SPATIAL CENSORED REGRESSION MODELS IN R: THE CENSSPATIAL PACKAGE

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ABSTRACT

CensSpatial is an R package for analyzing spatial censored data through linear models. It offers a set of tools for simulating, estimating, making predictions, and performing local influence diagnostics for outlier detection. The package provides four algorithms for estimation and prediction. One of them is based on the stochastic approximation of the EM (SAEM) algorithm, which allows easy and fast estimation of the parameters of linear spatial models when censoring is present. The package provides worthy measures to perform diagnostic analysis using the Hessian matrix of the completed log-likelihood function. This work is divided into two parts. The first part discusses and illustrates the utilities that the package offers for estimating and predicting spatial censored data. The second one describes the valuable tools to perform diagnostic analysis. Several examples in spatial environmental data are also provided.

1 Introduction

Spatial data can be described as observations that are spatially located in a coordinated area. Their analysis constitutes an area of statistics that is characterized by the spatial correlation among individuals where the interest is to predict the response at non-sampling sites. Usually, spatial analysis assumes that observations are fully observed, but this is not always possible. Sometimes data are subject to upper or lower (or both) detection limits, beyond which they are not quantifiable (Schelin and Sjöstedt-de Luna, 2014). An example of this can be seen in environmental (spatial) monitoring of different variables which often involves left-censored observations falling below the minimum limit of detection (LOD) of the instruments used to quantify them.

Methods for handling this type of data have been widely discussed in the statistical literature. Rathbun (2006) applied a Robbins-Monro stochastic approximation to estimate the parameters of a spatial regression model, obtain conditional simulations of left-censored observations and then obtain a predictor for data at unsampled sites by taking the weighted mean of kriging predictors via importance sampling. Schelin and Sjöstedt-de Luna (2014) proposed a semi-naive method that determines imputed values at censored locations in an iterative algorithm together with variogram estimation. They compared its predictive performance through a simulation study with Rathbun’s, Naive 1 and Naive 2 methods, which involve replacing the censored data by the LOD or the LOD/2 respectively. Militino and Ugarte (1999) developed an EM-type algorithm for maximum likelihood (ML) estimation in censored spatial data. However,
this approach suffers from several drawbacks that restrict its applicability, such as the computation of non-closed form expressions, which depend on high dimensional integrals.

The EM algorithm proposed by [Dempster et al. (1977)] is an iterative procedure for finding the maximum likelihood (ML) estimators of models that involves unobserved latent variables. In our context, an EM-type algorithm uses the observed and censored data to obtain estimates from the mean and correlation structures of a spatial model. In some cases, this algorithm cannot be carried out because the calculation of the expectations involved in E-step cannot be computed analytically. An alternative is to approximate those expectations by Monte Carlo methods to replace the traditional E-step. This procedure is the so-called Monte Carlo Markov chain (MCEM) algorithm. Even though it is possible to approximate these expectations, this can involve heavy computational effort when some generating methods are used to sample from the conditional latent variable distributions at each iteration.

An alternative to deal with this problem is the SAEM algorithm proposed by [Delignette et al. (1999)]. This algorithm replaces the E-step by a stochastic approximation obtained using simulated data, while the M-step remains unchanged. In the framework of spatial models, [Jank (2006)] showed that the computational effort of SAEM is much smaller and it reaches convergence in just a fraction of the simulation iterations when compared with the MCEM algorithm. This is due to the fact that a memory effect is contained in the SAEM method, in which the previous simulations are considered in the computation of the posterior ones.

Another important topic in spatial censored data analysis is the study of influential observations. Influence analysis is an important statistical tool because it can assess the goodness-of-fit and influential observations that can distort the parameter estimates leading in some cases to erroneous inference [Giménez and Galea (2013)]. For influential observation detection, we discuss two approaches; the first one is the case-deletion approach [Cook (1977)], an intuitively appealing method which has been applied to many statistical models. The second one is local influence diagnostics [Cook (1986)]. This technique assesses the stability of the estimation outputs with respect to the model inputs and has recently received attention in the literature on spatial models (see, e.g., [De Bastiani et al. (2014)]. [Zhu and Lee (2001)] developed an approach to perform local influence analysis for general statistical models with missing data, based on the $Q$-displacement function which is closely related to the conditional expectation of the complete-data log-likelihood in the E-step of the EM algorithm. This approach produces results very similar to those obtained from Cook’s method. Moreover, the case-deletion can be studied by the $Q$-displacement function, following the approach of [Zhu et al. (2001)]. So, we develop here methods to obtain case-deletion measures and local influence measures by using the method of [Zhu et al. (2001)] (see also [Lee and Xu (2004); Zhu and Lee (2001)]) in the context of spatial censored models.

The aim of this paper is to present the utilities of the R package CensSpatial. The rest of the paper is organized as follows. In Section 2 we derive all the theoretical framework for the estimation and prediction functions for spatial censored data implemented in the proposed package, along with the R functions for calling these methods. Also we present some useful graphical tools for interpretation as well as an application to real data. Section 3 presents the local influence diagnostic methods to detect influential observations. Graphical tools and an application are also provided in this section. Further details such as first and second derivatives of the spatial correlation, can be found in the Appendix.

## 2 Estimation and prediction utilities

First, we specify the spatial censored linear (SCL) model in a general sense and then we describe all the estimation and prediction tools implemented in the CensSpatial package.

### 2.1 Background of spatial linear models for censored response

As in [De Bastiani et al. (2014)], we will consider a Gaussian model with a linear specification for the spatial trend, which allows the inclusion of polynomial trends or generally, spatially referenced covariates. The linear model is defined as

$$Z = X\beta + \epsilon,$$

where $X$ and $\beta$ represent the trend matrix and the parameters considered for the mean structure respectively, and $\epsilon \sim N_n(0, \Sigma)$ is the stochastic component. We consider for the spatial case, $X$ is a full rank matrix and $\Sigma$ is a non-singular matrix with the form $\Sigma = \sigma^2\Psi$, with $\Psi = (\sigma^2I_n + R(\phi))$, where $I_n$ represents an $n \times n$ identity matrix, $R(\phi)$ is the correlation matrix for the error term and $\sigma^2 = \tau^2/\sigma^2$ (see Appendix). We also assume that the response $Z_i$ is not fully observed for all area $i$, so we call $(V_i, C_i)$ the observed data for the $i$th area, while $V_i$ represents either an uncensored observation ($V_i = V_{i0}$) or the LOD of censoring level ($V_i = (V_{i1}, V_{i2})$) and $C_i$ is the censoring indicator.
such that:

\[ C_i = \begin{cases} 
1 & \text{if } V_{1i} \leq Z_i \leq V_{2i}, \\
0 & \text{if } Z_i = V_{0i}. 
\end{cases} \tag{2} \]

Note that when \( C_i = 1 \), if \( Z_i \in (\infty, V_{2i}) \), then we get a left censored SCL model (Toscas, 2010), and if \( Z_i \in [V_{1i}, \infty) \), then we get a right censored SCL model. The model defined in (1)-(2) is called the spatial censored linear (SCL) model.

## 3 Likelihood and SAEM implementation

To compute the likelihood function associated with the SCL model, the observed and censored components of \( Z \) must be treated separately. Let \( Z^0 \) be the \( n^o \)-vector of observed responses and \( Z^c \) be the \( n^c \)-vector of censored observations, with \( (n = n^o + n^c) \), such that \( C_i = 0 \) for all elements in \( Z^0 \), and \( C_i = 1 \) for all elements in \( Z^c \). After reordering, \( Z \), \( V \), \( X \), and \( \Sigma \) can be partitioned as follows:

\[
Z = vec(Z^0, Z^c), \quad V = vec(V^0, V^c), \quad X^T = [X^o, X^c]^T \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma^{oo} & \Sigma^{oc} \\ \Sigma^{co} & \Sigma^{cc} \end{bmatrix},
\]

where \( vec(\cdot) \) denotes the function which stacks vectors or matrices having the same number of columns. Consequently, \( Z^0 \sim N_{n^0}(X^o \beta, \Sigma^{oo}) \), \( Z^c | Z^0 \sim N_{n^c}(\mu, S) \), where \( \mu = X^c \beta + \Sigma^{os}(\Sigma^{oo})^{-1}(Z^0 - X^o \beta) \) and \( S = \Sigma^{cc} - \Sigma^{co}(\Sigma^{oo})^{-1}\Sigma^{oc} \). Now, let \( \phi_n(u; a, A) \) be the pdf of \( N_n(a, A) \) evaluated at \( u \). From Vaida and Liu (2009) and Jacqmin-Gadda et al. (2000), the likelihood function (using conditional probability arguments) is given by

\[
L(\theta) = \phi_{n^o}(Z^0; X^o \beta, \Sigma^{oo}) P(Z^c \in V^c | Z^0), \tag{3}
\]

where

\[
V^c = \{Z^c = (Z^c_1, \ldots, Z^c_{n^c})^T | V_{11}^c \leq Z^c_1 \leq V_{21}^c, \ldots, V_{1n^c}^c \leq Z^c_{n^c} \leq V_{2n^c}^c \}
\]

and \( P(u \in A | Z^0) \) denotes the conditional probability of \( u \) being in the set \( A \) given the observed response. This function can be evaluated without much computational burden through the \texttt{mvtnorm} routine available in the namesake R package (see Genz et al., 2020). Now, we can compare models using a likelihood based criterion, such as the Akaike (AIC; Akaike, 1974) and Schwarz (BIC; Schwarz, 1978) information criteria.

### 3.1 SAEM algorithm for censored spatial data

We propose an EM-type (SAEM) algorithm by considering \( Z \) as a missing data or latent variable. In the estimation, the reading \( V \) at the censored observations (\( C_i = 1 \)) is treated as hypothetical missing data, and augmented with the observed dataset \( Z_n = (C^T, V^T, Z^T)^T \). Hence, the complete-data log-likelihood function is given by:

\[
\ell_c(\theta) \propto -\frac{1}{2} \left[ \log(|\Sigma|) + (Z - X \beta)^T \Sigma^{-1} (Z - X \beta) \right] + cte, \tag{4}
\]

with \( cte \) being a constant, independent of the parameter vector \( \theta \). Given the current estimate \( \hat{\theta}^{(k)} \), the E-step computes the conditional expectation of the complete data log-likelihood function, i.e., \( Q(\theta^{(k)}|\hat{\theta}^{(k)}) = E[\ell_c(\theta|Z_c)|V, C, \hat{\theta}^{(k)}] \). Denote the two conditional first moments for the response \( Z \) as \( \hat{Z}^{(k)} = E\{Z|V, C, \hat{\theta}^{(k)}\} \)

\[
\hat{Z}^{(k)} = \hat{A}^{(k)} = tr(\hat{Z}^{(k)} \Sigma^{-1}) - 2\hat{Z}^{(k)} \Sigma^{-1} X \beta + \beta^T X \Sigma^{-1} X \beta.
\]

\( \hat{A}^{(k)} \) is not fully observed, so the components of \( \hat{Z}^{(k)} \) and \( \hat{Z}^{(k)} \), corresponding to \( C_i = 1 \) will be estimated by the first two moments of the truncated normal distribution respectively. When \( C_i = 0 \), these components can be obtained directly from the observed values, i.e., \( \hat{Z}^{(k)} = Z^o \) and \( \hat{Z}^{(k)} = Z^o X^o \). Although these expectations exhibit closed forms (as functions of multinormal probabilities; for further details, please refer to Arismendi (2013)), the calculation
is computationally expensive requiring high-dimensional numerical integrations, resulting in convergence issues when the proportion of censored observations is non-negligible. At each iteration, the SAEM algorithm successively simulates from the conditional distribution of the latent variable, and updates the unknown parameters of the model. Thus, at iteration $k$, the SAEM algorithm proceeds as follows:

- **Step E-1 (sampling):** Sample $Z^c$ from a truncated normal distribution, denoted by $TN_{nc}(\mu, \Sigma^c)$, with $\hat{\Sigma}^c = \{Z^c = (Z^c_1, \ldots, Z^c_n)^T | V^c_{11} \leq Z^c_1 \leq V^c_{21}, \ldots, V^c_{n1} \leq Z^c_n \leq V^c_{2n} \}$, $\mu = X\beta + \Sigma^c(\Sigma^{nc})^{-1}(Z^c - X\alpha)$ and $\Sigma = \Sigma^{cc} - \Sigma^{c?}\Sigma^{nc}$. Here $TN_{nc}(\mu, \Sigma)$ denotes the $n$-variate truncated normal distribution in the interval $A$, where $A = A_1 \times \ldots \times A_n$. The new observation $Z^{(k,l)} = (Z^{(k,l)}_1, \ldots, Z^{(k,l)}_n)$, $Z^{c+1}_1, \ldots, Z^{c+1}_n$ is a sample generated for the $n^c$ censored cases and the observed values (uncensored cases), for $l = 1, \ldots, M$.

- **Step E-2 (stochastic approximation):** Since we have the sequence $Z^{(k,l)}$, at the $k$-th iteration we replace the conditional expectations $\hat{Z}^{(k)}$ and $\hat{ZZ}^{(k)}$ by the following stochastic approximations:

$$\hat{Z}^{(k)} = \hat{Z}^{(k-1)} + \delta_k \left[ \frac{1}{M} \sum_{l=1}^{M} Z^{(k,l)} - \hat{Z}^{(k-1)} \right],$$

$$\hat{ZZ}^{(k)} = \hat{ZZ}^{(k-1)} + \delta_k \left[ \frac{1}{M} \sum_{l=1}^{M} Z^{(k,l)} Z^{(k,l)^T} - \hat{ZZ}^{(k-1)} \right],$$

where $\delta_k$ is a smoothness parameter, i.e., a decreasing sequence of positive numbers as defined in Kuhn and Lavielle (2004), such that $\sum_{k=1}^\infty \delta_k = \infty$ and $\sum_{k=1}^\infty \delta_k^2 < \infty$. For the SAEM, the E-Step coincides with the MCEM algorithm at the expense of a significantly smaller number of simulations $M$ (suggested to be $M \leq 20$). For more details, see Delyon et al. (1999).

Finally, the conditional maximization (CM) step maximizes $Q(\theta | \hat{\theta}^{(k)})$ with respect to $\theta$ and obtains a new estimate $\hat{\theta}^{(k+1)}$, as follows:

**CM-Step (conditional maximization):**

$$\hat{\beta}^{(k+1)} = \left( X^T \hat{\Sigma}^{(k)} X \right)^{-1} X^T \hat{\Sigma}^{(k)} \hat{Z}^{(k)},$$

$$\hat{\sigma}^2^{(k+1)} = \frac{1}{n} \left[ tr(\hat{ZZ}^{(k)}^{-1}) - 2 \hat{Z}^{(k)} \hat{\psi}^{(k)} - 1 \right] + \hat{\beta}^{(k+1)} X^T \hat{\psi}^{(k)} X \hat{\beta}^{(k+1)},$$

$$\hat{\alpha}^{(k+1)} = \arg\max_{\alpha \in \mathbb{R}^{+x+}} \left( -\frac{1}{2} \log(|\Sigma|) - \frac{1}{2} \left[ tr(\hat{ZZ}^{(k)}^{-1}) - 2 \hat{Z}^{(k)} \hat{\Sigma}^{-1} \hat{Z}^{(k)} \right] \right) + \hat{\beta}^{(k+1)} X^T \Sigma^{-1} X \hat{\beta}^{(k+1)}.$$  

with $\alpha = (\nu^2, \phi)^T$. Note that $\hat{\sigma}^2$ can be recovered via $\hat{\sigma}^{2(k+1)} = \hat{\nu}^{2(k+1)} \hat{\phi}^{2(k+1)}$. The CM-step (7) can be easily accomplished using the optim routine in R. This process is iterated until some absolute distance between two successive evaluations of the actual log-likelihood $\ell(\theta)$, such as $|\ell(\hat{\theta}^{(k+1)}) - \ell(\hat{\theta}^{(k)})|$ or $|\ell(\hat{\theta}^{(k+1)})/\ell(\hat{\theta}^{(k)}) - 1|$, becomes small enough.

### 3.2 Prediction

Following Diggle and Ribeiro (2007), let $Z_{\text{obs}}$ denote a vector of random variables with observed realized values, and let $Z_{\text{pred}}$ denote another random variable whose realized values we would like to predict from the observed values of $Z_{\text{obs}}$. A prediction for $Z_{\text{pred}}$ could be any function of $Z_{\text{obs}}$, which we denote by $\hat{Z}_{\text{pred}} = \ell(Z_{\text{obs}})$. The mean square error (MSE) of $\hat{Z}_{\text{pred}}$ is given by $MSE(\hat{Z}_{\text{pred}}) = E((\hat{Z}_{\text{pred}} - Z_{\text{pred}})^2)$; so we can use $\hat{Z}_{\text{pred}} = E(Z_{\text{pred}} | Z_{\text{obs}})$ to find the predictor that minimizes $MSE(\hat{Z}_{\text{pred}})$. 


For a Gaussian process, we write $\mathbf{Z}_{pred} = (\mathbf{Z}_{pred}(s_1), \ldots, \mathbf{Z}_{pred}(s_n))$ for the unobserved values of the signal at the sampling locations $s_1, \ldots, s_n$. We want to predict the signal value at an arbitrary location; thus our target for prediction is $\hat{Z}(s)$.

Given that $(\mathbf{Z}_{obs}, \mathbf{Z}_{pred})$ is also a multivariate Gaussian process, we can use the results in Subsection 3.2 to minimize the $MSE(\mathbf{Z}_{pred})$, i.e., if $\mathbf{X}^{*} = (\mathbf{X}_{obs}, \mathbf{X}_{pred})$ is the $(n_{obs} + n_{pred}) \times p$ design matrix corresponding to $\mathbf{Z}^{*} = (\mathbf{Z}_{obs}^T, \mathbf{Z}_{pred}^T)$, then $\mathbf{Z}^{*} \sim N_n(n_{obs} + n_{pred}, \mathbf{X}^{*} \beta, \Sigma)$, where

$$\Sigma = \begin{bmatrix} \Sigma_{obs,obs} & \Sigma_{obs, pred} \\ \Sigma_{pred, obs} & \Sigma_{pred, pred} \end{bmatrix}$$

and $\mathbf{Z}_{pred} | \mathbf{Z}_{obs} \sim N_{n_{pred}}(\mu_p, \Sigma_p)$, where

$$\mu_p = \mathbf{X}_{pred} \beta + \Sigma_{obs, pred} \Sigma_{obs, obs}^{-1} (\mathbf{Z}_{obs} - \mathbf{X}_{obs} \beta)$$

and

$$\Sigma_p = \Sigma_{pred, pred} - \Sigma_{pred, obs} \Sigma_{obs, obs}^{-1} \Sigma_{obs, pred}.$$

Then, the predictor that minimizes the $MSE$ of prediction will be the conditional expectation in (8).

### 3.2.1 Prediction methods

Using the previous results, we will now describe the algorithm for each of the three prediction methods that are implemented in this paper.

(a) *Naive 1 and Naive 2 algorithms*

We proceed as follows:

1. Impute the censored observations using $LOD$ (called Naive 1 algorithm), or $LOD/2$ (called Naive 2 algorithm), depending on the type of censoring (left or right).
2. Compute the least squares or likelihood estimates for the mean (linear trend in our case) and covariance structure using the imputed data.
3. Evaluate the estimate $\theta$ from step 2 in expression (8) to obtain the predicted values.

(b) *Seminaive algorithm*

Here, we follow the method in Schelin and Sjöstedt-de Luna (2014). Let $C_i$ denote the indicator variable, indicating the presence ($C_i = 1$) or absence ($C_i = 0$) of a censored observation. Then we proceed as follows:

1. Set $\mathbf{Z}_0 = (\mathbf{Z}_{obs}^T, \tilde{\mathbf{v}}_0^T)$ where $\tilde{\mathbf{v}}_{0i} = 0$ if $C_i = 1$.
2. Obtain $\hat{\theta}_{0} = (\hat{\beta}, \hat{\sigma}^2, \hat{\phi})$ from $\mathbf{Z}_0$ by least squares.
3. Set $k = 0$. Let $\hat{\mathbf{Z}}^{-i}_k$ denote the data vector $\hat{\mathbf{Z}}_k$ where unit $i$ is removed.
4. Find the predictor $\hat{\mathbf{Z}}(h, \hat{\theta}_k)$ for all $i$ such that $C_i = 1$ using expression (8) and $\hat{\mathbf{Z}}^{-i}_k$, where $h$ denotes the distance between coordinates (isotropy).
5. Set $\hat{\mathbf{Z}}_{k+1} = (\mathbf{Z}_{obs}^T, \tilde{\mathbf{v}}_{k+1}^T)$ where $\tilde{\mathbf{v}}_{k+1} = (\max(0, \min(\hat{\mathbf{Z}}(h, \hat{\theta}_k), LOD)) : C_i = 1)$.
6. Update $\theta_k$ from the new data $\hat{\mathbf{Z}}_{k+1}$. 
7. For iteration $k + 1$, repeat steps 4-6 until the convergence criterion is satisfied for some constants $c_1, c_2$, and $c_3$ chosen by the user.
8. Once convergence is attained, evaluate the estimate $\hat{\theta}$ in expression (8) to obtain the predicted values.

Note that this algorithm only works for the left-censored case. The constants $c_1$, $c_2$, and $c_3$ are chosen such that the following three conditions are satisfied:

$$\left| \frac{\hat{\sigma}^2(\hat{\mathbf{Z}}_{k+1}) - \hat{\sigma}^2(\hat{\mathbf{Z}}_k)}{\sigma^2(\hat{\mathbf{Z}}_k)} \right| \leq c_1, \quad \hat{\sigma}^2(\hat{\mathbf{Z}}_{k+1}) \leq c_2 \hat{\sigma}^2(\mathbf{Z}_{obs}) \quad \text{and} \quad \hat{\xi}(\hat{\mathbf{Z}}_{k+1}) > c_3 \hat{\xi}(\mathbf{Z}_{obs})$$

where $\xi(X)$ represents the skewness of $X$. 

5
(c) **SAEM algorithm**

Here, we proceed as follows:

1. Obtain the estimates of the mean and covariance structure by SAEM procedure. For further details, see Ordonez et al. (2018).
2. Impute the censored observations by the approximate first moments \( \hat{Z}(k) \), obtained from the SAEM procedure.
3. Evaluate the above estimates in expression (8) to obtain predicted values for the unobserved locations. In this case, we use \( Z_{\text{obs}}^* \) to denote the observed data and estimate the censored observations via SAEM. This way, we can easily differentiate it from the fully observed response \( Z_{\text{obs}} \).

### 3.3 Application: Estimation and prediction tools

Now, we exemplify the estimation and prediction methods described previously and implemented in the CensSpatial package. We use the rainfall dataset from Parana state (Brazil) analyzed in Diggle and Ribeiro Jr (2002). We carried out some performance comparison for the four prediction methods by using a cross-validation study with the first 100 observations for estimation and the remaining 43 observations for comparison of the predictive power of the estimated methods.

The rainfall dataset does not contain censored data, so left censoring was generated artificially for the first 100 observations, following Schelin and Sjöstedt-de Luna (2014), i.e., we fixed the censoring level at \( \alpha \), sorting the data (in ascending order) and setting the LOD to the 100\( \alpha \)% percentile of the generated sample, i.e., the smallest \( \alpha n \) value of \( Z(s_1), \ldots, Z(s_n) \). This dataset, called paranacens25, is available in the package CensSpatial with 25% censoring level. The following instructions call the data, then observations that will be used for estimation are stored in the object `dataest` and those that will be used for prediction are stored in the object `datapred`.

```r
> data(paranacens25)
> dataest=paranacens25[1:100,]
> datapred=paranacens25[101:143,]
```

In order to specify the mean and covariance components for the SCL model, we use some descriptive graphs. Figure 1 depicts the rainfall dataset, where the left panel shows the average rainfall in each coordinate and the right panel shows the sample variogram for the observed data considering a Gaussian covariance structure. From this figure, we can conclude that it is plausible to assume a linear trend of the rainfall with respect to its coordinates. The behavior showed in the sample variogram seems to indicate that it is also plausible to consider a Gaussian covariance structure for the stochastic component. As an additional exercise, we compare Matérn covariance structures with different \( \kappa \) and the Gaussian structure. All these fits present similar results, but we choose the second one because it is simpler and has an easier interpretation.

![Rainfall dataset](image)

**Figure 1**: Rainfall dataset: (a) Average rainfall over different locations (b) Sample variogram and Gaussian correlation structure (solid line).
We fitted the left SCL model defined in (1)-(2) with the specifications previously discussed in the descriptive analysis. To carry out the fit, we used the object `dataest` defined previously and the four estimation methods available in the `CensSpatial` package: Seminaive, Naive 1 Naive 2 and SAEM algorithms. For the Seminaive algorithm, we chose the tuning parameter as $a_1 = 0.1$, $a_2 = 2$ and $a_3 = 5$ for the convergence criterion as suggested by Schelin and Sjöstedt-de Luna (2014). For the SAEM procedure, we performed a Monte Carlo simulation with a sample of size $M = 15$, a maximum number of iterations $W = 200$ and a cutoff point $c = 0.2$. We used the functions `variofit` and `likfit` from the `geoR` package to define the initial values for the SAEM algorithm. These values were used for the remaining algorithms in order to maintain the same conditions in the comparison of the four methods.

The estimation process via SAEM can be performed using the `SAEMSCL` function. Now, we detail how to call the routine and show a print summary of the results obtained through the `est` object. Arguments of the function are organized in such a way that the first line contains the parameters related to the censored observations, the second one the covariance structure, the third and fourth ones the model specification and finally the last one refers to the SAEM settings. The code below also provides a print summary with the parameter estimates, the linear trend, the covariance structure considered, some information criteria and specific details about the SAEM procedure. Finally, the output for this function is a list with all the information contained in the print summary and specific information of the estimation procedure as the estimates obtained in each iteration. The SAEM parameters for this function must not be changed unless the user knows how they work. We refer to Lachos et al. (2017) for details.

```r
> cov.ini=c(900,68); y=dataest$y1; cc=dataest$cc
> coords=dataest[,1:2]; coordspred=datapred[,1:2]

> est=SAEMSCL(cc,y,cens.type="left",trend="1st",coords=coords,
+ kappa=2, cov.model="gaussian", fix.nugget=F,nugget=170,
+ inits.sigmas=q=cov.ini[2],inits.phi=cov.ini[1], search=T,
+ lower=c(0.0001,0.0001),upper=c(50000,50000),
+ M=15,perc=0.25,MaxIter=200,pc=0.2)

-------------------------------------------------------------------------
Spatial Censored Linear regression with Normal errors (SAEM estimation)
-------------------------------------------------------------------------

*Type of trend: Linear function of its coordinates,
mu = beta0 + beta1*CoordX + beta2*CoordY

*Covariance structure: Gaussian

---------
Estimates
---------

    Estimated
bet 0  367.1364
beta 1 -0.0758
beta 2 -0.3124
sigma2  832.6448
phi  211.2341
tau2  360.1578

--------
Model selection criteria
--------

    Loglik AIC   BIC  AICcorr
Value -353.601 719.202 734.833  720.105

--------
Details
--------

Type of censoring = left
Convergence reached? = FALSE
Iterations = 200 / 200
MC sample = 15
Cut point = 0.2
```
For the Seminaive, Naive 1 and Naive 2 procedures, the estimation process can be carried out using the functions `Seminaive` and `algnaive12` (Naive 1 and Naive 2 algorithms) respectively. As for the `SAEMSCL` routine, the arguments are organized in such a way that parameters `data`, `y.col`, `coords.col`, `covar`, `cc` and `trend` are related to the censored data structure; parameters `cov.model`, `fix.nugget`, `nugget` and `kappa` refer to the covariance structure and `thetaini`, `cons` and `MaxIter` refer to the algorithm specifications as initial values and stopping criterion. The print summary and the output for these functions are quite similar to those offered by the `SAEMSCL` routine.

For our example, the output generated by this function was saved in the objects `r` and `s` for Seminaive and Naive 1, Naive 2 algorithms, respectively, as can be seen in the following instructions:

```r
> r=Seminaive(data=dataest,y.col=3,coords.col=1:2,covar=F,cc=cc,trend="1st", + cov.model="gaussian",fix.nugget=F,nugget=170,kappa=2, + thetaini=c(900,68),cons=c(0.1,2,0.5),MaxIter=200,copred=coordspred)
> s=algnaive12(data=dataest,y.col=3,coords.col=1:2,covar=F,cc=cc,trend="1st", + cov.model="gaussian",fix.nugget=F,nugget=170,kappa=2, + thetaini=c(900,68),copred=coordspred)
```

Table 1 shows the estimation results obtained through the three methods. The `SAEMSCL` method (in boldface) produced the best fit in terms of AIC and BIC values. The second best fitted method was Naive 1.

| Method  | $\hat{\beta}_0$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\sigma}^2$ | $\hat{\phi}$ | $\hat{\tau}^2$ | Loglik | AIC   | BIC   |
|---------|-----------------|-----------------|-----------------|-----------------|-------------|-------------|--------|-------|-------|
| Seminaive | 354.8862        | -0.0710         | -0.2261         | 771.4669        | 198.1979    | 277.8797    | -439.38 | 890.76 | 906.391|
| Naive 1  | 354.2134        | -0.0671         | -0.2383         | 878.7536        | 119.5233    | 232.2021    | -438.905| 889.811| 905.442|
| Naive 2  | 452.3357        | -0.2091         | -0.5186         | 1969.1575       | 49.1863     | 579.2321    | -501.812| 1015.625| 1031.256|
| SAEM     | 367.1364        | -0.0758         | -0.3124         | 832.6448        | 211.2341    | 360.1578    | -353.601| 719.202| 734.833|

Now we proceed to make predictions under each algorithm to compare their performance. To obtain predictions via SAEM, we call the `predSCL` function, which has three arguments: `xpred`, which indicates the covariate values at prediction locations; the prediction locations `coordspred`; and `est`, which is an object of the class "SAEMSpatialCens" and it is returned by the `SAEMSCL` function. This routine provides a list with the predicted values along with their standard deviations. The following instructions save the output of this function in the object `h` for the rainfall averages case. The prediction coordinates were specified using the object `coordspred` defined previously.

```r
> xpred=cbind(1,coordspred)
> h=predSCL(xpred=xpred,coordspred = coordspred,est = est)
```

The functions `Seminaive` and `algnaive12` already provide the predicted values for a set of locations `copred` of interest. For the rainfall average data, if `r` and `s` are objects of type `Seminaive` and `algnaive12`, the predicted values are provided as `r$predictions`, `s$predictions1` and `s$predictions2`.

We can use the results given by these functions to construct graphs that facilitate the prediction interpretation, e.g., panel (b) of Figure 2 was plotted using the output of the `predSCL` function following the next code lines:

```r
> reval=datapred$y1; predsaem= h$prediction; sdsaem=h$sdpred
> LI=predsaem - (1.96*sdsaem); LS=predsaem + (1.96*sdsaem)
> plot(reval,type="l",ylim=c(100,450),ylab="Rainfall Average",xlab="")
> lines(predsaem,col="gray48",type="both",pch=20)
> lines(LI,col="gray10",lty=2)
> lines(LS,col="gray10",lty=2)
> legend(locator(1),c("Real Data", "Prediction SAEM", "Confidence bands"), + col=c(1,"gray48","gray10"),lty=c(1,1,2),pch=c(NA,20,NA),bty="n")
```

Figure 2 shows the predicted values at the observed locations for the four methods. In the left panel (a) we can compare the methods with the true observed values in solid lines. All methods seem to follow the data properly. The right panel (b) shows the real data and a 95% confidence interval for the prediction obtained via SAEM.
Predictions for the SAEM algorithm may also be assessed graphically using the `predgraphics` function. This function has `grid1` as argument, which consists of a grid of coordinates to construct the plot for prediction, `xpred`, which are the covariate values defined for this grid. `est` is an object of the class "SAEMSpatialCens" and `sdgraph` is a logical argument which is TRUE by default, providing the standard deviation of the predictions. The following instructions illustrate how to use this routine (`predgraphics`) for the rainfall data.

```r
> coorgral=seq(min(coords[,1]),max(coords[,1]),length=50)
> coorgra2=seq(min(coords[,2]),max(coords[,2]),length=50)
> grid1=expand.grid(x=coorgral,y=coorgra2)
> xpred=cbind(1,grid1)
> graf=predgraphics(xpred=xpred,est=est,grid1=grid1,points=T,sdgraph=T,
+ colors = gray.colors(300),obspoints = sample(1:sum(cc==0),50),main2="")
```

Figure 3 is generated by the `predgraphics` function, i.e., two intensity plots, one for the predictors and another for the standard deviation. It also has options for including observed points for comparison purposes.

We used the mean square prediction error (MSPE) as a measure of prediction power. This is defined as (see, [Fridley and Dixon, 2007]):

$$MSPE = \frac{1}{n} \sum_{i=1}^{n} (Z_i - \hat{Z}_i)^2,$$

where $Z_i$ is the observed value, $\hat{Z}_i$ is the predicted value and $n$ is the number of samples predicted. Since the MSPE is an error measure, lower values lead to better models. Thus, using the $MSEP$ we can also conclude that the SAEM method presents the best fit, with a $\sqrt{MSEP}$ equal to 5.649, compared with its competitors, all with $\sqrt{MSEP}$ greater than 5.9.

## 4 Local influence diagnostic utilities

In this section, we describe all measures used for local diagnostics. All calculations are based in the Hessian matrix of log-likelihood conditional expectation and the SAEM estimates obtained through the `SAEMSCl` routine. We used two simulated processes to illustrate how the package works to detect influential observations.
We derive the normal curvature of the local influence for some common perturbation schemes that could be present

\[ \Delta \]

\( \omega \)

Consider a perturbation vector \( Q \) the complete data for the perturbed model will be denoted by \( \text{either in the model or the data. We will consider perturbations in response, scale matrix and explanatory variables.} \)

4.1 Local influence

Following Zhu and Lee (2001), we compute the Hessian matrix \( \frac{\partial^2 Q(\theta, \omega)}{\partial \alpha \partial \beta} \) in order to obtain local diagnostic measures for a particular perturbation scheme as described in later subsections. Let \( \theta = (\beta^\top, \alpha^\top) \) with \( \alpha = (\alpha_1, \alpha_2, \alpha_3)^\top \) where \( \alpha_1 = \sigma^2, \alpha_2 = \phi \) and \( \alpha_3 = \tau^2 \). Then, the necessary expressions to compute the Hessian matrix \( \hat{Q}(\theta) \) are given by:

\[
\begin{align*}
\frac{\partial^2 Q(\theta, \omega)}{\partial \beta \partial \beta} &= X^\top \Sigma^{-1} X, \\
\frac{\partial^2 Q(\theta, \omega)}{\partial \beta \partial \alpha_k} &= X^\top \frac{\partial \Sigma^{-1}}{\partial \alpha_k} \tilde{Z} - X^\top \frac{\partial \Sigma^{-1}}{\partial \alpha_k} X \beta, \\
\frac{\partial^2 Q(\theta, \omega)}{\partial \alpha_k \partial \alpha_l} &= \text{tr} \left( \frac{\partial \Sigma^{-1}}{\partial \alpha_k} \frac{\partial \Sigma^{-1}}{\partial \alpha_l} + \frac{\partial^2 \Sigma^{-1}}{\partial \alpha_k \partial \alpha_l} \Sigma^{-1} \right) \\
&- \frac{1}{2} \left[ \text{tr} \left( \tilde{Z} \Sigma^{-1} \frac{\partial \Sigma^{-1}}{\partial \alpha_k \alpha_l} \right) - 2 \beta^\top X^\top \frac{\partial^2 \Sigma^{-1}}{\partial \alpha_k \alpha_l} \tilde{Z} + \beta^\top X^\top \frac{\partial^2 \Sigma^{-1}}{\partial \alpha_k \alpha_l} X \beta \right].
\end{align*}
\]

4.1 Local influence

We derive the normal curvature of the local influence for some common perturbation schemes that could be present either in the model or the data. We will consider perturbations in response, scale matrix and explanatory variables.

Consider a perturbation vector \( \omega \) varying in an open region \( \Omega \subset \mathbb{R}^n \). The conditional log-likelihood expectation of the complete data for the perturbed model will be denoted by \( Q(\theta, \omega|\hat{\theta}) = E(\ell(\theta, \omega|Z_c)|V, C, \hat{\theta}) \) which reaches its maximum at \( \hat{\theta}(\omega) \). Assume there is an \( \omega_0 \) in \( \Omega \) such that \( \ell(\theta, \omega_0|Z_c) = \ell(\theta|Z_c) \) for all \( \theta \). We use the Q-displacement function defined as:

\[ f_{\omega}(\omega) = 2[Q(\theta, \omega|\hat{\theta}) - Q(\theta|\hat{\theta})], \]

to obtain the influence graph \( \delta(\omega) = (\omega^\top, f_{\omega}(\omega))^\top \). The local behavior of \( f_{\omega}(\omega) \) can be summarized using the curvature \( C_{f_{\omega},d} \) at \( \omega_0 \) in the direction of some unit vector \( d \). This curvature is defined by:

\[ C_{f_{\omega},d} = -2d^\top \hat{Q}_{\omega_0} d \quad \text{and} \quad \hat{Q}_{\omega_0} = \Delta_{\omega_0} - \hat{Q}(\theta|\hat{\theta})^{-1} \Delta_{\omega_0}, \]

where \( \Delta_{\omega} = \frac{\partial^2 Q(\theta, \omega)}{\partial \theta \partial \omega} \) and \( \hat{Q}(\theta|\hat{\theta}) \) is the Hessian matrix with elements as in expressions (11)-(13).
As described in [Cook (1986), the quantities $C_{fQ,a}$ and $\hat{Q}\omega_n$ are useful for detecting influential observations. Let the spectral (orthonormal) decomposition of $-2\hat{Q}\omega_n = \sum_{i=1}^{n} \lambda_i a_i^T$ with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$ eigenvalues and $a_i$, for $i = 1, \ldots, n$, eigenvectors of this matrix. Zhu and Lee (2001) proposed inspecting all eigenvectors corresponding to nonzero eigenvalues to capture the relevant information about influential observations.

Consider the aggregated contribution of each eigenvector with a corresponding nonzero eigenvalue as $\tilde{\lambda}_i = \lambda_i/(\lambda_1 + \cdots + \lambda_r)$. Let $a_i^T = (a^2_{i1}, \ldots, a^2_{in})^T$ and $M(0) = \sum_{i=1}^{r} \lambda_i a_i^2$ with $M(0)_l = \sum_{i=1}^{r} \lambda_i a^2_{il}$ for $l = 1, \ldots, n$, the $l$-th component of $M(0)$. The assessment of influential observations can be performed through visual inspection of $M(0)_l$ plotted against the index $l$. The $l$-th case may be considered influential if $M(0)_l$ is larger than a benchmark value. Zhu and Lee (2001) also considered the conformal normal curvature $B_{fQ,a}(\theta) = C_{fQ,a}(\theta)/tr[-2\hat{Q}\omega_n]$, whose computation is quite simple and has the property that $0 \leq B_{fQ,a}(\theta) \leq 1$. They also showed that for a perturbation vector $d_l$ whose $l$-th entry is 1 and the remaining entries are 0, $M(0)_l = B_{fQ,a_l}(\theta)$, so we can obtain $M(0)_l$ through $B_{fQ,a_l}(\theta)$.

In general, there is no rule to judge how influential a specific case in the data is. Some measures have been proposed to determine the benchmark value in the visual inspection of $M(0)$. Let $\tilde{M}(0)$ and $SM(0)$ be the mean and standard error of the vector $M(0)$ respectively. Poon and Poon (1999) proposed using $2\tilde{M}(0)$ as a benchmark value while Zhu and Lee (2001) proposed $\tilde{M}(0) + 2SM(0)$, for taking into account the standard deviation of $M(0)$. In the CensSpatial package, we use the benchmark proposed by Lee and Xu (2004) as $\tilde{M}(0) + c^*SM(0)$, where $c^*$ is a constant, which can vary depending on the specific application.

### 4.2 Perturbation schemes

We consider three types of perturbation schemes: perturbation in the response variable, which can indicate influence in their own predicted values, perturbation in the scale matrix $\Sigma$, which may reveal the individuals that are influential in the estimation of the scale parameter $\alpha$ and finally perturbation in explanatory variables. For all these schemes, we are interested in computing the $\Delta\omega$ matrix with elements:

$$
\Delta\beta = \frac{\partial^2 Q(\theta, \omega|\hat{\theta})}{\partial \beta \partial \omega} \quad \text{and} \quad \Delta\alpha = (\Delta_{\alpha_1}, \Delta_{\alpha_2}, \Delta_{\alpha_3}),
$$

where $\Delta_{\alpha_k} = \frac{\partial^2 Q(\theta, \omega|\hat{\theta})}{\partial \alpha_k \partial \omega}$, for $k = 1, 2, 3$.

#### 4.2.1 Response perturbation

A perturbation in response variable $V$ can be introduced by replacing it with $V(\omega) = V + \omega$, recalling the non-perturbed model $V_i = V_0$, for $C_i = 0$ and that $V_i$ belongs to $[V_{i1}, V_{i2}]$ for $C_i = 1$. In this way we can write the perturbed model as:

$$
C_i = \begin{cases} 
1 & \text{if } V_{i1}(\omega_i) \leq Z_i(\omega_i) \leq V_{i2}(\omega_i), \\
0 & \text{if } Z_i(\omega_i) = V_{0i}(\omega_i),
\end{cases}
$$

where $V_i(\omega) = V_i + \omega$, $V_2(\omega) = V_2 + \omega$ and $Z(\omega) = Z - \omega$. To obtain the perturbed function $Q(\theta|\hat{\theta}, \omega)$, we replace $\hat{Z}$ and $\hat{Z}Z^T$ with $\hat{Z} = \hat{Z} - \omega$ and $\hat{Z}_i\hat{Z}_i^T = \hat{Z}Z^T - \hat{Z}\omega - \omega\hat{Z}^T + \omega\omega^T$ in the non-perturbed $Q(\theta|\hat{\theta})$ from Subsection 2.3. Let $\omega_0$ be the vector that represents the case of no perturbation in response. Then, the matrix $\Delta\omega$, has the elements $\Delta\beta = -X^T\Sigma^{-1}$ and $\Delta\alpha = [\Delta_{\alpha_k}]$, where

$$
\Delta_{\alpha_k} = \frac{1}{2} \left( (Z_{(i)} + Z_{(i)}^T) \frac{\partial \Sigma^{-1}}{\partial \alpha_k} \right) + Z_{(i)}^T P^T \frac{\partial \Sigma^{-1}}{\partial \alpha_k},
$$

with $P = X(X^T\Sigma^{-1}X)^{-1}X^T\Sigma^{-1}$ and $Z_{(i)}$ the matrix with the $i$-th row being $\hat{Z}$ and the remaining rows being zero. Expressions for $\frac{\partial \Sigma^{-1}}{\partial \alpha_k}$ are defined in the Appendix.

#### 4.2.2 Scale matrix perturbation

Now, we consider the scheme perturbation of the form $\Sigma(\omega) = D(\omega)\Sigma$ to study the effects of perturbation on the scale matrix. In this case $D(\omega)$ is an $n \times n$ diagonal matrix with elements $\omega$. The non-perturbed model is obtained when
\(\omega_i = 1\) for all \(i = 1, \ldots, n\) and therefore \(\Delta \omega_0\) is a \((p+3) \times n\) matrix with components \(\Delta \beta_i = \mathbf{X}^\top \Sigma^{-1} \mathbf{d}(i) [\mathbf{I}_n - \mathbf{P}] \mathbf{Z}\) and \(\Delta \alpha_i = [\Delta \alpha_{ki}]\), with

\[
\Delta \alpha_{ki} = -\text{tr} \left( \Sigma^{-1} \mathbf{d}(i) \frac{\partial \Sigma}{\partial \alpha_k} \right) - \frac{1}{2} \left[ \text{tr} \left( \hat{\mathbf{Z}} \hat{\mathbf{Z}}^\top \left( \Sigma^{-1} \frac{\partial \Sigma}{\partial \alpha_k} \Sigma^{-1} \right) \mathbf{d}(i) \right) + \hat{\mathbf{Z}}^\top [\mathbf{P}^\top - 2\mathbf{I}] \left( \Sigma^{-1} \frac{\partial \Sigma}{\partial \alpha_k} \Sigma^{-1} \right) \mathbf{d}(i) \mathbf{P} \hat{\mathbf{Z}} \right],
\]

for \(k = 1, 2, 3\), where \(\mathbf{d}(i)\) is an \(n \times n\) matrix with the \(i\)-th diagonal element equal to 1 and the remaining elements equal to zero.

### 4.2.3 Explanatory variable perturbation

We consider for this scheme a perturbation of the form \(\mathbf{X}(\omega) = \mathbf{X} + \mathbf{W}\) and replace the perturbed \(Q(\theta|\hat{\theta}, \omega)\) function as in the response perturbation, in this case, \(\mathbf{W} = \omega \mathbf{1}^\top\). Under the non-perturbed model, \(\omega = 0\) and therefore \(\Delta \omega_0\) has elements \(\Delta \beta_i = \hat{\mathbf{Z}}^\top [\mathbf{I} - 2\mathbf{P}^\top] \Sigma^{-1} \mathbf{W}_i^{(1)}\) and \(\Delta \alpha_i = [\Delta \alpha_{ki}]\), with

\[
\Delta \alpha_{ki} = \hat{\mathbf{Z}}^\top [\mathbf{I} - \mathbf{P}^\top] \left( \Sigma^{-1} \frac{\partial \Sigma}{\partial \alpha_k} \mathbf{W}_i^{(1)} \right) (\mathbf{X}^\top \Sigma^{-1} \mathbf{X})^{-1} \mathbf{X}^\top \Sigma \hat{\mathbf{Z}},
\]

where \(\mathbf{W}_i^{(1)}\) is an \(n \times p\) matrix with the \(i\)-th row equal to 1 and the remaining rows equal to zero.

### 4.3 Application: Diagnostic tools

In this subsection, we use simulated data to illustrate the local diagnostic utilities that the \texttt{CensSpatial} package offers. The dataset corresponds to two linear left censored processes with the same mean structure but with different covariance structures for errors, i.e., we set a process as in equations (1)-(2) with spherical and Matérn (\(\kappa = 0.3\)) covariance structures and two covariates \(\mathbf{X}_1 \sim U(0, 1)\) and \(\mathbf{X}_2 \sim U(0, 3)\) for the mean structure. We set a censoring level of 15% and the observations \#91, \#126 and \#162 as atypical points for the two processes.

To generate the spatial random samples used in the simulations, we call the function \texttt{rspacens}, its principal arguments are \texttt{beta} and \texttt{cov.init} to fix the parameters of the mean and covariance structure respectively, \texttt{coords} to define the coordinates used for the sample, \texttt{cens} for the censoring level, \texttt{cens.type} to define the censoring type (left or right) and \texttt{cov.model} to define the spatial correlation function to use. The code below illustrates how we generate the samples and the atypical points used in this paper.

```r
> nobs=200; npred=100
> r1=sample(seq(1,30,length=400),n+n1); r2=sample(seq(1,30,length=400),n+n1)
> coords=cbind(r1,r2); cov.init=c(2,0.1); beta=c(5,3,1)
> xtot<-cbind(1,runif((n+n1)),runif((n+n1),2,3)); xobs=xtot[1:n,]
> obj=rspacens(cov.pars=c(3,3,0),beta=beta,x=xtot,coords=coords,cens=0.15,
+ n=(nobs+ npred),nl=npred,cov.model="matern",cens.type="left",kappa=0.3)

> y=obj$datare[,3]

> y[91]=y[91]+ 5*sd(y); y[126]=y[126]+ 5*sd(y); y[162]=y[162]+ 5*sd(y)
```

To assess the influence of outliers in the estimation process, we use the \texttt{localinfmeas} function. This uses an object of class "\texttt{SAEMSpatialCens}", i.e., an object as a result of the SAEM estimation procedure to compute the value of the vector \(\mathbf{M}(0)\) for each perturbation scheme and detect the influential points using the \texttt{Lee and Xu (2004)} criteria. The benchmark used for the routine is \(c^* = 3\) by default. On the other hand, the function \texttt{atypical} filters the results to get only the atypical points detected by the \texttt{localinfmeas} routine. The following instructions generate the output of the \texttt{localinfmeas} and \texttt{atypical} functions, considering a linear process with Matérn covariance structure and three perturbation schemes discussed previously.

```r
> sest=SAEMSCL(cc,y,cens.type="left",trend="other",x=xobs,coords=coords,
```
+ M=15, perc=0.25, MaxIter=5, pc=0.2, cov.model=type, kappa=0.3, fix.nugget=T, + nugget=0, inits.sigmae=cov.ini[1], inits.phi=cov.ini[2], search=T, + lower=0.00001, upper=50)

> w = localinfmeas(sest, fix.nugget = TRUE, c = 3)
> atypical(w)$RP

| obs   | m0    |
|-------|-------|
| 32    | 0.04478987 |
| 91    | 0.09086417 |
| 126   | 0.10935255 |
| 162   | 0.09978107 |

In these lines, \textit{w} is a localinfmeas object. Then \texttt{atypical(w)} will be a list with three elements: EP, SP and RP, containing the atypical detected observations for each perturbation scheme. By default, localinfmeas also generates a diagnostic plot with for the three perturbation schemes.

![Diagnostic plots](image)

**Figure 4:** Local influence diagnostic plots for spherical (first row) and Matérn (second row) processes under (a,d) response perturbation (b,e) scale matrix perturbation and (c,f) explanatory variable perturbation schemes

Figure 4 (provided by the localinfmeas function) shows the diagnostic plots for the two simulated datasets. It can be seen that the artificial outliers were detected, as expected. For the response perturbation, the procedure detected two atypical observations under the spherical covariance structure and one under the Matérn covariance structure. Thus, we can conclude that the proposed local influence measures work quite well to detect atypical observations in the context of censored spatial models.
5 Conclusions

This work introduces an R package for censored spatial data analysis that allows estimation, prediction and local influence diagnosis for censored Gaussian processes considering different linear trends and covariance structures. We discuss in detail all functions available in this package as well as the theoretical framework. Plots provided by the package constitute one of the most powerful tools for comparing models, assessing predictions and analyzing local influence. Applications and several examples can be found in the manual of the CensSpatial package, available in the CRAN repository. To the best of our knowledge, this R package is the first proposal, available to practitioners, to perform statistical analysis of censored spatial data.

Future extensions of this work include the use of scale mixtures of normal (SMN) distributions to accommodate heavy-tailed features (De Bastiani et al., 2014). We can also extend the proposal of Diggle et al. (1998) by including censoring in generalized linear mixed models (GLMMs).

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References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control* 19, 716–723.

Arismendi, J. (2013). Multivariate truncated moments. *Journal of Multivariate Analysis* 117(1), 41–75.

Cook, R. (1977). Detection of influential observation in linear regression. *Technometrics* 19, 15–18.

Cook, R. D. (1986). Assessment of local influence. *Journal of the Royal Statistical Society, Series B* 48, 133–169.

De Bastiani, F., A. H. M. de Aquino Cysneiros, M. A. Uribe-Opa zo, and M. Galea (2014). Influence diagnostics in elliptical spatial linear models. *TEST* 24(2), 322–340.

Delyon, B., M. Lavielle, and E. Moulines (1999). Convergence of a stochastic approximation version of the EM algorithm. *Annals of Statistics* 27(1), 94–128.

Dempster, A., N. Laird, and D. Rubin (1977). Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society, Series B* 39, 1–38.

Diggle, P. and P. Ribeiro (2007). *Model-Based Geostatistics*. Springer Series in Statistics.

Diggle, P. and P. Ribeiro Jr (2002). Bayesian inference in gaussian model-based geostatistics. *Geographical and Environmental Modelling* 6, 129–146.

Diggle, P. J., J. Tawn, and R. Moyeed (1998). Model-based geostatistics. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 47(3), 299–350.

Fridley, B. L. and P. Dixon (2007). Data augmentation for a Bayesian spatial model involving censored observations. *Environmetrics* 18, 107–123.

Genz, A., F. Bretz, T. Miwa, X. Mi, F. Leisch, F. Scheipl, and T. Hothorn (2020). mvtnorm: Multivariate Normal and t Distributions. R package version 1.1-1.

Giménez, P. and M. Galea (2013). Influence measures on corrected score estimators in functional heteroscedastic measurement error models. *Journal of Multivariate Analysis* 114, 1–15.

Jacqmin-Gadda, H., R. Thiebaut, G. Chene, and D. Commenges (2000). Analysis of left-censored longitudinal data with application to viral load in HIV infection. *Biostatistics* 1, 355–368.

Jank, W. (2006). Implementing and diagnosing the stochastic approximation EM algorithm. *Journal of Computational and Graphical Statistics* 15(4), 803–829.

Kuhn, E. and M. Lavielle (2004). Coupling a stochastic approximation version of EM with an MCMC procedure. *ESAIM: Probability and Statistics* 8, 115–131.

Lachos, V. H., L. A. Matos, T. S. Barbosa, A. M. Garay, and D. K. Dey (2017). Influence diagnostics in spatial models with censored response. *Environmetrics* 28(7), e2464.

Lee, S. Y. and L. Xu (2004). R influence analysis of nonlinear mixed-effects models. *Computational Statistics & Data Analysis* 45, 321–341.
Militino, A. F. and M. D. Ugarte (1999). Analyzing censored spatial data. *Mathematical Geology* 31(5), 551–561.

Ordonez, J. A., D. Bandyopadhyay, V. H. Lachos, and C. R. Cabral (2018). Geostatistical estimation and prediction for censored responses. *Spatial Statistics* 23, 109–123.

Poon, W. and Y. Poon (1999). Conformal normal curvature and assessment of local influence. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)* 61, 51–61.

Rathbun, S. L. (2006). Spatial prediction with left-censored observations. *Journal of Agricultural, Biological, and Environmental Statistics* 11(3), 317–336.

Schelin, L. and S. Sjöstedt-de Luna (2014). Spatial prediction in the presence of left-censoring. *Computational Statistics & Data Analysis* 74, 125–141.

Schwarz, G. (1978). Estimating the dimension of a model. *Annals of Statistics* 6(2), 461–464.

Toscan, P. J. (2010). Spatial modelling of left censored water quality data. *Environmetrics* 21(6), 632–644.

Vaida, F. and L. Liu (2009). Fast implementation for normal mixed effects models with censored response. *Journal of Computational and Graphical Statistics* 18, 797–817.

Zhu, H. and S. Lee (2001). Local influence for incomplete-data models. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)* 63, 111–126.

Zhu, H., S. Lee, B. Wei, and J. Zhou (2001). Case-deletion measures for models with incomplete data. *Biometrika* 88, 727–737.