Using the method of boundary states with perturbations to solve physically nonlinear problems of the theory of elasticity

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Abstract. The study looks upon the process of physically nonlinear deformation of isotropic and transversely isotropic homogeneous continuous solid bodies made from fibre composites where the reinforcing elements are far more rigid than the binder. The study offers an approach to writing out an explicit solution to a problem that effectively links a small parameter to the method of boundary states. Equations of the medium are presented as power series of small parameters. Each decomposition step calls for a solution of a linear elasticity problem, which is adequately addressed by the method of boundary states. Below are the results of solving test problems featuring an isotropic cube and a transversely isotropic cylinder with homogeneous boundary conditions. In both cases high accuracy is achieved as early on as the third iteration. Also presented is an axisymmetry problem for a transntrropic cylinder with inhomogeneous boundary conditions. In this case accuracy depends on the values of small parameters. For all of the problems described we provide a detailed accuracy analysis and draw conclusions as to convergence.

1. Introduction

Recent advances in the solution of mechanics problems give one a fairly realistic idea of the distribution of stresses within a solid body and enable the consideration of materials with more complex structure and rheological parameters, including anisotropic materials whose elastic properties are not fully symmetrical. When testing bodies made of such materials for single-axis tensioning or uniaxial shear, one also discovers that the resultant stress-deformation curve is non-linear. This is why trying to solve such problems by using equations typically applied to linearly elastic media would be inappropriate.

The solution of physically nonlinear problems is reduced to nonlinear differential equations that cannot realistically be solved by analysis other than in the simplest cases. This explains the wide spread of approximation methods applied to physically nonlinear elastostatics problems and related plasticity problems. These methods are based on the linearisation of differential equations and are reduced to the solution of problems of the theory of elasticity. Such methods include the elastic solutions method (proposed by A. Ilyushin [1]); the variable elasticity parameters method; the method where the solution of nonlinear problems is reduced to the solution of a series of linear problems of the theory of elasticity for inhomogeneous bodies; and the incremental stressing method (step-by-step
Physically nonlinear problems are addressed by quite an extensive body of research. Study [1] sets out the basic principles of the theory of plasticity of anisotropic materials and offers fairly new models of continuous media. Study [2] describes the process of elastoplastic deformation of transversely isotropic composites with cavities. Study [3] regards physical nonlinearity coupled with a material's inhomogeneity and solves a homogeneous and an inhomogeneous problem featuring a thick-wall cylinder in an axially symmetrical setting. Study [4] addresses a geometrically and physically nonlinear problem featuring a bending three-layer plate with a soft anisotropic filler. Study [5] offers a solution to a problem where plates come in contact with a nonlinear medium. Study [6] builds resolving equations for a plane-strain deformation theory of plasticity, which are described through mathematical models where stress-strain relations take the form of random cross-relationships between the invariants of stress and deformation tensors. Study [7] looks for a fundamental solution to a nonlinear problem.

Multiple studies address the evolution of methods of solving nonlinear problems. E.g., study [8] obtains, in a physically and geometrically nonlinear setting, integral representations of regular solutions to two-dimensional boundary value problems, which underpin the evolution of the boundary element method. In study [9], the perturbation method is used for the solution of eigenvalue problems arising in nonlinear fracture mechanics. The study solves a problem where a stress-strain state is determined in the vicinity of a crack apex, and another one addressing fatigue crack propagation. Study [10] deals with an averaging method used in physically nonlinear problems featuring layered plates in equilibrium and offers a solution to a problem featuring a layered plate bending under a dynamic strain. Study [11] compares different methods of integration of nonlinear differential equations and offers an accuracy analysis. Boundary value problems in mechanics with mass forces for a transversely isotropic cylinder were reviewed in studies [12] and [13]. In study [14] the method of boundary states is used for the solution of problems featuring bending and twisting anisotropic infinite bodies with irregular cross-section shapes.

In study [15] the small parameter method is used in an explicit solution of the first basic two-dimensional problem of the theory of elasticity for a weakly orthotropic material.

This study proposes an approach to the writing out of an explicit approximation for a problem featuring physically nonlinear isotropic and transversely isotropic media where the nonlinear deformation curve is little different from the linear one. The analytical form of the formulae makes it possible to promptly arrive to ready solutions of nonlinear problems featuring bodies of the given shape.

2. Reducing the model of a physically nonlinear isotropic medium by the perturbation method

The object of the research is a physically nonlinear medium from a solid isotropic material. The physically nonlinear theory, like the theory of plasticity, operates the following terms [16]: stress intensity

$$\sigma_i = \frac{1}{\sqrt{2}} \sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)};$$

and strain intensity

$$\varepsilon_i = \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + 3(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)}.$$

Below are the same values expressed via principal stresses:

$$\sigma_i = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}; \quad (2.1)$$
The relationship between stress intensity and strain intensity is shown in figure 1, where curve 1 represents a linear relationship and curve 2 represents a nonlinear relationship.

![Figure 1. Relationship between stress intensity and strain intensity.](image)

In figure 1 $E_0$ is the elastic modulus, and $E_c$ is the secant modulus, where

$$E_c = \frac{\sigma_i}{\varepsilon_i}. \tag{2.3}$$

Introduce small parameter $\beta$ that characterises the deviation of the secant modulus from the elastic modulus:

$$E_c = E_0 (1 - \beta). \tag{2.4}$$

Assume that the material has a nonlinear stress-strain diagram described by the equation

$$\sigma_i = A \varepsilon_i - B \varepsilon_i^k, \tag{2.5}$$

where $A$, $B$, and $k$ are experimentally defined constants of the material. From (2.4) it follows that

$$\beta = 1 - \frac{E_c}{E_0}. \tag{2.6}$$

By inserting relationships (2.3) and (2.5) in the last equation, we arrive at

$$\beta = 1 - \frac{A}{E_0} - \frac{B}{E_0} \varepsilon_i^{-k+1}. \tag{2.7}$$

The relationship between stress intensity and strain intensity is not stress-state dependent. From here it follows that the relationship $\sigma_i = f(\varepsilon_i)$ remains the same whatever stress-strain combinations we may use and can be defined by any relevant test, e.g., a uniaxial tensile test, where the principal stresses and strains are: $\sigma_1 = \sigma; \sigma_2 = \sigma_3 = 0; \varepsilon_1 = \varepsilon; \varepsilon_2 = \varepsilon_3 = -\nu \varepsilon$, where $\nu$ is the Poisson ratio. By inserting these values into (2.1) and (2.2), we arrive at

$$\sigma_i = \sigma; \varepsilon_i = \frac{2(1 + \nu)}{3} \varepsilon. \tag{2.7}$$

Having the relationship $\sigma \sim \varepsilon$ for uniaxial tensioning, we can use the formulae (2.7) we can postulate the relationship $\sigma_i \sim \varepsilon_i$. The maximal strain $\varepsilon$ is known from the uniaxial tensile test. By inserting it in the right-hand formula (2.7), we arrive at $\varepsilon_i$, and then use (2.6) to calculate the small parameter $\beta$.

The state of the isotropic medium is governed by Hooke’s law [16]:

$$\varepsilon_i = \frac{\sqrt{2}}{3} (\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2. \tag{2.2}$$
\[ \sigma_x = \lambda \theta + 2 \mu \varepsilon_x; \quad \sigma_y = 2 \mu \varepsilon_y; \]
\[ \sigma_z = \lambda \theta + 2 \mu \varepsilon_z; \quad \sigma_{xy} = 2 \mu \varepsilon_{xy}; \]
\[ \sigma_{xz} = \lambda \theta + 2 \mu \varepsilon_{xz}; \quad \sigma_{yz} = 2 \mu \varepsilon_{yz}, \]

where \( \lambda \) and \( \mu \) are Lamé parameters; \( \theta \) is the volume strain; and \( \theta = \varepsilon_x + \varepsilon_y + \varepsilon_z \).

If we substitute the secant modulus (2.4) for Young’s modulus, Hooke’s law will be as follows:
\[ \sigma_{xx} = \lambda \theta + 2 \mu \varepsilon_{xx} + 2 / 3 \mu \theta \beta - 2 \mu \varepsilon_{xx} \beta; \]
\[ \sigma_{yy} = \lambda \theta + 2 \mu \varepsilon_{yy} + 2 / 3 \mu \theta \beta - 2 \mu \varepsilon_{yy} \beta; \]
\[ \sigma_{zz} = \lambda \theta + 2 \mu \varepsilon_{zz} + 2 / 3 \mu \theta \beta - 2 \mu \varepsilon_{zz} \beta; \]
\[ \tau_{xy} = 2 \mu \varepsilon_{xy} - 2 \mu \varepsilon_{yx} \beta; \quad \tau_{yz} = 2 \mu \varepsilon_{yz} - 2 \mu \varepsilon_{zy} \beta; \quad \tau_{xz} = 2 \mu \varepsilon_{xz} - 2 \mu \varepsilon_{zx} \beta, \]

\[ \mu = \frac{E_0}{2(1 + \nu)}; \quad \lambda = \frac{E_\nu}{(1 + \nu)(1 - 2\nu)}. \]

Such an assignment allows us to describe real behaviour of a physically nonlinear medium through constants of a certain elastic medium and small parameter \( \beta \), whose zero value corresponds to a linear isotropic medium.

Introduce asymptotic series:
\[ u_j = \sum_{n=0}^{\infty} \beta^n u_j^{(n)}; \quad \varepsilon_{ij} = \sum_{n=0}^{\infty} \beta^n \varepsilon_{ij}^{(n)}; \quad \theta = \sum_{n=0}^{\infty} \beta^n \theta^{(n)}; \quad \sigma_{ij} = \sum_{n=0}^{\infty} \beta^n \sigma_{ij}^{(n)}. \]

The upper indices, which are equal to the exponents of the small parameter, serve to identify the numbers of elements in the asymptotic series.

After replacement of the summation and postulation variables with zero values for any formally non-existent element of expansion whose indices are negative \( (n < 0) \), Hooke’s law (2.8) leads to the following consequence:
\[ \sigma_{xx}^{(n)} = \lambda \theta^{(n)} + 2 \mu \varepsilon_{xx}^{(n)} + \tilde{\sigma}_{xx}^{(n)}; \quad \tilde{\sigma}_{xx}^{(n)} = 2 / 3 \mu \theta^{(n-1)} - 2 \mu \varepsilon_{xx}^{(n-1)}; \]
\[ \sigma_{yy}^{(n)} = \lambda \theta^{(n)} + 2 \mu \varepsilon_{yy}^{(n)} + \tilde{\sigma}_{yy}^{(n)}; \quad \tilde{\sigma}_{yy}^{(n)} = 2 / 3 \mu \theta^{(n-1)} - 2 \mu \varepsilon_{yy}^{(n-1)}; \]
\[ \sigma_{zz}^{(n)} = \lambda \theta^{(n)} + 2 \mu \varepsilon_{zz}^{(n)} + \tilde{\sigma}_{zz}^{(n)}; \quad \tilde{\sigma}_{zz}^{(n)} = 2 / 3 \mu \theta^{(n-1)} - 2 \mu \varepsilon_{zz}^{(n-1)}; \]
\[ \sigma_{xy}^{(n)} = 2 \mu \varepsilon_{xy}^{(n)} + \tilde{\sigma}_{xy}^{(n)}; \quad \tilde{\sigma}_{xy}^{(n)} = -2 \mu \varepsilon_{xy}^{(n-1)}; \]
\[ \sigma_{yz}^{(n)} = 2 \mu \varepsilon_{yz}^{(n)} + \tilde{\sigma}_{yz}^{(n)}; \quad \tilde{\sigma}_{yz}^{(n)} = -2 \mu \varepsilon_{yz}^{(n-1)}; \]
\[ \sigma_{xz}^{(n)} = 2 \mu \varepsilon_{xz}^{(n)} + \tilde{\sigma}_{xz}^{(n)}; \quad \tilde{\sigma}_{xz}^{(n)} = -2 \mu \varepsilon_{xz}^{(n-1)}. \]

After a change of notation (tensorial and indicial notation combined)
\[ s_{ij}^{(n)} = \sigma_{ij}^{(n)} - \tilde{\sigma}_{ij}^{(n)}, \]
we present the elements of the expansion in the form conventional for the generalized Hooke’s law for isotropic bodies:
\[ \varepsilon_{ij}^{(n)} = \lambda \theta^{(n)} + 2 \mu \varepsilon_{ij}^{(n)}. \]

The Cauchy formula is put in the same notation:
\[ \varepsilon_{ij}^{(n)} = \frac{1}{2} \left( u_{i,j}^{(n)} + u_{j,i}^{(n)} \right). \]
Using $X_i^0$ for volume forces and assuming as known the series $X_i^n = \sum_{k=0}^{\infty} \beta^n X_i^{0(n)}$, we rewrite the equilibrium equations in the following form:

$$\delta_{ij}^{(s)} + X_i^{(s)} = 0; \quad X_i^{(s)} = X_i^{0(n)} + \bar{\sigma}_{ij}^{(s)}. \quad (2.13)$$

Formulae (2.11) – (2.13) in their form correspond to the state of strain of a linear isotropic elastic body.

3. Reducing the model of a physically nonlinear transversely isotropic medium

The intensity of the tangent stresses (Huber-Mises stresses [17]) is

$$\tau_i = \frac{\sigma_i}{\sqrt{3}}. \quad (3.1)$$

The intensity of the shear strain is

$$\gamma_i = \sqrt{3} \varepsilon_i. \quad (3.2)$$

Consider the strain process in a isotropy plane $xy$ of a transversely isotropic body (where axis $z$ is perpendicular to the isotropy planes).

The relationship between tangent stress intensity and shear strain intensity is shown in figure 2, where curve 1 represents a linear relationship and curve 2 represents a nonlinear relationship.

In figure 2 $\sigma$ is the shear modulus for an isotropy plane, and $G_c$ is the secant shear modulus for the same plane, where

$$G_c = \frac{\tau_i}{\gamma_i}. \quad (3.3)$$

Introduce small parameter $\beta$ that characterises the deviation of the secant shear modulus from the shear modulus:

$$G_c = G(1 - \beta). \quad (3.4)$$

Assume that the material has a nonlinear pure shear diagram described by the equation

$$\tau_i = A \gamma_i - B \gamma_i^k, \quad (3.5)$$

where $A, B, \text{ and } k$ are constants of the material defined by a shear test in an isotropy plane. From (3.2) it follows that
\[ \beta = 1 - \frac{G}{G}. \]  

After inserting (3.3) and (3.6) in the relationship (3.5), we arrive at

\[ \beta = 1 - \frac{A}{G} - \frac{B}{G} \gamma_i^{l-1}. \]  

The relationship between tangent stress intensity and strain intensity is not stress-state dependent. From here it follows that the relationship \( \tau_i = f(\gamma_i) \), remains the same whatever stress-strain combinations we may use and can be defined by any relevant test, e.g., a pure shear test. Knowing the stress and strain values from a simple test, we can use formulae (2.1), (2.2), (3.1), and (3.2) to arrive at the relationship \( \tau_i \sim \gamma_i \) and calculate the small parameter using formula (3.7).

Similarly, we can introduce small parameter \( \alpha \) for planes that are perpendicular to isotropy planes:

\[ G^\varepsilon = G_z (1 - \alpha); \quad \alpha = 1 - \frac{C}{G_z} \left( \frac{D}{G_z} \gamma_i^{h-1} \right), \]  

where \( C, B, \) and \( h \) are constants of the material defined by a shear test on the plane perpendicular to the isotropy plane, while \( G^\varepsilon \) and \( G_z \) are the secant shear modulus and shear modulus in the same plane, respectively.

The state of the medium in the simplified theory of plasticity is governed by the generalized Hooke's law [18]:

\[
\begin{align*}
\sigma_{xx} &= (\lambda_2 + \lambda_4) \theta + \lambda_3 e_x + \lambda_4 (1 - \pi(p))(e_{xx} - e_{yy}); \\
\sigma_{yy} &= (\lambda_2 + \lambda_4) \theta + \lambda_3 e_y + \lambda_4 (1 - \pi(p))(e_{yy} - e_{zz}); \\
\sigma_{zz} &= \lambda_1 \theta + \lambda_1 e_z; \\
\sigma_{xy} &= 2\lambda_4 (1 - \pi(p)) e_{xy}; \\
\sigma_{xz} &= 2\lambda_5 (1 - \chi(q)) e_{xz}; \\
\sigma_{yz} &= 2\lambda_5 (1 - \chi(q)) e_{yz},
\end{align*}
\]

where \( \pi(p) \) and \( \chi(q) \) are Ilyushin plasticity functions that equal zero in an elastic zone, and \( \lambda_i \) is the parameters of the transversely isotropic medium that are linked to technical constants through the following expressions:

\[
\begin{align*}
\lambda_1 &= E_z (1 - \nu) / l; \\
\lambda_2 &= E(\nu + k \nu^2) / l[(1 + \nu)l]; \\
\lambda_3 &= E \nu / l; \\
\lambda_4 &= G = E / [2(1 + \nu)]; \\
\lambda_5 &= G_z; \\
l &= 1 - \nu - 2\nu^2 k; \\
k &= E / E_z; \quad \text{here, } E_z \text{ and } E \text{ are the moduli of elasticity for } z\text{-direction and for the isotropic plane, respectively; } \nu \text{ is the Poisson ratio that characterizes the compression along } r \text{ under tension along } z; \quad \nu \text{ is the Poisson ratio that characterises transverse compression in an isotropic plane under tension in the same planes; } G \text{ and } G_z \text{ are he shear modulus in the isotropy planes and the planes perpendicular to them, respectively.}
\end{align*}
\]

If we substitute the secant moduli (3.4) and (3.8) for shear moduli and give discrete values \( \beta \) and \( \alpha \) to functions \( \pi(p) \) and \( \chi(q) \), respectively, Hooke's law will be as follows:

\[
\begin{align*}
\sigma_{xx} &= [\lambda_2 + 2\lambda_4 (1 - \beta)] e_{xx} + \lambda_3 e_x + \lambda_5 e_{xz}; \\
\sigma_{yy} &= [\lambda_2 + 2\lambda_4 (1 - \beta)] e_{yy} + \lambda_3 e_y + \lambda_5 e_{yz}; \\
\sigma_{zz} &= \lambda_1 \theta + \lambda_1 e_z; \\
\sigma_{xy} &= 2\lambda_4 (1 - \beta) e_{xy}; \\
\sigma_{xz} &= 2\lambda_5 (1 - \alpha) e_{xz}; \\
\sigma_{yz} &= 2\lambda_5 (1 - \alpha) e_{yz}.
\end{align*}
\]

Such an assignment allows us to describe real behaviour of a physically nonlinear transversely isotropic medium through constants of a certain elastic medium and small parameters \( \beta \) and \( \alpha \), whose zero values correspond to a linear media.
We proceed to introduce asymptotic series (2.9). Acting by analogy with the "isotropic" case, we can reduce Hooke's law (3.10) to the following form:

\[
\begin{align*}
\sigma_{xx}^{(n)} &= \lambda_3 \theta^{(n)} + 2 \lambda_4 \varepsilon_{xx}^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)} + \sigma_{xx}^{(n)}; \\
\sigma_{yy}^{(n)} &= \lambda_3 \theta^{(n)} + 2 \lambda_4 \varepsilon_{yy}^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)} + \sigma_{yy}^{(n)}; \\
\sigma_{zz}^{(n)} &= \lambda_3 \theta^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)} + \sigma_{zz}^{(n)}; \\
\sigma_{xy}^{(n)} &= 2 \lambda_4 \varepsilon_{xy}^{(n)} + \sigma_{xy}^{(n)}; \\
\sigma_{yz}^{(n)} &= 2 \lambda_4 \varepsilon_{yz}^{(n)} + \sigma_{yz}^{(n)}; \\
\sigma_{xz}^{(n)} &= 2 \lambda_4 \varepsilon_{xz}^{(n)} + \sigma_{xz}^{(n)}; \\
\sigma_{zx}^{(n)} &= 2 \lambda_4 \varepsilon_{zx}^{(n)} + \sigma_{zx}^{(n)}. \\
\end{align*}
\]

After a change of notation (2.10), we arrive at a generalized Hooke's law for a transversely isotropic body expressed through constants \( \lambda_3 \):

\[
\begin{align*}
s_{xx}^{(n)} &= \lambda_3 \theta^{(n)} + 2 \lambda_4 \varepsilon_{xx}^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)}; \\
s_{yy}^{(n)} &= \lambda_3 \theta^{(n)} + 2 \lambda_4 \varepsilon_{yy}^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)}; \\
s_{zz}^{(n)} &= \lambda_3 \theta^{(n)} + \lambda_3 \varepsilon_{zz}^{(n)} + \sigma_{zz}^{(n)}; \\
s_{xy}^{(n)} &= 2 \lambda_4 \varepsilon_{xy}^{(n)} + \sigma_{xy}^{(n)}; \\
s_{yz}^{(n)} &= 2 \lambda_4 \varepsilon_{yz}^{(n)} + \sigma_{yz}^{(n)}; \\
s_{xz}^{(n)} &= 2 \lambda_4 \varepsilon_{xz}^{(n)} + \sigma_{xz}^{(n)}; \\
s_{zx}^{(n)} &= 2 \lambda_4 \varepsilon_{zx}^{(n)}. \\
\end{align*}
\] (3.11)

Formulae (2.12), (2.13), and (3.11) in their form correspond to the state of strain of a linear transversely isotropic elastic body.

4. Method of boundary states with perturbations for physically nonlinear media

Any internal state of a linear isotropic elastostatic medium is a set of displacements, strains, and stresses \( \xi = (u_i, \varepsilon_{ij}, \sigma_{ij}) \in \Xi \). Their traces on boundary \( \partial V \) of region \( V \) with an outer unit normal \( n_j \) contain information about the displacements and forces at work along boundary \( \gamma = \{u_i, p_j\} \in \Gamma \), \( p_j = \sigma_{ij} n_j \), and correspond to a boundary state. The spaces of the possible internal and boundary states are Hilbertian and isomorphic [19]: \( \Xi = \{\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}\} \leftrightarrow \Gamma = \{\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(n)}\} \).

Any well-defined problem is reduced to an infinite system of linear algebraic equations

\[
Q \xi = \eta.
\] (4.1)

relative to the vector of Fourier coefficients \( \xi \), the unknown state being expanded into a series with an orthonormal basis

\[
\xi = \sum_i c_i \xi^{(i)}.
\] (4.2)

Matrix \( Q \) is determined structurally only by the type of boundary conditions (BCs) and numerically through an orthonormal basis. Matrix \( Q \) in the first and second principal problems is a unit matrix. The vector of the right-hand members includes the information about the specific BCs.

An infinite system of equations (ISE) (4.1) is formulated at each step in accordance with the BC of the iteration. In practice, it is enough to factor in the BCs only when \( n = 0 \) and in all the following iterations solve only the principal problem with \( Q = E \), with the right-hand part adjusted for the occurrence of fictitious volume forces that in general are nonpotential but polynomial. A general method for searching for internal states for a class of such forces is known [20].
There are certain steps to be taken before the iterations: form the bases of spaces $\Xi$ and $\Gamma$ drawing from the general solution and functions that are harmonic in $V \cup \partial V$; form isomorphic orthonormal bases; and set terms $\chi_1^{(n)}$ of expansion series for $\chi_1^0$. Since the primitive basis is independent from small parameters, the orthonormal basis is formed only once and then used in each following iteration.

At step $n = 0$ we search for state $\xi^{(0)}$ determined by volume forces $\chi_1^{(n)}$; the real BCs are adjusted for this state to arrive at the ISE $Qe^{(0)} = q$; its solution and linear combination (4.2) produce internal state $\xi^{(0)}$; this value added to that of the state caused by volume forces prepares an initial approximation for $\xi$: $\xi = \xi^{(0)} + \xi^{(0)}$. Based on the previous (2.10) formulae, the tensor $\tilde{\sigma}_{\xi}^{(0)}$ is set.

At $n > 0$: form tensor $s_{ij}^{(n)} = \sigma_{ij}^{(n)} - \tilde{\sigma}_{ij}^{(n)}$ and vector $\chi_2^{(n)}$; search for state $\xi^{(n)}$ determined by volume forces $\chi_1^{(n)}$ in line with (2.13); adjust the BCs accordingly and solve the first principal problem for the set of equations (2.11) – (2.13) in the isotropic scenario and (2.12), (2.13), and (3.11) in the transversely isotropic scenario; use the result to add to the state caused by fictitious volume forces and adjust the stress fields in line with (2.10) $\sigma_{ij}^{(n)} = s_{ij}^{(n)} + \tilde{\sigma}_{ij}^{(n)}$ and so make the adjustment in the resultant state being accumulated with indices $\beta^*$ and $\beta^*, \alpha^*$, for the isotropic and the transversely isotropic scenario, respectively.

After performing a sufficient number of approximations it is necessary to make a final substitution of small parameter values and then reverse to dimensional values.

5. Solving a problem for an isotropic body

The method of boundary states with perturbations (MBSP) is herein tested on a fairly simple first principal problem featuring a cube. After a non-dimensionalisation similar to the one described in [21], the body occupies the region $V = \{(x, y, z) | -1 \leq x, y, z \leq 1\}$. The technical constants of the hypothetical isotropic material (2.4) – (2.6) are: $E_0 = 3$; $\mu = 0.5$; $A = 3$; $B = 2$; $k = 2$; $\varepsilon_i = 0.1$. Small parameter (2.6) $\beta = 1/15$.

Lateral faces $S_1$ and $S_2$ are subjected to forces (figure 3):

$$\{p_x, p_y, p_z\} = \begin{cases} (1, 0, 0), (x, y, z) \in S_1 \\ (-1, 0, 0), (x, y, z) \in S_2 \end{cases}.$$ 

There are no volume forces at work: $\chi_1^0 = 0$.

![Figure 3. Boundary conditions for the test problems.](image)

The MBSP applied allows to analyze an isotropic medium with a dimensionless Young's modulus of $E_0 = 3$ and Poisson ratio of $\nu = 0.5$.

Solution (2.9) involves the series:
The technical constants of the material are:

\[ \sigma_x = 1; \sigma_y = \sigma_z = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0. \]

After substitution of the small parameter at \( n = 3 \) the strains (calculated through the Cauchy formula [16]) equal:

\[ \varepsilon_x = 0.35713; \varepsilon_y = \varepsilon_z = -0.17856; \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \]

To estimate the error, compare the strains of the resultant state to those of the elastic state, where secant modulus (2.4) \( E_0 = E_z = 2.8 \) is used as elasticity modulus. In the last scenario,

\[ \varepsilon_x = 0.357143; \varepsilon_y = \varepsilon_z = -0.17857; \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \]

The error values for the strains are \( \varepsilon_x - 0.0036\%; \varepsilon_y \) and \( \varepsilon_z - 0.0056\% \). Three iterations are therefore enough for attaining high accuracy.

Let us research the convergence of the resultant series in case of a significant increase of the small parameter. Let us now assume that \( A = 3; B = 8; k = 2; \varepsilon_0 = 0.2 \); and \( \beta = 0.53333 \). Comparison is now to be made with the state at \( E_0 = E_z = 1.4 \). In this scenario the following is true:

\[ \varepsilon_x = 0.714286; \varepsilon_y = \varepsilon_z = -0.357143; \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \]

After substitution of the small parameter in series (5.1), the strains are:

at \( n = 3 \):

\[ \varepsilon_x = 0.65649; \varepsilon_y = \varepsilon_z = -0.32824; \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0; \]

at \( n = 16 \):

\[ \varepsilon_x = 0.71427; \varepsilon_y = \varepsilon_z = -0.357135; \varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = 0. \]

In the last scenario, the errors are: \( \varepsilon_x - 0.22\%; \varepsilon_y \) and \( \varepsilon_z - 0.22\% \). Greater accuracy is achievable by further iterations.

The material considered earlier was incompressible (\( \nu_0 = 0.5 \)). In the event of using a material whose Poisson ratio is other than 0.5, the calculations are less accurate. E.g., for \( E_0 = 3; \beta = 0.066666; n = 3 \), the accuracy of strain values at different Poisson ratios is presented in table 1.

| \( \nu_0 \) | 0.4 | 0.3 | 0.2 | 0.1 | 0.05 |
|------------|-----|-----|-----|-----|------|
| \( \varepsilon_x \) | 0.44 % | 0.89 % | 1.33 % | 1.78 % | 2 % |
| \( \varepsilon_y, \varepsilon_z \) | 1.1% | 2.96 % | 6.66 % | 17.77 % | 39.98 % |

It should be noted that the error already occurs at the first iteration, and so a greater \( n \) does not lead to better accuracy.

6. Solving a problem for a transversely isotropic body

1. Below is a test problem featuring a cubic body (in a Cartesian coordinate system). The body occupies the region \( V = \{(x, y, z) \mid -1 \leq x, y, z \leq 1\} \). The technical constants of the material are:

\( E = 1.3992; E_z = 2.6682; \nu = 0.0682; \nu_z = 0.248; G = 0.6549; G_z = 0.5396; A = 0.5; B = 1.2; \)
C = 0.4 ;  D = 1.1 ;  k = 2 ;  \varepsilon_i = 0.1 . Small parameters (3.7) and (3.8) are: \beta = 0.053339 ; \alpha = 0.054855 .

The lateral faces of the body are subjected to unit forces that cause an omniradial tensioning and a uniaxial shear. There are no volume forces at work: X^0_i = 0 .

Below are formulae for strains and stresses at n = 3:

\[ e_{xx} = e_{yy} = 0.573 + 0.49984 \beta + 0.43601 \beta^2 + 0.38033 \beta^3 ; \]
\[ e_{xx} = 0.18889 - 0.13952 \beta - 0.1217 \beta^2 - 0.10616 \beta^3 ; \]
\[ e_{yz} = e_{xz} = 0.92661 \sum_{n=1}^{3} \alpha^n ; e_{xy} = 0.76346 \sum_{n=1}^{3} \beta^n ; \]

\[ \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{xy} = 1 . \] (6.1)

After substitution of small parameters, the strains are:

\[ e_{x} = e_{y} = 0.600964 ; e_{z} = 0.181087 ; e_{yz} = e_{xz} = 0.980383 ; e_{xy} = 0.806445 . \]

To estimate the error, compare the strains of the resultant state to those of the elastic state with the following constants:

\[ E_r = \frac{4\lambda_4(\beta - 1)(\lambda_1 - \lambda_4(\lambda_2 + \lambda_4 - \beta \lambda_4)))}{\lambda_1^2 - \lambda_1 \lambda_2 + 2(\beta - 1)\lambda_4 \lambda_4} = 1.32899 ; E_r = \lambda_4 \frac{\lambda_3^2}{\lambda_2 - \lambda_1 \lambda_4 (1 - \beta)} = 2.65922 ; \]
\[ v_r = \frac{\lambda_3^2 - \lambda_3 \lambda_2}{\lambda_1^2 - \lambda_1 \lambda_2 + 2(\beta - 1)\lambda_4} = 0.07176 ; v_r = \frac{\lambda_3}{2(\lambda_2 + \lambda_4 (1 - \beta))} = 0.25922 ; \]
\[ G_r = \lambda_3 (1 - \alpha) = 0.51 ; G_r = \frac{E_r}{2(1 + \nu_{r,\theta})} = 0.62 . \] (6.2)

In the last scenario, \[ e_{x} = e_{y} = 0.600967 ; e_{z} = 0.181086 ; e_{yz} = e_{xz} = 0.980392 ; e_{xy} = 0.806452 . \]

The error values for the strains are: \[ e_{xx} \text{ and } e_{yy} = 0.00047 \% ; e_{z} = 0.00043\% , e_{yz} \text{ and } e_{xz} = 0.0009 \% ; e_{xy} = 0.00081\% . \]

Let us research the convergence of the resultant series in case of a significant increase of the small parameters. Let us now assume that \[ A = C = 0.3 ; B = 0.2 ; D = 0.1 ; k = 2 ; \alpha = 0.1 , \beta = 0.511401 \text{ and } \alpha = 0.4255 . \] Comparison is now to be made with the state for a material where \[ E = 0.723381 , E_r = 2.52729 , v = 0.13028 ; v_r = 0.424089 ; G_r = 0.31 ; G = 0.32 . \] In this scenario the following is true: \[ e_{xx} = e_{yy} = 1.03449 ; e_{z} = 0.60072 ; e_{yz} = e_{xz} = 1.6129 ; e_{xy} = 1.5625 . \]

After substitution of the small parameters in series (6.1), the strains are:

at \[ n = 3 : e_{xx} = e_{yy} = 0.99352 ; e_{z} = 0.060072 ; e_{yz} = e_{xz} = 1.56 ; e_{xy} = 1.45563 ; \]
at \[ n = 14 : e_{xx} = e_{yy} = 1.03449 ; e_{z} = 0.060074 ; e_{yz} = e_{xz} = 1.6129 ; e_{xy} = 1.56243 . \]

I.e., greater accuracy is achievable by further iterations.

2. Below is a test axially symmetrical problem featuring a cylinder (in a cylindrical coordinate system). The body occupies the region \[ V = \{(r, z) \mid 0 \leq r \leq 1 , -2 \leq z \leq 2 \} . \] The technical constants of the material are: \[ E = E_r = 1.3992 ; E_r = 2.6862 ; v = v_r = 0.0682 ; v_r = 0.248 ; G_r = G = 0.6549 ; G_r = 0.5396 ; A = 0.5 ; B = 0.2 ; C = 0.4 ; D = 0.1 ; k = 2 ; \varepsilon_i = 0.1 . \] The small parameters are: \[ \beta = 0.206026 ; \alpha = 0.240178 . \]
Lateral faces $S_1$ and $S_2$ of the body are subjected to forces:

$$\{p_r, p_z\} = \begin{cases} 
\{1, 0\}, & r = 1, -2 \leq z \leq 2; \\
\{0, 1\}, & z = 2, 0 \leq r \leq 1; \\
\{0, 1\}, & z = 2, 0 \leq r \leq 1.
\end{cases} \quad X_0 = 0.$$

Solution (13) of the third order ($n = 3$) involves the following series:

$$u = 0.573 r + 0.49984 r\beta + 0.43601 r\beta^2 + 0.38033 r\beta^3;$$

$$w = 0.18889 z - 0.13952 z\beta - 0.1217 z\beta^2 - 0.10616 z\beta^3;$$

$$\sigma_r = \sigma_\theta = \sigma_z = 1; \quad \sigma_r = \sigma_\theta = \sigma_{z\theta} = 0.$$ 

After substitution of the small parameter, the strains are:

$$\varepsilon_r = \varepsilon_\theta = 0.697819; \quad \varepsilon_z = 0.154051; \quad \varepsilon_{r\theta} = \varepsilon_{r\theta} = 0.$$ 

To estimate the error, use the same method as before, provided that comparison is now to be made with the state with the following technical constants (6.2): $E_r = 1.12778; \quad E_\theta = 2.62834; \quad \nu_r = 0.084403; \quad \nu_\theta = 0.297818; \quad G_r = 0.14; \quad G_\theta = 0.52.$

In the last scenario, $\varepsilon_r = \varepsilon_\theta = 0.698547; \quad \varepsilon_z = 0.153848; \quad \varepsilon_{r\theta} = \varepsilon_{r\theta} = 0.$ The error values for the strains are: $\varepsilon_r$ and $\varepsilon_\theta - 0.1 \%; \quad \varepsilon_z - 0.13\%.$

3. Below is an axially symmetrical calculation problem featuring a cylinder. Consider the first principal problem for a transversely isotropic cylinder assuming the same geometrical and physical medium parameters (figure 4).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Meridional section of a body of revolution and boundary conditions.}
\end{figure}

We propose a fifth-order solution of the problem. To get around the awkwardness of the formulas involved, we will provide the components of the displacement vector in a truncated form, focusing on the first terms at $\beta$ up to $n = 3$:

$$u_r \approx 0.6715 r - 0.00494 r^3 + \ldots + (0.5875 r - 0.00452 r^3 + \ldots) \beta +$$

$$+ (0.51095 r - 0.00415 r^3 + \ldots) \beta^2 + (0.4457 r - 0.00383 r^3 + \ldots) \beta^3 + \ldots;$$

$$u_z \approx -0.25894 z + 0.13193 z r^2 + \ldots + (-0.1635 z + 0.0024 z r^2 + \ldots) \beta +$$

$$+ (-0.1426 z + 0.0021 z r^2 + \ldots) \beta^2 + (-0.1244 z + 0.00183 z r^3 + \ldots) \beta^3 + \ldots.$$ 

We will assess the accuracy through the strains on the body's boundaries and present the result in a graphical form (figure 5). The dotted line reflects the accurate-solution strain, and the solid line
reflects the approximate solution strain. The strains on the graphs are scaled, i.e., the true value of $\varepsilon$, in figure 5 is equal to the graph value times $\kappa$.

As seen from the charts, approximate-solution strains are consistent with accurate-solution strains within a certain accuracy range ($\pm 10\%$ of exact value at any point on the body’s boundary). It should be noted that the error already occurs at the third iteration, and so a greater $n$ does not lead to better accuracy.

Figure 5. Verification of the solution of the calculation task.

Figure 6 provides a comparison of isolines of the components of the strain tensors of the cylinder made from a linear-elastic material and a nonlinear-elastic material with deviation parameters $\beta = 0.206026$ and $\alpha = 0.240178$. The top of figure 6 show strain isolines for a linear material, and the bottom show strain isolines for a nonlinear material.
Figure 6. Contours: a) deformation $\varepsilon_r$; b) deformation $\varepsilon_z$; c) deformation $\varepsilon_\theta$.

As seen from the figure 6, the two strain states present no difference in kind, as in effect the comparison is between two elastic states with similar BCs.

The foregoing analysis leads to the conclusion that MBSP proves to be an effective means of explicit full solution of physically nonlinear problems for isotropic and transversely isotropic media. To solve a physically nonlinear problem, one has to first solve a corresponding linear-elastic problem. That said, the accuracy of approximate solutions in the case of non-trivial boundary value problems heavily depends on the values of small parameters causing the nonlinear medium to deviate from the linear one.

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References
[1] Khaligitov A A., Khudazarov R S and Sagdullaeva D A. 2015 *Theories of plasticity and thermoplasticity of anisotropic bodies* (Tashkent: «Science and technology») p 320
[2] Polatov A M 2017 Modeling the effect of holes on the stress concentration in fibrous composites *Mathematics, its applications and mathematical education (MPMO17)* pp 303–308
[3] Andreev V I, Polyakova L S 2015 Analytical solution of a physically nonlinear problem for an inhomogeneous thick-walled cylindrical shell *Bulletin of MGSU* 11 pp 38–45
[4] Badriev I B, Makarov M V and Paimushin V N 2017 Numerical study of the physically
nonlinear problem of the longitudinal bending of a three-layer plate with a transversally soft aggregate. Bulletin of Perm National Research Polytechnic University. Mechanics 1 pp 39–51. DOI: 10.15593/perm.mech/2017.1.03

[5] Ivanov S P, Ivanov O G and Akhmetshin M N 2012 Application of solving a physically spatial nonlinear problem to the calculation of plates in contact with various media Journal of Structural Mechanics of Engineering Structures and Structures 4 pp 22–28

[6] Bakushe S V 2018 Resolving equations of plane strain in cylindrical coordinates for a physically nonlinear continuous medium Journal of Structural Mechanics of Engineering Structures and Structures 14 1 pp 38–45. DOI: 10.22363/1815-5235-2018-14-1-38-45

[7] Bochkarev A O 2005 Kelvin's nonlinear problem Vestnik SPbGU Ep. 10 Vol 2 pp 128–139

[8] Bochkarev A O 2004 Boundary integrals in a geometrically and physically nonlinear plane problem Vestnik SPbGU Ep. 10 3 pp 13–21

[9] Stepanova L V 2011 On methods for solving eigenvalue problems arising in nonlinear fracture mechanics Bulletin of the Nizhny Novgorod University. N.I. Lobachevsky 4(4) pp 1786–1788

[10] Sheshenin S V and Savenkova M I 2012 Averaging of nonlinear problems in the mechanics of composites (Moscow: Vestn. Mosk. un.-that. ser. 1, mathematics. Mechanics) 5 pp 58–62

[11] Kozlov M V, Sheshenin S V 2013 Comparative analysis of methods for integrating equations of the nonlinear theory of elasticity Vestn. Mosk. un.-that. ser. 1, mathematics. Mechanics 4 pp 61–65

[12] Ivanychev D A 2019 Solution of the contact problem of the theory of elasticity for anisotropic bodies of revolution with mass forces Bulletin of Perm National Research Polytechnic University. Mechanics 2 pp 49–62. DOI: 10.15593/perm.mech/2019.2.05.

[13] Ivanychев D A 2019 The boundary state method in solving the second main problem of the theory of anisotropic elasticity with mass forces Bulletin of the Tomsk state. un.-that. Mathematics and Mechanics 61 pp 45–60. DOI: 10.17223/19988621/61/5

[14] Ivanychev D A, Levina E Yu, Abdullah L S and Glazkova Yu A 2019 The method of boundary states in problems of torsion of anisotropic cylinders of finite length International Transaction Journal of Engineering, Management, & Applied Sciences & Technologies 10 2 pp 183-191. DOI: 10.14456/ITJEMAST.2019.18.

[15] Penkov V B, Ivanychev D A, Novikova O S and Levina L V 2018 An algorithm for full parametric solution of problems on the statics of orthotropic plates by the method of boundary states with perturbations IOP Conf. Series: Journal of Physics: Conf. Series 973 012015

[16] Lurie A I 1980 Nonlinear Theory of Elasticity (Moscow: Nauka) p 512

[17] Ishlinsky A Yu and Ivlev D D 2001 Mathematical Theory of Plasticity (Moscow: PHYSMATLITIS) p 704

[18] Pobedra B E 1984 Mechanics of Composite Materials (Moscow: Publishing House Mosk. university) p 336

[19] Penkov V B and Penkov V V 2001 The method of boundary states for solving problems of linear mechanics Far Eastern Mathematical Journal 2 pp 115–137

[20] Penkov V B, Kuzmenko V I, Kuzmenko V N and Levina L V 2019 A method for solving problems of the isotropic theory of elasticity with volume forces in a polynomial representation Applied Mathematics and Mechanics 83 1 pp 84–94

[21] Levina L V, Novikova O S and Penkov V B 2016 A full-parameter solution to the problem of the theory of elasticity of a simply connected bounded body Bulletin of Lipetsk State Technical University 2(28) pp 16–24