Second order perturbations of rotating bodies in equilibrium; the exterior vacuum problem.

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We study the exterior vacuum problem for first and second order stationary and axially symmetric perturbations of static bodies. The boundary conditions and their compatibility for the existence of an asymptotically flat exterior solution are discussed.

1 Introduction

Finding global models for rotating objects in general relativity has proven to be extremely difficult, even for axially symmetric configurations in equilibrium. So far there are no known explicit global models except for spherical stars (hence non-rotating) or the Neugebauer and Meinel disc of dust \cite{1}, where the matter source is encoded as suitable jumps on the first derivatives of the metric, which is otherwise vacuum everywhere.

Since a proper understanding of rotating objects in equilibrium within the context of general relativity is obviously fundamental for many astrophysical situations, alternative methods have been developed over the years. The two most important ones are, without doubt, the use of numerical methods and perturbation theory. The former can handle fully relativistic situations with intense gravitational fields and high velocities and has produced a large variety of very interesting results. However, the fact that closed expressions are never found in this setting leaves plenty of room for other approaches, like for instance perturbation theory. Its field of applicability is restricted to slowly rotating stars (so that the perturbation parameter can be taken to be the maximum angular velocity of the star, for instance) and also a wealth of results have been obtained (see e.g. \cite{2} and references therein).

This short note is a summary of a longer paper \cite{3} by the same authors where many more results and their proofs can be found. The object of this contribution is to explain in a few words the main motivation of this work, present the main result and give some indications of how it can be proven by analyzing the simplest possible situation.

The aim of this work is to study perturbation theory of rotating stars from a different perspective than normally done. We want to consider slowly rotating bodies with an arbitrary matter content and we wish to concentrate on the effect that the rotation has in the vacuum outside region. Thus, our background spacetime is composed of a static object with some non-vanishing energy-momentum tensor (typically a perfect fluid, but not necessarily). Although we have in mind the case when the static body is spherically symmetric, which is the physically most relevant one, all our results hold also for axially symmetric backgrounds. We assume, as it is usual in this context, that the object has a sharp boundary and that there is no matter outside it, so that the metric exterior to the body is vacuum. The metrics inside and outside the body must satisfy certain junction conditions (see \cite{4} for a detailed account) in order to produce a well-defined spacetime. Given this background, we
want to perturb it *arbitrarily* in the interior except for the restriction that the object is still in equilibrium and has axial symmetry (i.e. the perturbed metric is stationary and axially symmetric). Furthermore we want to do perturbations up to second order in perturbation theory. The necessity of going to second order comes from the fundamental results obtained in the seminal paper by Hartle [5] where rigidly rotating perturbations of spherically symmetric and static perfect fluids were analyzed and where it was found that to first order rotation only affects the $\phi$ crossed term in the metric (more technically, it only produces so-called axial or odd perturbations) without modifying, for instance, the spherical shape of the object, while second order perturbations, already affect the shape of the body and the rest of the metric components.

In Hartle’s paper some heuristic arguments were used at some places, specially regarding the matching procedure of the perturbed metrics. One of the aims of our paper is to set up a proper theoretical framework so that, in a future work, we can make rigorous all the arguments used by Hartle. This is important insofar Hartle’s paper has served as the basis of many developments in perturbation theory of rotating objects over the years.

2 Brief summary of stationary and axially symmetric rotating bodies

A spacetime describing a stationary and axially symmetric rotating body with a boundary surface is composed by two regions; one inside the body solving the Einstein field equations with matter, and another outside the body solving the vacuum field equations and being asymptotically flat (because we are dealing with an isolated body). Furthermore the two metrics must fulfill the so-called matching conditions on the boundary of the body, i.e. on a timelike, stationary and axially symmetric hypersurface $\Sigma$. The vacuum field equations outside the body can be written in terms of a coupled system of non-linear elliptic PDE for two scalars $U$ and $\Omega$ called the “norm” and the “twist” potentials respectively and which are defined in terms of the unique stationary Killing vector $\xi$ which is unit at infinity. The twist potential is defined only up to an arbitrary additive constant which is fixed by demanding $\Omega \to 0$ at infinity. With this choice, $\Omega$ vanishes if and only if the spacetime is static, hence the twist potential determines whether the body is rotating or not. In the vacuum region, there exist local coordinates $\{t, \phi, \rho, z\}$ called Weyl coordinates which are adapted to the stationary and axial Killing vectors and which locate the axis of symmetry at $\rho = 0$. This coordinate system is defined uniquely except for an additive constant in $z$, which can be fixed whenever the spacetime has an equatorial plane by choosing $z = 0$ on the equator. We will assume that the Weyl coordinates exist also globally in the vacuum exterior region (see [6] for global existence results of the Weyl coordinate system). In this setting $U$ and $\Omega$ are functions of $\rho$ and $z$ alone and satisfy the so-called Ernst equations, which read

$$\begin{align*}
\triangle_\gamma U + \frac{1}{2} e^{-4U} (d\Omega, d\Omega)_\gamma &= 0, \\
\triangle_\gamma \Omega - 4 (d\Omega, dU)_\gamma &= 0,
\end{align*}$$

(1)

where $\triangle_\gamma$ is the Laplacian of the flat metric $\gamma = d\rho^2 + dz^2 + \rho^2 d\phi^2$ and $(\ , \ )_\gamma$ denotes scalar product with respect to it. The asymptotic flatness condition demands $U = 1 - M/r + O(r^{-2})$, $\Omega = -2zJ/r^3 + O(r^{-3})$, for some constants $M, J$ and where $r^2 = \rho^2 + z^2$. The boundary of the rotating body as seen from the exterior region will be denoted by $\Sigma^E$ and will be defined by two functions $\rho = \rho(\mu)$, $z = z(\mu)$. If the metric inside the body is assumed to be known, then the matching conditions can be seen [7] (see [8] for the complete generalisation) to be equivalent to the following data: (i) the explicit form of the matching hypersurface in Weyl coordinates, i.e. the functions $\rho(\mu), z(\mu)$ and (ii) the values of $U$ and $\Omega$ (the latter except for an additive constant) *together with their normal derivatives* on $\Sigma^E$. Notice that
the Ernst equations are elliptic, which means that appropriate boundary data consist of fixing the value of the functions or their normal derivatives on the boundary (or perhaps a combination of both), but never the full values of the functions and their normal derivatives. In more technical terms the boundary conditions are of Cauchy type, which is unsuitable for elliptic problems. This property reflects the fact that given an arbitrary metric describing a rotating body in equilibrium, in general there will not exist a vacuum solution matching with the given metric and extending to infinity in an asymptotically flat manner. The problem of finding a global model of a rotating object is truly global in nature and cannot be broken into an interior and exterior problem without paying some price. In our case this translates into the necessity of dealing with an overdetermined boundary value problem for an elliptic system. Thus, one has to worry about existence of the solutions, i.e. which are the restrictions that the boundary data (and hence the interior metric) must satisfy so that they truly represent an isolated rotating body. Let us stress here that uniqueness is a much simpler problem which can be solved in full generality [7]. With regard to existence, there are results involving an infinite set of compatibility conditions on the boundary data which are necessary for existence to hold [9]. Whether they are also sufficient is still an open problem.

3 First and second order perturbations of the exterior region

After this brief reminder of the non-linear case, let us move into perturbation theory. Calling \( \epsilon \) the perturbation parameter, we consider a one-parameter family of spacetimes depending on \( \epsilon \). Since we take every element in the family to be a stationary and axially symmetric spacetime, vacuum outside some spatially compact region (defining the rotating body) and asymptotically flat, we have two families of functions \( U_\epsilon(\rho, z), \Omega_\epsilon(\rho, z) \), which for all \( \epsilon \) satisfy the Ernst equations (1). Defining \( U' \equiv \partial_\rho U_\epsilon|_{\epsilon=0}, U'' \equiv \partial_\rho \partial_\rho U_\epsilon|_{\epsilon=0} \) and similarly for \( \Omega' \) and \( \Omega'' \), and using now that the background spacetime is static, i.e. that \( \Omega_\epsilon|_{\epsilon=0} = 0 \), we find the first and second order perturbed field equations

\[
\begin{aligned}
\triangle_\gamma U'_{\epsilon 0} &= 0 \\
\triangle_\gamma \Omega'_{\epsilon 0} - 4 (d\Omega'_{\epsilon 0}, dU_0)_{\gamma} &= 0 \\
\triangle_\gamma U''_{\epsilon 0} + e^{-4U_0} (d\Omega''_{\epsilon 0}, d\Omega''_{\epsilon 0})_{\gamma} &= 0 \\
\triangle_\gamma \Omega''_{\epsilon 0} - 8 (d\Omega''_{\epsilon 0}, dU_0)_{\gamma} - 4 (d\Omega''_{\epsilon 0}, dU_0)_{\gamma} &= 0
\end{aligned}
\]

simply by taking first and second partial derivatives of (1) with respect to \( \epsilon \) and evaluating the result at \( \epsilon = 0 \).

Notice that we are performing the perturbation (i.e. derivative with respect to \( \epsilon \)) by considering \( \rho, z \) as independent of \( \epsilon \). This entails a suitable identification between the different spacetimes for different \( \epsilon \). Such an identification must always be made in order to define metric perturbations. However, the identification is not unique, as we could perform an arbitrary diffeomorphism on each element of the family \( V_\epsilon \) of spacetimes and the diffeomorphisms can obviously depend on \( \epsilon \). This freedom in the identification implies the gauge freedom which is inherent to perturbation theory in general relativity (and indeed in any geometric theory). Thus, when we take perturbations by performing derivatives with respect to \( \epsilon \) with fixed \( \{\rho, z\} \) we are effectively fixing the gauge.

We should consider now which is the domain where the equations (2) hold and what kind of boundary data need to be fulfilled. Given the interior family of metrics, the matching conditions fix (for every \( \epsilon \)) two functions \( \rho_\epsilon(\mu) \) and \( z_\epsilon(\mu) \) defining the matching hypersurface (i.e. the boundary of the body) for every \( \epsilon \). They can also be seen as two-surfaces defined in Euclidean 3-space \( \mathbb{E}^3 \) with the flat metric \( \gamma \) written in cylindrical coordinates. If \( \epsilon \) is close enough to \( \epsilon = 0 \) we have a family of axially symmetric surfaces \( \Sigma_\epsilon \) in \( \mathbb{E}^3 \), all of them diffeomorphic to the surface corresponding to the static background, which will be denoted by \( \Sigma_0 \). We will furthermore assume that the background surface is diffeomorphic to a sphere, which is physical reasonable (and certainly true whenever the static background is spherically symmetric).
Let us also choose the range of variation of $\mu$ so that there exist two fixed values $\mu_S$ and $\mu_N$ such that $\rho_\epsilon(\mu_S) = \rho_\epsilon(\mu_N) = 0$ for all $\epsilon$, i.e. that the intersection points of the surfaces with the axis of symmetry occurs at the same value of $\mu$. As explained before, the matching conditions together with the interior metrics provide us with four functions, which we denote $\mu$ with the axis of symmetry occurs at the same value of $r$. Asymptotic flatness demands $\lim_{r \to \infty} \psi = 0$ for any Cauchy data (i.e. for any interior perturbation). Thus we need to address the question of an overdetermined system and we should not expect asymptotically flat solutions to exist for problem involves an elliptic system of equations with Cauchy boundary data. Again this is

4 Compatibility conditions

In the previous section we have seen that, as in the non-linear case, the perturbed exterior problem involves an elliptic system of equations with Cauchy boundary data. Again this is an overdetermined system and we should not expect asymptotically flat solutions to exist for any Cauchy data (i.e. for any interior perturbation). Thus we need to address the question of what is the set of necessary and sufficient conditions that the boundary data must satisfy so that solutions exist. Asymptotic flatness demands $\lim_{r \to \infty} U_0 = \lim_{r \to \infty} \Omega_0 = \lim_{r \to \infty} U_0 = \lim_{r \to \infty} \Omega_0 = 0$. The perturbed Ernst equations can be collectively written as

$$\Delta_\gamma \psi = 0,$$

where $u = u(\rho, z)$ stands for $U_0$, $U_0'$, etc..., and $j = j(\rho, z)$ represents the inhomogeneous terms in the second order perturbation equations. The metric $\hat{\gamma}$ corresponds to either $\gamma$, for the $U$-equations, or $\hat{\gamma} \equiv e^{-U_0} \gamma$, for the $\Omega$-equations. The domain $(D_0, \hat{\gamma})$ is clearly unbounded because $\hat{\gamma}$ is an asymptotically flat metric. Thus, the compatibility conditions for the boundary values of $U_0'$, $U_0''$, $\Omega_0'$, $\Omega_0''$ can be studied as particular cases for the compatibility conditions of the Cauchy problem for the general inhomogeneous Poisson equation (3) defined on an unbounded asymptotically flat region $(D_0, \hat{\gamma})$ with boundary $\Sigma_0 = \{ \rho = \rho_0(\mu), z = z_0(\mu), \phi = \phi \}$, corresponding to the boundary of the static background metric. Furthermore, we will assume that $j$ tends to zero at infinity at least like $1/r^4$ (which follows in our case from asymptotic flatness).

In order to give a flavor of why Theorem 1 holds, let us concentrate on the simplest possible case, i.e. when $\hat{\gamma} = \gamma$ and the source term $j$ vanishes. A simple consequence of Gauss’ theorem is the so-called Green identity, which reads: for any compact domain $K \subset D_0$ and any function $\psi$ (both suitably differentiable)

$$\int_K (\psi \Delta_\gamma u - u \Delta_\gamma \psi) \eta_\gamma = \int_{\partial K} [\psi n_\gamma(u - u) \psi] dS_\gamma,$$

where $n_\gamma$ is a unit (with respect to $\gamma$) normal vector pointing outside $K$, $\eta_\gamma$ is the volume form of $(D_0, \gamma)$ and $dS_\gamma$ is the induced surface element of $\partial K$. We intend to apply this
identity to a function \( \psi \) that (i) solves the Laplace equation \( \triangle \gamma \psi = 0 \) on \( D_0 \), (ii) admits a \( C^1 \) extension to \( \partial D_0 = \Sigma_0 \) and (iii) it decays at infinity in such a way that \( r \psi \) is a bounded function on \( D_0 \). A function \( \psi \) satisfying these three properties is called a regular \( \gamma \)-harmonic function on \( D_0 \). If such a function satisfies (ii) and (iii) and is \( C^2 \) on \( D_0 \) we will call it regular. For such a function we may take \( K = D_0 \) in (4) because the integral at the boundary “at infinity” can be easily shown to vanish. Thus, denoting the boundary data of \( u \) on \( \Sigma_0 \) as \( u|_{\Sigma_0} \equiv f_0 \) and \( n_\gamma(u)|_{\Sigma_0} \equiv f_1 \) of \( u \), the expression above becomes

\[
\int_{\Sigma_0} \left[ \psi f_1 - f_0 n_\gamma(\psi) \right] \, dS_\gamma = 0. \tag{5}
\]

These are obviously necessary conditions that the overdetermined boundary data must satisfy in order for a regular solution \( u \) of the Laplace equation to exist. It is natural to ask whether such conditions are also sufficient. More precisely, give two arbitrary continuous functions \( f_0 \) and \( f_1 \) on \( \Sigma_0 \) which satisfy (5) for any choice of regular \( \gamma \)-harmonic function \( \psi \). We want to check whether there always exists a function \( u \) satisfying the Laplace equation \( \triangle u = 0 \), with \( ru \) bounded at infinity and such that the boundary equations \( u|_{\Sigma_0} = f_0 \), \( n_\gamma(u)|_{\Sigma_0} = f_1 \) are satisfied. The answer is yes as we show next. Consider the Dirichlet problem \( \triangle u = 0 \) with \( u|_{\Sigma_0} = f_0 \). Standard elliptic theory tells us that this problem always admits a unique solution \( u \) which tends to zero at infinity. Let us define \( \tilde{f}_1 \) on \( \Sigma_0 \) as \( \tilde{f}_1 \equiv n_\gamma(u)|_{\Sigma_0} \). Since \( u \) solves the Laplace equation, expression (5) must hold replacing \( f_1 \) with \( \tilde{f}_1 \). Furthermore, our assumption is that \( f_0 \) and \( f_1 \) satisfy (5) for any regular \( \gamma \)-harmonic function \( \psi \). Subtracting both expressions we get

\[
\int_{\Sigma_0} \psi(f_1 - \tilde{f}_1) dS_\gamma = 0.
\]

However, since the regular \( \gamma \)-harmonic function \( \psi \) is arbitrary and the Dirichlet problem for the Laplace equation always admits a solution, we have that \( \psi|_{\Sigma_0} \) is an arbitrary continuous function. This readily implies \( f_1 = \tilde{f}_1 \) and hence compatibility of the overdetermined boundary data.

However, condition (5) has a serious practical disadvantage, namely that it must be checked for an arbitrary decaying solution \( \psi \) of the Laplace equation. This makes it not useful in practical terms. Our aim is to reduce the number of solutions \( \psi \) that must be checked in (5) while still implying compatibility of \( f_0 \) and \( f_1 \). Here is where axial symmetry plays an essential role. This allows us to reduce the number of conditions to be checked to just a one-parameter family set. Without going into the details, let us just state that, in the axially symmetric case, conditions (5) restricted to the set of functions

\[
\psi_y(\rho, z) = 1/\sqrt{\rho^2 + (z - y)^2}, \tag{6}
\]

where \( y \) is a constant such that the point \( \{ \rho = 0, z = y, \phi \} \) lies outside the domain \( D_0 \), are already necessary and sufficient for existence of the solution \( u \). Thus the number of conditions reduces dramatically and becomes manageable. The main result we present in this contribution is the generalization of this result to the four cases corresponding to the perturbed Ernst equations. The integrals to be performed are, of course, more complicated in general but the idea is still the same. In order to write down our main theorem we need to introduce some notation. First of all, define \( z_S < z_N \) as the values of \( z \) at the intersection of \( \Sigma_0 \) with the axis of symmetry (i.e. the south and north poles respectively) and restrict \( y \) to the interval \( (z_S, z_N) \). We define an angle \( \gamma_y \in [0, 2\pi] \) by \( \cos \gamma_y(\rho, z) \equiv (z - y)/\sqrt{\rho^2 + (z - y)^2}, \sin \gamma_y(\rho, z) \equiv \rho/\sqrt{\rho^2 + (z - y)^2} \) and three functions \( W_y, Q_+ \) and \( Q_- \) as the unique vanishing at infinity solutions of the following compatible PDEs

\[
\begin{align*}
\text{d}W_y &= \cos \gamma_y \text{d}U_0 + \sin \gamma_y \star \text{d}U_0, \\
\text{d}Q_+ &= e^{-2U_0} e^{2W_y} \left[ (1 - \cos \gamma_y) \text{d}Q_0 - \sin \gamma_y \star \text{d}Q_0' \right].
\end{align*}
\]

Here \( \star \) means Hodge dual with respect to the metric \( dz^2 + d\rho^2 \). Then, we have
Theorem 1. Let $f_0, f_1$ be continuous axially symmetric functions on a $C^1$ simply connected, axially symmetric surface $\Sigma_0$ of $\mathbb{R}^3$. Let this surface be defined in cylindrical coordinates by \( \rho = \rho_0(\mu), \ z = z_0(\mu), \ \phi = \phi \), where $\mu \in [\mu_S, \mu_N]$ and $\mu_S < \mu_N$ are the only solutions of $\rho_0(\mu) = 0$. Call $z_S \equiv z(\mu_S)$ and $z_N \equiv z(\mu_N)$ and assume $z_S < z_N$ (i.e. that these values correspond to the “south” and “north” poles of the surface, respectively). Denote by $\Delta$ \( (i) \) the Cauchy boundary value problem

\[
\int_{\mu} \left[ \psi_y f_1 - f_0 n(\psi_y) \right] \rho_0 d\mu = 0, \quad \forall y \in (z_S, z_N),
\]

(ii) the Cauchy boundary value problem $\Delta_2 \Omega'_0 - 4 (d\Omega'_0, dU_0)_\gamma = 0, \Omega'_0|_{\Sigma_0} = f_0, n(\Omega'_0)|_{\Sigma_0} = f_1$ admits a regular solution on $D_0$ if and only if

\[
\int_{\mu} \left[ \psi_y f_1 - f_0 n(\psi_y) \right] \rho_0 e^{-4U_0}|_{\Sigma_0} d\mu = 0, \quad \forall y \in (z_S, z_N),
\]

(iii) the Cauchy boundary value problem $\Delta_2 U''_0 + e^{-4U_0} (d\Omega'_0, d\Omega'_0)_\gamma = 0, U''_0|_{\Sigma_0} = f_0, n(U''_0)|_{\Sigma_0} = f_1$ admits a regular solution on $D_0$ if and only if

\[
\int_{\mu} \left[ \psi_y f_1 - f_0 n(\psi_y) - T_1(n) \right] \rho_0 d\mu = 0, \quad \forall y \in (z_S, z_N),
\]

and (iv) the Cauchy boundary value problem $\Delta_2 \Omega''_0 - 8 (d\Omega'_0, dU'_0)_\gamma - 4 (d\Omega''_0, dU_0)_\gamma = 0, \Omega''_0|_{\Sigma_0} = f_0, n(\Omega''_0)|_{\Sigma_0} = f_1$ admits a regular solution on $D_0$ if and only if

\[
\int_{\mu} \left[ \left( \psi_y f_1 - f_0 n(\psi_y) \right) e^{-4U_0}|_{\Sigma_0} - T_2(n) \right] \rho_0 d\mu = 0, \quad \forall y \in (z_S, z_N),
\]

where $\psi_y$ is given in (6), $\psi_y = \frac{2U_0 - 2W_0}{\sqrt{\rho^2 + (z-y)^2}}$, $T_1 \equiv \frac{1}{2} \rho Q_+ \star dQ_-$ and $T_2 \equiv \frac{1}{2} \rho Q_- \star d(W'_0 + U'_0)$.

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