DISTRIBUTIVE BILATTICES FROM THE PERSPECTIVE OF NATURAL DUALITY THEORY

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Abstract. This paper provides a fresh perspective on the representation of distributive bilattices and of related varieties. The techniques of natural duality are employed to give, economically and in a uniform way, categories of structures dually equivalent to these varieties. We relate our dualities to the product representations for bilattices and to pre-existing dual representations by a simple translation process which is an instance of a more general mechanism for connecting dualities based on Priestley duality to natural dualities. Our approach gives us access to descriptions of algebraic/categorical properties of bilattices and also reveals how ‘truth’ and ‘knowledge’ may be seen as dual notions.

1. Introduction

This paper is the first of three devoted to bilattices, the other two being [11, 14]. Taken together, our three papers provide a systematic treatment of dual representations via natural duality theory, showing that this theory applies in a uniform way to a range of varieties having bilattice structure as a unifying theme. The representations are based on hom-functors and hence the constructions are inherently functorial. The key theorems on which we call are easy to apply, in black-box fashion, without the need to delve into the theory. Almost all of the natural duality theory we employ can, if desired, be found in the text by Clark and Davey [15].

The term bilattice, loosely, refers to a set \( L \) equipped with two lattice orders, \( \leq_t \) and \( \leq_k \), subject to some compatibility requirement. The subscripts have the following connotations: \( t \) measuring ‘degree of truth’ and \( k \) ‘degree of knowledge’. As an algebraic structure, then, a bilattice carries two pairs of lattice operations: \( \land_t \) and \( \lor_t \); \( \land_k \) and \( \lor_k \). The term distributive is applied when all possible distributive laws hold amongst these four operations; distributivity imposes strictly stronger compatibility between the two lattice structures than the condition known as interlacing. Distributive bilattices may be, but need not be, also assumed to have universal bounds for each order which are treated as distinguished constants (or, in algebraic terms, as nullary operations). In addition, a bilattice is often, but not always, assumed to carry in addition an involutory unary operation \( \neg \), thought of as modelling a negation. Historically, the investigation of bilattices (of all types) has been tightly bound up with their potential role as models in artificial intelligence and with the study of associated logics. We note, by way of a sample, the pioneering papers of Ginsberg [22] and Belnap [6, 7] and the more recent works [1, 29, 10]. We do not, except to a very limited extent in Section 11, address logical aspects of bilattices in our work.

In this paper we focus on distributive bilattices, with or without bounds and with or without negation. In [14] we consider varieties arising as expansions of those considered here, in particular distributive bilattices with both negation and a conflation operation. In [11] we move outside the realm of distributivity, and even outside the wider realm of interlaced bilattices, and study certain quasivarieties generated by finite non-interlaced bilattices arising in connection with default logics.

The present paper is organised as follows. Section 2 formally introduces the varieties we shall study and establishes some basic properties. Sections 4, 5 and 10 present our natural dualities for these varieties. We preface these sections by accounts of the relevant natural duality theory, tailored to our intended applications (Sections 3 and 9). Theory and practice are brought together in Sections 6 and 7, in which we demonstrate how our representation theory relates to, and

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illuminates, results in the existing literature. Section 8 is devoted to applications: we exploit our natural dualities to establish a range of properties of bilattices which are categorical in nature, for instance the determination of free objects and of unification type.

We emphasise that our approach differs in an important respect from that adopted by other authors. Bilattices have been very thoroughly studied as algebraic structures (see for example [29] and the references therein). Central to the theory of distributive bilattices, and more generally interlaced ones, is the theorem showing that such algebras can always be represented as products of pairs of lattices, with the structure determined from the factors (see [28] and [10] for the bounded and unbounded cases, respectively, and the informative historical survey by Davey [16] of the evolution of this oft-rediscovered result). The product representation is normally derived by performing quite extensive algebraic calculations. It is then used in a crucial way to obtain, for those bilattice varieties which have bounded distributive lattice reducts, dual representations which are based on Priestley duality [27, 24]. Our starting point is different. For each class $\mathcal{A}$ of algebras we study here and in [14], we first establish, by elementary arguments, that $\mathcal{A}$ takes the form $\mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is finite, or, more rarely, $\mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a set of two finite algebras. (In [11] we assume at the outset that $\mathcal{A}$ is the quasivariety generated by some finite algebra in which we are interested.) This gives us direct access to the natural duality framework. From this perspective, the product representation is a consequence of the natural dual representation, and closely related to it. For a reconciliation, in the distributive setting, of our approach and that of others and an explanation of how these approaches differ, see Sections 7 and 11.

We may summarise as follows what we achieve in this paper and in [11, 14]. For different varieties we call on different versions of the theory of natural dualities. Accordingly our account can, *inter alia*, be read as a set of illustrated tutorials on the natural duality methodology presented in a self-contained way. The examples we give will also be new to natural duality aficionados, but for such readers we anticipate that the primary interest of our work will be its contribution to the understanding of the interrelationship between natural and Priestley-style dualities for finitely generated quasivarieties of distributive lattice-based algebras. For this we exploit the piggybacking technique, building on work initiated in our paper [13] and our constructions elucidate precisely how product representations come about. All our natural dual representations are new, as are our Priestley-style dual representations in the unbounded cases. Finally we draw attention to the remarks with which we end the paper drawing parallels between the special role the knowledge order plays in our theory and the role this order plays in Belnap's semantics for a four-valued logic.

## 2. DISTRIBUTIVE PRE-BILATTICES AND BILATTICES

We begin by giving basic definitions and establishing the terminology we shall adopt henceforth. We warn that the definitions (bilattice, pre-bilattice, etc.) are not used in a consistent way in the literature, and that notation varies. Our choice of symbols for lattice operations enables us to keep overt which operations relate to truth and which to knowledge. Alternative notation includes $\vee$ and $\wedge$ in place of $\vee_t$ and $\wedge_t$, and $\oplus$ and $\otimes$ in place of $\vee_k$ and $\wedge_k$.

We define first the most general class of algebras we shall consider. We shall say that an algebra $A = (A; \vee_t, \wedge_t, \vee_k, \wedge_k)$ is an *unbounded distributive pre-bilattice* if each of the reducts $(A; \vee_t, \wedge_t)$ and $(A; \vee_k, \wedge_k)$ is a lattice and each of $\vee_t, \wedge_t, \vee_k$ and $\wedge_k$ distributes over each of the other three. The class of such algebras is a variety, which we denote by $\mathcal{DPB}_u$. Each of the varieties we consider in this paper and in [14] will be obtained from $\mathcal{DPB}_u$ by expanding the language by adding constants, or additional unary or binary operations.

Given $A \in \mathcal{DPB}_u$, we let $A_t = (A; \vee_t, \wedge_t)$ and refer to it as the *truth lattice reduct* of $A$ (or $t$-lattice for short); likewise we have a *knowledge lattice reduct* (or $k$-lattice) $A_k = (A; \vee_k, \wedge_k)$.

The following lemma is an elementary consequence of the definitions. We record it here to emphasise that no structure beyond that of an unbounded distributive pre-bilattice is involved.

**Lemma 2.1.** Let $A = (A; \vee_t, \wedge_t, \vee_k, \wedge_k) \in \mathcal{DPB}_u$. Then, for $a, b, c \in A$,

(i) $a \leq_k b \leq_k c$ implies $a \wedge_k c \leq_t a \vee_t c$;

(ii) $a \wedge_t b \leq_t a \ast_k b \leq_t a \vee_t b$, where $\ast_k$ denotes either $\wedge_k$ or $\vee_k$. 


Corresponding statements hold with $k$ and $t$ interchanged.

As we have indicated in the introduction, we shall wish to prove, for each bilattice variety $\mathcal{A}$ we study, that $\mathcal{A}$ is finitely generated as a quasivariety. This amounts to showing that there exists a finite set $\mathfrak{M}$ of finite algebras in $\mathcal{A}$ such that, for each $\mathcal{A} \in \mathcal{A}$ and $a \neq b$ in $\mathcal{A}$, there is $M \in \mathfrak{M}$ and a $\mathcal{A}$-homomorphism $h: A \to M$ with $h(a) \neq h(b)$. ($\mathfrak{M}$ will consist of a single subdirectly irreducible algebra or at most two such algebras.) This separation property is linked to the existence of particular quotients of the algebras in $\mathcal{A}$. Accordingly we are led to investigate congruences. We start with a known result. Our proof is direct and elementary: it uses nothing more than the distributivity properties of the $t$- and $k$-lattice operations, together with Lemma 2.1 and basic facts about lattice congruences given, for example, in [18, Chapter 6]. (Customarily the lemma would be obtained as a spin-off from the product representation theorem as this applies to distributive bilattices.)

**Proposition 2.2.** Let $A = (A; \vee_t, \wedge_t, \vee_k, \wedge_k)$ be an unbounded distributive pre-bilattice. Let $\theta \subseteq A^2$ be an equivalence relation. Then the following statements are equivalent:

(i) $\theta$ is a congruence of $A_t = (A; \vee_t, \wedge_t)$;

(ii) $\theta$ is a congruence of $A_k = (A; \vee_k, \wedge_k)$;

(iii) $\theta$ is a congruence of $A$.

**Proof.** It will suffice, by symmetry, to prove (i) $\Rightarrow$ (ii). So assume that (i) holds. Since $\theta$ is a congruence of $(A; \vee_t, \wedge_t)$, the $\theta$-equivalence classes are convex sublattices with respect to the $\leq_t$ order. We first observe that from Lemma 2.1(i) each equivalence class is convex with respect to the $\leq_k$ order, and from Lemma 2.1(ii) that each equivalence class is a sublattice of $(A; \vee_k, \wedge_k)$.

Finally we need to establish the quadrilateral property:

$$a \theta (a \wedge_k b) \iff b \theta (a \vee_k b).$$

For the forward direction observe that the distributive laws and Lemma 2.1(ii) (swapping $t$ and $k$) imply

$$a \wedge_t b = (a \vee_k b) \wedge_k (a \wedge_t b) = (a \wedge_k (a \wedge_t b)) \vee_k (b \wedge_k (a \wedge_t b)) = (a \vee_k b) \wedge_k (a \wedge_t b).$$

Combining this with $a \theta (a \wedge_k b)$ and with the fact that $\theta$ is a congruence of $(A; \vee_t, \wedge_t)$, we have $a \wedge_t b \theta (a \vee_k b) \wedge_t a$. Replacing $\wedge_t$ by $\vee_t$ in the previous argument, we obtain $a \vee_t b \theta (a \vee_k b) \vee_t a$. This proves that

$$[a]_{\theta} \vee_t [b]_{\theta} = [a]_{\theta} \wedge_t [a \vee_k b]_{\theta} \quad \text{and} \quad [a]_{\theta} \vee_t [b]_{\theta} = [a]_{\theta} \vee_t [a \wedge_k b]_{\theta}.$$ 

Since $(A; \wedge_t, \vee_t)/\theta$ is distributive, $[b]_{\theta} = [a \vee_k b]_{\theta}$, that is, $b \theta a \vee_k b$. \hfill $\square$

The following consequences of Proposition 2.2 will be important later. Take an unbounded distributive pre-bilattice $A$ and a filter $F$ of $A_t$. Then $F$ is a convex sublattice of $A_k$. If a map $h: A \to \{0, 1\}$ acts as a lattice homomorphism from $A_t$ into the two-element lattice $2$, then $h$ is a lattice homomorphism from $A_k$ into either $2$ or its dual lattice $2^D$. Hence each prime filter for $A_t$ is either a prime filter or a prime ideal for $A_k$ and vice versa. These results were first proved in [24, Lemma 1.11 and Theorem 1.12] and underpin the development of the duality theory presented there.

We now wish to consider the situation in which a distributive pre-bilattice has universal bounds with respect to its $\leq_t$ and $\leq_k$ orders. We recall a classic result, known as the $90^\circ$ Lemma. The result has its origins in [8] (see the comments in [23, Section 3] and also [28, Theorem 3.1]).

**Lemma 2.3.** Let $(L; \vee_t, \wedge_t, \vee_k, \wedge_k)$ be an unbounded distributive pre-bilattice. Assume that $(L; \leq_k)$ has a bottom element, $0_k$, and a top element, $1_k$.

(i) For all $a, b \in L$,

$$a \vee_k b = ((a \wedge_t b) \wedge_t 0_k) \vee_t ((a \vee_t b) \wedge_t 1_k),$$

$$a \wedge_k b = ((a \wedge_t b) \wedge_t 1_k) \vee_t ((a \vee_t b) \wedge_t 0_k).$$
(ii) For all $a \in L$,
\[
0_k \land_t 1_k \leq_t a \leq_t 0_k \lor_t 1_k,
\]
so that $(L, \leq_t)$ also has universal bounds, and in the lattice $(L; \lor_t, \land_t)$, the elements $0_k$ and $1_k$ form a complemented pair.

The import of Lemma 2.3(i) is that $\lor_k$ and $\land_k$ are term-definable from $\lor_t$ and $\land_t$ and the universal bounds of the $k$-lattice; henceforth when these bounds are included in the type we shall exclude $\lor_k$ and $\land_k$ from it. When we refer to an algebra $A = (A; \lor_t, \land_t, 0_t, 1_t)$ as being an unbounded distributive pre-bilattice we do not exclude the possibility that one, and hence both, of $A_k$ and $A_t$ has universal bounds; we are simply saying that bounds are not included in the algebraic language. We say an algebra $(A; \lor_t, \land_t, 0_t, 1_t, 0_k, 1_k)$ is a distributive pre-bilattice if $0_t$, $1_t$, $0_k$ and $1_k$ are nullary operations, and the algebra $(A; \lor_t, \land_t, \lor_k, \land_k)$ belong to $\mathcal{DPB}_u$, where $\lor_k$ and $\land_k$ are defined from $\lor_t$, $\land_t$, $0_k$ and $1_k$ as in Lemma 2.3(i), and $0_t$, $1_t$ and $0_k$, $1_k$ act as $0$, $1$ in the lattices $A_t$ and $A_k$, respectively.

We now add a negation operation. If $A = (A; \lor_t, \land_t, \lor_k, \land_k)$ belongs to $\mathcal{DPB}_u$ and carries an involutory unary operation $\neg$ which is interpreted as a dual endomorphism of $(A; \lor_t, \land_t)$ and an endomorphism of $(A; \lor_k, \land_k)$, then we call $(A; \lor_t, \land_t, \lor_k, \land_k, \neg)$ an unbounded distributive bilattice. Similarly, an algebra $(A; \lor_t, \land_t, \neg, 0_t, 1_t, 0_k, 1_k)$ is a distributive bilattice if the negation-free reduct is a distributive pre-bilattice, and $\neg$ is an involutory dual endomorphism of the bounded $t$-lattice reduct and endomorphism of the bounded $k$-lattice reduct. These conditions include the requirements that $\neg$ interchanges $0_t$ and $1_t$ and fixes $0_k$ and $1_k$.

For ease of reference we present a list of the varieties we consider in this paper, in the order in which we shall study them.

- $\mathcal{DB}$: distributive bilattices, for which we include in the type $\lor_t, \land_t, \neg, 0_t, 1_t, 0_k, 1_k$;
- $\mathcal{DB}_u$: unbounded distributive bilattices, having as basic operations $\lor_t, \land_t, \lor_k, \land_k, \neg$;
- $\mathcal{DPB}$: distributive pre-bilattices, having as basic operations $\lor_t, \land_t, 0_t, 1_t, 0_k, 1_k$;
- $\mathcal{DPB}_u$: unbounded distributive pre-bilattices, having as basic operations $\lor_t, \land_t, \lor_k, \land_k$.

We shall denote by $\mathcal{D}$ the variety of distributive lattices in which universal bounds are included in the type, and by $\mathcal{D}_u$ the variety of unbounded distributive lattices. For any $A \in \mathcal{DB}$ or $\mathcal{DPB}$, its bounded truth lattice $A_t = (A; \lor_t, \land_t, 0_t, 1_t)$ is a $\mathcal{D}$-reduct of $A$. Likewise the truth lattice $A_k = (A; \lor_t, \land_t)$ provides a reduct in $\mathcal{D}_u$ for any $A \in \mathcal{DB}_u$ or $\mathcal{DPB}_u$. We remark also that each member of $\mathcal{DB}$ has a reduct in the variety $\mathcal{DM}$ of De Morgan algebras, and that each algebra in $\mathcal{DB}_u$ has a reduct in the variety of De Morgan lattices; in each case the reduct is obtained by suppressing the knowledge operations. This remark explains the preferential treatment we always give to truth over knowledge when forming reducts.

Throughout we shall when required treat any variety as a category, by taking as morphisms all homomorphisms. Given a variety $\mathcal{A}$ whose algebras have reducts (or more generally term-reducts) in $\mathcal{D}$ obtained by deleting certain operations, we shall make use of the associated forgetful functor from $\mathcal{A}$ into $\mathcal{D}$, defined to act as the identity map on morphisms. (We shall later refer to $\mathcal{A}$ as being $\mathcal{D}$-based.) Specifically we define a forgetful functor $U: \mathcal{DB} \to \mathcal{D}$, for which $U(A) = A_t$ for any $A \in \mathcal{D}$. We also have a functor, again denoted $U$ and defined in the same way, from $\mathcal{DPB}$ to $\mathcal{D}$. Likewise there is a functor $U_u$ from $\mathcal{DB}_u$ or from $\mathcal{DPB}_u$ into $\mathcal{D}_u$ which sends an algebra to its truth lattice.

We now recall the best-known (pre-)bilattice of all, that known as $\text{FOUR}$. We consider the set $V = \{0, 1\}^2$ and, to simplify later notation, shall denote its elements by binary strings. We define lattice orders $\leq_t$ and $\leq_k$ on $V$ as shown in Figure 1; we draw lattices in the manner traditional in lattice theory. (In the literature of bilattices, the four-element pre-bilattice is customarily depicted via an amalgam of the lattice diagrams in Figure 1, with the two orders indicated vertically (for knowledge) and horizontally (for truth); virtually every paper on bilattices contains this figure and we do not reproduce it here.)
We may add truth constants \(0_t = 00\) and \(1_t = 11\) and knowledge constants \(0_k = 01\) and \(1_k = 10\) to \(\text{FOUR}\) to obtain a member of \(\text{DPB}\). The structure \(\text{FOUR}\) also supports a negation \(\neg\) which switches \(11\) and \(00\) and fixes \(01\) and \(10\). The four-element distributive bilattice and its unbounded counterpart play a distinguished role in what follows. Accordingly we define

\[
4 = \{\{00, 11, 01, 10\}; \lor_t, \land_t, \neg_0, 1_t, 0_k, 1_k\}\]

and

\[
4_u = \{\{00, 11, 01, 10\}; \lor_t, \land_t, \lor_k, \land_k, \neg_0\}.
\]

These belong, respectively, to \(\text{DB}\) and to \(\text{DB}_u\).

There are two non-isomorphic two-element distributive pre-bilattices without bounds. One, denoted \(2_u^+\), has underlying set \(\{0, 1\}\), and the \(t\)-lattice structure and the \(k\)-lattice structure both coincide with that of the two-element lattice \(2 = (\{0, 1\}; \lor, \land)\) in which \(0 < 1\). The other, denoted \(2_u^-\), has \(2\) as its \(t\)-lattice reduct and the order dual \(2^\partial\) as its \(k\)-lattice reduct. If we include bounds, we must have \(0_t = 0_k = 0\) and \(1_t = 1_k = 1\) if \(\leq_t\) and \(\leq_k\) coincide and \(0_t = 1_k = 0\) and \(1_t = 0_k = 1\) if \(\leq_k\) coincides with \(\geq_t\).

In neither the bounded nor the unbounded case do we have a two-element algebra which supports an involutory negation which preserves \(\land_k\) and \(\lor_k\) and interchanges \(\lor_t\) and \(\land_t\). Hence neither \(\text{DB}_u\) nor \(\text{DB}\) contains a two-element algebra. Similarly, if either variety contained a three-element algebra, having universe \(\{0, a, 1\}\), with \(0 <_t a <_t 1\), then \(\leq_k\) would have to coincide with either \(\leq_t\) or \(\geq_t\). The only involutory dual endomorphism of the \(t\)-reduct of the chain swaps 0 and 1 and fixes \(a\), and this map is not order-preserving with respect to \(\leq_k\). We conclude that, whether or not bounds are included in the type, there is no non-trivial distributive bilattice of cardinality less than four. Hence, the 90° Lemma implies that \(4\) and \(4_u\) are the only four-element algebras in \(\text{DB}\) and \(\text{DB}_u\), respectively.

As noted above, to derive a natural duality for any one of the varieties in which we are interested, we need to express the variety \(\mathcal{A}\) in question as a finitely generated quasivariety. Specifically, we need to find a finite set \(\mathcal{R}\) of finite algebras such that \(\mathcal{A} = \text{ISP}(\mathcal{R})\). We shall prove in subsequent sections, with the aid of Proposition 2.2, that

\[
\begin{align*}
\text{DB} & = \text{ISP}(4), & \text{DPB} & = \text{ISP}(2^+, 2^-), \\
\text{DB}_u & = \text{ISP}(4_u), & \text{DPB}_u & = \text{ISP}(2^+_u, 2^-_u).
\end{align*}
\]

Corresponding results hold for the varieties we consider in [14]. Such results are central to our enterprise. All are elementary in that the proofs use a minimum of bilattice theory and none of the algebraic structure theorems for bilattices is needed. (There is a close connection between our assertions above and the identification of the subdirectly irreducible algebras in the varieties concerned. The latter has traditionally been handled by first proving a product representation theorem. We reiterate that we prove our claims directly, by elementary means.)

3. THE NATURAL DUALITY FRAMEWORK

As indicated in Section 1, we shall introduce natural duality machinery in the form that is simplest to apply to each of the varieties we consider.

We first consider \(\mathcal{A} = \text{ISP}(\mathcal{M})\), where \(\mathcal{M}\) is a finite algebra with a lattice reduct. We shall aim to define an alter ego \(\bar{\mathcal{M}}\) for \(\mathcal{M}\) which will serve to generate a category \(\mathcal{X}\) dually equivalent to \(\mathcal{A}\). The alter ego will be a discretely topologised structure \(\bar{\mathcal{M}}\) on the same universe \(M\) as \(\mathcal{M}\) and will be equipped with a set \(R\) of relations which are algebraic in the sense that each member
of $R$ is a subalgebra of some finite power $M^n$ of $M$. (Later we shall need also to allow for nullary operations, but relations suffice in the simplest cases we consider.) We define $X$ to be the nullary operations, but relations suffice in the simplest cases we consider.) We define $X$ to be the alter ego is lifted pointwise in the obvious way. We denote the lifting of $r$ closed substructures of non-empty powers of $M$ of $X$ as follows: $$r\Downarrow_{M}\in X.$$ Here we have well-defined contravariant functors $D: A \rightarrow X$ and $E: X \rightarrow A$ defined as follows:

- on objects: $D: A \mapsto A(A, M)$,
- on morphisms: $D: x \mapsto \circ \, x$,

where $A(A, M)$ is seen as a closed substructure of $M^A$, and

- on objects: $E: X \mapsto X(X, M)$,
- on morphisms: $E: \phi \mapsto \circ \, \phi$,

where $X(X, M)$ is seen as a subalgebra of $M^X$.

Given $A \in \mathcal{A}$, we shall refer to $D(A)$ as the natural dual of $A$. We have, for each $A \in \mathcal{A}$, a natural evaluation map $e_A: A \rightarrow ED(A)$, given by $e_A(a)(x) = x(a)$ for $a \in A$ and $x \in D(A)$, and likewise there exists an evaluation map $e_X: X \rightarrow DE(X)$ for $X \in \mathcal{X}$. We say that $M$ yields a duality on $A$ if $e_A$ is an isomorphism for each $A \in \mathcal{A}$, and that $M$ yields a full duality on $A$ if in addition $e_X$ is an isomorphism for each $X \in \mathcal{X}$. Formally, if we have a full duality then $D$ and $E$ set up a dual equivalence between $A$ and $X$ with the unit and co-unit of the adjunction given by the evaluation maps. All the dualities we shall present in this paper are full and, moreover, in each case we are able to give a precise description of the dual category $X$. Better still, the dualities have the property that they are strong dualities. For the definition of a strong duality and a full discussion of this notion we refer the reader to [15, Section 3.2]. Strongness implies that $D$ takes injections to surjections and surjections to embeddings, facts which we shall exploit in Section 8.

Before proceeding we indicate, for the benefit of readers not conversant with natural duality theory, how Priestley duality fits into this framework. We have

- $\mathcal{A} = \mathcal{D}$, the class of distributive lattices with $0, 1$,
- $\mathcal{M} = 2$, the two-element chain in $\mathcal{D}$,
- $\mathcal{X} = \mathcal{P}$, the category of Priestley spaces,
- $\mathcal{M}^A = 2$, the discretely topologised two-element chain;
- $R = \{\leq\}$, where $\leq$ is the subalgebra $\{(0,0), (0,1), (1,1)\}$ of $2^2$.

This duality is strong [15, Theorem 4.3.2]. We later exploit it as a tool when dealing with bilattices having reducts in $\mathcal{D}$ and it is convenient henceforth to denote the hom-functors $D$ and $E$ setting it up by $H$ and $K$. When expedient, we view $KH(L)$ as the family of clopen up-sets of $L$, for $L \in \mathcal{D}$.

In accordance with our black-box philosophy we shall present without further preamble the first of the duality theorems we shall use. It addresses both the issue of the existence of an alter ego yielding a duality and that of finding one which is conveniently simple. Theorem 3.1 comes from specialising [15, Theorem 7.2.1] and the fullness assertion from [15, Theorem 7.1.2].

We deal with a quasivariety of algebras $\mathcal{A}$ generated by an algebra $M$ with a reduct in $\mathcal{D}$ and denote by $U$ the associated forgetful functor from $\mathcal{A}$ into $\mathcal{D}$. For $\omega_1, \omega_2 \in \Omega = \mathcal{D}(U(A), 2)$, we let $R_{\omega_1, \omega_2}$ be the collection of maximal $A$-subalgebras of sublattices of the form

$$(\omega_1, \omega_2)^{-1}(\leq) = \{(a, b) \in M^2 | \omega_1(a) \leq \omega_2(b)\}.$$

**Theorem 3.1.** (Piggyback Duality Theorem for $\mathcal{D}$-based algebras, single generator case) Let $\mathcal{A} = ISP(M)$, where $M$ is a finite algebra with a reduct in $\mathcal{D}$, and $\Omega = \mathcal{D}(U(A), 2)$. Let $M = (M; R, \mathcal{T})$ be the topological relational structure on the underlying set $M$ of $M$ in which $\mathcal{T}$ is the discrete topology and $R$ is the union of the sets $R_{\omega_1, \omega_2}$ as $\omega_1, \omega_2$ run over $\Omega$. Then $M$ yields a natural duality on $\mathcal{A}$. 
Moreover, if $\mathbf{M}$ is subdirectly irreducible, has no proper subalgebras and no endomorphisms other than the identity, then $\mathbf{M}$ as defined above determines a strong duality. So the functors $\mathbf{D} = \mathcal{A}(\_,- \mathbf{M})$ and $\mathbf{E} = \mathcal{X}(\_,- \mathbf{M})$ set up a dual equivalence between $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ and $\mathcal{X} = \mathbb{ISP}^+(\mathbf{M})$.

We now turn to the study of algebras which have reducts in $\mathbf{D}_a$ rather than in $\mathbf{D}$. We consider a class $\mathcal{A}$ of algebras for which we have a forgetful functor $\mathcal{U}$ from $\mathcal{A}$ into $\mathcal{D}_a$. The natural duality for $\mathbf{D}_a$ will take the place of Priestley duality for $\mathbf{D}$. This duality is less well known to those who are not specialists in duality theory, but it is equally simple. We have $\mathbf{D}_a = \mathbb{ISP}(\mathbf{2}_a)$, where $\mathbf{2}_a = \{(0,1); \sqcap, \lor\}$. The alter ego is $\mathbf{2}_{01} = \{(0,1); 0,1, \leq, \land\}$, where 0 and 1 are treated as nullary operations. It yields a strong duality between $\mathbf{D}_a$ and the category $\mathbf{P}_{01} = \mathbb{ISP}^+(\mathbf{2}_{01})$ of doubly-pointed Priestley spaces (bounded Priestley spaces in the terminology of [15, Theorem 4.3.2], where validation of the strong duality can also be found). The duality is set up by well-defined hom-functors $\mathbf{H}_a = \mathbf{D}_a(-,\mathbf{2}_a)$ and $\mathbf{K}_a = \mathbf{P}_{01}(-,\mathbf{2}_{01})$. A member $\mathbf{L}$ of $\mathbf{D}_a$ is isomorphic to $\mathbf{K}_a\mathbf{H}_a(\mathbf{L})$ and may be identified with the lattice of proper non-empty clopen up-sets of the doubly-pointed Priestley space $\mathbf{H}_a(\mathbf{L})$.

Most previous applications of the piggybacking theory have been made over $\mathbf{D}$ (see [15, Section 7.2]), or over the variety of unital semilattices. But one can equally well piggyback over $\mathbf{D}_a$; see [17, Theorem 2.5] and [15, Section 3.3 and Subsection 4.3.1]. (In [14] we extend the scope further: we handle bilattices with conflation by piggybacking over $\mathbf{D}\mathbf{B}$ and $\mathbf{D}\mathbf{B}_a$.)

**Theorem 3.2.** (Piggyback Duality Theorem for $\mathbf{D}_a$-based algebras, single generator case) Suppose that $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite algebra with a reduct in $\mathbf{D}_a$ but no reduct in $\mathbf{D}$. Let $\Omega = \mathbf{D}_a(\mathcal{U}(\mathbf{M}),\mathbf{2}_a)$ and $\mathbf{M} = (\mathbf{M}; R, \mathcal{T})$ be the topological relational structure on the underlying set $\mathbf{M}$ of $\mathbf{M}$ in which $\mathcal{T}$ is the discrete topology and $R$ contains the relations of the following types:

(a) the members of the sets $R_{\omega_1,\omega_2}$, as $\omega_1,\omega_2$ run over $\Omega$, where $R_{\omega_1,\omega_2}$ is the set of maximal $\mathcal{A}$-subalgebras of sublattices of the form

$$(\omega_1,\omega_2)^{-1}(\leq) = \{(a,b) \in \mathbf{M}^2 \mid \omega_1(a) \leq \omega_2(b)\};$$

(b) the members of the sets $R^+_\omega$, as $\omega$ runs over $\Omega$ and $i \in \{0,1\}$, where $R^+_\omega$ is the set of maximal $\mathcal{A}$-subalgebras of sublattices of the form

$$\omega^{-1}(i) = \{a \in \mathbf{M} \mid \omega(a) = i\}.$$  

Then $\mathbf{M}$ yields a natural duality on $\mathcal{A}$.

Assume moreover that $\mathbf{M}$ is subdirectly irreducible, that $\mathbf{M}$ has no non-constant endomorphisms other than the identity on $\mathbf{M}$ and that the only proper subalgebras of $\mathbf{M}$ are one-element subalgebras. Then the duality above can be upgraded to a strong, and hence full, duality by including in the alter ego all one-element subalgebras of $\mathbf{M}$, regarded as nullary operations. If $\mathcal{X} = \mathbb{ISP}^+(\mathbf{M})$, where $\mathbf{M}$ is upgraded as indicated, then the functors $\mathbf{D}_a = \mathcal{A}(\_,- \mathbf{M})$ and $\mathbf{E}_a = \mathcal{X}(\_,- \mathbf{M})$ yield a dual equivalence between $\mathcal{A}$ and $\mathcal{X}$.

**Proof.** Our claims regarding the duality follow from [17, Section 2]. For a discussion of the role played by the nullary operations in yielding a strong duality, we refer the reader to [15, Section 3.3], noting that our assumptions on $\mathbf{M}$ ensure that any non-extendable partial endomorphisms would have to have one-element domains. Hence it suffices to include these one-element subalgebras as nullary operations in order to obtain a strong duality.

We conclude this section with remarks on the special role of piggyback dualities. For quasi-varieties to which either Theorem 3.1 or Theorem 3.2 applies, we could have taken a different approach, based on the NU Strong Duality Theorem [15, Theorems 2.3.4 and 3.3.8], as it applies to a quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite algebra with a lattice reduct. This way, the set of piggybacking subalgebras would have been replaced by the set of all subalgebras of $\mathbf{M}^2$. But this has two disadvantages, one well known, the other revealed by our work in [13, Section 2]. Firstly, the set of all subalgebras of $\mathbf{M}^2$ may be unwieldy, even when $\mathbf{M}$ is small. In part to address this, a theory of entailment has been devised, which allows superfluous relations to be discarded from a duality; see [15, Section 2.4]. The piggybacking method, by contrast, provides alter egos which are much closer to being optimal. Secondly, as we reveal in Section 6, the piggyback relations
play a special role in translating natural dualities to ones based on the Priestley dual spaces of the algebras in \(U(A)\) or \(U_s(A)\), as appropriate. We shall also see that, even when certain piggyback relations can be discarded from an alter ego without destroying the duality, these relations do make a contribution in the translation process.

4. A NATURAL DUALITY FOR DISTRIBUTIVE BILATTICES

In this section we set up a duality for the variety \(\mathbf{D\mathbf{B}}\) and reveal the special role played on the dual side by the knowledge order.

**Proposition 4.1.** \(\mathbf{D\mathbf{B}} = \mathbf{ISP}(4)\).

**Proof.** Let \(A \in \mathbf{D\mathbf{B}}\). Let \(a \neq b\) in \(A\) and choose \(x \in \mathbf{D}(A, 2)\) such that \(x(a) \neq x(b)\). Define an equivalence relation \(\theta\) on \(A\) by \(p \theta q\) if and only if \(x(p) = x(q)\) and \(x(\neg p) = x(\neg q)\). Clearly \(\theta\) is a congruence of \(A\). By Proposition 2.2 it is also a congruence of \(A_k\), and by its definition it preserves \(\neg\). In addition, \(A/\theta\) is a non-trivial algebra (since \(x(a) \neq x(b)\)) and of cardinality at most four. Since the only such algebra in \(\mathbf{D\mathbf{B}}\), up to isomorphism, is 4, the image of the associated \(\mathbf{D\mathbf{B}}\)-homomorphism \(h: A \to A/\theta\) is (isomorphic to) 4, and separates \(a\) and \(b\).

It is instructive also to present \(h: A \to 4\), as above, more directly. We take

\[
h(c) = \begin{cases} 
    x(c)(1 - x(\neg c)) & \text{if } x(0_k) = 0, \\
    (1 - x(\neg c))x(c) & \text{if } x(0_k) = 1,
\end{cases}
\]

for all \(c\); here we are viewing the image \(h(c)\) as a binary string. In the case that \(x(0_k) = 0\), observe that \(h(0_k) = 01 = 0^k\) (note that \(\neg 0_k = 0_k\)). Since \(x(0_k) \wedge x(1_k) = x(0_k \wedge 1_k) = x(0) = 0\) and \(x(0_k) \vee x(1_k) = x(0_k \vee 1_k) = x(1) = 1\), we have \(x(1_k) = x(\neg 1_k) = 1\) and \(h_0(1_k) = 10 = 1^k\). It is routine to check that \(h\) preserves \(\vee\), \(\wedge\), and \(\neg\). Hence \(h\) is a \(\mathbf{D\mathbf{B}}\)-homomorphism and, by construction, \(h(a) \neq h(b)\). The argument for the case that \(x(0_k) = 1\) is similar.

In the following result we make use of the \(\mathbf{D}\)-morphisms from the \(\ell\)-lattice reduct of 4 into 2. These are the maps \(\alpha\) and \(\beta\) given respectively by \(\alpha^{-1}(1) = \{10, 11\}\) and \(\beta^{-1}(1) = \{01, 11\}\). Observe that \(\alpha\) and \(\beta\) correspond to the maps that assign to a binary string its first and second elements, respectively.

**Theorem 4.2.** (Natural duality for distributive bilattices) There is a dual equivalence between the category \(\mathbf{D\mathbf{B}}\) and the category \(\mathbf{P}\) of Priestley spaces set up by hom-functors. Specifically, let

\[
4 = (\{00, 11, 01, 10\}; \vee_t, \wedge_t, \neg, 0_t, 1_t, 0_k, 1_k)
\]

be the four-element bilattice in the variety \(\mathbf{D\mathbf{B}}\) of distributive bilattices and let its alter ego be

\[
A = (\{00, 11, 01, 10\}; \leq_k, \mathcal{T}).
\]

Then

\[
\mathbf{D\mathbf{B}} = \mathbf{ISP}(4) \quad \text{and} \quad \mathbf{P} = \mathbf{ISP}^+(4)
\]

and the hom-functors \(D = \mathbf{D\mathbf{B}}(-, 4)\) and \(E = \mathbf{P}(-, A)\) set up a dual equivalence between \(\mathbf{D\mathbf{B}}\) and \(\mathbf{P}\). Moreover, this duality is strong.

**Proof.** The proof involves three steps.

**Step 1: setting up the piggyback duality.**

We must identify the subalgebras of \(4^2\) involved in the piggyback duality supplied by Theorem 3.1 when \(A = \mathbf{D\mathbf{B}}\) and \(M = A\). Define \(\alpha\) and \(\beta\) as above.

We claim that the knowledge order \(\leq_k\) is the unique maximal \(\mathbf{D\mathbf{B}}\)-subalgebra of \((\alpha, \alpha)^{-1}(\leq)\). We first observe that it is immediate from order properties of lattices that \(\leq_k\) is a sublattice for the \(k\)-lattice structure. It also contains the elements \(01, 01\) and \(10, 10\). By the 90° Lemma (with \(k\) and \(t\) switched), \(\leq_k\) is also closed under \(\wedge_t\) and \(\vee_t\) (or this can be easily checked directly). Since \(\neg\) preserves \(\leq_k\), we conclude that \(\leq_k\) is a subalgebra of \(4^2\).

Now note that, for \(a = a_1a_2\) and \(b = b_1b_2\) binary strings in 4, we have \(\alpha(a) \leq \alpha(b)\) if and only if \(a_1 \leq b_1\) and that \(\alpha(\neg a) \leq \alpha(\neg b)\) if and only if \(1 - a_2 \leq 1 - b_2\) that is, if and only if \(b_2 \leq a_2\). It follows that if \((a, b)\) belongs to a \(\mathbf{D\mathbf{B}}\)-subalgebra of \((\alpha, \alpha)^{-1}(\leq)\) then \((a, b)\) belongs to the relation
\[\leq_k.\] Since we have already proved that \(\leq_k\) is a DB-subalgebra of \((\alpha, \alpha)^{-1}(\leq)\) we deduce that \(\leq_k\) is the unique maximal subalgebra contained in this sublattice. Likewise, the unique maximal DB-subalgebra of \((\beta, \beta)^{-1}(\leq)\) is \(\geq_k\).

We claim that no subalgebra of \(4^2\) is contained in \((\alpha, \beta)^{-1}(\leq)\). To see this we observe that 
\[
\alpha(0_k) = \alpha(10) = 1 \not\leq 0 = \beta(10) = \beta(0_k).
\]
Likewise, consideration of \(1_k\) shows that there is no DB-subalgebra contained in \((\beta, \alpha)^{-1}(\leq)\).

Following the Piggyback Duality Theorem slavishly, we should include both \(\leq_k\) and \(\geq_k\) in our alter ego. But it is never necessary to include a binary relation and also its converse in an alter ego, so \(\leq_k\) suffices.

**Step 2: describing the dual category.**

To prove that \(\mathcal{IP}(4)\) is the category of Priestley spaces it suffices to note that \(2 \in \mathcal{IP}(4)\) and that \(4 \in \mathcal{IP}(2)\). It follows that \(\mathcal{IP}(2) \subseteq \mathcal{IP}(4)\) and \(\mathcal{IP}(4) \subseteq \mathcal{IP}(2)\).

**Step 3: confirming the duality is strong.**

We verify that the sufficient conditions given in Theorem 3.1 for the duality to be strong are satisfied by \(M = 4\). We proved in Section 4 that there is no non-trivial algebra in DB of cardinality less than four. Hence 4 has no non-trivial quotients and no proper subalgebras. This implies, too, that 4 is subdirectly irreducible. Since every element of 4 is the interpretation of a nullary operation, the only endomorphism of 4 is the identity.

We might wonder whether there are alternative choices for the structure of the alter ego \(4\) of 4. We now demonstrate that, within the realm of binary algebraic relations at least, there is no alternative: it is inevitable that the alter ego contains the relation \(\leq_k\) (or its converse).

**Proposition 4.3.** The subalgebras of \(4^2\) are \(4^2, \Delta_{4^2}, \leq_k\) and \(\geq_k\). Here \(\Delta_{4^2}\) denotes the diagonal subalgebra \(\{(a, a) \mid a \in 4\}\).

**Proof.** We merely outline the proof, which is routine, but tedious. Assume we have a proper subalgebra \(r\) of \(4^2\), necessarily containing \(\Delta_4\) (since all the elements of 4 are constants in the language of DB) and assume that \(r\) is not \(\leq_k\). We must then check that \(r\) has to be \(\geq_k\). The proof relies on two facts: (i) an element belongs to \(r\) if and only if its negation does and (ii) if \(a = b \ast c\), where \(\ast \in \{\vee, \wedge, \vee_k, \wedge_k\}\) and \(c \in r\), then \(a \not\in r\) implies \(b \not\in r\).

The proposition allows us, if we prefer, to arrive at Theorem 4.2 without recourse to the piggyback method. As noted at the end of Section 3, it is possible to obtain a duality for a finitely generated lattice-based quasivariety \(A = \mathbb{ISP}(M)\) by including in the alter ego all subalgebras of \(M^2\). Applying this to \(DB = \mathbb{ISP}(4)\), we obtain a duality by equipping the alter ego with the four relations listed in Proposition 4.3. The subalgebras \(4^2\) and \(\Delta_{4^2}\) qualify as ‘trivial relations’ and can be discarded and we need only one of \(\leq_k\) and \(\geq_k\); see [15, Subsection 2.4.3]. Therefore the piggyback duality we presented earlier is essentially the only natural duality based on binary algebraic relations. (To have included relations of higher arity instead would have been possible, but would have produced a duality which is essentially the same, but artificially complicated.) We remark that the situation for DB is atypical, thanks to the very rich algebraic structure of 4.

5. A NATURAL DUALITY FOR UNBOUNDED DISTRIBUTIVE BILATTICES

We now focus on the variety \(DB_a\), to which we shall apply Theorem 3.2. We first need to represent \(DB_4\) as a finitely generated quasivariety.

**Proposition 5.1.** \(DB_a = \mathbb{ISP}(4_a)\).

**Proof.** We take \(A \in DB_a\) and \(a \neq b\) in \(A\) and use the Prime Ideal Theorem for unbounded distributive lattices to find \(x \in D_a(A, 2_a)\) with \(x(a) \neq x(b)\). We may then argue exactly as we did in the proof of Proposition 4.1, but now using the fact that \(4_a\) is, up to isomorphism, the only non-trivial algebra in \(DB_a\) of cardinality at most four.

We are ready to embark on setting up a piggyback duality for \(DB_a\). We find the piggybacking relations by drawing on the description of \(S(4^2)\) given in Proposition 4.3 to describe \(S(4_a^2)\). As a
byproduct, we shall see that among dualities whose alter egos contain relations which are at most binary, the knowledge order plays a distinguished role, just as it does in the duality for $\mathcal{DB}$.

Below, to simplify the notation, the elements of $4^2$ are written as pairs of binary strings. For example, 01 11 is our shorthand for (01, 11).

**Proposition 5.2.** The subalgebras of $4_u^2$ are of two types:

(a) the subalgebras of $4^2$, as identified in Proposition 4.3;

(b) decomposable subalgebras, in which each factor is $\{01\}$, $\{10\}$ or $4_u$.

**Proof.** The subalgebras of $4_u$ are $\{01\}$, $\{10\}$ and $4_u$. Any indecomposable subalgebra of $4_u^2$ must then be such that the projection maps onto each coordinate have image $4_u$. We claim that any indecomposable $\mathcal{DB}_u$-subalgebra $r$ of $4_u^2$ is a $\mathcal{DB}$-subalgebra of $4^2$. Suppose that $r \neq \Delta_{4_u^2}$, the diagonal subalgebra of $4_u^2$, and $r$ is indecomposable. Then $r$ would contain elements $a 01$ and $a' 10$ for some $a, a' \in 4_u$. If $a = 01$ and $a' = 10$. Then 11 11 and 00 00 are in $r$ and hence $r$ is a subalgebra of $4^2$. If $a \neq 01$, then also $(a \land_k \neg a) 01 \in r$. Any of the possibilities $a = 00, 11, 01$ implies that $10 01 \in r$. Therefore we must have $10 01 \in r$ and likewise $01 10 \in r$. Then, considering $\lor_k$ and $\land_k$, we get that 11 11 and 00 00 are in $r$. But this implies $01 10 \in r$, by considering $\land_k$. Similarly, $10 10 \in r$. The case $a' \neq 10$ follows by the same argument.

Figure ?? shows the lattice of subalgebras of $4_u^2$. In the figure the indecomposable subalgebras are unshaded and the decomposable ones are shaded.

![Figure 2. The subalgebras of $4_u^2$](image)

To list the piggybacking relations for $\mathcal{DB}_u$ we first need to establish some notation. For $\omega, \omega_1, \omega_2 \in \mathbb{H}_u(4_u)$ and $i \in \{0, 1\}$, let $R_{\omega_1, \omega_2}$ and $R^i_{\omega_2}$ be as defined in Theorem 3.2. We write $r_{\omega_1, \omega_2}$ respectively $r^i_{\omega_2}$, for the unique element of $R_{\omega_1, \omega_2}$ respectively $R^i_{\omega_2}$, whenever this set is a singleton. The set $\mathbb{H}_u(4_u)$ contains four elements: the maps $\alpha$ and $\beta$ defined earlier, and the constant maps onto 0 and 1, which we shall denote by $\mathbf{0}$ and $\mathbf{1}$, respectively. The following result is an easy consequence of Proposition 5.2.

**Proposition 5.3.** Consider $M = 4_u$. Then

(i) for the cases in which $R_{\omega_1, \omega_2}$ is a singleton,

(a) $r_{\alpha, \alpha}$ is $\leq_k$ and $r_{\beta, \beta}$ is $\geq_k$,

(b) $r_{\omega_1, \omega_2} = M^2$ whenever $\omega_1 = \emptyset$ or $\omega_2 = \mathbf{1}$,

(c) $r_{\alpha, \mathbf{0}} = \{01\} \times M$, $r_{\beta, \mathbf{0}} = \{10\} \times M$, $r_{\mathbf{1}, \alpha} = M \times \{01\}$, and $r_{\mathbf{1}, \beta} = M \times \{10\}$;

(ii) for the cases in which $R_{\omega_1, \omega_2}$ is not a singleton,

(a) $R_{\alpha, \beta} = \{\{01\} \times M, \{10 01\}\}$,

(b) $R_{\beta, \alpha} = \{\{10\} \times M, \{10 01\}\}$,

(c) $R_{\mathbf{1}, \mathbf{0}} = \emptyset$;

(iii) (a) $r^0_{\alpha} = r^1_{\beta} = \{01\}$ and $r^1_{\alpha} = r^0_{\beta} = \{10\}$,

(b) $r^0_{\mathbf{0}} = r^1_{\mathbf{1}} = M$ and $R^0_{\mathbf{0}} = R^1_{\mathbf{1}} = \emptyset$.

Below, when we describe the connections between the natural and Priestley-style dualities for $\mathcal{DB}_u$, we shall see that the subalgebras listed in Proposition 5.3 are exactly the relations we would expect to appear.

We now present our duality theorem for $\mathcal{DB}_u$. 
Theorem 5.4. (Natural duality for unbounded distributive bilattices) There is a strong, and hence full, duality between the category $\mathcal{DB}_u$ and the category $\mathcal{P}_{01}$ of doubly-pointed Priestley spaces set up by hom-functors. Specifically, let

$$4_u = \{(00, 01, 10)\}; \lor, \land, \land_k, \neg\}$$

be the four-element bilattice in the variety $\mathcal{DB}_u$ of distributive bilattices without bounds and let its alter ego be

$$4_y = \{(00, 11, 01); 01, 10, \leq_k, \top\},$$

where the elements 01 and 10 are treated as nullary operations. Then

$$\mathcal{DB}_u = \mathcal{ISP}(4_u) \quad \text{and} \quad \mathcal{P}_{01} = \mathcal{ISP}_c^+(4_y)$$

and the hom-functors $\mathcal{D} = \mathcal{DB}_u(-, 4_u)$ and $\mathcal{E} = \mathcal{P}_{01}(-, 4_y)$ set up the required dual equivalence between $\mathcal{DB}_u$ and $\mathcal{P}_{01}$.

Proof. Here we have included in the alter ego fewer relations than the full set of piggybacking relations as listed in Proposition 5.3 and we need to ensure that our restricted list suffices. To accomplish this we use simple facts about entailment as set out in [15, Subsection 2.4.3].

We have included as nullary operations both 01 and 10 and these entail the two one-element subalgebras $\{01\}$ and $\{10\}$ of 4_u. It then follows from Theorem 3.2 and Proposition 5.3 that 4_y yields a duality on $\mathcal{DB}_u$ (see [15, Section 2.4]). We now invoke the $M$-Shift Strong Duality Lemma [15, 3.2.3] to confirm that changing the alter ego by removing entailed relations does not result in a duality which fails to be strong.

Finally, we note that 4_y is a doubly-pointed Priestley space and hence a member of $\mathcal{P}_{01}$. In the other direction, 2_{01} is isomorphic to a closed substructure of 4_y and so belongs to $\mathcal{ISP}_c^+(4_y)$. Hence the dual category for the natural duality is indeed the category of doubly-pointed Priestley spaces. $\blacksquare$

6. HOW TO DISMOUNT FROM A PIGGYBACK RIDE

The piggyback method, applied to a class $\mathcal{A} = \mathcal{ISP}(M)$ of $\mathcal{D}$-based algebras, supplies an alter ego $M$ yielding a natural duality for $\mathcal{A}$, as described in Section 3. The relational structure of $M$ is constructed by bringing together 2 (the alter ego for Priestley duality for $\mathcal{ISP}(2)$) and $\mathcal{HU}(M)$ (the Priestley dual space of the distributive lattice reduct of the generating algebra of $\mathcal{A}$). This characteristic of the piggyback method has a significant consequence: it allows us, in a systematic way, to recover the Priestley dual spaces $\mathcal{HU}(A)$ of the $\mathcal{D}$-reducts of the algebras $A \in \mathcal{A}$. The procedure for doing this played a central role in [13], where it was used to study coproducts in quasivarieties of $\mathcal{D}$-based algebras. Below, in Theorem 6.1, we shall strengthen Theorem 2.3 of [13] by proving that the construction given there is functorial and is naturally equivalent to $\mathcal{HU}$.

Traditionally, dualities for $\mathcal{D}$-based (quasi)varieties have taken two forms: natural dualities, almost always for classes $\mathcal{A}$ which are finitely generated, and dualities which we dubbed $\mathcal{D}$-$\mathcal{P}$-based dualities in [13, Section 2]. In the latter, at the object level, the Priestley spaces of the $\mathcal{D}$-reducts of members of $\mathcal{A}$ are equipped with additional structure so that the operations of each algebra $A$ in $\mathcal{A}$ may be captured on $\mathcal{KHU}(A)$ (an isomorphic copy of $\mathcal{U}(A)$) from the structure imposed on the Priestley space $\mathcal{HU}(A)$. Now assume that $\mathcal{A} = \mathcal{ISP}(M)$, where $M$ is finite, so that a rival, natural, duality can be obtained by the piggyback method. Reconciliations of the two approaches appear rather rarely in the literature; we can however draw attention to [17, Section 3] and the remarks in [15, Section 7.4]. There are two ways one might go in order to effect a reconciliation. Firstly, we could use the fact that an algebra $A$ in $\mathcal{A}$ determines and is determined by its natural dual $D(A)$ and that $U(A)$ determines and is determined by $\mathcal{HU}(A)$. Given that, as we have indicated, we can determine $\mathcal{HU}(A)$ from $D(A)$, we could try to capitalise on this to discover how to enrich the Priestley spaces $\mathcal{HU}(A)$ to recapture the algebraic information lost in passage to the reducts. But this misses a key point about duality theory. The reason Priestley duality is such a useful tool is that it allows us concretely and in a functorial way to represent distributive lattices in terms of Priestley spaces. Up to categorical isomorphism, it is immaterial how the dual spaces are actually
constructed. An alternative strategy now suggests itself for obtaining a duality for $\mathcal{A}$ based on enriched Priestley spaces.

What we shall do in this section is to work with a version of Priestley duality based on structures directly derived from the natural duals $D(\mathcal{A})$ of the algebras $\mathcal{A}$, rather than one based on traditional Priestley duality applied to the class $U(\mathcal{A})$. This shift of viewpoint allows us to tap into the information encoded in the natural duality in a rather transparent way. We can hope thereby to arrive at a ‘Priestley-style’ duality for $\mathcal{A} = ISP(M)$. We shall demonstrate how this can be carried out in cases where the operations suppressed by the forgetful functor interact in a particularly well-behaved way with the operations which are retained. At the end of the section we also record how the strategy extends to $D_\omega$-based algebras.

In summary, we propose to base Priestley-style dualities on dual categories more closely linked to natural dualities rather than, as in the literature, seeking to enrich Priestley duality per se. The two approaches are essentially equivalent, but ours has several benefits. By staying close to a natural duality we are well placed to profit from the good categorical properties such a duality possesses. Moreover morphisms are treated alongside objects. Also, setting up a piggyback duality is an algorithmic process in a way that formulating a Priestley-style duality ab initio is not. Although we restrict attention in this paper to the special types of operation present in bilattice varieties, these could be handled by more traditional means, we note that our analysis has the potential to be adapted to other situations.

We now recall the construction of [13, Section 2] as it applies to the particular case of the piggyback theorem for the bounded case as stated in Theorem 3.1. Assume that $M$ and $R$ are as in that theorem. For a fixed algebra $A \in ISP(M)$, we define $Y_A = D(A) \times \Omega$, where $\Omega = D(U(A), 2)$, and equip it with the topology $T_{Y_A}$ having as a base of open sets

$$T_{Y_A} = \{ U \times V | U \text{ open in } D(A) \text{ and } V \subseteq \Omega \}$$

and with the binary relation $\preceq \subseteq Y_A^2$ defined by

$$(x, \omega_1) \preceq (y, \omega_2) \text{ if } (x, y) \in r^{D(A)} \text{ for some } r \in R_{\omega_1, \omega_2}.$$ 

In [13, Theorem 2.3], we proved that the binary relation $\preceq$ is a pre-order on $Y_A$. Moreover, if $\approx = \preceq \cap \succeq$ denotes the equivalence relation on $Y_A$ determined by $\preceq$ and $T_{Y_A}/\approx$ is the quotient topology, then $(Y_A/\approx; \preceq/\approx, T_{Y_A}/\approx)$ is a Priestley space isomorphic to $HU(A)$. This isomorphism is determined by the map $\Phi_A$ given by $\Phi_A([(x, \omega)]\approx) = \omega \circ x$.

**Theorem 6.1.** Let $\mathcal{A} = ISP(M)$, where $M$ is a finite algebra with a reduct in $D$. Then there exists a well-defined contravariant functor $L: \mathcal{A} \to P$ given by

- **on objects:** $A \mapsto L(A) = (Y_A/\approx; \preceq/\approx, T_{Y_A}/\approx),$
- **on morphisms:** $h \mapsto L(h): [(x, \omega)]\approx \mapsto [(D(h)(x), \omega)]\approx$.

Moreover, $\Phi$, defined on each $A$ by $\Phi_A: [(x, \omega)]\approx \mapsto \omega \circ x$, determines a natural isomorphism between $L$ and $HU$.

**Proof.** We have already noted that $L(A) \in P$. We confirm that $L$ is a functor. Let $h: A \to B$ and $(x, \omega), (y, \omega') \in Y_B$ be such that $(x, \omega) \preceq (y, \omega')$. Then there exists $r \in R_{\omega, \omega'}$ with $(x, y) \in r^{D(B)}$. Hence $(D(h)(x), D(h)(y)) \in r^{D(A)}$ and $(D(h)(x), \omega) \preceq (D(h)(y), \omega')$. Thus $L(h)$ is well defined and order-preserving. Since $D(h)$ is continuous and $Y_A/\approx$ carries the quotient topology, and since $L(h)^{-1}(U \times V) = D(h)^{-1}(U \times V)$, the map $L(h)$ is also continuous.

Theorem 3.1(c) in [13] proves that $\Phi_A: L(A) \to HU(A)$ is an isomorphism of Priestley spaces. We prove that $\Phi$ is natural in $\mathcal{A}$. Let $A, B \in \mathcal{A}, x \in D(B), h \in \mathcal{A}(A, B)$ and $\omega \in \Omega$. Then

$$\Phi_A(L(h)([(x, \omega)]\approx)) = \Phi_A([(D(h)(x), \omega)]\approx) = \Phi_A([(x \circ h, \omega)]\approx) = \omega \circ x \circ h = H(h)(\omega \circ x) = HU(h)(\omega \circ x) = HU(h)(\Phi_B([(x, \omega)]\approx)).$$

Therefore $\Phi$ is a natural isomorphism between the functors $L$ and $HU$. \hfill \Box

We take as before a $D$-based quasivariety $\mathcal{A} = ISP(M)$, with forgetful functor $U: \mathcal{A} \to D$, for which we have set up a piggyback duality. Theorem 6.1 tells us how, given an algebra $A \in \mathcal{A}$,
to obtain from the natural dual \(D(A)\) a Priestley space \(Y_A/\approx\) serving as the dual space of \(U(A)\). But it does not yet tell us how to capture on \(Y_A/\approx\) the algebraic operations not present in the reducts. However it should be borne in mind that the maps \(\omega\) in \(\Omega = HU(M)\) are an integral part of the natural duality construction and it is therefore unsurprising that these maps will play a direct role in the translation to a Priestley-style duality, if we can achieve this. We consider in turn operations of each of the types present in the bilattice context.

Assume first that \(f\) is a unary operation occurring in the type of algebras in \(A\) which interprets as a \(D\)-endomorphism on each \(A \in \mathcal{A}\). Then \(H(f^A) : HU(A) \rightarrow HU(A)\) is a continuous order-preserving map, given by \(H(f^A)(x) = x \circ f^A\), for each \(x \in HU(A)\). Conversely, \(f^A\) can be recovered from \(H(f^A)\) by setting \(f^A(a)\) for each \(a \in A\) to be the unique element of \(A\) for which \(x(f^A(a)) = (H(f^A) \circ x)(a)\) for each \(x \in HU(A)\). Denote \(H(f^A)\) by \(f^A\).

Then for each \(A \in \mathcal{A}\) the operation \(f^A\) is determined by \(f^M\). Dually, \(f^M\) should encode enough information to enable us, with the aid of Theorem 6.1, to recover \(f^A\). Define a map \(f_{Y_A} : Y_A \rightarrow Y_A\) by \(f_{Y_A}(x, \omega) = (x, \omega \circ f^M)\), for \(x \in D(A)\) and \(\omega \in \Omega\); here \(Y_A = D(A) \times \Omega\), as in Theorem 6.1. By definition of \((Y_A; \preceq, \mathcal{F}_A)\), the map \(f_{Y_A}\) is continuous. By Theorem 6.1(c), for every \(x, x' \in D(A)\) and \(\omega, \omega' \in \Omega\),

\[
(x, \omega) \approx (x', \omega') \iff \omega \circ x = \omega' \circ x' \\
\quad \iff \omega \circ f^M \circ x = \omega \circ f^A = \omega' \circ f^A = \omega' \circ f^M \circ x' \\
\quad \iff f_{Y_A}(x, \omega) \approx f_{Y_A}(x', \omega').
\]

Then the map \(\mathcal{F}_A : Y_A/\approx \rightarrow Y_A/\approx\) determined by \(\mathcal{F}_A([([x, \omega])_\approx]) = [f_{Y_A}(x, \omega)]_\approx\) is well defined and continuous. For each \((x, \omega) \in Y_A\) and \(a \in A\) we have

\[
f^A(\Phi_A([([x, \omega])_\approx])(a) = \omega \circ x(f^A(a)) = (\omega \circ f^M)(x(a)) = (\omega \circ f^M)(x(a)) = \Phi_A([([x, \omega \circ f^M])_\approx])(a) = \Phi_A(f^A([([x, \omega \circ f^M])_\approx]))(a).
\]

We have proved that \(f^A \circ \Phi_A = \Phi_A \circ \mathcal{F}_A\).

We now consider a unary operation \(h\) which interprets as a dual \(D\)-endomorphism on each \(U(A)\). As above, \(H(h^A) : HU(A) \rightarrow HU(A^\delta)\) is a continuous order-preserving map. Using the fact that the assignment \(x \mapsto 1 - x\) defines an isomorphism between the Priestley spaces \(HU(A)^\delta\) and \(HU(A^\delta)\), it is possible to define a map \(h^A : HU(A) \rightarrow HU(A)\) by \(h^A(x) = 1 - H(h^A)(x) = 1 - (x \circ h^A)\). Then \(h^A\) is continuous and order-reversing. Conversely, \(h^A\) is obtained from \(h^A\) by setting \(h^A(a)\) to be the unique element of \(A\) that satisfies \(x(h^A(a)) = (1 - (h^A(x)))(a)\) for each \(x \in HU(A)\). In the same way as before, we define a map \(h_{Y_A} : Y_A \rightarrow Y_A\) given by \(h_{Y_A}(x, \omega) = (x, 1 - \omega \circ h^M)\). Again we have an associated continuous (now order-reversing) map on \((Y_A; \preceq, \mathcal{F}_A)\) given by

\[
\mathcal{H}_A([([x, \omega])_\approx]) = [h_{Y_A}(x, \omega)]_\approx = [[x, 1 - \omega \circ h^M]]_\approx.
\]

Furthermore, \(\mathcal{H}_A \circ \Phi_A = \Phi_A \circ \mathcal{F}_A\).

Nullary operations are equally simple to handle. Suppose the algebras in \(A\) contain a nullary operation \(c\) in the type. Then for each \(A \in \mathcal{A}\) the constant \(c^A\) determines a clopen up-set \(c^A = \{x \in HU(A) \mid x(c^A) = 1\}\) in \(HU(A)\). Conversely, \(c^A\) is the unique element \(a\) of \(A\) such that \(x(a) = 1\) if and only if \(x \in c^A\). Now let \(c_{Y_A} = D(A) \times \{\omega \in \Omega \mid \omega(c^M) = 1\}\). In the same way as above we can move down to the Priestley space level and define

\[
c_{Y_A} = \{([x, \omega])_\approx \mid (x, \omega) \in c_{Y_A}\} = \{([x, \omega])_\approx \mid \omega(c^M) = 1\}.
\]

Then, for each \((x, \omega) \in Y_A\), we have

\[
\Phi_A([([x, \omega])_\approx]) \in c^A \iff 1 = (\omega \circ x)(c^A) = \omega(c^M) \\
\iff (x, \omega) \in c_{Y_A} \iff ([x, \omega])_\approx \in c_{Y_A}.
\]

That is, \(\Phi_A\) and its inverse interchange the sets \(c^A\) and \(c_{Y_A}\).
We sum up in the following theorem what we have shown on how enriched Priestley spaces may be obtained which encode the non-lattice operations of an algebra $A$ with a reduct $U(A)$ in $D$. Following common practice in similar situations, we shall simplify the presentation by assuming that only one operation of each kind is present. To state the theorem we need a definition. Let $Y$ be the category whose objects are the structures of the form $(Y; p, q, S)$, where $Y$ is a Priestley space, $p$ and $q$ are continuous self-maps on $Y$ which are respectively order-preserving and order-reversing, and $S$ is a distinguished clopen subset of $Y$. The morphisms of $Y$ are continuous order-preserving maps that commute with $p$ and $q$, and preserve $S$.

**Theorem 6.2.** Let $A = ISP(M)$ be a finitely generated quasivariety for which the language is that of $D$ augmented with two unary operation symbols, $f$ and $h$, and a nullary operation symbol $c$ such that, for each $A \in A$,

(i) $f^A$ acts as an endomorphism of $D$, and
(ii) $h^A$ acts as a dual endomorphism of $D$.

Then there exist well-defined contravariant functors $L^+$ and $HU^+$ from $A$ to $Y$ given by

- **on objects:** $L^+ : A \mapsto (L(A), \overline{f}_A, \overline{h}_A, \overline{c}_A)$,
- **on morphisms:** $L^+ : h \mapsto L(h)$;

and

- **on objects:** $HU^+ : A \mapsto (HU(A); \overline{f^A}, \overline{h^A}, \overline{c^A})$,
- **on morphisms:** $HU^+ : h \mapsto HU(h)$.

Moreover, $\Phi$, as defined in Theorem 6.1, is a natural equivalence between the functor $L^+$ and the functor $HU^+$.

Let $Y'$ denote the full subcategory of $Y$ whose objects are isomorphic to topological structures of the form $L^+(A)$ (or equivalently $HU^+(A)$) for some $A \in A$. the categories $A$ and $Y'$ are dually equivalent, with the equivalence determined by either $L^+$ or $HU^+$.

We now indicate the modifications that we have to make to Theorem 6.1 to handle the unbounded case. In Theorem 6.3, the sets of relations arising are as specified in Theorem 3.2.

Let $A = ISP(M)$, where $M$ is a finite algebra having a reduct $U_0(M)$ in $D_u$ and let $\Omega = H_uU_0(M)$. For each $A \in A$, let $Y_A = D_u(A) \times \Omega$ with the topology $F_Y$ having as a base of open sets $\{U \times V \mid U \text{ open in } D_u(A) \text{ and } V \subseteq \Omega\}$, and the binary relation $\leq \subseteq Y^2$ given by

$$(x,\omega) \leq (y,\omega) \text{ if } (x,y) \in r^{D_u(A)} \text{ for some } r \in R_{\omega_1,\omega_2}[.]$$

**Theorem 6.3.** Let $A = ISP(M)$, where $M$ is a finite algebra with a reduct in $D_u$. Then there exists a well-defined contravariant functor $L_u : A \rightarrow P_{01}$ given by

- **on objects:** $A \mapsto L_u(A) = (Y_A/\approx; \equiv/\approx, c_0, c_1, \overline{f}_Y, \overline{c}_Y)$,
- **on morphisms:** $h \mapsto L_u(h) : [(x,\omega)] \mapsto [(D_u(h)(x),\omega)]$.

Moreover, $\Phi$, defined on each $A$ by $\Phi_A([(x,\omega)]) = \omega \circ x$, determines a natural isomorphism between $L_u$ and $H_uU_0$.

**Proof.** The only new ingredient here as compared with the proof of Theorem 6.1 concerns the role of the constants. The argument used in the proof of that theorem, as given in [13, Theorem 2.3], can be applied directly to prove that $\Phi_A : (Y_A/\approx; \equiv/\approx, \overline{f}_Y, \overline{c}_Y) \rightarrow H_uU_0(A)$ defined by $\Phi_A([(x,\omega)]) = \omega \circ x$ is a well-defined homeomorphism which is also an order-isomorphism. To confirm that $L_u$ is well defined we shall show simultaneously that $\bigcup\{R_{\omega}^u \mid \omega \in \Omega\}/\approx$ is a singleton and that $\Phi_A$ maps its unique element to the corresponding constant map in $H_uU_0(A)$.

Thus $\{c_1\} = \bigcup\{R_{\omega}^u \mid \omega \in \Omega\}/\approx$ for $i \in \{0,1\}$.

Below we write $r$ rather than $r^{D_u(A)}$ for the lifting of a piggybacking relation $r$ to $D_u(A)$.

Let $\omega_1, \omega_2 \in \Omega$ and $r_1 \subseteq R_{\omega_1}^u, r_2 \subseteq R_{\omega_2}^u$, $x \in r_1$, and $y \in r_2$. For each $a \in A$, we have $\omega_1(a) = 1 = \omega_2(a)$. Then $\Phi_A([(x,\omega_1)]) = \Phi_A([(x,\omega_2)]) = 1$, where $1 : A \rightarrow \{0,1\}$ denotes the constant map $a \mapsto 1$. Since $\Phi_A$ is injective, $[(x,\omega_1)] = \nu([(x,\omega_2)])$. This proves that $|\bigcup\{R_{\omega}^u \mid \omega \in \Omega\}/\approx| \leq 1$ and that $\Phi_A(\bigcup\{R_{\omega}^u \mid \omega \in \Omega\}/\approx) \subseteq \{1\}$. Similarly, we obtain
From this and the definition of $\leq$ and $\Phi_A(\bigcup\{R_0^\omega \mid \omega \in \Omega\})/\approx \subseteq \{0\}$. Because $\Phi_A$ is surjective, there exists $x \in D_\omega(\mathcal{A})$ and $\omega \in \Omega$ such that $\omega \circ x = 1$. Then $x \in R_1^\omega$, which proves that $\bigcup\{R_1^\omega \mid \omega \in \Omega\} \neq \emptyset$. The same argument applies to $\bigcup\{R_0^\omega \mid \omega \in \Omega\}$. □

The arguments for handling additional operations in the bounded case carry over to piggyback dualities over $D_u$ with only the obvious modifications.

7. FROM A NATURAL DUALITY TO THE PRODUCT REPRESENTATION

The natural dualities in Theorems 4.2 and 5.4 combined with the Priestley dualities for bounded and unbounded distributive lattices, respectively, prove that $\mathcal{DB}$ is categorically equivalent to $\mathcal{D}$ and that $\mathcal{DB}_u$ is categorically equivalent to $\mathcal{D}_u$. These equivalences are set up by the functors $KD: \mathcal{DB} \to \mathcal{D}$ and $EH: \mathcal{D} \to \mathcal{DB}$, and $K_uD_u: \mathcal{DB}_u \to \mathcal{D}_u$ and $E_uH_u: \mathcal{D}_u \to \mathcal{DB}_u$.

With the aid of Theorem 6.1 we can give explicit descriptions of $EH$ and $KD$.

**Theorem 7.1.** Let $D: \mathcal{DB} \to \mathcal{P}$ and $E: \mathcal{P} \to \mathcal{DB}$ be the functors setting up the duality presented in Theorem 4.2. Then for each $\mathcal{A} \in \mathcal{DB}$ the Priestley dual $H(\mathcal{A}_t)$ of the t-lattice reduct of $\mathcal{A}$ is such that

$$H(\mathcal{A}_t) \cong D(\mathcal{A}) \bigsqcup D(\mathcal{A})^\beta,$$

where $\cong$ denotes an isomorphism of Priestley spaces.

**Proof.** Adopting the notation of Theorems 3.1 and 4.2, we note that in the proof of the latter we observed that

$$R_{\alpha,\beta} = R_{\beta,\alpha} = \emptyset, \quad r_{\alpha,\alpha} \leq_k \text{ and } r_{\beta,\beta} \geq_k$$

(here we have written $r_{\omega,\omega}$ for the unique element of $R_{\omega,\omega}$). As a result, for $\mathcal{A} \in \mathcal{DB}$, with $D(\mathcal{A}) = (X; \leq, \mathcal{T})$, we have

$$R_{\alpha,\beta}^{D(\mathcal{A})} = R_{\beta,\alpha}^{D(\mathcal{A})} = \emptyset, \quad r_{\alpha,\alpha}^{D(\mathcal{A})} \leq \text{ and } r_{\beta,\beta}^{D(\mathcal{A})} \geq .$$

From this and the definition of $\leq \subseteq Y_2^\mathcal{A}$ it follows that

$$(x, \omega_1) \leq (y, \omega_2) \iff \begin{cases} x \leq y \text{ and } \omega_1 = \omega_2 = \alpha, \text{ or} \\ x \geq y \text{ and } \omega_1 = \omega_2 = \beta. \end{cases}$$

Then $Y_\alpha = (D(\mathcal{A}) \times \Omega; \leq, \mathcal{T}_{Y_\alpha})$ is already a poset (no quotenting is required) for each $\mathcal{A} \in \mathcal{DB}$. And, order theoretically and topologically, $Y_\alpha$ is the disjoint union of ordered spaces $Y_\alpha$ and $Y_\beta$, where $Y_\alpha$ and $Y_\beta$ are the subspaces of $Y_\alpha$ determined by $D(\mathcal{A}) \times \{\alpha\}$ and $D(\mathcal{A}) \times \{\beta\}$, respectively. With this notation we also have $Y_\alpha \cong D(\mathcal{A})$ and $Y_\beta \cong D(\mathcal{A})^\beta$. The rest of the proof follows directly from Theorem 6.1 and the fact that finite coproducts in $\mathcal{P}$ correspond to disjoint unions [15, Theorem 6.2.4]. □

**Figure 3.** Obtaining $HU(\mathcal{A})$ from $D(\mathcal{A})$

Figure 3 shows the very simple way in which Theorem 7.1 tells us how to pass from the natural dual $D(\mathcal{A})$ of $\mathcal{A} \in \mathcal{DB}$ to the Priestley space $HU(\mathcal{A}) = H(\mathcal{A}_t)$. We start from copies $Y_\alpha$ and $Y_\beta$ of $D(\mathcal{A})$, indexed by the points $\alpha$ and $\beta$ of $\Omega = HU(\mathcal{A})$. The relation $\leq$ gives us the partial order on
$Y_\alpha \cup Y_\beta$ which restricts to $\leq_k$ on $Y_\alpha$ and $\geq_k$ on $Y_\beta$. The relation $\cong$ makes no identifications; in the right-hand diagram the two order comments are regarded as subsets of a single Priestley space; in the left-hand diagram they are regarded as two copies of the natural dual space. This very simple picture should be contrasted with the somewhat more complicated one we obtain below for the unbounded case; see Figure 4.

Theorem 7.1 shows us how to obtain $H(A_t)$ from $D(A)$. We conclude that for each $A \in \mathcal{A}$, the $t$-lattice reduct of $A$ is isomorphic to $L \times L^\prime$ where $L = KD(A)$. We will now see how to capture in $H(A_t)$ the algebraic operations suppressed by $U$. Drawing on Theorem 6.2 we have

$$\eta_A([(x,\alpha)]) = [(x,\beta)],$$
$$\eta_A((x,\beta)) = [(x,\alpha)];$$
$$\eta_A(\alpha \circ x) = \beta \circ x,$$
$$\eta_A(\beta \circ x) = \alpha \circ x;$$
$$\Gamma_k A = Y_\alpha,$$
$$\emptyset_k A = Y_\beta;$$
$$I_k^\alpha = \{ \alpha \circ x \mid x \in D(A) \},$$
$$I_k^\beta = \{ \beta \circ x \mid x \in D(A) \}.$$  

From this and Theorem 7.1, we obtain $KD(A) \cong A_t/\theta$ for each $A \in \mathcal{DB}$, where $\theta$ is the congruence defined by $a \theta b$ if and only if $a \wedge I_1 = b \wedge I_1$. Clearly $A_t/\theta$ is also isomorphic to the sublattice of $A_t$ determined by the set $\{ a \in A \mid a \leq_t I_1 \}$.

Since the duality we developed for $\mathcal{DB}$ was based on the piggyback duality using $A_t$ as the $\mathcal{D}$-reduct, Theorem 6.1 does not give us direct access to the $k$-lattice operations. Lemma 2.3 tells us that with the knowledge constants and the $t$-lattice operations we can access the $k$-lattice operations. But there is a way to recover the $k$-lattice operations directly from the dual space, and this can be adapted to cover the unbounded case too.

Take, as before, $\mathcal{A} = \mathcal{DB}$, $M = 4$ and $\Omega = \{ \alpha, \beta \}$. Let $A \in \mathcal{A}$ and $Y_A = D(A) \times \Omega$. Define a partial order $\preceq \subseteq Y_A$ by $(x,\omega) \preceq (y,\omega')$ if $\omega = \omega'$ and $x \preceq y$ in $D(A)$. It is clear that $(Y_A; \preceq, \tau_{Y_A}) \cong D(A) \prod_3 D(A)$. We claim that $H(A_k) \cong (Y_A; \preceq, \tau_{Y_A})$. To prove this, observe that, since $\alpha^{-1}(1) = \{11,01\}$ is a filter of the lattice $4_k$, the map $\alpha$ is a lattice homomorphism from $4_k$ into $2$. And since $\beta^{-1}(1) = \{11,10\}$ is an ideal in $4_k$ the map $\beta' = 1 - \beta$, is a lattice homomorphism from $4_k$ into $2$. It follows that we have a well-defined map $\eta_A : Y_A \rightarrow H(A_k)$ given by

$$\eta_A(x,\omega) = \begin{cases} \omega \circ x & \text{if } \omega = \alpha, \\ 1 - \omega \circ x & \text{if } \omega = \beta. \end{cases}$$

Assume that $(x,\omega) \preceq (y,\omega')$. Then $\omega = \omega'$ and for each $a \in A$ we have $x(a) \leq_k y(a)$ in $4$.

Since $\alpha$ is a $k$-lattice homomorphism, if $\omega = \omega'$, then

$$\eta_A(x,\alpha)(a) = \alpha(x(a)) \leq \alpha(y(a)) = \eta_A(y,\alpha)(a),$$

for each $a \in A$. If instead $\omega = \omega' = \beta$, we have $\beta_A(x(a)) \geq \beta_A(y(a))$ for each $a \in A$, then

$$\eta_A(x,\beta)(a) = 1 - \beta(x(a)) \leq 1 - \beta(y(a)) = \eta_A(y,\beta)(a).$$

Therefore $\eta_A$ preserves $\preceq$. To see that $\eta_A$ also reverses the order, assume $\eta_A(x,\omega) \leq \eta_A(y,\omega')$. Then $\eta_A(x,\omega)(a) \leq \eta_A(y,\omega')(a)$ in $2$, for each $a \in A$. Since $\alpha(1) = 1 \not< 0 = 1 - \beta(1)$ and $1 = \beta(0) = 1 \not< 0 = \alpha(1)$ it follows that $\omega = \omega'$. Now assume that $\omega = \omega' = \alpha$, then $\alpha(x(a)) \leq \alpha(y(a))$, for each $a \in A$, equivalently $(x(a),y(a)) \in r_{\alpha,\alpha} = \leq_k$ for each $a \in A$. By Theorem 5.4, $x \leq y$ in $D(A)$. We obtain $(x,\omega) \preceq (y,\omega)$. If $\omega = \omega' = \beta$ we argue in the same way, using the fact that $r_{\beta,\beta} = \geq_k$.

Finally, observe that for each $a \in A$, $b \in 4$ and $i \in 2$,

$$\eta_A(\{ x \in D(A) \mid x(a) = b \} \times \{ \alpha \}) = \{ z \in H(A_k) \mid z(a) = \alpha(b) \};$$

$$\eta_A(\{ x \in D(A) \mid x(a) = b \} \times \{ \beta \}) = \{ z \in H(A_k) \mid z(a) \neq \beta(b) \};$$

$$(\eta_A)^{-1}(\{ z \in H(A_k) \mid z(a) = i \}) = \{ x \in D(A) \mid x(a) \in \alpha^{-1}(1) \} \times \{ \alpha \};$$

$$\cup \{ x \in D(A) \mid x(a) \in \beta^{-1}(1 - i) \} \times \{ \beta \}. $$
Let \( \eta_A \) be a homeomorphism. Hence, as claimed, \( H(A_k) \cong (Y_A; \lesssim', T_{Y_A}) \). Since \( (Y_A; \lesssim', T_{Y_A}) \cong D(A) \coprod D(A) \), we conclude that \( A_k \cong L \times L \), where \( L \) denotes the lattice \( KD(A) \).

Theorem 7.1 can be seen as the product representation theorem for distributive bilattices expressed in dual form. We recall that, given a distributive lattice \( L = (L; \vee, \wedge, 0, 1) \), then \( L \odot L \) denotes the distributive bilattice with universe \( L \times L \) and lattice operations given by

\[
(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \wedge b_2), \quad (a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \vee b_2),
\]

negation is given by \( \neg (a) = (b, a) \) and the constants by \( 0_1 = (0, 1), 1_1 = (1, 0), 0_k = (0, 0) \) and \( 1_k = (1, 1) \). A well-known example is the representation of \( 4 \) as \( 2 \odot 2 \). More precisely, \( h: 4 \to 2 \odot 2 \) defined by \( h(ij) = (i, 1 - j) \), for \( i, j \in \{0, 1\} \), is an isomorphism.

As a consequence of Theorem 6.2 we obtain the following result.

**Theorem 7.2.** Let \( V: DB \to D \) and \( W: D \to DB \) be the functors defined by:

- on objects: \( A \mapsto V(A) = [0_k, 1_1] \),
- on morphisms: \( h \mapsto V(h) = h|_{[0_k, 1_1]} \),

where \( [0_k, 1_1] \) is considered as a sublattice of \( A_4 \) with bounds 0\( k \) and 1\(_1 \), and

- on objects: \( L \mapsto W(L) = L \odot L \),
- on morphisms: \( g \mapsto W(g): (a, b) \mapsto (g(a), g(b)) \).

Then \( V \) and \( W \) are naturally equivalent to \( KD \) and \( EH \), respectively.

**Corollary 7.3.** (The Product Representation Theorem for distributive bilattices) Let \( A \in DB \). Then there exists \( L = (L; \vee, \wedge, 0, 1) \in D \) such that \( A \cong L \odot L \).

We can now see the relationship between our natural duality for \( DB \) and the dualities presented for this class in [27, 24]. In [27], the duality for \( DB \) is obtained by first proving that the product representation is part of an equivalence between the categories \( DB \) and \( D \). The duality assigns to each \( A \in DB \) the Priestley space \( H([0_k, 1_1]) \), where the interval \( [0_k, 1_1] \) is considered as a sublattice of \( A_4 \). Then the functor from \( DB \) to \( P \) defined in [27, Corollaries 12 and 14] corresponds to \( HV \) where \( V: DB \to D \) is as defined in Theorem 7.2. The duality in [24], arrived at by a different route. At the object level, the authors consider first the De Morgan reduct of a bilattice and then enrich its dual structure by adding two clopen up-sets of the dual which represent the constants \( 0_k \) and \( 1_k \). In the notation of Theorem 6.2 their duality is based on the functor \( HU^+ \) by considering \( A = DB \) with only one lattice dual-endomorphism and two constants. The connection between their duality and ours follows from Theorems 6.1 and 6.2. Firstly, Theorem 6.1 tells us how to obtain \( L \) from \( D \). Then Theorem 6.2 shows how to enrich this functor to obtain \( L^+ \) and confirms that the latter is naturally equivalent to \( HU^+ \).

We now turn to the unbounded case, noting that, as regards dual representations, our results are entirely new, since neither [27] nor [24] considers duality for unbounded distributive bilattices. We shall rely on Theorem 6.3 to obtain a suitable description of \( K_nD_u \) and \( E_nH_u \). Fix \( A \in DB_u \) and let \( Y_\alpha = D(A) \times \{\omega\} \), for \( \omega \in \Omega = \{\alpha, \beta, 0, 1\} \). Let \( X \) be the doubly-pointed Priestley space obtained as in Theorem 6.3 by quotienting the pre-order \( \lesssim \) to obtain a partial order. Note that \( D(A) \) ordered by the pointwise lifting of \( \leq_k \) has top and bottom elements, viz. the constant maps onto 10 and onto 01, respectively. Hence, by Proposition 5.3(i)(c)–(d), \( Y_\alpha \) collapses to a single point and is identified with the bottom point of \( Y_\alpha \) and the top point of \( Y_\beta \). In the same way, \( Y_\top \) collapses to a point and is identified with the top point of \( Y_\alpha \) and with the bottom point of \( Y_\beta \). No additional identifications are made. This argument proves the following theorem.

**Theorem 7.4.** Let \( D_u: DB_u \to P_{01} \) and \( E_u: P_{01} \to DB_u \) be the functors setting up the duality presented in Theorem 5.4. Then for each \( A \in DB_u \) the Priestley dual \( H_u(A_i) \) of the t-lattice reduct of \( A \) is such that

\[
H_u(A_i) \cong D_u(A) \coprod P_{01} D_u(A) \circlearrowleft,
\]

where \( \cong \) denotes an isomorphism of doubly-pointed Priestley spaces.
Figure 4. Obtaining $H_u U_u (A)$ from $D_u (A)$

Figure 4 illustrates the passage from $(D(A) \times \Omega; \preceq; \mathcal{T})$ to $H_u U_u (A)$, including the way in which the union of the full set of piggybacking relations supplies a pre-order. The pre-ordered set $(Y_A; \preceq)$ has as its universe four copies of $D(A)$. Each copy is depicted in the figure by a linear sum of the form $1 \oplus P \oplus 1$; the top and bottom elements are depicted by circles. For $Y_A$, $P$ carries the lifting of the partial order $r_{\alpha\alpha}$, that is, $\preceq_k$ lifted to $D_B u (A, 4_u)$; for $Y_\beta$ the corresponding order is the lifting of $\geq_k$ to $D_B u (A, 4_u)$. Theorem 7.4 shows that $Y_T$, together with the top elements of $(Y_{\alpha}; \preceq_k)$ and of $(Y_{\beta}; \geq_k)$ form a single $\approx$-equivalence class, and likewise all elements of $Y_\bar{T}$ and the bottom elements of $Y_{\alpha}$ and of $Y_{\beta}$ form an $\approx$-equivalence class. These are the only $\approx$-equivalence class with more than one element. Thus the quotienting map which yields $H_u U_u (A)$ operates as shown. Topologically, the image $H_u U_u (A)$ carries the quotient topology, so that the top and bottom elements will both be isolated points if and only if $A_k$ is a bounded lattice.

Theorem 7.4 states that $H_u (A_k)$ is obtained as the coproduct of the doubly-pointed Priestley spaces $D_u (A)$ and $D_u (A)^\partial$. This coproduct corresponds to the product of unbounded distributive lattices $L = K_u D_u (A)$ and $L^\partial$, that is, $A_k \cong L \times L^\partial$. By the same argument as in the bounded case, $A_k \cong L \times L$. Moreover, using the analogue of Theorem 6.2, we have

$$\pi_A ([x, \alpha]) = [x, \beta], \quad \pi_A ([x, \beta]) = [x, \alpha];$$

$$\overline{\pi_A} (x \circ x) = \beta \circ x, \quad \overline{\pi_A} (\beta \circ x) = \alpha \circ x;$$

$$\overline{\pi_A} (1 \circ x) = 0 \circ x, \quad \overline{\pi_A} (0 \circ x) = 1 \circ x.$$

The construction of $L \otimes L$ for $L \in D$ applies equally well to $L \in D_u$; in this case the unbounded distributive bilattice $L \otimes L$ is defined on $L \times L$ by taking $(L \otimes L)_t = L \times L^\partial$, $(L \otimes L)_k = L \times L$ and $-L \otimes L (a, b) = (b, a)$, for each $a, b \in L$.

Given $A \in D_B u$, we define $L = K_u D_u (A)$. It follows from above that $A \cong L \otimes L$. Let $h: A \rightarrow L \otimes L$ denote the isomorphism between $A$ and $L \otimes L$. Then $L = A_1 / \ker (\rho)$ where $\rho (a) = a_1$ if $h(a) = (a_1, a_2)$. Using the $\circ$ construction we observe that $(a, b) \in \ker (\rho)$ if and only if $a \land_L b = a \lor_k b$. This can also be proved using the fact that closed subspaces of doubly-pointed Priestley spaces correspond to congruences and that

$$H_u (L) \cong Y_{\alpha} = D_u (A) \times \{ \alpha \} \cong Y_{\alpha} / \approx \subseteq Y_{\bar{A}} / \approx \cong D_u (A) \coprod_{\rho_1} D_u (A)^\partial \cong H_u (A_1).$$

Now observe that the isomorphism $Y_{\bar{A}} / \approx \cong H_u (A_1)$ is determined by the unique $P_{\alpha \beta}$-morphism such that $(x, \omega) \mapsto \omega \circ x$, for $\omega \in \{ \alpha, \beta \}$, and that $\alpha$ is a $D_u$-homomorphism from $A_1$ to $2_u$ and also from $A_k$ to $2_u$. We deduce that $(x \circ \alpha) (a) = (x \circ \alpha) (b)$ if and only if $a \land_L b = a \lor_k b$.

Our analysis yields the following theorem.
Theorem 7.5. For $A \in DB_u$ let $\theta_A = \{ (a, b) \in A^2 \mid a \land_k b = a \lor_k b \}$. Let $V_u: DB_u \to D_u$ and $W_u: D_u \to DB_u$ be the functors defined as follows:

- **on objects:** $A \mapsto V_u(A) = A_{/\theta_A}$,
- **on morphisms:** $h \mapsto V_u(h): [a]_{\theta_A} \mapsto [h(a)]_{\theta_B}$, where $h: A \to B$,

and

- **on objects:** $L \mapsto W_u(L) = L \com L$,
- **on morphisms:** $g \mapsto W_u(g): (a, b) \mapsto (g(a), g(b))$.

Then $V_u$ and $W_u$ are naturally equivalent to $K_uD_u$ and $E_uH_u$, respectively.

We have the following corollary; cf. [29, 9].

**Corollary 7.6.** (Product Representation Theorem for unbounded distributive bilattices) Let $A \in DB_u$. Then there exists a distributive lattice $L$ such that $A \cong L \com L$. Here the lattice $L$ may be identified with the quotient $A_{/\theta}$, where $\theta$ is the $D_u$-congruence given by $a \land \theta b$ if and only if $a \land_k b = a \lor_k b$.

Figure 5. The categorical equivalences in Theorems 7.2 and 7.5

8. Applications of the natural dualities for $DB$ and $DB_u$

In this section we demonstrate how the natural dualities we have developed so far lead easily to answers to questions of a categorical nature concerning $DB$ and $DB_u$. Using the categorical equivalence between $DB$ and $D$, and that between $DB_u$ and $D_u$, it is possible directly to translate certain concepts from one context to another. We shall concentrate on $DB$. Analogous results can be obtained for $DB_u$ and we mention these explicitly only where this seems warranted. We shall describe the following, in more or less detail: limits and colimits; free algebras; and projective and injective objects. These topics are very traditional, and our aim is simply to show how our viewpoint allows descriptions to be obtained, with the aid of duality, from corresponding descriptions in the context of distributive lattices. The results we obtain here are new, but unsurprising. We shall also venture into territory less explored by duality methods and consider unification type, and also admissible quasi-equations and clauses; here substantially more work is involved. It will be important for certain of the applications that we are dealing with strong, rather than merely full, dualities. Specifically we shall make use of the fact that if functors $D: A \to X$ and $E: X \to A$ set up a strong duality then surjections (injections) in $A$ correspond to embeddings (surjections) in $X$; see [15, Lemma 3.2.6]. On a technical point, we note that we always assume that an algebra has a non-empty universe.
Limits and colimits.

Since $\mathcal{DB}$ is a variety, the forgetful functor into the category $\mathcal{SET}$ of sets has a left adjoint. As a consequence all limits in $\mathcal{DB}$ are calculated as in $\mathcal{SET}$ (see [26, Section V.5]), and this renders them fairly easy to handle, with products being cartesian products and equalisers being calculated in $\mathcal{SET}$. (We refer the reader to [26, Section V.2] where the procedure to construct arbitrary limits from products and equalisers is fully explained.)

The calculation of colimits is more involved. The categorical equivalence between $\mathcal{DB}$ and $\mathcal{D}$ implies that if $S$ is a diagram in $\mathcal{DB}$ then

$$\text{Colim } S \cong \text{EH} (\text{Colim } \mathcal{KDS}) \cong W (\text{Colim } \mathcal{VS}).$$

This observation transfers the problem from one category to the other, but does not by itself solve it. However we can then use the natural duality derived in Theorem 4.2 in particular to compute finite colimits. We rely on the fact that colimits in $\mathcal{DB}$ correspond to limits in $\mathcal{P}$. Such limits are easily calculated, since cartesian products and equalisers of Priestley spaces are again in $\mathcal{P}$. (Corresponding statements hold for $\mathcal{DB}_u$ and $\mathcal{P}_01$ [15, Section 1.4].)

Congruences can be seen as particular cases of colimits, specifically as co-equalisers. This implies, on the one hand, that the congruences of an algebra in $\mathcal{DB}$ or in $\mathcal{DB}_u$ are in one-to-one correspondence with those substructures of its natural dual that arise as equalisers. Since $\mathcal{DB}$ is a variety and Theorem 4.2 supplies a strong duality, the lattice of congruences of an algebra $\mathcal{A}$ in $\mathcal{DB}$ is dually isomorphic to the lattice of closed substructures of its dual space (see [15, Theorem III.2.1]). Simultaneously, the lattice of congruences of $\mathcal{A} \in \mathcal{DB}$ is isomorphic to the lattice of congruences of $\mathcal{KD}(\mathcal{A}) \in \mathcal{D}$. Likewise, from Theorem 5.4, for each $\mathcal{A} \in \mathcal{DB}_u$ the congruence lattice of $\mathcal{A}$ is isomorphic to the congruence lattice of $\mathcal{K}_u \mathcal{D}_u(\mathcal{A}) \in \mathcal{D}_u$. The latter result was proved for interlaced bilattices in [29, Chapter II] using the product representation.

Free algebras.

A natural duality gives direct access to a description of free objects: If an alter ego $\mathcal{M}$ yields a duality on $\mathcal{A} = \mathcal{ISP}(\mathcal{M})$, then the power $\mathcal{M}^\lambda$ is the natural dual of the free algebra in $\mathcal{A}$ on $\lambda$ generators (see [15, Corollary II.2.4]). We immediately obtain $\mathcal{F}_D(\mathcal{A}) \cong \mathcal{E}_D(\mathcal{A}^\lambda)$ where $\lambda$ is a cardinal and $\mathcal{F}_D(\mathcal{A})$ denotes the free algebra on $\lambda$ generators in $\mathcal{DB}$; the free generators correspond to the projection maps.

Because $\mathcal{A} = \mathcal{2}^2$, we have $\mathcal{KD}(\mathcal{F}_D(\mathcal{A}^\lambda)) \cong \mathcal{F}_D(2\lambda)$. Therefore

$$\mathcal{F}_D(\mathcal{A}) \cong \mathcal{E}_D(\mathcal{F}_D(2\lambda)) \cong \mathcal{F}_D(2\lambda) \circ \mathcal{F}_D(2\lambda).$$

Hence $\mathcal{F}_D(2\lambda) \circ \mathcal{F}_D(2\lambda)$ is the free bounded distributive bilattice on $2\lambda$, the free generators being the pairs $\{x_{2i-1}, x_{2i}\}$ where $\{x_i \mid i \in 2\lambda\}$ is the set of free generators of $\mathcal{F}_D(2\lambda)$. Analogous results hold for $\mathcal{DB}_u$.

Injective and projective objects.

Injective, projective and weakly projective objects in $\mathcal{D}$ have been described (see [5] and the references therein; the definitions are given in Chapter I and results in Sections V.9 and V.10). The notions of injective and projective object are preserved under categorical equivalences. For categories which are classes of algebras with homomorphisms as the morphisms, weak projectives are also preserved under categorical equivalences. A distributive lattice $\mathcal{L}$ (with bounds) is injective in $\mathcal{D}$ (and in $\mathcal{DB}_u$ too) if and only it is complete and each element of $\mathcal{L}$ is complemented (see [5, Section V.9]). This implies that a distributive bilattice $\mathcal{A}$ is injective in $\mathcal{DB}$ if and only if $\mathcal{A}_k$ is complete (or equivalently $\mathcal{A}_k$ is complete) and each element of $\mathcal{A}$ is complemented in $\mathcal{A}_k$ (or equivalently $\mathcal{A}_k$ is complete and each element of $\mathcal{A}$ is complemented in $\mathcal{A}_k$). Moreover, since $\mathcal{D}$ has enough injectives, the same is true of $\mathcal{DB}$. Corresponding statements can be made for $\mathcal{DB}_u$.

The algebra $\mathcal{2}$ is the only projective of $\mathcal{D}$ [5, Section V.10]. Hence $\mathcal{4}$ is the only projective in $\mathcal{DB}$. The general description of weak projectives in $\mathcal{D}$ is rather involved (see [5, Section V.10]). But in the case of finite algebras there is a simple dual characterisation: a finite bounded distributive lattice is weakly projective in $\mathcal{D}$ if and only if its dual space is a lattice. This translates to bilattices: a finite distributive bilattice is weakly projective in $\mathcal{DB}$ if and only if its natural dual is a lattice, or equivalently if the family of homomorphisms into $\mathcal{4}$, ordered pointwise by $\leq_k$. 
forms a lattice. In the unbounded case we note that \( DB_a \) has no projectives since \( D_a \) has none, and that a finite member \( A \) of \( DB_a \) is weakly projective if and only if \( D_a(A) \) is a lattice.

**Unification type.**

The notion of unification was introduced by Robinson in [30]. Loosely, (syntactic) unification is the process of finding substitutions that equalise pairs of terms. When considering equivalence under an equational theory instead of equality the notion of unification evolves to encompass the concept of *equational unification*. We refer the reader to [3] for the general definitions and background theory of unification. To study the unification type of bilattices we shall use the notion of algebraic unification developed by Ghilardi in [21].

Let \( A \) be a finitely presented algebra in a quasivariety \( A \). A *unifier* for \( A \) in \( A \) is a homomorphism \( u: A \to P \), where \( P \) is a finitely generated weakly projective algebra in \( A \). (In [21] weakly projective algebras are called regular projective or simply projective.) An algebra \( A \) is said to be *solvable* in \( A \) if there exists at least one unifier for it. Let \( u_i: A \to P_i \) for \( i \in \{1, 2\} \) be unifiers for \( A \) in \( A \). Then \( u_1 \) is *more general* than \( u_2 \), in symbols, \( u_2 \leq u_1 \), if there exists a homomorphism \( f: P_1 \to P_2 \) such that \( f \circ u_1 = u_2 \). A unifier \( u \) for \( A \) is said to be a *most general unifier* (an mg-unifier) of \( A \) in \( A \) if \( u \leq u' \) implies \( u' \leq u \). For \( A \) solvable in \( A \) the *type* of \( A \) is defined as follows:

- **nullary** if there exists \( u \), a unifier of \( A \), such that \( u \not\subseteq v \) for each mg-unifier of \( A \) (in symbols, \( t_{type}(A) = 0 \));
- **unitary** if there exists a unifier \( u \) of \( A \) such that \( v \leq u \) for each unifier \( v \) of \( A \) (\( t_{type}(A) = 1 \));
- **finitary** if there exists a finite set \( U \) of mg-unifiers of \( A \) such that for each unifier \( v \) of \( A \) there exists \( u \in U \) with \( v \leq u \), and for each \( v \) of \( A \) there exists \( w \) unifier of \( A \) with \( w \not\subseteq v \) (\( t_{type}(A) = \omega \)); and
- **infinitary** otherwise (\( t_{type}(A) = \infty \)).

In [4], an algorithm to classify finitely presented bounded distributive lattices by their unification type was presented. Since the unification type of an algebra is a categorical invariant (see [21]), the results in [4] can be combined with the equivalence between \( DB \) and \( D \) to investigate the unification types of finite distributive bilattices.

Moreover, since the results in [4] were obtained using Priestley duality for \( D \), we can directly translate the results to bilattices and their natural duals. This yields the following characterisation. Let \( A \) be a finitely presented (equivalently, finite) bounded distributive bilattice. Then \( A \) is solvable in \( DB \) if and only if it is non-trivial and

\[
\text{type}_{DB}(A) = \begin{cases} 
1 & \text{if } DB(A) \text{ is a lattice, i.e., if } A \text{ is weakly projective,} \\
\omega & \text{if } DB(A) \text{ is not a lattice and for each } x, y \in DB(A) \text{ the interval } [x, y] \text{ is a lattice,} \\
0 & \text{otherwise.}
\end{cases}
\]

In [4] the corresponding theory for unbounded distributive lattices was not developed. With minor modifications to the proofs presented there, it is easy to extend the results to \( D_a \). Its translation to \( DB_a \) is as follows. Each finite algebra \( A \) in \( DB_a \) is solvable and

\[
\text{type}_{DB_a}(A) = \begin{cases} 
1 & \text{if } DB_a(A) \text{ is a lattice, i.e., if } A \text{ is weakly projective,} \\
0 & \text{otherwise.}
\end{cases}
\]

**Admissibility.**

The concept of admissibility was introduced by Lorenzen for intuitionistic logic [25]. Informally, a rule is admissible in a logic if when the rule is added to the system it does not modify the notion of theoremhood. The study of admissible rules for logics that admit an algebraic semantic has led to the investigation of admissible rules for equational logics of classes of algebras. For background on admissibility we refer the reader to [32].

A *clause* in an algebraic language \( L \) is an ordered pair of finite sets of \( L \)-identities, written \((\Sigma, \Delta)\). Such a clause is called a *quasi-identity* if \( \Delta \) contains only one identity. Let \( A \) be a quasivariety of algebras with language \( L \). We say that the \( L \)-clause \((\Sigma, \Delta)\) is *valid* in \( A \) (in
symbols $\Sigma \models_A \Delta$ if for every $A \in \mathcal{A}$ and homomorphism $h: \text{Term}_L \rightarrow A$, we have that $\Sigma \subseteq \ker h$ implies $\Delta \cap \ker h \neq \emptyset$, where $\text{Term}_L$ denotes the term (or absolutely free) algebra for $L$ over countably many variables (we are assuming that $\Sigma \models_A \Delta \cap \forall x \equiv \Delta$). For simplicity we shall work with the following equivalent definition of admissible clause: the clause $(\Sigma \Delta \Gamma)$ is called $\text{admissible in } A$ if it is valid in the free $\mathcal{A}$-algebra on countably many generators, $F\mathcal{A}(N_0)$.

Let $\mathcal{A}$ be a quasivariety. If a set of quasi-identities $\Lambda$ is such that $A \in \mathcal{A}$ belongs to the quasivariety generated by $F\mathcal{A}(N_0)$ if and only if $A$ satisfies the quasi-identities in $\Lambda$, then $\Lambda$ is called a basis for the admissible quasi-identities of $\mathcal{A}$. Similarly, $\Lambda$ is called a basis for the admissible clauses of $\mathcal{A}$ if $A$ satisfies the clauses in $\Lambda$ if and only if $A$ is in the universal class generated $F\mathcal{A}(N_0)$, that is, $A$ satisfies the same clauses as $F\mathcal{A}(N_0)$ does.

In the case of a locally finite quasivariety, checking that a set of clauses or quasi-identities is a basis can be restricted to finite algebras.

Lemma 8.1. [12] Let $\mathcal{A}$ be a locally finite quasivariety and let $\Lambda$ be a set of clauses in the language of $\mathcal{A}$.

(i) The following statements are equivalent:
   (a) for each finite $A \in \mathcal{A}$ it is the case that $A \in \text{ISP}(F\mathcal{A}(N_0))$ if and only if $A$ satisfies $\Lambda$;
   (b) $\Lambda$ is a basis for the admissible clauses of $\mathcal{A}$.

(ii) If the set $\Lambda$ consists of quasi-identities, then the following statements are equivalent:
   (a) for each finite $A \in \mathcal{A}$ it is the case that $A \in \text{ISP}(F\mathcal{A}(N_0))$ if and only if $A$ satisfies $\Lambda$;
   (b) $\Lambda$ is a basis for the admissible quasi-identities of $\mathcal{A}$.

In [12], using this lemma and the appropriate natural dualities, bases for admissible quasi-identities and clauses were presented for various classes of algebras—bounded distributive lattices, Stone algebras and De Morgan algebras, among others. Here we follow the same strategy using the dualities for $\mathcal{D}_B$ and $\mathcal{D}_B^*$ developed in Sections 4 and 5.

Lemma 8.2. Let $A$ be a finite distributive bilattice.

(i) $A \in \text{ISP}(F\mathcal{D}_B(N_0))$.

(ii) The following statements are equivalent:
   (a) $A \in \text{ISP}(F\mathcal{D}_B(N_0))$;
   (b) $D\mathcal{D}_B(A)$ is a non-empty bounded poset;
   (c) $A$ satisfies the following clauses:
      (1) $\{\{x \wedge_k y \approx y \approx 1\}, \{x \approx 1, y \approx 1\}\}$,
      (2) $\{\{x \lor_k y \approx y \approx 1\}, \{x \approx 1, y \approx 1\}\}$,
      (3) $\{0 \wedge 1\}$.

Proof. To prove (i) it is enough to observe that if $A$ is a subalgebra of any non-trivial algebra in $\mathcal{D}_B$, and therefore $\mathcal{D}_B = \text{ISP}(4) \subseteq \text{ISP}(F\mathcal{D}_B(N_0)) \subseteq \mathcal{D}_B$.

To prove (ii)(a)$\Rightarrow$(ii)(b), let $h: A \rightarrow F\mathcal{D}_B(N_0)$ be an injective homomorphism. Then the map $D\mathcal{D}_B(h): D\mathcal{D}_B(F\mathcal{D}_B(N_0)) \rightarrow D\mathcal{D}_B(A)$ is an order-preserving continuous and onto $D\mathcal{D}_B(A)$. Since $D\mathcal{D}_B(F\mathcal{D}_B(N_0)) \cong \mathbb{A}^{N_0}$ is bounded and non-empty, so is $D\mathcal{D}_B(A)$.

We next prove the converse, namely (ii)(b) $\Rightarrow$ (ii)(a). Let $t, b: A \rightarrow 4$ be the top and bottom elements of $D\mathcal{D}_B(A)$ and let $\{t, b, x_1, \ldots, x_n\}$ be an enumeration of the elements of the finite set $D\mathcal{D}_B(A)$. Let $P = \mathbb{A}^{n}$, then $E\mathcal{D}_B(P)$ is the free bounded distributive bilattice on $n$ generators. Then $E\mathcal{D}_B(P)$ belongs to $\text{ISP}(F\mathcal{D}_B(N_0))$. Now define $f: P \rightarrow D\mathcal{D}_B(A)$ by

$$f(c_1, \ldots, c_n) = \begin{cases} b & \text{if } c_i = 0_k \text{ for each } i \in \{1, \ldots, n\}, \\ x_i & \text{if } c_i \neq 0_k \text{ and } c_j = 0_k \text{ for each } j \in \{1, \ldots, n\} \setminus \{i\}, \\ t & \text{otherwise.} \end{cases}$$

It is easy to check that $f$ is order-preserving and maps $P$ onto $D\mathcal{D}_B(A)$. Since the natural duality of Theorem 4.2 is strong, the dual homomorphism $E\mathcal{D}_B(f): ED(A) \rightarrow E\mathcal{D}_B(P)$ is injective. Hence

$$A \cong ED(A) \in \text{ISP}(E\mathcal{D}_B(P)) \subseteq \text{ISP}(F\mathcal{D}_B(N_0)).$$

We now prove (ii)(b)$\Rightarrow$(ii)(c). Let $t: A \rightarrow 4$ be the top element of $D\mathcal{D}_B(A)$ and assume that $a, b \in A$ are such that $a \land_k b = 1_t$. If we assume that $a \neq 1_t \neq b$ then there exist $h_1, h_2: A \rightarrow 4$
such that $1_t <_k h_1(a)$ and $1_t <_k h_2(b)$. Since the order in $D_{\mathcal{DB}}(A)$ is determined pointwise by $\leq_k$, we then have $1_t <_k t(a), t(b)$. Then $t(a) = t(b) = 1_t$ and $t(a \land b) = 1_k$, a contradiction. Then $a = 1_t$ or $b = 1_t$. A similar argument proves that $D_{\mathcal{DB}}(A)$ having a lower bound implies that clause (2) is valid in $A$. If $A \in \mathcal{DB}$ is such that $0_t = 1_t$ then $A$ is trivial and $D_{\mathcal{DB}}(A)$ is empty. This proves that clause (3) is valid in any algebra $A$ whose natural dual $D_{\mathcal{DB}}(A)$ is non-empty.

Finally we prove (ii)(c) $\Rightarrow$ (ii)(b). Let $F = \{ c \in A \mid 1_t \leq_k c \}$. By clause (3), $A$ is non-trivial, so $0_t \notin F$. By clause (2), $F$ is a prime $t$-filter and it contains $1_t$. Thus it is a prime $t$-filter, as observed at the end of Section 2.

Let $x : A \to 2$ be the characteristic function of $F$. Then the map $f : A \to 4$ defined for each $a \in A$ by $f(a) = x(a)(1 - x(\neg a))$ is a well-defined bilattice homomorphism, as observed after Theorem 4.1. We shall prove that $f$ is the bottom element of $D_{\mathcal{DB}}(A)$. Let $a \in D_{\mathcal{DB}}(A)$ and $a \in A$. If $a \in F$ and $\neg a \notin F$, since $1_t \leq_k a$, then $f(a) = 1_t \leq_k h(a)$. If $a, \neg a \in F$, then $1_t \leq_k h(a), \neg h(a)$. Then $h(a) = 1_k = f(a)$. The other two cases follow by a similar argument, since $1_t \leq_k \neg a, \neg \neg a$. Then $f(a) \leq_k h(a)$ for each $a \in A$. This proves that $f \leq h$ in $D_{\mathcal{DB}}(A)$.

By a similar argument the validity of clause (1) implies that $D_{\mathcal{DB}}(A)$ is upper-bounded. □

Combining Lemmas 8.1 and 8.2 we obtain the following theorem.

**Theorem 8.3.** Every admissible quasi-equation in $\mathcal{DB}$ is also valid in $\mathcal{DB}$. Moreover the following clauses form a basis for the admissible clauses for $\mathcal{DB}$

$$
\{(x \land_k y \approx 1_k), (x \approx 1_t, y \approx 1_t)\}, \quad \{(x \lor_k y \approx 1_k), (x \approx 1_t, y \approx 1_t)\}
$$

and

$$
\{(0_t = 1_t), \emptyset\}.
$$

To simplify the proof of Lemma 8.2 the clauses presented in the previous theorem used the $k$-lattice operation. We can use Lemma 2.3 to rewrite the clauses using only constants and $t$-lattice operations.

**Lemma 8.4.** Every finite unbounded distributive bilattice $A$ is isomorphic to a subalgebra of $F_{\mathcal{DB}_u}(\aleph_0)$.

**Proof.** Let $D_u(A) = (X; \leq, \top, \bot, T)$. Since we assume that every algebra is non-empty, $X$ is non-empty. Let $X = \{ \top, \bot, x_1, \ldots, x_n \}$ be an enumeration of the elements of $X$. Let $Q = (\mathbb{A})^n$. Then $E_u(Q)$ is the free distributive bilattice on $n$ generators and it belongs to $\mathbb{S}(F_{\mathcal{DB}_u}(\aleph_0))$. Define $f : Q \to D_{\mathcal{DB}}(A)$ by

$$
f(c_1, \ldots, c_n) = \begin{cases} \bot & \text{if } c_i = 0_k \text{ for each } i \in \{1, \ldots, n\}, \\ x_i & \text{if } c_i \neq 0_k \text{ and } c_j = 0_k \text{ for each } j \in \{1, \ldots, n\} \setminus \{i\}, \\ \top & \text{otherwise.} \end{cases}
$$

Then $f$ is a continuous order-preserving map with image $D_u(A)$. Since the duality presented in Theorem 5.4 is strong, $E_u(f) : E_uD_u(A) \to E_u(Q)$ is injective. Then $A \in \mathbb{S}(F_{\mathcal{DB}_u}(\aleph_0))$. □

The following theorem follows directly from Lemmas 8.1 and 8.4.

**Theorem 8.5.** Every admissible clause in $\mathcal{DB}_u$ is also valid in $\mathcal{DB}_u$.

9. **Multisorted natural dualities**

We have delayed presenting dualities for pre-bilattice varieties because, to fit $\mathcal{DPB}$ and $\mathcal{DPB}_u$ into our general representation scheme, we shall draw on the multisorted version of natural duality theory. This originated in [17] and is summarised in [15, Chapter 7]. It is applicable in particular to the situation that interests us, in which we have a quasivariety $\mathcal{A} = \mathbb{S}(\mathcal{M}_1, \mathcal{M}_2)$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-isomorphic finite algebras of common type having a reduct in $\mathcal{D}$ or $\mathcal{D}_u$. We require the theory only for algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ of size two. We do not set up the machinery of piggybacking, opting instead to work with the multisorted version of the NU Duality Theorem, as given in [15, Theorem 7.1.2], in a form adequate to yield strong dualities for $\mathcal{DPB}$ and $\mathcal{DPB}_u$. 
We now give just enough information to enable us to formulate the results we require. The ideas parallel those presented in Section 3.

Given \( \mathcal{A} = \mathbb{ISP}(M_1, M_2) = \mathbb{ISP}(\mathcal{M}) \), we shall initially consider an alter ego for \( \mathcal{M} \) which takes the form \( \mathcal{M}' = (M_1 \cup M_2; R, \mathcal{T}) \), where \( R \) is a set of relations each of which is a subalgebra of some \( M_i \times M_j \), where \( i, j \in \{1, 2\} \). (To obtain a strong duality we may need to allow for nullary operations as well, but for simplicity we defer introducing this refinement.) The alter ego \( \mathcal{M}' \) is given the disjoint union topology derived from the discrete topology on \( M_1 \) and \( M_2 \). We may then form multisorted topological structures \( \mathcal{X} = X_1 \cup X_2 \) where each of the sorts \( X_i \) is a Boolean topological space, \( \mathcal{X} \) is equipped with the disjoint union topology and, regarded as a structure, \( \mathcal{X} \) carries a set \( R^X \) of relations \( r^X \); if \( r \subseteq M_i \times M_j \), then \( r^X \subseteq X_i \times X_j \). Given structures \( \mathcal{X} \) and \( \mathcal{Y} \) in \( \mathcal{X} \), a morphism \( \phi: \mathcal{X} \rightarrow \mathcal{Y} \) is a continuous map preserving the sorts, so that \( \phi(X_i) \subseteq Y_i \), and \( \phi \) preserves the relational structure. The terms isomorphism, embedding, etc., extend in the obvious way to the multisorted setting.

We define our dual category \( \mathcal{X} \) to have as objects those structures \( \mathcal{X} \) which belong to \( \mathbb{IS_P}^+(\mathcal{M}) \). Thus \( \mathcal{X} \) consists of isomorphic copies of closed substructures of powers of \( \mathcal{M} \); here powers are formed ‘by sorts’, and the relational structure is lifted pointwise to substructures of such powers in the expected way. We now define the hom-functors that will set up our duality. Given \( \mathcal{A} \in \mathcal{A} \) and we let \( D(A) = A(M_1, M_1) \cup A(M_1, M_2) \), where \( A(M_1, M_1) \cup A(M_1, M_2) \) is a (necessarily closed) substructure of \( M_1^1 \cup M_2^1 \) with the relational structure defined pointwise. Given \( X = X_1 \cup X_2 \in \mathcal{X} \), we may form the set \( \mathfrak{X}(X, \mathcal{M}) \) of \( \mathcal{X} \)-morphisms from \( \mathcal{X} \) into \( \mathcal{M} \). This set acquires the structure of a member of \( \mathcal{A} \) by virtue of viewing it as a subalgebra of the power \( M_1^X_1 \times M_2^X_2 \). We define \( E(X) = \mathfrak{X}(X, \mathcal{M}) \). Let \( D \) and \( E \) act on morphisms by composition in the obvious way. We then have well-defined functors \( D: \mathcal{A} \rightarrow \mathcal{X} \) and \( E: \mathcal{X} \rightarrow \mathcal{A} \). We say \( \mathcal{M} \) yields a multisorted duality if, for each \( \mathcal{A} \in \mathcal{A} \), the natural multisorted evaluation map \( e_\mathcal{A} \) given by \( e_\mathcal{A}(a): x \rightarrow x(a) \) is an isomorphism from \( \mathcal{A} \) to \( DE(A) \). The duality is full if each evaluation map \( e_\mathcal{X}: \mathcal{X} \rightarrow DE(X) \) is an isomorphism. As before we do not present the definition of strong duality, noting only that a strong duality is necessarily full. The following very restricted form of \([15, \text{Theorem 7.1.1}]\) will meet our needs.

**Theorem 9.1.** (Multisorted NU Strong Duality Theorem, special case) Let \( \mathcal{A} = \mathbb{ISP}(M_1, M_2) \), where \( M_1, M_2 \) are two-element algebras of common type having lattice reducts. Let \( \mathcal{M} = (M_1 \cup M_2; R, N, \mathcal{T}) \) where \( N \) contains all one-element subalgebras of \( M_i \), for \( i = 1, 2 \), treated as nullary operations, \( R \) is the set \( \bigcup \{ S(M_i \times M_j) \mid i, j \in \{1, 2\} \} \), and \( \mathcal{T} \) is the disjoint union topology obtained from the discrete topology on \( M_1 \) and \( M_2 \). Then \( \mathcal{M} \) yields a multisorted duality on \( \mathcal{A} \) which is strong.

**10. Dualities for distributive pre-bilattices**

Paralleling our treatment of other varieties, we first record the result on the structure of \( \mathcal{DPB}_u \) and \( \mathcal{DPB} \) we shall require.

**Proposition 10.1.** (i) \( \mathcal{DPB}_u = \mathbb{ISP}(2_u^+, 2_u^-) \) and (ii) \( \mathcal{DPB} = \mathbb{ISP}(2^+, 2^-) \).

**Proof.** Let \( \mathcal{A} \in \mathcal{DPB}_u \) and let \( a \neq b \) in \( \mathcal{A} \). Since \( \mathcal{D}_u = \mathbb{ISP}(2_u) \), there exists \( x \in \mathcal{D}_u(A_1, 2_u) \) with \( x(a) \neq x(b) \). The relation \( \theta \) given by \( x \theta d \) if and only if \( x(c) = x(d) \) is a \( t \)-lattice congruence and hence, by Proposition 2.2, a \( \mathcal{DPB}_u \)-congruence. The associated quotient algebra has two elements, and is necessarily (isomorphic to) either \( 2_u^+ \) or \( 2_u^- \). This proves (i). The same form of argument works for (ii), the only difference being that the map \( x \) now also preserves bounds. \( \square \)

The following two theorems are consequences of the Multisorted NU Duality Theorem. We consider \( \mathcal{DPB} \) first since the absence of one-element subalgebras makes matters particularly simple. We tag elements with \( \pm \) to indicate which 2-element algebra they belong to. In both cases we could use either \( \leq_k \) or \( \leq_t \) as the subalgebra of the square in either component. The choice we make mirrors that forced when negation is present. The choice will affect how the translation to the Priestley-style duality operates, but not the resulting duality.
Theorem 10.2. A strong natural duality for $\mathcal{DPB} = \mathbb{ISP}(2^+, 2^-)$ is obtained as follows. Take $\mathcal{M} = \{2^+, 2^-\}$ and as the alter ego

$$\mathcal{N} = \{\{0^+, 1^+\} \cup \{0^-, 1^-\}; r^+, r^-, T\},$$

where $r^+$ is $\leq_k$ on $2^+$ and $r^-$ is $\leq_k$ on $2^-$. Moreover $\mathcal{DPB}$ is dually equivalent to the category $\mathfrak{X} = \mathbb{ISP}^+(\mathcal{N})$.

Proof. The algebras $2^+, 2^-, 2^+ \times 2^-$ and $2^- \times 2^+$ have no proper subalgebras. The proper subalgebras of $2^+ \times 2^+$ are the diagonal subalgebra $\{(0,0), (1,1)\}$, and $\leq_k$ and its converse, and likewise for $2^- \times 2^-$. $\square$

Let $\mathfrak{M}$ and $\mathfrak{X}$ be as in Theorem 10.2. Since $r^+$ and $r^-$ are partial orders on the respective sorts, $(X_1, X_2; \leq_1, \leq_2, T)$ belongs to $\mathbb{ISP}^+(\mathcal{M})$ if and only if the topological posets $(X_1, \leq_1, T|_{X_1})$ and $(X_2, \leq_2, T|_{X_2})$ are Priestley spaces. Moreover, since the morphisms in $\mathfrak{X}$ are continuous maps that preserve the sorts and both relations, we deduce that a categorical equivalence between $\mathfrak{X}$ and $\mathcal{P} \times \mathcal{P}$ is set up by the functors $F: \mathfrak{X} \to \mathcal{P} \times \mathcal{P}$ and $G: \mathcal{P} \times \mathcal{P} \to \mathfrak{X}$ defined by

on objects: $\mathfrak{X} = (X_1 \cup X_2; \leq_1, \leq_2, T) \mapsto F(X) =$

$$((X_1; \leq_1, T|_{X_1}), (X_2; \leq_2, T|_{X_2})),$$

on morphisms: $h \mapsto F(h) = (h|_{X_1}, h|_{X_2}),$

and

on objects: $\mathcal{Z} = ((X; \leq_X, T_X), (Y; \leq_Y, T_Y)) \mapsto G(\mathcal{Z}) =$

$$(X \cup Y; \leq_X, \leq_Y, T),$$

on morphisms: $(f_1, f_2) \mapsto G(f_1, f_2) = f_1 \cup f_2,$

where $T$ is the topology on $X \cup Y$ generated by $T_X \cup T_Y$. Then the diagram in Figure 6 proves that $\mathcal{DPB}$ is categorically equivalent to $\mathcal{D} \times \mathcal{D}$, where $H \times H$ and $K \times K$ are the corresponding product functors.

![Figure 6. Equivalence between $\mathcal{DPB}$ and $\mathcal{D} \times \mathcal{D}$](image-url)

To obtain a strong duality for $\mathcal{DPB}_u$ we need first to determine $\mathbb{S}(\mathcal{M})$ and $\mathbb{S}(\mathcal{M} \times \mathcal{M}')$ where $\mathcal{M}, \mathcal{M}' \in \{2^+, 2^-\}$. To determine which binary relations to include we can argue in much the same way as for $\mathbb{S}(4, 2)$. Decomposable subalgebras of $\mathbb{S}(\mathcal{M} \times \mathcal{M}')$ can be discounted. It is simple to confirm that all indecomposable $\mathcal{DPB}_u$-subalgebras are $\mathcal{DPB}$-subalgebras, and such subalgebras have already been identified in the proof of Theorem 10.2. We omit the details.

Theorem 10.3. A strong, and hence full, duality for $\mathcal{DPB}_u = \mathbb{ISP}(2^+, 2^-)$ is obtained as follows. Take $\mathcal{M} = \{2^+, 2^-\}$ and as the alter ego

$$\mathcal{N} = \{\{0^+, 1^+\} \cup \{0^-, 1^-\}; r^+, r^-, 0^+, 0^-, 1^+, 1^-, T\},$$

where $r^+$ is $\leq_k$ on $2^+$ and $r^-$ is $\leq_k$ on $2^-$ and the constants are treated as nullary operations.

Reasoning as in the bounded case, $\mathfrak{X} = \mathbb{ISP}^+(\mathcal{M})$ is categorically equivalent to $\mathcal{P}_01 \times \mathcal{P}_01$. Then $\mathcal{DPB}_u$ is categorically equivalent to $\mathcal{D}_u \times \mathcal{D}_u$. We have an exactly parallel situation to that shown in the diagram in Figure 6.

As an aside, we remark that we could generate $\mathcal{DPB}_u$ as a quasivariety using the single generator $2^+ \times 2^-$ and apply Theorem 3.2. But there are some merits in working with the pair of algebras $2^+ \times 2^-$ and $2^- \times 2^-$. Less work is involved to formulate a strong duality and to confirm that it is indeed strong. More importantly for our purposes, the translation to a Priestley-style duality is more transparent in the multisorted framework.
As was done in Theorems 7.2 and 7.5 for $\mathcal{DB}$ and $\mathcal{DB}_u$, respectively, it is possible to develop a different presentation (naturally equivalent) of the functors that determine the equivalences between $\mathcal{D} \times \mathcal{D}$ and $\mathcal{DPB}$ and between $\mathcal{D}_u \times \mathcal{D}_u$ and $\mathcal{DPB}_u$. This will lead to the known product decomposition of distributive pre-bilattices with and without bounds. We choose not to develop this here, since we would need to introduce the multisorted version of the piggyback duality (see [15, Theorem 7.2.1]). The results could then be obtained just by modifying the arguments used to prove Theorems 7.2 and 7.5. Also the applications presented in Section 8 can be extended to $\mathcal{DPB}$ and $\mathcal{DPB}_u$ with the corresponding modifications.

11. Concluding remarks

With our treatment of representation theory for distributive bilattices now complete, we can take stock of what we have achieved.

The scope of our work is somewhat different from that of other investigators of bilattices. Throughout we have restricted attention to the distributive case. We have not ventured into the territory of logical bilattices in this paper, but we do observe that such bilattices are customarily assumed to be distributive. Nevertheless we should comment on the role of distributivity, as compared with the weaker condition of interlacing. Any interlaced (pre)-lattice has a product representation and, conversely, such a representation is available only if the two lattice structures are linked by interlacing. Accordingly the product representation features very strongly in the literature. As indicated in Section 7, the dual representations obtained in [27] and in [24] build on Priestley duality as it applies to the varieties $\mathcal{D}$ and $\mathcal{DM}$. The setting, perforce, is now that in which the bilattice structures are distributive and have bounds; the product representation is brought into play to handle the $k$-lattice operations.

We next comment on the role of congruences. In this paper, the core result is Proposition 2.2 asserting that the congruences of any distributive pre-bilattice coincide with the congruences of the $t$-lattice reduct and with the congruences of the $k$-lattice reduct. For the interlaced case, this result is obtained with the aid of the product representation and leads on to a description of subdirectly irreducible algebras; see [27, 29, 9]. We exploited Proposition 2.2 to obtain our ISP results for each of $\mathcal{DB}$, $\mathcal{DB}_u$, $\mathcal{DPB}$ and $\mathcal{DPB}_u$. These results are of course immediate once the subdirectly irreducible algebras are known, but our method of proof is much more direct. Conversely, our results immediately yield descriptions of the subdirectly irreducibles.

From what is said above it might appear that, in certain aspects our approach leads to the same principal results as previous approaches do, albeit by a different route. But we contend that we have done much more than this. In our setting we are able to harness the techniques of natural duality theory and to apply them in a systematic way to the best-known bilattice varieties. We hereby gain easy access to the applications presented, by way of illustration, in Section 8. It is true that the dualities developed in [27] and in [24] can be described using our dualities and vice versa. However the deep connections between congruences of lattice reducts, our ISP presentations, and the topological representation theory only becomes clear using natural dualities.

We end our paper with an interesting byproduct of our treatment which links back to the origins of bilattices. The theory of bilattices and the investigation of four-valued logics have been intertwined ever since the concept of a bilattice was first introduced. In his seminal paper [6] (also available in [7]), Belnap introduced two lattices over the same four-element set $\{F,T,Both,None\}$, the logical lattice $\mathcal{L}_4$ and the approximation lattice $\mathcal{A}_4$, the former admitting also a negation operation. With our notation, $\mathcal{L}_4 \cong \{\{00,01,10,11\}; \lor, \land, \neg\}$ and $\mathcal{A}_4 \cong \mathcal{A}_4$. Belnap defines a set-up as a map $s$ from a set $X$ of atomic formulas into $\{F,T,Both,None\}$, and extends $s$ in a unique way to a homomorphism $\mathcal{Fm}(X) \rightarrow \mathcal{L}_4$, where $\mathcal{Fm}(X)$ is the set of formulas in the language $\{\land, \lor, \neg\}$. He then introduces a logic, understood as an entailment relation between formulas based on set-ups and what is nowadays called a Gentzen system which is complete for this logic. The connection between Belnap’s logic and De Morgan lattices and De Morgan algebras, hinted at in the definition of the former, was unveiled in detail by Font in [20] in the context of abstract algebraic logic.
Belnap did more than just define his logic: he also presented a mathematical formulation of the epistemic dynamic of the logic. To do this, he defined epistemic states as sets of set-ups and lifted the order on $A^4$ to a pre-order, $\sqsubseteq$ (the approximation order), between epistemic states. He then considered the partial order obtained from $\sqsubseteq$ by quotienting by the equivalence relation $\sqsubseteq \cap \sqsubseteq$ and showed that the resulting poset is isomorphic to the family of upward-closed sets of set-ups; here set-ups are considered as elements of $A^4^\text{Fm}(X)$ and are ordered pointwise. This emphasises the importance of the poset structure, as opposed to the algebraic structure, of $A^4$. Furthermore, it is proved that, for each formula $A \in \text{Fm}(X)$, the assignment $A \mapsto \text{Tset}(A) = \{ s : \overline{A} \in \{ T, \text{Both} \} \}$ maps conjunctions to intersections, disjunctions to unions and $\neg A \mapsto \text{False}(A) = \{ s : \overline{A} \in \{ F, \text{Both} \} \}$. So we could interpret Belnap’s results as a representation of $\text{Fm}(X)$ as upward-closed subsets of homomorphisms from $\text{Fm}(X)$ to $A^4$ ordered pointwise by $A^4$.

Only a few steps are needed to connect Belnap’s representation, as outlined above, with the natural duality for De Morgan algebras; see [15, Section 4.3.15] and the references therein. We adopt the notation of [15] for the generating algebra, $\text{dM}$, and for the alter ego, $\text{dM}$. First observe that $L^4 \cong \text{dM}$ is a De Morgan algebra. Therefore each homomorphism $h: \text{Fm}(X) \rightarrow L^4$ factors through the free De Morgan algebra $F_{\text{dM}}(X)$. Hence the set of set-ups can be identified with $\text{dM}(F_{\text{dM}}(X), L^4)$. It is also necessary to check that for each formula $A$ the sets $\text{Tset}(A)$ and $\text{Fset}(A)$ are related by the involution of the dual space of a De Morgan algebra; more precisely, $g(\text{Tset}(A)) = \text{Fset}(A)$. And finally, of course, topology plays its role by enabling one to characterise those upward-closed sets (represented by maps) that correspond to formulas.

These observations serve to stress that $L^4$ and $A^4$ in Belnap’s works play quite different roles. Moreover, these structures are intimately related to the roles of $\text{dM}$ and $\text{dM}$ in the natural duality for De Morgan algebras. The idea of combining two lattices into one structure originated with Ginsberg [22]. The dualities presented in Theorem 4.2 and 5.4 can be seen as a bridge reconciling Belnap’s and Ginsberg’s approaches, the first considering two separated lattice structures $L^4$ and $A^4$ with different roles but based on the same universe, and the latter combining them into a single algebraic structure. We, in like manner, work with two different structures $4$ and $4$ (and $4.$ and $4.$ in the unbounded case) with different structures having distinctive roles: one logical with an algebraic structure, the other epistemic with a poset structure.

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