Low Mach number limit for the multi-dimensional full magnetohydrodynamic equations

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Abstract

The low Mach number limit for the multi-dimensional full magnetohydrodynamic (MHD) equations, in which the effect of thermal conduction is taken into account, is rigorously justified within the framework of classical solutions with small density and temperature variations. Moreover, we show that for a sufficiently small Mach number, the compressible MHD equations admit a smooth solution on the time interval where the smooth solution of the incompressible MHD equations exists. In addition, the low Mach number limit for the ideal MHD equations with small entropy variation is also investigated. The convergence rates are obtained in both cases.

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1. Introduction

The magnetohydrodynamic (MHD) equations govern the motion of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields. The full three-dimensional compressible MHD equations read as (see, e.g. [12, 15, 22, 23])

\begin{align}
\begin{aligned}
\partial_t \rho + \text{div}(\rho \mathbf{u}) &= 0, \\
\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p &= \frac{1}{4\pi} (\nabla \times \mathbf{H}) \times \mathbf{H} + \text{div} \Psi, \\
\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) &= -\nabla \times (\nu \nabla \times \mathbf{H}), \quad \text{div} \mathbf{H} = 0, \\
\partial_t \mathcal{E} + \text{div} \left( \mathbf{u} (\mathcal{E}' + p) \right) &= \frac{1}{4\pi} \text{div} \left( (\mathbf{u} \times \mathbf{H}) \times \mathbf{H} \right) \\
&\quad + \text{div} \left( \frac{\nu}{4\pi} \mathbf{H} \times (\nabla \times \mathbf{H}) + \mathbf{u} \Psi + \kappa \nabla \theta \right).
\end{aligned}
\end{align}
Here $x \in \Omega$, and $\Omega$ is assumed to be the whole $\mathbb{R}^3$ or the torus $T^3$. The unknowns $\rho$ denotes the density, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ the velocity, $H = (H_1, H_2, H_3) \in \mathbb{R}^3$ the magnetic field and $\theta$ the temperature; $\Psi$ is the viscous stress tensor given by
\[
\Psi = 2\mu D(u) + \lambda \text{div} u I_3
\]
with $D(u) = (\nabla u + \nabla u^\top)/2$, $I_3$ being the $3 \times 3$ identity matrix, and $\nabla u^\top$ the transpose of the matrix $\nabla u$; $E$ is the total energy given by $E = E' + |H|^2/(8\pi)$ and $E' = \rho(e + |u|^2/2)$ with $e$ being the internal energy, $\rho|u|^2/2$ the kinetic energy, and $|H|^2/(8\pi)$ the magnetic energy. The viscosity coefficients $\lambda$ and $\mu$ of the flow satisfy $2\mu + 3\lambda > 0$ and $\mu > 0$; $\nu > 0$ is the magnetic diffusion coefficient of the magnetic field, and $\kappa > 0$ is the heat conductivity. For simplicity, we assume that $\mu$, $\lambda$, $\nu$, and $\kappa$ are constants. The equations of state $p = p(\rho, \theta)$ and $e = e(\rho, \theta)$ relate the pressure $p$ and the internal energy $e$ to the density $\rho$ and the temperature $\theta$ of the flow.

The MHD equations have attracted a lot of attention of physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges, see, for example, [2, 4–6, 8, 9, 11–13, 23, 31] and the references cited therein. One of the important topics on equations (1.1)–(1.4) is to study its low Mach number limit. For the isentropic MHD equations, the low Mach number limit has been rigorously proved in [14, 16, 17, 20]. Nevertheless, it is more significant and difficult to study the limit for the non-isentropic models from both physical and mathematical points of view.

The main purpose of this paper is to present the rigorous justification of the low Mach number limit for the full MHD equations (1.1)–(1.4) within the framework of classical solutions.

Now, we rewrite the energy equation (1.4) in the form of the internal energy. Multiplying (1.2) by $u$ and (1.3) by $H/(4\pi)$, and summing them together, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \rho |u|^2 + \frac{1}{8\pi} |H|^2 \right) + \frac{1}{2} \text{div} \left( \rho |u|^2 u \right) + \nabla p \cdot u = \text{div} \Psi \cdot u + \frac{1}{4\pi} (\nabla \times H) \times H \cdot u + \frac{1}{4\pi} \nabla \times (u \times H) \cdot H
\]
\[
- \frac{\nu}{4\pi} \nabla \times (\nabla \times H) \cdot H.
\]
Using the identities
\[
\text{div} (H \times (\nabla \times H)) = |\nabla \times H|^2 - \nabla \times (\nabla \times H) \cdot H
\]
and
\[
\text{div} ((u \times H) \times H) = (\nabla \times H) \times H \cdot u + \nabla \times (u \times H) \cdot H,
\]
and subtracting (1.5) from (1.4), we obtain the internal energy equation
\[
\partial_t (\rho e) + \text{div} (\rho u e) + (\text{div} u) p = \frac{\nu}{4\pi} |\nabla \times H|^2 + \Psi : \nabla u + \kappa \Delta \theta,
\]
where $\Psi : \nabla u$ denotes the scalar product of two matrices:
\[
\Psi : \nabla u = \sum_{i,j=1}^{3} \frac{\mu}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 + \lambda |\text{div} u|^2 = 2\mu |D(u')|^2 + \lambda (\text{tr} D(u'))^2.
\]
In this paper, we shall focus our study on the ionized fluid obeying the perfect gas relations
\[
p = \mathcal{R} \rho \theta, \quad e = c_v \theta,
\]
where the constants $\mathcal{R}$, $c_v > 0$ are the gas constant and the heat capacity at constant volume, respectively. We point out here that our analysis below can be applied to more general equations of state for $p$ and $e$ by employing minor modifications in arguments.
To study the low Mach number limit of the system (1.1)–(1.3) and (1.8), we use its appropriate dimensionless form as follows (see the appendix for the details):

\[ \partial_t \rho + \text{div} (\rho \mathbf{u}) = 0, \]
\[ \frac{\nabla (\rho \theta)}{\epsilon^2} = (\nabla \times \mathbf{H}) \times \mathbf{H} + \text{div} \Psi, \]
\[ \partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times (\nabla \times \mathbf{H}), \quad \text{div} \mathbf{H} = 0, \]
\[ \rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \frac{\nabla (\rho \theta)}{\epsilon^2} = \epsilon \left[ \nabla^2 \mathbf{H} + \mathbf{e} \nabla \theta \right] + \frac{1}{\epsilon} \left[ \nabla^2 \mathbf{H} + \mathbf{e} \nabla \theta \right], \]
\[ \partial_t \mathbf{H} + \mathbf{u} \cdot \nabla \mathbf{H} + \text{div} \mathbf{u} \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{H}, \quad \text{div} \mathbf{H} = 0, \]
\[ (1 + \epsilon \mathbf{q}^2) \left( \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \frac{1}{\epsilon} \left[ (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} + (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} \right] - \mathbf{H} \nabla \mathbf{H} + \frac{1}{2} \nabla (|\mathbf{H}|^2) = 2 \mu \text{div} \left( \nabla \mathbf{u} \right) + \lambda \nabla \left( \text{tr} \nabla \mathbf{u} \right), \]
\[ (1 + \epsilon \mathbf{q}^2) \left( \partial_t \mathbf{H} + \mathbf{u} \cdot \nabla \mathbf{H} + \text{div} \mathbf{u} \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{H}, \quad \text{div} \mathbf{H} = 0, \]

where \( \epsilon = M \) is the Mach number and the coefficients \( \mu, \lambda, \nu \) and \( \kappa \) are the scaled parameters. \( \gamma = 1 + \Re/cV \) is the ratio of specific heats. Note that we have used the same notations and assumed that the coefficients \( \mu, \lambda, \nu \) and \( \kappa \) are independent of \( \epsilon \) for simplicity. Also, we have ignored the Cowling number in equations (1.10)–(1.13), since it does not create any mathematical difficulties in our analysis.

We shall study the limit as \( \epsilon \to 0 \) of the solutions to (1.10)–(1.13). We further restrict ourselves to the small density and temperature variations, i.e.

\[ \rho = 1 + \epsilon \gamma, \quad \theta = 1 + \epsilon \phi. \]

We first give a formal analysis. Putting (1.14) and (1.9) into the system (1.10)–(1.13), and using the identities

\[ \text{curl} \text{curl} \mathbf{H} = \nabla \text{div} \mathbf{H} - \Delta \mathbf{H}, \]

then we can rewrite (1.10)–(1.13) as

\[ \partial_t \mathbf{q}^2 + \mathbf{u}^2 \cdot \nabla \mathbf{q}^2 + \frac{1}{\epsilon} (1 + \epsilon \mathbf{q}^2) \text{div} \mathbf{u}^2 = 0, \]
\[ (1 + \epsilon \mathbf{q}^2) \left[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\epsilon} \left[ (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} + (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} \right] - \mathbf{H} \nabla \mathbf{H} + \frac{1}{2} \nabla (|\mathbf{H}|^2) = 2 \mu \text{div} \left( \nabla \mathbf{u} \right) + \lambda \nabla \left( \text{tr} \nabla \mathbf{u} \right), \]
\[ (1 + \epsilon \mathbf{q}^2) \left( \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\epsilon} \left[ (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} + (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} \right] - \mathbf{H} \nabla \mathbf{H} + \frac{1}{2} \nabla (|\mathbf{H}|^2) = 2 \mu \text{div} \left( \nabla \mathbf{u} \right) + \lambda \nabla \left( \text{tr} \nabla \mathbf{u} \right), \]
\[ (1 + \epsilon \mathbf{q}^2) \left( \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\epsilon} \left[ (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} + (1 + \epsilon \mathbf{q}^2) \nabla \mathbf{u} \right] - \mathbf{H} \nabla \mathbf{H} + \frac{1}{2} \nabla (|\mathbf{H}|^2) = 2 \mu \text{div} \left( \nabla \mathbf{u} \right) + \lambda \nabla \left( \text{tr} \nabla \mathbf{u} \right), \]

Here we have added the subscript \( \epsilon \) on the unknowns to stress the dependence of the parameter \( \epsilon \). Therefore, the formal limit as \( \epsilon \to 0 \) of (1.17)–(1.19) is the following incompressible MHD equations (suppose that the limits \( \mathbf{u}^2 \to \mathbf{w} \) and \( \mathbf{H}^2 \to \mathbf{B} \) exist.)

\[ \partial_t \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{w} + \nabla \pi + \frac{1}{2} \nabla (|\mathbf{B}|^2) = \mathbf{B} \cdot \nabla \mathbf{B} = \mu \Delta \mathbf{w}, \]
\[ \partial_t \mathbf{B} + \mathbf{w} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{w} = v \Delta \mathbf{B}, \]
\[ \text{div} \mathbf{w} = 0, \quad \text{div} \mathbf{B} = 0. \]

In this paper we shall establish the above limit rigorously. Moreover, we shall show that for sufficiently small Mach number, the compressible flows admit a smooth solution on the time interval where the smooth solution of the incompressible MHD equations exists. In addition, we shall also study the low Mach number limit of the ideal compressible MHD equations (namely, \( \mu = \lambda = \nu = \kappa = 0 \) in (1.1)–(1.4)) for which small pressure and entropy variations are assumed. The convergence rates are obtained in both cases.
We should point out here that it still remains to be an open problem to prove rigorously the low Mach number limit of the ideal or full non-isentropic MHD equations with large temperature variations within the framework of classical solutions, even in the whole space case, although the corresponding problems for the non-isentropic Euler and the full Navier–Stokes equations were solved in the whole space [1, 25] or the bounded domain in [27]. The reason is that the presence of the magnetic field and its interaction with hydrodynamic motion in the MHD flow of large oscillation cause serious difficulties. We cannot apply directly the techniques developed in [1, 25, 27] for the Euler and Navier–Stokes equations to obtain the uniform estimates for the solutions to the ideal or full non-isentropic MHD equations. In this paper, however, we shall employ an alternative approach, which is based on the energy estimates for symmetrizable quasilinear hyperbolic–parabolic systems and the convergence–stability lemma for singular limit problems [3, 30], to deal with the ideal or full non-isentropic MHD equations. There are two advantages of this approach: the first one is that we can rigorously prove the incompressible limit in the time interval where the limiting system admits a smooth solution. The second one is that the estimates we obtained do not depend on the viscosity and thermodynamic coefficients, compared with the results in [10] where all-time existence of smooth solutions to the full Navier–Stokes equations was discussed and the estimates depend on the parameters intimately.

For large entropy variation and general initial data, the authors have rigorously proved the low Mach number limit of the non-isentropic MHD equations with zero magnetic diffusivity in [18] by adapting and modifying the approach developed in [25]. We mention that the coupled singular limit problem for the full MHD equations within the framework of the so-called variational solutions were studied recently in [21, 26].

Before ending the introduction, we give the notations used throughout this paper. We use the letter $C$ to denote various positive constants independent of $\epsilon$. For convenience, we denote by $H^l(\Omega)$ ($l \in \mathbb{R}$) the standard Sobolev spaces and write $\| \cdot \|$ for the standard norm of $H^l$ and $\| \cdot \|$ for $\| \cdot \|_0$.

This paper is organized as follows. In section 2 we state our main results. The proof for the full MHD equations and the ideal MHD equations is presented in section 3 and section 4, respectively. Finally, an appendix is given to derive briefly the dimensionless form of the full compressible MHD equations.

2. Main results

We first recall the local existence of strong solutions to the incompressible MHD equations (1.21)–(1.23) in the domain $\Omega$. The proof can be found in [7, 28]. Recall here that $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$.

**Proposition 2.1 ([7, 28]).** Let $s > 3/2+2$. Assume that the initial data $(w, B)|_{t=0} = (w_0, B_0)$ satisfy $w_0 \in H^s$, $B_0 \in H^s$, and $\text{div} \, w_0 = 0$, $\text{div} \, B_0 = 0$. Then, there exist a $T^* \in (0, \infty]$ and a unique solution $(w, B) \in L^\infty(0, T^*; H^s)$ to the incompressible MHD equations (1.21)–(1.23), and for any $0 < T < T^*$,

$$
\sup_{0 \leq t \leq T} \{ \| (w, B)(t) \|_{H^s} + \| (\text{div} \, w, \partial_t B)(t) \|_{H^{s-2}} + \| \nabla \pi(t) \|_{H^{s-2}} \} \leq C.
$$

Denoting $U^* = (q^*, u^*, H^*, \phi^*)^T$, we rewrite the system (1.17)–(1.19) in the vector form

$$
A_0(U^*) \partial_t U^* + \sum_{j=1}^{3} A_j(U^*) \partial_j U^* = Q(U^*),
$$

(2.1)
where

\[ Q(U^\epsilon) = (0, F(u^\epsilon), \nu \Delta H^\epsilon, \kappa \Delta \phi^\epsilon + \epsilon (L(u^\epsilon) + G(H^\epsilon)))^\top, \]

with

\[
F(u^\epsilon) = 2\mu \text{div} (D(u^\epsilon)) + \lambda \nabla (\text{tr} D(u^\epsilon)), \\
L(u^\epsilon) = 2\mu |D(u^\epsilon)|^2 + \lambda (\text{tr} D(u^\epsilon))^2, \\
G(H^\epsilon) = v |\nabla \times H^\epsilon|^2,
\]

and the matrices \( A_j (U^\epsilon) \) \((0 \leq j \leq 3)\) are given by

\[
A_0 (U^\epsilon) = \begin{pmatrix}
1 + \epsilon q^\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 1 + \epsilon q^\epsilon & 0 & 0 & 0 & 0 \\
0 & 0 & 1 + \epsilon q^\epsilon & -H_1^\epsilon & 0 & 0 \\
0 & 0 & 0 & 1 + \epsilon q^\epsilon & -H_2^\epsilon & 0 \\
0 & 0 & 0 & 0 & 1 + \epsilon q^\epsilon & -H_3^\epsilon \\
0 & 0 & 0 & 0 & 0 & 1 + \epsilon q^\epsilon
\end{pmatrix},
\]

\[
A_1 (U^\epsilon) = \begin{pmatrix}
u \Delta \phi^\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & \nu \Delta \phi^\epsilon & 0 & 0 & 0 & 0 \\
0 & 0 & \nu \Delta \phi^\epsilon & -H_1^\epsilon & 0 & 0 \\
0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_2^\epsilon & 0 \\
0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_3^\epsilon \\
0 & 0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon
\end{pmatrix},
\]

\[
A_2 (U^\epsilon) = \begin{pmatrix}
u \Delta \phi^\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & \nu \Delta \phi^\epsilon & 0 & 0 & 0 & 0 \\
0 & 0 & \nu \Delta \phi^\epsilon & -H_1^\epsilon & 0 & 0 \\
0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_2^\epsilon & 0 \\
0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_3^\epsilon \\
0 & 0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon
\end{pmatrix},
\]

\[
A_3 (U^\epsilon) = \begin{pmatrix}
u \Delta \phi^\epsilon & 0 & 0 & 0 & 0 & 0 \\
0 & \nu \Delta \phi^\epsilon & 0 & 0 & 0 & 0 \\
0 & 0 & \nu \Delta \phi^\epsilon & -H_1^\epsilon & 0 & 0 \\
0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_2^\epsilon & 0 \\
0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon & -H_3^\epsilon \\
0 & 0 & 0 & 0 & 0 & \nu \Delta \phi^\epsilon
\end{pmatrix},
\]

It is easy to see that the matrices \( A_j (U^\epsilon) \) \((0 \leq j \leq 3)\) can be symmetrized by choosing

\[
\tilde{A}_0 (U^\epsilon) = \text{diag}(1 + \epsilon \phi^\epsilon)(1 + \epsilon q^\epsilon)^{-1},
\]

Moreover, for \( U^\epsilon \in \tilde{G}_1 \subset\subset G \) with \( G \) being the state space for the system (2.1), \( \tilde{A}_0 (U^\epsilon) \) is a positive definite symmetric matrix for sufficiently small \( \epsilon \).

Assume that the initial data \( U^\epsilon (x, 0) = U^\epsilon_0 (x) = (q^\epsilon_0 (x), \bar{u}^\epsilon_0 (x), \bar{H}^\epsilon_0 (x), \phi^\epsilon_0 (x))^\top \in H^1 \)
and \( U^\epsilon_0 (x) \in G_0, \tilde{G}_0 \subset\subset G \). The main theorem of this paper is the following.
Theorem 2.2. Let $s > \frac{3}{2} + 2$. Suppose that the initial data $U_0^\epsilon(x)$ satisfy
\[ \|U_0^\epsilon(x) - (0, u_0(x), B_0(x), 0)^\top\|_s = O(\epsilon). \]
Let $(w, B, \pi)$ be a smooth solution to (1.21)–(1.23) obtained in proposition 2.1. If $(w, \pi) \in C([0, T^*], H^{s+2}) \cap C^1([0, T^*], H^s)$ with $T^* > 0$ finite, then there exists a constant $\epsilon_0 > 0$ such that, for all $\epsilon \leq \epsilon_0$, the system (2.1) with initial data $U_0^\epsilon(x)$ has a unique smooth solution $U^\epsilon(x, t) \in C([0, T^*], H^s)$. Moreover, there exists a positive constant $K > 0$, independent of $\epsilon$, such that, for all $\epsilon \leq \epsilon_0$,
\[ \sup_{t \in [0, T^*]} \left\| U^\epsilon(\cdot, t) - \left( \frac{\epsilon}{2} \pi, w, B, \frac{\epsilon}{2} \pi \right)^\top \right\|_s \leq K \epsilon. \] (2.2)

Remark 2.1. From theorem 2.2, we know that for sufficiently small $\epsilon$ and well-prepared initial data, the full MHD equations (1.1)–(1.4) admit a unique smooth solution on the same time interval where the smooth solution of the incompressible MHD equations exists. Moreover, the solution can be approximated as shown in (2.2).

Remark 2.2. We remark that the constant $K$ in (2.2) is also independent of the coefficients $\mu, \nu$ and $\kappa$. This is quite different from the results by Hagstrom and Loranz [10], where the estimates do depend on $\mu$ intimately. Our approach is still valid for the ideal compressible MHD equations. However, we will give a particular analysis for the ideal model with more general pressure using the entropy form of the energy equation rather than the thermal energy equation in (1.8).

The ideal compressible MHD equations can be written as
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \quad (2.3) \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p &= \frac{1}{4\pi} (\nabla \times H) \times H, \quad (2.4) \\
\partial_t H - \nabla \times (u \times H) &= 0, \quad \text{div} H = 0, \quad (2.5) \\
\partial_t E + \text{div} (u(E' + p)) &= \frac{1}{4\pi} \text{div} ((u \times H) \times H). \quad (2.6)
\end{align*}

With the help of the Gibbs relation
\[ \theta \, dS = de + p \, d \left( \frac{1}{\rho} \right) \]
and the identity (1.7), the energy balance equation (2.6) is replaced by
\[ \partial_t (\rho S) + \text{div} (\rho Su) = 0, \quad (2.7) \]
where $S$ denotes the entropy. We reconsider the equation of state as a function of $S$ and $p$, i.e. $\rho = R(S, p)$ for some positive smooth function $R$ defined for all $S$ and $p > 0$, and satisfying $\frac{\partial R}{\partial p} > 0$. Then, by utilizing (2.3), (1.15) and (1.16), together with the constraint $\text{div} H = 0$, the system (2.3)–(2.5) and (2.7) can be written in the dimensionless form as follows (after applying the arguments similar to those in the appendix): 
\begin{align*}
A(S^\epsilon, p^\epsilon) (\partial_t p^\epsilon + u^\epsilon \cdot \nabla p^\epsilon) + \text{div} u^\epsilon &= 0, \quad (2.8) \\
R(S^\epsilon, p^\epsilon) (\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon) + \frac{\nabla p^\epsilon}{\epsilon^2} - H^\epsilon \cdot \nabla H^\epsilon + \frac{1}{2} \nabla (|H^\epsilon|^2) &= 0, \quad (2.9) \\
\partial_t H^\epsilon + u^\epsilon \cdot \nabla H^\epsilon + \text{div} u^\epsilon H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon &= 0, \quad \text{div} H^\epsilon = 0, \quad (2.10) \\
\partial_t S^\epsilon + u^\epsilon \cdot \nabla S^\epsilon &= 0, \quad (2.11)
\end{align*}
where $A(S^\epsilon, p^\epsilon) = \frac{1}{R(S^\epsilon, p^\epsilon)} \frac{\partial R(S^\epsilon, p^\epsilon)}{\partial p^\epsilon}$.
To study the low Mach number limit of the above system, we introduce the transformation
\[ p_e^\epsilon(x, t) = p e^{\epsilon q^\epsilon(x, t)}, \quad S^\epsilon(x, t) = S + \epsilon \Theta^\epsilon(x, t), \]  
where \( p \) and \( S \) are positive constants, to obtain that
\[ a(S + \epsilon \Theta^\epsilon, \epsilon q^\epsilon)(\partial_t q^\epsilon + u^\epsilon \cdot \nabla q^\epsilon) + \frac{1}{\epsilon} \div u^\epsilon = 0, \]  
\[ r(S + \epsilon \Theta^\epsilon, \epsilon q^\epsilon)(\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon) + \frac{1}{\epsilon} \nabla q^\epsilon - H^\epsilon \cdot \nabla H^\epsilon + \frac{1}{2} \nabla(|H^\epsilon|^2) = 0, \]  
\[ \partial_t H^\epsilon + u^\epsilon \cdot \nabla H^\epsilon + \div u^\epsilon H^\epsilon - H^\epsilon \cdot \nabla u^\epsilon = 0, \quad \div H^\epsilon = 0, \]  
\[ \partial_t \Theta^\epsilon + u^\epsilon \cdot \nabla \Theta^\epsilon = 0, \]  
where
\[ a(S^\epsilon, \epsilon q^\epsilon) = A(S^\epsilon, p e^{\epsilon q^\epsilon}) p e^{\epsilon q^\epsilon} = \frac{pe^{\epsilon q^\epsilon}}{R(S^\epsilon, p e^{\epsilon q^\epsilon})} \frac{\partial R(S^\epsilon, s)}{\partial s} \bigg|_{s = pe^{\epsilon q^\epsilon}}. \]

Making use of the fact that \( \curl \nabla = 0 \) and letting \( \epsilon \to 0 \) in (2.13) and (2.14), we formally deduce that \( \div v = 0 \) and
\[ \curl (r(S, 0)(\partial_t v + v \cdot \nabla v) - (\nabla \times J) \times J) = 0, \]  
where we have supposed that the limits \( u^\epsilon \to v \) and \( H^\epsilon \to J \) exist. Thus, we can expect that the limiting system of (2.13)–(2.16) takes the form
\[ r(S, 0)(\partial_t v + v \cdot \nabla v) - (\nabla \times J) \times J + \nabla \Pi = 0, \]  
\[ \partial_t J + v \cdot \nabla J = J \cdot \nabla v = 0, \]  
\[ \div v = 0, \quad \div J = 0 \]  
for some function \( \Pi \).

In order to state our result, we first recall the local existence of strong solutions to the ideal incompressible MHD equations (2.17)–(2.19) in the domain \( \Omega \). The proof can be found in [7, 28].

**Proposition 2.3 ([7, 28]).** Let \( s > 3/2 + 1 \). Assume that the initial data \((v, J)|_{t=0} = (v_0, J_0)\) satisfy \( v_0 \in H^s, J_0 \in H^s \) and \( \div v_0 = 0, \div J_0 = 0 \). Then, there exist a \( T^* \in (0, \infty] \) and a unique smooth solution \((v, J) \in L^\infty(0, T^*; H^s)\) to the incompressible MHD equations (1.21)–(1.23), and for any \( 0 < T < T^* \),
\[ \sup_{0 \leq t \leq T} \left[ ||(v, J)(t)||_{H^s} + ||(\partial_t v, \partial_t J)(t)||_{H^{s-1}} + ||\nabla \Pi(t)||_{H^{s-1}} \right] \leq C. \]

In the vector form, we arrive at, for \( V^\epsilon = (q^\epsilon, u^\epsilon, H^\epsilon, \Theta^\epsilon)^\top \), that
\[ A_0(\epsilon \Theta^\epsilon, \epsilon q^\epsilon) \partial_t V^\epsilon + \sum_{j=1}^3 \left[ u^\epsilon_j A_0(\epsilon \Theta^\epsilon, \epsilon q^\epsilon) + \epsilon^{-1} C_j + B_j(H^\epsilon) \right] \partial_j V^\epsilon = 0, \]  
where
\[ A_0(\epsilon \Theta^\epsilon, \epsilon q^\epsilon) = \text{diag}(a(S^\epsilon, \epsilon q^\epsilon), r(S^\epsilon, \epsilon q^\epsilon), r(S^\epsilon, \epsilon q^\epsilon), r(S^\epsilon, \epsilon q^\epsilon), 1, 1, 1, 1), \]
and \( C_j \) is symmetric constant matrix, and \( B_j(H^\epsilon) \) is a symmetric matrix of \( H^\epsilon \).

Assume that the initial data for the equations (2.20) satisfy
\[ V^\epsilon_0(x) = (\tilde{q}_0^\epsilon(x), \tilde{u}_0^\epsilon(x), \tilde{H}_0^\epsilon(x), \tilde{\Theta}_0^\epsilon(x))^\top \in H^s, \quad \text{and} \quad V^\epsilon_0(x) \in G_0, \quad \tilde{G}_0 \subset \subset G \]
Theorem 2.4. Let \( s > 3/2 + 1 \). Suppose that the initial data \( V_0^s(x) \) satisfy
\[
\|V_0^s(x) - (0, v_0(x), J_0(x), 0)^\top\|_{s} = O(\epsilon).
\]
Let \( (v, J, \Pi) \) be a smooth solution to (2.17)–(2.19) obtained in proposition 2.3. If \( (v, \Pi) \in C([0, \bar{T}_\epsilon], H^{s+1}) \cap C^1([0, \bar{T}_\epsilon], H^s) \) with \( \bar{T}_\epsilon > 0 \) finite, then there exists a constant \( \epsilon_1 > 0 \) such that, for all \( \epsilon \leq \epsilon_1 \), the system (2.20) with initial data \( V_0^s(x) \) has a unique solution \( V^s(x, t) \in C([0, \bar{T}_\epsilon], H^s) \). Moreover, there exists a positive constant \( K_1 > 0 \) such that, for all \( \epsilon \leq \epsilon_1 \),
\[
\sup_{t \in [0, \bar{T}_\epsilon]} \|V^s(\cdot, t) - (\epsilon \Pi, v, J, \epsilon \Pi)^\top\|_{s} \leq K_1 \epsilon.
\]

3. Proof of theorem 2.2

This section is devoted to proving theorem 2.2. First, following the proof of the local existence theory for the initial value problem of symmetrizable hyperbolic–parabolic systems by Volpert and Hudjaev in [29], we obtain that there exists a time interval \([0, T] \) with \( T > 0 \) so that the system (2.1) with initial data \( U_0^s(x) \) has a unique classical solution \( U^s(x, t) \in C([0, T], H^s) \) and \( U^s(x, t) \in G_2 \) with \( G_2 \subset \subset G \). We remark that the crucial step in the proof of local existence result is to prove the uniform boundedness of the solutions. See also [19] for some relative results.

Now, define
\[
T_\epsilon = \sup \{ T > 0 : U^s(x, t) \in C([0, T], H^s), U^s(x, t) \in G_2, \forall (x, t) \in \Omega \times [0, T] \}.
\]
Note that \( T_\epsilon \) depends on \( \epsilon \) and may tend to zero as \( \epsilon \) goes to 0.

To show that \( \lim_{\epsilon \to 0} T_\epsilon > 0 \), we shall make use of the convergence–stability lemma which was established in [3, 30] for hyperbolic systems of balance laws. It is also implied in [30] that a convergence–stability lemma can be formulated as a part of (local) existence theories for any evolution equations. For the hyperbolic–parabolic system (2.1), we have the following convergence–stability lemma.

Lemma 3.1. Let \( s > 3/2 + 2 \). Suppose that \( U_0^s(x) \in G_0, \tilde{G}_0 \subset \subset G \), and \( U_0^s(x) \in H^s \), and the following convergence assumption (A) holds.

(A) There exists \( T_* > 0 \) and \( U_\epsilon \in L^n(0, T_*; H^s) \) for each \( \epsilon \), satisfying
\[
\bigcup_{x, t, \epsilon} [U_\epsilon(x, t)] \subset \subset G,
\]
such that for \( t \in [0, \min\{T_*, T_\epsilon\}] \),
\[
\sup_{x, t} \|U^s(x, t) - U_\epsilon(x, t)\|_s = o(1), \quad \sup_{t} \|U^s(x, t) - U_\epsilon(x, t)\|_s = O(1), \quad \text{as } \epsilon \to 0.
\]
Then, there exist an \( \tilde{\epsilon} > 0 \) such that, for all \( \epsilon \in (0, \tilde{\epsilon}] \), it holds that
\[
T_\epsilon > T_*.
\]

To apply lemma 3.1, we construct the approximation \( U_\epsilon = (q_\epsilon, v_\epsilon, B_\epsilon, \phi_\epsilon)^\top \) with \( q_\epsilon = \epsilon \pi/2, v_\epsilon = w, B_\epsilon = B, \) and \( \phi_\epsilon = \epsilon \pi/2 \), where \((w, B, \pi)\) is the classical solution to the system (1.21)–(1.23) obtained in proposition 2.1. It is easy to verify that \( U_\epsilon \) satisfies
\[
\frac{\partial_t q_\epsilon + v_\epsilon \cdot \nabla q_\epsilon + \frac{1}{\epsilon} (1 + \epsilon q_\epsilon) \text{div } v_\epsilon}{\epsilon} = \frac{\pi_\epsilon + w \cdot \nabla \pi}{2},
\]

(3.1)
Low Mach number limit for full magnetohydrodynamic equations

\[
(1 + \epsilon q\epsilon)(\partial_t v_\epsilon + v_\epsilon \cdot \nabla v_\epsilon) + \frac{1}{\epsilon} \left[ (1 + \epsilon q\epsilon) \nabla \phi_\epsilon + (1 + \epsilon \phi_\epsilon) \nabla q_\epsilon \right] - B_\epsilon \cdot \nabla B_\epsilon + \frac{1}{2} \nabla [(B_\epsilon)_2^2] = \mu \Delta v_\epsilon + \frac{\epsilon^2}{2} \pi (w_\epsilon + w \cdot \nabla w + \nabla \pi),
\]

(3.2)

\[
\partial_t B_\epsilon + v_\epsilon \cdot \nabla B_\epsilon + \text{div } v_\epsilon B_\epsilon - B_\epsilon \cdot \nabla v_\epsilon = \nu \Delta B_\epsilon, \quad \text{div } B_\epsilon = 0,
\]

(3.3)

\[
(1 + \epsilon q\epsilon)(\partial_t \phi_\epsilon + v_\epsilon \cdot \nabla \phi_\epsilon) + \gamma - 1 \epsilon (1 + \epsilon q\epsilon)(1 + \epsilon \phi_\epsilon) \text{div } v_\epsilon = \left( \frac{\epsilon}{2} + \frac{\epsilon^3}{4\pi} \right) (\pi t + w \cdot \nabla \pi).
\]

(3.4)

We rewrite the system (3.1)–(3.4) in the following vector form:

\[
A_0(U_\epsilon) \partial_t U_\epsilon + \sum_{j=1}^3 A_j(U_\epsilon) \partial_j U_\epsilon = S(U_\epsilon) + R,
\]

(3.5)

with

\[
S(U_\epsilon) = (0, \mu \Delta v_\epsilon, \nu \Delta B_\epsilon, 0)^T
\]

and

\[
R = \begin{pmatrix}
\frac{\epsilon}{2} (\pi t + w \cdot \nabla \pi) \\
\frac{\epsilon^2}{2} (w_\epsilon + w \cdot \nabla w + \nabla \pi) \\
\frac{\epsilon}{2} + \frac{\epsilon^3}{4\pi} (\pi t + w \cdot \nabla \pi) \\
0
\end{pmatrix}.
\]

Due to the regularity assumptions on \((w, \pi)\) in theorem 2.2, we have

\[
\max_{t \in [0, T^*]} \| R(t) \|_{L^2} \leq C \epsilon.
\]

To prove theorem 2.2, it suffices to prove the error estimate in (2.2) for \(t \in [0, \min\{T^*, T_\epsilon\}]\) thanks to lemma 3.1. To this end, introducing

\[
E = U^\epsilon - U_\epsilon \quad \text{and} \quad A_j(U) = A_{0_j}(U) A_j(U),
\]

and using (2.1) and (3.5), we see that

\[
E_t + \sum_{j=1}^3 A_j(U^\epsilon) E_{x_j} = (A_j(U_\epsilon) - A_j(U^\epsilon)) U_{x_j} + A_{0_j}^{-1}(U^\epsilon) Q(U^\epsilon)
\]

\[
- A_{0_j}^{-1}(U_\epsilon) S(U_\epsilon) + R.
\]

(3.6)

For any multi-index \(\alpha\) satisfying \(|\alpha| \leq s\), we take the operator \(D^\alpha\) to (3.6) to obtain

\[
\partial_t D^\alpha E + \sum_{j=1}^3 A_j(U^\epsilon) \partial_{x_j} D^\alpha E = P_1^\alpha + P_2^\alpha + Q^\alpha + R^\alpha
\]

(3.7)

with

\[
P_1^\alpha = \sum_{j=1}^3 \{ A_j(U^\epsilon) \partial_{x_j} D^\alpha E - D^\alpha (A_j(U^\epsilon) \partial_{x_j} E) \},
\]

\[
P_2^\alpha = \sum_{j=1}^3 D^\alpha \{ (A_j(U_\epsilon) - A_j(U^\epsilon)) U_{x_j} \},
\]

\[
Q^\alpha = D^\alpha [ A_{0_j}^{-1}(U^\epsilon) Q(U^\epsilon) - A_{0_j}^{-1}(U_\epsilon) S(U_\epsilon) ],
\]

\[
R^\alpha = D^\alpha [ A_{0_j}^{-1}(U_\epsilon) R ].
\]
Define
\[ \tilde{A}_0(U^*) = \text{diag}\left( \frac{1 + \epsilon \phi^s}{(1 + \epsilon q^s)^2}, 1, 1, 1, \frac{1}{1 + \epsilon q^r}, \frac{1}{1 + \epsilon q^s}, \frac{1}{\gamma - 1}(1 + \epsilon \phi^r) \right), \]
and the canonical energy by
\[ \|E\|_c^2 := \int (\tilde{A}_0(U^*) E, E) \, dx. \]

Note that \( \tilde{A}_0(U^*) \) is a positive definite symmetric matrix and \( \tilde{A}_0(U^*) A_j(U^*) \) is symmetric.

Now, if we multiply (3.7) with \( \tilde{A}_0(U^*) \) and take the inner product between the resulting system and \( D^a E \), we arrive at
\[ \frac{d}{dt}\|D^a E\|_c^2 = 2 \int (\Gamma D^a E, D^a E) \, dx + 2 \int (D^a E)^T \tilde{A}_0(U^*) (P_q^a + P_\phi^a + Q^a + R^a), \quad (3.8) \]
where
\[ \Gamma = (\tilde{a}_i, \nabla) \cdot \left( \tilde{A}_0, \tilde{A}_0(U^*) A_1(U^*), \tilde{A}_0(U^*) A_2(U^*), \tilde{A}_0(U^*) A_3(U^*) \right). \]

Next, we estimate various terms on the right-hand side of (3.8). Note that our estimates only need to be done for \( t \in [0, \min(T^*, T_1)] \), in which both \( U^* \) and \( U_\epsilon \) are regular enough and take values in a convex compact subset of the state space. Thus, we have
\[ C^{-1} \int |D^a E|^2 \leq \|D^a E\|_c^2 \leq C \int |D^a E|^2 \quad (3.9) \]
and
\[ |(D^a E)^T \tilde{A}_0(U^*) (P_q^a + P_\phi^a + Q^a + R^a)| \leq C (|D^a E|^2 + |P_q^a|^2 + |P_\phi^a|^2 + |R^a|^2). \]

To estimate \( \Gamma \), we write \( A_j(U^*) = u^j I_k + \tilde{A}_j(U^*) \). Note that \( \tilde{A}_j(U^*) \) depends only on \( q^s, \phi^r \) and \( H^s \). Thus using (1.17) and (1.19), we have
\[ |\Gamma| = \left| \frac{\partial}{\partial t} \tilde{A}_0 + u^j \cdot \nabla \tilde{A}_0 + \tilde{A}_0 \text{div } u^j + \text{div } (\tilde{A}_0 \tilde{A}_j(U^*)) \right| \]
\[ = \left| \tilde{A}_0 \text{div } u^j - \tilde{A}_0 \text{div } u^j - \tilde{A}_0 \text{div } u^j - \tilde{A}_0 \text{div } u^j + \kappa (1 + \epsilon q^s)^{-1} \Delta \phi^r \right| \]
\[ \leq C + C (|\nabla u^j| + |\nabla q^s| + |\nabla \phi^r| + |\nabla H^s| + |\Delta \phi^r| + |\nabla u^j|^2 + |\nabla H^s|^2) \]
\[ \leq C + C (|\nabla |E|^2| + |E|^2| + C (|\Delta \phi^r| + |\nabla U_\epsilon| + |\nabla U_\epsilon|^2) \]
\[ \leq C + C (|E|_s + ||E||_c^2), \]
where we have used Sobolev’s embedding theorem and the fact that \( s > 3/2 + 2 \), and the symbols \( \tilde{A}_0' \) and \( \tilde{A}_0'' \) denote the differentiation of \( A_0 \) with respect to \( \rho^s \) and \( \theta^s \), respectively.

Since
\[ A_j(U^*) \partial_j D^a E = D^a (A_j(U^*) \partial_j E) = - \sum_{0 < \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \partial^\beta A_j(U^*) \partial^{a-\beta} E_{s_j}, \]
we obtain, with the help of the Moser-type calculus inequalities in Sobolev spaces, that
\[ \|P_q^a\| \leq C \left( \|u^j, H^s\|_s \|E_{s_j}\|_{\|\cdot\|^{-1}} + \|\epsilon^{-1} (\partial^\beta f(q^s, \phi^s) \partial^{a-\beta} E_{s_j})\| \right) \]
\[ + C \|\partial^\beta (1 + \epsilon q^s)^{-1} (H^s - B_{s_j}) + (1 + \epsilon q^s)^{-1} - (1 + \epsilon q_s)^{-1} B_{s_j} \| \partial^{a-\beta} E_{s_j} \| \]
\[ \leq C (1 + \|E\|_s + \|q^s, \phi^s\|_{\|\cdot\|}^2) \|E_{s_j}\|_{\|\cdot\|^{-1}} \]
\[ \leq C (1 + \|E\|_s + \|q^s, \phi^s\|_{\|\cdot\|}^2) \|E_{s_j}\|_{\|\cdot\|^{-1}}, \]
where \( f(q^s, \phi^s) = (1 + \epsilon q^s)^{-1} (1 + \epsilon \phi^r) + (1 + \epsilon q^s)^{-1} (1 + \epsilon \phi^r). \)
Similarly, utilizing the boundedness of \( \|U_\epsilon\|_{s+1} \), the term \( P_2^\alpha \) can be bounded as follows:

\[
\|P_2^\alpha\| \leq C\|U_{\epsilon, j}\|_s \|A_j(U_\epsilon) - A_j(U_j^\alpha)\|_{|\omega|}
\leq C\|(u_j^\alpha - v_j)I\|_s + \tilde{A}_j(U_\epsilon) - \tilde{A}_j(U_j^\alpha)\|_{|\omega|}
\leq C(1 + \|u^\alpha - v_\epsilon\|_{|\omega|}) + \|H^\epsilon - B_\epsilon\|_{|\omega|}) + C\|\epsilon^{-1}(f(q_\epsilon, \phi^\epsilon) - f(q_\epsilon, \phi))\|_{|\omega|}
\leq C(1 + \|q_\epsilon + \eta_\alpha(q^\epsilon - q_\epsilon) + \phi_\epsilon + \eta_\alpha(\phi^\epsilon - \phi_\epsilon)\|_{\|E\|_{|\omega|}})
\leq C(1 + \|E\|_{|\omega|})\|E\|_{|\omega|},
\]

where \( 0 \leq \eta_\alpha \leq 1 \) are constants.

The estimate of \( \int (D^\alpha E)^{\frac{1}{2}} \tilde{A}_0(U_j^\alpha) Q^\alpha \) is more complex and delicate. First, we can rewrite

\[
\int (D^\alpha E)^{\frac{1}{2}} \tilde{A}_0(U_j^\alpha) Q^\alpha = \int D^\alpha(u^\alpha - v_\epsilon)D^\alpha[(1 + \epsilon q^\alpha)^{-1} F(u^\alpha) - \mu(1 + \epsilon q_\epsilon)^{-1}\Delta v_\epsilon]
\]

\[+ \nu \int D^\alpha(H^\epsilon - B_\epsilon)(1 + \epsilon q^\alpha)^{-1} D^\alpha(\Delta H^\epsilon - \Delta B_\epsilon)
\]

\[+ \kappa(\gamma - 1)^{-1}\int D^\alpha(\phi^\epsilon - \phi_\epsilon)(1 + \epsilon \phi^\alpha)^{-1} D^\alpha[(1 + \epsilon q^\epsilon)^{-1}\Delta \phi^\epsilon - (1 + \epsilon q_\epsilon)^{-1}\Delta \phi_\epsilon]
\]

\[+ \epsilon(\gamma - 1)^{-1}\int D^\alpha(\phi^\epsilon - \phi_\epsilon)(1 + \epsilon \phi^\alpha)^{-1} D^\alpha[(1 + \epsilon q^\epsilon)^{-1}(L(u^\epsilon) + G(H^\epsilon))]
\]

= \( Q_a + Q_H + Q_\phi_1 + Q_\phi_2 \).

By integration by parts, the Cauchy and Moser-type inequalities, and Sobolev’s embedding theorem, we find that \( Q_a \) can be controlled as follows:

\[
Q_a = \int D^\alpha(u^\alpha - v_\epsilon)D^\alpha[(1 + \epsilon q^\alpha)^{-1} \mu \Delta (u^\alpha - v_\epsilon) + (\mu + \lambda)\nabla \text{div} (u^\alpha - v_\epsilon)]
\]

\[+ \mu \int D^\alpha(u^\alpha - v_\epsilon)D^\alpha[(1 + \epsilon q^\alpha)^{-1} - (1 + \epsilon q_\epsilon)^{-1}]\Delta v_\epsilon
\]

\[\leq - \int \frac{\mu}{1 + \epsilon q^\alpha} [D^\alpha \nabla (u^\alpha - v_\epsilon)]^2 - \int \frac{\mu + \lambda}{1 + \epsilon q^\alpha} [D^\alpha \text{div} (u^\alpha - v_\epsilon)]^2
\]

\[+ \int D^\alpha(u^\alpha - v_\epsilon) \sum_{0 < \beta \leq \alpha} D^\beta[(1 + \epsilon q^\alpha)^{-1} D^{\alpha-\beta} \{\mu \Delta (u^\alpha - v_\epsilon)
\]

\[+ (\mu + \lambda)\nabla \text{div} (u^\alpha - v_\epsilon)] + C\|E\|_{|\omega|}^2 + C\|E\|_{|\omega|}^2 + C\|D^\alpha \nabla (u^\alpha - v_\epsilon)\|_2
\]

\[\leq - C\int \mu |D^\alpha \nabla (u^\alpha - v_\epsilon)|^2 - C\int (\mu + \lambda) |D^\alpha \text{div} (u^\alpha - v_\epsilon)|^2
\]

\[+ C\|D^\alpha \nabla (u^\alpha - v_\epsilon)\|_2^2 + C\|E\|_{|\omega|}^4 + C\|E\|_{|\omega|}^2
\]

\[+ C\|D^\alpha \nabla (u^\alpha - v_\epsilon)\|_2^2 + C\|E\|_{|\omega|}^4 + C\|E\|_{|\omega|}^2 + \int D^\alpha(u^\alpha - v_\epsilon).
\]

\[
\sum_{1 < \beta \leq \alpha} D^\beta[(1 + \epsilon q^\alpha)^{-1} |D^{\alpha-\beta} \{\mu \Delta (u^\alpha - v_\epsilon) + (\mu + \lambda)\nabla \text{div} (u^\alpha - v_\epsilon)\}]
\]

\[\leq - C\int \mu |D^\alpha \nabla (u^\alpha - v_\epsilon)|^2 - C\int (\mu + \lambda) |D^\alpha \text{div} (u^\alpha - v_\epsilon)|^2
\]

\[+ C\|D^\alpha \nabla (u^\alpha - v_\epsilon)\|_2 + C\|E\|_{|\omega|}^2 + C\|E\|_{|\omega|}^2.
\]

Similarly, the terms \( Q_H, Q_\phi_1 \), and \( Q_\phi_2 \) can be bounded as follows:

\[
Q_H \leq - C\nu \int |D^\alpha \nabla (H^\epsilon - B_\epsilon)|^2 + C\epsilon \|D^\alpha \nabla (H^\epsilon - B_\epsilon)\|^2 + C\|E\|_{|\omega|}^4 + C\|E\|_{|\omega|}^2
\]

\[
Q_\phi_1 \leq - C\nu \int |D^\alpha \nabla (\phi^\epsilon - \phi_\epsilon)|^2 + C\epsilon \|D^\alpha \nabla (\phi^\epsilon - \phi_\epsilon)\|^2 + C\|E\|_{|\omega|}^4 + C\|E\|_{|\omega|}^2.
\]
and
\[ Q_{\phi} \leq C \epsilon \| D^\alpha \nabla (\phi^\epsilon - \phi) \|^2 + C \| E \|^2_\alpha + C \| E \|_\alpha^2. \]

Putting all the above estimates into (3.8) and taking \( \epsilon \) small enough, we obtain that
\[
\frac{d}{dt} \| D^\alpha E \|^2_e + \xi \int |D^\alpha \nabla (u^\epsilon - v^\epsilon)|^2 + \nu \int |D^\alpha \nabla (H^\epsilon - B^\epsilon)|^2 + \kappa \int |D^\alpha \nabla (\phi^\epsilon - \phi^\epsilon)|^2 \leq C \| R^\alpha \|^2 + C (1 + \| E \|^2_\alpha) \| E \|_\alpha^2 + \| E \|^2_e,
\]
(3.10)
where we have used the following estimate:
\[
\mu \int |D^\alpha \nabla (u^\epsilon - v^\epsilon)|^2 + (\mu + \lambda) \int |D^\alpha \text{div} (u^\epsilon - v^\epsilon)|^2 \geq \xi \int |D^\alpha \nabla (u^\epsilon - v^\epsilon)|^2
\]
for some positive constant \( \xi > 0 \).

Using (3.9), we integrate the inequality (3.10) over \((0, t)\) with \( t < \min \{ T^*, T^\epsilon \} \) to obtain
\[
\| D^\alpha E(t) \|^2_e \leq \| D^\alpha E(0) \|^2 + C \int_0^t \| R^\alpha(\tau) \|^2 d\tau + C \int_0^t \{ (1 + \| E(\tau) \|^2_\alpha) \| E \|_\alpha^2 + \| E \|^2_e \}(\tau) d\tau.
\]
Summing up this inequality for all \( \alpha \) with \( |\alpha| \leq s \), we obtain
\[
\| E(t) \|^2_\alpha \leq \| E(0) \|^2_\alpha + C \int_0^t \| R(\tau) \|^2 d\tau + C \int_0^t \{ (1 + \| E(\tau) \|^2_\alpha) \| E \|_\alpha^2 \}(\tau) d\tau.
\]
With the help of Gronwall’s lemma and the fact that
\[
\| E(0) \|^2_\alpha + \int_0^{T^*} \| R(\tau) \|^2 d\tau = O(\epsilon^2),
\]
we conclude that
\[
\| E(t) \|^2_\alpha \leq C \epsilon^2 \exp \left\{ C \int_0^t (1 + \| E(\tau) \|^2_\alpha) d\tau \right\} \equiv \Phi(t).
\]
It is easy to see that \( \Phi(t) \) satisfies
\[
\Phi'(t) = C (1 + \| E(t) \|^2_\alpha) \Phi(t) \leq C \Phi(t) + C \Phi^{s+1}(t).
\]
Thus, employing the nonlinear Gronwall-type inequality, we conclude that there exists a constant \( K \), independent of \( \epsilon \), such that
\[
\| E(t) \|^2_\alpha \leq K \epsilon,
\]
for all \( t \in [0, \min(\{ T^*, T^\epsilon \}) \) provided \( \Phi(0) = C \epsilon^2 < \exp(-CT^*) \). Thus, the proof is completed.

4. Proof of theorem 2.4

The proof of theorem 2.4 is essentially similar to that of theorem 2.2, and we only give some explanations here. The local existence of classical solution to the system (2.20) is given by the proof of theorem 2.1 in [24]. For each fixed \( \epsilon \), we assume that the maximal time interval of existence is \([0, T^\epsilon]\). To prove theorem 2.4, it is crucial to obtain the error estimates
in (2.21). For this purpose, we construct the approximation \( V_\epsilon = (q_\epsilon, v_\epsilon, J_\epsilon, \Theta_\epsilon) \) with \( q_\epsilon = \epsilon \Pi, v_\epsilon = v, J_\epsilon = J, \) and \( \Theta_\epsilon = \epsilon \Pi. \) It is then easy to verify that \( V_\epsilon \) satisfies

\[
\alpha(\tilde{\Sigma} + \epsilon \Theta_\epsilon, \epsilon q_\epsilon) (\partial_t q_\epsilon + v_\epsilon \cdot \nabla q_\epsilon) + \frac{1}{\epsilon} \nabla v_\epsilon = \epsilon \alpha(\tilde{\Sigma} + \epsilon^2 \Pi, \epsilon^2 \Pi)(\Pi_t + v \cdot \nabla \Pi),
\]

(4.1)

\[
r(\tilde{\Sigma} + \epsilon \Theta_\epsilon, \epsilon q_\epsilon) (\partial_t v_\epsilon + v_\epsilon \cdot \nabla v_\epsilon) + \frac{1}{\epsilon} \nabla q_\epsilon - J_\epsilon \cdot \nabla J_\epsilon + \frac{1}{2} (|J_\epsilon|^2)
\]

\[
= [r(\tilde{\Sigma} + \epsilon \Theta_\epsilon, \epsilon q_\epsilon) - r(\tilde{\Sigma}, 0)](v_t + v \cdot \nabla v),
\]

(4.2)

\[
\partial_t J_\epsilon + v_\epsilon \cdot \nabla J_\epsilon + \nabla v_\epsilon \cdot J_\epsilon - J_\epsilon \cdot \nabla v_\epsilon = 0, \quad \text{div} J_\epsilon = 0.
\]

(4.3)

\[
\partial_t \Theta_\epsilon + v_\epsilon \cdot \nabla \Theta_\epsilon = \epsilon (\Pi_t + v \cdot \nabla \Pi).
\]

(4.4)

Thus we can rewrite (4.1)–(4.4) in the vector form of (2.20) with a source term. Letting \( E = V^* - V_\epsilon, \) we can perform the energy estimates similar to those in the proof of theorem 2.2 to show theorem 2.4. Here we omit the details of the proof for conciseness.

Appendix

We give a dimensionless form of the system (1.1)-(1.3) and (1.8) for the ionized fluid obeying the perfect gas relations (1.9) by following the spirit of [12]. Introduce the new dimensionless quantities:

\[
x_\star = \frac{x}{L_0}, \quad t_\star = \frac{t}{L_0/u_0}, \quad u_\star = \frac{u}{u_0},
\]

\[
H_\star = \frac{H}{H_0}, \quad \rho_\star = \frac{\rho}{\rho_0}, \quad \theta_\star = \frac{\theta}{\theta_0},
\]

where the subscript 0 denotes the corresponding typical values and \( \star \) denotes dimensionless quantities. For convenience, all the coefficients are assumed to be constants. Thus, the dimensionless form of the system (1.1)–(1.3) and (1.8) is obtained by a direct computation:

\[
\frac{\partial \rho_\star}{\partial t_\star} + \text{div}_\star (\rho_\star u_\star) = 0,
\]

\[
\rho_\star \frac{\partial u_\star}{\partial t_\star} + \frac{1}{M^2} \nabla_\star (\rho_\star \theta_\star) = C(\nabla_\star \times H_\star) \times H_\star + \frac{1}{R} \text{div}_\star \Psi_\star,
\]

\[
\rho_\star \frac{\partial \theta_\star}{\partial t_\star} + (\gamma - 1) \rho_\star \theta_\star \text{div}_\star u_\star = \frac{(\gamma - 1)}{R_m} CM^2 |\nabla_\star \times H_\star|^2 + \frac{(\gamma - 1)M^2}{R} \Psi_\star : \nabla_\star u_\star + \frac{\gamma}{RP_\star} \Delta_\star \theta_\star,
\]

\[
\frac{\partial H_\star}{\partial t_\star} - \nabla_\star \times (u_\star \times H_\star) = \frac{1}{R_m} \nabla_\star \times (\nabla_\star \times H_\star), \quad \text{div}_\star H_\star = 0,
\]

where we have used the material derivative

\[
\frac{d}{dt_\star} = \frac{\partial}{\partial t_\star} + u_\star \cdot \nabla_\star,
\]

and the new viscous stress tensor

\[
\Psi_\star = 2\mathbb{D}_\star(u_\star) + \frac{\lambda}{\mu} \text{div}_\star u_\star I_3
\]

with \( \mathbb{D}_\star(u_\star) = (\nabla_\star u_\star + \nabla_\star u_\star^T)/2. \)
In the above dimensionless system, there are following dimensionless characteristic parameters:

Reynolds number: \( R = \frac{\rho_0 u_0 L_0}{\mu} \),
Mach number: \( M = \frac{u_0}{a_0} \),
Prandtl number: \( Pr = \frac{c_p \mu}{\kappa} \),
magnetic Reynolds number: \( R_m = \frac{v_0 L_0}{\nu} \),
Cowling number: \( C = \frac{\mu H_0^2 / 4 \pi \rho_0}{u_0^2} \),

where \( c_p \) is the specific heat at constant pressure and \( a_0 = \sqrt{\gamma \rho_0} \) is the sound speed. Note that \( \gamma = c_p - c_V \) and \( \gamma = c_p / c_V \).

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