Black hole entropy from Quantum Geometry

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Abstract

Quantum Geometry (the modern Loop Quantum Gravity using graphs and spin-networks instead of the loops) provides microscopic degrees of freedom that account for the black-hole entropy. However, the procedure for state counting used in the literature contains an error and the number of the relevant horizon states is underestimated. In our paper a correct method of counting is presented. Our results lead to a revision of the literature of the subject. It turns out that the contribution of spins greater then 1/2 to the entropy is not negligible. Hence, the value of the Barbero-Immirzi parameter involved in the spectra of all the geometric and physical operators in this theory is different than previously derived. Also, the conjectured relation between Quantum Geometry and the black hole quasi-normal modes should be understood again.

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I. INTRODUCTION AND RESULTS

Quantum Geometry [1-3] provides microscopic degrees of freedom that account for the black-hole entropy. The issue was raised in works by Krasnov [4] and Rovelli [5]. The final quantum Isolated Horizon framework was introduced by Ashtekar, Baez, Corichi and Krasnov [8]. The framework describes a black-hole in equilibrium surrounded by a quite arbitrary space-time. In the canonical formulation of the theory, the classical phase space is restricted to the set of space-times containing a black-hole of a fixed area $a$. After quantization, the kinematic Hilbert space is the tensor product of the horizon Hilbert space $\mathcal{H}_S$ and the bulk Hilbert space $\mathcal{H}_\Sigma$. Elements of $\mathcal{H}_S$ are identified with the black-hole kinematic quantum states. They represent quantum excitations of the $U(1)$ spin connection defined on the horizon. Mathematically, they are described by the quantum Chern-Simons theories on a punctured sphere, where all possible sets of punctures are admitted. Therefore, they are labeled by certain sequences of numbers (referred to as spins) assigned to the punctures. The bulk states, that is elements of $\mathcal{H}_\Sigma$, on the other hand, are described by Quantum Geometry, and define the quantum area of the horizon [6,7]. The quantum constraints commute with the quantum horizon area operator. Most importantly, all the solutions whose quantum areas fall into any given finite interval $[a - \delta a, a + \delta a]$ can be labeled by a finite number of the black-hole states (and by other labels corresponding to the bulk states). That number defines the Ashtekar-Baez-Corichi-Krasnov black-hole entropy. However, the procedure introduced in [8] for state counting contains a spurious constraint on admissible sequences of the labels, and the number of the relevant horizon states is underestimated. The goal of our work\(^1\) is to point out this error, to formulate a correct counting procedure, and to study the consequences. Several conclusions of [8] have to be re-examined:

(i) the statement that the leading contribution to the entropy comes entirely from the simplest states (characterized by punctures with labels $\pm 1/2$),

(ii) the confirmation of the Bekenstein and Hawking formula [9],

(iii) the determination of the value of the free parameter $\gamma$ introduced in Quantum Geometry by Barbero and Immirzi, as

$$\gamma_{\text{ABCK}} = \frac{\ln 2}{\pi \sqrt{3}}.$$  \hspace{1cm} (1)

The Barbero-Immirzi parameter labels different sectors of Quantum Geometry. The spectra of all the geometric and physical operators depend on $\gamma$. For example the quantized area is proportional to $\gamma$.

Therefore, in our paper we go back to the issue of calculating the entropy within the Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework. We formulate an exact combinatoric problem whose solution is the entropy: we carefully define a set of sequences freely labeling the black-hole quantum states which correspond to eigenvalues of $\gamma$.

\(^1\)This project was a part of the diploma thesis of one of the authors [14].
the quantum area operator equal or less than \(a\). The entropy is \(\ln N(a)\), where \(N(a)\) is the number of elements of this set. Next, we address the issue of the proportionality of the entropy to the area. We consider the ratio

\[
\frac{\ln N(a)}{a}
\]  

which is expected to be constant in the limit as \(a \to \infty\). We find that for large \(a\),

\[
\frac{\ln 2}{4\pi \gamma \ell_{\text{Pl}}^2} a \leq \ln N(a) \leq \frac{\ln 3}{4\pi \gamma \ell_{\text{Pl}}^2} a .
\]  

As before (in [8]), the Barbero-Immirzi parameter \(\gamma\) can be fixed by assuming the agreement with the Bekenstein-Hawking entropy (the proportionality of the entropy to the area in the leading order is not proved yet, at this point, but see below). Then, (3) provides lower and upper bounds for \(\gamma\),

\[
\frac{\ln 2}{\pi} \leq \gamma \leq \frac{\ln 3}{\pi} .
\]  

These results lead to a revision of the literature of the subject.

To begin with, they show that the number of states characterized by the lowest spin labels is essentially smaller than the number of all the states contributing to a given value of the horizon area. Therefore, the horizon states labeled by higher spins do contribute to the leading term of the entropy. In the consequence, the ABCK model of quantum black hole does not agree with Wheeler’s heuristic model ‘It from bit’. The (contradicted here) conclusion (\(i\)) listed above, had been used in [10] to point out a remarkable relation between the black hole quasi-normal modes, the Bekenstein-Hawking entropy and the ABCK entropy provided the minimal spin of the quantum excitations of the horizon is 1 rather then \(1/2\). In view of our correction, the meaning of that observation of [10] should be understood again.

Next, the conclusion (\(iii\)) listed above is not true: the value of the parameter \(\gamma\) can not be that of (1).

We were not able to verify the proportionality of the horizon entropy to the area ourselves. However, our combinatoric formulation of the problem of calculating the entropy derived in this paper has been later solved in an exact way in the accompanying paper [11]. The number \(\ln N(a)\) has been calculated rigorously therein, and the result is

\[
\ln N(a) = \frac{\gamma_M}{\gamma} \frac{a}{4\ell_{\text{Pl}}^2} - \frac{1}{2} \ln a + O(1)
\]  

in the limit of large horizon area \(a\), where the value of the parameter \(\gamma_M\) can be calculated at arbitrary accuracy. In conclusion, taking into account the results of [11], the proportionality of the entropy to the area in the leading order for large \(a\) is confirmed, and even the sub-leading terms are calculated.

In the last section we derive the probability distribution of the spin at a given puncture. The physical meaning of these new results will be discussed in a joint paper [12].
II. THE HORIZON QUANTUM STATES

Our departure point is the quantum Isolated Horizon theory introduced in [8]. A classical (weakly) isolated horizon is a null, non-expanding world-surface of a space-like 2-sphere $S$, equipped with a null vector field of a constant self-acceleration. This idea gave rise to a new quasi-local theory of black-hole in equilibrium interesting also from purely classical point of view [13]. The local degrees of freedom of space-time geometry out of the horizon are not restricted. An additional assumption satisfied by the isolated horizon geometry considered in [8] is the spherical symmetry (see [3] for a discussion of the general case). The classical phase space introduced in the Ashtekar-Baez-Corichi-Krasnov framework consists of all the gravitational fields which can exist in a neighborhood of an isolated horizon. The location and area $a$ of the horizon are fixed. In the corresponding $3+1$ Hamiltonian framework the horizon is represented by a 2-sphere $S$, whereas the remaining part of space under consideration (bulk) is the exterior region, a 3-manifold $\Sigma$ bounded by $S$. The only kinematic degrees of freedom of the horizon $S$ are given by a $U(1)$ connection defined on a spin bundle over $S$. The gravitational field data is defined on $\Sigma$. The bulk and the horizon data are subject to a consistency condition ensuring that the world-surface of $S$ is indeed an isolated horizon. In the quantum theory of this system, the bulk gravitational field is described by Quantum Geometry. The quantum consistency condition between the bulk and the horizon implies that the quantum degrees of freedom of the horizon should be described by a union of the Chern-Simons theories on arbitrarily punctured $S$.

Below, we outline the elements of the quantum theory which play a role in the entropy calculation. The kinematic Hilbert space of the quantum Isolated Horizon theory is contained in the tensor product $H_S \otimes H_\Sigma$, where $H_S$ is the Hilbert space of the quantum horizon degrees of freedom whereas $H_\Sigma$ is the Hilbert space of the quantum geometry defined in the bulk $\Sigma$.

To characterize the bulk quantum geometry Hilbert space $H_\Sigma$, it is convenient to use a certain orthogonal decomposition adapted to the boundary $S$. Consider a finite set of points,

$$\mathcal{P} = \{p_1, ..., p_n\} \subset S, \quad (6)$$

and a labeling of elements of $\mathcal{P}$ by numbers $j = (j_1, ..., j_n)$, and $m = m_1, ..., m_n$) where

$$j_i \in \frac{1}{2} \mathbb{N}, \quad m_i \in \{-j_1, ..., j_i\} \quad (7)$$

for $i = 1, ..., n$. The $j_i$ labels are assumed not to vanish,

$$j_i \neq 0. \quad (8)$$

The space $H_\Sigma$ is the orthogonal sum

$$H_\Sigma = \bigoplus_{(\mathcal{P},j,m)} H^\mathcal{P}_{\Sigma,j,m}, \quad (9)$$

where $\mathcal{P}$ runs through all the finite subsets of $S$, $(j, m)$ through all the finite labeling (7,8), and the empty set $\mathcal{P} = \emptyset$ is also associated a (non-trivial) single Hilbert space $H^\emptyset_{\Sigma}$. Each of the subspaces $H^\mathcal{P}_{\Sigma,j,m}$ is a certain (infinitely dimensional) Hilbert space. To understand
the (quantum) geometric meaning of these subspaces, consider any piece $S'$ of the horizon $S$ (more precisely, $S'$ is an arbitrary 2-sub-manifold in the 2-sphere $S$). The quantum area operator $\hat{A}_{S'}$ is defined in $\mathcal{H}_\Sigma$. Another operator of the quantum geometry in $\Sigma$ assigned to $S'$ is the flux $\hat{E}_{S'}$ of the vector field normal to $S$.\textsuperscript{2} Every subspace $\mathcal{H}_{\Sigma,j,m}$ is the eigenspace of the operators. The corresponding eigenvalues are:

\[
\begin{align*}
    a_{j,m}^{S',j} &= 8\pi\gamma\ell_P^2 \sum_{p_i \in P \cap S'} \sqrt{j_i(j_i+1)}, \\
    e_{j,m}^{S'} &= 8\pi\gamma\ell_P^2 \sum_{p_i \in P \cap S'} m_i
\end{align*}
\]

where only those points $p_i \in P$ contribute which are contained also in $S'$, and $\gamma > 0$ is a free parameter of Quantum Geometry known as Barbero-Immirzi parameter. In particular, if we take for $S'$ the whole horizon $S$, then the corresponding eigenvalue of the horizon area operator is (we drop $P$ because only the set of the values of $j$ matters)

\[
    a_{j}^{S} = 8\pi\gamma\ell_P^2 \sum_{i} \sqrt{j_i(j_i+1)}.
\]

Of course the quantum area and flux operators are defined also for 2-dimensional sub-manifolds of $\Sigma$ which do not intersect $S'$, and there are also other interesting operators of Quantum Geometry acting in $\mathcal{H}_\Sigma$, but they are not used in the Isolated Horizon framework.

We turn now to the horizon Hilbert space $\mathcal{H}_S$. Recall, that the ABCK framework requires fixing an arbitrary value for the classical horizon area $a$. In order to quantize consistently the horizon degrees of freedom, it is assumed in [8] that the fixed classical area $a$ is quantized in the following way,

\[
a = 4\pi\gamma\ell_P^2 k, \quad k \in \mathbb{N},
\]

where $k$ is arbitrary. The horizon Hilbert space can also be orthogonally decomposed in a way analogous to (9). In this case the labeling set consists of all the finite sequences $\vec{P}$ of points in $S$, labeled by arbitrary non-zero integers defined modulo $k$, and whose sum is zero modulo $k$,

\[
\vec{P} = (p_1, ..., p_n) \quad b = (b_1, ..., b_n),
\]

\[
b_i \in \mathbb{Z}_k, \quad i = 1, ..., n, \quad \sum_{i=1}^{n} b_i = 0.
\]

To every labeled sequence $(\vec{P}, b)$ there is assigned a 1-dimensional Hilbert space $\mathcal{H}_{\vec{P},b}$. The points $p_i$ are often called punctures, and for every $p_i$, the $U(1)$ element $e^{i\frac{2\pi b_i}{k}}$ is the eigenvalue\textsuperscript{2}.

\textsuperscript{2}There are two subtleties, which may be intriguing for the reader. The first one is that classically the flux of a vector field normal to a surface is the same as the area of the surface. In our quantum case however, the vector field is a quantum operator itself, in fact it is one of the $SU(2)$ spin operators. The area, on the other hand, is given by the total $SU(2)$ spin operator. The second subtlety is, that $S'$ contained in the horizon is assigned the area and flux by the bulk geometry.
of the holonomy transport along a circle containing \( p_i \). The full horizon Hilbert space can be written as the orthogonal sum

\[
\mathcal{H}_S = \bigoplus_{(\vec{P},b)} \mathcal{H}^{\vec{P},b}_S, \quad (16)
\]

where to the labeling set of pairs \((\vec{P}, b)\) we add the empty sequence which labels a single, 1-dimensional space \( \mathcal{H}^\emptyset_S \).

Now, the quantum condition that \( S \) be a section of a spherically symmetric isolated horizon is introduced in the product Hilbert space

\[
\mathcal{H}_S \otimes \mathcal{H}_\Sigma = \bigoplus_{(\vec{P},b)} \bigoplus_{(P',j,m)} \left( \mathcal{H}^{\vec{P},b}_S \otimes \mathcal{H}^{P',j,m}_\Sigma \right). \quad (17)
\]

It involves the holonomy operators acting in \( \mathcal{H}^{\vec{P},b}_S \) on the one hand and the flux operators acting in \( \mathcal{H}^{P',j,m}_\Sigma \) on the other hand (see [8] or [3] for a summary). The set of solutions is the orthogonal sum of products \( \mathcal{H}^{\vec{P},b}_S \otimes \mathcal{H}^{P',j,m}_\Sigma \) such that the points coincide modulo the ordering

\[
\vec{P} = (p_1, ..., p_n), \quad P = \{p_1, ..., p_n\} \quad (18)
\]

and the labelings agree with each other in the following way

\[
b_i = -2m_i \text{ mod } k. \quad (19)
\]

In conclusion, the kinematic Hilbert space of states in the ABCK quantum Isolated Horizon framework is the orthogonal sum with respect to all the labeled sequences of points in \( S \) which satisfy (18,19),

\[
\mathcal{H}_{\text{kin}} = \bigoplus_{\vec{P},j,m} \mathcal{H}^{\vec{P},b(m)}_S \otimes \mathcal{H}^{P',j,m}_\Sigma, \quad (20)
\]

where we emphasized the dependence of each sequence \( b = (b_1, ..., b_n) \) on a sequence \( m = (m_1, ..., m_n) \) via (19).

The final step of the ABCK framework is to take into account the quantum Einstein constraints [1–3] adjusted to the Isolated Horizon framework [8]. The vector constraints generate diffeomorphisms of \( \Sigma \) which preserve the boundary \( S \). The diffeomorphisms act naturally in \( \mathcal{H}_{\text{kin}} \) and this action is unitary. In the Quantum Geometry framework, the quantum vector constraints come down to the diffeomorphism invariance. Whereas there is only one normalizable diffeomorphism invariant state in \( \mathcal{H}_{\text{kin}} \), the invariance condition is imposed in a certain larger space (dual to an appropriate subspace of \( \mathcal{H}_{\text{kin}} \)). A Hilbert space of solutions is constructed via the diffeomorphism averaging procedure whose Quantum Geometry version is well described in [1,3]. It is particularly easy to explain the result of the analogous diffeomorphism averaging suitable for the space \( \mathcal{H}_S \) itself. Assign to every finite sequence \( b = (b_1, ..., b_n) \) defined in (15) a 1-dimensional Hilbert space \( \mathcal{H}^b_S \), and define

\[
\mathcal{H}_{S,\text{phys}} = \bigoplus_b \mathcal{H}^b_S. \quad (21)
\]
The averaging applied to the full space $\mathcal{H}_{\text{kin}}$ provides a Hilbert space of the following structure

$$\mathcal{H} = \bigoplus_{j,m} \mathcal{H}^{b(m),j,m},$$

(22)

where $b = (b_1, ..., b_n)$, $j = (j_1, ..., j_n)$ and $m = (m_1, ..., m_n)$ are arbitrary triples of finite sequences such that (7,15,19) holds. The remaining constraints are the scalar and the Gauss constraint. The Gauss constraint amounting to the invariance with respect to the local SU(2) rotations of the spin frames is already satisfied at $S$ due to (19). The scalar constraint, on the other hand, has been solved at the horizon already on the classical level. Since the quantum numbers $(j,m)$ used in (23) characterize the quantum geometry of bulk $\Sigma$ right at the boundary $S$, the plausible, but non-trivial assumption made in [8] is that the bulk scalar constraints with laps functions vanishing on $S$ do not affect $(j,m)$, and that for every choice of $(j,m)$ there is a solution of the bulk scalar constraints.

We are lead to a conclusion, that the final physical Hilbert $\mathcal{H}_{\text{phys}}$ of the quantum states can be characterized as

$$\mathcal{H}_{\text{phys}} = \bigoplus_{j,m} \mathcal{H}^{b(m),j,m},$$

(23)

where $b = (b_1, ..., b_n)$, $j = (j_1, ..., j_n)$ and $m = (m_1, ..., m_n)$ are arbitrary triples of finite sequences such that (7,15,19) is satisfied, and none of the spaces $\mathcal{H}^{b(m),j,m}$ is empty. Since the horizon area operator $\hat{A}_S$ commutes with all the constraints in this framework, it passes to the physical Hilbert space. It also follows from the details of the diffeomorphism averaging map, that each of the subspaces $\mathcal{H}^{b(m),j,m}$ is an eigenspace corresponding to the eigenvalue (12). In this way, the sequences $j$ are responsible for the area assigned to the 2-surface $S$ of the horizon by the bulk Quantum Geometry, whereas the sequences $b$ represent the intrinsic quantum degrees of freedom of the horizon. This is the key conclusion of the ABCK framework.

**III. THE COMBINATORIC FORMULATION OF THE PROBLEM**

From now our treatment starts to differ from that of [8]. Our aim is to define the number of the quantum states of the horizon which contribute to the entropy in a form of a clear and precise combinatoric formula.

Recall that the classical area $a$ of the horizon is fixed as

$$a = 4\pi \gamma \ell_P^2 k, \quad k \in \mathbb{N}. $$

(24)

Calculation of the horizon entropy consists in counting those quantum horizon states, elements of the Hilbert space $\mathcal{H}_{S,\text{phys}}$ (21) labeled by the finite sequences defined in (15) (the number $n$ of the entries can be arbitrary) which correspond to subspaces $\mathcal{H}^{b(m),j,m}$ in (23)
such that the area operator eigenvalue $a^j_S$ satisfies the following inequality\(^3\)

$$a^j_S = 8\pi\gamma\ell^2_{\text{Pl}} \sum_{i=1}^{n} \sqrt{j_i(j_i + 1)} \leq a.$$  \hspace{1cm} (25)

Combinatorially, the task amounts to counting the finite sequences $(b_1, \ldots, b_n)$ of elements of $\mathbb{Z}_k$, such that the following two conditions $(i)$ and $(ii)$ are satisfied:

$(i)$ \hspace{1cm} $\sum_{i=1}^{n} b_i = 0$, \hspace{1cm} (26)

$(ii)$ there exist two sequences: a sequence $(j_1, \ldots, j_n)$ consisting of non-zero elements of $\frac{1}{2}\mathbb{N}$ which satisfies the inequality (25), and a sequence $(m_1, \ldots, m_n)$ such that

$$b_i = -2m_i \mod k,$$

and $m_i \in \{-j_i, -j_i + 1, \ldots, j_i\}$, \hspace{1cm} (28)

for every $i = 1, \ldots, n$.

This recipe can be simplified. It is easy to eliminate the $j_i$'s by noting that, given a sequence $(m_1, \ldots, m_n)$, there is a sequence $(j_1, \ldots, j_n)$ such that (28) and (25) if and only if

$$8\pi\gamma\ell^2_{\text{Pl}} \sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} \leq a.$$  \hspace{1cm} (29)

Next, it follows from (26, 27) that

$$\sum_{i=1}^{n} m_i = n\frac{k}{2};$$ \hspace{1cm} (30)

with some $n' \in \mathbb{Z}$. But then,

$$a = 4\pi\gamma\ell^2_{\text{Pl}}k \geq 8\pi\gamma\ell^2_{\text{Pl}} \sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} >$$

$$> 8\pi\gamma\ell^2_{\text{Pl}} \sum_{i=1}^{n} |m_i| \geq 8\pi\gamma\ell^2_{\text{Pl}} \sum_{i=1}^{n} |m_i| =$$

$$= 4\pi\gamma\ell^2_{\text{Pl}}k|n'|.$$ \hspace{1cm} (33)

Hence

$$\sum_{i=1}^{n} m_i = 0, \text{ and also } \sum_{i=1}^{n} |m_i| \leq \frac{k}{2}. \hspace{1cm} (34)$$

\(^3\)Assuming just inequality instead of considering an interval $[a-\delta a, a+\delta a]$ simplifies the calculation. Knowing the result for all $a$ one can always consider the interval. The result, at least in our case, is the same.

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Finally, notice that given a sequence \((b_1, \ldots, b_n)\) of elements of \(\mathbb{Z}_k\) there is exactly one sequence \((m_1, \ldots, m_n)\) of elements of \(\frac{1}{2}\mathbb{Z}\) such that (27) and (34) holds. In this way, we have derived the following formula for the entropy counting:

The entropy \(S\) of a quantum horizon of the classical area \(a\) according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov [8] framework is

\[
S = \ln N(a),
\]

where \(N(a)\) is 1 plus the number of all the finite sequences \((m_1, \ldots, m_n)\) of non-zero elements of \(\frac{1}{2}\mathbb{Z}\), such that the following equality and inequality are satisfied:

\[
\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} \leq \frac{a}{8\pi\gamma\ell_P^2},
\]

where \(\gamma\) is the Barbero-Immirzi parameter of Quantum Geometry.

The extra term 1 above comes from the trivial sequence.

### IV. THE ENTROPY CALCULATIONS

To find an upper bound for the number \(N(a)\) introduced in the previous section (recall that \(a = 4\pi\gamma\ell_P^2k\), and \(k \in \mathbb{N}\)), consider the set \(M^+_k\) of all the finite sequences \((m_1, \ldots, m_n)\) of non-zero elements of \(\frac{1}{2}\mathbb{Z}\) which satisfy the inequality in (34), but do not necessarily sum to zero, that is,

\[
M^+_k := \{(m_1, \ldots, m_n) \mid n \in \mathbb{N}, 0 \neq m_1, \ldots, m_n \in \frac{1}{2}\mathbb{Z}, \sum_{i=1}^{n} |m_i| \leq \frac{k}{2}\}. \tag{37}
\]

Let \(N^+_k\) be the number of elements of \(M^+_k\) plus 1 (the empty sequence). Certainly,

\[
N(a) \leq N^+_k \tag{38}
\]

Next, since \(k\) is arbitrarily fixed integer, let it become a variable of the sequence \((N_0^+, N_1^+, \ldots, N_k^+,...)\). To establish a recurrence relation satisfied by the sequence \((N_0^+, N_1^+, \ldots, N_k^+,...)\), notice that if \((m_1, \ldots, m_n) \in M^+_{k-1}\), then both \((m_1, \ldots, m_n, \frac{1}{2})\), \((m_1, \ldots, m_n, -\frac{1}{2})\) \(\in M^+_k\). In the same way, for arbitrary natural \(0 < l \leq k\),

\[
(m_1, \ldots, m_n) \in M^+_{k-l} \Rightarrow (m_1, \ldots, m_n, \pm \frac{1}{2}l) \in M^+_k. \tag{39}
\]

Obviously, if we consider all \(0 < l \leq k\), and all the sequences \((m_1, \ldots, m_n) \in M^+_{k-l}\), then the resulting \((m_1, \ldots, m_n, \pm \frac{1}{2}l)\) form the entire set \(M^+_k\). Also, for two different \(l \neq l'\),

\[
(m_1, \ldots, m_n, \pm \frac{1}{2}l) \neq (m_1, \ldots, m_n, \pm \frac{1}{2}l'). \tag{40}
\]
This proves the following recurrence relation,

$$N_k^+ = 2N_{k-1}^+ + \ldots + 2N_0^+ + 1.$$  \hspace{1cm} (41)

The (unique) solution is

$$N_k^+ = 3^k.$$  \hspace{1cm} (42)

In conclusion,

$$N(a) \leq 3^k.$$  \hspace{1cm} (43)

To find a lower bound for $N(a)$, we use the inequality

$$\sqrt{|m_i|(|m_i| + 1)} \leq |m_i| + \frac{1}{2},$$  \hspace{1cm} (44)

and consider the number $N_k^-$ equal to 1 plus the number of elements in the set

$$M_k^- := \{(m_1, \ldots, m_n) \mid n \in \mathbb{N}, 0 \neq m_1, \ldots, m_n \in \frac{1}{2}\mathbb{Z}, \sum_{i=1}^n (|m_i| + \frac{1}{2}) \leq \frac{k}{2}\}.$$  \hspace{1cm} (45)

Notice that this time, ignoring the constraint that the elements $m_i$ of each sequence sum to zero (34) makes an in-equivalence relation between $N(a)$ and $N_k^-$ a priori not known. But let us postpone this problem for a moment. Using the same construction as above, we find the recurrence relation satisfied by $N_k^-,

$$N_k^- = 2N_{k-2}^- + \ldots + 2N_0^- + 1.$$  \hspace{1cm} (46)

The unique solution is

$$N_k^- = \frac{2}{3}2^k + \frac{(-1)^k}{3}.$$  \hspace{1cm} (47)

A lower bound for $N(a)$ is the number $N'_k^-$ of the elements of $M_k^-$ which additionally satisfy $m_1 + \ldots + m_n = 0,$

$$N'_k^- \leq N(a).$$  \hspace{1cm} (48)

A statistical physics argument giving the value of the desired number $N'_k^-$ is as follows (this argument is due to Meissner, who also provided an exact proof in [11]). We can think of each sequence $(m_1, \ldots, m_n)$ as of a sequence of random steps on a line. The total length of each path is bounded by $k$ owing to the inequality in the definition of the set $M_k^-.$ The number of sequences in $M_k^-$ of a given, fixed value of the sum

$$m_1 + \ldots + m_n = \delta$$  \hspace{1cm} (49)

depends on $\delta.$ The average value of $\delta$ is $\bar{\delta} = 0.$ For large values of $k$, the number of the paths corresponding to the random walk distance $\delta$ should be given by the Gaussian function

$$\frac{C}{\sqrt{k}}e^{-\frac{\delta^2}{2k}} N_k^-.$$  \hspace{1cm} (50)

In particular, the value

$$N'_k^- = \frac{C}{\sqrt{k}} N_k^-.$$  \hspace{1cm} (50)
corresponds to \( \delta = 0 \).

Summarizing,\[
\frac{C}{\sqrt{k}} N_k^- \leq N(a) \leq N_k^+, \tag{51}
\]
where the numbers \( N_k^- \) and \( N_k^+ \) were calculated in (43, 47). Therefore the entropy is bounded in the following way
\[
\ln 2 \frac{4}{\pi \gamma \ell^2_{\text{Pl}}}a + o(a) \leq S(a) \leq \ln 3 \frac{4}{\pi \gamma \ell^2_{\text{Pl}}}a. \tag{52}
\]

We are in a position now, to compare our results with those of [8]. The mistake made in [8] in the combinatorial formulation (see (44) in that paper) is an assumption, that one can order the \( j_i \) labels in (23) such that \( j_1 \leq ... \leq j_n \), and impose this as a constraint on ‘admissible’ states we count. As a consequence, the number of the states contributing to the entropy was underestimated: the analysis of [8] implied that for large \( a \) the leading contribution to the number of sequences comes only from the sequences \((-\frac{1}{2},..., \pm \frac{1}{2})\). The resulting number \( S_{\text{ABCK}}(a) \) proposed for the value of the entropy was
\[
S_{\text{ABCK}}(a) = \ln 2 \frac{4}{\pi \sqrt{3} \gamma \ell^2_{\text{Pl}}}a + O(a). \tag{53}
\]

The comparison with our (52) shows that the contribution from the higher half integers \(|m_i| > \frac{1}{2}\) can not be neglected.

The next consequence of the estimate (52) is a necessary condition for the agreement of the entropy \( S(a) \) with the Bekenstein-Hawking entropy \( a / 4 \ell^2_{\text{Pl}} \); the condition is that the value of the Barbero-Immirzi parameter \( \gamma \) is bounded in the following way,
\[
\ln \frac{2}{\pi} \leq \gamma \leq \ln \frac{3}{\pi}. \tag{54}
\]

On the other hand, the value proposed for the Barbero-Immirzi parameter by counting the sequences \((\pm 1/2,..., \pm 1/2)\) was
\[
\gamma_{\text{ABCK}} = \frac{\ln 2}{\pi \sqrt{3}}. \tag{55}
\]

In the accompanying paper [11] the recurrence relations (46,41) are generalized to an appropriate generalized recurrence relation satisfied by the desired number \( N(a) \) itself. Next, the combinatorial problem formulated in the previous section is solved completely. The method is remarkably powerful. The result of [11] is
\[
S(a) = \frac{\gamma_M}{\gamma} \frac{a}{4 \ell^2_{\text{Pl}}} - \frac{1}{2} \ln a + O(1), \tag{56}
\]
where \( \gamma_M \) is a constant defined by the following equation
\[
\frac{1}{2} - \sum_{0 \neq j \in \frac{1}{4} \mathbb{N}} e^{-2\pi \gamma_M \sqrt{j(j+1)}} = 0. \tag{57}
\]
The numerical value of $\gamma_M$ is calculated in [11].

An interesting historical remark is (it has been unraveled to us recently, after our results were derived) that before coming to their final result (53, 55), Ashtekar, Baez and Krasnov had established certain estimate. The inequality had been

$$S_{bh}(a) \leq \frac{\gamma_0}{\gamma} \frac{a}{4\ell_{p1}^2} + O(a)$$

(58)

where the constant $\gamma_0$ had been defined by the same equation as $\gamma_M$ above. But since this result emerged as an upper bound only, finally it was replaced by the improper one (53,55).

V. THE SPIN PROBABILITY DISTRIBUTION

Given any value $a$ of the classical horizon area, and the $N(a)$ quantum states of the horizon labeled by all the finite sequences $(m_1, ..., m_n)$ of the non-zero half-integers such that (36) one can fix any arbitrary

$$0 \neq m \in \frac{1}{2} \mathbb{Z}$$

(59)

and consider the subset of states corresponding to the sequences such that

$$m_1 = m.$$  

(60)

Denote the number of the elements of this subset by $N_{(a)}(m)$. The ratio

$$P_{(a)}(m) := \frac{N_{(a)}(m)}{N(a)}$$

(61)

can be considered as the probability that the first puncture is labeled by $m_1 = m$. The question is, what $P_{(a)}(m)$ is when $a$ is large compared to the minimal area $a_m$ created at the puncture$^4$,

$$a_m = \frac{8\pi \gamma \ell_{p1}^2}{\sqrt{|m|(|m|+1)}}.$$  

(62)

(Notice, that the answer could make no probabilistic sense, if the $\lim_{a \to \infty} P_{(a)}(m)$ were 0 for all the values of $m$, for example.) The number $N_{(a)}(m)$ defined above can be also thought of as a number of all the finite sequences $(m_2, ..., m_n)$ such that

$$\sum_{i=2}^{n} m_i = -m, \quad \sum_{i=2}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a - a_m}{8\pi \gamma \ell_{p1}^2}.$$  

(63)

By the same random walk argument as the one used above in the treatment of the number $N'_{k}$ (as before, see [11] for a proof) we have

$$N_{(a)}(m) = e^{-\frac{m^2}{a-a_m}} N(a-a_m) + ...$$  

(64)

$^4$This issue has been raised recently by John Baez.
where the other terms can be neglected when we go to the limit \( a \to \infty \). Since the number \( N(a) \) is given by exponentiating (56), we can see that in the limit \( a \to \infty \)

\[
P_a(m) \to e^{-2\pi \gamma \sqrt{|m|(|m|+1)}} =: P(m).
\]  

(65)

Now, the equality (57) means that the limits \( P(m) \) of the probabilities still sum to 1,

\[
\sum_{0 \neq m \in \frac{1}{2} \mathbb{Z}} P(m) = 1.
\]  

(66)

That (limit) probability distribution \( P(m) \) can also be written in terms of the entropy \( S(a_m) \) corresponding to the area \( a_m \), namely

\[
P(m) = e^{-S(a_m)}.
\]  

(67)

regardless of whether we fix the value of \( \gamma \) by the agreement with the Bekenstein-Hawking entropy or not.

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