Some new congruences for \((7, t)\)-regular bipartitions modulo \(t\)

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Abstract:
In this work, we study the function \(B_{s,t}(n)\), which counts the number of \((s, t)\)-regular bipartitions of \(n\). Recently, many authors proved infinite families of congruences modulo 11 for \(B_{3,11}(n)\), modulo 3 for \(B_{3,3}(n)\) and modulo 5 for \(B_{5,5}(n)\). Very recently, Kathiravan proved several infinite families of congruences modulo 11, 13 and 11 for \(B_{5,11}(n)\), \(B_{5,13}(n)\) and \(B_{2,8}(n)\). In this paper, we will prove infinite families of congruences modulo 5 for \(B_{2,15}(n)\), modulo 11 for \(B_{7,11}(n)\) and modulo 13 for \(B_{7,13}(n)\).

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1 Introduction

For \(n\) a positive integer, a partition of \(n\) is a non-increasing sequence of positive integers whose sum is \(n\). The number of partitions of \(n\) is denoted by \(p(n)\). The generating function for \(p(n)\), is given by

\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}. \tag{1.1}
\]

For a nonzero integer \(k\), we define the general partition function \(p_k(n)\) as the coefficient of \(q^n\) in the expansion of \((q; q)_k\). If \(k = -1\) we have usual partition function \(p(n)\). The generating function for \(p_k(n)\), is given by

\[
\sum_{n=0}^{\infty} p_k(n)q^n = (q; q)_{\infty}^k, \tag{1.2}
\]

where as customary, we define

\[
f_k := (q^k; q^k)_{\infty} = \prod_{m=1}^{\infty} (1 - q^{mk}).
\]

In [13, 19] Ramanujan obtained the beautiful identities

\[
\sum_{n=0}^{\infty} p(5n + 4)q^n = 5\frac{f_5^5}{f_1^5} \tag{1.3}
\]
and
\[
\sum_{n=0}^{\infty} p(7n + 5)q^n = \frac{f_3^2}{f_1^2} + 49q\frac{f_4^2}{f_1^2}.
\] (1.4)

Ramanujan [18] give a brief of the identities (1.3), he did not prove the identities (1.4) in [18], but [20] he did give a sketch of his proof of identities (1.4) in his unpublished manuscript of the partition and \( \tau \)-function. Note that (1.3) and (1.4) immediately yield the congruences \( p(5n + 4) \equiv 0 \pmod{5} \) and \( p(7n + 5) \equiv 0 \pmod{7} \).

Ramanujan partition congruences motivated an investigation of many classes of partitions, such as \( \ell \)-regular partitions. For a positive integer \( \ell \), a partition is said to be \( \ell \)-regular if none of its parts is divisible by \( \ell \). If \( b_\ell(n) \) denote the number of \( \ell \)-regular partitions of \( n \), then the generating function for \( b_\ell(n) \), is given by
\[
\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1^\ell}.
\] (1.5)

In recent years, many authors studied arithmetic properties of \( \ell \)-regular partitions [5, 6, 7, 9, 10, 12, 22, 23, 25].

Recall that, for a positive integers \( s > 1 \) and \( t > 1 \), a bipartition \((\lambda, \mu)\) of \( n \) is a pair of partitions \((\lambda, \mu)\) such that the sum of all the parts equals \( n \). A \((s, t)\)-regular bipartition of \( n \) is a bipartition \((\lambda, \mu)\) of \( n \) such that \( \lambda \) is a \( s \)-regular partition and \( \mu \) is a \( t \)-regular partition. If \( B_{s,t}(n) \) denote the number of \((s, t)\)-regular bipartitions of \( n \), then the generating function \( B_{s,t}(n) \), is given by
\[
\sum_{n=0}^{\infty} B_{s,t}(n)q^n = \frac{f_s f_t}{f_1}. 
\] (1.6)

Recently, Lin [16] proved infinite families of congruence modulo 3 for \( B_7(n) \), by using Ramanujan’s two modular equation of degree 7, and in [17], he proved infinite families of congruence modulo 3 for \( B_{13}(n) \). For more related works, see [14, 21].

Very recently, [18] Dou proved that, for \( n \geq 0 \) and \( \alpha \geq 2 \),
\[
B_{3,11}\left(3^\alpha n + \frac{5 \cdot 3^{\alpha-1} - 1}{2}\right) \equiv 0 \pmod{11}.
\]

Adiga and Ranganatha [1] proved infinite families of congruences modulo 3 for \( B_{3,7}(n) \) and Xia and Yao [24] proved several infinite families of congruences modulo 3 for \( B_{3,s}(n) \), modulo 5 for \( B_{5,s}(n) \) and modulo 7 for \( B_{3,7}(n) \). For example, let \( s \) be a positive integer and let \( p \geq 5 \) be a prime, for \( n \geq 0 \),
\[
B_{3,s}\left(p^{2\alpha+1}n + \frac{(1 + s)(p^{2\alpha+2} - 1)}{24}\right) \equiv 0 \pmod{3}.
\]

Very Recently, Kathiravan [15] proved several infinite families of congruences modulo 11, 13 and 11 for \( B_{5,11}(n) \), \( B_{5,13}(n) \) and \( B_{2,8}(n) \). For example, for all \( n \geq 0 \) and \( m \geq 0 \),
\[
B_{5,13}\left(5^{6m+5}(5n + k) + \frac{5^{6m+5} - 2}{3}\right) \equiv 0 \pmod{13}, \text{ where } k = 1, 5.
\]

In this paper, we will prove several infinite families of congruences modulo 5, 11 and 13 for \( B_{2,15}(n) \), \( B_{7,11}(n) \) and \( B_{7,13}(n) \). The main results of this paper are as follows,
Theorem 1.1. For all \( n \geq 0 \) and \( m \geq 0 \),
\[
B_{2,15} \left( 3^{2m+1} n + \frac{7 \cdot 3^{2m+1} - 5}{8} \right) \equiv 2^m B_{2,15}(3n+2) \pmod{5},
\]
(1.7)
\[
B_{2,15} \left( 3^{2m+2} n + \frac{23 \cdot 3^{2m+1} - 5}{8} \right) \equiv 0 \pmod{5},
\]
(1.8)
\[
B_{2,15} \left( 3^{2(m+1)+1} n + \frac{13 \cdot 3^{2(m+1)} - 5}{8} \right) \equiv 0 \pmod{5}.
\]
(1.9)

Theorem 1.2. For all \( n \geq 0 \) and \( m \geq 0 \),
\[
B_{7,11} \left( 7^{12m} n + \frac{2 \cdot 7^{12m} - 2}{3} \right) \equiv 3^m B_{7,11}(n) \pmod{11},
\]
(1.10)
\[
B_{7,11} \left( 7^{12m+11}(7n + k) + \frac{2 \cdot 7^{12m+11} - 2}{3} \right) \equiv 0 \pmod{11}, \text{ where } k = 1, 5, 6.
\]
(1.11)

Theorem 1.3. For all \( n \geq 0 \) and \( m \geq 0 \),
\[
B_{7,13} \left( 7^{40m} n + \frac{3 \cdot 7^{40m} - 3}{4} \right) \equiv 5^m B_{7,13}(n) \pmod{13},
\]
(1.12)
\[
B_{7,13} \left( 7^{40m+39}(7n + k) + \frac{7^{40m+39} - 3}{4} \right) \equiv 0 \pmod{13}, \text{ where } k = 2, 6, 7.
\]
(1.13)

2 The identities

In this section, we prove some lemmas to prove our main results. By the binomial theorem for any prime \( p \),
\[
f_p \equiv f_1^p \pmod{p}.
\]
(2.1)

Lemma 2.1. (Berndt [2, p.49]), we have
\[
\frac{f_2^2}{f_2} = \frac{f_1^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9},
\]
(2.2)
\[
\frac{f_3^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}.
\]
(2.3)

Lemma 2.2. (Hirschhorn and Sellers [11]), we have
\[
\frac{f_2^2}{f_2^2} = \frac{f_6^4 f_9^6}{f_3 f_{18}^4} + 2q \frac{f_3^3 f_{18}^3}{f_3^3} + 4q^2 \frac{f_2^2 f_{18}^2}{f_6^3}.
\]
(2.4)

Lemma 2.3. For \( n \geq 0 \), we have
\[
\sum_{n=0}^{\infty} p_5(7n + 3)q^n = 10 f_1^4 f_7 + 49 f_7^2 q
\]
(2.5)
Proof. Setting $k = 5$ in (1.2), we have
\[ \sum_{n=0}^{\infty} p_5(n)q^n = f_5^1 \] (2.6)

In [2, p. 303, Entry 17(v)], we have
\[ f_1 = f_{49} \left( \frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)} q - q^2 + \frac{C(q^7)}{A(q^7)} q^3 \right) \] (2.7)
where $A$, $B$ and $C$ are defined by
\[ A = A(q) := \frac{f(-q^3, -q^4)}{f(-q^2)}, \quad B = B(q) := \frac{f(-q^2, -q^5)}{f(-q^2)} \text{ and } C = C := \frac{f(-q, -q^6)}{f(-q^2)} \]
Substituting (2.7) into (2.6), we have
\[ \sum_{n=0}^{\infty} p_5(n)q^n = f_{49}^5 \left( B(q^7) \frac{A(q^7)}{B(q^7)} q - q^2 + C(q^7) q^3 \right)^5 \] (2.8)
If we extract those terms in which the power of $q$ is congruent to 3 modulo 7, divide by $q^3$ and replace $q^7$ by $q$, we have
\[ \sum_{n=0}^{\infty} p_5(7n + 3)q^n = f_7^5 \left( 20 \left( \frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3} \right) \right. \]
\[ \left. -10 \left( \frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2} \right) - 61q \right). \] (2.9)
From [3, P. 174, Entry 31] and [4, Eq. 3.11 and Eq. 3.15] in the terms of $A$, $B$ and $C$, we have
\[ \frac{B^5}{AC^4} - \frac{A^5}{B^4C} - \frac{C^5q^3}{A^4B} = 3q \] (2.10)
\[ \frac{AB^2}{C^3} + \frac{A^2Cq}{B^3} - \frac{BC^2q^2}{A^3} = f_7^4 + 8q \] (2.11)
\[ \frac{A^3}{BC^2} - \frac{B^3q}{A^2C} - \frac{C^3q^2}{AB^2} = f_7^4 + 5q \] (2.12)
\[ \frac{B^7}{C^4} - \frac{A^7q}{B^4} + \frac{C^7q^5}{A^4} = 14 \frac{f_7^4}{f_7^2} q + f_7^3 + 57q^2. \] (2.13)
Substituting (2.11) and (2.12) into (2.9) and simplifying, we completed the Lemma 2.3.

Lemma 2.4. [4, Theorem 3.2] For $n \geq 0$, we have
\[ \sum_{n=0}^{\infty} p_7(7n)q^n = \frac{f_7^8}{f_7^2} + 49 f_7^2 f_7^3 q \] (2.14)

Lemma 2.5. For $n \geq 0$, we have
\[ \sum_{n=0}^{\infty} p_9(7n + 4)q^n = -90 f_7^8 f_7^3 - 882 f_7^4 f_7^5 q - 2401 f_7^6 q^2 \] (2.15)
Proof. Setting $k = 9$ in (2.12), we have
\[
\sum_{n=0}^{\infty} p_9(n) q^n = f_9^0
\] (2.16)
Substituting (2.17) into (2.16), we have
\[
\sum_{n=0}^{\infty} p_9(n) q^n = f_9^0 \left( \frac{B(q)}{C(q)} \right)^3 - \frac{A(q)}{B(q)} q - q^2 + \frac{C(q)^3}{A(q)} q^3 \right)^9
\] (2.17)
If we extract those terms in which the power of $q$ is congruent to 4 modulo 7, divide by $q^4$ and replace $q^7$ by $q$, we have
\[
\sum_{n=0}^{\infty} p_9(7n+4) q^n = f_9^2 \left( \frac{36B^7}{C^7} - \frac{252A^2B^4}{C^6} + \frac{126A^4B}{C^5} \right)
\] (2.18)
Rearrange the above equation, we have
\[
\sum_{n=0}^{\infty} p_9(7n+4) q^n = f_9^3 \left( \frac{A^7q}{B^7} + \frac{C^7q^5}{A^7} \right) - 252 \left( \frac{A^2B^4}{C^6} - \frac{A^4C^2q^2}{B^6} - \frac{B^2C^4q^4}{A^6} \right)
\] (2.19)
From above follow that
\[
\sum_{n=0}^{\infty} p_9(7n+4) q^n = f_9^3 \left( \frac{A^7q}{B^7} + \frac{C^7q^5}{A^7} \right)
\]
Substituting (2.10), (2.11), (2.12) and (2.13) into (2.20), we have

\[
\sum_{n=0}^{\infty} p_q(7n+4)q^n = f_7^q \left( 36 \left( \frac{f_7^4}{f_7^2} + 14 \frac{f_7^4 q}{f_7^2} + 57 q^2 \right) - 252 \left( \left( \frac{f_7^4}{f_7^2} + 8q \right)^2 - 2q \left( \frac{f_7^4}{f_7^2} + 5q \right) \right) \\
+ 126 \left( (3q)q + \left( \frac{f_7^4}{f_7^2} + 5q \right) \left( \frac{f_7^4}{f_7^2} + 8q \right) \right) - 756q(3q) + 4284q \left( \frac{f_7^4}{f_7^2} + 8q \right) \\
- 3780q \left( \frac{f_7^4}{f_7^2} + 5q \right) - 9367q^2 \right). \tag{2.20}
\]

Simplify the equation (2.21), we completed the Lemma 2.5. \( \square \)

**Lemma 2.6.** For \( n \geq 0 \), we have

\[
\sum_{n=0}^{\infty} p_{11}(7n+1)q^n = -\frac{11f_{11}^{12}}{f_7^2} - 77f_7^{12} + 3773f_7^{14}q + 16807f_7^{14}q^3 \tag{2.22}
\]

**Proof.** Setting \( k = 11 \) in (1.2), we have

\[
\sum_{n=0}^{\infty} p_{11}(n)q^n = f_7^{11} \tag{2.23}
\]

Substituting (2.7) into (2.23), we have

\[
\sum_{n=0}^{\infty} p_{11}(n)q^n = f_7^{11} \left( \frac{B(q^2)}{C(q^2)} \frac{A(q^2)}{B(q^2)} q - q^2 + \frac{C(q^2)}{A(q^2)}q^3 \right)^{11} \tag{2.24}
\]

If we extract those terms in which the power of \( q \) is congruent to 1 modulo 7, divide by \( q \) and replace \( q^7 \) by \( q \), we have

\[
\sum_{n=0}^{\infty} p_{11}(7n+1)q^n = f_7^{11} \left( -\frac{11AB^9}{C_{10}} + q \left( -\frac{1320B^7}{C^7} + \frac{5940A^2B^4}{C^3} + \frac{6930A^4B}{C^5} - \frac{2310A^6}{B^2C^3} + \frac{165A^8}{B^5C^3} \right) \\
+ q^2 \left( -\frac{1320A^7}{B^7} + \frac{165B^8}{A^3C^5} - \frac{16632B^5}{AC^4} + \frac{63690A^2B^2}{C^3} - \frac{57750A^3}{BC^2} + \frac{16632A^5}{B^3C} + \frac{11A^6C}{B^{10}} \right) \\
+ q^3 \left( -\frac{134971}{A^5} - \frac{2310B^6}{A^4C^2} + \frac{57750B^3}{A^2C^2} + \frac{63690A^2C}{B^3} + \frac{5940A^4C^2}{B^6} \right) \\
+ q^4 \left( -\frac{6930B^4C}{A^5} + \frac{63690BC^2}{A^3} + \frac{57750C^3}{AB^2} - \frac{6930AC^4}{B^5} \right) \\
+ q^5 \left( -\frac{5940B^2C^4}{A^6} + \frac{16632C^5}{A^4B} - \frac{2310C^6}{A^2B^4} + q^6 \left( \frac{1320C^7}{A^7} - \frac{165C^8}{A^6B^3} + \frac{11BC^9}{A^{10}} \right) \right). \tag{2.25}
\]
Rearrange the above equation, we have
\[
\sum_{n=0}^{\infty} p_{11}(7n+1)q^n = f_{11}^{11} \left( -11 \left( \frac{AB^9}{C^{10}} - \frac{A^9 C q^2}{B^{10}} - \frac{BC^9 q^7}{A^{10}} \right) + 320q \left( \frac{B^7}{C^7} - \frac{A^7 q}{B^7} + \frac{C^7 q^5}{A^7} \right) 
\right.
\]
\[
- 5940q \left( \frac{A^2 B^4}{C^6} + \frac{A^4 C^2 q^2}{B^6} + \frac{B^2 C^4 q^4}{A^6} \right) + 6930q \left( \frac{A^4 B}{C^5} + \frac{B^4 C q^3}{A^5} - \frac{AC^4 q^3}{B^3} \right)
\]
\[
- 2310q \left( \frac{A^6}{B^4 C^2} + \frac{B^6 q^2}{A^4 C^2} + \frac{C^6 q^4}{A^4 B^2} \right) + 165q \left( \frac{A^8}{B^5 C^3} + \frac{B^8 q}{A^5 C^3} - \frac{C^8 q^5}{A^8 B^3} \right)
\]
\[
- 16632q^2 \left( \frac{B^5}{A^4 C^4} - \frac{A^5}{B^4 C^4} + \frac{C^5 q^3}{A^4 B^4} \right) + 63690q^2 \left( \frac{A^2 B}{C^3} + \frac{A^2 C q}{B^3} - \frac{BC^2 q^2}{A^3} \right)
\]
\[
- 57750q^2 \left( \frac{A^3}{BC^2} - \frac{B^3 q}{A^4 C} - \frac{C^3 q^2}{AB^2} \right) - 134971q^3 \right)
\]
\[(2.26)\]

From above follow that
\[
\sum_{n=0}^{\infty} p_{11}(7n+1)q^n = f_{11}^{11} \left( 11 \left( q \left( \frac{A^3}{BC^2} - \frac{B^3 q}{A^4 C} - \frac{C^3 q^2}{AB^2} \right) - \frac{C^5 q^3}{A^5 B} \right) - q \right)
\]
\[
- \left( \frac{B^7}{C^7} - \frac{A^7 q}{B^7} + \frac{C^7 q^5}{A^7} \right) \left( \frac{AB^2}{C^3} + \frac{A^2 C q^2}{B^3} - \frac{BC^2 q^2}{A^3} \right) + 1320q \left( \frac{B^7}{C^7} - \frac{A^7 q}{B^7} + \frac{C^7 q^5}{A^7} \right)
\]
\[
+ 5940q \left( \frac{A^2 B^4}{C^6} + \frac{A^4 C^2 q^2}{B^6} + \frac{B^2 C^4 q^4}{A^6} \right) + 6930q \left( \frac{A^4 B}{C^5} + \frac{B^4 C q^3}{A^5} - \frac{AC^4 q^3}{B^3} \right)
\]
\[
- 2310q \left( \frac{A^6}{B^4 C^2} + \frac{B^6 q^2}{A^4 C^2} + \frac{C^6 q^4}{A^4 B^2} \right) + 165q \left( \frac{A^8}{B^5 C^3} + \frac{B^8 q}{A^5 C^3} - \frac{C^8 q^5}{A^8 B^3} \right)
\]
\[
- 16632q^2 \left( \frac{B^5}{A^4 C^4} - \frac{A^5}{B^4 C^4} + \frac{C^5 q^3}{A^4 B^4} \right) + 63690q^2 \left( \frac{A^2 B}{C^3} + \frac{A^2 C q}{B^3} - \frac{BC^2 q^2}{A^3} \right)
\]
\[
- 57750q^2 \left( \frac{A^3}{BC^2} - \frac{B^3 q}{A^4 C} - \frac{C^3 q^2}{AB^2} \right) - 134971q^3 \right)
\]
\[(2.27)\]

Substituting \(2.10\), \(2.11\), \(2.12\) and \(2.13\) into \(2.27\), we have
\[
\sum_{n=0}^{\infty} p_{11}(7n+1)q^n = f_{11}^{11} \left( 11 \left( q \left( \frac{f_1}{f_2} + 5q \right) (3q - q) - \left( \frac{f_1}{f_2} + 14 \frac{f_1 q}{f_2} + 57q^2 \right) \left( \frac{f_1}{f_2} + 8q \right) \right) \right)
\]
\[
+ 1320q \left( \frac{f_1}{f_2} + 14 \frac{f_1 q}{f_2} + 57q^2 \right) + 5940q \left( 2q \left( \frac{f_1}{f_2} + 5q \right) - \left( \frac{f_1}{f_2} + 8q \right) \right)^2 \right)
\]
\[
+ 6930q \left( \left( \frac{f_1}{f_2} + 8q \right) \left( \frac{f_1}{f_2} + 5q \right) + (3q + q)^2 \right) - 2310q \left( \left( \frac{f_1}{f_2} + 5q \right)^2 + 2q \left( \frac{f_1}{f_2} + 8q \right) \right)
\]
\[
+ 165q \left( \frac{f_1}{f_2} + 8q \right)^2 - \left( \frac{f_1}{f_2} + 5q \right) (3q + q) \right) - 16632q^2 (3q) + 63690q^2 \left( \frac{f_1}{f_2} + 8q \right)\]
Simplify the equation (2.28), we completed the Lemma 2.6. $$\square$$

3 Congruence for \((2, 15)\)-regular bipartition

**Theorem 3.1.** For \(n \geq 0\), we have
\[
B_{2, 15}(9n + 8) \equiv 0 \pmod{5}, \\
B_{2, 15}(27n + 14) \equiv 0 \pmod{5}, \\
B_{2, 15}(27n + 23) \equiv 2B_{2, 15}(3n + 2) \pmod{5}.
\]

**Proof.** Setting \(s = 2\) and \(t = 15\) in (1.6), we have
\[
\sum_{n=0}^{\infty} B_{2, 15}(n)q^n = \frac{f_2 f_{15}}{f_1}.
\]
Substituting (2.4) into (3.4), we have
\[
\sum_{n=0}^{\infty} B_{2, 15}(n)q^n = \frac{f_2 f_{15} f_6}{f_3 f_{18}} + 2q f_6 f_3 f_3 + 4q^2 f_6 f_3 f_3 + (3.5)
\]
If we extract those terms in which the power of \(q\) is congruent to 2 modulo 3, divide by \(q^2\) and replace \(q^3\) by \(q\), we have
\[
\sum_{n=0}^{\infty} B_{2, 15}(3n + 2)q^n \equiv 4 \frac{f_2 f_3}{f_1} \pmod{5}.
\]
Substituting (4.10) into (3.6), we have
\[
\sum_{n=0}^{\infty} B_{2, 15}(3n + 2)q^n \equiv 4 f_3^2 \frac{f_6 f_3 f_3 f_3}{f_3 f_{18}} + q f_6 f_3 f_3 f_3 \pmod{5}.
\]
If we extract those terms in which the power of \(q\) is congruent to 1 modulo 3, divide by \(q\) and replace \(q^3\) by \(q\), we have
\[
\sum_{n=0}^{\infty} B_{2, 15}(9n + 5)q^n \equiv 4 \frac{f_2^6 f_3^3}{f_3} \pmod{5}.
\]
Entry 1(iv) on page 345 of [2] is Ramanujan’s cubic continued fraction
\[
f_1^3 = f_9^3(u^{-1} - 3q + 4q^3 u^2),
\]
where
\[
u = \frac{f_6 f_3 f_3 f_3}{f_3}.
\]
Now replacing \(q\) by \(q^2\) in (3.9), we have
\[
f_2^3 = f_{18}^3(u_1^{-1} - 3q^2 + 4q^6 u_1^2),
\]
Substituting (3.10) in (3.8) and extract those terms in which the power of \( q \) is congruent to 2 modulo 3, divide by \( q^2 \) and replace \( q^3 \) by \( q \), we have

\[
\sum_{n=0}^{\infty} B_{2,15}(27n + 23)q^n \equiv 3 \frac{f_2^2 f_6^3}{f_1} \pmod{5}.
\]  

(3.11)

This completed the proof Theorem 3.1, follow from (3.7), (3.11). Substituting (3.10) in (3.8) and extract those terms in which the power of \( q \) is congruent to 1 modulo 3, divide by \( q \) and replace \( q^3 \) by \( q \), we have (3.2).

**Proof Theorem 1.1**

Equation (1.7) follow from (3.3) by the mathematical induction. Employing (3.1) and (3.2) in (1.7) we obtain (1.8) and (1.9).

4 Congruence for \((7, 11)\)-regular bipartition

**Lemma 4.1.** For all \( n \geq 0 \) and \( m \geq 1 \),

\[
\sum_{n=0}^{\infty} B_{7,11}(7^m n + \frac{2(7^m - 1)}{3}) q^n \equiv A_{3m-2} f_7 f_1^3 + A_{3m-1} f_7^5 f_1^3 q + A_{3m} f_7 f_1^3 q^2. \quad (4.1)
\]

where

1. \( A_{3m-2} \equiv 9A_{3m-5} + 10A_{3m-4} + 10A_{3m-3} \),
2. \( A_{3m-1} \equiv 9A_{3m-5} + 5A_{3m-4} \),
3. \( A_{3m} \equiv 8A_{3m-5} \),

with \( A_1 = 9, A_2 = 9 \) and \( A_3 = 8 \).

**Proof.** Setting \( s = 7 \) and \( t = 11 \) in (1.6), we have

\[
\sum_{n=0}^{\infty} B_{7,11}(n) q^n \equiv f_7 f_1^3 \pmod{11}. \quad (4.2)
\]

Substituting (2.16) into (4.2), we have

\[
\sum_{n=0}^{\infty} B_{7,11}(n) q^n \equiv f_7 \sum_{n=0}^{\infty} p_9(n) q^n \pmod{11}. \quad (4.3)
\]

If we extract those terms in which the power of \( q \) is congruent to 4 modulo 7, divide by \( q^4 \) and replace \( q^7 \) by \( q \), we have

\[
\sum_{n=0}^{\infty} B_{7,11}(7n + 4) q^n \equiv f_1 \sum_{n=0}^{\infty} p_9(7n + 4) q^n \pmod{11}. \quad (4.4)
\]
Now from (2.15) and (4.4), we have
\[
\sum_{n=0}^{\infty} B_{7,11}(7n + 4)q^n \equiv f_1 \left( 9f_7^3 f_7 + 9f_1^4 f_7^5 q + 8f_7^9 q^2 \right),
\]
\[
\sum_{n=0}^{\infty} B_{7,11}(7n + 4)q^n \equiv 9f_1 f_7^3 + 9f_1^5 f_7^5 q + 8f_7^9 f_1 q^2.
\]
(4.5)

Substituting (2.6), (2.7) and (2.16) into (4.5), we have
\[
\sum_{n=0}^{\infty} B_{7,11}(7n + 4)q^n \equiv 9f_7^3 f_9^1 + 9f_1^5 f_7^5 q + 8f_7^9 f_1 q^2,
\]
(4.6)

If we extract those terms in which the power of \(q\) is congruent to 4 modulo 7, divide by \(q^4\) and replace \(q^7\) by \(q\), we have
\[
\sum_{n=0}^{\infty} B_{7,11}(49n + 32)q^n \equiv 9f_1^3 f_7^9 + 9f_1^5 f_7^5 q + 8f_7^9 f_1 q^2.
\]
(4.7)

Substituting Lemma 2.6 and Lemma 2.8 into (4.7), we have
\[
\sum_{n=0}^{\infty} B_{7,11}(49n + 32)q^n \equiv 9f_7 f_1^3 f_7^9 + 5f_7^3 f_7^5 q + 6f_7^9 f_1 q^2.
\]
(4.8)

By the induction the proof of the Lemma completed.

**Theorem 4.1.** For all \(n \geq 0\),
\[
B_{7,11} \left( 7^{12} n + \frac{2(7^{12} - 1)}{3} \right) \equiv 3B_{7,11}(n) \pmod{11}.
\]
(4.9)
\[
B_{7,11} \left( 7^{11}(7n + k) + \frac{2(7^{11} - 1)}{3} \right) \equiv 0 \pmod{11}, \text{ where } k = 1, 5, 6.
\]
(4.10)

**Proof.** From Lemma 4.1 put \(m = 11\), we find that, \(A_{31} = 0\) and \(A_{32} = 0\) and \(A_{33} = 8\)
\[
\sum_{n=0}^{\infty} B_{7,11} \left( 7^{11} n + \frac{2(7^{11} - 1)}{3} \right) q^n \equiv 8f_7^9 f_1 q^2.
\]
(4.11)

Substituting (2.7) into (4.11), we have (4.9) and (4.10).

**Proof of Theorem 1.2.** Equation (1.10) follow from (4.9) by mathematical induction. Employing (1.10) in (4.10), we obtain (1.11).
5 Congruence for \((7, 13)\)-regular bipartition

Lemma 5.1. For all \(n \geq 0\) and \(m \geq 1\),
\[
\sum_{n=0}^{\infty} B_{7,13} \left(7^{2m-1} n + \frac{7^{2m-1} - 3}{4}\right) q^n \equiv A_{4m-3} \frac{f_{13}}{f_7} + A_{4m-2} f_7^3 f_1^3 q + A_{4m-1} f_7^1 f_1^5 q^2 + A_{4m} f_7^{11} f_1 q^3 \pmod{13},
\]
where
1. \(A_{4m-3} \equiv 4A_{4m-6} + 4A_{4m-5} + 11A_{4m-4}\).
2. \(A_{4m-2} \equiv 6A_{4m-7} + 6A_{4m-6} + 6A_{4m-5} + 12A_{4m-4}\).
3. \(A_{4m-1} \equiv 3A_{4m-7} + 3A_{4m-6} + 4A_{4m-5} + 10A_{4m-4}\).
4. \(A_{4m} \equiv 11A_{4m-6} + 6A_{4m-5} + 2A_{4m-4}\).

with \(A_1 = 2, A_2 = 1, A_3 = 3\) and \(A_4 = 11\).

Proof. Setting \(s = 7\) and \(t = 13\) in \((1.6)\), we have
\[
\sum_{n=0}^{\infty} B_{7,13}(n) q^n \equiv f_7 f_1^{11} \pmod{13}.\]

Substituting \((2.23)\) into \((5.2)\), we have
\[
\sum_{n=0}^{\infty} B_{7,13}(n) q^n \equiv f_7 \sum_{n=0}^{\infty} p_{11}(n) q^n \pmod{13}.
\]

If we extract those terms in which the power of \(q\) is congruent to 1 modulo 7, divide by \(q\) and replace \(q^7\) by \(q\), we have
\[
\sum_{n=0}^{\infty} B_{7,13}(7n + 1) q^n \equiv f_1 \sum_{n=0}^{\infty} p_{11}(7n + 1) q^n \pmod{13}.
\]

Now from \((2.22)\) and \((5.4)\), we have
\[
\sum_{n=0}^{\infty} B_{7,13}(7n + 1) q^n \equiv f_1 \left(\frac{2 f_7^{12}}{f_7} + f_7^3 f_1^8 q + 3 f_7^2 f_1^4 q^2 + 11 f_7^{11} f_1 q^3\right),
\]
\[
\sum_{n=0}^{\infty} B_{7,13}(7n + 1) q^n \equiv \frac{f_{13}}{f_7} + f_7^3 f_1^3 q + 3 f_7^2 f_1^5 q^2 + 11 f_7^{11} f_1 q^3.
\]

Which is the \(m = 1\) case of \((5.1)\). Now suppose \((5.1)\) holds for some \(m \geq 1\). Substituting \((2.6)\), \((2.7)\) and \((2.10)\) in \((5.5)\), we have
\[
\sum_{n=0}^{\infty} B_{7,13} \left(7^{2m-1} n + \frac{7^{2m-1} - 3}{4}\right) q^n \equiv A_{4m-3} f_{13} f_7 \left(\frac{B_4(q^{91})}{C(q^{91})} - \frac{A_4(q^{91})}{B_4(q^{91})} q^{13} - q^{26} + \frac{C_4(q^{91})}{A_4(q^{91})} q^{65}\right) + A_{4m-2} f_7^3 \sum_{n=0}^{\infty} p_9(n) q^{n+1}
\]
\[
+ A_{4m-1} f_7^5 \sum_{n=0}^{\infty} p_5(n) q^{n+2} + A_{4m} f_7^{11} f_1 q^{10} \left(\frac{B(q^7)}{C(q^7)} - \frac{A(q^7)}{B(q^7)} q - q^2 + \frac{C(q^7)}{A(q^7)} q^3\right) q^3.
\]
If we extract those terms in which the power of \( q \) is congruent to 5 modulo 7, divide by \( q^5 \) and replace \( q^7 \) by \( q \), we have

\[
\sum_{n=0}^{\infty} B_{7,13} \left( \frac{7^{2m} n + 3 \cdot 7^{2m} - 3}{4} \right) q^n
\]

\[
\equiv 12 A_{4m-3} \frac{f_{91}}{f_1} q^3 + A_{4m-2} f_1^3 \sum_{n=0}^{\infty} p_9(7n + 4) q^n + A_{4m-1} f_1^5 \sum_{n=0}^{\infty} p_5(7n + 3) q^n + 12 A_{4m} f_1^{11} f_7, \tag{5.7}
\]

Now from Lemma 2.3, Lemma 2.5 and (5.7), we have

\[
\sum_{n=0}^{\infty} B_{7,13} \left( \frac{7^{2m} n + 3 \cdot 7^{2m} - 3}{4} \right) q^n
\]

\[
\equiv 12 A_{4m-3} \frac{f_{91}}{f_1} q^3 + A_{4m-2} f_1^3 \left( f_7^3 f_7 + 2 f_1^7 f_7^2 q + 4 f_7^2 q^2 \right) + A_{4m-1} f_1^5 \left( 10 f_1^4 f_7 + 10 f_7^2 q \right) + 12 A_{4m} f_1^{11} f_7,
\]

\[
\equiv \left( A_{4m-2} + 10 A_{4m-1} + 12 A_{4m} \right) f_1^{11} f_7 + \left( 2 A_{4m-2} + 10 A_{4m-1} \right) f_1^5 f_7^3 q + 4 A_{4m-2} f_1^3 q^2
\]

\[+ 12 A_{4m-3} \frac{f_{91}}{f_1} q^3 \]  \tag{5.8}

Substituting (1.11), (1.22), (2.7) and (2.23) in (5.8), we have

\[
\sum_{n=0}^{\infty} B_{7,13} \left( \frac{7^{2m} n + 3 \cdot 7^{2m} - 3}{4} \right) q^n
\]

\[
\equiv \left( A_{4m-2} + 10 A_{4m-1} + 12 A_{4m} \right) f_7 \sum_{n=0}^{\infty} p_{11}(n) q^n + \left( 2 A_{4m-2} + 10 A_{4m-1} \right) f_7^5 \sum_{n=0}^{\infty} p_{17}(n) q^{n+1}
\]

\[+ 4 A_{4m-2} f_7^3 f_7 q^2 \left( B(q^7) - \frac{A(q^7)}{B(q^7)} q - q^2 + \frac{C(q^7)}{A(q^7)} q^3 \right) q^2 + 12 A_{4m-3} f_{91} \sum_{n=0}^{\infty} p(n) q^{n+3} \]  \tag{5.9}

If we extract those terms in which the power of \( q \) is congruent to 1 modulo 7, divide by \( q \) and replace \( q^7 \) by \( q \), we have

\[
\sum_{n=0}^{\infty} B_{7,13} \left( \frac{7^{2m+1} n + 7^{2m+1} - 3}{4} \right) q^n
\]

\[
\equiv \left( A_{4m-2} + 10 A_{4m-1} + 12 A_{4m} \right) f_1 \sum_{n=0}^{\infty} p_{11}(7n + 1) q^n + \left( 2 A_{4m-2} + 10 A_{4m-1} \right) f_1^5 \sum_{n=0}^{\infty} p_{17}(7n) q^n
\]

\[+ 11 A_{4m-2} f_1^3 f_7 q + 12 A_{4m-3} f_{13} \sum_{n=0}^{\infty} p(7n + 5) q^{n+1}, \]  \tag{5.10}

Substituting (1.14), Lemma 2.4 and Lemma 2.6 in (5.10), we have

\[
\sum_{n=0}^{\infty} B_{7,13} \left( \frac{7^{2m+1} n + 7^{2m+1} - 3}{4} \right) q^n
\]

\[
\equiv \left( A_{4m-2} + 10 A_{4m-1} + 12 A_{4m} \right) f_1 \left( \frac{2 f_7^{12}}{f_7} + f_7^3 f_7 q + 3 f_7^2 f_7^2 q^2 + 11 f_7^{11} q^3 \right) \]
\[ (2A_{4m-2} + 10A_{4m-1})f_1^5 \left( \frac{f_7^8}{f_7} + 10f_7^2f_7^1q \right) + 11A_{4m-2}f_1^3f_1^2q + 12A_{4m-3}f_1^{13} \left( \frac{f_7^3}{f_7} + 10qf_7^2f_7^1 \right)q, \]

\[ \equiv (4A_{4m-2} + 4A_{4m-1} + 11A_{4m})f_1^{13}f_1^3q + (6A_{4m-3} + 6A_{4m-2} + 6A_{4m})f_7^2f_1^5q \]

\[ + (3A_{4m-3} + 3A_{4m-2} + 4A_{4m-1} + 10A_{4m})f_7^2f_1^5q^2 + (11A_{4m-2} + 6A_{4m-1} + 2A_{4m})f_7^{11}f_1q^3 \]

\[ \equiv A_{4m+1}f_1^{13} + A_{4m+1}f_1^3f_1^5q + A_{4m+3}f_7^2f_1^5q^2 + A_{4m+4}f_7^{11}f_1q^3 \quad (5.11) \]

This completed the proof of induction of (5.1).

**Theorem 5.1.** For all \( n \geq 0 \),

\[ B_{7,13} \left( 7^{40}n + \frac{3 \cdot 7^{40} - 3}{4} \right) \equiv 5B_{7,13}(n) \pmod{13}. \]  

(5.12)

\[ B_{7,13} \left( 7^{39}(7n + k) + \frac{7^{39} - 3}{4} \right) \equiv 0 \pmod{13}, \text{ where } k = 2, 6, 7. \]  

(5.13)

**Proof.** From Lemma 5.1 put \( m = 21 \), we find that, \( A_{81} = 0, A_{82} = 0, A_{83} = 0 \) and \( A_{84} = 8 \)

\[ \sum_{n=0}^{\infty} B_{7,13} \left( 7^{39}n + \frac{7^{39} - 3}{4} \right)q^n \equiv 8f_7^{11}f_1q^3. \]  

(5.14)

Substituting (2.7) into (5.14), we have (5.12) and (5.13).

**Proof of Theorem 1.3.** Equation (1.10) follow from (5.12) by mathematical induction. Employing (1.10) in (5.13), we obtain (1.11).

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