Combinatorics

Synteties on plabic graphs and associated polytopes

Symétries dans les graphes plan biclores et les polytopes associés

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\textbf{A B S T R A C T}

For Grassmann varieties, we explain how the duality between the Gelfand–Tsetlin polytopes and the Feigin–Fourier–Littelmann–Vinberg polytopes arises from different positive structures.
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\textbf{R É S U M É}

Nous expliquons, pour les variétés grasmanniennes, comment la dualité entre les polytopes de Gelfand–Tsetlin et les polytopes de Feigin–Fourier–Littelman–Vinberg émerge dans différentes structures positives.
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\section{Introduction}

Plabic graphs (planar bicoloured graphs) were introduced by Postnikov \cite{Post} to parametrize cells in the totally non-negative (TTN) Grassmannians \((\text{Gr}_{k,n}(\mathbb{R}))_{\geq 0}\). These graphs are drawn inside a disk with boundary vertices labelled by \(1, 2, \ldots, n\) in a fixed orientation and internal vertices coloured black and white. For a reduced plabic graph \(\mathcal{G}\) corresponding to the top cell in the TTN-Grassmannian \((\text{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}\), Rietsch and Williams \cite{RW} constructed a family of polytopes for positive integers \(r\) as Newton–Okounkov bodies \[5,7\] associated with the line bundle \(r \in \mathbb{Z} \cong \text{Pic}(\text{Gr}_{n-k,n}(\mathbb{C})).\]

When the plabic graph \(\mathcal{G} := \text{Gr}_{k,n}^{\text{rec}}\) is chosen as in \cite{RW} (see Section 4.2), the corresponding Newton–Okounkov body \(\text{NO}_{\mathcal{G}}\) is unimodularly equivalent to the Gelfand–Tsetlin polytope \(\text{GT}_{k,n}^1\).

The Newton–Okounkov body is by definition a closed convex hull of points; even when it is a polytope, to read off its defining inequalities is a hard problem. In \cite{RW}, the authors used mirror symmetry of Grassmannians to obtain these inequalities from the tropicalization of the super-potential on an open set of the mirror Grassmannian arising from the Landau–Ginzburg model. By applying this symmetry, they give explicit defining inequalities of \(\text{NO}_{\mathcal{G}}\).

Lattice points in Gelfand–Tsetlin polytopes parametrize the bases of finite-dimensional irreducible representations of the Lie algebra \(\mathfrak{sl}_n\). Motivated by a conjecture of Vinberg, another family of polytopes, called FFLV polytopes, is found by Feigin,
the second author, and Littelmann [3], whose lattice points also parametrize the bases of finite-dimensional irreducible representations of $s_{kn}$.

For a plabic graph $\mathcal{G}$, its mirror $\mathcal{G}^\vee$ is defined by swapping the black/white colouring of internal vertices in $\mathcal{G}$. When the plabic graph $\mathcal{G}$ corresponds to the top cell in $(Gr_{n-k,n}(\mathbb{R}))_{\geq 0}$, $\mathcal{G}^\vee$ parametrizes the top cell in $(Gr_{k,n}(\mathbb{R}))_{\geq 0}$.

**Theorem 1.** The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to $FFLV_{k,n}^1$ (see Section 4.1 for definition).

Another way to relate Gelfand–Tsetlin polytopes to FFLV polytopes is via a connection between the corresponding clusters in different cluster algebras. Each reduced plabic graph $\mathcal{G}$ gives a cluster $\mathcal{C}$ consisting of Plücker coordinates $\Delta_{I_1}, \ldots, \Delta_{I_m}$ where $I_1, \ldots, I_m$ are some $(n-k)$-element subsets of $[n] = \{1, 2, \ldots, n\}$.

For $I \subset [n]$, let $I^c$ denote its complement. Then the set $\mathcal{C}' = \{\Delta_{I_1}, \ldots, \Delta_{I_m}\}$ is a cluster for $Gr_{k,n}(\mathbb{C})$, corresponding to a plabic graph $\mathcal{G}^\vee$.

**Corollary 1.** The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to $FFLV_{k,n}^1$.

2. Plabic graphs

We recall the definition and basic properties of plabic graphs, following [8,10].

**Definition 1.** A plabic graph is an undirected planar graph $\mathcal{G}$ satisfying:

1. $\mathcal{G}$ is embedded in a closed disk and considered up to homotopy;
2. $\mathcal{G}$ has $n$ vertices on the boundary of the disk, called boundary vertices, which are labelled clockwise by $1, 2, \ldots, n$;
3. all other vertices of $\mathcal{G}$ are strictly inside the disk, they are called internal vertices and coloured in black and white;
4. each boundary vertex is incident to a single edge.

In [8] (see also [10]), there are three local moves defined on plabic graphs: gluing two vertices of the same colour, removing redundant vertices, and mutating a square. For a plabic graph $\mathcal{G}$, let $\mathcal{F}(\mathcal{G})$ denote the set of its faces, which is invariant under the local moves.

**Definition 2.** A plabic graph $\mathcal{G}$ is called reduced if there are no parallel edges $\mathcal{G}$ after applying any sequences of local moves.

**Definition 3.** Let $\mathcal{G}$ be a reduced plabic graph. The trip $T_i$ starting from a boundary vertex $i$ is the path going through the edges of $\mathcal{G}$, obeying the following rules:

1. at each internal black vertex, the path turns to the rightmost direction;
2. at each internal white vertex, the path turns to the leftmost direction.

The trip $T_i$ ends at a boundary vertex $\pi(i)$. We associate in this way a trip permutation $\pi_\mathcal{G} := (\pi(1), \ldots, \pi(n))$ with $\mathcal{G}$. Let $\pi_{k,n} = (n-k+1, n-k+2, \ldots, n, 1, 2, \ldots, n-k)$. The face labelling of $\mathcal{G}$ is the injective map $\lambda_{\mathcal{G}} : \mathcal{F}(\mathcal{G}) \rightarrow \binom{[n]}{k}$ (the set of $k$-element subsets of $\{1, \ldots, n\}$) defined as follows: for a face $F \in \mathcal{F}(\mathcal{G})$, $\lambda_{\mathcal{G}}(F)$ consists of those $i$ such that $F$ is to the left of the trip $T_i$. We set $V_{\mathcal{G}} := \lambda_{\mathcal{G}}(\mathcal{F}(\mathcal{G}))$.

See Fig. 1 for an example.

3. Polytopes arising from plabic graphs

We associate polytopes with plabic graphs following [10]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the base field.

3.1. Positive Grassmannians

For $0 < k < n$, let $Mat_{k,n}$ denote the set of $k \times n$-matrices with entries in $\mathbb{K}$. For $J \in \binom{[n]}{k}$ and $A \in Mat_{k,n}$, let $\Delta_{J}(A)$ denote the maximal minor of $A$ corresponding to columns in $J$.

Let $Gr_{k,n}$ be the Grassmann variety embedded into $\mathbb{P}^{N-1}$ via the Plücker embedding where $N = \binom{n}{k}$. The minors $\{\Delta_{J} | J \in \binom{[n]}{k}\}$ give the Plücker coordinates on $Gr_{k,n}$. When the base field is $\mathbb{R}$, the totally non-negative (resp. totally positive) Grassmannian $(Gr_{k,n}(\mathbb{R}))_{\geq 0}$ consists of those elements in $Gr_{k,n}$ having non-negative (resp. positive) Plücker coordinates.
3.2. Perfect orientations

To study flow models on plabic graphs, we fix a perfect orientation \( \mathcal{O} \) on \( \mathcal{G} \). Such an orientation requires that, at each black (resp. white) internal vertex, there is exactly one edge going out (resp. going in). It is shown in [9] that each reduced plabic graph admits an acyclic perfect orientation. Once such an orientation is fixed, we denote the source set by \( I_{\mathcal{O}} := \{i \in [n] \mid i \) is a boundary source of \( \mathcal{O} \}; its complement \( I_{\mathcal{O}}^\circ \) is the set of boundary sinks.

For \( i \in \binom{[n]}{k} \), let \( x_i \) be a variable. For \( i \in I_{\mathcal{O}} \) and \( j \in I_{\mathcal{O}}^\circ \), let \( \mathcal{P}_{i,j} \) be the set of directed paths from \( i \) to \( j \). For such a directed path \( \gamma \), let \( \mathcal{F}_\gamma(\mathcal{G}) \) denote the set of faces to the left of \( \gamma \). A flow \( \mathfrak{F} \) from \( I_{\mathcal{O}} \) to \( J \in \binom{[n]}{k} \) is a collection of pairwise vertex-disjoint directed paths in \( \mathcal{G} \) going from \( I_{\mathcal{O}} \setminus (I_{\mathcal{O}} \cap J) \) to \( J \setminus (I_{\mathcal{O}} \cap J) \).

For a directed path \( \gamma \in \mathcal{P}_{i,j} \), we define the weight of \( \gamma \) in \( C[x_i \mid i \in \binom{[n]}{k}] \) by:

\[
\text{wt}(\gamma) := \prod_{F \in \mathcal{F}_\gamma(\mathcal{G})} x_{\mathcal{G}}(F).
\]

The weight of a flow is the product of the weights of the paths it contains. For \( J \in \binom{[n]}{k} \), we define \( P_J \) to be the sum of the weights of all flows from \( I_{\mathcal{O}} \) to \( J \).

For a reduced plabic graph \( \mathcal{G} \) of trip permutation \( \pi_{n-k,k} \) with perfect orientation \( \mathcal{O} \), there exists only one face \( F_0 \) to the right of all directed paths with \( \lambda_{\mathcal{G}}(F_0) = [n-k+1, \ldots, n] \). We set \( \mathcal{V}_0^\circ := \mathcal{V}_0 \setminus \{\lambda_{\mathcal{G}}(F_0)\} \), \( \Delta_{\mathcal{G}} := \{x_i \mid i \in \mathcal{V}_0^\circ\} \) and \( \Delta_{\mathcal{G}}^\circ := \{x_i \mid i \in \mathcal{V}_0\} \).

**Theorem 2 ([8,12]):** Let \( \mathbb{X} := G_{n-k,k}(\mathbb{C}) \) and \( \mathbb{C}(\mathbb{X}) \) be the field of rational functions on \( \mathbb{X} \). There exists an isomorphism of fields:

\[
\mathbb{C}(\mathbb{X}) \cong C(x_i \mid x_i \in \Delta_{\mathcal{G}}^\circ), \quad \Delta_J \mapsto P_J.
\]

The choice of the perfect orientation \( \mathcal{O} \) will only change the formula of \( P_J \) by a scalar. We always assume that the choice \( I_{\mathcal{O}} = \{1, 2, \ldots, k\} \) is made.

Let \( \prec \) be a total order on \( \Delta_{\mathcal{G}} \). It induces a term order \( \prec \) on monomials in \( \Delta_{\mathcal{G}} \) by taking the lexicographic order. Let \( f \) be a polynomial in Plücker coordinates of \( \mathbb{X} \). By Theorem 2, \( f \) can be written as a polynomial in \( \Delta_{\mathcal{G}}^\circ \):

\[
f = \sum_{\mathbf{a} \in \mathbb{Z}^{\mathcal{V}_0^\circ}} c_{\mathbf{a}} x^{\mathbf{a}}, \text{ where } x^\mathbf{a} = \prod_{i \in \mathcal{V}_0^\circ} x_i^{a_i} \text{ if } \mathbf{a} = (a_i)_{i \in \mathcal{V}_0^\circ}.
\]

Let \( v_{\mathcal{G}} : \mathbb{C}(\mathbb{X})^* \to \mathbb{Z}^{\mathcal{V}_0^\circ} \) be the minimal term valuation on \( \mathbb{C}(\mathbb{X}) \) with respect to the above total order.

Let \( L_k \) denote the very ample line bundle on \( \mathbb{X} \) generating \( \text{Pic}(\mathbb{X}) \). It gives the Plücker embedding. The space of global sections \( H^0(\mathbb{X}, L_k) \), as a representation of \( \text{GL}_k(\mathbb{C}) \), is isomorphic to \( V(r \sigma_k)^* \), where the latter is the dual of the finite-dimensional irreducible representation of highest weight \( r \sigma_k \) (\( \sigma_k \) is the \( k \)-th fundamental weight). The homogeneous coordinate ring \( \mathbb{C}[\mathbb{X}] := \bigoplus_{r \geq 0} H^0(\mathbb{X}, L_k^r) \) is embedded into \( \mathbb{C}(\mathbb{X}) \) by sending \( s \in H^0(\mathbb{X}, L_k^r) \) to \( s/\Delta_{\mathcal{G}}^\circ \).

**Definition 4.** The Newton–Okounkov body associated with \( L_k \), \( v_{\mathcal{G}} \) and the lexicographic order is defined by:

\[
\text{NO}_{\mathcal{G}} := \text{conv} \left( \bigcup_{r \geq 1} \{v_{\mathcal{G}}(s)/r \mid s \in H^0(\mathbb{X}, L_k^r) \setminus \{0\}\} \right).
\]
We set $\text{NO}_G^\downarrow := \text{conv}(\{(s) \mid s \in \mathbb{H}(X, L_k) \cup \{0\}\}) \subseteq \text{NO}_G$. For the issue on whether this inclusion is proper (i.e., whether $\text{NO}_G^\downarrow$ is integral), see [10, Theorem 15.17].

4. Duality between Newton–Okounkov bodies

4.1. Order polytopes and chain polytopes

Let $(P, \leq_P)$ be a poset with covering relation $\prec$. Stanley [11] associated two Ehrhart equivalent polytopes, the order polytope and the chain polytope, with this poset. We recall here a dilated version of them.

For $r \in \mathbb{N}_0$, we denote the dilated order polytope $O^r(P)$ to be the representation of the poset $P$ on the interval $[0, r]$ with the order on real numbers:

$$O^r(P) := \text{Hom}_{\text{Poset}}((P, \leq_P), ([0, r], \leq)) \subseteq \mathbb{R}^P.$$

The dilated chain polytope $C^r(P)$ has the following facets: for any $p \in P$, $x_p \geq 0$; for any maximal chain $p_1 < \cdots < p_s$, $x_{p_1} + \cdots + x_{p_s} \leq r$, where $x_p$ is the coordinate of $p \in P$ in $\mathbb{R}^P$.

Stanley [11] showed that the integral points of the chain polytope $C^r(P)$ are given by the characteristic functions of the anti-chains in $P$. In particular, the element $p \in P$ gives an integral point $\chi_p$ in $C^r(P)$.

In the following, we fix $1 < k \leq n - 1$, and let $(P_{k,n}, \leq)$ be the poset given by the elements $p_{i,j}$, where $1 \leq i \leq k$ and $1 < j \leq n$, with covering relations

$$p_{i,j+1} \prec p_{i,j}, \quad p_{i,j+1} \prec p_{i,j}.$$

The polytope $O(P_{k,n}, r)$ is the Gelfand–Tsetlin polytope $G^r_{k,n}$ for the representation $V(\tau_0 \sigma_1)$ of $\mathfrak{sl}_n (\{4\})$; while $C(P_{k,n}, r)$ is the Feigin–Fourier–Littelmann–Vinberg polytope $\text{FFLV}^r_{k,n}$ (\{1,3\}) of the same representation.

For a polytope $\mathcal{Q} \subseteq \mathbb{R}^m$, let $\text{S}(\mathcal{Q}) := \mathcal{Q} \cap \mathbb{Z}^m$ denote the set of integral points in it. The following integer decomposition properties hold: the $r$-fold Minkowski sum of $\text{S}(O(P_{k,n}, 1))$ resp. $\text{S}(C(P_{k,n}, 1))$ coincides with $\text{S}(O(P_{k,n}, r))$ resp. $\text{S}(C(P_{k,n}, r))$.

Moreover, if $a = \{p_{i,1,j}, \ldots, p_{i,k,j}\}$ is an anti-chain in $P_{k,n}$, then one has, for the corresponding lattice points, $\chi_a = \chi_{p_{i,1,j}} + \cdots + \chi_{p_{i,k,j}} \in \chi(P_{k,n}, 1)$.

**Proposition 1.** Suppose that $Q$ is an integral polytope in $\mathbb{R}^{P_{k,n}}$ such that

- $\#S(Q) = \#S(\text{FFLV}^1_{k,n})$;
- there is a parametrization of the lattice points in $Q$ by anti-chains in $P_{k,n}$ sending an anti-chain $a$ to $y_a \in \mathbb{R}^{P_{k,n}}$ such that, for any anti-chain $a = \{p_{i,1,j}, \ldots, p_{i,k,j}\}$, the relation $y_a = y_{p_{i,1,j}} + \cdots + y_{p_{i,k,j}}$ holds;
- there is a linear map of determinant 1 expressing $y_{p_{i,j}}$ in terms of $\chi_{p_{i,j}}$.

Then the assignment $\chi_{p_{i,j}} \mapsto y_{p_{i,j}}$ induces a unimodularly equivalence from $\text{FFLV}^1_{k,n}$ to $Q$.

4.2. Duality of polytopes from positive structures

We refer the reader to [10, Section 7.1] for the definition of the rec-plabic graph $G^\text{rec}_{k,n}$. For example, the plabic graph in Fig. 1 is $G^\text{rec}_{4,7}$.

The following has been shown in [10, Lemma 15.2]:

**Proposition 2.** The Newton–Okounkov body $\text{NO}_{G^\text{rec}}_{k,n}$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $G^1_{n-k,n}$.

We define the dual rec-plabic graph $(G^\text{rec}_{k,n})^\vee$ by swapping the black/white colour of the internal vertices, reversing the perfect orientation and changing the boundary labelling $r \mapsto r + n - k \mod n$. The dual rec-plabic graph is a polytope graph of trip permutation $\pi_{k,n}$ with a perfect orientation. The face labelling in $(G^\text{rec}_{k,n})^\vee$ of a face $F$ in $G^\text{rec}_{k,n}$ is given by the complement:

$$\lambda^\vee_{G^\text{rec}_{k,n}}(F) = (\lambda_{G^\text{rec}_{k,n}}(F))^\vee.$$

Notice that in $(G^\text{rec}_{k,n})^\vee$, for a boundary source $i$ and a boundary sink $j$, the flow from $i$ to $j$ of strongly minimal weight (we borrow the notion of strongly minimal from [10, Definition 5.13]) is given by a “vertical” path starting from $i$ followed by a “horizontal” path ending in $j$. We denote this path by $\gamma^\min_{i,j}$ (see Fig. 2 for an example for $\gamma^\min_{3,6}$).

**Proposition 3.** In the dual rec-plabic graph $(G^\text{rec}_{k,n})^\vee$, let $\{i_1 \prec \cdots \prec i_r\}$ be a subset of the sources and $\{j_1 > \cdots > j_t\}$ be a subset of the sinks. Let $J = \{i_1, \ldots, i_r, j_1, \ldots, j_t\}$. Then the unique flow $F(J)$ of strongly minimal weight is given by $\gamma^\min_{i_1,j_1}, \ldots, \gamma^\min_{i_r,j_t}$. 

Proof. Since the paths of strongly minimal weight do not intersect, the flow of minimal weight is given by the union of these paths. □

Theorem 3. The Newton–Okounkov body \( NO(G_{k,n}^{rec}) \) is unimodularly equivalent to the FFLV polytope \( FFLV_{k,n}^1 \).

Proof. We first set \( Q = NO(\bigwedge_{k,n}^{rec}) \) and verify the conditions in Proposition 1 to show that \( Q \) is unimodularly equivalent to \( FFLV_{k,n}^1 \) by a linear map.

The polytope \( Q \) is a lattice polytope satisfying \( \#S(Q) = \#S(FFLV_{k,n}^1) \) (the valuation images of the Plücker coordinates are different). Let \( f_{i \times j} := v_{G_{k,n}^{rec}}(y_{i,j}^{\min}) \). We define a linear map

\[
\psi : FFLV_{k,n}^1 \longrightarrow Q, \quad x_{p_{i,j}} \mapsto f_{i \times j}.
\]

We label a basis on the right-hand side indexed by the faces of the plabic graph and a basis on the left-hand side indexed by the elements \( p_{i,j} \). Using row operations, one can show straightforwardly, that the matrix of \( \psi \) corresponding to these bases has determinant 1.

Since \( \psi \) is linear, \( NO(G_{k,n}^{rec}) \) is unimodularly equivalent to \( FFLV_{k,n}^1 \).

Remark 1. We set \( (G_{k,n}^{rec})_{\omega_{i,j}} \) to be the plabic graph obtained from \( G_{k,n}^{rec} \) by replacing each \( I = \{i_1, \ldots, i_{n-k}\} \) by \( \omega_{i,j} = [n + 1 - i_{n-k}, \ldots, n + 1 - i_1] \). This is nothing but applying a maximal Green sequence of mutations [6] to the cluster variables in \( G_{k,n}^{rec} \).

Then one can show similarly to the theorem above, that the Newton–Okounkov body \( NO(G_{k,n}^{rec})_{\omega_{i,j}} \) is unimodularly equivalent to \( FFLV_{n-k,n}^1 \).

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