Birth and death process with bounded jumps in random environment

Hua-Ming WANG
Department of Mathematics, Anhui Normal University, Wuhu 241003, China
E-mail: hmking@mail.ahnu.edu.cn

Abstract

Let $\omega = (\omega_i)_{i \in \mathbb{Z}} = (\mu_i^L, ..., \mu_i^1, \lambda_i^1, ..., \lambda_i^R)_{i \in \mathbb{Z}}$, which serves as the environment, be an ergodic sequence of random nonnegative vectors, with $L, R$ two positive integers. We study birth and death process $\{N_t\}_{t \geq 0}$ in random environment, which, given the environment $\omega$, waits at a state $n$ an exponentially distributed time with parameter $\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n$ and then jumps to $n - i$ with probability $\mu_i^L_n / (\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n)$, $i = 1, ..., L$ or to $n + j$ with probability $\lambda_j^R_n / (\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n)$, $j = 1, ..., R$. Firstly, by an approach known as “the environment viewed from particle”, we prove a law of large numbers of the $h$-skeleton process $\{N_{nh}\}_{n \geq 1}$, which is indeed a random walk in random environment with unbounded jumps. Then by approximation, we prove the law of large numbers for the process $\{N_t\}_{t \geq 0}$. At last, using the branching structure of the embedded process, we give the explicit form of the velocity when $L = R = 2$.

Keywords: birth and death process; random environment; random walk; skeleton process.

MSC 2010: 60K37; 60J80

1 Introduction

1.1 Model and background

The aim of this paper is to study the birth and death process with bounded jumps in random environment. To construct the environment, fix $1 \leq L, R \in \mathbb{Z}$ and let $\Omega$ be the collection of $\omega = (\omega_i)_{i \in \mathbb{Z}} = (\mu_i^L, ..., \mu_i^1, \lambda_i^1, ..., \lambda_i^R)_{i \in \mathbb{Z}}$, where $\mu_i^L, \lambda_r^R \geq 0$ for all $i \in \mathbb{Z}$, $l = 1, ..., L$ and $r = 1, ..., R$. Equip $\Omega$ with the Borel $\sigma$-algebra $\mathcal{F}$ and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. The so-called random environment is a random element of $\Omega$ chosen according to $\mathbb{P}$. Then the realization of $\omega$, let $\{N_t\}_{t \geq 0}$ be a continuous time Markov chain, which waits at a state $n$ an exponentially distributed time with parameter $\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n$ and then jumps to $n - i$ with probability $\mu_i^L_n / (\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n)$, $i = 1, ..., L$ or to $n + j$ with probability $\lambda_j^R_n / (\sum_{i=1}^L \mu_i^L_n + \sum_{r=1}^R \lambda_r^R_n)$.
We call the process \( \{N_t\}_{t \geq 0} \) a birth and death process with bounded jumps in random environment.

Such processes are the continuous time analogue of random walk with bounded jumps in random environment which was introduced in Key [10] and further developed in Letchikov [13, 14], Brémont [2, 3, 4], Hong and Zhang [9], Hong and Wang [7, 8] etc.

The nearest neighbour setting (\( L = R = 1 \)) was studied in Ritter [16], where the existence, the criteria for recurrence and the Law of Large Numbers (LLN for short) were studied. The work of Ritter [16] could be carried out because the bilateral birth and death process (with jump size exactly one) was well developed. Especially, for the nearest setting, the LLN could be proved because under the annealed probability, the first passage times \( T_n = \inf\{t : N_t = n\}, n \geq 1 \) are stationary and mixing. However, for the bounded-jump setting, under the annealed probability, the ladder times \( T_0 = 0, T_n = \inf\{t : N_t > N_{T_{n-1}}\}, n \geq 1 \) are not stationary any longer. So to prove the LLN, one needs some other consideration.

In this paper, we adopt the method known as “the environment viewed from particle”. But such method does not work directly for \( \{N_t\} \), since process is of continuous time parameter. We consider firstly the LLN of \( h\)-skeleton process \( \{N_{nh}\}_{n \geq 0} \) and then pass to the LLN of \( \{N_t\} \) by approximation. It is interesting that the \( h\)-skeleton process is indeed a random walk in random environment with unbounded jumps.

To figure out the explicit asymptotic velocity of LLN, we use the branching structure within the embedded process revealed in Hong and Wang [8]. The idea is as follows. The invariant density for the “environment viewed from particle” is closely related to the quenched mean of the ladder time \( T_1 := \inf\{t > 0 : N_t > 0\} \). By the branching structure within the embedded process \( \{\chi_n\} \), one could use a multitype branching process in random environment to count how many times \( \{N_t\} \) has ever visited state \( i \) before \( T_1 \). But every time it visits \( i \), it would wait here an exponentially distributed time period. In this way, we could study the distribution of \( T_1 \) and consequently give the explicit velocity for LLN of \( \{N_t\} \) when \( L = R = 2 \).

1.2 Main results

Next we introduce some further notations. For a typical realization of \( \omega \), \( P^x_\omega \) denotes the law induced by the process \( \{N_t\} \) starting from \( x \). The measure \( P^x_\omega \) is usually related as the quenched probability. Define another probability measure \( P^x \) by \( P^x(\cdot) = \int_\Omega P^x_\omega(\cdot)P(d\omega) \), which is usually called the annealed probability. The notations \( E^x_\omega, E^x \) and \( E \) will be used to denote the expectation operators with respect to \( P^x_\omega, P^x \) and \( P \) respectively. The superscript \( x \) will be omitted if it is 0. Let operator \( \theta \) be the canonical shift on \( \Omega \) defined by \( (\theta\omega)_i = \omega_{i+1} \). The following two conditions are always assumed to be satisfied.

(C1) \( (\Omega, \mathcal{F}, P, \theta) \) forms a stationary and ergodic system.

(C2) the measure \( P \) is uniformly elliptic, that is,

\[
P(\varepsilon < \mu_l, \mu_0 < M, 1 \leq l \leq L, 1 \leq r \leq R) = 1
\]

for some small \( \varepsilon > 0 \) and large \( M > 0 \).

Under condition (C2), the process \( \{N_t\} \) exists \( P \)-a.s.. For details, see the appendix section.
Given $\omega$, define for $i \in \mathbb{Z}$,

$$b_i(k) = \begin{cases} 
\frac{\sum_{j=R-k+1}^{R} \lambda_j}{\mu_i} & \text{if } 1 \leq k \leq R, \\
-\frac{\sum_{j=R-k}^{R-1} \lambda_j}{\mu_i} & \text{if } R+1 \leq k \leq R+L-1,
\end{cases}$$

and let

$$A_i = \begin{pmatrix} 
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
b_i(1) & b_i(2) & \cdots & b_i(L+R-1)
\end{pmatrix}$$

be an $(L+R-1) \times (L+R-1)$ matrix.

Since $A_i$ depends only on $\omega$, $\{A_i\}_{i \in \mathbb{Z}}$ is an ergodic sequence of random matrices under $\mathbb{P}$. Moreover, under condition (C2), $\mathbb{E} \ln \|A_0^{-1}\| + \mathbb{E} \ln \|A_0\| < \infty$. Hence one could use Oseledec’s multiplicative ergodic theorem (see [15]) to the sequence $\{A_i\}_{i \in \mathbb{Z}}$. Consequently, we get the Lyapunov exponents of the sequence $\{A_i\}_{i \in \mathbb{Z}}$ which we write in increasing order as

$$-\infty < \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{R+L-1} < \infty.$$

**Proposition 1** (Recurrence/transience criteria). Suppose that conditions (C1) and (C2) are satisfied. Let $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{R+L-1}$ be the Lyapunov exponents of the sequence $\{A_i\}_{i \in \mathbb{Z}}$ under the probability measure $\mathbb{P}$. Then

1. $\gamma_R > 0 \Rightarrow P(\lim_{t \to \infty} N_t = \infty) = 1$;
2. $\gamma_R = 0 \Rightarrow P(-\infty = \liminf_{t \to \infty} N_t < \limsup_{t \to \infty} N_t = \infty) = 1$;
3. $\gamma_R < 0 \Rightarrow P(\lim_{t \to \infty} N_t = -\infty) = 1$.

**Proof.** Since the recurrence criteria for $\{N_t\}$ is the same as the embedded process $\{\chi_n\}$ defined below. Proposition 1 is just a corollary of Theorem A in Letchikov [14].

Let $\tau_0 = 0$, and for $n \geq 1$ define $\tau_n = \inf\{t > \tau_{n-1} : N_t \neq N_{\tau_{n-1}}\}$. Since the process $\{N_t\}$ exists, $P$-a.s., $\tau_n < \infty$ for all $n$. Indeed $\tau_n$, $n \geq 0$ are the consecutive discontinuities of $\{N_t\}$. Let $\chi_n = N_{\tau_n}$, $n \geq 0$. Then $\{\chi_n\}$ is called the embedded process of $\{N_t\}$. Next we study the LLN of $\{N_t\}$. Let $T_0 = 0$ and for $n \geq 1$, define recursively

$$T_n = \inf\{t > T_{n-1} : N_t > N_{T_{n-1}}\}.$$

We call $T_n$ the ladder time of the process $\{N_t\}$. Define

$$v_p = \mathbb{E}\left(\frac{\sum_{r=1}^{R} \sum_{k \leq 0} E_{\omega}^{-k} \omega (\sum_{j=1}^{U_k} \xi_{kj} | N_{T_1} = r)(\sum_{l=1}^{L}(-l)\mu_l^0 + \sum_{r=1}^{R} r\lambda_r^0)}{\sum_{r=1}^{R} E(T_1 | N_{T_1} = r)}\right)$$

where

$$U_k := \#\{n : N_{\tau_n} = k, \tau_n < T_1\}$$

is the number of times the embedded process $\{\chi_n\}$ has ever visited $k$ before it hits $[1, \infty)$, and given $\omega$, $\xi_{kj}$, $k \leq 0$, $j \geq 0$ are independent random variables which are also all independent of $U_k$ such that $P_{\omega}(\xi_{kj} > t) = e^{-t(\sum_{l=1}^{L} \mu_l^0 + \sum_{r=1}^{R} \lambda_r^0)}$, $t \geq 0$.  

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Theorem 1 (LLN of \{N_t\}). Suppose that conditions (C1) and (C2) are satisfied and \(\gamma_R \geq 0\). Then
(a) \(ET_1 < \infty \Rightarrow \lim_{t \to \infty} \frac{N_t}{t} = v_\omega > 0, P\text{-a.s.}\);
(b) \(ET_1 = \infty \Rightarrow \lim_{t \to \infty} \frac{N_t}{t} = 0, P\text{-a.s.}\).

Remark 1. (i) The velocity \(v_\omega\) of LLN is not given directly in the language of \(\omega\). One could calculate \(E_{\theta-k_\omega}(\sum_{j=1}^k \xi_{j}(N_{T_1}=r))\) and \(E_\omega(T_1|N_{T_1}=r)\) to give the explicit form of \(v_\omega\) by the branching structure constructed in \[8\]. We treat the case \(L = R = 2\) to explain the idea in Section 3. The special case \(R = 1, L > 1\) is discussed in Wang \[18\].
(ii) The case \(\gamma_R \leq 0\) could be treated in a similar way. We omit this part in this paper.
(iii) In the proof of part (a) of Theorem 1, we only use the condition “\(C2'\)” for some small \(\kappa > 0\) and large \(K > 0\), \(P(\kappa < \sum_{i=1}^L \mu_i + \sum_{r=1}^R \lambda^r < K) = 1\). Under condition “\(C2'\),” \(P(\lambda^r, \mu_i = 0\) for some \(1 \leq r \leq R, 1 \leq l \leq L > 0\) is permitted. So “\(C2'\)” is weaker than “\(C2\).” When proving part (b) of Theorem 1, we borrow some results from Brémond \[3\] where uniform ellipticity of “\(C2\)” is used.

Fix a number \(h > 0\), which will be assumed to be small enough. For \(n \geq 0\), define \(X_n = N_{nh}\). Then \(\{X_n\}\) is a discrete time random walk in random environment with unbounded jumps, which is usually called the \(h\)-skeleton process of \(\{N_t\}\). For \(i, j \in \mathbb{Z}\) let
\[
p_\omega(h, i, j) := P_\omega(N_h = i + j|N_0 = i)
\]
be the transition probabilities of \(\{X_n\}\). Define \(T_1^h = \inf\{k : X_k > 0\}\), and let
\[
U_k^h = \#\{0 \leq n < T_1^h : X_n = k\}.
\]

Theorem 2 (LLN of skeleton process). Suppose that conditions (C1) and (C2) hold. Then \(P\text{-a.s.}, \{X_n\}\) is transient to the right, recurrent or transient to the left according as \(\gamma_R \geq 0, \gamma_R = 0\) or \(\gamma_R \leq 0\). Moreover, if \(\gamma_R \geq 0\), then
\[
E(T_1^h) = \infty \Rightarrow P\text{-a.s.}, \lim_{n \to \infty} \frac{X_n}{n} = 0;
\]
\[
E(T_1^h) < \infty \Rightarrow P\text{-a.s.}, \lim_{n \to \infty} \frac{X_n}{n} = v_\omega^h > 0,
\]
where
\[
v_\omega^h = \frac{E\left(\sum_{i=1}^{\infty} \sum_{n \leq 0} E_{\theta-k_\omega}(U_k^h|X_{T_1^h=0}) \sum_{j \in \mathbb{Z}} p_\omega(h, 0, j)\right)}{\sum_{i=1}^{\infty} E(T_1^h|X_{T_1^h=i})}.
\]

Remark 2. (i) The \(h\)-skeleton process \(\{X_n\}\) is a random walk in random environment with unbounded jumps. We prove Theorem 2 based on an argument “the environment viewed from particles” which dates back to Kozlov \[11\]. But in Kozlov \[11\], this approach is limited to treat the bounded-jump setting. By estimating \(p_\omega(h, i, j)\) we could use the large deviation of a sequence of martingale differences with unbounded range. Hence LLN of \(\{X_n\}\) could be proved. (ii) One sees easily from Theorem 1 that \(v_\omega^h\) is indeed independent of \(h\) and \(v_\omega^h = v_\omega\). For details, see the proof of Theorem 1 below.

The paper is organized as follows. The LLN of \(h\)-skeleton process \(\{N_{nh}\}_{n \geq 0}\) is proved in Section 3 while the LLN for \(\{N_t\}\) is proved in Section 4 by approximation. We devote Section 4 to give the explicit form of the velocity \(v_\omega\) for the case \(L = R = 2\) by using the branching structure in the embedded process constructed in \[8\]. An Appendix is also given at the end of the paper to discuss the existence of the process \(\{N_t\}\).
2 LLN of the skeleton process-Proof of Theorem 2

The recurrence criteria for \( \{X_n\} \) in Theorem 2 follows directly from Proposition 1. The zero speed regime of the LLN for \( \{X_n\} \) follows from the counterpart of Theorem 1. But it will be a long journey to prove the non-zero speed regime of the LLN in Theorem 2. To begin with, we estimate the tail probability of the transition probability of \( \{X_n\} \). The following lemma shows that for fixed \( i \in \mathbb{Z}, \mathbb{P}\text{-a.s., } p_\omega(h, i, j) \) decays exponentially to 0 as \( |j| \to \infty \).

**Lemma 1.** Suppose that Condition (C2) is satisfied. Then for \( \mathbb{P}\text{-a.a. } \omega \), there exist \( 0 < c_0 < \infty \) and \( 0 < c_1 < \infty \), which are independent of \( h \) and \( \omega \), such that for \( |j| > \max\{L, R\} \),

\[
p_\omega(h, i, j) < e^{c_0 h} e^{-c_1 |j|},
\]

where \( c_1 \) could be made arbitrarily large by adjusting \( c_0 \). Consequently, for \( \mathbb{P}\text{-a.a. } \omega \) and \( \lambda \in \mathbb{R} \),

\[
E_\omega(e^{\lambda[N_h-i]}) < \infty.
\]

**Proof.** We prove only the case \( j > \max\{L, R\} \). The case \( j < -\max\{L, R\} \) follows similarly. Let \( m = \lceil \frac{h}{R} \rceil \). Since at each discontinuity, the process \( \{N_t\} \) jumps at most a distance \( R \) to the right, then starting from \( i \), in order to reach \( i+j \), there must be at least \( m \) discontinuities in a time period of length \( h \). Let \( \eta_k, k = 1, \ldots, m \) be these discontinuities and \( \tau_k \) be the waiting time after \( \eta_k \) until the process \( \{N_t\} \) leaves \( N_{\eta_k} \). Then we have that

\[
p_\omega(h, i, j) = P_\omega(N_h = i + j | N_0 = i) \leq P_\omega^i(\tau_1 + \ldots + \tau_m \leq h).
\]

Note that \( P_\omega^i(\tau_k > t | N_{\eta_k} = i_k) = e^{-s_k t} \) for \( t > 0 \) where \( s_k = \left( \sum_{l=1}^{L} \mu_{i_k}^l + \sum_{r=1}^{R} \lambda_{i_k}^r \right) \). Moreover under \( P_\omega^i \), \( \tau_k, k = 1, \ldots, m \) are mutually independent. Then by Chebycheff’s bound, it follows that for \( \lambda < 0, K = (L+R)M, \kappa = (L+R)\epsilon \),

\[
P_\omega^i(\tau_1 + \ldots + \tau_m \leq h)
= \sum_{i_1, \ldots, i_k} P_\omega^i \left( \sum_{k=1}^{m} \tau_k \leq h \mid N_{\eta_k} = i_k, 1 \leq k \leq m \right) P_\omega^i(N_{\eta_k} = i_k, 1 \leq k \leq m)
\leq \sum_{i_1, \ldots, i_k} e^{-\lambda h} E_\omega^i \left( e^{\lambda \sum_{k=1}^{m} \tau_k} \mid N_{\eta_k} = i_k, 1 \leq k \leq m \right) P_\omega^i(N_{\eta_k} = i_k, 1 \leq k \leq m)
= e^{-\lambda h} \prod_{i_1, \ldots, i_k} \frac{s_k}{s_k - \lambda} P_\omega^i(N_{\eta_k} = i_k, 1 \leq k \leq m)
\leq e^{-\lambda h} \left( \frac{K}{\kappa - \lambda} \right)^m, \mathbb{P}\text{-a.s.,}
\]

where the last inequality follows from condition (C2). Substituting the above estimation to (3), we have that \( \mathbb{P}\text{-a.s.,} \)

\[
p_\omega(h, i, j) \leq e^{-\lambda h} \left( \frac{K}{\kappa - \lambda} \right)^m \leq e^{-\lambda h} e^{\frac{\lambda h}{R} (\log K - \log (\kappa - \lambda))}.
\]

By choosing \( \lambda < 0 \) properly and letting \( c_0 = -\lambda, c_1 = (\log (\kappa - \lambda) - \log K)/R \), (2) is proved. Of course, we could make \( c_1 \) arbitrarily large by adjusting the value of \( \lambda \).
Lemma 2. Under the measure \( \mathbb{P} \), \( \{p_\omega(h,i,j)\}_{i \in \mathbb{Z}} \) is a stationary and ergodic sequence.

For the proof of the lemma, refer to Durrett [9].

For \( n \geq 0 \), define \( \overline{\pi}(n) = \theta^{X_n} \omega \). \( \{\overline{\pi}(n)\}_{n \geq 0} \) is an \( \Omega^N \)-valued process. It is usually called “the environment viewed from particle”. Let

\[
K^h(\omega, d\omega') = \sum_{j \in \mathbb{Z}} p_\omega(h, 0, j)\delta_{\omega' = \theta^j \omega}.
\]

Lemma 3. Under either \( P \) or \( P_\omega \), \( \{\overline{\pi}(n)\}_{n \geq 0} \) is a Markov chain with transition kernel \( K^h(\omega, \omega') \).

**Proof.** For test function \( f_1, ..., f_n, f_{n+1} \) we have that

\[
E_\omega \left( \prod_{k=1}^{n+1} f_k(\overline{\pi}(k)) \right) = E_\omega \left( \prod_{k=1}^{n} f_k(\overline{\pi}(k)) E_\omega^{X_n} f_{n+1}(\theta^{X_1} \omega) \right)
\]

\[
= E_\omega \left( \prod_{k=1}^{n} f_k(\overline{\pi}(k)) \sum_{j \in \mathbb{Z}} p_\omega(X_n, j, h) f_{n+1}(\theta^{X_n+j} \omega) \right)
\]

\[
= E_\omega \left( \prod_{k=1}^{n} f_k(\overline{\pi}(k)) \sum_{j \in \mathbb{Z}} K^h f_{n+1}(\overline{\pi}(n)) \right).
\]

Consequently, \( \{\overline{\pi}(n)\}_{n \geq 0} \) is a Markov chain under \( P_\omega \). Taking expectation, the above equations also yield the Markov property of \( \{\overline{\pi}(n)\}_{n \geq 0} \) under \( P \). \( \square \)

Whenever \( E(T_1^h) < \infty \), define the measures

\[
Q^h(\omega) := E \left( \sum_{i \geq 1} \frac{1_{X_{T_1^h} = i}}{P_\omega(X_{T_1^h} = i)} \prod_{k=0}^{T_1^h-1} 1_{\overline{\pi}(k) \in d\omega} \right), \quad \overline{Q}^h(\omega) = \frac{Q^h(\omega)}{E(T_1^h)}.
\]

Lemma 4. Suppose that conditions (C1), (C2) hold and \( E(T_1^h) < \infty \). Then \( Q \) is invariant under the kernel \( K^h \), that is

\[
Q(B) = \int \int 1_{\omega' \in B} K^h(\omega, \omega') Q^h(\omega) d\omega.
\]

Moreover, \( Q \sim \mathbb{P} \) and

\[
\frac{dQ}{d\mathbb{P}} = \sum_{k \leq 0} \sum_{i \geq 1} E_{\theta^{-k} \omega}(U_k^h | X_{T_1^h} = i) =: \pi^h(\omega),
\]

where \( U_k^h = \# \{ n \leq T_1^h : X_n = k \} \).

**Proof.** By the definition of \( Q^h \), we have that

\[
\int \int 1_{\omega' \in B} K^h(\omega, \omega') Q^h(\omega) d\omega = \int P_\omega(\theta^{X_1} \omega \in B) Q^h(\omega) d\omega
\]

\[
= E \left( \sum_{i \geq 1} \frac{1_{X_{T_1^h} = i}}{P_\omega(X_{T_1^h} = i)} \prod_{k=0}^{T_1^h-1} P_\omega(\theta^{X_1} \overline{\pi}(k) \in B) \right)
\]
Then it follows from (4) that

\[ E \left( \sum_{i \geq 1} \sum_{k \geq 0} P_\omega(X_{T_i^h} = i) \sum_{k=0}^{T_i^h-1} P_\omega(k)(\omega(k + 1) \in B) \right) \]

\[ = E \left( \sum_{i \geq 1} \sum_{k \geq 0} P_\omega \left( T_i^h \geq k + 1, \omega(k + 1) \in B \mid X_{T_i^h} = i \right) \right) \]

\[ = E \left( \sum_{i \geq 1} \sum_{k \geq 1} P_\omega \left( T_i^h > k, \omega(k) \in B \mid X_{T_i^h} = i \right) \right) \]

\[ + E \left( \sum_{i \geq 1} P_\omega \left( T_i^h < \infty, \theta^i \omega \in B \mid X_{T_i^h} = i \right) \right). \]

Since \( E(T_i^h) < \infty \), then \( P(T_i^h < \infty) = P(T_i^h > 0) = 1 \). This fact together with the stationarity implies that the right-most hand of the last equations equals to

\[ E \left( \sum_{i \geq 1} \sum_{k \geq 0} P_\omega \left( T_i^h > k, \omega(k) \in B \mid X_{T_i^h} = i \right) \right) + E \left( \sum_{i \geq 1} P_\omega \left( T_i^h > 0, \omega(0) \in B \mid X_{T_i^h} = i \right) \right) \]

\[ = E \left( \sum_{i \geq 0} \sum_{k \geq 0} P_\omega \left( T_i^h > k, \omega(k) \in B \mid X_{T_i^h} = i \right) \right) = Q^h(B). \]

The first part of the lemma follows. To prove the second part, for testing function \( f(\omega) \), we have that

\[ \int f dQ^h = E \left( \sum_{i \geq 1} \frac{1_{X_{T_i^h} = i}}{P_\omega(X_{T_i^h} = i)} \sum_{k=0}^{T_i^h-1} f(\omega(k)) \right) = E \left( \sum_{i \geq 1} \frac{1_{X_{T_i^h} = i}}{P_\omega(X_{T_i^h} = i)} \sum_{k=0}^{T_i^h-1} f(\theta^k \omega) \right) \]

\[ = E \left( \sum_{i \geq 1} \frac{1_{X_{T_i^h} = i}}{P_\omega(X_{T_i^h} = i)} \sum_{k \leq 0} U_k^h f(\theta^k \omega) \right) = E \left( \sum_{i \geq 1} E_\omega \left( \sum_{k \leq 0} U_k^h f(\theta^k \omega) \mid X_{T_i^h} = i \right) \right) \]

\[ = E \left( f(\omega) \sum_{i \geq 1} \sum_{k \leq 0} E_{\theta^{-k} \omega} \left( U_k^h \mid X_{T_i^h} = i \right) \right). \] (4)

By stationarity, we have that

\[ E \left( \sum_{i \geq 1} \sum_{k \leq 0} E_{\theta^{-k} \omega} \left( U_k^h \mid X_{T_i^h} = i \right) \right) = \sum_{i \geq 1} \sum_{k \leq 0} E \left( U_k^h \mid X_{T_i^h} = i \right) \leq E(T_i^h) < \infty. \]

Then it follows from (4) that \( Q^h \sim \mathbb{P} \) and

\[ \frac{dQ^h}{d\mathbb{P}} = \sum_{k \leq 0} \sum_{i \geq 1} E_{\theta^{-k} \omega} (U_k^h \mid X_{T_i^h} = i). \]

\( \square \)

**Lemma 5.** Under the conditions of Lemma 4, \( \{\omega(n)\} \) is stationary and ergodic under the probability measure \( Q^h \times P_\omega \).
With Lemma 4 in hand, Lemma 5 follows similarly as Sznitman [17], Theorem 1.2 or Zeitouni [19], Corollary 2.1.25.

Define the local drift \( d(x, \omega) = E_\omega^x (X_1 - X_0) \) and set

\[
M_n = X_n - X_0 - \sum_{k=0}^{n-1} d(X_n, \omega).
\]

**Lemma 6.** Under \( P_\omega \), \( \{M_n\} \) is a martingale and \( P\text{-a.s.}, \lim_{n \to \infty} \frac{M_n}{n} = 0 \).

**Proof.** Note that

\[
E_\omega(M_n - M_{n-1} | M_{n-1}, ..., M_0) = E_\omega(X_n - X_{n-1} - d(X_{n-1}, \omega) | X_{n-1}, ..., X_0) = 0.
\]

Then \( \{M_n\} \) is a martingale under \( P_\omega \). By Lemma 1, there exists some constant \( c_2 \) and \( c_3 \) such that

\[
E_\omega(|X_n - X_{n-1}|) < c_2 \quad \text{and} \quad E_\omega(e^{\lambda |X_n - X_{n-1}|}) < c_3.
\]

Therefore

\[
E_\omega(e^{\lambda |M_n - M_{n-1}|}) < \infty.
\]

Then one follows from Theorem 3.2 in [12] that there exists constant \( c_4 > 0 \) such that for \( \lambda > 0 \), and \( n \) large enough,

\[
P(|M_n| > \sqrt{n\lambda}) \leq e^{-c_4 \lambda^{\frac{2}{3}}}.\]

Then for \( \epsilon \) small, we have that

\[
P(|M_n| > \frac{n}{2} + \epsilon) \leq e^{-c_4 n^{\frac{2}{3}}}.\]

An application of Borel-Cantelli’s lemma yields that \( P\text{-a.s.}, \lim_{n \to \infty} \frac{M_n}{n} = 0 \).

**Proof of Theorem 2:** By Lemma 5, \( \{\varpi(n)\} \) is a stationary and ergodic sequence under the measure \( \overline{Q}^h \times P_\omega \). Using Birkhoff’s ergodic theorem, we have that for \( \overline{Q}^h \)-a.a. or \( P\)-a.a. \( \omega \), \( P_\omega \)-a.s.,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} d(X_k, \omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} d(0, \varpi(k)) = \int d(0, \omega) d\overline{Q}^h. \tag{5}
\]

We conclude from Lemma 6 and (5) that \( P\text{-a.s.}, \)

\[
\lim_{n \to \infty} \frac{X_n}{n} = \int d(0, \omega) d\overline{Q}^h =: v^h_\varpi.
\]

It follows from Lemma 4 that

\[
v^h_\varpi = \int d(0, \omega) d\overline{Q}^h = \frac{\mathbb{E} \left( \sum_{i=1}^{\infty} \sum_{h \leq 0} E_{\theta^{-k} \omega}(U_{\theta^{-k} h}^h | X_{T_i^h = \iota}) \sum_{j \in \mathbb{Z}} j p_\omega(h, 0, j) \right)}{\sum_{i=1}^{\infty} E(T_i^h | X_{T_i^h} = \iota)}.
\]

\[\square\]

3 The LLN of \( \{N_t\} \)-Proof of Theorem 1

In this section, using the LLN (nonzero speed regime) of the \( h \)-skeleton process proved in Section 2, we proof the LLN of the process \( \{N_t\} \). Firstly, we show that for \( h \) small enough, \( E(T_1^h) < \infty \) whenever \( E(T_1) < \infty \). Therefore, under the condition of part (a) of Theorem 1, the condition of the nonzero speed regime of LLN in Theorem 2 is also satisfied for \( h \) small enough.
Lemma 7. Fix $s > 1$. Suppose that condition (C2) holds. Then, for $n$ large enough, $P$-a.s.,
\[ T_1^{\frac{1}{n^s}} \leq T_1 + \frac{1}{n^s}. \]
Consequently, we have that for $n$ large, if $E(T_1) < \infty$, then $E\left(T_1^{\frac{1}{n^s}}\right) < \infty$ and $P$-a.s.,
\[ \lim_{n \to \infty} T_1^{\frac{1}{n^s}} = T_1. \]

Proof. Given $\omega$, we have that
\[
P_\omega\left(T_1^{\frac{1}{n^s}} \leq T_1 + \frac{1}{n^s}\right) \leq P_\omega\left(\{N_t\} has at least one jump in [T_1, T_1 + \frac{1}{n^s}]\right)
\]
\[
\leq P_\omega\left(\{N_t\} leaves N_{T_1} within time \frac{1}{n^s}\right)
\]
\[
= \sum_{r=1}^{R} P_\omega(N_{T_1} = r)P_\omega\left(\nu_r < \frac{1}{n^s} \mid N_{T_1} = r\right)
\]
where $\nu_r$ is the waiting time at state $r$ until the next jump of $\{N_t\}$ happens. Since $\nu_r$ is exponentially distributed with parameter $\zeta_r := \sum_{l=1}^{L} \mu_l^r + \sum_{k=1}^{R} \lambda_k^r$. Using condition (C2), with $K = (L + R)M$, it follows that $P$-a.s., the right-most hand of (6) equals to
\[
\sum_{r=1}^{R} P_\omega(N_{T_1} = r)(1 - e^{-\zeta_r \frac{1}{n^s}}) \leq 1 - e^{-K \frac{1}{n^s}}.
\]
Noting that $s > 1$, then we have that $P$-a.s.,
\[
\sum_{n=1}^{\infty} P\left(T_1^{\frac{1}{n^s}} \leq T_1 + \frac{1}{n^s}\right) < \infty.
\]
An application of Borel Cantelli’s lemma yields that for $n$ large, $P$-a.s.,
\[
T_1^{\frac{1}{n^s}} \leq T_1 + \frac{1}{n^s}.
\]

Since $P$-a.s., $T_1^{h} > T_1$, then one follows from (7) that $P$-a.s.,
\[ \lim_{n \to \infty} T_1^{\frac{1}{n^s}} \frac{1}{n^s} = T_1. \]

Proof of Theorem 1: We prove part (a) first. Suppose $\gamma_R \geq 0$ and $E(T_1) < \infty$. Fix $h > 0$ small enough. For any $t > 0$, there is a unique number $n_t$ such that $n_t h \leq t < (n_t + 1)h$. Let $J_t$ be the number of jumps of $\{N_t\}$ in the time interval $[n_t h, (n_t + 1)h)$. Then, under condition (C2), a similar argument as the proof of Lemma 1 yields that there exist some positive constants $c_5$ and $c_6$ such that
\[
P(J_t > n) \leq e^{c_5 h} e^{-n c_6}.
\]
Note that the bound in (8) is independent of $t$. Applying Borel-Cantelli lemma, we have that $P$-a.s.,
\[ \lim_{n \to \infty} \frac{J_t}{n} = 0 \]
uniformly in $t$. Since at each discontinuity, $\{N_t\}$ jumps at most a distance $L$ to the left or a distance $R$ to the right, we have that
\[
\frac{N_{n_t h} - J_t L}{(n_t + 1)h} \leq \frac{N_t}{t} \leq \frac{N_{n_t h} + J_t R}{n_t h}.
\]
For $h$ small enough, by Lemma 4, $E(T_1^h) < \infty$. Then we have from the nonzero speed regime of LLN in Theorem 2 (9) and (10) that P-a.s.,

$$\lim_{t \to \infty} \frac{N_t}{t} = \frac{v_p^h}{h}. \quad (11)$$

To prove part (a) Theorem 1, it suffices to show that, for all $h > 0$,

$$\frac{v_p^h}{h} = v_p. \quad (12)$$

Indeed, since $\frac{N}{t}$ is independent of $h$ and the limit in (11) exists, then $\frac{v_p^h}{h}$ is independent of $h$. Therefore we have that

$$\frac{v_p^h}{h} = \lim_{h \to 0} \frac{v_p^h}{h} = \lim_{h \to 0} \frac{\mathbb{E}\left( \sum_{i=1}^{\infty} \sum_{k \leq 0} E_{\theta-k}\omega(\sum_{j \in \mathbb{Z}} j \omega(h,0,j)) \right)}{h \sum_{i=1}^{\infty} E(T_1^h | X_{T_1^h} = i) \mathbb{E} \left( \sum_{i=1}^{\infty} \sum_{k \leq 0} E_{\theta-k}\omega(hU_k | N_{hT_1} = i) \right)}$$

$$= \lim_{h \to 0} \frac{\mathbb{E}\left( \sum_{i=1}^{\infty} \sum_{k \leq 0} E_{\theta-k}\omega(hU_k | N_{hT_1} = i) \right)}{\sum_{i=1}^{\infty} \mathbb{E} \left( hU_k | N_{hT_1} = i \right)} \quad (13)$$

Note that by Lemma 7 under condition (C2), P-a.s., $\sum_{j \in \mathbb{Z}} j \omega(h,0,j)$ is uniformly bounded from above. Moreover, by Lemma 7 P-a.s., $hT_1^h$ is bounded by $T_1 + h$ for $h$ small, and by stationarity

$$\mathbb{E}\left( \sum_{i=1}^{\infty} \sum_{k \leq 0} E_{\theta-k}\omega(hU_k | N_{hT_1} = i) \right) = \sum_{i=1}^{\infty} \sum_{k \leq 0} \mathbb{E} \left( E_{\theta-k}\omega(hU_k | N_{hT_1} = i) \right)$$

$$= \sum_{i=1}^{\infty} \sum_{k \leq 0} \mathbb{E} \left( hU_k | N_{hT_1} = i \right) = \sum_{i=1}^{\infty} \mathbb{E} \left( hT_1^h | N_{hT_1} = i \right).$$

Since $P$-a.s., $\lim_{h \to \infty} hT_1^h = T_1$, then by the above discussion, the condition of dominated convergence theorem is satisfied. Note also that $P$-a.s.,

$$\lim_{h \to 0} hU_k = \lim_{h \to 0} \sum_{j=0}^{T_1^h-1} h \mathbb{1}_{N_{ih} = k} = \int_0^{T_1} \mathbb{1}_{N_t = k} dt \overset{D}{=} \sum_{i=1}^{U_k} \xi_{ki}, \quad (14)$$

where “$\overset{D}{=}$” means equal in $P_\omega$ distribution. Since $P$-a.s., $\lim_{h \to 0} hT_1^h = T_1$, the path of $\{N_t\}$ is right continuous and $hT_1^h \geq T_1$, then by dominated convergence, it follows from (13) and (14) that

$$\frac{v_p^h}{h} = \lim_{h \to 0} \frac{\mathbb{E}\left( \sum_{i=1}^{\infty} \sum_{k \leq 0} E_{\theta-k}\omega(hU_k | N_{hT_1} = i) \right)}{\sum_{i=1}^{\infty} \mathbb{E} \left( hT_1^h | N_{hT_1} = i \right)}$$

$$= \frac{\mathbb{E}\left( \sum_{r=1}^{R} \sum_{k \leq 0} E_{\theta-k}\omega(\sum_{j=1}^{U_k} \xi_{kj} | NT_1 = r) \right)}{\sum_{r=1}^{R} \mathbb{E} \left( hT_1 | NT_1 = r \right)}$$

$$= v_p > 0,$$
where in the second equality, we use the facts $P(N_{T_1} = r) = 0$ for $r > R$ and
\[
\lim_{h \to 0} \frac{p_\omega(h, 0, j)}{h} = \begin{cases} 
\lambda_0^j & \text{if } j = 1, ..., R, \\
\mu_0^j & \text{if } j = 1, ..., L, \\
0 & \text{if } j < -L \text{ and } j > R.
\end{cases}
\]
Consequently, (12) is proved and part (a) of Theorem 1 follows.

Next, we turn to prove part (b) of Theorem 1. Suppose $\gamma R \geq 0$ and $E(T_1) = \infty$. Recall that $\tau_n, n \geq 0$ are the consecutive discontinuities of $\{N_t\}$ and $\{\chi_n\} = \{N_{\tau_n}\}$ is the embedded process. Let $\overline{T}_1 := \inf\{k : \chi_k > 0\}$. As the first step, we show that
\[
E(\overline{T}_1) = \infty. \tag{15}
\]
For this purpose, for $k \leq 0$ let $\overline{U}_k = \#\{n < \overline{T}_1 : \chi_n = k\}$. Then Wald’s equation implies that,
\[
E_\omega(T_1) = \sum_{k \leq 0} E_\omega\left(\sum_{i=1}^{\overline{U}_k} \xi_{ki}\right) = \sum_{k \leq 0} E_\omega(\overline{U}_k) E_\omega(\xi_{k1}).
\]
But condition (C2) implies that $\mathbb{P}$-a.s.,
\[
\frac{1}{K} < E_\omega(\xi_{k1}) = \frac{1}{\sum_{i=1}^L \mu_k^i + \sum_{r=1}^R \lambda_k^r} < \frac{1}{\kappa}
\]
with $K = (L + R)M$ and $\kappa = (L + R)\varepsilon$. Then it follows that $\mathbb{P}$-a.s.,
\[
\frac{1}{K} \sum_{k \leq 0} E_\omega(\overline{U}_k) \leq E_\omega(T_1) \leq \frac{1}{\kappa} \sum_{k \leq 0} E_\omega(\overline{U}_k).
\]
Taking expectation, we have that
\[
\frac{E(\overline{T}_1)}{K} \leq E(T_1) \leq \frac{E(\overline{T}_1)}{\kappa}.
\]
Therefore (15) follows. Then it follows from Brémont [3] (See, Proposition 9.1, Theorem 9.2 and Corollary 9.3 therein.) that $P$-a.s.,
\[
\lim_{n \to \infty} \frac{\chi_n}{n} = \lim_{n \to \infty} \frac{N_{\tau_n}}{n} = 0. \tag{16}
\]
As the second step, we show that for $h > 0$ small enough and $n$ large
\[
P \text{-a.s., } \frac{\tau_n}{n} > h. \tag{17}
\]
For $k \geq 1$, let $\nu_k = \tau_k - \tau_{k-1}$. Then under $P_\omega$, $\nu_k, k \geq 1$ is mutually independent and exponentially distributed. By Chebycheff’s bound, it follows that for $\lambda < 0$,
\[
P_\omega(\tau_n < nh) = e^{-\lambda nh} E_\omega(e^{\lambda \sum_{k=1}^n \nu_k}) = e^{-\lambda nh} \prod_{k=1}^n E_\omega(e^{\lambda \nu_k}) \leq e^{-n(\lambda h - \log K + \log(\kappa - \lambda))}
\]
with $\kappa = (L + R)\varepsilon$ and $K = (L + R)M$. Choosing properly $\lambda < 0$ and $h$ small enough, we could make $c(h) := \lambda h - \log K + \log(\kappa - \lambda)$ a strictly positive number. Consequently we have $\sum_{n=1}^{\infty} P_\omega(\tau_n < nh) < \infty$. By applying Borel-Cantelli lemma, we have (17).

Finally, (16) and (17) imply that $P$-a.s.,

\[
\lim_{n \to \infty} N_{\tau_n} = 0.
\]

For $t > 0$, there is a unique random number $n_t$ such that $\tau_{n_t} \leq t < \tau_{n_t+1}$. Since at each discontinuity, $\{N_t\}$ jumps at most a distance $L$ to the left or $R$ to the right, we have

\[
\frac{N_{\tau_{n_t}} - L}{\tau_{n_t+1}} \leq \frac{N_{\tau_{n_t}}}t \leq \frac{N_{\tau_{n_t}} + R}{\tau_{n_t}}.
\]

It follows from (18) and (19) that $P$-a.s.,

\[
\lim_{t \to \infty} \frac{N_t}t = 0.
\]

Part (b) of Theorem 1 is proved.

\section{Discussion of the velocity $v_\mathbb{P}$}

In this section, we let $R = L = 2$ and discuss the asymptotic velocity $v_\mathbb{P}$. We have

\begin{theorem}
Let $\pi(\omega)$ and $D(\omega)$ be as in (23) and (24) below. Suppose $L = R = 2$ and $E(\pi(\omega)) < \infty$. Then $P$-a.s.,

\[
\lim_{t \to \infty} \frac{N_t}t = \frac{E(\pi(\omega)(2\lambda_0^2 + \lambda_1^0 + \mu_1^0 - 2\mu_2^0))}{E(D(\omega))}.
\]

\end{theorem}

\begin{proof}
Theorem 3 is just a special case of Theorem 1. We need only to calculate $v_\mathbb{P}$. Recall that $T_1 = \inf\{n \geq 0 : \chi_n > 0\}$ is the first ladder time of the embedded process. Define

\[
\overline{U}_k = \#\{n \leq T_1 : \chi_n = k\}
\]

which is the occupation time at state $k$ of the embedded process. In [8], the authors show that $\overline{U}_k$ could be written as the functional of a multitype branching process $\{Z_n\}_{n \leq 1}$ whose offspring matrices are the function of the environment $\omega$. To introduce these results, we need to introduce some notations.

Fix $a < b$. Let $\partial^+[a, b] = \{b, b+1\}$ and $\partial^-[a, b] = \{a, a-1\}$ be the positive and negative boundaries of $[a, b]$ correspondingly. For $k \in (a, b)$, $\zeta \in \partial^+[a, b] \cup \partial^-[a, b]$, define

\[
P_k(a, b, \zeta) = P^k_\omega(\{\chi_a\} \text{ exits the interval } [a+1, b-1] \text{ at } \zeta).
\]

For $j = 1, 2$ let $p^j_k = \frac{\lambda_j^k}{\mu_1^k + \mu_2^k + \lambda_1^k + \lambda_2^k}$ and $q^j_k = \frac{\mu_1^k}{\mu_1^k + \mu_2^k + \lambda_1^k + \lambda_2^k}$.

Writing $P_k(a, b, \zeta)$ as $P_k(\zeta)$ temporarily, one follows from Markov property that

\[
P_k(\zeta) = p^2_k P_{k+2}(\zeta) + p^1_k P_{k+1}(\zeta) + q^1_k P_{k-1}(\zeta) + q^2_k P_{k-2}(\zeta),
\]
which leads to the following matrix form

\[ V_k(\zeta) = M_k V_{k+1}(\zeta) \]

where

\[
V_k(\zeta) := \begin{pmatrix}
(P_{k-1} - P_{k-2})(\zeta) \\
(P_k - P_{k-1})(\zeta) \\
(P_{k+1} - P_k)(\zeta)
\end{pmatrix},
\]

\[
M_k = \begin{pmatrix}
-\frac{\mu_1^2 + \mu_2^2}{\mu_1^2} & \frac{\lambda_1^2 + \lambda_2^2}{\mu_1^2} & \frac{\lambda_2^2}{\mu_1^2} \\
\lambda_1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

Then \( P_k(a, b, b) \) and \( P_k(a, b, b + 1) \) follows from some standard procedure. They could be expressed in terms of \( \{M_i\}_{i \in \mathbb{Z}} \). For \( k \leq i \), let

\[ f_k(i, i + 1) = P_k(-\infty, i + 1, i + 1) \quad \text{and} \quad f_k(i, i + 2) = P_k(-\infty, i + 1, i + 2). \]

For \( i \in \mathbb{Z} \), let

\[
\alpha_{i,1} = \frac{q_i^1 p_{i-1}^1}{1 - q_i^1 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\alpha_{i,3} = \frac{q_i^2 p_{i-1}^1}{1 - q_i^1 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\beta_{i,1} = \frac{q_i^2 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)}{1 - q_i^2 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\beta_{i,3} = \frac{q_i^2 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)}{1 - q_i^2 f_{i-1}^2(i - 2, i - 1) - q_i^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\gamma_{i,1} = \frac{q_{i+1}^1 p_{i-1}^1}{1 - q_{i+1}^1 f_{i-1}^2(i - 2, i - 1) - q_{i+1}^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\gamma_{i,3} = \frac{q_{i+1}^2 p_{i-1}^1}{1 - q_{i+1}^1 f_{i-1}^2(i - 2, i - 1) - q_{i+1}^2 f_{i-1}(i - 2, i - 1)},
\]

\[
\alpha_{i,2} := q_i^1 - \alpha_{i,1} - \alpha_{i,3}; \quad \beta_{i,2} := q_i^2 - \beta_{i,1} - \beta_{i,3}; \quad \gamma_{i,2} := q_{i+1}^2 - \gamma_{i,1} - \gamma_{i,3}.
\]

Set

\[
u_1 := \begin{pmatrix}
\alpha_{i,1} \\
\alpha_{i,1} + \alpha_{i,2} + \alpha_{i,3} \\
\alpha_{i,1} + \alpha_{i,2} + \alpha_{i,3} \\
\alpha_{i,1} + \alpha_{i,2} + \alpha_{i,3} \\
\alpha_{i,1} + \alpha_{i,2} + \alpha_{i,3} \\
0, \ldots, 0
\end{pmatrix} \in \mathbb{R}^9,
\]

and for \( i \leq 0 \) define

\[
x_i = \frac{\alpha_{i,1}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}, \quad
y_i = \frac{\alpha_{i,2}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}, \quad
z_i = \frac{\alpha_{i,3}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}},
\]

\[
w_i = \frac{\beta_{i,1}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}, \quad
v_i = \frac{\beta_{i,2}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}, \quad
s_i = \frac{\beta_{i,3}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}, \quad
\]

\[
t_i = \frac{\gamma_{i,1}}{1 - \alpha_{i,1} - \alpha_{i,2} - \alpha_{i,3}}.
\]

Define also the matrices

\[
Q_i = \begin{pmatrix}
x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\
x_i & y_i & s_i & z_i & w_i & 1 - s_i & 0 & 0 & 0 \\
x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\
x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\
x_i v_i & y_i v_i & s_i v_i & z_i v_i & w_i v_i & (1 - s_i) v_i & t_i v_i & (1 - t_i) v_i & 1 - v_i \\
x_i & y_i & 0 & z_i & w_i & 0 & t_i & 1 - t_i & 0 \\
x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0 \\
x_i & y_i & 0 & z_i & w_i & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

We have the following proposition which could be found in [8].

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Proposition 2. Suppose that \( \gamma_R \geq 0 \). For \( \mathbb{P} \)-a.a. \( \omega \), there exists a 9-type branching process \( \{Z_n\} \), whose mean offspring matrices are as in (21) and initial distribution mean is \( u_1 \) in (24), such that in \( P_\omega \)-distribution, with \( v_1 = (1, 1, 1, 0, 0, 0, 1, 1, 1)^T \) and \( v_2 = (1, 1, 0, 1, 1, 0, 1, 1, 0)^T \),
\[
U_k = Z_{k+1}v_1 + Z_kv_2, \ k \leq 0.
\]
Moreover, with empty product being identity,
\[
E_\omega(U_k|\chi_{T_1} = 2) + E_\omega(U_k|\chi_{T_1} = 1) = \left( \frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 1, 0, ..., 0 \right) (Q_0Q_{-1} \cdots Q_{k+1}v_1 + Q_0Q_{-1} \cdots Q_kv_2). \tag{22}
\]
Since after the \( i \)th visit of state \( k \), \( \{N_i\} \) will wait here an exponentially distributed time \( \xi_{ki} \) with parameter \( \mu_i^1 + \mu_i^2 + \lambda_i^1 + \lambda_i^2 \), then given \( \omega \), in \( P_\omega \) distribution,
\[
T_1 = \sum_{k \leq 0} \sum_{i=1}^{\gamma_k} \xi_{ki}.
\]
Hence by Ward’s equation, it follows from (22) that
\[
\sum_{r=1}^2 E_\omega(T_1|N_1 = r) = \sum_{r=1}^2 \sum_{k \leq 0} E_\omega(U_k|N_1 = r)E_\omega(\xi_{k1})
= \sum_{k \leq 0} \frac{1}{\mu_k^1 + \mu_k^2 + \lambda_k^1 + \lambda_k^2} \times \left( \frac{\alpha_{1,1}}{\alpha_{1,1} + \alpha_{1,2}}, \frac{\alpha_{1,2}}{\alpha_{1,1} + \alpha_{1,2}}, 1, 0, ..., 0 \right) (Q_0Q_{-1} \cdots Q_{k+1}v_1 + Q_0Q_{-1} \cdots Q_kv_2)
= : D(\omega) \tag{23}
\]
and
\[
\sum_{r=1}^2 \sum_{k \leq 0} E_{\theta^{-k}\omega}( \sum_{j=1}^{\gamma_k} \xi_{kj}|N_1 = r)
= \sum_{k = 0}^{\infty} \frac{1}{\mu_0^1 + \mu_0^2 + \lambda_0^1 + \lambda_0^2} \times \left( \frac{\alpha_{k+1,1}}{\alpha_{k+1,1} + \alpha_{k+1,2}}, \frac{\alpha_{k+1,2}}{\alpha_{k+1,1} + \alpha_{k+1,2}}, 1, 0, ..., 0 \right) (Q_kQ_{k-1} \cdots Q_1v_1 + Q_kQ_{k-1} \cdots Q_0v_2)
= : \pi(\omega). \tag{24}
\]
Substituting (23) and (24) to (1), we conclude that
\[
v_\mathbb{P} = \frac{\mathbb{E}\left( \sum_{r=1}^2 \sum_{k \leq 0} E_{\theta^{-k}\omega}( \sum_{j=1}^{\gamma_k} \xi_{kj}|N_1 = r) (2\lambda_0^2 + \lambda_0^1 - \mu_0^1 - 2\mu_0^2) \right)}{ \sum_{r=1}^R \mathbb{E}(T_1|N_1 = r)} = \frac{\mathbb{E}(\pi(\omega)(2\lambda_0^2 + \lambda_0^1 - \mu_0^1 - 2\mu_0^2))}{\mathbb{E}(D(\omega))}.
\]
Appendix: On the existence of \( \{N_t\} \)

Given \( \omega \), let \( Q = (q_{ij}) \) be a matrix with

\[
q_{ij} = \begin{cases} 
\lambda_r^i, & \text{if } j = i + r, \ r = 1, \ldots, R; \\
\mu_l^i, & \text{if } j = i - l, \ l = 1, \ldots, L; \\
-(\sum_{i=1}^L \mu_l^i + \sum_{r=1}^R \lambda_r^i), & \text{if } j = i; \\
0, & \text{else.}
\end{cases}
\]

Then \( Q \) is obviously a conservative Q-matrix. Note that under (C2), \( Q \) is bounded from above. Hence the process \( \{N_t\} \) exists (See for example Anderson [1], Proposition 2.9, Chapter 2.). Next we give a condition which implies the existence of \( \{N_t\} \) but is weaker than (C2). We have from classical argument that there exists at least one transition matrix \( (p_\omega(t, i, j)) \) such that

\[
\lim_{t \to 0} \frac{p_\omega(t, i, j) - \delta_{ij}}{t} = q_{ij}, \ i, j \in \mathbb{Z}.
\] (25)

Let \( (p_\omega(h, i, j)) \) be a standard transition matrix satisfying (25). Let \( \{N_t\} \) be a continuous time Markov chain with transition matrix \( (p_\omega(h, i, j)) \). Let \( \tau_0 = 0 \) and define \( \tau_n := \inf\{t > \tau_{n-1} : N_t \neq N_{\tau_{n-1}}\} \) recursively for \( n \geq 1 \). Then \( \tau_n, n \geq 0 \) are the consecutive discontinuities of the process \( \{N_t\} \).

Proposition 3. Suppose that \( \mathbb{P}(\sum_{l=1}^L \mu_l^0 + \sum_{r=1}^R \lambda_r^0 > 0) = 1 \) and

\[
\mathbb{P}\left(\sum_{n=1}^\infty \left(\max_{1 \leq k \leq R} \left\{ \sum_{r=1}^R \lambda_r^{nR-k} + \sum_{l=1}^L \mu_l^{nR-k} \right\} \right)^{-1} = \infty \right) = 1,
\]

\[
\mathbb{P}\left(\sum_{n=-\infty}^0 \left(\max_{1 \leq k \leq L} \left\{ \sum_{r=1}^L \lambda_r^{nL-k} + \sum_{l=1}^L \mu_l^{nL-k} \right\} \right)^{-1} = \infty \right) = 1.
\]

Then for \( \mathbb{P}\)-a.a. \( \omega \), there is a unique transition matrix \( (p_\omega(h, i, j)) \) which satisfies (25).

Proof. From the discussion above we have \( \mathbb{P}\)-a.s., \( \tau_n < \infty, n \geq 1 \). By the classical argument of the uniqueness of the Q-process, if

\[
P(\lim_{n \to \infty} \tau_n = \infty) = 1,
\]

then the minimal solution \( p_\omega(t, i, j) \) is the unique Q-transition matrix. Let \( q_i = -q_{ii}, \ i \in \mathbb{Z} \). If

\[
P\left(\sum_{n=0}^\infty q_{\chi_n}^{-1} = \infty \right) = 1,
\] (26)

then we have (see Chung [5], Theorem 1 in II.19) that \( P(\lim_{n \to \infty} \tau_n = \infty) = 1 \).
Next we show (26). In fact, if the process $\{\chi_n\}_{n \geq 0}$ is recurrent or transient to the right, it must visit at least one state of each of the sets $B_n := \{nR - k\}_{k=1}^R$ for $n = 0, 1, 2, \ldots$. Then $P$-a.s.,

$$
\sum_{n=0}^{\infty} q_n^{-1} \geq \sum_{n=1}^{\infty} \left( \max_{1 \leq k \leq R} \left\{ \sum_{r=1}^{R} \lambda_r n_{R-k} + \sum_{l=1}^{L} \mu_l n_{R-k} \right\} \right)^{-1} = \infty.
$$

Else if the process $\{\chi_n\}_{n \geq 0}$ is transient to the left, it must visit at least one state of each of the sets $A_n := \{nL - k\}_{k=1}^L$ for $n = 0, -1, -2, \ldots$. It follows that $P$-a.s.,

$$
\sum_{n=0}^{\infty} q_{\chi_n}^{-1} \geq \sum_{n=-\infty}^{0} \left( \max_{1 \leq k \leq L} \left\{ \sum_{r=1}^{R} \lambda_r n_{L-k} + \sum_{l=1}^{L} \mu_l n_{L-k} \right\} \right)^{-1} = \infty.
$$

Consequently (26) follows. \hfill \Box

**Acknowledgements:** The author would like to thank Professor Wenming Hong for his useful comments on the paper.

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