Generalized Clustering Conditions of Jack Polynomials at Negative Jack Parameter $\alpha$

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We present several conjectures on the behavior and clustering properties of Jack polynomials at negative parameter $\alpha = -\frac{1}{2}$, of partitions that violate the $(k,r,N)$ admissibility rule of Feigin et. al. We find that "highest weight" Jack polynomials of specific partitions represent the minimum degree polynomials in $N$ variables that vanish when $s$ distinct clusters of $k+1$ particles are formed, with $s$ and $k$ positive integers. Explicit counting formulas are conjectured. The generalized clustering conditions are useful in a forthcoming description of fractional quantum Hall quasiparticles.

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I. INTRODUCTION

The Jack polynomials are a family of symmetric homogeneous multivariate polynomials characterized by a dominant partition $\lambda$ and a rational number parameter $\alpha$. They appeared in physics in the context of the Calogero-Sutherland model [2] for positive coupling $\alpha$.

In a recent paper, Feigin et. al. [1] initiated the study of Jack polynomials (Jacks) in $N$ variables, at negative rational parameter $\alpha_{k,r} = -\frac{k+1}{r-1}$ ($k+1$, $r-1$ relatively prime), of certain $(k,r,N)$-admissible partitions $\lambda$ : $\lambda_i - \lambda_{i+k} \geq r$, by proving they form a basis of a differential ideal $I_{N}^{k,r}$ in the space of symmetric polynomials. They showed that the set of Jacks with parameter $\alpha_{k,2}$ and $(k,2,N)$-admissible $\lambda$ are a basis for the space of symmetric homogeneous polynomials that vanish when $k$ variables $z_i$ coincide. In [3], we found that these Jacks naturally implement a type of “generalized Pauli principle” on a generalization of Fock spaces for abelian and non-abelian fractional statistics [4]. We found that (bosonic) Laughlin, Moore-Read, and Read-Rezayi Fractional Quantum Hall (FQH) wavefunctions (as well as others, such as the state Simon et al. [5] have called the “Gaffnian”) can be explicitly written as single Jack symmetric polynomials. We identified $r$ as the minimum power (cluster angular momentum) with which the admissible Jacks vanish as a cluster of $k + 1$ particles come together.

In [3] we adopted a physical perspective to the Jack problem and uniquely obtained the abelian and non-abelian FQH ground states and the admissibility rule on partitions by imposing, on an arbitrary Jack polynomial, a highest and lowest weight condition common in FQH studies on the sphere [6]. However, in imposing only the highest weight condition on the Jacks we obtained another infinite series of polynomials of partitions which violate the Feigin et. al. admissibility rule. The $(k,r,N)$-admissible configurations of [1] do not exhaust the space of well-behaved Jack polynomials at negative $\alpha_{k,r}$. We find an infinite series of Jack polynomials, with partitions characterized by a different integer $s$ which are still well-behaved at negative rational $\alpha_{k,r}$. This paper is devoted to analyzing the clustering properties and the counting of such polynomials. We note that these polynomials can be interpreted as lowest Landau level (LLL) many-body wavefunctions and have applications in the construction of FQH quasiparticle excitations [7].

We obtain a characterization of the symmetric polynomials in $N$ variables $P(z_1, z_2, ..., z_N)$ satisfying the following set of generalized clustering (vanishing) conditions: (i) $P(z_1, z_2, ..., z_N)$ vanishes when we form $s$ distinct clusters, each of $k+1$ particles, but remains finite when $s-1$ distinct clusters of $k+1$ particles are formed and (ii) $P(z_1, z_2, ..., z_N)$ does not vanish when a large cluster of $s(k+1)-1$ particles is formed. $s$ and $k$ are integers greater or equal to 1 and the case $s = 1$ is the case described by Feigin et. al. [1]. More precisely, let $F$ be the space of all polynomials satisfying the clustering condition:

$$P(z_1 = ... = z_{k+1}, z_{k+2} = ... = z_{2(k+1)}, ..., z_{(s-1)(k+1)+1} = ... = z_{s(k+1)}, z_{s(k+1)+1}, z_{s(k+1)+2}, ..., z_N) = 0 \quad (1)$$

Let $F_1$ be the space of all polynomials satisfying both Eq.(1), and the clustering condition $P(z_1 = ... = z_{s(k+1)-1}, z_{s(k+1)-1}, z_{s(k+1)+1}, ..., z_N) = 0$. In this paper we look at the coset space $F/F_1$ and focus on two problems: (1) We find the generators of the ideal above, which are the minimum degree polynomials (for $N$ particles) satisfying the clustering conditions above, and (2) We give a (conjectured) analytic expression for the number
of linearly independent polynomials in $N$ variables, of momentum (total degree of the polynomial) $M$ and of flux (maximum power in each variable) $N \Phi$ that span the coset $F/F_1$. Our motivation is to find new properties of Jack polynomials at negative $\alpha$ which are applicable to the study of FQH quasiparticles [5]. Our results are also related to the Cayley-Sylvester problem of coincident loci [3,4].

II. PROPERTIES OF JACK POLYNOMIALS

The Jacks $J^\alpha_N(z)$ are symmetric homogeneous polynomials in $z \equiv \{z_1, z_2, \ldots, z_N\}$, labeled by a partition $\lambda$ with length $\ell_\lambda \leq N$, and a parameter $\alpha$; the partition $\lambda$ can be represented as a (bosonic) occupation-number configuration $n(\lambda) = \{n_m(\lambda), m = 0, 1, 2, \ldots\}$ of each of the lowest Landau level (LLL) orbitals with angular momentum $L_z = \hbar k$ (see Fig.1), where, for $m > 0, n_m(\lambda)$ is the multiplicity of $m$ in $\lambda$. When $\alpha \to \infty$, $J^\alpha_N \to m_N$, which is the monomial wavefunction of the free boson state with occupation-number configuration $n(\lambda)$; a key property of the Jack $J^\alpha_N(z)$ is that its expansion in terms of monomials only contains terms $m_N$ with $\mu \leq \lambda$, where $\mu < \lambda$ means the partition $\mu$ is dominated by $\lambda$ [10]. Jacks are also eigenstates of a Laplace-Beltrami operator $\mathcal{H}_{LB}(\alpha)$ given by

$$\sum_i \left( z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i < j} \frac{z_i + z_j}{z_i - z_j} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right).$$

(2)

A partition $\lambda$ is "$(k,r,N)$-admissible" [1] if $n(\lambda)$ obeys a "generalized Pauli principle" where, for all $m \geq 0$, $\sum_{j=1}^{\ell_\lambda} n_{m+j-1} \leq k$, so $r$ consecutive "orbitals" contain no more than $k$ particles [6].

Partitions $\lambda$ can be classified by $\lambda_1$, their largest part. When $J^\alpha_N(z)$ is expanded in occupation-number states (monomials), no orbital with $m > \lambda_1$ is occupied, and Jacks with $\lambda_1 \leq N \Phi$ form a basis of FQH states on a sphere surrounding a monopole with charge $N \Phi$ [7]. Uniform states on the sphere satisfy the conditions $L^+ \psi = 0$ (highest weight, HW) and $L^- \psi = 0$ (lowest weight, LW) where:

$$L^+ = E_0; \quad \lambda\psi = N \Phi Z - E_2; \quad L^z = \frac{1}{2} N N \Phi - E_1$$

where $Z \equiv \sum_i z_i$. When both conditions are satisfied, $E_0 \psi \equiv M \psi = \frac{1}{4} N N \Phi \psi$. The $L^+, L^-, L^z$ operators endow the polynomial space with an angular momentum structure which we use to characterize the polynomials. Any homogeneous polynomial is an eigenstate of the $L^z$ operator; let the $L^z$ eigenvalue of the HW Jacks be $l^{max}_\alpha$. The HW states then have $L^2 = \frac{1}{4} (L^+ L^- + L^- L^+ + L^z L^z) = l^{max}_\alpha (l^{max}_\alpha + 1)$, and hence the HW polynomials are the $\{|l\rangle, \langle l|\}$ states of a $2l^{max}_\alpha + 1$ angular momentum multiplet of linearly independent polynomials. The non-HW states of $\{|l\rangle, \langle l|\}$ are states of a $2l^{max}_\alpha + 1$ angular momentum multiplet that span $\Phi^\alpha_Z$ of each of $\lambda$. Our motivation is to find new properties as the HW Jacks satisfy the same cluster-algebraic framework as the non-HW polynomials. Any non-HW polynomial is an eigenstate of $L^z$, so that the orbital occupation partition is dominated by $\lambda$. Applying the lowering operator $(L^z)^{-1}$ on the HW states. They are linearly independent by virtue of having the same $N \Phi$ but different total degree $M$. Applying the $L^-$ operator $2l^{max}_\alpha + 1$ times kills the state. This angular momentum structure is extensively used in studies of FQH states on the sphere and we find it to be extremely valuable in the empirical polynomial counting presented in the following sections.

It is very instructive to find the conditions for a Jack to satisfy the HW condition, $E_0 J^\alpha_N = 0$. The action of $E_0$ on a Jack can be obtained from a formula due to Lassalle [11]. In [3] we found that the condition $E_0 J^\alpha_N = 0$ places severe restrictions on both the Jack parameter $\alpha$ and on the partition $\lambda$. We found the following conditions $\alpha < 0, n_0 = N - \ell_\lambda > 0$ (non-zero occupancy of the $m = 0$ "orbital"), as well as

$$N - \ell_\lambda + 1 + \alpha (\lambda_\ell - 1) = 0,$$

(4)

where $\lambda_\ell$ is the smallest (non-zero) part in $\lambda$. This imposes the following two conditions: (i) $\alpha$ is a negative rational, which we can choose to write as $- (k + 1)/(r - 1)$, with $(k + 1)$ and $(r - 1)$ both positive, and relatively prime; (ii) $\lambda_\ell = (r - 1)s + 1$, and $n_0 = (k + 1)s - 1$, where $s > 0$ is a positive integer. The remaining HW conditions require that both in $\lambda$ have multiplicity $k$, so that the orbital occupation partition is $n(\lambda^0_{k,r,s}) = [n_0 0^s (r-1) k 0^r (r-1) k 0^r (r-1) k_0, \ldots]$, (i.e., the $(k,r,N)$-admissibility condition is satisfied as an equality for orbitals $m \geq \lambda_\ell$).

We call these Jacks HW $(k, r, s, N)$ states $J^\alpha_N^{k,r}$. Non-HW $(k, r, s, N)$ states with $n_0$ particles in the zeroth orbital can be obtained by inserting zeroes (holes) in the partition to the right of the $\lambda_\ell$'s orbital. This defines a set of partitions whose Jacks satisfy the same clustering properties as the HW Jacks $J^\alpha_N^{k,r}$ (see Section III). These partitions are $\lambda_{k, r, s} : n(\lambda_{k, r, s}) = [n_0 0^s (r-1) n(\lambda_{k, r})]$.
Observe that when s > 1 states are both HW and LW states on the sphere, and satisfy the clustering property that they vanish as the r'th power of the distance between k + 1-particles. The s > 1 states satisfy generalized clustering conditions.

where \( n(\lambda_{k,r}) \) is a \((k, r, N - n_0)\) admissible configuration in the sense of Feigin et. al.. This set of partitions do not exhaust the number of polynomials with the clustering property Eq.(1). The case \( s = 1 \) gives the generators of the ideals obtained by Feigin et. al and are related to FQH ground states and their quasihole excitations \( [3] \). The cases \( s > 1 \) are new and violate the admissibility conditions of Feigin et. al. (see Fig.2).

III. GENERALIZED CLUSTERING CONDITIONS OF JACK POLYNOMIALS

We now present the two generalized clustering conditions satisfied by the (HW and non-HW) Jacks of the \((k, r, s, N)\) partitions \( J_{\lambda_{k,r,s}}^\alpha \).

A. First Clustering Property

The Jacks \( J_{\lambda_{k,r,s}}^\alpha \) allow \( s - 1 \), but not \( s \), different clusters of \( k + 1 \) particles. First form \( s - 1 \) clusters of \( k + 1 \) particles \( z_1 = \ldots = z_{k+1}(=Z_1); z_{k+2} = \ldots = z_{2(k+1)} (= Z_2); \ldots; z_{(s-2)(k+1)+1} = \ldots = z_{(s-1)(k+1)} (= Z_{s-1}) \) can be different. Then, form a \( k \) (not \( k + 1 \)) particle cluster \( z_{(s-1)(k+1)+1} = \ldots = z_{s(k+1)-1}(=Z_F) \) (a final \( s'th \) cluster of \( k + 1 \) particles would make the polynomial vanish), see Fig.4. With the above conditions on the coordinates, the clustering condition reads:

\[
J_{\lambda_{k,r,s}}^\alpha (z_1, \ldots, z_N) \propto \prod_{i=s(k+1)}^{N} (Z_{F} - z_i)^r
\]

(5)

Observe that when \( s = 1 \) Eq.(5) reduces to the usual clustering condition satisfied by the \((k, r)\) sequence, given in \( [3] \).

B. Second Clustering Property

The Jacks \( J_{\lambda_{k,r,s}}^\alpha \) allow a large cluster of \( n_0 = (k + 1)s - 1 \) particles at the same point. As a particular case of Eq.(4), they cannot allow \( n_0 + 1 \) particles to come at the same point, as this would involve the formation of \( s \) clusters of \( k + 1 \) particles, which Eq.(5) forbids. Clustering \( n_0 \) particles at the same point \( z_1 = \ldots = z_{(k+1)s-1} = Z \) we find the following property (see Fig.3):

\[
J_{\lambda_{k,r,s}}^\alpha (z_1, \ldots, z_N) \propto \prod_{i=s(k+1)}^{N} (Z - z_i)^{(r-1)s} + 1
\]

(6)

The HW Jacks of partitions \( \lambda_{k,r,s}^0 \) satisfy an even more stringent property; with \( z_1 = \ldots = z_{(k+1)s-1} = Z \), we find:

\[
J_{\lambda_{k,r,s}^0}^\alpha (z_1, \ldots, z_N) = \prod_{i=s(k+1)}^{N} (Z - z_i)^{(r-1)s} + 1 \times \prod_{i=s(k+1)}^{N} (Z - z_i)^{(r-1)s} + 1 \times \prod_{i=s(k+1)}^{N} (Z - z_i)^{(r-1)s} + 1
\]

(7)

where \( n(\lambda_{k,r,s}^0) = [k0^{r-1}0^{s-1} \ldots k] \) is the maximum density \((k, r, N - n_0)\)-admissible partition. For \( s = 1 \), Eq.(7) also reduces to the usual clustering condition satisfied by the \((k, r)\)-admissible sequence \( [3] \). We have performed an extensive numerical checks of the above conjectured clustering conditions. We also note that the LHS and RHS of Eq.(7) match in both total momentum \( M^0 \) (total degree of the polynomial):

\[
E_1 J_{\lambda_{k,r,s}^0}^\alpha \equiv M^0 J_{\lambda_{k,r,s}^0}^\alpha = (N - (k + 1)s + 1) \times \\
\times [(r - 1)s + 1 + \frac{1}{2}(N - (k + 1)s - k + 1)] J_{\lambda_{k,r,s}^0}^\alpha
\]

(8)

and in flux (maximum degree in each variable) \( N_\Phi^0 \):

\[
N_\Phi^0 = \frac{r}{k} (N - k - (k + 1)(s - 1)) + (r - 1)(s - 1).
\]

(9)
The superscript denotes the fact that we are considering the momentum and flux of Jack polynomials of HW partitions $\lambda_{k,r,s}^0$.

C. Additional Clustering Condition

We empirically find that the HW Jacks $J_{\lambda_{k,r,s}^0}^{0k,r}$ satisfy a third type of clustering which has no correspondence in the $s = 1$ case. Forming $s - 1$ clusters of $2k + 1$ particles together: $z_1 = \ldots = z_{2k+1}(= Z_1); \ z_{(2k+1)+1} = \ldots = z_{2(2k+1)}(= Z_2); \ldots; z_{(s-1)(2k+1)+1} = \ldots = z_{(s-1)(2k+1)}(= Z_{s-1})$ we find that the HW Jacks satisfy, up to a numerical proportionality constant, the clustering:

$$J_{\lambda_{k,r,s}^0}^{0k,r} (z_1, \ldots, z_N) = \prod_{i<j}^{r-1} (Z_i - Z_j)^{k(3r-2)} \times \prod_{i=1}^{r-1} \prod_{l=(s-1)(2k+1)+1}^{N} (Z_i - z_l)^{2r-1} \times J_{\lambda_{k,r,s}^0}^{0k,r} (z_{(s-1)(2k+1)+1}, \ldots, z_N)$$

Some slightly tedious algebra proves that the total momentum and flux match between the LHS and RHS of Eq.(10).

D. Angular Momentum Structure

The $J_{\lambda_{k,r,s}^0}^{0k,r} (z_1, \ldots, z_N)$ are the HW states of an angular momentum multiplet of $l(\lambda_{k,r,s}^0) = l_{\text{max}}^0$ and $l_z = l_{\text{max}}^0$, ..., $l_{\text{max}}^0$. The LW states of the multiplet $(L^-)^{2l_{\text{max}}^0} J_{\lambda_{k,r,s}^0}^{0k,r} (z_1, \ldots, z_N)$ are also single Jack polynomials of the “symmetric” partition to $\lambda_{k,r,s}^0$ in orbital notation (see Fig.4). The value of $l_{\text{max}}^0$ is:

$$L_z^2 J_{\lambda_{k,r,s}^0}^{0k,r} = l_{\text{max}}^0 J_{\lambda_{k,r,s}^0}^{0k,r} = \frac{1}{2} \left[ ((r-1)s+1)(2(k+1)s-2-N) + \frac{r}{4} ((k+1)s-1)(N-(k+1)s+1-k) \right] J_{\lambda_{k,r,s}^0}^{0k,r}$$

Most importantly, we find that powers of the operator $L^- = \sum_i \left( z_i^2 \frac{\partial}{\partial z_i} - N_\Phi z_i \right)$, acting on $J_{\lambda_{k,r,s}^0}^{0k,r} (z_1, \ldots, z_N)$ create linearly independent polynomials with the same vanishing conditions Eq.(3), Eq.(4) as the HW $J_{\lambda_{k,r,s}^0}^{0k,r} (z_1, \ldots, z_N)$. There are $2l_{\text{max}}^0 + 1$ such polynomials.

IV. $J_{\lambda_{k,r,s}^0}^{0k,r}$ : SMALLEST DEGREE POLYNOMIALS WITH GENERALIZED CLUSTERING

In the remainder of the paper we focus on the case $r = 2$. We empirically find the following property: given $N$ particles (with $N > n_0$ for the clustering condition Eq.(1) to be well defined), we find that the $r = 2$ HW Jacks $J_{\lambda_{k,2,s}^0}^{0k,2} (z_1, \ldots, z_N)$ are the smallest degree (smallest momentum $M$ - Eq.(5) and flux $N_\Phi$ - Eq.(6)) polynomials satisfying the clustering conditions Eq.(5), Eq.(6). There are exactly $2 \cdot l(\lambda_{k,2,s}^0) + 1$ polynomials in $N$ variables of $N_\Phi^0$ (Eq.(1)) and of unrestricted total dimensions, satisfying the clustering Eq.(5), Eq.(6). A basis for this ideal is, explicitly:

$$(L^-)^m J_{\lambda_{k,2,s}^0}^{0k,2} ; \quad m = 0, 1, \ldots, 2 \cdot l(\lambda_{k,2,s}^0);$$

We find that $(L^-)^{2l(\lambda_{k,2,s}^0)+1} J_{\lambda_{k,2,s}^0}^{0k,2} = 0$. One can easily understand this counting by looking at the orbital occupation numbers of the relevant partitions. The occupation number of the $J_{\lambda_{k,2,s}^0}^{0k,2}$ is $[n_0 n^0 \cdots k k \cdots k k]$. This is the lowest weight partition (smallest degree polynomials) where the clustering conditions Eq.(5), Eq.(6) are satisfied. Interpreting this as orbital occupation num-
by Kasatani et al. for the specific \((k, r, s) = (1, 2, 2)\) case of the problem studied here. As we reach higher \(k\) and \(s\) integers, the number of ’modified’ Jacks appearing in the expansion of \((L^-)^m J_{\lambda}^{\alpha,k,r}_{X^k_{r,s}}\) grows large. We therefore prefer to characterize the basis of these polynomials by the HW Jack and the polynomials that result from it by successive application of the \(L^-\) operator.

We now relax the constraint \(N_p = N^d_{\Phi}\) and focus on the counting of the polynomials satisfying Eq.\(5\) and Eq.\(6\).

V. COUNTING POLYNOMIALS

We want to provide the counting of the number of linearly independent polynomials in the ideal \(F/F_1\). We start by counting the \(s = 1\) polynomials. These are related to the admissible partitions of \([1]\) or the generalized Pauli principle of \([4, 3]\).

A. Counting of \((k, r, s) = (1, r, 1)\) Polynomials

We first obtain a counting of linearly independent polynomials in \(N\) particle coordinates \(z_1, ..., z_N\), of total momentum \(M\), with the degree in each coordinate \(\leq N_{\Phi}\) satisfying the condition \(P(z_1, z_2, z_3, z_4, ... \sim (z_i - z_j)^r\). We believe this result was previously known, although we could not explicitly find it in the literature. From the work of Feigin et al. \([1]\), this number is equal to the number of \((k, r) = (1, r)\) admissible partitions of \(N_{\Phi}\) orbitals, related by the squeezing rule \([3]\) (so as to keep the partition weight \(M\) constant). Call this number \(p_{1,r,1}(N, M, N_{\Phi})\). From the theory of partitions \([12]\), such numbers are most easily obtained from a generating function \(G(q)\), and we can analytically prove that:

\[
p_{1,r,1}(N, M, N_{\Phi}) = \frac{1}{M!} \frac{\partial^M G_{1,r,1}(N, N_{\Phi}, q)}{\partial q^M} \bigg|_{q=0} \tag{13}
\]

where the generating function \(G(q)\) reads:

\[
G_{1,r,1}(N, N_{\Phi}, q) = \frac{q^{N(N-1)} \prod_{i=1}^{N} \prod_{j=1}^{N_{\Phi}-r(N-1)} (1 - q^i)}{\prod_{i=1}^{N} (1 - q^i) \prod_{i=1}^{N_{\Phi}-r(N-1)} (1 - q^i)}
\]

if \(N_{\Phi} \geq r(N - 1)\) and \(p_{1,r,1}(N, M, N_{\Phi} < r(N - 1)) = 0\).

\(p_{1,r,1}(N, M, N_{\Phi})\) represents a building block for future results. We have numerically checked, by building the null-space of polynomials satisfying the clustering condition \(P(z_1, z_2, z_3, z_4, ... \sim 0\), that \(p_{1,r,1}(N, M, N_{\Phi})\) gives the right polynomial counting. In the context of FQH, it reproduces the right counting for quasihole states. For example, it is known that the Laughlin state with \(x\) number of quasiholes has \((N + x)!/(N!x!))\) independent states and one can numerically check the identity:

\[
\binom{N + x}{x} = \frac{q^{N(N-1)+xN}}{\prod_{M=\frac{1}{2}}^{\frac{N(N-1)}{2}} p_{1,r,1}(N, M, N_{\Phi} = r(N-1)+x)}
\]
We can now provide a formula for the number of polynomials in $N$ variables, of total dimension $M$, of any maximum power of each coordinate (any $N_\Phi$), which satisfy the clustering condition $P(z_1, z_2, \ldots) \sim (z_i - z_j)^r$. We take $N_\Phi \rightarrow \infty$ to obtain the simpler expression:

$$p_{1, r, 1}(N, M) = \frac{1}{M!} \frac{\partial^M \phi_{1, r, 1}(N, q)}{\partial q^M} \bigg|_{q = 0}$$

(14)

To find out the total number of symmetric polynomials satisfying the vanishing $P(z_1, z_2, z_3, \ldots, z_N) = 0$ we must particularize to the lowest vanishing power possible, $r = 2$. In this case, Eq.(14) is identical to the formula of Kasatani et. al. [8], although Eq.(13) represents a more comprehensive counting of the polynomials as it contains information on the allowed maximum degree in each variable, $N_\Phi$.

Our aim to conjecture similar expressions for the counting of the dimension space of the polynomials in the coset space $F/F_1$.

B. Counting of $(k, r, s) = (1, 2, s)$ Polynomials

Using $p_{1, 2, 1}(N, M, N_\Phi)$, we now obtain the counting of polynomials in the ideal $F/F_1$ with $k = 1, r = 2$ and $s > 1$. We first reproduce the result of Kasatani et. al. [8] which is the case $(k, r, s) = (1, 2, 2)$ of our problem. Kasatani et. al. [8] obtained the dimension of the linear space of polynomials satisfying the clustering conditions $P(z_1, z_2, z_3, \ldots, z_N) = 0$ and $P(z_1, z_1, z_2, z_3, \ldots, z_N) \neq 0$, of total dimension $M$, of any allowed maximum degree in each of the coordinates. We then derive the general case, which contains information about $N_\Phi$.

Define $p_{k=1, r=2, s=2}(N, M) = p_{1, 2, 2}(N, M)$ as the number of polynomials in $N$ variables of total momentum (degree) $M$, with any allowed $N_\Phi (\leq \infty)$, satisfying the clustering condition Eq.(3), Eq.(6) (with $k = 1, r = 2, s = 2$). This number can be found as follows: start with the partition $a_1(1, 2, 2)$, of total dimension (weight) $M$ which has $N - 1$ particles all pushed maximally to the left of the orbitals, while the $N$th particle is pushed as far as needed to the right so that the polynomial has dimension $M$. This partition reads $[0010101…10 0 0 1 0 … 1]$. Note that, by $(k, r, s, N)$ admissibility, we cannot push the first $N - 1$ particles anymore to the left than they already are. Then $p_{1, 2, 2}(N, M)$ is the sum of two terms: First, we can form $(k, r, s, N) = (1, 2, 2, N)$ admissible partitions by keeping the occupation of the zeroth orbital to be 3 and by squeezing on the remainder partition $[0010101…10 0 0 1 0 … 1]$ to form all the $(k, r) = (1, 2)$-admissible partitions. As discussed before, this gives Jack polynomials with the same clustering condition as the HW Jack, and their number is the same as the number of $(k, r) = (1, 2)$ admissible partitions of $N - 3$ variables and total momentum $M - 3(N - 3)$, i.e.: $p_{1, 2, 1}(N - 3, M - 3(N - 3))$. Second, we can form polynomials with the same $(k, r, s, N) = (1, 2, 2, N)$ clustering by taking some particles out of the zeroth-orbital, although these now involve divergent Jacks (with compensating vanishing coefficients). We can form all the polynomials (of dimension $M$ in $N$ variables, with less than 3 particles in the zeroth orbital, and that satisfy the clustering conditions $(k, r, s, N) = (1, 2, 2, N)$) by acting with $L^- 2$ on all the polynomials of dimension $M - 1$, that satisfy the same clustering conditions. This number is then $p_{1, 2, 2}(N, M - 1)$, and we find the recursion relation:

$$p_{1, 2, 2}(N, M) = p_{1, 2, 2}(N, M - 1) + p_{1, 2, 1}(N - 3, M - 3(N - 3))$$

(15)

To find the generating function, multiply Eq.(15) by $q^M$, sum over $M$, re-shift variables in the sum and obtain:

$$p_{1, 2, 2}(N, M) = \frac{1}{M!} \frac{\partial^M \phi_{1, 2, 2}(N, q)}{\partial q^M} \bigg|_{q = 0}$$

(16)

This reproduces a formula obtained by Kasatani et. al. [8] through different methods.

We now use the same reasoning to count the dimension of the ideal $F/F_1$ with $(k = 1, r = 2)$ and general $s$. The number of polynomials of $N$ variables of momentum (total degree) $M$, with unrestricted $N_\Phi$, which satisfy the clustering conditions Eq.(3) and Eq.(6) with $k = 1, r = 2$ and any $s > 1$ is $p_{1, 2, s}(N, M)$:

$$p_{1, 2, s}(N, M) = \frac{1}{M!} \frac{\partial^M \phi_{1, 2, s}(N, q)}{\partial q^M} \bigg|_{q = 0};$$

$$G_{1, 2, s}(N, q) = \frac{q^{(N-s)(N-1)}}{(1-q) \prod_{i=1}^{N-s}(1-q^i)}$$

(17)

where $n_0 = 2s - 1$.

We now introduce information on the maximum degree in each coordinate separately (flux). Define $p_{k=1, r=2, s}(N, M, N_\Phi) = p_{1, 2, s}(N, M, N_\Phi)$ as the number of polynomials in $N$ variables of total momentum (degree) $M$, with flux $\leq N_\Phi$, satisfying the clustering conditions Eq.(3) and Eq.(6) with $k = 1$ and $s = 2$. We briefly present the reasoning used to conjecture a count of these polynomials. The smallest dimension and flux correspond to the partition $[n_0 0 \ldots 1010101 \ldots 101010]$. Define $p_{k=1, r=2, s}(N, M, N_\Phi)$ as the number of polynomials in $N$ variables of total momentum (degree) $M$, with flux $\leq N_\Phi$, satisfying the clustering conditions Eq.(3) and Eq.(6) with $k = 1$ and $s = 2$. We briefly present the reasoning used to conjecture a count of these polynomials. The smallest dimension and flux correspond to the partition $[n_0 0 \ldots 1010101 \ldots 101010]$. The number of zeroes on the right is just right to make the total number of orbitals $N_\Phi + 1$. Some of the orbitals to the right might be unoccupied. This “padding” to the right has the effect of allowing $L^-$ to move particles up to the rightmost orbital. By symmetry in orbital space, the highest partition corresponds to $[0000000000010101010]$. 
satisfying the clustering conditions Eq. (5) and Eq. (6).

Immediately see that \( p_{1,2,s}(N, M, N_\phi) = 0 \) for \( M < (s + 1)(N - n_0) + (N - n_0)(N - n_0 - 1) \) or \( N_\phi < s + 1 + 2(N - n_0 - 1) \). Also, \( p_{1,2,s}(N, M, N_\phi) = 0 \) for \( M > NN_\phi - (N - n_0)(N - n_0 + s) \). There is also an "intermediate" total degree that is important in the counting, which corresponds to the partition \([01010\ldots00\ldots n_0]\) of total degree \( n_0 N_\phi + (N - n_0)(N - n_0 - 1) \) when the rightmost orbital has been occupied by the maximum number of particles possible, \( n_0 \). Then \( p_{1,2,s}(N, M, N_\phi) \) reads:

\[
p_{1,2,s}(N, M, N_\phi) = \begin{cases} 
0 & \text{if } M < (s + 1)(N - n_0) + (N - n_0)(N - n_0 - 1) \text{ or } N_\phi < s + 1 + 2(N - n_0 - 1) \\
= \sum_{i=0}^{M-(s+1)(N-n_0)} p_{1,2,1}(N-n_0, i, N_\phi - (s + 1)) & \text{if } M \leq n_0 N_\phi + (N - n_0)(N - n_0 - 1) \\
= \sum_{i=0}^{NN_\phi - (N - n_0)(s + 1) - M} p_{1,2,1}(N-n_0, i, N_\phi - (s + 1)) & \text{if } n_0 N_\phi + (N - n_0)(N - n_0 - 1) < M \leq NN_\phi - (N - n_0)(N - n_0 + s) \\
= 0 & \text{if } M > NN_\phi - (N - n_0)(N - n_0 + s)
\end{cases}
\]

\( p_{1,2,1}(N, M, N_\phi) \) was explicitly given in a previous subsection, and \( n_0 = 2s - 1 \).

By summing the previous expression over all \( M \) we can find the number of polynomials of \( N \) variables, with degree in each variable \( N_\phi \) and of unrestricted momentum (total degree) \( p_{1,2,s}(N, N_\phi) = \sum_{M=0}^{\infty} p_{1,2,s}(N, M, N_\phi) \), satisfying the clustering conditions Eq. (5) and Eq. (6) with \( k = 1 \) and \( s \) arbitrary integer. However, by applying an empirical rule we observed, based on the multiplet nature of these polynomials, we find an alternate simpler formula, which is not obviously equal to \( \sum_{M=0}^{\infty} p_{1,2,s}(N, M, N_\phi) \); extensive numerical checks have however confirmed their equivalence:

\[
p_{1,2,s}(N, N_\phi) = \begin{cases} 
0 & \text{if } N_\phi < s + 1 + 2(N - n_0 - 1) \\
= (NN_\phi - 2(s + 1)N + 2n_0(s + 1) + 1) \sum_{i=0}^{NN_\phi} p_{1,2,1}(N-n_0, i, N_\phi - (s + 1)) - 2 \sum_{i=0}^{NN_\phi} i \cdot p_{1,2,1}(N-n_0, i, N_\phi - (s + 1)) & \text{if } N_\phi \geq s + 1 + 2(N - n_0 - 1)
\end{cases}
\]

\( p_{1,2,1}(N, M, N_\phi) \) was explicitly given in subsection IV A., and \( n_0 = 2s - 1 \).

C. Counting of \((k, r, s) = (k, 2, 1)\) Polynomials

We now move to the \( k > 1 \) case. We first obtain a count of the polynomials satisfying the \((k, r) = (k, 2)\) statistics, i.e. of the Read-Rezayi \( Z_k \) states. We want to count the number of polynomials in \( N \) variables, of momentum (total degree) \( M \) with maximum flux (maximum degree in each coordinate) \( N_\phi \), that vanish when \( k + 1 \) particles come together. We call this number \( p_{k,2,1}(N, M, N_\phi) \). As we know [1], this is equal to the number of \((k, 2)\)-admissible partitions of weight \( M \), made out of at most \( N \) parts, and with \( \lambda_1 \leq N_\phi \). We can derive this by perform-
ing a slight modification of a formula due to Feigin and Loktev \[13\] (see also Andrews \[12\]). \(p_{k,2,1}(N,M,N_\Phi) = 0\) for \(N_\Phi < \frac{2}{k}(N-k)\) or for \(M < \frac{1}{k}N(N-k)\) but otherwise is:

\[
p_{k,2,1}(N,M,N_\Phi) = \frac{1}{M!} \left[ \frac{\partial^M}{\partial q^M} \right] \times \\
\times (q^{-N}G_{k,2,1}(N_\Phi,q,z) ) \mid_{q=0,z=0}
\]

where the generating function \(G_{k,2,1}(N_\Phi,q,z)\) is \[13,12\]:

\[
G_{k,2,1}(N_\Phi,q,z) = \sum_{m_1,n_1=0}^{N_\Phi+1} \sum_{m_1+n_1 \leq N_\Phi+1} q^{k(m_1^2-n_1^2+n_1(N_\Phi+2)+\frac{1}{m_1+1}(N_\Phi+3-n_1)} \times \\
\times \frac{z^{(k+1)(m_1+n_1)}(1-q^i)(zq^{i+1}+1)}{\prod_{i=1}^{m_1+1}(1-zq^i)\prod_{i=1}^{n_1+1}(1-q^i)(zq^{N_\Phi-i-n_1+3-1})}
\]

(19)

We have numerically performed extensive checks of the compatibility of the formula above, in the case \(k = 1\), with the simpler expression of \(p_{1,2,1}(N,M,N_\Phi)\) obtained earlier. The formula above also correctly gives the dimension of the quasihole Hilbert space in the \(Z_k\) parafermions sequence. For one-quasihole, this is known to be:

\[
\binom{N+k}{k} = \sum_{i=0}^{\left\lfloor \frac{N-k}{k} \right\rfloor} p_{k,2,1}(N,i,N_\Phi) = \frac{2}{k}(N-k)+1
\]

Extensive numerical checks prove the above identity. Moreover, for the \(k = 2\) Read-Moore state with 2 quasi-particles:

\[
\binom{N+2}{4} + \binom{(N-2)/2+4}{4} = \\
= \sum_{i=0}^{\left\lfloor \frac{N-k}{k} \right\rfloor} p_{k,2,1}(N,i,N_\Phi) = \frac{2}{k}(N-k) + 2
\]

Eq.(19) for \((k,2)\) admissible partitions found by Feigin and Loktev \[13\] and prior to them by Andrews \[12\] gives the most information possible about the counting of the Read-Rezayi wavefunctions and quasiholes. It provides information about the total degree of the polynomial (multiplet structure), which the usual counting \[13\] of quasiholes does not since it sums over all the possible total dimensions of the polynomials subject to a flux \(N_\Phi\) upper bound.

D. Counting of the \((k,r,s) = (k,2,s)\) Polynomials

Using \(p_{k,2,1}(N,M,N_\Phi)\) we obtain the counting of polynomials in the \(F/F_1\) ideal with arbitrary \(k\) and \(s\). Following a line of reasoning similar to the one used in the \(k = 1\) case, we find the number \(p_{k,2,s}(N,M,N_\Phi)\) of polynomials in \(N\) variables, of momentum (total degree) \(M\), of flux \(N_\Phi\) that have the clustering conditions Eq.(15) and Eq.(16) for general \(k\) and \(s > 1\) integers reads:

\[
p_{k,2,s}(N,M,N_\Phi) = \\
= 0; \text{ if } M < (s+1) \cdot (N-n_0) + \frac{1}{k}(N-n_0)(N-n_0-k) \text{ or } N_\Phi < s + 1 + \frac{2}{k}(N-n_0-k) \\
= \sum_{i=0}^{M-(s+1)(N-n_0)} p_{k,2,1}(N-n_0,i,N_\Phi -(s+1)); \text{ if } 0 \leq n_0 N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k) \\
= \sum_{i=0}^{NN_\Phi-(N-n_0)(s+1)-M} p_{k,2,1}(N-n_0,i,N_\Phi -(s+1)); \text{ if } n_0 N_\Phi + \frac{1}{k}(N-n_0)(N-n_0-k) < M \leq NN_\Phi-(N-n_0)(s+\frac{N-n_0}{k})
\]

(20)
where \( n_0 = (k + 1)s - 1 \).

By summing the previous expression over all \( M \) we can find the number of polynomials of \( N \) variables, with flux \( N_\Phi \) and of unrestricted momentum (total degree) \( p_{k,2,s}(N, N_\Phi) = \sum_{M=0}^{\infty} p_{k,2,s}(N, M, N_\Phi) \), satisfying the clustering conditions Eq. (5) and Eq. (6) with \( k \) and \( s \) arbitrary integers. However, by using the angular momentum multiplet structure of these polynomials, we find an alternate formula which is not obviously equal to \( \sum_{M=0}^{\infty} p_{k,2,s}(N, M, N_\Phi) \); extensive numerical checks have however confirmed their equivalence:

\[
p_{k,2,s}(N, N_\Phi) =
\]

\[
= 0 \text{ if } N_\Phi < s + 1 + \frac{2}{k}(N - n_0 - k)
\]

\[
= (N N_\Phi - 2(s+1)N + 2n_0(s+1) + 1) \sum_{i=0}^{N N_\Phi} p_{k,2,1}(N - n_0, i, N_\Phi - (s+1)) - 2 \sum_{i=0}^{N N_\Phi} i \cdot p_{k,2,1}(N - n_0, i, N_\Phi - (s+1))
\]

\[
(22)
\]

\( p_{k,2,1}(N, M, N_\Phi) \) is given in VC., and \( n_0 = (k + 1)s - 1 \).

VI. SUBIDEALS OF THE CAYLEY-SYLVESTER PROBLEM

So far we have focused on the ideal \( F/F_1 \). We can systematically characterize the ideal \( F \) of polynomials \( P(z_1, z_2, ..., z_N) \) which vanish when we form \( s \) clusters of \( k + 1 \) particles in the following way: let the sub-ideals \( F_i \) be the polynomials that satisfy (Eq. (1)) but that also vanish when \( (k + 1)s - i \) particles are brought at the same point. Then \( F = \bigcup_{i=0}^{k+1} F_i/F_{i+1} \), where \( F_0 = F \) and \( F_{k+1} \) is the ideal of polynomials that vanish when \( s \) \(-1 \) clusters of \( k + 1 \) particles are formed (\( F_{k+2} = \emptyset \)). Hence the polynomial ideal \( F_i \) is defined by the two clustering conditions:

\[
P(z_1 = ... = z_{k+1}, z_{k+2} = ... = z_{2(k+1)}, ..., z_{(s-1)(k+1)+1} = ... = z_{s(k+1)}, z_{s(k+1)+1}, z_{s(k+1)+2}, ..., z_N) = 0
\]

and

\[
P(z_1 = ... = z_{s(k+1)-i}, z_{s(k+1)-i+1}, z_{s(k+1)-i+2}, ..., z_N) = 0
\]

We have not found the generators for the \( F_i/F_{i+1} \) ideals, nor have we able to find their counting rules for the general case. However, we have solved the problem for several specific cases which we present below.

A. Subideals of the \((k, r, s) = (k, 2, 2)\)

We now give the partitions for the generators (smallest degree highest weight polynomials) for the subideals \( F_i/F_{i+1} \) for the infinite series \((k, r, s) = (k, 2, 2)\). The first smallest degree polynomials that vanish when 2 distinct clusters of \( k + 1 \) particles are formed, but does not vanish when one large cluster of \( 2k+1 \) particles is formed, is dominated by the root partition:

\[
P_{2k + 100k; 0k; 0k ... 0k}(z_1, ..., z_{k+1}, z_{k+2}, ..., z_{2k+1}) \not= 0
\]

\[
P_{2k+1}(z_1, ..., z_{2k+1}) \not= 0
\]

\[
P_{2k}(z_1, ..., z_{2k}) = 0
\]

\[
P_{k+1}(z_1, ..., z_{k+1}) = 0
\]

\[
P_{k}(z_1, ..., z_{k}) = 0
\]

\[
P_{s}(z_1, ..., z_{s}) = 0
\]

\[
P_{1}(z_1) = 0
\]

\[
P_{0}(z_0) = 0
\]
FIG. 7: Cayley-Sylvester subideals for polynomials satisfying $P(z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, z_4, \ldots) = 0$

The polynomial above, as well as the ones we introduce below, can be written as a linear combinations of monomials of partitions dominated by the root partition above, with coefficients that are uniquely defined by the HW and clustering conditions. These are of course, the Jacks. Then the smallest degree polynomial that vanishes when either 2 distinct clusters of $k + 1$ particles are formed or when a large single cluster of $2k + 1$ particles is formed, but does not vanish when one large cluster of $2k$ particles if formed, is dominated by the partition (see Fig [7]):

$$|2k01k - 11k - 11k - 1...1k - 1) : P(z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, z_4, \ldots) = 0 \quad (26)$$

$$P(z_1, \ldots, z_1, z_2, z_3, \ldots) = 0, \quad P(z_1, \ldots, z_1, z_2, z_3, \ldots) \neq 0;$$

The smallest degree polynomial that vanishes when either 2 distinct clusters of $k + 1$ particles are formed or when a large single cluster of $2k + 1$ particles is formed but does not vanish when one large cluster of $2k - 1$ particles if formed, is dominated by the partition (see Fig 7):

$$|2k - 102k - 22k - 22k - 2...2k - 2) : P(z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, z_4, \ldots) = 0 \quad (27)$$

$$P(z_1, \ldots, z_1, z_2, z_3, \ldots) = 0, \quad P(z_1, \ldots, z_1, z_2, z_3, \ldots) \neq 0;$$

and so on until. At last, the smallest degree polynomial that vanishes when either 2 distinct clusters of $k + 1$ particles are formed or when a single cluster of $k + 2$ particles is formed but does not vanish when one cluster of $k + 1$ particles if formed, is dominated by the partition (see Fig 7):

$$|k + 10k0k0...k0) : P(z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, z_4, \ldots) = 0 \quad (28)$$

$$P(z_1, \ldots, z_1, z_2, z_3, \ldots) = 0, \quad P(z_1, \ldots, z_1, z_2, z_3, \ldots) \neq 0;$$

The polynomials of the last subideal, Eq. (29) are related to the quasiparticle excitations of abelian and non-abelian FQH states [7]. They perform well under $k + 2$-body repulsive interactions. We can also find the smallest weight partitions of polynomials that vanish when $s$ clusters of $k + 1$ particles come together when $k + 2$ particles come together, but do not vanish when $k + 1$ particles form a cluster:

$$|k + 10k + 10...k + 10k0k0...k0) : P(z_1, \ldots, z_1, z_2, \ldots, z_2, z_3, z_k+1, z_k+2, \ldots) = 0 \quad (29)$$

$$P(z_1, \ldots, z_1, z_2, z_3, \ldots) = 0, \quad P(z_1, \ldots, z_1, z_2, z_3, \ldots) \neq 0;$$

### B. Counting of the $F_k/F_{k+1}$ Subideal

The counting of dimension of the subideals above is a rather difficult (but tractable) problem. The “easy” exceptions are the first subideal, whose generator is $|2k + 100k0k0...k0k\rangle$ and whose counting formulas we have already conjectured in the body of this manuscript, and the last subideal whose generator is $|k + 10k0k0...k0k\rangle$, and whose counting formula we conjecture below. The number $p_{k+1}(N, M, \Phi)$ of polynomials, satisfying the clusterings of Eq. (29), of $N$ variables, of momentum (total degree $M$) and of flux (maximum separate degree in any variable) $\Phi$ is:

$$p_{k,2,2}(N, M, \Phi) =$$

$$= 0; \quad \text{if} \quad M < 2 \cdot (N - n_0) + \frac{1}{k} (N - n_0)(N - n_0 - k) \quad \text{or} \quad N_\Phi < 2 + \frac{2}{k} (N - n_0 - k)$$
\[ \sum_{i=0}^{M-2(N-n_0)} p_{k,2,1}(N-n_0, i, N_\Phi - 2); \text{ if } 0 \leq M \leq n_0 N_\Phi + \frac{1}{k} (N-n_0)(N-n_0 - k) \]

\[ NN_\Phi - (N-n_0)2 - M \]

\[ \sum_{i=0}^{N N_\Phi - (N-n_0)2 - M} p_{k,2,1}(N-n_0, i, N_\Phi - 2); \text{ if } n_0 N_\Phi + \frac{1}{k} (N-n_0)(N-n_0 - k) < M \leq NN_\Phi - (N-n_0)(1 + \frac{N-n_0}{k}) \]

\[ = 0 \text{ if } M > NN_\Phi - (N-n_0)(1 + \frac{N-n_0}{k}) \]

where \( n_0 = k + 1 \).

By summing the previous expression over all \( M \) we can find the number of polynomials of \( N \) variables, with degree in each variable at most \( N_\Phi \) and of unrestricted momentum (total degree) \( p_{F_k/F_{k+1}}(N, N_\Phi) = \sum_{M=0}^{\infty} p_{k,2,2}(N, M, N_\Phi) \). By applying a rule based on the multiplet nature of these polynomials, we find an alternate formula which is not obviously equal to \( \sum_{M=0}^{\infty} p_{k,2,2}(N, M, N_\Phi) \); extensive numerical checks have however confirmed their equivalence:

\[ p_{\text{subideal}}(N, N_\Phi) = \]

\[ = 0 \text{ if } N_\Phi < 2 + \frac{2}{k} (N-n_0 - k) \]

\[ = (NN_\Phi - 4N + 4n_0 + 1) \sum_{i=0}^{NN_\Phi} p_{k,2,1}(N-n_0, i, N_\Phi - 2) - 2 \sum_{i=0}^{NN_\Phi} i \cdot p_{k,2,1}(N-n_0, i, N_\Phi - 2) \]

where \( n_0 = k + 1 \).

**VII. CONCLUSIONS**

In this paper we have made several new conjectures about the behavior of Jack polynomials at negative Jack parameter \( \alpha \). By applying a HW condition, we find that the \((k, r)\)-admissible partitions of Feigin et al. [1] do not exhaust the space of partitions for which the Jack polynomials are well-behaved. We find a new infinite series of Jacks, described by a positive integer \( s \) that vanish when \( s \) distinct clusters of \( k + 1 \) particles are formed, but do not vanish when a large cluster of \( s(k + 1) - 1 \) particles is formed. We conjecture an empirical counting of polynomials with such clustering properties. We also find the dominant partitions and counting of polynomials that vanish when either \( s \) distinct clusters of \( k + 1 \) particles are formed or a cluster of \( k + 2 \) particles are formed, but do not vanish when a large cluster of \( k + 1 \) particles is formed. These results will be of physical use in the description of the quasiparticle excitations of the abelian and non-abelian Fractional Quantum Hall states [7].

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1. B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, Int. Math. Res. Not. 23, 1223 (2002).
2. B. Sutherland, Phys. Rev. A 4, 2019 (1971).
3. B.A. Bernevig and F.D.M. Haldane, arXiv:0707.3637.
4. F.D.M. Haldane, Bull. Am. Phys. Soc. 51, 633 (2006).
5. S.H. Simon, E.H. Rezayi, N.R. Cooper, and I. Berdnikov, PRB 75, 075317 (2007).
6. F.D.M. Haldane, Phys. Rev. Lett. 51, 605 (1983).
7. B.A. Bernevig and F.D.M. Haldane, in preparation.
8. M. Kasatani and et.al., arXiv:math/0404079v1.
9. J.V. Chipalkatti, Arch. Math. 83, 422 (2004).
10. R.P. Stanley, Adv.Math. 73, 76 (1989).
11 M. Lassalle, J. Func. Anal. 158, 289 (1998).
12 G.E. Andrews, Theory of Partitions, Cambridge University Press (July 28, 1998).
13 B. Feigin and S. Loktev, arXiv:math/0006221v3.
14 N. Read and E. Rezayi, Phys. Rev. B 59, 8084 (1999).