Landau-Zener transition with energy-dependent decay rate of the excited state

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A remarkable feature of the Landau-Zener transition is insensitivity of the survival probability to the decay rate, $\tau^{-1}$, of the excited state. Namely, the probability for a particle which is initially (at $t \to -\infty$) in the ground state to remain at $t \to \infty$ in the same state is insensitive to $\tau^{-1}$ which is due to e.g. coupling to continuum [V. M. Akulin and W. P. Schleich, Phys. Rev. A 46, 4110 (1992)].

This insensitivity was demonstrated for the case when the density of states in the continuum is energy-independent. We study the opposite limit when the density of states in the continuum is a step-like function of energy. As a result of this step-like behavior of the density of states, the decay rate of a driven excited level experiences a jump as a function of time at certain moment $t_0$. We take advantage of the fact that the analytical solution at $t < t_0$ and at $t > t_0$ is known. We show that the decay enters the survival probability when $t_0$ is comparable to the transition time.

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I. INTRODUCTION

According to the celebrated paper\textsuperscript{1,2} by Landau, Zener, Majorana, and Stueckelberg, as two levels are swept by each other with velocity, $v$, a particle remains in the level, which it occupied before the crossing, with probability $\exp\left(-\frac{2\pi J^2}{\dot{v}}\right)$, where $J$ is the tunneling amplitude between the levels at the point of crossing.

In the contemporary research the crossing of the driven levels is implemented in qubits based on superconducting circuits and quantum dot interferometers, see e.g. Refs. 5, 6 for review.

It is remarkable that the Landau-Zener formula is robust with respect to the possibility of tunneling of a particle out of the excited state. As it was first pointed out in Ref. 7 see also Ref. 8 for any $J$ the Landau-Zener formula remains applicable even when the tunneling time, $\tau$, is much shorter than $\frac{\dot{v}}{v}$, which is the characteristic time of the transition. In the latter limit, each time an electron enters the excited state, it typically does not return to the initial state, but rather directly proceeds to the continuum. While the portion of events when the electron still returns is exponentially small, the Landau-Zener formula still holds despite the interference oscillations in the occupation of the excited state are completely washed out.\textsuperscript{9}

While the result\textsuperscript{2} seems counter-intuitive, it can be interpreted as a manifestation of the profound property of multistate Landau-Zener transition established in Refs. 9, 10. The key to this interpretation lies in the observation that the tunnel decay of the excited state can be modeled by the broadening of the level, corresponding to the excited state, into a multiplet of the closely spaced discrete levels. Then the scenario of crossing of two levels translates into the crossing of the driven ground-state level by the numerous excited-state levels. For this arrangement, it was demonstrated in Refs. 9, 10 that the survival probability is equal to the product of partial probabilities regardless of how close the levels forming the multiplet are spaced. With matrix element, $J$, being distributed between the sublevels, the width of the multiplet drops out of the product.

The fact that survival probability is independent of the tunneling width of the excited state poses a fundamental question: will the decay enter the survival probability if it switches on only above a certain energy, $\varepsilon_0$? Another way to pose this question is: since the excited level is driven, the decay of the excited state switches on at certain time moment, $t_0$. Will then $t_0$ enter the survival probability? This question is the focus of the present paper. To address it, we take advantage of the fact that the analytical solutions at $t < t_0$ and at $t > t_0$ are known. The prime question is, certainly, whether switching on the decay enhances or suppresses the survival probability. Note also, that here is a natural scale for $t_0$, which is the characteristic time of the transition, $\frac{\dot{v}}{v}$.

II. THE MODEL

It is convenient to reason within a concrete model of levels which are swept passed each other with a relative velocity, $v$. This model is illustrated in Fig. 1. Assume that the right level evolves with time as

$$\varepsilon_r(t) = \frac{vt}{2}$$

and does not decay. The left level evolves with time as

$$\varepsilon_l(t) = -\frac{vt}{2}.$$  \hfill (2)

As illustrated in Fig. 1, at time $t > t_0$ an electron can tunnel out of the left level into the continuum, while at $t < t_0$ the tunneling channel is shut off.

Initially, at $t \to -\infty$ an electron is located in the right level. According to Refs. 7 and 8 at large positive $t_0 \gg \frac{\dot{v}}{v}$, the tunnel decay drops out of the survival probability, $\exp\left(-\frac{2\pi J^2}{\dot{v}}\right)$. 


FIG. 1: (Color online) Schematic illustration of the Landau-Zener transition with decaying excited state. Two levels are swept past each other with relative velocity \( v \). Depending on the position of the left level, it can decay into continuum. At time \( t < t_0 \) the left level does not decay at all. At \( t > t_0 \) the electron in the left level can tunnel out into the continuum. The tunneling time is \( \tau \). The tunneling time \( \tau \) is expressed as 

\[
\tau = \frac{\sqrt{v^2 - 4a^2}}{2a}.
\]

Concerning the amplitude to find electron in the left level, we view the survival probability as a function of \( t \). Thus, if the electron in the left level can tunnel out into the continuum, it can decay into continuum. At \( t < t_0 \) time the position of the left level, it can decay into continuum. At \( t > t_0 \) the survival probability on \( t_0 \). In the limit \( t_0 \rightarrow -\infty \), the electron in the left level can tunnel out at all times. Then, as it was first demonstrated in Ref. 7, the tunneling time drops out from the survival probability.

Equally, at large negative \( t_0 \ll -\frac{J}{v} \), the Landau-Zener transition proceeds as if there was no decay. Thus, if we view the survival probability as a function of \( t_0 \), we expect that the decay will enter this function only if \( t_0 \) falls into the interval \( \sim \frac{J}{v} \).

At all times, evolution of the amplitude to find the electron in the right level is described by the equation

\[
i\dot{a} = \frac{vt}{2}a + Jb.
\]

Concerning the amplitude to find electron in the left level, before and after switching evolves as

\[
i\dot{b} = -\frac{vt}{2}b + Ja, \quad t < t_0
\]

\[
i\left(\frac{b + b}{\tau}\right) = -\frac{vt}{2}b + Ja, \quad t > t_0.
\]

The system Eqs. (3) and (4) should be solved under the condition \( b(-\infty) = 0 \), i.e. under the condition that the electron is initially in the right level. The solution at \( t < t_0 \) has the textbook form

\[
a(t) = D_{\nu}(z), \quad b(t) = -i\sqrt{v}D_{\nu-1}(z),
\]

where the argument of the parabolic cylinder function, \( D_{\nu} \), is defined as

\[
z = t\sqrt{ve^{\pi i/4}},
\]

while the parameter \( \nu \) is expressed via \( J \) and \( v \) as

\[
\nu = -\frac{iJ^2}{v}.
\]

To find the solution at \( t > t_0 \) we make the following substitution in Eqs. (3) and (4)

\[
a(t) = a_0(t) \exp\left(-\frac{t}{2\tau}\right), \quad b(t) = b_0(t) \exp\left(-\frac{t}{2\tau}\right).
\]

Upon this substitution, we arrive to the following system of equations for \( a_0(t) \) and \( b_0(t) \)

\[
i\dot{a}_0 = \left(\frac{i}{2\tau} + \frac{vt}{2}\right)a_0 + Jb_0,
\]

\[
i\dot{b}_0 = -\left(\frac{i}{2\tau} + \frac{vt}{2}\right)b_0 + Ja_0.
\]

We see that the system Eq. (9) reproduces the system for \( a(t) \) and \( b(t) \) upon the replacement \( t \rightarrow t + \frac{\pi i}{4\nu} \).

General solution of the system Eq. (9) is given by the linear combination of the parabolic cylinder functions of the shifted argument

\[
a_0(t) = AD_{\nu} \left[ \left( t + \frac{i}{\nu\tau} \right) \sqrt{ve^{\pi i/4}} \right] + BD_{\nu} \left[ - \left( t + \frac{i}{\nu\tau} \right) \sqrt{ve^{\pi i/4}} \right],
\]

\[
b_0(t) = -i\sqrt{v} \left[ AD_{\nu-1} \left( t + \frac{i}{\nu\tau} \right) \sqrt{ve^{\pi i/4}} \right] - BD_{\nu-1} \left[ - \left( t + \frac{i}{\nu\tau} \right) \sqrt{ve^{\pi i/4}} \right].
\]

Constants \( A \) and \( B \) are determined from the continuity of \( a(t) \) and \( b(t) \) at \( t = t_0 \). The corresponding system reads

\[
D_{\nu}(z_0)e^{\frac{i\theta}{2\pi\nu}} = AD_{\nu}(z'_0) + BD_{\nu}(-z'_0),
\]

\[
D_{\nu-1}(z_0)e^{\frac{i\theta}{2\pi\nu}} = AD_{\nu-1}(z'_0) - BD_{\nu-1}(-z'_0).
\]

where the arguments \( z_0 \) and \( z'_0 \) are defined as

\[
z_0 = t_0 \sqrt{ve^{\pi i/4}}, \quad z'_0 = z_0 + \frac{i}{\sqrt{v}}e^{\pi i/4}.
\]
Solving the system Eq. (12) yields the following expressions for the constants \( A \) and \( B \)
\[
A = e^{\frac{t_0}{\nu}} D_v(-z_0^t)D_{v-1}(z_0) + D_v(z_0)D_{v-1}(-z_0^t) \\
B = -e^{\frac{t_0}{\nu}} D_v(-z_0^t)D_{v-1}(z_0) - D_v(z_0)D_{v-1}(-z_0^t),
\] (14)

In the absence of decay, \( \tau \to \infty \), the dependence on the moment, \( t_0 \), should drop out. Indeed, setting \( z_0 = z_0^t \) in Eqs. (14), (15) we find \( A = 1, B = 0 \).

The quantity we are interested in is the survival probability, which we define as the ratio of probabilities to find the particle in the left level at \( z \to \infty \) and \( z \to -\infty \). With the help of the explicit form of \( a(t) \) given by Eq. (10), we find
\[
Q_{LZ} \left| \frac{A \left( e^{-\frac{t_0}{\nu}} D_v(z) \right)_{t \to \infty} + B \left( e^{-\frac{t_0}{\nu}} D_v(-z) \right)_{t \to \infty}}{D_v(z)_{t \to -\infty}} \right|^2.
\] (16)

Using the large-argument asymptotes of the parabolic-cylinder functions, we obtain
\[
Q_{LZ} = |A|^2e^{-2\pi|\nu|} + (A^*B + B^*A)e^{-\pi|\nu|} + |B|^2.
\] (17)

Equations Eqs. (14), (15), and (17) yields a formal solution to the problem. We will analyze it in the most interesting limit \( |\alpha_0| < \frac{\nu}{2} \), when the decay switches on in the course of the transition. Then, as demonstrated in the Appendix, the expressions for the amplitudes \( a(t) \) and \( b(t) \) before switching of the decay can be simplified as
\[
a(t) = \lambda e^{iJt} + \lambda^{-1}e^{-iJt}, \\
b(t) = -\lambda e^{iJt} + \lambda^{-1}e^{-iJt}.
\] (18)

Here \( \lambda = \exp \left( -\frac{\pi|\nu|^2}{2\nu} \right) = \exp \left( -\frac{\pi\nu}{2} \right) \).

It is convenient to cast \( a(t_0^+) \) and \( b(t_0^+) \), which are the amplitudes at \( t = t_0^+ \), in a more concise form by introducing a notation
\[
\kappa_0 = \lambda e^{iJt_0}.
\] (19)

Then Eq. (18) takes the form
\[
a(t_0^+) = \kappa_0 + \frac{1}{\kappa_0}, \\
b(t_0^+) = -\kappa_0 + \frac{1}{\kappa_0}.
\] (20)

Similarly, upon switching on the tunneling, the simplified expressions for \( a(t) \) and \( b(t) \) read
\[
a(t) = e^{-\frac{t}{\nu}} \left[ A \left\{ \lambda e^{iJ(t + \frac{\nu}{2})} + \lambda^{-1}e^{-iJ(t + \frac{\nu}{2})} \right\} \\
+ B \left\{ \lambda e^{-iJ(t + \frac{\nu}{2})} + \lambda^{-1}e^{iJ(t + \frac{\nu}{2})} \right\} \right],
\]
\[
b(t) = e^{-\frac{t}{\nu}} \left[ -A \left\{ \lambda e^{iJ(t + \frac{\nu}{2})} - \lambda^{-1}e^{-iJ(t + \frac{\nu}{2})} \right\} \\
+ B \left\{ \lambda e^{-iJ(t + \frac{\nu}{2})} - \lambda^{-1}e^{iJ(t + \frac{\nu}{2})} \right\} \right].
\] (21)

Again, the above expressions assume a concise form with the help of the auxiliary notations
\[
\kappa = \lambda \exp \left\{ iJ \left( t_0 + \frac{i}{\nu} \right) \right\}, \\
\mu = \lambda \exp \left\{ -iJ \left( t_0 + \frac{i}{\nu} \right) \right\}.
\] (22)

With the help of these notations, the amplitudes \( a(t) \) and \( b(t) \) at \( t = t_0^+ \) can be written as
\[
a(t_0^+) = e^{-\frac{t}{\nu}} \left[ A f_1 + B g_1 \right], \\
b(t_0^+) = e^{-\frac{t}{\nu}} \left[ A f_2 + B g_2 \right],
\] (23)

where the coefficients \( f_1, f_2, g_1, \) and \( g_2 \) are given by
\[
f_1 = \kappa + \frac{1}{\kappa}, \quad f_2 = -\kappa + \frac{1}{\kappa}, \\
g_1 = \mu + \frac{1}{\mu}, \quad g_2 = \mu - \frac{1}{\mu}.
\] (24)

From the continuity conditions \( a(t_0^-) = a(t_0^+) \), \( b(t_0^-) = b(t_0^+) \) we infer the following expressions for \( A \) and \( B \)
\[
A = e^{\frac{t_0}{\nu}} g_1 b(t_0^-) - g_2 b(t_0^-), \\
B = e^{\frac{t_0}{\nu}} f_2 a(t_0^-) - f_1 b(t_0^-). \] (25)

Note that the combination in denominators in Eq. (25) is equal to
\[
f_2 g_1 - f_1 g_2 = 2\mu \left( \frac{1}{\kappa} - \kappa \right). \] (26)

Then the system Eq. (12) yields the following results for the coefficients \( A \) and \( B \)
\[
A = \frac{\lambda^4 e^{\frac{2t_0}{\nu}} - 1}{\lambda^4 - 1} \exp \left( \frac{t_0}{2\nu} - \frac{J}{v\nu} \right). \] (27)
\[ B = -2\lambda^2 \exp\left(\frac{t_0}{2\tau}\right) \frac{\sinh\left(\frac{B}{2\tau}\right)}{B^2 - 1}. \]  

(28)

Substituting these values into Eq. (16), we arrive at the final result
\[ Q_{LZ} = \left[\lambda^2 A + B^2\right] = e^{-2\pi|v|} \exp\left[\frac{2}{\tau} \left(\frac{J}{v} + \frac{t_0}{2}\right)\right]. \]  

(29)

While \( e^{-2\pi|v|} \) is a conventional Landau-Zener result, the second exponent describes the effect of the tunnel decay on the survival probability and on the moment, \( t_0 \), of the switching on the decay.

### III. CONCLUDING REMARKS

An interesting feature of the result obtained is that there is a special time moment \( t_0 = -\frac{2J}{v} \) of switching on the decay. For this \( t_0 \) the survival probability does not decay on the tunnel decay at all. Qualitatively, the meaning of the ratio \( Jv \) is the characteristic time of the Landau-Zener transition. As seen from Eqs. (3) and (4), the ratio \( \frac{J}{v} \) is the time when the amplitudes \( a(t) \) and \( b(t) \) are of same order. Concerning the dependence of \( Q_{LZ} \) on the tunneling rate \( \tau^{-1} \), it is seen from Eq. (29) the characteristic \( \tau \) is also \( \frac{J}{v} \). Note that, see Eq. (18) the probabilities \( |a(t)|^2 \) and \( |b(t)|^2 \) contain a constant part \( \frac{1}{2v} + \lambda^2 \) plus oscillatory parts \( \pm 2 \cos Jt \). When the survival probability is small, \( \lambda \ll 1 \), oscillatory parts constitute a small correction. At \( \tau \lesssim \frac{J}{v} \) the oscillations are completely washed out by the decay. If \( \tau \) is constant, this washout does not affect \( Q_{LZ} \). On the contrary, the second factor in Eq. (29) is entirely due to the abrupt switching on of the decay.

It is instructive to compare our results with Ref. [13] where the decay also enters into the survival probability. In Ref. [13] the left and right levels formed doublets. The states of the right doublet, emulating the initial state of the Landau-Zener transition, did not decay, while both states of the right doublet, emulating the final state, did decay into continuum. It was shown in Ref. [13] that, unlike the conventional transition, the decay rates of the final states entered the survival probability, if they are different. By contrast, in the present manuscript we consider a conventional Landau-Zener transition, but allow both mutually crossing levels to decay. Calculation in Ref. [13] illustrates the difference between the integrable and non-integrable Landau-Zener models: in integrable models the tunnel decay drops out of the survival probability, while in non-integrable models the decay enters the survival probability, \( Q_{LZ} \), explicitly. In this regard, the message of the present paper can be formulated as follows: even an integrable model, with a tunnel rate being a function of energy, it enters explicitly into \( Q_{LZ} \).

Appendix: derivation of Eq.18

We start from the integral representation of the parabolic cylinder function
\[ D_\nu(z) = \left(\frac{2}{\pi}\right)^{1/2} e^{\frac{z^2}{4}} \int_0^\infty du\ e^{-\frac{u^2}{2} + izu} \cos \left(\nu u - \frac{\pi u}{2}\right). \]  

(30)

Following Ref. [12], it is convenient to divide the integral Eq. (30) into two contributions
\[ D_\nu(z) = I_+(z)e^{iz} + I_-(z)e^{-iz}, \]  

(31)

where the functions \( I_+(z) \) and \( I_-(z) \) are defined as
\[ I_\pm(z) = \left(\frac{1}{2\pi}\right)^{1/2} e^{\frac{z^2}{4}} \int_0^\infty du\ u e^{-\frac{u^2}{2} \pm iuz}. \]  

(32)

Both integrals are evaluated with the help of steepest descent approach. The corresponding saddle points are given by
\[ u_\pm = \frac{iz}{2} \pm \frac{(4\nu - z^2)^{1/2}}{2}. \]  

(33)

Performing the Gaussian integration around the saddle points, we arrive to the following asymptotic expressions for \( I_+, I_- \)
\[ I_+(z) \approx (4\nu - z^2)^{-1/4} \exp \left[\frac{1}{4}iz(4\nu - z^2)^{1/2} - \frac{1}{2\nu}\right] \times \left[\frac{1/2}{2} + \frac{1}{2}(4\nu - z^2)^{1/2}\right]^{\nu + \frac{1}{2}}, \]  

(34)

\[ I_-(z) \approx (4\nu - z^2)^{-1/4} \exp \left[-\frac{1}{4}iz(4\nu - z^2)^{1/2} - \frac{1}{2\nu}\right] \times \left[-\frac{1}{2} + \frac{1}{2}(4\nu - z^2)^{1/2}\right]^{\nu + \frac{1}{2}}. \]  

(35)

The condition of applicability of the steepest descent method is that the typical value of \( (u - u_+) \) contributing to the integral Eq. (32) is much smaller than \( u_+ \). With characteristic time of the transition being \( \frac{J}{v} \), we conclude that characteristic \( z \) is \( \sim \frac{J}{v\tau} \sim |\nu|^{1/2} \). Thus, for \( |\nu| \gg 1 \), when the position of the saddle point is \( u_+ \sim |\nu|^{1/2} \gg 1 \) the saddle-point result applies not only for large, but for arbitrary \( z \).

To derive Eq. (18), we substitute Eqs. (34), (35) into
Eq. (31). This yields
\[
D_\nu(z) = e^{\frac{z}{W(0)^2}} \left(e^{-\frac{\pi i}{8}} + iJ^2 \frac{v}{2} \right)^{1/4} \left(e^{-\frac{\pi}{4} v^{1/2}} + \frac{\pi i}{4} v^{1/2} \right)^{1/2} \left(e^{-\frac{\pi i}{4} v^{1/2}} + \frac{\pi i}{4} v^{1/2} \right)^{1/2} \times
\left\{ e^{\frac{i}{4} W(t) \left(-vt + W(t)\right)} e^{\frac{\pi i}{4} \frac{v^2}{t^2}} + e^{\frac{-i}{4} W(t) \left(vt + W(t)\right)} e^{\frac{-\pi i}{4} \frac{v^2}{t^2}} \right\},
\]
(36)
where \( W(t) = (4J^2 + v^2t^2)^{1/2} \). Taking the limit \( vt \ll J \)
and recalling that \( \exp \left(-\frac{\pi J^2}{2vt}\right) = \lambda \), we recover Eq. (18).

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