Linear quadratic mean field games with a major player: 
The multi-scale approach

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Abstract
This paper considers linear-quadratic (LQ) mean field games with a major player and analyzes an asymptotic solvability problem. It starts with a large-scale system of coupled dynamic programming equations and applies a re-scaling technique introduced in Huang and Zhou (2018a, 2018b) to derive a set of Riccati equations in lower dimensions, the solvability of which determines the necessary and sufficient condition for asymptotic solvability. We next derive the mean field limit of the strategies and the value functions. Finally, we show that the two decentralized strategies can be interpreted as the best responses of a major player and a representative minor player embedded in an infinite population, which have the property of consistent mean field approximations.

Key words: asymptotic solvability, linear-quadratic, mean field game, major and minor players, re-scaling, Riccati differential equation

1 Introduction

Mean field game theory has undergone a phenomenal growth. It provides a powerful methodology for handling complexity in noncooperative mean field decision problems. The readers are referred to (Caines, Huang, and Malhamé, 2017) for an overview. Most existing analysis has been developed based on two routes called the direct approach and the fixed point approach. By the direct approach, one starts by formally solving an $N$-player game to obtain a large coupled solution equation system, and next derives a simple liming equation system by taking $N \rightarrow \infty$ (Lasry and Lions, 2007). By the fixed point approach, one determines the best response of a representative agent to a mean field of an infinite population, and next all the agents’ best responses should regenerate that mean field (Huang, Malhamé, and Caines, 2006). This procedure formalizes a fixed point problem, which can be solved and further used to design decentralized strategies. For LQ mean field games, the recent work (Huang and Zhou, 2018b) shows the exact relationship of the two approaches. In general, the fixed point approach has more flexibilities and can be implemented in diverse models (Huang, Caines, and Malhamé, 2007; Li and Zhang, 2008; Bensoussan et al, 2013; Huang and Ma, 2016; Carmona and Delarue, 2018). Further convergence analysis in the direct approach can be found in (Cardaliaguet et al, 2015; Lacker, 2016; Fischer, 2017). Mean field games have found applications in traffic routing (Bauso, Zhang, and Papachristodoulou, 2017), smart grids (Couillet, et al, 2012; Ma, Callaway, and Hiskens, 2013; Kizilkale, Salhab, and Malhamé, 2019) and production planning (Wang and Huang, 2019), among others. A notable feature of the early literature of mean field games is that all players in the model are comparably small, and can be called peers.

Huang (2010) introduces an LQ mean field game model with a major player which has strong influence. A motivating example is the interaction between a large corporation and many much smaller firms. There has been a rapid increase of literature on mean field games with a major and many minor players. In the setting of LQ models, Nguyen and Huang (2012a) consider continuum parametrized minor players, and Nguyen and Huang (2012b) extend to mass behavior directly impacted by the major player. Kordonis and Papavassilopoulos (2015) analyze minor players with random entrance. Major players with leadership are studied by Bensoussan et al (2017), Moon and Basar (2018). Partial state observation is considered by Caines and Kizilkale...
(2017), and Firoozi and Caines (2015). Huang, Wang and Wu (2016) take linear backward stochastic differential equations to model the dynamics of the players. Huang, Jaimungal, and Nourian (2015) present an application of the major player mean field game theory to an optimal execution model with an institutional trader and a large number of small traders.

Major-minor player games with nonlinear diffusion dynamics are an important class of modelling; see Nourian and Caines (2013), Buckdahn, Li and Peng (2014), Bensoussan, Chau and Yam (2016), Carmona and Zhu (2016). Leader-follower interaction is adopted by Bensoussan et al (2015), Fu and Horst (2018). To deal with this nonlinear modelling, forward-backward stochastic differential equations provide a vital analytical tool. Sen and Caines (2016) apply nonlinear filtering when the major player’s state is partially observed. More recently, Lasry and Lions (2018) introduce master equations for mean field games with major and minor players. They may be viewed as a pair of abstract dynamic programming equations. Cardaliaguet, Cirant, and Porretta (2018) prove the convergence of the Nash equilibria by use of the master equations when the number of minor players tends to infinity. A mean field principal-agent model is formulated by Elie, Mastrolia, and Possamai (2019).

For major player models with discrete states, see (Huang 2012; Kolokoltsov, 2017; Carmona and Wang, 2017).

Huang (2010) applies a state space augmentation approach by adding the mean field dynamics into the two decision problems, one for the major player and one for a representative minor player. This Markovianizes the problem and enables the use of dynamic programming. The procedure of Huang (2010) is based on the fixed point approach and the associated consistent mean field approximations, and that work only assumes existence of the solution.

This paper analyzes the LQ mean field game with a major player and homogeneous (or symmetric) minor players and takes the direct approach by starting with the solution for $N + 1$ players. Specifically, we will extend an asymptotic solvability notion introduced in a recent work Huang and Zhou (2018a) for LQ mean field games without a major player. With or without a major player, asymptotic solvability can be informally stated as the existence of Nash equilibria with complete state information for all sufficiently large population sizes, in addition to some boundedness property of the solution. We exploit the multi-scale nature of the optimization solution and use a re-scaling method in Huang and Zhou (2018a, 2018b) so that the key information in some higher order terms in the solution of the Riccati equation can be registered. We derive the necessary and sufficient condition for asymptotic solvability and evaluate the value function. The re-scaling method gives a set of ordinary differential equations (ODEs) for nine matrix functions. To reveal the special structure underlying these functions, we will further relate them to the best responses of the major player and a representative minor player staying an infinite population, where consistent mean field approximations hold. The latter is a key feature of the fixed point approach in mean field games. Our mean field limit analysis shares similarity to (Cardaliaguet, Cirant, and Porretta, 2018) which performs convergence analysis in a nonlinear system via the master equation. But we explicitly exploit the multi-scale phenomena in our model to identify a lower dimensional object which governs the asymptotic behavior of the system when the number of minor players tends to infinity. Similar methods appear in the statistical physics literature on mean field models (Ott and Antonsen, 2008; Pazo and Montbrio, 2014).

We mention other related LQ models of finding mean field limits via analyzing large scale equations. Papavassilopoulos (2014) uses large algebraic Riccati equations in mean field games and analyzes existence by an implicit function theorem. Priuli (2015) considers coupled Hamilton-Jacobi-Bellman and Kolmogrov equations with decentralized information. Mean field social optimal control is analyzed in (Huang 2003, Chap. 6; Herty, Pareschi, and Steffensen, 2014).

The organization of the paper is as follows. Section 2 describes the LQ Nash game with $N + 1$ players together with its solution via dynamic programming and Riccati equations. Section 3 extends the formulation of asymptotic solvability in Huang and Zhou (2018a, 2018b) to the LQ model with a major player. Section 4 presents further mean field limits and the performance. Section 5 formulates two optimal control problems under a mean field generated by an infinite number of minor players and addresses the relation to the asymptotic solvability problem. Section 6 concludes the paper.

Notation: For symmetric matrix $S \geq 0$, we may write $x^T S x = \|x\|_S^2$. For a matrix $Z = (z_{jk}) \in \mathbb{R}^{l \times m}$, denote the $l_1$-norm $\|Z\|_1 = \sum_{j,k} |z_{jk}|$. Let the function $g(\delta, x)$ be defined for $x$ in a subset $D_x$ of a Euclidean space and parameter $\delta \in (0, p]$ for some $p > 0$. We say $g$ is compactly of $O(\delta)$ if for each compact subset $D_0 \subset D_x$, there exists a constant $c_0$ depending on $D_0$ such that $\sup_{x \in D_0} |g(\delta, x)| \leq c_0 \delta$.

## 2 The LQ game with major and minor players

We consider the LQ game with a major player $\phi_0$ and $N$ minor players $\phi_i$, $1 \leq i \leq N$. At time $t \geq 0$, the states of $\phi_0$ and $\phi_i$ are, respectively, denoted by $X_0(t)$ and $X_i(t)$, $1 \leq i \leq N$. The dynamics of the $N + 1$ players are given by a system of linear stochastic differential equations (SDEs):

$$dX_0(t) = (A_0X_0(t) + B_0u_0(t) + F_0X^{(N)}(t))dt + D_0dw_0(t),$$

$$dX_i(t) = (AX_i(t) + Bu_i(t) + FX^{(N)}(t) + GX_0(t))dt + Dw_i(t), \quad 1 \leq i \leq N, \quad t \geq 0,$$
where we have state $X_t \in \mathbb{R}^n$, control $u_t \in \mathbb{R}^{n_1}$, and $X^{(N)} = \frac{1}{N} \sum_{k=1}^N X_k(t)$. The initial states $\{X_j(0), 0 \leq j \leq N\}$ are independent with $EX_j(0) = x_j(0)$ and finite second moment. The $N + 1$ standard $n_2$-dimensional Brownian motions $\{W_j, 0 \leq j \leq N\}$ are independent and also independent of the initial states. The constant matrices $A_0, A, B_0, B, D_0, D, F_0, F, G$ have compatible dimensions. The costs of players $\mathcal{A}_k$, $0 \leq k \leq N$, are given by

$$J_0 = E \int_0^T \left[ |X_t(t) - \Gamma_0X^{(N)}(t) - \eta_0|^2_{\mathcal{Q}_0} + |u_t|^2_{\mathcal{R}_0} \right] dt$$

$$+ E|X_0(T) - \Gamma_0X^{(N)}(T) - \eta_0|^2_{\mathcal{Q}_0}, \quad (3)$$

$$J_i = E \int_0^T \left[ |X_t(t) - \Gamma_iX_0(t) - \Gamma_2X^{(N)}(t) - \eta_i|^2_{\mathcal{Q}} + |u_t|^2_{\mathcal{R}} \right] dt$$

$$+ E|X_t(T) - \Gamma_iX_0(T) - \Gamma_2X^{(N)}(T) - \eta_i|^2_{\mathcal{Q}}, \quad 1 \leq i \leq N. \quad (4)$$

The constant matrices (or vectors) $Q_0, R_0, R_0, Q_0, \Gamma_0, \Gamma_0, \eta_0, Q, R, Q, \Gamma_0, \Gamma_0, \eta_0$ above have compatible dimensions, and $Q_0 \geq 0, Q_0 \geq 0, Q \geq 0, R_0 > 0, R > 0$. For notational simplicity, we only consider constant parameters for the model. Our analysis can be easily extended to the case of time-dependent parameters. Define

$$x = (x^{(N)}_0, x^{(N)}_1, \ldots, x^{(N)}_N)^T \in \mathbb{R}^{(N+1)n},$$

$$X(t) = \begin{bmatrix} X_0(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{(N+1)n}, \quad W(t) = \begin{bmatrix} W_0(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{(N+1)n_2},$$

$$\hat{A} = \text{diag} \{ A_0, A, \ldots, A \} + \begin{bmatrix} 0, & 0 \end{bmatrix},$$

$$\hat{D} = \text{diag} \{ D_0, D, \ldots, D \} \in \mathbb{R}^{(N+1)n \times (N+1)n_2},$$

$$B_0 = e_1^{N+1} \otimes B_0 \in \mathbb{R}^{(N+1)n \times n_1},$$

$$B_k = e_k^{N+1} \otimes B \in \mathbb{R}^{(N+1)n \times n_1}, \quad 1 \leq k \leq N.$$

We denote by $1_{k \times l}$ a $k \times l$ matrix with all entries equal to 1, by $\otimes$ the Kronecker product, and by the column vectors $\{e_1^{N+1}, \ldots, e_k^{N+1}\}$ the canonical basis of $\mathbb{R}^k$. We may use a subscript $n$ to indicate the identity matrix $I_n$ to be $n \times n$.

Now we write (1) and (2) in the form

$$dX(t) = \left( \hat{A}X(t) + \sum_{k=0}^N B_ku_k(t) \right) dt + \hat{D}dW(t). \quad (5)$$

Under closed-loop perfect state (CLPS) information, we denote the value function of $\mathcal{A}_j$ by $V_j(t, x), 0 \leq j \leq N$, which corresponds to the initial condition $X(0) = x = (x_0^T, \ldots, x_N^T)^T$. The set of value functions is determined by the system of Hamilton-Jacobi-Bellman (HJB) equations

$$0 = \frac{\partial V_0}{\partial t} + \min_{u_0 \in \mathbb{R}^{n_1}} \left( \frac{\partial^T V_0}{\partial x} \left( \hat{A}x + \sum_{k=0}^N B_ku_k \right) + u_0^T R_0 u_0 \right. \left. + |x_0 - \Gamma_0x^{(N)} - \eta_0|^2_{\mathcal{Q}} + \frac{1}{2} \text{Tr}(\hat{D}^T (V_0)x\hat{D}) \right), \quad (6)$$

$$V_0(T, x) = |x_0 - \Gamma_0x^{(N)} - \eta_0|^2_{\mathcal{Q}}, \quad 0 \leq j \leq N.$$
Denote
\[
K_0 = [I_0, 0, \ldots, 0] - \frac{1}{N}[0, I_0, \ldots, I_0],
\]
\[
Q_0 = K_0^T Q_0 K_0,
\]
\[
K_i = [I_0, 0, \ldots, 0] - [I_i, 0, \ldots, 0] - \frac{1}{N}[0, I_i, \ldots, I_i],
\]
\[
Q_i = K_i^T Q K_i,
\]
where \(I_i\) is the \((i+1)\)th submatrix in (16). We have \(K_0, K_i \in \mathbb{R}^{n \times (N+1)n}\) and \(Q_0, Q_i \in \mathbb{R}^{(N+1)n \times (N+1)n}\). We write
\[
|x_0 - I_0 x^{(N)} - \eta_0|^2 = x^T Q_0 x - 2 x^T K_0^T Q_0 \eta_0 + \eta_0^T Q_0 \eta_0,
\]
\[
|x_i - I_i x^{(N)} - \eta_i|^2 = x^T Q_i x - 2 x^T K_i^T Q_i \eta_i + \eta_i^T Q_i \eta_i, \quad 1 \leq i \leq N.
\]
We may write \(V_j(T, x), 0 \leq j \leq N,\) in a similar form.

We substitute (13) into (10) and derive the equation systems:
\[
\begin{aligned}
P_0(t) &= - (P_0 \tilde{A} + \tilde{A}^T P_0) + P_0 B_0 R_0^{-1} B_0^T P_0 \\
&\quad + P_0 \sum_{k=1}^N B_k R_k^{-1} B_k^T P_k \\
&\quad + \sum_{k=1}^N P_k B_k R_k^{-1} B_k^T P_0 - Q_0, \\
P_0(T) &= Q_0 f_0,
\end{aligned}
\]
\[
\begin{aligned}
S_0(t) &= - \tilde{A}^T S_0 + P_0 B_0 R_0^{-1} B_0^T S_0 \\
&\quad + \sum_{k=1}^N P_k B_k R_k^{-1} B_k^T S_0 \\
&\quad + P_0 \sum_{k=1}^N B_k R_k^{-1} B_k^T S_0 + K_0^T Q_0 \eta_0, \\
S_0(T) &= - K_0^T Q_0 f_0 \eta_0,
\end{aligned}
\]
\[
\begin{aligned}
\dot{r}_0(t) &= S_0^T B_0 R_0^{-1} B_0^T S_0 + 2 S_0^T \sum_{k=1}^N B_k R_k^{-1} B_k^T S_k \\
&\quad - \eta_0^T Q_0 \eta_0 - \text{Tr}(\tilde{D}^T P_0 \tilde{D}), \\
\dot{r}_0(T) &= \eta_0^T Q_0 f_0 \eta_0.
\end{aligned}
\]

By (11) and (13), we derive the equation systems:
\[
\begin{aligned}
P_i(t) &= - (P_i \tilde{A} + \tilde{A}^T P_i) + (P_i B_i R_i^{-1} B_i^T P_i + P_i B_i R_i^{-1} B_i^T P_i) \\
&\quad + P_i \sum_{k=1}^N B_k R_k^{-1} B_k^T P_i \\
&\quad + \sum_{k=1}^N P_k B_k R_k^{-1} B_k^T P_i - Q_i, \\
P_i(T) &= Q_i f_i, \quad 1 \leq i \leq N,
\end{aligned}
\]
\[
\begin{aligned}
\dot{S}_i(t) &= - \tilde{A}^T S_i + P_i B_i R_i^{-1} B_i^T S_i \\
&\quad + P_i \sum_{k=1}^N B_k R_k^{-1} B_k^T S_i + K_i^T Q_i, \\
S_i(T) &= - K_i^T Q_i f_i, \quad 1 \leq i \leq N,
\end{aligned}
\]
\[
\begin{aligned}
\dot{r}_i(t) &= 2 S_i^T B_i R_i^{-1} B_i^T S_i \\
&\quad + 2 S_i^T \sum_{k=1}^N B_k R_k^{-1} B_k^T S_k \\
&\quad - S_i^T B_i R_i^{-1} B_i^T S_i - \eta_i^T Q_i \eta_i - \text{Tr}(\tilde{D}^T P_i \tilde{D}), \\
\dot{r}_i(T) &= \eta_i^T Q_i f_i, \quad 1 \leq i \leq N.
\end{aligned}
\]

Remark 1 If (20) and (23) have a solution \((P_0, \ldots, P_N)\) on \([t,T] \subseteq [0,T],\) such a solution is unique due to the local Lipschitz continuity of the vector field; see (Hale, 1969). The ODE guarantees each \(P_i, 0 \leq i \leq N,\) to be symmetric. If (20) and (23) have a unique solution \((P_0, \ldots, P_N)\) on \([0,T],\) then we can uniquely solve \((S_0, \ldots, S_N)\) and \((r_0, \ldots, r_N).\)

We consider CLPS information, so that the state vector \(X(t)\) is available to each player.

Theorem 1 Suppose that (20) and (23) have a unique solution \((P_0, \ldots, P_N)\) on \([0,T].\) Then we can uniquely solve (21), (22), (24), (25), and the Nash game of \(N+1\) players has a set of feedback Nash strategies given by
\[
\begin{aligned}
\dot{u}_0 &= - R_0^{-1} B_0^T (P_0 X(t) + S_0), \\
\dot{u}_i &= - R_i^{-1} B_i^T (P_i X(t) + S_i), \quad 1 \leq i \leq N.
\end{aligned}
\]

Proof. This theorem follows the standard results in (Basar and Olsder, 1999, Theorem 6.16, Corollary 6.5). \qed

By Theorem 1 and Remark 1, the solution of the feedback Nash strategies completely reduces to the study of (20) and (23).

3 Asymptotic solvability

Define the \((N+1)n \times (N+1)n\) identity matrix
\[
I_{(N+1)n} = \begin{bmatrix}
I_n & 0 & \cdots & 0 \\
0 & I_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_n
\end{bmatrix}
\]
For \(1 \leq i \neq j \leq N+1,\) exchanging the \(i\)th and \(j\)th rows of submatrices in \(I_{(N+1)n},\) let \(J_{ij}\) denote the resulting matrix.
For instance, we have

\[
J_{12} = \begin{bmatrix}
0 & I_n & \cdots & 0 \\
I_n & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_n
\end{bmatrix}.
\]

It is easy to check that \(J_{ij}^T = J_{ij}^{-1} = J_{ij}\).

**Theorem 2** \(P_0(t)\) and \(P_1(t)\) have the representation:

\[
P_0(t) = \begin{bmatrix}
\Pi_0^t & \Pi_2^t & \cdots & \Pi_0^t \\
\Pi_2^t & \Pi_0^t & \cdots & \Pi_0^t \\
\vdots & \vdots & \ddots & \vdots \\
\Pi_0^t & \Pi_0^t & \cdots & \Pi_0^t
\end{bmatrix},
\]

and

\[
P_1(t) = \begin{bmatrix}
\Pi_0^t & \Pi_a^t & \Pi_b^t & \cdots & \Pi_b^t \\
\Pi_a^t & \Pi_0^t & \Pi_b^t & \cdots & \Pi_b^t \\
\vdots & \vdots & \ddots & \vdots \\
\Pi_b^t & \Pi_b^t & \cdots & \Pi_b^t
\end{bmatrix},
\]

where each submatrix depends on \(t\) and is \(n \times n\). Moreover, \(P_i(t) = J_{2i+1}^T P_i(t) J_{2i+1}\) for \(i \geq 2\).

**Proof.** See Appendix A. \(\square\)

**Definition 3** The sequence of Nash games (1)-(4) has asymptotic solvability if there exists \(N_0\) such that for all \(N \geq N_0\), \((P_0, \cdots, P_N)\) has a solution on \([0, T]\) and

\[
\sup_{N \geq N_0, 0 \leq t \leq T} \left( \|P_0(t)\|_{l_1} + \|P_1(t)\|_{l_1} \right) < \infty.
\]

Note that (30) is equivalent to

\[
\sup_{N \geq N_0, 0 \leq t \leq T} \left[ \|P_0(t)\|_{l_1} + N\|\Pi_0^t(t)\| + N^2\|\Pi_1^t(t)\| \right] < \infty,
\]

\[
\sup_{N \geq N_0, 0 \leq t \leq T} \left[ \|P_0(t)\|_{l_1} + N\|\Pi_0(t)\| + N\|\Pi_a(t)\| + N\|\Pi_b(t)\| \right]
+ N\|\Pi_2(t)\| + N^2\|\Pi_3(t)\| < \infty.
\]

Denote

\[
M_0 = B_0R_0^{-1}B_0^T, \quad M = BR^{-1}B^T.
\]

Define

\[
\begin{align*}
A_0^0 &= \Pi_0^0, \quad A_0^N = N\Pi_0^0, \quad A_3^0 = N^2\Pi_3^0, \\
A_0^N &= \Pi_0, \quad A_0^N = N\Pi_0, \\
A_3^N &= N^2\Pi_3, \quad A_3^N = \Pi_a, \\
A_0^N &= N\Pi_0, \quad A_0^N = N\Pi_0.
\end{align*}
\]

For (20) and (23) we write the ODE system for the set of variables \((A_0^0, A_0^N, \ldots, A_0^N)\) in (33); see Appendix B. The following ODE system is obtained as the limit of the above ODE system:

\[
\begin{align*}
\dot{A}_0^0 &= A_0^0 M_0 A_0^0 - (A_0^0 A_0 + A_0^0 A_0^0) \\
&+ A_0^0 (M M_0 G - G) A_0^0 - Q_0, \\
\dot{A}_0^N &= (A_0^N - A_0^N) A_0^N + A_0^N (M A_1 + A_2) - A - F \\
&- A_0^N F_0 + (M M_0 - G) A_0^N + Q_0 G_0, \\
\dot{A}_0^N &= A_0^N M_0 A_0^N - A_0^N F_0 - F_0 A_0^N \\
&+ (A_0^N (M A_1 + A_2) - A - F) \\
&+ (A_0^N (M A_1 + A_2) - A - F) A_0^N - Q_0 F_0 G_0, \\
\dot{A}_0^N &= A_0^N M_0 A_0^N - A_0^N F_0 - F_0 A_0^N + A_0^N F_0 \\
&- (A_0^N (M A_1 + A_2) - A - F) \\
&+ (A_0^N (M A_1 + A_2) - A - F) A_0^N - Q_0 F_0 G_0.
\end{align*}
\]

\[
\begin{align*}
\dot{A}_1 &= A_1 M A_1 - A_1 A - A T A_1 - Q, \\
\dot{A}_2 &= A_2 (M_0 A_0^0 - F_0) - A_1 F + (A_1 M - A T) A_2 \\
&+ A_2 (M A_1 + A_2) - A - F + Q F_2, \\
\dot{A}_3 &= A_3 (M_0 A_0^0 - A^T A_2) - A_2 T A_2 \\
&- A_2 (M A_1 + A_2) - A - F \\
&+ (A_1 A_2 - A T - F_2) A_3 - T_2 Q F_2, \\
\dot{A}_4 &= A_3 (M_0 A_0^0 - A^T A_2) - A_2 T A_2 \\
&- A_2 (M A_1 + A_2) - A - F \\
&+ (A_1 A_2 - A T - F_2) A_3 - T_2 Q F_2, \\
\dot{A}_5 &= A_5 (M_0 A_0^0 - A^T A_2) - A_2 T A_2 \\
&- A_2 (M A_1 + A_2) - A - F \\
&+ (A_1 A_2 - A T - F_2) A_3 - T_2 Q F_2.
\end{align*}
\]

**Theorem 4** The sequence of games in (1)-(4) has asymptotic solvability if and only if (34) has a solution on \([0, T]\).
Proof. See Appendix B. □

Due to the quadratic terms in its right hand sides, we call (34) a system of Riccati ODEs. As it turns out later in Section 5, this set of solution functions can be interpreted according to two optimal control problems in an infinite population.

4 Equilibrium costs and decentralized control

For this section, we assume (34) has a solution on [0, T]. Therefore there exists $N_0 > 0$ such that for all $N \geq N_0$, (20) and (23) have a solution $(P_0, \ldots, P_N)$ on [0, T].

Proposition 1 Let $(S_0, \ldots, S_N)$ be the solution of (21) and (24). We have the representation

\[
S_0(t) = [\theta_0^T, \theta_1^T, \ldots, \theta_N^T],
\]

\[
S_i(t) = [\theta_0^T, \theta_1^T, \ldots, \theta_{i-1}^T, \theta_i^T, \theta_{i+1}^T, \ldots, \theta_N^T], \quad 1 \leq i \leq N,
\]

where each vector of $\theta_0^T(i), \theta_1^T(i), \theta_i^T, \theta_{i+1}(i), \ldots, \theta_N^T$ is in $\mathbb{R}^n$ and $\theta_i^T$ is the $(i+1)$th component of $S_i$.

Proof. The method is similar to proving Theorem 2, and we omit the detail. □

Define

\[
\alpha_0^{ON} = \theta_0, \quad \alpha_i^{ON} = N\theta_1, \quad \alpha_0^N = \theta_0, \quad \alpha_i^N = \theta_i, \quad \alpha_2^N = N\theta_2.
\]

We derive a set of ODEs for $(\alpha_0^{ON}, \alpha_1^{ON}, \alpha_0^N, \alpha_1^N, \alpha_2^N)$; see Appendix C. By taking the limit form of these equations, we introduce the ODE system:

\[
\begin{align*}
\dot{\alpha}_0^0 &= (L_0^0 \alpha_0 - A_0^0) \alpha_0 + (A_0 M - G^T) \alpha_0^0 + \Lambda_0^0 \alpha_0 + Q_0 \eta_0, \\
\dot{\alpha}_0^1 &= (L_0^1 \alpha_0 - A_0^1) \alpha_0 + (A_1 M - G^T) \alpha_1 + \Lambda_0^1 \alpha_0 + \Lambda_0^1 \alpha_0 + Q_0 \eta_0, \\
\dot{\alpha}_0 &= (L_0 \alpha_0 - A_0 \alpha_0) \alpha_0 + (A_M M - G^T) \alpha_1 + \Lambda_0 \alpha_0 + \Lambda_0 \alpha_0 + Q_0 \eta, \\
\dot{\alpha}_0 &= (L_0 \alpha_0 - A_0 \alpha_0) \alpha_0 + (A_M M - G^T) \alpha_1 + \Lambda_0 \alpha_0 + \Lambda_0 \alpha_0 + Q_0 \eta, \\
\dot{\alpha}_2 &= (L_2 \alpha_0 - A_2 \alpha_0) \alpha_0 + (A_2 \alpha_2) \alpha_2 + Q_0 \eta, \\
\end{align*}
\]

where

\[
\begin{align*}
\alpha_0^0(T) &= \eta_0^T, \quad \alpha_0^1(T) = \Gamma_0^T \eta_0 \\
\alpha_0(T) &= \Gamma_0^T \eta_0, \quad \alpha_1(T) = \eta_0, \quad \alpha_2(T) = \Gamma_0^T \eta_0.
\end{align*}
\]

After (34) is solved, $(\alpha_0^0, \ldots, \alpha_2)$ satisfies a linear ODE system and can be uniquely solved on [0, T].

Proposition 2 We have

\[
\sup_{0 \leq t \leq T} \{ |\theta_0(t) - \alpha_0^0(t)| + |N \theta_1(t) - \alpha_0^1(t)| + |\theta_i(t) - \alpha_0(t)| \\
+ |\theta_1(t) - \alpha_1(t)| + |N \theta_2(t) - \alpha_2(t)| \} = O(\frac{1}{N}).
\]

Proof. We consider the first ODE system for $(A_0^{ON}, \ldots, A_N^{ON})$ and $(\alpha_0^{ON}, \ldots, \alpha_2)$, and the second ODE system for $(A_0^{N}, \ldots, A_N^{N})$ and $(\alpha_0^N, \ldots, \alpha_2^N)$. By (Huang and Zhou, 2018, Theorem 4), we obtain the error bound. □

In view of (22) and (25), we obtain

\[
\dot{r}_0 = (\alpha_0^{ON})^T M_0 \alpha_0^{ON} + 2N \alpha_1^{ON} M_1 \alpha_1^{ON} - \eta_0^T Q_0 \eta_0 \\
- \text{Tr}(D_0^0 \alpha_1^{ON} D_0) - \frac{1}{N} \text{Tr}(D^T A_N^T D_N),
\]

\[
\dot{r}_i = 2 \alpha_0^{ON} M_0 \alpha_0^{ON} + 2N \alpha_i^{ON} M_i \alpha_i^{ON} - \eta_0^T Q_0 \eta_0 + \text{Tr}(D_0^0 \alpha_i^{ON} D_0) + \text{Tr}(D^T A_N^T D_N),
\]

\[
- \frac{1}{N} \text{Tr}(D_0^0 \alpha_i^{ON} D_0) + \text{Tr}(D^T A_N^T D_N).
\]

where $r_0(T) = \eta_0^T Q_0 \eta_0$ and $r_i(T) = \eta_i^T Q_i \eta_i$. For $N \geq N_0$, we can uniquely solve $r_0$ and $r_i$. It is clear that $r_i$ does not depend on $i$. We rewrite

\[
\dot{r}_0 = (\alpha_0^{ON})^T M_0 \alpha_0^{ON} + 2N \alpha_1^{ON} M_1 \alpha_1^{ON} - \eta_0^T Q_0 \eta_0 \\
- \text{Tr}(D_0^0 \alpha_1^{ON} D_0) - \frac{1}{N} \text{Tr}(D^T A_N^T D_N),
\]

\[
\dot{r}_i = 2 \alpha_0^{ON} M_0 \alpha_0^{ON} + 2N \alpha_i^{ON} M_i \alpha_i^{ON} - \eta_0^T Q_0 \eta_0 + \text{Tr}(D_0^0 \alpha_i^{ON} D_0) + \text{Tr}(D^T A_N^T D_N),
\]

\[
- \frac{1}{N} \text{Tr}(D_0^0 \alpha_i^{ON} D_0) + \text{Tr}(D^T A_N^T D_N).
\]

As the approximation of (35) and (36), we introduce the ODE system

\[
\begin{align*}
\dot{\chi}_0 &= \alpha_0^T \eta_0 + 2 \alpha_1^T \eta_0 - \eta_0^T Q_0 \eta_0 - \text{Tr}(D_0^0 \Lambda_1^T D_0), \\
\dot{\chi}_i &= 2 \alpha_i^T \eta_0 + 2 \alpha_i^T \eta_0 + \eta_i^T Q_i \eta_i + \text{Tr}(D_0^0 \alpha_i^{ON} D_0) + \text{Tr}(D^T A_N^T D_N),
\end{align*}
\]

where $\chi_0(T) = \eta_0^T Q_0 \eta_0$ and $\chi(T) = \eta_i^T Q_i \eta_i$. We solve $(\chi_0, \chi)$ on [0, T].

Proposition 3 We have

\[
\sup_{0 \leq t \leq T} \{ |r_0(t) - \chi_0(t)| + |r_i(t) - \chi(i)| \} = O(\frac{1}{N}).
\]

Proof. The proof is similar to that of Proposition 2. □

Assumption (H): The initial states $X_1(0), X_2(0), \ldots$ are i.i.d. and $X_1(0)$ has mean $\mu$ and covariance $\Sigma$. In addition, $X_0(0)$ has mean $\mu_0$ and covariance $\Sigma_0$.

Denote the set of Nash equilibrium strategies $\hat{u} = (\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_N)$ given by (26)-(27).
Proposition 4 The costs under the set of strategies \( \hat{\mu} \) have the asymptotic form

\[
\lim_{N \to \infty} J_0(\hat{\mu}) = \mu_0^T \Lambda_0(0) \mu_0 + 2 \mu_0^T \Lambda_0(0) \mu + \mu^T \Lambda_0 \mu \\
+ 2(\mu_0^T \alpha_0(0) + \mu^T \alpha_1(0)) + \text{Tr}(\Lambda_2(0) \Sigma_0) + \chi_0(0),
\]

\[
\lim_{N \to \infty} J_1(\hat{\mu}) = \mu_0^T \Lambda_0(0) \mu_0 + 2 \mu_0^T \Lambda_0(0) \mu_0 + 2 \mu_0^T \Lambda_0(0) \mu \\
+ \mu^T (\Lambda_1(0) + 2 \Lambda_2(0) + \Lambda_3(0)) \mu \\
+ 2(\mu_0^T \alpha_0(0) + \mu^T \alpha_1(0) + \mu^T \alpha_2(0)) \\
+ \text{Tr}(\Lambda_0(0) \Sigma_0 + \Lambda_1(0) \Sigma) + \chi_1(0).
\]

Proof. Note that

\[
J_0(\hat{\mu}) = E[X^T(0) P_0(0) X(0) + 2 S_0^T(0) X(0)] + r_0(0),
\]

and

\[
X^T(0) P_0(0) X(0) = X_0^T(0) \Pi_0(0) X_0(0) \\
+ 2 X_0^T(0) \Pi_0(0) X_0(0) \\
+ X^T(0) \left[ \begin{array}{ccc}
\Pi_0^T(0) & \cdots & \Pi_0^T(0)
\end{array} \right] X_0(0),
\]

where \( X_0(0) = [X_1^T(0), X_2^T(0), \ldots, X_N^T(0)]^T \). Similarly we have

\[
J_1(\hat{\mu}) = E[X^T(0) P_1(0) X(0) + 2 S_1^T(0) X(0)] + r_1(0).
\]

We complete the proof by elementary computations and taking limits. \( \square \)

Substituting \( \tilde{u}_0 \) and \( \tilde{u}_i \) into (1) and (2), we have

\[
dX_0 = (A_0 X_0 + M_0 (\Pi_1 X_0 + \Pi_2 X_0 + \theta_0)) \\
+ F_0 X^N dt + D_0 dW_0,
\]

\[
dX^N = (A X^N - M (\Pi_1 X + \Pi_2 X + (N-1) \Pi_0 X^N) \\
+ \theta_0) + F X^N + G X_0 dt + \frac{1}{N} \sum_{i=1}^{N} dW_i. \tag{37}
\]

When \( N \to \infty \), we obtain a limit form of the strategies

\[
\tilde{u}_0 = -R_0^{-1} B_0^T (\Lambda_0^0 X_0 + \Lambda_0^0 \overline{X} + \alpha_0^0), \tag{38}
\]

\[
\tilde{u}_i = -R^{-1} B^T (\Lambda_i^0 X_0 + \Lambda_1 X_0 + \Lambda_2 \overline{X} + \alpha_i), \tag{39}
\]

where \( \overline{X} \) is the infinite population limit of the state average \( \overline{X}^N \) of the minor players. Under the decentralized strategies (38)-(39) in the \( N+1 \) player game, we write the closed-loop system of equations:

\[
dx_0(t) = \left[ A_0 x_0 - M_0 (\Lambda_0^0 x_0 + \Lambda_0^0 \overline{x} + \alpha_0^0) \\
+ F_0 X^N \right] dt + D_0 dW_0, \tag{40}
\]

\[
dx_i(t) = \left[ A x_i - M (\Lambda_i^0 x_0 + \Lambda_1 x_0 + \Lambda_2 \overline{x} + \alpha_i) \\
+ F X^N + G x_0 \right] dt + dW_i, \quad 1 \leq i \leq N, \tag{41}
\]

\[
dx(t) = \left[ (A-M(\Lambda_1+\Lambda_2)+F) \overline{x} \\
+ (G-M\Lambda_0^0) x_0 - M \alpha_1 \right] dt, \quad t \geq 0. \tag{42}
\]

Following the standard mean square error estimate for \( |X^N - \overline{X}| \), under assumption (H) we can show that \( (\tilde{u}_0, \ldots, \tilde{u}_N) \) is an \( \epsilon \)-Nash equilibrium for the \( N+1 \) player game, where \( \epsilon = O(1/\sqrt{N}) \) and each player may use centralized state information \( X(t) \); see related methods in (Huang, 2010).

By the re-scaling technique, we derive the mean field limits of the costs and strategies. The feasibility condition is determined by (34) directly based on the model parameters in (1)-(4). This is different from (Huang, 2010), where the existence condition is described in an augmented state space in \( 3n \) dimensions and imposes consistency requirements on \( 3n \times 3n \) matrices.

5 The limiting control problems and best responses

For this section, we assume (34) has a solution on \( [0,T] \).

An interesting question is whether the above two limit strategies in (38)-(39) have the interpretation as best responses in appropriately constructed optimal control problems. Finding the best response of a single agent in an infinite population model has been a key step in the fixed point approach in mean field games; see (Huang, Caines, and Malhamé, 2007, Huang, 2010). We introduce two optimal control problems.

Problem (P0): The dynamics are given by

\[
dX_0(t) = (A_0 X_0 + B_0 u_0 + F_0 \overline{X}) dt + D_0 dW_0, \tag{43}
\]

\[
d\overline{X}(t) = [(A-M(\Lambda_1+\Lambda_2)+F) \overline{X} \\
+ (G-M\Lambda_0^0) x_0 - M \alpha_1] dt, \tag{44}
\]

where \( x_0(0) \) and \( \overline{X}(0) = \mu_0 \) are given. Equation (44) may be viewed as the limit of (37) but now \( x_0 \) is indirectly controlled by \( u_0 \) in (43). The cost is

\[
\mathcal{J}_0 = E \left[ \int_0^T \left( |x_0(t) - \Gamma_0 \overline{X}(t) - \eta_0|^2 + |u_0(t)|^2 \right) dt \right. \\
+ \left. E |x_0(T) - \Gamma_0 \overline{X}(T) - \eta_0|^2 \right]_{\tilde{u}_0}. \tag{45}
\]
Problem (P1): The dynamics are given by
\[
\begin{align*}
    dX_1(t) &= (AX_1 + Bu_1 + F\overline{X} + GX_0) dt + DdW_1, \\
    dX_0(t) &= [A_0X_0 - M_0(\Lambda_1^T A_0 + \Lambda_2^T \overline{X} + \alpha_0^2)] dt \\
    &\quad + \Phi_0 d\overline{X} dt + D_0 dW_0, \\
    d\overline{X}(t) &= [(A - M(\Lambda_1 + \Lambda_2) + F)\overline{X} \\
    &\quad + (G - MA_0^T)X_0 - MA_1] dt,
\end{align*}
\]
where \(X_1(0), X_0(0)\) and \(\overline{X}(0) = \mu_0\) are given. The notation \((X_0, \overline{X})\) is reused in Problem (P1), where \(u_0\) has taken a specific form in (47). Equation (47) can be viewed as a limit form of (40) when \(N \to \infty\). Since the two problems will be solved separately, this should cause no risk of confusion. The cost is
\[
\begin{align*}
    \mathcal{T}_1 &= E \int_0^T \left( |X_1(t) - \Gamma_1 X_0(t) - \Gamma_2 \overline{X}(t) - \eta_1^2 + |u_1|^2_R \right) dt \\
    &\quad + E|X_1(T) - \Gamma_1 X_0(T) - \Gamma_2 \overline{X}(T) - \eta_1^2|_\mathcal{Q}_f^2, \\
\end{align*}
\]
(49)

Since \(R_0 > 0, Q_0, Q_0^2 \geq 0, R > 0, Q, Q_f \geq 0\), both (P0) and (P1) can be solved uniquely. The resulting optimal control laws will also be called best responses.

Below, we start with the solution of Problem (P0). Denote
\[
\begin{align*}
    A_0 &= \begin{bmatrix} A_0 & F_0 \\
    -MA_0^T & A + F - M(\Lambda_1 + \Lambda_2) \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\
    0 \end{bmatrix}, \\
    Q_0 &= \begin{bmatrix} I_n & \Gamma_0^T \\
    -\Gamma_0^T & Q_0[I_n, -\Gamma_0] \end{bmatrix}, \\
    \mathbb{E}_0 &= \begin{bmatrix} \Phi^0_1 & \Phi^0_2 \\
    \Phi^0_2^T & \Phi^0_3 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad S_0 = \begin{bmatrix} \beta^0_1 \\
    \beta^0_2 \end{bmatrix},
\end{align*}
\]
where the terminal condition can be determined as
\[
\begin{align*}
    \mathbb{E}_0(T) &= \begin{bmatrix} I_n & -\Gamma_0^T \\
    -\Gamma_0^T & 0 \end{bmatrix} Q_0[I_n, -\Gamma_0], \\
    S_0(T) &= \begin{bmatrix} I_n & \eta_0 \\
    -\Gamma_0^T & 0 \end{bmatrix} Q_0[\eta_0].
\end{align*}
\]
We uniquely solve \(\mathbb{E}_0\) and \(S_0\) on \([0, T]\). Note that \(\mathbb{E}_0\) is a \(2n \times 2n\) matrix. The optimal control law is
\[
    u_0^* = -R_0^{-1}\mathbb{E}_0^{-1}(\mathbb{E}_0X_0(t) + S_0).
\]
Denote
\[
\begin{align*}
    \mathbb{E}_0 &= \begin{bmatrix} \Phi^0_1 & \Phi^0_2 \\
    \Phi^0_2^T & \Phi^0_3 \end{bmatrix}, \quad \Phi^0_0(t) \in \mathbb{R}^{n \times n}, \quad S_0 = \begin{bmatrix} \beta^0_1 \\
    \beta^0_2 \end{bmatrix},
\end{align*}
\]
where \(\Phi^0_1\) and \(\Phi^0_3\) are symmetric. Then by (50), we derive the ODE system:
\[
\begin{align*}
    \Phi^0_1 &= \Phi^0_3 M_0 \Phi^0_2 - \Lambda_1^T \Phi^0_0 - \Phi^0_A \Lambda_1 - (G^T - \Lambda_0 M) \Phi^0_2, \\
    \Phi^0_2 &= -\Lambda_1^T \Phi^0_2 - (G^T - \Lambda_0 M) \Phi^0_3 - \Phi^0_1 F_0, \\
    \Phi^0_3 &= \Phi^0_0 M_0 \Phi^0_2 - \Phi^0_2 F_0, \\
    \Phi^0_1 F_0 &= (A + F - M(\Lambda_1 + \Lambda_2)) + \Phi^0 M_0 \Phi^0_2 + Q_0 \Gamma_1, \\
    \Phi^0_0 F_0 &= (A + F - M(\Lambda_1 + \Lambda_2)) - \Phi^0_0(F_0) - \Phi^0_0 T M_0 \Phi^0_2 - \Phi^0_2 F_0, \\
    \Phi^0_1 T M_0 \Phi^0_2 &= \Phi^0_0 T M_0 \Phi^0_2 - \Phi^0_2 F_0, \\
    \Phi^0_0 T M_0 \Phi^0_2 &= \Phi^0_0 T M_0 \Phi^0_2 - \Phi^0_2 F_0, \\
    \Phi^0_0 T M_0 \Phi^0_2 &= \Phi^0_0 T M_0 \Phi^0_2 - \Phi^0_2 F_0.
\end{align*}
\]

We further write
\[
\begin{align*}
    |X_0 - \Gamma_0 \overline{X} - \eta_0|^2_{\mathcal{Q}_0} &= (X_0^T, \overline{X}^T)Q_0[X_0 \overline{X} - 2\eta_0^T Q_0[I_n, -\Gamma_0][X_0 \overline{X}] + \eta_0^T Q_0 \eta_0.
\end{align*}
\]

By dynamic programming for the optimal control problem (P0), we introduce
\[
\begin{align*}
    \mathbb{P}_0 &= -\Phi^0_0 \mathbb{P}_0 - \mathbb{P}_0 A_0 + \mathbb{P}_0 B_0 R_0^{-1} \mathbb{E}_0^{-1} \mathbb{P}_0 - \mathbb{Q}_0, \\
    \mathbb{S}_0 &= -A_0^T \mathbb{S}_0 + \mathbb{P}_0 B_0 R_0^{-1} \mathbb{P}_0 \mathbb{S}_0 + \mathbb{P}_0 \left[ 0 \begin{bmatrix} I_n & -\Gamma_0^T \end{bmatrix} Q_0 \eta_0
\end{align*}
\]
where
\[
\begin{align*}
    \beta^0_0 &= (\Phi^0_0 M_0 - \Lambda_1^T) \beta^0_0 - (G^T - \Lambda_0 M) \beta^0_1 + \Phi^0_0 M_0 \Lambda_1 + Q_0 \eta_0, \\
    \beta^0_1 &= (\Phi^0_0 M_0 - F_0^T) \beta^0_2 - (A^T + F^T - (\Lambda_1 + \Lambda_2^T) M) \beta^0_1 + \Phi^0_0 M_0 \Lambda_1 - \Gamma_0^T Q_0 \eta_0.
\end{align*}
\]

Finally, we rewrite \(u_0^*\) as
\[
    u_0^* = -R_0^{-1} \mathbb{P}_0^T \Phi^0_0 X_0 + \Phi^0_2 \overline{X} + \beta^0_0.
\]
Now we give the solution of Problem (P1). Denote
\[
A = \begin{bmatrix}
A_0 - M_0\lambda_1^0 & 0 & F_0 - M_0\lambda_2^0 \\
G & A & F \\
G - MA_1^0 & 0 & A + F - M(A_1 + A_2)
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 \\
B \\
0
\end{bmatrix}, \quad f = -\begin{bmatrix}
M\alpha_0^0 \\
0 \\
M\alpha_1^n
\end{bmatrix},
\]
\[
Q = [-I_\Gamma, I_n, -I_2]Q[-I_\Gamma, I_n, -I_2],
\]
\[
Q_f = [-I_{\Gamma f}, I_n, -I_{2f}]Q_f[-I_{\Gamma f}, I_n, -I_{2f}].
\]

The dynamics can be given as
\[
\begin{bmatrix}
\frac{dX_0}{dt} \\
\frac{dX_1}{dt} \\
\frac{dX}{dt}
\end{bmatrix} = \begin{bmatrix}
X_0 \\
X_1 \\
X
\end{bmatrix} dt + (\mathbb{B}u_1 + f) dt + \begin{bmatrix}
D_0 dW_0 \\
D_1 dW_1 \\
0
\end{bmatrix}.
\]

By dynamic programming, we introduce the two ODEs:
\[
\dot{P} = -A^T P - PA + \mathbb{B}R^{-1}B^T P - Q, \quad (52)
\]
\[
\dot{S} = -A^T S + \mathbb{B}R^{-1}B^T S \\
+ \begin{bmatrix}
M_0\alpha_0^0 \\
0 \\
M\alpha_1^n
\end{bmatrix} Q_f \eta,
\]
where
\[
\mathbb{P}(T) = Q_f, \quad \mathbb{S}(T) = -\begin{bmatrix}
-I_{\Gamma f} \\
I_n \\
-I_{\Gamma_{2f}}
\end{bmatrix} Q_f \eta_f.
\]

We uniquely solve \(\mathbb{P}\) and \(\mathbb{S}\) on \([0, T]\). Denote
\[
\mathbb{P} = \begin{bmatrix}
\Phi_0 & \Phi_a & \Phi_b \\
\Phi_a^T & \Phi_1 & \Phi_2 \\
\Phi_b^T & \Phi_2^T & \Phi_3
\end{bmatrix}, \quad \mathbb{S} = \begin{bmatrix}
\beta_0 \\
\beta_1 \\
\beta_2
\end{bmatrix},
\]
where \(\Phi_0, \Phi_1\) and \(\Phi_3\) are symmetric. By (52), we derive the ODE system:
\[
\begin{aligned}
\dot{\Phi}_0 &= \Phi_a M \Phi_a^T - G^T \Phi_a^T - \Phi_a G - \Gamma_{T_1}^T Q I_1 \\
-(\Phi_a^T - \Phi_1^0 M_0) \Phi_0 - \Phi_0 (A_0 - M_0\lambda_2^0) \\
-(G^T - L_\alpha M) \Phi_b - \Phi_b (G - M\lambda_1^0), \\
\dot{\Phi}_1 &= \Phi_1 M \Phi_1 - \Phi_2 F, \\
\dot{\Phi}_2 &= -A^T \Phi_2 - \Phi_1 F - \Phi_a^T (F_0 - M_0\lambda_2^0) \\
&- \Phi_2 (A + F - M(A_1 + A_2)) + \Phi_1 M \Phi_2 + Q I_2, \\
\dot{\Phi}_3 &= \Phi_3 M \Phi_2 - F^T \Phi_2 - \Phi_2^T F \\
-(F_0^T - \lambda_2^0 M_0) \Phi_3 - \Phi_3 (F_0 - M_0\lambda_2^0) \\
-(A^T + F^T - (A_1 + \lambda_2^0 M_0) \Phi_3 \\
\Phi_3 (A + F - M(A_1 + A_2)) - \Gamma_{T_2}^T Q I_2, \\
\dot{\Phi}_a &= -\Phi_a A - G^T \Phi_a - (\Phi_1^0 M_0) \Phi_a \\
-(G^T - L_\alpha M) \Phi_b - \Phi_b + \Phi_1 M \Phi_1 + \Gamma_{T_1}^T Q, \\
\dot{\Phi}_b &= -\Phi_b (F_0 - M_0\lambda_2^0) - G^T \Phi_2 - \Phi_a F + \Phi_a M \Phi_2 \\
-(\Phi_1^0 M_0) \Phi_b - (G^T - L_\alpha M) \Phi_3 \\
-\Phi_b (A + F - M(A_1 + A_2)) - \Gamma_{T_1}^T Q I_2,
\end{aligned}
\]
where
\[
\begin{aligned}
\Phi_0(T) &= \Gamma_{T_1}^T Q_f \Gamma_{f_1}, \\
\Phi_1(T) &= Q_f, \quad \Phi_2(T) = -Q_f \Gamma_{f_2}, \\
\Phi_3(T) &= \Gamma_{T_2}^T Q_f \Gamma_{f_2}, \\
\Phi_a(T) &= -\Gamma_{f_1}^T Q_f, \\
\Phi_b(T) &= \Gamma_{T_2}^T Q_f \Gamma_{f_2}.
\end{aligned}
\]

Now by (53), we derive
\[
\begin{aligned}
\dot{\beta}_0 &= (\Phi_a M - G^T) \beta_1 - (\Phi_1^0 - \lambda_2^0 M_0) \beta_0 \\
&- (G^T - \lambda_\alpha M) \beta_2 + \Phi_0 M_0 \alpha_0^0 + \Phi_2 M_\alpha_1 + \Gamma_{T_1}^T Q \eta, \\
\dot{\beta}_1 &= (\Phi_1 M - A^T) \beta_2 + \Phi_0 M_0 \alpha_0^0 + \Phi_2 M_\alpha_1 + Q \eta, \\
\dot{\beta}_2 &= (\Phi_2^T M - F^T) \beta_1 - (F_0 - \lambda_2^0 M_0) \beta_0 \\
&- (A^T + F^T - (A_1 + \lambda_2^0 M_0) \beta_2 \\
&+ \Phi_2 M_0 \alpha_0^0 + \Phi_2 M_\alpha_1 - \Gamma_{T_2}^T Q \eta,
\end{aligned}
\]
where
\[
\begin{aligned}
\beta_0(T) &= \Gamma_{T_1}^T Q_f \eta_f, \\
\beta_1(T) &= -Q_f \eta_f, \quad \beta_2(T) = \Gamma_{T_2}^T Q_f \eta_f.
\end{aligned}
\]

The optimal control law is given by
\[
u_1 = -R^{-1}B^T (\Phi_a X_0 + \Phi_1 X_1 + \Phi_2 X + \beta_1).
\]

**Theorem 5** We have
\[
\mathbb{P}_0 = \begin{bmatrix}
\lambda_1^0 & \alpha_0^0 \\
\lambda_2^0 & \alpha_1^0
\end{bmatrix}, \quad \mathbb{S}_0 = \begin{bmatrix}
\alpha_0^0 \\
\alpha_1^0
\end{bmatrix},
\]
and
\[
\mathbb{P} = \begin{bmatrix} A_0, & A_a, & A_b \\ A_a^T, & A_1, & A_2 \\ A_b^T, & A_2^T, & A_3 \end{bmatrix}, \quad S = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}.
\]

Proof. We can directly show that
\[
(\Phi_1^0, \Phi_2^0, \Phi_3^0, \Phi_0, \Phi_1, \Phi_2, \Phi_3, \Phi_a, \Phi_b)
\]
is a solution of (34). Similarly, \((\beta_0^0, \beta_1^0, \beta_0, \beta_1, \beta_2)\) satisfies the ODE of \((\alpha_0^0, \alpha_1^0, \alpha_0, \alpha_1, \alpha_2)\). Therefore, we obtain the representation of \((\mathbb{P}_0, \mathbb{P}, S_0, S)\).

It is now clear that \((\tilde{u}_0, \tilde{u}_1)\) agrees with \((\tilde{u}_0, \tilde{u}_1)\) given by (38)-(39). Moreover, \(\tilde{X}\) in the closed-loop system of (P1) is actually generated by an infinite number of minor players applying the control law \(\tilde{u}_i\) while \(\tilde{u}_1\) is a best response to \(\tilde{X}\). This suggests a consistent means field approximation, which is well known in mean field games (Caines, Huang, and Malhamé, 2017).

6 Concluding remarks

We study an asymptotic solvability problem for LQ Nash games involving a major player and \(N\) minor players, where \(N\) tends to infinity. We obtain the necessary and sufficient condition of asymptotic solvability via a system of Riccati ODEs and evaluate the equilibrium costs. The system of Riccati ODEs has close relation with a limit control model of two players: the major player and a representative minor player.

Appendix A: Proof of Theorem 2

Lemma A.1 Assume that (20) and (23) have a solution \((P_0(t), \ldots, P_N(t))\) on \([0, T]\). Then the following holds.

i) \(P_0(t)\) has the representation
\[
P_0(t) = \begin{bmatrix} \Pi_0^0 & \Pi_2^0 & \Pi_4^0 & \cdots & \Pi_N^0 \\ \Pi_2^0 & \Pi_3^0 & \Pi_4^0 & \cdots & \Pi_N^0 \\ \Pi_4^0 & \Pi_4^0 & \Pi_5^0 & \cdots & \Pi_N^0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^0 & \Pi_4^0 & \Pi_4^0 & \cdots & \Pi_N^0 \end{bmatrix},
\]

where \(\Pi_0^0(t), \Pi_2^0(t)\) and \(\Pi_4^0(t)\) are \(n \times n\) symmetric matrix functions. The matrix \(\Pi_0^0\) appears for \(N^2 - N\) times.

ii) \(P_1(t)\) has the representation
\[
P_1(t) = \begin{bmatrix} \Pi_0 & \Pi_a & \Pi_b & \cdots & \Pi_b \\ \Pi_1^T & \Pi_1 & \Pi_2 & \cdots & \Pi_2 \\ \Pi_1^T & \Pi_2 & \Pi_3 & \cdots & \Pi_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_1^T & \Pi_4 & \Pi_4 & \cdots & \Pi_3 \end{bmatrix},
\]

where \(\Pi_0(t), \Pi_1(t), \Pi_2(t)\) and \(\Pi_3(t)\) are \(n \times n\) symmetric matrix functions. The matrix \(\Pi_4\) appears for \((N - 1)(N - 2)\) times.

iii) For \(i > 1\), \(P_i(t) = J_{2i+1}^T P_i(j_{2i+1})\), and we obtain \(P_2^{12} = P_0^{33}, P_0^{03} = P_1^{13}\).

Proof. i) For \(0 \leq l \leq N\), denote \(P_l = (P_{ij}^l)_{l \leq i, j \leq N+1}\), where \(P_{ij}^l\) is an \(n \times n\) matrix. Let \(\hat{P}_l = J_{23}^T P_l J_{23}\). By elementary matrix computations, we can verify that \((\hat{P}_0, \hat{P}_2, \hat{P}_3, \cdots, \hat{P}_N)\) satisfies (20) and (23). Hence,
\[
\hat{P}_0 = P_0, \quad \hat{P}_2 = P_1, \quad \hat{P}_3 = P_2, \quad \hat{P}_k = P_k, \quad k \geq 3.
\]

Then \(P_2 = J_{23}^T P_1 J_{23}\) and \(P_0 = J_{23}^T P_0 J_{23}\), and we obtain \(P_0^{12} = P_0^{33}, P_0^{03} = P_1^{13}\).

Taking \(J_{k,k+1}, k \geq 3\) in place of \(J_{23}\) and following the method in (Huang and Zhou, 2018b), we obtain the representation of \(P_0\).

ii) Now denote \(\hat{P}_l = J_{54}^T P_l J_{34}\), and we can verify that
\[
(\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4, \cdots, \hat{P}_N)
\]
is a solution of (20) and (23). Hence,
\[
\hat{P}_1 = P_1, \quad \hat{P}_2 = P_2, \quad \hat{P}_3 = P_3.
\]

This yields \(P_1^{13} = P_1^{14}, P_2^{13} = P_2^{14}\) and \(P_3^{13} = P_3^{14}\). In addition, \(P_1^{13} = P_1^{34}\). Since \(P_1\) is symmetric, \((P_1^{13})^T = P_1^{14}\). So \(P_1^{34}\) is symmetric. Similarly, by the relation \(J_{45}^T P_1 J_{45} = P_1\), we obtain \(P_1^{34} = P_1^{35}\). Now, repeatedly using the relation \(J_{k,k+1}^T P_k J_{k,k+1} = P_k\) for all \(k \geq 3\), we obtain the representation of \(P_1\). Note that \(P_4\) is symmetric.

iii) This equality can be shown as in the case \(i = 2\) in i). □

Proof of Theorem 2: By Lemma A.1, we have
\[
P_1^0(t) = \Pi_0^0 M_0 P_1^0 + N P_2^0 M P_1^0 + N P_3^0 M P_1^0 - (\Pi_1^0 A_0 + A_0^T \Pi_1^0) - N(\Pi_0^0 G + G^T \Pi_2^0) - Q_0,
\]
\[
P_1^0(T) = Q_0 f_1.
\]
It follows that
\[ \Pi_0(T) = \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}} Q_f(I - \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}}), \tag{A.6} \]

and

\[ \Pi_1(T) = \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}} Q_f(I - \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}}), \tag{A.7} \]

and

\[ \Pi_2(T) = \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}} Q_f(I - \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}}), \tag{A.8} \]

and

\[ \Pi_3(T) = \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}} Q_f(I - \frac{\Gamma_T}{\tilde{N}^{\frac{1}{2}}}), \tag{A.9} \]

By Lemma A.1, we have

\[ \Pi_0(t) = \Pi_0(t) + \Pi_2(t) - \Pi_3(t) + \Pi_4(t), \]

and

\[ \Pi_1(t) = \Pi_1(t) + \Pi_2(t) + \Pi_3(t) - \Pi_4(t), \]

and

\[ \Pi_2(t) = \Pi_2(t) + \Pi_3(t) + \Pi_4(t) - \Pi_1(t), \]

and

\[ \Pi_3(t) = \Pi_3(t) + \Pi_4(t) - \Pi_1(t) + \Pi_2(t), \]

\[ \Pi_4(t) = \Pi_4(t) - \Pi_1(t) - \Pi_2(t) + \Pi_3(t), \]

\[ \Pi_0(t) = \Pi_0(t) + \Pi_2(t) - \Pi_3(t) + \Pi_4(t), \]

\[ \Pi_1(t) = \Pi_1(t) + \Pi_2(t) + \Pi_3(t) - \Pi_4(t), \]

\[ \Pi_2(t) = \Pi_2(t) + \Pi_3(t) + \Pi_4(t) - \Pi_1(t), \]

\[ \Pi_3(t) = \Pi_3(t) + \Pi_4(t) - \Pi_1(t) + \Pi_2(t), \]

\[ \Pi_4(t) = \Pi_4(t) - \Pi_1(t) - \Pi_2(t) + \Pi_3(t), \]

\[ \Pi_0(t) = \Pi_0(t) + \Pi_2(t) - \Pi_3(t) + \Pi_4(t), \]

\[ \Pi_1(t) = \Pi_1(t) + \Pi_2(t) + \Pi_3(t) - \Pi_4(t), \]

\[ \Pi_2(t) = \Pi_2(t) + \Pi_3(t) + \Pi_4(t) - \Pi_1(t), \]

\[ \Pi_3(t) = \Pi_3(t) + \Pi_4(t) - \Pi_1(t) + \Pi_2(t), \]

\[ \Pi_4(t) = \Pi_4(t) - \Pi_1(t) - \Pi_2(t) + \Pi_3(t), \]
and
\[ \Pi_4(t) = \Pi_0^T M_0 \Pi_0^0 + \Pi_1^0 M_0 \Pi_0 + \Pi_2^0 M_2 \Pi_2 + \Pi_3^0 M_3 \Pi_3 + (N-2)(\Pi_0^T M_1 \Pi_2 + \Pi_1^T M_1 \Pi_1) \]
\[-\frac{1}{K}(\Pi_0^T f_0 + F^T \Pi_0) \]
\[ - \left( \Pi_0 (A + \frac{N^2 - 2}{AN} F) + (A^T + \frac{N^2 - 2}{AN} F^T) \Pi_0 \right) \]
\[ - \frac{1}{K}(\Pi_1^T F + F^T \Pi_1 + \Pi_2^T F + F^T \Pi_2) - \frac{1}{\sqrt{K}} Q \frac{\Pi}{\sqrt{K}}. \]
\[ \Pi_4(T) = \frac{\Gamma^T}{K} Q J \frac{\Gamma}{\sqrt{K}}. \] (A.10)

and
\[ \Pi_6(t) = \Pi_0^0 M_0 \Pi_0^0 + \Pi_1^0 M_0 \Pi_1 + \Pi_2^0 M_2 \Pi_2 + \Pi_3^0 M_3 \Pi_3 + (N-1)(\Pi_0^T M_1 \Pi_2 + \Pi_1^T M_1 \Pi_1) \]
\[-\frac{1}{K}(\Pi_0^T f_0 + F^T \Pi_0) \]
\[ - \left( \Pi_0 (A + \frac{N^2 - 2}{AN} F) + (A^T + \frac{N^2 - 2}{AN} F^T) \Pi_0 \right) \]
\[ - (A_0^T \Pi_0 + G^T \Pi_1 + (N-1)G^T \Pi_2^2) \]
\[ + \Gamma^T Q \frac{\Pi}{\sqrt{K}}. \]
\[ \Pi_6(T) = -\Gamma^T Q J \frac{\Gamma}{\sqrt{K}}, \] (A.11)

and
\[ \Pi_8(t) = \Pi_0^0 M_0 \Pi_0^0 + \Pi_1^0 M_0 \Pi_1 + \Pi_2^0 M_2 \Pi_2 + \Pi_3^0 M_3 \Pi_3 + (N-2)(\Pi_0^T M_1 \Pi_2 + \Pi_1^T M_1 \Pi_1) \]
\[-\frac{1}{K}(\Pi_0^T f_0 + F^T \Pi_0) \]
\[ - \left( \Pi_0 (A + \frac{N^2 - 2}{AN} F) + (A^T + \frac{N^2 - 2}{AN} F^T) \Pi_0 \right) \]
\[ - (A_0^T \Pi_0 + G^T \Pi_1 + (N-2)G^T \Pi_4) \]
\[ - \Gamma^T Q \frac{\Pi}{\sqrt{K}}. \]
\[ \Pi_8(T) = -\Gamma^T Q J \frac{\Gamma}{\sqrt{K}}. \] (A.12)

We have \( \Pi_3(T) = \Pi_4(T) \), and
\[ \Pi_3(t) - \Pi_4(t) = (\Pi_3 - \Pi_4)(M_1 M_1 - M_2)^2 - A) \]
\[ + (\Pi_1 M - \Pi_2^T M - A^T)(\Pi_3 - \Pi_4). \]

Therefore,
\[ \Pi_3(t) = \Pi_4(t), \quad \forall t \in [0,T]. \] (A.13)

This completes the proof. \( \square \)

**Appendix B: Proof of Theorem 4**

**Step 1.** By (A.1), (A.3) and (A.5), we determine
\[ \dot{A}_2^{TN} = A_1^{TN} M_0 A_2^{TN} + A_2^{TN} M_0 A_1^{TN} + A_0^{TN} M (A_1^{TN})^T - (A_0^{TN} g + G^T (A_0^{TN})^T) - Q_0. \] (B.1)

\[ A_2^{QN} = A_1^{QN} M_0 A_2^{QN} + A_2^{QN} (M A_1^{QN} + M A_2^{QN} - F - A) \]
\[- A_0^{TN} A_2^{QN} - A_1^{TN} F_0 + (A_0^{TN} M - G^T) A_2^{QN} \]
\[ + Q_0 - g_3^0 (1/N, M A_2^{QN}) \]
\[ \text{(B.2)} \]

\[ A_1^{QN} = (A_0^{TN})^T M_0 A_1^{QN} + A_0^{TN} M_0 A_1^{QN} - (A_2^{TN})^T F_0 \]
\[- F_0 A_2^{QN} - A_3^{QN} (A + F) - (A^T + F^T) A_3^{QN} \]
\[ + A_3^{QN} M_2^{TN} + A_2^{TN} M_3^{TN} \]
\[- (F_0 - F_0) A_2^{QN} + g_3^0 (1/N, A_3^{QN}) \]
\[ \text{(B.3)} \]

where \( A_1^{QN}(T) = Q_f, \quad A_2^{QN}(T) = -Q_f T_f, \quad A_3^{QN}(T) = \Gamma_0^T Q_f T_f \), and
\[ g_3^0 (1/N, A_2^{QN}) = -(1/N) (A_2^{QN} M A_2^{TN}) \]

For reasons of space, the expression of \( g_3^0 \) is not displayed. We further obtain
\[ \dot{A}_0^N = A_0^{TN} M_0 A_1^N - A_0^{TN} M_0 A_0^{TN} \]
\[- A_0^{TN} A_0 - A_0^{TN} A_0 - (A_0^{TN} + A_0^{TN}) G - G^T (A_0^{TN} + A_0^{TN})^T \]
\[ + A_0^{TN} M_0 A_0^{TN} + A_0^{TN} M_0 A_0^{TN} \]
\[- \Gamma^T Q_1 + g_0^1 (1/N, A_0^{TN} A_0^{TN}) \]
\[ \text{(B.4)} \]

\[ \dot{A}_0^N(T) = \Gamma^T J_1 Q_1 \Gamma_f, \]

\[ \dot{A}_1^N = A_1^{TN} A_1^N - A_1^{TN} A_1^N - Q \]
\[ + g_1 (1/N, A_1^{TN} A_1^N) \]
\[ \text{(B.5)} \]

\[ \dot{A}_2^N = A_0^{TN} M_0 A_2^{TN} - F_0 + A_0^{TN} M_0 A_2^{TN} \]
\[- A_2^{TN} (A + F) - A^T A_2^{TN} - A_2^{TN} F + A_2^{TN} M_2^{TN} \]
\[ + Q_I^2 + g_2^2 (1/N, A_2^{TN} A_2^{TN}) \]
\[ \text{(B.6)} \]

\[ \dot{A}_3^N(T) = - (1 - \Gamma^T) Q_1 J_2 \Gamma_f, \]

\[ \text{Step 2.} \quad \dot{A}_2^N = A_0^{TN} T_0 M_0 A_2^{TN} + (A_2^{TN})^T M_0 A_2^{TN} \]
\[- A_2^{TN} M_0 A_2^{TN} + A_2^{TN} M_2^{TN} + A_2^{TN} M_2^{TN} \]
\[- A_0^{TN} F_0 - F_0 A_2^{TN} - A_2^{TN} F - F_2 A_2^{TN} \]
\[- A_2^{TN} (A + F) - (A^T + F^T) A_2^{TN} - \Gamma^T Q_2 \]
\[ + g_3^0 (1/N, A_2^{TN}) \]
\[ \text{(B.7)} \]

\[ \dot{A}_3^N = A_0^{TN} T_0 M_0 A_2^{TN} + (A_2^{TN})^T M_0 A_2^{TN} \]
\[- A_2^{TN} M_0 A_2^{TN} + A_2^{TN} M_2^{TN} + A_2^{TN} M_2^{TN} \]
\[- A_0^{TN} F_0 - F_0 A_2^{TN} - A_2^{TN} F - F_2 A_2^{TN} \]
[-A_2^T (A + F) - (A^T + F^T) A_2^T - \Gamma^T Q_2 \]
\[ + g_3^0 (1/N, A_2^{TN}, A_2^{TN}) \]
where $g_0, \cdots, g_b$ are not displayed and are compactly of $O(1/N)$.

Step 2. Proof of Theorem 4.

Denote $\xi^N = (A^{ON}, A_2^{ON}, \cdots, A_b^{ON})$ for (33), and we view each of $g_0, \cdots, g_b$ as a function of $\xi^N$ with parameter $1/N$. They are all compactly of $O(1/N)$. For some $C > 0$, we further have

$$
\| \xi^N(t) \| \leq C/N.
$$

Subsequently, we view the ODE system (B.1)-(B.9) as a slightly perturbed form of (34). The remaining proof is similar to that of (Huang and Zhou, 2018b, Theorem 5) and we only give its sketch. If asymptotic solvability holds, we solve (B.1)-(B.9) for all sufficiently large $N$. By taking some increasing subsequence of population sizes $N_1 < N_2 < \cdots$, we can ensure that as $k \to \infty$, their solutions $\{\xi^N_k(t), k = 1, 2, \cdots\}$ have a limit as a vector function on $[0, T]$ which satisfies the limit ODE (34) on $[0, T]$. Conversely, if (34) has a solution on $[0, T]$, there exists $N_0 > 0$ such that (B.1)-(B.9) has a solution on $[0, T]$ for all $N \geq N_0$; all these solutions are uniformly bounded. Accordingly we obtain $(P_{0}, \cdots, P_N)$ to satisfy (30) for all $N \geq N_0$. So asymptotic solvability holds.

\[\square\]

Appendix C

In view of (21) and (24), by Proposition 1 we have

$$
\begin{align*}
\theta_0(t) &= -A_0^T \eta_0 - N \hat{G}^T \theta_0 + \Pi Q_0 \theta_0 + N \hat{M} \theta_0 \\
&\quad + N \hat{P} \theta_1 + Q_0 \eta_0, \\
\theta_1(t) &= -\hat{F}_{\hat{N}}^T \theta_0 - \hat{A} \theta_1 + \hat{F}^T \theta_0 + \Pi Q_0 \theta_0 + N \hat{M} \theta_0 \\
&\quad + (N-1) \Pi \hat{M} \theta_0 + N \hat{P} \theta_1 - \hat{F}_{\hat{N}} Q_0 \eta_0, \\
\end{align*}
$$

where

$$
\theta_0(T) = -Q_0 \eta_0, \quad \theta_1(T) = -\hat{F}_{\hat{N}} Q_0 \eta_0,
$$

and

$$
\begin{align*}
\hat{\theta}_0(t) &= -A_0^T \theta_0 - G^T \theta_1 - (N-1) G^T \theta_0 + \Pi Q_0 \theta_0 \\
&\quad + \Pi Q_0 \theta_0 + N \hat{M} \theta_0 + (N-1) \Pi \hat{M} \theta_0 \\
&\quad + (N-1) \Pi \hat{M} \theta_0 - \hat{F}_{\hat{N}} Q_0 \eta_0, \\
\hat{\theta}_1(t) &= -\hat{F}_{\hat{N}}^T \theta_0 - (\hat{A}^T + \hat{F}^T) \theta_1 - (N-1) \hat{F}^T \theta_0 + \Pi Q_0 \theta_0 \\
&\quad + \Pi Q_0 \theta_0 + N \hat{M} \theta_0 + (N-1) \Pi \hat{M} \theta_0 \\
&\quad + (N-1) \Pi \hat{M} \theta_0 - \hat{F}_{\hat{N}} Q_0 \eta_0, \\
\hat{\theta}_2(t) &= -\hat{F}_{\hat{N}}^T \theta_0 - \hat{F}^T \theta_1 - \hat{F} T \theta_0 - (N-1) \hat{F} T \theta_0 + \Pi Q_0 \theta_0 \\
&\quad + \Pi Q_0 \theta_0 + N \hat{M} \theta_0 + \Pi \hat{M} \theta_0 + (N-2) \Pi \hat{M} \theta_0 \\
&\quad + (N-1) \Pi \hat{M} \theta_0 - \hat{F}_N Q_0 \eta_0,
\end{align*}
$$

where

$$
\begin{align*}
\theta_0(T) &= \Gamma_T Q_0 \eta_0, \quad \theta_1(T) = -(I - \hat{F}_{\hat{N}} Q_0) \eta_0, \\
\theta_2(T) &= \Gamma_T Q_0 \eta_0.
\end{align*}
$$

By (C.1), we have the relation

$$
\begin{align*}
\hat{\alpha}_0^{ON} &= (A_0^{ON} M_0 - A_0^T) \alpha_0^{ON} + (A_0^N M - G^T) \alpha_1^{ON} \\
&\quad + A_2^{ON} \alpha_1^N + Q_0 \eta_0, \\
\alpha_1^{ON} &= (A_1^{ON})^T M_0 - \Pi Q_0 \alpha_1^{ON} \\
&\quad + (A_1^N M + A^{-T}_N M - A^T - F^T) \alpha_1^{ON} \\
&\quad + A_2^{ON} \alpha_1^N - \Gamma_T Q_0 \eta_0 + h_1(1/N, \alpha_1^{ON}, \alpha_1^N),
\end{align*}
$$

where $\alpha_0^{ON}(T) = -Q_0 \eta_0$, $\alpha_1^{ON}(T) = \Gamma_T Q_0 \eta_0$, and

$$
\begin{align*}
\alpha_0^N &= (A_0^N M_0 - A_0^T) \alpha_0^N + (A_0^N M - G^T) \alpha_1^N \\
&\quad + (A_2^N M - G^T) \alpha_1^N + \alpha_1^N M_0 \alpha_0^{ON} - \Gamma_T Q \eta \\
&\quad + h_0(1/N, \alpha_1^N, \alpha_1^N, \alpha_0^N), \\
\alpha_1^N &= A_2^N M_0 \alpha_0^{ON} - A^T \alpha_1^N + (A_1^N + A_2^T M - G^T) \alpha_1^N \\
&\quad + h_1(\alpha_1^N, \alpha_0^N, \alpha_1^N, \alpha_2^N, \alpha_0^N, \alpha_1^N), \\
\alpha_2^N &= (A_2^{ON})^T M_0 - \Pi Q_0 \alpha_2^N + ((A_3^N + A_2^T M - G^T) \alpha_1^N \\
&\quad + (A_2^N + A_2^T M - G^T) \alpha_1^N + A_2^T M_0 \alpha_0^{ON} \\
&\quad - \Gamma_T Q \eta + h_2(\alpha_1^N, \alpha_1^N, \alpha_0^N, \alpha_1^N),
\end{align*}
$$

where the terminal condition is

$$
\begin{align*}
\alpha_0^N(T) &= \Gamma_T Q_0 \eta_0, \quad \alpha_1^N(T) = -(I - \hat{F}_{\hat{N}} Q_0) \eta_0, \\
\alpha_2^N(T) &= \Gamma_T Q_0 \eta_0,
\end{align*}
$$

and $h_0$, $h_1$, $h_2$ are compactly of $O(1/N)$.
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