Complexity and ordinary life, and some mathematics in both

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What is mathematics, exactly? This is a somewhat complicated question, with no simple answer. In any event, mathematics is like a large place, with many regions and villages, and many different ways of doing things. One can also try to make up new ways of doing things, in connection with whatever might be of interest.

Let us look here at a few points which can come up naturally in ordinary life, as well as involve some substantial mathematics (in some of their forms).

Differences between answers of “yes” and “no”

Imagine the following kind of question: “Does so-and-so have a pencil in his or her office?”

If someone finds a pencil in the office, then that provides a way in which an answer of “yes” can be clearly established. The pencil can simply be shown.

An answer of “no” is quite different, and apparently more complicated. How can one establish an answer of “no”, without just going through all of the contents of the office?

This type of distinction is sometimes described technically through phrases like effective witnesses or succinct certificates. Roughly speaking, an effective witness or succinct certificate is something which would be used to establish an answer to a given question, which is reasonably small or manageable, and for which the verification of this as providing an answer would be fairly easy and definite.

This general idea can be given precise forms in suitable mathematical contexts, and in particular it has a basic role in theoretical computer science.
In the example above, an actual pencil in the office would serve as an effective witness or succinct certificate for an answer of “yes”, to the question of whether there is a pencil in so-and-so’s office. For an answer of “no”, it is not clear what might make sense as an effective witness. There may not be one.

This is a very general issue, and one that comes up in many ways. Instead of a pencil, the question might be more interesting, concerning the possibility that so-and-so is in possession of items of value to sports fans, or nice jewelry, or whether so-and-so remembers about Wednesday evening. One might really want to be able to have a definite answer to the question, and that might not be so easy to come by. (So-and-so might thus be in a good position to be evasive.)

Even if there are effective witnesses for a given answer (“yes” or “no”), this is not the same as saying that it is easy to find an effective witness. In other words, a pencil or other object might serve as an effective witness for an answer of “yes” if one can find one, but that does not mean that it is easy to do so, or that one has a good method for doing so. One might be faced once more with the prospect of something like an exhaustive search, and that might not be appealing or feasible. (Again, this could be helpful for so-and-so.)

In an argument, this might come up in a slightly different (but equivalent) way, in which the roles of “yes” and “no” are reversed. One person might want to put forward a general statement, while another person might want to look in to its correctness. The statement could be something like “Members of the Oblidian Club are all sipifsts!” An effective witness for an answer of “no” could be a person in the Oblidian Club who is manifestly not a sipifst. It may not be easy to locate such an individual even if he or she exists, but at least the possibility is there. For an answer of “yes”, there might not be any simple effective witness like this at all.

These are phenomena which occur all the time, in arguments between people in particular. In theoretical computer science, there are mathematically-precise versions of these notions, and in fact there are famous unsolved problems pertaining to them. In technical terms, one of these is the problem that asks whether the “complexity class” NP is equal to the complementary class co-NP. The NP class involves questions for which effective witnesses for answers of “yes” always exists, while the co-NP class entails questions for which effective witnesses for “no” exist.
Even if effective witnesses exist, it might not be easy to find them, and the complexity class P consists of questions for which there is a method to figure out an answer of “yes” or “no” in a limited amount of time (namely, in “polynomial time”). It is a famous unsolved problem to know whether the class P might actually be the same as NP, even though P seems to be significantly more restrictive than NP.

See [1, 3, 4, 5] for more about these classes and unsolved problems.

At any rate, versions of these issues come up a lot, and in various forms, in both ordinary activities, and in more technical or mathematical situations. We shall see a bit more of this later.

Hamburgers and exponentiation

Some years ago, there was a restaurant (with special emphasis on hamburgers) which had a sign that said something like the following:

We have 256 different kinds of hamburgers.

This might seem to be a rather remarkable statement, but in fact we can look at it mathematically and see how it makes a lot of sense.

The first point is to understand something about what the number 256 actually is. It is not just any number, but a very special one.

One might first notice that 256 is an even number, meaning that it is divisible by 2. One can see this quickly because its last digit is an even number. (One is told in school, at least in years gone by, that a number is even if its last digit is even. This is not hard to verify anyway.) Thus 256 can be written as 2 times something. One can check that that something is 128:

\[ 256 = 2 \times 128. \]

This is a bit curious, because 128 is also an even number. In fact 128 is 2 times 64, and so we get that

\[ 256 = 2 \times 2 \times 64. \]

Now again, 64 is even. It is 2 times 32. And 32 is even, 32 = 2 \times 16. And then 16 = 2 \times 8, 8 = 2 \times 4, and 4 = 2 \times 2.

In the end, we get that 256 can be obtained by multiplying together a bunch of 2’s, without needing any other numbers. Specifically, 256 is the
product of eight 2’s, i.e.,

\[ 256 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2. \]

This is sometimes written as

\[ 256 = 2^8. \]

In general, \(2^m\) means the number that one gets by multiplying together \(m\) 2’s. This process is called exponentiation, and the number \(m\) in \(2^m\) is called the exponent.

This is analogous to the relationship between multiplication and addition. That is, \(m \times 2\) is the same as adding together \(m\) 2’s. With \(2^m\), one multiplies the 2’s instead of adding them.

One of the interesting features of exponentiation is that it leads to pretty big numbers rather quickly. We shall see more of this soon. For the moment, let us continue with the question of the hamburgers. Now that we know that 256 is the same as eight 2’s multiplied together, what does that suggest about the restaurant having 256 different kinds of hamburgers?

In fact, there is a rather simple answer to this. Consider the following 8 choices that one might be offered, in having a hamburger at the restaurant:

1. Do you want pickles?
2. Do you want lettuce?
3. Do you want tomatoes?
4. Do you want onions?
5. Do you want cheese?
6. Do you want ketchup?
7. Do you want mustard?
8. Do you want mayonnaise?

Each of these choices represents 2 possibilities, “yes” or “no”. With all 8 choices, where one is free to give answers of “yes” or “no” for each one, independently of the rest, one obtains a total of 256 possible types of hamburgers to have. This comes from the fact that 256 is really the same as multiplying eight 2’s together, where each of the eight 2’s corresponds to the 2 options
for one of the questions above. (This is a standard observation, and it is not too hard to check.)

This is not to say that the restaurant maintains a supply of all 256 different types of hamburgers at any given moment. Instead, the restaurant has a supply of each of the 8 different ingredients mentioned in the questions above. They might add the ingredients in preparing the hamburger to be served, or they might have a place where a customer can get his or her own ingredients.

One might say that the restaurant has 256 different kinds of hamburgers implicitly. This is as opposed to having them all explicitly, with 256 actual hamburgers present, covering all 256 possible different ways of answering the 8 questions above.

**Hamburgers and exponentiation, part 2**

Now let us imagine that there might be more than the 8 choices mentioned above. There might be options for oregano, bean sprouts, or mushrooms, for instance, or a whole-wheat bun. There might also be an option for a vegetarian burger instead of a usual hamburger.

Suppose that there are 20 yes-or-no choices for the burgers, rather than 8. This means that there would be $2^{20}$ different kinds of burgers, i.e., a product of twenty 2’s. This number is equal to

$$1,048,576$$

(i.e., a little more than a million).

What if there were 30 such choices instead? This would mean that there would be $2^{30}$ different kinds of burgers. This number is equal to

$$1,073,741,824$$

(a little more than a billion).

Each time that one adds ten more choices, one should multiply the total number of burgers by $2^{10}$, which is 1024 (a little more than a thousand). With 40 choices one would get about a trillion different kinds of hamburgers, and with 50 one would get about a quadrillion.

Relatively speaking, it would not be that hard to make a place where one really could have 20, 30, or more choices for what to have on the burger. It might be a little difficult to put in all 30 types of ingredients on a single
hamburger, but still, one can imagine things like this. (There are modest variants of this scenario which might be easier for making the hamburgers, and which still lead to large numbers of different types.)

With larger numbers of choices like this, the *implicitness* becomes more and more of an issue. One can imagine a place where a large number of options for the different hamburgers is possible, so that very many different kinds of hamburgers are possible, but this is very different from actually having all of those millions or billions of hamburgers right there. The range of all of these hamburgers would be represented *implicitly* in this way, if not *explicitly*.

To put this into perspective, how many seconds are there in a year? There are 60 seconds in a minute, 60 minutes in an hour, 24 hours in a day, and 365 days in a year. Thus the total number of seconds in a year is

$$60 \times 60 \times 24 \times 365 = 31,536,000.$$

One might also think about issues of storage for the millions or billions of hamburgers. Having a facility in which one can make hamburgers with any of the various choices is much easier by comparison.

All of this makes sense much more generally. A modest number of independent choices or options can lead to huge numbers of total different outcomes, like different possible hamburgers, as above. This works just as well for other types of objects, in addition to hamburgers.

As another basic scenario, imagine having a bunch of places in some area, like towns and villages, and some roads connecting them. One can then have a lot of different paths that one could take, following roads between the different places. I.e., does one go here first and then there, or the other way around, and so on.

In general situations like this, the number of different paths that would be possible can grow exponentially, as in the case of hamburgers. Now it would grow exponentially as compared to the number of places and roads in the region, rather than the number of ingredients, like pickles, cheese, and mustard. With a modest number of places and roads, there can be an enormous number of different paths that one could take, just as before.

For this statement, we might agree to only count paths that do not go in circles anywhere. Otherwise, one might simply go around circles over and over again, which leads to numerous paths by itself. Without ever going around in circles in any given path, one can still have very many paths, with
exponential size for the number of paths. If one includes paths that can go around in circles (for a while, say), then one gets even more.

In this scenario the idea of implicitness comes up too. One might depict the places and roads on a map or a diagram, and this implicitly gives a way to represent all of the paths on them. Any given path can be traced out on the map or diagram, for instance. An actual listing of all of the individual paths would be quite different. In general there could be too many of them to list in a reasonable or practical way.

### Big sets and effective witnesses

Once one has very big sets like this, as in the illustrations of hamburgers and paths, one has again issues like those from before, with effective witnesses, complexity, and so on.

A classical example of this occurs with the case of paths. Imagine that one has a list of places in the region, and that one asks the following type of question:

> Is there a path along these roads, which visits all of the locations on this list, but where the total distance travelled is at most 230 miles?

For an answer of “yes”, there is a clear effective witness for this problem, namely a path with the given properties. In particular, for a given path, it is not too hard to check whether or not it has all of the required features.

However, this is not to say that it is easy to find such a path when it exists, or to determine whether such a path does exist (without necessarily exhibiting it). Also, it is not clear that there should be reasonable effective witnesses for an answer of “no”. In other words, how might one be convinced that such a path does not exist, without simply checking all of the individual paths of total length at most 230 miles?

The problem of finding a yes-or-no answer to the question above is sometimes called “the travelling salesperson problem”. It has been much studied, and it is a famous example of a problem which is “complete” for the class NP that was mentioned earlier. See [1, 3, 4, 5].

There are similar issues that one can see in the context of hamburgers. One could have a list of conditions that one would like a hamburger to satisfy (in terms of the different ingredients that are available, as before), and one might ask whether such a hamburger exists. For any given hamburger, it
could be easy to decide whether that one enjoys the right properties, but this is very different from being able to say much about what happens among all hamburgers. In particular, it may not be clear how to find a hamburger with the given features, except for just searching through them directly. There may not be any good way to know when there is no such hamburger, without searching through them all. (When there is a suitable hamburger, one can stop searching as soon as one finds it, but when there is not, one may not know when to stop, without going to the end.)

This type of question about hamburgers is a version of the “satisfiability” problem, discussed in [1, 3, 4, 5].

There are a lot of questions and phenomena along these lines. In technical versions, one can see this in [1, 2, 3, 4, 5], for instance. There are also plenty of forms of this which come up in ordinary life.

In some ways this is not really a big deal. I.e., it is just fair, and reasonable. On the other hand, it is basic and substantial, and can be quite tricky.

References

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