Global residue harmonic balance method for strongly nonlinear oscillator with cubic and harmonic restoring force

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Abstract

This paper focuses on the numerical investigation of a strongly nonlinear oscillator with cubic and harmonic restoring force. We transform this oscillator as a free damped cubic-quintic Duffing oscillator equation by Taylor approximation. The approximated solutions with high accuracy are provided by using the global residue harmonic balance method (GRHBM) without any discretization or restrict assumptions. The sensitive analysis of the approximation or the frequency with respect to the amplitude is considered in detail. Numerical comparisons with Runge–Kutta method and harmonic balance method are given to show the efficiency and stability of GRHBM.

Keywords

Oscillator, approximation, frequency, global residue harmonic balance method

Introduction

Nonlinear oscillations have wide applications in physics, mathematics, mechanics and engineering areas. Generally, the nonlinear differential equations (NDEs) can be used to model the nonlinear oscillators. Due to the strong nonlinearity of NDEs, the study of accurate approximations has been paid much attention. In the past decades, many perturbation and analytical approaches were proposed for solving NDEs. Traditional perturbation methods are not effective for the strong nonlinear oscillators or NDEs and do not give the approximations with sufficient accuracy. In order to overcome these limitations, some analytical methods were investigated including harmonic balance method, variational iteration method, homotopy analysis method, homotopy perturbation method, Li–He’s modified homotopy perturbation method, enhanced homotopy perturbation method, asymptotic method, energy balance method, differential transformation method, parameter expansion method, parameter expansion method, variational principle, frequency-amplitude formulation and so forth.

In this paper, we consider a strongly nonlinear oscillator with cubic and harmonic restoring force

\[
\frac{d^2 u}{dt^2} + u + au^3 + b \sin(u) = 0,
\]  

where a and b are given constants. The initial conditions for (1) are defined as follows

\[
u(0) = A, \quad \frac{du}{dt}(0) = 0,
\]
with a given constant $A$. The above nonlinear oscillator has wide applications in mechanical engineering and transmission in porous media.\textsuperscript{1,34,35} It can be used to model the vibration of the system consisting of a mass together with a spring with cubic nonlinearity and the driving force as shown in Figure 1. We remark that $u(t)$ is the system response, $M$ is the mass, $k$ is the stiffness coefficient of the spring and $\sin(u)$ can be seen as the term resulting from the driving force.

Hosen and Chowdhury proposed an analytical technique based on the harmonic balance method (HBM) for solving this strongly nonlinear oscillator with cubic and harmonic restoring force.\textsuperscript{1} A modified energy balance method (MEBM) was proposed for obtaining the higher-order approximations.\textsuperscript{34} We remark that MEBM may perform less efficient than HBM in some special cases. El-Dib and Matoog pointed out that the period solution of (1) is available when the coefficients of (1) satisfy some constrained condition.\textsuperscript{35} Recently, harmonic balance method proposed by Borges et al.\textsuperscript{36} was also applied for solving the nonlinear oscillators.\textsuperscript{4,16,17} There are some modifications and improvements of HBM, such as modified harmonic balance method,\textsuperscript{37} residue harmonic balance method\textsuperscript{38} and spreading residue harmonic balance method.\textsuperscript{39} In fact, for this strongly nonlinear oscillator (1), the accuracy of the HBM solution in Hosen and Chowdhury (2015)\textsuperscript{1} may deteriorate with the increase of the amplitude or the time. The motivation of this paper lies in that an approximation with high accuracy will be helpful for understanding the vibration behaviour of this nonlinear oscillator. For this purpose, we will consider an analytical approach based on the global residue harmonic balance method (GRHBM) and the Taylor approximation. The GRHBM is an approximation method combined the ideas of the residue harmonic balance method and the homotopy perturbation technology. Different with HBM and MEBM, the residual part can be used to modify the previous approximations of the nonlinear equations. Due to the efficiency of GRHBM, it has been applied to various kinds of the strongly nonlinear oscillators and nonlinear systems arising from mechanics and engineering areas.\textsuperscript{3,40–44} Together with GRHBM and Taylor approximation, an efficient technique can be proposed for solving the oscillator (1). By applying the Taylor approximation of the nonlinear term $\sin(u)$, we first obtain a free damped cubic-quintic Duffing oscillator equation.\textsuperscript{45,46} Then the approximations with high accuracy are given with the help of GRHBM without any discretization. For illustrating the effectiveness of GRHBM, an example resulting from the initial value problem of (1) is discussed in detail. We further consider the sensitive analysis of the approximated solution or the frequency about different amplitude $A$. Numerical comparisons with Runge–Kutta method (RK) and the harmonic balance method are provided. Finally, some conclusions are given.

**Analysis of the strongly nonlinear oscillator (1) by GRHBM**

Denote $f(u) = u + au^3 + b \sin(u)$, the nonlinear system (1) can be simplified as

$$\frac{d^2u}{dt^2} + f(u) = 0.$$

(3)

It is easy to verify that $f(u) = -f(-u)$. As shown in the literatures,\textsuperscript{3,40–44} the global residue harmonic balance method can give the approximated solutions or frequency with high accuracy for the nonlinear systems arising from the science and engineering areas. We will apply the GRHBM for (3). The main difficulty lies in the nonlinearity of $\sin(u)$ arising from $f(u)$. We first introduce Taylor approximation of $\sin(u)$ to release the difficulty of formulating the approximation of the nonlinear equation (3). Theoretically, with the higher nonlinearity of the unknown function, the higher-order polynomials are required for ensuring the accuracy of the approximations. Different with the approximated approach in Hosen and Chowdhury (2015),\textsuperscript{1} we consider the following approximation of $\sin(u)$
\[ \sin(u) \approx u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5, \]

and the approximated equation to (3) can be given by

\[
\frac{d^2u}{dt^2} + (1 + b)u + \left( a - \frac{b}{6} \right) u^3 + \frac{b}{120} u^5 = 0. \tag{4}
\]

We remark that the above equation can be seen as a free damped cubic-quintic Duf
di
g oscillator equation.\textsuperscript{45,46}

We then consider GRHBM for solving (4). By introducing an auxiliary variable \( \tau = \omega t \), we have

\[
\frac{du}{dt} = \omega \frac{du}{d\tau}, \quad \frac{d^2u}{dt^2} = \omega^2 \frac{d^2u}{d\tau^2}.
\]

The approximated equation (4) can be rewritten as

\[
\omega^2 \frac{d^2u}{d\tau^2} + (1 + b)u + \left( a - \frac{b}{6} \right) u^3 + \frac{b}{120} u^5 = 0. \tag{5}
\]

where the initial conditions are given by

\[ u(0) = A, \quad \frac{du}{d\tau}(0) = 0. \]

Assume that the initial approximation to (5) is defined by

\[ u_1(\tau) = A \cos \omega_1 \tau, \quad \tau = \omega_1 t, \]

where \( \omega_1 \) is an unknown frequency to be determined later.

Substituting \( u_1(\tau) \) into (5) yields that

\[
-\omega_1^2 A \cos \tau + (1 + b) A \cos \tau + \left( a - \frac{b}{6} \right) A^3 \cos^3 \tau + \frac{b}{120} A^5 \cos^5 \tau = 0. \tag{6}
\]

One can rewrite the system (6) as a nonlinear system with respect to \( \cos \tau, \cos 3\tau \) and \( \cos 5\tau \). We give the coefficient of \( \cos \tau \), and let it be zero resulting in the following equation

\[
\omega_1^2 - 1 - b - \frac{3}{4} A^2 + \frac{b}{8} A^4 - \frac{b}{192} A^4 = 0.
\]

Thus, the approximated frequency \( \omega_1 \) is given by

\[
\omega_1 = \sqrt{1 + b - \frac{3}{4} A^2 + \frac{b}{8} A^4 - \frac{b}{192} A^4}. \tag{7}
\]

and the first order approximated solution can be obtained by \( u_1(\tau) = A \cos \omega_1 \tau \). Recalling (6), the nonlinear terms including \( \cos 3\tau \) and \( \cos 5\tau \) can be seen as the residual part of (6). For convenience, we denote the residual part of (6) by

\[ R_1(\tau) = \frac{1}{4} \left( a - \frac{b}{6} \right) A^3 \cos 3\tau + \frac{1}{384} A^4 b \cos 3\tau + \frac{b}{1920} A^5 \cos 5\tau. \]

We next show the construct of the second order approximation to (5). According to GRHBM, the second order approximated solution to (5) can be expressed as

\[
u(\tau) = u_1(\tau) + pu_2(\tau), \quad \omega_2 = \omega_1^2 + p\omega_2, \tag{8}
\]

where \( u_2(\tau) = \lambda(\cos \tau - \cos 3\tau) \) with two unknown constants \( \omega_2 \) and \( \lambda \).

Different with HBM,\textsuperscript{4,16,17} all the residual errors of the present approximation are used to improve the higher-order approximations in GRHBM. By GRHBM, the nonlinear function \( F_1(\tau, \omega_2, u_2) \) can be formulated by substituting the
approximation (8) into (5) and taking the coefficients of \( p \). By combining \( F_1(\tau, \omega_2, u_2) \) with \( R_1(\tau) \), we have the nonlinear system as follows

\[
F_1(\tau, \omega_2, u_2) + R_1(\tau) = 0. \tag{9}
\]

In order to obtain the unknown parameters in (9), we further equate the coefficients of \( \cos \tau \) and \( \cos 3\tau \) defined by (9) to zero. It follows the nonlinear equations

\[
A\omega_2 + \Gamma_1\lambda = 0, \tag{10}
\]

\[
\Gamma_2\lambda + \Gamma_3 = 0, \tag{11}
\]

with the coefficients as follows

\[
\Gamma_1 = \omega_1^2 - 1 - b - \frac{3}{2}A^2 a - \frac{1}{4}A^2 b - \frac{5}{384}A^4 b,
\]

\[
\Gamma_2 = 9\omega_1^2 - 1 - b - \frac{3}{4}A^2 a + \frac{1}{8}A^2 b - \frac{1}{384}A^4 b,
\]

\[
\Gamma_3 = \frac{1}{4}A^2 a - \frac{1}{24}A^3 b + \frac{1}{384}A^5 b.
\]

It is easy to obtain \( \lambda \) and \( \omega_2 \) as

\[
\lambda = -\frac{\Gamma_3}{\Gamma_2}, \quad \omega_2 = \frac{\Gamma_1\Gamma_3}{A\Gamma_2}. \tag{12}
\]

By (8), we have the following second order frequency

\[
\tilde{\omega}_2 = \sqrt{\omega_1^2 + \frac{\Gamma_1\Gamma_3}{A\Gamma_2}}. \tag{13}
\]

We can finally obtain the second order approximation to (1) given by (8), where \( \lambda \) and \( \tilde{\omega}_2 \) are defined by (12) and (13), respectively. We remark that the third or higher-order approximations can be given by GRHBM in a similar manner. Generally, after two steps of GRHBM, one can obtain the approximated solutions to (1) with sufficient accuracy.

**Numerical results**

In this section, we test an initial value problem associated with (1) to show the effectiveness of GRHBM. We compare it with some existing methods including Runge–Kutta method and harmonic balance method.\(^1\)

We focus on the numerical behaviour of the strongly nonlinear oscillator (1) with \( a = b = 1 \) and different \( A \). Four cases including \( A = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3} \) and \( \pi \) are considered in this example. When \( A = \frac{\pi}{3} \), the approximated frequency \( \omega_1 \) and \( \tilde{\omega}_2 \) given by GRHBM are 1.64062569 and 1.63834708, respectively. We remark that these frequency parameters can be given by

![Figure 2](image-url). Numerical comparisons of the approximations to (1) with \( A = \frac{\pi}{3} \) (left) and \( \frac{\omega}{\omega_1} \) (right).
mathematics software. Here we refer the second order approximation by HBM as $HBM_2$. The second order frequency $\omega_{HBM_2}$ in this example is equal to 1.63653323. Comparisons of the approximations obtained by RK, HBM and GRHBM are shown in Figure 2 (left). Figure 2 (right) plots the numerical behaviour of $\frac{du}{dt}$. We also define two kinds of errors, one is the relative error defined by

$$error = \frac{|u_{RK} - \bar{u}|}{|u_{RK}|},$$

the other is the log error given by

**Figure 3.** Comparisons of errors of HBM and GRHBM with $A = \frac{\pi}{3}$ error (left) and log error (right).

**Figure 4.** Numerical comparisons of the approximations to (1) with $A = \frac{\pi}{2}$: $u$ (left) and $\frac{du}{dt}$ (right).

**Figure 5.** Comparisons of errors of HBM and GRHBM with $A = \frac{\pi}{2}$ error (left) and log error (right).
with $\hat{u}$ given by HBM or GRHBM. Numerical errors of HBM and GRHBM are plotted in Figure 3. It is easy to see that GRHBM and HBM work well in this example. Due to the advantage that the residual errors are used for improving the accuracy of the approximations, the errors of the second order approximation by GRHBM (named as $GRHBM_2$) are less than the errors of $HBM_2$. This implies that GRHBM performs better than HBM. Figures 4 and 5 show the numerical results of the nonlinear oscillator (1) with $A = \frac{2}{3}\pi$. We also see that GRHBM can give the approximations with high accuracy without any discretization or restrict assumptions. We further consider the sensitive analysis of the tested algorithms with respect to...
the amplitude $A$, and plot the results in Figures 6, 7, 8 and 9. For the fixed $a$ and $b$ in this example, the error of GRHBM solutions is monotonic increasing about the amplitude $A$. Compared with HBM, GRHBM is less sensitive to the parameter $A$. We see that some HBM solutions deviate from the RK solutions, but the GRHBM solutions agree well with the RK solutions. We also consider the behaviour of the approximated frequency $\omega$ given by GRHBM. Figure 10 plots the curve of $\omega$ with $0 < A < \pi$, which shows its monotone increasing property. Therefore, we can conclude that GRHBM is a powerful and effective method for the nonlinear oscillator (1).

**Conclusions**

This paper dealt with the strongly nonlinear oscillator with cubic and harmonic restoring force by using the global residue harmonic balance method (GRHBM). The approximations were provided without complicated calculations, which agree well with the solutions given by Runge–Kutta method. The sensitive analysis of the approximation and the approximated frequency about the amplitude were also investigated. Comparisons with HBM and Runge–Kutta method confirmed that GRHBM is an effective and reliable method for this nonlinear oscillator. However, there still remains an open problem on choosing the optimal approximation of the nonlinear term $\sin(\nu)$ so that the accuracy of the approximations and the computational cost can be as small as possible. Our future work will focus on considering this open problem and extending GRHBM to other nonlinear oscillators.

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