An inner model theoretic proof of Becker’s theorem

Grigor Sargsyan

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Abstract

We re-prove Becker’s theorem from [1] by showing that $AD^{L(R)}$ implies that $L(R) \models \omega_2$ is $\delta_1^2$-supercompact. Our proof uses inner model theoretic tools instead of Baire category. We also show that $\omega_2$ is $\Theta$-strongly compact.

This article draws inspiration from the work of Neeman ([5]) who, using inner model theoretic tools, showed that under $AD^{L(R)}$, $\omega_1$ is $\Theta$-supercompact. We have also been influenced by the work of Becker ([1]), Becker-Jackson ([2]) and Jackson ([4]). In [1], Becker showed that assuming $AD+V=L(R)$, $\omega_2$ is $\delta_1^2$-supercompact. In [2], Becker and Jackson showed that, under $AD+V=L(R)$, all projective cardinals are $\delta_1^2$-supercompact. Finally, in [4], Jackson showed that under $AD+V=L(R)$ all Suslin cardinals and their successors are $\delta_1^2$-supercompact.

In this short note, we re-prove Becker’s theorem using inner model theoretic tools. The paper assumes familiarity with what is commonly called HOD analysis. The reader can find this background exposited in [5] and in [8]. The point of re-proving such results is to find more applications of inner model theory in descriptive set theory. In particular, we strongly believe that connecting iteration sets with Kechris-Woodin generic codes will yield many applications, and thus invite the community to consider Conjecture 3.1.

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1 Measures on $\wp_{\omega_2}(\lambda)$

We do not want to make the paper artificially long. The paper is aimed at experts of inner model theory, those who are familiar with the terminology of [8].
We assume $AD + V = L(\mathbb{R})$. Fix $\lambda < \Theta$. Let $A$ be an OD set of reals such that $\gamma_{A,\infty} \geq \lambda$. Suppose $\mathcal{R}$ is a suitable premouse that is $A$-iterable. It is customary to let $\delta^\mathcal{R}$ be the Woodin cardinal of $\mathcal{R}$. Assume that $\lambda \in \text{rng}(\pi(\mathcal{R},A),\infty)$. We then let $\lambda^\mathcal{R}$ be such that $\pi(\mathcal{R},A),\infty(\lambda^\mathcal{R}) = \lambda$.

We let $\text{Code}(A, \lambda) \subseteq \mathbb{R}$ be the set of reals $x$ such that $x$ codes a pair $(\mathcal{R}_x, \alpha_x)$ such that $\mathcal{R}_x$ is an $A$-iterable suitable pre-mouse such that $\lambda^\mathcal{R}$ is defined and $\alpha_x < \lambda^\mathcal{R}$. Let $\leq_{A,\lambda}$ be the natural pre-wellordering of $\text{Code}(A, \lambda)$ given by: $x \leq_{A,\lambda} y$ if and only if whenever $S$ is an $A$-iterate of $\mathcal{R}_x$ and an $A$-iterate of $\mathcal{R}_y$, $\pi(\mathcal{R}_x,A)(S,A)(\alpha_x) \leq \pi(\mathcal{R}_y,A)(S,A)(\alpha_y)$. We have that $\leq_{A,\lambda}$ has length $\lambda$. Given $x \in \text{Code}(A, \lambda)$ let

$$c(x) = \pi(\mathcal{R}_x,A),\infty(\alpha_x) = |x|_{\leq_{A,\lambda}}.$$ 

Let $S$ be a tree of a $\Sigma^2_1$-scale on a universal $\Sigma^2_1$-set. Given $x$ and $y$ we write $x \sim_S y$ if and only if $x \in L[S,y]$ and $y \in L[S,x]$. We then say that $d$ is an $S$-degree if $d$ is an $\sim_S$-class. We write $d \leq_S e$ if $d \in L[S,e]$. Let now $C(A, \lambda) = \{d : \text{Code}(A, \lambda) \cap HC^{L[S,d]} \neq \emptyset\}$. The following are two key points to keep in mind:

1. $\sim_S$ is an equivalence relation.
2. $C(A, \lambda)$ contains an $S$-cone, i.e., there is an $S$-degree $e$ such that whenever $e \leq_S d, d \in C(A, \lambda)$.

The following is a corollary to the Harrington-Kechris theorem (see [3], and see [6] and the references there for some uses of it).

**Corollary 1.1** There is a formula $\phi$ such that whenever $d \in C(A, \lambda)$, $g$ is $\omega_1^V$-generic over $L[S,d]$ and $\mathcal{R} \in L_{\omega_1^V}[S,d][g]$,

$$\mathcal{R} \text{ is a suitable premouse if and only if } L[S,d][g] \models \phi[\mathcal{R}].$$

Moreover, there is a formula $\psi$ such that for any $A$-iterable suitable $\mathcal{Q}$, $\mathcal{R} \in L_{\omega_1^V}[S][g]$ and for any $\pi$,

$$\mathcal{R} \text{ is an } A \text{-iterate of } \mathcal{Q} \text{ and } \pi : H^\mathcal{Q}_A \rightarrow H^\mathcal{R}_A \text{ is the } A \text{-iteration embedding if and only if } L[S,d][g] \models \psi[\mathcal{Q}, \mathcal{R}, \pi, \tau_A],$$

where $\tau_A$ is the term relation for $A$ in $L[S,d]^{\text{Coll}(\omega, < \omega_1^V)}$.

The formulas $\phi$ and $\psi$ essentially repeat the definitions of suitability and $A$-iterability. Another important lemma that we need is a consequence of what is usually called *generic comparisons* (see [8]). The proof is a standard generic comparison argument which we leave to the reader.
Lemma 1.2 Suppose $d \in C(A, \lambda)$ and $g$ is $< \omega_1^\omega$-generic over $L[S, d]$. Suppose $\phi$ is as in Corollary 1.1, and for some $R \in L[S, d][g]$, $L[S, d][g] \models \phi[Q, R]$. Then there is an $\emptyset$-iterate $S$ of $R$ such that $S \in L_{\omega_1^\omega}[S, d]$.

Given $d \in C(A, \lambda)$ we let $B_d$ be the set of $\beta$ such that there is $x \in Code(A, \lambda)$ with the property that $(R_x, \alpha_x) \in L[S, d]$ and $c(R_x, \alpha_x) = \beta$. As $|L_{\omega_1^\omega}[S, d]| = \omega_1^\omega$, we have that $B_d \in \wp_2(\lambda)$. Lemma 1.2 has the following easy corollary.

Corollary 1.3 Suppose $d_0 \in C(A, \lambda)$ and $d$ is a $S$-degree such that $L[S, d]$ is a $< \omega_1^\omega$-generic extension of $L[S, d_0]$. Then $B_{d_0} = B_d$.

We now define $\mu(A, \lambda)$ on $\wp_2(\lambda)$ by setting $B \in \mu(A, \lambda)$ if and only if for an $S$-cone of $d$, $B_d \in B$.

Lemma 1.4 $\mu(A, \lambda)$ is an $\omega_2$-complete ultrafilter on $\wp_2(\lambda)$.

Proof. Clearly $\mu(A, \lambda)$ is an ultrafilter. Let $(B_\xi : \xi < \omega_1)$ be such that $B_\xi \in \mu(A, \lambda)$ for all $\xi < \omega_1$. Let $WO$ be the set of reals coding a countable ordinal. Using the coding lemma we can find $y \in \mathbb{R}$ and a $\Sigma^1_2(y)$-set $D \subseteq WO \times \mathbb{R}$ such that

1. $[y]_S \in C(A, \lambda)$,
2. for every $x \in WO$, $D_x \neq \emptyset$ (here $D_x = \{z : (x, z) \in D\}$),
3. for every $x \in WO$, $D_x \subseteq \{z : [z]_S$ is a base of a cone witnessing that $B_{|x|} \in \mu(A, \lambda)\}^1$.

Let $d \in C$ be such that $y \in L[S, d]$. We claim that $B_d \in B_\xi$ for every $\xi < \omega_1$. To see this, fix $\xi < \omega_1$. Let $g \subseteq Coll(\omega, \xi)$ be $L[S, d]$-generic and $u$ be a real such that $L[S, d][g] = L[S, u]$. Let $x \in \mathbb{R}^{L[S, u]}$ be such that $|x| = \xi$. Because $D$ is $\Sigma^1_2(y)$ we have that there is $z \in D_x \cap L[S, u]$. Because $[z]_S \preceq [u]_S$, we must have that $B_{|u|} \in B_\xi$. However, it follows from Corollary 1.3 that $B_d = B_{|u|}$. Hence, $B_d \in B_\xi$.

As $d$ was arbitrary, we have shown that for any $d$ that is $S$-above $[y]_S$, $B_d \in \cap_{\xi < \omega_1} B_\xi$. It follows that $\cap_{\xi < \omega_1} B_\xi \in \mu(A, \lambda)$.

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1 where $[z]_S$ is the $S$-degree given by $z$ and $|x|$ is the ordinal coded by $x$. 

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2 \ \ \omega_2 \ \text{is} \ \delta_1^2 \text{-supercompact and} \ < \Theta\text{-strongly compact}

Proposition 2.1 \ For every \lambda < \Theta \ \text{and an ordinal definable} \ A \subseteq \mathbb{R} \ \text{such that} \ \gamma_{A,\infty} \geq \lambda, \ \mu(A, \lambda) \ \text{is superfine, i.e., for every} \ B \in \mathcal{P}(\lambda),
\{D \in \mathcal{P}(\lambda) : B \subseteq D\} \in \mu(A, \lambda).

Proof. Fix \ B \ \text{and let} \ f : \omega_1 \rightarrow B \ \text{be a bijection. Let} \ B_\xi = \{x \in \text{Code}(A, \lambda) : c(x) = f(\xi)\}. \ \text{Using the coding lemma find} \ y \in \mathbb{R} \ \text{and} \ D \subseteq \text{WO} \times \mathbb{R} \ \text{such that}

1. \ [y]_S \in C(A, \lambda),
2. \ D \in \Sigma_2(y),
3. \ \text{for every} \ x \in \text{WO}, \ D_x \neq \emptyset,
4. \ \text{for every} \ x \in \text{WO}, \ D_x \subseteq \{z \in \text{Code}(A, \lambda) : c(z) = f(|x|)\}.

\text{We claim that for every} \ d \ \text{such that} \ [y]_S \leq_S d, \ B \subseteq B_d. \ \text{To see this, fix} \ d \ \text{such that} \ [y]_S \leq_S d. \ \text{Fix} \ \zeta \in B. \ \text{We want to see that} \ \zeta \in B_d. \ \text{Let} \ \xi = f^{-1}(\zeta), \ \text{and fix} \ u \in \mathbb{R} \ \text{such that} \ L[S,u] \ \text{is a generic extension of} \ L[S,d] \ \text{and} \ \xi \ \text{is countable in} \ L[S,u]. \ \text{Fix} \ x \in \text{WO} \cap L[S,u] \ \text{such that} \ |x| = \xi. \ \text{Because} \ D \in \Sigma_2(y), \ \text{we have that} \ D_x \cap L[S,u] \neq \emptyset. \ \text{Fix then} \ z \in \text{Code}(A, \lambda) \cap D_x \in L[S,u]. \ \text{It follows that} \ c(z) = f(\xi). \ \text{Since} \ c(z) \in B_{|u|} = B_d, \ \text{we have that} \ \zeta \in B_d. \ \square

Putting Proposition 1.4 and Proposition 2.1 we get the following corollary.

Corollary 2.2 \ Assume \ AD + V = L(\mathbb{R}). \ Then \ \omega_2 \ \text{is a} \ < \Theta\text{-strongly compact. More precisely, for every} \ \lambda < \Theta \ \text{there an} \ \omega_2\text{-complete superfine ordinal definable ultrafilter on} \ \mathcal{P}(\lambda).

Theorem 2.3 (Becker, [1]) \ Assume \ AD + V = L(\mathbb{R}). \ Then \ \omega_2 \ \text{is} \ \delta_1^2\text{-supercompact.}

Proof. \ Set \ \lambda = \delta_1^2. \ \text{Suppose} \ \mathcal{R} \ \text{is an} \ \emptyset\text{-iterable suitable pre-mouse. Recall that if} \ \nu \ \text{is the least cardinal that is} \ < \delta^\mathcal{R}\text{-strong in} \ \mathcal{R} \ \text{then} \ \pi_{(\mathcal{R},\emptyset)}(\nu) = \lambda \ \text{(see [7, Chapter 8])}. \ \text{We now want to show that} \ \mu =_{\text{def}} \mu(\emptyset, \lambda) \ \text{is an} \ \omega_2\text{-supercompactness measure.}
\text{Proposition 1.4 shows that} \ \mu \ \text{is} \ \omega_2\text{-complete and Proposition 2.1 shows that} \ \mu \ \text{is fine. It remains to show that} \ \mu \ \text{is normal. The following lemma is the first step towards normality. Set} \ \text{Code} =_{\text{def}} \ \text{Code}(\emptyset, \lambda) \ \text{and} \ \leq^* =_{\emptyset,\lambda}.

Lemma 2.4 \ Suppose \ F : \mathcal{P}(\lambda) \rightarrow \lambda \ \text{is such that for an} \ S\text{-cone of} \ d, \ F(B_d) \in B_d. \ \text{Then for an} \ S\text{-cone of} \ d \ \text{there is} \ x \in (\mathbb{R}^{L[S,d]} \cap \text{Code}) \ \text{such that} \ c(x) = F(B_d).
Proof. Assume not. Fix an $S$-degree $d_0$ such that whenever $d$ is $S$-above $d_0$, for every $x \in (\mathbb{R}^L[S,d] \cap \text{Code})$, $c(x) \neq F(B_d)$. Fix $(\mathcal{R}, \alpha) \in L[S, d_0]$ such that $\pi_{(\mathcal{R}, \emptyset), \infty}(\alpha) = F(B_{d_0})$.

Let $\nu < \omega_1$ be any cardinal of $L[S, d_0]$ such that $(\mathcal{R}, \alpha) \in L_{\nu}[S, d_0]$ and let $g \subseteq \text{Coll}(\omega, (\nu^+)L[S, d_0])$ be $L[S, d_0]$-generic. Let $x \in \mathbb{R}$ be such that $L[S, d_0][g] = L[S, x]$. We then have that $B_{d_0} = B[x]_S$ (see Corollary 1.1). This is a contradiction as we can find $y \in L[S, x] \cap \mathbb{R}$ coding $(\mathcal{R}, \alpha)$. □

Lemma 2.5 $\mu$ is normal.

Proof. Suppose $\mu$ is not normal. Let $F : \varphi_{\omega_2}(\lambda) \to \lambda$ be such that for an $S$-cone of $d$, $F(B_d) \in B_d$ but $F$ is not constant on a $\mu$-measure one set. Let $e_0 \in C$ be a base for the cone of the previous sentence.

Let $e \in C$ be $S$-above $e_0$ and such that for every $d$ such that $e \in L[S, d]$, there is $x \in (\mathbb{R}^L[S,d] \cap \text{Code})$ with the property that $c(x) = F(B_d)$. We now follow an idea of Becker from [1].

Given an ordinal $\xi < \lambda$ let $D_\xi = \{d : F(B_d) \neq \xi\}$. We have that for each $\xi$, $D_\xi$ contains an $S$-cone. Let then $C_\xi = \{x \in \mathbb{R} : [x]_S$ is a base of a cone contained in $D_\xi\}$. It follows from the coding lemma that there is a real $y$ and a set $D$ such that

1. $e \leq_S [y]_S$,
2. $H \subseteq \text{Code} \times \mathbb{R}$ is $\Sigma^3_1(y)$,
3. if $(x, z) \in H$ then $z \in C_{c(x)}$,
4. for every $x \in \text{Code}$ there is $z$ such that $(x, z) \in H$.

Set $d = [y]_S$. Let $x \in \text{Code} \cap L[S, d]$ be such that $c(x) = F(B_d)$. Because $H$ is $\Sigma^3_1(y)$ we have that $H_x \cap L[S, d] \neq \emptyset$. Let then $z \in H_x \cap L[S, d]$. It follows that $[z]_S \leq_S d$. Hence, $d \in H_{c(x)}$. It follows that $F(B_d) \neq c(x) = F(B_d)$, contradiction. □
3 A covering conjecture

Again we assume $AD + V = L(\mathbb{R})$. Suppose $\kappa < \lambda < \Theta$ and $A$ is an ordinal definable set of reals such that $\gamma_{A,\infty} \geq \lambda$. Given $X \in \mathcal{V}_\kappa(\lambda)$ we say that $X$ is an $A$-iteration set if for every $\alpha \in X$ there is an $A$-iterable $Q$ such that $\alpha \in \text{rng}(\pi(Q,A,\infty))$ and $\pi(Q,A,\infty)[\gamma_Q^A] \cap \lambda \subseteq X$.

**Conjecture 3.1** Assume $AD + V = L(\mathbb{R})$. Suppose $\kappa$ is either a Suslin cardinal or a successor of a Suslin cardinal. Then for every $\lambda < \Theta$, an OD set $A \subseteq \mathbb{R}$ such that $\lambda \leq \gamma_{A,\infty}$, and $B \in \mathcal{V}_\kappa(\lambda)$ there is an $A$-iteration set $X \in \mathcal{V}_\kappa(\lambda)$ such that $B \subseteq X$.

The conjecture is clearly true for $\kappa = \omega_1$. Proposition 2.1 shows that the conjecture is true for $\kappa = \omega_2$. We expect that the validity of the full conjecture will follow once a link is made between Kechris-Woodin generic codes and iteration sets.

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