A no-go theorem for accelerating cosmologies from M-theory compactifications

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Abstract

It is known that four-dimensional cosmologies exhibiting transient phases of acceleration can be obtained by compactifications of low-energy effective string or M-theory on time-varying manifolds. In the four-dimensional theory, the acceleration can be attributed to a quintessential scalar field with a positive effective potential. Recently, Townsend has conjectured that the potentials obtained by such compactifications cannot give rise to late-time accelerating universes which possess future event horizons. Such a ‘no-go’ result would be desirable, since current string or M-theory seems unable to provide an adequate description of space-times with future event horizons. In this letter, we provide a proof of this conjecture for a class of warped compactifications with a single scalar modulus parametrising the volume of the compactification manifold.
Recent observations indicate that the universe is undergoing an accelerated expansion, but the nature and origin of the ‘dark energy’ driving this acceleration remains elusive. One of the more popular candidates is a scalar field with a positive potential, also known as quintessence. While it is easy to construct ad hoc potentials which give either transient or eternal acceleration, it remains a problem to obtain suitable potentials from a more fundamental theory, such as string or M-theory. Part of the problem stems from the existence of a ‘no-go’ theorem [1] for (possibly warped) compactifications of 10- or 11-dimensional string or M-theory on a time-independent manifold. This no-go theorem states that if the strong energy condition (SEC) is satisfied in the higher-dimensional theory, as it is in low-energy effective string or M-theory, then it is also satisfied in the compactified theory. The latter fact will rule out the possibility of accelerating universes in the compactified theory, since acceleration requires a violation of the SEC.

A way to circumvent the no-go theorem was found by Townsend and Wohlfarth [2], who considered hyperbolic compactification manifolds that have a time-varying volume. They showed that the resulting four-dimensional universe will exhibit a transient phase of acceleration. This result was subsequently extended by others to include product space compactifications [3], as well as flux compactifications, i.e., compactifications of higher-dimensional antisymmetric tensor fields with non-zero flux [4] (see also [5] for some earlier work, and [6] for a recent review). Thus, it appears that accelerating universes are quite generic in time-dependent compactifications of higher-dimensional supergravity theories, which usually contain such antisymmetric tensor fields.

In the reduced four-dimensional space-time, the volume modulus of the compactification manifold appears as a scalar field $\phi$, with a potential $V(\phi)$ whose exact form depends on the details of the compactification. For acceleration to occur, this potential has to be positive, or at least positive in some region of the space of scalar fields. An example of a potential which commonly arises is a single exponential of the form

$$V = \Lambda e^{-a\phi}, \quad (1)$$

where $\Lambda$ and $a$ are positive constants. It is known that compactifications on hyperbolic manifolds of constant curvature yield exponential potentials with $1 < a < \sqrt{3}$, while flux compactifications yield exponential potentials with $\sqrt{3} \leq a < 3$. However, no conventional supergravity compactifications are known to give exponential potentials with $a \leq 1$.\footnote{More exotic types of compactifications, such as when the compactification manifold is non-compact,}
It has been shown in [9] that exponential potentials (1) with $a < 1$ will give rise to eternally accelerating cosmologies with future event horizons. The presence of future event horizons means that current string or M-theory may not be able to provide an adequate description of such cosmologies, in the same way that it is unable to provide a description of de Sitter space (see, e.g., [10]). This has prompted Townsend [11] to make the conjecture that universes with future event horizons cannot arise from compactifications of low-energy effective string or M-theory. In particular, this will rule out exponential potentials with $a < 1$.

In this letter, we will provide a proof of this conjecture, which may be regarded as a type of no-go theorem that generalises the original one of [1] to time-dependent compactifications.

Our starting point is a $D$-dimensional theory satisfying the SEC. The specific examples we have in mind are the $D = 10$ or 11 supergravity actions that serve as the low-energy effective actions for string or M-theory. Since our intention is to compactify this theory to four dimensions, we consider a space-time metric of the form:

$$ds^2_D = \Omega^2(y) e^{-\sqrt{\frac{n}{n+2}}\phi(x)} ds^2_4(x) + e^{\sqrt{\frac{n}{n+2}}\phi(x)} ds^2_n(y),$$

where $n = D - 4$. Here, $ds^2_4$ is the metric of the four-dimensional space-time, while $ds^2_n$ is the metric of some compact, non-singular $n$-manifold $\mathcal{M}$ without boundary. $\Omega$ is a smooth, non-vanishing field on $\mathcal{M}$, known as a warp factor. On the other hand, $\phi$ is a four-dimensional scalar field which parametrises the volume of $\mathcal{M}$. The exponents in (2) have been chosen to give a four-dimensional metric that is in the Einstein conformal frame, as well as a canonically normalised kinetic term for $\phi$ in the four-dimensional effective action. In the case when $\phi$ is identically zero, we recover the ansatz used in [1].

As we are specifically interested in four-dimensional cosmological solutions arising from time-dependent compactifications, let us write the four-dimensional metric in the standard FRW form:

$$ds^2_4 = -d\tau^2 + S^2(\tau) \left( \frac{dr^2}{1-kr^2} + r^2d\Omega_2^2 \right),$$

where $S(\tau)$ is a time-dependent scale factor, and $k = +1, 0, -1$, corresponding to closed, flat and open universes, respectively. Also, the scalar field $\phi = \phi(\tau)$ is assumed to be a function of time only.

The four-dimensional action resulting from the compactification of the $D$-dimensional
theory will take the form (possibly after consistent truncations of irrelevant fields):

\[
\frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right],
\]

(4)

for some effective scalar potential \( V(\phi) \). The exact form of this potential depends on the details of the compactification; examples that are known to arise from string or M-theory include exponential ones of the form (1) as discussed above, and the sum of two exponential terms as studied in [11, 12, 13, 14]. The equations of motion that follow from the action (4), after substituting the form of the metric (3), are

\[
\ddot{S} \frac{\dot{S}}{S} = \frac{1}{6} (-\dot{\phi}^2 + V),
\]

(5)

\[
\ddot{\phi} = -3\dot{\phi} \frac{\dot{S}}{S} - V',
\]

(6)

where \( \dot{} \equiv \frac{d}{d\tau} \) and \( \prime \equiv \frac{d}{d\phi} \). We also have a third equation of motion, the so-called Friedmann equation, but it is not needed here [c.f. (14) below].

Now, the SEC on the matter content of the \( D \)-dimensional theory implies that the \( D \)-dimensional Ricci tensor has a time-time component which satisfies \( R^{(D)}_{00} \geq 0 \). A calculation using the metrics (2) and (3) yields the explicit expression:

\[
R^{(D)}_{00} = -3 \frac{\dot{S}}{S} - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \sqrt{\frac{n+2}{n}} \left( 3 \dot{\phi} \frac{\dot{S}}{S} + \ddot{\phi} \right) + \frac{1}{4} e^{-\sqrt{\frac{n+2}{n}} \phi} \Omega^{-2}(y) \nabla^2 \Omega^4(y).
\]

(7)

Multiplying this by \( \Omega^2 \) and integrating over the higher-dimensional \( n \)-manifold \( M \), we obtain

\[
\int_M \Omega^2 R^{(D)}_{00} = \left[ \int_M \Omega^2 \right] \left[ -3 \frac{\dot{S}}{S} - \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \sqrt{\frac{n}{n+2}} \left( 3 \dot{\phi} \frac{\dot{S}}{S} + \ddot{\phi} \right) \right]
\]

\[
= \left[ \int_M \Omega^2 \right] \left( -\frac{1}{2} V - \frac{1}{2} \sqrt{\frac{n}{n+2}} V' \right),
\]

(8)

where we have used the two equations of motion (5) and (6) to rewrite the right-hand side in the second line. The \( D \)-dimensional SEC then becomes a very simple condition on the potential \( V \):

\[
V' \leq -\sqrt{\frac{n+2}{n}} V.
\]

(9)

Although we are primarily interested in string or M-theory (for which \( n = 6 \) or 7), we shall keep \( n \) arbitrary. For the case of an exponential potential (1), this translates to the condition
that $a > 1$. This immediately explains the observation made in [11] that exponential potentials with $a < 1$ do not seem to arise from compactifications of string or M-theory. If such potentials could arise, they would imply a violation of the SEC in these higher-dimensional theories.

Our task now is to show that the condition (9) will ensure that a four-dimensional universe which solves the equations of motion coming from (4), will not possess any future event horizons even if it undergoes late-time acceleration. We start off by recalling what is known about the general behaviour of $V$. It has been argued in [15] that $V$ must generically vanish as $\phi$ approaches infinity, but that it could still have local minima; a few possible shapes for $V$ are illustrated in [15]. In particular, a positive minimum would imply the presence of a metastable de Sitter region, while a negative minimum would correspond to an anti-de Sitter ‘basin of attraction’. The latter would cause the universe to evolve into a big crunch singularity, and so can be excluded from our consideration without any loss of generality.

It can be seen that the condition (9) immediately rules out the possibility of any potential arising from compactification having stationary points with $V > 0$. Potentials having stationary points with $V < 0$ are not ruled out by this condition—indeed they are present in flux compactifications on a sphere (see, e.g., [12])—but we shall not consider such potentials any further as they will have one or more AdS basins of attraction. Finally, potentials having stationary points with $V = 0$ are also allowed by (9), but they are necessarily either local maxima or saddle points. Both cases will lead to AdS basins of attraction, and so again we shall exclude such potentials from consideration. Hence, we are left with strictly positive potentials that have no stationary points.

Since $V$ is assumed to be positive from now on, the condition (9) can be recast as

$$\frac{V'}{V} < -1.$$  \hspace{1cm} (10)

Note that this implies that $V$ is falling to zero faster than the $a = 1$ exponential potential [which saturates the bound in (10)]. It is now possible to see, at least intuitively, why Townsend’s conjecture is true, if we recall that cosmological solutions can be interpreted as a ball rolling, with friction, up and down the potential $V$ [12]. A steeper descent will imply a late-time evolution in which the scalar field generally has a larger kinetic energy as compared to its potential energy, thus leading to a universe with a lower rate of acceleration by (5).

\footnote{This point was also noted in [11], and references therein.}
Since it is known that the $a = 1$ exponential potential gives rise to universes that are free of future event horizons [9, 16], it follows that any positive potential $V$ satisfying (10) will also give rise to universes without future event horizons.\(^3\)

In the remainder of this letter, we shall see in detail how the preceding result follows from a phase-space analysis of the equations of motion. This approach has the added bonus in that it will allow us to classify the various asymptotic behaviour that can arise from different $V$. Let us define a new time coordinate $t$, related to the proper time $\tau$ by

$$dt = \sqrt{V} \, d\tau.$$  \hspace{1cm} (11)

If we further write $S = e^\alpha$, then the equations of motion (5) and (6) become

$$\ddot{\alpha} = \frac{1}{6}(1 - \dot{\phi}^2) - \dot{\alpha}^2 - \frac{V'}{2V} \dot{\alpha} \dot{\phi},$$  \hspace{1cm} (12)

$$\ddot{\phi} = -3\dot{\alpha} \dot{\phi} - \frac{V'}{2V} (2 + \dot{\phi}^2),$$  \hspace{1cm} (13)

where we denote $\dot{\cdot} \equiv \frac{d}{dt}$ from now on. Note that the definition (11) has deliberately been chosen to ensure that the potential appears in (12) and (13) only as the ratio $\frac{V'}{V}$. These two differential equations govern the time evolution of the solution, and the result can be plotted as a trajectory in the $(\dot{\phi}, \dot{\alpha})$-phase space. We also have the Friedmann equation:

$$\dot{\alpha}^2 - \frac{1}{12} \dot{\phi}^2 = \frac{1}{6} - \frac{k}{V} e^{-2\alpha}.$$  \hspace{1cm} (14)

When $k = 0$, (14) describes a hyperbola in the phase space, which divides the latter into regions where $k = \pm 1$. We would mainly be interested in the upper half of the phase space in which $\dot{\alpha} > 0$, corresponding to expanding universes. Furthermore, since the second derivative of the scale factor $S$ with respect to proper time $\tau$ is given by

$$\frac{d^2 S}{d\tau^2} = \frac{V S}{6} (1 - \dot{\phi}^2),$$  \hspace{1cm} (15)

we see that positive acceleration occurs when $|\dot{\phi}| < 1$, corresponding to a vertical open strip in the phase space.

The dynamical system (12) and (13) is particularly simple for the case of an exponential potential (1), and has been well-studied by various authors (see, e.g., [18, 19, 14]). In this

\(^3\)The $a = 1$ exponential potential was also used as a ‘reference potential’ in [17] to study the asymptotic behaviour of cosmologies with more general potentials.
case, there are two fixed points in the upper-half phase space, given by

\[
(\dot{\phi}_1, \dot{\alpha}_1) = \left( \frac{\sqrt{2} a}{\sqrt{3 - a^2}}, \frac{1}{\sqrt{2(3 - a^2)}} \right), \tag{16}
\]

\[
(\dot{\phi}_2, \dot{\alpha}_2) = \left( 1, \frac{a}{2} \right). \tag{17}
\]

Note that the first lies on the \( k = 0 \) hyperbola, while the second lies on the boundary between the acceleration and deceleration regions. They coincide when \( a = 1 \). Table 1 summarises the late-time behaviour of all solutions that are attracted to either of these fixed points, specifically whether they undergo accelerated and/or decelerated expansion. Whenever a universe undergoing accelerated expansion possesses a future event horizon, this is indicated in the table. As can be seen, such horizons are only present in the case when \( a < 1 \) \([9, 16]\), corresponding to the situation where the fixed point lies on the part of the hyperbola inside the acceleration region. Other solutions that are not attracted to either of these fixed points need not be considered, as they correspond to universes which will either undergo late-time decelerated expansion or contract into a big crunch singularity \([14]\).

We return to the case of a general positive potential \( V \) satisfying \([10]\). Since it has no stationary points, \( \phi \) will exhibit a runaway behaviour \( \phi \to \infty \) at late times. Suppose the dynamical equations \([12, 13]\) have fixed points\(^4\) in this limit. This means that the ratio

\(^4\)Called quasi fixed points in the terminology of \([13]\).
\( \frac{V'}{V} \) will approach some constant value:

\[
\lim_{\phi \to \infty} \frac{V'(\phi)}{V(\phi)} \equiv -a_\infty, \tag{18}
\]

which satisfies \( a_\infty \geq 1 \). Thus, solutions that are attracted to these fixed points have the same asymptotic behaviour as those for the exponential potential \( (11) \) with \( a = a_\infty \). In particular, the fixed points \( (16) \) and \( (17) \) will continue to serve as attractors for expanding universes with such potentials, and the late-time behaviour of these solutions coincide with those listed in Table 1 for \( a \geq 1 \). It follows that those universes which are undergoing late-time acceleration are free of future event horizons. Again, we need not consider solutions that are not attracted to these fixed points, as they correspond to universes which will either undergo late-time decelerated expansion or contract into a big crunch singularity.

We now turn to the possibility that \( (12) \) and \( (13) \) do not admit any fixed points; this would occur when \( \frac{V''}{V} \) does not have a well-defined asymptotic limit (e.g., it is asymptotically periodic) or when it diverges to \( -\infty \). A phase-space trajectory in this case will therefore not terminate. However, its general direction of flow can still be inferred from those of the exponential potential with exponents \( a > 1 \), since the former can be viewed as arising from an exponential potential with an effective exponent \( a(\phi) \) that varies along the trajectory according to the value of \( -\frac{V''}{V} \) [which is always greater than 1 by \( (10) \)]. With this fact in mind, we consider the three possibilities \( k = 0, \pm 1 \) separately.

The situation is simplest for closed \( (k = +1) \) universes, for which it is known that any trajectory with exponential potential \( a > 1 \) will flow to the lower-half of the phase space (c.f. the phase portraits in \[18, 19, 14\]), corresponding to a universe that is contracting into a big crunch singularity. It therefore follows that any solution with positive potential satisfying \( (10) \) will also do likewise, and therefore can be excluded from consideration.

For flat \( (k = 0) \) universes, any trajectory is restricted to flow along the (upper) hyperbola described below \[13\]. If \( a(\phi) \geq \sqrt{3} \) asymptotically, then the trajectory will flow to right infinity, corresponding to an expanding but decelerating universe. Otherwise, the trajectory would asymptotically oscillate back and forth along the right side of the hyperbola. In any case, it cannot enter the acceleration region \( |\dot{\phi}| < 1 \) from the right, since it follows from \( (10) \) and \( (13) \) that \( \ddot{\phi} > 0 \) at \( \dot{\phi} = 1 \). Thus, these trajectories will also describe expanding but decelerating universes, which would not possess any future event horizons.

The most interesting situation occurs for open \( (k = -1) \) universes. If \( a(\phi) \to \infty \) asymptotically, then it can be seen from the phase portraits in \[18, 19, 14\] that the trajectory
will flow to the top of the phase space, approaching $\dot{\phi} = 1$ from the right. This corresponds to an expanding but decelerating universe. Otherwise, the trajectory will asymptotically oscillate between the acceleration and deceleration regions in the top-right quadrant of the phase space, corresponding to a universe undergoing late-time oscillating acceleration and deceleration. To show that such a universe would not possess a future event horizon, it suffices to show that the integral $\int_{\tau_0}^{\infty} \frac{d\tau}{S(\tau)}$ diverges (see, e.g., the second reference in [9] or [20]). Indeed, it follows from (11) and (14) that

$$\int_{\tau_0}^{\infty} \frac{d\tau}{S} = \int_{\phi_0}^{\infty} \frac{d\phi}{(d\phi/d\tau) S} = \int_{\phi_0}^{\infty} \frac{1}{\dot{\phi} \sqrt{\dot{\alpha}^2 - \frac{1}{12} \dot{\phi}^2 - \frac{1}{6}}} d\phi,$$

(19)

which is clearly divergent since the term under the square-root does not vanish asymptotically. Hence, none of these universes will possess a future event horizon, thereby completing the proof of Townsend’s conjecture.

Now, since the condition (10) is valid for all times, it can tell us more than just the late-time behaviour of the acceleration of the universe. For example, it can be used to constrain the amount of cosmological inflation that is possible in the early universe. Indeed, (10) immediately implies that the ‘slow-roll’ condition $|V'| V \ll 1$ cannot be satisfied for such models. In other words, the potential $V$ is falling to zero too quickly and any inflationary era will be too short to be realistic. This is consistent with observations made previously in [3, 21] that the number of e-foldings that can be obtained from such models is only of order 1. However, this is still adequate to describe the current phase of acceleration that our universe seems to be undergoing.

In this letter, we have only considered compactifications of low-energy effective string or M-theory. Although this has been the focus of much of the research in string cosmology, there has been some recent progress to include string corrections, both perturbative and non-perturbative, as well as the presence of extended sources such as branes. This would certainly modify the form of $V$. In some cases, the resulting potentials have been shown to possess a metastable de Sitter region [22]. The no-go theorem presented in this letter is not surprisingly violated for such examples, and it would be interesting to see if a more general one exists that would encompass these cases.

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