Hypergraph Lagrangians: Resolving the Frankl-Füredi conjecture

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Abstract

An old conjecture of Frankl and Füredi states that the Lagrangian of an $r$-uniform hypergraph on $m$ edges is maximised by an initial segment of colex. In this paper we prove this conjecture for a wide range of sufficiently large $m$. In particular, we confirm the conjecture in the case $r = 3$ for all sufficiently large $m$. In addition, we find an infinite family of counterexamples for each $r \geq 4$ and provide a new proof for large $t$ of a related conjecture of Nikiforov.

1 Introduction

The notion of the Lagrangian of a graph was originally introduced in 1965 by Motzkin and Strauss [13] to provide a beautiful new proof of Turán’s theorem. This concept was later generalised to uniform hypergraphs, where the study of Lagrangians has played an important role in the advancement of our understanding of hypergraph Turán problems. Notably, hypergraph Lagrangians were used by Frankl and Rödl [7] to disprove a conjecture of Erdős [5] on jumps of hypergraph Turán densities. See, for example, [8], [20] and the excellent survey of Keevash [9] for further applications.

In many of these results, the Turán problem can be converted into the problem of determining (or finding good bounds for) the Lagrangian of a particular hypergraph. In this paper we are

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interested in determining the maximum value of the Lagrangian over all \( r \)-graphs (namely, \( r \)-uniform hypergraphs) with a fixed number of hyperedges.

In order to introduce our main results, we require some technical definitions. For \( t \in \mathbb{N} \), say that \( w = (w(1), \ldots, w(t)) \in \mathbb{R}^t \) is a weighting of \([t]\) if: \( w(i) \geq 0 \), for all \( i \in [t] \); and \( \sum_{v \in V} w(v) = 1 \). Let \( G \subseteq [t]^{(r)} \) be a hypergraph on vertex set \([t]\). For \( e \in G \) and a weighting \( w \) of \([t]\), define
\[
w(e) := \prod_{i \in e} w(i),
\]
and for \( F \subseteq G \) define
\[
w(F) := \sum_{e \in F} w(e).
\]

For \( w \) a weighting of \([t]\) and \( G \subseteq [t]^{(r)} \), we may also say that \( w \) is a weighting of \( G \). Define the Lagrangian of \( G \), denoted \( \lambda(G) \), as follows.
\[
\lambda(G) := \max \{ w(G) : w \text{ is a weighting of } [t] \}.
\]

Say that a weighting \( w \) of \([t]\) is maximal for \( G \) if \( w(G) = \lambda(G) \). Also define
\[
\Lambda(m, r) := \max \{ \lambda(G) : G \subseteq \mathbb{N}^{(r)}, |G| = m \}.
\]

For a graph \( G \), it is a simple exercise to show that \( \lambda(G) \) is achieved by equally distributing the weight over a largest clique in \( G \). However, there is no easy way known for calculating the Lagrangian of a given \( r \)-graph (when \( r \geq 3 \)).

Recall that the colexicographic or colex order on \( \mathbb{N}^{(r)} \) is the ordering in which \( A < B \) if \( \sum_{i \in A} 2^i < \sum_{i \in B} 2^i \). Define \( \mathcal{C}(m, r) \) to be the family containing the first \( m \) sets in the colex order on \( \mathbb{N}^{(r)} \). A conjecture of Frankl and Füredi [6] from 1989 states that \( \mathcal{C}(m, r) \) has the largest Lagrangian of any family of cardinality \( m \).

**Conjecture 1.1** (Frankl and Füredi [6]). Let \( G \subseteq \mathbb{N}^{(r)} \) such that \( |G| = m \). Then
\[
\lambda(G) \leq \lambda(\mathcal{C}(m, r)).
\]

In other words, this conjecture says that there exists some weighting \( w \) such that \( w(\mathcal{C}(m, r)) = \Lambda(m, r) \). An interesting special case of this conjecture, which we (following Tyomkyn [21]) refer to as the principal case, is when \( m = \binom{t}{r} \), i.e. when \( \mathcal{C}(m, r) \) is the clique \([t]^{(r)} \).
When discussing results in support of Conjecture 1.1 it is helpful to simultaneously consider the ranges \((\binom{t-1}{r-1}) \leq m < \binom{t-1}{r-1}\), for \(t \in \mathbb{N}\) (where \(t\) is sometimes taken to be sufficiently large). This is a natural partition as the colex graph on \((\binom{t}{r})\) edges is a clique. Understandably, the difficulty of the problem varies depending on how ‘far away’ \(m\) is from \((\binom{t-1}{r-1})\), i.e. how far away \(G\) must be from being a clique. In fact, it will also be convenient for us to express \(m := \binom{t}{r} - a\), for \(1 \leq a \leq \binom{t-1}{r-1}\). In other words, \(a\) is the number of edges that are ‘absent’ from \([t]^r\) in \(C(m, r)\). We will use \(m\) and \(a\) interchangeably throughout the paper.

For a summary of progress made in support of Conjecture 1.1 see Table 1. The results organized in Table 1 show that the principal case of Conjecture 1.1 has been resolved by Talbot \[18\] in the case \(r = 3\) and, for \(r \geq 4\) and \(t\) sufficiently large, by Tyomkyn \[21\]. Nikiforov \[14\] has recently proved the principal case of Conjecture 1.1 for \(3 \leq r \leq 5\). He also showed that the principal case holds whenever \(t \geq 4(r - 1)(r - 2)\), thus providing an explicit bound on \(t\). Interestingly, the results of Nikiforov rely on the analysis of elementary symmetric functions and provide very novel methods for studying Lagrangians of hypergraphs. The principal case has recently been proved in all cases by Lu \[12\].

| Author(s)                  | \(r\) | Bounds on \(a\)                        |
|----------------------------|-------|----------------------------------------|
| Motzkin and Strauss        | 2     | all \(a\)                              |
| Talbot \[18\]              | 3     | \(a \geq 2t - 3\) and \(a \in \{1, 2\}\) |
| Tang, Peng, Zhang and Zhao | 3     | \(a \geq \frac{3t}{2} - \frac{5}{2}\) and \(a \in \{3, 4\}\) |
| Tyomkyn \[21\]            | \(\geq 4\) | \(a \geq t + \delta \cdot t^{3/4}\) |
| Tyomkyn \[21\]            | \(\geq 4\) | \(a \geq \gamma_r \cdot t^{r-2}\) |
| Lei, Lu, Peng \[11\]      | 3     | \(a \geq t + \zeta \cdot t^{2/3}\) |

Table 1: In this table we summarise the main progress made towards Conjecture 1.1. Here, \(\delta\) and \(\zeta\) are absolute constants and \(\gamma_r\) is an absolute constant depending on \(r\). Recall that, by definition, \(a \leq \binom{t-1}{r-1}\).

Our first main result improves upon the results in Table 1 to extend the range of \(a\) for which Conjecture 1.1 is known to hold.

**Theorem 1.2.** For \(r \geq 3\) let \(G\) be an \(r\)-graph with \((\binom{t}{r}) - a\) edges. Then, for \(t\) sufficiently large, \(\lambda(G) \leq \lambda(C(m, r))\) whenever

\[
\begin{align*}
(i) & \quad \binom{t-2}{r-2} \leq a \leq \binom{t-1}{r-1}, \\
(ii) & \quad r + 2 \leq a \leq t - (r - 1).
\end{align*}
\]

In particular, Theorem 1.2 shows that Conjecture 1.1 holds for sufficiently large \(m\) when
\( a \notin [4] \) in the case \( r = 3 \). Given the results in Table 1, we immediately obtain the following corollary.

**Corollary 1.3.** Let \( m \) be sufficiently large. Then \( \lambda(G) \leq \lambda(C(m, 3)) \) for any 3-graph \( G \) on \( m \) edges.

As a first step to proving Theorem 1.2, we show that if there exists a counterexample to Conjecture 1.1, then there exists one that is supported on \([t]\) vertices. This is expressed in the following Corollary to Theorem 1.2. The precise statement of the theorem is more technical, and will be given in Section 3.

**Corollary 1.4.** Let \( r \geq 3 \). There exists \( t_0 := t_0(r) \) such that the following statement holds for all \( t \geq t_0 \). If there exists an \( r \)-graph \( G \) with cardinality \( m \), where \( \binom{t-1}{r} < m \leq \binom{t}{r} \), such that \( \lambda(G) > \lambda(C(m, r)) \), then there exists an \( r \)-graph \( G' \subseteq \binom{t}{r} \) with cardinality \( m \) such that \( \lambda(G') > \lambda(C(m, r)) \).

An analogous result for \( r = 3 \) was proved by Talbot [18] and was used in [11, 18, 20, 21] to prove that Conjecture 1.1 holds for certain ranges (see Table 1).

Our final main result shows that, for each \( r \geq 4 \), there exists an infinite family of counterexamples to Conjecture 1.1.

**Theorem 1.5.** Let \( r \geq 4 \) and let \( t \) be sufficiently large. Let \( m := \binom{t-1}{r} + \binom{t-2}{r-1} + r \). Then

\[
\max \{ \lambda(G) : |G| = m \} > \lambda(C(m, r)).
\]

In fact, Theorem 1.5 is a special case of a much stronger result, which relates the problem of maximising the Lagrangian with the problem of maximising the sum of degrees squared (see Theorem 7.1). This theorem shows that for a wide range of \( t \) and \( m \), the colex graphs are quite far from maximising \( \Lambda(m, r) \).

Corollary 1.3 and Theorem 1.5 together resolve Conjecture 1.1 for large \( m \) and \( r \geq 3 \) (the case \( r = 2 \) having been resolved by Motzkin and Strauss [13]): the conjecture holds for \( r = 3 \) and is false when \( r \geq 4 \).

Nikiforov [14] noted that Conjecture 1.1 does not provide an explicit expression for \( \lambda(C(m, r)) \); in light of this he made the following conjecture and proved it for \( 3 \leq r \leq 5 \) and sufficiently large \( m \), using analytic arguments.

**Conjecture 1.6** (Nikiforov [14]). Let \( r \geq 3 \). If \( m = \binom{t}{r} \), for some real \( x \) which satisfies \( x \geq r - 1 \), then \( \Lambda(m, r) \leq mx^{r-1} \), with equality if and only if \( x \in \mathbb{Z} \).
This conjecture was recently proven for all $r$ and $m$ by Lu [12] using a different analytic approach to Nikiforov. We have an independent proof of this conjecture for $r \geq 3$ and sufficiently large $t$, which follows directly from the methods we use to prove Theorem 1.3. We include this result in an appendix; we believe that, as we use very different techniques to Lu, our proof is of independent interest.

The paper is organised as follows. In Section 2 we introduce some notation and preliminary results that will be used throughout the paper. Theorem 3.2 (which implies Corollary 1.4) will be proved in Section 3. Then the proof of Theorem 1.2 is divided into three regimes based on the size of $a$: Sections 4-6 prove the theorem in each of these regimes. In Section 7 we prove Theorem 1.5 and our result towards Conjecture 1.6 is given in Appendix A.

2 Preliminaries

In this section we will define some notation that will be used throughout the paper and introduce some preliminary lemmas that are helpful in the proof.

First, let us introduce some notation. Let $G \subseteq [t]^{(r)}$. We define $\overline{G} = [t]^{(r)} \setminus G$. Given a set $S \subseteq [t]$, define $N_G(S) := \{e \setminus S : e \cup S \in G\}$; whenever $G$ is clear from the context we omit the subscript $G$. We may sometimes abuse notation and write $N(v_1, \ldots, v_s)$ when $S = \{v_1, \ldots, v_s\}$. For $i \in [t]$, define $G \setminus \{i\}$ to be the hypergraph on vertex set $[t] \setminus \{i\}$ and edge set $\{e \in G : i \notin e\}$. For vertices $i, j \in V(G)$, we define $N_j(i) := N(i) \setminus \{j\}$. Recall that a weighting $w$ of $G$ is called a maximal weighting if $w(G) = \lambda(G)$.

The first lemma we present gives some properties that any maximal weighting of $G$ satisfies. As the proof is not long, we include it for completeness.

**Lemma 2.1** (Frankl and Rödl [7]). Let $G \subseteq [t]^{(r)}$ and let $w$ be a maximal weighting of $G$.

(i) For all $i, j \in [t]$ with $w(i), w(j) > 0$, we have $w(N(i)) = w(N(j))$.

(ii) If $i, j \in [t]$ are such that there is no hyperedge of $G$ containing $\{i, j\}$, then $\lambda(G) \leq \max\{\lambda(G \setminus \{i\}), \lambda(G \setminus \{j\})\}$.

**Proof.** For (i), suppose, in order to obtain a contradiction, that $w(N(i)) > w(N(j))$. Let $0 < \varepsilon < \min\{w(i), w(j)\}$. Define another weighting $w'$ of $[t]$ as follows. Set $w'(i) := w(i) + \varepsilon$, $w'(j) := w(j) - \varepsilon$, and for all $e \in G$, let $w'(e) := w(e)$. Then $w'(G) > w(G)$, which is a contradiction. Therefore, $w(N(i)) = w(N(j))$.

For (ii), suppose there is no hyperedge of $G$ containing $\{i, j\}$. Let $\alpha := \max\{\lambda(G \setminus \{i\}), \lambda(G \setminus \{j\})\}$. Then $\lambda(G) \leq \max\{\lambda(G \setminus \{i\}), \lambda(G \setminus \{j\})\}$.


\( w'(j) := w(j) - \varepsilon \) and, for \( k \in [t] \setminus \{i, j\} \), set \( w'(k) := w(k) \). As

\[
\begin{align*}
    w'(N(i)) &= w(N(i)) - \varepsilon \cdot w(N(i, j)) \\
    w'(N(j)) &= w(N(j)) + \varepsilon \cdot w(N(i, j)),
\end{align*}
\]

we have

\[
    w'(G) - w(G) = \varepsilon (w(N(i)) - w(N(j))) - \varepsilon^2 \cdot w(N(i, j)). \tag{2.1}
\]

Choosing \( \varepsilon \) to be sufficiently small gives \( w'(G) - w(G) > 0 \), contradicting the choice of \( w \) as a maximal weighting of \( G \). This completes the proof of \( \text{[i]} \).

Suppose, without loss of generality, that \( w(i) \geq w(j) \) and that there is no hyperedge containing \( i \) and \( j \) (and so \( N(i, j) = \emptyset \)). Define \( w' \) as above with \( \varepsilon = w(j) \). From (2.1) and (i) we see that \( w' \) is a maximal weighting of \( G \), where \( w'(j) = 0 \). But defining \( G' := G \setminus \{j\} \), we get that \( w' \) is a weighting of \( G' \) such that \( w'(G') = w'(G) \). This proves \( \text{[ii]} \).

The following corollary will be very useful throughout this paper.

**Corollary 2.2.** Let \( G \subseteq [t]^{(r)} \) and let \( w \) be a maximal weighting of \( G \), and let \( i, j \in [t] \) be such that \( w(i), w(j) > 0 \). Then

\[
    w(N(i, j))(w(i) - w(j)) = w(N_j(i)) - w(N_i(j)). \tag{2.2}
\]

**Proof.** Using the relation \( w(N(i)) = w(j)w(N(i, j)) + w(N_j(i)) \) and Lemma 2.1[,1], the proof follows. \( \Box \)

Given statement \( \text{[i]} \) of Lemma 2.1, it is easy to see that for \( G \subseteq [t]^{(r)} \), any maximal weighting \( w \) of \( G \), and any \( j \in [t] \) we can write

\[
    w(G) = \frac{1}{r} \sum_{i \in [t]} w(i)w(N(i)) = \frac{w(N(j))}{r}. \tag{2.3}
\]

This property will be used throughout the rest of the paper.

In order to state our next preliminary lemma, we should first state some definitions. Recall that for \( 1 \leq i < j \leq t \), the \( ij\)-compression of \( F \in [t]^{(r)} \) is defined to be

\[
    C_{ij}(F) := \begin{cases} 
        (F \setminus j) \cup i & \text{if } i \in F, j \notin F, \\
        F & \text{otherwise.}
    \end{cases}
\]
For \( \mathcal{F} \subseteq [t]^{(r)} \) we define

\[
C_{ij}(\mathcal{F}) := \{C_{ij}(F) : F \in \mathcal{F} \} \cup \{F : C_{ij}(F) \in \mathcal{F}\}.
\]

\( \mathcal{F} \) is said to be left-compressed if \( C_{ij}(\mathcal{F}) = \mathcal{F} \) for all \( 1 \leq i < j \leq t \).

The next lemma (originally observed by Frankl and Füredi [6]) will tell us that to find the maximum value of \( \lambda(G) \) over all hypergraphs with \( m \) hyperedges, it suffices to consider left-compressed hypergraphs. As above, we include the simple proof for completeness. We say that a weighting \( w \) of \([t]\) is decreasing if \( w(1) \geq \ldots \geq w(t) \).

**Lemma 2.3** (Frankl and Füredi [6]). Let \( G \subseteq [t]^{(r)} \). For any decreasing weighting \( w \) of \([t]\) and any \( 1 \leq i < j \leq t \), we have \( w(C_{ij}(G)) \geq w(G) \).

**Proof.** We have

\[
w(C_{ij}(G)) - w(G) = \sum_{e \in G, i \in e, j \notin e} (w(i) - w(j)) w(e \setminus \{j\}).
\]

As \( w \) is decreasing, the right hand side is non-negative. This completes the proof. \( \square \)

We now collect together some simple deductions about Lagrangians.

**Lemma 2.4.** Let \( r \geq 3 \). Then there exists \( t_0 := t_0(r) \) such that, for all \( t \geq t_0 \) and \( G \subseteq [t]^{(r)} \), the following statements hold.

(i) \( \lambda([t]^{(r)}) = \frac{1}{t^{r-1}} (t-1) \cdots (t-r+1) = \frac{1}{r!} \left( 1 - \frac{(r-1)r}{2t} + O(t^{-2}) \right) \).

(ii) For \( s = O(t) \), we have \( |\lambda([t+s]^{(r)}) - \lambda([t]^{(r)})| = O\left(\frac{s}{t^r}\right)\).

(iii) \( \lambda(G) \leq 1/r! \).

(iv) For any weighting \( w \) of \([t]\) and \( i \in [t] \), we have \( w(N(i)) \leq (1 - w(i))^{r-1} \lambda(N(i)) \).

**Proof.** Let \( w \) be a maximal weighting of \([t]^{(r)} \). By Lemma 2.1 [4], we have \( w(N(i)) = w(N(j)) \) for all \( i, j \in [t] \). By Corollary 2.2, as \( N_i(j) = N_j(i) \) for every \( i, j \in [t] \), we have \( w(i) = w(j) \) for all \( i, j \in [t] \). Hence every vertex has weight \( 1/t \). So \( \lambda([t]^{(r)}) = \binom{t}{r} \frac{1}{t^r} \).
\[
\begin{align*}
&= \frac{1}{r!} t(t-1) \cdots (t-r+1) \\
&= \frac{1}{r!} t^r - \sum_{\ell=1}^{r-1} \ell \cdot t^{r-1} + O(t^{r-2}) \\
&= \frac{1}{r!} \left( 1 - \frac{(r-1)r}{2t} + O(t^{-2}) \right),
\end{align*}
\]

as required for (i).

Now for (ii), using (i) we have

\[
\left| \lambda([t+s]^{(r)}) - \lambda([t]^{(r)}) \right| = \frac{1}{r!} \left( \frac{(r-1)r}{2t} - \frac{(r-1)r}{2(t+s)} + O(t^{-2}) \right) \\
= O\left( \frac{s}{t(t+s)} \right) \\
= O\left( \frac{s}{t^2} \right),
\]

as required.

For (iii) let \( w \) be a maximal weighting of \( G \). If there exists some \( e \in [t]^{(r)} \setminus G \), then \( w(G+e) \geq w(G) \). So

\[
\lambda(G) = w(G) \leq w([t]^{(r)}) \leq \lambda([t]^{(r)}),
\]

which is less than \( 1/r! \) by (i).

If \( w(i) = 1 \), then \( w(N(i)) = 0 \) and (iv) follows. So suppose \( w(i) < 1 \) and define \( w' \) such that \( w'(i) = 0 \) and \( w'(j) = \frac{w(j)}{1-w(i)} \) for all \( j > 1 \). Note that \( w' \) is a weighting of \( [t] \setminus \{i\} \) and that, by definition, \( w'(j) > w(j) \) for all \( j \in [t] \setminus \{i\} \). Therefore we have

\[
\frac{w(N(i))}{(1-w(i))^{r-1}} = w'(N(i)) \leq \lambda(N(i)),
\]

and (iv) follows. \( \square \)

### 3 Bounding the support of \( G \)

For a finite \( G \subseteq \mathbb{N}^{(r)} \) and \( w \) a weighting of \( G \), say that \( (w,G) \) is optimal if \( w(G) = \Lambda(|G|, r) \) and subject to this the number of vertices of \( G \) is minimal (where the vertices of \( G \) are the vertices that are contained in some edge of \( G \)). Note that if \( (w,G) \) is optimal, then every vertex of \( G \) has non-zero weight.
Remark 3.1. Throughout the paper, we often wish to consider \((w, G)\) satisfying particular properties and so we say that \((w, G)\) is well-behaved if \((w, G)\) is optimal, \(w\) is decreasing (i.e. \(w(1) \geq w(2) \geq \ldots\)) and \(G\) is left-compressed. Note that, for any optimal \((w, G)\), by relabelling we can assume \(w\) is decreasing and hence, by Lemma 2.3, also left-compressed. So whenever we have \((w, G)\) such that \(w(G) = \Lambda(|G|, r)\), we may assume \((w, G)\) is well-behaved. We say that \(G\) is well-behaved if there exists a weighting \(w\) of \(G\) such that \((w, G)\) is well-behaved.

The aim of this section is to show that if there exists a counterexample to Conjecture 1.1, then there exists one that is supported on \(t\) vertices. We will prove the following theorem, which clearly implies Corollary 1.4.

**Theorem 3.2.** Let \(r \geq 3\). There exists \(t_0 := t_0(r)\) such that the following statement holds for all \(t \geq t_0\). Let \(w\) be a weighting of an \(r\)-graph \(G\) with cardinality \(m\), where \(\binom{t-1}{r} < m \leq \binom{t}{r}\), such that \((w, G)\) is well-behaved. Then \(G \subseteq [t]^{(r)}\).

We observe that Theorem 3.2 immediately implies Conjecture 1.1 in the principal case, and also when the number of edges is 1 or 2 below a principal case.

**Observation 3.3.** Let \(r \geq 3\), \(a \in \{0, 1, 2\}\) and \(G\) be an \(r\)-graph on \(m := \binom{t}{r} - a\) edges with a weighting \(w\) such that \((w, G)\) is well-behaved. By applying Theorem 3.2 we have that \(G \subseteq [t]^{(r)}\) and so \(G\) is \([t]^{(r)}\) with \(a\) edges removed. As \((w, G)\) is optimal, so in particular \(G\) is left-compressed, it is clear that \(G \simeq C(m, r)\).

Throughout the section let \(r \geq 3\). We begin by proving some simple facts about a well-behaved pair \((w, G)\). We note that statements similar to (i) and (iii) were also proved in [21] (in Section 3).

**Proposition 3.4.** Let \(G \subseteq N^{(r)}\) such that \(|G| = m\), where \(\binom{t-1}{r} \leq m \leq \binom{t}{r}\) and let \(w\) be a weighting of \(G\) such that \((w, G)\) is well-behaved. There exist constants \(\rho, \kappa > 0\) depending on \(r\) and \(t_0\) such that, for all \(t \geq t_0\), the following statements hold.

(i) For all \(i \in V(G)\), we have \(w(i) \leq \frac{r+1}{t}\).

(ii) For all \(i \in V(G)\), we have \(|N(i)| \geq \rho t^{r-1}\).

(iii) \(|V(G)| \leq \kappa t|\).

(iv) \(\Omega(t)\) vertices of \(G\) have weight \(\Omega(1/t)\).
Proof. As \((w, G)\) is optimal. As we may assume \(w\) is decreasing, it suffices to show that \(w(1) \leq \frac{r+1}{t}\). Using (2.3) and statements (i), (iii) and (iv) of Lemma 2.4 (as \(N(1) \subseteq V(G)\)) we have
\[
r \cdot w(G) = w(N(1)) \leq (1 - w(1))^{r-1} \lambda(N(1)) \leq \frac{(1 - w(1))^{r-1}}{(r-1)!}.
\] (3.1)

As \(|G| = m > \binom{t}{r}\), using Lemma 2.4 (i) gives
\[
w(G) \geq \lambda(\binom{t}{r}) = 1.
\] (3.2)

Putting (3.1) together with (3.2) gives
\[
r! \cdot \frac{1}{t^r} \binom{t}{r} \leq (1 - w(1))^{r-1}.
\]

It is not difficult to check this implies that \(w(1) < \frac{r+1}{t}\), as required (set \(w(1) = \frac{r+1}{t}\), rearrange and obtain a contradiction). This proves (i).

Now for (ii). Note that as \(w\) is decreasing and \(a = O(t^{r-1})\), (i) gives that \(w(i) = O(t^{r-1})\) for all \(i \in V(G)\). Using this and (2.3) gives for each \(i \in V(G)\):
\[
\frac{1}{(t-1)^r} \binom{t-1}{r} = \lambda([t-1]^{(r)}) \leq w(G) = \frac{w(N(i))}{r} = O(|N(i)|t^{r-1}).
\] (3.3)

By rearranging, we find that \(|N(i)| = \Omega(t^{r-1})\), as required for the proof of (ii).

We have
\[
\frac{1}{r} \sum_{i \in [n]} |N(i)| = m \leq \binom{t}{r}.
\]

Using (ii) to bound \(|N(i)|\) and rearranging gives that there exists \(\kappa > 0\) such that \(|V(G)| \leq \kappa t\), as required for (iii).

Pick \(\delta := \min\{\kappa, (2\kappa \cdot w(1))^{-1}\}\) and note that as \(w(1) = O(t^{-1})\), we have that \(\delta = O(1)\). Suppose, for a contradiction, that at most \(\delta t\) vertices have weight at least \(\frac{1}{2\kappa^2 t}\). Then using (i) to bound the weight of these vertices gives
\[
\sum_{i \in V(G)} w(i) \leq \delta t \cdot w(1) + (\kappa - \delta) t \cdot \frac{1}{2\kappa^2 t} < \frac{1}{2\kappa} + \frac{1}{2\kappa} < 1,
\]
a contradiction. This proves (iv).
Throughout the remainder of the section, define \( t_0, \rho \) and \( \kappa \) to be the constants from Proposition 3.4 and assume that \( t \geq t_0 \) (so Proposition 3.4 can be applied). The next lemma shows that removing \( O(t^{-1}) \) edges from a hypergraph will change the Lagrangian by \( \Omega(1/t^2) \).

**Lemma 3.5.** There exist \( t_1, \alpha > 0 \) such that

\[
\Lambda(m, r) - \Lambda(m - \alpha t^{-1}, r) = \Omega(t^{-2}).
\]

**Proof.** Define \( \alpha := 2\kappa r^{-1} \) and let \( t \) be large enough so that

\[
m' := \binom{t-1}{r} - \alpha t^{-1} > \binom{1}{r} \geq \binom{t_0}{r}. \tag{3.4}
\]

Let \( H \subseteq \mathbb{N}(r) \) such that \( |H| = m' \) and let \( w \) be a weighting of \( H \) such that \((w, H)\) is optimal.

By Proposition 3.4 (iii) and definition of \( m' \) (3.4), we have \( |V(H)| \leq \kappa t \). This implies, in particular, that for all \( x \in V(H) \), we (crudely) have \( |N(x)| \leq \kappa r^{-1} t^{-1} \). Let \( x \in V(H) \) be such that \( w(x) \geq w(y) \) for any \( y \in V(H) \). (Note that as \((w, H)\) is optimal, every vertex of \( H \) has non-zero weight.) Let \( u \) be a new vertex and define \( H' := (V', E') \), where \( V' := V(H) \cup \{u\} \) and

\[
E' := E(H) \cup \{u \cup f : f \in N(x)\} \cup \{\{x, u\} \cup S : S \subseteq V(H) \setminus \{x\}, |S| = r - 2\}.
\]

Define \( w' \) as follows. For all \( v \in V' \setminus \{x, u\} \), set \( w'(v) = w(v) \), and set \( w'(x) = w'(u) = w(x)/2 \).

So \( w' \) is a weighting of \( H' \).

For \( t \) sufficiently large, we have

\[
|H'| - |H| \leq \kappa r^{-1} t^{-1} + \kappa r^{-2} t^{-2} < \alpha t^{-1}.
\]

We also have that \( w(H') - w(H) \) is precisely the weight of the edges containing \( x \) and \( u \). As \( |H'| \geq m' \), by Proposition 3.4 (iv) and (3.4), \( \Omega(t) \) vertices of \( H' \) have weight \( \Omega(1/t) \). Let \( B \) be the set of these vertices and let \( E_B \subseteq E' \) be the edges that contain \( \{x, u\} \) and are contained within \( B \cup \{x, u\} \). So we have \( w'(E_B) = \Omega(t^{-2}) \).

By choice of \( \alpha \), we have \( |E(H')| < m \). So putting this all together gives that

\[
\Lambda(m, r) - \Lambda(m - \alpha t^{-1}, r) \geq \lambda(H') - \lambda(H) = \Omega(t^{-2}),
\]

as required. \qed
Throughout the remainder of the section, let \( t \) be sufficiently large, let \( G \) be an \( r \)-graph with cardinality \( m \) for \((t-1) < m \leq \binom{t}{r}\) and let \( w \) be a weighting of \( G \) such that \((w,G)\) is well-behaved. Let \( V(G) := [n] \). We will show that \( n \leq t \), which will prove Theorem 3.2. We now give a lower bound on the weight of all but \( O(1) \) vertices of \( G \).

**Lemma 3.6.** The following statements hold.

(i) There exist constants \( \beta, \gamma > 0 \) such that at most \( \gamma \) vertices \( i \) in \( G \) satisfy \( w(i) < \frac{\beta}{t} \).

(ii) For all \( i \in [n] \), we have \( ||N(1)| - |N(i)|| = O(t^{r-2}) \)

**Proof.** Let \( \alpha \) be the constant from Lemma 3.5 Using Proposition 3.4 (ii), for \( \gamma := 2\alpha \rho \) we have \( |N(n)| + \ldots + |N(n-\gamma+1)| \geq 2\alpha t^{r-1} \). We will show that \( w(n-\gamma) = \Omega(1/t) \), from which (i) will follow as \( w \) is decreasing.

Let \( S := \{n-\gamma, \ldots, n\} \subseteq V(G) \) and let \( G' \) be the graph obtained from \( G \) by deleting edges incident to at least two vertices of \( S \). Let \( W \) be the weight of the edges containing at least two vertices of \( S \). As \( w \) is decreasing, the total weight of these edges is at most

\[
W \leq \sum_{i,j \in S} w(i)w(j)w(N(i,j)) \leq \frac{|S|^2}{2} w(n-\gamma)w(n-\gamma+1). \tag{3.5}
\]

There are no edges in \( G' \) containing any pair of vertices from \( S \). So by Lemma 2.1 (iii), there exists some \( U \subseteq S \) with \( |U| = |S| - 1 \) such that \( \lambda(G') \leq \lambda(G'') \), where \( G'' := G \setminus U \).

As all the \( r \)-tuples that are edges in \( G \) but not in \( G'' \) are incident with \( S \), by choice of \( \gamma \) and by letting \( t \) be sufficiently large,

\[
e(G'') \leq e(G) - 2\alpha t^{r-1} + O(t^{r-2}) < e(G) - \alpha t^{r-1}.
\]

So using Lemma 3.5 we have that

\[
\lambda(G) - \lambda(G'') = \Omega(t^{-2}). \tag{3.6}
\]

Combining (3.5) and (3.6) gives

\[
\left(\frac{\gamma + 1}{2}\right)w(n-\gamma)w(n-\gamma+1) \geq W \geq \lambda(G) - \lambda(G') \geq \lambda(G) - \lambda(G'') = \Omega(t^{-2}).
\]

This shows that there exists a constant \( \beta > 0 \) such that \( w(n-\gamma) \geq \frac{\beta}{t} \), as required for (i).
Now consider (ii). As $G$ is left-compressed, $N(i) \subseteq N(1)$ and hence $N_1(i) \subseteq N_1(1)$. Thus we have

$$0 \leq |N(1)| - |N(i)| = |N_1(1)| - |N_1(i)| = |N_1(1) \setminus N_1(i)|.$$

Let $T := N_1(1) \setminus N_1(i)$. It suffices to show that $|T| = O(t^{r-2})$.

Let $E_1 := \{ e \in T : w(j) \geq \frac{\beta}{r} \text{ for all } j \in e \}$ and let $E_2 := T \setminus E_1$. We will show that $|E_1|, |E_2| = O(t^{r-2})$, which will imply the claim.

First consider $|E_2|$. From (i) we know that at most $\gamma$ vertices have weight less than $\frac{\beta}{r}$. By Proposition 3.4 (iii), $G$ contains at most $\kappa t$ vertices. So $|E_2| \leq \gamma \cdot (\kappa t)^{r-2} = O(t^{r-2})$, as required.

Now let us bound $|E_1|$. Using Proposition 3.4 (i) shows that we can bound the left hand side of (2.2) by

$$w(N(1,i))(w(1) - w(i)) \leq |N(1,i)| \cdot O(t^{-(r-1)}) \leq \left( \frac{\kappa t}{r-2} \right) \cdot O(t^{-(r-1)}) = O(t^{-1}). \quad (3.7)$$

By definition of $E_1$, we have $w(E_1) \geq \left( \frac{\beta}{r} \right)^{r-1} |E_1|$ and so we can bound the right hand side of (2.2) by

$$w(N_1(1)) - w(N_1(i)) = w(T) \geq w(E_1) \geq \left( \frac{\beta}{r} \right)^{r-1} |E_1|. \quad (3.8)$$

Combining (3.7) and (3.8) gives that $|E_1| = O(t^{r-2})$, as required. This completes the proof of (ii) and the proof of the lemma.

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Let us suppose, in order to obtain a contradiction, that $n = |V(G)| = t + s$, for $s \geq 1$. We will show that there exist constants $\mu, \nu > 0$ such that

$$|\{ e \in \overline{G} : w(e) \geq \mu t^{-r} \}| \geq s \cdot \nu t^{r-1}. \quad (3.9)$$

Before proving (3.9), let us show how this implies the theorem.

By (3.9), we have

$$\lambda([n]^{(r)}) - \lambda(G) \geq w(\overline{G}) \geq s \cdot \frac{\mu \nu}{t} = \Omega(s/t).$$

However, by choice of $G$ we have $\lambda(G) \geq \lambda([t-1]^{(r)})$, so this contradicts Lemma 2.4 (ii). It
follows that $V(G) = t$, as required.

It remains to prove (3.9). As $|V(G)| = t + s$, we have

$$\binom{t + s}{r} - \binom{t}{r} \leq |G| \leq \binom{t + s}{r} - \binom{t - 1}{r}.$$ 

And so as $|V(G)| \leq kt$ (by Proposition 3.4 (iii)), $s = O(t)$, and there are constants $\gamma_1$ and $\gamma_2$ such that, for $t$ sufficiently large,

$$\gamma_1 st^{r-1} \leq |G| \leq \gamma_2 st^{r-1}.$$ 

(3.10)

Let

$$U := \{ e \in G : w(e) \geq (\beta/t)^r \} \quad \text{and} \quad U' := E(G) \setminus U,$$

where $\beta$ is the constant from Lemma 3.6 (i). Now suppose, in order to obtain a contradiction, that $|U| < (\gamma_1/2) \cdot s \cdot t^{r-1}$.

Let $S := \{ i \in V(G) : w(i) < \beta/t \}$. By Lemma 3.6 (ii), $|S| \leq \gamma$. Each set in $U'$ contains a vertex of $S$ and so, by the pigeonhole principle, some vertex $j \in S$ is contained in at least $\gamma_1^{-1} |U'|$ members of $U'$. So in particular, there are $\Omega(s \cdot t^{r-1})$ sets of $G$ that contain $j$.

However, using Lemma 3.6 (ii) and letting $t$ be sufficiently large, gives that for all $i \in V(G)$ there are $\Omega(s \cdot t^{r-1})$ sets of $G$ containing $i$. So in total, as $|V(G)| \geq t - 1$, we have

$$|G| \geq |V(G)| \cdot \Omega(s \cdot t^{r-1}) = \Omega(s \cdot t^{r}) > \gamma_2 st^{r-1},$$

for $t$ sufficiently large. This contradicts (3.10). This completes the proof of (3.9) and hence the proof of Theorem 3.2. \qed

4 Proof of Theorem 1.2 (i)

The main goal of this section is to prove the following theorem.

**Theorem 4.1.** Let $G$ be an $r$-uniform hypergraph with $\binom{t-1}{r} + \binom{t-2}{r-1}$ edges, where $t$ is large. Then $\lambda(G) \leq \lambda([t - 1]^{(r)})$.

In particular, this implies Theorem 1.2 (i) (as adding edges to a graph cannot decrease its Lagrangian). We begin by proving some simple bounds. The proof of Theorem 4.1 will then
be given in the following subsection. Our hope is that, by initially separating out these basic bounds, the key ideas within the proof will not be obfuscated and will be clearer to the reader.

Throughout the section let \( r \geq 3 \) and let \( t \) be large. Let \( G \) be an \( r \)-graph with \( \binom{t-1}{r} \leq m \leq \binom{t}{r} \) edges and let \( w \) be a weighting such that \((w,G)\) is well-behaved (which we may do by Remark 3.1). By Theorem 3.2 we may also assume that \( V(G) \subseteq [t] \).

### 4.1 Preliminaries

The following proposition will be used both throughout the proof of Theorem 4.1 and also in the following two sections to complete the proof of Theorem 1.2. For a graph \( G \) and vertex \( x \in G \), define \( e(x) \) to be the number of edges of \( G \) that contain \( x \). As \( G \) is left-compressed, we have \( e(1) \leq \ldots \leq e(t) \). Recall that \( a = \binom{t}{r} - m \), i.e. \( a \) is the number of ‘absent edges’.

**Proposition 4.2.** The following statements hold for \((w,G)\).

1. \( w(1) = \frac{1}{t} + O(at^{-r}) \).
2. \( e(1) \leq \frac{rn}{t} \).
3. \( w(t-1) \geq \frac{1}{6t} + O(t^{-2}) \).
4. For \( x < t \), \( w(x) = w(1) - \Theta \left( \frac{e(x)-e(1)}{t^r-1} \cdot w(t) \right) \).
5. \( w(t) = w(1) - \Theta \left( \frac{e(t)-e(1)}{t^r-1} \right) \).

**Proof.** Note that

\[
\lambda(G) \geq \left( \binom{t}{r} - a \right) \frac{1}{t^r},
\]
as this is the weight of \( G \) with respect to the uniform weight on \([t]\). Moreover, using (2.3) and Lemma 2.4 (iv) and observing that \( N(1) \) is a subgraph of the complete graph \([t-1]^{(r-1)}\), we have

\[
\lambda(G) = \frac{w(N(1))}{r} \leq \frac{1}{r} \cdot (1 - w(1))^{r-1} \binom{t-1}{r-1} \cdot \frac{1}{(t-1)^{r-1}}.
\]

Putting the two inequalities together, we find that

\[
(1 - w(1))^{r-1} \geq r \cdot (t-1)^{r-1} \cdot \frac{1}{r} \cdot \left( \binom{t}{r} - a \right) \frac{1}{t^r}
\]

\[
= \left( \frac{t-1}{t} \right)^{r-1} \left( 1 - \frac{a}{\binom{t}{r}} \right) \tag{4.1}
\]
Hence,
\[
     w(1) \leq 1 - \left(1 - \frac{1}{t}\right) \left(1 - \frac{a}{t}\right)^{1/(r-1)} = \frac{1}{t} + O(at^{-r}).
\]

This completes the proof of (iii).

The statement (iii) follows from the fact \(e(1) \leq e(x)\) for every \(x \in [t]\), which is a consequence of the fact that \(G\) is left-compressed.

Now consider (iii). Let \(G'\) be obtained from \(G\) by removing any edge containing at least two of the three vertices \(t, t-1, t-2\). By Lemma 2.4 (iii), \(\lambda(G') \leq [t-2]^{(r-2)}\). Since \((G, w)\) is well-behaved and \(G\) has at least \(\binom{t-1}{r}\) edges, we have \(w(G) \geq \lambda([t-1]^{(r)})\). Hence, by Lemma 2.4 (i), \(w(G) - w(G')\) is at least \(\frac{1}{2(r-2)!} \cdot \frac{1}{t^2} + O(t^{-3})\), but is also at most
\[
     \frac{1}{(r-2)!} \cdot \left( w(t)w(t-1) + w(t)w(t-2) + w(t-1)w(t-2) \right) \leq \frac{3}{(r-2)!} \cdot w(t-1)w(t-2).
\]
Since \(w(t-2) \leq 1/(t-2)\), we find that \(w(t-1) \geq \frac{1}{6t} + O(t^{-2})\), as required for (iii).

We now prove (iv). First, we claim that \(w(N(1, x)) = \Theta(1)\). Indeed, by (ii), there are at most \(O(t^{-3})\) non-edges in \(N(1, x)\) (as a graph on vertex set \([t] \setminus \{1, x\}\)), and by (iii), the weight of each edge in \(N(1, x)\), except for possibly those containing \(t\), is \(\Omega(t^{-(r-2)})\). Hence,
\[
     w(N(1, x)) = \Omega\left(\left(\frac{t}{r} - \frac{3}{2} - O(t^{-3})\right) \cdot t^{-(r-2)}\right) = \Omega(1).
\]

For an upper bound, it follows from Lemma 2.4 (i) that
\[
     w(N(1, x)) \leq \lambda([t-2]^{(r-2)}) \leq 1/(r-2)! = O(1).
\]

Claim 4.3. The weight of each missing edge is \(\Theta(w(t)t^{-(r-1)})\).

Proof. First consider the case where \(w(t) > \frac{1}{2(6)^{r-1}}\). By (ii) and (iii), all vertices other than \(t\) have weight between \(1/6t + O(t^{-2})\) and \(1/t + O(t^{-2})\). It follows that, in this case, the weight of each missing edge is \(\Theta(w(t)t^{-(r-1)})\).

We now show that if \(w(t) \leq \frac{1}{2(6)^{r-1}}\) then all missing edges contain \(t\). Indeed, by maximality of \(w(G)\), the weight of any non-edge is at most the weight of any existing edge. Now, if there is a missing edge that does not contain \(t\) then it has weight at least \(\frac{1}{(6)^{r-1}} + O(t^{-(r+1)})\), by (iii), which is larger than the weight of any \(r\)-set that contains \(t\) (as the weight of any such \(r\)-set is at most \(w(t) \cdot w(1)^{r-1} \leq \frac{1}{2(6)^{r-1}} + O(t^{-(r+1)})\)), so all \(r\)-sets that contain \(t\) are missing edges of
G. But this implies that the number of missing edges is larger than \((t−1)/(r−1)\), a contradiction. So in this case, all missing edges contain \(t\) as required. It follows that the weight of each missing edge is \(\Theta(w(t)t^{-(r−1)})\).

As \(G\) is left-compressed, \(N_x(1) \setminus N_1(x)\) is a collection of \((r−1)\)-sets that are edges missing from \(N(x)\) but not \(N(1)\). Thus \(|N_x(1) \setminus N_1(x)| = e(x) − e(1)|. By Claim 4.3, each missing edge has weight \(\Theta(w(t)t^{-(r−2)})\). So we have \(w(N_x(1)) − w(N_1(x)) = \Theta \left( \frac{\epsilon(x)−\epsilon(1)}{t^{r−1}} \cdot w(t) \right)\). So as \(w(N(1,x)) = \Theta(1)\), using Corollary 2.2 we have

\[ w(x) = w(1) − \Theta \left( \frac{\epsilon(x)−\epsilon(1)}{t^{r−1}} \cdot w(t) \right), \]

as required for (iv).

Statement (iv) follows from Corollary 2.2 and (iii), using the fact that \(G\) is left-compressed, so \(N_1(1) \subseteq N_1(t)\), hence \(N_1(t) \setminus N_1(1)\) contains \(e(t) − e(1)\) edges, each of which has weight \(\Theta(t^{−(r−1)})\).

\[ \text{Proof of Theorem 4.1} \]

We are now ready to complete the proof of Theorem 4.1. Before giving the details, we provide a brief overview. Define \(H\) to be the \(r\)-graph on vertex set \([t]\) whose non-edges are exactly the \(r\)-tuples that contain \(t−1\) and \(t\).

As \(G\) and \(H\) have the same number of edges, we can pair the elements of \(E(G) \setminus E(H)\) with the elements of \(E(H) \setminus E(G)\), and think of \(H\) as obtained from \(G\) by swapping edges and non-edges that form pairs (we will see that each pair consists of an edge of \(G\) that does not contain both \(t−1\) and \(t\) and a non-edge of that does). We evaluate \(w(G) − w(H)\) by thinking of \(H\) in this way and evaluating the contribution of each swap. Then we use the symmetry of \(H\) to show that by slightly modifying \(w\), we are able to regain more weight than we lost, thus showing that \(\lambda(H) > \lambda(G)\), a contradiction to the choice of \(G\).

**Proof of Theorem 4.1.** We may assume that some non-edge of \(G\) does not contain both \(t\) and \(t−1\); otherwise, by Lemma 2.1(ii) we may remove one of \(t\) and \(t−1\) without decreasing \(\lambda(G)\), but then we obtain a graph on \(t−1\) vertices, hence \(\lambda(G) \leq \lambda([t−1]^{(r)})\), as required.

**Claim 4.4.** \(w(G) − w(H) = O(t^{−0.1}w(t)^2)\).

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**Proof.** For \( x_1 < \ldots < x_r \) and \( y_1 < \ldots < y_{r-2} \), let \( x := (x_1, \ldots, x_r) \) be a non-edge of \( G \) not containing \( \{t-1, t\} \) and let \( y := (y_1, \ldots, y_{r-2}, t-1, t) \) be an edge of \( G \). The weight lost by swapping \( x \) for \( y \) is bounded from above by the following expression.

\[
\begin{align*}
w(y_1, \ldots, y_{r-2}, t-1, t) - w(x_1, \ldots, x_r) \\
\leq (w(1)^{r-2} - w(x_{r-2})^{r-2}) \cdot w(t-1)w(t) \\
\leq \left( w(1)^{r-2} - \left( w(1) - O\left( \frac{e(x_{r-2})}{t^{r-2}} \cdot w(t) \right) \right)^{r-2} \right) \cdot w(t)w(t-1)
\end{align*}
\]

(4.2)

Here we used Proposition 4.2 (i), (iii) and Proposition 4.2 (iv).

If \( e(t-1) \leq t^{r-2-0.1} \), then also \( e(x_{r-2}) \leq t^{r-2-0.1} \) (as \( x_{r-2} < t-1 \)), so the loss from one swap is \( O(t^{-r-0.1}w(t)^2) \). Thus in total \( O(t^{-0.1}w(t)^2) \) is lost from all the swaps, as required.

Now suppose that \( e(t-1) \geq t^{r-2-0.1} \). Let \( S := \{ x \in [t] : e(x) \geq t^{r-2-0.2} \} \). We have \( |S| = O(t^{0.2}) \), as the total number of missing edges is \( O(t^{r-2}) \). We claim that every non-edge of \( G \) contains at least two vertices from \( S \). Indeed, let \( (x_1, \ldots, x_r) \) be a missing edge, and suppose it contains at most one vertex from \( S \). Then

\[
\begin{align*}
w(x_1, \ldots, x_r) &\geq w(t) \cdot \left( w(1) - O\left( \frac{t^{0.1} \cdot e(t-1)}{t^{r-2}} \cdot w(t) \right) \right)^{r-1} \\
&= w(t) \cdot \left( w(1) - O\left( \frac{t^{0.1} \cdot e(t-1)}{t^{r-2}} \cdot w(t) \right) \right) \cdot w(1)^{r-2}
\end{align*}
\]

(4.3)

It follows that the weight of \( G \) can be increased by swapping \( (x_1, \ldots, x_r) \) with any existing edge of \( G \) that contains \( t \) and \( t-1 \), a contradiction.

We now consider two types of missing edges: non-edges with exactly two vertices in \( S \), and non-edges with at least three vertices in \( S \).

By (4.2) and as \( x_{r-2} \notin S \), the loss per swap of a non-edge of the first type is \( O(t^{-(r-2)-0.2}w(t)^2) \). Hence the total loss from swaps of non-edges of the first type is \( O(t^{-0.2}w(t)^2) \). The loss from a swap of a non-edge of the second type is \( O(t^{-(r-2)}w(t)^2) \). Note that, as the number of non-edges containing any fixed three vertices from \( S \) is at most \( \binom{t-3}{r-3} \), the number of non-edges of the second type is \( O(|S|^3 \cdot t^{r-3}) = O(t^{r-3+0.6}) \). Hence the total loss from such edges is \( O(t^{-0.4}w(t)^2) \). So the total loss is \( O(t^{-0.2}w(t)^2) \), as required. \( \square \)
We define a new weight function $w'$ by

$$w'(x) = \begin{cases} 
  w(x) & x \neq t, t - 1 \\
  w(t - 1) + w(t) & x = t - 1 \\
  0 & x = t.
\end{cases}$$

As in $H$ there are no edges containing both $t$ and $t - 1$ and the neighbourhoods of $t$ and $t - 1$ in $[t - 2]$ are the same, $w'(H) = w(H)$.

**Claim 4.5.** There is a vertex $x \in [t - 2]$ such that $|w(x) - w'(t - 1)| = \Omega(t^{-0.02}w(t))$.

**Proof.** If $e(t - 1) \leq t^{r-2-0.01}$, then using Proposition 4.2 [iv] we have $w(t - 1) = w(1) - O(t^{-0.01}w(t))$. Hence $w'(t - 1) - w(1) = w(t) + w(t - 1) - w(1) = \Omega(w(t))$. Otherwise, if $e(t - 1) \geq t^{r-2-0.01}$, then $e(t - 2) \geq t^{r-2-0.02}$ (otherwise, similarly to the calculation in (4.3), all non-edges contain $t - 1$ and $t$, a contradiction). Hence $w(t - 2) = w(1) - \Omega(t^{-0.02}w(t))$. So either $|w(1) - w'(t - 1)| = \Omega(t^{-0.02}w(t))$, or $|w(t - 2) - w'(t - 1)| = \Omega(t^{-0.02}w(t))$. \qed

Let $x_0$ be as in Claim 4.5. We define a new weight function $w''$ by

$$w''(x) = \begin{cases} 
  w'(x) & x \neq x_0, t - 1 \\
  \frac{1}{2} \cdot (w'(t - 1) + w'(x_0)) & x = t - 1 \text{ or } x_0.
\end{cases}$$

Note that the edges in $[t - 1]$ with at most one vertex in $\{x_0, t - 1\}$ have the same contribution to the weight of $H$ under $w'$ and $w''$. Hence

$$w''(H) - w'(H) = w(N_H(x_0, t - 1)) \cdot (w''(x_0)w''(t - 1) - w'(x_0)w'(t - 1))$$

$$= w(N_H(x_0, t - 1)) \cdot \left((w'(x_0) + w'(t - 1))^2/4 - w'(x_0)w'(t - 1)\right)$$

$$= w(N_H(x_0, t - 1)) \cdot \left((w(x_0) - w'(t - 1))^2\right)$$

$$= \Omega \left(t^{-0.04}w(t)^2\right).$$

To bound $N_H(x_0, t - 1)$ we used the fact that $H$ contains a clique on $[t - 1]$, so $|N(x_0, t_1)| = \Omega(t^{r-2})$, and moreover every vertex in $[t - 1]$ has weight $\Omega(t^{-1})$ (by definition of $H$ and Proposition 4.2 [iii]). In the final line we apply Claim 4.5.

Using Claim 4.4 we have $w(G) - w'(H) = w(G) - w(H) = O(t^{-0.1}w(t)^2)$, hence $w''(H) > w(G)$, a contradiction to the assumption that some non-edge does not contain $t - 1$ and $t$. This completes the proof of Theorem 4.1. \qed
5 Proof of Theorem 1.2 (ii) for \( a \geq t^{0.01} \)

The main theorem of the section is the following.

**Theorem 5.1.** Let \( r \geq 3 \) and \( 2 \leq i \leq r - 1 \). For \( t \) sufficiently large, let \( G \subseteq [t]^{(r)} \) be a well-behaved graph with \( \binom{t}{i} - a \) edges, where \( t^{0.01} \cdot \binom{t-(i+1)}{r-(i+1)} \leq a \leq \binom{t-i}{r-i} \). Then every edge of \( G \) contains \( \{ t - (i - 1), \ldots, t \} \).

It is not difficult to see that Theorem 5.1 implies Theorem 1.2 (ii) when \( a \geq t^{0.01} \).

**Proof of Theorem 1.2 (ii) when \( a \geq t^{0.01} \).** Let \( r \geq 3 \) and let \( t \) be sufficiently large. Let \( G \) be an \( r \)-graph on \( \binom{t}{r} - a \) edges, where \( t^{0.01} \cdot (t - i) \cdot \binom{t-i}{r-i} \leq a \leq (t - i) \cdot \binom{t-i}{r-i} \) for some \( 2 \leq i \leq r - 1 \). It is not difficult to see that Theorem 5.1 implies Theorem 1.2 (ii) when \( a \geq t^{0.01} \).

Analogously to above, throughout this section \( r \geq 3 \), \( t \) is sufficiently large and \( G \) is an \( r \)-graph with \( \binom{t}{i} - a \) edges, where \( t^{0.01} \cdot \binom{t-(i+1)}{r-(i+1)} \leq a \leq \binom{t-i}{r-i} \) for some \( 2 \leq i \leq r - 1 \). Let \( w \) be a weighting of \( G \) such that \((w, G)\) is well-behaved. As before, we may assume \( V(G) \subseteq [t] \).

We first prove some preliminary bounds before presenting the proof of Theorem 5.1. Recall that for \( x \in V(G) \), we defined \( e(x) \) to be the number of edges of \( G \) that contain \( x \).

**Proposition 5.2.** The following statements hold for \((w, G)\).

(i) \( e(t) = \Omega(a) \).

(ii) If \( a < \binom{t-2}{r-2} \), then \( w(t) \geq \frac{1}{2t} (1 + O(t^{-1})) \).

(iii) If \( \frac{7}{8} \cdot \binom{t-2}{r-2} \) \leq a \leq \binom{t-2}{r-2} \) then \( e(t) = \Omega(a) \).

**Proof.** We first prove a couple of claims about the Lagrangians of colex graphs.

**Claim 5.3.** For \( 0 \leq i \leq \binom{t-2}{r-2} \), let \( H_i \) denote \( C(r, \binom{t-1}{r} + \binom{t-2}{r-1} + i) \). Then

\[
\lambda(H_i) \geq \lambda([t-1]^{(r)}) + \frac{i}{4(t-1)^r}.
\]
Proof. Define \( w' \) to be the weight function such that \( w'(t) = w'(t-1) = \frac{1}{t-1} \) and \( w'(x) = \frac{1}{t-1} \) for every \( x \in [t-2] \). Then
\[
\lambda(H_i) \geq w'(H_i) = \frac{i}{4(t-1)^2} = \lambda([t-1]^{(r)}) + \frac{i}{4(t-1)^2},
\]
where \( \bar{w} \) is the uniform weighting on \([t-1]\). This completes the proof of the claim. \( \square \)

Claim 5.4. For \( 0 \leq i \leq \binom{t-2}{r-2} \), let \( F_i \) denote \( C(r, (\binom{t}{r}) - i) \). Then
\[
\lambda(F_i) \geq \lambda([t]^{(r)}) - \frac{i}{4r} + \Omega\left(\frac{i^2}{t^{2(r-1)}}\right).
\]

Proof. Note that \( F_{i+1} \) can be obtained by removing one edge from \( F_i \); denote this edge by \( f_i \), and let \( w_i \) be a weighting of \([t]\) such that \( w_i(F_i) = \lambda(F_i) \). Since \( e(t) = i \) in \( F_i \) (i.e. all non-edges contain \( t \)), it follows from Proposition 4.2(i) and (v) that \( w_i(t) = w_i(1) - \Omega(it^{-(r-1)}) = \frac{1}{t} - \Omega(it^{-(r-1)}) \). Therefore
\[
\lambda(F_{i+1}) \geq w_i(F_{i+1}) = w_i(F_i) - w_i(f_i) = \lambda(F_i) - \frac{1}{4r} + \Omega(it^{-2(r-1)}),
\]
as \( w_i(f_i) \geq w_i(t)^r = \frac{1}{t} - \Omega(it^{-2(r-1)}) \). Hence
\[
\lambda([t]^{(r)}) - \lambda(F_i) = \sum_{0 < j \leq i} (\lambda(F_i) - \lambda(F_{i+1}))
\leq \sum_{0 < j \leq i} \left( \frac{1}{4r} - \Omega(it^{-2(r-1)}) \right)
= \frac{i}{4r} - \Omega\left(\frac{i^2}{t^{2(r-1)}}\right).
\]
It follows that \( \lambda(F_i) \geq \lambda([t]^{(r)}) - \frac{i}{4r} + \Omega\left(\frac{i^2}{t^{2(r-1)}}\right) \), as required. \( \square \)

We now prove (ii). As the weight of each non-edge is at least \( w(t)^r \),
\[
\lambda([t]^{(r)}) \geq w([t]^{(r)}) = w(G) + w(\overline{G}) \geq w(G) + a \cdot w(t)^r.
\]
On the other hand, as \( G \) is well-behaved and by Claim 5.4
\[
w(G) \geq \lambda(H_a) \geq \lambda([t]^{(r)}) - \frac{a}{4r} + \Omega\left(\frac{a^2}{t^{2(r-1)}}\right).
\]
It follows that $\frac{1}{t} - w(t) = \Omega(\frac{a}{r^{2r}})$. Since $w(1) \geq 1/t$ and by Proposition 4.2 (iv), we have $1/t - w(t) \leq w(1) - w(t) = O(\frac{e(1) - e(t)}{t})$. Hence $e(t) = \Omega(a)$ as required.

Now we prove (iii). Write $e(G) = (\binom{t-1}{r} + \binom{t-2}{r-1} + i$ (so $i \geq 1$). Note that

$$\lambda(G) \geq \lambda(F_i) \geq \lambda([t-1]^{(r)}) + \frac{i}{4(t-1)^r}, \quad (5.1)$$

where the first inequality holds as $\lambda(G) = \Lambda(|G|, r)$, and the second inequality follows from Claim 5.3.

Let $G'$ be a graph obtained from $G$ by removing any $i$ edges that contain both $t$ and $t-1$ (note that such edges exist). By Proposition 4.2 (i), we have $w(1) \leq \frac{1}{t}(1 + O(t^{-1}))$, hence the weight of the edges removed is at most

$$i \cdot w(t)w(t-1)^{r-2} \leq i \cdot w(t)w(t-1) \cdot \frac{1}{t^{r-2}}(1 + O(t^{-1})),$$

and at least

$$w(G) - w(G') \geq w(G) - \lambda([t]^{(r)}) \geq \frac{i}{4(t-1)^r},$$

by Theorem 4.1 and (5.1). Combining these inequalities gives that $w(t)w(t-1) \geq \frac{1}{4t^2}(1 + O(t^{-1}))$. In particular, $w(t) \geq \frac{1}{10}(1 + O(t^{-1}))$, as required.

Finally we turn our attention to (iii). By Proposition 4.2 (i) and (iv), $w(t) \leq \frac{1-\delta}{t}$, for some constant $\delta > 0$, and we may assume that $\delta < 1/10$. If $w(t-1) \leq \frac{1-\delta/r}{t}$, then $w(t) \geq 1/t$ and by Proposition 4.2 (iv) we have $e(t-1) = \Omega(a)$, as required. So suppose otherwise. Since an $r$-tuple that does not contain $t$ has weight at least $\frac{(1-\delta/r)^r}{t} \geq w(t) \cdot w(1)^{r-1}$, every non-edge contains $t$, i.e. $e(t) = a$. It follows that

$$w(1) - w(t) = \frac{w(N_t(1)) - w(N_1(t))}{w(N(1,t))} \geq (e(t) - e(1)) \cdot w(t-1)^{r-1} \cdot (r-2)!$$

$$\geq a \cdot (1 + O(t^{-1})) \cdot (1 - \delta/r)^{r-1} \cdot r^{-(r-1)}(r-2)!$$

$$\geq a \cdot (1 - \delta)(r - 2)! \cdot t^{-(r-1)}$$

$$\geq a \cdot \frac{9}{10}(r - 2)! \cdot t^{-(r-1)},$$

where the equality follows from Corollary 2.2. The first inequality follows since $w(t-1)^{r-1}$ is a lower bound on the weight of any $(r-1)$-tuple in $[t] \setminus \{1,t\}$ and $w(N(1,t)) \leq \lambda([t - 2]^{(r-2)}) \leq \frac{1}{(r-2)!}$; the second inequality follows from the assumptions on $w(t-1)$ and $e(t)$.
and from Proposition 4.2 (ii); and the final inequality follows since \( \delta < 1/10 \). Since \( w(t) \geq \frac{1}{10}(1 + O(t^{-1})) \) and \( w(1) \leq \frac{1}{10}(1 + O(t^{-1})) \) (by (ii) and Proposition 4.2 (i)), it follows that

\[
a \leq \frac{3}{4} \cdot \frac{10}{9} \cdot \frac{t^{r-2}}{(r-2)!} \cdot (1 + O(t^{-1})) < \frac{7}{8} \left( \frac{t-2}{r-2} \right),
\]

a contradiction. \( \square \)

5.1 Proof of Theorem 5.1

Say that \( x \neq y \in V(G) \) are twins if \( \{e \in N(x) : y \notin e\} = \{e \in N(y) : x \notin e\} \). Note that if \( w \) is a maximal weighting of \( G \), then we may assume that \( w(x) = w(y) \) if \( x, y \) are twins by Corollary 2.2.

As in the proof of Theorem 4.1, we will compare \( G \) with a graph \( H \) on the same number of edges in which all non-edges contain \( \{t - (i - 1), \ldots, t\} \). However, for technical reasons, in this case we require that 1 and \( t - i \) are twins in \( H \).

**Proof of Theorem 5.1.** The case where \( a = \binom{t-2}{r-2} \) is covered by Theorem 4.1 hence we may assume that \( a < \binom{t-2}{r-2} \). Let \( I := \{t - (i - 1), \ldots, t\} \). Suppose that there exists an edge of \( G \) that does not contain \( I \).

**Claim 5.5.** There exists an \( r \)-graph \( H \) on vertex set \([t]\) with \( |H| = |G| \) such that every edge of \( \overline{H} \) contains \( I \), the vertices 1 and \( t - i \) are twins in \( H \) and all but \( O(t^{r-i-1}) \) \( r \)-tuples in \( E(H) \setminus E(G) \) do not contain \( I \).

**Proof.** Let \( F \) be a graph obtained from \( G \) by swapping each edge of \( G \) that does not contain \( I \) with an edge of \( G \) that does; note that such a graph exists by our assumption on the number of non-edges. So every edge of \( \overline{G} \) contains \( I \).

We will show that there exists a graph \( H \) with \( |H| = |F| \) in which the vertices 1 and \( t - i \) are twins, and \( E(H) \Delta E(F) \) contains only \( r \)-tuples that contain \( I \) and at least one of 1 and \( t - i \). This suffices to prove the claim as the condition on \( E(H) \Delta E(F) \) ensures both that every edge of \( \overline{H} \) contains \( I \), and that all but \( O(t^{r-i-1}) \) \( r \)-tuples in \( E(H) \setminus E(G) \) do not contain \( I \) (as \( |E(H) \Delta E(F)| = O(t^{r-i-1}) \)).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be defined as follows.

\[
\mathcal{A} = \{ A \subseteq [t], |A| = r - 1 : A \cup \{1\} \in E(F) \text{ and } A \cup \{t - i\} \notin E(F) \}
\]
\[ B = \{ B \subseteq [t], |B| = r - 1 : B \cup \{ t - i \} \in E(F) \text{ and } B \cup \{1\} \notin E(F) \}. \]

Denote \( a = |A| \) and \( b = |B| \). If \( a \) and \( b \) have the same parity, fix any subset \( S \subseteq A \cup B \) of size \( \frac{a+b}{2} \), and define

\[
T_1 := \{ A \cup \{1\} : A \in A \} \cup \{ B \cup \{ t - i \} : B \in B \}, \\
T_2 := \{ C \cup \{1\} : C \in S \} \cup \{ C \cup \{ t - i \} : C \in S \}.
\]

If \( a \) and \( b \) have different parities, we consider two cases. If there is an edge \( e \) in \( F \) that contains \( I \cup \{1, t - i\} \), then let \( S \subseteq A \cup B \) have size \( \frac{a+b+1}{2} \) and let \( e \) be any \( r \)-set containing \( I \cup \{1, t - i\} \). Set

\[
T_1 := \{ A \cup \{1\} : A \in A \} \cup \{ B \cup \{ t - i \} : B \in B \} \cup \{ e \}, \\
T_2 := \{ C \cup \{1\} : C \in S \} \cup \{ C \cup \{ t - i \} : C \in S \} \cup \{ e \}.
\]

Otherwise, if there are no edges that contain \( I \cup \{1, t - i\} \), then let \( S \subseteq A \cup B \) have size \( \frac{a+b-1}{2} \) and let \( e \) be any \( r \)-set containing \( I \cup \{1, t - i\} \). Set

\[
T_1 := \{ A \cup \{1\} : A \in A \} \cup \{ B \cup \{ t - i \} : B \in B \}, \\
T_2 := \{ C \cup \{1\} : C \in S \} \cup \{ C \cup \{ t - i \} : C \in S \} \cup \{ e \}.
\]

For each case, define \( H \) to be the graph obtained from \( F \) by replacing the edges \( T_1 \) by \( T_2 \). Note that \( |T_1| = |T_2| \) and so \( |H| = |F| = |G| \). By definition, the vertices \( 1 \) and \( t - i \) are twins in \( H \), and every member of \( E(H) \triangle E(F) \) contains \( I \) and at least one of \( \{1, t - i\} \). This completes the proof of the claim.

As in the proof of Claim 4.1, we first find an upper bound on the weight lost when replacing \( G \) with \( H \), and later we show that a modification of \( w \) allows us to gain more weight than we lost, thus reaching a contradiction.

**Claim 5.6.** \( w(H) - w(G) = O(t^{-0.001/r} \cdot \frac{a^2}{t^{(r-1)}}) \).

As the proof is very similar to the proof of Claim 4.4, we do not include all the details.

**Proof.** Again we pair the edges of \( E(G) \setminus E(H) \) with those of \( E(H) \setminus E(G) \), so that we consider \( H \) as being obtained from \( G \) by a series of swaps, such that in all but \( O(t^{r-i-1}) \) swaps an edge in \( G \) that contains \( I \) is swapped with a non-edge that does not contain \( I \). Furthermore, the remaining swapped pairs consist of two \( r \)-tuples that contain \( I \).
Let \( (x_1, \ldots, x_r) \) be a non-edge in \( G \) (that may or may not contain \( I \)) and suppose that it is swapped with the edge \( (y_1, \ldots, y_{r-i}, t - (i - 1), \ldots, t) \) in \( G \). Then

\[
w(y_1, \ldots, y_{r-i}, t - (i - 1), \ldots, t) - w(x_1, \ldots, x_r) \\
\leq (w(1)^{r-i} - w(x_{r-i})^{r-i}) \cdot w(t - (i - 1)) \cdots w(t) \\
= O \left( \frac{e(x_{r-i})}{t^{2(r-1)}} \right).
\]

Here we used (iii) and (iv) of Proposition 4.2 and (iii) of Proposition 5.2.

First consider the contribution from pairs consisting of an edge and a non-edge that both contain \( I \). Since there are \( O(t^{r-i-1}) = O(t^{-0.01}a) \) of them, their contribution is \( O\left(\frac{t^{-0.01}a^2}{t^{2(r-1)}}\right) \).

Next consider the remaining swapped pairs. We distinguish two cases. If \( e(t - (i - 1)) \leq t^{-0.001/r} \cdot a \), then \( e(x_{r-i}) \leq t^{-0.001/r} \cdot a \), which implies that the loss from each swap is \( O(t^{-0.001/r} \cdot \frac{a}{t^{2(r-1)}}) \), and \( O(t^{-0.001/r} \cdot \frac{a^2}{t^{2(r-1)}}) \) in total, as required.

Otherwise, i.e. when \( e(t - (i - 1)) \geq t^{-0.001/r} \cdot a \), we define \( S = \{ x \in [t] : e(x) \geq t^{-0.002\cdot a} \} \). Then \( |S| = O(t^{0.002/r}) \), as the total number of missing edges is \( a \). By a calculation similar to (4.3), every non-edge contains at least \( i \) vertices of \( S \). We consider two types of non-edges: those that contain exactly \( i \) vertices of \( S \), and those that contain at least \( i + 1 \) vertices of \( S \). The number of non-edges of the second type is at most \( O(|S|^{i+1}) \cdot \frac{(t^{-i})^{(i+1)}}{(r-i)^{i+1}} \) = \( O(a \cdot t^{-0.008}) \), hence the loss from swaps involving non-edges of the second type is \( O(\frac{t^{-0.002/r} \cdot a^2}{t^{2(r-1)}}) \). The loss per swap that involves an edge of the first type is \( O(\frac{t^{-0.002/r} \cdot a^2}{t^{2(r-1)}}) \), hence in total \( O(\frac{t^{-0.002/r} \cdot a^2}{t^{2(r-1)}}) \). The claim follows.

We now show that by changing the weight \( w \) of the vertices of \( H \) slightly, we can obtain a graph whose weight is larger than the weight of \( G \), thus reaching a contradiction to the choice of \( G \) and \( w \).

**Claim 5.7.** There exist two vertices \( x, y \in [t] \) that are twins in \( H \), such that \( |N_H(x, y)| = \Omega(t^{r-2}) \) and \( w(x) - w(y) = \Omega(t^{-0.0001/r} \cdot \frac{a}{t^{r-2}}) \).

**Proof.** We consider two cases. First suppose that \( e(t - (i - 1)) \leq t^{-0.0001/r} \cdot a \). If \( i \geq 3 \), then \( a = O(t^{r-3}) \). If \( i = 2 \), then by Proposition 5.2 (iii), \( a \leq \frac{7}{8} \frac{(t-2)}{t^{r-2}} \). By Corollary 2.2 we have

\[
w(t - 1) - w(t) = \frac{w(N_{t-1}(t) - w(N_t(t - 1)))}{w(N(t - 1, t))} \\
= \Omega(w(N_{t-1}(t) \setminus N_t(t - 1)))
\]
\[
\begin{align*}
\Omega \left( \frac{e(t) - e(t-1)}{t^{r-1}} \right) &= \Omega \left( \frac{a}{t^{r-1}} \right),
\end{align*}
\]

For the second equality, we used the fact that \(w(N(t-1, t))\); indeed, at least \(\frac{1}{8} (\tbinom{t-2}{r-2})\) \((r-2)\)-tuples are present in \(N(t-1, t)\) and by Proposition 5.2 each of them has weight \(\Omega(t^{-(r-2)})\). The third equality follows as \(N_t(t-1) \subseteq N_{t-1}(t)\), as the graph is left-compressed, and the final equality follows from the assumption on \(e(t-1)\) and by Proposition 5.2 (ii). Hence, we may take \(x = t - (i-1), y = t\); indeed, note that by choice of \(H\) these vertices are twins in \(H\) and since \(a \leq \frac{7}{8} (\tbinom{t-2}{r-2})\) we have \(|N_H(x, y)| \geq \frac{1}{8} (\tbinom{t-2}{r-2})\).

Now suppose that \(e(t - (i-1)) \geq t^{-0.0001/r} \cdot a\). It follows that \(e(t - i) = \Omega(t^{-0.0001/r} \cdot a)\), as otherwise all non-edges contain \(I\) (similarly to the calculation of \((4.3)\)). Hence \(w(t-1) = w(t-i) = \Omega(t^{-0.0001/r} \cdot a)\) by Proposition 4.2 (ii, v) and Proposition 5.2 (ii). Then we may take \(x = 1, y = t - i\); indeed, by Claim 5.5 we have that 1 and \(t-i\) are twins in \(H\), and \(|N(1, t-i)| = \tbinom{t-2}{r-2} - O(t^{r-2})\) (because all non-edges contain \(I\)).

Let \(x, y\) be as in Claim 5.7. Define \(w'\) as follows.

\[
w'(z) = \begin{cases} 
  w(z) & z \neq x, y \\
  \frac{1}{2} \cdot (w(x) + w(y)) & z \in \{x, y\}.
\end{cases}
\]

Note that \(w'\) is a legal weight function. Since \(x\) and \(y\) are twins in \(H\), the contribution of edges that contain none or exactly one of them to the weight of \(H\) is the same in \(w\) and in \(w'\). We thus have the following.

\[
w'(H) - w(H) = w(N_H(x, y)) \cdot \left( \frac{(w(x) + w(y))}{2} - w(x)w(y) \right) - \frac{w(x) - w(y)}{2} \\
= w(N_H(x, y)) \cdot \frac{(w(x) - w(y))}{2} \\
= \Omega \left( \frac{t^{-0.0002/r} a^2}{t^{2(r-1)}} \right),
\]

where the last equality follows as \(w(N(x, y)) = \Omega(1)\) (as \(|N(x, y)| = \Omega(t^{r-2})\), and each edge in \(N(x, y)\) has weight \(\Omega(t^{-(r-2)})\) and by the assumption on \(x\) and \(y\). It follows from Claim 5.6 that \(w(G) < w'(H)\), a contradiction to the choice of \(G\) and \(w\). \(\square\)
6 Understanding the structure when $a$ is small

The main result of the section is the following lemma, which evaluates the Lagrangian of $G$ when $G$ is a left-compressed subgraph of $[t]^{(r)}$ with not too many non-edges.

**Lemma 6.1.** Let $G$ be a left-compressed $r$-graph on vertex set $[t]$ with $a \leq \binom{t-2}{r-2}$ non-edges. Then

$$\lambda(G) = \mu_0 - \frac{a}{t^r} + \frac{1}{2\mu_2t^{2(r-1)}} \cdot \sum_i e(i)^2 - \frac{r^2a^2}{2\mu_2t^{2(r-1)}} + O\left(a^3t^{-3r+4}\right),$$

where $\mu_i = \binom{t-i}{r-i} \frac{1}{t^r}$ and $e(i)$ is the number of non-edges incident with vertex $i$.

Note that this bound becomes effective when $a = o(t^{r-2})$, as then the error term is smaller than the third term.

Before proving Lemma 6.1 we will state some corollaries. First, it will be helpful to introduce some definitions and notation that will be used throughout the section.

Given a hypergraph $H$, denote the degree of a vertex $x$ by $d(x)$, and let $P_2(H) := \sum_{x \in V(H)} d(x)^2$.

Define

$$P_2(r, m) := \max\{P_2(H) : H \subseteq N^{(r)}, |H| = m\},$$

$$P_2(r, m, t) := \max\{P_2(H) : H \subseteq [t]^{(r)}, |H| = m\}.$$

Below (in Proposition 6.3) we characterise the $r$-graphs $H$ that satisfy $P_2(H) = P_2(r, |H|)$. We are not aware of an existing solution to this problem, but it is not implausible that such a solution exists. In contrast, the problem of characterising $r$-graphs $H$ on $t$ vertices for which $P_2(H) = P_2(r, |H|, t)$ has drawn considerable attention. For $r = 2$, Ahlswede and Katona [2] and Olpp [16] independently showed that for every $m$ and $t$ either the colex graph $C(m, 2)$ or the lex graph $L(m, t, 2)$ graph are maximisers of $P_2(H)$, among $t$-vertex graphs with $m$ edges.

(Here the lex graph $L(m, t, r)$ is defined as follows: given sets $A, B \in [t]^{(r)}$, $A <_{\text{lex}} B$ if and only if $\min\{A, B\} \in A$. The graph $L(m, t, r)$ is the initial segment according to $<_{\text{lex}}$ of $[t]^{(r)}$ of size $m$). Characterising the maximisers is a surprisingly delicate task (see, e.g. [12, 17]).

For $r \geq 3$, the task of calculating $P_2(r, m, t)$ seems out of reach; in particular, it is not the case that for every $r, m, t$ either the corresponding colex or lex graph are maximisers of $P_2(r, m, t)$ contrary to a conjecture from [2]. Nevertheless, some upper bounds on $P_2(r, m, t)$

---

1Take $r = 3$, $m = t = 6$. Then the sum of degrees squared of the lex graph $L(r, m, t)$ and the colex graph $C(r, m)$ is 70, whereas the graph defined by the following edges has sum of degrees squared 72: \{123, 124, 125, 134, 135, 145\}. 27
have been proved \[3, 4, 15\].

The main conclusion that we draw from Lemma 6.1 is the following immediate corollary (where we use the fact that \(P_2(G) = O(a^2)\)).

**Corollary 6.2.** Let \(a \leq (t-2)/(r-2)\) and let \(G\) be a left-compressed \(r\)-graph on \(t\) vertices with \(m := \binom{t}{2} - a\) edges such that \(\lambda(G) = \Lambda(m, r)\). Then

\[
P_2(G) \in (1 \pm O(at^{-r-2})) \cdot P_2(r, a, t).
\]

In particular, if \(a = o(t^{r-2}/3)\) then \(P_2(G) = P_2(r, a, t)\).

Given this corollary, the following proposition will allow us to determine \(G\) when \(a\) is relatively small.

**Proposition 6.3.** Let \(H\) be an \(r\)-graph such that \(P_2(H) = P_2(|H|, r)\). Then one of the following holds.

(a) \(H\) is a subgraph of a clique \([r+1]^r\),

(b) there is a set \(S\) of \(r-1\) vertices such that all edges in \(H\) contain \(S\).

**Proof.** First, we claim that \(|e \cap f| = r-1\) for every pair of distinct edges \(e, f \in E(H)\). Indeed, let \(m = e(H)\). Let \(H'\) be an \(r\)-graph with \(m\) edges that satisfies (B). Then \(P_2(H') = (r-1)m^2 + m\) (as the vertices in \(S\) have degree \(m\) and the remaining \(m\) non-isolated vertices have degree 1). Hence, \(P_2(H) \geq (r-1)m^2 + m\). Also,

\[
P_2(H) = \sum_{x \in V(H)} d(x)^2
\]

\[
= \sum_{e \in E(H)} \sum_{x \in e} d(x)
\]

\[
= \sum_{e, f \in E(H)} |e \cap f|
\]

\[
\leq \sum_{e \in E(H)} (r + (m - 1) \cdot (r - 1))
\]

\[
\leq (r-1)m^2 + m.
\]

Here, the first inequality follows as for any edge \(e\), for every other edge \(f\), we have \(|e \cap f| \leq r-1\).

In fact, since \(P_2(H) \geq (r-1)m^2 + m\), we must have equality. Hence, \(|e \cap f| = r-1\) for every distinct \(e, f \in E(H)\), as desired.
We now conclude that $H$ satisfies |(3) or (10). Let $e, f \in E(H)$ be distinct; so $|e \cap f| = r - 1$. If $e \cap f \subseteq g$ for every $g \in E(H)$, then (10) holds. So let us assume that there is an edge $g$ such that $e \cap f \notin g$. We claim that $h \subseteq e \cup f$ for every $h \in E(H)$. We first show this for $h = g$. Write $S = g \cap (e \cup f)$; if $g \notin e \cup f$ then $|S| \leq r - 1$ and $|S \cap e|, |S \cap f| \geq r - 1$, which implies that $S = e \cap f$, a contradiction to the choice of $g$. Now let $h \in E(H)$. Suppose that $h \notin e \cup f$, and denote $T = h \cap (e \cup f)$. Then $|T| \leq r - 1$ and $|T \cap e|, |T \cap f|, |T \cap g| \geq r - 1$. It follows that $|e \cap f \cap g| \geq r - 1$, a contradiction. Hence, all edges of $H$ are contained in $e \cup f$, a set of size $r + 1$, so (11) holds.

Using this, we can complete the proof of Theorem 1.5 (iii) via the following corollary.

**Corollary 6.4.** Let $a \notin \{3, \ldots, r + 1\}$ be such that $a = o(t^{1/3})$, and let $G$ be an $r$-graph on $m := \binom{t}{a} - a$ edges such that $\lambda(G) = \Lambda(m, r)$. Then $G \simeq G(|G|, r)$.

**Proof.** Let $w$ be a weighting of $G$ such that $(w, G)$ is well-behaved (which we may assume by Remark 3.1). Hence by Theorem 3.2, we may assume that $G \subseteq [t]^{(r)}$. Then using Corollary 6.2 we have that $P_2(G) = P_2([G], r)$. Now by Proposition 6.3 $G$ is a **star** (as (13) cannot arise or is equivalent to (10) when $a$ is 1 or 2), i.e. all its edges contain $r - 1$ fixed vertices, namely $t - (r - 2), \ldots, t$ (as $G$ is left-compressed). This determines $G$ uniquely and implies that $G \simeq G(|G|, r)$.

The remainder of the section is devoted to proving Lemma 6.1. Our plan is as follows: we first obtain good estimates for $w(x)$ in terms of $e(x)$ (the number of edges of $G$ incident with $x$), and then we estimate the weight of the non-edges (i.e. we estimate $w(\overline{G}) = w([t]^{(r)}) - w(G)$), and also the difference between $w([t]^{(r)})$ and $\lambda([t]^{(r)})$. Putting these two estimates together, we obtain an estimate for $w(G)$.

**Proof of Lemma 6.1.** Let $w$ be a decreasing weighing of $G$ (and suppose $G$ is left-compressed with respect to $w$). Write $w(i) = \alpha - \delta(i)$, and define

$$
\mu_i := \binom{t - i}{r - i} t^{-(r - i)} \quad \delta(1) := \frac{e(1)}{\mu_2 t^{r-1}}
$$

(6.1)

Note that $1 = \sum_i w(i) = \sum_i (\alpha - \delta(i)) = t\alpha - \sum_i \delta(i)$, which implies that

$$
\alpha = \frac{1}{t} \left( 1 + \sum_i \delta(i) \right).
$$

(6.2)
Claim 6.5. \( \delta(i) = \frac{e(i)}{\mu_2 t^{r-1}} \left( 1 + O(at^{-(r-2)}) \right) \) for \( i \in [t] \).

Proof. By Proposition 4.2(ii) we have \( w(1) = 1/t + O(at^{-r}) \), hence, using \( e(1) \leq ar/t \) (see Proposition 4.2(ii)) and the definition of \( \delta(1) \), we have \( \alpha = \frac{1}{2} \left( 1 + O(at^{-(r-1)}) \right) \). It follows from (6.2) that \( \sum_i \delta(i) = O(at^{-(r-1)}) \). We can calculate \( w(N(1,i)) \) by first expressing the weight of all \( (r-2) \)-tuples in \( [t] \setminus \{1, i\} \) and then taking away the weight of the \( (r-2) \)-tuples \( f \) such that \( f \cup \{1, i\} \in G \). Hence, the following holds (recall the definition of \( \mu_2 \) from (6.1))

\[
w(N(1,i)) = \left( \frac{t-2}{r-2} \right) a^{r-2} + O \left( \sum_i \delta(i) \cdot \left( \frac{t-3}{r-3} \right) a^{r-3} \right) + O \left( e(1,i)t^{-(r-2)} \right) = \mu_2 + O(at^{-(r-1)}),
\]

where we use the fact that \( e(1,i) \leq e(1) \leq ar/t \) (using Proposition 4.2(ii)). Let \( \{i_1, \ldots, i_s\} \) be a subset of \( [t] \), where \( s \leq r \). Then

\[
w(i_1) \cdots w(i_s) = (\alpha - \delta(i_1)) \cdots (\alpha - \delta(i_s)) = a^s - \alpha^{s-1} \sum_j \delta(i_j) + O \left( t^{-(s-2)} \left( at^{-(r-1)} \right)^2 \right)
\]

For the second equality, we use the fact that \( \delta(x) = O \left( \frac{a}{t^{r-2}} \right) = O(t^1) \) for every \( x \in [t] \), by Claim 6.5. By Corollary 2.2 \( \delta(i) - \delta(1) = w(1) - w(i) = \frac{w(N_1(i)) - w(N_1(1))}{w(N(1,i))} \). Since \( G \) is left-compressed, it follows from (6.4) that

\[
w(N_1(i)) - w(N_1(1)) = w(N_1(i) \setminus N_i(1)) = (e(i) - e(1)) \cdot t^{-(r-1)} \cdot \left( 1 + O(at^{-(r-2)}) \right)
\]

Thus the following holds.

\[
\delta(i) = \frac{1}{w(N(1,i))} \cdot \frac{e(i) - e(1)}{t^{r-1}} \left( 1 + O(at^{-(r-2)}) \right) + \delta(1) = \frac{e(i)}{\mu_2 t^{r-1}} \left( 1 + O(at^{-(r-2)}) \right).
\]

This is the required estimate for \( \delta(i) \). \( \Box \)

In order to estimate \( \lambda(G) \), we evaluate the difference \( \lambda([t]^{(r)}) - \lambda(G) = \lambda([t]^{(r)}) - w(G) \). We begin with \( w(G) \). Note that \( w(G) = w([t]^{(r)}) - w(G) \). We first evaluate both terms on the
right hand side of this expression, starting with the weight of $G$.

$$w(G) = \sum_{(i_1, \ldots, i_r) \in E(G)} w(i_1) \cdots w(i_r)$$

$$= \sum_{(i_1, \ldots, i_r) \in E(G)} \left( \alpha^r - \alpha^{r-1} \sum_{j \in [r]} \delta(i_j) + O \left( t^{-(r-2)} \left( at^{-(r-1)} \right)^2 \right) \right)$$

$$= \sum_{(i_1, \ldots, i_r) \in E(G)} \left( \frac{1}{t^r} - \frac{s^2 a}{\mu_2 t^{2r-1}} - \frac{1}{\mu_2 t^{2(r-1)}} \sum_{j \in [r]} e(i_j) + O \left( a^2 t^{-3r+4} \right) \right)$$

$$= \frac{a}{t^r} + \frac{s^2 a^2}{\mu_2 t^{2r-1}} - \frac{1}{\mu_2 t^{2(r-1)}} \cdot \sum_i e(i)^2 + O \left( a^3 t^{-3r+4} \right).$$

(6.5)

Here we used (6.4) and the following estimate.

$$\alpha^s = \frac{1}{t^s} \left( 1 + \sum_i \delta(i) \right)^s$$

$$= \frac{1}{t^s} \left( 1 + s \cdot \sum_i \delta(i) + O \left( (at^{r-1})^2 \right) \right)$$

$$= \frac{1}{t^s} \left( 1 + s \cdot \sum_i e(i) \right) + O \left( a^2 t^{-2r+3-s} \right)$$

$$= \frac{1}{t^s} \left( 1 + \frac{sra}{\mu_2 t^{r-1}} \right) + O \left( a^2 t^{-2r+3-s} \right).$$

We also have the following.

$$w([t]^{(r)}) = \left( \frac{t}{r} \right)^r \alpha^r - \left( \frac{t-1}{r-1} \right) \alpha^{r-1} \sum_i \delta(i) + \left( \frac{t-2}{r-2} \right) \alpha^{r-2} \sum_{i<j} \delta(i) \delta(j) + O \left( (at^{-(r-1)})^3 \right)$$

Combining this with (6.5) gives an expression for $w(G)$, as required.

We now express $\lambda([t]^{(r)})$ in a way that will make it easy to compare it with $w(G)$. Here we use the fact that the Lagrangian of the clique $[t]^{(r)}$ is the weight of the clique with respect to the uniform weighting, where every vertex has weight $\frac{1}{t} = \alpha - \frac{1}{t} \sum_i \delta(i)$.

$$\lambda([t]^{(r)}) = \left( \frac{t}{r} \right)^r \left( \alpha - \frac{1}{t} \sum_i \delta(i) \right)^r$$

$$= \left( \frac{t}{r} \right)^r \alpha^r - \left( \frac{t}{r} \right)^r \alpha^{r-1} \cdot r \cdot \frac{1}{t} \sum_i \delta(i) + \left( \frac{t}{r} \right)^{r-2} \left( \frac{r}{2} \right) \left( \frac{1}{t} \sum_i \delta(i) \right)^2 + O \left( (at^{-(r-1)})^3 \right)$$
Lemma 6.1 follows, as by (6.5) and (6.6), we have the following.

In this section we find an infinite family of counterexamples to the Frankl-Füredi conjecture for each $r \geq 4$. We will also reveal another connection between the Lagrangian of an $r$-graph $G$ and $P_2(G)$. We now present the main result of this section, which we shall use to give counterexamples to Conjecture 1.1.

Now let us estimate the difference between the Lagrangian and the weight of $[t]^{(r)}$ with respect to $w$.

\[
\lambda([t]^{(r)}) - w([t]^{(r)}) = (t - 2) \alpha^{r-2} \cdot \frac{1}{2} \left( \sum_i \delta(i)^2 - \frac{1}{t} \left( \sum_i \delta(i) \right)^2 \right) + O \left( a^3 t^{-3r+4} \right)
\]

By (6.5) and (6.6), we have the following.

\[
\lambda([t]^{(r)}) - \lambda(G) = \lambda([t]^{(r)}) - w([t]^{(r)}) + w(G)
\]

\[
= \frac{1}{2 \mu_2 t^{2(r-1)}} \left( \sum_i e(i)^2 - \frac{r^2 a^2}{t} \right) + \left( \frac{a}{t^r} + \frac{r^2 a^2}{\mu_2 t^{2(r-1)}} - \frac{1}{\mu_2 t^{2(r-1)}} \cdot \sum_i e(i)^2 \right) + O \left( a^3 t^{-3r+4} \right)
\]

Lemma 6.1 follows, as $\lambda([t]^{(r)}) = \mu_0$. \qed

7 Counterexamples for the Frankl-Füredi conjecture

In this section we find an infinite family of counterexamples to the Frankl-Füredi conjecture for each $r \geq 4$. We will also reveal another connection between the Lagrangian of an $r$-graph $G$ and $P_2(G)$. We now present the main result of this section, which we shall use to give counterexamples to Conjecture 1.1.

Theorem 7.1. Let $r \geq 4$ and $2 \leq i \leq r - 2$. Let $G$ be a well-behaved $r$-graph on vertex set $[t]$ such that every member of $\overline{G}$ contains $I := \{t - (i - 1), \ldots, t\}$. Let $H \subseteq [t]^{(r-i)}$ have edge set $\{e \setminus I : e \in G, I \subseteq e\}$. Then $P_2(H) \in (1 + O(t^{-(i-1)}))P_2(|H|, r - i, t - i)$. 

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Remark 7.2. In Theorem 7.1 if \( e(H)t^{r-i-1} \cdot t^{-(i-1)} = o(1) \) (i.e. if \( e(H) = o(t^{2i-r}) \)) then \( H \) maximises (precisely, not just asymptotically) \( P_2(H') \) among \((r-i)\)-graphs \( H' \) with \( t-i \) vertices and the same number of edges as \( H \) (as \( P_2(H) = O(e(H) \cdot t^{r-i-1}) \)).

Note that if \( G \) is a colex graph as in Theorem 7.1 then \( H \) is a colex \((r-i)\)-graph. However, in many cases the colex graph does not maximise, nor is it close to maximising, the sum of degrees squared among \((r-i)\)-graphs with order \( t-i \) and the same size. Therefore, Theorem 7.1 allows us to find a wide range of counter examples to Conjecture 1.1. In the next corollary we give such an example.

**Corollary 7.3.** Let \( r \geq 4 \), let \( t \) be sufficiently large and let \( m := \binom{t-1}{r} + \binom{t-2}{r-1} + r \). Then \( \Lambda(m, r) > \lambda(C(m, r)) \).

**Proof.** Let \( G := C(m, r) \) edges. Then, in the language of Theorem 7.1 we have \( i = 2 \) and \( H \simeq C(r, r-2) \), i.e. it consists of a clique \([r-2]^{(r-1)}\) and an additional edge that intersects the clique in \( r-3 \) vertices. However, by Theorem 7.1 and Remark 7.2 if \( \lambda(G) = \Lambda(|G|, r) \), then \( P_2(H) = P_2(r, r-2, t-i) \). However, applying Proposition 6.3, shows that every edge of any \((r-2)\)-graph \( H' \) such that \( P_2(H') = P_2(r, r-2) \) contains \( r-3 \) fixed vertices, a contradiction.

Remark 7.4. Recall that when \( r=3 \) there are no counter examples for large \( t \). This is understandable in light of Theorem 7.1 as we would have \( i = 2 \) so \( H \) is a 1-graph, i.e. a collection of singletons, and 1-graphs are uniquely determined (up to relabelling) by the number of vertices and edges.

We now turn to the proof of Theorem 7.1. The proof is somewhat similar to the proof of Theorem 5.1. We first estimate \( w(x) \) in terms of the number of edges of \( H \) incident with \( x \), and then we use these estimates to compare \( w(G) \) and \( w'(G') \), where \( G' \) is obtained by replacing \( H \) be a graph with the same number of edges and which maximises \( P_2(\cdot) \), and \( w' \) is a suitable weight function.

**Proof of Theorem 7.1.** Let \( w \) be a decreasing maximal weighting of \( G \). Since the vertices \( t-(i-1), \ldots, t \) are twins in \( G \) they have the same weight; denote it by \( \beta \). Write

\[
\Delta = \frac{\beta^i w(1)^{r-i-1}}{w(N(1, t-i))}
\]

\[
\alpha = w(t-i) - d(t-i) \cdot \Delta,
\]

where \( d(x) \) denotes the degree of \( x \) in \( H \). For \( x \in \{t-i\} \) write \( w(x) = \alpha + \delta(x) \).
Claim 7.5. \( \alpha = 1/t + O(t^{-i}) \) and \( \delta(x) = d(x) \cdot \Delta(1 + O(t^{-(i-1)})) \) for \( x \in [t-i] \).

Proof. Recall that \( w(x) = \Omega(1/t) \) for every \( x \in [t-1] \) (see Proposition 4.2(i)). Hence, since the number of non-edges in \( N(1,x) \) is \( O(t^{r-i-2}) \) (as all non-edges contain \( \{ t-(i-1), \ldots, t \} \)), we have \( w(N(1,x)) = \Omega(1) \). Hence, by Corollary 2.2, \( w(1)-w(x) = O(w(N_x(1))-w(N_1(x))) = O(t^{-i}) \) for \( x \in [t-i] \), where the second inequality follows as the number of missing edges that contain any fixed vertex is \( O(t^{r-i-1}) \), and the weight of each \( (r-1) \)-set is \( O(t^{-(r-1)}) \). It follows that \( w(x) = w(1)(1 + O(t^{-(i-1)})) \).

We now show that \( w(N(x,t-i)) = w(N(1,t-i)) + O(t^{-i}) \) for \( x \in [t-i] \). Indeed, \( w(N(x,t-i)) - w(N(1,t-i)) \) is equal to

\[
(w(1) - w(x)) \cdot w(N(1,x,t-i)) + w(N(1,t-i) \setminus N(x,t-i)) = O(t^{-i}),
\]

as \( w(N(1,x,t-i)) = 1/(r-3)! \) (by Lemma 2.4) and all missing edges contain \( \{ t-(i-1), \ldots, t \} \). Using Corollary 2.2 again, we find that

\[
w(x) - w(t-i) = \frac{1}{w(N(1,x,t-i))} \left( w(N_{t-i}(x)) - w(N_{t-i}(x)) \right)
\]
\[
= \frac{1}{w(N(1,t-i))} \cdot (d(x) - d(t-i)) \cdot \beta^i w(1)^{r-i-1} \cdot \left( 1 + O(t^{-(i-1)}) \right)
\]
\[
= (d(x) - d(t-i)) \cdot \Delta \cdot (1 + O(t^{-(i-1)}))
\]
\[
= (d(x) - d(t-i)) \cdot \Delta + O(d(x) \cdot \Delta \cdot t^{-(i-1)}),
\]

where the second equality holds as \( N_x(t-i) \subseteq N_{t-i}(x) \), and the weight of each edge in \( N_{t-i}(x) \) is \( \beta^i w(1)^{r-i-1}(1 + O(t^{-(i-1)})) \) and the last equality holds as \( d(x) \geq d(t-i) \). It follows that

\[
w(x) = w(t-i) - d(t-i) \cdot \Delta + d(x) \cdot \Delta(1 + O(t^{-(i-1)}))
\]
\[
= \alpha + d(x) \cdot \Delta(1 + O(t^{-(i-1)})),
\]

as required. Also, by Proposition 4.2, \( w(1) = \frac{1}{t} + O(t^{-i}) \), hence \( \alpha = \frac{1}{t} + O(t^{-i}) \).

Let \( G' \) be another \( r \)-graph on vertex set \([t]\) for which every member of \( \overline{G'} \) contains \( \{ t-(i-1), \ldots, t \} \) and suppose that \( G' \) has the same number of edges as \( G \). Define \( H' \) analogously to \( H \), and denote by \( d'(x) \) the degree of a vertex \( x \) in \( H' \). By assumption, \( \lambda(G) \geq \lambda(G') \). Our aim is to show that \( P_2(H) \geq P_2(H')(1 + O(t^{-(i-1)})) \). Denote the number of edges of \( H \) by \( m \); then \( H' \) also has \( m \) edges.
In order to compare $\lambda(G)$ with $\lambda(G')$ we define a modified weight function $w'$ as follows.

$$w'(x) = \begin{cases} 
\beta & x \in \{t - (i - 1), \ldots, t\} \\
\alpha + d'(x) \cdot \Delta(1 + O(t^{-i-1})) & \text{otherwise.}
\end{cases}$$

Also, for $x \in [t - i]$, write

$$w(x) = \alpha + d(x) \cdot \Delta(1 + \varepsilon(x)).$$

Then by Claim 7.5, we have $\varepsilon(x) = O(t^{-i-1})$. Let

$$\zeta = \sum_{x \in [t - i]} \delta'(x) \cdot d'(x) \cdot \Delta(1 + O(t^{-i-1})).$$

and note that $\zeta = O(t^{-i-1})$. For $x \in [t - i]$, set

$$w'(x) = \alpha + \delta'(x) \text{ and } \delta'(x) = d'(x) \cdot \Delta(1 + \zeta).$$

Observe that $\sum_{x \in [t - i]} \delta'(x) = \sum_{x \in [t - i]} \delta(x)$, from which it follows that $w'$ is a legal weight function.

Now we wish to compare $w(G)$ with $w'(G')$. We start be estimating difference in the weight of the edges that contain $\{t - (i - 1), \ldots, t\}$.

**Claim 7.6.** The difference between the weight, with respect to $w'$, of edges in $G'$ that contain $\{t - (i - 1), \ldots, t\}$ and the weight, with respect to $w$, of edges in $G$ that contain $\{t - (i - 1), \ldots, t\}$ is

$$\frac{\Delta^2}{(r - 2)!} \cdot \left( P_2(H') - P_2(H) + O\left(t^{-i-1}(P_2(H) + P_2(H'))\right) \right)(1 + O(t^{-1})).$$

**Proof.** Note that the required quantity is $\beta^i (w'(H') - w(H))$. Let us evaluate $w(H)$.

$$w(H) = \sum_{(x_1, \ldots, x_{r-i})} w(x_1) \cdots w(x_{r-i})$$

$$= \sum_{(x_1, \ldots, x_{r-i})} (\alpha + \delta(x_1)) \cdots (\alpha + \delta(x_{r-i}))$$

$$= \sum_{0 \leq j \leq r-i} \alpha^{r-i-j} \sum_{1 \leq x_1 \leq \ldots \leq x_j \leq t-i} \delta(x_1) \cdots \delta(x_j) \cdot d(x_1, \ldots, x_j)$$

$$= m\alpha^{r-i} + \alpha^{r-i-1}\Delta \cdot P_2(H) \left(1 + O(t^{-i-1})\right)$$

where $d(x_1, \ldots, x_j)$ is the number of edges in $H$ that contain $\{x_1, \ldots, x_j\}$. Indeed, we
used the facts that $\delta(x) = d(x) \cdot \Delta(1 + O(t^{-i+1}))$, $d(x_1, \ldots, x_j) \leq d(x_1)$ and $d(x) \leq (t_{r-i-1})$ to obtain the bound $\sum_{x_1, \ldots, x_j} \delta(x_1) \cdots \delta(x_j) : d(x_1, \ldots, x_j) = O(\sum_{x \in [t-i]} \Delta \cdot d(x)^2).

\((t_{r-i-1})^j \Delta^j \left( \sum_{x_{r-i}} \delta(x_1) \cdots \delta(x_j) \right) = O \left( \Delta \cdot P_2(H) t^{-(i-1)} \right) \) for $j \geq 2$. A similar argument shows that

$$w'(H') = m\alpha^{r-i} + \alpha^{r-i-1} \Delta \cdot P_2(H') \left( 1 + O(t^{-i}) \right).$$

hence

$$\beta^i(w'(H') - w(H)) = \frac{\Delta^2}{(r-2)!} \cdot \left( P_2(H') - P_2(H) + O \left( t^{-(i-1)} \left( P_2(H) + P_2(H') \right) \right) \right) \left( 1 + O(t^{-1}) \right),$$

where we used the definition of $\Delta$ and the fact that $\alpha = w(1)(1 + O(t^{-i}))$, and $w(N(1, t-i)) = 1 + O(t^{-1}).$

Next we evaluate the contribution of edges of $G$ that do not contain $\{t-(i-1), \ldots, t\}$.

**Claim 7.7.** The difference between the weight of $r$-tuples that do not contain $\{t-(i-1), \ldots, t\}$ with respect to $w$ and $w'$ is

$$\frac{\Delta^2}{2(r-2)!} \cdot \left( P_2(H') - P_2(H) + O \left( t^{-(i-1)} \left( P_2(H) + P_2(H') \right) \right) \right) \left( 1 + O(t^{-1}) \right).$$

**Proof.** Given $0 \leq l \leq i-1$, the difference between the weight of edges that contain exactly $l$ vertices from $\{t-(i-1), \ldots, t\}$ in $G$ and in $G'$ is the following times $\binom{i}{l} \beta^l$.

$$\sum_{x_1 < \ldots < x_{r-l}} \left( w(x_1) \cdots w(x_{r-l}) - w'(x_1) \cdots w'(x_{r-l}) \right)$$

$$= \sum_{x_1 < \ldots < x_{r-l}} \left( (\alpha + \delta(x_1)) \cdots (\alpha + \delta(x_{r-l})) - (\alpha + \delta'(x_1)) \cdots (\alpha + \delta'(x_{r-l})) \right)$$

$$= \sum_{0 \leq j \leq r-1} \alpha^{r-j} \binom{t-i-j}{r-i-j} \sum_{x_1 < \ldots < x_j} \left( \delta(x_1) \cdots \delta(x_j) - \delta'(x_1) \cdots \delta'(x_j) \right)$$

$$= \alpha^{r-l-2} \binom{t-l-2}{r-l-2} \cdot \frac{1}{2} \left( \sum_x \delta(x)^2 - \sum_x \delta'(x)^2 + O \left( t^{-(i-1)} \sum_x (\delta(x)^2 + \delta'(x)^2) \right) \right)$$

$$= \frac{\Delta^2}{2(r-l-2)!} \cdot \left( P_2(H') - P_2(H) + O \left( t^{-(i-1)} \left( P_2(H) + P_2(H') \right) \right) \right) \left( 1 + O(t^{-1}) \right).$$

Indeed, for the penultimate equality we used the fact that $\sum_{x \in [t-i]} \delta(x) = \sum_{x \in [t-i]} \delta'(x)$.
which implies that the summands with $j = 0$ and $j = 1$ are 0. Furthermore, for $j = 2$, we used the equation $\sum_{x < y} \delta(x)\delta(y) = \frac{1}{2} \left( (\sum_x \delta(x))^2 - \sum_x \delta(x)^2 \right)$, which implies that

$$\sum_{x < y} \delta(x)\delta(y) - \sum_{x < y} \delta'(x)\delta'(y) = \frac{1}{2} \left( \sum_x \delta(x)^2 - \sum_x \delta'(x)^2 \right).$$

Similarly, for $j \geq 3$ we use the fact that

$$\sum_{x_1 < \ldots < x_j} \delta(x_1) \cdot \ldots \cdot \delta(x_j) = \frac{1}{j!} \left( \sum_x \delta(x) \right)^j + O \left( \sum_x \delta(x)^2 \cdot t^{j-2} \cdot t^{-(j-2)i} \right)$$

to conclude that

$$\sum \delta(x_1) \cdot \ldots \cdot \delta(x_j) - \sum \delta'(x_1) \cdot \ldots \cdot \delta'(x_j) = O \left( t^{-(i-1)} \left( \sum_x \delta(x)^2 + \sum_x \delta'(x)^2 \right) \right).$$

For the last equality we used the fact that $\delta(x) = d(x) \cdot \Delta(1 + O(t^{-i-1}))$ and similarly for $\delta'(x)$; also, we used the fact that $\alpha = \frac{1}{t} + O(t^{-i})$.

Claim 7.7 follows: for $1 \leq l \leq i - 1$, the contribution of the edges with exactly $l$ vertices from $\{t-(i-1), \ldots, t\}$ is accounted for in the $O(t^{-1})$ error term, and the main term accounts for the edges with no vertices in $\{t-(i-1), \ldots, t\}$ (unsurprisingly, as there are much more of the latter type of edges than the former).

By Claims 7.6 and 7.7 we have

$$w'(G') - w(G) = \frac{\Delta}{2(r-2)!} \cdot \left( P_2(H') - P_2(H) + O(t^{-(i-1)} \left( P_2(H) + P_2(H') \right)) \right) \left( 1 + O(t^{-1}) \right).$$

Hence, since $\lambda(G) = \Lambda(|G|, r)$, we have $P_2(H) \geq P_2(H')(1 + O(t^{-(i-1)}))$. Since $H'$ can be chosen arbitrarily, by taking $H'$ such that $P_2(H') = P_2(m, r - i, t - i)$, we find that $P_2(H) \in (1 + O(t^{-(i-1)}))P_2(m, r - i, t - i)$, as required.

8 Conclusion

In this paper we prove that, in some sense, well-behaved $r$-graphs $G$ (this means that $G$ is left-compressed, maximises $\Lambda(|G|, r)$, and that the number of vertices of $G$ is minimal with respect to these properties) are close to colex graphs. Indeed, in Theorem 3.2 we show that if $m \leq \binom{t}{i}$, where $m = |G|$, then $G$ is supported on $\lfloor t \rceil$, i.e. $G \subseteq \lfloor t \rceil^r$, similarly to colex graphs.
Moreover, in Theorem 5.1 we show that if \( t^{0.01} \cdot \binom{t-i}{r-(i+1)} \leq a \leq \binom{t-i}{r-i} \), where \( a = \binom{t}{r} - m \), then all edges of \( \overline{G} \) (defined as \( \overline{G} = [t]^{(r)} \setminus G \)) contain the last \( i \) vertices \( \{t-(i-1), \ldots, t\} \), a property that holds for colex graphs. In fact, we are able to extend this to all \( a > (r+1) \binom{t-i}{r-(i+1)} \) (using a more careful analysis of the arguments in the proof of Theorem 5.1 as well as a different argument for very small \( a \)’s that uses Lemma 6.1). However, we did not wish to make the paper more rebuscated and for the sake of brevity, we do not include this proof here.

On the other hand, looking more closely at the structure of the graph, we see that colex graphs are often far from the maximisers. Indeed, let us focus on left-compressed \( r \)-graphs \( G \) on vertex set \([t]\), with \( a \) missing edges, where \( \binom{t-(i+1)}{r-(i+1)} \leq a \leq \binom{t-i}{r-i} \), such that all edges of \( \overline{G} \) contain \( I := \{t-(i-1), \ldots, t\} \). In Theorem 7.1 we show that the graph \( H \), of \( (r-i) \)-tuples in \([t-i]\) whose union with \( I \) is an edge of \( G \), is close to maximising (or, in certain ranges, exactly maximises) \( P_2(m, t-i, r-i) \) among all \( (r-i) \)-graph with \( t-i \) vertices and \( |H| \) edges (recall that \( P_2(H) = \sum_{x \in V(H)} d_H(x)^2 \) and \( P_2(m, t, r) = \max\{P_2(H) : H \subseteq [t]^{(r)}, |H| = m\} \); see Section 6). If \( G \) is a colex graph, then \( H \) is also a colex graph. However, in many ranges of \( r, m, t \), the sum of degrees squared of, say, a lex graph (with suitable size and order) is much larger than that of the colex graph.

It would, of course, be interesting to find \( r \)-graphs \( G \) of size \( m \) that maximise \( \lambda(G) \) among \( r \)-graphs with the same size, for all \( r \) and \( m \). However, this problem seems very hard. An indication to the difficulty is the relation to the problem of maximising \( P_2(H) \), among all \( r \)-graphs with certain size and order, which in itself appears very hard.

**Note added before submission.** Right before submitting the paper, we noticed the very recent preprint by Lei and Lu [10]. They improve the results in Table 1 and prove, independently from us, a statement as in Theorem 3.2.

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\(^2\)The irony is not lost on us.
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A Proof of Nikiforov’s Conjecture

Theorem A.1. Let \( (t-1)_r < m \leq (t)_r \) and let \( x \) be such that \( (x)_r = m \) (so \( x \) need not be an integer). For \( t \) sufficiently large, we have \( \Lambda(m, r) \leq mx^{-r} \), with equality if and only if \( x \) is an integer.

Proof. Let \( G \) be a well-behaved \( r \)-graph with \( m \) edges. Our aim is to show that \( \lambda(G) \leq mx^{-r} \). By Theorem 3.2, we may assume that the vertex set of \( G \) is \([t]\). Write \( m = \binom{t}{r} - a \).

By Theorem 4.1 if \( a \leq \binom{t-2}{r-2} \), then

\[
\lambda(G) \leq \lambda([t-1]) \leq \left( \frac{t-1}{r} \right) \left( \frac{1}{t-1} \right)^r \leq mx^{-r},
\]

as \( m \geq \binom{t-1}{r} \) and \( x \geq t-1 \). We may thus assume that \( a < \binom{t-2}{r-2} \). Again by Theorem 4.1 we may assume that \( a > 0 \).

Now, let us look more closely at the expression \( mx^{-r} \). Write \( x = t - \varepsilon \), then clearly \( 0 \leq \varepsilon < 1 \). Let \( f \) be the function defined by \( f(s) = \binom{s}{r} \) for \( s \geq r \). Then

\[
a = \binom{t}{r} - m = \binom{t}{r} - \binom{x}{r} = f(t) - f(x) \leq (t-x)f'(t) \leq \varepsilon \binom{t}{r-1}.
\]

Here we used the fact that \( f'(s) \) is increasing and that \( f'(t) \leq \binom{t}{r-1} \). We now obtain a lower
bound on $mx^{-r}$.

$$mx^{-r} = mt^{-r} \left( \frac{1}{1 - \varepsilon/t} \right)^r$$

$$\geq mt^{-r}(1 + r\varepsilon/t)$$

$$\geq mt^{-r} \left( 1 + \frac{ra}{(r-t)} \right)$$

$$= \binom{t}{r} t^{-r} - at^{-r} \left( 1 - \frac{\binom{t}{r}}{t} r + \frac{ra}{(r-t)} \right)$$

$$\geq \lambda([t]^{(r)}) - a \cdot t^{-(r+1)} \cdot (r-1)(1 - r/t).$$

In contrast, recall that by Proposition 5.2(ii) $w(t) = \Omega(1/t)$, hence the weight of every non-edge of $G$ is at least $\Omega(t^{-r})$. We thus have

$$w(G) = w([t]^{(r)}) - w(G) \leq \lambda([t]^{(r)}) - \Omega(at^{-r}) \leq mx^{-r},$$

as required. Note that we get equality if and only if $a = 0$, that is, when $x = t$. \qed