Exponential Slowdown for Larger Populations: The $(\mu + 1)$-EA on Monotone Functions

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Abstract

Pseudo-Boolean monotone functions are unimodal functions which are trivial to optimize for some hillclimbers, but are challenging for a surprising number of evolutionary algorithms. A general trend is that evolutionary algorithms are efficient if parameters like the mutation rate are set conservatively, but may need exponential time otherwise. In particular, it was known that the $(1 + 1)$-EA and the $(1 + \lambda)$-EA can optimize every monotone function in pseudolinear time if the mutation rate is $c/n$ for some $c < 1$, but that they need exponential time for some monotone functions for $c > 2.2$. The second part of the statement was also known for the $(\mu + 1)$-EA.

In this paper we show that the first statement does not apply to the $(\mu + 1)$-EA. More precisely, we prove that for every constant $c > 0$ there is a constant $\mu_0 \in \mathbb{N}$ such that the $(\mu + 1)$-EA with mutation rate $c/n$ and population size $\mu_0 \leq \mu \leq n$ needs superpolynomial time to optimize some monotone functions. Thus, increasing the population size by just a constant has devastating effects on the performance. This is in stark contrast to many other benchmark functions on which increasing the population size either increases the performance significantly, or affects performance only mildly.

The reason why larger populations are harmful lies in the fact that larger populations may temporarily decrease selective pressure on parts of the population. This allows unfavorable mutations to accumulate in single individuals and their descendants. If the population moves sufficiently fast through the search space, then such unfavorable descendants can become ancestors of future generations, and the bad mutations are preserved. Remarkably, this effect only occurs if the population renews itself sufficiently fast, which can only happen far away from the optimum.

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This is counter-intuitive since usually optimization becomes harder as we approach the optimum. Previous work missed the effect because it focused on monotone functions that are only deceptively close to the optimum.

1 Introduction

Population-based evolutionary algorithms (EAs) are general-purpose heuristics for optimization. Having a population may be helpful, because it allows for diversity in the algorithm’s states. Such diversity may be helpful for escaping local minima, and it is a necessary ingredient for crossover operations as they are used in genetic algorithms (GAs). Theoretical and practical analysis of population-based algorithms have indeed found positive effects, and showed a general trend that larger populations are often better [20]. The only (mild) observed negative effect is, intuitively speaking, that maintaining a population of size \( \mu \) may slow down the optimization time by a factor of at most \( \mu \). Only few, highly artificial examples are known [18] in which a \((\mu + 1)\)-EA or \((\mu + 1)\)-GA with time budget \( \mu t \) performs significantly worse than a \((1 + 1)\)-EA with time budget \( t \). In this sense, it is easy to believe that a \((\mu + 1)\) algorithm is at least as good as a \((1 + 1)\) algorithm, except for the slightly higher runtime since each individual only has probability \( 1/\mu \) per round of creating an offspring.

Our results challenge this belief, and show that it fails badly for some monotone functions. Our main results show that increasing \( \mu \) from 1 to a larger constant can increase the runtime from quasilinear to exponential.

A monotone\(^1\) pseudo-Boolean function is a function \( f : \{0,1\}^n \to \mathbb{R} \) such that for every \( x, y \in \{0,1\}^n \) with \( x \neq y \) and \( x_i \geq y_i \) for all \( 1 \leq i \leq n \) it holds \( f(x) > f(y) \). Monotone functions are easy benchmark functions for optimization techniques, since they always have a unique local and global optimum at the all-one string. Moreover, from every search point there are short, fitness-increasing paths to the optimum, by flipping zero-bits into one-bits. Consequently, there are many algorithms which can easily optimize every monotone function. A particular example is random local search (RLS), which is the \((1 + 1)\) algorithm that flips in each round exactly one bit, uniformly at random. RLS can never increase the distance from the optimum for a monotone function, and it optimizes any such function in time \( O(n \log n) \) by a coupon collector argument. Thus monotone functions should be regarded as an easy benchmark for evolutionary algorithms. Nevertheless it was shown in [2,3,12,16] that a surprising number of evolutionary algorithms need exponential time to optimize some monotone functions, especially if they mutate too aggressively, i.e., the mutation parameter \( c \) is too large (see Section 1.2 for a detailed discussion). However, in all considered cases the algorithms were efficient if the mutation parameter satisfied \( c < 1 \).

\(^1\)Following [12,16], we call them monotone functions, although strictly monotone functions would be slightly more accurate.
1.1 Our Results

We show that the $(\mu+1)$ Evolutionary Algorithm, $(\mu+1)$-EA, becomes inefficient even if the mutation strength is smaller than 1. More precisely, we show that for every $c > 0$ there is a $\mu_0 = \mu_0(c) \in \mathbb{N}$ such that for all $\mu_0 \leq \mu \leq n$ there are some monotone functions for which the $(\mu+1)$-EA with mutation rate $c/n$ needs superpolynomial time to find the optimum. If $\mu$ is $O(1)$ then this time is even exponential in $n$. Note that for $0 < c \leq 1$, it is known that the $(1+1)$-EA finds the optimum in quasilinear time for any monotone functions [14]. Thus increasing the population size only slightly (from 1 to $\mu_0$) makes the optimization time explode from quasilinear to exponential.

The monotone functions that are hard to optimize are due to Lengler and Steger [16], and were dubbed HotTopic functions in [12]. These functions look locally like linear functions in which all bits have some positive weights. However, in each region of the search space there is a specific subset of bits (the ‘hot topic’), which have very large weights, while all other bits have only small weights. If an algorithm improves in the hot topic, then it will accept the offspring regardless of whether the other bits deteriorate. In [12, 13, 16] it was shown that an algorithm like the $(1+1)$-EA with $c > 2.13...$ will mutate too many of these bits outside of the hot topic, and will thus not make progress towards the global optimum.

The key insight of our paper is that for such weighted linear functions with imbalanced weights, populations may also lead to an accumulation of bad mutations, even if the mutation rate is small. Here is the intuition. For a search point $x$, we call the number of one-bits in the hot topic in $x$ the rank of $x$. Consider a $(\mu+1)$-EA close to the optimum, and assume for simplicity that all search points in the population $S_0$ have the same rank $i$. At some point one of them will improve in the hot topic by flipping a zero-bit there. Let us call the offspring $x$, and let us assume that its rank is $i + 1$. Then $x$ is fitter than all other search points in the population because it has a higher rank. Moreover, every offspring or descendant of $x$ will also be fitter than all the other points in the population, as long as they maintain rank $i + 1$. Thus for a while the $(\mu+1)$-EA will accept all (or most) descendants of $x$, and remove search points of rank $i$ from the population. This goes on until some time $t_0$ at which search points of rank $i$ are completely eliminated from the population. Note that at time $t_0$, most descendants $x'$ of $x$ have considerably smaller fitness than $x$, since the algorithm accepts every type of mutation outside of the hot topic, and most mutations are detrimental. If some descendant $x'$ of $x$ creates an offspring $y$ of even higher rank, then $y$ is accepted and the cycle repeats with $y$ instead of $x$. The crucial point is that $y$ is an offspring of $x'$, which has accumulated a lot of bad mutations compared to $x$. So typically, $x'$ is considerably less fit than $x$, but still it passes on its bad genes.

The above effect needs that the probability of improving in the hot topic has the right order. If the probability is too large (close to one), then $x$ will already spawn an offspring of rank $i + 1$ before it has spawned many descendants with the same rank. On the other hand, if the probability is too small then there will
be no rank-improving mutations until time $t_0$, and after time $t_0$ the algorithm starts to remove the worst individuals of rank $i + 1$ from the population. We remark that this latter regime was already studied in [12], for the extreme case in which the improvement probability is so small that typically the population of rank $i + 1$ collapses into copies of $x$ before a further improvement is made. (In the terminology of [12], it was the assumption that the parameter $\varepsilon$ of the HOTTOPIC function was sufficiently small.) However, there is a rather large range of improvement probabilities that lead to the aforementioned effect, i.e., they typically yield an offspring $y$ from some inferior search point $x'$ of rank $i + 1$.

1.2 Related Work

The analysis of EAs on monotone functions started in 2010 by the work of Doerr, Jansen, Sudholt, Winzen and Zarges [2, 3]. Their contribution was twofold: firstly, they showed that the $(1+1)$-EA, which flips each bit independently with static mutation rate $c/n$, needs time $O(n \log n)$ on all monotone functions if the mutation parameter $c$ is a constant strictly smaller than one. This result was already implicit in [6].

On the other hand, it was also shown in [2, 3] that for large mutation rates, $c > 16$, there are monotone functions for which the $(1+1)$-EA needs exponential time. The construction of hard monotone functions in [2, 3] was later simplified by Lengler and Steger [16], who improved the range for $c$ from $c > 16$ to $c > c_0 = 2.13...$. Their construction was later called HOTTOPIC functions in [12], and it will also be the basis for the results in this paper.

For a long time, it was an open question whether $c = 1$ is a threshold at which the runtime switches from polynomial to exponential. On the presumed threshold $c = 1$, a bound of $O(n^{3/2})$ was known due to Jansen [6], but it was unclear whether the runtime is quasilinear. Finally, Lengler, Martinsson and Steger [14] could show that $c = 1$ is not a threshold, showing by an information compression argument an $O(n \log^2 n)$ bound for all $c \in [1, 1 + \varepsilon]$ for some $\varepsilon > 0$.

Recently, the limits of our understanding of monotone functions were pushed significantly by Lengler [12, 13], who analyzed monotone functions for a manifold of other evolutionary and genetic algorithms. In particular, he analyzed the algorithms on HOTTOPIC functions, and found sharp thresholds in the parameters, such that on one side of the threshold the runtime on HOTTOPIC was $O(n \log n)$, while on the other side of the threshold it was exponential. These algorithms include the $(1+1)$-EA, the $(1+\lambda)$-EA, the $(\mu+1)$-EA, for which the threshold condition was $c < c_0$, where $c_0 = 2.13...$ and it further included the $(1+(\lambda,\lambda))$-GA, and the so-called ‘fast $(1+1)$-EA’ and ‘fast $(1+\lambda)$-EA’. Surprisingly, for the genetic algorithms $(\mu+1)$-GA and the ‘fast $(\mu+1)$-GA’, any parameter range leads to runtime $O(n \log n)$ on HOTTOPIC if the population size $\mu$ is large enough, showing that crossover is strongly beneficial in these cases.

For some of the algorithms, Lengler in [12, 13] also complemented the results on HOTTOPIC functions by statements asserting that for less aggressive choices
of the parameters the algorithms optimize every monotone function efficiently. For example, he proved that for mutation parameter $c < 1$ and for every constant $\lambda \in \mathbb{N}$, with high probability the $(1 + \lambda)$-EA optimizes every monotone function in $O(n \log n)$ steps. Analogous statements were proven for the ‘fast $(1 + 1)$-EA’ and ‘fast $(1 + \lambda)$-EA’, and for the $(1 + (\lambda, \lambda))$-GA, but the condition $c < 1$ needs to be replaced by analogous conditions on the parameters of the respective algorithms. Moreover, in the case of the ‘fast $(1 + \lambda)$-EA’, the result was only proven if the algorithm starts sufficiently close to the optimum. Lengler did not prove any results for general monotone functions for the population-based algorithms $(\mu + 1)$-EA and $(\mu + 1)$-GA, and for their ‘fast’ counterparts. Our result shows that at least for the $(\mu + 1)$-EA, this gap had a good reason. As mentioned before, we will show that for every (constant) mutation parameter $c > 0$, there are monotone functions on which the $(\mu + 1)$-EA needs superpolynomial time if the population size $\mu$ is larger than some constant $\mu_0 = \mu_0(c)$. It also shows that the $(\mu + 1)$-EA and the $(1 + \lambda)$-EA behave completely differently on the class of monotone functions, since the $(1 + \lambda)$-EA is efficient for all constant $\lambda$ whenever $c < 1$.

Surprisingly, our instance of a hard monotone function is again a HotTopic function. This may appear contradictory to the result in [12, 13] that the $(\mu + 1)$-EA is efficient on HotTopic functions if $c < c_0$. The reason why there is no contradiction is that all the results in [12, 13] on HotTopic come with an important catch. The HotTopic functions come with several parameters, and we will give the formal definition and a more detailed discussion in Section 2.3. For now it suffices to know that one of the parameters, $\varepsilon$, essentially determines how close the algorithm needs to come to the optimum before the fitness function starts switching between different hot topics. In [12, 13], only small values of $\varepsilon$ were considered. More precisely, it was shown that for every $\mu \in \mathbb{N}$ there is an $\varepsilon_0 > 0$ such that the results for the $(\mu + 1)$-EA hold for all HotTopic functions with parameter $\varepsilon \leq \varepsilon_0$, and there were similar restrictions for other parameters of the HotTopic function. In a nutshell, the effect of switching hot topics was only studied close to the optimum. Arguably, this was a natural approach since usually the hardest region for optimization is close to the optimum. In this paper, we consider HotTopic functions in a different parameter regime: we study relatively large values of the parameter $\varepsilon$, which is a regime of the HotTopic functions in which the action happens far away from the optimum. Consequently, the results from [12, 13] on the $(\mu + 1)$-EA on HotTopic do not carry over to the version of HotTopic functions that we consider in this paper. We stress this point to resolve the apparent contradiction between our results and the results in [12, 13].

The above discussion also shows a rather uncommon phenomenon. Consider a small mutation parameter, e.g., $c = 1/2$. Our results show that the $(\mu + 1)$-EA fails to make progress if the HotTopic function starts switching hot topics far away from the optimum. On the other hand, by the results in [12], the $(\mu + 1)$-EA is not deceived if the HotTopic function starts switching hot topics close to the optimum. Thus, we have found an example where optimization close to the optimum is easier than optimization far away from the optimum, quite the
opposite of the usual behavior of algorithms. This strange effect occurs because the problem of the \((\mu + 1)\)-EA arises from having a non-trivial population. However, close to the optimum, progress is so hard that the population tends to degenerate into multiple copies of a single search point, which effectively decreases the population size to one and thus eliminates the problem (see also the discussion in Section 1.1 above).

Most other work on population-based algorithms has shown benefits of larger population sizes, especially when crossover is used \([4, 8, 9, 17]\). The only exception in which a population has theoretically been proven to be severely disadvantageous is on Ignoble Trails. This rather specific function has been carefully designed to lead into a trap for crossover operators \([18]\), and it is deceptive for \(\mu = 2\) if crossover is used, but not for \(\mu = 1\). Arguably, the HotTopic functions are also rather artificial, although they were not specifically designed to be deceptive for populations. However, regarding the larger and more natural framework of monotone functions, our results imply that a \((\mu + 1)\)-EA with mutation parameter \(c = 1\) does not optimize all monotone functions efficiently if \(\mu\) is too large, while the corresponding \((1 + 1)\)-EA is efficient.

Moreover, Lengler and Schaller pointed out an interesting connection between HotTopic functions and a dynamic optimization problem in \([15]\), which is arguably more natural. In that paper, the algorithm should optimize a linear function with positive weights, but the weights of the objective function are re-drawn each round (independently and identically distributed). This setting is similar to monotone functions, since a one-bit is always preferable over a zero-bit, and the all-one string is always the global optimum. However, the weight of each bit changes from round to round, which somewhat resembles that the HotTopic function switches between different hot topics as the algorithm progresses. In \([15]\) the \((1 + 1)\)-EA was studied, and the behavior in the dynamic setting is very similar to the behavior on HotTopic functions. It remains open whether the effects observed in our paper carry over to this dynamic setting.

2 Preliminaries and Definitions

2.1 Notation

Throughout the paper we will assume that \(f : \{0, 1\}^n \to \mathbb{R}\) is a monotone function, i.e., for every \(x, y \in \{0, 1\}^n\) with \(x \neq y\) and such that \(x_i \geq y_i\) for all \(1 \leq i \leq n\) it holds \(f(x) > f(y)\). We will consider algorithms that try to maximize \(f\), and we will mostly focus on the runtime of an algorithm, which we define as the number of function evaluations before the first evaluation of the global maximum of \(f\).

For \(n \in \mathbb{N}\), we denote \([n] := \{1, \ldots, n\}\). We use the notation \(x = y \pm z\) to abbreviate \(x \in [y - z, y + z]\). For a search point \(x\), we write \(\text{OM}(x)\) for the ONEMAX-value of \(x\), i.e., the number of one-bits in \(x\). For \(x \in \{0, 1\}^n\) and \(\emptyset \neq I \subseteq [n]\), we denote by \(d(I, x) := |\{i \in I \mid x_i = 0\}|/|I|\) the density of zero-bits in \(I\). In particular, \(d([n], x) = 1 - \text{OM}(x)/n\). Landau notation like
$O(n), o(n), \ldots$ is with respect to $n \to \infty$. An event $E = E(n)$ holds \textit{with high probability or whp} if $\Pr[E(n)] \to 1$ for $n \to \infty$. A function $f : \mathbb{N} \to \mathbb{R}$ grows \textit{quasi-exponentially} if there is $\delta > 0$ such that $f(x) = \exp\{\Omega(n^\delta)\}$, and it grows \textit{quasilinearly} if there is $C > 0$ such that $f(x) = O(n \log^C n)$.

Throughout the paper we will use $n$ for the dimension of the search space, $\mu$ for the population size, and $c$ for the mutation parameter. We will assume that $c = \Theta(1)$, but we will allow that $\mu = \mu(n)$ depends on $n$.

### 2.2 Algorithm

We will consider the $(\mu + 1)$-EA with population size $\mu \in \mathbb{N}$ and mutation parameter $c > 0$ for maximizing a pseudo-boolean fitness function $f : \{0, 1\}^n \to \mathbb{R}$. This algorithm maintains a population of $\mu$ search points. In each round, it picks one of these search points uniformly at random, the \textit{parent} $x^t$ for this round. From this parent it creates an \textit{offspring} $y^t$ by flipping each bit in $x^t$ independently with probability $c/n$, and adds it to the population. From the $\mu + 1$ search points, it then discards the one with lowest fitness from the population, breaking ties randomly. We will always assume that the mutation parameter $c$ is a constant independent of $n$, but the population size $\mu = \mu(n)$ may depend on $n$.

\begin{algorithm}
1 \textbf{Initialization:} \\
2 $S_0 \leftarrow \emptyset$; \\
3 \hspace{1em} for $i = 1, \ldots, \mu$ do \\
4 \hspace{2em} Sample $x^{(0,i)}$ uniformly at random from $\{0, 1\}^n$; \\
5 \hspace{2em} $S_0 \leftarrow S_0 \cup \{x^{(0,i)}\}$; \\
6 \textbf{Optimization:} \\
7 \hspace{1em} for $t = 1, 2, 3, \ldots$ do \\
8 \hspace{2em} \textbf{Mutation:} \\
9 \hspace{3em} Choose $x^t \in S_{t-1}$ uniformly at random; \\
10 \hspace{3em} Create $y^t$ by flipping each bit in $x^t$ independently with probability $c/n$; \\
11 \hspace{1em} \textbf{Selection:} \\
12 \hspace{2em} Set $S_t \leftarrow S_{t-1} \cup \{y^t\}$; \\
13 \hspace{2em} Select $x \in \arg \min \{f(x) \mid x \in S_t\}$ (break ties randomly) and update $S_t \leftarrow S_t \setminus \{x\}$;
\end{algorithm}
2.3 HotTopic Functions

In this section we give the construction of hard monotone functions by Lengler and Steger [16], following closely the exposition in [12]. The functions come with five parameters $n \in \mathbb{N}$, $0 < \beta < \alpha < 1$, $0 < \varepsilon < 1$, and $L \in \mathbb{N}$, and they are given by a randomized construction. Following [12], we call the corresponding function $\text{HotTopic}_{n,\alpha,\beta,\varepsilon,L} = HT_{n,\alpha,\beta,\varepsilon,L} = HT$.

For $1 \leq i \leq L$ we choose sets $A_i \subseteq [n]$ of size $\alpha n$ independently and uniformly at random, and we choose subsets $B_i \subseteq A_i$ of size $\beta n$ uniformly at random. We define the level $\ell(x)$ of a search point $x \in \{0,1\}^n$ by

$$\ell(x) := \max \{ \ell' \in [L] : |\{j \in B_{\ell'} : x_j = 0\}| \leq \varepsilon \beta n\},$$

where we set $\ell(x) = 0$, if no such $\ell'$ exists. Then we define $f : \{0,1\}^n \rightarrow \mathbb{R}$ as follows:

$$HT(x) := \ell(x) \cdot n^2 + \sum_{i \in A_{\ell(x)+1}} x_i \cdot n + \sum_{i \in R_{\ell(x)+1}} x_i,$$

where $R_{\ell(x)+1} := [n] \setminus A_{\ell(x)+1}$, and where we set $A_{L+1} := B_{L+1} := \emptyset$. One easily checks that this function is monotone [12].

So the set $A_{\ell+1}$ defines the hot topic while the algorithm is at level $\ell$, where the level is determined by the sets $B_i$. Following up on the discussion in the introduction, observe that the level $\ell$ increases if the density of zero-bits in $B_{\ell'}$ drops below $\varepsilon$ for some $\ell' > \ell$. From the analysis we will see that with high probability this only happens if the density of one-bits in $A_{\ell+1}$ and in the whole string is also roughly $\varepsilon$, up to some constant factors. Hence, the parameter $\varepsilon$ determines how far away the algorithm is from the optimum when the level changes.

Throughout the paper we will assume that $\alpha$ and $\beta$ are independent of $n$, whereas we will choose small constants $\eta, \rho > 0$ and set $\varepsilon = \mu^{-1+\eta}$ and $L = \exp\{\rho n / \log^2 \mu\}$, i.e., $\varepsilon$ and $L$ may depend of $n$, since we also allow $\mu$ to depend on $n$.\footnote{In the papers [12,13,16] the parameter $L$ was replaced by a constant parameter $\rho$ such that $L = e^{\rho n}$. This had the advantage that their parameters were all independent of $n$, but since our parameters depend on $n$ anyway, it is more convenient to use the parameter $L$. However, both versions are equivalent.}

2.4 Tools

To obtain good tail bounds, we often apply Chernoff’s inequality.

**Theorem 1 (Chernoff Bound [1]).** Let $X_1, \ldots, X_n$ be independent random variables (not necessarily i.i.d.) that take values in $[0,1]$. Let $S = \sum_{i=1}^n X_i$ and $\mu = \mathbb{E}[S]$. Then for all $0 \leq \delta \leq 1$,

$$\Pr[S \leq (1-\delta)\mu] \leq e^{-\delta^2\mu/2},$$

and for all $\delta \geq 0$,

$$\Pr[S \geq (1+\delta)\mu] \leq e^{-\min(\delta^2,\delta)\mu/3}.$$
Since we consider the \((\mu + 1)\)-EA with mutation rate \(c/n\), the number of bit flips in a mutation is a Binomial random variable with parameters \(n\) and \(c/n\). We will use frequently that this distribution can be approximated by a Poisson distribution with expectation \(c\). This approximation is quantified by Le Cam’s theorem.

**Theorem 2** (Le Cam, Proposition 1 in [10]). Suppose \(X_1, \ldots, X_n\) are independent Bernoulli random variables s.t. \(\Pr[X_i = 1] = p_i\) for \(i \in [n]\), \(\lambda_n = \sum_{i \in [n]} p_i\) and \(S_n = \sum_{i \in [n]} X_i\). Then

\[
\sum_{k=0}^{\infty} \left| \Pr[S_n = k] - \frac{\lambda_n^k e^{-\lambda_n}}{k!} \right| < 2 \sum_{i=1}^{n} p_i^2.
\]

In particular, if there is a constant \(c > 0\) such that \(p_i = c/n\) for all \(i \in [n]\), then \(\Pr[S_n = k] = c^k e^{-c}/k! \pm O(1/n)\) for all \(k \geq 0\).

We will also use the following lemma, which estimates the probability to improve on the current hot topic if the density of zero-bits is roughly \(\varepsilon\).

**Lemma 3.** Let \(\alpha, c > 0\) be constants. Consider a set \(A \subseteq [n]\) of size \(\alpha n\), and consider any \(x \in \{0, 1\}^n\) such that \(\varepsilon/2 \leq d(A, x) \leq 2\varepsilon\) for some \(0 < \varepsilon = \varepsilon(n) < 1\). Then the probability \(p_I\) that a standard bit mutation with rate \(c/n\) strictly increases the number of one-bits in \(A\) satisfies

\[
\frac{\varepsilon}{2} \cdot \alpha c e^{-\alpha c} - O(1/n) \leq p_I \leq 2\varepsilon\alpha c + O(\varepsilon^2 + 1/n),
\]

where the \(O\)-notation is with respect to \(n \to \infty\). The probability \(p_R\) that the number of one-bits in \(A\) does not decrease is bounded by

\[
e^{-\alpha c} - O(1/n) \leq p_R \leq e^{-\alpha c} + O(\varepsilon + 1/n).
\]

We can use Le Cam’s theorem to prove Lemma 3.

**Proof of Lemma 3.** For the lower bound we observe that

\[p_I \geq \Pr[\text{flip 1 zero-bit and 0 one-bits in } A].\]

Assume that the density is exactly \(d\). Since the number of zero-bit flips is independent of the number of one-bit flips, we may bound separately by Le Cam’s theorem, \(\Pr[\text{flip 1 zero-bit}] \geq d \alpha c e^{-\alpha c} - O(1/n)\) and \(\Pr[\text{flip 0 one-bits}] \geq e^{-(1-d)\alpha c} - O(1/n)\). Multiplying both terms yields \(d \alpha c e^{-\alpha c} - O(1/n)\), and the lower bound follows by plugging in \(d \geq \varepsilon/2\).

For the upper bound, we use

\[p_I \leq \Pr[\text{flip 1 zero-bit and 0 one-bits}] + \Pr[\text{flip at least 2 zero-bits}].\]

We may bound the first term as before by \(d \alpha c e^{-\alpha c} + O(1/n)\), but now we plug in the bound \(d \leq 2\varepsilon\) and \(e^{-\alpha c} \leq 1\). This gives the main term. It remains to
show that the second term is $O(\varepsilon^2 + 1/n)$, and this follows immediately by Le Cam’s theorem, since
\[
\sum_{k=2}^{\infty} \Pr[\text{flip } k \text{ zero-bits}] \leq O(1/n) + \sum_{k=2}^{\infty} \frac{(d\alpha c)^k e^{-d\alpha}}{k!} = O(d^2 + 1/n).
\]
For the probability $p_R$, one way of creating an offspring of the same rank is to flip no bits at all in $A$, which is $e^{-\alpha c} + O(1/n)$ by Le Cam’s theorem. This already proves the lower bound. All other possibilities to leave the rank unchanged require to flip at least one zero-bit in $A$, which has probability $O(\varepsilon)$ by the same arguments as above.

3 Formal Statement of the Result

The main result of this paper is the following.

**Theorem 4.** For every constant $c > 0$ and $0 < \beta < \alpha < 1$ there exist constants $\mu_0 = \mu_0(c) \in \mathbb{N}$ and $\eta, \rho > 0$ such that the following holds for all $\mu_0 \leq \mu \leq n$ where $n$ is sufficiently large. Consider the $(\mu + 1)$-EA with population size $\mu$ and mutation rate $c/n$ on the $n$-bit HotTopic function $HT_{n, \alpha, \beta, \varepsilon, L}$, where $\varepsilon = \mu^{-1+\eta}$ and $L = \lfloor \exp\{\rho n / \log^2 \mu\} \rfloor$. Then with high probability the $(\mu + 1)$-EA visits every level of the HT function at least once. In particular, it needs at least $L$ steps to find the optimum, with high probability and in expectation.

In particular, if $\mu \geq \mu_0$ is a constant (independent of $n$) then with high probability the optimization time is exponential.

We remark that the requirement $\mu \leq n$ is not tight, and we conjecture that the runtime is always superpolynomial for $\mu \geq \mu_0$, also for much larger values of $\mu$. However, we did not undertake big efforts to extend the range of $\mu$ since we do not feel that it adds much to the statement. For larger values of $\mu$, e.g., $\mu = n^2$, our proof does not go through unmodified. With our definition of $\varepsilon = \mu^{-1+\eta}$, we only get error probabilities of the form $\exp\{-\Omega(\varepsilon n / \log^2 \mu)\}$, which are not $o(1)$ if e.g. $\mu = n^2$. Hence we would need to choose larger values of $\varepsilon$, and then we lose a very convenient property, namely that for every fixed $i$, with high probability no individual of rank at most $i - 1$ creates an individual of rank at least $i + 1$. To avoid these complications, we only consider $\mu \leq n$.

4 Proof Overview

The next three sections are devoted to proving Theorem 4. The key ingredient is to analyze the drift of the density $d([n], x)$ for search points $x$ which have roughly density $\varepsilon$. We first start by giving an informal overview, and by discussing similarities and differences to the situation in [16] and [12].

We will analyze the algorithm in the regime where the fittest search point $x$ in the population satisfies
\[
d(A\ell + 1, x) \in [\varepsilon/2, 2\varepsilon] \quad \text{and} \quad d(R\ell + 1, x) \in [\varepsilon/2, 2\varepsilon],
\]
where \( \ell = \ell(x) \) and \( \varepsilon = \mu^{-1+\eta} \) is the parameter of the HotTopic function. It will turn out that for large \( \mu \), the algorithm already needs quasi-exponential time to escape this situation.

The main idea is similar to \([12, 16]\), in which the \((1 + 1)\)-EA and other algorithms were analyzed. We first sketch the main argument for the \((1 + 1)\)-EA, and explain afterwards which parts must be replaced by new arguments. The crucial ingredient is that while the density \( d(A_{\ell+1}, x) \) of zero-bits on the hot topic decreases from \( 2\varepsilon \) to \( \varepsilon \), the total density \( d([n], x) \) has a positive drift, i.e., a drift away from the optimum. Moreover, the probability to change \( k \) bits in one step has a tail that decays exponentially with \( k \). Therefore, it was shown that with high probability \( d([n], x) \) stays above \( \varepsilon + \gamma \) for an exponential number of steps, where \( \gamma \) is a small constant. Then it was argued that as long as \( d([n], x) \) stays bounded away from \( \varepsilon \), it is exponentially unlikely that the level ever increases by more than one. Since there are an exponential number of levels, this implies an exponential runtime.

The analysis of \((\mu+1)\)-EA and \((\mu+1)\)-GA for constant \( \mu \) in \([12]\) was obtained by reducing it to the analysis of a related \((1 + 1)\) algorithm. This was possible since the choice of parameters in \([12]\) (choosing the parameter \( \varepsilon = \varepsilon(\mu) \) sufficiently small) made the algorithm operate close to the optimum. In this range, there are only few zero-bits, and thus it is rather unlikely that a mutation improves the fitness. On the other hand, there is always a constant probability (if \( \mu \) is constant) to create a copy of the fittest individual. In such a situation, the population degenerates frequently into a collection of copies of a single search point. Thus, the population-based algorithms behave similarly to a \((1 + 1)\) algorithm. This \((1 + 1)\) algorithm has essentially the same mutation parameter as the \((\mu + 1)\)-EA, while for the \((\mu + 1)\)-GA it has a much smaller mutation parameter (less than one), which is the reason why the \((\mu + 1)\)-GA is efficient on all HotTopic instances with small parameter \( \varepsilon \). For us, the situation is more complex since we consider larger values of \( \varepsilon \). As a consequence, it is easier to find a search point with better fitness, and the population does not collapse. Hence, it is not possible to represent the population by a single point.

Instead, we proceed as follows. Fix a fitness level \( \ell \), and consider the auxiliary fitness function

\[
\ell(x) := n \sum_{j \in A_{\ell+1}} x_j + \sum_{j \in R_{\ell+1}} x_j.
\]  

(4)

We will first study the behavior of the \((\mu + 1)\)-EA on \( \ell \). Considering this fitness function is essentially the same as assuming that the level remains the same. We will see in the end that this assumption is justified, by the same arguments as in \([12, 16]\). For a search point \( x \), we define the rank \( \text{rk}(x) := |\{ j \in A_{\ell+1} \mid x_j = 1 \}| \) of \( x \) as the number of correct bits in the current hot topic. Note that by construction of \( \ell \), a search point with higher rank is always fitter than a search point with smaller rank.

Now we define \( X_i \) to be the set of search points of rank \( i \) that are visited by the \((\mu + 1)\)-EA, and we define \( Z_i \) to be the OneMax-value (the number of
one-bits) of the last search point in $X_i$ that the algorithm deletes out of its population. Note that due to elitist selection, this search point is also (one of) the fittest search point(s) in $X_i$ that the algorithm ever visits, and hence it has the largest OneMax-value among all search points in $X_i$ that the algorithm ever visits. Then our goal is to show that $\mathbb{E}[Z_{i+1} - Z_i] = -\Omega(1)$, under the assumption that the population satisfies (3), i.e., that the density of the fittest search point is close to $\varepsilon$. This assumption can be justified by a coupling argument as in [12, 16]. Computing the drift of $Z_i$ is the heart of our proof, and the main technical contribution of this paper. In fact, to simplify the analysis we only prove the slightly weaker statement that $\mathbb{E}[Z_{i+K} - Z_i] = -\Omega(1)$ for a suitable constant $K$, which is equally suited. Once we have established this negative drift, the remainder of the proof as in [12, 16] carries over almost unchanged.

To estimate the drift $\Delta := \mathbb{E}[Z_{i+K} - Z_i]$, we will assume for this exposition that $\mu = \omega(1)$, so that we may use $O$-notation. (In the formal proof we will use the weaker assumption $\mu \geq \mu_0$ for a sufficiently large constant $\mu_0 = \mu_0(c).$) We distinguish between good and bad events. Good events will represent the typical situation: they will occur with high probability, and if they occur $K$ times in a row, then it will deterministically follow that $Z_{i+K} - Z_i \leq -1$. On the other hand, bad events may lead to a positive difference, but they are unlikely and thus they contribute only a lower order term to the drift. We will discriminate two types of bad events. Firstly, we will show that the probability $\Pr[Z_{i+K} - Z_i > \lambda \log \mu]$ drops exponentially in $\lambda$. This implies that the events in which $Z_{i+K} - Z_i > \log^{3/2} \mu$ contribute at most a term $o(1)$ to the drift, where the exponent $3/2$ is rather arbitrary. Hence, we can restrict ourselves to the case that $Z_{i+K} - Z_i \leq \log^{3/2} \mu$. Now assume that we have any event of probability $o(\log^{-3/2} \mu)$. In the case $Z_{i+K} - Z_i \leq \log^{3/2} \mu$, this event can contribute at most a $o(1)$ term to the drift. Hence, we may declare any such event as a bad event, and conclude that all bad events together only contribute a $o(1)$ term to the drift.

As we have argued, we may neglect any event with probability $o(\log^{-3/2} \mu)$. This is a rather large error probability, which allows us to dub many events as ‘bad’, and to use rather coarse estimates on the error probability. In particular, in several cases we may use Chebyshev’s inequality to get simple estimates of order roughly $\log^{-2} \mu$ (in the proofs we give us a bit of slack and show $O(\log^{-7/4} \mu)$ instead), although more sophisticated methods might give better error bounds. We conclude this overview by describing how a good event, and thus a typical situation, looks like. In what follows, all claims hold with probability at least $1 - o(\log^{-3/2} \mu)$.

Let us call $t_i$ the first round in which an individual of rank at least $i$ is created, and $T_i$ the round in which the last individual of rank at most $i$ is eliminated. Then typically $T_i - t_i = O(\mu \log \mu) \cap \Omega(\mu)$. Let $|X_i| = |X_i(t)|$ denote the number of search points in the population of rank $i$. We want to study the family forest $F_i$ of $X_i$, which is closely related to the family trees and family graphs that have been used in other work on population-based EAs, e.g. [11, 19, 20]. The vertices of this forest are all individuals of rank at least $i$
that are ever included into the population. A vertex is called a root if its parent has rank less than \( i \). Otherwise, the forest structure reflects the creation of the search points, i.e., vertex \( u \) is a child of vertex \( v \) if the individual \( u \) was created by a mutation of \( v \).

As \( X_i \) grows, eventually the first few search points of rank \( i + 1 \) are created, and form the first roots of the family forest. Then the forest starts growing, both because new roots may appear and because the vertices in the forest may create offspring. At some point we have \( |X_{i+1}| = \mu^\delta \) for some (suitably small) \( \delta > 0 \). At this point, we still have typically \( |X_i| = O(\mu^\delta/\varepsilon) = O(\mu^{1+\delta-n}) = o(\mu) \), where the latter holds if \( \delta \) is small enough. Moreover, at this point there are no search points of rank strictly larger than \( i + 1 \). The sets \( X_i \) and \( X_{i+1} \) both continue to grow with roughly the same speed until the search points of rank at most \( i - 1 \) are eliminated from the population. Afterwards, the search points of rank \( i \) are eliminated from the population, until only search points of rank at least \( i + 1 \) remain. Crucially, up to this point every search point of rank at least \( i + 1 \) is accepted into the population. In other words, there is no selective pressure on the search points of rank \( i + 1 \), and every mutation of a search point of rank \( i + 1 \) enters the family tree, as long as the rank \( i + 1 \) is preserved. Therefore, we can sandwich the family forest \( F_{i+1} \) of rank \( i + 1 \) up to this point between two random forests \( F', F'' \) which are obtained by certain forest growth processes (generally known as recursive trees) in which no vertex is ever eliminated and all vertices continue to spawn offspring with a fixed rate.

We want to understand the set of individuals in \( X_{i+1} \) that spawn offspring in \( X_{i+2} \), and thus spawn the roots for the family forest \( F_{i+2} \). As before we can argue that no individuals of rank at least \( i + 2 \) are created before the family forest of rank \( i + 1 \) reaches size \( \mu^\delta \), so the family forest \( F_{i+1} \) is bounded from below by a random forest \( F' \) of at least this size. Moreover, we can show that the time \( T_{i+1} \) at which all individuals of rank \( i + 1 \) are eliminated from the population satisfies \( T_{i+1} - t_{i+1} \leq C \mu \log \mu \) for a suitable constant \( C > 0 \). Hence, the \( F_{i+1} \) is bounded from above by the random forest \( F'' \) at time \( t_{i+1} + C \mu \log \mu \). This forest is only polynomially large in \( \mu \).

The recursive trees that we use to sandwich \( F_{i+1} \) are well understood. In particular, it is known that even in \( F'' \) only a small fraction \( \mu^\delta \) of the vertices are in depth at most \( \delta \log \mu \), where \( \delta, \phi > 0 \) are suitable constants. Since each such vertex creates an offspring of strictly larger rank with probability \( \varepsilon/\mu \) per round, the expected number of offspring of rank \( i + 2 \) of these vertices is at most \( O(\mu^\delta \varepsilon/\mu \cdot (T_{i+1} - t_{i+1})) \). With the right choice of parameters, this is \( \mu^{-O(1)} \), and we may conclude that no vertices of depth at most \( \phi \log \mu \) create roots of rank \( i + 2 \). On the other hand, since we do not truncate any vertices in the creation of \( F'' \), they are obtained from their parents by unbiased mutations of \( [n] \setminus A_\ell \), and we can show that most (all but at most \( \mu^\delta \)) vertices of depth at least \( \phi \log \mu \) in \( F'' \) have accumulated \( \epsilon' \log \mu \) more bad than good bit-flips when compared to their roots, for a suitable \( \epsilon' > 0 \). For the \( \mu^\delta \) exceptional vertices, none of them will create a root of rank \( i + 2 \) in \( T_{i+1} - t_{i+1} \) rounds, even if they are in \( F_{i+1} \).

To summarize, good events consist of the following four main points. Firstly,
no vertex of rank at most \( i \) creates an offspring of rank at least \( i + 2 \). Secondly, every vertex in \( X_{i+1} \) that creates an offspring in \( X_{i+2} \) has at least depth \( \phi \log \mu \) in the family forest. Thirdly, every vertex in \( X_{i+1} \) of depth at least \( \phi \log \mu \) that creates an offspring in \( X_{i+2} \) has a OneMax value that is at least \( c' \log \mu \) smaller than that of its root. Finally, we also require that no vertex in \( F'' \) exceeds the OneMax value of its root by more than \( C \log \mu \), for some \( C > 0 \). The complete list in the proof contains even more requirements, but these four already imply a decline in \( Z_i \) if they hold over \( K \) consecutive steps. In this case, inductively the OneMax values of all roots in \( F_{i+K} \) are at most \( Z_i - Kc' \log \mu \). Moreover, \( Z_{i+K} \) exceeds the OneMax value of the corresponding root in \( X_{i+K} \) by at most \( C \log \mu \), so we have \( Z_{i+K} \leq Z_i - Kc' \log \mu + C \log \mu \). Choosing \( K \) sufficiently large shows that \( Z_i \) must decrease in these typical situations.

5 Drift of \( Z_i \)

In this main section of the proof, we show that the random variable \( Z_i \) has negative drift. We will use the same notation as in the proof outline. In particular, \( X_i \) denotes the set of all search points of rank \( i \) that the algorithm visits, and \( Z_i \) denotes the OneMax-value of the last search point from \( X_i \) that the algorithm keeps in its population. If \( X_i \) is empty (which, as we will see, is very unlikely), then we set \( Z_i := Z_{i-1} \). Moreover, we define \( X_{\geq i} := \bigcup_{i' \geq i} X_{i'} \), and the definition of terms like \( X_{>i} \) follows analogously. For a given parent individual \( x \), we denote by \( p_I \) (by \( p_R \)) the probability that an offspring of \( x \) has rank which is strictly larger than (at least as large as) the rank of \( x \).

Throughout this section, we fix a level \( \ell \) and consider the \((\mu + 1)\)-EA on the linear function \( f_\ell \) defined in (4). In this section, we will study the case that \( i \in \left( (1 - 2\varepsilon/\alpha n), (1 - \varepsilon/2)\alpha n \right] \), where \( \varepsilon = \mu^{-1+\eta} \). Note that this is a weaker form of Condition (3), i.e., we consider search points for which the density in \( A \) is close to \( \varepsilon \).

5.1 Preliminaries

5.1.1 Growth of \( |X_{\geq i}| \)

In this section we give bounds on the time that the set \( X_{\geq i} \) needs to grow from size 1 to size \( \mu^\kappa \).

**Lemma 5.** For all \( 0 < \alpha < 1 \), \( c > 0 \), \( 0 < \eta < \kappa \leq 1 \), there exists a constant \( \mu_0 \) such that the following holds for all \( \mu_0 \leq \mu \leq n \). Let \( i \in \left( (1 - 2\varepsilon)\alpha n, (1 - \varepsilon/2)\alpha n \right] \), where \( \varepsilon = \mu^{-1+\eta} \). Consider the \((\mu + 1)\)-EA with mutation rate \( c/n \) on the linear function \( f_\ell \). Denote by \( T_i^\kappa = T^\kappa \) the number of rounds until \( |X_{\geq i}| \) reaches \( \mu^\kappa \) after the algorithm visits the first point \( x^i \) in \( X_i \). With probability \( 1 - O(\log^{-7/4} \mu) \),

\[
(1 - O(\log^{-1/8} \mu))(\kappa - \eta)\log \mu \leq T_i^\kappa \leq (1 + O(\log^{-1/8} \mu)\kappa \mu^{\alpha \mu} \mu \log \mu).
\]
Moreover,
\[ E[T^\kappa] \leq (1 + O(\log^{-1} \mu)) \kappa e^{\alpha_c} \mu \log \mu. \]

**Proof.** By the definition of \( f_\ell \), all individuals in \( X_{\geq i} \) are fitter than those in \( X_{<i} \). So no points in \( X_{\geq i} \) will be discarded until \( X_{<i} \) becomes empty, and we are interested in the growth of \( |X_{\geq i}| \) during this period. Let \( T_j \) be the time needed for \( |X_{\geq i}| \) to grow from \( j \) to \( j + 1 \). By definition we have \( T^\kappa = \sum_{j=1}^{\mu - 1} T_j \). Denote by \( x^t \) the point picked by the algorithm in round \( t \) and denote by \( y^t \) its offspring. The probability that both \( x^t \) and \( y^t \) belong to \( X_{\geq i} \) is at least \( p_j = \left( \frac{j}{\mu} \right) \cdot \frac{1}{p_R} \), where \( j \) is the size of \( X_{\geq i} \) at the beginning of round \( t \). It is clear that we can dominate \( T_j \) by the random variable \( \bar{T}_j \) that follows a geometric distribution with parameter \( p_j \). So we have
\[ E[T_j] \leq E[\bar{T}_j] = \frac{1}{p_j} \quad \text{and} \quad \text{Var}[\bar{T}_j] = \frac{1 - p_j}{p_j^2}. \]

We also give an upper bound for \( \text{Var}[T_j] \), which will be useful in the proof of Lemma 6:
\[ \text{Var}[T_j] = E[T^2_j] - E[T_j]^2 \leq \sum_{k=1}^{\infty} k^2 \Pr(T_j = k) \leq \sum_{k=1}^{\infty} k^2 \Pr(\bar{T}_j = k) = \frac{1}{p_j} \quad \text{and} \quad \text{Var}[\bar{T}_j] = O\left( \frac{\mu^2}{j^2} \right). \]

It will turn out helpful to define \( \bar{T}^\kappa = \sum_{j=1}^{\mu - 1} \bar{T}_j \), since we can use it later to apply Chebyshev’s inequality to it. By the bound on \( p_R \) in Lemma 3, it holds that
\[ E[T^\kappa] \leq E[\bar{T}^\kappa] = \sum_{j=1}^{\mu - 1} E[\bar{T}_j] \leq \sum_{j=1}^{\mu - 1} (1 + O(1/m)) \frac{\mu e^{\alpha_c}}{j}. \]

For Harmonic series, we have \( \sum_{j=1}^{m} \frac{1}{j} = \log m + \gamma + O(1/m) \), where \( \gamma \) is the Euler-Mascheroni constant and \( \log \) denotes the natural logarithm. Therefore,
\[ E[\bar{T}^\kappa] \leq (1 + O(\log^{-1} \mu)) \kappa e^{\alpha_c} \mu \log \mu. \]

Next we derive a lower bound of \( E[T^\kappa] \). Consider the probability that \( X_{\geq i} \) gets a new offspring \( y^t \) in a round where \( |X_{\geq i}| = j \):
\[ \Pr[y^t \in X_{\geq i}] = \Pr[x^t \in X_{\geq i} \land y^t \in X_{\geq i}] + \Pr[x^t \notin X_{\geq i} \land y^t \in X_{\geq i}] \leq j/\mu \cdot 1 + (\mu - j)/\mu \cdot p_t \leq j/\mu + p_t. \]

Let \( p'_t = j/\mu + p_t \), similarly as for the upper bound on \( T^\kappa \), we can couple \( T^\kappa \) with a random variable \( \bar{T}^\kappa = \sum_{j=1}^{\mu - 1} \bar{T}_j \), where the \( \bar{T}_j \) are independent and
geometrically distributed with parameter $p'_j$, respectively. Then

$$E[T^\kappa] \geq E[\hat{T}^\kappa] \geq \sum_{j=1}^{\mu^\kappa-1} \frac{1}{p'_j} = \sum_{j=1}^{\mu^\kappa-1} \frac{\mu}{j + \mu p_I}$$

$$\geq \sum_{j=1}^{\mu^\kappa-1} \frac{\mu}{j + \lceil \mu p_I \rceil} = \sum_{j=1}^{\mu^\kappa-1} \frac{\mu}{j} - \sum_{j'=1}^{\lceil \mu p_I \rceil} \frac{\mu}{j'}$$

$$= \mu \log (\mu^\kappa - 1 + \lceil \mu p_I \rceil) - \log (\lceil \mu p_I \rceil) - O(1/\lceil \mu p_I \rceil))$$

$$\geq \mu \log (\mu^\kappa/\lceil \mu p_I \rceil) - O(\mu/\lceil \mu p_I \rceil).$$

Since $\varepsilon = \mu^{-1+n}$ for $0 < \eta < \kappa$, we have $\lceil \mu p_I \rceil = \Theta(\mu^\eta)$. Hence,

$$E[T^\kappa] \geq E[\hat{T}^\kappa] \geq (1 - O(\log^{-1} \mu))(\kappa - \eta) \mu \log \mu.$$

Since $\hat{T}^\kappa$ is the sum of independent random variables, we can bound its variances by bounding $p_j = j \mu R / \mu$ using Lemma 3:

$$\text{Var}[\hat{T}^\kappa] = \sum_{j=1}^{\mu^\kappa} \text{Var}[\hat{T}_j] \leq \sum_{j=1}^{\mu^\kappa} \left( \frac{1}{p'_j^2} - \frac{1}{p_j} \right) \leq \sum_{j=1}^{\mu^\kappa} \frac{1}{p'_j^2}$$

$$\leq (1 + O(1/n)) \sum_{j=1}^{\mu^\kappa} \frac{\mu^{2\alpha c}}{j^2} = (1 + O(1/n)) \frac{\mu^{2\alpha c} \pi^2}{6}$$

where the last step follows $\sum_{j=1}^{\infty} 1/j^2 = \pi^2/6$.

Given the bounds on $E[\hat{T}^\kappa]$ and $\text{Var}[\hat{T}^\kappa]$, by Chebyshev’s inequality, for any $\sigma > 0$,

$$\Pr [\hat{T}^\kappa - E[\hat{T}^\kappa] \geq \sigma \mu \log \mu] \leq \Pr [\hat{T}^\kappa - E[\hat{T}^\kappa] \geq \sigma \mu \log \mu]$$

$$\leq \frac{\text{Var}[\hat{T}^\kappa]}{\sigma^2 \mu^2 \log^2 \mu} \leq \frac{(1 + O(1/n)) \frac{\pi^2}{6}}{\sigma^2 \mu^2 \log^2 \mu}.$$ Choosing $\sigma = \log^{-1/8} \mu$ shows the upper bound. For the lower bound, we proceed similarly with $\hat{T}^\kappa$ instead of $\hat{T}^\kappa$. We use that $p'_j \geq j/\mu = p_j / p_R$, and thus $\text{Var}[\hat{T}_j] \leq \text{Var}[\hat{T}_j]/p_R^2$. Hence, $\text{Var}[\hat{T}^\kappa]/p_R^2 = O(\mu^2)$, and the lower bound follows analogously to the upper bound. This concludes the proof.

5.1.2 Improvements

In the following lemma, we give a lower bound on $|X_{2i+1}|$ when $X_{2i}$ reaches a certain size.

**Lemma 6.** Let $\alpha, \kappa \in (0, 1)$, $c > 0$, $\eta < 1/3$ be constants such that $\kappa > 1 - \eta/6$. Consider the $(\mu + 1)$-EA with $\mu \leq n$ and mutation rate $c/n$ on the linear function $f_1$. Let $\varepsilon = \mu^{-1+n}$ and let $i \in [(1 - 2\varepsilon) \alpha n, (1 - \varepsilon/2) \alpha n]$. Denote
by $Y_{i+1}^\kappa = Y^\kappa$ the size of $X_{\geq i+1}$ when $|X_{\geq i}|$ reaches $\mu^\kappa$. Then with probability $1 - O(\mu^{6(1-\kappa)-\eta} \log \mu)$,

$$Y^\kappa = \Omega(\varepsilon \mu^{3\kappa - 2}) = \Omega(\mu^{3(\kappa - 1) + \eta}) = \mu^{O(1)}.$$  

For the proof, Let $Y_j$ be an indicator random variable denoting whether a parent from $X_{\geq i}$ is picked and produces an offspring in $X_{\geq i+1}$, given that $|X_{\geq i}| = j$ at the beginning of this round. Then $Y^\kappa$ dominates the number $\bar{Y}^\kappa$ of offspring that are produced in this way, i.e.,

$$Y^\kappa \geq \bar{Y}^\kappa := \sum_{j=1}^{\mu^\kappa - 1} \sum_{j'=1}^{T_j - 1} Y_j,$$

where $T_j$ is the time needed for $|X_{\geq i}|$ to grow from $j$ to $j+1$ and the inner sum denotes the sum of $T_j - 1$ independent random variables that are identically distributed as $Y_j$. Then we can bound the expectation and variance of $\bar{Y}^\kappa$, and obtain a lower bound for it (and for $Y^\kappa$) by Chebyshev’s inequality. We omit the details due to space restrictions.

### 5.2 Tail Bounds

In this section, we will give rather loose tail bounds to show that it is unlikely that $Z_i$ is much larger than $Z_{i-1}$. All constants in this section are independent of $\mu$. This includes all hidden constants in the $O$-notation.

#### 5.2.1 Tail Bound on the Lifetime of $X_i$

As before, let $t_i$ be the first round in which an individual of rank at least $i$ is created, and let $T_i$ be the round in which the last individual of rank at most $i$ is eliminated.

**Lemma 7.** For all $0 < \alpha, \eta < 1$, $c > 0$, there is a constant $\mu_0 \in \mathbb{N}$ such that the following holds for all $\mu_0 \leq \mu \leq n$. Let $i \in [(1 - 2c)\alpha n, (1 - \varepsilon/2)\alpha n]$, where $\varepsilon = \mu^{-1+\eta}$. Consider the $(\mu + 1)$-EA with mutation rate $c/n$ on the linear function $f_\mu$. Then for all $\beta \geq 1$ and $C \geq 64\varepsilon c$,

$$\Pr[T_{i} - t_i \geq \beta \cdot C \mu \log \mu] \leq 2^{-\beta}.$$  

*Proof. We first show that $\Pr[T_{i} - t_i \geq C' \mu \log \mu] \leq 1/2$ for a suitable constant $C' > 0$. Let $x^{\geq i}$ be the first individual of rank at least $i$ and let $x^j$ with rank $j$ be the first individual of rank strictly larger than $i$. We can divide the process from $t_i$ to $T_i$ into two parts. The first part ends when $x_j$ is created, and we denote by $t_j$ the round when this happens. The second part starts after $t_j$ and ends when $X_{\geq j}$ reaches size $\mu$. Since we are proving an upper bound of the tail, we can consider the second part ends when $|X_{\geq j}|$ reaches $\mu$ for simplicity.

Denote by $\mathcal{E}$ the event that $T_{i} - t_i \geq C' \mu \log \mu$. Define $\mathcal{E}_1$ as the event that $t_j - t_i \geq C'/2 \cdot \mu \log \mu$ and $\mathcal{E}_2$ as the the event that $T_{i} - t_j \geq C'/2 \cdot \mu \log \mu$. 

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Clearly, \( \mathcal{E} \) implies \( \mathcal{E}_1 \) or \( \mathcal{E}_2 \), since \( \neg \mathcal{E}_1 \) and \( \neg \mathcal{E}_2 \) imply \( \neg \mathcal{E} \). Therefore, it holds that
\[
\Pr[\mathcal{E}] \leq \Pr[\mathcal{E}_1 \lor \mathcal{E}_2] \leq \Pr[\mathcal{E}_1] + \Pr[\mathcal{E}_2].
\]

Now we bound the probabilities of \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) separately. By Lemma 5, if \( \mu \) is sufficiently large then the expected time for \( |X_{\geq j}| \) growing from 1 to \( \mu \) is at most \( 2e^{ac} \mu \log \mu \). Hence, by Markov’s inequality,
\[
\Pr[\mathcal{E}_2] = \Pr\left[T_i - t_j \geq \frac{C'}{2} \mu \log \mu\right] \leq \frac{\mathbb{E}[T_i - t_j]}{C'/2 \cdot \mu \log \mu} \leq \frac{4e^{ac}}{C'} \leq \frac{1}{4},
\]
where the last step holds if we choose \( C' \geq 16e^{ac} \). For \( \mathcal{E}_1 \) to happen, a necessary condition is \( \text{rk}(x^{\geq i}) = i \), otherwise we have \( t_j = t_i \). So we assume \( \text{rk}(x^{\geq i}) = i \) in the following analysis. Denote by \( \mathcal{E}' \) the event that it takes at least \( T^\kappa \geq C'/2 \cdot \mu \log \mu \) rounds until \( |X_{\geq i}| = \mu^\kappa \) for some constant \( 0 < \kappa < 1 \) to be chosen later. Then
\[
\Pr[\mathcal{E}_1] = \Pr[\mathcal{E}'_1] \Pr[\mathcal{E}_1 | \mathcal{E}'_1] + \Pr[\neg \mathcal{E}'_1] \Pr[\mathcal{E}_1 | \neg \mathcal{E}'_1] 
\leq \Pr[\mathcal{E}'_1] \cdot 1 + 1 \cdot \Pr[\mathcal{E}_1 | \neg \mathcal{E}'_1].
\]
Again by Lemma 5, \( \mathbb{E}[T^\kappa] \leq 2ke^{ac} \mu \log \mu \) if \( \mu \) is sufficiently large, so as before
\[
\Pr[\mathcal{E}'_1] = \Pr\left[T^\kappa \geq \frac{C'}{2} \mu \log \mu\right] \leq \frac{\mathbb{E}[T^\kappa]}{C'/2 \cdot \mu \log \mu} \leq \frac{4ke^{ac}}{C'}.
\]

By Lemma 6, with probability at least \( 1 - O(\mu^{6(1-\kappa)-\eta} \log \mu) \) there are offspring of rank at least \( i + 1 \) when \( X_{\geq i} \) reaches size \( \mu^\kappa \). Therefore, we have
\[
\Pr[\mathcal{E}_1 | \neg \mathcal{E}'_1] = O(\mu^{6(1-\kappa)-\eta} \log \mu).
\]

By choosing \( \kappa > 1 - \eta/6 \), we can make this probability less than \( 1/8 \). Moreover, assuming \( C' \geq 32ke^{ac} \) then we also have \( \Pr[\mathcal{E}'_1] \leq 1/8 \) by (6) and thus \( \Pr[\mathcal{E}_1] \leq 1/4 \) by (5). This proves that \( \Pr[T_i - t_i \geq C' \mu \log \mu] \leq 1/2 \) for \( C' \geq \max\{16e^{ac} \cdot 2ke^{ac}\} \).

To conclude the proof, we set \( C := 2C' \). Then for all integral \( \beta' \in \mathbb{N} \) we consider \( \beta' \) phases and repeat the same argument. This shows \( \Pr[T_i - t_i \geq \beta' \cdot C' \mu \log \mu] \leq 2^{-\beta'} \). Hence, for \( C \geq 64e^{ac} \) it holds
\[
\Pr[T_i - t_i \geq \beta \cdot C \mu \log \mu] \leq \Pr[T_i - t_i \geq [\beta] C' \mu \log \mu] \\
\leq 2^{-[\beta]} \leq 2^{-\beta}.
\]

### 5.2.2 Tail Bound on Steps of \( Z_i \)

**Lemma 8.** For all \( 0 < \alpha, \eta < 1 \), \( c > 0 \) there is a constant \( \mu_0 \in \mathbb{N} \) such that the following holds for all \( \mu_0 \leq \mu \leq n \), where \( n \) is sufficiently large. Let \( i \in [(1-2\varepsilon)n, (1-\varepsilon/2)n] \), where \( \varepsilon = \mu^{-1+\eta} \). Assume that the \((\mu+1)\)-EA with
proof. added.

If on the other hand \( Z_i - 1 < (1 - 4\varepsilon)n \) then \( Z_i < (1 - 2\varepsilon)n \) with probability
\[
1 - \exp\{-\Omega(\varepsilon n / \log^2 \mu)\} \text{ for } \beta = \Theta(\varepsilon n / \log^2 \mu).
\]
Moreover, fix some round \( t \geq 0 \), assume that \( i_0 \in [(1 - 2\varepsilon)\alpha n, (1 - \varepsilon/2)\alpha n] \) is the smallest rank in the population \( S_t \) at time \( t \), and assume that there is at least one individual of rank at least \( i_0 + 1 \) in \( S_t \) and at least one individual \( x \) (possibly the same) such that \( \Omega M(x) \geq (1 - 4\varepsilon)n \). Then for all \( 1 \leq \beta \leq \varepsilon n / \log^2 \mu \),
\[
\Pr[Z_{i_0} - \max\{\Omega M(x) \mid x \in S_t\} \geq \beta \cdot C_2 \log \mu] \leq 2^{-\beta}.
\]

If on the other hand \( \Omega M(x) < (1 - 4\varepsilon)n \) for all \( x \) in the population then \( Z_{i_0} < (1 - 2\varepsilon)n \) with probability \( 1 - \exp\{-\Omega(\varepsilon n / \log^2 \mu)\} \) for \( \beta = \Theta(\varepsilon n / \log^2 \mu) \).

Proof. By Lemma 7, there is \( C = 64\e^{ac} \) such that
\[
\Pr[T_i - t_i \geq (\beta + 2) \cdot C_2 \mu \log \mu] \leq \frac{1}{4}2^{-\beta}.
\]
Consider a forest \( F' \) obtained by the following forest growing process. In round 0 there is a single root. Each subsequent round, each node flips a coin and creates a new child with probability \( 1/\mu \). Moreover, each round a new root is added.

We claim that we can couple the family forest \( F \) of rank at least \( i \) to the forest \( F' \) as follows. Throughout the process we will maintain \( F \subseteq F' \). The first point \( x^i \) that the algorithm visits in \( X_i \) (in round \( t_i \)) corresponds to the only root \( r_0 \) in round 0 in \( F' \). In every round \( t > t_i \), a point \( x^i \) in the current population is selected to create an offspring \( y^i \). For each \( x \in X_{\geq i} \), if \( x^i = x \) (which happens with probability \( 1/\mu \) if \( x \) is still in the current population, and with probability zero otherwise) then we attach a child to \( x \) in \( F' \) if \( y^i \in X_{\geq i} \) then we attach \( y^i \) to \( x \) in \( F' \), otherwise we attach an arbitrary dummy child to \( x \) in \( F' \). If \( x^i \) is not in \( X_{\geq i} \) while \( y^i \) is, we add \( y^i \) as a new root \( r_i \) to \( F' \), otherwise we add a new dummy root to \( F' \). For every node \( x \in F' \) that is a dummy node (that has no corresponding node in \( F \)) or whose copy in \( F \) has been removed from the population, we add another dummy node as its child with probability \( 1/\mu \). In this way, for each vertex in \( F' \) we create a new child with probability \( 1/\mu \) in each round. On the other hand, by construction, we have \( F \subseteq F' \) at all times. Therefore, the claims that we will prove for all vertices in \( F' \) will also hold for all vertices in \( F \).

Claim 1. Let \( s_t \) denote the number of vertices in \( F' \) in round \( t = (\beta + 2) \cdot C_2 \mu \log \mu \). Then
\[
\Pr[s_t \geq \mu^{2(\beta + 2)C}] \leq \frac{1}{4}2^{-\beta}.
\]
In particular, the probability that the algorithm visits at least \( \mu^{2(\beta + 2)C} \) vertices in \( X_i \) is at most \( \frac{1}{4}2^{-\beta} \).
Proof. In \( t \) rounds we add \( t \) roots to the forest, and we will give a uniform bound for all of them. So we fix a root and denote by \( \sigma_\tau \) the number of vertices in this tree in round \( \tau \), where \( 0 \leq \tau \leq t \). To simplify notation, we may assume \( t_i = 0 \), and we further assume pessimistically that the root is introduced in round 0. Then we have \( \sigma_0 = 1 \) and \( \mathbb{E}[\sigma_{\tau+1} \mid \sigma_\tau] = (1 + 1/\mu)\sigma_\tau \) for \( 0 \leq \tau \leq t - 1 \). By linearity of expectation, we have \( \mathbb{E}[\sigma_t] \leq (1 + 1/\mu)^t \). Since there are \( t \) roots, and using that \( (1 + 1/\mu)^\mu \leq e \), we obtain

\[
\mathbb{E}[s_t] \leq t \mathbb{E}[\sigma_t] \leq t(1 + 1/\mu)^t \leq (\beta + 2)C\mu^{1+\beta+2C} \log \mu.
\]

By Markov’s inequality, it holds that

\[
\Pr[s_t \geq \mu^{2(\beta+2)C}] \leq \frac{\mathbb{E}[s_t]}{\mu^{2(\beta+2)C}} \leq (\beta + 2)C\mu^{1-\beta} \log \mu < \frac{1}{4}2^{-\beta},
\]

where the last step holds for all \( \mu \geq \mu_0 \) if \( \mu_0 \) is sufficiently large. \( \square \)

Claim 2. Consider \( F' \) at a time when it has at most \( \mu^{2(\beta+2)C} \) vertices and assume \( \mu_0 \leq \mu \leq n \) for a suitably large \( \mu_0 \). Let \( x \) be a search point that corresponds to a vertex in \( F' \) of depth at most \( d := \beta C_2 \log \mu \) with root \( y \), where \( C_2 = 400C/e \). Then for \( C_2 = 1600C \),

\[
\Pr[x \text{ and } y \text{ differ in more than } \beta C_2 \log \mu \text{ bits}] \leq \frac{1}{4}2^{-\beta} \cdot \mu^{-4(\beta+2)C}.
\]

In particular, the probability that there exists such a vertex \( x \) is at most \( \frac{1}{4}2^{-\beta} \).

Proof. Let \( y_i \) be the \( i \)-th bit in \( y \), the event \( y_i \neq x_i \) implies that the \( i \)-th bit is flipped at least once. Denote by \( d' \) the distance between \( x \) and \( y \). Using \( (1 - c/n) \geq e^{-2c/n} \) and \( e^{-x} \geq 1 - x \), it holds

\[
\Pr[y_i \neq x_i] \leq \Pr[\text{bit } i \text{ is flipped at least once}]
= 1 - \left(1 - \frac{c}{n}\right)^{d'} \leq 1 - \left(1 - \frac{c}{n}\right)^{d}
\leq 1 - e^{-2cd/n} \leq 2c\beta C_2' \log \mu/n.
\]

Let \( D = \{i \in [n] \mid y_i \neq x_i\} \) be the set of bits that \( y \) and \( x \) disagree. Then its expected size is \( \mathbb{E}[|D|] \leq 2c\beta C_2' \log \mu \). Since \( \beta C_2 \log \mu = 4c\beta C_2' \log \mu \geq 2 \mathbb{E}[|D|] \), and the bits are modified independently, we can apply Chernoff’s inequality:

\[
\Pr[|D| \geq \beta C_2 \log \mu] \leq e^{-\beta C_2 \log \mu/6} = \mu^{-\beta C_2/6} = \mu^{-1600\beta C/6} \leq \frac{1}{4}2^{-\beta} \cdot \mu^{-4(\beta+2)C}.
\]

We remark that the factor 1600 from \( C_2 \) is more than what we need here, but it will be convenient later. The last statement of the claim follows by a union bound over all vertices in \( F' \). \( \square \)
Claim 3. Consider $F'$ at a time when it has at most $\mu2^{(\beta+2)C}$ vertices and assume $\mu_0 \leq \mu \leq n$ for a suitably large $\mu_0$. With $d = \beta C_2 \log \mu$ as in Claim 2, let $x$ be a search point that corresponds to a vertex in $F'$ of depth larger than $d$ with root $y$. If $\Omega(y) \geq (1 - 8\varepsilon)n$ then

$$\Pr[x \text{ has more one-bits than } y] \leq \frac{1}{8}2^{-\beta} \cdot \mu^{-4(\beta+2)C}. \quad (7)$$

In particular, the probability that there exists such a vertex $x$ is at most $\frac{1}{8}2^{-\beta}$.

On the other hand, if $n$ is sufficiently large then the following holds. If $\Omega(y) \leq (1 - 8\varepsilon)n$ then $\Pr[\Omega(x) \geq (1 - 4\varepsilon)n] \leq 2e^{-\varepsilon n/6}$. Moreover, the probability that such a vertex $x$ exists is at most $\frac{1}{8}2^{-\varepsilon n/6} \leq \frac{1}{8}2^{-\beta}$, where $\beta \leq \varepsilon n/\log^2 \mu$.

Proof. Let the depth of $x$ be $d' \geq d$. First we argue that we may assume $d \leq n/(2c)$. If $d \geq n/(2c)$, then consider just the last $n/(2c)$ steps. In these, every bit has a constant probability to be touched exactly once, and a constant probability not to be touched at all. If the number of one-bits before the last $n/(2c)$ steps was at least $n/2$, then with probability $1 - e^{-\Omega(n)}$, $x$ has at least $8\varepsilon n$ zero-bits due to the first case, and if the number of one-bits was at most $n/2$ then the second case gives at least $8\varepsilon n$ zero-bits. In either case, $x$ has more zero-bits than $y$ with sufficiently large probability. So we may assume $d \leq n/(2c)$.

We consider first the case $\Omega(y) \geq (1 - 8\varepsilon)n$. Let $B_{01}$ be the number of bits flipped from 0 to 1. Then similarly as for Claim 2 we bound $\mathbb{E}[B_{01}]$ by

$$|\{i \mid y_i = 0\}| \cdot \Pr[x_i = 1 \mid y_i = 0]$$

$$\leq |\{i \mid y_i = 0\}| \cdot \Pr[\text{bit } i \text{ flipped at least once in } d' \text{ mutations}]$$

$$\leq 8\varepsilon n \cdot (1 - (1 - c/n)^d) \leq 8\varepsilon n \cdot (2cd'/n) = 16\varepsilon cd'.$$  \quad (8)

Similarly, let $B_{10}$ be the number of bits flipped from 1 to 0 in $d'$ mutations, its expectation $\mathbb{E}[B_{10}]$ is

$$|\{i \mid y_i = 1\}| \cdot \Pr[x_i = 0 \mid y_i = 1]$$

$$\geq |\{i \mid y_i = 1\}| \cdot \Pr[\text{bit } i \text{ flipped exactly once in } d' \text{ mutations}]$$

$$\geq \frac{n}{2} \left( \frac{d'}{1} \right) \frac{c}{n} \left( 1 - \frac{c}{n} \right)^{d'-1} \geq \frac{d'c}{4} \left( 1 - \frac{c}{n} \right) \geq \frac{d'c}{4}. \quad (9)$$

Since all bits contribute independently, we may apply the Chernoff bound. If $\mu_0$ is sufficiently large, then with probability at least $1 - e^{-d'c/32}$ each, we have $B_{01} \leq cd'/8$ and $B_{10} \geq cd'/8$. Both inequalities together imply that $\Omega(x) \leq \Omega(y)$ as desired, and the probability that at least one of the inequalities is violated is at most $2e^{-d'c/32} \leq 2e^{-dc/32} = 2\mu^{-c\beta C_2/32} = 2\mu^{-25\beta C/2}$, which is small enough.

For the second statement, assume $\Omega(y) \leq (1 - 8\varepsilon)n$, and consider the first vertex $x'$ on the path from $y$ to $x$ such that $\Omega(x') \geq (1 - 6\varepsilon)n$. The probability that more than $\varepsilon n$ bits were flipped in the creation of $x'$ is at most
e^{-e n/6} by the Chernoff bound, since by definition of \( x' \) the parent of \( x' \) has an OM-value smaller than \((1 - 6\varepsilon)n\), we may assume that \( \text{OM}(x') \leq (1 - 5\varepsilon)n \). Then, starting from \( x' \) we may use the same calculation as above, only that we need to bound the probability that \( \varepsilon n \) more zero-bits than one-bits are flipped. This is bounded by the probability that \( E[B_{01}] \geq \varepsilon n \), which is at most \( e^{-e n/6} \) by the Chernoff bound. For the final union bound over all vertices, note that due to the different basis, the claimed probability \( 1/8 \cdot 2^{-e n/6} \) is much larger than \( 2e^{-e n/6} \), and their ratio can absorb a union bound over \( \mu^{2(\beta+2)C} = e^{O(e n/\log \mu)} \) points if \( \mu_0 \) is sufficiently large.

To summarize, we have shown that the probability of the following four events can all be bounded by \( 1/4 \cdot 2^{-\beta} \).

- \( \mathcal{E}_1 \): \( T_i - t_i \geq (\beta + 2)C\mu \log \mu \).
- \( \mathcal{E}_2 \): \( s_t \geq \mu^{2(\beta+2)C} \) at time \( t = (\beta + 2)C\mu \log \mu \).
- \( \mathcal{E}_3 \): Among the first \( \mu^{2(\beta+2)C} \) vertices in \( F' \), there exists a search point \( x \) with a distance at most \( \beta C' \mu \log \mu \) to its root \( y \) such that \(|\{ i \in [n] \mid y_i \neq x_i \}| > \beta C' \mu \log \mu \).
- \( \mathcal{E}_4 \): Among the first \( \mu^{2(\beta+2)C} \) vertices in \( F' \), there exists a search point \( x \) with a distance larger than \( \beta C' \mu \log \mu \) to its root \( y \) such that either \( \text{OM}(y) \leq (1 - 8\varepsilon)n \) and \( \text{OM}(x) > \text{OM}(y) \) or \( \text{OM}(y) \leq (1 - 8\varepsilon)n \) and \( \text{OM}(x) \geq (1 - 4\varepsilon)n \).

Moreover, the probability that there exists a search point \( x \) with a distance larger than \( \beta C' \mu \log \mu \) to its root \( y \) such that \( \text{OM}(y) \leq (1 - 8\varepsilon)n \) and \( \text{OM}(x) \geq (1 - 4\varepsilon)n \) can alternatively be bounded by \( e^{-O(e n)} \) by Claim 3. We want to argue how the bounds for these events imply the lemma. Assume first \( Z_{i-1} \geq (1 - 4\varepsilon)n \). Then unless \( \mathcal{E}_1 \) or \( \mathcal{E}_2 \) occur, we may restrict ourselves to the first \( \mu^{2(\beta+2)C} \) vertices in \( F' \). If \( \mathcal{E}_3 \) does not occur, there are no offspring in distance at most \( \beta C' \mu \log \mu \) from their root that have OM-value larger than \( Z_{i-1} = \beta C' \mu \log \mu \). For vertices in larger distance from their roots, we need to discriminate two cases. Either the root has OM-value at least \((1 - 8\varepsilon)n\) in which case the offspring do not exceed their root unless the first part of \( \mathcal{E}_4 \) occurs. Or the root has OM-value at most \((1 - 8\varepsilon)n\), in which case the offspring do not exceed a OM-value of \((1 - 4\varepsilon)n\) unless the second part of \( \mathcal{E}_4 \) occurs. In both cases, the OM-values of the offspring do not exceed \( Z_{i-1} \). Hence, we have shown that \( Z_i - Z_{i-1} > \beta \cdot C_2 \mu \log \mu \) is only possible if one of the events \( \mathcal{E}_1 - \mathcal{E}_4 \) occurs, and thus

\[
\Pr[Z_i - Z_{i-1} > \beta \cdot C_2 \log \mu] \leq \sum_{j=1}^{4} \Pr[\mathcal{E}_j] \leq 2^{-\beta}.
\]

If \( Z_{i-1} < (1 - 4\varepsilon)n \) then we use the same case distinction, but we apply \( \mathcal{E}_1 - \mathcal{E}_4 \) with \( \beta = \varepsilon n/\log^2 \mu \). Finally, the same argument also applies in the second setting where \( i_0 \) is the smallest rank in the population \( S_i \), and there is at least one individual of rank at least \( i_0 + 1 \). This concludes the proof of Lemma 8. \( \square \)
5.3 Typical Situations

As outlined in the overview, our analysis of the drift will be based on studying what happens in 'typical' situations. To characterize these, we use the following definition of 'good' events. Again we consider the \((\mu + 1)\)-EA on the linear function \(f_t\). For parameters \(\phi, c_d, c_c > 0\) we define the event \(\mathcal{E}_{\text{good}}(i) := \mathcal{E}_a \cap \mathcal{E}_b \cap \ldots \cap \mathcal{E}_e\), where \(\mathcal{E}_a\) etc. are the following events about the family forest \(F_i\) of rank \(i\). Recall the family forest consists of all \(x \in X_{\geq 1}\), and a vertex \(u\) is a child of \(v\) if \(u\) was created as an offspring of \(v\). We will be concerned about those vertices in the family forest in \(X_i\), i.e., vertices of rank exactly \(i\).

- \(\mathcal{E}_a\): No vertex in \(X_{\leq i-1}\) creates offspring in \(X_{\geq i+1}\).
- \(\mathcal{E}_b\): There are at most \(\varepsilon \mu \log^3 \mu\) roots in \(F_i\).
- \(\mathcal{E}_c\): No vertex in \(X_i\) of depth at most \(\phi \log \mu\) in \(F_i\) creates offspring.
- \(\mathcal{E}_d\): For every vertex \(x \in X_i\) that creates an offspring in \(X_{\geq i+1}\), if the root \(r\) of \(x\) has \(\text{Om}(r) \geq (1 - \delta) n\) then \(\text{Om}(x) \leq \text{Om}(r) - c_d \log \mu\), and if \(\text{Om}(r) \leq (1 - \delta) n\) then \(\text{Om}(x) \leq (1 - 4\varepsilon) n\). Moreover, the mutation changes at most \(c_d/2 \cdot \log \mu\) bits.
- \(\mathcal{E}_e\): No vertex in \(X_i\) has an Om-value which exceeds the Om-value of its root in \(F_i\) by more than \(c_c \log \mu\).

**Lemma 9.** For every \(0 < \alpha < 1, c > 0\) there are \(c_d, c_c > 0\) such that the following holds. For any constant parameters \(0 < \phi < 1\) and \(\eta > 0\) that satisfy the following conditions, where \(C_4 := 2e^{\alpha \phi} + 3\) and \(f(\phi) = \phi(1 - \log \phi)\),

\[
\eta < \min \left\{ \frac{1}{2}, \frac{1}{C_4 f(\phi)}, \frac{1}{2} - 2C_4 f(\phi), \frac{c_d}{128}, \frac{c_c}{6} \right\}, \tag{10}
\]

there exists \(\mu_0\) such that for all \(\mu_0 \leq \mu \leq n\) and all \(i \geq (1 - \delta) n\), the \((\mu + 1)\)-EA on \(f_t\) satisfies

\[
\Pr[\mathcal{E}_{\text{good}}(i)] \geq 1 - O\left(\log^{-7/4} \mu\right).
\]

We remark that for all \(\alpha\) and \(c\) it is possible to set the parameters to satisfy the above conditions since we can make \(f(\phi)\) arbitrarily small by choosing \(\phi\) small.

**Proof.** We need to show that \(\Pr[E] = 1 - O(\log^{-7/4} \mu)\) holds for \(E = \mathcal{E}_a, \ldots, \mathcal{E}_e\). Thus we split the proof into five parts.

\(\mathcal{E}_a\): No vertex in \(X_{\leq i-1}\) creates offspring in \(X_{\geq i+1}\) for \(\eta < 1/2\).

We consider the number of offspring that are created from points in \(X_{\leq i-1}\) and are members of \(X_{\geq i+1}\) after the first point \(x\) in \(X_{\geq i}\) is created. We first argue that the probability that \(x \in X_i\) is \(1 - O(\varepsilon)\).

Assume the parent of \(x\) is \(y\) and the rank of \(y\) is \(i - j\), where \(j \geq 1\). Since flipping at most \(j\) zero-bits implies that \(x \in X_{\leq i}\), so \(x \in X_{\geq i+1}\) is only possible if at least \(j + 1\) zero-bits are flipped, and thus \(\Pr[x \in X_{\geq i+1}] \leq \frac{1}{2^j} \leq \frac{1}{8}\).
\[ \Pr[\text{flip at least } j + 1 \text{ zero-bits}] \]. Moreover, flipping \( j \) zero-bits and 0 one-bits implies \( x \in X_i \), so \( \Pr[x \in X_i] \geq \Pr[\text{flip } j \text{ zero-bits and 0 one-bits}] \). Therefore,

\[ \frac{\Pr[x \in X_{\geq i+1}]}{\Pr[x \in X_i]} \leq \frac{\Pr[\text{flip at least } j + 1 \text{ zero-bits}]}{\Pr[\text{flip } j \text{ zero-bits and 0 one-bits}]} . \]

Let \( d \) be the density of zero-bits in the \( A_{\ell+1} \) part of \( y \), and note that \( d = O(\varepsilon) \).

Consider

\[ \frac{\Pr[\text{flip } j + 1 \text{ zero-bits}]}{\Pr[\text{flip } j \text{ zero-bits}]} = \frac{(\frac{\alpha n}{j+1})^{j+1} (1 - \frac{c}{n})^{\alpha n - j - 1}}{(\frac{\alpha n}{j})^{j} (1 - \frac{c}{n})^{\alpha n - j}} \]

\[ = \left( \frac{\alpha n}{j+1} \right) \frac{c}{n} (1 - \frac{c}{n})^{-1} \]

\[ \leq \alpha n \cdot \frac{c}{n} \cdot \left( 1 - \frac{1}{2} \right)^{-1} \]

\[ = 2\alpha c d \]

\[ \leq 1/2 \tag{11} \]

and

\[ \Pr[\text{flip 0 one-bits}] = (1 - c/n)^{(1-d)n} \geq e^{-2c(1-d)n} \geq e^{-2\alpha c} \]

Altogether we have

\[ \frac{\Pr[x \in X_{\geq i+1}]}{\Pr[x \in X_i]} \leq \frac{\sum_{k=j+1}^{\infty} \Pr[\text{flip exactly } k \text{ zero-bits}]}{\Pr[\text{flip } j \text{ zero-bits}]} \frac{\Pr[\text{flip 0 one-bits}]}{\Pr[\text{flip 0 one-bits}]} \]

\[ \leq \frac{\sum_{k=j+1}^{\infty} \left( \left( \frac{1}{2} \right)^{k-1} \Pr[\text{flip } j + 1 \text{ zero-bits}]ight)}{\Pr[\text{flip } j \text{ zero-bits}] \Pr[\text{flip 0 one-bits}]} \frac{2 \Pr[\text{flip } j + 1 \text{ zero-bits}]}{\Pr[\text{flip } j \text{ zero-bits}] \Pr[\text{flip 0 one-bits}]} \]

\[ \leq 4\alpha c d e^{2\alpha c} = O(\varepsilon). \tag{12} \]

Therefore, the probability that the first point \( x \) in \( X_{\geq i} \) belongs to \( X_i \) is

\[ \Pr[x \in X_i \mid x \in X_{\geq i}] = \frac{\Pr[x \in X_i]}{\Pr[x \in X_i] + \Pr[x \in X_{\geq i+1}]} \]

\[ = \frac{1}{1 + \frac{\Pr[x \in X_{\geq i+1}]}{\Pr[x \in X_i]}} = 1 - O(\varepsilon). \]

By Lemma 7 with a suitable \( \kappa = \Theta(\log \log \mu) \), after the first search point in \( X_i \) is created, with probability \( 1 - O(\log^{-7/4} \mu) \) it takes at most \( T = O(\mu \log \mu \log \log \mu) \) rounds until the set \( X_{\leq i} \) is completely deleted. By Lemma 3, the probability that an offspring of a search point in \( X_{\leq i-1} \) is in \( X_{\geq i+1} \) is \( O(\varepsilon^2) \), and hence the expected number of offspring in \( X_{\geq i+1} \) created from \( X_{\leq i-1} \) is at most
\[ O(\varepsilon^2 T) = O(\varepsilon^2 \mu \log^2 \mu) = O(\mu^{-1+2\eta} \log^2 \mu). \] Since \( \eta < 1/2 \), by Markov’s inequality, the probability that the number of such offspring is at least 1 can be bounded by \( O(\mu^{-1+2\eta} \log^2 \mu) = O(\log^{-7/4} \mu) \), as required.

\( \mathcal{E}_b \): There are at most \( \varepsilon \mu \log^3 \mu \) roots in \( F_1 \).

We know from \( \mathcal{E}_a(i-1) \) that we may assume that no points in \( X_i \) are created from \( X_{<i-2} \). Hence, it suffices to count the number of roots in \( X_i \) that are created from \( X_{i-1} \). As in the proof for \( \mathcal{E}_a \), by Lemma 7, after the first search point in \( X_{i-1} \) is created, with probability \( 1 - O(\log^{-7/4} \mu) \) it takes at most \( T = O(\mu \log \mu \log \log \mu) \) rounds until the set \( X_{<i-1} \) is completely deleted. In each round we have a probability of at most \( p_T = O(\varepsilon) \) to create a new root in \( X_i \), so the expected number of roots in \( X_i \) is \( O(\varepsilon T) = O(\varepsilon \mu \log \mu \log \mu) \). By Markov’s inequality, the number of roots is at most \( \varepsilon \mu \log^3 \mu \) with probability at most \( O(\varepsilon T (\varepsilon \mu \log^3 \mu)) = O(\log^{-7/4} \mu) \).

\( \mathcal{E}_c \): No vertex in \( X_i \) of depth at most \( \phi \log \mu \) in \( F_i \) creates offspring.

We start similarly as in Lemma 8. In particular, we couple \( F_i \) with a recursive tree \( F' \supseteq F_i \). We start with \( \varepsilon \mu \log^3 \mu \) roots in \( F' \), since the number of roots in \( F_i \) is at most \( \varepsilon \mu \log^3 \mu \) by \( \mathcal{E}_b \). In each round, each vertex in \( F' \) creates an offspring with probability \( 1/\mu \). Recall that by Lemma 5, the lifetime of \( X_i \) is at most \( T := 2^e \alpha \mu \log \mu \) with probability at least \( 1 - O(\log^{-7/4} \mu) \), if \( \mu \geq \mu_0 \) for a sufficiently large \( \mu_0 \). Hence, it suffices to study \( F' \) after \( T \) rounds. The size \( s_t \) at time \( t \) of \( F' \) satisfies \( s_0 = \varepsilon \mu \log^3 \mu \) and the recursion \( \mathbb{E}[s_{t+1} | s_t] = (1 + 1/\mu) s_t \). Inductively, we obtain

\[ \mathbb{E}[s_T] \leq s_0 (1 + 1/\mu)^T \leq \mu \log^3 \mu \cdot e^{2^e \alpha \log \mu} \leq \mu^{C_4-1} \]

for a suitable constant \( C_4 := 2^{e \alpha} + 3 \). By Markov’s inequality, with probability at least \( 1 - 1/\mu \) it holds that \( s_T \leq \mu^{C_4} \). So in the following we may assume that \( s_T \leq \mu^{C_4} \).

We want to bound the number of vertices in depth at most \( \phi \log \mu \). We fix a root, and consider the tree attached to this root. As we aim for an upper bound, we may pessimistically assume that this tree has exactly \( m = \mu^{C_4} \) vertices. According to [5], for a recursive tree with a single root and \( m = \mu^{C_4} \) vertices, the expected number of vertices at depth \( k \) is

\[ \mathbb{E}[D_k(m)] = (1 + O(\log^{-1} m)) \frac{\log^k m}{k! \cdot \Gamma(1 + k/\log m)} \]

uniformly for \( 1 \leq k \leq K \log m \) and any \( K > 0 \), where \( \Gamma \) denotes the Gamma function. To understand the asymptotics of this expression, let us momentarily assume that \( \mu \to \infty \) (and thus, \( m \to \infty \)). For \( k \leq \log \log m \), it is easy to see that \( \mathbb{E}[D_k(m)] = (\log m)^{o(1)} \), and it remains \( (\log m)^{o(1)} \) if we sum over all \( 1 \leq k \leq \log \log m \). For \( k > \log \log m \), let us parametrize \( k = \phi' \log m \). Since we are interested in \( \phi' \leq \phi < 1 \), we may restrict our analysis to the range \( 0 < \phi' < 1 \). Since \( \Gamma(x) \geq 1/2 \) for \( 1 < x < 2 \), by Stirling’s approximation we
have
\[
\mathbb{E}[D_k(m)] = (1 + O(\log^{-1} m)) \frac{(\log m)^{\phi' \log m}}{(\phi' \log m)!} \Gamma(1 + \phi') \\
\leq (1 + O(\log^{-1} m)) \frac{(\log m)^{\phi' \log m}}{\sqrt{2\pi \phi' \log m (\phi' \log m/e)^{\phi' \log m} \cdot 1/2}} \\
= (1 + O(\log^{-1} m)) \frac{2/(\pi \phi' \log m)}{m^{\phi'(1-\log \phi')}} \\
= o(m^{\phi'(1-\log \phi')}).
\]

Let \( f(\phi) = \phi(1-\log \phi) \), then for any \( k = \phi' \log m \leq \phi \log m \), we have \( \mathbb{E}[D_k(m)] = o(m^{f(\phi')}) = o(m^{f(\phi)}) \) since \( f(\phi) \) is monotonously increasing when \( 0 < \phi < 1 \). Therefore, by Markov’s inequality,
\[
\Pr \left[D_k(m) \geq m^{2f(\phi)} \right] \leq \frac{\mathbb{E}[D_k(m)]}{m^{2f(\phi)}} \leq o(m^{-f(\phi)})
\]
for any \( k \leq \phi \log m \). Applying union bound, with probability \( 1 - o(km^{-f(\phi)}) = 1 - o(\mu^{-C_4 f(\phi) \log \mu}) \), \( D_k(m) < m^{2f(\phi)} \) holds for all \( k \leq \phi \log m \) and thus
\[
\sum_{i=1}^{\phi \log \mu} D_i(\mu^{C_4}) < \sum_{i=1}^{\phi \log \mu} \mu^{2C_4 f(\phi)} = \phi \mu^{2C_4 f(\phi)} \log \mu.
\]

All this was derived under the assumption that \( \mu \to \infty \). However, spelling out the \( o \)-notation, there is an \( \mu_0 \in \mathbb{N} \) such that for all \( \mu \geq \mu_0 \) with probability at least \( 1 - \mu^{-C_4 f(\phi) \log \mu} \) we have \( \sum_{i=1}^{\phi \log \mu} D_i(\mu^{C_4}) \leq \phi \mu^{2C_4 f(\phi) \log \mu} \). So from now on we may continue with the weaker assumption \( \mu \geq \mu_0 \).

Since we have \( \varepsilon \mu^{3 \log^3 \mu} = \mu^{3 \log^3 \mu} \) roots in \( F' \), and each tree in \( F' \) has at most \( \mu^{C_4} \) vertices, by a union bound over all roots, with probability at least \( 1 - \mu^{-\eta-C_4 f(\phi) \log^4 \mu} \) the number of vertices in depth at most \( \phi \log \mu \) is at most
\[
\mu^{\eta \log^3 \mu} \sum_{i=0}^{\phi \log \mu} D_i(\mu^{C_4}) \leq \phi \mu^{\eta+2C_4 f(\phi) \log^4 \mu}.
\]

Note that error probability is \( \mu^{\eta-C_4 f(\phi) \log^4 \mu} = \mu^{-\Omega(1)} \) since we assumed that \( \eta < C_4 f(\phi) \).

In each round, every such vertex has a probability of at most \( p_1/\mu = O(\varepsilon/\mu) \) to create an offspring of strictly larger rank: it must be selected as parent and its offspring must have strictly larger rank. Since the vertices in \( X_i \) are present for at most \( T = O(\mu \log \mu) \) rounds, the expected number of offspring in \( X_{i+1} \) created by vertices in \( X_i \) of depth at most \( \phi \log \mu \) is \( O(T \varepsilon/\mu \cdot \mu^{\eta+2C_4 f(\phi) \log^4 \mu}) = O(\mu^{-1+2C_4 f(\phi) \log^5 \mu}) \). By Markov’s inequality, the probability that the number of such offspring is at least 1 is \( O(\mu^{-1+2C_4 f(\phi) \log^5 \mu}) \). Since \( \eta < 1/2 - C_4 f(\phi) \), this probability is \( \mu^{-\Omega(1)} \). Hence, we have shown that with sufficiently small probability the vertices in depth at most \( d = \phi \log \mu \) do not create offspring in \( X_{i+1} \).
\(\mathcal{E}_e\): For every vertex \(x \in X_i\) that creates an offspring in \(X_{2^{i+1}}\), if the root \(r\) of \(x\) has \(\text{OM}(r) \geq (1-8\varepsilon)n\) then \(\text{OM}(x) \leq \text{OM}(r) - c_d \log \mu\), and if \(\text{OM}(r) \leq (1-8\varepsilon)n\) then \(\text{OM}(x) \leq (1-4\varepsilon)n\). Moreover, the mutation changes at most \(c_d/2 \cdot \log \mu\) bits.

If \(\mathcal{E}_e\) holds, the vertices in \(X_i\) that create offspring in \(X_{2^{i+1}}\) must be of distance at least \(d = \phi' \log \mu\) where \(\phi' > \phi\) from their roots. Consider a root \(r\) with \(\text{OM}(r) \geq (1-8\varepsilon)n\). Similar to the proof of Claim 3 in Lemma 8, let \(B_{10}\) and \(B_{01}\) be the number of bits flipped from 1 to 0 and the number of bits flipped from 0 to 1, respectively. As in the proof of Lemma 8, we may argue that it suffices to consider \(d \leq n/(2c)\). For such \(d\), again as in the proof of Lemma 8 we have \(\mathbb{E}[B_{01}] \leq 16\varepsilon cd \leq cd/16\) and \(\mathbb{E}[B_{10}] \geq cd/4\), see (8) and (9), respectively, and the probability that they deviate more than \(cd/16\) from their expectation is at most \(e^{-cd/128}\) (for \(B_{01}\) to undershoot its expectation) and \(e^{-cd/48}\) (for \(B_{01}\) to overshoot its expectation). If we choose \(c_d < c\phi'/16\), then \(\mathbb{E}[B_{10}] - \mathbb{E}[B_{01}] \geq 2 \cdot cd/16 + cd \log \mu\). In this case, \(B_{01} \geq B_{10} - c_d \log \mu\) is only possible if either \(B_{01} \leq B_{10}\) or \(B_{10}\) deviates at least by \(cd/16\) from its expectation, and thus \(\text{Pr}[B_{01} \geq B_{10} - c_d \log \mu] \leq 2e^{-cd/128} \leq 2\mu^{-M/4},\) with \(M := c\phi'/128\).

If \(\mathcal{E}_e(i+1)\) holds, the number of offspring in \(X_{2^{i+1}}\) created by points in \(X_i\) is \(\Theta(\varepsilon \mu \log^3 \mu)\), which means the number of points in \(X_i\) that create offspring in \(X_{2^{i+1}}\) is at most \(\varepsilon \mu \log^3 \mu\). By a union bound, with probability at least \(1 - \varepsilon \mu \log^3 \mu \cdot 2\mu^{-M} = 1 - 2\mu^{-M + \eta \log^3 \mu}\), a vertex in \(X_i\) that creates an offspring in \(X_{2^{i+1}}\) has a OM-value which is at least \(c_d \log \mu\) smaller than that of its root.

Since we assumed \(\eta < M\), this probability is \(\mu^{-\Omega(1)}\), and thus sufficiently small. This concludes the case that the root has OM-value at least \((1-8\varepsilon)n\).

If a vertex \(x\) has a root which has at most OM-value \((1-8\varepsilon)n\), we consider the first vertex \(x'\) of OM-value at least \((1-6\varepsilon)n\) on the path from the root to \(x\). Then we know that \(x'\) has OM-value at most \((1-5\varepsilon)n\) (since its direct parent had OM-value less than \((1-6\varepsilon)n\)). Then by similar arguments as above, the probability that a descendant of \(x'\) has OM-value which is \(\varepsilon\eta\) larger than \(x'\) is \(e^{-\Omega(\varepsilon \eta)} = \mu^{-\omega(1)}\), and thus we can easily apply a union bound over all vertices in the \(F'_c\)-forest.

Finally, we come to the number of bit flips in the improving mutation. In one mutation the expected number of changed bits is \(c\). Let \(c_d/2 \cdot \log \mu = (1+\delta')c\) for some \(\delta' > 1\), by Chernoff bound, the probability that the number of changed bits is larger than \(c_d/2 \cdot \log \mu\) can be bounded by \(e^{-\delta' c/3} = \Theta(\mu^{-c_d/6})\).

Similarly, by union bound, the error probability is at most \(O(\varepsilon \mu \log^3 \mu \mu^{-c_d/6}) = O(\mu^{\eta - c_d/6} \log^3 \mu),\) which is \(\mu^{-\Omega(1)}\) since \(\eta < c_d/6\), as required.

\(\mathcal{E}_c\): No vertex in \(X_i\) has an OM-value which exceeds the OM-value of its root in \(F_i\) by more than \(c_e \log \mu\).

We set \(c_e = 1600C\). Claim 2 and Claim 3 in the proof of Lemma 8 state that for all \(\beta \geq 1\), for vertex \(x\) and its root \(y\), if the distance between them is at most \(\beta C_2 \log \mu\), then the probability that they differ in at least \(\beta C_2 \log \mu\) bits is at most \(1/4 \cdot 2^{-\beta} \cdot \mu^{-4(\beta+2)C}\); if the distance is more than \(\beta C_2 \log \mu\), the probability that \(x\) has more one-bits than \(y\) is \(1/4 \cdot 2^{-\beta} \cdot \mu^{-4(\beta+2)C}\), where \(C \geq 64e^{ac}, C_2 = 400C/c,\) and \(C_2 = 1600C\). Moreover, by Claim 1 there are
at most $\mu^{2(\beta+2)C}$ vertices in the recursive tree $F'$. Hence, a union bound over all these vertices gives a probability of at most $1/4 \cdot 2^{-\beta} \cdot \mu^{-2(\beta+2)C}$ for both cases. We choose $\beta = 1$ and $c_e = C_2 = 102400e^{ac_c}$, then the probability that $x$ has $c_e \log \mu$ more one-bits than $y$ is at most $2 \cdot 1/4 \cdot 2^{-1} \cdot \mu^{-6C} = \Theta(\mu^{-6C})$. Therefore, $\Pr[\mathcal{E}_c] = 1 - \Theta(\mu^{-6C}) \geq 1 - O(\log^{-7/4} \mu)$. \hfill \Box

5.4 Estimating the Drift

**Lemma 10.** Let $\ell \in [L]$ and $i \in \mathbb{N}$. Consider the $(\mu + 1)$-EA on the linear auxiliary function $f_\ell(x) := n \sum_{j \in A_i} x_j + \sum_{j \in R_i} x_j$. Assume that in some step $t \geq 0$ the highest rank in the population is $i$, and that $\mathcal{E}_{\text{good}}(i), \ldots, \mathcal{E}_{\text{good}}(i + K)$ hold, where $K := \lceil 2(c_e + 1)/c_d \rceil$. Then $Z_{i+K} \leq Z_i - \log \mu$.

**Proof.** Let $j \in \{i + 1, \ldots, i + K\}$, and let $x \in X_j$ be any root in the $j$-th population forest. By $\mathcal{E}_d(j - 1)$, the parent individual $x'$ of $x$ is in $X_{j-1}$. By $\mathcal{E}_d(j - 1)$, the root $y$ of $x'$ in the $(j-1)$-th family forest satisfies $\Omega(y) \geq \Omega(x') + c_d \log \mu \geq \Omega(x) + c_d/2 \cdot \log \mu$. By induction, we obtain that for every root $x \in X_j$ there exists a root $\tilde{y} \in X_i$ such that $\Omega(x) \leq \Omega(\tilde{y}) - (j-i)c_d/2 \cdot \log \mu \leq Z_i - (j-i)c_d/2 \cdot \log \mu$, where the second step holds since $\Omega(\tilde{y}) \leq Z_i$ by definition of $Z_i$. Now consider any individual $\tilde{x} \in X_{i+K}$, and let $x \in X_j$ be its root. By $\mathcal{E}_e(i + K)$, we have

$$\Omega(\tilde{x}) \leq \Omega(x) + c_e \log \mu \leq Z_i^j - Kc_d/2 \cdot \log \mu + c_e \log \mu \leq Z_i - \log \mu,$$

where the latter inequality follows from the definition of $K$. Since (13) holds for all $\tilde{x} \in X_{i+K}$, we obtain $Z_{i+K} \leq Z^j - \log \mu$, as required. \hfill \Box

We are now ready to prove the main theorem on the drift of $Z_i$. As outlined in the introduction, we do not consider $Z_{i+1} - Z_i$, but rather $Z_{i+K} - Z_i$ for a suitable constant $K \in \mathbb{N}$. Moreover, in order to apply the negative drift theorem later, we show that the drift is even negative if we truncate the difference $Z_{i+K} - Z_i$ at $-\log \mu$.

**Theorem 11.** For every $c > 0$ there is a $\mu_0 \in \mathbb{N}$ and a $K \in \mathbb{N}$ such that for all $\mu_0 \leq \mu \leq n$ where $n$ is sufficiently large the following holds for the $(\mu + 1)$-EA with mutation parameter $c$ on the auxiliary function $f_\ell$. Assume that in some generation the fittest search point satisfies (3). Then

$$\mathbb{E} [\max \{ Z_{i+K} - Z_i, -\log \mu \}] \leq -1.$$

**Proof.** Let $K$ be the constant from Lemma 10. Recall from Lemma 9 that the event $\mathcal{E}_{\text{good}}$ has probability $1 - O(\log^{-7/4} \mu)$, which is at least $1/2$ if $\mu$ is sufficiently large. By Lemma 10, the event $\mathcal{E}_{\text{good}}$ implies $Z_{i+K} - Z_i \leq -\log \mu$, so in this case the term $\max \{ Z_{i+K} - Z_i, -\log \mu \}$ evaluates to $-\log \mu$. Hence,
we may compute

\[ E_{\text{good}} := \sum_{j=-\infty}^{\infty} \max\{j, -\log \mu\} \cdot \Pr[Z_{i+K} - Z_i = j \text{ and } \mathcal{E}_{\text{good}}]\]

\[ = (-\log \mu) \cdot \sum_{j=-\infty}^{\infty} \Pr[Z_{i+K} - Z_i = j \text{ and } \mathcal{E}_{\text{good}}]\]

\[ = -\log \mu \cdot \Pr[\mathcal{E}_{\text{good}}] \leq -2,\]

where the last step follows from \(\Pr[\mathcal{E}_{\text{good}}] \geq 1/2\) if \(\mu\) is sufficiently large. In the remainder, we will show that the term \(E_{\text{good}}\) is very close to \(E[\max\{Z_{i+K} - Z_i, -\log \mu\}]\). In fact, the difference is

\[ E[\max\{Z_{i+K} - Z_i, -\log \mu\}] - E_{\text{good}} \leq \sum_{j=1}^{\infty} j \cdot \Pr[Z_{i+K} - Z_i = j \text{ and } \neg \mathcal{E}_{\text{good}}]. \tag{14} \]

For an arbitrary constant \(C > 0\) we may define \(j_0 := [C \log \mu \log \log \mu]\). Then we bound \(j\) by \(j_0\) in the range \(j \leq j_0\), and we bound \(\Pr[Z_{i+K} - Z_i = j \text{ and } \neg \mathcal{E}_{\text{good}}]\) by \(\Pr[Z_{i+K} - Z_i = j]\) for \(j > j_0\). We obtain

\[ (14) \leq \sum_{j=1}^{j_0} j_0 \cdot \Pr[Z_{i+K} - Z_i = j \text{ and } \neg \mathcal{E}_{\text{good}}] + \sum_{j=j_0+1}^{\infty} j \cdot \Pr[Z_{i+K} - Z_i = j]\]

\[ \leq j_0 \cdot \Pr[\neg \mathcal{E}_{\text{good}}] + \sum_{j=j_0+1}^{\infty} j \cdot \Pr[Z_{i+K} - Z_i = j] \]

\[ \leq j_0 \cdot \frac{\Pr[\neg \mathcal{E}_{\text{good}}]}{O(\log^{-7/4} \mu)} + \sum_{j=j_0+1}^{\infty} j \cdot \Pr[Z_{i+K} - Z_i = j] \]

\[ = O\left(j_0 \log^{-7/4} \mu\right) + O\left(\frac{\log \mu \cdot j_0 2^{-j_0/(C_2 \log \mu)}}{O(\log^{-3/4} \mu)}\right)\]

The right term is \(O(\log^{-1} \mu)\) if we choose the constant \(C > 0\) in the definition of \(j_0 := [C \log \mu \log \log \mu]\) appropriately. The left term is \(O(\log \log \mu \cdot \log^{-3/4} \mu)\). Hence, by choosing \(\mu\) sufficiently large, we can make both terms smaller than 1/2, and obtain that \(\mathbb{E}[\max\{Z_{i+K} - Z_i, -\log \mu\}] \leq E_{\text{good}} + 1 \leq -1\), as desired.

\[ \square \]

6 Proof of Theorem 4

In this section we will show how our main result, the lower runtime of the \((\mu + 1)\)-EA, follows from the negative drift of \(Z_i\). The proof follows similar ideas
as in [16] and [12]. We start with one more lemma that describes the behavior of the \((\mu + 1)\)-EA on \(f_t\).

**Lemma 12.** For every constant \(0 < \delta < 2/7\) the following holds. Let \(\ell \in [L]\) and consider the \((\mu + 1)\)-EA on \(f_\ell\) under the assumption that \(d([n], x) \geq \varepsilon(1 + 2\delta)\) and \(\varepsilon(1 + \delta/2) \leq d(A_{\ell+1}, x) \leq d([n], x) + \delta\varepsilon\) hold for all \(x\) in the initial population. For \(t \geq 0\), let \(x^t\) be the offspring in round \(t\). Then with probability \(1 - \exp\{-\Omega(\varepsilon n/\log^2 \mu)\}\), the following holds for all \(t \leq L\).

- \(d([n], x^t) \geq \varepsilon(1 + \delta)\).
- \(d([n], x^t) \geq \varepsilon(1 + 2\delta)\) or \(d(A_{\ell+1}, x^t) \geq \varepsilon(1 + \delta/4)\).

**Proof.** Let \(i_0\) be the largest rank in the initial population, i.e., the largest number of one-bits in \(A_{i_0+1}\) in the initial population. We fix an offset \(a \in \{0, \ldots, K - 1\}\) and consider the sequence of random variables \(Y_{i,a} := Z_{i_0+a+iK}/\log \mu\), where \(i\) is a non-negative integer. In the initial population, each individual has at most \(n(1 - \varepsilon(1 + 2\delta))\) one-bits by assumption. Hence, we also have \(Z_{i_0+a} \leq n(1 - \varepsilon(1 + 3\delta/2))\) with probability \(1 - \exp\{-\Omega(\varepsilon n)\}\) for all offsets \(a \in \{0, \ldots, K - 1\}\), since otherwise at least one of the \(K\) mutations would need to flip \(\Omega(\varepsilon n)\) bits, which happens only with probability \(\exp\{-\Omega(\varepsilon n)\}\) by the Chernoff bound. Thus for the first statement it suffices to show that \(Y_{i,a} \leq Y_{0,a} + \varepsilon\delta n/(2 \log \mu)\) for all \(i \geq 0\). Since that is equivalent to \(Z_{i_0+a+iK} \leq Z_{i_0+a} + \varepsilon\delta n/2\) for all \(i \geq 0\), and we already have \(Z_{i_0+a} \leq n(1 - \varepsilon(1 + 3\delta/2))\) for all \(a\) with high probability, altogether it implies \(Z_{i'} \leq n(1 - \varepsilon(1 + \delta))\) for all \(i' \geq i_0\). As \(Z_{i'}\) denotes the maximum number of one-bits in rank \(i'\), we conclude that \(d([n], x^t) \geq \varepsilon(1 + \delta)\) holds for any individual \(x^t\) of rank \(i' \geq i_0\). For the second statement, we distinguish between two cases. Note that the index \(i\) counts, up to the factor \(K\), the increase in one-bits in \(A_{\ell+1}\). If \(i \leq \alpha\varepsilon\delta/(4K)\) then for any \(x^t\) of rank \(i_0 + a + iK\), \(d(A_{\ell+1}, x^t) = (\alpha\varepsilon(1 + \delta/2) - a - iK)/\alpha n \geq \varepsilon(1 + \delta/2) - (i + 1)K/\alpha n \geq \varepsilon(1 + \delta/4)\). For \(i > \alpha\varepsilon\delta/(4K)\) we aim to show that \(Y_{i,a} \leq Y_{0,a}\).

We would like to apply the negative drift theorem to \(Y_{i,a}\) for the range \([1 - \varepsilon(1 + 3\delta/2)n/\log \mu, 1 - \varepsilon(1 + \delta)n/\log \mu]\). First note that we study a linear function, and that the bits in \(A\) have larger weights than the remaining bits. Thus, it can be shown by a coupling argument (Lemma 4.2 in [16]) that if \(d(A_{\ell+1}, x) \leq d([n], x) + \delta \varepsilon\) holds initially, then the slightly weaker condition \(d(A_{\ell+1}, x) \leq d([n], x) + 2\delta \varepsilon\) remains true for all individuals in the population for the next \(L\) rounds, with probability at least \(1 - Le^{-\Omega(\varepsilon n)}\). By choosing the constant parameter \(\rho\) in the definition of \(L = \exp\{\rho\varepsilon n/\log^2 \mu\}\) small enough, the factor \(L\) can be swallowed by the term \(e^{-\Omega(\varepsilon n)}\). Thus we may assume that whenever \(Y_{i,a}\) is in the range \([1 - \varepsilon(1 + 3\delta/2)n/\log \mu, 1 - \varepsilon(1 + \delta)n/\log \mu]\) then \(d(A_{\ell+1}, x) \leq d([n], x) + 2\delta \varepsilon\). In addition, we have \(d(A_{\ell+1}, x) \geq \varepsilon/2\) before the level changes, since otherwise with probability \(1 - e^{-\Omega(\varepsilon n)}\) it holds that \(d(B_{\ell+1}, x) < \varepsilon\), which implies an increase of level. Thus the conditions in (3) are satisfied, and thus Lemma 6 is applicable.

So let us study the drift of \(Y_{i,a}\) in the range \([1 - \varepsilon(1 + 3\delta/2)n/\log \mu, 1 - \varepsilon(1 + \delta)n/\log \mu]\). First note that the probability to jump over this interval
(or over more than, say, half of it) is \( \exp\{-\Omega(n/\log^2 \mu)\} \): for \( Y_{i,a} \geq (1 - 4\varepsilon)n/\log \mu \) this follows from the first statement in Lemma 8, for \( Y_{i,a} < (1 - 4\varepsilon)n/\log \mu \) it follows from the second statement in Lemma 8. So we may assume that \( Y_{i,a} \) is contained in the interval at some point. Inside of the interval, by Lemma 8 the sequence of random variables \( (Y_{i,a})_{i \geq 0} \) has an upper exponential tail bound, i.e., \( \Pr[Y_{i+1,a} - Y_{i,a} > K \beta C_2] \leq K2^{-\beta} \) for all \( 1 \leq \beta \leq \varepsilon n/\log^2 \mu \). (In particular, the probability that there is ever a jump larger than \( K \beta C_2 \varepsilon n/\log^2 \mu \) within \( L \) steps is at most \( O(L \cdot 2^{-\varepsilon n/\log^7 \mu}) = o(1) \), so we may assume that such jumps never occur.) However, the negative drift theorem requires exponential tail bounds in both directions, so we need to truncate the downwards steps of \( Y_{i,a} \) as follows. We set \( \bar{Y}_{0,a} := Y_{0,a} \), and we define \( \bar{Y}_{i,a} \) recursively by \( \bar{Y}_{i+1,a} - \bar{Y}_{i,a} := \max\{Y_{i+1,a} - Y_{i,a}, -1\} \). Then \( \bar{Y}_{i,a} \) satisfies the tail bound condition in the negative drift theorem, and clearly we have \( \bar{Y}_{i,a} \geq Y_{i,a} \) for all \( i \geq 0 \). Moreover, by Theorem 11 the truncated random variable \( \bar{Y}_{i,a} \) has a negative drift of amplitude at least \( 1/\log \mu \). Hence, by the negative drift theorems in [7], for every fixed \( i \geq 0 \), with probability \( 1 - \exp\{-\Omega(\varepsilon n/\log^2 \mu)\} \) we have \( Y_{i,a} \leq \bar{Y}_{0,a} + \varepsilon n/(2 \log \mu) \), and for every fixed \( i > \alpha n \delta/(4K) - 1 \) we have \( Y_{i,a} \leq \bar{Y}_{0,a} \). The proof is concluded by a union bound over all possible \( i \). Since there are at most \( n \) possible values, this increases the error probability by a factor of \( n \), which we can swallow in the expression \( \exp\{-\Omega(\varepsilon n/\log^2 \mu)\} \). \( \square \)

Finally, we have collected all ingredients to prove our main result.

**Proof of Theorem 4.** Let \( L := \exp\{\rho \varepsilon n/\log^2 \mu\} \) be the number of levels. For the proof, we will consider an auxiliary run of the \((\mu + 1)\)-EA with a dynamic fitness function \( \check{f} \) in which we only allow the levels to increase by one. In particular, the function \( \check{f} \) does not only depend on the current state of the algorithm, but also on the algorithm’s history. More precisely, we define an auxiliary level \( \check{\ell}(x,t) \) of a search point \( x \), which we only allow to increase by at most one per round. Recall that \( \ell(x) \) was defined in (1) as \( \ell(x) = \max\{\ell' \in [L] : d(B_{\ell'}, x) \leq \varepsilon\} \). For \( \check{\ell}(t) \), we use the same definition except that we let the maximum go over only \( \ell' \leq \min\{\check{\ell}(t - 1) + 1, L\} \). I.e., we set \( \check{\ell}(0) := 0 \), and if an offspring \( y' \) of \( x' \) enters the population in round \( t \), then we set \( \check{\ell}(y', t) := \max\{\ell' \in [\min\{\check{\ell}(x', t - 1) + 1, L\}] : d(B_{\ell'}, y') \leq \varepsilon\} \). (If the population stays the same in round \( t \), then we leave \( \check{\ell} \) unchanged.) Then we define the auxiliary fitness of \( y' \) as

\[
\check{f}(y') := \check{\ell}(y', t) \cdot n^2 + \sum_{i \in A_{\check{\ell}(y', t) + 1}} y_i^t \cdot n + \sum_{i \in R_{\check{\ell}(y', t) + 1}} y_i^t,
\]

i.e., we use the same definition as for the HOTTopic function except that we replace \( \ell(y') \) by \( \check{\ell}(t) \). Then we proceed as the \((\mu + 1)\)-EA, i.e., in each round we compute and store the auxiliary fitness of the new offspring (which may depend on the whole history of the algorithm), and we remove the search point for which we have stored the lowest auxiliary fitness. This definition does not make much sense from an algorithmic perspective, but we will see in hindsight
that the auxiliary process behaves identical to the actual \((\mu + 1)\)-EA. We will next argue why this is the case.

For the auxiliary process, it is obvious that we only need to uncover the set \(A_{\ell+1}\) and \(B_{\ell+1}\) when we reach level \(\ell(t) = i\). As we will show later for the auxiliary process, with high probability the density \(d([n], x^i)\) stays strictly above \(\varepsilon \cdot (1 + \delta)\) for a suitable constant \(\delta > 0\). Now fix any round \(t\) with auxiliary level \(\ell(t)\). Since we do need to uncover \(B_{\ell(t)+2}\) at some point after time \(t\), its choice does not influence the behavior of the auxiliary process until time \(t\). Hence, we can first let the auxiliary process run until time \(t\), and afterwards uncover the set \(B_{\ell(t)+2}\). Since \(B_{\ell(t)+2} \subseteq [n]\) is a uniformly random subset of size \(\beta n\), it contains at least \(\beta \varepsilon (1 + \delta)n\) zero-bits in expectation, and the probability that \(B_{\ell(t)+2}\) contains at most \(\beta \varepsilon n\) zero-bits is \(\exp\{-\Omega(\beta \varepsilon n)\}\). The same argument also holds for \(B_{\ell(t)+3}, \ldots, B_L\). Since \(L = \exp\{\rho \varepsilon n / \log^2 \mu\}\) with desirably small \(\rho > 0\), we can afford a union bound over all such sets and all times \(t \leq L\), which is a union bound over less than \(L^2 = \exp\{2\rho \varepsilon n / \log^2 \mu\}\) terms. Hence, with high probability we have \(d(B_1, x^i) > \varepsilon\) for all \(1 \leq t \leq L\) and all \(\ell(t) + 2 \leq i \leq L\). A straightforward induction shows that this implies \(\ell(t) = \ell(t)\) for all \(t \leq L\), and thus the \((\mu + 1)\)-EA behaves identical to the auxiliary process. Note that this already implies that the \((\mu + 1)\)-EA visits each of the \(L\) levels, which implies the desired runtime bound. It only remains to show that there is a constant \(\delta > 0\) such that the auxiliary process satisfies \(d([n], x^i) > \varepsilon \cdot (1 + \delta)\) for all \(t \leq L\).

The advantage of the auxiliary process is that we may postpone drawing \(A_{\ell+1}\) until we reach level \(\ell = \ell\). In particular, since \(A_{\ell+1} \subseteq [n]\) is a uniformly random subset, we may use the same argument as before and conclude that \(|d(A_{\ell+1}, x) - d([n], x)| < \delta \varepsilon\) holds with probability \(1 - e^{-\Omega(\varepsilon n)}\) for any constant \(\delta > 0\) that we desire, and for all members \(x\) of the population when we reach level \(\ell\). In fact, we have exponentially small error probability, so we may afford a union bound and conclude that with high probability the same holds for all \(\ell\).

We want to show that the auxiliary process, if running on level \(\ell\) and starting with a population that initially satisfies \(d(A_{\ell+1}, x) < \varepsilon(1 + \delta)\) for \(\delta < 2/7\), maintains \(d([n], x^i) \geq \varepsilon(1 + \delta)\) for all new search points \(x^i\) until \(t > L\). By the first conclusion from Lemma 12, \(d([n], x^i) \geq \varepsilon(1 + \delta)\) holds as long as the level remains to be \(\ell\) and \(t \leq L\). When a point \(x\) reaches level \(\ell + 1\), by definition we have \(d(B_{\ell+1}, x) < \varepsilon\). Since \(B_{\ell+1}\) is a uniformly random subset of \(A_{\ell+1}\), by the Chernoff bound \(d(A_{\ell+1}, x) < \varepsilon(1 + \delta/4)\) holds with probability \(1 - e^{-\Omega(\varepsilon n)}\). So we apply the second conclusion of Lemma 12 to \(x\) and conclude that \(d([n], x) \geq \varepsilon(1 + 2\delta)\). With high probability, it holds that \(d(A_{\ell+2}, x) \geq \varepsilon(1 + 2\delta) - \varepsilon \delta\) and the conditions in Lemma 12 are satisfied again for level \(\ell + 1\). By induction we obtain \(d([n], x^i) \geq \varepsilon(1 + \delta)\) for all \(t \leq L\). As the choice of \(\ell\) is arbitrary, we start with \(\ell = 0\) and \(d([n], x^i) \geq \varepsilon(1 + \delta)\) holds for all \(t \leq L\). This concludes the proof.
7 Conclusion

We have shown that the \((\mu + 1)\)-EA with arbitrary mutation parameter \(c > 0\) needs exponential time on some monotone functions if \(\mu\) is too large. This is one of the very few known situations in which even a slightly larger population size \(\mu\) can lead to a drastic decrease in performance. The main reason is that, if progress is steady enough that the population does not degenerate, the search points that produce offspring are typically not the fittest ones. We believe that this is an interesting phenomenon which deserves further investigations, also in less artificial contexts.

For example, consider the \((\mu + 1)\)-EA on weighted linear functions with a skewed distribution (e.g., on BinVal), and with a fixed time budget (so that the action happens away from the optimum). It is quite conceivable that the same effect hurts performance, i.e., if the algorithm flips a high-weight bit, it will allow (almost) any offspring of this individual into the population, even though this offspring has probably fewer correct bits than other search points in the population. Does that mean that the fixed-budget performance of the \((\mu + 1)\)-EA on \text{BinVal} deteriorates with increasing \(\mu\)? Are the resulting individuals further away from the optimum?

An even more pressing question is about crossover. We have studied the \((\mu + 1)\)-EA, but do the same results also apply for the \((\mu + 1)\)-GA? In [12] it was shown that close to the optimum (for small values of the HotTopic parameter \(\varepsilon\)) crossover helps dramatically, and that a large population size can even counterbalance large mutation parameters \(c\). So, close to the optimum, for the \((\mu + 1)\)-GA the effect of large population size was beneficial, while for the \((\mu + 1)\)-EA it was neutral and did not affect the threshold \(c_0\). Thus if we study the \((\mu + 1)\)-GA on HotTopic functions with large \(\varepsilon\), then a beneficial effect of large populations is competing with a detrimental effect. Understanding this interplay would be a major step towards a better understanding of crossover in general.

Similarly, since the problems originate in non-trivial populations, what happens if we equip the \((\mu + 1)\)-EA with a diversity mechanism (duplication avoidance, genotypical or phenotypical niching), and study it close to the optimum? Does it fall for the same traps? This question was already asked in [12], but our results shed additional light on the question.

Finally, it is open whether the \((\mu + 1)\)-EA is fast on any monotone function if it starts close enough to the optimum. i.e., for every \(\mu \in \mathbb{N}\), does there exist an \(\varepsilon = \varepsilon(\mu)\) such that the \((\mu + 1)\)-EA, initialized with a random search point with \(\varepsilon n\) zero-bits, has runtime \(O(n \log n)\) for every monotone function? Of course, the same question also applies to other algorithms like the \((\mu + 1)\)-GA and the ‘fast’ counterparts of the \((\mu + 1)\)-EA and the \((\mu + 1)\)-GA. Interestingly, the result in [12] that the ‘fast (1+\(\lambda\))-EA’ with good parameters is efficient for every monotone function was only proven under this assumption, that the algorithm starts close to the optimum. So this also raises the question whether there are traps for the ‘fast (1+\(\lambda\))-EA’ that only take effect far away from the optimum.
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