Further progress in control of localized nonlinear waves

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Abstract. Previously found mechanical system is studied which consists of the nonlinearly elastic layer subjected to external loading. The last is assumed to be a distributive control which causes in particular, localization of longitudinal nonlinear strain waves. Various types of localized nonlinear waves are achieved due to a suitable choice of the external loading. Wave profiles arising at variations around this optimal choice of the control are studied.

1. Introduction

The localized wave solutions to nonlinear equations contain restrictions concerning their shapes, velocities and initial phases. Only particular solutions of nonintegrable nonlinear wave equations may be obtained by direct methods \cite{1,2}. More general localized wave solutions can be obtained only numerically, however, their stable propagation depends on many factors including choice of initial and boundary conditions.

Recently, a method of control of nonlinear waves has been developed \cite{4–6}. The algorithm of distributed feedback control allows us to achieve the desired profile of localized wave and its stable propagation of permanent shape and velocity. In particular, it was found \cite{4} that the most efficient control of the solution of the sine-Gordon equation is achieved when the control is organized as follows,

\[
U_{tt} - U_{xx} + \sin U + u(x,t) = 0
\]  

(1)

where the control function is

\[
u(x,t) = \gamma \left( \alpha_1(U - U^*) - \alpha_2(U_t - U^*_t) \right),
\]

where \(U(x,t)^*\) is the target function or a desired wave solution.

A natural question arises whether such control terms relate to reality. For this purpose, a mechanical system has been described in \cite{6} whose model equation is Eq. (1). This system is a horizontal nonlinearly elastic layer laying on a visco-elastic foundation and subjected to an external loading from the upper lateral surface. Then propagation of shear strain waves along the layer is described by Eq. (1) where the control function is described by external loading.
However, propagation of longitudinal waves in the layer is described by another model equation with a control. In this paper, the new algorithm of control following from the mechanical problem is examined. A tendency to some localized waves of permanent shape and velocity is revealed, and suitable loading is found. Variations in the wave shape due to the variations in the loading around its suitable form are studied.

2. Statement of the problem

The model nonlinear equation for longitudinal strain waves in a layer lying on the visco-elastic foundation and subjected to an external loading is [6]:

\[ \rho U_{tt} - \frac{4\mu(\lambda + \mu)}{(\lambda + 2\mu)^2} U_{xx} + \beta \sin \left( \frac{2}{h} U \right) + v(x, t) = 0, \]  

where \( v(x, t) \) is the control function,

\[ v(x, t) = \frac{\lambda}{\lambda + 2\mu} k_d \left( \alpha_1 (U - U^*)_x + \alpha_2 (U_t - U^*_t)_x \right). \]

At the same time, the target function \( U^* \) is connected with the tangentional external stress,

\[ f_\tau = \frac{\lambda}{\lambda + 2\mu} k_d \left( \alpha_1 U^*_x + \alpha_2 U^*_t \right), \]

Other notations are: \( \lambda \) and \( \mu \) are the Lamé coefficients, \( k_d > 0 \) is a friction coefficient at the upper lateral surface, \( \alpha_1, \alpha_2 \) are the coefficients of the visco-elastic foundation model.

Equation (2) can rewritten in a canonical form (1) by introducing scales for the variables,

\[ x = \sqrt{\frac{2\mu h(\lambda + \mu)}{\beta(\lambda + 2\mu)^2}} \tilde{x}, \quad t = \sqrt{\frac{\rho h}{2\beta}} \tilde{t}, \quad U = \frac{h}{2} \tilde{U}. \]

Let us introduce

\[ \tilde{\gamma} = \frac{\lambda k_d h}{2\beta(\lambda + 2\mu)}, \quad \tilde{\alpha}_1 = \sqrt{\frac{\beta(\lambda + 2\mu)^2}{2\mu h(\lambda + \mu)}} \alpha_1, \quad \tilde{\alpha}_2 = \frac{\beta(\lambda + 2\mu)}{h} \sqrt{\frac{1}{\rho\mu(\lambda + \mu)}} \alpha_2. \]

Then omitting tildes for simplicity, one obtains the control function of the form

\[ u(x, t) = \gamma \left( \alpha_1 (U - U^*)_x + \alpha_2 (U_t - U^*_t)_x \right). \]  

We will consider the control problem of achievement of the desired wave profile from initially undisturbed state which differs from our previous studies where transformation of some nonzero initial disturbance was studied.

The first choice of the target function is

\[ U^*(x, t) = A \text{sech}^2(k(x - Wt - x_1)) - \frac{1}{2} A \text{sech}^2(k(x - Wt - x_2)), \]

where \( W \) is a constant phase velocity, \( A \) and \( k \) are the profile parameters. The profile of this wave consists of a hump and a hole parts that correspond to the tensile and compression parts respectively, see the dashed profile in the first sketch in Fig.1.

Another choice of the target function corresponds to the breather-shaped wave:

\[ U^* = 4 \text{sech}(x - a Vt - x_0) \sin(b x - Vt - x_1) \]
Figure 1. Generation of the localized wave with compression and tensile parts at $\gamma = 5$. Shown by dashed line is the desired traveling wave (4).

Both (4) and (5) do not correspond to analytical solutions of the sine-Gordon equation. The waves with such profiles do not exist without control.

In both cases zero initial conditions are assumed,

$$U(x, 0) = 0, \quad U_t(x, 0) = 0,$$

and zero boundary conditions are

$$U \rightarrow 0 \quad at \quad x \rightarrow -\infty$$

$$U \rightarrow 0 \quad at \quad x \rightarrow \infty.$$

3. Achievement of the desired wave profile

Numerical simulations of Eq. (1) with the control function defined by Eq. (3) were performed using the tools of Wolfram Mathematica with the NDSolve procedure.

The parameters of the target function (4) are chosen equal to $W = 1.1, A = 1, k = 0.5, x_1 = 12, x_2 = 17$. The control parameters depend on the features of the layer material and foundation. We formally assume fixed values for $\alpha_1 = 50, \alpha_2 = 40$ and vary the value of $\gamma$. One can otherwise vary the values of foundation parameters keeping the fixed value of $\gamma$.

Shown in Fig. 1 is the generation of the wave (4) due to the control at $\gamma = 5$. One can see that initially undisturbed state gradually fills in the desired wave shape shown by dashed line. At time $t = 10$ the waves coincide and further calculations demonstrate stable propagation of the wave shown in the last sketch in Fig. 1 with the velocity $W$.

The similar study is performed for the target function (5). The target function parameters are assumed to be equal to $a = 0.5, b = 5, x_0 = 0, x_1 = 0, V = 0.95$, while the control parameters $\alpha_i$ remain of the same value. The simulations at $\gamma = 1.5$ are shown in Fig. 2. One can see that again the profile of the nonlinear wave is generated from undisturbed initial state by the control, the wave gradually achieves the desired wave shape of the breather and further propagates keeping its shape and velocity according the target function (5).
Figure 2. Generation of the breather by the control at $\gamma = 1.5$. Shown by dashed line is the desired traveling wave (5).

Figure 3. Generation of compression and tensile wave with $\gamma = 1$. Shown by dashed line is the desired traveling wave (4).

4. Variations in the desired wave
We achieve the desired localized waves propagation at special values of the control parameters. One can note that variation in these values result in temporal arising of additional parts in the wave profile. Thus, for $\gamma_1$ the generation of the wave (4) is accompanied by an additional variations in the wave profile as shown in Fig. 3. One can see that this defect visible at $t = 0.5$ and $t = 1$ disappears at $t = 10$ and further propagation of the wave is similar to the case $\gamma = 5$. 
The dynamics of the defect evolution for different values of $\gamma$ is shown in Fig. 4 for $\gamma = 0.75$ and $\gamma = 1.5$. In both cases one can see that the defect is developing for some time and goes away from the desired wave but then it gradually disappears as shown in the last sketch in Fig. 3.

5. Conclusion
The new algorithm of distributed feedback control arising from the real mechanical system, is examined. It is shown that it works efficiently similar to our previous algorithms. variations in the desired shape of the wave may appear for some time but they decay as time further goes on. Future work concerns the study of a similar mechanical system but for the materials having di-atomic crystalline structure. In this case one anticipates the control of the coupled nonlinear wave equations [5].

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