General Kastler-Kalau-Walze Type Theorems for Manifolds with Boundary II *

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Abstract In this paper, we establish some general Kastler-Kalau-Walze type theorems for any dimensional manifolds with boundary which generalize the results in [WW1].

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1 Introduction

The noncommutative residue found in [Gu],[Wo] plays a prominent role in noncommutative geometry. For one-dimensional manifolds, the noncommutative residue was discovered by Adler [Ad] in connection with geometric aspects of nonlinear partial differential equations. For arbitrary closed compact $n$-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [Wo] using the theory of zeta functions of elliptic pseudodifferential operators. In [Co1], Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogue. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in [Co2]. Let $s$ be the scalar curvature and $W_{\text{res}}$ denote the noncommutative residue. Then the Kastler-Kalau-Walze theorem gives an operator-theoretic explanation of the gravitational action and says that for a 4–dimensional closed spin manifold, there exists a constant $c_0$, such that

$$W_{\text{res}}(D^{-2}) = c_0 \int_M s d\text{vol}_M. \quad (1.1)$$

In [Ka], Kastler gave a brute-force proof of this theorem. In [KW], Kalau and Walze proved this theorem in the normal coordinates system simultaneously. And then, Ackermann proved that the Wodzicki residue $W_{\text{res}}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of $D^2$ in [Ac].

On the other hand, Fedosov etc. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace in [FGLS]. In [Sc],

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Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. We [Wa2] proved a Kastler-Kalau-Walze type theorem for 4-dimensional spin manifolds with boundary. Furthermore, We [Wa3] generalized the definition of lower dimensional volumes in [Po] to manifolds with boundary and found a Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary. In [WW2], we established a general Kastler-Kalau-Walze type theorem for any dimensional manifolds with boundary which generalized the results in [Wa3]. In [WW1], we computed the lower dimensional volume $\text{Vol}_{6}^{1,3}$ for 6-dimensional spin manifolds with boundary and got another Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary. The motivation of this paper is to establish a general Kastler-Kalau-Walze type theorem for any dimensional manifolds with boundary which extends the theorem in [WW1]. Our main result is as follows.

**Theorem 1.1** Let $M$ be a $n = 2m + 2$-dimensional compact spin manifold with the boundary $\partial M$, then

$$\widetilde{\text{Wres}}[\pi^+D^{-1} \circ \pi^+D^{(-2m+1)}] = \frac{(2 - n)(2\pi)^{n/2}}{12\Gamma(n/2)} \int_M s\text{dvol}_M + \frac{2}{1 - n}L_0\text{Vol}(S_{n-2}) \int_{\partial M} K\text{dvol}_{\partial M},$$

(1.2)

where $\text{Vol}(S_{n-2})$ is the canonical volume of the $n - 2$ dimensional sphere $S_{n-2}$ and $s$ is the scalar curvature on $M$ and $K$ is the extrinsic curvature on $\partial M$ as well as $L_0$ is a constant (see (3.42)).

This paper is organized as follows: In Section 2, we recall the definition of the lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, we establish some general Kastler-Kalau-Walze type theorems for any dimensional manifolds with boundary which extend the theorem in [WW1].

## 2 Lower-Dimensional Volumes of Spin Manifolds with boundary

In this section we consider an $n = 2m + 2$-dimensional oriented Riemannian manifold $(M, g^M)$ with boundary $\partial M$ equipped with a fixed spin structure. We assume that the metric $g^M$ on $M$ has the following form near the boundary

$$g^M = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2,$$

(2.1)

where $g^{\partial M}$ is the metric on $\partial M$. Let $U \subset M$ be a collar neighborhood of $\partial M$ which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $h(x_n) \in C^\infty([0, 1))$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric $\tilde{g}$ on $\tilde{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has
the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\hat{g} = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,$$  \hspace{1cm} (2.2)

such that $\hat{g}|_M = g$. We fix a metric $\hat{g}$ on the $\hat{M}$ such that $\hat{g}|_M = g$. We can get the spin structure on $\hat{M}$ by extending the spin structure on $M$. Let $D$ be the Dirac operator associated to $\hat{g}$ on the spinors bundle $S(T\hat{M})$. We want to compute

$$\widetilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D(-2m+1)]$$

(for the related definitions, see , Section 2, 3 [Wa1]).

To define the lower dimensional volume, some basic facts and formulae about Boutet de Monvel’s calculus which can be found in Sec.2 in [Wa1] are needed. Let

$$F : L^2(R_t) \to L^2(R_v); \quad F(u)(v) = \int e^{-ivt} u(t) dt$$

denote the Fourier transformation and $\Phi(\mathbb{R}^+)$, similarly define $\Phi(\mathbb{R}^-)$, where $\Phi(\mathbb{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^+); \quad f \to f_{\mid \mathbb{R}^+}; \quad \mathbb{R}^+ = \{ x \geq 0; x \in \mathbb{R} \}.$$

We define $H^+ = F(\Phi(\mathbb{R}^+)); \quad H^-_0 = F(\Phi(\mathbb{R}^-))$ which are orthogonal to each other. We have the following property: $h \in H^+ \cap H^-$ iff $h \in C^\infty(\mathbb{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{ \text{Im} \xi < 0 \}$ ($\{ \text{Im} \xi > 0 \}$) such that for all nonnegative integer $l$,

$$\frac{d^l h}{d \xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l c_k}{d \xi^l}(\xi^k)$$

as $|\xi| \to +\infty, \text{Im} \xi \leq 0 \ (\text{Im} \xi \geq 0)$.

Let $H'$ be the space of all polynomials and $H^- = H^-_0 \oplus H'$; $H = H^+ \oplus H^-$. Denote by $\pi^+$ ($\pi^-$) respectively the projection on $H^+$ ($H^-$). For calculations, we take $H = \hat{H} = \{ \text{rational functions having no poles on the real axis} \}$ ($\hat{H}$ is a dense set in the topology of $H$). Then on $\hat{H}$,

$$\pi^+ h(\xi) = \frac{1}{2\pi i} \lim_{u \to 0^+} \int_{\text{Im}(\xi) > 0} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi,$$  \hspace{1cm} (2.3)

where $\Gamma^+$ is a Jordan close curve included $\text{Im}(\xi) > 0$ surrounding all the singularities of $h$ in the upper half-plane and $\xi_0 \in \mathbb{R}$. Similarly, define $\pi'$ on $\hat{H}$,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi.$$  \hspace{1cm} (2.4)

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(R)$, $\pi' h = \frac{1}{2\pi} \int_R h(v) dv$ and for $h \in H^+ \cap L^1(R)$, $\pi' h = 0$. 

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Let $M$ be an $n$-dimensional compact oriented manifold with boundary $\partial M$. Denote by $\mathcal{B}$ Boutet de Monvel’s algebra, we recall the main theorem in [FGLS].

**Theorem 2.1 (Fedosov-Golse-Leichtnam-Schrohe)** Let $X$ and $\partial X$ be connected, $\dim X = n \geq 3$, $A = \left( \pi^+P + G \begin{pmatrix} K \\ T \\ S \end{pmatrix} \right) \in \mathcal{B}$, and denote by $p$, $b$ and $s$ the local symbols of $P, G$ and $S$ respectively. Define:

$$\tilde{\text{Wres}}(A) = \int_X \int_S \text{tr}_E \left[ p_{-n}(x, \xi) \right] \sigma(\xi) dx + 2\pi \int_{\partial X} \int_{S'} \left\{ \text{tr}_E \left[ (\text{tr}b_{-n})(x', \xi') \right] + \text{tr}_E \left[ s_{1-n}(x', \xi') \right] \right\} \sigma(\xi') dx' , \quad (2.5)$$

Then

a) $\tilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$;  
b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Let $p_1, p_2$ be nonnegative integers and $p_1 + p_2 \leq n$. Then by Sec 2.1 of [Wa2], we have

**Definition 2.2.** Lower-dimensional volumes of spin manifolds with boundary are defined by

$$\text{Vol}^{(p_1, p_2)}M := \text{Wres}[\pi^+D^{-p_1} \circ \pi^+D^{-p_2}] . \quad (2.6)$$

Denote by $\sigma_l(A)$ the $l$-order symbol of an operator $A$. Similar to (2.1.7) in [Wa2], we have that

$$\text{Wres}[\pi^+D^{-p_1} \circ \pi^+D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)}[\sigma_{-n}(D^{-p_1-p_2})] \sigma(\xi) dx + \int_{\partial M} \Phi , \quad (2.7)$$

where

$$\Phi = \int_{|\xi|=1} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j, k=0}^{+\infty} \frac{(-j)^{|\alpha|+j+k+1}}{\alpha! (j+k+1)!} \text{trace}_{S(TM)} \left[ \partial_{x^n}^j \partial_{\xi^n}^j \partial_{\xi'}^k \sigma_r(D^{-p_1})(x', 0, \xi', \xi_n) \right. \left. \times \partial_{x'^n}^r \partial_{\xi'^n}^r \partial_{\xi'^n}^r \sigma_r(D^{-p_2})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') dx' , \quad (2.8)$$

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$. Since $[\sigma_{-n}(D^{-n+2})]_M$ has the same expression as $\sigma_{-n}(D^{-n+2})$ in the case of manifolds without boundary, by (2.5) in [Ac], we have

$$\text{Wres}(D^{-n+2}) = \frac{(2 - n)(2\pi)^{n/2}}{12 \Gamma(n/2)} \int_M s \text{vol}_M . \quad (2.9)$$

So we only need to compute $\int_{\partial M} \Phi$. 

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3 Kastler-Kalau-Walze type theorems for any dimensional manifolds with boundary

In this section, we compute the lower dimensional volume for any dimensional compact manifolds with boundary and get a Kastler-Kalau-Walze type formula in this case. From now on we always assume that $M$ carries a spin structure so that the spinor bundle and the Dirac operator are defined on $M$.

Let $\tilde{E}_n = \frac{\partial}{\partial x_n}$, $\tilde{E}_j = \sqrt{h(x_n)}E_j$ $(1 \leq j \leq n-1)$, where $\{E_1, \cdots, E_{n-1}\}$ are orthonormal basis of $T\partial M$. Let $\nabla^L$ denote the Levi-civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{E}_1, \cdots, \tilde{E}_n\}$, the connection matrix $(\omega, t)$ is defined by

$$\nabla^L(\tilde{E}_1, \cdots, \tilde{E}_n)^t = (\omega, t)(\tilde{E}_1, \cdots, \tilde{E}_n)^t.$$ 

The Dirac operator is defined by

$$D = \sum_{j=1}^{n} c(E_j)[\tilde{E}_j + \frac{1}{4} \sum_{s,t} \omega, t(E_j)c(E_s)c(E_t)]. \quad (3.1)$$

Let $g^{ij} = g(dx_i, dx_j)$ and

$$\nabla^L, \partial_j = \sum_k \Gamma^k_i \partial_k; \quad \Gamma^k_i = g^{ij} \Gamma^k_{ij},$$

Let the cotangent vector $\xi = \sum \xi_j dx_j$ and $\xi^i = g^{ij} \xi_i$. Let the symbols of $D^2$ and $D$ be

$$\sigma(D^2) = \sigma_2(D^2) + \sigma_1(D^2) + \sigma_0(D^2); \quad \sigma(D^1) = \sigma_1(D^1) + \sigma_0(D^1). \quad (3.2)$$

By the composition formula of pseudodifferential operators, then we have

$$\sigma_m(D^{-m}) = |\xi|^{-2}, \quad \sigma_{-2m-2}(D^{-2m-2}) = (|\xi|^2)^{1-m}, \quad \sigma_{-2m+1}(D^{-2m+1}) = \sqrt{-1} c(\xi)^{-2m}. \quad (3.3)$$

By (3.8) in [WW2], we have:

$$\sigma_{-2m}^{-1}(D^{-2m}) = m\sigma_2(D^2)^{-m+1}\sigma_3(D^{-2})$$

$$-\sqrt{-1} \sum_{k=0}^{m-2+2} \sum_{\mu=1}^{m-k+2} \partial_\mu \sigma_2^{-m+k+1}(D^2)\partial_\mu \sigma_2^{-1}(D^2)(\sigma_2(D^2))^{-k}. \quad (3.4)$$

By (3.4), we know that

$$\sigma_{-2m}(D^{1-2m}) = \sigma_{-2m}(D^{-2m} \cdot D) = \left\{\sum_{|\alpha|=0}^{+\infty} \left(\sqrt{-1}\right)^{|\alpha|} \frac{1}{\alpha!} \partial_\alpha^{\epsilon} [\sigma(D^{-2m})] \partial_\alpha^{\epsilon} [\sigma(D)] \right\}_{-2m}$$

$$= \sigma_{-2m}(D^{-2m})\sigma_0(D) + \sigma_{-2m-1}(D^{-2m})\sigma_1(D) + \sum_{|\alpha|=1} \left(\sqrt{-1}\right)^{|\alpha|} \partial_\alpha^{\epsilon} [\sigma_{-2m}(D^{-2m})] \partial_\alpha^{\epsilon} [\sigma_1(D)]$$
By Lemma 2.2 in [Wa2] and (3.12) in [WW2], we have for case a) I)

\[ \pi_\xi^{-2m+2} \sigma_0(D) + \sum_{j=1}^{2m+2} \partial_{\xi_j}(|\xi|^{-2m}) \partial_{x_j} c(\xi) + \left[ m \sigma_2(D)(-m+1) \sigma_{-3}(D^{-2}) \right] \]

\[-\frac{1}{\sqrt{-1}} \sum_{k=0}^{m-2m+2} \sum_{\mu=1}^{2m+2} \partial_{\xi_k} \sigma_2^{-m-k+1}(D^2) \partial_{x_k} \sigma_2^{-1}(D^2)(\sigma_2(D^2))^{-k} \sqrt{-1} c(\xi). \tag{3.5} \]

Since \( \Phi \) is a global form on \( \partial M \), so for any fixed point \( x_0 \in \partial M \), we can choose the normal coordinates \( U \) of \( x_0 \) in \( \partial M \) (not in \( M \)) and compute \( \Phi(x_0) \) in the coordinates \( \tilde{U} = U \times [0,1) \) and the metric \( h(x_n)g^{\partial M} + dx_n^2 \). The dual metric of \( g^M \) on \( \tilde{U} \) is \( h(x_n)g^{\partial M} + dx_n^2 \). Now we can compute \( \Phi \) (see formula (2.8) for the definition of \( \Phi \)). Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -n, r \leq -1, \ell \leq 1 - 2m \), then we have the \( \int_{\partial M} \Phi \) is the sum of the following five cases:

**case a) I)** \( r = -1, \ l = 1 - 2m, \ k = j = 0, \ |\alpha| = 1 \)

\[
\text{case a) I}) = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi_n}^\alpha \pi_\xi^+ \sigma_{-1}(D^{-1}) \right] \times \partial_{x_n} \sigma_{1-2m}(D^{1-2m}) \bigg| (x_0) \ d\xi_n \sigma(\xi') d\xi'. \tag{3.6} \]

By Lemma 2.2 in [Wa2] and (3.12) in [WW2], we have for \( j < n \)

\[
\partial_{x_j} \sigma_{1-2m}(D^{1-2m})(x_0) = \partial_{x_j} [\sqrt{-1} c(\xi)] |\xi|^{-2m} \]

\[
= \sqrt{-1} \partial_{x_j} c(\xi)(x_0) |\xi|^{-2m} + \sqrt{-1} c(\xi) \partial_{x_j} (|\xi|^{-2m})(x_0) = 0. \tag{3.7} \]

so **case a) I** vanishes.

**case a) II)** \( r = -1, \ l = 1 - 2m, \ k = |\alpha| = 0, \ j = 1 \)

By (2.8), we get

\[
\text{case a) II}) = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[ \sigma_{-1}(D^{-1}) \right] \times \partial_{x_n}^2 \sigma_{1-2m}(D^{1-2m}) \bigg| (x_0) \ d\xi_n \sigma(\xi') d\xi'. \tag{3.8} \]

By (2.2.23) in [Wa2], we have

\[
\pi_\xi^+ \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} c(\xi')(x_0)}{2(\xi_n - i)} + \sqrt{-1} h'(0) \left[ \frac{ic(\xi')}{4(\xi_n - i)} + \frac{c(\xi') + ic(d\xi_n)}{4(\xi_n - i)^2} \right]. \tag{3.9} \]
So
\[ \partial_{\xi_n}^2 \pi_n^+ \partial_{x_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{4(\xi_n - i)^3} + \sqrt{-1} h'(0) \left[ \frac{ic(\xi')}{8(\xi_n - i)^3} + \frac{c(\xi') + ic(dx_n)}{24(\xi_n - i)^4} \right]. \]

We know that
\[ \sigma_{1-2m}(D^{1-2m}) = \frac{\sqrt{-1}[c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^m}, \] (3.11)

By the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \), then we have the equalities:
\[ \text{tr}[c(\xi')c(dx_n)] = 0; \quad \text{tr}[c(dx_n)^2] = -2^{m+1}; \quad \text{tr}[c(\xi')^2(x_0)|_{|\xi'|=1} = -2^{m+1}; \]
\[ \text{tr}[\partial_{x_n}c(\xi')c(dx_n)] = 0; \quad \text{tr}[\partial_{x_n}c(\xi')(\xi')](x_0)|_{|\xi'|=1} = -2^m h'(0). \] (3.12)

By (3.10),(3.11) and (3.12), we have
\[
\text{trace}[\partial_{\xi_n}^2 \partial_{x_n} \pi_n^+ \sigma_{-1}(D^{-1}) \times \sigma_{1-2m}(D^{1-2m})](x_0)|_{|\xi'|=1} = \frac{\sqrt{-1}[c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^m} \times \frac{\sqrt{-1}[c(\xi') + \xi_n c(dx_n)]}{(1 + \xi_n^2)^m} = \frac{2^m h'(0) i}{12(\xi_n + i)^m (\xi_n - i)^{m+3}}. \] (3.13)

By (3.13) and the Cauchy integral formula, we have
\[ \text{case a) II} = \text{Vol}(S_{n-2}) \frac{2^m h'(0) \pi}{12(m + 2)!} [(\xi_n + i)^{-m}]^{m+2} |_{\xi_n = i} dx', \] (3.14)

where \( \text{Vol}(S_{n-2}) \) is the canonical volume of \( S_{n-2} \) and denote the \( p \)-th derivative of \( f(\xi_n) \) by \([f(\xi_n)]^{(p)}\).

case a) III) \( r = -1, \ l = 1 - 2m, \ j = |\alpha| = 0, \ k = 1 \)

By (2.8) and an integration by parts, we get
\[
\text{case a) III} = -\frac{1}{2} \int_{|\xi'|=1}^{\int_{-\infty}^{+\infty}} \text{trace}[\partial_{\xi_n}^+ \pi_n^+ \sigma_{-1}(D^{-1}) \times \partial_{x_n} \sigma_{1-2m}(D^{1-2m})] (x_0) d\xi_n \sigma(\xi') dx' \\
= \frac{1}{2} \int_{|\xi'|=1}^{\int_{-\infty}^{+\infty}} \text{trace}[\partial_{\xi_n}^2 \pi_n^+ \sigma_{-1}(D^{-1}) \times \partial_{x_n} \sigma_{1-2m}(D^{1-2m})] (x_0) d\xi_n \sigma(\xi') dx'. \] (3.15)
By (2.2.29) in [Wa2], we have
\[
\partial_{\xi_n}^2 \pi_{-1}(D^{-1})(x_0)|_{\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{\xi_n - i}.
\]

By (3.3), direct computations show that
\[
\partial_{x_n} \sigma_1(D^{1-2m})(x_0)|_{\xi'|=1} = \frac{\sqrt{-1} \partial_{x_n}[c(\xi')](x_0) - \sqrt{-1} mh'(0)c(\xi)}{(1 + \xi_n^2)^{m+1}}.
\]

So by (3.16),(3.17) and (3.12) and the Cauchy integral formula, we have
\[
\text{case a) III)} = \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left\{ \frac{c(\xi') + ic(dx_n)}{\xi_n - i} \right\} dx'\partial_{x_n} \sigma(\xi')dx\]
\[
= \frac{1}{2} \int_{|\xi'|=1}^{+\infty} 2^m h'(0) \left\{ \frac{-i \xi_n^2 - 2m \xi_n + 2mi - i}{(\xi_n + i)^{m+1}} \right\} \partial_{x_n} \sigma(\xi')dx'\]
\[
= \pi i h'(0) 2^m \text{Vol}(S^{n-2})dx'\left[ -i \xi_n^2 - 2m \xi_n + 2mi - i \right]^{(m+3)}|_{\xi_n=i}.
\]

\textbf{case b)} \ r = -2, l = 1 - 2m, k = j = |\alpha| = 0

By (2.8), we get
\[
\text{case b)} = -i \int_{|\xi'|=1}^{+\infty} \text{trace} \left\{ \pi_{-1} \sigma(D^{-1}) \times \partial_{x_n} \sigma_1(D^{1-2m})(x_0) \partial_{x_n} \sigma(\xi') \right\} dx'.
\]

By (2.2.34)-(2.2.37) in [Wa2], we have
\[
\pi_{-1} \sigma(D^{-1})(x_0)|_{\xi'|=1} = B_1 - B_2,
\]
where
\[
B_1 = -\frac{A_1}{4(\xi_n - i)} - \frac{A_2}{4(\xi_n - i)^2},
\]
and
\[
A_1 = ic(\xi')\sigma_0(D)c(\xi') + ic(dx_n)\sigma_0(D)c(dx_n) + ic(\xi')c(dx_n)\partial_{x_n}[c(\xi')];
\]
\[
A_2 = [c(\xi') + ic(dx_n)]\sigma_0(D)[c(\xi') + ic(dx_n)] + c(\xi')c(dx_n)\partial_{x_n}c(\xi') - i\partial_{x_n}[c(\xi')].
\]

\[
B_2 = \frac{h'(0)}{2} \left[ \frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^2}[ic(\xi') - c(dx_n)] \right] .
\]
Similar to (2.2.38) in [Wa2], we have

\[
\partial_{\xi_n} \sigma_{1-2m}(D^{1-2m})(x_0)|_{\xi'|=1} = \sqrt{-1} \left[ \frac{c(dx_n)}{(1 + \xi_n^2)^m} - m \times \frac{2\xi_n c(\xi') + 2\xi_n^2 c(dx_n)}{(1 + \xi_n^2)^{m+1}} \right].
\]

(3.24)

By (3.23), (3.24) and (3.12), we have

\[
\text{tr}[B_2 \times \partial_{\xi_n} \sigma_{1-2m}(D^{1-2m})(x_0)]|_{\xi'|=1} = \frac{\sqrt{-1}}{2} h'(0) \text{trace} \left\{ \left[ \frac{1}{4i(\xi_n - i)} + \frac{1}{8(\xi_n - i)^2} - \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \right] c(dx_n) + \left[ \frac{-1}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} \right] ic(\xi') \right\} \\
\times \left\{ \left[ \frac{1}{(1 + \xi_n^2)^m} - \frac{2m\xi_n^2}{(1 + \xi_n^2)^{m+1}} \right] c(dx_n) - \frac{2m\xi_n}{(1 + \xi_n^2)^{m+1}} c(\xi') \right\} \\
= h'(0)2^{m-2} \times \frac{(2m - 1)\xi_n^3 - 2i(2m - 1)\xi_n^2 - (6m - 1)\xi_n + 4i}{(\xi_n - i)^2(1 + \xi_n^2)^{m+1}}.
\]

(3.25)

By (2.2.40) in [Wa2], we have

\[
B_1 = \frac{-1}{4(\xi_n - i)^2} [(2 + i\xi_n) c(\xi') \sigma_0(D) c(\xi') + i\xi_n c(dx_n) \sigma_0(D) c(dx_n)] \\
+ (2 + i\xi_n) c(\xi') c(dx_n) \partial_{\xi_n} c(\xi') + ic(dx_n) \sigma_0(D) c(\xi') + ic(\xi') \sigma_0(D) c(dx_n) - i\partial_{\xi_n} c(\xi')].
\]

(3.26)

By (3.24), we have

\[
\partial_{\xi_n} \sigma_{1-2m}(D^{1-2m})(x_0)|_{\xi'|=1} = \sqrt{-1} \left[ \frac{1 + (1 - 2m)\xi_n^2}{(1 + \xi_n^2)^{m+1}} c(dx_n) - \frac{2m\xi_n}{(1 + \xi_n^2)^{m+1}} c(\xi') \right].
\]

(3.27)

Similar to Lemma 2.4 in [Wa2], we have

\[
\sigma_0(D)(x_0) = c_0 c(dx_n), \quad \text{where } c_0 = \frac{1 - n}{4} h'(0).
\]

(3.28)

By the relation of the Clifford action and trAB = trBA, then we have the equalities:

\[
\text{tr}[c(\xi') \sigma_0(D) c(dx_n)] = -c_0 2^{m+1}; \quad \text{tr}[c(dx_n) \sigma_0(D) c(dx_n)^2] = c_0 2^{m+1}; \\
\text{tr}[c(\xi') c(dx_n) \partial_{\xi_n} c(\xi') c(dx_n)]|_{\xi'|=1} = -2^m h'(0); \quad \text{tr}[c(dx_n) \sigma_0(D) c(\xi')^2] = c_0 2^{m+1}.
\]

(3.29)

By (3.26)-(3.29), considering for \(i < n\), \(\int_{[\xi'|=1}\text{odd number product of } \xi_i} \sigma(\xi') = 0\), then

\[
\text{tr}[B_1 \times \partial_{\xi_n} \sigma_{1-2m}(D^{1-2m})(x_0)]|_{\xi'|=1} = \frac{2^m i h'(0)}{4(\xi_n - i)^2(1 + \xi_n^2)^{m+1}} \cdot \left( (n - 1)(2m - 1)\xi_n^2 - 2im\xi_n - 1 \right)
\]

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By (2.2.44) in [Wa2] and similar to (3.29) in [WW2], we have

\begin{align*}
\sum_{j=1}^{2m+2} \partial_{\xi_j} |(\xi|^{-2m}) \partial_{\xi_j} (c(\xi))(x_0)||_{\xi|^{-1}} = -2m \xi_n(1+\xi_n^2)^{-m} \partial_{\xi_n} |c(\xi')(x_0)||_{\xi|^{-1}}.
\end{align*}

By (2.8) and an integration by parts, we get

\begin{align*}
case c) = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-2m}(D^{-2m+1})](x_0) | d\xi_n \sigma(\xi') dx' \\
= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \sigma_{-1}(D^{-1}) \times \sigma_{-2m}(D^{-2m+1})](x_0) | d\xi_n \sigma(\xi') dx' \tag{3.32}
\end{align*}

By (2.2.44) in [Wa2] and similar to (3.29) in [WW2], we have

\begin{align*}
\partial_{\xi_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{-c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}; \tag{3.33}
\end{align*}

By Lemma 2.2 in [Wa2], we have

\begin{align*}
\sum_{j=1}^{2m+2} \partial_{\xi_j} \sigma_{-2m}(D) \partial_{\xi_j} (c(\xi))(x_0)||_{\xi|^{-1}} = -2m \xi_n(1+\xi_n^2)^{-m-1} \partial_{\xi_n} |c(\xi')(x_0)||_{\xi|^{-1}}.
\end{align*}

By (3.26) in [WW2], we have

\begin{align*}
\sigma_{-3}(D^{-2})(x_0)|_{|\xi'|=1} = \frac{-i}{(1+\xi_n^2)^2} \left( \frac{1}{2} h'(0) \sum_{k<n} \xi_k c(\xi_k) c(\xi_n) + \frac{n-1}{2} h'(0) \xi_n \right) - \frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3} \\
= \frac{-i}{(1+\xi_n^2)^2} \left( - \frac{1}{2} h'(0) c(\xi') c(dx_n) + \frac{n-1}{2} h'(0) \xi_n \right) - \frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3}. \tag{3.36}
\end{align*}
So by (3.5), (3.34)-(3.36), we have
\[
\begin{align*}
\sigma_{-2m}(D^{1-2m})(x_0)|_{|\xi'|=1} &= \frac{(1-n)h'(0)c(dx_n)}{4(1+\xi_n^2)^m} - 2m\xi_n (1+\xi_n^2)^{-m-1}\partial_{x_n}[c(\xi')](x_0) \\
+mi(1+\xi_n^2)^{-m+1}[c(\xi') + \xi_n c(dx_n)] & \times \left[ \frac{-ih'(0)c(\xi')c(dx_n)}{2(1+\xi_n^2)^2} + \frac{(-n+1)h'(0)i\xi_n}{2(1+\xi_n^2)^2} - \frac{2ih'(0)\xi_n}{(1+\xi_n^2)^3} \right] \\
& \times \left[ (2nm - 2m - n + 1)i\xi_n^3 + (-2m + 1 + 2\pi)\xi_n^2 \\
& + (2nm - 2m + 4m^2 - n + 1)i\xi_n + (n - 1 + 2\pi) \right].
\end{align*}
\]
By (3.33) and (3.37), we have
\[
\begin{align*}
\text{trace}[\partial_{\xi_n}^+\partial_{\xi_n}^{-1}(D^{-1}) \times \sigma_{-2m}(D^{-2m+1})]|_{|\xi'|=1} &= \frac{2^{m}h'(0)}{4(\xi_n - i)^{m+3}(\xi_n + i)^{m+2}} \\
& \times \left[ (2nm - 2m - n + 1)i\xi_n^3 + (-2m + 1 + 2\pi)\xi_n^2 \\
& + (2nm - 2m + 4m^2 - n + 1)i\xi_n + (n - 1 + 2\pi) \right].
\end{align*}
\] (3.38)
Then by the Cauchy integral formula, we get
\[
\begin{align*}
case c) &= -\frac{\pi 2^{m-1}h'(0)\text{Vol}(S^{n-2})dx'}{(m+2)!} \left\{ \frac{1}{(\xi_n + i)^{m+2}} \right. \\
& \left. \times \left[ (2nm - 2m - n + 1)i\xi_n^3 + (-2m + 1 + 2\pi)\xi_n^2 \\
& + (2nm - 2m + 4m^2 - n + 1)i\xi_n + (n - 1 + 2\pi) \right] \right\} |_{|\xi'|=1}. 
\end{align*}
\] (3.39)
Similar to (3.41) in [WW2], we have for the extrinsic curvature
\[
K(x_0) = \frac{1-n}{2}h'(0). 
\] (3.40)
By (3.14),(3.18),(3.31), (3.39) and (3.40), we have
\[
\Phi = \frac{2}{1-n} \text{Vol}(S^{n-2})L_0 \int_{\partial M} K \text{dvol}_{\partial M},
\] (3.41)
where
\[
L_0 = \frac{2^m\pi}{12(m+2)!} |[(\xi_n + i)^{-m}]^{(m+2)}|_{\xi_n=i} + \frac{\pi 2^{m}}{(m+3)!} \left[ \frac{-i\xi_n^2 - 2m\xi_n - 2m-i}{(\xi_n + i)^{m+1}} \right]^{(m+3)} |_{\xi_n=i} \\
+ \frac{2^{m+1}\pi i}{(m+2)!} \left[ \frac{(4m^2 - 1)\xi_n^2 + (-4m^2 - 6m + 2)i\xi_n - (n+1)}{4(\xi_n + i)^{m+1}} \right]^{(m+2)} |_{\xi_n=i} \\
+ \frac{-\pi 2^{m-1}}{(m+2)!} \left\{ \frac{1}{(\xi_n + i)^{m+2}} \right. \\
& \left. \times \left[ (2nm - 2m - n + 1)i\xi_n^3 + (-2m + 1 + 2\pi)\xi_n^2 \\
& + (2nm - 2m + 4m^2 - n + 1)i\xi_n + (n - 1 + 2\pi) \right] \right\}^{(m+2)} |_{\xi_n=i}. 
\] (3.42)
By (2.9), (3.41) and (3.42), we prove Theorem 1.1.

Nextly, for \( n = 2m + 1 \)-dimensional spin manifolds with boundary, we compute \( \text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{(1-2m)}] \). By Proposition 3.5 in [WW2], we have

\[
\text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{(1-2m)}] = \int_{\partial M} \Phi. \tag{3.43}
\]

From the formula (2.8) for the definition of \( \Phi \), and the sum is taken over \( r - k + |\alpha| + \ell - j - 1 = -(2m + 1), \ r \leq -1, \ \ell \leq 1 - 2m \), then \( r = -1, \ \ell = 1 - 2m, \ k = |\alpha| = j = 0, \)

\[
\Phi = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)}[\pi^+_{\xi_n} \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{1-2m}(D^{1-2m})]d\xi_n \sigma(\xi')dx'.
\]

By (5.3) in [Wa2], we have

\[
\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = -\frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)^2}. \tag{3.45}
\]

So

\[
\text{trace}_{S(TM)}[\partial_{\xi_n} \pi^+_{\xi_n} \sigma_{-1}(D^{-1}) \times \sigma_{1-2m}(D^{1-2m})](x_0)|_{|\xi'|=1}
\]

\[
= \frac{-i}{2(\xi_n - i)^2(1 + \xi_n^2)^m} \text{trace}_{S(TM)} \{ [c(\xi') + ic(dx_n)] [c(\xi') + \xi_n c(dx_n)] \}
\]

\[
= \frac{-2m^{-1}}{(\xi_n - i)^{m+1}(\xi_n + i)^m}. \tag{3.46}
\]

Then

\[
\int_{\partial M} \Phi = \frac{2^{-m}m(m+1)\cdots(2m-1)\text{Vol}_{\partial M}\text{Vol}(S^{n-2})\pi}{m!}. \tag{3.47}
\]

So we have

**Theorem 3.1** Let \( M \) be a \( n = 2m + 1 \)-dimensional compact spin manifold with the boundary \( \partial M \), then

\[
\text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{(1-2m)}] = \frac{2^{-m}m(m+1)\cdots(2m-1)\text{Vol}_{\partial M}\text{Vol}(S^{n-2})\pi}{m!}. \tag{3.48}
\]

Nextly for \( n = 2m + 1 \)-dimensional spin manifolds with boundary, we compute \( \text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{(2-2m)}] \). Now we can compute \( \Phi \) (see formula (2.8) for the definition of \( \Phi \)). Since the sum is taken over \( r + \ell - k - j - |\alpha| - 1 = -(2m + 1), \ r \leq -1, \ \ell \leq 2 - 2m \), then we have the \( \int_{\partial M} \Phi \) is the sum of the following five cases:
case a) I) $r = -1, \ l = 2 - 2m, \ k = j = 0, \ |\alpha| = 1$

$$\text{case a) I) } = - \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[ \partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n}^2 \sigma_{2-2m}(D^{2-2m}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.49)$$

By (3.12) in [WW2], we have for $j < n, \partial_x \sigma_{2-2m}(D^{2-2m})(x_0) = 0$. So case a) I) = 0.

case a) II) $r = -1, \ l = 2 - 2m, \ k = |\alpha| = 0, \ j = 1$

By (2.8), we get

$$\text{case a) II) } = - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n}^2 \sigma_{1-2m}(D^{1-2m}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.50)$$

By (3.9) and (3.16) in [WW2] and

$$\text{tr}[c(\xi')] = \text{tr}[c(dx_n)] = \text{tr}[\partial_{\xi_n} c(\xi')] = 0, \quad (3.51)$$

we have case a) II) = 0.

case a) III) $r = -1, \ l = 2 - 2m, \ j = |\alpha| = 0, \ k = 1$

By (2.8) and an integration by parts, we get

$$\text{case a) III) } = - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \partial_{\xi_n} \sigma_{2-2m}(D^{2-2m}) \right] (x_0) d\xi_n \sigma(\xi') dx'$$

$$= \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_{\xi_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{2-2m}(D^{2-2m}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.52)$$

By (3.16) and (3.21) in [WW2] and (3.51), we have case a) III) = 0.

case b) $r = -2, \ l = 2 - 2m, \ k = j = |\alpha| = 0$

By (2.8), we get

$$\text{case b) } = - i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-2}(D^{-1}) \times \partial_{\xi_n} \sigma_{2-2m}(D^{2-2m})] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.53)$$

By (3.33) in [WW2] and (3.20)-(3.23), (3.51), (3.53) and

$$\text{tr}[c(\xi') \sigma_0(D) c(\xi')] = \text{tr}[c(dx_n) \sigma_0(D) c(dx_n)] = \text{tr}[c(\xi') c(dx_n) \partial_{\xi_n} c(\xi')] = 0, \quad (3.54)$$
we have case $b) = 0$.

case $c)$ $r = -1$, $l = 1 - 2m$, $k = j = |\alpha| = 0$

By (2.8) and an integration by parts, we get

case $c) = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi^+\sigma_1(D^{-1}) \times \partial_{\xi_n}\sigma_1(D^{-2m+2})](x_0)d\xi_n\sigma(\xi')dx'\]

By (3.33) and (3.8),(3.26),(3.29) in [WW2] and (3.51),(3.54), (3.55), we have case $c) = 0$. So we get $\int_{\partial M} \Phi = 0$.

**Theorem 3.2** Let $M$ be a $n = 2m + 1$-dimensional compact spin manifold with the boundary $\partial M$, then

$$\tilde{Wres}[\pi^+D^{-1} \circ \pi^+D(-2m+2)] = \frac{(2 - n)(2\pi)^{n/2}}{12\Gamma(n/2)} \int_M s d\text{vol}_M. \quad (3.56)$$

Nextly, for $n = 2m$-dimensional spin manifolds with boundary, we compute $\tilde{Wres}[\pi^+D^{-1} \circ \pi^+D(2-2m)]$. By Proposition 3.5 in [WW2], we have

$$\tilde{Wres}[\pi^+D^{-1} \circ \pi^+D(2-2m)] = \int_{\partial M} \Phi. \quad (3.57)$$

From the formula (2.8) for the definition of $\Phi$, and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -2m$, $r \leq -1$, $\ell \leq 2 - 2m$, then $r = -1$, $\ell = 2 - 2m$, $k = |\alpha| = j = 0$,

$$\Phi = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)}[\pi^+_{\xi_n}\sigma_1(D^{-1}) \times \partial_{\xi_n}\sigma_2(-2m)(D^{2-2m})]d\xi_n\sigma(\xi')dx'. \quad (3.58)$$

By (2.2.44) in [Wa2] and (3.47) in [WW2] and (3.51),(3.58), we get $\Phi = 0$.

**Theorem 3.3** Let $M$ be a $n = 2m$-dimensional compact spin manifold with the boundary $\partial M$, then

$$\tilde{Wres}[\pi^+D^{-1} \circ \pi^+D(-2m+2)] = 0. \quad (3.59)$$

Similar to Theorem 3.2, we can get

**Theorem 3.4** Let $M$ be a $n = 2m + 3$-dimensional compact spin manifold with the boundary $\partial M$, then

$$\tilde{Wres}[\pi^+D^{-2} \circ \pi^+D(-2m+1)] = \frac{(2 - n)(2\pi)^{n/2}}{12\Gamma(n/2)} \int_M s d\text{vol}_M. \quad (3.60)$$
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