Cellular approximations of fusion systems

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Abstract

In this paper we study the cellularization of classifying spaces of saturated fusion systems over finite $p$-groups with respect to classifying spaces of finite $p$-groups. We give explicit criteria to decide when a classifying space is cellular and we explicitly compute the cellularization for a family of exotic examples.

1 Introduction

The transfer plays an important role both in stable homotopy theory of classifying spaces and group cohomology. Given a finite group $G$ and a prime $p$, if $S$ is a Sylow $p$-subgroup, the properties of the transfer imply that the mod $p$ cohomology of $G$ injects into the mod $p$ cohomology of $S$. In stable homotopy theory, the spectrum of $BG^\wedge_p$ is a retract of the spectrum of $BS$ and the splitting is constructed by an idempotent in stable selfmaps of the spectrum of $BS$.

In 1990s, E. Dror-Farjoun and W. Chachólski generalized the concept of CW-complex, spaces build from spheres by means pointed homotopy colimits. Let $A$ be a pointed space and let $C(A)$ denote the smallest collection of pointed spaces that contains $A$ and it is closed by weak equivalences and pointed homotopy colimits. A pointed space $X$ is $A$-cellular if $X \in C(A)$. These concepts are also defined in stable homotopy category.

In the stable context, the fact that $BG^\wedge_p$ is a stable retract of $BS$ implies that that $\Sigma^\infty_+ BG^\wedge_p$ belongs to $C(\Sigma^\infty_+ BS)$. In unstable homotopy theory of classifying spaces, $BG^\wedge_p$ is not a retract of $BS$, but we can ask ourselves whether $BG^\wedge_p$ is in the cellular class $C(BS)$, or more generally, given a finite $p$-group $P$, $BG^\wedge_p \in C(BP)$?

The homotopy type of $BG^\wedge_p$ is determined by the $p$-local structure of $G$, the fusion system associated to $G$. Given a finite $p$-group $S$, $p$ a prime, a fusion system over $S$ is a subcategory of the category of groups whose objects are the subgroup of $S$ and morphisms are given a set of injective homomorphisms, containing those which are induced by conjugation by elements of $S$. A fusion system $F$ is saturated if it verifies certain axioms such as would be holded if $S$ were a Sylow $p$-subgroup of a finite group. These ideas were developed by L. Puig in an unpublished notes. Afterwards, D. Benson suggested the idea of associating a

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“classifying space” to each saturated fusion system (see [Ben98]). The notion of classifying space was formulated by C. Broto, R. Levi and B. Oliver in [BLO03b], where the notion of “centric linking system” (or “p-local finite group”) associated to saturated fusion systems appears. At that time, it was not known if every saturated fusion has an associated linking system.

K. Ragnarsson [Rag06] constructed a classifying space spectrum $B\mathcal{F}$ associated to $\mathcal{F}$ by splitting the spectrum of $BS$ via an idempotent stable selfmap. Analogously to the situation for finite groups, we have then $B\mathcal{F} \in C(BS)$.

Recently, A. Chermak [Che13] has proved the existence and uniqueness of centric linking systems, that means, each saturated fusion system $\mathcal{F}$ has a unique (up to isomorphism) centric linking system associated to $\mathcal{F}$, and so a unique (up to homotopy equivalence) classifying space $B\mathcal{F}$. See Section 2 for specific definitions, details and main results which we will use in the rest of the paper about fusion systems.

Previous works in finite groups (see [Flo07], [FS07] and [FF11]) suggest the strong relationship between the cellularity properties of $BG \vee p$ with respect to classifying spaces of finite $p$-groups and the fusion structure of $G$ at the prime $p$.

Let $P$ be a finite $p$-group, we will denote by $\text{Cl}_\mathcal{F}(P)$ the smallest strongly $\mathcal{F}$-closed subgroup in $S$ which contains all the images of homomorphisms $P \to S$ (see Section 3 below). The main result of this paper is the following theorem.

**Theorem 5.1.** Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$ and let $P$ be a finite $p$-group. Then $B\mathcal{F}$ is $BP$-cellular if and only if $S = \text{Cl}_\mathcal{F}(P)$.

**Corollary 5.2.** Let $(S, \mathcal{F})$ be a saturated fusion system and $P$ a finite $p$-group.

(a) The classifying space $B\mathcal{F}$ is $BS$-cellular.

(b) Let $(S, \mathcal{F}')$ be a saturated fusion system with $\mathcal{F} \subset \mathcal{F}'$. If $B\mathcal{F}'$ is BP-cellular then $B\mathcal{F}'$ is also BP-cellular.

(c) Let $A$ be a pointed connected space. If $\text{Cl}_\mathcal{F}((\pi_1A)_\text{ab}) = S$, then $B\mathcal{F}$ is $A$-cellular.

(d) Let $\Omega_{p^m}(S)$ be the (normal) subgroup of $S$ generated by its elements of order $p^i$, which $i \leq m$. Then $B\mathcal{F}$ is $\mathbb{B}Z/p^m$-cellular if and only if $S = \text{Cl}_\mathcal{F}(\Omega_{p^m}(S))$. In particular, there is a non-negative integer $m_0 \geq 0$ such that $B\mathcal{F}$ is $\mathbb{B}Z/p^m$-cellular for all $m \geq m_0$.

There exists an augmented idempotent endofunctor $CW_A: \text{Spaces} \to \text{Spaces}$, such that for all pointed space $X$, the space $CW_AX$ is $A$-cellular and the augmentation map $c_X: CW_AX \to X$ is an $A$-equivalence, that means, it is induced a weak equivalence in pointed mapping space $(c_X)_*: \text{map}_*(A,CW_AX) \to \text{map}_*(A,X)$. Roughly speaking, $CW_AX$ is the best $A$-cellular approximation of $X$. We will say that $CW_AX$ is the $A$-cellularization of $X$ and the map $c_X: CW_AX \to X$ is the $A$-cellular approximation of $X$. See [DF96] for more details about the construction and main properties of the functor $CW_A$.

The strategy of proof of Theorem 5.1 is the analysis of the Chachólski fibration to compute $CW_{BP}(B\mathcal{F})$ which is described in [Cha96]. Let $C$ be the homotopy cofibre of the evaluation map $ev: \vee_{[BP,B\mathcal{F}]} BP \to B\mathcal{F}$, then $CW_{BP}(B\mathcal{F})$ is the homotopy fibre of the
composite \( r : B\mathcal{F} \rightarrow C \rightarrow P_{\Sigma BP} C \), where \( P_{\Sigma BP} \) denotes the \( \Sigma BP \)-nullification functor defined by A. K. Bousfield in [Bou94]. We prove that \( CW_{BP}(B\mathcal{F}) \simeq B\mathcal{F} \) if and only if the map \( r^\wedge_p \) is null-homotopic (see the proof of Theorem 5.1). Therefore, the \( BP \)-cellularity of \( B\mathcal{F} \) is equivalent to the homotopy nullity of the map \( r^\wedge_p \).

To approach this question, we study the kernel of \( r^\wedge_p \) in the sense of D. Notbohm introduced in [Not94] for maps from classifying spaces of compact Lie groups (Section 3). Given a map \( r : B\mathcal{F} \rightarrow Z \), where \( Z \) is a connected \( p \)-complete and \( \Sigma B\mathcal{Z}/p \)-null space (that is, \( Z^\wedge_p = Z \) and map, \( (\Sigma B\mathcal{Z}/p, Z) \simeq * \)), then \( \ker(f) := \{ x \in S \mid f_{|_{B(S)}} = * \} \). We show that \( \ker(f) \) is a strongly \( \mathcal{F} \)-closed subgroup of \( S \) (Proposition 3.5) with the following main property.

**Theorem 3.6.** Let \((S, \mathcal{F})\) be a saturated fusion system. Let \( Z \) be a connected \( p \)-complete and \( \Sigma B\mathcal{Z}/p \)-null space. A map \( f : B\mathcal{F} \rightarrow Z \) is null-homotopic if and only if \( \ker(f) = S \).

Furthermore, any strongly \( \mathcal{F} \)-closed subgroup \( K \leq S \) is the kernel of a map from \( B\mathcal{F} \) to \( B(\Sigma S/K) \langle \Sigma \rangle \) for certain \( m \geq 0 \) (Proposition 3.7).

In Section 4 we study homotopy properties of \( CW_{BP}(B\mathcal{F}) \) needed for the proof of the main theorem. Section 5 contains the proof of Theorem 5.1. A key step is the computation of the kernel of \( r^\wedge_p \).

**Proposition 5.5.** Let \((S, \mathcal{F})\) be a saturated fusion system. Then \( \ker(r^\wedge_p) = Cl_{\mathcal{F}}(P) \).

The last two sections are devoted to give explicit examples. Concretely, in Section 6 we describe a strategy to compute the \( BP \)-cellularization of \( B\mathcal{F} \) when \( S \neq Cl_{\mathcal{F}}(P) \). This is the case when we have a homotopy factorization of \( r^\wedge_P : B\mathcal{F} \rightarrow (P_{\Sigma BP} C)^\wedge_p \) by a map \( r^\wedge_p : B\mathcal{F} \rightarrow (P_{\Sigma BP} C)^\wedge_p \) with trivial kernel (verifying certain technical conditions in Proposition 6.2). This is the case when \( Cl_{\mathcal{F}}(S) \) is normal in \( \mathcal{F} \).

**Corollary 6.5.** Let \((S, \mathcal{F})\) be a fusion system and let \( P \) be a finite \( p \)-group. If \( Cl_{\mathcal{F}}(P) \triangleleft \mathcal{F} \), then \( CW_{BP}(B\mathcal{F}) \) is homotopy equivalent to the homotopy fibre of \( B\mathcal{F} \rightarrow B(\mathcal{F}/Cl_{\mathcal{F}}(P)) \).

This result allow us compute, for all \( r \geq 1 \), the \( B\mathcal{Z}/p^r \)-cellularization of the classifying space of \( \mathbb{Z}/p^n \wr \mathbb{Z}/q \), with \( p \neq q \), and of the Suzuki group \( Sz(2^n) \), with \( n \) an odd integer at least 3 (Example 6.6).

The last section contains an explicit description of the \( B\mathcal{Z}/3^l \)-cellularization of the classifying space of a family of exotic fusion systems over a finite 3-group given in [DRV07].

**Corollary 7.1.** Let \( \mathcal{F} \) be an exotic fusion system over \( B(3, r; 0, \gamma, 0) \) such that \( \mathcal{F} \) has at least one \( \mathcal{F} \)-Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then

(i) If \( \gamma = 0 \), then \( B\mathcal{F} \) is \( B\mathcal{Z}/3^l \)-cellular for all \( l \geq 1 \).

(ii) Assume \( \gamma \neq 0 \). Then \( B\mathcal{F} \) is \( B\mathcal{Z}/3^l \)-cellular if and only if \( l \geq 2 \). If \( l = 1 \), \( Cl_{\mathcal{F}}(\mathbb{Z}/3) = \langle s, s_2 \rangle \).

Moreover, when \( \gamma \neq 0 \) and \( l = 1 \), we show that \( CW_{BP}(B\mathcal{F}) \) is the homotopy fibre of a map \( B\mathcal{F} \rightarrow (B\Sigma_3)^{s_2} \).

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2 Preliminaries on the homotopy theory of fusion systems

A saturated fusion system is a small subcategory of the category of groups which encodes fusion/conjugacy data between subgroups of a fixed finite $p$-group $S$, as formalized by L. Puig (see [Pui06] and also [AKO11]). Such objects have classifying spaces which satisfy many of the rigid homotopy theoretic properties of $p$-completed classifying spaces of finite groups, as generalized by C. Broto, R. Levi and B. Oliver (see [BLO03b]).

Definition 2.1. Let $S$ be a finite $p$-group. A saturated fusion system on $S$ is a subcategory $\mathcal{F}$ of the category of groups with $\text{Ob}(\mathcal{F})$ the set of all subgroups of $S$ and such that it satisfies the following properties. For all $P, Q \leq S$:

$f.1$) $\text{Hom}_S(P, Q) \subseteq \text{Hom}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q)$; and

$f.2$) each $\varphi \in \text{Hom}_\mathcal{F}(P, Q)$ is the composite of an isomorphism in $\mathcal{F}$ followed by an inclusion.

For all $P \leq S$ and all $P' \leq S$ which is $\mathcal{F}$-conjugate to $P$ ($P$ and $P'$ are isomorphic as objects in $\mathcal{F}$):

$s.1$) For all $P \leq S$ which is fully normalized in $\mathcal{F}$ (i.e. $|N_S(P)| \geq |N_S(P')|$ for all $P' \mathcal{F}$-conjugate to $P$), $P$ is also fully centralized in $\mathcal{F}$ ($|C_S(P)| \geq |C_S(P')|$ for all $P' \mathcal{F}$-conjugate to $P$), and $\text{Out}_S(P) \in \text{Syl}_p(\text{Out}_\mathcal{F}(P))$.

$s.2$) Let $P \leq S$ and $\varphi \in \text{Hom}_\mathcal{F}(P, S)$ be such that $\varphi(P)$ is fully centralized. If we set $N_{\varphi} = \{g \in N_S(P) \mid \varphi c g \varphi^{-1} \in \text{Aut}_S(\varphi(P))\}$,

then there is $\bar{\varphi} \in \text{Hom}_\mathcal{F}(N_{\varphi}, S)$ such that $\bar{\varphi}|_P = \varphi$.

The standard example is given by the fusion category of a finite group $G$. Given a finite group $G$ with a fixed Sylow $p$-subgroup $S$, let $\mathcal{F}_S(G)$ be the category with $\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$ for all $P, Q \leq S$, where $\text{Hom}_G(P, Q) = \{\varphi \in \text{Hom}(P, Q) \mid \varphi = c g \text{ for some } g \in G\}$. This category $\mathcal{F}_S(G)$ satisfies the saturation axioms (see [BLO03b, Proposition 1.3]).

In order to recover the homotopy type of the Bousfiel-Kan $p$-completion of the classifying space $BG_p$, Broto-Levi-Oliver [BLO03a] introduce a new category defined using the group $G$. A $p$-subgroup $P \leq G$ is $p$-centric if $Z(P')$ is a Sylow $p$-subgroup of $C_G(P)$ for all $P' G$-conjugate to $P$. Let $\mathcal{L}_S(G)$ be the category whose objects are $p$-centric subgroups of $G$ with $\text{Mor}_{\mathcal{L}_S(G)}(P, Q) = \{x \in G \mid x P^{-1} x \leq Q\}/\text{Out}(C_G(P))$ for all $P, Q \leq S$. In [BLO03a] the authors proved that the classifying space of this $p$-local structure is $|\mathcal{L}_S(G)|_p = BG_p$. This new structure one can associate to a finite group can also be generalized in the context of abstract fusion systems.
**Definition 2.2.** [BLO03b] Let $\mathcal{F}$ be a saturated fusion system over a finite $p$-group $S$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$ (i.e., the subgroups $P \leq S$ such that $C_S(P) = Z(P)$), together with a functor $\pi: \mathcal{L} \to \mathcal{F}^\times$, and “distinguished” monomorphisms $\delta_P: P \to Aut_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfies the following conditions:

(l.1) $\pi$ is the identity on objects and surjective on morphisms. For each pair of objects $P, Q \leq \mathcal{L}$, $Z(P)$ acts freely on $\text{Mor}_\mathcal{L}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq Aut_\mathcal{L}(P)$), and $\pi$ induces a bijection

$$\text{Mor}_\mathcal{L}(P, Q)/Z(P) \xrightarrow{\cong} \text{Hom}_\mathcal{F}(P, Q).$$

(l.2) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, $\pi$ sends $\delta_P(g) \in Aut_\mathcal{L}(P)$ to $c_g \in Aut_\mathcal{F}(P)$.

(l.3) For each $f \in \text{Mor}_\mathcal{L}(P, Q)$ and each $g \in P$, $f \circ \delta_P(g) = \delta_Q(\pi f(g)) \circ f$.

Recently, in 2013, A. Chermak [Che13] proved the existence and uniqueness of centric linking systems associated to saturated fusion systems (see also [Oli13]).

**Theorem 2.3.** ([Che13],[Oli13]) Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Then there exists a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$. Moreover, $\mathcal{L}$ is uniquely determined by $\mathcal{F}$ up to isomorphism.

**Definition 2.4.** The classifying space $B\mathcal{F}$ of a saturated fusion system $(S, \mathcal{F})$ is the Bousfield-Kan $p$-completion of the nerve of the associated centric linking system $|\mathcal{L}|^\wedge_p$. We denote by $\Theta: BS \to B\mathcal{F}$ the map induced by the distinguished monomorphism $\delta_S$.

Given a map $f: B\mathcal{F} \to X$ and $P \leq S$, we denote by $f|_{BP}$ the composite $f \circ \Theta \circ Bi$ where $i: P \to S$.

We describe some results concerning the homotopy type of the classifying space $B\mathcal{F}$ and mapping spaces $\text{map}(BP, B\mathcal{F})$ which will be used in the rest of the paper.

**Proposition 2.5** ([BLO03b],[BCG*07],[CL09]). For any saturated fusion system $(S, \mathcal{F})$, $B\mathcal{F}$ is a $p$-complete space and $\pi_i(B\mathcal{F})$ are finite $p$-groups for all $i \geq 1$. The fundamental group $\pi_1(B\mathcal{F}) \cong S/\mathcal{O}_F^p(S)$, where $\mathcal{O}_F^p(S) := \langle [Q, \mathcal{O}^p(Aut_\mathcal{F}(Q))] \mid Q \leq S \rangle$.

**Proof.** The description of the fundamental group $\pi_1(B\mathcal{F})$ is given in [BCG*07, Theorem B]. The fact that $B\mathcal{F}$ is $p$-complete follows from [BK72, Proposition I.5.2] since the nerve of the associated linking system $|\mathcal{L}|$ is $p$-good by [BLO03b, Proposition 1.12]. Finally, $\pi_i(B\mathcal{F})$ are finite $p$-groups for all $i \geq 1$ by [CL09, Lemma 7.6].

**Proposition 2.6.** [BLO03b, Theorem 6.3] Let $(S, \mathcal{F})$ be a saturated fusion system. If $P$ is a finite $p$-group and $\rho: P \to S$ a group homomorphism such that $\rho(P)$ is fully $\mathcal{F}$-centralized, there is a saturated fusion system $(C_S(\rho(P)), C_\mathcal{F}(\rho(P)))$ and a homotopy equivalence $BC_\mathcal{F}(\rho(P)) \xrightarrow{\sim} \text{map}(BP, B\mathcal{F})_\rho$. In particular, the evaluation map $\text{map}(BP, B\mathcal{F})_\rho \to B\mathcal{F}$ is a homotopy equivalence.
Moreover, any finite covering of the classifying space of a saturated fusion system is again the classifying space of a saturated fusion system.

**Theorem 2.7** ([BCG+07, Theorem A]). Let \((S, \mathcal{F})\) be a saturated fusion system and \(\mathcal{L}\) be the associated linking system. Then there is a normal subgroup \(H \triangleleft \pi_1|\mathcal{L}|\) which is minimal among all those whose quotient is finite and \(p\)-solvable. Any covering space of the geometric realization \(|\mathcal{L}|\) whose fundamental group contains \(H\) is homotopy equivalent to \(|\mathcal{L}'|\) for some linking system \(\mathcal{L}'\) associated to a saturated fusion system \((S', \mathcal{F}')\), where \(S' \leq S\) and \(\mathcal{F}' \leq \mathcal{F}\).

In this work it is important to understand the homotopy properties of mapping spaces between classifying spaces. This question was already studied in [BLO03b]. One of the standard techniques used when studying maps between \(p\)-completed classifying spaces of finite groups is to replace them by the \(p\)-completion of a homotopy colimit of classifying spaces of subgroups.

**Definition 2.8.** Let \(\mathcal{F}\) be a saturated fusion system over a finite \(p\)-group \(S\). The orbit category of \(\mathcal{F}\) is the category \(O(\mathcal{F})\) whose objects are the subgroups of \(S\), and whose morphisms are defined by

\[
\text{Mor}_{O(\mathcal{F})}(P, Q) = \text{Rep}_\mathcal{F}(P, Q) := \text{Inn}(Q) / \text{Hom}_\mathcal{F}(P, Q).
\]

We let \(O'(\mathcal{F})\) denote the full subcategory of \(O(\mathcal{F})\) whose objects are the \(\mathcal{F}\)-centric subgroups of \(S\). If \(\mathcal{L}\) is a centric linking system associated to \(\mathcal{F}\), then \(\tilde{\pi}\) denotes the composite functor

\[
\tilde{\pi} : \mathcal{L} \xrightarrow{\pi} \mathcal{F} \xrightarrow{} O'(\mathcal{F})
\]

The homotopy type of the nerve of a centric linking system can be described as a homotopy colimit over the orbit category.

**Proposition 2.9** ([BLO03b, Proposition 2.2]). Fix a saturated fusion system \(\mathcal{F}\) over a finite \(p\)-group \(S\) and an associated centric linking system \(\mathcal{L}\), and let \(\tilde{\pi} : O'(\mathcal{F}) \to \text{Top}\) be the projection functor. Let \(\tilde{B} : O'(\mathcal{F}) \to \text{Top}\) be the left homotopy Kan extension over \(\tilde{\pi}\) of the constant functor \(\mathcal{L} \xrightarrow{\pi} \text{Top}\). Then \(\tilde{B}\) is a lift of the classifying space functor \(P \mapsto \text{BP}\) to the category of topological spaces, and

\[
|\mathcal{L}| \cong \text{hocolim}_{O'(\mathcal{F})}(\tilde{B}).
\]

Given fusion systems \(\mathcal{F}\) and \(\mathcal{F}'\) on \(S\) and \(S'\) respectively, a homomorphism \(\psi : S \to S'\) is called fusion preserving if for every \(\varphi \in \mathcal{F}(P, Q)\) there exists some \(\varphi' \in \mathcal{F}'(\psi(P), \psi(Q))\) such that \(\psi \circ \varphi = \varphi' \circ \psi\).

**Theorem 2.10** ([CL09, Theorem 1.3]). Let \((S, \mathcal{F})\) and \((S', \mathcal{F}')\) be saturated fusion systems. Suppose that \(\rho : S \to S'\) is a fusion preserving homomorphism. Then there exists some \(m \geq 0\) and a map \(\tilde{f} : B\mathcal{F} \to B(\mathcal{F}' \wr \Sigma_m)\) such that the diagram below commutes up to homotopy

\[
\begin{array}{ccc}
BS & \xrightarrow{\Theta} & B\mathcal{F} \\
\downarrow{B\rho} & & \downarrow{\tilde{f}} \\
BS' & \xrightarrow{\Theta'} & B(\mathcal{F}' \wr \Sigma_m).
\end{array}
\]

where \(\mathcal{F}' \wr \Sigma_m\) is the saturated fusion system whose classifying space is \(((B\mathcal{F}')^m_h)\).
3 The kernel of a map from a classifying space

Given a connected space $A$, we say that a space $X$ is $A$-null if the evaluation map $ev: \text{map}(A,X) \to X$ is a weak equivalence (see [DF96]). If $X$ is connected, $X$ is $A$-null if and only if $\text{map},(A,X)$ is weakly contractible. There is a nullification functor $P_A: \text{Spaces} \to \text{Spaces}$ with a natural transformation $\eta_X: X \to P_A(X)$ which is initial with respect to maps into $A$-null spaces. We say that $X$ is $A$-acyclic if $P_A X \cong \ast$.

The kernel of a map $f: BG^A \to Z$, where $G$ is a compact Lie group and $Z$ is a connected $p$-complete $\Sigma B\mathbb{Z}/\mathbb{Z}$-null space, is defined by D. Notbohm in [Not94]. We adapt his definition to our context.

**Definition 3.1.** Let $(S,F)$ be a saturated fusion system and let $Z$ be a connected $p$-complete $\Sigma B\mathbb{Z}/\mathbb{Z}$-null space. If $f: B\mathcal{F} \to Z$ is a pointed map, we define the kernel of $f$

$$\ker(f) := \{ g \in S \mid f|[B(g)] \cong \ast \}.$$ 

**Remark 3.2.** By Proposition 2.6, we have $\text{map}(B\mathbb{Z}/\mathbb{Z}, B\mathcal{F})_c \cong B\mathcal{F}$. It follows then by looping that $\Omega B\mathcal{F}$ is $B\mathbb{Z}/\mathbb{Z}$-null, or equivalently, that $B\mathcal{F}$ is $\Sigma B\mathbb{Z}/\mathbb{Z}$-null ([DF96, 3.A.1]).

**Remark 3.3.** If $X$ is a $B\mathbb{Z}/\mathbb{Z}$-null space, then $X$ is $B\mathbb{Z}$-null for any finite $p$-group $P$. There are weak equivalences $\text{map}_P(BP, X) \simeq \text{map}_P(BP, X) \simeq \ast$, where the last equivalence follows from Lemma 6.13 in [Dwy96] which states that $BP$ is $B\mathbb{Z}/\mathbb{Z}$-acyclic. A direct proof can be obtained by induction using the central extension of a $p$-group and Zabrodsky’s Lemma [Dwy96, Proposition 3.4].

We will show that $\ker(f)$ is a subgroup of $S$ with some important properties.

**Definition 3.4.** Let $\mathcal{F}$ be a fusion system over a finite $p$-group $S$. Then a subgroup $K \leq S$ is strongly $\mathcal{F}$-closed if for all $P \leq K$ and all morphism $\varphi: P \to S$ in $\mathcal{F}$ we have $\varphi(P) \leq K$.

If $G$ is a finite group and $S \in \text{Syl}_p(G)$, $K \leq S$ is strongly $\mathcal{F}_5(G)$-closed if and only if $K$ is strongly closed in $G$, i.e., if for all $k \in K$ and $g \in G$ such that $c_k(s) \in S$, then $c_k(s) \in K$.

Since the intersection of strongly $\mathcal{F}$-closed subgroups is again strongly $\mathcal{F}$-closed, given a finite $p$-group $P$, we define $\text{Cl}_P(P)$ to be the smallest strongly $\mathcal{F}$-closed subgroup of $S$ that contains $f(P)$ for all $f \in \text{Hom}(P, S)$.

**Proposition 3.5.** Let $f: B\mathcal{F} \to Z$ be a pointed map as in Definition 3.1. The kernel $\ker(f)$ is a strongly $\mathcal{F}$-closed subgroup of $S$.

**Proof.** Since $\langle x \rangle = \langle x^{-1} \rangle$, in order to show that $\ker(f)$ is a subgroup of $S$ it is sufficient to prove that if $x, y \in \ker(f)$, then $xy \in \ker(f)$. The composite $B \langle x, y \rangle \to BS \to B\mathcal{F} \to Z$ is null-homotopic by [Not94, Proposition 2.4]. Since $\langle xy \rangle \subseteq \langle x, y \rangle$, $f|_{B\langle xy \rangle} \cong \ast$.

Let $P \leq \ker(f)$ and $\varphi \in \text{Hom}_\mathcal{F}(P, S)$. We have the following homotopy commutative diagram

$$
\begin{array}{c}
BP \\
\text{B}^\varphi(P) \\
\end{array} \xrightarrow{B\phi} \begin{array}{c}
B\mathcal{F} \\
\text{B}^\varphi(P) \\
\end{array} \xrightarrow{f} Z
$$

which shows that $f|_{BP}$ is null-homotopic if and only if $f|_{B\phi(P)}$ is null-homotopic. \(\square\)
W. Dwyer shows in [Dwy96, Theorem 5.1] that if we have a finite group $G$, a map $f: BG_p \to Z$ as in Definition 3.1 is null-homotopic if and only if $\ker(f) = S$. This statement is also true for classifying spaces of saturated fusion systems.

**Theorem 3.6.** Let $(S, F)$ be a saturated fusion system. Let $Z$ be a connected $p$-complete $\Sigma B\mathbb{Z}/p$-null space. Then a map $f: BF \to Z$ is null-homotopic if and only if $\ker(f) = S$.

**Proof.** If $f \simeq \ast$, then $f|_{BS} \simeq \ast$ and therefore $\ker(f) = S$. Now assume that $f|_{BS} \simeq \ast$, we will show that $f \simeq \ast$.

**Step 1:** Assume that $\pi_1(Z)$ is abelian. By Proposition 2.9, $BF \simeq (\operatorname{hocolim}_{O(F)} \Theta BP)^\wedge$, where $\Theta BP \simeq BP$ for $P \in F^c$. Since any map $BP \to BF$ factors through $\Theta: BS \to BF$ by [BLO03b, Theorem 4.4], $f|_{BP} \simeq \ast$ for all $P \in F^c$. Therefore we have two maps

$$\operatorname{hocolim}_{O(F)}(\Theta BP) \to (\operatorname{hocolim}_{O(F)}(\Theta BP))^\wedge \xrightarrow{f} Z,$$

such that both are nullhomotopic when restricted to $BP$ for all $P \in F^c$.

The obstructions for these maps to be homotopic are in $\lim^i_{O(F)} \pi_i(\operatorname{map}(BP, Z)_c)$, for $i \geq 1$ (see [Woj87]). Since a $B\mathbb{Z}/p$-null space is $BQ$-null for any finite $p$-group $Q$ (Remark 3.3), $Z$ is $\Sigma BP$-null and hence $\operatorname{map}_i(BP, Z)$ is homotopically discrete, therefore $\operatorname{map}_i(BP, Z)_c \simeq \ast$ and, from the fibration map $\operatorname{map}_i(BP, Z)_c \to \operatorname{map}(BP, Z)_c \to Z$, we obtain $\operatorname{map}(BP, Z)_c \simeq Z$.

Given a morphism $\varphi: P \to Q$ in $F^c$, $B\varphi$ is a pointed map and induces a commutative diagram

$$\operatorname{map}(BQ, Z)_c \xrightarrow{B\varphi^\ast} \operatorname{map}(BP, Z)_c \xrightarrow{\sigma} Z \xrightarrow{id} Z,$$

which shows that the obstructions are in $\lim^i_{O(F)} \pi_i Z$, where $\pi_i Z$ is a constant functor in $O^c(F)$. Since $Z$ is $p$-complete and $\pi_1(Z)$ is abelian, the constant functors $\pi_i(Z)$ all take values in $Z_{(p)} - \operatorname{Mod}$.

Let $F := \pi_n Z$ be the constant functor. Fix $P$ in $O^c(F)^{\text{op}}$ and consider the atomic functors $F_P: O^c(F)^{\text{op}} \to Z_{(p)} - \operatorname{Mod}$

$$F_P(Q) := \begin{cases} \pi_n Z, & \text{if } Q = P, \\ 0, & \text{if } Q \neq P. \end{cases}$$

and $\tilde{F}_P(Q) := F(Q)/F_P(Q)$. Filtering $F$ as a series of extensions of functors $F_P$, one for each object $P$, and using the long exact sequence for higher limits (see proof of Theorem 1.10 of [JMO95]), if $\lim^i F_P = 0$ for all $P$ and $i > 0$, then $\lim^i F = 0$ for $i > 0$. By [JMO92, Proposition 6.1 (i),(ii)], if $p \nmid |\operatorname{Out}_F(P)|$, then

$$\lim^i F_P = \begin{cases} (\pi_n Z)^{\operatorname{Out}_F(P)} = \pi_n Z, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$
and if \( p \mid |\text{Out}_F(P)| \), then \( \lim^i F_p = 0 \) for \( i \geq 0 \). Therefore \( \lim^i_{\text{O}(\mathcal{F})} \pi_i Z = 0 \) for \( i > 0 \), and finally \( f \simeq \ast \).

**Step 2:** We will prove that a map \( f: B\mathcal{F} \to B\mathcal{G} \) is null-homotopic if \( f|_{BS}: BS \to BG \) is null-homotopic where \( G \) is a discrete group.

We will apply Zabrodsky’s lemma [Dwy96, Proposition 3.5] to the map \( f|_{BS} \) and the fibre sequence \( F \to BS \to B\mathcal{F} \) where \( F \) is connected since \( \pi_1(\Theta) \) is an epimorphism (Lemma 2.5). Note that map \( (F, BG) \) is homotopically discrete, then Zabrodsky’s lemma shows that there is a homotopy equivalence map \( (B\mathcal{F}, BG) \simeq \text{map}(BS, BG) \) where that last mapping space corresponds to the components which are null-homotopic when restricted to \( F \). The bijection between components implies that \( f \simeq \ast \).

Finally, we can prove the theorem by looking at the universal cover \( Z \) of \( Z \). Let \( p: Z \to B\pi_1(Z) \). The composite \( p \circ f \) is null-homotopic by Step 2. Therefore there is a lift \( \tilde{f}: B\mathcal{F} \to \tilde{Z} \) to the universal cover of \( Z \). In order to apply Step 1, we need to check that \( \tilde{f}|_{BS}: BS \to \tilde{Z} \) is null-homotopic. Since both \( Z \) and \( B\pi_1(Z) \) are \( \Sigma B\mathbb{Z}/p \)-null, applying mapping spaces from \( BS \) to the universal cover fibration shows that map \( (BS, \tilde{Z})|_c \simeq \tilde{Z} \) where \( \{c\} \) is the set of maps which are null-homotopic when postcomposed with \( \tilde{Z} \to Z \). Note that \([\tilde{f}|_{BS}] \in \{c\} \). Since \( \tilde{Z} \) is connected, the set \( \{c\} \) only consists on the constant map and then \( \tilde{f}|_{BS} \) is nullhomotopic.

Given a strongly \( \mathcal{F} \)-closed subgroup \( K \leq S \) in a saturated fusion system \( \mathcal{F} \), there exists a map between classifying spaces \( f: B\mathcal{F} \to BG_p^\infty \) for a finite group \( G \) such that \( \ker(f) = K \).

**Proposition 3.7.** Let \((S, \mathcal{F})\) be a saturated fusion system and \( K \) be a strongly \( \mathcal{F} \)-closed subgroup and \( \mathcal{L} \) be the associated centric linking system. Let \( \rho \) be the composite \( S \xrightarrow{\pi} S/K \xrightarrow{\text{reg}} \Sigma_{S/K} \), where \( \pi \) is the quotient homomorphism and reg is the regular representation of \( S/K \). Then there is a non-negative integer \( m \geq 0 \) and a map \( f: \mathcal{L} \to B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty \) such that the following diagram

\[
\begin{array}{ccc}
BS & \xrightarrow{(\Delta \rho)^\infty_p} & (B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty_p) \\
\downarrow \cong & & \downarrow \Delta^\infty_p \\
\mathcal{L} & \xrightarrow{f} & B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty_p
\end{array}
\]

is commutative up to homotopy. Moreover, \( \ker(f^\infty_p) = K \).

**Proof.** Let \( n = |S/K| \). According to [CL09, Theorem 1.2], if \( \rho \) is fusion invariant then there is a non-negative integer \( m \geq 0 \) and a map \( f: \mathcal{L} \to B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty_p \) such that \( f|_{BS} \) is homotopic to the composite \( BS \xrightarrow{(\Delta \rho)^\infty_p} (B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty_p) \xrightarrow{\Delta^\infty_p} B(\Sigma_{S/K}^{\Sigma_{p^m}})^\infty_p \). Therefore, it is sufficient to show that \( \rho \) is fusion invariant: for all \( P \leq S \) and \( \varphi: P \to S \) in \( \mathcal{F} \) there is \( \omega \in \Sigma_n \) such that \( \rho|_P \circ \varphi = c_\omega \circ \rho|_P \).

The homomorphisms \( \rho|_P \) and \( c_\omega \rho|_P \) equip \( S/K \) with a structure of a \( P \)-set which are isomorphic. Hence, to prove the above equality, we only need to show that \((S/K, \leq)\) and \((S/K, \leq_\varphi)\) are equivalent as \( P \)-sets.
Note that for any $\varphi: P \to S \in \mathcal{F}$,
\[(S/K, \leq \varphi) \cong \text{Iso}^*(\varphi) \text{Res}^S_{\varphi(P)}(S/K) \cong \text{Iso}^*(\varphi) \text{Res}^S_{\varphi(P)} \text{Ind}^S_K(*)\]

Applying the Mackey formula to $\text{Res}^S_{\varphi(P)} \text{Ind}^S_K$, we get
\[
(S/K, \leq \varphi) = \prod_{[x] \in \varphi(P)/S/K} \text{Iso}^*(\varphi) \text{Ind}^P_{\varphi(P) \cap K} \text{Iso}^*(c_x) \text{Res}^K_{(\varphi(P) \cap K)^*}(*),
\]

where the second equality comes from the commutativity of isogation and induction and, where $K^\perp = K$ because $K$ is strongly $\mathcal{F}$-closed and $c_x: K \to S$ is in $\mathcal{F}$, hence $c_x(K) = K^\perp \leq K$ and since $c_x$ is an isomorphism, $K^\perp = K$.

Next we prove that $\varphi^{-1}(\varphi(P) \cap K) = P \cap K$. On the one hand, $\varphi^{-1}(\varphi(P) \cap K) \leq P \cap K$ because $\varphi^{-1}|_{\varphi(P) \cap K}: \varphi(P) \cap K \to S$ is in $\mathcal{F}$, $\varphi(P) \cap K \leq K$, $K$ is strongly $\mathcal{F}$-closed and $\varphi^{-1}(\varphi(P) \cap K) \leq P$. The equality will follow if $|\varphi^{-1}(\varphi(P) \cap K)| = |P \cap K|$. Since $\varphi$ is an isomorphism, it is enough to check $|\varphi(P) \cap K| = |\varphi(P \cap K)|$. We already know that $|\varphi(P) \cap K| = |\varphi^{-1}(\varphi(P) \cap K)| \leq |P \cap K| = |\varphi(P \cap K)|$. Since $\varphi|_{P \cap K}: P \cap K \to S$ is in $\mathcal{F}$, $P \cap K \leq K$ and $K$ is strongly $\mathcal{F}$-closed, $\varphi(P \cap K) \leq K$ but also $\varphi(P \cap K) \leq \varphi(P)$, hence $\varphi(P \cap K) \leq \varphi(P) \cap K$ and therefore $|\varphi(P \cap K)| \leq |\varphi(P) \cap K|$.

Therefore in the above formula, since $\text{Iso}^*(\varphi) \text{Iso}^*(c_x) \text{Res}^K_{(\varphi(P) \cap K)^*}(*) = *$ as $(P \cap K)$-set and $\varphi^{-1}(\varphi(P) \cap K) = P \cap K$, we get for all $\varphi: P \to S \in \mathcal{F}$
\[
(S/K, \leq \varphi) \cong \prod_{\varphi \in \varphi(P)/S/K} \text{Ind}^P_{\varphi(P) \cap K}(*),
\]

where $l_\varphi = |\varphi(P)|/S/K = |S/\varphi(P) \cdot K| = |S|/|\varphi(P) \cdot K|$ since $K \triangleleft S$. In particular, if $\varphi = id_P$, then
\[
(S/K, \leq \varphi) \cong \prod_{P \cap K} \text{Ind}^P_{P \cap K}(*) \cong \prod_{P \cap K} P/P \cap K,
\]

where $l = |S|/|P \cdot K|$.

Therefore, $\rho$ will be fusion invariant if $l = l_\varphi$. It is enough to show $|\varphi(P) \cap K| = |P \cap K|$ and, in fact, this equality has been already proved in a paragraph above.

Finally, $f|_{BS}$ is the induced map on classifying spaces of the homomorphism
\[
\xymatrix{S \ar[r]^-{\pi} & S/K \ar[r]^-{\text{res}} & \Sigma[S/K] \ar[r] & \Sigma[S/K] \text{ ! } \Sigma[P^w],}
\]

whose kernel is $K$ by construction.

\[\square\]

**Question:** Given a saturated fusion system $(S, \mathcal{F})$ and a map $f: B\mathcal{F} \to Z$ where $Z$ is a connected $\Sigma BZ/p$-null $p$-complete space. Does $f$ factors, up to homotopy, through $\tilde{f}: B\mathcal{F}' \to Z$ with trivial kernel, where $\mathcal{F}'$ is a saturated fusion system?

Under some hypothesis we can give a positive answer to the previous question.
Proposition 3.8. Let \((S, \mathcal{F})\) be a saturated fusion system with associated linking system \(\mathcal{L}\). Let \(K\) be the kernel of \(f : B\mathcal{F} \to Z\), where \(Z = \Sigma \mathbb{Z}/p\)-null \(p\)-complete space.

If \(K\) is normal in \(\mathcal{F}\), then there exist a saturated fusion system \((S/K, \mathcal{F}/K)\) with associated linking system \(\mathcal{L}/K\) and a map \(pr : |\mathcal{L}| \to |\mathcal{L}/K|\), whose homotopy fibre is \(BK\), such that \(f\) factors via \(\tilde{f} : B(\mathcal{F}/K) \to Z\) with trivial kernel.

Proof. In [OV07, Section 2], the authors prove that if \(K\) is normal in \(\mathcal{F}\), then there is a saturated fusion system \((S/K, \mathcal{F}/K)\) with linking system \(\mathcal{L}/K\) and a map \(pr : |\mathcal{L}| \to |\mathcal{L}/K|\) whose homotopy fibre is \(BK\).

Since \(Z\) is \(p\)-complete, one can consider the composite \(g : |\mathcal{L}| \to B\mathcal{F} \to Z\) such that \(g_p^\wedge \simeq f\). By assumption \(Z\) is \(\Sigma \mathbb{Z}/p\)-null, and also \(BK\) is \(\mathbb{Z}/p\)-acyclic, then we get a unique factorization, up to homotopy, \(\tilde{g} : |\mathcal{L}/K| \to Z\) of \(g\) by applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) to the homotopy fibre sequence \(BK \to |\mathcal{L}| \to |\mathcal{L}/K|\) and the map \(g\). Take \(\tilde{f} = \tilde{g}_p^\wedge\). The same argument applied to the fibration \(BK \to BS \xrightarrow{\pi} B(S/K)\) and \(g_{\mathbb{B}S}\) shows that \(f_{B(S/K)}^B \simeq *\) by uniqueness.

Let \([x] \in \ker(\tilde{f}) \leq S/K\), then \(f_{B(S/K)}^B \simeq *\) implies that \(f_{B(S/K)}^B \simeq *\) and finally \(x \in K\). \(\square\)

Corollary 3.9. Let \((S, \mathcal{F})\) be a saturated fusion system with associated linking system \(\mathcal{L}\). Let \(K\) be the kernel of \(f : B\mathcal{F} \to Z\), where \(Z\) is a connected \(\Sigma \mathbb{Z}/p\)-null \(p\)-complete space.

If \(K\) is abelian or \(S\) is resistant, then there exist a saturated fusion system \((S/K, \mathcal{F}/K)\) with associated linking system \(\mathcal{L}/K\) and a map \(pr : |\mathcal{L}| \to |\mathcal{L}/K|\) whose homotopy fibre is \(BK\) and a homotopy factorization \(f : B\mathcal{F}/K \to Z\) with trivial kernel.

Proof. Recall that \(K\) is strongly \(\mathcal{F}\)-closed by Proposition 3.5. If \(K\) is abelian and strongly \(\mathcal{F}\)-closed, then it is normal in \(\mathcal{F}\) by [Cra10, Proposition 3.14]. If \(S\) is resistant, each strongly \(\mathcal{F}\)-closed subgroup is also normal in \(\mathcal{F}\). Finally, apply Proposition 3.8. \(\square\)

4 Homotopy properties of \(BP\)-cellular approximations

In this context, it is natural to study homotopy properties of \(CW_{BP}X\) which are inherited by those of \(X\). We will show in this section that \(CW_{BP}(B\mathcal{F})\) is a \(p\)-good nilpotent space whose fundamental group is a finite \(p\)-group.

Proposition 4.1. Let \((S, \mathcal{F})\) be a saturated fusion system. Then both \(B\mathcal{F}\) and \(CW_{BP}(B\mathcal{F})\) are nilpotent. Furthermore, there exists a homotopy fibre sequence

\[
CW_{BP}(B\mathcal{F}) \to CW_{BP}(B\mathcal{F})^\wedge_p \to (CW_{BP}(B\mathcal{F})^\wedge_p)_Q.
\]

Proof. The classifying space \(B\mathcal{F}\) is \(p\)-complete and \(\pi_0(B\mathcal{F})\) are finite \(p\)-groups by Lemma 2.5, hence \(B\mathcal{F}\) is nilpotent according to [BK72, Proposition VII.4.3(ii)]. Then \(CW_{BP}(B\mathcal{F})\) is also nilpotent [CF15, Corollary 3.2].

Moreover, it follows from [CF15, Lemma 2.8] that \(R_*CW_{BP}(B\mathcal{F}) = *\) for \(R = Q\) and \(R = \mathbb{F}_q, q \neq p\), since \(H_*(BP; R) = 0\). Finally, the Sullivan’s arithmetic square applied to the nilpotent space \(CW_{BP}(B\mathcal{F})\) gives the desired result. \(\square\)
The $R$-completion functor is a homological localization functor when we restrict to $R$-good spaces (see [BK72, p. 205]). By [BK72, Proposition VII.5.1], if the fundamental group of a pointed space $X$ is finite, then $X$ is $p$-good for any prime $p$. Therefore, in order to show that $\mathrm{CW}_{BP}(BF)$ is a $p$-good space, we will prove that its fundamental group is a finite $p$-group.

**Proposition 4.2.** Let $(S,F)$ be a saturated fusion system and let $P$ be a finite $p$-group. Then $\pi_1\mathrm{CW}_{BP}(BF)$ is a finite $p$-group. Therefore, $\mathrm{CW}_{BP}(BF)$ is a $p$-good space.

**Proof.** The proof will be divided in several steps. Let $C$ be the homotopy cofibre of the evaluation map $ev: \vee_{[BP, BF]}, BP \to BF$.

**Step 1:** Assume $C$ is simply connected. The Chacholski’s homotopy fibre sequence $\mathrm{CW}_{BP}(BF) \to BF \to P_{\Sigma BP}C$ induces a long exact sequence of homotopy groups

$$\ldots \to \pi_2(P_{\Sigma BP}C) \to \pi_1\mathrm{CW}_{BP}(BF) \to \pi_1(BF) \to \ldots$$

where $\pi_1(BF)$ is a finite $p$-group by Lemma 2.5. Therefore, we are reduced to prove that $\pi_2(P_{\Sigma BP}C)$ is a finite $p$-group.

Hurewicz’s theorem shows that $H_2(P_{\Sigma BP}C; \mathbb{Z}) \cong \pi_2(P_{\Sigma BP}C)$, since $P_{\Sigma BP}C$ is simply connected ([Bou94, 2.9]). Moreover, since $\Sigma BP$ is 1-connected, we obtain an epimorphism $H_2(C; \mathbb{Z}) \to H_2(P_{\Sigma BP}C; \mathbb{Z})$ by [CGR15, Proposition 3.2]. Then it is enough to prove that $H_2(C; \mathbb{Z})$ is a finite $p$-group.

The cofibration sequence $\vee_{[BP, BF]}, BP \to BF \to C$ induces a long exact sequence of homology groups

$$\ldots \to H_2(BF; \mathbb{Z}) \overset{f_1}{\longrightarrow} H_2(C; \mathbb{Z}) \overset{f_2}{\longrightarrow} H_1(\vee_{[BP, BF]}, BP; \mathbb{Z}) \longrightarrow \ldots$$

which allows to describe $H_2(C; \mathbb{Z})$ by a short exact sequence

$$0 \to \ker f_2 \to H_2(C; \mathbb{Z}) \to \text{Im } f_2 \to 0.$$ 

By exactness, $\ker f_2 = \text{Im } f_1$ is a quotient of $H_2(BF; \mathbb{Z})$, where $H_2(BF; \mathbb{Z})$ is a finite $p$-group since $H^*(BF; \mathbb{Z}) \to H^*(BS; \mathbb{Z})$ by [BLO03b, Theorem B]. Hence $\ker f_2$ is a finite $p$-group.

Now note that

$$\text{Im } f_2 \subset H_1(\bigvee_{[BP, BF]} BP; \mathbb{Z}) \cong \pi_1(\bigvee_{[BP, BF]} BP)_{ab} \cong \oplus_{[BP, BF]} P_{ab},$$

where $[BP, BF]$, is a finite set because there is an epimorphism of sets $[BP, BS] \to [BP, BF]$, by [BLO03b, Theorem 4.4], and $[BP, BS] \cong \text{Hom}(P, S)$ is finite. Finally, $H_2(C; \mathbb{Z})$ is a finite $p$-group.

**Step 2:** We will show that there exists a saturated fusion system $(S', F')$ and a $BP$-equivalence $f: BF' \to BF$ such that the homotopy cofibre of $ev: \vee_{[BP, BF]}, BP \to BF'$ is 1-connected.

Let $N$ be the normal subgroup of $\pi_1 BF'$ generated by the image of homomorphisms $P \to \pi_1 BF'$ such that the induced pointed map $BP \to B \pi_1 BF'$ lifts to $BF$. Let $X$ be the
pullback of the universal cover \( \overline{B\mathcal{F}} \to B\mathcal{F} \to B\pi_1B\mathcal{F} \) along \( BN \to B\pi_1B\mathcal{F} \). Then we have a homotopy commutative diagram

\[
\begin{array}{c}
\overline{B\mathcal{F}} \\
\downarrow f \\
X \\
\downarrow \\
BN \\
\end{array}
\Rightarrow
\begin{array}{c}
\overline{B\mathcal{F}} \\
B(\pi_1B\mathcal{F} / N) \\
\end{array}
\]

where the vertical arrows are homotopy fibrations and the horizontal arrows are principal homotopy fibrations. By Theorem 2.7, there is a saturated fusion system \((S', \mathcal{F}')\) such that \(S' \leq S, \mathcal{F}' \leq \mathcal{F}\) and \(X \approx B\mathcal{F}'\). Furthermore, \(f: B\mathcal{F}' \approx X \to B\mathcal{F}\) is a BP-equivalence by Proposition 4.3.

Let \( C \) be the homotopy cofibre of \( ev: \vee_{[BP,BF']} BP \to B\mathcal{F}'\). By Seifert-Van Kampen’s theorem, \( C \) will be 1-connected if \( \pi_1(ev): \vee_{[BP,BF']} P \to \pi_1(B\mathcal{F}') \approx N \) is an epimorphism.

Since \( N \) is generated by the image of homomorphisms \( f: P \to \pi_1B\mathcal{F} \) such that the induced map \( Bf: BP \to B\pi_1B\mathcal{F} \) lifts to \( f: BP \to B\mathcal{F} \), it is enough to show that for each of those \( f, \text{Im}(f) \leq \text{Im}(\pi_1(ev)) \).

By definition, there is a factorization \( f: P \to N \leq \pi_1B\mathcal{F} \) which lifts to \( B\mathcal{F} \). There exists a unique \( \beta: BP \to B\mathcal{F}' \) (up to homotopy) such that the diagram

\[
\begin{array}{c}
BP \\
\downarrow f \\
\downarrow \\
BN \\
\end{array}
\Rightarrow
\begin{array}{c}
B\mathcal{F}' \\
\downarrow \beta \\
B\mathcal{F} \\
\end{array}
\]

is homotopy commutative. Therefore \( \text{Im}(\pi_1(\beta)) = \text{Im}(f) \leq \text{Im}(\pi_1(ev)) \).

Now we are ready to complete the proof. By Step 2 there exists a saturated fusion system \((S', \mathcal{F}')\) and a BP-equivalence \( f: B\mathcal{F}' \to B\mathcal{F} \) such that the homotopy cofibre of \( ev: \vee_{[BP,BF']} BP \to B\mathcal{F}' \) is 1-connected. Hence \( \text{CW}_{BP}(B\mathcal{F}) = \text{CW}_{BP}(B\mathcal{F}') \) and \( \pi_1\text{CW}_{BP}(B\mathcal{F}) \approx \pi_1\text{CW}_{BP}(B\mathcal{F}') \) is a finite \( p \)-group by Step 1.

\( \square \)

For the proof of Proposition 4.2 we need the next version of a result of N. Castellana, J.A. Crespo and J. Scherer.

**Proposition 4.3** ([CCS07, Proposition 2.1]). Let \( P \) be a finite \( p \)-group and let \( F \to E \to BG \) be a fibration, where \( G \) is a discrete group. Let \( N \) be the normal subgroup of \( G \) generated by the image of homomorphisms \( P \to G \) such that the induced pointed map \( BP \to BG \) lifts to \( E \). Then the pullback of the fibration along \( BN \to BG \)

\[
\begin{array}{c}
E' \to E \to B(G/N) \\
\downarrow f \\
BN \to BG \\
\end{array}
\]

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induces a BP-equivalence \( f : E' \to E \) on the total space level.

**Proof.** We want to show that \( f \) induces a BP-equivalence. The top fibration in the diagram yields a fibration

\[
\text{map}_*(BP, E') \xrightarrow{f_*} \text{map}_*(BP, E) \xrightarrow{pr_*} \text{map}_*(BP, B(G/N)).
\]

Since the base space is homotopically discrete, we only need to check that all components of \( \text{map}_*(BP, E) \) are sent by \( pr_* \) to \( \text{map}_*(BP, B(G/N)) \). Thus consider a pointed map \( h : BP \to E \).

The composite \( pr \circ h \) is homotopy equivalent to a map induced by a group homomorphism \( \alpha = \pi_1(pr \circ h) : P \to G \) whose image is in \( N \) by construction. Therefore \( pr \circ h \simeq pr' \circ \pi \circ h \) is null-homotopic. \( \square \)

**Remark 4.4.** In the original version the authors consider \( P = BZ/p^m \) for \( m > 1 \) and \( \bar{N} \) to be the (normal) subgroup generated by all elements \( g \in G \) of order \( p^i \) for some \( i \leq r \) such that the inclusion \( f_g : B(g) \to BG \) lifts to \( E \), but this subgroup does not have the desired properties. Consider the fibration \( BZ/2 \to BZ/4 \to BZ/2 \), this fibration has no section and hence \( \bar{N} = \{0\} \). Then \( E' \simeq BZ/2 \) and \( CW_{BZ/4}(BZ/2) \simeq BZ/2 \neq BZ/4 = CW_{BZ/4}(BZ/4) \), contradicting the proposition. However, \( N \equiv Z/2 \equiv \langle g \rangle \), since \( f_g = pr \). Hence \( E' \simeq BZ/4 \) and \( f : E' \to BZ/4 \) is an equivalence, in particular it is a \( BZ/4 \)-equivalence.

## 5 Cellular properties of the classifying space of a saturated fusion system

The goal of this section is to prove the main theorem of the paper. Given a finite \( p \)-group \( P \), this result characterizes the property of being BP-cellular for classifying spaces of saturated fusion systems in terms of the fusion data.

**Theorem 5.1.** Let \( (S, F) \) be a saturated fusion system and let \( P \) be a finite \( p \)-group. Then \( BF \) is BP-cellular if and only if \( S = Cl_F(P) \).

**Corollary 5.2.** Let \( (S, F) \) be a saturated fusion system and let \( P \) be a finite \( p \)-group.

(a) The classifying space \( BF \) is BS-cellular.

(b) Let \( (S, F') \) be a saturated fusion system with \( F \subset F' \). If \( BF \) is BP-cellular then \( BF' \) is also BP-cellular.

(c) Let \( A \) be a pointed connected space. If \( Cl_F(\pi_1 A_{ab}) = S \), then \( BF \) is \( A \)-cellular.

(d) Let \( \Omega_{p^m}(S) \) be the (normal) subgroup of \( S \) generated by its elements of order \( p^i \), with \( i \leq m \). Then \( BF \) is \( BZ/p^m \)-cellular if and only if \( S = Cl_F(\Omega_{p^m}(S)) \). In particular, there is a non-negative integer \( m_0 \geq 0 \) such that \( BF \) is \( BZ/p^m \)-cellular for all \( m \geq m_0 \).

**Proof.** (a) Direct from Theorem 5.1 since \( Cl_F(S) = S \).
(b) It follows from the inclusions $\text{Cl}_F(P) \leq \text{Cl}_{F'}(P) \leq S$.

(c) Notice that $SP^\infty A \simeq \prod_{i \geq 1} K(H_i(A; \mathbb{Z}), i)$ is $A$-cellular by [DF96, Corollary 4.A.2.1], so $B(\pi_1 A)_G \simeq K(H_1(A; \mathbb{Z}), 1)$ is also $A$-cellular from [DF96, 2.D]. Then, $BF$ is $B(\pi_1 A)_ab$-cellular using (a).

(d) We have $\text{Cl}_F(\Omega_p^n(S)) \cong \text{Cl}_F(\mathbb{Z}/p^n)$. Note that $B\Omega_p^n(S)$ is $B\mathbb{Z}/p^n$-cellular and that there exists an $m_0 \geq 0$ such that $S$ is generated by elements of order a power of $p$ less than $p^{m_0}$.

The strategy of proof for Theorem 5.1 goes by analyzing the fibre sequence given in [Cha96, Theorem 20.5]

$$
\text{CW}_{BP}(BF) \xrightarrow{c} BF \xrightarrow{r} P_{\Sigma BP}C,
$$

where $C$ is the homotopy cofibre of the evaluation map $ev: \vee_{[BP, BF]} BP \to BF$ and $r$ is the composite $BF \to C \to P_{\Sigma BP}C$.

The first goal is to compute the kernel of $r^\wedge_p$. In order to apply the theory of kernels, developed in Section 3, the target of the map needs to be a connected $p$-complete $\Sigma\mathbb{Z}/p$-null space.

Since $\pi_1 BF$ is a finite $p$-group, the same holds for $P_{\Sigma BP}C$ [Bou94, 2.9], therefore Bousfield-Kan $p$-completion of the previous homotopy fibration is a homotopy fibre sequence ([BK72, II.5.1])

$$
\text{CW}_{BP}(BF) \leftarrow BF \xrightarrow{r^\wedge_p} (P_{\Sigma BP}C)^\wedge_p.
$$

**Lemma 5.3.** If $X$ is a $1$-connected space and $P$ is a finite $p$-group, then $(P_{\Sigma BP}X)^\wedge_p$ is $\Sigma\mathbb{Z}/p$-$null$. 

**Proof.** We have the following weak homotopy equivalences

$$
\text{map}^\wedge_p(\Sigma\mathbb{Z}/p, (P_{\Sigma BP}X)^\wedge_p) \simeq \text{map}^\wedge_p(B\mathbb{Z}/p, \Omega(P_{\Sigma BP}X)^\wedge_p) \simeq \text{map}^\wedge_p(B\mathbb{Z}/p, (\Omega P_{\Sigma BP}X)^\wedge_p)
$$

where the last equivalence holds by [BK72, V.4.6 (ii)] since $X$ is $1$-connected (and so is $P_{\Sigma BP}X$ by [Bou94, 2.9]).

The commutation rules between nullification functors and loops in [DF96, 3.A.1] show that $\Omega P_{\Sigma BP}X \simeq P_{BP}(\Omega X)$. Finally, $\text{map}^\wedge_p(B\mathbb{Z}/p, (P_{BP} \Omega X)^\wedge_p) \simeq \text{map}^\wedge_p(B\mathbb{Z}/p, (P_{BP} \Omega X)) \simeq *$ where the first equivalence follows from Miller’s theorem [Mil84, Thm 1.5] and the second from the fact that $BP$ is $B\mathbb{Z}/p$-acyclic ([Dwy96, Lemma 6.13]).

Next we describe a criteria for detecting when a map from an $A$-cellular space is null-homotopic which will be useful later.

**Proposition 5.4.** Let $X$ and $Y$ be pointed connected spaces. Assume that $X$ is $A$-cellular and $Y$ is $\Sigma A$-$null$. Then a pointed map $f: X \to Y$ is null-homotopic if and only if for any map $g: A \to X$ the composite $f \circ g$ is null-homotopic.
Proof. If \( f \) is null-homotopic then for any map \( g: A \to X \) the composite \( f \circ g \) is null-homotopic.

Assume that for any map \( g: A \to X \), the composite \( f \circ g \) is null-homotopic. Let \( C \) be the homotopy cofibre of \( ev: \bigvee_{[A,X]} A \to X \). Since \( f \circ ev \simeq \ast \) by assumption, there is a map \( \tilde{f}: C \to Y \) such that the following diagram is homotopy commutative

\[
\begin{array}{ccc}
\bigvee_{[A,X]} A & \xrightarrow{ev} & X \\
\downarrow f & & \downarrow f \\
C & \xleftarrow{\sim} & Y
\end{array}
\]

Then \( f \simeq \ast \) if and only if \( \tilde{f} \simeq \ast \).

Since \( X \) is \( A \)-cellular, \( \Sigma P \Sigma A C \simeq \ast \). Finally \( \tilde{f} \simeq \ast \) because map \( \ast \left( C, Y \right) \simeq \text{map} \left( \Sigma P \Sigma A C, Y \right) \simeq \text{map} \left( \ast, Y \right) \simeq \ast \), where the first equivalence follows from the fact that \( Y \) is \( \Sigma A \)-null.\( \square \)

A key step in the proof of Theorem 5.1 is the following computation of the kernel of the map \( r_p \).\( \mathcal{F} \)

**Proposition 5.5.** Let \( (S, F) \) be a saturated fusion system. Then \( \ker(r_p) = Cl_F(P) \).

**Proof.** We will show that \( \ker(r_p) \leq Cl_F(P) \) and that \( f(P) \leq \ker(r_p) \) for all \( f: P \to S \), then since \( \ker(r_p) \) is strongly \( \mathcal{F} \)-closed, the definition of \( Cl_F(P) \) implies the equality.

By universal properties of cellularization and \( p \)-completion, any map \( BP \to BS \) lifts to \( \text{CW}_{BP}(BF)^\wedge_p \), then \( f(P) \leq \ker(r_p) \) for any homomorphism \( f: P \to S \).

According to Proposition 3.7, there exist \( m \geq 0 \) and a pointed map between classifying spaces \( k: BF \to B(\Sigma |S/K| \wr \Sigma p^m)^\wedge_p \) such that \( \ker(k) = Cl_F(P) \). Let \( \iota: B \ker(r_p) \to BF \) be the composite \( B \ker(r_p) \to BS \to BF \). It is enough to show that \( k \circ \iota \) is nullhomotopic.

There is a lift \( \iota: B \ker(r_p) \to \text{CW}_{BP}(BF) \) such that following diagram is homotopy commutative

\[
\begin{array}{ccc}
\text{CW}_{BP}(BF)^\wedge_p & \xrightarrow{c} & B(\Sigma |S/K| \wr \Sigma p^m)^\wedge_p \\
\downarrow \iota & & \downarrow k \circ \iota \\
B \ker(r_p) & \xleftarrow{\sim} & (P_{\Sigma BP}C)^\wedge_p
\end{array}
\]

We will show that \( k \circ c \) is nullhomotopic by applying Proposition 5.4. Recall that any map \( \overline{f}: BP \to BF \) is homotopic to \( \Theta \circ Bp: BP \to BS \to BF \) where \( p \in \text{Hom}(P, S) \) (see [BLO03b, Theorem 4.4]). Then, the map \( k \) has the property that for any \( f: BP \to BF \), the composite \( k \circ f \) is nullhomotopic.

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In particular, if \( c : CW_{BP}(B\mathcal{F}) \to B\mathcal{F} \) is the augmentation, for any \( f : BP \to CW_{BP}(B\mathcal{F}) \) we also have \( (k \circ c) \circ f \simeq * \). Therefore \( k \circ c \simeq * \) by Proposition 5.4 (since \( B(\Sigma[S/\mathcal{K}] \wr \Sigma_\mathcal{P}^m)^p \) is \( \Sigma BZ/p \)-null and hence \( \Sigma BP \)-null by Remark 3.3). \( \square \)

An immediate consequence of Proposition 5.5 is one implication in Theorem 5.1, that is, if \( B\mathcal{F} \) is \( BP \)-cellular then \( S = \text{Cl}_\mathcal{F}(P) \). In order to prove the theorem, we will need a couple of technical results.

**Lemma 5.6.** Let \((S, \mathcal{F})\) be a saturated fusion system. If \( S = \text{Cl}_\mathcal{F}(P) \), then homotopy cofibre \( C \) of the evaluation \( ev : \sqrt{[BP,B\mathcal{F}]} \to BP \to B\mathcal{F} \) is 1-connected.

**Proof.** From the cofibration sequence and Seifert-Van Kampen theorem, we have that \( \pi_1 C \simeq \pi_1 B\mathcal{F} / N \), where \( N \) is the minimal normal subgroup of \( \pi_1 B\mathcal{F} \) containing \( \text{Im}(\pi_1(\text{ev})) \).

Given \( f : BP \to B\mathcal{F} \), there is group homomorphism \( g : P \to S \) such that \( f \simeq \Theta \circ Bg \) (see [BLO03, Theorem 4.4]). Let \( \bar{N} \) be normal subgroup of \( S \) generated by all \( g(P) \), where \( g \in \text{Hom}(P, S) \). Then the fundamental group can be described as \( \pi_1(C) \simeq S/\bar{N}O_\mathcal{P}^p(S) \).

First, \( \bar{N}O_\mathcal{P}^p(S) \) is strongly \( \mathcal{F} \)-closed by [DGPS11, Proposition A.9]. Moreover it contains \( g(P) \) for all \( g \in \text{Hom}(P, S) \). Therefore we have inclusions \( \text{Cl}_\mathcal{F}(P) \subseteq \bar{N}O_\mathcal{P}^p(S) \subseteq S \). Since we are assuming \( S = \text{Cl}_\mathcal{F}(P) \), the previous inclusions are all equalities and \( C \) is then 1-connected. \( \square \)

**Proposition 5.7.** Let \( Z \) be a \( \Sigma^i BP \)-null space where \( P \) is a finite \( p \)-group and \( i \geq 0 \). If \( \pi_1 Z \) is a finite \( p \)-group then \( Z_\mathcal{P}^\wedge \) is also \( \Sigma^i BP \)-null.

**Proof.** Since \( \pi_1 Z \) is a finite \( p \)-group, there is a homotopy commutative diagram of fibre sequences

\[
\begin{array}{ccc}
F & \longrightarrow & Z(1) \longrightarrow Z_\mathcal{P}^\wedge(1) \\
\downarrow & & \downarrow \\
F & \longrightarrow & Z \longrightarrow Z_\mathcal{P}^\wedge \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & B\pi_1(Z) \xrightarrow{\sim} B\pi_1(Z_\mathcal{P}^\wedge)
\end{array}
\]

From the proof of Proposition 3.1 in [DZ87], it is easy to see that \( [\Sigma^i BP, Z_\mathcal{P}^\wedge] \simeq [\Sigma^i BP, Z] \simeq * \).

Then it is enough to check that \( \text{map}_*(\Sigma^i BP, \Omega Z_\mathcal{P}^\wedge) \simeq * \). Consider the homotopy fibre sequence \( \Omega Z_\mathcal{P}^\wedge \to F \to Z \). Since \( Z \) is \( \Sigma^i BP \)-null, \( \text{map}_*(\Sigma^i BP, \Omega Z_\mathcal{P}^\wedge) \simeq \text{map}_*(\Sigma^i BP, F) \). This last mapping space is homotopy equivalent to \( \text{map}_*(\Sigma^i BP, F_\mathcal{P}^\wedge) \) by Miller’s theorem [Mil84, Thm 1.5] since \( F \) is nilpotent. Finally \( F_\mathcal{P}^\wedge \simeq * \). \( \square \)

We are now ready to prove the main theorem of this paper.

**Proof of Theorem 5.1.** First assume that \( B\mathcal{F} \) is \( BP \)-cellular. Then \( P_{\Sigma BP}C \) is contractible and \( r \simeq * \). This implies that \( \ker(r_\mathcal{P}^\wedge) = S \), but also \( \ker(r_\mathcal{P}^\wedge) = \text{Cl}_\mathcal{F}(P) \) by Proposition 5.5. Therefore \( S = \text{Cl}_\mathcal{F}(P) \).
Now assume that $S = Cl_F(P)$. Since $C$ is 1-connected by Lemma 5.6 (and therefore $P_{ΣBP}C$ is so by [Bou94, 2.9]), consider the $p$-completed homotopy fibre sequence ([BK72, II.5.1])

$$CW_{BP}(BF)^P \xrightarrow{r_P^\wedge} BF \xrightarrow{r_P^\wedge}(P_{ΣBP}C)^P.$$

We will show that $(P_{ΣBP}C)^P$ is weakly contractible. For this we apply first the theory of kernels developed in Section 3 to $r_P^\wedge$ (see Lemma 5.3 and Lemma 5.6). By assumption and Proposition 5.5, ker($r_P^\wedge$) = $S$ and therefore $r_P^\wedge$ is null-homotopic by Theorem 3.6. Then there is a splitting $CW_{BP}(BF)^P \cong BF \times Ω(ΣBPΣBP)$.  

Let $X = CW_{BP}(BF)^P$ and $Y = Ω(P_{ΣBP}C)^P$ for simplicity. We will show that $P_{BP}(X)^P \cong *$ and that $P_{BP}(Y)^P \cong Y$. Since both nullification and $p$-completion functors commute with products, the previous splitting $X \cong BF \times Y$ shows that $Y \cong *$. Since $(P_{ΣBP}C)^P$ is 1-connected, then it follows that it is contractible.

First, we have that $P_{BP}(X)^P = P_{BP}(CW_{BP}(BF))^P \cong *$ by checking the hypothesis of [CF15, Lemma 3.9]. One only needs to check that $P_{BP}(X)$ is $p$-good and that $P_{BP}(Y)$ is $BP$-null. Notice that $τ_1X$ is a finite $p$-group since $C$ is 1-connected by Lemma 5.6. Therefore the space $P_{BP}(X)^P$ is $BP$-null by Proposition 5.7 and the space $P_{BP}(X)$ is $p$-good by Proposition 4.2. Finally, recall that $BP$-cellular spaces are $BP$-acyclic ([DF96, 3.B.1]).

Now, 

$$P_{BP}(Y)^P = P_{BP}(Ω(P_{ΣBP}C)^P)^P \cong (ΩP_{ΣBP}((P_{ΣBP}C)^P))^P = (ΩP_{ΣBP}(C)^P)^P = Ω(ΣBPΣBP)^P = Y$$

where the second equivalence follows from commutation rules [DF96, 3.A.1], the third holds because $(P_{ΣBP}C)^P$ is $ΣBP$-null by Millers’s theorem [Mil84, Thm 1.5], and the forth is a commutation of taking loops and $p$-completion [BK72, V.4.6 (ii)].

Summarizing, we have proved that $c: CW_{BP}(BF) \rightarrow BF$ is a mod $p$ equivalence. Finally, using Proposition 4.1 we get that $c$ is a weak equivalence since $CW_{BP}(BF)^P \cong BF$ and $BFQ \cong *$. \hfill $\square$

6 Examples

Let $G$ be a finite group. The situation when $G$ is generated by elements of order $p$ is well studied by R. Flores and R. Foote in [FF11]. We start by giving a simple example where $G$ is not generated by elements of order $p^i$.

Example 6.1. Let $G = Σ_3$, the permutation group of 3 elements. $Σ_3$ is generated by transpositions, i.e., by elements of order 2, but the Sylow 3-subgroup of $Σ_3$ is $S = Z/3$. Therefore, $BS$ is $BZ/3'$-cellular for all $r \geq 1$ and hence $(Σ_3)^3$ is $BZ/3'$-cellular for all $r \geq 1$ by Corollary 5.2.

Notice that $BS_3$ is not $BZ/3'$-cellular for any $r \geq 1$: applying map, $(BZ/3', -)$ to the homotopy fibre sequence $BZ/3 \xrightarrow{Bi} BS_3 \rightarrow BZ/2$, we see that $Bi$ is a $BZ/3'$-equivalence for any $r$, since map, $(BZ/3', BZ/2) \cong *$. Then $CW_{BZ/3}(Σ_3) \cong CW_{BZ/3}(BZ/3) \cong BZ/3$. 

We are now interested in the study of $CW_{BP}(BF)$ when $S ≠ Cl_F(P)$. Our aim is to get tools to identify $P_{BP}(C)^P$ in the $p$-completed Chacholsky’s fibration describing $CW_{BP}(BF)$.
**Proposition 6.2.** Let \((S,F)\) be a saturated fusion system, \(\mathcal{L}\) be the associated centric linking system and let \(P\) be a finite \(p\)-group. Assume that there is a saturated fusion system \((S',F')\), with associated linking system \(\mathcal{L}'\), and a factorization (up to homotopy)

\[
\begin{array}{ccc}
|\mathcal{L}| & \xrightarrow{\eta} & BF' \\
\pi & & \downarrow \pi_p' \\
|\mathcal{L}'| & \xrightarrow{\eta_p'} & (P_{\Sigma BP}C)_p^{\wedge}
\end{array}
\]

such that

(i) the map \(\tilde{r}\) is injective, i.e., \(\ker(\tilde{r}) = \{e\}\),

(ii) and or \(F\), the homotopy fibre of \(\pi : |\mathcal{L}| \to |\mathcal{L}'|\), or \(F_p\), the homotopy fibre of \(\pi_p^{\wedge}\), is \(BP\)-acyclic.

Then \(\tilde{r}\) is a homotopy equivalence. In particular, \(CW_{BP}(BF') \simeq F_p\).

**Proof.** We will proceed as follows: first we will construct a map \(\hat{\pi}_p^{\wedge} : (P_{\Sigma BP}C)_p^{\wedge} \to BF'\) under \(BF\), and then we will prove that \(\hat{\pi}_p^{\wedge}\) is a homotopy inverse of \(\tilde{r}_p^{\wedge}\). This will give us that \(CW_{BP}(BF')_p^{\wedge} \simeq F_p\). Finally we will show, in this situation, that \(CW_{BP}(BF')\) is \(p\)-complete and hence \(CW_{BP}(BF') \simeq F_p\).

(a) Construction of \(\hat{\pi}_p^{\wedge} : (P_{\Sigma BP}C)_p^{\wedge} \to BF'\). In order to apply Zabrodsky lemma [Dwy96, Proposition 3.4] to the following situation

\[
\begin{array}{ccc}
BF & \xrightarrow{\pi_p^{\wedge}} & BF' \\
\downarrow c^{\wedge}_p & & \downarrow r_p^{\wedge} \\
(P_{\Sigma BP}C)_p^{\wedge}
\end{array}
\]

we need to check that two assumptions are satisfied:

- \(\pi_p^{\wedge} \circ c^{\wedge}_p \simeq \ast\): or equivalently, \(\pi_p^{\wedge} \circ c : CW_{BP}(BF) \to BF'\) is null-homotopic since \(BF'\) is \(p\)-complete and \(CW_{BP}(BF)\) is \(p\)-good.

  According to Proposition 5.4 it is sufficient to show that the composite \(\pi_p^{\wedge} \circ c \circ f\) is null-homotopic for all \(f \in \text{map}_e(BP,CW_{BP}(BF))\). But if we postcompose with \(\tilde{r}\), we have \(\tilde{r} \circ \pi_p^{\wedge} \circ c \circ f \simeq r_p^{\wedge} \circ c \circ f \simeq \ast\). There exists \(\rho \in \text{Hom}(P,S')\) such that \(\pi_p^{\wedge} \circ c \circ f \simeq \Theta' \circ BP\), then \(\rho(\tilde{r}) \leq \ker(\tilde{r}) = \{e\}\).

- \(\text{map}_e(CW_{BP}(BF)_p^{\wedge}, \Omega BF'') \simeq \ast\): since \(BF'\) and \(CW_{BP}(BF)\) are \(p\)-good space we have \(\text{map}_e(CW_{BP}(BF)_p^{\wedge}, BF') = \text{map}_e(CW_{BP}(BF), BF')\) and hence, taking loops,

\[
\text{map}_e(CW_{BP}(BF)_p^{\wedge}, \Omega BF'') = \text{map}_e(CW_{BP}(BF), \Omega BF').
\]

The last mapping space is contractible since \(\Omega BF'\) is \(BP\)-null (\(p\):mapping space) and \(CW_{BP}(BF)\) is \(BP\)-acyclic ([DF96, 3.B.1]).
(b) $\tilde{r}$ and $\tilde{\pi}_p^\wedge$ are homotopy inverse. First note that both $\text{Id}_{(P_{\Sigma BP}C)^\wedge_p}$ and $\tilde{r} \circ \tilde{\pi}_p^\wedge$ factor $r_p^\wedge$ since $\text{Id}_{(P_{\Sigma BP}C)^\wedge_p} \circ r_p^\wedge \simeq \tilde{r} \circ \pi \simeq \tilde{\pi}_p^\wedge \circ r_p^\wedge$. In order to show that $\tilde{r} \circ \tilde{\pi}_p^\wedge \simeq \text{Id}_{(P_{\Sigma BP}C)^\wedge_p}$, we will apply Zabrodsky lemma to the fibration $\text{CW}_{BP}(B\mathcal{F})^\wedge_p \to B\mathcal{F} \to (P_{\Sigma BP}C)^\wedge_p$ and the composite map $r_p^\wedge \circ \pi : B\mathcal{F} \to (P_{\Sigma BP}C)^\wedge_p$. Then uniqueness up to homotopy of the factorization will give the desired equivalence.

We check that $\text{map}_*(\text{CW}_{BP}(B\mathcal{F})^\wedge_p, \Omega(P_{\Sigma BP}C)^\wedge_p) \simeq \ast$ and $\tilde{r}^\wedge \circ \tilde{\pi}_p^\wedge \circ c_p^\wedge \simeq \ast$. First, by Proposition 5.7, $(P_{\Sigma BP}C)^\wedge_p$ is $\Sigma BP$-null, then $\Omega(P_{\Sigma BP}C)^\wedge_p$ is $BP$-null ([DF96, 3.A.1]). Now, since $\text{CW}_{BP}(B\mathcal{F})^\wedge_p$ and $\Omega(P_{\Sigma BP}C)^\wedge_p$ are $p$-good and $\text{CW}_{BP}(B\mathcal{F})$ is $BP$-acyclic, we have

$$\text{map}_*(\text{CW}_{BP}(B\mathcal{F})^\wedge_p, \Omega(P_{\Sigma BP}C)^\wedge_p) \simeq \text{map}_*(\text{CW}_{BP}(B\mathcal{F}), \Omega(P_{\Sigma BP}C)^\wedge_p) \simeq \ast$$

Finally, $r_p^\wedge \circ \pi \circ c_p^\wedge \simeq r_p^\wedge \circ c_p^\wedge \simeq \ast$.

It remains to prove that $\tilde{r}_p^\wedge \circ \tilde{\pi}_p^\wedge \simeq \text{Id}_{B\mathcal{F}}$. First note that $\text{Id}_{B\mathcal{F}} \circ \pi \simeq \tilde{r} \circ \pi = \tilde{\pi} \circ \tilde{r}$, then both $\text{Id}_{B\mathcal{F}}$ and $\tilde{\pi} \circ \tilde{r}$ factor $\pi$. In order to show that $\text{Id}_{B\mathcal{F}} \simeq \tilde{r} \circ \tilde{\pi}_p^\wedge$, we will apply again Zabrodsky’s lemma to the fibration $F_p \to B\mathcal{F} \to B\mathcal{F}'$ (resp. $F \to |\mathcal{L}| \to |\mathcal{L}'|$, depending on whether $F$ or $F_p$ is $BZ/p$-acyclic) and the composite $\tilde{\pi}_p^\wedge r_p^\wedge$ (resp. $\tilde{\pi} \circ r_p^\wedge \circ \eta$).

If $i : F_p \to B\mathcal{F}$, then $r^\wedge \circ i \simeq \tilde{r} \circ \pi \circ i \simeq \ast$. Then since $\Omega B\mathcal{F}'$ is $BZ/p$-null and $F_p$ is $BZ/p$-acyclic (resp. $F$ is $BZ/p$-acyclic), $\text{map}_*(F_p, \Omega B\mathcal{F}') \simeq \ast$ (resp. $\text{map}_*(F, \Omega B\mathcal{F}') \simeq \ast$).

(c) $\text{CW}_{BP}(B\mathcal{F})$ is $p$-complete. By Proposition 4.1 we obtain the homotopy fibration

$$\text{CW}_{BP}(B\mathcal{F}) \to F_p \to (F_p)_Q,$$

where $(F_p)_Q \simeq \ast$, since $\pi_i(F_p)$ are finite groups for $i \geq 1$.

Remark 6.3. Proposition 6.2 also holds without assuming the existence of $\pi$ and only considering the factorization on $p$-completed classifying spaces if we know that the homotopy fiber $F_p$ is $BZ/p$-acyclic.

Corollary 6.4. Let $(S, \mathcal{F})$ be a saturated fusion system and let $P$ be a finite $p$-group. If $O^0_p(S) \lhd \text{Cl}_P$, then $\text{CW}_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'$, where $B\mathcal{F}'$ is the connected cover of $B\mathcal{F}$ with $\pi_1 B\mathcal{F}' \simeq \text{Cl}_P / O^0_P(S)$.

Proof. The connected cover $B\mathcal{F}'$ is the classifying space of a saturated fusion system over $\text{Cl}_P$ by [BCG'07, Theorem A]. So $B\mathcal{F}'$ is $BP$-cellular by Corollary 5.2. We get the following commutative diagram of homotopy fibrations

$$\begin{array}{ccc}
B\mathcal{F} & \longrightarrow & B\mathcal{F}' \\
\downarrow & & \downarrow \circ f \\
B\mathcal{F}' & \longrightarrow & B\mathcal{F} & \longrightarrow & B(S/\text{Cl}_P) \\
B(\text{Cl}_P/O^0_P(S)) & \longrightarrow & B(S/O^0_P(S)) & \longrightarrow & B(S/\text{Cl}_P) \\
\end{array}$$
In order to apply Proposition 6.2, we have to find a factorization of \(r_p^\gamma : B\mathcal{F} \to (P_{\Sigma BP}C)^\wedge\) by a map \(\tilde{r} : B(S/\text{Cl}_\mathcal{F}(P)) \to (P_{\Sigma BP}C)^\wedge\) with trivial kernel, where we can assume that \(C\) is 1-connected by step 2 in the proof of Proposition 4.2.

By Lemma 5.3 and Remark 3.3, \(\Omega(P_{\Sigma BP}C)^\wedge\) is \(BP\)-null and \(B\mathcal{F}'\) is \(BP\)-cellular space (hence \(BP\) acyclic), then \(\text{map}_\mathcal{F}(B\mathcal{F}, \Omega(P_{\Sigma BP}C)^\wedge) \simeq \ast\). Moreover, \(r_p^\gamma \circ f \mid_{\text{Cl}_\mathcal{F}(P)} \simeq \ast\) since \(\ker(r_p^\gamma) = \text{Cl}_\mathcal{F}(P)\). We are in the conditions of applying Zabrodsky lemma (see [Dwy96, Proposition 3.4]) which shows that there is a map \(\tilde{r} : B(S/\text{Cl}_\mathcal{F}(P)) \to (P_{\Sigma BP}C)^\wedge\) such that the diagram

\[
\begin{array}{ccc}
B\mathcal{F}' & \xrightarrow{f} & B\mathcal{F} \\
\downarrow & & \downarrow r_p^\gamma \\
B(S/\text{Cl}_\mathcal{F}(P)) & \xrightarrow{\tilde{r}} & (P_{\Sigma BP}C)^\wedge
\end{array}
\]

commutes up to homotopy.

Let \([x] \in \ker(\tilde{r}) \subseteq S/\text{Cl}_\mathcal{F}(P)\), where \(x \in S\). Since the above diagram commutes, \(r_p^\gamma|_{\text{Cl}(\gamma)} \simeq \bar{r}|_{B[Sx]} \simeq \ast\), then \(x \in \ker(r_p^\gamma) = \text{Cl}_\mathcal{F}(P)\) and hence \([x] = e\). Finally, Proposition 6.2 gives us the equivalence \(\text{CW}_{BP}(B\mathcal{F}) \simeq B\mathcal{F}'\).

\[\square\]

**Corollary 6.5.** Let \((S, \mathcal{F})\) be a fusion system and let \(P\) be a finite \(p\)-group. If \(\text{Cl}_\mathcal{F}(P) \sim \mathcal{F}\), then \(\text{CW}_{BP}(B\mathcal{F})\) is homotopy equivalent to the homotopy fibre of \(B\mathcal{F} \to B(\mathcal{F}/\text{Cl}_\mathcal{F}(P))\).

**Proof.** Let \(K = \text{Cl}_\mathcal{F}(P)\). Since \(K\) is normal in \(\mathcal{F}\), there is a saturated fusion system \((S/K, \mathcal{F}/K)\), and a map defined between the nerve of the associated linking systems \(\pi : |\mathcal{L}| \to |\mathcal{L}/K|\) whose homotopy fibre is \(BK\), and an injective factorization of \(r_p^\gamma\) by \(B\mathcal{F}/K\) from Proposition 3.8. Then the result follows from Proposition 6.2.

\[\square\]

**Example 6.6.** Let \(G\) be a finite group and let \(S\) be a \(p\)-Sylow subgroup of \(G\). Assume that \(N_G(S)\) controls fusion in \(G\). Then \(BN_G(S)^\wedge = BG_S^\wedge\) and \(S\) is normal in \(\mathcal{F}_{S}(N_G(S))\). On account of Corollary 6.5, for all finite \(p\)-group \(P\), \(\text{CW}_{BP}(BG_S^\wedge)\) is equivalent to the homotopy fibre of \(BN_G(S)^\wedge \to B(N_G(S)/\text{Cl}_{\mathcal{F}_{S}(N_G(S))}(P))_{p}^\wedge\).

An example is given by \(G = \mathbb{Z}/p^n \wr \mathbb{Z}/q = (\mathbb{Z}/p^n) \rtimes \mathbb{Z}/q\), when \(p \neq q\) and \(n \geq 1\). The Sylow \(p\)-subgroup of \(G\) is \(S = (\mathbb{Z}/p^n)\) and \(\text{Cl}_{\mathcal{F}_{S}(G)}(\mathbb{Z}/p^n) = (\mathbb{Z}/p^n)\) is abelian and hence normal in \(\mathcal{F}_{S}(G)\) by Corollary 3.9. Therefore \(BG_P^\wedge\) is \(B\mathbb{Z}/p^n\)-cellular if and only if \(r \geq n\) by Theorem 5.1. Then \(\text{CW}_{B\mathbb{Z}/p^n}(BG_P^\wedge)\) is equivalent to the homotopy fibre of \(BG_P^\wedge \to B(G/(\mathbb{Z}/p^n))_{p}^\wedge\) by Corollary 6.5.

Other explicit examples appear in [FS07, Example 5.2]. The authors proved that the normalizer of the Sylow of the Suzuki group \(Sz(2^n)\), with \(n\) an odd integer at least \(3\), is \(N_{Sz(2^n)}(S) = S \rtimes \mathbb{Z}/(2^n - 1)\) and it controls fusion in \(Sz(2^n)\). In this case, \(S\) is \(B\mathbb{Z}/2^{n}\)-cellular for all \(m \geq 2\) and hence \(BSz(2^n)^\wedge\) is so. Moreover, \(\text{Cl}_{\mathcal{F}_{S}(Sz(2^n))}(\mathbb{Z}/2) \simeq (\mathbb{Z}/2)^n\) and hence \(\text{CW}_{B\mathbb{Z}/2}(BSz(2^n))^\wedge\) is equivalent to the homotopy fibre of \(BN_{Sz(2^n)}(S)^\wedge \to B(N_{Sz(2^n)}(S)/(\mathbb{Z}/2)^n)_{2}^\wedge\).
7 The cellularization of the classifying spaces of a family of exotic fusion systems at $p = 3$

Let $S = B(3, r; 0, \gamma, 0)$, for $r \geq 4$ and $\gamma = 0, 1, 2$, be the family of finite 3-groups of order $3^l$ (see [DRV07, Theorem A.2, Proposition A.9]) generated by $\{s, s_1, \ldots, s_{r-1}\}$ where

- $s_i = [s_{i-1}, s]$ for all $i \in \{2, \ldots, r - 1\}$,
- $[s_1, s_i] = 1$ for all $i \in \{2, \ldots, r - 1\}$,
- $s_1^2 s_3 s_3 = s_{r-1}'$,
- $s_i^2 s_{i+1}^2 = 1$ for all $i \in \{2, \ldots, r - 1\}$,
- $s^3 = 1$.

The center of $S$ is $\langle s_{r-1} \rangle$. The normal subgroup $\gamma_1 = \langle s_1, \ldots, s_{r-1} \rangle$ of $S$ is of index 3 and the corresponding group extension is split. There are group isomorphisms

$$B(3, r; 0, \gamma, 0) = \langle s_1, s_2 \rangle \rtimes \langle s \rangle = \begin{cases} (\mathbb{Z}/3^m \times \mathbb{Z}/3^m) \rtimes \mathbb{Z}/3 & \text{if } r = 2m + 1, \\ (\mathbb{Z}/3^m \times \mathbb{Z}/3^{m+1}) \rtimes \mathbb{Z}/3 & \text{if } r = 2m. \end{cases}$$

In [DRV07, Theorem 5.10], the authors construct families of exotic 3-local finite groups $\mathcal{F}$ whose Sylow 3-subgroup is $S = B(3, r; 0, \gamma, 0)$.

**Proposition 7.1.** Let $\mathcal{F}$ be an exotic fusion system over $B(3, r; 0, \gamma, 0)$ such that $\mathcal{F}$ has at least one $\mathcal{F}$-Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then

(i) If $\gamma = 0$, then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular for all $l \geq 1$.

(ii) Assume $\gamma \neq 0$. Then $B\mathcal{F}$ is $B\mathbb{Z}/3^l$-cellular if and only if $l \geq 2$. If $l = 1$, $\text{Cl}_F(\mathbb{Z}/3) = \langle s, s_2 \rangle$.

**Proof.** By Theorem 5.1 we are reduced to the computation of $\text{Cl}_F(\mathbb{Z}/3^l)$. Let $N := \langle s_2, s \rangle$ which is a proper normal subgroup of $S$. This subgroup $N$ contains the center of $S$, $Z(S) = \langle s_{r-1} \rangle \cong \mathbb{Z}/3$ by [DRV07, Lemma A.10]. Moreover, the description of $\mathcal{F}$ and the computation of automorphisms of $S$ in [DRV07, Theorem 5.10, Lemma A.14] show that $N$ is strongly $\mathcal{F}$-closed.

First $s \in \text{Cl}_F(\mathbb{Z}/3^l)$ for all $l \geq 1$. If $\text{Cl}_F(\mathbb{Z}/3^l)$ is a proper normal subgroup of $S$, again by [DRV07, Lemma A.10], $\text{Cl}_F(\mathbb{Z}/3^l) = N$ with $[N : S] = 3$, and

$$N \cong \begin{cases} 3^{l+2} & \text{if } r = 4, \\ B(3, r - 1; 0, 0, 0) & \text{if } r > 4. \end{cases}$$

We have then inclusions $N \subset \text{Cl}_F(\mathbb{Z}/3^l) \subset S$. And $\text{Cl}_F(\mathbb{Z}/3^l) = S$ if and only if there exists $x \in S \setminus N$ such that $x^{3^l} = 1$.

Let $x = s^i s_{i}^j s_k^k$ for $s_1, s_2 \in S$, where $i = 0, 1, 2$. Since the index $[\text{Cl}_F(\mathbb{Z}/3^l) : S] \in \{1, 3\}$, $x \in \text{Cl}_F(\mathbb{Z}/3^l)$ if and only if $x^2 \in \text{Cl}_F(\mathbb{Z}/3^l)$. But if $x = s^i s_{i}^j s_k^k$, then $x^2 = s^i s_{i}^j s_k^k = s s_{i}^j s_k^k$, therefore we
can assume \( i = 0, 1 \). Note that \( s_i = [s_{i-1}, s] \in (s, s_2) \subset N \) for all \( i \in \{3, \ldots, r - 1 \} \) and \( s_1^3 = s_3^{-1}s_2^{-3}N \in N \). If \( j \equiv 0 \mod 3 \) then \( x \in N \).

If \( i = 0 \) then \( x = s_1^j \) and \( [s_1, s_2] = 1 \). Then \( o(x) = 3 \) if and only if \( o(s_1^j) = 3 \) and \( o(s_2^j) = 3 \).

In particular, \( 3j3^n \), that is, \( 3j \). But then \( j \equiv 0 \mod 3 \) and \( x \in N \).

If \( i = 1 \) and \( x = ss_1^j \) one can compute \( x^3 = (s_{r-1})^j \) using relations \( s^j_1s = ss_1^j, s^j_2 = ss_1^j, s^j_3 = ss_1^j, s^j_4 = 1 \) with \( a, b, c > 0 \) ([DRV07, Proposition A.9]). We have that \( x^3 = (s_{r-1})^j3^{j-1} \) and \( x^{3j} = 1 \) if and only if \( r \equiv 0 \mod 3 \). That is, \( x^9 = 1 \) always, and \( x^3 = 1 \) if \( r = 0 \) or \( i = 0 \mod 3 \), but in this last case \( x \in N \).

Summarizing, if \( r = 0 \) then \( x = ss_1 \in S \setminus N \) is of order 3 and then \( Cl_F(\mathbb{Z}/3^j) = S \) for all \( j \geq 1 \). If \( y \neq 0 \) then \( x = ss_1 \in S \setminus N \) is of order 9 and then \( Cl_F(\mathbb{Z}/3^j) = S \) for all \( j \geq 2 \). Finally if \( y = 0 \), \( Cl_F(\mathbb{Z}/3^j) = N \).

We will finish this section by describing \( \text{CW}_{B \mathbb{Z}/3}(B \mathcal{F}) \) when \( \mathcal{F} \) is an exotic fusion system over \( B(3, r; 0, y; 0) \) with \( y \neq 0 \) such that \( \mathcal{F} \) has at least one \( \mathcal{F} \)-Alperin rank two elementary abelian 3-subgroup given in [DRV07].

**Lemma 7.2.** Let \( \mathcal{F} \) an exotic fusion system under the hypothesis of Proposition 7.1, \( B \mathcal{F} \) is 1-connected.

**Proof.** By Proposition 2.5, the fundamental group of \( B \mathcal{F} \) is \( \pi_1(B \mathcal{F}) \equiv S/O^\mathcal{F}_p(S) \), where \( O^\mathcal{F}_p(S) := \langle \langle Q, O^\mathcal{F}_p(\text{Aut}_\mathcal{F}(Q)) \mid Q \leq S \rangle \rangle \). The subgroup \( O^\mathcal{F}_p(S) \) is a strongly \( \mathcal{F} \)-closed subgroup of \( S \), and the arguments in the proof in [DRV07, page 1751] show that it must contain \( N = (s, s_2) < S \). We will show that \( O^\mathcal{F}_p(S) = S \) by proving that \( s_1 \in O^\mathcal{F}_p(S) \). Checking tables in [DRV07, Theorem 5.10, Lemma A.14], we see that the automorphisms of order two \( \eta \) and/or \( \omega \) are group elements in \( \text{Aut}_\mathcal{F}(S) \). By the description given there \( \eta(s_1) = s_1s_2^j \) and \( \omega(s_1) = s_1s_2^j \). Then \( s_1^{-1}\eta(s_1) = s_1^{-1}s_2^j \) or \( s_1^{-1}\omega(s_1) = s_1^{-1}s_2^j \) are elements of \( O^\mathcal{F}_p(S) \), since \( s_2, s_1 \in N \subset O^\mathcal{F}_p(S) \), then \( s_1 \in O^\mathcal{F}_p(S) \).

**Proposition 7.3.** Let \( \mathcal{F} \) be an exotic fusion system satisfying the hypothesis of Proposition 7.1 with \( y \neq 0 \). Let \( N = (s, s_2) < S \), then there exists a unique map (up to homotopy) \( f : B \mathcal{F} \to (B \Sigma_3)^\wedge \) whose kernel is \( N \).

**Proof.** The proof of Proposition 3.7 shows that the quotient morphism \( S \to S/N \cong \mathbb{Z}/3 \) gives a fusion preserving homomorphism \( \rho : S \to \Sigma_3 \). We want to show that this morphism extends to a map \( f : B \mathcal{F} \to (B \Sigma_3)^\wedge \).

By Proposition 2.9, \( B \mathcal{F} \cong (\text{hocolim}_{\mathcal{F}}(BP))^\wedge \), where \( BP \cong BP \in F \in \mathcal{F}^c \). The fusion preserving property of \( \rho \) shows that \( B \rho \in \text{lim}_{\mathcal{F}}(BP, (B \Sigma_3)^\wedge) \).

The obstructions for rigidifying the homotopy commutative diagram in the category of spaces lie in \( \text{lim}_{\mathcal{F}}^{i+1}(\pi_i(\text{map}(BP, (B \Sigma_3)^\wedge), \mathcal{F}^c \rho) \mathcal{F}^c \rho) \), for \( i \geq 1 \) (see [Woj87]). Note that since the 3-Sylow subgroup of \( \Sigma_3 \) is abelian, we have \( \pi_1(\text{map}(BP, (B \Sigma_3)^\wedge), \mathcal{F}^c \rho) \mathcal{F}^c \rho) \) is abelian being a quotient of \( C_{\mathbb{Z}/3}(\rho(P)) \). In fact, it will be trivial or \( \mathbb{Z}/3 \) (Proposition 2.6).

We will show that for any \( F : O(\mathcal{F}) \to \mathbb{Z}(P) - \text{Mod} \), \( \text{lim}_{\mathcal{F}} F = 0 \) for \( i > 1 \). From [BLO03b, Proposition 3.2, Corollary 3.3], we are reduced to show that derived limits of atomic functors have the same vanishing property. Note that from [DRV07, Theorem
5.10, Lemma A.14], the relevant outomorphism groups \( \text{Out}_F(P) \) are \( SL_2(F_3) \) or \( GL_2(F_3) \). In both cases the 3-Sylow subgroup is of order 3, and then \([\text{JMO}92, \text{Proposition 6.2(i)}]\) implies the result.

The obstructions to uniqueness lie in \( \lim^i_{\Omega(F)} \pi_i(\text{map}(BP, (B\Sigma_3)_3^\wedge)^\wedge) \), for \( i \geq 1 \) (see [Woj87]). By the previous paragraph we have to look at the first derived functor of atomic functors with value \( Z/3 \). But since \( \text{Aut}(Z/3) \approx Z/2 \), by \([\text{JMO}92, \text{Proposition 6.1(ii)}]\) any element of order 3 will act trivially on \( Z/3 \). Finally note that if a map \( g : BF \rightarrow (B\Sigma_3)_3^\wedge \) has kernel \( N \), its restriction \( g|_{BS} : BS \rightarrow (B\Sigma_3)_3^\wedge \) has to be homotopic to \( BP \).

**Proposition 7.4.** Let \( F \) be an exotic fusion system over \( B(3, r; 0, \gamma, 0) \) with \( \gamma \neq 0 \) such that \( F \) has at least one \( F \)-Alperin rank two elementary abelian 3-subgroup given in [DRV07, Theorem 5.10]. Then there exists a map \( f : BF \rightarrow (B\Sigma_3)_3^\wedge \) such that \( \text{CW}_{BZ/3}(BF) \) is the homotopy fiber of \( f \).

**Proof.** Let \( f \) be the map constructed in Proposition 7.3 with \( \text{ker}(f) = \text{Cl}_F(S) \). Precisely because of this, \( f \circ ev \simeq * \) where \( ev : \sqrt{[BZ/3,BF]}_3 BZ/3 \rightarrow BF \). Then \( f \) factors through the cofibre \( C \) of \( ev \) and, since \( (B\Sigma_3)_3^\wedge \) is \( BZ/3 \)-null, we obtain a factorization of \( f \), \( f' : P_{\Sigma BZ/3}(C) \rightarrow (B\Sigma_3)_3^\wedge \) such that the following diagram is homotopy commutative

\[
\begin{array}{ccc}
BF & \xrightarrow{f'} & BF' \\
\downarrow & & \downarrow_{r_3'} \\
P_{\Sigma BZ/3}(C) & \xrightarrow{f'} & (B\Sigma_3)_3^\wedge.
\end{array}
\]

The strategy if to construct a homotopy inverse of \( f' \), \( \Theta : (B\Sigma_3)_3^\wedge \rightarrow P_{\Sigma BZ/3}(C)_3^\wedge \), up to 3-completion, which fits in the previous diagram up to homotopy.

Since \( \Sigma_3 \) has an abelian normal 3-Sylow subgroup \( Z/3 \), we have that \( (BZ/3)_{\eta Z/2} \rightarrow B\Sigma_3 \) is an equivalence. Consider the fibre sequence \( BN \rightarrow BS \rightarrow BZ/3 \) and the map \( r_3'_{|BS} : BS \rightarrow P_{BZ/3}(C)_3^\wedge \), by Zabrodsky’s lemma [Dwy96, Proposition 3.4], \( r_3'_{|BS} \) factors (uniquely up to homotopy) via \( \Theta' : BZ/3 \rightarrow P_{BZ/3}(C)_3^\wedge \). In order to get \( \Theta \), we only need check that \( \Theta' \) is \( Z/2 \)-equivariant up to homotopy. For any \( F \) in the hypothesis of the proposition, note that there is an element in \( \omega' \in \text{Out}_F(S) \) which project to \( \omega \in \text{Out}_{C_3}(Z/3) \) (they are called \( \eta \) or \( \omega \) in the tables [DRV07, Theorem 5.10]). Since \( \Theta' \) is unique up to homotopy factoring \( r_3' \circ \Theta \), and \( \Theta \circ \omega' = \Theta \), it follows \( \omega \circ \Theta' = \Theta' \).

Next we check that \( \Theta \) is a homotopy inverse to \( f \). First consider the following homotopy commutative diagram:

\[
\begin{array}{ccc}
\text{CW}_{BZ/3}(BF) & \xrightarrow{f'} & BF' \\
\downarrow & & \downarrow_{r_3'} \\
P_{\Sigma BZ/3}(C) & \xrightarrow{f'} & (B\Sigma_3)_3^\wedge \text{P}_{\Sigma BZ/3}(C)_3^\wedge.
\end{array}
\]

Since 3-completion on the bottom line also gives a homotopy commutative diagram, unicity on Zabrodsky’s lemma (see [Dwy96, Proposition 3.4]) shows that \( \Theta \circ f' \) is 3-completion. So \( \Theta \circ (f')_3^\wedge \simeq \text{id} \).

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Now consider the following homotopy commutative diagram

\[
\begin{array}{cccc}
BK & \rightarrow & F & \\
\downarrow & & \downarrow & \\
BS \Theta & \rightarrow & BF & B^F F \\
\downarrow & & \downarrow & \downarrow & f \\
BZ/3 & \Theta & (B\Sigma_3)^3 & \rightarrow (B\Sigma_3)^3.
\end{array}
\]

The map \((f')^3 \Theta\) is determined by its restriction to the Sylow 3-subgroup \(Z/3\) by Proposition 7.3. Again \(\iota \circ (f')^3 \Theta\) and \(\iota\) give homotopy commutative diagrams when placed in the bottom line, then unicity on Zabrodsky’s lemma (see [Dwy96, Proposition 3.4]) shows that they are homotopic.

\[\square\]

References

[AKO11] M. Aschbacher, R. Kessar, and B. Oliver. *Fusion systems in algebra and topology*, volume 391 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2011.

[BCG+07] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver. Extensions of \(p\)-local finite groups. *Trans. Amer. Math. Soc.*, 359(8):3791–3858 (electronic), 2007.

[Ben98] D. Benson. *Cohomology of sporadic groups, finite loop-spaces, and the Dickson invariants*, Geometry and cohomology in group theory. London Matth. Soc. Lecture Notes, ser. 252. Cambrige Univ. Press, 1998.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[BLO03a] C. Broto, R. Levi, and B. Oliver. Homotopy equivalences of \(p\)-completed classifying spaces of finite groups. *Invent. Math.*, 151(3):611–664, 2003.

[BLO03b] C. Broto, R. Levi, and B. Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16(4):779–856, 2003.

[Bou94] A. K. Bousfield. Localization and periodicity in unstable homotopy theory. *J. Amer. Math. Soc.*, 7(4):831–873, 1994.

[CCS07] N. Castellana, J. A. Crespo, and J. Scherer. Postnikov pieces and \(BZ/p\)-homotopy theory. *Trans. Amer. Math. Soc.*, 359(3):1099–1113, 2007.

[CF15] N. Castellana and R. J. Flores. Homotopy idempotent functors on classifying spaces. *Trans. Amer. Math. Soc.*, 367(2):1217–1245, 2015.

[CGR15] N. Castellana and A. Gavira-Romero. Cellular approximations of infinite loop spaces. *J. Lond. Math. Soc.*, 2015.
W. Chachólski. On the functors $\text{CW}_A$ and $P_A$. *Duke Math. J.*, 84(3):599–631, 1996.

A. Chermak. Fusion systems and localities. *Acta Math.*, 211(1):47–139, 2013.

N. Castellana and A. Libman. Wreath products and representations of $p$-local finite groups. *Adv. Math.*, 221(4):1302–1344, 2009.

D. A. Craven. Control of fusion and solubility in fusion systems. *J. Algebra*, 323(9):2429–2448, 2010.

E. Dror-Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

A. Díaz, A. Glesser, S. Park, and R. Stancu. Tate’s and Yoshida’s theorems on control of transfer for fusion systems. *J. Lond. Math. Soc. (2)*, 84(2):475–494, 2011.

A. Díaz, A. Ruiz, and A. Viruel. All $p$-local finite groups of rank two for odd prime $p$. *Trans. Amer. Math. Soc.*, 359(4):1725–1764 (electronic), 2007.

W. G. Dwyer. The centralizer decomposition of $BG$. In *Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994)*, volume 136 of *Progr. Math.*, pages 167–184. Birkhäuser, Basel, 1996.

W. G. Dwyer and A. Zabrodsky. Maps between classifying spaces. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 106–119. Springer, Berlin, 1987.

R. J. Flores and R. M. Foote. The cellular structure of the classifying spaces of finite groups. *Israel J. Math.*, 184:129–156, 2011.

R. J. Flores. Nullification and cellularization of classifying spaces of finite groups. *Trans. Amer. Math. Soc.*, 359(4):1791–1816 (electronic), 2007.

R. J. Flores and J. Scherer. Cellularization of classifying spaces and fusion properties of finite groups. *J. Lond. Math. Soc. (2)*, 76(1):41–56, 2007.

S. Jackowski, J. McClure, and B. Oliver. Homotopy classification of self-maps of $BG$ via $G$-actions. II. *Ann. of Math. (2)*, 135(2):227–270, 1992.

S. Jackowski, J. McClure, and B. Oliver. Maps between classifying spaces revisited. In *The Čech centennial (Boston, MA, 1993)*, volume 181 of *Contemp. Math.*, pages 263–298. Amer. Math. Soc., Providence, RI, 1995.

H. Miller. The Sullivan conjecture on maps from classifying spaces. *Ann. of Math. (2)*, 120(1):39–87, 1984.

D. Notbohm. Kernels of maps between classifying spaces. *Israel J. Math.*, 87(1-3):243–256, 1994.
[]Oli13  Bob Oliver. Existence and uniqueness of linking systems: Chermak’s proof via obstruction theory. *Acta Math.*, 211(1):141–175, 2013.

[]OV07  B. Oliver and J. Ventura. Extensions of linking systems with $p$-group kernel. *Math. Ann.*, 338(4):983–1043, 2007.

[]Pui06  L. Puig. Frobenius categories. *J. Algebra*, 303(1):309–357, 2006.

[]Rag06  Kári Ragnarsson. Classifying spectra of saturated fusion systems. *Algebr. Geom. Topol.*, 6:195–252, 2006.

[]Woj87  Z. Wojtkowiak. On maps from $\varinjlim F$ to $\mathbf{Z}$. In *Algebraic topology, Barcelona, 1986*, volume 1298 of *Lecture Notes in Math.*, pages 227–236. Springer, Berlin, 1987.

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