Solving efficiently the dynamics of many-body localized systems at strong disorder

Giuseppe De Tomasi,1 Frank Pollmann,2 and Markus Heyl1

1 Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187-Dresden, Germany
2 Technische Universität München, 85747 Garching, Germany

We introduce a method to efficiently study the dynamical properties of many-body localized systems in the regime of strong disorder and weak interactions. Our method reproduces qualitatively and quantitatively the real-time evolution with a polynomial effort in system size and independent of the desired time scales. We use our method to study quantum information propagation, correlation functions, and temporal fluctuations in one- and two-dimensional MBL systems. Moreover, we outline strategies for a further systematic improvement of the accuracy and we point out relations of our method to recent attempts to simulate the real-time dynamics of quantum many-body systems in classical or artificial neural networks.

Introduction— Experiments in quantum simulators, such as ultra-cold atoms in optical lattices and trapped ions can access the dynamical properties of closed quantum many-body systems far from equilibrium [1–4]. Therefore it is now possible to experimentally study intrinsically dynamical phenomena that are challenging to realize and probe on other platforms. One prominent example constitutes the many-body localized phase in systems with strong disorder, whose signatures have been observed in a series of recent experiments [5–9]. Many-body localization (MBL) describes an insulating and non-ergodic phase of matter, in which interacting particles are localized due to the presence of a strong disorder potential [10–13], extending the phenomenon of Anderson localization [14] to interacting particles. Importantly, the presence of interactions makes the dynamical properties much richer [15–21]. In particular, they give rise to an additional dephasing mechanism, allowing entanglement and quantum information propagation even though particle and energy transport is absent [15–18, 22]. Describing, however, quantitatively this interaction-induced propagation theoretically for large systems beyond exact numerical methods has remained as one of the main challenges.

In this work, we introduce an efficient numerical method to study the dynamics of weakly-interacting fermions in a strongly-localized MBL phase revealing the essential mechanisms for the characteristic quantum information transport. Importantly, our approach not only captures the qualitative aspects of the quantum real-time dynamics but is also quantitatively accurate. While it is based on an expansion in terms of the interaction strength, we find that errors remain bounded as a function of time and therefore our approximation remains controllable even over many temporal decades. Our method is efficient in that the unitary dynamics for local observables can be computed for any desired time with polynomial effort in system size \( L \) only. We make use of this to not only study the dynamics of interacting fermions in one dimension (1D) but also two dimensions (2D) for up to 200 lattice sites. After benchmarking our approach by comparing the entanglement entropy with exact diagonalization, we study the characteristic quantum information transport on the basis of the quantum Fisher information [7, 23–28], the logarithmic light-cone in correlation functions [22, 29, 30], and temporal fluctuations of observables, all for 1D and 2D. Finally, we point out a connection between our approach and recent ideas to encode quantum states into classical and artificial neural networks.

Models & Methods— At sufficiently strong disorder the MBL eigenstates are expected to be adiabatically connected to the non-interacting ones [31, 32]. In such a case the system is fully described by an extensive number of quasi-local integral of motions \( \{ \hat{I}_l \} \) [33–38], which emphasize an emerging weak form of integrability [31, 35] such that the Hamiltonian of the system can be expanded in the following form:

\[
\hat{H} = \sum_l J_l^{(1)} \hat{I}_l + \sum_{l,m} J_{l,m}^{(2)} \hat{I}_l \hat{I}_m + \ldots ,
\]

where \( l = 1, \ldots, L \) enumerates the \( L \) sites of the underlying lattice. For the considered weakly interacting case, higher-order couplings between the integrals of motion \( \hat{I}_l \) become exponentially suppressed, so that we can terminate the expansion as done in Eq. (1). Moreover, it is expected on general grounds that \( J_{l,m}^{(2)} \sim e^{-d(l,m)/\xi} \) with \( d(l, m) \) the spatial distance of the two involved lattice sites \( l \) and \( m \) and \( \xi \) denoting the localization length. While it is expected that this so-called l-bit representation exists, it has remained as a central challenge (i) to construct explicitly the integrals of motion \( \{ \hat{I}_l \} \) and (ii) to make use of the l-bit Hamiltonian to compute its dynamics.

In this work, we show that in the limit of weakly interacting fermions at strong disorder both of these challenges can be solved for efficiently. In this limit we can decompose the Hamiltonian \( H = H_0 + \hat{V} \) with \( H_0 \) a non-interacting Anderson-localized system and \( \hat{V} \) the interaction part, whose strength we denote by \( V \). We take as the \( \hat{I}_l \)’s the integrals of motion of \( \hat{H}_0 = \sum_l \epsilon_l \hat{I}_l \) with \( \hat{I}_l = \eta_l^\dagger \eta_l \) and \( \eta_l^\dagger \) (\( \eta_l \)) denoting the creation (annihilation) operator for a single-particle Anderson eigenstate.
we neglect all contributions that do not commute with the expectation values of time-evolved local observables and with the entanglement entropy calculated with $H$ the one calculated with $\hat{H}^{\text{eff}}$. (b): QFI for the 1D MBL system for several system size compared with exact results.

\[ \psi \text{ with eigenvalue } \epsilon_l. \] As a second step, we express

\[ V = \sum_{lmnk} B_{lmnk} \hat{n}_l \hat{n}_m \hat{n}_n \hat{n}_k \text{ in terms of the } \{ \hat{n}_l \}. \]

Then we neglect all contributions that do not commute with the $\{ \hat{l}_l \}$ so that we arrive at the following desired $l$-bit Hamiltonian:

\[ \hat{H}^{\text{eff}} = \sum_i \epsilon_i \hat{n}_i^\dagger \hat{n}_i + \sum_{l,m} B_{l,m} \hat{n}_l^\dagger \hat{n}_m \hat{n}_l \hat{n}_m, \tag{2} \]

with $B_{l,m} = B_{lmnm} - B_{lmlm}$. In the concluding discussion we will outline how one can improve systematically the accuracy of the $l$-bits by accounting for higher orders $V$. For the following, we will use the representation above and show that it is already sufficient to capture quantitatively the dynamics for small $V$.

Having discussed the construction of the $l$-bit Hamiltonian, we now how this can be used to study dynamics, which is based on two main properties. First, the real-time evolution of $\hat{n}_l$ and $\hat{n}_l^\dagger$ can be determined analytically via $\hat{n}_l(t) = \exp[i t c \epsilon_l] \hat{n}_l$, where $c \epsilon_l$ is always real. Second, for an initial state $|\psi\rangle$, which is a product states in terms of the bare fermions, i.e. Gaussian, the expectation values of time-evolved local observables and correlation functions can be reduced to the evaluation of Slater determinants, which can be done very efficiently. For example, for a generic local observable $A = \sum_{l,m} a_{l,m} \hat{n}_l^\dagger \hat{n}_m$, we need only to calculate $\langle \hat{n}_l \hat{n}_m \rangle(t) = \langle \hat{n}_l \hat{n}_m \rangle_{\psi} e^{it c \epsilon_l} = \langle \hat{n}_l \hat{n}_m \rangle_{\psi} \langle \hat{n}_l \hat{n}_m \rangle_{\psi}$. The term $\langle \hat{n}_l \hat{n}_m \rangle_{\psi} e^{it \sum_{p} (B_{l,p} - B_{p,l}) \hat{n}_p \hat{n}_p}$ can be efficiently computed using Wick’s theorem [39], interpreting $e^{it \sum_{p} (B_{l,p} - B_{p,l}) \hat{n}_p \hat{n}_p}$ as an effective time-evolution operator of the quadratic Hamiltonian $\hat{H}^{(l,m)} = \sum_{p} (B_{l,p} - B_{m,p}) \hat{n}_p \hat{n}_p$. Importantly, such initial conditions are typically chosen in theory [15, 20, 21, 40–44] and have been realized in the MBL context experimentally [5, 6, 8, 9].

For concreteness, we demonstrate our method for the Hamiltonian

\[ \hat{H} := -\frac{1}{2} \sum_{(i,j)} (\hat{c}_i^\dagger \hat{c}_j + \text{h.c.}) + \sum_j \hat{n}_j \hat{n}_j + V \sum_{(i,j)} \hat{n}_i \hat{n}_j \tag{3} \]

where $\hat{c}_i^\dagger (\hat{c}_i)$ is the fermionic creation (annihilation) operator at site $i$ and $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$. $\{ \hat{n}_j \}$ are random fields uniformly distributed between $[-W, W]$, and $V$ is the interaction strength.

We study the system both in a 1D lattice of size $L$ with periodic boundary conditions and defined in a rectangular lattice (2D) of size $S = L \times L$ with periodic and open boundary conditions respectively in the $x$ and in the $y$ direction. We focus on half-filling $N/L = 1/2$ ($N/|S| = 1/2$) with $N$ the number of fermions. The 1D system is believed to have an MBL-phase at strong-disorder [45–49]. The 2D case on the other hand has largely remained elusive due to the lack of efficient methods to simulate sufficiently large system sizes. However, let us note that a recent experiment gives evidence of an MBL phase in a bosonic 2D system [6].

Following our prescription outlined before, we first diagonalize the noninteracting model by introducing $\hat{n}_l = \sum_{i,j} \phi_i(i) \hat{c}_i^\dagger$. This then leads to $B_{l,m} = V \sum_{(g,j)} [\langle \phi_i(i) \rangle^2 |\phi_m(g)\rangle^2 - \phi_i(i) \phi_m(g) \delta_{i,m} \langle \phi_m(g) \rangle]$. In the remainder, we choose staggered initial states of charge-density type both for 1D $|\psi\rangle = \prod_{L=1}^{L/4} \hat{c}_{2s}^\dagger |0\rangle$ and for 2D $|\psi\rangle = \prod_{L=1}^{L/2} \prod_{L/4}^{L/4} \hat{c}_{2s(g)}^\dagger |0\rangle$, motivated by recent experiments [6]. Disorder averaged quantities will be indicated with an overline, e.g. $\langle \hat{n}_l \rangle$. 

FIG. 1. (a): Bipartite half-chain entanglement entropy $S(t)$ after a global quantum quench for several systems sizes ($L$) in 1D. $S(t)$ has been calculated using the exact Hamiltonian $\hat{H}$ (exact) and the effective model $\hat{H}^{\text{eff}}$ (approx.). The inset shows the relative error $\delta S(t) = |S - S^{\text{approx}}(t)|/S(t)$, between the entanglement entropy calculated with $\hat{H}$ the one calculated with $\hat{H}^{\text{eff}}$. (b): Disorder averaged QFI-density $\langle \tilde{\mathcal{Q}}(t) \rangle$ for the 2D model for several system sizes (S) and a fixed disorder and interaction strength. The inset shows that also in this case $\langle \tilde{\mathcal{Q}}(t) \rangle \sim \log \log t$. For both panels the evolution has been obtained using $\hat{H}^{\text{eff}}$. Dashed-lines are for the non-interacting case ($V = 0$) for the largest system size in each panels.
Benchmark for Quantum-Information propagation—

We now compare the exact dynamics by $\hat{H}$ with the one generated by $\hat{H}^\text{eff}$. For the benchmark we choose to study quantum information (entanglement) propagation which inherits one of the central and nontrivial features of MBL phases. In Fig. 1 we show data for two measures both obtained using exact diagonalization and via our effective Hamiltonian \[50\]. First, this includes the half-chain entanglement entropy

$$S(t) = -\text{Tr} \rho_{L/2}(t) \log \rho_{L/2}(t),$$

where $\rho_{L/2}(t)$ denotes the reduced density matrix of half of the system. Second, we study the quantum Fisher information (QFI) related to the initial charge-density pattern defined by

$$\mathcal{F}_Q(t) = 4 \left[ \langle \hat{\mathcal{O}}(t)^2 \rangle - \langle \hat{\mathcal{O}}(t) \rangle^2 \right], \quad \hat{\mathcal{O}} = \sum_x (-1)^x \hat{n}_x.$$ \[5\]

The QFI probes the propagation of quantum correlations and is an entanglement witness \[7, 23–28\], that has been also measured in a recent MBL experiment \[7\].

As we can see from Fig. 1(a) the effective model reproduces not only qualitatively the unbounded logarithmic growth of the entanglement entropy, but even more importantly also quantitatively correctly in the long-time limit. In particular, the inset in Fig. 1(a) shows that the relative error $\delta S(t) = |S(t) - S_{\text{approx}}(t)|/S(t)$ is a bounded function of time and remains smaller than 3% for all times. Let us note that the results for $\delta S(t)$ gives evidence that our method not only reproduces the logarithmic growth after disorder averaging but even for all random configurations. We computed $S(t)$ using our method by reconstructing the full reduced density matrix $\rho_{L/2}$, which is not scalable for large systems and is used here only for the benchmark against exact diagonalization. Similarly, also for the QFI the dynamics generated by the effective Hamiltonian follows closely the exact one, see Fig. 1(b) where we define the QFI-density $f_Q = \mathcal{F}_Q/2N$. While the entanglement entropy serves as a prime example for MBL properties, it’s computation within our method is not scalable to large system sizes. This, however, is different for the QFI which can still be computed efficiently even for large systems, which allows us to also access it in 2D, see below. It is important to note that despite being perturbative in the interaction strength $V$, our method reproduces the exact dynamics also for longer times than the expected range of validity of perturbation theory ($\sim 1/V$).

Results—

Having shown that our method reproduces quantitatively the exact dynamics at a controlled error, we now aim to further demonstrate the capabilities of our method. We target this goal by addressing several aspects of MBL systems which up to now have not been accessible or could not be settled due to system size limitations. This includes aspects of quantum information propagation, logarithmic light cones, and temporal fluctuations of local observables both in 1D and 2D. In the following, we choose a larger interaction strength $V = 0.1$ instead of $V = 0.01$ as used for Fig. 1, which increases slightly the relative error in the computed quantities, but in the same time allows us to amplify the influence of interaction effects.

Figure 2 show the QFI-density $f_Q(t) = \mathcal{F}_Q(t)/2N$ (Eq. 5) for the 1D (a) and the 2D case (b), respectively, now computed for much larger systems than done for the benchmark in Fig. 1. For the 2D model we choose the QFI along the x-direction, i.e., $\hat{\mathcal{O}} = \sum_x (-1)^x \hat{n}_x$ with $\hat{n}_x := \hat{n}_{x,t}$. For comparison we also include the results for the noninteracting models, which show quick satu-
rations to a system-size independent value. For nonvanishing interactions, the behavior of $\mathcal{F}_Q(t)$ changes completely and we observe a slow growth, which is consistent with $\mathcal{F}_Q(t) \sim \log \log t$ (insets) over many decades in time and almost independent of system size. As a consequence, we are capable to demonstrate slow quantum information propagation in 2D MBL systems, which up to now has not been possible by other means. In a recent experiment in trapped ions implementing a long-range disordering Ising model evidence for an intermediate $\mathcal{F}_Q(t) \sim \log t$ growth has been found [7], which, however, might be due to the presence of long-range hopping, that can lead to $B_{0j} \sim 1/t^3$ with power-law instead of exponential dependence [51].

As a next step we aim at studying quantum correlation spreading via the two-point connected correlation function, defined by

$$C_x(t) = \langle \hat{n}_x(t) \hat{n}_0(0) \rangle - \langle \hat{n}_x(t) \rangle \langle \hat{n}_0(0) \rangle,$$

where $t$ and real-space dependence is contained in Fig. 3. The 1D case we address in Fig. 3(a) where we show a color plot of $C_x(t)$ displaying the logarithmic light-cone [20, 30] over many decades with quantum correlations spreading in space only logarithmically slowly in time. Interestingly, however, we find that there exists a time scale $t^* \propto x$ beyond which $C_x(t)$ starts to decrease again, see Fig. 3 (b), an effect which has not yet been recognized before. Remarkably, this indicates that quantum correlations are eventually scrambled in the long-time limit in an MBL system, which might be consistent and even necessary with the expectation to reach in the asymptotic long-time limit a state with volume-law entanglement entropy. From the rescaling of the time axis used in Fig. 3(b) we find evidence that this correlation time $t^* \propto x$ scales exponentially with the distance $x$ ($\log t^* \sim x$).

For the case of an Anderson insulator with $V = 0$ the long-time evolved state obeys an area-law state [15, 36] with frozen quantum correlations in the long-time limit $C_x(t)$ (Fig. 3(b) 3 dashed-line). Finally, in Fig. 3(c) we study correlation spreading in 2D, where we again find a logarithmic light-cone.

As opposed to an Anderson insulator it has been argued that an MBL system can show relaxation [49, 52] in the sense that expectation values of local observables reach at long time a stationary value in the thermodynamic limit for a single disorder realization with decaying temporal fluctuations. Here, we use our method to re-examine the temporal fluctuations in 1D and to study them also for 2D systems. These are defined for $\hat{n}_x$ via

$$\Delta n_x^2(t) = \frac{1}{L} \sum_n \Delta n_x^2(n) = \langle (\hat{n}_x(t) - \langle \hat{n}_x \rangle_{\text{tav}})^2 \rangle,$$

where $\langle \hat{n}_x \rangle_{\text{tav}}$ denotes the long-time average of $\langle \hat{n}_x(t) \rangle$. The results we show in Fig. 4. Both in 1D and 2D the temporal fluctuations exhibit an algebraic decay with time, $\Delta n^2_x(t) \sim t^{-\alpha}$. As a reference we have included also the data for the noninteracting cases ($V = 0$, dashed-lines), where temporal fluctuations remain nonvanishing for all times. We find that the exponent $\alpha$ is proportional to the single-particle localization length $\xi_{loc}$ [53], for which we now aim to give an analytical argument. This shows that our method not only can be used for numerically computing quantities but also for analytical predictions. For this purpose we consider a special initial state $|\psi\rangle = \prod_{l=1}^L \frac{\hat{n}_l + \hat{n}_l^\dagger}{\sqrt{2}} |0\rangle$ for which the calculations are simplified but which gives qualitatively the same decay of the temporal fluctuations. [54]. For this state we find $\Delta n_x^2(t) = |\sum_{l \neq m} \phi_l(x) \phi_m(x)e^{i(e_l - e_m)}Q_{lm}|^2$ with $Q_{lm} = 2^{-2} \sum_{k=1}^{N} \cos(A_{k}^{lm}) \sum_{s \neq k} \cos(A_{s}^{lm})$ and $A_{k}^{lm} = (V/2)(B_{m,k} - B_{l,k}) \sim V\sqrt{\min(|m-k|, |l-k|)} \xi_{loc}^4$. The sum over $(l, m)$ can be restricted only to eigenstates, whose centers are located within a distance $\xi_{loc}$ away from $x$. Each term of the cos’s and sin’s with argument $A_{k}^{lm}$ decays exponentially in $k$, which leads to a power-law in time [42] with an exponent proportional to $\xi_{loc}$. This gives $\Delta n_x^2(t) \sim t^{-\xi_{loc}^4}$ [20].

Conclusions—In this work, we have formulated a method which allows to efficiently study the dynamics of weakly-interacting localized fermions. The accuracy of the approach can be further increased systematically by taking into account those contributions to the interaction term, which are not commuting with the bare integrals of motion $I_1$ and which have been completely neglected in the present study. For example, to lowest order they can be eliminated by a Schrieffer-Wolff transformation. Alternatively, one might start from self-consistently determined single-particle states including also corrections from the interaction potential.

Our method can be applied to any weakly interacting
MBL system, which exhibits an \( l \)-bit representation, not only limited to the quantum quench dynamics studied here. Thus, it can be used also to study, for example, also driven Floquet MBL systems [55] such as they appear in discrete time crystals [56, 57]. However, let us note that even in cases where an MBL phase might not be stable asymptotically for infinite system sizes and infinite times, our method might still provide a description on intermediate time scales. For example, it is currently under debate whether MBL can be stable at all in 2D [6, 8, 58–60]. A proposed mechanism for the breakdown of MBL relies on rare resonances which, however, only manifest on very long time scales, below which our \( l \)-bit description can be still accurate.

Overall, our method maps the dynamical quantum many-body problem onto a system of classical degrees of freedom of mutually commuting operators, similar in spirit to recent works where dynamical problems have been solved using classical [61] or artificial neural networks [62]. Instead of solving the problem in the basis of the bare particles, our work shows that a simple basis transformation onto more convenient degrees of freedom can improve the accuracy and efficiency dramatically, which might also be of relevance for the aforementioned approaches.

We thank J.H. Bardarson, S. Bera, A. Burin, A. Eckardt, I. M. Khaymovich, D. Trapin for several illuminating discussions. FP acknowledges the support of the DFG Research Unit FOR 1807 through grants no. PO 1370/2–1, TRR80, the Nanosystems Initiative Munich (NIM) by the German Excellence Initiative, and the European Research Council (ERC) under the European Commission (NIM) by the German Excellence Initiative, and the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement no. 771537). This research was conducted in part at the KITP, which is supported by NSF Grant No. NSF PHY-1748958. MH acknowledges support from the Deutsche Forschungsgemeinschaft via the Gottfried Wilhelm Leibniz Prize program.

---

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[2] R. Blatt and C. F. Roos, Nature Physics 8, 277 (2012).
[3] I. Bloch, J. Dalibard, and S. Nascimbène, Nature Physics 8, 267 (2012).
[4] I. M. Georgescu, S. Ashhab, and F. Nori, Reviews of Modern Physics 86, 153 (2014), arXiv:1308.6253 [quant-ph].
[5] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science 349, 842 (2015).
[6] J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science 352, 1547 (2016).
[7] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nat. Phys. advance online publication (2016), letter.
[8] P. Bordia, H. P. Lüschen, S. S. Hodgman, M. Schreiber, I. Bloch, and U. Schneider, Phys. Rev. Lett. 116, 140401 (2016).
[9] H. P. Lüschen, P. Bordia, S. Scherg, F. Alet, E. Altman, U. Schneider, and I. Bloch, Phys. Rev. Lett. 119, 260401 (2017).
[10] D. Basko, I. Aleiner, and B. Altshuler, Annals of Physics 321, 1126 (2006).
[11] R. Nandkishore and D. A. Huse, Annual Review of Condensed Matter Physics 6, 15 (2015), https://doi.org/10.1146/annurev-conmatphys-031214-041726.
[12] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, ArXiv e-prints (2018), arXiv:1804.11065 [cond-mat.dis-nn].
[13] P. Alet and N. Lafforencie, Comptes Rendus Physique (2018), https://doi.org/10.1016/j.crhy.2018.03.003.
[14] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[15] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. 109, 017202 (2012).
[16] M. Žnidarič, T. c. v. Prosen, and P. Prelovšek, Phys. Rev. B 77, 064426 (2008).
[17] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett. 110, 260601 (2013).
[18] M. Žnidarič, Phys. Rev. B 97, 214202 (2018).
[19] E. Canovi, D. Rossini, R. Fazio, G. E. Santoro, and A. Silva, Phys. Rev. B 83, 094431 (2011).
[20] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. B 90, 174302 (2014).
[21] R. Singh, J. H. Bardarson, and F. Pollmann, New Journal of Physics 18, 023046 (2016).
[22] M. C. Bañuls, N. Y. Yao, S. Choi, M. D. Lukin, and J. I. Cirac, Phys. Rev. B 96, 174201 (2017).
[23] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994).
[24] P. Hauke, M. Heyl, L. Tagliacozzo, and P. Zoller, Nature Physics 12, 778 EP (2016), article.
[25] D. Petz and C. Ghinea, Quantum Probability and Related Topics , 261 (2011), arXiv:1008.2417 [quant-ph].
[26] H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi, and M. K. Oberthaler, Science 345, 424 (2014).
[27] G. Tóth, Physical Review A 85, 022322 (2012), arXiv:1006.4368 [quant-ph].
[28] P. Hyllus, W. Laskowski, R. Krischek, C. Schwenner, W. Wieczorek, H. Weinfurter, L. Pezzè, and A. Smerzi, Physical Review A 85, 022321 (2012), arXiv:1006.4366 [quant-ph].
[29] D.-L. Deng, X. Li, J. H. Pixley, Y.-L. Wu, and S. Das Sarma, Phys. Rev. B 95, 024202 (2017).
[30] B. Swingle and D. Chowdhury, Phys. Rev. B 95, 060201 (2017).
[31] J. Z. Imbrie, Journal of Statistical Physics 163, 998 (2016).
[32] B. Bauer and C. Nayak, Journal of Statistical Mechanics: Theory and Experiment 2013, P09005 (2013).
[33] A. Chandran, I. H. Kim, G. Vidal, and D. A. Abanin, Phys. Rev. B 91, 085425 (2015).
[34] V. Ros, M. Muller, and A. Scardicchio, Nuclear Physics B 891, 420 (2015).
[35] D. A. Huse, R. Nandkishore, and V. Oganesyan, Phys. Rev. B 90, 174202 (2014).
[36] M. Serbyn, Z. Papić, and D. A. Abanin, Phys. Rev. Lett.
[37] N. Pancotti, M. Knap, D. A. Huse, J. I. Cirac, and M. C. Baǐnils, Phys. Rev. B 97, 094206 (2018).
[38] L. Rademaker, M. Ortuo, and A. M. Somoza, Annalen der Physik 529, 1600322.
[39] M. E. Peskin and D. V. Schroeder, An Introduction to quantum field theory (Addison-Wesley, Reading, USA, 1995).
[40] F. Iemini, A. Russomanno, D. Rossini, A. Scardicchio, and R. Fazio, Phys. Rev. B 94, 214206 (2016).
[41] R. Vasseur, S. A. Parameswaran, and J. E. Moore, Phys. Rev. B 91, 140202 (2015).
[42] S. Vardhan, G. De Tomasi, M. Heyl, E. J. Heller, and F.Pollmann, Phys. Rev. Lett. 119, 016802 (2017).
[43] E. V. H. Doggen, F. Schindler, K. S. Tikhonov, A. D. Mirlin, T. Neupert, D. G. Polyakov, and I. V. Gornyi, ArXiv e-prints (2018), arXiv:1807.05051 [cond-mat.dis-nn].
[44] M. Žnidarič, T. e. v. Prosen, and P. Prelovšek, Phys. Rev. B 77, 064426 (2008).
[45] G. De Tomasi, S. Bera, J. H. Bardarson, and F. Pollmann, Phys. Rev. Lett. 118, 016804 (2017).
[46] S. Bera, H. Schomerus, F. Heidrich-Meisner, and J. H. Bardarson, Phys. Rev. Lett. 115, 046603 (2015).
[47] S. Bera and A. Lakshminarayan, Phys. Rev. B 93, 134204 (2016).
[48] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[49] G. De Tomasi, S. Bera, J. H. Bardarson, and F. Pollmann, Phys. Rev. Lett. 115, 046603 (2015).
[50] S. Bera and A. Lakshminarayan, Phys. Rev. B 93, 134204 (2016).
[51] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[52] M. Schmitt and M. Heyl, Phys. Rev. Lett. 117, 120402 (2016).
[53] For additional comparing data see the Supplemental material.
[54] G. De Tomasi in preparation.
[55] S. Inglis and L. Pollet, Phys. Rev. Lett. 117, 120402 (2016).
[56] For additional data see the Supplemental material.
[57] For additional data see the Supplemental material.
[58] P. Bordia, H. Lüschen, U. Schneider, M. Knap, and I. Bloch, Nature Physics 13, 460 (2017).
[59] S. Choi, J. Choi, R. Landig, G. Kucsko, H. Zhou, J. Isoya, F. Jelezko, S. Onoda, H. Sumiya, V. Khemani, C. von Keyserlingk, N. Y. Yao, E. Demler, and M. D. Lukin, Nature (London) 543, 221 (2017).
[60] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, Nature (London) 543, 217 (2017).
[61] T. B. Wahl, A. Pal, and S. H. Simon, ArXiv e-prints (2017), arXiv:1711.02678 [cond-mat.dis-nn].
[62] W. De Roeck and J. Z. Imbrie, Philosophical Transactions of the Royal Society of London Series A 375, 20160422 (2017), arXiv:1705.00756 [math-ph].
[63] I.-D. Potirniche, S. Banerjee, and E. Altman, ArXiv e-prints (2018), arXiv:1805.01475 [cond-mat.dis-nn].
[64] M. Schmitt and M. Heyl, ArXiv e-prints, arXiv:1707.06656 (2017), arXiv:1707.06656 [cond-mat.str-el].
[65] I. Peschel and V. Eisler, Journal of Physics A: Mathematical and Theoretical 42, 504003 (2009).
[66] B. Kramer and A. MacKinnon, Reports on Progress in Physics 56, 1469 (1993).
SUPPLEMENTAL MATERIAL TO SOLVING EFFICIENTLY THE DYNAMICS OF MANY-BODY LOCALIZED SYSTEMS AT STRONG DISORDER

Using Free-Fermion Techniques— The effective model to describe an MBL-phase in the weak-interactions regime reads

$$\hat{H}^{\text{eff}} = \sum_l \epsilon_l \hat{\eta}_l^+ \hat{\eta}_l + \sum_{l,m} B_{l,m} \hat{\eta}_l^+ \hat{\eta}_m \hat{\eta}_m^+, \quad \tag{8}$$

where $$\hat{\eta}_l^+ = \sum_p \phi_l(i) \hat{c}_p$$ with $$\{\phi_l\}$$ and $$\{\epsilon_l\}$$ respectively the single-particle wavefunctions and eigenvalues. The coefficient $$\{B_{l,m}\}$$ are given by

$$B_{l,m} = V \sum_{i,j} \langle \phi_l(i) \phi_m(i) \phi_j(j) \phi_m(j) \rangle - \langle \phi_l(i) \rangle^2 \langle \phi_m(j) \rangle^2. \quad \tag{9}$$

In this section, we give an example to show how to calculate efficiently the expectation values of local observables if the quantum evolution is performed using the Hamiltonian $$\hat{H}^{\text{eff}}$$. Let’s consider the density-operator

$$\langle \hat{n}_x \rangle = \sum_{l,m} \phi_l(x) \phi_m(x) e^{it(\epsilon_l - \epsilon_m)} \langle \hat{\eta}_l^+ e^{it} \sum_p (\tilde{B}_{l,p} - \tilde{B}_{p,m}) \hat{\eta}_p \hat{\eta}_m \rangle,$$

where $$\tilde{B}_{l,m} = B_{l,m} + B_{m,l}$$. In Eq. 10 we have used the exact time-dependence of the operators $$\{\hat{\eta}^b\}$$

$$\hat{\eta}^b(t) = e^{it\epsilon_l} e^{it} \sum_m \tilde{B}_{l,m} \hat{\eta}_m \hat{\eta}_m \hat{\eta}_m^+. \quad \tag{11}$$

Now, the expectation value $$\langle \hat{n}_x \rangle$$ can be calculated with standard free-fermion technique [63], seeing the operator $$e^{it \sum_p (\tilde{B}_{l,p} - \tilde{B}_{p,m}) \hat{\eta}_p \hat{\eta}_m}$$ a quantum-evolution operator for the quadratic Hamiltonian defined by

$$\hat{H}^{(l,m)} = \sum_p (\tilde{B}_{l,p} - \tilde{B}_{p,m}) \hat{\eta}_p \hat{\eta}_p. \quad \tag{12}$$

FIG. 5. $$\mathcal{S}(t)$$ evolved with $$\hat{H}$$ and with $$\hat{H}^{\text{eff}}$$ (dashed-line) for the one-dimensional case for $$W = 6$$, $$V = 0.1$$ and several system sizes $$L$$. 

FIG. 6. $$\mathcal{S}(t)$$ evolved with $$\hat{H}$$ and with $$\hat{H}^{\text{eff}}$$ (dashed-line) for fixed $$L = 14$$ and $$V = 0.1$$ and for several disorder strengths $$W$$.

FIG. 7. The top-panel shows the time fluctuation $$\Delta n^2(t)$$ for several $$W$$’s in the strong disorder limit and fixed $$V$$ and $$L$$. The initial state is given by $$|\psi\rangle = \prod_{s=\pm L/4-2}^{L/4} \hat{c}_s^\dagger |0\rangle$$ (charge-density state). The bottom-panel shows $$\xi_{\text{loc}}$$ calculated with transfer-matrix technique at the band-center of the single-particle problem, giving evidence that $$\xi_{\text{loc}} \sim \log W^{-1}.$$
Comparison with exact the exact results— Here, we show further data, comparing the quantum dynamics computed with the exact Hamiltonian $\hat{H}$ and the effective Hamiltonian $\hat{H}^{\text{eff}}$. Figure 5 shows the entanglement entropy $S(t)$ for the one-dimensional system for $V = 0.1$ and $W = 6$, which are the values that have been used in the main text, for several system sizes $L$. $S(t)$ independently if calculate with $\hat{H}$ or $\hat{H}^{\text{eff}}$ presents the typical log-growth propagation in an MBL-phase ($\sim \xi \log(t)$). Although, the prefactor $\xi$ is different, since our approximation assumes that the localization length in the interacting case ($\xi$) is the same as the non-interacting one ($\xi^{\text{loc}}$). In other words, for the approximated dynamics we have $S^{\text{approx}}(t) \sim \xi^{\text{loc}} \log t$, while for the exact one $S(t) \sim \xi \log t$ with $\xi \sim \xi^{\text{loc}} + \mathcal{O}(V/W)$, making the relative error $\delta S(t)$ a bounded function of time. Figure 6 shows $S(t)$ for fixed interaction strength $V = 0.1$ and system size $L = 14$ for several disorder strengths $W$. As expected our approximation works better for larger disorder strength, in any case independently of $W$ we have $S^{\text{approx}}(t) \sim \log t$.

Time fluctuations— In this section, we show further data concerning the time fluctuation of local observables (i.e. $\hat{n}_x$), which is defined by

$$\Delta n^2(t) = \frac{1}{L} \sum_x \Delta n^2_x(t),$$

(13)

where

$$\Delta n^2_x(t) = (\langle \hat{n}_x \rangle(t) - \langle \hat{n}_x \rangle_{\text{time ave.}})^2,$$

(14)

and

$$\langle \hat{n}_x \rangle_{\text{time ave.}} = \lim_{T \to \infty} \frac{1}{T} \int_0^T ds \langle \hat{n}_x \rangle(s),$$

(15)

is the long-time average of $\langle \hat{n}_x \rangle(t)$. In the main text we show that the time fluctuation decay algebraically $\Delta n^2(t) \sim t^{-\alpha}$. Moreover, we claim that $\alpha \propto \xi^{\text{loc}}$ [20]. Figure 7 shows $\Delta n^2(t)$ for several $W$‘s in the strong disorder limit. For all inspected disorder strengths $\Delta n^2(t)$ decays algebraically for several order of magnitude, the curves have been rescaled to underline that $\alpha \propto \log W^{-1}$. Indeed in the strong disorder limit $\xi^{\text{loc}} \sim \log W^{-1}$, as shown in Fig. 7, where $\xi^{\text{loc}}$ has been calculated using standard transfer-matrix technique [64]. Furthermore, in the main text we support the result $\Delta n^2(t) \sim t^{-\xi^{\text{loc}}}$ with an analytical argument starting from a different initial state $|\psi\rangle = \prod_l^{L} \frac{\hat{n}_l + \hat{n}_{l+1}}{\sqrt{2}} |0\rangle$. Figure 8 shows $\Delta n^2(t)$ starting from $|\psi\rangle$ for several disorder strengths $W$, giving evidence that $\Delta n^2(t) \sim t^{-\xi^{\text{loc}}}$, with $\xi^{\text{loc}} \sim \log W^{-1}$. 

FIG. 8. $\Delta n^2(t)$ calculated using $\hat{H}^{\text{eff}}$ for several $W$‘s in the strong disorder limit and fixed $V$ and $L$ with initial state $|\psi\rangle = \prod_l^{L} \frac{\hat{n}_l + \hat{n}_{l+1}}{\sqrt{2}} |0\rangle$. 

\[\log \frac{\Delta n^2(t)}{\left(\log(W)\right)^{-1}} \}



\begin{figure}
\centering
\includegraphics[width=\columnwidth]{figure8}
\caption{$\Delta n^2(t)$ calculated using $\hat{H}^{\text{eff}}$ for several $W$‘s in the strong disorder limit and fixed $V$ and $L$ with initial state $|\psi\rangle = \prod_l^{L} \frac{\hat{n}_l + \hat{n}_{l+1}}{\sqrt{2}} |0\rangle$.}
\end{figure}