Simulating nonlinear steady-state traveling waves on the falling liquid film entrained by a gas flow

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Abstract. The article is devoted to the simulation of nonlinear waves on a liquid film flowing under gravity in the known stress field at the interface. The paper studies nonlinear waves on a liquid film, flowing under the action of gravity in a known stress field at the interface. In the case of small Reynolds numbers the problem is reduced to the consideration of solutions of the nonlinear integral-differential equation for film thickness deviation from the undisturbed level. The periodic and soliton steady-state traveling solutions of this equation have been numerically found. The analysis of branching of new families of steady-state traveling solutions has been performed. In particular, it is shown that this model equation has solutions in the form of solitons-humps.

1. Introduction and problem statement

The joint flow of liquid and gas is a classical problem of hydrodynamics. Flows of thin liquid films in the presence of counter- or co-current gas flows often occur in various technological applications. Two stages of simulation are often distinguished: determining gas stresses on the film surface and further calculating the evolution of waves in the liquid. The possibility of the problem staging is justified in particular in [1, 2].

In this paper, we consider the dynamics of nonlinear waves on a fluid film entrained by the cocurrent gas flow and falling under the action of gravity along the vertical plane in a known stress field at the interface. In [2] for the system of hydrodynamic equations written in tensor form, invariant to coordinate systems, for the considered flow in the case of small flow rates ($\text{Re} \approx 1$) an evolution equation for the film thickness $h$ was obtained:

$$
\frac{\partial h}{\partial t} + \text{Re} \frac{h_0^2}{\text{Fr}} \frac{\partial h}{\partial x} + \text{Re} \tau_0 \frac{\partial h}{\partial x} + \left( \frac{1}{3} \epsilon \text{ReW} h^3 \frac{\partial^3 h}{\partial x^3} + \frac{2}{15} \frac{\text{Re}^3}{\text{Fr}^2} h^5 \frac{\partial h}{\partial x} \right) \left( h + \tau_0 \text{Fr} \right) + \frac{1}{2} \text{Re} h_0^2 \tau_0 \int h_k(k) e^{ikx} dk = 0. \quad (1.1)
$$

Here, $\text{Re} = \rho h_0 u_0^2 / \mu$ is Reynolds number, $\text{Fr} = u_0^2 / gh_0$ is Froude number, $\text{W} = \sigma / \rho l_0^2$ is Weber number and $\epsilon = h_0 / l_0$ is relation of specific film thickness $h_0$ to specific wave length $l_0$. Direction of coordinate $x$ coincides with the direction of gravity vector. In addition, in the equation (1.1) and in the dimensionless complexes we use the characteristic scales of velocity $u_0$ and time $l_0 / u_0$. Here, $\sigma$ is the surface tension, $\rho$ is the density, $\mu \mu$ is the dynamic viscosity of liquid, $g$ is the acceleration of gravity, $\tau_0$ is undisturbed component of gas shear stress on the film surface, $\tau(k) = \tau_0(k) + i \tau_{\text{im}}(k)$
are the Fourier components of tangential stresses of gas due to the boundary curvilinearity, and \( \hat{h}(k,t) \) are the Fourier components of the surface form expansion:
\[
\hat{h}(k,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x,t) e^{-ikx} dx.
\]

Here it was assumed that \( We \sim 1 \). Equation (1.1) accurate to notations coincides with the equation obtained in [1]. Some differences in the coefficients are due to the fact that it is the countercurrent flow of gas that was considered in [1].

Restricting to perturbations of small but finite amplitude, using the transformation: \( h = 1 + \epsilon h_1 \), and introducing slow \( (t_0 = \tau_0) \) and fast \( (t_1 = \epsilon \tau_1) \) times into consideration in [2] it was shown that in the first approximation (fast times) perturbations of small but finite amplitude propagate with a characteristic constant velocity \( c_0 = \text{Re}/\text{Fr}(1 + \text{Fr}\tau_0) \). The equation describing the nonlinear evolution of the perturbations at large (slow) times \( (t_1 = \epsilon \tau_1) \) has the form [2]:
\[
\frac{\partial h_1}{\partial \tau_1} + \frac{\text{Re}}{\text{Fr}}(2 + \text{Fr}\tau_0) h_1 + \frac{W\text{Re}e \partial^4 h_1}{3} + \frac{2}{15} \frac{\text{Re}^4 \partial^2 h_1}{\text{Fr}^2} (1 + \tau_0\text{Fr}) + \frac{1}{2} \text{Re} \tau_0 \int \hat{h}_0 k^2 \tau(k) e^{ikx} dk = 0.
\]

After replacement \( t = b_1 \), \( h_1 = AH \), \( b = W\text{Re}e/3 \), \( \epsilon = 0.4\text{Re}^2(1 + \text{Fr}\tau_0)/\left(W\text{Fr}^2\right) \), \( A = 2\text{Fr}b/\left(\text{Re}(2 - \text{Fr}\tau_0)\right) \) the equation (1.2) takes the form [2]:
\[
\frac{\partial H}{\partial t} + 2H \frac{\partial H}{\partial x} + \frac{\partial^2 H}{\partial x^2} + \frac{\partial^4 H}{\partial x^4} + B \int_{-\infty}^{\infty} ik^2 \tau(k) \hat{H}(k,t) e^{ikx} dk = 0.
\]

Here \( B = \frac{\text{Re}\tau_0}{2b} \equiv \frac{3\tau_0}{2W\epsilon} \).

The aim of this work is to find steady-state traveling periodic and soliton solutions of equation (1.3).

It is easy to study the linear stability of the unperturbed solution \( H = 0 \). Indeed, neglecting the nonlinear term in (1.3) for linear solutions \( H = \exp(i k(x - \epsilon t)) \), we obtain the dispersion relation:
\[
c = c_r + ic_i = Bk\tau(k) + i(k - k_0^2). \]

For unstable perturbations, the imaginary part of the phase velocity \( c_i \) will be greater than zero. They are long-wave disturbances. Their wave numbers are smaller than the neutral wave number \( k_n \) satisfying the equation:
\[
1 - k_n^2 + B \tau_{im}(k_n) = 0. \tag{1.4}
\]

As it can be seen from (1.4), for the freely falling film \( (B = 0) \) the neutral wave number \( k_n = 1 \). We will choose the parameters of the undisturbed flow, so that the neutral wave number \( k_n \) different from the unit. At the same time, it is required for this value of \( k_n \) to correspond to a definite value of \( \tau_{im} \).

Then for the value of the parameter \( B \) from (1.4) we have:
\[
B = \frac{k_n^2 - 1}{\tau_{im}(k_n)}.
\]
Obtaining results presented below, we used data on friction pulsations obtained by the model of boundary conditions transfer to the unperturbed level in the paper [2].

In the paper [3] it was shown that in the vicinity of the neutral wave number \( k_0 \) there is the soft type of branching: the wave numbers of the periodic steady-state traveling regimes of small but finite amplitude lie in the area of linear instability. For steady-state traveling solutions

\[ H(x,t) = H(\xi), \quad \xi = x - ct \]

and equation (1.3) is written in the form

\[
-c \frac{dH}{d\xi} + 2H \frac{d^2H}{d\xi^2} + \frac{d^4H}{d\xi^4} + B \int_{-\infty}^{\infty} ik^2 \tau(k) \hat{H}(k) e^{ik\xi} dk = 0. \tag{1.5}
\]

This equation was solved numerically.

Equation (1.5) is invariant relative to the transform

\[ H(\xi) \rightarrow H(\xi) + \text{const}, \quad c = c + 2\text{const}. \]

Taking this into account, we will further restrict to consider the periodic solutions, where

\[
\langle H(\xi) \rangle = \int_{0}^{2\pi/k} H(\xi) d\xi = 0. \tag{1.6}
\]

The use of solutions with normalization (1.6) means that the characteristic thickness scale is the average layer thickness in the wave flow regime.

2. Results of nonlinear wave calculation on the model equation

For finding solutions of equation (1.3) periodic on \( \xi \) the function \( H \) is represented as a spatial Fourier series:

\[ H(\xi) = \sum_{n} H_n \exp(i kn \xi). \tag{2.1} \]

Since \( H \) is real function, then \( \overline{H_{-n}} = H_{n}. \) The overbar means the operation of complex conjugation. After substituting (2.1) into equation (1.5), given that in this case the integral term in (1.3) is transformed to the corresponding series, obtain an infinite system of algebraic equations for Fourier harmonics \( H_n. \) Supposing that all \( H_n(t) \) with indices \( |n| \geq N \) are equal to zero, obtain its finite-dimensional analogue.

For finding new families, we use the method of stability theory. The study of stability of the steady-state periodic traveling solutions \( H_0(\xi) \) with respect to linear perturbations allows finding branching points for new solutions. The method of stability investigation is similar to those described in [4]. Below is its brief description.

Let \( H_0(\xi) \) is the periodic solution of equation (1.5) with wave number \( k. \) To study its stability, we substitute \( H(\xi) = H_0(\xi) + h(\xi,t) \) in (1.3) and linearize it on the perturbation \( h(\xi,t). \) As a result, we obtain a linear equation with periodic coefficients. Taking into account that the variable \( \iota \) does not enter the equation explicitly, its solution has the form:

\[ h(\xi,t) = \exp(-\gamma t) h_1(\xi) + C.C. \]

Here \( C.C. \) is complex conjugated expression. From the Floquet theorem it follows that the physically reasonable solutions \( h_1(\xi) \) bounded at infinities have the following form:

\[ h_1(\xi) = \exp(i Q k \xi \varphi(\xi)), \]

where \( \varphi(\xi) \) is the periodic function of the same period as \( H_0(\xi), \) and \( Q \) is the real parameter. Therefore, the study of the stability of steady-state traveling solutions of equation (1.3) is reduced to investigation of the spectrum of eigenvalues \( \gamma \) at various values \( Q. \) For these eigenvalues the resulting equation has periodic solutions \( \varphi(\xi) \) of the same period as \( H_0(\xi). \) It is sufficient to consider only the
changes of \( Q \) in a unit interval, for example \( 0 \leq Q \leq 1 \). The wave is stable if for any values of \( Q \), in all \( \gamma \) the real part \( \gamma_r > 0 \). New steady-state traveling modes bifurcate from the solution \( H_0(\xi) \), if at some point \((k, Q)\) one of the eigenvalues \( \gamma \) is equal to zero:

\[ \gamma(k, Q) = 0. \]

If \( Q = p/r \) is the rational number, then there is a mode periodic on \( \xi \) with a new wave number \( k_{\text{new}} = k/r \), and if \( Q = 0 \), then the new regime with the same period as \( H_0(\xi) \) can branch off.

In the case of \( B = 0 \), the equation (1.3) is transformed into equation which is now called "Kuramoto-Sivashinski equation (K-S)". For the falling film it was derived in [5]. It is well known that the family (family I) of wavy steady-state traveling solutions branches from the trivial one in the point \( k_n = 1 \) (in our normalization). It can be continued into the region of instability to the value \( k_n = 0.4979 \). At this point every odd-numbered harmonics is equal to zero, and the solution with the wave number \( k = 2k_n = 0.9958 \) is obtained. Thus, this family is closed on itself and cannot be continued to the region of smaller wave numbers. For all values of wave numbers the phase velocity of the waves of the family \( I \) \( c = 0 \). At the point \( k_n = 0.554 \) there is a bifurcation with \( Q = 0 \), and from the first family two new families with the same wavelength and \( c \neq 0 \) branch [4]. These families in the limit \( k \to 0 \) transform into the localized solitary waves (Depending on what is larger in the value, the value of the maximum or minimum, these solitons are often called positive or negative, respectively). For the first time these solitons were found in [6]. The solution with \( c > 0 \) transforms to a known positive soliton (soliton-hump) with the characteristic oscillations in the forefront.

![Figure 1](image1.png)  
**Figure 1.** Dependences of the amplitudes of steady-state traveling solutions of the families on the wave number \( k \): curve I - family I and curve II - family II. Neutral wave number \( k_n = 1.1 \), \( B = 0.0037 \).

![Figure 2](image2.png)  
**Figure 2.** Surface profiles for the solitons: curve I - family I and curve II - family II. Neutral wave number \( k_n = 1.1 \), \( B = 0.0037 \).

The emergence of a new term with \( B \neq 0 \) leads to a change in the bifurcation picture. Thus, for small values of the parameter \( B \) the first family extends to the region of small wave numbers and in the limit becomes a negative soliton. Family II branches from the first family with period doubling (\( Q = 0.5 \)). The limit of this family at \( k \to 0 \) is a positive soliton (soliton-elevation). The example of this case is presented in Figure 1, which shows the behavior of amplitudes \( A = H_{\text{max}} - H_{\text{min}} \) for the first two families of steady-state traveling solutions depending on the wave number. Here, the value of the parameter \( B = 0.0037 \), i.e. in this case the influence of the gas flow is weak. The first family branches off from the trivial solution \( H_0 = 0 \) in the neutral point \( k_n = 1.1 \). The new family II branches off from the first family at the point \( k_l = 0.5472 \). The limits of these families at \( k \to 0 \) are a negative
soliton with velocity $c = -1.53$ and a positive one with velocity $c = 1.87$, respectively. The profiles of these solitons are shown in Figure 2. This result is fairly predictable. However, with further increase of the parameter $B$ the pattern of branches is changing more radically.

Figure 3. Dependences of the amplitudes of steady-state traveling solutions of the families on the wave number $k$; curve I - family I, curve II - family II and curve III - family III. Neutral wave number $k_n = 1.5$, $B = 0.222$.

As an example, this paper presents the results, obtained at a sufficiently large value of the parameter $B = 0.222$. In this case, the impact of the gas flow blowing the film is significant.

Figure 3 shows the behavior of amplitudes for the first three families of steady-state traveling solutions depending on the wave number. The first family branches off from the trivial solution in the neutral point $k_n = 1.5$. Families II and III arise from the first family at the points $k_1 = 0.7433$ and $k_2 = 0.6304$, respectively. They bifurcate as a result of the fact that at points $b_{11} = 1.4866$ and $b_{21} = 1.2608$ in the solutions of the first family at the parameter value $Q = 0.5$ one of the eigenvalues

Figure 4. Surface profiles for the solitons: curve I - family I, curve II - family II and curve III - family III. Neutral wave number $k_n = 1.5$, $B = 0.222$.

Figure 5. Surface profile for the one-hump positive soliton at $B = 0.222$.

Figure 6. Surface profile for the one-hump positive soliton at $B = 0$ (solution to the K-S equation).
\[ \gamma(k_{bif}, Q) = 0. \] All three families transform to soliton solutions, but all these solitons are negative. Family I transforms into the negative one-hump soliton, and families II and III transform into different two-hump solitons, respectively. Their profiles are shown in Figure 4. In contrast to the case of small \( B \), for a given fixed value of the parameter \( B = 0.222 \) a family of steady-state traveling solutions, which limit is a positive one-humped soliton, can not be found by successive bifurcations. The corresponding families of solutions are found, moving along the continuity from small to large values of the parameter \( B \). This soliton is shown in figure 5. For comparison, figure 6 shows the one-humped soliton being a solution to the K-S equation.

3. Conclusion

Nonlinear waves on the liquid film flowing under the action of gravity in the known stress field at the interface have been considered. In the case of small Reynolds numbers, the problem is reduced to considering solutions of a nonlinear integro-differential equation for the film thickness deviation from the undisturbed level. Steady-state traveling periodic and soliton solutions to this equation have been considered. Using the results of the stability analysis of the steady-state periodic traveling solutions we have found new families of waves, including those that transform into positive solitons in the limit of small wave numbers.

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