ON THE CONCEPT OF EPR STATES AND THEIR STRUCTURE

In memory of our friend Moshe Flato

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In this paper the notion of an EPR state for the composite $S$ of two quantum systems $S_1, S_2$, relative to $S_2$ and a set $\mathcal{O}$ of bounded observables of $S_2$, is introduced in the spirit of the classical examples of Einstein–Podolsky–Rosen and Bohm. We restrict ourselves mostly to EPR states of finite norm. The main results are contained in Theorem 3, 4, 5, 6 in section III and imply that if EPR states of finite norm relative to $(S_2, \mathcal{O})$ exist, then the elements of $\mathcal{O}$ have discrete probability distributions and the Von Neumann algebra generated by $\mathcal{O}$ is essentially imbeddable inside $S_1$ by an antiunitary map. The EPR states then correspond to the different imbeddings and certain additional parameters, and are explicitly given by formulae which generalize the famous example of Bohm. If $\mathcal{O}$ generates all bounded observables, $S_2$ must be of finite dimension and can be imbedded inside $S_1$ by an antiunitary map, and the EPR states relative to $S_2$ are then in canonical bijection with the different imbeddings of $S_2$ inside $S_1$; moreover they are then given by formulae which are exactly those of the generalized Bohm states. The notion of EPR states of infinite norm is also explored and it is shown that the original state of Einstein–Podolsky–Rosen can be realized as a renormalized limit of EPR states of finite quantum systems considered by Weyl, Schwinger, and many others. Finally, a family of states of infinite norm generalizing the Einstein–Podolsky–Rosen example is explicitly given.

I. Introduction

Let $S_1, S_2$ be two quantum systems, for example, those of two one–dimensional particles. The famous example, first introduced by Einstein,
Podolsky, and Rosen in 1935\textsuperscript{1}, describes a state $\sigma$ of the composite system $S = S_1 \times S_2$ with the following property. Let $X_i, P_i$ be the position and momentum coordinates of the $i^{th}$ particle ($i = 1, 2$); then, if a measurement of $X_1$ (resp. $P_1$) is known to have a definite value when $S$ is in the state $\sigma$, the value of $X_2$ (resp. $P_2$) can be predicted with certainty. The conclusions that these authors drew from this example about the completeness of the quantum mechanical description of physical reality, and their refutation by Bohr in 1935\textsuperscript{2}, are well known, and the reader may refer to the papers of these authors and other related articles on quantum measurement theory reprinted in the well known reprint collection of Wheeler–Zurek\textsuperscript{3}.

The Einstein–Podolsky–Rosen state has infinite norm and so does not lie in Hilbert space; indeed in their example both systems are infinite dimensional and the state in question is actually a distribution state. In an effort to simplify the discussion of Einstein et al, Bohm introduced spin (or polarization) states of particle pairs with the same properties as their states. Bohm’s example deals with 2–dimensional quantum systems and his computations of the probabilities and discussions of gedankenexperiments eventually led to experimental tests whether these probabilities could be derived from a local hidden variable theory. For all this the reader may consult Bohm’s famous book\textsuperscript{4} as well as the nice discussion in\textsuperscript{5}.

In this paper we introduce the concept of a state $\sigma$ of the composite $S$ of two quantum systems $S_i (i = 1, 2)$ being EPR relative to $(S_2, \mathcal{O})$ where $\mathcal{O}$ is any set of bounded observables of $S_2$. Briefly, this is the case if there is, for each $A_2 \in \mathcal{O}$, a bounded observable $A_1$ of $S_1$ such that the measured value of $A_1$ in the state $\sigma$ determines with certainty the value of $A_2$ in $S_2$. We determine completely the relationship between $\mathcal{O}$ and $\sigma$ (Theorems 3, 5, III), and, for a fixed state $\sigma$ with this property, show that this predictive map $A_2 \to A_1$ extends to a map $B_2 \to B_1$ for all bounded observables $B_2$ lying in an algebra canonically associated to $\sigma$, and for no others; and further that the map that takes $B_2$ to $B_1$ is an antilinear algebra homomorphism which is an essential imbedding (which means the kernel consists of elements that are 0 in the state $\sigma$ (Theorem 4, III). Special cases of this result have been obtained in the literature, for instance in\textsuperscript{8,9}. Moreover, when such states exist relative to $\mathcal{O}$, the elements of $\mathcal{O}$ have discrete probability distributions in those states. If we now suppose, as was done by Einstein et al, that the state $\sigma$ has the EPR property relative to $(S_2, X_2)$ and $(S_2, P_2)$ where $X_2, P_2$ are two
bounded observables that generate the algebra of all bounded operators (or equivalently, if the only bounded operators commuting with both $X_2$ and $P_2$ are the scalars), then $S_2$ has finite dimension $d \leq \dim S_1$ and the EPR states are in bijection with the set of antiunitary isomorphisms of $S_2$ as a subsystem of $S_1$; moreover, the associated states are essentially of the form in the example of Bohm (suitably generalized). Of course, if we assume that the two systems have the same finite dimension, the EPR states are completely symmetrical with respect to the two systems, and they are exactly the generalized Bohm states (Theorem 6, III).

It turns out that our definition of the EPR states forces the distributions of the selected observables $A_2$ to be discrete. Thus the original state of Einstein et al cannot be subsumed under our framework although it has the same formal structure. For a rigorous discussion of this state from the point of view of operator algebras see\textsuperscript{11}. Nevertheless one can use the theory of approximations of quantum systems by finite quantum systems developed in\textsuperscript{13,14,15,16,17} to show that the Einstein–Podolsky–Rosen state is the limit of suitably renormalized EPR states associated to a particle moving in a large cyclic group as the order of the cyclic group goes to infinity. For another treatment of a similar limiting process see\textsuperscript{10}. We also mention a recent paper\textsuperscript{12} where multipartite states that are maximally EPR correlated are characterized, although this appears to go in a direction different from the line of discussion pursued in this paper.

**II. The concept of an EPR state**

We begin with a brief discussion of the Bohm state and follow the discussion in pp 69—72 of\textsuperscript{5}. The Bohm state is that of a composite of two spin $1/2$ systems, say that of an electron and a positron, and has the form

$$\Phi = \frac{1}{\sqrt{2}} (\varphi_+ \otimes \psi_- - \varphi_- \otimes \psi_+)$$

$\pm$ referring to the spin up or spin down states of the electron and positron respectively. Let $A_1$ (resp. $A_2$) denote the electron (resp. positron) spin observable with values $\pm 1$ and corresponding eigenstates $\varphi_\pm$ (resp. $\psi_\pm$). It is then a simple calculation that if in the state $\Phi$ we know $A_1$ is observed to have a given value $\pm 1$, then the value of $A_2$ is determined with certainty to be $\mp 1$, and vice versa. Furthermore, let $B_2$ be the observable in the spin system of the positron corresponding to the spin in an arbitrary direction,
so that $B_2$ has the values $\pm 1$ with corresponding eigenstates $\eta_{\pm}$. Another simple calculation shows that $\Phi$ can be expressed in the form

$$\Phi = \frac{1}{\sqrt{2}}(\chi_+ \otimes \eta_- - \chi_- \otimes \eta_+)$$

where $\chi_{\pm}$ is an orthonormal basis for the space of the electron uniquely determined by $\eta_{\pm}$. Indeed, if $\eta_{\pm}$ are defined by

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix}$$

where $(a_{ij})$ is a unitary matrix, then $\chi_{\pm}$ are determined by

$$\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}$$

So, if $B_2$ is the observable in the system of the positron with values $\pm 1$ and (orthonormal) eigenstates $\eta_{\pm}$, then the pair of observables $(B_1, B_2)$ has the same property as $(A_1, A_2)$, namely, that in the state $\Phi$ if the value of $B_1$ is observed to have a given value $\pm 1$, then the value of $B_2$ is determined with certainty to be $\mp 1$ and vice versa. In other words, $\Phi$ has the remarkable property that if $B_2$ is any observable in the positron system with values $\pm 1$, there is a uniquely associated observable $B_1$ in the electron system such that an observation of $B_1$ that yields a value of $B_1$ predicts the value of $B_2$ and vice versa.

The example of Bohm generalizes immediately to arbitrary finite-dimensional systems. Let $\mathcal{H}_j (j = 1, 2)$ be two Hilbert spaces of the same finite dimension $N$ and let $(\varphi_i)_{1 \leq i \leq N}$ and $(\psi_i)_{1 \leq i \leq N}$ be orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. $\mathcal{H}_1$ and $\mathcal{H}_2$ are the Hilbert spaces corresponding to two systems $S_1$ and $S_2$ respectively. Let

$$\Phi = \frac{1}{\sqrt{N}} \sum_{1 \leq i \leq N} \varphi_i \otimes \psi_i$$

Then exactly as in the case of the Bohm example we can show that if $(\eta_i)_{1 \leq i \leq N}$ is any orthonormal basis of $\mathcal{H}_2$, there is an orthonormal basis $(\chi_i)_{1 \leq i \leq N}$ of $\mathcal{H}_1$ such that $\Phi$ can be expressed in the form

$$\Phi = \frac{1}{\sqrt{N}} \sum_{1 \leq i \leq N} \chi_i \otimes \eta_i$$
Indeed,

\[ \psi_i = \sum_j a_{ij} \eta_j \rightarrow \chi_i = \sum_j a_{ji} \varphi_j \]

It follows from this as in the Bohm example that if \( B_2 \) is any observable with \( N \) distinct values in the system \( S_2 \), there is an observable \( B_1 \) in the system \( S_1 \) with the following property: if in the state \( \Phi \) for the compound system an observation of \( B_1 \) in the system \( S_1 \) yields an exact value, the value of \( B_2 \) in \( S_2 \) can be predicted with certainty. It is also remarkable that in this and the earlier example the roles of \( B_1 \) and \( B_2 \) can be interchanged.

Any definition of an EPR state in the general context of two arbitrary quantum systems will of course depend on what features of the examples of Bohm and Einstein et al that one wishes to focus on. In order to formulate our notion and justify its reasonableness we begin with some preliminaries.

Let \( S_1, S_2 \) be two quantum systems and let \( \mathcal{H}_i \) be the Hilbert space of \( S_i \). As usual \( \mathcal{H}_i \) is complex and separable. Then the Hilbert space of the composite system \( S_1 \times S_2 \) is the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). An observable \( A_1 \) of \( S_1 \) is considered as an observable of \( S \) via the identification \( A_1 \mapsto A_1 \otimes 1 \); similarly observables \( A_2 \) of \( S_2 \) are considered as observables of \( S \) via the identification \( A_2 \mapsto 1 \otimes A_2 \). Given a state of \( S_1 \times S_2 \), i.e., a unit vector \( \sigma \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), the commuting observables \( A_1 \otimes 1 \) and \( 1 \otimes A_2 \) have a joint probability distribution \( P_{\sigma,A_1,A_2} \) in the state \( \sigma \). We shall often write \( P \) or \( P_{\sigma} \) when it is clear what \( \sigma, A_1, A_2 \) are. Then \( P \) is a probability measure on \( \mathbb{R}^2 \); the probability measures \( P_1, P_2 \) induced on \( \mathbb{R} \) by the projections \( (x_1, x_2) \mapsto x_1, x_2 \) are the distributions of \( A_1 \otimes 1, 1 \otimes A_2 \) in the state \( \sigma \). For borel sets \( E, F \) the probability of the event \( \{ A_1 \otimes 1 \in E, 1 \otimes A_2 \in F \} \) is \( P(E \times F) \). We also have the family \((q_a)_{a \in \mathbb{R}}\) of conditional probability measures on \( \mathbb{R} \), with the interpretation that \( q_a(F) \) is the conditional probability of the value of \( 1 \otimes A_2 \) belonging to the borel set \( F \) when \( A_1 \otimes 1 \) is known to have the value \( a \). Mathematically, \((q_a)_{a \in \mathbb{R}}\) is characterized as the family, unique almost everywhere with respect to \( P_1 \), with the property that for all borel sets \( M \subset \mathbb{R}^2 \),

\[ P(M) = \int_{\mathbb{R}} q_a(M[a])dP_1(a) \quad (M[a] = \{ b \mid (a,b) \in M \}) \]

We wish to focus on the fact that the examples of Bohm and Einstein et al feature observables \( A_i \) in \( S_i \) such that a measurement of \( A_1 \) in \( S_1 \)
predicts with certainty the value of $A_2$ in $S_2$. Indeed, in the classical argumentation of Einstein et al, this property was interpreted to mean that we can measure the observable $A_2$ in $S_2$ without disturbing the system $S_2$. Without making this interpretation we shall first formulate this in precise mathematical terms. Since the value of $A_2$ is determined with certainty by the value of $A_1$ we must have a function $g$ such that if $A_1$ is observed to have the value $a$, $A_2$ has the value $g(a)$. For general reasons we shall assume that $g$ is a borel function. This can be formulated in either of two ways: either that

$$q_a(g(\{a\}) = 1 \quad \text{for } P_1 - \text{almost all } a$$

or in the apparently weaker form where only $P$ and not the $q_a$ intervenes:

$$P(A_1 \otimes 1 \in E, 1 \otimes A_2 \in F) = 0 \quad \text{if } g(E) \cap F = \emptyset \quad (E \cap g^{-1}(F) = \emptyset)$$

Indeed, if the value of $A_1 \otimes 1$ is $a \in E$, then the value of $1 \otimes A_2$ cannot be in $F$ if $g(E) \cap F = \emptyset$. Actually, these two formulations are equivalent as the following lemma shows.

**Lemma 1.** Let $P$ be the probability measure on $\mathbb{R}^2$ as above and let $g$ be a borel map of $\mathbb{R}$ into $\mathbb{R}$. Let $G$ be the graph of of $g$, namely,

$$G = \{(x, g(x)) \mid x \in \mathbb{R}\}$$

Then the following statements are equivalent.

(a) $P(E \times F) = 0$ if $E \cap g^{-1}(F) = \emptyset$, i.e., if $(E \times F) \cap G = \emptyset$

(b) $P(\mathbb{R}^2 \setminus G) = 0$

(c) For $P_1$—almost all $a$,

$$q_a(\{g(a)\}) = 1$$

**Proof.** (b)$\iff$(c): It is known that $G$ is a borel set. By general results in measure theory, $P$, which can be viewed as a probability measure on $G$ by the condition (b), can be fibered with respect to the projection $(x_1, x_2) \mapsto x_1$. The fibers are the points $\{g(a)\}$ and so the fiber measures are delta functions at the points $g(a)$ which is (c). If (c) is assumed, then

$$p(G) = \int_\mathbb{R} q_a(\{g(a)\})dP_1(a) = 1$$
which is (b).

\[ (b) \iff (a) \] The implication \( (b) \implies (a) \) is trivial. The reverse implication requires a more delicate argument. However, if \( P \) is discrete, i.e., if all its mass is concentrated in a countable set, then \( (a) \implies (b) \) is easy. In fact, in this case, the probability measures of \( x_1 \) and \( x_2 \) are both discrete. Let \( D_i \) be the set of points where \( P_i \) has positive mass. Since \( P(\{a\} \times (R \setminus \{g(a)\})) = 0 \) for \( a \in D_1 \) by (a), we have \( P(x_1 = a, x_2 = g(a)) = P(x_1 = a) \). Summing over \( a \) one sees that \( P(g(D_1)) = 1 \) and hence \( P(G) = 1 \) which is (b). Note that in this case \( P(x_2 = g(a)) \geq P(x_2 = g(a), x_1 = a) = P(x_1 = a) > 0 \) so that \( g \) maps \( D_1 \) into \( D_2 \); as \( P(x_2 \in g(D_1)) = 1 \) we must have \( g(D_1) = D_2 \).

In the general case the argument for showing that \( (a) \implies (b) \) is more technical but it is not needed for this paper (the point is that we shall use only the apparently weaker form (a), and as (a) is a trivial consequence of (b) and hence also of (c), this does not affect the argumentation of the rest of the paper). Using a general result on borel maps (see\(^6\) p. 137) we may assume that we are in the situation of separable metric spaces \( X \) and \( Y \) and a continuous map \( g \) of \( X \) into \( Y \). The probability measure \( P \) is defined on \( X \times Y \) and we are given that \( P(E \times F) = 0 \) for borel sets \( E, F \) if \( (E \times F) \cap G = \emptyset \) where \( G \) is the graph of \( g \). Note that the graph is now a closed set as \( g \) is continuous (this is also a proof that the graph of a borel map is a borel set). If \( (a, b) \) is a point not in \( G \), there are open sets \( E, F \) respectively containing \( a, b \) such that \( E \times F \) is disjoint from \( G \), and so \( P(E \times F) = 0 \). By separability, \( X \times Y \setminus G \) can be covered by a countable collection of sets \( E_i \times F_i \) where \( E_i, F_i \) are open and \( P(E_i \times F_i) = 0 \), and so \( P(X \times Y \setminus G) = 0 \). This proves that \( (a) \implies (b) \).

**Corollary 2.** Suppose that the equivalent conditions of the lemma are satisfied. Then there is a borel set \( F \) such that \( F \subset g(R) \) and \( P_2(F) = 1 \). If \( P \) is discrete, and \( D_i \) is the set of positive mass points of \( P_i \), then \( g(D_1) = D_2 \).

**Proof.** The second statement was established in the course of the above proof. To prove the first note that we can find a sequence of compact sets \( G_i \subset G \) such that \( P(\cup_i G_i) = 1 \). If \( K_i \) is the image of \( G_i \) under the projection \( (x_1, x_2) \mapsto x_2 \), then \( K_i \) is compact and \( P_2(\cup_i K_i) = 1 \). Obviously \( \cup_i K_i \subset g(R) \).

We shall now make our definition of an EPR state.
Definition 1: Let $A_2$ be a bounded observable of $S_2$ and $\sigma \in H_1 \otimes H_2$ a unit vector. Then $\sigma$ is said to be an EPR state of $S_1 \times S_2$ relative to $(S_2, A_2)$ if there is a bounded observable $A_1$ of $S_1$ such that $P = P^{\sigma, A_1, A_2}$ has the following property: there is a borel map $g: \mathbb{R} \to \mathbb{R}$ such that

$$P(A_1 \otimes E, 1 \otimes A_2 \in F) = 0 \quad \text{whenever} \ E \cap g^{-1}(F) = \emptyset$$

If there is a set $O$ of bounded observables of $S_2$ such that $\sigma$ is EPR relative to $(S_2, A_2)$ for each $A_2 \in O$, we say that $\sigma$ is an EPR state of $S_1 \times S_2$ relative to $(S_2, O)$.

III. The main results

Our aim now is to explore the consequences of our definition of an EPR state $\sigma$ relative to $(S_2, A_2)$ for the structural relationships between $\sigma, A_2, A_1$. Before we can formulate and prove our main results we need some preliminaries. Note that all our scalar products are linear in the first argument and conjugate linear in the second. Our entire argument depends on a canonical identification of $H_1 \otimes H_2$ with the space of conjugate linear maps of $H_2$ into $H_1$ (equally of $H_1$ into $H_2$) that are of the Hilbert–Schmidt class. This identification is well known, but as conjugate linear maps are somewhat less familiar than linear ones we go into this in some detail. Let $C_{21}$ be the linear space of bounded conjugate linear maps $L(H_2 \to H_1)$ such that $Tr(L^\dagger L) < \infty$. Here $L^\dagger$, defined by the relation $(Lu, v) = (L^\dagger v, u)$, is also a conjugate linear map, from $H_1$ into $H_2$, so that $L^*L$ is a linear map of $H_2$. The scalar product

$$(L, M) = Tr(M^\dagger L) \quad (L, M \in C_{21})$$

then converts $C_{21}$ into a Hilbert space. The space $C_{21}$ contains as a dense subspace the set $C_{21,f}$ of $L$ of finite rank.

Lemma 1. There is a canonical unitary isomorphism

$$\sigma \mapsto L_\sigma, \quad H_1 \otimes H_2 \simeq C_{21}$$

such that for any $\sigma \in H_1 \otimes H_2$ and any ON basis $(e_n)$ of $H_2$,

$$\sigma = \sum_n L_\sigma e_n \otimes e_n$$
**Proof.** The simplest way to construct this canonical isomorphism is to first fix an ON basis \((e_n)\) for \(H_2\). Then the elements of \(H_1 \otimes H_2\) are precisely those of the form

\[
\sigma = \sum_n v_n \otimes e_n \quad (v_n \in H_1, \sum_n ||v_n||^2 < \infty) \quad (2)
\]

We define \(L_\sigma\) as the unique *conjugate linear* map of Hilbert–Schmidt class of \(H_2\) into \(H_1\) such that \(L_\sigma e_n = v_n\). The point is that \(L_\sigma\) *depends only on \(\sigma\) and not on the orthonormal basis \((e_n)\) that enters the representation (2) of \(\sigma\). Indeed, if \((f_m)\) is another ON basis of \(H_2\), we can write \(e_n = \sum_m u_{nm} f_m\) where \((u_{nm})\) is a unitary matrix. Then

\[
\sigma = \sum L_\sigma e_n \otimes e_n
= \sum_{np} \overline{u_{np}} L_\sigma f_p \otimes \sum u_{nm} f_m
= \sum_{mp} (\sum_n u_{nm} \overline{u_{np}}) L_\sigma f_p \otimes f_m
= \sum_m L_\sigma f_m \otimes f_m
\]

since

\[
\sum_n u_{nm} \overline{u_{np}} = \delta_{mp}
\]

Finally

\[
||\sigma||^2 = \sum_m ||v_n||^2 = \sum_n ||L_\sigma e_n||^2 = Tr(L_\sigma^\dagger L_\sigma) \quad (3)
\]

**Remark 1.** It should be noted that had we defined \(L_\sigma\) as the *linear* map such that \(L_\sigma e_n = v_n\) then it will not be independent of the ON basis chosen. So to guarantee the canonical nature it is essential to choose \(L_\sigma\) as the *conjugate linear map* taking \(e_n\) to \(v_n\).

**Remark 2.** The representation of vectors in \(H_1 \otimes H_2\) in the form

\[
\sum v_n \otimes e_n \quad ((e_n)\) an ON basis of \(H_2\), \(\sum_n ||v_n||^2 < \infty)\]


is well known, see for instance the discussion of Von Neumann in Chapter VI of\textsuperscript{12} where reference is made to the work of E. Schmidt. However Von Neumann, concerned as he was about other aspects of the quantum theory of composite systems, does not remark on the use of conjugate linear operators that makes the representation independent of the ON basis, a fact that is absolutely crucial for us.

\textbf{Remark 3.} The construction of the isomorphism

\[ \sigma \mapsto L_\sigma \]

is perhaps not esthetically nice since we use a basis for its definition. An alternative way is to proceed as follows. Let $\mathcal{H}'$ be the \textit{algebraic tensor product} of $\mathcal{H}_1$ and $\mathcal{H}_2$. Then one knows that $\mathcal{H}'$ is canonically isomorphic to the space of linear maps of finite rank from $\mathcal{H}_2^\ast$ to $\mathcal{H}_1$; but $\mathcal{H}_2^\ast$ is in canonical \textit{antiunitary isomorphism} with $\mathcal{H}_2$ and so we have a canonical linear isomorphism of $\mathcal{H}'$ with the space of \textit{conjugate linear maps} of finite rank from $\mathcal{H}_2$ to $\mathcal{H}_1$. Explicitly,

\[ \sigma = \sum_{1 \leq j \leq m} a_j \otimes b_j \implies L_\sigma u = \sum_{1 \leq j \leq m} (b_j, u)a_j \]

Then

\[ L_\sigma^\dagger w = \sum_{1 \leq j \leq m} (a_j, w)b_j \quad (w \in \mathcal{H}_1) \]

Taking the $(b_j)$ to be orthonormal, we see that $L_\sigma b_j = a_j$ and $L_\sigma u = 0$ if $u$ is orthogonal to the $b_j$. Hence

\[ \text{Tr}(L_\sigma^\dagger L_\sigma) = \sum_j ||a_j||^2 = ||\sigma||^2 \]

The required isomorphism is then obtained by extending the map $\sigma \mapsto L_\sigma$ from $\mathcal{H}'$ to $C_{21, f}$ by completion since $\mathcal{H}'$ (resp. $C_{21, f}$) is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$ (resp. $C_{21}$).

The operator $L_\sigma^\dagger L_\sigma$, being of trace class, has a discrete spectrum with eigenvalues $\lambda_j > 0$ ($j \geq 1$) of finite multiplicity, and possibly 0 as an eigenvalue whose multiplicity could be infinite. If $\mathcal{H}_2(\lambda_j)$ is the eigenspace corresponding to $\lambda_j$ and $d_j = \dim(\mathcal{H}_2(\lambda_j))$, then

\[ \text{Tr}(L_\sigma^\dagger L_\sigma) = \sum_j d_j \lambda_j < \infty \]

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We have the orthogonal decomposition
\[ \mathcal{H}_2 = \mathcal{H}_2^\sigma \oplus \mathcal{H}_2^0 \]
where
\[ \mathcal{H}_2^\sigma = \bigoplus_{j \geq 1} \mathcal{H}_2(\lambda_j), \quad \mathcal{H}_2^0 = \text{the kernel of } L_\sigma^\dagger L_\sigma \]
We shall use these notations a little later. At this moment we note a simple fact.

**Lemma 2.** Fix a unit vector \( \sigma \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). Let \( B_2 \) be a bounded observable of \( \mathcal{H}_2 \) commuting with \( L_\sigma^\dagger L_\sigma \). Then, \( B_2 \) leaves the \( \mathcal{H}_2(\lambda_j) \) invariant. In particular, in the state \( \sigma \) the probability distribution of \( 1 \otimes B_2 \) is discrete and is concentrated on the set of eigenvalues of \( B_2 \) on \( \mathcal{H}_2^\sigma \).

**Proof.** It is obvious that \( B_2 \) leaves the \( \mathcal{H}_2(\lambda_j) \) invariant, and as these are finite dimensional, \( B_2 \) has discrete spectrum on each of these and hence on \( \mathcal{H}_2^\sigma \). Let \((e_{jp})_{1 \leq p \leq d_j}\) be an ON basis of \( \mathcal{H}_2(\lambda_j) \) consisting of eigenstates of \( B_2 \), \( B_2 e_{jp} = b_{jp} e_{jp} \). By the previous lemma we can write
\[
\sigma = \sum_{j \geq 1} \sum_{1 \leq p \leq d_j} L_\sigma e_{jp} \otimes e_{jp}
\]
and so, if \( \beta \) is the set of all the numbers \( b_{jp} \),
\[
P^\sigma(1 \otimes B_2 \in \beta) \geq \sum_{j \geq 1} \sum_{1 \leq p \leq d_j} ||L_\sigma e_{jp}||^2 = Tr(L_\sigma^\dagger L_\sigma) = 1
\]
We now come to the result which is the basis for everything that we can say about EPR states. Its proof depends essentially on the possibility of using any ON basis of \( \mathcal{H}_2 \) in the decomposition of \( \sigma \).

**Theorem 3.** Let \( \mathcal{O} \) be any set of bounded observables of \( \mathcal{H}_2 \) and let \( \sigma \) be an element of unit norm in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \). If \( L_\sigma \) is the element of \( C_{21} \) that corresponds to \( \sigma \) under the canonical isomorphism of Lemma 1, then \( \sigma \) is an EPR state relative to \( (S_2, \mathcal{O}) \) if and only if \( L_\sigma^\dagger L_\sigma \) commutes with every element of \( \mathcal{O} \).

**Proof.** It is obviously enough to do this for each element of \( \mathcal{O} \) separately. Fix \( B_2 \in \mathcal{O} \) and assume that \( \sigma \) is EPR relative to \( (S_2, B_2) \). Let \( B_1 \) be
a bounded observable of \( \mathcal{H}_1 \) with the following property: there is a borel map \( g \) of \( \mathbb{R} \) into \( \mathbb{R} \) such that

\[
P(B_1 \otimes 1 \in E, 1 \otimes B_2 \in F) = 0 \quad (E \cap g^{-1}(F) = \emptyset)
\]

We should prove that \( L_\sigma^\dagger L_\sigma \) commutes with \( B_2 \). The proof is slightly simpler if \( B_1 \) and \( B_2 \) have discrete spectra, but not by much. Still it may be worthwhile to give the argument separately in this case.

**Case of discrete spectra**: Let \( \beta_1 \) (resp. \( \beta_2 \)) be the set of eigenvalues of \( B_1 \) (resp. \( B_2 \)). For \( a \in \beta_1 \) (resp. \( b \in \beta_2 \)) let \( E_a \) (resp. \( F_b \)) be the corresponding eigenspace. Then \( g \) is a map \( \beta_1 \to \beta_2 \).

We are given that

\[
P(B_1 \otimes 1 = a, 1 \otimes B_2 = b) = 0 \quad (b \neq g(a))
\]

Fix \( b \in \beta_2 \). Select an ON basis \( (e_i) \) of \( F_b \) and an ON basis \( (f_j) \) of \( F_b^\perp \) and write

\[
\sigma = \sum_i L_\sigma e_i \otimes e_i + \sum_j L_\sigma f_j \otimes f_j
\]

Let \( Q_a \) be the orthogonal projection \( \mathcal{H}_1 \to E_a \). Then

\[
P(B_1 \otimes 1 = a, 1 \otimes B_2 = b) = || \sum_i Q_a L_\sigma e_i \otimes e_i ||^2 = \sum_i || Q_a L_\sigma e_i ||^2
\]

Since this is zero for \( b \neq g(a) \), we must have

\[
Q_a L_\sigma e_i = 0 \quad (b \neq g(a))
\]

In other words, if we write

\[
E[b] = \bigoplus_{a:g(a)=b} E_a
\]

then

\[
L_\sigma [F_b] \subset E[b]
\]

Suppose now that \( b' \neq b \), and let \( u \in F_b, v \in F_{b'} \). Then

\[
(L_\sigma^\dagger L_\sigma u, v) = (L_\sigma v, L_\sigma u) = 0
\]
since
\[ E[b] \perp E[b'] \]
Thus
\[ L_\sigma^\dagger L_\sigma u \in F_b \]
This proves that \( L_\sigma^\dagger L_\sigma \) leaves all the \( F_b \) invariant and hence that it commutes with \( B_2 \).

**General case**: We must prove that \( L_\sigma^\dagger L_\sigma \) commutes with all the spectral projections of \( B_2 \). Since \( L_\sigma^\dagger L_\sigma \) is self adjoint, this is equivalent to showing that \( L_\sigma^\dagger L_\sigma \) leaves the spectral subspaces of \( B_2 \) invariant. For any borel set \( B \subset \mathbb{R} \) let \( F_B \) (resp. \( E_B \)) be the corresponding spectral subspace of \( B_2 \) (resp. \( B_1 \)). Write \( Q_B \) for the orthogonal projection \( \mathcal{H}_1 \rightarrow E_B \). Fix a borel set \( B \subset \mathbb{R} \). Select ON bases \((e_i)\) for \( F_B \) and \((f_j)\) for \( F_B^\perp \). Then

\[ \sigma = \sum_i L_\sigma e_i \otimes e_i + \sum_j L_\sigma f_j \otimes f_j \]

If \( C = g^{-1}(B) \), then

\[
0 = P(B_1 \otimes 1 \in \mathbb{R} \setminus C, 1 \otimes B_2 \in B) \\
= \| \sum_i Q_{\mathbb{R}\setminus C} L_\sigma e_i \otimes e_i \|^2 \\
= \sum_i \| Q_{\mathbb{R}\setminus C} L_\sigma e_i \|^2
\]

and hence
\[ Q_{\mathbb{R}\setminus C} L_\sigma e_i = 0 \quad (\text{for all } i) \]
Thus
\[ L_\sigma [F_B] \subset E_{g^{-1}(B)} \]
We now calculate \((L_\sigma^\dagger L_\sigma u, v)\) for \( u \in F_B, v \in F_{\mathbb{R}\setminus B} \). We have
\[
(L_\sigma^\dagger L_\sigma u, v) = (L_\sigma v, L_\sigma u) = 0
\]
since \( g^{-1}(\mathbb{R} \setminus B) = \mathbb{R} \setminus g^{-1}(B) \) and
\[
E_{g^{-1}(B)} \perp E_{\mathbb{R}\setminus g^{-1}(B)}
\]

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Thus

\[ L_\sigma^\dagger L_\sigma [F_B] \perp F_{g^{-1}(B)} \]

which gives

\[ L_\sigma^\dagger L_\sigma [F_B] \subset F_B \]

This is what we wanted to prove.

We now take up the converse. We assume that \( B_2 \) commutes with \( L_\sigma^\dagger L_\sigma \) and wish to find a bounded observable \( B_1 \) of \( \mathcal{H}_1 \) such that the EPR property is satisfied for the pair \((B_1, B_2)\). We use Lemma 2 above. On \( \mathcal{H}_2(\lambda_j) \) we can write \( L_\sigma \) as \( \lambda_j^{1/2} U_j \) where \( U_j \) is an antiunitary imbedding of \( \mathcal{H}_2(\lambda_j) \) into \( \mathcal{H}_1 \). If \( \mathcal{H}_1(\lambda_j) = L_\sigma[\mathcal{H}_2(\lambda_j)] \), it is then easy to check that the \( \mathcal{H}_1(\lambda_j) \) are mutually orthogonal. Let

\[ \mathcal{H}_1^\sigma = \oplus_j \mathcal{H}_1(\lambda_j) \]

We define \( U \) as the antiunitary isomorphism of \( \mathcal{H}_2^\sigma \) with \( \mathcal{H}_1^\sigma \subset \mathcal{H}_1 \) which is equal to \( U_j \) on \( \mathcal{H}_2(\lambda_j) \). If now \( (e_{jp})_{1 \leq p \leq d_j} \) is any ON basis of \( \mathcal{H}_2(\lambda_j) \), we have the representation

\[ \sigma = \sum_{j \geq 1} \lambda_j^{1/2} \sum_{1 \leq p \leq d_j} U e_{jp} \otimes e_{jp} \]

We take the \( e_{jp} \) to be the eigenstates of \( B_2 \), \( B_2 e_{jp} = b_{jp} e_{jp} \). Let us define

\[ B_1 = U B_2 U^\dagger \]

It is easy to check that \( B_1 U e_{jp} = b_{jp} U e_{jp} \). We may therefore conclude that the distribution of \( B_1 \otimes 1 \) is discrete in the state \( \sigma \) with its mass concentrated on the set \( \beta_2 \) of eigenvalues of \( B_2 \) in \( \mathcal{H}_2^\sigma \). It is immediate that

\[ P(B_1 \otimes 1 = b_1, 1 \otimes B_2 = b_2) = 0 \quad (b_1 \neq b_2, b_i \in \beta_2) \]

This completes the proof of the theorem.

**Remark.** Note the obvious symmetry between the roles of \( B_1 \) and \( B_2 \) as revealed in the last relation.

For any set \( \mathcal{O} \) of bounded observables in \( \mathcal{H}_2 \) we write \( \mathcal{O}' \) for the set of bounded observables commuting with \( \mathcal{O} \) and \( \mathcal{O}'' = (\mathcal{O}')' \). Theorem 3 leads at once to the following results.
Theorem 4. Let notation be as above and let $\sigma$ be a unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let $L_\sigma \in C_{21}$ correspond to $\sigma$ and let $B^\sigma$ be the Von Neumann algebra of bounded operators commuting with $L_\sigma \dagger L_\sigma$. Then $\sigma$ is EPR relative to the observables in $B^\sigma$ and to no others. These observables all have discrete probability distributions in the state $\sigma$, which are concentrated on the set of their eigenvalues in $\mathcal{H}_2^\sigma$ (on which they have discrete spectra). The state $\sigma$ induces an antilinear homomorphism $B_2 \mapsto B_1$ of $B_2$ into the algebra of bounded operators of $\mathcal{H}_1$, and the distributions of $B_1$ and $B_2$ are the same for all observables in $B_2 \in B^\sigma$. Moreover we have

$$P(B_1 \otimes = b_1, 1 \otimes B_2 = b_2) = 0 \quad (b_1 \neq b_2, \ b_i \in \beta^\sigma)$$

where $\beta^\sigma$ is the spectrum of $B_2$ on $\mathcal{H}_2^\sigma$.

Remark. The map $B_2 \mapsto B_1$ need not be an imbedding. However all observables in its kernel vanish on $\mathcal{H}_2^\sigma$ and so vanish with probability 1 in the state $\sigma$. We may therefore say that it is an essential imbedding.

Intuitively, the existence of the essential imbedding of $B^\sigma$ inside $S_1$ is reasonable because, as $B_1$ determines $B_2$, the propositions of $B_2$ must be found within those of $B_1$, and so, by Wigner’s theorem, this map should be effected by a symmetry. The technical point which goes beyond this heuristic reasoning is that this symmetry is antiunitary.

Theorem 5. If $\mathcal{O}$ is any set of bounded observables of $\mathcal{H}_2$, there exist EPR states relative to $(S_2, \mathcal{O})$ if and only if there are projections $Q$ commuting with $\mathcal{O}$ whose ranges have dimensions $\leq \dim \mathcal{H}_1$. If $Q_j$ is a family of such projections which are mutually orthogonal, $F_j = \text{range of } Q_j$, $d_j = \dim F_j$, and if $\dim \oplus F_j \leq \dim \mathcal{H}_1$, then for any set of numbers $d_j$ such that $\sum_j d_j \lambda_j = 1$ and any antiunitary imbedding $U$ of $\oplus F_j$ into $\mathcal{H}_1$ the state

$$\sigma = \sum_{j \geq 1} \lambda_j^{1/2} \sum_{1 \leq p \leq d_j} U e_{jp} \otimes e_{jp}$$

where the $e_{jp}$ are any ON basis for $\mathcal{H}_2(\lambda_j)$ is EPR relative to $(S_2, \mathcal{O})$. Every state EPR relative to $(S_2, \mathcal{O})$ is obtained this way, and any such is EPR relative to $(S_2, [\mathcal{O}])$ where $[\mathcal{O}]$ is the set of observables in the Von Neumann algebra generated by $\mathcal{O}$.

Remark. The fact that a state which has the EPR property with respect to some observables has that property for infinitely many others has been known for a long time; see $^8,^9$.  

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Suppose we assume, as is the case in the Einstein–Podolsky–Rosen example, that the bounded observables $X_2$ and $P_2$ of $\mathcal{H}_2$ have the property that the only bounded observables simultaneously measurable with both of them are the scalars. Then $L_\sigma^\dagger L_\sigma$ must be a scalar, there is only one $j$ in the above formulae, $\mathcal{H}_2 = \mathcal{H}_2(\lambda_1)$ is finite dimensional, and $\dim \mathcal{H}_1 \geq \dim \mathcal{H}_2$. Then $U$ is an antiunitary injection of $\mathcal{H}_2$ into $\mathcal{H}_1$ and there is a bijection between EPR states of $S$ relative to $(S_2, \{X_2, P_2\})$ and the equivalence classes of antiunitary imbeddings of $\mathcal{H}_2$ into $\mathcal{H}_1$. In particular, if, as is true in many examples, that $S_1$ and $S_2$ are identical, then the EPR states are the same relative to each system, and are in canonical bijection with the set of antiunitary symmetries between the two systems. We thus have the following theorem.

**Theorem 6.** Let $\sigma$ be a unit vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ and let it be an EPR state relative to $(S_2, \{X_2, P_2\})$ where $X_2, P_2$ are bounded observables with the property that only the scalars in $S_2$ commute with both of them. Then $\dim \mathcal{H} = d < \infty$, $\dim \mathcal{H}_1 \geq d$, and there is an antiunitary imbedding $U$ of $\mathcal{H}_1$ into $\mathcal{H}_2$ such that

$$\sigma = d^{-1/2} \sum_{1 \leq j \leq d} U e_j \otimes e_j$$

where $(e_j)$ is any ON basis of $\mathcal{H}_2$. The correspondence $\sigma \rightarrow U$ induces a bijection between the set of states of $S_1 \times S_2$ that are EPR relative to $(S_2, X_2, P_2)$ and the set of equivalence classes of antiunitary imbeddings of $\mathcal{H}_2$ into $\mathcal{H}_1$. In this case, if $\mathcal{B}$ is the set of all bounded observables of $S_2$ and, for $B_2 \in \mathcal{B}$ we define $B_1 = U B U^\dagger$, then $B_1 \otimes 1$ and $1 \otimes B_2$ have identical distributions in $\sigma$ which are concentrated on the (finite) set $\alpha$ of eigenvalues of $B_2$, and

$$P(B_1 \otimes 1 = b_1, 1 \otimes B_2 = b_2) = 0 \quad (b_1 \neq b_2, b_1, b_2 \in \alpha)$$

If further $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$, the EPR states relative to $(S_2, \{X_2, P_2\})$ are precisely those that are EPR relative to $(S_1, X_1, P_1)$, and are in bijective correspondence with the set of antiunitary isomorphisms of $\mathcal{H}_1$ and $\mathcal{H}_2$.

**Remark.** We see that in this case the EPR states are essentially the same as the ones discussed at the beginning of this paper as generalizations of the Bohm states.

**IV. Examples**
We have mentioned already that one can construct the analogues of the original Einstein–Podolsky–Rosen states in certain finite quantum systems. These finite systems were first introduced by Weyl\textsuperscript{13} and explored subsequently by Schwinger\textsuperscript{14}, Digernes et al\textsuperscript{15}, Husstad\textsuperscript{16}, and Digernes et al\textsuperscript{17}. In their most general form they treat a particle which moves not in the real line $\mathbb{R}$ but in a finite abelian group $G$ (for Weyl and Schwinger this group was $\mathbb{Z}_N$, the group of integers modulo $N$ while for Digernes et al it was a more general finite abelian group). When $G$ is a large cyclic group it serves as an approximation to $\mathbb{R}$, which is the point of view of the papers loc. cit. Indeed, the group $\mathbb{Z}_N$ is identified by a grid of equidistant points symmetric about the origin in $\mathbb{R}$, with the intergrid distance of the order of $N^{-1/2}$, so that when $N \to \infty$ the kinematics of the system go over in the limit to the kinematics of the usual one particle system in quantum mechanics. We shall take up this approximation point of view in the next section. Here we shall keep our discussion to the structure of some specific EPR states. We take $\mathcal{H}_1 = \mathcal{H}_2 = L^2(G)$ where $G$ is a finite abelian group whose order will be denoted by $|G|$. The scalar product is given by

$$ (f, g) = \frac{1}{|G|} \sum_{x \in G} f(x) g(x)^{\text{conj}} \quad (f, g \in L^2(G)) $$

For simplicity we consider the antiunitary isomorphism

$$ U : f \mapsto f^{\text{conj}} $$

of $\mathcal{H}_2$ with $\mathcal{H}_1$. If $\{e_n\}$ is any ON basis of $\mathcal{H}_1$ we have the representation of the corresponding state $\sigma_U$ as

$$ \sigma_U = d^{-1/2} \sum e_n^{\text{conj}} \otimes e_n $$

Now we have two ON bases of $\mathcal{H}_2$, namely

$$ \{ \sqrt{|G|} \delta_x \}_{x \in G}, \quad \{ \xi \}_{\xi \in \hat{G}} $$

where $\delta_x$ is the delta function at $x$ and $\hat{G}$ is the group of characters of $G$. So

$$ \sigma_U = \sqrt{|G|} \sum_{x \in G} \delta_x \otimes \delta_x = \frac{1}{\sqrt{|G|}} \sum_{\xi \in \hat{G}} \xi^{-1} \otimes \xi \quad (*) $$

The equality of the last two expressions in $(*)$ can also be verified directly using the orthogonality relations in $G$ and $\hat{G}$. Let $X_2$ be an observable in
$S_2$ with distinct values $a_x (x \in G)$ and corresponding eigenstates $\delta_x$. Then $UX_2U^\dagger = X_1$ has the same definition in $\mathcal{H}_1$. Clearly $X_i$ are the position observables in the two systems. For $Y_2$ we take the observable in $S_2$ with distinct values $b_\xi$ and eigenstates $\xi$. Then $Y_1 = UY_2U^\dagger$ is the observable in $\mathcal{H}_1$ with values $b_\xi$ and eigenstates $\xi^{-1}$. The $Y_i$ are the momentum observables in the two systems. It is then a simple calculation to verify the EPR property. For the pair $(\mathcal{H}_1, \mathcal{H}_2)$ these are summarized by the symmetrical relations

$$P(X_1 \otimes 1 = a_x, 1 \otimes X_2 = a'_x) = 0 \quad (a_x \neq a'_x)$$

$$P(Y_1 \otimes 1 = b_\xi, 1 \otimes Y_2 = b'_\xi) = 0 \quad (b_\xi \neq b'_\xi)$$

As a second example let us take $\mathcal{H}_1 = \mathcal{H}_2 = L^2(G) \otimes \mathbb{C}^N$ and let the bounded observables $X'_2, P'_2$ of $\mathcal{H}_2$ be defined by $X'_2 = X_2 \otimes 1, P'_2 = P_2 \otimes 1$, the observables $X_2, P_2$ in $L^2(G)$ being as in the preceding example (all this inside $\mathcal{H}_2$, the tensor products here should not be confused with the one involving $\mathcal{H}_1$ and $\mathcal{H}_2$). One may view this as a quantum system of a particle with $N$ spin states moving in the finite abelian group $G$. The commutator of $\{X'_2, P'_2\}$ is then the algebra $\mathcal{A} = 1 \otimes \mathcal{M}$ where $\mathcal{M}$ is the matrix algebra in $\mathbb{C}^N$. Although $X'_2$ and $P'_2$ generate an algebra without any dispersion states, nevertheless there is a wide choice of EPR states relative to $\{X'_2, P'_2\}$ since the choice of $L^\dagger_\sigma L_\sigma$ within $\mathcal{A}$ is arbitrary, so that they will depend on more than just an antiunitary imbedding of $\mathcal{H}_2^\sigma$ into $\mathcal{H}_1$. This example shows that the structure of EPR states relative to a set $\mathcal{O}$ does not depend exclusively on the structure of the algebra generated by $\mathcal{O}$ but also on its commutator in $\mathcal{H}_2$.

**V. States of infinite norm**

Theorem 5 of IV shows that if an observable $B_2$ in $\mathcal{H}_2$ has a continuous spectrum, there is no EPR state relative to it. Strictly speaking therefore, the original state of Einstein et al is not subsumed under our results since their state is defined by a tempered distribution which does not have finite norm. Nevertheless in the approximation scheme of Weyl, Schwinger and others mentioned in the previous section, in the limit when $G = \mathbb{Z}_N$ approaches $\mathbb{R}$, the states considered there might be expected to go over to the original Einstein–Podolsky–Rosen state after a renormalization. We shall see now that this is the case. Since the calculations are similar to those found in^14,15 we shall be very brief. Indeed, in this
example, $\mathcal{H}_1 = \mathcal{H}_2 = L^2(\mathbb{R})$ and the identification is taken to be, as in the finite case, the map

$$U : \psi \mapsto \psi^{\text{conj}} \quad (\psi \in \mathcal{H}_2)$$

Let $X_i$ (resp. $P_i$) be the position (resp. momentum) of the $i^{th}$ particle.

The original state of Einstein et al is

$$\sigma_U = \sigma = \int e^{i(x_1 - x_2)p} dp \quad (\hbar = 1)$$

which can be written as

$$\int U e_p \otimes e_p dp, \quad e_p(x) = e^{-ixp}$$

Of course the integrals have to be interpreted as tempered distributions and so have to be paired with Schwartz functions. In this state, if $P_1 \otimes 1 = p$, then $1 \otimes P_2 = -p$. A simple calculation using Fourier analysis then shows that we also have the representation

$$\sigma = \int \delta_x \otimes \delta_x dx$$

Thus

$$\sigma = \int U e_p \otimes e_p dp = 2\pi \int \delta_x \otimes \delta_x dx$$

The analogy with $(\ast)$ of section IV is now clear. This representation shows that in this state, if $X_1 \otimes 1 = x$, then $1 \otimes X_2 = x$.

To exhibit $\sigma$ as the limit of renormalized EPR states associated to the cyclic group $\mathbb{Z}_N$ we use the imbedding of $L^2(\mathbb{Z}_N)$ into $L^2(\mathbb{R})$ given by (see\textsuperscript{15})

$$N^{1/2} \delta_x \mapsto \varepsilon^{-1/2} \chi_{r\varepsilon}$$

where $\varepsilon = (2\pi/N)^{1/2}$, and $\chi_{r\varepsilon}$ is the characteristic function of the interval $((r - 1/2)\varepsilon, (r + 1/2)\varepsilon)$ and $x$ runs through the congruences classes of $r (r = 0, \pm 1, \pm 2, \ldots, \pm (N - 1)/2)$ (we take $N$ to be odd, which is of no consequence as we let $N \to \infty$). Then the EPR state associated to $\mathbb{Z}_N$ is

$$N^{1/2} \sum_x \delta_x \otimes \delta_x$$

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which goes over under our imbedding to

\[ \sigma_N = (2\pi)^{-1/2} \sum_{|r| \leq (N-1)/2} \chi_{r\varepsilon} \otimes \chi_{r\varepsilon} \]

Since, for any Schwartz function \( f \) we have

\[ \langle \chi_{r\varepsilon}, f \rangle = \int_{(r-1/2)\varepsilon}^{(r+1/2)\varepsilon} f(x)dx = \varepsilon f(r\varepsilon) + O(\varepsilon^3) \]

we find that

\[ N^{1/2} \sigma_N (f \otimes g) = \sum_{|r| \leq (N-1)/2} \varepsilon f(r\varepsilon)g(r\varepsilon) + O(\varepsilon) \]

\[ \rightarrow \int_{\mathbb{R}} f(x)g(x)dx \]

Hence

\[ N^{1/2} \sigma_N \rightarrow \sigma \quad (N \rightarrow \infty) \]

Note that the norm of the state on the left goes to infinity as it should, since the left side is a state of infinite norm.

As we mentioned in the introduction, the paper\(^{10}\) contains a detailed discussion of the original EPR state of Einstein et al as a limit of normalized states with very sharp correlations between the position and momentum variables in the two systems, while the paper\(^{11}\) contains a rigorous characterization of the Einstein–Podolsky–Rosen state from the point of algebraic quantum theory.

It is easy to see that we can generalize the original example of Einstein et al by taking other choices of \( U \). If \( U \) is an antiunitary isomorphism of the Schwartz space with itself in the Schwartz topology then we obtain a class of states generalizing the example of Einstein et al. For instance we may take

\[ (Uf)(x) = e^{i\theta(x)} f(x)^{\text{conj}} \]

where \( \theta \) is a smooth real function whose derivatives have polynomial growth at most. We shall take up the properties of these states on a later occasion.

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\[ 20 \]
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