KÄHLER MANIFOLDS WITH NUMERICALLY EFFECTIVE RICCI CLASS AND MAXIMAL FIRST BETTI NUMBER ARE TORI

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Abstract. Let $M$ be an $n$-dimensional Kähler manifold with numerically effective Ricci class. In this note we prove that, if the first Betti number $b_1(M) = 2n$, then $M$ is biholomorphic to the complex torus $T^n_C$.

1. Introduction

Let $M$ be a compact complex manifold with a fixed hermitian metric $\omega$. By [DPS93] [DPS94] a holomorphic line bundle $L$ over $M$ is called numerically effective (abb. nef) if for every $\varepsilon > 0$, there is a smooth hermitian metric $h_\varepsilon$ on $L$ such that the curvature satisfies

$$\Theta_{h_\varepsilon} \geq -\varepsilon \omega$$

If $M$ is projective, $L$ is nef precisely if $L \cdot C \geq 0$ for all curves $C \subset M$. We say a Kähler manifold $M$ is nef if the anticanonical bundle $-K_M$ is nef. In [DPS93] it is conjectured, for a nef Kähler manifold, both of the following holds:

(A1) the fundamental group $\pi_1(M)$ has polynomial growth.
(A2) the Albanese map $\alpha : M \to Alb(M)$ is surjective.

If $M$ is projective, (A2) was proved by Q. Zhang [Zh]. In [Pa1] Paun proved (A1), assuming a conjecture of Gromov concerning the fundamental group of Riemannian manifold with almost non-negative Ricci curvature (compare [ChCo]).

By the Aubin-Calabi-Yau theorem [Ab][Ya], [DPS93] proved that $M$ is nef if and only if there exist a sequence of Kähler metrics $\{\omega_k\}$ on $M$ such that, for each $k > 0$, the metric $\omega_k$ belongs to a fixed cohomology class $\{\omega\}$, and the Ricci curvature of $\omega_k$ is bounded from below by $-1/k$.

A Bochner type theorem for the first Betti number was obtained by Paun [Pa2], namely, for every nef Kähler manifold $M$ of complex dimension $n$ it holds that $b_1(M) \leq 2n$. The main result of this note is the following:

Theorem 1.1. Let $M$ be a nef Kähler manifold of dimension $n$. If the first Betti number $b_1(M) = 2n$, then $M$ is biholomorphic to the complex torus $T^n_C$.

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Theorem 1.1 may be considered as a complex version of a conjecture of Gromov, proved by T. Colding [Co], asserts that a Riemannian $n$-manifold of almost non-negative Ricci curvature and first Betti number $n$ is homeomorphic to the torus $T^n$.

Obviously, Theorem 1.1 implies conjecture (A2) in the case of $b_1(M) = 2n$. In the proof of Theorem 1.1, in fact we will prove that there is a uniform upper bound for the diameters. But this does not hold if $b_1(M) = 2n - 2$ (the first Betti number of a Kähler manifold is always even). Since there is a sequence of Kähler metrics on $S^2$ in the same Kähler class with positive Ricci curvature but converge to a non-compact space of dimension 1, thus the product $T_c^{n-1} \times S^2$ serves as an example.

The following result verifies (A2) for manifold with $b_1(M) = 2n - 2$, provided $G_1/[G_1, G]$ has rank at least two, where $G = \pi_1(M)$, $G_1 = [G, G]$.

**Theorem 1.2.** Let $M$ be a nef Kähler manifold of dimension $n$. Let $G = \pi_1(M)$. If the first Betti number $b_1(M) = 2n - 2$, and $G_1/[G_1, G]$ has rank at least two where $G_1 = [G, G]$, then the Albanese map $\alpha : M \to T_c^{n-1}$ is surjective.

**Remark 1.3.** By Theorems 1.1 one confirms immediately conjecture (A2) for $n = 2$. This was first obtained in [DPS93] using algebraic geometry methods.

The proof of our Theorems uses the deep results in Riemannian geometry, including the equivariant Gromov-Hausdorff convergence [FY], a splitting theorem of Cheeger-Colding for limit spaces [ChCo], and a result of Paun [Pa2]. It would be interesting if Theorem 1.1 could be proved in a way of pure algebraic geometry. Indeed, if the Albanese map $\alpha$ is surjective, by the Poincare-Lelong equation, one obtains easily that a nef Kähler manifold $M$ of dimension $n$ with $b_1(M) = 2n$ is biholomorphic to the complex torus $T^n_c$ (compare [Mo]).

By Campana [Ca] the above conjectures (A1) and (A2) together with Gromov’s celebrated theorem [Gr] implies that the fundamental group of a nef Kähler manifold is almost abelian. By our approach, it seems plausible to prove the following:

**Conjecture 1.4.** Let $M$ be a nef Kähler manifold of dimension $n$. If there is an epimorphism $\varphi : \pi_1(M) \to \Gamma$ where $\Gamma$ is a torsion free nilpotent group of rank at least $2n$, then $\Gamma \cong \mathbb{Z}^{2n}$ and $M$ is biholomorphic to the complex torus $T^n_c$. 
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2. Proof of Theorems 1.1 and 1.2

By [DPS93], a nef Kähler manifold $M$ admits a family of Kähler metrics $\omega_\varepsilon$ in the same Kähler class $[\omega]$ with Ricci curvature $\text{Ric}.(\omega_\varepsilon) \geq -\varepsilon \omega$, where $\varepsilon \in (0,1)$. The diameters of this family may not have a uniform upper bound. In other words, the pointed Gromov-Hausdorff limit of $(M,\omega_\varepsilon)$ may not be compact. Because of this, many techniques in metric geometry do not apply to this situation. To overcome this difficulty, [DPS93] obtained the following key lemma.

Lemma 2.1 (DPS93). Let $M$ be a nef Kähler manifold. Let $U \subset \tilde{M}$ (the universal covering of $M$) be a compact subset. Then, $\forall \delta > 0$, there exists a closed subset $U_{\varepsilon,\delta} \subset U$ such that
\begin{align*}
(2.1.1) \quad \text{vol}_{\omega}(U - U_{\varepsilon,\delta}) < \delta; \\
(2.1.2) \quad \text{diam}_{\omega}(U_{\varepsilon,\delta}) \leq C/\sqrt{\delta}.
\end{align*}
where $C$ is a constant independent of $\varepsilon$ and $\delta$.

For convenience let us recall the definition of equivariant Gromov-Hausdorff distance (cf. [FY] for details).

Let $\mathfrak{M}$ (resp. $\mathfrak{M}_{eq}$) denote the set of all isometry classes of pointed metric spaces $(X,p)$ (resp. triples $(X,\Gamma,p)$), such that, for any $D$, the metric ball $B_p(D,X)$ of radius $D$ is relatively compact and such that $X$ is a length space [GLP][FY] (resp. $(X,p) \in \mathfrak{M}$ and $\Gamma$ is a closed subgroup of isometries of $X$).

Let $\Gamma(D) = \{\gamma \in \Gamma : d(\gamma p, p) < D\}$.

Definition 2.2. Let $(X,\Gamma,p),(Y,\Lambda,q) \in \mathfrak{M}_{eq}$. An $\varepsilon$-equivariant pointed Hausdorff approximation stands for a triple $(f,\varphi,\psi)$ of maps $f : B_p(\frac{1}{\varepsilon},X) \to Y$, $\varphi : \Gamma(\frac{1}{\varepsilon}) \to \Lambda(\frac{1}{\varepsilon})$, and $\psi : \Lambda(\frac{1}{\varepsilon}) \to \Gamma(\frac{1}{\varepsilon})$ such that
\begin{align*}
(2.2.1) \quad f(p) = q; \\
(2.2.2) \quad \text{the } \varepsilon\text{-neighborhood of } f(B_p(\frac{1}{\varepsilon},X)) \text{ contains } B_q(\frac{1}{\varepsilon},Y);
\end{align*}
\[(2.2.3) \text{ if } x, y \in B_p \left( \frac{1}{\varepsilon}, X \right), \text{ then } \]
\[|d(f(x), f(y)) - d(x, y)| < \varepsilon;\]
\[(2.2.4) \text{ if } \gamma \in \Gamma \left( \frac{1}{\varepsilon} \right), x \in B_p \left( \frac{1}{\varepsilon}, X \right), \gamma x \in B_p \left( \frac{1}{\varepsilon}, X \right), \text{ then } \]
\[d(f(\gamma x), \varphi(\gamma)(f(x))) < \varphi;\]
\[(2.2.5) \text{ if } \mu \in \Lambda \left( \frac{1}{\varepsilon} \right), x \in B_p \left( \frac{1}{\varepsilon}, X \right), \psi(\mu)(x) \in B_p \left( \frac{1}{\varepsilon}, X \right), \text{ then } \]
\[d(f(\psi(\mu)(x)), \mu f(x)) < \varepsilon.\]

The equivariant pointed Gromov-Hausdorff distance \(d_{eGH}(\langle X, \Gamma, p \rangle, \langle Y, \Lambda, q \rangle)\) stands for the infimum of the positive numbers \(\varepsilon\) such that there exist \(\varepsilon\)-equivariant pointed Hausdorff approximations from \(\langle X, \Gamma, p \rangle\) to \(\langle Y, \Lambda, q \rangle\) and from \(\langle Y, \Lambda, q \rangle\) to \(\langle X, \Gamma, p \rangle\).

\textbf{Proof of Theorem 1.1.} Let \(\omega_k\) be a sequence of Kähler metrics on \(M\) in the same Kähler class with Ricci curvature \(\geq -\frac{1}{k}\omega\). Let \(\tilde{M}_k\) be the Riemannian covering space of \(M_k\) (the manifold \(M\) with the Kähler metric \(\omega_k\)). Using Lemma 2.1 Paun [Pa2] proved that there is an open subset \(\tilde{U}_k \subset \tilde{M}_k\) of diam \(\omega_k(\tilde{U}_k) \leq C\) such that the homomorphism \(\pi_1(U_k) \to \pi_1(M_k)\) is surjective, where \(U_k\) is the image of \(\tilde{U}_k\) in \(M_k\), \(C\) is a universal constant independent of \(k\).

For convenience let \(G = \pi_1(M)\), and let \(\Gamma = G/[G,G]\). Consider \(\tilde{M}_k = \tilde{M}_k/[G,G]\). By assumption \(\mathbb{Z}^{2n} \subset \Gamma\) acts on \(\tilde{M}_k\) by isometry. By a lemma of Gromov [GLP] (compare [Pa 2]) it follows that there is a finite index torsion free subgroup \(\Gamma_k\) of \(\Gamma\) such that, fixing a base point \(p_k \in \tilde{U}_k \subset \tilde{M}_k\),

\[(2.3.1) \text{ the geometric norm of any non-trivial element of } \Gamma_k \text{ is at least } C.\]

\[(2.3.2) \Gamma_k \text{ is generated by } \gamma_1, \cdots, \gamma_{2n} \text{ so that the geometric norm of every } \gamma_i \text{ is at most } 2C.\]

Since \(\Gamma_k\) acts on \(\tilde{M}_k\) by isometry, the quotient space \(\tilde{M}_k/\Gamma_k\) is a finite Riemannian covering space of \(M_k\). Because the Ricci curvature of \(\tilde{M}_k\) is bounded from below, by the Gromov compactness theorem (cf. [FY]) the pointed spaces converge

\[(\tilde{M}_k, \Gamma_k, p_k) \xrightarrow{d_{eGH}} (X, \Gamma_\infty, q)\]

in the equivariant Gromov-Hausdorff topology when \(k\) tends to infinity. By (2.3.1) it is easy to see that the isometric action of \(\Gamma_\infty\) on \(X\) is discrete and effective. By the splitting theorem [ChCo] the limit space \(X = Y \times \mathbb{R}^{\ell}\), where \(Y\) contains no line. By [GLP] it is well-known the Hausdorff dimension of \(X\) is at most \(2n\), therefore \(\ell \leq 2n\). We first need

\textbf{Lemma 2.3.} \(\Gamma_\infty \cong \mathbb{Z}^{2n}\).
Proof of Lemma 2.3. By definition, there are maps \( \varphi_k : \Gamma_k(k) \to \Gamma_\infty(k) \), \( \psi_k : \Gamma_\infty(k) \to \Gamma_k(k) \) and a \( \frac{1}{k} \)-Hausdorff approximation \( f_k : B_{p_k}(k, \bar{M}_k) \to B_{q}(k, X) \) satisfying (2.2.1)-(2.2.5).

We first claim that \( \varphi_k \) is injective for sufficiently large \( k \). If not, there are two elements \( \gamma_k \neq \lambda_k \in \Gamma_k(k) \) such that \( \varphi(\gamma_k) = \varphi(\lambda_k) \) for any \( k \). Let \( \mu_k = \varphi(\gamma_k) = \varphi(\lambda_k) \). Put \( x = p_k \). By (2.2.4) we get that \( d(f_k(\lambda_k x), \mu_k f_k(x)) < \frac{1}{k} \) and \( d(f_k(\gamma_k x), \mu_k f_k(x)) < \frac{1}{k} \). Therefore, \( d(f_k(\lambda_k x), f_k(\gamma_k x)) < \frac{2}{k} \) and so \( d(\lambda_k \gamma_k^{-1} x, x) = d(\lambda_k x, \gamma_k x) < \frac{4}{k} \) since \( f_k \) is a \( \frac{1}{k} \)-Hausdorff approximation. A contradiction to (2.3.1).

Secondly, we claim that \( \varphi_k(\gamma_i \gamma_j) = \varphi_k(\gamma_i) \varphi_k(\gamma_j) = \varphi_k(\gamma_j) \varphi_k(\gamma_i) \) for any \( \gamma_i, \gamma_j \in \Gamma_k(k) \) so that \( \gamma_i \gamma_j \in \Gamma_k(k) \). In fact, by (2.2.4) again we get that \( d(\varphi_k(\gamma_i \gamma_j) f_k(x), f_k(\gamma_i \gamma_j x)) < \frac{1}{k} \); \( d(\varphi_k(\gamma_i) \varphi_k(\gamma_j) f_k(x), \varphi_k(\gamma_i) f_k(\gamma_j x)) < \frac{1}{k} \) and \( d(f_k(\gamma_i \gamma_j x), \varphi_k(\gamma_i) f_k(\gamma_j x)) < \frac{1}{k} \). Thus, \( d(\varphi_k(\gamma_i \gamma_j) f_k(x), f_k(\gamma_i) \varphi_k(\gamma_j) f_k(x)) < \frac{2}{k} \). For the same reason as above, by (2.3.1) it follows that \( \varphi_k(\gamma_i \gamma_j) = \varphi_k(\gamma_i) \varphi_k(\gamma_j) \). The claim follows.

Similar argument applies to show that \( \varphi_k(\gamma_i^{-1}) = \varphi_k(\gamma_i)^{-1} \), if \( \gamma_i, \gamma_i^{-1} \in \Gamma_k(k) \).

Next we verify that \( \varphi_k : \Gamma_k(k) \to \Gamma_\infty(k) \) is also surjective.

We argue by contradiction. Assume such an element \( \mu_k \in \Gamma_\infty(k) \). By (2.2.5) \( d(f_k(\psi(\mu_k)(x), \mu_k f_k(x)) < \frac{1}{k} \). By (2.2.4) \( d(f_k(\psi(\mu_k)(x), \varphi_k(\psi(\mu_k)) f_k(x)) < \frac{1}{k} \). Therefore, \( d(\varphi_k(\psi(\mu_k)) f_k(x), \mu_k f_k(x)) < \frac{2}{k} \). By (2.3.1) this implies that \( \mu_k = \varphi_k(\psi(\mu_k)) \). A contradiction.

For sufficiently large \( k \), let \( \Gamma_0 \) be the subgroup of \( \Gamma_\infty \) generated by \( \varphi_k(\gamma_1), \ldots, \varphi_k(\gamma_{2n}) \). It may be verified easily that this does not depend on the choice of \( k \). By (2.3.2) and the above \( \Gamma_0 \) is a commutative group of rank \( 2n \). Since \( \varphi_k \) is surjective, \( \Gamma_0 = \Gamma_\infty \). The desired result follows. \( \square \)

To continue the proof of Theorem 1.1, we first prove that \( X = \mathbb{R}^{2n} \). It suffices to show that \( \ell = 2n \).

We argue by contradiction. Assume \( \ell < 2n \).

Since \( \Gamma_\infty \) preserves the splitting, there is a well-defined homomorphism \( p : \Gamma_\infty \to \text{Isom}(\mathbb{R}^\ell) \). Let \( \Gamma_{0, \infty} \) denote the kernel of \( p \). By the generalized Bieberbach theorem (cf. [FY]) the image \( p(\Gamma_{0, \infty}) \) has rank at most \( \ell \). By Lemma 2.3 \( \Gamma_{0, \infty} \) has rank \( \geq 1 \). For a nontrivial element of \( \mu \in \Gamma_{0, \infty} \), by (2.2.5) there is a sequence of element \( \gamma_k = \psi_k(\mu) \in \Gamma_k \) (of infinite order) such that the \( \gamma_k \)-action on \( \bar{M}_k \) converges to the action of \( \mu \) on \( Y \times \mathbb{R}^\ell \). Observe that a minimal geodesic representation in \( \bar{M}_k/\Gamma_k \) gives rise to a line in \( \bar{M}_k \), on which \( \gamma_k \) acts by deck transformation. This sequence of lines converges to a line in \( Y \).
on which \( \mu \) acts by translation. Therefore the line lies in \( Y \) since \( \mu \in \Gamma_{0,\infty} \) acts trivially on the factor \( \mathbb{R}^\ell \). A contradiction to the assumption that \( Y \) has no line. Hence \( \ell = 2n \).

Finally, by (2.3.2) we see that \( \mathbb{R}^{2n}/\Gamma_\infty \) is compact. By [FY] Lemma 3.4 \( \tilde{M}_k/\Gamma_k \) converges to \( \mathbb{R}^{2n}/\Gamma_\infty \). This shows that \( \tilde{M}_k/\Gamma_k \) has uniformly bounded diameter. Therefore, \( \tilde{M}_k/\Gamma_k \) has almost non-negative Ricci curvature in Gromov’s sense [GLP]. By [Co] we conclude that \( \tilde{M}_k/\Gamma_k \) is homeomorphic to a torus \( T^{2n} \), and so is \( M \). By Poincare-Lelong equation it follows that the Albanese map has no zeros and is actually a biholomorphism. This completes the proof of Theorem 1.1. \( \square \)

Remark 2.4. The above proof actually shows that a sequence of Kähler metrics on \( T^{2n} \) in the same Kähler class \([\omega]\) with Ricci curvature \( \geq -\varepsilon \omega \) has uniformly bounded diameter, and so the metrics do not collapse.

Let \( G = \pi_1(M) \). Consider the lower central series
\[
\cdots G_2 \subset G_1 \subset G_0 = G
\]
where \( G_1 = [G,G] \) and \( G_2 = [G_1,G] \). Let \( G'_2 \subset G \) be the normal subgroup such that \( G/G'_2 = (G/G_2)/\text{torsion} \). Assume \( H_1(G)/\text{torsion} \cong \mathbb{Z}^{2n-2} \), and \( \text{rank}(G/G'_2) = 2n - 2 + m \). By [Pa2] we may assume elements \( \gamma_1, \ldots, \gamma_{2n-2}; \alpha_1, \ldots, \alpha_m \in G \) which generate a finite index subgroup \( \Gamma'_k \subset G/G'_2 \) and satisfy (2.3.1), (2.3.2) and

(2.5.1) the geometric norms of \( \alpha_1, \ldots, \alpha_m \) are all less than \( 2C \).

We warn that it is not true if we require that \( \alpha_1, \ldots, \alpha_m \) satisfy (2.3.2).

Now we start the proof of Theorem 1.2. We will only sketch the main steps since the proof follows the same line as the previous one.

**Proof of Theorem 1.2.** Let \( \tilde{M}'_k = \tilde{M}_k/G'_2 \). Consider the triple \( (\tilde{M}'_k,\Gamma'_k,p_k) \). The pointed spaces converge
\[
(\tilde{M}'_k,\Gamma'_k,p_k) \xrightarrow{d_G} (X,\Gamma'_\infty,q)
\]

Exactly the same argument in the previous proof implies that \( X = Y \times \mathbb{R}^{2n-2} \) and \( Y \) contains at least a line since the group generated by \( \{\alpha_1, \ldots, \alpha_m\} \) converges to a non-trivial isometry group acting on \( X \) acting trivially on \( \mathbb{R}^{2n-2} \), where \( Y \) is a length space of Hausdorff dimension at most two. However, since (2.3.2) is not satisfied for the \( \alpha_i \)'s, the limit group \( \Gamma'_\infty \) may not be discrete (compare [FY] example 3.11). Therefore, \( X = Y_0 \times \mathbb{R}^{2n-1} \) where the Hausdorff dimension of \( Y_0 \) is at most 1.

If \( Y_0 \) is compact, e.g., zero dimensional, by (2.3.2) and (2.5.1) it follows that the limit space \( Y_0 \times \mathbb{R}^{2n-1}/\Gamma'_\infty \) is compact. Therefore, the diameters of the sequence \( \tilde{M}_k/\Gamma_k \).
have a uniform upper bound, so are the diameters of $M_k$ (since $M_k$ is a finite isometric quotient of $M_k'/\Gamma_k'$). By [Pa3] it follows that the Albanese map is surjective.

If $Y_0$ is 1-dimensional and non-compact, clearly, $Y_0$ has two ends and thus $Y_0$ contains a line. By [ChCo] once again $Y_0 = \mathbb{R}$. This proves that $X \cong \mathbb{R}^{2n}$. Since $m \geq 2$, the rank of $\Gamma'_\infty$ is at least $2n$ (may be non-discrete). By the generalized Bieberbach theorem the quotient $\mathbb{R}^{2n}/\Gamma'_\infty$ has to cocompact. For the same reasoning as above the desired result follows.

□

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