The $G_{\text{Newton}} \to 0$ Limit of Euclidean Quantum Gravity

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Abstract

Using the Ashtekar formulation, it is shown that the $G_{\text{Newton}} \to 0$ limit of Euclidean or complexified general relativity is not a free field theory, but is a theory that describes a linearized self-dual connection propagating on an arbitrary anti-self-dual background. This theory is quantized in the loop representation and, as in the full theory, an infinite dimensional space of exact solutions to the constraints are found. An inner product is also proposed. The path integral is constructed from the Hamiltonian theory and the measure is explicitly computed nonperturbatively, without relying on a semiclassical expansion. This theory could provide the starting point for a new approach to a perturbation theory in $G_{\text{Newton}}$ that does not rely on a background field expansion and in which full diffeomorphism invariance is satisfied at each order.
1 Introduction

The starting point for all contemporary work on the problem of quantum gravity is the non-renormalizability of the standard perturbation theory which is understood as an expansion, in powers of $l_{\text{plank}} = \sqrt{\hbar G / c^3}$, of the metric $g_{\mu\nu}$ around the flat metric $\eta_{\mu\nu}$. In this semiclassical approach to perturbation theory, the quantum field theory of gravitons is defined by making a shift in the definition of the field operator

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + l_{\text{plank}} \hat{h}_{\mu\nu} \quad (1)$$

and then constructing a quantum field theory for the deviations $\hat{h}_{\mu\nu}$ from the fixed classical background $\eta_{\mu\nu}$.

Formulated this way, the $G_{\text{Newton}} \to 0$ limit of quantum gravity is just a free field theory. However, in this standard formulation, the limit $G_{\text{Newton}} \to 0$ has two troublesome features. First, the interpretation of the gauge symmetry as diffeomorphism invariance is lost. Second, properties of the background spacetime which do not extend to the full theory, such as its Poincare invariance, determine the operator ordering prescription used in defining the free field theory around which the perturbation theory is constructed.

The purpose of this paper is to show that if we do not base the quantization on the shifted field (1) there is a different $G_{\text{Newton}} \to 0$ limit of quantum gravity that, in the Euclidean case, is not a free field theory. It is instead an interacting, chirally asymmetric, theory that retains the exact diffeomorphism invariance of the full theory. The significance of this result is that this formulation could be the starting point of a new approach to perturbation theory that is completely quantum mechanical in that it does not rely on a semiclassical expansion around a classical background metric and so retains, at each order, the full diffeomorphism invariance of general relativity.

More particularly, I will show first that, for the the Euclidean or complexified classical theory, there is a $G_{\text{Newton}} \to 0$ limit of general relativity that consists of the full anti-selfdual, or left handed, sector of the theory, together with a linearized $U(1)^3$ connection. These latter fields may be interpreted as the linearization of the self-dual part of the Weyl tensor, that propagates on the anti-selfdual solution. This chirally asymmetric result follows from taking the $G_{\text{Newton}} \to 0$ limit within the new-variables formulation of Ashtekar\(^1\). It shows that what $G_{\text{Newton}}$ measures is the coupling between the left and right handed degrees of freedom of the theory.

This theory, which I will call the "googly" theory, as it is reminiscent of the googly\(^2\) problem of Penrose’s twistor program\(^3\), is not a free field theory, but it may be an integrable system. As I will show in section 3 it can be quantized

\(^1\)For the linearization of the Ashtekar formalism, which is not equivalent to the theory described here, see \(^4\).

\(^2\)
using the loop representation which allows us to construct exact solutions to the quantum constraints, as in the case of the full theory. Furthermore, in the case of the googly theory, the quantization overcomes several obstacles that are still unresolved in the full theory, in that a larger set of solutions to the constraints may be immediately obtained and a finite dimensional subalgebra of the physical observables can be found by inspection.

Finally, the paper closes with a new definition of the Euclidean path integral in which the measure is determined nonperturbatively from the canonical theory. It is proposed that this integral, rather than the free field theory, should be the starting point for a perturbative analysis of quantum gravity.

2 The Classical Theory

In the Ashtekar formulation, Newton’s constant (which we will from now on call $G$) appears only in association with the self-dual connection $A_a$ as in

$$F_{ab} = \partial_a A^i_b - \partial_b A^i_a + G\epsilon^{ijk} A^j_a A^k_b, \quad D_a \tilde{E}^b_i = \partial_a E^{bik} + G\epsilon^{ijk} A^j_a \tilde{E}^{bk}$$

(2)

The googly theory is arrived at by simply setting $G = 0$ in all such expressions, while keeping $A_a$ and $\tilde{E}^{a}_{b}$ fixed. (It will be useful to recall that $A^a_{i}$, has dimension of $(length)^{-3}$ while $\tilde{E}^{a}_{i}$, the conjugate momenta to $A^a_{i}$, is dimensionless.) No assumption that $\tilde{E}^{a}_{i}$ is close to a flat, or any other, background metric is made.

We may in fact begin by setting $G = 0$ in the action for general relativity in the self-dual form\[10\], which then becomes,

$$S = \int \epsilon^{\mu\nu\alpha\beta} e^\mu_i e^\nu_j 4 f^{ij}$$

where $4 f^{ij}_\mu = \partial^\mu 4 A_i^j - \partial^\mu 4 A^i_j$ and the $e^i_\mu$ and $4 A_i^j$ are 1-forms on spacetime. $A_i^j_\mu$ is the self-dual connection, which satisfies $A_i^j_\mu = \frac{1}{4} \epsilon^{ijkl} A_i^k_\mu$. For the moment, all fields will be assumed to be complex, reality conditions will be discussed shortly.

The Hamiltonian analysis may be performed as usual\[10, 3, 3, 14\] on a three surface $\Sigma$, leading to canonical variables $A_i^a$, which are three one forms on $\Sigma$, and $\tilde{E}^{a}_{j}$, which are three vector densities on $\Sigma$, with Poisson brackets,

$$\{A_i^a(x), \tilde{E}^{b}_{j}(y)\} = \delta^b_a \delta^{ij} \delta^3(x, y).$$

(4)

The constraints of the theory are

$$g^i = \partial_a \tilde{E}^{a}_{i}$$

(5)

$$C_a = \tilde{E}^{a}_{i} f^{i}_{ab}$$

(6)
\[ C = \epsilon^{ijk} \tilde{E}^{ai} \tilde{E}^{bj} f^k_{ab} \]  
(7)

with \( f^k_{ab} = 2\partial_{[a} A^k_{b]} \). We may note that the \( g^i \) generate a \( U(1)^3 \) internal gauge group, thus setting \( G \) to zero may be thought of as a contraction in which \( SU(2) \to U(1)^3 \). The remaining constraint algebra is identical to that of the Ashtekar formalism. The equation of motion are then deduced from the Hamiltonian

\[ H = \int_{\Sigma} NC. \]  
(8)

(we do not consider here boundary conditions or the boundary terms which appear in the asymptotically flat case\[1, 2, 3\].) If we set \( V^{ai} = N \tilde{E}^{ai} \) and choose \( N \) such that \( \partial_a N = 0 \) the equations of motion are

\[ \dot{V}^{ai} = \epsilon^{ijk} [V^j, V^k]^a \]  
(9)

\[ \dot{A}^{ai} = \epsilon^{ijk} V^{jk} f^k_{ab} \]  
(10)

Note that the \( V^{ai} \) decouple. They are three divergence free vector fields on \( \Sigma \) which evolve according to (9). This is exactly the anti-self dual sector of the theory, as is easily seen from the fact that \( F^a_{ab} \), which is the self-dual part of the Weyl tensor, vanishes\[12\]. The \( A^{ai} \) then satisfy linear constraint and evolution equations in the presence of the anti-selfdual background \( V^{ai} \). Because \( f^k_{ab} = 2\partial_{[a} A^k_{b]} \) is the linearization of the self-dual Weyl tensor we see that the \( A^{ai} \) can be interpreted as the first order selfdual fields propagating on exact anti-selfdual metrics. However, it should be emphasized that they do not satisfy the linearization of Einstein’s equations around the anti-self dual backgrounds, setting \( G = 0 \) in the Einstein equations is not the same thing as linearizing them\[2\].

Note that all these equations are well defined even when \( \tilde{E}^{ai} \) are degenerate. Degenerate solutions to the full theory have been studied in \[11\] and it is easy to see that there are degenerate solutions in this case as well. Around non-degenerate solutions the theory has 4 phase space degrees of freedom, which can be seen by counting as well as by expanding around flat space \( \tilde{E}^{ai} = \delta^{ai} + h^{ai} \). We may then fix symmetric traceless transverse gauge for \( h^{ai} \) and \( A^{ai} \) and taking the fourier transform, following treatements of the linearized theory\[4, 5\] write the physical degrees of freedom as,

\[ h^{ai}(p) = h^+(p)m^a(p)m^i(p) + h^-(p)\tilde{m}^a(p)\bar{m}^i(p) \]  
(11)

\[ A^{ai}(p) = A^+(p)m^a(p)m^i(p) + A^-(p)\tilde{m}^a(p)\bar{m}^i(p) \]  
(12)

\footnote{I would like to thank Abhay Ashtekar, Carlos Kozenmeh, Carlo Rovelli and Joseph Samuel for a discussion that clarified this point and in particular for the remark that this result is not in conflict with the theorem of Samuel\[5\] on the non-existence of purely self-dual perturbations.}
where the $m(p)^a$ are the complex polarization vectors that satisfy $p^a m_a = 0$, $m_a m^a = 0$ and $\tilde{m}^a = 1$. From (9) and (10), the physical modes evolve as,

$$\dot{h}_\pm = \pm |p| h_\pm$$

$$\dot{A}_\pm = \mp |p| A_\pm$$

So far, we have been using complex fields, to define the Euclidean or the Minkowskian case we have to impose reality conditions. To define the Euclidean theory, we impose the conditions

$$\tilde{E}^{ai} = \tilde{E}^{ai}$$

$$A_{ai} = \tilde{A}_{ai}$$

so that in the nondegenerate case we have 4 real phase space degrees of freedom per point. Full Euclidean general relativity may now be constructed by a power series in $G$, by writing

$$\tilde{E}^{ai} = \tilde{E}_{0}^{ai} + G \tilde{E}_{1}^{ai} + G^2 \tilde{E}_{2}^{ai} + ...$$

$$A_{ai} = A_{0}^{ai} + GA_{1}^{ai} + ...$$

and continuing the expansion of the constraints and equations of motion.

The situation is rather different in the Minkowskian theory because there the reality conditions involve $G$. The Minkowskian reality conditions of the full theory are that $\tilde{q}^{ab}$ and its time derivative should be real. The latter condition can be expressed as,

$$G(A_{ai} + \tilde{A}_{ai}) = 2 \Gamma(\tilde{E})_{ai}$$

where $\Gamma(\tilde{E})_{ai}$ is the three dimensional Christofel connection of the frame fields $\tilde{E}^{ai}$. If we take $G$ to zero with $A_{ai}$ and $\tilde{E}^{ai}$ held fixed we see that $\Gamma(\tilde{E})_{ai} = 0$, which means the spatial geometry is flat. This can be seen also by another route, in which we use the fact that the second reality condition (19) expresses the fact that the time derivative of $\tilde{q}^{ab}$ should be real. To work this out we must recall that in the Minkowskian theory the Poisson brackets (4) and (9) are multiplied by $i$ (or alternately, we take $\tilde{q}^{ab}$ to be negative definite so that the $E^{ai}$ are pure imaginary.) We then find that

$$\frac{d\tilde{q}^{ab}}{dt} = iN^{-2} \epsilon_{ijk} [V_j, V_k]^{(a} V_i^{b)}$$

so that the requirement that this be real leaves us with $\epsilon_{ijk}[V_j, V_k]^{(a} V_i^{b)} = 0$, which implies, in the nondegenerate case, both that the metric of three space is flat and that its time derivative is zero. This is just the well known fact that Minkowski spacetime is the only real self-dual solution of the Einstein equations.
The $A_{ai}$ are then not restricted by the Minkowskian reality conditions of the $G \to 0$ limit of the theory we have considered here, and they then carry four real phase space degrees of freedom. We may note that these dynamics are different from the linearization of the theory around flat spacetime, in which (19) couples the linear fluctuations in $A_{ai}$ to the linear fluctuations in the metric around a background. One way to see this is to consider a different limit of the theory in which we first expand $\tilde{E}^{ai} = \delta^{ai} + G h^{ai}$, and then take the $G \to 0$ limit. We then see that to first order the Minkowskian reality conditions mix $A_{ai}$ and $h_{ai}$.

For the physical modes they are

$$\tilde{h}_{\pm}(p) = h_{\pm}(-p)$$

$$G(A_{\pm}(p) + \tilde{A}_{\pm}(p)) = \mp 2G |p|h_{\pm}(p)$$

Consistent expansion of the constraints, equations of motion and the reality conditions in power of $G$ then reproduces the usual perturbation expansion.

Finally, we may note that for this googly theory it is immediately possible to write down physical observables, which are also constants of motion for the compact case. The reader may verify that

$$Q = \int_{\Sigma} \tilde{E}^{ai} A_{ai}$$

and

$$Q^i = \int_{\Sigma} \epsilon^{ijk} \tilde{E}^{aj} A_{ak}$$

commute with the constraints. They generate the global scale transformations and $SO(3)$ rotations, respectively. It is interesting to note that these are also observables for the opposite, strong-coupling, limit of the theory. If the theory is integrable, it may be possible to write down an explicit infinite dimensional algebra of observables, but this has not yet been done.

3 The Quantum Theory

A natural set of variables to use to quantize the theory are the loop observables, which are invariant under the internal gauge transformations. (For background and applications of the loop representation, see [8, 9, 14, 15, 16, 17].) The $U(1)^3$ loop observables have already been studied for linearized gravity, they are:

$$l^k[\gamma] \equiv e^{i l} \int_{\Sigma} A^{a}_{k}[\gamma(s)] \bar{\gamma}^{a}(s) ds$$

where $l$ is a length which is not related to $G_{\text{Newton}}$, and is only included because $A^{a}_{k}$ has dimensions of $(\text{length})^{-3}$. We may note that, unlike the case of linearized
gravity, in this model the frame fields are invariant under internal gauge transformations.

The Poisson algebra is
\[
\{ \tilde{E}^{ai}(x), t^k[\gamma] \} = -i^2 \Delta^a[x, \gamma] \delta^{ik} t^k[\gamma] \tag{26}
\]
where
\[
\Delta^a[x, \gamma] = \int ds \delta^3(x, \gamma(s)) \dot{\gamma}^a(s). \tag{27}
\]

Finally, we may note that \( t^k[\gamma] \) of a trivial loop is equal to 1.

The quantization is performed on the triple nonparametric loop space \((\mathcal{H}L_1)^3\), which is the same space on which the loop representation for linearized gravity is constructed. We first construct the nonparametric loop space \(\mathcal{L}^1\) (which is also called the \(U(1)\) holonomic loop space) which is the space of sets of parametrized, loops on \(\Sigma\) on which we have imposed the following equivalence relations: i) invariance under monotonic reparametrizations, ii) for two loops \(\alpha\) and \(\beta\) that have a common base point the set \(\alpha \cup \beta \approx \alpha \circ \beta\), iii) for a loop \(\alpha\) and an open line \(\eta\) with a common base point \(\alpha \circ \eta \circ \eta^{-1} \approx \alpha\). We call this space \(\mathcal{L}^1\), as it incorporates all of the relations among \(U(1)\) holonomies. We then consider the elements of the product \((\mathcal{L}^1)^3\), which we will denote \(\vec{\alpha} = \{\alpha_1, \alpha_2, \alpha_3\}\).

We then quantized the theory by defining the (unphysical) Hilbert space, \(\mathcal{S}\), consisting of complex functions \(\psi[\vec{\alpha}]\) on \(\mathcal{L}^1\), with the inner product
\[
\langle \Psi | \chi \rangle = \int d\mu[\alpha_1] d\mu[\alpha_2] d\mu[\alpha_3] \bar{\Psi}[\vec{\alpha}] \chi[\vec{\alpha}] \tag{29}
\]
where \(d\mu[\alpha]\) is defined to be the discrete measure on \(\mathcal{L}^1\). Thus, a state is normalizable only if it is nonzero on a countable set of loops. We may note that this inner product is introduced only to define the kinematical quantization. The physical states, which are those in the kernal of the constraints, will not be normalizable with respect to it.

We may now define the operators,
\[
\hat{t}^k[\gamma] \psi[\vec{\alpha}] = \psi[\vec{\alpha} \cup_k \gamma] \tag{30}
\]
where \(\vec{\alpha} \cup_1 \gamma = \{\alpha_1 \cup \gamma, \alpha_2, \alpha_3\}\) (and similarly for \(k = 2, 3\)) and
\[
\hat{E}^{ak}(x) \psi[\vec{\alpha}] = i\hbar \Delta^a[x, \alpha_k] \psi[\vec{\alpha}] \tag{31}
\]
The reader may verify that the commutator algebra of these operators is \(i\hbar\) times the corresponding Poisson brackets and that \(\hat{E}^{ak}(x)\) is hermitian and
\( t^k[\gamma] \) is unitary with respect to the inner product (29). Thus we see that the reality conditions are satisfied by this choice of the inner product.

As in full general relativity\(^8\), the diffeomorphism constraint (6) can be represented by

\[
D(v)\Psi[\alpha_1, \alpha_2, \alpha_3] = \frac{d}{dt}\Psi[\phi_t \circ \alpha_1, \phi_t \circ \alpha_2, \phi_t \circ \alpha_3]
\]  

(32)

where \( v \) is the vector field which generates the one parameter family of diffeomorphisms \( \phi_t \). The diffeomorphism invariant states, which satisfy

\[
D(v)\Psi[\vec{a}] = 0
\]

(33)

are

\[
\Psi[\alpha_1, \alpha_2, \alpha_3] = \Psi[L(\alpha_1, \alpha_2, \alpha_3)]
\]  

(34)

where \( L(\alpha_1, \alpha_2, \alpha_3) \) are the diffeomorphism equivalence classes of \( \mathcal{H}L_1 \), which we will call generalized, labeled, link classes.

One may also find a large subset of the solution space of the Hamiltonian constraint (7). Following methods developed in\(^{18,8}\) one easily shows that the solution space to the Hamiltonian constraint includes the following three types of labeled graphs:

1) All characteristic functions of labeled loops with intersections in which loops of only one type meet.

2) All superpositions of labeled loops of the form

\[
|A> = A_{i_1...i_n} t^{i_1}[\alpha_1]...t^{i_n}[\alpha_n]|0>
\]

(35)

where \( |0> \) is the state that has support (and is equal to unity) only on trivial loops and the coefficients \( A_{i_1...i_n} \) are symmetric in the \( i \) and \( j \) indices any time the loops \( \alpha_i \) and \( \alpha_j \) have a point of intersection at which they are non-parallel.

3) Any intersections in which loops with different labels exchange tangent vectors is allowed.

It is not presently known if these represent all of the solutions to the Hamiltonian constraint, this problem is under investigation. The physical state space then consists of functions on \( L_c[\alpha_1, \alpha_2, \alpha_3] \), which we define to be the diffeomorphism equivalence classes of labeled knotted graphs consisting of loops which satisfy the Hamiltonian constraint at each intersection. We will call these the hamiltonian knotted graphs.

As usual, the physical states are not normalizable in the unphysical inner product (29), and a new inner product needs to be introduced on the physical state space. The most natural candidate for a physical inner product is

\[
<\Psi|\chi> = \sum_{L_c} \bar{\Psi}[L_c]|\chi[L_c]
\]

(36)
where $L_c$ are the Hamiltonian Knotted graphs. This form must be considered a conjecture, which is to be verified by showing that it satisfies the reality conditions for the physical observables as described in [3, 4, 8, 9]. That this may be the case is suggested from the fact that it can be constructed from (29) by a formal operation in which the amplitude for each loop is normalized by the volume of its orbit under diffeomorphisms in the discrete measure $d\mu[\alpha]$.

This completes the Hamiltonian quantization of the system. The states are given by the functions on the Hamiltonian knotted graphs, normalizable under the inner product (36). As in the full theory, the key problem remains the construction of the physical observables, and the verification of the conjecture for the form of the inner product. In particular, it is a nontrivial problem to construct the known physical observables (23) and (24) on this Hilbert space. This suggests the physical interpretation of the theory. Finally, we may also note that the physical meaning of a Hilbert space associated with the Euclidean quantum theory is obscure. This problem is certainly connected with the general problem of time in quantum cosmology [20].

Finally, it would be interesting to try to extend these methods to the Minkowskian case. The only difference at the Hilbert space level is that the inner product must now be chosen to implement the Minkowskian reality conditions. This is a non-trivial problem, as, at least for the nondegenerate sector, the Minkowskian reality conditions reduce the classical field theory to a topological field theory with no local degrees of freedom. To carry out this program will then be to raise the important issue of the role of the degenerate configurations in the quantum theory.

4 The Euclidean path integral

We may note that for the Googly theory, as well as for the full theory, the Hamiltonian and diffeomorphism constraints may be solved using the method of Capovilla, Dell and Jacobson [21]. This allows us to construct the measure for the path integral non-perturbatively, resolving an issue that has been the subject of some discussion [22]. The path integral may be derived from the Hamiltonian formulation in the usual way [23]. The expression for the partition function is,

$$Z = \int [dA_{\alpha}] [d\tilde{E}^{b}] e^{-\int d^4x \tilde{E}^{ab} \dot{A}_{ab} \prod_x \delta[C(x)] \delta[C'_a(x)] \delta[g'^l(x)] \delta(\xi) \Delta_{F-P}}$$

(37)

where $\xi$ represents seven gauge fixing conditions per point and $\Delta_{F-P}$ is the Faddeev-Popov determinant [24]. The solutions to the Hamiltonian and dif-

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3 We may note that this issue arises in the $2 + 1$ case, however there it is resolved by the fact that the degenerate and nondegenerate metrics are linked by the gauge transformations of the theory (See [23]). This is the case for the $3 + 1$ theory for some, but not all degenerate metrics.
Feomorphism constraints are given, just as in the full theory, by
\[ \tilde{E}^{ai} = [\phi^{-1}]^i_j \tilde{B}^{aj} \] (38)
where \( \tilde{B}^{aj} = \epsilon^{abc} f^i_{bc} \) and \( [\phi^{-1}]^i_j = \phi^i_k \phi^k_j - 1 / 2 \delta^i_j (\phi^k_k)^2 \). We may note that this solution fails to hold for cases where \( \det(\tilde{B}^{ai}) \neq 0 \), but as these are of measure zero in the original canonical measure in (40), this should not affect the path integral. Using these solutions to eliminate the Hamiltonian and diffeomorphism constraints, and introducing \( A^\mu_a \) as a lagrange multiplier to exponentiate the Gauss’s law constraint we find that
\[
Z = \int [dA^\mu_a][d\phi^i_j] J[A,\phi] e^{\int d^4x \epsilon^{\mu\nu\alpha\beta} f_{\mu\nu\alpha} A^\mu \partial \phi^\nu \partial \phi^\alpha} \right] \Delta F - P
\]
where \( A^\mu_a \) is now the spacetime connection and \( J[A,\phi] \) is the determinant that comes from solving the constraints, which is,
\[
J[A,\phi] = \prod_x \det \left[ \frac{\partial \tilde{E}^{ai}(x)}{\partial \phi^j_m} \right] = \prod_x \det \left[ - (\phi^i_k \phi^m_j) \tilde{B}^{aj} \right] = \prod_x [-(\det \tilde{B})^3 (\det \phi)^6] \]
(40)
The definition of the path integral is then completed by specifying a choice of the seven gauge fixing functions \( \xi \), and by introducing ghosts to exponentiate the resulting determinants. The evaluation of this path integral will be discussed in a future publication, for the present, we may note the following points.

1) With a choice of gauge fixing functions which is linear in \( A^\mu_a \), the path integral is Gaussian in \( A^\mu_a \), if we use ghosts to exponentiate the determinants in \( J[A,\phi] \) and \( \Delta F - P \). After the Gaussian integral is done one is left with a path integral over \( \phi^i_j \), and the ghost fields. As it comes from a theory whose classical solutions are the self-dual metrics (on which are propagating the linearized antiself-dual modes), it seems possible that this expression may be interesting from a topological point of view, in view of the role that self-dual connections have recently played in topology.

2) The path integral (43) is completely diffeomorphism invariant, unlike the \( G \to 0 \) limit of the standard perturbative path integral. Thus it seems a better candidate on which to base a perturbation theory in \( G \) than the standard perturbative definition, in which the leading order term looses the gauge symmetry of the full theory.

3) The degenerate metrics found in the Ashtekar formulation are explicitly taken into account in the formulation of (43), and seem to cause no particular problems. This is interesting on account of the role that the degenerate metrics play in the solvable formulation of 2+1 quantum gravity invented by Witten[25].

4) As the basic variable is a connection, one may attempt to use (43), or its extension to the full theory, as the starting point for a Monte-Carlo evaluation of the path integral for quantum gravity. The principle technical problem
to be faced is that the use of fermionic variables seems essential, in order to include correctly the Faddeev-Popov determinants that are necessary to correctly handle the diffeomorphism invariance.

5) Finally, the action in the exponential (43) is not manifestly bounded. However, at least in the asymptotically flat case, in which the Hamiltonian is bounded from below, the path integral should be convergent as the measure we have constructed implements the constraints that insure positive energy [29].

6) Essentially the same construction can be used to construct the path integral for the full theory. This, and the issue of the convergence of these path integrals, will be discussed elsewhere.

5 Conclusions

The googly theory is a reduction of general relativity that may provide an interesting tool for studying problems in both classical and quantum gravity. At the classical level, it is interesting to have a Hamiltonian system whose solution space is essentially the self-dual metrics [26]. This may play an interesting role in studies of the geometry and topology of self-dual spacetimes [12, 27]. In addition, the speculations that the self-dual sector may be integrable can now be pursued within a Hamiltonian framework.

At the quantum level, the googly theory provides a diffeomorphism invariant quantum field theory that is non-trivial, but still simpler than quantum gravity. Its quantization seems to be interesting in both the canonical and path integral frameworks. For example, as more can be learned about the physical observables in this case it becomes possible to test the conjecture [3, 8] that the physical inner product is determined by reality conditions on the physical observables. Furthermore, if the classical theory is integrable than there then ought to be an infinite dimensional algebra of physical observables, or constants of the motion. If they can be constructed, then one may attempt to define their action in the quantum theory in terms of the representation defined here, or more directly via Isham’s group quantization program [28].

However, what is even more interesting is the possibility that this theory provides a starting point for a new definition of perturbative quantum gravity in which exact diffeomorphism invariance is respected at every order. Such a perturbation theory would be purely quantum mechanical, rather than semi-classical, in that no shift of the operators around a purely classical background is made—an assumption that is surely suspect at the Planck scale. Instead, the zero’th order states and observables must be exactly diffeomorphism invariant. This may be expected to have profound consequences for the problem of the ultra-violet divergences.
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