SOME OPTIMIZATIONS FOR (MAXIMAL) MULTIPLIERS IN $L^p$

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Abstract. Using the Multi-Frequency Calderón-Zygmund decomposition of [6], and a discretization suggested in the arguments of [4] we lower estimates of [6] and [4]. We also use these techniques to improve the results of [3].

1. Introduction

In his celebrated paper [1], Bourgain proved the following:

Theorem 1.1 (Lemma 4.11 of [1]). For any collection of $N$ frequencies, $\Sigma := \{\xi_1, \ldots, \xi_N\} \subset \mathbb{R}$, then

$$\sup_j \left\| \left( \hat{f} 1_{R_j} \right)^\vee (x) \right\|_{L^2(\mathbb{R})} \lesssim \log^2 N \|f\|_{L^2(\mathbb{R})},$$

where $R_j$ is the $2^{-j}$ neighborhood of $\Sigma$. (We refer the reader to §1.2 for the precise definition of the $\lesssim$ notation)

This beautiful lemma was proven via a robust argument which combined Fourier analysis and a metric entropy approach; it proved a key ingredient in his treatment of pointwise polynomial ergodic theorems. We further remark that this result is essentially sharp [2], since it has been shown that at least a power of $\log^{1/4} N$ is needed on the right-hand side.

In recent years, this result has been generalized in a number of ways. In [5, Theorem 8.7] a weighted ($L^2$-) version of the above result was used to extend Bourgain’s Return Times theorem. In [4, Theorem 1.5], the weighted result was extended to the lower $L^p$ regime, $1 < p \leq 2$ by using elegant time-frequency techniques. The boundary $p = 1$ case remained out of reach, however, until Nazarov, Oberlin and Thiele developed a Multi-Frequency Calderón Zygmund decomposition [6]. There, they further extended the $L^2$-weighted result of [5] to the variational setting [6, Theorem 1.2], and additionally studied the $L^1$ endpoint [6, Theorem 5.1].

In this note, we improve the main results of the above papers [4, Theorem 1.5], [6, Theorem 1.2] (see below for the precise statements). We do so by establishing an optimal estimate for some metric entropy calculations (see Lemma 2.3 for a precise statement) and through more delicate use of the Multi-Frequency Calderón Zygmund decomposition (see §3).

Our main $L^2$-result (§2) is the following
Proposition 1.2. Suppose that \( \Sigma := \{\xi_1, \ldots, \xi_N\} \subset \mathbb{R} \) are 1-separated frequencies, and define
\[
D_k(f) := \sum_{k=1}^{N} \int \hat{f}(\xi) \overline{\phi_{2^k}(\xi - \xi_j)} e(\xi x) \, d\xi,
\]
where we abbreviate \( e(t) := e^{2\pi it} \). Then, for \( q > 2 \),
\[
\|V^q(D_k(f)) \|_{L^2(\mathbb{R})} \lesssim (\log N)^2 \cdot \left( 1 + \frac{q}{q-2} \right)^2 \cdot \|f\|_{L^2(\mathbb{R})},
\]
where the \( q \)-variation is over finite increasing sequences; we refer the reader below for a more involved discussion of the \( q \)-variation norm.

We define the (non-homogeneous) \( q \)-variation of the sequence of functions \( \{D_k(f)\} \)
\[
V^q(D_k(f))(x) := \sup_k |D_k(f)| + \left( \sup_{k_1 < k_2 < \cdots < k_m} \sum_m |D_{k,m} f - D_{k,m+1} f|^q \right)^{1/q}(x),
\]
where the supremum is taken over finite increasing sequences; we refer the reader below for a more involved discussion of the \( q \)-variation norm.

We also include a simplified – and optimized – proof of [6, Theorem 5.1]; this proof is easily adapted to study the \( L^1 \) endpoint treated in [4]. We prove the following

Proposition 1.3. Suppose \( \Sigma := \{\xi_1, \ldots, \xi_n\} \subset \mathbb{R} \) are not necessarily 1-separated, and define \( D_k(f) \) as above. Then, for \( q > 2 \),
\[
\|V^q(D_k(f))\|_{L^1(\mathbb{R})} \lesssim \sqrt{N} \cdot \log^3 N \cdot \left( 1 + \frac{q}{q-2} \right)^2 \|f\|_{L^1(\mathbb{R})}.
\]

Remark 1.4. The prefactor \( \sqrt{N} \) is sharp, and though suggested in the work of [6] Theorem 5.1, was (narrowly) missed there.

The techniques used in the proof of the above proposition robust enough to prove useful in studying the behavior of Fourier multipliers with bounded \( r \)-variation. We briefly discuss our main result in this direction (see §4 for greater depth):

We define the (non-homogeneous) \( r \)-variation norm, the \( V^r \)-norm, of a function
\[
\|h\|_{V^r(\mathbb{R})} := \|h\|_{L^\infty(\mathbb{R})} + \left( \sup_{\xi_1 < \xi_2 < \cdots < \xi_k} \sum_k |h(\xi_k) - h(\xi_{k+1})|^r \right)^{1/r},
\]
where the supremum is over all strictly increasing finite sequences of real numbers.

For \( X \) a finite collection of disjoint subintervals of \( \mathbb{R} \), and \( \{m_\omega : \omega \in X\} \) a collection of functions with \( \hat{m}_\omega \) supported in \( \omega \), and \( \|\hat{m}_\omega\|_{V^r(\mathbb{R})} \) uniformly bounded in \( \omega \), we consider operators of the form
\[
\mathcal{M}f := \left( \sum_\omega \hat{m}_\omega \hat{f} \right)^\vee.
\]

In [4], it was proven that for any \( r \geq 2 \), \( \left| \frac{1}{q} - \frac{1}{2} \right| < \frac{1}{r} \), and \( \epsilon > 0 \), there exist absolute constants \( C_{q,r,\epsilon} \) so that
\[
\|\mathcal{M}f\|_{L^q(\mathbb{R})} \leq C_{q,r,\epsilon} |X|^\left\lfloor \frac{1}{r} - \frac{1}{2} + \epsilon \right\rfloor \sup_\omega \|\hat{m}_\omega\|_{V^r(\mathbb{R})} \|f\|_{L^q(\mathbb{R})}.
\]
We prove that the $\epsilon$ is extraneous:

**Theorem 1.5.** There exist absolute constants $C_{r,q}$ so that

$$\|Mf\|_{L^q(R)} \leq C_{q,r} |X|^{\frac{1}{2} - \frac{1}{q}} \|\hat{m}_\omega\|_{V^r(R)} \|f\|_{L^q(R)}.$$ 

We prove this result by studying the $L^1$-endpoint behavior of simpler multiplier operators (§4). Our approach, however, just fails to shed light on the endpoint question raised in [9]:

**Problem 1.6.** For $\|\hat{m}_\omega\|_{V^2(R)} \leq 1$ uniformly bounded, what is the best $|X|$-dependent bound, $C_1(|X|)$, in the weak-type inequality

$$\|Mf\|_{1,\infty} \leq C_1(|X|) \|f\|_1?$$

We look forward to addressing this problem in further research.

The structure of the paper is as follows:
Below, we present our improvements to [6] and [4].
In §2, we develop our $L^2$-theory;
In §3, we turn to the end-point $L^1$ question; and
In §4, we discuss multipliers with bounded $r$-variation.

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1.2. **Notation.** Throughout, we will make use of the exponential notation

$$e(t) := e^{2\pi it}.$$ 

For an interval $I$, we let $CI$ denote the concentric interval dilated by a factor of $C$:

$$I = (-l + c, c + l) \Rightarrow CI = (-C + c, c + Cl).$$ 

In the first sections, with $q > 2$ fixed, we also shall assume $N = O_q(1)$ is large – say $\log N \frac{q}{q - 2} \geq 10$; in what follows, the modifications necessary to lift this assumption are straightforward.

We will also make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$ to denote the estimate $X \leq CY$ for an absolute constant $C$. If we need $C$ to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_r Y$ denotes the estimate $X \leq C_r Y$ for some $C_r$ depending on $r$. We use $X \approx Y$ as shorthand for $X \lesssim Y \lesssim X$.

1.3. **Preliminaries.** Begin by fixing a band-limited Schwartz function $\phi \in S(\mathbb{R})$, with supp $\hat{\phi} \subset [-1/2, 1/2]$, and let $\phi_t(x) := \frac{1}{t} \phi(x/t)$ denote the usual $L^1$-normalized dilation.

We recall two families of operators which measure the fluctuation of a sequence of vectors:

For a collection of vectors in a Hilbert space $\{c_k\} \subset H$, let $M_\lambda(\{c_k\})$ denote the $H$-$\lambda$-entropy of the $\{c_k\}$, i.e. the fewest number of $\lambda$-balls inside of $H$ required to cover the $\{c_k\}$. 
We also introduce the homogeneous and non-homogeneous variation operators, respectively defined below:

\[
\tilde{V}_q^q \left( \{ c_k \} \right) := \sup_{k \geq k_1 < k_2 < \cdots < k_M} \left( \sum_{m} \| c_{k_m} - c_{k_{m-1}} \|_H^q \right)^{1/q},
\]

\[
V_q^q \left( \{ c_k \} \right) := \sup_k \| c_k \|_H + \sup_{k \geq k_1 < k_2 < \cdots < k_M} \left( \sum_{m} \| c_{k_m} - c_{k_{m-1}} \|_H^q \right)^{1/q}.
\]

When \( H = \mathbb{C} \), we shall omit the subscript. We note that for any \( \lambda > 0 \),

\[
\lambda M_\lambda \left( \{ c_k \} \right)^{1/q} \lesssim \tilde{V}_q^q \left( \{ c_k \} \right).
\]

We will prove the following

**Proposition 1.7.** Suppose that \( \Sigma := \{ \xi_1, \ldots, \xi_N \} \subset \mathbb{R} \) are \( 1 \)-separated frequencies, and define

\[
D_k(f) := \sum_{k=1}^N \int \hat{f}(\xi) \hat{\phi}_{2^k}(\xi - \xi_j) e(\xi x) \, d\xi
\]

\[
= \sum_{k=1}^N e(\xi_j x) \int \hat{f}(\xi + \xi_j) \hat{\phi}_{2^k}(\xi) e(\xi x) \, d\xi.
\]

Then, with \( q > 2 \),

\[
\left\| \nabla^q(D_k(f) \mid k \geq 1) \right\|_{L^2(\mathbb{R})} \lesssim (\log N)^2 \cdot \left( 1 + \frac{q}{q-2} \right)^2 \cdot \| f \|_{L^2(\mathbb{R})}.
\]

By using a Rademacher-Menshov style argument used in [1, Lemma 4.32] or [6, Theorem 4.3], one may lift the frequency-separation hypothesis at the cost of a further logarithm:

**Proposition 1.8.** For any frequencies \( \Sigma := \{ \xi_1, \ldots, \xi_N \} \) and any dyadic (frequency) interval \( \omega \), let \( \xi_\omega \) denote the smallest frequency \( \xi_j \in \Sigma \cap \omega \) which lies in \( \omega \), and let

\[
\hat{\phi}_\omega(\xi) := \hat{\phi} \left( \frac{\xi - \xi_\omega}{|\omega|} \right).
\]

In this instance, define

\[
D_k(f) := \sum_{|\omega| = 2^{-k}, \omega \cap \Sigma \neq \emptyset} \int \hat{f}(\xi) \hat{\phi}_\omega(\xi) e(\xi x) \, d\xi.
\]

Then, with \( q > 2 \),

\[
\| \nabla^q(D_k(f)) \|_{L^2(\mathbb{R})} \lesssim (\log N)^3 \cdot \left( 1 + \frac{q}{q-2} \right)^2 \cdot \| f \|_{L^2(\mathbb{R})}.
\]

For the sake of clarity we have contented ourselves with the above results; it should be noted, however, that by arguing as in [6 §4.1-4.2], an improvement of [6 Proposition 4.2] can be achieved:

If

\[
\Delta_k(f) := \sum_{|\omega| = 2^{-k}, \omega \cap \Sigma \neq \emptyset} f \ast m_\omega,
\]

where \( \{ m_\omega \} \) are Schwartz functions with \( \widehat{m}_\omega \subset \omega \),
then we have the following

**Corollary 1.9.** For any choice of \( N \) frequencies, \( \Sigma := \{ \xi_1, \ldots, \xi_N \} \subset \mathbb{R} \), there exists a \( t = t(q) > 2 \) so that

\[
\| \mathcal{V}(\Delta_k(f)) \|_{L^2_x(\mathbb{R})} \lesssim (\log N)^3 \cdot \left( 1 + \frac{q}{q-2} \right)^2 (D_2' + V^t) \| f \|_{L^2_x(\mathbb{R})},
\]

where \( V^t := \sup_j \mathcal{V}^t \left( \sum_{|\omega|=2-k} \hat{m}_\omega(\xi_j) \right) \), and where \( D_2' := \sup_{\omega, \xi} |\omega|^2 |\hat{m}_\omega(2)\omega(\xi)| \).

We also consider the \( L^1 \)-endpoint behavior of the (non-frequency separated) operator \( \mathcal{V}(D_k(f)) \), \( q > 2 \). We prove

**Proposition 1.10.** Under the non-frequency separated hypotheses of Proposition \( L.8 \) with \( q > 2 \)

\[
\| \mathcal{V}(D_k(f)) \|_{L^{1,\infty}(\mathbb{R})} \lesssim \sqrt{N} \cdot \log^3 N \cdot \left( 1 + \frac{q}{q-2} \right)^2 \| f \|_{L^1_x(\mathbb{R})}.
\]

Our proof relies on the Multi-Frequency Calderón-Zygmund decomposition of \( \mathcal{V}, \) and indeed is similar to that of \( L.8 \) Theorem 5.1. Our modifications were motivated by the outstanding question of \( D.5 \) concerning the \( L^1 \)-endpoint behavior of the operator

\[
\sup_{k \geq 5} |\Delta_k f|.
\]

Indeed, an easy adaptation of our methods yields that for \( 1 < p \leq 2 \)

\[
\left\| \sup_{k \geq 5} |\Delta_k f| \right\|_{L^p(\mathbb{R})} \lesssim t \cdot N^{1/p-1/2} \log^2 N \| f \|_{L^p(\mathbb{R})},
\]

where the implicit constants depend on a certain \( V^t \)-quantity in the above notation. This result represents a modest improvement over the current bound of (essentially) \( \lesssim \cdot N^{1/p-1/2+\epsilon} \) for \( \epsilon > 0 \). We refer the reader to \( D.5 \) Theorem 1.5] for the precise statement.

2. The \( L^2 \) Estimate

In this section, we work under the frequency-separated hypothesis. The modifications to pass to the general case can be found in \( L.5 \) Theorem 4.3].

2.1. Preliminaries. We shall make use of the following variational results which appear in \( D.3 \).

**Lemma 2.1** (Lemma 3.30 of \( D.5 \)). With \( f := (f_1, f_2, \ldots, f_N) \) an \( N \)-tuple of functions, define pointwise

\[
M_\lambda(x) := M_\lambda(f * \phi_t(x)|t > 0).
\]

Then, with \( C_\phi := \|x\phi'(x)\|_1 \), and \( r > 2 \), we have

\[
\left\| \sup_{\lambda > \phi} \lambda M_\lambda^{1/r}(x) \right\|_{L^2(\mathbb{R})} \leq C_\phi \frac{1}{r-2} \| f \|_H = C_\phi \frac{1}{r-2} \left( \sum_{j=1}^N |f_j|^2 \right)^{1/2} \left\| \frac{1}{r-2} \left( \sum_{j=1}^N |f_j|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R})}.
\]
Lemma 2.2 (Lemma 3.33 of [1]). Maintaining the notation of the previous lemma, with \( N > 0 \) we have the bound
\[
\left\| \int_0^\infty \min \left\{ M_\lambda^{1/2}(x), N^{1/2} \right\} \, d\lambda \right\|_{L_2^2(\mathbb{R})} \lesssim C_0 (\log N)^2 \left\| \left( \sum_{j=1}^N |f_j|^2 \right)^{1/2} \right\|_{L_2^2(\mathbb{R})}.
\]

To the best of our knowledge, the following technical result has yet to appear in print.

Lemma 2.3. With the above notation,
\[
\left\| \int_0^\infty \min \left\{ M_\lambda(1/2), N^{1/2} M_\lambda(x)^{1/q} \right\} \, d\lambda \right\|_{L_2^2(\mathbb{R})} \lesssim \left( \log N - \frac{q-2}{q} \right)^2 \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L_2^2(\mathbb{R})}.
\]

Proof. Let
\[
F(x) := K_\phi \cdot \left( \sum_{j=1}^N |\mathcal{M}(f_j)|^2 \right)^{1/2}
\]
denote an appropriate amplification of the vector-valued maximal function, where \( K_\phi \) depends only on the least radially-decreasing majorant of the Schwartz function \( \phi \).

Set
\[
\theta := \frac{1}{\log N}, \quad a := \frac{q-2}{2q}, \quad C := N^{\frac{q}{2}} = N^{\frac{q-2}{q}};
\]
and majorize
\[
\int_0^\infty \min \left\{ M_\lambda(1/2), N^{1/2} M_\lambda(x)^{1/q} \right\} \, d\lambda
\]
\[
\leq N^{\theta/2} \int_0^{F(x)/C} \lambda^{1-a} M_\lambda(x)^{1-a + \frac{\theta}{q}} \, d\lambda + N^{\theta/2} \int_{F(x)/C}^{F(x)} \lambda M_\lambda(x)^{1-a + \frac{\theta}{q}} \, d\lambda
\]
\[
=: I_1(x) + I_2(x).
\]

We begin with \( I_1(x) \):
\[
I_1(x) \leq N^{\theta/2} \cdot \left( \sup_{\lambda > 0} \lambda^{1-a} M_\lambda(x) \left( \frac{2q}{q-q\theta + 2\theta} \right)^{-1} \right) \cdot \int_0^{F(x)/C} \frac{d\lambda}{\lambda^{1-a}}
\]
\[
\leq N^{\theta/2} \cdot \left( \sup_{\lambda > 0} \lambda^{1-a} M_\lambda(x) \left( \frac{2q}{q-q\theta + 2\theta} \right)^{-1} \right) \cdot \frac{1}{a} \left( F(x)/C \right)^a
\]
\[
= \frac{1}{a} \left( \sup_{\lambda > 0} \lambda^{1-a} M_\lambda(x) \left( \frac{2q}{q-q\theta + 2\theta} \right)^{-1} \right) \cdot (F(x))^a
\]
\[
\leq \frac{1}{a} \left( \sup_{\lambda > 0} \lambda M_\lambda(x) \left( \frac{2q-2a}{(q-q\theta + 2\theta)} \right)^{-1} \right) + F(x),
\]
where we used Young’s inequality with exponents \( \left( \frac{1}{q}, \frac{1}{t} \right) \) in the final line. With
\[
t := \frac{2q(1-a)}{q-q\theta + 2\theta} = \frac{2q \log N - (q-2)}{q \log N - (q-2)} > 2,
\]
\[
\frac{1}{t-2} = \frac{q-q\theta + 2\theta}{2q\theta - 2qa - 4\theta} = \frac{q \log N - (q-2)}{2(q-2 - \frac{q-2}{2})} \leq \log N \frac{q}{q-2}
\]
and Lemma 2.1 in mind, we take $L^2$-norms and estimate

$$
\|I_1\|_2 \lesssim \frac{1}{a} \left( \log N \frac{q}{q-2} \right) \left( \sum_j \|f_j\|_{L^2_x}^2 \right)^{1/2} + \frac{1}{a} \left( \sum_j \|f_j\|_{L^2_x}^2 \right)^{1/2} 
$$

$$
\lesssim \left( \log N \frac{q}{q-2} \right)^2 \left\| \sum_j |f_j|^2 \right\|_{L^2_x}^{1/2}.
$$

We now turn to the similar, but slightly simpler, second component:

$$
I_2(x) \leq N^{1/2} \left( \sup_{\lambda > 0} \lambda M_{1/2}(x) \left( \frac{2q}{q-q\theta+2\theta} \right)^{-1} \right) \log C
$$

$$
\lesssim \frac{\theta}{2a} \log N \left( \sup_{\lambda > 0} \lambda M_{1/2}(x) \left( \frac{2q}{q-q\theta+2\theta} \right)^{-1} \right)
$$

$$
= \frac{q}{q-2} \log N \left( \sup_{\lambda > 0} \lambda M_{1/2}(x) \left( \frac{2q}{q-q\theta+2\theta} \right)^{-1} \right).
$$

The argument is concluded as above, this time with

$$
t := \frac{2q}{q-q\theta+2\theta} = \frac{2q \log N}{\log N - (q-2)} > 2,
$$

$$
\frac{1}{t-2} = \frac{q-q\theta+2\theta}{2q\theta-4\theta} = \frac{q \log N - (q-2)}{2(q-2)} \leq \frac{\log N}{q-2},
$$

we have

$$
\|I_2\|_2 \lesssim \left( \log N \frac{q}{q-2} \right)^2 \left\| \sum_j |f_j|^2 \right\|_{L^2_x}^{1/2}.
$$

The previous lemma will be used in conjunction with the lemma below to yield Proposition 1.7.

**Lemma 2.4.** For a sequence

$$\{c_k(1), \ldots, c_k(N)\}_{k \geq 1} \subset \mathbb{R}^N,$$

let $M_{\lambda} = M_{\lambda}(\{c_k\})$ denote the $L^2([N])$-entropy of the collection, where $L^2([N])$ is the usual $N$-dimensional Hilbert space. Suppose that $\{\xi_1, \ldots, \xi_N\}$ are 1-separated frequencies.

Then, for $q > 2$,

$$
\left\| \sum_{j=1}^N c_k(j)e(\xi_jy) \ |k \geq 1 \right\|_{L^q_x[0,1]}
$$

$$
\lesssim \int_0^\infty \min \left\{ M_{\lambda}^{1/2}, N^{1/2}M_{\lambda}^{1/q} \right\} \ d\lambda + \int_0^\infty \min \left\{ M_{\lambda}^{1/2}, N^{1/2} \right\} \ d\lambda.
$$

**Proof.** The first term on the right hand side is from the homogeneous $q$-variation, which comes from the proof of [3, Lemma 3.2]. For the second term, which extends the result to non-homogeneous $q$-variation, see the proof of [5, Lemma 8.4].
We are now ready for the proof of our $L^2$ result.

2.2. The Proof.

**Proposition 2.5.** Suppose that $\{\xi_1, \ldots, \xi_N\}$ are 1-separated frequencies, and define

$$D_k(f) := \sum_{k=1}^{N} \int \hat{f}(\xi) \phi_{2^k}(\xi - \xi_j)e(\xi x) \, d\xi$$

$$= \sum_{k=1}^{N} e(\xi_j x) \int \hat{f}(\xi + \xi_j) \phi_{2^k}(\xi)e(\xi x) \, d\xi.$$ 

Then

$$\|\mathcal{V}^q(D_k(f) \mid k \geq 1)\|_{L^2_2(\mathbb{R})} \lesssim (\log N)^2 \cdot \left(1 + \frac{q}{q - 2}\right) \cdot \|f\|_{L^2_2(\mathbb{R})}.$$ 

This proposition, and its proof, are very similar to [6, Proposition 4.1], which is in turn inspired by [1, Lemma 4.11]. The only novelty is the use of Lemma 2.3.

**Proof.** Define, via the Fourier transform,

$$\hat{f}_j(\xi) := \hat{f}(\xi + \xi_j)\hat{\psi}(\xi),$$

where $1_{[-1/4,1/4]} \lesssim \hat{\psi} \lesssim 1_{[-1/2,1/2]}$ (say), and note that by the separation of the $\{\xi_j\}$ and Plancherel,

$$\left\| \left(\sum |f_j|^2\right)^{1/2} \right\|_{L^2} \leq \|f\|_2.$$

With this notation in hand, we may express

$$D_k(f) = \sum_{k=1}^{N} e(\xi_j x) \int \hat{f}_j(\xi) \phi_{2^k}(\xi)e(\xi x) \, d\xi = \sum_{k=1}^{N} e(\xi_j x)f_j * \phi_{2^k}(x);$$

it is therefore enough to prove the vector-valued estimate

$$\|\mathcal{V}^q \left( \sum_{k=1}^{N} e(\xi_j x)f_j * \phi_{2^k} \mid k \geq 1 \right)\|_{L^2_2} \lesssim (\log N)^2 \cdot \left(1 + \frac{q}{q - 2}\right)^2 \cdot \left\| \left(\sum_j |f_j|^2\right)^{1/2} \right\|_{L^2_2}.$$ 

[1, Lemma 4.13] allows us to reduce the matter to the homogeneous case:

$$(*) \quad \left\| \mathcal{V}^q \left( \sum_{k=1}^{N} e(\xi_j x)f_j * \phi_{2^k} \mid k \geq 1 \right)\right\|_{L^2_2} \lesssim (\log N)^2 \cdot \left(\frac{q}{q - 2}\right)^2 \cdot \left\| \left(\sum_j |f_j|^2\right)^{1/2} \right\|_{L^2_2},$$

where without loss of generality each $f_j$ has frequency support inside $[-1/2,1/2]$; we use Bourgain’s averaging argument to do so.

Let $B$ denote the best a priori constant satisfying $(*)$, which we know is finite – and indeed $\lesssim \sqrt{N}$ by Fefferman-Stein’s vector-valued Hardy-Littlewood maximal inequality [8, §2.1]. The job is to prove

$$B \lesssim (\log N)^2 \cdot \left(\frac{q}{q - 2}\right)^2.$$
With $T_y(g)(x) := g(x + y)$, we majorize, for $0 \leq y < \frac{1}{100}$

$$
\left\| \hat{\mathcal{V}}^q \left( \sum_{k=1}^{N} e(\xi_j x) f_j \ast \phi_{2^k} \ | k \geq 1 \right) \right\|_{L_y^2} \\
\leq \left\| \hat{\mathcal{V}}^q \left( \sum_{k=1}^{N} e(\xi_j x) (T_y f_j) \ast \phi_{2^k} \ | k \geq 1 \right) \right\|_{L_y^2} + \left\| \hat{\mathcal{V}}^q \left( \sum_{k=1}^{N} e(\xi_j x) (f_j - T_y f_j) \ast \phi_{2^k} \ | k \geq 1 \right) \right\|_{L_y^2} \\
=: S_1(y) + B \left\| \left( \sum_j |(f_j - T_y f_j)|^2 \right)^{1/2} \right\|_{L_y^2} < S_1(y) + \frac{B}{2} \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L_y^2},
$$

where we used the band-limited nature of the $\{f_j\}$ in the last inequality.

Averaging this inequality over $0 \leq y < \frac{1}{100}$, we see that we just need to prove

$$
\left( \int_0^{\frac{1}{100}} |S_1(y)|^2 \ dy \right)^{1/2} \leq \left\| \hat{\mathcal{V}}^q \left( \sum_{k=1}^{N} e(\xi_j y) \cdot (e(\xi_j x)(f_j) \ast \phi_{2^k}(x)) \ | k \geq 1 \right) \right\|_{L_y^2[0,1]} \bigg\|_{L_y^2} \\
\lesssim (\log N)^2 \cdot \left( \frac{q}{q - 2} \right)^2 \cdot \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L_y^2}.
$$

If we apply Lemma 2.21 to the inner integral, we get the bound

$$
\left\| \hat{\mathcal{V}}^q \left( \sum_{k=1}^{N} e(\xi_j y) \cdot (e(\xi_j x)(f_j) \ast \phi_{2^k}(x)) \ | k \geq 1 \right) \right\|_{L_y^2[0,1]} \leq \int_0^{\infty} \min\{M_\lambda^{1/2}(x), N^{1/2} M_\lambda^{1/q}(x)\} \ d\lambda,
$$

where $M_\lambda(x)$ is the (pointwise) $l^2[N]-\lambda$-entropy of

$$(e(\xi_1 x)(f_1) \ast \phi_{2^k}(x), e(\xi_2 x)(f_2) \ast \phi_{2^k}(x), \ldots, e(\xi_N x)(f_N) \ast \phi_{2^k}(x))_{k \geq 1}.$$

Taking $L_y^2$-norms and using Lemma 2.23 now yields the result. \hfill $\square$

3. The Weak-Type $(1 - 1)$ Estimate

As announced, in this section we prove Proposition 1.10 restated below for the reader’s convenience:

**Proposition 3.1.** With $D_k(f) := \sum_{|\omega| = 2^{-k} \omega \in \Sigma \neq 0} \int \hat{f}(\xi) \phi_\omega(\xi) e(\xi x) \ d\xi$,

$$
\left\| \mathcal{V}^q(D_k(f)) \right\|_{L,1,\infty} \lesssim \sqrt{N} \cdot \log^3 N \cdot \left( 1 + \frac{q}{q - 2} \right)^2 \|f\|_{L^1}.
$$

Before turning to the proof proper, we briefly recall the multi-frequency Calderón-Zygmund Decomposition:
3.1. The Multi-Frequency Calderón-Zygmund.

**Theorem 3.2** ([6], Theorem 1.1). For any $f \in L^1(\mathbb{R})$, $\lambda > 0$, there exists a decomposition

$$f = g + b = g + \sum_J b_J$$

for a disjoint collection of (dyadic) intervals $\{J\}$ according to the following properties:

1. $\|f_J\|_1 := \|f I_J\|_1 \lesssim \frac{|J|}{\sqrt{N}}$;
2. $\|g_J\|_2 := \|f_J - b_J\|_2 \lesssim |J|^{1/2} \lambda$;
3. $\|g_J\|_2 \lesssim \sqrt{N} \lambda \|f\|_1$;
4. $\mathrm{supp} \ b_J \subset 3J$, $\sum_J |J| \lesssim N \|f\|_1$;
5. $\|b_J\|_1 \lesssim \lambda |J|$; and
6. $b$ is orthogonal to the frequencies: $\int b_J(x) e(-\xi x) = 0$ for each $J, n$.

It will be convenient to discretize our operator. We do so as follows:

3.2. A Discretization. With $\phi_\omega$ as above, let $\eta \in \mathcal{S}(\mathbb{R})$ be a positive mean-one mollifier with $\mathrm{supp} \ \eta \subset [-0.1,0.1]$ (say), and let $A$ be a Schwartz function with $1_{[-1.4,1.4]} \leq A(\xi) \leq 1_{[-1.6,1.6]}$.

With $B := \eta^\nu$, we use windowed Fourier series as in [6, §6] to express

$$\hat{f}_\varphi_\omega(\xi) = \frac{1}{|\omega|} \sum_{l \in \mathbb{Z}} \left( \int \hat{f}(t) \hat{\varphi}_\omega(t) e\left(\frac{l}{|\omega|} t\right) dt \right) e\left(-\frac{l}{|\omega|} \xi \hat{A}(B)\right) \left(\frac{\xi - \xi_\omega}{|\omega|}\right).$$

Set

$$\phi_{l,\omega}(z) := e(-z \xi_\omega) \frac{1}{|J|} \hat{g} \left(\frac{z - l |J|}{|J|}\right)$$

where $\hat{g}(z) := g(-z)$ denotes reflection about the origin, and $|I| = |\omega|^{-1}$; we note that in the special case $l = 0$, we have

$$\phi_{l,\omega}(z) = e(-z \xi_\omega) \hat{g}_{l}(z).$$

After an argument with the Fourier transform, we have

$$f * \varphi_\omega(x) = \sum_{l \in \mathbb{Z}} \langle f, \phi_{l,\omega} \rangle e(\xi_\omega x) \cdot (AB) \left(\frac{x - l |J|}{|J|}\right) = \sum_{l \in \mathbb{Z}} \langle f, \phi_{l,\omega} \rangle e(\xi_\omega x) \cdot T_{l,|J|}(AB)(x).$$

Consequently, with $2^k = |I| = |\omega|^{-1}$, we may express

$$D_k(f) := \sum_{l \in \mathbb{Z}} \left( \sum_{|\omega| = 2^{-k} |\omega|: \omega \neq 0} \langle f, \phi_{l,\omega} \rangle e(\xi_\omega x) \right) T_{l,|J|}(AB)(x),$$

as a sum of projections.

**Proof of Theorem 1.7.** By homogeneity, it suffices to prove the weak-type estimate at height $\lambda = 1$. We therefore apply the Multi-Frequency Calderón-Zygmund decomposition 3.2 at height 1. By standard arguments, it suffices to show that for each $J$ selected

$$\|\mathcal{Y}(D_k(b_J))\|_{L^1(J^\nu)} \lesssim |J|,$$
where \( J^* := 100J \). By scale and translation invariance, we may assume \( J = [0, 1) \); we proceed to majorize

\[
V^q(D_k(b_J)) \leq \sum_{k \geq 0} |D_k(b_J)| + \sum_{k < 0} |D_k(b_J)|,
\]

according to scale, and estimate each term separately in \( \| - \|_{L^1((J^*)^c)} \).

We begin with the more demanding

**Case 1**: \( k \geq 0 \). We use our discretization to bound

\[
\left\| \sum_{k \geq 0} |D_k(b_J)| \right\|_{L^1} \leq \sum_{l \in \mathbb{Z}} \sum_{|I| = 2^k, k \geq 0} \left( \sum_{|\omega| = 2^{-k} \omega \cap \Sigma \neq \emptyset} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} A \right) \cdot \left( T_{l,|I|} B \right) \|_{L^1} \leq \sum_{l \in \mathbb{Z}} \sum_{|I| = 2^k, k \geq 0} \| T_{l,|I|} B \|_{L^2} \left( \sum_{|\omega| = 2^{-k} \omega \cap \Sigma \neq \emptyset} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} A \right) \|_{L^2} \lesssim \sum_{l \in \mathbb{Z}} \sum_{|I| = 2^k, k \geq 0} |I|^{1/2} \left( \sum_{|\omega| = 2^{-k} \omega \cap \Sigma \neq \emptyset} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} A(x) \right) \|_{L^2}.
\]

We now split our sum

\[
\sum_{|\omega| = 2^{-k} \omega \cap \Sigma \neq \emptyset} := \sum_{i=1}^{10} \sum_{\omega, i} := \sum_{i=1}^{10} \left( \sum_{\omega \cap \Sigma \neq \emptyset} \omega = 2^{-k} [m, m+1), m \equiv i \mod 10 \right),
\]

so that we may bound

\[
\left\| \sum_{|\omega| = 2^{-k} \omega \cap \Sigma \neq \emptyset} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} A(x) \right\|_{L^2} \leq \sum_{i=1}^{10} \left( \sum_{\omega, i} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} A(x) \right) \|_{L^2} = \sum_{i=1}^{10} \left( \sum_{\omega, i} |b_J, \phi_{l,\omega}|^2 |T_{l,|I|} A|^2 \right)^{1/2} \|_{L^2} \leq \frac{1}{|I|^{3/4}},
\]

taking into account the band-limited nature of our function \( A \). Our bounds will be uniform in \( 1 \leq i \leq 10 \), so we suppress \( i \)-dependence throughout.

We begin with the top term, \( l = 0 \), and suppress \( l \)-dependence as well; we will show that for each \( |I| \),

\[
\left\| \left( \sum_{\omega} |\langle b_J, \phi_{\omega} \rangle|^2 |T_{l,|I|} A|^2 \right)^{1/2} \right\|_{L^2} \lesssim \frac{1}{|I|^{3/4}},
\]

since a sum over \( |I| \geq 1 \) will yield the \( l = 0 \) bound

\[
\sum_{k \geq 0} \left\| \sum_{\omega} \langle b_J, \phi_{l,\omega} \rangle e(\xi_\omega x) T_{l,|I|} (AB) \right\|_{L^1} \lesssim 1.
\]
Regarding our scale $|I| = 2^k$ as fixed, we abbreviate
\[
\rho(z) := \tilde{\phi}_{|I|}(z) - \phi_{|I|}(0) = \frac{1}{|I|} \tilde{\phi}(\frac{z}{|I|}) - \frac{1}{|I|} \tilde{\phi}(0),
\]
and use the orthogonality of the $\{b_J\}$ to our frequencies $\{e(-z \xi)\}$ to re-express the integrand as
\[
\left( \sum_\omega |<b_J, \rho(z) e(-z \xi)>|^2 |T_{|I|}A|^2(x) \right)^{1/2};
\]
we remark the mean-value theorem yields the bound
\[
(*) \quad |\rho(z)| \lesssim \frac{1}{|I|^2}
\]
pointwise near $3J$, with implicit constant depending on $\|\phi'\|_{L^\infty}$.

Now, using the triangle inequality, it suffices to estimate the $L^2$ norm of each of the following functions separately:
\[
F(x) := \left( \sum_\omega |<f_J, \rho(z) e(-z \xi)>|^2 |T_{|I|}A|^2(x) \right)^{1/2},
\]
\[
G(x) := \left( \sum_\omega |<g_J, \rho(z) e(-z \xi)>|^2 |T_{|I|}A|^2(x) \right)^{1/2}.
\]

$F$ is easy, since we have the pointwise bound
\[
F(x) \leq \left( \sum_\omega \left( \|f_J\|_1 \cdot \frac{1}{|I|^2} \right)^2 \right)^{1/2} \ |T_{|I|}A|(x) \leq \sqrt{N} \|f_J\|_1 \frac{1}{|I|^2} |T_{|I|}A|(x).
\]

Consequently,
\[
\|F\|_{L^2} \lesssim \sqrt{N} \|f_J\|_1 \frac{1}{|I|^2} |I|^{1/2} \lesssim \frac{1}{|I|^{3/2}}.
\]

We use duality to handle $\|G\|_{L^2}$. Specifically, for an appropriate $\sum \|\psi_\omega\|_{L^2}^2 \leq 1$, we estimate
\[
\|G\|_2 \lesssim \left( \sum_\omega \langle \psi_\omega, T_{|I|}A \rangle \cdot \langle g_J, \rho(z) e(-z \xi) \rangle \right)
\]
\[
= \langle g_J, \sum_\omega \rho(z) e(-z \xi) \cdot \langle \psi_\omega, T_{|I|}A \rangle \rangle
\]
\[
\lesssim \left( \sum_\omega \rho(z) e(-z \xi) \cdot \langle \psi_\omega, T_{|I|}A \rangle \right)_{L^2(3J)}
\]
\[
\leq \left( \sum_\omega \chi(z) \rho(z) e(-z \xi) \cdot \langle \psi_\omega, T_{|I|}A \rangle \right)_{L^2(\mathbb{R})},
\]
where $1_{[-4,4]} \leq \chi \leq 1_{[-5,5]}$ (say) is a smooth bump function, and we have made use of the normalization $\|g_J\|_{L^2} \lesssim 1$, along with the support condition $\text{supp } g_J \subset 3J$. 
We will bound the above expression by the square root of
\[
\sum_\omega \| \rho(z)e(-z\xi_\omega)\chi \|^2_{L^2} \cdot |(\psi_\omega, T_{|I|}A)|^2 + \sum_{\omega \neq \omega'} |(\psi_\omega, T_{|I|}A)K(\omega, \omega')|(\psi_{\omega'}, T_{|I|}A)|
\]
\[\lesssim |I| \left( \sum_\omega \| \rho(z)\chi(z)e(-z\xi_\omega)\|^2_{L^2} \| \psi_\omega \|^2 + \sum_{\omega \neq \omega'} \| \psi_\omega \|_{L^2}K(\omega, \omega')\| \psi_{\omega'} \|_{L^2} \right)
\]
\[= |I|(D + O),\]
where
\[K(\omega, \omega') := \int |\rho(z)\chi(z)|^2e((\xi_{\omega'} - \xi_\omega)z) \, dz.\]
Using (*), the diagonal estimate is straightforward:
\[D \leq \max_\omega \| \rho(z)\chi(z)e(-\xi_\omega z)\|^2_{L^2} \lesssim \frac{1}{|I|^2}.\]

We will use the bound
\[K(\omega, \omega') \lesssim \min \left\{ \frac{1}{|\xi_\omega - \xi_{\omega'}|}, \frac{1}{|\xi_\omega - \xi_{\omega'}|^2} \right\} \frac{1}{|I|^4} \leq \frac{1}{|\xi_\omega - \xi_{\omega'}|^{3/2}} \frac{1}{|I|^3}\]
to control the off-diagonal term. We obtain this estimate by integrating by parts once and twice, and by using that on supp \( \chi \) we have the bounds
\[|\rho| + |\rho'| \lesssim \frac{1}{|I|^2}, \quad |\rho''| \lesssim \frac{1}{|I|^3}.\]
Using the fact that for \( \omega \neq \omega' \)
\[|\xi_\omega - \xi_{\omega'}| > |\omega| = |I|^{-1},\]
we may bound, for each \( \omega \),
\[\sum_{\omega' \neq \omega} K(\omega, \omega') \leq \frac{1}{|I|^3} \sum_{\omega' \neq \omega} \frac{1}{|\xi_\omega - \xi_{\omega'}|^{3/2}} \lesssim \frac{|I|^{3/2}}{|I|^3} \sum_{n=1}^N \frac{1}{n^{3/2}} \lesssim \frac{1}{|I|^{3/2}}.\]

With this and Young’s inequality leads to the bound
\[O \lesssim \sum_{\omega \neq \omega'} \| \psi_\omega \|^2_{L^2}|K(\omega, \omega')| \lesssim \sum_\omega \| \psi_\omega \|^2_{L^2} \frac{1}{|I|^{3/2}} \lesssim \frac{1}{|I|^{3/2}}.\]

Putting things together, we have obtained the estimate
\[|I|(D + O) \lesssim \frac{1}{|I|^3/2};\]
taking a square-root leads to the bound
\[\| G \|_{L^2} \lesssim \frac{1}{|I|^{1/4}}.\]

Combining all our estimates, we may bound
\[\left\| \left( \sum_\omega |(b_J, \phi_\omega)|^2|T_{|I|}A|^2 \right)^{1/2} \right\|_{L^2} \leq \| F \|_{L^2} + \| G \|_{L^2} \lesssim \frac{1}{|I|^{1/4}},\]
which concludes the \( l = 0 \) case.
The remaining terms \( l \neq 0 \) are handled similarly, using the analogous mean-value estimate on \( J \)

\[
(*)_l \left| \frac{1}{|l|} \hat{\phi} \left( \frac{z - l|I|}{|I|} \right) - \frac{1}{|l|} \hat{\phi} \left( \frac{-l|I|}{|I|} \right) \right| \lesssim \frac{1}{|l|^2 |I|^{-2}},
\]

and similarly

\[
|\partial_{\alpha}^2 \left( \frac{1}{|I|} \hat{\phi} \left( \frac{z - l|I|}{|I|} \right) \right) | \lesssim_{\phi} \frac{l^{-2}}{|I|^{1+\alpha}}, \quad \alpha = 1, 2,
\]

where the implicit constants depend further on the decay of \( \phi', \phi'' \).

This concludes the first case, \( |I| \geq |J| = 1 \).

□

We next turn to

Case 2: \( k < 0 \). With \( J = [0, 1) \) as above, and regarding the scale \( |I| = 2^k \) as fixed, we use the decay of the functions \( \phi_\omega \) to estimate

\[
\left\| \sum_{|\omega|=2^{-k}: \omega \cap \Sigma \neq \emptyset} \phi_\omega \ast b_J \right\|_{L^1((J^*)^c)} \leq N \max_\omega \| \phi_\omega \ast b_J \|_{L^1((J^*)^c)} \\
\leq N \|b_J\|_1 \left\| \sup_{y \in J} |\phi_\omega(z-y)| \right\|_{L^1((J^*)^c)} \\
\lesssim N |I|^{M-1} e^{1-M},
\]

where we may take

\[
M = M(N) = \log N,
\]

and the implicit constant depends on the best bound, \( D_M \), satisfying \( |\phi(z)| \leq D_M |z|^{-M} \) as \( |z| \to \infty \). Summing over \( |I| \) concludes the proof.

□

Remark 3.3. As mentioned above, this argument is easily adapted to handle the (simpler) operator

\[
Vf(x) := \left( \sum_{n=1}^N V_{k \geq 5}^n (1_I(x)\langle f, \phi_{I,n} \rangle)^2 \right)^{1/2}
\]

of \([4, \S 5]\). Indeed, one can similarly prove that

\[
\|Vf\|_{L^{1,\infty}} \lesssim \sqrt{N}\|f\|_1,
\]

with implicit constants depending on various (scalar) variational quantities discussed in \([4]\).

Remark 3.4. This proof further generalizes to the case of multiple weights, i.e. the \( \{D_k\} \) are replaced with \( \{\Delta_k\} \) of \([6]\).
4. Multipliers with Bounded $r$-Variation

This section will be devoted to Theorem 1.5.

We will use the following decomposition lemma [3]:

**Lemma 4.1.** Suppose $1 \leq r < \infty$, and $\|g\|_{V^r(R)} < \infty$. Then there exist collections of intervals $I_j$ so that we may decompose

$$g = \sum_{j \geq 0} \sum_{I \in I_j} d_I 1_I,$$

where $|I_j| \leq 2^j$, $|d_I| \leq 2^{-j/r} \|g\|_{V^r(R)}$ for each $I \in I_j$, and the intervals $I \in I_j$ are pairwise disjoint.

Using this lemma, it is enough to establish the following:

**Proposition 4.2.**

$$\left\| \left( \sum_{\omega \in X} d_\omega 1_\omega \hat{f} \right)^\vee \right\|_{L^q(R)} \lesssim \sup |d_\omega| |X|^{\frac{1}{2} - \frac{2}{q}} \|f\|_{L^q(R)};$$

by duality, it is enough to consider the $1 < q \leq 2$ case. We will establish the weak-type estimate

**Proposition 4.3.**

$$\left\| \left( \sum_{\omega \in X} d_\omega 1_\omega \hat{f} \right)^\vee \right\|_{L^{1,\infty}(R)} \lesssim \sup |d_\omega||X|^{1/2}\|f\|_{L^1(R)}.$$

So, viewing $\{\omega\}$ and $\{d_\omega\}$ as fixed, without loss of generality $|d_\omega| \leq 1$, we let

$$Tf = T_{\{\omega\}, \{d_\omega\}} := \left( \sum_{\omega \in X} d_\omega 1_\omega \hat{f} \right)^\vee$$

denote the relevant operator.

The proof of Proposition 4.3 will follow along similarly to that of Proposition 1.10; technical complications arise from the fact that our multipliers are given by (rough) indicator functions, rather than by (smooth) Schwartz weights.

We therefore begin with a smooth decomposition of each $1_\omega$.

Let $\{u(\omega)\}$ be (dyadic) Whitney intervals so that for each $\omega$

- $100u(\omega) \subset \omega$;
- $\bigcup u(\omega) = \omega$ is an almost disjoint union;
- there exists an absolute $K = O(1)$ so that $\sum_{u(\omega)} 1_{20u(\omega)} \leq K$, i.e. the collection $\{20u(\omega)\}$ has bounded overlap.

Associated to this decomposition, we define two families of smooth functions,

$$\{\phi_{u(\omega)}\} \text{ and } \{A_{u(\omega)}\}.$$

The $\{\phi_{u(\omega)}\}$ satisfy

$$1_\omega = \sum_{u(\omega)} \hat{\phi}_{u(\omega)}, \text{ and } \operatorname{supp} \hat{\phi}_{u(\omega)} = I(u(\omega)),$$
where $I(u(\omega))$ is an interval concentric with $u(\omega)$ of without loss of generality dyadic length and $2 < \frac{|I(u(\omega))|}{|u(\omega)|} \leq 4$. By our Whitney decomposition, we may assume that there exists an absolute $R = O(1)$ so that for each $\omega$ and each $j$,

$$\#\{u(\omega) : I(u(\omega)) = 2^{-j}\} \leq R.$$

We similarly define $\hat{A}_u(\omega)$ to be smooth functions supported in $15u(\omega)$ and identically equal to one on $10u(\omega)$.

We shall require uniform decay on $\{\phi_u(\omega)\}$ and $\{A_u(\omega)\}$: if $I(u(\omega)) = 2^{-j}$, then we demand that

$$\left\| \hat{\phi}_u^{(M)}(\omega) \right\|_{\infty} + \left\| \hat{A}_u^{(M)}(\omega) \right\|_{\infty} \leq C M 2^{Mj}$$

for $M = \log_2 N$.

Finally, we consider a family of smooth, positive, mean-one mollifiers,

$$\int \eta_j = 1, \supp \eta \subset \{ \xi : |\xi| \leq \frac{1}{1000} 2^{-j} \},$$

so that for $|I(u(\omega))| = 2^{-j}$,

$$\eta_j * \hat{A}_u(\omega) = 1$$

on $5I(u(\omega))$ (say).

Next, if $c_\omega$ denotes the center of each interval $\omega$, we define rescaled, shifted-to-the-origin versions of our multipliers $\{\phi_u(\omega), A_u(\omega)\}$:

For $|I(u(\omega))| = 2^{-j}$, we define

$$\hat{\phi}_u(\omega) * (2^j(\xi - c_\omega)) := \hat{\phi}_u(\omega)(\xi) \text{ i.e. } \hat{\phi}_u(\omega)(\xi) := \hat{\phi}_u(\omega)\left(\frac{\xi}{2^j}\right) + c_\omega,$$

and similarly for $A_u(\omega)*$.

Collect

$$\mathcal{F}_j := \{ u(\omega) : |I(u(\omega))| = 2^{-j} \}$$

and sparsify

$$\mathcal{F}_j = \bigcup_{i=1}^{RK} \mathcal{F}^k_j$$

into $RK$ families, so that

- Each $\mathcal{F}^k_j$ has at most one $u(\omega)$ with $|I(u(\omega))| = 2^{-j}$ from each $\omega$ (i.e. if $u(\omega), u'(\omega') \in \mathcal{F}^k_j$ with $|I(u(\omega))| = |I(u'(\omega'))|$, then necessarily $\omega \neq \omega'$;

- For each $u(\omega) \in \mathcal{F}^k_j$, the supports of each $A_u(\omega)$ are disjoint.

Since our estimates will be uniform in $k$, in what follows, we shall suppress all dependence of $k$ in our $\mathcal{F}^k_j$.

With this in mind, we decompose

$$\hat{T}\hat{f} := \sum_j \hat{T}_j \hat{f} := \sum_j \left( \sum_{\mathcal{F}^k_j} d_{u(\omega)} \hat{\phi}_u(\omega) \hat{f} \right),$$

where we denote $d_{u(\omega)} = d_\omega$ for $u(\omega) \subset \omega$.

Following along the lines of the previous section we discretize each $T_j$:

With $|I(u(\omega))| = 2^{-j}$ we used windowed Fourier series to express

$$\hat{\phi}_u(\omega)(\xi) \hat{f}(\xi) = 2^{j} \sum_{l \in \mathbb{Z}} \left( \int \hat{f}(t) \hat{\phi}_u(\omega)(t) e(2^j l t) \, dt \right) e(-2^j l \xi) \hat{A}_u(\omega) * \eta_j(\xi).$$
After an argument with the Fourier transform, we have
\[ d_u(\omega) \hat{\phi}_{u(\omega)} * f(x) \]
\[ = \sum_{l \in \mathbb{Z}} (f(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)*}(2^{-j} x - l)) \cdot (e(c_\omega x)A_{u(\omega)*}(2^{-j} x - l)) \cdot (d_{u(\omega)} \cdot \eta_j^*(x - 2^j l)) \]
\[ = \sum_{l \in \mathbb{Z}} (f(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)*}(2^{-j} x - l)) \cdot (E_{u(\omega),j,l}(x)) \cdot (d_{u(\omega)} \cdot \eta_j^*(x - 2^j l)) ; \]
for convenience, we remark that
\[ E_{u(\omega),j,l}(\xi) := (e(c_\omega x)A_{u(\omega)*}(2^{-j} x - l))(\xi) \]
\[ = e(-\xi 2^j l) - e(c_\omega 2^j l) \hat{A}_{u(\omega)}(\xi). \]

We now apply the Multi-Frequency Calderón-Zygmund decomposition at height \( \lambda = 1 \), and extract \( b = \sum_j b_j \) orthogonal to \( \{ e(-c_\omega x) : \omega \in X \} \). By standard arguments, and scale and translation invariance, our task is once again to show that
\[ \|T b_j\|_{L^1(|x| \geq 100)} \lesssim 1 \]
where we may assume that \( J = [0, 1) \).

We will majorize
\[ \|T b_j\|_{L^1(|x| \geq 100)} \leq \sum_j \| T_j b_j \|_{L^1(|x| \geq 100)} \]
according to scale and estimate each term separately.

We turn to

4.1. **The Proof of Theorem 1.5**. As mentioned, the strategy here is very similar to that of Proposition 1.10.

We begin with the more involved case, \( j \geq 0 \).

The **Proof, \( j \geq 0 \)**. We further decompose
\[ T_j b_j := \sum_l T_j b_j \]
\[ := \sum_l \left( \sum_{F_j} (b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)*}(2^{-j} x - l)) \cdot (e(c_\omega x)A_{u(\omega)*}(2^{-j} x - l)) \right) \cdot (d_{u(\omega)} \eta_j(x - 2^j l)) \]
\[ = \sum_l \left( \sum_{F_j} (b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)*}(2^{-j} x - l)) \cdot E_{u(\omega),j,l}(x) \right) \cdot (d_{u(\omega)} \eta_j(x - 2^j l)) ; \]
we will estimate each \( \|T_j b_j\|_1 \), and sum over \( (j,l) \); by the arguing as above, and using the uniform decay of the \( \{ \phi_{u(\omega)*} \} \), it will be enough to estimate the top \( l = 0 \) term. We will therefore suppress \( l \)-dependence in what follows.

We begin by applying Cauchy-Schwarz to estimate
\[ \|T_j b_j\|_1 \lesssim 2^{j/2} \left\| \sum_{F_j} (b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)*}(2^{-j} x)) E_{u(\omega),j}(x) \right\|_2 , \]
then use the orthogonality of
\[ \{ E_{u(\omega),j}(x) \} \]
(see above) to express
\[
\left\| \sum_{F_j} \langle b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)^*}(2^{-j}x) \rangle E_{u(\omega),j}(x) \right\|_2
\]
\[
= \left\| \sum_{F_j} |\langle b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)^*}(2^{-j}x) \rangle|^2 |E_{u(\omega),j}(x)|^2 \right\|_2^{1/2}
\]
\[
= \left\| \sum_{F_j} |\langle b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)^*}(2^{-j}x) \rangle|^2 |A_{u(\omega)^*}(2^{-j}x)|^2 \right\|_2^{1/2}.
\]
Since \(b_j\) is orthogonal to each \(e(c_\omega x)\), we may replace the inner products
\[
\langle b_j(x), e(-c_\omega x)2^{-j} \phi_{u(\omega)^*}(2^{-j}x) \rangle \equiv \langle b_j(x), e(-c_\omega x) (2^{-j} \phi_{u(\omega)^*}(2^{-j}x) - 2^{-j} \phi_{u(\omega)^*}(0)) \rangle
\]
\[
=: \langle b_j(x), e(-c_\omega x) \rho_{u(\omega),j}(x) \rangle.
\]
We observe that for \(|x| \leq 5\) (say) we have the \(\rho_{u(\omega),j}\)-uniform bounds
\[
|\rho_{u(\omega),j}| + |\rho'_{u(\omega),j}| \lesssim 2^{-2j}
\]
\[
|\rho''_{u(\omega),j}| \lesssim 2^{-3j},
\]
the first point following by the mean-value theorem.

We now majorize
\[
\left( \sum_{F_j} |\langle b_j(x), e(-c_\omega x) \rho_{u(\omega),j}(x) \rangle|^2 |A_{u(\omega)^*}(2^{-j}x)|^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{F_j} |\langle f_j(x), e(-c_\omega x) \rho_{u(\omega),j}(x) \rangle|^2 |A_{u(\omega)^*}(2^{-j}x)|^2 \right)^{1/2}
\]
\[
+ \left( \sum_{F_j} |\langle g_j(x), e(-c_\omega x) \rho_{u(\omega),j}(x) \rangle|^2 |A_{u(\omega)^*}(2^{-j}x)|^2 \right)^{1/2}
\]
\[
=: F(x) + G(x),
\]
and estimate each function separately in \(L^2\).

Using our uniformity conditions on the \(\{\rho_{u(\omega),j}\}, \{A_{u(\omega)}\}\), we now find ourselves in the situation of Proposition 1.10 and we may argue as above to conclude

- \(\|F\|_2 \lesssim 2^{-3j/2}\); and
- \(\|G\|_2 \lesssim 2^{-3j/4}\).

Consequently, we have achieved the upper bound on
\[
\|T_j b_j\|_1 \lesssim 2^{j/2} \cdot 2^{-3j/4} \lesssim 2^{-j/4}.
\]
We may similarly deduce
\[
\|T_j b_j\|_1 \lesssim 2^{-j/4} \cdot (|l| + 1)^{-2};
\]
since this is summable over \(l \in \mathbb{Z}, j \geq 0\), we have completed this part of the proof. \(\square\)
We next turn to the simpler $j < 0$ case, where we rely solely on the decay of our functions.

The Proof, $j < 0$. Using our (uniform) decay estimates on $\phi_u(\omega)$, we simply estimate, with $M = \log_2 N$

$$
\|T_j b_j\|_{L^1((100J)^c)} \leq N \max_{u(\omega) \in F_j} \|\phi_{u(\omega)} * b_j\|_{L^1((100J)^c)}
$$

$$
\lesssim N \|b_j\|_1 2^{-j} \min \left\{1, \left(\frac{2^j}{|x|}\right)^M\right\} \|1_{|x| \geq 50}\|_{L^1}\n$$

$$
\lesssim N(2^j)^{M-1}.
$$

Summing the foregoing in $j$ yields an upper estimate of $N^{2^{1-M}} \lesssim 1$ for $M = \log_2 N$.

\[ \square \]

**Remark 4.4.** An interesting question, raised in [7], concerns the $L^p$-behavior of Bourgain’s maximal (rough) singular integral $\mathcal{M}$.

**Problem 4.5.** For any collection of $N$ frequencies, $\Sigma := \{\xi_1, \ldots, \xi_N\} \subset \mathbb{R}$, consider the maximal operator

$$
\mathcal{M}^* f := \sup_j \left| \left( \hat{f} 1_{R_j} \right)^\vee \right| (x),
$$

where $R_j$ is the $2^{-j}$ neighborhood of $\Sigma$. For $p \neq 2$, what are the best $N$-dependent constants, $C_p(N)$, so that

$$
\|\mathcal{M}^* f\|_{L^p(\mathbb{R})} \leq C_p(N)\|f\|_{L^p(\mathbb{R})}?
$$

Although the operator $\mathcal{M}^* f$ is too rough to be handled by present technique, we look forward to pursuing this line of inquiry in future work.

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