Spectral decomposition of some non-self-adjoint operators

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J.F. and N. Frantz, Spectral decomposition of some non-self-adjoint operators, Annales Henri Lebesgue, to appear.
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Introduction

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Some motivations (I)

Nuclear optical model

- Feshbach, Porter and Weisskopf (’54): nuclear optical model describing both elastic scattering and absorption of a neutron targeted onto a nucleus
- “Pseudo-Hamiltonian” on $L^2(\mathbb{R}^3)$

$$H = -\Delta + V(x) - iW(x)$$

with $V$ and $W$ real-valued, bounded and compactly supported, $W \geq 0$
- $-iH$ generates a strongly continuous semigroup of contractions. Dynamics described by the Schrödinger equation

$$\begin{cases}
i \partial_t u_t = Hu_t \\
 u_0 \in D(H)
\end{cases}$$

- Probability that the neutron, initially in the normalized state $u_0$ (supposed to be orthogonal to bound states), eventually escapes from the nucleus:

$$p_{\text{scatt}}(u_0) = \lim_{t \to \infty} \| e^{-itH} u_0 \|^2$$

- Probability of absorption:

$$p_{\text{abs}}(u_0) = 1 - \lim_{t \to \infty} \| e^{-itH} u_0 \|^2$$

- Empirical model widely used in Nuclear Physics
Some motivations (II)

\[ \mathcal{PT} \text{-symmetric operators} \]

- [Bender, Boettcher '98]: large class of ‘\( \mathcal{PT} \)-invariant Hamiltonians’ have real spectra
- For Schrödinger operators \( H = -\Delta + V(x) \) on \( L^2(\mathbb{R}^d) \), \( \mathcal{PT} \)-symmetry means that
  \[
  \overline{V(-x)} = V(x)
  \]
- [Borisov, Krejcirik '08, '12], [Wen, Bender '20]: examples of \( \mathcal{PT} \)-symmetric Schrödinger operators having continuous spectra

\[ \text{Non-self-adjoint operators in Quantum Mechanics} \]

- Holomorphic families of closed operators [Dereziński and collaborators]
- [Bagarello, Gazeau, Szafraniec, Znojil '15]: *Non-Selfadjoint Operators in Quantum Physics. Mathematical Aspects.*
- [Krejcirik '17]: *Mathematical aspects of quantum mechanics with non-self-adjoint operators.*
Abstract framework
Abstract model

The model

- \( \mathcal{H} \) complex Hilbert space
- Hamiltonian
  \[ H = H_0 + V = H_0 + CW, \]
  with \( H_0 \geq 0, \ C \in \mathcal{B}(\mathcal{H}), \ C > 0 \) and relatively compact with respect to \( H_0, \ W \in \mathcal{B}(\mathcal{H}) \) arbitrary
- \( H \) is a closed operator with domain
  \[ D(H) = D(H_0) \]
- \( -iH \) generates a strongly continuous group \( \{ e^{-itH} \}_{t \in \mathbb{R}} \) s.t.
  \[ \| e^{-itH} \| \leq e^{\| V \| |t|}, \ t \in \mathbb{R} \]
- \( H^* = H_0 + CW^* C \) with domain \( D(H^*) = D(H_0) \)
- \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) \) and \( \sigma(H) \setminus \sigma_{\text{ess}}(H) \) consists of an at most countable number of eigenvalues of finite algebraic multiplicities that can only accumulate at points of \( \sigma_{\text{ess}}(H) \)
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Figure: Spectrum of $H$
Point spectral subspace (I)

Point spectrum

\[ \sigma_p(H) := \{ \lambda \in \mathbb{C}, \ \text{Ker}(H - \lambda) \neq \{0\} \} \]

Algebraic multiplicity of an eigenvalue \( \lambda \in \sigma_p(H) \):

\[ m_\lambda := \dim \left( \bigcup_{k \geq 1} \text{Ker}((H - \lambda)^k) \right) \]

Discrete spectrum and discrete spectral subspace

\[ \sigma_{\text{disc}}(H) := \sigma(H) \setminus \sigma_{\text{ess}}(H) \subset \sigma_p(H) \]

- For \( \lambda \in \sigma_{\text{disc}}(H) \), Riesz projection defined by
  \[ \Pi_\lambda = -\frac{1}{2i\pi} \int_{\gamma} R_H(z)dz, \quad R_H(z) = (H - z)^{-1}, \]
  where \( \gamma \) is a circle centered at \( \lambda \), of sufficiently small radius
- \( \text{Ran}(\Pi_\lambda) \) spanned by generalized eigenvectors of \( H \) associated to \( \lambda \), \( u \in D(H^k) \) s.t. \( (H - \lambda)^k u = 0 \)
- Discrete spectral subspace:
  \[ \mathcal{H}_{\text{disc}}(H) = \text{Span} \left\{ u \in \text{Ran}(\Pi_\lambda), \ \lambda \in \sigma_{\text{disc}}(H) \right\}^{\text{cl}} \]
Point spectral subspace (II)

**Set of embedded eigenvalues**

$$\sigma_{\text{emb}}(H) := \sigma_p(H) \cap \sigma_{\text{ess}}(H)$$

**To define spectral projections corresponding to embedded eigenvalues**

- We suppose the existence of a **conjugation operator** $J \in \mathcal{B}(\mathcal{H})$ satisfying
  
  $$ JD(H_0) \subset \mathcal{D}(H_0) \quad \text{and} \quad \forall u \in \mathcal{D}(H_0), \quad JHu = H^*Ju $$

- If $\lambda \in \sigma_{\text{emb}}(H)$, we suppose that $m_\lambda < \infty$ and that the symmetric bilinear form
  
  $$ \text{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle $$
  
  is non-degenerate

- Under these conditions, there exists a basis $(\varphi_k)_{1 \leq k \leq m_\lambda}$ of $\text{Ker}((H - \lambda)^{m_\lambda})$ such that $\langle J\varphi_i, \varphi_j \rangle = \delta_{ij}$, $1 \leq i, j \leq m_\lambda$. Then

  $$ \Pi_\lambda u := \sum_{k=1}^{m_\lambda} \langle J\varphi_k, u \rangle \varphi_k, \quad u \in \mathcal{H} $$

- 

  $$ \mathcal{H}_{\text{emb}}(H) := \text{Span} \{ u \in \text{Ran}(\Pi_\lambda), \lambda \in \sigma_{\text{emb}}(H) \}^{\text{cl}} $$
Point spectral subspace

$$\mathcal{H}_p(H) = \mathcal{H}_{\text{disc}}(H) \oplus \mathcal{H}_{\text{emb}}(H)$$
Asymptotically disappearing states

Subspaces of asymptotically disappearing states

$$\mathcal{H}_{\text{ads}}^{\pm}(H) := \left\{ u \in \mathcal{H}, \lim_{t \to \pm \infty} \| e^{-itH} u \| = 0 \right\}_{\text{cl}}$$

Relation with discrete generalized eigenstates

- Easy to see that
  $$\text{Span} \{ \text{Gen. eigenstates associated to } \lambda, \pm \text{Im}(\lambda) < 0 \}_{\text{cl}} \subset \mathcal{H}_{\text{ads}}^{\pm}(H)$$

- **Question:** conditions implying that the previous inclusion becomes an equality?
  - [Kato '66] For small perturbations, $H$ and $H_0$ are similar, hence $\mathcal{H}_{\text{ads}}^{\pm}(H) = \{0\}$
  - For dissipative operators, $\text{Im}(V) \leq 0$, the question was left as an open problem in [Davies '80], with an answer given in [F., Fröhlich '18]
Absolutely continuous spectral subspace

\[ \mathcal{H}_{ac}(H) := \left\{ u \in \mathcal{H}, \exists c_u > 0, \forall v \in \mathcal{H}, \int_{\mathbb{R}} |\langle e^{-itH}u, v \rangle|^2 dt \leq c_u \|v\|^2 \right\} \]

Relation with point spectral subspace of \( H^* \)

- Not difficult to verify that

\[ \mathcal{H}_{ac}(H) \subset \mathcal{H}_p(H^*)^\perp \]

- **Question**: conditions implying that the previous inclusion becomes an equality?

- Other definitions considered in the literature: [Davies '79] for dissipative operators, [Naboko '76] using the theory of dilations of dissipative operators

- Under suitable assumption, coincides with the space of ‘scattering states’
**Definition: Spectral singularity**

1. \( \lambda \in \sigma_{\text{ess}}(H) \) is an **outgoing/incoming regular spectral point** of \( H \) if \( \lambda \) is not an accumulation point of eigenvalues located in \( \lambda \pm i(0, \infty) \) and if the limit
   \[
   CR_H(\lambda \pm i0^+)CW := \lim_{\varepsilon \to 0^+} CR_H(\lambda \pm i\varepsilon)CW
   \]
   exists in the norm topology of \( \mathcal{B}(\mathcal{H}) \).

2. \( \lambda \) is a **regular spectral point** of \( H \) if it is both an incoming and an outgoing regular spectral point of \( H \).

3. **Spectral singularity** = not regular spectral point.

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**Diagram:**
- Regular spectral point: \( \lim CR_H(z)CW \) exists both from above and from below.
- Outgoing spectral singularity: incoming regular spectral point.
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Remarks

For dissipative operators, similar definition in [F, Fröhlich '18]. Other related notions:

- [Dunford '58] (theory of spectral operators), [Schwartz '59] (spectral singularity = singular point of a ‘spectral resolution’ for non-self-adjoint operators)
- [F, Nicoleau '19] (for dissipative operators, spectral singularity = point of the essential spectrum where the scattering matrix is non-invertible)
- For one-dimensional Schrödinger operators (spectral singularity = zero of the Jost function)

Some properties [F, Frantz]

- \( \lambda \) embedded eigenvalue \( \Rightarrow \) \( \lambda \) both outgoing and incoming spectral singularity
- At thresholds, outgoing and incoming spectral singularities coincide
Spectral singularities (III)

Proposition (Schrödinger operators in 3-dimension)

Suppose that $V$ is a complex-valued potential such that $\langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^3)$ with $\sigma > 1$. Let $C(x) = \langle x \rangle^{-\sigma/2}$. Then for all $\lambda > 0$, the following conditions are equivalent:

1. $\lambda$ is an outgoing/incoming spectral singularity of $H = -\Delta + V$
2. There exists $\Psi \neq 0$, $\langle x \rangle^{-\sigma/2}\Psi \in L^2$, $\Psi$ satisfying the outgoing/incoming Sommerfeld radiation condition, such that
   $$( -\Delta + V(x) - \lambda ) \Psi = 0$$

The same holds at the threshold $\lambda = 0$ if $\langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^3)$ with $\sigma > 2$

Remarks

- There is an abstract version of this proposition involving the Gelfand triple $\text{Ran}(C) \hookrightarrow \mathcal{H} \hookrightarrow (\text{Ran}(C))'$.
- [Wang '12]: For any $\lambda > 0$, one can construct a smooth compactly supported potential $V$ such that $\lambda$ is a spectral singularity of $H$.
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Hypotheses (I)

(H1) Limiting absorption principle for $H_0$

$$\sup_{\pm \text{Im}(z) > 0} \| CR_0(z) C \| < \infty$$

Remark

Satisfied for $H_0 = -\Delta$, $C(x) = \langle x \rangle^{-\sigma/2}$, $\sigma > 2$ in dimension $d \geq 3$

Consequences

- The spectrum of $H_0$ is purely absolutely continuous, i.e.
  $$\sigma_{pp}(H_0) = \emptyset, \quad \sigma_{ac}(H_0) = \sigma(H_0), \quad \sigma_{sc}(H_0) = \emptyset$$

- The limits $CR_0(\lambda \pm i0^+) C$ exist for almost every $\lambda \in \sigma_{\text{ess}}(H)$, in the norm topology of $\mathcal{B}(\mathcal{H})$
Hypotheses (II)

(H2) Eigenvalues of $H$

$H$ only has a **finite number of eigenvalues with finite algebraic multiplicities**

Remark

- Satisfied for Schrödinger operators $H = -\Delta + V(x)$ in $L^2(\mathbb{R}^3)$ if $V$ is exponentially decaying [Frank, Laptev, Safronov '16]
- Does **not** exclude embedded eigenvalues
(H3) Spectral singularities for $H$

$H$ only has a finite number of spectral singularities $\{\lambda_1, \ldots \lambda_n\} \subset \sigma_{\text{ess}}(H)$ and there exist $\varepsilon_0 > 0$ and integers $\nu_1, \ldots, \nu_n, \nu_\infty \geq 0$ such that

$$\sup_{\text{Re}(z) \in \sigma_{\text{ess}}(H), \pm \text{Im}(z) \in (0, \varepsilon_0)} |r(z)| \|CR_H(z)CW\| < \infty,$$

where $z_0$ is an arbitrary complex number such that $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ and

$$r(z) := \frac{1}{(z - z_0)^\nu_\infty} \prod_{j=1}^n \frac{(z - \lambda_j)^{\nu_j}}{(z - z_0)^{\nu_j}}$$
Hypotheses (IV)

Remarks

- The factors \((z - \lambda_j)^{\nu_j}\) ‘regularize’ the singularities of \(z \mapsto CR_H(z)CW\) as \(z\) approaches \(\lambda_j\). Dividing them by \((z - z_0)^{\nu_j}\) produces bounded terms.
- The factor \(\frac{1}{(z - z_0)^{\nu_\infty}}\) ‘regularize’ a possible singularity at \(\infty\).
- For Schrödinger operators \(H = -\Delta + V(x)\) in \(L^2(\mathbb{R}^3)\) with \(V\) compactly supported, (H3) is satisfied with \(\nu_j\) the multiplicity of the resonance \(\lambda_j\) and \(\nu_\infty = 0\).
Asymptotically disappearing states

Recall that
\[
\mathcal{H}_{\text{ads}}^\pm (H) := \left\{ u \in \mathcal{H}, \lim_{t \to \pm \infty} \| e^{-itH} u \| = 0 \right\}_{\text{cl}}
\]

**Theorem [F, Frantz]**

Suppose that Hypotheses (H1)–(H3) hold. Then
\[
\mathcal{H}_{\text{ads}}^\pm (H) = \text{Span} \{ \text{Gen. eigenstates associated to } \lambda, \pm \text{Im}(\lambda) < 0 \}_{\text{cl}}
\]

**Remark**

For dissipative operators, analogous result proven in [F. Fröhlich ’18]. The proof in [F. Fröhlich ’18] relies on the existence and properties of wave operators. Our proof does not rely on scattering theory.

**Theorem (Consequence for Schrödinger operators)**

Suppose that \( V \) is a complex-valued potential such that \( V \in L^\infty_c(\mathbb{R}^3) \). Then the previous theorem applies to \( H = -\Delta + V \).
Hypotheses (IV)

(H4) Conjugation operator

There exists an anti-linear continuous map \( J : \mathcal{H} \rightarrow \mathcal{H} \) such that

1. \( JD(H_0) \subset D(H_0) \) and \( \forall u \in D(H_0), \ JH_0u = H_0Ju \)
2. \( JC = CJ \) and \( JW = W^*J \)

Moreover, for all embedded eigenvalues \( \lambda \in \sigma_{\text{ess}}(H) \), the symmetric bilinear form

\[ \text{Ker}((H - \lambda)^{m\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle \]

is non-degenerate

Remark

For Schrödinger operators \( H = -\Delta + V(x) \), \( J \) is the complex conjugation and Hypothesis (H4) means that for all embedded eigenvalues \( \lambda \in [0, \infty) \),

\[ \text{Ker}((H - \lambda)^{m\lambda}) \ni (u, v) \mapsto \int_{\mathbb{R}^3} u(x)v(x)dx \]

is non-degenerate
Absolutely continuous spectral subspace

Recall that

\[ \mathcal{H}_{ac}(H) := \left\{ u \in \mathcal{H}, \exists c_u > 0, \forall v \in \mathcal{H}, \int_{\mathbb{R}} | \langle e^{-itH} u, v \rangle |^2 dt \leq c_u \|v\|^2 \right\} ^{c_1} \]

**Theorem [F, Frantz]**

Suppose that Hypotheses (H1)–(H3) hold. If \( H \) has embedded eigenvalues, suppose in addition that (H4) holds. Then

\[ \mathcal{H}_{ac}(H) = \mathcal{H}_p(H^*)^\perp \]

**Remark: comparable results in the literature (only for \( H \) dissipative)**

[Simon '79] for dissipative Schrödinger operators, [Davies '80] for abstract dissipative operators using the theory of dilations, with a different definition of \( \mathcal{H}_{ac} \) and a different result (\( \mathcal{H}_{ac} \) coincides with the orthogonal complement of ‘bound states’)

**Theorem (Consequence for Schrödinger operators)**

Suppose that \( V \) is a complex-valued potential such that \( V \in L^\infty_c(\mathbb{R}^3) \) and the previous hypothesis on embedded eigenvalues is satisfied. Then the previous theorem applies to \( H = -\Delta + V \)
Consequence of the previous two theorems

Suppose that Hypotheses (H1)–(H4) hold. Then we have the following $J$-orthogonal direct sum decompositions of the Hilbert space:

\[ \mathcal{H} = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_p(H) \]
\[ = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{disc}(H) \oplus \mathcal{H}_{emb}(H) \]
\[ = \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{ads}^+(H) \oplus \mathcal{H}_{ads}^-(H) \oplus \mathcal{H}_b(H), \]

where $\mathcal{H}_b(H)$ is the space of ‘bound states’, i.e. the closure of the vector space spanned by all generalized eigenvectors of $H$ corresponding to real eigenvalues (either isolated or embedded).
Regularized functional calculus
Regularized Stone formula

Stone’s formula for self-adjoint operators

Suppose $H$ is a self-adjoint operator without embedded eigenvalues. Then

$$\text{Id} = \sum_{\lambda \in \sigma_{\text{disc}}(H)} \Pi_\lambda + \text{w-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} (R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon)) d\lambda$$

Regularized version

Recall that we have assumed

$$\sup_{\text{Re}(z) \in \sigma_{\text{ess}}(H)} |r(z)| \|CR_H(z)CW\| < \infty, \quad r(z) = \frac{1}{(z - z_0)^{\nu_\infty}} \prod_{j=1}^{n} \frac{(z - \lambda_j)^{\nu_j}}{(z - z_0)^{\nu_j}}$$

Then, under our assumptions, we have

$$r(H) = \sum_{\lambda \in \sigma_{\text{disc}}(H)} r(H) \Pi_\lambda + \text{w-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} r(\lambda)(R_H(\lambda + i\varepsilon) - R_H(\lambda - i\varepsilon)) d\lambda$$
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**Riesz-Dunford functional calculus**

\[
  r(H) = -\frac{1}{2i\pi} \int_{\Gamma_\varepsilon} r(z) R_H(z) \, dz \quad \text{then} \quad \varepsilon \to 0^+
\]

**Figure:** The contour \( \Gamma_\varepsilon \).
Recall Hypothesis (H1):
\[ \sup_{\pm \text{Im}(z) > 0} \| CR_0(z)C \| < \infty \]

**Proposition (Functional calculus in intervals not containing spectral singularities)**

Suppose (H1). Let \( I \subset \mathbb{R} \) be a closed interval and suppose that there exists \( \epsilon_0 > 0 \) such that
\[ \sup_{\text{Re}(z) \in I, \pm \text{Im}(z) \in (0, \epsilon_0)} \| CR_H(z)CW \| < \infty. \]

Then the map
\[ C_b(I) \ni f \mapsto f(H) := \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_I f(\lambda) \left( R_H(\lambda + i\epsilon) - R_H(\lambda - i\epsilon) \right) d\lambda \in \mathcal{B}(\mathcal{H}) \]

**Remark**

Related to the Dunford-Schwartz theory of spectral operators
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Functional calculus (II)

Proposition (Regularized functional calculus)

Suppose (H1). Let \( I \subset \mathbb{R} \) be a closed interval and suppose that there exists \( \varepsilon_0 > 0 \) and a bounded holomorphic function \( h \) such that

\[
\sup_{\text{Re}(z) \in I} |h(z)| \|CR_H(z)CW\| < \infty, \quad \lambda \mapsto \sup_{0 < \varepsilon < \varepsilon_0} |h'(\lambda \pm i\varepsilon)| \in L^2(I).
\]

Then the map

\[
C_{b, \text{reg}}(I) \ni f \mapsto f(H) := w\text{-lim}_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_I g(\lambda)(h(\lambda + i\varepsilon)R_H(\lambda + i\varepsilon)
- h(\lambda - i\varepsilon)R_H(\lambda - i\varepsilon))d\lambda \in \mathcal{B}(\mathcal{H})
\]

is an algebra morphism and there exists \( c > 0 \) such that

\[
\|f(H)\|_{\mathcal{B}(\mathcal{H})} \leq c\|g\|_{L^\infty}.
\]

Here

\[
C_{b, \text{reg}}(I) := \{f : I \to \mathbb{C}, \ \exists g \in C_b(I), \ f = hg\}
\]

Remark

Other functional calculi for operators on Banach spaces under an assumption of polynomial growth of the resolvent near the real axis: [Davies '95] (general theory), [Georgescu, Gérard, Hafner '13] (Krein spaces)
Thank you!
## Spectral singularities and resonances (I)

\[ \mathcal{H} = L^2(\mathbb{R}^d), \quad H_0 = -\Delta, \quad V \text{ compactly supported} \]

**Resonance** may be defined as a pole of the map

\[ \mathbb{C} \ni z \mapsto (H - z^2)^{-1} : L^2_c(\mathbb{R}^3) \to L^2_{\text{loc}}(\mathbb{R}^3), \]

Then

\[ \text{Spectral singularity at } \lambda > 0 = \text{resonance at } \pm \lambda^{1/2} \]

### Remarks

- Resonances theory: [Sjöstrand '02], [Dyatlov-Zworski '18]

- [Wang '12]: For any \( \lambda > 0 \), one can construct a smooth compactly supported potential \( V \) such that \( \lambda \) is a spectral singularity of \( H \)
Spectral singularities and resonances (II)

Example: $\mathcal{H} = L^2(\mathbb{R}^3)$, $H_0 = -\Delta$, $V$ short-range

If $V(x) = O(\langle x \rangle^{-\delta})$ with $\delta > 1$, then $\pm \lambda^{1/2}$ (with $\lambda > 0$) may be called a resonance of $H$ if the equation $(H - \lambda)u = 0$ admits a distributional solution (called a resonant state)

$$u \in \bigcap_{\sigma > 1} L^2_{-\sigma/2} \setminus L^2$$

satisfying the Sommerfeld radiation condition

$$u(x) = |x|^{-1} e^{\pm i \lambda^{1/2} |x|} \left( a\left(\frac{x}{|x|}\right) + o(1)\right), \quad |x| \to \infty,$$

with $a \in L^2(S^2)$, $a \neq 0$. Here $L^2_{-\sigma/2} = \{ f : \mathbb{R}^3 \to \mathbb{C}, x \mapsto \langle x \rangle^{-\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^3) \}$
**Spectral singularities: Characterization (I)**

**Assumption (given $\lambda \in \sigma_{\text{ess}}(H)$, with $R_0 := R_{H_0}$)**

$$CR_0(\lambda \pm i0^+)C := \lim_{\varepsilon \to 0^+} CR_0(\lambda \pm i\varepsilon)C$$ exist in the topology of $B(H)$ \(\ast\)

**Free Schrödinger operator in $\mathcal{H} = L^2(\mathbb{R}^3)$**

- For $\lambda > 0$, the limits
  $$\langle x \rangle^{-s} \left( -\Delta - (\lambda \pm i0^+) \right) \langle x \rangle^{-s},$$
  exist in the norm topology of $B(H)$, for any $s > \frac{1}{2}$, where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$

- If $\lambda = 0$, the limits
  $$\langle x \rangle^{-s} (-\Delta \pm i0^+) \langle x \rangle^{-s},$$
  exist (and coincide) for any $s > 1$
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Spectral singularities: Characterization (II)

Extension of the Hilbert space

- Let $\mathcal{H}_C := \text{Ran}(C)$ equipped with $\langle u, v \rangle_{\mathcal{H}_C} := \langle C^{-1}u, C^{-1}v \rangle$
- Let $\mathcal{H}'_C$ be the anti-dual of $\mathcal{H}_C$. We obtain the Gelfand triple

$$\mathcal{H}_C \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}'_C$$

- Assuming that $\mathcal{D}(H_0|_{\mathcal{H}_C}) := \{ u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, H_0u \in \mathcal{H}_C \}$ is dense in $\mathcal{H}_C$, $H$ extends to

$$H' = H'_0 + \text{CWC}' : \mathcal{H}'_C \rightarrow \mathcal{H}'_C$$

- $CR_0(\lambda \pm i0^+)C := \lim_{\varepsilon \rightarrow 0^+} CR_0(\lambda \pm i\varepsilon)C$ exists in $\mathcal{B}(\mathcal{H})$ is equivalent to

$$R_0(\lambda \pm i0^+) := \lim_{\varepsilon \rightarrow 0^+} R_0(\lambda \pm i\varepsilon) \text{ exist in } \mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$$

Incoming/outgoing resonant states

Let $\lambda \in \sigma_{\text{ess}}(H)$ be a spectral singularity of $H$. The space $\mathcal{H}'_{\mathcal{H}_C}(\lambda) \subset \mathcal{H}'_C$ of outgoing/incoming resonant states corresponding to $\lambda$ is defined by

$$\mathcal{H}'_{\mathcal{H}_C}(\lambda) := \text{Ker} \left( \text{Id} + R_0(\lambda \pm i0^+)\text{CWC}' \right)$$
Spectral singularities: Characterization (III)

**Theorem [F, Frantz]**

Suppose that \((\ast)\) holds. The following conditions are equivalent:

1. \(\lambda\) is an **outgoing/incoming spectral singularity** of \(H\)
2. \(\lambda\) is an **eigenvalue** of \(H'\) associated to an **eigenvector** \(\Psi \in \mathcal{H}_C^{\pm}(\lambda)\)

**Some consequences**

- \(\lambda\) **embedded eigenvalue** \(\Rightarrow\) \(\lambda\) both outgoing and incoming spectral singularity
- At **thresholds**, outgoing and incoming spectral singularities coincide
- Suppose that \(V\) is a complex-valued potential such that \(\langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^3)\) with \(\sigma > 1\). Let \(C(x) = \langle x \rangle^{-\sigma/2}\). Then for all \(\lambda > 0\), the following conditions are equivalent:
  1. \(\lambda\) is an outgoing/incoming spectral singularity of \(H\)
  2. There exists \(\Psi \in \mathcal{H}_C^{\pm}(\lambda) \subset L^2_{-\sigma/2}, \Psi \neq 0\), such that
     \[
     (-\Delta + V(x) - \lambda)\Psi = 0
     \]
- The same holds at the threshold \(\lambda = 0\) if \(\langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^3)\) with \(\sigma > 2\)
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Spectral singularities: Ingredient of the proof

Proposition (Birman-Schwinger principle for spectral singularities)

Suppose that (*) holds. Then the following conditions are equivalent:

1. \( \lambda \) is an outgoing/incoming regular spectral point of \( H \)
2. \( \text{Id} + CR_0(\lambda \pm i0^+)CW \) is invertible in \( \mathcal{B}(\mathcal{H}) \)

Remark

Birman-Schwinger principle recently studied in abstract non-self-adjoint settings:

- [Behrndt, ter Elst, Gesztesy '20]
- [Hansmann, Krejcirik '20]