Optimal Focusing for Monochromatic Scalar and Electromagnetic Waves

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Abstract

For monochromatic solutions of D’Alembert’s wave equation and Maxwell’s equations, we obtain sharp bounds on the sup norm as a function of the far field energy. The extremizer in the scalar case is radial. In the case of Maxwell’s equation, the electric field maximizing the value at the origin follows longitude lines on the sphere at infinity. In dimension $d = 3$ the highest electric field for Maxwell’s equation is smaller by a factor $2/3$ than the highest corresponding scalar waves.

The highest electric field densities on the balls $B_R(0)$ occur as $R \to 0$. The density dips to half max at $R$ approximately equal to one third the wavelength. The extremizing fields are identical to those that attain the maximum field intensity at the origin.

Key words: Maxwell equations, focusing, energy density, extreme light initiative.

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1 Introduction.

The problem we address is to find for fixed frequency $\omega/2\pi$, the monochromatic solutions of the wave equation and of Maxwell’s equation which achieve the highest field values at a point or more generally the greatest electrical
energy in ball of fixed small radius. They are constrained by the energy at $|x| \sim \infty$. This leads to several variational problems:

- Maximize the field strength at a point.
- For fixed $R$, maximize the energy in a ball of radius $R$.

A third problem is,

- Find $R_{\text{optimal}}$ so that the energy density is largest.

We show that the last is degenerate, the maximum occurring at $R = 0$.

There are experiments in course whose strategy is to focus a number of coherent high power laser beams on a small volume to achieve very high energy densities. The problem was proposed to me by G. Mourou because of his leadership role in the European Extreme Light Initiative. If by better focusing one can reduce the size of the incoming lasers there would be significant benefits. Identification of the extrema guides the deployment of the lasers. With the experimental design as motivation the maximization for the Maxwell equations is interpreted as maximization of the energy in the electric field $E$, ignoring the magnetic contribution. Including the magnetic contribution creates an analogous problem amenable to the techniques introduced here.

As natural as these questions appear, I have been unable to find previous work on them.

Study solutions of the scalar wave equation and of Maxwell equations,

$$v_{tt} - \Delta v = 0, \quad E_t = \text{curl} \, B, \quad B_t = -\text{curl} \, E, \quad \text{div} \, E = \text{div} \, B = 0,$$

for spatial dimensions $d \geq 2$. Units are chosen so that the propagation speed is 1.

**Definition 1.1** A solution $v$ of the wave equation is monochromatic if it is of the form

$$v = \psi(t) u(x), \quad \text{with} \quad \psi'' + \omega^2 \psi = 0, \quad \omega > 0. \tag{1.1}$$

Monochromatic solutions of Maxwell's equation are those of the form

$$\psi(t) \left( E(x), B(x) \right), \quad \text{with} \quad \psi'' + \omega^2 \psi = 0, \quad \omega > 0. \tag{1.2}$$
They are generated by
\[ e^{\pm i\omega t} u(x), \quad \text{and} \quad e^{\pm i\omega t} \begin{pmatrix} E(x) \\ B(x) \end{pmatrix}. \] (1.3)

Scaling \( t, x \to \omega t, \omega x \) reduces the study to the case \( \omega = 1 \). In that case, the **reduced wave equations** are satisfied,
\[ (\Delta + 1) u(x) = 0, \quad (\Delta + 1) E(x) = (\Delta + 1) B(x) = 0. \] (1.4)

**Notation.** The absolute value sign \(| |\) is used to denote the modulus of complex numbers, the length of vectors in \( \mathbb{C}^d \), surface area, and, volume. Examples: \(|S^{d-1}| \) and \(|B_R(0)|\).

**Example 1.2** The plane waves \( e^{i(\pm \omega t + \xi x)} \) with \( |\xi| = \omega \) is a monochromatic solution of the wave equation. Its period and wavelength are equal to \( 2\pi/\omega \). For Maxwell’s equations the analogue is \( E = e^{i(\pm \omega t + \xi x)} e \) with \( e \in \mathbb{C}^d \) satisfying \( \xi \cdot e = 0 \) to guarantee the divergence free condition.

The solutions that interest us tend to zero as \( |x| \to \infty \).

**Example 1.3** When \( d = 3 \), \( u(x) := \sin |x|/|x| \) is a solution of the reduced wave equation (see also Example 3.8). The corresponding solutions of the wave equation is
\[ v = e^{\pm i} \frac{\sin |x|}{|x|} = \frac{1}{2} \left( e^{i(\pm t+|x|)} - e^{i(\pm t-|x|)} \right). \]

For the plus sign, the first term represents an incoming spherical wave and the second outgoing. To create such a solution it suffices to generate the incoming wave. The outgoing wave with the change of sign is then generated by that wave after it focuses at the origin.

**Example 1.4** Finite energy solutions of Maxwell’s equations are those for which \( \int_{\mathbb{R}^d} |E|^2 + |B|^2 \, dx < \infty \). They satisfy
\[ \forall R > 0, \quad \lim_{t \to \infty} \int_{|x| \leq R} |E(t,x)|^2 + |B(t,x)|^2 \, dx = 0. \]

Therefore, the solution \( (E, B) = 0 \) is the only monochromatic solution of finite energy.
The solutions \((E(x), B(x))\) that tend to zero as \(x \to \infty\) define tempered distributions on \(\mathbb{R}^d\). When \((E(x), B(x))\) is a tempered solution of the reduced wave equation, the Fourier Transforms satisfy

\[(1 - |\xi|^2) \hat{E}(\xi) = (1 - |\xi|^2) \hat{B}(\xi) = 0.\]

Therefore the support of \(\hat{E}\) is contained in the unit sphere \(S^{d-1} := \{|\xi| = 1\}\). Since \(1 - |\xi|^2\) has nonvanishing gradient on this set it follows that the value of \(\hat{E}\) on a test function \(\psi(\xi)\) is determined by the restriction of \(\psi\) to \(S^{d-1}\). Therefore there is a distribution \(e \in \mathcal{D}'(\{|\xi| = 1\})\) so that

\[E(x) := \int_{|\xi|=1} e^{ix\xi} e(\xi) \, d\sigma,\]

where we use the usual abuse of notation indicating as an integral the pairing of the distribution \(e\) with the test function \(e^{ix\xi}|_{|\xi|=1}\). Conversely, every such expression is a tempered vector valued solution of the reduced wave equation. If \(e\) is smooth, the principal of stationary phase (see §2.2) shows that as \(|x| \to \infty\),

\[E(x) = \frac{1}{|x|^{(d-1)/2}} \left( e^{-i|x|} e((-x/|x|)) + e^{i\pi(d-1)/4} e^{i|x|} e(x/|x|) + O(1/|x|) \right).\]

The field in \(O(|r|^{-(d-1)/2})\). In particular,

\[
\sup_{R \geq 1} R^{-1} \int_{|x| \leq R} |E(x)|^2 \, d\sigma < \infty, \quad \text{and.} \tag{1.7}
\]

\[
\lim_{R \to \infty} \int_{R \leq |x| \leq 2R} |E(x)|^2 \, dx = c_d \int_{|\xi|=1} |e(\xi)|^2 \, d\sigma. \tag{1.8}
\]

For a field defined by a distribution \(e\), \((1.7)\) holds if and only if \(e \in L^2(S^{d-1})\). In that case \((1.8)\) holds and stationary phase approximation holds in an \(L^2\) sense. This is the class of solutions of the reduced wave equation that we study. Equation \((1.8)\) shows that \(\|e\|_{L^2(S^{d-1})}\) is a natural measure of the strength of the field at infinity.

The divergence free condition in Maxwell’s equations is satisfied if and only if \(\xi \cdot e(\xi) = 0\) on \(S^{d-1}\). In that case, the solutions \(e^{\pm it} E(x)\) of the time dependent equation are linear combinations of the plane waves in Example 1.2. Denote by \(H\) the closed subspace of \(e \in L^2(S^{d-1}; \mathbb{C}^d)\) with \(\xi \cdot e = 0\).
For \( e \in H \) and \( x = r\xi \) with \( |\xi| = 1 \) and \( r >> 1 \), the solution \( e^{it}E(x) \) of Maxwell’s equations satisfies

\[
E(x) \approx \frac{1}{\sqrt{2\pi}} \left( e^{i(t-r)} e(-\xi) + \frac{e^{i\pi(d-1)/4}}{r^{(d-1)/2}} e^{i(t+r)} e(\xi) \right).
\]

In practice, the incoming wave

\[
\frac{1}{\sqrt{2\pi}} \int_{S^{d-1}} e^{i\pi(d-1)/4} \frac{e^{i(t+r)}}{r^{(d-1)/2}} e(\xi) \, d\sigma
\]

is generated at large \( r \) and the monochromatic solution is observed for \( t >> 1 \). The phase factor \( e^{i\pi(d-1)/4} \) corresponds to the phase shift from the focusing at the origin.

The first two variational problems for Maxwell’s equations seek to maximize

\[
J_1(e) := |E(0)|^2, \quad \text{and} \quad J_2(e) := \int_{|x| \leq R} |E(x)|^2 \, dx.
\]

among \( e \in H \) with \( \int_{S^{d-1}} |e(\xi)|^2 \, d\sigma = 1 \).

Theorems 3.1 and 3.2 compute the maxima of \( J_1 \) in the scalar and electromagnetic cases. The maximum in the scalar case and also the vector case without divergence free condition is \( |S^{d-1}| \). It is attained when and only when \( e \) is constant. For the electromagnetic case, \( \xi \cdot e(\xi) = 0 \) so the constant densities are excluded. The maximum is achieved at multiplies and rotations of the field \( \ell(\xi) \) from the next definition.

**Definition 1.5** For \( \xi \in S^{d-1} \) denote by \( \ell(\xi) \) the projection of the vector \((1,0,\ldots,0)\) orthogonal to \( \xi \),

\[
\ell(\xi) := (1,0,\ldots,0) - \left( \xi \cdot (1,0,\ldots,0) \right) \xi = (1,0,\ldots,0) - \xi_1 \xi. \quad (1.9)
\]

\( \ell \) is a vector field whose integral curves are the lines of longitude connecting the pole \((-1,0,\cdots,0)\) to opposite pole \((1,0,\cdots,0)\). The maximum value of \( J_1 \) for electromagnetic waves is smaller by \((d-1)/d\) than the extremum in the scalar case. The same functions also solve the \( J_2 \) problem when \( R \) is not too large. The study of \( J_1 \) is reduced to an application of the Cauchy-Schwartz inequality.

In §4 the maximization of \( J_2 \) is transformed to a problem in spectral theory. Maximizing \( J_2 \) is equivalent to finding the norm of an operator. In the
scalar case we call the operator $L$. Finding the norm is equivalent to finding
the spectral radius of the self adjoint operator $L^*L$. The operator $L^*L$ is
compact and rotation invariant on $L^2(S^{d-1})$. Its spectral theory is reduced
by the spaces of spherical harmonics of order $k$. On the space of spherical
harmonics of degree $k$, $L^*L$ is multiplication by a constant $\Lambda_{d,k}(R)$ computed
exactly in terms of Bessel functions in Theorem 5.2. Theorem 7.1 shows that
in the scalar case $\Lambda_{d,0}(R)$ is the largest for $R \leq \pi/2$.

For the Maxwell problem, the corresponding operator is denoted $L_M^*L_M$. We
do not know all its eigenvalues. However two explicit eigenvalues are
$(2/3)(\Lambda_{d,0}(R) - \Lambda_{d,2}(R))$, and, $\Lambda_{d,1}(R)$. When $R \leq \pi/2$, Theorem 7.3 proves
that they are the largest and second largest eigenvalues. The proof uses
the minimax principal. The eigenfunctions for the largest eigenvalue are
rotates of multiples of $\ell$.

For $R > \pi/2$ we derive in §8 rigorous sufficient conditions guaranteeing that
the same functions provide the extremizers. The conditions involve the $\Lambda_{d,k}$.
To verify them we evaluate the integrals defining the $\Lambda_{3,k}$ approximately.
By such evaluations we show that for $d = 3$ and $R \leq 2.5$ the solutions
maximizing $J_1$ also maximize $J_2$.

The energy density is equal to the largest eigenvalue divided by $|B_R(0)|$. In
the range $d = 3, R \leq 2.5$ in both the scalar and electromagnetic case this quantity is a decreasing functions of $R$. This shows that the third problem
at the start, of finding the radius with highest energy density is solved by
$R = 0$. However the graph is fairly flat. The density dips to about 1/2 its
maximum at about $R = 2$ which is about a third the wavelength.

For focusing of electromagnetic waves to a ball of radius $R$ no larger than one
third of a wavelength the optimal strategy is to choose $e(\xi)$ a multiple of a
rotate of $\ell(\xi)$. The extremizing electric fields are as polarized as a divergence
free field can be. When $d = 3$ formula (1.6) and Example 3.10 show that the
far field for this choice is equal up to rotations by

$$c \frac{\sin |x|}{|x|} \ell \left( \frac{x}{|x|} \right).$$

This field is cylindrically symmetric with axis of symmetry along the $x_1$-axis.
The restriction of $\ell$ to the unit sphere is cylindrically symmetric given by
rotating the following figure about the horizontal axis.

For the problem of focusing a family of lasers, this suggests using linearly
polarized sources concentrated near vertical equator and sparse near the poles
on the horizontal axis. In contrast, for scalar waves one should distribute 
sources as uniformly as possible.

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2 Monochromatic waves.

2.1 Electromagnetic waves and their transforms.

Proposition 2.1 i. $E$ given by (1.5) satisfies $\text{div} E = 0$ if and only if 

$$e(\xi) \cdot \xi = 0, \quad \text{on} \quad \{||\xi|| = 1\}.$$ 

(2.1)

ii. If a monochromatic solution of the Maxwell equations has electric field 
given by (1.3) with $\omega = 1$ and $E$ is given by (1.5) then the magnetic field is 
equal to $e^{it} B(x)$ with 

$$B(x) = -\int_{||\xi|| = 1} e^{ix\xi} \xi \wedge e(\xi) \, d\sigma,$$ 

(2.2)
Proof. Differentiating (1.5) yields,
\[ \text{div } E = \int_{|\xi|=1} e^{ix\xi} i\xi \cdot e(\xi) d\sigma, \quad \text{curl } E = \int_{|\xi|=1} e^{ix\xi} i\xi \wedge e(\xi) d\sigma. \]
The first formula proves i.
The Maxwell equations together with (1.3) yield
\[-\text{curl } E = B_t = iB.\]
Therefore, the second formula proves ii. \[\blacksquare\]

Remark 2.2 The condition (2.1) asserts that \(e(\xi)\) is tangent to the unit sphere. Brouwer’s Theorem asserts that if \(\xi \mapsto e(\xi)\) is continuous then there must be a \(\xi\) where \(e(\xi) = 0\).

Example 2.3. If \(d = 3\) and \(E\) is given (1.5) with \(e(\xi) = \ell(\xi)\) then on \(|\xi| = 1\),
\[ \xi \wedge \ell(\xi) = \xi \wedge \left( (1,0,0) - \xi_1 \xi \right) = \xi \wedge (1,0,0) = (0,\xi_3,-\xi_2) \]
is the tangent field to latitude lines winding around the \(x_1\)-axis. Since this is an odd function, the magnetic field vanishes at the origin, \(B(0) = 0\).

2.2 The competing solutions.

First verify the stationary phase formula (1.6) from the appendix. Consider \(E\) given by (1.5) with \(e \in C^\infty(\{|\xi| = 1\})\). For \(x\) large, the integral (1.5) has two stationary points, \(\xi = \pm x/|x|\). At \(\xi = x/|x|\) parameterize the surface by coordinates in the tangent plane at \(x\) to find that the phase \(x\xi\) has a strict maximum equal to \(|x|\) and hessian equal to the \(-I_{(d-1)\times(d-1)}\). At \(\xi = -x/|x|\) the phase has a minimum with value \(-|x|\) and hessian equal to the identity. The stationary phase method yields (1.6). The energy in the electric field satisfies (1.7) and (1.8).

Theorem 2.4 Suppose that \(E \in S'(\mathbb{R}^d)\) is a tempered solution of the reduced wave equation given by (1.5) with \(e \in D'(S^{d-1})\). Then, (1.7) holds if and only
if \( e \in L^2(S^{d-1}; \mathbb{C}^d) \). In that case (1.8) holds. In addition the stationary phase approximation holds in the sense that as \( R \to \infty \),

\[
\int_{R \leq |x| \leq 2R} \left| E(x) - \frac{1}{|x|^{(d-1)/2}} \left( e^{-i|x|} e((-x/|x|) + e^{i\pi(d-1)/4} e^{i|x|/2}) \right) \right|^2 \, dx = o(R) .
\]

**Proof.** The first two assertions are consequences of Hörmander [1] Theorems 7.1.27 and 7.1.28. The Theorem 7.1.28 also implies that

\[
\sup_{R \geq 1} \frac{1}{R} \int_{R \leq |x| \leq 2R} |E(x)|^2 \, dx \leq c(d) \int_{|\xi| = 1} |e(\xi)|^2 \, d\sigma .
\]

This estimate shows that to prove the third assertion it suffices to prove it for the dense set of \( e \in C^\infty(S^{d-1}) \). In that case the result is a consequence of the stationary phase formula (1.6).

**Definition 2.5** \( H \) is the closed subspace of \( e \in L^2(S^{d-1}; \mathbb{C}^d) \) consisting of \( e(\xi) \) so that \( \xi \cdot e(\xi) = 0 \). Denote by \( \Pi \) the orthogonal projection of \( e \in L^2(S^{d-1}; \mathbb{C}^d) \) on \( H \).

The next example explains a connection between the solutions of the reduced equation that we consider and those satisfying the Sommerfeld radiation conditions.

**Example 2.6** If \( g \in \mathcal{E}'(\mathbb{R}^d; \mathbb{C}^d) \) is a distribution with compact support, then there are unique solutions of the reduced wave equation

\[
(\Delta + 1) E_{out} = g, \quad \text{(resp. } (\Delta + 1) E_{in} = g \text{)}
\]

satisfying the outgoing (resp. incoming) radiation conditions. The difference \( E := E_{out} - E_{in} \) is a solution of the homogeneous reduced wave equation. The field \( F := e^{it} E(x) \) is the unique solution of the initial value problem

\[
\Box F = 0, \quad F|_{t=0} = g, \quad F_t|_{t=0} = ig .
\]

When \( g \in L^2 \) the formula \( E(x) \delta(\tau - 1) = c \int_{-\infty}^{\infty} e^{-i\tau t} F dt \) together with the solution formula for the Cauchy problem imply that (1.7) holds (or see [1] Theorem 14.3.4 showing that both incoming and outgoing fields satisfy (1.7)). More generally, the Fourier transforms in time of solutions of Maxwell’s equations with compactly supported divergence free square integrable initial data yield examples of monochromatic solutions in our class.
2.3 Spherical symmetry is impossible.

It is natural to think that focusing is maximized if waves come in equally in all directions. For the scalar wave equation that is the case. However, such waves do not exist for Maxwell’s equations. Whatever is the definition of spherical symmetry, such a field must satisfy the hypotheses of the following theorem.

**Theorem 2.7** If \( E(x) \in C^1(\mathbb{R}^d) \) satisfies \( \text{div} \, E = 0 \) and for \( x \neq 0 \) the angular part

\[
E - \left( \frac{E \cdot x}{|x|} \right) \frac{x}{|x|}
\]

has length that depends only on \( |x| \) then \( E \) is identically equal to zero.

**Proof.** The restriction of the angular part of \( E \) to each sphere \( |x| = r \) is a \( C^1 \) vector field tangent to the sphere and of constant length. Brouwer’s Theorem asserts that there is a point \( x \) on the sphere where the tangent vector field vanishes. Therefore the constant length is equal to zero and \( E \) is radial.

Therefore in \( x \neq 0 \),

\[
E(x) = \phi(|x|) \, x.
\]

Since \( E \in C^1 \) it follows that \( \phi \in C^1(\{|x| > 0\}) \).

Compute for those \( x \),

\[
\text{div} \, E = \phi \, \text{div} \, x + (\nabla_x \phi) \cdot x = d \, \phi + r \, \phi_r.
\]

Therefore in \( x \neq 0 \), \( r \, \phi_r + d \, \phi = 0 \) so \( \phi = c \, r^{-d} \).

Since \( E \) is continuous at the origin it follows that \( c = 0 \) so \( E = 0 \) in \( x \neq 0 \). By continuity, \( E \) vanishes identically.

3 Maximum field strengths.

We solve the variational problems associated to the functional \( J_1 \) to yield sharp pointwise bounds on monochromatic waves. The fact that the bounds for electromagnetic fields are smaller shows that focusing effects are weaker. The extremizing fields are first characterized by their Fourier Transforms. Explicit formulas in \( x \)-space are given in §3.4.
3.1 Scalar waves.

Theorem 3.1 If

\[ u(x) = \int_{|\xi|=1} e^{ix\xi} f(\xi) \, d\sigma, \quad f \in L^2(S^{d-1}), \]

and \( x \in \mathbb{R}^d \), then

\[ |u(x)| \leq |S^{d-1}|^{1/2} \|f\|_{L^2(S^{d-1})} \]

with equality achieved if and only if \( f \) is a scalar multiple of \( e^{-ix\xi} \).

**Proof.** The quantity to maximize is the \( L^2(S^{d-1}) \) scalar product of \( f \) with \( e^{-ix\xi} \). The result is exactly the Cauchy-Schwartz inequality. \( \blacksquare \)

3.2 Electromagnetic waves.

From Definition \( \ref{def:electric-field} \) \( \ell(\xi) \) is tangent to the longitude lines on the unit sphere connecting the pole \( (-1,0,\ldots,0) \) to the pole \( (1,0,\ldots,0) \). It is the gradient of the restriction of the function \( \xi_1 \) to the unit sphere.

Theorem 3.2 If \( d \geq 2 \) and

\[ E(x) = \int_{|\xi|=1} e^{ix\xi} e(\xi) \, d\sigma, \quad e \in H. \]

Then

\[ |E(0)| \leq \left( \frac{d-1}{d} |S^{d-1}| \right)^{1/2} \|e\|_{L^2(S^{d-1})}. \] (3.1)

Equality holds if and only if \( e \) is equal to a constant multiple of a rotate of \( \ell(\xi) \).

**Proof.** By homogeneity it suffices to consider \( \|e\|_{L^2(S^{d-1})} = 1 \). Rotation and multiplication by a complex number of modulus one reduces to the case \( E(0) = |E(0)|(1,0,\ldots,0) \) and \( |E(0)| = \int e_1(\xi) \, d\sigma \). Need to study,

\[ \sup \left\{ \int e_1(\xi) \, d\sigma : \xi \cdot e(\xi) = 0, \int_{|\xi|=1} \|e(\xi)\|^2 \, d\sigma = 1 \right\}. \]
The quantity to be maximized is

$$\int_{|\xi|=1} e_1(\xi) \, d\sigma = (e, (1, 0, \cdots, 0))_{L^2(S^{d-1})}$$

The constant function, $(1, 0, \cdots, 0)$ does not belong to the subspace $H$. The projection theorem shows that the quantity is maximized for $e$ proportional to the projection of $(1, 0, \cdots, 0)$ on $H$. Equivalently, using (1.9) together with $e \cdot \xi = 0$ yields

$$e_1 = e \cdot (1, 0, \cdots, 0) = e \cdot (\ell(\xi) + \xi_1 \xi) = e \cdot \ell.$$  

which is equal to the $H$ scalar product of $e$ and $\ell$. Since one has the orthogonal decomposition

$$(1, 0, \cdots, 0) = \ell(\xi) + \xi_1 \xi, \quad \text{one has, } \quad 1 = |\ell(\xi)|^2 + \xi_1^2.$$  

The Cauchy-Schwartz inequality shows that the quantity (3.2) is

$$\leq \|e\|_{L^2(S^{d-1})} \|\ell\|_{L^2(S^{d-1})} = \|e\|_{L^2(S^{d-1})} \left( \int_{|\xi|=1} (1 - \xi_1^2) \, d\sigma \right)^{1/2}. \quad (3.3)$$  

The extremum is attained uniquely when $e = z \ell/\|\ell\|$ with $|z| = 1$.

To evaluate the integral on the right of (3.3) compute,

$$\int_{|\xi|=1} \xi_1^2 \, d\sigma = \int_{|\xi|=1} \xi_j^2 \, d\sigma = \frac{1}{d} \int_{|\xi|=1} \sum_j \xi_j^2 \, d\sigma = \frac{1}{d} \int_{|\xi|=1} 1 \, d\sigma = \frac{|S^{d-1}|}{d}.$$

Therefore

$$\int_{|\xi|=1} (1 - \xi_1^2) \, d\sigma = |S^{d-1}| - \frac{|S^{d-1}|}{d} = |S^{d-1}| \frac{d-1}{d}.$$  

Together with (3.3) this proves (3.1).

\[\square\]

**Remark 3.3** If one constrains $e$ to have support in a subset $\Omega$ then with $\chi$ denoting the characteristic function of $\Omega$,

$$\int_{|\xi|=1} e(\xi) \cdot \ell(\xi) \, d\sigma = \int_{|\xi|=1} e(\xi) \cdot \ell(\xi) \chi(\xi) \, d\sigma$$

and $E_1$ is maximized by the choice $e = \ell(\xi)\chi(\xi)$. In the extreme light initiative $\Omega$ is a small number of disks distributed around the equator $x_1 = 0$. 

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3.3 Derivative bounds.

Corollary 3.4 If \( d \geq 2 \) and \( E \) satisfies

\[
E(x) = \int_{|\xi|=1} \xi e(\xi) \, d\sigma, \quad e \in H,
\]

then for all \( \alpha \in \mathbb{N}^d \) and \( x \in \mathbb{R}^d \),

\[
|\partial_x^\alpha E(x)| \leq \left( \frac{d-1}{d} |S^{d-1}| \right)^{1/2} \|e\|_{L^2(S^{d-1})}.
\] (3.4)

**Proof.** The case \( \alpha = 0 \) follows from Theorem 3.2 applied to

\[
\tilde{E}(x) := E(\alpha + x) = \int_{|\xi|=1} e^{i\xi \cdot x} e^{i\xi \cdot \tilde{e}(\xi)} \, d\sigma := \int_{|\xi|=1} e^{i\xi \cdot \tilde{e}(\xi)} \, d\sigma.
\]

Compute for \( |\alpha| > 0 \)

\[
\partial_x^\alpha E = \partial_x^\alpha \int_{|\xi|=1} e^{i\xi \cdot e(\xi)} \, d\sigma = \int_{|\xi|=1} e^{i\xi \cdot (i\xi)^\alpha \tilde{e}(\xi)} \, d\sigma,
\]

which is of the same form as \( E \) with density \( (i\xi)^\alpha \tilde{e}(\xi) \) orthogonal to \( \xi \). Since \( |\xi_j| \leq 1 \) it follows that \( |\xi^\alpha| \leq 1 \) so \( \| (i\xi)^\alpha e \|_{L^2(S^{d-1})} \leq \| e \|_{L^2(S^{d-1})} \). Therefore the general case follows from the case \( \alpha = 0 \).

**Remark 3.5** Derivative bounds for the scalar case are derived in the same way. They lack the factor \((d-1)/d\).

3.4 Formulas for the extremizing fields.

The electric field corresponding to the extremizing density \( \ell \) is explicitly calculated. The computation relies on relations between Bessel functions, spherical harmonics, and, the Fourier Transform. These relations are needed to analyse \( J_2 \).

Start from identities in Stein-Weiss [2]. Their Fourier transform is defined on page 2,

\[
\int f(x) e^{-i2\pi x \cdot \xi} \, dx, \quad n.b. \text{the } 2\pi \text{ in the exponent.}
\]
We will not follow this convention, so adapt their identities. The Bessel function of order \( k \) is (page 153),
\[
J_k(t) = \frac{(t/2)^k}{\Gamma[(2k + 1)/2] \Gamma(1/2)} \int_{-1}^{1} e^{its} (1 - s^2)^{(2k-1)/2} \, ds, \quad -1/2 < k \in \mathbb{R}.
\] (3.5)

Theorem 3.10 (page 158) is the following.

**Theorem 3.6** If \( x \in \mathbb{R}^d \), \( f = f(|x|)P(x) \in L^1(\mathbb{R}^d) \) with \( P \) a homogeneous harmonic polynomial of degree \( k \), then
\[
\int f(x)e^{-2\pi i x \cdot \eta} \, dx = F(|\xi|)P(\xi) \text{ with}
\]
\[
F(r) = 2\pi i^{-k} r^{-(d+2k-2)/2} \int_{0}^{\infty} f(s) J_{(d+2k-2)/2}(2\pi rs) s^{(d+2k)/2} \, ds.
\]

This theorem is equivalent, by scaling and linear combination, to the same formula with \( f = \delta(r-1) \). That case is the identity,
\[
\int_{|x|=1} e^{-i2\pi x \cdot \eta} \, d\sigma = 2\pi i^{-k} |\xi|^{-(d+2k-2)/2} J_{(d+2k-2)/2}(2\pi |\xi|) P(\xi).
\] (3.6)

**Remark 3.7 i.** For \( |\xi| \to \infty \), \( J(|\xi|) = O(|\xi|^{-1/2}) \), and \( P(\xi) = O(|\xi|^k) \) so the right hand side is \( O(|\xi|^{-(d-2)/2-1/2}) = O(|\xi|^{-(d-1)/2}) \) as required by the principle of stationary phase.

**ii.** For \( |\xi| \to 0 \), \( J_{(d-2k-2)/2}(|\xi|) = O(|\xi|^{(d-2k-2)/2}) \) so the right hand side of (3.5) is \( O(|\xi|^k) \). The higher the order of \( P \) the smaller is the Fourier transform near the origin.

To adapt to the Fourier transform without the \( 2\pi \) in the exponent, use the substitution \( \eta = 2\pi \xi, \, |\eta| = 2\pi |\xi| \) to find,
\[
\int_{|x|=1} e^{-ix\eta} \, d\sigma = 2\pi i^{-k} (|\eta|/2\pi)^{-(d+2k-2)/2} J_{(d+2k-2)/2}(|\eta|) P(\eta/2\pi).
\]

Using the homogeneity of \( P \) yields
\[
= (2\pi)^{1-k} i^{-k} (|\eta|/2\pi)^{-(d+2k-2)/2} J_{(d+2k-2)/2}(|\eta|) P(\eta).
\]

The exponent of \( 2\pi \) is equal to \( d/2 \) yielding,
\[
\int_{|x|=1} e^{-ix\eta} \, d\sigma = (2\pi)^{d/2} i^{-k} |\eta|^{-(d+2k-2)/2} J_{(d+2k-2)/2}(|\eta|) P(\eta). \quad (3.7)
\]
Since, $|\eta|^{-k} P(\eta) = P(\eta/|\eta|)$ (3.7) equivalent to,

$$
\int_{|x|=1} e^{-i\xi x} P(x) \, d\sigma = (2\pi)^{d/2} i^{-k} |\eta|^{-(d-2)/2} J_{(d+2k-2)/2}(|\eta|) \, P(\eta/|\eta|). \tag{3.8}
$$

The change of variable $\eta \mapsto -\eta$ yields,

$$
\int_{|x|=1} e^{i\xi x} P(x) \, d\sigma = (2\pi)^{d/2} (-i)^{-k} |\eta|^{-(d-2)/2} J_{(d+2k-2)/2}(|\eta|) \, P(\eta/|\eta|). \tag{3.9}
$$

Finally interchange the role of $x$ and $\eta$ to find,

$$
\int_{|\eta|=1} e^{i\xi \eta} P(\eta) \, d\sigma = (2\pi)^{d/2} (-i)^{-k} |x|^{-(d-2)/2} J_{(d+2k-2)/2}(|x|) \, P(x/|x|). \tag{3.10}
$$

**Example 3.8** The second most interesting example is the extremizing field for the scalar case when $d = 3$. In that case $P = \text{constant}$ and there is a short derivation. The function $u(x) := \int_{|\xi|=1} e^{i\xi x} \, d\sigma$ is a radial solution of $(\Delta + 1) u = 0$. In $x \neq 0$ these are spanned for $d = 3$ by $e^{\pm ix}/r$. Smoothness at the origin forces $u = A \sin r/r$. Since $u(0) = |S^{d-1}|$ it follows that $A = |S^{d-1}|$.

The most interesting case for us is $d = 3$ and the extremizing field $E$ with $e(\xi) = \ell(\xi)$. Since $\ell$ is not a spherical harmonic, the preceding result does not apply directly. To find the exact electric field, decompose $\ell$ in spherical harmonics.

**Lemma 3.9** The spherical harmonic expansion of the restriction of $\ell(\xi)$ to the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is

$$
\ell(\xi) = \left( \frac{d-1}{d} - \sum_{j=2}^{d} \frac{\xi_j^2 - \xi_j^2}{d} \right) , -\xi_1 \xi_2 , \cdots , -\xi_1 \xi_d \right) \quad \text{on} \quad |\xi| = 1. \tag{3.11}
$$

**Proof of lemma.** When $|\xi| = 1$,

$$
\ell(\xi) = (1, 0, \ldots, 0) - \xi_1 \xi = (1, -\xi_1 \xi_2 , \cdots , -\xi_1 \xi_d) - (\xi_1^2, 0, \ldots, 0). \tag{3.12}
$$

The first summand has coordinates that are spherical harmonics.
Decompose

\[ \xi_i^2 = \frac{\xi_1^2 + \cdots + \xi_d^2}{d} + \sum_{j=2}^{d} \frac{\xi_1^2 - \xi_j^2}{d}, \quad \xi \in \mathbb{R}^d, \]

to find the expansion in spherical harmonics of the restriction of \( \xi_i^2 \) to the unit sphere \( \xi_1^2 + \cdots + \xi_d^2 = 1 \),

\[ \xi_1^2 = \frac{1}{d} + \sum_{j=2}^{d} \frac{\xi_1^2 - \xi_j^2}{d} \quad \text{on} \quad \xi_1^2 + \cdots + \xi_d^2 = 1. \tag{3.13} \]

Using (3.13) in (3.12) proves (3.11).

**Example 3.10** The stationary phase formula (1.6) applied to the extremizing \( e = \ell(\xi) \) which is an even function yields for \( d = 3 \)

\[ E(x) = \frac{-1}{\sqrt{2\pi}} \ell \left( \frac{x}{|x|} \right) \frac{\sin |x|}{|x|} + O(|x|^{-2}). \]

This is the sum of incoming and outgoing waves with spherical wave fronts and each with profile on large spheres proportional to \( \ell(x/|x|) \). The desired incoming wave is such an \( \ell \)-wave.

## 4 Equivalent selfadjoint eigenvalue problems.

The section introduces eigenvalue problems equivalent to the maximization of \( J_2 \).

### 4.1 The eigenvalue problem for focusing scalar waves.

**Definition 4.1** For \( R > 0 \) define the compact linear operator \( L : L^2(S^{d-1}) \rightarrow L^2(B_R(0)) \) by

\[ (Lf)(x) := \int_{|\xi|=1} e^{ix\xi} f(\xi) \, d\sigma. \]

The operator \( L \) commutes with rotations. The adjoint \( L^* \) maps \( L^2(B_R(0)) \rightarrow L^2(S^{d-1}) \).
Proposition 4.2 The following four problems are equivalent.

i. Maximize the functional \( J_2 \) on scalar monochromatic waves.

ii. Find \( f \in L^2(S^{d-1}) \) with \( \|f\|_{L^2(S^{d-1})} = 1 \) so that \( \|Lf\|_{B^R(0)} \) is largest.

iii. Find the norm of \( L \).

iv. Find the largest eigenvalue of the positive compact self adjoint operator \( L^*L \) on \( L^2(S^{d-1}) \).

Proof. The equivalence of the first three follows from the definitions. The equivalence with the third follows from the identity
\[
\|Lf\|_{L^2(B^R)}^2 = (Lf, Lf)_{L^2(B^R)} = (L^*Lf, f)_{L^2(S^{d-1})}.
\]

Definition 4.3 Define a rank one operator
\[
L^2(S^{d-1}; \mathbb{C}) \ni f \to L_0 f := \int_{|\xi|=1} f(\xi) \, d\sigma \in \mathbb{C}.
\]

Remark 4.4 i. The problem of maximizing \( J_1 \) for scalar waves is equivalent to finding the norm of \( L_0 \) and also finding the largest eigenvalue of \( L_0^*L_0 \).

ii. The same formula defines an operator from \( L^2(S^{d-1}) \to L^2(B^R(0)) \) mapping \( f \) to a constant function. With only small risk of confusion we use the same symbol \( L_0 \) for that operator too.

Definition 4.5 The vector valued version of \( L \) and \( L_0 \) are defined by
\[
(Le)(x) := \int_{|\xi|=1} e^{ix\xi} e(\xi) \, d\sigma, \quad e \in L^2(S^{d-1}; \mathbb{C}^d),
\]
\[
L_0 e := \int_{|\xi|=1} e(\xi) \, d\sigma, \quad e \in L^2(S^{d-1}; \mathbb{C}^d).
\]
Denote by \( L_M \) and \( L_{0,M} \) the restriction of \( L \) and \( L_0 \) to \( H \).

Remark 4.6 i. The problem of maximizing the functional \( J_1 \) for monochromatic electromagnetic waves is equivalent to finding the largest eigenvalue of \( L_{0,M}^*L_{0,M} \).

ii. The problem of maximizing the functional \( J_2 \) for monochromatic electromagnetic waves is equivalent to finding the largest eigenvalue of \( L_M^*L_M \).

iii. The operator \( L\Pi \) is equal to \( L_M \) on \( H \) and equal to zero on \( H^\perp \). Therefore \( (L\Pi)^*(L\Pi) \) is equal to \( L_M^*L_M \) on \( H \) and equal to zero on \( H^\perp \).
5 Exact eigenvalue computations.

5.1 The operators $L_0^*L_0$, $L_0^*L_0^*$ and $L_{0,M}^*L_{0,M}^*$.

**Theorem 5.1**

i. The spectrum of $L_0^*L_0$ contains one nonzero eigenvalue, $|S^{d-1}|$, with multiplicity one. The eigenvectors are the constant functions.

ii. The spectrum of the operator $L_0^*L_0$ contains one nonzero eigenvalue, $|S^{d-1}|$, with multiplicity $d$. The eigenvectors are the $\mathbb{C}^d$-valued constant functions.

iii. The spectrum of $L_{0,M}^*L_{0,M}$ contains one nonzero eigenvalue, $|S^{d-1}|(d-1)/d$ with multiplicity $d$. The corresponding eigenspace consists of $\ell(\xi)$ and functions obtained by rotation and scalar multiplication.

**Proof iii.** Suppose that $f$ is an eigenfunction in $H$ so that the norm of $\int f \ d\sigma$ is maximal. Rotating $f$ yields a function in $H$ with the same $\|L_0f\|$ and with $L_0f$ parallel to $(1,0,\ldots,0)$. Therefore maximizing $L_0f$ and maximizing $L_0f \cdot (1,0,\ldots,0)$ yield the same extreme value.

Since $f \in H$,

$$\int f_1 d\sigma = \int f \cdot (1,0,\ldots,0)) \ d\sigma = \int f \cdot \ell \ d\sigma.$$ 

The extreme value is attained for $f$ parallel to $\ell$. This shows that $\ell$ is an eigenfunction corresponding to the largest eigenvalue.

Rotating and taking scalar multiples yields a complex eigenspace of dimension $d$. Since rank $L_0 = d$ these are all the eigenfunctions.

If $\|f\|_{L^2(S^{d-1})} = 1$, then the maximization of $J_0$ shows that $|L_{0,M}f|^2 = |S^{d-1}|(d-1)/d$ proving the formula for the eigenvalue.

5.2 The eigenfunctions and eigenvalues of $L^*L$ and $L^*L$.

**Theorem 5.2** In dimension $d$, the spherical harmonics of order $k$ are eigenfunctions of $L^*L$ with eigenvalue

$$\Lambda_{d,k}(R) := (2\pi)^{d/2} |S^{d-1}| \int_0^R r \left[J_{(d+2k-2)/2}(r)\right]^2 dr. \quad (5.1)$$
Remark 5.3 From the point of view of focusing of energy into balls, all spherical harmonics of the same order are equivalent.

Proof. If $P$ is a homogeneous harmonic polynomial of degree $k$ formula (3.10) shows that

$$(LP)(x) = \phi_{d,k}(|x|) P(x),$$

defining the function $\phi$.

The operator $L^*$ is an integral operator from $L^2(B_R) \to L^2(S^{d-1})$ with kernel $e^{-ix\xi}$. Therefore

$$L^*L(P) = \int_{|x| \leq R} e^{-ix\xi} \phi_{d,k}(|x|) P(x) \, dx.$$  

Introduce polar coordinates $x = ry$ with $|y| = 1$ to find

$$L^*L(P) = |S^{d-1}| \int_0^R \int_{|y| = 1} r^{d-1} e^{-iry\xi} \phi_{d,k}(r) r^k P(y) \, d\sigma(y) \, dr.$$  

Formula (3.10) shows that

$$\int_{|y| = 1} e^{-iry\xi} P(y) \, d\sigma(y) = \phi_{d,k}(r) P(-r \xi) = (-r)^k \phi_{d,k}(r) P(\xi).$$

Therefore

$$L^*L(P) = \int_0^R (-1)^k |S^{d-1}| r^{d+2k-1} \phi_{d,k}(r)^2 \, dr \, P(\xi).$$

This proves that the harmonic polynomial are eigenvectors with eigenvalue depending only on $d$ and $k$.

The formula for the eigenvalue follows on noting that the eigenvalue is equal to the square of the $L^2(B_R(0))$ norm of

$$F(x) := \int_{|\eta| = 1} e^{i\eta x} P(\eta) \, d\sigma, \quad \int_{|\eta| = 1} |P(\eta)|^2 \, d\sigma = 1.$$  

Using polar coordinates and (3.10) yields,

$$\|F\|_{L^2(B_R(0))}^2 = (2\pi)^{d/2} |S^{d-1}| \int_0^R r^{-(d-2)} \left[J_{(d+2k-2)/2}(r)\right]^2 r^{d-1} \, dr$$

$$= (2\pi)^{d/2} |S^{d-1}| \int_0^R r \left[J_{(d+2k-2)/2}(r)\right]^2 \, dr.$$  

(5.3)
This proves [5.1].

The spectral decomposition of $L$ is nearly identical to that of $L$. The next result is elementary.

**Corollary 5.4** The eigenvalues of $L^*L$ are the same as the eigenvalues of $L^*L$. The eigenspaces consists of vector valued functions each of whose components belongs to the corresponding eigenspace of $L^*L$.

### 5.3 Some eigenfunctions and eigenvalues of $L^*_M L_M$.

The situation for $L^*_M L_M$ is more subtle. Our first two results show that there are eigenfunctions intimately related to the eigenvalues $\Lambda_{d,1}(R)$ and $\Lambda_{d,0}(R)$.

**Theorem 5.5** The d dimensional space of functions $e(\xi) := \zeta \wedge \xi$ with $\zeta \in \mathbb{C}^d \setminus \mathbf{0}$ are eigenfunctions of $L^*_M L_M$. The eigenvalue is $\Lambda_{d,1}(R)$.

**Remark 5.6** These $e(\xi)$ are the vector valued spherical harmonics of degree 1 that belong to $H$, that is, that satisfy $\xi \cdot e(\xi) = 0$.

**Proof.** Follows from $Le = \Lambda_{d,1}(R)e$ and $e \in H$.

Though the constant functions which are eigenvectors of $L$ do not belong to $H$, their projection on $H$ yield eigenvectors of $L^*_M L_M$.

**Theorem 5.7** The d dimensional space consisting of scalar multiples of rotates of $\ell(\xi)$ consists of eigenfunctions of $L^*_M L_M$ with eigenvalue equal to

$$
\left( \Lambda_{d,0}(R) - \Lambda_{d,2}(R) \right) \frac{d-1}{d}.
$$

(5.4)

**Proof of theorem.** Use (3.11). Since the spherical harmonics are eigenfunctions of $L^*L$ one has, suppressing the $R$ dependence of $\Lambda$,

$$
L^*L \ell = \left( \Lambda_{d,0} \frac{d-1}{d} - \Lambda_{d,2} \sum_{j=2}^{d} \frac{\xi_1^2 - \xi_j^2}{d} , -\Lambda_{d,2} \xi_1 \xi_2 , \cdots , -\Lambda_{d,2} \xi_1 \xi_d \right).
$$
Multiply (3.11) by $\Lambda_{d,2}$ to find on $\xi_1^2 + \cdots + \xi_d^2 = 1$, 

$$\Lambda_{d,2} \ell = \left( \Lambda_{d,2} \frac{d-1}{d} - \Lambda_{d,2} \sum_{j=2}^{d} \frac{\xi_1^2 - \xi_j^2}{d}, -\Lambda_{d,2} \xi_1 \xi_2, \cdots, -\Lambda_{d,2} \xi_1 \xi_d \right).$$

Subtract from the preceding identity to find, 

$$L^* L \ell = \left( \Lambda_{d,0} - \Lambda_{d,2} \right) \frac{d-1}{d} (1, 0, \cdots, 0).$$

Projecting perpendicular to $\xi$ using $\Pi (1, 0, \cdots, 0) = \ell$ yields 

$$\Pi L^* L \Pi \ell = \Pi L^* L \ell = \left( \Lambda_{d,0} - \Lambda_{d,2} \right) \frac{d-1}{d} \ell.$$

This proves that $\ell$ is an eigenfunction of $(L\Pi)^*(L\Pi)$ with eigenvalue $(\Lambda_{d,0} - \Lambda_{d,2}) (d-1)/d$.

Remark 4.6.iii shows that it is an eigenfunction of $L^*_M L_M$ with the same eigenvalue.

By rotation invariance the same is true of all scalar multiples of rotates of $\ell$.

They form a $d$ dimensional vector space. spanned by the projections tangent to the unit sphere of the unit vectors along the coordinate axes.

As in Theorem 5.5 if one defines $H_k$ to consist of spherical harmonics of degree $k$ that belong to $H$, then $H_k$ are orthogonal eigenspaces of $L^*_M L_M$ with eigenvalue $\Lambda_{d,k}(R)$.

**Example 5.8** In $\mathbb{R}^2$ the homogeneous $\mathbb{R}^2$ valued polynomials of degree two whose radial components vanish are spanned by $(-x_1 x_2, x_1^2)$ and $(x_2^2, -x_1 x_2)$. There are no harmonic functions is their span proving that when $d = 2$, $H_2 = 0$. It is clear that $H_0 = 0$.

Though there are a substantial number of eigenvectors of $L^*_M L_M$ accounted for by the $H_k$ they are far from the whole story.

### 6 Spectral asymptotics.

#### 6.1 Behavior of the $\Lambda_{d,k}(R)$.

**Proposition 6.1** i. As $R \to 0$, $\Lambda_{d,k}(R) = O(R^{d+2k})$. 

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ii. As $R \to 0$, $\Lambda_{d,0}(R) = |S^{d-1}| |B_R(0)| (1 + O(R))$.

iii. $\lim_{k \to \infty} \Lambda_{d,k}(R) = 0$ uniformly on compact sets of $R$.

Proof. i. Formula (3.5) shows that $J_k(t) = O(t^k)$ as $t \to 0$. Assertion i then follows from (5.1).

ii. By definition, $\Lambda_{d,0}(R)$ is the square of the $L^2(B_R)$ norm of $\int_{|\xi|=1} e^{ix\xi} f(\xi) d\sigma$ for $f$ a constant function of norm 1. Take $f = |S^{d-1}|^{-1/2}$. For $R$ small $e^{ix\xi} \approx 1$ showing that for $R$ small the three operators are approximated by $L_0^0$, $L_0^0$, and $L_0^0$, $M^0$, respectively. We know the exact spectral decomposition of the approximating operators. Each has exactly one nonzero eigenvalue. In performing the approximation some care must be exercised since the operator to be approximated has norm $O(R^{d/2})$ tending to zero as $R \to 0$.

Proposition 6.2 i. Each of the operators $L$, $L$, and $L_M$ has norm no larger than $(|B_R(0)| |S^{d-1}|)^{1/2}$.

ii Each of the differences $L - L_0$, $L - L_0$, and $L_M - L_0,M$ has norm no larger than

$$\frac{|S^{d-1}| R^{(d+2)/2}}{(d+2)^{1/2}}.$$  \hfill (6.1)

Proof. i. Treat the case of $L$. For $\|e\| = 1$, the Cauchy-Schwartz inequality implies that for each $x$, $\|Le(x)\|_{C^d}^2 \leq |S^{d-1}|$. Integrating over the ball of radius $R$ proves i.
ii. The Cauchy-Schwarz inequality estimates the difference by

\[ |(L - L_0)e| = \int_{|\xi|=1} |e^{ix\xi} - 1| e(\xi) d\sigma \leq \int_{|\xi|=1} |x| |e| d\sigma \]

\[ \leq |x| |S^{d-1}|^{1/2} \|e\|_{L^2(S^{d-1})}. \]

For \(e\) of norm one this yields

\[ \|L - L_0\|_{L^2(B_R(0))} \leq |S^{d-1}| \int_E |x|^2 dx \]

\[ = |S^{d-1}| \int_{|\omega|=1} \int_0^R r^2 r^{d-1} dr d\sigma(\omega) = |S^{d-1}|^2 \frac{R^{d+2}}{d+2}, \]

completing the proof.

Theorem 6.3 i. For each \(d\) there is an \(R_L(d) > 0\) so that for \(0 \leq R < R_L(d)\)
the eigenvalue \(\Lambda_{d,0}(R)\) is the largest eigenvalue of \(L^*L\). It has multiplicity one. The eigenfunctions are constants.

ii. For \(0 \leq R < R_L(d)\) the eigenvalue \(\Lambda_{d,0}(R)\) is the largest eigenvalue of \(L^*L\). It has multiplicity \(d\). The eigenfunctions are constant vectors.

iii. For each \(d\) there is an \(R_M(d) > 0\) so that for \(0 \leq R < R_M(d)\) the eigenvalue

\[ \left(\Lambda_{d,0}(R) - \Lambda_{d,2}(R)\right) \frac{d - 1}{d} \]

is the largest eigenvalue of \(L_M^*L_M\). It has multiplicity \(d\). The eigenfunctions are rotates of constant multiples of \(\ell\).

iv. In all three cases, the other eigenvalues are \(O(R^{d+1})\).

Proof. We prove iii and iv for the operator \(L_M^*L_M\). Proposition 6.2 implies that the compact self adjoint operators \(L_M^*L_M\) and \(L_{0,M}^*L_{0,M}\) differ by \(O(R^{d+1})\) in norm.

Part iii of Theorem 6.1 shows that the spectrum of \(L_{0,M}^*L_{0,M}\) contains one positive eigenvalue, \(\lambda_+ := |B_R(0)| |S^{d-1}|(d-1)/d\). The factor \(|B_R(0)|\) arises because \(L_{0,M}\) in the present context is viewed as an operator with values in the functions on \(B_R(0) \subset \mathbb{R}^d\). The eigenfunctions are scalar multiplies of rotates of \(\ell\). The rest of the spectrum is the eigenvalue 0.
It follows that the spectrum of $L^*_M L_M$ lies in the union of disks of radius $O(R^{d+1})$ centered at zero and $\lambda_+$. For $R$ small these disks are disjoint and the eigenspace associated to the disk about $\lambda_+$ has dimension $d$.

Theorem 5.7 shows that the eigenfunctions of $L^*_0 M L_0 M$ with eigenvalue $\lambda_+$ are eigenfunctions of $L^*_M L_M$. The eigenvalue is given by (5.4).

It follows that for $R$ small, the scalar multiples of rotates of $\ell$ is an eigenspace of $L^*_M L_M$ of dimension $d$ and eigenvalue in the disk about $\lambda_+$. This completes the proof of iii.

The fact that the other eigenvalues lie in a disk of radius $O(R^{d+1})$ centered at the origin proves iv.

The proofs for the operators $L$ and $L$ are similar.

7 Largest eigenvalues for $R \leq \pi/2$.

7.1 Monotonicity of $\Lambda_{d,k}(R)$ in $k$.

Recall that the wavelength is equal to $2\pi$.

**Theorem 7.1** For $0 \leq R \leq \pi/2$, $\Lambda_{d,k}(R)$ is strictly monotonically decreasing in $k = 0, 1, 2, \ldots$. In particular the largest eigenvalue of $L^* L$ and $L^* L$ is $\Lambda_{d,0}(R)$. The corresponding eigenfunctions are constant scalar and constant vector functions respectively.

**Proof.** Write,

$$J_k(r) = \frac{2 (r/2)^k}{\Gamma[(2k+1)/2] \Gamma(1/2)} \int_0^1 \cos(rs) \left(1 - s^2\right)^{(2k-1)/2} ds. \quad (7.1)$$

For $0 \leq r \leq \pi/2$ the cosine factor in the integral is positive. Since $(1 - s^2)^{(2k-1)/2}$ is decreasing in $k$ for $s \in [0, 1]$, the integral is decreasing in $k$.

$\Gamma[(2k+1)/2]$ is increasing in $k$. Since $r \leq 2$, $(r/2)^k$ decreases with $k$. The proof is complete.

**Remark 7.2** The first figure in §8.1 shows that the graphs of the functions $\Lambda_{3,0}(R)$ and $\Lambda_{3,1}(R)$ cross close to $R = \pi$. The proof shows that this cannot happen for $R \leq \pi/2$. 

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7.2 Largest eigenvalues of $L_M^*L_M$ when $d = 3$, $R \leq \pi/2$.

Theorem 7.3 When $d = 3$ and $R \leq \pi/2$ the strictly largest eigenvalue of $L_M^*L_M$ is $(\Lambda_{3,0}(R) - \Lambda_{3,2}(R))2/3$. The eigenfunctions are the scalar multiples of rotates of $\ell$, and, $\Lambda_{3,1}(R)$ is the next largest eigenvalue.

The proof uses the following criterion valid for all $d, R$. For ease of reading, the $R$ dependence of $\Lambda_{d,k}(R)$ is often suppressed.

Theorem 7.4 i. The eigenvalue $(\Lambda_{d,0} - \Lambda_{d,2})(d - 1)/d$ of $L_M^*L_M$ is strictly larger than all others only if

$$(\Lambda_{d,0} - \Lambda_{d,2})(d - 1)/d > \Lambda_{d,1}. \quad (7.2)$$

ii. If in addition to (7.2), the two largest eigenvalues of $L^*L$ are $\Lambda_{d,0}$ and $\Lambda_{d,1}$, then the eigenvalue $(\Lambda_{d,0} - \Lambda_{d,2})(d - 1)/d$ of $L_M^*L_M$ is strictly larger than the others. The eigenfunctions are the scalar multiples of rotates of $\ell$, and, $\Lambda_{d,1}$ is the next largest eigenvalue of $L_M^*L_M$.

Remark 7.5 Equation (7.2) implies that $\Lambda_{d,0} > \Lambda_{d,1}$. The additional condition in ii is that for all $k \geq 1$, $\Lambda_{d,1} \geq \Lambda_{d,k}$.

Proof of Theorem 7.4. i. Since $\Lambda_{d,1}$ is also an eigenvalue of $L_M^*L_M$, necessity is clear.

ii. Under these hypotheses Theorem 5.2 shows that $\Lambda_{d,0}$ is the largest eigenvalue of $L^*L$ with one dimensional eigenspace consisting of constant functions. Corollary 5.4 shows that $L^*L$ has the same largest eigenvalue with $d$ dimensional eigenspace consisting of $\mathbb{C}^d$ valued constant functions. The next largest eigenvalue of $L^*L$ is $\Lambda_{d,1}$. In particular, $L^*L$ has exactly $d$ eigenvalues counting multiplicity that are greater than $\Lambda_{d,1}$.

Since $L_M$ is the restriction of $L$ to a closed subspace, the minmax principal implies that $L_M^*L_M$ has at most $d$ eigenvalues counting multiplicity that are greater than $\Lambda_{d,1}$.

Theorem 5.7 provides a $d$ dimensional eigenspace with eigenvalue given by the left hand side of (7.2) and therefore greater than $\Lambda_{d,1}$. In particular there are exactly $d$ eigenvalues counting multiplicity that are greater than $\Lambda_{d,1}$.

Theorem 5.5 shows that $\Lambda_{d,1}$ is an eigenfunction of $L_M^*L_M$ so it must be the next largest.
Example 7.6 Parts i and ii of Proposition 6.1 show that the sufficient condition is satisfied for small $R$. This gives a second proof that for small $R$, $\ell$ is an extreme eigenfunction for $L_M^*L_M$. The first proof is part iii of Theorem 6.3.

Proof of Theorem 7.3. Verify the sufficient condition of Theorem 7.4 ii.

Since $R \leq \pi/2$, $\Lambda_{3,k}$ are strictly decreasing in $k$. Therefore $\Lambda_{3,0}$ and $\Lambda_{3,1}$ are the two largest eigenvalues of $L^*L$.

It remains to verify (7.2). Use formulas (5.1) and (7.1). Formula (5.1) with $d = 3$ and $k = 0, 1, 2$ involves $J_{1/2}, J_{3/2}, J_{5/2}$.

Since the integral in (7.1) is decreasing in $k$ it follows that

$$
\frac{J_{k+1}(r)}{J_k(r)} \leq \frac{r}{2} \frac{\Gamma((2k+1)/2)}{\Gamma((2k+1)/2 + 1)}.
$$

The functional equation $\Gamma(n + 1) = (n + 1)\Gamma(n)$ yields

$$
\frac{J_{k+1}(r)}{J_k(r)} < \frac{r}{2} \frac{1}{(2k+3)/2} = \frac{r}{2k+3}.
$$

Therefore,

$$
\frac{J_{3/2}(r)}{J_{1/2}(r)} < \frac{r}{4}, \quad \text{and,} \quad \frac{J_{5/2}(r)}{J_{3/2}(r)} \leq \frac{r}{6}.
$$

Injecting these estimates in (5.1) yields

$$
\frac{\Lambda_{3,1}(R)}{\Lambda_{3,0}(R)} < \frac{R^2}{4^2}, \quad \text{and,} \quad \frac{\Lambda_{3,2}(R)}{\Lambda_{3,1}(R)} < \frac{R^2}{6^2}.
$$

(7.3)

Therefore,

$$
\Lambda_{3,1} \leq \frac{R^2}{4^2} \Lambda_{3,0}, \quad \text{and,} \quad \Lambda_{3,2} \leq \frac{R^2}{6^2} \Lambda_{3,1} \leq \frac{R^2}{4^2} \frac{R^2}{6^2} \Lambda_{3,0}, \quad \text{so,}
$$

$$
(\Lambda_{3,0} - \Lambda_{3,2}) \frac{2}{3} - \Lambda_{3,1} > \Lambda_{3,0} \left(\frac{2}{3} - \frac{2}{3} \frac{R^4}{4^2} - \frac{R^2}{4^2}\right) := \Lambda_{3,0} h(R).
$$

The polynomial $h(R)$ is equal to $2/3$ when $R = 0$ and decreases as $R$ increases. To verify (7.2) it suffices to show that $h(\pi/2) > 0$. Since $2 > \pi/2$, $h(\pi/2) > h(2) = 43/108 > 0$.\end{flushright}
8 Numerical simulations to determine largest eigenvalues.

Recall that the wavelength is equal to $2\pi$. In this section the dimension $d = 3$.

Theorem 5.2, Corollary 5.4, and Theorem 7.3 allow one in favorable cases to find the largest eigenvalues of $L^*L$, $L^*L$, and $L^*_M L_M$, by evaluating the integrals defining $\Lambda_{3,k}(R)$ for $k = 0, 1, 2, \ldots$. These quantities decrease rapidly with $k$ so to compute the largest ones requires little work.

8.1 Simulations for scalar waves.

For scalar waves the eigenvalues are exactly the $\Lambda_{3,k}(R)$. For $R$ small, they are monotone in $k$ so the optimal focusing is for $k = 0$. Our first simulation (performed with the aid of Matlab) computes approximately the integrals defining $\Lambda_{3,k}(R)$ for $R \leq 2\pi$ and $k = 0, 1, 2, 3$. The resulting graphs are in the figure on the left. The horizontal axis is $R$ and on the vertical axis is plotted the integral on the right hand side of (5.1), that is,

$$\frac{\Lambda_{3,k}(R)}{(2\pi)^{3/2}|S^2|} = \frac{\Lambda_{3,k}(R)}{2^{7/2} \pi^{5/2}}, \quad k = 0, 1, 2, 3.$$ 

The four curves correspond to the four values of $k$. The graph with the leftmost hump is $\Lambda_{3,0}(R)$. The graph with the hump second from the left is $\Lambda_{3,1}(R)$ and so on. The conclusion is that $\Lambda_{3,0}(R)$ crosses transversely the graph $\Lambda_{3,1}(R)$ just to the right of $R = 3$. At that point, $\Lambda_{3,1}(R)$ becomes
the largest. On the right is a zoom showing that the crossing is suspiciously close to $R = \pi$.

The graphs of the $\Lambda_{3,k}(R)$ are a little misleading since it is not the total energy but the energy density that is of interest. The next figure plots as a function of $R \frac{\Lambda_{3,0}}{(2^{7/2} \pi^{5/2} |B_R(0)|)}$. The small gap near $R = 0$ is because the division by $|B_R(0)|$ is a sensitive operation and leads to numerical errors in that range.

The energy density is greatest for balls with radius close to $R = 0$. The density drops to half its maximum value at about $R = 2$ which is about $1/3$ of the wavelength.

8.2 Simulations for electromagnetic waves.

Using Theorem 7.3 one can investigate the analogous questions for Maxwell’s equations by manipulations of the $\Lambda_{3,k}(R)$.

The simulations of the preceding subsection show that for $R \leq 3$ one has $\Lambda_{3,0}(R) > \Lambda_{3,1}(R)$. To show that the eigenvalue corresponding to $\ell(\xi)$ is the optimum it suffices to verify (7.2). To do so one needs to verify the positivity of

$$2^{-7/2} \pi^{-5/2} \left( \frac{2 \Lambda_{3,0}(R)}{3 |B_R(0)|} - \frac{2 \Lambda_{3,2}(R)}{3 |B_R(0)|} - \frac{\Lambda_{3,1}(R)}{|B_R(0)|} \right).$$

This is a linear combination of quantities computed in the preceding subsection. Its graph is plotted on the left. The graph crosses from positive to negative near $R = 2.5$. The criterion is satisfied for all $R$ to the left of this crossing.
For $R < 2.5$ the energy density for the optimizing monochromatic electromagnetic fields associated with $\ell(\xi)$ is equal to

$$\frac{2}{3} \Lambda_{3,0}(R) - \frac{2}{3} \Lambda_{3,2}(R)$$

Because of the factor $2/3$ it is smaller than the density in the scalar case by that factor. The subtraction in the formula shows that the density drops off more rapidly in the electromagnetic case than in the scalar case. The graph of $2^{-7/2} \pi^{-5/2}$ times this quantity is plotted next.

As in the scalar case the maximal energy density occurs on balls with radius near zero. The intensity drops to one half of this value to the left of $R = 1.9$. This is close to the corresponding value for scalar waves, about one third of a wavelength.
References

[1] L. Hörmander, *The Analysis of Linear Partial Differential Operators I, II*, Springer-Verlag, Berlin, 1983.

[2] E. Stein and G. Weiss, *Fourier Analysis on Euclidean Space*, Princeton University Press, Princeton, 1971.