Dynamic response analysis of nonlinear secondary oscillators to idealised seismic pulses

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Summary
The paper deals with the seismic response analysis of nonlinear secondary oscillators. Bilinear, sliding and rocking single-degree-of-freedom dynamic systems are analysed as representative of a wide spectrum of secondary structures and nonstructural components. In the first stage, the equations governing their full dynamic interaction with linear multi-degree-of-freedom primary structures are formulated, and then conveniently simplified using primary-secondary two-degree-of-freedom systems and dimensionless coefficients. In the second stage, the cascade approximation is applied, whereby the feedback action of the secondary oscillator on the primary structure is neglected. Owing to the piecewise linearity of the secondary systems being considered, efficient semi-analytical and step-by-step numerical solutions are presented. The semi-analytical solutions allow the direct evaluation of the seismic response under pulse-type ground excitations and are also used to validate step-by-step numerical schemes, which in turn can be used for general-type seismic excitations. In the third stage, a set of decoupling criteria are proposed for the pulse-type base excitations, identifying the conditions under which a cascade analysis is admissible from an engineering standpoint. Finally, the influence and relative dependencies between the input parameters of the ground motion and the primary-secondary assembly are quantified on the response of the secondary systems through nonlinear floor response spectra, and general trends are identified and discussed.

KEYWORDS
cascade approximation, floor spectra, nonlinear oscillators, primary-secondary dynamic interaction, semi-analytical solutions, step-by-step numerical solutions

1 INTRODUCTION
Understanding the seismic dynamics of secondary structures, such as mechanical, electrical or architectural components and contents, is of paramount importance to develop engineering solutions for community resilience. Observations following the aftermath of major earthquakes have indicated that failure of secondary structures may pose serious...
life safety concerns and disrupt the operational continuity of buildings, resulting in onerous losses to owners, occupants and insurance companies, which may often exceed the ones associated with structural damage. Acceleration-sensitive rigid components (ARCs) are an important class of systems that can be modelled either as linear or nonlinear single-degree-of-freedom (SDoF) secondary oscillators. Examples include medical and telecommunications equipment, emergency generators, transformers, compressors, chillers and computer cabinets that are all of practical interest in civil and industrial buildings.

ARCs tend to be vulnerable to earthquakes. Indeed, they are designed to perform functions other than resisting seismic loading and are often manufactured out of brittle materials, sensitive to vibration. Moreover, they are usually positioned at high elevations and are thus subjected to severe amplification of the ground shaking. Their mass and stiffness are relatively low, which can result in natural frequencies close to those of their supporting structures; additionally, they tend to possess damping ratios lower than those required for protection against sharp resonant effects. Further challenges arise as their behaviour is characterised by nonlinearities and failure modes such as sliding, rocking or overturning. Nevertheless, owing to the nonexhaustive list of ARCs and their strong dependence on both the ground shaking and the load-bearing structure, there are no universally accepted standards in place for modelling their behaviour, and the determination of their seismic response is not straightforward.

A large body of research has been devoted to the nonlinear seismic analysis of SDoF systems. Amongst other investigations, the dynamics of bilinear oscillators have been studied in presence of sinusoidal excitation by Caughey, who identified that such systems exhibit soft-type resonance and unbounded resonance beyond a critical value of the excitation. Makris and Black showed that, for a given dimensionless yield displacement and strength, the response of pulse-driven bilinear blocks is self-similar, regardless of the pulse duration and intensity. More recently, Voyagaki et al proposed a transformation method capable of estimating the portion of the excitation associated with plastic deformations and the duration of each yielding branch.

The behaviour of sliding systems has been examined by Westermo and Udwadia, who studied the harmonic response of a sliding rigid mass, and then the periodic solution to an oscillating block on a rough surface. Makris and Constantinou investigated the transient and steady-state response of harmonically driven frictional oscillators. Hong and Liu refined the frictional oscillator model formulation, deriving exact solutions to simple harmonic loading. More recently, Voyagaki et al proposed a new response evaluation method and interpretation of rigid-plastic block dynamics by imposing a shift in the ordinates and the abscissa of the excitation function. The authors also presented analytical and numerical solutions for the response to idealised near-fault acceleration pulses.

The dynamics of rocking blocks have been investigated by Yim et al, who studied the response of free-standing blocks to earthquake ground motions, uncovering pronounced sensitivity to variations in the slenderness ratio and size of the block, as well as the ground motion characteristics. Spanos and Koh examined the response of harmonically excited rocking blocks resting on rigid foundations, identifying safe and unsafe regions. Zhang and Makris as follows investigated the transient response of pulse-driven free-standing rocking blocks, showing that the block can overturn in two distinct modes, later extending their analyses to the case of anchored blocks. Palmeri and Makris examined the response of rigid structures rocking on viscoelastic and yielding foundation. More recently, Dimitrakopoulos and DeJong derived new closed-form solutions and similarity laws for the free-standing block. Voyagaki et al showed that the full nonlinear equations tend to result in a more stable response when compared with their linearised counterparts. In a subsequent paper, closed-form solutions were presented for a two-dimensional rigid block driven by idealised acceleration pulses.

All the above-referenced studies deal with the nonlinear analysis of oscillators directly excited by the ground shaking, rather than by the seismic response of a primary supporting structure, as in the case of ARCs. As a matter of fact, coupled analysis techniques for primary-secondary interacting systems have been termed cumbersome, due to the excessive number of degrees of freedom, dissimilarities in their inertia and stiffness and the need to study such systems in various configurations, for example, due to uncertainties in the exact position of the secondary systems in the primary structure. Analysis methods have therefore evolved mainly for linear secondary systems, including analytical-based methods, substructuring techniques and other simplified methods. The cascade approximation (also admissible for nonlinear systems) represents a practical simplified approach, whereby the response of the secondary structure is sequentially evaluated at the points of attachment, neglecting the feedback action on the primary structure, based on the premise that the ARC is sufficiently ‘light’ and not in-tune with the primary system. If these conditions are met, and if the ARC is assumed linear, decoupling criteria can be employed for deciding whether this approach is acceptable in practice.

In presence of a heavy ARC or when the ARC vibrates close to, or is tuned with, the primary structure, this analysis approach can lead to unrealistic
results\textsuperscript{32} while, to the best of the authors’ knowledge, no decoupling criteria exist to aid the designer in the case of nonlinear systems.

In this paper, the seismic response analysis of bilinear, sliding and free-standing rocking secondary blocks, representative of a wide spectrum of building components of engineering interest, is addressed. The equations governing their full dynamic interaction with linear primary structures are first derived, through a convenient reduction of the primary-secondary assembly to a two-degree-of-freedom system with dimensionless coefficients. By exploiting the piecewise linear form of the secondary systems, semi-analytical and numerical solutions are derived for evaluating the cascade response in presence of an idealised pulse-type base excitation. The semi-analytical solutions allow a rigorous quantification of the response as a function of the input parameters, while the numerical ones permit the exact determination of the dynamic response under a general type of excitation (the only source of approximation being the time discretisation of the ground shaking besides rounding numerical errors). A set of decoupling criteria are then identified in relation to the cumulative error in the response history of each system, for various combinations of the input parameters, thus providing indications on the validity regions for the cascade approximation. Finally, the relative influence and dependencies between the characteristics of the ground motion and primary-secondary assembly are quantified on the response of light secondary systems through floor response spectra.

2  \hspace{1cm} \textbf{COMBINED PRIMARY-SECONDARY SYSTEM VIBRATION}

Let us consider the case of a combined dynamic system consisting of an SDoF nonlinear secondary oscillator (simply ‘S’ in the following) attached to a multi-degree-of-freedom (MDoF) linear primary structure (denoted with ‘P’ in the following). The latter represents herein a multi-storey structure designed to respond within its linear-elastic regime to the seismic excitation (thus meeting, e.g., the ‘damage limitation’ requirement of EC8\textsuperscript{33} or the ‘fully operational’ performance design objective in the ‘Vision 2000’ report\textsuperscript{34}), while the S oscillator describes a nonlinear attachment, whose seismic performance is potentially critical to the overall resilience of the building or facility being analysed.

The equation of motion of the P structure, assumed at rest for \( t \leq 0 \), reads as follows:

\[
M_p \cdot \ddot{u}_p(t) + C_p \cdot \dot{u}_p(t) + K_p \cdot u_p(t) = -M_p \cdot \tau_{pg} \cdot \ddot{u}_g(t) - \tau_{ps} m_n \left\{ \ddot{u}_s(t) + \tau_{ps}^T \cdot \left\{ \dddot{u}_p(t) + \tau_{pg} \cdot \ddot{u}_g(t) \right\} \right\};
\]

\[
u_p(0) = 0_n; \quad \dot{u}_p(0) = 0_n,
\]

where \( M_p, C_p \) and \( K_p \) are the mass, damping and stiffness matrices of the P structure in the geometrical space, respectively; \( m_n \) is the mass of the S oscillator; \( u_p(t) = \{ u_{p1}(t), \ldots, u_{pn}(t) \}^T \) is the vector listing the \( n \) dynamically significative DoFs of the P structure relative to the ground, the superscripted \( ^T \) being the transpose operator; \( u_s(t) \) is the translational DoF of the S oscillator relative to the motion of the P structure; \( \ddot{u}_g(t) = \{ \ddot{u}_{g1}(t), \ldot{u}_{g2}(t) \}^T \) is the \( (2 \times 1) \) vector containing the horizontal components of the ground acceleration in two orthogonal directions, say \( x \) and \( y \), the over-dot denoting differentiation with respect to the time \( t \); \( \tau_{ps} \) is the corresponding \( (n \times 2) \) matrix of seismic incidence, while \( \tau_{ps} \) is the \( (n \times 1) \) incidence vector of the feedback action of the S oscillator on the P structure; \( u_p(0) \) and \( \dot{u}_p(0) \) are the vectors of the initial conditions, \( 0_n \) being a zero vector of dimensions \( (n \times 1) \). Additionally, the P structure is assumed in the following to be classically damped, meaning that the Caughey-O’Kelly condition \( K_p \cdot M_p^{-1} \cdot C_p = C_p \cdot M_p^{-1} \cdot K_p \) is satisfied.\textsuperscript{35} Furthermore, it could be noted that \( \dddot{u}_p^{(a)}(t) = \dddot{u}_p(t) + \tau_{pg} \cdot \ddot{u}_g(t) \) is the vector listing the absolute accelerations of the P DoFs; \( \dddot{u}_p^{(a)}(t) = \tau_{ps} \cdot u_p^{(a)}(t) \) is the absolute acceleration of the P structure at the location and direction of the S oscillator and \( \dddot{u}_s^{(a)}(t) = \dddot{u}_s(t) + \dddot{u}_p^{(a)}(t) \) is the absolute acceleration of the S mass, so that \( F_s(t) = m_s \ddot{u}_s^{(a)}(t) \) is the feedback force that the S oscillator exerts on the P structure.

2.1  \hspace{1cm} \textbf{Modal transformation of coordinates}

The dimensions of the P-S combined dynamic system may be reduced by projecting the equations of motion of the P structure onto a reduced modal subspace. Accordingly, a solution to the real-valued eigenproblem \( M_p \cdot \Phi_p \cdot \Omega_p^2 = \Phi_p \cdot \Omega_p^2 = K_p \cdot \Phi_p \) is sought, where \( \Phi_p = \{ \phi_{p1}, \ldots, \phi_{pn} \} \) is the \( (n \times n) \) modal matrix, whose columns are the \( n \) modal shapes of the P structure, and \( \Omega_p = \text{diag} \{ \alpha_{p1}, \ldots, \alpha_{pm} \} \) is its diagonal spectral matrix, listing the associated modal circular frequencies, ordered from the lowest to the highest. If the P-S dynamic interaction has its largest effects on the \( i \)th mode of vibration of the P structure, the following transformation can be adopted:
\( u_p(t) = \phi_{pl} q_{pl}(t), \)

where \( q_{pl}(t) \) is the \( i \)th modal coordinate of the P structure. For an S oscillator acting on the \( x \) or \( y \) direction, for instance, \( i \) may be the mode of vibration with the largest modal participation factor in that direction. Without loss of generality, the modal shape \( \phi_{pl} \) can be normalised such that the largest horizontal displacement is set equal to 1. Also, if needed, more than one mode can be successively considered for the P structure.

Upon substitution of equation (2) in equation (1), premultiplication of the result by \( \phi_{pl}^T \) and further manipulations, one obtains the following:

\[
\ddot{u}_{ps}(t) + 2 \zeta_p \omega_p \dot{u}_{ps}(t) + \omega_p^2 u_{ps}(t) = -\phi_{ps} \beta \phi_{pl}^T \dot{u}_{pl} - \phi_{ps} \gamma \phi_{pl}^T \ddot{u}_{pl}(t) + \ddot{u}_{pl}(t); \quad u_{ps}(0) = 0; \quad \dot{u}_{ps}(0) = 0, \tag{3}
\]

where \( u_{ps}(t) = \phi_{ps} q_{pl}(t) \) denotes the contribution of the \( i \)th mode of vibration to the seismic response of the P structure at the position of the S oscillator, \( \phi_{ps} = \tau_{ps}^T \phi_{pl} \) being the corresponding modal displacement; \( \ddot{u}_{pl}(t) = \dot{u}_{pl}(t) + \tau_{pl}^T \tau_{pg} \ddot{u}_{pg}(t) \) is the absolute acceleration of the S mass, considering for the P structure only the contribution of the \( i \)th mode of vibration; \( \zeta_p \) and \( \omega_p \) are the \( i \)th modal values of equivalent viscous damping ratio and circular frequency of the P modal oscillator, respectively; \( \beta = m_p^{-1} \phi_{pl}^T \cdot M_p \cdot \tau_{pg} = \{ \beta_{1x}, \beta_{1y}, \beta_{2x}, \beta_{2y} \}^T \) is the \((2 \times 1)\) vector collecting the \( i \)th dimensionless modal factors for the seismic input, with \( m_p = \phi_{pl}^T \cdot M_p \cdot \phi_{pl} \) being the \( i \)th modal mass for the P structure; similarly, \( \gamma = \dot{m}_p^{-1} \phi_{pl}^T \cdot \tau_{ps} \cdot m_s \) is the \( i \)th dimensionless modal factor for the S feedback on the P structure.

For the sake of simplifying the notation, in the following, the generalised P coordinate \( u_{ps}(t) \) will be replaced with \( u_p(t) \); additionally, the modal index \( i \) will be dropped in all the other quantities in equation (3) and only the case of one-directional ground acceleration will be considered. As a result, equation (3) simplifies as follows:

\[
\ddot{u}_p(t) + 2 \zeta_p \omega_p \dot{u}_p(t) + \omega_p^2 u_p(t) = -\phi_{ps} \beta \dot{u}_p(t) - \phi_{ps} \gamma \dot{u}_p(t) + \ddot{u}_p(t) + \ddot{u}_s(t); \quad u_p(0) = 0; \quad \dot{u}_p(0) = 0. \tag{4}
\]

Notably, setting \( \gamma = 0 \) decouples equation (4) from the equation governing the seismic response of the S oscillator: that is, the approach is identical to the so-called ‘cascade approximation,’ where the P-S dynamic interaction is neglected. It is worth noting here that the condition \( \gamma = 0 \) does not imply a massless S system, but merely the fact that the feedback action of the S oscillator on the P structure is negligible; neither does it refer to the case of an S oscillator whose mass is negligible in comparison with the mass of the floor that is placed upon, as \( \gamma \) takes into account the overall modal mass of the system, and it is thus a stronger assumption, but nonetheless applicable to several systems of engineering interest. Furthermore, in the special case where P is a SDoF oscillator, that is, \( n = 1 \), then, \( \phi_{ps} = \beta = 1 \) and \( \gamma = m_s/m_p \) is the S-to-P mass ratio.

### 2.2 Nonlinear secondary oscillators

S systems of engineering interest may show a whole range of different nonlinear behaviours. In this paper, bilinear, sliding and rocking oscillators are considered as representative of distinct nonlinearities, all characterised by piecewise force-displacement (or moment-rotation) constitutive law with positive, zero or negative slopes. The equations governing the combined seismic response of the resulting P-S systems are presented in the following.

#### 2.2.1 Bilinear oscillator

For the bilinear elastoplastic subsystem of Figure 1A, the dynamic equilibrium of the mass \( m_s \) in the horizontal direction gives the following:

\[
\ddot{u}_s(t) + 2 \zeta_s \omega_s \dot{u}_s(t) + \frac{f_b(u_s(t), \dot{u}_s(t))}{m_s} = -\ddot{u}_p(t); \quad u_s(0) = 0; \quad \dot{u}_s(0) = 0, \tag{5}
\]
where \( u_s(t) \) is the displacement of the S oscillator relative to its point of attachment to the P structure; \( \zeta_s \) and \( \omega_s = \sqrt{k_s/m_s} \) are the viscous damping ratio and natural circular frequency of the S oscillator in its elastic range, \( k_s \) being the elastic stiffness of the S oscillator; \( \ddot{u}_p(t) + \dot{u}_p(t) \) is the absolute horizontal acceleration of the P structure at the position and direction of the S oscillator, in which \( \ddot{u}_p(t) \) satisfies equation (4); furthermore, \( f_s \) represents the bilinear restoring force of the S oscillator, which can be expressed as follows:

\[
f_b(u_s(t), \dot{u}_s(t)) = \psi_s \omega_s^2 m_s u_s(t) + a_s m_s (1 - \psi_s) z_s(t),
\]

(6)

where \( \psi_s \) in the range \( 0 \leq \psi_s \leq 1 \) is the post-to-pre-yield stiffness ratio; \( a_s \) represents the specific strength of the system, that is, the yield force divided by \( m_s \); \( z_s(t) \) is an auxiliary state variable satisfying \( |z_s(t)| \leq 1 \) and ruled by the nonlinear differential equation\(^{36} \) as follows:

\[
z_s(t) = \frac{\ddot{u}_s(t)}{a_s} \frac{\omega_s^2}{a_s} \left[ 1 - H(\dot{u}_s(t)) H(z_s(t) - 1) - H(\dot{u}_s(t)) H(1 - z_s(t)) \right].
\]

(7)

\( H(\cdot) \) being the Heaviside unit step function, such that \( H(x) = 1 \) if \( x \geq 0 \) and \( H(x) = 0 \) if \( x < 0 \).

Notably, setting \( \psi_s = 0 \) in equation (6) corresponds to an elastic-perfectly-plastic S oscillator, while the condition \( \psi_s = 1 \) results in the linear-elastic regime of motion, with \( f_b(t) = \omega_s^2 m_s u_s(t) \).

### 2.2.2 Sliding block

The rigid-perfectly plastic SDoF S system depicted in Figure 1B can be considered as a limiting case of the bilinear restoring force of equation (6), namely by setting \( \psi_s = 0 \) and \( \omega_s \to +\infty \). Accordingly, the system exhibits infinite preyielding stiffness and infinite ductility, and the ‘sliding’ restoring force so obtained can be expressed in the following form:

\[
\begin{align*}
(f_s(\dot{u}_s(t))) &= a_s m_s \text{sgn}(\dot{u}_s(t)), & |\dot{u}_s(t)| > 0; \\
(f_s(\dot{u}_s(t))) &= |a_s m_s|, & \dot{u}_s(t) = 0,
\end{align*}
\]

(8)

in which (\( \star \)) is the signum function (i.e., \( \text{sgn}(x) = +1 \) if \( x > 0 \), \( \text{sgn}(x) = -1 \) if \( x < 0 \) and \( \text{sgn}(x) = 0 \) if \( x = 0 \)), and the specific strength is given by \( a_s = \mu_s g \), \( \mu_s \) being the coefficient of sliding friction assuming horizontal contact surface, and \( g \) is the acceleration of gravity. Notably, the formalism used in equation (8) contains information about both the sliding phase, when the velocity \( \dot{u}_s(t) \) is different than zero, and the ‘sticking’ phase, when \( \dot{u}_s(t) = 0 \), in which case \( f_s \) may take any value between \( -a_s m_s \) and \( a_s m_s \), as required to sustain equilibrium\(^{11} \).

If now the viscous damping is neglected in the vibration of the S sliding oscillator (i.e., \( \zeta_s = 0 \)), the dynamic equilibrium under seismic excitation reads as follows:
\[ \ddot{u}_s(t) + a_s \text{sgn}(\dot{u}_s(t)) = -u_p^{(a)}(t). \]  

Evidently, equation (9) is only valid within the sliding phases (i.e., when \( \dot{u}_s(t) \neq 0 \)), while no relative P-S motion occurs in the sticking phase (i.e., \( u_s = \dot{u}_s = 0 \)); in the latter case, \( \dot{u}_s(t) = 0 \), and thus equation (4) represents a P structure, whose floor mass at the S oscillator's point of attachment is augmented by \( m_s \), and whose modal frequencies and shapes are consequently altered.

The initiation condition for the sliding regime of motion can simply be expressed as \( |\ddot{u}_p^{(a)}(t)| = a_s \). Following initiation, an instantaneous stop or a full stop can occur in the system once the velocity relative to the P structure drops to zero (i.e., \( \dot{u}_s(t) = 0 \)); in the former case, the motion will reverse or it will continue in the same direction, while in the latter case, the system will remain at rest until the initiation condition is met again.

### 2.2.3 Rocking block

Let us now consider the case of a rectangular S block standing free on a P structure. If the coefficient of sliding friction is sufficiently large (i.e., \( \mu_s \rightarrow +\infty \)), the block can experience pure rocking motion, oscillating about its centres of rotation \( O \) and \( O' \), as illustrated in Figure 1C. Based on the assumption of zero vertical ground acceleration, the equilibrium of moments about these pivot points allows deriving the equation governing its seismic response during the rocking motion:

\[
\ddot{\theta}_s(t) = \frac{-p^2}{g} \left[ \sin(A(t)) + \frac{\dot{u}_p^{(a)}(t)}{g} \cos(A(t)) \right]; \quad \theta_s(0) = 0; \quad \dot{\theta}_s(0) = 0.
\]  

\( A(t) = \alpha \text{sgn}(\theta_s(t)) - \theta_s(t), \) \( \theta_s(t) \) being the rotational response of the rigid S block; \( \alpha = \tan^{-1}(b/h) \) is its slenderness angle, which depends on width \( b \) and height \( h \) of the block; \( p = \sqrt{3g/(4R)} \) is a geometrical parameter (e.g., \( p \approx 2 \text{ rad/s} \) for an electrical transformer) and \( R = 0.5\sqrt{b^2 + h^2} \) is half the block’s diagonal. Noticeably, equation (10) coincides with the seismic equation of motion presented by Zhang and Makris,17 with the only difference that in the present case, the dynamic excitation at the base of the rocking block includes the seismic response of the P structure. Once the block’s rotation, \( \theta_s(t) \), is known, the translational horizontal displacement of its centre of mass, \( u_s(t) \), relative to the P structure can be readily determined as follows:

\[ u_s(t) = \text{sgn}(\theta_s(t))R\sin(\alpha) - R\sin(A(t)). \]  

It can be observed that the S block will initially possess infinite rotational stiffness until the applied moment about one of the pivot points \( O \) and \( O' \) reaches the value \( |M_s| = m_s g R \sin(\alpha) \), and a softening branch (negative stiffness) initiates, reaching \( M_s = 0 \) at the tipping condition \( |\theta_s| = \alpha \). Interestingly, this behaviour reminiscences the sliding S block (Figure 1B), except that the behaviour of the latter is hysteretic (i.e., energy is dissipated when the direction of motion reverses) and the stiffness is zero in the motion phase. It is further noted here that equations (10) and (11) are analogous to the ones presented by Vassiliou and Makris37 for base-isolated rigid rocking blocks.

The initiation condition for equation (10) is \( |\ddot{u}_p^{(a)}(t)| = g \tan(\alpha) \). Similar to the sliding case, \( \dot{u}_p^{(a)}(t) \) is obtained by directly solving equation (4) after setting \( \dot{u}_s(t) = 0 \). Once the rocking motion begins, a change in the sign of the rotation \( \theta_s(t) \) will correspond to an impact and (assuming that the block does not bounce back) the pivot point will switch from \( O \) to \( O' \) (or vice versa).

Importantly, impacts are the only form of energy dissipation considered in this type of oscillations. It is assumed that the rotational velocity after the impact, \( \dot{\theta}_s^+ \), is a fraction of the velocity before the impact, \( \dot{\theta}_s^- \), that is, \( \dot{\theta}_s^+ = \epsilon_s \dot{\theta}_s^- \), where \( \epsilon_s \) is the coefficient of energy restitution, with \( 0 < \epsilon_s \leq \epsilon_{s,\text{max}} < 1 \), \( \epsilon_{s,\text{max}} \) being the maximum allowed value.

Considering the special case where \( P \) is a SDoF oscillator with \( n = 1 \) and \( \phi_{ps} = \beta = 1 \), conservation of linear momentum (for the P-S system) and angular momentum (for S), gives the following:

\[
\dot{u}_p^+ = \dot{u}_p^- + \frac{6h\gamma \sin^2(\alpha)}{8 + 5\gamma - 3\gamma \cos(2\alpha)} \dot{\theta}_s^- = \dot{u}_p^- + \epsilon_{p,\text{max}} \dot{\theta}_s^-; \quad \dot{\theta}_s^+ = \frac{-2 + \gamma - 3(2 + \gamma) \cos(2\alpha)}{-8 - 5\gamma + 3\gamma \cos(2\alpha)} \dot{\theta}_s^- = \epsilon_{s,\text{max}} \dot{\theta}_s^- ,
\]  

where \( \dot{u}_p^- \) and \( \dot{\theta}_s^- \) are the negative phases of the velocity and angular velocity, respectively, and \( \epsilon_{p,\text{max}} \) and \( \epsilon_{s,\text{max}} \) are the coefficients of energy restitution for the P-S and S-P impacts, respectively.
where \( \varepsilon_{p,\text{max}} \) accounts for a residual contribution in the translational velocity of the P oscillator after the impact. These expressions are equivalent (although arranged in a different form) with respect to those presented by Vassiliou and Makris.\(^{37}\)

Notably, \( \varepsilon_{p,\text{max}}/h \) can take values between zero (for slender blocks, when \( \alpha \to 0 \)) and 0.3, while \( \varepsilon_{s,\text{max}} \) assumes values between one (as \( \alpha \to 0 \)) and zero. Evidently, when \( \gamma > 0 \) additional energy is lost during impact due to the translational velocity of P, while as \( \gamma \to 0 \), the behaviour approaches the case of a block rocking on a rigid foundation (i.e., \( \varepsilon_{p,\text{max}} \to 0 \) and \( \varepsilon_{s,\text{max}} \to 1 - 3\sin^2(\alpha/2) \)).

It is worth emphasising here that the impact model presented in equation (12), and consequently the postimpact response, are limited to the special case where P is modelled as a SDoF oscillator (i.e., \( n = 1 \) and \( \phi_{ps} = \beta = 1 \)), or to the case where P is modelled as an \( n \) DoF system but the analysis of the rocking S is carried out in cascade (i.e., \( \gamma = 0 \)). However, equations (4) and (10) are still applicable to the combined P-S system arising from the S block standing free on an \( n \) DoF P structure provided that no impact occurs, the latter case being only significant if the ground shaking causes overturning without impact. A general impact model for such a system would require explicit consideration of more modes of vibration, which falls beyond the scope of this work.

### 3 | CASCADE RESPONSE OF PIECEWISE LINEAR SECONDARY OSCILLATORS

In the preceding section, the combined vibration (i.e., full dynamic interaction) of a class of nonlinear SDoF S oscillators mounted on a linear MDoF P structure has been addressed. Notably, the parameter \( \gamma \) in equation (4) accounts for the relative contribution of the feedback action on P, due to the presence of the S oscillator. In practical applications, however, particularly in the case of nonlinear systems, the resulting equations of motion that govern the full dynamic system can be cumbersome. For this reason, the feedback action on P is usually neglected, based on the assumption that its contribution can be regarded as ‘small enough,’ leading to the ‘cascade approximation.’ The latter approach is acceptable when the S-to-P mass ratio is sufficiently low, that is, \( \phi_{ps} m_s \ll \bar{m}_p \) and therefore \( \gamma \ll 1 \). Setting then \( \gamma = 0 \), the equations of motion decouple and can be considered one after the other, that is, this coincides to solving first equation (4) to evaluate the response of the P structure, and then equations (5), (9) and (10) to determine the response of each S oscillator. In this way, various types of nonlinear ‘light’ components can be considered without the need to reanalyse each time the P load-bearing structure, or consider the interaction between multiple nonlinear S oscillators.

In this section, the cascade analysis is addressed in detail for the specific case of the seismic response evaluation of the piecewise linear S oscillators described in the previous section. To enable the derivation of closed-form analytical expressions, the case of a ground acceleration comprising of an idealised full-cycle sinusoidal pulse is considered as follows:

\[
\ddot{u}_g(t) = \begin{cases} 
\alpha_g \sin(\omega_g t), & 0 \leq t \leq 2\pi/\omega_g, \\
0, & \text{otherwise},
\end{cases}
\]  

where \( \alpha_g \) and \( \omega_g \) are the amplitude and frequency of the pulse, respectively. This type of excitation has been extensively considered in the literature to quantify the effects of near-fault ground motions on various structural systems and to shed light on the key parameters affecting their performance.\(^{7,17,37}\)

#### 3.1 | Closed-form solutions for pulse-driven nonlinear secondary oscillators

The combination of pulse-type harmonic excitation (see equation (13)) and piecewise linearity of the S systems enables the derivation of analytical solutions, which are presented in the following. These solutions are derived separately for each regime of motion and are expressed in terms of initial conditions defined at the transition times between piecewise linear branches. They are then ‘pieced together’ to construct the whole response time history. Details on the solution implementation procedure are provided in Appendix C. The purpose of deriving closed-form solutions is twofold: (i) they allow directly expressing the dynamic response as a function of the input parameters of the ground motion and the P structure, conveniently permitting the examination of the relative dependencies and contributions, and (ii) they can be used to validate numerical solutions for general-type excitations.
3.1.1  Bilinear oscillator

The case of the bilinear S oscillator is considered first. Integrating equation (5) twice with respect to the time \( t \) and enforcing the initial conditions, the analytical solution in terms of displacements and velocities can be expressed as follows:

\[
\begin{align*}
\ddot{u}_s(t) & = e^{-\bar{\zeta}_s \bar{\omega}_s t} \left[ C_1^b \sin(\bar{\omega}_s t) + C_2^b \cos(\bar{\omega}_s t) \right] + C_3^b \sin(\omega_g t + \phi) + C_4^b \cos(\omega_g t + \phi) \\
& \quad + e^{-\bar{\zeta}_p \bar{\omega}_p t} \left[ C_5^b \sin(\bar{\omega}_p t) + C_6^b \cos(\bar{\omega}_p t) \right] + C_7^b(t); \\
\end{align*}
\]

in which \( \bar{\omega}_s = \omega_s \sqrt{\psi - \bar{\omega}_s^2} \) and \( \bar{\omega}_p = \omega_p \sqrt{1 - \bar{\omega}_p^2} \) are the damped natural circular frequencies for S and P subsystems, respectively, and \( C_1^b - C_7^b \) are a set of coefficients provided in Appendix A.1.1. 

Notably, setting \( \psi = 1 \), \( \bar{C}_3 = -a_q \), \( \bar{C}_4 = \bar{C}_5 = \bar{C}_6 = 0 \) in the above, and making the substitutions \( \bar{\omega}_s = \bar{\omega}_p = \omega_s = \omega_p \), \( \bar{\zeta}_s = \bar{\zeta}_p \), \( u_{s,0} = u_{p,0} \) and \( \dot{u}_{s,0} = \dot{u}_{p,0} \), one can recover the response \( \ddot{u}_s(t) \), \( \ddot{u}_p(t) \) of the linear SDoF P oscillator.

Also, it is worth emphasising here that the complexity of the closed-form expressions presented in this section is due to the fact that the dynamic input acting on the nonlinear S oscillators, that is, \( \ddot{u}_p(t) \) in the right-hand side of equations (5), (9) and (17), does not consist simply of the ground shaking of equation (13), but rather \( \ddot{u}_p(t) = \ddot{u}_g(t) + \ddot{u}_p(t) \), where the second term is in turn the superposition of two contributions, namely the homogenous and particular solutions of equation (4) particularised for \( \ddot{u}_s(t) = 0 \).

3.1.2  Sliding block

The pertinent solution to equation (9) for the sliding S reads as follows:

\[
\begin{align*}
\ddot{u}_s(t) & = -\frac{1}{2} a_s t^2 \text{sgn}(\ddot{u}_s(t)) + C_1^s + C_2^s t + C_3^s \sin(\omega_g t + \phi) + C_4^s \cos(\omega_g t + \phi) \\
& \quad + e^{-\bar{\zeta}_p \bar{\omega}_p t} \left[ C_5^s \sin(\bar{\omega}_p t) + C_6^s \cos(\bar{\omega}_p t) \right]; \\
\end{align*}
\]

\[
\begin{align*}
\dot{u}_s(t) & = \frac{d}{dt} u_s(t) = -a_s t \text{sgn}(\dot{u}_s(t)) + C_1^s + C_2^s \omega_g \cos(\omega_g t + \phi) - C_3^s \omega_g \sin(\omega_g t + \phi) \\
& \quad + e^{-\bar{\zeta}_p \bar{\omega}_p t} \left[ (C_5^s \bar{\omega}_p - C_6^s \bar{\omega}_p) \cos(\bar{\omega}_p t) - (C_5^s \bar{\omega}_p + C_6^s \bar{\omega}_p) \sin(\bar{\omega}_p t) \right],
\end{align*}
\]

where the associated coefficients \( C_1^s - C_6^s \) are reported in Appendix A.1.2.

Upon determination of an analytical expression for \( \ddot{u}_p(t) \) as detailed in Section 3.1.1, differentiation with respect to time and substitution of the resulting expression in the motion initiation condition \( \ddot{u}_p(t) = a_s \) gives the following:

\[
\begin{align*}
-C_3^s \sin(\omega_g t) - C_4^s \cos(\omega_g t) - e^{-\bar{\zeta}_p \bar{\omega}_p t} \left[ C_5^s \sin(\bar{\omega}_p t) + C_6^s \cos(\bar{\omega}_p t) \right] = a_s
\end{align*}
\]

(in which it is assumed \( \phi = 0 \) and \( u_{p,0} = \dot{u}_{p,0} = 0 \)). Determination of the initiation time instant for sliding motion requires a solution to the above transcendental equation, which can be done using a numerical scheme such as the bisection method.38

3.1.3  Rocking block

Considering now the rocking S block, if the latter is slender enough, the angle \( \theta_s(t) \) can be assumed to be sufficiently small in the range of practical interest (i.e., \( |\theta_s(t)| \leq \alpha \)) for equation (10) to be linearised by assuming \( \sin(\pm \alpha - \theta_s(t)) \approx \pm \alpha - \theta_s(t) \) and \( \cos(\pm \alpha - \theta_s(t)) \approx 1 \). The equation of the linearised system then reads as follows:
\[ \dot{\theta}_s(t) = -p^2 \left\{ \alpha \text{sgn}(\theta_s(t)) - \theta_s(t) + \frac{\ddot{u}_p(t)}{g} \right\}; \quad \theta_s(0) = \dot{\theta}_s(0) = 0, \]  

which is used in the following in place of equation (10). Solution to equation (17) gives the rotation response of the rocking S block as follows:

\[ \theta_s(t) = C_1^s \sinh(p t) + C_2^s \cosh(p t) + C_3^s \sin(\omega_g t + \phi) + C_4^s \cos(\omega_g t + \phi) \]

\[ + e^{-\zeta_p t} [C_5^s \sin(\omega_p t) + C_6^s \cos(\omega_p t)] + \alpha \text{sgn}(\theta_s(t)); \]

\[ \dot{\theta}_s(t) = \frac{d}{dt} \theta_s(t) = C_1^s \ p \ \cosh(p t) + C_2^s \ p \ \sinh(p t) + C_3^s \ \omega_g \cos(\omega_g t + \phi) - C_4^s \ \omega_g \sin(\omega_g t + \phi) \]

\[ + e^{-\zeta_p t} \left[ (C_5^s \ \omega_p - C_6^s \ \omega_p) \cos(\omega_p t) - (C_5^s \ \omega_p + C_6^s \ \omega_p) \sin(\omega_p t) \right] , \]

where the coefficients \( C_1^s - C_6^s \) are provided in Appendix A.1.3.

Notably, setting \( C_5 = -g \) and \( C_4^s = C_6^s = 0 \) in the above expressions returns the seismic response of a free-standing block rocking on a rigid base.\(^{17}\) Similar to the sliding S, a numerical scheme is utilised in identifying the initiation time instant and the time of impact.

### 3.2 Numerical solutions for general-type excitation

In the preceding subsection, new semi-analytical solutions were derived for the S subsystems under consideration, in presence of an idealised full cycle pulse-type base excitation. Owing to the piecewise linear form of the systems considered, a highly efficient numerical procedure for a general-type excitation is presented in what follows.

Let us consider a piecewise linear input, for example, a ground shaking known through a tabulated record of ground acceleration, with a certain sampling frequency \( \Delta \tau^{-1} \). Denoting \( t_i = i \Delta t \), with \( i = 0, 1, \ldots, N \), the discretised solution of each component analysed in cascade can be developed by interpolating the excitation over each time interval. The solution, valid within each regime of motion, takes the general form as follows:

\[ y(t_{i+1}) = \Theta(\Delta t) \cdot y(t_i) + \Gamma_0(\Delta t) \cdot \eta(t_i) + \Gamma_1(\Delta t) \cdot \eta(t_{i+1}) , \]

where \( y(t) \) denotes the vector of state variables of each system (i.e., \( y(t) = \{ u(t), \dot{u}(t) \}^T \) for the bilinear and sliding S oscillators, and \( y(t) = \{ \theta(t), \dot{\theta}(t) \}^T \) for the rocking one); \( \Theta(\Delta t) \) is the \( 2 \times 2 \) transition matrix; \( \Gamma_0(\Delta t) \) and \( \Gamma_1(\Delta t) \) are vectors that depend on \( \Delta t \), which is assumed sufficiently small, so that the interpolation of the force is satisfactory; and \( \eta(t_i) \) is a function depending on the \( i \)th time instant of the input excitation. The associated expressions are reported in Appendix B for the bilinear, sliding and rocking S subsystems, while implementation details are provided in Appendix C.

Notably, the simple recurrence formula in equation (19) is derived from exact evaluation of the equation of motion of each system, and therefore the only restriction on the size of the time step within each piecewise linear branch is that the dynamic excitation is adequately approximated and closely spaced intervals are used such that response peaks are captured.

The validity of the solutions presented in Sections 3.1 and 3.2 has been confirmed through comparisons with standard numerical integration tools. Although it falls beyond the scope of this work to provide a detailed performance comparison, implementation of the derived solutions in MATLAB\(^{39}\) and comparison with its built-in ordinary differential equations (ODEs) resulted in substantial performance improvements for the former. Implementation in C++ was also found straightforward, leading to further performance improvements. Such implementations are not only suitable for use on high-performance computing (HPC) clusters, but pave the way for the deployment of container-based structural analysis software applications, in-line with a modern cloud-computing paradigm.
4 | ANALYSIS OF PIECEWISE LINEAR SECONDARY OSCILLATORS

The vibration of pulse-driven nonlinear S oscillators has been addressed in the preceding sections, and analytical as well as numerical solutions were presented for this purpose. In this section, the dynamic interaction effects are investigated, and the influence and relative dependencies of the input parameters are quantified on the response through floor response spectra.

4.1 | Dimensional analysis

To analyse the subsystems under consideration, the response quantities of interest are recast in dimensionless form, based on the principle of dimensional homogeneity, such that the number of terms required to fully characterise the response is smaller than the number \( v_{11} \) of engineering design variables defining the problem. In doing so, the response for each combined P-S system, in line with Section 2, depends on a total of \( v_{11} - 2 \) variables as follows:

\[
\{ u_p(t), u_s(t) \} = f(\beta, \gamma, \phi_{ps}, \omega_g, \omega_p, \phi, \omega_p, \zeta_p, \omega_s, \zeta_s, \psi, \alpha, \varepsilon, g);
\]

\[ (20a) \]

\[
\{ u_p(t), u_s(t) \} = f(\beta, \gamma, \phi_{ps}, \omega_g, \omega_p, \phi, \omega_p, \zeta_p, a_s);
\]

\[ (20b) \]

\[
\{ u_p(t), \theta_s(t) \} = f(\beta, \gamma, \phi_{ps}, \omega_g, \omega_p, \phi, \omega_p, \zeta_p, \alpha, p, \varepsilon, g);
\]

\[ (20c) \]

where equations (20a) to (20c) correspond to the bilinear \( (v_{11} = 14) \), sliding \( (v_{11} = 11) \) and rocking \( (v_{11} = 14) \) S subsystems, respectively; and the variables \( a_g = [L][T]^{-2}, \omega_g = [T]^{-1}, \omega_p = [T]^{-1}, \omega_s = [L][T]^{-2}, p = [T]^{-1} \) and \( g = [L][T]^{-2} \) involve two reference dimensions \( (r_{11} = 2) \), namely length \( [L] \) and time \( [T] \), while \( \beta, \gamma, \phi_{ps}, \phi, \zeta_p, \zeta_s, \psi, \alpha \) and \( \varepsilon \) are all dimensionless.

Adopting Buckingham's Π-theorem,\(^{40}\) there exist \( n_{11} = v_{11} - r_{11} \) independent dimensionless products \( (\Pi_1, \Pi_2, \ldots, \Pi_{n_{11}}) \), giving rise to a reduced set of \( n_{11} - 2 \) variables fully characterising the response of each system. Herein, \( a_g \) and \( \omega_p \) are chosen as repeating variables, leading to the dimensionless response as follows:

\[
\{ u_p^* (t^*), u_s^* (t^*) \} = f^*(\beta, \gamma, \phi_{ps}, \omega_g^*, \phi, \zeta_p, \omega_s^*, \zeta_s, \psi, a_s^*);
\]

\[ (21a) \]

\[
\{ u_p^* (t^*), u_s^* (t^*) \} = f^*(\beta, \gamma, \phi_{ps}, \omega_g^*, \phi, \zeta_p, a_s^*) ;
\]

\[ (21b) \]

\[
\{ u_p^* (t^*), \theta_s^* (t^*) \} = f^*(\beta, \gamma, \phi_{ps}, \omega_g^*, \phi, \zeta_p, \alpha, p^*, \varepsilon, a_s^*);
\]

\[ (21c) \]

where \( u^* = u \omega_p^2 / a_g \), in which the normalising factor \( a_g / \omega_p^2 \) represents a measure of the ‘persistence’ of the excitation;\(^7\) \( \theta_s^* = \theta_s / \alpha \) is the rotation response for the rocking S block, normalised such that \( \theta_s^* = 1 \) identifies the condition where overturning is incipient; \( t^* = \omega_p t \) is the dimensionless time. Furthermore, \( \omega_g^* = \omega_g / \omega_p \) and \( \omega_s^* = \omega_s / \omega_p \) are the frequencies of the pulse and the bilinear S oscillator, respectively, relative to \( \omega_p^* \) (i.e., the frequency of the P structure); \( \alpha_s^* = a_s / a_g \) is the dimensionless strength (for the bilinear S subsystem, yielding occurs when \( |u_s^*| |\omega_s^*| > |\alpha_s^*| \)); \( p^* = p / \omega_p \) is the dimensionless dynamic parameter of the rocking block and \( \alpha_g^* = a_g / g \) the dimensionless pulse amplitude. Notably, the ductility ratio \( \mu_d = (a_g^* |u_s^*_{max}|) / \alpha_g^* \) can be used in place of \( \alpha_s^* \), for the bilinear S oscillator, if the seismic response is to be characterised in terms of ductility spectra.

4.2 | Criteria pertaining decoupled analyses

The case of a two-degree-of-freedom (2DoF) P-S system is considered, comprising of a linear SDoF P modal oscillator and an S oscillator of piecewise linear form. The P system considered herein does not represent a MDoF structure, but
merely a SDoF oscillator, which arises after setting \( n = 1 \) and \( \phi_{ps} = \beta = 1 \) in equation (4). This simple representation serves the purpose of exploring the conditions whereby the cascade approximation is admissible for evaluating the response of the S attachment.

A full-cycle sine pulse is considered (i.e., \( \phi = 0 \)) with a dimensionless frequency \( \omega_s^* = 0.14 \), based on a circular frequency \( \omega_p = 14.76 \text{ rad/s} \), chosen such that the pulse temporal duration is \( t_f^* = 3 \omega_p \). Each P-S assembly is simulated using the expressions developed in Section 2 over selected design parameter combinations, and the range \( 0 \leq \gamma \leq 0.5 \) is considered for dimensionless modal factor for the S feedback on the P structure. This permits investigating the admissibility of the cascade approximation via comparisons of the full-dynamic interaction (i.e., \( \gamma \neq 0 \)) with the cascade approximation (i.e., \( \gamma = 0 \)). It has been verified that the responses obtained with the expressions in Section 2 for \( \gamma = 0 \) coincide with the solutions developed in Section 3. To facilitate assessment, the error of the cascade approximation has been quantified over the response time history duration \( t_f^* \) through the following cumulative expression:

\[
\epsilon_\gamma = \frac{\int_0^{t_f^*} |x(t^*) - x_{\text{casc}}(t^*)| \, dt^*}{\int_0^{t_f^*} |x(t^*)| \, dt^*} \times 100,
\]

where \( x_{\text{casc}} \) represents the cascade response of the quantity of interest with respect to its reference value \( x \).

Figure 2 shows the influence of parameter \( \gamma \) on the response histories of each S component, for \( t_f^* = 60 \), where the red solid line represents the cascade approximation (\( \gamma = 0 \)), the blue solid line marks the upper bound \( \gamma = 0.5 \) and the grey area denotes the envelope of responses, that is, \( 0 \leq \gamma \leq 0.5 \), while Table 1 summarises the peak response and cumulative errors.

The bilinear S oscillator is first examined in the left column of Figure 2, and the reference values \( e_p = 0.05 \) and \( e_s = 0.02 \) are assumed. Figure 2A shows the response in the linear limit, when there is no plastic deformation (\( a_s^* \rightarrow \infty \), \( \psi = 1 \)), at resonance (\( \omega_p^* = 1 \)). As expected, the oscillations indicate strong dynamic interaction between the two systems, and the cumulative error \( \epsilon \) increases with \( \gamma \). Specifically, the values \( \gamma = 0.001, \gamma = 0.01, \gamma = 0.1 \) and \( \gamma = 0.5 \) correspond to cumulative errors of \( e_{0.001} = 4\%, e_{0.01} = 29\%, e_{0.1} = 62\% \) and \( e_{0.5} = 63\% \), respectively, suggesting that a value of \( \gamma \leq 0.001 \) is required for the cascade approximation to be satisfactory. For the elastoplastic system in Figure 2D with \( \omega_p^* = 0.5, a_s^* = 0.7 \) and \( \psi = 0 \), the peak error of \( e_{0.5} = 3\% \) highlights that the cascade approximation can be satisfactory for 'heavy and relatively flexible' S subsystems. Setting the bilinear system in resonance, \( \omega_p^* = 1 \) in Figure 2G, increases the dynamic interaction and the resulting errors; however, these tend to be an order of magnitude less than for the linear S oscillator in Figure 2A (see Table 1), as expected due to the P-S detuning occurring when the S oscillator exceeds the elastic limit. Interestingly, setting \( \omega_p^* = 1.5 \) (Figure 2I) tends to increase the errors for the elastic-perfectly plastic S oscillator with respect to \( \omega_p^* = 1 \), indicating that the cascade approximation is less effective for 'heavy and relatively stiff' S subsystems.

The sliding S subsystem is next considered (see central column of Figure 2), which corresponds to the case where the bilinear S oscillator approaches the rigid-plastic limit, that is, \( a_s^* \rightarrow \infty \). Figure 2B shows the effect of modal feedback parameter \( \gamma \) on the seismic response of the S block when its relative frictional strength is as small as \( a_s^* = 0.3 \). Interestingly, all curves closely resemble each other and the approximation results in errors lower than \( e_{0.5} = 2.4\% \). The reason for this behaviour is that the relatively low frictional force at the P-S interface results in a form of seismic isolation of the block itself: the feedback force on the P structure is indeed limited, and the S block experiences relatively large displacements (i.e., in the order of \( u_{s,\text{max}}^* = \max |u_s^*(t^*)| \approx 150 \)). This is confirmed by inspection of Figure 2E,H,K, showing that the higher the relative frictional strength \( a_s^* \), the less the relative maximum displacement \( u_{s,\text{max}}^* \) exhibited by the sliding block, the larger the effects of the P-S dynamic interaction and, therefore, the larger the inaccuracy when the cascade approximation is adopted. In particular, for \( a_s^* = 0.9 \) (see Figure 2K), the red line (cascade approach) and blue line (dynamic interaction with \( \gamma = 0.5 \)) show significant differences (with a prominent grey area associated with various intermediate values of \( \gamma \)), although in this case, the relative maximum displacement is comparatively small (\( u_{s,\text{max}}^* < 4 \)). It can be concluded then that cascade decoupling is particularly effective in the case of sliding S blocks with a relatively low value of \( a_s^* \).

The results of the rocking S block are considered next (see right column of Figure 2). A reference value of \( p^* = 0.136 \) (i.e., \( p = 2 \)) has been assumed throughout analysis, as well as the coefficients \( e_p = 0 \) and \( e_s = 0.85 \). In all cases, the two extreme values considered for the model feedback factor (i.e., \( \gamma = 0 \) and \( \gamma = 0.5 \)) do not correspond to the upper and lower bounds of the response envelope. Figure 2C shows the rotation time history for \( \alpha = 0.20 \) and
$a^*_{g} = 0.219$. As shown, the approximation remains satisfactory provided that $\gamma \leq 0.01$, with a peak error of $\epsilon_{0.01} = 10\%$, while large cumulative errors are otherwise predicted (e.g., $\epsilon_{0.1} = 85\%$). A small increase in the dimensionless pulse amplitude ($a^*_{g} = 0.223$, i.e., +1.8%, in Figure 2F) has a profound effect on the trajectory of the rocking response, which is brought to rest when $\gamma \leq 0.01$ and $t^* > 30$ (in our analyses, it is assumed that the rest occurs when the total energy of the block is sufficiently small, i.e., < $10^{-7}$). For higher values of $\gamma$, the response increases, leading to large errors (e.g., $\epsilon_{0.1} = 474\%$). Figure 2I shows that a further 1.8\% increase in the intensity of the ground shaking, that is, $a^*_{g} = 0.227$, causes the overturning of the block provided that $\gamma \leq 0.01$. In this case, the associated errors are $\epsilon_{0.001} = 0.1\%$, $\epsilon_{0.01} = 0.6\%$ and $\epsilon_{0.1} = 8.6\%$. For larger values of $\gamma$ (e.g., $\gamma = 0.5$), the block does not overturn, resulting in large errors in the cascade approximation, e.g., $\epsilon_{0.5} = 70\%$). Figure 2L shows that when $a^*_{g} = 0.280$, no impact and no oscillatory motion are exhibited, and for all the values of $\gamma \leq 0.5$, the response increases monotonically until overturning is reached; in this case, the resulting peak error due to the cascade approximation in negligible, being $\epsilon_{0.5} = 3\%$. Overall, the results reveal that the cascade approximation is acceptable even for relatively large value of the modal parameter $\gamma$ (i.e., relatively heavy S blocks) in two extreme cases, namely (i) when short oscillation cycles are exhibited (e.g., at low values of $a^*_{g}$) or (ii) when the response increases monotonically following initiation leading to overturning (e.g., at large vales of $a^*_{g}$). On the contrary, for intermediate cases of $a^*_{g}$, when the block is close to overturning, the value of the modal parameter $\gamma$ can have a significant influence on the S response, leading to the accumulation of errors in the cascade approximation; in these cases, only light S will be satisfactorily analysed with the cascade approximation (say, $\gamma < 0.01$).
the floor level. The fundamental period of vibration in the horizontal direction of interest is their own plane, while the self-weight and super-dead load are the two sources of mass for the structure and are lumped generic floor, modelled as a light (i) bilinear, (ii) sliding and (iii) rocking SDoF oscillator. Floors are assumed rigid in their own plane, while the self-weight and super-dead load are the two sources of mass for the structure and are lumped at the floor level. The fundamental period of vibration in the horizontal direction of interest is \( T_{p,1} = 0.426 \text{ s} \) corresponding to a participation of 84% of the modal mass and modal factor \( \beta = 1.3 \).

A series of floor response spectra have been evaluated using the cascade approximation and are presented in the form of region plots, to understand the influence of the various input parameters on the seismic response of the S components. Each contour plot comprises of a grid of 100×100 points (i.e., 10,000 simulations). The computational effort has been mitigated by the application of the analytical solutions presented in Section 3.1. The dimensionless pulse frequency range \( \pi / (3 \omega_{p,1}) \leq \omega_{g} \leq 1 \) has been considered throughout, so that the temporal duration of the ground acceleration pulse, \( T_g = 2\pi/\omega_g \), satisfies conditions \( T_g \geq T_{p,1} \) and \( T_g \leq 6 \text{ s} \).

For illustration purposes, the top row of Figure 3 shows the seismic response of the P structure in terms of absolute acceleration for constant viscous damping ratio \( \zeta_p = 0.05 \) and selected values of \( \omega_{g} \) (plotted with different colours) and \( \phi_{ps} \) (corresponding to different columns), while the resulting responses of case-study bilinear (\( \omega_{g} = 1.2, \zeta_p = 0.02, \mu = 3, \psi = 0 \)), sliding (\( \alpha = 0.8 \)) and rocking (\( \alpha = 0.25, \beta = 0.136, \epsilon = 0.7, \alpha_{s} = 0.43 \)) S oscillators are plotted in the second, third and bottom rows, respectively. Furthermore, (i) the horizontal dashed and dotted grey lines, drawn, respectively, at \( \dot{u}_{g}^{(a)}(t) = \alpha_{s} = 0.8 \) and \( \ddot{u}_{g}^{(a)}(t) = \tan(\alpha) / \alpha_{s}^{2} = 0.59 \) in Figure 3A-C, denote the reference levels of initiation for the sliding and rocking motion for the chosen S blocks; (ii) the absolute acceleration \( \ddot{u}_{g}^{(a)}(t) \) in these figures is effectively the sum of two components, namely the single-cycle ground pulse, \( \ddot{u}_{g}^{(a)}(t) = \sin(\omega_{g}^{*} t) \), and the acceleration \( \ddot{u}_{g}^{(a)}(t) \) of the P structure relative to the ground. The latter contribution increases with both \( \phi_{ps} \) and \( \omega_{g}^{*} \) and becomes negligible for each of the two limiting conditions \( \phi_{ps} \to 0 \) and \( \omega_{g}^{*} \to 0 \). While the interpretation of the first condition is straightforward (i.e., \( \phi_{ps} = 0 \) means that the S subsystem is mounted directly at the ground level, see Figure 3A), the second
condition is a consequence of the quasi-static nature of the P response when $\omega_g / C^2 p$ (see blue lines in Figure 3B,C, plotted for $\omega^*_g = 0$).

Apart from two limiting cases mentioned above, the dynamics of the P structure significantly affect the seismic action on the S subsystem. In this respect, the selection of viscous damping ratio $\zeta_p$ is potentially critical, as it dictates the rate of exponential decay in the amplitude of the oscillations experienced by the P structure: that is, the smaller $\zeta_p$, the larger the number of ‘high-amplitude’ oscillations in $u_p^*(t^*)$. This can be clearly seen in relation to the dotted and dashed lines in Figure 3A-C, where any acceleration peak and trough exceeding those thresholds can result in a further initiation of motion for the sliding and rocking S blocks. This is confirmed by examining the seismic response of sliding S block: indeed, while only two distinct sliding phases can be observed for $\phi_{ps} = 0$ (Figure 3G), independently of the choice of $\omega^*_g$, more S oscillation cycles occur for $\phi_{ps} > 0$ and $\omega^*_g \geq 0.7$ (see Figure 3H,I).

Another interesting observation is that, while the seismic response of the bilinear S oscillator tends to increase ordinately with $\phi_{ps}$ due to the amplified motion of the P structure at a higher level (see Figure 3D,F), the same does not always happen for the rocking subsystem. For instance, for $\omega^*_g = 0.4$ (orange lines), the same S block appears to overturn when located at the base of the P frame (see Figure 3J, with $\phi_{ps} = 0$) and at an intermediate height (see Figure 3K, with $\phi_{ps} = 0.5$), while it is shown to survive the seismic action at the top floor (see Figure 3L, with $\phi_{ps} = 1$).

To summarise, just looking at the selected case-study time histories, it can be concluded that the combination of nonlinearity in the S subsystems and transient vibration in the pulse-driven P structure can give rise to very complex scenarios.

Aimed at identifying key trends in the seismic response of the three different types of nonlinear S oscillators, the contour plots of Figures 4 to 6 show the maximum seismic responses for selected design parameters. Specifically, Figure 4 illustrates the response of the bilinear S oscillator in terms of constant-ductility inelastic spectra, computed using an interpolative procedure with 2% error tolerance. The set of 15 plots (from Figure 4A-O) quantifies the effects of the dimensionless frequencies $\omega^*_b$ (on the horizontal axis) and $\omega^*_g$ (on the vertical axis) on the

**FIGURE 3** Effect of $\omega^*_g$ and $\phi_{ps}$ on the following: the absolute acceleration response of P (A, B, C) for $\zeta_p = 0.05$; the displacement of the bilinear S (D, E, F) for $\alpha^*_s = 1.2$, $\zeta_s = 0.02$, $\mu = 3$ and $\psi = 0$; the displacement of the sliding S (G, H, I) for $a^*_s = 0.8$; and the rotation of the rocking S (J, K, L) for $\alpha = 0.25$, $p^* = 0.136$, $\epsilon = 0.7$ and $a^*_g = 0.43$.
dimensionless strength $a_*^s$, while each column is associated with a different value of the ductility ratio $\mu$ and each row represents a different combination of P-S parameters. As shown, all plots are ordered, with some well-defined regions clearly emerging. For instance, for the linear case ($\mu = 1$, left column), peak spectral regions (shown in yellow) are
centred at \( \omega_s^* = 1 \), where the S subsystem is in resonance with the P frame, and at relatively lower pulse frequency \( (\omega_g^* = 0.9) \). As \( \mu \) increases, these regions enlarge and are shifted towards higher frequencies for the S oscillator (i.e., \( \omega_s^* > 1 \)). Furthermore, increasing the values of the parameters \( \zeta_s, \zeta_p \) and \( \psi_{ps} \) appears to increase the required strength of the S subsystem, while the opposite happens when the parameter \( \psi \) increases. The effects of all these parameters are similar, in the sense that they influence the magnitude of the response, while the overall shape of the contour plot tends to remain unaffected.

Figure 5 plots the effect of parameters \( \omega_g^* \) and \( a_s^* \) on the peak displacement response \( u_{s,max}^* = \max|u_s^*(t^*)| \) of the sliding S block. The normalised strength considered in the analyses, \( 0.1 \leq a_s^* \leq 3.5 \), is chosen to cover a broad range of response cases. Overall, the response is a smooth function of the governing parameters, comprising of distinct regions. As signified by the white area (i.e., \( u_{s,max}^* = 0 \)), sliding motion is initiated provided \( a_s^* \) is sufficiently low. Interestingly, as the normalised pulse frequency \( \omega_g^* \) increases, the minimum strength required for the system to exhibit sliding motion increases for low spectral ordinates (i.e., \( 0 < u_{s,max}^* \leq 20 \)) and reduces for higher ordinates (i.e., \( u_{s,max}^* > 20 \)). In other words, higher spectral ordinates live in a narrow low-frequency low-strength band and are not encountered beyond a certain value of \( \omega_g^* \). Also, fluctuations of lower values of \( a_s^* \) and \( \omega_g^* \) can result in significant changes in the peak spectra; this does not happen for relatively large values of these quantities. Finally, a reduction from \( \psi_{ps} = 1 \) (Figure 5A) to \( \psi_{ps} = 0.5 \) (Figure 5B) at \( \zeta_p = 0.05 \), and similarly, a reduction from \( \zeta_p = 0.05 \) (Figure 5A) to \( \zeta_p = 0.02 \) (Figure 5C) at \( \psi_{ps} = 1 \) will only have a dominant effect on low-response ordinates (i.e., \( u_{s,max}^* \leq 20 \)).

**FIGURE 6** Rotation response spectra for rocking S: effect of parameters \( \omega_g^* \) and \( a_s^* \) on the peak rotation
Figure 6 shows the effect of parameters $\omega_g^*$ and $a_g^*$ on the peak rotation response $\theta_s^{\max} = \max|\theta_s(t^*)|$ for the rocking S block. A set of 15 contour plots has been produced (from Figure 6A-O), in which the white regions denote no rocking motion (i.e., $\theta_s^{\max} = 0$), the yellow regions mark overturning (i.e., $\theta_s^{\max} \geq 1$) and the remaining colours indicate intermediate conditions where the S block experiences rocking without overturning. Each column of Figure 6 is associated with a different value of slenderness angle $\alpha$, and each row represents a different combination of system parameters. A general observation is that the various plots tend to follow similar trends, where larger rocking S motion tends to occur when the pulse amplitude $a_g^*$ increases and the pulse frequency $\omega_g^*$ decreases. Furthermore, unsurprisingly, increasing the size $R$ (i.e., reducing $p^* \propto R^{-1}$) and the slenderness angle $\alpha$ improves the seismic stability of S block; the same can generally be said for the viscous damping ratio of the P frame, while increasing the coefficient of energy restitution $\varepsilon$ tends to have the opposite effects. Despite these general trends, the complex combination of pulse-driven P vibration and rocking S motion means that, for instance, the contour plots of Figure 6D,G,M reveal the unexpected presence of ‘overturning islands’ (yellow) roughly centered at $\omega_g^* = 0.65$ and $a_g^* = 0.47$ in areas of otherwise ‘moderate rocking’ (green), where $\theta_s^{\max} \approx 0.5$ (corresponding to $\theta_s^{\max} \approx 4^\circ$). Another counterintuitive outcome is that, comparing Figure 6E,K for $\omega_g^* \geq 0.4$ and $a_g^* \geq 0.3$, the overturning region (yellow) is bigger for $\phi_{ps} = 0.5$ (mid P floor) than for $\phi_{ps} = 1$ (top P floor), confirming the findings of Figure 3K,L. Indeed, depending on the design input parameters and the excitation characteristics, the overturning tendency of a rocking S block may reduce or increase with the location in the P structure, expressed herein through the dimensionless modal amplitude $\phi_{ps}$.

### 5 CONCLUSIONS

The seismic response analysis of bilinear, sliding and free-standing rocking pulse frequency range secondary (S) oscillators has been addressed. The equations governing their dynamic interaction with linear pulse frequency range primary (P) structures were derived, followed by a simplified P-S assembly into a 2DoF system with dimensionless coefficients. The process permits to investigate systematically the coupled P-S governing equations, thus facilitating an a priori quantification and assessment of the approximation associated with the cascade analysis, largely used in the engineering practice.

Analyses of coupled pulse-driven 2DoF P-S systems revealed pronounced sensitivity to combinations of the design input parameters. Although decoupling criteria are commonly used in the cascade analysis of linear systems (e.g., limits in the mass and frequency S-to-P ratios), they may significantly underestimate or overestimate the effects on the seismic response of nonlinear subsystems. In particular, it was found that the effectiveness of the cascade approximation tends to increase when fewer oscillations are exhibited in the response of the S subsystem: that is, for lower strength values $a_g^*$ in the bilinear and sliding S oscillators, and for the two extreme cases of the rocking S block, that is, for short oscillation cycles or for overturning without impacts.

Closed-form piecewise linear semi-analytical solutions were derived for the cascade analysis of the nonlinear S oscillators considered under pulse-type excitations. They permit direct evaluation of the response as a function of the parameters of the S-P system assembly and the ground excitation. Step-by-step piecewise linear numerical solutions were further derived by interpolating the excitation over each time interval and solving the equation of motion in closed form. This is particularly important in the context or performance-based (or resilience-based) earthquake engineering, enabling the accurate and efficient quantification of the effects that different design choices on the P structures can have on the seismic response of a given array of S components, and how sensitive they are to the uncertain design parameters, including the modal properties of the P structure, the location of the S components and their mechanical properties such as yielding and frictional strength and coefficient of energy restitution.

For demonstration purposes, the derived solutions were exploited for the cascade analysis of nonlinear S oscillators mounted on a pulse-driven linear P structure, presented in the form of time-history plots and contour plots. Due to complexities of the nonlinear behaviour of the S components combining with the transient motion of the P structure, some peculiar behaviours have been identified and discussed, for example, the role of the number of ‘high-amplitude’ cycles experienced by the P structure and the counterintuitive phenomenon for which, under certain circumstances, the same S rocking block could survive when placed at the top floor of a building but overturn when placed at a lower level.

The analyses presented as part of this study therefore demonstrate the importance of rigorously: (i) assessing whether a cascade approach is acceptable from an engineering standpoint; (ii) quantifying the sensitivity of nonlinear S dynamic responses to changes in the key design parameters.
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APPENDIX A: Coefficients for the semi-analytical solutions

Bilinear oscillator

For the case of the bilinear S oscillator (Section 3.1.1), the coefficients $c^b_1$ to $c^b_8$ are defined as follows:

\[ c^b_1 = \begin{cases} 0, & \text{if } \psi = \zeta_s = 0 \\ \frac{1}{\omega^s} \left( (C^b_4 \omega_g - C^b_3 \zeta_s \omega_h) \sin(\phi) - (C^b_5 \omega_g + C^b_6 \zeta_s \omega_h) \cos(\phi) \right), & \text{otherwise} \end{cases} \]  

\[ c^b_2 = \ u_{s,0} - C^b_0 \sin(\phi) - C^b_4 \cos(\phi) - C^b_6 - C^b_8; \]  

\[ c^b_3 = \ \frac{C_3 \left( \psi \omega_s^2 - \omega_g^2 \right) + 2 \zeta_s \omega_g \omega_h C_4}{2(2 \zeta_s^2 - \psi) \omega_s^2 \omega_h^2 + \omega_s^2 + \psi^2 \omega_h^2}; \]  

\[ c^b_4 = \ \frac{C_4 \left( \psi \omega_s^2 - \omega_g^2 \right) - 2 \zeta_s \omega_g \omega_h C_3}{2(2 \zeta_s^2 - \psi) \omega_s^2 \omega_h^2 + \omega_s^2 + \psi^2 \omega_h^2}; \]  

\[ C^b_5 = \ \frac{C_5 \left( C - \alpha_p^2 + \psi \omega_s^2 \right) + 2 \alpha_p \beta \zeta_s \omega_h - \zeta_s \omega_h \psi - \omega_h}{2 \alpha_p^2 \left[ C + (2 \zeta_s^2 - \psi) \omega_s^2 \right] + (C + \psi \omega_s^2)^2 + \omega_s^2}; \]  

\[ C^b_6 = \ \frac{2 \alpha_p \beta \zeta_s \omega_h - \zeta_s \omega_h \psi - \omega_h}{2 \alpha_p^2 \left[ C + (2 \zeta_s^2 - \psi) \omega_s^2 \right] + (C + \psi \omega_s^2)^2 + \omega_s^2}; \]  

\[ C^b_7(t) = \begin{cases} \frac{t \left( \alpha_p \left( \zeta_s \omega_h - \alpha_p \omega_h \right) \right) + 2 \alpha_p \beta \zeta_s \omega_h - \zeta_s \omega_h \psi - \omega_h}{2 \alpha_p^2 \left[ C + (2 \zeta_s^2 - \psi) \omega_s^2 \right] + (C + \psi \omega_s^2)^2 + \omega_s^2}, & \psi = \zeta_s = 0 \\ \frac{a_s \omega_s \left( 1 - e^{-2 \zeta_s \omega_h t} - 2 \zeta_s \omega_h t \right)}{4 \zeta_s^2 \omega_s^2}, & \psi = 0, \zeta_s \neq 0; \end{cases} \]  

\[ c^b_8 = \ \begin{cases} 0, & \text{if } \psi = 0, \\ \frac{z_s \left( \psi - 1 \right) a_s}{\psi \omega_s^2}, & \text{otherwise} \end{cases} \]  

where $C = \zeta_s \omega_h^2 - 2 \zeta_p \zeta_s \omega_h \omega_h$ and $A = \zeta_s^2 \omega_h^2 + \omega_s^2$, $c^b_1$, $c^b_2$ depend on the initial conditions $u_{s,0}$, $\dot{u}_{s,0}$. Furthermore, $z_s = \pm 1$ is a discrete variable that replaces the continuous state variable $|z_s(t)| \leq 1$ in the piecewise linear case, after its time dependency is dropped, and whose value depends on whether the oscillator is at the positive or negative extreme of its elastic domain.

The above coefficients also depend on the following:

\[ C_1 = \ \frac{1}{\alpha_p^2} \left( (C_8 \omega_h - C_7 \zeta_p \omega_p) \sin(\phi) - (C_7 \omega_h + C_8 \zeta_p \omega_p) \cos(\phi) + \zeta_p \omega_p \alpha_p u_{p,0} + \dot{u}_{p,0} \right); \]  

\[ C_2 = \ u_{p,0} - C_7 \sin(\phi) - C_8 \cos(\phi); \]  

\[ C_3 = \ C_7 \omega_h^2 - \alpha_h; \]  

\[ C_4 = \ C_8 \omega_h^2; \]  

\[ C_5 = \ C_1 \left( \alpha_p - \zeta_p \omega_p \right) \left( \zeta_p \omega_p + \omega_p \right) - 2 \zeta_p \alpha_p \omega_p C_2; \]  

\[ C_6 = \ 2 C_1 \zeta_p \omega_p \alpha_p + C_2 \left( \alpha_p + \zeta_p \omega_p \right) \left( \alpha_p + \zeta_p \omega_p \right); \]  

\[ C_7 = \ \frac{a_g \beta \left( \alpha_p - \omega_h \right) \left( \alpha_p + \omega_h \right)}{2 \left( \zeta_p^2 - 1 \right) \omega_h^2 \alpha_p^2 + \omega_h^4 + \alpha_p^4}; \]  

\[ C_8 = \ \frac{2 a_g \beta \zeta_p \omega_h \omega_p}{2 \left( \zeta_p^2 - 1 \right) \omega_h^2 \alpha_p^2 + \omega_h^4 + \alpha_p^4}; \]  

which are functions of the parameters of the P structure and the ground excitation, and $C_1$ and $C_2$ depend on $u_{p,0}$ and $\dot{u}_{p,0}$. 
Additionally, The coefficients \( C_1^s - C_6^s \) for the expressions presented in Section 3.1.2 take the form as follows:

\[
C_1^s = u_{s,0} - C_3^s \sin(\phi) - C_4^s \cos(\phi) - C_6^s; \\
C_2^s = \dot{u}_{s,0} - C_3^s \omega_p \cos(\phi) + C_4^s \omega_p \sin(\phi) - C_5^s \omega_p + C_6^s \omega_p; \\
C_3^s = -\frac{C_3}{\omega_p^2}; \\
C_4^s = -\frac{C_4}{\omega_p^2}; \\
C_5^s = \frac{C_5}{\left(\omega_p^2 + \zeta_p^2 \omega_p^2\right)^2} - 2 \zeta_p \omega_p \omega_p C_6; \\
C_6^s = \frac{C_6}{\left(\omega_p^2 + \zeta_p^2 \omega_p^2\right)^2} + 2 \zeta_p \omega_p \omega_p C_5,
\]

(A3a)

(A3b)

(A3c)

in which \( C_1^s \) and \( C_2^s \) depend on the initial conditions \( u_{s,0} \) and \( \dot{u}_{s,0} \), respectively, and \( C_3 \) to \( C_6 \) retain the same form as in Section A.1.1.

**Sliding block**

The coefficients \( C_1^r - C_6^r \) for the rocking S block (§ 3.1.3) are as follows:

\[
C_1^r = \frac{1}{p} \left[ \dot{\theta}_{s,0} - C_3^r \omega_p \cos(\phi) + C_4^r \omega_p \sin(\phi) - C_5^r \omega_p + C_6^r \omega_p \right]; \\
C_2^r = \theta_{s,0} - C_3^r \sin(\phi) - C_4^r \cos(\phi) - C_6^r - \alpha_s \text{ sgn}[\theta_s(t)]; \\
C_3^r = -\frac{p^2 C_3}{g \left( p^2 + \omega_p^2 \right)}; \\
C_4^r = -\frac{p^2 C_4}{g \left( p^2 + \omega_p^2 \right)}; \\
C_5^r = -\frac{p^2 \left[ p^2 C_5 + C_5 \left( \omega_p - \zeta_p \omega_p \right) \left( \omega_p + \zeta_p \omega_p \right) + 2 \zeta_p \omega_p \omega_p C_6 \right]}{g \left[ p^4 + 2 p^2 \left( \omega_p - \zeta_p \omega_p \right) \left( \omega_p + \zeta_p \omega_p \right) + \left( \omega_p^2 + \zeta_p^2 \omega_p^2 \right)^2 \right]}; \\
C_6^r = \frac{p^2 \left[ 2 \zeta_p \omega_p \omega_p C_5 + C_6 \left( \zeta_p^2 \omega_p^2 - p^2 - \omega_p^2 \right) \right]}{g \left[ p^4 + 2 p^2 \left( \omega_p - \zeta_p \omega_p \right) \left( \omega_p + \zeta_p \omega_p \right) + \left( \omega_p^2 + \zeta_p^2 \omega_p^2 \right)^2 \right]},
\]

where \( C_3 \) to \( C_6 \) are likewise identical to the ones in Section A.1.1.

**APPENDIX B: Coefficients for the numerical solutions**

**Bilinear oscillator**

The expressions for \( \Theta(\Delta t) \), \( \Gamma_0(\Delta t) \), \( \Gamma_1(\Delta t) \) and \( \eta(t_i) \) are reported. For the bilinear S, \( \eta(t) = \tilde{u}_p^{(3)}(t) + a_s z_s (1 - \psi) \).

Additionally,
Evidently, the response of the P structure is readily available by setting \( y(t) = \{ u_p(t), \ddot{u}_p(t) \}^\top \), \( \eta(t) = \phi_{ps} \beta \rho(t) \) and \( \psi = 1 \), as well as substituting the parameter set \( \{ \omega_p, \zeta_p, \alpha_p, \omega_{\text{ref}} \} \) in place of \( \{ \omega_s, \zeta_s, \alpha_s, \omega_{\text{ref}} \} \).

### Sliding block

In the case of the sliding S block, \( \eta(t) = \ddot{u}_p^{(a)}(t) + \mu_s \text{sgn}(\dot{u}_s(t)) \). Furthermore,

\[
\Theta(\Delta t) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} ; \quad \Gamma_0(\Delta t) = \begin{bmatrix} -\Delta t^2/3 \\ -\Delta t/2 \end{bmatrix} ; \quad \Gamma_1(\Delta t) = \begin{bmatrix} -\Delta t^2/6 \\ -\Delta t/2 \end{bmatrix} .
\]  

\[ (B5a) \]

\[
\Theta(\Delta t) = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix} = \begin{bmatrix} e^{-\zeta_s \omega_s \Delta t} (\zeta_s r_1 + r_2) & r_1 e^{-\zeta_s \omega_s \Delta t} \\ \psi r_1 \omega_s e^{-\zeta_s \omega_s \Delta t} & e^{-\zeta_s \omega_s \Delta t} (r_2 - \zeta_s r_1) \end{bmatrix} ;
\]

\[ (B5b) \]

\[
r_1 = \begin{cases} \frac{\omega_s \Delta t}{\omega_s \sin(\omega_{\text{ref}} \Delta t)}, & \text{if } \psi = \zeta_s = 0 \\ \text{otherwise} \end{cases} ; \quad r_2 = \cos(\omega_{\text{ref}} \Delta t) ;
\]

\[ (B5c) \]
Rocking block

For the rocking S block, \( \eta(t) = p^2 \left( \alpha \, \text{sgn}(\theta_s(t)) + \frac{\ddot{u}_s(t)}{g} \right) \). Additionally,

\[
\begin{align*}
\Theta(\Delta t) & = \begin{bmatrix}
\cosh(p \Delta t) & \sinh(p \Delta t) \\
\frac{p}{p \sinh(p \Delta t)} & \cosh(p \Delta t)
\end{bmatrix}, &
\Gamma_0(\Delta t) & = \begin{bmatrix}
\frac{\sinh(p \Delta t) - p \Delta t \cosh(p \Delta t)}{p^3 \Delta t} \\
-\frac{1 - \cosh(p \Delta t) - \cosh(p \Delta t) - 1}{p^2 \Delta t}
\end{bmatrix}, &
\Gamma_1(\Delta t) & = \begin{bmatrix}
\frac{p - \sinh(p \Delta t)}{\Delta t} \\
-\frac{1 - \cosh(p \Delta t)}{p^2 \Delta t}
\end{bmatrix}.
\end{align*}
\]

APPENDIX C: Solution implementation details

The cascade analysis solutions presented in Sections 3.1 and 3.2 are implemented in a computer program and are separately evaluated for each regime of motion. An iterative scheme, such as the bisection method, is commonly utilised for piecewise solutions, is adopted to identify state events (i.e., initiation and change in the regime of motion), facilitating the decomposition of the solution to individual segments that are then synthesised to construct the time history of the response. Specifically, the same piecewise linear solution is assumed valid for subsequent time steps, until a change in the regime is detected. A change in sign of the velocity \( \dot{u}_s \) indicates transitioning from the postyield to the preyield branch, for the bilinear S, and from the slipping to the sticking regime for the sliding S; similarly, a change in the sign of rotation \( \theta_s \) indicates the occurrence of an impact and switching of the pivot point for the rocking S. When such a change in sign occurs, say during time step \( t_i \) to \( t_{i+1} \), the time step is split and the state event time \( t_\varepsilon \), \( t_i < t_\varepsilon < t_{i+1} \) is iteratively determined to a prescribed tolerance level. The solution, is then retained up to \( t_\varepsilon \). Solution proceeds thereafter, assuming as initial conditions the state variable values at \( t_\varepsilon \), and using the piecewise linear equation other than the one valid at \( t_i \), which is evaluated over the time interval \( t_\varepsilon \) to \( t_{i+1} \) and subsequent intervals until a state event is identified again. Notably, the same procedure is adopted for determining the times of initiation for sliding and rocking motion, except that rather than a change in sign, the initiation condition is checked (see Sections 2.2.2 and 2.2.3 and equation (16)). To this end, it is emphasised that time discretisation of the semi-analytical solutions does not render them as numerical, but is merely required for identification of the state events. Specifically for the case of the rocking S block, a step discontinuity is introduced in the rotational velocity at the time of impact through \( \dot{\theta}_s(t_\varepsilon) = \varepsilon \, \dot{\theta}_s(t_\varepsilon) \), which accounts for the energy dissipation via the coefficient of energy restitution \( \varepsilon \), at which time the block’s rotation is zero. Further, as the oscillations of the rocking S tend to reduce in amplitude once the strong motion phase of the seismic event is finished, the system will continuously transition between the two regimes of motion with an increasing frequency. As a result, the solution procedure has to be interrupted several times to identify state events, on occasions where the response of the block can be regarded as negligible. Hence, rather than continuing integrating the equations until the end of the duration of motion (which is theoretically possible), a stopping criterion may be used to bring the system to rest when its total energy is sufficiently low.