Spin-3 Mielke-Baekler gravity

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Abstract

Mielke-Baekler gravity consists in the usual Einstein-Hilbert action with a cosmological term and rotational and translational Chern-Simons terms with arbitrary couplings. For a particular choice of these couplings, we can obtain Einstein-Hilbert action, its teleparallel equivalent, and the exotic Witten’s gravity.

In this work, we use the Chern-Simons formalism to generalize the three dimensional Mielke-Baekler gravity theory in order to introduce spin-3 fields. We study its asymptotic symmetries, black hole solution, and also analyse its canonical structure at its singular point.

Keywords: Mielke-Baekler gravity, anti-de Sitter, higher-spin.

1 Introduction

Outside local sources, three-dimensional Einstein-Hilbert (EH) gravity has constant curvature and no local degrees of freedom. However, as shown by Brown and Henneaux \cite{1}, the AdS\textsubscript{3} asymptotic symmetries are two copies of the Virasoro algebra. This result is the precursor of the AdS\textsubscript{3}/CFT\textsubscript{2} correspondence and also showed that three-dimensional gravity is far from trivial.

The study of the black hole solutions and asymptotic symmetries of three-dimensional gravity has not been exclusive to EH gravity. It has been extended to other three-dimensional models such as Conformal Gravity \cite{2}, Topologically Massive Gravity \cite{3}, \cite{4}, New Massive Gravity and General Massive Gravity \cite{5}, \cite{6}. These models have some common features: They can be written as a Chern-Simons-like action \cite{7}, and they have a vanishing torsion. The Mielke-Baekler (MB) gravity model \cite{8}, \cite{9} consists in the usual EH action with a cosmological term plus a rotational and a translational Chern-Simons (CS) term. For a particular choice of parameters, we can recover EH gravity, Teleparallel gravity, and the parity-odd “exotic” Witten’s gravity \cite{10}. Therefore, MB gravity is a good laboratory to test the role of curvature and torsion in the AdS\textsubscript{3}/CFT\textsubscript{2} correspondence. In references \cite{11}-\cite{19} the relation of MB gravity with the CS action, black hole solutions, canonical structure and asymptotic symmetries has been explored and a supersymmetric extension of MB gravity was constructed in \cite{20}.

On the other hand, there has been a lot of interest in the study of higher-spin gravity models due to, between others, their non-linear dynamics \cite{21} and their relation with String Theory \cite{22}. In three dimensions, higher-spin EH gravity can be written as the difference of two CS actions under the SL(N) x SL(N) group \cite{23}, \cite{24}. This construction was generalized to spin-3 topologically massive gravity \cite{25}, \cite{26} and CS-like gravity \cite{27}.

In this work, we build the spin-3 MB gravity. Different from other spin-3 gravity models, we do not need to impose any constraints on torsions and therefore we can build a teleparallel equivalent spin-3 EH action. We also use its CS formulation to obtain black-hole solutions and asymptotic symmetries, which consists of two copies of the \textit{W}\textsubscript{3} algebra with different CS central charges. The pure MB gravity theory has a singular point and we are going to show that this point is preserved in the presence of the spin-3 fields and analyse its canonical structure.

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This paper is organized as follows: In section 2, we give a brief review of MB gravity and its CS formulation. In section 3, we use the CS formalism in order to introduce spin-3 fields in the MB gravity, analyse its equations of motion, show its black-hole solution, and also the asymptotic symmetries. In section 4, we present the singular point of the spin-3 MB theory and analyse its canonical structure using Dirac’s algorithm. Finally, in section 5 we comment on our results and give future perspectives of this work.

2 The AdS sector of Mielke-Baekler gravity theory

In the first-order formalism of three-dimensional gravity, the independent fields are the dreibein $e^a = e^a_\mu dx^\mu$ and the self-dual spin connection $\omega^a = \omega^a_\mu dx^\mu$. The curvature $R_a$ and torsion $T_a$ are defined by

$$ R_a = d\omega_a + \frac{1}{2} \epsilon_{abc} \omega_b \wedge \omega_c, \quad T_a = de_a + \epsilon_{abc} \omega_b \wedge e_c. \quad (1) $$

We lower and rise indices with the Minkowsky metric $\eta_{ab} = \text{diag} (-1, 1, 1)$. The Levi-Civita symbol is denoted by $\epsilon_{abc}$ and by convention $\epsilon_{012} = 1$.

The action for the three-dimensional Mielke-Baekler (MB) gravity theory \[8\] is given by

$$ I_{MB} = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 \quad (2) $$

where $(a, \Lambda, \alpha_3, \alpha_4)$ are free parameters and

$$ I_1 = \int e^a \wedge R_a \quad (3) $$
$$ I_2 = -\frac{1}{3} \int \epsilon_{abc} e^a \wedge e^b \wedge e^c \quad (4) $$
$$ I_3 = \int \left( \omega^a \wedge d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) \quad (5) $$
$$ I_4 = \int e^a \wedge T_a. \quad (6) $$

The term $I_1$ represents the Einstein-Hilbert (EH) action, $I_2$ is a cosmological term, the third term $I_3$ is a CS action for the spin-connection and $I_4$ is a translational CS term.

The equations of motion of the MB action are:

$$ aR_a + \alpha_4 T_a - \frac{1}{2} \Lambda \epsilon_{abc} e^b \wedge e^c = 0 \quad (7) $$
$$ \alpha_3 R_a + aT_a + \frac{1}{2} \alpha_4 \epsilon_{abc} e^b \wedge e^c = 0. \quad (8) $$

For an specific choice of parameters, the above equations degenerate to a single one \[9\]. This is a singular point of the MB theory, and it has some similarities with the chiral point of Topological Massive Gravity \[18\].

For the non-degenerated sector, $\alpha_3 \alpha_4 - a^2 \neq 0$, we can determine the curvature and torsion forms. The vacuum solution is characterized by a constant curvature

$$ R_a = \frac{q}{2} \epsilon_{abc} e^b \wedge e^c = 0, \quad q = \frac{-a\Lambda + (\alpha_4)^2}{\alpha_3 \alpha_4 - a^2} \quad (9) $$

and a constant torsion

$$ T_a = \frac{p}{2} \epsilon_{abc} e^b \wedge e^c = 0, \quad p = \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}. \quad (10) $$

We can identify three particular cases of the MB theory: First, the EH gravity with cosmological term. Here we set $\alpha_3 = \alpha_4 = 0$, such that the torsion vanishes

$$ R_a = \frac{\Lambda}{2a} \epsilon_{abc} e^b \wedge e^c = 0, \quad T_a = 0. \quad (11) $$
The importance of the CS formulation lies in that we have a machinery to explore boundary conditions, black hole solutions and asymptotic symmetries [30], [31]. In the next section we will make use the the CS action to couple MB with spin-3 fields.

In addition, we fixed \( a = 1/16\pi G \). Then, the curvature will be positive, negative or zero, depending on the value of \( \Lambda \).

A second case corresponds to \( a \Lambda + (\alpha_4)^2 = 0 \), where we have a vanishing curvature (\( q = 0 \)) and constant torsion

\[
T_a + \frac{\alpha_4}{a} \epsilon_{abc} e^b \wedge e^c = 0, \quad R_a = 0.
\]

This case corresponds to the three-dimensional Teleparallel theory. Note that the parameter \( \alpha_3 \) does not appear either in the field equations or in the condition for zero curvature, then, it can be set to zero without loss of generality.

The third case is the Witten’s “exotic” gravity [10], where \( a = \Lambda = 0 \). This model have a vanishing torsion, a curvature that is proportional to \( q = -\alpha_4/\alpha_3 \), and reversed roles for mass and angular momentum [28].

It is convenient to write the self-dual spin connection in terms of the self-dual Levi-Civita connection \( \tilde{\omega}^a \) and the contorsion \( \tau^a \): \( \omega^a = \tilde{\omega}^a + \tau^a \). For MB gravity we have \( \tau^a = \frac{p^2}{4} e^a \), this can be directly computed from the field equation (10). The curvature and torsion forms corresponding to the self-dual Levi-Civita connection are

\[
\tilde{R}_a = \frac{1}{2} \Lambda_{\text{eff}} \epsilon_{abc} e^b \wedge e^c, \quad \tilde{T}_a = 0, \quad \Lambda_{\text{eff}} \equiv q - \frac{p^2}{4}.
\]

In the above equation, \( \Lambda_{\text{eff}} \) acts as an effective cosmological constant. We can further restrict ourselves to the AdS sector, where \( \Lambda_{\text{eff}} \) is negative

\[
\Lambda_{\text{eff}} = q - \frac{p^2}{4} = -\frac{1}{l^2} < 0.
\]

In the following table we show a choice of parameters \( (a, \Lambda, \alpha_3, \alpha_4) \) for three particular cases which belong to the AdS sector of the MB theory:

|                | \( a \) | \( \Lambda \) | \( \alpha_3 \) | \( \alpha_4 \) | \( q \) | \( p \) |
|----------------|--------|-------------|-------------|-------------|-----|-----|
| Einstein – Hilbert | $\frac{1}{16\pi G}$ | $-\frac{1}{16\pi G}$ | 0            | 0           | 0   | 0   |
| Teleparallel    | $\frac{1}{16\pi G}$ | $-\frac{1}{4\pi G}$ | 0            | $-\frac{1}{8\pi G}$ | 0   | $\frac{2}{7}$ |
| Exotic         | 0      | 0           | $-\frac{1}{16\pi G}$ | $-\frac{1}{16\pi G}$ | $-\frac{1}{l^2}$ | 0   |

The values for the exotic theory are in agreement with [29].

### 2.1 Chern-Simons formulation

In [11] it was shown that the AdS sector of MB theory can be written as the difference of two CS actions with different CS levels. To achieve this, we need to define two Lie algebra valued gauge connections \( A^\pm \) as the linear combination of the self-dual spin connection and the dreibein:

\[
A^\pm = A_a^\pm J_a^\pm = \left[ \omega^a + \left( \frac{p}{2} + \frac{1}{l} \right) e^a \right] J_a^\pm.
\]

where \( J_a^\pm \) are \( so(2,1) \sim sl(2) \) generators: \([ J_a^+, J_b^- ] = \epsilon_{ab}^c J_c^\pm \). Then, we have that the AdS sector of MB can be written as

\[
I_{\text{MB}} = I_{\text{CS}}[ A^+ ] - I_{\text{CS}}[ A^- ]
\]

(16)
where
\[ I_{CS}^{\pm} [A^{\pm}] = \frac{k^{\pm}}{4\pi} \int_M \text{tr} \left[ A^{\pm} \wedge dA^{\pm} + \frac{2}{3} A^{\pm} \wedge A^{\pm} \wedge A^{\pm} \right] + B^{\pm} [A^{\pm}] . \] (17)

In the above equation, \( B^{\pm} \) are boundary terms, the traces are \( \text{tr} (J^{\pm}_a J^{\pm}_b) = \eta_{ab}/2 \), and \( \text{tr} (J^{\pm}_a J^{\pm}_b) = 0 \). The CS levels \( k^{\pm} \) are given by:
\[ k^{\pm} = 4\pi \left( a + \alpha_3 \frac{p}{2} \right) l \pm \alpha_3 \] . (18)

For EH and Teleparallel gravity, the CS levels become equal to \( k^+ = k^- = l/4G \). For “exotic” gravity, the CS levels are \( k^\pm = \mp l/4G \). Furthermore, the CS action (16) has the equation of motion
\[ F^{\pm} = dA^{\pm} + A^{\pm} \wedge A^{\pm} = 0 \] (19)

and it is invariant under the gauge transformation
\[ \delta_\lambda A^{\pm} = d\lambda^{\pm} + [A^{\pm}, \lambda^{\pm}] = 0 \] . (20)

The importance of the CS formulation lies in that we have a machinery to explore boundary conditions, black hole solutions and asymptotic symmetries [30], [31], [32]. In the next section we will make use the CS action to couple MB with spin-3 fields.

3 Spin-3 generalization of Mielke-Baekler gravity

Three-dimensional EH gravity coupled with symmetric tensor fields of spin N can be written as the difference of two CS actions under the group SL(N) \( \times \) SL(N) [23]. Analogously, we state that the spin N Mielke-Baekler theory is the difference of two CS action under the group SL (N) \( \otimes \) SL (N) with CS levels given by (18) 3.

In the following we consider \( N = 3 \) in order to have an explicit expression for the higher-spin Mielke-Baekler action. In this case, the gauge connection is given by
\[ A^{\pm} = \left[ \omega^{a} + \left( -\frac{P}{2} \pm \frac{1}{L} \right) e^{a} \right] J^{\pm}_a + \left[ \omega^{ab} + \left( -\frac{P}{2} \pm \frac{1}{L} \right) e^{ab} \right] T^{\pm}_{ab} \] (21)

where \( e^{ab} \) is a vielbein-like field and \( \omega^{ab} \) an auxiliary field. These new quantities are traceless and symmetric in the flat indices. The generators of the \( sl(3) \) algebra, denoted by \( J^{\pm}_a, T^{\pm}_{ab} \), satisfy
\[ [J^{\pm}_a, J^{\pm}_b] = \epsilon^{c}_{ab} J^{\pm}_c \] \[ [J^{\pm}_a, T^{\pm}_{bc}] = \epsilon^{d}_{a(\eta T_{c)d}} \] \[ [T^{\pm}_{ab}, T^{\pm}_{cd}] = \sigma \left[ \eta_{(c}\epsilon_{d)b} \right] J^{\pm}_e \] (22)
(23)
(24)

where a pair of parentheses on the indices denotes a complete symmetrization without normalization factor, and \( \sigma \) is a negative parameter (Positive values of \( \sigma \) are related to the \( su(1,2) \) algebra). Furthermore, we have that
\[ \text{tr} (J^{\pm}_a J^{\pm}_b) = \frac{1}{2} \eta_{ab}, \quad \text{tr} (J^{\pm}_a T^{\pm}_{bc}) = 0, \quad \text{tr} (T^{\pm}_{ab} T^{\pm}_{cd}) = -\frac{\sigma}{2} \left( \eta_{(c}\eta_{d)b} - \frac{2}{3} \eta_{ab}\eta_{cd} \right) . \] (25)

Replacing (21) in (16), we obtain the spin-3 Mielke-Baekler action
\[ I^{\text{Spin-3}}_{\text{MB}} = a_1^{(3)} + A_2^{(3)} + \alpha_3 L_3^{(3)} + \alpha_4 L_4^{(3)} \] (26)

3In [24] was also proposed a higher-spin generalization of MB gravity. However, we will not restrict to the AdS sector, as we will see in the next section
Note that in the absence of spin-3 fields, the quantities \( R \) hold. Then, the two sets of equations above can be solved for the curvatures and torsions:

\[
R^{(3)}_a = 2 \int \left( e^a \wedge R^{(3)}_a - 2\sigma e^{ab} \wedge R^{(3)}_{ab} \right) \tag{27}
\]

\[
R^{(3)}_{ab} = -\frac{1}{3} \int \left( \epsilon_{abc} e^a \wedge e^b \wedge e^c - 12\sigma \epsilon_{abc} e^a \wedge e^b \wedge e^d \right) \tag{28}
\]

\[
T^{(3)}_a = \int \left( \omega^a \wedge d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c - 2\sigma \omega^{ab} \wedge d\omega_{ab} - 4\epsilon_{abc} \omega^a \wedge \omega^b \wedge \omega^c \right) \tag{29}
\]

\[
T^{(3)}_{ab} = \int \left( e^a \wedge T^{(3)}_a - 2\sigma e^{ab} \wedge T^{(3)}_{ab} \right). \tag{30}
\]

These terms are the spin-3 generalization of (5)-(9), respectively. We still have four couplings: \((a, \Lambda, \alpha_3, \alpha_4)\), reminiscent of the supersymmetric generalization \([20]\). However, they are constrained to the condition \(a^2 - \alpha_3\alpha_4 \neq 0\) since the connection \([21]\) depends on \(l, p, q\). The curvatures are defined by

\[
R^{(3)}_a = \frac{1}{2} \epsilon_{abc} \omega^b \wedge e^c - 2\epsilon_{abc} \omega^b \wedge e^d \tag{31}
\]

\[
R^{(3)}_{ab} = \epsilon_{cda} \omega^c \wedge \omega^d \tag{32}
\]

and the torsions are given by

\[
T^{(3)}_a = d\omega_a + \epsilon_{abc} \omega^b \wedge e^c - 4\epsilon_{abc} \omega^b \wedge \omega^d \tag{33}
\]

\[
T^{(3)}_{ab} = d\omega_{ab} + \epsilon_{cda} \omega^c \wedge \omega^d \tag{34}
\]

Note that in the absence of spin-3 fields, the quantities \( R^{(3)}_a \) and \( T^{(3)}_a \) coincide with the usual curvature and torsion \([1]\) and \( R^{(3)}_{ab} = T^{(3)}_{ab} \) = 0. Also notice that, for \(a = 0, \alpha = \alpha_4 = 0\), the action \([26]\) reduces to the usual spin-3 EH action with cosmological constant.

The variation of the action with respect to \(e_a\) and \(\omega_a\) fields gives the following equations of motion

\[
aR^{(3)}_a + \alpha_4 T^{(3)}_a - \Lambda \left[ \frac{1}{2} \epsilon_{abc} \omega^b \wedge e^c - 2\epsilon_{abc} \omega^b \wedge e^d \right] = 0 \tag{35}
\]

\[
\alpha_3 R^{(3)}_a + aT^{(3)}_a + \alpha_4 \left[ \frac{1}{2} \epsilon_{abc} \omega^b \wedge e^c - 2\epsilon_{abc} \omega^b \wedge e^d \right] = 0 \tag{36}
\]

which are generalizations of (7), (8). The variations with respect to \(e_{ab}, \omega_{ab}\) give

\[
aR^{(3)}_{ab} + \alpha_4 T^{(3)}_{ab} - \Lambda \epsilon_{cda} e^c \wedge e^d = 0 \tag{37}
\]

\[
\alpha_3 R^{(3)}_{ab} + aT^{(3)}_{ab} + \alpha_4 \epsilon_{cda} e^c \wedge e^d = 0. \tag{38}
\]

Equations \((35)-(38)\) are equivalent to zero curvature equation \((19)\) in the CS formalism.

Now, since we build the spin-3 MB theory from the CS formulation, the condition \(a^2 - \alpha_3\alpha_4 \neq 0\) holds. Then, the two sets of equations above can be solved for the curvatures and torsions:

\[
R^{(3)}_a = \frac{1}{2} q \epsilon_{abc} e^b \wedge e^c + 2\sigma q \epsilon_{abc} e_{m}^b \wedge e_{m}^c = 0 \tag{39}
\]

\[
T^{(3)}_a = \frac{1}{2} p \epsilon_{abc} e^b \wedge e^c + 2\sigma p \epsilon_{abc} e_{m}^b \wedge e_{m}^c = 0 \tag{40}
\]

\[
R^{(3)}_{ab} = q \epsilon_{cda} e^c \wedge e_{b}^d = 0 \tag{41}
\]

\[
T^{(3)}_{ab} = p \epsilon_{cda} e^c \wedge e_{b}^d = 0. \tag{42}
\]

Note that for spin-3 EH or exotic gravity, where \(p = 0\), we have the zero torsion constraints: \(T^{(3)}_a = 0 = T^{(3)}_{ab}\). However, the torsion defined in \([1]\) is not zero, which resemble the supergravity
generalization of MB gravity \[20\]. On the other hand, for the spin-3 teleparallel gravity \( q = 0 \), we have that the curvatures are zero \( R^{(3)}_a = 0 = R^{(3)}_{ab} \). In the next section, we will show that the spin-3 teleparallel gravity has the same central charge that the spin-3 EH action. In this sense, they are equivalent.

Now, for the pure MB gravity theory, the AdS sector can be explicitly realized when we write the spin-connection as the sum of the Levi-Civita connection and a contorsion. Analogously, we can perform a map that leads us from the family of the four parametric model to the AdS sector with and effective cosmological constant. This transformation is given by

\[
\omega_a = \tilde{\omega}_a + \frac{p}{2} e_a, \quad \omega_{ab} = \tilde{\omega}_{ab} + \frac{p}{2} e_{ab}
\]

where \( \tilde{\omega}_a \) and \( \tilde{\omega}_{ab} \) satisfy \( T^{(3)}_a (\tilde{\omega}) = 0 = T^{(3)}_{ab} (\tilde{\omega}) \). The remaining equations of motion become

\[
R^{(3)}_a (\tilde{\omega}) - \frac{1}{l^2} \left( \frac{1}{2} \epsilon^{abc} e^b \wedge e^c - 2 \sigma \epsilon^{abc} e^b_e \wedge e^d \right) = 0 \quad (44)
\]

\[
R^{(3)}_{ab} (\tilde{\omega}) + \frac{1}{l^2} \epsilon^{cd(a e^c \wedge e^d)} = 0 \quad (45)
\]

which are exactly the equations of motion for spin-3 EH action with and effective cosmological constant \(-l^{-2}\).

### 3.1 Asymptotic Symmetries and Spin-3 MB Black Hole

The asymptotic symmetries and black hole solution of pure MB gravity has been explored using the CS formulation in \[12, 13\]. The same machinery can be used for the spin-3 MB theory. We will closely follow the spin-3 EH case found in \[33\]. First, let us consider the manifold \( M \) with the topology of a cylinder with radial, angular and temporal coordinates denoted by \((\rho, \varphi, t)\), respectively.

The equation of motion \([19]\) allows the following decomposition of the gauge connection:

\[
A^\pm = b^{-1}_\pm (\rho) \left[ d + a^\pm_\varphi (t, \varphi) d\varphi + a^\pm_\rho (t, \varphi) dt \right] b_\pm (\rho)
\]

where \( b_\pm \) is a group element and \( a^\pm_\varphi \) are auxiliary connections. Before specify the explicit form of these connections, it is convenient to transform the generators from \((J_a, J_{ab})\) to the \((L^\pm_i, W^\pm_m)\) basis, where

\[
\begin{align*}
[L^\pm_i, L^\pm_j] &= (i-j) L^\pm_{i+j} \\
[L^\pm_i, W^\pm_m] &= (2i-m) W^\pm_{i+m} \\
[W^\pm_m, W^\pm_n] &= \frac{\sigma}{3} (m-n) (2m^2 + 2n^2 - mn - 8) L^\pm_{m+n}
\end{align*}
\]

and \( i, j = \pm 1, 0 \), and \( m, n = \pm 2, \pm 1, 0 \). Now, we impose the following boundary conditions on the angular component

\[
a^\pm_\varphi (t, \varphi) = L_\pm - \frac{2\pi}{k_\pm} L_\pm L_{\mp 1} + \frac{\pi}{2k_\pm} W_\pm W_{\mp 2}
\]

and for the temporal component, we write \( a^\pm_\rho = \Lambda^\pm (\mu^\pm, \nu^\pm) \), where:

\[
\Lambda^\pm (\mu^\pm, \nu^\pm) \equiv \pm (\pm L_{\pm 1} - \mu'_{\pm} L_{0} = \frac{1}{2} \left( \mu''_{\pm} - \frac{4\pi}{3k_\pm} L_{\pm} \mu_{\pm} + \frac{8\pi}{k_\pm} W_{\pm} \nu_{\pm} \right) L_{\pm 1} + \nu_{\pm} W_{\pm 2} - \nu'_{\pm} W_{\pm 1}
\]

\[
\pm \left[ \frac{1}{24} \nu''_{\pm} - \frac{4\pi}{3k_\pm} L_{\pm} \nu'_{\pm} - \frac{7}{6k_\pm} L_{\pm} \nu'_{\pm} - \frac{\pi}{3k_\pm} L_{\pm} \nu_{\pm} + \frac{4\pi^2}{k_\pm} L_{\pm} \nu_{\pm} + \frac{\pi}{2k_\pm} W_{\pm} \mu_{\pm} \right] W_{\pm 2}
\]

\[
\pm \left[ \nu''_{\pm} - \frac{8\pi}{k_\pm} L_{\pm} \nu_{\pm} \right] W_{0} - \frac{1}{6} \left( \mu''_{\pm} - \frac{20\pi}{k_\pm} L_{\pm} \nu'_{\pm} - \frac{8\pi}{k_\pm} L_{\pm} \nu_{\pm} \right) W_{\mp 1}.
\]
The function $\Lambda$ depends on the charges $\mathcal{L}_\pm, \mathcal{W}_\pm$ (arbitrary functions of $t, \varphi$) and the so called chemical potentials $\mu^\pm, \nu^\pm$. In the above equation the prime denotes angular derivatives. The boundary conditions (50), (51) are in agreement with the equation of motion (19).

The gauge parameter in (20) can be decomposed as $\lambda^\pm = b_{\pm}^{-1}(\rho)\eta^\pm(t, \varphi)b_{\pm}(\rho)$. The asymptotic form of $\eta$ that preserves the boundary condition (50) is $\eta^\pm = \Lambda^\pm(\epsilon^\pm, \xi^\pm)$, provided that the charges transform as

\[
\delta_{\epsilon, \xi} \mathcal{L}_\pm = \frac{k_\pm}{4\pi} \epsilon''_\pm \pm 2\mathcal{L}_\pm \epsilon'_\pm \pm \mathcal{L}'_\pm \epsilon_\pm \mp 3\mathcal{W}_\pm \epsilon'_\pm \mp 2\mathcal{W}'_\pm \epsilon_\pm \tag{52}
\]
\[
\delta_{\epsilon, \xi} \mathcal{W}_\pm = \pm \frac{k_\pm}{12\pi} \nu^{(5)}_\pm \pm \frac{10}{3} \sigma \mathcal{L}_\pm \nu''_\pm \pm 5\sigma \mathcal{L}'_\pm \nu'_\pm \mp 3\sigma \left( \mathcal{L}'_\pm - \frac{64\pi}{9k_\pm} \mathcal{L}^2_\pm \right) \xi'_\pm \\
\pm \frac{2\sigma}{3} \left( \xi''_\pm - \frac{16\pi}{k_\pm} (L^2_\pm)' \right) \xi_\pm \pm \mathcal{W}'_\pm \epsilon'_\pm \pm 3\mathcal{W}_\pm \epsilon_\pm \tag{53}
\]

where $\epsilon^\pm, \xi^\pm$ are arbitrary functions. We can decompose the charges $\mathcal{L}, \mathcal{W}$ in Fourier modes, and show that equations (52), (53) represent two copies of the $W_3$ algebra with two different central charges

\[
c_\pm = 6k_\pm = 24\pi \left[ \left( a + \alpha_3 \frac{b}{2} \right) l \pm \alpha_3 \right] . \tag{54}
\]

Since the values of $c_\pm$ for spin-3 EH and Teleparallel gravity coincide, as in the pure MB case, we say that the teleparallel model is equivalent to the EH action in the sense that the curvatures $R^{(3)}_a, R^{(3)}_{ab}$ are zero.

Now, the black-hole solution are obtained when the charges and chemical potentials are considered constants. Under these condition the temporal auxiliary component reduces to

\[
a^\pm_t = \pm \mu_\pm \left( L_{\pm 1} - \frac{2\pi}{k_\pm} \mathcal{L}_\pm L_{\mp 1} \mp \frac{\pi}{2k_\pm \sigma} \mathcal{W}_\pm W_{\mp 2} \right) \\
\pm \nu_\pm \left( W_{\pm 2} \mp \frac{4\pi}{k_\pm} \mathcal{W}_\pm L_{\mp 1} - \frac{4\pi}{k_\pm} L_\pm W_0 \mp \frac{4\pi^2}{k_\pm^2} L^2_\pm W_{\mp 2} \right) . \tag{55}
\]

Note that for $\nu_\pm = 0$, we have that the temporal and angular components are proportional, this is a common choice in holography since it allows to integrate the boundary term on the action, and also allows to redefine the boundary condition on the null-directions [23], [34], [35]. The boundary conditions (50), (55) in addition to the spin-connections $\omega_a, \omega_{ab}$ represent the MB black-hole.

### 4 Singular point of spin-3 Mielke-Baekler gravity:

In the previous section, we built the spin-3 MB theory from the CS formulation. This imposes a condition on the parameters: $\alpha_3\alpha_4 - a^2 \neq 0$, since the parameters $l, p, q$ must exist. Now, let us change the perspective and consider as starting point the action (20) with no restriction on the parameters $a, \Lambda, \alpha_3, \alpha_4$. This is similar to the pure MB gravity theory (2) where the parameters were free.

The equations of motion (35)-(38) are still valid. However, when

\[
\frac{a}{\alpha_3} = \frac{\alpha_4}{a} = \frac{\Lambda}{\alpha_4} = \mu \equiv \mu \tag{56}
\]

they will reduce to only two:

\[
R^{(3)}_a + \mu T^{(3)}_a + \mu^2 \epsilon_{abc} \left[ \frac{1}{2} \epsilon^b \wedge \epsilon^c - 2\sigma \epsilon^{bd} \wedge \epsilon^c d \right] = 0 \tag{57}
\]
\[
R^{(3)}_{ab} + \mu T^{(3)}_{ab} + \mu^2 \epsilon_{cd(a} \epsilon^c \wedge \epsilon^d b) = 0 \tag{58}
\]
meaning that we need further information to determine \( R_a^{(3)}, T_a^{(3)} \) and \( R_{ab}, T_{ab}^{(3)} \). Furthermore, for the singular point \([50]\) we have \( \alpha_3 \alpha_4 - a^2 = 0 \). Therefore, it is outside the AdS sector and cannot be written as the difference of two CS actions.

We can follow a similar procedure as the one performed in [18] in the context of pure MB gravity. Since at the singular point there is not enough information about the curvature and torsion components, we can impose vanishing torsion \( p = 0 \). This can be accomplished from the CS sector if we write

\[
\alpha_3 \alpha_4 - a^2 = \varepsilon, \quad a \Lambda + (\alpha_4)^2 = \frac{\varepsilon}{\mu}, \quad \alpha_3 \Lambda + \alpha_4 a = 0
\]

where \( \varepsilon \) is a parameter, that will eventually go to zero to reach the singular point. Under \([52]\), the CS levels become, when \( \varepsilon \to 0 \): \( k_+ = 8 \pi a l \) and \( k_- = 0 \), meaning that the two CS levels of the original theory degenerate to one. In addition, we still need to introduce Lagrange’s multipliers due to the zero torsion condition, making this model similar to the chiral point of spin-3 topologically massive gravity. In the following section, we will use the canonical formalism to understand the singular point without imposing further information on the torsion.

### 4.1 Canonical analysis at the singular point:

Since the structure of spin-3 MB gravity is no longer the difference of two CS actions, we need to analyse its canonical structure. The canonical analysis of pure MB gravity has been performed in [36], at its singular point in [18] and the canonical analysis of topologically massive gravity at the chiral point in [37]. It is also worth noticing that the canonical charges of topologically massive gravity has been computed in [38].

Let us begin the canonical analysis introducing a foliation of space-time: We denote the temporal coordinate by \( x^0 = t \) and by \( x^i \) the angular and radial coordinates. After eliminating some boundary terms, it is possible to write the spin-3 MB Lagrangian as

\[
\mathcal{L}_{\text{Spin}^{-3}}^{\text{MB}} = a \mu e^{ij} e_0^j \partial_0 e_{ai} - 2 \sigma a \mu e^{ij} e^b_j \partial_0 e_{abi} + 2 a \varepsilon^{ij} \left( e_0^i + \frac{1}{2 \mu} \omega_0^i \right) \partial_0 \omega_{ai} - 4 a \varepsilon^{ij} \sigma \left( e_{ab}^0 + \frac{1}{2 \mu} \omega_{ab}^0 \right) \partial_0 \omega_{abi} - \left( e_{0}^a + \frac{1}{\mu} \omega_0^a \right) J_a - \left( e_{0}^{ab} + \frac{1}{\mu} \omega_0^{ab} \right) J_{ab}
\]

where the quantities \( J_a, J_{ab} \) are defined by

\[
J_a \equiv - a \varepsilon^{ij} \left[ R_{aij} + \mu T_{aij} + 2 \mu^2 \epsilon_{abc} \left( \frac{1}{2} e_j^0 e_j^c - 2 \sigma e_j^b \epsilon_j^d e_j^c + \frac{1}{2} \right) \right] \quad (61)
\]

\[
J_{ab} \equiv 2 a \varepsilon^{ij} \left[ R_{bij} + \mu T_{bij} + \mu^2 \epsilon_{mn(a} \epsilon^m_{i} \epsilon_{b)}^n + \mu^2 \epsilon_{mn(a} e_j^m e_j^n b)_{ij} - \mu^2 \epsilon_{mn(a} e_j^m e_j^n b)_{i} \right].
\]

Note that these quantities are related to the left side of the equations of motion \([50], [51]\). Then, we must expect, from the Hamiltonian analysis, that \( J_a, J_{ab} \) are zero.

Let us denote \( (\pi^a, \pi^{ab}, \Pi^a, \Pi^{ab}) \) as the canonical momenta of \( (e_{ai}, e_{abi}, \omega_{ai}, \omega_{abi}) \), respectively. Then, since there are no temporal derivatives of the fields \( (e_{ai}, e_{abi}, \omega_{ai}, \omega_{abi}) \), we have the following primary constraints:

\[
\phi^a \equiv \pi^a \equiv \pi^{ab} \equiv \Pi^a \equiv \Pi^{ab} \approx 0, \quad (63)
\]

and from the definition of canonical momenta of the remaining fields we have

\[
\phi^{ai} \equiv \pi^{ai} - a \mu e^{ij} e_j^a \approx 0, \quad \Phi^{ai} \equiv \Pi^{ai} - 2 a \varepsilon^{ij} \left( e_0^j + \frac{1}{2 \mu} \omega_0^j \right) \approx 0 \quad (64)
\]

\[
\phi^{abi} \equiv \pi^{abi} + 2 a \sigma \mu \varepsilon^{ij} e_j^{ab} \approx 0, \quad \Phi^{abi} \equiv \Pi^{abi} + 4 a \sigma \varepsilon^{ij} \left( e_0^{ab} + \frac{1}{2 \mu} \omega_0^{ab} \right) \approx 0. \quad (65)
\]
As we see, all canonical momenta will define primary constraints, which simplifies the canonical analysis. The primary Hamiltonian is given by:

\[ \mathcal{H}_T = \left( e_0^a + \frac{1}{\mu} \omega_0^a \right) \mathcal{J}_a + \left( e_0^b + \frac{1}{\mu} \omega_0^b \right) \mathcal{J}_{ab} + u_{a\alpha} \phi^{a\alpha} + u_{aba} \phi^{aba} + v_{a\alpha} \Phi^{a\alpha} + v_{aba} \Phi^{aba} \] (66)

where the \( u \)'s and \( v \)'s are Lagrange multipliers. The consistency condition of the four constraints (63) lead us to only two secondary constraints

\[ \mathcal{J}_a \approx 0, \quad \mathcal{J}_{ab} \approx 0 \] (67)
as predicted, and in perfect agreement with the equations of motion. The consistency condition for (64), (65) gives, again, only two relations:

\[ \partial_0 \omega_i^a + \mu \partial_i e_i^a - \mu u_i^a - v_i^a \approx 0 \] (68)
\[ \partial_0 \omega_i^b + \mu \partial_i e_i^b - \mu u_i^b - v_i^b \approx 0. \] (69)

where we used the equations of motions to simplify the expressions. Note that we can solve the above equations for \( (v_i^a, v_i^b) \) or for \( (u_i^a, u_i^b) \). This indeterminacy is related to the singular matrix build with the Poisson brackets of the constraints (64), (65). The next step in the Dirac’s algorithm is to test the consistency conditions for the secondary constraints (67). Since the Poisson Brackets between \( (\mathcal{J}_a, \mathcal{J}_{ab}) \) with the constraints (64), (65) are non-zero, it looks like there will be new relations between the parameters. However, when we replace (68), (69), we notice that the consistency conditions are satisfied. Therefore, the algorithm is closed.

We can now replace (68), (69) in the Primary Hamiltonian. After eliminating some boundary terms we obtain

\[ \mathcal{H}_T = \left( e_0^a + \frac{1}{\mu} \omega_0^a \right) \tilde{\mathcal{J}}_a + \left( e_0^b + \frac{1}{\mu} \omega_0^b \right) \tilde{\mathcal{J}}_{ab} + u_{a\alpha} \chi^{a\alpha} + u_{abi} \chi^{abi} + u_{a0} \phi^{a0} + u_{a0} \phi^{a0} + v_{a0} \Phi^{a0} + v_{a0} \Phi^{a0} \] (70)

where

\[ \tilde{\mathcal{J}}_a = \mathcal{J}_a - \mu \partial_i \Phi^i_a - \mu \epsilon_{abc} \left( \omega_i^b + \mu e_i^b \right) \Phi^{ci} - 2 \mu \epsilon_{abc} \left( \omega_i^{ab} + \mu e_i^{ab} \right) \Phi^{abi} \] (71)
\[ \tilde{\mathcal{J}}_{ab} = \mathcal{J}_{ab} - \mu \partial_i \Phi^i_{ab} - \mu \epsilon_{mn(a} \left( \omega_i^{mb} + \mu e_i^{mb} \right) \Phi^{ni]} + 2 \sigma \mu \epsilon_{mn(a} \left( \omega_i^{mb} + \mu e_i^{mb} \right) \Phi^{ni]} \] (72)

and

\[ \chi^{a\alpha} \equiv \phi^{a\alpha} - \mu \Phi^{a\alpha} \quad \chi^{abi} \equiv \phi^{abi} - \mu \Phi^{abi}. \] (73)

Note that the constraints (73) appear naturally after replacing the Lagrange’s multipliers \( v_i^a, v_i^b \).

Now, it is time to classify the constraints. The set \( (\Phi^{a\alpha}, \Phi^{abi}) \) is second-class, since

\[ \left\{ \phi^{ai}(x), \phi^{bj}(y) \right\} = -\frac{2a}{\mu} \varepsilon^{ij} \eta^{ab}, \quad \left\{ \phi^{abi}(x), \phi^{cde}(y) \right\} = \frac{2a\sigma}{\mu} \varepsilon^{ijkl} \eta^{(a} \eta^{c(l} \eta^{d)k} \] (74)
The set \( (\phi^{a0}, \phi^{a0}, \Phi^{a0}, \Phi^{ab0}, \chi^{ai}, \chi^{abi}, \mathcal{J}^a, \mathcal{J}^{ab}) \) is first-class, since the non-zero Poisson brackets close an algebra

\[ \left\{ \mathcal{J}_a(x), \mathcal{J}_b(y) \right\} = -\mu \epsilon_{ab} \mathcal{J}_c(x) \] (75)
\[ \left\{ \mathcal{J}_a(x), \mathcal{J}_c(y) \right\} = -\mu \epsilon_{ab} \mathcal{J}_d(y) \] (76)
\[ \left\{ \mathcal{J}_{ab}(x), \mathcal{J}_{cd}(y) \right\} = -\mu \sigma \left( \eta^{a(c} \epsilon^{d)k} + \eta^{b(c} \epsilon^{d)k} \right) \mathcal{J}^{km}(x). \] (77)

Finally, the Poisson Brackets between the two sets are

\[ \left\{ \mathcal{J}^a(x), \Phi^{ai}(y) \right\} = -\mu \epsilon^{ab} \Phi^{ci}(x), \quad \left\{ \mathcal{J}^a(x), \Phi^{bc}(y) \right\} = -\mu \epsilon^{ab} \Phi^{cd}(x) \] (78)
\[ \left\{ \mathcal{J}^{ab}(x), \Phi^{ci}(y) \right\} = \mu \epsilon^{ab} \Phi^{cd}(x), \quad \left\{ \mathcal{J}^{ab}(x), \Phi^{cd}(y) \right\} = \sigma \mu \epsilon^{ab} \Phi^{cd}(x) \] (79)
where the Dirac’s delta has been omitted in the above brackets. The second-class constraints can be eliminated by building Dirac’s brackets. This way \( \Phi^{ai} = 0 \) and \( \Phi^{abi} = 0 \) can be used as strong equations. We will not perform that task since the important result lies in the algebra of the first-class constraints \((75)-(77)\) which, after an appropriate scaling, becomes a single copy of the \( sl(3) \) algebra \((22)-(24)\). This was expected from the limit \((59)\). However, with the canonical analysis we did not impose any further condition on the torsion components, as shown explicitly by the constraints \((61)-(62)\).

Finally, notice that the dimensions of the phase-space is \( N = 108 \), which is the counting of the variables of the spin-2 sector \((e_{a\mu}, \omega_{a\mu})\), spin-3 sector \((e_{ab\mu}, \omega_{ab\mu})\), and their respective canonical momenta. We have \( S = 18 \) second-class constraints, and \( M = 45 \) first-class constraints. This gives a total of \( N - 2M - S = 0 \) degrees of freedom, as expected from a topological theory.

5 Final considerations

In this work we presented the spin-3 Mielke-Baekler gravity theory \((26)\) with four couplings, which allow us to build the higher-spin versions of the Einstein-Hilbert, Teleparallel and Exotic gravity. We performed this generalization with the aid of the Chern-Simons formulation of Mielke-Baekler pure gravity, following a similar approach to the spin-3 generalizations of Einstein-Hilbert gravity and Topologically Massive gravity. We showed that, in the presence of non-vanishing torsion components \( T^{(3)}_a \) and \( T^{(3)}_{ab} \), the asymptotic symmetries are two copies of the \( W_3 \) algebra with different central charges \((54)\). It is worth noticing that boundary condition \((50)\) satisfies the so-called highest weight ansatz. However, as shown in \((39)\) in the context of EH gravity (and also in flat gravity \((10)\), higher-spin \((11)\) and supergravity \((32)\), it is possible to consider a different ansatz that will lead to new asymptotic symmetries, such as a higher-spin generalization of \((19)\).

Since the pure Mielke-Gravity theory is not restricted to its AdS sector, we eliminated the previous conditions on the couplings for the spin-3 theory. Then, outside the AdS sector we found a singular point where the number of equations of motions is reduced in half and we cannot determine neither torsion or curvature components. This degeneracy is also present in the canonical analysis, where some consistency conditions are redundant, since they reproduce the same constraints. The final counting of degrees of freedom is zero, meaning that the theory at the singular point is still topological.

As a future perspective, we will study the thermodynamic properties of the spin-3 MB black-hole and analyse how torsion modifies the black-hole entropy. In particular, it is known that Exotic gravity has reversed roles for mass and angular momentum, it will be interesting to see if this characteristic is still preserved in the presence of spin-3 fields.

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References

[1] J.D. Brown, M. Henneaux, Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity, Commun. Math. Phys. 104 (1986) 207.

[2] J. H. Horne, E. Witten, Conformal Gravity in Three-dimensions as a Gauge Theory, Phys.Rev.Lett. 62, (1989) 501;
[3] S. Deser, R. Jackiw, S. Templeton, Topologically Massive Gauge Theories, Annals Phys. 140 (1982) 372, Annals Phys. 281 (2000) 409, Erratum: Annals Phys. 185 (1988) 406.

[4] S. Deser, R. Jackiw, Three-Dimensional Massive Gauge Theories, Phys.Rev.Lett. 48 (1982) 975.

[5] E. A. Bergshoeff, O. Hohm, P. K. Townsend, Massive Gravity in Three Dimensions, Phys. Rev. Lett.102 (2009) 201301.

[6] E. A. Bergshoeff, O. Hohm, P. K. Townsend, More on Massive 3D Gravity, Phys. Rev. D79 (2009) 124042.

[7] E. A. Bergshoeff, O. Hohm, W. Merbis, A. J. Routh, P. K. Townsend, Chern-Simons-like Gravity Theories, Lect. Notes Phys. 892 (2015) 181.

[8] E. W. Mielke, P. Baekler, Topological gauge model of gravity with torsion, Phys. Lett. A 156, (1991) 399.

[9] P. Baekler, E. W. Mielke, F. W. Hehl, Dynamical symmetries in topological 3 − D gravity with torsion, Nuovo Cim. B107 (1992) 91.

[10] E. Witten, 2+1 dimensional gravity as an exactly soluble system, Nucl. Phys. B311, 46 (1988).

[11] M. Blagojević, M. Vasiljić, Asymptotic symmetries in 3d gravity with torsion, Phys. Rev. D67 (2003) 084032.

[12] M. Blagojević, M. Vasiljić, 3D gravity with torsion as a Chern-Simons gauge theory, Phys. Rev. D 68, (2003) 104023.

[13] M. Blagojević, M. Vasiljić, Asymptotic dynamics in 3D gravity with torsion, Phys. Rev. D68 (2003) 124007.

[14] M. Blagojević, B. Cvetković, Black hole entropy in 3 − D gravity with torsion, Class. Quant. Grav. 23 (2006) 4781

[15] M. Blagojević, B. Cvetković, Black hole entropy from the boundary conformal structure in 3D gravity with torsion, JHEP 0610 (2006) 005.

[16] S. L. Cacciatori, M. M. Caldarelli, A. Giacomini, D. Klemm, D. S. Mansi, Chern-Simons formulation of three-dimensional gravity with torsion and nonmetricity, J. Geom. Phys. 56 (2006) 2523.

[17] D. Klemm, G. Tagliabue, The CFT dual of AdS gravity with torsion, Class. Quant. Grav. 25 (2008) 035011.

[18] R. C. Santamaria, J. D. Edelstein, A. Garbarz, G. E. Giribet, On the addition of torsion to chiral gravity, Phys. Rev. D 83, (2011) 124032.

[19] B. Cvetković, D. Simić, Near-horizon geometry with torsion, Phys.Rev. D99 (2019) 024032.

[20] A. Giacomini, R. Troncoso, S. Willson, Three-dimensional supergravity reloaded, Class. Quant. Grav. 24 (2007) 2845.

[21] M. A. Vasiliev, Nonlinear equations for symmetric massless higher spin fields in (A)dS(d), Phys. Lett. B567 (2003) 139.

[22] A. Sagnotti, Notes on Strings and Higher Spins, J.Phys. A46 (2013) 214006.
[23] A. Campoleoni, S. Fredenhagen, S. Pfenninger, S. Theisen, \textit{Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields}, JHEP 1011 (2010) 007.

[24] A. Campoleoni, \textit{Higher Spins in D = 2 + 1}, Subnucl. Ser. 49 (2013) 385.

[25] B. Chen, J. Long, J.-b. Wu, \textit{Spin-3 topologically massive gravity}, Phys. Lett. B705 (2011).

[26] A. Bagchi, S. Lal, A. Saha, B. Sahoo, \textit{Topologically Massive Higher Spin Gravity}, JHEP 1110 (2011) 150.

[27] M. R. Setare, H. Adami, \textit{Entropy Formula and Conserved Charges of Spin-3 Chern-Simons-Like Theories of Gravity}, Adv. Theor. Math. Phys. 23 (2019) 593.

[28] P. K. Townsend, B. Zhang, \textit{Thermodynamics of “Exotic” Bañados-Teitelboim-Zanelli Black Holes}, Phys. Rev. Lett. 110, 241302 (2013).

[29] M. Bañados, \textit{Global charges in Chern-Simons theory and the 2 + 1 Black Hole}, Phys. Rev. D52, 5816 (1996).

[30] M. Bañados, \textit{Notes on Black Holes and Three dimensional Gravity}, AIP Conf. Proc. 490, 198 (1999).

[31] O. Miskovic, R. Olea, \textit{On boundary conditions in three-dimensional AdS gravity}, Phys.Lett. B640 (2006) 101-107

[32] C. Bunster, M. Henneaux, A. Perez, D. Tempo, R. Troncoso, \textit{Generalized Black Holes in Three-dimensional Spacetime}, JHEP 1405 (2014) 031.

[33] M. Gutperle, P. Kraus, \textit{Higher Spin Black Holes}, JHEP1105, 022 (2011).

[34] M. Ammon, M. Gutperle, P. Kraus, E. Perlmutter, \textit{Spacetime Geometry in Higher Spin Gravity}, JHEP1110, 053 (2011).

[35] M. Blagojević, B. Cvetković, \textit{Canonical structure of 3D gravity with torsion}, Published in “Trends in General Relativity and Quantum Cosmology”. Volume 2. Edited by Charles V. Benton. N.Y., Nova Science Publishers, 85 (2006), arXiv:gr-qc/0412134.

[36] D. Grumiller, R. Jackiw, N. Johansson, \textit{Canonical analysis of cosmological topologically massive gravity at the chiral point}, Published in "Fundamental Interactions: A Memorial Volume for Wolfgang Kummer", World Scientific, 363 (2010), arXiv:0806.4185.

[37] O. Miskovic, R. Olea, \textit{Background-independent charges in Topologically Massive Gravity}, JHEP 0912 (2009) 046.

[38] D. Grumiller, M. Riegler, \textit{Most general AdS$_3$ boundary conditions}, JHEP 1610 (2016) 023.

[39] D. Grumiller, W. Merbis, M. Riegler, \textit{Most general flat space boundary conditions in three-dimensional Einstein gravity}, Class. Quant. Grav. 34 (2017) no.18, 184001.

[40] C. Krishnan, A. Raju, \textit{Chiral Higher Spin Gravity}, Phys. Rev. D95 (2017) no.12, 126004.

[41] C. E. Valcárcel, \textit{New boundary conditions for (extended) AdS$_3$ supergravity}, Class. Quant. Grav. 36 (2019) 065002.