GROWTH OF HYPERBOLIC CELLS

PRITHA CHAKRABORTY

Abstract. We consider a hyperbolic polygon in the unit disk \( \{ z : |z| < 1 \} \) with all its vertices on the unit circle \( \{ z : |z| = 1 \} \) and a growth process of such polygons when each \( n \)-gon generates an \( n(n - 1) \)-gon by inverting itself across all of its sides. In this paper, we prove some general monotonicity results of inversion for convex hyperbolic \( n \)-gons and solve an extremal problem that, among all convex hyperbolic 4-gons containing the origin, the inverted side length of the longest side of the given hyperbolic 4-gon is minimal for the regular hyperbolic 4-gon.

1. Introduction

We consider the Poincaré model of the hyperbolic plane, that is the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) supplied with the metric \( d_{\sigma_D}(z) = |dz|/(1 - |z|^2), z \in \mathbb{D} \). In this model, the hyperbolic geodesics are circular arcs that are orthogonal to the unit circle \( \mathbb{T} = \{ z : |z| = 1 \} \). A set \( S \subset \mathbb{D} \) is hyperbolically convex if for any two points \( z_1 \) and \( z_2 \) in \( S \), the hyperbolic geodesic connecting \( z_1 \) to \( z_2 \) lies entirely inside \( S \). A hyperbolic \( n \)-gon is a simply connected subset of \( \mathbb{D} \), which contains the origin and which is bounded by a Jordan curve consisting of \( n \) hyperbolic geodesics and arcs of the unit circle \( \mathbb{T} \) which form the sides of a hyperbolic \( n \)-gon. In this paper, the considered \( \mathbb{D}_n, n \geq 3 \) is a convex hyperbolic \( n \)-gon on the unit disk \( \mathbb{D} \) having all its vertices \( A_1, A_2, \ldots, A_n \) on \( \mathbb{T} \) and ordered in the positive direction of \( \mathbb{T} \) such that

\[
A_1 = 1, \quad A_j = \exp \left( 2\pi i \sum_{k=1}^{j-1} \alpha_k \right), \quad j = 2, 3, \ldots, n.
\]

where \( 0 < \alpha_k < 1/2 \) is the angle corresponding to the side \( A_k A_{k+1} \) and having its sides on circles orthogonal to \( \mathbb{T} \). By \( \mathbb{D}_n^* \), we denote the regular hyperbolic \( n \)-gon on the unit disk \( \mathbb{D} \) having all its vertices \( A_1^*, A_2^*, A_3^*, \ldots, A_n^* \) on \( \mathbb{T} \) and ordered in the positive direction of \( \mathbb{T} \) such that

\[
A_1^* = 1, \quad A_j^* = \exp \left( 2\pi i \sum_{k=1}^{j-1} \alpha_k^* \right), \quad j = 2, 3, \ldots, n.
\]

where \( \alpha_k^* = 1/n \) is the angle corresponding to the side \( A_k^* A_{k+1}^* \) and having its sides on circles orthogonal to \( \mathbb{T} \).

We consider the following realization model of \( \mathbb{D}_n \) as a biological cell which replicates itself at a discrete time \( s = 0, 1, 2, \ldots \) (see Figure 1).

- The cell \( \mathbb{D}_n^{(0)} = \mathbb{D}_n \) is the only cell of generation 0.
- If we reflect \( \mathbb{D}_n^{(0)} \) with respect to its sides we get \( n \) new non-overlapping hyperbolic \( n \)-gons \( \mathbb{D}_{ns}, 1 \leq s \leq n \) of generation 1.
Now every $D_{ns}$ can be reflected with respect to each of its $n - 1$ “free” sides to get new $n - 1$ $n$-gons of generation 2. Altogether we have $n(n - 1)$ $n$-gons of generation 2.

Continuing we will have $n(n - 1)^2$ $n$-gons of generation 3, $n(n - 1)^3$ $n$-gons of generation 4, etc.

Let $D^{(s)}_n$ be the body of all generations $\leq s$ (i.e. $D^{(s)}_n$ is again a convex hyperbolic polygon which is precisely the union of all convex polygons of generations $j, 0 \leq j \leq s$). In particular, $D^{(1)}_n$ is a convex hyperbolic $n(n-1)$-gon on the unit disk $\mathbb{D}$ having all its vertices $B_1, B_2, \ldots, B_{n(n-1)}$ on $T$ and having its sides on circles orthogonal to $T$. Let $\alpha_{j,k}$ be the angle corresponding to the side of $D^{(1)}_n$ obtained by reflecting the vertex $A_k$ with respect to the side $A_j A_{j+1}$ of the given $D_n$ where $j = 1, 2, \ldots, n$. Similarly, by $(D^{*}_n)^{(1)}$ we denote a regular hyperbolic $n(n-1)$-gon on the unit disk $\mathbb{D}$ having all its vertices $B^*_1, B^*_2, \ldots, B^*_{n(n-1)}$ on $T$ and having its sides on circles orthogonal to $T$. Let $\alpha^*_{j,k}$ be the angle corresponding to the side of $(D^*_n)^{(1)}$ obtained by reflecting the vertex $A^*_k$ with respect to the side $A^*_j A^*_{j+1}$ of the given $D^*_n$ where $j = 1, 2, \ldots, n$.

Hyperbolic polygons play not only a significant role in hyperbolic geometry and trigonometry but also in the flourishing theory of Fuschian groups, Riemann surfaces and automorphic functions \cite{2,3,6}. As a matter of fact, hyperbolic polygons are dense in the class of hyperbolically convex regions. Thus on a brighter note, approximation of these regions by hyperbolic $n$-gons combined with variational arguments serve as a primary tool to solve several extremal problems. The regular hyperbolic $n$-gon is extremal for many functionals in the hyperbolic plane. Geometrical results include that the regular one maximizes the hyperbolic area among all hyperbolic $n$-gons with a given hyperbolic perimeter. Recent results involving functionals of non-geometrical nature such as, eigenvalues of the Laplacian, capacities, conformal radius, harmonic measure etc. can be found in \cite{1,8,9,10}.

In this paper, the main result is,

**Theorem 1.1.** Let $\alpha_{j,k}$ and $\alpha^*_{j,k}$ be as defined before for $D_4$. Then

\[
\max_{j,k} \alpha_{j,k} \geq \max_{j,k} \alpha^*_{j,k}.
\]

Equality in (1.1) is attained only in the case when $D_4$ is a regular 4-gon.
The proof of Theorem 1.1 given in Section 4 is geometric and is based on the monotonicity results of the inverted sides of the hyperbolic n-gon discussed in Section 3. Some well known preliminary results which assist the proof of the results described in the later sections are summarized in Section 2. Section 3 contains several interesting results related to hyperbolic n-gons as in the considered model, the inversion of a regular hyperbolic n-gon does not always produce a regular hyperbolic n-gon and the inversion of a non-regular hyperbolic n-gon never produce a regular hyperbolic n-gon. In Section 5, we discuss a more general geometric problem posed by A. Solynin.

2. Preliminaries

2.1. Circular Inversion. Let \( \Gamma \) be a fixed circle in the plane with center \( O \) and radius \( r \). Then the inverse of a point \( P \) with respect to \( \Gamma \) is the point \( P' \) lying on the ray from \( O \) through \( P \) such that \( |OP||OP'| = r^2 \). Then, we say \( P' \) is obtained from \( P \) by circular inversion with respect to circle \( \Gamma \). This is extensively studied in [4]. We use the following two theorems to obtain some interesting results for hyperbolic n-gons. The proofs can be found in [4], Chapter 7, Section 37.

**Proposition 2.1.** If a circle \( \gamma \) is orthogonal to \( \Gamma \) (at its intersection points), then \( \gamma \) is transformed into itself by circular inversion in \( \Gamma \). Conversely, if a circle \( \gamma \) contains a single pair \( A, A' \) of inverse points, then \( \gamma \) is orthogonal to \( \Gamma \) and is sent into itself.

**Proposition 2.2.** If \( P, P' \) and \( Q, Q' \) are pair of inverse points with respect to some circle with center at \( O \) and radius \( r \), then
\[
|P'Q'| = \frac{r^2|PQ|}{|OP||OQ|}.
\]

2.2. Majorization. Let \( x, y \) be the vectors in \( \mathbb{R}^n \). Let \( x^{(1)} \) be the largest element in \( x \), \( x^{(2)} \) be the second largest element, and so on. The vector \( x \) is said to be majorize the vector \( y \) (denoted \( x \succ y \)) if \( \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \) for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \).

A real-valued function \( \phi \) defined on a set \( A \subset \mathbb{R}^n \) is said to be Schur-convex (Schur-concave) on \( A \) if
\[
x \succ y \text{ on } A \Rightarrow \phi(x) \leq \phi(y) \quad (\phi(x) \geq \phi(y)).
\]

A detailed exposition of the properties of majorization and the proof of the following well known theorem by Schur, Hardy, Littlewood, Pólya is given in [7]. We shall apply Theorem 2.3 to study a more general problem in Section 5.

**Theorem 2.3.** If \( U \subset \mathbb{R}^n \), where \( U \) is an open set in \( \mathbb{R}^n \), and \( g : U \to \mathbb{R} \) is convex (concave), then
\[
\phi(x) = \sum_{i=1}^{n} g(x_i)
\]
is Schur-convex (Schur-concave) on \( U \), where \( x = (x_1, x_2, \ldots, x_n) \). Consequently,
\[
x \prec y \text{ (} x \succ y \text{) on } U \Rightarrow \phi(x) \leq \phi(y)
\]
Figure 2. Formula of the inverse point $C'$

3. General results for hyperbolic $n$-gons

In this section, we shall first generate a formula of an inverse point with respect to a side of a hyperbolic $n$-gon. Suppose $C' := e^{2\pi ix}$ is the inverse point of $C := e^{2\pi i\beta}$ with respect to the side $AB$ lying on the circle whose center is at $P$ and radius $r$. Define $A := e^{2\pi ia}$ and $B := e^{2\pi ib}$ where $a = \sum_{m=1}^{\infty} \alpha_i$ for some fixed $m$ and $b = a + \frac{\pi}{2}$. Thus, in order to obtain a general formula, we apply the transformation $z \mapsto e^{-2\pi ib}z$ which symmetrically places $A$ and $B$ with respect to the real axis. To ease our notations, we define $\beta' := \beta - b$ and $x' := x - b$ for the transformed polygon (See Figure 2). Clearly, $OP = \sec \pi \alpha$, $r = \tan \pi \alpha$. Let us denote the vectors $\overrightarrow{OC}$, $\overrightarrow{OC'}$ by complex numbers $z$ and $z'$ respectively. Note that, $z' = e^{2\pi ix'}$ and $z = e^{2\pi i\beta'}$. Then $\overrightarrow{OP} + \overrightarrow{PC} = \overrightarrow{OC'} = z' - \sec \pi \alpha$ and similarly, $\overrightarrow{OC} = z - \sec \pi \alpha$. Thus, we have $|\overrightarrow{PC'}|/|\overrightarrow{PC}| = \tan^2 \pi \alpha$. This implies $|z' - \sec \pi \alpha||z - \sec \pi \alpha| = \tan^2 \pi \alpha$. Thus $|z' - \sec \pi \alpha| = \tan^2 \pi \alpha/|z - \sec \pi \alpha|$. Suppose $\mu$ denotes the unit vector in the direction of $\overrightarrow{PC}$ (and so for $\overrightarrow{PC'}$). Therefore, we obtain

$$e^{2\pi i x'} = z' = \overrightarrow{OC'} = \overrightarrow{OP} + \overrightarrow{PC'} = \sec \pi \alpha + \mu|\overrightarrow{PC'}|$$

$$= \sec \pi \alpha + \frac{z - \sec \pi \alpha}{|z - \sec \pi \alpha|} |z' - \sec \pi \alpha| = \sec \pi \alpha + \frac{\tan^2 \pi \alpha}{z - \sec \pi \alpha}$$

$$= \sec \pi \alpha + \frac{\tan^2 \pi \alpha}{e^{-2\pi i \beta'} - \sec \pi \alpha} = \frac{1}{\cos \pi \alpha} + \frac{e^{2\pi i \beta'} \sin^2 \pi \alpha}{(\cos \pi \alpha)(\cos \pi \alpha - e^{2\pi i \beta'})}$$

Hence, we obtain

$$e^{2\pi i x'} = -e^{2\pi i \beta'} \cos \pi \alpha - e^{-2\pi i \beta'}$$

Substituting $\beta'$ by $\beta - b$ and $x'$ by $x - b$ in (3.1), we obtain

$$e^{2\pi i x} = -e^{2\pi i \beta} \cos \pi \alpha - e^{-2\pi i (\beta - b)}$$

(3.2)
If $\beta - b > 0$, considering the argument on both sides of (3.2), we obtain

$$2\pi x = -\pi + 2\pi \beta + 2 \tan^{-1} \left( \frac{\sin 2\pi (\beta - b)}{\cos \pi \alpha - \cos 2\pi (\beta - b)} \right).$$

If $\beta - b < 0$, considering the argument on both sides of (3.2), we obtain

$$2\pi x = \pi + 2\pi \beta - 2 \tan^{-1} \left( \frac{\sin 2\pi (b - \beta)}{\cos \pi \alpha - \cos 2\pi (b - \beta)} \right).$$

Thus, the inverse point with respect to a side of a hyperbolic $n$-gon is given by $e^{2\pi ix}$ where $2\pi x$ is given as in (3.3) and (3.4).

**Lemma 3.1.** For fixed $j$, where $1 \leq j \leq n$, $\alpha_{j,k}^*$ is monotonically decreasing for $k \neq j$ and $1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil$ (1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil), if $n$ is even (odd).

**Proof.** It is sufficient to prove the theorem for $j = n$ and when $n$ is even (see Figure 3). We can repeat the arguments to prove the result when $n$ is odd. Let $A_1^*, A_2^*, \ldots, A_n^*$ be the side of $D_n^*$ corresponding to the angle $\alpha_n^*$ which lies on the circle whose center is at P and has radius $r$. Due to symmetry it is sufficient to consider $l_1, l_2, \ldots, l_{\left\lceil \frac{n+1}{2} \right\rceil}$, the lines joining $P$ to the vertices of $D_n^*$, $A_1^*, A_2^*, \ldots, A_n^*$ respectively. Let $B_2^*, B_3^*, \ldots, B_{\left\lceil \frac{n+1}{2} \right\rceil}$ be the inverse points of the points $A_2^*, A_3^*, \ldots, A_{\left\lceil \frac{n+1}{2} \right\rceil}$ respectively with respect to the side $A_1^* A_n^*$. Let $m_1, m_2, \ldots, m_{\left\lceil \frac{n+1}{2} \right\rceil}$ denote $|A_1^* B_2^*|, |B_2^* B_3^*|, \ldots, |B_{\left\lceil \frac{n+1}{2} \right\rceil} B_{\left\lceil \frac{n+1}{2} \right\rceil}|$ respectively. Since the sides of the hyperbolic $n$-gon are orthogonal to $T$, it is sufficient to show that $m_1 > m_2 > \ldots > m_{\left\lceil \frac{n+1}{2} \right\rceil}$.

This will further imply that $\alpha_{n,2}^* > \alpha_{n,3}^* > \ldots > \alpha_{n,\left\lceil \frac{n+1}{2} \right\rceil}^*$. Since $D_n^*$ is a regular hyperbolic $n$-gon, then $|A_1^* A_2^*| = |A_2^* A_3^*| = \ldots = |A_{n-1}^* A_n^*|$. Also, it is straightforward to note that $l_1 < l_2 < \ldots < l_{\left\lceil \frac{n+1}{2} \right\rceil}$. So, for any $k$, where $1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil - 1$, we obtain

$$m_k = \frac{r^2 |A_k^* A_{k+1}^*|}{l_k l_{k+1}} \geq \frac{r^2 |A_{k+1}^* A_{k+2}^*|}{l_{k+1} l_{k+2}} = m_{k+1}.$$ 

Hence, the result follows. □
We discuss the monotonicity of the argument of the inverse point and the monotonicity of the inverted sides with respect to two adjacent and non-adjacent sides of a hyperbolic \( n \)-gon in Lemma 3.2. This serves as an important tool to prove the main theorem of this paper.

**Lemma 3.2.** (i) (Monotonicity with respect to one point) Let \( e^{2\pi i x} \) be the inverse point of \( e^{2\pi i \beta} \) with respect to the side of \( D_n \) whose corresponding angle is \( \alpha \). Then \( x \) is a decreasing function of \( \beta \) for \( 0 \leq \beta < 1 \).

(ii) (Monotonicity with respect to two adjacent sides) Let \( \alpha_k, \alpha_l \) be the corresponding angles of two adjacent sides of \( D_n \) and \( \alpha_k < \alpha_l \). Then \( \alpha_{k,l} < \alpha_{l,k} \).

(iii) (Monotonicity with respect to two non-adjacent sides) Let \( \alpha_k, \alpha_l \) for some \( k \neq l \) be the corresponding angles of two non-adjacent sides of \( D_n \) and \( \alpha_k < \alpha_l \). Then \( \alpha_{k,l} < \alpha_{l,k} \).

**Proof.** (i) We normalize the initial end point (in the positive direction of \( T \)) of the side corresponding to \( \alpha \) at 1. Using (3.3), for a fixed \( \alpha \), the argument of \( e^{2\pi i x} \) is given by a \( 2\pi \)-multiple of

\[
 f(\beta) := -\pi + 2\pi \beta + 2 \tan^{-1} \left( \frac{\sin 2\pi (\beta - \alpha/2)}{\cos \pi \alpha - \cos 2\pi (\beta - \alpha/2)} \right).
\]

We show that \( f \) is a decreasing function of \( \beta \), where \( \alpha \leq \beta < 1 \). We obtain,

\[
 \frac{\partial f}{\partial \beta} = \frac{-4\pi \sin^2 \pi \alpha}{3 - 2 \cos 2\pi \beta + \cos 2\pi \alpha - 2 \cos 2\pi (\beta - \alpha)}.
\]

Note that, \( f(\alpha) = 2\pi \alpha > 0 \) and \( f(1) = 2\pi > 0 \). Note that the numerator of (3.5) is clearly negative and therefore it suffices to show that

\[
 g(\beta) := 3 - 2 \cos 2\pi \beta + \cos 2\pi \alpha - 2 \cos 2\pi (\beta - \alpha) > 0.
\]

Clearly \( g(\alpha) = 1 - \cos 2\pi \alpha > 0 \), \( g(1) = 1 - \cos 2\pi \alpha > 0 \). We therefore complete the proof by showing that \( g \) is a concave function of \( \beta \), where \( \alpha \leq \beta < 1 \). Note that,

\[
 \frac{\partial g}{\partial \beta} = 2\pi [\sin 2\pi \beta + \sin 2\pi (\beta - \alpha)].
\]
Let $C_{AB}$ and $C_{BC}$, where $\beta = \alpha$.

Therefore, we show that for a fixed $0 < \alpha < 1$, we have $\pi y = 2 \pi \beta - 2 \tan^{-1} \left( \frac{\sin 2 \pi (b_1 - \beta_1)}{\cos \pi \alpha_2 - \cos 2 \pi (b_1 - \beta_1)} \right),$

where $\beta_1 = 0$, $b_1 = \alpha_1 + \frac{\pi}{2}$. Also, $C'$ is the inverse point of $C$ with respect to the side $AB$. Using (3.3), the argument of $C'$ is,

$$2 \pi x = \pi + 2 \pi \beta_1 - 2 \tan^{-1} \left( \frac{\sin 2 \pi (b_1 - \beta_1)}{\cos \pi \alpha_2 - \cos 2 \pi (b_1 - \beta_1)} \right),$$

where $\beta_1 = 0$, $b_1 = \alpha_1 + \frac{\pi}{2}$. Then, $|BC'| = 2 \pi (\alpha_1 - y)$, $|A'B| = 2 \pi (x - \alpha_1)$.

Therefore, we show that $|BC'| = |A'B| = 2 \pi (x + y - 2 \pi \alpha_1) > 0$. Equivalently, it suffices to show that for a fixed $0 < \alpha_2 < 1/2$,

$$f(\alpha_1) := -2 \left[ (\alpha_1 - \alpha_2) \pi + \tan^{-1} \left( \frac{\sin \pi (2 \alpha_1 + \alpha_2)}{\cos \pi \alpha_2 - \cos \pi (2 \alpha_1 + \alpha_2)} \right) \right]$$

$$- \tan^{-1} \left( \frac{\sin \pi (\alpha_1 + 2 \alpha_2)}{\cos \pi \alpha_1 - \cos \pi (2 \alpha_2 + \alpha_1)} \right)$$

is a concave function of $\alpha_1$ where $0 < \alpha_1 < \alpha_2$. Note that, $f(0) = 0$, $f(\alpha_2) = 0$.

We obtain

$$\frac{\partial f}{\partial \alpha_1} = -4 \sin^2 \pi \alpha_2 \left( \frac{2}{\Lambda_2} - 1 \right) = -4 \sin^2 \pi \alpha_2 \left( \frac{2 \Lambda_1 - \Lambda_2}{\Lambda_1 \Lambda_2} \right),$$

where

$$\Lambda_1 = 3 + \cos 2 \pi \alpha_2 - 2 \cos 2 \pi \alpha_1 - 2 \cos 2 \pi (\alpha_1 + \alpha_2),$$

$$\Lambda_2 = 3 + \cos 2 \pi \alpha_1 - 2 \cos 2 \pi \alpha_2 - 2 \cos 2 \pi (\alpha_1 + \alpha_2).$$

It is straightforward to observe that as in (3.6), for fixed $0 < \alpha_2 < 1/2$, $\Lambda_1, \Lambda_2 > 0$ for $0 < \alpha_1 < \alpha_2$. Also, $\frac{\partial f}{\partial \alpha_1} = 0$ implies either $\alpha_2 = 0$ or $2 \Lambda_1 - \Lambda_2 = 0$. But $\alpha_2$ can never be zero, therefore, all the points satisfying $2 \Lambda_1 - \Lambda_2 = 0$, for a fixed $\alpha_2$, are the critical points of $f$. Next, we claim that there exists only one critical point of $f$ in $(0, \alpha_2)$. Equivalently, it is sufficient to show that there exists only one zero of the function $g(\alpha_1) := 2 \Lambda_1 - \Lambda_2$ inside $(0, \alpha_2)$ for a fixed $\alpha_2$. Note that
\(g(0) = -4 \sin^2 \pi \alpha_2 < 0, \ g(\alpha_2) = -4 \cos^2 \pi \alpha_2 - \cos \pi \alpha_2 + 5 > 0\). By the Intermediate Value Theorem, there exists at least one zero inside \((0, \alpha_2)\). To show, there exists only one, we claim that \(g'(\alpha_1) := 2\Lambda_1 - \Lambda_2\) is a strictly increasing function of \(\alpha_1\), where \(0 < \alpha_1 < \alpha_2 < \frac{1}{2}\). Consider
\[
\frac{\partial g}{\partial \alpha_1} = 2\pi (5 \sin 2\pi \alpha_1 + 2 \sin 2\pi (\alpha_1 + \alpha_2)).
\]

We shall show that \(\frac{\partial g}{\partial \alpha_1} > 0\). We can rewrite it as, \(5 \sin \alpha_1 + 2 \sin(\alpha_1 + \alpha_2) = \kappa_1 \sin \alpha_1 + \kappa_2 \cos \alpha_1\) where \(\kappa_1 = 5 + 2 \cos \alpha_2, \ \kappa_2 = 2 \sin \alpha_2\). Then, using the elementary trigonometry formula,
\[
\kappa_1 \sin \alpha_1 + \kappa_2 \cos \alpha_1 = \sqrt{\kappa_1^2 + \kappa_2^2} \sin(\alpha_1 + H),
\]
where \(H = \tan^{-1}\left(\frac{\kappa_2}{\kappa_1}\right)\), it is sufficient to show that \(0 < \alpha_1 + H < \pi\) (since \(\sqrt{\kappa_1^2 + \kappa_2^2} > 0\)). It is clear that \(\alpha_1 + H > 0\). Since \(\alpha_1 < \alpha_2\),
\[
\alpha_1 + H = \alpha_1 + \tan^{-1}\left(\frac{2 \sin \alpha_2}{5 + 2 \cos \alpha_2}\right) < \alpha_2 + \tan^{-1}\left(\frac{2 \sin \alpha_2}{5 + 2 \cos \alpha_2}\right).
\]

It can be easily shown that by substituting \(\sin \alpha_2\) by \(w\) that
\[
h(w) := \sin^{-1}(w) + \tan^{-1}\left(\frac{2w}{5 + 2\sqrt{1 - w^2}}\right)
\]
is an increasing function in \((0, 1)\). Therefore, for all \(w \in (0, 1),
\[
h(w) < h(1) = \sin^{-1}(1) + \tan^{-1}\left(\frac{2(1)}{5 + 2(0)}\right) = \frac{\pi}{2} + \tan^{-1}(2/5) < \frac{\pi}{2} + \frac{\pi}{4} < \pi.
\]

Therefore, \(\alpha_1 + H < \pi\). Thus, \(\frac{\partial f}{\partial \alpha_1} > 0\) for \(2\Lambda_1 - \Lambda_2 < 0\) and \(\frac{\partial f}{\partial \alpha_1} < 0\) for \(2\Lambda_1 - \Lambda_2 > 0\). Therefore, \(f\) is a concave function of \(\alpha_1\) which completes the proof.

(iii) As shown in Figure 4b let \(\alpha_k, \ \alpha_l\) be the corresponding angles of two non-adjacent sides \(AB\) and \(CD\) of \(D_n\) respectively and \(|AB| < |BC|\). We normalize one end point of the side \(AB\) at 1. Notice that the inverse points move continuously on \(T\). Hence by rotating the side \(CD\) adjacent to \(AB\) will reduce this case to (ii) and therefore the result follows. \(\Box\)

To study further properties of the inversion of an hyperbolic \(n\)-gon, we ask the following questions. (a) If we start with a regular polygon with \(n\) sides and reflect it with respect to its sides, what can be said about the new polygon? Is it regular or non-regular? (b) If we start with a non-regular polygon with \(n\) sides and reflect it with respect to its sides, what can be said about the new polygon? Is it ever regular or always non-regular? The answers to these questions are given by Lemma 3.3 and 3.4.

**Lemma 3.3.** Let \(D_n\) be a regular hyperbolic polygon with \(n \geq 4\) sides. Then \(\left(D_n^s\right)^{(s)}\) is non-regular for \(s \geq 1\). Further, \(\left(D_6^1\right)^{(1)}\) is a regular hyperbolic 6-gon and \(\left(D_5^s\right)^{(s)}\) is a non-regular hyperbolic polygon for \(s \geq 2\).
Proof. If \( n = 3 \), then \((D_3^n)^{(1)}\) is a regular 6-gon by Lemma 3.2(ii). Suppose, if possible, \((D_n^n)^{(s)}\) is regular for \( s \geq 1 \) and for all \( n \geq 4 \). Then Lemma 3.2(ii) guarantees that \((D_n^n)^{(s-1)}\) is also regular for \( s \geq 1 \). Continuing in a similar fashion, we conclude that \((D_n^n)^{(1)}\) is regular for \( n \geq 4 \). Let \( e^{2\pi i\beta} \) be the inverse point of \( e^{2\pi i\alpha} \) on the side corresponding to the angle \( \alpha \). We normalize one endpoint of the side at 1. Using (3.3) for \( D_n^n \) with \( \alpha = \frac{2\pi}{n} \), \( \beta = \frac{4\pi}{n} \) and \( b = \pi \), we have

\[
2\pi x = -\pi + \frac{4\pi}{n} + 2\tan^{-1}\left(\frac{\sin 3\pi/n}{\cos \pi/n - \cos 3\pi/n}\right).
\]

Then we shall show that \( \frac{2\pi}{n} - 2\pi x \neq \frac{2\pi}{n(n-1)} \) for \( n \geq 4 \). Suppose if possible,

\[
\frac{2\pi}{n} - 2\pi x = \frac{2\pi}{n(n-1)}
\]

\[
\Rightarrow \pi - \frac{2\pi}{n} = 2\tan^{-1}\left(\frac{\sin 3\pi/n}{\cos \pi/n - \cos 3\pi/n}\right)
\]

\[
\Rightarrow \tan\left(\frac{\pi}{2} - \frac{\pi}{n} - 1\right)\cos \pi/n - \cos 3\pi/n
\]

\[
\Rightarrow \cos \frac{\pi}{n-1} - \cos \frac{3\pi}{n} + \sin \frac{\pi}{n-1} \sin \frac{3\pi}{n} = 0
\]

\[
\Rightarrow \frac{1}{2} \left(\cos \frac{\pi}{n(n-1)} + \cos \frac{(2n-1)\pi}{n(n-1)} - 2\cos \frac{(2n-3)\pi}{n(n-1)}\right) = 0
\]

\[
\Rightarrow \frac{1}{2} \left(\cos \frac{\pi}{n(n-1)} - \cos \frac{(2n-3)\pi}{n(n-1)}\right) + \frac{1}{2} \left(\cos \frac{(2n-1)\pi}{n(n-1)} - \cos \frac{(2n-3)\pi}{n(n-1)}\right) = 0
\]

\[
\Rightarrow \sin \frac{\pi}{n} \left(\sin \frac{(n-2)\pi}{n(n-1)} - 2\sin \frac{\pi}{n(n-1)} \cos \frac{\pi}{n}\right) = 0
\]

\[
\Rightarrow \sin \frac{\pi}{n} \left(2\sin \frac{(n-2)\pi}{n(n-1)} - \sin \frac{\pi}{n-1}\right) = 0
\]

Clearly \( \sin \frac{\pi}{n} \neq 0 \) for \( n \geq 4 \). So, consider \( f(n) := 2\sin \frac{(n-2)\pi}{n(n-1)} - \sin \frac{\pi}{n-1} \). Clearly, \( f(4) = 1 - \frac{\sqrt{3}}{2} > 0 \), contradicting our assumption. For \( n \geq 5 \), \( \frac{(n-2)\pi}{n(n-1)} > \frac{\pi}{n+2} \). Therefore,

\[
f(n) > 2\sin \frac{\pi}{n+2} - \sin \frac{\pi}{n-1}
\]

\[
= 2 \left[\frac{\pi}{n+2} - \frac{1}{3!} \left(\frac{\pi}{n+2}\right)^3\right] - \frac{\pi}{n-1}
\]

\[
= \frac{\pi}{3(n-1)(n+2)^3} \left[3n^3 - 2\pi n^2 - (\pi^2 + 36)n + 2(\pi^2 - 24)\right].
\]

A straightforward calculus argument will suggest that the above expression is strictly positive for \( n \geq 5 \), which is a contradiction. Thus, \((D_n^n)^{(1)}\) is not regular for \( n \geq 4 \). Similar arguments will show that \((D_3^n)^{(s)}\), \( s \geq 2 \) is non-regular. \( \square \)

**Lemma 3.4.** Let \( D_n \) be a non-regular hyperbolic polygon with \( n \geq 3 \) sides. Then \((D_n^n)^{(s)}\) is non-regular for \( s \geq 1 \) and for all \( n \geq 3 \).

**Proof.** Suppose, if possible, \((D_n^n)^{(s)}\) is a regular hyperbolic \( n \)-gon with \( s \geq 1 \), \( n \geq 3 \). Then Lemma 3.2(ii) guarantees that \((D_n^n)^{(s-1)}\) is a regular polygon. However
Lemma 3.3 confirms that the inversion of a regular polygon always results in a non-regular polygon, which contradicts our hypothesis. Hence, the result follows. Similar arguments will conclude that \((D_3)^{(s)}\) is non-regular for \(s \geq 2\). Also, \((D_3)^{(1)}\) is non-regular by Lemma 3.2(ii).

4. PROOF OF THEOREM 1.1

In this section, we illustrate the proof of Theorem 1.1 by using variation of vertices of \(D_4\) or their continuous movement on the unit circle \(T\) to exhaust the possible configurations to the extremal one, which is \(D_4^*\) as shown in Figure 8 where the longest inverted sides of \(D_4^*\) are the eight corner ones due to Lemma 3.1. We start with an assumed extremal configuration with a fixed number of longest inverted sides. The movement of vertices results in change of side-lengths of \(D_4\) but preserves the number of sides to produce a “new” \(D_4\). We vary the vertices in such a way that causes an increment in the side-length of the “new” \(D_4\) corresponding to an inverted non-longest side in \(D_4^{(1)}\) and thus a decrement in the side-length corresponding to the inverted longest side. We next argue that there are only two possible movements of the vertices which cause the increment/decrement which further contradicts the assumed extremal configuration due to Lemma 3.2(ii) and (iii). These concurrently guarantee that there cannot be only one longest side in the extremal configuration.

Suppose first that there is only one longest side in the extremal configuration. The possible configurations are listed in Figure 5. Due to the symmetry of the 4-gon the remaining cases of one longest side configuration will be the same as what we discuss here. Suppose \(\alpha_{1,4}\) is the angle corresponding to the longest side (see Figure 5a). Then by Lemma 3.2(ii), \(\alpha_1 > \alpha_4\). We move A continuously such that \(\alpha_4 > \alpha_1\). This results in \(\alpha_{4,1} > \alpha_{1,4}\) (by Lemma 3.2(ii)), which gives a contradiction. Suppose \(\alpha_{1,3}\) is the angle corresponding to the longest side (see Figure 5b). Then by Lemma 3.2(iii), \(\alpha_1 > \alpha_3\). We move C continuously such that \(\alpha_3 > \alpha_1\). This results in \(\alpha_{3,1} > \alpha_{1,3}\) (by Lemma 3.2(iii)), which again gives a contradiction. Thus, there are more than one longest side in the extremal configuration. The following Lemmas 4.1 and 4.2 explain two particular configurations when the adjacent and opposite sides of extremal \(D_4\) are equal respectively.
Lemma 4.1. Let $D_4$ be an extremal hyperbolic polygon such that pair of adjacent sides are equal and has at least two adjacent corner longest sides after inversion. Then $D_4 = D_4^\ast$.

Proof. Without loss of generality, suppose $\alpha_1 > \frac{\pi}{2}$. Given that $\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$ which forces $\alpha_{1,4} = \alpha_{4,1}$ and $\alpha_{2,3} = \alpha_{3,2}$. Suppose $\alpha_{1,4}$ and $\alpha_{4,1}$ are the angles corresponding to the longest sides in the extremal configuration (see Figure 6(a)). By Lemma 3.2(i), $x$ is a decreasing function of $\alpha_1$, where $e^{2\pi ix}$ is the inverse point of $e^{2\pi i\alpha_1}$. As $\alpha_1$ decreases, $-\alpha_1$ increases. Therefore, $\alpha_{1,4} < \alpha_{2,3}$ and $\alpha_{4,1} < \alpha_{3,2}$, which is a contradiction. Thus, all the sides are equal and thus $D_4^\ast$ gives the extremal configuration.

Lemma 4.2. Let $D_4$ be an extremal hyperbolic polygon such that pair of the opposite sides are equal and has at least two opposite longest sides after inversion. Then $D_4 = D_4^\ast$.

Proof. Given that $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$. This forces $\alpha_{1,3} = \alpha_{3,1}$ and $\alpha_{2,4} = \alpha_{4,2}$. Suppose $\alpha_{1,3}$ and $\alpha_{3,1}$ are the angles corresponding to longest sides in the extremal configuration (see Figure 6(b)). By Lemma 3.2(i), if $\alpha_1$ decreases then $\pi + \alpha_1$
Figure 8. Extremal Configuration

decreases and therefore $\alpha_{1,3} < \alpha_{2,4}$ and $\alpha_{3,1} < \alpha_{4,2}$, which is a contradiction. Thus, all the sides are equal which forces $D_4 = D_4^*$. □

We further describe here the remaining two types of variations which demonstrates the exhaustion of all possible configurations that can be considered to reach the conclusion that the extremal configuration is when $D_4 = D_4^*$. The analysis of variation described here works identically for any number of longest sides concerned in the assumed extremal configuration.

(a) The first type of variation is when we vary a “free” vertex in a sense that it affects only one inverted longest side in $D_4^{(1)}$ and the others remain unaffected. This variation is already shown in Figure 5. To get a better view of this case for more number of longest sides, suppose that $\alpha_{1,4}$ and $\alpha_{1,2}$ are the angles corresponding to longest sides (see Figure 7a). Then by Lemma 3.2(ii), $\alpha_1 > \alpha_2$. We move C such that $\alpha_2 > \alpha_1$. This results in $\alpha_{2,1} > \alpha_{1,2}$ (by Lemma 3.2(ii)), a contradiction to the assumption. Thus $\alpha_{1,2} = \alpha_{2,1}$. Thus by Lemma 4.2 the extremal configuration is $D_4^*$ as shown in Figure 8.

(b) The second type of variation is when we vary a “non-free” vertex in a sense that it affects some or all inverted longest sides in $D_4^{(1)}$. For the better understanding of this case, suppose that $\alpha_{1,4}$, $\alpha_{1,3}$, $\alpha_{2,4}$ are the angles corresponding to longest sides (see Figure 7b). Then by Lemma 3.2(ii), (iii) $\alpha_1 > \alpha_2 > \alpha_3$, $\alpha_4$. We move C such that max{$\alpha_1, \alpha_2$} < min{$\alpha_3, \alpha_4$} which results in $\alpha_{4,2} > \alpha_{2,4}$ and $\alpha_{3,1} > \alpha_{1,3}$ (by Lemma 3.2(ii), (iii)). This is however a contradiction. Thus, the only possibility is when $\alpha_{1,3} = \alpha_{3,1}$ and $\alpha_{2,4} = \alpha_{4,2}$. Therefore, by Lemma 4.2 the extremal configuration is $D_4^*$ as shown in Figure 8.

5. Discussion

Consider a hyperbolic $n$-gon $D_n$ discussed in Section 1. Since all the vertices of $D_n$ are on $T$, it is well known that the maximal hyperbolic area of $D_n$ is $\pi(n-2)$. With an aim to find the explicit formula for the Euclidean area of $D_n$, let $\alpha$ be the angle corresponding to one such side of $D_n$. Then, as shown in Figure 9 it is sufficient to find the area of the region $OAD$. Using $\Delta OAB$, we obtain $r = \tan(\pi \alpha)$. Therefore, the area of the sector $ABC$ is equal to $r^2 \theta/2 = (\tan^2(\pi \alpha)/2)(\pi/2 - \pi \alpha)$ and the area of $\Delta OAB = \frac{1}{2} |OA||OB| = \frac{1}{2} \tan(\pi \alpha)$. Thus, the area of the sector
Therefore, the area of the required region $OAD$ is given by
\[
F(\alpha) := \tan(\pi \alpha) \left( 1 - \tan(\pi \alpha) \left( \frac{\pi}{2} - \pi \alpha \right) \right).
\]

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the angles corresponding to the sides of $D_n$. Then the Euclidean area of $D_n$ is given by
\[
\sum_{k=1}^n F(\alpha_k) = \sum_{k=1}^n \tan \pi \alpha_k \left[ 1 - \pi \tan \pi \alpha_k \left( \frac{1}{2} - \alpha_k \right) \right] .
\]

**Lemma 5.1.** For $\alpha \in \mathbb{R}$,
\[
F(\alpha) = \tan \pi \alpha \left[ 1 - \pi \tan \pi \alpha \left( \frac{1}{2} - \alpha \right) \right]
\]
is a concave function of $\alpha$ for $0 < \alpha < \frac{1}{2}$.

**Proof.** To show that $F(\alpha)$ is a concave function of $\alpha$, it is sufficient to show that $\frac{dF^2}{d\alpha^2} < 0$. We have,
\[
\frac{dF^2}{d\alpha^2} = -\pi^2 \sec^4 \pi \alpha \left[ \pi (1 - 2\alpha)(2 - \cos 2\pi \alpha) - 3 \sin 2\pi \alpha \right].
\]
Again it suffices to show that
\[
g(\alpha) := \pi (1 - 2\alpha)(2 - \cos 2\pi \alpha) - 3 \sin 2\pi \alpha
\]
is a strictly decreasing function of $\alpha$, which in turn proves that $g(\alpha) > g \left( \frac{1}{2} \right) = 0$. Notice that
\[
\frac{dg}{d\alpha} = 2 \pi \left[ -2 - 2 \cos 2\pi \alpha + \pi (1 - 2\alpha) \sin 2\pi \alpha \right]
\]
\[
\leq 2 \pi \left( -2 - 2 \cos 2\pi \alpha \right)
\]
\[
= -2 \pi \sin^2 \pi \alpha < 0.
\]
in $(0, 1/2)$. Therefore, the result follows. \qed

**Theorem 5.2.** Let $D^*_n$ and $D_n$, $n \geq 3$ be as defined before. Then
\[
\text{area} \left( D_n \right) \leq n \tan \frac{\pi}{n} \left[ 1 - \frac{\pi (n - 2)}{2n} \tan \frac{\pi}{n} \right],
\]
where $\text{area} \left( D_n \right)$ is the Euclidean area of $D_n$. Equality in (5.3) is attained only if $D_n$ is a rotation of $D^*_n$ about the origin.

**Proof.** We maximize the area of $D_n$ given in (5.1) subject to the condition $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. By Lemma 5.1 for each $0 < \alpha_k < 1/2$, $F(\alpha_k)$ is a concave function of $\alpha_k$. Thus using Jensen’s inequality \cite{7}, we obtain
\[
F \left( \frac{\sum_{k=1}^n \alpha_k}{n} \right) \geq \frac{\sum_{k=1}^n F(\alpha_k)}{n} \Rightarrow \sum_{k=1}^n F(\alpha_k) \leq n F \left( \frac{1}{n} \right).
\]
where $F(1/n)$ is computed using (5.2) to get the right hand side expression in (5.3). \qed
A. Solynin suggested an immediate geometric question “How does the Euclidean area of $D_n^{(s)}$ grow for $s \geq 0$?” In particular,

**Conjecture 5.3.** Let $D_n^{(s)}$ and $(D_n^*)^{(s)}$, $s \geq 0$ and $n \geq 3$ be as defined before. Then

$$\text{area}(D_n^{(s)}) \leq \text{area}((D_n^*)^{(s)}).$$

with the sign of equality only if $D_n$ is a rotation of $D_n^*$ about the origin.

The case $s = 0$ for any $n \geq 3$ of Conjecture 5.3 is proved in Theorem 5.2 that the Euclidean area for $D_n$ is maximal for the regular $n$-gon $D_n^*$. Also, by Lemma 3.3 it is known that $(D_5^*)^{(1)}$ is a regular 6-gon and hence the conjecture 5.3 is proved for $s = 1$ and $n = 3$. However, it is not straightforward to prove the result for any $n \geq 4$ and $s \geq 1$. It is clear that the method applied in Theorem 5.2 does not work for $s \geq 1$ and $n \geq 4$ due to the complexity of the problem and in particular the involvement of a large number of sides. In support of the inequality conjectured in (5.4) we discuss here the application of majorization techniques discussed in Section 2 which may serve as a promising line of attack to prove Conjecture 5.3 for $s = 1$ and for any $n \geq 4$.

Let $B := (\beta^{(1)}_1, \beta^{(2)}_1, \ldots, \beta^{(n(n-1))}_1)$ be a decreasing rearrangement of $\{\alpha_{j,k}\}$ and $B^* := (\beta^{*(1)}_1, \beta^{*(2)}_1, \ldots, \beta^{*(n(n-1))}_1)$ be a decreasing rearrangement of $\{\alpha^{*}_{j,k}\}$, that is, $\beta^{(1)}_1 = \max_j \{\alpha_{j,k}\}$, $\beta^{(2)}_1 = \text{second largest } \alpha_{j,k}, \ldots, \beta^{(n(n-1))}_1 = \min_j \{\alpha_{j,k}\}$ and so on for $B^*$. Notice that, to prove Conjecture 5.3 for $s = 1$, it is sufficient to prove the following conjecture:

**Conjecture 5.4.** Let $B$ and $B^*$ be as defined before. Then $B \succ B^*$.

Notice that to prove Conjecture 5.4 we need to show

$$\sum_{i=1}^{m} \beta^{(i)} \geq \sum_{i=1}^{m} \beta^{*(i)}$$

for all $m = 1, 2, \ldots, n(n-1)$. In particular, we already proved in Theorem 1.1 that $\beta^{(1)}_1 \geq \beta^{*(1)}_1$ for $D_4$. We strongly believe that repeating similar arguments, one can generalize the result of Theorem 1.1 for any $D_n$. However, we need a different tool.
to prove the remaining cases of (5.5). By substituting $\alpha$ by $\alpha_{j,k}$ in Lemma 5.1, we obtain the function $F(\alpha_{j,k})$. So, to prove Conjecture 5.3 for $D_n^{(1)}$, we maximize area$(D_n^{(1)})$ which is,

$$G(B) := \sum_{j=1}^{n(n-1)} F(\beta_{(j)})$$

subject to $\sum_{k=1}^{n-1} \alpha_{j,k} = \alpha_j$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} \alpha_j = 1$. Lemma 5.1 affirms that $G(B)$ is a concave function and Theorem 2.3 confirms that $G$ is a Schur-concave function. Therefore, $G(B) \leq G(B^*)$ which supports the inequality conjectured in Conjecture 5.4 for $s = 1$ is true.

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