Convergence Rate of the (1+1)-Evolution Strategy with Success-Based Step-Size Adaptation on Convex Quadratic Functions

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ABSTRACT

The (1+1)-evolution strategy (ES) with success-based step-size adaptation is analyzed on a general convex quadratic function and its monotone transformation, that is, $f(x) = g((x - x^*)^T H (x - x^*))$, where $g: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, $H$ is a positive-definite symmetric matrix, and $x^* \in \mathbb{R}^d$ is the optimal solution of $f$. The convergence rate, that is, the decrease rate of the distance from a search point $x_t$ to the optimal solution $x^*$, is proven to be in $O(\exp(-L/\text{Tr}(H)))$, where $L$ is the smallest eigenvalue of $H$ and $\text{Tr}(H)$ is the trace of $H$. This result generalizes the known rate of $O(\exp(-1/d))$ for the case of $H = I_d$ ($I_d$ is the identity matrix of dimension $d$) and $O(\exp(-1/((d \cdot \xi)))$ for the case of $H = \text{diag}(\xi \cdot I_d^{1/2}, I_d^{1/2})$. To the best of our knowledge, this is the first study in which the convergence rate of the (1+1)-ES is derived explicitly and rigorously on a general convex quadratic function, which depicts the impact of the distribution of the eigenvalues in the Hessian $H$ on the optimization and not only the impact of the condition number of $H$.

KEYWORDS

(1+1)-evolution strategy, Convergence rate, Convergence analysis, Convex quadratic function

1 INTRODUCTION

Background. The evolution strategy (ES) is one of the most competitive classes of randomized algorithms in the field of continuous black-box optimization (BBO), where the objective function $f: \mathbb{R}^d \to \mathbb{R}$ can only be accessed through a black-box query $x \mapsto f(x)$, and neither the gradient $\nabla f$ nor the characteristic constants such as Lipschitz constants are available.

In particular, variants of the covariance matrix adaptation evolution strategy (CMA-ES) [17–19] demonstrate the prominent efficiency in overcoming a variety of BBO difficulties in both real-world and benchmark problems [10–12, 16, 25, 30–33]. Many studies were conducted to improve the competitiveness of CMA-ES [3, 4, 6, 23]. However, evolutionary approaches, including the above-mentioned references, are often developed on the basis of empirical evaluation on sets of benchmark problems, and their theoretical guarantee is yet to be developed sufficiently.

As a step towards the convergence guarantee of the state-of-the-art CMA-ES variants, in this study, we analyze a variant of ESs, the so-called (1+1)-ES with success-based step-size adaptation, which is the oldest variant of ES and was originally proposed by Rechenberg in 1973 [29]. Although the (1+1)-ES is drastically simpler than CMA-ES variants, especially because of the lack of covariance matrix adaptation mechanisms, they share several core features, such as randomness and heuristic step-size adaptation mechanisms. These features are surely the key to success but make it impossible to guarantee that the step size stays in a bounded range where a sufficient decrease in the objective function value in one step is guaranteed, which is the basis of the analysis of deterministic mathematical programming approaches [9, 27]. There is always a (possibly small) probability that the step size goes out of the desired range during the optimization. This makes it difficult to analyze the convergence properties of ES variants, including CMA-ES variants.

We study the convergence rate of the (1+1)-ES on convex quadratic functions and their monotone transformation, that is, $f(x) = g((x - x^*)^T H (x - x^*)/2)$, where $g: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, $H$ is a positive definite symmetric matrix, and $x^* \in \mathbb{R}^d$ is the optimal point. In particular, we are interested in obtaining the dependency of the convergence rate on the search space dimension $d$ and the condition number $\text{Cond}(H)$ of the Hessian matrix, that is, the ratio between the greatest and smallest eigenvalues of $H$. Because an arbitrary twice continuously differentiable function can be approximated by a convex quadratic function around its local optimal points, the asymptotic analysis on a general convex quadratic function is ubiquitous in the local convergence analysis on a broader class of functions. Previous studies [5, 26] showed that the (1+1)-ES converges linearly towards the global optimum on function classes including general convex quadratic functions; however, the convergence speed has not been given explicitly.
Related Work. Akimoto et al. [1] proved that the expected runtime of the (1+1)-ES until it finds an $\epsilon$-neighbor of the optimum is in $\Theta(d \cdot \log(1/\epsilon))$ on the spherical function $f: x \mapsto g(\|x\|^2)$.

Jägersküpper [20–22] carried out analyses on a spherical function and a specific convex quadratic function $f(x_1, \ldots, x_d) = \frac{\xi}{d} \sum_{i=1}^d x_i^2 + \frac{\xi}{2d} \sum_{j=d/2+1}^d x_j^2$ (note $\xi = \text{Cond}(H)$ in our notation), where $1/\xi \to 0$ as $d \to \infty$. The number of function evaluations to halve the function value is proven to be in $\Theta(d \cdot \xi)$ with an overwhelming probability. Loosely speaking, these results translate to the convergence rate of $\Theta(\exp\left(-\frac{1}{d \cdot \text{Cond}(H)}\right))$.

Linear convergence of the (1+1)-ES was proven to be a wider class of functions. Auger and Hansen [5] performed the Markov chain analysis of the (1+1)-ES on positively homogeneous functions with smooth levelsets, where a positively homogeneous function is a class of functions, all levelsets of which are similar in shape. Morinaga and Akimoto [26] extended the scope of the analysis of [1] and showed the linear convergence of the (1+1)-ES, including strongly convex and Lipschitz smooth functions and positively homogeneous functions. However, in these studies, the convergence rate was not explicitly derived, and hence its dependency on $d$ and $\text{Cond}(H)$ was not revealed.

Glasmacher [14] showed the convergence of the (1+1)-ES to a stationary point, even on a broader class of functions. Because the objective of [14] was not to show linear convergence, the convergence was not guaranteed to be linear.

To sum up, there is no prominent result that explicitly estimates the impact of an ill-conditioned problem on the convergence rate of the (1+1)-ES on a general convex quadratic function. Nevertheless, for the specific convex quadratic functions [21], there are clues that the convergence rate of the (1+1)-ES is in $\Theta\left(\exp\left(-\frac{1}{d \cdot \text{Cond}(H)}\right)\right)$. The objective of the current study is to extend the scope of the latter analysis to a general convex quadratic function.

Contribution. We prove that the convergence rate of the (1+1)-ES on a convex quadratic function and its monotone transformation is in $O\left(\exp\left(-\frac{L}{\text{Tr}(H)}\right)\right)$, where $L$ is the smallest eigenvalue of a positive definite Hessian $H$ and $\text{Tr}(H)$ is the trace of $H$. Because $\frac{L}{\text{Tr}(H)} \geq \frac{1}{d \cdot \text{Cond}(H)}$ for any positive definite symmetric $H$, the convergence rate is indicated to be in $O\left(\exp\left(-\frac{1}{d \cdot \text{Cond}(H)}\right)\right)$.

The order $O\left(\exp\left(-\frac{L}{\text{Tr}(H)}\right)\right)$ is consistent with the upper bound obtained for a specific $H$ in previous studies [1, 20–22]. Our result is more rigorous than those of the previous studies in that the convergence rate becomes slower not only as the condition number increases but also as the distribution of large eigenvalues of the Hessian becomes heavier. In addition, we provide a lower bound of the convergence rate, which is in $\Omega\left(\exp\left(-\frac{1}{d \cdot \text{Cond}(H)}\right)\right)$. In the case of a spherical function, where $H$ is the identity matrix, the upper and lower bounds admit, and the result is consistent with the consequence of [1].

Organization. In Section 2, the algorithm to be studied is described in detail, and the concept of the algorithm design and important features are also discussed. Next, the definition of the convergence rate and a technique to bound it are presented. Then, the goal of the current research is formulated. In Section 3, we analyze the probability that the algorithm succeeds in advancing the optimization. Based on this, the expected progress of the optimization is estimated. In Section 4, we show that the convergence rate is in $O\left(\exp\left(-L/\text{Tr}(H)\right)\right)$ with a potential function approach proposed in [1, 26]. Furthermore, in Section 5, the convergence rate is revealed to be in $\Omega\left(\exp\left(-\text{Cond}(H)/d\right)\right)$. In Section 6, we summarize the results and offer implications for a deeper understanding of our findings. Finally, a perspective on future research is given. Most of the proofs of technical lemmas are provided in the supplementary material.

2 FORMULATION

2.1 Algorithm

We analyze the (1+1)-ES with success-based step-size adaptation, as described in Algorithm 1. The adaptation mechanism of the step size $\sigma$ is a generalized variant of the well-known $1/5$-success rule, first presented by Rechenberg [29] and then simplified by Kern et al. [24].

Algorithm 1 (1+1)-ES with success-based $\sigma$-adaptation

1: input $m_0 \in \mathbb{R}^d$, $\sigma_0 > 0$, $f: \mathbb{R}^d \to \mathbb{R}$
2: parameter $\alpha_+ > 1$, $\alpha_- < 1$
3: for $t = 0, 1, \ldots$, until the stopping criterion is met do
4:    $x_t \sim m_t + \sigma_t \cdot z_t$, where $z_t \sim N(0, I)$
5:    if $f(x_t) \leq f(m_t)$ then
6:        $m_{t+1} \leftarrow x_t$
7:        $\sigma_{t+1} \leftarrow \sigma_t \cdot \alpha_+$
8:    else
9:        $m_{t+1} \leftarrow m_t$
10:       $\sigma_{t+1} \leftarrow \sigma_t \cdot \alpha_-$
11:    end if
12: end for

The main components of this algorithm are greedy selection of a candidate solution and step-size adaptation. For each iteration $t \geq 0$, a candidate $x_t$ is sampled from the isotropic Gaussian distribution $\mathcal{N}(m_t, \sigma^2\cdot I)$ (Line 4), the objective values of the candidate and current solutions (Lines 5 and 8) are compared, the current solution is updated with the candidate solution if the candidate solution is not worse than the current one (Lines 6, 9), and the step size is adapted in response to the result of the comparison (lines 7 and 10).

The step-size adaptation is designed to maintain the probability of sampling a candidate solution better than or equally good as $p_{\text{target}}$, defined as

$$p_{\text{target}} := \frac{\log(1/\alpha_-)}{\log(\alpha_+/\alpha_-)}.$$ (1)
If the probability to obtain a better solution is always $p_{\text{target}}$, the expected step size $\sigma_t$ keeps the same value, and if not, the algorithm attempts to control the probability by increasing or decreasing the step size $\sigma_t$. In fact, if $f$ is continuously differentiable, the probability of $f(x_t) \leq f(m_t)$ can get arbitrarily closer to 1/2 by taking $\sigma_t \rightarrow 0$ for any noncritical $m_t$, that is, $\nabla f(m_t) \neq 0$ [14].

A mathematical model of the (1+1)-ES is defined as follows.

**Definition 1.** Let $\Theta = \mathbb{R}^{d+1}$ be the state space and $\theta_t = (m_t, \log(\sigma_t))$ be the state of Algorithm 1 at iteration $t$. Let $\{z_t\}_{t \geq 0}$ be the sequence of independent and $N(0, I)$-distributed random vectors. For a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, let $\theta_0 = (m_0, \log(\sigma_0))$ and $\theta_{t+1} = \theta_t + \mathcal{G}(\theta_t, z_t; f)$, where

$$
\mathcal{G}(\theta, z; f) := (\sigma \cdot z, \log(\sigma_t)) - \frac{1}{2} \left( f(m + \sigma \cdot z) - f(m) \right). 
$$

(2)

Let $\{T_t\}_{t \geq 0}$ be the natural filtration of $\{\theta_t\}_{t \geq 0}$. We write

$$
\{(\theta_t, T_t)\}_{t \geq 0} = \mathcal{E}(f, \theta_0, \{z_t\}_{t \geq 0}).
$$

(3)

The state sequence defined in Definition 1 is invariant to any strictly increasing transformation of the objective function and to any translation with a corresponding translation of the initial search point. This is formally stated in the following proposition. Its proof is provided in Appendix A.1.

**Proposition 1.** Given $\{z_t\}_{t \geq 0}$ and $\theta_0 \in \Theta$, the following hold.

1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary strictly increasing function, i.e., $g(x) < g(y)$ for $x < y$. Subsequently, $\theta_t = \theta_0$ for all $t \geq 0$.

2. Let $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be an arbitrary translation of the state vector $x$ for $x \rightarrow x^*$. Define $S_T : (m, \log(\sigma)) \rightarrow (T^{-1}(m), \log(\sigma))$. Subsequently, $\theta_t = S_T(\theta_t)$ for all $t \geq 0$, for $(\theta_t, T_{t})_{t \geq 0} = \mathcal{E}(f, \theta_0, \{z_t\}_{t \geq 0})$ and $(\theta_t, T_{t})_{t \geq 0} = \mathcal{E}(f \circ T, S_T(\theta_0), \{z_t\}_{t \geq 0})$.

2.2 Convergence Rate

Previous studies [1, 2, 5, 20–22, 26] suggested that the (1+1)-ES converges not faster than or linear convergence on a broad class of functions including convex quadratic functions. Loosely speaking, linear convergence implies that the logarithmic distance $\log(\|m_t - x^*\|)$ from $m_t$ to the optimal point $x^*$ decreases by a constant. We formally define the linear convergence of the (1+1)-ES as follows.

**Definition 2.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function with the global optimum at $x^*$ and $\{\theta_t\}_{t \geq 0}$ be the sequence of the state vectors defined in Definition 1 with $m_t \neq x^*$. If there exists a constant $A > 0$ satisfying

$$
\mathbb{P} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|} = -A \right) = 1,
$$

then $\exp(-A)$ is called the convergence rate of the (1+1)-ES on $f$. The upper convergence rate $\exp(-A^{\text{up}})$ and the lower convergence rate $\exp(-A^{\text{inf}})$ are defined as constants satisfying

$$
\mathbb{P} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|} = -A^{\text{up}} \right) = 1,
$$

$$
\mathbb{P} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|} = -A^{\text{inf}} \right) = 1.
$$

Note that $A^{\text{up}}$ and $A^{\text{inf}}$ always exist in $\mathbb{R}_{\geq 0}$, whereas $A$ does not necessarily exist. $A$ exists if and only if $A^{\text{up}} = A^{\text{inf}}$. Otherwise, $\exp(-A^{\text{inf}}) < \exp(-A^{\text{up}}) < 0$. Our objective is to derive an upper bound of $\exp(-A^{\text{up}})$ and a lower bound of $\exp(-A^{\text{inf}})$.

We bound the lower convergence rate and the upper convergence rate from below and above, respectively, by using the following proposition, which relies on the strong law of large numbers on Martingale [8]. Its proof is provided in Appendix A.2.

**Proposition 2.** Let $\{T_t\}_{t \geq 0}$ be a filtration of a $\sigma$-algebra and $\{X_t\}_{t \geq 0}$ be a Markov chain adapted to $\{T_t\}_{t \geq 0}$. Consider the following conditions:

1. $\mathbb{E}(X_t - X_t | T_t) \leq -B$ for all $t \geq 0$.
2. $\mathbb{E}(X_t - X_t | T_t) \geq -C$ for all $t \geq 0$.

If C1 and C3 hold, then

$$
\mathbb{P} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}(X_t - X_0) \leq -B \right) = 1.
$$

(7)

If C2 and C3 hold, then

$$
\mathbb{P} \left( \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}(X_t - X_0) \geq -C \right) = 1.
$$

(8)

2.3 Problem

We analyze the (1+1)-ES (Definition 1) on convex quadratic functions and its monotone transformations, defined as follows:

**Definition 3.** $Q$ is a set of all functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ that can be expressed as $f(x) = g((x - x^*)^T H (x - x^*))$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ represents a strictly increasing function, $H \in \mathbb{R}^{d \times d}$ represents a positive definite symmetric matrix, and $x^* \in \mathbb{R}^d$ is the global optimal point. For simplicity, we call $H$ the Hessian matrix of $f$.

The property of convex quadratic functions allows us to investigate $\log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|}$ instead of $\log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|}$ to derive the upper convergence rate and the lower convergence rate. This is advantageous because $f(m_t)$ is not increasing in $x$, whereas $\|m_t - x^*\|$ is. Moreover, Proposition 1 shows that it is sufficient to work on $f : x \mapsto \frac{1}{2} x^T H x$ to represent the analysis on $h \in Q$. It is formally stated in the following proposition, the proof of which is provided in Appendix A.3.

**Proposition 3.** Let $f \in Q$ and $g, h, x^*$, as given in Definition 3. Let $h : x \mapsto \frac{1}{2} x^T H x$ and $S : (m, \log(\sigma)) \mapsto (m + x^*, \log(\sigma))$. We define $(\theta_t, T_t)_{t \geq 0} = \mathcal{E}(f, \theta_0, \{z_t\}_{t \geq 0})$ and $(\theta_t, T_t)_{t \geq 0} = \mathcal{E}(h, S(\theta_0), \{z_t\}_{t \geq 0})$. Next, for any $\theta_t \in \Theta$, with probability one (note that the probability is taken for $\{z_t\}_{t \geq 0}$), we obtain that

$$
\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|} = \lim_{t \rightarrow \infty} \frac{1}{2t} \log \frac{h(m_t)}{h(m_0)},
$$

(9)

$$
\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\|m_t - x^*\|}{\|m_0 - x^*\|} = \lim_{t \rightarrow \infty} \frac{1}{2t} \log \frac{h(m_t)}{h(m_0)}.
$$

(10)

3 LEMMAS

In this section, we assume that $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex quadratic function $f(x) = \frac{1}{2} x^T H x$, where $H$ is a positive definite symmetric matrix, the smallest and greatest eigenvalues of which are denoted
by $L$ and $U$, respectively. We investigate the expected one-step decrease in $\log(f(m_l))$, that is, the expectation of $\log(f(m_{l+1})) - \log(f(m_l))$, and the success probability, probability of the event $f(x_t) \leq f(m_l)$.

Exploiting the assumptions that $f$ is convex quadratic, an upper bound of the expected one-step decrease in $\log(f(m_l))$ is derived in the form of a product of a quadratic form of $\sigma_l$ and the success probability. The upper convergence rate bound is derived on the basis of the following result. Its proof is provided in Appendix A.4.

**Lemma 5.** Suppose that $f$ is convex quadratic, an upper bound of the expected one-step decrease in $\log(f(m_l))$ is derived in the form of a product of a quadratic form of $\sigma_l$ and the success probability. The upper convergence rate bound is derived on the basis of the following result. Its proof is provided in Appendix A.4.

**Lemma 4.** Let $z$ be a random vector that follows the $d$-dimensional standard normal distribution. For any $m \in \mathbb{R}^d$ and $\sigma > 0$,

$$E \left[ \log \left( \frac{f(m + \sigma z)}{f(m)} \right) 1 \{ f(m + \sigma z) \leq f(m) \} \right] \leq$$

$$\sigma \| \nabla f(m) \| \left( \frac{1}{4 \| \nabla f(m) \|} - \frac{1}{\sqrt{2 \pi}} \right) \cdot \Pr \left[ f(m + \sigma z) \leq f(m) \right].$$

(11)

The following lemma is used to bound the variance of the one-step decrease in $\log(f(m_l))$, which will be used to prove condition C3 of Proposition 2. It is also used to lower-bound the expected one-step decrease in $\log(f(m_l))$. Its proof is provided in Appendix A.5.

**Lemma 5.** Suppose that $d > 3$ and let $z$ be a random variable that follows the $d$-dimensional standard normal distribution. For any $m \in \mathbb{R}^d$ and $\sigma > 0$,

$$E \left[ \exp \left( \log \left( \frac{f(m + \sigma z)}{f(m)} \right) 1 \{ f(m + \sigma z) \leq f(m) \} \right) \right] \leq 1 + \frac{1}{d - 3} U.$$  

(12)

If $\text{Tr}(H^2)/\text{Tr}(H)^2$ is sufficiently small, which occurs if $\text{Cond}(H)$ is bounded and $d \to \infty$ because $\text{Tr}(H^2)/\text{Tr}(H)^2 \leq \text{Cond}(H)/d$, the success probability can be approximated by $\Phi \left( \frac{1}{2 \| \nabla f(m) \|} \right)$ with the cumulative distribution function of the one-dimensional standard normal distribution. It is formally stated in the following lemma, the proof of which is provided in Appendix A.6.

**Lemma 6.** Let $z$ denote a random vector that follows the $d$-dimensional standard normal distribution and $\Phi(\cdot)$ denote the cumulative distribution function of the one-dimensional standard normal distribution. For any $m \in \mathbb{R}^d$, $\sigma > 0$, and $\epsilon > 0$,

$$\Phi \left( \frac{1}{2 \| \nabla f(m) \|} \right) \cdot (1 + \epsilon) = \frac{2}{\epsilon^2} \frac{\text{Tr}(H)^2}{\text{Tr}(H)^2} \cdot \text{Pr} \left[ f(m + \sigma z) \leq f(m) \right]$$

$$< \Phi \left( \frac{1}{2 \| \nabla f(m) \|} \right) \cdot (1 - \epsilon) + \frac{2}{\epsilon^2} \frac{\text{Tr}(H)^2}{\text{Tr}(H)^2}.$$  

(13)

As a corollary of Lemma 6, we can derive sufficient conditions on $\frac{\sigma \text{Tr}(H)}{\| \nabla f(m) \|}$ to upper-bound and lower-bound the success probability. The proof is provided in Appendix A.7.

**Corollary 7.** Define $B^\text{high}_H : \left( 0, \frac{U}{2} \right) \to \mathbb{R}_{>0}$ as

$$B^\text{high}_H(q) := \frac{1}{q - \frac{\epsilon^2}{4} \frac{\text{Tr}(H)^2}{\text{Tr}(H)^2}}.$$  

(14)

Subsequently,

$$\frac{\Phi(\text{Tr}(H))}{\| \nabla f(m) \|} \leq B^\text{high}_H(q) \Rightarrow \Pr \left[ f(m + \sigma z) \leq f(m) \right] < q.$$  

(15)

Suppose that $\text{Tr}(H^2) < \text{Tr}(H)^2/4$. We define $B^\text{low}_H : \left( 2 \cdot \frac{\text{Tr}(H)^2}{\| \nabla f(m) \|}, \frac{U}{2} \right) \to \mathbb{R}_{>0}$ as

$$B^\text{low}_H(q) := \Phi \left( \frac{1}{2 \| \nabla f(m) \|} \right) \cdot \frac{1}{q - \frac{\epsilon^2}{4} \frac{\text{Tr}(H)^2}{\text{Tr}(H)^2}}.$$  

(16)

Subsequently,

$$\frac{\Phi(\text{Tr}(H))}{\| \nabla f(m) \|} \geq B^\text{low}_H(q) \Rightarrow \Pr \left[ f(m + \sigma z) \leq f(m) \right] > q.$$  

(17)

$B^\text{low}_H$ is left-continuous, strictly decreasing, and $B^\text{low}_H(q) \geq 2\Phi^{-1}(1 - q)$ for all $q \in \left( 0, \frac{U}{2} \right)$.

Lemma 8, derived from Lemma 4 and corollary 7, splits the state space $\Theta$ of the (1+1)-ES into three distinct areas. Case (i) corresponds to a situation in which $\sigma$ is so small that the success probability is close to 1/2. Case (ii) corresponds to a situation in which $\sigma$ is so large that the success probability is close to 0. In both cases, the expected one-step decrease in $\log(f(m))$ is close to zero. Case (iii) corresponds to a situation in which $\sigma$ is in a reasonable range, where we can guarantee that the expected decrease in $\log(f(m))$ is sufficient. The proof is provided in Appendix A.8.

**Lemma 8.** Suppose that

$$\frac{\text{Tr}(H^2)}{\text{Tr}(H)^2} < \frac{1}{8} \min \left\{ \Phi \left( \frac{1}{2 \sqrt{2 \pi}} \right), 1 - \Phi \left( \frac{3}{2 \sqrt{2 \pi}} \right) \right\}.$$  

(18)

Let $B^\text{high}_H$ and $B^\text{low}_H$ be defined in Corollary 7. Subsequently, there exists $q^\text{low}$ satisfying

$$2 \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H)^2} < q^\text{low} < \frac{1}{2},$$  

(19)

$$\frac{4}{\sqrt{2 \pi}} > B^\text{low}_H(q^\text{low}).$$  

(20)

Fix such $q^\text{low}$, and then there exists $q^\text{high}$ such that $q^\text{low} < q^\text{high} < 1/2$. We define

$$Q_H = \sup \left\{ Q : B^\text{high}_H(Q) > B^\text{low}_H(q^\text{low}) \right\}.$$  

(21)

Thus, $Q_H > 0$. Moreover, the following statements hold.

(i) If $\sigma < B^\text{high}_H(q^\text{high}) \cdot \sqrt{2Lf(m)}/\text{Tr}(H)$,

$$\Pr \left[ f(m + \sigma z) \leq f(m) \right] > q^\text{high}.$$  

(22)
In this section, the upper convergence rate of the (1+1)-ES on convex quadratic functions is defined as follows. This form of the potential function is defined in Definition 4. Let \( \{ (\theta_t, T_t) \}_{t>0} = \text{ES}(f, \theta_0, \{z_t\}_{t>0}) \) for \( \theta_0 \in \Theta \). Next, we apply Proposition 2 with \( X_t = V(\theta_t) \). Conditions C1 and C3 in Proposition 2 are derived in Section 4.2. Subsequently, we obtain \( B > 0 \) such that
\[
\Pr \left[ \limsup_{t \to \infty} \frac{1}{t} V(\theta_t) \leq B \right] = 1 .
\]
Because \( \log(f(m)) \leq V(\theta) \), using Proposition 3, we obtain the upper bound of the upper convergence rate. Finally, we evaluate the dependency of \( B \) on \( d \) and \( \text{Cond}(H) \) in Section 4.3.

### 4.1 Potential Function

The potential function \( V(\theta) \) on a convex quadratic function \( f(x) = \frac{1}{2} x^T H x \) is defined as follows. This form of the potential function was first proposed by [1] for the sphere function and was generalized in [26]. The following definition is specialized for a convex quadratic function:

**Definition 4 (Potential function).** Let \( f : x \mapsto \frac{1}{2} x^T H x \) with a positive definite symmetric \( H \). Let \( L \) and \( U \) be the smallest and greatest eigenvalues of \( H \), respectively. The potential function \( V(\theta) \) for the (1+1)-ES solving \( f \) is defined as
\[
V(\theta) = \log(f(m)) + v \cdot \log^+ \left( \frac{b_H \sqrt{L f(m)}}{\text{Tr}(H) \sigma} \right) + v \cdot \log^+ \left( \frac{\sqrt{\text{Tr}(H) \sigma}}{b_H \sqrt{L f(m)}} \right) \quad (27)
\]
where \( v \in (0, 2) \) and \( 0 < b_s < b_L \) are constants, and \( \log^+ (x) := \log(x) \cdot 1 \{ x > 1 \} \).

The first term measures the main quantity that we would like to decrease. The second and third terms measure the progress of \( \sigma \)-adaptation when \( \sigma \) is too small and too large, respectively. See [1] and [26] for a more detailed description.

We define \( b_s, b_L, \) and \( v \) as follows. Suppose that (18) holds. We choose \( q^\text{low} \) and \( q^\text{high} \) such that
\[
2 \cdot \frac{\text{Tr}(H_2)}{\text{Tr}(H)^2} < q^\text{low} < p_{\text{target}} < q^\text{high} < \frac{1}{2} , \quad (28)
\]
\[
\frac{4}{\sqrt{2\pi}} > q^\text{low} > q^\text{high} \cdot \frac{a_1}{a^*_1} \cdot q^\text{high} \cdot \frac{a^*_1}{a_1} . \quad (29)
\]
If \( a_1 \) and \( a^*_1 \) are set so that
\[
\frac{4}{\sqrt{2\pi}} > q^\text{high} (p_{\text{target}}) \quad (30)
\]
holds, as we see in Lemma 8, we can find such \( q^\text{low} \). Because \( b^\text{high}(q) < 2b^{-1} (1 - q) \to 0 \) as \( q \to 1/2 \), we see that we can find a pair \( (q^\text{low}, q^\text{high}) \) satisfying the conditions above. Then, we set \( b_L \) as follows
\[
b_s = \sqrt{2} \cdot b^\text{high}(q^\text{high}) \cdot a_1, \quad (31)
\]
\[
b_L = \sqrt{2} \cdot b^\text{high}(q^\text{low}) \cdot a^*_1. \quad (32)
\]
It is easy to see from eqs. (29), (31) and (32) that \( 0 < b_s < b_L \). Let
\[
w = \frac{L \cdot b^\text{high}(q^\text{high})}{2 \text{Tr}(H)} \left( \frac{4}{\sqrt{2\pi}} - \frac{b^\text{low}(q^\text{low})}{\text{Tr}(H)} \right) \cdot Q_H. \quad (33)
\]
so that the left-hand side of (25) is upper-bounded by \(-w\). Finally, we set \( v \) as
\[
v = \min \left\{ \frac{w}{4 \log(\alpha_1/\alpha^*_1)}, 1 \right\}. \quad (34)
\]

### 4.2 Expected Potential Decrease

The following three lemmas guarantee that the potential function \( V(\theta_t) \) decreases sufficiently in any case of the step size in Lemma 8. In the following lemmas, let \( f(x) = \frac{1}{2} x^T H x \) with a positive definite symmetric \( H \) satisfying (18). Let \( \{ (\theta_t, T_t) \}_{t>0} = \text{ES}(f, \theta_0, \{z_t\}_{t>0}) \) be the state sequence of the (1+1)-ES solving \( f \) with \( \theta_0 \in \Theta \). The potential function \( V \) is defined in Definition 4.

The first lemma is for the case in which the step size is too small to expect a sufficient decrease in \( \log(f(m_t)) \). In this situation, however, we can lower-bound the success probability by \( q^\text{high} \) in light of Lemma 8. This leads to a sufficient expected decrease in the second term in (27) as \( \sigma \) is increased by \( \alpha_1 > 1 \) with a probability no less than \( q^\text{low} \). The proof is provided in Appendix A.9.

**Lemma 9 (Small-step-size case).** If \( \sigma_t < \frac{b_L \sqrt{f(m_t)}}{\text{Tr}(m) \text{Tr}(H)} \), the following holds:
\[
E[V(\theta_{t+1}) - V(\theta_t) | T_t] \leq - \min \left\{ \frac{w}{4} \log(\alpha_1/\alpha^*_1), (q^\text{high} - p_{\text{target}}) \right\}. \quad (35)
\]

The second lemma is for the case in which the step size is too large. In this situation, Lemma 8 ensures that the success probability is no greater than \( q^\text{low} \). The expected decrease in \( \log(f(m_t)) \) can be arbitrarily small, as the success probability is close to zero. However, because \( \sigma \) is decreased by \( \alpha_1 < 1 \) with a probability no less than \( 1 - q^\text{low} \), the third term in (27) will be decreased sufficiently in expectation. The proof is provided in Appendix A.10.
Finally, we attain the main result. The upper convergence rate is satisfying (28) and (29). Subsequently, for each $t \geq 0$ and $\vartheta \in \Theta$, we have $B > 0$ for all $\vartheta \in \Theta$ satisfying (18).}

Let $\{\vartheta_t, \tilde{F}_t\}_{t \geq 0} = \mathbb{E}(f(\vartheta_0, \vartheta_1 \mid \tilde{F}_1) \mid \tilde{F}_1)$ be the state sequence of the (1+1)-ES solving $h$ with $\vartheta_0 = S(\vartheta_0) \in \Theta$. Next, in light of Proposition 3, the upper convergence rate of $(\tilde{h}_t)_{t \geq 0}$ is equal to

$$\limsup_{t \to \infty} \frac{1}{2t} \log \frac{h(\tilde{h}_t)}{h(\tilde{h}_0)}$$

with probability one. Therefore, without loss of generality, we assume that $f(x) = \frac{1}{2} x^T H x$ in the rest of the proof.

Under condition (30), we can find a pair $(q_{\text{low}}, q_{\text{high}})$ satisfying eqs. (28) and (29), as described in Section 4.1. Subsequently, from Lemma 8, it is easy to see that $w \in \Theta$ in eq. (33) and, hence, $\nu$ in eq. (34) are strictly positive. Moreover, because $q_{\text{high}} > \nu_{\text{target}}$ and $q_{\text{low}} < \nu_{\text{target}}$, we have $B > 0$ for $B$ defined in eq. (38).

Let $X_t = V(\vartheta_t)$ in Proposition 2 with $V$ defined in Definition 4 and $b, b_r$, and $a$ defined in eqs. (31), (32) and (34), respectively. Condition $C1 - E[V(\vartheta_{t+1}) - V(\vartheta_t) | \tilde{F}_t] \leq -B$ for all $t \geq 0$ is satisfied with $B$ defined in eq. (38) in light of Lemmas 9 to 11. Condition $C3 - \sum_{t=1}^{\infty} \text{Var}[V(\vartheta_t) - \tilde{F}_1]^{1/2} < \infty$ is satisfied for $d > 3$ in light of Lemma 12. Therefore, with probability one, we obtain that

$$\limsup_{t \to \infty} \frac{1}{2t} \log \frac{f(\vartheta_t)}{f(\vartheta_0)} \leq -B$$

Because $f(m) = V(\vartheta)$ for all $\vartheta \in \Theta$, we have

$$\limsup_{t \to \infty} \frac{1}{2t} \log \frac{f(\vartheta_t)}{f(\vartheta_0)} \leq -B$$

Hence, we get $\exp(-A_{\text{sup}}) \leq \exp(-B/2) < 1$.

Finally, we prove (39). Let $q_{\text{low}}$ and $q_{\text{high}}$ be set so that

$$0 < q_{\text{low}} < \nu_{\text{target}} < q_{\text{high}} < \frac{1}{2}$$

$$\sqrt{\frac{2}{\pi}} > \Phi^{-1}(1-q_{\text{high}}) > \frac{\alpha_1}{\alpha_1} \Phi^{-1}(1-q_{\text{high}}).$$

For a function $f \in Q_{d,k}$, we have

$$\frac{\text{Tr}(H^2)}{\text{Tr}(H)^2} < \frac{\text{Cond}(H)}{d} \to 0 \text{ as } d \to \infty.$$ (45)

Subsequently, for each $q \in (0, (1/2), B_{\text{low}}(q) \to 2\Phi^{-1}(1-q)$ and $\nu_{\text{high}}(q) \to 2\Phi^{-1}(1-q)$. Therefore, for any $\kappa > 1$, there exists $D > 0$ such that all $f \in Q_{d,k}$ with $d \geq D$ satisfy (28) and (29). Considering that $Q_H = q_{\text{low}}$ in the limit of $\frac{\text{tr}(H^2)}{\text{tr}(H)} \to 0$, we have $w \in \Omega_{d \to \infty} \left(\frac{1}{\text{tr}(H)}\right)$. Finally, as $A_{\text{sup}} \geq B/2$, we obtain (39). This completes the proof.

Note that (18) is the only condition that restricts the scope of the analysis in terms of the class of functions. Furthermore, in light of (45), condition (18) is satisfied for any Hessian matrix $H$ with a bounded condition number if $d$ is sufficiently large. Therefore, Theorem 13 asymptotically provides the upper bound of the convergence rate on general convex quadratic function for sufficiently large $d$.

We remark on the consequences. The hyper-parameters $\alpha_1$ and $\alpha_1$ are often set depending on the search space dimension $d$, or typically chosen such that $\log(\alpha_1/\alpha_1) \in \Theta_{d \to \infty}/(1/d)$. The theorem ensures that as long as $\log(\alpha_1/\alpha_1) \in \Theta_{d \to \infty}/(1/d)$, the upper convergence rate is in $\Omega_{d \to \infty} \left(\frac{1}{\text{tr}(H)}\right)$. In contrast, if $\log(\alpha_1/\alpha_1) \in \alpha_{d \to \infty}/(1/d)$, we have $A_{\text{sup}} \in$...
$O_{d \to \infty} \left( \exp \left( - \log \left( \frac{m_{t}}{m_{0}} \right) \right) \right) = O_{d \to \infty} \left( \frac{m_{t}}{m_{0}} \right) = O_{d \to \infty} \left( a_{1}^{1/P_{\text{target}}} \right)$.

This is rather intuitive for the following reason. $\|m_{t} - x^{*}\|$ does not converge faster than $\sigma_t$ because $\sigma_t$ needs to be proportional to $\|m_{t} - x^{*}\|$ to produce a sufficient decrease. The speed of the decrease in $\sigma_t$ is $a_{1}$. Therefore, the upper convergence rate should not be smaller than $a_{1}$.

Bounding the upper convergence rate with $\frac{L}{d \cdot \text{Cond}(H)}$ is more informative than bounding it with $\frac{L}{d \cdot \text{Cond}(H)}$. As mentioned in the introduction, we have $\frac{L}{d \cdot \text{Cond}(H)} \geq \frac{1}{d \cdot \text{Cond}(H)}$. Therefore, the bound $O_{d \to \infty} \left( \exp \left( - \frac{1}{d \cdot \text{Cond}(H)} \right) \right)$ immediately implies that $O_{d \to \infty} \left( \exp \left( - \frac{1}{d \cdot \text{Cond}(H)} \right) \right)$. However, even for the same condition numbers $\text{Cond}(H) = \xi \geq 1$, the bound with $\frac{1}{d \cdot \text{Cond}(H)}$ can be significantly different, depending on the distribution of the eigenvalues of $H$. For example, let us consider the following two situations:

$H_{\text{cigar}} = \text{diag}(\xi, \cdots, \xi, 1)$ \quad $\Rightarrow \quad \frac{L}{\text{Tr}(H_{\text{cigar}})} = \frac{1}{(d-1)\xi + 1}$, \quad (46)

$H_{\text{discuss}} = \text{diag}(\xi, 1, \cdots, 1)$ \quad $\Rightarrow \quad \frac{L}{\text{Tr}(H_{\text{discuss}})} = \frac{1}{\xi + (d-1)}$.

For $H_{\text{cigar}}$, we have $O_{d \to \infty} \left( \exp \left( - \frac{1}{d \cdot \text{Cond}(H)} \right) \right)$, whereas for $H_{\text{discuss}}$, we have $O_{d \to \infty} \left( \exp \left( - \frac{1}{d} \right) \right)$. In other words, if only a small portion of the axes are sensitive to the objective function value (i.e., directions corresponding to the eigenvalues of $\xi$), the upper convergence rate on the ill-conditioned $\text{Cond}(H) \gg 1$ convex quadratic function can be as good as the upper convergence rate on the spherical $\text{Cond}(H) = 1$ convex quadratic function.

5 LOWER CONVERGENCE RATE BOUND

The lower bound of the lower convergence rate, $\exp(-A_{\text{inf}}^{\text{low}})$, is obtained immediately from Proposition 2 and Lemma 5.

**Theorem 14 (Lower convergence rate bound).** Assume that the objective function $f : \mathbb{R}^{d} \to \mathbb{R}$ satisfies the following: $f \in Q$ defined in Definition 3 and $d \geq 4$.

Let $\{(\theta_{0}, \mathcal{F}) \}_{t \geq 0} = ES(f, \theta_{0}, \{x_{t}\}_{t \geq 0})$ be the state sequence of the (1+1)-ES solving $f$ with $\theta_{0} \in \Theta$.

Subsequently, the lower convergence rate (Definition 2) of the (1+1)-ES solving $f$ is lower-bounded as

$$\exp \left( -A_{\text{inf}}^{\text{low}} \right) \geq \exp \left( \frac{\text{Cond}(H)}{2(d-3)} \right)$$

for all $\theta_{0} \in \Theta \setminus \{x^{*}, \log(\sigma) : \log(\sigma) \in \mathbb{R} \}$.

**Proof of Theorem 14.** As discussed in the proof of Theorem 13, we can assume without loss of generality that $f(x) = \frac{1}{2} x^{T} H x$.

We apply Proposition 2 with $X_{t} = \log f(m_{t})$. The LHS of (12) is $\mathbb{E}[\exp(\{X_{t+1} - X_{t}\} \mid \mathcal{F}_{t})]$. Using the fact that $x \leq \exp(x) - 1$ for all $x \geq 0$, we obtain $\mathbb{E}[|X_{t+1} - X_{t}| \mid \mathcal{F}_{t}] \leq \mathbb{E}[\exp(|X_{t+1} - X_{t}|) \mid \mathcal{F}_{t}] - 1 \leq \frac{1}{d^{2}} \xi^{2}$ with Lemma 5. Note that $X_{t+1} - X_{t} = -|X_{t+1} - X_{t}|$, as it is non-positive. Hence, we obtain C2 of Proposition 2 with $C = \frac{1}{d^{2}} \xi^{2}$. Using the fact that $x^{2} \leq 2(\exp(x) - 1) \leq 2(\exp(x) - 1)$ for all $x \geq 0$, we obtain $\mathbb{E}[|X_{t+1} - X_{t}| \mid \mathcal{F}_{t}] \leq \frac{2d^{2}}{\xi^{2}}$. From Var$[X_{t+1} \mid \mathcal{F}_{t}] \leq \mathbb{E}[|X_{t+1} - X_{t}| \mid \mathcal{F}_{t}]$, we obtain Var$[X_{t+1} \mid \mathcal{F}_{t}] \leq \frac{2d^{2}}{\xi^{2}}$. C3 in Proposition 2 is then satisfied. Therefore, with probability one, we have that

$$\liminf_{t \to \infty} \frac{1}{t} \log \left( \frac{m_{t}}{m_{0}} \right) \geq -C.$$ \quad (49)

In light of Proposition 3, with probability one, we have that

$$\liminf_{t \to \infty} \frac{1}{t} \log \left( \frac{\|m_{t} - x^{*}\|}{\|m_{0} - x^{*}\|} \right) \geq -C \frac{2}{d}.$$ \quad (50)

This completes the proof.

6 DISCUSSION

**Conclusion.** In this work, the convergence rate of the (1+1)-ES on the potentially ill-conditioned function, general convex quadratic function, is analyzed. It is revealed that the upper convergence rate is $O_{d \to \infty} \left( \exp \left( \frac{1}{d \cdot \text{Cond}(H)} \right) \right)$ and that the lower convergence rate is $\Omega_{d \to \infty} \left( \exp \left( \frac{1}{d \cdot \text{Cond}(H)} \right) \right)$. The order of the upper convergence rate in terms of both the dimension $d$ and the Hessian $H$ is derived for the first time. Our analysis on $\text{Tr}(H)/d$ is superior to that on $d \cdot \text{Cond}(H)$ (partly shown in [21]) in that it reveals the impact of the distribution of the eigenvalues of $H$. Namely, it theoretically suggests that the ill-conditioned problem is not only in the ratio between the greatest and smallest eigenvalues but also in heaviness of the distribution of the eigenvalues, at least for the (1+1)-ES.

**Discussion.** Further, we clarify the limitation of the current analysis. As $\text{Tr}(H^{2})/\text{Tr}(H)^{2} \to 0$, with the definition of $B_{H}^{\text{low}}$ (16), condition (30) is rewritten as

$$\frac{4}{\sqrt{2\pi}} > 2\Phi^{-1} \left( 1 - p_{\text{target}} \right).$$ \quad (51)

Hence, $p_{\text{target}} > \Phi^{-1} \left( \frac{\sqrt{2}}{2} \right)$ is significant for the existence of $\gamma > 0$, which satisfies the inequality above. Note that $\Phi \left( \frac{\sqrt{2}}{2} \right) \approx 0.212$, and then the classic 1/5-success rule[24, 29] is slightly out of the scope of the current study.
The previous work [26] takes almost the same approach, in particular, the same potential function as ours. The most distinct difference is the separation method of the state space $\Theta$ on the step size $\sigma$ in Lemmas 8 to 11. In [26], if ignoring the setting of the constants, they defined a reasonable step-size range in the form of $[1 \cdot \sigma/\sqrt{f(m_1)}, u - \sigma/\sqrt{f(m_1)}]$, while we defined it in the form of $[1 \cdot \sigma/\sqrt{f(m_1)}, u - \sigma/\sqrt{f(m_1)}]]$. This change made it possible to bound the success probability in each scenario and the expected one-step progress in the case of the reasonable step-size more tightly. Our approach, analyzing the expected decrease in the potential function (27) on each proportion of $\Theta$ separated by $[1 \cdot \sigma/\sqrt{f(m_1)}, u - \sigma/\sqrt{f(m_1)]}]$, does not have such problems and leads to a tighter upper convergence rate bound, as proved in Section 4.

The analysis of the $(1+1)$-ES is also important in terms of demonstrating the potential and limitation of the continuous BBO algorithms. In fact, many theoretical studies [7, 13, 15, 28] on derivative-free algorithms adopt settings that utilize the properties of the objective function other than the function value for theoretical analysis. The current study on the $(1+1)$-ES exploits only the function value in the optimization and does not bring additional properties of the objective function (such as the Lipschitz constant or the condition number of the Hessian) into the algorithm parameters. In other words, the current study is performed in a purely black-box setting.

Future Work. The existence of a constant bound of the upper convergence rate of the $(1+1)$-ES on the $\alpha$-strongly convex and $\gamma$-Lipschitz smooth function is clearly shown in [26], as mentioned. Considering that such a function is a superset of convex quadratics, it might be possible to state that the upper convergence rate is $O_{\Delta \to 0} \exp(-\frac{\alpha}{\gamma})$ to match our result; however, it is still unclear what is missing for its proof.

If we bring the CMA mechanism into Algorithm 1, intuitively, the convergence rate improves on a severely ill-conditioned convex quadratic function, i.e., the case $\text{Cov}(H)$ is considerably larger than 1. However, to date, a promising approach to estimate the convergence rate of the CMA-ES theoretically is scarcely established for any class of the function. Exploring the possibility of expanding the applicable range of the analysis scheme is also an important future work in terms of the class of algorithm.

Regarding the trace or the condition number of $H$, the derived lower convergence rate still does not match the upper convergence rate, which seems to be rigorous, although it matches with respect to $d$. However, previous works [20–22] attain the matching order of the convergence rate with respect to the condition number of $H$ on a specific convex quadratic function with an overwhelming probability. There is room for consideration as to which method is suitable for estimating the lower convergence rate, although our method seems to be potent in estimating the upper convergence rate.

REFERENCES

[1] Youhei Akimoto, Anne Auger, and Tobias Glasmachers. 2018. Draft theory in continuous search spaces: expected hitting time of the $(1+1)$-ES with 1/5 success rule. In Proceedings of the Genetic and Evolutionary Computation Conference. 801–808.

[2] Youhei Akimoto, Anne Auger, Tobias Glasmachers, and Daiki Morinaga. 2020. Global Linear Convergence of Evolution Strategies on More Than Smooth Strongly Convex Functions. arXiv preprint arXiv:2009.06647 (2020).

[3] Youhei Akimoto, Anne Auger, and Nikolaus Hansen. 2014. Comparison-based natural gradient optimization in high dimension. In Proceedings of the 2014 Annual Conference on Genetic and Evolutionary Computation. 373–380.

[4] Youhei Akimoto and Nikolaus Hansen. 2020. Diagonal acceleration for covariance matrix adaptation evolution strategies. Evolutionary computation 28, 3 (2020), 405–435.

[5] Anne Auger and Nikolaus Hansen. 2013. Linear convergence on positively homogeneous functions of a comparison based step-size adaptive randomized search: the $(1+1)$ ES with generalized one-fifth success rule. arXiv preprint arXiv:1310.8397 (2013).

[6] Anne Auger, Marc Schoenauer, and Nicolas Vanhacque. 2004. LS-CMA-ES: A second-order algorithm for covariance matrix adaptation. In International Conference on Parallel Problem Solving from Nature. Springer. 182–191.

[7] Krishnakumar Balasubramanian and Saeed Ghadimi. 2018. Zeroth-order (non)-convex stochastic optimization via conditional gradient and gradient updates. In Advances in Neural Information Processing Systems. 3455–3464.

[8] Yuan Shih Chow et al. 1967. On a strong law of large numbers for martingales. The Annals of Mathematical Statistics 38, 2 (1967), 610–610.

[9] Olivier Devolder, François Glineur, Yurii Nesterov, et al. 2013. First-order methods with exact oracle: the strongly convex case. CORE Discussion Papers 2013016 (2013), 47.

[10] Yinpeng Dong, Hang Su, Baoyuan Wu, Zhifeng Li, Wei Liu, Tong Zhang, and Jun Zhu. 2019. Efficient decision-based black-box adversarial attacks on face recognition. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition. 7714–7722.

[11] Garuda Fuji, Masayuki Takahashi, and Youhei Akimoto. 2018. CMA-ES-based structural topology optimization using a level set boundary expression—Application to optical and carpet cloaks. Computer Methods in Applied Mechanics and Engineering 332 (2018), 624–643.

[12] Thomas Geitzenbeiek, Michael Van De Panne, and A Frank Van Der Stappen. 2013. Flexible muscle-based locomotion for bipedal creatures. ACM Transactions on Graphics (TOG) 32, 6 (2013), 1–11.

[13] Saeed Ghadimi and Guanghui Lan. 2013. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. SIAM Journal on Optimization 23, 4 (2013), 2341–2368.

[14] Tobias Glasmachers. 2020. Global convergence of the $(1+1)$ evolution strategy to a critical point. Evolutionary computation 28, 1 (2020), 27–53.

[15] Daniel Golovin, John Karro, Greg Kochanski, Chansoo Lee, Xingyou Song, and Quyi Zhang. 2019. Gradientless Descent: High-Dimensional Zeroth-Order Optimization. In International Conference on Learning Representations.

[16] David Ha and Jürgen Schmidthuber. 2018. Recurrent world models facilitate policy evolution. In Advances in neural information processing systems. 2450–2462.

[17] Nikolaus Hansen and Anne Auger. 2014. Principled design of continuous stochastic search: From theory to practice. In Theory and principled methods for the design of metaheuristics. Springer. 145–180.

[18] Nikolaus Hansen, Sibylle D Müller, and Petros Koumoutsakos. 2003. Reducing the time complexity of the derandomized evolution strategy with covariance matrix adaptation (CMA-ES). Evolutionary computation 11, 1 (2003), 1–18.

[19] Nikolaus Hansen and Andreas Ostermeier. 2001. Completely derandomized self-adaptation in evolution strategies. Evolutionary computation 9, 2 (2001), 159–195.

[20] Jens Jägersköpper. 2003. Analysis of a simple evolutionary algorithm for minimization in Euclidean spaces. In International Colloquium on Automa- to, Languages, and Programming. Springer. 1068–1079.

[21] Jens Jägersköpper. 2006. How the $(1+1)$-ES using isotropic mutations minimizes positive definite quadratic forms. Theoretical Computer Science 361, 1 (2006), 38–56.

[22] Jens Jägersköpper. 2007. Algorithmic analysis of a basic evolutionary algorithm for continuous optimization. Theoretical Computer Science 379, 3 (2007), 329–347.

[23] Grahame A Jastrzkebski and Dirk V Arnold. 2006. Improving evolution strategies through active covariance matrix adaptation. In 2006 IEEE international conference on evolutionary computation. IEEE. 2814–2821.

[24] Stefan Kern, Sibylle D Müller, Nikolaus Hansen, Dirk Büche, Jiri Ocenasek, and Petros Koumoutsakos. 2004. Learning probability distributions in continuous evolutionary algorithms—a comparative review. Natural Computing 3, 1 (2004), 77–112.

[25] Iris Kriest, Volkmar Sauerland, Samar Khatiwala, Anand Srivastav, and Andreas Oschlies. 2017. Calibrating a global three-dimensional biogeochemical ocean model (MOPS-1.0). Geoscientific Model Development 10 (2017), 127–154.

[26] Daiki Morinaga and Youhei Akimoto. 2019. Generalized drift analysis in continuous domain: linear convergence of $(1+1)$-ES on strongly convex functions with Lipschitz continuous gradients. In Proceedings of the 15th ACM/SIGEVO Conference on Foundations of Genetic Algorithms. 13–24.
Yu Nesterov. 2013. Gradient methods for minimizing composite functions. Mathematical Programming 140, 1 (2013), 125–161.

Yuri Nesterov and Vladimir Spokoiny. 2017. Random gradient-free minimization of convex functions. Foundations of Computational Mathematics 17, 2 (2017), 527–566.

Ingo Rechenberg. 1973. Evolution strategy: Optimization of technical systems by means of biological evolution. Fromman-Holzboog, Stuttgart 104 (1973), 15–16.

Luis Miguel Rios and Nikolaos V Sahinidis. 2013. Derivative-free optimization: a review of algorithms and comparison of software implementations. Journal of Global Optimization 56, 3 (2013), 1247–1293.

Jannis Uhlendorf, Agnès Miermont, Thierry Delaveau, Gilles Chavvin, François Fages, Samuel Bottani, Gregory Batt, and Pascal Hersen. 2012. Long-term model predictive control of gene expression at the population and single-cell levels. Proceedings of the National Academy of Sciences 109, 35 (2012), 14271–14276.

Konstantinos Varelas, Anne Auger, Dimo Brockhoff, Nikolaus Hansen, Ouassim Elhara, Yann Smet, Rami Kassab, and Frédéric Barbaresco. 2020. A Comparative Study of Large-scale Variants of CMA-ES. (2020).

Vanessa Volz, Jacob Schrum, Jialin Liu, Simon M Lucas, Adam Smith, and Sebastian Risi. 2018. Evolving mario levels in the latent space of a deep convolutional generative adversarial network. In Proceedings of the Genetic and Evolutionary Computation Conference. 221–228.
A PROOF

A.1 Proof of Proposition 1

Proof of Proposition 1. For the first claim, it is sufficient to show that \( G(\theta, z; f) = G(\theta, z; g \circ f) \) for all \( \theta \in \Theta \) and \( z \in \mathbb{R}^d \). This is trivial because \( f(m + \sigma \cdot z) \leq f(m) \Leftrightarrow g(f(m + \sigma \cdot z) \leq g(f(m)) \).

For the second claim, we first show that \( \tilde{G}(\tilde{m}, z; f) = G((m + x', \sigma); z; f \circ T) \) for all \( \tilde{m} \in \mathbb{R}^d \). Because \( f(T(m + x' + \sigma \cdot z)) \leq f(T(m + x')) \Leftrightarrow f(m + \sigma \cdot z) \leq f(m) \), it is obvious that \( \tilde{G}((m, \sigma); z, f) = G((m + x', \sigma); z; f \circ T) \). Assume that \( \tilde{m}_t = m_t + x' \) and \( \tilde{\sigma}_t = \sigma_t \). Next, we have \( \tilde{G}(\tilde{m}_t, z; f) = G(\tilde{m}_t, z; f \circ T) \), and hence \( \tilde{m}_{t+1} = m_{t+1} + x' \) and \( \tilde{\sigma}_{t+1} = \sigma_{t+1} \). Because the assumption holds for \( t = 0 \), by mathematical induction, we obtain the second claim.

\( \square \)

A.2 Proof of Proposition 2

Proof of Proposition 2. Let \( Z_{t+1} = X_{t+1} - \mathbb{E}[X_{t+1} | F_t] \). Next, \( (Z_t)_{t \geq 1} \) is a Martingale difference sequence adapted to \( (F_t) \). Suppose that C3 holds. C3 immediately implies that \( \sum_{t=1}^{\infty} \mathbb{E}[Z_t^2 / F_{t-1}] < \infty \). Subsequently, from the strong law of large numbers of Martingale [8], we obtain \( \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} Z_t = 0 \) almost surely. Next, we have

\[
\frac{1}{t} (X_t - X_0) = \frac{1}{t} \sum_{i=1}^{t} (X_i - X_{i-1})
\]

(52)

\[
= \frac{1}{t} \sum_{i=1}^{t} (Z_i + \mathbb{E}[X_i - X_{i-1} | F_{i-1}])
\]

(53)

\[
= \frac{1}{t} \sum_{i=1}^{t} Z_i + \frac{1}{t} \sum_{i=1}^{t} \mathbb{E}[X_i - X_{i-1} | F_{i-1}].
\]

(54)

We obtain (7) by taking \( \lim sup \) of the equation above and using C1 as well as \( \lim sup_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} Z_t = 0 \). Similarly, we obtain (8) by taking \( \lim inf \) and using C2 as well as \( \lim inf_{t \to \infty} \frac{1}{t} \sum_{i=1}^{t} Z_t = 0 \). This completes the proof.

\( \square \)

A.3 Proof of Proposition 3

Proof of Proposition 3. In light of Proposition 1, we have \( \theta_t = S(\hat{\theta}_t) \) for all \( t \geq 0 \). Next, \( ||m_t - x'|| = ||m_t|| \) for all \( t \geq 0 \). Let \( L \) and \( U \) be the smallest and greatest eigenvalues of \( H \), respectively.

Next, for any \( x \in \mathbb{R}^d \), we have

\[
2h(x) / U \leq ||x||^2 \leq h(x) / L.
\]

(55)

In particular, we obtain

\[
\log \left( \frac{L}{U} \right) \leq 2 \log \left( \frac{||m_t||}{||m_0||} \right) - \log \left( \frac{h(m_t)}{h(m_0)} \right) \leq \log \left( \frac{U}{L} \right).
\]

(56)

Taking \( \lim sup \) and \( \lim inf \) after multiplying all terms by \( \frac{1}{t} \), we obtain Equations (9) and (10).

\( \square \)

A.4 Proof of Lemma 4

Proof of Lemma 4. For a general convex function \( f : \mathbb{R}^d \to \mathbb{R} \), we have

(1) \( I(f(m + \sigma \cdot z) \leq f(m)) \leq I(\langle \nabla f(m), z \rangle \leq 0) \), hence \( I(f(m + \sigma \cdot z) \leq f(m)) = I(\langle \nabla f(m), z \rangle \leq 0) / I(f(m + \sigma \cdot z) \leq f(m)) \);

(2) \( h(z) = (f(m + \sigma \cdot z) - f(m)) / I(\langle \nabla f(m), z \rangle \leq 0) \) and \( g(z) = I(f(m + \sigma \cdot z) \leq f(m)) / I(f(m + \sigma \cdot z) \leq f(m)) \) are negatively correlated, i.e., \( (h(z_1) - h(z_2)) / (g(z_1) - g(z_2)) \leq 0 \) for all \( z_1, z_2 \in \mathbb{R}^d \).

Because \( h \) and \( g \) are negatively correlated, we have \( \mathbb{E}[h(z)g(z)] \leq \mathbb{E}[h(z)\mathbb{E}[g(z)]] \). Moreover, we have \( \mathbb{E}[g(z)] = \mathbb{E}[I(f(m + \sigma \cdot z) \leq f(m))] = \mathbb{P}[I(f(m + \sigma \cdot z) \leq f(m))] \).

Now, we suppose that \( f \) is a convex quadratic function, \( \nabla^2 f(x) = H \). Letting \( e = \langle \nabla f(m), ||\nabla f(m)|| \rangle \) and \( z_e = (e, z) \), we have

\[
\mathbb{E}[h(z)] = \mathbb{E}[I(f(m + \sigma \cdot z) \leq f(m))], I(\langle \nabla f(m), z \rangle \leq 0)].
\]

(57)

\[
= \mathbb{E}\left[\sigma \|\nabla f(m)\|z_e + \frac{\sigma^2}{2} \right] \mathbb{P}[I(\langle \nabla f(m), z \rangle \leq 0)]
\]

(58)

\[
= \sigma \|\nabla f(m)\| z_e + \frac{\sigma^2}{2} \mathbb{P}[I(\langle \nabla f(m), z \rangle \leq 0)]
\]

(59)

\[
= \sigma \|\nabla f(m)\| \left(1 - \frac{1}{\sqrt{2\pi}} + \frac{1}{4} I(\mathbb{P}[I(\langle \nabla f(m), z \rangle \leq 0)])
\]

(60)

\[
= \sigma \|\nabla f(m)\| \left(1 - \frac{1}{\sqrt{2\pi}} + \frac{1}{4} I(\mathbb{P}[I(\langle \nabla f(m), z \rangle \leq 0)])
\]

(61)

This completes the proof.

\( \square \)

A.5 Proof of Lemma 5

Proof of Lemma 5. Let \( \lambda_i(H) \) be the \( i \)-th greatest eigenvalue of \( H \) for \( i = 1, \ldots, d \), and \( \sigma_z^2 = \arg \min_{\sigma_z > 0} f(m + \sigma z) \). We have

\[
\log \left( \frac{f(m + \sigma \cdot z)}{f(m)} \right) \cdot \mathbb{I} \{ f(m + \sigma \cdot z) \leq f(m) \}
\]

(62)

\[
\leq \log \left( \frac{\min_{\sigma_z > 0} f(m + \sigma z)}{f(m)} \right)
\]

(63)

\[
= \log \left( 1 + \frac{\sigma^2}{2} \mathbb{E}[Hz^T Hz] + \frac{(x')^2}{2} f(m) \right).
\]

(64)

\[
= \log \left( 1 - \frac{\min(m^T H \cdot m)}{z^T H \cdot m} \right)
\]

(65)

\[
\leq \frac{1}{2} \frac{(m^T H^2 z)^2}{z^T H \cdot m} \cdot \mathbb{I} \{ f(m + \sigma z) \leq f(m) \}
\]

(66)

\[
= \log \left( 1 - \frac{\sqrt{\mathbb{E}[H^2]}}{\sqrt{\mathbb{E}[H^2]}} \frac{z^T H \cdot m}{\mathbb{I} \{ f(m + \sigma z) \leq f(m) \}} \right)
\]

(67)

The right-hand side (RHS) is stochastically dominated by

\[
Z := -\log \left( 1 - \frac{\lambda_1(H)N_i^2}{\sum_{i=1}^{d} \lambda_1(H)N_i^2} \right)
\]

(68)

\[
= \log \left( 1 + \frac{\lambda_1(H)N_i^2}{\sum_{i=1}^{d} \lambda_1(H)N_i^2} \right)
\]

(69)

i.e., the cumulative density of the RHS of (67) is upper-bounded by the cumulative density of \( Z \), where \( N_1, \ldots, N_d \) are independently...
and standard normally distributed random variables. Next,

\[ \begin{align*}
\mathbb{E}[\exp(Z)] &= 1 + \mathbb{E} \left[ \frac{\lambda_1(H)N_r^2}{\sum_{i=2}^{d} \lambda_i(H)N_i^2} \right], \\
&\leq 1 + \frac{\lambda_1(H)}{(d-1)\lambda_d(H)} \mathbb{E} \left[ \frac{N_r^2}{\frac{1}{d-1} \sum_{i=2}^{d} N_i^2} \right], \\
&= 1 + \frac{\lambda_1(H)}{(d-3)\lambda_d(H)}. 
\end{align*} \]

(70)  
(71)  
(72)

where for the last equality, we utilized the fact that \( \frac{N_r^2}{\frac{1}{d-1} \sum_{i=2}^{d} N_i^2} \) is \( F \)-distributed with degrees of freedom of \((1, d-1)\) and its expected value is \((d-1)/(d-3)\) if \(d > 3\).

\[ \square \]

### A.6 Proof of Lemma 6

**Proof of Lemma 6.** First, by using Chebyshev’s inequality and \( \mathbb{E}[z^THz] = \text{Tr}(H) \), we obtain

\[ \begin{align*}
\Pr \left[ \frac{z^THz}{\text{Tr}(H)} - 1 \geq \epsilon \right] &\leq \frac{1}{\epsilon^2} \cdot \text{Var} \left[ \frac{z^THz}{\text{Tr}(H)} \right] = \frac{2}{\epsilon^2} \cdot \text{Tr}(H^2) .
\end{align*} \]

(73)

Let \( \epsilon = \frac{\nabla f(m)}{\|
abla f(m)\|} \) and \( z_e = \langle e, z \rangle \). We can write the success probability as

\[ \begin{align*}
\Pr\{f(m + \sigma z) < f(m)\} &= \Pr\left[ \sigma(\nabla f(m), z) < -\frac{1}{2} z^THz \right] \\
&= \Pr\left[ \sigma\|
abla f(m)\| z_e < -\frac{1}{2} z^THz \right] \\
&= \Pr\left[ z_e < -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} .
\end{align*} \]

(74)  
(75)  
(76)  
(77)

Next, we derive the upper bound.

\[ \begin{align*}
\Pr\{f(m + \sigma z) < f(m)\} &= \Pr\left[ z_e < -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right] \\
&= \Pr\left[ z_e < -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right] \cup \left( \frac{z^THz}{\text{Tr}(H)} \geq 1 - \epsilon \right) \\
&\leq \Pr\left[ z_e < -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right] \cup \left( \frac{z^THz}{\text{Tr}(H)} < 1 - \epsilon \right) \\
&\leq \Pr\left[ \frac{z^THz}{\|
abla f(m)\|} - (1 - \epsilon) \right] + \Pr\left[ \frac{z^THz}{\text{Tr}(H)} < 1 - \epsilon \right] \\
&\leq \Phi \left( -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right) + \frac{2}{\epsilon^2} \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H)^2} .
\end{align*} \]

(78)  
(79)  
(80)  
(81)  
(82)  
(83)

Finally, we derive the lower bound.

\[ \begin{align*}
\Pr\{f(m + \sigma z) > f(m)\} &= \Pr\left[ z_e > -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right] \\
&= \Pr\left[ \left( \frac{z^THz}{\|
abla f(m)\|} \right) \cup \left( \frac{z^THz}{\text{Tr}(H)} \leq 1 + \epsilon \right) \right] \\
&\leq \Pr\left[ \left( \frac{z^THz}{\|
abla f(m)\|} \right) \cup \left( \frac{z^THz}{\text{Tr}(H)} > 1 + \epsilon \right) \right] .
\end{align*} \]

(84)  
(85)  
(86)  
(87)

Hence, we have

\[ \begin{align*}
\Pr\{f(m + \sigma z) < f(m)\} &= \Phi \left( -\frac{1}{2} \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \right) - \frac{2}{\epsilon^2} \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H)^2} .
\end{align*} \]

(88)  
(89)

This completes the proof.  \( \square \)

### A.7 Proof of Corollary 7

**Proof of Corollary 7.** First, we prove eq. (15). In light of Lemma 6, for each \( q \in (0, \frac{1}{2}) \), it is sufficient to show that there exists an \( \epsilon > 0 \) such that the LHS of (13) is no smaller than \( q \). By solving this inequality for \( \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \), we obtain (using \( -\Phi^{-1}(x) = \Phi^{-1}(1 - x) \))

\[ \frac{\sigma\|
abla f(m)\| \text{Tr}(H)}{z^THz} \leq \frac{2\Phi^{-1}(1 - \frac{q + \frac{2}{\epsilon^2} \cdot \text{Tr}(H^2)}{\text{Tr}(H)^2})}{1 + \epsilon} .\]

(90)

That is, if there exists an \( \epsilon > 0 \) such that the condition above holds, then we have \( \Pr\{f(m + \sigma z) < f(m)\} > q \). In contrast, for each \( q \in (0, \frac{1}{2}) \), we can easily see that there exists \( \epsilon_{H}^{\text{high}}(q) > \sqrt{\frac{4 - 2q}{\text{Tr}(H)^2}} \text{Tr}(H^2) \) such that \( B_{H}^{\text{high}}(q) = B_{H}^{\text{high}}(q, \epsilon_{H}^{\text{high}}(q)) \) (0.1). Hence, we obtain eq. (15).

Next, we prove the properties of \( B_{H}^{\text{high}} \). First, note that \( B_{H}^{\text{high}}(q, \epsilon) \) is continuous and strictly decreasing with respect to \( q \) for each \( \epsilon \). We prove the right-continuity and strict-decrease of \( B_{H}^{\text{high}} \) by contradiction. If \( B_{H}^{\text{high}} \) does not strictly decrease at a point \( q \) in the domain, then there must exist \( q' > q \) such that \( B_{H}^{\text{high}}(q') = B_{H}^{\text{high}}(q') > B_{H}^{\text{high}}(q, \epsilon) \). However, \( B_{H}^{\text{high}}(q') = B_{H}^{\text{high}}(q' ; \epsilon_{H}^{\text{high}}(q')) < B_{H}^{\text{high}}(q, \epsilon_{H}^{\text{high}}(q')) \leq B_{H}^{\text{high}}(q, \epsilon_{H}^{\text{high}}(q')) \), which contradicts \( B_{H}^{\text{high}}(q) < B_{H}^{\text{high}}(q') \). Hence, \( B_{H}^{\text{high}} \) is strictly decreasing. If \( B_{H}^{\text{high}} \) is not right-continuous at a point \( q \), there must exist \( \epsilon > 0 \) such that \( B_{H}^{\text{high}}(q) < B_{H}^{\text{high}}(q, \epsilon) \).
whereas from the continuity of $B_{H}^\text{high}(\cdot; e^\text{high}(q))$, the RHS satisfies $\lim_{q \to 0} \sqrt{\frac{1}{\ell H^2} (q; e^\text{high}(q)) - F_{H}(q; e^\text{high}(q))} = 0$, which is a contradiction. Hence, $B_{H}^\text{high}$ is right-continuous. It is trivial to see that $B_{H}^\text{high}(q) \leq 2\Phi^{-1}(1 - q)$ for all $q \in \left(0, \frac{1}{2}\right)$.

Similarly, we prove eq. (17). In light of Lemma 6, for each $q \in \left(2 \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H)}, \frac{1}{2}\right)$, it is sufficient to show that there exists an $\varepsilon \in (0, 1)$ such that the RHS of (13) is no greater than $q$. The condition $\varepsilon < 1$ is necessary to have the RHS of (13) smaller than $1/2$. By solving this inequality for $\sigma_{\delta_{\varepsilon}}$, we obtain

$$\sigma_{\delta_{\varepsilon}} \left(\frac{\varepsilon}{\|\nabla f(m)\|}\|\nabla f(m)\|\right) \geq \frac{2.e^{-1}}{1 - \varepsilon} \left(1 - \frac{\varepsilon}{\delta_{\varepsilon}} \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H^2)}\right) =: B_{H}^\text{low}(q; e) \cdot (92)$$

That is, if there exists an $\varepsilon \in (0, 1)$ such that the condition above holds, then we have $\frac{\text{Pr}[f(m + \sigma_{z}) < f(m)]}{\text{Pr}[f(m + \sigma_{z}) > f(m)]} \leq 1 - \varepsilon$ and $\varepsilon_{H}^\text{low}(q) \leq \left(\frac{2.e^{-1}}{1 - \varepsilon} \cdot \frac{\text{Tr}(H^2)}{\text{Tr}(H^2)}\right)$ such that $B_{H}^\text{low}(q) = B_{H}(q; e^\text{low}(q))$. Hence, we obtain eq. (15).

Finally, we prove the properties of $B_{H}^\text{low}$. Note that $B_{H}^\text{high}(q; e)$ is continuous and strictly decreasing with respect to $q$ for each $e$. We prove the left-continuity and strict decrease of $B_{H}^\text{low}$ by contradiction. If $B_{H}^\text{low}$ is not strictly decreasing at a point $q$ in the domain, then there must exist $q' < q$ such that $B_{H}^\text{low}(q') \geq B_{H}^\text{low}(q'')$. However, $B_{H}^\text{low}(q') = B_{H}^\text{low}(q'; e^\text{low}(q')) > B_{H}^\text{low}(q'; e^\text{low}(q')) = B_{H}^\text{low}(q; e^\text{low}(q'))$, which contradicts $B_{H}^\text{low}(q') \geq B_{H}^\text{low}(q'')$. Hence, $B_{H}^\text{high}$ is strictly decreasing. If $B_{H}^\text{low}$ is not left-continuous at a point $q$, there must exist $\delta > 0$ such that $B_{H}^\text{low}(q + \delta) \geq B_{H}^\text{low}(q')$ for all $q' < q$. Because $B_{H}^\text{low}(q; e^\text{low}(q)) = B_{H}^\text{low}(q' + \delta; e^\text{low}(q')) = B_{H}^\text{low}(q'; e^\text{low}(q'))$, we have $B_{H}^\text{low}(q + \delta) \geq B_{H}^\text{low}(q'; e^\text{low}(q'))$. The LHS must be no smaller than any $q' < q$, whereas from the continuity of $B_{H}^\text{low}(\cdot; e^\text{low}(q))$, the RHS satisfies $\lim_{q' \to q^{-}} B_{H}^\text{low}(q'; e^\text{low}(q')) = B_{H}^\text{low}(q; e^\text{low}(q)) = 0$, which is a contradiction. Hence, $B_{H}^\text{low}$ is left-continuous. It is trivial to see that $B_{H}^\text{high}(q) \geq 2\Phi^{-1}(1 - q) = (1/2)$.
Proof of Lemma 9. If $\sigma_t < \frac{b_t \sqrt{L} f(m_t)}{\text{Tr}(H) \sigma_t}$, we have $b_t \sqrt{L} f(m_t) > \alpha_t > 1$ and $\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{L} f(m_t)} < \frac{b_t}{\alpha_t} < 1$. The potential function at $t$ is

$$V(\theta_t) = \log(f(m_t)) + v \cdot \log \left( \frac{b_t \sqrt{L} f(m_t)}{\text{Tr}(H) \sigma_t} \right), \quad (96)$$

and a one-step difference of $V(\theta)$ under the current condition is written as follows:

$$V(\theta_{t+1}) - V(\theta_t) = \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) + v \cdot \log \left( \frac{b_t \sqrt{L} f(m_{t+1})}{\text{Tr}(H) \sigma_{t+1}} \right) + \frac{b_t}{\alpha_t} \cdot \frac{\text{Tr}(H) \sigma_{t+1}}{\text{Tr}(H) \sigma_t} \cdot I_t + v \cdot \log \left( \frac{b_t \sqrt{L} f(m_{t+1})}{\text{Tr}(H) \sigma_{t+1}} \right), \quad (107)$$

The RHS is deformed by extracting $\log(f(m_{t+1})/f(m_t))$ and by rewriting $\sigma_{t+1}$ into $\alpha_t \sigma_t$ and $\alpha_t' \sigma_t$ using the conditions of $I_1$ and $I_1'$. An upper bound of $V(\theta_{t+1}) - V(\theta_t)$:

$$V(\theta_{t+1}) - V(\theta_t) \leq v \cdot \log \left( \frac{b_t \sqrt{L} f(m_{t+1})}{\text{Tr}(H) \sigma_{t+1}} \right) \cdot I_\uparrow + v \cdot \log \left( \frac{b_t \sqrt{L} f(m_{t+1})}{\text{Tr}(H) \sigma_{t+1}} \right) \cdot I_\downarrow \cdot \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{L} f(m_t)}, \quad (108)$$

To the last bound of the expression above, we use $b_t \sqrt{L} f(m_t) > \alpha_t$. The first term of the RHS of the expression above is non-positive under the current condition $\sigma_t < \frac{b_t \sqrt{L} f(m_t)}{\text{Tr}(H) \sigma_t}$, which is upper-bounded by $1 - \eta_{\text{high}}$ in light of Lemma 8. By taking the expectation on both sides of the inequality above, we obtain

$$\mathbb{E}[V(\theta_{t+1}) - V(\theta_t) | F_t] \leq v \cdot \log \left( \frac{\alpha_1}{\alpha_1'} \right) \cdot \left( 1 - \eta_{\text{high}} \frac{\log(\alpha_t)}{\log(\alpha_1')} \right). \quad (109)$$

Because $v \cdot \log(\alpha_t/\alpha_1') = \min\{w/4, \log(\alpha_t/\alpha_1')\}$, we obtain (35). This completes the proof.

A.10 Proof of Lemma 10

Proof of Lemma 10. If $\sigma_t > \frac{b_t \sqrt{L} f(m_t)}{\text{Tr}(H) \sigma_t} \leq \frac{b_t \sqrt{L} f(m_t)}{\sqrt{2} \text{Tr}(H) \sigma_t} < 1$. Naturally $I_\downarrow = 0$. Because

$$\frac{b_t \sqrt{L} f(m_{t+1})}{\text{Tr}(H) \sigma_{t+1}} \leq \frac{b_t \sqrt{L} f(m_t)}{\text{Tr}(H) \sigma_t} \leq \frac{b_t \sqrt{L} f(m_t)}{\sqrt{2} \text{Tr}(H) \sigma_t} < 1. \quad (110)$$
As \( \sigma_t > \frac{b_t}{\sqrt{2a_t} \| \nabla f(m_t) \|} \Rightarrow \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} < 1 \), the potential function at \( t \) under the current condition is

\[
V(\theta_t) = \log f(m_t) + v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right). \tag{120}
\]

Next, a one-step difference of the potential function under \( \sigma_t > \frac{b_t}{\sqrt{2a_t} \| \nabla f(m_t) \|} \) is

\[
V(\theta_{t+1}) - V(\theta_t) = \left(1 - (\hat{I}_s - I_t) \frac{v}{2} \right) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \tag{121}
\]

\[
+ v \cdot \log \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot \hat{I}_s \hat{I}_t \tag{122}
\]

\[
+ v \cdot \log \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot I_s I_t \tag{123}
\]

\[
+ v \cdot \log \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot I_s \hat{I}_t \tag{124}
\]

\[
+ v \cdot \log \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot \hat{I}_s I_t \tag{125}
\]

\[
- v \cdot \log^+ \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \tag{126}
\]

\[
= \left(1 - (\hat{I}_s - I_t) \frac{v}{2} \right) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \tag{128}
\]

\[
+ v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \hat{I}_t + v \cdot \log \left( \alpha_1 \right) \tag{129}
\]

\[
\leq v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \hat{I}_t + v \cdot \log \left( \alpha_1 \right). \tag{130}
\]

Note that \( \mathbb{E}[\hat{I}_t] = \Pr[f(m_t) + \sigma_1 z_t < f(m_t) | \mathcal{F}_t] \), which is upper-bounded by \( q^{\text{low}} \) in light of Lemma 8. By taking the expectation on both sides of the inequality above, we obtain

\[
\mathbb{E}[V(\theta_{t+1}) - V(\theta_t) | \mathcal{F}_t] \tag{131}
\]

\[
\leq v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \left( q^{\text{low}} - \frac{\log(1/\alpha_1)}{\log(\alpha_t/\alpha_1)} \right) \tag{132}
\]

\[
= v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \left( q^{\text{low}} - \rho_{\text{target}} \right). \tag{133}
\]

Because \( v \cdot \log(\alpha_t/\alpha_1) = \min\{w/4, \log(\alpha_t/\alpha_1)\} \), we obtain (36). This completes the proof. □

### A.11 Proof of Lemma 11

Proof of Lemma 11. Extracting \( \log(f(m_{t+1})/f(m_t)) \), \( \log(\alpha_t) \) and \( \log(\alpha_1) \) from \( V(\theta_{t+1}) \).

\[
V(\theta_{t+1}) = \log f(m_{t+1}) \tag{134}
\]

\[
+ v \cdot \log \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot \hat{I}_s \hat{I}_t \tag{135}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot I_s I_t \tag{136}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot I_s \hat{I}_t \tag{137}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot \hat{I}_s I_t \tag{138}
\]

\[
= \log f(m_{t+1}) \tag{139}
\]

\[
- (\hat{I}_s - I_t) \frac{v}{2} \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \tag{140}
\]

\[
+ v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \hat{I}_s I_t - v \cdot \log \left( \alpha_1 \right) \cdot I_s \tag{141}
\]

\[
+ v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot I_s \hat{I}_t + v \cdot \log \left( \alpha_1 \right) \cdot I_t \tag{142}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot I_s \tag{143}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot \hat{I}_t. \tag{144}
\]

Next,

\[
V(\theta_{t+1}) - V(\theta_t) = \left(1 - (\hat{I}_s - I_t) \frac{v}{2} \right) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \tag{145}
\]

\[
+ v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot \hat{I}_s I_t - v \cdot \log \left( \alpha_1 \right) \cdot I_s \tag{146}
\]

\[
+ v \cdot \log \left( \frac{\alpha_t}{\alpha_1} \right) \cdot I_s \hat{I}_t + v \cdot \log \left( \alpha_1 \right) \cdot I_t \tag{147}
\]

\[
+ v \cdot \log \left( \frac{Tr(H)\sigma_t}{b_t \sqrt{L_f(m_t)}} \right) \cdot I_s \tag{148}
\]

\[
- v \cdot \log^+ \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot I_t \tag{149}
\]

\[
- v \cdot \log^+ \left( \frac{b_t \sqrt{L_f(m_t)}}{Tr(H)\sigma_t} \right) \cdot \hat{I}_t \tag{150}
\]

In order to inspect the sum of (148) ~ (151), we calculate the products of \( \hat{I}_s, \hat{I}_t, I_t \), and \( I_s \). Note \( f(m_t) \geq f(m_{t+1}) \) for all \( t > 0 \) in
Algorithm 1.

\begin{align}
I_{\ell}I_1 &= \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1, \\
I_{\ell}I_1 &= \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1, \\
\frac{\ell}{b_1\sqrt{L_f(m_1)}} \cdot I_1 &= 1 \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1 \\
&= \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1. \\
\frac{\ell}{b_1\sqrt{L_f(m_1)}} \cdot I_1 &= 1 \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1 \\
&= \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1.
\end{align}

Note that $I_{\ell}I_1 = 1 \Rightarrow 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)}$ and $I_{\ell}I_1 = 1 \Rightarrow 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)}$. Exploiting these compositions of the indicator functions, the sum of (148)–(151) is proved to be non-positive in general as follows:

\begin{align}
\frac{\ell}{b_1\sqrt{L_f(m_1)}} \cdot I_1 &= 1 \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1 \\
&= 1 \{ 1 \leq \frac{b_1}{\sqrt{\ell}L_f(m_1)} \} \cdot I_1.
\end{align}

In light of Lemma 8, we have

\begin{align}
\mathbb{E} \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) | F_t \leq -w.
\end{align}

Taking the expectation on both sides of the inequality above, we obtain

\begin{align}
\mathbb{E}[V(\theta_{t+1}) - V(\theta_t) | F_t] &\leq -\left( 1 - \frac{\alpha}{\alpha_1} \right) \cdot I_{\ell}I_1 - \alpha \cdot \log \left( \frac{\alpha}{\alpha_1} \right) \cdot I_1, \\
&\leq -\left( 1 - \frac{\alpha}{\alpha_1} \right) \cdot I_{\ell}I_1 - w \cdot \log \left( \frac{\alpha}{\alpha_1} \right).
\end{align}

This completes the proof.

A.12 Proof of Lemma 12

Proof.

\begin{align}
\text{Var}[V(\theta_{t+1}) | F_t] &\leq \mathbb{E}[(V(\theta_{t+1}) - V(\theta_t))^2 | F_t]
\end{align}

holds. Next, we prove that

\begin{align}
V_{t+1} - V_t \leq \alpha \cdot \log \left( \frac{\alpha}{\alpha_1} \right)
\end{align}
and

\[ V(\theta_{t+1}) - V(\theta_t) \geq (1 + \theta) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) - 2\theta \cdot \log (\alpha_t) + \theta \cdot \log (\alpha_t^\dagger). \]  

(177)

First, the discussion in Appendix A.11 until (168) holds for any \( m_t \) and \( \sigma_t \). Therefore, upper bound (176) is straightforward from (168).

Next, we use the lower bounds of (162) and (163) to prove lower bound (177). From the discussion in Appendix A.11 until (151), for any \( \theta_t \),

\[ V(\theta_{t+1}) - V(\theta_t) = \left( 1 - (\mathbb{I}_s - \mathbb{I}_f \frac{\theta}{2}) \right) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) + \theta \cdot \log (\frac{a_t}{a_t^\dagger}) \cdot \mathbb{I}_s, \]  

(178)

\[ + \theta \cdot \log \left( \frac{b_t \sqrt{\log f(m_t)}}{\text{Tr}(H) \sigma_t} \right) \cdot \mathbb{I}_f, \]  

(181)

\[ + \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) \cdot \mathbb{I}_s, \]  

(182)

\[ - \theta \cdot \log^*(\frac{b_t \sqrt{\log f(m_t)}}{\text{Tr}(H) \sigma_t}), \]  

(183)

\[ - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}). \]  

(184)

To make a lower bound of the expression above, we use the lower bounds of (162) and (163). (162):

\[ \theta \cdot \log \left( \frac{b_t \sqrt{\log f(m_t)}}{\text{Tr}(H) \sigma_t} \right) - \theta \cdot \log^*(\frac{b_t \sqrt{\log f(m_t)}}{\text{Tr}(H) \sigma_t}) \cdot \mathbb{I}_s = \theta \cdot \log (\frac{a_t}{a_t^\dagger}) \cdot \mathbb{I}_s, \]  

(185)

\[ \geq \theta \cdot \log (\frac{a_t}{a_t^\dagger}), \]  

(187)

(163):

\[ \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \cdot \mathbb{I}_f = \theta \cdot \log (\frac{a_t^\dagger}{a_t^\dagger}), \]  

(188)

\[ \geq \theta \cdot \log (\frac{a_t^\dagger}{a_t^\dagger}), \]  

(189)

Further, the lower bounds on each indicator function of (189) are

\[ \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \]  

(190)

\[ \geq \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \cdot \mathbb{I}_f \]  

(192)

\[ \geq \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \cdot \mathbb{I}_s \]  

(194)

\[ \geq \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \cdot \mathbb{I}_s \]  

(196)

and

\[ \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \]  

(197)

\[ \geq \theta \cdot \log \left( \frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}} \right) - \theta \cdot \log^*(\frac{\text{Tr}(H) \sigma_t}{b_t \sqrt{\log f(m_t)}}) \cdot \mathbb{I}_f \]  

(199)

Therefore, the lower bound of the sum of eq. (181)~(184) is

\[ \theta \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) - \theta \cdot \log (\alpha_t). \]  

(200)

Additionally, the lower bound of the sum of the remaining terms (178) ~ (180) is

\[ \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) - \theta \cdot \log (\alpha_t) - \theta \cdot \log (\alpha_t^\dagger). \]  

(201)

With (200) and (201), we see that for any \( m_t \) and \( \sigma_t \), (177) holds. Integrating (177) and (168), we obtain

\[ |V(\theta_{t+1}) - V(\theta_t)| \leq \max \left\{ \frac{\theta}{a_t^\dagger}, \frac{\theta}{a_t^\dagger + \theta} \right\}. \]  

(203)

\[ (1 + \theta) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) + 2\theta \cdot \log (\alpha_t) - \theta \cdot \log (\alpha_t^\dagger). \]  

(204)

\[ (1 + \theta) \cdot \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) + 2\theta \cdot \log (\alpha_t) - \theta \cdot \log (\alpha_t^\dagger). \]  

(205)

As the only random variable in the RHS of the expression above is \( \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \), to prove \( \mathbb{E}[|V(\theta_{t+1}) - V(\theta_t)|^2 | \mathcal{F}_t] \), we need to prove that

\[ \mathbb{E} \left[ \left| \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \right| \mathcal{F}_t \right] < \infty. \]  

(206)
and
\[
\mathbb{E} \left[ \left( \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \right)^2 \bigg| \mathcal{F}_t \right] < \infty. \tag{207}
\]
By exploiting the fact that \( x \leq \exp(x) - 1 \) and \( x^2 \leq 2(\exp(x) - 1) \), both (206) and (207) are straightforward for \( d > 3 \) with Lemma 5.

Namely,
\[
\mathbb{E} \left[ \left| \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \right| \bigg| \mathcal{F}_t \right] \leq \frac{1}{d - 3} \cdot \frac{U}{L} < \infty, \tag{208}
\]
and
\[
\mathbb{E} \left[ \left( \log \left( \frac{f(m_{t+1})}{f(m_t)} \right) \right)^2 \bigg| \mathcal{F}_t \right] \leq \frac{2}{d - 3} \cdot \frac{U}{L} < \infty. \tag{209}
\]
This completes the proof. \( \Box \)