ON A GLOBAL LAGRANGIAN CONSTRUCTION FOR
ORDINARY VARIATIONAL EQUATIONS ON 2-MANIFOLDS

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Abstract. Locally variational systems of differential equations on smooth manifolds, having certain de Rham cohomology group trivial, automatically possess a global Lagrangian. This important result due to Takens is, however, of sheaf-theoretic nature. A new constructive method of finding a global Lagrangian for second-order ODEs on 2-manifolds is described on the basis of solvability of exactness equation for Lepage 2-forms, and the top-cohomology theorems. Examples from geometry and mechanics are discussed.

1. Introduction

In this paper, we address the problem of finding a concrete global Lagrangian for variational second-order ODEs on smooth fibered manifolds. The existence of a global variational principle for given equations is essentially influenced by topology of the underlying space: for ordinary variational equations of arbitrary order it depends on the second de Rham cohomology group $H^{n+1}_{dR}(Y; \mathbb{R}) = 0$. The following theorem belongs to important global results achieved within the calculus of variations on smooth manifolds.

Theorem (Takens, 1979). Each locally variational source equation is globally variational provided $H^{n+1}_{dR}(Y; \mathbb{R}) = 0$.

In [19], Takens obtained this result within the framework of a variational bicomplex theory over sheaves of differential forms on infinite jet prolongations of fibered manifolds over general n-dimensional bases. An analogous global result to this theorem was also obtained by Vinogradov [23, 24] (see [25] for more detail exposition as a part of C-spectral sequences using Spencer cohomology). Variational bicomplex theories have been developed since late seventieth by many authors with the aim to study a complex, analogous to the de Rham complex, where the Euler–Lagrange mapping is included as one of its morphisms in an exact sequence; see Anderson and Duchamp [1], Dedecker and Tulczyjew [3], and Tulczyjew [21].

A different approach to similar ideas was developed in the variational sequence theory by Krupka [6, 7], who considered the quotient sequence of the de Rham sequence over finite-order jet prolongations of fibered manifolds with respect to its contact subsequence. Thus, a basic concept of the calculus of variations, the Euler–Lagrange mapping, can be constructed (for variational functionals on fibered spaces) as the quotient mapping of the exterior derivative operator $d$, acting on differential forms, by the restriction of $d$ to the so-called contact forms. The main

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global result of Krupka [6] reads: The variational sequence of order \( r \) over \( Y \) is an acyclic resolution of the constant sheaf \( \mathbb{R}_Y \) over \( Y \). From this theorem and the well-known abstract de Rham theorem, we immediately obtain the following corollary: Let \( \varepsilon \) be a locally variational source form on \( J^r Y \). If \( H^{n+1} Y \) is trivial, then \( \varepsilon \) is also globally variational. This is a finite-order analogue of the result due to Takens [19].

One should notice, however, that the cohomology conditions in the variational sequence and the variational bicomplex theory have different meaning. The relationship of these two theories can be found in Pommaret [17], Vitolo [26], and Krupka et. al. [10].

A common feature of the previously mentioned works [1, 3, 6, 7, 19, 21, 23, 24, 25, 26] is an absence of a concrete global variational principle, whose existence is guaranteed by cohomology conditions. To the authors’ knowledge, there is no general method how to construct a global Lagrangian for locally variational equations. Note that simple examples show that the well-known Vainberg–Tonti formula (cf. Tonti [20], Krupka [5]) fails to produce Lagrangians which are defined globally.

In the framework of global variational theory on finite-order jet prolongations of fibered manifolds (cf. Krupka [2], and references therein), the formulation of our main problem is the following: let \( \varepsilon \) be a given locally variational source form (also known as a dynamical form in Lagrangian mechanics) on the second jet prolongation \( J^2 Y \) of a fibered manifold \( Y \) over a 1-dimensional base. Then we search for a (concrete) global Lagrangian \( \lambda \), which is a horizontal differential 1-form defined (globally) on \( J^1 Y \), such that \( \varepsilon \) coincides with the image of \( \lambda \) in the Euler–Lagrange mapping, that is, with the Euler–Lagrange form associated with \( \lambda \).

Section 2 is devoted to basic facts of second-order variational ODEs in accordance with the general theory (cf. Krupková and Prince [15, 14], Krupková [13]), including necessary and sufficient conditions for local variationality, namely the Helmholtz conditions. The geometry of second-order PDEs has been studied recently by Saunders, Rossi and Prince [18]. For local variational principles based on Lepage forms, see Brajerčík and Krupka [2].

Recall that a Lepage form represents a far-going generalization of a 1-form, introduced by E. Cartan within the framework of the calculus of variations (see Krupka [4, 5]). Roughly speaking, a Lepage form gives a geometric description of the associated variational functional: its variations, extremals and invariance are characterized by means of the geometric operations as the exterior derivative, and the Lie derivative of differential forms. The meaning of Lepage forms for the calculus of variations and their basic properties have recently been summarized by Krupka, Krupková and Saunders [9].

In Section 3, we develop the main idea of this paper: to tackle the problem on the basis of solvability of the global exactness equation for the Lepage equivalent \( \alpha_{\varepsilon} \) of a source form \( \varepsilon \). Globally defined 2-form \( \alpha_{\varepsilon} \) represents an example of a Lepage 2-form in Lagrangian mechanics (see Krupková [12, 13]), and it satisfies the condition \( \alpha_{\varepsilon} = d\Theta_{\lambda} \), which is a subject of solution with respect to unknown \( \lambda \), where \( \Theta_{\lambda} \) is the well-known Cartan form. Recall that \( \Theta_{\lambda} \) depends on the choice of a Lagrangian \( \lambda \) whereas \( d\Theta_{\lambda} \) does not. As a result, we reduce the global exactness of Lepage equivalent \( \alpha_{\varepsilon} \) of \( \varepsilon \) to global exactness of a certain 2-form \( \omega \), defined on the underlying fibered manifold \( Y \).
In Section 4, we apply the standard de Rham top-cohomology theory (see Lee [16]) to solve global exactness of the differential 2-form \( \omega \), and combine it with results obtained in Section 3, see Theorem [17]. Indeed, this step can be proceeded on smooth manifolds of dimension two only, and our general global Lagrangian construction for locally variational source forms is therefore restricted to 2-manifolds with trivial the second de Rham cohomology group of \( Y \). We point out that the equation \( \omega = dy \) need not have a global solution, and even if solvability of this equation is assured, no general construction of its solution is known on smooth \( m \)-dimensional manifolds. This circumstance makes our problem difficult in general. Note that topic of this paper is closely related to the inverse problem of the calculus of variations, discussed by most of the authors mentioned above, and solved by J. Douglas in his seminal paper (1941) for systems of two ordinary equations of two dependent variables.

Section 5 contains two examples of mechanical systems, namely, the kinetic energy on the open M"obius strip, and a gyroscopic type system on the punctured torus in the Euclidean space \( \mathbb{R}^3 \), where the corresponding global variational principles are discussed. We emphasize, however, that the theory can not be applied to general \( m \)-manifolds; e.g. for \( m = 3 \), \( H_3^{de}S^3 \) of the 3-sphere \( S^3 \) is trivial, nevertheless we can proceed only under additional requirements (cf. Corollary [10]).

Notation and underlying geometric structures are coherent with our recent work Krupka, Urban, and Volná [11], where general higher-order formulas can be found. Throughout, we consider fibered manifolds the Cartesian products \( Y = \mathbb{R} \times M \) over the real line \( \mathbb{R} \) and projection \( \pi \), where \( M \) is a smooth connected 2-manifold. Thus, the jet prolongations \( J^1Y \) and \( J^2Y \) of \( Y \) can be canonically identified with the product \( \mathbb{R} \times T^1M \) and \( \mathbb{R} \times T^2M \), respectively, where \( T^1M \) is the tangent bundle of \( M \), and \( T^2M \) denotes the manifold of second-order velocities over \( M \). Recall that elements of \( T^2M \) are 2-jets \( J^2_2\pi \in J^2(\mathbb{R}, M) \) with origin \( 0 \in \mathbb{R} \) and target \( \zeta(0) \in M \). The jet prolongations are considered with its natural fibered manifold structure: if \( (V, \psi) \), \( \psi = (t, x, y) \), is a fibered chart on \( \mathbb{R} \times M \), the associated chart on \( J^2Y \) (respectively, \( J^1Y \)) reads \( (V^2, \psi^2) \), \( \psi^2 = (t, x, y, \dot{x}, \dot{y}) \) (respectively, \( (V^1, \psi^1) \), \( \psi^1 = (t, x, y, \dot{x}, \dot{y}) \)), where \( V^2 \) (respectively, \( V^1 \)) is the preimage of \( V \) in the canonical jet prolongation \( \pi^{2,0} : J^2Y \to Y \) (respectively, \( \pi^{1,0} : J^1Y \to Y \)).

Let \( W \) be an open set in \( Y \), and \( \Omega^1W \) the exterior algebra of differential forms on \( W \). By means of charts, we put \( h\mathrm{d}t = dt, \ h\mathrm{d}x = \dot{x}dt, \ h\mathrm{d}y = \dot{y}dt, \ h\dot{x} = \ddot{x}dt, \ h\dot{y} = \ddot{y}dt, \) and for any function \( f : W^1 \to \mathbb{R} \), \( h\dot{f} = f \circ \pi^{2,1} \), where \( \pi^{2,1} \) denotes the canonical jet prolongation \( J^2Y \to J^1Y \). These formulas define a global homomorphism of exterior algebras \( h : \Omega^1W \to \Omega^2W \), called the \( \pi \)-horizontalization. A 1-form \( \rho \) on \( W^1 \) is called contact, if \( h\rho = 0 \). In a fibered chart \( (V, \psi) \), \( \psi = (t, x, y) \), on \( W \), every contact 1-form \( \rho \) has an expression \( \rho = A_x\omega^x + A_y\omega^y \), for some functions \( A_x, A_y : V^1 \to \mathbb{R} \), where \( \omega^x = dx - \dot{x}dt, \ \omega^y = dy - \dot{y}dt \). For any differential 1-form \( \rho \) on \( W^1 \), the pull-back \( (\pi^{2,1})^* \rho \) has a unique decomposition \( (\pi^{2,1})^* \rho = h\rho + p\rho \), where \( h\rho \), resp. \( p\rho \), is \( \pi^2 \)-horizontal (respectively, contact) 1-form on \( W^2 \). This decomposition can be directly generalized to arbitrary \( k \)-forms. For \( k = 2 \), if \( \rho \) is a 2-form on \( W^1 \), then we get \( (\pi^{2,1})^* \rho = p_1\rho + p_2\rho \), where \( p_1\rho \) (resp. \( p_2\rho \)) is the 1-contact (respectively, 2-contact) component of \( \rho \), spanned by \( \omega^x \wedge dt, \ \omega^y \wedge dt \) (respectively, \( \omega^x \wedge \omega^y \)). Analogously, we employ these concepts on \( W^2 \).

The results of this work can be generalized to higher-order variational differential equations by means of similar methods.
2. Second-order ordinary variational equations

Let \( \varepsilon \) be a source form on \( \mathbb{R} \times T^2 M \). By definition, \( \varepsilon \) is a 1-contact and \( \pi^{2,0} \)-horizontal 2-form. In a chart \((V, \psi), \psi = (t, x, y)\), on \( \mathbb{R} \times M \), \( \varepsilon \) is expressed as

\[
\varepsilon = (\varepsilon_x \omega^x + \varepsilon_y \omega^y) \wedge dt,
\]

where the coefficients \( \varepsilon_x, \varepsilon_y \) are differentiable functions on \( V^2 \), and \( \omega^x = dx - \dot{x} dt, \omega^y = dy - \dot{y} dt \) are contact 1-forms on \( V^1 \). To simplify further considerations, but without loss of generality, we suppose that \( \varepsilon_x, \varepsilon_y \) do not depend on the time variable \( t \) explicitly, that is \( \varepsilon_x, \varepsilon_y \) are functions of \( x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y} \) only. Source form (2.1) associates a system of two second-order differential equations

\[
\varepsilon_x (x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = 0, \quad \varepsilon_y (x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = 0,
\]

for unknown differentiable curves \( \zeta \) in \( M \), \( t \to \zeta(t) = (x \circ \zeta(t), y \circ \zeta(t)) \), defined on an open interval of \( \mathbb{R} \).

2-form (2.1) (or system (2.2)) is called locally variational, if there exists a real-valued function \( \mathcal{L} \) on a chart neighborhood \( V^1 \), \( \mathcal{L} = \mathcal{L}(t, x, y, \dot{x}, \dot{y}) \), such that (2.2) coincide with the Euler–Lagrange equations, i.e. \( \varepsilon_x = E_x(\mathcal{L}) \) and \( \varepsilon_y = E_y(\mathcal{L}) \) are the Euler-Lagrange expressions, associated with \( \mathcal{L} \),

\[
E_x(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial x} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}}, \quad E_y(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial y} \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}}.
\]

A Lagrangian of order 1 for \( Y \) is by definition a \( \pi^1 \)-horizontal 1-form \( \lambda \) on \( W^1 \subset \mathbb{R} \times T^1 M \). In a fibered chart, \( \lambda = \mathcal{L} dt \), where \( \mathcal{L} : V^1 \to \mathbb{R} \) is a real-valued function called the (local) Lagrange function associated with \( \lambda \). Every Lagrangian \( \lambda \) associates a source form \( E_\lambda \), locally expressed by

\[
E_\lambda = E_x(\mathcal{L}) \omega^x \wedge dt + E_y(\mathcal{L}) \omega^y \wedge dt.
\]

\( E_\lambda \) is called the Euler-Lagrange form, associated with \( \lambda \). Thus, locally variational forms belong to image of the Euler–Lagrange mapping \( \lambda \to E_\lambda \). Note that a Lagrangian is a representative of class of 1-forms, whereas a source form is a representative of class of 2-forms in the (quotient) variational sequence over \( W \subset Y \); for further remarks see Krupka [7].

From the definition, it is easy to observe that the coefficients of locally variational form (2.1) coincide with the Euler–Lagrange expressions of a Lagrange function with respect to every chart. Nevertheless, such local Lagrange functions, defined on chart neighborhoods in \( \mathbb{R} \times T^1 M \), need not define a (global) function on \( \mathbb{R} \times T^1 M \). If there exists a Lagrange function \( \mathcal{L} \) for \( \varepsilon \) defined on \( \mathbb{R} \times T^1 M \), then \( \varepsilon \) is called globally variational.

In the following theorem, we give necessary and sufficient conditions for locally variational source forms.

**Theorem 1.** Let \( \varepsilon \) be a source form on \( \mathbb{R} \times T^2 M \), locally expressed by (2.1) with respect to a chart \((V, \psi), \psi = (t, x, y)\), on \( \mathbb{R} \times M \). The following conditions are equivalent:

(a) \( \varepsilon \) is locally variational.
(b) The functions \( \varepsilon_x, \varepsilon_y \) satisfy identically the system
\[
\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_y}{\partial x} = 0, \quad \frac{d \partial \varepsilon_x}{dt \partial x} = 0, \quad \frac{\partial \varepsilon_y}{\partial y} - \frac{d \partial \varepsilon_y}{dt \partial y} = 0,
\]
(2.4)
\[
\frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} - \frac{d}{dt} \left( \frac{\partial \varepsilon_x}{\partial y} + \frac{\partial \varepsilon_y}{\partial x} \right) = 0,
\]
\[
\frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_y}{\partial x} - \frac{1}{2} \frac{d}{dt} \left( \frac{\partial \varepsilon_x}{\partial y} - \frac{\partial \varepsilon_y}{\partial x} \right) = 0.
\]
(c) The functions \( \varepsilon_x, \varepsilon_y \) are of the form
\[
\varepsilon_x = A_x + B_{xx} \dot{x} + B_{xy} \dot{y}, \quad \varepsilon_y = A_y + B_{yx} \dot{x} + B_{yy} \dot{y},
\]
where the functions \( A_x, A_y, B_{xx}, B_{xy}, B_{yx}, B_{yy} \) depend on \( x, y, \dot{x}, \dot{y} \) only, and satisfy
\[
\begin{align*}
B_{xy} &= B_{yx}, & \frac{\partial B_{xx}}{\partial y} &= \frac{\partial B_{xy}}{\partial x}, & \frac{\partial B_{xx}}{\partial x} &= \frac{\partial B_{yx}}{\partial y}, \\
\frac{\partial A_x}{\partial x} - \frac{\partial B_{xx}}{\partial y} \dot{x} - \frac{\partial B_{xy}}{\partial y} \dot{y} &= 0, & \frac{\partial A_y}{\partial y} - \frac{\partial B_{yx}}{\partial x} \dot{x} - \frac{\partial B_{yy}}{\partial y} \dot{y} &= 0,
\end{align*}
\]
(2.6)
\[
\begin{align*}
\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x} - 2 \frac{\partial B_{xy}}{\partial x} \dot{x} - 2 \frac{\partial B_{xy}}{\partial y} \dot{y} &= 0, \\
\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \dot{x} - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \dot{y} &= 0.
\end{align*}
\]
(d) The function
\[
\mathcal{L} = \mathcal{L}_T - \frac{d}{dt} \left( x \int_0^1 C_x (sx, sy, s\dot{x}, s\dot{y}) \, ds + y \int_0^1 C_y (sx, sy, s\dot{x}, s\dot{y}) \, ds \right),
\]
where functions \( C_x, C_y \) are given by conditions (2.6) as \( B_{xy} = \partial C_x/\partial y = \partial C_y/\partial x \), and
\[
\mathcal{L}_T = x \int_0^1 \varepsilon_x (sx, sy, s\dot{x}, s\dot{y}) \, ds + y \int_0^1 \varepsilon_y (sx, sy, s\dot{x}, s\dot{y}) \, ds,
\]
(2.8)
is a Lagrange function for \( \varepsilon \) defined on \( V^1 \).

(e) To every point of \( \mathbb{R} \times T^2 \mathcal{M} \) there is a neighborhood \( W \) and a 2-contact 2-form \( F_W \) on \( W \) such that the form \( \alpha_W = \varepsilon |_W + F_W \) is closed.

(f) There exists a closed 2-form \( \alpha_\varepsilon \) on \( \mathbb{R} \times T^1 \mathcal{M} \) such that \( \varepsilon = p_1 \alpha_\varepsilon \). If \( \alpha_\varepsilon \) exists, it is unique and it has a chart expression given by
\[
\alpha_\varepsilon = (\varepsilon_x \omega^x + \varepsilon_y \omega^y) \wedge dt + \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \omega^x \wedge \omega^y + B_{xx} \omega^x \wedge \omega^x + B_{xy} \omega^x \wedge \omega^y + B_{yy} \omega^y \wedge \omega^y,
\]
where \( \omega^x = dx - \dot{x} dt \), \( \omega^y = dy - \dot{y} dt \), \( \omega^x = dx - \dot{x} dt \), \( \omega^y = dy - \dot{y} dt \), are contact 1-forms on \( V^2 \).

The identities expressed by Theorem 11 (b), or equivalently (c), are the well-known *Helmholtz conditions* of local variationality (cf. Krupková and Prince, and references therein). Formula (2.8) yields the Vainberg–Tonti Lagrange function for a locally variational source form \( \varepsilon \) (see Tonti [20]), which is defined on \( V^2 \) and can always be reduced to first-order Lagrange function (2.7) on \( V^1 \). Note that the
Euler–Lagrange form associated with Lagrangian (2.7) coincides with source form \( \varepsilon \), provided the Helmholtz conditions are satisfied.

A relationship between locally variational source forms and closed forms was studied by Krupka [5], and is given by Theorem 1 (e). Theorem 1 (f), represents global generalization of condition (e) due to Krupková [12]. A straightforward coordinate transformations applied to formula (2.9) verify that \( \alpha \) defines a global form on \( \mathbb{R} \times T^1M \), and it represents an example of a Lepage 2-form in mechanics (see also Krupková and Prince [14, 15]). \( \alpha \) is called a Lepage equivalent of locally variational source form \( \varepsilon \).

A \( 1 \)-form \( \vartheta \) on \( W^1 \subset \mathbb{R} \times T^1M \) is called a Lepage form (of first-order), if the contraction \( i_\xi d\vartheta \) is contact \( 1 \)-form for every \( \pi_1, 0 \)-vertical vector field \( \xi \) on \( W^1 \). In addition, if \( h\vartheta = \lambda \) for a Lagrangian \( \lambda \) on \( W^1 \), \( \vartheta \) is called the Lepage equivalent of \( \lambda \). The concepts of a Lepage form and the Lepage equivalent of a Lagrangian are introduced for finite-order jet prolongations of fibered manifolds over \( n \)-dimensional basis (see Krupka [5]), and also the Grassmann fibrations (see Urban and Krupka [22]); for \( n = 1 \) (fibered mechanics), the Lepage equivalent of a Lagrangian is unique. The following theorem recalls the well-known Cartan form \( \Theta_\lambda \) of a first-order Lagrangian \( \lambda \), which represents an example of a Lepage form. For variational principles in mechanics based on the Cartan form and its generalizations, see Krupka, Krupková and Saunders [9].

**Theorem 2.** Every first-order Lagrangian \( \lambda \in \Omega^1_{1,X}W \) has a unique Lepage equivalent \( \Theta_\lambda \). If \( \lambda \) has a chart expression \( \lambda = \mathcal{L} dt \), then

\[
\Theta_\lambda = \mathcal{L} dt + \frac{\partial \mathcal{L}}{\partial \dot{x}} \omega^x + \frac{\partial \mathcal{L}}{\partial \dot{y}} \omega^y, \tag{2.10}
\]

and

\[
p_1 d\Theta_\lambda = E_\lambda, \tag{2.11}
\]

where

\[
E_\lambda = (E_x(\mathcal{L})\omega^x + E_y(\mathcal{L})\omega^y) \wedge dt. \tag{2.12}
\]

Combining Theorem 1 (f), with the Lepage equivalent property (2.11), we get a straightforward observation for globally variational source forms.

**Corollary 3.** Let \( \varepsilon \) be a source form on \( \mathbb{R} \times T^2M \), which is locally variational. If the equation

\[
\alpha_\varepsilon = d\Theta_\lambda \tag{2.13}
\]

has a (global) solution \( \lambda \in \Omega^1_{1,X} (\mathbb{R} \times T^1M) \), then \( \varepsilon \) is also globally variational, and vice versa.

**Proof.** Theorem 1 (f), assures a unique 2-form \( \alpha_\varepsilon \) on \( \mathbb{R} \times T^1M \) which is closed and satisfies \( \varepsilon = p_1\alpha_\varepsilon \). Applying the operator \( p_1 \) onto (2.13), we get \( \varepsilon = E_\lambda \) for some \( \lambda \in \Omega^1_{1,X} (\mathbb{R} \times T^1M) \).

\( \square \)

3. **The exactness equation for Lepage 2-form**

Let \( \varepsilon \) be a locally variational source form defined on \( \mathbb{R} \times T^2M \), and consider the Lepage equivalent \( \alpha_\varepsilon \) of \( \varepsilon \) (Theorem 1 (f)). The Poincaré lemma implies that \( \alpha_\varepsilon \) is locally exact. We observe that the main problem of finding a global Lagrangian
for \( \varepsilon \) is closely related with global exactness of Lepage 2-form \( \alpha_\varepsilon \) or, in other words, with finding a solution \( \mu \) defined on \( \mathbb{R} \times T^1 M \) of the equation

\[
\alpha_\varepsilon = d\mu.
\]

A solution \( \mu \) of (3.1) produces Lagrangian \( h_\mu \), which is equivalent to \( \lambda = \mathcal{L} dt \) given by equation (2.1). Indeed, if \( \mu \) solves (3.1) and \( \alpha_\varepsilon = d\Theta_\lambda \), then \( \mu = \Theta_\lambda + df \), hence we get \( h_\mu = \lambda + h(df) = (\mathcal{L} + df/dt) dt \), which differs from \( \lambda \) by means of total derivative of a function.

Recall that equation (3.1) need not have a global solution on \( T^1 M \) and, moreover, if a solution exists, there is no construction of this solution on general \( m \)-dimensional manifolds; see Remark 1.4.

The next lemma allows global canonical decomposition of \( \alpha_\varepsilon \) into closed forms.

**Lemma 4.** Let \( \alpha_\varepsilon \) be the Lepage equivalent of a locally variational source form \( \varepsilon \) on \( \mathbb{R} \times T^2 M \). Then there is a unique decomposition of \( \alpha_\varepsilon \) on \( \mathbb{R} \times T^1 M \),

\[
\alpha_\varepsilon = \alpha_0 \wedge dt + \alpha',
\]

where \( \alpha_0 \) and \( \alpha' \) are closed forms defined on \( T^1 M \), and \( \alpha' \) does not contain \( dt \). In a fibered chart \((V, \psi)\), \( \psi = (t, x, y) \), on \( \mathbb{R} \times M \), we have

\[
\alpha_0 = \left( A_x - \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) y \right) dx + \left( A_y - \frac{1}{2} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \dot{x} \right) dy
\]

\[
+ (B_{xx} \dot{x} + B_{xy} \dot{y}) d\dot{x} + (B_{xy} \dot{x} + B_{yy} \dot{y}) d\dot{y},
\]

and

\[
\alpha' = \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) dx \wedge dy
\]

\[
+ (B_{xx} dx + B_{xy} dy) \wedge d\dot{x} + (B_{xy} dx + B_{yy} dy) \wedge d\dot{y}.
\]

**Proof.** Since the Lepage 2-form \( \alpha_\varepsilon \) is closed, it is sufficient to show that also \( \alpha' \) is closed. This can be, however, directly verified in a fibered chart employing the Helmholtz conditions (2.6). Indeed, from (3.3) we have a chart expression

\[
d\alpha' = \left( \frac{\partial B_{xy}}{\partial x} - \frac{\partial B_{xx}}{\partial y} + \frac{1}{2} \frac{\partial}{\partial \dot{x}} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \right) dx \wedge dy \wedge d\dot{x}
\]

\[
+ \left( \frac{\partial B_{yy}}{\partial x} - \frac{\partial B_{xy}}{\partial y} + \frac{1}{2} \frac{\partial}{\partial \dot{y}} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \right) dx \wedge dy \wedge d\dot{y}
\]

\[
+ \left( \frac{\partial B_{xx}}{\partial y} - \frac{\partial B_{xy}}{\partial y} \right) dx \wedge d\dot{x} \wedge d\dot{y} + \left( \frac{\partial B_{xy}}{\partial y} - \frac{\partial B_{yy}}{\partial x} \right) dy \wedge d\dot{x} \wedge d\dot{y},
\]

where the last two summands vanish using

\[
\frac{\partial B_{xx}}{\partial y} = \frac{\partial B_{xy}}{\partial x}, \quad \frac{\partial B_{yy}}{\partial x} = \frac{\partial B_{xy}}{\partial y},
\]

and from the Helmholtz conditions (2.6) it is easy to see that the following identities hold

\[
\frac{\partial B_{xy}}{\partial x} - \frac{\partial B_{xx}}{\partial y} + \frac{1}{2} \frac{\partial}{\partial \dot{x}} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) = 0,
\]

\[
\frac{\partial B_{yy}}{\partial x} - \frac{\partial B_{xy}}{\partial y} + \frac{1}{2} \frac{\partial}{\partial \dot{y}} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) = 0.
\]
Hence \( d\alpha' = 0 \), as required. It remains to show that both local forms \( \alpha_0 \) and \( \alpha' \) define global forms on \( \mathbb{R} \times T^1M \). Since \( \varepsilon (2.1) \) is defined on \( \mathbb{R} \times T^1M \), we get for an arbitrary chart transformation \( x = x(\tilde{x}, \tilde{y}), \ y = y(\tilde{x}, \tilde{y}), \) on \( M \), the relations

\[
\frac{\partial A_x}{\partial \tilde{y}} - \frac{\partial A_y}{\partial \tilde{x}} = \left( \frac{\partial A_x}{\partial \tilde{y}} - \frac{\partial A_y}{\partial \tilde{x}} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{x}}{\partial y} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{x}}{\partial y} \right)
\]

\[
+ 2B_{\tilde{x}\tilde{y}} \left( \left( \frac{\partial^2 \tilde{x}}{\partial x \partial y} \frac{\partial \tilde{x}}{\partial y} - \frac{\partial^2 \tilde{x}}{\partial x^2} \frac{\partial \tilde{x}}{\partial y} \right) \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{x}}{\partial y} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{x}}{\partial y} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{x}}{\partial y} \right)
\]

\[
(3.7)
\]

\[
+ 2B_{\tilde{y}\tilde{y}} \left( \left( \frac{\partial^2 \tilde{y}}{\partial x \partial y} \frac{\partial \tilde{y}}{\partial y} - \frac{\partial^2 \tilde{y}}{\partial x^2} \frac{\partial \tilde{y}}{\partial y} \right) \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{y}}{\partial y} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{y}}{\partial y} \right) \left( \frac{\partial \tilde{y}}{\partial x} \frac{\partial \tilde{y}}{\partial y} \right)
\]

and

\[
B_{\tilde{x}\tilde{x}} = B_{\tilde{x}\tilde{y}} \left( \frac{\partial \tilde{x}}{\partial x} \right)^2 + 2B_{\tilde{y}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial y} \right)^2 + B_{\tilde{y}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial x} \right)^2,
\]

\[
B_{\tilde{x}\tilde{y}} = B_{\tilde{x}\tilde{y}} \left( \frac{\partial \tilde{x}}{\partial x} \right)^2 + B_{\tilde{y}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial y} \right)^2 + B_{\tilde{y}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial x} \right)^2,
\]

\[
B_{\tilde{y}\tilde{y}} = B_{\tilde{x}\tilde{x}} \left( \frac{\partial \tilde{x}}{\partial x} \right)^2 + 2B_{\tilde{x}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial y} \right)^2 + B_{\tilde{y}\tilde{y}} \left( \frac{\partial \tilde{y}}{\partial x} \right)^2.
\]

With respect to the given chart transformation,

\[
\frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) dx \wedge dy + (B_{\tilde{x}\tilde{x}} dx + B_{\tilde{x}\tilde{y}} dy) \wedge \dot{d} \tilde{x} + (B_{\tilde{y}\tilde{x}} dx + B_{\tilde{y}\tilde{y}} dy) \wedge \dot{d} \tilde{y}
\]

\[
= \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) dx \wedge dy + (B_{\tilde{x}\tilde{x}} \dot{d} \tilde{x} + B_{\tilde{x}\tilde{y}} \dot{d} \tilde{y}) \wedge \dot{d} \tilde{x} + (B_{\tilde{y}\tilde{x}} \dot{d} \tilde{x} + B_{\tilde{y}\tilde{y}} \dot{d} \tilde{y}) \wedge \dot{d} \tilde{y},
\]

proving that \( \alpha' \) given by local formula \( 3.4 \) is a \( 2 \)-form on \( \mathbb{R} \times T^1M \). By similar arguments, the same is true for \( \alpha_0 \).

**Lemma 5.** The equation \( \alpha_0 \wedge dt = d\mu_0 \) has always a solution \( \mu_0 \) defined on \( \mathbb{R} \times T^1M \). In a fibered chart \( (V, \psi), \psi = (t, x, y), \) on \( \mathbb{R} \times M \), \( \mu_0 \) is expressed by

\[
\mu_0 = -\left( A_x - \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \dot{y} \right) t dx - \left( A_y - \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) \dot{x} \right) t dy
\]

\[
(3.9)
\]

and the horizontal component \( h\mu_0 \) of \( \mu_0 \), defined on \( \mathbb{R} \times T^2M \), is expressed by formula

\[
h\mu_0 = -(\epsilon_x \dot{x} + \epsilon_y \dot{y}) dt,
\]

where \( \epsilon_x, \epsilon_y \) are given by \( 2.5 \).

**Proof.** Using the Helmholtz conditions \( 2.6 \) and formulas \( 3.7, 3.8 \), the chart transformation shows that \( \mu_0 \) given by \( 3.9 \) defines a \( 1 \)-form on \( \mathbb{R} \times T^1M \). To verify that \( \mu_0 \) satisfies \( \alpha_0 \wedge dt = d\mu_0 \), we proceed by a straightforward calculation in a chart. \( \square \)
Now we analyze the equation $\alpha' = d\mu'$, where $\alpha'$ is given by formula (3.4). To this purpose we define canonical local sections as follows. Let $(V, \psi) = (t, x, y)$, be a fixed fibered chart on $\mathbb{R} \times M$, and $(V^1, \psi^1) = (t, x, y, \dot{x}, \dot{y})$, be the associated chart on $\mathbb{R} \times T^1M$. Put

$$\pi_1^1 (t, x, y, \dot{x}, \dot{y}) = (t, x, y, \dot{y}), \quad s_{1,v}^1 (t, x, y, \dot{y}) = (t, x, y, \nu, \dot{y}),$$

and

$$\pi_2^1 (t, x, y, \dot{y}) = (t, x, y), \quad s_{2,\sigma}^1 (t, x, y) = (t, x, y, \sigma).$$

$\pi_1^1$ maps the chart domain $V_1^1 \subset \mathbb{R} \times T^1M$ onto its open subset $V_1^1 \subset V_1^1$, given by the equation $\dot{x} = 0$, whereas $\pi_2^1$ maps $V_1^1$ onto the chart domain $V$ in $\mathbb{R} \times M$.

Let $K_1$ be a local homotopy operator, acting on (local) differential forms on $V_1^1 \subset \mathbb{R} \times T^1M$ as

$$K_1 \rho = \int_0^{\dot{x}} (\pi_1^1)^* (s_{1,v}^1)^* \left( i_{\pi_2^1} \rho \right) dv,$$

and $K_2$ acts on forms on $V_1^1 \subset V^1$ as

$$K_2 \varphi = \int_0^{\dot{y}} (\pi_2^1)^* (s_{2,\sigma}^1)^* \left( i_{\pi_2^1} \varphi \right) d\sigma,$$

where in formulas (3.11), (3.12), the integration operation is applied on coefficients of the corresponding differential form.

**Theorem 6.** Let $\alpha_x$ be the Lepage equivalent of $\varepsilon$, and $\alpha'$ is 2-form on $T^1M$ given by means of (3.4). If $(V, \psi) = (t, x, y)$, is a fibered chart on $\mathbb{R} \times M$, then

$$\alpha' - \omega = d\kappa,$$

where

$$\omega = \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)_{(x, y, 0, 0)} dx \wedge dy,$$

and

$$\kappa = K_1 \alpha' + (\pi_1^1)^* K_2 \left( (s_{1,1}^1)^* \alpha' \right)$$

$$= - \left( \int_0^{\dot{x}} B_{xx} (x, y, \nu, \dot{y}) \ d\nu + \int_0^{\dot{y}} B_{xy} (x, y, 0, \sigma) \ d\sigma \right) \ dx$$

$$- \left( \int_0^{\dot{x}} B_{xy} (x, y, \nu, \dot{y}) \ d\nu + \int_0^{\dot{y}} B_{yy} (x, y, 0, \sigma) \ d\sigma \right) \ dy.$$

**Proof.** First, we prove the identity

$$\alpha' - (\pi_1^1)^* (s_{1,0}^1)^* \alpha' = d(K_1 \alpha').$$
Using \[ (3.14) \], the left-hand side of \[ (3.10) \] is expressed as
\[
\alpha' - (\pi_1^1)^* (s_{1,0}^1)^* \alpha' = 2 \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right) dx \land dy + (B_{xx}dx + B_{xy}dy) \land dx + (B_{xy}dx + B_{yy}dy) \land dy
\]
and
\[
- \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)_{(x,y,0,0)} dx \land dy - (B_{xy}(x,y,0,0) dx + B_{yy}(x,y,0,0) dy) \land dy.
\]
From the definition of \[ K_1 \] \[ (3.11) \], the right-hand side of \[ (3.10) \] reads
\[
d\left( K_1 \alpha' \right) = d \left( \int_0^\infty (\pi_1^1)^* (s_{1,0}^1)^* \left( i_{\alpha'} \right) dv \right)\]
\[
= \left( \int_0^\infty \left( \frac{\partial B_{xx}}{\partial y} - \frac{\partial B_{xy}}{\partial x} \right)_{(x,y,v,\dot{y})} dv \right) dx \land dy + (B_{xx}dx + B_{xy}dy) \land dx
\]
\[
- \left( \int_0^\infty \frac{\partial B_{xx}}{\partial y} \dot{y} dx \land dy - \left( \int_0^\infty \frac{\partial B_{xy}}{\partial y} \dot{y} \right) dx \land dy \right)
\]
Now, we apply the Helmholtz conditions \[ (2.6) \] to \[ d\left( K_1 \alpha' \right) \]. Namely, the identities \[ (3.9) \], and
\[
\frac{\partial B_{xx}}{\partial y} - \frac{\partial B_{xy}}{\partial x} = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right),
\]
implies that
\[
d\left( K_1 \alpha' \right)
\]
\[
= \frac{1}{2} \left( \int_0^\infty \frac{\partial}{\partial x} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)_{(x,y,v,\dot{y})} dv \right) dx \land dy + (B_{xx}dx + B_{xy}dy) \land dx
\]
\[
- \left( \int_0^\infty \frac{\partial B_{xy}}{\partial x} \dot{y} dx \land dy - \left( \int_0^\infty \frac{\partial B_{yy}}{\partial x} \dot{y} \right) dx \land dy \right)
\]
\[
= \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)_{(x,y,0,0)} dx \land dy - \frac{1}{2} \left( \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} \right)_{(x,y,0,0)} dx \land dy
\]
\[
+ (B_{xx}dx + B_{xy}dy) \land dx + (B_{xy}dx + B_{yy}dy) \land dy
\]
\[
- (B_{xy}(x,y,0,0) dx + B_{yy}(x,y,0,0) dy) \land dy,
\]
as required to show \[ (3.10) \].

By similar arguments we observe that the following formula holds
\[
(s_{1,0}^1)^* \alpha' = (\pi_1^1)^* (s_{1,0}^1)^* (s_{1,0}^1)^* \alpha' + d \left( K_2 \left( (s_{1,0}^1)^* \alpha' \right) \right).
\]
Hence
\[
(\pi_1^1)^* (s_{1,0}^1)^* \alpha' = (s_{1,0}^1 \circ s_{1,0}^1 \circ \pi_1^1)^* \alpha' + d \left( (\pi_1^1)^* K_2 \left( (s_{1,0}^1)^* \alpha' \right) \right)
\]
\[
= \omega + d \left( (\pi_1^1)^* K_2 \left( (s_{1,0}^1)^* \alpha' \right) \right)
\]
and substituting this formula into \[ (3.10) \], we get the identity \[ (3.13) \].
Remark 7. The identity (3.13) is formulated by Theorem 8 in an arbitrary chart. By means of chart transformations, we show that this formula holds also globally. However, we emphasize that even if \( \omega (\lambda) \) defines a differential form on \( T^1M \), this need not be longer true in general for a solution \( \kappa \) of the equation (3.13). The well-known example of a differential form with an analogous property is the Cartan form \( \Theta \), which depends on the choice of a Lagrangian \( \lambda \) whereas \( d\Theta \) does not.

Theorem 8. Both \( \kappa (\lambda) \) and \( \omega (\lambda) \) define (global) differential 1-forms on \( T^1M \).

Proof. As for the form \( \omega \), the transformation property (3.14) shows that (3.14) defines a 2-form on \( T^1M \subset \mathbb{R} \times T^1M \). We now prove that also \( \kappa \), given in an arbitrary fibered chart by means of the formula (3.13), is a global form. To this purpose, consider two overlapping fibered charts on \( T^1M \) with coordinates functions \((x, y, \dot{x}, \dot{y})\) and \((\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}})\), and the coordinate transformation \( \Psi \circ \Psi^{-1} (\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}) = (x (\bar{x}, \bar{y}), y (\bar{x}, \bar{y}), \dot{x} (\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}), \dot{y} (\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}})) \). Putting

\[
\begin{align*}
f (x, y, \dot{x}, \dot{y}) &= -\int_0^\lambda B_{xx} (x, y, \nu, \dot{\nu}) \, d\nu - \int_0^\lambda B_{xy} (x, y, 0, \sigma) \, d\sigma, \\
g (x, y, \dot{x}, \dot{y}) &= -\int_0^\lambda B_{yx} (x, y, \nu, \dot{\nu}) \, d\nu - \int_0^\lambda B_{yy} (x, y, 0, \sigma) \, d\sigma,
\end{align*}
\]

we get an expression of \( \kappa \) by means of the coordinates \((x, y, \dot{x}, \dot{y})\) as

\[
(3.17) \quad \kappa = f (x, y, \dot{x}, \dot{y}) \, dx + g (x, y, \dot{x}, \dot{y}) \, dy.
\]

Using the change of variables theorem for integrals, formulas (3.8), and the transformation described by

\[
(\Psi \circ \Psi^{-1}) (x, y, \nu, \dot{\nu}) = (\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}), \quad \dot{\bar{x}} = \dot{x} (x, y), \quad \dot{\bar{y}} = \dot{y} (x, y),
\]

\[
\begin{align*}
\hat{\nu} &= \frac{\partial \bar{x}}{\partial x} \nu + \frac{\partial \bar{x}}{\partial y} \dot{\nu}, \quad \hat{\mu} = \frac{\partial \bar{y}}{\partial x} \nu + \frac{\partial \bar{y}}{\partial y} \dot{\nu}, \\

\begin{align*}
(\Psi \circ \Psi^{-1}) (x, y, 0, \sigma) &= (\bar{x}, \bar{y}, \bar{\sigma}, \bar{\sigma}), \\
\bar{\sigma} &= \frac{\partial \bar{y}}{\partial y} \sigma, \quad 0 = \frac{\partial \bar{x}}{\partial \tau} \bar{\sigma} + \frac{\partial \bar{x}}{\partial y} \bar{\sigma}, \quad \sigma = \frac{\partial \bar{y}}{\partial \tau} \bar{\sigma} + \frac{\partial \bar{y}}{\partial y} \bar{\sigma},
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial y}{\partial \bar{x}} \, d\sigma + \frac{\partial y}{\partial \bar{y}} \, d\bar{\sigma} &= 0, \quad \frac{\partial x}{\partial \bar{\tau}} \, \bar{\sigma} + \frac{\partial x}{\partial \bar{y}} \, \bar{\sigma} = 0,
\end{align*}
\]

the integral (3.14) over segments transforms into the line integral

\[
(\Psi \circ \Psi^{-1})^* (f (x, y, \dot{x}, \dot{y}) \, dx + g (x, y, \dot{x}, \dot{y}) \, dy)
\]

\[
= -\int_{\hat{\nu}=\bar{\mu}=\bar{\sigma}} (B_{\bar{x}\bar{x}} (\bar{x}, \bar{y}, \hat{\nu}, \hat{\mu}) \, d\nu + B_{\bar{x}\bar{y}} (\bar{x}, \bar{y}, \hat{\nu}, \hat{\mu}) \, d\mu) \cdot d\bar{x}
\]

\[
- \int_{\hat{\nu}=\bar{\mu}=\bar{\sigma}} (B_{\bar{x}\bar{x}} (\bar{x}, \bar{y}, \nu, \mu) \, d\nu + B_{\bar{x}\bar{y}} (\bar{x}, \bar{y}, \nu, \mu) \, d\mu) \cdot d\bar{y},
\]

whereas

\[
\begin{align*}
\kappa &= f (x, y, \dot{x}, \dot{y}) \, dx + g (x, y, \dot{x}, \dot{y}) \, dy, \\
\end{align*}
\]
Now we observe that both summands in the previous expression of \((\Psi \circ \bar{\Psi}^{-1})^\ast \kappa\) represent line integrals that are independent upon the choice of a path as the integrands satisfy the Helmholtz conditions \((3.5)\), cf. Theorem 1. To this purpose we may consider, without loss of generality, rectangular charts. Hence 
\[
(\Psi \circ \bar{\Psi}^{-1})^\ast \left( f(x, y, \dot{x}, \dot{y}) \, dx + g(x, y, \dot{x}, \dot{y}) \, dy \right)
\]
\[
= - \int_0^\hat{\bar{\gamma}} B_{\bar{x}\bar{y}}(\bar{x}, \bar{y}, \nu, \hat{\bar{\gamma}}) \, d\nu \cdot d\bar{x} - \int_0^\hat{\bar{\gamma}} B_{\bar{y}\bar{x}}(\bar{x}, \bar{y}, 0, \hat{\bar{\gamma}}) \, d\mu \cdot d\bar{x}
\]
\[
- \int_0^\hat{\bar{\gamma}} B_{\bar{x}\bar{y}}(\bar{x}, \bar{y}, \nu, \hat{\bar{\gamma}}) \, d\nu \cdot d\bar{x} - \int_0^\hat{\bar{\gamma}} B_{\bar{y}\bar{x}}(\bar{x}, \bar{y}, 0, \hat{\bar{\gamma}}) \, d\mu \cdot d\bar{x}
\]
\[
= f(\bar{x}, \bar{y}, \dot{x}, \dot{y}) \, dx + g(\bar{x}, \bar{y}, \dot{x}, \dot{y}) \, dy,
\]
as required.

\[\square\]

Remark 9. Theorems 9 and 8 show, according to our expectation from the cohomology results of the global variational theory by Takens [19] and others, that the global exactness of Lepage equivalent \(\alpha\) on \(J^2(\mathbb{R} \times M)\) reduces to global exactness of a 2-form defined on \(M\).

Corollary 10. If the 2-form \(\omega\) \((3.14)\) vanishes, i.e. if the coefficients of \(\omega\) satisfy 
\[
\left( \frac{\partial A_x}{\partial \dot{y}} - \frac{\partial A_y}{\partial \dot{x}} \right)_{(x,y,0,0)} = 0
\]
in every chart, then source form \(\varepsilon\) is globally variational and it admits a Lagrangian on \(\mathbb{R} \times T^1 M\), namely 
\[
\lambda = h(\mu_0 + \kappa),
\]
where \(\mu_0\) and \(\kappa\) are given by \((3.9)\) and \((3.15)\), respectively.

Proof. This is an immediate consequence of Theorem 6 and Lemma 5. \[\square\]

4. A GLOBAL CONSTRUCTION ON 2-MANIFOLDS: TOPO-COHOMOLOGY

Recall now two theorems, characterizing the top de Rham cohomology groups of connected smooth manifolds, that is \(H^m_{dR} M\), where \(\dim M = m\). Our main reference is Lee [16].

Theorem 11 (Orientable, top-cohomology). Let \(M\) be a connected orientable smooth \(m\)-manifold.

(a) If \(M\) is compact, then \(H^m_{dR} M\) is one-dimensional and is spanned by the cohomology class of any smooth orientation form.

(b) If \(M\) is noncompact, then \(H^m_{dR} M = 0\).

Theorem 12 (Nonorientable, top-cohomology). Let \(M\) be a connected nonorientable smooth \(n\)-manifold. Then \(H^m_{dR} M = 0\).

Remark 13. In other words, Theorems 11 and 12 characterize manifolds with trivial de Rham top-cohomology groups as follows: If a connected smooth \(m\)-manifold \(M\) obeys \(H^m_{dR} M = 0\), then \(M\) is either nonorientable, or orientable and noncompact. Moreover, the proof of Theorem 11 (b), and also of Theorem 12, is constructive.
Remark 14. In general, if $M$ is an $m$-dimensional smooth manifold and $\rho$ is a closed differential $k$-form on $M$, $k \leq m$, then the equation $\rho = d\mu$ need not have a (global) solution $\mu$ on $X$. Indeed, it is the $k$-th de Rham cohomology group $H^k_{\text{dR}}(M)$ which decides about solvability of the exactness equation $\rho = d\mu$. Clearly, if $H^k_{\text{dR}}(M)$ is trivial, then $\rho = d\mu$ has always a solution $\mu$ on $X$. Nevertheless, in this case ($H^k_{\text{dR}}(M) = 0$) there is no general constructive procedure of finding a solution $\mu$ for a given closed $k$-form $\rho$, where $k < m$; if $k = m$, to find a solution one can apply the top-cohomology theorems.

Our problem of solving the exactness equation $\alpha_x = d\mu$ (4.1) globally concerns a given 2-form $\alpha_x$ on $\mathbb{R} \times T^1M$. However, Theorem 6 and Lemma 5 reduce this problem to a 2-form, which is defined on $M$. Indeed, applying formulas (3.9) and (3.13) we obtain

$$\alpha_x = \alpha_0 + \alpha' = \omega + d(\mu_0 + \kappa),$$

where $\mu_0$ (3.9) and $\kappa$ (3.15) are 1-forms on $\mathbb{R} \times T^1M$, and $\omega$ (3.14) is a 2-form on $M$.

Following Lee [16], for orientable noncompact manifolds we briefly describe construction of a solution of the exactness equation for $\omega$,

$$\omega = d\eta.$$

To this purpose, recall the Poincaré lemma for compactly supported forms.

Lemma 15. Let $\rho$ be a compactly supported closed $k$-form on $\mathbb{R}^m$, where $1 \leq k \leq n$. If $k = m$, suppose in addition that $\int_{\mathbb{R}^m} \rho = 0$. Then there exists a compactly supported $(k-1)$-form $\vartheta$ on $\mathbb{R}^m$ such that $d\vartheta = \rho$.

Let us consider the case when $M$ is a connected orientable noncompact smooth 2-manifold (Theorem 11, (b)). The solution $\eta$ of (4.1) is determined by means of an appropriate covering of $M$.

Lemma 16. If $M$ is a noncompact connected manifold, then there exists countable, locally finite open cover $\{V_j\}$ of $M$ such that each $V_j$ is connected and precompact, and for each $j$, there exists an index $k > j$ such that $V_j \cap V_k \neq \emptyset$.

Proof. See Lee [16], Errata 2018.

Consider $\{V_j\}$ satisfying Lemma 16. For each $j$, denote $K(j)$ the least index $k > j$ such that $V_j \cap V_k \neq \emptyset$. Let $\theta_j$ be a 2-form, compactly supported in $V_j \cap V_{K(j)}$ such that $\int_{M} \theta_j = 1$. Let $\{\psi_j\}$ be a smooth partition of unity subordinate to $\{V_j\}$. For each $j$, put $\omega_j = \psi_j \omega$. Put $c_1 = \int_{V_1} \omega_1$. Then the 2-form $\omega_1 - c_1 \theta_1$ is compactly supported in $V_1$, and $\int_{M} (\omega_1 - c_1 \theta_1) = 0$. With loss of generality, we may assume that $V_j$ is star-shaped for every $j$. By Lemma 15 there exists a compactly supported 1-form $\eta_1$ on $V_1$ such that $d\eta_1 = \omega_1 - c_1 \theta_1$. Further, we proceed by induction as follows. Suppose we have compactly supported 1-forms $\eta_j$ on $V_j$ and constants $c_j$, where $1 \leq j \leq m$, such that

$$d\eta_j = \left(\omega_j + \sum_{i: K(i) = j} c_i \theta_i\right) - c_j \theta_j.$$

Let

$$c_{j+1} = \int_{V_{j+1}} \left(\omega_{j+1} + \sum_{i: K(i) = j+1} c_i \theta_i\right).$$

(4.2)
Then

\[
\int_{V_{j+1}} \left( \left( \omega_{j+1} + \sum_{i: K(i) = j+1} c_i \theta_i \right) - c_{j+1} \theta_{j+1} \right) = 0,
\]

and Lemma 15 assures existence of a compactly supported 1-form \( \eta_{j+1} \) on \( V_{j+1} \) which satisfies formula (4.2) with \( j \) replaced by \( j + 1 \).

Now, extending each \( \eta_j \) on \( M \setminus V_j \) to be zero, we set

\[
\eta = \sum_j \eta_j.
\]

Since the open covering \( \{V_j\} \) is locally finite, formula (4.3) is correct and defines a differential 1-form on \( M \) such that \( d\eta = \omega \), as required.

The preceding procedure of solving the exactness equation (4.1), together with Lemma 4 and Theorems 6, 8, imply our main result, the following application of the de Rham top-cohomology in the theory of variational equations.

**Theorem 17.** Let \( M \) be a connected smooth 2-manifold, and let \( \varepsilon \) be a locally variational source form defined on \( \mathbb{R} \times T^2 M \). If \( H^2_{dR}M = 0 \), then \( \varepsilon \) is globally variational and admits a global Lagragian

\[
\lambda = h (\mu_0 + \kappa + \eta),
\]

where 1-form \( \eta \) is a solution of equation (4.1) on \( M \), \( \mu_0 \) and \( \kappa \) are given by formulas (3.9) and (3.15), respectively.

5. **Examples**

We analyze two examples of globally variational source forms on \( \mathbb{R} \times T^2 M \) over \( \mathbb{R} \) (one-dimensional basis), where \( M \) is two-dimensional smooth manifold, the open Möbius strip \( M_{r,a} \) and the punctured torus \( T^2 P \), both with trivial the second de Rham cohomology group, \( H^2_{dR}M = 0 \). Another classical examples of smooth manifolds with this property include namely the punctured plane \( \mathbb{R}^2 \setminus \{0\} \), and the Klein bottle \( K \).

**Example 18** (Kinetic energy on \( M_{r,a} \)). Consider the open subset \( W \subset \mathbb{R}^4 \) of the Euclidean space, where \( W = \mathbb{R} \times (\mathbb{R}^3 \setminus \{(0,0,z)\}) \), endowed with its open submanifold structure and global Cartesian coordinates \((t,x,y,z)\). We introduce an atlas on \( W \), adapted to fibered open Möbius strip \( \mathbb{R} \times M_{r,a} \) of radius \( r \) and width \( 2a \), \( 0 < a < r \). Let \( V \) and \( \bar{V} \) be an open covering of \( W \), given by

\[
V = \mathbb{R} \times (\mathbb{R}^3 \setminus ((-\infty,0] \times \{0\} \times \mathbb{R})), \quad \bar{V} = \mathbb{R} \times (\mathbb{R}^3 \setminus ([0,\infty) \times \{0\} \times \mathbb{R})),
\]

and define coordinate functions \((t,\varphi,\tau,\vartheta)\) on \( V \) by \( t = t \),

\[
\varphi = \text{atan2}(y,x),
\]

\[
\tau = \frac{1}{\sqrt{2}} \left( \sqrt{x^2 + y^2} - r \right) \sqrt{1 + \frac{x}{\sqrt{x^2 + y^2}}} + \frac{1}{\sqrt{2}} \text{sgn}(y)z \sqrt{1 - \frac{x}{\sqrt{x^2 + y^2}}},
\]

\[
\vartheta = -\frac{1}{\sqrt{2}} \left( \sqrt{x^2 + y^2} - r \right) \text{sgn}(y) \sqrt{1 - \frac{x}{\sqrt{x^2 + y^2}}} + \frac{1}{\sqrt{2}} z \sqrt{1 + \frac{x}{\sqrt{x^2 + y^2}}}.
\]
and \((\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\theta})\) on \(\bar{V}\) by \(\bar{t} = t\), and

\[
\bar{\varphi} = \begin{cases} 
\arctan(2)(y, x), & y \geq 0, \\
\arctan(2)(y, x) + 2\pi, & y < 0,
\end{cases}
\]

where \(\arctan(2)(y, x)\) is the arctangent function with two arguments. One can directly check that the pairs \((V, \Psi), \Psi = (t, \varphi, \tau, \vartheta)\), and \((\bar{V}, \bar{\Psi}), \bar{\Psi} = (\bar{t}, \bar{\varphi}, \bar{\tau}, \bar{\theta})\), are charts on \(W\) adapted to \(\mathbb{R} \times M_{r,a}\), constituting a smooth atlas on \(W\). In the chart \((V, \Psi)\) (resp. \((\bar{V}, \bar{\Psi})\)), \(\mathbb{R} \times M_{r,a}\) has the equation \(\vartheta = 0\) with \(-a < \tau < a\) (resp. \(\bar{\theta} = 0\) with \(-a < \bar{\tau} < a\)). On the intersection \(V \cap \bar{V}\), the chart transformations between \((V, \Psi)\) and \((\bar{V}, \bar{\Psi})\) is given by

\[
\Psi \circ \bar{\Psi}^{-1} : \bar{\Psi}(\bar{V}) \setminus \{\bar{\varphi} = \pi\} \to \Psi(V) \setminus \{\varphi = 0\},
\]

where \(\epsilon\) be a source form on \(\mathbb{R} \times T^1M_{r,a}\), locally expressed by

\[
\epsilon = \epsilon_{\varphi} \omega^\varphi \wedge dt + \epsilon_{\tau} \omega^\tau \wedge dt,
\]

where \(\epsilon_{\varphi} = A_{\varphi} + B_{\varphi \tau} \dot{\varphi}\), \(\epsilon_{\tau} = A_{\tau} + B_{\tau \tau} \dot{\tau}\), and

\[
B_{\varphi \tau} = -\left(\frac{\left(r + \tau \cos \frac{\varphi}{2}\right)^2}{2} + \frac{\tau^2}{4}\right), \quad B_{\tau \tau} = -1,
\]

\[
A_{\varphi} = \frac{1}{2} \varphi^2 \tau \sin \frac{\varphi}{2} \left(r + \tau \cos \frac{\varphi}{2}\right) - \frac{1}{4} \dot{\varphi}^2 \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right),
\]

\[
A_{\tau} = \frac{1}{4} \varphi^2 \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right).
\]

Using the chart transformation \((\bar{V}, \bar{\Psi})\), it is easy to verify that \(\epsilon\) defines a 2-form on \(\mathbb{R} \times T^2M_{r,a}\). Since

\[
\left(\frac{\partial A_{\varphi}}{\partial \tau} - \frac{\partial A_{\tau}}{\partial \varphi}\right)_{(\varphi, \tau, 0, 0)} = -\left(-\dot{\varphi} \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right)\right)_{(\varphi, \tau, 0, 0)} = 0,
\]

Corollary \(\text{[10]}\) implies that \(\epsilon\) admits a global Lagrangian

\[
\lambda = h \left(\mu_0 + \kappa\right),
\]

where \(\mu_0\) \((\text{[3.9]})\) reads

\[
\mu_0 = -\left(A_{\varphi} + \frac{1}{2} \dot{\varphi}^2 \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right)\right) t d\varphi
\]

\[
- \left(A_{\tau} - \frac{1}{2} \varphi^2 \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right)\right) t d\tau
\]

\[
- B_{\varphi \tau} \dot{\varphi} t d\varphi - B_{\tau \tau} \dot{\tau} t d\tau,
\]

and \(\kappa\) \((\text{[3.14]}\)) is given by

\[
\kappa = K_1 \alpha' + \left(\pi_1^1\right)^s K_2 \left(s_{1,0}^1\right)^s \alpha',
\]

where from \((\text{[3.4]}\),

\[
\alpha' = -\frac{1}{2} \dot{\varphi} \left(4 \cos \frac{\varphi}{2}\left(r + \tau \cos \frac{\varphi}{2}\right) + \tau\right) d\varphi \wedge d\tau + B_{\varphi \varphi} d\varphi \wedge d\varphi + B_{\tau \tau} d\tau \wedge d\tau.
\]
From \[3.11, \ 3.12\], we get
\[
\kappa = \left( \left( r + \tau \cos \frac{\varphi}{2} \right)^2 + \frac{\tau^2}{4} \right) \phi \, d\varphi + \tau \, dt.
\]
Hence a global Lagrangian on \( \mathbb{R} \times T^2 M_{r,a} \) reads
\[
\lambda = \mathcal{L} \, dt,
\]
where \( \mathcal{L} : \mathbb{R} \times T^2 M_{r,a} \to \mathbb{R} \) is the Lagrange function, given by
\[
\mathcal{L} = -\frac{1}{2} \phi^2 \tau \sin \frac{\varphi}{2} \left( r + \tau \cos \frac{\varphi}{2} \right) \dot{\varphi} \, t + \frac{1}{4} \phi^2 \left( 4 \cos \frac{\varphi}{2} \left( r + \tau \cos \frac{\varphi}{2} \right) + \tau \right) \dot{\tau} + \left( r + \tau \cos \frac{\varphi}{2} \right)^2 + \frac{\tau^2}{4} \phi^2 \, \dot{\phi}^2 + \dot{\tau}^2.
\]
Note that \( \varepsilon \) admits also the kinetic energy Lagrangian \( \mathcal{L}_{\text{kin}} \, dt \), given by a global function on \( \mathbb{R} \times T^1 M_{r,a} \),
\[
\mathcal{L}_{\text{kin}} = \frac{1}{2} \left( \dot{\tau}^2 + \left( r + \tau \cos \frac{\varphi}{2} \right)^2 + \frac{\tau^2}{4} \right) \phi^2,
\]
which arises as a pull-back of the standard kinetic Lagrangian on \( J^1 (\mathbb{R} \times \mathbb{R}^3) \) with respect to the canonical embedding of \( M_{r,a} \) into the Euclidean space \( \mathbb{R}^3 \), and which is equivalent to \( \lambda \) (i.e. the associated Euler-Lagrange forms coincide).

**Example 19** (Gyroscopic equations on punctured torus). In this example, we study gyroscopic type equations on the punctured torus \( T_P \). The torus \( T = S^1_R \times S^1_r \) (Cartesian products of circles of radius \( R \) and \( r \), respectively) in the Euclidean space \( \mathbb{R}^3 \) is endowed with its smooth manifold structure as the Cartesian product of smooth structures on \( S^1 \), and the punctured torus \( T_P = T \setminus \{(R + r, 0, 0)\} \) has the open submanifold structure. The parametric equations of \( T \subset \mathbb{R}^3 \) reads
\[
x = (R + r \cos \vartheta) \cos \varphi, \quad y = (R + r \cos \vartheta) \sin \varphi, \quad z = r \sin \vartheta,
\]
where \( 0 < r < R, \varphi \in [0, 2\pi) \), and \( \vartheta \in [0, 2\pi) \). The point \( P \) with Euclidean coordinates \( (R + r, 0, 0) \) arises for \( \varphi = 0 = \vartheta \). A smooth atlas on \( T_P \) can be chosen as the following four charts, \( (U_{\varphi\vartheta}, \Phi_{\varphi\vartheta}), \Phi_{\varphi\vartheta} = (\varphi, \vartheta) \), where \( \varphi \) and \( \vartheta \) are the angle coordinates on \( S^1_R \) and \( S^1_r \), respectively, and \( (U_{\varphi\vartheta}, \Phi_{\varphi\vartheta}), \Phi_{\varphi\vartheta} = (\hat{\varphi}, \hat{\vartheta}), (U_{\varphi\vartheta}, \Phi_{\varphi\vartheta}), \Phi_{\varphi\vartheta} = (\check{\varphi}, \check{\vartheta}), (U_{\varphi\vartheta}, \Phi_{\varphi\vartheta}), \Phi_{\varphi\vartheta} = (\bar{\varphi}, \bar{\vartheta}) \), where \( -\pi < \varphi, \vartheta < \pi \), and \( 0 < \hat{\varphi}, \hat{\vartheta} < 2\pi \).

Consider the system
\[
(5.2) \quad \varepsilon_{\varphi} = A_\varphi + B_{\varphi\varphi} \check{\varphi} + B_{\varphi\vartheta} \check{\vartheta}, \quad \varepsilon_{\vartheta} = A_\vartheta + B_{\varphi\vartheta} \check{\varphi} + B_{\vartheta\vartheta} \check{\vartheta},
\]
where \( B_{\varphi\varphi} = (R + r \cos \vartheta)^2, B_{\varphi\vartheta} = r^2, B_{\vartheta\vartheta} = 0 \), and
\[
A_\varphi = -r (R + r \cos \vartheta) \left( 2 \check{\varphi} \sin \vartheta + a \sin \vartheta - b \sin \varphi \cos \vartheta + c \cos \varphi \cos \vartheta \right) \check{\vartheta},
\]
\[
A_\vartheta = r (R + r \cos \vartheta) \left( \check{\varphi} \sin \vartheta + a \sin \vartheta - b \sin \varphi \cos \vartheta + c \cos \varphi \cos \vartheta \right) \check{\varphi},
\]
and \( a, b, c \) are some functions depending on \( \varphi, \vartheta \). System \( (5.2) \) defines a differential form \( \varepsilon = (\varepsilon_{\varphi} \omega^2 + \varepsilon_{\vartheta} \omega^3) \wedge dt \) on \( \mathbb{R} \times T^2 T_P \) globally, and arises as the pull-back of the gyroscopic type system in the Euclidean space \( \mathbb{R} \times \mathbb{R}^3 \),
\[
\varepsilon = ay + bz, \quad \varepsilon = -ax + cz, \quad \varepsilon = -bx - cy.
\]
It is straightforward to verify with the help of Theorem 11 \([2.0]\), that \((5.2)\) is locally variational (cf. Krupka, Urban, and Volná \([11]\)). Theorem 17 implies that \((5.2)\) is also globally variational and admits a global Lagrangian of the form

\[
\lambda = h(\mu_0 + \kappa + \eta),
\]

where \(\mu_0\) and \(\kappa\) are given by \((3.9)\) and \((3.13)\), and \(\eta\) is a solution of the equation

\[
\omega = d\eta \text{ on } T_P, \text{ where } 2\text{-form } \omega \text{ is given by } (3.14).
\]

Thus, we have

\[
\begin{align*}
\mu_0 &= r(R + r \cos \vartheta) \varphi^2 \sin \vartheta dt d\vartheta - (R + r \cos \vartheta)^2 \varphi t d\varphi - r^2 \dot{\vartheta} d\vartheta, \\
\kappa &= -(R + r \cos \vartheta)^2 \varphi d\varphi - r^2 \dot{\vartheta} d\vartheta,
\end{align*}
\]

and

\[
\omega = -r(R + r \cos \vartheta)(a \sin \vartheta - b \sin \varphi \cos \vartheta + c \cos \varphi \cos \vartheta) d\varphi \wedge d\vartheta.
\]

Clearly, for a particular choice of the functions \(a, b, c\), for instance \(a = \cos \vartheta \sin \varphi, b = \sin \vartheta + \cos \varphi, c = \sin \varphi\), the 2-form \(\omega\) vanishes, hence by Corollary 10 we get a global Lagrangian \(\lambda = h(\mu_0 + \kappa)\).

In case that the 2-form \(\omega\) on \(T_P\) does not vanish, we may proceed as described in Section 4 to construct a solution \(\eta = d\eta\), defined on \(T_P\). To this purpose, for instance, one may consider an open *star-shaped* cover \(\{V_j\}\) of \(T_P\) satisfying conditions of Lemma 16 such that in addition each \(V_j\) is a preimage of rectangles with respect to coordinate mappings and obeys the condition \(V_j \cap V_{j+1} \neq \emptyset\); e.g. \(\{V_j\}\) arises as a refinement of an open cover \(\{\tilde{V}_k\}\) of \(T_P\), where \(\tilde{V}_0 = T_P \setminus (cD_2)\), \(\tilde{V}_k = D_k \setminus (cD_{k+2})\), \(k = 1, 2, \ldots\).

According to the proof of Theorem 12 (Lee \([16]\); cf. Section 4), let \(\{\psi_j\}\) be a smooth partition of unity, subordinate to this cover, and \(\theta_j\) be a 2-form compactly supported in \(V_j \cap V_{j+1}\) for each \(j\) (obtained with the help of a smooth bump function). For every \(j\), we put \(\omega_j = \psi_j \omega, c_1 = \int_{V_j} \omega_1, c_{j+1} = \int_{V_{j+1}} \omega_{j+1} + c_j \theta_j\). Then there exist a compactly supported 1-form \(\eta_1\) such that \(d\eta_1 = \omega_1 - c_1 \theta_1\), and for each \(j\), a compactly supported 1-form \(\eta_j\) such that \(d\eta_j = \omega_{j+1} + c_j \theta_j - c_{j+1} \theta_j\).

1-form \(\eta = \sum_j \eta_j\) is then a solution of \(\omega = d\eta\) on \(T_P\).

Note that for constant functions \(a, b,\) and \(c\), we get a global Lagrangian \((5.3)\), where

\[
\eta = -r(R + r \cos \vartheta) \cos \vartheta (b \cos \varphi + c \sin \varphi) d\vartheta - r \left(aR \cos \vartheta + \frac{1}{4} a \cos 2\vartheta\right) d\varphi.
\]

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