Basis Light-Front Quantization Approach to Positronium

Paul Wiecki, Yang Li, Xingbo Zhao, Pieter Maris and James P. Vary
Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011 USA
(Dated: April 25, 2014)

We present the first application of the recently developed Basis Light-Front Quantization (BLFQ) method to self-bound systems in quantum field theory, using the positronium system as a test case. Within the BLFQ framework, we develop a two-body effective interaction, operating only in the lowest Fock sector, that implements photon exchange, neglecting fermion self-energy effects. We then solve for the mass spectrum of this interaction. The resulting spectrum is compared to the expected Bohr spectrum of non-relativistic quantum mechanics. We examine in detail the dependence of the results on the infrared and ultraviolet cutoffs of the theory.

I. INTRODUCTION

The \textit{ab initio} calculation of hadron mass spectra and observables in terms of their underlying quark and gluon degrees of freedom remains a significant challenge to theoretical physics. Bound state problems in a quantum field theory such as QCD are inherently non-perturbative. Hadron problems are also many-body in nature and must be solved at strong coupling. Due to the complexity of the problem, seemingly simple quantities, such as the proton magnetic moment, have not yet been accurately calculated.

Observables of interest include the mass spectrum of the hadronic system (in particular “exotic” states with quantum numbers beyond the constituent quark model), along with properties of the corresponding eigenstates. These include transition and decay rates, electric and magnetic moments, form factors, structure functions and generalized parton distributions (GPDs).

Basis Light-Front Quantization (BLFQ) is a promising tool for tackling hadron problems from first principles. BLFQ is a Hamiltonian-based approach that combines the advantages of light-front dynamics with modern developments in \textit{ab initio} nuclear structure calculations, such as the No-Core Shell Model (NCSM). The similarity of light-front Hamiltonian quantum field theory to non-relativistic quantum many-body theory allows the quantum field theoretical bound state problem to be formulated as a large, sparse matrix diagonalization problem. State-of-the-art methods developed for NCSM calculations can then be used to address hadronic systems. The diagonalization of the light-front Hamiltonian in a Fock-space basis yields the mass eigenstates of the system, along with amplitudes for evaluating non-perturbative observables.

Recent works have successfully applied BLFQ to the single-electron problem in QED in order to evaluate the electron anomalous magnetic moment both with and without an external trap. Another recent application evaluates the electron GPDs. In addition, BLFQ has been extended to time-dependent strong external field problems (tBLFQ) such as non-linear Compton scattering.

Here, we investigate the positronium system as a test case for applying BLFQ to self-bound systems. Our primary purpose is to confirm that BLFQ is capable of generating the expected Bohr spectrum of positronium, including relativistic effects such as the hyperfine splitting of the ground state.

For this initial test case, we implement a two-body effective interaction that operates only in the lowest Fock sector. The effective interaction implements the one-photon-exchange kernel necessary for Coulomb binding, but neglects the fermion self energy. The model, therefore, requires no renormalization. In future applications involving dynamical photons in the basis, a non-perturbative renormalization scheme will be needed. We intend to use the results of this calculation as a benchmark for implementing a Fock-sector dependent renormalization scheme.

We begin by introducing the basic elements of BLFQ, such as our choice of basis and truncation scheme. We then detail the derivation of the two-body effective interaction and solve for the spectrum of positronium using this interaction. Finally, we examine in detail the dependence of the results on the regulators of our theory. Preliminary results were reported in Refs. \cite{15, 16}.

II. BASIS LIGHT-FRONT QUANTIZATION

In principle, hadron observables can be evaluated by solving the eigenvalue equation

\begin{equation}
P_{\mu}P_{\mu}\ket{\Psi} = M^2 \ket{\Psi}, \tag{1}
\end{equation}

where $P_{\mu}$ is the energy-momentum 4-vector operator. In BLFQ, we express the operator $P^2$ in light-cone gauge. The operator $P^2$ then plays the role of the Hamiltonian operator in non-relativistic Quantum Mechanics. As such, it is sometimes referred as the “light-cone Hamiltonian” $H_{LC} \equiv P^2$. (Note that in this convention the “Hamiltonian” has energy squared units.) This operator can be derived from any field theoretical Lagrangian via the Legendre transform. In BLFQ, Eq. (1) is expressed in a truncated basis, and the resulting finite-dimensional matrix is diagonalized numerically. One then examines the trends in observables as the basis truncation is relaxed to estimate the results in the infinite matrix (or “continuum”) limit.
Of course $H_{LC}$, being field theoretical in origin, contains terms which change particle number. Thus the basis space for performing a diagonalization must be expanded to include states with any number or species of particles. For example, the positronium wavefunction could be expressed schematically as

$$\langle e^+e^- \rangle_{\text{phys}} = a \langle e^+e^- \rangle + b \langle e^+e^-\gamma \rangle + c \langle e^+e^-\gamma\gamma \rangle + d \langle \gamma \rangle + f \langle e^+e^-e^+e^- \rangle + \cdots. \quad (2)$$

When $H_{LC}$ is derived from the QED or QCD Lagrangian, the resulting interactions change particle number by at most two. The resulting matrix will then be extremely sparse for a many-body calculation.

### A. Basis and Truncation Scheme

In order to numerically diagonalize $H_{LC}$, the infinite dimensional basis must be truncated down to a finite dimension. In BLFQ, three separate truncations are made.

First, the number of Fock sectors in the basis is truncated. This truncation will be based on physical as well as practical considerations. For instance, positronium is expected to be fairly well described by the lowest few sectors. Thus, in this introductory work, we limit ourselves to only the $|e^+e^-\rangle$ and $|e^+e^-\gamma\rangle$ sectors.

Secondly, we discretize the longitudinal momentum by putting our system in a longitudinal box of length $L$ and applying periodic boundary conditions (BCs). Specifically, we choose periodic BCs for bosons and anti-periodic BCs for fermions. Thus

$$P^+ = \frac{2\pi}{L} j,$$

where $j$ is an integer for bosons, or a half-integer for fermions. For bosons, we exclude the “zero modes”, i.e. $j \neq 0$. In the many-body basis, all basis states are selected to have the same total longitudinal momentum $P^+ = \sum_i p^+_i$, where the sum is over the particles in a particular basis state. We then parameterize $P^+$ using a dimensionless variable $K = \sum_i j_i$, such that $P^+ = \frac{2\pi}{L} K$. For a given particle $i$, the longitudinal momentum fraction $x$ is defined as

$$x_i = \frac{p^+_i}{P^+} = \frac{j_i}{K}. \quad (4)$$

Due to the positivity of longitudinal momenta on the light-front, fixing $K$ also serves as a Fock space cutoff and makes the number of longitudinal modes finite. It is easy to see that $K$ determines our “resolution” in the longitudinal direction, and thus our resolution on parton distribution functions. The longitudinal continuum limit corresponds to the limit $L, K \to \infty$.

Finally, in the transverse direction, we employ a 2D Harmonic Oscillator (HO) basis. On the light-front, the generating operator for the 2D HO basis can be expressed as

$$P^\Omega_+ = \frac{\Omega}{2} \sum_i \left( \frac{p^2_i}{2p^+_i} + \frac{\Omega^2}{2p^+_i} \right). \quad (5)$$

Here, and elsewhere, the boldface type is reserved for 2D transverse vectors. Each value of $\Omega$ in $\mathbb{R}$ determines a unique complete and orthonormal basis. By defining the new coordinates $\mathbb{R}$

$$q \equiv \frac{p}{\sqrt{x}},$$

$$s \equiv \sqrt{x} r,$$

we can write the generating operator as

$$P^\Omega_+ = \frac{\Omega}{2} \sum_i \left[ \left( \frac{q_i}{\sqrt{P^+\Omega}} \right)^2 + \left( \sqrt{P^+\Omega} s_i \right)^2 \right]. \quad (6)$$

We see that $P^\Omega_+$ generates a basis of energy scale $b = \sqrt{P^+\Omega}$. In this work, we use Eq. (6) to define the 2D HO basis states. The momentum-space eigenfunctions of Eq. (6) are

$$\psi^m_n(q) = \frac{1}{b} \sqrt{\frac{4\pi \times n!}{(m + |m|)!}} e^{i m \phi} \rho^{m} \rho^{m-2} L^m_n(\rho^2), \quad (8)$$

where $\rho \equiv |q|$, and $\phi = \text{arg}(q)$. $L^m_n(x)$ is the generalized (or “associated”) Laguerre polynomial.

The basis is made finite by restricting the number of allowed oscillator quanta in each many-body basis state according to

$$\sum_i (2n_i + |m_i| + 1) \leq N_{\text{max}}. \quad (9)$$

The transverse continuum limit corresponds to $N_{\text{max}} \to \infty$. In addition, we use an “M-scheme” basis. That is, our many body states have well defined values of the total angular momentum projection

$$M_j = \sum_i (m_i + s_i), \quad (10)$$

where $s = \pm \frac{1}{2}$ is the fermion spin, but they do not have a well-defined total angular momentum $J$.

The choice of the 2D HO basis for BLFQ was made with the hadrons in mind. The HO potential is a confining potential, and therefore its wavefunctions should form an ideal basis for systems subject to QCD confinement. However, it is well known that the HO basis is not ideal for the Coulomb problem, due to the mismatch of the asymptotic behaviors of the wavefunctions (see, for example, Ref. [18]): the HO wavefunctions have Gaussian tails, while the hydrogen wavefunctions have a slower exponential decay. We therefore expect slow convergence in the positronium problem.
B. Infrared and Ultraviolet Cutoffs

Our truncations implicitly impose both infrared and ultraviolet cutoffs on our theory. The ultraviolet cutoff scale of the 2D HO basis of energy scale $b$ is estimated to be $\Lambda_{UV} \sim b\sqrt{N_{\text{max}}}$, while the infrared cutoff scale is $\lambda_{IR} \sim b/\sqrt{N_{\text{max}}}$. The basis energy scale $b$ is given above as $b = \sqrt{\frac{\pi}{K}}$. Therefore, we estimate that the infrared and ultraviolet cutoffs of our basis are

$$\Lambda_{UV} \sim b_0\sqrt{K N_{\text{max}}}$$

$$\lambda_{IR} \sim b_0\frac{K}{N_{\text{max}}}$$  \hspace{1cm} (11)

where $b_0 = \sqrt{\frac{2\pi}{K}}$.

C. Center-of-Mass Factorization

In BLFQ, we construct our many-body basis in single-particle coordinates. The rationale for doing this is its straightforward generalization to a basis of many particles. In principle, relative (Jacobi) coordinates could be used, but this process rapidly becomes intractable as the particle number is increased, due to the need for proper symmetrization of the basis states. Of course, for the two-particle positronium system Jacobi coordinates would be ideal. However, viewing the positronium problem only as a test case for our larger framework, here we nonetheless choose to work in single-particle coordinates.

Since our basis is constructed in single-particle coordinates, the center-of-mass (CM) motion of the system is contained in our solutions. The use of the HO basis combined with the $N_{\text{max}}$ truncation is a great advantage here since it allows for the exact factorization of the wavefunction into “intrinsic” and “CM” components, even within a truncated basis. The CM motion can then be removed from the low-lying spectrum by introducing a Lagrange multiplier proportional to the CM motion acting only in the $|e^+e^-\gamma\rangle$ space and $Q$ space to be the $|e^+e^-\gamma\rangle$ space. Let $\mathcal{P}$ be the operator that projects onto the $P$ space, and $\mathcal{Q}$ be the operator that projects onto the $Q$ space.

In addition to the effective interaction, the complete Hamiltonian will also include those terms from $H = H_{LC}$ which act directly in the two-particle space: $\mathcal{P}H\mathcal{P}$. This contains two pieces. First, it contains the two-particle kinetic energy. Secondly, it contains the light-front instantaneous photon exchange interaction. Thus the total Hamiltonian can be expressed as

$$\langle f | \mathcal{P} H_{\text{tot}} \mathcal{P} | i \rangle = \langle f | \mathcal{P} (H_0 + H_{\text{inst}} + H_{\text{eff}}) \mathcal{P} | i \rangle,$$  \hspace{1cm} (14)

where states $|i\rangle$ and $|f\rangle$ are states in $P$ space ($|e^+e^-\rangle$). The basis states $|i\rangle$ and $|f\rangle$ are eigenstates of the free Hamiltonian (i.e. $H_0 |i\rangle = \epsilon_i |i\rangle$) with eigenvalue

$$\epsilon_i = \frac{\sum_j p_j^2 + m_j^2}{x_j^2}$$  \hspace{1cm} (15)

where the sum runs over particles (of mass $m_j$) in the state $|i\rangle$.

B. Two-Body Effective Interaction

We choose the Bloch form of the effective Hamiltonian. The Bloch Hamiltonian is given by:

$$\langle f | H_{\text{eff}} | i \rangle = \frac{1}{2} \sum_{n} \langle f | \mathcal{P} H Q | n \rangle \langle n | \mathcal{Q} H \mathcal{P} | i \rangle$$

$$\times \left[ \frac{1}{\epsilon_i - \epsilon_n} + \frac{1}{\epsilon_f - \epsilon_n} \right].$$  \hspace{1cm} (16)

Here, $H = H_{LC} = P^2$ is the light-cone Hamiltonian introduced above. States $|i\rangle$ and $|f\rangle$ are states in $P$ space ($|e^+e^-\rangle$), while state $|n\rangle$ is in the $Q$ space ($|e^+e^-\gamma\rangle$). We have written Eq. (16) in the notation traditionally used for effective potentials in non-relativistic Quantum Mechanics calculations. One must remember, though, that both the “Hamiltonian” $H$ and the “unperturbed energy” $\epsilon$ have energy squared units in our formalism. Note that if $|i\rangle = |f\rangle$ this reduces to the usual formula
from second-order energy shift in perturbation theory. The derivation of (16), based on a perturbative expansion of the Okubo-Lee-Suzuki effective Hamiltonian [23–31], is given in Ref. [24].

Since we are interested in primarily the effects of repeated photon exchange, we will only include those combinations of terms in $\mathcal{P}HQ$ and $QHP\mathcal{P}$ which generate the photon exchange. We neglect the combinations which result in the photon being emitted and absorbed by the same fermion. That is, we do not incorporate the fermion self-energy, and therefore no fermion mass renormalization is necessary in this model. In addition, we work with unit-normalized eigenstates and a fixed value of the coupling constant. Details are presented in the Appendix.

1. Light-front small-\(x\) singularities

The instantaneous photon exchange interaction $H_{\text{inst}}$ contains a singularity of the form $\frac{1}{(x_1 - x_1')^2}$, where $x_1$ ($x_1'$) is the longitudinal momentum fraction of the incoming (outgoing) fermion (see Eq. (19)). In light-front $S$-matrix perturbation theory, the amplitude for electron-positron scattering via a dynamical photon contains a term identical to the instantaneous photon exchange interaction, but with opposite sign. Thus the instantaneous interaction is cancelled in its entirety, leaving behind the familiar equal-time Feynman amplitude [2, 3]. From this perspective, the instantaneous photon exchange interaction exists only to cancel this singularity in the light-front electron-positron scattering amplitude. This singularity is an artifact of the use of light-cone gauge, and must be cancelled.

In our non-perturbative calculation, this cancellation of small-\(x\) singularities does not occur in general. To remove the unphysical singularity in $H_{\text{inst}}$, we introduce a counterterm of the form

\[
\langle f | H_{\text{ct}} | i \rangle = - \sum_n \sum_n (\langle f | PHQ | n \rangle \langle n | QHP | i \rangle)
\]

\[
\times \left[ \frac{(a - b)^2}{2ab(a + b)} \right]
\]

where $a = \epsilon_i - \epsilon_n$ and $b = \epsilon_f - \epsilon_n$. The resulting regulated effective potential is

\[
\langle f | H_{\text{eff}}^{\text{reg}} | i \rangle = \langle f | (H_{\text{eff}} + H_{\text{ct}}) | i \rangle
\]

\[
= \sum_n \sum_n \langle f | PHQ | n \rangle \langle n | QHP | i \rangle
\]

\[
\times \left[ \frac{1}{2} (\epsilon_i - \epsilon_n) + (\epsilon_f - \epsilon_n) \right]
\]

(18)

In this form the cancellation of the instantaneous diagram does occur, and $H_{\text{inst}} + H_{\text{eff}}^{\text{reg}}$ is free of unphysical light-front small-\(x\) singularities. Details are given in the Appendix. We note that our choice of counterterm is equivalent to the prescription used in previous work in Discretized Light-Cone Quantization (DLCQ) [31, 32]. Refs. [31, 32] also provide arguments for the plausibility of this prescription.

2. Final Expression

Our Hamiltonian is then

\[
(f | \mathcal{P}H_{\text{tot}}\mathcal{P} | i) = (f | \mathcal{P} (H_0 + H_{\text{inst}} + H_{\text{eff}}^{\text{reg}}) \mathcal{P} | i)
\]

where, after canceling the instantaneous interaction, we are left with ($\alpha = g^2/4\pi$)

\[
H_{\text{inst}} + H_{\text{eff}}^{\text{reg}} = \frac{\alpha}{K} \sum_{\vec{a}_i'\vec{a}_i'\vec{a}_i} d^j_{\vec{a}_i'} b_{\vec{a}_i'} d^k_{\vec{a}_i} b_{\vec{a}_i}
\]

\[
\times \sqrt{x_1 x_2 x_1' x_2'} \int \frac{d^4 q_1}{(2\pi)^2} \frac{d^4 q_2}{(2\pi)^2} \frac{d^4 q_3}{(2\pi)^2}
\]

\[
\times (2\pi) \delta(\vec{q}_1 + \vec{q}_2 - \vec{x}_1 - \vec{x}_2') \delta(\vec{x}_1 - \vec{x}_2')
\]

\[
\times \Psi^{\alpha_{i1}}(q_1) \Psi^{\alpha_{i1}^\dagger}(q_1) \Psi^{\alpha_{i2}^\dagger}(q_2) \Psi^{\alpha_{i2}}(q_2)
\]

\[
\times S^{\alpha_{i1},\alpha_{i2}}(\sqrt{x_1 q_1}, \sqrt{x_1 q_2}, \sqrt{x_2 q_1}, \sqrt{x_2 q_2})
\]

(19)

with $\vec{a}_i$ representing the set of discrete quantum numbers $(j_i, s_i, n_i, m_i)$ and $\alpha_i$ representing the subset $(j_i, s_i)$. If the fermions have mass $m_f$ and the photon has mass $\mu$, the parts of the energy denominator are given by

\[
\epsilon_i - \epsilon_n = \frac{\sqrt{x_1 q_1^2} + m_f^2}{x_1} - \frac{\sqrt{x_1 q_1^2} + m_f^2}{x_1'}
\]

\[
- \frac{\sqrt{x_2 q_2^2} + m_f^2}{x_2} - \frac{\sqrt{x_2 q_2^2} + m_f^2}{x_2'}
\]

\[
- (\epsilon_f - \epsilon_n) = \frac{\sqrt{x_1 q_1^2} - \sqrt{x_1 q_1'^2} + \mu^2}{x_1 - x_1'}
\]

\[
- \frac{\sqrt{x_2 q_2^2} - \sqrt{x_2 q_2'^2} + \mu^2}{x_2 - x_2'}
\]

(20)

The explicit expression for the spinor part, $S^{\alpha_{i1},\alpha_{i2}}(\sqrt{x_1 q_1}, \sqrt{x_1 q_2}, \sqrt{x_2 q_1}, \sqrt{x_2 q_2})$, is provided in the Appendix (Table I). The highly oscillatory 8D integration can be evaluated using repeated 2D Talin-Moshinsky transformations [33]. The integral can be reduced down to a single 2D integral, which is evaluated numerically. The details are presented in the Appendix.

Note that a fictitious photon mass $\mu$ has been introduced to regulate the expected Coulomb singularity that, while integrable, introduces numerical difficulties. For convenience in removing regulators, we tie the fictitious photon mass to our estimated infrared cutoff of the basis. That is, we select

\[
\mu = \lambda_{IR}
\]

with $\lambda_{IR}$ given by (11). We will later examine the limit $\mu = \lambda_{IR} \rightarrow 0$.

C. Modified Effective Interaction

Previous authors investigating the problem of positronium on the light-front with a one-photon exchange kernel have noted a very small, but noticeable, dependence
on the ultraviolet cutoff of the theory. Both Refs. [31, 35] state that the origin of the instability can be traced to a particular term in the effective interaction (or one-photon exchange kernel) which tends to a non-zero constant (in momentum space) at asymptotically large momentum transfer, corresponding to a Dirac delta potential in coordinate space. Since the 2D Dirac delta potential well has no bound states of finite binding energy [36], this leads to a divergence.

Ref. [31] argues that this divergent piece of the effective interaction will be cancelled when higher Fock sectors are included. They therefore suggest that by simply dropping this term, one would obtain a sensible one photon interaction will be cancelled when higher Fock sectors are included. They therefore suggest that by simply dropping this term, one would obtain a sensible one photon exchange model for positronium on the light-front. Specifically, they suggest modifying the spinor part of the interaction:

\[
S \to S - 2 \left[ \frac{p_{\text{rel}}^2}{x_1 x_2} + \frac{p_{\text{rel}}^2}{x_1' x_2'} \right] \delta s_1^i \delta s_2^j (\delta s_1^+ \delta s_2^- + \delta s_1^- \delta s_2^+),
\]  

(23)

where \( s_i = \pm \) represents the fermion spin. In single-particle coordinates, we have \( p_{\text{rel}} = x_2 p_1 - x_1 p_2 \) and \( p_{\text{rel}}' = x_2' p_1' - x_1' p_2' \). If one carefully examines our expression for the spinor part (given in the Appendix), one finds the same term (with opposite sign), only expressed in single-particle coordinates, and therefore also containing a CM piece.

We note that, in our longitudinal momentum weighted coordinates, the modification can be expressed conveniently as

\[
S \to S - 2 \left[ q_{\text{rel}}^2 + q_{\text{rel}}'^2 \right] \delta s_1^i \delta s_2^j (\delta s_1^+ \delta s_2^- + \delta s_1^- \delta s_2^+),
\]  

(24)

where \( q_{\text{rel}} = \sqrt{x_2} q_1 - \sqrt{x_1} q_2 \) and \( q_{\text{rel}}' = \sqrt{x_2'} q_1' - \sqrt{x_1'} q_2' \).

Using this “modified” interaction, Ref. [31] finds that the results depend less strongly on \( \Lambda_{UV} \) and are in reasonable agreement with the predictions of non-relativistic quantum mechanics.

In the next section, we will examine the convergence of both the “modified” and “unmodified” effective interactions.

### IV. NUMERICAL RESULTS

In non-relativistic Quantum Mechanics, the hyperfine splitting between the \( ^1S_0 \) and \( ^3S_1 \) states of positronium scales as \( \alpha^4 \), where \( \alpha \) is the fine structure constant. At physical coupling, the expected hyperfine splitting and even the binding energy are then uncomfortably small relative to the precision of our numerical integrals. Since we would like to use the hyperfine splitting to test our BLFQ results, we use a large coupling of \( \alpha = 0.3 \) to exaggerate both the binding energy and the hyperfine splitting. We then compare our results not to experiment, but to the predictions of non-relativistic Quantum Mechanics at this unphysical value of \( \alpha \). This value of \( \alpha \) also allows a direct comparison to the DLCQ results of Ref. [31, 32].

![FIG. 1: Representative spectrum of positronium (\( \alpha = 0.3 \)) calculated in BLFQ at \( K = N_{\text{max}} = 19 \) and \( b_0 = \mu = 0.1 m_f \), using the unmodified (top) and modified (bottom) effective interactions. The exact energies shown should not be interpreted as final converged results. Using the unmodified interaction (top), the approximate rotational invariance allows for the clear identification of the \( ^1S_0, ^1S_1, ^2S_0, ^2S_1, ^1P_1, ^2P_0, ^2P_1 \) and \( ^2P_2 \) states of the positronium system (see text for details). Using the modified interaction (bottom), the approximate rotational invariance is more strongly broken.](image)

The numerical results were obtained using the Hopper Cray XE6 and Edison Cray XC30 at NERSC. ScALAPACK software [37] was used for the diagonalization. In this particular implementation of BLFQ, the resulting matrix is quite dense. However, in future applications involving multiple Fock sectors, the matrix will be extremely sparse.

We obtain exact CM factorization for both the modified and unmodified interactions. The spurious CM motion is removed by using a Lagrange multiplier, as discussed in Sec. II C, so that all states shown below are in the ground state of CM motion.
A. Spectrum

A representative spectrum is shown in Fig. 1. These results are produced with $K = N_{\text{max}} = 19$ and $b_0 = \mu = 0.1 m_f$. The energies shown are only representative and should not be considered converged or final. The general features of the spectrum shown here are common to any calculation with $K = N_{\text{max}} = 19$ and above. Convergence will be considered below.

On the light-front, the total angular momentum operator is dynamical. In addition, manifest rotational invariance is lost in our calculation due to Fock sector truncation, as well as the different discretizations in the longitudinal and transverse directions. However, the total $J$ of the states can be extracted by examining the multiplet structure of the spectrum as it appears in the top panel of Fig. 1. The ground state, for example, appears only in the $M_J = 0$ calculation, suggesting that it has $J = 0$. We also see a triplet of states above the ground state with $M_J = -1, 0, 1$, suggesting that these states form a $J = 1$ multiplet. The lack of manifest rotational invariance is seen only in the lack of exact degeneracy between the states in this multiplet. The difference, however, is quite small, being approximately 1% of the binding energy, and is nearly invisible on this scale. We therefore feel confident extracting $J$ by examining the number of states in each approximately degenerate multiplet.

In the non-relativistic quantum mechanics description of the positronium system, the states are labelled using spectroscopic notation) and is the total intrinsic spin, $L$ is the total orbital angular momentum (expressed in spectroscopic notation) and $J$ is the total angular momentum. (Note that the $n$ of atomic physics is related to the radial node quantum number of nuclear physics by $n_{\text{atomic}} = n_{\text{radial}} + L + 1$.) The lowest two states are then $1^1S_0$ and $1^3S_1$, the singlet and triplet ground state configurations respectively. The splitting between them is referred to as the hyperfine splitting. In a relativistic theory such as BLFQ, $L$ and $S$ are no longer good quantum numbers. We identify the low-lying $J = 0$ and $J = 1$ multipoles in the BLFQ spectrum as the expected $1^1S_0$ and $1^3S_1$ states of positronium. For simplicity, we will label the states according to their non-relativistic description, but we stress that we can only extract $J$ (approximately) from our BLFQ calculation, and not $L$ or $S$. Nevertheless, the BLFQ wavefunction of the state we identify as the $1^1S_0$ state is indeed anti-symmetric with respect to spin exchange, as one would expect for an $S = 0$ state. (We reached this conclusion by examining the amplitude by eye at $N_{\text{max}} = 4$.) Similarly, the $1^3S_1$ wavefunction is symmetric with respect to spin exchange, consistent with the non-relativistic expectation.

A similar grouping of states appear higher up in the spectrum. We identify these as the $2^1S_0$ and $2^3S_1$ states of positronium. The remaining four states can be identified, via similar reasoning, to be one $J = 0$ multiplet, two $J = 1$ multiplets, and one $J = 2$ multiplet. (The highest line in the $M_J = 0$ calculation is thicker to indicate that it represents two nearly degenerate states.) This is exactly what we expect for the $2P$ levels of positronium in non-relativistic quantum mechanics. We therefore identify these states with the expected $2^1P_1$, $2^3P_0$, $2^3P_1$ and $2^3P_2$ positronium states. Note that in our BLFQ calculation, we cannot distinguish the two $J = 1$ states because we do not know $L$ and $S$. (In a relativistic notation, the states are distinguished by their charge conjugation quantum number.)

In the lower portion of Fig. 1 the spectrum of the modified interaction is shown. The only difference is that the rotational invariance is more severely broken. Compared to the basic interaction, the $M_J = 0$ states are shifted upwards, while the $M_J = \pm 1$ states remain essentially unchanged. The reason for this can be easily seen from the expression (23), in which it is clear that the modification only affects those states in which the electron and positron have opposite spins. Despite the lack of (near) rotational symmetry in the modified interaction, we will continue to use the same state identifications used in the unmodified case.

We conclude that, although the modified interaction mitigates a divergence in the effective interaction, it does not result in a fully consistent model for positronium. A fully consistent model would need to include higher Fock sectors. However, at the current level of truncation, we believe the modified interaction offers the best chance at achieving reasonably converged results.

B. Ground State and Hyperfine Splitting

We now consider the dependence of the singlet and triplet ground states of positronium on the regulators of our theory. This is shown in Fig. 2 where the singlet and triplet ground state energies are plotted as a function of the ultraviolet cutoff $\Lambda_{UV}$ for a series of values of the infrared cutoff $\Lambda_{IR}$. The ground state energy and cutoffs are expressed in fermion mass units throughout. The infrared cutoff $\Lambda_{IR} = b_0 \sqrt{K/N_{\text{max}}}$ is kept constant along each curve by fixing $K = N_{\text{max}}$. The infrared cutoff is then adjusted by changing $b_0$. Note that, since $\Lambda_{UV} = b_0 \sqrt{K/N_{\text{max}}}$ is also proportional to $b_0$, as the infrared cutoff is lowered the same values of $K$ and $N_{\text{max}}$ will give lower values of $\Lambda_{UV}$. On each curve, the point with the highest $\Lambda_{UV}$ corresponds to a calculation at $K = N_{\text{max}} = 43$. Also shown in Fig. 2 are the same results using the modified interaction introduced above.

Since we fix $b_0$ along each curve in Fig. 2 the energy scale $b$ of the basis is a function of $K$ ($b = b_0 \sqrt{K}$). Now, for the positronium system, optimal convergence should be achieved if the natural length of the basis corresponds to the Bohr radius of positronium. Indeed, by treating $b$ as a variational parameter, we have found that the optimal basis energy scale is $b \sim m_f/2$. However, we also find that the varying basis energy scale does not affect our conclusions at high $K = N_{\text{max}}$. 
Throughout this work we calculate only at odd \( K = N_{\text{max}} \). The reason is as follows. The kinetic energy term of \( H_{LC} \) \([15]\) has a term of the form \( m_f^2 \left( \frac{1}{x_1} + \frac{1}{x_2} \right) \). Since we require \( x_1 + x_2 = 1 \), this term in the kinetic energy takes its minimum value when \( x_1 = x_2 = \frac{1}{2} \). This leads to a binding threshold of \( 2m_f \) as expected. If \( K \) is odd, states with \( x_1 = x_2 \) are present in the basis. If, however, \( K \) is even, equal longitudinal momentum splitting is not present in the basis and the ground state energy is increased by a term proportional to \( m_f^2 \). Since, in positronium, the fermion mass scale is much larger than the binding energy scale, the ground state energy is unnecessarily far from convergence if \( K \) is even.

Returning to Fig. 2 we see that, as a function of \( \Lambda_{UV} \), the singlet ground state energy increases rapidly before reaching a peak, after which it slowly decreases. This behavior is due to competing dependences of the ground state energy on \( K \) and \( N_{\text{max}} \): the ground state energy increases with increasing \( K \), but decreases with increasing \( N_{\text{max}} \). For the unmodified interaction, the ground state has a decreasing linear dependence on \( \Lambda_{UV} \) beyond the peak. The slope is small, but nonetheless it is clear that the results are not converging. This is the same slight dependence on \( \Lambda_{UV} \) found in previous studies \([31, 33, 35]\).

The slight dependence of the unmodified singlet ground state energy on \( \Lambda_{UV} \) is greatly weakened by using the modified interaction. This behavior was also seen in Ref. \([31]\), which suggested the modification (see Sec. III C). At our current highest truncation of \( K = N_{\text{max}} = 43 \), it remains unclear whether the singlet ground state of the modified interaction actually converges or retains a residual dependence on the ultraviolet cutoff. Nevertheless, to check for consistency with the predictions of non-relativistic quantum mechanics, we plot our results at the highest cutoff (\( K = N_{\text{max}} = 43 \)) against \( \mu \) to examine the physical limit \( \mu \to 0 \). This is done in Sec. IV D.

We also note that the convergence with respect to \( \Lambda_{UV} \) is slower as \( \mu = \lambda_{IR} \) is decreased. This makes sense since as \( \mu \) is decreased, the problem evolves from a finite-range Yukawa interaction into the infinite-range Coulomb interaction. Our HO basis thus becomes less ideal as \( \mu \) is decreased towards zero.

In the lower panel of Fig. 2, we show the same plot for the triplet ground state, identified as the first excited state of the \( M_J = 0 \) calculation. The dependence on the ultraviolet cutoff is much weaker than that of the singlet ground state, even using the unmodified interaction. This observation is also consistent with the existing literature.

C. \( 2^3P_2 \) State

As a representative excited state, we also consider the \( 2^3P_2 \) state. Being the lowest \( J = 2 \) state in the positronium spectrum, this state arises in our calculation as the lowest state of the \( M_J = 2 \) calculation.

The dependence of the \( 2^3P_2 \) state on the regulators \( \Lambda_{UV} \) and \( \lambda_{IR} \) is shown in Fig. 3 using the modified interaction. For this state, the modified interaction produces very little change. In fact, a plot of the results using the unmodified interaction would be indistinguishable from that shown. Note that, for \( \lambda_{IR} = \mu = 0.1m_f \) and \( 0.06m_f \), the energy of this state is above the binding threshold of \( 2m_f \). The \( 2^3P_2 \) state is thus a quasi-bound continuum state, and only becomes bound as \( \mu = \lambda_{IR} \) decreases towards zero.

The convergence with respect to \( \Lambda_{UV} \) is significantly slower than it is for the ground state. For example, when \( \lambda_{IR} = 0.1m_f \), the modified interaction results for the ground state are nearly independent of \( \Lambda_{UV} \) above \( \Lambda_{UV} = 2m_f \), as shown in Fig. 2. However, Fig. 3 shows that the \( 2^3P_2 \) energy retains a significant \( \Lambda_{UV} \) dependence in this region.

In addition, the curves in Fig. 2 all display a peak followed by a slow decrease. Recall that this peak is due
D. $\lambda_{IR} \to 0$ Limit

Previous literature indicates that we cannot expect to achieve true convergence for the positronium system on the light-front with our current Fock sector truncation. Our results are consistent with this conclusion. While we cannot claim that our results are convergent, the results are nonetheless reasonably stable within the range of regulators accessible at this time; any dependence on $\Lambda_{UV}$ is very weak. Any remaining weak divergences are likely the result of our minimal Fock sector truncation. Therefore, as a consistency check, we compare the BLFQ results at large $\Lambda_{UV}$ and small $\lambda_{IR}$ to the predictions of non-relativistic quantum mechanics.

These predictions are (expressed in fermion mass units) [38]:

$$M_{\text{singlet}} = 2 - \frac{\alpha^2}{4} \left( 1 + \frac{63}{48} \alpha^2 \right)$$

$$M_{\text{triplet}} = 2 - \frac{\alpha^2}{4} \left( 1 - \frac{1}{48} \alpha^2 \right)$$

$$M_{3P_2} = 2 - \frac{\alpha^2}{16} \left( 1 + \frac{43}{960} \alpha^2 \right).$$

In these formulas, the $\alpha^2$ term corresponds to the Bohr energies and the $\alpha^4$ term incorporates first-order perturbative relativistic corrections, neglecting the possibility of a temporary annihilation of the electron and positron into a virtual photon. Since we neglect the $|\gamma\rangle$ Fock sector, our BLFQ results should be comparable to these predictions.

To estimate the ground state energy in the limit of large ultraviolet cutoff and small infrared cutoff, we plot our modified interaction results at $K = N_{\text{max}} = 43$ against $\mu = \lambda_{IR}$, as shown in Fig. [5] (Recall that the ground state energy is much more stable if the modified interaction is used.) The curves fit nicely to second-order polynomials, allowing us to estimate their values at $\mu = \lambda_{IR} \to 0$.

For comparison, the predicted energies from non-relativistic quantum mechanics are indicated as crosses along the $\mu = 0$ axis. We find that the results are largely in agreement. First, all the expected states of positronium are present in our calculation, as discussed in Sec. [IVA] The binding energy differs by only ~ 10% and the hyperfine splitting is the correct order of magnitude as the expected result of $\alpha^4/3 = 2.7 \times 10^{-3}$, expressed in fermion mass units. The $3P_2$ state, as discussed in Sec. [IVC] is not yet converged with respect to $\Lambda_{UV}$ at $K = N_{\text{max}} = 43$, and therefore comes in well above the expected value.

For the singlet and triplet ground states, the two extrapolated curves appear to be coming together in the $\mu \to 0$ limit. However, this is mostly due to the $\mu = 0.02m_f$ point. For the other four points, the hyperfine splitting is consistently near $2.0 \times 10^{-3}m_f$ (a 25% difference from the NRQM expectation), but at $\mu = 0.02m_f$ it drops off to $1.3 \times 10^{-3}m_f$ (a 50% difference). We attribute the trend to a lack of convergence with respect to $\Lambda_{UV}$ of the modified $\lambda = 0.02m_f$ results. As is clearly seen in Fig. [4] the $\lambda = 0.02m_f$ curves have not yet reached a stable region. The singlet still has a decreasing trend at $K = N_{\text{max}} = 37$, while the triplet still has an increasing trend. These trends will tend to increase the hyperfine splitting for $\lambda = 0.02m_f$. For example, the extrapolated hyperfine splitting at $K = N_{\text{max}} = 43$ is larger than that at $K = N_{\text{max}} = 37$ by $2.0 \times 10^{-4}m_f$, or 25% of the $K = N_{\text{max}} = 37$ hyperfine splitting, a significant increase.

We also stress that at our current level of Fock sector truncation the positronium ground state does not converge. Although the modified interaction has provided us with a reasonably stable value for the ground state energy, we remind the reader that the modified interaction breaks the rotational invariance of $H_{LC}$ more severely than the unmodified interaction.

V. CONCLUSIONS AND OUTLOOK

We have presented here the first application of BLFQ to self-bound systems in quantum field theory, using positronium as the test case. In this work, we have
truncated the Fock space to include only the $|e^+e^-\gamma\rangle$ sectors. We further restricted the basis to the $|e^+e^-\rangle$ sector only by developing a two-body effective interaction, incorporating the photon exchange effects generated by the $|e^+e^-\gamma\rangle$ sector. Diagonalization of the light-cone Hamiltonian including this interaction results in a repeated iteration of the effective interaction, giving a non-perturbative solution to the bound state problem (equivalent to light-front ladder truncation with one-photon exchange kernel). In this initial work, we have neglected fermion-self energy effects arising from the $|e^+e^-\gamma\rangle$ sector. Our model, then, requires no renormalization.

As with previous studies of positronium on the light-front at the same Fock sector truncation, we find that the ground state is unstable against increasing ultraviolet cutoff. Nonetheless, the terms in the effective interaction that cause the divergence can be dropped, resulting in a more stable ground state. Using this modified interaction, the results can be compared for consistency with the formulas of non-relativistic quantum mechanics. All of the expected $J$ multiplets arise in our calculation and the ground state binding energy differs by only $\sim 10\%$ from the NRQM value. The hyperfine splitting of the ground state is also reproduced approximately. The higher excited states converge more slowly in our HO basis, which is not ideal for the long tails of the Coulomb wavefunctions.

A straightforward extension of this work is to include a confining interaction between the fermions. The model should then be applicable to heavy quarkonia systems. A natural choice for the confining potential in BLFQ is a quadratic confinement. Such a quadratic potential is motivated by the phenomenological success of the “soft wall” AdS/QCD model\cite{11,42}. The effective interaction implemented here would then be interpreted as providing QCD corrections to the semiclassical approximation provided by the AdS/QCD model.

The ultimate goal, however, is to incorporate the higher Fock sectors into the basis, thereby including dynamical bosons. This will be required to obtain realistic QCD results without an ad hoc confining interaction. The main challenge is the development and implementation of a non-perturbative renormalization scheme, such as the Fock sector dependent scheme of Refs. \cite{32,40}. The recent successful renormalization of the free electron problem in BLFQ \cite{11} with a dynamical photon in the basis suggests the possibility of “embedding” the renormalized, physical electron into the positronium system. Such developments would make BLFQ a powerful tool for addressing the ab initio properties of the hadrons.

**Acknowledgments**

We thank S. J. Brodsky, H. Honkanen, D. Chakrabarti and V. A. Karmanov for fruitful discussions. This work was supported in part by the Department of Energy under Grant Nos. DE-FG02-87ER40371 and DESC0008485 (SciDAC-3/NUCLEI) and by the National Science Foundation under Grant No. PHY-0904782. A portion of the computational resources were provided by the National Energy Research Scientific Computing Center (NERSC), which is supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

**Appendix: Derivation of Two-Body Effective Interaction**

In this Appendix, we present the detailed derivation of the two-body effective interaction, including the necessary integrals. Our starting point is the Bloch effective Hamiltonian \cite{24}:

$$\langle f | H_{\text{eff}} | i \rangle = \frac{1}{2} \sum_n \langle f | P H Q | n \rangle \langle n | Q H P | i \rangle \times \left[ \frac{1}{\epsilon_i - \epsilon_n} + \frac{1}{\epsilon_f - \epsilon_n} \right].$$  \hspace{1cm} (A.1)

For this BLFQ application, the $P$ space is the $|e^+e^-\rangle$ sector, while the $Q$ space is the $|e^+e^-\gamma\rangle$ sector. Also, in BLFQ applications, $H \equiv H_{\text{LC}} = P^\mu P_\mu = P^+ P^- - \mathbf{P}^2$. Note that $\mathbf{P}^2$ does not couple the sectors, and so does not contribute to this summation. Therefore, here we will use $H = P^+ P^-$. The sum is over the complete $Q$ space. The notation $\epsilon_i$ denotes the eigenvalue of the free Hamiltonian $H_0$ for the state $|i\rangle$ (see Eq. (15)):

$$\epsilon_i = \sum_j \frac{P_j^2 + m_j^2}{x_j}.$$  \hspace{1cm} (A.2)
where the sum runs over all particles (of mass $m_j$) in the state $|i\rangle$.

After adding the counterterm presented in the main text, the effective Hamiltonian takes the form:

$$
\langle f | H_{\text{eff}}^{\text{reg}} | i \rangle = \sum_n \langle f | \mathcal{P} \mathcal{H} Q | n \rangle \langle n | \mathcal{Q} \mathcal{H} P | i \rangle \frac{1}{2} \left[ (\epsilon_i - \epsilon_n) + (\epsilon_f - \epsilon_n) \right].
$$

(A.3)

We will see that this choice, and only this choice, results in the cancellation of the light-front small-$x$ divergences.

1. Sum Over Intermediate States

The basic interaction of the LFQED Hamiltonian that connects the sectors is the vertex interaction, given by

$$
P^- = g \int dx^{+} d^{2}x \Psi(x) \gamma_{\mu} A^{\mu}(x) \Psi(x),
$$

(A.4)

evaluated at $x^+ = 0$.

The free field mode expansions in BLFQ are

$$
\Psi(x) = \sum_{\alpha} \frac{1}{\sqrt{2L}} \int \frac{d^{2}p}{(2\pi)^2} \left[ b_{\alpha}(p) u_{s}(p)e^{-ipx} + d^\dagger_{\alpha}(p) v_{s}(p)e^{ipx} \right],
$$

(A.5)

and

$$
A^{\mu}(x) = \frac{1}{\sqrt{2Lk^+}} \int \frac{d^{2}k}{(2\pi)^2} \left[ a_{\beta}(k) \epsilon^\kappa_{\alpha}(k)e^{-ikkx} + \text{h.c.} \right],
$$

(A.6)

where the greek subscripts are shorthand for the set of quantum numbers $\alpha = (j,s)$ and $\beta = (j,\lambda)$, where $s = \pm \frac{1}{2}$ is the fermion spin and $\lambda = \pm 1$ is the photon helicity.

Plugging these mode expansions into $P^-$ yields terms of the form $P_{e^{-}\rightarrow e}^{-} \sim b^\dagger b a$, $P_{e^{-}\rightarrow \gamma e}^{-} \sim b^\dagger b a$, $P_{\gamma^{-}\rightarrow e}^{-} \sim d^\dagger d a$, $P_{\gamma^{-}\rightarrow \gamma e}^{-} \sim d^\dagger d a$. These are the only terms which will connect the $|e^+ e^-\rangle$ states to the $|e^+ e^-\rangle$ states and survive the operation of the projection operators. For example we have the basic interaction vertices shown in Fig. 5 which are given by

$$
P_{e^{-}\rightarrow \gamma e}^{-} = g \sum_{\alpha_{i},\alpha_{j}} \theta(p_{1}^{+} - p_{2}^{+}) \delta_{z_{i}z_{j}} \times \int \frac{d^{2}p_{1}}{(2\pi)^2} \frac{d^{2}p_{2}}{(2\pi)^2} (2\pi)^{2}\delta^{(2)}(p_{1} + k - p_{2}) \times \bar{u}_{s_{i}}(p_{1}^{+}) \gamma_{\mu} \epsilon^{\mu\kappa}_{\alpha_{i}}(k) u_{s_{j}}(p_{2}^{+}) \bar{d}_{\alpha_{j}}(p_{1}^{+}) d_{\alpha_{i}}(p_{2}^{+})
$$

(A.7)

and

$$
P_{\gamma^{-}\rightarrow e}^{-} = -g \sum_{\alpha_{i},\alpha_{j}} \theta(p_{2}^{+} - p_{1}^{+}) \delta_{z_{i}z_{j}} \times \int \frac{d^{2}p_{2}}{(2\pi)^2} \frac{d^{2}p_{1}}{(2\pi)^2} (2\pi)^{2}\delta^{(2)}(p_{2} + k - p_{1}) \times \bar{u}_{s_{2}}(p_{2}^{+}) \gamma_{\mu} \epsilon^{\mu\kappa}_{\alpha_{i}}(k') u_{s_{1}}(p_{1}^{+}) \bar{d}_{\alpha_{j}}(p_{2}^{+}) d_{\alpha_{i}}(p_{1}^{+}),
$$

(A.8)

The overall minus in Eq. (A.8) has arisen from normal ordering, dropping a vacuum bubble. The theta functions appear since the light-front longitudinal momentum must be positive.

In the effective interaction, these basic interaction vertices are stitched together to generate both fermion self-energy loops and exchanges of photons between the electron and positron as shown in Fig. 6. From here on, we neglect these fermion self energy terms to focus on the photon exchange terms. The effective Hamiltonian is then:

$$
H_{\text{eff}}^{\text{reg}} = \sum_{n} P_{e^{-}\rightarrow \gamma e}^{+} |n\rangle \langle n| P_{e^{-}\rightarrow \gamma e}^{-} + \sum_{n} P_{\gamma^{-}\rightarrow e}^{+} |n\rangle \langle n| P_{\gamma^{-}\rightarrow e}^{-} = (H_{\text{eff}}^{\text{reg}})_{e^{-}\rightarrow \gamma e} + (H_{\text{eff}}^{\text{reg}})_{e^{-}\rightarrow \gamma e}.
$$

(A.9)

We now turn to a particular time ordering of the photon exchange, namely the one in which the photon is first emitted by the electron and later absorbed by the positron. This corresponds to the combination $(H_{\text{eff}}^{\text{reg}})_{e^{-}\rightarrow \gamma e}$ described above. Let us consider the expression for the effective Hamiltonian in BLFQ. Instead of doing an infinite summation over the complete $Q$ space in the H.O. basis, we perform the summation in momen-
tum space, where the sum is an integral and we can write

\[
(H_{\text{eff}})_{\epsilon \rightarrow \epsilon'} = \sum_n \frac{P^+ P^-_{\epsilon' - \epsilon}}{2} |n\rangle \langle n| P^+ P^-_{\epsilon' - \epsilon}
\]

\[
= (P^+)^2 \int d^2 \mathbf{x} \int d^2 \mathbf{t} \int d^2 \mathbf{s} \int d^2 \mathbf{t}' \frac{P^-_{\epsilon' - \epsilon}(\mathbf{r}) d^2 \mathbf{s} a^\dagger_\alpha(\mathbf{t}) |0\rangle \langle 0| a_\alpha(\mathbf{t}') b_\beta(\mathbf{r}) P^-_{\epsilon' - \epsilon}}{D_{\delta, \epsilon, \xi}(\mathbf{r}, \mathbf{s}, \mathbf{t})}
\]

(A.10)

The energy denominator is given by

\[
D_{\delta, \epsilon, \xi}(\mathbf{r}, \mathbf{s}, \mathbf{t}) = \frac{1}{2} \int (\epsilon_i - \epsilon_n) = \frac{\epsilon_i + \epsilon_n}{2} - \epsilon_n
\]

\[
= \frac{1}{2} \left[ \left( \frac{p^2_1 + m^2_1}{x_1} + \frac{p^2_2 + m^2_2}{x_2} \right) \left( \frac{p'^2_1 + m^2_1}{x'_{1}} + \frac{p'^2_2 + m^2_2}{x'_{2}} \right) \right]
\]

\[
- \left( \frac{r^2 + m^2_2}{x_r} + \frac{s^2 + m^2_2}{s^2} + \frac{t^2 + m^2_2}{t^2} \right).
\]

(A.11)

In the final equality we have used the expression for the kinetic energy appropriate for \( H = P^+ P^- \). For simplicity of notation, below we will write \( D_{\delta, \epsilon, \xi}(\mathbf{r}, \mathbf{s}, \mathbf{t}) \equiv D(\mathbf{r}, \mathbf{s}, \mathbf{t}) \).

We can now substitute Eqs. \( (A.7) \) and \( (A.8) \) into Eq. \( (A.10) \). With the help of the canonical commutation relations

\[\begin{align*}
[a_\alpha(\mathbf{k}), a^\dagger_\beta(\mathbf{k}')_\mu] & = (2\pi)^3 \delta^{(2)}(\mathbf{k} - \mathbf{k}') \delta_\alpha^\beta
\end{align*}\]

(A.12)

\[\begin{align*}
[b_\alpha(\mathbf{p}), b^\dagger_\beta(\mathbf{p}')_\mu] & = (2\pi)^3 \delta^{(2)}(\mathbf{p} - \mathbf{p}') \delta_\alpha^\beta
\end{align*}\]

\[\begin{align*}
[d_\alpha(\mathbf{p}), d^\dagger_\beta(\mathbf{p}')_\mu] & = (2\pi)^3 \delta^{(2)}(\mathbf{p} - \mathbf{p}') \delta_\alpha^\beta
\end{align*}\]

(A.13)

it is straightforward to obtain (\( \alpha = \frac{2\pi}{\tau} \)):

\[
(H_{\text{eff}})_{\epsilon \rightarrow \epsilon'} = -\frac{\alpha}{K} \sum_{\alpha_1 \alpha_2} \delta^{j_1 + j_2}_r \theta(p^+_1 - p'^+_1)
\]

\[
\times \int d^2 \mathbf{p}_1 d^2 \mathbf{p}'_1 d^2 \mathbf{P}_1 d^2 \mathbf{P}'_1 \frac{(2\pi)^2 \delta^{(2)}(\mathbf{p}_1 + \mathbf{P}_1 - \mathbf{P}'_1)}{(2\pi)^2 (2\pi)^2 (2\pi)^2 (2\pi)^2 (x_1 - x'_1) D(p^+_1, p^+_2, p^-_1, p^-_2)}
\]

\[
\times (P^+)^2 \bar{u}_{j_1}(p^+_1) \gamma_\mu u_{j_2}(p^-_1) \bar{v}_{j_2}(p^-_2) \gamma_\nu v_{j_1}(p^+_2)
\]

\[
\times \left( \sum_\lambda e^{\mu}_\lambda(k) e^{\nu}_\lambda(k) \right) b^\dagger_\lambda(p^+_1) d^\dagger_\lambda(p^-_2) |0\rangle \langle 0| d_{\alpha_2}(\mathbf{p}_2) b_{\alpha_1}(\mathbf{p}_1),
\]

(A.14)

where \( \mu = (k^-, k^+, k) \)

\[
\left( \frac{p^+_1 - p'^+_1}{p^+_2 - p'_2} \right) \left( p^+_1 - p'^+_1, p^-_1 - p'_2 \right)
\]

Note that \( \mu \neq (p^+_1 - p'^+_1, p^-_1 - p'_2) \) due to the minus component. (Light-front energy is not conserved.) Note that in momentum space the state of the photon is completely determined by the external legs, except for the helicity. Thus there remains a sum over helicity states. In the HO basis there would have been an infinite sum over the quantum number \( n \) of the oscillator.

Similarly, the other time ordering (corresponding to photon emission by the positron and absorption by the electron) gives

\[
(H_{\text{eff}})_{\epsilon' \rightarrow \epsilon} = -\frac{\alpha}{K} \sum_{\alpha_1 \alpha_2} \delta^{j_1 + j_2}_r \theta(p^+_2 - p'^+_2)
\]

\[
\times \int d^2 \mathbf{p}_1 d^2 \mathbf{p}'_1 d^2 \mathbf{P}_1 d^2 \mathbf{P}'_1 \frac{(2\pi)^2 \delta^{(2)}(\mathbf{p}_1 + \mathbf{P}_2 - \mathbf{P}'_2)}{(2\pi)^2 (2\pi)^2 (2\pi)^2 (2\pi)^2 (x_2 - x'_2) D(p^+_2, p^+_1, p^-_2, p^-_1)}
\]

\[
\times (P^+)^2 \bar{u}_{j_1}(p^+_1) \gamma_\mu u_{j_2}(p^-_2) \bar{v}_{j_2}(p^-_1) \gamma_\nu v_{j_1}(p^+_2)
\]

\[
\times \left( \sum_\lambda e^{\mu}_\lambda(k) e^{\nu}_\lambda(k) \right) b^\dagger_\lambda(p^+_1) d^\dagger_\lambda(p^-_2) |0\rangle \langle 0| d_{\alpha_2}(\mathbf{p}_2) b_{\alpha_1}(\mathbf{p}_1).
\]

(A.15)

Now, in light-cone gauge, the polarization sum is given by

\[
\sum_\lambda \epsilon^{\mu}_\lambda(k) \epsilon^{\nu}_\lambda(k) = -g^{\mu \nu} + \eta^\mu k^\nu + \eta^\nu k^\mu
\]

(A.16)

where \( \eta^\mu = (\eta^-, \eta^+, \eta) = (2, 0, 0) \) is a unit vector in the light-front + direction \( (k^u \eta^u = k^+) \). The delta function requires \( p^+_2 - p'^+_2 = -2 \left( p^+_1 - p'^+_1 \right) \) (similarly for momentum fractions). Also, one can easily show that \( D(p^+_2, p^+_1, p^-_2, p^-_1) = -D(p^+_1, p^+_2, p^-_1, p^-_2) \). Therefore, the two time orderings can be combined to obtain our result:

\[
H_{\text{eff}} = -\frac{\alpha}{K} \sum_{\alpha_1 \alpha_2} \delta^{j_1 + j_2}_r \theta(p^+_1 - p'^+_1)
\]

\[
\times \int d^2 \mathbf{p}_1 d^2 \mathbf{p}'_1 d^2 \mathbf{P}_1 d^2 \mathbf{P}'_1 \frac{(2\pi)^2 \delta^{(2)}(\mathbf{p}_1 + \mathbf{P}_1 - \mathbf{P}'_1)}{(2\pi)^2 (2\pi)^2 (2\pi)^2 (2\pi)^2 (x_1 - x'_1) D(p^+_1, p^+_2, p^-_1, p^-_2)}
\]

\[
\times (P^+)^2 \bar{u}_{j_1}(p^+_1) \gamma_\mu u_{j_2}(p^-_1) \bar{v}_{j_2}(p^-_2) \gamma_\nu v_{j_1}(p^+_2)
\]

\[
\times \left( \sum_\lambda \epsilon^{\mu}_\lambda(k) \epsilon^{\nu}_\lambda(k) \right) b^\dagger_\lambda(p^+_1) d^\dagger_\lambda(p^-_2) |0\rangle \langle 0| d_{\alpha_2}(\mathbf{p}_2) b_{\alpha_1}(\mathbf{p}_1).
\]

(A.17)

Note that the \( |0\rangle \langle 0| \) present in Eqs. \( (A.14) \) and \( (A.15) \) is redundant within the \( |e^+\rangle \) Fock sector, and has therefore been dropped from Eq. \( (A.17) \).

2. Cancelling the Instantaneous Photon Exchange Term

The instantaneous photon exchange term in LFQED is given by

\[
P_{\text{inst}} = \frac{g^2}{2} \int_{-L}^{L} dx^- d^2 \mathbf{x} \bar{\Psi}(x) \gamma^+ \Psi(x) \frac{1}{(i\partial^+)^2} \bar{\Psi}(x) \gamma^+ \Psi(x),
\]

(A.18)

evaluated at \( x^+ = 0 \). This is shown diagrammatically in Fig. 4. Again, we substitute in the free field mode
expansions and expand. The term of interest is:

\[
H_{\text{inst}} = P^{+}P_{\text{inst}}^{-} = -\frac{4\alpha}{K}\sum_{\alpha,\alpha_{1},\alpha_{2}\alpha_{2}'} \delta_{j_{1}j_{2}'} \delta_{s_{1}s_{2}'} \left( \frac{d^{2}p_{1}}{(2\pi)^{2}} \frac{d^{2}p_{1}'}{(2\pi)^{2}} \frac{d^{2}p_{2}}{(2\pi)^{2}} \frac{d^{2}p_{2}'}{(2\pi)^{2}} \right) \delta^{(2)}(p_{1} + p_{2} - p_{1}' - p_{2}') \\
\times \frac{1}{(x_{1} - x_{1}')^2} \delta_{\gamma_{\mu}u_{s_{1}}(p_{1})\bar{v}_{s_{2}'}(p_{2})\gamma_{\nu}v_{s_{2}'}(p_{2}')} - \frac{\delta_{s_{1}s_{2}'}^{\prime}}{(x_{1} - x_{1}')^2}.
\]

(A.19)

We will now show that the second term in the parentheses of \( H_{\text{eff}}^{\text{reg}} \) (Eq. (A.17)) exactly cancels \( H_{\text{inst}} \). Comparing, we see that cancellation will be achieved if

\[
A = \frac{(P^{+})^2}{4(x_{1} - x_{1}')^2} \frac{\eta^{\mu}k^{\nu} + \eta^{\nu}k^{\mu}}{k^{+}} \\
\times \frac{\delta_{s_{1}s_{2}'}^{\prime}}{(x_{1} - x_{1}')^2}.
\]

(A.20)

This is indeed the case, but before we proceed we need a few results. Consider the four-vector \( l_{\mu}^{\nu} \equiv (k + p_{1}')_{\mu} - (p_{1})_{\mu} \). By the definition of \( k^{\mu} \) (just after Eq. (A.14)), we have \( l_{\mu}^{\nu} = 0 \) and \( l_{\mu}^{\nu} = 0 \). Thus \( l_{\mu}^{\nu} \sim \eta^{\mu} \). In fact, if we define \( Q = l_{\mu}^{\nu} \), then we have \( l_{\mu}^{\nu} = \frac{Q}{2}\eta^{\mu} \) (recall \( \eta^{\mu} = 2 \)). Combining this with the definition of \( l_{\mu}^{\nu} \), we see that

\[
k^{\mu} = \frac{Q}{2}\eta^{\mu} + (p_{1}' - p_{1})_{\mu},
\]

(A.21)

where \( Q = l_{\mu}^{\nu} = k^{+} + p_{1}' - p_{1} \). Similarly, starting from the vector \( l_{\mu}^{\nu} \equiv (k + p_{2} - p_{2}')_{\mu} \), we can show that

\[
k^{\mu} = \frac{Q}{2}\eta^{\mu} + (p_{2} - p_{2}')_{\mu},
\]

(A.22)

where \( Q = l_{\mu}^{\nu} = k^{+} + p_{2} - p_{2}' \). Using this information, we see that

\[
\bar{u}_{s_{1}'}(p_{1}')\gamma_{\nu}k^{\mu}\bar{u}_{s_{1}}(p_{1}) = \frac{Q}{2}\bar{u}_{s_{1}'}(p_{1}')\gamma^{\nu}u_{s_{1}}(p_{1}) = \frac{Q}{2}(2\delta_{s_{1}s_{1}'}^{\prime})
\]

(A.23)

\[
\bar{v}_{s_{2}'}(p_{2}')\gamma_{\nu}k^{\mu}\bar{v}_{s_{2}}(p_{2}) = \frac{Q}{2}\bar{v}_{s_{2}'}(p_{2}')\gamma^{\nu}v_{s_{2}}(p_{2}') = \frac{Q}{2}(2\delta_{s_{2}s_{2}'}^{\prime}).
\]

(A.24)

We are now ready to consider

\[
A = \frac{(P^{+})^2}{4(x_{1} - x_{1}')^2} \frac{\eta^{\mu}k^{\nu} + \eta^{\nu}k^{\mu}}{k^{+}} \\
\times \bar{u}_{s_{1}'}(p_{1}')\gamma_{\mu}u_{s_{1}}(p_{1})\bar{v}_{s_{2}'}(p_{2}')\gamma_{\nu}v_{s_{2}'}(p_{2}') \\
= \frac{(2\delta_{s_{1}s_{1}'}^{\prime})(2\delta_{s_{2}s_{2}'}^{\prime})}{(x_{1} - x_{1}')^2} \left[ \frac{Q + \bar{Q}}{2} P^{+} \right].
\]

(A.25)

By writing out the expressions for \( Q, \bar{Q} \) and \( D \), it is readily seen that the factor in brackets is \( -1 \). Therefore

\[
A = \frac{\delta_{s_{1}s_{1}'}^{\prime}\delta_{s_{2}s_{2}'}^{\prime}}{(x_{1} - x_{1}')^2}
\]

(A.26)

as required for the cancellation of the instantaneous term. Therefore we obtain:

\[
H_{\text{eff}}^{\text{reg}} + H_{\text{inst}} = \frac{\alpha}{K}\sum_{\alpha,\alpha_{1}\alpha_{2}\alpha_{2}'} \delta_{j_{1}j_{2}'} \delta_{s_{1}s_{2}'} \\
\times \int \frac{d^{2}p_{1}}{(2\pi)^{2}} \frac{d^{2}p_{1}'}{(2\pi)^{2}} \frac{d^{2}p_{2}}{(2\pi)^{2}} \frac{d^{2}p_{2}'}{(2\pi)^{2}} \delta^{(2)}(p_{1} + p_{2} - p_{1}' - p_{2}') \\
\times \left( P^{+} \right)^{2} \bar{u}_{s_{1}'}(p_{1}')\gamma_{\mu}u_{s_{1}}(p_{1})\bar{v}_{s_{2}'}(p_{2}')\gamma_{\nu}v_{s_{2}'}(p_{2}') \\
\times b_{\alpha_{1}}(p_{1}')d_{\alpha_{2}}(p_{2}')d_{\alpha_{2}}(p_{2})b_{\alpha_{1}}(p_{1}).
\]

(A.27)

where we have contracted the \( g^{\mu\nu} \) term.

3. Translation Back to Harmonic Oscillator Basis

To translate the result (A.26) to the HO basis, we use

\[
b_{\alpha}(p) = \sum_{nm} b_{\alpha}\Psi_{n}^{m}(p),
\]

(A.27)

where \( \alpha \) is now shorthand for the set of quantum numbers \((j, s, n, m)\). We therefore obtain the result

\[
H_{\text{eff}}^{\text{reg}} + H_{\text{inst}} = \frac{\alpha}{K}\sum_{\alpha,\alpha_{1}\alpha_{2}\alpha_{2}'} \delta_{j_{1}j_{2}'} \delta_{s_{1}s_{2}'} \\
\times \int \frac{d^{2}p_{1}}{(2\pi)^{2}} \frac{d^{2}p_{1}'}{(2\pi)^{2}} \frac{d^{2}p_{2}}{(2\pi)^{2}} \frac{d^{2}p_{2}'}{(2\pi)^{2}} \delta^{(2)}(p_{1} + p_{2} - p_{1}' - p_{2}') \\
\times \Psi_{n_{1}}^{m_{1}}(p_{1})\Psi_{n_{2}}^{m_{2}}(p_{2})\Psi_{n_{1}'}^{m_{1}'}(p_{1}')\Psi_{n_{2}'}^{m_{2}'}(p_{2}') \\
\times \left( P^{+} \right)^{2} \bar{u}_{s_{1}'}(p_{1}')\gamma_{\mu}u_{s_{1}}(p_{1})\bar{v}_{s_{2}'}(p_{2}')\gamma_{\nu}v_{s_{2}'}(p_{2}').
\]

(A.28)

The spinor part \((P^{+})^{2} \bar{u}_{s_{1}'}(p_{1}')\gamma_{\mu}u_{s_{1}}(p_{1})\bar{v}_{s_{2}'}(p_{2}')\gamma_{\nu}v_{s_{2}'}(p_{2}')\) contains 16 different spin combinations. These are enumerated in Table II.

To translate (A.28) into the longitudinal momentum weighted coordinates, we make the substitutions \( p \rightarrow \sqrt{x}q \), \( \int d^{2}p \rightarrow x \int d^{2}q \), \( b_{\alpha} \rightarrow b_{\alpha}/\sqrt{x} \) and \( \Psi_{n}^{m}(p) \rightarrow \Psi_{n}^{m}(q) \).
\[ \Psi_m^\alpha(q), \text{ Eq. (A.28)} \] now reads

\[
\begin{aligned}
H_{\text{eff}} + H_{\text{inst}} &= \frac{\alpha}{K} \sum_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} d_{\alpha_1} d_{\alpha_2} b_{\alpha_3} b_{\alpha_4} \\
&\times \sqrt{x_1 x_3 x_4 x_2} \int \frac{d^2 q_1}{(2\pi^2)^2} \frac{d^2 q_2}{(2\pi^2)^2} \frac{d^2 q_3}{(2\pi^2)^2} \frac{d^2 q_4}{(2\pi^2)^2} \\
&\times (2\pi)^2 \delta(2) (\sqrt{x_1 q_1} + \sqrt{x_2 q_2} - \sqrt{x_3 q_3} - \sqrt{x_4 q_4}) \\
&\times (x_1 - x_4)^D.
\end{aligned}
\]

Therefore we only consider the integral

\[
\begin{aligned}
I_{ED} &= \int \frac{d^2 q_1}{(2\pi^2)^2} \frac{d^2 q_2}{(2\pi^2)^2} \frac{d^2 q_3}{(2\pi^2)^2} \frac{d^2 q_4}{(2\pi^2)^2} \\
&\times (2\pi)^2 \delta(2) (\sqrt{x_1 q_1} + \sqrt{x_2 q_2} - \sqrt{x_3 q_3} - \sqrt{x_4 q_4}) \\
&\times (x_1 - x_4)^D.
\end{aligned}
\]

This integral, now expressed in single-particle coordinates, can be simplified by transforming to relative coordinates, using the Talmi-Moshinsky (TM) transform \[23, 34]\:

\[
\Psi_{m_1}^\alpha(q_1) \Psi_{m_2}^\alpha(q_2) = \sum_{N M n m} \mathcal{M}_{n_1 n_2 m_2 m}^{N M n m} \Psi_N^m(Q) \Psi_n^m(q).
\]

The quantities \(\mathcal{M}_{n_1 n_2 m_2 m}^{N M n m}\) are known as TM brackets, and the new relative coordinates are

\[
\begin{align*}
q &= \sqrt{x_2 q_1 - \sqrt{x_1 q_2}} \\
\sqrt{x_1 + x_2} \\
Q &= \sqrt{x_1 q_1 + \sqrt{x_2 q_2}}.
\end{align*}
\]

We now perform two separate TM transformations from the variables \(q_1, q_2 \to Q, q\) and \(q_1', q_2' \to Q', q'\). \(Q'\) and \(q'\) are also given by \(\text{[A.35]}\), with all quantities primed. In terms of the relative coordinates, the denominator can be expressed as

\[
(x_1 - x_4)D = \frac{1}{2(x_1 + x_2)} \left[ (\sqrt{x_1 x_2} q + \sqrt{x_1 x_2} q')^2 \\
+ (\sqrt{x_1 x_2} q' + \sqrt{x_1 x_2} q)^2 + (x_1 + x_2) \Delta \right],
\]

where \(\Delta = m_j^2 (x_1 - x_4)^2 (\frac{1}{x_1 + x_2} \sqrt{x_1 x_2} q + \frac{1}{x_1 + x_2} \sqrt{x_1 x_2} q')^2 + 2 \mu^2 \geq 0\). Note the denominator is independent of \(Q\) and \(Q'\).

Due to the \(x\) dependence of the coordinates \(q_1, q_2\), the TM phase \[23, 34]\ \(\delta\) is given by \(\tan \delta = \sqrt{x_2/x_1}\) for the \(q_1, q_2 \to Q, q\) transform and by \(\tan \delta =\)
\[ \sqrt{x_1'x_2'} \] for the \( q_1', q_2' \rightarrow Q', q' \) transform. The integral then takes the form:

\[ I_{ED} = \sum_{n,M} \sum_{n',M'} M_{n,M}^{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

which simplifies to

\[ I_{ED} = \frac{1}{x_1 + x_2} \sum_{n,M} \sum_{n',M'} M_{n,M}^{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

We will refer to the remaining integral as

\[ I_{nmn'm'} = \int \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

The eightfold sum can be reduced significantly to three folds. Each TM transform comes with two Kronecker deltas:

\[ \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \sim \delta_{nmn'}^{nmn'} \delta_{n'm'n'}^{n'M'n'M'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \]

The final result is:

\[ I_{nmn'm'} = \frac{1}{2\pi} \frac{1}{x_1 + x_2} \sum_{n,M} \sum_{n',M'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \int d^2 \mathbf{q} \cdot d^2 \mathbf{q'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

where \( c = (\sqrt{x_1'x_2'} - \sqrt{x_1x_2})^2/(x_1 + x_2) \) and \( c' = (\sqrt{x_1'x_2'} + \sqrt{x_1x_2})^2/(x_1 + x_2) \). If one now substitutes in the expressions for the wavefunctions, the angular integrations are trivial and it straightforward to obtain

\[ I_{nmn'm'} = \frac{1}{2\pi} \frac{1}{x_1 + x_2} \sum_{n,M} \sum_{n',M'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \int d^2 \mathbf{q} \cdot d^2 \mathbf{q'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

The eightfold sum can be reduced significantly to three folds. Each TM transform comes with two Kronecker deltas:

\[ \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \sim \delta_{nmn'}^{nmn'} \delta_{n'm'n'}^{n'M'n'M'} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \]

One of the six Kronecker deltas is left over as an angular dependence. The integral is given by

\[ I = \int \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \left[ \frac{d^2 \mathbf{q} \cdot d^2 \mathbf{q'}}{(2\pi)^2} \right] \mathcal{M}^{nmn'}_{n,M} \mathcal{M}^{n'M'}_{n',M'} \times \left[ (x_1' - x_1)D \right] \]

where \( m = \mu(m) \equiv n + n + \frac{1}{2} \left( |m| + |n| - |m| - |\lambda - m| \right) / 2 \), \( \nu = \nu(m) \equiv n' + n' + \frac{1}{2} \left( |m'| + |n'| - |m'| - |\lambda - m'| \right) / 2 \), and \( \lambda = m + m' \).
[1] J. P. Vary, H. Honkanen, Jun Li, P. Maris, S. J. Brodsky, A. Harindranath, G. F. de Teramond, P. Sternberg, E. G. Ng, C. Yang, Phys. Rev. C 81, 035205 (2010).
[2] S. J. Brodsky, H.-C. Pauli and S. S. Pinsky, Physics Reports 301, 299 (1998).
[3] A. Harindranath, “Light Front QCD: Lecture notes”, Saha Institute of Nuclear Physics, Kolkata (2005).
[4] P. Navrátil, J. P. Vary and B. R. Barrett, Phys. Rev. Lett. 84, 5728 (2000).
[5] P. Navrátíl, J. P. Vary and B. R. Barrett, Phys. Rev. C 62, 054311 (2000).
[6] P. Sternberg, E. G. Ng, C. Yang, P. Maris, J. P. Vary, M. Sosonkina and H. V. Le, in Proceedings of the 2008 ACM/IEEE Conference on Supercomputing, Austin, 2008, (IEEE, New York, 2008), article number 15.
[7] P. Maris, M. Sosonkina, J.P. Vary, E.G. Ng, C. Yang, Procedia Computer Science 1, 97 (2010).
[8] H. Honkanen, P. Maris, J. P. Vary and S. J. Brodsky, Phys. Rev. Lett. 106, 061603 (2011) [arXiv:1008.0068 [hep-ph]].
[9] H. Honkanen, P. Maris, J. P. Vary and S. J. Brodsky, Phys. Rev. Lett. 106, 061603 (2011) [arXiv:1008.0068 [hep-ph]].
[10] X. Zhao, H. Honkanen, P. Maris, J. P. Vary and S. J. Brodsky, Few Body Syst. 52, 339 (2012) [arXiv:1110.0553 [hep-ph]].
[11] X. Zhao, H. Honkanen, P. Maris, J. P. Vary and S. J. Brodsky, arXiv:1402.4195 [nucl-th].
[12] D. Chakrabarti, X. Zhao, H. Honkanen, R. Manohar, P. Maris and J. P. Vary, arXiv:1403.0704 [hep-ph].
[13] X. Zhao, A. Ilderton, P. Maris and J. P. Vary, Phys. Rev. D 88, 065014 (2013) [arXiv:1303.3273 [nucl-th]].
[14] X. Zhao, A. Ilderton, P. Maris and J. P. Vary, Phys. Lett. B 726, 856 (2013) [arXiv:1309.5338 [nucl-th]].
[15] P. Maris, P. W. Wiecki, Y. Li, X. Zhao and J. P. Vary, Acta Phys. Polon. Supp. 6, 321 (2013).
[16] P. W. Wiecki, Y. Li, X. Zhao, P. Maris and J. P. Vary, in Proc. Int. Conf. Nucl. Theor. Supercomputing Era (NTSE-2013), Ames, IA, USA, May 1-17, 2013, edited by A. M. Shirokov and A. I. Mazur. Pacific National University, Khabarovsk, 2013, p. 136.
[17] K. Hornbostel, S. J. Brodsky and H.-C. Pauli, Phys. Rev. D 41, 3814 (1990).
[18] M. A. Caprio, P. Maris and J. P. Vary, Phys. Rev. C 86, 034312 (2012).
[19] S. A. Coon, M. I. Avetian, M. K. G. Kruse, U. van Kolck, P. Maris and J. P. Vary, Phys. Rev. C 86, 054002 (2012), arXiv:1205.3230 [nucl-th] (2012).