A far-from-CMC existence result for the constraint equations on manifolds with ends of cylindrical type

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Abstract
We extend the study of the vacuum Einstein constraint equations on manifolds with ends of cylindrical type initiated by Chruściel and Mazzeo (2012) and Chruściel et al (2012 Adv. Theor. Math. Phys. at press) by finding a class of solutions to the fully coupled system on such manifolds. We show that given a Yamabe positive metric $g$ which is conformally asymptotically cylindrical on each end and a 2-tensor $K$ such that $(\text{tr} g K)^2$ is bounded below away from zero and asymptotically constant, then we may find an initial data set $(\bar{g}, \bar{K})$ such that $\bar{g}$ lies in the conformal class of $g$.

Keywords: Einstein constraint equations, conformal method, non-constant mean curvature
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1. Introduction
Let $(M, g, K)$ be an initial data set where $(M, g)$ is a complete Riemannian $n$-manifold and $K$ is a symmetric 2-tensor. Throughout this paper, we will assume $n \geq 3$. The Cauchy problem in general relativity asks whether $(M, g)$ can be embedded as a spacelike submanifold of some $(n + 1)$-dimensional Lorentzian manifold $(\mathcal{V}, \hat{g})$ which satisfies Einstein’s equation, and in such a way that $g$ is the metric induced by $\hat{g}$ and the second fundamental form of this embedding is given by $K$. In the absence of matter fields, this is equivalent (see, for example, [2] or [13]) to requiring that $(g, K)$ solve the Einstein constraint equations

$$\begin{cases} R(g) = 2\Lambda + |K|^2_{\hat{g}} - (\text{tr}_{\hat{g}} K)^2 \\ \text{div}_{\hat{g}} K - \nabla_{\hat{g}} \text{tr}_{\hat{g}} K = 0 \end{cases}$$

(1.1)
where $\Lambda$ is a cosmological constant. Defining $\tau = \text{tr}_{g}K$, the only restriction we place on $\Lambda$ is that there is some positive constant $\beta_0$ such that
\begin{equation}
\beta := \frac{n-2}{4n} \tau^2 - \frac{n-2}{2(n-1)} \Lambda \geq \beta_0, \tag{1.2}
\end{equation}

Beyond equation (1.2), $\Lambda$ will not play a central role in our analysis.

1.1. Asymptotically cylindrical and periodic ends

Suppose the manifold $(M, g)$ has a finite number of ends $E_\ell$, each of which is of cylindrical type. This is to say that $E_\ell$ is diffeomorphic to the half cylinder $\mathbb{R}^+ \times N_\ell$ where $N_\ell$ is a closed $(n-1)$-manifold. We say that a metric $\hat{g}_\ell$ is asymptotically cylindrical (AC) on $E_\ell$ if $\hat{g}_\ell$ and its derivatives decay exponentially to the standard product metric on $E_\ell$. More precisely, for some $\omega > 0$ we have that $|\nabla^k[\hat{g}_\ell - (dt^2 + \hat{g}_\ell)]| = O(e^{-\omega t})$ for all $k$, where $\hat{g}_\ell$ is some Riemannian metric on $N_\ell$, and $\nabla$ is the connection associated to the cylindrical metric $dt^2 + \hat{g}_\ell$ (we may, for our purposes, weaken this condition by only requiring this decay in $m > 2$ derivatives). More generally, we say that the metric $g$ is conformally AC if $g = \omega^{4/(n-2)} \hat{g}$ where $\hat{g}$ is AC on each end. Observe that the substitutions $\omega \rightarrow \omega t_i$ as $t \rightarrow \infty$ on $E_\ell$, where $\omega t_i$ is a smooth positive function on $N_\ell$, and that the derivatives converge at the same rate. We also define a closely related class of metrics on $M$, namely those which are asymptotically periodic (AP) on each end. Such metrics and their derivatives decay exponentially to a periodic metric $\hat{g}_\ell$ with period $T_\ell > 0$, which is the lift to $\mathbb{R} \times N_\ell$ of a metric on $(\mathbb{R}/T_\ell \mathbb{Z}) \times N_\ell$. For convenience, assume that all rates of decay are $e^{-\omega t}$, and all periods are equal to $T_i$.

1.2. The conformal method

A number of approaches have been taken to solve the constraint equations under varying mean curvature assumptions (for an excellent survey, see [1]), but perhaps none has been so successful as the conformal method. The basic idea is to fix a metric $g$ on $M$ and to look for a solution $(\tilde{g}, \tilde{K})$ of the constraint equations with the form
\begin{align}
\tilde{g}_{ij} &= \phi_n^{-(n-2)} g_{ij} \\
\tilde{K}^{ij} &= \frac{\tau}{n} \phi^{-2} g^{ij} + \tilde{g}^{-2\alpha} (\sigma^{ij} + (D\phi)^{ij}),
\end{align}

Here $\alpha = (n+2)/(n-2)$, $\tilde{\phi}$ is a positive function, $\sigma$ is a given transverse traceless 2-tensor, and $D$ is the conformal Killing operator, which maps a vector field $X$ to a symmetric 2-tensor by
\begin{equation}
(DX)^{ij} = \nabla^i X^j + \nabla^j X^i - \frac{2}{n} \nabla_k X^k g^{ij}.
\end{equation}

Note that $DX$ is simply the trace-free part of the deformation tensor associated to $X$. We call any vector field in the nullspace of $D$ a conformal Killing field. Observe that the substitutions (1.3)–(1.4) effectively replace the unknowns $(g, K)$ in the constraint equations with new unknowns $(\phi, W)$, and the task of solving the constraint equations is then reduced to solving the coupled system
\begin{equation}
\begin{cases}
\Delta_n \tilde{\phi} - c_n R_g \tilde{\phi} - \beta \tilde{\phi}^{\alpha} + c_n \sigma + DW^2 g^{\gamma} \tilde{\phi}^{-\gamma} = 0 \\
\Delta_{\tilde{\gamma}} W = \frac{n-1}{n} \tilde{\phi}^{\gamma-1} \nabla \tau
\end{cases}
\end{equation}

where
\begin{equation}
c_n = \frac{n-2}{4(n-1)}, \quad \gamma = \frac{3n-2}{n-2}.
\end{equation}
and $\Delta_L := -\text{div}_y D$ (see, for example, [6]). With the assumptions above, we seek a solution to the system (1.5) with the hypothesis that on each end $|\sigma|^2_g$ converges to some function $\tilde{\sigma}_t^2$ which is not identically zero. Assume that each $\tilde{\sigma}_t^2$ is independent of $t$ (in the AC case) or periodic with period $T$ (in the AP case).

1.3. Notation and preliminary results

Before stating our main results, we now define the weighted Sobolev and Hölder norms used throughout this paper. For any $1 \leq p < \infty$, define the $W^{k,p}_\gamma$-norm by

$$\|X\|_{W^{k,p}_\gamma} = \left( \sum_{j \leq k} \int_M |\nabla^j X|^p e^{-p\gamma} \, dV_g \right)^{1/p}.$$  

(1.6)

The smooth function $t$ is said to be ‘radial’ in the sense that on each end there are positive constants $C_1 \leq C_2 \leq C_3$ such that

$$C_1 + t \leq C_2 + \text{dist}_y(x, \partial E_\ell) \leq C_3 + t.$$  

On each end $E_\ell$, the set $\{t = a\}$ is diffeomorphic to $N_\ell$ for any $a > 0$, and we define $t$ to be identically zero on the compact part of the manifold which is disjoint from the ends. In keeping with the convention of [5], we will denote by $H^k_q$ the function space $W^{k,\infty}_q$. For any $q \in M$, define $B_1(q)$ to be the ball of radius 1 about $q$ (where we assume without loss of generality that the radius of injectivity for $M$ is greater than 1). We define the Hölder norms as

$$\|X\|_{k,\mu;B_1(q)} = \sup_{x \in B_1(q)} \frac{|\nabla^k X(x) - \nabla^k X(y)|}{d_q(x, y)^\mu}$$  

(1.7)

and also the weighted Hölder norms by

$$\|X\|_{k,\mu,\delta} = \sup_{q \in M} \|X\|_{k,\mu;B_1(q)},$$  

(1.8)

We similarly denote the standard sup-norm on $C^0$ by $\| \cdot \|_0$ and the corresponding weighted sup-norm by $\| \cdot \|_{0,\delta}$. We shall denote by $C^0_q$ the space of all functions (or tensor fields) $f$ with $\|f\|_{k,\mu,\delta} < \infty$. The space $C^0_q$ is defined similarly.

The operator $\Delta_L$ appearing in (1.5) is called the conformal vector Laplacian. This elliptic operator and its properties on AC/AP manifolds were studied extensively in [5], and the results in that paper are central to the proof of our main theorem below. In particular, in order to say something about the solutions of (1.5), we will need some information about the solvability of the equation $\Delta_L W = V$ when the underlying metric is AC/AP. For this we first define $\mathcal{Y}_\ell$ to be the set of all globally bounded conformal Killing fields of the associated exactly cylindrical or periodic metric on $E_\ell$. In the AP case, assume every $Y \in \mathcal{Y}_\ell$ also satisfies $\Gamma_* Y = \text{Re} \, e^{\theta Y}$ for some $\theta \in \mathbb{R}$, where $\Gamma_*$ is the pull-back of a generator $\Gamma$ of the group of deck transformations on the exactly periodic end (see section 5 in [5]). For each $\ell$ we define a smooth cutoff function $\chi_\ell$ which vanishes outside of $E_\ell$ and is identically 1 where $t \geq 1$. We let $\mathcal{Y} = \bigoplus \chi_\ell Y : Y \in \mathcal{Y}_\ell$. Theorem 6.1 in [5] then spells out the Fredholm properties of $\Delta_L$, acting between weighted Sobolev spaces:

**Theorem 1.1.** Let $(M,g)$ be a Riemannian $n$-manifold with a finite number of ends which are either AC or AP. Suppose further that there is no nontrivial globally defined solution to $\Delta_L Y = 0$ which lies in $L^2(TM)$. Then there exists a number $\delta_* > 0$ such that if $0 < \delta < \delta_*$, the map

$$\Delta_L : H^{k+2}_\delta(TM) \to H^k_\delta(TM)$$  

(1.10)
is surjective and the map
\[ \Delta_\lambda : H^{k+2}_{-\delta}(TM) \to H^k_{-\delta}(TM) \] (1.11)
is injective for every \( k \geq 0 \). Moreover, if \( 0 < \delta < \delta_\ast \), then for all \( k \geq 0 \) the map
\[ \Delta_\lambda : H^{k+2}_{-\delta}(TM) \oplus \mathcal{Y} \to H^k_{-\delta}(TM) \] (1.12)
is surjective with finite dimensional nullspace.

The proof of this theorem also shows that \( \mathcal{Y} \) is a finite-dimensional subspace of \( H^{k+2}_{-\delta} \) for all \( k \geq 0 \) (and thus lies in \( C^1_{\alpha,\mu} \) for any \( 0 < \mu < 1 \)). It then follows from a standard parametrix construction that we can find a bounded generalized inverse \( G : H^k_{-\delta}(TM) \to H^{k+2}_{-\delta}(TM) \oplus \mathcal{Y} \) satisfying \( \Delta_\lambda \circ G = I \), and analogous results can also be proved for weighted Hölder spaces as well as weighted \( L^p \)-Sobolev spaces using only a slight modification of the proof of theorem 1.1. To be more specific, the authors of [5] cite the results in [10] (for the AC case) and [11] (for the AP case) to construct parametrices for the operator \( \Delta_\lambda \) acting between weighted Sobolev spaces of functions on \( M \). These same operators are also bounded maps between the appropriate Hölder or \( L^p \)-Sobolev spaces, so the arguments in [10, 11] apply equally well in those cases.

2. Summary of main results

Our goal is to prove the following theorem.

**Theorem 2.1.** Let \((M,g)\) be a complete Riemannian n-manifold with a finite number of ends which are AC (resp. AP), and with scalar curvature satisfying \( R \geq R_0 > 0 \). Suppose that \( \tau \) is a scalar function such that \( \nabla \tau \in C^0_{\alpha,\mu}(TM) \) for some \( 0 < \mu < 1 \), and \( \sigma \in C^1_{\alpha,\mu} \) is a symmetric 2-tensor with \( \|\sigma\|_0 \) sufficiently small relative to the constant \( R_0 \) and \( \|\nabla \tau\|_{0,-\delta} \). If \( \sigma \neq 0 \), \( \beta \geq \beta_0 > 0 \), and \( g \) admits no nontrivial globally defined \( L^2 \) conformal Killing fields, then the system (1.5) has a solution. Moreover, the initial data corresponding to this solution is conformally AC (resp. AP).

We will prove theorem 2.1 by using a fixed point argument modeled on the ones by Holst, Nagy, and Tsogtgerel in [7] and by Maxwell in [9]. The strategy in those papers was to choose a function space in which one would reasonably expect to find a solution \( \phi \) (associated to some \( W_\phi \) via the momentum constraint equation) and to construct global barriers to define a certain subset of this function space. The authors then considered a composition of the solution maps of the respective equations in (1.5) as a map from this subset to itself. An application of the Schauder fixed point theorem picks out a distinguished \( \tilde{\phi} \) such that \((\tilde{\phi}, W_{\tilde{\phi}})\) solves the system. Our strategy is essentially the same, though we rely heavily on the recent results of [4, 5] regarding the solvability of the Lichnerowicz and momentum constraint equations. Section 3 is dedicated to constructing suitable global barriers in the AC/AP case under the assumption of positive scalar curvature, and in section 4 we describe the corresponding fixed point construction leading to a proof of theorem 2.1.

From the discussion in the previous section, theorem 2.1 implies that given any AC/AP initial data set \((g,K)\) satisfying the given conditions, we may find a solution \((\overline{g},\overline{K})\) to the constraint equations (1.1) such that \( \overline{g} \) is conformally equivalent to \( g \). For AP metrics we construct a conformal factor solving (1.5) which asymptotically has the same period as the underlying metric, and therefore the AP initial data we consider admit AP solutions to (1.1). However, if the underlying metric is AC, generally the metric \( \overline{g} \) will not be, instead lying in the larger class of conformally AC metrics. This leads us to ask whether our existence result
can be extended to hold if we only assume that $g$ is conformally AC. In section 5 we analyze the operator $\Delta_{L}$ associated to a conformally AC metric and show that the constructions of sections 3 and 4 extend in a natural way to handle this case.

**Theorem 2.2.** Let $(M, g)$ be a complete Riemannian $n$-manifold with a finite number of conformally AC ends which admits no nontrivial globally defined $L^2$ conformal Killing fields. If $(M, g)$ is Yamabe positive and $\tau, \sigma$ are as in theorem 2.1 with $\|\sigma\|_0$ sufficiently small relative to $\|\nabla \tau\|_0^{-\epsilon}$, then (1.5) has a solution. Moreover, the initial data corresponding to this solution is conformally AC.

We will see in the proof below that the convergence of the solution metric to its limit on the ends may be slower than the convergence of the original metric in both the AP and conformally AC cases.

### 3. Global barriers

The system (1.5) consists of a scalar equation and a vector equation. The former is commonly known as the Lichnerowicz equation and the latter as the momentum constraint equation. Given a positive function $\phi$, let $W_\phi$ be the solution to the corresponding momentum constraint equation (we henceforth drop the tildes from our notation). Now suppose $\phi_+$ is a positive function with the property that for any $\phi$ satisfying $0 < \phi \leq \phi_+$, we have $\text{Lich}_\phi(\phi_+) \leq 0$, where we are using the notation

$$\text{Lich}_\phi(\phi) := \Delta \phi - c_\alpha R \phi - \beta \phi^\mu + c_\alpha |\sigma + DW_\phi|^2 \phi^{-\nu}.$$  \hspace{1cm} (3.1)

Note that we have dropped the subscript of ‘$g$’ from all metric-dependent operators and functions since there is no ambiguity going forward. A function $\phi_+$ with this property is called a global supersolution of the system (1.5). If the inequality $\text{Lich}_\phi(\phi_+) \leq 0$ only holds in the weak sense, we say that $\phi_+$ is a weak global supersolution. Given a weak global supersolution, we define a continuous function $\phi_-$ to be a weak global subsolution if, for all $\phi_- \leq \phi \leq \phi_+$, we have $\text{Lich}_{\phi_-}(\phi_-) \geq 0$ in the weak sense.

The utility of global sub-/supersolutions is realized in the following proposition, a proof of which can be found in [4].

**Proposition 3.1.** Let $(M, g)$ be a smooth Riemannian manifold and $F$ a locally Lipschitz function. Suppose that $\phi_- \leq \phi \leq \phi_+$ are continuous functions which satisfy

$$\Delta_\phi \phi \geq F(x, \phi), \quad \Delta_\phi \phi \leq F(x, \phi)$$

weakly. Then there exists a smooth function $\tilde{\phi}$ on $M$ such that

$$\Delta_\phi \tilde{\phi} = F(x, \tilde{\phi}), \quad \phi_- \leq \tilde{\phi} \leq \phi_+.$$  

Thus given a pair of weak global sub-/supersolutions and a function $\phi$ satisfying $\phi_- \leq \phi \leq \phi_+$, we can find a solution $\tilde{\phi}$ to the Lichnerowicz equation with $W = W_\phi$. If $\phi = \phi_+$, then this function solves the system (1.5).

It would be quite useful to know whether the solution $\tilde{\phi}$ guaranteed by proposition 3.1 is unique. While we have no reason to suspect that uniqueness holds for an arbitrary pair of global sub-/supersolutions, one may show that uniqueness holds in certain special cases. For example, if we require that $\phi - \phi \leq C e^{-\omega t}$ for some constant $C$ and look for solutions in an appropriate Hölder space, uniqueness follows from a slight adaptation of an argument in [3], which states the analogous uniqueness result when $M$ is asymptotically Euclidean.
Proposition 3.2. Let $\phi_1, \phi_2 \in C_{-\mu}$ be a pair of positive solutions of the Lichnerowicz equation

$$\Delta \phi - c_{\tau} R \phi - \beta \phi^p + c_{\omega} |\phi|^{2p} \phi^{-p} = 0$$  \hspace{1cm} (3.2)$$

which are bounded away from zero, and suppose the underlying manifold $(M, g)$ and the function $\beta$ are as in theorem 2.1 and $|\tilde{\sigma}|^2 \rightarrow \delta^2_{\tau}$ on $E_\ell$ at the rate $e^{-\omega t}$ for some $\omega > 0$. If $\phi_2 - \phi_1 \in C_{-\omega}$, then $\phi_1 = \phi_2$.

Proof sketch. Defining new metrics on $M$ by $g_i = \phi^{4/(n-2)}_i$, so that $g_2 = (\phi_2 \phi_1^{-1})^{4/(n-2)} g_1$, an argument nearly identical to the proof of theorem 8.3 in [3] proves that we have

$$(\Delta_{\mu} - \lambda)(\phi_2 \phi_1^{-1} - 1) = 0$$  \hspace{1cm} (3.3)$$

for some nonnegative continuous function $\lambda$. One easily checks that, under the given hypothesis, $\phi_2 \phi_1^{-1} - 1 \in C_{-\omega}(M)$. Since $\Delta_{\mu} - \lambda$ is seen to be injective as a map $C_{-\omega}(M) \rightarrow C_{-\omega}(M)$ by the maximum principle, we conclude that $\phi_2 \phi_1^{-1} - 1 = 0$, or $\phi_1 = \phi_2$ as claimed.

We note that Chruściel and Mazzeo proved in [4] that one may always solve (3.2) with the given hypotheses by constructing sub-/supersolutions $\phi \leq \tilde{\phi}$ which depend on $\sigma$ and both approach a limit function $\phi$ on the ends at a rate of $e^{-\omega t}$. Proposition 3.2 implies that the solution they obtain is unique among functions with the same asymptotic limit. If we thus fix limiting functions $\phi$, $\beta$ and $\tilde{\delta}^2_{\tau}$, and let $\Sigma$ be the set of tensors $\tilde{\sigma}$ satisfying the hypothesis of proposition 3.2, then theorem 4.7 in [4] gives us a well-defined map $Q : \Sigma \rightarrow C_{-\mu}$ sending $\tilde{\sigma}$ to the unique solution of (3.2) with asymptotic limit $\phi$ (note that the proof of this theorem implies the hypothesis $\tilde{\delta}^2_{\tau} > 0'$ can be weakened to $\tilde{\delta}^2_{\tau} \neq 0'$). In particular, if we construct a pair of global sub-/supersolutions $\phi_- \leq \phi_+$ such that $\phi_\pm \rightarrow \phi$ on the ends at the rate $e^{-\omega t}$, then for any $\phi_- \leq \phi \leq \phi_+$, we have $\phi_\pm \leq Q(\sigma + DW_\phi) \leq \phi_+$ for some fixed $\sigma$ defined as in theorem 2.1 so long as $DW_\phi$ decays at a sufficiently fast exponential rate. This observation will be crucial to our fixed point construction below. To find such global barriers, we first find a constant global supersolution by applying what is essentially the argument in proposition 14 of [9], which is itself a variant of an argument given in [7]. However, as we are working in the noncompact case, we first establish some control over the behavior of $DW_\phi$ on the ends in terms of $\|\phi\|_0$.

Lemma 3.3. Let $\phi$ be a bounded continuous function on $M$ and $\nabla \tau \in C^{0}_{-\tau}(TM)$ for some positive $\delta' < \delta$. If $W_\phi$ is the solution of the corresponding momentum constraint equation, then for any $\delta < \delta'$, there exists some constant $K$ which does not depend on $\phi$ or $\tau$ such that the following pointwise estimate holds:

$$|DW_\phi| \leq K \|\nabla \tau\|_{0,-\delta'} \|\phi\|_0^{-1} e^{-\delta t}.$$  \hspace{1cm} (3.4)$$

Proof. First note that $W_\phi$ is given by the image of $(n-1)n^{-1}\phi^{p-1} \nabla \tau$ under the generalized inverse $G$, which is a bounded map $C_{-\delta} \rightarrow C_{-\delta}^{1,\mu} \oplus \mathcal{Y}$ (one way to see this is to use its boundedness as a map $L^p_{-\delta} \rightarrow W_{-\delta}^{2,p} \oplus \mathcal{Y}$ and apply standard embedding arguments for sufficiently large $p$). Moreover, we have

$$|DW_\phi| = |DW_\phi| e^{\delta t} e^{-\delta t} \leq \|DW_\phi\|_{0,-\delta} e^{-\delta t} \leq \|DW_\phi\|_{0,\mu,-\delta} e^{-\delta t}.$$  \hspace{1cm} (3.5)$$

Endowing $\mathcal{Y}$ with the induced $C^{1,\mu}_{-\delta}$-norm, which is equivalent to any other choice of norm since $\mathcal{Y}$ is finite-dimensional, we see that $D$ is a bounded map $C^{1,\mu}_{-\delta} \oplus \mathcal{Y} \rightarrow C_{-\delta}^{0,\mu}$ and therefore

$$|DW_\phi| \leq C \|W_\phi\|_{C^{1,\mu}_{-\delta} \oplus \mathcal{Y}} e^{-\delta t}.$$  \hspace{1cm} (3.6)$$
Hence the boundedness of $G$ implies
\[
|DW_\phi| \leq K\|\phi\|_{0.\gamma-1} |\nabla \tau|_{0,-\delta'} e^{-\delta t} \leq K\|\nabla \tau\|_{0,-\delta'} \|\phi\|_{0}^{\gamma-1} e^{-\delta t},
\]  
proving the assertion. \hfill \Box

Using lemma 3.3, finding a constant global supersolution becomes a relatively simple matter under the hypothesis of theorem 2.1.

**Proposition 3.4.** Suppose, as in the hypothesis of theorem 2.1, that there are constants $R_0$, $\beta_0$ which are positive lower bounds for the scalar curvature $R$ and $\beta$ respectively. Then if $\|\sigma\|_0$ is sufficiently small relative to $R_0$ and $\|\nabla \tau\|_{0,-\delta'}$, the system (1.5) admits a constant global supersolution.

**Proof.** Let $K_\tau = K\|\nabla \tau\|_{0,-\delta'}$ where $K$ is as in lemma 3.3. We will argue that if $\|\sigma\|_0$ is sufficiently small, then there exists an open interval $I$ in the positive real line for which any $\epsilon_+ \in I$ is a constant global supersolution of (1.5). As $R \geq R_0$, for any $0 < \phi \leq \epsilon_+$ we have
\[
\text{Lich}_\phi(\epsilon_+) = -c_nR\epsilon_+ - \beta \epsilon_+^\alpha + c_n|DW_\phi| + \sigma^2 \epsilon_+^{-\gamma} \\
\leq -c_nR\epsilon_+ + 2c_n|DW_\phi|^2 \epsilon_+^{-\gamma} + 2c_n|\sigma|^2 \epsilon_+^{-\gamma} \\
\leq -c_nR_0\epsilon_+ + 2c_nK_\tau^2 \epsilon_+^{2\gamma-2-\gamma} + 2c_n|\sigma|^2 \epsilon_+^{-\gamma} \\
= -c_nR_0\epsilon_+ + 2c_nK_\tau^2 \epsilon_+^{\alpha} + 2c_n|\sigma|^2 \epsilon_+^{-\gamma},
\]
where we have used lemma 3.3 and the fact that $\phi \leq \epsilon_+$ in the third inequality, and the identity $\gamma - 2 = \alpha$ in the final equality. Now suppose that we may find some $\epsilon_+$ such that
\[
\left(\frac{\|\sigma\|_0}{K_\tau}\right)^{1/(\alpha+1)} < \epsilon_+ < \left(\frac{R_0}{4K_\tau^2}\right)^{1/(\alpha-1)}.
\]  
(3.8)

The second inequality in (3.8) implies that
\[
- c_nR_0\epsilon_+ + 2c_nK_\tau^2 \epsilon_+^{\alpha} < -2c_nK_\tau^2 \epsilon_+^{\alpha}.
\]  
(3.9)

On the other hand, the first inequality in (3.8) implies that
\[
\epsilon_+ > \left(\frac{\|\sigma\|_0^2}{K_\tau^2}\right)^{1/(\alpha+\gamma)} \Rightarrow 2c_nK_\tau^2 \epsilon_+^{\alpha+\gamma} > 2c_n\|\sigma\|_0^2
\]  
(3.10)

where we have used the fact that $2(\alpha+1) = \alpha + \gamma$. Together these inequalities imply
\[
- c_nR_0\epsilon_+ + 2c_nK_\tau^2 \epsilon_+^{\alpha} + 2c_n\|\sigma\|_0^2 \epsilon_+^{-\gamma} < -2c_nK_\tau^2 \epsilon_+^{\alpha} + 2c_n\|\sigma\|_0^2 \epsilon_+^{-\gamma} \\
= \epsilon_+^{\alpha+\gamma} (-2c_nK_\tau^2 \epsilon_+^{\alpha+\gamma} + 2c_n\|\sigma\|_0^2) \\
< 0,
\]
proving that such an $\epsilon_+$ is a global supersolution of (1.5). Finally, observe that the smallness condition (3.8) on $\|\sigma\|_0$ is equivalent to
\[
\|\sigma\|_0 < K_\tau \epsilon_+^{\alpha+1} = K_\tau (\epsilon_+^{\alpha-1})^{(\alpha+1)/(\alpha-1)} < K_\tau (R_0/4K_\tau^2)^{\alpha/2} \\
= 2^{-n} K_\tau^{1-n} R_0^{n/2}.
\]

This along with the definition of $K_\tau$ proves that this smallness condition is the same as in the statement of the proposition, so we are finished. \hfill \Box

A constant global supersolution is not particularly desirable, for unless we construct a global subsolution which approaches this constant asymptotically, we would be able to say little about the asymptotics of a solution provided by proposition 3.1. Because we would like to obtain better information on the asymptotics of solutions to (1.5), we would like to find a
pair of global sub-supersolutions which converge to the same limits on the ends, trapping the asymptotic behavior of the solution obtained by applying proposition 3.1. To accomplish this, we use the constant global supersolution constructed above to define a limiting function \( \phi_t \) on each \( \mathcal{N}_i \), and then construct an appropriate pair of global sub-supersolutions to guarantee that the solution of the Lichnerowicz equation between these approaches \( \phi_t \) asymptotically on \( \mathcal{E}_i \).

With this goal in mind, consider the following construction. As in [4], we define the reduced Lichnerowicz equation to be the semilinear equation on each \( \mathcal{N}_i \) given by

\[
\Delta \hat{\phi}_t - c_i \hat{R}_t \phi_t - \beta_i \phi_t^\gamma + c_n \phi_t^\gamma = 0
\]

where \( \hat{g}_t \) is defined in section 1.1. \( \hat{R}_t \) and \( \hat{\beta}_t \) are the asymptotic limits of \( R \) and \( \beta \) respectively on \( \mathcal{E}_i \). Note that in the AC case this is the same as the Lichnerowicz equation on \( \mathcal{N}_i \) since the dimensional constants in the two equations differ.

Due to our assumptions on \( R, \sigma \), and \( \beta \), one easily checks that \( \epsilon_+ \) is a supersolution to the reduced Lichnerowicz equation on either \( \mathcal{N}_i \) or \( \mathcal{S}^1 \times \mathcal{N}_i \). Moreover, a bounded subsolution \( \rho u_t \) of this equation is constructed in [4], where \( \rho \) is any sufficiently small positive number. We may thus choose \( \rho < \epsilon_+ (\max_t \max_{\mathcal{N}_i} | u_t |)^{-1} \) so that proposition 3.1 provides a solution \( \phi_t \leq \epsilon_+ \) of (3.11). We then replace \( \epsilon_+ \) with some \( \epsilon_+ > \epsilon_+ \) in the interval \( I \) described in the proof of proposition 3.4. Continuing to call this new global supersolution ‘\( \epsilon_+ \)’, we thus have \( \phi_t < \epsilon_+ \) for each \( \ell \), and we extend \( \phi_t \) to all of \( \mathcal{E}_i \) by defining it to be translation invariant in the AC case and invariant under t-translations by multiples of \( T \) in the AP case. We extend these functions using cutoffs to a positive \( C^{2,\mu} \) function \( \phi \) on all of \( M \).

Given \( \delta_\epsilon > 0 \) as in theorem 1.1, assume that \( \nabla \tau \in C^{0,\mu}_{-\hat{\phi}} \) where \( 0 < \delta' < \delta_\epsilon \). Now choose any positive \( \delta < \delta' \), select some positive number \( \nu < \delta/(2\gamma - 2) \) and let \( u \) be the unique solution in \( C^{2,\mu}_{-\hat{\phi}} (M) \) of the equation

\[
\Delta u - c_i R u = -e^{-\nu t}.
\]

We ultimately define a weak global supersolution \( \phi_+ \) to equal \( \epsilon_+ \) on some compact set \( \mathcal{K} \) and \( \min \{ \epsilon_+, \phi + bu \} \) outside of \( \mathcal{K} \) where \( b \) is a large constant. This is a weak global supersolution provided that \( \phi + bu \) is a global supersolution on \( M \setminus \mathcal{K} \) and that \( \phi + bu \geq \epsilon_+ \) on \( \mathcal{K} \). We will next show that, with the usual assumption that \( \beta \) is bounded below away from zero, we may find a compact set \( \mathcal{K} \subset M \) such that \( \phi + bu \) is a global supersolution outside of \( \mathcal{K} \) for any sufficiently large \( b \). If we then choose \( b \) large enough so that \( \phi + bu > \epsilon_+ \) on all of \( \mathcal{K} \), \( \phi_+ = \min \{ \epsilon_+, \phi + bu \} \) is a continuous weak global supersolution of (1.5) on \( M \) which asymptotically approaches \( \phi \) on the ends.

**Proposition 3.5.** Let \( 0 < \nu < \delta/(2\gamma - 2) \) be so small that \( \text{Lich}_0(\hat{\phi}) \) decays faster than \( e^{-\nu t} \). Let \( u \in C^{2,\mu}_{-\hat{\phi}} \) be defined as in (3.12) and suppose that \( \beta \geq \beta_0 > 0 \) where \( \beta_0 \) is a constant.

Then there exists a compact set \( \mathcal{K} \subset M \) such that \( \phi + bu \) is a global supersolution on \( M \setminus \mathcal{K} \) for any sufficiently large \( b \).

**Proof.** First note that the maximum principle and the asymptotics of the metric imply that there are positive constants \( k_1 \) and \( k_2 \) such that \( k_1 e^{-\nu t} \leq u \leq k_2 e^{-\nu t} \). If \( 0 < \phi < \hat{\phi} + bu \), we have

\[
\text{Lich}_\phi(\hat{\phi} + bu) = (\Delta - c_i R) \hat{\phi} - be^{-\nu t} - \beta (\hat{\phi} + bu)^\gamma + c_n |\text{DW}_\phi + \sigma |^2 (\hat{\phi} + bu)^{\gamma}. \]

Because \( \alpha > 1 \), by convexity we may bound this expression above by

\[
(\Delta - c_i R) \hat{\phi} - be^{-\nu t} - \beta \hat{\phi}^\gamma - \beta (bu)^\gamma + c_n (|\text{DW}_\phi| + |\sigma |)^2 (\hat{\phi} + bu)^{\gamma}. \]

We next use the fact that \( \gamma > 1 \) to note that \( (\hat{\phi} + bu)^{\gamma} < \min(\hat{\phi}^{\gamma}, (bu)^{\gamma}) \), bounding the previous expression above by

\[
(\Delta - c_i R) \hat{\phi} - \beta \phi^\gamma + c_n |\phi |^2 (\phi - b e^{-\nu t} - \beta_0 (bu)^\gamma + c_n (|\text{DW}_\phi|^2 + 2|\text{DW}_\phi||\sigma |)(bu)^{\gamma}. \]
The first three terms in this expression equal \( \text{Lich}_0(\phi) \), so we may use lemma 3.3 to bound this last expression above by

\[
\text{Lich}_0(\phi) - b e^{-\nu t} - \beta_0 b^\alpha u^\alpha + c_n |D\phi|^2 b^{-\gamma} u^{-\gamma} + 2c_n |D\phi||\sigma| b^{-\gamma} u^{-\gamma} \\
\leq \text{Lich}_0(\phi) - b e^{-\nu t} - c_1 \beta_0 b^\alpha e^{-a \nu t} + c_2 \|\phi + bu\|_0^{2} - 2 b^{-\gamma} e^{(\nu - 2b)\gamma} \\
+ c_3 \|\phi + bu\|_0^{\gamma} - \|\sigma||b^{-\gamma} e^{(\nu - 3b)\gamma}}
\]

where \( c_1 = k_1^\nu, c_2 = c_n k_1^{-\gamma} K_2, \) and \( c_3 = 2c_n k_1^{-\gamma} K_2. \) Now choose \( b \) so large that \( \|\phi/b + u\|_0 \leq 2\|u\|_0. \) Then letting \( \tilde{c}_2 = c_2(2\|u\|_0)^\gamma - 1 \) and \( \tilde{c}_3 = c_3(2\|u\|_0)^\gamma - 1, \) we see that the final expression above is bounded above by

\[
\text{Lich}_0(\phi) - b e^{-\nu t} - c_1 \beta_0 b^\alpha e^{-a \nu t} + \tilde{c}_2 b^\alpha e^{(\gamma - 2b)\gamma} + \tilde{c}_3 \|\sigma||b^{-\gamma} e^{(\gamma - 3b)\gamma}},
\]

where we have used that \( \gamma - 2 = \alpha. \) One easily sees that the final two terms in the expression above decay faster than the middle term since \( v < \delta/(2\gamma - 2) \) by assumption. By hypothesis, there is some \( t_0 \) such that \( \text{Lich}_0(\phi) < e^{-\nu t} \) whenever \( t > t_0, \) so we may choose some \( t_1 > t_0 \) such that

\[
-c_1 \beta_0 b^\alpha e^{-a \nu t} + \tilde{c}_2 b^\alpha e^{(\gamma - 2b)\gamma} + \tilde{c}_3 \|\sigma||b^{-\gamma} e^{(\gamma - 3b)\gamma} < 0
\]

whenever \( t > t_1. \) Clearly \( \tilde{c}_3 b^{-1}\|\sigma||b^{-\gamma} e^{(\gamma - 3b)\gamma} \leq \tilde{c}_3 b^{-\gamma} \|\sigma||b^{-\gamma} e^{(\gamma - 3b)\gamma}, \) so we may set \( \mathcal{K} = \{t \leq t_1\} \) to prove the proposition.

We may similarly construct a weak global subsolution which is asymptotic to \( \hat{\phi}. \) Because for every \( \phi \leq \hat{\phi}, \) we have that \( |D\phi| \leq K \|\phi\|_0^{\gamma - \delta} e^{-\delta t} \) by lemma 3.3, outside of some large compact set of the form \( \{t \leq t_2\} \) (which we will also denote by \( \mathcal{K} \)) we have

\[
\text{Lich}_0(\hat{\phi}) + e^{-\nu t} - c_n |D\phi|(|D\phi| + 2|\sigma|)(\min_M \hat{\phi})^{-\gamma} \geq 0
\]

where \( \nu \) is as in the previous proposition. Now let \( \nu \) be the unique solution to

\[
\begin{cases}
(\Delta - c_n R)\nu = -e^{-\nu t} \\
\nu|_{\partial \mathcal{K}} = \phi|_{\partial \mathcal{K}}
\end{cases}
\]

on \( M \setminus \mathcal{K} \) and define a function \( \varphi_- \) on \( M \setminus \mathcal{K} \) by \( \varphi_- = \hat{\phi} - \nu \) (note that \( \nu \) is positive and hence \( \varphi_- < \hat{\phi}. \)) Wherever \( \varphi_- \) is positive on \( M \setminus \mathcal{K}, \) we have

\[
\text{Lich}_\phi(\varphi_-) = (\Delta - c_n R)\varphi_- - \beta \varphi_- + c_n |D\phi| + \sigma^2 \varphi_-^{-\gamma} \\
\geq (\Delta - c_n R)\hat{\phi} + e^{-\nu t} - \beta \hat{\phi} + c_n |D\phi| + \sigma^2 \hat{\phi}^{-\gamma} \\
\geq (\Delta - c_n R)\hat{\phi} - \beta \hat{\phi} + c_n |\sigma|^2 \hat{\phi}^{-\gamma} + e^{-\nu t} + c_n |D\phi|(|D\phi| - 2|\sigma|)|\hat{\phi}^{-\gamma} \\
\geq \text{Lich}_\phi(\hat{\phi}) + e^{-\nu t} - c_n |D\phi|(|D\phi| + 2|\sigma|)(\min_M \hat{\phi})^{-\gamma} \\
\geq 0
\]

where the final inequality follows from the definition of \( \mathcal{K}. \) Since \( \nu \) decays on the ends and \( \hat{\phi} \) is \( t \)-invariant or periodic on the ends, we know that \( \varphi_- = \hat{\phi} - \nu \) is strictly positive outside of some compact set containing \( \mathcal{K}. \) Define such a set \( \mathcal{K}' \supset \mathcal{K} \) where \( \varphi_- > \frac{1}{2} \min_M \hat{\phi} \) on \( M \setminus \mathcal{K}'. \) We then let \( \eta \) be the solution on \( \mathcal{K}' \) to the following boundary value problem:

\[
\begin{cases}
(\Delta - c_n R - \beta)\eta = 0 \\
\eta|_{\partial \mathcal{K}'} = \frac{1}{2} \min(1, \min_M \hat{\phi})
\end{cases}
\]

One easily checks that \( \eta \) is a global subsolution for the Lichnerowicz equation on \( \mathcal{K}', \) is positive by the strong maximum principle, and is less than \( \varphi_- \) on \( \partial \mathcal{K}'. \) We thus conclude that \( \phi_- = \max(\varphi_- \eta) \) is a weak global subsolution for the Lichnerowicz equation on \( M \).
4. Fixed point argument

The asymptotics of our global sub-/supersolution pair \( \phi_- \leq \phi_+ \) suggest that we look for a solution of the coupled system (1.5) with the form \( \bar{\psi} + \psi \) where \( \psi \in C^{2,\mu}_{-\gamma}(M) \). Namely, if we treat \( \bar{\psi} \) as fixed, we can write (1.5) as a nonlinear elliptic system for \( (\psi, W) \). Similar to the approach in [7, 9], we will find a solution for this system by applying Schauder’s fixed point theorem to a well-chosen function on a weighted Hölder space. Let us first state the fixed point theorem we have in mind.

**Theorem 4.1.** Let \( X \) be a Banach space, and let \( U \subset X \) be a non-empty, convex, closed, bounded subset. If \( T : U \to U \) is a compact operator, then there exists a fixed point \( u \in U \) such that \( T(u) = u \).

A proof of theorem 4.1 can be found in [8]. Let \( 0 < \nu' < \nu \). We will find a solution \((\bar{\psi}, W_{\bar{\psi}})\) to the \((\psi, W)\)-system as a fixed point of a composition of suitably chosen solution operators on the set

\[
U = \{ \psi \in C^{0,\nu'} : \phi_- \leq \bar{\psi} \leq \phi_+ \}. \tag{4.1}
\]

This set clearly meets all the criteria of theorem 4.1. Below we define a map \( T \) which preserves \( U \) and has the properties stated in the theorem.

Let us first define a map \( \mathcal{W} : U \to C^{1,\mu}_{-\gamma}(TM) \oplus \mathcal{W} \) which sends \( \psi \) to the vector field

\[
W_{\bar{\psi}} := G(n^{-1}(n - 1)(\bar{\psi} + \psi)^{\nu - 1}\nabla \tau),
\]

where \( G \) is the bounded generalized inverse for the conformal vector Laplacian \( \Delta_{\gamma} \). We next define a map \( S_\sigma : C^{0,\mu}_{-\gamma}(S^2_0(M)) \to C^{2,\mu}_{-\gamma}(M) \) by

\[
\pi \mapsto Q(\sigma + \pi) - \phi
\]

where \( Q(\tilde{\sigma}) \) is defined to be the unique solution \( \phi_- \leq \tilde{\phi} \leq \phi_+ \) of

\[
\Delta \phi - c_n R \phi - \beta \phi^\nu + c_n |\sigma|^{1/2} \phi^{-\gamma} = 0 \tag{4.3}
\]

which satisfies \( Q(\tilde{\sigma}) \to \tilde{\phi} \) on the ends. \( Q \) is well-defined by the discussion in section 3. We show that \( S_\sigma \) is continuous using an argument very similar to the proof of proposition 13 in [9].

**Lemma 4.2.** The map \( S_\sigma : C^{0,\mu}_{-\gamma}(S^2_0(M)) \to C^{2,\mu}_{-\gamma}(M) \) is continuous.

Note that here the domain of \( S_\sigma \) is the set of traceless symmetric 2-tensors in the indicated weighted Hölder space.

**Proof.** We invoke the implicit function theorem (see [12], for example). We define a map \( F : C^{2,\mu}_{-\gamma}(M) \times C^{0,\mu}_{-\gamma}(S^2_0(M)) \to C^{2,\mu}_{-\gamma}(M) \) by

\[
F(\psi, \pi) = (\Delta - c_n R)(\bar{\psi} + \psi) - \beta (\bar{\psi} + \psi)^\nu + c_n |\sigma + \pi|^{1/2} (\bar{\psi} + \psi)^{-\gamma},
\]

so that \( F(\tilde{S}_\sigma(\pi), \pi) = 0 \) by definition. The Fréchet derivative \( DF(\psi, \pi)(h, k) \) of \( F \) at the point \((\psi, \pi)\) acting on pairs \((h, k) \in C^{2,\mu}_{-\gamma}(M) \times C^{0,\mu}_{-\gamma}(S^2_0(M)) \) is given by

\[
\begin{aligned}
(\Delta - c_n R)h - \alpha \beta (\bar{\psi} + \psi)^{\nu-1} h + 2c_n |\sigma + \pi, k| (\bar{\psi} + \psi)^{-\gamma} - c_n \gamma |\sigma + \pi|^{1/2} (\bar{\psi} + \psi)^{-\gamma - 1} h.
\end{aligned}
\]

In particular,

\[
DF(\psi, \pi)(h, 0) = (\Delta - c_n R + \alpha \beta (\bar{\psi} + \psi)^{\nu-1} + c_n \gamma |\sigma + \pi|^{1/2} (\bar{\psi} + \psi)^{-\gamma - 1}) h.
\]

Hence given any pair \((\psi, \pi)\) for which \( \bar{\psi} + \psi > 0 \) everywhere, which is certainly true for all \( \psi \) in the image of \( S_\sigma \), we see that \( DF(\psi, \pi)(\cdot, 0) : C^{2,\mu}_{-\gamma}(M) \to C^{0,\mu}_{-\gamma}(M) \) is an isomorphism. Since
of theorem 4.1, and so straightforward computation reveals that the associated conformal vector Laplacian

\[ \tilde{\omega} \]

First note that if \( g \) is an AC metric on \( M \) and \( w \in C^{2,\mu} \) has the asymptotic properties described in section 1.1, we define a conformally AC metric \( \tilde{g} = w^{4/(n-2)}g \) and seek to solve (1.5) with respect to this metric. It should be noted that one can view the problem in the conformally AC setting as a degenerate case of the AP problem, and can thus likely prove the results below via a slight modification of the techniques developed to find a solution to (1.5) in the AP case. However, we believe that the existence proof obtained by modifying the techniques developed for the AC case is more transparent, and we shall take this approach below.

5. The conformally AC case

Thus far we have assumed that our Riemannian metric is only AC or AP. We have deferred our discussion of solving (1.5) in the conformally AC case because finding a solution requires only a slight modification of the arguments above. In particular, if \( g \) is an AC metric on \( M \) and \( w \in C^{2,\mu} \) has the asymptotic properties described in section 1.1, we define a conformally AC metric \( \tilde{g} = w^{4/(n-2)}g \) and seek to solve (1.5) with respect to this metric. It should be noted that one can view the problem in the conformally AC setting as a degenerate case of the AP problem, and can thus likely prove the results below via a slight modification of the techniques developed to find a solution to (1.5) in the AP case. However, we believe that the existence proof obtained by modifying the techniques developed for the AC case is more transparent, and we shall take this approach below.

5.1. Indicial roots of \( \Delta_{\tilde{g}} \) for conformally AC metrics

First note that if \( g \) were exactly cylindrical and \( w = \hat{w} \) (so that \( w \) does not depend on \( t \)), a straightforward computation reveals that the associated conformal vector Laplacian \( \tilde{\Delta}_{\tilde{g}} \) can be expressed with respect to the conformal Killing operator of the exactly cylindrical metric via

\[
(\tilde{\Delta}_{\tilde{g}}X)^i = \tilde{w}^{-4/(n-2)}(\Delta_{\tilde{g}}X)^i - \frac{R}{2} \nabla_j(\tilde{w}^{-4/(n-2)})(DX)^{ij}.
\]  

(5.1)

As in this equation, throughout this subsection only the operators denoted with a tilde are defined with respect to the conformally AC metric \( \tilde{g} \) while all others are defined with respect to the AC metric \( g \). As \( \tilde{w} \) is bounded above and below, this operator is uniformly elliptic. Now return to the general case. Since \( \tilde{g} \) decays exponentially to \( \hat{w}^{4/(n-2)}g \), \( \tilde{\Delta}_{\tilde{g}} := \tilde{D}^\ast \tilde{D} \) can be expressed as the operator in (5.1) (where all the operators on the right are defined with respect to the exactly cylindrical metric) plus a perturbation term whose coefficients decay like \( e^{-\alpha t} \) for some \( \alpha' > 0 \). In particular, we may regard \( \tilde{\Delta}_{\tilde{g}} \) as an elliptic \( b \)-operator so that, with the obvious change of coordinates \( x = e^{-t} \), we may cite results from [10] to analyze this operator just as in the proof of theorem 1.1.

To be specific, as in [5] for each \( N_f \) we may define the indicial family \( I_{\lambda} \) to be the family of operators on the cylinder \( \mathbb{R} \times N_f \) given by \( I_{\lambda} \) is the Fourier transform in \( t \) and \( \tilde{\Delta}_{\tilde{g}} \) is the operator on the right hand side of (5.1) with an exactly cylindrical underlying metric \( g \). One easily checks that \( I_{\lambda} \) is a second-order elliptic operator depending polynomially on \( \lambda \), and note that our definition of \( I_{\lambda} \) is equivalent to \( I_{\lambda}(\tilde{\Delta}_{\tilde{g}}) \) in [10]. As in [5], one may use the analytic Fredholm theorem to show that this operator is invertible away from a discrete set of complex numbers \( \Lambda(\tilde{\Delta}_{\tilde{g}}) = \{ \lambda_j \} \) which we define to be the indicial roots of the conformally AC operator \( \tilde{\Delta}_{\tilde{g}} \).
It follows from the results of [10] that the indicial roots of an elliptic $b$-operator such as $\Delta_L$ are precisely the decay rates of elements in its nullspace (so that only real $\lambda$ correspond to bounded solutions of $\Delta_L Y = 0$), and there are only finitely many in any horizontal strip \( \{ a < \text{Im} \xi < b \} \). This implies that we may choose $\delta_\ast$ so small that the only indicial roots in the strip \( \{ -\delta_\ast < \text{Im} \xi < \delta_\ast \} \) are real. It then follows from theorem 4.26 in [10] and an argument analogous to the proof of theorem 1.1 that if $\delta$ belongs to the punctured interval \( (-\delta_\ast, \delta_\ast) \setminus \{0\} \), the map

$$\hat{\Delta}_\delta : C_{k+2,\mu}^0(TM) \to C_{k,\mu}^0(TM)$$

is Fredholm for any $k \geq 0$. Injectivity and surjectivity arguments can then be made exactly as in the proof of theorem 1.1, and theorem 7.14 in [10] implies that the generalized inverse $G$ maps $C_{-\delta,\mu}^{k,\mu}(TM)$ into $C_{-\delta,\mu}^{k+2,\mu}(TM) \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}_m$ where $\mathcal{H}_i$ is a subspace of vector fields whose restrictions to each end are periodic solutions of $\hat{\Delta}_\delta Y = 0$. However, we claim that these vector fields actually restrict to conformal Killing fields of the exactly conformally cylindrical metric on each end. For if $T_{\delta}i$ is the period of $Y \in \mathcal{H}_i$ restricted to $E_i$, quotienting the cylinder by $T_{\delta}i\mathbb{Z}$ gives us a solution to $\Delta_L Y = 0$ on the closed manifold $S^1 \times N_i$. We may thus integrate by parts to find that, in fact, $\tilde{D}Y \equiv 0$, implying that $Y$ is a conformal Killing field on each end. The difference $\hat{D} - \tilde{D}$ is a first order operator with coefficients decaying like $e^{-\omega t}$, and this means that if we take $\delta < \omega$, the image of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ under $\hat{D}$ belongs to $C_{-\delta,\mu}^{k+1,\mu}(S^1_i(TM))$. Hence the composition $\hat{D} \circ G$ is a bounded map from $C_{-\delta,\mu}^{k,\mu}(TM)$ into $C_{-\delta,\mu}^{k+1,\mu}(S^1_i(TM))$. This mapping property implies that lemma 3.3 holds in the conformally AC case, and so we may bound the magnitude of $\tilde{D}W_\phi$ in terms of $\|\phi\|_0$, allowing us to construct a pair of global sub-/supersolutions.

5.2. Global barriers in the conformally AC case

To prove theorem 2.2, we must show that we can replace the condition $R \geq R_0 > 0$ with the weaker condition that $(M, g)$ is Yamabe positive. This is to say that the Yamabe invariant of the conformal class $[g]$ is positive, recalling that the Yamabe invariant for a manifold with cylindrical ends is defined as

$$Y(M,[g]) = \inf_{\mathcal{D}M \ni \phi} \int_M \left( \frac{1}{2} \int_M (|\nabla u|^2 + c_R u^2) \right)^\frac{1}{2} u^{n+2+(n-2)}.$$ \hspace{1cm} (5.3)

We have the following proposition, which is direct consequence of the proof of proposition 4.6 in [4]:

**Proposition 5.1.** Suppose that $(M, g)$ has conformally AC or AP ends, and that $(M, g)$ is Yamabe positive. Then there exists a positive function $u \in C^\infty(M)$ such that on each end $E_i$, $u \to u_t$ as $t \to \infty$, where $u_t \in C^\infty(N_i)$ in the conformally AC case and $u_t \in C^\infty(S^1 \times N_i)$ in the AP case, such that $\tilde{g} = e^{4/(n-2)}g$ has $\tilde{R} \geq \tilde{R}_0 > 0$ everywhere.

In particular, so long as $(M, g)$ is Yamabe positive, we may conformally transform $g$ to a metric which has scalar curvature bounded below by a positive constant. This fact allows us to construct global sub-/supersolutions much in the same way as in section 3.

**Proposition 5.2.** Let $(M, g)$ be a manifold with conformally AC ends with $Y(M,[g]) > 0$, and suppose $\tau$ is as in theorem 2.2. If $\tilde{\theta}$ is the conformal factor given by proposition 5.1, then we may find some constant $\epsilon > 0$ such that $\epsilon \tilde{\theta}$ is a global supersolution of the system (1.5) if $\|\sigma\|_0$ is sufficiently small.
Proof. The proof is again essentially the same as in proposition 14 of [9]. From the identity 
\[ \Delta \vartheta - c_n R \vartheta = -c_n \tilde{R} \vartheta^\alpha \] we see that given any \( \phi \leq \vartheta \),
\[
\text{Lich}_\phi (\vartheta) = -c_n \tilde{R} \vartheta^\alpha - \beta e^\alpha \vartheta^\alpha + c_n |\sigma + DW| e^{-\gamma} \vartheta^\gamma \leq -c_n \tilde{R} \vartheta^\alpha - \beta e^\alpha \vartheta^\alpha + 2 c_n |\sigma| 2 e^{-\gamma} \vartheta^\gamma + 2 c_n |DW| 2 e^{-\gamma} \vartheta^\gamma.
\]
Using lemma 3.3 (which we showed to be valid in the conformally AC case), it is then easy to see that this quantity is bounded above by
\[
(-c_n \tilde{R} e - \beta \varepsilon) (\inf \vartheta) - 2 c_n \|\sigma\|_0^2 e^{-\gamma} + 2 c_n K^2 e^\alpha (\sup \vartheta) (2 \gamma - 2) (\inf \vartheta)^{-\gamma}.
\]
The rest of the argument proceeds exactly as in the proof of proposition 3.4, though now the smallness condition on \( \|\sigma\|_0 \) will depend on \( K, \tilde{R}_0, \inf \vartheta, \) and \( \sup \vartheta \).

Next observe that if \( \tilde{g} = u^{4/(\alpha - 2)} g \), where \( g \) is an AC metric, the conformal covariance of the Lichnerowicz equation and the previous result imply that \( \varepsilon \tilde{\vartheta} \tilde{w} \) is a supersolution of the reduced equation (3.11) (assume without loss of generality that \( M \) has only one end). We may construct a subsolution \( \phi_- < \varepsilon \tilde{\vartheta} \tilde{w} \) as in section 3 to extract a solution \( \tilde{w} \) to (3.11). Hence if our putative solution has an asymptotic limit, we expect it to be \( \tilde{\vartheta} \), and we extend this function to all of \( M \) as above. Just as in the AC case, we will seek to construct a solution \( (\tilde{\vartheta}, W_\tilde{\vartheta}) \) where \( \tilde{\vartheta} = \phi + \psi \) for some \( \psi \in C^2, \nu \).

Proposition 5.3. Let \( (M, g) \) and \( \vartheta \) be as in the previous proposition. If \( \nu > 0 \) is sufficiently small, then we may find a compact set \( \mathcal{K} \subset M \) such that \( \tilde{\vartheta} + b \vartheta e^{-\nu \vartheta} \) is a global supersolution on \( M \setminus \mathcal{K} \) for any sufficiently large \( b \).

Proof. One easily checks that there is a constant \( A = A(n, g, \|\vartheta\|_C^1) \) satisfying
\[
|2 \nabla \vartheta \cdot \nabla e^{-\nu \vartheta} + \vartheta \Delta e^{-\nu \vartheta}| \leq \nu A e^{-\nu \vartheta},
\]
and we then observe that
\[
(\Delta - c_n R) (\vartheta e^{-\nu \vartheta}) = -c_n \vartheta^\alpha \tilde{R} e^{-\nu \vartheta} + 2 \nabla \vartheta \cdot \nabla e^{-\nu \vartheta} + \vartheta \Delta e^{-\nu \vartheta} \leq (-c_n (\inf \vartheta) \tilde{R}_0 + \nu A) e^{-\nu \vartheta}.
\]
From this it follows that there is some \( \nu_0 > 0 \) such that if \( \nu \leq \nu_0 \), there is a positive constant \( k_\nu \) such that \( (\Delta - c_n R) (\vartheta e^{-\nu \vartheta}) < -k_\nu e^{-\nu \vartheta} \). For any \( \phi \leq \tilde{\vartheta} + b \vartheta e^{-\nu \vartheta} \) with \( \nu \) sufficiently small, we thus have that \( \text{Lich}_\phi (\tilde{\vartheta} + b \vartheta e^{-\nu \vartheta}) \) is bounded above by
\[
(\Delta - c_n R) (\phi - b k_\nu e^{-\nu \vartheta} - \beta (\phi + b \vartheta e^{-\nu \vartheta}) e^{-\nu \vartheta}) + c_n |\sigma + DW| 2 (\phi + b \vartheta e^{-\nu \vartheta}) e^{-\nu \vartheta}.
\]
One then argues exactly as in the proof of proposition 3.5 to prove the result.

Just as in the AC case, we may choose \( \nu \) sufficiently small and \( b \) sufficiently large so that \( \min [\hat{\vartheta}, \phi + b \vartheta e^{-\nu \vartheta}] \) is a weak global supersolution.

The proofs of the previous two propositions suggest how we might construct a global subsolution.

Proposition 5.4. Let \( \nu < \delta/(2 \gamma - 2) \) be so small that \( (\Delta - c_n R) (\vartheta e^{-\nu \vartheta}) < -k_\nu e^{-\nu \vartheta} \), and let \( \phi_+ \) be defined as above. Then there is some \( \alpha > 0 \) such that if \( \phi - a \vartheta e^{-\nu \vartheta} \leq \phi \leq \phi_+ \), we have that
\[
\text{Lich}_\phi (\phi - a \vartheta e^{-\nu \vartheta}) \geq 0
\]
wherever \( \phi - a \vartheta e^{-\nu \vartheta} > 0 \).
Proof. Carrying out computations nearly identical to those in section 3, we find that \( \text{Lich}_0(\phi - a\theta e^{-\alpha t}) \) is bounded below by
\[
\text{Lich}_0(\phi) + a_k e^{-\alpha t} - c_\alpha |DW| (|DW| + 2|\sigma|)(\inf \phi)^{-\gamma}.
\]
Since \(|\text{Lich}_0(\phi)| \leq C_1 e^{-\alpha t} \) by construction, the upper bound \( \phi \leq \phi_+ \) and lemma 3.3 imply that for some constant \( C_2 > 0 \) we have
\[
\text{Lich}_0(\phi - a\theta e^{-\alpha t}) \geq a_k e^{-\alpha t} - C_1 e^{-\alpha t} - C_2 e^{-\beta t}.
\]
Therefore, as \( \nu < \beta < \omega \), we need only choose \( a \) so large that \( a_k > C_1 + C_2 \) to ensure that (5.4) holds.

It remains to construct a global subsolution on the bounded region where \( \phi - a\theta e^{-\alpha t} \leq 0 \). Choose \( T > 0 \) so large that \( \phi - a\theta e^{-\alpha t} > \frac{1}{2} \inf \phi \) whenever \( t \geq T \). Since \( R \) is bounded below and \( \beta \geq C_0 > 0 \), we may find some constant \( K \) such that \( c_\alpha R + K\beta > 0 \) everywhere. Letting \( \mathcal{K} = \{ t \leq T \} \), we may thus solve the boundary value problem
\[
\begin{cases}
(\Delta - c_\alpha R - K\beta)\eta = 0 \\
\eta|_{\partial \mathcal{K}} = \frac{1}{2} \min(1, \inf \phi)
\end{cases}
\]
The maximum principle implies that \( \eta < 1 \), and the strong maximum principle implies that \( \eta > 0 \). One easily verifies that \( \eta \) is a global subsolution on \( \mathcal{K} \), and that if we extend \( \eta \) to be identically zero outside of \( \mathcal{K} \), we find that \( \max(\eta, \phi - a\theta e^{-\alpha t}) \) is a weak global supersolution.

5.3. Modifications to the fixed point argument

Having constructed a pair of global sub-supersolutions with the correct asymptotic limits, we may apply a fixed point argument similar to that found in section 4 to find a solution to (1.5). For this we define the set \( U \) and the maps \( V \) and \( \mathcal{Q} \) just as in section 4. The map \( W \) is well-defined by the discussion in section 5.1, and proposition 3.2 along with the conformal covariance of the Lichnerowicz equation imply that the solution obtained via proposition 3.1 in the conformally AC case is unique. Hence the map \( \mathcal{Q} \) is well-defined. We note next that \( \text{Lich}_0(\mathcal{Q}(\sigma)) = 0 \), and so we simply replace ‘\( \phi \)’ with ‘\( \mathcal{Q}(\sigma) \)’ in the definition of the map \( S_\phi(\pi) \). The advantage of this replacement is that the conformal covariance of the Lichnerowicz equation gives us the identity
\[
S_\phi(\pi) = \mathcal{Q}(\pi + \pi) - \mathcal{Q}(\pi) = \hat{\theta} \hat{\mathcal{Q}}(\theta^{-2}(\pi + \pi)) - \hat{\theta} \hat{\mathcal{Q}}(\theta^{-2}\pi)
\]
where \( \hat{\mathcal{Q}} \) is the solution operator for the Lichnerowicz equation with respect to the metric \( g^{\text{AC}} \).

The proof of theorem 2.2 reads exactly as the proof of theorem 2.1 once we have established the analogue of lemma 4.2. For this we use an implicit function theorem argument very similar to the proof of that lemma, but we must deal with the fact that we no longer have \( R > 0 \). We shall get around this difficulty using a trick employed by Maxwell in [9], which essentially boils down to the identity (5.5).

Lemma 5.5. The map \( S_\phi : C_{\phi,0}^1(S^2_0(M)) \to C_{\phi,0}^1(M) \), defined with respect to a conformally AC metric, is continuous.

Proof. Given \( \tau_0 \in C_{\phi,0}^1(S^2_0(M)) \), we set \( \theta_0 = \mathcal{Q}(\pi + \pi_0) \). If we also set \( \hat{\theta} = \theta_0^{-2}\pi_0 \) and \( \hat{\tau}_0 = \theta_0^{-2}\pi_0 \), and define \( \hat{S}_\phi \) as in (5.5), we have \( \hat{S}_\phi(\hat{\tau}_0) = 1 - \hat{\mathcal{Q}}(\hat{\theta}) \). We next define the map
\[
F: C_{\phi,0}^1(M) \times C_{\phi,0}^1(S^2_0(M)) \to C_{\phi,0}^1(M)
\]
by
\[
F(\psi, \pi) = (\hat{\Delta} - c_\alpha \hat{R})(\hat{\mathcal{Q}}(\hat{\theta}) + \psi) - \beta(\hat{\mathcal{Q}}(\hat{\theta}) + \psi) + c_\alpha |\pi|^2(\hat{\mathcal{Q}}(\hat{\theta}) + \psi)^{-\gamma}.
\]
The Fréchet derivative is computed exactly as in the proof of lemma 4.2, and we find that at
the point $(\hat{\psi}_0, \hat{\pi}_0) = (\hat{S}_0(\hat{\sigma}_0), \hat{\pi}_0) = (1 - \hat{Q}(\hat{\sigma}), \hat{\pi}_0)$, we have
\[DF_{(\hat{\psi}_0, \hat{\pi}_0)}(h, 0) = [\hat{\Delta} - (c_n \hat{R} + \alpha \beta + c_n \gamma |\hat{\sigma} + \hat{\pi}_0|^2)]h.\]
However, as $\hat{Q}(\hat{\sigma} + \hat{\pi}_0) = 1$, we have
\[-c_n \hat{R} = \beta - c_n |\hat{\sigma} + \hat{\pi}_0|^2,\]
so that in fact
\[DF_{(\hat{\psi}_0, \hat{\pi}_0)}(h, 0) = [\hat{\Delta} - ((\alpha - 1) \beta + c_n (\gamma + 1)|\hat{\sigma} + \hat{\pi}_0|^2)]h.\]

The map $DF_{(\hat{\psi}_0, \hat{\pi}_0)}(\cdot, 0) : C^2(\hat{S}_0(\hat{\sigma})(\hat{\pi}), M) \to C^0(\hat{\sigma}(\hat{\pi}), M)$ is thus an isomorphism, and so we conclude
by the implicit function theorem that $\hat{S}_\delta \hat{\sigma}$ is continuous. The lemma now follows from (5.5),
since we have $\hat{S}_0(\hat{\pi}(\hat{\sigma})) = \hat{\theta}_0 \hat{S}_0(\hat{\theta}_0^{-2} \hat{\pi})$ for all $\hat{\pi} \in C^0(\hat{\sigma}(\hat{\pi}), (\hat{S}_0^2(M))).$ □

As remarked above, the proof of theorem 2.2 now goes through exactly as the proof of theorem 2.1.

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