Amplitudes in the N=4 SYM from Quantum Geometry of the Momentum Space

A. Gorsky

Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya 25, Moscow, 117259, Russia

Abstract

We discuss multiloop MHV amplitudes in the $N = 4$ SYM theory in terms of effective gravity in the momentum space with IR regulator branes as degrees of freedom. Kinematical invariants of external particles yield the moduli spaces of complex or Kahler structures which are the playgrounds for the Kodaira-Spencer(KS) or Kahler type gravity. We suggest fermionic representation of the loop MHV amplitudes in the $N = 4$ SYM theory assuming the identification of the IR regulator branes with KS fermions in the B model and Lagrangian branes in A model. The two-easy mass box diagram is related to the correlator of fermionic currents on the spectral curve in B model or hyperbolic volume in the A model and it plays the role of a building block in the whole picture. The BDS-like anzatz has the interpretation as the semiclassical limit of a fermionic correlator. It is argued that fermionic representation implies a kind of integrability on the moduli spaces. We conjecture the interpretation of the reggeon degrees of freedom in terms of the open strings stretched between the IR regulator branes.
1 Introduction

The $N = 4$ SYM theory provides a possibility to recognize some features of the theories with less amount of SUSY. While $N = 4$ SYM is far from the QCD-like theories in the infrared because of the lack of confinement it shares common features in UV region where physics in asymptotically free theories is described within a perturbation theory. That is considering the perturbative expansion in $N = 4$ SYM coupling constant which does not run we could try to clarify some generic properties of the perturbative expansion in the gauge theories.

It is of prime importance to discover any hidden symmetries at high energies or equivalently hidden integrable structures providing the nontrivial conservation laws which restrict the form of the scattering amplitudes. In the four-dimensional setup the integrability behind the amplitudes is known only at the Regge limit when the $SL(2, C)$ spin chain gets materialized [1, 2](see [3] for review).

The simplest objects at generic kinematics are the MHV amplitudes which are the perfect starting point for any discussion since at the planar limit they can be described in terms of the single kinematical function. Even at the tree level MHV amplitudes [4] enjoy some remarkable properties. They are localized on the complex curves in the twistor space [5] (see [6] for review) and can be described as the correlators of chiral bosons on the genus zero Riemann surface [7]. It turns out that the generating function for the tree MHV amplitudes is just the particular solution to the self-duality equation in YM theory [8, 9]. It substitutes the naive superposition of the plane waves of the same chirality in a nonlinear theory. Moreover this solution provides the symplectic transformation [10](see also [11]) of the YM theory in the light-cone gauge into the so-called tree MHV Lagrangian formulated in [12] which to some extend is the analogue of the t’Hooft effective vertex generated by instantons. However this approach becomes less clear when going to higher loops. Indeed, the attempt to formulate the one-loop MHV amplitudes in a twistor-like manner was not successful enough [13] and certainly calls for additional insights on the problem.

One more line of development based on a first quantized picture for the loop calculations was initiated in [14]. It was shown that the three-point amplitude written in the Schwinger parametrization implies the identification of the Schwinger proper time parameter with the radial coordinate in the AdS geometry providing some rationale for the appearance of the AdS space. The similar interpretation holds true for the calculation of the one-loop effective actions in the different backgrounds [15]. However starting from one loop four-point amplitude the situation becomes more subtle because of the emerging moduli space. It was suggested in [18] that the Feynman diagram can be presented in terms of the skeleton graph parameterized by the set of Schwinger parameters. This set of parameters can be mapped for a planar limit of n-point amplitude into the manifold $M_{0,n} \times \mathbb{R}^2$ where $M_{0,n}$ is the moduli space of the n-punctured sphere. The mapping of Schwinger parameters into the coordinates
on the moduli space is quite nontrivial and for instance does not respect the special conformal transformations [16]. The gluing of the segments of the skeleton diagram is subtle but some arguments based on OPE supporting this picture were presented [17]. Hence we could expect that loop amplitudes in the SYM theory can be expressed as correlators of some vertex operators on the moduli space of the complex structures which is the framework of the B model.

Another important starting point for the multiloop generalizations was provided by the geometrical picture found in [19]. It was argued that the one-loop amplitudes can be identified with the hyperbolic volume of the ideal tetrahedron in the space of the Feynman parameters. The corresponding Kahler moduli are fixed by the kinematical invariants. The hyperbolic volumes of three-dimensional manifolds are the natural playground for the A-model and the $SL(2, C)$ gravity provides the natural generalizations of the one-loop answer. It is possible to consider the partition function of $SL(2, C)$ gravity in terms of the proper gluing the three-dimensional manifold from the ideal tetrahedrons [20].

More recently Bern, Dixon and Smirnov (BDS) have formulated the conjecture [21] that all-loop MHV amplitudes get exponentiated and factorize into IR divergent and finite parts. Moreover it was conjectured that the finite part of all-loop amplitude involves only all-loop cusp anomalous dimension $\Gamma_{cusp}(\alpha)$ and finite part of one-loop amplitude. Inspired by this conjecture Alday and Maldacena have calculated the amplitude at strong coupling regime via minimal surfaces in $AdS$-type geometry with the proper boundary conditions [22]. They have found unexpected relation between the MHV amplitudes in planar limit of $N = 4$ SYM theory and Wilson polygons in the momentum space.

The Wilson polygon-amplitude duality refreshes the problem but deserves for the explanation itself. It was originally formulated at strong coupling when the Wilson loop is calculated in terms of minimal surface in the $AdS_5$ geometry upon a kind of T-duality transform. Later it was shown that duality holds true at the perturbative regime as well [28, 29] which puts it on more firm ground. Recently the explicit derivation of the one-loop duality has been presented [25]. The important anomalous Ward identity for the special conformal transformations with respect to the dual conformal group has been derived. It fixes the kinematical dependence of the amplitudes up to five external legs [24, 23]. However Ward identities tell nothing about the functional form of the amplitudes starting from six external legs. Recently the dual superconformal group was identified as the symmetry of the worldsheet theory of the superstring in $AdS_5 \times S^5$ geometry [26, 27].

Finally it was recognized that BDS anzatz fails at weak coupling at two loop level for six external legs [31, 30] and at strong coupling [33, 32] for infinitely large number of external legs. Moreover the BDS anzatz seems not fit well with the Regge limit [34](see however [35]). On the other hand at two loop level the duality between Wilson polygon and MHV amplitude survives. The current status of the whole problem has
been reviewed in [36].

There are a lot of pressing questions to be answered. Just mention a few;

- Is there some geometrical picture behind the BDS-like anzatz which would suggest the way of its necessary generalization?
- Is there the generalization of the dual conformal Ward identity which would fix the functional form of the one-loop amplitude for any number of external legs?
- Is there the fermionic representation for the loop amplitudes which would imply the hidden integrability?
- Is there clear geometrical picture behind the reggeization of the gluon?

In what following we shall suggest the answers to some of these questions and make a couple of conjectures.

To some extend we shall try to generalize the geometrical picture for the tree amplitudes suggested in [5]. At the tree level in [5] the Euclidean D1 ”instanton” branes with the attached open strings have been considered in the twistor space. The D1 brane is localized at the point in the Minkowski space in agreement with the locality of the vertex generating tree MHV amplitude in the MHV formalism. To describe the loop amplitudes we shall adopt a little bit different picture and consider $C^4$ manifold in the B model. The B branes substitute ”D1 instantons” and are embedded in $C^4$. The somewhat similar objects were also introduced as the IR regulator branes in the Alday-Maldacena calculation. Indeed the dilaton field gets changed upon the T-duality in the RG radial coordinate which means that D-instanton is added to the background. After the Fourier transform along flat four-dimensions D-instanton gets transformed into the D3 brane we shall work with. The Wilson polygon which corresponds to the boundary of the string worldsheet and is presumably dual to the amplitude is located just on these IR regulator branes. Contrary to the previous considerations the positions of the regulator branes will not be free but determined dynamically in terms of the cross-ratios of the external momenta.

The emerging moduli space of IR regulator branes plays the central role in the picture. Contrary to the tree level the KS degrees of freedom in the B model defined on the moduli space do not decouple and provide the phase space for the corresponding integrable system. The essential point in our approach concerns the quantization of the emerging moduli spaces and the identification of the corresponding Planck constant with some function of the YM coupling constant. Therefore to some extend we could tell that the loop MHV amplitudes emerge upon a kind of gravitational dressing of tree ones within the Kodaira- Spencer type gravity in the ”momentum” or twistor space.
The physics of the scattering at the loop level can be treated from the different perspectives. From the point of view of the KS gravity on the moduli space we are calculating the correlator of the fermions or the fermionic currents which can be identified with the tau-function of the 2d integrable system. The second viewpoint concerns a consideration of the gauge theory on the IR regulator branes whose number is fixed by the number of the external particles. Finally one could consider the worldsheet viewpoint where the regulator branes provide the proper boundary conditions for the string. These viewpoints are complimentary and allow to check the self-consistency of approach.

Within the KS perspective we shall discuss the fermionic representation behind the loop MHV amplitudes which generalizes the Nair’s fermionic representation for the tree amplitudes. The fermionic picture is a heart of the integrability which admits the representation in terms of the chiral fermions on the Riemann surface in the external gauge field. The gauge field on the Riemann surface represents the “point of Grassmanian” or in physical terms the particular Bogolyubov transformation between the fermionic vacua. Such emerging fermionic picture is known to be quite generic in the set of examples which involves minimal string models [38], $c = 1$ string [39] and crystal melting problem [40]. This approach was summarized in [41] where it was argued that fermions in the KS gravity correspond to mirror of Lagrangian branes in the A model. These B branes are also referred to as Kontsevich or noncompact branes and their positions on the Riemann surface yield the “times” in the corresponding integrable systems. Note that in the framework of the topological strings in A-model we discuss the Kahler geometry while in B-model the complex geometry is captured by the Kodaira-Spencer [42] theory.

The natural question concerns the role of the Riemann surface in the B model where the KS fermions lives on. In the several examples it collects the information about the the infinite set of the anomalous Ward identities in the theory [41] and encodes the unbroken part of the $W_\infty$ symmetry. More qualitatively it means that if we introduce the IR regularization of the theory corresponding to the infinite blow-up of $\mathbb{C}^3$ in the A model the ”anomaly” survives upon the regularization removed. Note some analogy with the description of the Seiberg-Witten solution to the low-energy $N = 2$ SYM theory [43]. In that case we have first to perform the summation over the point-like instantons [44] which amounts to the particular blow-ups and desingularizes the target space theory geometry providing the nontrivial Riemann surface. The physical correlators after all are calculated in terms of the fermions on this emerging Riemann surface.

The fermion one-point function corresponds to the Baker-Akhiezer function in the integrability framework and to the single regulator brane insertion at some point on the moduli space. Since generically we are interested in the quantum integrable system the Riemann surface gets quantized and yields the corresponding Baxter equation [47]. The semiclassical solutions to the Baxter equation which are the generating
functions for the Lagrangian sub-manifolds in the particular integrable system play important role in the analysis. They serve as the building blocks for the correlators in the $N = 4$ YM theory and can be considered as the “semiclassical B brane wave function” or as the effective action in the gauge theory on the brane worldvolume. From the moduli space viewpoint the solution to the Baxter equation provides the generating function of the Lagrangian sub-manifold. The natural integrable system on the moduli space can be identified with the 3-KP system however similar to the $N = 2$ SYM one could expect the pair of integrable systems - 2D field theory and finite dimensional one. The natural finite dimensional integrable system which is responsible for the hidden symmetries at the generic kinematics is conjectured to be related to the Faddeev-Volkov model [48] and the corresponding statistical model [75] based on the discrete quantum conformal transformations.

Since we are trying to sum the perturbation series the YM coupling constant is expected to be involved into some algebraic structure behind all-loop answers. It is this hidden symmetry which provides the choice of the particular solution to the Yang-Baxter equation. The Faddeev-Volkov solution to the Yang-Baxter implies that we are actually trying to relate the YM coupling constant with the parameter $q$ of $U_q(SL(2, R))$. The proper identification turns out to be nontrivial problem since in particular it has to respect the S-duality group in $N = 4$ theory. It will be argued that the BDS ansatz corresponds to the limit $q \to 1$ while the Regge limit seems to be related to the opposite “strong coupling regime” of the quantum group.

The consideration of the gauge theories on the regulator brane worldvolume is useful as well. The theory can be thought of as in the Coulomb phase and the position of the regulator brane in the particular complex plane corresponds to the coordinate on the Coulomb moduli space. Since all regulator branes are at different positions on the moduli space the theory generically has the gauge group $U(1)^k$ where $k$ is related to the number of the external gluons however there are possible enhancements to the nonabelian factors at some kinematical regions. The effective action of each $U(1)$ gauge theory on the regulator brane plays the role of the wave function of the two-dimensional fermion or B brane in KS gravity on the B model side. Similarly the worldvolume theory on the Lagrangian regulator D2 branes can be considered on the A-model side. In this case we shall consider the twisted superpotentials in the worldvolume theory. The minimization of the effective superpotential amounts to the selection of the positions of the Lagrangian branes at the base manifold.

The Riemann surface involved has the interpretation in the regulator brane worldvolume theory as well. To this aim note that the coordinate on the moduli space plays the role of the complex scalar in the B brane worldvolume theory. One can consider the change of variables in the theory corresponding to the reparametrization of this scalar field. Such transformation is familiar in the $N = 1$ SYM theory as the generalized Konishi transformations which are anomalous. One can collect all Konishi transformations or Virasoro constraints in the Dijkgraaf-Vafa approach
It is important to discuss separately the special Regge kinematical region where the hidden symmetries of the amplitudes were found for the first time. The hidden symmetries were captured at one loop by the $SL(2,\mathbb{C})$ spin chains [1, 2]. It was shown in [52] that the $N$-reggeon dynamics belongs to the same universality class as conformal $N = 2$ SQCD with $N_f = 2N$ at the strong coupling orbifold point. We shall argue that the brane geometry in the reggeon case is similar to the one in SQCD which provides the qualitative explanation of the same universality class for both theories. The new object is the open string stretched between two regulator branes and is the analogue of the massive vector bosons and monopoles in the conventional $N = 2$ SYM theory. Here we shall tempt to interpret these open strings as the ”reggeons”. The ”masses” of these effective degrees of freedom correspond to the differences of the positions of the regulator branes on the proper Riemann surface and therefore depend on the kinematical invariants. We shall comment on the possible link of this picture with the effective Reggeon field theory [53].

The paper is organized as follows. In Section 2 we remind the main features concerning the loop MHV amplitudes. In Section 3 we briefly consider the example of the $c = 1$ noncritical string which provides some intuition on the calculation of the amplitudes from the target space perspective. In Section 4 we review the relevant properties of the quantum dilogarithm. Section 5 is devoted to the formulation of our conjecture for the $N = 4$ MHV amplitudes. In Section 6 we make conjecture on the pair of the underlying integrable systems. Some arguments concerning the Regge limit of the amplitudes are present in Section 7. In the last Section we collect the main points of our proposal and fix the open questions to be answered.

2 The loop results for the MHV amplitudes

Let us remind the main results concerning the loop MHV amplitudes. The MHV gluon amplitudes involve two gluons of the negative chiralities and the rest of gluons have positive chiralities. Consider the ratio of all-loop and tree answers. The following form of the all-loop amplitudes has been suggested in [21]

$$\log\left(\frac{M_{all=\text{loop}}}{M_{\text{tree}}}\right) = (F_{\text{div}} + \Gamma_{\text{cusp}}(\lambda)M_{\text{one-loop}})$$

which involves only the all-loop answer for the cusp anomaly $\Gamma_{\text{cusp}}$ and one-loop MHV amplitude. The IR divergent part $F_{\text{div}}$ gets factorized in the all-loop answer. The cusp anomaly measures UV behavior of the contour with cusp [54]. Recently the closed integral equation has been found for the cusp anomalous dimension in $N = 4$ SYM theory [55] which correctly reproduces the weak and strong coupling expansions.
The finite part of the one-loop MHV which presumably defines the all-loop answer can be written in terms of the finite part of the so-called two-mass easy box function \( F_{2em} \) [29]

\[
M_{\text{one-loop, finite}} = \sum_{p,q} F_{2em,f}^2(p, q, P, Q)
\]  

(2)

This function can be expressed in terms of the dilogarithms only

\[
F_{2em,f}^2(p, q, P, Q) = \text{Li}_2(1 - aP^2) + \text{Li}_2(1 - aQ^2) - \text{Li}_2(1 - a(q + P)^2) - \text{Li}_2(1 - a(p + P)^2)
\]  

(3)

where

\[
a = \frac{P^2 + Q^2 - (q + P)^2 - (p + P)^2}{P^2Q^2 - (q + P)^2(q + P)^2}
\]  

(4)

and \( p + q + P + Q = 0 \). One more expression for the function \( F_{2em,f}^2 \) can be written in terms of the variables \( x_{i,k} = p_i - p_k \) as the sums [28]

\[
\sum_i \sum_r \text{Li}_2(1 - \frac{x_{i,i+r}^2}{x_{i,i+r+1}^2x_{i-1,i+r+1}^2})
\]  

(5)

where

\[
x_i = p_{i+1} - p_i
\]  

(6)

Since all external momenta are on the mass shell the arguments of dilogarithms are expressed in terms of the cross-ratios of the scalar products of the momenta only.

Since we shall aim to get the geometrical interpretation of the BDS anzatz we need first the clear geometry behind one loop. It is provided by the observation in [19] that box diagram with all external particles off-shell just calculate the volume of the three-dimensional ideal hyperbolic tetrahedron in the space of the Feynman parameters (see Appendix). The Kahler modulus \( z \) of the tetrahedron is fixed by the kinematical invariants and in the two-mass easy box it reduces to the conformal ratios. The appearance of the hyperbolic volume implies that the topological string approach or CS with \( SL(2, C) \) group are relevant [20]. Indeed we can consider the ideal tetrahedron as the knot complement and calculate its volume via the Chern-Simons theory action with the complex group. To some extend the exponentiation of the one-loop answer in the BDS manner corresponds to the calculation of the classical partition function in \( SL(2, C) \) CS action which involves \( \exp(Vol(z)) \) factor. Let us emphasize that the volume is finite if we consider off-shell particles particles only and IR divergence of the amplitude corresponds to the divergence of the volume. Some development along such interpretation of the higher loops can be found in [37].

The topological string picture usually can be represented both in the A-model and B-model sides. The hyperbolic volume calculation evidently corresponds to the
Kahler gravity in A-side so one could ask about the one-loop geometry in the B-model language. It can be uncovered indeed if we recall that dilogarithm is the natural object on the moduli space $M_{(0,n)}$ [57] where it provides the proper canonical transformations. The natural arguments of dilogs as functions on the moduli space are just the conformal ratios appeared in the one-loop integrals. We shall try to interpret BDS-like structure in the B-model language as the result of the quantization of the Teichmuller space when the one-loop dilogs are substituted by the quantum dilogs. This viewpoint will be useful when we shall search for the proper “degrees of freedom” involved into higher loops calculations. They will be conjectured to be IR regulator branes or equivalent fermions.

3 The c=1 Example

The useful example which shares some essential features with our problem is provided by the $c = 1$ model. The noncritical $c = 1$ model describes the string in the one dimensional compact target space and because of the non-criticality the target geometry becomes two-dimensional due to the Liouville direction. The only physical modes are massless tachyons generically gravitationally dressed by the Liouville modes.

The theory enjoys two natural types of branes which provide the boundary conditions for the strings; so called ZZ and FZZT branes. The compact ZZ branes correspond to the unstable D0 branes localized in the Liouville direction. On the other hand the noncompact branes correspond to the stable D1 branes extended along the Liouville coordinate till the Liouville wall at the cosmological constant scale $\mu$ [39].

The explicit target space description is more appropriate for our purposes. This approach is natural in the topological string setup and was developed in [39]. The crucial point is the existence of the so-called chiral ring in the theory. In the c=1 case it collects the information about the set of certain anomalous relations in the theory. The most useful description of the chiral ring involves the Riemann surface supplied with some meromorphic differential. For $c = 1$ theory it reads as

$$x^2 - y^2 = const$$

where $x, y \in C$. In terms of the Riemann surface a compact ZZ brane corresponds to the tunneling state in the inverted oscillator or equivalently to the closed cycle on the surface pinched at the degeneration point. On the other hand noncompact FZZT brane corresponds to the open path on the surface. The corresponding semiclassical “wave function”

$$\Psi_{FZZT} \propto exp(i \int y dx)$$

transforms nontrivially when going to different coordinate paths on the surface.
Another useful language is provided by the matrix model approach. The random matrix model is usually built on the set of infinite number of ZZ branes whose coordinates correspond to the eigenvalues of the matrix of the infinite size triangulating the string worldsheet. On the other hand one can consider the matrix model of the Kontsevich type on the N noncompact FZZT branes. In this case one deals with the $N \times N$ matrix model with the source term $TrA$ and the eigenvalues of the matrix $\Lambda$ encode the positions of the FZZT branes. The appropriate object in the matrix model is its resolvent

$$W(z) = < Tr \frac{1}{z - M} >$$

obeying the loop equation which in the semiclassical limit coincides with the equation of the spectral curve.

All languages yield the same important feature of $c=1$ model - its hidden integrability. It turns out that tau-function of the Toda hierarchy serves as the generating function for the tachyonic amplitudes [46]. This objects can be naturally described in terms of the chiral boson $\phi(z)$ on the spectral curve or as the corresponding fermion

$$\Psi(z) = exp(g^{-1}\phi(z))$$

More precisely one considers the following matrix element in the theory of the chiral boson or its fermionized version

$$\tau(t, A) = < t | exp(\psi^A \psi) | 0 > = < t | V >$$

where $| t > = exp(\sum_k t_k \alpha_k)$ is generically the coherent state of the chiral boson with modes $\alpha_k$ on our Riemann surface. The matrix $\Lambda$ encodes the scattering of fermions off the Liouville wall which essentially provides the whole answer for the tachyonic amplitudes [46]. The integrability encodes the infinite number of the Ward identities in the theory followed from the symplectic invariance of the Riemann surface which is the complex Liouville torus for the complex Hamiltonian system. Some part of the symmetries is spontaneously broken yielding the corresponding Ward identities. Some of them are unbroken yielding the equation of the quantum spectral curve [41]. The set of Ward identities can be formulated in terms of the fermionic bilinears on the surface and provides the exact answers for the correlators in the theory.

The FZZT or noncompact branes parameterize the moduli space of the complex structures in the target geometry and can be naturally treated within the KS gravity in the target space. The positions of the noncompact FZZT branes $z_i$ yield the following "times" in the Toda integrable system

$$T_k = \frac{1}{k} \sum_{i=1}^{N} z_i^{-k}$$

The "wave function" of the FZZT brane itself can be considered as the Baker-Akhiezer function in the integrable systems. Quantum mechanically the Riemann
surface gets quantized yielding the Baxter equation for the eigenvalue of the Baxter operator. Solution to the Baxter equation corresponds to the wave function of the single separated variable. The brane interpretation of the separated variables has been suggested in [56]. The correlator of the Kontsevich branes inserted at points \( z_i \) has the following structure

\[
< 0|\Psi(z_1) \ldots \Psi(z_n)|V > = \prod_{i,j}(z_i - z_j)e^{\sum_i \phi(z_i)}
\]  

(13)

where one can clearly distinguish the classical and quantum components of the answer. Another point to be mentioned is the identification of the quantization parameter. In the commutation relation on the quantized Riemann surface

\[
[x, y] = g_s
\]  

(14)

the Planck constant is just the string constant or the graviphoton field.

To summarize, \( c = 1 \) model provides the example when the non-perturbative structure of the theory is stored is the Riemann surface in the B model which collects the information about the chiral ring. This Riemann surface has to be considered as the energy level of some complex Hamiltonian and the set of canonical transformations of the phase space amounts to the set of Ward identities in the initial model which fix the scattering amplitudes. The deformations of the complex structures which are dynamical degrees of freedom in the Kodaira-Spencer theory are parameterized by the fermions on the Riemann surface which upon quantization yield the Baxter equation of the corresponding integrable system.

4 Quantum dilogarithm

The one-loop answer is expressed in terms of the dilogarithms hence in this Section we shall briefly review some relevant properties of the quantum dilogarithm defined as the following integral

\[
\Psi_b(z) = exp\left(\frac{1}{4} \int \frac{e^{-2ix}dx}{x \sinh(bx) \sinh(b^{-1}x)}\right)
\]  

(15)

The integration contour is chosen in such way that the integral reduced to the infinite sum via residue calculation

\[
\Psi_b(z) = exp\left(\sum_n \frac{e^{-nx}}{n[n]}\right)
\]  

(16)

It obeys the functional equations

\[
\Psi_b(z)\Psi_b(-z) = e^{i\pi z^2 - i\pi(1+2c_q^2)/6}
\]  

(17)
and
\[ \Psi_b(z - b^\pm 1/2) = (1 + e^{2\pi z b^\pm 1}) \Psi_b(z + b^\pm 1/2) \] (18)
where \( c_b = \frac{1}{2}(b + b^{-1}) \), as well as the unitarity condition
\[ \Psi_b(z) = \Psi_b(\bar{z})^{-1} \] (19)

The quantum dilogarithm can be represented as the ratio of two \( q \)-exponentials
\[ \Psi_b(z) = \frac{(e^{2\pi(z + c_b)b}; q^2)_{\infty}}{(e^{2\pi(z - c_b)b^{-1}}; \bar{q}^2)_{\infty}} \] (20)
where \( q = e^{i\pi b^2} \), \( \bar{q} = e^{-i\pi b^{-2}} \) and
\[ (x, q)_{\infty} = \prod_n (1 - q^n x). \] (21)

The dilogarithm enjoys the duality
\[ \Psi_b(z) = \Psi_{b^{-1}}(z) \] (22)
which is essential when it is involved into gluing of the conformal blocks in the Liouville model [60, 61]. Note that the central charge in the corresponding Liouville theory reads as
\[ c_{\text{Liouv}} = 1 + 6(b + b^{-1})^2 \] (23)

The quantum dilogarithm is natural object from the viewpoint of the quantum torus algebra defined by the relation
\[ \hat{U}\hat{V} = q\hat{V}\hat{U} \] (24)
where \( \hat{U} = \exp(i\hat{x}) \) and \( \hat{V} = \exp(i\hat{p}) \). It is assumed that the variables \( x, p \) obey the canonical phase space commutation relations. In terms of the quantum torus the quantum dilogarithm is defined via the relation
\[ \Psi(\hat{V})\Psi(\hat{U}) = \Psi(\hat{U})\Psi(-\hat{V})\Psi(\hat{V}) \] (25)
This property represents the so-called quantum pentagon relation [59] and reduces to the classical Rogers identity for \( Li_2(z) \) in the semiclassical limit.

Quantum mechanically the dilogarithm defines the operator with the kernel
\[ K(x, z) = \Psi_q(z)e^{\frac{zx}{2\pi q}} \] (26)
which serves as the generating function of the following canonical Backlund type transformations
\[ U \rightarrow ((1 + qU)V \quad V \rightarrow U^{-1} \]

which belongs to the outer automorphisms of the algebra of functions on the quantum torus. The important property of this transformation is the operator version of the pentagon relation

\[ \hat{K}^5 = 1 \]

which states that automorphism is of the fifth order.

Dilogarithm plays an essential role in the symplectic treatment of the Teichmüller space. Remind that the Teichmüller space \( T(S) \) is the space of the complex structures on the Riemann surface \( S \) modulo trivial diffeomorphisms homotopy equivalent to the identity while the moduli space \( M(S) \) is obtained via the factorization of the Teichmüller by the action of the mapping class group. The quantum dilogarithm plays the role of the quantum generating function for the particular element of the mapping class group - flip transformation. The flip transformations are responsible for the generic transition maps between coordinate systems corresponding to the different triangulations. The quantization of the Teichmüller space was developed in [58, 57] (see [60] for review).

There are different ways to introduce the coordinates on the Teichmüller space of the punctured spheres we are interested in. Of particular interest are the coordinates related to the geodesic lengths and the corresponding classical Poisson structure can be written in a simple way in terms of triangulations. These shear coordinates can be introduced in terms of the cross-ratio of the four points on the real line connected by the geodesic circles in the upper half-plane

\[ e(z) = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_2)(x_4 - x_1)} \]

The natural arguments of the dilogarithms defined on the moduli space are just cross-ratios which we meet in the answers for the MHV amplitude. In the semiclassical limit we have

\[ \Psi_b(x) \rightarrow exp\left(\frac{1}{b}L_{t_2}(e^x)\right) \]

In the context of integrability dilogarithms appear as the ingredients of the fundamental R-matrix involved into the description of the Liouville and sin-Gordon theories in the discrete space-time. In terms of the discretized version of the Kac-Moody currents with the commutation relation similar to the quantum torus

\[ \omega_n \omega_{n+1} = q^2 \omega_{n+1} \omega_n \]

\[ \omega_n \omega_m = \omega_m \omega_n, \quad |n - m| \geq 2 \]

The periodicity condition for the current variables is assumed \( \omega_n = \omega_{n+N} \). In terms of these dynamical variables one can define the solution to the Yang-Baxter equation
depending on the spectral parameter $\lambda$ \[64\]

$$ R(\lambda, \omega) = \frac{\Psi_b(\omega)\Psi_b(\omega^{-1})}{\Psi_b(\lambda\omega)\Psi_b(\lambda\omega^{-1})} \quad (33) $$

The product of the R-matrixes over the lattice cites yields the evolution operator for the massive model on the space-time lattice.

The finite-dimensional system discussed in the B-model framework should carry some information on the S-duality of the $N = 4$ gauge theory. The modular parameter of the gauge theory can be identified with the quantization parameter of the integrable system. However to get the full modular symmetry the modular double \[63\] has to be included into the game. It unifies two quantum tori with the S-dual moduli. It turns out that the modular double plays the crucial role in the quantum integrable systems providing the self-consistency of the local and nonlocal integrals of motion.

That is integrable system has to be supplied with the following symmetry $U_q(SL(2, R) \otimes U_{\tilde{q}}(SL(2, R))$. The corresponding R-matrix acting of the modular double reads as

$$ R = e^{\frac{\pi}{2}(p_3 + p_4) \otimes (p_1 + p_4)}\Psi(p_{13})\Psi(p_{34})\Psi(p_{23})\Psi(p_{24}) \quad (34) $$

$$ p_{ik} = p_i \otimes I + I \otimes p_k \quad (35) $$

and the variables $p_i, i = 1 \ldots 4$ obey the commutation relations

$$ [p_k, p_{k+1}] = -2\pi i I \quad (36) $$

The integrable system with such symmetry was found in \[48\]. It was argued in \[75\] that using the R matrix for the modular double one can define the positive weights which provide the unitary model.

Such type of R-matrix emerges naturally within the discrete quantum Liouville theory related to the discrete conformal transformations \[48, 75, 58\]. It was argued that the structure of the modular double is necessary for the self-consistent description of the Liouville theory at the strong coupling region $1 < c < 25$. The integrability of the model to some extent is equivalent to its very quantum existence \[75\]. The link with the dilogarithms involved into the description of the Teichmuller space goes as follows. The universal Teichmuller space is known to be identified with the coadjoint Virasoro orbit. On the other hand the Liouville action plays the role of the geometrical action on this orbit \[78\] therefore it is no surprise that the triangulation of the moduli space is related with the discrete quantum Liouville theory.

In the physical setup the quantum dilogarithm corresponds to the probability of the charged pair creation in the constant external field in four-dimensional scalar QED theory. It is assumed that both electric (E) and magnetic (H) fields are switched then the one-loop effective action reads as

$$ L_{\text{one-loop}} = \frac{1}{16\pi^2} \int \frac{dt}{t} e^{-m^2 t} \left( \frac{e^2 ac}{\sinh(ect)\sinh(eat)} - \frac{1}{t^2} - \frac{e^2}{6}(a^2 - c^2) \right) \quad (37) $$
where $a^2 - c^2 = E^2 - H^2$ and $ac = EH$. The last two terms provide the proper substraction to get the finite answer. The "strong coupling limit" $|b| \to 1$ of the quantum dilogarithm corresponds to the almost self-dual external field while the semiclassical limit corresponds to the strong deviation from the self-dual regime.

5 Finite part of $N = 4$ SYM MHV amplitudes and "momentum space" geometry

5.1 The brane picture

Let us now formulate our proposal for finite part of the MHV loop amplitudes. Recall that the tree amplitudes were described in terms of the D1 string instanton embedded into the twistor manifold [5]. The instanton is localized at point in the Minkowski space and open strings representing gluons are attached to it. To describe the loop amplitude we shall substitute D1 brane by the IR regulator brane embedded into the proper manifold. The gluons are attached to the regulator branes whose embedding coordinates are considered as dynamical degrees of freedom. Contrary to tree case regulator branes are localized at the sub-manifold of the complexified Minkowski space. The loop amplitudes can be considered from the different perspectives: in terms of the KS gravity on the particular Riemann surface, within the worldvolume theory on the regulator branes and in the theory on the string worldsheet. Let us emphasize that the embedding of the IR regulator branes nontrivially depends on the external momenta.

The starting point is the representation of the $N = 4$ theory via geometrical engineering [65] as the IIA superstring compactified on the three-dimensional Calabi-Yau manifold which was identified as the $K3 \times T^2$ geometry in the singular limit. One has to consider the singular limit of K3 manifold when it develops $A_{N-1}$ singularity, where $N$ becomes the rank of the gauge group, and upon blowing up procedure it can be represented as $ALE_N$ geometry. On the other hand the Kahler class of the $T^2$ can be identified with the coupling constant

$$\text{Area}(T^2) = 1/g_{YM}^2$$

(38)

At weak coupling the torus is large and can be approximated by the complex plane. That is the geometry can be roughly approximated by $C^3$ upon the particular blow-ups.

As we have seen the one-loop answer for the MHV amplitude determining the BDS form of the amplitude involves the sum of the dilogarithms depending on the cross-ratios of the $x_i$ variables. Below we shall try to explain how such functions with cross-ratio arguments emerge naturally both in A-model and B-model frameworks. As is well-known the A-model captures the information about the Kahler moduli
while the B-model about the complex moduli and we shall see where these moduli comes from. The brane description of the scattering amplitude involves the set of the Lagrangian branes in the A-model and the corresponding B-model branes. It is these branes which provide the corresponding moduli spaces.

5.1.1 B-model

First we shall discuss B-model approach. There are two natural B-model setups. The first one follows from the topological S-duality [69] and corresponds to the same manifold with the S-dual moduli. Since in our case area of the torus is defined in terms of the YM coupling constant we end up with the small dual torus in the S-dual model at weak coupling. This topological B-model in the S-dual geometry can be described in terms of the noncommutative $U(1)$ gauge theory in D=6 which is naturally defined on the D5 brane worldvolume. Another viewpoint is provided by the mirror symmetry which maps A-model to B-model on the different manifold. It is convenient to consider the dual mirror geometry upon the infinite blowup of $C^3$.

Let us interpret the BDS anzatz in terms of the correlator of the noncompact Euclidean B branes embedded into the four dimensional complex space. Consider 3d complex manifold which is mirror to the topological vertex [67]. This manifold classically is described by the equation in the $C^4$ with coordinates $x, y, u, v$

$$xy = e^u + e^v + 1$$

(39)

At the discriminant locus it defines the Riemann surface

$$H(v, u) = e^u + e^v + 1 = 0$$

(40)

of genus zero with three different asymptotic regions.

This Riemann surface emerges from the infinite blow-ups of the origin of the toric fibration of $C^3$ upon the mirror transform and provides the part of IR regularization of the theory. We shall try to argue that the loop MHV amplitudes can be identified with the fermionic correlators on the Riemann surface (40). Fermions on the surface (40) represent the degrees of freedom in the KS gravity. They are identified with the IR regulator branes imbedded into $C^4$ geometry.

There are two B branes defined by the equations

$$x = 0 \quad H(v, u) = 0$$

(41)

and

$$y = 0 \quad H(v, u) = 0$$

(42)

which intersect along the Riemann surface. The intersecting branes provides the natural fermionic degrees of freedom on the intersection surface [76] from the open strings.
stretched between them. The fermions are in external field amounted from the world-volume gauge connections on the intersecting branes. This gauge field representing the point of the Grassmanian which can be read off from the topological vertex. In addition to two branes intersecting along the Riemann surface we introduce the set of Kontsevich -like branes classically localized at the points \((v_i, u_i)\) at the Riemann surface. The number of such branes is fixed by the number of the external gluons and the coordinates of these branes on the surface are defined by some particular cross-ratios. The cross-ratios are the natural coordinates on the moduli space of the punctured spheres that is the \((u, v)\) space is related to the \(T^*M_{0,4}\). Hence we are in the framework of the KS gravity and the fermions on the Riemann surface represent the KS gravity degree of freedom.

The Riemann surface gets quantized and the branes-fermions should obey the equation of the quantum Riemann surface that is Baxter equation which provides the wave functions depending on the separated variables \([56]\). The Baxter equation in our problem reads as

\[
(e^{\hbar \partial_v} + e^v + 1)Q(v) = 0 \tag{43}
\]

Its solution corresponds to the vev of fermionic bilinear \(J\)

\[
Q(v) = \langle 0|J(v)|V \rangle \tag{44}
\]

and turns out to be the quantum dilogarithm \([41]\). Note that the solution to the Baxter equation in our case can not be presented in the polynomial form that is we have infinite number of the Bethe roots.

To get the MHV all-loop amplitude in the BDS form we take the semiclassical limit of the fermionic correlator on this surface. Indeed using the semiclassical limit for the quantum dilogarithm we can represent the four-point fermionic current correlator as

\[
\langle \bar{J}(z_1)\bar{J}(z_2)J(z_3)J(z_4) \rangle \propto \exp(\hbar^{-1}(Li_2(z_3) + Li_2(z_4) - Li_2(z_1) - Li_2(z_2))) \tag{45}
\]

This expression exactly coincides with the expression for the finite contribution of the single 2-easy mass box diagram hence upon the identification of the Planck constant

\[
\hbar^{-1} = \Gamma_{\text{cusp}}(\lambda) \tag{46}
\]

we reproduce BDS anzatz for the finite part of the amplitude. Indeed the one-loop answer for the MHV amplitude can be expressed purely in terms of the sum of 2-mass easy box diagrams with different grouping of the gluon momenta and therefore in terms of the fermionic correlators.

Since the regulator brane (D1 “instanton”) yielding the tree amplitude is localized in the complexified Minkowski space \(M^c\) \([5]\) one could ask about similar localization of regulator branes responsible for the higher loop calculations. To this aim
recall that the complexified Minkowski space $M^c$ is equivalent to the Grassmanian $Gr(2, 4)$. On the other hand the factor of the Grassmanian by the maximal torus action is related to the compactified moduli space \[82]\]

$$Gr(2, 4)/T = \tilde{M}_{0,4}$$ \hfill (47)

This representation allows to represent the complexified Minkowski space itself as the fancy divisor of the $\tilde{M}_{0,4}$ \[81]. We suggest that this realization implies the localization of the regulator branes on the submanifold of $T^*(\tilde{M}^c/T)$ . It is natural to identify this manifold with the Riemann surface where the KS degrees of freedom live.

### 5.1.2 A-model

Let us present the qualitative arguments concerning the corresponding A-model picture. In the A-model we introduce the set of Lagrangian branes with topology $S^1 \times \mathbb{R}^2$. They can be also thought of as D6 branes if we add the conventional $\mathbb{R}^3, 1$ piece of geometry. The emergence of the dilogarithm as the wave function of the Lagrangian brane has been discovered in the $C^3$ geometry in \[68]. The brane/antibrane can be considered as the insertion of the fermion/antifermion \[68] in the fermionic representation of the topological vertex picture \[67].

In this case we get the Kahler gravity as the target space description of our geometry. The Lagrangian of the Kahler gravity was conjectured to reduce to the $SL(2, \mathbb{C})$ CS theory on the Lagrangian branes which describes the quantum geometry of the hyperbolic space. Since we are trying to identify the amplitude as the wave function of the Lagrangian branes, its argument in the proper polarization should be Kahler modulus of the ideal tetrahedron. This is indeed consistent with the loop calculations since the box diagram yields the hyperbolic volume in the space of the Feynman parameters \[19].

The intersecting Lagrangian branes naturally provide the set of knots. The knot complements are the natural hyperbolic manifolds and we can consider the triangulation of the three-dimensional cusped hyperbolic space by the ideal tetrahedron (see, for instance \[20])

It turns out that quantum dilogarithm with the proper argument can be attributed to each tetrahedron and the main problem turns out to be the gluing the whole manifold from the several ideal tetrahedra. The gluing conditions have the form of the Bethe Anzatz-type equations if one attributes to the each tetrahedron a kind of S-matrix \[20]. That is in the A-model picture the all-loop answer deserves the accurate gluing of the submanifolds in the hyperbolic spaces.

### 5.2 The regulator brane worldvolume theory

Since fermions in KS framework are identified as the regulator B branes the natural question concerns their four-dimensional worldvolume theory. The theory on the
regulator branes share many features with $N = 2$ and $N = 1$ SYM low-energy sectors. The number of the regulator branes is fixed by the number of the external gluons so naively one could expect a kind of $SU(K)$ gauge theory. The worldsheet theory on the regulator branes enjoys the complex scalar corresponding to the complex coordinate $z$ of the brane on the Riemann surface (40). This is similar to the situation when the vev of the scalar field corresponds to the position of the D4 branes on the $u$-plane in the IIA realization of the $N = 2$ SYM theory [45].

Since the different regulator branes are at the different points on the Riemann surface we can speak about the Coulomb branch of the regulator worldvolume theory. However their positions on the Riemann surface are fixed that is we could say about the localization of the B branes at the points of the moduli space $M_{(0,4)}$. Similar to the $N = 1$ SYM theory when branes are localized at positions corresponding to the discrete vacua the regulator branes are localized at some points parameterized by the cross-ratios. These points correspond to the local rapidities in the framework of integrability and simultaneously have to correspond to the minima of the effective superpotentials $W_{\text{eff}}(z_i)$ in the regulator worldvolume theory.

Since we identify dilogaritms as the regulator brane wave functions it is necessary to explain where they come from in the worldvolume theory. The qualitative arguments looks as follows. In the worldvolume theory there are massive excitations corresponding to the open strings stretched between two regulator branes. They are analogue of the massive W-bosons in $N = 1$ SYM theory on the Coulomb branch. In our case the masses of these "particles" are related to the cross-ratios. To recover the dilogaritms let us remind that usually in the external field the effective action develops the imaginary part corresponding to the pair creation. The probability of the pair creation in the external field is described by the classical trajectory in the Euclidean space and in the leading approximation reads as

$$\text{Im}S_{\text{eff}} \propto e^{-\frac{m^2}{cE}}$$

(48)

for a particle of the mass $m$ in the external field $E$. Upon taking into account the multiple wrapping and the quadratic fluctuations one gets for the scalar particle Schwinger pair production

$$\text{Im}S_{\text{eff}} \propto \sum_n \frac{1}{n^2} e^{-\frac{nm^2}{cE}}$$

(49)

that is dilog plays the role of the decay probability. Hence one can say that we are considering the Euclidean version of the regulator worldvolume theory and the amplitude from this viewpoint is described via bounce type configuration corresponding to the creation of the pairs of the effective massive degrees of freedom. Note that the real part of the effective action corresponds to the summation over the loop contributions of the same degrees of freedom.

We have a $N = 1$ type theory on the regulator branes with the finite number of vacua and the complex scalar field whose vacuum expectation value corresponds
to the coordinates of the B branes on the Riemann surface. Hence we can discuss the role of the symmetries implied by the area preserving diffeomorphisms of the Riemann surface. From the regulator worldvolume theory this transformation is the symmetry of the target space analogous to the transformations of the scalar field in $N = 1$ gauge theory with adjoint scalar. It $N = 1$ SYM case the generalized Konishi anomaly responsible for these transformations [50, 51]

$$\Phi \rightarrow f(\Phi)$$

(50)
captures the unbroken part of $W_\infty$.

In the A-model one can similarly consider the worldvolume theory on the D2(or D6) Lagrangian regulator branes. In this case the corresponding dilogarithm functions emerge upon summation over the disc instantons with boundaries located at the corresponding Lagrangian branes

$$W_{\text{eff}} \propto \sum_n d_n e^{-nA}$$

(51)

where A-is the corresponding area of the target disc. The effective twisted superpotentials in the three-dimensional worldvolume theory with one compact dimension were discussed in [74]. Roughly speaking the wave function of D2 brane has the form

$$\Psi(z) \propto e^{W_{\text{twisted}}(z)}$$

(52)

Note that in the A model D2 branes can be considered as wrapped around the ideal tetrahedrons whose Kahler modulus are defined by the cross-ratios providing the masses of the same effective "W-bosons" as in B-model. The issue of the gluing of the tetrahedra gets reformulated in terms of the minimization of the total twisted superpotential which is equivalent to the solution to the Bethe Anzatz equations in the XXZ spin chain model [74]. The solution to the Bethe Anzatz equations fixes the positions of the Lagrangian branes. To fit this arguments with the complex moduli remark that Bethe Anzatz equations for the XXZ model is related to the solution to the classical equations of motion in the discrete Liouville model [88].

5.3 $\hbar^{-1} = \Gamma_{\text{cusp}}(\alpha)$?

Let us comment on the identification of the Planck constant for the quantization of the KS gravity as the inverse cusp anomalous dimension inspired by the BDS anzatz. At the first glance it looks completely groundless however the argument supporting this identification goes as follows. The emergence of the cusp anomaly in the exponent means that it plays the role of the effective string tension or equivalently the inverse Planck constant. Such effective tension emerges if one considers the string whose boundary is extended along the light-like contours. It was shown [72] that in the
limit suggested in [70] the string worldsheet action can be identified with $O(6)$ sigma model and the energy of the ground state in $O(6)$ model is proportional to the length of the string multiplied by the $\Gamma_{\text{cusp}}(\alpha)$. That is indeed $\Gamma_{\text{cusp}}(\alpha)$ plays the role of the effective tension of the string in this special kinematics. Since in our case the boundary of the string worldsheet lies on the Wilson polygon the effective tension involving the cusp anomalous dimension is natural.

However certainly this point is far from being clarified. For instance in the Ward identity for the special conformal transformation $\Gamma_{\text{cusp}}$ enters as the multiplier in the anomalous contribution. This claim has been explicitly checked at the first loops in the gauge theory calculations and the arguments that it holds true at all orders have been presented. The anomalous Ward identity reads as [23]

$$K^\nu W(x_1, \ldots, x_N) = \sum_{i=1}^{n} (2x_i^\nu x_i \partial_i - x_i^2 \partial_i^\nu) W(x_1, \ldots, x_N) = \frac{1}{2} \Gamma_{\text{cusp}}(\alpha) \sum_{i=1}^{n} \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu$$

and it has been proved at strong coupling as well [73].

In this equation $\Gamma_{\text{cusp}}$ plays the role of the Planck constant not the inverse one. To match both arguments we could suggest that in the Ward identity we are considering the S-dual formulation and therefore the D1 string worldsheet action instead of the F1 one in $O(6)$ sigma model. This would imply that the Wilson polygon equivalent to the MHV amplitude could be considered as the boundary of the D1 string as well. Similarly the $\Gamma_{\text{cusp}}$ enters the loop equation for the Wilson loops with cusps [71]

$$\Delta_C < W(C) > = \sum_{\text{cusps}} \Gamma_{\text{cusp}}(\alpha, \theta_i) < W(C) >$$

where $\Delta_C$ is the Laplace operator in the loop space and the summation goes over all cusps along the contour $C$. We assume that there are no self-intersections. In this case the $\Gamma_{\text{cusp}}(\alpha)$ has the natural interpretation as the inverse Planck constant since the loop equation has the form of the Schrödinger equation.

In more general setup it is highly desirable to realize the meaning of the relation of such type in the first quantized language. Since the cusp anomalous dimension is just the renormalization factor for the self-crossing of the worldline it is very interesting to understand if such self-crossing is involved into the quantization issue. In particular in the Ising model the effect of the self-crossing is captured by the topological term and in the description of the topological string on $C^3$ somewhat similar $\theta$ term in six dimensions plays the role of the quantization parameter indeed [66]. In the gauge theory language such objects are related to the renormalization of the double-trace operators couplings.

Note that generically the relation between the YM coupling and string coupling
involves the $B_{NS}$ field

$$\frac{1}{g_{YM}^2} = \sqrt{\text{Vol}_{T^2} + B_{NS}^2} \quad (55)$$

One could try to speculate that the self-crossing could be sensitive to the $B_{NS}$ field. Anyway it is clear that precise identification of the relation between the quantization parameter in the KS gravity and the YM coupling is one of the necessary steps in improving the BDS anzatz.

6 Integrability behind the scattering amplitudes

6.1 General remarks

In this Section we shall discuss the hidden integrability behind the scattering amplitudes and present the arguments that similarly to the integrability pattern behind effective actions in N=2 SYM theory (see [79] for the review) two integrable systems are involved. The degrees of freedom of both integrable systems are related to the coordinates of the regulator branes. One of these systems which we identify as the Whitham-like 3-KP one plays the role of RG flows in the regulator brane world-sheet theory or equivalently the motion of the regulator brane along the ”radial” RG-coordinate. The second integrable system generalizing the Hitchin-like or spin chain models involves the effective interactions between the regulator branes. We shall give arguments that this system is based on the Faddeev-Volkov solution to the Yang-Baxter equation for the infinite-dimensional representations of the noncompact $SL(2, \mathbb{R})$ group.

Recall how two integrable systems are involved into the description of the low energy effective actions of $N = 2$ SYM theories. The first finite dimensional system is of the Hitchin or spin chain type and its complex Liouville tori are identified with the Seiberg-Witten curves. This spectral curve emerges in the gauge theory upon the summation over the infinite number of instantons [44].

Following [80] one can canonically define the dual integrable system whose phase space is built on the integrals of the motion of the first one. In the simplest case of $SU(2)$ theory the phase space for the dual system has the symplectic structure $[84]$

$$\omega = da \wedge da_D \quad (56)$$

where the variables $a, a_D$ are the standard variables in N=2 SYM framework [43]. The prepotential $F$ can be identified with the generating function of the Lagrangian sub-manifold in the dual system with the $a, a_D$ phase space

$$H(a(u), \frac{\partial F}{\partial a}) = u \quad (57)$$

21
and obeys the Hamilton-Jacobi equation
\[
\frac{\partial F}{\partial \log \Lambda} = H
\] (58)

In the brane setup the prepotential defines the semiclassical "wave function" of the D4 brane \( \Psi(a) \propto \exp(h^{-1}F(a)) \) in the IIA brane picture where perturbatively the argument of the wave function can be identified with coordinate of the D4 brane on the NS5 brane. The total perturbative prepotential in \( SU(N_c) \) can be considered as a sum of the exponential factors in the product of the wave functions of \( N_c \) D4 branes. At the A-model side these wave functions can be considered in the Kahler gravity framework and the arguments of the wave function have to be treated as the Kahler classes of the blow-upped spheres.

The integrals of motion provide the moduli space of the complex structures in the Calabi-Yau geometry in the B model hence we are precisely in the KS framework. In this B-model formulation we consider the argument of the brane wave function as the coordinate on the moduli space of the complex structures. The dual Whitham-type integrable system naturally defines the \( \tau \)-function of the 2d Toda theory formulated in terms of the chiral fermions on the Riemann surface with two marked points. Upon perturbing \( N = 2 \) theory down to \( N = 1 \) the moduli space disappears and the number of vacua becomes finite. In the integrability framework this is treated in the following manner. The Hamiltonian of the first finite-dimensional system turns out to coincide with the superpotential of the \( N = 1 \) system [85]. That is the vacua of the gauge theory at the classical level correspond to the equilibrium points in the Hamiltonian system \( W' = 0 \).

### 6.2 3-KP system

Let us turn to the integrable structure relevant for the scattering amplitudes at generic kinematics and first identify the degrees of freedom and evolution "times". As we have described above the fermionic degrees of freedom correspond to the noncompact branes localized on the Riemann surface. The two-dimensional field theory corresponds to the reduction of the Kodaira-Spencer theory on the two-dimensional surface. The coordinate on the Riemann surface is related with the coordinate on the moduli space \( M_{0,4} \). The Kodaira- Spencer theory is described by the two dimensional Lagrangian
\[
L_{KS} = \int (\partial \phi \bar{\partial} \phi + \frac{1}{\lambda} \omega \bar{\partial} \phi + \frac{\lambda}{\omega} (\partial \phi)^{2} \bar{\partial} \phi)
\] (59)

where \( \phi \) is the basic scalar in KS theory \( \omega \) is one-form on the surface and \( \lambda \) is the topological string coupling constant. It was argued recently [76] that the cubic interaction term in the KS Lagrangian can be formulated as the screening operator in the two-dimensional conformal theory. The fields on the surface are in the external
abelian connection of the Berry type which tells how the B-branes transform under the change of the complex structure fixed by the momenta of external particles.

As we have mentioned in the $c = 1$ example there are two possible set of "times", "compact" and "noncompact" ones. The compact ones correspond to the variation of the complex structure at infinities and are responsible for the insertion of the vertex operators of the "tachyonic" degrees of freedom while the noncompact ones correspond to the insertions of the noncompact B-branes at the particular values of the cross-ratios. The gluon vertex operators in this framework correspond to the tachyonic vertex operators in $c = 1$ model. The set of Kontsevich times determined by the positions of the B-branes are defined by (12) where $z_i$ are the corresponding cross ratios.

The form of the Riemann surface $H(u, v) = 0$ dictates that there are three infinities and therefore we are dealing with the particular solution to 3-KP integrable system. To describe the integrable system it is convenient to introduce the chiral fermions with the following mode expansion

$$\psi(x_i) = \sum_n \psi_{n+1/2}^{-n-1}, \quad \psi^*(x_i) = \sum_n \psi_{n+1/2}^{-n-1}$$

around the $i$-th infinity, $i = 1, 2, 3$, and the commutation relations

$$\{\psi_i^n, \psi_m^j\} = \delta^{ij} \delta_{n+m,0}$$

Defining the vacuum state by relations

$$\psi_n|0> = 0, \quad \psi_n^*|0> = 0, \quad n > 0$$

the generic state $|V>$ can be presented in the form

$$|V> = \exp(\sum_{i,j} \sum_{n,m} a_{nm}^{ij} \psi_{n-1/2}^{i} \psi_{m-1/2}^{j})|0>$$

where the point of Grassmanian representing the topological vertex was derived in [66]. for instance the diagonal coefficients read as

$$a_{nm}^{ii} = (-1)^n \frac{q^{m(m+1)-n(n+1)}}{[m+n+1][m][n]}$$

The tau-function of the 3-KP system plays the role of the generating function for the MHV amplitudes. In the semiclassical approximation we can safely consider the differential on the classical Riemann surface

$$dS = v du$$

which yields the semiclassical brane wave function

$$\Psi_{qs} \propto \exp(-\hbar^{-1} \int^x v(u) du)$$
involving the dilogarithms. The tau-function obeys the 3-KP equation and there are the additional $W_{1+\infty}$ Ward identity written in terms of the fermions

$$\oint_u \psi^*(u)e^{nu}\psi(u) + (-1)^n \oint_v \psi^*(v)e^{nv}\psi(v) + \oint_s \psi^*(s)e^{ns}\psi(s) = 0$$  \hspace{1cm} (67)

where the sum over three asymptotic regions is considered.

The quantization of the system can be done most effectively in terms of the Baxter equation. The Baxter equation implies that the regulator branes are localized on the surface. Hence the whole set of the equations determining amplitudes involves the dual conformal transformations on the regulator worldvolume and the set of Ward identities for the coordinate of regulator brane in the transverse moduli space. It is these Ward identities which fix the dependence of the amplitude on the conformal invariants for large number of external legs.

The precise form of higher Hamiltonians from $W_{1+\infty}$ responsible for the higher conservation laws in the scattering amplitude problem can be written as the fermionic bilinears \cite{41}. Generically as was discussed in \cite{41} one has some unbroken part of $W_\infty$ which annihilates the $\tau$-function corresponding to the topological vertex and therefore the scattering amplitude in the form of BDS-like anzatz.

### 6.3 On the Faddeev-Volkov model

Let us turn now to the description of the second integrable system representing the particular solitonic sector of the infinite-dimensional integrable system. We shall conjecture that the integrable system at the generic kinematics is the generalization of the $SL(2,C)$ spin chain relevant for the Regge limit of the amplitudes.

The finite-dimensional integrable systems can be usually defined in terms of the R-matrix. The Faddeev-Volkov model is defined via the Drinfeld solution to the Yang-Baxter equation which provides the universal R-matrix acting on $U_q(SL(2,R)) \otimes U_{\bar{q}}(SL(2,R))$. The corresponding statistical model describes the discrete quantum Liouville theory \cite{75} with the following partition function

$$Z = \int \prod_{ij} W_{p-q}(S_i - S_j) \prod_{kl} \tilde{W}_{p-q}(S_k - S_l) \prod_i dS_i$$  \hspace{1cm} (68)

where the Boltzmann weights depend only on the differences of the spins $S_k$ at the neighbor cites and rapidity variables at the ends of the edge. The first product is over the horizontal edges $(i,j)$ while the second product is over the vertical edges $(k,l)$. The integral is over all internal spin degrees of freedom. In the fundamental R-matrix the cross-ratios of the relative rapidities of the particles play the role of the local inhomogeneities in the lattice model and Boltzmann weights are defined as \cite{75}

$$W_\theta(s) = F(\theta)^{-1}e^{2i\eta \theta s} \frac{\Psi(s + ic_\theta \theta \pi)}{\Psi(s - ic_\theta \theta \pi)}$$  \hspace{1cm} (69)
where spin and local rapidity variables are combined together in the argument of the function $\Psi$ and $F(\theta)$ is some normalization factor. The relative importance of the spin variables and the local inhomogeneities depends on the value of the YM coupling constant and the kinematical region.

Semiclassically when $b \to 0$ the spin variables are frozen and the Boltzmann weight behaves as

$$W_\theta(\rho/2\pi b) = \exp(-\frac{A(\theta|\rho)}{2\pi b^2} + ...)$$

where

$$A(\theta|\rho) = iLi_2(-e^{\rho+i\theta}) - iLi_2(-e^{\rho-i\theta})$$

The extremization of the semiclassical action yields the Bethe Anzatz type equations connecting the dynamical spin variables with the local rapidities

$$\prod_i e^{\rho_i} + e^{\rho_i+\theta_{ij}} e^{\rho_j} + e^{\rho_j+\theta_{ij}} = 1$$

Let us try to compare the brane geometry behind two integrable systems behind the low-energy $N = 2$ SYM and in the $N = 4$ scattering problem. In $N = 2$ case in the IIA picture we have $N_c$ D4 branes stretched between two NS5 branes and coordinates of D4 branes on the NS5 brane correspond to the vacuum expectation values of the scalars. The second set of degrees of freedom is provided by the low-dimensional branes on D4 branes with attached strings connecting D4 branes. The first ”fast” integrable system describes the dynamics of the strings while the second ”slow” integrable system describes the dynamics of D4 branes on the moduli space of the vacua. Very similarly in the scattering geometry the D4 branes are substituted by the B-model branes localized on the moduli space $M_{0,4}$ and their dynamics is described by the Whitham-type 3-KP hierarchy while the second integrable system with $N$ degrees of freedom corresponds to the dynamics of the open strings attached to the B- branes.

Completing this Section let us make comment on how the interplay between ”soft” and ”regulator” degrees of freedom is captured by the integrable dynamics. To this aim remind the description of the KdV hierarchy in terms of the Liouville field. The KdV equation can be considered as the rotator on the coadjoint orbit of the Virasoro algebra. The coadjoint orbit is the symplectic manifold and the geometrical action on the Virasoro orbit is the Liouville action [78]. It can be derived upon integration of the chiral fermion in the external gravitational field. On the other hand the KdV Hamiltonians are provided by the integration of the ”heavy” non-relativistic degree of freedom in the same gravitational field that is $\log \det(d^2 + T)$, where $T$ is the two-dimensional energy stress tensor. In the Lax representation we consider the isospectral evolution of the Baker-Akhiezer function which is the eigenfunction of the Schrodinger operator and can be attributed to the ”heavy” degree of freedom.
This is similar to our case since the wave functions of the noncompact branes can be considered as the BA functions of the integrable system.

7 Comments on the Regge limit

In this Section we shall discuss some features which hopefully could help in the explanation of the Reggeization of the amplitude. The interpretation of the Reggeon in the dual picture was discussed in [52] where its identification as the singleton representation in \(AdS_3\) was suggested and the universality class of the multireggeon system was clarified. The dual picture behind the pomeron state was discussed in [83]. We shall conjecture on the interpretation of the reggeon degrees of freedom in the KS gravity framework.

Remind that the Reggeon field \(V(x)\) enters the effective Lagrangian being coupled to the semi-infinite light-like Wilson line [53] playing the role of the source

\[
L_{\text{int}} = -\frac{1}{g} \partial_+ P \exp(-\frac{g}{2} \int_{-\infty}^{x^+} A_+ dx_-) \partial^2 V_- - \frac{1}{g} \partial_- P \exp(-\frac{g}{2} \int_{-\infty}^{x^-} A_- dx_+) \partial^2 V_+ \quad (73)
\]

where \(x_+, x_-\) are the light-cone coordinates and \(A\) is the conventional gluon field. That is according to the gauge/string duality it is natural to lift the reggeon field to the field in the bulk. Hence the correlator of the light-like Wilson lines could be derived by differentiation of the bulk reggeon action with respect to the boundary values of the reggeon field. It is this line of reasoning was implied in [52] when interpreting the Reggeon as the singleton in the bulk action.

The reggeon field does not transform under the local gauge transformation however carries the global color charge and therefore interacts with the conventional gluon. It is this interaction amounts to the BFKL hamiltonian governing the \(t = \log s\) evolution of the pomeron state [86] and the corresponding multireggeon BKP generalization [87]. Note that the situation is reminiscent to the standard interplay between the local and global symmetries in the brane picture. In the color \(N_c\) branes worldvolume theory the gauge symmetry on the flavor \(N_F\) branes is seen as the global flavor symmetry. The open strings connecting the color and flavor branes carry the global flavor number.

We could conjecture that the reggeized gluon can be identified with the open string stretched between two regulator branes that is it can be considered as the massive vector ”gauge” particle for the ”flavor” gauge group on the set of the IR regulator branes. Such reggeon field indeed plays the role of the source on the regulator brane worldvolume and therefore the Wilson polygon on the regulator brane worldvolume presumably can be derived upon the differentiating the classical action in the bulk over the boundary values of the reggeon field.

The reggeized gluon emerges upon the re-summation of the perturbative series
hence one could try to identify the particular limit in the KS framework which could provide the multireggeon dynamics of the all-loop amplitude in the generic kinematics. To this aim it is useful to compare the integrable structures at the generic kinematics we discussed above and the one responsible for the Regge limit [1, 2].

The Regge limit is described in terms of the $SL(2,C)$ spin chains when the number of sites in the chain corresponds to the number of reggeons. The possible limit which could yield such spin chain from the Faddeev-Volkov model or statistical model [75] looks as follows. In the model [75] the statistical weights depend on the sum of the local rapidities and the spin variables. It is clear that one can not expect the semiclassical limit of the quantum dilogarithm to be relevant since the reggeization of the gluon happens upon the nontrivial resummation of the perturbation series.

Fortunately there is the limit [75] corresponding to the strong coupling region in the Liouville theory when the quantum dilogarithms reduce to the ratio of gamma functions depending on the $SL(2,R)$ spin variables

$$\Psi_{c_b→0}(s + \eta x) \propto \frac{\Gamma(1 - s + ix/2)}{\Gamma(1 - s - ix/2)} \quad (74)$$

where $|b| = 1$ and $x$ is the rescaled local rapidity. The leading argument depends on the difference of two infinite-dimensional representations in the neighbor sites and the expression coincides with the fundamental R-matrix involved into the $SL(2,R)$ spin chains. That is in this particular limit we get the statistical weights or R-matrixes depending only on the $SL(2,R)$ spins similar to the BFKL-type Hamiltonian while the local rapidity yields the "time" variable $\log s$. Note that clearly this suggestive argument needs for further clarification.

Another possible limit which can be compared with is the semiclassical limit of the multi-reggeon system which is described in terms of the higher genus Riemann surface of the type

$$y^2 = P_N^2(x) - 4x^{2N} \quad (75)$$

where $N$- is the number of Reggeons and $P_N$ is the N-th order polynomial depending on the higher integrals of motion. It was shown [52] that the Reggeon system belongs to the same universality class as $N=2$ SYM with $N_f = 2N_c$ at strong coupling orbifold point. In that case the brane geometry behind the low-energy effective action is known [5] and the theory is realized on the M5 brane with worldvolume $(R_{3,1}, \Sigma)$ where the surface $\Sigma$ lies in the internal space.

In the scattering geometry it is known [52] that the spectral curve of the integrable spin chain is embedded into the complexified $(x,p)$ space where $x$ is the coordinate in the conventional Minkowski space. On the other hand we have discussed the geometry in the internal space $(u,v)$ which can be roughly thought of as the $T^*M(0,4)$. The two viewpoints can be matched if we use the realization of the $T^*M(0,4)$ as the $T^*(M^c//T)$ hence we indeed can try to treat the spectral surface as the sub-manifold in the complexified phase space.
Some comments on the role of $SL(2,C)$ group is in order. It is just the group of the Lorentz rotations which act both in the coordinate and the momentum space. In the A model the set of Lagrangian branes yields the knots and there is the natural action of $SL(2,C)$ on the knot complement. That is the $SL(2,C)$ holonomies around the boundary torus yields the degrees of freedom in the spin chain model. In the B model side one can try to relate the group with the $SL(2,R)$ structure which has been considered within the lifting of KS theory on the Riemann surface to the three-dimensional CS theory. The derivation of the spin chain system from the set of Wilson lines in the CS theory has been discussed long time ago [89] and the similar derivation could be expected here as well.

8 Discussion

In this paper we have suggested the relation between the multiloop MHV amplitudes and effective gravity on the moduli spaces provided by the kinematical invariants of the scattering particles. This viewpoint allowed us to suggest the relevant integrability pattern and amplitudes were treated as the fermionic current correlators on the moduli spaces. The key idea is that the scattering process induces the back-reaction on the geometry of the "momentum space" through the nontrivial dynamics on the emerging moduli space. That is one can say that the tree amplitude is dressed by the effective gravitational degrees of freedom which can be treated within the Kahler gravity in the A type geometry or KS gravity in the type B model. They are identified with the coordinates of Lagrangian branes in the A model or the corresponding non-compact branes in the B model. These branes serve as the effective IR regulators in the theory. On the field theory side the four fermion currents correlator on the moduli space is identified with the two-mass easy box amplitude which is the basic block in the whole answer. Within the conventional calculation of the Feynman diagrams the relevant moduli spaces are parameterized by the Schwinger or Feynman parameters.

The BDS anzatz corresponds to the semiclassical limit in the effective gravity and $\Gamma_{\text{cusp}}$ has to be identified with the effective inverse Planck constant in KS gravity. The anzatz has to be modified and our proposal suggests several natural directions of its generalizations. First, one could imagine that the quantization parameter can be generalized to more complicated function than the cusp anomalous dimension which would respect the S-duality of $N = 4$ theory. The next evident point concerns the full quantum theory in the gravity framework which effectively substitutes the dilogarithm function in the BDS anzatz by the quantum dilogarithm. However these modifications do not produce proper higher polylogarithms which are known to appear in higher loop calculations of the amplitudes and Wilson polygons. The most natural way to get desired higher polylogarithms in our picture is to take into account the nontrivial Feynman diagrams in the two-dimensional KS theory probably involving loops. Indeed increasing the number of vertexes in the KS tree diagrams we increase
the transcendentality of the answer. We expect that all mentioned generalizations are necessary to be taken into account to get the correct all-loop answer.

We have identified the most natural integrable structure behind the scattering amplitudes which are considered as a kind of the "wave functions" in the particular model. The KS gravity in our case naturally involves the 3-KP hierarchy and the roles of the "time" variables are played by the combination of the conformal cross-ratios expressed in terms of the external momenta. The second finite-dimensional integrable system is conjectured to be related to the Faddeev-Volkov model however this point deserves for further investigation. The integrability is responsible for the conservation laws in addition to the dual superconformal symmetry. The relevant Ward identities correspond to the area preserving symplectomorphisms of the spectral curve similar the considerations discussed previously in $c = 1$ model.

The additional IR regulator branes added into the picture are responsible for the blow up of the internal momentum space in the manner dictated by the scattering process. The blow up of the internal geometry physically corresponds to the IR regularization of the field theory and the anomaly in the transformations in the momentum space tells that the regularization does not decouple completely. This a little bit surprising picture implies that we have to take into account the dynamics of the regulator degrees of freedom as well. It is highly desirable to develop the microscopic derivation of these IR branes. One can imagine that such branes emerge upon the peculiar summation of the noncommutative instantons in the effective abelian target space description of our simplified $C^3$ geometry.

Naively IR regulators are treated semiclassically but generically the fermion currents representing the regulator branes obey the quantum Baxter equation. It is clear that the discrete Liouville model plays the important role in the whole picture providing the particular gravitational dressing of the operators involved. We expect that these discrete Liouville modes correspond to the remnant of the reparametrization of the Wilson polygons coming from the cusps. The regulator branes to some extend play the role similar to the Liouville walls in the $c = 1$ model.

We expect that our treatment of the scalar box function imply that the hidden structure behind the gauge box diagrams holds for the non-MHV case as well. In particular we expect that non-vanishing all-plus amplitude in the usual YM theory which anticipated to be of the anomalous nature since long corresponds to the purely anomalous part of the algebra of the symplectomorphisms of the spectral curve.

One of the most inspiring findings of the paper is the appearance of the hidden "new massive degree of freedom". They correspond on the A model side to the D2 brane wrapped around the 2-cycle created by the scattering states or the open string stretched between two IR regulator branes in the B model. It is somewhat similar to the W-boson or monopole states in the Seiberg-Witten description of low-energy effective action of $N = 2$ theory however its "mass" is fixed by the kinematical invariants of the scattering particles. It would be very interesting to develop these
reasoning further and determine the corresponding walls of marginal stability in the space of the kinematical invariants. In the Regge limit we anticipate its important role in the Reggeon field theory. We plan to elaborate this issue further elsewhere.

It is evident that our proposal requires clarifications in many respects. In particular the clear understanding of the amplitudes of the gluon scattering with generic chiralities is absent yet and our conjecture for the improvement of the BDS anzatz deserves for further evidences. Nevertheless we believe that the dual representation of amplitudes in terms of the dynamical systems on the moduli space of the regulator branes we have suggested is the useful step towards the derivation of the dual geometry responsible for the summation of the perturbative series in SYM theory.

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Appendix

To discuss the multi-loop calculations it is useful to utilize the geometrical picture behind the one-loop calculations which we shall review following [19]. There exists the explicit map of the box diagram to the hyperbolic volume of the particular simplex build from the kinematical invariants of the external momenta. Introduce the Feynman parametrization of the internal generically massive propagators with the parameters $\alpha_i$. If one considers the one-loop N-point function with the external momenta $p_i$ in D space-time dimensions it can be brought into the usual form

$$J(D, p_1, \ldots p_N) \propto \int_0^1 \ldots \int_0^1 \prod d\alpha_i \delta(\sum \alpha_i - 1) \left| \sum \alpha_i^2 m_i^2 + \sum_{i<j} \alpha_i \alpha_j m_j m_i C_{ij} \right|^{D/2-N}$$

where

$$C_{jl} = \frac{m_i^2 + m_l^2 - k_{jl}^2}{2m_j m_i}, \quad k_{ij} = p_i - p_j$$

(76)

(77)
and \( m_i \) is the mass in the \( i \)-th propagator.

It is possible \([19]\) to organize for the generic one-loop diagram the \( N \) dimensional simplex defined as follows. First introduce the \( N \) mass vectors \( m_i, a_i \), where \( a_i \) are the unit vectors. The length of the side connecting the \( i \)-th and \( j \)-th mass vectors is \( \sqrt{k_{ij}} \) that is one can define the momentum side of the simplex. Therefore the \( N \)-dimensional simplex involves \( \frac{N(N+1)}{2} \) sides including \( N \) mass sides as well as \( \frac{N(N-1)}{2} \) momentum sides. At each vertex \( N \) sides meet and at all vertices but one there are one mass side and \( (N-1) \) momentum sides. The volume of such \( N \)-dimensional simplex is given as follows

\[
V^{(N)} = \frac{(\prod m_i) \sqrt{\text{det} C}}{N!} \quad (78)
\]

There are \( (N+1) \) hypersurfaces of dimension \( (N-1) \) one of which contains only momentum sides and can be related with the massless \( N \)-point function.

Upon the change of variables the loop integral get transformed into the following form

\[
J(D, p_1, \ldots, p_N) \propto \prod m_i^{-1} \int_0^\infty \ldots \int_0^\infty \prod d\alpha \delta(\alpha^TC\alpha - 1)(\sum \frac{\alpha_i}{m_i})^{N-D} \quad (79)
\]

It is useful to introduce the content of the \( N \)-dimensional solid angle \( \Omega^{(N)} \) subtended by the hypersurfaces at the mass meeting point. It turns out that \( \Omega^{(N)} \) coincides with the content of the \( (N-1) \) dimensional simplex in the hyperbolic space whose sides are equal to the hyperbolic angles \( \tau_{ij} \) defined at small masses as follows

\[
C_{ij} = \cosh\tau_{ij} \quad (80)
\]

Then the integral for the case \( D = N \) acquires the following form

\[
J(N, p_1, \ldots, p_N) = i^{1-2N} \frac{\pi^{N/2} \Gamma(N/2)\Omega^{(N)}}{N!V^{(N)}} \quad (81)
\]

hence the calculation of the Feynman integral is nothing but the calculation of the hyperbolic volume in the proper space. The case \( N \neq D \) can be treated similarly with some modification \([19]\).

To avoid IR divergence it is useful to start with the box diagram with all off-shell particles that is \( D = N \) simplices in the hyperbolic space.

\[
J(4, p_1, p_2, p_3, p_4) = \frac{2i\pi^2 \Omega^{(4)}}{m_1m_2m_3m_4 \sqrt{\text{det} C}} \quad (82)
\]

Since all internal propagators are massless in our case we get the ideal hyperbolic tetrahedron whose all vertices are at infinity. In the massless limit we get

\[
(m_i^2m_j^2m_k^2m_l^2\text{det} C)_{m_i \to 0} = \frac{1}{16} \lambda(k_{12}^2k_{34}^2, k_{13}^2k_{24}^2, k_{14}^2k_{23}^2) \quad (83)
\]
where the Källen function $\lambda(x, y, z)$ is defined as
\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx
\] (84)

and $\sqrt{-\lambda}$ is just the area of the triangle with sides $\sqrt{k_{12}^2 k_{34}^2}$, $\sqrt{k_{23}^2 k_{24}^2}$, $\sqrt{k_{31}^2 k_{23}^2}$. The hyperbolic volume of the ideal tetrahedron under consideration reads as
\[
2i\Omega^{(4)} = Cl_2(\psi_{12}) + Cl_2(\psi_{13}) + Cl_2(\psi_{23})
\] (85)

where the dihedral angles are defined via the kinematical invariants
\[
-cos\psi_{12} = \frac{k_{13}^2 k_{24}^2 + k_{14}^2 k_{23}^2 - k_{12}^2 k_{34}^2}{\sqrt{k_{13}^2 k_{23}^2 k_{14}^2 k_{43}^2}}
\] (86)
\[
-cos\psi_{13} = \frac{k_{14}^2 k_{23}^2 + k_{12}^2 k_{43}^2 - k_{13}^2 k_{24}^2}{\sqrt{k_{14}^2 k_{23}^2 k_{12}^2 k_{43}^2}}
\] (87)
\[
-cos\psi_{14} = \frac{k_{12}^2 k_{34}^2 + k_{13}^2 k_{24}^2 - k_{14}^2 k_{32}^2}{\sqrt{k_{12}^2 k_{34}^2 k_{13}^2 k_{24}^2}}
\] (88)

and $\psi_{12} = \psi_{34}$, $\psi_{13} = \psi_{24}$, $\psi_{14} = \psi_{32}$. The functions involved are defined as
\[
Cl_2(x) = Im[Li_2(e^{ix})] = -\int_0^x dy ln|2siny/2|
\] (89)

In the case of the two mass-easy box diagram defining the one-loop MHV amplitude the additional simplification of the kinematical invariants happens since two external particles are on the mass shell. In this case the arguments of the $Li_2$ function degenerate to the conformal ratios of four points.

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