TWISTED ELLIPTIC GENUS FOR K3 AND BORCHERDS PRODUCT

TOHRU EGUCHI AND KAZUHIRO HIKAMI

Abstract. We further discuss the relation between the elliptic genus of K3 surface and the Mathieu group $M_{24}$. We find that some of the twisted elliptic genera for K3 surface, defined for conjugacy classes of the Mathieu group $M_{24}$, can be represented in a very simple manner in terms of the $\eta$ product of the corresponding conjugacy classes. It is shown that our formula is a consequence of the identity between the Borcherds product and additive lift of some Siegel modular forms.

1. Introduction

Studies of the K3 surface based on the representation theory of superconformal algebras (SCA) was initiated some time ago [19]. Therein, the elliptic genus of K3 surface was decomposed into a sum of characters of $\mathcal{N} = 4$ SCA. Recently, an intriguing property of this decomposition was discovered [18]; multiplicities of the non-BPS representations are given by the sum of dimensions of irreducible representations of the Mathieu group $M_{24}$. This observation is a variant of the famous Monstrous moonshine [8] where the Fourier coefficients of the modular $J$-function is expressed as the sum of dimensions of representations of the Monster group. In our case of “Mathieu moonshine”, multiplicities of non-BPS representations are Fourier coefficients of a certain mock theta function [14, 15].

The Mathieu group $M_{24}$ has 26 conjugacy classes. To each conjugacy class we can introduce a twisted version of the elliptic genus of K3 surface which is an analogue of McKay–Thompson series in the Monstrous moonshine. One can determine uniquely the decomposition of the multiplicities of non-BPS representations once one has a complete list of twisted genera for all conjugacy classes.

Recently a completed list of twisted genera became available [5, 23], [24, 17] and by making use them the decomposition has been carried out up to a very high level $\approx1000$. This result produces a very strong support for the Mathieu moonshine conjecture.

In the study of the Mathieu group $M_{24}$, the $\eta$-products associated with various cycle shapes play an important role [13, 31, 33]. Purpose of this paper is to present simple relationships between the twisted elliptic genera of K3 and the $\eta$-products for various conjugacy classes of $M_{24}$. Our result follows from the identity of Siegel modular forms of degree-2 which are expressed both as an infinite sum (additive lift) and as an infinite product (Borcherds lift) simultaneously.

Date: December 26, 2011. Revised on May 7, 2012.

2010 Mathematics Subject Classification. 58J26, 81Txx, 20C34, 14J28.

Key words and phrases. elliptic genus, superconformal algebra, moonshine, Mathieu group, Jacobi form, mock theta function.
2.1 Character Decomposition

The elliptic genus of $K3$ surface is known to be a Jacobi form with weight 0 and index 1, and it is decomposed as

$$Z_{K3}(z; \tau) = 20 \text{ch}_4 R_0(z; \tau) - 2 \text{ch}_1 R_{1/2}(z; \tau) + \sum_{n=1}^{\infty} A(n) \text{ch}_{n+1/2}^R(z; \tau),$$

(2.1)

where $\text{ch}_{\ell}^R(z; \tau)$, ($\ell = 0, 1/2$), $\text{ch}_{\ell+1/2}^R(z; \tau)$ are characters of BPS and non-BPS representations of $\mathcal{N} = 4$ SCA in $R$ sector (with $(-1)^F$ insertion). $\ell$ denotes the iso-spin. See



| $g$ | cycle shape | permutation |
|-----|-------------|-------------|
| 1A  | $1^{24}$    | ( )         |
| 2A  | $1^8 \cdot 2^8$ | (1,8)(2,12)(4,15)(5,7)(9,22)(11,18)(14,19)(23,24) |
| 3A  | $1^6 \cdot 3^6$ | (3,18,20)(4,22,24)(5,19,17)(6,11,8)(7,15,10)(9,12,14) |
| 5A  | $1^4 \cdot 5^4$ | (2,21,13,16,23)(3,5,15,22,14)(4,12,20,17,7)(9,18,19,10,24) |
| 4B  | $1^4 \cdot 2^2 \cdot 4^4$ | (1,17,21,9)(2,13,24,15)(3,23)(4,14,5,8)(6,16)(12,18,20,22) |
| 7A  | $1^3 \cdot 7^3$ | (1,17,5,21,24,10,6)(2,12,13,9,4,23,20)(3,8,22,7,18,14,19) |
| 7B  | $1^3 \cdot 7^3$ | (1,21,6,5,10,17,24)(2,9,20,13,23,12,4)(3,7,19,22,14,8,18) |
| 8A  | $1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$ | (1,13,17,24,21,15,9,2)(3,16,23,6)(4,22,14,12,5,18,8,20)(7,11) |
| 6A  | $1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$ | (1,8)(2,24,11,12,23,18)(3,20,10)(4,15)(5,19,9,7,14,22)(6,16,13) |
| 11A | $1^2 \cdot 11^2$ | (1,3,10,4,14,15,5,24,13,17,18)(2,21,23,9,20,19,6,12,16,11,22) |
| 15A | $1^1 \cdot 3^3 \cdot 15^1$ | (2,13,23,21,16)(3,7,9,5,4,18,15,12,19,22,20,10,14,17,24)(6,8,11) |
| 15B | $1^1 \cdot 3^3 \cdot 15^1$ | (2,23,16,13,21)(3,12,24,15,17,18,14,4,10,5,20,9,22,7,19)(6,8,11) |
| 14A | $1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$ | (1,12,17,13,5,9,21,4,24,23,10,20,6,2)(3,18,8,14,22,19,7)(11,15) |
| 14B | $1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$ | (1,13,21,23,6,12,5,4,10,2,17,9,24,20)(3,14,7,8,19,18,22)(11,15) |
| 23A | $1^1 \cdot 23^1$ | (1,7,6,24,14,4,16,12,20,9,11,5,15,10,19,18,23,17,3,2,8,22,21) |
| 23B | $1^1 \cdot 23^1$ | (1,4,11,18,8,6,12,15,17,21,14,9,19,2,7,16,5,23,22,24,20,10,3) |
| 2B  | $1^2 \cdot 2^2$ | (1,12,24,23,10,8,18,6,3,21,2,7)(4,9,11,15,13,16,20,5,22,17,14,19) |
| 4C  | $6^4$ | (1,24,10,18,3,2)(4,11,13,20,22,14)(5,17,19,9,15,16)(6,21,7,12,23,8) |
| 3B  | $6^3$ | (1,23,18,21)(2,12,10,6)(3,7,24,8)(4,15,20,17)(5,14,9,13)(11,16,22,19) |
| 1B  | $6^2$ | (1,10,3)(2,24,18)(4,13,22)(5,19,15)(6,7,23)(8,21,12)(9,16,17)(11,20,14) |
| 21B | $6^3 \cdot 11^1$ | (1,8)(2,10)(3,20)(4,22)(5,17)(6,11)(7,15)(9,13)(12,14)(16,18)(19,23)(21,24) |
| 10A | $2^2 \cdot 10^2$ | (1,8)(2,18,21,19,13,10,16,24,23,9)(3,4,5,12,15,20,22,17,14,7)(6,11) |
| 21A | $3^1 \cdot 21^1$ | (1,3,9,15,5,12,2,13,20,23,17,4,14,10,21,22,19,6,7,11,16)(8,18,24) |
| 21B | $3^1 \cdot 21^1$ | (1,12,17,22,16,5,23,21,11,15,20,7,9,13,14,6,3,2,4,19)(8,24,18) |
| 4A  | $2^1 \cdot 4^4$ | (1,4,8,15)(2,9,12,22)(3,6)(5,24,7,23)(10,13)(11,14,18,19)(16,20)(17,21) |
| 12A | $2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$ | (1,15,8,4)(2,19,24,9,11,7,12,14,23,22,18,5)(3,13,20,6,10,16)(17,21) |

Table 1. Representatives of conjugacy classes $g$. 

The Siegel modular form is physically interpreted as the partition function of $\frac{1}{2}$ BPS states in the CHL model [31,12] of a heterotic string theory compactified on $K3 \times S^2$. The twisted elliptic genera of $K3$ for conjugacy classes 2A, 3A, 5A, 7A are closely related to the $Z_N$ orbifold partition function of CHL models [31,25,29,10].

This paper is organized as follows: in section 2 we recall the twisted elliptic genera. By use of the Hecke operator, we express the twisted elliptic genus in terms of the $\eta$-product for a number of conjugacy classes. We show that this identity can be derived by using the Borcherds product in section 3. The last section is devoted to concluding remarks. Notations on modular forms are summarized in the Appendix.

2. Twisted Elliptic Genus and Cycle Shape

2.1 Character Decomposition

The elliptic genus of $K3$ surface is known to be a Jacobi form with weight 0 and index 1, and it is decomposed as

$$Z_{K3}(z; \tau) = 20 \text{ch}_4 R_0(z; \tau) - 2 \text{ch}_1 R_{1/2}(z; \tau) + \sum_{n=1}^{\infty} A(n) \text{ch}_{n+1/2}^R(z; \tau),$$

(2.1)
Appendix for their explicit forms. $A(n)$ is the multiplicity of the non-BPS representation with $h = n + 1/4$, and is given as follows:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|-----|---|---|---|---|---|---|---|---|---|-----|
| $A(n)$ | 90 | 462 | 1540 | 4554 | 11592 | 27830 | 61686 | 131100 | 265650 | ... |

Numbers $A(n)$ are the expansion coefficients of a certain mock theta function with a shadow $\eta(\tau)^3$ [14]. Observation in [18] is that $A(n)$ is decomposed into a sum of dimensions of the irreducible representations $R$ of $M_{24}$,

$$\sum_R \text{mult}_R(n) \dim R = A(n),$$

and that the multiplicities $\text{mult}_R(n)$ are conjectured to be positive integers. By introducing a vector space

$$V(n) = \bigoplus_R \text{mult}_R(n) R,$$

we can write $A(n)$ as

$$A(n) = \text{Tr}_{V(n)} 1.$$  \hspace{1cm} (2.4)

**2.2 Twisted Elliptic Genus of $K3$**

Corresponding to each conjugacy class $g$ of the Mathieu group $M_{24}$ we can define a variant of $A(n)$ by

$$A_g(n) = \text{Tr}_{V(n)} g = \sum_R \text{mult}_R(n) \chi_R^g.$$  \hspace{1cm} (2.5)

Here $\chi_R^g = \text{Tr}_R g$ denotes the character of $M_{24}$ for the representation $R$ and conjugacy class $g$. We define the twisted elliptic genus $Z_g(z; \tau)$ for a class $g$ by the decomposition

$$Z_g(z; \tau) = (\chi_g - 4) \text{ch}_{h = n, \ell = 0}^R(z; \tau) - 2 \text{ch}_{h = n + \frac{1}{4}, \ell = \frac{1}{2}}^R(z; \tau) + \sum_{n=1}^\infty A_g(n) \text{ch}_{h= n + \frac{1}{4}, \ell = \frac{1}{2}}^R(z; \tau)$$

$$= \left[\frac{\theta_{11}(z; \tau)}{[\eta(\tau)]^3}\right] (\chi_g \mu(z; \tau) - \Sigma_g(\tau)).$$  \hspace{1cm} (2.6)

Here $\chi_g \in \mathbb{Z}$ is the Witten index of twisted genus, i.e. $Z_g(z = 0; \tau)$.

We have used

$$- q^{\frac{1}{2}} \Sigma_g(\tau) = - 2 + \sum_{n=1}^\infty A_g(n) q^n.$$  \hspace{1cm} (2.7)

This decomposition reduces to (2.1) when $g = 1A$.

We classify conjugacy classes into type I and type II; conjugacy classes of type I contain a cycle of length $1$, i.e. a fixed point under permutation of 24 elements and thus they arise from those of $M_{23}$. See Table II. On the other hand conjugacy classes of type II are those which are intrinsically $M_{24}$. These two types of conjugacy classes have a qualitatively different behavior. It should be noted that, in our following studies on twisted elliptic genera, there is no difference between conjugacy classes with the same cycle shape such as 7A and 7B, and we use a notation 7AB for such a pair of conjugacy classes.
The twisted elliptic genera were first obtained for (mainly) type I classes \[5, 23\] and then obtained for type II classes \[24, 17\], and their complete list is available now. See Table 2.

It turns out that their Witten indices are given by

| $g$ | $1A$ | $2A$ | $3A$ | $5A$ | $4B$ | $7AB$ | $8A$ | $6A$ | $11A$ | $15AB$ | $14AB$ | $23AB$ | others |
|-----|------|------|------|------|------|-------|------|------|-------|-------|-------|-------|-------|
| $\chi_g$ | 24   | 8    | 6    | 4    | 4    | 3     | 2    | 2    | 2     | 1     | 1     | 1     | 0      |

Note that

$$\chi_g = \text{Tr}_{\rho_1} g + \text{Tr}_{\rho_{23}} g$$

where $\rho_1, \rho_{23}$ denote 1 and 23-dimensional representation of $M_{24}$, respectively.

Hereafter we mainly work with the conjugacy classes $g$ of type I which possess non-vanishing indices $\chi_g \neq 0$.

### 2.3 Twisted Elliptic Genus from the $\eta$-Products of Cycle Shape

Given a conjugacy class $g$ of $M_{24}$ and its cycle shape $1^{r_1}2^{r_2}3^{r_3} \cdots$, the corresponding $\eta$-product $\eta_g(\tau)$ is defined by

$$\eta_g(\tau) = \prod_i [\eta(i \tau)]^{r_i}.$$  \hspace{1cm} (2.9)

For instance,

- $\eta_{1A}(\tau) = \eta(\tau)^{24}$,
- $\eta_{2A}(\tau) = \eta(\tau)^8 \eta(2\tau)^8$,
- $\eta_{4A}(\tau) = \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4$,
- $\eta_{8A}(\tau) = \eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2$,
- $\cdots$

It is well-known that the type I classes of $M_{24}$ have a balanced shape $r_i = r_{N/i}$ where $N$ is the order of the element $g$. As observed in Refs. \[13, 31, 33\], these $\eta$-products are cusp forms, $\eta_g(\tau) \in S_k(\Gamma_0(N), \chi)$, i.e., weight $k$ cusp form on $\Gamma_0(N)$ with character $\chi$, and they are the Hecke eigenforms. See Table 3 for data of $k$, $N$, and $\chi$.

For our later convention, we define a Jacobi form

$$\varphi_g(z; \tau) = \eta_g(\tau) \phi_{-2,1}(z; \tau),$$  \hspace{1cm} (2.10)

which belongs to $\mathbb{J}_{k-2,1}(\Gamma_0(N), \chi)$, i.e., weight $(k-2)$ Jacobi form on $\Gamma_0(N)$ with index=1 and character $\chi$. See Appendix \[A\] for a fundamental property of Jacobi form and the definition of $\phi_{-2,1}(z; \tau)$.

In order to relate the $\eta$-product to the twisted elliptic genus, we introduce the Hecke operator. On the Jacobi form $\phi \in \mathbb{J}_{k,m}(\Gamma_0(N), \chi)$, the Hecke operator $T_n$ acts as \[22\]

$$\left(T_n\phi\right)(z; \tau) = n^{k-1} \sum_{a \geq 0, \frac{ad=n}{(a,N)=1}} \sum_{b=0}^{d-1} \frac{1}{d^k} \chi(a) \phi \left( a z; \frac{a\tau + b}{d} \right).$$  \hspace{1cm} (2.11)

The character satisfies $\chi(a) = 0$ unless $(a, N) = 1$, so we may omit a condition in the summation. We note that $T_n \phi$ belongs to $\mathbb{J}_{k,mn}(\Gamma_0(N), \chi)$. 

We find that the twisted elliptic genus $Z_g(z;\tau)$ for type I conjugacy class has a simple expression in terms of the $\eta$-product. Derivation is given in the next section.

For $g = 1A, 2A, 3A, 5A, 7AB, 4B, 6A,$ and $8A$, we have

\[
Z_g(z;\tau) = -\frac{(T_2\varphi_g)(z;\tau)}{\varphi_g(z;\tau)}.
\] (2.12)

We need a correction term for the remaining type I cases. The $\varphi$-functions (2.10) for $g = 11A, 14AB,$ and $15AB,$ are weight 0 modular forms, and we find

\[
Z_{11A}(z;\tau) = -\frac{(T_2\varphi_{11A})(z;\tau)}{\varphi_{11A}(z;\tau)} - \frac{11}{2} \varphi_{11A}(\tau),
\] (2.13)
Table 3. $\eta$-product for conjugacy class $g$. $\eta_g$ is a weight $k$ modular form on $\Gamma_0(N)$ with character $\chi$. $\chi$ is written only if non-trivial.

| $g$   | $\eta_g$ | $k$ | $N$ | $\chi$ |
|-------|-----------|-----|-----|---------|
| 1A    | $1^{24}$  | 12  | 1   |         |
| 2A    | $1^8 \cdot 2^8$ | 8   | 2   |         |
| 3A    | $1^6 \cdot 3^6$ | 6   | 3   |         |
| 5A    | $1^4 \cdot 5^4$ | 4   | 5   |         |
| 4B    | $1^4 \cdot 2^2 \cdot 4^4$ | 5   | 4   | $\frac{1}{d}$ |
| 7AB   | $1^3 \cdot 7^3$ | 3   | 7   |         |
| 8A    | $1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$ | 3   | 8   | $\frac{1}{d}$ |
| 6A    | $1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$ | 4   | 6   |         |
| 11A   | $1^2 \cdot 11^2$ | 2   | 11  |         |
| 15AB  | $1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$ | 2   | 15  |         |
| 14AB  | $1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$ | 2   | 14  |         |
| 23AB  | $1^1 \cdot 23^1$ | 1   | 23  | $\frac{-23}{d}$ |
| 12B   | $12^2$    | 1   | 144 | $\frac{1}{d}$ |
| 6B    | $6^4$     | 2   | 36  |         |
| 4C    | $4^6$     | 3   | 16  | $\frac{1}{d}$ |
| 3B    | $3^8$     | 4   | 9   |         |
| 2B    | $2^{12}$  | 6   | 4   |         |
| 10A   | $2^2 \cdot 10^2$ | 2   | 20  |         |
| 21AB  | $3^1 \cdot 21^1$ | 1   | 63  | $\frac{-23}{d}$ |
| 4A    | $2^4 \cdot 4^4$ | 4   | 8   |         |
| 12A   | $2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$ | 2   | 24  |         |

\[ Z_{14AB}(z; \tau) = -\frac{(T_2 \varphi_{14AB})(z; \tau)}{\varphi_{14AB}(z; \tau)} - 7 \varphi_{14AB}(z; \tau), \]  
\[ Z_{15AB}(z; \tau) = -\frac{(T_2 \varphi_{15AB})(z; \tau)}{\varphi_{15AB}(z; \tau)} - \frac{15}{2} \varphi_{15AB}(z; \tau). \]  

For $g = 23AB$ case, the twisted genus can not be expressed only in terms of the function $\varphi_{23AB}$, and we have
\[ Z_{23AB}(z; \tau) = -\frac{(T_2 \varphi_{23AB})(z; \tau)}{\varphi_{23AB}(z; \tau)} - \frac{23}{8} (f_{23,1}(\tau) + 3 f_{23,2}(\tau)) \phi_{-2,1}(z; \tau). \]  

See Appendix for new forms of $\Gamma_0(23)$, $f_{23,1}(\tau)$ and $f_{23,2}(\tau)$.

3. Borcherds Products

3.1 Hilbert Scheme of Points on $K3$

We first recall the elliptic genus of the Hilbert scheme of points on the $K3$ surface \cite{12}. \cite{11}. We consider a “second-quantized” version of $K3$ elliptic surface
\[ M(\Omega) = \sum_{m=0}^{\infty} Z_{K3}^{[m]}(z; \tau) p^m, \]
where \( p = e^{2\pi i \sigma} \), and \( \Omega = (\frac{\tau}{z}, \frac{\sigma}{z}) \) is in the Siegel upper half-plane. \( Z_{K3}^{[m]} \) denote the elliptic genus of the \( m \)-th symmetric product of \( K3 \).

As was pointed out in [11], a generating function \( M(\Omega) \) has an infinite product representation

\[
M(\Omega) = \prod_{m=1}^{\infty} \prod_{n=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \frac{1}{(1 - p^m q^n \zeta^\ell)^{c(nm, \ell)}}
\]  

where \( c(n, \ell) \) is the Fourier coefficients of the \( K3 \) elliptic genus

\[
Z_{K3}(z; \tau) = \sum_{n=0}^{\infty} \sum_{\ell \in \mathbb{Z}} c(n, \ell) q^n \zeta^\ell.
\]

This construction is based on the fact that \( T_m Z_{K3} \) is a weak Jacobi form with weight 0 and index \( m \). Recalling generators of the Siegel upper half plane given in Appendix A.4, we need to symmetrize the infinite product (3.2) in \( \tau \) and \( \sigma \) to generate a Siegel modular form. After symmetrization and demanding a good modular behavior we obtain

\[
\Phi(\Omega) = p q \zeta \prod_{n, \ell, m \in \mathbb{Z}} (1 - p^m q^n \zeta^\ell)^{c(nm, \ell)}. \tag{3.4}
\]

Here a condition \((n, \ell, m) > 0\) means

\[
n \geq 0, \ m \geq 0, \text{ and } \begin{cases} \ell \in \mathbb{Z}, & \text{when } m + n > 0, \\ \ell < 0, & \text{when } m, n = 0. \end{cases}
\]

The Siegel modular form \( \Phi(\Omega) \) is the Igusa cusp form with weight 10, and equals \( M(\Omega)^{-1} \) up to an extra factor

\[
\Phi(\Omega) M(\Omega) = p q \zeta (1 - \zeta^{-1})^2 \prod_{n=1}^{\infty} (1 - q^n \zeta)^2 (1 - q^n)^{20} (1 - q^n \zeta^{-1})^2
\]

\[
= p \varphi_{1A}(z; \tau), \tag{3.5}
\]

which is called the Hodge anomaly [27, 26].

It is well-known that besides being an infinite product, the Igusa cusp form \( \Phi(\Omega) \) can also be expressed as an infinite sum or Saito–Kurokawa–Maass additive lift, from a weak Jacobi form with weight 10, i.e. \( \eta(\tau)^{24} \phi_{-2, 1}(z; \tau) = \varphi_{1A}(z; \tau) \)

\[
\Phi(\Omega) = \Phi_{1A}(\Omega) = \sum_{m=1}^{\infty} p^m (T_m \varphi_{1A}) (z; \tau). \tag{3.6}
\]

The identity between the infinite product and infinite sum follows from the Koecher principle that Siegel modular form with weight 0 is a constant. This relation is interpreted as the analogue of the Weyl denominator formula for generalized Kac–Moody algebra [2, 27, 28].

Our formula of the elliptic genus (2.12) in the case \( g = 1A \)

\[
Z_{1A}(z; \tau) = -\frac{T_2 \varphi_{1A}(z; \tau)}{\varphi_{1A}(z; \tau)}.
\]
simply follows from (3.1), (3.5) and (3.6).

The inverse of the Borcherds product is a partition function of $\frac{1}{4}$ BPS states, and at a pole $z = 0$ its residue factorizes into a pair of $\frac{1}{2}$ BPS partition functions [12]

$$\Phi^A_1(\Omega) \approx z \rightarrow 0 (2\pi i z)^2 \eta(\tau)^{24} \eta(\sigma)^{24} = \frac{1}{(2\pi i z)^2} \frac{1}{\eta^A(\tau)} \frac{1}{\eta^A(\sigma)}.$$ (3.7)

It is to be noted that the Hodge anomaly (3.5) is related to the coefficients of BPS characters in the character decomposition of the elliptic genus for $K3^{[m]}$, 

$$\prod_{n=1}^{\infty} \frac{1}{(1-p^n \zeta)^2 (1-p^n \zeta^{-1})^2} = \sum_{m=0}^{\infty} p^m \sum_{s=0}^{\infty} \gamma_{m,s} \frac{\zeta^{s+1} - \zeta^{-s-1}}{\zeta - \zeta^{-1}},$$ (3.10)

where $\gamma_{m,s}$ is the number of the BPS characters of isospin $\frac{s}{2}$ representation in the elliptic genus of $K3^{[m]}$ [14] [16]. See also [30].

3.2 Borcherds Product for Twisted Elliptic Genus

In order to construct an infinite product representation and derive (2.12) for general conjugacy classes, we have to introduce another Hecke operator acting on Jacobi forms of the congruence subgroup $\Gamma_0(N)$. We set the Hecke operator $V_n$ on $\phi \in J_{k,m}(\Gamma_0(N))$ defined as [1, 6]

$$(V_n \phi)(z; \tau) = n^{k-1} \sum_{(a \ b \ c \ d) \in \Gamma_0(N) \setminus \text{Mat}_n(N)} (c \tau + d)^{-k} e^{-2\pi i mn \frac{cz^2}{c\tau + d}} \phi \left( \frac{nz}{c \tau + d}; \frac{a \tau + b}{c \tau + d} \right),$$ (3.8)

where

$$\text{Mat}_n(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, a d - b c = n, c \equiv 0 \mod N \right\}.$$ 

We have $(V_n \phi)(z; \tau) \in J_{k,mn}(\Gamma_0(N))$. The representatives of cosets are given by cusps of $\Gamma_0(\text{ord}(g))$ as [1]

$$\Gamma_0(N) \setminus \text{Mat}_n(N) = \bigsqcup_{f/e \in \text{Cusp}(\Gamma_0(N))} M_{f/e} \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) \mid a d = n, a e \equiv 0 \mod N, 0 \leq b < h_e d \right\},$$ (3.9)

where $\text{Cusp}(\Gamma_0(N))$ denotes the set of cusps of $\Gamma_0(N)$, and $M_{f/e} = (f^* \ e^*) \in SL(2; \mathbb{Z})$ with a positive divisor $e$ of $N$. $h_e$ is the width of cusp $f/e$,

$$h_e = \frac{N}{(e^2, N)}.$$ 

When $N$ is prime, the Hecke subgroup $\Gamma_0(N)$ has two cusps, $\tau = i\infty$ and 0. The width is 1 and $N$, respectively.

By use of $Z_g(z; \tau) \in J_{0,1}(\Gamma_0(\text{ord}(g)))$, we introduce an analogue of (3.2)

$$M_g(\Omega) = \exp \left[ \sum_{m=1}^{\infty} p^m (V_m Z_g)(z; \tau) \right].$$ (3.10)
In order to rewrite this into an infinite product form, we introduce the Fourier expansion of \( Z_g(z; \tau) \) at each cusp \( f/e \) of \( \Gamma_0(\text{ord}(g)) \),

\[
(Z_g|_{0,1}M_{f/e})(z; \tau) = \sum_{n \in \mathbb{Z}/h_e} \sum_{\ell \in \mathbb{Z}} c_{g,f/e}(n, \ell) q^n \zeta^\ell.
\] (3.11)

(For the definition of slash operator, see Appendix A). Note that the Fourier expansion at a cusp is a power series in \( q^{\frac{1}{h_e}} \).

It is well-known that \( c_{g,f/e}(n, \ell) \) depends only on \( 4n - \ell^2 \) \([22]\) and we may write

\[
c_{g,f/e}(n, \ell) = c_{g,f/e}(4n - \ell^2).
\] (3.12)

We have

\[
\log M_g(\Omega) = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{f/e \in \text{Cusp}(\Gamma_0(\text{ord}(g)))} \sum_{ad=m \ mod \ \text{ord}(g)} \sum_{a=0}^{h_e-1} \sum_{b=0}^{h_e-1} \sum_{d=1}^{h_e} c_{g,f/e}(n, \ell) \frac{1}{N_e} \left( p^d q^n \zeta^\ell \right)^{N_e a'}.
\]

where we have used \( N_e = \frac{\text{ord}(g)}{e} \). (3.13)

As a result, we obtain \([1, 6]\)

\[
M_g(\Omega) = \prod_{f/e \in \text{Cusp}(\Gamma_0(\text{ord}(g)))} \prod_{m=1}^{\infty} \prod_{n=0}^{\infty} \prod_{\ell \in \mathbb{Z}} \left( 1 - (p^m q^n \zeta^\ell)^{N_e} \right)^{\frac{1}{N_e} c_{g,f/e}(mn, \ell)}.
\] (3.14)

For the contribution of the cusp at \( \infty \) we set \( c_{g,\infty}(n, \ell) = c_g(n, \ell) \) which is the Fourier expansion coefficients of the elliptic genus \( Z_g \) and \( N_e = h_e = 1 \).

It should be noted that only the Fourier coefficients at integral powers in (3.11) contribute to the infinite product (3.14). In Table 4 we have tabulated values of the Fourier coefficients \( c_{g,f/e}(m) \) for integers \( m \leq 32 \).

By inspection we notice an interesting relation

\[
\sum_{f/e \in \text{Cusp}(\Gamma_0(\text{ord}(g)))} h_e c_{g,f/e}(n) = c_{1A}(n)
\] (3.15)

for all \( g \) of type I. Namely, sum of the expansion coefficients for each cusp weighted by the width reproduces the original Fourier coefficients of \( K3 \) elliptic genus.\(^1\)

\(^1\) There exist more linear relations among Fourier coefficients such as (we abbreviate \( c_{g,\infty} \) to \( c_g \))

\[
\begin{align*}
c_{2A} - c_{4B} &= c_{4B,4}, & c_{3A} - c_{6A} &= 2c_{6A,4}, & c_{1A} - c_{2A} &= 8c_{6A,0}, \\
c_{7AB} - c_{14AB} &= 2c_{14AB,4}, & c_{3A} - c_{15AB} &= 5c_{15AB,4},
\end{align*}
\]

Making use of these relations and (3.15) one can derive a different expression for \( M_g(\Omega) \) for all \( g \) of type I

\[
M_g(\Omega) = \exp \left[ \sum_{m,n,\ell} \sum_{k=1}^{\infty} \frac{1}{k} c_{g/k}(4mn - \ell^2) \left( p^m q^n \zeta^\ell \right)^k \right],
\]

which is used in some literature (see, e.g., \([25],[5],[6]\)).
Using the Table 4 we note that the $\eta$ product for all type I class $g$ can be uniformly written as

$$
\eta_g(\tau) = \prod_{f/e \in \text{Cusp}(\Gamma_0(\text{ord}(g)))} \eta(N_e \tau)^{\frac{h_e}{N_e} [c_{g,f/e}(0) + 2c_{g,f/e}(-1)]},
$$

(3.16)

Here we see how the cycle decomposition of an element $g$ corresponds to the decomposition into cusps of $\Gamma_0(\text{ord}(g))$. Length of each cycle equals $N_e$ and its power is given by $h_e [c_{g,f/e}(0) + 2c_{g,f/e}(-1)]$.

As before, by symmetrizing $p$ and $q$ in (3.14) we set

$$
\Phi_g(\Omega) = p^\frac{1}{2} \sum_\ell c_{g,0,\ell} q^\frac{1}{2} \sum_\ell c_{g,0,\ell} \zeta^\frac{1}{2} \sum_{\ell > 0} c_{g,0,\ell} \prod_{f/e \in \text{Cusp}(\Gamma_0(\text{ord}(g)))} \prod_{n,m,\ell \in \mathbb{Z}} (1 - (p^n q^m \zeta^\ell)^{N_e} h_e c_{g,f/e}(nm,\ell)).
$$

(3.17)

By checking transformation properties under generators of a congruence subgroup given in Appendix A.4, we see that $\Phi_g(\Omega)$ is the Siegel modular form on $\Gamma_0(2)$ [1, 6]. Comparing with (3.14), we find the Hodge anomaly

$$
\Phi_g(\Omega) M_g(\Omega) = p \varphi_g(z; \tau).
$$

(3.18)

Furthermore as shown in [6], the Borcherds product can be written as an additive lift of the $\eta$-product

$$
\Phi_g(\Omega) = \sum_{m=1}^{\infty} p^m (T_m \varphi_g)(z; \tau),
$$

(3.19)

for $g = 1A, 2A, 3A, 5A, 7AB, 4B, 6A, \text{and} 8A$. Our formula (2.12) follows from (3.10), (3.18), and (3.19).

In the case of remaining conjugacy classes, the weight of Jacobi form $\varphi_g$ becomes zero or negative and the situation is somewhat complex. For $g = 11A$, we conjecture the following relation for the Borcherds product (3.17)

$$
\Phi_{11A}(\Omega) = \frac{1}{11} \left[ \exp \left( 11 \sum_{m=1}^{\infty} p^m (T_m \varphi_{11A})(z; \tau) \right) - 1 \right].
$$

(3.20)

Similar relations are conjectured for 14AB and 15AB,

$$
\Phi_{14AB}(\Omega) = \frac{1}{14} \left[ \exp \left( 14 \sum_{m=1}^{\infty} p^m (T_m \varphi_{14AB})(z; \tau) \right) - 1 \right],
$$

(3.21)

$$
\Phi_{15AB}(\Omega) = \frac{1}{15} \left[ \exp \left( 15 \sum_{m=1}^{\infty} p^m (T_m \varphi_{15AB})(z; \tau) \right) - 1 \right].
$$

(3.22)

We do not have an analogous expression for 23AB.

We note that exactly the same exponent as (3.16) appears in the RHS of (3.17). Then again the inverse of $\Phi_g(\Omega)$ factorizes at its pole into $\eta$ products, and we obtain

$$
\frac{1}{\Phi_g(\Omega)} \approx \frac{1}{(2 \pi i)^2} \frac{1}{\eta_g(\tau)} \frac{1}{\eta_g(\sigma)},
$$

(3.23)

for all $g \in \text{type I}$.
| $g$ | $\text{cusp}$ | $h_n$ | $N_g$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $18$ | $19$ | $20$ | $21$ | $22$ | $23$ | $24$ | $25$ | $26$ |
|-----|-------------|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 1A  | $\infty$   | 1     | 1    | 2    | 20   | 128  | 216  | -1026| 1616 | -5504 | 8032 | -23550| 33048| -86400| 117280| -288652| 376608 | -854528 | 1112832| -2402298| 3082192|
| 2A  | $\infty$   | 1     | 1    | 2    | 4    | 0    | -8   | -4   | 16   | 0    | -32  | 2    | 56   | 0    | -96   | 4    | 160   | 0    | -256  | 6    | 400  |
| 3A  | $\infty$   | 1     | 1    | 0    | 8    | -64  | 112  | -512 | 800  | -2752 | 4032  | -1176 | 16496| -43200| 58688 | -141824| 188224 | -427264 | 556544 | -1201152| 1540896|
| 4B  | $\infty$   | 1     | 1    | 2    | 2    | -2   | 0    | 0    | 4    | 4    | -6   | 0    | -8   | 10   | 12    | -14  | 0    | 0    | -20  |
| 5A  | $\infty$   | 1     | 1    | 2    | 4    | 0    | -8   | 0    | 16   | 0    | -32  | 2    | 56   | 0    | -96   | 4    | 160   | 0    | -256  | 6    | 400  |
| 6A  | $\infty$   | 1     | 1    | 0    | 6    | 2    | -20  | 36   | -168 | 264  | -912  | 1336  | -3912 | 5484  | -14376| 19536  | -47232 | 626888 | 143240 | 185424 | -400248 | 513480 |
| 7AB | $\infty$   | 1     | 1    | 2    | -1   | 5    | -8   | 17   | -22  | 47   | -60   | 110   | -132  | 239   | -292  | 492   | -580   | 963   | -1134  | 1810  | -2106  |
| 8A  | $\infty$   | 1     | 1    | 0    | 7    | 3    | -19  | 32   | -149 | 234   | -793  | 1156  | -3380 | 4740  | -12377 | 16796  | -40592 | 53884  | -122213 | 159138 | -343444 | 406614 |
| 11A | $\infty$   | 1     | 1    | 2    | -2   | 4    | -8   | 12   | -8   | 18   | -20  | 34    | -40   | 60    | -68   | 104   | -120   | 172   | -192   | 278   | -316   |
| 14A | $\infty$   | 1     | 1    | 0    | 11   | 2    | -20  | 12   | -94  | 148   | -502  | 732   | -2144  | 3008   | -7860  | 10668  | -25796  | 34248  | -77700  | 101184 | -218416 | 280228 |
| 15A | $\infty$   | 1     | 1    | 2    | -3   | 7    | -8   | 19   | -26  | 49   | -60   | 114   | -140  | 245   | -292  | 500   | -596   | 973    | -1138  | 1826   | -2134  |
| 15B | $\infty$   | 1     | 1    | 0    | 15   | 1    | -9   | 16   | -73  | 114   | -303  | 576   | -1682  | 2356   | -6171  | 8384   | -20260  | 268888 | -61037  | 79506  | -171592 | 220126 |
| 23A | $\infty$   | 1     | 1    | 2    | -3   | 10   | -14  | 32   | -40  | 85   | -110  | 209   | -256  | 471   | -572  | 996   | -1190  | 2015   | -2392  | 3916   | -4592  |
| 23B | $\infty$   | 1     | 1    | 0    | 23   | 1    | -6   | 10   | -46  | 72   | -243  | 354   | -1033  | 1448   | -3777  | 5124   | -12376  | 16426  | -37241  | 48488  | -104618 | 134208 |

Table 4. The Fourier coefficient $c_{h,n}(g) = c_{h,n}(\tau|g)$ of the twisted elliptic genus $f_\ell(g) = f_\ell(\tau|g)$.
4. Concluding Remarks

In this paper we tried to obtain a simple and direct relationship between \( \eta \)-products of various conjugacy classes of \( M_{24} \) and the corresponding twisted elliptic genus of \( K3 \) surface. It seems that the simplest way to derive such a relation is to use the identity of Siegel modular forms which may be constructed either from Borcherds products or Saito–Kurokawa additive lifts of Jacobi forms.

Relationship (2.12) seems to exhibit some deep relation between \( M_{24} \) and \( K3 \) surface. RHS is based purely on \( M_{24} \) and has nothing to do with \( K3 \). It, however, coincides with LHS which is the twisted genus of \( K3 \).

We have so far discussed (2.12) and its variation only in the case of type I classes. If one tries to find a similar relation for type II classes, one obtains

\[
Z_{2B}(z; \tau) = -\frac{(T_2\varphi_{2B})(z; \tau)}{\varphi_{2B}(z; \tau)},
\]
\[
Z_{4A}(z; \tau) = -\frac{(T_2\varphi_{4A})(z; \tau)}{\varphi_{4A}(z; \tau)},
\]
\[
Z_{10A}(z; \tau) = -\frac{(T_2\varphi_{10A})(z; \tau)}{\varphi_{10A}(z; \tau)} + 10 \varphi_{10A}(z; \tau),
\]
\[
Z_{12A}(z; \tau) = -\frac{(T_2\varphi_{12A})(z; \tau)}{\varphi_{12A}(z; \tau)} - 12 \varphi_{12A}(z; \tau),
\]

which has the same form as the type I classes.

Unfortunately, in the case of other type II classes we obtain expressions which do not seem to clarify the relationship between \( M_{24} \) and \( K3 \) surface

\[
Z_{4C}(z; \tau) = -\frac{(T_2\varphi_{4C})(z; \tau)}{\varphi_{4C}(z; \tau)} - 16 \frac{\eta(2\tau)^4 \eta(8\tau)^4}{\eta(4\tau)^4} \phi_{-2,1}(z; \tau),
\]
\[
Z_{3B}(z; \tau) = -\frac{(T_2\varphi_{3B})(z; \tau)}{\varphi_{3B}(z; \tau)} - 18 \frac{\eta(\tau)^3 \eta(9\tau)^3}{\eta(3\tau)^2} \phi_{-2,1}(z; \tau),
\]
\[
Z_{6B}(z; \tau) = -\frac{(T_2\varphi_{6B})(z; \tau)}{\varphi_{6B}(z; \tau)} - 2 \left( \frac{\eta(\tau)^3 \eta(9\tau)^3}{\eta(3\tau)^2} + 6 \frac{\eta(2\tau)^3 \eta(18\tau)^3}{\eta(6\tau)^2} + 8 \frac{\eta(4\tau)^3 \eta(36\tau)^3}{\eta(12\tau)^2} \right) \phi_{-2,1}(z; \tau),
\]

As a whole, type I classes are reasonably under good control while we still know very little about type II classes. Twisted genera of type II classes have vanishing Witten index and appear to have a little contact with the classical geometry of \( K3 \) surface. On the other hand its character expansion is described in terms of modular forms and should be easier to handle than the type I classes. We hope to report progress on these issue in the near future.
Acknowledgments

T.E. would like thank California Institute for Technology and profs. H. Ooguri and J.H. Schwarz for Moore distinguished scholarship during the fall of 2011 and kind hospitality. K.H. thanks the Simons Center for Geometry and Physics for hospitality in the summer of 2011. Authors would like to thank H. Aoki for sending them an unpublished manuscript. Thanks are also to M. Kaneko for communications. This work is supported in part by Grant-in-Aid from the Ministry of Education, Culture, Sports, Science and Technology of Japan.
We collect facts about modular forms. See, e.g., [3, 22], for details.

A.1 Jacobi Theta Functions and Dedekind \( \eta \)-function

The Jacobi theta functions are defined by

\[
\begin{align*}
\theta_{11}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2})(z+\frac{1}{2})}, \\
\theta_{10}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2})z}, \\
\theta_{00}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{n^2} e^{2\pi inz}, \\
\theta_{01}(z; \tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{3}{2}n^2} e^{2\pi in(z+\frac{1}{2})}.
\end{align*}
\]

(A.1)

The Dedekind \( \eta \)-function is

\[
\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]

(A.2)

We set the Eisenstein series by

\[
\phi_2^{(N)}(\tau) = \frac{24}{N-1} q \frac{\partial}{\partial q} \log \left( \frac{\eta(N\tau)}{\eta(\tau)} \right).
\]

(A.3)

We introduce new forms of level-23

\[
\begin{align*}
f_{23,1}(\tau) &= \frac{1}{4} [\Theta_1(\tau)]^2 + \frac{1}{2} \Theta_1(\tau) \Theta_2(\tau) - \frac{3}{4} [\Theta_2(\tau)]^2, \\
f_{23,2}(\tau) &= \left[ \eta(\tau) \eta(23\tau) \right]^2 \\
&= \frac{1}{4} [\Theta_1(\tau)]^2 - \frac{1}{2} \Theta_1(\tau) \Theta_2(\tau) + \frac{1}{4} [\Theta_2(\tau)]^2.
\end{align*}
\]

(A.4)

where the \( \Theta \)-functions are

\[
\begin{align*}
\Theta_1(\tau) &= \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+6n^2}, \\
\Theta_2(\tau) &= \sum_{m,n \in \mathbb{Z}} q^{2m^2+mn+3n^2}.
\end{align*}
\]

A.2 Modular Form

We set the slash operator

\[
(f|_k \gamma)(\tau) = (\text{det } \gamma)^{\frac{k}{2}} (c\tau + d)^{-k} f(\gamma \tau),
\]

where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is an integral matrix. A modular form \( f(\tau) \in M_k(\Gamma_0(N), \chi) \) satisfies

\[
(f|_k \gamma)(\tau) = \chi(d) f(\tau)
\]

(A.6)
for $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$, and $\chi$ is a Dirichlet character modulo $N$.

### A.3 Jacobi Form

We define the slash operator

$$
(\phi|_{k,m}\gamma)(z;\tau) = (\det \gamma)^{\frac{k}{2}} (c\tau + d)^{-k} e^{2\pi im\frac{cz^2}{c\tau + d}} \phi \left( \frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d} \right),
$$

for an integral matrix $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$. Then the Jacobi form $\phi(z;\tau)$ with weight $k$ and index $m \in \mathbb{Z}$ on $\Gamma_0(N)$ with $\chi(d)$ fulfills

$$
(\phi|_{k,m}\gamma)(z;\tau) = \chi(d) \phi(z;\tau),
$$

where $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$ and $s, t \in \mathbb{Z}$. We set such space as $\mathbb{J}_{k,m}(\Gamma_0(N),\chi)$. Examples of the Jacobi forms $\mathbb{J}_{k,m}(\Gamma(1))$ are as follows $[22]$;

$$
\phi_{-2,1}(z;\tau) = -\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^6},
$$

$$
\phi_{0,1}(z;\tau) = 4 \left[ \left( \frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)} \right)^2 + \left( \frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)} \right)^2 + \left( \frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)} \right)^2 \right].
$$

### A.4 Siegel Modular Form

The Siegel modular form of degree-2 and weight $k$ is a function of $\Omega = \left( \begin{array}{cc} z & \sigma \\ \sigma^T & \tau \end{array} \right)$ in the Siegel upper half plane satisfying

$$
F \left( (A\Omega + B)(C\Omega + D)^{-1} \right) = \det(C\Omega + D)^k F(\Omega).
$$

Here $M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp(2;\mathbb{Z})$ fulfills $M^T J M = J$ for $J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. We use the congruence subgroup $\Gamma_0^{(2)}(N)$ given by

$$
\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\mathbb{Z}) \left| C = 0 \mod N \right. \right\},
$$

Its generators are $[11]$:

$$
\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ n & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & 0 & b \\ cN & 0 & d \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 12 & B \\ 0 & 12 \end{pmatrix},
$$

where $n \in \mathbb{Z}$, $\left( \begin{array}{cc} a & b \\ cN & d \end{array} \right) \in \Gamma_0(N)$, and $B^T = B$. Note that the first one corresponds to symmetrization of $\tau$ and $\sigma$.

### B. $\mathcal{N} = 4$ Superconformal Characters

The characters are defined by

$$
ch^R_{h,k}(z;\tau) = \text{Tr}_R \left( (-1)^F e^{4\pi i T_0^3} q^{L_0 - \frac{c}{24}} \right),
$$

(B.1)
where $R$ means the Ramond sector of the theory. They are explicitly given as follows \cite{Eguchi:2010uu, Eguchi:2010ub}:

- **BPS (massless) representations ($h = \frac{1}{4}$)**
  \[
  \text{ch}^{R}_{h=\frac{1}{4}, \ell=0}(z; \tau) = \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3} \mu(z; \tau),
  \]
  \[
  \text{ch}^{R}_{h=\frac{1}{4}, \ell=\frac{1}{2}}(z; \tau) + 2 \text{ch}^{R}_{h=\frac{1}{4}, \ell=0}(z; \tau) = q^{-\frac{1}{4}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3},
  \]

  where
  \[
  \mu(z; \tau) = i e^{\pi i z} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2} n(n+1)} e^{2 \pi i n z}. \quad \text{(B.4)}
  \]

- **non-BPS (massive) representations ($h > \frac{1}{4}$)**
  \[
  \text{ch}^{R}_{h > \frac{1}{4}, \ell=\frac{1}{2}}(z; \tau) = q^{h-\frac{1}{4}} \frac{[\theta_{11}(z; \tau)]^2}{[\eta(\tau)]^3}. \quad \text{(B.5)}
  \]

**References**

[1] H. Aoki and T. Ibukiyama, *Simple graded rings of Siegel modular forms, differential operators and Borcherds products*, Int. J. Math. 16, 249–279 (2005).
[2] R. E. Borcherds, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. 120, 161–213 (1995).
[3] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier, *The 1-2-3 of Modular Forms*, Springer, Berlin, 2008.
[4] S. Chaudhuri, G. Hockney, and J. D. Lykken, *Maximally supersymmetric string theories in $D < 10$*, Phys. Rev. Lett. 75, 2264–2267 (1995) [hep-th/9505054].
[5] M. C. N. Cheng, *K3 surfaces, $N = 4$ dyons, and the Mathieu group $M_{24}$*, Commun. Number Theory Phys. 4, 623–657 (2010), arXiv:1005.5415 [hep-th].
[6] F. Cléry and V. Gritsenko, *Siegel modular forms of genus 2 with the simplest divisor*, Proc. London Math. Soc. (3) 102, 1024–1052 (2011), arXiv:0812.3962 [math.NT].
[7] J. H. Conway, R. T. Curtis, R. A. Wilson, S. P. Norton, and R. A. Parker, *ATLAS of Finite Groups*, Clarendon Press, Oxford, 1985.
[8] J. H. Conway and S. P. Norton, *Monstrous moonshine*, Bull. London Math. Soc. 11, 308–339 (1979).
[9] A. Dabholkar and S. Nampuri, *Spectrum of dyons and black holes in CHL orbifolds using Borcherds lift*, J. High Energy Phys. 2007:11, 077 (2007), 20 pages [hep-th/0603056].
[10] J. R. David, P. D. Jatkar, and A. Sen, *Product representation of dyon partition function in CHL model*, J. High Energy Phys. 2006:06, 064 (2006), 31 pages, hep-th/0602254.
[11] R. Dijkgraaf, G. Moore, E. Verlinde, and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, Commun. Math. Phys. 185, 197–209 (1997), hep-th/9608096.
[12] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Counting dyons in $N = 4$ string theory*, Nucl. Phys. B 484, 543–561 (1996), hep-th/9607026.
[13] D. Dummit, H. Kisilevsky, and J. McKay, *Multiplicative products of $\eta$-functions*, in J. McKay, ed., *Finite Groups — Coming of Age*, vol. 45 of Contemp. Math., pp. 89–98, Amer. Math. Soc., Providence, 1985.
[14] T. Eguchi and K. Hikami, *Superconformal algebras and mock theta functions*, J. Phys. A: Math. Theor. 42, 304010 (2009), 23 pages, arXiv:0812.1151 [math-ph].
[15] ———, *Superconformal algebras and mock theta functions 2. Rademacher expansion for K3 surface*, Commun. Number Theory Phys. 3, 531–554 (2009), arXiv:0904.0911 [math-ph].
[16] ———, *$N = 4$ superconformal algebra and the entropy of hyperKähler manifolds*, J. High Energy Phys. 2010:02, 019 (2010), 28 pages, arXiv:0909.0410 [hep-th].
[17] ———, *Note on twisted elliptic genus of K3 surface*, Phys. Lett. B 694, 446–455 (2011), arXiv:1008.4924 [hep-th].
TWISTED ELLIPTIC GENUS FOR K3 AND BORCHERDS PRODUCT

[18] T. Eguchi, H. Ooguri, and Y. Tachikawa, Notes on the K3 surface and the Mathieu group M24, Exp. Math. 20, 91–96 (2011), arXiv:1004.0956 [hep-th].

[19] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, Superconformal algebras and string compactification on manifolds with SU(n) holonomy, Nucl. Phys. B 315, 193–221 (1989).

[20] T. Eguchi and A. Taormina, Unitary representations of the N = 4 superconformal algebra, Phys. Lett. B 196, 75–81 (1986).

[21] Character formulas for the N = 4 superconformal algebra, Phys. Lett. B 200, 315–322 (1988).

[22] M. Eichler and D. Zagier, The Theory of Jacobi Forms, vol. 55 of Progress in Mathematics, Birkhäuser, Boston, 1985.

[23] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, Mathieu twining characters for K3, J. High Energy Phys. 2010:09, 058 (2010), 20 pages arXiv:1006.0221 [hep-th].

[24] ———, Mathieu moonshine in the elliptic genus of K3, J. High Energy Phys. 2010:10, 062 (2010), 24 pages arXiv:1008.3778 [math.AG].

[25] S. Govindarajan and K. G. Krishna, Generalized Kac–Moody algebras from CHL dyons, J. High Energy Phys. 2009, 032 (2009), 39 pages arXiv:0807.4451 [hep-th].

[26] V. Gritsenko, Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms, Algebra i Analiz 11, 100–125 (1999), math/9906190.

[27] V. A. Gritsenko and V. V. Nikulin, Siegel automorphic corrections of some Lorentzian Kac–Moody Lie algebras, Amer. J. Math. 119, 181–224 (1997), alg-geom/9506017.

[28] ———, Automorphic forms and Lorentzian Kac–Moody algebra 2, Int. J. Math. 9, 201–275 (1998) alg-geom/9611028.

[29] P. D. Jatkar and A. Sen, Dyon spectrum in CHL models, J. High Energy Phys. 2006:04, 018 (2006), 31 pages hep-th/0510147.

[30] S. Katz, A. Klemm, and C. Vafa, M-theory, topological strings, and spinning black holes, Adv. Theor. Math. Phys. 3, 1445–1537 (1999), hep-th/9910181.

[31] M. Koike, On McKay’s conjecture, Nagoya Math. J. 95, 85–89 (1984).

[32] T. Kondo and T. Tasaka, The theta functions of sublattices of the Leech lattice, Nagoya Math. J. 101, 151–179 (1986).

[33] G. Mason, M24 and certain automorphic forms, in J. McKay, ed., Finite Groups — Coming of Age, vol. 45 of Contemp. Math., pp. 223–244, Amer. Math. Soc., Providence, 1985.

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS, KYOTO UNIVERSITY, KYOTO 606–8502, JAPAN, CALIFORNIA INSTITUTE OF TECHNOLOGY, E. 1200 CALIFORNIA BLVD., PASADENA CA 91125, U.S.A.

E-mail address: eguchi@yukawa.kyoto-u.ac.jp

FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA 819-0395, JAPAN.

E-mail address: KHikami@gmail.com