COUNTING PSEUDO-HOLOMORPHIC DISCS
IN CALABI-YAU 3 FOLD

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ABSTRACT. In this paper we define an invariant of a pair of 6 dimensional symplectic manifold with vanishing 1st Chern class and its Lagrangian submanifold with vanishing Maslov index. This invariant is a function on the set of the path connected components of the bounding cochains (solution of A infinity version of Maurer-Cartan equation of the filtered A infinity algebra associated to the Lagrangian submanifold). In the case when the Lagrangian submanifold is a rational homology sphere, it becomes a numerical invariant. This invariant depends on the choice of almost complex structure. The way how it depends on the almost complex structure is described by a wall crossing formula which involves moduli space of pseudo-holomorphic spheres.

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1. INTRODUCTION

This paper is a continuation of [7] Subsection 3.6.4 and [5]. Let \((M, \omega)\) be a symplectic manifold of (real) dimension \(2 \times 3\). We assume that \(c^1(M) = 0\) in \(H^2(M; \mathbb{Q})\). (Here we use compatible almost complex structure of tangent bundle to define \(c^1(M, L)\).) Let \(L \subset M\) be a relatively spin Lagrangian submanifold and \(\mu_L : \mathbb{H}_2(M, L; \mathbb{Z}) \rightarrow 2\mathbb{Z}\) its Maslov index homomorphism. (See [7] Subsection 2.1.1.) We assume that \(\mu_L = 0\). In this paper we consider such

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a pair \((M, L)\). A typical example is a pair of a Calabi-Yau 3 fold \(M\), and its special Lagrangian submanifold \(L\). This is one of the most interesting cases of (homological) mirror symmetry. Our main purpose of this paper is to define and study an invariant of such \((M, L)\). It is independent of various choices involved in the construction but depends on the almost complex structure \(J\) of \(M\).

We consider \(\mathcal{M}(L; J; \Lambda_+)\) the set of \(\Lambda_+\)-valued points of Maurer-Cartan formal scheme’ of the filtered \(A_\infty\) structure associated to \(L\). This is the set of gauge equivalence classes of bounding cochains and defined in \(\cite{7}\) Section 4.3. (Here we include \(J\) in the notation since \(J\) dependence is rather crucial in this paper.) We study cyclic filtered \(A_\infty\) algebra \(\Lambda(\langle \cdot \rangle, \{m^J_{k,\beta}\})\) produced in \(\cite{5}\) by modifying the construction of \(\cite{7}\). In our case where \(\mu_L = 0\) we can reduce the coefficient ring to \(\Lambda_0 = \Lambda_0^{(0)}\), that is the degree 0 part of the universal Novikov ring with \(\mathbb{R}\) coefficient. (The universal Novikov ring is defined at \(\cite{7}\) beginning of Subsection 1.2.) We denote by \(\Lambda_+\) its maximal ideal. Let \([b] \in \mathcal{M}(L; J; \Lambda_+)\). We define a superpotential (without leading term) by:

\[
\Psi'(b; J) = \sum_{k=0}^{\infty} \sum_{\beta} \frac{T^{2\beta \omega}}{k + 1} \langle m^J_{k,\beta}(b, \ldots, b), b \rangle.
\]

To obtain a superpotential which is independent of the perturbation and other choices involved, we need to add the constant term to it. Note \(m^J_{-1,\beta}\) is defined by using moduli space \(\mathcal{M}_{k+1}(\beta; J)\) of pseudo-holomorphic discs with \(k + 1\) marked points and of homology class \(\beta \in H_2(M, L; \mathbb{Z})\). We use \(\mathcal{M}_{0}(\beta; J)\), the moduli space of \(J\) holomorphic discs, of homology class \(\beta\) without marked point, to define

\[
\Psi(b; J) = \Psi'(b; J) + \sum_{\beta \in H_2(L, \mathbb{Z})} T^{2\beta \omega} m^J_{-1,\beta}.
\]

More precisely we assume that our almost complex structure \(J\) satisfies the following:

**Assumption 1.** There exists no nontrival \(J\)-holomorphic sphere \(v: S^2 \to M\) such that \(v(S^2) \cap L \neq \emptyset\).

By dimension counting we find that the set of such \(J\) is dense.

**Theorem 1.1.**

1. If \(J\) satisfies the Assumption 1, then there exists a function \(\Psi: H^1(L; \Lambda_+) \to \Lambda_+\) which depends not only on \(J\) but also on perturbation etc.
2. There exists an isomorphism between the set \(\mathcal{M}(L; J; \Lambda_+)\) and the set of critical points of \(\Psi\).
3. The restriction of \(\Psi\) to its critical point set \(\mathcal{M}(L; J; \Lambda_+)\) depends only on \(M, L, J\) and is independent of the choice of perturbation etc.

We call \(\Psi\) the superpotential. The value \(\Psi(b)\) depends only on the path connected component of \(b \in \mathcal{M}(L; J)\). See Proposition 2.3.

**Corollary 1.1.** If \(L\) is a rational homology sphere in addition, then \(\mathcal{M}(L; J; \Lambda_+)\) is one point. So the value of \(\Psi\) at that point is an invariant of \(M, L, J\).
In Sections 2 and 3 we develop the theory of superpotential of cyclic filtered $A_\infty$ algebra of dimension 3 with additional data corresponding to $m^J_{1, \beta}$. In Section 2 we fix our cyclic filtered $A_\infty$ algebra and review the construction of superpotential and its gauge invariance. We next study its relation to pseudo-isotopy of cyclic filtered $A_\infty$ algebra to complete the algebraic part of the proof of Theorem 1.1 in Section 3. The algebraic structure we assumed in Sections 2 and 3 are realized in Section 4, where the proof of Theorem 1.1 is completed.

We can extend the domain $H^1(L; \Lambda_+)$ of the definition of $\Psi(b; J)$ as follows. Let $e_i, i = 1, \ldots, b_1$ be the basis of $H^1(L; \mathbb{Z})/\text{Torsion}$. We put $b = \sum x_i e_i$ where $x_i \in \Lambda_0$. We put $y_i = e^{x_i} = \sum_{k=0}^{\infty} \frac{1}{k!} x_i^k$.

We define the strongly convergent Laurent power series ring (See [2].)

\[ \Lambda_0 \langle \langle y_1, \ldots, y_{b_1}, y_1^{-1}, \ldots, y_{b_1}^{-1} \rangle \rangle \]

as the set of formal sums

\[ f(y_1, \ldots, y_{b_1}) = \sum_{i=1}^{\infty} T^{\lambda_i} P_i(y_1, \ldots, y_{b_1}) \]

where $\lambda_i \in \mathbb{R}_{\geq 0}$ with $\lim_{i \to \infty} \lambda_i = \infty$ and $P_i$ are Laurent polynomial. We remark that for each $f$ as in (4) and $y_1, \ldots, y_{b_1} \in \Lambda_0$ with $v(y_i) = 0$, the sum

\[ \sum_{i=1}^{\infty} T^{\lambda_i} P_i(y_1, \ldots, y_{b_1}) \]

converges in $T$-adic topology. Therefore $f(y_1, \ldots, y_{b_1})$ is well defined.

**Theorem 1.2.**

1. $\Psi(b, J) \in \Lambda_0 \langle \langle y_1, \ldots, y_{b_1}, y_1^{-1}, \ldots, y_{b_1}^{-1} \rangle \rangle$.
2. There exists $\delta > 0$ such that $\Psi$ is extended to

\[ \{(y_1, \ldots, y_{b_1}) \mid -\delta < v(y_i) < \delta \} \]

3. Its critical point set is identified with $M(L; J)_\delta$ which is introduced in Theorem 1.2 [4].
4. The restriction of $\Psi$ to $M(L; J)_\delta$ is independent of the perturbation etc. and depends only on $M, L, J$.

Here $v(\cdot)$ is defined by

\[ v \left( \sum a_i T^{\lambda_i} \right) = \inf \{ \lambda_i \mid a_i \neq 0 \} \]

We prove Theorem 1.2 in Section 7.

In Section 5 we use canonical model constructed in [7] Section 4.5 and [5] Section 10, to rewrite the definition of $\Psi$.

In Section 6 we discuss the way how superpotential $\Psi$ depends on almost complex structure. The main result is Theorem 1.3 below. We assume that $J_0$ and $J_1$ satisfy Assumption 1. We take a path $\mathcal{J} = \{J_t \mid t \in [0, 1] \}$ of tame almost complex structures joining them. Let $\mathcal{M}^c_t(\alpha; J)$ be the moduli space of $J$ holomorphic stable maps of genus zero in $M$ of homology class $\alpha \in H_2(M; \mathbb{Z})$ and with one marked point. It has a Kuranishi structure of (virtual) dimension 2. We put

\[ \mathcal{M}^c_t(\alpha; J) = \bigcup_{t \in [0, 1]} \{t\} \times \mathcal{M}^c_t(\alpha; J_t). \]
Using evaluation map $ev : \mathcal{M}_1^3(\alpha; J) \to M$ we obtain a virtual fundamental chain $ev_*([\mathcal{M}_1^3(\alpha; J)])$ of dimension 3. Since $J_0$ and $J_1$ satisfy Assumption 4 it follows that

$$L \cap ev(\partial \mathcal{M}_1^3(\alpha; J)) = \emptyset.$$  

Therefore

$$n(L; \alpha; J) = [L] \cap ev_*([\mathcal{M}_1^3(\alpha; J)]) \in \mathbb{Q}$$

is well-defined. Moreover it depends only on $M, L, \alpha, J_0, J_1$ and is independent of the path $\mathcal{J}$.

**Theorem 1.3.** Let $[b] \in \mathcal{M}(L; J_0)$. We take canonical isomorphism $I_* : \mathcal{M}(L; J_0) \to \mathcal{M}(L; J_1)$ in [7] Section 4.3. Then we have:

$$\Psi(I_*(b), J_1) - \Psi(b, J_0) = \sum_{\alpha \in H_2(M; \mathbb{Z})} T^{\alpha \cap \omega} n(L; \alpha; J).$$

Theorem 1.3 is proved in Section 8. In Section 8 we discuss some conjectures, open problems, and relations to various related topics.

**Remark 1.**

1. Superpotential of the form (1) appears in the physics literature [17, 20].
2. The idea to include the 2nd term of (3) to obtain a numerical invariant of Lagrangian submanifold is due to D. Joyce. It was communicated to the author by P. Seidel around 2002, who also explained him the importance of cyclic symmetry for this purpose. (However the appearance of nontrivial wall crossing by the change of $J$ was unknown at that time.)
3. The appearance of the nonzero wall crossing term in the right hand side of (8) is closely related to the phenomenon discussed in [7] Section 3.8 and Subsection 7.4.1. Around the same time as the authors of [7] found this phenomenon, a similar observation was done independently by M. Liu [16].
4. A related homological algebra was discussed before by [3, 14]. The part concerning the second term of (3) is not discussed there.
5. All the $A_\infty$ algebras and pseudo-isotopies between them which appear in the geometric situation in this paper, are unital. We omit the argument on unitality since it is a straightforward analog of one in [5].

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2. **Superpotential and its gauge invariance**

Let $(C, \langle \cdot \rangle, \{m_{k, \beta}\})$ be a $G$-gapped cyclic filtered $A_\infty$ algebra of dimension 3. Recall that $G \subset \mathbb{R}_{\geq 0} \times 2\mathbb{Z}$ is a discrete submonoid in the sense of [7] Condition 3.1.6, [5] Definition 6.2. In this paper we always assume

$$G \subset \mathbb{R}_{\geq 0} \times \{0\}.$$  

Namely $G \subset \mathbb{R}_{\geq 0}$. In this case

$$m_{k, \beta} : B_k(C[1]) \to C[1]$$

is always of degree 1 (after degree shift). We put $C_+ = \overline{C} \otimes_{\mathbb{R}} \Lambda_+$. 
Definition 2.1. We define 
\[ \Psi' : C^1_+ \to \Lambda_0 \]
by
\[ \Psi'(b) = \sum_{k=0}^{\infty} \sum_{\beta \in G} \frac{T^{E(\beta)}}{k+1} (m_{k,\beta}(b, \ldots, b), b). \]  

Remark 2. (1) More precisely the right hand side of (10) converges in \( T \) adic topology. In various cases, it converges in the topology of [5] Definition 13.1. (It converges in the case of filtered \( A_\infty \) algebra of Lagrangian Floer theory by [5] Theorem 1.2.) See Section 7 on the convergence.

(2) Since \( \deg' b = \deg b - 1 = 0 \). We have
\[ \deg' m_{k,\beta}(b, \ldots, b) = 1. \]
Namely \( \deg m_{k,\beta}(b, \ldots, b) + \deg b = 3 \). Therefore in case the dimension of our cyclic filtered \( A_\infty \) algebra is 3, the inner product in the right hand side of (10) is well defined.

We fix a basis \( e_i \in \mathcal{C} \) and put \( b = \sum x_i e_i \). Then \( \Psi'(b) = \sum P_{\beta}(x_1, \ldots) \) where \( P_{\beta} \) is a formal power series. Therefore we can differentiate \( \Psi' \) formally. We have:

Proposition 2.1. If \( b \in C^1_+ \) then the differential of \( \Psi' \) vanishes at \( b \) if and only if
\[ \sum_{\beta \in H_2(M,L;\mathbb{Z})} T^{E(\beta)} m_{k,\beta}(b, \ldots, b) = 0. \]  
This is [7] Proposition 3.6.50. (11) is called the \( A_\infty \) Maurer-Cartan equation.

Definition 2.2. \( \tilde{M}(C;\Lambda_+) \) is the set of all \( b \in C^1_+ \) satisfying (11).

We next review the definition of gauge equivalence from [7] Section 4.3. We consider
\[ b(t) = \sum_{\beta : E(\beta) > 0} T^{E(\beta)} b_{\beta}(t), \quad c(t) = \sum_{\beta : E(\beta) > 0} T^{E(\beta)} c_{\beta}(t) \]  
where \( b_{\beta}(t), c_{\beta}(t) \) are polynomial with coefficient in \( \mathbb{C}^1, \mathbb{C}^0 \) respectively.

Definition 2.3 (See [7] Proposition 4.3.5). We say \( b_0 \in \tilde{M}(C;\Lambda_+) \) is gauge equivalent to \( b_1 \in \tilde{M}(C;\Lambda_+) \) if there exists \( b(t), c(t) \) as in (12) such that:
(1) \( b(0) = b_0, b(1) = b_1 \).
(2) \[ \frac{d}{dt} b(t) + \sum_{k=1}^{\infty} m_k(b(t), \ldots, b(t), c(t), b(t), \ldots, b(t)) = 0. \]

It is proved in [7] Lemma 4.3.4 that gauge equivalence is an equivalence relation. We denote by \( \mathcal{M}(C;\Lambda_+) \) the set of gauge equivalence classes.

Remark 3. It follows from 1.2 that \( b(t) \in \tilde{M}(C;\Lambda_+) \) for any \( t \). (7) Lemma 4.3.7.)

Proposition 2.2. If \( b_0 \in \tilde{M}(C;\Lambda_+) \) is gauge equivalent to \( b_1 \in \tilde{M}(C;\Lambda_+) \) then
\[ \Psi'(b_0) = \Psi'(b_1). \]
Proof. We have

\[
\frac{d}{dt} \Psi'(b) = \frac{d}{dt} \sum_{k=0}^{\infty} \sum_{\beta \in G} \frac{T_E(\beta)}{k+1} \left< m_{k,\beta}(b(t), \ldots, b(t)), b(t) \right>
\]

(14)

\[
+ \sum_{k=0}^{\infty} \sum_{\beta \in G} \frac{T_E(\beta)}{k+1} \left< m_{k,\beta}(b(t), \ldots, b(t)), \frac{db(t)}{dt} \right>.
\]

Since \(b(t) \in \widetilde{\mathcal{M}}(C; \Lambda_+)\), it follows that (14) is zero. □

By Proposition 2.2 we obtain

(15) \(\Psi' : \mathcal{M}(C; \Lambda_+) \to \Lambda_0\).

We remark that in the proof of Proposition 2.2 we only use the existence of families \(b(t)\) in \(\widetilde{\mathcal{M}}(C; \Lambda_+)\) joining \(b_0\) and \(b_1\). In other words, we did not use the existence of \(c(t)\). Therefore we have:

**Proposition 2.3.** If the map \(t \mapsto b(t) \in \widetilde{\mathcal{M}}(C; \Lambda_+)\) is a \(C^1\) map then

\[\Psi(b(0)) = \Psi(b(1)).\]

**Remark 4.** Proposition 2.3 may imply that superpotential is locally constant on \(\mathcal{M}(C; \Lambda_+)\) and so \(\Psi\) depends only on the ‘irreducible component’ of \(\mathcal{M}(C; \Lambda_+)\). Since the property of \(\mathcal{M}(C; \Lambda_+)\) as a topological space can be rather complicated, we do not try to study this point in this paper.

## 3. Pseudo-isotopy invariance

In [5] Definition 8.5, it is defined that \((C, \langle \cdot \rangle, \{m_{k,\beta}^t\}, \{c_{k,\beta}^t\})\) is a pseudo-isotopy of cyclic filtered \(A_\infty\) algebra if:

(1) \(m_{k,\beta}^t\) and \(c_{k,\beta}^t\) are smooth. Namely

\[t \mapsto m_{k,\beta}^t(x_1, \ldots, x_k)\]

is smooth. (That is the coefficient is a smooth function of \(t \in [0, 1]\).)

(2) For each (but fixed) \(t\), the triple \((C, \langle \cdot \rangle, \{m_{k,\beta}^t\})\) defines a cyclic filtered \(A_\infty\) algebra.

(3) For each (but fixed) \(t\), and \(x_i \in \bar{C}[1]\), we have

(16) \[\langle c_{k,\beta}^t(x_1, \ldots, x_k), x_0 \rangle = (-1)^* \langle c_{k,\beta}^t(x_0, x_1, \ldots, x_{k-1}), x_k \rangle\]

\[* = (\deg x_0 + 1)(\deg x_1 + \ldots + \deg x_k + k).\]
(4) For each \( x_i \in C \)
\[
\frac{d}{dt}m^t_{k,\beta}(x_1, \ldots, x_k)
\] (17)
\[+
\sum_{k_1+k_2=k_1+\beta_1+\beta_2=\beta}(-1)^{k_2}c_{k_1,\beta}(x_1, \ldots, m^t_{k_2,\beta_2}(x_i, \ldots), \ldots, x_k)
\] (17)
\[\sum_{k_1+k_2=k_1+\beta_1+\beta_2=\beta}m^t_{k_1,\beta_1}(x_1, \ldots, \beta_2(x_i, \ldots), \ldots, x_k)
\] (17)\[= 0.
\] (17)
Here \( * = \deg x_1 + \ldots + \deg x_{i-1} \).
(5) \( m^t_{k_i(0,0)} \) is independent of \( t \). \( c_{k_i(0,0)} = 0 \).

**Definition 3.1.**

1. \((C, \langle \cdot \rangle, \{m_{k,\beta}\}, \{m_{-1,\beta}\})\) is said to be an **inhomogeneous cyclic filtered** \( A_\infty \) **algebra** if \((C, \langle \cdot \rangle, \{m_{k,\beta}\})\) is cyclic filtered \( A_\infty \) algebra and \( m_{-1,\beta} \in \mathbb{R} \).
2. \((C, \langle \cdot \rangle, \{m^t_{k,\beta}\}, \{c_{k,\beta}\}, \{m^t_{-1,\beta}\})\) is said to be a **pseudo-isotopy of inhomogeneous cyclic filtered** \( A_\infty \) **algebra** if \((C, \langle \cdot \rangle, \{m^t_{k,\beta}\}, \{c_{k,\beta}\})\) is a pseudo-isotopy of cyclic filtered \( A_\infty \) algebra,
\[
t \mapsto m^t_{-1,\beta}
\] (18)
is a real valued smooth function and if
\[
\frac{d}{dt}m^t_{-1,\beta} + \sum_{\beta_1+\beta_2=\beta} \langle c^t_{0,\beta_1}(1), m^t_{0,\beta_2}(1) \rangle = 0.
\] (18)

Let \((C, \langle \cdot \rangle, \{m^t_{k,\beta}\}, \{c_{k,\beta}\})\) be a pseudo-isotopy of cyclic filtered \( A_\infty \) algebra. We consider cyclic filtered \( A_\infty \) algebras \((C, \langle \cdot \rangle, \{m^0_{k,\beta}\})\) and \((C, \langle \cdot \rangle, \{m^1_{k,\beta}\})\). By \[5\] Theorem 8.2 there exists an isomorphism
\[
c = c(1; 0) : (C, \langle \cdot \rangle, \{m^0_{k,\beta}\}) \to (C, \langle \cdot \rangle, \{m^1_{k,\beta}\})
\] (19)
of cyclic filtered \( A_\infty \) algebra. It induces
\[
c_* : \mathcal{M}(C, \{m^0_{k,\beta}\}) \to \mathcal{M}(C, \{m^1_{k,\beta}\})
\] by \[7\] Theorem 4.3.22. The main result of this section is as follows.

**Theorem 3.1.** We have
\[
\Psi'(c_*(b)) + \sum_\beta T^{E(\beta)}m^t_{-1,\beta} = \Psi'(b) + \sum_\beta T^{E(\beta)}m^0_{-1,\beta}.
\] (20)

**Proof.** We also constructed
\[
c(t; 0) : (C, \langle \cdot \rangle, \{m^0_{k,\beta}\}) \to (C, \langle \cdot \rangle, \{m^t_{k,\beta}\})
\] (20)\[\text{in} \[5\] Definition 9.4. It is an isomorphism and depends smoothly on } t. \text{ We put}
\[
b(t) = c(t; 0)_*(b) = \sum_{k,\beta} c_{k,\beta}(t; 0)(b, \ldots, b)
\] (20)
and

\[ f(t) = \Psi'(b(t)) + \sum_{\beta} T^{E(\beta)} m_{-1, \beta} \]

(21)

\[ = \sum_{k=0}^{\infty} \frac{1}{k+1} \langle m_k^t (b(t), \ldots, b(t)), b(t) \rangle + \sum_{\beta} T^{E(\beta)} m_{-1, \beta}. \]

We calculate the derivative of \( f(t) \). The derivative of the first term is:

\[ \sum_{k=0}^{\infty} \frac{1}{k+1} \left\langle \frac{dm_k^t}{dt} (b(t), \ldots, b(t)), b(t) \right\rangle \]

(22)

\[ + \sum_{k=0}^{\infty} \frac{1}{k+1} \left\langle m_k^t (b(t), \ldots, \frac{db(t)}{dt}, \ldots, b(t)), b(t) \right\rangle \]

\[ + \sum_{k=0}^{\infty} \frac{1}{k+1} \left\langle m_k^t (b(t), \ldots, b(t)), \frac{db(t)}{dt} \right\rangle \]

The sum of 2nd and the 3rd terms of (22) is:

\[ \sum_{k=0}^{\infty} \left\langle m_k^t (b(t), \ldots, b(t)), \frac{db(t)}{dt} \right\rangle = 0 \]

by cyclic symmetry and Maurer-Cartan equation of \( b(t) \).

We calculate the 1st term by using (17) and obtain:

\[ -\sum_{k=0}^{\infty} \sum_{k_1+k_2=k+1}^{k_2-1} \sum_{i=0}^{k_1-1} \frac{1}{k+1} \left\langle c_{k_1}^i (b(t), \ldots, \underline{b(t)}, \overline{m_{k_2}^t} (b(t), \ldots, \underline{b(t)})), b(t) \right\rangle \]

(23)

\[ + \sum_{k=0}^{\infty} \sum_{k_1+k_2=k+1}^{k_2-1} \sum_{i=0}^{k_1-1} \frac{1}{k+1} \left\langle \underline{m_{k_2}^t} (b(t), \ldots, \underline{b(t)}, \overline{c_{k_1}^i} (b(t), \ldots, \underline{b(t)})), b(t) \right\rangle \]

We have

\[ \left\langle c_{k_1} \ldots m_{k_2}^t (b(t), \ldots, \ldots), b(t) \right\rangle = \langle c_{k_1}^i (b(t), \ldots, \underline{b(t)}, \overline{m_{k_2}^t} (b(t), \ldots, \underline{b(t)})), b(t) \rangle \]

and

\[ \langle m_{k_2} \ldots c_{k_1}^i (b(t), \ldots, \ldots), b(t) \rangle = \langle m_{k_2}^t (b(t), \ldots, \underline{c_{k_1}^i} (b(t), \ldots, \underline{b(t)})), b(t) \rangle \]

\[ = -\langle c_{k_2}^i (b(t), \ldots, \underline{b(t)}, \overline{m_{k_1}^t} (b(t), \ldots, \underline{b(t)})), b(t) \rangle \]

by cyclic symmetry and \([5]\) (56). Therefore (23) is equal to

(24)

\[ -\sum_{k=0}^{\infty} \sum_{k_1+k_2=k+1} \left\langle c_{k_2}^i (b(t), \ldots, \underline{b(t)}, \overline{m_{k_1}^t} (b(t), \ldots, \underline{b(t)})) \right\rangle. \]

Using Maurer-Cartan equation for \( b(t) \) we find that (21) is equal to

\[ \langle c'(0), m'(0) \rangle. \]

By (18) this cancels with the derivative of the 2nd term of (21). Namely \( f(t) \) is independent of \( t \). \( \square \)
Definition 3.2. Let \((C, \langle \cdot, \cdot \rangle, \{m_{k, \beta}\}, \{m_{-1, \beta}\})\) be an inhomogeneous cyclic filtered \(A_\infty\) algebra. We call the function \(\Psi : \mathcal{M}(C; \Lambda_+) \rightarrow \Lambda_+\), defined by
\[
\Psi(b) = \Psi'(b) + \sum_\beta T^{E(\beta)} m_{-1, \beta}^0
\]
its superpotential.

4. Geometric realization

Let \(M\) be a \(3 \times 2\) dimensional symplectic manifold with \(c^1(M) = 0\) and \(L\) its relatively spin Lagrangian submanifold with vanishing Maslov index.

In [5] Theorem 1.1, we defined a \(G\)-gapped cyclic filtered \(A_\infty\) algebra \((\Lambda(L), \langle \cdot, \cdot \rangle, \{m_{J, \beta}'\}, \{m_{J-1, \beta}'\})\) on its de Rham complex. We also proved that its pseudo-isotopy type is independent of the choice of \(J\), perturbation etc. The main result of this section is as follows.

Theorem 4.1. If \(J\) satisfies Assumption [6] then there exists \(m_{J-1, \beta}' \in \mathbb{R}\) such that \((\Lambda(L), \langle \cdot, \cdot \rangle, \{m_{J, \beta}'\}, \{m_{J-1, \beta}'\})\) is an inhomogeneous cyclic and gapped filtered \(A_\infty\) algebra.

Moreover the pseudo-isotopy type of it depends only on \(M, L, J\) and is independent on other choices involved in the definition.

Proof. For \(\beta \in H_2(M, L; \mathbb{Z})\) let \(M_k(\beta; J)\) be the moduli space of stable \(J\) holomorphic maps \(v : (\Sigma, \partial \Sigma) \rightarrow (M, L)\) from bordered Riemann surface \(\Sigma\) of genus 0 with connected nonempty boundary \(\partial \Sigma\), and with \(k\) boundary marked points, such that \(v\) is of homology class \(\beta\). Let \(ev = (ev_0, \ldots, ev_{k-1}) : M_k(\beta; J) \rightarrow L^k\) be the evaluation maps at the boundary marked points. (See [7] Subsection 2.1.1.)

In [5] Theorem 3.1 and Corollary 3.1, we proved an existence of its Kuranishi structure with the following properties:

(1) It is compatible with the forgetful map
\[
\text{forget}_{k,0} : M_k(\beta; J) \rightarrow M_0(\beta; J).
\]
(See [5] Section 3 for the precise definition of this compatibility.)
(2) For \(k \geq 1\) the evaluation map \(ev_0 : M_k(\beta; J) \rightarrow L\) is weakly submersive, in the sense of [7] Definition A1.13.
(3) It is invariant under the cyclic permutation of the boundary marked points.
(4) We consider the decomposition of the boundary:
\[
\partial M_{k+1}(\beta) = \bigcup_{1 \leq i \leq j \leq k + 1} \bigcup_{\beta_1 + \beta_2 = \beta} M_{j-i+1}(\beta_1)_{ev_0} \times_{ev_0} M_{k-j+i}(\beta_2).
\]
(See [7] Subsection 7.1.1.) Then the restriction of the Kuranishi structure of \(M_{k+1}(\beta)\) to the left hand side coincides with the fiber product Kuranishi structure in the right hand side.
(5) We consider the decomposition
\[
\partial M_0(\beta) = \bigcup_{\beta_1 + \beta_2 = \beta} (M_1(\beta_1)_{ev_0} \times_{ev_0} M_1(\beta_2)) / \mathbb{Z}_2.
\]
Then, the fiber product Kuranishi structure on \(M_1(\beta_1)_{ev_0} \times_{ev_0} M_1(\beta_2)\) (which is well-defined by 2) coincides with the pull back of the Kuranishi structure to \(\partial M_0(\beta)\).
We remark that in general the decomposition of the boundary of $\partial M_0(\beta)$ is given by
\[
\partial M_0(\beta) = \bigcup_{\beta_1 + \beta_2 = \beta} (M_1(\beta_1)_{ev_0} \times_{ev_0} M_1(\beta_2)) / \mathbb{Z}_2
\]
\[
\cup \bigcup_{\beta} M_1^{cl}(\beta)_{ev_0} \times_M L.
\]
(28)

Here $M_1^{cl}(\beta)$ is the moduli space of stable maps of genus zero without boundary, one marked point and of homology class $\tilde{\beta} \in H_2(M; \mathbb{Z})$. The sum is taken over all $\tilde{\beta} \in H_2(M; \mathbb{Z})$ which goes to $\beta$ by $i_* : H_2(M; \mathbb{Z}) \to H_2(M; \mathbb{Z})$. By Assumption 1, the 2nd term of the right hand side of (28) is an empty set.

Let $E_0 > 0$. Then in [5] Theorem 5.1 and Corollary 5.1, we proved the existence of system of continuous families of multisections on the above Kuranishi spaces $M_k(\beta; J)$ with $\beta \cap [\omega] < E_0$ with the following properties:

1. The families of multisections are transversal to 0.
2. It is compatible with the forgetful map (25). (See [5] Section 5 for the precise definition of this compatibility.)
3. For $k \geq 1$ the evaluation map $ev_0$ induces a submersion of its zero set, in the sense of [5] Definition 4.1.4.
4. It is invariant under the cyclic permutation of the boundary marked points.
5. It is compatible with the identification (26).
6. It is compatible with the identification (27).

Let $\rho_i \in \Lambda(L)$ ($i = 1, \ldots, k$) be the differential forms on $L$. In [5] Section 6 we defined
\[
\mathcal{M}_{k-1,\beta}(\rho_1, \ldots, \rho_k) = Corr(M_{k+1}(\beta; J); (ev_1, \ldots, ev_k, ev_0))(\rho_1 \times \cdots \times \rho_k).
\]

Here the right hand side is the smooth correspondence associated to the above continuous family of perturbations. (See [5] Section 4.) (Note that (29) depends on the choice of family of multisections. The symbol $s$ is put to clarify this dependence.)

We next define $\mathcal{M}_{k-1,\beta}$. Let $pt$ be the space consisting of one point. We have an obvious map $\Lambda(\beta; J) \to pt$. Note $\Lambda(pt) = \mathbb{R}$. Moreover
\[
\dim M_0(\beta; J) = \dim L - 3 + \mu(\beta) = 0.
\]

Therefore we have an $\mathbb{R}$ linear map:
\[
Corr(M_{k}^{\pm}(\beta); (\triangledown, \triangledown)) : \mathbb{R} \to \mathbb{R}.
\]

**Definition 4.1.** For $\beta \cap [\omega] < E_0$, we put
\[
\mathcal{M}_{k,\beta}^{\pm} = Corr(M_{k}^{\pm}(\beta); (\triangledown, \triangledown))(1) \in \mathbb{R}.
\]

**Definition 4.2.**

1. An inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T^{E_0}$ is $(C, \langle \cdot , \cdot \rangle, \{m_{k,\beta} | E(\beta) < E_0\}, \{m_{k-1,\beta} | E(\beta) < E_0\})$ such that $(C, \langle \cdot , \cdot \rangle, \{m_{k,\beta} | E(\beta) < E_0\})$ is a cyclic filtered $A_\infty$ algebra modulo $T^{E_0}$ and $m_{k-1,\beta} \in \mathbb{R}$.

2. A pseudo-isotopy of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T^{E_0}$ is $(C, \langle \cdot , \cdot \rangle, \{m_{k,\beta} | E(\beta) < E_0\}, \{\chi_{k,\beta} | E(\beta) < E_0\}, \{m_{k-1,\beta} | E(\beta) < E_0\})$, if $(C, \langle \cdot , \cdot \rangle, \{m_{k,\beta} | E(\beta) < E_0\}, \{\chi_{k,\beta} | E(\beta) < E_0\})$ is a pseudo-isotopy of cyclic filtered $A_\infty$ algebra modulo $T^{E_0}$ (namely (16) (17) hold for $E(\beta) < E_0$) and (18) holds for $E(\beta) < E_0$. 

The modulo $T^E_0$ version of Proposition 2.2 and Theorem 3.1 can be proved by the same proof.

$(\Lambda(L), \langle \cdot \rangle, \{m^{J, \beta}_{k, r}\}, \{m^{J, \beta}_{-1, r}\})$ which we defined above is an inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T^E_0$.

**Proposition 4.1.** $(\Lambda(L), \langle \cdot \rangle, \{m^{J, \beta}_{k, r}\}, \{m^{J, \beta}_{-1, r}\})$ is independent of the choice of Kuranishi structure and family of multisections $s$ satisfying the properties listed in this section, up to pseudo-isotopy of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T^E_0$.

**Proof.** Let us take two different choices of system of Kuranishi structures and of families of multisections. We consider $[0,1] \times \mathcal{M}_k(\beta; J)$ and evaluation maps $ev = (ev_0, \ldots, ev_{k-1}) : [0,1] \times \mathcal{M}_k(\beta; J) \to L^k$, $ev_t : [0,1] \times \mathcal{M}_k(\beta; J) \to [0,1]$.

As in [5] Section 11 Lemmas 11.1, 11.2, we have a system of Kuranishi structures and continuous families of multisections on $[0,1] \times \mathcal{M}_k(\beta; J)$ with the following properties:

1. The families of multisections are transversal to $0$.
2. It is compatible with the forgetful map $[0,1] \times \mathcal{M}_k(\beta; J)$.
3. For $k \geq 1$ the evaluation map
   $$(ev_t, ev_0) : [0,1] \times \mathcal{M}_k(\beta; J) \to [0,1] \times L$$
   is weakly submersive and induces a submersion of the zero set of family of multisections, in the sense of [5] Definition 4.1.4.
4. They are invariant under the cyclic permutation of the boundary marked points.
5. It is compatible with the identification $[20]$.
6. It is compatible with the identification $[27]$.
7. $ev_t : [0,1] \times \mathcal{M}_0(\beta) \to [0,1]$ is weakly submersive and induces a submersion on the zero set of family of multisections, in the sense of [5] Definition 4.1.4.
8. At $t_0 = 0, 1$ the induced Kuranishi structure and families of multisections on $\{t_0\} \times \mathcal{M}_k(\beta)$ coincides with given two choices of Kuranishi structures and of families of multisections.

In [5] Section 11, we defined a pseudo-isotopy of cyclic filtered $A_\infty$ algebra as follows. Let $\rho_1, \ldots, \rho_k \in \Lambda(L)$. We put

$$\text{Corr}_x([0,1] \times \mathcal{M}_{k+1}(\beta; J); (ev_1, \ldots, ev_k), (ev_1, ev_t))(\rho_1 \times \ldots \times \rho_k)$$

$$= \rho(t) + dt \wedge \sigma(t),$$

and define

$$m^{k, \beta}_{k, \beta}(\rho_1, \ldots, \rho_k) = \rho(t), \quad c^{k, \beta}_{k, \beta}(\rho_1, \ldots, \rho_k) = \sigma(t).$$

We next define $m^{k, \beta}_{1, \beta}$. Let $\text{tri} : [0,1] \times \mathcal{M}_0(\beta; J) \to \text{pt}$ be an obvious map to a point. We take $1 \in \Lambda^0(\text{pt}) = \mathbb{R}$ and put

$$\text{Corr}_x([0,1] \times \mathcal{M}_0(\beta; J); \text{tri}, ev_t)(1) = \rho(t) + dt \wedge \sigma(t).$$

We then define

$$m^{k, \beta}_{1, \beta} = \rho(t).$$
Lemma 4.1. $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{c_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ above defines a pseudo-isotopy of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_0}$.

Proof. In [5] Section 11 it is proved that $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{c_{k, \beta}^i\})$ is a pseudo-isotopy of cyclic filtered $A_\infty$ algebra modulo $T_{E_0}$. Therefore it suffices to check (18).

Let $0 \leq t_1 < t_2 \leq 1$. We have:

$$\partial(t_2, t_2) = \left(\{t_1, t_2\} \times M_0(\beta; J)\right)$$

$$\cup \bigcup_{\beta_1 + \beta_2 = \beta} \left(\{t_1, t_2\} \times M_1(\beta_1)\right) \times (t_1, t_2) / \mathbb{Z}_2.$$ We then can use Lemma 4.2 in the same way as [5] Section 12 and [7] Section 7.2, to extend $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ to an inhomogeneous cyclic filtered $A_\infty$ algebra.

Proof. We may assume that $A_\infty$ algebra modulo $T_{E_0}$ extended to a pseudo-isotopy of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_0}$ between them.

Then, $(C, \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ can be extended to a $G$-gapped inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_0}$ between them.

The proof of Proposition 4.1 is now complete. □

We thus proved mod $T_{E_0}$ version of Theorem 4.1. We next prove the following inhomogeneous version of Theorem 8.1 [5].

Lemma 4.2. Let $0 < E_0 < E_1$ and $(C, \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ be $G$-gapped inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_i}$, for $i = 0, 1$. Let $(C, \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{c_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ be a pseudo-isotopy of $G$-gapped inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_0}$ between them.

Then, $(C, \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ can be extended to a $G$-gapped inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_1}$ and $(C, \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{c_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ can be extended to a pseudo-isotopy of $G$-gapped inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_1}$ between them.

Proof. We may assume that $G \cap [E_0, E_1] = \{E_0\}$. In [5] Theorem 8.1 the extension to cyclic filtered $A_\infty$ algebra mod $T_{E_1}$ and extension to pseudo-isotopy of cyclic filtered $A_\infty$ algebra mod $T_{E_1}$ are obtained. So it suffices to find $m_{-1, \beta}$ for $E(\beta) = E_0$. We define

$$m_{-1, \beta}^i = m_{-1, \beta}^i + \sum_{\beta_1 + \beta_2 = \beta} \int_{t_1}^{t_2} \langle c_{0, \beta_1}^i (1), m_{0, \beta_2}^i (1) \rangle dt.$$

It is easy to check (18). □

We next construct gapped inhomogeneous cyclic filtered $A_\infty$ algebra $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$. Let $E_i$ be sequence $0 < \ldots < E_i < E_{i+1} < \ldots$. We obtain a sequence $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_i}$ for each $i$. By Proposition 4.1 we have a pseudo-isotopy of inhomogeneous cyclic filtered $A_\infty$ algebra modulo $T_{E_i}$ between $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{c_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$ and $(\Lambda(L), \langle \cdot \rangle, \{m_{k, \beta}^i\}, \{m_{-1, \beta}^i\})$.
We thus proved Theorem 1.1.3. The proof of Theorem 1.1 is now complete.

Furthermore, the construction of pseudo-isotopy of pseudo-isotopies in [5] Section 14 imply (34) \( \Psi((f_i)_*(b)) \equiv \Psi(b) \mod T^{E_i} \).

For \( i > j \), let

\begin{equation}
(f_i)_*: \mathcal{M}(\Lambda(L), \{m^i_{k,j}\}; \Lambda^+) \cong \mathcal{M}(\Lambda(L), \{m^i_{k,j}\}; \Lambda^+).
\end{equation}

and

\begin{equation}
(c_{i,j}')_*: \mathcal{M}(\Lambda(L), \{m^{i,j}_{k,j}'\}; \Lambda^+) \cong \mathcal{M}(\Lambda(L), \{m^{i,j}_{k,j}'\}; \Lambda^+).
\end{equation}

be the isomorphisms induced by the pseudo-isotopies. We have

\begin{equation}
\Psi((c_{i,j}')_*(b)) = \Psi((f_i)_*(b)).
\end{equation}

Furthermore, the construction of pseudo-isotopy of pseudo-isotopies in [5] Section 14 imply

\begin{equation}
(f_j)_* \circ (c_{i,j})_* = (c_{i,j}')_* \circ (f_i)_*.
\end{equation}

(34), (35), (36) immediately imply

\begin{equation}
\Psi((f_i)_*(b)) = \Psi(b).
\end{equation}

We thus proved Theorem 1.1.3. The proof of Theorem 1.1 is now complete. □

5. Relation to canonical model

In [7] Subsection 5.4.4 and [5] Section 10, we defined canonical model \((H, \langle \cdot, \cdot \rangle, \{m^{\text{can}}_{k,j}\})\) of \( G \)-gapped cyclic filtered \( A^\infty \) algebra \((C, \langle \cdot, \cdot \rangle, \{m_{k,j}\})\). (We assumed \( \mathcal{C} \) is either finite dimensional or de Rham complex \( \Lambda(L) \).) We also constructed a \( G \)-gapped cyclic filtered \( A^\infty \) homomorphism \( f: H \to C \), which is a homotopy equivalence. Suppose that \((C, \langle \cdot, \cdot \rangle, \{m_{k,j}\}; \{m_{-1,j}\})\) is an inhomogeneous \( G \)-gapped cyclic filtered \( A^\infty \) algebra. In this section, we will define \( m_{-1,j}^{\text{can}} \) so that \( f_*: \mathcal{M}(\Lambda; \Lambda^+) \to \mathcal{M}(C; \Lambda^+) \) preserves superpotential.
To define $m^{can}_{-1, \beta}$ we need some notations. We use results and notations of [5] Sections 9 and 10 in this section.

Let $T$ be a ribbon tree. Let $C_0(T)$ be the set of vertices. We assume that we have its decomposition $C_0(T) = C_0^\text{int}(T) \sqcup C_0^\text{ext}(T)$ to interior vertices and exterior vertices. Let $\beta(\cdot) : C_0^\text{int}(T) \to G$ be a map to a discrete submonoid $G$ of $\mathbb{R}_{\geq 0}$.

**Definition 5.1.** We denote by $Gr^-(k, \beta)$ the set of $\Gamma = (T, C_0^\text{int}(T), C_0^\text{ext}(T), \beta(\cdot))$ such that: (1) $\sum_{v \in C_0^\text{int}(T)} \beta(v) = \beta$. (2) $\# C_0^\text{ext}(T) = k$. (3) If $\beta(v) = 0$, then $v$ has at least 3 edges.

The automorphism group $\text{Aut}(\Gamma)$ of an element $\Gamma = (T, C_0^\text{int}(T), C_0^\text{ext}(T), \beta(\cdot))$ of $Gr^-(k, \beta)$ is the set of isomorphisms $\phi : T \to T$ of ribbon tree which preserves the decomposition $C_0(T) = C_0^\text{int}(T) \sqcup C_0^\text{ext}(T)$ and such that $\beta(\phi(v)) = \beta(v)$.

We remark that $k = 0, 1, \ldots$ in $Gr^-(k, \beta)$. The case $k = 0$ is included. We also remark that the automorphism of rooted ribbon tree is trivial.

Let $(v, e)$ be a flag of $\Gamma$, that is a pair of an interior vertex $v$ and an edge $e$ containing $v$. Let $b \in \overline{C}$. We are going to define $m(\Gamma; b) \in \mathbb{R}$.

Let $T_0, \ldots, T_l$ be the irreducible components of $\Gamma \setminus \{v\}$. We enumerate them so that $e \in T_0$ and they respect counter clockwise cyclic order of $\mathbb{R}^2$. Together with the data induced from $\Gamma$, the tree $T_i$ defines an element $\Gamma_i \in Gr(k_i, \beta_i)$. Here $Gr(k_i, \beta_i)$ is as in [5] Definition 9.1. Namely its element is an element of $Gr^-(k_i, \beta_i)$ together with a choice of a base point which is an exterior vertex. In our situation the base points of $\Gamma_i$ are $v$ for all $i$.

**Definition 5.2.**

$$m(\Gamma, v, e; b) = (m_{\ell, \beta(v)}(f_1(b, \ldots, b), \ldots, f_\ell(b, \ldots, b)), f_0(b, \ldots, b)).$$

Here $f_\ell$ is defined in [5] section 10.

We remark that there is no sign in Definition 5.2 since the degree of $b$ after shifted is even.

**Lemma 5.1.** $m(\Gamma, v, e; b)$ is independent of $v$ and $e$ and depends only on $\Gamma$ and $b$.

This is Proposition 10.1 [5]. Hereafter we write $m(\Gamma; b)$ in place of $m(\Gamma, v, e; b)$.

**Definition 5.3.**

$$m^{can}_{-1, \beta} = \sum_{\Gamma \in Gr^-(0, \beta)} \frac{m(\Gamma)}{\# \text{Aut}(\Gamma)}.$$  

We remark that we write $m(\Gamma)$ instead of $m(\Gamma; b)$, since in the case of $\Gamma \in Gr^-(0, \beta)$ there is no exterior vertex and hence $b$ never appears.

$(H; \langle \cdot, \cdot \rangle, \{m^{can}_k\}, \{m^{can}_{-1, \beta}\})$ is an inhomegeneous $G$-gapped cyclic filtered $A_\infty$ algebra. Let

$$\Psi^{can} : \mathcal{M}(H; \Lambda_+) \to \Lambda_+$$

be its superpotential. The filtered $A_\infty$ homomorphism $f : H \to C$ induces $f_* : \mathcal{M}(H; \Lambda_+) \to \mathcal{M}(C; \Lambda_+)$ by

$$f_*(b) = \sum_{k=0}^{\infty} \sum_{\beta \in G} T^{E(\beta)} f_{k, \beta}(b, \ldots, b).$$

The main result of this section is:
Theorem 5.1.

\( (38) \quad \Psi(f^*(b)) = \Psi^{\text{can}}(b). \)

Remark 5. We consider the case of \( C = \Lambda(L) \) with \( H^1(L; \mathbb{R}) = 0 \). Then since \( H^1 = 0 \), the set \( \mathcal{M}(L; \Lambda_+) \) consists of one point \( 0 \). Therefore \( \mathcal{M}(C; \Lambda_+) \) also consists of one point. The invariant of Corollary 1.1 is the value of superpotential at this point.

Theorem 5.1 implies that this invariant is

\( (39) \quad \sum_{\beta \in G} T^{E(\beta)} m_{-1,\beta}^{\text{can}} = \sum_{\beta \in G} \sum_{\Gamma \in \text{Gr}^{-1}(0,\beta)} T^{E(\beta)} m(\Gamma). \)

Proof of Theorem 5.1. Let \( b \in H^1 = H^1 \otimes \Lambda_+ \). We define

\( (40) \quad \Phi(b) = \sum_{k=0}^{\infty} \sum_{\beta \in G} \sum_{\Gamma \in \text{Gr}^{-1}(k,\beta)} T^{E(\beta)} m(\Gamma; b) \# \text{Aut}(\Gamma). \)

Lemma 5.2.

\( \Phi(b) = \Psi^{\text{can}}(b). \)

Proof. In view of Definition 5.3 it suffices to prove:

\( (41) \quad \langle m_{k,\beta}(b, \ldots, b), b \rangle = (k + 1) \sum_{\Gamma \in \text{Gr}^{-1}(k+1,\beta)} m(\Gamma; b) \# \text{Aut}(\Gamma). \)

We will prove (41) below.

Let \( \Gamma \in \text{Gr}^{-1}(k+1,\beta) \). Let \( \{v_0, \ldots, v_k\} = C_0^\text{ext}(\Gamma) \) such that \( v_0, \ldots, v_k \) respects the counter clockwise cyclic order of \( \mathbb{R}^2 \). Let \( e_i \) be the unique edge containing \( v_i \).

We define \( v'_i \) by \( \partial e_i = \{v_i, v'_i\} \).

By definition we have:

\[ m_{k,\beta}(b, \ldots, b) = \sum_{\Gamma \in \text{Gr}^{-1}(k+1,\beta)} \sum_{i=0}^{k} m(\Gamma, v_i; b, \ldots, b) \# \text{Aut}(\Gamma). \]

This is because \( (\Gamma, v_i) \in \text{Gr}(k+1,\beta) \) and \( (\Gamma, v_i) \) is the same element as \( (\Gamma, v_j) \) in \( \text{Gr}(k+1,\beta) \) if and only if there exists an element of \( \phi \in \text{Aut}(\Gamma) \) such that \( \phi(v_i) = v_j \).

Moreover

\[ \langle m_{r,\epsilon_i}(b, \ldots, b), b \rangle = m(\Gamma, v_i, e_i; b, \ldots, b), \]

where the right hand side in defined in Definition 10.1 [5]. By Proposition 10.1 [5], \( m(\Gamma, v_i, e_i; b, \ldots, b) \) is independent of \( i \) and is \( m(\Gamma; b) \). This implies (41). The proof of Lemma 5.2 is complete.

The next proposition completes the proof of Theorem 5.1.

Proposition 5.1. If \( b \in \widetilde{\mathcal{M}}(L; \Lambda_+) \), then we have:

\( (42) \quad \Phi(b) = \Psi^{\text{can}}(f^*(b)). \)

Proof.
Lemma 5.3.

\[
\sum_{(\ell,\beta) \neq (1,0)} \frac{TE(\beta)}{\ell + 1} (m_{\ell,\beta}(f_*(b), \ldots, f_*(b)), f_*(b))
\]

\[= \sum_{k=0}^{\infty} \sum_{\beta \in \Gamma} \sum_{\Gamma \in Gr^{-}(k, \beta)} \frac{TE(\beta)}{\#Aut(\Gamma)} \#C_{1}^{\text{int}}(\Gamma)m(\Gamma; b). \tag{43}\]

Proof. Let \(\Gamma \in Gr^{-}(k, \beta)\) and \((v, e)\) its flag. We obtain the irreducible components \(\Gamma_0, \ldots, \Gamma_\ell\) of \(\Gamma \setminus v\) as before. By definition we have

\[\langle m_{\ell,\beta(v)}(f_1(b), \ldots, b), f_1(b), \ldots, b\rangle = m(\Gamma; b). \tag{44}\]

We remark that the right hand side is independent of \((v, e)\) by Proposition 10.1 [5]. If we take the sum of (44) over all \(\Gamma, v\) with weight \(TE(\beta)/\#Aut(\Gamma)\) then we obtain the right hand side of (43). On the other hand, if we take the sum of the right hand side of (44) over all \(\Gamma, v, e\) with weight \(TE(\beta)\) we obtain

\[\sum_{(\ell,\beta) \neq (1,0)} \frac{TE(\beta)}{\ell + 1} (m_{\ell,\beta}(f_*(b), \ldots, f_*(b)), f_*(b)). \tag{45}\]

Since the choice of \(e\) for given \(\Gamma, v\) is \(\ell + 1\), we obtain (43). \(\square\)

Lemma 5.4.

\[\langle m_{1,0}(f_*(b)), f_*(b)\rangle = -2 \sum_{k=0}^{\infty} \sum_{\beta \in \Gamma} \sum_{\Gamma \in Gr^{-}(k, \beta)} \frac{TE(\beta)}{\#Aut(\Gamma)} \#C_{1}^{\text{int}}(\Gamma)m(\Gamma; b). \tag{46}\]

Here \(C_{1}^{\text{int}}(\Gamma)\) is the set of interior edges.

Proof. Let \((v, e)\) be a flag of \(\Gamma \in Gr^{-}(k, \beta)\) such that \(e\) is an interior edge. We define \(m'(\Gamma, e, v; b)\) as follows. Let \(T_{(0)}^{(1)}\) be the irreducible components of \(\Gamma \setminus e\) such that \(T_{(0)}\) contains \(v\). We put \(T_{(1)} = T_{(1)}^{(1)} \cup e\). Using the data induced from \(T_{(0)}\), the trees \(T_{(0)}, T_{(1)}\) induce \(\Gamma_{(0)} \in Gr(k_{(0)}, \beta_{(0)}), \Gamma_{(1)} \in Gr(k_{(1)}, \beta_{(1)})\). (The roots of \(\Gamma_{(0)}, \Gamma_{(1)}\) are \(v\).) We define

\[m'(\Gamma, e, v; b) = \langle m_{1,0}(f_{(1)}(b), \ldots, b), f_{(0)}(b), \ldots, b\rangle. \tag{47}\]

Let \(v_0, \ldots, v_k\) be the set of exterior vertices of \(\Gamma\). Let \(e_i\) be the edge containing \(v_i\) and and \(\partial e_i = \{v_i, v'_i\}\).

Sublemma 5.1. If \(v \neq v'_i\), \((i = 0, \ldots, k)\) then

\[m'(\Gamma, e, v; b) = -m(\Gamma; b). \tag{48}\]

If \(v = v'_i\) then

\[m'(\Gamma, e, v; b) = -m(\Gamma; b) + (m_{|v_i,v_j}, b, \ldots, b, b). \tag{49}\]

Here \((\Gamma, v_i) \in Gr(k, \beta)\) and \(m(\Gamma, v_i)\) is defined in [5] Section 10.

Proof. We use Lemma 10.1 [5], its proof and notations there, during the proof of Sublemma 5.1.

Let \(\Gamma, v, e\) be as in Sublemma 5.1. We put \(\partial e = \{v, v'\}\). Let \(T_0, \ldots, T_m\) be the irreducible components of \(\Gamma \setminus v'\). We enumerate them so that \(v \in T_0\) and it respects
counter clockwise cyclic order of $\mathbb{R}^2$. $T_i$ together with the data induced from $\Gamma$ becomes $\Gamma_i$, whose root is $v'$. By definition

$$\Gamma_{(i)} = \Gamma_1 \cup \ldots \cup \Gamma_m \cup e.$$ 

Therefore by the definition in [5] Section 10, we have

$$(49) \quad f_{\Gamma_{(1)}}(b, \ldots, b) = (G \circ m_{m,\beta(v')})(f_{\Gamma_1}(b, \ldots, b), \ldots, f_{\Gamma_m}(b, \ldots, b)).$$

Therefore

$$(50) \quad m'(\Gamma, e; v; b) = \langle (m_{1,0} \circ G \circ m_{m,\beta(v')})(f_{\Gamma_1}(b, \ldots, b), \ldots, f_{\Gamma_m}(b, \ldots, b), f_{\Gamma_{(0)}}(b, \ldots, b)),$$

By Lemma 10.1 [5] we have

$$(51) \quad m_{1,0} \circ G = -G \circ m_{1,0} + \Pi - \text{identity}.$$ 

We first assume $v \neq v'$. Then $\Gamma_{(0)} \in Gr(k_{(0)}, \beta_{(0)})$ with $(k_{(0)}, \beta_{(0)}) \neq (1, 0)$. It follows that $f_{\Gamma_{(0)}}(b, \ldots, b) \in \text{Im} G$. We remark that

$$\langle \text{Im} G, \text{Im} G + \text{Im} \Pi \rangle = 0.$$

Therefore

$$m'(\Gamma, e; v; b) = \langle -m_{m,\beta(v')}(f_{\Gamma_1}(b, \ldots, b), \ldots, f_{\Gamma_m}(b, \ldots, b), f_{\Gamma_{(0)}}(b, \ldots, b)),\rangle$$

as required.

If $v = v_i$ then $f_{\Gamma_{(i)}}$ is identity. Therefore

$$m'(\Gamma, e; v; b) = \langle -m_{m,\beta(v')}(f_{\Gamma_1}(b, \ldots, b), \ldots, f_{\Gamma_m}(b, \ldots, b)),\rangle$$

The proof of sublemma is complete. □

Using Maurer-Cartan equation for $b$ we find

$$\sum_{k=0}^{\infty} \sum_{\beta \in G} \sum_{\Gamma \in Gr^-(k, \beta)} \sum_{i=0}^{k} \frac{T^{E(\beta)}}{\# \text{Aut}(\Gamma)} m_{m,\beta(v')}(h_i, b) = 0.$$

Therefore the sum of the second term of (48) vanishes. Lemma 5.4 now follows from Sublemma 5.1. □

Since $\Gamma$ is a tree we have $\# C_0^{\text{int}}(\Gamma) - \# C_1^{\text{int}}(\Gamma) = 1$. Therefore Lemmas 5.3 and 5.4 imply Proposition 5.1. □

Using the proof of Theorem 5.1 and [5] Section 9, we can prove the following:

**Theorem 5.2.** If two gapped inhomogeneous cyclic filtered $A_\infty$ algebras are pseudo-isotopic to each other, then so are their canonical models.

We omit the proof since it is a straightforward analog and we do not use Theorem 5.2 in this paper.
6. Wall crossing formula

In this section we prove Theorem 1.3. We first review the definition of the number (8) in more detail.

We remark that (8) is a rational number since we can use multi (but finitely many) valued section of \( M^{cl}_1(\alpha; J) \) to define it. (The argument to do so is the same as [10].)

On the other hand, to prove Theorem 1.3 we need to choose a perturbation of \( M^{cl}_1(\alpha; J) \) so that it is compatible with one in \( M_k(\beta; J) \). Here

\[
M_k(\beta; J) = \bigcup_{t \in [0,1]} \{t\} \times M_k(\beta; J_t).
\]

Since we use \textit{continuous family of} multi-sections to perturb \( M_k(\beta; J) \), we need to use \textit{continuous family of} multi-sections also for \( M^{cl}_1(\alpha; J) \). Actually this is the way taken in [5] Sections 3 and 5.

There exists a Kuranishi structure and \textit{continuous family of} multi-sections on \( M^{cl}_1(\alpha; J) \) with the following properties:

(1) The evaluation map

\[
(ev_t, ev^{\text{int}}) : M^{cl}_1(\alpha; J) \to [0,1] \times M
\]

is \textit{weakly submersive}.

(2) Continuous family of multi-sections is transversal to 0 and \( (ev_t, ev^{\text{int}}) \) induces \textit{submersion} on its zero set.

(3) The image of the restriction of \( (ev_t, ev^{\text{int}}) \) to the zero set of \textit{continuous family of} multi-sections is disjoint from \( \{0,1\} \times L \).

This is Lemmas 3.2 and 5.3 of [5]. Let \( \text{tri} : M^{cl}_1(\alpha; J) \to \text{pt} \) be the trivial map. We use the above \textit{continuous family of} multi-sections and define

\[
\text{Corr}(M^{cl}_1(\alpha; J); \text{tri}, ev^{\text{int}})(1) \in \Lambda(M).
\]

\textbf{(54)} is a smooth differential form of degree

\[
\dim \mathbb{R} M - \dim \mathbb{R} M^{cl}_1(\alpha; J) = 6 - 6 + c^1(M) \cap [\alpha] + 2 - 6 + 1 = 3.
\]

**Definition 6.1.** We put:

\[
n(L; \alpha; J) = \int_L \text{Corr}(M^{cl}_1(\alpha; J); \text{tri}, ev^{\text{int}})(1) \in \mathbb{R}.
\]

We also define:

\[
n(L; \alpha; J; t) = \int_L \text{Corr}(M^{cl}_1(\alpha; J) \cap ev^{-1}_t([0,t]); \text{tri}, ev^{\text{int}})(1) \in \mathbb{R}.
\]

The submersivity of \( (ev_t, ev^{\text{int}}) \) implies that \( n(L; \alpha; J; t) \) is a smooth function of \( t \).

**Theorem 6.1.** \textit{In the situation of Theorem 1.3} \( (\Lambda(L), \langle \cdot , \cdot \rangle, \{m_{\lambda_0}^L\}, \{m_{-1,\lambda}^L\}) \) is pseudo-isotopic to \( (\Lambda(L), \langle \cdot , \cdot \rangle, \{m^L_{\lambda_0}\}, \{m^L_{-1,\lambda} + \Delta(\beta)\}) \) as \textit{inhomogeneous gapped cyclic filtered} \( A_\infty \) algebras. Here

\[
\Delta(\beta) = \sum_{\beta_{\lambda_0} = \beta} n(L; \beta; J).
\]
Proof. We consider the moduli space \((52)\) and evaluation map
\[(ev_t, ev) = (ev_t, ev_0, \ldots, ev_{k-1}) : \mathcal{M}_k(\beta; \mathcal{J}) \rightarrow [0, 1] \times L^k.\]
By \([5]\) Section 11 we have a system of Kuranishi structures and families of multisections on \(\mathcal{M}_k(\beta; \mathcal{J})\) for \(\beta \cap \omega < E_0\), with the following properties:

1. The families of multisections are transversal to 0.
2. They are compatible with the forgetful map \(\text{forget}_{k, \beta} : \mathcal{M}_k(\beta; \mathcal{J}) \rightarrow \mathcal{M}_0(\beta; \mathcal{J})\).
3. For \(k \geq 1\) the evaluation map
\[(ev_t, ev_0) : \mathcal{M}_k(\beta; \mathcal{J}) \rightarrow [0, 1] \times L\]
is weakly submersive and induces a submersion of the zero set of family of multisections, in the sense of \([5]\) Definition 4.1.4.
4. They are invariant under the cyclic permutation of the boundary marked points.
5. They are compatible with the identification \((26)\).
6. We consider the decomposition:
\[
\partial \mathcal{M}_0(\beta; \mathcal{J}) = \bigcup_{\beta_1 + \beta_2 = \beta} (\mathcal{M}_1(\beta_1; \mathcal{J})_{ev_t, ev_0} \times (ev_t, ev_0) \mathcal{M}_1(\beta_2; \mathcal{J})) / \mathbb{Z}_2
\]
\[
\bigcup \bigcup_{t \in [0, 1]} \bigcup_{\beta_3, \beta_4(\beta) = \beta} \{t\} \times \left(M^c_1(\beta; J_t)_{ev_0} \times_M L\right).
\]

Then the Kuranishi structures and the families of multisections are compatible with \((59)\). We use the Kuranishi structure and families of multisections on \(\mathcal{M}^c_1(\tilde{\beta}; J_t)\) which is explained in this section for the second term of the right hand side of \((56)\).

7. The evaluation map, \(ev_t : \mathcal{M}_0(\beta; \mathcal{J}) \rightarrow [0, 1]\) is weakly submersive and induces a submersion of the zero set of family of multisections, in the sense of \([5]\) Definition 4.1.4.
8. At \(t_0 = 0, 1\) the induced Kuranishi structure and families of multisections on \(\mathcal{M}_k(\beta; \mathcal{J}) \cap ev_t^{-1}\{t_0\}\) coincides with given choices Kuranishi structures and families of multisectons on \(\mathcal{M}_k(\beta; J_{t_0})\).

This is mostly the same as one we used in the proof of Proposition 4.1. The only difference is the second term of \((59)\). It appears since the fiber product \(\mathcal{M}^c_1(\tilde{\beta}; J_t)_{ev_0} \times_M L\) can be nonempty in the situation where we consider one parameter family of complex structures.

We now define \(m^t_{k, \beta}, c^t_{k, \beta}\) for \(k \geq 0\) in the same way as \((30), (31)\) using \(\mathcal{M}_k(\beta; \mathcal{J})\) in place of \([0, 1] \times \mathcal{M}_k(\beta; \mathcal{J})\).

We finally define \(m^t_{-1, \beta}\) as follows. We put:
\[
(57) \quad \text{Corr}_* (\mathcal{M}_0(\beta; \mathcal{J}); \text{tri}, ev_t)(1) = \rho(t) + dt \wedge \sigma(t)
\]
and define
\[
(58) \quad m^t_{-1, \beta} = \rho(t) + \sum_{\beta_3, \beta_4(\beta) = \beta} n(L; \tilde{\beta}; \mathcal{J}; t).
\]
We can prove \((\Lambda(L), \langle \cdot, \cdot \rangle, \{m^t_{k, \beta}\}, \{c^t_{k, \beta}\})\) is a pseudo-isotopy of gapped cyclic filtered \(A_\infty\) algebra mod \(T E_0\) in the same way as \([5]\) Section 11.
To prove \((\Lambda(L), \langle \cdot, \cdot \rangle, \{m^i_{k, \beta}\}, \{c^i_{k, \beta}\}, \{m^f_{k, \beta}\})\) is an \textit{inhomogeneous} pseudo-isotopy of gapped cyclic filtered \(A_\infty\) algebra mod \(T^{E_0}\) it suffices to prove (18). Let \(0 \leq t_1 < t_2 \leq 1\). We have:
\[
\partial \left( M_0(\beta; J) \cap ev_t^{-1}([t_1, t_2]) \right) = (\{t_1\} \times M_0(\beta; J_{t_1})) \cup (\{t_2\} \times M_0(\beta; J_{t_2}))
\]
\[
(62) \quad \Psi(b, \beta)
\]
\[
(61)
\]
Let \(y = \exp(x)\). We regard the superpotential \(\Psi(b; J)\) as a function of \(x_i\) then we have:
\[
\Psi(b; J) = \sum_{\beta \in G} T^{\beta \cap \phi} m^j_{-1, \beta} y^{\beta \beta}.
\]
\[
(60)
\]
We remark \(m^1_{-1, \beta} = m^j_{-1, \beta} + \sum_{\beta \cap \phi(\beta) = \beta} n(L; \beta; J)\). The proof of Theorem 6.1 is complete. (Actually we need to go from modulo \(T^{E_0}\) version to Theorem 6.1 itself. We omit this part since it is the same as one for Theorems 1.1 and 4.1.)

7. Convergence

In this section we prove Theorem 1.2. Actually most of the ideas of the proof is in [5] Section 13. Let \(b = \sum_{i=1}^{b_1} x_i e_i\), where \(e_i\) is a basis of \(H^1(L; \mathbb{R})\). We put \(y_i = e^{x_i}\). For \(\beta \in H_2(X, L; \mathbb{Z})\) we define \(\partial_i \beta \in \mathbb{Z}\) by \(\partial \beta = \sum_{i=1}^{b_1} \partial_i \beta e_i\) and define
\[
y^{\beta \beta} = \prod_{i=1}^{b_1} y_i^{\beta \beta}.
\]

**Theorem 7.1.** We regard the superpotential \(\Psi(b; J)\) as a function of \(x_i\) then we have:
\[
\Psi(b; J) = \sum_{\beta \in G} T^{\beta \cap \phi} m^j_{-1, \beta} y^{\beta \beta}.
\]

Theorem 1.2 follows immediately from Theorem 7.1.

**Proof.** Let \(\rho\) be a closed one form on \(L\). By definition we have
\[
\langle m_{k, \beta}^j(\rho, \ldots, \rho), \rho \rangle = \text{Corr}(M_k(\beta; J); (ev_1, \ldots, ev_k, ev_0), \text{tri})(\rho \times \cdots \times \rho) \in \Lambda^0(\pt) = \mathbb{R}.
\]
Then, by the same argument as the proof of Lemma 13.1 [5], we have
\[
\langle m_{k, \beta}^j(\rho, \ldots, \rho), \rho \rangle = \frac{1}{k!} (\rho \cap \partial \beta)^{k+1} m^j_{-1, \beta}.
\]

Theorem 7.1 follows easily. \(\square\)
We turn to the proof of Theorem 1.2. We take a Weinstein neighborhood $U$ of $L$. Namely $U$ is symplectomorphic to a neighborhood $U'$ of zero section in $T^* L$. We choose $\delta_1$ so that for $c = (c_1, \ldots, c_n) \in [-\delta_1, +\delta_1]^{\mathbb{N}}$ the graph of the closed one form $\sum_{i=1}^{n} c_i e_i$ is contained in $U'$. We send it by the symplectomorphism to $U$ and denote it by $L(c)$. We may take $\delta_2 < \delta_1$ such that if $c = (c_1, \ldots, c_n) \in [-\delta_2, +\delta_2]^{\mathbb{N}}$ then there exists a diffeomorphism $F_c : M \to M$ such that

\[
F_c(L) = L(c),
\]

\[
(F_c)_* J \text{ is tamed by } \omega.
\]

Then we have an isomorphism

\[
\mathcal{M}_0(L(c); (F_c^{-1})^*(\beta), (F_c)_* J) \cong \mathcal{M}_0(L; \beta, J).
\]

We can extend this isomorphism to their Kuranishi structures and family of multisections on them. We can then use Proposition 5.1 and (65) to obtain:

\[
m^J_{-1, \beta, L} = m^J_{-1, \beta, L(c)}.
\]

Here we include $L$ and $L(c)$ in the notation to clarify the Lagrangian submanifold we study. Theorem 7.1 and [5] Lemma 13.5 then implies:

\[
\Psi(y; L(c); (F_c)_* (J)) = \Psi(y(c); L; J),
\]

where we put $y(c)_i = T^{-c_i, \partial_i, \beta} y_i$. In (67) we include $L$ in the notation of superpotential to clarify the Lagrangian submanifold we study. We regard superpotential as a function of $y_i$ by using Theorem 7.1.

Since the right hand side converges in $\Lambda \langle y_1, \ldots, y_{b_1}, y_1^{-1}, \ldots, y_{b_1}^{-1} \rangle$, it follows that $\Psi(y(c); L; J)$ converges for $c = (c_1, \ldots, c_n)$ with $|c_i| < \delta$. This implies Theorem 7.1.2.

3 and 4 of Theorem 7.1 follows from Theorem 1.1. The proof of Theorem 7.1 is complete.

Once the convergence is established, Propositions 2.1, 2.2 and Theorems 3.1, 4.1, 5.1 are generalized in the same way to our larger domain of convergence.

8. Concluding remarks

8.1. Rationality and integrality.

Conjecture 8.1. In the situation of Corollary 1.1, we have $\Psi^{\text{can}}(0; J) \in \mathbb{Q}$.

We remark that filtered $A_\infty$ structure on $H(L)$ is constructed in [7] over $\Lambda^Q_{0, \text{nov}}$. In [5] and in this paper, we work over $\mathbb{R}$ coefficient to use continuous family of multisections and de Rham theory for construction. This is the reason why we can not prove Conjecture 8.1 by the method of this paper.

Conjecture 8.2. There exist integers $\theta^j_\beta \in \mathbb{Z}$ for each $\beta \in H_2(M; L; \mathbb{Z})$ such that

\[
\Psi^{\text{can}}(0; J) = \sum_{d \in \mathbb{Z}^+; |\beta| = H_2(M; L; \mathbb{Z})} d^{-2} \theta^j_\beta / d.
\]

This is an analog of the corresponding conjecture for Gromov-Witten invariant of genus zero. (See [11].) The factor $d^{-2}$ is discussed in [10].
8.2. Bulk deformation and generalization to non Calabi-Yau case etc. In this paper we assumed $\dim C = 3$, $c^1(M) = 0$, $\mu_L = 0$. This assumption is used to define $m^J_{-1,\beta}$. Namely it is used to show that the (virtual) dimension of $M_0(\beta; L; J)$ is 0. We may use bulk deformation ([7] Section 3.8) to obtain a numerical invariant in some other cases, as follows.

We consider the moduli space $M_{\ell,k}(\beta; L; J)$ of bordered stable $J$-holomorphic curve of genus zero with $\ell$ interior marked points and $k$ boundary marked points, one boundary component and of homology class $\beta$. Let $\sigma_1, \ldots, \sigma_\ell$ be closed forms on $M$. We may consider

$$\text{Corr}(M_{\ell,0}(L; \beta; J); (ev^\text{int}, \text{tri}))(\sigma_1, \ldots, \sigma_\ell) \in \Lambda^*(\text{pt}) = \mathbb{R}$$

if $* = n + \mu(\beta) - 3 + 2\ell - \sum \deg \sigma_i = 0$.

We obtain similar numbers by considering $M_{\ell,k}(L; \beta; J)$ and differential forms on $L$. The algebraic structure behind this ‘invariant’ is not yet clear to the author. So the study of them is a problem for future research. Another case where numerical invariant is defined is the case when $M$ is a toric manifold and $L$ is its $T^n$ orbit. In that case $M_{\ell,k}(L; \beta; J)$ of $\beta \in H_2(M, L; \mathbb{Z})$ with Maslov index $\geq 2$ only is related to our structures. See [9] and references therein for this case.

8.3. The case of real point. We assume $\dim C = 3$ and let $\tau: M \to M$ be $J$-anti holomorphic involution. We assume that $L = \{x \in M \mid \tau(x) = x\}$ is nonempty. Then it becomes a Lagrangian submanifold. We assume $L$ is $\tau$-relatively spin (See [6] Chapter 8 for its definition. [6] Chapter 8 will become [8].) (If $L$ is spin then it is $\tau$-relatively spin.) Then in [6] Chapter 8 Sections 34 and 38, we constructed $m^J_k,\beta$ such that

$$(69) \quad m^J_{k,\tau,J}(x_1, \ldots, x_k) = (-1)^{k+\sum \deg x_i} m^J_k(x_1, \ldots, x_k)$$

where $* = \sum 0 \leq i < j \leq k \deg x_i \deg x_j$. ([8] Theorem 34.20.) We can combine the construction of [8] with one in [5] and can define inhomogeneous cyclic filtered $A_\infty$ algebra $(\Lambda(L), \langle, \rangle, \{m^J_k, \beta\}, \{m^J_{-1,\beta}\})$ satisfying (69). Moreover $m^J_{-1,\beta}$ satisfies

$$(70) \quad m^J_{-1,\tau,J} = m^J_{-1,\beta}.$$ 

Then its superpotential satisfies

$$(71) \quad \Psi(-b; L; J) = \Psi(b; L; J).$$

In particular $b = 0$ is a critical point.

**Conjecture 8.3.** The critial value $\Psi(0; L; J)$ is equivalent to a particular case of the invariant by Solomon [18].

We can prove $\Psi(0; L; J_0) = \Psi(0; L; J_1)$ if there exists a family of almost complex structures $J_t$ such that $\tau_t J_t = -J_t$. In fact we can show

$$(72) \quad n(\hat{\beta}; L; \mathcal{J}) = -n(\tau_0 \hat{\beta}; L; \mathcal{J})$$

for such $\mathcal{J} = \{J_t\}$.

If we can generalize this construction in a way suggested in Subsection [8.2] it seems likely that we can reproduce the invariants of Solomon and Welschinger [22].

The superpotential we defined in this paper is also likely to be related to the numbers studied by Walcher [21]. (For such a purpose we need to include flat bundle on $L$. In fact in [21] it seems that several flat connections are used to cancel the wall crossing term which appears in [8].)
8.4. Generalization to higher genus and Chern-Simons perturbation theory. The right hand side of the formula \((39)\) has obvious similarity with the invariant of Chern-Simons perturbation theory \((\Pi)\). It seems very likely that we can combine two stories to obtain an invariant counting the number of stable maps from bordered Riemann surface with arbitrary many boundary components and of arbitrary genus. Its rigorous definition is not known at the time of writing of this paper. The author is unable to do it at the time of writing of this paper because of the transversality problem. Here we describe some ideas and explain the difficulty to make it rigorous.

Let \(T\) be a ribbon graph. Namely it is a graph together with a choice of cyclic order of the sets of edges containing each vertices. It uniquely determines a compact oriented 2 dimensional manifold \(\Sigma(T)\) without boundary and an embedding \(i : T \to \Sigma(T)\) such that the cyclic order of the edges are induced by the orientation of \(\Sigma(T)\) and that the connected component of the complement \(\Sigma(T) \setminus T\) are all discs. (We do not assume that \(T\) or \(\Sigma(T)\) is connected.)

Let \(C_0(T)\) be the set of vertices and let \(\ell = \#C_0(T)\). For \(v_i \in C_0(T)\), let \(k_i\) be the number of edges containing \(v_i\). Let \(e_{i,1}, \ldots, e_{i,k_i}\) be the set of such edges. The set of the pair \((v_i, e_{i,j})\) where \(i = 1, \ldots, \ell, j = 1, \ldots, k_i\) is called a flag. Let \(\text{Fl}(T)\) be the set of flags.

We next consider a compact oriented 2 dimensional manifold \(\Sigma\) with boundary \(\partial \Sigma\). We assume \(\partial \Sigma\) has at least \(\ell\) connected components \(\partial_i \Sigma, i = 1, \ldots, \ell\) and on \(\partial_i \Sigma\) we put \(k_i\) boundary marked points. There may be other component of \(\partial \Sigma\), on which we do not put boundary marked points. (We remark that we do not assume that \(\Sigma\) is connected.) Each of the boundary marked points thus corresponds to an element of \(\text{Fl}(T)\).

Let \(\beta \in H_2(M, L; \mathbb{Z})\) where \(M\) is a 6 dimensional symplectic manifold with \(c_1(M) = 0\) and \(L\) its Lagrangian submanifold such that \(H^1(L; \mathbb{Q}) = 0\). We consider the pair \((j, v)\) where \(j\) is a complex structure on \(\Sigma\) and \(v : (\Sigma, \partial \Sigma) \to (M, L)\) is a \(j - J\) holomorphic map. Let \(\mathcal{M}(\Sigma; \beta; L; J)\) be the moduli space of such pair. (We take stable map compactification. It has a Kuranishi structure of dimension \(\#\text{Fl}(T)\).) Evaluation map at each boundary marked points gives

\[
ev : \mathcal{M}(\Sigma; \beta; L; J) \to L^{\#\text{Fl}(T)}.
\]

We next consider the operator \(G : \Lambda(L) \to \Lambda(L)\) of degree +1 as in Lemma 10.1 \([5]\). We can associate a distributional form \(\tilde{G}\) on \(L \times L\) or degree 2 such that

\[
\langle G(u), v \rangle = \int \tilde{G} \wedge (u \times v).
\]

(See \([1]\).) For each edge \(e\) of \(T\) we have \(\pi_e : L^{\#\text{Fl}(T)} \to L^2\), that is the projection to the factors corresponding to \((v, e), (v', e)\) where \(\partial e = \{v, v'\}\). We now ‘define’

\[
\mathfrak{m}(T; \Sigma; \beta; L; J) = \int_{\mathcal{M}(\Sigma; \beta; L; J)} \ev^* \left( \prod_{e \in C_1(T)} \pi_e^*(\tilde{G}) \right).
\]

To define the right hand side of \((74)\) rigorously, we need to take an appropriate perturbation of our moduli space \(\mathcal{M}(\Sigma; \beta; L; J)\) and use it to define its virtual fundamental chain.

The case when the genus of \(\Sigma\) is 0, \(\Sigma\) has only one boundary component, and \(T\) is a tree, is worked out in this paper and \([5]\). In that case, it is important to
find a perturbation so that it is compatible with the process to forget boundary marked points. As we remarked in [3] Remark 3.2, the way we constructed such a continuous family of multisections in this paper and in [4] uses the fact that the genus of \( \Sigma \) is 0. So it cannot be directly generalized to higher genus case.

If we can find appropriate way to rigorously define (74), we then put

\[(75) \quad \Psi(S, T; L; J) = \sum_{T, \Sigma} S^{2 - \chi(\Sigma \# \Sigma(T))} T^{\beta \cap \omega} m(T; \Sigma; \beta; L; J).\]

This is expected to become an invariant of \( M, L, J \).

Here \( \Sigma \# \Sigma(T) \) is defined as follows. For each \( v \in C_0(T) \) we remove a small ball \( B(v) \) centered at \( v \) from \( \Sigma(T) \). We then glue \( \partial B(v_i) \) with the \( i \)-th boundary component of \( \Sigma \). We thus obtain \( \Sigma \# \Sigma(T) \) which is a compact oriented 2 dimensional manifold with or without boundary. \( \chi(\Sigma \# \Sigma(T)) \) is its Euler number. We take the sum for \( T, \Sigma \) such that \( \Sigma \# \Sigma(T) \) is connected. (Here the sum is over topological type of \( \Sigma \) and \( T \). We actually need to divide each term by the order of appropriate automorphism group in a way similar to \( (39) \).) \( S \) is a formal parameter which is called string coupling constant in physics literature.

**Problem 8.1.** Let \( M, L, J \) be a triple of symplectic manifold \( M \), its Lagrangian sub manifold \( L \), and its tame almost complex structure \( J \), such that \( \dim L = 3 \), \( c^1(M) = 0 = \mu_L \), \( H^1(L; \mathbb{Q}) = 0 \). Define an invariant \( \Psi(S, T; L; J) \) such that at \( T = 0 \) it becomes perturbative Chern-Simons invariant and at \( S = 0 \) it becomes the invariant of Corollary 1.1.

**Remark 6.** The study of Chern-Simons perturbation theory suggests that we need to fix framing of \( L \) in order to obtain an appropriate perturbation.

When we generalize the story to the case \( H^1(L; \mathbb{Q}) \neq 0 \), we need to consider the case when \( T \) has exterior vertices and \( \Sigma \) has a boundary marked point on the component other than \( k_i \) components \( \partial_i \Sigma \). In that case we expect to obtain certain algebraic structure on \( H^1(L; \Lambda_0) \). We believe that involutive-bi-Lie infinity structure (4) is appropriate for this purpose. More precisely this is the case when at least one element of \( H^1(L; \mathbb{Q}) \) is assigned to each of the connected component of the boundary. (In genus 0 it corresponds to \( m_{k, \beta} \) with \( k \geq 0 \).) If we restrict to such cases, the wall crossing phenomenon (the \( J \) dependence) does not seem to occur. Namely the algebraic structure is expected to be independent of \( J \) up to homotopy equivalence. (This is certainly the case of genus zero as is proved in [5].)

**8.5. Mirror to Donaldson-Thomas invariant.** Let \( M \) be a symplectic manifold of dimension 6 and \( c^1(M) = 0 \). We consider the set \( \mathcal{Lag}(M) \) of pairs \( (L, [b]) \) such that \( L \) is a relatively spin Lagrangian submanifold with \( \mu_L = 0 \) and \( [b] \in \mathcal{M}(L; \Lambda_0) \).

We say \( (L, [b]) \sim (L', [b']) \) if there exists a Hamiltonian diffeomorphism \( F : M \to M \) such that \( L' = F(L) \) and \( F_*(b) \) is Gauge equivalent to \( b' \). Let \( \mathcal{Lag}(M) \) be the quotient space. The quotient topology on \( \mathcal{Lag}(M) \) is rather pathological. Namely it is likely to be non-Hausdorff in general. We also need to take appropriate compactification of this moduli space by including singular Lagrangian submanifolds, for example. (Such a compactification is not known at the time of writing this paper.)

On the other hand, we can define a ‘local chart’ of \( \mathcal{Lag}(M) \) as follows. Let \( (L, [b_0]) \in \mathcal{Lag}(M) \). We take \( \delta > 0 \) small such that for \( L(c) \) with \( c = \sum c_i e_i \),
$|c_i| < \delta$, there exists $F_c$ as in (63), (64). We consider

$$A(\delta) = \{(y_1, \ldots, y_n) \mid y_i \in \Lambda, |v(y_i)| < \delta\}.$$

Then a neighborhood of $(L, [b_0])$ is identified with the set of $(y_1, \ldots, y_m) \in A(\delta)$ satisfying the Maurer-Cartan equation

$$(76) \quad \sum_{k=0}^{\infty} \sum_{\beta} m_{k,\beta}(y_1, \ldots, y_m) = 0.$$ 

We remark the equation (76) is well defined by Theorem 1.2.

$y_i = e^{x_i} = T^{c_i}y_i$ with $c_i = v(y_i)$ then $b' = \sum \log y'_i e_i$ and $L(c)$ defines an element of $\text{Lag}(M; (F_c) \ast J)$. (See Section 7 and [5] Section 13.) Using the independence of Maurer-Cartan scheme of almost complex structure, we obtain an element of $\text{Lag}(M) = \text{Lag}(M; J)$. Thus, one may regard $\text{Lag}(M)$ as a kind of ‘non-separable rigid analytic stack’.

We remark that the equation (76) is equivalent to

$$\nabla_y \Psi = 0.$$ 

Thus our situation is similar to one which appears in Donaldson-Thomas invariant. (Thomas [19], Joyce [13], Kontsevich-Soibelman [15].) The role of superpotential is taken by the holomorphic Chern-Simons invariant.

**Problem 8.2.**

1. Find an appropriate stability condition for the pair $(L, [b])$ and use it to construct a moduli space $\text{Lag}^{st}(M)$ of stable pairs $(L, [b])$ which has better properties than $\text{Lag}(M)$.

2. Define an invariant which is the ‘order’ of $\text{Lag}^{st}(M)$ in the sense of virtual fundamental cycle.

3. Prove that it coincides with Donaldson-Thomas invariant of the Mirror manifold of $M$.

It seems to the author that this problem is very difficult to study at this stage.

**Remark 7.** After [5] had been put on an arXiv, and at the time of final stage of writing this article, a paper [12] was put on an arXiv, where a different construction of a similar invariant as one in Corollary 1.1 (over $\mathbb{Q}$) is sketched.

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