Quantum metrology out of equilibrium

Abstract
We address open quantum systems out-of-equilibrium as effective quantum probes for the characterisation of their environment. We discuss estimation schemes for parameters driving a de-phasing evolution of the probe and then focus on qubits, establishing a relationship between the quantum Fisher information and the residual coherence of the probe. Finally, we apply our results to the characterisation of the ohmicity parameter of a bosonic environment.

Keywords
Quantum metrology, Quantum probes, Open quantum systems

1. Introduction

In this paper we address open quantum systems out-of-equilibrium [1, 2], employed as quantum probes to precisely characterise some relevant properties of their environment [3–5]. The probing scheme we are going to discuss is the following (see Fig. 1): a quantum probe, i.e. a simple quantum system like a qubit, is prepared in a known initial state and then made to interact with a larger, and possibly complex, system, which represents the environment of the quantum probe. The environment usually induces decoherence, to an amount which depends on its temperature, spectral density and internal correlations [6]. The final state of the quantum probes thus carries information about the properties of the environment. In turn, any measurement performed on the probe may be exploited to infer the values of some relevant environment parameters [7–11]. In this situation, the inherent fragility of quantum systems to decoherence represents a resource, making quantum probes a very effective technique, able to provide enhanced precision compared to classical (thermal) probes [14, 15]. In addition, quantum probes are usually small and do not perturb the system under investigation, thus representing a non-invasive technique suitable for delicate samples [16, 17].

The interaction time between the probe and the system is usually a tunable parameter, and a question thus arises on whether it may be used to further optimise the estimation precision. In a classical setting, e.g. thermometry, this is not the case, since one prepares the probe, leave it interacting with the sample, and then read the environment parameter by measuring the probe when it has reached its stationary state, i.e. it is at equilibrium with its environment. On the other hand, it has been recently shown that optimal estimation by quantum probes may be achieved also at finite time, i.e. when the probe has not

Fig 1. A quantum probe is prepared in a known initial state and then made to interact with a larger, and possibly complex, system, which represents the environment of the probe. The output state of the probe thus carries information about the properties of the environment, e.g. the parameter $\gamma$, which may be extracted by performing measurements at the output.
reached stationarity, and it is still in an out-of-equilibrium state [18–23]. Following this results, we address here open quantum systems out-of-equilibrium as possible quantum probes for the characterisation of their environment. At first, we discuss a general scheme to estimate parameters driving the dephasing evolution of a probe. We then devote attention to qubit probes, and establish a relationship between the quantum Fisher information and the residual coherence of the probe. The paper is structured as follows. In Section 2, we establish notation, describe the dynamics of a probe subject to dephasing, and introduce the notion of residual coherence. In Section 3, we briefly review the tools of quantum parameter estimation. In Section 4, we address quantum probes out-of-equilibrium, establish a relationship between the quantum Fisher information and the residual coherence of the probe, and apply our results to the characterisation of the ohmlicity parameter of a bosonic environment. Section 5 closes the paper with some concluding remarks.

2. Probing by dephasing

Let us consider a generic quantum system interacting with its environment. No assumptions is made on the dimension $d$ of the Hilbert space of the probe. We also do not assume any specific form for the interaction Hamiltonian, but assume that the resulting dynamics corresponds to a pure dephasing, i.e. to a Von Neumann-Liouville equation of the form

$$\dot{\rho} = -i[H, \rho] - \kappa [H, [H, \rho]] \quad \text{(1)}$$

$$\dot{\rho} = -i[H, \rho] + 2 \kappa L[H] \rho, \quad \text{(2)}$$

where $\rho$ is the density matrix describing the state of the system, $H$ is its free Hamiltonian, $\kappa > 0$ is a dephasing rate and

$$L[O] = -\frac{i}{2} \{O^T O, \bullet\} + O \bullet O^T$$

is a superoperator in the Lindblad form. Moving to the interaction picture, i.e. to a reference frame rotating with $H$, the equation of motion reduces to

$$\dot{\rho} = 2 \kappa L[H] \rho, \quad \text{(3)}$$

Upon writing the initial state in the Hamiltonian basis, i.e. $\rho_0 = \sum_{n_k} \rho_{n_k} |e_n\rangle \langle e_n|$ with $H |e_n\rangle = E_n |e_n\rangle$, the state after the interaction with the environment is given by

$$\rho_t = \sum_{n_k} \rho_{n_k} e^{-\gamma \Omega_{n_k}} |e_n\rangle \langle e_n|, \quad \text{(4)}$$

$$= \int d\zeta g(\zeta; 0, 2\gamma) e^{-i\zeta H} \rho_0 e^{i\zeta H} \quad \text{(5)}$$

where $\gamma = \kappa t$, $\Omega_{n_k} = E_n - E_k$ and $g(\zeta; 2, \sigma^2)$ is a Gaussian distribution in the variable $\zeta$ with mean $2$ and variance $\sigma^2$. The coherence of the probe after the interaction is given by

$$C_Y = \sum_{n_k} |\rho_{n_k}| e^{-\gamma \Omega_{n_k}^2} = 2 \sum_{n_k} |\rho_{n_k}| e^{-\gamma \Omega_{n_k}^2}, \quad \text{(6)}$$

and is always smaller than the initial coherence $C_0 = 2 \sum_{n_k} |\rho_{n_k}|$. Since the precision of quantum probes is strictly related to their sensitivity to decoherence, it is quite natural to start from a probe initially prepared in a maximally coherent state, i.e. $\rho_0 = |\psi_0\rangle \langle \psi_0|$ where

$$|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_{n_k} |e_n\rangle \langle e_n| \quad C_0 = 1. \quad \text{(7)}$$

For the sake of simplicity we also assume equi-spaced levels for the probe, i.e. $\Omega_{n_k}^2 = \Omega^2(n-k)^2$. In this case, the residual coherence after the interaction may be expressed as

$$C_Y = \frac{2}{d} \sum_{j=1}^{d-1} e^{-\gamma \Omega^2 j^2}. \quad \text{(8)}$$

3. Quantum parameter estimation

After the interaction with the environment the state of the probe depends on the parameter we would like to estimate. In order to optimize the inference strategy, i.e. to optimize the extraction of information, we employ the tools of quantum estimation theory [27, 28], which provides recipes to find the best detection scheme and to evaluate the corresponding lower bounds to precision. The precision also depends on the interaction time or, equivalently, on the residual coherence of the probe.

Let us consider the family of quantum states $\rho_\lambda$ and assume that the dephasing rate depends on some parameter of interest, i.e. $\gamma = \gamma(\lambda)$. We perform measurements on repeated preparations of the probe and then process the overall sample of outcomes in order to estimate $\lambda$. Let us denote by $Z$ the observable we measure on the probe $Z|z\rangle = z|z\rangle$, $P_z = |z\rangle \langle z|$, and by $p(x|\lambda) = \text{Tr}[\rho_{x|\lambda} P_x]$ the distribution of its outcomes for a given value of $\lambda$. After choosing a certain observable $Z$, we perform $M$ repeated measurements, collecting the data $z = \left\{ z_1, \ldots, z_M \right\}$. This set is then processed by an estimator $\hat{\lambda} \equiv \hat{\lambda}(z)$, i.e. a function from the space of data to the set of possible values of the parameter. The estimate value of the parameter is the mean value of the estimator over data, i.e.

$$\bar{\lambda} = \int dz \ p(z|\lambda) \hat{\lambda}(z). \quad \text{(9)}$$
where \( p(x|\lambda) = \prod_{i=1}^{N} p(z_i|\lambda) \) since the repeated measurements are independent on each other. The precision of this estimation strategy corresponds to the variance of the estimator i.e.

\[
V_\lambda \equiv \text{Var} \lambda = \int dz \ p(z|\lambda) \left[ \lambda - \overline{\lambda} \right]^2.
\]

The smaller is \( V_\lambda \), the more precise is the is the estimation strategy. In fact, the precision of any unbiased estimator (i.e. an estimator such that \( \lambda \to \lambda \) for \( M \gg 1 \)), is bounded by the so-called Cramèr-Rao (CR) inequality:

\[
V_\lambda \geq \frac{1}{MF_\lambda},
\]

(11)

where \( F_\lambda \) is the Fisher information (FI) of \( Z \)

\[
F_\lambda = \int dz \ p(z|\lambda) \left[ \partial_{\lambda} \log p(z|\lambda) \right]^2,
\]

(12)

i.e. the information that can be extracted on \( \lambda \) by performing measurements of \( Z \) on \( \varrho_\lambda \). The best, i.e. more precise, measurement to infer the value of \( \lambda \) is the measurement maximising the FI, where the maximization is performed over all the possible probe observables. As a matter of fact, the maximum is achieved for any observable having the same spectral measure of the so-called symmetric logarithmic derivative \( L_\lambda \), i.e. the selfadjoint operator satisfying the equation

\[
2 \partial_{\lambda} \varrho_{\lambda} = L_\lambda \varrho_{\lambda} + \varrho_{\lambda} L_\lambda.
\]

(13)

The corresponding FI is usually referred to as the quantum Fisher information (QFI) and may be expressed as \( H_\lambda = \text{Tr}[\varrho_{\lambda} L_\lambda^2] \). Since \( F_\lambda \leq H_\lambda \), the ultimate bound to precision in estimating \( \lambda \) by performing quantum measurements on \( \varrho_{\lambda} \) is given by the quantum CR bound

\[
V_\lambda \geq \frac{1}{MH_\lambda}.
\]

(14)

In terms of the eigenvalues and eigenvectors of \( \varrho_\lambda = \sum_i \varrho_i |\phi_i\rangle \langle \phi_i| \) the QFI may be written as

\[
H_\lambda = \sum_i \left( \frac{\partial_{\lambda}\varrho_i}{\varrho_i} \right)^2 + 2 \sum_{i \neq j} \frac{(\varrho_i - \varrho_j)^2}{\varrho_i + \varrho_j} \left| \langle \phi_i | \partial_{\lambda} \phi_j \rangle \right|^2.
\]

(15)

In order to evaluate analytically the QFI, we need to diagonalise the density matrix of the probe after the interaction, i.e. that in Eq. (4). This can be done easily for low dimensional probes (qubit and qutrits), whereas numerical solutions are unavoidable for higher dimension.

4. Quantum probes out of equilibrium

Let us now consider the simplest quantum probe, i.e. a qubit system used to characterize its environment, which itself induces dephasing on the qubit. In this case we may consider a generic pure initial state \( |\psi_0\rangle = \cos \phi |e_1\rangle + \sin \phi |e_2\rangle \) and use the notation \( \Omega = \Omega_2 = E_2 - E_1 \). The initial coherence is given by \( C_0 = \sin 2\phi \) and final one by \( C_\lambda = C_{\psi(0)} = e^{-\lambda^2/\omega^2} \sin 2\phi \). In order to evaluate the QFI, we leave the qubit to evolve, then diagonalise the state, and finally use Eq. (15). After some algebra, we obtain a remarkably compact formula

\[
H_\lambda = \Omega^4 \left( \partial_{\lambda} \gamma \right)^2 \frac{C_0^2 C_\lambda^2}{C_0^2 - C_\lambda^2},
\]

(16)

which is valid for any \( \phi \) and expresses the QFI in terms of the dependence of the dephasing rate on the parameter of interest, i.e. the susceptibility \( \partial_{\gamma} \gamma \), and on the relationship between the initial and the final coherence of the probe. As it may easily proved, the maximum of the QFI is achieved for \( \phi = \pi/4 \), thus confirming the intuition, already mentioned in Section 2, that the optimal initial state of the probe corresponds to a maximally coherent state \( |\rangle = (|e_1\rangle + |e_2\rangle)/\sqrt{2} \). We remark that Eq. (16) is valid for any initial preparation of the probe, any kind of environment, and any parameter. The only assumption is that the interaction leads to a pure dephasing evolution of the probe.

\[\text{Fig 2.} \quad \text{A qubit is prepared in a maximally coherent state } |\rangle = (|e_1\rangle + |e_2\rangle)/\sqrt{2} \text{ and then is made to interact with a Ohmic environment made of bosonic modes, characterised by a spectral density } J_s(\omega). \text{ The output state of the probe carries information about the ohmicity parameter } s, \text{ which may be estimated by performing measurements at the output.}\]

As an application, let us now consider a qubit, used to probe the nature of an Ohmic environment made of bosonic modes (see Fig. 2). In order to introduce the problem, let us write the Hamiltonian of the whole system. We use the natural system of units \( (\hbar = 1) \), and also write the Hamiltonian in unit of the qubit frequency \( \Omega \), making it adimensional

\[
\mathcal{H} = \frac{1}{2} \sigma_3 + \sum_k \omega_k b_k^T b_k + \sigma_3 \sum_k (g_k b_k^T + g_k^* b_k),
\]

(17)
where \( \omega_k \) is the (dimensionless) frequency of the \( k \)-th environment modes. The \( \sigma \)'s are the Pauli matrices and \([\sigma_j, \sigma_k] = \delta_{jk}\) describe the modes of the environment. The \( g_k \)'s are coupling constants, describing the interaction of each mode with the qubit probe. Their distribution determines the spectral density of the environment, according to the expression \( J(\omega) = \sum_k |g_k|^2 \delta(\omega_k - \omega) \). The spectral density is the crucial quantity to describe the system–environment interaction, and it does depend on the specific features of the environment. In turn, the characterisation of the spectral density is crucial to understand, and possibly control, quantum decoherence\(^{29–34}\).

A large class of structured reservoirs is characterised by an Ohmic spectral density of the form

\[
J_\omega(\omega) = \omega \left( \frac{\omega}{\omega_c} \right)^s \exp \left\{ - \frac{\omega}{\omega_c} \right\},
\]

where the frequencies are in unit of \( \Omega \). The cutoff frequency describes a natural boundary in frequency response of the system. As we will see, it determines the timescale of the evolution. The quantity \( s \) is a real positive number, which governs the behaviour of the spectral density at low frequencies. Upon varying \( s \) we move from the so-called sub-Ohmic regime \((s < 1)\), to Ohmic \((s = 1)\), and to super-Ohmic one \((s > 1)\). Different values of the ohmicity parameter \( s \) often corresponds to radically different kind of dynamics, and therefore it would be highly desirable to have an estimation scheme for its precise characterisation.

Such a scheme may be obtained from the results of the previous Sections, since the dynamics induced on the qubit is a pure dephasing. In order to prove this result, one assumes that the system is initially in the state \(|\psi\rangle \otimes |0\rangle\) (i.e. a generic state for the probe and, assuming to be at zero temperature, the ground state for the environment), then evolve the whole system according to the Hamiltonian in Eq. (17), and finally trace out the environment. The resulting evolution is that of Eq. (4) where the dephasing rate is given by

\[
\gamma_\tau(\tau) = \int_0^\infty \frac{1 - \cos(\omega \tau/\omega_c)}{\omega^2} J_\omega(\omega) \, d\omega,
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{2} \log (1 + \tau^2) & s = 1 \\
\frac{1 - \cos(\pi s \tan^{-1} \tau)}{(1 + \tau^2)^{s/2}} & \Gamma[s - 1] \, s \neq 1
\end{array} \right.
\]

where \( \tau = \omega_c t \) and \( \Gamma[x] = \int_0^\infty t^{x-1}e^{-t} \, dt \) is the Euler Gamma function. Upon preparing the qubit in the initial state \(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \) and using Eq. (16) the QFI for the estimation of \( s \) is given by

\[
H_s = \frac{\partial, \gamma_\tau(\tau)}{e^{2\gamma_\tau(\tau)} - 1},
\]

where the interaction time \( \tau \) is a free parameter that can be used to further optimise precision.

In Fig. 3 we show the behaviour of \( H_s \) as a function of \( s \) and \( \tau \). In order to emphasise the non trivial features of this function, we show a 3D plot illustrating the behaviour for limited interaction time (\( \tau \leq 7 \)) and a contour plot with a longer time range (\( \tau \leq 35 \)). The time value \( \tau \), maximising \( H_s \) at fixed \( s \), increases with \( s \) when \( s \) is small and then jump to a smaller, almost constant, value for larger values of \( s \). In the intermediate region, \( H_s \) is an increasing function of \( \tau \).

In Fig. 4 we show the optimised value of \( H_s \) as a function of \( s \). The physical meaning of this plot is that estimation of intermediate values of \( s \), corresponding to slightly super-Ohmic environments \((s \simeq 1.5)\), is inherently more precise than the estimation of smaller or larger values.
5. Conclusions

In this paper we have addressed the use of open quantum systems out-of-equilibrium as possible quantum probes for the characterisation of their environment. In particular, we have discussed estimation schemes involving parameters governing a de-phasing evolution of the probe. For qubit probe we have found a simple relation linking the quantum Fisher information to the residual coherence of the probe. Finally, we have addressed in some details the estimation of the ohmicity parameter of a bosonic environment, finding that depending on the values of $s$, one may achieve optimal estimation at finite interaction time, i.e. when the probe is in an out-of-equilibrium state, or for large time, when, presumably, the qubit has reached its stationary state. Overall, our results pave the way for further investigation of quantum metrology out-of-equilibrium, perhaps exploiting memory effects [35], and confirm that pure de-phasing is an effective mechanism to imprint information on quantum probes without exchanging energy with the system under investigation.

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