(HOPF) ALGEBRA AUTOMORPHISMS OF THE HOPF ALGEBRA $\mathcal{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$

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Abstract. In this paper, we completely determine the group of algebra automorphisms for the two-parameter Hopf algebra $\mathcal{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$. As a result, the group of Hopf algebra automorphisms is determined for $\mathcal{U}_{r,s}^{\leq 0}(\mathfrak{sl}_3)$ as well. We further characterize all the derivations of the subalgebra $U_{r,s}^{+}(\mathfrak{sl}_3)$, and calculate its first degree Hochschild cohomology group.

Introduction

Motivated by the study of down-up algebras [2], a two-parameter quantized enveloping algebra (quantum group) $U_{r,s}(\mathfrak{sl}_n)$ has recently been investigated by Benkart and Witherspoon in the references [3, 4]. The two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_n)$ remains as a close analogue of the one-parameter quantized enveloping algebra $U_q(\mathfrak{sl}_n)$ associated to the finite dimensional complex simple special linear Lie algebra $\mathfrak{sl}_n$. As a matter of fact, the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_n)$ shares many similar properties with its one-parameter analogue.

For instance, the two-parameter quantized enveloping algebra $U_{r,s}(\mathfrak{sl}_n)$ is also a Hopf algebras and it admits a triangular decomposition. Besides having a similar representation theory to that of the one-parameter quantum group $U_q(\mathfrak{sl}_n)$, the Hopf algebra $U_{r,s}(\mathfrak{sl}_n)$ can be realized as the Drinfeld double of its Hopf sub-algebras $U_{r,s}^{\geq 0}(\mathfrak{sl}_n)$ and $U_{r,s}^{\leq 0}(\mathfrak{sl}_n)$. However, the algebra $U_{r,s}(\mathfrak{sl}_n)$ does have some different ring-theoretic features in that $U_{r,s}(\mathfrak{sl}_n)$ is more rigid and possesses less symmetries. In addition, the center of the algebra $U_{r,s}(\mathfrak{sl}_n)$ shows a different picture as well [1].

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To better understand the structure and properties of the algebra \( U_{r,s}(\mathfrak{sl}_n) \), one naturally starts with the investigations of its subalgebra \( U^+_{r,s}(\mathfrak{sl}_n) \) and Hopf subalgebra \( U^\geq_{r,s}(\mathfrak{sl}_n) \). In this paper, we will study the algebra \( U^+_{r,s}(\mathfrak{sl}_3) \) in terms of its derivations, and its augmented Hopf algebra \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \) in terms of its algebra automorphisms and Hopf algebra automorphisms. We shall determine all the algebra automorphisms and Hopf algebra automorphisms of \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \). In addition, we will characterize all the derivations of \( U^+_{r,s}(\mathfrak{sl}_3) \). As a result, we will compute the first Hochschild cohomology group of the algebra \( U^+_{r,s}(\mathfrak{sl}_3) \).

Now let me briefly mention the methods which we shall follow. In order to determine the algebra and Hopf algebra automorphisms of \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \), we shall closely follow the approach used in [5]. In order to characterize the derivations of \( U^+_{r,s}(\mathfrak{sl}_3) \), we shall embed the algebra \( U^+_{r,s}(\mathfrak{sl}_3) \) into a quantum torus, whose derivations had been explicitly described in [7]. We would like to point out that this embedding will allow us to extend the derivations of \( U^+_{r,s}(\mathfrak{sl}_3) \) to the derivations of the associated quantum torus. Therefore, via this embedding, we shall be able to pull the information on derivations back to the algebra \( U^+_{r,s}(\mathfrak{sl}_3) \).

Based on a result on the derivations of quantum torus established in [7], we will be able to determine all the derivations of the algebra \( U^+_{r,s}(\mathfrak{sl}_3) \) modulo its inner derivations. As an immediate application, we show that the first Hochschild cohomology group \( HH^1(U^+_{r,s}(\mathfrak{sl}_3)) \) of \( U^+_{r,s}(\mathfrak{sl}_3) \) is a 2-dimensional vector space over the base field \( \mathbb{C} \).

The paper is organized as follows. In Section 1, we recall some basic definitions and properties on the two-parameter quantized enveloping algebras \( U^+_{r,s}(\mathfrak{sl}_3) \) and its augmented Hopf algebra \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \). In Section 2, we determine the algebra automorphism group and Hopf algebra automorphism group of \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \). In Section 3, we characterize the derivations of \( U^+_{r,s}(\mathfrak{sl}_3) \), and compute the first Hochschild cohomology group \( HH^1(U^+_{r,s}(\mathfrak{sl}_3)) \).

1. Definitions and Basic Properties of \( U^+_{r,s}(\mathfrak{sl}_3) \) and \( \tilde{U}^\geq_{r,s}(\mathfrak{sl}_3) \)

1.1. Definition and basic properties of \( U^+_{r,s}(\mathfrak{sl}_3) \). Motivated by the study of down-up algebras, a two-parameter quantized enveloping algebra \( U_{r,s}(\mathfrak{sl}_n) \) associated to the finite dimensional simple complex Lie algebra \( \mathfrak{sl}_n \) has been recently studied by Benkart and Witherspoon in [3] and the references therein. For the purpose of this paper, we shall only recall the definitions of the algebras \( U^+_{r,s}(\mathfrak{sl}_3) \) and \( U^\geq_{r,s}(\mathfrak{sl}_3) \), which are indeed subalgebras of \( U_{r,s}(\mathfrak{sl}_n) \). One easily sees that the algebra
$U_{r}^{+}(\mathfrak{sl}_{n})$ can be regarded as a two-parameter quantized enveloping algebra of a maximal nilpotent Lie subalgebra of the Lie algebra $\mathfrak{sl}_{n}$.

Let $C = (a_{ij})$ denote the Cartan matrix associated to the Lie algebra $\mathfrak{sl}_{n}$. Let us define the following notation:

$$< i, j > = a_{ij} \text{ for } i < j;$$
$$< i, i > = 1 \text{ for } i = 1, \cdots, n - 1;$$
$$< i, j > = 0 \text{ for } i > j.$$

Suppose that $r, s \in \mathbb{C}$ such that $r^{m}s^{n} = 1$ implies $m = n = 0$. We recall the following definition:

**Definition 1.1.** The two-parameter quantized enveloping algebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_{n})$ is defined to be the $\mathbb{C}$-algebra generated by $E_{i}, W_{i}$ subject to the following relations:

$$W_{i}^{\pm 1}W_{j}^{\pm 1} = W_{j}^{\pm 1}W_{i}^{\pm 1};$$
$$W_{i}^{\pm 1}W_{i}^{\mp 1} = 1;$$
$$W_{i}E_{j} = r^{<j,i>} s^{-<i,j>} E_{j}W_{i};$$
$$E_{i}^{2}E_{i+1} - (r + s)E_{i}E_{i+1}E_{i} + rsE_{i+1}E_{i}^{2} = 0;$$
$$E_{i}E_{i+1}E_{i} - (r^{-1} + s^{-1})E_{i+1}E_{i}E_{i+1} + r^{-1}s^{-1}E_{i}E_{i+1}^{2} = 0.$$

And the algebra $U_{r,s}(\mathfrak{sl}_{n})$ is defined to be the subalgebra of $U_{r,s}^{\geq 0}(\mathfrak{sl}_{n})$ generated by $E_{i}$.

From [3], we know that the two-parameter quantized enveloping algebra $U_{r,s}^{\geq 0}(\mathfrak{sl}_{n})$ has a Hopf algebra structure, which is defined by the following coproduct, counit and antipode:

$$\Delta(W_{i}^{\pm 1}) = W_{i}^{\pm 1} \otimes W_{i}^{\pm 1};$$
$$\Delta(E_{i}) = E_{i} \otimes 1 + W_{i} \otimes E_{i};$$
$$\epsilon(W_{i}^{\pm 1}) = 1;$$
$$\epsilon(E_{i}) = 0;$$
$$S(W_{i}^{\pm 1}) = W_{i}^{\mp 1};$$
$$S(E_{i}) = -W_{i}^{-1}E_{i}.$$
Definition 1.2. The algebra $U_{r,s}^+(\mathfrak{sl}_3)$ is defined to be the $\mathbb{C}$–algebra generated by the generators $E_1, E_2$ subject to the following relations:

\[
E_1^2E_2 - (r + s)E_1E_2E_1 + rsE_2E_1^2 = 0; \\
E_1E_2^2 - (r + s)E_1E_2E_1 + rsE_2E_2E_1 = 0.
\]

Naturally, one may think of the algebra $U_{r,s}^+(\mathfrak{sl}_3)$ as a two-parameter quantum Heisenberg algebra. Indeed, the algebra $U_{r,s}^+(\mathfrak{sl}_3)$ shares many similar properties as the algebra $U_q^+(\mathfrak{sl}_3)$, which has been traditionally called the quantum Heisenberg algebra.

In addition, we recall the definition of the following Hopf subalgebra of $U_{r,s}(\mathfrak{sl}_3)$:

Definition 1.3. The Hopf algebra $U_{r,s}^>(\mathfrak{sl}_3)$ is defined to be the $\mathbb{C}$–algebra generated by $E_1, E_2, W_1, W_2$ subject to the following relations:

\[
W_1W_1^{-1} = 1 = W_2W_2^{-1}; \\
W_1W_2 = W_2W_1; \\
W_1E_1 = rs^{-1}E_1W_1; \\
W_1E_2 = sE_2W_1; \\
W_2E_1 = r^{-1}E_1W_2; \\
W_2E_2 = rs^{-1}E_2W_2; \\
E_1^2E_2 - (r + s)E_1E_2E_1 + rsE_2E_1^2 = 0; \\
E_1E_2^2 - (r + s)E_2E_1E_2 + rsE_2^2E_1 = 0.
\]

However, we should mention that we will not study the Hopf algebra $U_{r,s}^>(\mathfrak{sl}_3)$ in this paper. Instead, we will study its augmented version $U_{r,s}^=0(\mathfrak{sl}_3)$, which we shall define in the next subsection.

Before we introduce the Hopf algebra $U_{r,s}^=0(\mathfrak{sl}_3)$, let us mention some basic properties of the algebra $U_{r,s}(\mathfrak{sl}_3)$ in the rest of this subsection. It is easy to see that the two-parameter quantized enveloping algebra $U_{r,s}^+(\mathfrak{sl}_n)$ can also be presented as an iterated skew polynomial ring and a PBW-basis can be constructed for $U_{r,s}(\mathfrak{sl}_n)$ as well. For conveniences, we shall only recall the skew polynomial presentation for the algebra $U_{r,s}^+(\mathfrak{sl}_3)$ here. For the general construction, we refer the reader to references [1, 8].

First of all, Let us fix some notation by setting the following new variables:

\[
E_1 = E_1, \quad E_2 = E_2, \quad E_3 = E_1E_2 - sE_2E_1.
\]
Then it is easy to see that we have the following relations between these new variables:

\[ E_1 E_3 = r E_3 E_1, \quad E_2 E_3 = r^{-1} E_3 E_2. \]

Now let us further define some algebra automorphisms \( \tau_2, \tau_3 \) and some derivations \( \delta_2, \delta_3 \) as follows:

\[
\begin{align*}
\tau_2(E_1) &= r^{-1} E_1; \\
\delta_2(E_1) &= 0; \\
\tau_3(E_1) &= s^{-1} E_1; \\
\tau_3(E_3) &= r^{-1} E_3; \\
\delta_3(E_1) &= -s^{-1} E_3; \\
\delta_3(E_3) &= 0.
\end{align*}
\]

Then it is easy to see that we have the following result

**Theorem 1.1.** The algebra \( U_{r,s}^+(\mathfrak{sl}_3) \) can be presented as an iterated skew polynomial ring. In particular, we have the following result

\[ U_{r,s}^+(\mathfrak{sl}_3) \cong \mathbb{C}[E_1][E_3, \tau_2, \delta_2][E_2, \tau_3, \delta_3]. \]

Based on the previous theorem, we have an obvious corollary as follows:

**Corollary 1.1.** The set \( \{ E_1^i E_2^j E_3^k | i, j, k \geq 0 \} \) forms a PBW-basis of the algebra \( U_{r,s}^+(\mathfrak{sl}_3) \). In particular, \( U_{r,s}^+(\mathfrak{sl}_3) \) has a GK-dimension of 3.

1.2. The Augmented Hopf algebra \( \tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3) \). In this subsection, we shall introduce an augmented Hopf algebra \( \tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3) \), which contains the algebra \( U_{r,s}^+(\mathfrak{sl}_3) \) as a subalgebra and enlarges the Hopf algebra \( U_{r,s}^{\geq 0}(\mathfrak{sl}_3) \).

First of all, we need to define the following new variables:

\[ K_1 = W_1^{2/3} W_2^{1/3}, \quad K_2 = W_1^{1/3} W_2^{2/3}. \]

Then we have the following definition of \( \tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3) \).
Definition 1.4. The algebra $\tilde{U}^{\geq 0}_{r,s}(sl_3)$ is a $\mathbb{C}$-algebra generated by $E_1, E_2, K_1^{\pm 1}, K_2^{\pm 1}$ subject to the following relations:

$K_1 K_1^{-1} = 1 = K_2 K_2^{-1}$;
$K_1 K_2 = K_2 K_1$;
$K_1 E_1 = r^{1/3} s^{-2/3} E_1 K_1$;
$K_1 E_2 = r^{1/3} s^{1/3} E_2 K_1$;
$K_2 E_1 = r^{-1/3} s^{-1/3} E_1 K_2$;
$K_2 E_2 = r^{2/3} s^{-1/3} E_2 K_2$;
$E_1^2 E_2 - (r + s) E_1 E_2 E_1 + r s E_2 E_1^2 = 0$;
$E_1 E_2^2 - (r + s) E_2 E_1 E_2 + r s E_2^2 E_1 = 0$.

Then it is straightforward to verify the following result:

Proposition 1.1. The algebra $\tilde{U}^{\geq 0}_{r,s}(sl_3)$ is a Hopf algebra with the coproduct, counit and antipode defined as above.

Recall that we have $E_3 = E_1 E_2 - s E_2 E_1$, then it is easy to see that we have the following result

Theorem 1.2. The algebra $\tilde{U}^{\geq 0}_{r,s}(sl_3)$ has a $\mathbb{C}$-basis

$\{K_1^m K_2^n E_1^i E_2^j E_3^k \mid m, n \in \mathbb{Z}, i, j, k \in \mathbb{Z}_{\geq 0}\}$.

In particular, one can see that all the invertible elements of $\tilde{U}^{\geq 0}_{r,s}(sl_3)$ are of the form $\lambda K_1^m K_2^n$ for some $\lambda \in \mathbb{C}^*$ and $m, n \in \mathbb{Z}$. 

\qed
2. ALGEBRA AND HOPF ALGEBRA AUTOMORPHISMS OF $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$

In this section, we will first determine the algebra automorphism group of the algebra $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$. As a result, we are able to determine the Hopf algebra automorphism group of $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ as well. We will closely follow the approach used in [5].

2.1. The algebra automorphism group of $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$

Suppose that $\theta \in \text{Aut}_C(\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3))$ is an algebra automorphism of the algebra $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$. Note that the elements $K_1, K_2$ are invertible in $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$ and $\theta$ is an algebra automorphism, then the elements $\theta(K_1)$ and $\theta(K_2)$ are invertible too. Therefore, we have the following

$$\theta(K_1) = a_1 K_1^x K_2^y, \quad \theta(K_2) = a_2 K_1^z K_2^w$$

for some $a_1, a_2 \in \mathbb{C}^*$ and $x, y, z, w \in \mathbb{Z}$.

Let $M_\theta = (M_{ij})$ denote the corresponding $2 \times 2$—matrix associated to the algebra automorphism $\theta$. Specifically, we will set $M_{11} = x, M_{12} = y, M_{21} = z$ and $M_{22} = w$. Since $\theta$ is an algebra automorphism, we know that the matrix $M_\theta$ is an invertible matrix with integer coefficients. In particular, we have the following

$$xw - yz = \pm 1.$$

Suppose that for $l = 1, 2$, we have

$$\theta(E_l) = \sum_{m_l, n_l} a_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3} K_1^{m_l} K_2^{n_l} E_1^{\beta_l^1} E_2^{\beta_l^2} E_3^{\beta_l^3}$$

where $a_{m_l, n_l, \beta_l^1, \beta_l^2, \beta_l^3} \in \mathbb{C}^*$ and $m_l, n_l \in \mathbb{Z}$ and $\beta_l^1, \beta_l^2, \beta_l^3 \in \mathbb{Z}_{\geq 0}$.

Then we have the following

**Proposition 2.1.** Let $\theta \in \text{Aut}_C(\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3))$ be an algebra automorphism of $\tilde{U}_{r,s}^{\geq 0}(\mathfrak{sl}_3)$, then we have $M_\theta \in \text{GL}(2, \mathbb{Z}_{\geq 0})$.

**Proof:** The proof of **Proposition 2.1** in [5] can essentially be adopted here word by word with the replacement of $s$ by $r^{-1}$. For the reader’s convenience, we will present the detailed proof here.

Since $K_1 E_1 = r^{1/3} s^{-2/3} E_1 K_1$ and $K_2 E_1 = r^{-1/3} s^{-1/3} E_1 K_2$, we have the following

$$\theta(K_1) \theta(E_1) = r^{1/3} s^{-2/3} \theta(E_1) \theta(K_1);$$
$$\theta(K_2) \theta(E_1) = r^{-1/3} s^{-1/3} \theta(E_1) \theta(K_2).$$

Via a detailed calculation, we further have the following

$$(\beta_1^1 + \beta_2^1) x + (\beta_1^2 + \beta_2^3) y = 1;$$
$$(\beta_1^1 + \beta_2^3) z + (\beta_1^2 + \beta_2^3) w = 0.$$


Similarly, we also have the following
\[(\beta_1^2 + \beta_3^2)x + (\beta_2^2 + \beta_3^2)y = 0\]
\[(\beta_2^3 + \beta_3^2)z + (\beta_2^2 + \beta_3^2)w = 1.\]

Now let us set a $2 \times 2$ matrix $B = (b_{ij})$ with the following entries
\[b_{11} = (\beta_1^1 + \beta_3^1);\]
\[b_{21} = (\beta_1^2 + \beta_3^1);\]
\[b_{12} = (\beta_2^1 + \beta_3^2);\]
\[b_{22} = (\beta_2^2 + \beta_3^3).\]

Then we have the following system of equations
\[b_{11}x + b_{21}y = 1;\]
\[b_{12}x + b_{22}y = 0;\]
\[b_{11}z + b_{21}w = 0;\]
\[b_{12}z + b_{22}w = 1.\]

This implies that the product $M_\theta B$ of matrices $M_\theta$ and $B$ is equal to the identity matrix. Therefore, we have $M_{\theta^{-1}} = B$, where $M_{\theta^{-1}}$ is the corresponding matrix associated to the algebra automorphism $\theta^{-1}$. Note that the entries $b_{11}, b_{12}, b_{21}, b_{22}$ are all nonnegative integers, thus we have $M_{\theta^{-1}} \in GL(2, \mathbb{Z}_{\geq 0})$.

Applying similar arguments to the algebra automorphism $\theta^{-1}$, we can prove that $M_\theta \in GL(2, \mathbb{Z}_{\geq 0})$ as desired.

To proceed, we now recall an important lemma (**Lemma 2.2** from [5]), which characterizes the matrix $M_\theta$:

**Lemma 2.1.** If $M$ is a matrix in $GL(n, \mathbb{Z}_{\geq 0})$ such that its inverse matrix $M^{-1}$ is also in $GL(n, \mathbb{Z}_{\geq 0})$, then we have $M = (\delta_{i\sigma(j)})_{i,j}$, where $\sigma$ is an element of the symmetric group $S_n$.

Based the previous Proposition and Lemma, we immediately have the following result:

**Corollary 2.1.** Let $\theta \in Aut_\mathbb{C}(\hat{U}_{r,s}^{\geq 0}(sl_3))$ be an algebra automorphism of $\hat{U}_{r,s}^{\geq 0}(sl_3)$. Then for $l = 1, 2$, we have
\[\theta(K_l) = a_lK_{\sigma(l)}\]
where $\sigma \in S_2$ and $a_l \in \mathbb{C}^*$. 

Furthermore, we can further prove the following result:
Proposition 2.2. Let \( \theta \in Aut_C(\mathcal{U}_{r,s}^{>0}(\mathfrak{sl}_3)) \) be an algebra automorphism of \( \mathcal{U}_{r,s}^{>0}(\mathfrak{sl}_3) \). Then for \( l = 1, 2 \), we have
\[
\theta(E_l) = b_lK_1^{m_l}K_2^{n_l}E_{\sigma(l)}
\]
where \( b_l \in \mathbb{C}^* \) and \( m_l, n_l \in \mathbb{Z} \).

Proof: Let \( \theta \in Aut_C(\mathcal{U}_{r,s}^{>0}(\mathfrak{sl}_3)) \) be an algebra automorphism of \( \mathcal{U}_{r,s}^{>0}(\mathfrak{sl}_3) \). To prove the proposition, there are two cases to consider:

Case 1: Suppose that \( \theta(K_1) = a_1K_1 \) and \( \theta(K_2) = a_2K_2 \), then we need to prove that
\[
\theta(E_1) = b_1K_1^{m_1}K_2^{n_1}E_1, \quad \theta(E_2) = b_2K_1^{m_2}K_2^{n_2}E_2.
\]
Since \( K_1E_1 = r^{1/3}s^{-2/3}E_1K_1 \) and \( K_2E_1 = r^{-1/3}s^{-1/3}E_1K_2 \), we have the following
\[
\theta(K_1)\theta(E_1) = r^{1/3}s^{-2/3}\theta(E_1)\theta(K_1);
\]
\[
\theta(K_2)\theta(E_1) = r^{-1/3}s^{-1/3}\theta(K_2)\theta(K_1).
\]
Thus we have the following
\[
a_1K_1(\sum_{m_1,n_1,\beta_1^1,\beta_1^2,\beta_1^3} b_{m_1,n_1,\beta_1^1,\beta_1^2,\beta_1^3}K_1^{m_1}K_2^{n_1}E_1^{\beta_1^1}E_2^{\beta_1^2}E_3^{\beta_1^3})K_1.
\]
We also have the following
\[
a_2K_2(\sum_{m_1,n_1,\beta_2^1,\beta_2^2,\beta_2^3} b_{m_1,n_1,\beta_2^1,\beta_2^2,\beta_2^3}K_1^{m_1}K_2^{n_1}E_1^{\beta_2^1}E_2^{\beta_2^2}E_3^{\beta_2^3})K_2.
\]

Via detailed calculations and simplifications, we have the following
\[
\beta_1^1 + \beta_1^2 = 1, \quad \beta_1^2 + \beta_1^3 = 0.
\]
Based on the fact that \( \beta_i^3 \) are nonnegative integers, we have that
\[
\beta_1^1 = 1, \quad \beta_2^2 = 0 = \beta_1^3.
\]
Similarly, we can also have the following
\[
\beta_2^1 = \beta_2^3 = 0, \quad \beta_2^3 = 1.
\]
Thus we have proved Case 1.

Case 2: Suppose that \( \theta(K_1) = a_1K_2 \) and \( \theta(K_2) = a_2K_1 \), we need to prove that \( \theta(E_1) = b_1K_1^{m_1}K_2^{n_1}E_2 \) and \( \theta(E_2) = b_2K_1^{m_2}K_2^{n_2}E_1 \). Since the
proof goes the same as in Case 1, we will not repeat the detail here. □

Furthermore, we can easily verify that $E_1, E_2$ can not be exchanged. In particular, we have the following result

**Corollary 2.2.** Let $\theta \in \text{Aut}_C(\tilde{U}_{r,s}^0(sl_3))$ be an algebra automorphism. Then for $l = 1, 2$, we have

$$\theta(K_l) = a_lK_l, \quad \theta(E_l) = b_lK_1^{m_l}K_2^{n_l}E_l$$

where $a_l, b_l \in \mathbb{C}^*$ and $m_l, n_l \in \mathbb{Z}$.

□

Now we are going to prove one of the main results of this paper, which describes the algebra automorphism group of $\tilde{U}_{r,s}^0(sl_3)$:

**Theorem 2.1.** Let $\theta \in \text{Aut}_C(\tilde{U}_{r,s}^0(sl_3))$ be an algebra automorphism of $\tilde{U}_{r,s}^0(sl_3)$. Then for $l = 1, 2$, we have the following

$$\theta(K_l) = a_lK_l, \quad \theta(E_1) = b_1 K_1^a K_2^b E_1, \quad \theta(E_2) = b_2 K_1^c K_2^d E_2$$

where $a_l, b_l \in \mathbb{C}^*$ and $a, b, c, d \in \mathbb{Z}$ such that $b = c, a + b + d = 0$.

**Proof:** Let $\theta$ be an algebra automorphism of $\tilde{U}_{r,s}^0(sl_3)$ and suppose that

$$\theta(E_1) = b_1 K_1^a K_2^b E_1, \quad \theta(E_2) = b_2 K_1^c K_2^d E_2.$$

Then we have the following

$$(K_1^a K_2^b E_1)(K_1^a K_2^b E_1)(K_1^c K_2^d E_2) = (r^{-1/3}s^{2/3})^a(r^{1/3}s^{-1/3})^b(r^{-1/3}s^{2/3})^{2c}(r^{1/3}s^{-1/3})^{2d}K_1^{a+c}K_2^{b+d}E_1^2E_2^2$$

$$= r^{(-a+b-2c+2d)/3}s^{(2a+b+4c+2d)/3}K_1^{2a+c}K_2^{2b+d}E_1^2E_2^2;$$

and

$$(K_1^a K_2^b E_1)(K_1^c K_2^d E_2)(K_1^a K_2^b E_1) = (r^{-1/3}s^{2/3})^c(r^{1/3}s^{-1/3})^d(r^{-1/3}s^{2/3})^a(r^{1/3}s^{-1/3})^b K_1^{2a+c}K_2^{2b+d}E_1E_2E_1$$

$$= r^{(-2a-b-c+d)/3}s^{(a+2b+2c+d)/3}K_1^{2a+c}K_2^{2b+d}E_1E_2E_1;$$
and
\[(K_1^a K_2^b E_2)(K_1^a K_2^b E_1)(K_1^a K_2^b E_1) = (r^{1/3}s^{-1/3})^a (r^{-2/3}s^{1/3})^b (r^{-2/3}s^{1/3})^a
(K_1^2)^{2a+c} K_2^{2b+d} E_2 E_1^2 = r^{(-3a-3b)/3} s^{3b/3} K_1^{2a+c} K_2^{2b+d} E_2 E_1^2.
\]

Applying the automorphism \(\theta\) to the first two-parameter quantum Serre relation
\[E_1^2 E_2 - (r + s) E_1 E_2 E_1 + rs E_2 E_1^2 = 0,
\]
we have the following system of equations
\[-a + b - 2c + 2d = -2a - b - c + d;
-3a - 3b = -2a - b - c + d;
2a + b + 4c + 2d = a + 2b + 2c + d;
3b = a + 2b + 2c + d.
\]

It is easy to see that the previous system of equations is reduced to the following system of equations
\[a + 2b - c + d = 0;
(a - b + 2c + d = 0.
\]

Similarly, from the second two-parameter quantum Serre relation
\[E_1 E_2^2 - (r + s) E_2 E_1 E_2 + rs E_2^2 E_1 = 0,
\]
we also have the same system of equations as follows
\[a + 2b - c + d = 0;
a - b + 2c + d = 0.
\]

Solving the system
\[a + 2b - c + d = 0;
a - b + 2c + d = 0;
\]
we have that \(b = c\) and \(a + b + d = 0\). Thus we have proved the theorem as desired.

2.2. Hopf algebra automorphisms of \(\bar{U}_{r,s}^{>0}(sl_3)\). In this subsection, we further determine all the Hopf algebra automorphisms of the Hopf algebra \(\bar{U}_{r,s}^{>0}(sl_3)\). Let us denote by \(\text{Aut}_{\text{Hopf}}(\bar{U}_{r,s}^{>0}(sl_3))\) the group of all Hopf algebra automorphisms of \(\bar{U}_{r,s}^{>0}(sl_3)\).

First of all, we have the following result
Theorem 2.2. Let \( \theta \in \text{Aut}_{Hopf}(\bar{U}_{r,s}^{\geq 0}(sl_3)) \). Then for \( l = 1, 2 \), we have the following

\[
\theta(K_l) = K_l, \quad \theta(E_l) = b_l E_l,
\]

where \( b_l \in \mathbb{C}^* \). In particular, we have

\[
\text{Aut}_{Hopf}(\bar{U}_{r,s}^{\geq 0}(sl_3)) \cong (\mathbb{C}^*)^2.
\]

**Proof:** Let \( \theta \in \text{Aut}_{Hopf}(\bar{U}_{r,s}^{\geq 0}(sl_3)) \) be a Hopf algebra automorphism of \( \bar{U}_{r,s}^{\geq 0}(sl_3) \), then we have \( \theta \in \text{Aut}_{\mathbb{C}}(\bar{U}_{r,s}^{\geq 0}(sl_3)) \). Therefore, we have the following

\[
\theta(K_l) = a_l K_l;
\]
\[
\theta(E_1) = b_1 K_1^a K_2^b E_1;
\]
\[
\theta(E_2) = b_2 K_1^c K_2^d E_2;
\]

where \( a_l, b_l \in \mathbb{C}^* \) for \( l = 1, 2 \) and \( a, b, c, d \in \mathbb{Z} \) such that \( b = c \) and \( a + b + d = 0 \).

First of all, we need to prove that \( a_l = 1 \) for \( l = 1, 2 \). Since \( \theta \) is a Hopf algebra automorphism, we have the following

\[
(\theta \otimes \theta)(\Delta(K_l)) = \Delta(\theta(K_l))
\]

for \( l = 1, 2 \), which imply the following

\[
a_l^2 = a_l
\]

for \( l = 1, 2 \). Thus we have \( a_l = 1 \) for \( l = 1, 2 \) as desired.

Second of all, we need to prove that \( a = b = c = d = 0 \). Note that we have the following

\[
\Delta(\theta(E_1)) = \Delta(b_1 K_1^a K_2^b E_1)
\]
\[
= \Delta(b_1 K_1^a K_2^b) \Delta(E_1)
\]
\[
= b_1 (K_1^a K_2^b \otimes K_1^b K_2^a)(E_1 \otimes 1 + K_1^2 K_2^{-1})
\]
\[
= b_1 K_1^a K_2^b E_1 \otimes K_1^a K_2^b + b_1 K_1^a K_2^b K_1^2 K_2^{-1} \otimes K_1^a K_2^b E_1
\]
\[
= \theta(E_1) \otimes K_1^a K_2^b + K_1^a K_2^b K_1^2 K_2^{-1} \otimes \theta(E_1)
\]

and

\[
(\theta \otimes \theta)(\Delta(E_1)) = (\theta \otimes \theta)(E_1 \otimes 1 + K_1^2 K_2^{-1} \otimes E_1)
\]
\[
= \theta(E_1) \otimes 1 + \theta(K_1^2 K_2^{-1}) \otimes \theta(E_1)
\]
\[
= \theta(E_1) \otimes 1 + K_1^2 K_2^{-1} \otimes \theta(E_1).
\]

Since \( \Delta(\theta(E_1)) = (\theta \otimes \theta)\Delta(E_1) \), we have \( a = b = 0 \). Since \( b = c \) and \( a + b + d = 0 \), we have \( a = b = c = d = 0 \).
In addition, it is obvious that the algebra automorphism \(\theta\) defined by \(\theta(K_l) = K_l\) and \(\theta(E_l) = b_l E_l\) for \(l = 1, 2\) is a Hopf algebra automorphism of \(U_{r,s}^{\geq 0}(\mathfrak{sl}_3)\) as well. Therefore, we have proved the theorem.

\[\blacksquare\]

3. Derivations and the first Hochschild cohomology group of \(U_{r,s}^{+}(\mathfrak{sl}_3)\)

In this section, we determine all the derivations of \(U_{r,s}^{+}(\mathfrak{sl}_3)\). In particular, we prove each derivation of \(U_{r,s}^{+}(\mathfrak{sl}_3)\) can be uniquely written as the sum of an inner derivation and a linear combination to certain specifically defined derivations. As a result, we are able to prove that the first Hochschild cohomology group of \(U_{r,s}^{+}(\mathfrak{sl}_3)\) is a two-dimensional vector space over the base field \(\mathbb{C}\). All these will be done through an embedding of the algebra \(U_{r,s}^{+}(\mathfrak{sl}_3)\) into a quantum torus, whose derivations had been described in [7]. This method has also been successfully used to compute the derivations of the algebra \(U_q(\mathfrak{sl}_3^+)\) in [6].

3.1. The embedding of \(U_{r,s}^{+}(\mathfrak{sl}_3)\) into a quantum torus. In this subsection, we embed the algebra \(U_{r,s}^{+}(\mathfrak{sl}_3)\) into a quantum torus, which enables us to extend the derivations of \(U_{r,s}^{+}(\mathfrak{sl}_3)\) to derivations of the quantum torus. Note that the algebra \(U_{r,s}^{+}(\mathfrak{sl}_3)\) has a Goldie quotient ring, which we shall denote by \(Q(U_{r,s}^{+}(\mathfrak{sl}_3))\). Inside the Goldie quotient ring \(Q(U_{r,s}^{+}(\mathfrak{sl}_3))\) of \(U_{r,s}^{+}(\mathfrak{sl}_3)\), let us define the following new variables

\[T_1 = E_1, \quad T_2 = E_2 - \frac{1}{r - s}E_3E_1^{-1}, \quad T_3 = E_3.\]

Concerning the relationships between the variables \(T_i, i = 1, 2, 3\), it is easy to see that we have the following proposition

**Proposition 3.1.** The following identities hold:

1. \(T_1T_2 = sT_2T_1\);
2. \(T_1T_3 = rT_3T_1\);
3. \(T_2T_3 = r^{-1}T_3T_2\).

\[\blacksquare\]

Let us denote by \(A^3\) the subalgebra of \(Q(U_{r,s}^{+}(\mathfrak{sl}_3))\) generated by \(T_1^{\pm 1}, T_2, T_3\), then we have the following

**Proposition 3.2.** The subalgebra \(A^3\) is the same as the subalgebra of \(Q(U_{r,s}^{+}(\mathfrak{sl}_3))\) generated by \(E_1^{\pm 1}, E_2, E_3\). In particular, \(A^3\) is a free module over the subalgebra generated by \(E_2, E_3\).
Furthermore, let us denote by $A^2$ the subalgebra of $Q(U_{r,s}^+(\mathfrak{sl}_3))$ generated by $T_1^{\pm 1}, T_2, T_3^{\pm 1}$. Then we have the following proposition

**Proposition 3.3.** The subalgebra $A^2$ is the same as the subalgebra of $Q(U_{r,s}^+(\mathfrak{sl}_3))$ generated by $E_1^{\pm 1}, E_2, E_3^{\pm 1}$.

Similarly, let us denote by $A^1$ the subalgebra of $Q(U_{r,s}^+(\mathfrak{sl}_3))$ generated by $T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}$. Thanks to **Proposition 3.1**, we know that the indeterminates $T_1, T_2, T_3$ generate a quantum torus, which we shall denote by $Q_3 = \mathbb{C}_{r,s}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}]$. In particular, we have the following

**Proposition 3.4.** The algebra $A^1 = Q_3 = \mathbb{C}_{r,s}[T_1^{\pm 1}, T_2^{\pm 1}, T_3^{\pm 1}]$ is a quantum torus.

Now let us define a linear map

$$\mathcal{I}: U_{r,s}^+(\mathfrak{sl}_3) \rightarrow Q_3$$

from $U_{r,s}^+(\mathfrak{sl}_3)$ into $A^1 = Q_3$ as follows

$$\mathcal{I}(E_1) = T_1, \quad \mathcal{I}(E_2) = T_2 + \frac{1}{r-s}T_3T_1^{-1}, \quad \mathcal{I}(E_3) = T_3.$$

It is easy to see that the linear map $\mathcal{I}$ can be extended to an algebra monomorphism from $U_{r,s}^+(\mathfrak{sl}_3)$ into $A^1 = Q_3$. Furthermore, it is straightforward to prove the following result:

**Theorem 3.1.** Let us set $A^4 = U_{r,s}^+(\mathfrak{sl}_3), \Sigma_4 = \{T_i^1 | i \in \mathbb{Z}_{\geq 0}\}, \Sigma_3 = \{T_3^i | i \in \mathbb{Z}_{\geq 0}\}, \Sigma_2 = \{T_2^i | i \in \mathbb{Z}_{\geq 0}\}$, then we have the following

1. $A^3 = A^4\Sigma_4^{-1}$;
2. $A^2 = A^3\Sigma_3^{-1}$;
3. $A^1 = A^2\Sigma_2^{-1}$;
4. $A^4 \subset A^3 \subset A^2 \subset A^1$;
5. The center of $A^i$ is the base field $\mathbb{C}$ for $i = 1, 2, 3, 4$.

From the reference [7], one knows that a derivation $D$ of the quantum torus $A^1 = Q_3$ is of the form $D = \text{ad}_t + \delta$ where $\text{ad}_t$ is an inner derivation
determined by some \( t \in A^4 \), and \( \delta \) is a central derivation which acts on the variables \( T_i, i = 1, 2, 3 \) as follows:

\[
\delta(T_i) = \alpha_i T_i
\]

for \( \alpha_i \in \mathbb{C} \).

Let \( D \) be a derivation of \( U^+_{r,s}(sl_3) = A^4 \). According to the previous theorem, one can extend the derivation \( D \) to a derivation of \( A^i \) for \( i = 3, 2, 1 \), and thus a derivation of the quantum torus \( A^1 \). We still denote the extension by \( D \). Therefore, as a derivation of the quantum torus \( A^1 \), the derivation \( D \) can be decomposed as follows

\[
D = ad_t + \delta
\]

where \( ad_t \) is an inner derivation determined by some \( t \in A^1 \), and \( \delta \) is a central derivation of \( A^1 \) which is defined by \( \delta(T_i) = \alpha_i T_i \) for \( \alpha_i \in \mathbb{C}, i = 1, 2, 3 \).

Now we are going to prove that the element \( t \) can actually be chosen from the algebra \( U^+_{r,s}(sl_3) \) and the scalars \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) are related to each other. In particular, we have the following key lemma.

**Lemma 3.1.** The following is true:

1. The element \( t \) can be chosen from \( U^+_{r,s}(sl_3) \);
2. We have \( \alpha_3 = \alpha_1 + \alpha_2 \);
3. We have \( D(E_i) = ad_t(E_i) + \alpha_i E_i \) for \( i = 1, 2, 3 \).

**Proof:** First of all, we show that \( t \in A^2 \). Suppose that we have

\[
t = \sum_{i,j,k} a_{i,j,k} T_1^i T_3^j T_2^k \in A_1.
\]

If \( k \geq 0 \) for all \( k \), then we are done. Otherwise, let us set

\[
t_+ = \sum_{k \geq 0} a_{i,j,k} T_1^i T_3^j T_2^k.
\]

Then we have that \( t = t_- + t_+ \).

First of all, we have the following

\[
D(T_1) = ad_t(T_1) + \delta(T_1)
\]

\[
= (t_- T_1 - T_1 t_-) + (t_+ T_1 - T_1 t_+) + \alpha_1(T_1)
\]

for some \( \alpha_1 \in \mathbb{C} \).

Since \( D \) is a derivation of the algebra \( U^+_{r,s}(sl_3) \) and the variable \( T_1 \) is in the algebra \( A^4 = U^+_{r,s}(sl_3) \), we have that the element \( D(T_1) \) is in the
algebra $A^4$, and furthermore in the algebra $A^2$. Note that the elements of $A^2$ don’t involve negative powers of the variable $T_2$, thus we have the following

$$t_- T_1 = T_1 t_-.$$

Therefore, we have the following

$$T_1(\sum_{k<0} a_{i,j,k}T_1^iT_3^jT_2^k) = (\sum_{k<0} a_{i,j,k}r^j s^k T_1^iT_3^jT_2^k)T_1$$

$$= (\sum_{k<0} a_{i,j,k}T_1^iT_3^jT_2^k)T_1.$$

Thus we have $k = 0$. Since we are supposed to have $k < 0$ in the expression of $t_-$, we have run into a contradiction. Therefore, we have that $t_- = 0$ and $t = t_+ \in A^2$. Similarly, we can also prove that $t \in A^3$.

Since the algebra $A^3$ is also generated by the elements $E_1^{\pm 1}, E_2, E_3$, we have the following

$$t = \sum_{i,j \geq 0, k \geq 0} a_{i,j,k}E_1^iE_2^jE_3^k.$$

Applying the derivation $D$ to the variable $E_3$, we can further prove that $i \geq 0$. Therefore, we have proved that $t \in A^4 = U_{r,s}(sl_3)$ as desired.

Since we have $D = ad_t + \delta$ for some $t \in U_{r,s}(sl_3)$, we have

$$D(E_2) = ad_t(E_2) + \delta(E_2)$$

$$= (tE_2 - E_2 t) + \delta(T_2 + \frac{1}{r - s} T_3 T_1^{-1})$$

$$= (tE_2 - E_2 t) + \alpha_2 (T_2 + \frac{1}{r - s} T_3 T_1^{-1})$$

$$+ \frac{1}{r - s} (\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1}$$

$$= (tE_2 - E_2 t) + \alpha_2 E_2 + \frac{1}{r - s} (\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1}.$$

Note that $D(E_2) \in U_{r,s}(sl_3)$, we have $\frac{1}{r - s} (\alpha_3 - \alpha_1 - \alpha_2) T_3 T_1^{-1} \in A^4$.

Thus we have the following

$$\alpha_3 = \alpha_1 + \alpha_2$$

and

$$D(E_2) = ad_t(E_2) + \alpha_2 E_2.$$

So we have the proved the lemma as desired. \qed
Now let us define two derivations $D_1, D_2$ of the algebra $U^+_{r,s}(sl_3)$ as follows:

$D_1(E_1) = E_1, \quad D_2(E_2) = 0, \quad D_1(E_3) = E_3; \\
D_2(E_1) = 0, \quad D_2(E_2) = E_2, \quad D_2(E_3) = E_3.

Based on the previous lemma, we have the following

**Theorem 3.2.** Let $D$ be a derivation of $U^+_{r,s}(sl_3)$. Then we have

$D = \text{ad}_t + \mu_1 D_1 + \mu_2 D_2$

for some $t \in U^+_{r,s}(sl_3)$ and $\mu_i \in \mathbb{C}$ for $i = 1, 2$.

Recall that the Hochschild cohomology group in degree 1 of $U^+_{r,s}(sl_3)$ is denoted by $HH^1(U^+_{r,s}(sl_3))$, which is defined as follows

$HH^1(U^+_{r,s}(sl_3)) = \text{Der}(U^+_{r,s}(sl_3))/\text{InnDer}(U^+_{r,s}(sl_3)).$

where $\text{InnDer}(U^+_{r,s}(sl_3)) = \{\text{ad}_t \mid t \in U^+_{r,s}(sl_3)\}$ is the Lie algebra of inner derivations of $U^+_{r,s}(sl_3)$. It is well known that $HH^1(U^+_{r,s}(sl_3))$ is a module over $HH^0(U^+_{r,s}(sl_3)) = \mathbb{C}$.

Now we state the structural result for the first Hochschild cohomology of $U^+_{r,s}(sl_3)$.

**Theorem 3.3.** The following is true:

1. Every derivation $D$ of $U^+_{r,s}(sl_3)$ can be uniquely written as follows:

$D = \text{ad}_t + \mu_1 D_1 + \mu_2 D_2$

where $\text{ad}_t \in \text{InnDer}(U^+_{r,s}(sl_3))$ and $\mu_1, \mu_2 \in \mathbb{C}$.

2. The first Hochschild cohomology group $HH^1(U^+_{r,s}(sl_3))$ of $U^+_{r,s}(sl_3)$ is a two-dimensional vector space spanned by $D_1$ and $D_2$.

**Proof:** Suppose that we have $\text{ad}_t + \mu_1 D_1 + \mu_2 D_2 = 0$, then we need to prove that $\mu_1 = \mu_2 = \text{ad}_t = 0$. Let us set $\delta = \mu_1 D_1 + \mu_2 D_2$. Then $\delta$ is a derivation of the algebra $U^+_{r,s}(sl_3)$.

Note that we can extend the derivation $\delta$ to a derivation of $A^1$, and we also have $\text{ad}_t + \delta = 0$ as a derivation of $A^1$. Furthermore, we have the following

$\delta(T_1) = \mu_1 T_1, \quad \delta(T_2) = \mu_2 T_2, \quad \delta(T_3) = (\mu_1 + \mu_2)T_3.$

Thus the derivation $\delta$ is a central derivation of the quantum torus $A^1$. According to the result in [7], we have that $\text{ad}_t = 0 = \delta$. Thus we have $\mu_1 = \mu_2 = 0$ as desired.
So we have proved the uniqueness for the decomposition of $D$, which further implies the second statement of the theorem.

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