SET-THEORETIC SOLUTIONS TO THE YANG-BAXTER EQUATION AND GENERALIZED SEMI-BRACES

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ABSTRACT

This paper aims to introduce a construction technique of set-theoretic solutions of the Yang-Baxter equation, called strong semilattice of solutions. This technique, inspired by the strong semilattice of semigroups, allows one to obtain new solutions. In particular, this method turns out to be useful to provide non-bijective solutions of finite order. It is well-known braces, skew braces and semi-braces are closely linked with solutions. Hence, we introduce a generalization of the algebraic structure of semi-braces based on this new construction technique of solutions.

Keywords  Quantum Yang-Baxter equation · set-theoretical solution · brace · skew brace · semi-brace · generalized semi-brace

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1 Introduction

The quantum Yang-Baxter equation appeared in the work of Yang [48] and Baxter [2]. It is one of the basic equations in mathematical physics, and it laid the foundations of the theory of quantum groups [32]. Solutions of the Yang-Baxter equation are instrumental in the construction of semisimple Hopf algebras [23, 42] and provide examples of colouring invariants in knot theory [40]. More recently, the Yang-Baxter solution popped up in the theory of quantum computation [33, 49], where solutions of the Yang-Baxter equation provide so-called universal gates. One of the central open problems is to find all solutions of the Yang-Baxter equation. Let $V$ be a vector space over a field $K$. Then, a solution of the Yang-Baxter equation is a linear map $R : V \otimes V \to V \otimes V$, for which the following holds on $V^{\otimes 3}$: $(R \otimes \text{id}_V)(\text{id}_V \otimes R)(R \otimes \text{id}_V) = (\text{id}_V \otimes R)(R \otimes \text{id}_V)(\text{id}_V \otimes R)$. The simplest solutions are the solutions $R$ induced by a linear extension of a mapping $r : X \times X \to X \times X$, where $X$ is a basis for $V$, satisfying the
set-theoretic version of the Yang-Baxter equation, i.e., satisfying the following on $X^3$:

$$(r \times \text{id}_X)(\text{id}_X \times r)(r \times \text{id}_X) = (\text{id}_X \times r)(r \times \text{id}_X)(\text{id}_X \times r).$$

In this case, $r$ is said to be a set-theoretic solution of the Yang–Baxter equation (briefly a solution). Drinfel’d, in [22], posed the question of finding these set-theoretic solutions. Denote for $x, y \in X$, the element $r(x, y) = (\lambda_x(y), \rho_y(x))$.

One says that a set-theoretic solution $r$ is left non-degenerate if $\lambda_x$ is bijective, for every $x \in S$, right non-degenerate if $\rho_y$ is bijective, for every $y \in S$, and non-degenerate if $r$ is both left and right non-degenerate. If a solution is neither left nor right non-degenerate, then it is called degenerate. The first papers on set-theoretic solutions are those of Etingof, Schedler and Soloviev [24] and Gateva–Ivanova and Van den Bergh [26]. Both papers considered involutive solutions, i.e., solutions $r$ where $r^2 = \text{id}$. Rump [43] introduced a new algebraic structure, braces, that generalizes radical rings and provides an algebraic framework. We provide the equivalent definition formulated by Cedó, Jespers and Okniński [12]. A triple $(B, +, \circ)$ is called a left brace if $(B, +)$ is an abelian group and $(B, \circ)$ is a group, such that for any $a, b, c \in B$ it holds that

$$a \circ (b + c) = a \circ b + a \circ c.$$

This new structure showed connections between the Yang-Baxter equation and ring theory, flat manifolds, orderability of groups, Garside theory, regular subgroups of the affine group, see for instance [5, 6, 14, 21, 25, 45]. Lu, Yan, and Zhu [37] and Soloviev [46] started the study of non-degenerate bijective solutions, not necessarily involutive. Almost all of the ideas used in the theory of involutive solutions can be transported to non-involutive solutions. The algebraic framework now is provided by skew left braces [27]. Let $(B, +)$ and $(B, \circ)$ be groups on the same set $B$. If, for any $a, b, c \in B$, condition (⋄) holds, the triple $(B, +, \circ)$ is called a skew left brace. Skew left braces and some of their applications are intensively studied, see for instance [8, 13, 20, 34, 44].

In [36], Lebed drew the attention on idempotent solutions. Indeed, using idempotent solutions and graphical calculus from knot theory, she provides a unifying tool to deal with several diverse algebraic structures, such as free monoids, free commutative monoids, factorizable monoids, plactic monoids and distributive lattice. Examples and classifications of these solutions have been provided by Matsumoto and Shimizu [38] and by Stanovský and Vojtěchovský [37]. Moreover, Cvetko-Vah and Verwimp, [19] provided cubic solutions with skew lattices. A cubic solution $r$ is a solution such that $r^3 = r$; hence, this class includes both involutive and idempotent solutions. More in general, a more systematic approach to the study of solutions with finite order can be found recently in [9, 10, 11]. Catino, Colazzo, and Stefanelli [7] and Jespers and Van Antwerpen [31] introduced the algebraic structure called left semi-brace to deal with solutions that are not necessarily non-degenerate or that are idempotent. Let $(B, +)$ be a semigroup and $(B, \circ)$ be a group. Then $(B, +, \circ)$ is called a left semi-brace if, for any $a, b, c \in B$, it holds that

$$a \circ (b + c) = a \circ b + a \circ (a^- + c),$$
where $a^-$ denotes the inverse $a$ in $(B, \circ)$. If $(B, +)$ is a left cancellative semigroup, then we call $(B, +, \circ)$ a left cancellative left semi-brace. This was the original definition by Catino, Colazzo and Stefanelli [7]. It has been shown that left semi-braces, under some mild assumption, provide set-theoretic solutions of the Yang-Baxter equation. Moreover, the associated solution is left non-degenerate if and only if the left semi-brace is left cancellative.

Out of algebraic interest Brzeziński introduced left trusses [4] and left semi-trusses [3]. A quadruple $(B, +, \circ, \lambda)$ is called a left semi-truss if both $(B, +)$ and $(B, \circ)$ are semigroups and $\lambda : B \times B \to B$ is a map such that $a \circ (b + c) = (a \circ b) + \lambda(a, c)$. Clearly, the class of left semi-trusses contains all left semi-braces, rings, associative algebras and distributive lattices. This entails that it will prove difficult to present deep results on this class. However, one may examine large subclasses. In particular, Brzeziński [4] focused on left semi-trusses with $(B, +)$ a left cancellative semigroup and $(B, \circ)$ a group, and showed that such a left semi-truss is equivalent with a left cancellative semi-brace, thus providing set-theoretic solutions of the Yang-Baxter, albeit known ones. In [39] Miccoli introduced almost left semi-braces, a particular instance of left semi-trusses, and constructed set-theoretic solutions associated with this algebraic structure. In [13], Colazzo and Van Antwerpen continue this study focusing on the subclass of brace-like left semi-trusses, i.e., left semi-trusses in which the multiplicative semigroup is a group and which includes almost left semi-braces. Concerning solutions, they showed that the solution one can associate with an almost left semi-brace is already the associated solution of a left semi-brace. In particular, this shows that brace-like left semi-trusses will not yield a universal algebraic structure that produces set-theoretic solutions.

In this paper, we focus on a new algebraic structure that includes left semi-braces and is an instance of left semi-trusses, which is on a different path with respect to brace-like left semi-trusses, called generalized left semi-brace. A triple $(S, +, \circ)$ is called a generalized left semi-brace if $(S, +)$ is a semigroup, $(S, \circ)$ is a completely regular semigroup (or union of groups), and such that, for any $a, b, c \in S$, it holds

$$a \circ (b + c) = a \circ b + a \circ (a^- + c),$$

where $a^-$ denotes the (group) inverse of $a$ in $(B, \circ)$. We prove that under some mild assumptions, generalized left semi-braces provide solutions. In particular, elementary examples of generalized left semi-braces produce cubic solutions that cannot be obtained by skew lattices and left semi-brace. Also, we introduce a construction technique that provides generalized left semi-braces called the strong semilattice of generalized left semi-braces. This technique is inspired by the description of semigroups which are unions of groups due to Clifford [16].

Furthermore, we introduce a construction technique for solutions called the strong semilattice of solutions. This technique takes a family of disjoint sets $\{X_\alpha \mid \alpha \in Y\}$ indexed by a semilattice $Y$ and solutions defined on these sets, then, under some assumptions of compatibility, it allows one to construct a solution on the union of the sets $X_\alpha$. We prove that the solutions provide by the strong semilattice of left semi-braces are a particular instance of strong semilattice of solutions.

Finally, we prove that the strong semilattice of solutions is a useful tool to provide solutions of finite order. Indeed the strong semilattice of solutions of finite order is a solution of finite order. Moreover, a solution $r$ is of finite order if
there exist a non-negative integer \( i \) and a positive integer \( p \) such that \( r^{p+i} = r^i \) and the minimal integers that satisfy such relation are said to be index and period respectively. We show that it is possible to determine the index and the period of the semilattice of solutions \( \{ r_\alpha \mid \alpha \in Y \} \) as a function of the indexes and periods of \( r_\alpha \). As a corollary of this result, we prove that solutions associated with strong semilattices of left semi-braces are not bijective, so they are clearly different from solutions obtained by left semi-braces.

2 Basic tools on left semi-braces

Let us briefly present some basic background information regarding left semi-braces. Most of the content of this section appear in [31]. In particular, we provide different proof of [31, Corollary 2.9] based on a result in semigroup theory due to Hickey [28] that gives a clear description of completely regular semigroup with middle units. Moreover, we add further information on the behavior of middle units of the additive semigroup of a left semi-brace. Finally, we present concrete examples of left semi-braces.

Let's start by recalling the definition of left semi-braces.

**Definition 1.** Let \( B \) be a set with two operations \( + \) and \( \circ \) such that \((B, +)\) is a semigroup and \((B, \circ)\) is a group. Then, \((B, +, \circ)\) is said to be a left semi-brace if

\[
a \circ (b + c) = a \circ b + a \circ (a^- + c),
\]

for all \( a, b, c \in B \), where \( a^- \) is the inverse of \( a \) in \((B, \circ)\).

Throughout, 0 denotes the identity of the group \((B, \circ)\). Moreover, we call \((B, +)\) and \((B, \circ)\) the additive semigroup and the multiplicative group of the left semi-brace \((B, +, \circ)\), respectively. Furthermore, if the semigroup \((B, +)\) has a pre-fix, pertaining to some property of the semigroup, we will also use this pre-fix with the left semi-brace. Hence, the left semi-braces introduced in [7], where one works under the restriction that the semigroup \((B, +)\) is left cancellative, will be called left cancellative left semi-braces.

Now, we recall that an element \( u \) of an arbitrary semigroup \((S, +)\) is a middle unit of \( S \) if \( a + u + b = a + b \), for all \( a, b \in S \). Thus, \( u + u \) is idempotent but \( u \) itself need not be idempotent (see [17, p. 98]). This is not the case for the element 0 in the additive semigroup of a left semi-braces: the following proposition shows that 0 is an idempotent middle unit.

**Proposition 2.** Let \( B \) be a left semi-brace. Then, the following hold:

1. 0 is a middle unit of \((B, +)\);
2. 0 is an idempotent of \((B, +)\);
3. \( B = B + B \);
4. \( B + 0 \) is a subgroup of \((B, \circ)\);
5. \( 0 + B \) is a subsemigroup of \((B, \circ)\).

**Proof.** 1. See [31] Lemma 2.4(1)].
(2) Since, by (1), \(0 + 0 + 0 = 0 + 0\), we have that
\[
0 = (0 + 0)^- \circ (0 + 0) = (0 + 0)^- \circ 0 + (0 + 0)^- \circ (0 + 0)
\]
\[
= (0 + 0)^- + (0 + 0)^- \circ (0 + 0) = (0 + 0)^- + 0.
\]
Thus, since 0 is a middle unit,
\[
0 + 0 = (0 + 0)^- + 0 + 0 = (0 + 0)^- + 0 = 0.
\]

(3) If \(b \in B\), by (2) we have that
\[
b = b \circ 0 = b \circ (0 + 0) = b \circ 0 + b \circ (b^- + 0) \in B + B.
\]

(4) By (2) it is clear that \(B + 0\) is not empty. Moreover, by (1), if \(a, b \in B\), it holds that
\[
(a + 0)^- \circ (b + 0) = (a + 0)^- \circ b + (a + 0)^- \circ (a + 0 + 0)
\]
\[
= (a + 0)^- \circ b + (a + 0)^- \circ (a + 0) = (a + 0)^- \circ b + 0 \in B + 0.
\]

(5) See [31, Lemma 2.6(iii)].

To show the following theorem, let us recall that an arbitrary semigroup \((S, +)\) is said to be a rectangular group if it is isomorphic to the direct product of a group and a rectangular band. For background and details on this topic we refer the reader to [17].

**Theorem 3.** Let \(B\) be a completely simple left semi-brace. Then, the additive semigroup \((B, +)\) of \(B\) is a rectangular group.

**Proof.** The thesis follows by [28, Corollary 3.5], which states that any completely simple semigroup with a middle unit is a rectangular group.

A special case in which the additive semigroup is completely simple is when \(0 + B\) is a subgroup of \((B, \circ)\) (see [31, Theorem 2.8]). For instance, this is the case when \(B\) is finite. As a consequence of Theorem 3 we have that the additive semigroup \((B, +)\) can be written as
\[
B = I + G + \Lambda,
\]
that is the direct sum of the left zero semigroup \(I := E(B + 0)\), the group \(G := 0 + B + 0\), and the right zero semigroup \(\Lambda := E(0 + B)\). Moreover, the set of idempotents is \(E(B) = I + \Lambda\) and it is a rectangular band. Where \(E(S)\) denotes the set of idempotents of \(S\), for any semigroup \(S\).
For the sake of completeness, let us introduce a further property of middle units of left semi-braces that holds without restrictions on the additive semigroup. At first, we recall that the additive semigroup of a left semi-brace $B$ does not contain a zero element if $B$ has at least two elements (see \[31\] Lemma 2.3).

In the following, we prove that middle units of the additive structure of an arbitrary left semi-brace are idempotents.

**Proposition 4.** Let $B$ be a left semi-brace and $e \in B$. Then, if $e$ is a middle unit then $e$ is an idempotent of the semigroup $(B, +)$.

**Proof.** Since $0$ is idempotent, we have that

$$e = e \circ 0 = e \circ (0 + 0) = e \circ 0 + e \circ (e^- + 0) = e + e \circ (e^- + 0).$$

Since $e$ is a middle unit, it follows that

$$e + e = e + e \circ (e^- + 0) = e + e \circ (e^- + 0) = e,$$

which is our assertion. \qed

Following Ault’s paper \[1\] Theorem 1.8, by Proposition 4 we have that the additive semigroup of any left semi-brace contains a subsemigroup $M_B$, called the *semigroup of middle units of $B$*, that is explicitly given by

$$M_B = \{x \mid x \in B, x \text{ has inverse } x' \text{ with } x + x', x' + x \text{ middle units}\},$$

where the inverse $x'$ of $x$ means that $x' = x' + x + x'$ and $x = x + x' + x$.

Then, $M_B$ is a subsemigroup of $(B, +)$ that is a rectangular group. This is an interesting substructure of a left semi-brace, which is beyond the purpose of this paper and shall be studied elsewhere.

Now, having as reference the \[7\] Example 2, we provide the following class of examples of completely simple left semi-braces that allow one to obtain solutions.

**Example 5.** Let $(B, \circ)$ be a group with identity 0, $f, g$ idempotent endomorphisms of $(B, \circ)$ such that $fg = gf$. Let us consider the following operation

$$a + b := b \circ fg (b^-) \circ f (a),$$

for all $a, b \in B$. It is easy to check that $(B, +, \circ)$ is a completely simple left semi-brace. Now, observe that the map $\rho$ is an anti-homomorphism from the group $(B, \circ)$ into the monoid $B^B$, where $B^B$ denotes the monoid of the functions from $B$ into itself. Indeed, $\rho_b$ is given by

$$\rho_b (a) = (a^- + b)^- \circ b = (b \circ fg (b^-) \circ f (a^-))^- \circ b = f (a) \circ fg (b),$$

where the inverse $x'$ of $x$ means that $x' = x' + x + x'$ and $x = x + x' + x$. Then, $M_B$ is a subsemigroup of $(B, +)$ that is a rectangular group. This is an interesting substructure of a left semi-brace, which is beyond the purpose of this paper and shall be studied elsewhere.
moreover
\[
\rho_b \rho_a (c) = \rho_b (f (c) \circ f g (a)) = f^2 (c) \circ f^2 g (a) \circ f g (b) = f (c) \circ f g (a \circ b) = \rho_{a \circ b} (c),
\]
for all \(a, b, c \in B\). Thus, by [31, Proposition 2.14], the semigroup \((B, +)\) is completely simple.

In addition, since \(\rho\) is an anti-homomorphism of the group \((B, \circ)\), we obtain that the map \(r : B \times B \to B \times B\) defined by \(r (a, b) = (\lambda_a (b), \rho_b (a))\) in [31, Theorem 5.1] is a solution. Note that \(\lambda_a (b) = a \circ b \circ f g (b^-) \circ f (a^-)\), therefore, \(r\) is explicitly given by
\[
r (a, b) = (a \circ b \circ f (g (b^-) \circ a^-), f (a \circ g (b))),
\]
for all \(a, b \in B\).

Let us examine special cases of the previous class of examples. Firstly, observe that if \(f \neq \text{id}\) and \(g\) is not the constant map of value 0, then the semigroup \((B, +)\) is not neither left nor right cancellative. Moreover, note that:

**Case 1:** if \(g\) is the constant map of value 0, then
\[
a + b = b \circ f (0) \circ f (a) = b \circ f (a),
\]
for all \(a, b \in B\), i.e., \(B\) coincides with the left cancellative left semi-brace provided in [7, Example 2]. Moreover, the solution associated to \(B\) is given by
\[
r (a, b) = (a \circ b \circ f (a^-), f (a)).
\]

**Case 2:** if \(f = \text{id}\), then
\[
a + b = b \circ g (b^-) \circ a,
\]
for all \(a, b \in B\). In this case, the semigroup \((B, +)\) is right cancellative. Indeed, if \(a + b = c + b\), it follows that \(b \circ g (b^-) \circ a = b \circ g (b^-) \circ c\), and so \(a = c\). Moreover, it is easy to check that \(B\) is both right and left semi-brace. In addition, note that the solution associated to \(B\) is given by
\[
r (a, b) = (a \circ b \circ g (b^-) \circ a^-, a \circ g (b)).
\]

**Case 3:** if \(f = g\), then
\[
a + b = b \circ f^2 (b^-) \circ f (a) = b \circ f (b^-) \circ f (a),
\]
for all \(a, b \in B\). Note that if \(a \in B\), it holds
\[
a + a = a \circ f (a^-) \circ f (a) = a,
\]
i.e., every element is idempotent with respect to the sum. In this case, the semigroup \((B, +)\) is a rectangular band where \(\ker f = 0 + B\) and \(\text{Im} f = B + 0\). Moreover, the solution \(r\) associated to \(B\), given by

\[
 r(a, b) = \left(a \circ b \circ f \left(b^{-1} \circ a^{-1}\right), f(a \circ b)\right),
\]

is an idempotent solution, consistently with [31, Theorem 5.1] in the case in which the group \(G = \{0\}\).

The following is a class of examples of completely simple left semi-braces such that, under suitable assumptions, give rise to solutions.

**Example 6.** Let \(G, H\) be two groups, \(B := G \times H\) and consider the group \((B, \circ)\) where

\[
(a, u) \circ (b, v) = (a^u b, u^v),
\]

for all \((a, u), (b, v) \in G \times H\), i.e., the classical Zappa-Szép product of \(G\) and \(H\) (see [35]) that has identity \((1, 1)\). Let \(\varphi\) be a map from \(G\) into \(H\) such that \(\varphi(1) = 1\) and define the following operation on \(B\)

\[
(a, u) + (b, v) = (a, u \varphi(b) v),
\]

for all \((a, u), (b, v) \in G \times H\). It is easy to check that the structure \((B, +, \circ)\) is a left semi-brace. Let us note that \((B, +)\) is a left group. Moreover, \((a, u)\) in \(G \times H\) is idempotent with respect to the sum if and only if \(\varphi(a) = u^{-1}\). In addition, if \((a, u), (b, v) \in E(B)\), we have that

\[
(a, u) + (b, v) = (a, u \varphi(b) v) = (a, u),
\]

hence \(E(B)\) is a sub-semigroup of \((B, +)\) and it is also a left zero semigroup. Note also that \((a, u) + (1, 1) = (a, u \varphi(1) 1) = (a, u)\), i.e., \((1, 1)\) is a right identity with respect to the sum. Furthermore, we have that

\[
\rho_{(b,v)}(a,u) = \left(\left(\varphi(b)v\right)^{-1} a, u^b v\right),
\]

for all \((a, u), (b, v), (c, w) \in B\). One can check that \(\rho\) is an anti-homomorphism if and only if it holds

\[
\varphi(b) v \varphi(c) = \varphi(b^c v) v^c, \tag{1}
\]

for all \(a, b \in G\) and \(u \in H\). Moreover, by the characterization [10, Theorem 3], one can verify that the map \(r\) in [31, Theorem 5.1] associated to the left semi-brace \(B\) is a solution if and only if

\[
\varphi(a) u \varphi(b) = \varphi(b) v \varphi\left(\left(\varphi(b)v\right)^{-1} a^u b\right) \left(\left(\varphi(b)v\right)^{-1} u\right)^b \tag{2}
\]
holds, for all $a, b \in G$ and $u, v \in H$. On the other hand, note that, if $a, b \in G$ and $u \in H$, considered $v = \varphi(b)^{-1}$ in $\mathbb{G}$, we obtain
\[
\varphi(a) u \varphi(b) = v^{-1} v \varphi\left((v^{-1} v)^{-1} (a u b) \left((v^{-1} v)^{-1} \right)^a u \right)^b = \varphi(a u b) u^b,
\]
hence (1) is satisfied.

Now, let us consider $G$ the cyclic group $C_2$ of 2 elements, $H$ the cyclic group $C_3$ of 3 elements, and $\varphi$ the constant map of value 1 from $G$ into $H$. Hence, if $(B, \circ)$ is the cyclic group $C_2 \times C_3$, then the condition (1) trivially holds, hence $r$ is a solution. Instead, if $(B, \circ)$ is the symmetric group $C_2 \rtimes C_3$, then (1) is not satisfied, equivalently (2) does not hold, hence $r$ is not a solution.

3 Definitions and examples

Braces, skew braces, and semi-braces were introduced to study set-theoretic solutions of the Yang-Baxter equation. The following definition generalizes these structure to the case in which the multiplicative structure is no more a group.

At first, we recall that a semigroup $(S, \circ)$ is completely regular if for any element $a$ of $S$ there exists a (unique) element $a^-$ of $S$ such that
\[
a = a \circ a^- \circ a, \quad a^- = a^- \circ a \circ a^-, \quad a \circ a^- = a^- \circ a.
\]
Conditions (3) imply that $a^0 := a \circ a^- = a^- \circ a$ is an idempotent element of $(S, \circ)$.

**Definition 7.** Let $S$ be a set with two operations $+$ and $\circ$ such that $(S, +)$ is a semigroup (not necessarily commutative) and $(S, \circ)$ is a completely regular semigroup. Then, we say that $(S, +, \circ)$ is a generalized left semi-brace if
\[
a \circ (b + c) = a \circ b + a \circ (a^- + c),
\]
for all $a, b, c \in S$. We call $(S, +)$ and $(S, \circ)$ the additive semigroup and the multiplicative semigroup of $S$, respectively. A generalized right semi-brace is defined similarly, replacing condition (4) by
\[
(a + b) \circ c = (a + a^-) \circ c + b \circ c,
\]
for all $a, b, c \in S$.

A generalized two-sided semi-brace is a generalized left semi-brace that is also a generalized right semi-brace with respect to the same pair of operations.

Let us note that if $S$ is a generalized left semi-brace and $a \in S$, then the map
\[
\lambda_a : S \longrightarrow S, \quad b \longmapsto a \circ (a^- + b)
\]
is an endomorphism of the semigroup \((S, +)\) and \(\lambda_{a \circ b}(x) = (a \circ b)^0 + \lambda_a \lambda_b(x)\), for all \(a, b, x \in S\). Indeed, if \(a, b, x, y \in S\), we have that

\[
\lambda_a(x + y) = a \circ (a^- + x + y) = a \circ (a^- + x) + a \circ (a^- + y) = \lambda_a(x) + \lambda_a(y).
\]

and

\[
\lambda_{a \circ b}(x) = (a \circ b) \circ ((a \circ b)^- + x) = a \circ (b \circ (a \circ b)^- + b \circ (b^- + x))
\]

\[
= a \circ b \circ (a \circ b)^- + a \circ (a^- + \lambda_b(x))
\]

\[
= (a \circ b)^0 + \lambda_a \lambda_b(x).
\]

Of course, left semi-braces \([7, 31]\) are examples of generalized left semi-braces. Moreover, a generalized left semi-brace can be obtained from every completely regular semigroup.

**Example 8.** If \((S, \circ)\) is an arbitrary completely regular semigroup and \((S, +)\) is a right zero semigroup (or a left zero semigroup), then \((S, +, \circ)\) is a generalized two-sided semi-brace.

Unlike left semi-braces, a generalized left semi-brace \(S\) can have a zero element even if \(S\) has more than one element. Examples of such generalized left semi-braces can be easily obtained by any Clifford semigroup.

**Example 9.** If \((S, \circ)\) is a Clifford semigroup, which is a completely regular semigroup where all idempotent elements are central, then \((S, +, \circ)\), where \(a + b = a \circ b\) for all \(a, b \in S\), is a generalized two-sided semi-brace.

More in general, the previous generalized left semi-braces can be obtained through the following construction.

**Proposition 10.** Let \(Y\) be a (lower) semilattice, \(\{S_\alpha \mid \alpha \in Y\}\) a family of disjoint generalized left semi-braces. For each pair \(\alpha, \beta\) of elements of \(Y\) such that \(\alpha \geq \beta\), let \(\phi_{\alpha, \beta} : S_\alpha \to S_\beta\) be a homomorphism of generalized left semi-braces such that

\[
(1) \quad \phi_{\alpha, \alpha}\text{ is the identical automorphism of } S_\alpha, \text{ for every } \alpha \in Y
\]

\[
(2) \quad \phi_{\beta, \gamma} \phi_{\alpha, \beta} = \phi_{\alpha, \gamma}, \text{ for all } \alpha, \beta, \gamma \in Y \text{ such that } \alpha \geq \beta \geq \gamma.
\]

Then, \(S = \bigcup \{S_\alpha \mid \alpha \in Y\}\) endowed by the addition and the multiplication defined by

\[
a + b = \phi_{\alpha, \alpha \beta}(a) + \phi_{\beta, \alpha \beta}(b), \quad (6)
\]

\[
a \circ b = \phi_{\alpha, \alpha \beta}(a) \circ \phi_{\beta, \alpha \beta}(b), \quad (7)
\]

for any \(a \in S_\alpha\) and \(b \in S_\beta\), is a generalized left semi-brace. Such a generalized left semi-brace is said to be the strong semilattice \(Y\) of generalized left semi-brace \(S_\alpha\) and is denoted by \(S = [Y; S_\alpha, \phi_{\alpha, \beta}]\).
Proof. First note that \((S, +)\) is a semigroup and \((S, \circ)\) is a completely regular semigroup. Now, let \(a \in S_\alpha, b \in S_\beta,\) and \(c \in S_\gamma.\) Set \(\delta := \alpha\beta, \varepsilon := \beta\gamma, \zeta := \alpha\gamma,\) and \(\eta := \alpha\beta\gamma.\) It follows that

\[
a \circ (b + c) = a \circ (\phi_{\beta, \varepsilon} (b) + \phi_{\gamma, \varepsilon} (c))
\]

\[
= \phi_{\alpha, \eta} (a) \circ \phi_{\varepsilon, \eta} (\phi_{\beta, \varepsilon} (b) + \phi_{\gamma, \varepsilon} (c))
\]

\[
= \phi_{\alpha, \eta} (a) \circ (\phi_{\beta, \eta} (b) + \phi_{\gamma, \eta} (c))
\]

\[
= \phi_{\alpha, \eta} (a) \circ \phi_{\beta, \eta} (b) + \phi_{\alpha, \eta} (a) \circ (\phi_{\gamma, \eta} (c))
\]

where the last equality holds since \(S_\eta\) is a generalized left semi-brace. Moreover,

\[
a \circ b + a \circ (a^- + c) = \phi_{\alpha, \delta} (a) \circ \phi_{\beta, \delta} (b) + a \circ \left( (\phi_{\alpha, \zeta} (a))^- + \phi_{\gamma, \zeta} (c) \right)
\]

\[
= \phi_{\alpha, \delta} (a) \circ \phi_{\beta, \delta} (b) + \phi_{\alpha, \zeta} (a) \circ \left( (\phi_{\alpha, \zeta} (a))^- + \phi_{\gamma, \zeta} (c) \right)
\]

\[
= \phi_{\alpha, \eta} (\phi_{\alpha, \delta} (a) \circ \phi_{\beta, \delta} (b)) + \phi_{\alpha, \eta} (\phi_{\alpha, \zeta} (a) \circ \left( (\phi_{\alpha, \zeta} (a))^- + \phi_{\gamma, \zeta} (c) \right)) \quad \text{since } \delta \zeta = \eta
\]

\[
= \phi_{\alpha, \eta} (a) \circ \phi_{\beta, \eta} (b) + \phi_{\alpha, \eta} (a) \circ \left( (\phi_{\alpha, \eta} (a))^- + \phi_{\gamma, \eta} (c) \right)
\]

by 2.

Therefore, \(S\) is a generalized left semi-brace. \(\square\)

Remark 11. If \(S = \{Y; S_\alpha, \phi_{\alpha, \beta}\}\) is a strong semilattice \(Y\) of left semi-braces \(S_\alpha,\) then \((S, \circ)\) is a strong semilattice of groups, and hence, by [29] Theorem 4.2.1, \((S, \circ)\) is a Clifford semigroup.

4 Solutions related to generalized left semi-braces

This section is devoted to provide a sufficient condition to obtain solutions through a generalized left semi-brace. To this end, we recall that if \(S\) is a left cancellative left semi-brace, then the map \(r : S \times S \rightarrow S \times S\) given by

\[
r (a, b) := \left( a \circ (a^- + b), (a^- + b)^- \circ b \right)
\]

for all \(a, b \in S,\) is a solution. Moreover, [31] Theorem 5.1] gives a sufficient condition to obtain that the map in [30] is still a solution for a left semi-brace, not necessarily left cancellative. In addition, in [10] Theorem 3], we state a necessary and sufficient condition to ensure that \(r\) is a solution.

Specifically, if \((S, +, \circ)\) is a left semi-brace, then, the map \(r : S \times S \rightarrow S \times S,\) defined by \(r (a, b) := \left( a \circ (a^- + b), (a^- + b)^- \circ b \right),\) for all \(a, b \in S,\) is a solution if and only if

\[
a + \lambda_b (c) \circ (0 + \rho_c (b)) = a + b \circ (0 + c)
\]

holds, for all \(a, b, c \in S.\)
Let us remark that if $S$ is a generalized left semi-brace with $(S, +)$ a right zero semigroup, then the map $r$ as in (3) is a solution if and only if it holds $(a \circ b)^0 = (a^0 \circ b^0)$, for all $a, b \in S$. Observe that semigroups $(S, \circ)$ satisfying such a condition, lie in the wide class of right cryptogroups, see [41]. In this way, we get new idempotent solutions that are of the form $r(a, b) = (a^0, a \circ b)$, different from those obtained in [36, 38, 19, 47]. Moreover, note that if $(S, +, \circ)$ is the generalized left semi-brace of Example 9 we obtain that $r(a, b) = (a^0 \circ b, b^- \circ a \circ b)$ is a solution. In particular, if $(S, \circ)$ is commutative, clearly $r(a, b) = (a^0 \circ b, b^0 \circ a)$ and it is easy to verify that $r$ is a cubic solution, i.e., $r^3 = r$.

Our aim is to show that if $S = [Y; S_\alpha, \phi_{\alpha, \beta}]$ is a strong semilattice of generalized left semi-braces such that every $S_\alpha$ satisfies condition (9), then the map in (3) is a solution. This result is a consequence of a more general construction technique on solutions we introduce in the following theorem.

**Theorem 12.** Let $Y$ be a (lower) semilattice, $\{X_\alpha \mid \alpha \in Y\}$ a family of disjoint sets indexed by $Y$, and $r_\alpha$ a solution on $X_\alpha$, for every $\alpha \in Y$. For each pair $\alpha, \beta$ of elements of $Y$ such that $\alpha \geq \beta$, let $\phi_{\alpha, \beta} : X_\alpha \rightarrow X_\beta$ be a map. If the following conditions are satisfied

1. $\phi_{\alpha, \alpha}$ is the identity map of $X_\alpha$, for every $\alpha \in Y$,
2. $\phi_{\beta, \gamma} \phi_{\alpha, \beta} = \phi_{\alpha, \gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$,
3. $(\phi_{\alpha, \beta} \times \phi_{\alpha, \beta}) r_\alpha = r_\beta (\phi_{\alpha, \beta} \times \phi_{\alpha, \beta})$, for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$,

set $X = \bigcup \{X_\alpha \mid \alpha \in Y\}$, then the map $r : X \times X \rightarrow X \times X$ defined by

$$r(x, y) := r_{\alpha, \beta} (\phi_{\alpha, \beta} (x), \phi_{\beta, \gamma} (y)),$$

for all $x \in X_\alpha$ and $y \in X_\beta$, is a solution on $X$. We call the pair $(X, r)$ the strong semilattice $Y$ of solutions $(X_\alpha, r_\alpha)$.

The proof of Theorem 12 is technical, and for the sake of clarity, we present it in the next section. Now, as a consequence of this theorem, we obtain the following result.

**Theorem 13.** Let $S = [Y; S_\alpha, \phi_{\alpha, \beta}]$ be a strong semilattice of generalized left semi-braces. Then, if $S_\alpha$ satisfies (9), for every $\alpha \in Y$, then the map $r_S : S \times S \rightarrow S \times S$ defined by

$$r(a, b) := \left(a \circ (a^- + b), (a^- + b)^- \circ b\right),$$

for all $a, b \in S$, is a solution.

**Proof.** For any $\alpha \in Y$, let $r_\alpha : S_\alpha \times S_\alpha \rightarrow S_\alpha \times S_\alpha$ be the solution associated to the left semi-brace $S_\alpha$, i.e., the map defined by $r_\alpha(x, y) = (x \circ (x^- + y), (x^- + y)^- \circ y)$. Since $S$ is a strong semilattice of left semi-braces, by Proposition 10 $\phi_{\alpha, \alpha}$ is the identical automorphism of $S_\alpha$ and $\phi_{\beta, \gamma} \phi_{\alpha, \beta} = \phi_{\alpha, \gamma}$, for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$. Hence, conditions 1. and 2. in Theorem 12 are satisfied. Moreover, let $a, b \in Y$ such that $\alpha \geq \beta$. Since,
\( \phi_{\alpha,\beta} \) is a homomorphism of left semi-braces, for all \( x, y \in S_{\alpha} \) it follows that

\[
(\phi_{\alpha,\beta} \times \phi_{\alpha,\beta}) r_{\alpha} (x, y) = \left( \phi_{\alpha,\beta} (x \circ (x^{-} + y)), \phi_{\alpha,\beta} (x^{-} + y \circ y) \right)
= \left( \phi_{\alpha,\beta} (x) \circ \left( (\phi_{\alpha,\beta} (x))^{-} + \phi_{\alpha,\beta} (y) \right), \left( (\phi_{\alpha,\beta} (x))^{-} + \phi_{\alpha,\beta} (y) \right)^{-} \circ \phi_{\alpha,\beta} (y) \right)
= r_{\beta} (\phi_{\alpha,\beta} (x)), \phi_{\alpha,\beta} (y))
= r_{\beta} (\phi_{\alpha,\beta} \times \phi_{\alpha,\beta}) (x, y).
\]

Hence (3) in Theorem 12 holds. Therefore, according to Theorem 12 we shall consider the strong semilattice \( Y \) of solutions \( r_{\alpha} \), i.e., the map \( r \) defined by

\[
r(x, y) = r_{\alpha\beta} (\phi_{\alpha,\alpha\beta} (x), \phi_{\beta,\alpha\beta} (y)),
\]

for all \( x \in S_{\alpha}, y \in S_{\beta} \). Finally, note that, by Proposition 10

\[
r(x, y) = \left( \phi_{\alpha,\alpha\beta} (x) \circ \left( (\phi_{\alpha,\alpha\beta} (x))^{-} + \phi_{\beta,\alpha\beta} (y) \right), \left( (\phi_{\alpha,\alpha\beta} (x))^{-} + \phi_{\beta,\alpha\beta} (y) \right)^{-} \circ \phi_{\beta,\alpha\beta} (y) \right)
= \left( x \circ (x^{-} + y), (x^{-} + y)^{-} \circ y \right),
\]

for all \( x \in S_{\alpha}, y \in S_{\beta} \).

5 Strong semilattices of set-theoretical solutions

This section aims to provide a proof of Theorem 12 and to give some examples of strong semilattices of solutions. Furthermore, we focus on analyzing strong semilattices of solutions with finite order.

Proof of Theorem 12 At first, note that if \( \lambda_{x}^{[\omega]} \) and \( \rho_{y}^{[\omega]} \) are the maps from \( X_{\omega} \) into itself that define every solution \( r_{\omega} \), i.e., \( r_{\omega} \) is written as \( r_{\omega} (x, y) = \left( \lambda_{x}^{[\omega]} (y), \rho_{y}^{[\omega]} (x) \right), \) for all \( x, y \in X_{\omega} \), then the condition 3 is equivalent to the following equalities

\[
\phi_{\omega,\epsilon} \lambda_{x}^{[\omega]} (y) = \lambda_{\phi_{\omega,\epsilon} (x)}^{[\epsilon]} \phi_{\omega,\epsilon} (y) \quad (10)
\phi_{\omega,\epsilon} \rho_{y}^{[\omega]} (x) = \rho_{\phi_{\omega,\epsilon} (y)}^{[\epsilon]} \phi_{\omega,\epsilon} (x) \quad (11)
\]

for all \( \omega, \epsilon \in Y \) such that \( \omega \geq \epsilon \) and \( x, y \in X_{\omega} \). In addition, let us observe that if \( x \in X_{\omega} \) and \( y \in Y_{\epsilon} \), then the two components of the map \( r \), i.e.,

\[
\lambda_{x} (y) = \lambda_{\phi_{\omega,\epsilon} (x)}^{[\omega]} \phi_{\omega,\epsilon} (y) \quad \rho_{y} (x) = \rho_{\phi_{\omega,\epsilon} (y)}^{[\omega]} \phi_{\omega,\epsilon} (x),
\]

for all \( \omega, \epsilon \in Y \) such that \( \omega \geq \epsilon \).
lie in $X_\omega$, consistently with the second part of the subscript of the maps $\phi$. To avoid overloading the notation, hereinafter we will write the previous elements as

$$\lambda_x(y) = \lambda_{\phi_\omega,\omega}(x)\phi_i,\omega(y) \quad \rho_y(x) = \rho_{\phi_\omega,\omega}(y)\phi_\omega,\omega(x).$$

Now, we verify that $\tau$ is a solution proving that the relations

$$L_1 := \lambda_x\lambda_y(z) = \lambda_{\lambda_x,\lambda_y}(z) = L_2$$
$$C_1 := \lambda_{\rho_x,\rho_y}(z) = \lambda_{\rho_x,\rho_y}(z)\lambda_x(y) = C_2$$
$$R_2 := \rho_{\rho_x,\rho_y}(z) = \rho_{\rho_x,\rho_y}(z)\rho_y(x) = R_1$$

are satisfied, for all $x, y, z \in X$. At this purpose, let $x, y, z$ be elements of $X_\omega, X_\iota, X_\kappa$ respectively, assume $\nu := \omega\kappa$ and

$$X := \phi_\omega,\nu(x) \quad Y := \phi_\iota,\nu(y) \quad Z := \phi_\kappa,\nu(z) \quad \text{in } X_\nu.$$

Setting

$$U := \rho_{\phi_\omega,\nu}(x) \quad V := \lambda_{\phi_\omega,\nu}(y) \quad \text{in } X_\omega$$

we have that

$$L_2 = \lambda_x(y)\lambda_{\rho_y}(z) = \lambda_{\lambda_x\lambda_y}(z) = \lambda_{\lambda_x\lambda_y}(z)$$
$$\quad = \lambda_{\lambda_x\lambda_y}(z)\phi_{\nu,\omega}(x) \quad \nu = \omega\kappa$$
$$\quad = \lambda_{\phi_\omega,\nu}(V)\phi_{\nu,\omega}\lambda_{\phi_\omega,\nu}(U) (Z) \quad \omega\nu = \nu$$
$$\quad = \lambda_{\phi_\omega,\nu}(V)\phi_{\nu,\omega}\lambda_{\phi_\omega,\nu}(U) (Z) \quad \phi_{\nu,\omega} = \text{id}_{X_\nu}$$

Since

$$\phi_{\omega,\nu}(V) = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \text{by (10)}$$
$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \phi_{\omega,\nu,\omega}\phi_\omega,\omega = \phi_{\omega,\nu,\omega}$$
$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \phi_{\omega,\nu,\omega,\nu} = \phi_{\omega,\nu,\omega,\nu} = \phi_{\omega,\nu}$$

$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y)$$

$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y)$$

Since

$$\phi_{\omega,\nu}(V) = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \text{by (10)}$$
$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \phi_{\omega,\nu,\omega}\phi_\omega,\omega = \phi_{\omega,\nu,\omega}$$
$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y) \quad \phi_{\omega,\nu,\omega,\nu} = \phi_{\omega,\nu,\omega,\nu} = \phi_{\omega,\nu}$$

$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y)$$

$$\quad = \lambda_{\phi_\omega,\nu}(x)\phi_\omega,\omega(y)$$

Since
and
\[ \phi_{\omega, \nu} (U) = \rho_{\phi_{\omega, \nu}, \omega}(y) \phi_{\omega, \nu}(x) \]
\[ = \rho_{\phi_{\omega, \nu}}(y) \phi_{\omega, \nu}(x) \]
\[ = \rho_{\omega}(x), \]
by (11)

it follows that
\[ L_2 = \lambda x(\nu) \lambda \rho_{\nu}(x)(z) \]
\[ = \lambda x \lambda \rho_{\nu}(z). \]

Moreover, it holds
\[ L_1 = \lambda x \lambda (z) = \lambda x \lambda (y) \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \phi_{\kappa, \kappa}(y) \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \lambda \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \lambda \phi_{\kappa, \kappa}(z), \]

hence we obtain that \( L_1 = L_2 \). Now, setting
\[ W := \rho_{\phi_{\nu, \kappa}, \kappa}(y) \]
\[ Z := \phi_{\phi_{\nu, \kappa}, \kappa}(z) \]

observe that
\[ C_1 = \lambda p_{\phi_{\nu, \kappa}}(x) \rho_{\nu}(y) \]
\[ = \lambda p_{\phi_{\nu, \kappa}, \nu}(y) \phi_{\nu}(x) \]
\[ = \lambda p_{\nu}(x)(W) \]
\[ = \lambda p_{\phi_{\nu, \kappa}, \nu}(x)(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu} \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu} \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W). \]

Moreover, it holds
\[ L_1 = \lambda x \lambda (z) = \lambda x \lambda (y) \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \phi_{\kappa, \kappa}(y) \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \lambda \phi_{\kappa, \kappa}(z) \]
\[ = \lambda x \lambda \phi_{\kappa, \kappa}(z), \]

hence we obtain that \( L_1 = L_2 \). Now, setting
\[ W := \rho_{\phi_{\nu, \kappa}, \kappa}(y) \]
\[ Z := \phi_{\phi_{\nu, \kappa}, \kappa}(z) \]

observe that
\[ C_1 = \lambda p_{\phi_{\nu, \kappa}}(x) \rho_{\nu}(y) \]
\[ = \lambda p_{\phi_{\nu, \kappa}, \nu}(y) \phi_{\nu}(x) \]
\[ = \lambda p_{\nu}(x)(W) \]
\[ = \lambda p_{\phi_{\nu, \kappa}, \nu}(x)(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu} \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu} \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W) \]
\[ = \lambda \phi_{\nu, \nu, \nu}(x) \phi_{\nu, \nu, \nu}(W). \]
Since
\[
\phi_{k,\nu}(Z) = \lambda_{\phi_{k,\nu}\phi_{k,\kappa}(y)}\phi_{k,\kappa}(Z)
\]
\[
= \lambda_{\phi_{k,\nu}(y)}\phi_{k,\nu}(z)
\]
\[
= \lambda_Y(Z)
\]
by (10)
\[
\phi_{k,\nu}(\lambda_{\phi_{k,\nu}\phi_{k,\kappa}}(y)) = \phi_{k,\nu}\phi_{k,\kappa} = \phi_{k,\nu}
\]
and
\[
\phi_{k,\nu}(W) = \rho_{\phi_{k,\nu}\phi_{k,\kappa}(z)}\phi_{k,\nu}(y)
\]
\[
= \rho_{\phi_{k,\nu}(z)}\phi_{k,\nu}(y)
\]
\[
= \rho_Z(Y)
\]
by (11)
\[
\phi_{k,\nu}(\phi_{k,\kappa}(W)) = \phi_{k,\nu}\phi_{k,\kappa} = \phi_{k,\nu}
\]
it follows that
\[
C_1 = \lambda_{\rho_Z(Y)}(x)\rho_Z(Y).
\]
in \(X_\nu\)

Furthermore, since \(U\) and \(V\) lie in \(X_\omega\), we have
\[
C_2 = \rho_{\phi_{\omega,\nu}(z)}\lambda_X(y)
\]
\[
= \rho_{\phi_{\omega,\nu}(y)}\phi_{\omega,\nu}(z)\lambda_X(z)
\]
\[
= \rho_{\lambda_X}(V)
\]
\[
= \rho_{\phi_{\omega,\nu}(\lambda_X)}(V)
\]
\[
= \rho_{\phi_{\omega,\nu}(\lambda_X)}(V)
\]
\[
= \rho_{\phi_{\omega,\nu}(\lambda_X)}(V)
\]
\[
= \rho_{\phi_{\omega,\nu}(\lambda_X)}(V)
\]
\[
\phi_{\omega,\nu} = id_{X_\nu}
\]
As seen before, \(\phi_{\omega,\nu}(U) = \rho_Y(X)\) and \(\phi_{\omega,\nu}(V) = \lambda_X(Y)\), thus
\[
C_2 = \rho_{\lambda_Y(X)}(z)\lambda_X(Y).
\]
in \(X_\nu\)
Consequently, since $r_\nu$ is a solution, we obtain that $C_1 = C_2$.

Finally, since $W$ and $Z$ lie in $X_\nu$, note that

$$\rho_2 = \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(x))$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(y))\rho_{\phi_{\omega_\nu}}(x)$$

$$= \rho_W\rho_Z(x)$$

$$= \rho_W\phi_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(y))\rho_{\phi_{\omega_\nu}}(x)$$

$$= \rho_{\phi_{\omega_\nu},\nu}(W)\phi_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(x))$$

$$= \rho_{\phi_{\omega_\nu},\nu}(W)\rho_{\phi_{\omega_\nu}}(x)(X)$$

$$= \rho_{\phi_{\omega_\nu},\nu}(W)\rho_{\phi_{\omega_\nu}}(x)(X)$$

$$\phi_{\phi_{\omega_\nu}} = \text{id}_{X_\nu}$$

As seen before, $\phi_{\omega_\nu}(W) = \rho_Z(Y)$. In addition we have

$$\phi_{\omega_\nu}(Z) = \lambda(\phi_{\omega_\nu},\phi_{\omega_\nu})(\phi_{\omega_\nu},\phi_{\omega_\nu})(z)$$

$$= \lambda(\phi_{\omega_\nu},\phi_{\omega_\nu})(z)$$

$$\phi_{\omega_\nu}(W)\phi_{\omega_\nu} = \phi_{\omega_\nu}, \phi_{\omega_\nu}\phi_{\omega_\nu} = \phi_{\omega_\nu}$$

$$\phi_{\omega_\nu}(Z) = \lambda(Y)$$.

It follows that

$$\rho_2 = \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(x))$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(y))\phi_{\omega_\nu}(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu}(y))\phi_{\omega_\nu}(x)$$

Moreover, it holds

$$\rho_1 = \rho_{\phi_{\omega_\nu}}(x) = \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$= \rho_{\phi_{\omega_\nu}}(\phi_{\omega_\nu},\phi_{\omega_\nu})(x)$$

$$\phi_{\omega_\nu}(\phi_{\omega_\nu},\phi_{\omega_\nu}) = \phi_{\omega_\nu}, \phi_{\omega_\nu}(\phi_{\omega_\nu},\phi_{\omega_\nu}) = \phi_{\omega_\nu}$$

$$\rho_1 = \rho_2$$.

Therefore, the map $r_\nu$ is a solution.

Strong semilattices of solutions $(X_\omega, r_\omega)$ allows one to produce examples of solutions with finite order if solutions $r_\alpha$ are.
Examples 14. 1. Let $X$ be a semilattice of sets indexed by $Y$ such that conditions 1. and 2. of Theorem 12 are satisfied and let $r_\alpha$ be the twist map on $X_\alpha$, for every $\alpha \in Y$. Then, if $\alpha \geq \beta$ we have that

$$\phi_{\alpha,\beta} \lambda_x (y) = \phi_{\alpha,\beta} (x) = \lambda_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (y))$$

$$\phi_{\alpha,\beta} \rho_y (x) = \phi_{\alpha,\beta} (x) = \rho_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (x),$$

for all $x, y \in X_\alpha$, i.e., condition 3. of Theorem 12 holds. Hence, the strong semilattice of solutions $(X, r)$ is such that $r^3 = r$. Indeed, if $x \in X_\alpha$ and $y \in X_\beta$, assuming $\nu := \alpha \beta$ we have that $r (x, y) = r_\nu (\phi_{\alpha,\nu} (x), \phi_{\beta,\nu} (y))$, hence

$$r^2 (x, y) = (\phi_{\alpha,\nu} (x), \phi_{\beta,\nu} (y))$$

and so

$$r^3 (x, y) = r_\nu (\phi_{\alpha,\nu} (x), \phi_{\beta,\nu} (y)) = r (x, y).$$

Consequently, $r^3 = r$.

2. Let $X := X_\alpha \cup X_\beta$ be a semilattice of sets such that $\alpha > \beta$, $c$ a fixed element of $X_\beta$, and $\phi_{\alpha,\beta} (x) = c$, for every $x \in X_\alpha$. Let $r_\alpha$ be the twist map on $X_\alpha$ and $r_\beta$ the idempotent solution on $X_\beta$ defined by $r_\beta (x, y) := (x, c)$, for all $x, y \in X_\beta$. Then, if $x, y \in X_\alpha$, we have that

$$\phi_{\alpha,\beta} \lambda_x (y) = c = \phi_{\alpha,\beta} (x) = \lambda_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (y))$$

$$\phi_{\alpha,\beta} \rho_y (x) = c = \rho_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (x),$$

hence the assumptions of Theorem 12 are satisfied. Moreover, the strong semilattice of solutions $(X, r)$ is such that $r^3 = r$. Indeed, if $x \in X_\alpha$ and $y \in X_\beta$, since $r (x, y) = r_\beta (\phi_{\alpha,\beta} (x), y) = r_\beta (c, y) = (c, c)$, it follows that

$$r^2 (x, y) = r_\beta (c, c) = (c, c) = r (x, y).$$

Therefore, we obtain that $r^3 = r$.

3. Let $X := X_\alpha \cup X_\beta$ be a semilattice of sets such that $\alpha > \beta$, $c$ a fixed element of $X_\beta$, and $\phi_{\alpha,\beta} (x) = c$, for every $x \in X_\alpha$. Let $f$ be an idempotent map from $X_\beta$ into itself, $f \neq \text{id}_{X_\beta}$, and $r_\alpha$ the map from $X_\alpha \times X_\alpha$ into itself defined by $r_\alpha (x, y) := (f (x), x)$, for all $x, y \in X_\alpha$. Thus, $r_\alpha$ is a solution such that $r^2_\alpha = r^3_\alpha$. Let $r_\beta$ be the idempotent solution defined by $r_\beta (x, y) = (x, c)$, for all $x, y \in X_\beta$. Then, if $x, y \in X_\alpha$, we obtain that

$$\phi_{\alpha,\beta} \lambda_x (y) = c = \phi_{\alpha,\beta} (x) = \lambda_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (y))$$

$$\phi_{\alpha,\beta} \rho_y (x) = c = \rho_{\phi_{\alpha,\beta}} (\phi_{\alpha,\beta} (x),$$

thus the hypotheses of Theorem 12 are satisfied. Moreover, the strong semilattice of solutions $(X, r)$ is such that $r^3 = r^2$. Indeed, if $x \in X_\alpha$ and $y \in X_\beta$, since $r (x, y) = r_\beta (\phi_{\alpha,\beta} (x), y) = r_\beta (c, y) = (c, c)$, it follows that

$$r^2 (x, y) = r_\beta (c, c) = (c, c) = r (x, y).$$
and clearly $r^3(x, y) = r^2(x, y)$. Therefore, $r^3 = r^2$.

To investigate strong semilattices of solutions with finite order, we need the notions of the index and the period of a solution $r$ that are

\[
i(r) := \min \{ j \mid j \in \mathbb{N}_0, \exists l \in \mathbb{N} \ r^j = r^l, \ j \neq l \} \]
\[
p(r) := \min \{ k \mid k \in \mathbb{N}, r^{i(r)+k} = r^{i(r)} \},
\]
respectively. These definitions of the index and the period are slightly different from the classical ones (cf. [30, p. 10]), but they are functional to distinguish bijective solutions from non-bijective ones. For more details, we refer the reader to [10].

In the following theorem we show that, given a semilattice $Y$ of finite cardinality, the strong semilattice of solutions $(X, r)$ indexed by $Y$ is of finite order if and only if solutions $r_\alpha$ are. Furthermore, it allows for establishing the order of the strong semilattice of solutions $(X, r)$ if the index and the period of solutions $r_\alpha$ are known. Conversely, the index and the period of a strong semilattice of solutions $(X, r)$ give us upper bounds of the indexes and periods of solutions $r_\alpha$.

**Theorem 15.** Let $(X, r)$ be a strong semilattice of solutions indexed by a finite semilattice $Y$. Then, $r_\alpha$ is a solution with finite order on $X_\alpha$, for every $\alpha \in Y$, if and only if $r$ is a solution with finite order. More precisely, the index of $r$ is

\[
i(r) = \max \{ 1, i(r_\alpha) \mid \alpha \in Y \}
\]

and the period is

\[
p(r) = \text{lcm} \{ p(r_\alpha) \mid \alpha \in Y \}.
\]

**Proof.** At first suppose that $r_\alpha$ is a solution with finite order, for every $\alpha \in Y$. Let $n := \text{lcm} \{ p(r_\alpha) \mid \alpha \in Y \}$ and $i := \max \{ i(r_\alpha) \mid \alpha \in Y \}$. If $x \in X_\alpha$ and $y \in X_\beta$, setting $\nu := \alpha \beta$ we have that

\[
r^{n+i}(x, y) = r^{n+i}_\nu(\phi_{\alpha,\nu}(x), \phi_{\beta,\nu}(y)) = r^i_\nu(\phi_{\alpha,\nu}(x), \phi_{\beta,\nu}(y)).
\]

Consequently, if $i = 0$, i.e., $r_\alpha$ is bijective for every $\alpha \in Y$, we obtain that

\[
r^{n+1}(x, y) = r_\nu(\phi_{\alpha,\nu}(x), \phi_{\beta,\nu}(y)) = r(x, y).
\]

Therefore, $r^{n+1} = r$ and clearly the index of $r$ is 1 by the assumption on $i$. Now, assume that $i \neq 0$ and that $i = i(r_\gamma)$, for a certain $\gamma \in Y$. If $r^n = r^h$, for a positive integer $h$, in particular we have that $r^n_\gamma = r^h_\gamma$. Since $n = p(r_\gamma)q + i$ for a certain natural number $q$, it follows that

\[
r^{p(r_\gamma)+i} = r^i = r^{p(r_\gamma)q+i} = r^n_\gamma = r^h_\gamma,
\]
hence $i \leq h$ and so $i(r) = i$.

Now, we proceed to determine the period of $r$. If $r^{m} = r^{i(r)}$ for a natural number $m$, then $r^{m} = r^{i(r)}$, for every $\alpha \in Y$. Consequently, $p(r_{\alpha})$ divides $m - i(r)$, for every $\alpha \in Y$, thus $\gcd \{ p(r_{\alpha}) \mid \alpha \in Y \}$ divides $m - i(r)$, i.e., $n - i(r)$ divides $m - i(r)$. Therefore, $n - i(r) \leq m - i(r)$ and hence $p(r) = n - i(r)$. Conversely, suppose that the solution $r$ is with finite order, set $i := i(r)$, and $p := p(r)$. If $\alpha \in Y$, since $\phi_{\alpha,\alpha} = \text{id}_{X_{\alpha} \times X_{\alpha}}$ we obtain that $r^{p+1}_{\alpha} = r^{i}_{\alpha}$. Therefore, $r_{\alpha}$ is a solution with finite order. Clearly, we have that $i(r_{\alpha})$ is less than $i$ and $p(r_{\alpha})$ divides $p$.

Let us note that if $(X, r)$ is a strong semilattice of non-bijective solutions $r_{\alpha}$ such that $i(r_{\alpha}) = i$ and $p(r_{\alpha}) = n$, for every $\alpha \in Y$, then $r$ is still a solution of index $i$ and period $n$, also in the case of an infinite semilattice $Y$. Indeed, one can prove this statement by similar computations used for the non-bijective case in the proof of Theorem 15.

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