Investigation of variable thermo-physical properties of viscoelastic rheology: A Galerkin finite element approach

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Galerkin finite element (GFEM) algorithm is implemented to investigate the variable viscosity, variable thermal conductivity and variable mass diffusion coefficient on viscoelasticity and non-Newtonian rheology of Maxwell fluid. Computer code is developed for weak form of FEM equations and validated with already published benchmark (a special case of present work). Theoretical results for velocities, temperature and concentration are displayed to analyze the effects of arising parameters including variable Prandtl number and variable Schmidt number. Shear stresses (only for Newtonian case) heat and mass fluxes at the elastic surface are computed and recorded in tabular form. © 2018 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). https://doi.org/10.1063/1.5032171

I. INTRODUCTION

Transport of heat and mass in fluid flows plays a vital role in many processes which frequently occur at industry and engineering. There are many mechanisms for the transport of both heat and mass in fluids flows induced by moving surfaces (moving with constant or variable velocity). Flows induced by elastic sheets have been studied by the several researchers because such flows are encountered in many industrial and engineering applications. Stretching phenomenon in the extrusion of polymers, glass-fiber, paper production, drawing of plastic/elastic films is commonly cited in the literature. Stretching phenomenon also plays a significant role during the production of sheets. Therefore different stretching rates (linear, nonlinear, one dimensional and multidimensional wall stretching rates) are considered.1,5

During manufacturing processes, fluids with different rheologies are used. Maxwell fluid is one of the viscoelastic fluids and possesses the dual property of both the viscosity and elasticity. Several studies on Maxwell fluids are available. Some recent and relevant investigations are described here. For instance, Nawaz et al.6 theoretically discussed Hall and ion slip effects on the flow of Maxwellian of plasma over a vertical surface. The effect as discussed in Ref. 6 are studied by Hayat et al.7 considering non-Newtonian rheology of Jaffrey fluid when Hall and ion slip effects are of considerable order of magnitude. Khan et al.8 visualized the effects of microorganisms on the stratification in the Maxwellian fluid subjected to magnetic field. Awais et al.9 examined three dimensional flow of Maxwell fluid over an exponentially stretching surface. The simultaneous effects including chemical reaction on mass transfer in the flow of Maxwell fluid are investigated by Awais et al.10 The theoretical study on heat transfer in MHD Maxwellian fluid using Cattaneo-Christov heat...
flux model is carried out by Sleem et al.\textsuperscript{11} and noted the significant effect of non-Fourier parameter (relaxation parameter associated with Cattaneo-Charistov model) on the temperature and heat flux.

Experimental and theoretical data shows that the viscosity, the thermal conductivity and the mass diffusion coefficient remain constant under some specific scenarios. Such scenarios are rare. However, there may be some real time physical situations where change in viscosity, thermal conductivity and mass diffusion coefficient are very small and negligible. In such cases assumption of constant physical properties (viscosity thermal conductivity, mass diffusion coefficient) may be considered but in a general sense, the assumption of constant properties is not valid and may lead to erroneous results. Due to this reason, researchers have given mathematical models for variable thermo-physical properties. However these studies are fewer and most of them are restricted to the Newtonian fluid. For example, Lai and kulacki\textsuperscript{12} studied the effects of variable viscosity on the flow in the porous medium by considering all the physical properties to be constant except dynamic viscosity. Prasad et al.\textsuperscript{13} analyzed heat characteristics in the flow of viscoelastic fluid in the presence of variable viscosity. The effects of variable Prandtl number on the flow of fluid of variable viscosity and variable thermal conductivity are discussed by Singh and Agarwal.\textsuperscript{14} Mokhopadhyay and Layek\textsuperscript{15} studied the effect of temperature on the viscosity of the fluid but they considered the thermal conductivity of the fluid to be constant. Pop et al.\textsuperscript{16} analyzed the effects of variable viscosity on the flow of fluid over a moving of surface. Chaim.\textsuperscript{17} analyzed the effects of temperature dependent thermal conductivity of the transfer of heat in the flow over stretching surface. Gary et al.\textsuperscript{18} studied stagnation point flow of viscosity varying fluid. Mehta and Sood\textsuperscript{19} examined the behavior of the variation of the dynamic viscosity of the fluid in the presence of porous medium. The effect of variable viscosity on the boundary layer flow over a stretching body is studied by Makhopadhyay et al.\textsuperscript{20} Aziz\textsuperscript{21} investigated the behavior of thermo-physical properties on the transport of heat and mass in three-dimensional flow subjected to magnetic fluid. Sandeep et al.\textsuperscript{22} numerically investigated simultaneously the effects of Brownian motion and thermospheres on the transport of heat and mass in non-Newtonian Carreau liquid.

Literature review reveals that no study dealing with variable viscosity, variable thermal conductivity and variable diffusion coefficient in the flow of Maxwell fluid is available yet. This investigation fills this gap in the literature. The governing problems are solved by a powerful numerical technique called FEM and has been used by many researchers working in the field of engineering and science. Some most relevant studies can be mentioned through Refs. 5 and 23–25 and refs. therein. This investigation consists of five sections. Section I contains literature review. Section II is designated for discussion about rheological and thermo-physical properties of the Maxwell fluid. Problem statement and its mathematical formulation is also given in this section. Galerkin finite element derivation is given in Section III. The key observations are given in Section V.

### II. BACKGROUND AND RHEOLOGY

#### A. Viscoelastic fluid model and its rheology

Serval models for non-Newtonian viscoelastic fluids have been proposed. Maxwell model is a viscoelastic model which exhibits dual property of viscosity and elasticity. Due to this nature, it behaves like solid as well as fluid. The constitutive equations for Maxwell (viscoelastic) fluid are given by\textsuperscript{10,11,13,14}

\[
\tau = -pI + S, \quad (1)
\]

where \(p\) is the pressure, \(I\) is the identity tensor and \(S\) is the extra stress tensor defined by

\[
S + \lambda_1 \frac{DS}{Dt} = \mu A_1, \quad (2)
\]

where \(A_1\) is the first Rivilon-Ericksen tensor, \(\frac{DS}{Dt}\) is the convective derivative, \(\lambda_1\) is the relaxation time and \(\mu\) is the dynamic viscosity. For \(\lambda_1 = 0\), the constitutive equation (2) reduces to Newtonian case.
B. Variable viscosity, variable thermal conductivity, variable mass diffusion coefficient

Most of the studies consider constant physical properties including viscosity, thermal conductivity and mass conductance. The assumption of constant viscosity, thermal conductivity and mass conductance is valid only in some rare cases. Especially when mass transport occurs in isothermal circumstances, when heat transfer takes place due to high temperature differences, the assumptions of constant viscosity, constant thermal conductivity and constant mass diffusion coefficient is not valid. Therefore, various models for variable viscosity, variable thermal conductivity and variable mass diffusion coefficient are in practice. The most common model exhibits the variation of viscosity, thermal conductivity and mass diffusion coefficient as a function of temperature. Experimental data shows that the viscosity varies as an inverse function of temperature difference \( T - T_\infty \), where \( T \) is the temperature of the fluid and \( T_\infty \) is the temperature of the ambient fluid. However, thermal conductivity and mass diffusion are linear coefficient functions of temperature. Mathematical models for viscosity and thermal conductivity are

\[
\frac{1}{\mu} = \frac{1}{\mu_\infty}[1 + l + r(T - T_\infty)] = a(T - T_r),
\]

\[
K(T) = K_\infty[1 + \epsilon(T - T_\infty)],
\]

where \( a = rl/\mu_\infty, T_r = T_\infty - 1/r \). Further \( a \) and \( T_\infty \) are constant and \( \epsilon \) is very small constant. For liquids \( a > 0 \) whereas for gasses \( a < 0 \). \( K_\infty \) and \( \mu_\infty \), respectively, are the thermal conductivity and the viscosity of the fluid outside the boundary layer region. Diffusion of mass in liquids is an analogy of heat conduction in solids, liquids. Conduction of heat is governed by Fourier law of heat conduction whereas mass diffusion is governed by Fick’s first law. Due to this analogy, mass diffusion coefficient may be consider as a linear function of temperature i.e.

\[
D(T) = D_\infty[1 + \epsilon_1(T - T_\infty)],
\]

where \( D_\infty \) is the mass diffusion coefficient for the solute in the fluid outside the boundary layer region and \( \epsilon_1 \) is very small parameter. It is important to note that for \( \epsilon_1 = \epsilon = 0 \), \( D = D_\infty \) and \( K = K_\infty \), the case of constant thermal conductivity and mass diffusion coefficient.

C. Physical model and coordinate system

Consider diffusion of solute with variable diffusion coefficient in the flow of Maxwell fluid of variable viscosity and variable thermal conductivity (see Eqs. (3) and (4)) over an elastic sheet. The sheet is stretchable and can be stretched with velocity \( U(x) = U_0 e^{x/L} \). Also consider that the sheet is permeable. Through pores of sheet, fluid can be sucked out or can be injected into the flow regime with velocity \( V_w(x) = V_0 e^{x/L} \). As sheet is stretching exponentially, therefore, it is not reasonable to assume the temperature of the sheet to be constant and it seems to be logical to take the temperature of the sheet equal to \( T_w(x) = T_\infty + B e^{x/L} \) where \( U_0 \) and \( V_0 \) are the reference velocities \( L \) is the reference length and \( B \) is the constant which depend on the thermal property of the fluid. Further, \( V_0 < 0 \) is the case for which fluid is being sucked out from the flow regime. However, for \( V_0 > 0 \), Maxwell fluid can be injected into the flow regime. The concentration of solute at the surface of stretching surface is taken of the form \( C_w(x) = C_\infty + B e^{x/L} \). Flow is induced by the stretching of the sheet and is fully developed. The solute of temperature dependent mass diffusion coefficient is being transported by both the convection and diffusion. Flow configuration is shown in Fig. 1.

Above stated scenario suggests the following steady two dimensional flow fields

\[
V = [u(x, y), v(x, y), 0], \quad T = T(x, y), \quad C = C(x, y)
\]

Moving through Eqs. (1)–(6) and using the boundary layer approximations, one gets the following boundary layer equations

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

where...
where \( \rho \) is the density and \( c_p \) is the specific heat of the fluid.

No slip assumption suggests the following boundary conditions
\[
\begin{align*}
    u &= U_4(x), \\
    \frac{\partial u}{\partial y} &= 0, \quad T = T_4(x), \\
    \frac{\partial C}{\partial y} &= 0, \quad C = C_4(x) \quad \text{at} \ y = 0
\end{align*}
\]
\[
\begin{align*}
    u &\to 0, \quad T \to T_{\infty}, \quad C \to C_{\infty} \quad \text{as} \ y \to \infty
\end{align*}
\]

D. Dimensional analysis

Since flow is two dimensional, therefore, single steam system exists and has direct relation with velocity components \( u \) and \( v \). Existence of stream function, reference velocities and reference length suggest the following new variables
\[
\begin{align*}
    \Psi &= \sqrt{2\nu_{\infty} L U_0 f(\eta)} e^{\frac{\Psi}{2}}, \quad \eta = \frac{y}{\sqrt{2\nu_{\infty} L}} e^{\frac{\Psi}{2}}, \quad \theta(\eta) = \frac{T - T_{\infty}}{T_{w} - T_{\infty}}, \\
    \phi(\eta) &= \frac{C - C_{\infty}}{C_w - C_{\infty}}, \quad u = \frac{\partial \Psi}{\partial x}, \quad v = -\frac{\partial \Psi}{\partial y},
\end{align*}
\]
and reduce the boundary layer Eqs. (7)–(10) and the boundary conditions (11) into the couple non linear boundary value problems
\[
\begin{align*}
    2[(f')^2 + k'\frac{f'^3}{2} + \frac{1}{2}f^2f'' - 3ff'''] &= \frac{1}{(1+\theta')(1+\theta)} \left[ f'''' + \frac{3}{2}f''' - f'' \frac{\theta'}{\partial \theta} \right] \\
    f(0) &= s, \quad f'(0) = 1, \quad f''(\infty) = 0 \quad \text{(13)}
\end{align*}
\]
\[
\begin{align*}
    (1 + \epsilon \theta)\theta'' + Pr_{\infty} f' \theta' - b Pr_{\infty} f' \theta + \epsilon \theta^2 &= 0 \\
    \theta(0) &= 1, \quad \theta(\infty) = 0, \quad \text{(14)}
\end{align*}
\]
\[
\begin{align*}
    (1 + \epsilon \theta)\phi'' - Sc_{\infty} f' \phi b - f' \phi - b Pr_{\infty} f' \phi + \epsilon \theta \phi' &= 0 \\
    \phi(0) &= 1, \quad \phi(\infty) = 0 \quad \text{(15)}
\end{align*}
\]
where \( k' \) is the Deborah number, \( \theta_r (= 1/r(T - T)) \) is the fluid viscosity parameter, \( Pr_{\infty} \) and \( Sc_{\infty} \) are Prandtl and Schamidt numbers for the ambient regime. \( s \) is suction/injection parameter. It is worth mentioning that \( \epsilon = 0 \) is the case of constant diffusion coefficients (mass diffusion and thermal conductivity). Further \( \theta_r \to \infty \) is the case of constant viscosity. However, \( \theta_r = -1 \) is for liquids. This discussion concludes that \( \epsilon_1 = \epsilon = 0 \) and \( \theta_r \to \infty \), represents the case of constant thermophysical
properties. It is also observed that \( k^+ = 0, \epsilon = \epsilon_1 = 0 \) and \( \theta_r \to \infty \), is the case of Newtonian fluid with constant properties.

**E. Variable Prandtl and variable Schmidt numbers**

According to the definitions of Prandtl and Schamidt numbers \( \text{Pr} = \mu c_p/k \) and \( \text{Sc} = \mu/\rho D \), here \( \mu \), \( k \) and \( D \) are variables and dependent upon temperature (see Eqs. (3)–(5)). Thus Eqs. (13)–(15) gives the following variable forms of Prandtl and Schamidt numbers.

\[
\begin{align*}
\text{Pr} &= \frac{\rho c_p}{(1 + \epsilon\theta)(1 + \theta/\theta_r)} , \quad \text{Pr}_w = (1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r}) \text{Pr} \\
\text{Sc} &= \frac{\mu}{(1 + \epsilon\theta)(1 + \theta/\theta_r)} , \quad \text{Sc}_w = (1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r}) \text{Sc}
\end{align*}
\]

(16)

using above expression in Eqs. (14) and (15), we have

\[
(1 + \epsilon\theta)\theta'' + \text{Pr}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f \theta' - b \text{Pr}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f' \theta + \epsilon\theta^2 = 0
\]

(17)

\[
(1 + \epsilon_1\theta)\phi'' + \text{Sc}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f \phi' - b \text{Sc}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f' \phi + \epsilon\theta' \phi' = 0
\]

(18)

The Skin friction coefficient \( C_f \) at the surface is defined by

\[
C_f = \tau_{y|y=0}^\text{wall} = \frac{1}{\rho(U_w)^2} \frac{\theta_r}{(\theta_r - 1)} f''(0),
\]

The dimensionless form of heat flux at the surface (Nusselt number) is defined by:

\[
Nu = \frac{xq_w}{k_w(T_w - T_\infty)} = -\frac{1}{\sqrt{\text{Re}}} (1 + \epsilon\theta) \theta'(0),
\]

The Sherwood number \( Sh \) (mass flux at the surface) is defined by

\[
Sh = \frac{xm_w}{D_w(C_w - C_\infty)} = -\frac{1}{\sqrt{\text{Re}}} (1 + \epsilon_1) \phi'(0),
\]

where \( m_w = -D(\partial C/\partial y)|_{y=0} \) and \( \text{Re} \) is the local Reynolds number.

**III. NUMERICAL METHOD: GALERKIN FINITE ELEMENT FORMULATION**

**A. Weighted residual integral**

Highly nonlinear boundary value problems (13), (17), and (18) are solved by Galerkin finite element method (GFEM). The elements of stiffness matrix and boundary vector are given below:

\[
\int_{\eta_r}^{\eta_f} \eta f^+ - h \eta d\eta = 0 \quad (19)
\]

\[
\int_{\eta_r}^{\eta_f} \eta [h^+ + \frac{3}{2} h' - \frac{\theta'}{\theta - \theta_r} h' - (1 - \frac{\theta}{\theta_r}) (2h^2 - fh' + k^+ (2h^3 + f^2h'' - 3fh'))] d\eta = 0 \quad (20)
\]

\[
\int_{\eta_r}^{\eta_f} \eta [(1 + \epsilon\theta)\theta'' + \text{Pr}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f \theta' - b \text{Pr}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f' \theta + \epsilon\theta^2] d\eta = 0 \quad (21)
\]

\[
\int_{\eta_r}^{\eta_f} \eta [(1 + \epsilon\theta)\phi'' + \text{Sc}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f \phi' - b \text{Sc}(1 + \epsilon\theta)(1 - \frac{\theta}{\theta_r})f' \phi + \epsilon\theta' \phi'] d\eta = 0 \quad (22)
\]

\[
f = \sum \psi_i f_i, h = \sum \psi_i h_i, \theta = \sum \psi_i \theta_i, \phi = \sum \psi_i \phi_i
\]

(23)

and \( w_1, w_2, w_3, w_4 = \psi_i \) for \( (i = 1, 2) \)
B. Stiffness matrix elements computation

\[
K_{ij}^{11} = \int_{\eta_r}^{\eta_s} \psi_i \frac{d\psi_j}{d\eta} d\eta,
K_{ij}^{12} = -\int_{\eta_r}^{\eta_s} \psi_i \psi_j d\eta,
K_{ij}^{13} = 0, K_{ij}^{14} = 0,
\]

\[
K_{ij}^{21} = 0, K_{ij}^{22} = \int_{\eta_r}^{\eta_s} \left[ -\frac{d\psi_i}{d\eta} \frac{d\psi_j}{d\eta} + \left(\frac{k^2}{\theta_r^2} \frac{d^2\psi_i}{d\eta^2} - \frac{k^2}{\theta_r^2} \frac{d^2\psi_j}{d\eta^2} \right) \right] d\eta,
\]

\[
+ \frac{3}{2} \frac{d\psi_i}{d\eta} - \bar{\theta_r} \psi_i \frac{d\psi_j}{d\eta} - 2\bar{\theta_r} \psi_i \psi_j + \frac{1}{2} \bar{\theta_r} \psi_j \frac{d\psi_j}{d\eta},
+ 2k^2 \bar{\theta_r} \psi_i \psi_j - \frac{2k^2 \bar{\theta_r} \psi_i \psi_j}{\theta_r} d\eta,
\]

\[
K_{ij}^{23} = 0, K_{ij}^{24} = 0, K_{ij}^{31} = 0, K_{ij}^{32} = 0,
\]

\[
K_{ij}^{33} = \int_{\eta_r}^{\eta_s} \left[ - (1 + \epsilon_0) \frac{d\psi_i}{d\eta} \frac{d\psi_j}{d\eta} + Pr(1 + \epsilon_0)(1 - \bar{\theta_r}) \frac{d\psi_j}{d\eta} \right] d\eta,
\]

\[
- b Pr(1 + \epsilon_0)(1 - \bar{\theta_r}) \frac{d\psi_j}{d\eta} \right] d\eta, K_{ij}^{34} = 0
\]

\[
K_{ij}^{41} = 0, K_{ij}^{42} = 0, K_{ij}^{43} = 0,
\]

\[
K_{ij}^{44} = \int_{\eta_r}^{\eta_s} \left[-(1 + \epsilon_1 \bar{\theta}) \frac{d\psi_i}{d\eta} \frac{d\psi_j}{d\eta} + Sc(1 + \epsilon_0)(1 - \bar{\theta_r}) \frac{d\psi_j}{d\eta} \right] d\eta,
\]

\[
\left[ \bar{\theta_r} \frac{d\psi_j}{d\eta} \right] - b \bar{\theta_r} \psi_j \right] d\eta + \epsilon_0 \bar{\theta_r} \psi_j \frac{d\psi_j}{d\eta} \right] d\eta,
\]

\[
b_1^1 = 0, b_1^2 = -\int_{\eta_r}^{\eta_s} \psi_i \psi_j' d\eta, b_1^3 = -\int_{\eta_r}^{\eta_s} (1 + \epsilon_0) \psi_i \psi_j' d\eta, b_1^4 = -\int_{\eta_r}^{\eta_s} (1 + \epsilon_1 \theta) \psi_i \psi_j' d\eta
\]

It is important to note that the elements of the stiffness matrix are functions of unknown nodal values. Thus assembly process gives birth to a nonlinear system of algebraic equations which are converted into the linear system of algebraic equation using Picard linearization scheme and the resulting linear system of algebraic equations is solved iteratively by Guass Seidal approach.

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**TABLE I.** Numerical values of $-\theta'(0)$ and $-\phi'(0)$ for different elements (different grid sizes).

| Grid | $(Re)^{\frac{1}{2}} Nu$ | $(Re)^{\frac{1}{2}} Sh$ |
|------|-------------------|-------------------|
| 10   | 1.8529            | 2.5726            |
| 50   | 2.4904            | 3.5556            |
| 100  | 2.5523            | 3.6539            |
| 150  | 2.5664            | 3.6760            |
| 200  | 2.5717            | 3.6844            |
| 250  | 2.5743            | 3.6884            |
| 300  | 2.5757            | 3.6907            |
| 350  | 2.5766            | 3.6920            |
| 400  | 2.5771            | 3.6929            |
| 450  | 2.5775            | 3.6936            |
| 500  | 2.5778            | 3.6940            |
| 550  | 2.5780            | 3.6943            |
| 600  | 2.5782            | 3.6946            |
| 700  | 2.5783            | 3.6948            |
| 750  | 2.5784            | 3.6950            |
The numerical experiments are performed to search $\eta_{\text{max}}$ with computational tolerance $10^{-6}$ with 200 elements. Numerical computations showed that $\eta_{\text{max}} = 3$ (see graphical representations of velocity, temperature and concentration).

1. Grid independent study

The unknown nodal values also depend on the number elements (grid size) but this is not desirable. The computed values correspond to the realistic case only when these are grid independent. Therefore, grid independent analysis is performed and is presented in Table I. This table shows that only 200 elements of the computational domain $[0, 3]$ are sufficient for the grid independent solutions.

IV. RESULTS AND DISCUSSION

Extensive simulations are carried out in order to examine the behavior of velocity, temperature and concentration when fluid parameters are varied. Fig. 2 shows the variation of velocity for different values of Deborah number $k'$ when $\epsilon = \epsilon_1 = 0.1, b = 2, Pr = 3, Sc = 5$ and $s = 0.2$.

![FIG. 2. Velocity profile for different values of Deborah number $k'$ when $\epsilon = \epsilon_1 = 0.1, b = 2, Pr = 3, Sc = 5$ and $s = 0.2.$](image)

Fig. 3 shows the velocity profile for different values of suction/injection parameter $s$ when $\epsilon = \epsilon_1 = 0.1, b = 2, Pr = 3, Sc = 5$ and $k' = 0.2$.

![FIG. 3. Velocity profile for different values of suction/injection parameter $s$ when $\epsilon = \epsilon_1 = 0.1, b = 2, Pr = 3, Sc = 5$ and $k' = 0.2.$](image)
values of Deborah number $k^*$ when $\theta_0 = -1$ and $\theta_1 \to \infty$. An increase in Deborah number corresponds to an increase in the elastic nature of the fluids. It is quite evident from this graphical representation that velocity of the viscoelastic fluid decreases when Deborah number is increased. This decrease in velocity is due to the elastic nature of the Maxwell fluid as it tries to restore its deformation during its flow. This Fig. also shows that an increase in the Deborah number momentum boundary layer thickness decreases. This reveals that Maxwell fluid has less boundary layer thickness as compare to the Newtonian fluid. Fig. 2 also predicts that the velocity of viscoelastic fluid (Maxwell fluid) with variable viscosity is less than the velocity of the Maxwell fluid with constant viscosity. The momentum boundary layer thickness for Newtonian fluid is higher than that of the Maxwell fluid.

The effect of non-uniform suction/injection on the flow of Maxwellian fluid in the flow regime for both cases $\theta_0 = -1$ and $\theta_1 \to \infty$ is displayed in Fig. 3. This graphical representation predicts that the velocity increases when injection parameter $s(>0)$ increases. However, opposite behavior of velocity is noted when nonuniform suction of the fluid becomes more and more strong. The effect of suction/injection parameter on the momentum boundary layer thickness is higher when fluid viscosity is constant as compare to the case when fluid viscosity is the function of the temperature.
The Fig. 3 also illustrates that the momentum boundary layer thickness for suction case is less than that in the case of injection. The effects of fluid viscosity parameter $\theta_r$ on velocity is represented in Fig. 4.

The effect of elastic nature of fluid on the temperature is given in Fig. 5 for both $\theta_r = -1$ and $\theta_r \to \infty$. It is noted from this display of the temperature curves that the temperature of viscous fluid ($k^* = 0$) is less than that of the visco elastic fluid ($k^* \neq 0$). From this Fig. it is also observed that temperature of the fluid with variable viscosity is less than that of the fluid with constant viscosity. Fig. 5 predicts that thermal boundary layer thickness is an increasing function of Deborah number. It is also clear from Fig. 5 that the thermal boundary layer thickness for viscous fluid is less than that of the Maxwell fluid. Fig. 6 demonstrates the behavior of the temperature of the fluid (for both the cases of the constant viscosity ($\theta_r \to \infty$) and temperature dependent viscosity ($\theta_r = -1$)) by varying suction/injection parameter $s$. The behavior of temperature and thermal boundary layer thickness with respect to the variation of the power index parameter $b$ for both constant and variable viscosity

FIG. 7. Temperature profile for different values of power index $b$ when $k^* = 0.2$, $b = 2$, $\epsilon = \epsilon_1 = 0.1$, $Pr = 3$ and $s = 0.2$. 
FIG. 8. Temperature profile for different values of Prandtl number Pr when \(k' = 0.2, b = 2, \epsilon = \epsilon_1 = 0.1\) and \(s = 0.2\).

is sketched in Fig. 7. As well as temperature and thermal boundary layer thickness decreases when power index parameter \(b\) is increased. The effect of Prandtl number Pr on the temperature of the fluid for both the cases \(\theta_0 = -1\) and \(\theta_0 \to \infty\) is displayed in Fig. 8. As the Prandtl Pr is the ratio of the momentum diffusion to the thermal diffusion and an increase in Pr corresponds to the decrease in thermal diffusion coefficient. So there is a decrease in the temperature with respect to the Prandtl number and this for both the cases \(\theta_0 = -1\) and \(\theta_0 \to \infty\). The variation of temperature of the fluid for various values of \(\epsilon\) for both the cases \(\theta_0 = -1\) and \(\theta_0 \to \infty\) is given in Fig. 9. The \(\epsilon = 0\) is the case when thermal conductivity of the Maxwell fluid is constant \(i.e.,\) does not depend upon the temperature. However for \(\epsilon \neq 0\) is the case when thermal conductivity is variable and is a function of temperature. It is noted that Fig. 9 that the temperature increases when \(\epsilon\) is increased. The effect of fluid viscosity parameter \(\theta_r\) on the dimensionless temperature is given in Fig. 10. This Fig. reflects that the temperature decreases as the fluid viscosity parameter \(\theta_r\) is increased. Eventually, this decrease in temperature causes a decrease in thermal boundary layer thickness. Thus, thermal boundary layer thickness can be controlled through the fluid viscosity parameter \(\theta_r\).

FIG. 9. Temperature profile for different values of \(\epsilon\) when \(k' = 0.2, b = 2, \text{Pr} = 3,\) and \(s = 0.2\).
The effect of power index parameter $b$ on the concentration of solute diffusing in the Maxwellian fluid is displayed in Fig. 11. It is clear from this Fig. that the effect of power index parameter $b$ on the concentration for $\theta_\tau = -1$ is less than that of the concentration in the Maxwell fluid when viscosity is not varying with respect to temperature. There is a prominent effect of elasticity on the diffusion phenomenon of the solute in Maxwell fluid for both the cases $\theta_\tau = -1$ and $\theta_\tau \to \infty$. This behavior of elasticity on the concentration is represented by the Fig. 12. The effect of suction parameter $s$ on concentration profile is displayed in Fig. 13. It is found that concentration profile decreases when $s$ is increased for both constant and variable viscosity. The variation of dimensionless concentration field under the influence of fluid viscosity parameter $\theta_\tau$ is given in Fig. 14. Figs. 4, 10, and 14 shows viscosity parameter $\theta_\tau$ has similar effects on velocity, temperature and concentration. Comparison of Figs. 15 and 17 reveals that the effect of Prandtl number Pr on the concentration field is opposite to the effect of Schmidt number on the concentration field. The variation of thermal conductivity and mass
FIG. 12. Concentration profile for different values of Deborah number $k^*$ when $Sc = 5$, $b = 2$, $\epsilon_1 = \epsilon = 0.1$, $Pr = 3$ and $s = 0.2$.

FIG. 13. Concentration profile for different values of non-uniform suction/injection parameter $s$ when $k^* = 0.2$, $Sc = 5$, $b = 2$, $\epsilon_1 = \epsilon = 0.1$ and $Pr = 3$.

FIG. 14. Concentration profile for different values of fluid viscosity parameter $\theta$, when $k^* = 0.2$, $Sc = 5$, $b = 2$, $\epsilon = 0.1$, $Pr = 3$ and $s = 0.2$. 
diffusion coefficient is represented by varying the parameter $\epsilon$. The behavior of concentration solute under the variation of $\epsilon$ is displayed in Fig. 16. Concentration field increases when $\epsilon$ is increased for both the cases $\theta_r = -1$ and $\theta_r \rightarrow \infty$. The concentration boundary layer thickness for the case of constant viscosity ($\theta_r \rightarrow \infty$) is greater than that for the case of temperature dependent viscosity. The influence of variation of thermal and mass diffusion coefficients on the concentration of the solute in the flow of Maxwell fluid is demonstrated by the Fig. 16. Concentration boundary layer thickness has an increasing behavior when diffusion coefficients varies with temperature. The effect of Schamidt number $Sc$ on the concentration of solute diffusing in the Maxwellian fluid is displayed in Fig. 17. It is clear from this Fig. that the effect of Schamidt number $Sc$ on the concentration for $\theta_r = -1$ is less than that of the concentration in the Maxwellian when viscosity is not varying with respect to temperature.

Numerical values of skin friction coefficient and heat and mass fluxes for both cases of constant and temperature dependent viscosity are computed and are recovered in Table II. From this it can be observed that shear stress at the surface increases when Prandtl number is increased whereas heat and mass fluxes are decreasing functions of Prandtl number. As an increase in Prandtl number is
due to the decrease in the thermal diffusivity which causes less transfer of heat from the surface into the fluid regime. Therefore, a reduction in rate of heat transfer (when Prandtl number is increased) is observed (see Table II.) As Schmidt $Sc$ number is an analogue of Prandtl number $Pr$, therefore, Schmidt number $Sc$ has the effects on Sherwood as those of $Pr$ on Nusselt number. The rate of heat and mass transfer when thermal conductivity and mass conductance increases due to the rise in temperature. In qualitative sense, those observations are same for the both cases of constant and temperature dependent viscosity.

TABLE II. Behavior of skin friction coefficient, Nusselt number and Sherwood number.

| $\theta_r$ | $-\frac{1}{(Re)^2 C_f}$ | $\frac{1}{(Re)^2 Nu}$ | $\frac{1}{(Re)^2 Sh}$ |
|------------|--------------------------|----------------------|----------------------|
| $\theta_r \to -10$ | | | |
| $Pr$ | | | |
| 2 | 2.3298 | 1.9781 | 3.8453 |
| 3 | 2.3430 | 2.6226 | 3.7680 |
| 5 | 2.3591 | 3.6893 | 3.6663 |
| 0.1 | 2.3430 | 2.6226 | 3.7631 |
| 0.4 | 2.3383 | 2.9289 | 4.2798 |
| 0.8 | 2.3340 | 3.3990 | 4.9707 |
| 0 | 2.3430 | 2.6226 | 3.7512 |
| $\epsilon$ | | | |
| 0.4 | 2.3383 | 2.9289 | 4.2798 |
| 0.8 | 2.3340 | 3.3990 | 4.9707 |
| 0 | 2.3430 | 2.6226 | 3.7512 |
| $\epsilon_1$ | | | |
| 0.5 | 2.3430 | 2.6226 | 3.7566 |
| 0.9 | 2.3430 | 2.6226 | 3.7128 |
| 1 | 2.3430 | 2.6226 | 1.0953 |
| $Sc$ | | | |
| 2 | 2.3430 | 2.6226 | 1.9113 |
| 3 | 2.3430 | 2.6226 | 2.6011 |
| $\theta_r \to \infty$ | | | |
| $Pr$ | | | |
| 2 | 2.4607 | 1.9108 | 3.6906 |
| 3 | 2.4607 | 2.5393 | 3.6332 |
| 5 | 2.4607 | 3.5795 | 3.5571 |
| 0.1 | 2.4607 | 2.5393 | 3.6285 |
| 0.4 | 2.4607 | 2.8271 | 4.1138 |
| 0.8 | 2.4607 | 3.2726 | 4.7649 |
| 0 | 2.4607 | 2.5393 | 3.6171 |
| $\epsilon$ | | | |
| 0.5 | 2.4607 | 2.5393 | 3.6232 |
| 0.9 | 2.4607 | 2.5393 | 3.5826 |
| 1 | 2.4607 | 2.5393 | 1.0701 |
| $\epsilon_1$ | | | |
| 0.5 | 2.4607 | 2.5393 | 3.6232 |
| 0.9 | 2.4607 | 2.5393 | 3.5826 |
| 1 | 2.4607 | 2.5393 | 1.0701 |
| $Sc$ | | | |
| 2 | 2.4607 | 2.5393 | 1.8571 |
| 3 | 2.4607 | 2.5393 | 2.5186 |
V. CONCLUSION

The effects of variable Prandtl and Schmidt numbers on non-Newtonian rheology of viscoelastic fluid are investigated by Galerkin finite element method (GFEM). The behavior of velocity temperature and concentration in boundary layer region under the variation of physical parameters is displayed. The effects of dimensionless parameters on the boundary layer thickness (momentum, thermal and concentration) are studied. The behavior of shear stress (at the surface) and heat flux are analyzed by varying emerging parameters. The most significant observation are listed below.

1. The velocity of viscous fluid is greater than that of the viscoelastic fluid for both the cases of constant viscosity ($\theta_r \to \infty$) and temperature dependent viscosity. Consequently, momentum boundary layer thickness is greater than that of viscoelastic fluid.
2. The behavior of non-uniform injection on velocity is opposite to the behavior of non-uniform suction of velocity. Injection increases the momentum boundary layer thickness. However, suction reduces the momentum boundary layer thickness.
3. The temperature of viscous fluid is less than that of the viscoelastic fluid for both the cases of constant viscosity and variable viscosity. Similar observations are noted for the thermal boundary thickness when $\theta_r = -1$ and $\theta_r \to \infty$.
4. The dimensionless temperature decreases as Prandtl number is increased. Thermal boundary layer thickness is reduced by increasing Prandtl number.
5. Concentration boundary layer thickness is a decreasing function of $b, s, \theta_r$ and $Sc$.

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1. E. M. A. Elbashbeshy, “Heat transfer over an exponentially stretching continuous surface with suction,” Arch. Mech. 53(6), 643–651 (2001).
2. M. Nawaz and T. Hayat, “Axisymmetric stagnation-point flow of nanofluid over a stretching surface,” Adv. Appl. Math. Mech. 6, 22–32 (2014).
3. T. Hayat, A. Shafiq, and A. Alsaedi, “Effect of Joule heating and thermal radiation in the flow of third grade fluid over radiative surface,” PLOS One. 9(1), 83–153 (2014).
4. S. Das, S. Chakraborty, R. Jana, and O. Makinde, “Entropy analysis of unsteady magnetonanofluid flow past accelerating stretching sheet with convective boundary conditions,” Appl. Math. Mech. 36(12), 1593–1610 (2015).
5. M. Nawaz and T. Zubair, “Finite element study of three dimensional radiative nano-plasma flow subject to Hall and ion slip currents,” Results in Physics. 7, 4111–4122 (2017).
6. M. Nawaz, T. Hayat, and A. Alsaedi, “Mixed convection three-dimensional Maxwell fluid flow in the presence of Hall and ion-slip effects,” J. Heat Transfer. 135(4), 042–502 (2013).
7. T. Hayat, M. Awaits, M. Nawaz, S. Irum, and S. Obaidat, “Mixed convection three-dimensional flow with Hall and ion-slip effects,” Int. J. Nonlinear Scien. Numer. Simulat. 14(3–4), 167–177 (2013).
8. M. I. Khan, M. Waqasa, T. Hayat, M. I. Khan, and A. Alsaedi, “Behavior of stratification phenomenon in flow of Maxwell nanomaterial with motile gyrotactic microorganisms in the presence of magnetic field,” Int. J. of Mech. Scien. 131–132, 426–434 (2017).
9. M. Awaits, T. Hayat, and A. Ali, “3-D Maxwell fluid flow over an exponentially stretching surface using 3-stage Lobatto II A formula,” AIP Advances. 6(5), 055–121 (2016).
10. M. Awaits, N. Muhammad, T. Hayat, and A. Alsaedi, “Chemical reaction effects in Maxwell fluid flow over a permeable surface: Dual solutions,” Int. J. Nonlinear Scien. Numer. Simulat. 16, 123–128 (2015).
11. S. Saleem, M. Awaits, S. Nadeend, N. Sandeep, and M. T. Mustafa, “Theoretical analysis of upper convected Maxwell fluid flow with Cattaneo–Christov heat flux model,” Chin. J. Phys. 55, 1615–1625 (2017).
12. F. C. Lai and F. A. Kulacki, “The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium,” Int. J. Heat Mass Transfer. 33(5), 1028–1031 (1990).
13. K. V. Prasad, P. Duhal, V. Umesh, and N. S. Prasanna Rao, “The effect of variable viscosity on MHD viscoelastic fluid flow and heat transfer over a stretching sheet,” Commun Nonlinear Sci. Numer. Simulat. 15, 331–344 (2010).
14. V. Singh and S. Agarwal, “Flow and heat transfer of Maxwell fluid with variable viscosity and thermal conductivity over an exponentially stretching sheet,” American J. Fluid Dyn. 8(4), 87–95 (2013).
15. S. Mukhopadhyay and G. C. Layek, “Effects of variable fluid viscosity on flow past a heated stretching sheet embedded in a porous medium in presence of heat source/sink,” Meccanica. 47, 863–876 (2012).
16. I. Pop, R. S. R. Gorla, and M. Rashidi, “The effect of variable viscosity on flow and heat transfer to a continuous moving flat plate,” Int. J. Eng. Sci. 30(1), 1–6 (1992).
17. T. C. Chaim, “Heat transfer with variable thermal conductivity in a stagnation-point flow towards a stretching sheet,” Int. Commun. Heat Mass Transfer. 23, 239–248 (1996).
18 J. Gary, D. R. Kassoy, H. Tadjeran, and A. Zebib, “The effects of significant viscosity variation on convective heat transport in water saturated porous medium,” J. Fluid Mech. 117, 233–249 (1982).
19 K. N. Mehta and S. Sood, “Transient free convection flow with temperature dependent viscosity in a fluid saturated porous medium,” Int. J. Eng. Sci. 30(8), 1083–1087 (1992).
20 S. Mukhopadhyay, G. C. Layek, and S. A. Samad, “Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity,” Int. J. Heat Mass Transfer. 48, 4460–4466 (2005).
21 M. A. El-Aziz, “Temperature dependent viscosity and thermal conductivity effects on combined heat and mass transfer in MHD three-dimensional flow over a stretching surface with Ohmic heating,” Meccanica. 42, 375–386 (2007).
22 N. Sandeep, G. Kumaran, and S. Saleem, “Effect of cross diffusion on magnetohydrodynamic parabolic flow of Carreau liquid in the presence of buoyancy force,” J. Integr. Neurosci. (2018).
23 J. Reddy, An introduction to the finite element method (McGraw-Hill International Edition, 1984).
24 J. Reddy, An introduction to the nonlinear finite element analysis (Oxford University Press, 2005).
25 Z. Li, M. Sheikholeslami, A. J. Chamkhad, Z. A. Raizah, and S. Saleem, “Control volume finite element method for nanofluid MHD natural convective flow inside a sinusoidal annulus under the impact of thermal radiation,” Comp. Meth. Appl. Mech. Eng. 338, 618–633 (2018).