Lanczos Algorithm for 2DPCA

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Abstract. The traditional PCA algorithm (Principle Component Analysis) can obtain the feature space of face image and realize face recognition by expanding the face image matrix into vectors in face recognition. 2DPCA (Two-dimensional Principle Component Analysis) doesn’t need to spread the image matrix into one-dimensional vectors. The covariance matrix is constructed directly by using two-dimensional image matrix, so that the calculation of eigenvalue and eigenvector is simplified. We apply the Lanczos algorithm to 2DPCA in this paper for Indian faces. The numerical experiments show that our method runs much faster and gets better recognition rates than the traditional 2DPCA algorithm.

Keywords. Lanczos algorithm; 2DPCA; face recognition.

1. Introduction

Since the 21st century, with the development of computer hardware, the research of face recognition has made great progress. Compared with other human features, such as infrared, palmprint and fingerprint, the face recognition has more direct and convenient features. Therefore, the face recognition has been widely used in public security, information security and commercial fields. Among them, PCA is widely used in face recognition because of its high efficiency in feature extraction and data description. However, PCA algorithm needs to transform the matrix produced by face image into one-dimensional vector when processing face image, which brings about the problem of large computation quantity and small sample matrix singularity. Therefore, in reference [1-6], the authors proposed two DPCA methods based directly on 2 face image matrices, which effectively solved the singular problem of matrix.

When the traditional 2DPCA reduces the dimension of two-dimensional data, the singular value or eigenvalue decomposition algorithm is generally used, because the singular value decomposition and eigenvalue decomposition algorithm need the cubic operation, when the image pixel is high, the 2DPCA method based on singular value decomposition or eigenvalue decomposition algorithm cannot meet the real-time requirement of the system. To solve this problem, in this paper, we apply the Lanczos algorithm to the dimensionality reduction of 2DPCA and compare it with the traditional 2DPCA algorithm. The new algorithm will greatly improve the operation efficiency under the premise of the recognition rate of the guarantor’s face.

This paper first illustrates the traditional algorithm of 2DPCA, then the Lanczos algorithm is introduced and applied in the data dimensionality reduction of 2DPCA. Finally, the numerical experiments prove the efficiency and validity of our method.

2. 2DPCA Algorithm Brief Introduction

In this section, we introduce DPCA 2 algorithm. 2DPCA is a common algorithm in linear model parameter estimation. The basic idea is to normalize the input image matrix, find out its covariance
matrix, and find out the eigenvalues and eigenvectors of the covariance matrix by eigenvalue
decomposition algorithm. Then the eigenvectors corresponding to the larger eigenvalues are selected to
form the eigen subspace. Finally, we can obtain the features of the image by projecting the original
image matrix into the feature subspace, which contains the main information needed for recognition.

Next, we briefly review DPCA 2 algorithm. First, we assume that the
training sample set we chose is

\[ T = \{X_1, X_2, \ldots, X_N\} \]

where \( N \) is the number of training samples, So, for all the training \( \bar{X} \) mean
value of the sample is \( X_i \in \mathbb{R}^{m \times n} \)

\[
\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i
\]  

(1)

Similar to the PCA, we can construct the covariance matrix \( G \):

\[
G = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})^T
\]  

(2)

Therefore, in 2 D-PCA, we need to decompose the covariance matrix \( G \) as eigenvalues and select \( k \)
of the most eigenvector corresponding to a large eigenvalue and forms a projection subspace matrix.

\[ U_k \in \mathbb{R}^{m \times k}, U_k = \{u_1, u_2, \ldots, u_k\} \]

For any given image sample matrix \( X_i \), set

\[
Y_i = (X_i - \bar{X})U
\]  

(3)

So, \( Y_i \) is the original image matrix \( X_i \) feature matrix obtained by projection \( U \) subspace. If projected
into the subspace matrix \( U \), then we can get the corresponding feature matrix of each sample, and then
we can identify and classify the test samples as long as we choose a reasonable classification method.

Now, we can give 2 DPCA algorithms.

**2DPCA algorithm:**

Input: \( N \) two-dimensional matrix of training samples \( X_i \in \mathbb{R}^{m \times n} \) \((i = 1, 2, \ldots, N)\).

Output: projection Space Matrix \( U_k \in \mathbb{R}^{m \times k} \).

(1) Calculate the mean matrix \( X \) of training samples:

\[
\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i
\]

(2) Calculate the covariance matrix \( G \):

\[
G = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})^T
\]

(3) Eigenvalue decomposition of \( G \):

\[ G = U \Sigma U^T \]

The eigenvectors corresponding to the first \( k \) maximum eigenvalues were selected to form the
projection space matrix \( U_k : U_k = \{u_1, u_2, \ldots, u_k\} \).

3. **LANCZOS Algorithm in 2 DPCA Application**

We obtain DPCA projection subspace matrix \( U_k \) by eigenvalue decomposition of the covariance matrix
\( G \) in the 2DPCA algorithm. However, eigenvalue decomposition requires the cubic operations. When
the dimension of the matrix has higher dimension, it is difficult to achieve real-time requirements.
Lanczos algorithm, as a projection subspace method, has been widely used in eigenvalue decomposition,
least square problem and matrix dimension reduction [7-10].
Firstly, we briefly introduce the Lanczos algorithm. For a given symmetric matrix \( G \in \mathbb{R}^{m \times m} \) and the initial unitary vector \( u_1 \), Lanczos algorithm constructs a Krylov subspace composed of orthogonal bases [8]:

\[
\kappa_k = \text{range}(u_1, Gu_1, G^2u_1, \ldots, G^{k-1}u_1)
\]

(4)

Secondly, we give the tri-diagonal process of the Lanczos algorithm. For a given symmetric matrix \( G \in \mathbb{R}^{m \times m} \), initial unitary vector \( u_1 \) and \( i = 1, 2, \ldots, k \), we calculate:

\[
\alpha_i = u_i^T Gu_i, \quad \alpha_i = u_i^T Gu_i, \quad \beta_{i+1} u_{i+1} = Gu_i - \alpha_i u_i - \beta_i u_{i-1}
\]

(5)

Here \( u_i \) is Lanczos vector \( (i = 1, 2, \ldots, k) \), and \( \beta_{i+1} \) should met the Constraints \( \langle u_i, u_{i+1} \rangle = 0 \) and \( \| u_i \| = 1 \). For exact calculations, Lanczos vectors \( u_{i+1} \) and \( u_i \) \((i = 1, 2, \ldots, k)\) are completely orthogonal. Hence Lanczos vector \( u_i \) constitutes Krylov subspace \( \kappa_k(G, u_1) \), a set of orthogonal bases.

In equation (5), the matrix may be expressed as:

\[
GU_k = U_{k+1}T_{k+1}(\cdot; 1: k)
\]

(6)

Here, \( T_{k+1} \in \mathbb{R}^{(k+1) \times (k+1)} \) is a triangular. \( U_{k+1} \) is an orthogonal matrix. So,

\[
T_{k+1} = \begin{bmatrix}
\alpha_1 & \beta_2 & & \\
\beta_2 & \alpha_2 & \ddots & \\
& \ddots & \ddots & \beta_{k+1} \\
& & \beta_{k+1} & \alpha_k
\end{bmatrix}, \quad U_{k+1} = [u_1, u_2, \ldots, u_{k+1}].
\]

among which, \( T_{k+1}(\cdot; 1: k) \) is obtained by removing the matrix obtained from the last column of \( T_{k+1} \).

Therefore, if we need to get a few maximum eigenvalues \( G \) of the symmetric matrix, we only need to do eigenvalue decomposition \( T_k \) the symmetric triangular matrix. We can treat the \( T_k \) eigenvalues as approximate eigenvalues \( G \) the matrix and the \( T_k \) features. The vector is combined with the Lanczos vector to form the approximate eigenvector \( G \) the matrix. From the above analysis, we can apply Lanczos algorithm to 2 DPCA.

**Lanczos-2DPCA algorithm:**

Input: two-dimensional matrix \( X \) of \( N \) training samples \( X_i \in \mathbb{R}^{m \times n} \) \((i = 1, 2, \ldots, N)\), initial unitary vector \( u_1 \), and the number of iteration steps \( k \).

Output: projection Space Matrix \( U_k \in \mathbb{R}^{m \times k} \).

1. Calculate the mean matrix \( \bar{X} \) of training samples \( \bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i \).

2. Calculate the covariance matrix \( G \): \( G = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})(X_i - \bar{X})^T \).

3. Initialize \( \alpha_i = u_i^T Gu_i, \quad \beta_i = 0 \).

For \( i = 1, 2, \ldots, k \),
Calculate \( w_i, \alpha_i \) satisfy, \( w_i = Gu_i - \beta_i u_{i-1}, \alpha_i = \langle w_i, u_i \rangle \);

Calculate \( w_i, \beta_{i+1} \) satisfy, \( w_i = w_i - \alpha_i u_i, \beta_{i+1} = \|w_i\| \);

if \( \beta_{i+1} = 0 \) then Stop;
end if

Calculate \( u_{i+1} \), satisfy \( u_{i+1} = \frac{w_i}{\beta_{i+1}} \);

End for

(4) \( T = \text{tridiag} \{\alpha_1, \beta_2, \beta_3, \alpha_4, \beta_5, \ldots, \beta_k, \alpha_k\} \), Calculate \( U_T, S_T \), satisfy \([U_T, S_T, U_T] = \text{svd}(T)\);

(5) Calculate the Projection Subspace Matrix \( U_k \), \( U_k = UU_T \).

From the above algorithm, we can set Lanczos-2DPCA time complexity of the algorithm as \( O(2Nmn + mnk + k^3 + mk^2) \). However, the time complexity of the traditional 2 DPCA algorithm is \( O(2Nmn + m^3) \). In biometric recognition, we usually take dimension \( k \ll m \). So, the time complexity of Lanczos-2DPCA algorithm is much lower than that of traditional DPCA.

Orthogonal:
Lanczos vector often loses the orthogonality after multi-step iterations due to the influence of rounding error in Lanczos algorithm, which affects its convergence and the accuracy [8]. Therefore, we must reorthogonalize the Lanczos vector; since the values \( k \) in 2DPCA are usually small, so we take the completely re-orthogonal approach here.

In Step 3 of Lanczos-2DPCA algorithm, for \( u_{i+1} = \frac{w_i}{\beta_{i+1}} \), we need to add one step:

\[
 u_{i+1} = u_{i+1} - \sum_{j=1}^{i} \langle u_{i+1}, u_j \rangle u_j
\]

(7)

Classifier selection:
The projection subspace matrix can be obtained by calculation in 2 DPCA and Lanczos-2DPCA. So the training samples \( X_i \) principal element \( Y_i \) can be expressed as:

\[
 Y_i = (X_j - \bar{X})U_k
\]

(8)

The testing samples \( X_j \) principal element \( Y_j \) can be expressed as:

\[
 Y_j = (X_j - \bar{X})U_k
\]

(9)

Therefore, we can calculate \( Y \) Euclidean distance between the principal element of the training sample \( Y_i \) and test sample master \( Y_j \):

\[
 D(Y_i, Y_j) = \|Y_i - Y_j\|
\]

(10)

Using the nearest proximity principle, we can realize the recognition and classification of faces.
4. Numerical Experiments
In order to verify the efficiency of our algorithm, we will give a specific numerical example in this section. In Matlab 2012, all experiments were run on a computer with Intel (R) CPU@2.30GHz, 12GB memory.

In this example, we consider the INDIAN face database as shown in figure 1 (which is derived from [11]), which has a pixel of 480*640 for face images. The database contains 59 human faces, including 22 women and 37 men, each with 11 images. In this case, we choose the ten front faces as experimental data, the first five are used as training sets, and the remaining five are used as test sets to verify the efficiency of the two algorithms mentioned in this paper.

We show the running rate and recognition rate of DPCA 2 algorithm and the Lanczos-2DPCA algorithm in different rank cases in figure 2. On the left of figure 2, we can see that compared with DPCA 2 algorithm, the Lanczos-2DPCA algorithm given in this paper is very well in time efficiency. On the right one of figure 2, we can see that the recognition rate of DPCA algorithm is higher when the rank is low, but when the rank is greater than 6, the recognition rate of Lanczos-2DPCA is higher than that of the traditional DPCA algorithm. From the image, we can see that the recognition rate of Lanczos-2DPCA increases with the that of rank DPCA2 and the algorithm should be more stable.

Figure 1. Indian face database.

Figure 2. INDIAN: (left) run time VS rank; (right) recognition rate VS rank.
5. Conclusion
In this paper, for the feature that the traditional 2 DPCA is inefficient in time, we propose the usage of the Lanczos algorithm in 2DPCA data dimensionality reduction. Although Lanczos can only get approximate solutions in theory, from the experimental results, the new algorithm is much higher than that of traditional 2 DPCA algorithm in time efficiency, and the recognition rate increases with the increase of rank. It is also more stable than the traditional 2 DPCA algorithm.

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