \( \delta \in (0, \frac{1}{2}) \), with probability at least \( 1 - 2\delta \), we have

\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\theta^*) \\
\leq \frac{2}{n} E|x_1| + \left( \frac{2^{\alpha-1}\varepsilon}{\alpha} E|x_1|^{\alpha} + \frac{2^{\alpha-1} + 1}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + 1 \right) \left( \frac{1}{n} (d \log (6nr) + \log \frac{1}{\delta^2}) \right)^{\frac{\alpha-1}{\alpha}} \\
= O\left( \left( \frac{d \log n}{n} \right)^{\frac{\alpha-1}{\alpha}} \right).
\]

4.2. **Proof of Theorem 4.2 and Corollary 4.3.** Before proving the Theorem 4.2, we first give the following auxiliary lemmas.

**Lemma 4.4.** Keep the same notations and assumptions as in Theorem 4.2. Then, the following inequality

\[
\hat{R}_{\varphi, \ell_1}(\theta^*) - R_{\ell_1}(\theta^*) \leq \frac{\beta^{\alpha-1} \alpha}{\alpha} R_{\ell_\alpha}(\theta^*) + \frac{1}{n\beta} \log \frac{1}{\delta}
\]

holds with probability at least \( 1 - \delta \).

**Proof.** Noticing that \( (x_i, y_i), i = 1, \cdots, n \), are i.i.d., by (2.1), we have

\[
E \left[ \exp \left( n\beta \hat{R}_{\varphi, \ell_1}(\theta^*) \right) \right] = E \left[ \exp \left( \sum_{i=1}^{n} \varphi(\beta |y_i - x_i^T \theta^*|) \right) \right] \\
= \left[ E \left[ \exp \left( \varphi(\beta |y_i - x_i^T \theta^*|) \right) \right] \right]^n \\
\leq \left[ E \left[ 1 + \beta |y_1 - x_1^T \theta^*| + \frac{\beta^{\alpha} |y_1 - x_1^T \theta^*|^{\alpha}}{\alpha} \right] \right]^n,
\]
then, by the inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$, we have
\[
\mathbb{E}[\exp(n\beta \hat{R}_{\varphi,\ell_1}(\theta^*))] \leq [1 + \beta R_{\ell_1}(\theta^*) + \frac{\beta^\alpha}{\alpha} R_{\ell_\alpha}(\theta^*)]^n
\leq \exp\left(n\beta R_{\ell_1}(\theta^*) + \frac{\beta^\alpha}{\alpha} R_{\ell_\alpha}(\theta^*)\right).
\]
Therefore, by Markov inequality, we have
\[
\mathbb{P}\left(n\beta \hat{R}_{\varphi,\ell_1}(\theta^*) \geq n\beta R_{\ell_1}(\bar{\theta}) + \frac{\beta^\alpha}{\alpha} R_{\ell_\alpha}(\theta^*) + \log \frac{1}{\delta}\right)
\leq \frac{\mathbb{E}[\exp(n\beta \hat{R}_{\varphi,\ell_1}(\theta^*))]}{\exp\left(n\beta R_{\ell_1}(\theta^*) + \frac{\beta^\alpha}{\alpha} R_{\ell_\alpha}(\theta^*) + \log \frac{1}{\delta}\right)} \leq \delta.
\]
The proof is complete. \hfill \square

**Lemma 4.5.** For any $\varepsilon > 0$, let $\mathcal{N}(\Theta, \varepsilon)$ be an $\varepsilon$-net of $\Theta$ with cardinality $N(\Theta, \varepsilon)$. Then, with probability at least $1 - \delta$, the following inequality
\[
- \frac{1}{n\beta} \sum_{i=1}^n \varphi(\beta |y_i - x_i^T \bar{\theta}| - \varepsilon |x_i|)
\leq -R_{\ell_1}(\bar{\theta}) + \varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha-1}}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + \frac{(2\beta)^{\alpha-1}\varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha + \frac{1}{n\beta} \log \frac{N(\Theta, \varepsilon)}{\delta}
\]
holds for all $\bar{\theta} \in \mathcal{N}(\Theta, \varepsilon)$.

**Proof.** For a fixed $\bar{\theta} \in \mathcal{N}(\Theta, \varepsilon)$, noticing that $(x_i, y_i)$, $i = 1, \ldots, n$, are i.i.d., by (2.1), we have
\[
\mathbb{E}[\exp\left(-\sum_{i=1}^n \varphi(\beta |y_i - x_i^T \bar{\theta}| - \varepsilon |x_i|)\right)]
= \left[\mathbb{E}\left[\exp\left(-\varphi(\beta |y_i - x_i^T \bar{\theta}| - \varepsilon |x_i|)\right)\right]\right]^n
\leq \left[\mathbb{E}\left[1 - \beta |y_i - x_i^T \bar{\theta}| + \beta \varepsilon |x_1| + \frac{\beta^\alpha}{\alpha} (|y_i - x_i^T \bar{\theta}| - \varepsilon |x_1|)^\alpha\right]\right]^n
= \left[1 - \beta R_{\ell_1}(\bar{\theta}) + \beta \varepsilon \mathbb{E}|x_1| + \frac{\beta^\alpha}{\alpha} \mathbb{E}\left[^{|y_i - x_i^T \bar{\theta}| - \varepsilon |x_1| |^\alpha]\right]\right]^n,
\]
then, by (2.5) and the inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$, we have
\[
\mathbb{E}[\exp\left(-\sum_{i=1}^n \varphi(\beta |y_i - x_i^T \bar{\theta}| - \varepsilon |x_i|)\right)]
\leq \left[1 - \beta R_{\ell_1}(\bar{\theta}) + \beta \varepsilon \mathbb{E}|x_1| + \frac{\beta^\alpha 2^{\alpha-1}}{\alpha} R_{\ell_\alpha}(\bar{\theta}) + \frac{\beta^\alpha \varepsilon^\alpha 2^{\alpha-1}}{\alpha} \mathbb{E}|x_1|^\alpha\right]^n
\leq \exp\left[n\left(-\beta R_{\ell_1}(\bar{\theta}) + \beta \varepsilon \mathbb{E}|x_1| + \frac{\beta^\alpha 2^{\alpha-1}}{\alpha} R_{\ell_\alpha}(\bar{\theta}) + \frac{\beta^\alpha \varepsilon^\alpha 2^{\alpha-1}}{\alpha} \mathbb{E}|x_1|^\alpha\right)\right].
\]
Therefore, by Markov inequality, we have
\[
\mathbb{P}\left(-\sum_{i=1}^n \varphi(\beta |y_i - x_i^T \bar{\theta}| - \varepsilon |x_i|)\right)
= \frac{\mathbb{E}[\exp(n\beta \hat{R}_{\varphi,\ell_1}(\theta^*))]}{\exp\left(n\beta R_{\ell_1}(\theta^*) + \frac{\beta^\alpha}{\alpha} R_{\ell_\alpha}(\theta^*) + \log \frac{1}{\delta}\right)} \leq \delta.
\]
\[
\begin{align*}
&\geq n(-\beta R_{\ell_1}(\hat{\theta}) + \beta \varepsilon \mathbb{E}|x_1| + \frac{\beta^2 \alpha^{-2}}{\alpha} R_{\ell_1}(\hat{\theta}) + \frac{\beta \varepsilon \alpha^{-2}}{\alpha} \mathbb{E}|x_1|^\alpha) + \log \frac{1}{\delta'} \\
&\leq \exp \left[n(-\beta R_{\ell_1}(\hat{\theta}) + \beta \varepsilon \mathbb{E}|x_1| + \frac{\beta^2 \alpha^{-2}}{\alpha} R_{\ell_1}(\hat{\theta}) + \frac{\beta \varepsilon \alpha^{-2}}{\alpha} \mathbb{E}|x_1|^\alpha) + \log \frac{1}{\delta'} \right] \leq \delta',
\end{align*}
\]

where \(\delta' \in (0, 1)\), which will be chosen later. Hence, for a fixed \(\tilde{\theta} \in \mathcal{N}(\Theta, \varepsilon)\), with probability at most \(\delta'\), we have

\[
-\frac{1}{n\beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \tilde{\theta}| - \beta \varepsilon|x_i|) = -R_{\ell_1}(\tilde{\theta}) + \varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha^{-1}}}{\alpha} R_{\ell_1}(\tilde{\theta}) + \frac{(2\beta)^{\alpha^{-1}}\varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha) + \frac{1}{n\beta} \log \frac{1}{\delta'}
\]

Therefore, since the set \(\mathcal{N}(\Theta, \varepsilon)\) has \(N(\Theta, \varepsilon)\) elements, we have

\[
\mathbb{P} \left( \bigcap_{\tilde{\theta} \in \mathcal{N}(\Theta, \varepsilon)} \left\{ -\frac{1}{n\beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \tilde{\theta}| - \beta \varepsilon|x_i|) \leq -R_{\ell_1}(\tilde{\theta}) + \varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha^{-1}}}{\alpha} R_{\ell_1}(\tilde{\theta}) + \frac{(2\beta)^{\alpha^{-1}}\varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha) + \frac{1}{n\beta} \log \frac{1}{\delta'} \right\} \right) \geq 1 - N(\Theta, \varepsilon) \delta'.
\]

Finally, taking \(\delta' = \frac{\delta}{N(\Theta, \varepsilon)}\), with probability at least \(1 - \delta\), the following inequality

\[
-\frac{1}{n\beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \tilde{\theta}| - \beta \varepsilon|x_i|) \leq -R_{\ell_1}(\tilde{\theta}) + \varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha^{-1}}}{\alpha} \sup_{\theta \in \Theta} R_{\ell_1}(\theta) + \frac{(2\beta)^{\alpha^{-1}}\varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha) + \frac{1}{n\beta} \log \frac{N(\Theta, \varepsilon)}{\delta}
\]

holds for all \(\tilde{\theta} \in \mathcal{N}(\Theta, \varepsilon)\). The proof is complete. \qed

Based on Lemma 4.5, we have the following lemma.

**Lemma 4.6.** Keep the same notations and assumptions as in Theorem 4.2. Then, for any \(\varepsilon > 0\), the following inequality

\[
R_{\ell_1}(\tilde{\theta}) - \tilde{R}_{\varphi, \ell_1}(\tilde{\theta}) \leq 2\varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha^{-1}}}{\alpha} \sup_{\theta \in \Theta} R_{\ell_1}(\theta) + \frac{(2\beta)^{\alpha^{-1}}\varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha) + \frac{1}{n\beta} \log \frac{N(\Theta, \varepsilon)}{\delta}
\]

holds with probability at least \(1 - \delta\).

**Proof.** Since \(\tilde{\theta} \in \Theta\), there exists a \(\tilde{\theta} \in \mathcal{N}(\Theta, \varepsilon)\) such that

\[
|\tilde{\theta} - \hat{\theta}| \leq \varepsilon,
\]

which implies

\[
|y_i - x_i^T \tilde{\theta}| \geq |y_i - x_i^T \hat{\theta}| - |x_i^T (\hat{\theta} - \tilde{\theta})| \geq |y_i - x_i^T \hat{\theta}| - \varepsilon|x_i|.
\]
Then, since \( \varphi(\cdot) \) is non-decreasing, we have
\[
\hat{R}_{\varphi, \ell_1}(\hat{\theta}) = \frac{1}{n \beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \hat{\theta}|) \geq \frac{1}{n \beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \hat{\theta}| - \beta \varepsilon |x_i|),
\]
by Lemma 4.5, with probability at least \( 1 - \delta \), we have
\[
\hat{R}_{\varphi, \ell_1}(\hat{\theta}) \geq R_{\ell_1}(\hat{\theta}) - [\varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha-1}}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + \frac{(2\beta)^{\alpha-1} \varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha + \frac{1}{n \beta} \log \frac{N(\Theta, \varepsilon)}{\delta}].
\]
What’s more, by (4.4) and triangle inequality, we have
\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\hat{\theta}) = \mathbb{E}[|x_1^T \hat{\theta} - y_1| - |x_1^T \hat{\theta} - y_1|] \leq \mathbb{E}[|x_1^T \hat{\theta} - x_1^T \hat{\theta}|] \leq \varepsilon \mathbb{E}|x_1|,
\]
which further implies that with probability at least \( 1 - \delta \), the inequality
\[
\hat{R}_{\varphi, \ell_1}(\hat{\theta}) \geq R_{\ell_1}(\hat{\theta}) - [2\varepsilon \mathbb{E}|x_1| + \frac{(2\beta)^{\alpha-1}}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + \frac{(2\beta)^{\alpha-1} \varepsilon^\alpha}{\alpha} \mathbb{E}|x_1|^\alpha + \frac{1}{n \beta} \log \frac{N(\Theta, \varepsilon)}{\delta}]
\]
holds. The proof is complete. \( \square \)

Now, we can give the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Recall
\[
\hat{R}_{\varphi, \ell_1}(\theta) = \frac{1}{n \beta} \sum_{i=1}^{n} \varphi(\beta|y_i - x_i^T \theta|),
\]
since \( \hat{\theta} \) is the minimizer of (4.2), we have
\[
\hat{R}_{\varphi, \ell_1}(\hat{\theta}) - \hat{R}_{\varphi, \ell_1}(\theta^*) \leq 0,
\]
which implies
\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\theta^*)
= (R_{\ell_1}(\hat{\theta}) - \hat{R}_{\varphi, \ell_1}(\theta^*)) + (\hat{R}_{\varphi, \ell_1}(\hat{\theta}) - \hat{R}_{\varphi, \ell_1}(\theta^*)) + (\hat{R}_{\varphi, \ell_1}(\theta^*) - R_{\ell_1}(\theta^*))
\leq (R_{\ell_1}(\hat{\theta}) - \hat{R}_{\varphi, \ell_1}(\theta)) + (\hat{R}_{\varphi, \ell_1}(\theta^*) - R_{\ell_1}(\theta^*)).
\]
By Lemma 4.6 and Lemma 4.4, we immediately obtain the desired result. \( \square \)

**Proof of Corollary 4.3.** For any \( \varepsilon \in (0, 1] \), by [17, Corollary 4.2.13] we have
\[
N(B_1, \varepsilon) \leq (1 + \frac{2}{\varepsilon})^d \leq (\frac{3}{\varepsilon})^d,
\]
where \( B_1 = \{x \in \mathbb{R}^d : |x| \leq 1 \} \). Since \( \Theta \subseteq B_r \), we have
\[
(4.5) \quad \log N(\Theta, \varepsilon) \leq \log N(B_r, \frac{\varepsilon}{2}) \leq d \log \frac{6r}{\varepsilon}.
\]
Therefore, by (4.3) with \( \varepsilon = \frac{1}{n} \), we have
\[
R_{\ell_1}(\hat{\theta}) - R_{\ell_1}(\theta^*)
\leq 2\varepsilon \mathbb{E}|x_1| + (\frac{2^{\alpha-1}}{\alpha} \mathbb{E}|x_1|^\alpha + \frac{2^{\alpha-1} + 1}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + 1) \left( \frac{1}{n} (d \log \frac{6r}{\varepsilon} + \log \frac{1}{\delta^2}) \right)^{\frac{\alpha-1}{\alpha}}
\]
\[
= 2\varepsilon \mathbb{E}|x_1| + (\frac{2^{\alpha-1}}{\alpha n^\alpha} \mathbb{E}|x_1|^\alpha + \frac{2^{\alpha-1} + 1}{\alpha} \sup_{\theta \in \Theta} R_{\ell_\alpha}(\theta) + 1) \left( \frac{1}{n} (d \log (6nr) + \log \frac{1}{\delta^2}) \right)^{\frac{\alpha-1}{\alpha}}.
\]
The proof is complete.

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A GENERALIZED CATONI’S M-ESTIMATOR UNDER FINITE α-TH MOMENT ASSUMPTION

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