ANOMALY MATCHING AND LOW ENERGY THEORIES AT HIGH MATTER DENSITY

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I provide new arguments supporting the validity of the t’Hooft anomaly conditions at non zero quark chemical potential. These constraints strengthen the quark-hadron continuity scenario. Finally I review the 2SC effective Lagrangian for color superconductivity.

1. Introduction

To understand the low energy physics of strongly interacting theories such as QCD and QCD like theories, where perturbation theory is not applicable, effective Lagrangians constructed with the aid of the global symmetries of the underlying theory play a relevant role. To constrain the low energy dynamics at zero quark chemical potential t’Hooft anomaly matching conditions are much used.

While low energy effective Lagrangians at non zero quark chemical potential for QCD like theories have been widely used in literature, the consequences of the t’Hooft anomaly conditions in this regime are still not fully explored. Here I provide new arguments, not discussed in literature, supporting the validity of the anomaly matching conditions at non zero chemical potential. These constraints strengthen the quark-hadron continuity scenario. Finally I briefly review the 2SC effective low energy theory.

2. Color Superconductivity and the QCD Phase Diagram

At zero temperature but very high quark chemical potential strong interactions favor the formation of quark-quark condensates in the color antisymmetric channel \(^1\). Possible physical applications are related to the physics of compact objects \(^1\), supernovae explosions \(^2\) as well as to the Gamma Ray Bursts puzzle\(^3\). Recently these ideas have been investigated in detail in \(^4\).
According to the number of light flavors in play we have different phases. A theoretical picture of the QCD phase diagram is presented in the second panel of Fig. 1 while in the first panel the relevant experiments and possible physical applications are displayed.

![Figure 1](image)

**Figure 1.** Left Panel: Possible physical applications and region of the QCD phase diagram explored by different experiments. Right Panel: An oversimplified cartoon of the theoretical QCD phase diagram. The crystalline phase may exists only if we have different chemical potentials for the up and down quarks. The deconfined $SU_c(2)$ corresponds to the phase where we have gapped up and down quarks and the remaining unbroken $SU_c(2)$ of color deconfined.

### 2.1. Color Flavor Locked Phase

For $N_f = 3$ light flavors at very high chemical potential dynamical computations suggest that the preferred phase is a superconductive one and the following ansatz for a quark-quark type of condensate is energetically favored:

$$\epsilon^{\alpha\beta} < q_{L\alpha,a,i} q_{L\beta,b,j}> \sim k_1 \delta_{ai} \delta_{bj} + k_2 \delta_{aj} \delta_{bi}. \quad (1)$$

A similar expression holds for the right transforming fields. The Greek indices represent spin, $a$ and $b$ denote color while $i$ and $j$ indicate flavor. The condensate breaks the gauge group completely while locking the left/right transformations with color. The final global symmetry group is $SU_{c+L+R}(3)$, and the low energy spectrum consists of 9 Goldstone bosons.

The low energy effective theory for 3 flavors (CFL) has been developed in $^6$. We refer to $^1, ^7$ for a complete summary and review of this phase.
2.2. 2 SC General Features

QCD with 2 massless flavors has gauge symmetry $SU_c(3)$ and global symmetry $SU_L(2) \times SU_R(2) \times U_V(1)$. At very high quark density the ordinary Goldstone phase is no longer favored compared with a superconductive one associated to the following type of diquark condensates:

$$\langle L^a \rangle \sim \langle \epsilon^{abc} \epsilon^{ij} q^{q_L}_{Lb,i} q^{q_L}_{Lc,j}; \alpha \rangle, \quad \langle R^a \rangle \sim -\langle \epsilon^{abc} \epsilon^{ij} q^{q_R}_{Rb,i} q^{q_R}_{Rc,j}; \dot{\alpha} \rangle,$$

If parity is not broken spontaneously, we have $\langle L_a \rangle = \langle R_a \rangle = f_\delta^3$, where we choose the condensate to be in the 3rd direction of color. The order parameters are singlets under the $SU_L(2) \times SU_R(2)$ flavor transformations while possessing baryon charge $\frac{2}{3}$. The vev leaves invariant the following symmetry group:

$$[SU_c(2)] \times SU_L(2) \times SU_R(2) \times \tilde{U}_V(1),$$

where $[SU_c(2)]$ is the unbroken part of the gauge group. The $\tilde{U}_V(1)$ generator $\tilde{B}$ is the following linear combination of the previous $U_V(1)$ generator $B$ and the broken diagonal generator of the $SU_c(3)$ gauge group $T_8$:

$$\tilde{B} = B - \frac{2\sqrt{3}}{3} T_8 = \text{diag}(0, 0, 1).$$

The quarks with color 1 and 2 are neutral under $\tilde{B}$ and consequently so is the condensate.

3. Anomaly Matching Conditions

The superconductive phase for $N_f = 2$ possesses the same global symmetry group as the confined Wigner-Weyl phase. The ungapped fermions have the correct global charges to match the 't Hooft anomaly conditions as shown in $^8$. Specifically the $SU(2)_{L/R}^2 \times U(1)_V$ global anomaly is correctly reproduced in this phase due to the presence of the ungapped fermions. This is so since a quark in the 2SC case is surrounded by a diquark medium (i.e. $q \langle qq \rangle$) and behaves as a baryon.

$$\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}_{\text{color}=3} \sim \begin{pmatrix} p \\ n \end{pmatrix}.$$
structure). This pole is then interpreted as due to a goldstone boson when chiral symmetry is spontaneously broken.

One might be worried that, since the chemical potential explicitly breaks Lorentz invariance, the gapless (goldstone) pole may disappear modifying the infrared structure of the three point function. This is not possible. Thanks to the Nielsen and Chadha theorem\(^{10}\), not used in\(^9\), we know that gapless excitations are always present when some symmetries break spontaneously even in the absence of Lorentz invariance\(^a\). Since the quark chemical potential is associated with the barionic generator which commutes with all of the non abelian global generators the number of goldstone bosons must be larger or equal to the number of broken generators. Besides all of the goldstones must have linear dispersion relations (i.e. are type I\(^{10}\)). This fact not only guarantees the presence of gapless excitations (justifying the analysis made in\(^9\) on the infrared behavior of the form factors) but demonstrates that the pole structure due to the gapless excitations needed to saturate the triangle anomaly is identical to the zero quark chemical potential one in the infrared.

It is also interesting to note that the explicit dependence on the quark chemical potential is communicated to the goldstone excitations via the coefficients of the effective Lagrangian (see\(^7\) for a review). For example \(F_\pi\) is proportional to \(\mu\) in the high chemical potential limit and the low energy effective theory is a good expansion in the number of derivatives which allows to consistently incorporate in the theory the Wess-Zumino-Witten term\(^8\) and its corrections.

The validity of the anomaly matching conditions have far reaching consequences. Indeed, in the three flavor case, the conditions require the goldstone phase to be present in the hadronic as well as in the color superconductive phase supporting the quark-hadron continuity scenario\(^{11}\). At very high quark chemical potential the effective field theory of low energy modes (not to be confused with the goldstone excitations) has positive Euclidean path integral measure\(^{12}\). In this limit the CFL is also shown to be the preferred phase. Since the fermionic theory has positive measure only at asymptotically high densities one cannot use this fact to show that the CFL is the preferred phase for moderate chemical potentials. This is possible using the anomaly constraints.

While the anomaly matching conditions are still in force at non zero

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\(^a\)Under specific assumptions which are met when Lorentz invariance is broken via the chemical potential.
quark chemical potential \cite{8} the persistent mass condition \cite{13} ceases to be valid. Indeed a phase transition, as function of the strange quark mass, between the CFL and the 2SC phases occurs.

We recall that we can saturate the t’Hooft anomaly conditions either with massless fermionic degrees of freedom or with gapless bosonic excitations. However in absence of Lorentz covariance the bosonic excitations are not restricted to be fluctuations related to scalar condensates but may be associated, for example, to vector condensates \cite{14}.

4. 2SC Effective Low Energy Theory

The spectrum in the 2SC state is made of 5 massive Gluons with a mass of the order of the gap, 3 massless Gluons confined (at zero temperature) into light glueballs and gapless up and down quarks in the direction (say) 3 of color.

4.1. The 5 massive Gluons

The relevant coset space \( G/H \) \cite{15,16} with

\[
G = SU_c(3) \times U_V(1) , \quad \text{and} \quad H = SU_c(2) \times \tilde{U}_V(1) 
\]

is parameterized by:

\[
V = \exp(i\xi^i X^i) , \tag{6}
\]

where \( \{X^i\} \ i = 1, \ldots, 5 \) belong to the coset space \( G/H \) and are taken to be \( X^i = T^{i+3} \) for \( i = 1, \ldots, 4 \) while

\[
X^5 = B + \frac{\sqrt{3}}{3} T^8 = \text{diag}(\frac{1}{2}, \frac{1}{2}, 0). \tag{7}
\]

\( T^a \) are the standard generators of \( SU(3) \). The coordinates

\[
\xi^i = \frac{\Pi^i}{f} \quad i = 1, 2, 3, 4 , \quad \xi^5 = \frac{\Pi^5}{f}
\]

via \( \Pi \) describe the Goldstone bosons which will be absorbed in the longitudinal components of the gluons. The vevs \( f \) and \( \tilde{f} \) are, at asymptotically high densities, proportional to \( \mu \). \( V \) transforms non linearly:

\[
V(\xi) \rightarrow u_V g V(\xi) h^V(\xi, g, u) h^V(\xi, g, u) \tag{8}
\]

with

\[
u_V \in U_V(1) , \quad g \in SU_c(3) ,
\]

\[
h(\xi, g, u) \in SU_c(2) , \quad h_{\tilde{V}}(\xi, g, u) \in \tilde{U}_V(1) . \tag{9}
\]
It is convenient to define the following differential form:

\[
\omega_\mu = i V^\dagger D_\mu V \quad \text{with} \quad D_\mu V = (\partial_\mu - ig_s G_\mu) V ,
\]

with \( G_\mu = G^m_\mu T^m \) the gluon fields while \( g_s \) is the strong coupling constant. \( \omega \) transforms according to:

\[
\omega_\mu \rightarrow h(\xi, g, u) \omega_\mu h^\dagger(\xi, g, u) + i h(\xi, g, u) \partial_\mu h^\dagger(\xi, g, u)
+ i h_V^\dagger(\xi, g, u) \partial_\mu h_V^\dagger(\xi, g, u).
\]

We decompose \( \omega_\mu \) into

\[
\omega^\parallel_\mu = 2 S^a \text{Tr} [S^a \omega_\mu] \quad \text{and} \quad \omega^\perp_\mu = 2 X^i \text{Tr} [X^i \omega_\mu] ,
\]

\( S^a \) are the unbroken generators of \( H \), while \( S^{1,2,3} = T^{1,2,3} \) and \( S^4 = \bar{B} / \sqrt{2} \).

The most generic two derivative kinetic Lagrangian for the goldstone bosons is:

\[
L = f^2 a_1 \text{Tr} \left[ \omega^\perp_\mu \omega^{\mu\perp} \right] + f^2 a_2 \text{Tr} \left[ \omega^\perp_\mu \right] \text{Tr} \left[ \omega^{\mu\perp} \right] .
\]

The double trace term is due to the absence of the condition for the vanishing of the trace for the broken generator \( X^5 \). It emerges naturally in the non linear realization framework at the same order in derivative expansion with respect to the single trace term. In the unitary gauge these two terms correspond to the five gluon masses. \(^{15}\)

4.2. ... and don’t relax yet!

For the fermions it is convenient to define the dressed fermion fields

\[
\tilde{\psi} = V^\dagger \psi ,
\]

transforming as \( \tilde{\psi} \rightarrow h_{\tilde{\psi}}(\xi, g, u) h(\xi, g, u) \tilde{\psi} \). \( \psi \) has the ordinary quark transformations (i.e. is a Dirac spinor). Pictorially \( \tilde{\psi} \) can be viewed as a constituent type field or alternatively as the bare quark field \( \psi \) immersed in the diquark cloud represented by \( V \). The non linearly realized effective Lagrangian describing in medium fermions, gluons and their self interactions.
up to two derivatives is:

\[ L = f^2 a_1 \text{Tr} \left[ \omega^I_{\mu} \omega^\perp_{\mu} \right] + f^2 a_2 \text{Tr} \left[ \omega^I_{\mu} \right] \text{Tr} \left[ \omega^\perp_{\mu} \right] \]
\[ + b_1 \bar{\psi} i \gamma^\mu (\partial_\mu - i \omega^\perp_\mu) \psi + b_2 \bar{\psi} \gamma^\mu \omega^\perp_\mu \psi \]
\[ + m_M \bar{\psi}^C C_i^\gamma_5 (iT^2) \psi + h.c. , \]  

(14)

where \( \bar{\psi}^C = i \gamma^2 \bar{\psi}^* \), \( i, j = 1, 2 \) are flavor indices and

\[ T^2 = S^2 = \frac{1}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & 0 \end{pmatrix} . \]

(15)

Here \( a_1, a_2, b_1 \) and \( b_2 \) are real coefficients while \( m_M \) is complex. From the last two terms, representing a Majorana mass term for the quarks, we see that the massless degrees of freedom are the \( \psi_{a=3,i} \). The latter possesses the correct quantum numbers to match the 't Hooft anomaly conditions 8.

5. The \( SU_c(2) \) Glueball Lagrangian

The \( SU_c(2) \) gauge symmetry does not break spontaneously and confines. Calling \( H \) a mass dimension four composite field describing the scalar glueball we can construct the following lagrangian 17:

\[ S_{G-ball} = \int d^4 x \left\{ \frac{\xi}{2} \sqrt{\hat{b}} H^{-\frac{1}{2}} \left[ \partial^0 H \partial^0 H - v^2 \right] \right\} \]
\[ - \frac{b}{2} H \log \left[ \frac{H}{\Lambda^4} \right] . \]  

(16)

This Lagrangian correctly encodes the underlying \( SU_c(2) \) trace anomaly. The glueballs move with the same velocity \( v \) as the underlying gluons in the 2SC color superconductor. \( \Lambda \) is related to the intrinsic scale associated with the \( SU_c(2) \) theory and can be less than or of the order of few MeVs 18 b. Once created, the light \( SU_c(2) \) glueballs are stable against strong interactions but not with respect to electromagnetic processes 17. Indeed, the glueballs couple to two photons via virtual quark loops.

\[ \Gamma [h \rightarrow \gamma \gamma] \approx 1.2 \times 10^{-2} \left[ \frac{M_h}{1 \text{ MeV}} \right]^5 \text{eV} , \]  

(17)

where \( \alpha = e^2/4\pi \simeq 1/137 \). For illustration purposes we consider a glueball mass of the order of 1 MeV which leads to a decay time \( \tau \approx 5.5 \times 10^{-14} \text{s} \).

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b According to the present normalization of the glueball field \( \hat{A}^4 \) is \( v \Lambda^4 \) with \( \Lambda \) the intrinsic scale of \( SU_c(2) \) after the coordinates have been appropriately rescaled 18,17 to eliminate the \( v \) dependence from the action 19.
This completes the effective Lagrangian for the 2SC state which corresponds to the Wigner-Weyl phase.

Using this Lagrangian one can estimate the $SU_c(2)$ glueball melting temperature to be

$$T_c \leq \sqrt{ \frac{90e^3}{2\pi} \bar{\Lambda} } < T_{CSC}.$$ 

(18)

Where $T_{CSC}$ is the color superconductive transition temperature. The deconfining/confining $SU_c(2)$ phase transition within the color superconductive phase is second order.

6. Conclusions

The Phase Diagram in Fig. 1 is just an educated guess of the true QCD phase diagram. Indeed other interesting phases may emerge. For example in the CFL phase the $K^+$ and $K^0$ modes may be unstable for large values of the strange quark mass signaling the formation of a kaon condensate. Vortex solutions in dense quark matter due to kaon condensation have been explored.

Another interesting avenue is the possibility of higher spin condensates which can enrich the phase diagram structure of QCD and QCD-like theories. Recent lattice simulations seem to support these predictions for 2 color QCD. If these results are confirmed then for the first time we observe spontaneous rotational breaking solely due to strongly interacting matter.

An interesting part of the QCD phase diagram which will be covered elsewhere is the temperature driven confining-deconfining phase transition. Thanks to lattice simulations we have a great deal of information. Only very recently new methods have been proposed which might help studying...
the phase diagram at non zero chemical potential via lattice simulations. However still much is left to be understood about the nature of the transition of hot hadronic matter to a plasma of deconfined quarks and gluons. New effective Lagrangians for the Polyakov loops and the Polyakov loops together with the hadronic states for the pure Yang-Mills theory lead to a deeper understanding of the properties of the underlying, temperature driven, deconfinement transition.

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