NOTE ON EQUIVALENCES FOR DEGENERATIONS OF
CALABI-YAU MANIFOLDS

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ABSTRACT. This note studies the equivalencies among convergences of
Ricci-flat Kähler-Einstein metrics on Calabi-Yau manifolds, cohomology
classes and potential functions.

1. INTRODUCTION

A Calabi-Yau manifold $X$ is a simply connected complex projective man-
ifold with trivial canonical bundle $K_X \cong \mathcal{O}_X$, and the Hodge numbers
$h^{n,0}(X) = 1$, $h^{i,0}(X) = 0$, $0 < i < n$. A polarized Calabi-Yau manifold
$(X,L)$ is a Calabi-Yau manifold $X$ with an ample line bundle $L$.

In [30], S.-T. Yau proved the Calabi’s conjecture which asserts the exis-
tence of Ricci-flat Kähler-Einstein metrics on Calabi-Yau manifolds. More
explicitly, if $X$ is a Calabi-Yau manifold, then for any Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$, there exists a unique Ricci-flat Kähler-Einstein metric $\omega \in \alpha$, i.e.

$$\text{Ric}(\omega) \equiv 0.$$ 

The Riemannian holonomy group of such metric is $SU(n)$. Conversely, one
can show that a simply connected compact Riemannian manifold with ho-
lonomy group $SU(n)$ is a Calabi-Yau manifold in our definition, and the
metric is a Ricci-flat Kähler-Einstein metric (cf. [32]).

A degeneration of Calabi-Yau $n$-manifolds $\pi : \mathcal{X} \rightarrow \Delta$ is a flat morphism
from a variety $\mathcal{X}$ of dimension $n + 1$ to a disc $\Delta \subset \mathbb{C}$ such that for any
t $t \in \Delta^* = \Delta \setminus \{0\}$, $X_t = \pi^{-1}(t)$ is a Calabi-Yau manifold, and the central
fiber $X_0 = \pi^{-1}(0)$ is singular. If there is a relative ample line bundle $\mathcal{L}$ on
$\mathcal{X}$, we call it a degeneration of polarized Calabi-Yau manifolds, denoted by
$(\pi : \mathcal{X} \rightarrow \Delta, \mathcal{L})$. A Calabi-Yau variety $X_0$ is a normal projective Goren-
stein variety with trivial canonical sheaf $K_{X_0} \cong \mathcal{O}_{X_0}$, and having at worst canonical singularities, i.e. for any resolution $\bar{\pi} : \bar{X}_0 \rightarrow X_0$,

$$K_{\bar{X}_0} \cong_Q \bar{\pi}^* K_{X_0} + \sum_{E} a_E E,$$

where $E$ are effective exceptional prime divisors, and $\cong_Q$ stands for the $\mathbb{Q}$-linear equivalence.

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There are several recent works, due to C.-L. Wang, V. Tosatti, and S. Takayama etc. ([28, 29, 26, 23]), establishing the equivalences of various properties along degenerations of Calabi-Yau manifolds. It begins as Candelas, Green and Hübsch discovered that some nodal degenerations of Calabi-Yau 3-folds have finite Weil-Petersson distances (cf. [2]). In general, [28] proves that for a degeneration of Calabi-Yau n-manifolds, the Weil-Petersson distance between general fibers and the central fiber is finite, if the central fiber is a Calabi-Yau variety. Wang conjectured in [29] that the converse is also true, and proposed to attack the conjecture by using the minimal model program. Eventually, [26, 23] prove Wang’s conjecture, and obtain the equivalency between the finiteness of Weil-Petersson metric and the central filling-in with Calabi-Yau varieties under various conditions. In [23], the further equivalence to the properties of Ricci-flat Kähler-Einstein metrics is also established. The equivalencies are used in [33] to construct completions for moduli spaces for polarized Calabi-Yau manifolds. The goal of this note is to add two more perspectives to this picture.

If \((\pi : X \to \Delta, L)\) is a degeneration of polarized Calabi-Yau manifolds, then all fibers \(X_t, t \in \Delta^*\), are diffeomorphic to each other, and we denote \(X\) the underlying differential manifold. Let \(\mathbb{H} \to \Delta^*\) be the universal covering given by \(w \mapsto t = \exp 2\pi \sqrt{-1}w\), where \(\mathbb{H} = \{w \in \mathbb{C} | \text{Im}(w) > 0\}\), and \(\tilde{\pi} : \tilde{X} \to \mathbb{H}\) be the pull-back family of \(X_{\Delta^*} = \pi^{-1}(\Delta^*) \to \Delta^*\). The total space \(\tilde{X}\) is diffeomorphic to \(X \times \mathbb{H}\), and we identify \(H^n(X_t, \mathbb{C}), t \neq 0\), canonically with \(H^n(X, \mathbb{C})\). Under this setup, we have the first theorem of this paper.

**Theorem 1.1.** Let \((\pi : X \to \Delta, L)\) be a degeneration of polarized Calabi-Yau manifolds. Let \(\Omega_t\) be the holomorphic volume form on \(X_t, t \in \Delta^*\), i.e. a nowhere vanishing section of the canonical bundle \(K_{X_t}\), such that

\[
(-1)^{n^2} \int_{X_t} \Omega_t \wedge \overline{\Omega_t} \equiv 1,
\]

and \(\omega_t\) be the unique Ricci-flat Kähler-Einstein metric in \(c_1(L)|_{X_t} \in H^{1,1}(X_t, \mathbb{R})\). Then the following statements are equivalent.

i) When \(t \to 0\), the cohomology classes

\([\Omega_t] \to \beta \quad \text{in} \quad H^n(X, \mathbb{C}).\)

ii) When \(t \to 0\),

\[(X_t, \omega_t) \xrightarrow{dGH} (Y, dY),\]

in the Gromov-Hausdorff sense, where \((Y, dY)\) is a compact metric space.

iii) The origin \(0 \in \Delta\) is at finite Weil-Petersson distance from \(\Delta^*\).

The purpose of this theorem is to present the equivalences obtained in [28, 29, 26, 23] from a more Riemannian geometric point of view. We regards \(X_t, t \neq 0\), as \(X\) equipped with a complex structure \(J_t\), and the metric \(\omega_t\) as
a Riemannian metric with holonomy group $SU(n)$. The holomorphic volume form $\Omega_t$ is parallel with respect to any Ricci-flat Kähler-Einstein metric on $X_t$, and after a certain normalization, $\Omega_t$ gives two calibration $n$-forms $\text{Re}(\Omega_t)$ and $\text{Im}(\Omega_t)$ in the sense of [11]. Therefore, Theorem 1.1 gives a criterion of Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics via the cohomology classes of calibration forms in the context of special holonomy Riemannian geometry. It is desirable to remove the algebro-geometric conditions, for example $X_t$ fitting into an algebraic family with respect to the parameter $t$, and to prove it directly without using the sophisticated algebraic geometry and PDE.

Now we switch to a more analytic point of view. We recall that a degeneration of polarized Calabi-Yau manifolds $(\pi : \mathcal{X} \to \Delta, L)$ is called having only log-canonical singularities, if $\mathcal{X}$ is normal, $X_0$ is reduced, $K_{\mathcal{X}} + X_0$ is $\mathbb{Q}$-Cartier, and for any log-resolution $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}$ of singularities,

$$K_{\tilde{\mathcal{X}}} + X'_0 \equiv_{\mathbb{Q}} \tilde{\pi}^*(K_{\mathcal{X}} + X_0) + \sum_E a_E E, \quad \text{and } a_E \geq 0,$$

where $E$ are effective exceptional prime divisors, and $X'_0$ is the strict transform of $X_0$ (cf. [13]).

Let $(\pi : \mathcal{X} \to \Delta, L)$ be a degeneration of polarized Calabi-Yau $n$-manifolds. There is an integer $m \geq 1$ such that $L^m$ is relative very ample, which induces a relative embedding $\Phi : \mathcal{X} \hookrightarrow \mathbb{CP}^N \times \Delta$ such that $L^m \cong \Phi^*O_{\mathbb{CP}^N}(1)$, and the restriction $\Phi_t = \Phi|_{X_t}$ embeds $X_t$ into $\mathbb{CP}^N$ for any $t \in \Delta$. Note that the choice of $\Phi$ is not unique. If $\omega_t$ is the unique Ricci-flat Kähler-Einstein metric in $c_1(L^m)|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, and $\omega_{FS}$ denotes the Fubini-Study metric on $\mathbb{CP}^N$, then for any $t \in \Delta^*$, there is a unique function $\varphi_t$, called the potential function, on $X_t$ such that

$$\omega_t = \Phi_t^*\omega_{FS} + \sqrt{-1} \partial\bar{\partial} \varphi_t, \quad \sup_{X_t} \varphi_t = 0.$$

The second theorem is the following.

\textbf{Theorem 1.2.} Let $(\pi : \mathcal{X} \to \Delta, L)$ be a degeneration of polarized Calabi-Yau $n$-manifolds such that $\mathcal{X}$ is normal, the relative canonical bundle $K_{\mathcal{X}/\Delta}$ is trivial, i.e. $K_{\mathcal{X}/\Delta} \cong O_{\mathcal{X}}$, and $\mathcal{X} \to \Delta$ has at worst log-canonical singularities. Let $\omega_t$ be the unique Ricci-flat Kähler-Einstein metric presenting $c_1(L^m)|_{X_t} \in H^{1,1}(X_t, \mathbb{R})$, $t \in \Delta^*$, where $m \geq 1$ such that $L^m$ is relative very ample. Then the following statements are equivalent.

i) There is a relative embedding $\Phi : \mathcal{X} \hookrightarrow \mathbb{CP}^N \times \Delta$ induced by $L^m$ such that the potential functions $\varphi_t$ determined by $\Phi$ and $\omega_t$ satisfy

$$\inf_{X_t} \varphi_t \geq -C,$$

for a constant $C > 0$ independent of $t$.

ii) The central fiber $X_0$ is a Calabi-Yau variety.

iii) When $t \to 0$,

$$(X_t, \omega_t) \overset{dGH}{\to} (Y, d_Y),$$
in the Gromov-Hausdorff sense, where \((Y,d_Y)\) is a compact metric space.

Note that in the condition i) of Theorem 1.2, if we choose a different embedding \(\Phi'\), then the new potential function \(\varphi'_t = \varphi_t + \xi\), where \(\xi\) is a continuous function on \(\mathcal{X}\) such that \(\Phi'^*\omega_{FS} = \Phi^*\omega_{FS} - \sqrt{-1}\partial \bar{\partial} \xi\) on any \(X_t\). Hence the uniformly boundedness of \(\varphi_t\) is equivalent to the boundedness of \(\varphi'_t\), and the precise bound is not essential. We can replace i) by saying that for any embedding induced by \(L^m\), the boundedness of potential functions holds. This theorem shows the equivalency between such boundedness condition and many other equivalent properties studied in [28, 26, 23].

Section 2 gives an exposition of a result due to C.-L. Wang, which shows the equivalency between the convergence of cohomology classes and the finiteness of Weil-Petersson distances. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

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2. Hodge theoretic criterion

Let \((\pi : \mathcal{X} \to \Delta, \mathcal{L})\) be a degeneration of polarized Calabi-Yau \(n\)-manifolds, and \((\pi : \mathcal{X}^* = \pi^{-1}(\Delta^*) \to \Delta^*, \mathcal{L})\) be the base change by the inclusion \(\Delta^* \hookrightarrow \Delta\). A natural Kähler metric, possibly degenerated, is defined on \(\Delta^*\), called the Weil-Petersson metric, which measures the deformation of complex structures of fibers \(X_t\). If \(\Theta_t\) is a relative holomorphic volume form, i.e. a no-where vanishing section of the relative canonical bundle \(K_{\mathcal{X}^*/\Delta^*}\), then the Weil-Petersson metric is

\[
\omega_{WP} = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \int_{X_t} (-1)^{n^2} \Theta_t \wedge \bar{\Theta}_t \geq 0,
\]

(cf. [24, 25]). The Weil-Petersson metric \(\omega_{WP}\) is the curvature of the first Hodge bundle \(\pi_* \mathcal{K}_{\mathcal{X}^*/\Delta^*}\) with a natural Hermitian metric.

We rephrase Corollary 1.2 in [28] as the following.

**Proposition 2.1.** Let \(X\) be the underlying differential manifold of general fibers \(X_t, t \in \Delta^*,\) and let \(\Omega_t\) be the holomorphic volume form on \(X_t, t \in \Delta^*,\) such that

\[
(-1)^{n^2} \int_{X_t} \Omega_t \wedge \bar{\Omega}_t = 1.
\]

Then the following statements are equivalent.

i) When \(t \to 0\),

\(\left[\Omega_t\right] \to \beta\) in \(H^n(X, \mathbb{C})\).

ii) The origin \(0 \in \Delta\) is at finite Weil-Petersson distance from \(\Delta^*\).
Remark 2.2. Note that $\Omega_t$ does not vary holomorphically with respect to the variable $t$ due to the normalization condition, and it is only a smooth section of the relative canonical bundle.

Proof. We recall the argument in Section 1 of [28] with only a minor modification, which relies on the classical theory of Hodge degenerations (See [9] and [10] for the necessary background knowledge in the proof). If $\mathbb{H} \to \Delta^*$ is the universal covering given by $w \mapsto t = \exp 2\pi \sqrt{-1}w$, and $\tilde{\pi} : \tilde{X} \to \mathbb{H}$ is the base change of $X^* \to \Delta^*$, then $\tilde{X}$ is diffeomorphic to $X \times \mathbb{H}$. We identify $H^n(X_t, \mathbb{C})$, $t \neq 0$, canonically with $H^n(X, \mathbb{C})$, and we always write $\tilde{\pi}^{-1}(w)$ as $X_t$ for $t = \exp 2\pi \sqrt{-1}w$. If $\tilde{L}$ denotes the pull-back of $L$ to $\tilde{X}$, then the first Chern class $c_1(\tilde{L}) \in H^2(X, \mathbb{Z})$.

The family of polarized Calabi-Yau $n$-manifolds $(\pi : X^* \to \Delta^*, L)$ gives a polarized variation of Hodge structures of weight $n$. The Hodge filtrations are $F_1^t \subset \cdots \subset F_n^t = V$, for any $t \in \Delta^*$, where $V \subset H^n(X, \mathbb{C})$ is the primitive cohomology with respect to $c_1(L)$, and $F_1^t = H^{n,0}(X_t)$. The polarization is the Hodge-Riemann bilinear form

$$Q(\phi, \psi) = (-1)^{\frac{n(n-1)}{2}} \int_X \phi \wedge \psi,$$

for any $\phi$ and $\psi \in V$, which satisfies

$$Q(F^p, F^{n+1-p}) = 0,$$

for any $\xi \in V \cap H^{p,q}(X_t)$.

Let $T : H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})$ be the monodromy operator induced by the parallel transport of the local system $R^n_{\pi_*} \mathbb{Z}$ along the loop generating $\pi_1(\Delta^*)$. Since the polarization is invariant under parallel transports of $R^n_{\pi_*} \mathbb{Z}$, $T$ acts on $V$. Note that $T$ is quasi-unipotent, and therefore, we assume that $T$ is unipotent by passing to a certain base change, i.e. $(T-I)^k = 0$ for a $k \leq n+1$ (cf. [9]). We let $N = \log T : H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C})$, which satisfies $N^k = 0$ and $Q(N\phi, \psi) = -Q(\phi, N\psi)$ for any $\phi, \psi \in V$ (cf. [9]).

Since $h^{n,0}(X_t) = 1$, we only consider the period map $P : \mathbb{H} \to \mathbb{P}(V)$ of the $n$th-flag, which is defined by $P(w) = \langle [\Theta_w] \rangle$, where $\Theta_w$ is a holomorphic volume form on $X_t$, and $\langle [\Theta_w] \rangle$ denotes the complex line in $V$ determined by $[\Theta_w]$. If we define $\Xi : \mathbb{H} \to \mathbb{P}(V)$ by $\Xi(w) = e^{-wN} P(w)$, then $\Xi$ descends to a map $\Xi : \Delta^* \to \mathbb{P}(V)$. The Schmid’s nilpotent orbit theorem asserts that $\Xi$ extends to a holomorphic map from $\Delta$, denoted still by $\Xi : \Delta \to \mathbb{P}(V)$ (cf. [21], Chapter IV in [9], and Section 16.3 in [10]).

If $A : \Delta \to V$ is a holomorphic map such that $A(t) \neq 0$ for any $t \in \Delta$, and $\langle A(t) \rangle \in \Xi(t)$, then $\langle e^{wN} A(t) \rangle \in \mathbb{P}(w)$, for any $w \in \mathbb{H}$ with $t = \exp 2\pi \sqrt{-1}w$. Let $\theta_t$ be a relative holomorphic volume form on $X^*$ such that $\theta_t = e^{wN} A(t)$ on $X_t$. Note that $A(t) = a_0 + h(t)$ where $a_0 = A(0)$ and $h(t)$ is a holomorphic vector valued function on $\Delta$ with $|h(t)| \leq C|t|$ for a constant $C > 0$. Here $| \cdot |$ denotes a fixed Euclidean norm on the finite dimensional
Theorem 3.1. Let $\converge$, which does not appear explicitly in the literature.

The calculation in Section 1 of \cite{28} shows that $d>\max\{1|N^t_0a_0\neq 0\}$.

Before we prove Theorem 1.1, we show a result about the Gromov-Hausdorff convergence, which does not appear explicitly in the literature.

\textbf{Theorem 3.1.} Let $($,$\Delta,L)$ be a degeneration of polarized Calabi-Yau manifolds, and $\omega_t$ be the unique Ricci-flat Kähler-Einstein metric in
\( c_1(L)|_{X_t} \in H^{1,1}(X_t, \mathbb{R}) \). If the origin 0 ∈ Δ is at finite Weil-Petersson distance from \( \Delta^* \), then when \( t \to 0 \),

\[
(X_t, \omega_t) \xrightarrow{dGH} (Y, d_Y),
\]
in the Gromov-Hausdorff sense, where \( (Y, d_Y) \) is a compact metric space, and is homeomorphic to a Calabi-Yau variety.

This theorem could be proved by the argument in the proof of Lemma 6.9 in [14], and we provide an independent proof here.

**Proof.** By Corollary 1.7 in [23], the diameters

\[
\text{diam}_{\omega_t}(X_t) \leq D,
\]
for a constant \( D > 0 \) independent of \( t \). The Gromov precompactness theorem (cf. [16, 17]) asserts that for any sequence \( t_k \to 0 \),

\[
(X_{t_k}, \omega_{t_k}) \xrightarrow{dGH} (Y, d_Y),
\]
by passing to a subsequence, in the Gromov-Hausdorff sense, where \( Y \) is a compact metric space. Now we improve the convergence to along \( t \to 0 \), i.e. without passing to any subsequences. We follow the arguments in the proof of Theorem 1.4 in [23].

Let \( P \) be the Hilbert polynomial of the general fibers, i.e. \( P = P(\mu) = \chi(X_t, L^i_t) \) where \( L_t = L|_{X_t} \). By Matsusaka’s Big Theorem (cf. [15]), there is an \( m_0 > 0 \) depending only on \( P \) such that for any \( m \geq m_0 \), \( L^m_t \) is very ample, and \( H^i(X_t, L^m_t) = \{0\} \), \( i > 0 \), \( t \in \Delta^* \). A basis \( \Sigma \) of \( H^0(X_t, L^m_t) \) induces an embedding \( \Phi_t : X_t \hookrightarrow \mathbb{CP}^N \) such that \( L^m_t = \Phi^*_t \mathcal{O}_{\mathbb{CP}^N}(1) \). We regard \( \Phi_t(X_t) \) as a point in the Hilbert scheme \( \mathcal{Hil}^{P_m} \) parametrizing the subshemes of \( \mathbb{CP}^N \) with Hilbert polynomial \( P_m(\mu) = \chi(X_t, \mathcal{L}^m_t) \), where \( N = P_m(1) - 1 \). For any other choice \( \Sigma', \Phi'_t(X_t) = \varrho(u, \Phi_t(X_t)) \) for a \( u \in SL(N + 1) \) where \( \varrho : SL(N + 1) \times \mathcal{Hil}^{P_m} \to \mathcal{Hil}^{P_m} \) is the \( SL(N + 1) \)-action on \( \mathcal{Hil}^{P_m} \) induced by the natural \( SL(N + 1) \)-action on \( \mathbb{CP}^N \) (See [27] for the background knowledge). We choose \( m \gg 1 \) such that \( \mathcal{L}^m \) is relative ample on \( X^* = X \setminus X_0 \).

Theorem 1.2 of [8] asserts that by taking \( m \gg 1 \), we have a subsequence \( X_{t_k} \) satisfying the following. For any \( t_k \), there is an orthonormal basis \( \Sigma_k \) of \( H^0(X_{t_k}, L^m_{t_k}) \) with respect to the \( L^2 \)-norm induced by the Hermitian metric \( H_k \) on \( L_{t_k} \) giving \( \omega_{t_k} \), i.e. \( \omega_{t_k} = -i \partial\bar{\partial} \log H_k \), which defines an embedding \( \Phi_{t_k} : X_{t_k} \hookrightarrow \mathbb{CP}^N \) with \( L_{t_k} = \Phi_{t_k}^* \mathcal{O}_{\mathbb{CP}^N}(1) \). And \( \Phi_{t_k}(X_{t_k}) \) converges to \( X_\infty \) in the reduced Hilbert scheme \( \mathcal{Hil}^{P_m}_{\text{red}} \) with respect to the natural analytic topology. Furthermore \( X_\infty \) is homeomorphic to the Gromov-Hausdorff limit \( Y \). By Proposition 4.15 of [8], \( X_\infty \) is a projective normal variety with only log-terminal singularities. Note that we can choose holomorphic volume forms \( \Omega_{t_k} \) converging to a holomorphic volume form \( \Omega_\infty \) on the regular locus \( X_{\infty, \text{reg}} \) along the Gromov-Hausdorff convergence. Thus the canonical sheaf \( K_{X_\infty} \) is trivial, i.e. \( K_{X_\infty} \cong \mathcal{O}_{X_\infty} \), and \( X_\infty \) is 1-Gorenstein,
which implies that $X_\infty$ has at worst canonical singularities. Consequently, $X_\infty$ is a Calabi-Yau variety.

For any $p \in \mathcal{Hil}^P_m$, we denote $O_p$ the $SL(N + 1)$-orbit, i.e. $O_p = \{g(u, p) | u \in SL(N + 1)\}$, and $\overline{p}$ the Zariski closure of $O_p$ in $\mathcal{Hil}^P_m$. If $\mathcal{H}^o \subset \mathcal{Hil}^P_m$ denotes the open subscheme parameterizing smooth projective manifolds with Hilbert polynomial $P_m$, then $\overline{p} \cap \mathcal{H}^o$ is clearly closed in $\mathcal{H}^o$, which works as the following.

Let $q \in \overline{p} \cap \mathcal{H}^o$, and $\iota : \Delta \to \overline{p}$ such that $q = \iota(0)$ and $\iota(\Delta^*) \subset O_p$. We obtain a family of Calabi-Yau manifolds $Z \to \Delta$ as the base change, i.e. $Z = U^P_m \times_{\mathcal{Hil}^P_m} \Delta$, where $U^P_m \to \mathcal{Hil}^P_m$ denotes the universal family. Note that all fibers $Z_z, z \in \Delta^*$, are isomorphic to each other as $\iota(z) \in O_p$ for any $z \neq 0$. Thus the image of the period map $P : \Delta \to \mathcal{D}$ is one point, where $\mathcal{D}$ denotes the classifying space for the polarized Hodge structure of weight $n$ (cf. [9]). The differential of the period map $dP_z : T_z \Delta \to T_{P(z)}(\mathcal{D})$ is a composition of the Kodaira-Spencer map $T_z \Delta \to H^{n-1,1}(Z_z)$ and a map $\eta : H^{n-1,1}(Z_z) \to T_{P(z)}(\mathcal{D})$ (cf. Chapter III in [9] and Section 16.2 in [10]). The local Torelli theorem for Calabi-Yau manifolds says that $\eta$ is injective (cf. Proposition 3.6 in [8]), and thus the Kodaira-Spencer map of $Z \to \Delta$ is trivial. Therefore all fibers $Z_z, z \in \Delta$, are biholomorphic to each other, denoted by $Z$. Since $Z$ is simply connected, any restriction $\mathcal{O}_{CP^N(1)}|Z_z$ is isomorphic to the same ample line bundle $L_Z$, and any $Z_z \subset CP^N$ is the image of the embedding given by a basis of $H^0(Z, L_Z)$. Hence $q = \iota(0) \in O_p$, and $\overline{p} \cap \mathcal{H}^o = O_p \cap \mathcal{H}^o$.

Now we continue the proof. By the universal property of the universal family $U^P_m \to \mathcal{Hil}^P_m$, we have a morphism $\lambda : \Delta \to \mathcal{Hil}^P_m$ such that $\pi : \mathcal{X} \to \Delta$ is the pull-back family of $U^P_m$, i.e. the base change $\mathcal{X} = U^P_m \times_{\mathcal{Hil}^P_m} \Delta$. The Zariski closure of orbits

$$\mathcal{A} = \bigcup_{t \in \Delta} \{t\} \times O_{\lambda(t)} \subset \Delta \times \mathcal{Hil}^P_m$$

is studied in the part 4) of the proof of Theorem 1.4 in [23]. It is proved in [23] that $\mathcal{A}$ is an irreducible and projective variety over $\Delta$, and if $a : \mathcal{A} \to \Delta$ is the restriction of the natural projection, the fiber $a^{-1}(t) = \{t\} \times \overline{O_{\lambda(t)}}$ for any $t \in \Delta^*$. However the central fiber $a^{-1}(0)$ may be reducible. If we let

$$a^o = a|_{\mathcal{A}^o} : \mathcal{A}^o = \mathcal{A} \cap (\Delta^* \times \mathcal{H}^o) \to \Delta^*,$$

then $\mathcal{A}^o$ is a Zariski open set of $\mathcal{A}$, and

$$a^{o-1}(t) = \{t\} \times (\overline{O_{\lambda(t)}} \cap \mathcal{H}^o) = \{t\} \times (O_{\lambda(t)} \cap \mathcal{H}^o), \quad t \in \Delta^*.$$
necessary. We denote the pull-back family
\[ \pi' : X' = (\mathcal{U}^{P_m} \times Hilb_{N}^{P_m} \Delta)_{\text{red}} \to \Delta \]
by \( p \circ \nu \), where \( p : \mathcal{A} \to Hilb_{N}^{P_m} \) is the restriction of the natural projection map. Let \( \mathcal{L}' \) be the pull-back bundle of \( \Omega_{CP^{N}}(1) \), which is a relative very ample line bundle on \( X' \), where \( \Omega_{CP^{N}}(1) \) is the line bundle on \( \mathcal{U}^{P_m} \) induced by \( O_{\mathbb{P}^{N}}(1) \). Note that the central fiber \( X'_{0} = \pi'^{-1}(0) = X_{\infty} \), and for any \( t = s^{l} \), \( X'_{t} \) is isomorphic to \( X_{t} \), since \( \nu(s) \) belongs to \( \mathcal{X}_{\nu}^{-1}(t) = \{ t \} \times O_{\mathcal{X}_{\nu}}(t) \).

More explicitly, the isomorphism is given by an element \( u_{s} \in SL(N+1) \) such that \( \varrho(u_{s}, p \circ \nu(s)) = \lambda(t) \). The restricted bundle \( L'_{X'} \cong L_{\mathcal{V}}^{m} \).

We have a new polarized degeneration of Calabi-Yau manifolds \( (\pi' : X' \to \Delta, \mathcal{L}') \) with a Calabi-Yau variety \( X_{\infty} \) as the central fiber. Since \( X_{\infty} \) is normal, the total space \( X' \) is normal, and thus the relative canonical sheaf \( K_{X'/\Delta} \) is defined, i.e. \( K_{X'/\Delta} \cong K_{X'} \otimes \pi'^*K_{\Delta}^{-1} \), and is trivial, i.e. \( K_{X'/\Delta} \cong O_{X'} \).

If \( \omega' \) is the unique Ricci-flat Kähler-Einstein metric presenting \( c_{1}(\mathcal{L}'|_{X'}) \), then \( \omega' = m\omega_{0} \) after we identify \( X'_{s} \) and \( X_{t} \) with \( t = s^{l} \). In this case, the convergence of \( \omega'_{s} \) is studied in [20], [18], [19]. It is proved in [18] that
\[ F_{s}^{*}\omega'_{s} \to \omega \quad \text{when} \quad s \to 0, \]
in the \( C^{\infty} \)-sense on any compact subset \( K \) belonging to the regular part \( X'_{0,\text{reg}} \) of \( X'_{0} \), where \( F_{s} : X'_{0,\text{reg}} \to X'_{s} \) is a smooth family of embeddings with \( F_{0} = \text{Id}_{X'_{0}} \), and \( \omega \) is a Ricci-flat Kähler-Einstein metric on \( X'_{0,\text{reg}} \) with \( \omega \in c_{1}(\mathcal{L}')|_{X'_{0}} \), which was obtained previously in [4]. Furthermore, [19] proves
\[ (X'_{s}, \omega'_{s}) \xrightarrow{d_{GH}} (Y', d_{Y'}), \]
when \( s \to 0 \), in the Gromov-Hausdorff sense, where \( (Y', d_{Y'}) \) is the metric completion of \( (X'_{0,\text{reg}}, \omega) \), which is a compact metric space. Note that \( t \to 0 \) if and only if \( s \to 0 \), and \( Y \) is homeomorphic to \( Y' \), and \( d_{Y'} = \sqrt{md_{Y}} \). We obtain the conclusion. \( \square \)

**Remark 3.2.** One crucial step in the proof is to replace the original degeneration \( (\pi : \mathcal{X} \to \Delta, \mathcal{L}) \) by a new one \( (\pi' : \mathcal{X}' \to \Delta, \mathcal{L}') \), which satisfies that \( \mathcal{X}' \) contains all smooth fibers of \( \mathcal{X} \), and the new central fiber \( X'_{0} \) is a Calabi-Yau variety. Furthermore, under the identification of two general fibers \( X_{t} \cong X'_{t} \), \( t = s^{l} \), we have \( \mathcal{L}'|_{X_{t}} \cong \mathcal{L}'|_{X'_{t}} \). By Corollary 2.3 in [33], the Calabi-Yau variety \( X'_{0} \) is the unique choice as the filling-in in the following sense. If \( (\pi'' : \mathcal{X}'' \to \Delta, \mathcal{L}'') \) is another degeneration with the Calabi-Yau variety \( X''_{0} \) as the central fiber, and if there is a sequence of points \( s_{k} \to 0 \) in \( \Delta \) such that \( X''_{s_{k}} \cong X''_{s_{k}} \) and \( \mathcal{L}'|_{X''_{s_{k}}} \cong \mathcal{L}'|_{X''_{s_{k}}} \), then \( X''_{0} \) is isomorphic to \( X''_{0} \) (see also [4], [14], [17], [22]). Such property is called the separatedness condition, and is used to construct certain completions of moduli spaces (14, 22, 33, 34).

The other way to find a Calabi-Yau variety as the filling-in is to use the minimal model program as proposed by Wang ([29]), and carried out in
In this case, we further assume that the degeneration \((\mathcal{X} \to \Delta, \mathcal{L})\) comes from a quasi-projective family. More explicitly, there is a flat family of polarized \(n\)-varieties \((\mathcal{X} \to C, \mathcal{L})\) over a smooth curve \(C\) with a marked point \(y\) such that both of \(\mathcal{X}\) and \(C\) are quasi-projective, \(\mathcal{X}\) is the pull-back family of \(\mathcal{X}\) for an embedding \(\Delta \hookrightarrow C\) mapping 0 to \(y\), and \(\mathcal{L}\) is the pull-back bundle of \(\mathcal{L}\). All examples the author know satisfy this assumption.

Now we follow the arguments in \cite{26, 23}, and by taking the Mumford’s semi-stable reduction, i.e. a sequence of base changes and blow-ups, we obtain a degeneration \(\tilde{\pi} : \tilde{\mathcal{X}} \to \Delta\) with a normal crossing central fiber \(\tilde{\mathcal{X}}_0\) and \(\tilde{\mathcal{X}} \setminus \tilde{\mathcal{X}}_0\) being a base change of \(\mathcal{X} \setminus \mathcal{X}_0\) by \(s \mapsto s^j = t\). Then \cite{26, 23} use the recent results in the minimal model program, for example \cite{5, 16}, to show that \(\tilde{\mathcal{X}} \to \Delta\) is birational to a family \(\tilde{\mathcal{X}}' \to \Delta\) such that \(\tilde{\mathcal{X}} \setminus \tilde{\mathcal{X}}_0 \sim \tilde{\mathcal{X}}' \setminus \tilde{\mathcal{X}}'_0\), and the relative canonical divisor \(K_{\tilde{\mathcal{X}}'/\Delta}\) is \(\mathbb{Q}\)-Cartier, and \(\mathbb{Q}\)-linearly trivial.

A further argument proves that \(K_{\tilde{\mathcal{X}}'/\Delta}\) is Cartier, and linearly trivial (see the proof of Theorem 1.2 in \cite{26} for details). Thus we have a Calabi-Yau variety \(\tilde{\mathcal{X}}_0'\) as the central filling-in.

It is known that the minimal model \(\tilde{\mathcal{X}}'\) is not uniquely chosen, and it is proved in \cite{12} that any other choice \(\tilde{\mathcal{X}}''\) connects to \(\tilde{\mathcal{X}}'\) by a sequence of flops. Therefore the central Calabi-Yau variety \(\tilde{\mathcal{X}}_0'\) obtained by the minimal model program is not unique. Comparing to the unique chosen \(\mathcal{X}_0'\) in the proof of Theorem 3.1 what happens is the following. If \(\tilde{\mathcal{L}}\) is the pull-back bundle of \(\mathcal{L}\) on \(\tilde{\mathcal{X}} \setminus \tilde{\mathcal{X}}_0\) and therefore on \(\tilde{\mathcal{X}}' \setminus \tilde{\mathcal{X}}'_0\), then \(\tilde{\mathcal{L}}\) is relative ample. If one minimal model \(\tilde{\mathcal{X}}'\) allows that \(\tilde{\mathcal{L}}\) extends to a relative ample line bundle on \(\tilde{\mathcal{X}}'\) crossing the central fiber \(\tilde{\mathcal{X}}'_0\) after taking a certain power, then \(\tilde{\mathcal{X}}'\) is the only minimal model among many possible choices allowing the ample extension of \(\tilde{\mathcal{L}}\) by the separatedness condition (cf. Theorem 2.1 in \cite{1}). And this uniquely chosen \(\tilde{\mathcal{X}}_0'\) would coincide with the Gromov-Hausdorff limit of the Ricci-flat Kähler-Einstein metrics representing the polarization on the nearby fibers.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Firstly, we show that i) implies ii). By Proposition 2.1 \(0 \in \Delta\) is at finite Weil-Petersson distance from \(\Delta^*\), and thus ii) is the consequence of Theorem 3.1.

Now if we assume that ii) is true, then clearly there is a diameter upper bound, i.e. \(\text{diam}(\omega_t) \leq D\) for a constant \(D > 0\), which is equivalent to the volume non-collapsing property, i.e. for any \(r < \text{diam}(\omega_t)\),

\[
\text{Vol}_{\omega_t}(B_{\omega_t}(r)) \geq \kappa r^{2n},
\]

for a constant \(\kappa > 0\), by the Bishop-Gromov comparison theorem, and the polarization condition

\[
\text{Vol}_{\omega_t}(X_t) = \frac{1}{n!} \int_{X_t} c_1(\mathcal{L})^n > 0.
\]
By Theorem 1.4 of [23], $0 \in \Delta$ is at finite Weil-Petersson distance from $\Delta^*$. We obtain i) by Proposition 2.1.

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Firstly, the $C^0$-estimate in Section 3 of [18] shows that ii) implies i). Now we assume that i) is true, and we denote $\omega_t^o = \Phi_t^* \omega_{FS} \in c_1(\mathcal{L}^m)|_{X_t}$ on $X_t$, $t \in \Delta^*$.

Let $\Psi_t$ be a nowhere vanishing section of $\mathcal{K}_{X/\Delta}$, i.e. the divisor $\text{div}(\Psi_t) = 0$. Note that the codimension of the singular set $\mathcal{S}_X$ of $\mathcal{X}$ is bigger or equal to 2, since $\mathcal{X}$ is normal, and any irreducible component of $X_0$ has multiplicity one as $X_0$ is reduced. If $p \in X_{reg} \cap X_{0,reg}$, where $X_{reg} = \mathcal{X}\setminus \mathcal{S}_X$ and $X_{0,reg}$ denotes the regular set of $X_0$, then there is a neighborhood $U \subset X_{reg}$ of $p$ such that $U \cap X_{0,reg} \subset X_{0,reg}$, and there are coordinates $z_0, z_1, \ldots, z_n$ on $U$ satisfying that $X_t \cap U = \{z_0 = t\}$, and $z_1, \ldots, z_n$ are coordinates on $X_t \cap U$. Therefore, there is a nowhere vanishing holomorphic function $h_U$ on $U$ such that $\Psi_t = h_U dz_1 \wedge \cdots \wedge dz_n$. Since $\omega_t^o$ is smooth on $U$, we have

$$(-1)^n \frac{n^2}{\pi} \Psi_t \wedge \overline{\Psi}_t \leq C_U (\omega_t^o)^n,$$

on $X_t \cap U'$ for a constant $C_U > 0$, and a smaller $U' \subset U$.

The Ricci-flat condition is equivalent to that the potential function $\varphi_t$ satisfies the Monge-Ampère equation

$$\omega_t^o = (\omega_t^o + \sqrt{-1} \partial \overline{\partial} \varphi_t)^n = e^{\rho_t} (-1)^n \frac{n^2}{\pi} \Psi_t \wedge \overline{\Psi}_t,$$

where $\rho_t$ is a constant function when restricted on $X_t$. The argument in Section 3 of [18] shows a generalized Yau-Schwartz lemma, i.e.

$$\omega_t^o \leq C \omega_t$$

for a constant $C > 0$ independent of $t$. The proof is as the following. If $\Phi_t: (X_t, \omega_t) \to (\mathbb{C}P^N, \omega_{FS})$ is the inclusion map induced by $\mathcal{X} \subset \mathbb{C}P^N \times \Delta$, the Chern-Lu inequality says

$$\Delta_{\omega_t} \log |\partial \Phi_t|^2 \geq \frac{\text{Ric}_{\omega_t}(\partial \Phi_t, \overline{\partial} \Phi_t)}{|\partial \Phi_t|^2} - \frac{\text{Sec}(\partial \Phi_t, \overline{\partial} \Phi_t, \partial \Phi_t, \overline{\partial} \Phi_t)}{|\partial \Phi_t|^2},$$

where $\text{Sec}$ denotes the holomorphic bi-sectional curvature of $\omega_{FS}$ (cf. [31]). Note that $\Phi_t^* \omega_{FS} = \omega_t^o$, $|\partial \Phi_t|^2 = \text{tr}_{\omega_t} \omega_t^o = n - \Delta_{\omega_t} \varphi_t$ and $\text{Ric}_{\omega_t} = 0$. Thus we have that

$$\Delta_{\omega_t} (\log \text{tr}_{\omega_t} \omega_t^o - 2 \overline{F}_t \varphi_t) \geq -2 \overline{R} n + \overline{R} \text{tr}_{\omega_t} \omega_t^o.$$

where $\overline{F}_t$ is a constant depending only the upper bound of Sec. By the maximum principle, there is an $x \in X_t$ such that $\text{tr}_{\omega_t} \omega_t^o(x) \leq 2n$, and

$$\text{tr}_{\omega_t} \omega_t^o \leq 2n e^{2 \overline{F}_t (\varphi_t - \varphi_t(x))} \leq C,$$

by the assumption i), where $C > 0$ is a constant independent of $t$, and we obtain (4.3).
By (4.2) and (4.3), we have
\[ C^{-n}(\omega_o^n)^n \leq \omega_t^n = e^{\rho t}(-1)^{n/2} \Psi_t \wedge \overline{\Psi}_t, \]
and after we restrict this inequality on \( U \), (4.1) asserts that
\[ e^{\rho t} \geq C_1 \]
for a constant \( C_1 \) independent of \( t \). We have
\[ C_1 \int_{X_t} (-1)^{n/2} \Psi_t \wedge \overline{\Psi}_t \leq e^{\rho t} \int_{X_t} (-1)^{n/2} \Psi_t \wedge \overline{\Psi}_t = \int_{X_t} \omega_t^n = \int_{X_t} c_1(L^m)^n, \]
and we obtain ii) by Corollary 1.5 of [23].

Under the assumption ii), [19] proves
\[ (X_t, \omega_t) \overset{dGH}{\to} (Y, dY), \]
when \( t \to 0 \), in the Gromov-Hausdorff sense, where \((Y, d_Y)\) is the metric completion of \((X_{0, reg}, \omega)\), which is a compact metric space (See also the survey paper [34] for more discussions of Gromov-Hausdorff topology in the current circumstances). Therefore ii) implies iii).

If we view iii) as the assumption, then Theorem 1.1 and Proposition 2.1 show that the origin \( 0 \in \Delta \) is at finite Weil-Petersson distance from \( \Delta^* \). We obtain ii) by Theorem 1.3 in [23]. \( \square \)

Finally, we collect some earlier results mainly in [28, 29, 26, 23, 18] for the reader’s convenience, and refer readers to these papers for more detailed discussions.

**Theorem 4.1** ([28, 29, 23, 26]). Let \((\pi : \mathcal{X} \to \Delta, \mathcal{L})\) be a degeneration of polarized Calabi-Yau manifolds, and \( \omega_t \) be the unique Ricci-flat Kähler-Einstein metric in \( c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R}) \). Then the following statements are equivalent.

i) The origin \( 0 \in \Delta \) is at finite Weil-Petersson distance from \( \Delta^* \).

ii) The diameters
\[ \text{diam}_{\omega_t}(X_t) \leq D, \]
for a constant \( D > 0 \) independent of \( t \).

iii) If we further assume that \( \mathcal{X} \to \Delta \) comes from a quasi-projective family, then by passing to a finite base change, \( \mathcal{X} \to \Delta \) is birational to a new family \( \mathcal{X}' \to \Delta \) such that the new central fiber \( X^*_0 \) is a Calabi-Yau variety, and \( X' \setminus X_0 \simeq X^' \setminus X'_0 \).

**Theorem 4.2** ([28, 29, 26, 23, 18]). Let \((\pi : \mathcal{X} \to \Delta, \mathcal{L})\) be a degeneration of polarized Calabi-Yau \( n \)-manifolds such that \( \mathcal{X} \) is normal, the relative canonical bundle \( K_{\mathcal{X}/\Delta} \) is trivial, i.e. \( K_{\mathcal{X}/\Delta} \simeq \mathcal{O}_{\mathcal{X}} \), and \( \mathcal{X} \to \Delta \) has at worst log-canonical singularities. Let \( \omega_t \) be the unique Ricci-flat Kähler-Einstein metric presenting \( c_1(\mathcal{L})|_{X_t} \in H^{1,1}(X_t, \mathbb{R}) \), \( t \in \Delta^* \). Then the following statements are equivalent.

i) The origin \( 0 \in \Delta \) is at finite Weil-Petersson distance from \( \Delta^* \).
ii) The central fiber $X_0$ is a Calabi-Yau variety.

iii) When $t \to 0$,

$$(X_t, \omega_t) \underset{dGH}{\longrightarrow} (Y, d_Y),$$

in the Gromov-Hausdorff sense, where $(Y, d_Y)$ is a compact metric space.

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