UNIQUE DETERMINATIONS IN INVERSE SCATTERING PROBLEMS WITH PHASELESS NEAR-FIELD MEASUREMENTS

DEYUE ZHANG
School of Mathematics, Jilin University
Changchun 130012, China

YUKUN GUO*
School of Mathematics, Harbin Institute of Technology
Harbin 150001, China

FENGLIN SUN
School of Mathematics, Jilin University
Changchun 130012, China

HONGYU LIU*
Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong SAR, China

(Communicated by Habib Ammari)

Abstract. In this paper, we establish the unique determination results for several inverse acoustic scattering problems using the modulus of the near-field data. By utilizing the superpositions of point sources as the incident waves, we rigorously prove that the phaseless near-fields collected on an admissible surface can uniquely determine the location and shape of the obstacle as well as its boundary condition and the refractive index of a medium inclusion, respectively. We also establish the uniqueness in determining a locally rough surface from the phaseless near-field data due to superpositions of point sources. These are novel uniqueness results in inverse scattering with phaseless near-field data.

1. Introduction. Inverse scattering theory is concerned with the determination of an unknown or inaccessible target scatterer from the incident wave and the corresponding measured near-field or far-field data. In particular, the scattering and inverse scattering of time-harmonic waves are of great importance in various applications such as radar detection, sonar inspection, nondestructive testing and modern medical diagnostics. The time-harmonic inverse scattering problems are typically based on complex-valued data. Hence, in terms of the accessibility to the corresponding phase information, the measured data in inverse scattering problems can be classified into two types: phased/full data and intensity-only/modulus-only or phaseless data. Over the past several decades, the inverse scattering problems...
with full measured data (both phase and intensity) have been mathematically and numerically studied intensively in the literature (see, e.g., the research monographs \cite{1, 13, 23} and \cite{14, 15, 34, 35, 53, 54} for recently proposed interesting applications). Recently, a great deal of effort has been devoted to phaseless inverse scattering problems \cite{2, 5, 6, 24, 25, 29, 30}. The motivation for investigating phaseless inverse problems is mainly due to the fact that such phase information is extremely difficult to be measured accurately or even completely unavailable in a rich variety of realistic scenarios. As a result, only the phaseless data can be practically obtained in these cases.

The inverse scattering problem with one incident plane wave and phaseless far-field data is challenging due to the \textit{translation invariance property}, namely, the modulus of the far field pattern is invariant under translations \cite{31, 39}. Specifically speaking, the location of scatterer cannot be uniquely determined by the phaseless far-field data. Nevertheless, shape recovery from phaseless data is still possible. Actually, quite a number of inversion schemes have been proposed to reconstruct the shape of the scatterer from the modulus-only far-field data with a single incident plane wave, see \cite{19, 20, 21, 32, 33, 36}. We also refer to \cite{12, 18} for the relevant numerical studies.

It is often desirable to develop corresponding techniques to tackle the difficulty of translation invariance. An effective attempt in this direction is the superposition of distinct incident plane waves proposed in \cite{61}. Recently, the reference ball technique was introduced in \cite{57} to break the translation invariance in phaseless inverse acoustic scattering problem. By incorporating a suitably chosen ball into the scattering system as well as the superposition of incident plane wave and point sources, the authors in \cite{57} rigorously prove that the location and shape of the obstacle as well as its boundary condition or the refractive index can be uniquely determined by the modulus of far-field patterns. We would like to point out that the idea of adding a reference ball to the scattering system was first proposed in \cite{37} to numerically enhance the resolution of the linear sampling method. The reference ball technique was later also adopted in \cite{56} in dealing with phaseless inverse scattering problems. Similar strategies of adding reference objects or sources to the scattering system have also been extensively applied to the theoretical analysis and numerical approaches for different models of phaseless inverse scattering problems \cite{17, 16, 22, 58}. In the absence of any additional reference object, the uniqueness can be established by the superposition of incident point sources and phaseless far-field data, see \cite{51}.

In this paper, we deal with the uniqueness issue concerning the inverse acoustic scattering problems with incident point sources and phaseless near-field data. In the areas of optics and engineering sciences, the phaseless inverse scattering with near-field data is also known as phase retrieval problem \cite{40, 41}. The inverse scattering problems with phaseless near-field data have been studied numerically (see, e.g., \cite{8, 9, 10, 12, 11, 46, 52}), and few studies have been made on the theoretical aspects of uniqueness for the inverse scattering problems. A recent result on uniqueness in \cite{24} was related to the reconstruction of a potential with the phaseless near-field data for point sources on a spherical surface and an interval of wavenumbers, which was extended in \cite{25} to determine the wave speed in generalized 3-D Helmholtz equation. We also refer to \cite{49} for the relevant studies on the Maxwell case. The uniqueness of a coefficient inverse scattering problem with phaseless near-field data has been established in \cite{30}. By using the superposition of incident point sources
and supplementing a reference ball of impedance type, the uniqueness for an inverse interior scattering problem with phaseless data was recently established in [60]. We also refer to [29, 44, 45] for some recovery algorithms for the inverse medium scattering problems with phaseless near-field data. The stability analysis for linearized near-field phase retrieval in X-ray phase contrast imaging can be found in [42].

In this work, we establish the uniqueness via superposition of incident point sources, which does not rely on any additional reference/interfering scatterer. By introducing the concept of an admissible surface, together with the superposition of point sources, we rigorously prove that the bounded scatterer (impenetrable obstacle or medium inclusion) and the locally perturbed half-plane (a.k.a locally rough surface) could be uniquely determined from the phaseless near-field measurements. For the uniqueness of inverse scattering by locally rough surfaces with phaseless far-field data, we refer to [56]. A key feature of this study is that we make use of the limited-aperture phaseless near-field data co-produced by the scatterer and point sources, thus the configuration is practically more feasible than the cases of using the phaseless scattered data. Moreover, only a single frequency is utilized in the problems under consideration. Some studies on phaseless inverse scattering problems in the multi-frequency regime can be found in [26, 27, 28]. Finally, we refer to [50] for the similar investigations using interference waves and an interval of frequencies.

The rest of this paper is arranged as follows. Section 2 is devoted to the inverse scattering problem of uniquely determining a bounded scatterer. Then in section 3, we study the uniqueness results on phaseless inverse scattering by locally perturbed half-planes.

2. Uniqueness for inverse scattering by bounded scatterers.

2.1. Problem setting. We begin this section with the acoustic scattering problems for an incident plane wave. Assume \( D \subset \mathbb{R}^3 \) is an open and simply-connected domain with \( C^2 \) boundary \( \partial D \). Denote by \( \nu \) the unit outward normal to \( \partial D \) and by \( S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \} \) the unit sphere in \( \mathbb{R}^3 \). Let \( u^i(x, d) = e^{ikx \cdot d} \) be a given incident plane wave, where \( d \in S^2 \) and \( k > 0 \) are the incident direction and wavenumber, respectively. Then, the obstacle scattering problem can be formulated as: to find the total field \( u = u^i + u^s \) which satisfies the following boundary value problem (see [13]):

\[
\begin{align*}
(1) & \quad \Delta u + k^2 u = 0 \quad \text{in} \ \mathbb{R}^3 \setminus D, \\
(2) & \quad \mathcal{B} u = 0 \quad \text{on} \ \partial D, \\
(3) & \quad \lim_{r \to |x| \to \infty} r \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0,
\end{align*}
\]

where \( u^s \) denotes the scattered field and 3 is the Sommerfeld radiation condition. Here \( \mathcal{B} \) in 2 is the boundary operator defined by

\[
\mathcal{B} u = \begin{cases} 
\partial u & \text{for a sound-soft obstacle,} \\
\frac{\partial u}{\partial \nu} + ik \lambda u & \text{for an impedance obstacle,}
\end{cases}
\]

where \( \lambda \) is a real parameter. This boundary condition 4 covers the Dirichlet/sound-soft boundary condition, the Neumann/sound-hard boundary condition (\( \lambda = 0 \)), and the impedance boundary condition (\( \lambda \neq 0 \)).
The medium scattering problem is to find the total field \( u = u^i + u^s \) that fulfills
\[
\Delta u + k^2 n(x) u = 0 \quad \text{in} \; \mathbb{R}^3,
\]
\[
\lim_{r=|x| \to \infty} r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0,
\]
where the refractive index \( n(x) \) of the inhomogeneous medium is piecewise continuous such that \( \text{Re}(n) > 0, \text{Im}(n) \geq 0 \) and \( 1 - n(x) \) is supported in \( D \).

The direct scattering problems 1–3 and 5–6 admit a unique solution (see, e.g., [7, 13, 43]), respectively, and the scattered wave \( u^s \) has the following asymptotic behavior
\[
u^s(x, d) = \frac{e^{i k |x|}}{|x|} \left\{ u^\infty(\hat{x}, d) + \mathcal{O}\left( \frac{1}{|x|} \right) \right\}, \; |x| \to \infty
\]
uniformly in all observation directions \( \hat{x} = x/|x| \in \mathbb{S}^2 \). The analytic function \( u^\infty(\hat{x}, d) \) defined on the unit sphere \( \mathbb{S}^2 \) is called the far field pattern or scattering amplitude (see [13]).

Now, we turn to introducing the inverse acoustic scattering problem for incident point sources with limited-aperture phaseless near-field data. To this end, we first introduce the following definition of admissible surfaces.

**Definition 2.1** (Admissible surface). An open surface \( \Gamma \) is called an admissible surface with respect to domain \( \Omega \) if
(i) \( \Omega \subset \mathbb{R}^3 \setminus \overline{D} \) is bounded and simply-connected;
(ii) \( \partial \Omega \) is analytic homeomorphic to \( \mathbb{S}^2 \);
(iii) \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \);
(iv) \( \Gamma \subset \partial \Omega \) is a two-dimensional analytic manifold with nonvanishing measure.

**Remark 1.** We would like to point out that this requirement for the admissibility of \( \Gamma \) is quite mild and thus can be easily fulfilled. For instance, \( \Omega \) can be chosen as a ball whose radius is less than \( \pi/k \) and \( \Gamma \) is chosen as an arbitrary corresponding hemisphere.

For a generic point \( z \in \mathbb{R}^3 \setminus \overline{D} \), the incident field due to the point source located at \( z \) is given by
\[
\Phi(x, z) := \frac{e^{i k |x-z|}}{4\pi |x-z|}, \; x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{ z \}),
\]
which is also known as the fundamental solution to the Helmholtz equation. We denote by \( v^i_D(x, z) \) and \( v^\infty_D(\hat{x}, z) \) the near-field and far-field pattern generated by \( D \) corresponding to the incident field \( \Phi(x, z) \). Define
\[
v(x, z) := v^i_D(x, z) + \Phi(x, z), \; x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{ z \})
\]
and
\[
v^\infty(\hat{x}, z) := v^\infty_D(\hat{x}, z) + \Phi^\infty(\hat{x}, z), \; \hat{x} \in \mathbb{S}^2,
\]
where \( \Phi^\infty(\hat{x}, z) := e^{-i k \hat{x} \cdot z} / (4\pi) \) is the the far-field pattern of \( \Phi(x, z) \).

For two generic and distinct source points \( z_1, z_2 \in \mathbb{R}^3 \setminus \overline{D} \), we denote by
\[
v^i(x; z_1, z_2) := \Phi(x, z_1) + \Phi(x, z_2), \; x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{ z_1 \} \cup \{ z_2 \}),
\]
the superposition of these point sources. Then, by the linearity of direct scattering problem, the near-field co-produced by \( D \) and the incident wave \( v^i(x; z_1, z_2) \) is given by
\[
v(x; z_1, z_2) := v(x, z_1) + v(x, z_2), \; x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{ z_1 \} \cup \{ z_2 \}).
\]
With these preparations, we formulate the phaseless inverse scattering problems as the following.

**Problem 1** (Phaseless inverse scattering by a bounded scatterer). Assume that $\Gamma$ and $\Sigma$ are admissible surfaces with respect to $\Omega$ and $G$, respectively, such that $\Omega \cap G = \emptyset$. Let $D$ be the impenetrable obstacle with boundary condition $\mathcal{B}$ or the inhomogeneous medium with refractive index $n$. Given the phaseless near-field data

\[
\{ |v(x, z_0)|: x \in \Sigma \}, \\
\{ |v(x, z)|: x \in \Sigma, \ z \in \Gamma \}, \\
\{ |v(x, z_0) + v(x, z)|: x \in \Sigma, \ z \in \Gamma \}
\]

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in \mathbb{R}^3 \setminus (D \cup \Gamma \cup \Sigma)$. Then we have

**Theorem 2.2.** Assume that $\Gamma$ and $\Sigma$ are admissible surfaces with respect to $\Omega$ and $G$, respectively, such that $\Omega \cap G = \emptyset$. For two scatterers $D_1$ and $D_2$, suppose that the corresponding near-fields satisfy that

\[
|v_1(x, z_0)| = |v_2(x, z_0)|, \quad \forall x \in \Sigma, \\
|v_1(x, z)| = |v_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Gamma
\]

and

\[
|v_1(x, z_0) + v_1(x, z)| = |v_2(x, z_0) + v_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Gamma
\]

for an arbitrarily fixed $z_0 \in \mathbb{R}^3 \setminus (D \cup \Gamma \cup \Sigma)$. Then we have
(i) If $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.

(ii) If $D_1$ and $D_2$ are two medium inclusions with refractive indices $n_1$ and $n_2$ respectively, then $n_1 = n_2$.

Proof. From 8, 9 and 10, we have for all $x \in \Sigma$, $z \in \Gamma$

\begin{equation}
\text{Re} \left\{ v_1(x, z_0) \overline{v_1(x, z)} \right\} = \text{Re} \left\{ v_2(x, z_0) \overline{v_2(x, z)} \right\},
\end{equation}

where the overline denotes the complex conjugate. According to 8 and 9, we denote

\begin{equation}
v_j(x, z_0) = r(x, z_0) e^{i\alpha_j(x, z_0)}, \quad v_j(x, z) = s(x, z) e^{i\beta_j(x, z)}, \quad j = 1, 2,
\end{equation}

where $r(x, z_0) = |v_j(x, z_0)|$, $s(x, z) = |v_j(x, z)|$, $\alpha_j(x, z_0)$ and $\beta_j(x, z)$ are real-valued functions, $j = 1, 2$.

Since $\Sigma$ is an admissible surface of $G$, by Definition 2.1 and the analyticity of $v_j(x, z)$ with respect to $x$, we have $s(x, z) \neq 0$ for $x \in \Sigma$, $z \in \Gamma$. Further, the continuity yields that there exists open sets $\tilde{\Sigma} \subset \Sigma$ and $\Gamma_0 \subset \Gamma$ such that $s(x, z) \neq 0$, $\forall (x, z) \in \tilde{\Sigma} \times \Gamma_0$. Similarly, we have $r(x, z_0) \neq 0$ on $\tilde{\Sigma}$. Again, the continuity leads to $r(x, z_0) \neq 0$ on an open set $\Sigma_0 \subset \Sigma$. Therefore, we have $r(x, z_0) \neq 0$, $s(x, z) \neq 0$, $\forall (x, z) \in \Sigma_0 \times \Gamma_0$. In addition, taking 11 into account, we derive that

\begin{equation}
\cos[\alpha_1(x, z_0) - \beta_1(x, z)] = \cos[\alpha_2(x, z_0) - \beta_2(x, z)], \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0.
\end{equation}

Hence, either

\begin{equation}
\alpha_1(x, z_0) - \alpha_2(x, z_0) = \beta_1(x, z) - \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0
\end{equation}

or

\begin{equation}
\alpha_1(x, z_0) + \alpha_2(x, z_0) = \beta_1(x, z) + \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0
\end{equation}

holds with some $m \in \mathbb{Z}$.

First, we consider the case 12. Since $z_0$ is fixed, let us define $\gamma(x) := \alpha_1(x, z_0) - \alpha_2(x, z_0) - 2m\pi$ for $x \in \Sigma_0$, and then, we deduce for all $(x, z) \in \Sigma_0 \times \Gamma_0$

\begin{equation}
v_1(x, z) = s(x, z) e^{i\beta_1(x, z)} = s(x, z) e^{i\beta_2(x, z) + i\gamma(x)} = v_2(x, z) e^{i\gamma(x)}.
\end{equation}

From the reciprocity relation [3, Theorem 3] for point sources, we have

\begin{equation}
v_1(z, x) = e^{i\gamma(x)} v_2(z, x), \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0.
\end{equation}

Then, for every $x \in \Sigma_0$, by using the analyticity of $v_j(z, x) (j = 1, 2)$ with respect to $z$, we have $v_1(z, x) = e^{i\gamma(z)} v_2(z, x), \forall z \in \partial \Omega$. Let $w(z, x) = v_1(z, x) - e^{i\gamma(z)} v_2(z, x)$, then

\begin{equation}
\begin{cases}
\Delta w + k^2 w = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}

By the assumption of $\Omega$ that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$, we find $w = 0$ in $\Omega$. Now, the analyticity of $v_j(z, x) (j = 1, 2)$ with respect to $z$ yields

\begin{equation}
v_1(z, x) = e^{i\gamma(z)} v_2(z, x), \quad \forall z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\})\text{.}
\end{equation}

i.e., for all $z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\})$

\begin{equation}
v_{D_1}^s(z, x) + \Phi(z, x) = e^{i\gamma(z)} \left[v_{D_2}^s(z, x) + \Phi(z, x)\right].
\end{equation}

By Green’s formula [13, Theorem 2.5], one can readily deduce that $v_{D_j}^s(z, x) (j = 1, 2)$ are bounded for $z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2})$, which, together with 14, implies that
Further, noticing

$$v_{\mathcal{D}_1}^\circ(z, x) = v_{\mathcal{D}_2}^\circ(z, x), \ \forall (x, z) \in \Sigma_0 \times \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2} \cup \{ x \}).$$

And thus, the far-field patterns coincide, i.e.

$$v_{\mathcal{D}_1}^\circ(\hat{z}, x) = v_{\mathcal{D}_2}^\circ(\hat{z}, x), \ \forall (x, \hat{z}) \in \Sigma_0 \times \mathbb{S}^2.$$  

Now, from the mixed reciprocity relation [13, Theorem 3.16] for the obstacle or [47, Theorem 2.2.4] for the inhomogeneous medium, we have

$$u_1^\circ(x, -\hat{z}) = u_2^\circ(x, -\hat{z}), \ \forall (x, \hat{z}) \in \Sigma_0 \times \mathbb{S}^2.$$  

Further, the analyticity of $u_j^\circ(x, d)(j = 1, 2)$ with respect to $x$ yields $u_1^\circ(x, d) = u_2^\circ(x, d), \ \forall (x, d) \in \partial G \times \mathbb{S}^2$. By the similar discussion of 14 for $u_1^\circ(x, d) - u_2^\circ(x, d)$ on $G$, we have

$$u_1^\circ(x, d) = u_2^\circ(x, d), \ \forall (x, d) \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2}) \times \mathbb{S}^2.$$  

Therefore, we obtain

$$(15) \quad u_1^\circ(\hat{x}, d) = u_2^\circ(\hat{x}, d), \ \forall \hat{x}, d \in \mathbb{S}^2.$$  

Next we are going to show that the case 13 does not hold. Suppose that 13 is true, then following a similar argument, we see that there exists $\eta(x)$ such that $v_1(z, x) = e^{i\eta(x)}\overline{v_2(z, x)}$ for $x \in \Sigma_0, z \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2} \cup \{ x \}),$ i.e.

$$v_{\mathcal{D}_1}^\circ(z, x) + \Phi(z, x) = e^{i\eta(x)}[\overline{v_{\mathcal{D}_2}^\circ(z, x)} + \Phi(z, x)].$$

Then, from the boundedness of $v_{\mathcal{D}_2}^\circ(z, x)$, it can be seen that $\Phi(z, x) - e^{i\eta(x)}\overline{\Phi(z, x)}$ is bounded for all $z \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2} \cup \{ x \})$. Since

$$\Phi(z, x) - e^{i\eta(x)}\overline{\Phi(z, x)} = \left[ 1 - e^{i\eta(x)} \right] \frac{\cos(k|z - x|)}{4\pi |z - x|^2} + i \left[ 1 + e^{i\eta(x)} \right] \frac{\sin(k|z - x|)}{4\pi |z - x|^2},$$

by letting $z \to x$, we deduce $e^{i\eta(x)} = 1$, and thus, $v_1(z, x) = \overline{v_2(z, x)}$ for $z \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \overline{\mathcal{D}_2} \cup \{ x \})$. We claim that $v_1^\circ(\hat{z}, x) \neq 0$ for $\hat{z} \in \mathbb{S}^2$. Otherwise, if $v_1^\circ(\hat{z}, x) \equiv 0$ for $\hat{z} \in \mathbb{S}^2$, then from Rellich Lemma [13, Theorem 2.14], we have $v_1(z, x) = 0$ for all $z \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \{ x \})$. Further, from $v_1(z, x) = v_{\mathcal{D}_1}^\circ(z, x) + \Phi(z, x)$ and the boundedness of $v_{\mathcal{D}_1}^\circ(z, x)$, we deduce $\Phi(z, x)$ is bounded for all $z \in \mathbb{R}^3 \setminus (\overline{\mathcal{D}_1} \cup \{ x \})$, which is a contradiction. Then, the continuity leads to $v_1^\circ(\hat{z}, x) \neq 0, \forall \hat{z} \in S$, where $S \subset \mathbb{S}^2$ is an open set. By taking $\hat{z} \in S, z = \rho \hat{z}$, and using the definition of far-field pattern (see [13, Theorem 2.6]), we obtain

$$\lim_{\rho \to \infty} e^{-ik\rho}v_1(\rho \hat{z}, x) = v_1^\circ(\hat{z}, x)$$

and

$$\lim_{\rho \to \infty} e^{ik\rho}v_2(\rho \hat{z}, x) = v_2^\circ(\hat{z}, x).$$

Further, noticing $v_1(\rho \hat{z}, x) = v_2(\rho \hat{z}, x)$ and $v_1^\circ(\hat{z}, x) \neq 0$, we have

$$\lim_{\rho \to \infty} e^{2ik\rho} = \frac{v_{\mathcal{D}_1}^\circ(\hat{z}, x)}{v_1^\circ(\hat{z}, x)},$$

which is a contradiction. Hence, the case 13 does not hold.

Having verified 15, we can complete our proof as the consequences of two existing uniqueness results. For the inverse obstacle scattering, by Theorem 5.6 in [13], we
have \( D_1 = D_2 \) and \( B_1 = B_2 \), and for inverse medium scattering, Theorem 10.5 in [13] leads to \( n_1 = n_2 \).

**Remark 2.** We would like to point out that an analogous uniqueness result in two dimensions remains valid after appropriate modifications of the fundamental solution, the radiation condition and the admissible surface. So we omit the 2D details.

**Remark 3.** We would like to remark that a similar result on uniqueness can be also obtained by using the superposition of a fixed plane wave and some point sources as the incident fields.

3. Uniqueness for inverse scattering by locally perturbed half-planes.

3.1. Problem statement. We begin this section with the precise formulations of the model scattering problem. Assume that the real-valued function \( f \in C^2(\mathbb{R}) \) has a compact support. Let \( \Gamma = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = f(x_1), x_1 \in \mathbb{R} \} \) be the locally perturbed curve and \( D = \{ x \in \mathbb{R}^2 \mid x_2 > f(x_1), x_1 \in \mathbb{R} \} \) be the locally perturbed half-plane above curve \( \Gamma \). Denote by \( \Gamma_c = \{ x \in \mathbb{R}^2 \mid x_2 = 0 \} \) and by \( \Gamma_p = \Gamma \setminus \Gamma_c \) the local perturbation. For a generic point \( z \in D \), the incident field \( u^i \) due to the point source located at \( z \) is given by

\[
u^i(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|) = \frac{i}{4} J_0(k|x - z|) - \frac{1}{4} Y_0(k|x - z|), \quad x \in D \setminus \{z\},
\]

which is also known as the fundamental solution to the Helmholtz equation with wavenumber \( k > 0 \), where \( J_0 \) and \( Y_0 \) are the Bessel functions of the first kind and the second kind of order 0, respectively. Then, the scattering problem can be formulated as: find the scattered field \( u^s \), such that

\[
\begin{align*}
\Delta u^s + k^2 u^s &= 0 \quad \text{in } D, \\
B_c u &= 0 \quad \text{on } \Gamma_c, \\
B_p u &= 0 \quad \text{on } \Gamma_p, \\
\lim_{r=|x| \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - i k u^s \right) &= 0,
\end{align*}
\]

where \( u = u^i + u^s \) denotes the total field. Here \( B_c \) and \( B_p \) in 17-18 are the boundary operators defined by

\[
B_c u = \begin{cases} 
  u, & \text{for a sound-soft perturbed half-plane,} \\
  \frac{\partial u}{\partial \nu}, & \text{for a sound-hard perturbed half-plane},
\end{cases}
\]

and

\[
B_p u = \begin{cases} 
  u, & \text{on } \Gamma_{p,D}, \\
  \frac{\partial u}{\partial \nu} + \lambda u, & \text{on } \Gamma_{p,I},
\end{cases}
\]

where \( \nu \) is the unit normal on \( \Gamma \) directed into \( D \), \( \Gamma_{p,D} \cup \Gamma_{p,I} = \Gamma_p \), \( \Gamma_{p,D} \cap \Gamma_{p,I} = \emptyset \), \( \lambda \in C(\Gamma_{p,I}) \) and \( \text{Im} \lambda \geq 0 \). The mixed boundary condition 21 is rather general in the sense that it covers the usual Dirichlet/sound-soft boundary condition \( (\Gamma_{p,I} = \emptyset) \), the Neumann/sound-hard boundary condition \( (\Gamma_{p,D} = \emptyset \text{ and } \lambda = 0) \), and the impedance boundary condition \( (\Gamma_{p,D} = \emptyset \text{ and } \lambda \neq 0) \).

The existence of a unique solution to the scattering problem 16–19 by a sound-soft perturbed half-plane with \( \Gamma_{p,I} = \emptyset \) could be established by a variational method [59, 4], and the well-posedness of the problem 16–19 by a sound-hard perturbed...
half-plane with $\Gamma_{p,D} = \emptyset$ and $\lambda = 0$ was studied in [48] by the integral equation method. In three-dimension case, we refer to [55] for the integral equation method.

The well-posedness of the problem 16–19 by a sound-soft perturbed half-plane with $\Gamma_{p,i} \neq \emptyset$ can be obtained similarly by the variational method as shown in [59, 4], while the existence and uniqueness of solutions to the problem 16–19 by a sound-hard perturbed half-plane can be established by the variational method with an even expansion and extension of the solution (an odd expansion and extension of the solution in [59, 4]) and a simple modification of the Dirichlet-to-Neumann operator in [59, 4] which does not affect the properties.

In the following we are going to consider the inverse scattering problem by the locally perturbed half-plane for incident point sources with limited-aperture phaseless near-field data. Similar to Definition 2.1, we first introduce the description of admissible curves.

**Definition 3.1 (Admissible curve).** An open curve $\Lambda$ is called an admissible curve with respect to domain $\Omega$ if

(i) $\overline{\Omega} \subset D$ is bounded and simply-connected;

(ii) $\partial \Omega$ is analytic homeomorphic to $S$;

(iii) $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$;

(iv) $\Lambda \subset \partial \Omega$ is a one-dimensional analytic manifold with nonvanishing measure.

**Remark 4.** It can be readily seen that Definition 3.1 is the one-dimensional version of Definition 2.1. Thus it is easy to find an admissible pair of $(\Omega, \Lambda)$. For example, $\Omega$ can be chosen as a disk whose radius is less than $2.4048/k$ and $\Lambda$ is chosen as an arbitrary corresponding semicircle.

Analogous to the arguments in the previous section, for two generic and distinct source points $z_1, z_2 \in D$, we denote by

\[
u_i(x; z_1, z_2) := u_i(x, z_1) + u_i(x, z_2), \quad x \in D\setminus(\{z_1\} \cup \{z_2\}),
\]

the superposition of these point sources. Then, by the linearity of direct scattering problem, the total near-field is given by

\[
u(x; z_1, z_2) := u(x, z_1) + u(x, z_2), \quad x \in D\setminus(\{z_1\} \cup \{z_2\}).
\]

We are now in the position to formulate the phaseless inverse scattering problems under consideration.

**Problem 2 (Phaseless inverse scattering by locally perturbed half-planes).** Let $\Gamma$ be the locally perturbed curve with boundary condition $\mathcal{B}_c$ and $\mathcal{B}_p$. Assume that $\Lambda$ and $\Sigma$ are admissible curves with respect to $\Omega$ and $G$, respectively. Given the phaseless near-field data

\[
\begin{align*}
\{|u(x, z_0)| : x \in \Sigma\}, \\
\{|u(x, z)| : x \in \Sigma, \ z \in \Lambda\}, \\
\{|u(x, z_0) + u(x, z)| : x \in \Sigma, \ z \in \Lambda\},
\end{align*}
\]

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in D\setminus(\Lambda \cup \Sigma)$, determine the locally perturbed curve $\Gamma$ as well as the boundary condition $\mathcal{B}_c$ and $\mathcal{B}_p$.

For an illustration of the above problem, we refer to Figure 2. The next subsection will be devoted to the uniqueness issue of this problem.
3.2. **Uniqueness results.** Before we present the uniqueness result on phaseless inverse scattering, the following reciprocity relation for the total fields is needed.

**Lemma 3.2.** Let $u^s(x, z)$ be the scattered field satisfying 16–19. Then we have

$$u(x, z) = u(z, x), \quad \forall x, z \in D, \quad x \neq z.$$  \hfill (23)

**Proof.** The proof of the reciprocity relation is similar to that of Theorem 3.1.4 in [38], so the details are omitted.

Let $\Gamma_j = \{ x \in \mathbb{R}^2 \mid x_2 = f_j(x_1), x_1 \in \mathbb{R} \}$ be the locally perturbed curve with the real-valued function $f_j \in C^2(\mathbb{R})$ having a compact support, $j = 1, 2$. Denote by $D_j = \{ x \in \mathbb{R}^2 \mid x_2 > f_j(x_1), x_1 \in \mathbb{R} \}$ the domain above $\Gamma_j$, $j = 1, 2$, and by $D_0 = D_1 \cap D_2$.

Denote by $u^s_j$ and $u_j$ the scattered field and the total field generated by $\Gamma_j$, respectively, corresponding to the incident field $u^i(x, z)$, $j = 1, 2$. Now, the following theorem shows that Problem 2 admits a unique solution, namely, the geometrical and physical information of the locally perturbed plane can be simultaneously and uniquely determined from the modulus of total near-fields.

**Theorem 3.3.** Let $\Gamma_1$ and $\Gamma_2$ be locally perturbed curves with boundary conditions $\mathcal{B}_{c,1}, \mathcal{B}_{p,1}$ and $\mathcal{B}_{c,2}, \mathcal{B}_{p,2}$, respectively. Assume that $\Lambda$ and $\Sigma$ are admissible curves with respect to $\Omega$ and $G$, respectively, such that $\overline{\Omega} \subseteq D_0, \overline{\Sigma} \subseteq D_0$ and $\overline{\Omega} \cap \overline{\Sigma} = \emptyset$.

Suppose that the corresponding total near-fields satisfy that

$$|u_1(x, z)| = |u_2(x, z)|, \quad \forall x \in \Sigma,$$  \hfill (24)

$$|u_1(x, z)| = |u_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Lambda$$  \hfill (25)

and

$$|u_1(x, z_0) + u_1(x, z)| = |u_2(x, z_0) + u_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Lambda$$  \hfill (26)

for an arbitrarily fixed $z_0 \in D_0 \setminus (\Lambda \cup \Sigma)$. Then we have $\Gamma_1 = \Gamma_2$, $\mathcal{B}_{c,1} = \mathcal{B}_{c,2}$ and $\mathcal{B}_{p,1} = \mathcal{B}_{p,2}$.

**Proof.** In terms of 24, 25 and 26, we have for all $x \in \Sigma, z \in \Lambda$

$$\text{Re} \left\{ u_1(x, z_0)\overline{u_1(x, z)} \right\} = \text{Re} \left\{ u_2(x, z_0)\overline{u_2(x, z)} \right\}.$$  \hfill (27)

Using 24 and 25, we denote

$$u_j(x, z_0) = r(x, z_0)e^{i\alpha_j(x, z_0)}, \quad u_j(x, z) = s(x, z)e^{i\beta_j(x, z)}, \quad j = 1, 2,$$
where \( r(x, z_0) = |u_j(x, z_0)| \), \( s(x, z) = |u_j(x, z)| \), \( \alpha_j(x, z_0) \) and \( \beta_j(x, z) \), are real-valued functions, \( j = 1, 2 \).

Due to the fact that \( \Sigma \) is an admissible curve of \( G \), Definition 3.1 and the analyticity of \( u_j(x, z) \) with respect to \( x \) imply that \( s(x, z) \neq 0 \) for \( x \in \Sigma, z \in \Lambda \). Moreover, by the continuity we deduce that there exists open sets \( \Sigma \subset \Sigma \) and \( \Lambda_0 \subset \Lambda \) such that \( s(x, z) \neq 0 \), \( \forall (x, z) \in \Sigma \times \Lambda_0 \). Analogously, we obtain \( r(x, z_0) \neq 0 \) on \( \Sigma \). The continuity also leads to \( r(x, z_0) \neq 0 \) on an open set \( \Sigma_0 \subset \Sigma \). Hence, we have

\[
\cos[\alpha_1(x, z_0) - \beta_1(x, z)] = \cos[\alpha_2(x, z_0) - \beta_2(x, z)], \quad \forall (x, z) \in \Sigma_0 \times \Lambda_0.
\]

Therefore, either

\[
\alpha_1(x, z_0) - \alpha_2(x, z_0) = \beta_1(x, z) - \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Lambda_0
\]

or

\[
\alpha_1(x, z_0) + \alpha_2(x, z_0) = \beta_1(x, z) + \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Lambda_0
\]

holds with some \( m \in \mathbb{Z} \).

We first deal with the case (28). Note that \( z_0 \) is fixed, we can define \( \gamma(x) := \alpha_1(x, z_0) - \alpha_2(x, z_0) - 2m\pi \) for \( x \in \Sigma_0 \), thus we deduce for all \( (x, z) \in \Sigma_0 \times \Lambda_0 \)

\[
u_1(z, x) = s(x, z)e^{i\beta_1(x, z)} = s(x, z)e^{i\beta_2(x, z) + \gamma(x)} = u_2(x, z)e^{i\gamma(x)}.
\]

By the reciprocity relation 23, we arrive at

\[
u_1(z, x) = e^{i\gamma(x)}u_2(z, x), \quad \forall (x, z) \in \Sigma_0 \times \Lambda_0.
\]

Now, for every \( x \in \Sigma_0 \), the analyticity of \( u_j(z, x)(j = 1, 2) \) with respect to \( z \) leads to \( u_1(z, x) = e^{i\gamma(x)}u_2(z, x), \forall z \in \partial\Omega \). Define \( w(z, x) = u_1(z, x) - e^{i\gamma(x)}u_2(z, x) \), then

\[
\Delta w + k^2w = 0 \quad \text{in } \Omega, \\
w = 0 \quad \text{on } \partial\Omega.
\]

Since \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), (30) implies that \( w = 0 \) in \( \Omega \). So, the analyticity of \( u_j(z, x)(j = 1, 2) \) with respect to \( z \) yields

\[
u_1(z, x) = e^{i\gamma(x)}u_2(z, x), \quad \forall z \in D_0 \setminus \{x\},
\]

namely, for all \( z \in D_0 \setminus \{x\} \),

\[
u_1^*(z, x) + u^*(z, x) = e^{i\gamma(x)}[u_2^*(z, x) + u^*(z, x)].
\]

From the boundedness of \( u_j^*(z, x) \) for \( z \in D_j(j = 1, 2) \) and 14, we see \( 1 - e^{i\gamma(x)}u^*(z, x) \) is bounded for \( z \in D_0 \setminus \{x\} \). Therefore, by letting \( z \to x \), we find \( e^{i\gamma(x)} = 1 \). Furthermore, the continuity of the scattered fields and the reciprocity relation 23 lead to

\[
u_1^*(x, z) = u_2^*(x, z), \quad \forall x \in \Sigma_0, z \in D_0.
\]

By a similar argument of (30) for \( \varpi(x, z) = u_1^*(x, z) - u_2^*(x, z) \) on \( G \), we have

\[
u_1^*(x, z) = u_2^*(x, z), \quad \forall x, z \in D_0.
\]

Next we shall show that the case (29) does not hold. Suppose that (29) is true, then following a similar argument, we find that for every \( x \in \Sigma_0 \), there exists \( \eta(x) \) such that

\[
u_1(x, z) = e^{i\eta(x)}u_2(z, x) \]

for all \( z \in D_0 \setminus \{x\} \), that means

\[
u_1^*(x, z) + u^*(z, x) = e^{i\eta(x)}[u_2^*(z, x) + u^*(z, x)].
\]
Then, it can be seen from the boundedness of \( u_j^i(z, x) \) that \( u^i(z, x) - e^{i\eta(x)}u^i(z, x) \) is bounded for all \( z \in D_0 \setminus \{x\} \). By letting \( z \to x \) in

\[
\frac{u^i(z, x) - e^{i\eta(x)}u^i(z, x)}{z - x} = \frac{\varepsilon^{i\eta(x)} - 1}{4} Y_0(k|x - z|) + i\frac{\varepsilon^{i\eta(x)} + 1}{4} J_0(k|x - z|),
\]

we find that \( \varepsilon^{i\eta(x)} = 1 \), which implies \( u_1(z, x) = u_2(z, x) \) for \( z \in D_0 \setminus \{x\} \). Now, following the same ideas in Theorem 2.2, we deduce that the case 29 does not hold.

Once the case 32 is verified, we would conclude that \( \Gamma_1 = \Gamma_2 \). Otherwise, assume that \( \Gamma_1 \neq \Gamma_2 \). Then, without loss of generality, there exists \( x^* \in \partial D_0 \) such that \( x^* \in \Gamma_1 \) and \( x^* \in \Gamma_2 \). Define

\[
z_n := x^* - \frac{1}{n} \nu(x^*), \quad n = 1, 2, ...
\]

such that \( z_n \in D_0 \) for sufficiently large \( n \). Then, from the reciprocity relation and the smoothness of \( u_2^i(x^*, z) \) in \( D_2 \), we have

\[
\lim_{n \to \infty} B_{p,1}u_2^i(x^*, z_n) = \lim_{n \to \infty} B_{p,1}u_2^i(z_n, x^*) = B_{p,1}u_2^i(x^*, x^*).
\]

On the other hand, the boundary conditions 18 and 32 imply that

\[
\lim_{n \to \infty} B_{p,1}u_2^i(x^*, z_n) = \lim_{n \to \infty} B_{p,1}u_2^i(x^*, z_n) = - \lim_{n \to \infty} B_{p,1}u_1^i(x^*, z_n) = \infty,
\]

which is a contradiction. Therefore \( \Gamma_1 = \Gamma_2 \). Further, similar to the proof of Theorem 5.6 in [13], we obtain \( \mathcal{R}_{c,1} = \mathcal{R}_{c,2} \) and \( \mathcal{R}_{p,1} = \mathcal{R}_{p,2} \).

**Remark 5.** We want to point out that an analogous uniqueness result in three dimensions remains valid subject to some modifications of the fundamental solution and the admissible curve.

**Acknowledgments.** We would like to thank the anonymous referees for the useful comments and suggestions.

**REFERENCES**

[1] H. Ammari and H. Kang, *Polarization and Moment Tensors, With Applications to Inverse Problems and Effective Medium Theory*, Springer, New York, 2007.

[2] H. Ammari, Y. T. Chow and J. Zou, *Phased and phaseless domain reconstructions in the inverse scattering problem via scattering coefficients*, *SIAM J. Appl. Math.*, **76** (2016), 1000–1030.

[3] C. Athanasiadis, P. A. Martin, A. Spyropoulos and I. G. Stratis, *Scattering relations for point sources: Acoustic and electromagnetic waves*, *J. Math. Phys.*, **43** (2002), 5683–5697.

[4] G. Bao and J. Lin, *Imaging of local surface displacement on an infinite ground plane: The multiple frequency case*, *SIAM J. Appl. Math.*, **71** (2011), 1733–1752.

[5] G. Bao, P. Li and J. Lv, *Numerical solution of an inverse diffraction grating problem from phaseless data*, *J. Opt. Soc. Am. A.*, **30** (2013), 293–299.

[6] G. Bao and L. Zhang, *Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data*, *Inverse Problems*, **32** (2016), 085002.

[7] F. Cakoni, D. Colton and P. Monk, *The direct and inverse scattering problem for partially coated obstacles*, *Inverse Problems*, **17** (2001), 1997–2015.

[8] E. J. Candès, T. Strohmer and V. Voroninski, *PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming*, *Commun. Pure Appl. Math.*, **66** (2013), 1241–1274.

[9] E. J. Candès, X. Li and M. Soltanolkotabi, *Phase retrieval via Wirtinger flow: Theory and algorithms*, *IEEE Trans. Information Theory*, **61** (2015), 1985–2007.

[10] S. Caorsi, A. Massa, M. Pastorino and A. Randazzo, *Electromagnetic detection of dielectric scatterers using phaseless synthetic and real data and the memetic algorithm*, *IEEE Trans. Geoscience Remote Sensing*, **41** (2003), 2745–2753.
[11] Z. Chen, S. Fang and G. Huang, A direct imaging method for the half-space inverse scattering problem with phaseless data, *Inverse Probl. Imaging*, 11 (2017), 901–916.

[12] Z. Chen and G. Huang, Phaseless imaging by reverse time migration: Acoustic waves, *Numer. Math. Theor. Meth. Appl.*, 10 (2017), 1–21.

[13] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd edition, Springer, New York, 2013.

[14] Y. Deng, J. Li and H. Liu, On identifying magnetized anomalies using geomagnetic monitoring, *Arch. Ration. Mech. Anal.*, 231 (2019), 153–187.

[15] Y. Deng, H. Liu and G. Uhlmann, On an inverse boundary problem arising in brain imaging, *J. Differential Equations*, 267 (2019), 2471–2502.

[16] H. Dong, J. Lai and P. Li, Inverse obstacle scattering problem for elastic waves with phased or phaseless far-field data, *SIAM J. Imaging Sci.*, 12 (2019), 809–838.

[17] H. Dong, D. Zhang and Y. Guo, A reference ball based iterative algorithm for imaging acoustic obstacle from phaseless far-field data, *Inverse Probl. Imaging*, 13 (2019), 177–195.

[18] P. Gao, H. Dong and F. Ma, Inverse scattering via nonlinear integral equations method for a sound-soft crack from phaseless data, *Appl. Math.*, 63 (2018), 149–165.

[19] O. Ivanyshyn, Shape reconstruction of acoustic obstacles from the modulus of the far field pattern, *Inverse Probl. Imaging*, 1 (2007), 609–622.

[20] O. Ivanyshyn and R. Kress, Identification of sound-soft 3D obstacles from phaseless data, *Inverse Probl. Imaging*, 4 (2010), 131–149.

[21] O. Ivanyshyn and R. Kress, Inverse scattering for surface impedance from phase-less far field data, *J. Comput. Phys.*, 230 (2011), 3443–3452.

[22] X. Ji, X. Liu and B. Zhang, Target reconstruction with a reference point scatterer using phaseless far field patterns, *SIAM J. Imaging Sci.*, 12 (2019), 372–391.

[23] A. Kirsch and N. Grinberg, *The Factorization Methods for Inverse Problems*, Oxford University Press, Oxford, 2008.

[24] M. V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, 74 (2014), 392–410.

[25] M. V. Klibanov, A phaseless inverse scattering problem for the 3-D Helmholtz equation, *Inverse Probl. Imaging*, 11 (2017), 263–276.

[26] M. V. Klibanov, N. A. Koshev, D.-L. Nguyen, L. H. Nguyen, A. Brettin and V. Astratov, A numerical method to solve a phaseless coefficient inverse problem from a single measurement of experimental data, *SIAM J. Imaging Sci.*, 11 (2018), 2339–2367.

[27] M. V. Klibanov, D.-L. Nguyen and L. H. Nguyen, A coefficient inverse problem with a single measurement of phaseless scattering data, *SIAM J. Appl. Math.*, 79 (2019), 1–27.

[28] M. V. Klibanov, L.-H. Nguyen and K. Pan, Nanostructures imaging via numerical solution of a 3-D inverse scattering problem without the phase information, *Appl. Numer. Math.*, 110 (2016), 190–203.

[29] M. V. Klibanov and V. G. Romanov, Reconstruction procedures for two inverse scattering problems without the phase information, *SIAM J. Appl. Math.*, 76 (2016), 178–196.

[30] M. V. Klibanov and V. G. Romanov, Uniqueness of a 3-D coefficient inverse scattering problem without the phase information, *Inverse Problems*, 33 (2017), 095007.

[31] R. Kress and W. Rundell, Inverse obstacle scattering with modulus of the far field pattern as data, in *Inverse Problems in Medical Imaging and Nondestructive Testing (Oberwolfach, 1996)*, Springer, Vienna, 1997, 75–92.

[32] K. M. Lee, Shape reconstructions from phaseless data, *Eng. Anal. Bound. Elem.*, 71 (2016), 174–178.

[33] J. Li and H. Liu, Recovering a polyhedral obstacle by a few backscattering measurements, *J. Differential Equations*, 259 (2015), 2101–2120.

[34] J. Li, H. Liu and H. Sun, On a gesture-computing technique using electromagnetic waves, *Inverse Probl. Imaging*, 12 (2018), 677–696.

[35] J. Li, H. Liu, W.-Y. Tsai and X. Wang, An inverse scattering approach for geometric body generation: A machine learning perspective, *Mathematics in Engineering*, 1 (2019), 800–823.

[36] J. Li, H. Liu and Y. Wang, Recovering an electromagnetic obstacle by a few phaseless backscattering measurements, *Inverse Problems*, 33 (2017), 035001.

[37] J. Li, H. Liu and J. Zou, Strengthened linear sampling method with a reference ball, *SIAM J. Sci. Comput.*, 31 (2009), 4013–4040.

[38] C. Lines, *Inverse Scattering by Unbounded Obstacles*, Ph.D thesis, Brunel University, 2003.
[39] J. Liu and J. Seo, On stability for a translated obstacle with impedance boundary condition, *Nonlinear Anal.*, 59 (2004), 731–744.

[40] M. H. Maleki, A. J. Devaney and A. Schatzberg, Tomographic reconstruction from optical scattered intensities, *J. Opt. Soc. Am. A.*, 9 (1992), 1356–1363.

[41] M. H. Maleki and A. J. Devaney, Phase-retrieval and intensity-only reconstruction algorithms for optical diffraction tomography, *J. Opt. Soc. Am. A.*, 10 (1993), 1086–1092.

[42] S. Maretszke and T. Hohage, Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging, *SIAM J. Appl. Math.*, 77 (2017), 384–408.

[43] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[44] R. G. Novikov, Formulas for phase recovering from phaseless scattering data at fixed frequency, *Bull. Sci. Math.*, 139 (2015), 923–936.

[45] R. G. Novikov, Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions, *J. Geom. Anal.*, 26 (2016), 346–359.

[46] L. Pan, Y. Zhong, X. Chen and S. P. Yeo, Subspace-based optimization method for inverse scattering problems utilizing phaseless data, *IEEE Trans. Geoscience and Remote Sensing*, 49 (2011), 981–987.

[47] R. Potthast, *Point Sources and Multipoles in Inverse Scattering Theory*, Chapman & Hall/CRC, Boca Raton, 2001.

[48] F. Qu, B. Zhang and H. Zhang, A novel integral equation for scattering by locally rough surfaces and application to the inverse problem: The Neumann case, arXiv:1901.08703v1, 2019.

[49] V. G. Romanov, A problem on determining the permittivity coefficient in a stationary system of Maxwell equations, *Dokl. Math.*, 95 (2017), 230–234.

[50] V. G. Romanov and M. Yamamoto, Phaseless inverse problems with interference waves, *J. Inverse Ill-Posed Probl.*, 26 (2018), 681–688.

[51] F. Sun, D. Zhang and Y. Guo, Uniqueness in phaseless inverse scattering problems with known superposition of incident point sources, *Inverse Problems*, 35 (2019), 105007.

[52] T. Takenaka, D. J. N. Wall, H. Harada and M. Tanaka, Reconstruction algorithm of the refractive index of a cylindrical object from the intensity measurements of the total field, *Microwave Opt. Tech. Lett.*, 14 (1997), 182–188.

[53] X. Wang, Y. Guo, J. Li and H. Liu, Mathematical design of a novel input/instruction device using a moving acoustic emitter, *Inverse Problems*, 33 (2017), 105009.

[54] X. Wang, Y. Guo, J. Li and H. Liu, Two gesture-computing approaches by using electromagnetic waves, *Inverse Probl. Imaging*, 13 (2019), 879–901.

[55] A. Willers, *The Helmholtz equation in disturbed half-spaces*, Math. Methods Appl. Sci., 9 (1987), 312–323.

[56] X. Xu, B. Zhang and H. Zhang, Uniqueness in inverse scattering problems with phaseless far-field data at a fixed frequency II, *SIAM J. Appl. Math.*, 78 (2018), 3024–3039.

[57] D. Zhang and Y. Guo, Uniqueness results on phaseless inverse acoustic scattering with a reference ball, *Inverse Problems*, 34 (2018), 085002.

[58] D. Zhang, Y. Guo, J. Li and H. Liu, Retrieval of acoustic sources from multi-frequency phaseless data, *Inverse Problems*, 34 (2018), 094001.

[59] D. Zhang, F. Ma and M. Fang, A finite element method with perfectly matched absorbing layers for the wave scattering from a cavity, *Chinese Journal of Computational Physics*, 25 (2008), 301–308.

[60] D. Zhang, Y. Wang, Y. Guo and J. Li, Uniqueness in inverse cavity scattering problems with phaseless near-field data, *Inverse Problems*, 35 (2019), 39 pp.

[61] B. Zhang and H. Zhang, Recovering scattering obstacles by multi-frequency phaseless far-field data, *J. Comput. Phys.*, 345 (2017), 58–73.

Received for publication November 2019.

*E-mail address:* dyzhang@jlu.edu.cn

*E-mail address:* ykguo@hit.edu.cn

*E-mail address:* sunfl18@mails.jlu.edu.cn

*E-mail address:* hongyu.liuip@gmail.com; hongyliu@cityu.edu.hk