Wave Propagation in Nontrivial Backgrounds

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It is well known that waves propagating in a nontrivial medium develop “tails”. However, the exact form of the late-time tail has so far been determined only for a narrow class of models. We present a systematic analysis of the tail phenomenon for waves propagating under the influence of a general scattering potential \(V(x)\). It is shown that, generically, the late-time tail is determined by spatial derivatives of the potential. The central role played by derivatives of the scattering potential appears not to be widely recognized. The analytical results are confirmed by numerical calculations.

One of the most remarkable features of wave dynamics in curved spacetimes is the development of “tails”. Gravitational waves (or other fields) propagate not only along light cones, but also spread inside them. This implies that at late times waves do not cut off sharply but rather die off in tails. In particular, it is well established that the late-time evolution of massless fields propagating in black-hole spacetimes is dominated by an inverse power-law behaviour.

Price \([1]\) was the first to provide a detailed analysis of the mechanism by which the spacetime outside of a (nearly spherical) collapsing star divests itself of all radiative multipole moments, and leaves behind a Schwarzschild black hole. The evolution of such waves (gravitational, electromagnetic, and scalar) in a curved spacetime is governed by the Klein-Gordon (KG) equation

\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{l(l+1)}{x^2} + \frac{1}{x_s^2}V(x/x_s) \right] \Psi = 0 ,
\]

(1)

where the term \(l(l+1)/x^2\) is the well-known centrifugal barrier \(l\) is the multipole order of the field, and \(V(x)\) is an effective curvature potential (we henceforth take \(x_s = 1\) without loss of generality). It was demonstrated that all radiative perturbations decay asymptotically as an inverse power of time, the power indices equal 2\(l + 3\). Physically, these inverse power-law tails are associated with the backscattering of waves off the effective curvature potential, \(V(x)\), at asymptotically far regions.

The analysis of Price has been extended by many authors. Bičák \([2]\) generalized the results to the case of an electrically neutral scalar field propagating in a charged Reissner-Nordström spacetime. Leaver \([3]\) demonstrated that the late-time tail can be associated with the existence of a branch cut in the Green’s function for the wave propagation problem.

Gundlach, Price, and Pullin \([4]\) showed that power-law tails are a genuine feature of gravitational collapse – the existence of these tails was demonstrated in full non-linear numerical simulations of the spherically symmetric collapse of a self-gravitating scalar field (this was later reproduced in \([5]\)). Moreover, since the late-time tail is a direct consequence of the scattering of the waves at asymptotically far regions, it has been suggested that power-law tails would develop independently of the existence of an horizon. This implies that tails should appear in perturbations of stars as well. Moreover, the KG wave equation has a much wider range of physical applicability. For instance, electromagnetic waves propagating in an optical cavity can be described by this equation as well \([6,7]\). A nontrivial dielectric constant distribution, \(n^2(x)\), provides a nontrivial medium, and is analogous to the presence of a scattering potential \(V(x)\).

For other related works, see e.g., \([8,9]\), and references therein.

The dynamics of waves in rotating black-hole spacetimes has received much attention recently \([10,11]\). It has been demonstrated that power-law tails present in the Kerr spacetime as well (However, the damping indices are generally different from those found in the Schwarzschild case.)

Yet, a thorough understanding of the fascinating phenomenon of wave tails is far from being complete. In particular, the exact form of the late-time tail has so far been determined only for a narrow class of models. In fact, most of previous analyses are restricted to “logarithmic potentials” of the form \(V(x) \sim \ln^\beta x/x^\alpha\) (where \(\alpha > 2\) and \(\beta = 0, 1\) are parameters). In a brilliant work, Ching et. al. \([10]\) have shown that, generically, the late-time behaviour of waves propagating under the influence of these specific potentials has the form \(\Psi \sim \ln^\beta t/t^{2l+\alpha}\) (with the well-known exception of a “pure” power-law decay in the Schwarzschild spacetime).

The purpose of this paper is to present a systematic analysis of the tail phenomenon with a general scattering potential \(V(x)\). Ching et. al. \([10]\) provided a heuristic picture of the scattering problem which is an important first step in this direction: Consider a wave from a source point \(y\). The late-time tail observed at a fixed spatial location, \(x\), is a consequence of the wave first propagating to a distant point \(x' >> y, x\), being scattered by \(V(x')\), and then returning to \(x\) at a time \(t \simeq (x' - y) + (x' - x) \simeq 2x'\). Hence, according to this scenario, the scattering amplitude (and thus the late-time tail itself) are expected to be proportional to \(V(x') \simeq V(t/2)\).

However, it is well-known, at least in the specific case of
logarithmic potentials, that this simple picture requires two modifications. First, there is an extra suppression of the late-time tail by a factor of $t^{-2l}$. Second, if $\alpha$ is an odd integer less than $2l + 3$, the leading term in the late-time tail vanishes \([4]\), and one should consider subleading terms (The well-known Schwarzschild spacetime belongs to this case.)

There are several interesting and important open questions regarding the tail phenomenon in a general analysis. What determines the late-time tail – is it simply the asymptotic form of the scattering potential itself, as suggested by the heuristic picture? How generic is the suppression of waves with $l \geq 1$? Is it always by a factor of $t^{-2l}$? What is the (most) general class of scattering potentials for which the leading term in the late-time tail vanishes? (The Schwarzschild spacetime is only one specific example to these potentials.) What is the general form of the late-time behaviour in these cases? These questions, and several others call for a study of the general properties of wave tails. In this paper we present our main results for this fascinating phenomena.

We consider the evolution of a wave field whose dynamics is governed by a KG-type equation $\Phi''_U + V(x)\Phi = 0$. Resolving the field into spherical harmonics $\Phi = \sum_l m \Psi_m(t, r)Y_l^m(\theta, \phi)/r$ (where $r$ being the circumsferential radius), one obtains a wave equation of the form Eq. (1) for each multipole moment \([32]\) (For brevity we henceforth suppress the indices $l, m$ on $\Psi$.)

It proofs useful to introduce the double-null coordinates $u \equiv t - x$ and $v \equiv t + x$, which are a retarded time coordinate and an advanced time coordinate, respectively. The initial data is in the form of some compact outgoing pulse in the range $u_0 \leq u \leq u_1$, specified on an ingoing null surface $v = v_0$.

The general solution to the wave-equation \([3]\) can be written as a series depending on two arbitrary functions $F$ and $G$ \([4]\)

$$\Psi = \sum_{k=0}^{l} A_k x^{-k} \left[ G^{(l-k)}(u) + (-1)^k F^{(l-k)}(v) \right] + \sum_{k=0}^{\infty} B_k(x) \left[ G^{(l-k-1)}(u) + (-1)^k F^{(l-k-1)}(v) \right], \quad (2)$$

where $A_k = (l+k)!/2^k k!(l-k)!$. For any function $H$, $H^{(k)}$ is its $k$th derivative; negative-order derivatives are to be interpreted as integrals. The functions $B_k(x)$ satisfy the recursion relation

$$B_k' = \frac{1}{2} \left[ B_{k-1}' - l(l+1)x^{-2}B_{k-1} - V(A_kx^{-k} + B_{k-1}) \right], \quad (3)$$

for $k \geq 1$, where $B' = dB/dx$, and $B_0' = -V(x)/2$.

For the first Born approximation to be valid the scattering potential $V(x)$ should approach zero faster than $1/x^2$ as $x \to \infty$, see e.g., \([10,16]\). Otherwise, the scattering potential cannot be neglected at asymptotically far regions [see Eq. \(3\) below]. It is useful to classify the scattering potentials into three groups, according to their asymptotic behaviour:

- Group I: $|V'|$ approaches zero slower than $|V|/x$, and faster than $|V|$ as $x \to \infty$.
- Group II: $|V'|$ approaches zero at the same rate as $|V|$ as $x \to \infty$.
- Group III: $|V'|$ approaches zero at the same rate as $|V|/x$ as $x \to \infty$.

Group I. — For this case the recursion relation, Eq. \([3]\), yields $B_k(x) = -2^{-(k+1)}V^{(k-1)}(x)$.

The first stage of the evolution is the scattering of the field in the region $u_0 \leq u \leq u_1$. The first sum in Eq. \([3]\) represents the primary waves in the wave front (i.e., the zeroth-order solution, with $V \equiv 0$), while the second sum represents backscattered waves. The interpretation of these integral terms as backscatter comes from the fact that they depend on data spread out over a section of the past light cone, while outgoing waves depend only on data at a fixed $x$.

After the passage of the primary waves there is no outgoing radiation for $u > u_1$, aside from backscattered waves. This means that $G(u_1) = 0$. Hence, for a large $x$ at $u = u_1$, the dominant term in Eq. \([3]\) is

$$\Psi(u = u_1, x) = B_l(x)G^{(-1)}(u_1). \quad (4)$$

This is the dominant backscatter of the primary waves.

With this specification of characteristic data on $u = u_1$, we shall next consider the asymptotic evolution of the field. We confine our attention to the region $u > u_1$, $x \gg x_s$. To a first Born approximation, the spacetime in this region is approximated as flat \([33]\). Thus, to first order in $V$ (that is, in a first Born approximation) the solution for $\Psi$ can be written as

$$\Psi = \sum_{k=0}^{l} A_k x^{-k} \left[ g^{(l-k)}(u) + (-1)^k f^{(l-k)}(v) \right], \quad (5)$$

Comparing Eq. \([3]\) with the initial data on $u = u_1$, Eq. \([4]\), one finds $f(v) = -2^{-1}V^{(-1)}(v/2)G^{(-1)}(u_1)$.

For late times $t \gg x$ one can expand $g(u) = \sum_{n=0}^{\infty} (-1)^n g^{(n)}(t) x^n/n!$ and similarly for $f(v)$. With these expansions, Eq. \([4]\) can be rewritten as

$$\Psi = \sum_{n=-l}^{\infty} K^n_l x^n \left[ f^{(l+n)}(t) + (-1)^n g^{(l+n)}(t) \right], \quad (6)$$

where the coefficients $K^n_l$ are those given in \([33]\). They vanish for $-l \leq n \leq l$ if $l - n$ is odd.
Using the boundary conditions for small \( r \) (regularity as \( x \to -\infty \), at the horizon of a black hole, or at \( x = 0 \), for a stellar model), one finds that at late times \( g(t) = (-1)^{l+1}f(t) \) to first order in the scattering potential \( V \) (see e.g., [3] for additional details). That is, the incoming and outgoing parts of the tail are equal in magnitude at late-times. This almost total reflection of the ingoing waves at small \( r \) can easily be understood on physical grounds — it simply manifests the impenetrability of the barrier to low-frequency waves [3] (which are the ones to dominate the late-time evolution [3]).

We therefore find that the late-time behaviour of the field at a fixed radius (\( x \ll t \)) is dominated by [see Eq. (3)]

\[
\Psi \simeq 2K_{l}^{l+1}f^{(2l+1)}(t)x^{l+1},
\]

which implies

\[
\Psi \simeq \Psi_{0}G^{(-1)}(u_{l})x^{l+1}V^{(2l)}(t/2),
\]

where \( \Psi_{0} = -2^{-(2l+1)}K_{l}^{l+1} \). Hence, the late-time tail is determined by the \( 2l \)th derivative of the scattering potential.

The analysis for Groups II and III proceeds along the same lines. That is, one should first solve Eq. (3) for \( B_{k}(x) \to f(v) \) using Eq. (3) \( \to \) which finally yields \( \Psi(t \to \infty) \) through Eq. (3). In the following we present the main results for groups I and II.

**Group II.** — The dominant backscatter of the primary waves is \( \Psi(u = u_{l}, x) = \sum_{k=l}^{\infty}B_{k}(x)G^{(l-k-1)}(u_{l}) \), where the \( B_{k} \) are the same as for group I. Using an analysis along the same lines as before, one finds

\[
\Psi \simeq -\sum_{n=\alpha l+1,l+3,...}^{\infty}2^{-n}K_{l}^{n}x^{n}\sum_{k=l}^{\infty}2^{-k}G^{(l-k-1)}(u_{l})V^{(n+k-1)}(t/2),
\]

at late-times. Note that Eq. (3) is merely a generalization of Eq. (3), and reduces to it if \( |V'| \) approaches zero faster than \( |V| \) (in which case \( V^{(2l)} \) dominates at late-times).

**Group III.** — Let \( V(x) \equiv W(x)/x^{\alpha} \), where \( \alpha > 2 \), and \( W(x)/x^{\gamma} \to 0 \) as \( x \to \infty \) for any \( \gamma > 0 \). The general solution for \( B_{k}(x) \) is now given by \( B_{k}(x) \simeq \sum_{n=0}^{\infty}b_{n}(k, \alpha)W^{(n)}(x)x^{\alpha+k-n-1} \), where the \( b_{n}(k, \alpha) \) are complicated numerical coefficients, which are determined by the recursion relation Eq. (3). The analysis now proceeds along the same lines as before; one finds that the late-time behaviour of the wave is given by

\[
\Psi \simeq G^{(-1)}(u_{l})x^{l+1}\sum_{n=0}^{\infty}c_{n}(l, \alpha)W^{(n)}(t/2)x^{-(\alpha+2l-n)},
\]

**TABLE I.** Late-time behaviour for various scattering potentials.

| Group | \( V(x) \) | \( \Psi^{l}(t \to \infty) \) |
|-------|-------------|------------------|
| I     | \( e^{-x^{\beta}} \), \( 0 < \beta < 1 \) | \( e^{-t/(\alpha+2l-2)}t^{-(\alpha+2l)} \) |
| II    | \( \sin(x^{\beta})/x^{\alpha} \), \( 0 < \beta < 1 \) | \( \sin[(t/2)^{\alpha}]t^{-(\alpha+2l)} \) |
| IIIa  | \( \sin(x)/x^{\alpha} \) | \( \text{periodic function} \times t^{-\alpha} \) |
| IIIb  | \( x^{1/2}/x^{\alpha} \), \( \beta > 0 \) | \( t^{-\alpha+2l} \) |
| VIa   | \( x^{1/2}/x^{\alpha} \), \( \alpha < 2l+3 \), \( \beta > 0 \) | \( t^{-(\alpha+2l+3)} \ln t \) |

where the coefficients \( c_{n}(l, \alpha) \) are constructed from the \( b_{n}(l, \alpha) \).

Group III is divided into three subgroups according to the asymptotic behaviour of the function \( W(x) \), and the value of \( \alpha \):

Subgroup IIIa: \( |W'| \) approaches zero faster than \( |W|/x \) as \( x \to \infty \), and \( \alpha \) is not an odd integer less than \( 2l + 3 \). In this case, the dominant term in Eq. (10) is \( W(t/2) t^{-(\alpha+2l)} \). Recall now that \( V' \sim W(t^{2l+1}) \) as \( t \to \infty \), which implies

\[
\Psi \sim G^{(-1)}(u_{l})x^{l+1}V^{(2l)}(t/2),
\]

at late-times.

Subgroup IIIb: \( |W'| \) approaches zero faster than \( |W|/x \) as \( x \to \infty \), and \( \alpha \) is an odd integer less than \( 2l + 3 \). (This subgroup of scattering potentials includes the Schwarzschild spacetime as a special case.) In this case one finds that the leading term of \( B_{l}(x) \) [proportional to \( V^{(-1)}(x) \) vanishes, and sub-leading terms should therefore be considered. Hence, the late-time behaviour of the wave is dominated by

\[
\Psi \sim G^{(-1)}(u_{l})x^{l+1}V^{(2l)}_{st}(t/2),
\]

Here \( V^{(2l)}_{st} \) is the (first) \( l \)th sub-leading term in the asymptotic derivative. Namely, \( V^{(2l)}_{st} \sim W'/\ln \beta x \). [Note that the results of [10] for the specific family of logarithmic potentials (with \( W \sim \ln \beta x \)) coincide with the general expressions, Eqs. (11) and (12).]

Subgroup IIIc: \( |W'| \) approaches zero at the same rate as \( |W|/x \) as \( x \to \infty \). In this case, the late-time dynamics of the field is given by Eq. (11).

*Numerical calculations.* — It is straightforward to integrate Eq. (11) using the methods described in [13].

The late-time evolution of the field is independent of the form of the initial data used. The results presented here are for a Gaussian pulse.

Table I gives a selected list of scattering potentials, chosen as representative of the various different groups (We have studied other potentials as well, which are not shown here.)

The temporal evolutions of the waves (under the influence of the various scattering potentials) are shown in Figs. 1 and 2. We find an excellent agreement between the analytical results and the numerical calculations.
Summary and physical implications. — We have given a systematic analysis of the tail phenomena for waves propagating under the influence of a general scattering potential.

It was shown that the late-time tail is governed by spatial derivatives of the scattering potential (generically, by the $2l$th derivative). In particular, the potential function itself does not enter into the expression of the late-time tail (with the exception of the monopole case). The central role played by derivatives of the scattering potential appears not to be widely recognized. The analytical results are in excellent agreement with numerical calculation.

In addition, we have demonstrated that the (extra) suppression of waves by a factor of $t^{-2l}$ (which adds to the basic late-time decay), a phenomena well-known in black-hole spacetimes, is actually not a generic feature of the scattering problem. In particular, for scattering potentials that belong to group I the suppression of the waves is smaller, while for scattering potentials that belong to group II there is no (extra) suppression at all.

Moreover, it was shown that the familiar case of the Schwarzschild spacetime belongs to a wider group of scattering potentials, in which the leading term in the tail [proportional to $V^{(2l)}(t/2)$] vanishes (and thus, sub-leading terms dominate the late-time dynamics).

We are at present extending the analysis to include: (i) time-dependent scattering potentials, and (ii) scattering potentials that lack spherical symmetry (in which case the scattering problem is of $2 + 1$ dimensions).

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