Landau–Kolmogorov inequality revisited

A. Shadrin
DAMTP, University of Cambridge, UK

1 Introduction

The Landau–Kolmogorov problem consists of finding the upper bound $M_k$ for the norm of intermediate derivative $\|f^{(k)}\|$, when the bounds $\|f\| \leq M_0$ and $\|f^{(n)}\| \leq M_n$, for the norms of the function and of its higher derivative, are given.

Here, we consider the case of a finite interval when $f \in W_n^\infty[-1,1]$ and all the norms are the max-norms, $\|\cdot\| = \|\cdot\|_{L_\infty[-1,1]}$. Precisely, given $n, k \in \mathbb{N}$ and $\sigma \geq 0$, we define the functional class

$$W_n^\infty(\sigma) := \{f : f \in W_n^\infty[-1,1], \|f\| \leq 1, \|f^{(n)}\| \leq \sigma\}$$

and consider the problem of finding the values

$$m_k(x, \sigma) := \sup_{f \in W_n^\infty(\sigma)} |f^{(k)}(x)|, \quad x \in [-1,1],$$

$$M_k(\sigma) := \sup_{f \in W_n^\infty(\sigma)} \|f^{(k)}\| = \sup_{x \in [-1,1]} m_k(x, \sigma).$$

Our interest to that particular case is motivated by the fact that there are good chances to add this case to a short list of Landau–Kolmogorov inequalities where a complete solution exists, i.e., a solution that covers all values of $n, k \in \mathbb{N}$ (and, for a finite interval, all values of $\sigma > 0$). The main guideline in finding how good these chances are is the following conjecture.

**Conjecture 1.1 (Karlin [4])** For all $n, k \in \mathbb{N}$ and all $\sigma > 0$,

$$m_k(1, \sigma) = \sup_{x \in [-1,1]} m_k(x, \sigma). \quad (1.1)$$

If (1.1) is true for particular set $\{n, k, \sigma\}$, then the function $f \in W_n^\infty(\sigma)$ that provides extremum $M_k(\sigma)$ to the value $\|f^{(k)}\|$ over $W_n^\infty(\sigma)$ is the same as the solution to the pointwise problem at the end-point of the interval. The latter solution is however known to be a certain Chebyshev or Zolotarev spline $Z_n(\cdot, \sigma)$ (which is just a polynomial for small $\sigma$), and thus we have a characterization of the extremal function.

**Corollary 1.2** If equality (1.1) is valid for particular $\{n, k, \sigma\}$, then for that set of parameters we have

$$M_k(\sigma) = \|Z_n^{(k)}(\cdot, \sigma)\| = Z_n^{(k)}(1, \sigma). \quad (1.2)$$

So far, Karlin’s conjecture has been proved for small $n$ with all $\sigma$, and for all $n$ with particular $\sigma$, namely in the following cases:

1. $n = 2$, all $\sigma$, Chui–Smith [11] ($\sigma \leq \sigma_n$), Landau [5] ($\sigma > \sigma_n$);
2. $n = 3$, all $\sigma$, Sato [8], Zvyagintsev–Lepin [12];
3. $n = 4$, all $\sigma$, Zvyagintsev [11] ($\sigma \leq \sigma_n$), Naidenov [7] ($\sigma > \sigma_n$);
4. $n \in \mathbb{N}$, $\sigma = \sigma_n$, Eriksson [3].
\[ \sigma_n := \| T_n^{(n)} \| = 2^{n-1}n! \],

where \( T_n \) is the Chebyshev polynomial of degree \( n \) on the interval \([-1, 1]\).

The value \( \sigma = \sigma_n \) serves as a borderline between two types of the extremal Zolotarev functions \( Z_n(\cdot, \sigma) \): if \( \sigma \leq \sigma_n \), then \( Z_n \) is a polynomial of degree \( n \), while for \( \sigma > \sigma_n \) it is a perfect spline of degree \( n \) with \( r \) knots. There are further borderlines \( \sigma_{n,r} \) (with \( \sigma_{n,1} := \sigma_n \)) which indicate that the spline \( Z_n(\cdot, \sigma) \) has exactly \( r \) knots if \( \sigma_{n,r} < \sigma \leq \sigma_{n,r+1} \), but that distinction is hardly of any use, since for \( n > 3 \) there are no reasonable estimates for perfect splines even with one knot. In this respect, we may apply more or less developed polynomial tools to tackle the problem for \( \sigma \leq \sigma_n \), and then may try to use polynomial estimates in the spline case, when \( \sigma > \sigma_n \).

In this paper, we prove Karlin’s conjecture in several further subcases.

1) The first result closes the “polynomial” case and proves that, for \( \sigma \leq \sigma_n \), the extremum value of the \( k \)-th derivative of \( f \in W^n_\infty(\sigma) \) is provided by the corresponding Zolotarev polynomial.

**Theorem 1.3** If

\[ n \in \mathbb{N}, \quad 1 \leq k \leq n - 1, \quad 0 \leq \sigma \leq \sigma_n, \quad (1.3) \]

then Karlin’s conjecture \((1.1)-(1.2)\) is true.

2) For the “spline” case, we managed to advance only up to the second derivative.

**Theorem 1.4** If

\[ n \in \mathbb{N}, \quad k = 1, 2, \quad \sigma_n < \sigma < \infty, \]

then Karlin’s conjecture \((1.1)-(1.2)\) is true.

The further advance depends mostly on improving the lower bound for the exact constant \( C_{n,k} \) in Landau-Kolmogorov inequality on the half-line:

\[ \| f^{(k)} \|_{\mathbb{R}_+} \leq C_{n,k} \| f \|_{1/n}^{1-k/n} \| f^{(n)} \|_{1/n}^{k/n} \]

The existing lower bounds for \( C_{n,k} \), which are due to Stechkin, are not very satisfactory for general \( n \) and \( k > 2 \).

3) However, for small \( n \), these bounds can be improved, thus leading to one more extension.

**Theorem 1.5** If

\[
\begin{align*}
\text{n} & = 5, 6, & 1 \leq k \leq n - 2, \\
\text{n} & = 7, 8, 9, & 1 \leq k \leq n - 3, & \sigma_n < \sigma < \infty, \\
\text{n} & = 10, 11, & 1 \leq k \leq 6,
\end{align*}
\]

then Karlin’s conjecture \((1.1)-(1.2)\) is true.

In all the cases, the proof is based on comparing the upper bound for the local extrema of the function \( m_k(\cdot, \sigma) \) with the lower bound for the value \( m_k(1, \sigma) \). The technique we use is not working for the value \( k = n - 1 \), what explains restriction in \((1.4)\). In \((1.3)\), i.e., for \( 0 \leq \sigma < \sigma_n \), we managed to cover the case \( k = n - 1 \) by different means.

The upper bounds are given in terms of Zolotarev polynomials and these estimates may be viewed as a generalization to higher derivatives of Markov-type results of Schur \([9]\) and Erdős-Szego \([2]\). These bounds demonstrate once again, if we borrow the words of Shoenenberg said about cubic splines, “the brave behaviour of Zolotarev polynomials under difficult circumstances”.  

\[ 2 \]
2 Main ingredients of the proof

Karlin’s conjecture states that the function $m_k(\cdot, \sigma)$ (which is a positive even function) reaches its maximal value at the end-points of the interval $[-1, 1]$. To establish this fact it is sufficient to check that, at any point $x_0$ inside the interval $(-1, 1)$ where $m_k(\cdot, \sigma)$ takes its local maximum, we have

$$m_k(x_0, \sigma) < m_k(1, \sigma).$$

If $f$ is the function from $W_\infty(\sigma)$ that attains a locally maximal value $m_k(x_0, \sigma)$, then clearly

$$m_k(x_0, \sigma) = |f^{(k)}(x_0)|, \quad f^{(k+1)}(x_0) = 0,$$

and it makes sense to introduce the following quantity:

$$m_k^*(x_0, \sigma) := \sup \{ |f^{(k)}(x_0)| : f \in W_\infty(\sigma), f^{(k+1)}(x_0) = 0 \}, \quad x_0 \in [-1, 1].$$

The next statement follows immediately.

**Claim 2.1** If, for a given $n, k \in \mathbb{N}$ and $\sigma > 0$, we have

$$\sup_{x_0 \in [-1, 1]} m_k^*(x_0, \sigma) \leq m_k(1, \sigma), \quad (2.1)$$

then Karlin’s conjecture is true.

In order to verify inequality (2.1), we split it into two parts

$$m_k^*(x_0, \sigma) \leq A(n, k, \sigma), \quad A(n, k, \sigma) \leq m_k(1, \sigma), \quad (2.2)$$

and then check whether $A \leq B$. So, we need two different estimates:

a) a good lower bound for the end-point value $m_k(1, \sigma) = \sup \{ |f^{(k)}(1)| : f \in W_\infty(\sigma) \}$,

b) a good upper bound for $|f^{(k)}(x_0)|$, where $f$ is from $W_\infty(\sigma)$ and satisfies $f^{(k+1)}(x_0) = 0.$

Actually, if $x = x_0$ stays sufficiently far away from the end-points $x = \pm 1$, then a reasonable upper bound for $|f^{(k)}(x_0)|$ can be established irrespectively of whether $f^{(k+1)}(x_0)$ vanishes or not. Therefore, for the upper bounds for $|f^{(k)}(x)|$, we will consider two cases

b1) $m_k^*(x, \sigma) \leq A_{n,k}^*(\sigma), \quad \omega_k < |x_0| \leq 1, \quad$ b2) $m_k(x, \sigma) \leq A_{n,k}(\sigma), \quad |x| \leq \omega_k < 1,$

with an appropriately chosen value $\omega_k$.

We will distinguish between the cases $\sigma \leq \sigma_n$ and $\sigma > \sigma_n$.

1) The case $\sigma \leq \sigma_n$.

1a) Lower estimates for $m_k(1, \sigma)$. Clearly, $m_k(1, \sigma)$ is monotoniously increasing with $\sigma$, therefore, we have the trivial estimate

$$m_k(1, \sigma) \geq m_k(1, \sigma_0) = T_{n-1}^{(k)}(1).$$

However, this estimate is too rough when $k = O(n)$, so we will use a finer one.

**Proposition 2.2** We have

$$m_k(1, \sigma) \geq B_{n,k}(\sigma) := \left(1 - \frac{\sigma}{\sigma_n}\right) T_{n-1}^{(k)}(1) + \frac{\sigma}{\sigma_n} T_n^{(k)}(1), \quad 0 \leq \sigma \leq \sigma_n. \quad (2.3)$$
Proof. Let us show that \( m_k(x, \sigma) \) as a function of \( \sigma \) is concave. For any \( x \in [-1, 1] \), and for any \( \sigma' < \sigma'' \), let \( f_1 \) and \( f_2 \) be the functions such that
\[
m_k(x, \sigma^{(i)}) = f_i^{(k)}(x), \quad f_i \in W^n_{\infty}(\sigma^{(i)}), \quad i = 1, 2.
\]
It is clear that, for any \( \sigma \in [\sigma', \sigma''] \), with \( t \) such that \( \sigma = (1 - t)\sigma' + t\sigma'' \), the function \( f := (1 - t)f_1 + tf_2 \) belongs to \( W^n_{\infty}(\sigma) \), hence we have
\[
m_k(x, \sigma) \geq f^{(k)}(x) = (1 - t)f_1^{(k)}(x) + tf_2^{(k)}(x) = (1 - t)m_k(x, \sigma') + tm_k(x, \sigma'').
\]
In particular, with \( \sigma_0 := T^{(n)}_{n-1} = 0 \) and \( \sigma_n = T^{(n)}_n \), we have
\[
m_k(1, \sigma) \geq \left( 1 - \frac{\sigma}{\sigma_n} \right) m_k(1, \sigma_0) + \frac{\sigma}{\sigma_n} m_k(1, \sigma_n),
\]
But \( m_k(1, \sigma_0) = T^{(k)}_{n-1}(1) \) and \( m_k(1, \sigma_n) = T^{(k)}_n(1) \), hence the result. \( \square \)

1b) Upper estimate for \( m_k^*(x_0, \sigma) \). We will use a comparison lemma of the kind similar to the one that was used by Matorin [6] in (actually) proving that \( m_k(1, \sigma_n) \leq T^{(k)}_n(1) \).

Lemma 2.3 Let \( p \in \mathcal{P}_n[-1, 1] \) be a polynomial that satisfies the following conditions:
1) \( p^{(k+1)}(x_0) = 0 \), 2) \( p \) has an \( n \)-alternance on \([-1, 1]\), 3) \( \|p^{(n)}\| \geq \sigma \). (2.4)
Then, for any \( f \in W^n_{\infty}[-1, 1] \) and for any \( x_0 \in [-1, 1] \) such that
1') \( f^{(k+1)}(x_0) = 0 \), 2') \( \|f\| \leq 1 \), 3') \( \|f^{(n)}\| \leq \sigma \),
we have
\[
|f^{(k)}(x_0)| \leq |p^{(k)}(x_0)|.
\]
Proof. Assume the contrary, i.e., that \( f^{(k)}(x_0) = p^{(k)}(x_0)/\gamma \) with some \( \gamma \) such that \( |\gamma| < 1 \). Then the function \( g := \gamma f \) satisfies
2'') \( \|g\| < 1 \), 3'') \( \|g^{(n)}\| < \sigma \),
and moreover
1'') \( g^{(k)}(x_0) = p^{(k)}(x_0), \quad g^{(k+1)}(x_0) = p^{(k+1)}(x_0) = 0 \).
Consider the difference \( h = p - g \). By the \( n \)-alternation property (2) of \( p \), since \( \|g\| < 1 \), the function \( h \) has at least \( n - 1 \) distinct zeros on \([-1, 1]\), hence \( H := h^{(k-1)} \) has at least \( n - k \) distinct zeros strictly inside \((-1, 1)\), and by (1'), we also have \( H'(x_0) = H''(x_0) = 0 \). It follows that \( H' = h^{(k)} \) has at least \( n - k + 1 \) zeros on \([-1, 1]\] counting multiplicities, therefore
\[
h^{(n)} \text{ has at least one sign change on } [-1, 1].
\]
On the other hand, by (3) and (3'') we have \( |g^{(n)}(x)| < \sigma \) and \( |p^{(n)}(x)| \equiv \text{const} \geq \sigma \), hence
\[
|h^{(n)}(x)| = |p^{(n)}(x) - g^{(n)}(x)| > 0 \text{ for all } x \in [-1, 1], \quad \text{a contradiction.}
\]

Corollary 2.4 We have
\[
m_k^*(x_0, \sigma) \leq |p^{(k)}(x_0)| \quad (2.5)
\]
where \( p \) is any polynomial of degree \( n \) that satisfies conditions (1)-(3) in (2.4).
Let \( \{Z_n(\cdot, \theta)\} \) be the family of the Zolotarev polynomials parametrized with respect to the value of its highest derivative \( \theta := Z_n^{(n)}(\cdot, \theta) \) (see Sect. 3 for details). Given \( x_0 \), our choice for \( p \) in Proposition 2.5 is the dilated Zolotarev polynomial \( Z_n(\cdot, \theta x_0) \) such that \( Z_n^{(k+1)}(x_0, \theta x_0) = 0 \). An advantage of choosing such a \( p \) is that, for \( x_0 \in [\omega_k, 1] \), the value of \( p^{(k)}(x_0) \) can be further bounded in terms of the single Zolotarev polynomial \( Z_n(\cdot, \theta_k) \) such that
\[
Z_n^{(k+1)}(1, \theta_k) = 0.
\]

Namely, as we show in Sects. 3.4
\[
\sup_{x \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq \max\{1, \sigma \theta_k\}^{k/n} \max\{T_n^{(k)}(\omega_k), Z_n^{(k)}(1, \theta_k)\}
\]
In Sects. 5.6 we provide the estimates for the values appeared here on the right-hand side and, thus, arrive at the following statement.

**Proposition 2.5** We have
\[
\sup_{x \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq A_{n,k}^*(\sigma) := \begin{cases} T_n^{(k)}(1), & 0 \leq \frac{\sigma}{\sigma_n} \leq \eta_k; \\ \lambda_k T_n^{(k)}(1) \left(\frac{1}{\eta_k \sigma_n}\right)^{k/n}, & \eta_k \leq \frac{\sigma}{\sigma_n} \leq 1. \end{cases} \tag{2.6}
\]
where
\[
\lambda_k = \frac{1}{k+1} \frac{n-1}{n-1+k}, \quad \eta_k = \frac{n-(k+1)}{2(2n-(k+1))}.
\]

1b2) Upper estimate for \( m_k(x, \sigma) \). We use a technique based on the Lagrange interpolation. Let \( \ell_\Delta \in P_{n-1} \) be the polynomial of degree \( n-1 \) that interpolates \( f \in W_\infty^n(\sigma) \) on a mesh \( \Delta = (t_i)_{i=1}^n \).

From the identity \( f^{(k)}(x) = \ell_\Delta^{(k)}(x) + (f^{(k)}(x) - \ell_\Delta^{(k)}(x)) \) it follows that
\[
|f^{(k)}(x)| \leq \Lambda_k(x)\|f\| + \Omega_k(x)\|f^{(n)}\|
\]
where
\[
\Lambda_k(x) = \sup_{\|p\|_\Delta = 1} |p^{(k)}(x)|, \quad \Omega_k(x) = \sup_{\|f^{(n)}\| = 1} |f^{(k)}(x) - \ell_\Delta^{(k)}(x)|,
\]
whence
\[
\sup_{x \in [\omega_k]} m_k(x, \sigma) \leq \sup_{x \in [\omega_k]} \Lambda_k(x) + \sup_{x \in [\omega_k]} \Omega_k(x)\sigma.
\]
In Sect. 7 we prove that calculation of the suprema on the right-hand side is reduced to computing the largest local maxima of two specific polynomials and that leads to the following estimate.

**Proposition 2.6** We have
\[
\sup_{x \in [\omega_k]} m_k(x, \sigma) \leq A_{n,k}(\sigma) := \frac{3}{2k+1} T_n^{(k)}(1) + \frac{2}{2k+1} \frac{2k+1}{n+k} T_n^{(k)}(1) \frac{\sigma}{\sigma_n}. \tag{2.7}
\]

The latter estimate is not particularly good for \( k = 1 \) and \( k = 2 \), so for such \( k \) we also use another one
\[
\sup_{x \in [\omega_k]} m_k(x, \sigma_n) \leq \left(1 + \sin \frac{\pi}{2n}\right)^k \frac{1}{2k+1} T_n^{(k)}(1). \tag{2.8}
\]

1c) The constants in estimates (2.3), (2.6) and (2.7) are easy to compare (they are simple functions of \( t = \sigma/\sigma_n \)) and, in Sect. 8 we prove that if \( n \in \mathbb{N}, 1 \leq k \leq n-2 \) and \( 0 \leq \sigma \leq \sigma_n \), then
\[
\max(A_{n,k}(\sigma), A_{n,k}^*(\sigma)) \leq B_{n,k}(\sigma),
\]
and that implies
\[ m_k(1, \sigma) = \sup_{x \in [-1,1]} m_k(x, \sigma), \quad 0 \leq \sigma \leq \sigma_n, \quad 1 \leq k \leq n - 2. \]

2) The case \( \sigma > \sigma_n \).

For that case, it is more convenient to reformulate the original problem. Namely, instead of
considering functions from the class
\[ W^\infty_n(\sigma) := \{ f : f \in W^\infty_n[-1,1], \| f \|_{[-1,1]} \leq 1, \| f^{(n)} \|_{[-1,1]} \leq \sigma \}, \quad \sigma_n < \sigma < \infty, \]
i.e., functions on a fixed interval \( I_1 = [-1,1] \) with increasing norms \( \| f^{(n)} \|_{[-1,1]} \leq \sigma \), we will
consider functions from the class
\[ W^\infty_n(I_s) := \{ f : f \in W^\infty_n[-s,1], \| f \|_{[-s,1]} \leq 1, \| f^{(n)} \|_{[-s,1]} \leq \sigma_n \}, \quad 2 < |I_s| < \infty, \quad (2.9) \]
i.e., functions with a fixed norm \( \| f^{(n)} \|_{[-s,1]} = \sigma_n \) on the intervals \( I_s := [-s,1] \) of increasing length
\( |I_s| > |I_1| = 2 \). The pointwise Landau-Kolmogorov problem consists then of finding the value
\[ m_k(x, I_s) := \sup_{f \in W^\infty_n(I_s)} \| f^{(k)}(x) \|, \]
and Karlin’s conjecture states that \( m_k(x, I_s) \) is maximal at \( x = 1 \).

2a) Lower estimate for \( m_k(1, I_s) \). Denote by \( B_{n,k}^- \) the best constant in the Landau-Kolmogorov
inequality on the half-line for the normalized functions:
\[ B_{n,k}^- := \sup \{ \| f^{(k)}(1) \| : \| f \|_{[-\infty,1]} \leq 1, \| f^{(n)} \|_{[-\infty,1]} \leq \sigma_n \}. \quad (2.10) \]

Proposition 2.7 For all \( |I_s| > |I_1| = 2 \) we have
\[ m_k(1, I_s) \geq B_{n,k}^+, \quad (2.11) \]
Proof. Clearly, with \( n \) and \( \sigma_n \) fixed, the spaces defined in \([2.9]\) are embedded into each other,
namely \( W^\infty_n(I_s) \supset W^\infty_n(I_t) \) for \( s < t \), therefore for the suprema \( m_k(1, I_s) := \sup \| f^{(k)}(1) \| \) over those
spaces, we have the inequalities
\[ m_k(1, I_s) \geq m_k(1, I_t), \quad s < t. \]
Letting \( t = -\infty \), we obtain \([2.11]\). \( \square \)

2b). Upper estimates for \( m_k(x, I_s) \) and \( m_k^*(x_0, I_s) \). Similar arguments show that the upper
bounds for \( m_k(x, I_s) \) and \( m_k^*(x, I_s) \) are majorized by those of \( m_k(x, I_1) \) and \( m_k^*(x, I_1) \), respectively.
Namely, moving the interval \( I = [a,b] \) of length \( |I| = 2 \) inside any \( I_s \), we see that \( W^\infty_n(I_s) \subset W^\infty_n(I) \), hence
\[ \sup_{x \in [\omega_s,1]} m_k^*(x, I_s) \leq \sup_{x \in \omega_s,1} m_k^*(x, I_1) \]
\[ \sup_{x \in [0,\omega_k]} m_k(x, I_s) \leq \sup_{x \in [0,\omega_k]} m_k(x, I_1). \]
where \( s_0 \) is the middle of the interval \([-s,1]\). The right-hand sides are equivalent to the values
\( m_k^*(x, \sigma_n) \) and for those we have the upper estimates \([2.6]-[2.7]\).
Proposition 2.8 For all $|I_s| > |I_1| = 2$ we have

$$\sup_{x \in [\omega_1,1]} m_k^s(x, I_s) \leq A^*_n,k(\sigma_n),$$  \hspace{1cm} (2.12)

$$\sup_{x \in [s_0,\omega_k]} m_k(x, I_s) \leq A_{n,k}(\sigma_n).$$  \hspace{1cm} (2.13)

2c) Final step. In Sect. [11] we prove that the constants in (2.11)-(2.13) satisfy the inequality

$$\max (A_{n,k}(\sigma_n), A^*_n,k(\sigma_n)) \leq B^{+}_{n,k}, \quad k = 1, 2$$

and that proves that

$$m_k(1, I_s) = \sup_{x \in [-s,1]} m_k(x, I_s), \quad |I_s| \geq 2,$$

or, equivalently,

$$m_k(1, \sigma) = \sup_{x \in [-1,1]} m_k(x, \sigma), \quad \sigma_n < \sigma < \infty.$$

3 Zolotarev polynomials

Here, we remind some facts about Zolotarev polynomials taking some extracts from our survey [10, p.240-242]. Note that we use a slightly different parametrization for $Z_n$.

Definition 3.1 A polynomial $Z_n \in \mathcal{P}_n$ is called Zolotarev polynomial if it has at least $n$ equioscillations on $[-1,1]$, i.e. if there exist $n$ points

$$-1 \leq \tau_1 < \tau_2 < \cdots < \tau_{n-1} < \tau_n \leq 1$$

such that

$$(-1)^{n-i} Z_n(\tau_i) = \|Z_n\| = 1.$$

There are many Zolotarev polynomials, for example the Chebyshev polynomials $T_n$ and $T_{n-1}$ of degree $n$ and $n - 1$, with $n + 1$ and $n$ equioscillation points, respectively. One needs one parameter more to get uniqueness. We will use parametrization through the value of the $n$-th derivative of $Z_n$:

$$\|Z_n^{(n)}\| = \theta \iff Z_n(x, \theta) := \frac{\theta}{n!} x^n + \sum_{i=0}^{n-1} a_i(\theta)x^i.$$

By Chebyshev’s result, $\|p^{(n)}\| \leq \|T_n^{(n)}\| \|p\|$, so the range of the parameter is

$$-\sigma_n \leq \theta \leq \sigma_n, \quad \sigma_n = \|T_n^{(n)}\| = 2^{n-1} n!.$$

As $\theta$ traverses the interval $[-\sigma_n, \sigma_n]$, Zolotarev polynomials go through the following transformations:

$$-T_n(x) \to -T_n(ax + b) \to Z_n(x, \theta) \to T_{n-1}(x) \to Z_n(x, \theta) \to T_n(cx + d) \to T_n(x).$$

Zolotarev polynomials subdivide into 3 groups depending on the structure of the set $\mathcal{A} := (\tau_i)$ of their alternation points.

1) $\mathcal{A}$ contains $n + 1$ points: then $Z_n$ is the Chebyshev polynomial $T_n$.

2) $\mathcal{A}$ contains $n$ points but only one of the endpoints: then $Z_n$ is a stretched Chebyshev polynomial $T_n(ax + b)$, $|a| < 1$.

3) $\mathcal{A}$ contains $n$ points including both endpoints: then $Z_n$ is called a proper Zolotarev polynomial and it is either of degree $n$, or the Chebyshev polynomial $T_{n-1}$ of degree $n - 1$. 

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For a proper Zolotarev polynomial \( Z_n \), besides the interior alternation points \( (\tau_i)_{i=2}^{n-1} \), there is a point \( \beta = \beta(\theta) \) outside \([-1, 1]\) where its first derivative vanishes.

V. Markov proved that zeros of \( Z_n'(\cdot, \theta) \) are monotonically increasing functions of \( \theta \in [-\sigma_n, \sigma_n] \), with \( \beta \) going through the infinity as \( \theta \) passes the zero. It follows that, for any \( \theta_1, \theta_2 \), zeros of \( Z_n'(\cdot, \theta_1) \) and \( Z_n'(\cdot, \theta_2) \) interlace with each other, hence by the Markov interlacing property the same is true for their derivatives of any order. In particular, the following lemma is true.

**Lemma 3.2** Let \((\alpha_i)_{i=1}^{M-1}\) be the zeros of \( T_{n-1}^{(m)} \) in increasing order, and, for any given \( \theta \), let \((\tau_i)_{i=1}^{M}\) be the zeros of \( Z_n^{(m)}(\cdot, \theta) \). Then, \((\alpha_i) \) and \((\tau_i) \) interlace, i.e.,

\[\tau_1 < \alpha_1 < \tau_2 < \alpha_2 < \tau_3 < \cdots < \alpha_{M-1} < \tau_M.\]

Another consequence of the interlacing property is the following observation.

**Lemma 3.3** Let \( \omega_k \) be the rightmost zero of \( T_n^{(k+1)} \), and let \( Z_n(\cdot, \theta_k) \) be the Zolotarev polynomials whose \((k+1)\)st derivative vanishes at \( x = 1 \), i.e.,

\[T_n^{(k+1)}(\omega_k) = 0, \quad Z_n^{(k+1)}(1, \theta_k) = 0.\]

Further, for a given \( x_0 \in (\omega_k, 1) \), let \( Z_n(\cdot, \theta_{x_0}) \) be the Zolotarev polynomial such that

\[Z_n^{(k+1)}(x_0, \theta_{x_0}) = 0, \quad x_0 \in [\omega_k, 1].\]

Then

\[|\theta_k| < |\theta_{x_0}| < \sigma_n.\]

**Proof.** According to our parametrization, we have \(-T_n(x) = Z_n(x, -\sigma_n)\), and as \( \theta \) increases from \(-\sigma_n \) to \(-0 \), the rightmost zero of \( Z_n^{(k+1)}(\cdot, \theta) \) increases from \( \omega_k \) to \(+\infty\), passing through the value \( 1 \) for some \( \theta := \theta_k \). Therefore

\[\omega_k < x_0 < 1 \iff -\sigma_n < \theta_{x_0} < \theta_k.\]

2) Here we give some upper estimates for the values \( T_n^{(k)}(\omega_k) \) relative to the value \( T_n^{(k)}(1) \). The estimates for \( T_n^{(k)}(\omega_k) \) has been given on several occasions, we summarize what we need in the following statement.

**Lemma 3.4** Let \( \omega_k := \omega_{n,k} \) be the rightmost zero of \( T_n^{(k+1)} \). Then

1) \[|T_n^{(k)}(\omega_k)| \leq \frac{1}{2k+1} T_n^{(k)}(1), \quad n \in \mathbb{N}, \quad 1 \leq k \leq n-1;\]

2) \[|T_n'(\omega_1)| \leq \frac{1}{4} T_n'(1), \quad n \geq 5;\]

3) \[|T_n''(\omega_2)| \leq \frac{8}{59} T_n''(1), \quad n \geq 10.\]

**Proof.** The first inequality was proved by Eriksson \([3]\) who actually derived a stronger estimate:

\[|T_n^{(k)}(\omega_k)| \leq \frac{F_k(\omega_k)}{2k+1} T_n^{(k)}(1),\]

where

\[F_k(x) := \frac{2(1+x)^2}{(2k+5)x+2} \leq 1, \quad x \in [0, 1].\]

The second inequality is due to Erdős–Szegő \([2], p.464\). To derive the third one, we note that the function \( F_2(\cdot) \) has the single minimum at \( x_s = \frac{2k+1}{2k+5} = \frac{5}{9} \), therefore, if \( x_s < \omega_2 < 1 \), then

\[F_2(\omega_2) < F_2(1) = \frac{8}{11}.\]

But \( \omega_2 \) is the largest zero of the third derivative of \( T_n \), therefore it is greater than the third largest zero of \( T_n' \), i.e., \( \omega_2 > \cos \frac{3\pi}{n} \), so \((3.2)\) is valid if \( \cos \frac{3\pi}{n} \geq \frac{5}{9} \), and the latter holds for \( n \geq 10. \]
Corollary 3.5  We have
\[
\max_{x \in [0, \omega_{k-1}]} |T_n^{(k)}(x)| \leq \frac{1}{2k+1} T_n^{(k)}(1)  
\]  \hspace{1cm} (3.3)

Proof. The values of local maxima of $|T_n^{(k)}(x)|$ increase with $|x|$, and since $\omega_k = \max_i |\xi_i|$, we have
\[
\max_{x \in [0, \omega_k]} |T_n^{(k)}(x)| \leq |T_n^{(k)}(\omega_k)| \leq \frac{1}{2k+1} T_n^{(k)}(1)
\]
On the interval $[\omega_k, \omega_{k-1}]$ the value $|T_n^{(k)}(x)|$ decreases monotonically from the rightmost maximum $T_n^{(k)}(\omega_k)$ to the rightmost zero $T_n^{(k)}(\omega_{k-1}) = 0$, hence the inequality for such $x$. \hfill \Box

4 A generalization of Erdős–Szegő result

By $Q_n$ we denote the unit ball in the space $P_n$, i.e., the set of polynomials $p \in P_n$ such that $\|p\| \leq 1$. According to the well-known Markov inequality
\[
\sup_{p \in Q_n} |p'(x)| \leq n^2, \quad x \in [-1, 1],
\]
and equality is attained at $x = 1$ for $p = T_n$.

In 1913, Schur [9] considered the problem of finding the maximum of $|p'(x_0)|$ under additional assumption that $p''(x_0) = 0$. Let $Q_n^k(x_0)$ be the unit ball of polynomials such that $p^{(k+1)}(x_0) = 0$. Schur proved that
\[
\sup_{p \in Q_n^k(x_0)} |p'(x_0)| < 1 \frac{1}{2} n^2. \quad (4.1)
\]
Moreover, he showed that if $\lambda_n$ is the least constant in front of $n^2$, then, for $\lambda_\infty := \limsup_{n \to \infty} \lambda_n$, we have
\[
0.217 \cdots \leq \lambda_\infty \leq 0.465 \cdots.
\]

In 1942, Erdős and Szegő [2] refined Shur’s result by showing that the limit $\lambda_\infty = \lim_{n \to \infty} \lambda_n$ exists and it is equal to
\[
\lambda_\infty = \kappa^{-2}(1 - E/K)^2 = 0.3124 \cdots \quad (4.2)
\]
where $E, K$ are the complete elliptic integrals associated with the modulus $\kappa$. (They did not improve the uniform bound [11] though.)

They also showed that, for any $x_0 \in [-1, 1]$, the supremum of $|p'(x_0)|$ is attained when $p$ is a Zolotarev polynomial $Z_n(\cdot, \theta)$, and that the maximum over $x_0$ is attained at $x_0 = 1$ for $n \geq 4$, and at $x_0 = 0$ for $n = 3$.

In this section, we generalize these results to the derivatives of order $k \geq 2$.

Denote by
\[
\mu_k(x) := \max_{p \in Q_n^k(x)} |p^{(k)}(x)|, \quad x \in [-1, 1],
\]
the best constant in the pointwise Markov inequality, and by
\[
\mu_k^*(x_0) := \max_{p \in Q_n^k(x_0)} |p^{(k)}(x_0)| \quad x_0 \in [-1, 1],
\]
the best constant in the pointwise Schur-type inequality. It is clear that
\[
\mu_k^*(x_0) \leq \mu_k(x), \quad x_0 \in [-1, 1],
\]
and that equality occurs only if $\mu_k^*(x_0) = 0$, i.e. if $x_0$ is a point of local extremum (maximum or minimum) of the function $\mu_k(\cdot)$ inside $(-1, 1)$.

The next two lemmas are straightforward extensions of the arguments given in [2] pp.461-462, from $k = 1$ to $k \geq 2$. 

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**Lemma 4.1** For any \( \theta \), if \( Z_n^{(k+1)}(x_0, \theta) = 0 \), then

\[
\mu_k^*(x_0) = Z_n^{(k)}(x_0, \theta). \tag{4.3}
\]

Conversely, for any \( x_0 \in [-1, 1] \), with some \( \theta = \theta_{x_0} \) there is a polynomial \( Z_n(\cdot, \theta) \) such that (4.3) is true.

**Lemma 4.2** Let \( x_0 \) be a point such that

\[
\mu_k^*(x_0) < \mu_k(x_0) \quad \text{and} \quad x_0 \neq \pm 1.
\]

Then, for small \( \delta > 0 \), there is a point \( x_1 \in [x_0 - \delta, x_0 + \delta] \), such that

\[
\mu_k^*(x_0) < \mu_k^*(x_1).
\]

**Proof.** Let \( \mu_k^*(x_0) = Z_n^{(k)}(x_0) \), where \( Z_n^{(k+1)}(x_0) = 0 \) and let \( p \in \mathcal{Q}_n \) be the polynomial such that

\[
p^{(k)}(x_0) > Z_n^{(k)}(x_0) > 0.
\]

Then the polynomial \( q = (1 - \epsilon)Z_n + cp \) satisfies

\[
|q| < 1, \quad q^{(k)}(x_0) > Z_n^{(k)}(x_0) = \mu_k^*(x_0),
\]

and, for small \( \epsilon \), its \( k \)-th derivative has a local maximum in the neighbourhood of \( x_0 \) (because \( Z_n^{(k)} \) has). Let \( x_1 \) be the point of that maximum, i.e., \( q^{(k+1)}(x_1) = 0 \). Then \( q^{(k)}(x_1) > q^{(k)}(x_0) \), and respectively

\[
\mu_k^*(x_0) < q^{(k)}(x_0) < q^{(k)}(x_1) \leq \mu_k^*(x_1),
\]

the latter inequality by definition of \( \mu_k^*(\cdot) \).

**Corollary 4.3** Let \( \eta \) be a point of local maximum of the function \( \mu_k^*(\cdot) \). Then

\[
\mu_k^*(\eta) = \mu_k(\eta).
\]

**Theorem 4.4** Let \( Z_n(x, \theta_k) \) be the Zolotarev polynomial such that

\[
Z_n^{(k+1)}(1, \theta_k) = 0.
\]

Then

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}.
\]

**Proof.** Let \( \eta_k \) be the points of local maxima of \( \mu_k^*(\cdot) \) inside the interval \((-1, 1)\). Then

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{\mu_k^*(\eta_k), \mu_k^*(1)\}
\]

The corollary shows that, inside \((-1, 1)\), the local maxima of \( \mu_k^*(\cdot) \) coincide with the extrema (maxima or minima) of \( \mu_k(\cdot) \). On the other hand, V. Markov proved that the local maxima of \( \mu_k(\cdot) \) coincide with those of \( |T_n^{(k)}| \). Hence

\[
\max_{x_0 \in [-1,1]} \mu_k^*(x_0) = \max \{|T_n^{(k)}(\xi_i)|, \mu_k^*(1)\}, \quad \text{where} \quad T_n^{(k+1)}(\xi_i) = 0.
\]

Further, it is known that the local maxima of \( |T_n^{(k)}| \) are increasing as \( |\xi_i| \) increases, i.e.,

\[
\max_i |T_n^{(k)}(\xi_i)| = |T_n^{(k)}(\omega_k)|,
\]

where \( \omega_k \) is the rightmost zero of \( T_n^{(k+1)} \). Finally, by Lemma 4.3

\[
\mu_k^*(1) = |Z_n^{(k)}(1, \theta_k)|,
\]

and that completes the proof.
Theorem 4.5 Let $Z_n(x, \theta_k)$ be the Zolotarev polynomial such that
\[ Z_n^{(k+1)}(1, \theta_k) = 0. \]
Then
\[ \max_{x_0 \in [\omega_k, 1]} m_k^*(x_0, \sigma) \leq \max\{1, S \}^{k/n} \max \{|T_n^{(k)}(\omega_k)|, |Z_n^{(k)}(1, \theta_k)|\}. \]

Proof. According to Corollary 2.3
\[ m_k^*(x_0, \sigma) \leq |p^{(k)}(x_0)|, \]
where $p$ is any polynomial of degree $n$ such that
1) $p^{(k+1)}(x_0) = 0$, 2) $p$ has an $n$-alternation in $[-1, 1]$, 3) $\|p^{(n)}\| \geq \sigma$.

We take $p$ as a dilated Zolotarev polynomial $Z_n(\cdot, \theta_{x_0})$ such that $Z_n^{(k+1)}(x_0, \theta_{x_0}) = 0$. The latter satisfies conditions (1)-(2), and its highest derivative has the value $\theta_{x_0}$. So, if $\theta_{x_0} \geq \sigma$, then condition (3) is fulfilled with $p = Z_n(\cdot, \theta_{x_0})$, but if $\theta_{x_0} < \sigma$, then we have to scale $Z_n$ to ensure (3). So we set
\[ p(x) := Z_n(x_0 + \frac{1}{n}(x - x_0), \theta_{x_0}), \quad \gamma_0 := \max\{1, \frac{\sigma}{\theta_{x_0}}\}, \]
whence
\[ m_k^*(x_0, \sigma) \leq p^{(k)}(x_0) = \max\{1, (\frac{\sigma}{\theta_{x_0}})^{k/n}\} Z_n^{(k)}(x_0, \theta_{x_0}). \]

Finally,
\[ \omega_k \leq x_0 \leq 1 \implies \begin{cases} 1) |Z_n^{(k)}(x_0, \theta_{x_0})| \leq \max\{T_n^{(k)}(\omega_k), Z_n^{(k)}(1, \theta_k)\}, \\ 2) |\theta_k| \leq |\theta_{x_0}| \leq \sigma_n, \end{cases} \]
where the first inequality us due to Theorem 4.3 and the second one is due to Lemma 5.3. \qed

5 Upper estimates for $Z_n^{(k)}(1, \theta_k)$ and generalization of Schur inequality

Recall that by Markov’s inequality
\[ \sup_{\|p^{(k)}\| \leq 1} |p^{(k)}(x)| \leq |T_n^{(k)}(1)|, \quad x \in [-1, 1], \]
so we will give some upper estimates for the constant $\lambda_k$ such that
\[ Z_n^{(k)}(1, \theta_k) \leq \lambda_k T_n^{(k)}(1) \]
We will get those estimates using the following lemma.

Lemma 5.1 Let $p \in \mathcal{P}_n$ be any polynomial that satisfies the following conditions:
1) $p^{(k+1)}(1) = 0$, 2) $p$ has an $n$-alternation on $[-1, 1]$. \hspace{1cm} (5.1)

If $Z_n^{(k+1)}(1, \theta_k) = 0$, then
\[ |Z_n^{(k)}(1, \theta_k)| \leq |p^{(k)}(1)|. \hspace{1cm} (5.2) \]

Proof. The proof is parallel to the proof of Lemma 2.3 since $Z_n$ satisfies $\|Z_n\| \leq 1$. Assuming the contrary to (5.2), we derive that the $n$-th derivative of $h := p - \gamma Z_n$ should change its sign which is impossible as $h$ is a polynomial of degree $n$. \qed

2a) We will construct several $p$ that satisfy (5.1) using alternation properties of $T_n$ and $T_{n-1}$. We start with the simplest one.
Lemma 5.2 We have
\[ |Z_n^{(k)}(1, \theta_k)| \leq \frac{1}{k+1} T_n^{(k)}(1). \] (5.3)

**Proof.** Take
\[ p(x) = T_n(x) - cq(x), \quad q(x) := (x - 1)T'_n(x), \quad p^{(k+1)}(1) := 0, \]
so that \( p \) has an \( n \)-alternance on \([ - \cos \frac{\pi}{n}, 1] \) for any \( c \), and where the last equality defines particular \( c := \frac{T_n^{(k+1)}(1)}{q^{(k+1)}(1) T_n^{(k)}(1)} \). Then
\[ p^{(k)}(1) = T_n^{(k)}(1) - cq^{(k)}(1) = \left( 1 - \frac{T_n^{(k+1)}(1)}{q^{(k+1)}(1) T_n^{(k)}(1)} \right) T_n^{(k)}(1), \]
and since \( q^{(m)}(1) = mT_n^{(m)}(1) \), it follows that
\[ p^{(k)}(1) = \left( 1 - \frac{k}{k+1} \right) T_n^{(k)}(1) = \frac{1}{k+1} T_n^{(k)}(1). \]
\[ \square \]

2b) The next lemma improves the previous estimate for \( k = \mathcal{O}(n) \).

**Lemma 5.3** We have
\[ |Z_n^{(k)}(1, \theta_k)| \leq T_n^{(k)}(1), \] (5.4)
\[ |Z_n^{(k)}(1, \theta_k)| \leq \frac{n-1}{k+1 n-1+k} T_n^{(k)}(1). \] (5.5)

**Proof.** Take
\[ p(x) = T_{n-1}(x) - cq(x), \quad q(x) := (x^2 - 1)T'_{n-1}(x), \quad p^{(k+1)}(1) := 0. \]
Then
\[ p^{(k)}(1) = T_{n-1}^{(k)}(1) - cq^{(k)}(1) = \left( 1 - \frac{T_{n-1}^{(k+1)}(1)}{q^{(k+1)}(1) T_{n-1}^{(k)}(1)} \right) T_{n-1}^{(k)}(1) =: \hat{\lambda}_{n,k} T_{n-1}^{(k)}(1). \]
Since \( q'(x) = (x^2 - 1)T''_{n-1}(x) + 2xT'_{n-1}(x) = xT'_{n-1}(x) + (n-1)^2 T_{n-1}(x) \), we have
\[ q^{(m)}(1) = T_{n-1}^{(m)}(1) + ((n-1)^2 + (m-1))T_{n-1}^{(m-1)}(1), \]
and using
\[ T_n^{(k+1)}(1) = \frac{n^2 - k^2}{2k+1} T_n^{(k)}(1), \quad T_n^{(k-1)}(1) = \frac{2k-1}{n^2 - (k-1)^2} T_n^{(k)}(1), \]
we obtain, after some simplifications,
\[ \hat{\lambda}_{n,k} = 1 - \frac{k}{k+1} \frac{(n-1)^2 - k^2}{2(n-1)^2 + (k+1)(n-1)^2 - (k-1)^2} \]
\[ = \frac{1}{k+1} \frac{k}{k+1} \frac{4k(n-1)^2 + (k-1)}{(n-1)^2 - (k-1)^2}(2(n-1)^2 + (k+1)) \]
\[ \leq \frac{1}{k+1} \frac{k}{k+1} \frac{1}{n-k} \leq 1 \]
and that proves the first inequality \([5.4]\). Using

\[
T_{n-1}^{(k)}(1) = \gamma T_n^{(k)}(1), \quad \gamma = \frac{n-1}{n} \frac{n-k}{n-1+k},
\]

we obtain

\[
\lambda_{n,k} = \hat{\lambda}_{n,k} \gamma \leq \frac{1}{k+1} \frac{n}{n-k} \gamma = \frac{1}{k+1} \frac{n-1}{n-1+k}
\]

and that proves \([5.5]\). \(\square\)

2c) In the next lemma, we get further improvements for \(k = 1\) and \(k = 2\).

**Lemma 5.4** We have

\[
Z_n'(1, \theta_1) \leq \frac{1}{3} T_n'(1), \quad Z_n''(1, \theta_2) \leq \frac{3}{\pi^2} \frac{n^2 - 6}{15 - \pi^2} < 0.23 T_n''(1). \quad (5.6)
\]

**Proof.** Set \(\xi := \cos \frac{\pi}{n}\), and let

\[
r(x) = T_n(x) - cq(x), \quad q(x) := (x + 1)T_n'(x), \quad r^{(k+1)}(\xi) := 0.
\]

The polynomial \(r\) has an \(n\)-alternance on \([-1, \xi]\), so that, after finding \(r^{(k)}(\xi)\) we will transform it to the polynomial \(p(x) := r \left( -1 + (x + 1) \frac{1+\xi}{2} \right)\), which has an \(n\)-alternance on \([-1, 1]\) and satisfies

\[
p^{(k)}(1) = \left( \frac{1+\xi}{2} \right)^k r^{(k)}(\xi).
\]

Let us find \(r^{(k)}(\xi)\). We have

\[
r^{(k)}(\xi) = T_n^{(k)}(\xi) - c q^{(k)}(\xi) = T_n^{(k)}(\xi) - \frac{q^{(k)}(\xi)}{q^{(k+1)}(\xi)} T_n^{(k+1)}(\xi),
\]

where

\[
q^{(m)}(\xi) = (1 + \xi) T_n^{(m+1)}(\xi) + m T_n^{(m)}(\xi),
\]

so that setting \(a_k := T_n^{(k)}(\xi)\), we obtain

\[
r^{(k)}(\xi) = a_k - \frac{(1 + \xi) a_{k+1} + k a_k}{(1 + \xi) a_{k+2} + (k + 1) a_{k+1}} a_{k+1}.
\]

Further, we have

\[
a_0 = T_n(\xi) = -1, \quad a_1 = T_n'(\xi) = 0,
\]

and, for \(k \geq 2\), the values \(a_k\) can be computed from the recurrence relation

\[
(\xi^2 - 1) a_{k+2} + (2k + 1) \xi a_{k+1} = (n^2 - k^2) a_k.
\]

In particular, we find

\[
a_2 = \frac{n^2}{1 - \xi^2}, \quad a_3 = \frac{3\xi}{1 - \xi^2} a_2, \quad a_4 = \frac{5\xi}{1 - \xi^2} a_3 - \frac{n^2 - 2^2}{1 - \xi^2} a_2.
\]

For \(k = 1\), this gives

\[
r'(\xi) = -\frac{(1 + \xi) a_2}{(1 + \xi) a_3 + 2a_2} a_2 = -\frac{n^2}{2 + \xi} \quad \Rightarrow \quad |p'(\xi)| = \frac{1 + \xi}{2(2 + \xi)} T_n'(1) < \frac{1}{3} T_n'(1).
\]
For $k = 2$, we obtain
\[
r''(\xi) = a_2 - \frac{(1 + \xi)a_3 + 2a_2}{(1 + \xi)a_4 + 3a_3}\ a_3 \Rightarrow p''(1) = c(n, \xi)T''_n(1),
\]
where
\[
c(n, \xi) = \left(\frac{1 + \xi}{2}\right)^2 \left(\frac{6\xi + 3\xi^2}{2\xi^2 + 9\xi + 4} - n^2(1 - \xi^2) - 1\right) \frac{1}{1 - \xi^2} \frac{n}{n^2 - 1}.
\]
One can show that $c(n, \xi) = c(n, \cos \frac{\pi}{n})$ is increasing with $n$ to its limit value given in (5.6).
\[\square\]

**Remark 5.5** We checked two other possibilities to construct $p$.
1) The option
\[
p(x) = T_n(x) - cq(x), \quad q(x) := (x + 1)T'_n(x), \quad p^{(k+1)}(1) := 0,
\]
results in
\[
|p^{(k)}(1)| = \frac{1}{2k + 1} \frac{4n^2 - 1}{(2n^2 + (k + 1))} T^{(k)}_n(1),
\]
which is slightly worse than (5.3).
2) The option
\[
p(x) = \frac{x - \gamma}{1 - \gamma} T_{n-1}(x), \quad p^{(k+1)}(1) := 0,
\]
is very poor for small $k$, and for large $k = O(n)$ it is slightly worse than (5.5).

### 6 Lower bound for $Z^{(n)}(\cdot, \theta_k)$

**Lemma 6.1** Let $Z_n(x, \theta_k)$ be a Zolotarev polynomial such that
\[
Z^{(k+1)}_n(-1, \theta_k) = 0.
\]
Then
\[
\theta_k := \|Z^{(n)}_n\| \geq \eta_{n,k} \sigma_n, \quad \eta_{n,k} := \frac{n - (k + 1)}{2(n - (k + 1))}.
\]

**Proof.** Set $m = k + 1$ and $M = n - m$, and denote by $(\tau_i)_{i=1}^M$ the zeros of $Z^{(m)}_n$ in increasing order:
\[-1 = \tau_1 < \tau_1 < \cdots < \tau_M < 1.\]
Then
\[
Z^{(m)}_n(x) = A(x + 1)(x - \tau_2)\cdots(x - \tau_M),
\]
where
\[
A = \frac{Z^{(m)}_n(1)}{2(1 - \tau_2)\cdots(1 - \tau_M)} = \frac{1}{2} \frac{A_1}{A_2},
\]
and respectively
\[
\|Z^{(n)}_n\| = AM! = \frac{M!}{2} \frac{A_1}{A_2}. \tag{6.1}
\]
Let us find lower bounds for the constants $A_1$ and $1/A_2$.
1) Let $(\alpha_i)_{i=1}^{M-1}$ be the zeros of $T^{(m)}_{n-1}$ in increasing order. They interlace with zeros of $Z^{(m)}_n$, i.e.
\[-1 = \tau_1 < \alpha_1 < \tau_2 < \alpha_2 < \tau_3 < \cdots < \alpha_{M-1} < \tau_M < 1,\]

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therefore
\[ \frac{1}{A_2} := \frac{1}{(1 - \tau_2) \cdots (1 - \tau_M)} > \frac{1}{(1 - \alpha_1) \cdots (1 - \alpha_{M-1})}. \]

On the other hand,
\[ T^{(m)}_{n-1}(x) = \frac{\|T^{(n-1)}_{n-1}\|}{(M-1)!} (x - \alpha_1) \cdots (x - \alpha_{M-1}) \Rightarrow T^{(m)}_{n-1}(1) = \frac{\|T^{(n-1)}_{n-1}\|}{(M-1)!} (1 - \alpha_1) \cdots (1 - \alpha_{M-1}), \]

and respectively
\[ \frac{1}{A_2} > \frac{1}{(1 - \alpha_1) \cdots (1 - \alpha_{M-1})} = \frac{1}{(M-1)!} \frac{\|T^{(n-1)}_{n-1}\|}{T^{(m)}_{n-1}(1)}. \] (6.2)

2) The lower bound for \( A_1 \) is provided by
\[ A_1 := Z_{\alpha}^{(m)}(1, \theta_k) \geq T^{(m)}_{n-1}(1) \frac{\sigma_n - \theta_k}{\sigma_n} + T^{(m)}_{n-1}(1) \frac{\theta_k}{\sigma_n} = T^{(m)}_{n-1}(1) \frac{\theta_k}{\sigma_n} \left( \frac{\sigma_n - \theta_k}{T^{(m)}_{n-1}(1)} \right). \] (6.3)

3) Combining estimates (6.1) and (6.3), we obtain
\[ \theta_k \geq \frac{n - m}{2} \left( \frac{\|T^{(n-1)}_{n-1}\|}{\sigma_n} \right) \left( \frac{\sigma_n - \theta_k}{T^{(m)}_{n-1}(1)} \right). \]

From the relations
\[ \frac{\|T^{(n-1)}_{n-1}\|}{\sigma_n} := \frac{\|T^{(n-1)}_{n-1}\|}{\|T^{(n)}_{n}\|} = \frac{1}{2n}, \quad \frac{T^{(m)}_{n-1}(1)}{T^{(m)}_{n-1}(1)} = n - 1 \frac{n - 1 + m}{n - m} > n + m, \]

it follows that
\[ \theta_k > \frac{n - m}{4n} \left( \sigma_n - \theta_k + \frac{n + m}{n - m} \theta_k \right) = \frac{n - m}{4n} \left( \sigma_n + \frac{2m}{n - m} \theta_k \right). \]

So, \((1 - \frac{m}{2n})\theta_k \geq \frac{n - m}{4n} \sigma_n\), and finally
\[ \theta_k > \frac{n - m}{2(2n - m)} \sigma_n, \quad m = k + 1. \]

**Proposition 6.2** We have
\[ \sup_{x \in [\omega_k, 1]} m^*_k(x_0, \sigma) \leq A_{n,k}^*(\sigma) := \begin{cases} T^{(k)}_{n-1}(1), & 0 \leq \frac{\sigma}{\sigma_n} \leq \eta_k; \\ \lambda_k T^{(k)}_{n}(1) \left( \frac{1}{\eta_k} \frac{\sigma}{\sigma_n} \right)^{k/n}, & \eta_k \leq \frac{\sigma}{\sigma_n} \leq 1. \end{cases} \] (6.4)

where
\[ \lambda_k = \frac{1}{k + 1} \frac{n - 1}{n - 1 + k}, \quad \eta_k = \frac{n - (k + 1)}{2n - (k + 1)}. \]

**7 Upper estimates for** \( m_k(x) \) **for** \( x \in [0, \omega_k] \)

**Lemma 7.1** We have
\[ m_k(x) \leq \frac{3}{2k + 1} T^{(k)}_{n-1}(1) + \frac{2}{2k + 1} \frac{2(k + 1)}{n + k} T^{(k)}_{n}(1) \frac{\sigma}{\sigma_n}. \]
\textbf{Proof.} For \( f \in W_{\infty}^n(\sigma) \), let \( l \in \mathcal{P}_n \) be the Lagrange polynomial of degree \( n \) that interpolates \( f \) at the points of local extrema of \( T_{n-1}' \) on the interval \([-1, 1] \), i.e.
\[ l(x) = f(x), \quad (x^2 - 1)T_{n-1}'(x) = 0. \]
Then
\[ f^{(k)}(x) = l^{(k)}(x) + (f^{(k)}(x) - l^{(k)}(x)) \leq D_k(x)\|f\| + \Omega_k(x)\|f^{(n)}\|, \]
where
\[ D_k(x) := \sup_{x \in [0, 1]} \left| p_{n-1}^{(k)}(x) \right|, \quad \Omega_k(x) := \sup_{x \in [0, 1]} \left| f^{(k)}(x) - l^{(k)}(x) \right|. \]

1) For the first constant, we have the estimate
\[ D_k(x) \leq \max\{U(x), V(x)\}, \]
where \( U(x) := |T_{n-1}^{(k)}(x)| \) and
\[
V(x) := \frac{1}{k} (x^2 - 1) T_{n-1}^{(k+1)}(x) + x T_{n-1}^{(k)}(x) \\
\leq \frac{k - 1}{k} |T_{n-1}^{(k)}(1)| + \frac{(n - 1)^2 - (k - 1)^2}{k} |T_{n-1}^{(k-1)}(1)|.
\]
We have
\[
U(x) \leq \frac{1}{2k + 1} T_{n-1}^{(k+1)}(1), \\
V(x) \leq \frac{k - 1}{k} \frac{1}{2k + 1} T_{n-1}^{(k)}(1) + \frac{(n - 1)^2 - (k - 1)^2}{k} \frac{1}{2k - 1} T_{n-1}^{(k-1)}(1) \\
= \left( \frac{k - 1}{k} \frac{1}{2k + 1} + \frac{1}{k} \right) T_{n-1}^{(k)}(1) \\
= \frac{3}{2k + 1} T_{n-1}^{(k)}(1).
\]

2) For the second constant, we have
\[ \Omega_k(x) \leq \max \left| \frac{1}{n!} \omega^{(k)}(x) \right|. \]
where \( \omega(x) = c(x^2 - 1)T_{n-1}'(x) \), with its leading coefficient equal to one, i.e., \( c = \frac{1}{2n-2} \frac{1}{n-1} \). Set
\[ q(x) := (x^2 - 1)T_{n-1}'(x). \]
Then
\[ \Omega_k(x) \leq \frac{1}{2n-2} \frac{1}{n!} \frac{1}{n-1} \max |q^{(k)}(x)| = \frac{2}{\sigma_n} \frac{1}{n!} \frac{1}{n-1} \max |q^{(k)}(x)|. \]
Since \( q'(x) = (n - 1)^2 T_{n-1}(x) + x T_{n-1}'(x) \), we have
\[ q^{(k)}(x) = (n - 1)^2 + (k - 1)T_{n-1}^{(k-1)}(x) + x T_{n-1}^{(k)}(x) \leq \frac{(n - 1)^2 + (k - 1)}{2k - 1} T_{n-1}^{(k-1)}(1) + \frac{1}{2k + 1} T_{n-1}^{(k)}(1) \]
\[ = \left( \frac{(n - 1)^2 + (k - 1)}{(n - 1)^2 - (k - 1)^2} + \frac{1}{2k + 1} \right) T_{n-1}^{(k)}(1) = \frac{c_{n,k}}{2k + 1} T_{n}^{(k)}(1), \]
where
\[ c_{n,k} = \frac{2(k + 1)(n - 1)^2 + (k + 2)(k - 1) n^{-1}}{n (n + k) (n - 1 + (k - 1))} \leq 2(k + 1) \frac{n - 1}{n - 1 + k} \frac{n - 1}{n} \leq 2(k + 1) \frac{n - 1}{n + k}. \]
Thus
\[
\Omega_k(x) \leq \frac{2}{2k+1} \frac{2(k+1)}{n+k} \frac{1}{\sigma_n} T_n^{(k)}(1).
\]

\[
\square
\]

**Corollary 7.2** We have
\[
m_k(x, \sigma_n) \leq \frac{3}{2k+1} T_n^{(k)}(1), \quad k \geq 2.
\]

**Proof.** We have
\[
m_k(x, \sigma_n) \leq \alpha_{n,k} T_n^{(k)}(1),
\]
where
\[
\alpha_{n,k} = \frac{3}{2k+1} \frac{n-1}{n} \frac{n-k}{n-1+k} + \frac{2}{2k+1} \frac{2(k+1)}{n+k} \leq \frac{3}{2k+1} \frac{n-k}{n+k} + \frac{2}{2k+1} \frac{2(k+1)}{n+k} \leq \frac{3}{2k+1}.
\]

**Lemma 7.3** We have
\[
\max_{x \in [0, \omega_k]} m_k(x, \sigma_n) \leq \frac{1}{(1-\delta_k/2)^k} T_n^{(k)}(\omega_k),
\]
where \( \delta_k \) is the maximal distance between two consecutive zeros of \( T_n^{(k+1)} \).

**Proof.** We will use the following estimate. Let \( f \in W^n(\sigma_n) \), i.e., \( \|f\| \leq 1 \) and \( \|f^{(n)}\| \leq \|T_n^{(n)}\| \).

Then
\[
T_n^{(k+1)}(\xi_i) = 0 \Rightarrow |f^{(k)}(\xi_i)| \leq |T_n^{(k)}(\xi_i)| \leq T_n^{(k)}(\omega_k).
\]
Let \( (\xi_i) \) be the zeros of \( T_n^{(k+1)} \), and let \( \delta_k = \max_i |\xi_i - \xi_{i+1}| \). Set
\[
\hat{T}_n(x) = T_n(\gamma x), \quad \gamma = \frac{1}{1-\delta_k/2} > 1.
\]
Then
\[
\hat{T}_n^{(k+1)}(\xi) = 0 \Rightarrow |f^{(k)}(\xi)| \leq |\hat{T}_n^{(k)}(\xi)| \leq \gamma^k T_n^{(k)}(\omega_k).
\]

**Corollary 7.4** We have
\[
\max_{x \in [0, \omega_1]} m_k(x, \sigma_n) \leq \left( \frac{1}{1 - \sin \frac{\pi(k+1)}{2n}} \right)^k T_n^{(k)}(\omega_k).
\]

**Proof.** Since \( \cos \frac{\pi i}{n} \) are zeros of \( T_n' \), the zeros \( \xi_i \) of \( T_n^{(k+1)} \) are located in the intervals \( \cos \frac{\pi i}{n} < \xi_i < \cos \frac{\pi i}{n} \), and for the distance between two consecutive \( \xi_i \) we have
\[
\delta_k = \max_i |\xi_i - \xi_{i+1}| \leq \max_i \left| \cos \frac{\pi i}{n} - \cos \frac{\pi i + (k+1)}{n} \right| \leq 2 \sin \frac{\pi (k+1)}{2n}.
\]

**Corollary 7.5** We have
\[
\max_{x \in [0, \omega_1]} m_1(x, \sigma_n) \leq \frac{1}{2} T_n'(1). \quad (7.1)
\]
**Proof.** a) For $n = 4$, we have

$$T_4(x) = 8x^4 - 8x^2 + 1, \quad T'_4(x) = 16(2x^3 - x), \quad T''_4(x) = 16(6x^2 - 1),$$

so that

$$\omega_1 = 1/\sqrt{6}, \quad \delta_1/2 = 1/\sqrt{6}, \quad T'_4(\omega_1) = 32/3\sqrt{6} = 2/3\sqrt{6}T''_4(1),$$

hence

$$\alpha_4 = \frac{1}{1 - 1/\sqrt{6}} \frac{2}{3\sqrt{6}} < 0.46 \leq 0.5.$$

b) For $n = 5$, we have

$$T_5(x) = 16x^5 - 20x^3 + x, \quad T'_5(x) = 5(16x^4 - 12x^2 + 1), \quad T''_5(x) = 40x(8x^3 - 3),$$

so that

$$\omega_1 = \sqrt{3}/8, \quad \delta_1/2 = \frac{1}{2}\sqrt{3}/8, \quad T'_5(\omega_1) = \frac{25}{4} = \frac{1}{4}T''_5(1),$$

and

$$\alpha_5 = \frac{1}{1 - 1/2\sqrt{3}/4} < 0.361 \leq 1/2.$$

c) For $n \geq 6$, we have $T'_n(\omega_1) \leq \frac{1}{4}T''_n(1)$, hence

$$\alpha_n \leq \frac{1}{1 - \sin \frac{\pi}{4}} \frac{1}{4} \leq 1/2.$$

□

8 Proof of Theorem 1.3, the case $k \leq n - 2$

**Theorem 8.1** We have

$$\max_{x_0 \in [\omega_k, 1]} m^*_k(x_0, \sigma) \leq m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n.$$

**Proof.** 1) The case $\sigma \leq \theta_k$. By Lemma 5.3 we have

$$m^*_k(x_0, \sigma) \leq T''_{n-1}(1),$$

while

$$m_k(1, \sigma) > m_k(1, \sigma_0) = T_{n-1}(1).$$

2) The case $\sigma > \theta_k$. In this case

$$m^*_k(x_0, \sigma) \leq \frac{1}{k + 1} \frac{n - 1}{n - 1 + k} \left( \frac{\sigma}{\theta_k} \right)^{k/n} T''_{n-1}(1) = \gamma \left( \frac{t}{\alpha} \right)^{k/n} T''_{n-1}(1),$$

and

$$m_k(1, \sigma) \geq (1 - t)T''_{n-1}(1) + tT''_{n}(1) = (\beta(1 - t) + t)T''_{n}(1),$$

where

$$\alpha := \frac{n - (k + 1)}{2(n - (k + 1))}, \quad \beta := \frac{T''_{n-1}(1)}{T''_{n}(1)} = \frac{n - 1}{n} \frac{n - k}{n - 1 + k}, \quad t := \sigma/\sigma_n.$$
So, we need to prove that
\[ f(t) := \gamma \left( \frac{t}{\alpha} \right)^{k/n} \leq \beta(1 - t) + t =: g(t), \quad t \in [\alpha, 1]. \]

The function \( f \) is concave, therefore it is bounded from above by its tangent \( \ell \) at \( t = 2\alpha \), i.e.
\[ f(t) \leq \ell(t) = \gamma 2^{k/n} \left( 1 + \frac{k t - 2\alpha}{n} \right). \]

So, we are done, once we prove that
\[ \ell(t) \leq g(t) \quad \text{on} [\alpha, 1]. \]

Both functions are straight lines, so we need to check this inequality only at the end-points.

1) At \( t = \alpha \), we have
\[ \ell(\alpha) = \gamma 2^{k/n} \left( 1 - \frac{k}{2n} \right) \leq \gamma \left( 1 + \frac{k}{n} \right), \quad g(\alpha) = g(0) = \beta. \]

2) At \( t = 1 \), we have \( g(1) = 1 \), while
\[ \ell(1) = \gamma 2^{k/n} \left( 1 + \frac{k}{n} \frac{1 - 2\alpha}{2\alpha} \right) = \gamma \left( 1 + \frac{k}{n} \frac{n - 1}{n - 1 + k} \right) \left( 1 + \frac{k}{n (n - (k + 1))} \right). \]

Expression in the parenthesis is less than 1 + \( k \), so
\[ \ell(1) \leq 2^{k/n} \frac{n - 1}{n - 1 + k} \leq \frac{n + k}{n} \frac{n - 1}{n - 1 + k} < 1. \]

Theorem 8.2 We have
\[ \max_{x \in [0, \omega_k]} m_k(x, \sigma) \leq m_k(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n. \]

**Proof.** 1) For \( k \geq 2 \), we use the estimates
\[ m_k(x, \sigma) \leq \frac{3}{2k + 1} T_{n-1}^{(k)}(1) + \frac{2}{2k + 1} 2^{k+1} T_{n}^{(k)}(1) =: \ell_1(t). \]

and
\[ m_k(1, \sigma) \geq (1 - t)T_{n-1}^{(k)}(1) + tT_{n}^{(k)}(1) =: \ell_2(t), \]

To prove that \( \ell_1(t) \leq \ell_2(t) \) it is sufficient to compare their values at the end-points:
\[ \ell_1(0) = \frac{3}{2k + 1} T_{n-1}^{(k)}(1) \leq T_{n-1}^{(k)}(1) = \ell_2(0), \]
\[ \ell_1(1) \leq \frac{5}{2k + 1} T_{n}^{(k)}(1) \leq T_{n}^{(k)}(1) = \ell_2(1). \]

2) For \( k = 1 \), we use the following estimates:
\[ m_k(1, \sigma) \geq m_k(1, \sigma_0) = T_{n-1}^{\prime}(1) = \frac{(n - 1)^2 T_n(1)}{n^2} \geq \frac{9}{16} T_n(1), \]
and
\[ m_k(x, \sigma) \leq m_k(x, \sigma_n) \leq \frac{1}{2} T_n(1). \]
9 Proof of Theorem 1.3: the case \( k = n - 1 \)

Here we cover the case \( k = n - 1 \) for \( 0 \leq \sigma \leq \sigma_n \).

**Theorem 9.1** We have

\[
m_{n-1}(x, \sigma) \leq m_{n-1}(1, \sigma) = Z_n^{(n-1)}(1, \sigma), \quad 0 \leq \sigma \leq \sigma_n.
\]

**Proof.** For \( f \in W_{\infty}^n(\sigma) \), let \( l \in \mathcal{P}_{n-1} \) be the Lagrange polynomial of degree \( n-1 \) that interpolates \( f \) at the points of local extrema of \( Z_n(\cdot, \sigma) \) on the interval \([-1, 1]\), i.e.

\[
l(\tau_i, \sigma) = f(\tau_i), \quad -1 = \tau_0 < \tau_1 < \cdots < \tau_{n-2} < \tau_{n-1} = 1.
\]

Then

\[
f^{(n-1)}(x) = l^{(n-1)}(x, \sigma) + (f^{(n-1)}(x) - l^{(n-1)}(x, \sigma)) \leq D_{n-1}(x, \sigma) \| f \| + \Omega_{n-1}(x, \sigma) \| f^{(n)} \|,
\]

where

\[
D_{n-1}(x, \sigma) := \sup_{\| p_{n-1} \|_{\infty} = 1} |p_{n-1}^{(n-1)}(x)|, \quad \Omega_{n-1}(x, \sigma) := \sup_{\| f^{(n)} \|_{1} = 1} |f^{(n-1)}(x) - l^{(n-1)}(x, \sigma)|.
\]

Therefore,

\[
m_{n-1}(x, \sigma) \leq D_{n-1}(x, \sigma) + \Omega_{n-1}(x, \sigma) \sigma. \tag{9.1}
\]

1) It is known that the extremum value \( D_{n-1}(x, \sigma) \) (which is a constant, since \( p^{(n-1)} \equiv \text{const} \)) is attained by the polynomial \( p \in \mathcal{P}_{n-1} \) such that

\[
p(\tau_i, \sigma) = (-1)^i, \quad i = 0, \ldots, n - 1. \tag{9.2}
\]

It is easy to see that, with

\[
\omega(x, \sigma) := \prod (x - \tau_i),
\]

we have

\[
p(x) = Z_n(x, \sigma) - \frac{\sigma}{n!} \omega(x, \sigma).
\]

Indeed, \(9.2\) is clearly fulfilled, and \( p \) is of degree \( n - 1 \) because the leading coefficients of both polynomials on the right-hand side are equal to \( \sigma/n! \). Therefore

\[
D_{n-1}(x, \sigma) = p^{(n-1)}(1, \sigma) = Z_n^{(n-1)}(1, \sigma) - \frac{\sigma}{n!} \omega^{(n-1)}(1, \sigma) > 0. \tag{9.3}
\]

2) For \( \Omega_{n-1}(x, \sigma) \) we show below that

\[
\Omega_{n-1}(x, \sigma) \leq \Omega_{n-1}(1, \sigma) = \frac{1}{n!} \omega^{(n-1)}(1, \sigma). \tag{9.4}
\]

Thus, from \(9.1-9.4\), we obtain

\[
m_{n-1}(x, \sigma) \leq |Z_n^{(n-1)}(1, \sigma) - \frac{\sigma}{n!} \omega^{(n-1)}(1, \sigma)| + |\frac{\sigma}{n!} \omega^{(n-1)}(1, \sigma)| = Z_n^{(n-1)}(1, \sigma),
\]

and theorem is proved. \( \square \)

**Lemma 9.2** We have

\[
\Omega_{n-1}(x, \sigma) \leq \Omega_{n-1}(1, \sigma) = \frac{1}{n!} \omega^{(n-1)}(1, \sigma). \tag{9.5}
\]
Proof. For $\Omega_{n-1}(x, \sigma)$ we have the convex majorant
\[
\Omega_{n-1}(x, \sigma) \leq \Omega^*_n(x, \sigma) = \frac{1}{n} \sum_{i=0}^{n-1} |x - \tau_i(\sigma)|,
\]
so that
\[
\Omega_{n-1}(x, \sigma) \leq \max \{\Omega^*_n(0, \sigma), \Omega^*_n(1, \sigma)\}
\]
We note that
\[
\Omega^*_n(1, \sigma) = 1 - \frac{1}{n} \sum_{i} \tau_i(\sigma) = \frac{1}{m!} |\omega^{(n-1)}(1, \sigma)| = \Omega_{n-1}(1, \sigma),
\]
so we need to prove that
\[
c_1(\sigma) := \frac{1}{n} \sum_{i=0}^{n} |\tau_i(\sigma)| \leq 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma) =: c_2(\sigma).
\]
For large $n$, this inequality is self-evident because the alternation points $\tau_i(\sigma)$ are spread sufficiently uniform in the interval $[-1, 1]$, therefore $c_1(\sigma) < 1$ while $c_2 \to 1$. But we need it for all $n \geq 2$.

We will use the monotonicity property of $\tau_i(\sigma)$ as functions of $\sigma$. We have
\[
\tau_i(\sigma_0) \leq \tau_i(\sigma) \leq \tau_i(\sigma_n) \quad (9.6)
\]
Here, $\tau_i(\sigma_0)$ are zeros of $(x^2 - 1)T'_{n-1}(x)$ and $\tau_i(\sigma_n)$ are zeros of $(x - 1)T'_n(x)$, therefore
\[
\cos \frac{\pi ((n-i) - 1)}{n} \leq \tau_i(\sigma) \leq \cos \frac{\pi (n-i)}{n}, \quad i = 1, \ldots, n-1, \quad \tau_n(\sigma) = 1.
\]
It follows that
\[
c_2(\sigma) = 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma) \geq 1 - \frac{1}{n} \sum_{i=1}^{n} \tau_i(\sigma_n) = 1 - \frac{1}{n}.
\]
On the other hand, with $m = |\frac{\pi}{2}|$,
\[
\sum_{i=1}^{n-1} |\tau_i(\sigma_0)| + \sum_{i=n+1}^{n} |\tau_i(\sigma_n)| = \sum_{i=0}^{m-1} \cos \frac{\pi i}{n} + \sum_{i=m+1}^{m-1} \cos \frac{\pi i}{n} \leq 1 + \frac{1}{\sin \frac{\pi}{2n}},
\]
where we used the inequality
\[
\sum_{i=0}^{m-1} \cos i x = \frac{1}{2} + \left(\frac{1}{2} + \sum_{i=1}^{m-1} \cos i x\right) = \frac{1}{2} + \frac{\sin (m - \frac{1}{2}) x}{2 \sin \frac{x}{2}} \leq \frac{1}{2} + \frac{1}{2 \sin \frac{\pi}{2n}}, \quad x \in \left\{\frac{\pi}{n}, \frac{\pi}{n-1}\right\}.
\]
a) For $n \geq 6$ we have
\[
c_1(\sigma) \leq \frac{1}{n} + \frac{1}{n \sin \frac{\pi}{2n}} \leq \frac{1}{6} + \frac{1}{6 \sin \frac{\pi}{2}} = 0.81 < \frac{5}{6} < 1 - \frac{1}{n} \leq c_2(\sigma).
\]
b) For $n = 5$,
\[
c_1(\sigma) \leq \frac{1}{5} \left(1 + \cos \frac{\pi}{4} + 1 + \cos \frac{\pi}{5} + \cos \frac{2\pi}{5} \right) = 0.76 < \frac{4}{5} = \Omega_{n-1}(1, \sigma).
\]
c) For $n = 3$ and $n = 4$, we cannot obtain the inequality $c_1(\sigma) \leq c_2(\sigma)$ through the estimates (9.3). In these cases we split the interval $[\sigma_0, \sigma_n]$ into two parts:
1) $\tau_i(\sigma_0) \leq \tau_i(\sigma) \leq \tau_i(\tilde{\sigma}_n), \quad \sigma \in [\sigma_0, \tilde{\sigma}_n]$; 2) $\sigma \in [\tilde{\sigma}_n, \sigma_n]$,
where the second interval contains $\sigma$ such that $Z_n(\cdot, \sigma)$ are the Chebyshev polynomials stretched from the interval $[-\cos \frac{\pi}{n}, 1]$ to a slightly larger interval $[-\cos \phi, 1]$ up to $[-1, 1]$, i.e.

$$Z_n(x, \sigma) = T_n(1 + s(x - 1)), \quad s \in [s_n, 1], \quad s_n := \frac{1 + \cos \frac{\pi}{2n}}{2} = \cos^2 \frac{\pi}{2n}.$$  

The alternation points of such $Z_n$ are given by

$$\tau_i(\sigma) = (1 + t) \cos \frac{(n - i)\pi}{n} - t, \quad t \in [0, t_n], \quad t_n = \tan^2 \frac{\pi}{2n}.$$

c_1) Consider first the case $\sigma \in [0, \hat{\sigma}_n]$.  

For $n = 2$,

$$\tau_1(\sigma) = -1, \quad \tau_2(\sigma) = 1.$$

For $n = 3$, we have

$$\tau_1(\sigma) = -1, \quad 0 \leq \tau_2(\sigma) \leq \frac{1}{3}, \quad \tau_3(\sigma) = 1,$$

so that

$$c_1(\sigma) = \frac{1}{3} \sum_{i=1}^{3} |\tau_i(\sigma)| \leq \frac{7}{9}, \quad c_2(\sigma) \geq 1 - \frac{1}{3} \sum_{i=1}^{3} \tau_i(\hat{\sigma}_n) = \frac{1}{9}.$$

For $n = 4$,

$$\tau_1(\sigma) = -1, \quad -\frac{1}{2} \leq \tau_2(\sigma) \leq -(3 - 2\sqrt{2}), \quad \frac{1}{2} \leq \tau_3(\sigma) \leq 4\sqrt{2} - 5, \quad \tau_4(\sigma) = 1,$$

so that

$$c_1(\sigma) \leq \frac{1}{4} \sum_{i=1}^{4} |\tau_i(\sigma)| > 0.78, \quad \Omega_{n-1}(1, \sigma) \geq 1 - \frac{1}{4} \sum_{i=1}^{3} \tau_i(\hat{\sigma}_n) = 0.87.$$

c_2) In the case $\sigma \in [\hat{\sigma}_n, \sigma]$, we have

$$\tau_i(\hat{\sigma}) = (1 + t) \cos \frac{(n - i)\pi}{n} - t, \quad t \in [0, \tan^2 \frac{\pi}{2n}],$$

and, for $n = 2$,

$$c_1(\sigma) = \frac{1}{2} \sum_{i=1}^{4} |\tau_i(\sigma)| = 1 + t, \quad c_2(\sigma) = 1 - \frac{1}{2} \sum_{i=1}^{4} \tau_i(\sigma) = \frac{1 + t}{2},$$

while for $n = 3$,

$$c_1(\sigma) = \frac{1}{3} \sum_{i=1}^{3} |\tau_i(\sigma)| = 2 + t, \quad c_2(\sigma) = 1 - \frac{1}{3} \sum_{i=1}^{3} \tau_i(\sigma) = \frac{2 + 2t}{3},$$

whereas for $n = 4$,

$$c_1(\sigma) = \frac{1}{4} \sum_{i=1}^{4} |\tau_i(\sigma)| = \frac{\sqrt{2} + 1}{4}(1 + t), \quad c_2(\sigma) = 1 - \frac{1}{4} \sum_{i=1}^{4} \tau_i(\sigma) = \frac{3 + 3t}{4}.$$
10 Lower bounds for $B_{n,k}$

In Lemma 2.11, we proved that
\[
m_k(1, I_s) \geq B_{n,k},
\]
where $B_{n,k}$ is the best constant in the Landau-Kolmogorov inequality on the half-line subject to normalization as given below:
\[
B_{n,k} = \sup \{ |f^{(k)}(1)| : \|f\|_{[-\infty, 1]} = \|T_n\|, \|f^{(n)}\|_{[-\infty, 1]} = \|T_n^{(n)}\| \}.
\]
So, any lower bound for $B_{n,k}$ serves as a lower bound for $m_k(1, I_s)$.

If $g$ is an arbitrary function from $W_{-\infty, 1}^m$, then its linear transformation
\[
f(x) := \frac{\|T_n\|}{\|g\|} g \left( x \cdot \left( \frac{\|g\| \|T_n^{(n)}\|}{\|T_n\| \|g^{(n)}\|} \right)^{1/n} \right)
\]
is a properly normalized function, and
\[
B_{n,k} \geq \sup_\{ f \} |f^{(k)}(1)| = \sup_g \frac{|g^{(k)}(1)|}{\|g\|_{1-k/n} \|g^{(n)}\|^{k/n}} \|T_n\|^{1-k/n} \|T_n^{(n)}\|^{k/n} =: \gamma_{n,k} T_n^{(k)}(1),
\]
where
\[
\gamma_{n,k} = C_{n,k}/T_{n,k}, \quad C_{n,k} := \sup_g \frac{|g^{(k)}(1)|}{\|g\| \|g^{(n)}\|^{k/n}} \quad \text{and} \quad T_{n,k} := \frac{|T_n^{(k)}(1)|}{\|T_n\|^{1-k/n} \|T_n^{(n)}\|^{k/n}}.
\]
The constant $C_{n,k}$ is the best constant in the LK-inequality on the half-line in the homogeneous form
\[
|g^{(k)}(1)| \leq C_{n,k} \|g\|_{1-k/n} \|g^{(n)}\|^{k/n}.
\]
Stechkin proved that
\[
C_{n,k} \geq \frac{k!}{(2k)!} \left( \frac{(2n)!}{n!} \right)^{k/n} \quad \text{and} \quad C_{n,k} \geq \frac{(2n)!^{1-k/n}}{(n-k)!},
\]
whichever is preferable. He also showed that
\[
a \left( \frac{n}{p} \right)^p \leq C_{n,k} \leq T_{n,k} \leq A \left( \frac{2n}{p} \right)^p, \quad p = \min(k, n-k).
\]

**Lemma 10.1** We have
\[
B_{n,k} \geq \gamma_{n,k} T_n^{(k)}(1),
\]
where
\[
\gamma_{n,k} \geq (2/e)^{2k}.
\]

**Proof.** We have
\[
C_{n,k} = \frac{k!}{(2k)!} \left( \frac{(2n)!}{n!} \right)^{k/n}, \quad T_{n,k} = \frac{2^k k! n^2 (n^2 - 1^2) \cdots (n^2 - (k-1)^2)}{(2k)! (2n-1)!^{k/n}},
\]
so that
\[
\gamma_{n,k} \geq C_{n,k}/T_{n,k} = \frac{2^{-k/n}}{n^2 (n^2 - 1^2) \cdots (n^2 - (k-1)^2) (2n)^{k/n}} \quad (10.1)
\]
\[
> \frac{n^{2k}}{n^2 (n^2 - 1^2) \cdots (n^2 - (k-1)^2) (2/e)^{2k}} > (2/e)^{2k}, \quad (10.2)
\]
where we used
\[
(2n)^{k/n} > \left( \frac{\sqrt{4\pi n (2n/e)^2}}{2/e} \right)^{k/n} > 2^{k/n} n^{2k} (2/e)^{2k}.
\]
Lemma 10.2 For \( n \leq 15 \), and \( 1 \leq k \leq n - 1 \), we have
\[
C_{n,k} > \frac{T_{n+m}^{(k)}(1)}{T_{n+m}^{(n)}(1)^{k/n}},
\]
where
\[
m = 1, \quad 3 \leq n \leq 6, \quad m = 2, \quad 7 \leq n \leq 10, \quad m = 3, \quad 11 \leq n \leq 14.
\]

Proof. For \( x \in [-1, 1] \), consider the function
\[
g(x) := g_{n,m}(x) := \phi(x)T_{n+m}(x), \quad \phi(x) = c_n \int_{-1}^{x} (1 - t^2)^n dt, \quad \phi(1) = 1,
\]
where the last equality defines the constant \( c_n \). We extend it to the half-line \([-\infty, 1] \) by setting \( g_{n,m}(x) = 0 \) for \( x < -1 \). Then
\[
g \in W_{\infty}^n[-\infty, 1], \quad g^{(k)}(1) = T_{n+m}^{(k)}(1), \quad k = 1, \ldots, n,
\]
and
\[
C_{n,k} \geq \frac{|g^{(k)}(1)|}{\|g\|^{1-k/n}\|g^{(n)}\|^{k/n}} = \frac{|g^{(k)}(1)|}{\|g^{(n)}\|^{k/n}[-1,1]}.
\]
So, we are done once we prove that \( \|g^{(n)}\|[-1,1] = g^{(n)}(1) \). The latter is proved numerically: the graph of the function \( g^{(n)} \) (provided by MAPLE) shows that, on \([-1, 1] \), for the values \( n \) and \( m \) given above, it attains its maximum at \( x = 1 \).

Corollary 10.3 We have
\[
m_k(1, I_s) > \gamma_{n,k} T_n^{(k)}(1)
\]
where
\[
\gamma_{n,k} = \frac{T_{n+m}^{(k)}(1)}{T_{n+m}^{(n)}(1)} \left( \frac{T_n^{(n)}(1)}{T_n^{(n)}(1)} \right)^{k/n}.
\]

11 Proof of Theorem 1.4

1) For \( k = 1 \), we have the inequality
\[
m_1(1, I_s) \geq B_{n,1} = \gamma_{n,1} T_n^{(n)}(1),
\]
where, by (10.1)-(10.2),
\[
\gamma_{n,1} = \frac{2^{-1/n}}{n^2} (2n)^{1/n} > (2/e)^2 > 0.541, \quad \gamma_{3,1} > 0.79.
\]
We proved in (7.1) that
\[
m_1(x, \sigma_n) \leq \frac{1}{2} T_n^{(n)}(1), \quad x \in [0, \omega_1],
\]
and we also have
\[
m_1^*(x, \sigma_n) \leq \alpha_{n,1} T_n^{(n)}(1), \quad x \in [\omega_1, 1],
\]
where
\[
\alpha_{n,1} = \frac{1}{3} \left( \frac{2(2n-2)}{n-2} \right)^{1/n} \leq \alpha_{4,1} = \frac{1}{3} 6^{4/4} < 0.522, \quad n \geq 4, \quad \alpha_{3,1} = 2/3.
\]

2) For \( k = 2 \), we have
\[
m_2(1, I_s) \geq B_{n,2} \geq \gamma_{n,2} T_n^{(n)}(1),
\]
where
\[ \gamma_{n,2} = \frac{2^{-2/n}}{n^2(n^2 - 1)}(2n)^{2/n} > (2/e)^4 > 0.293. \]

For the upper bounds, we have
\[ m_x^2(x, \sigma_n) \leq \alpha_{n,2} T_n''(1), \quad \alpha_{n,2} = 0.23 \left( \frac{2(2n - 3)}{(n - 3)} \right)^{2/n} \]
and
\[ m_2(x, \sigma_n) \leq \beta_{n,2} T_n''(1), \]
where
\[ \beta_{n,2} = \frac{1}{5} \frac{8}{11} \frac{1}{(1 - \sin \frac{3\pi}{2n})^2} < 0.28, \quad n \geq 16, \quad \beta_{n,2} = \frac{3}{5}, \quad n < 16. \]

We put all the values in the table.

| \( n \) | \( n = 4 \) | \( 5 \leq n \leq 15 \) | \( n \geq 16 \) |
|--------|-------------|----------------|--------------|
| \( \alpha_{n,2} \) | 0.72 | \( \leq 0.50 \) | \( \leq 0.277 \) |
| \( \beta_{n,2} \) | – | \( \leq 0.60 \) | \( \leq 0.288 \) |
| \( \gamma_{n,2} \) | 0.79 | \( \geq 0.63 \) | \( \geq 0.293 \) |
### 12 Proof of Theorem 1.5

The values of $\gamma_{n,k}$ in (10.3):

| $k/n$ | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1     | 0.87| 0.87| 0.87| 0.82| 0.82| 0.82| 0.82| 0.79| 0.79| 0.79| 0.80| 0.80|
| 2     | 0.79| 0.77| 0.77| 0.67| 0.67| 0.68| 0.68| 0.63| 0.63| 0.64| 0.64| 0.65|
| 3     | 0.72| 0.70| 0.57| 0.57| 0.57| 0.57| 0.57| 0.50| 0.51| 0.51| 0.52| 0.52|
| 4     | 0.66| 0.51| 0.49| 0.49| 0.49| 0.49| 0.41| 0.41| 0.42| 0.42| 0.43| 0.43|
| 5     | 0.50| 0.45| 0.43| 0.43| 0.43| 0.34| 0.34| 0.34| 0.35| 0.35| 0.35| 0.35|
| 6     | 0.46| 0.41| 0.39| 0.30| 0.29| 0.29| 0.29| 0.29| 0.29| 0.29| 0.29| 0.29|
| 7     |     |     |     | 0.43| 0.38| 0.27| 0.26| 0.25| 0.25| 0.25| 0.25| 0.25|
| 8     |     |     |     | 0.41| 0.27| 0.25| 0.23| 0.22| 0.22| 0.22| 0.22| 0.22|
| 9     |     |     |     |     | 0.31| 0.25| 0.22| 0.21| 0.20| 0.20| 0.20| 0.20|
| 10    |     |     |     |     |     |     | 0.29| 0.23| 0.20| 0.19| 0.19| 0.19|
| 11    |     |     |     |     |     |     |     | 0.28| 0.22| 0.19| 0.19| 0.19|
| 12    |     |     |     |     |     |     |     |     | 0.27| 0.20| 0.20| 0.20|
| 13    |     |     |     |     |     |     |     |     |     | 0.26| 0.26| 0.26|

The values of

$$\alpha_{n,k} = \frac{1}{k+1} \frac{n-1}{n-1+k} \left( \frac{2(2n-(k+1))}{n-(k+1)} \right)^{k/n}$$

| $k/n$ | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1     | 0.58| 0.55| 0.54| 0.53| 0.53| 0.52| 0.52| 0.52| 0.51| 0.51| 0.51| 0.51|
| 2     | 0.63| 0.48| 0.43| 0.40| 0.39| 0.38| 0.37| 0.36| 0.36| 0.36| 0.35| 0.35|
| 3     | 0.63| 0.44| 0.37| 0.34| 0.32| 0.30| 0.30| 0.29| 0.28| 0.28| 0.28| 0.28|
| 4     | 0.64| 0.42| 0.34| 0.30| 0.28| 0.26| 0.25| 0.24| 0.24| 0.24| 0.23| 0.23|
| 5     | 0.65| 0.40| 0.32| 0.28| 0.25| 0.24| 0.22| 0.22| 0.22| 0.21| 0.21| 0.21|
| 6     | 0.67| 0.40| 0.31| 0.26| 0.24| 0.22| 0.21| 0.21| 0.20| 0.20| 0.20| 0.20|
| 7     | 0.68| 0.40| 0.30| 0.25| 0.22| 0.20| 0.19| 0.19| 0.19| 0.19| 0.19| 0.19|
| 8     | 0.69| 0.39| 0.29| 0.24| 0.21| 0.19| 0.19| 0.19| 0.19| 0.19| 0.19| 0.19|
| 9     |     |     |     |     |     |     |     |     |     | 0.70| 0.39| 0.29| 0.24| 0.21|
| 10    |     |     |     |     |     |     |     |     |     | 0.71| 0.39| 0.29| 0.29| 0.23|
| 11    |     |     |     |     |     |     |     |     |     | 0.72| 0.39| 0.28| 0.28| 0.28|
| 12    |     |     |     |     |     |     |     |     |     |     | 0.73| 0.39| 0.28| 0.28|
| 13    |     |     |     |     |     |     |     |     |     |     |     | 0.74| 0.39| 0.28|

It is readily seen that, for the values of $k/n$ above and on the shadowed cells, we have

$$\alpha_{n,k} \leq \gamma_{n,k}.$$
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