Supersymmetric Duality in Superloop Space

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Abstract

In this paper we constructed superloop space duality for a four dimensional supersymmetric Yang-Mills theory with $N = 1$ supersymmetry. This duality reduces to the ordinary loop space duality for the ordinary Yang-Mills theory. It also reduces to the Hodge duality for an abelian gauge theory. Furthermore, the electric charges, which are the sources in the original theory, appear as monopoles in the dual theory. Whereas, the magnetic charges, which appear as monopoles in the original theory, become sources in the dual theory.

1 Introduction

An important concept in the electromagnetism is the existence of the Hodge duality. The symmetry and topological concepts inherent in field theories have been analysed using this duality [1]-[4]. In fact, this duality has been thoroughly studied and many interesting physical consequences arising from this duality have also been analysed [5]-[13]. It is known that electrodynamics is dual under Hodge star operation, $\ast F_{\mu\nu} = -\epsilon_{\mu\nu\tau\rho}F_{\tau\rho}/2$. This is because the field tensor for pure electrodynamics, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, satisfies, $\partial^\nu F_{\mu\nu} = 0$. This field tensor also satisfies the Bianchi identity, $\partial^\nu \ast F_{\mu\nu} = 0$. This field equation for pure electrodynamics can be interpreted as the Bianchi identity for $\ast F_{\mu\nu}$, because the Hodge star operation is reflexive, $(\ast F_{\mu\nu}) = -F_{\mu\nu}$. So, we can express $\ast F_{\mu\nu}$ in terms of a dual potential, $\tilde{A}_\mu$, such that $\ast F_{\mu\nu} = \partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu$. It has also been known that the existence of magnetic monopoles is equivalent to (electric) charge quantization which in turn is equivalent to the electromagnetic gauge group being compact (i.e. $U(1)$) [14]. However, a non-abelian version of this duality and its consequences for non-abelian monopoles can only be analysed in the framework of loop space [15]-[16].

For Yang-Mills theory, the field tensor, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu]$, again satisfies, $D^\nu F_{\mu\nu} = 0$, where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative. It also satisfies the Bianchi identity, $D^\nu \ast F_{\mu\nu} = 0$. However, now this does not imply the existence of a dual potential because the covariant derivative in
the Bianchi identity involves the potential $A_\mu$ and not some dual potential $\tilde{A}_\mu$, appropriate to $^* F_{\mu\nu} = 0$. In fact, it has been demonstrated that in certain cases no such solution for such a dual potential exist even for the ordinary Yang-Mills theory [15]-[16]. Thus, the Yang-Mills theory is not dual under the Hodge star operation. However, it is possible to construct a generalized duality transformation for the ordinary Yang-Mills theory in the loop space, such that for the abelian case, it reduces to the Hodge star operation [17]-[18].

This duality has been used for studying 't Hooft’s order-disorder parameters [19]. For any two spatial loops $C$ and $C'$ with the linking number $n$ between them, and the gauge symmetry generated by the gauge group $su(N)$, the order-disorder parameters satisfy, $A(C)B(C') = B(C')A(C)exp(2\pi in/N)$. Thus, it has been demonstrated that in certain cases no such solution for such a dual potential exist even for the ordinary Yang-Mills theory [15]-[16]. Thus, the Yang-Mills theory is not dual under the Hodge star operation. However, it is possible to construct a generalized duality transformation for the ordinary Yang-Mills theory in the loop space, such that for the abelian case, it reduces to the Hodge star operation [17]-[18].

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parameterized by $\xi^A = (\xi^a, \xi^s, \xi^{\mu})$, along a curve $C$,

$$C : \{\xi^A(s) : s = 0 \rightarrow 2\pi, \xi^A(0) = \xi^A(2\pi)\}, \quad (1)$$

where $\xi^A(0) = \xi^A(2\pi)$ is a fixed point on this curve. The space of all such superfunctions parameterizes the superloop space. A functional on this superloop space can be constructed as

$$\Phi[\xi] = P_s \exp i \int_0^{2\pi} \left[ \Gamma^{xa}(\xi(s)) \frac{d\xi^a(s)}{ds} + \Gamma^a(\xi(s)) \frac{d\xi^s(s)}{ds} + \Gamma^a(\xi(s)) \frac{d\xi^\mu(s)}{ds} \right]$$

$$= P_s \exp i \int_0^{2\pi} \Gamma^A(\xi(s)) \frac{d\xi^A(s)}{ds}, \quad (2)$$

here $P_s$ denotes ordering in $s$ increasing from right to left and the derivative in $s$ is taken from below. This loop space variable is a scalar superfield from the supersymmetric point of view, and can be projected to component superloops. In particular, we have $[\Phi[\xi]] = \phi[\xi]$, which in Wess-Zumino gauge is given by

$$\phi[\xi] = P_s \exp i \int_0^{2\pi} A^\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds} \quad (3)$$

We can also define the parallel transport from a point $\xi(s_1)$ to $\xi(s_2)$ along path parametrized by $\xi$ as

$$\Phi[\xi : s_1, s_2] = P_s \exp i \int_{s_1}^{s_2} \Gamma^A(\xi(s)) \frac{d\xi^A}{ds} \quad (4)$$

Now using $\Phi[\xi]$, we can define a gauge Lie algebra valued $F_A[\xi][s]$ as

$$F_A[\xi][s] = \sqrt{2} \Phi^{-1}[\xi] \delta_A(\xi) \Phi[\xi]$$

$$= \Phi^{-1}[\xi : s, 0][H^{AB}(\xi(s)) \Phi[\xi : s, 0] \frac{d\xi^B(s)}{ds}], \quad (5)$$

where $\delta_A(\xi) = \delta/\delta \xi^A(s) = (\delta/\delta \xi^a(s), \delta/\delta \xi^s(s))$. Here we first followed a path to $s$ and then turn backwards along the same path. Thus, the phase factor for the segment of the superloop beyond $s$ did not contribute and $H^{AB}(\xi(s))$ was obtained because of the infinitesimal circuit generated at $s$.

It is convenient at this stage to define a functional curl and a functional divergence for these superloop space variables as

$$\text{curl} F[\xi][s]_{AB} = \delta_A(\xi) F_B[\xi][s] - \delta_B(\xi) F_A[\xi][s],$$

$$\text{div} F[\xi][s] = \delta^A(\xi) F_A[\xi][s]. \quad (6)$$

These superloop variables are highly redundant and have to be constrained by an infinite set of conditions which can be expressed by the vanishing of the superloop space curvature $G_{AB}[\xi][s] = (\text{curl} F[\xi][s])_{AB} + i[F_A[\xi][s], F_B[\xi][s]] = 0$. Now we construct $E_A[\xi][s]$ from $F_A[\xi][s]$ as follows,

$$E_A[\xi][s] = \Phi[\xi : s, 0] F_A[\xi][s] \Phi^{-1}[\xi : s, 0], \quad (7)$$

So, $E_A[\xi][s]$ is obtained from a parallel transport of $F_A[\xi][s]$. Thus, $E_A[\xi][s]$ depends only on a segment of the loop $\xi(s)$ around $s$ and is therefore a segmental
variable rather than a full loop variable. However, when this segment shrinks to a point, we have $E^A[A_\xi][s] \to H^{AB}(\xi(s))d\xi_B(s)/ds$. This limit has to be taken only after all the loop operations such as loop differentiation has been performed. This is because all these loop operations require a segment of the loop on which they can operate. Now we can define a functional curl and a functional divergence for $E_A[A_\xi][s]$ as

$$(\text{curl } E[A_\xi])_{AB} = \delta_A(s)E_B[A_\xi][s] - \delta_B(s)E_A[A_\xi][s],$$

$$\text{div } E[A_\xi] = \delta_A(s)E_A[A_\xi][s].$$

We first note that

$$\delta_A(s')E_B[A_\xi][s] = \Phi[\xi : s, 0][\delta_A(s')]F_B[A_\xi][s] + i\Theta(s - s')[F_A[A_\xi][s], F_B[A_\xi][s]]\Phi^{-1}[\xi : s, 0],$$

where $i\Theta(s - s')$ is the Heaviside function. So, the superloop space curvature can now be written as $G_{AB}[\xi, s] = \Phi[\xi : s, 0](\text{curl } E[A_\xi])_{AB}\Phi^{-1}[\xi : s, 0]$ and thus the constraints can be fixed as $(\text{curl } E[A_\xi])_{AB} = 0$.

3 Duality

For ordinary gauge theories, it is possible to construct a duality using loop space formalism, such that it reduces to the Hodge star operation for the abelian case [17]-[18]. In this section we will further generalize this duality from a ordinary Yang-Mills theory to a supersymmetric Yang-Mills theory. In order to achieve this we define a new variable $\tilde{E}_A[\eta][t]$ which is dual to $E_A[A_\xi][s]$. Here $\eta$ is another parameter loop which is parameterized by $t$, $\eta(t) = \eta^a(t)\theta_a + (\gamma^\mu\eta_\mu(t))\theta_a\theta_b$. This dual variable is constructed as follows,

$$\omega^{-1}[\eta(t)]\tilde{E}^A[\eta][t] = -\frac{2}{N}\epsilon^{ABCD}d\eta_B(t)\int dt d\xi ds E_C[A_\xi][s]d\xi_D(s)/ds \times \left[\frac{d\xi^F(s)}{ds} - \frac{d\xi_F(s)}{ds}\right]^{-2}\delta(\xi(s) - \eta(t)),$$

where $N$ is a normalization constant. Here $\omega[\eta(t)]$ is a local rotational matrix which accounts for transforming the quantities from a direct frame to the dual frame. In the integral $E_C[A_\xi][s]$ depends on a little segment from $s_-$ to $s_+$, such that the limit $\epsilon \to 0$ is taken only after integration, where $\epsilon = s_+ - s_-$. As we may need to calculate the loop derivative of $\tilde{E}^A[\eta][t]$, so we regard $\tilde{E}^A[\eta][t]$ as a segmental quantity depending on a segment from $t_-$ to $t_+$ and only after differentiatation the limit $\epsilon' \to 0$ is taken, where $\epsilon' = t_+ - t_-$. This limit is taken before the limit $\epsilon \to 0$ for the integral. Thus, we can take $\epsilon' < \epsilon$, and the $\delta$-function now ensures that $\xi(s)$ coincides from $s = t_-$ to $s = t_+$ with $\eta(t)$. After the limit is taken and the segment shrinks to a point, we have $E^A[\eta][t] \to \tilde{H}^{AB}[\eta(t)]d\eta_B(t)/dt$. Here $\tilde{H}^{AB}$ can be constructed from a dual potential. Thus, this superloop space duality implies the existence of a dual potential $\tilde{\Gamma}_A = (\Gamma_{oa}, \Gamma_a, \Gamma_a)$, such that $[\nabla_A, \nabla_B] = \tilde{H}^{AB}$ where $\nabla_A = D_A - i\Gamma_A$.

It may be noted that if we used $[\Phi[\xi]] = \phi[\xi]$ as the loop space variable, then this duality would reduce to the ordinary duality for the ordinary Yang-Mills fields. So, if we use $[\Phi[\xi]] = \phi[\xi]$ as the loop space variable, then we
can construct $E_\mu[\xi|s]$ from $F_\mu[\xi|s]$, where $F_\mu[\xi|s]$ is the loop space connection corresponding to loop variable $[\Phi[\xi]] = \phi[\xi]$, in the Wess-Zumino gauge. Then $\tilde{E}_\mu|t|t$, which is dual to $E_\mu[\xi|s]$, is given by

$$\omega^{-1}[\eta(t)]\tilde{E}_\mu|\eta|t|\omega[\eta(t)] = -\frac{2}{N} \epsilon^{\mu\nu\tau\rho} \frac{d\eta_\mu(t)}{dt} \int D\xi ds E_\tau[\xi|s] \frac{d\xi_\rho(s)}{ds} \times \left[ \frac{d\xi^\sigma(s)}{ds} \frac{d\xi_\rho(s)}{ds} \right]^{-2} \delta(\xi(s) - \eta(t)), \quad (11)$$

If we let the segmental width of $\tilde{E}_\mu|\eta|t$ go to zero, then we can write

$$\omega^{-1}[x]\tilde{F}_{\mu\nu}[x]\omega[x] = -\frac{2}{N} \epsilon^{\mu\nu\tau\rho} \int D\xi ds E_\tau[\xi|s] \frac{d\xi_\rho(s)}{ds} \times \left[ \frac{d\xi^\sigma(s)}{ds} \frac{d\xi_\rho(s)}{ds} \right]^{-2} \delta(x - \eta(t)), \quad (12)$$

Here we first do the integration before taking the limit to zero. Thus, in the abelian case, when we take the the limit $\epsilon \to 0$, we obtain [15]

$$\tilde{F}_{\mu\nu}[x] = -\frac{2}{N} \epsilon^{\mu\nu\tau\rho} \int D\xi ds E_\tau[\xi|s] \frac{d\xi_\rho(s)}{ds} \times \left[ \frac{d\xi^\sigma(s)}{ds} \frac{d\xi_\rho(s)}{ds} \right]^{-2} \delta(x - \eta(t)) = -\frac{1}{2} \epsilon^{\mu\nu\tau\rho} F_{\tau\rho}[x]. \quad (13)$$

Now identifying $\tilde{F}_{\mu\nu}$ with $\ast F_{\mu\nu}$, we obtain the Hodge star operation for ordinary electrodynamics. Thus, for the ordinary abelian gauge theory, this duality reduces to the usual Hodge duality.

### 4 Sources and Monopoles

We will shown in this section that this duality in the superloop space transforms the electric charges, which are the sources in the original theory, into monopoles in the dual theory. It also transforms the magnetic charges, which are monopoles in the original theory, into sources in the dual theory. In order to prove this result, it is useful to first show that this duality is invertible. This can be demonstrated by first defining $E_A[\zeta|u]$ as,

$$\omega^{-1}[\zeta(u)]E_A[\zeta|u]\omega[\zeta(u)] = -\frac{2}{N} \epsilon^{ABCD} \frac{d\zeta_B(u)}{du} \int D\eta dt \tilde{E}_C[\eta|t] \frac{d\eta_\rho(t)}{dt} \times \left[ \frac{d\eta^\sigma(t)}{dt} \frac{d\eta_\rho(t)}{dt} \right]^{-2} \delta(\eta(t) - \zeta(u)), \quad (14)$$

where $\zeta_B(u)$ is a new loop parameterized by $u$. Now we define $A^A[\zeta(u)]$ as

$$A^A[\zeta(u)] = \frac{2}{N} \epsilon^{ABCD} \frac{d\zeta_B(u)}{du} \int D\eta dt \omega^{-1}[\eta(t)] \tilde{E}_C[\eta|t]\omega[\eta(t)]$$

Here we first do the integration before taking the limit to zero. Thus, in the abelian case, when we take the the limit $\epsilon \to 0$, we obtain [15]
Thus, we obtain,

\[
\begin{align*}
\times \frac{d\eta(t)}{dt} \left[ \frac{d\eta^F(t)}{dt} \frac{d\eta^G(t)}{dt} \right]^{-2} & \delta(\eta(t) - \zeta(u)) \\
= -\frac{4}{N} \epsilon^{ABCD} \frac{d\zeta_{B}(u)}{du} \int D\eta D\xi dt ds \frac{d\eta_D(t)}{dt} \frac{d\eta^Q(t)}{dt} \\
\times \left[ \frac{d\eta^F(t)}{dt} \frac{d\eta^G(t)}{dt} \right]^{-2} & \delta(\eta(t) - \zeta(u)) E^W[\xi|s] \\
\times \frac{dE^F(s)}{ds} \left[ \frac{dE^Y(s)}{ds} \frac{d\xi_Y(s)}{ds} \right]^{-2} & \delta(\xi(s) - \eta(t)) e_{CQWE}. \quad (15)
\end{align*}
\]

Thus, we obtain,

\[
\omega^{-1}[\zeta(u)] E^A[\xi|u] \omega[\zeta(u)] = -\frac{2}{N} \epsilon^{ABCD} \frac{d\zeta_{B}(u)}{du} \int D\eta dt \tilde{E}_C[\eta|t] \frac{d\eta_D(t)}{dt} \\
\times \left[ \frac{d\eta^F(t)}{dt} \frac{d\eta^G(t)}{dt} \right]^{-2} & \delta(\eta(t) - \zeta(u)). \quad (16)
\]

Now identifying \(\zeta(u)\) with \(\xi(s)\), we obtain the desired result that this duality is invertible.

The color electric charge is the source term in the supersymmetric Yang-Mills theory. Thus, it can be defined as the non-vanishing of \(\nabla^C H_{BC}\). Alternately, it can also be defined as the non-vanishing of \(\text{div} F[\xi|s]\). Furthermore, as \(\text{div} E[\xi|s] = \Phi[\xi : s_1, 0] \text{div} F[\xi|s]] \Phi^{-1}[\xi : s_1, 0]\), so the color electric charge can also be defined as the non-vanishing of \(\text{div} E[\xi|s]\). Similarly, as the the color magnetic charge is a monopole in the supersymmetric Yang-Mills theory, it is characterized by non-vanishing of \(G_{AB}[\xi, s]\). So, the color magnetic charge can be defined as the non-vanishing of \((\text{curl} E[\xi|s])_{AB}\). A monopole in the dual theory is also characterized by non-vanishing of \((\text{curl} \tilde{E}[\eta|t])_{AB}\), and a source in the dual theory is defined as the non-vanishing of \(\text{div} \tilde{E}[\eta|t]\). So, under the duality transformation a electric charge in the original theory should appear as a magnetic monopole in the dual theory. So, the non-vanishing of \(\text{div} E[\xi|s]\) should imply the non-vanishing of \((\text{curl} \tilde{E}[\eta|t])_{AB}\). Furthermore, a magnetic monopole in the original theory should appear as the source term in the dual theory. So, the non-vanishing of \((\text{curl} E[\xi|s])_{AB}\) should imply the non-vanishing of \(\text{div} \tilde{E}[\eta|t]\). Now as \(\eta(t)\) coincides with \(\xi(s)\) from \(s = t-\) to \(s = t+\), so we can write

\[
\frac{\delta}{\delta\eta_M(t)} \left( \omega^{-1}[\eta(t)] E^A[\eta|t] \omega[\eta(t)] \right) \epsilon_{MANP} = -\frac{2}{N} \epsilon^{ABCD} \frac{d\eta_B(t)}{dt} \int D\xi ds \delta E_C[\xi|s] \frac{d\eta_D(s)}{ds} \\
\times \left[ \frac{dE^F(s)}{ds} \frac{d\xi_F(s)}{ds} \right]^{-2} & \delta(\xi(s) - \eta(t)) \epsilon_{MANP}. \quad (17)
\]

Here we have performed the integration by parts with respect to \(D\xi\). This expression can be simplified to the following expression,

\[
\left( \omega^{-1}[\eta(t)](\text{curl} \tilde{E}[\eta|t])_{AB} \omega[\eta(t)] \right) = -\frac{1}{N} \int D\xi ds A_{AB}(t, s) \text{div} E[\xi|s] \\
\times \left[ \frac{d\xi_F(s)}{ds} \frac{d\xi_F(s)}{ds} \right]^{-2} & \delta(\xi(s) - \eta(t)), \quad (18)
\]

6
where
\[ A_{AB}(t, s) = \left[ \frac{d\eta^C(t)}{dt} \frac{d\xi^D(s)}{ds} - \frac{d\eta^D(t)}{dt} \frac{d\xi^C(s)}{ds} \right] \epsilon_{ABCD}. \]

(19)

Now if \( \text{div}E[\xi|s] = 0 \), then \( \text{curl} E[\eta|t]_{AB} = 0 \). As the duality is invertible, we can also show that if \( \text{div} \tilde{E}[\xi|s] = 0 \), then \( \text{curl} E[\eta|t]_{AB} = 0 \). So, an electric charge which is a source in the original theory appears as a monopole in the dual theory, and magnetic charge which is a source in the dual theory appears as a monopole in the original theory.

5 Conclusion

In this paper we have analysed a four dimensional pure Yang-Mills theory with \( \mathcal{N} = 1 \) supersymmetry in superloop space formalism. We have constructed a generalized duality in superloop space, for this theory. Under this generalized duality transformation the electric charges which appear as sources in the original theory become monopoles in the dual theory. Furthermore, the magnetic charges which appear monopoles in the original theory become sources in the dual theory. This duality reduces to the ordinary loop space duality for ordinary Yang-Mills theory. As the loop space duality for ordinary Yang-Mills theory reduces to the Hodge star operation in the abelian case, so, this generalized duality transformation also reduces to the Hodge star operation for ordinary electrodynamics.

It may be noted that the existence of a duality for ordinary Yang-Mills theory has many interesting physical consequences [21]-[29]. It will be interesting to construct a supersymmetric version of these results using the results of this paper. Thus, the results of this paper can be used to construct a supersymmetric Dualized Standard Model. It will also be interesting to analyses the phenomenological consequences of this model. In particular, we expect to have a dual symmetry corresponding to the super-gauge symmetries of the supersymmetric Standard Model. It will also be interesting to generalize the results of this paper to theories with greater amount of supersymmetry. The results obtained in this paper can also be used for analysing monopoles in the ABJM theory [30]. It may be noted that the supersymmetry of the ABJM theory is expected to get enhanced because of monopole operators [37]-[38]. Thus, the formalism developed in this paper can be find application in the supersymmetry enhancement of the ABJM theory.

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