A simulation of the Schwinger model in the overlap formalism

Rajamani Narayanan\textsuperscript{a}, Herbert Neuberger\textsuperscript{b} and Pavlos Vranas\textsuperscript{c}

\textsuperscript{a}School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540

\textsuperscript{b}Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849

\textsuperscript{c}Department of Physics, Columbia University, New York, NY 10027

Abstract

In the continuum, the single flavor massless Schwinger model has an exact global axial $U(1)$ symmetry in the sector of perturbative gauge fields. This symmetry is explicitly broken by gauge fields with nonzero topological charge inducing a nonzero expectation value for the bilinear $\bar{\psi}\psi$. We show that a lattice formulation of this model, using the overlap formalism to treat the massless fermions, explicitly exhibits this phenomenon. A Monte Carlo simulation of the complete system yields the correct value of the fermion condensate and shows unambiguously that it originates from the sector of topological charge equal to unity.
Introduction

In chiral gauge theories fermions appear bilinearly in the action and interact with each other only by the exchange of gauge bosons. The bilinearity in fermion fields is preserved in most methods of regularization. Then the regularization becomes defined for any single (complex) irreducible representation carried by the chiral fermions, be it anomalous or not. Using this definition one can construct, in particular, vector gauge theories where the multiplets are assembled in left–handed/right–handed pairs per representation. Massless QCD is an example of such a vector gauge theory and it has an anomalous global $U(1)$ symmetry that is violated by topology induced ’t Hooft vertices [1]. Therefore, any regularization of chiral fermions that preserves bilinearity should break the anomalous $U(1)$, but conserve the nonanomalous global symmetries. Some of the existing proposals may have problems in this respect; we know of no examples where this point has been tested with quantum fluctuations of gauge bosons taken into account.

Here we subject the overlap formalism to this test. For smooth, classical, gauge fields we already know that the overlap formalism works correctly [2,3]. The purpose of this paper is to report a Monte Carlo simulation of the single flavor, strictly massless Schwinger model [4] formulated on the lattice. Each of the two chiral components of the Dirac fermion are realized using the overlap formula. Of course, this model can also be formulated on the lattice using conventional techniques [5], but then a massless limit has to be taken, there is difficulty in realizing a single flavor, there is no factorization of the theory into left and right sectors for fixed gauge fields, and the role of topology is obscured. Anyhow, our main point is not to present yet another solution of the Schwinger model, but, to establish a useful test for proposals to regularize chiral gauge theories and show that the overlap passes this test successfully.

In order to make it absolutely clear that the overlap succeeds in computing something that no previous approach to regularizing chiral fermions could, we picked as our observable just the ’t Hooft vertex $\bar{\psi}_R \psi_L$ which together with its hermitian conjugate makes up the mass bilinear $\bar{\psi} \psi$ more familiar in this context [6]. Our observable receives contributions only from gauge fields of topological charge one. The Schwinger model is a good model to pick because it has the necessary physics ingredients, its study is not particular demanding computationally and, above all, it is exactly soluble in the continuum, even for finite size systems. Various aspects of a two flavor version of the model have been investigated in detail in recent years both in the continuum and on the lattice by Azakov, Joos and Dilger [7]. Here we rely mostly on the continuum analysis of Sachs and Wipf [6].
The model

The continuum action in Euclidean space is:

\[
S[\bar{\psi}, \psi, A_\mu] = \frac{1}{2g^2} \int \sum_{\mu, \nu} (\epsilon_{\mu \nu} \partial_\mu A_\nu)^2 - \int \bar{\psi}_L \sigma_\mu (\partial_\mu + i A_\mu) \psi_L - \int \bar{\psi}_R \sigma_\mu^* (\partial_\mu + i A_\mu) \psi_R
\]

where \(\mu, \nu = 1, 2\), \(\sigma_1 = 1\) and \(\sigma_2 = i\). \(\bar{\psi}_R, L, \psi_R, L\) are Grassmann variables and \(A_\mu\) is a \(U(1)\) gauge field. The model is defined on a torus of fixed physical size \(l \times l\) with periodic boundary conditions for the electric field, \(E = \epsilon_{\mu \nu} \partial_\mu A_\nu\), and the fermions. The topological charge \(q\) is an integer given by \(q = \frac{1}{2\pi} \int E\). The expectation value of the fermion bilinear is:

\[
<\bar{\psi}\psi> = \int [d\bar{\psi}d\psi] \frac{1}{\int [d\bar{\psi}d\psi] e^{-S(\bar{\psi}, \psi, A_\mu)}} \equiv \frac{N}{D}
\]

The lattice formulation begins by replacing the continuum torus with a toroidal square lattice consisting of \(L\) sites in both directions. The lattice spacing \(a = \frac{l}{L}\). Gauge fields \(U_\mu(x) = e^{iaA_\mu(x)}\) are associated with the links of the lattice. The overlap formalism [2] is used to deal with the chiral fermions on the lattice. This results in the following expressions for \(N\) and \(D\) appearing in the expectation value of the fermion bilinear:

\[
N = \frac{1}{L^2} \sum_{x, \alpha} \prod U_\mu(x) e^{-S_s(U)} \left[ |U < L - |a_{x\alpha}|L+ > U|^2 + |U < L - |a^{\dagger}_{x\alpha}|L+ > U|^2 \right]
\]

\[
D = \int \prod U_\mu(x) e^{-S_s(U)} |U < L - |L+ > U|^2
\]

The \(a\)-term in \(N\) comes from gauge fields carrying topological charge +1 and is the overlap expression for \(<\bar{\psi}_R\psi_L>\) while the \(a^{\dagger}\)-term comes from charge -1 and represents \(<\bar{\psi}_L\psi_R>\). In our simulations we could have looked at only one of these terms; we took both in order to increase statistics. Analytically, the two overlap matrix elements have exactly equal magnitudes. The CP violating angle \(\theta\) is set to zero and therefore only the square of the absolute value appears in the above formula. The states \(|L\pm > U\) are the ground states of many body Hamiltonians

\[
\mathcal{H}^{\pm} = \sum_{x, \alpha, \beta} a^{\dagger}_{x, \alpha} \mathcal{H}^{\pm}(x, \alpha, \beta; U) a_{y, \beta} \quad \{a^{\dagger}_{x, \alpha}, a_{y, \beta}\} = \delta_{\alpha, \beta} \delta_{x, y}.
\]

\(\alpha, \beta = 1, 2\) and \(x = (x_1, x_2)\) with \(x_\mu = 0, 1, 2, \ldots, L - 1\). The single particle hermitian...
Hamiltonians $H^\pm$ are given by:

$$H^\pm = \begin{pmatrix} B^\pm & C \\ C^\dagger & -B^\pm \end{pmatrix};$$

$$C(x, y) = \frac{1}{2} \sum_\mu \sigma_\mu (\delta_{y,x+\mu} U_\mu(x) - \delta_{x,y+\mu} U_\mu^*(y));$$

$$B^\pm(x, y) = \frac{1}{2} \sum_\mu (2 \delta_{x,y} - \delta_{y,x+\mu} U_\mu(x) - \delta_{x,y+\mu} U_\mu^*(y)) \pm \delta_{x,y}$$

The parameter $m$ can be chosen in the range $0 < m < 1$ and is set to $m = 0.9$ here. Note that topological charge $\pm 1$ means that the number of single particle states filled in $|L+>_U$ and $|L->_U$ differ by $\pm 1$.

We need a gauge action and an associated Monte Carlo method that will give reasonable estimates for the topological charge distribution. For this reason, we would like to avoid correlations between configurations as far as possible. With this in mind, we pick a single plaquette action,

$$S_g(U) = \sum_p s_g(U_p),$$

for the gauge action appearing in the formulae for $N$ and $D$. $U_p$ is the oriented product of the four link elements making up the plaquette $p$. For such an action the plaquettes are almost independent variables in the pure gauge theory. To minimize the finite lattice spacing effects arising from the gauge action, we chose the “heat-kernel” action [8]:

$$e^{-s_g(U_p)} = \sum_{m \in \mathbb{Z}} e^{-s^2 m^2} [U_p]^m$$

This choice assures exact scaling of the string tension with the coupling $g$ in the pure gauge theory.

**The Monte Carlo method**

To compute the bilinear, we use a Monte Carlo method where configurations are drawn independently of each other. This avoids correlations between sequential samplings. To this end we replace $N$ by $N'$ and $D$ by $D'$ with $N' = \frac{N}{Z}$ and $D' = \frac{D}{Z}$ where

$$Z = \int \prod dU_\mu(x) e^{-S_g'(U)}$$
Gauge configurations will be generated stochastically using the distribution \( e^{-S'_g(U)} \) and, denoting averages w.r.t. \( e^{-S'_g(U)} \) by \( \langle \ldots \rangle \), we evaluate probabilistically

\[
N' = \frac{1}{L^2} \sum_{x, \alpha} \langle e^{-S_g(U)} + S'_g(U) \left[ |U - a_{x, \alpha}|L^+ > U|^2 + |U - a_{x, \alpha}^\dagger|L^+ > U|^2 \right] \rangle
\]

\[
D' = \langle e^{-S_g(U)} + S'_g(U) |U - |L^+ > U|^2 \rangle.
\]

\( S'_g(U) \) is picked to be

\[
S'_g(U) = S_g(U) - s_g(U_{p^*})
\]

where \( p^* \) is some arbitrarily chosen plaquette. The probability distribution \( e^{S'_g(U)} \) factorizes into one factor of \( e^{-s_g(U_p)} \) for each \( p \neq p^* \). To generate a configuration, one draws a set of \((L^2 - 1) U_p\) variables, each according to the plaquette factor \( e^{-s_g(U_p)} \). This fixes \((L^2 - 1)\) out of the \((L^2 + 1)\) link variables remaining after choosing a gauge on the \( L \times L \) lattice. Two more link variables determining Polyakov loops winding around the torus are drawn randomly since the gauge action is unaffected by these loops. Since we are computing gauge invariant quantities the exact nature of gauge fixing is irrelevant and there is no need to average over gauge transformations.

The parameters

Having described the Monte Carlo method, we now proceed to determine the parameters of the model. Since we are treating the overlap factors as observables in a pure gauge theory, this method will work only if the fluctuations in these factors are not too rapid. Technically, they are of the order of \( e^{cL^2} \) and since we are simulating a system of finite physical size \( l \), \( c \) is also dependent on \( L \). Based on the exact solution in the continuum we choose \( l \) so that the fluctuations in the fermionic determinant are small. We feel confident in using the continuum results for our estimates since we have already shown that the overlap agrees with the continuum very well for sufficiently smooth fields \([2,3]\). The zero momentum mode of the electrical field is quantized by topology, so the dependence on it is special. The fermionic determinant also depends on the zero momentum component of the gauge fields, but these are only two variables and the dependence though sizable is not too violent. Previous work assures us that the overlap will follow continuum behavior faithfully also for these variables \([9]\). There remain now the nonzero momentum modes of the electrical field \( \tilde{E}(k) \equiv \phi(k) k^2 \). The bosonic action per mode is \( e^{-\frac{1}{2g^2} |\phi(k)|^2 k^4} \) yielding \( |\phi(k)|^2 \approx \frac{2g^2}{k}. \) The fermion determinant factorizes among the modes \([2]\) and, per mode, is
given by $e^{-\frac{1}{\pi^2}|\phi(k)|^2k^2} \approx e^{-\frac{\mu^2}{\pi^2}k^2}$ where $\mu = \frac{g}{\sqrt{\pi}}$. The smallest nonzero momentum is $k = \frac{2\pi}{L}$ so we get a typical contribution of the order $e^{-\frac{\mu^2}{4\pi^2}k^2}$. For $\mu l \approx 3$, the typical number in the exponent is expected to be quite small. If we sum over the higher modes the result changes a little but we do not need to complete this exercise for our simple estimates.

Apart from the overlap factor, we also have the factor $e^{-S_g(U)+S'_g(U)} = e^{-s_g(U_p^*)}$ with $U_p^* = \prod_{p \neq p^*} U_p$. We should note that our method does not implement importance sampling for this factor. We were willing to accept this drawback in exchange for the unhindered generation of the topological charge distribution. Since our physical volume is fixed, the coupling becomes weaker as the lattice gets finer and $e^{s_g(U_p)}$ gets more peaked. Its sampling however has a roughly constant and quite broad width, so, unlike in the usual case where statistical errors per lattice decrease with size due to self averaging, here the statistical error will stay more or less constant as a function of size and therefore the computing cost per given accuracy will increase dramatically with size. Although this can be remedied to some extent, it is not necessary for the computation presented here.

Gauge configurations contributing to $N'$ have unit lattice topological charge and those that contribute to $D'$ have zero lattice topological charge. In all other cases the respective overlap vanishes identically. There is a probability $p(0) \equiv p$ to generate gauge field configurations of zero lattice topological charge. In a sample of $n$ lattices there will be $j \approx pn$ lattices contributing to $D'$. $\frac{j}{n}$ is Poisson distributed with standard deviation $\sigma(0) = \sqrt{\frac{p(1-p)}{n}}$. In the continuum one has $p(k) = Ne^{-\frac{2\pi}{(\mu l)^2}k^2}$. For $\mu l \sim 3$ for pure gauge (ignoring the single plaquette $p^*$ in the continuum) the decay at higher $k$'s is sufficiently rapid to ignore all but $k = 0, \pm 1$ as a first approximation for the expected errors. Hence there is a factor of $\frac{p}{1-p}$ in the ratio $\frac{N'}{D'}$. This factor will be distributed with a relative error $\delta = \frac{1}{\sqrt{np(1-p)}}$. To minimize it we would like to work with $p \approx \frac{1}{2}$. This is indeed the case for $\mu l \approx 3$.

In view of the above considerations, we picked $\mu l = 3$. The continuum limit is approached by scaling the lattice coupling $g$ with the lattice spacing via $g = \frac{3\sqrt{\pi}}{L}$. Simulations were performed on lattices ranging from $L = 3$ to $L = 10$. Since $\delta \approx \frac{1}{\sqrt{n}}$ is a lower bound on the relative error in $\frac{N'}{D'}$, we see that in order to reach an accuracy of a few percent, we need a few thousand lattices at least. This is quite acceptable if the lattice sizes $L$ can be kept small.

The continuum result for the dimensionless quantity $\frac{\bar{\psi}_R \bar{\psi}_L}{\mu} \equiv f(\mu l)$ is known exactly from the work of Sachs and Wipf [6]: $f(3) \approx 0.236$ and $f(\infty) \approx .284$. While we have no reason to try to extrapolate numerically to infinite physical size, we like the fact that we are working on a system that is reasonably large in physical terms. The finite size effect
is seen from the exact solution to be of the order of 20% and it is nice that we don’t have to eliminate it numerically.

**Algorithm**

We used very direct and unsophisticated algorithms. Double precision was used throughout the algorithm. The diagonalization of $H^\pm$ was done using Jacobi’s method and the needed determinants evaluating the overlaps were computed using $LU$ decomposition. Since half of the configurations require the evaluation of $L^2$ determinants the algorithm goes, asymptotically, as $L^8$. In practice, the larger prefactor in front of the $L^6$ corresponding to Jacobi diagonalization versus the one corresponding to determinant computation seems to be important for our sizes and the algorithm scaled slightly worse than $L^6$. The worst case, $L = 10$, took an average of 5 minutes CPU time per lattice on an 100 MHz α–workstation. No attempts were made to speed up the code. The results of this paper were obtained using the equivalent of a few hundred hours on such a machine.

**Results**

Our results are summarized in the table below where the numerical results are listed as a function of $L$. Errors were estimated in several ways and all gave consistent answers. We see that within statistical errors the correct answer is obtained already at $L = 5$, so most of the computer time we used was not really needed. We felt, however, that it was necessary to ascertain that nothing bizarre was happening when one went closer to the continuum limit.

| $L$ | $f_L(3)$ | # of confs. |
|-----|----------|-------------|
| 3   | 0.0812(32) | 5000        |
| 4   | 0.1638(66) | 5000        |
| 5   | 0.224(11)  | 5000        |
| 6   | 0.223(13)  | 5000        |
| 7   | 0.235(14)  | 6000        |
| 8   | 0.238(12)  | 8173        |
| 10  | 0.224(18)  | 4919        |
| $\infty$ | 0.236 | Continuum |

**Table:** $f_L(\mu l) \equiv \frac{L^N}{(\mu l)^D}$ for various lattice spacings $\frac{1}{L}$ at $\mu l = 3$.
Conclusions and Outlook

In this paper we have shown that the overlap formalism for chiral fermions passed a certain “vector test”. This test can be used as a check for any other proposal that regularizes a chiral gauge theory. It is simple, not very costly to apply and, at present, we believe it would be a good selection method. It is hard for us to see how a bilinear method for the regularization of chiral gauge theories could succeed without passing this vector test. To be sure, let us stress that the numerical result we have obtained here should be obtainable by traditional methods [5]. In these methods one does not have exact left right factorization. When one starts from a chiral method this factorization is built in.

The next step would be to study a chiral Schwinger model in the overlap formalism. Here we do not have exact gauge invariance on the lattice in the anomaly free case although this is achieved in the continuum for smooth gauge fields. We have to check whether the Foerster–Nielsen–Ninomya [10] mechanism of gauge invariance restoration in the continuum limit indeed works on the lattice as we conjectured in [2]. Preliminary studies we have done are encouraging. Then the road to exhibiting fermion number violation on the lattice is open.

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