On the Calculation of \( \text{gl.dim} G^N(A) \) and \( \text{gl.dim} \tilde{A} \) by Using Gröbner Bases

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Abstract. Let \( A = K \langle X_1, \ldots, X_n \rangle / \langle G \rangle \) be a \( K \)-algebra defined by a finite Gröbner basis \( G \). It is shown how to use the Ufnarovski graph \( \Gamma(\text{LM}(G)) \) and the graph of \( n \)-chains \( \Gamma_C(\text{LM}(G)) \) to calculate \( \text{gl.dim} G^N(A) \) and \( \text{gl.dim} \tilde{A} \), where \( G^N(A) \), respectively \( \tilde{A} \), is the associated \( N \)-graded algebra of \( A \), respectively the Rees algebra of \( A \) with respect to the \( N \)-filtration \( FA \) of \( A \) induced by a weight \( \mathbb{N} \)-grading filtration of \( K \langle X_1, \ldots, X_n \rangle \).

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Let \( K \langle X \rangle = K \langle X_1, \ldots, X_n \rangle \) be the free algebra generated by \( X = \{X_1, \ldots, X_n \} \) over a field \( K \), \( I \) an arbitrary (two-sided) ideal of \( K \langle X \rangle \), and \( A = K \langle X \rangle / I \) the corresponding quotient algebra. Fixing a positive weight \( \mathbb{N} \)-gradation \( \{K \langle X \rangle_p \}_{p \in \mathbb{N}} \) for \( K \langle X \rangle \) by assigning to each \( X_i \) a positive degree \( n_i \), \( 1 \leq i \leq n \), so that \( K \langle X \rangle = \bigoplus_{p \in \mathbb{N}} K \langle X \rangle_p \), and considering the \( \mathbb{N} \)-filtration \( FA \) of \( A \) induced by the weight \( \mathbb{N} \)-grading filtration \( FK \langle X \rangle = \{F_p K \langle X \rangle = \bigoplus_{i \leq p} K \langle X \rangle_i \}_{p \in \mathbb{N}} \) of \( K \langle X \rangle \), then it is well-known that \( FA \) determines two \( \mathbb{N} \)-graded \( K \)-algebras, namely the associated \( \mathbb{N} \)-graded algebra \( G^N(A) = \bigoplus_{p \in \mathbb{N}} (F_p A / F_{p-1} A) \) of \( A \) and the Rees algebra \( \tilde{A} = \bigoplus_{p \in \mathbb{N}} F_p A \) of \( A \), both are intimately related to the structure theory of \( A \). Let \( \preceq_{gr} \) be an \( \mathbb{N} \)-graded monomial ordering on the standard \( K \)-basis \( B \) of \( K \langle X \rangle \), and let \( \langle \text{LM}(I) \rangle \) be the monomial ideal of \( K \langle X \rangle \) generated by the set \( \text{LM}(I) \) of leading monomials of \( I \) with respect to \( \preceq_{gr} \). If \( \Omega \) is the unique reduced monomial generating set of \( \langle \text{LM}(I) \rangle \) such that the graph \( \Gamma_C(\Omega) \) of \( n \)-chains (in the sense of [1], [11]) does not contain any \( d \)-chains, then, based on ([2], Theorem 4), it was proved in [7],

\[ \text{gl.dim} G^N(A) \]

\[ \text{gl.dim} \tilde{A} \]
without any extra assumption, that
\[ \text{gl.dim} A \leq \text{gl.dim} G^N(A) \leq \text{gl.dim}(K \langle X \rangle / \langle \text{LM}(I) \rangle) \leq d, \]
\[ \text{gl.dim} \tilde{A} \leq d + 1, \]
where gl.dim abbreviates the phrase “global dimension”. This note aims to strengthen the above results, that is, firstly we will show further that if \( K \langle X \rangle / \langle \text{LM}(I) \rangle \) has the polynomial growth of degree \( m \), then

(i) the following two equalities hold:
\[ \text{gl.dim} G^N(A) = \text{gl.dim}(K \langle X \rangle / \langle \text{LM}(I) \rangle) = m, \quad \text{gl.dim} \tilde{A} = m + 1; \]

(ii) \( I \) is generated by a finite Gröbner basis \( \mathcal{G} \), and consequently both \( G^N(A) \) and \( \tilde{A} \) are defined by finite Gröbner bases;

and secondly, we demonstrate, by examining interesting examples, that bringing the problem of determining (i) above down-to-earth, if we start with (ii), i.e., with a finite Gröbner basis for \( I \), then an effective solution to (i) may be achieved. Moreover, the last example given in section 3 will indicate in passing that under the assumption of ([2], Theorem 6), the Hilbert series of a finitely presented monomial algebra does not always have the form \( \prod_{i=1}^{d} (1 - z^{e_i})^{-1} \) as asserted in loc. cit.

Throughout this paper we let \( K \langle X \rangle \) denote the free \( K \)-algebra \( K \langle X_1, \ldots, X_n \rangle \), and let \( \mathcal{B} \) be the standard \( K \)-basis of \( K \langle X \rangle \) consisting of words in the alphabet \( X = \{X_1, \ldots, X_n\} \) (including the empty word which gives the identity element 1). Unless otherwise stated, the \( \mathbb{N} \)-gradation of \( K \langle X \rangle \) means any positive weight \( \mathbb{N} \)-gradation of \( K \langle X \rangle \) by assigning to each \( X_i \) a positive degree \( n_i \), \( 1 \leq i \leq n \). Moreover, ideals mean two-sided ideals, and if \( M \subset K \langle X \rangle \), then we use \( \langle M \rangle \) to denote the ideal of \( K \langle X \rangle \) generated by \( M \). For a general theory on Gröbner bases in \( K \langle X \rangle \), we refer to [9].

1. Preliminaries

In this section we recall several well-known algorithmic results from [10], [2], and [5], that will be used in deriving the main results of this note.

Adopting the commonly used terminology in computational algebra, let us call elements in \( \mathcal{B} \) the monomials. Given a monomial ordering \( \prec \) on \( \mathcal{B} \), as usual we write \( \text{LM}(f) \) for the leading monomial of \( f \in K \langle X \rangle \); and if \( S \) is any subset of \( K \langle X \rangle \), then we write \( \text{LM}(S) \) for the set of leading monomials of \( S \), i.e., \( \text{LM}(S) = \{\text{LM}(f) \mid f \in S\} \). If \( \mathcal{G} \) is a Gröbner basis in \( K \langle X \rangle \) with respect to \( \prec \), then it is well-known that we may always assume that \( \mathcal{G} \) is \( LM \)-reduced, that is, \( g_1, g_2 \in \mathcal{G} \) and \( g_1 \neq g_2 \) implies \( \text{LM}(g_1) \not\div \text{LM}(g_2) \). Consequently, if \( \Omega \) is a subset of \( \mathcal{B} \) satisfying \( u_i \not\div u_j \) for all \( u_i, u_j \in \Omega \) with \( i \neq j \), then we just say simply that \( \Omega \) is reduced.
Let $\Omega = \{u_1, ..., u_s\}$ be a reduced finite subset of $B$. For each $u_i \in \Omega$, say $u_i = X_{i_1}^{\alpha_1} \cdots X_{i_s}^{\alpha_s}$ with $X_{i_j} \in X$ and $\alpha_j \in \mathbb{N}$, we write $l(u_i) = \alpha_1 + \cdots + \alpha_s$ for the length of $u_i$. Put

$$\ell = \max \{l(u_i) \mid u_i \in \Omega\}.$$ 

Then the Ufnarovski graph of $\Omega$ (introduced by V. Ufnarovski in [10]), denoted $\Gamma(\Omega)$, is defined as a directed graph, in which the set of vertices $V$ is given by

$$V = \{v_i \mid v_i \in B - \langle \Omega \rangle, \ l(v_i) = \ell - 1\},$$

and the set of edges $E$ contains the edge $v_i \rightarrow v_j$ if and only if there exist $X_i, X_j \in X$ such that $v_iX_i = X_jv_j \in B - \langle \Omega \rangle$. Thus, for an LM-reduced finite Gröbner basis $G = \{g_1, \ldots, g_s\}$ in $K\langle X \rangle$, the Ufnarovski graph of $G$ is defined to be the Ufnarovski graph $\Gamma(\text{LM}(G))$ of the reduced subset of monomials $\text{LM}(G) = \{\text{LM}(g_1), \ldots, \text{LM}(g_s)\}$.

**Remark** To better understand the practical application of $\Gamma(\Omega)$, it is essential to notice that the Ufnarovski graph is defined by using the length $l(u)$ of the monomial (word) $u \in B$ instead of using the degree of $u$ as an $\mathbb{N}$-homogeneous element in $K\langle X \rangle$, though both notions coincide when each $X_i$ is assigned to degree 1.

The first effective application of $\Gamma(\Omega)$ was made to determine the growth of the monomial algebra $K\langle X \rangle/\langle \Omega \rangle$.

**1.1. Theorem** ([10], 1982) Let $\Omega = \{u_1, ..., u_s\}$ be a reduced finite subset of $B$, and let $\Gamma(\Omega)$ be the Ufnarovski graph of $\Omega$ as defined above. Then the growth of $K\langle X \rangle/\langle \Omega \rangle$ is alternative. It is exponential (i.e., the Gelfand-Kirillov dimension of $K\langle X \rangle/\langle \Omega \rangle$ is $\infty$) if and only if there are two different cycles with a common vertex in the graph $\Gamma(\Omega)$; Otherwise, $K\langle X \rangle/\langle \Omega \rangle$ has the polynomial growth of degree $m$ (i.e., the Gelfand-Kirillov dimension of $K\langle X \rangle/\langle \Omega \rangle$ is equal to $m$), where $m$ is, among all routes of $\Gamma(\Omega)$, the largest number of distinct cycles occurring in a single route.

Let the free $K$-algebra $K\langle X \rangle = K\langle X_1, ..., X_n \rangle$ be equipped with the augmentation map $\varepsilon$ sending each $X_i$ to zero. For any ideal $J$ contained in the augmentation ideal $\langle X_1, ..., X_n \rangle$ (i.e., the kernel of $\varepsilon$), by using the $n$-chains determined by $\text{LM}(J)$, D. J. Anick constructed in [1] a free resolution of the trivial module $K$ over the quotient algebra $K\langle X \rangle/J$ which gave rise to several efficient applications to the homological aspects of associative algebras ([1], [2]). Let $\Omega \subset B$ be a reduced (finite or infinite) subset of monomials. Following [1] and [2], V. Ufnarovski constructed in [11] the graph of $n$-chains of $\Omega$ as a directed graph $\Gamma_C(\Omega)$, in which the set of vertices $V$ is defined as

$$V = \{1\} \cup X \cup \{\text{all proper suffixes of } u \in \Omega\},$$
and the set of edges $E$ consists of all edges

$$1 \rightarrow X_i \text{ for every } X_i \in X$$

and edges defined by the rule: for $u, v \in V - \{1\}$,

$$u \rightarrow v \text{ in } E \iff \text{there is a unique } w = X_{i_1} \cdots X_{i_{m-1}} X_{i_m} \in \Omega \text{ such that } uv = \{ w, \text{ or } sw \text{ with } s \in B, sX_{i_1} \cdots X_{i_{m-1}} \in B - \langle \Omega \rangle \}$$

For $n \geq -1$, an $n$-chain of $\Omega$ is a monomial (word) $v = v_1 \cdots v_n v_{n+1}$ given by a route of length $n + 1$ starting from 1 in $\Gamma_C(\Omega)$:

$$1 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n \rightarrow v_{n+1}$$

Writing $C_n$ for the set of all $n$-chains of $\Omega$, it is clear that $C_{-1} = \{1\}$, $C_0 = X$, and $C_1 = \Omega$.

For an LM-reduced (finite or infinite) Gröbner basis $G$ of $K\langle X \rangle$, $\Gamma_C(\text{LM}(G))$ is referred to as the graph of $n$-chains of $\Omega$.

**Remark** As with the Ufnarovski graph $\Gamma(\Omega)$ defined in section 2, to better understand the practical application of the graph $\Gamma_C(\Omega)$ of $n$-chains determined by $\Omega$ in the subsequent sections and the next chapter, it is essential to notice that an $n$-chain is defined by a route of length $n + 1$ starting with 1, as described above, instead of by the degree of the $\mathbb{N}$-homogeneous element $v = v_1 \cdots v_n v_{n+1}$ read out of that route.

**1.2. Theorem** ([2], Theorem 4) Let $\Omega \subset B$ be a reduced subset of monomials. Then $\text{gl.dim}(K\langle X \rangle / \langle \Omega \rangle) \leq m$ if and only if the graph $\Gamma_C(\Omega)$ of $n$-chains of $\Omega$ does not contain any $m$-chains.

□

**1.3. Theorem** ([2], Theorem 6) Let $\Omega$ be as in Theorem 1.2. Suppose $K\langle X \rangle / \langle \Omega \rangle$ has finite global dimension $m$. If $K\langle X \rangle / \langle \Omega \rangle$ does not contain a free subalgebra of two generators, then the following statements hold.

(i) $K\langle X \rangle / \langle \Omega \rangle$ is finitely presented, that is, $\langle \Omega \rangle$ is finitely generated.

(ii) $K\langle X \rangle / \langle \Omega \rangle$ has the polynomial growth of degree $m$.

(iii) The Hilbert series of $K\langle X \rangle / \langle \Omega \rangle$ is of the form $H_{K\langle X \rangle / \langle \Omega \rangle}(t) = \prod_{i=1}^{m} (1 - t^{e_i})^{-1}$, where each $e_i$ is a positive integer, $1 \leq i \leq m$.

□

By using the above theorem, the following result was derived by T. Gateva-Ivanova in [5].

**1.4. Theorem** ([5], Theorem II) Let $J$ be an $\mathbb{N}$-graded ideal of $K\langle X \rangle$ and $R = K\langle X \rangle / J$ the corresponding $\mathbb{N}$-graded algebra defined by $J$. Suppose that the associated monomial algebra
\( \overline{R} = K \langle X \rangle / \langle \text{LM}(J) \rangle \) of \( R \) has finite global dimension and the polynomial growth of degree \( m \), where \( \text{LM}(J) \) is taken with respect to a fixed \( \mathbb{N} \)-graded monomial ordering \( \prec_{\text{gr}} \) on \( B \) (see the definition in the next section). Then the following statements hold.

(i) \( \text{gl.dim} R = \text{gl.dim} \overline{R} = m \).

(ii) The ideal \( J \) has a finite Gröbner basis.

(iii) The Hilbert series of \( R \) is of the form \( H_R(t) = \prod_{i=1}^{m} (1 - t^{e_i})^{-1} \), where each \( e_i \) is a positive integer, \( 1 \leq i \leq n \).

\( \Box \)

2. The Main Results

In this section we show how to obtain the results of (i) – (ii) announced in the beginning of this note.

Concerning the first equality \( \text{gl.dim} G^\mathbb{N}(A) = \text{gl.dim}(K \langle X \rangle / \langle \text{LM}(I) \rangle) = m \), let us first recall some results from [8], [6], and [7]. Note that with respect to the fixed weight \( \mathbb{N} \)-gradation of \( K \langle X \rangle \), every element \( f \in K \langle X \rangle \) has a unique expression \( f = \sum_{p=1}^{p} F_p \) with \( F_p \neq 0 \), where each \( F_i \) is an \( \mathbb{N} \)-homogeneous element of degree \( i \) in \( K \langle X \rangle \), \( 1 \leq i \leq p \). We call \( F_p \) the \( \mathbb{N} \)-leading homogeneous element of \( f \), denoted \( \text{LH}(f) \), i.e., \( \text{LH}(f) = F_p \). For a subset \( S \subset K \langle X \rangle \), we then write \( \text{LH}(S) \) for the set of \( \mathbb{N} \)-leading homogeneous elements of \( S \), that is, \( \text{LH}(S) = \{ \text{LH}(f) \mid f \in S \} \).

2.1. Proposition ([8], Proposition 2.2.1; [7], Theorem 1.1) Let \( I \) be an arbitrary ideal of \( K \langle X \rangle \) and \( A = K \langle X \rangle / I \). Considering the \( \mathbb{N} \)-filtration \( FA \) of \( A \) induced by the weight \( \mathbb{N} \)-grading filtration \( FK \langle X \rangle \) of \( K \langle X \rangle \), if \( G^\mathbb{N}(A) \) is the associated \( \mathbb{N} \)-graded algebra of \( A \) determined by \( FA \), then there is an \( \mathbb{N} \)-graded \( K \)-algebra isomorphism

\[ K \langle X \rangle / \langle \text{LH}(I) \rangle \cong G^\mathbb{N}(A). \]

\( \Box \)

Also recall that any well-ordering \( \prec \) on \( B \) may be used to define a new ordering \( \prec_{\text{gr}} \): for \( u, v \in B \),

\[ u \prec_{\text{gr}} v \iff d(u) < d(v) \]

or \( d(u) = d(v) \) and \( u \prec v \),

where \( d(\ ) \) is referred to the degree function on homogeneous elements of \( K \langle X \rangle \). If \( \prec_{\text{gr}} \) is a monomial ordering on \( B \), then it is called an \( \mathbb{N} \)-graded monomial ordering. Typical \( \mathbb{N} \)-graded monomial ordering on \( B \) is the well-known \( \mathbb{N} \)-graded (reverse) lexicographic ordering.

2.2. Theorem ([8], Theorem 2.3.2; [7], Proposition 3.2.) Let \( I \) be an arbitrary ideal of \( K \langle X \rangle \) and \( G \subset I \). Then \( G \) is a Gröbner basis of \( I \) with respect to some \( \mathbb{N} \)-graded monomial ordering.
\(\prec\) on \(\mathcal{B}\) if and only if the set of \(\mathbb{N}\)-leading homogeneous elements \(\mathbf{LH}(\mathcal{G})\) of \(\mathcal{G}\) is a Gröbner basis for the \(\mathbb{N}\)-graded ideal \(\langle \mathbf{LH}(I) \rangle\) with respect to \(\prec\).

\[\Box\]

We are ready to obtain the first result of this section.

**2.3. Theorem** Let \(I\) be an arbitrary ideal of \(K\langle X\rangle\), \(A = K\langle X\rangle/I\), and \(\overline{A} = K\langle X\rangle/(\mathbf{LM}(I))\) the associated monomial algebra of \(A\) with respect to a fixed \(\mathbb{N}\)-graded monomial ordering \(\prec\) on \(\mathcal{B}\). Suppose that \(\text{gl.dim}\overline{A} < \infty\), and that \(\overline{A}\) has the polynomial growth of degree \(m\). With notation as before, the following statements hold.

(i) \(\text{gl.dim}\mathcal{G}^\mathbb{N}(A) = \text{gl.dim}\overline{A} = m\).

(ii) The ideal \(I\) has a finite Gröbner basis \(\mathcal{G}\), and \(\mathbf{LH}(\mathcal{G})\) is a finite homogeneous Gröbner basis in \(K\langle X\rangle\) such that \(\mathcal{G}^\mathbb{N}(A) \cong K\langle X\rangle/(\mathbf{LH}(\mathcal{G}))\).

**Proof** (i) Since we are using the \(\mathbb{N}\)-graded monomial ordering \(\prec\), it is straightforward that \(\mathbf{LM}(I) = \mathbf{LM}(\langle \mathbf{LH}(I) \rangle)\). Hence, both algebras \(A = K\langle X\rangle/I\) and \(K\langle X\rangle/(\mathbf{LH}(I))\) have the same associated monomial algebra \(\overline{A} = K\langle X\rangle/(\mathbf{LM}(I))\). So, by making use of the unique reduced monomial generating set \(\Omega\) of the monomial ideal \(\langle \mathbf{LM}(I) \rangle\), the equality in (i) follows from Theorem 1.3(ii), Theorem 1.4(i), and Proposition 2.1.

(ii) It is a well-known fact that if \(\langle \mathbf{LM}(I) \rangle\) is finitely generated, then \(I\) is generated by a finite Gröbner basis. That \(I\) has a finite Gröbner basis \(\mathcal{G}\) is guaranteed by Theorem 1.3(i). Thus, the second assertion of (ii) concerning \(\mathbf{LH}(\mathcal{G})\) follows from Proposition 2.1 and Theorem 2.2.

\[\Box\]

It remains to show that under the same assumption as in Theorem 2.3, the Rees algebra \(\overline{A}\) of \(A\) has the properties listed in the beginning of this paper. To this end, we need a little more preparation.

Let \(\prec\) be a fixed \(\mathbb{N}\)-graded monomial ordering on \(\mathcal{B}\) with respect to the positive weight \(\mathbb{N}\)-gradation of \(K\langle X\rangle\). If \(f \in K\langle X\rangle\) has the linear presentation by elements of \(\mathcal{B}\):

\[f = \mathbf{LC}(f)\mathbf{LM}(f) + \sum_i \lambda_i w_i \text{ with } \mathbf{LM}(f) \in \mathcal{B} \cap K\langle X\rangle_{p^i}, \; w_i \in \mathcal{B} \cap K\langle X\rangle_{q_i},\]

where \(\mathbf{LC}(f)\) is the leading coefficient of \(f\), then \(f\) corresponds to a unique homogeneous element in the free algebra \(K\langle X, T \rangle = K\langle X_1, \ldots, X_n, T \rangle\), i.e., the element

\[\overline{f} = \mathbf{LC}(f)\mathbf{LM}(f) + \sum_i \lambda_i T^{p^i q_i} w_i.\]

Assigning to \(T\) the degree 1 in \(K\langle X, T \rangle\) and using the fixed positive weight of \(X\) in \(K\langle X\rangle\), we get the weight \(\mathbb{N}\)-gradation of \(K\langle X, T \rangle\) which extends the weight \(\mathbb{N}\)-gradation of \(K\langle X\rangle\). Consequently, writing \(\mathcal{B}\) for the standard \(\mathbb{N}\)-basis of \(K\langle X, T \rangle\), we may extend \(\prec\) to an \(\mathbb{N}\)-graded monomial ordering \(\prec_{T-\mathcal{B}}\) on \(\mathcal{B}\) such that

\[T \prec_{T-\mathcal{B}} X_i, \; 1 \leq i \leq n,\]

and hence \(\mathbf{LM}(f) = \mathbf{LM}(\overline{f})\).
If \( I \) is an ideal of \( K\langle X \rangle \) generated by the subset \( S \), then we put
\[
\widetilde{I} = \{ f | f \in I \} \cup \{ X_i T - TX_i | 1 \leq i \leq n \},
\]
\[
\widetilde{S} = \{ f | f \in S \} \cup \{ X_i T - TX_i | 1 \leq i \leq n \}.
\]
The assertion of the next proposition was inferred in ([8], Theorem 2.3.1; [6], CH.III, Corollary 3.8) and a detailed proof was given in ([7], section 8)

2.4. Proposition With the preparation made above, let \( \mathcal{G} \) be a Gröbner basis of the ideal \( I \) in \( K\langle X \rangle \) with respect to \( \prec_{gr} \), and \( A = K\langle X \rangle / I \). Then \( \widetilde{\mathcal{G}} \) is a Gröbner basis for the \( \mathbb{N} \)-graded ideal \( \langle \tilde{I} \rangle \) in \( K\langle X, T \rangle \) with respect to \( \prec_{\tau_{\mathbb{N}gr}} \), and consequently \( \tilde{A} \cong K\langle X, T \rangle / \langle \tilde{I} \rangle = K\langle X, T \rangle / \langle \widetilde{\mathcal{G}} \rangle \), where \( \tilde{A} \) is the Rees algebra of \( A \) defined by the \( \mathbb{N} \)-filtration \( FA \) induced by the weight \( \mathbb{N} \)-grading filtration \( FK\langle X \rangle \) of \( K\langle X \rangle \).

\[ \Box \]

Now, the following result is established.

2.5. Theorem Let \( I \) be an arbitrary ideal of \( K\langle X \rangle \), \( A = K\langle X \rangle / I \), and \( \overline{A} = K\langle X \rangle / (\text{LM}(I)) \) the associated monomial algebra of \( A \) with respect to a fixed \( \mathbb{N} \)-graded monomial ordering \( \prec_{gr} \) on \( B \). Suppose that \( \text{gl.dim} \overline{A} < \infty \), and that \( \overline{A} \) has the polynomial growth of degree \( m \). With notation as before, the following statements hold.

(i) \( \text{gl.dim} \overline{A} = m + 1 \).

(ii) The ideal \( I \) has a finite Gröbner basis \( \mathcal{G} \), and \( \widetilde{\mathcal{G}} \) is a finite homogeneous Gröbner basis of \( K\langle X, T \rangle \) such that \( \tilde{A} \cong K\langle X, T \rangle / (\widetilde{\mathcal{G}}) \).

Proof By the assumption, \( \overline{A} \) has Gelfand-Kirillov dimension \( m \). Hence \( A \) has Gelfand-Kirillov dimension \( m \). It follows from ([7], Theorem 8.3(i)) that \( \tilde{A} \) has Gelfand-Kirillov dimension \( m + 1 \). But by the assumption and Theorem 2.3, \( I \) has a finite Gröbner basis \( \mathcal{G} \). So by Proposition 2.4, \( \widetilde{\mathcal{G}} \) is a finite Gröbner basis for the ideal \( \langle \tilde{I} \rangle \) and \( \tilde{A} \cong K\langle X, T \rangle / (\widetilde{\mathcal{G}}) \). Thus, \( \tilde{A} \) must have the polynomial growth of degree \( m + 1 \) by Theorem 1.1. Therefore, the associated monomial algebra \( K\langle X, T \rangle / (\text{LM}(\widetilde{\mathcal{G}})) \) of \( \tilde{A} \) has the polynomial growth of degree \( m + 1 \). In order to finish the proof, by Theorem 1.4, it remains to show that \( K\langle X, T \rangle / (\text{LM}(\widetilde{\mathcal{G}})) \) has finite global dimension. To see this clearly, we quote the argument from the proof of ([7], Theorem 8.5) as follows.

Note that by the definition of \( \prec_{\tau_{\mathbb{N}gr}} \), we have
\[
\text{LM}(\widetilde{\mathcal{G}}) = \{ \text{LM}(g), X_i T | g \in \mathcal{G}, 1 \leq i \leq n \}.
\]

Thus, it is easy to see that the graph of \( n \)-chains \( \Gamma_C(\text{LM}(\mathcal{G})) \) of \( \mathcal{G} \) is a subgraph of the graph of \( n \)-chains \( \Gamma_C(\text{LM}(\widetilde{\mathcal{G}})) \) of \( \widetilde{\mathcal{G}} \), and that

(a) the graph \( \Gamma_C(\text{LM}(\widetilde{\mathcal{G}})) \) of \( \widetilde{\mathcal{G}} \) has no edge of the form \( T \to v \) for all \( v \in \widetilde{V} \), where \( \widetilde{V} \) is the set of vertices of \( \Gamma_C(\text{LM}(\widetilde{\mathcal{G}})) \);
(b) if \( v \in \tilde{V} \) is of the form \( v = sX_j, s \in B \), then \( \Gamma_C(\text{LM}(\tilde{G})) \) contains the edge \( v \to T \);
(c) any \( d + 1 \)-chain in \( \Gamma_C(\text{LM}(\tilde{G})) \) is of the form

\[
1 \to X_i \to v_1 \to v_2 \to \cdots \to v_{d-1} \to T,
\]

where

\[
1 \to X_i \to v_1 \to v_2 \to \cdots \to v_{d-1}
\]
is a \( d \)-chain in \( \Gamma_C(\text{LM}(G)) \).

Therefore, if the graph \( \Gamma_C(\text{LM}(G)) \) does not contain any \( d \)-chain, then \( \Gamma_C(\text{LM}(\tilde{G})) \) does not contain any \( d+1 \)-chain. Hence, if \( \text{gl.dim}A = K\langle X \rangle /\langle \text{LM}(I) \rangle < \infty \), then \( \text{gl.dim}K\langle X, T \rangle /\langle \text{LM}(\tilde{G}) \rangle < \infty \) by Theorem 1.2, as desired. \( \square \)

3. Examples of Calculating \( \text{gl.dim}G^N(A) \) and \( \text{gl.dim}\tilde{A} \)

Let \( I, A = K\langle X \rangle /I, \tilde{A} = K\langle X \rangle /\langle \text{LM}(I) \rangle, G^N(A), \) and \( \tilde{A} \) be as in section 2. Combining Theorem 1.1, Theorem 1.2, Proposition 2.1 and Proposition 2.4, it is now clear that if, with respect to a fixed \( \mathbb{N} \)-graded monomial ordering \( \prec_{gr} \) on \( B \), we start with a finite Gröbner basis \( G = \{ g_1, \ldots, g_s \} \) for the ideal \( I \), then the equalities of Theorem 2.3(i) and Theorem 2.5(i) may be determined in a computational way:

1. Determine whether \( \tilde{A} \) has polynomial growth by checking the Ufnarovski graph \( \Gamma(\text{LM}(G)) \); if \( \tilde{A} \) has polynomial growth, then the degree \( m \) is read out of the graph simultaneously.
2. Determine whether \( \tilde{A} \) has finite global dimension by checking the set \( C_n \) of \( n \)-chains in the graph \( \Gamma_C(\text{LM}(G)) \); if \( C_d = \emptyset \) for some \( d \), then \( \text{gl.dim}A \leq d \).
3. If both (1) and (2) have a positive result, i.e., \( \tilde{A} \) has the polynomial growth of degree \( m \) and \( \text{gl.dim}A < \infty \), then immediately we can write down the following:

\[
\text{gl.dim}G^N(A) = m, \quad \text{gl.dim}\tilde{A} = m + 1.
\]

Let us point out incidentally that when the above (2) is done, the Hilbert series for both \( G^N(A) \) and \( \tilde{A} \) may also be written down just by using the \( n \)-chains in \( \Gamma_C(\text{LM}(G)) \), respectively the \( n \)-chains in \( \Gamma_C(\text{LM}(\tilde{G})) \) as described in the proof of Theorem 2.5, and ([1], formula 16), that is,

\[
H_{G^N(A)}(t) = \left( 1 - \sum_{i=0} (-1)^i H_{C_i}(t) \right)^{-1}, \quad H_{\tilde{A}}(t) = \left( 1 - \sum_{i=0} (-1)^i H_{\tilde{C}_i}(t) \right)^{-1},
\]

where \( H_{C_i}(t) \) denotes the Hilbert series of the \( \mathbb{N} \)-graded \( K \)-module spanned by the set \( C_i \) of \( i \)-chains in \( \Gamma_C(\text{LM}(G)) \), and similarly, \( H_{\tilde{C}_i}(t) \) denotes the Hilbert series of the \( \mathbb{N} \)-graded \( K \)-module spanned by the set \( \tilde{C}_i \) of \( i \)-chains in \( \Gamma_C(\text{LM}(\tilde{G})) \).

We illustrate the computational procedure mentioned above by examining several examples. Notations are maintained as before.
Let \( \Omega = \{ X_j X_i \mid 1 \leq i < j \leq n \} \subset B \subset K(X) \). Then by ([2], Example 3), \( \text{gr.dim}(K(X) / \langle \Omega \rangle) = n \) is the degree of the polynomial growth of \( K(X) / \langle \Omega \rangle \). Note that \( B - \langle \Omega \rangle = \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N} \} \), which gives rise to a PBW \( K \)-basis for \( K(X) / \langle \Omega \rangle \). Enlightened by this fact, our first example will be the algebra \( A = K(X) / I \), which, with respect to a fixed monomial ordering \( \prec \) on \( B \), has the property that \( B - \langle \Omega \rangle = \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N} \} \) yields a PBW \( K \)-basis for both \( A \) and \( K(X) / \langle \Omega \rangle \).

Below let us describe first the ideal \( I \) by Gröbner basis (probably a known result but the author has lack of a proper reference).

3.1. Proposition Let \( I \) be an ideal of \( K(X) \) and \( A = K(X) / I \). The following two statements are equivalent with respect to a fixed monomial ordering \( \prec \) on \( B \):

(i) \( B - \langle \Omega \rangle = \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N} \} \) and hence \( A \) has the PBW \( K \)-basis \( \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N} \} \), where each \( X_i \) is the image of \( X_i \) in \( A \);

(ii) \( I \) is generated by a reduced Gröbner basis of the form

\[
G = \left\{ R_{ji} = X_j X_i - F_{ji} \mid F_{ji} \in K(X), \ 1 \leq i < j \leq n \right\}
\]

satisfying \( \text{LM}(R_{ji}) = X_j X_i, \ 1 \leq i < j \leq n \), and

\[
F_{ji} = \sum_{p=1}^{m} \lambda_p w_p \text{ with } \lambda_p \in K^*, \ w_p \in \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N} \}.
\]

Proof (i) \( \Rightarrow \) (ii) First, we show that under the assumption of (i) \( I \) has a finite Gröbner basis \( G = \{ R_{ji} \mid 1 \leq i < j \leq n \} \) of the described form such that \( \text{LM}(R_{ji}) = X_j X_i \). By classical Gröbner basis theory, it is sufficient to prove that the reduced monomial generating set \( \Omega \) of \( \langle \text{LM}(I) \rangle \) is of the form \( \Omega = \{ X_j X_i \mid 1 \leq i < j \leq n \} \). To see this, recall that

\[
\Omega = \{ w \in \text{LM}(I) \mid \text{if } u \in \text{LM}(I) \text{ and } u | w \text{ then } u = w \},
\]

and consequently \( B - \langle \Omega \rangle \) is obtained by the division by \( \Omega \). Since \( X_i \in B - \langle \Omega \rangle \), and \( X_j X_i \notin B - \langle \Omega \rangle \), it follows that \( X_j X_i \notin \Omega \), \( 1 \leq i < j \leq n \). Furthermore, noticing the feature of monomials in \( B - \langle \Omega \rangle \), it is clear that the only monomials of length 2 contained in \( \Omega \) are \( X_j X_i, \ 1 \leq i < j \leq n \), and that \( \Omega \) cannot contain monomials of length \( \geq 3 \). Therefore, \( \Omega \) has the form as we claimed above, and \( \Omega \) determines a Gröbner basis \( G = \{ R_{ji} \mid 1 \leq i < j \leq n \} \) with \( \text{LM}(R_{ji}) = X_j X_i \). It follows from classical Gröbner basis theory that \( G \) can be reduced further to a Gröbner basis such that \( F_{ji} = R_{ji} - \text{LM}(R_{ji}) \) is a normal element, which is clearly a linear combination of monomials in \( \{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_j \in \mathbb{N} \} \), but this means actually that \( G \) is a reduced Gröbner basis.

(ii) \( \Rightarrow \) (i) If \( G \) is a Gröbner basis of \( I \) as described, then since \( \langle \text{LM}(I) \rangle = \langle \text{LM}(G) \rangle \) and \( \text{LM}(R_{ji}) = X_j X_i, \ 1 \leq i < j \leq n \), the division by \( \text{LM}(G) \) yields

\[
B - \langle \text{LM}(I) \rangle = \left\{ X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N} \right\},
\]

as desired. \( \square \)
Remark The last proposition tells us that in order to have a PBW $K$-basis in terms of Gröbner basis in $K\langle X \rangle$, it is necessary to consider a finite subset $G$ of $K\langle X \rangle$ as described in Proposition 3.1(ii).

In light of Proposition 3.1, the next result is now obtained by using Theorem 2.3, Theorem 2.5 and ([2], Example 3).

3.2. Theorem Fixing a positive weight $\mathbb{N}$-gradation for the free $K$-algebra $K\langle X \rangle = K\langle X_1, ..., X_n \rangle$, let $\prec_{gr}$ be an $\mathbb{N}$-graded monomial ordering on the standard $K$-basis $B$ of $K\langle X \rangle$. With notation as in section 2, if $G$ is a Gröbner basis in $K\langle X \rangle$ with respect to $\prec_{gr}$, such that the ideal $I = \langle G \rangle$ has the property that $B - LM(I) = \{X_1^{\alpha_1}X_2^{\alpha_2}\cdots X_n^{\alpha_n} \mid \alpha_1, ..., \alpha_n \in \mathbb{N}\}$, then with respect to the $\mathbb{N}$-filtration $FA$ of $A = K\langle X \rangle/I$ induced by the weight $\mathbb{N}$-grading filtration $FK\langle X \rangle$,

$$\text{gl.dim}\,G^{\mathbb{N}}(A) = \text{gl.dim}(K\langle X \rangle/\langle LM(G) \rangle) = \text{gl.dim}(K\langle X \rangle/\langle LM(G) \rangle) = n;$$
$$\text{gl.dim}\,\tilde{A} = \text{gl.dim}(K\langle X, T \rangle/\langle LM(G) \rangle) = n + 1.$$

\[\square\]

Example (1) Rather than quoting those well-known Gröbner bases that give rise to a PBW $K$-basis, let us look at a small one. Consider the ideal $I = \langle R_{21} \rangle$ of the free $K$-algebra $K\langle X \rangle = K\langle X_1, X_2 \rangle$ generated by the single element

$$R_{21} = X_2X_1 - qX_1X_2 - \alpha X_2 - f(X_1),$$

where $q, \alpha \in K$, and $f(X_1)$ is a polynomial in the variable $X_1$. Assigning to $X_1$ the degree 1, then in either of the following two cases:
(a) $\deg f(X_1) \leq 2$, and $X_2$ is assigned to the degree 1;
(b) $\deg f(X_1) = n \geq 3$, and $X_2$ is assigned to the degree $n$,

$G = \{R_{21}\}$ forms a Gröbner basis for $I$ with respect to the $\mathbb{N}$-graded lexicographic ordering $X_1 \prec_{gr} X_2$, such that $LM(G) = \{X_2X_1\}$. Putting $A = K\langle X_1, X_2 \rangle/I$, and noticing that in both gradations $LH(R_{21}) = X_2X_1 - qX_1X_2$ and $\tilde{R}_{21} = X_2X_1 - qX_1X_2 - \alpha TX_2 - f(X_1)$, it follows from Theorem 3.2 that $\text{gl.dim}G^{\mathbb{N}}(A) = 2$, $\text{gl.dim}\,\tilde{A} = 3$.

Example (2) Let the free $K$-algebra $K\langle X \rangle = K\langle X_1, X_2 \rangle$ be equipped with a positive weight $\mathbb{N}$-gradation, such that $d(X_1) = n_1$ and $d(X_2) = n_2$, and let $G = \{g_1, g_2\}$ be any Gröbner basis with respect to some $\mathbb{N}$-graded monomial ordering $\prec_{gr}$ on the standard basis $B$ of $K\langle X \rangle$, such that $LM(g_1) = X_1^2X_2$ and $LM(g_2) = X_1X_2^2$. For instance, with respect to the $\mathbb{N}$-graded lexicographic ordering $X_2 \prec_{gr} X_1$,

$$g_1 = X_1^2X_2 - \alpha X_1X_2X_1 - \beta X_2X_1^2 - \lambda X_2X_1 - \gamma X_1, \quad \alpha, \beta, \lambda, \gamma \in K,$$
$$g_2 = X_1X_2^2 - \alpha X_2X_1X_2 - \beta X_2^2X_1 - \lambda X_2^2 - \gamma X_2,$$
Consider the algebra $A = K\langle X \rangle/\langle \mathcal{G} \rangle$ which is equipped with the $\mathbb{N}$-filtration $FA$ induced by the weight $\mathbb{N}$-grading filtration $FK\langle X \rangle$. With notation as before, the following statements hold.

(i) All three algebras $A$, $G^\mathbb{N}(A)$ and $K\langle X \rangle/\langle \text{LM}(G) \rangle$ have the polynomial growth of degree 3, while $\tilde{A}$ and $K\langle X_1, X_2, T \rangle/\langle \text{LM}(\tilde{G}) \rangle$ have the polynomial growth of degree 4.

(ii) $\text{gl.dim}G^\mathbb{N}(A) = \text{gl.dim}K\langle X \rangle/\langle \text{LM}(G) \rangle = 3$, and $\text{gl.dim}\tilde{A} = \text{gl.dim}K\langle X_1, X_2, T \rangle/\langle \text{LM}(\tilde{G}) \rangle = 4$.

In particular, all results mentioned above hold for the down-up algebra $A(\alpha, \beta, \gamma)$ (in the sense of [3]) which is defined by the relations

$$
g_1 = X_1^2X_2 - \alpha X_1X_2X_1 - \beta X_2X_1^2 - \gamma X_1, \quad \alpha, \beta, \gamma \in K.
$$

$$
g_2 = X_1X_2^2 - \alpha X_1X_2X_2 - \beta X_2^2X_1 - \gamma X_2.
$$

**Proof** Put $\Omega = \{X_1^2X_2, X_1X_2^2\}$. Then the Ufnarovski graph $\Gamma(\Omega)$ of $\Omega$ is presented by

$$
\begin{array}{cccc}
X_1X_2 & \overset{\alpha}{\longrightarrow} & X_2X_1 & \overset{\beta, \gamma}{\longrightarrow} X_1^2
\end{array}
$$

which shows that $K\langle X \rangle/\langle \text{LM}(G) \rangle$ has the polynomial growth of degree 3. Hence, by referring to the proof of Theorem 2.5, the assertions of (i) are determined. Next, the graph of $n$-chains $\Gamma_C(\Omega)$ of $\Omega$ is presented by

$$
\begin{array}{cccc}
X_1^2 & \overset{\gamma}{\longrightarrow} & X_1 & \overset{\alpha, \beta}{\longrightarrow} X_1X_2 & \overset{\beta, \beta, \beta}{\longrightarrow} X_2
\end{array}
$$

which shows that

$$
C_{i-1} = \begin{cases} 
\{X_1, X_2\}, & i = 1, \\
\{X_1X_2, X_1^2X_2\}, & i = 2, \\
\{X_1^2X_2\}, & i = 3, \\
\emptyset, & i \geq 4.
\end{cases}
$$

Note that $\text{LM}(G) = \{X_1^2X_2, X_1X_2^2, X_1T, X_2T\}$. By referring to the proof of Theorem 2.5, it is then straightforward that the set $\tilde{C}_{i-1}$ consisting of $i - 1$-chains from $\Gamma_C(\text{LM}(G))$ is

$$
\tilde{C}_{i-1} = \begin{cases} 
\{X_1, X_2, T\}, & i = 1, \\
\{X_1T, X_2T, X_1X_2^2, X_1^2X_2\}, & i = 2, \\
\{X_1^2X_3^2, X_1X_2^2T, X_1^2T\}, & i = 3, \\
\{X_1^2T\}, & i = 4, \\
\emptyset, & i \geq 5.
\end{cases}
$$
So the conditions of Theorem 2.3 and Theorem 2.5 are satisfied, and consequently the assertions of (ii) are determined.

Example (3) Let the free $K$-algebra $K\langle X \rangle = K\langle X_1, X_2 \rangle$ be equipped with a positive weight $\mathbb{N}$-gradation such that $d(X_1) = n_1$ and $d(X_2) = n_2$, and for any positive integer $n$, let $G = \{ g \}$ with $g = X_2^nX_1 - qX_1X_2^n - F$ with $q \in K$ and $F \in K\langle X \rangle$. Consider the algebra $A = K\langle X \rangle / \langle G \rangle$ which is equipped with the $\mathbb{N}$-filtration $FA$ induced by the weight $\mathbb{N}$-grading filtration $FK\langle X \rangle$.

With notation as before, the following statements hold.

(i) All three algebras $A$, $G^{\mathbb{N}}(A)$ and $K\langle X \rangle / \langle \text{LM}(G) \rangle$ have the polynomial growth of degree 2, while $\tilde{A}$ and $K\langle X_1, X_2, T \rangle / \langle \text{LM}(\tilde{G}) \rangle$ have the polynomial growth of degree 3.

(ii) $\text{gl.dim} G^{\mathbb{N}}(A) = \text{gl.dim} K\langle X \rangle / \langle \text{LM}(G) \rangle = 2$, and $\text{gl.dim} \tilde{A} = \text{gl.dim} K\langle X_1, X_2, T \rangle / \langle \text{LM}(\tilde{G}) \rangle = 3$.

(iii) If $n_1 = n_2 = 1$, then $G^{\mathbb{N}}(A)$, respectively $\tilde{A}$, has Hilbert series

$$H_{G^{\mathbb{N}}(A)}(t) = \frac{1}{1 - 2t + t^{n+1}}$$

respectively

$$H_{\tilde{A}}(t) = \frac{1}{1 - 3t + 2t^2 + t^{n+1} - t^{n+2}},$$

which, in the case of $n \geq 2$, cannot be always the form

$$\frac{1}{(1 - t^{e_1})(1 - t^{e_2})},$$

respectively

$$\frac{1}{(1 - t^{e_1})(1 - t^{e_2})(1 - t^{e_3})},$$

as claimed in ([2], Theorem 6).

Proof Consider the $\mathbb{N}$-graded lexicographic ordering $X_1 \prec_{gr} X_2$ with respect to the natural $\mathbb{N}$-gradation of $K\langle X \rangle$. For any positive integer $n$, if $g = X_2^nX_1 - qX_1X_2^n - F$ with $q \in K$ and $F \in K\langle X \rangle$ such that $\text{LM}(F) \prec_{gr} X_2^nX_1$, then it is easy to check that $G = \{ g \}$ forms a Gröbner basis for the ideal $I = \langle G \rangle$. Put $\Omega = \{ \text{LM}(G) = X_2^nX_1 \}$. Then it is straightforward to verify that for $n = 1$, the Ufnarovski graph $\Gamma(\Omega)$ is presented by

\[ X_1 \xrightarrow{\cap} X_2 \]

for $n = 2$, the Ufnarovski graph $\Gamma(\Omega)$ is presented by

\[ X_1^2 \xrightarrow{\cap} X_2^2 \]

\[ X_2X_1 \xrightarrow{\cap} X_1X_2 \]

and for $n \geq 3$, the Ufnarovski graph $\Gamma(\Omega)$ is presented by

\[ X_1^{n-1}X_2 \rightarrow X_1^{n-2}X_2^2 \rightarrow \cdots \rightarrow X_1X_2^{n-1} \rightarrow X_2^n \]

\[ X_1^n \xrightarrow{\cap} X_2X_1^{n-1} \xrightarrow{\cap} X_2^{n-2}X_1 \xrightarrow{\cap} X_2^{n-1}X_1 \]
Hence \( K_\langle X \rangle / \langle \mathrm{LM}(\mathcal{G}) \rangle \) has the polynomial growth of degree 2, and then (i) follows. Since the graph \( \Gamma_C(\Omega) \) of \( n \)-chains of \( \Omega \) is presented by

\[
\begin{align*}
X_1 & \leftarrow X_2 & X_1 & \rightarrow X_2 \\
1 & \quad 1 & (n = 1) & (n \geq 2)
\end{align*}
\]

where \( q \leq n - 2 \) for the vertices \( X_2^q X_1 \), it is clear that

\[
C_{i-1} = \begin{cases} 
\{X_1, X_2\}, & i = 1, \\
\{X_2^q X_1\}, & i = 2, \\
\emptyset, & i \geq 3.
\end{cases}
\]

Also note that \( \mathrm{LM}(\widetilde{\mathcal{G}}) = \{X_2^q X_1, X_2 T, X_1 T\} \). By referring to the proof of Theorem 2.5, it is then straightforward that the set \( \widetilde{C}_{i-1} \) consisting of \( i - 1 \)-chains from \( \Gamma_C(\mathrm{LM}(\mathcal{G})) \) is

\[
\widetilde{C}_{i-1} = \begin{cases} 
\{X_1, X_2, T\}, & i = 1, \\
\{X_1 T, X_2 T, X_2^n X_1\}, & i = 2, \\
\{X_2^n X_1 T\}, & i = 3, \\
\emptyset, & i \geq 4.
\end{cases}
\]

So the conditions of Theorem 2.3 and Theorem 2.5 are satisfied, and consequently the assertions of (ii) and (iii) are determined.

\[\square\]

**Remark** For every positive integer \( N = n + 1 \geq 2 \), it was shown in [7] that if \( q \neq 0 \) and \( F \neq 0 \) with total degree \( < n \), then the algebras defined by \( \mathcal{G} \) and \( \mathrm{LH}(\mathcal{G}) \) in the last example provide a (non-monomial) homogeneous \( N \)-Koszul algebra and a non-homogeneous \( N \)-Koszul algebra (in the sense of [4]), respectively.

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