Abstract

We present some methods and results in the application of algebraic geometry and computer algebra to the study of algebraic vector bundles, foliations and zeta functions. A connection of the methods and results with noncommutative geometry will be considered.

Introduction

This is a work on perspective. In the survey I want to discuss algebraic and computer algebra aspects of vector bundles, foliations and zeta functions from commutative and noncommutative geometry points of view. I shall do it on the base of some parts of papers of A. Connes [1, 2], H. Moriyoshi [4] and N. Nekrasov and A. Schwarz [5] on noncommutative torus, operator algebras, and instantons on noncommutative $\mathbb{R}^4$, of A. Connes and D. Kreimer [3] on renormalization and the Hopf algebra structure of graphs and D. Broadhurst and D. Kreimer [6] on renormalization automated by Hopf algebra. Algebraic varieties appears in many mathematical and physical problems (strings as algebraic curves, some Calabi-Yau manifolds). So I want mention some novel papers on computer algebra [7, 8, 9] which have strong connections with algebraic geometric.

In section 1 we give formulae for computation of local charts, tangent bundles and transition functions of two dimensional sphere. These formulae (and some formulae of the next subsection) can be implemented on computer algebra systems (Reduce, Maple) straightforward. Most formulae are tested by author on different versions of Reduce and Maple. Then we discuss vector bundles over algebraic curves, Kronecker foliation, formal groups, moduli spaces and connections.

As have noted by A. Connes [1], there is "a spectral interpretation of zeros of zeta and L-functions in terms of constructions involving adeles, more specifically the noncommutative space of adele classes". In section 2 we give formulae for
computation of values of some zeta and L-functions.

Section 3 presents very short discussion of the algebraic context of noncommutative geometry.

For some problems it is necessary to programming them from the beginning (elementary data structures, efficient algorithm). A useful standpoint of solving a variety of problems is to model them in terms of some graphs (still 1959 F. Harary has proposed a graph method for complete reduction of a matrix with a view toward finding its eigenvalues [10]). These include trees, branching and connectivity with cutset of graphs, covering problems, networks and flows, matching and maximal matching. Graphs can modeling commutative diagrams and complexes as well as Feynman diagrams. Under interaction of strings manifolds of varying dimension can appear. Solving differential equations on the manifolds by an iterative method (for instance, by Newton map $N_f(x) = x - [Df(x)]^{-1} f(x)$) we have to compute Jacobian. The efficient algorithm can be implemented by the cutset method. In the Appendix we give our implementation of the cutset algorithm.

Side by side with works of above-mentioned authors, the survey based on my talks on SCAN2000 [33], Banach Center (Warsaw) conference [34], SAGP’99 (Luminy, France), IMACS-ACA [35], Mittag-Leffler seminar ”Geometry and Physics” (1998), Math. Institute of Stockholm University seminar ”Algebra and Geometry” [10], Dubna’98 [38] and on SNADE’97 [37].

So the purpose of the talk is two-fold: first to review algebraic geometric and computer algebra aspects of vector bundles, foliations and zeta functions; second to discuss some noncommutative context of the structures and methods in the frame of algebraic and analytic computations.

1 Geometry, Topology and Dynamics

As elements of differential geometry [13] and differential topology [11] play an important role in our consideration we recall for specialists in computer science notions of local charts, atlases, differentiable manifold and tangent bundles by the example of two dimensional sphere $S^2 : x_1^2 + x_2^2 + x_3^2 = 1$.

Local charts as half-spheres of $S^2$. 

In the case local charts are $(U_i, \varphi_i)$, where

$$x = (x_1, x_2, x_3), \ U_i = (x \in S^2 : x_i > 0), \ U_{3+i} = (x \in S^2 : x_i < 0),$$

$i = 1, 2, 3; \bigcup_{j=1}^6 U_j = S^2$ and maps $\varphi_i : U_i \to \mathbb{R}^2$ are follows

$$\varphi_1(x_2, x_3) : u_1 = x_2, u_2 = x_3;$$

$$\varphi_2(x_1, x_3) : v_1 = x_1, v_2 = x_3;$$

$$\varphi_2^{-1}(v_1, v_2) = (v_1, \sqrt{1 - v_1^2 - v_2^2}, v_2);$$

$$\varphi_3(x_1, x_2, x_3) : w_1 = x_2, w_2 = x_3;$$

$$\varphi_3^{-1}(w_1, w_2) = (w_1, w_2, \sqrt{1 - w_1^2 - w_2^2}).$$
Therefore $TS^1$ is a trivial bundle and manifold $S^1$ is parallelizable. Tangent bundle $TS^1$ of $S^1$ is the product $S^1 \times \mathbb{R}$. Therefore $TS^1$ is a trivial bundle and manifold $S^1$ is parallelizable. Tangent bundle $TS^1$ of $S^2$ can be defined by 3 local charts (in the sense of vector bundles; see below). Let $U_i = (x \in S^2 ||x|| < 1)$. Then $\bigcup_{i=1}^{3} U_i = S^2$. Let $E$ be the set $(x,t)$ in $\mathbb{R}^3 \times \mathbb{R}^3$ with $x \in S^2$ as above and $t = (u,v,w)$ such that scalar product $<x,t> = 0$. Let $p : (x,t) \rightarrow x$ be the projection $E$ on $S^2$. Homeomorphisms

$$\Phi_1 : p^{-1}(U_1) \rightarrow (U_1 \times \mathbb{R}^3)$$
(x_1, x_2, x_3; u, v, w) \mapsto (x_1, x_2, x_3; vx_3 - wx_2, u),
\Phi_2 : p^{-1}(U_2) \to (U_2 \times \mathbb{R}^2)
(x_1, x_2, x_3; u, v, w) \mapsto (x_1, x_2, x_3; wx_1 - ux_3, v),
\Phi_3 : p^{-1}(U_3) \to (U_3 \times \mathbb{R}^2)
(x_1, x_2, x_3; u, v, w) \mapsto (x_1, x_2, x_3; ux_3 - vx_2, w),
\intertext{define atlas \((U_i, \Phi_i)\) of tangent bundle \(TS^2\). There is well known fact that \(TS^2\) is not trivial bundle and therefore the manifold \(S^2\) is not parallelizable. The above tangent bundles of spheres are really vector bundles. This means that in the case of \(S^2\) there are unique continuous mappings}
\begin{align*}
g_{ij} : U_i \cap U_j &\to GL(2, \mathbb{R}),
\end{align*}
\intertext{such that the mapping}
\begin{align*}
\psi_{ij} = \Phi_i \circ \Phi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^2 &\to (U_i \cap U_j) \times \mathbb{R}^2,
\end{align*}
\intertext{satisfies \(\psi_{ij} = (x, g_{ij}(x)y)\) for every \((x, y) \in (U_i \cap U_j) \times \mathbb{R}^2\). In the case \(TS^2\) the transition functions \(g_{ij}\) are follows:}
\begin{align*}
g_{21}(x) &= \frac{-1}{x_2^2 + x_3^3} \begin{vmatrix} x_1 x_2 & x_3 \\ -x_3 & x_1 x_2 \end{vmatrix},
g_{32}(x) &= \frac{-1}{x_1^2 + x_3^3} \begin{vmatrix} x_2 x_3 & x_1 \\ -x_1 & x_2 x_3 \end{vmatrix},
g_{13}(x) &= \frac{-1}{x_1^2 + x_2^3} \begin{vmatrix} x_1 x_3 & x_2 \\ -x_2 & x_1 x_3 \end{vmatrix}.
\end{align*}
\subsection{1.1 Vector Bundles}
More generally we can give a manifold by charts and local diffeomorphisms. \textit{Local chart} or a \textit{system of coordinates} on a topological space \(M\) is a pair \((U, \varphi)\) where \(U\) is an open set in \(M\) and \(\varphi : U \to \mathbb{R}^m\) is a homeomorphism from \(U\) to an open set \(\varphi(U)\) in \(\mathbb{R}^m\). An \textit{atlas} \(\Phi\) of dimension \(m\) is a collection of local charts whose domains cover \(M\) and such that if \((U, \varphi), (U_1, \varphi_1) \in \Phi\) and \(U \cap U_1 \neq 0\) then the map
\[\varphi_1 \circ \varphi^{-1} : \varphi(U \cap U_1) \to \varphi_1(U \cap U_1)\]
is a \(C^r\)-diffeomorphism between open sets in \(\mathbb{R}^m\).
\textit{Fibre space} is the object \((E, p, B)\), where \(p\) is the continuous surjective (= on) mapping of a topological space \(E\) onto a space \(B\) (in our consideration \(B = M\) is a differential manifold), and \(p^{-1}(b)\) is called the \textit{fibre} above \(b \in B\). Both the notation \(p : E \to B\) and \((E, p, B)\) are used to denote a \textit{fibration}, a \textit{fibre space}, a
fibre bundle or a bundle.

Vector bundle is fibre space each fibre \( p^{-1}(b) \) of which is endowed with the structure of a (finite dimension) vector space \( V \) over skew-field \( K \) such that the following local triviality condition is satisfied: each point \( b \in B \) has an open neighborhood \( U \) and a \( V \)-isomorphism of fibre bundles \( \phi : p^{-1}(U) \rightarrow U \times V \) such that \( \phi|_{p^{-1}(b)} : p^{-1}(b) \rightarrow b \times V \) is an isomorphism of vector spaces for each \( b \in B \).

\( \dim V \) is said to be the dimension of the vector bundle.

An Hermitian bundle over algebraic variety \( X \) consists of a vector bundle over \( X \) and a choice of \( \mathcal{C}^\infty \) Hermitian metric on the vector bundle over complex manifold \( X(\mathbb{C}) \), which is invariant under antiholomorphic involution of \( X(\mathbb{C}) \).

**Tangent bundle**

The tangent space to a differentiable manifold \( M \) at point \( a \in M \) can be defined as the set of tangency classes of smooth paths in \( M \) based at \( a \). It will be denoted by \( T_aM \). Elements of \( T_aM \) are called tangent vectors to \( M \) at \( a \).

The tangent bundle of \( M \), denoted by \( TM \), is the union of the tangent spaces at all the points of \( M \). By well known way \( TM \) can be made into a smooth manifold.

Recall well known facts about \( TM \):

(i) if \( M \) is \( C^r \) then \( TM \) is \( C^{r-1} \);

(ii) if \( M \) is \( C^\infty \) or \( C^\omega \) then the same holds for \( TM \);

(iii) if \( M \) has dimension \( n \) then \( TM \) has dimension \( 2n \);

(iv) there is a natural map \( p : TM \rightarrow M \) called the projection map, taking \( T_aM \) to \( a \) for each \( a \) in \( M \), i.e. \( p \) takes all tangent vectors at \( a \) to the point \( a \) itself.

Thus \( p^{-1}(a) = T_aM \) (fibre of the bundle over \( a \)). The projection \( p \) is a smooth map \( C^{r-1} \) if \( M \) is \( C^r \);

**Vector fields and Flows.**

A vector field on a smooth manifold \( M \) is a map \( F : M \rightarrow TM \) which satisfies \( p \circ F = id_M \), where \( p \) is the natural projection \( TM \rightarrow M \). By its definition a vector field is a section of the bundle \( TM \).

Let \( F \) be a vector field on \( M \). A solution curve to \( F \), based at \( a \) on \( M \) is a path \( c : I \rightarrow M \) (where \( I \) is some open interval \( (a, b) \) with \( a < 0 < b \) ) satisfying \( c(0) = a \) and \( T_t c = F(c(t)) \) for all \( t \) in \( I \). (Here \( TI = I \times \mathbb{R} \) and so \( T_t c \) is a map \( \mathbb{R} \rightarrow TM \).) This demonstrate the well known fact that a vector field on a manifold is the global version of a first order autonomous system of \( n \) ordinary differential equations on \( \mathbb{R}^n \).

Let

\[
\dot{x} = Ax, \tag{2}
\]

where \( x \in \mathbb{R}^n \), \( A \) is \( n \times n \) matrix, be a linear system of ordinary differential equations. It is well known that the solution of the system (2) together with the initial condition \( x(0) = x_0 \) is given by

\[
x(t) = e^{At}x_0,
\]
where $e^{At}$ is an $n \times n$ matrix function defined by its Taylor series. The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow of linear system (2).

Recall the definition of the flow $\phi_t$ of the nonlinear system $\dot{x} = F(x)$, (3)

Let $I(x_0)$ be the maximal interval of existence of the solution $\phi(t, x_0)$ of (2) with the initial value $x(0) = x_0$. Let $X$ be an open subset of $\mathbb{R}^n$ and let $F \in C^1(X)$. For $x_0 \in X$, let $\phi(t, x_0)$ be the solution of (3) with initial value problem $x(0) = x_0$ defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the mapping $\phi_t : X \to X$ defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the flow of the differential equation (1) or the flow of the vector field $F(x)$.

### 1.2 Vector Bundles over Projective Algebraic Curves

Let $X$ be a projective algebraic curve over algebraically closed field $k$ and $g$ the genus of $X$. Let $VB(X)$ be the category of vector bundles over $X$. Grothendieck have shown that for a rational curve every vector bundle is a direct sum of line bundles. Atiyah have classified vector bundles over elliptic curves. The main result is

**Theorem [12]**. Let $X$ be an elliptic curve, $A$ a fixed base point on $X$. We may regard $X$ as an abelian variety with $A$ as the zero element. Let $E(r, d)$ denote the set of equivalence classes of indecomposable vector bundles over $X$ of dimension $r$ and degree $d$. Then each $E(r, d)$ may be identified with $X$ in such a way that

$$det : E(r, d) \to E(1, d)$$

corresponds to $H : X \to X$, where $H(x) = hx = x + x + \cdots + x$ ($h$ times), and $h = (r, d)$ is the highest common factor of $r$ and $d$.

Curve $X$ is called a configuration if its normalization is a union of projective lines and all singular points of $X$ are simple nodes. For each configuration $X$ can assign a non-oriented graph $\Delta(X)$, whose vertices are irreducible components of $X$, edges are its singular and an edge is incident to a vertex if the corresponding component contains the singular point. Drozd and Greuel have proved:

**Theorem [13]**. 1. $VB(X)$ contains finitely many indecomposable objects up to shift and isomorphism if and only if $X$ is a configuration and the graph $\Delta(X)$ is a simple chain (possibly one point if $X = \mathbb{P}^1$).

2. $VB(X)$ is tame, i.e. there exist at most one-parameter families of indecomposable vector bundles over $X$, if and only if either $X$ is a smooth elliptic curve or it is a configuration and the graph $\Delta(X)$ is a simple cycle (possibly, one loop if $X$ is a rational curve with only one simple node).

3. Otherwise $VB(X)$ is wild, i.e. for each finitely generated $k$–algebra $\Lambda$ there
exists a full embedding of the category of finite dimensional $\Lambda$–modules into $\mathcal{VB}(X)$.

Let $X$ be an algebraic curve. How to normalize it? There are several methods, algorithms and implementations for this purpose. A new algorithm and implementation is presented in [9].

### 1.3 Foliations

A simplest example of foliation is a trivial $k$ dimensional foliation or a trivial codimension $n - k$ foliation of Euclidean space $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, 0 \leq k \leq n$:

$$\mathbb{R}^n = \bigcup_{(x_{k+1}, \ldots, x_n)} \mathbb{R}^k \times (x_{k+1}, \ldots, x_n),$$

that is, $\mathbb{R}^n$ decomposes into a union of $\mathbb{R}^k \times (x_{k+1}, \ldots, x_n)$'s each of which is $C^\infty$ diffeomorphic to $\mathbb{R}^k$. $\mathbb{R}^k \times (x_{k+1}, \ldots, x_n)$ is called a leaf of the foliation.

Still one example can be obtained from consideration of nonsingular vector fields on the torus [13, 18]. More generally, a nonsingular flow on a manifold corresponds to a foliation on the manifold by one dimensional leaves where the leaves are provided with Riemannian metrics and directed.

Consider vector field

$$Q(x, y)\partial x + P(x, y)\partial y, \tag{1}$$

where $P$ and $Q$ are complex polynomials of degree $n$ in two variables. The vector field (or corresponding dual 1-form $\omega = P(x, y)dx - Q(x, y)dy$) gives rise to a foliation $\mathcal{F}$ of degree $n$ of two dimensional projective space $\mathbb{P}^2$ over $\mathbb{C}$ by Riemann surfaces and singular points. It is naturally to ask about limit cycles and multivalued first integrals of (1) and a topology of the foliation. The investigation of (1) began in [19], where the case $n = 2$ considered. For the investigation of (1) and $\mathcal{F}$ it is naturally to introduce of algebraic geometric methods and algebraic geometric invariants [20, 21]. These include blow-ups, divisors, indexes of singular points, Chern classes of vector bundles over some Riemann surfaces and computation of holonomy (monodromy) groups. Already in the case of [13] the complex dimension of the space of coefficients of (1) is equal 12 (real dimension is equal 24). This defines the application of computer algebra in the case.

#### 1.3.1 Kronecker foliation

Let $\theta$ be a irrational number. Consider a vector field $F = \partial x + \theta \partial y$ on the two dimensional torus $T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. The Kronecker foliation $\mathcal{F}_\theta$ is defined by the flow of $F$ in the following way: each leaf is labeled by a point of circle $S^1 = \mathbb{R}/\mathbb{Z}$. So the leaf space is the foliated bundle

$$\left(\mathbb{R} \times \mathbb{R}/\mathbb{Z}\right)/\mathbb{Z}, (x, y) \sim (x + 1, y + \theta),$$
where the equivalence relation on \((x, y) \in \mathbb{R} \times S^1\) is defined by
\[
R_\theta(x, y) = (x, y + \theta) \text{ mod } 1.
\]
By the foliation \(F_\theta\) the \(C^*\)-algebra \(A_\theta\) is constructed. \(A_\theta\) is generated by a pair of unitary symbols subject to the relation
\[
UV = \exp(2\pi i \theta) VU. \tag{2}
\]

1.3.2 Formal Groups

Formal groups was introduced by J. Dieudonne and M. Lazard around 1954. During 1968-71 and later were found interesting connections between topology and formal groups [22, 23]. Also was found connection between formal groups and zeta functions [25]. There is a connection of formal groups with characteristic classes of foliations and \(K\)-theory [24]. The construction of the Grothendiesk group \(K_0\) for algebraic monoid is rather simple [27]. What is the \(K_0\) functor in the case of formal groups? Let us consider a simple example.

**Proposition.** Let \(\mathcal{F}_k\) be the category of commutative formal groups over field \(k\) such that every formal group in the category is the product of one dimensional formal groups. Then \(K_0(\mathcal{F}_k)\) is a free abelian group with infinite number of generators.

1.3.3 Moduli spaces

The theory of moduli spaces [28, 29] has, in recent years, become the meeting ground of several different branches of mathematics and physics - algebraic geometry, instantons, differential geometry, string theory and arithmetics. Here we recall some underlying algebraic structures of the relation. In previous subsection we have reminded the situation with vector bundles on projective algebraic curves \(X\). On \(X\) any first Chern class \(c_1 \in H^2(X, \mathbb{Z})\) can be realized as \(c_1\) of vector bundle of prescribed rank (dimension) \(r\). How to classify vector bundles over algebraic varieties of dimension more than 1? This is one of important problems of algebraic geometry and the problem has closed connections with gauge theory in physics and differential geometry. Manford [23] and others have formulated the problem about the determination of which cohomology classes on a projective variety can be realized as Chern classes of vector bundles? Moduli spaces are appeared in the problem. What is moduli? Classically Riemann claimed that \(3g - 3\) (complex) parameters could be for Riemann surface of genus \(g\) which would determine its conformal structure (for elliptic curves, when \(g = 1\), it is needs one parameter). From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance,
for an elliptic curve the invariant is the modular invariant of the elliptic curve. Let $B$ be a class of objects. Let $S$ be a scheme. A family of objects parametrized by the $S$ is the set of objects

$$X_s : s \in S, X_s \in B$$

equipped with an additional structure compatible with the structure of the base $S$. Parameter varieties is a class of moduli spaces. These varieties is very convenient tool for computer algebra investigation of objects that parametrized by the parameter varieties. We have used the approach for investigation of rational points of hyperelliptic curves over prime finite fields [30].

1.4 Connection

Consider the connection in the context of algebraic geometry. We shall base at one I. Shafarevitch’s seminar on algebraic geometry. Let $S/k$ be the smooth scheme over field $k$, $U$ an element of open covering of $S$, $\mathcal{O}_S$ the structure sheaf on $S$, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on $U$. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1–dimension differentials, $\mathcal{F}$ be a coherent sheaf. The connection on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S)$, $g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$ 

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma(U, \Theta^1_{S/k})$. The connection is the homomorphism

$$\rho : \Theta^1_{S/k} \to \text{End}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}), \rho(\partial)(fg) = \partial(f)g + f\rho(\partial).$$

1.4.1 Integration of connection

Let $\Omega^i_{S/k}$ be the sheaf of germs of $i$–differentials,

$$\nabla^i(\alpha \otimes f) = d\alpha \otimes f + (-1)^i\alpha \wedge \nabla(f)$$

Then $\nabla$, $\nabla^i$ define the sequence of homomorphisms:

$$\mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F} \to \Omega^2_{S/k} \otimes \mathcal{F} \to \cdots,$$

(3)

A connection is integrable if (3) is a complex. This is equivalent to $\nabla \otimes \nabla^1 = 0$. 

9
2 Zeta Functions

We have considered some validated numerics aspects of evaluation of zeta functions in papers [33, 32]. Here we shall consider some computer algebra aspects of computation and evaluation of values of zeta functions.

2.1 Riemann zeta

Consider the series \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) for complex values of \( s \) with \( \text{Re}(s) \geq 1 \). Analytical formula for computation of values of \( \zeta(s) \) can be taken from the result of Backlund (a little bit reformulated). The result is based of Euler-Maclaurin summation.

**Proposition** (Backlund) Let \( N \) be natural > 1. Let \( s = \sigma + it \) and let \( \sigma \geq 1 \). Let \( B_{2k} \) be the Bernoulli numbers in even numeration, \( S(N-1,s) = \sum_{n=1}^{N-1} \frac{1}{n^s} + \frac{\Delta^{1-s}}{s-1} \), \( B(N,k,s) = \frac{1}{2^N} B_{2k} s^N N^{-s-1} + \cdots + \frac{B_{2k}}{(2k)!} s(s+1) \cdots (s+2k-2) N^{-2k+1} \).

Then

\[
\zeta(s) = S(N-1,s) + B(N,k,s) + R_{2k},
\]

where

\[
|R_{2k-2}| \leq \left| \frac{s + 2k-1}{\sigma + 2k-1} \right| B_{2k} \text{ term of } (4).
\]

2.2 Dirichlet series

Let \( K = \mathbb{Q}(\sqrt{D}) \) be a real quadratic field with positive integer squarefree \( D \) and \( \chi(n) = (\frac{\Delta}{n}) \) be the Kronecker symbol. Here

\[
\Delta = \begin{cases} 
D, & D \equiv 1 \pmod{4}, \\
4D, & D \equiv 2,3 \pmod{4}.
\end{cases}
\]

Let \( A = \pi/\Delta, \ E(x) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{\sqrt{t}} dt, \ \text{erfc}(x) = \frac{2}{\pi} \int_{x}^{\infty} e^{-t^2} dt. \)

**Proposition** [35, 34]

\[
L(1,\chi) = \frac{1}{\sqrt{\Delta}} \sum_{n=1}^{m} \chi(n) E(An^2) + \sum_{n=1}^{m} \left( \frac{\chi(n)}{n} \right) \text{erfc}(n\sqrt{A}) + R_m,
\]

where \( |R_m| < \Delta^{3/2} e^{-A} n^2 \).

2.3 Remarks about \( L \)-functions of elliptic curves

[31]. Let \( E/\mathbb{Q} \) be an elliptic curve given in Weierstrass form by an equation

\[
E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,
\]

(5)
and let \( b_2, b_4, b_6, b_8, c_4, c_6, \Delta, j \) be the usual associated quantities \([31]\). Let now (5) be a global minimal Weierstrass equation for \( E \) over \( \mathbb{Z} \). For each prime \( p \) the reduction (5) \( \pmod{p} \) defines a curve \( E_p \) over the prime field \( \mathbb{F}_p \). Let \( A_p \) denote the number of points of \( E_p \) rational over \( \mathbb{F}_p \). Let

\[
t_p = 1 + p - A_p.
\]

If \( p \nmid \Delta \), then \( t_p \) is the trace of Frobenius and satisfies \( |T_p| \leq 2 \sqrt{p} \). In the case

\[
\zeta_{E_p}(s) = \frac{1 - t_p p^{-s} + p^{1-2s}}{(1 - p^{-s})(1 - p^{1-s})}.
\]

If \( p \mid \Delta \), then \( E_p \) is not an elliptic curve and has a singularity \( S \). In the case

\[
t_p = \begin{cases} 
0, & \text{if } S \text{ is a cusp,} \\
1, & \text{if } S \text{ is a node,} \\
-1, & \text{if } S \text{ is a node with tangent quadratic over } \mathbb{F}_p 
\end{cases}
\]

The Hasse-Weil \( L \)–function of \( E/\mathbb{Q} \) is defined by equation

\[
L(E)(s) = \prod_{p \mid \Delta} \frac{1}{(1 - t_p p^{-s})} \prod_{p \nmid \Delta} \frac{1}{1 - t_p p^{-s} + p^{1-2s}}.
\]

From the work of A. Wiles, R. Taylor and A. Wiles, and work of F. Diamond it is known that (semistable) elliptic curves over \( E/\mathbb{Q} \) are modular. Knowing the modularity of \( E/\mathbb{Q} \) is equivalent to the existence of a modular form \( f \) on \( \Gamma_0(N) \) for some natural value \( N \), which we write \( f = \sum a_n q^n \). The \( L \)–function of \( E \) is thus given by the Mellin transform of \( f \),

\[
L(f,s) = \sum a_n q^n. 
\]

In particular, the behavior of \( L(E,s) \) at \( s = 1 \) can be deduced from modular properties of \( E \).

Let \( E/\mathbb{Q} \) be a modular elliptic curve and the global minimal model of the \( E/\mathbb{Q} \) has prime conductor \( l \). Let \( p \) be a prime and \( A_p \) be the number of points of \( E_p \) in \( \mathbb{F}_p \). Then there exists a modular form \( f \) on \( \Gamma_0(l) \),

\[
f = \sum_{n=1}^{\infty} a_n q^n, 
\]

where \( a_p, p \neq l \), equals \( p + 1 - A_p \).

3 Some Algebraic Methods and Structures of Noncommutative Geometry

At first remind two definitions.

**Hopf algebras.**

Let \( H \) be an algebra with unit \( e \) over field \( k \). Let \( a, b \in H \), \( \mu(a,b) = ab \) the product in \( H \), \( \alpha \in k \), \( p : k \to H \), \( p(\alpha) = \alpha e \) the unit, \( \varepsilon : H \to k \), \( \varepsilon(a) = 1 \), the counit, \( \Delta(a) = a \otimes a \) the coproduct, \( S : H \to H \) the antipode, such that the axioms
1)\((1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta\) (coassociativity);
2)\(\mu \circ (1 \otimes S) \circ \Delta = \mu \circ (S \otimes 1) \circ \Delta = p \circ \varepsilon\) (antipode).

Then the system \((H, \mu, p, \varepsilon, \Delta, S)\) is called the **Hopf algebra**.

Let \(A\) and \(B\) be \(C^*\)-algebras. Let \(K\) be the \(C^*\)-algebra of compact operators. If \(A \otimes K\) is isomorphic to \(B \otimes K\) then \(A\) is called **Morita equivalent** to \(B\).

We give here some remarks to algebraic setting of papers \([1, 2, 4]\). In a general context this algebraic setting includes:

- Operator algebras.
- Representation theory.
- K-theory.
- Algebraic geometry.

More specifically:
- \(C^*\)-algebras; finite projective modules; holonomy groupoid;
- groupoid for transformation groups; group \(C^*\)-algebras; Morita equivalence; K-theory;
- von Neumann algebras; cyclic cohomology; invariant transversal measures and the Ruelle-Sullivan current; Godbillon-Vey class; Hopf algebras.

One of the paradigm of noncommutative geometry is to describe the geometry of ordinary space in terms of the algebra of functions and then deforming to the noncommutative case. By deformation theory as the part of moduli space theory some constructions of section 1 can be embedded into noncommutative geometry.

4 Appendix. The Common Lisp text of the packages CUTSET and CUTSETDG

Rooted graph \(G\) is the graph that each node of \(G\) is reachable from a node \(r \in G\). By `cutset` of the graph we shall understand an appropriated subset of nodes (called cutpoints) such that any cycle of the graph contains at least one cutpoint. DFS is the abbreviation of Depth First Search method. During DFS we numbering nodes and label (mark) edges. By Tarjan [13] the DFS method has linear complexity. The packages are implemented on Allegro CL 3.0.2 [11, 12]. The main function `Cutsetdg` of the package CUTSETDG computes Cutset (a subset of vertices which cut all cycles in the graph) of arbitrary rooted directed graph. It uses 3 functions: `Adjarcn`, `Cutpoints` and `Unicut`. For this program the author developed rather simple and efficient algorithm that based on DFS-method. The function is the base for Cutset methods for systems of procedures. The main function `Cutset` of the Package CUTSET computes `Cutset` of arbitrary rooted graph. It also uses 3 functions: `Adjedgn`, `Cutpoints` and `Unicut`. An algorithm for this function that based on DFS-method is also developed by the author. The algorithm is rather simple and efficient. The function is the base for Cutset methods. Package CUTSET

Title: Cutset of rooted graph

Summary: This package implements computation of a cutset of rooted graph.
The graph have to defined by adjacency list.
Example of Call: (Cutset 'a) where "a" is a root of exploring graph.

;; Name: Cutset
;;
;; Title: Cutset of rooted graph
;;
;; Author: Nikolaj M. Glazunov
;;
;; Summery:
;; This package implements computation of a cutset of rooted graph. The graph have to defined by adjacency list
;;
;; Allegro CL 3.0.2
;;
(DEFUN Cutset (startnode)
  ;; startnode is a root of exploring graph
  ;; DFS - Depth First Search method for connected graph
  ;; cuts - cutset of the graph
  ;; st - stack
  ;; v - exploring node
  ;; sv - son of the exploring node
  ;; inl - inverse edges list
  ;; df - DFS-numbering of the node (property)
  ;; Adjed - adjacency list of the node (property)
  ;; 1, 2 - lables

  (SETF (GET startnode 'df ) 1) ;DFS-numbering is equal 1
  (SETF (GET startnode 'Adjed)
    (Adjedgn (LIST startnode))) ;;startnode obtained the a-list
  ;; property Adjed (adjacency edges)

  (PROG ( st v sv inl cuts)
    (PUSH startnode st)
    1  (SETQ v (CAR st)) ;;explored node received value from stack of nodes

    2  (COND ((NOT (EQ NIL (GET v 'Adjed)))
      (SETQ sv (CAAR (GET v 'Adjed)))

      (COND ((EQ NIL (GET sv 'df))
        (SETF (GET sv 'df) (+ 1 (GET v 'df))))))

13
;;; modification of the edge of son's adj-list
(SETF (GET sv 'Adjed)
  (Adjedgn (LIST sv))) ;; node sv obtained the adj-list property
;; Adjed (adjesency edges)
(SETF (GET v 'Adjed) (REMOVE (LIST sv v) (GET v 'Adjed) :TEST 'EQUAL))
(SETF (GET sv 'Adjed) (REMOVE (LIST v sv) (GET sv 'Adjed) :TEST 'EQUAL))
(PUSH sv st)
(GO 1)
)
(T ;; the node has number.
  ;; the edge is inverse
  (SETQ inl (CONS (CAR (GET v 'Adjed)) inl))
(SETF (GET v 'Adjed) (REMOVE (LIST sv v) (GET v 'Adjed) :TEST 'EQUAL))
(SETF (GET sv 'Adjed) (REMOVE (LIST v sv) (GET sv 'Adjed) :TEST 'EQUAL))
(GO 2)
)
)
)
)
(POP st)
(COND ((NOT (NULL st))
  (GO 1))
(T ;; end
  (SETQ cuts (Unicut (Cutpoints inl))))
  (RETURN cuts))
)
)
)

(DEFUN Adjedgn (PATH)
  (MAPCAN #'(LAMBDA (E)
    (COND ((MEMBER (CAR E) (CDR E)) NIL)
      (T (LIST E)))))
  (MAPCAR #'(LAMBDA (E)
    (CONS E PATH))
      (GET (CAR PATH) 'NEIGHBORS))))

(DEFUN Cutpoints (inl)
  (MAPCAR 'CAR inl))
(DEFUN Unicut (cpl)
  (COND ((NULL cpl) NIL)
    (T (CONS (CAR cpl)
Package CUTSETDG
Title: Cutset of a rooted directed graph
Summary: This package implements computation of a cutset of a rooted directed graph. The graph have to defined by adjacency list.
Example of Call: (Cutsetdg 'a) where "a" is a root of exploring graph.

The Common Lisp text of the package CUTSETDG

;;;; Name: Cutsetdg
;;;; Title: Cutset of a rooted directed graph
;;;; Author: Nikolaj M. Glazunov
;;;; Summary:
;;;; This package implements computation of a cutset
;;;; of a rooted directed graph. The graph have to defined by adjacency
;;;; list
;;;; Example of Call: (Cutsetdg 'a)
;;;; where "a" is a root of exploring graph
;;;; Allegro CL 3.0.2
;;;;
(defun cutsetdg (startnode)
  ;; startnode is a root of exploring graph
  ;; DFS - Depth First Search method for connected graph
  ;; st - stack
  ;; v - exploring node
  ;; sv - son of the exploring node
  ;; inl - inverse edges list
  ;; df - DFS-numbering of the node (property)
  ;; Outarcs - adjacency list of Output arcs of the node (property)
  ;; 1, 2 - lables

  (setf (get startnode 'df) 1) ;DFS-numbering is equal 1
  (setf (get startnode 'Outarcs)
(Adjarcn (LIST startnode)); startnode obtained the a-list
    ;; property Outarcn (Output arcs of the
    ;; node)

(PROG (st v sv inl cuts)
    (PUSH startnode st)
1  (SETQ v (CAR st)); explored node received value from stack of nodes
2  (COND ((NOT (EQ NIL (GET v 'Outarcs)))
               (SETQ sv (CAAR (GET v 'Outarcs)))
               (COND ((EQ NIL (GET sv 'df))
                          (SETF (GET sv 'df) (+ 1 (GET v 'df)))
                   ;; modification of the edge of son's adj-list
                   (SETF (GET sv 'Outarcs)
                          (Adjarcn (LIST sv)))); node sv obtained the adj-list property
                          ;; Outarcs (adjacency Outarcs)
                          (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                          :TEST 'EQUAL))
               (PUSH sv st)
               (GO 1)
    )

(T
       (COND ((AND (> (GET v 'df) (GET sv 'df))
                   (NOT (EQUAL v sv)))
               (SETQ inl (CONS (CAR (GET v 'Outarcs)) inl))
               (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                          :TEST 'EQUAL))
               (GO 2)
       )
    )

(T
 (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                          :TEST 'EQUAL))
 (GO 2)
)
)
)
)
)
)
)
)
)
)
)(T
  (POP st)
  (COND ((NOT (NULL st))
          (GO 1)
  )
  (T ;; end
   (SETQ cuts (Unicut (Cutpoints inl)))
   (RETURN cuts)
16
(DEFUN Adjarcn (PATH)
  (MAPCAN #'(LAMBDA (E)
    (COND ((MEMBER (CAR E) (CDR E)) NIL)
           (T (LIST E)))))
  (MAPCAR #'(LAMBDA (E)
    (CONS E PATH))
           (CAR (GET (CAR PATH) 'NEIGHBORS))))
(DEFUN Cutpoints (inl)
  (MAPCAR 'CAR inl))
(DEFUN Unicut (cpl)
  (COND ((NULL cpl) NIL)
         (T (CONS (CAR cpl)
                  (Unicut (REMOVE (CAR cpl) cpl)))))
  )
)

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