Asymptotic expansions for functions of the increments of certain Gaussian processes

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Abstract

Let $G = \{G(x), x \geq 0\}$ be a mean zero Gaussian process with stationary increments and set $\sigma^2(|x-y|) = E(G(x) - G(y))^2$. Let $f$ be a function with $Ef^2(\eta) < \infty$, where $\eta = N(0,1)$. When $\sigma^2$ is regularly varying at zero and

$$\lim_{h \to 0} \frac{h^2}{\sigma^2(h)} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sigma^2(h)}{h} = 0 \quad \text{but} \quad \left( \frac{d^2}{ds^2} \sigma^2(s) \right)^{j_0}$$

is locally integrable for some integer $j_0 \geq 1$, and satisfies some additional regularity conditions,

$$\int_a^b f \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) dx$$

$$= \sum_{j=0}^{j_0} (h/\sigma(h))^j \frac{E(H_j(\eta)f(\eta))}{\sqrt{j!}} : (G')^j : (I_{[a,b]}) + o \left( \frac{h}{\sigma(h)} \right)^{j_0}$$

in $L^2$. Here $H_j$ is the $j$-th Hermite polynomial. Also : $(G')^j : (I_{[a,b]})$ is a $j$-th order Wick power Gaussian chaos constructed from the Gaussian field $G'(g)$, with covariance

$$E(G'(g)G'(\tilde{g})) = \iint \rho(x-y)g(x)\tilde{g}(y) \, dx \, dy,$$

where $\rho(s) = \frac{1}{2} \frac{d^2}{ds^2} \sigma^2(s)$.

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1 Introduction

Let $G = \{G(x), x \in R_+\}, \ G(0) = 0,$ be a mean zero Gaussian process with stationary increments, and set

$$E(G(x) - G(y))^2 = \sigma^2(x - y) = \sigma^2(|x - y|). \quad (1.1)$$

The function $\sigma^2$ is referred to as the increment's variance of $G$. Clearly $\sigma^2(0) = 0$.

In this paper we are primarily concerned with Gaussian processes that are smoother than Brownian motion but not so smooth that they have mean square derivatives.

Let $d\mu(x) = (2\pi)^{-1/2} \exp(-x^2/2)\ dx$ denote standard Gaussian measure on $R^1$. Let $f \in L^2(R^1, \ d\mu)$, i.e., $Ef^2(\eta) < \infty$, where $\eta$ is a normal random variable with mean zero and variance one, (i.e. $\eta = N(0, 1)$). To avoid trivialities we assume that $\sigma^2(h) \not\equiv 0$ and $f(x) \not\equiv 0$. In all that follows $0 \leq a < b < \infty$.

We obtain an $L^2$ asymptotic expansion for

$$\int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) \ dx, \quad (1.2)$$

as $h \to 0$, that holds for a large class of Gaussian processes and for all $f \in L^2(R^1, \ d\mu)$. The asymptotic expansion involves a generalized derivative $G'$ of the Gaussian process $G$.

We impose the following conditions on the Gaussian processes considered here:

$\sigma^2(h)$ is regularly varying at zero of index $1 \leq \beta \leq 2; \quad (1.3)$

$$\lim_{h \to 0} \frac{h^2}{\sigma^2(h)} = 0 \quad \text{and} \quad \lim_{h \to 0} \frac{\sigma^2(h)}{h} = 0; \quad (1.4)$$

$$\frac{|\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)|}{h^2} \leq C\frac{\sigma^2(s)}{s^2} \quad \text{for} \ h \leq \frac{s}{8}; \quad (1.5)$$

$\sigma^2(s)$ has a second derivative for each $s \neq 0. \quad (1.6)$

Set

$$\rho(s) := \frac{1}{2} \frac{d^2}{ds^2} \sigma^2(s), \quad s \neq 0. \quad (1.7)$$
It follows from (1.4) that
\[ \frac{d\sigma^2(0)}{ds} = 0 \quad \text{and} \quad \rho(0) := \lim_{h \to 0} \frac{\sigma^2(h)}{h^2} = \infty. \] (1.8)

The next theorem is the main result in this paper.

**Theorem 1.1** Let \( f \in L^2(R^1, d\mu) \) and let \( G = \{G(x), x \in R_+\} \), \( G(0) = 0 \), be a mean zero Gaussian process with stationary increments satisfying (1.3)–(1.6), and assume that there exists a \( \zeta > 0 \) such that for all \( 0 < M < \infty \) we can find \( C_M < \infty \) with
\[
|\rho(x)| \leq C_M \frac{|x|^\zeta}{|x|^\zeta} := C_M \varphi(x), \quad |x| \leq M
\] (1.9)
and
\[
|\rho(x + h) - \rho(x)| \leq C_M \frac{|h|}{|x|} |\rho(x)|, \quad 4|h| \leq |x| \leq M.
\] (1.10)
Then for all integers \( j_0 \), such that \( j_0 \zeta < 1 \), and for all for \( b \geq a \),
\[
\int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) \, dx
\] (1.11)
\[
= \sum_{j=0}^{j_0} \left( \frac{h}{\sigma(h)} \right)^j \frac{E(H_j(\eta)f(\eta))}{\sqrt{j!}} : (G')^j : (I_{[a,b]}) + o \left( \frac{h}{\sigma(h)} \right)^{j_0}
\]
in \( L^2 \).

There are many terms in (1.11) that require definition. The functions \( \{H_k(x)\}_{k=0}^\infty \) are the Hermite polynomials. The process \( G' = \{G'(f), f \in B_0(R_+)\} \) is a mean zero Gaussian field with
\[
E \left( G'(f)G'(\tilde{f}) \right) = \int \int \rho(t-s) f(s) \tilde{f}(t) \, ds \, dt \quad \forall f, \tilde{f} \in B_0(R_+),
\] (1.12)
where \( B_0(R_+) \) is the set of bounded Lebesgue measurable functions on \( R_+ \) with compact support. We construct \( G' \) in Section [2] (We use the notation \( G' \) because it is a generalized derivative of the Gaussian process \( G \). This is also explained in Section [2]).
The random variable $(G')^{k_0} : (I_{[a,b]})$ is the ‘value’ of the $k_0$-th order Wick power Gaussian chaos process $\{ : (G')^{k_0} : (g), g \in \mathcal{B}_0(R_+) \}$, at $g = I_{[a,b]}$. This process is constructed from $G'$ in Section 3 and has second moment

$$E \left( : (G')^{k_0} : (g) \right)^2 = k_0! \int\int \rho^{k_0}(x - y)g(x)g(y) \, dx \, dy. \quad (1.13)$$

It is well known that $(G')^{k_0} : (I_{[a,b]})$ can also be expressed as a multiple Wiener-Itô integral. We discuss this in Section 3.

The $k$-th order Wick power of a mean zero Gaussian random variable $X$ is

$$X^k := \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{2j} E(X^{2j}) \, X^{k-2j}. \quad (1.14)$$

When $X = N(0,1)$, $X^k := \sqrt{k!} H_k(X)$. Therefore

$$X^k := \sqrt{k!} \sigma^k X H_k \left( \frac{X}{\sigma X} \right). \quad (1.15)$$

$\sigma^2_X$ denotes the variance of $X$. We show in Theorem 3.1 that when

$$\left( \frac{d^2}{ds^2} \sigma^2(s) \right)^{k_0}$$

is locally integrable \quad (1.16)

and satisfies an additional very mild regularity condition then

$$\lim_{h \to 0} \int : \frac{G(x + h) - G(x)}{h} \overbrace{k_0}^{k_0} : g(x) \, dx = : (G')^{k_0} : (g) \quad (1.17)$$

in $L^2$.

When $\rho(0) < \infty$, $G$ has a mean square derivative and one would expect \(1.17\) to hold with $G'$ being the mean square derivative. Theorem 3.1 shows that this holds for all $f \in L^2(R^1, d\mu)$ and for a much more general class of Gaussian processes.

The class of Gaussian processes satisfying the hypotheses of Theorem 3.1 is very rich. This is illustrated in the next proposition.

**Proposition 1.1** Let $h$ be any function that is regularly varying at infinity with negative index or is slowly varying at infinity and decreasing. Then, for any $1 < \beta < 2$, there exists a Gaussian process with stationary increments
for which the increments variance $\sigma^2(x)$ satisfies the hypotheses of Theorem \ref{thm:1.1} and is such that

$$\sigma^2(x) \sim |x|^\beta h(\log 1/|x|) \quad \text{as} \quad x \to 0. \quad (1.18)$$

Other examples are given in Section \ref{sec:5}.

For any function $f \in L^2(R^1, d\mu)$,

$$f(x) = \sum_{k=0}^{\infty} a_k H_k(x) \quad \text{in} \quad L^2(R^1, d\mu), \quad (1.19)$$

where

$$a_k = \int f(x) H_k(x) \, d\mu(x) = E H_k(\eta) f(\eta) \quad (1.20)$$

and

$$\sum_{k=0}^{\infty} a_k^2 = \int |f(x)|^2 \, d\mu(x) < \infty. \quad (1.21)$$

For a given $f \in L^2(R^1, d\mu)$ let

$$k_0 := k_0(f) = \inf_{k \geq 1} \{k | a_k \neq 0\}. \quad (1.22)$$

The integer $k_0$ is known as the Hermite rank of $f$.

We have the following corollary of Theorem \ref{thm:1.1}:

**Corollary 1.1** For a given $f \in L^2(R^1, d\mu)$ let $k_0$ be as defined in (1.22) and let $G = \{G(x), x \in R_+\}$, $G(0) = 0$, be a mean zero Gaussian process with stationary increments satisfying (1.3)–(1.6). Assume that (1.16) holds with $k = k_0$ and that

$$h = o \left( \frac{h^2}{\sigma^2(h)} \right)^{k_0}. \quad (1.23)$$

Then, for $b \geq a$

$$\lim_{h \downarrow 0} \int_a^b f \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) \, dx = (b-a) Ef(\eta) \frac{E(H_{k_0}(\eta)f(\eta))}{\sqrt{k_0!}} : (G')^{k_0} : (I_{[a,b]}) \quad (1.24)$$

in $L^2$.

(Note that (1.23) is implied by (1.9) if $k_0 \zeta < 1$).
Example 1.1 It follows immediately from (1.15) and (1.24) that
\[ \lim_{h \to 0} \int_a^b: \left( \frac{G(x + h) - G(x)}{h} \right)^k: dx = \frac{1}{\sqrt{k!}} : (G^n)^k : (I_{[a,b]}) \quad (1.25) \]
in \( L^2 \). Remarkably, we show in [8], that under the hypotheses of Theorem 1.1, the limit in (1.25) is also almost sure.

It is clear from (1.17) that when \( k_0 > 1 \), the limit in (1.24) is not a normal random variable. We do get a normal limit when \( k_0 = 1 \), as we state in the next corollary of Theorem 1.1.

Corollary 1.2 Let \( f \in L^2(R^1, d\mu) \) be such that \( E(\eta f(\eta)) \neq 0 \). Let \( G = \{G(x), x \in R_+\}, G(0) = 0, \) be a mean zero Gaussian process with stationary increments satisfying (1.3)–(1.6). Assume \( \rho(s) \) is locally integrable. Then
\[ \lim_{h \to 0} \int_a^b \frac{f(G(x + h) - G(x))}{\sigma(h)} \frac{dx}{h/\sigma(h)} - (b - a) Ef(\eta) \Phi(h) \text{ law} = N(0,1), \quad (1.26) \]
in \( L^2 \).

It is interesting to compare Corollary 1.2 with the normal central limit theorem obtained in [7, Theorem 1.1] that holds for all Gaussian processes with concave increment’s variance and for some Gaussian processes with convex increment’s variance but where (1.16) does not hold for \( k_0 = 2 \).

Theorem 1.2 [7, Theorem 1.1] Assume that \( \sigma^2(h) \) is concave or that \( \sigma^2(h) = h^r \), \( 1 < r \leq 3/2 \). Then for all symmetric functions \( f \in L^2(R^1, d\mu) \)
\[ \lim_{h \to 0} \int_a^b f \frac{G(x + h) - G(x)}{\sigma(h)} \frac{dx}{\Phi(h)} - (b - a) Ef(\eta) \text{ law} = N(0,1), \quad (1.27) \]
where \( \Phi^2(h) \) is the variance of the numerator.

Theorem 1.2 appears similar to Corollary 1.2. In fact under the conditions of Corollary 1.2
\[ \Phi(h) \sim h\sigma(b - a) Ef(\eta)/\sigma(h) \quad (1.28) \]
as \( h \to 0 \). However, there are important differences between these results. Theorem 1.2 applies to symmetric functions \( f \) whereas in Corollary 1.2 we
require that \( E(\eta f(\eta)) \neq 0 \), which excludes symmetric functions \( f \). Indeed we see from Corollary 1.1 that if \( f \) is symmetric and \( E(\eta^2 f(\eta)) \neq 0 \) the dominant term on the right in (1.24) is
\[
\frac{E(H_2(\eta) f(\eta))}{\sqrt{2}} : (G')^2 : (I_{[a,b]}),
\]
(1.29)
as long as (1.16) holds with \( k_0 = 2 \). The hypotheses of Theorem 1.2 excludes processes for which (1.16) holds with \( k_0 = 2 \). It is clear that the integrability of powers of \( \rho \) at the origin play a critical role in whether or not we get normal central limit theorems.

Also note that in Corollary 1.2, we have convergence in \( L^2 \). (See Remark 2.2 for further discussion along this line.)

We use \( f \sim g \) at zero to indicate that \( \lim_{h \downarrow 0} f(h)/g(h) = 1 \) and \( f \approx g \) at zero to indicate that there exist \( 0 < C_1 \leq C_2 < \infty \) such that \( \liminf_{h \downarrow 0} f(h)/g(h) \geq C_1 \) and \( \limsup_{h \downarrow 0} f(h)/g(h) \leq C_2 \).

1.1 Motivation

The motivation for this paper comes from our work [6] on the local times \( \{L^x_t, (t,x) \in R_+ \times R\} \) of the real valued symmetric Lévy process \( X = \{X(t), t \in R_+\} \) with characteristic function \( E^{i\lambda X(t)} = e^{-t\psi(\lambda)} \). We show that if
\[
\sigma^2_0(x) = \frac{4}{\pi} \int_0^\infty \frac{\sin^2 \frac{\lambda x}{2}}{\psi(\lambda)} d\lambda
\]
is concave, and satisfies some additional very weak regularity conditions, then for any \( p \geq 1 \), and all \( t \in R_+ \)
\[
\lim_{h \downarrow 0} \int_a^b \frac{|L^x_{t + h} - L^x_t|}{\sigma_0(h)}^p dx = 2^{p/2} E|\eta|^p \int_a^b |L^x_t|^{p/2} dx
\]
for all \( a, b \) in the extended real line almost surely, and also in \( L^m, m \geq 1 \).

This result is obtained via the Eisenbaum Isomorphism Theorem and depends on a related result for Gaussian processes \( \{G(x), x \in R^d\} \) with stationary increments. If the increments variance \( \sigma^2_0(x) \) is concave, and satisfies some additional very weak regularity conditions we show in [6] that
\[
\lim_{h \to 0} \int_a^b \frac{|G(x + h) - G(x)|}{\sigma_0(h)}^p dx = E|\eta|^p (b - a)
\]
(1.30)
for all \(a, b \in \mathbb{R}^1\), almost surely. Viewing this as a strong law we then obtained
the corresponding central limit theorem, \([7, \text{Theorem 1.1}]\), which we repeat
as Theorem [1.2] in this paper. Initially, the motivation for this paper was to
see what happens for Gaussian processes that are smoother than those that
satisfy the hypotheses of Theorem [1.2] but are not so smooth that they are
mean square differentiable. However, now that we have the results of [6, 7]
and this paper, we have an overview that enables us to present this work
as method for finding limits of a natural sequence of stationary Gaussian
processes.

Let \(G\) be the Gaussian process with stationary increments introduced at
the very beginning this section. Since

\[
E \left( \frac{(G(x + h) - G(x))(G(y + h) - G(y))}{\sigma^2(h)} \right) = \frac{\sigma^2(x - y + h) + \sigma^2(x - y - h) - 2\sigma^2(x - y)}{\sigma^2(h)}
\]

we see that

\[
G_h(x) \overset{\text{def}}{=} \frac{G(x + h) - G(x)}{\sigma(h)} \quad x \in \mathbb{R}^1
\]

is a stationary Gaussian process with \(E(G_h^2(0)) = 1\). A natural question is
to ask whether

\[
G_0(x) \overset{\text{def}}{=} \lim_{h \to 0} G_h(x)
\]

exists. (The natural limit would be in \(L^2\).) A necessary condition for such a
limit is that the limit of the covariance \(E(G_h(x)G_h(y))\) should exist. This is
given in (1.31) which we write as

\[
E(G_h(x)G_h(y)) = \frac{\sigma^2(x - y + h) + \sigma^2(x - y - h) - 2\sigma^2(x - y)}{h^2} \frac{h^2}{\sigma^2(h)}
\]

When \(x - y \neq 0\) and \(\sigma^2(s)\) has a second derivative for \(s \neq 0\)

\[
\lim_{h \to 0} E(G_h(x)G_h(y)) = (\sigma^2''(x - y) \lim_{h \to 0} \frac{h^2}{\sigma^2(h)}).
\]

(Note that even when \(\sigma^2(s)\) is not differentiable at zero most Gaussian
processes that one can think of have the property that \(\sigma^2(s)\) has a second
derivative for \(s \neq 0\). For example \(\sigma^2(s) = |s|^r, 0 < r \leq 2\) or
\(\sigma^2(s) = (\log 1/|s|)^{-r} \wedge 1\),

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Thus for (1.33) to hold \( \sigma^2(s) \) must also have a second derivative at \( s = 0 \).

When \( \sigma^2(s) \) does not have a second derivative at \( s = 0 \) we consider a weak limit for (1.32),

\[
\lim_{h \to 0} \int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) dx, \tag{1.36}
\]

for \( f \in L^2(R^1, d\mu) \). It is natural to approach this by first taking \( f = H_k(x) \) the \( k \)-th Hermite polynomial.

Under the hypotheses of Theorem 1.1 for \( k \geq 2 \)

\[
\lim_{h \to 0} \int_a^b H_k \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) dx = 0 \tag{1.37}
\]

whereas

\[
\lim_{h \to 0} \int_a^b \left( \frac{G(x + h) - G(x)}{h} \right)^k : dx = : (G^k) : (I_{[a,b]}), \tag{1.38}
\]

a well defined random variable, as we show in (3.27) and Theorem 3.1. Thus we see that when the hypotheses of Theorem 1.1 are satisfied it is quite natural to write the right-hand side of (1.11) in terms of Wick powers.

In Theorem 1.2 we show that for \( \sigma^2 \) relatively ‘large’ at zero

\[
\int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) dx - (b - a)Ef(\eta), \tag{1.39}
\]

divided by it’s variance, has a normal limit, in distribution, as \( h \to 0 \). We see this, heuristically, as a result of the fact that in these cases the increments of \( G \) are only slightly correlated so that, writing the integral as a sum, we are in the standard situation of a normal central limit theorem. (Note that when \( \sigma^2 \) is concave the increments of \( G \) are negatively correlated.)

On the other hand for \( f \) and \( \sigma^2(h) \) sufficiently smooth

\[
\lim_{h \to 0} f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) = f \left( \frac{G'(x)}{\sigma'(0)} \right) \quad a.s. \tag{1.40}
\]
as stochastic processes for \( x \in [-T, T] \) for any \( T > 0 \), where \( G' \) is the actual derivative of \( G \). In this case if we expand the right-hand side of (1.40) in Hermite polynomials we get

\[
\lim_{h \to 0} \int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) \, dx = \sum_{j=0}^{\infty} (1/\sigma'(0))^j \frac{E(H_j(\eta)f(\eta))}{\sqrt{j!}} \int_a^b : (G'(x))^j : \, dx \quad a.s.
\]

We now see that (1.11) lies somewhere between (1.27) and (1.41). What distinguishes the hypotheses of Theorem 1.1 is that although \( \sigma^2 \) is not twice differentiable at zero, nevertheless

\[
\int_0^T |(\sigma^2)^{(n)}(x)| \, dx < \infty.
\]

We see in (1.11) what looks like the beginning of the power series expansion in (1.41). We see this even more dramatically in Example 5.2 in which we show that for \( \sigma^2(u) \approx Cu^2 \log^2 \frac{1}{u} \) and \( (\sigma^2)^{(n)}(u) \approx \log \frac{1}{u} \),

\[
\int_a^b f \left( \frac{G(x + h) - G(x)}{\sigma(h)} \right) \, dx \sim \sum_{j=0}^{\infty} (h/\sigma(h))^j \frac{E(H_j(\eta)f(\eta))}{\sqrt{j!}} \int_a^b : (G'(x))^j : (I_{[a,b]}) \quad (1.43)
\]

in \( L^2 \), as \( h \to 0 \).

By considering a full range of Gaussian processes we can appreciate how the asymptotic behavior of (1.36) changes as the increments variance of \( G \) becomes smoother.

In Section 2 we define the generalized derivative \( G' \). In Section 3 we construct the \( k \)-th order Wick power process. This is used in Section 4 to prove Theorem 1.1 and Corollaries 1.1 and 1.2. In Section 5 we give examples of Gaussian processes that satisfy the hypotheses of Theorem 1.1.

There are many papers about non normal central limit theorems for non-linear functionals of Gaussian processes. See for example [2, Dobrushin and Major], [4, Major], [11, Taqqu] and [10, Surgailis]. The focus of these papers differs significantly from what is considered in this paper. They consider long–range dependence and the limiting distributions that are obtained are self–similar. In this paper we are concerned with local phenomena. The generalized derivative \( G'' \) of the Gaussian process \( G \), appears in the limit and
it is clear from (1.13) that the limiting distributions we obtain are not, in
general, self–similar.

Moreover because of the nature of the problems considered in the above
references, they only consider weak convergence. In contrast we obtain
asymptotic expansions in $L^2$. This remark also applies to more recent re-
results on the non normal weak convergence of multiple Wiener–Itô integrals;
see, for example, [9, Nourdin and Peccati], and the references therein.

2 Generalized derivatives

The second condition in (1.4) implies that $G$ has a version with continuous
sample paths. (Clearly it implies that $\sigma^2(h) \leq Ch$, for $h \in [0,h_0]$ for some
constant $C$ and $h_0 > 0$. Therefore, continuity follows from [4, Lemma 6.4.6].)
We work with this version. (It follows from the first condition in (1.4) that
the paths of $G$ are not mean square differentiable.)

Lemma 2.1 Let $G = \{G(x), x \geq 0\}$ be a mean zero Gaussian process with
stationary increments and $G(0) = 0$, and with increments variance $\sigma^2$ sat-
ifying the second condition in (1.4). If $\rho$ is locally integrable there exists a
mean zero Gaussian field $\{G'(g), g \in B_0(R_+)\}$ with covariance

$$E (G'(g)G'(\tilde{g})) = \int \int \rho(t - s) g(s) \tilde{g}(t) \, ds \, dt. \quad (2.1)$$

We use the following simple lemma which follows by simply doing the integration.

Lemma 2.2 Let $\phi$ be a symmetric function on $R^1$. Suppose that $\phi''$ is locally
integrable on $R^1$ and $\phi(0) = \phi'(0) = 0$. Then

$$\frac{1}{2} \int_a^b \int_a^b \phi''(x - y) \, dx \, dy = \phi(b - a). \quad (2.2)$$

Proof of Lemma 2.1 It follows from Lemma 2.2 that

$$\sigma^2(x) = \int_0^x \int_0^x \rho(t - s) \, dt \, ds. \quad (2.3)$$
Also, since \( G(0) = 0 \), \( EG^2(x) = \sigma^2(x) \). Consequently for \( x \leq y \)

\[
EG(x)G(y) = \frac{1}{2} \left\{ \sigma^2(x) + \sigma^2(y) - \sigma^2(y - x) \right\}
\]

\[
= \frac{1}{2} \int \int \left\{ I_{[0,x]} + I_{[0,y]} - I_{[x,y]} \right\} \rho(t - s) \, ds \, dt
\]

\[
= \frac{1}{2} \left\{ \int_0^y \int_0^y \rho(t - s) \, ds \, dt + \int_0^y \int_x^y \rho(t - s) \, ds \, dt \right\}.
\]

Since \( \sigma^2 \) is symmetric, so is \( \rho \). Therefore

\[
EG(x)G(y) = \int_0^x \int_0^y \rho(t - s) \, ds \, dt.
\]

(2.5)

It follows from this that for \( x' \leq x \), and \( y' \leq y \)

\[
E (G(x) - G(x')) (G(y) - G(y')) = \int_{x'}^x \int_{y'}^y \rho(t - s) \, ds \, dt.
\]

(2.6)

Let \( \mathcal{E}(R_+) \) be the set of elementary functions on \( R_+ \) of the form \( g(x) = \sum_{i=1}^n g_i I_{(a_i,b_i]}(x) \). For such functions \( g(x) \) we define the stochastic integral

\[
\int g(x) \, dG(x) := \sum_{i=1}^n g_i (G(b_i) - G(a_i)).
\]

(2.7)

Note that by (2.6), for these functions,

\[
\int \int \rho(t - s) \, g(s) \, g(t) \, ds \, dt
\]

\[
= \sum_{i,j=1}^n g_i g_j \int_{a_i}^{b_i} \int_{a_j}^{b_j} \rho(t - s) \, ds \, dt = E \left( \int g(x) \, dG(x) \right)^2 \geq 0.
\]

(2.8)

It follows from this that the inner product

\[
(g, \bar{g})_G := \int \int \rho(t - s) \, g(s) \, \bar{g}(t) \, ds \, dt
\]

is positive definite on \( \mathcal{E}(R_+) \).
Let $\mathcal{G}$ be the closure of $\mathcal{E}(R_+)$ in the norm
$$
\|g\|_{\mathcal{G}} = \left( \int \int \rho(t-s) g(s) g(t) \, ds \, dt \right)^{1/2}.
$$
(2.10)

Note that $\mathcal{G}$ is a Hilbert space. It follows from (2.8) that the stochastic integral extends from $\mathcal{E}(R_+)$ to a mean zero Gaussian field $\{G'(g), g \in \mathcal{G}\}$ with covariance
$$
E(G'(g)G'\tilde{g})) = (g, \tilde{g})_{\mathcal{G}}.
$$
(2.11)

It is easy to see that $\mathcal{G}$ contains $B_0(R_+)$. □

**Remark 2.1** There are several possible definitions of stochastic integrals for general Gaussian processes. See the discussion in [1] for the special case of fractional Brownian motion.

We intend the notation $G'$ to suggest the derivative. If $G$ itself is differentiable then $G'(g)$ could be written as
$$
\int G'(x)g(x) \, dx,
$$
in which case the notation $G'(g)$ would be completely appropriate. However, even though the Gaussian processes that concern us are not differentiable we may think of them as having generalized derivatives for several reasons, which we give in the remainder of this section.

**Theorem 2.1** Let $G$ be a Gaussian process of the type described in Lemma 2.1. Then for any $g \in B_0(R_+)$
$$
\lim_{h \to 0} \int \left( \frac{G(x+h) - G(x)}{h} \right) g(x) \, dx = G'(g) \quad \text{in } L^2.
$$
(2.13)

**Proof** Let
$$
X_h(g) := \int \left( \frac{G(x+h) - G(x)}{h} \right) g(x) \, dx.
$$
(2.14)

We show that
$$
\lim_{h \to 0} E\left( X_h(g) - G'(g) \right)^2 = 0
$$
(2.15)

by showing that all the terms of the expectation have the same limit as $h \to 0$. 13
Using the fact that $G(x + h) - G(x) = \int I_{\{x,x+h]\}}(y) \, dG(y)$, it follows by Fubini’s Theorem and (2.1) that
\[
E \{X_h(g)G'(g)\} = \frac{1}{h} \int E \{(G(x + h) - G(x)) \, G'(g)\} \, g(x) \, dx
\]
\[
= \frac{1}{h} \int \left\{ \int \int \rho(t-s) I_{\{x,x+h]\}}(s) \, g(t) \, ds \, dt \right\} g(x) \, dx
\]
\[
= \int \int \left\{ \frac{1}{h} \int_{s-h}^{s} g(x) \, dx \right\} \rho(t-s) \, g(t) \, ds \, dt.
\]
By Lebesgue’s theorem on differentiation
\[
\lim_{h \to 0} \frac{1}{h} \int_{s-h}^{s} g(x) \, dx = g(s) \quad \text{for almost all } s.
\] (2.17)

Using this and the Dominated Convergence Theorem we see that
\[
\lim_{h \to 0} E \left( X_h(g)G'(g) \right) = \int \int \rho(t-s) \, g(s) \, g(t) \, ds \, dt. \tag{2.18}
\]

Considering (2.1) we see that to complete the proof of this theorem it suffices to show that
\[
\lim_{h \to 0} E \left( X_h^2(g) \right) = \int \int \rho(t-s) \, g(s) \, g(t) \, ds \, dt. \tag{2.19}
\]

Using (2.6) we have
\[
E \left( X_h(g)X_{h'}(\bar{g}) \right) = \frac{1}{h} \int \int \rho(t-s) \, g(s) \, \bar{g}(y) \, ds \, dt
\]
\[
= \frac{1}{h} \int \int \left\{ \int_{x}^{x+h} \int_{y}^{y+h} \rho(t-s) \, ds \, dt \right\} \, g(x) \, dx \, \bar{g}(y) \, dy
\]
\[
= \int \int \left\{ \frac{1}{h} \int_{t-h}^{t} g(x) \, dx \right\} \left\{ \frac{1}{h'} \int_{s-h'}^{s} \bar{g}(y) \, dy \right\} \rho(t-s) \, ds \, dt.
\]
It now follows from the Dominated Convergence Theorem and (2.17), that (2.19) holds.

**Remark 2.2** When $g = I_{\{(a,b]\}}$, (2.13) and the construction of $G'$ show that
\[
\lim_{h \to 0} \int_{a}^{b} \left( \frac{G(x + h) - G(x)}{h} \right) \, dx = G'(I_{\{(a,b]\}}) = G(b) - G(a) \quad \text{in } L^2. \tag{2.21}
\]
It is easy to see that this limit actually holds almost surely. Since $G$ has
continuous paths almost surely,
\[
\lim_{h \to 0} \int_a^b \frac{G(x + h) - G(x)}{h} \, dx \quad (2.22)
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left\{ \int_a^b G(x + h) \, dx - \int_a^b G(x) \, dx \right\}
\]
\[
= \lim_{h \to 0} \frac{1}{h} \int_b^{b+h} G(x) \, dx - \frac{1}{h} \int_a^{a+h} G(x) \, dx \quad \text{a.s.}
\]
More generally, for all $g \in E(R_+)$ we actually have almost sure convergence
in (2.13).

Finally we note that we can consider $G$ to be a (random) distribution
defined by
\[
G(f) = \int G(x) f(x) \, dx, \quad f \in C_0^\infty(R_+). \quad (2.23)
\]
In this case $G$ has a distributional derivative $DG$. Using the fact that for
any $f \in C_0^\infty(R_+)$, $(f(x + h) - f(x))/h$ converges to $f'(x)$ uniformly, we have
\[
DG(f) := -G(f') \quad (2.24)
\]
\[
= \lim_{h \to 0} \int G(x) \frac{f(x + h) - f(x)}{h} \, dx
\]
\[
= \lim_{h \to 0} \int \frac{G(x + h) - G(x)}{h} f(x) \, dx
\]
almost surely. Therefore, it follows from Theorem 2.1 that for $f \in C_0^\infty(R_+)$,
$G'(f) \overset{L^2}{=} DG(f)$.

3 Wick powers of generalized derivatives

Let $(X, Y)$ be a two dimensional Gaussian random variable. By [3] Theorem
3.9]
\[
E(: X^k :: Y^j :) = k!(E(XY))^k \delta_{k,j}. \quad (3.1)
\]
(: $X^k$ : is defined in (1.14).) It follows from (1.15) and (3.1) that if $X$ and $Y$
are $N(0, 1)$ and $(X, Y)$ is a two dimensional Gaussian random variable then
\[
E(H_k(X)H_j(Y)) = (E(XY))^k \delta_{k,j}. \quad (3.2)
\]
We say that a function $\varrho(x)$ is weakly positive definite if

$$\int \int \varrho(s-t)g(s)g(t)\,ds\,dt \geq 0 \quad (3.3)$$

for all $g \in B_0(R_+)$. Let $\varrho(x)$ be a symmetric, weakly positive definite function that is locally integrable on $R^1$. Consider the mean zero Gaussian field $F = \{F(g), g \in B_0(R_+)\}$ with covariance

$$E(F(f)F(g)) = \int \int \varrho(s-t)f(s)g(t)\,ds\,dt \quad f, g \in B_0(R_+). \quad (3.4)$$

(We are particularly interested in the case in which $\varrho(0) = \infty$, in which case it is not the covariance of a stationary Gaussian process.)

Let $f_\delta(s)$ be a continuous positive symmetric function on $(s, \delta) \in R_+ \times (0, 1]$, with support in the ball of radius $\delta$ centered at the origin, with $\int f_\delta(y)\,dy = 1$. That is, $f_\delta$ is a continuous approximate identity. Set $f_{x,\delta}(s) = f_\delta(s-x)$.

Assume that

$$\varrho \in L^k_{\text{loc}}(R^1). \quad (3.5)$$

We now define, what we call, the $k$-th Wick power Gaussian chaos associated with $F$.

**Lemma 3.1** Let $\{f_\delta, \delta \in (0, \delta_0]\}$ be a family of approximate identities and assume that $\varrho$ is a symmetric, weakly positive definite function that satisfies (3.5). Then for all $g \in B_0(R_+)$

$$:F^k:(g) \overset{\text{def}}{=} \lim_{\delta \to 0} \int : (F(f_{x,\delta}))^k : g(x)\,dx \quad \text{exists in } L^2. \quad (3.6)$$

and

$$E(:F^k:(g))^2 = k! \int \int \varrho^k(x-y)g(x)g(y)\,dx\,dy. \quad (3.7)$$

**Proof** Consider the mean zero Gaussian process $\{F(f_{x,\delta}), (x, \delta) \in R_+ \times (0, 1]\}$ with covariance

$$E(F(f_{x,\delta})F(f_{y,\delta'})) = \int \int \varrho(x'-y')f_{x,\delta}(x')f_{y,\delta'}(y')\,dx'\,dy' \quad (3.8)$$

$$= \int \int \varrho(x'+x-y'-y)f_\delta(x')f_{\delta'}(y')\,dx'\,dy' \overset{\text{def}}{=} \varrho_{\delta,\delta'}(x,y).$$

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It follows from (3.1) that
\[ E((\mathcal{F}(f_{x,\delta}))^k : (\mathcal{F}(f_{y,\delta'}))^k) = k! (\rho_{\delta,\delta'}(x, y))^k. \] (3.9)

Let \( g \in \mathcal{B}_0(R_+) \). It follows from (3.8), (3.9) and Fubini’s theorem that
\[
E \left( \int \int : (\mathcal{F}(f_{x,\delta}))^k : (\mathcal{F}(f_{y,\delta'}))^k : g(x)g(y) \, dx \, dy \right)
\tag{3.10}
\]
\[
= k! \int \int (\rho_{\delta,\delta'}(x, y))^k g(x)g(y) \, dx \, dy
\]
\[
= k! \int \int \left( \prod_{j=1}^k \rho(x + v_j - y - w_j) \prod_{j=1}^k f_{\delta}(v_j) f_{\delta'}(w_j) \, dv_j \, dw_j \right) \, g(x)g(y) \, dx \, dy
\]
\[
= k! \int \ldots \int \left( \prod_{j=1}^k \rho(x - y + v_j - w_j) g(x)g(y) \, dx \, dy \right) \prod_{j=1}^k f_{\delta}(v_j) f_{\delta'}(w_j) \, dv_j \, dw_j.
\]

Since \( \rho \in L^k_{\text{loc}}(R^1) \) and translation is continuous in \( L^k_{\text{loc}}(R^1) \), the double integral in parentheses immediately above is continuous in \((v_1 - w_1, \ldots, v_k - w_k)\) and goes to
\[
\int \int \rho^k(x - y)g(x)g(y) \, dx \, dy
\] (3.11)
as \( \sup_{1 \leq j \leq n} |v_j - w_j| \to 0 \). Consequently
\[
\lim_{\delta, \delta' \to 0} E \left( \int \int : (\mathcal{F}(f_{x,\delta}))^k : (\mathcal{F}(f_{y,\delta'}))^k : g(x)g(y) \, dx \, dy \right) = k! \int \int \rho^k(x - y)g(x)g(y) \, dx \, dy.
\] (3.12)

It follows from this that
\[
\lim_{\delta, \delta' \to 0} E \left( \int : (\mathcal{F}(f_{x,\delta}))^k : g(x) \, dx - \int : (\mathcal{F}(f_{x,\delta'}))^k : g(x) \, dx \right)^2 = 0. \tag{3.13}
\]

This implies (3.6). The relation in (3.7) follows from (3.12). \( \square \)

**Remark 3.1** Suppose that \( \tilde{F} \) is a mean zero Gaussian field, with covariance \( \tilde{\rho} \in L^k_{\text{loc}}(R^1) \), and that \( \mathcal{F} \) and \( \tilde{\mathcal{F}} \) are jointly Gaussian with
\[
E(\mathcal{F}(f)\tilde{\mathcal{F}}(f')) = \int \int \psi(x - y)f(x)f'(y) \, dx \, dy
\] (3.14)
for some $\psi \in L^k_{\text{loc}}(R^1)$. If we return to (3.8) and replace $\mathcal{F}(f_{y,\delta})$ by $\tilde{\mathcal{F}}(f_{y,\delta'})$ and continue the argument in the proof of Lemma 3.1 we see that

$$E(\mathcal{F}^k : (g) : \tilde{\mathcal{F}}^k : (g')) = k! \int \int \psi^k(x - y)g(x)g'(y) \, dx \, dy.$$  (3.15)

**Remark 3.2** Although we say that we are particularly interested in the case in which $g(0) = \infty$ in (3.4), Lemma 3.1 also applies when $g$ is the covariance of a stationary Gaussian process. Given a mean zero stationary Gaussian process $\tilde{G} = \{\tilde{G}(x), x \geq 0\}$, with continuous covariance $\varphi(s)$, we can define a Gaussian field $\mathcal{G} = \{G(g), g \in B_0(R_+)\}$ by

$$\mathcal{G}(f) = \int \tilde{G}(x) f(x) \, dx.$$  (3.16)

Clearly

$$E(G(f)G(g)) = \int \int \varphi(s - t) f(s) g(t) \, ds \, dt \quad f,g \in B_0(R_+).$$  (3.17)

It follows from Lemma 3.1 that we can construct a $k$–th order Wick power chaos $\mathcal{G}^k = \{G^k(g), g \in B_0(R_+)\}$ with

$$E(\mathcal{G}^k : (g))^2 = k! \int \int \varphi^k(x - y) g(x)g(y) \, dx \, dy.$$  (3.18)

However, we do not really need Lemma 3.1 when we are dealing with a mean zero stationary Gaussian process, since we can simply form the $k$–th order Wick power chaos $\tilde{G}^k = \{\tilde{G}^k(g), g \in B_0(R_+)\}$ by setting

$$\tilde{G}^k : (g) = \int \left(\tilde{G}(x)\right)^k : g(x) \, dx.$$  (3.19)

It is easy to see that these two processes, $\mathcal{G}^k$ and $\tilde{G}^k$, are equivalent, (in $L^2$). To do this we now show that

$$\lim_{\delta \to 0} \int \left(\mathcal{G}(f_{x,\delta})\right)^k : g(x) \, dx = \int \left(\tilde{G}(x)\right)^k : g(x) \, dx, \quad \text{in } L^2.$$  (3.20)

Note that by (3.10) and the fact that $\tilde{G}$ has covariance $\varphi$,

$$E(G(f_{x,\delta})\tilde{G}(y)) = \int \int \varphi(x' - y) f_{x,\delta}(x') \, dx' \quad \text{def} = \int \int \varphi(x' + x - y) f_{\delta}(x') \, dx'$$

$$= \varphi(x,y).$$

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Therefore, it follows from (3.1) that
\[
E(\langle G(f_{x,\delta}) \rangle^k : \langle \tilde{G}(y) \rangle^k) = k! (\varphi_\delta(x, y))^k.
\] (3.22)

We use this in place of (3.9) and continue with the argument in (3.10)–(3.12), with obvious modifications, to see that
\[
\lim_{\delta \to 0} E \left( \int \int : \langle G(f_{x,\delta}) \rangle^k : \langle \tilde{G}(y) \rangle^k : g(x) g(y) \, dx \, dy \right)
\]
\[
= \int \int \varphi^k(x - y) g(x) g(y) \, dx \, dy.
\] (3.23)

Using this, (3.12) with \( g \) replaced by \( \varphi \), and the obvious fact that
\[
E \left( \int \int : \langle \tilde{G}(x) \rangle^k : \langle \tilde{G}(y) \rangle^k : g(x) g(y) \, dx \, dy \right)
\]
\[
= k! \int \int \varphi^k(x - y) g(x) g(y) \, dx \, dy,
\] we get (3.20).

**Remark 3.3** In this paper we consider Wick powers of Gaussian fields, \( \{ : \langle G' \rangle^k : (g), g \in B_0(R_+) \} \). It is well known that \( : \langle G' \rangle^k : (g) \) can also be expressed as a multiple Wiener-Itô integral. (See, e.g. [4].) We briefly explain this for the benefit of those familiar with multiple Wiener-Itô integrals:

Since \( \rho(x) \) is symmetric and weakly positive definite, it follows from the Bochner–Schwartz Theorem that \( \rho(x) = \int e^{i \lambda x} d\mu(\lambda) \) for some positive Radon measure \( \mu \). When \( \rho(0) = \infty \), \( \mu \) is not a finite measure.

Let \( Z_\mu \) be the (complex valued) Gaussian random spectral measure corresponding to \( \mu \). Then
\[
: \langle G' \rangle^k : (g) = \int \cdots \int \hat{g}(\lambda_1 + \cdots + \lambda_k) \, dZ_\mu(\lambda_1) \cdots dZ_\mu(\lambda_k)
\] (3.25)

where \( \hat{g} \) is the Fourier transform of \( g \). (This is the end of Remark 3.3.)

We now apply the above results about constructing Gaussian chaos to the processes that concern us. In Lemma 2.1 we define the Gaussian field \( \{ G'(g), g \in B_0(R_+) \} \). When \( \rho \in L^k_{\text{loc}}(R^4) \) the procedure that leads to (3.6) and (3.7) enables us to define \( k \)-th Wick power chaos
\[
: \langle G' \rangle^k : (g) = \lim_{\delta \to 0} \int : \langle G'(f_{x,\delta}) \rangle^k : g(x) \, dx
\] (3.26)
as a limit in $L^2$, with
\[ E(: \langle G' \rangle^k : \langle g \rangle)^2 = k! \int \int \rho^k(x - y)g(x)g(y) \, dx \, dy. \] (3.27)

Note also that the Gaussian field $X_h(g)$ defined in (2.14) is of the form of (3.16). Therefore, it follows from (3.19) that
\[ : X^k_h : \langle g \rangle = \int \left( \frac{G(x + h) - G(x)}{h} \right)^k : g(x) \, dx. \] (3.28)

The next theorem is a critical result in this paper.

**Theorem 3.1** Let $G = \{G(x), x \in R_+\}$, $G(0) = 0$, be a mean zero Gaussian process with stationary increments. Let $\rho$ be as defined in (1.7) and assume that $\rho(x)$ is locally integrable and that $\rho(|x|)$ is bounded on $[\delta, M]$ for each $0 < \delta < M < \infty$. Then for all $g \in B_0(R_+)$,
\[ \lim_{h \to 0} \int : \left( \frac{G(x + h) - G(x)}{h} \right)^k : g(x) \, dx = : \langle G' \rangle^k : \langle g \rangle \text{ in } L^2. \] (3.29)

**Proof** By (2.20)
\[ E(X_h(g)X_{h'}(g)) \] (3.30)
\[ = \int \int \left( \frac{1}{h} \int_x^{x+h} \frac{1}{h'} \int_y^{y+h'} \rho(s - t) \, dt \, ds \right) g(x)g(y) \, dx \, dy. \]
Consequently by (3.14) and (3.15)
\[ E(: X^k_h : \langle g \rangle : X^k_{h'} : \langle g \rangle) \] (3.31)
\[ = k! \int \int \left( \frac{1}{h} \int_x^{x+h} \frac{1}{h'} \int_y^{y+h'} \rho(s - t) \, dt \, ds \right)^k g(x)g(y) \, dx \, dy. \]

In a similar vein by (2.13) and (2.20), Lebesgue’s Theorem and a change of variables
\[ E(X_h(g)G'(g)) \] (3.32)
\[ = \lim_{h' \to 0} E(X_h(g)X_{h'}(g)) \]
\[ = \lim_{h' \to 0} \int \int \left\{ \frac{1}{h} \int_{t-h}^{t} g(x) \, dx \right\} \left\{ \frac{1}{h'} \int_{s-h'}^{s} g(y) \, dy \right\} \rho(t - s) \, ds \, dt \]
\[ = \int \int \left\{ \frac{1}{h} \int_{t-h}^{t} g(x) \, dx \right\} \rho(t - s) \, g(s) \, ds \, dt \]
\[ = \int \int \left( \frac{1}{h} \int_{x-h}^{x} \rho(s - y) \, ds \right) g(x)g(y) \, dx \, dy. \]
Therefore, by (3.14) and (3.15)

\[
E(\cdot; X^k_h: (g) : (G')^k : (g)) = k! \int \int \left( \frac{1}{h} \int_{x}^{x+h} \rho(s - y) \, ds \right)^k g(x)g(y) \, dx \, dy
\]

(3.33)

where

\[
A_{h,\delta} = k! \int \int_{|x - y| < \delta} \left( \frac{1}{h} \int_{x}^{x+h} \rho(s - y) \, ds \right)^k g(x)g(y) \, dx \, dy
\]

(3.34)

and

\[
B_{h,\delta} = k! \int \int_{|x - y| \geq \delta} \left( \frac{1}{h} \int_{x}^{x+h} \rho(s - y) \, ds \right)^k g(x)g(y) \, dx \, dy.
\]

(3.35)

Fix \( \delta > 0 \). Using the fact that \( \rho \) is bounded away from the origin, the Dominated Convergence Theorem, and Lebesgue’s theorem on differentiation, we see that

\[
\lim_{h \to 0} B_{h,\delta} = k! \int \int_{|x - y| \geq \delta} \rho^k(x - y) g(x)g(y) \, dx \, dy.
\]

(3.36)

On the other hand, using the Hölder or Jensen inequality, we see that for \( h \leq \delta \)

\[
|A_{h,\delta}| \leq k! \int \int_{|x - y| < \delta} \left( \frac{1}{h} \int_{x}^{x+h} |\rho^k(s - y)| \, ds \right) |g(x)| |g(y)| \, dx \, dy
\]

\[
\leq k! \int \int_{|x - y| < 2\delta} |\rho^k(s - y)| \left( \frac{1}{h} \int_{s-h}^{s} |g(x)| \, dx \right) |g(y)| \, dy \, ds.
\]

\[
\leq C \int_{|s| < 2\delta} |\rho^k(s)| \, ds.
\]

(3.37)

Since by assumption \( \rho^k(s) \) is locally integrable we can make this arbitrarily small by choosing \( \delta > 0 \) sufficiently small. Thus we have shown that

\[
\lim_{h \to 0} E(\cdot; X^k_h: (g) : (G')^k : (g)) = k! \int \int \rho^k(s - y)g(s)g(y) \, dy \, ds.
\]

(3.38)

Similar reasoning shows that

\[
\lim_{h \to 0} E(\cdot; X^k_h: (g) : (G')^k : (g))^2 = k! \int \int \rho^k(s - y)g(s)g(y) \, dy \, ds
\]

(3.39)

for all \( g \in B_0(R_+) \). Using (3.38), (3.39) and (3.27) we get (3.29).
Remark 3.4 In Section 2 we explain why we think of the field $G'$ as a generalized derivative of the Gaussian process $G = \{G(x), x \in R^1\}$. In (3.26) we construct the $k$-th Wick power chaos $(G')^k : (g)$. When $G$ itself is mean square differentiable, i.e. when

$$\lim_{h \to 0} E \left( \frac{G(x + h) - G(x)}{h} \right)^2 = \lim_{h \to 0} \frac{\sigma^2(h)}{h^2} := \frac{1}{2} (\sigma^2)''(0) < \infty,$$  

(3.40)

$\{\frac{d}{dx} G(x), x \in R^1\}$ is a stationary Gaussian process with covariance $\frac{1}{2} (\sigma^2)''(x - y)$. In this case, as we show in (3.19),

$$(G')^k : (g) = \int : (\frac{d}{dx}G(x))^k : g(x) dx.$$  

(3.41)

This further motivates the description of $G'$ as a generalized derivative.

However, for Gaussian processes satisfying (1.4), $\lim_{h \to 0} \sigma^2(h) = \infty$, and consequently $\frac{d}{dx} G(x)$ is not a stochastic process. (In these cases, formally taking $g$ in (2.1) to be the delta ‘function’ $\delta_x$, gives $E(G'(x))^2 = \rho(0) := \lim_{h \to 0} \frac{\sigma^2(h)}{h^2} = \infty$.)

4 $L^2$ asymptotic expansion

For each $h$ we consider the symmetric positive definite kernel

$$\tau_h(x, y) = \frac{1}{\sigma^2(h)} E(G(x + h) - G(x))(G(y + h) - G(y))$$  

(4.1)

$$= \frac{1}{2\sigma^2(h)} \left( \sigma^2(x - y + h) + \sigma^2(x - y - h) - 2\sigma^2(x - y) \right)$$

$$:= \tau_h(x - y) = \tau_h(y - x),$$

(see (2.4)). Note that since $G$ has stationary increments it follows from the Cauchy–Schwarz inequality that

$$|\tau_h(x - y)| \leq 1 \quad \forall x, y \in R^1.$$  

(4.2)

To continue we need some estimates of the integrals of powers of $\tau_h$. 

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Lemma 4.1 Suppose that \(\sigma^2\) satisfies (1.3)–(1.6) and \(\rho(s)\) is locally integrable and is bounded in compact neighborhoods excluding the origin. Then

\[
\lim_{h \to 0} \int_{a}^{b} \int_{a}^{b} \frac{\tau_h(x - y) \, dx \, dy}{h^2/\sigma^2(h)} = \sigma^2(b - a) \tag{4.3}
\]

and

\[
\lim_{h \to 0} \frac{\int_{a}^{b} \int_{a}^{b} \tau_h^2(x - y) \, dx \, dy}{\int_{a}^{b} \int_{a}^{b} \tau_h(x - y) \, dx \, dy} = 0. \tag{4.4}
\]

More generally if, in addition, \(\rho^k(s)\) is locally integrable for some integer \(k \geq 1\) and

\[
h = o\left(h^2/\sigma^2(h)\right)^k \tag{4.5}
\]

then

\[
\lim_{h \to 0} \int_{a}^{b} \int_{a}^{b} \frac{\tau_h^k(x - y) \, dx \, dy}{(h^2/\sigma^2(h))^k} = \int_{a}^{b} \int_{a}^{b} \rho^k(x - y) \, dx \, dy \tag{4.6}
\]

and

\[
\lim_{h \to 0} \frac{\int_{a}^{b} \int_{a}^{b} \tau_h^{k+1}(x - y) \, dx \, dy}{\int_{a}^{b} \int_{a}^{b} \tau_h^k(x - y) \, dx \, dy} = 0. \tag{4.7}
\]

**Proof** It follows from (4.1) and (2.14) that

\[
\int_{a}^{b} \int_{a}^{b} \frac{\tau_h(x - y) \, dx \, dy}{h^2/\sigma^2(h)} = \int_{a}^{b} \int_{a}^{b} \frac{\sigma^2(x - y + h) + \sigma^2(x - y - h) - 2\sigma^2(x - y)}{2h^2} \, dx \, dy
\]

\[
= E(X_h(I_{[a,b]}))^2.
\]

By (2.21),

\[
\lim_{h \to 0} E(X_h(I_{[a,b]}))^2 = E(G'(I_{[a,b]}))^2 = E(G(b) - G(a))^2 = \sigma^2(b - a). \tag{4.9}
\]

Thus we get (4.3).

The statement in (4.6) follows as above using (3.31) and then (3.29) which implies that

\[
\lim_{h \to 0} E(\cdot \, X_h^k : (I_{[a,b]})) = E\left(\cdot \, (G')^k : (I_{[a,b]})\right)^2. \tag{4.10}
\]
We now obtain (4.7), which, considering (1.4), includes (4.4). We are given that $\rho^k(s)$ is locally integrable. Suppose that $\rho^{k+1}(s)$ is also locally integrable. Then, by (4.6)

$$
\lim_{h \to 0} \left( \frac{\sigma^2(h)}{h^2} \right)^{k+1} \int_a^b \int_a^b \tau_h^{k+1}(x - y) \, dx \, dy = \int_a^b \int_a^b \rho^{k+1}(x - y) \, dx \, dy. \tag{4.11}
$$

The statement in (4.7) clearly follows from this, (1.4) and (4.6).

Suppose $\rho^{k+1}(s)$ is not integrable over neighborhoods of the origin. By a change of variables, and with $c = b - a$,

$$
\int_a^b \int_a^b (\tau_h(x - y))^{k+1} \, dx \, dy \tag{4.12}
$$

$$
= 2 \int_0^c \left( \frac{\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)}{2\sigma^2(h)} \right)^{k+1} (c - s) \, ds
$$

$$
\leq 16ch + \int_{sh}^c \left( \frac{\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)}{2\sigma^2(h)} \right)^{k+1} (c - s) \, ds,
$$

where, for the last line we use (4.2). Also

$$
\int_{sh}^c \left( \frac{\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)}{\sigma^2(h)} \right)^{k+1} \, ds
$$

$$
= \left( \frac{h^2}{\sigma^2(h)} \right)^{k+1} \int_{sh}^c \left( \frac{\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)}{h^2} \right)^{k+1} \, ds
$$

$$
\leq c \left( \frac{h^2}{\sigma^2(h)} \right)^{k+1} \int_{sh}^c \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} \, ds,
$$

where, for the last line we use (1.5). Let $a > 0$. Using (1.5) again we see that

$$
\int_a^c \rho^{k+1}(s) \, ds \tag{4.14}
$$

$$
= \frac{1}{2^{k+1}} \int_a^c \lim_{h \to 0} \left( \frac{\sigma^2(s + h) + \sigma^2(s - h) - 2\sigma^2(s)}{h^2} \right)^{k+1} \, ds
$$

$$
\leq C \int_a^c \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} \, ds.
$$

Consequently, since $\rho^{k+1}(s)$ is not locally integrable, the final integral in (4.13) goes to infinity as $h \downarrow 0$. Since $\sigma^2(s)$ is regularly varying at zero, this
means that \( \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} \) is regularly varying at zero with index less than or equal to \(-1\).

Suppose that its index is equal to \(-1\). This implies that \( \left( \frac{h^2}{\sigma^2(h)} \right)^k \) is regularly varying with index equal to \( k/(k+1) < 1 \) and that the integral in the last line of (4.13) is slowly varying. Consequently

\[
\left( \frac{h^2}{\sigma^2(h)} \right)^{k+1} \int_{sh}^{c} \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} ds
\]  

is regularly varying with index equal to 1. Taking (4.6), (4.12) and (4.13) into account we see that (4.7) holds in this case.

Finally, suppose that the index of \( \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} \) is less than \(-1\). In this case

\[
\left( \frac{h^2}{\sigma^2(h)} \right)^{k+1} \int_{sh}^{c} \left( \frac{\sigma^2(s)}{s^2} \right)^{k+1} ds \sim Ch
\]

at 0, for some constant \( C \). Taking (4.6) and (4.5) into account we again get (4.7).

**Proof of Theorem 1.1** It follows from (1.19) and (1.20) that

\[
\int_{a}^{b} f \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) dx
\]  

\[
= \sum_{j=0}^{j_0} E(H_j(\eta)f(\eta)) \int_{a}^{b} H_j \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) dx
\]

\[+
\sum_{j=j_0+1}^{\infty} a_j \int_{a}^{b} H_j \left( \frac{G(x+h)-G(x)}{\sigma(h)} \right) dx.
\]

Denote the last line in (4.17) by \( Z(h) \). Using (3.2) and then (4.2) we have

\[
EZ^2(h) = \sum_{j=j_0+1}^{\infty} a_j^2 \int_{a}^{b} \int_{a}^{b} \tau^2_h(x-y) dx dy
\]  

\[
\leq \int_{a}^{b} \int_{a}^{b} \tau^2_{j_0+1}(x-y) dx dy \sum_{j=j_0+1}^{\infty} a_j^2.
\]

It follows easily from (1.9) with \( j_0 \zeta < 1 \) and (2.3) that (4.5) holds with \( k \) replaced by \( j_0 \). It then follows from (4.7) and (4.6) that the last line in (4.18)
is \( o \left( \frac{h}{\sigma(h)} \right)^{j_0} \). Since

\[
\begin{align*}
\sum_{j=0}^{j_0} E(H_j(f(h))) & \int_a^b H_j \left( \frac{G(x+h) - G(x)}{\sigma(h)} \right) \, dx \\
& = \sum_{j=0}^{j_0} \left( \frac{h}{\sigma(h)} \right)^j \frac{E(H_j(f(h)))}{\sqrt{j!}} \int_a^b \left( \frac{G(x+h) - G(x)}{h} \right)^j \, dx,
\end{align*}
\]

we see that (1.11) follows from (4.21) in the next lemma.

**Lemma 4.2** Let \( G = \{G(x), x \in R_+\} \), \( G(0) = 0 \), be a mean zero Gaussian process with stationary increments that satisfies the hypotheses of Theorem [1.1]. Then for \( 1 \leq j \leq j_0 \) and \( g \in B_0(R_+) \)

\[
\| : X_h^j : (g) - : (G')^j : (g) \|_2 \leq C(|h| \varphi^j(h))^{1/2}
\]

and

\[
\lim_{h \to 0} \| : X_h^j : (g) - : (G')^j : (g) \|_2 \left( \frac{\sigma^2(h)}{h^2} \right)^{(j_0-j)/2} = 0.
\]

**Proof** To obtain (4.20) we use (3.31) for \( \| : X_h^j : (g) \|_2 \), (3.7) for \( \| : (G')^j : (g) \|_2 \) and (3.33) for \( E \left( X_h^j : (g) : (G')^j : (g) \right) \) to see that

\[
\begin{align*}
\frac{1}{j!} \| : X_h^j : (g) - : (G')^j : (g) \|_2^2 \\
& = \int \int \left\{ \left( \frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds \right)^j - \left( \frac{1}{h} \int_x^{x+h} \rho(s-y) \, ds \right)^j \\
& \quad - \left( \frac{1}{h} \int_y^{y+h} \rho(x-t) \, dt \right)^j + \rho(x-y))^j \right\} g(x)g(y) \, dx \, dy.
\end{align*}
\]

Set \( z = x - y \). We write

\[
\frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds = \rho(z) + \left( \frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds - \rho(z) \right).
\]
By (1.10), for $4|h| \leq |z| \leq M$,
\[
\left| \frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds - \rho(z) \right| \leq \frac{1}{h} \int_0^h \frac{1}{h} \int_0^h |\rho(z+s-t) - \rho(z)| \, dt \, ds \leq C_M \frac{|h|}{|z|} |\rho(z)|.
\] (4.24)

Note that given an integer $j_0$, there exists a $u_{j_0} > 0$, such that
\[
\frac{1}{2} \leq 1 - 2j|u| \leq (1 + u)^j \leq 1 + 2|u| \leq 2
\] (4.25)
for all $0 \leq |u| \leq u_{j_0}$ and $1 \leq j \leq j_0$. Therefore, if we take $C_M \frac{|h|}{|z|} \leq u_{j_0}$ we see that when $4|h| \leq z \leq M$,
\[
\left( \frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds \right)^j = \rho^j(z) + O \left( \frac{|h|}{|z|} |\rho^j(z)| \right)
\] (4.26)
where the last term is independent of $z$, (but depends on $M$ and $j_0$).

We estimate the other two integrals in the bracket in (4.22) similarly to see that there exists a constant $C'$ such that for all $h$ sufficiently small
\[
\left| \int \int_{|x-y| \geq C'h} \{ \cdots \} g(x)g(y) \, dx \, dy \right| \leq K \int \int_{C'h \leq |z| \leq M} \frac{|h|}{|z|} |\rho^j(z)| \, dz
\] (4.27)
\[
\leq K' |h|^{1-j \zeta} = K'' |h| \varphi^j(h).
\]
where, in addition to other dependencies, $K$ and $K'$ depend on the support of $g$.

By (2.6) and the second inequality in (2.4), with $z = x - y$,
\[
\frac{1}{h} \int_x^{x+h} \frac{1}{h} \int_y^{y+h} \rho(s-t) \, dt \, ds = \frac{1}{h^2} \int_0^h \int_0^h \rho(z + u - v) \, dv \, du.
\] (4.28)

We conclude the proof by considering the integral in (4.22) when $|x-y| \leq C'h$. Note that
\[
\frac{1}{h} \left| \int_x^{x+h} \rho(s-y) \, ds \right| = \frac{1}{h} \left| \int_0^h \rho(z+s) \, ds \right|
\] (4.29)
\[
\leq C_M \frac{1}{h} \int_0^h \frac{1}{|z+s|} \, ds
\]
\[
\leq 2C_M \frac{1}{h} \int_0^{h/2} \frac{1}{|s|} \, ds
\]
\[
\leq \frac{C}{|h|^\zeta} = C\varphi(h),
\]

where for the third line we use the symmetry of the integrand. Therefore

$$\left| \left( \frac{1}{h} \int_0^h \rho(z + s) \, ds \right)^j \right| \leq C\varphi^j(h). \quad (4.30)$$

Similarly

$$\left| \left( \frac{1}{h} \int_0^h \frac{1}{h} \int_0^h \rho(z + s - t) \, dt \, ds \right)^j \right| \leq C\varphi^j(h). \quad (4.31)$$

Consequently, the integral of the first three terms in the bracket in (4.22), taken over the region $|z| < C'h$ is bounded by $C''h\varphi^j(h)$. As for the integral of the last term in the bracket in (4.22), taken over the region $|z| < C'h$, consider

$$\left| \int \int_{|x-y| \leq C'h} \rho^j(|x-y|)g(x)g(y) \, dx \, dy \right|. \quad (4.32)$$

Using (1.9) it is easy to see that this also has the bound $C''h\varphi^j(h)$. Thus we obtain (4.20).

To obtain (4.21) we first note that by (2.3), a change of variables followed by one integration, and (1.9)

$$\sigma^2(h) = 2 \int_0^h (h-s)\rho(s) \, ds \leq Ch^2\varphi(h). \quad (4.33)$$

Therefore

$$\left( \frac{\sigma^2(h)}{h^2} \right)^j \leq C'\varphi^j(h). \quad (4.34)$$

The statement in (4.21) follows immediately from this and (1.20). □

**Proof of Corollary 1.1.** This follows immediately from Theorem 1.1 once we observe that the conditions (1.9) and (1.10) are only used in two places: the proof of Lemma 4.2 which is not need here, and in the proof of (4.5) which is now assumed in condition (1.23). □

**Proof of Corollary 1.2.** Note that $H_1(x) = x$. Consequently, for $f \in L^2(R^1, d\mu), a_1 = E(\eta f(\eta))$. Therefore, in Corollary 1.2 $k_0(f) = 1$. Hence (1.23) is given by the second condition in (1.4). Also, By (2.21), $(G^f)^1 : (I_{[a,b]}) = G^f(I_{[a,b]}) = G(b) - G(a)$. Thus (1.26) is a special case of (1.24). □
5 FB-mixtures and other examples

Set

\[ \phi(u) = 2 \int_{-\infty}^{\infty} (1 - \cos 2\pi \lambda u) \nu(d\lambda) \] (5.1)

where

\[ \int_{-\infty}^{\infty} (1 \wedge \lambda^2) \nu(d\lambda) < \infty. \] (5.2)

It is well known, (see e.g. [5, page 236]), that \( \phi \) can be the increments variance of Gaussian process with stationary increments that is zero at zero.

We construct a wide class of examples of Gaussian processes that satisfy the hypotheses of Theorem 1.1 based on the ideas underlying “stable mixtures” considered in [5, Section 9.6]. For \( 1 < \beta < 2 \) let

\[ \psi(\lambda) = \int_{\beta}^{2} |\lambda|^s d\mu(s), \] (5.3)

where \( \mu \) is a finite positive measure on \([\beta, 2]\) such that

\[ \int_{\beta}^{2} d\mu(s) < \infty. \] (5.4)

We show in [5, Section 9.6] that \( \psi \) can be represented as in (5.1) for some measure \( \nu \) satisfying (5.2). Therefore, as we point out in the preceding paragraph, \( \psi \) is the increments variance of a Gaussian process with stationary increments that is zero at zero.

In [5, Section 9.6] we refer to \( \psi \) as a stable mixture because we were studying Lévy processes and \( |\lambda|^s \) is the Lévy exponent of a symmetric stable process. Here, since we are concerned with Gaussian processes, we refer to \( \psi \) as an FB-mixture because \( |\lambda|^s \) is the increments variance of fractional Brownian motion.

In [5, Section 9.6] we study \( \psi(\lambda) \) as \( \lambda \to \infty \). The proofs of [5] Lemma 9.6.1 and Remark 9.6.2], with obvious modifications, give the proof of the next lemma.

**Lemma 5.1** The function \( \psi(\lambda) \) is a normalized regularly varying function at zero with index \( \beta \). Moreover for \( n = 1, 2, \ldots \),

\[ \lambda^n \psi^{(n)}(\lambda) / \psi(\lambda) \to \beta(\beta - 1) \ldots (\beta - n + 1) \quad \text{as} \quad \lambda \to 0, \] (5.5)

where \( \psi^{(n)} \) denotes the \( n \)-th derivative of \( \psi \).
It follows from (5.3) that $\psi(\lambda)$ is twice differentiable for all $\lambda \neq 0$ and

$$
\psi''(\lambda) = \int_\beta^2 s(s-1)|\lambda| s^{-2} \, d\mu(s).
$$

(5.6)

We note that $\psi$ is convex and bounded away from the origin. In addition, by (5.5)

$$
\psi''(\lambda) \sim \beta(\beta - 1) \frac{\psi(\lambda)}{\lambda^2} \quad \text{as } \lambda \to 0.
$$

(5.7)

It follows that $\psi''$ is a regularly varying function at zero with index $-(2 - \beta)$. Therefore, for any integer $j_0 \geq 1$ we can find a $1 < \beta < 2$ such that (1.9) holds with $j_0 \zeta < 1$. (Clearly $\frac{1}{2} \psi''$ takes the role of $\rho$ in (1.7).)

Lastly, we note that

$$
|\psi''(\lambda + h) - \psi''(\lambda)| = \int_\beta^2 s(s-1)|\lambda| s^{-2} \left(1 + \frac{h}{\lambda} \right)^{s-2} \, d\mu(s).
$$

(5.8)

which implies (1.5). Thus we see that FB-mixtures $\psi(\lambda)$ are the increments variance of Gaussian processes that satisfy the hypotheses of Theorem 1.1.

We give some concrete examples of FB-mixtures.

**Example 5.1** A simple one that follows immediately from (5.3) is

$$
\psi(\lambda) = \sum_{k=0}^\infty a_k \lambda^{\beta_k} \quad a_k \geq 0, \quad \{a_k\} \in \ell_1,
$$

(5.10)

where $\beta_0 = \beta$ and $\{\beta_k\}$ is increasing with $\sup_k \beta_k < 2$.

As a slight modification of this, it is easy to see that

$$
\tilde{\psi}(\lambda) = \sum_{k=0}^n a_k \lambda^{\beta_k} \quad a_k > 0,
$$

(5.11)

where $\beta_0 = \beta$ and $\{\beta_k\}$ is increasing with $\beta_n = 2$, is the increments variance of a Gaussian process, $\tilde{G}$, that satisfies the hypotheses of Theorem 1.1. We get this by taking $\tilde{G}$ to be the sum of two independent Gaussian processes. One with increments variance the FB-mixture, $\psi(\lambda) = \sum_{k=0}^{n-1} a_k \lambda^{\beta_k}$ and the other $\{\sqrt{a_n} t \eta, t \in R_+\}$. 

30
Lemma 5.2 Let \( \rho(s) \) be a bounded increasing function on \([0, 2 - \beta]\), \( 1 < \beta < 2 \) satisfying
\[
\int_0^{2 - \beta} \frac{d\rho(v)}{2 - \beta - v} < \infty. \tag{5.12}
\]
Then we can find an FB-mixture with increments variance
\[
\psi(\lambda) = \lambda^\beta \hat{\rho}(\log 1/\lambda) \tag{5.13}
\]
where
\[
\hat{\rho}(v) = \int_0^\infty e^{-vs} d\rho(s). \tag{5.14}
\]

Proof Let \( \mu(s) \) in (5.3) be a measure with distribution function \( \rho(s - \beta) \). It follows from (5.12) that (5.4) holds. Therefore, for \( 1 < \beta < 2 \),
\[
\psi(\lambda) = \int_\beta^2 \lambda^s d\rho(s - \beta) \tag{5.15}
\]
\[
= \lambda^\beta \left( \int_0^{2 - \beta} \lambda^s d\rho(s) \right) \tag{5.16}
\]
\[
= \lambda^\beta \left( \int_0^{2 - \beta} e^{-\log 1/\lambda}s d\rho(s) \right) \tag{5.17}
\]
\[
= \lambda^\beta \hat{\rho}(\log 1/\lambda). \tag{5.18}
\]

Proof of Proposition 1.1 We use the fact that we can find an FB-mixture of the form (5.13). For \( p > 0 \), let \( \rho(s) \sim s^p L(1/s)/\Gamma(1+p) \) at zero, in (5.14). Then by [5, Theorem 14.7.6], \( \hat{\rho}(s) \sim s^{-p} L(s) \) as \( s \to \infty \). Thus (1.18) follows from Lemma 5.2. For \( p = 0 \) if \( \rho(s) \sim L(1/s) \) we must have \( L(1/s) \) increasing as \( s \) increases. In this case \( \hat{\rho}(s) \sim L(s) \) and \( L(s) \) is decreasing.

Example 5.2 Let
\[
g(\lambda) = \frac{\log |\lambda| - 1}{|\lambda|^3} I_{\{|\lambda| \geq e\}} \tag{5.16}
\]
and consider (5.1) with \( \nu(d\lambda) = g(\lambda) d\lambda \). Then
\[
\sigma^2(u) \approx Cu^2 \log^2 1/u \quad \text{and} \quad (\sigma^2)'(u) \approx \log^2 1/u \tag{5.17}
\]
at zero. We show this immediately below, and also, that \( \sigma^2(u) \) satisfies the hypotheses of Theorem 1.1. Therefore, we get (1.43).
By (5.1) we have
\[
\sigma^2(u) = 8 \int_e^\infty \sin^2 \pi \lambda u \left( \log \lambda - \frac{1}{\lambda^3} \right) d\lambda
\]  
(5.18)
\[
\leq 8\pi^2 u^2 \int_e^{1/(\pi u)} \log \lambda \frac{1}{\lambda} d\lambda + 8 \int_{1/(\pi u)}^\infty \log \lambda \frac{1}{\lambda^3} d\lambda
\]
and
\[
\sigma^2(u) \geq C u^2 \int_e^{1/(\pi u)} \log \lambda \frac{1}{\lambda} d\lambda.
\]  
(5.19)
The inequalities in (5.18) and (5.19) give the first estimate in (5.17).

Write \(\sin^2 \pi \lambda u = (1 - \cos 2\pi \lambda u)/2\) in (5.18) and use trigonometric identities to write
\[
\frac{\sigma^2(u + h) - \sigma^2(u)}{h} = 4 \int_e^\infty \left\{ \cos 2\pi \lambda u (1 - \cos 2\pi \lambda h) + \sin 2\pi \lambda u \sin 2\pi \lambda h \right\} \left( \log \lambda - \frac{1}{\lambda^3} \right) d\lambda
\]  
(5.20)
Note that the absolute value of the term in the bracket
\[
\leq 2 \left| \sin \pi \lambda h \right| + \left| \sin 2\pi \lambda h \right| \leq 6\pi \lambda.
\]  
(5.21)
Therefore we can use the Dominated Convergence Theorem to see that
\[
(\sigma^2(u)')' = 8\pi \int_e^\infty \sin 2\pi \lambda u \left( \log \lambda - \frac{1}{\lambda^2} \right) d\lambda.
\]  
(5.22)
Using integration by parts we have
\[
(\sigma^2(u)')' = 8\pi \int_e^\infty \left( \log \lambda - \frac{1}{\lambda^2} \right) \frac{1}{u} \sin 2\pi \lambda u \left( 2 \log \lambda - 3 \right) d\lambda.
\]  
(5.23)
Exactly the same argument used in (5.20) and (5.21) shows that
\[
(\sigma^2(u)')'' = -4 \int_e^\infty (1 - \cos 2\pi \lambda u) \left( \frac{2 \log \lambda - 3}{\lambda^3} \right) d\lambda + \frac{8\pi}{u} \int_e^\infty \sin 2\pi \lambda u \left( \frac{2 \log \lambda - 3}{\lambda^2} \right) d\lambda
\]  
(5.24)
\[
= I + II.
\]
Considering (5.18) we see that

\[ |I| \approx C' \sigma'^2(u) \left( u^2 \right), \quad (5.25) \]

and by the same methods used to obtain the first estimate in (5.17) we get

\[ II \approx (\log 1/u)^2. \quad (5.26) \]

Thus we get the second estimate in (5.17).

It follows from (5.18) that

\[ \sigma^2(au) = 8a^2 \int_{ea}^{\infty} \sin^2 \pi \lambda u \left( \frac{\log(\lambda/a) - 1}{\lambda^3} \right) d\lambda \quad (5.27) \]

\[ = a^2 \left( \sigma^2(u) - 8 \int_{e}^{ea} \sin^2 \pi \lambda u \left( \frac{\log \lambda - 1}{\lambda^3} \right) d\lambda \right. \]

\[ + 8 \log 1/a \int_{ea}^{\infty} \sin^2 \pi \lambda u \lambda^2 d\lambda \].

It is easy to see that the last two integrals in (5.27) are \( o(\sigma^2(u)) \) at zero. This shows that \( \sigma^2(u) \) is regularly varying at zero with index 2.

Note that by (5.17)

\[ (\sigma^2)'(s) = O \left( \frac{\sigma^2(s)}{s^2} \right) \quad (5.28) \]

at zero. This implies that \( \sigma^2 \) satisfies (1.5).

We now show that \( \sigma^2 \) satisfies (1.10). We write the integral II in (5.24) as

\[ \frac{8\pi}{u} \left( \int_{e}^{\infty} \sin 2\pi \lambda u \left( \frac{2 \log \lambda - 2}{\lambda^2} \right) d\lambda - \int_{e}^{\infty} \frac{\sin 2\pi \lambda u}{\lambda^2} d\lambda \right). \quad (5.29) \]

Clearly

\[ Q(u) = \frac{8\pi}{u} \int_{e}^{\infty} \frac{\sin 2\pi \lambda u}{\lambda^2} d\lambda = 16\pi^2 \int_{2\pi u e}^{\infty} \frac{\sin s}{s^2} ds. \quad (5.30) \]

Furthermore, by integration by parts, as in (5.23),

\[ \frac{8\pi}{u} \int_{e}^{\infty} \sin 2\pi \lambda u \left( \frac{2 \log \lambda - 2}{\lambda^2} \right) d\lambda \]

\[ = \frac{8}{u^2} \int_{e}^{\infty} \left( 1 - \cos 2\pi \lambda u \right) \left( \frac{2 \log \lambda - 3}{\lambda^3} \right) d\lambda. \]

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Substituting this in (5.24) we see that

\[(\sigma^2(u))'' = \frac{4}{u^2} \int_{e}^{\infty} (1 - \cos 2\pi \lambda u) \left( \frac{2 \log \lambda - 3}{\lambda^3} \right) d\lambda + Q(u). \quad (5.32)\]

As in (5.23) we can differentiate under the integral sign to get

\[(\sigma^2(u))''' = -\frac{8}{u^3} \int_{e}^{\infty} (1 - \cos 2\pi \lambda u) \left( \frac{2 \log \lambda - 3}{\lambda^3} \right) d\lambda \quad (5.33)\]

\[+ \frac{8\pi}{u^2} \int_{e}^{\infty} \sin 2\pi \lambda u \left( \frac{2 \log \lambda - 3}{\lambda^2} \right) d\lambda + O(1/u).\]

Separating the integral as in (5.18) we see that

\[|(\sigma^2(u))'''| \leq C \left( \frac{1}{u} \right)^{2} \left( \frac{1}{\log 1/u} \right). \quad (5.34)\]

This implies that \(\sigma^2\) satisfies (1.10). Thus the Gaussian process determined by (5.1) and (5.16) satisfies the hypotheses of Theorem 1.1.

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