Angular Momentum of the BTZ Black Hole in the Teleparallel Geometry

A. A. Sousa*, R. B. Pereira
Departamento de Matemática
Instituto de Ciências e Letras do Médio Araguaia
Universidade Federal de Mato Grosso
78698-000 Pontal do Araguaia, MT, Brazil
Departamento de Física, ICET-UFMT, MT, Brazil
and
J. F. da Rocha-Neto
Fundação Universidade Federal do Tocantins
Campus Universitário de Arraias,
Rua Universitária, s/n centro Arraias-TO, Brazil
77330-000

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Abstract

We carry out the Hamiltonian formulation of the three-dimensional gravitational teleparallelism without imposing the time gauge condition, by rigorously performing the Legendre transform. Definition of the gravitational angular momentum arises by suitably interpreting the integral form of the constraint equation $\Gamma^{ik} = 0$ as an angular momentum equation. The gravitational angular momentum is evaluated for the gravitational field of a rotating BTZ black hole.

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(*) E-mail: adellane@cpd.ufmt.br
1 Introduction

The search for a consistent expression for the gravitational energy and angular momentum of a self-gravitating distribution of matter is undoubtedly a long-standing problem in general relativity. The gravitational field does not possess the proper definition of an energy momentum tensor and an angular momentum tensor and one usually defines some energy-momentum and angular momentum as Bergmann [1] or Landau-Lifschitz [2] which are pseudo-tensors. The Einstein’s general relativity can also be reformulated in the context of the teleparallel (Weitzenböck) geometry. In this geometrical setting the dynamical field quantities correspond to orthormal tetrad fields $e^a_\mu$ ($a, \mu$ are $\text{SO}(3,1)$ and space-time indices, respectively). These fields allow the construction of the Lagrangian density of the teleparallel equivalent of general relativity (TEGR), which offers an alternative geometrical framework for Einstein’s equations. The Lagrangian density for the tetrad field in the TEGR is given by a sum of quadratic terms in the torsion tensor $T^a_\mu\nu = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu$, which is related to the anti-symmetric part of Cartan’s connection $\Gamma^\lambda_\mu\nu = e^{a\lambda} \partial_\mu e^a_\nu$. The curvature tensor constructed out of the latter vanishes identically. This connection defines a space with teleparallelism, or absolute parallelism [3].

The Hamiltonian formulation, when consistently established, not only guarantees that field quantities have a well defined time evolution, but also allow us to understand physical theories from a different perspective. We have learned from the work of Arnowitt, Deser and Misner (ADM) [4] that the Hamiltonian analysis of Einstein’s general relativity reveals the intrinsic structure of the theory: the time evolution of field quantities is determined by the Hamiltonian and vector constraints. Thus four of the ten Einstein’s equations acquire a prominent status in the Hamiltonian framework. Ultimately this is an essential feature for the canonical approach to the quantum theory of gravity. In the framework of the TEGR it is possible to make definite statements about the energy and momentum of the gravitational field. This fact constitutes the major motivation for considering this theory. In the 3+1 formulation of the TEGR [5], and by imposing Schwinger’s time gauge condition [6] for the tetrad field by fixing $e^{(k)}_0 = 0$ which implies $e^{(0)}_k = 0$, it is found that the Hamiltonian and vector constraints contain each one a divergence in the form of scalar and vector densities, respectively, that can be identified with the energy and momentum densities of the gravitational field.
field [7]. This identification has proved to be consistent, and it has been
demonstrated that teleparallel theories provide a natural instrument for the
investigation of gravitational energy. The most relevant application is the
evaluation of the irreducible mass of the Kerr black hole [8].

In the Hamiltonian formulation of the TEGR, with no \textit{a priori} restriction
on the tetrad fields, there arises a set of primary constraints $\Gamma^{ik}$ that
satisfy the angular momentum algebra [9]. Following the prescription for
defining the gravitational energy, the definition of the gravitational angular
momentum arises by suitably interpreting the integral form of the constraint
equation $\Gamma^{ik} = 0$ as an angular momentum equation. It has been applied
this definition to the gravitational field of a thin, slowly rotating mass shell.

On the other hand, gravity theories in three dimensions have gained con-
siderable attention in the recent years. The expectation is that the study of
lower-dimensional theories will provide relevant information about the corre-
sponding theory in four dimensions. A family of three-parameter teleparallel
theories in 2+1 dimensions are proposed by Kawai [11], and a Hamiltonian
formulation was developed [12]. The definition of energy of the gravitational
field [13] was proposed with the use of the orthonormal triads fields $e^a_\mu$
($a, \mu$ are SO(2,1) and space-time indices, respectively) in Schwinger’s time
gauge [6].

In this paper we carry out the Hamiltonian formulation of this three-
parameter theory in three-dimensions without imposing the time gauge con-
dition, by rigorously performing the Legendre transform. We have not found
it necessary to establish a 2+1 decomposition for the triad field. Follow-
ing Dirac’s method [14], we find that the three-parameter family is reduced
to one-parameter family. We conclude that the Legendre transform is well
deﬁned only if certain conditions on the parameters are satisfied. These con-
ditions are $c_1 + 8 \gamma c_3 = 0$ and $3c_1 + 4c_2 = 0$ (These results appeared in the
canonical formulation by imposing the time gauge condition). These are the
conditions for the formulation to be well deﬁned from the point of view of the
initial value problem. The value of this unique parameter was ﬁxed in pre-
vious works of the author [13] as -2/3, leading to the solution of the so-called
three-dimensional BTZ black hole solution [15] in the teleparallel geometry.
Since its discovery in 1992, BTZ black hole solution has often been used in
current literature as a simple but realistic model for black hole physics.

The BTZ solution has been used to study quantum models in 2+1 dimen-
sions because the corresponding models in 3+1 dimensions are very complicated [16]. The constraint algebra of the theory suggests that certain momentum components are related to the gravitational angular momentum. It turns out to be possible to define, in this context, the angular momentum of the gravitational field. We apply this definition to the gravitational field of a rotating BTZ black hole. It turns out, however, that consistent values for the gravitational angular momentum are achieved by requiring the triad field to satisfy (a posteriori) the time gauge condition (this seems to be a requirement to obtain the irreducible mass of the Kerr black hole, in the quadridimensional case) [10]. In the region near to the surface of the BTZ black hole $r > r_0$, we arrive at $M^{12} \simeq -\frac{J}{2} \left(1 - \frac{r_0}{r}\right)$ where $J$ is the angular momentum of the source and $r_0$ is the external horizon of the BTZ black hole. We compare our expression of gravitational angular momentum with the result obtained by means of Komar method (this approach assumes the existence of certain Killing vector fields that allow the construction of conserved integral quantities) [17]. We find that the results are different. However, it is noted that the result obtained by means of our method is much simpler and does not depend on Killing vectors. We confirm that we are dealing with the angular momentum of the gravitational field and not of the source of the field. These results indicate that the alternative geometrical formulation provided by the TEGR allows to obtain expressions for the energy, momentum and angular momentum of the gravitational field.

The article is divided as follows. In the section 2, we review the Lagrangian formulation and carry out the Hamiltonian formulation of arbitrary teleparallel theories without fixing gauge. In the section 3, we obtain a realistic measure of the angular momentum of the field in terms of the angular momentum of the source. In the section 4, we compare our result with the gravitational angular momentum of BTZ black hole calculated by means of the Komar method. Finally, in the section 5, we present our conclusions.

We employ the following notation. Space-time indices $\mu, \nu, \ldots$ and SO(2,1) indices $a, b, \ldots$ run from 0 to 2. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The triad field $e^a{}_{\mu}$ yields the definition of the torsion tensor: $T^a{}_{\mu\nu} = \partial_{[\mu}e^a{}_{\nu]} - \partial_{[\nu}e^a{}_{\mu]}$. The flat Minkowski space-time metric is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (- + +)$. 
2 Hamiltonian formulation

We begin by introducing the three basic postulates that the Lagrangian density of the gravitational field in empty space, in the teleparallel geometry, must satisfy. It must be invariant under (i) coordinate transformations, (ii) global $[SO(2, 1)]$ Lorentz’s transformations, and (iii) parity transformations. In the present formulation, we add a negative cosmological constant $\Lambda = -\frac{2}{l^2}$ to the Lagrangian density. The most general Lagrangian density quadratic in the torsion tensor is written

$$L = e \left( c_1 t_{abc} t_{abc} + c_2 v^a v_a + c_3 a_{abc} a^{abc} \right)$$  \hspace{1cm} (1)

where $c_1, c_2$ and $c_3$ are constants, $e = \det (e^a_{\mu})$, and

$$t_{abc} = \frac{1}{2} (T_{abc} + T_{bac}) + \frac{1}{4} (\eta_{ca} v_b + \eta_{cb} v_a) - \frac{1}{2} \eta_{ab} v_c,$$  \hspace{1cm} (2)

$$v_a = T_{ba}^b,$$  \hspace{1cm} (3)

$$a_{abc} = \frac{1}{3} (T_{abc} + T_{cab} + T_{bca}) ;$$  \hspace{1cm} (4)

$$T_{abc} = e^\mu_b e_\nu_c T_{a\mu\nu}.$$  \hspace{1cm} (5)

The definitions given above correspond to the irreducible components of the torsion tensor [11]. The field equations for the Lagrangian density (1) are obtained in Ref. [11].

The Hamiltonian formulation is obtained by writing the Lagrangian density in first-order differential form. For this purpose we introduce an auxiliary field quantity $\Delta_{abc} = -\Delta_{acb}$ that will be related to the torsion tensor. The first-order differential Lagrangian formulation in empty space-time reads

$$L(e, \Delta) = -e \Lambda_{abc} (\Delta_{abc} - 2T_{abc}),$$  \hspace{1cm} (6)

where

$$\Lambda_{abc} = c_1 \Theta_{abc} + c_2 \Omega_{abc} + c_3 \Gamma_{abc},$$

and $\Theta_{abc}, \Omega_{abc}$ and $\Gamma_{abc}$ are defined as,

$$\Theta_{abc} = \frac{1}{2} \Delta_{abc} + \frac{1}{4} \Delta_{bac} - \frac{1}{4} \Delta_{cab} + \frac{3}{16} \left( \eta^{ca} \Delta^b - \eta^{ba} \Delta^c \right),$$  \hspace{1cm} (7)
\[ \Omega^{abc} = \frac{1}{2} \left( \eta^{ab} \Delta_c - \eta^{ac} \Delta^b \right), \quad (8) \]

\[ \Gamma^{abc} = \frac{1}{3} \left( \Delta^{abc} + \Delta^{bca} + \Delta^{cab} \right), \quad (9) \]

where \( T_{abc} = e_{b\mu} e_{c\nu} T_{a\mu\nu} \) and \( \Delta_b = \Delta^a_{ab} \).

Variation of the action constructed out of (6) with respect to \( \Delta_{abc} \) yields an equation that can be reduced to \( \Delta_{abc} = T_{abc} \). This equation can be split into two equations:

\[ \Delta_{a0k} = T_{a0k} = \partial_0 e_{ak} - \partial_k e_{a0}, \quad (10) \]

\[ \Delta_{ai0} = T_{ai0} = \partial_i e_{ak} - \partial_k e_{ai} \cdot \quad (10) \]

The Hamiltonian density will be obtained by the standard prescription \( L = p\dot{q} - H_0 \) and by properly identifying primary constraints. We have not found it necessary to establish any kind of 2+1 decomposition for the triad field. Therefore in the following both \( e_{a\mu} \) and \( g_{\mu\nu} \) are space-time fields. The analysis developed here is similar to that developed in Refs. [9, 10].

The Lagrangian density can be expressed as

\[ L(e, \phi) = 4 e \Lambda^{a0k} \dot{e}_{ak} - 4 e \Lambda^{a0k} \partial_k e_{a0} + 2 e \Lambda^{aij} T_{aij} + e \Lambda^{abc} \dot{\phi}_{abc}, \quad (11) \]

where the dot indicates time derivative, and \( \Lambda^{a0k} = \Lambda^{abc} e_b^0 e_c^k \), \( \Lambda^{aij} = \Lambda^{abc} e_b^i e_c^j \).

Therefore the momentum canonically conjugated to \( e_{ak} \) is given by

\[ P^{ak} = 4 e \Lambda^{a0k}. \quad (12) \]

In terms of (12) expression (11) reads

\[ L = P^{ak} \dot{e}_{ak} - P^{ak} \partial_k e_{a0} - e \Lambda^{aij}(-2T_{aij} + \Delta_{aij}) - 2e \Lambda^{a0i} \Delta_{a0i}. \quad (13) \]

The last term on the right hand side of equation (13) is identified as

\[ 2e \Lambda^{a0i} \Delta_{a0i} = \frac{1}{2} P^{ai} \Delta_{a0i}. \]
The Hamiltonian formulation is established once we rewrite the Lagrangian density (13) in terms of $e_{ak}$, $\Pi_{ak}$ and further nondynamical field quantities. It is carried out in two steps. First, we take into account equations (10) and (13) so that half of the auxiliary fields, $\phi_{aij}$, are eliminated from the Lagrangian density by means of the identification
\begin{equation}
\Delta_{aij} = T_{aij}.
\end{equation}
Therefore we have
\begin{equation}
L(e_{ak}, P^{ak}, e_{a0}, \Delta_{a0k}) = P^{ak} e_{ak} + e_{a0} \partial_k P^{ak} - \partial_k (e_{a0} P^{ak})
\end{equation}
\begin{equation}
- e \frac{1}{4} g^{ik} g^{jl} T^a_{kl} T_{aij} \left( \frac{c_1}{2} + \frac{c_3}{3} \right) + \left( -\frac{1}{2} c_1 + \frac{2}{3} c_3 \right) g^{il} T^j_{kl} T^k_{ij} +
\end{equation}
\begin{equation}
+ \left( -\frac{3}{4} c_1 + c_2 \right) g^{ik} T^j_{ji} T^n_{nk}
\end{equation}
\begin{equation}
- \frac{1}{2} \Delta_{a0k} \left\{ P^{ak} - 2 e \left[ g^{i0} g^{jk} T^a_{ij} \left( c_1 + \frac{2}{3} c_3 \right) +
\end{equation}
\begin{equation}
+ e^{ai} \left( g^{i0} T^k_{ij} - g^{jk} T^o_{ij} \right) \left( -\frac{1}{2} c_1 + \frac{2}{3} c_3 \right) +
\end{equation}
\begin{equation}
\left( \frac{3}{4} c_1 - c_2 \right) \left( e^{ak} g^{i0} - e^{a0} g^{ik} \right) T^j_{ji} \right\}\}.
\end{equation}

2.1 The canonical momentum
The second step consists of expressing the remaining auxiliary field quantities, the “velocities” $\Delta_{a0k}$, in terms of the momenta $P^{ak}$. This is the nontrivial step of the Legendre transform. Denoting (..) and [..] as the symmetric and anti-symmetric parts of field quantities, respectively, we decompose $P^{ak}$ into irreducible components plus terms of excess:
\begin{equation}
P^{ak} = e^a_i P^{(ik)} + e^a_i P^{[ik]} + e^a_0 P^{0k} + e \left\{ e^a_i \left( c_1 + \frac{8}{3} c_3 \right) g^{00} g^{ik} \Delta^j_{0j}
\end{equation}
\begin{equation}
+ \frac{1}{2} \left( 3c_1 + 4c_2 \right) g^{00} g^{ik} \Delta^j_{0j} +
\end{equation}
\[ + \left( c_1 + \frac{8}{3} c_3 \right) g^{0i} g^{0k} \Delta^j_{0j} - \frac{1}{2} (3c_1 + 4c_2) g^{0i} g^{0k} \Delta^j_{0j} + \\
+ \left( c_1 + \frac{8}{3} c_3 \right) g^{ik} g^{0j} \Delta^0_{0j} - \frac{1}{2} (3c_1 + 4c_2) g^{ik} g^{0j} \Delta^0_{0j} + \\
+ \left[ \left( c_1 + \frac{8}{3} c_3 \right) - \frac{1}{2} (3c_1 + 4c_2) \right] g^{00} g^{ik} T^j_{jl} \right] + \\
+ \left( c_1 + \frac{8}{3} c_3 \right) g^{0i} g^{jk} \Delta^0_{0j} - \left( c_1 + \frac{8}{3} c_3 \right) g^{0k} g^{0j} T^i_{0j} + \\
+ \frac{1}{2} \left[ (3c_1 + 4c_2) - \left( c_1 + \frac{8}{3} c_3 \right) \right] g^{0i} g^{jk} T^0_{0j} + \\
+ \left( c_1 + \frac{8}{3} c_3 \right) g^{0i} g^{jk} T^i_{0j} - \\
- \left[ \left( c_1 + \frac{8}{3} c_3 \right) - \frac{1}{2} (3c_1 + 4c_2) \right] g^{0i} g^{kl} T^j_{jl} + \\
e^a \left\{ \left( c_1 + \frac{8}{3} c_3 \right) - \left( c_1 + \frac{8}{3} c_3 \right) \right\} g^{00} g^{ki} \Delta^0_{0i} - \\
- \frac{1}{2} (3c_1 + 4c_2) g^{0k} g^{0i} \Delta^0_{0i} + \left( c_1 + \frac{8}{3} c_3 \right) \left( g^{00} g^{kl} - g^{0l} g^{0k} \right) T^j_{ij} + \\
+ \frac{1}{2} (3c_1 + 4c_2) \left( g^{0l} g^{0k} - g^{00} g^{kl} \right) T^j_{lj} + \\
\left( c_1 + \frac{8}{3} c_3 \right) g^{00} g^{ik} \Delta^0_{0i} + \left( c_1 + \frac{8}{3} c_3 \right) g^{0i} g^{jk} T^0_{ij} \right\}, \tag{16} \]

where

\[ P^{(ik)} = - \left( c_1 - \frac{4}{3} c_3 \right) e \left\{ g^{0i} (3g^{jk} \Delta^0_{0j} - g^{ji} \Delta^k_{0j} + 2g^{ik} \Delta^j_{0j}) + \\
+ g^{0k} (g^{0j} \Delta^0_{0j} + g^{kj} \Delta^0_{0j} - g^{0i} \Delta^j_{0j}) + \\
+ g^{0i} (g^{0j} \Delta^k_{0j} + g^{kj} \Delta^0_{0j} - g^{0i} \Delta^j_{0j}) - 2g^{ik} g^{0j} \Delta^0_{0j} - N^{ik} \right\}, \tag{17} \]

\[ N^{ik} = - g^{0l} (g^{jk} T^l_{ij} + g^{ij} T^k_{lj} - 2g^{ik} T^j_{lj}) - (g^{kl} g^{0i} + g^{ij} g^{0k} T^j_{lj} \right) \tag{18} \]
\[ P^{[ik]} = - (c_1 - \frac{4}{3} c_3) e \left\{ - g^{ij} g^{kj} T^0_{ij} + (g^{ik} g^{0k} - g^{kl} g^{0l}) T^j_{ij} \right\} , \quad (19) \]

\[ P^{0k} = 2 \left( c_1 - \frac{4}{3} c_3 \right) e \left[ g^{0i} g^{ik} T^0_{ij} + (g^{00} g^{kl} - g^{0l} g^{0k}) T^j_{ij} \right] , \quad (20) \]

where we have already identified \( \Delta_{aij} = T_{aij} \).

### 2.2 Conditions on the free parameters

Before carrying out the Legendre transform we can establish the conditions under which the Lagrangian density will be exempt of the “velocities” \( \Delta_{a0k} \).

We see that the excess terms, which contains several \( \Delta_{a0k} \) type terms, is discarded if we require

\[ \begin{align*}
    c_1 + \frac{8}{3} c_3 &= 0, \\
    3c_1 + 4c_2 &= 0.
\end{align*} \quad (21) \]

These results also appeared in the canonical formulation imposing the time gauge condition [12]. The crucial point in this analysis is that only the symmetrical components \( P^{(ij)} \) depend on the “velocities” \( \Delta_{a0k} \). The other three components, \( P^{[ij]} \) and \( P^{0k} \) depend solely on \( T_{aij} \). Therefore we can express only three of the “velocity” fields \( \Delta_{a0k} \) in terms of the components \( P^{(ij)} \). With the purpose of finding out which components of \( \Delta_{a0k} \) can be inverted in terms of the momenta we decompose \( \Delta_{a0k} \) identically as

\[ \Delta^{a}_{0j} = e^{ai} \psi_{ij} + e^{ai} \sigma_{ij} + e^{a0} \lambda_j , \quad (22) \]

where \( \psi_{ij} = \frac{1}{2}(\Delta_{0ij} + \Delta_{j0i}) \), \( \sigma_{ij} = \frac{1}{2}(\Delta_{0ij} - \Delta_{j0i}) \), \( \lambda_j = \Delta_{00j} \), and \( \Delta_{\mu0j} = e^{a}_{\mu} \Delta_{a0j} \) (like \( \Delta_{abc} \), the components \( \psi_{ij} \), \( \sigma_{ij} \) and \( \lambda_j \) are also auxiliary field quantities). By defining

\[ \Pi^{ik} = \frac{1}{e} P^{(ik)} + \left( c_1 - \frac{4}{3} c_3 \right) M^{ik} , \quad (23) \]

we find that \( \Pi^{ik} \) depends only on \( \psi_{ij} \):
\[ \Pi^{ik} = -2g^{00}(g^d g^{jk} \psi_{lj} - g^{ik} \psi) + \\
+2(g^0 g^{kl} g^{0j} + g^0 k^l g^{0j}) \psi_{lj} - 2(g^{ik} g^{0l} g^{0j} \psi_{lj} + g^{0i} g^{0k} \psi) , \quad (24) \]

where \( \psi = g^{ik} \psi_{ik} \).

We can now invert \( \psi_{mn} \) in terms of \( \Pi^{ik} \). After a number of manipulations we arrive at

\[ \psi_{mn} = -\frac{1}{2g^{00}} \left( g_{im} g_{kn} \Pi^{ik} - g_{mn} \Pi \right) , \quad (25) \]

where \( \Pi = g_{ik} \Pi^{ik} \).

At last we need to rewrite the third line of the Lagrangian density in terms of canonical variables. By making use of (16), (21), (22) and (25) we can rewrite

\[ -\frac{1}{2} \Delta_{a0k} \left\{ P^a_k - \frac{3}{2} e_c \left\{ g^{0i} g^{jk} T_{ij} - \\
- e^{a i} (g^{0j} T_{i j} - g^{kj} T_{ij}) + \\
+2(e^{ak} g^{0i} - e^{0a} g^{ki}) T_{ji} \right\} \right\} = \\
-\frac{1}{2} \Delta_{a0k} \left\{ \frac{3}{2} e_c \left\{ g^{00} \left( -g^{ik} \Delta_{a0i} - e^{ai} \Delta^k_{0i} + 2e^{ak} \Delta i_{0i} \right) + \\
+g^{0k} (g^{i0} \Delta_{a0i} + e^{ai} \Delta^0_{0i}) + e^{a0} (g^{0i} \Delta^k_{0i} + g^{ki} \Delta^0_{0i}) - \\
-2(e^{a0} g^{ko} \Delta^i_{0i} + e^{ak} g^{0i} \Delta^0_{0i}) \right\} \right\} , \quad (26) \]

in the form

\[ -\frac{3}{2} e_c \left( \frac{1}{4g^{00}} \right) \left( g_{mi} g_{nj} \Pi^{mn} \Pi^{ij} - \Pi^2 \right) . \quad (27) \]
2.3 Total Hamiltonian density

Thus we finally obtain the primary Hamiltonian density $H_0 = P^{ak} \dot{e}_{ak} - L$,

$$H_0(e_{ak}, P^{ak}, e_{a0}) = e_{a0} \partial_k P^{ak} + \frac{3}{2} ec_1 \left( \frac{1}{4g^{00}} \right) \left( g_{mi} g_{nj} \Pi^{mn} \Pi^{ij} - \Pi^2 \right) -$$

$$- \frac{3}{2} ec_1 \left( \frac{1}{4} g^{ik} g^{ji} T^a_{ki} T_{aij} - \frac{1}{2} g^{il} T^j_{ki} T^k_{ij} - g^{ik} T^j_{ij} T^m_{ik} n_k \right). \quad (28)$$

We may now write the total Hamiltonian density. For this purpose we have to identify the primary constraints. They are given by expressions (19) and (20), which represent relations between $e_{ak}$ and the momenta $\Pi^{ak}$. Thus we define (primary constraints)

$$\Gamma^{ik} = -\Gamma^{ki} = P^{[ik]} + \frac{3}{2} ec_1 e \left\{ -g^{il} g^{kj} T^0_{li} + (g^{il} g^{0k} - g^{kl} g^{0i}) T^j_{lj} \right\}, \quad (29)$$

$$\Gamma^k = P^{0k} - 3ec_1 e \left[ g^{ik} g^{0i} T^0_{ij} + (g^{00} g^{kl} - g^{0l} g^{0k}) T^j_{ij} T^j_{lj} \right]. \quad (30)$$

Therefore the total Hamiltonian density is given by

$$H(e_{ak}, P^{ak}, e_{a0}, \alpha_{ik}, \beta_k) = H_0 + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k + \partial_k (e_{a0} P^{ak}), \quad (31)$$

where $\alpha_{ik}$ and $\beta_k$ are Lagrange multipliers.

3 Algebra of the gravitational angular momentum

The calculations of the Poisson brackets between these constraints are exceedingly complicated. We describe here only the algebra of angular momentum. The Poisson bracket algebra closes in quadridimensional formulation, since all the constraints of the theory are first class.
The Poisson bracket between two quantities \( F \) and \( G \) is defined by

\[
\{F, G\} = \int d^3x \left( \frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta P_{ai}(x)} - \frac{\delta F}{\delta P_{ai}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right),
\]

by means of which we can write down the evolution equations.

The Hamiltonian density determines the time evolution of any field quantity \( f(x) \):

\[
\dot{f}(x) = \int d^3y \{f(x), H(y)\}|_{\Gamma^{ik} = \Gamma^k = 0}. \tag{32}
\]

The Poisson brackets of primary constraints \( \Gamma^{ik} \) are given by

\[
\{\Gamma^{ij}(x), \Gamma^{kl}(y)\} = \frac{1}{2} \left( g^{il} \Gamma^{jk} + g^{jk} \Gamma^{il} - g^{ik} \Gamma^{jl} - g^{jl} \Gamma^{ik} \right) \delta(x - y), \tag{33}
\]

that resemble the ordinary algebra of angular momentum.

Finally, we would like to remark that the Hamiltonian formulation of the theory can be described more succinctly in terms of the constraints \( H_0, \Gamma^{ik} \) and \( \Gamma^k \), by the Hamiltonian density in the form

\[
H(e_{ak}, P^{ak}, e_{a0}, \alpha_{ik}, \beta_k) = H_0 + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k. \tag{34}
\]

without the surface term.

4 Gravitational angular momentum of a rotating BTZ black hole

The main motivation for considering the angular momentum of the gravitational field in the present investigation resides in the fact that the constraints \( \Gamma^{ik} \), satisfy the algebra of angular momentum. Indeed, Ref. [10] presented some progress in this respect.

Following the prescription for defining the gravitational energy out of the Hamiltonian constraint of the TEGR and considering \( c_1 = -2/3 \), we interpret the integral form of the constraint equation \( \Gamma^{ik} = 0 \) as an angular
momentum equation, and therefore we define the angular momentum of the gravitational field \( M^{ik} \) according to
\[
M^{ik} = \int_S d^2 x \, P^{[ik]} = \int_S d^2 x \, e \left[ -g^{li} g^{kj} T^0_{lj} + (g^{il} g^{0k} - g^{kl} g^{0i}) T^i_j \right], \tag{35}
\]
for an arbitrary surface \( S \) of the bi-dimensional space.

### 4.1 The determination of triad fields of the BTZ black hole

Since the definition of the above equation is a bi-dimensional integral we will consider a space-time metric that exhibits rotational motion. One exact solution that is everywhere regular in the exterior region of the rotating source is the metric associated to a rotational BTZ black hole. The main motivation for considering this metric is the construction of a source for the exterior region of space-time. The metric reads [15]
\[
ds^2 = -N^2 dt^2 + f^{-2} dr^2 + r^2 (N^\theta dt + d\theta)^2, \tag{36}\]
where
\[
N(r) = f(r) = \sqrt{\left(-8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right)}, \tag{37}\]
\[
N^\theta(r) = -\frac{4G}{r^2} J, \tag{38}\]
\( J \) can be identified with the angular momentum of the source. The gravitational constant \( G \) has the dimensions of an inverse mass [15]. The space-time geometry of the BTZ black hole is one of constant negative curvature and therefore it is, locally, that of anti-de Sitter (AdS) space. Thus, the BTZ black hole can only differ from the anti-de Sitter space in its global properties.

The set of triad fields that satisfy the metric is given by
\[
e_{a\mu} = \begin{pmatrix}
\left(-8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right)^{1/2} & 0 & 0 \\
0 & 0 & \left(-8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right)^{-1/2} \\
-\frac{4GJ}{r} & \left(-8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right)^{-1/2} & r
\end{pmatrix}, \tag{39}\]
where the determinant of $e_{a\mu}$ is

$$e = r.$$  

(40)

The only nonvanishing component of the torsion tensor that is needed in the following reads

$$T^{(2)}_{12} = 1.$$  

(41)

The anti-symmetric components $P^{[ik]}$ can be easily evaluated. We obtain

$$P^{[12]}(r) = e T^{(2)}_{12} g^{02} e^{(2)} 2 g^{11},$$

(42)

where

$$g^{02} = -\frac{4GJ}{r^2 \left(-M + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right)},$$

$$e^{(2)} = \frac{1}{r},$$

$$g^{11} = \left(-8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}\right).$$

The only nonvanishing component of the angular momentum is given by

$$M^{[12]} = \frac{1}{16\pi G} \int_S d^2 x \ P^{[12]} = -\frac{1}{4\pi} \int_0^{2\pi} d\theta \int_{r_0}^r \frac{J}{r} dr.$$  

(43)

The integral is calculated in the region near to the surface of the BTZ black hole $r > r_0$ [18], because the metric possess regular solution in the exterior region of the rotating source. In the limit $r \to r_0$, the calculation is straightforward. Since $\left|\frac{r_0}{r} - 1\right| < 1$, we find

$$M^{[12]} = \frac{J}{2} \ln \left(1 + \frac{r_0 - r}{r}\right) \simeq -\frac{J}{2} \left(1 - \frac{r_0}{r}\right),$$

(44)

where $r_0$ is the external horizon of the BTZ black hole. We identify $M^{[12]}$ as the angular momentum of the gravitational field. This quantity is less than the angular momentum of the source. In the limit $r \to \infty$, we recover the anti-de Sitter space and $M^{[12]} \to -\infty$, because the gravitational field is more intense at points far from the black hole. Moreover we know that the energy density of the gravitational field diverges in the limit $r \to \infty$ [19].
5 Integral of Komar

In order to assess the significance of the above result, we present here the angular momentum associated to the metric tensor by means of Komar’s integral $Q_K$ [17],

$$Q_K = \frac{1}{8\pi} \int_S \sqrt{-g} \varepsilon_{\alpha\beta\mu\nu} \nabla^{[\alpha} \xi^{\beta]} dx^\mu \wedge dx^\nu,$$  \hspace{1cm} (45)

where $S$ is a surface of the “disk” of radius $R \to \infty$, $\xi^\mu$ is the Killing vector field $\xi^\mu = \delta_2^\mu$ and $\nabla$ is the covariant derivative constructed out of the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$. Introducing the gravitational constant $G$, the integral $Q_K$ reduces to

$$Q_K = -\frac{J}{2}.$$  \hspace{1cm} (46)

where $J$ is again the angular momentum of the source.

One expects the gravitational angular momentum to be of the order of magnitude of the intensity of the gravitational field. We observe that Komar’s integral yields a value proportional to the angular momentum of the source, whereas $M^{12}$ is smaller than $Q_K$. In similarity to Ref. [10], we observe that $M^{12}$ yields the angular momentum of the gravitational field, not of the source, in contrast to $Q_K$.

6 Conclusions

In the context of Einstein’s general relativity rotational phenomena is certainly not a completely understood issue. The prominent manifestation of a purely relativistic rotational effect is the dragging of inertial frames. If the angular momentum of the gravitational field of isolated systems has as meaningful notion, than it is reasonable to expect the latter to be somehow related to the rotational motion of the sources, but not equal to the angular momentum of the sources. The Hamiltonian formulation of TEGR in three dimensions can provide an easy expression to calculate the gravitational angular momentum with the use of the triad fields obtained from the metric. It resulted to be smaller than Komar’s expression in the region near the black hole. Komar’s expression is independent of the surface of integration, and therefore its value does represent the angular momentum of the source, not of
the field. It is necessary to test our expression of angular momentum to other configurations of the gravitational field, to verify its consistency. Efforts in this respect will be carried out.

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