Probability distributions of extremes of self-similar Gaussian random fields

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July 2, 2014

Abstract

We have obtained some upper bounds for the probability distribution of extremes of a self-similar Gaussian random field with stationary rectangular increments that are defined on the compact spaces. The probability distributions of extremes for the normalized self-similar Gaussian random fields with stationary rectangular increments defined in $\mathbb{R}_+^2$ have been presented. In our work we have used the techniques developed for the self-similar fields and based on the classical series analysis of the maximal probability bounding from below for the Gaussian fields.

Keywords: distribution of extremes, self-similar random field, finite dimensional distributions, fractional Brownian sheet.

AMS MSC 2010: 60E05, 60E15.

1 Introduction

A self-similar process is a stochastic process that is invariant in distribution under the suitable scaling of time and space. A random process $\{X(t), t \in \mathbb{R}\}$ is self-similar with index $H > 0$ if for all $a > 0 \{X(at), t \in \mathbb{R}\} \overset{d}{=} \{a^{-H}X(at), t \in \mathbb{R}\}$, where $\overset{d}{=}$ denotes equality of the finite-dimensional distributions. We refer to Embrechts and Maejima [5] and Samorodnitsky and Taqqu [12] for the extensive surveys on results and techniques for self-similar processes.

In this paper we consider the self-similar random fields that are an extension of the self-similar stochastic processes. More precisely, we deal with anisotropic self-similar random fields which means that their indexes of self-similarity are different for different coordinates. We denote $\mathbb{R}_+ = [0, +\infty)$.

Definition 1. A real valued random field $\{X(t), t = (t_1, \ldots, t_n) \in \mathbb{R}_+^n\}$ is self-similar with index $H = (H_1, \ldots, H_n) \in (0, +\infty)^n$ if

$$\{X(a_1 t_1, \ldots, a_n t_n), \ t \in \mathbb{R}_+^n\} \overset{d}{=} \{a_1^{H_1} \cdots a_n^{H_n} X(t), \ t \in \mathbb{R}_+^n\},$$

for all $a_1 > 0, \ldots, a_n > 0$. 

An interest to the anisotropic self-similar random fields is motivated by the applications coming from the climatological and environmental sciences (see [10, 11]). Several authors have proposed to apply such random fields for modelling phenomena in spatial statistics, stochastic hydrology and image processing (see [2, 3, 4]).

**Definition 2.** The normalized fractional Brownian sheet with Hurst index $H = (H_1, \ldots, H_n), 0 < H_i < 1, i = 1, n$ is the centered Gaussian random field $B_H = \{B_H(t), t \in \mathbb{R}^n_+\}$ with a covariance function

$$
E(B_H(t)B_H(s)) = 2^{-n} \prod_{i=1}^{n} (|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}), \quad t, s \in \mathbb{R}^n_+.
$$

This field is self-similar with index $H = (H_1, \ldots, H_n)$ by Definition 1. Moreover, we shall consider only the case $n = 2$ since switching to the parameter of the higher dimension is rather technical.

Denote $0 = (0, 0)$.

**Definition 3.** Let $X = \{X(t), t \in \mathbb{R}^2_+\}$ be a self-similar field with index $H = (H_1, H_2) \in (0, +\infty)^2$. For any $u = (u_1, u_2) \in \mathbb{R}^2_+$ and any $v = (v_1, v_2) \in \mathbb{R}^2_+$ such that $v_1 > u_1, v_2 > u_2$ define

$$
\Delta_u X(v) = X(v_1, v_2) - X(u_1, v_2) - X(v_1, u_2) + X(u_1, u_2).
$$

The field $X$ admits stationary rectangular increments if for any $u = (u_1, u_2) \in \mathbb{R}^2_+

$$
\{\Delta_u X(u + h), h \in \mathbb{R}^2_+\} \overset{d}{=} \{\Delta_0 X(h), h \in \mathbb{R}^2_+\}.
$$

The fractional Brownian sheet has the stationary rectangular increments. The proof of this property for the $\mathbb{R}^2_+$ case can be found in the paper [1]. A similar property for the case $n > 2$ can be easily proved as well.

The aim of the paper is to obtain the upper bound for probability distributions of extremes of normalized self-similar Gaussian random fields with the stationary rectangular increments. These probabilities can be used for the estimation of asymptotic growth of sample paths of the fractional Brownian sheet. Furthermore, these probabilities can be applied to investigate the asymptotic behavior of the fractional derivative of the fractional Brownian motion, which is used in the analysis of a non-standard maximum likelihood estimate for the unknown drift parameter in the stochastic differential equations driven by fractional Brownian motion (see Kozachenko et al. [8]).

To achieve this goal we use the results from the theory of extremes for the Gaussian processes (Kozachenko et al. [9]). This theory, in turn, is based on the theory of metric spaces. To apply these results we need to choose the appropriate compact metric space $(\mathcal{T}, \rho)$ and to estimate the variance of the increments. Since we work with anisotropic field we expect that the chosen metric has the different geometric characteristics along different directions. So, we use two metrics $\rho_1(t, s) = \max_{i=1,2} |t_i - s_i|, t, s \in \mathcal{T} \subset \mathbb{R}^2_+$ and $p_2(t, s) = \sum_{i=1,2} |t_i - s_i|^{H_i}, t, s \in \mathcal{T} \subset \mathbb{R}^2_+$, where $H = (H_1, H_2) \in (0, 1)^2$ is the index of self-similarity of the corresponding random field. The second metric has played an important role in the studying the anisotropic Gaussian fields and the self-similar random fields (see [14]).

The main point in the proofs of this paper is the self-similar property of the fields. This yields the similar behavior of sample paths on compact subsets. From the theory of extremes for the Gaussian processes we get the upper bounds for the probabilities defined in the compact sets. Whence we expand $\mathbb{R}^2_+$ into the union of the compact subsets and apply the inequalities for probabilities in
each subset. We use the techniques of the self-similar fields based on the classical series analysis for finding the bound from below of the maximal probabilities for the Gaussian fields. Several results in this paper are obtained by the optimization procedure.

The paper is organized as follows. In Section 2, we present the probability distributions of extremes of the Gaussian fields defined on the compact spaces and a bound for the variance of its increments in the case of self-similar field. In Section 3 we establish the probability distributions of extremes of the fields defined on compact metric space \((T, \rho_1)\) and derive the upper bounds for such probabilities of the normalized field defined on \(\mathbb{R}^2_+\). In Section 4 we obtain the probability distribution of extremes of the normalized self-similar Gaussian field defined on the metric space \((T, \rho_2)\).

## 2 Probability distributions of extremes of a Gaussian field defined on a compact space

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space satisfying the standard assumptions. It is assumed that all processes under consideration are defined on this space.

The next theorem follows from Theorem 2.8 of [13] or it could be obtained form Lemma 3.2 of [9].

**Theorem 2.1.** Let \((T, \rho)\) be a metric compact space and \(X = \{X(t), t \in T\}\) be a separable centered Gaussian process. Suppose there exists such a continuous monotonically increasing function \(\sigma : \mathbb{R}_+ \to (0, +\infty), \sigma(0) = 0\) that the following inequality holds

\[
\sup_{\rho(t,s) \leq h} (E(X(t) - X(s))^2)^{1/2} \leq \sigma(h).
\]

Let

\[
\beta = \sigma \left( \inf_{s \in T} \sup_{t \in T} \rho(t,s) \right), \quad \gamma = \sup_{u \in T} \left( E[X^2(u)] \right)^{1/2}.
\]

We denote as \(N(\varepsilon)\) the minimal number of closed \(\rho\)-balls with radius \(\varepsilon\) needed to cover the space \((T, \rho)\). Let \(r : [1, +\infty) \to (0, +\infty)\) be such a continuous function that a function \(r(e^t), t > 0\) is convex. If

\[
\int_{0}^{+\infty} r(N(\sigma^{(-1)}(u))) du < \infty,
\]

then for all \(\lambda > 0, 0 < p < 1, \varepsilon > 0\)

\[
I_T(\varepsilon) := P \left\{ \sup_{t \in T} |X(t)| > \varepsilon \right\} \leq 2 \exp \left\{ \frac{\lambda^2 \gamma^2}{2(1-p)} + p \frac{\lambda^2 \beta^2}{2(1-p)^2} - \lambda \varepsilon \right\} \times
\]

\[
\times r^{(-1)} \left( \frac{1}{\beta p} \int_{0}^{\beta p} r \left( N(\sigma^{(-1)}(u)) \right) du \right).
\]

We shall minimize the right-hand side of (3) with respect to \(\lambda > 0\).

**Corollary 2.2.** Under the conditions of Theorem 2.1, we have

\[
I_T(\varepsilon) \leq 2 \exp \left\{ -\frac{\varepsilon^2 (1-p)}{2(\gamma^2 + \beta^2 p)} \right\} r^{(-1)} \left( \frac{1}{\beta p} \int_{0}^{\beta p} r \left( N(\sigma^{(-1)}(u)) \right) du \right).
\]
Proof. Consider the right-hand side of (3). To prove the corollary it is sufficient to minimize the following value

\[ \frac{1}{2} \lambda^2 \gamma^2 \frac{\lambda^2 \beta^2}{2 (1 - p)} + p \frac{\lambda^2 \beta^2}{2 (1 - p)^2} - \lambda \varepsilon. \]

Differentiating this expression with respect to \( \lambda \), we get

\[ \frac{d}{d\lambda} \left( \frac{1}{2} \lambda^2 \gamma^2 \frac{\lambda^2 \beta^2}{2 (1 - p)} + p \frac{\lambda^2 \beta^2}{2 (1 - p)^2} - \lambda \varepsilon \right) = \lambda \gamma^2 \frac{\lambda^2 \beta^2}{2 (1 - p)} + p \frac{\lambda^2 \beta^2}{2 (1 - p)^2} - \varepsilon. \]

Then, the minimum is achieved if

\[ \lambda = \lambda^* = -\frac{\varepsilon (1 - p)}{2 \left( \gamma^2 + \frac{\beta^2 p}{1 - p} \right)}. \]

If we replace \( \lambda \) by \( \lambda^* \) in (3), we obtain (4).

Throughout the paper the field \( X = \{ X(t), t \in \mathbb{R}^2_+ \} \) is a Gaussian self-similar random field with index \( H = (H_1, H_2) \in (0, 1)^2 \) and with stationary rectangular increments. Denote \( \mathbf{1} = (1, 1) \). Evidently,

\[ E[X(t)]^2 = t_1^{2H_1} t_2^{2H_2} E[X^2(\mathbf{1})], \quad t = (t_1, t_2) \in \mathbb{R}^2_+. \]

In what follows we need some auxiliary results.

**Lemma 2.3.** For all \( s = (s_1, s_2) \in \mathbb{R}^2_+, \ t = (t_1, t_2) \in \mathbb{R}^2_+ \) we have

\[ E[X(t) - X(s_1, t_2)]^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2} E[X^2(\mathbf{1})], \quad (5) \]

\[ E[X(s_1, t_2) - X(s)]^2 = |t_2 - s_2|^{2H_2} s_1^{2H_1} E[X^2(\mathbf{1})]. \quad (6) \]

**Proof.** Without loss of generality suppose that \( s_1 \leq t_1 \). It follows from self-similarity that for any \( s \in \mathbb{R}_+ : X(s, 0) = X(0, s) = 0 \) a.s. Then the left-hand side of (5) equals

\[ E \left( X(t) - X(t_1, 0) - X(s_1, t_2) + X(s_1, 0) \right)^2 = E \left( \Delta_{s_1, 0} X(t) \right)^2. \]

Stationarity of the increments implies that

\[ E \left( \Delta_{s_1, 0} X(t) \right)^2 = E \left( \Delta_0 X(t_1 - s_1, t_2) \right)^2 = E \left( X(t_1 - s_1, t_2) \right)^2. \]

Further, self-similarity implies that

\[ E \left( X(t) - X(s_1, t_2) \right)^2 = E \left( X(t_1 - s_1, t_2) \right)^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2} E[X^2(\mathbf{1})]. \]

The proof of the equality (5) can be done in a similar way.

**Lemma 2.4.** Assume that \( E X^2(\mathbf{1}) = 1 \). For all \( s = (s_1, s_2) \in \mathbb{R}^2_+, \ t = (t_1, t_2) \in \mathbb{R}^2_+ \) we have

\[ \left( E \left[ X(t) - X(s) \right]^2 \right)^{1/2} \leq |t_1 - s_1|^{H_1} t_2^{H_2} + |t_2 - s_2|^{H_2} s_1^{H_1}. \quad (7) \]
Proof. Using the Minkowski inequality, we get
\[
\left( \mathbb{E} \left[ X(t) - X(s) \right]^2 \right)^{1/2} = \left( \mathbb{E} \left[ X(t) - X(s_1, t_2) + X(s_1, t_2) + X(s) \right]^2 \right)^{1/2}
\]
\[
\leq \left( \mathbb{E} \left[ X(t) - X(s_1, t_2) \right]^2 \right)^{1/2} + \left( \mathbb{E} \left[ X(s_1, t_2) - X(s) \right]^2 \right)^{1/2}.
\]
It follows from Lemma 2.3 that
\[
\mathbb{E} \left[ X(t) - X(s_1, t_2) \right]^2 = |t_1 - s_1|^{2H_1} t_2^{2H_2},
\]
and
\[
\mathbb{E} \left[ X(s_1, t_2) - X(s) \right]^2 = |t_2 - s_2|^{2H_2} s_1^{2H_2}.
\]
Hence, inequality (7) holds.

3 Random fields on space \((T, \rho_1)\)

In this section we put
\[
\rho(t, s) = \rho_1(t, s) = \max_{i=1,2} |t_i - s_i|, t, s \in T \subset \mathbb{R}_+^2.
\]

Corollary 3.1. Let \(\sigma(h) = Ch^\alpha, 0 < \alpha \leq 1, C > 0\) and \(T = [0, T]^2\) in Theorem 2.1. Then
\[
I_{[0,T]^2}(\varepsilon) \leq 8 \exp \left\{ -\frac{\varepsilon^2 (1 - p)}{2 \left( \gamma^2 + \frac{C^2 T^{2\alpha} \mu}{2^{2\alpha}(1-p)} \right)} \right\} \left( \frac{\varepsilon}{p} \right)^{2/\alpha}
\]
for all \(0 < p < 1\) and \(\varepsilon > 0\).

Proof. We have
\[
\beta = C \left( \frac{T}{2} \right)^{\alpha}, \quad N(u) \leq \left( \frac{TC^{1/\alpha}}{2^{1/\alpha}} + 1 \right)^2.
\]
Put \(r(v) = v^\mu, v \in \mathbb{R}_+, 0 < \mu < \alpha/2\). It follows from Corollary 2.2 that
\[
I_{[0,T]^2}(\varepsilon) \leq 2 \exp \left\{ -\frac{\varepsilon^2}{2 \left( \gamma^2 + \frac{C^2 T^{2\alpha} \mu}{2^{2\alpha}(1-p)} \right)} \right\} Z(p),
\]
where
\[
Z(p) = \left( \frac{1}{\beta p} \int_0^{\beta p} \left( N\left( (\sigma^{-1})(u) \right) \right)^\mu du \right)^{1/\mu}.
\]
Since \(u \leq \beta p\), we have
\[
\frac{TC^{1/\alpha}}{2^{1/\alpha}} \geq \frac{TC^{1/\alpha}}{2(\beta p)^{1/\alpha}} > \frac{2TC^{1/\alpha}}{2TC^{1/\alpha}} = 1.
\]
Therefore, we obtain
\[
Z(p) \leq \left( \frac{1}{\beta p} \int_0^{\beta p} \left( \frac{TC^{1/\alpha}}{2^{1/\alpha}} + 1 \right)^{2\mu} du \right)^{1/\mu} \leq \left( \frac{1}{\beta p} \int_0^{\beta p} \left( \frac{TC^{1/\alpha}}{u^{1/\alpha}} \right)^{2\mu} du \right)^{1/\mu} = \]
As \( \mu \to 0 \), we have
\[
Z(p) \leq T^2 C^{2/\alpha} \frac{1}{(\beta p)^{2/\alpha}} e^{2/\alpha} = 4 \left( \frac{\mu}{p} \right)^{2/\alpha}.
\]
The last inequality completes the proof. \( \square \)

From now on we denote \( H = \min\{H_1, H_2\} \), where \( \mathbf{H} = (H_1, H_2) \in (0, 1)^2 \) is the index of self-similarity.

**Proposition 3.2.** Let \( T = [0, 1]^2, \rho = \rho_1 \), and \( X = \{X(t), t \in \mathbb{R}^2_+\} \) be a centered Gaussian self-similar random field of order \( \mathbf{H} = (H_1, H_2) \in (0, 1)^2 \) with stationary rectangular increments. Then for all \( 0 < p < 1 \) we have
\[
P \left\{ \sup_{t \in [0,1]^2} |X(t)| > \varepsilon \right\} \leq 8 \exp \left\{ - \frac{\varepsilon^2 (1 - p)}{2 \left( 1 + \frac{4p}{2\gamma(H_1p)_p} \right)} \right\} \left( \frac{\varepsilon}{p} \right)^{2/H}, \quad \varepsilon > 0. \tag{10}
\]

**Proof.** We have from inequality \( \mathbb{7} \) that for all \( t, s \in [0, 1]^2 \)
\[
\left( E |X(t) - X(s)|^2 \right)^{1/2} \leq |t_1 - s_1|^{H_1} |t_2|^{H_2} + |t_2 - s_2|^{H_2} |s_1|^{H_1}
\leq |t_1 - s_1|^{H_1} + |t_2 - s_2|^{H_2} \leq 2 \max_{i=1,2} |t_i - s_i|^{H_i} \leq 2 \max_{i=1,2} |t_i - s_i|^{H} = 2[\rho(s, t)]^H.
\]
Therefore, it follows from \( \mathbb{11} \) that \( \sigma(h) = 2h^H \) and \( \gamma = 1 \), where \( \gamma \) is defined in \( \mathbb{2} \). Thus, inequality \( \mathbb{10} \) follows from \( \mathbb{5} \), where \( C = 2, T = 1, \alpha = H. \) \( \square \)

Denote \( S_{T_1T_2} = [0, T_1] \times [0, T_2] \subset \mathbb{R}_+^2, \ T_1 > 0, T_2 > 0. \) The self-similarity of random field gives a correspondence between the probability distribution of extremes that defined in \( [0, 1]^2 \) and in \( S_{T_1T_2}. \)

**Corollary 3.3.** Under the conditions of Proposition 3.2, we have
\[
P \left\{ \sup_{t \in S_{T_1T_2}} \frac{|X(t)|}{T_1^{H_1}T_2^{H_2}} > \varepsilon \right\} = P \left\{ \sup_{t \in [0,1]^2} |X(t)| > \varepsilon \right\}
\leq 8 \exp \left\{ - \frac{\varepsilon^2 (1 - p)}{2 \left( 1 + \frac{4p}{2\gamma(H_1p)_p} \right)} \right\} \left( \frac{\varepsilon}{p} \right)^{2/H}, \tag{11}
\]
where \( \varepsilon > 0, p \in (0, 1). \)

**Proof.** It follows from self-similarity that \( \{T_1^{-H_1}T_2^{-H_2}X(T_1t_1, T_2t_2), t \in \mathbb{R}_+\} \) and \( \{X(t), t \in \mathbb{R}_+\} \) have the same finite dimensional distributions. Therefore,
\[
\sup_{t \in S_{T_1T_2}} \frac{|X(t)|}{T_1^{H_1}T_2^{H_2}} \overset{d}{=} \sup_{t \in [0,1]^2} |X(t)|.
\]
Hence, inequality \( \mathbb{11} \) follows from Proposition 3.2. \( \square \)
Corollary 3.4. Let $\varepsilon > 2$. Under the conditions of Proposition 3.2 we have

$$
P \left\{ \sup_{t \in [0,1]^2} |X(t)| > \varepsilon \right\} \leq 8e^{\frac{3\varepsilon^2}{2(4^1-H + 3)}} e^{\frac{\varepsilon^2}{2(1 + 4^1-H(\varepsilon^2 - 1))}} e^{\frac{3}{2}\varepsilon^4/H}. \tag{12}$$

Proof. Put $p = 1/\varepsilon^2$ in (10). Then

$$
P \left\{ \sup_{t \in [0,1]^2} |X(t)| > \varepsilon \right\} \leq 8 \exp \left\{ -\frac{\varepsilon^2 - 1}{2(1 + 4^1-H(\varepsilon^2 - 1))} \right\} e^{\frac{3}{2}\varepsilon^4/H} \leq 8e^{\frac{3\varepsilon^2}{2(4^1-H + 3)}} e^{\frac{\varepsilon^2}{2(1 + 4^1-H(\varepsilon^2 - 1))}} e^{\frac{3}{2}\varepsilon^4/H}. \tag{13}$$

The corollary is proved. \(\Box\)

We obtained the upper bound for the probability of exceeding of a self-similar Gaussian random field above the level $\varepsilon > 2$ that defined in $[0,1]^2$.

For normalized fields we now prove the upper bound for such probabilities defined in $\mathbb{R}^2_+$. Denote $x \lor y = \max\{x,y\}$.

Theorem 3.5. Let $X = \{X(t), t \in \mathbb{R}^2_+\}$ be a centered Gaussian self-similar random field with index $\mathbf{H} = (H_1, H_2) \in (0,1)^2$ and stationary rectangular increments. Let a function $c : (0, +\infty) \to (0, +\infty)$ and a sequence $\{b_n, n \in \mathbb{N}\}$ satisfy the following conditions

(i) $c$ is increasing on $[1, +\infty)$, $c(t) \to \infty$, $t \to \infty$, and $c\left(\frac{1}{t}\right) = c(t), t \geq 1$;

(ii) $b_0 = 1$, $b_n < b_{n+1}, n \in \mathbb{N}$, $b_n \to \infty$, $n \to \infty$, and $M := \inf_{k \in \mathbb{N}} \left(\frac{b_k}{b_{k+1}}\right)^{H_1+H_2} c(b_k) > 0$;

(iii) for all $D > 0$ the following series converges

$$
\sum_{k=1}^{\infty} \exp \left\{ -D \left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right) \right\} < +\infty.
$$

Then for all $\varepsilon > 2/M$ we have

$$
P \left\{ \sup_{t \in \mathbb{R}^2_+} \frac{|X(t)|}{(t_1 \lor t_2)^{H_1+H_2} c(t_1 \lor t_2)} > \varepsilon \right\} \leq 16e^{\frac{3\varepsilon^2}{2(4^1-H + 3)}} \sum_{k=0}^{\infty} \exp \left\{ -\frac{3\varepsilon^2}{2(4^1-H + 3)} \left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^2 \right\} \left(\frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k)\right)^{\frac{4}{H}} =: \tilde{Z}(\varepsilon). \tag{13}$$

In this case with probability 1 for all $t \in (0, +\infty)^2$ the inequality holds:

$$
|X(t)| < \xi(t_1 \lor t_2)^{H_1+H_2} c(t_1 \lor t_2),
$$

where $\xi$ is a random variable such that for all $\varepsilon > 2/M$ : $P\{\xi > \varepsilon\} \leq \tilde{Z}(\varepsilon)$. 

7
Let us remark that

\[ T \sim X \]

Firstly, consider \( E \). Evidently, we get

\[ \bar{P}(T, \varepsilon) = \mathbb{P} \left\{ \sup_{t \in (b_t, t_2)} |X(t)| > \varepsilon \right\} \]

Evidently, we get

\[ \bar{P}(\mathbb{R}_+, \varepsilon) = \bar{P}(\mathbb{R}_+^2 \setminus [0, 1]^2, \varepsilon) + \bar{P}([0, 1]^2, \varepsilon) \]

Firstly, consider \( \bar{P}(\mathbb{R}_+^2 \setminus [0, 1]^2, \varepsilon) \). Note that, if \( t \in B_k, k \geq 0 \) then \( b_k \leq t_1 \vee t_2 \leq b_{k+1} \) and \( c(b_k) \leq c(t_1 \vee t_2) \leq c(b_{k+1}), k \geq 1 \). Therefore, we get

\[ \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} \left( \frac{|X(t)|}{b_k + 1} \right) > \varepsilon \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} \left( \frac{|X(t)|}{b_k + 1} \right) > \varepsilon \right\} \]

From corollaries 3.3 and 3.4 we obtain that for \( \varepsilon > 2/M \)

\[ \bar{P}(\mathbb{R}_+ \setminus [0, 1]^2, \varepsilon) \leq \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} |X(t)| > \varepsilon \right\} \]

\[ \leq 8 \varepsilon + \varepsilon |X(t)| \exp \left\{ -\frac{3\varepsilon^2}{2(4^{1-H} + 3) \left( b_k^{1+H} c(b_k) \right)^2} \left( \frac{b_k^{1+H} c(b_k)}{b_{k+1}^{1+H}} \right)^H \right\} \]

Consider \( \bar{P}([0, 1]^2, \varepsilon) \). Note that, if \( t \in B_{-k}, k \geq 1 \) then \( \frac{1}{b_k + 1} \leq t_1 \vee t_2 \leq \frac{1}{b_k} \) and \( c(b_k) \leq c(t_1 \vee t_2) = c(\frac{1}{b_k + 1}) \leq c(b_{k+1}), k \geq 1 \). Therefore, we have

\[ \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} \left( \frac{|X(t)|}{b_k + 1} \right) > \varepsilon \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} \left( \frac{|X(t)|}{b_k + 1} \right) > \varepsilon \right\} \]

From corollaries 3.3 and 3.4 we obtain that for \( \varepsilon > 2/M \)

\[ \bar{P}([0, 1]^2, \varepsilon) \leq \mathbb{P} \left\{ \sup_{t \in [0, 1]^2} |X(t)| > \varepsilon \right\} \]
\[ \leq 8e^{\frac{\pi}{2}+\frac{1}{2}e^{A/H}} \sum_{k=1}^{\infty} \exp \left\{ -\frac{3\varepsilon^2}{2(4^{-1}+3)} \left( \frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k) \right)^2 \right\} \left( \frac{b_k}{b_{k+1}} \right)^{\frac{4H_1+H_2}{H}} (c(b_k))^{4/H}. \]

The theorem is proved. \( \square \)

The following corollary is an immediate consequence of Theorem 3.5.

**Corollary 3.6.** Let \( M = \inf_{k \in \{0\} \cup \mathbb{N}} \left( \frac{b_k}{b_{k+1}} \right)^{H_1+H_2} c(b_k) > 0. \) Denote

\[ u = \frac{3}{(4^{-1}+3)} \frac{M^2}{4} \quad \text{and} \quad v_k = \frac{2}{M^2} \left( \frac{b_k^{H_1+H_2}}{b_{k+1}^{H_1+H_2}} c(b_k) \right)^2, k \geq 0. \]

If for any \( H \in (0,1)^2 \) the series \( \sum_{k=0}^{\infty} \frac{v_k^{2/H}}{e^{v_k}} \) converges, then for any \( \varepsilon > \frac{2}{M} \sqrt{\frac{2}{3} (4^{-1}+3)} \)

\[ \mathbb{P} \left\{ \sup_{t \in \mathbb{R}^2_+} \frac{|X(t)|}{(t_1 \vee t_2)^{H_1+H_2} c(t_1 \vee t_2)} > \varepsilon \right\} \leq 16\sqrt{2} \left( \frac{\varepsilon}{2} \right)^{2/H} \frac{e^{4/H}}{e^{\ln(1+e)}} \left( \sum_{k=0}^{\infty} \frac{v_k^{2/H}}{e^{v_k}} \right) M^{4/H} e^{-u\varepsilon^2}. \quad (14) \]

**Proof.** It is clear that \( uc^2 > 2 \) and \( v_k > 2 , k \geq 0. \) Recall that for \( u \varepsilon^2 , v_k > 2 \) we have \( u \varepsilon^2 + v_k \leq u \varepsilon^2 v_k. \) It follows from (12) that for \( \varepsilon > \frac{2}{M} \sqrt{\frac{2}{3} (4^{-1}+3)} > \frac{2}{s} \) we have

\[ \tilde{Z}(\varepsilon) = 16e^{\frac{\pi}{2}+\frac{1}{2}e^{A/H}} \sum_{k=0}^{\infty} \frac{s^{4/H}}{2^{2/H} \exp\{u \varepsilon^2 v_k\}} \leq 16\sqrt{2} \left( \frac{\varepsilon}{2} \right)^{2/H} \frac{e^{4/H}}{e^{\ln(1+e)}} \left( \sum_{k=0}^{\infty} \frac{v_k^{2/H}}{e^{v_k}} \right) M^{4/H} e^{-u\varepsilon^2}. \]

The corollary is proved. \( \square \)

Consider an example of applying Corollary 3.6.

**Example 1.** Put \( b_k = e^k , k = 0,1,\ldots, \) and \( c(t) = \sqrt{\ln(1+e)} t \geq 1 \) in Theorem 3.5. Then \( M = \inf_{k \in \{0\} \cup \mathbb{N}} \left( \frac{b_k}{b_{k+1}} \right)^{H_1+H_2} c(b_k) = e^{-(H_1+H_2)} , \) and

\[ u = \frac{3}{(4^{-1}+3)} \frac{s^2}{4} = \frac{3}{4(4^{-1}+3)} e^{-2(H_1+H_2)} , \]

\[ v_k = 2e^{2(\varepsilon+H_2)} \frac{\ln(k+e)}{e^{2(H_1+H_2)}} = 2 \ln(k+e), k \geq 0. \]

Then inequality (14) has the form

\[ \mathbb{P} \left\{ \sup_{t \in \mathbb{R}^2_+} \frac{|X(t)|}{(t_1 \vee t_2)^{H_1+H_2} \sqrt{\ln(1+e)}} > \varepsilon \right\} \leq 16\sqrt{2} e^{2/H} \frac{e^{4/H}}{e^{\ln(1+e)}} \left( \sum_{k=0}^{\infty} \frac{\ln(k+e)}{e^{2(H_1+H_2)}} \right) e^{-2(H_1+H_2)} \exp\{-u\varepsilon^2\} \]

\[ \leq 16\sqrt{2} e^{2/H} \frac{e^{4/H}}{e^{\ln(1+e)}} \left( \sum_{k=0}^{\infty} \frac{\ln(k+e)}{e^{2(H_1+H_2)}} \right) \exp\left\{ -\frac{3\varepsilon^2}{4(4^{-1}+3)} e^{-2(H_1+H_2)} \right\}. \]

Thus, we obtain the upper bound for probability distribution of extremes of normalized self-similar Gaussian random field with stationary rectangular increments that defined in \( \mathbb{R}^2_+. \)
4 Random fields on \((T, \rho_2)\)

Recall the notation of the metric \(\rho_2(t, s) = \sum_{i=1,2} |t_i - s_i|^H, t = (t_1, t_2) \in \mathbb{R}_+^2, s = (s_1, s_2) \in \mathbb{R}_+^2\), where \(H = (H_1, H_2) \in (0, 1)\) is the index of self-similarity of a field \(X\). Now we want to obtain result which is similar to Proposition 3.2 but with metric \(\rho_2\).

Let us remember that \(N(u)\) is the minimal number of closed \(\rho\)-balls with radius \(u\) needed to cover space \((T, \rho)\). First let us prove the estimate for \(N(u)\) in the case \(\rho = \rho_2\) and \(T = S_{T_1T_2}\).

**Lemma 4.1.** Let \(\rho = \rho_2\) and \(T = S_{T_1T_2}\). Then

\[
N(u) \leq 2 \left( \frac{T_1}{4K_1u^\frac{1}{H_1}} + \frac{3}{2} \right) \left( \frac{T_2}{4K_2u^\frac{1}{H_2}} + \frac{3}{2} \right), u > 0,
\]

where

\[
K_1 = \left( \frac{H_2}{H_1 + H_2} \right)^\frac{1}{H_1}, \quad K_2 = \left( \frac{H_1}{H_1 + H_2} \right)^\frac{1}{H_2}.
\]

**Proof.** Consider an auxiliary metric \(\rho_3 = \{\rho_3(x, y) = \frac{|y_1 - x_1|}{a_1} + \frac{|y_2 - x_2|}{a_2}, x = (x_1, x_2) \in \mathbb{R}_+, y = (y_1, y_2) \in \mathbb{R}_2\}, \) with \(a_1 > 0, a_2 > 0\). A closed \(\rho_3\)-ball with radius 1 in space \((T, \rho_3)\) is a set \(V_{\rho_3}(1) = \{ x = (x_1, x_2) \in \mathbb{R}_+, \frac{|x_1|}{a_1} + \frac{|x_2|}{a_2} \leq 1 \}\). The minimum number of \(V_{\rho_3}(1)\) needed to cover space \((T, \rho_3)\) is less then \(2 \left( \frac{T_1 + a_1}{2a_1} + 1 \right) \left( \frac{T_2 + a_2}{2a_2} + 1 \right)\).

Put

\[
a_1 = 2 \left( \frac{H_2}{H_1 + H_2} \right)^\frac{1}{H_1} \varepsilon^\frac{1}{H_1} = 2K_1 \varepsilon^\frac{1}{H_1},
\]

\[
a_2 = 2 \left( \frac{H_1}{H_1 + H_2} \right)^\frac{1}{H_2} \varepsilon^\frac{1}{H_2} = 2K_2 \varepsilon^\frac{1}{H_2}.
\]

It is not hard to prove that \(V_{\rho_3}(1) \subset V_{\rho_2}(\varepsilon)\). Hence,

\[
N_{\rho_2}(\varepsilon) \leq N_{\rho_3}(1) \leq 2 \left( \frac{T_1}{4K_1 \varepsilon^\frac{1}{H_1}} + \frac{3}{2} \right) \left( \frac{T_2}{4K_2 \varepsilon^\frac{1}{H_2}} + \frac{3}{2} \right).
\]

To prove the next statement, we need some notation. Denote

\[
T_H = \max\{T_1^{H_1}, T_2^{H_2}\}, \quad H = \min\{H_1, H_2\}, \quad Q = \frac{1}{H_1} + \frac{1}{H_2},
\]

\[
N_1 = \left( \frac{H_1 + H_2}{H_2} \right)^\frac{1}{H_1} + 3, \quad N_2 = \left( \frac{H_1 + H_2}{H_1} \right)^\frac{1}{H_2} + 3.
\]

**Proposition 4.2.** Let \((T, \rho) = (S_{T_1T_2}, \rho_2)\), \(T_1 \geq 1, T_2 \geq 1\) and \(X = \{X(t), t \in \mathbb{R}_+^2\}\) be a centered self-similar Gaussian random field with stationary rectangular increments. Under the conditions of Theorem 3.1 for all \(0 < p < 1\) we have

\[
I_T(\varepsilon) = P \left\{ \sup_{t \in T} |X(t)| > \varepsilon \right\} \leq N_1 N_2 \left( \frac{\varepsilon}{p} \right)^Q \exp \left\{ -\frac{\varepsilon^2 (1 - p)}{2 \left( T_1^{2H_1} T_2^{2H_2} + \frac{p}{1-p} T_1^{-H_1 T_2^H} \right)} \right\}, \varepsilon > 0. \tag{15}
\]
Proof. Recall that \( \rho_2(s, t) = |s_1-t_1|^{H_1} + |s_2-t_2|^{H_2}, s = (s_1, s_2), t = (t_1, t_2), s, t \in T. \) From Lemma 2.4 we get

\[
\sup_{\rho(s, t) \leq h} \left( \mathbb{E}(X(t) - X(s))^2 \right)^{1/2} \leq \sup_{\rho(s, t) \leq h} \left( t_2 H_2 |s_1-t_1|^{H_1} + t_1 H_1 |s_2-t_2|^{H_2} \right) \leq T_\eta h.
\]

Thus, we can put \( \sigma(h) = T_\eta h \) in Theorem 2.1. From (2) we have

\[
\beta = \sigma \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right) = T_\eta \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right).
\]

It is clear that

\[
\gamma^2 = \sup_{t \in T} \mathbb{E}X^2(t) = T_1^{2H_1} T_2^{2H_2} \mathbb{E}X^2(1) = T_1^{2H_1} T_2^{2H_2}.
\]

From Lemma 4.1 we have

\[
N(u) \leq 2 \left( \frac{T_1}{4K_1 u^{\eta_1}} + \frac{3}{2} \right) \left( \frac{T_2}{4K_2 u^{\eta_2}} + \frac{3}{2} \right),
\]

and therefore

\[
N(\sigma^{-1}(u)) \leq 2 \left( \frac{T_1 T_\eta^{\eta_1}}{4K_1 u^{\eta_1}} + \frac{3}{2} \right) \left( \frac{T_2 T_\eta^{\eta_2}}{4K_2 u^{\eta_2}} + \frac{3}{2} \right).
\]

It follows from \( \beta > \beta_p \geq u \) that

\[
1 < \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{\eta_1} \frac{T_\eta^{\eta_1}}{u^{\eta_1}}, \quad i = 1, 2.
\]

Recall that \( 0 < H_i < 1 \) and

\[
\frac{T_i}{2} = \left( \left( \frac{T_i}{2} \right)^{H_i} \right)^{\frac{1}{H_i}} \leq \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{\frac{1}{H_i}} = \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{\frac{1}{H_i}}, \quad i = 1, 2.
\]

Then

\[
\left( \frac{T_i T_\eta^{\eta_i}}{4K_i u^{\eta_i}} + \frac{3}{2} \right) \leq \left( \frac{T_i T_\eta^{\eta_i}}{4K_i u^{\eta_i}} + \frac{3}{2} \right) \frac{T_\eta^{\eta_i}}{u^{\eta_i}} + \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \frac{T_\eta^{\eta_i}}{u^{\eta_i}} \frac{3(\eta_\eta^{\eta_i})}{2u^{\eta_i}}
\]

\[
\leq \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{\frac{1}{H_i}} \left( \frac{T_\eta^{\eta_i}}{u^{\eta_i}} \right) \left( \frac{1}{2K_i} + \frac{3}{2} \right).
\]

Therefore, we have the following inequality for \( Z(p) \), where \( Z(p) \) is defined in (3). For each \( 0 < \mu < 1/Q \) we obtain

\[
Z(p) \leq \left( \frac{1}{\beta_p} \int_0^{\beta_p} \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{Q \left( \frac{T_\eta^{\eta_i}}{u^{\eta_i}} \right) \frac{N_1 N_2}{2}} \right)^{\mu} du \right)^{1/\mu}
\]

\[
= 2N_1 N_2 \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^{Q \left( \frac{T_\eta^{\eta_i}}{u^{\eta_i}} \right)} \left( \int_0^{\beta_p} \frac{1}{u^{Q\mu}} \right)^{1/\mu}
\]

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\[
\begin{align*}
= \frac{N_1N_2}{2} \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right) Q \frac{T_\eta^Q}{(\beta p)^Q} \left( \frac{1}{1 - Q \mu} \right)^{1/\mu}.
\end{align*}
\]

As \( \mu \to 0 \), we have
\[
Z(p) \leq \frac{N_1N_2}{2} \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right) Q \frac{T_\eta^Q}{(\beta p)^Q} \left( \frac{1}{1 - Q \mu} \right)^{1/\mu}.
\]

Finally, from (14) we obtain
\[
I_T(\varepsilon) \leq N_1N_2 \left( \frac{\varepsilon}{p} \right)^Q \exp \left\{ -\frac{\varepsilon^2(1 - p)}{2 \left( T_1^{2H_1}T_2^{2H_2} + \frac{p}{1 - p}T_\eta^2 \left( \left( \frac{T_1}{2} \right)^{H_1} + \left( \frac{T_2}{2} \right)^{H_2} \right)^2 \right)} \right\}
\]
\[
\leq N_1N_2 \left( \frac{\varepsilon}{p} \right)^Q \exp \left\{ -\frac{\varepsilon^2(1 - p)}{2 \left( T_1^{2H_1}T_2^{2H_2} + \frac{p}{1 - p}4^{1-H}T_\eta^4 \right)} \right\}.
\]

\[\square\]

**Corollary 4.3.** Under the conditions of Proposition 4.2 we have
\[
P \left\{ \sup_{t \in \mathbb{T}} |X(t)| > \varepsilon \right\} \leq N_1N_2 \varepsilon^{2Q} \exp \left\{ Q + \frac{3}{2T_1^{2H_1}T_2^{2H_2}(3+4^{1-H})} \right\} \times \exp \left\{ -\frac{3\varepsilon^2}{2(3+4^{1-H})} \right\}, \quad \varepsilon > 2.
\]

**Proof.** The corollary follows from (15) if we put \( p = 1/\varepsilon^2 \). \[\square\]

**Corollary 4.4.** Let \((\mathbb{T}, \rho) = ([0,1]^2, \rho_2)\). Under the conditions of Proposition 4.2 we have
\[
P \left\{ \sup_{t \in [0,1]^2} |X(t)| > \varepsilon \right\} \leq N_1N_2 \varepsilon^{2Q} \exp \left\{ Q + \frac{3}{2(3+4^{1-H})} \right\} \exp \left\{ -\frac{3\varepsilon^2}{2(3+4^{1-H})} \right\}, \quad \varepsilon > 2.
\]

**Proof.** In this case \( T_1 = T_2 = 1 \), so the corollary follows from (16). \[\square\]

We want to find an upper bound for probability distribution of extremes defined in \([1, +\infty)^2\). For this goal we obtain probabilities defined in \([1,2]^2\).

**Proposition 4.5.** Let \( \mathbb{T} = [1,2]^2 \), \( p = \rho_2 \) and \( X = \{X(t), t \in \mathbb{R}_+^2\} \) be a centered self-similar Gaussian random field with stationary rectangular increments. Under the conditions of Theorem 2.1 for all \( 0 < p < 1 \) we have
\[
I_{[1,2]^2}(\varepsilon) = P \left\{ \sup_{t \in [1,2]^2} |X(t)| > \varepsilon \right\} \leq N_1N_2 \left( \frac{\varepsilon}{p} \right)^Q \exp \left\{ -\frac{\varepsilon^2(1 - p)}{2(4^{H_1+H_2} + \left( 1 + 2|H_1-H_2| \right)^2)} \right\}.
\]

(17)
**Proof.** We prove the proposition in the same way as Proposition 4.2. Denote \( \eta = \max\{H_1, H_2\} \) and \( H = \min\{H_1, H_2\} \). It is clear that \( \sigma(h) = 2^\eta h \) and

\[
\beta = \sigma \left( \left( \frac{1}{2} \right)^{H_1} + \left( \frac{1}{2} \right)^{H_2} \right) = 2^\eta \left( 2^{-H_1} + 2^{-H_2} \right) = 1 + 2^{H_1 - H_2},
\]

\[
\gamma^2 = 4^{H_1 + H_2}.
\]

From Lemma 4.1 we have

\[
N(\sigma^{-1}(u)) \leq 2 \left( \frac{2^\eta H_1}{4K_1 u^{1/H_1}} + \frac{3}{2} \right) \left( \frac{2^\eta H_2}{4K_2 u^{1/H_2}} + \frac{3}{2} \right).
\]

It follows from \( \beta > \beta_p \geq u > 0 \) that

\[
1 \leq \frac{\beta^{1/H_i}}{u^{1/H_i}} = \frac{(1 + 2^{H_1 - H_2})^{1/H_i}}{2u^{1/H_i}}.
\]

Then for \( i = 1, 2 \)

\[
\left( \frac{2^\eta H_i}{4K_i u^{1/H_i}} + \frac{3}{2} \right) \leq \frac{2^\eta H_i}{4K_i u^{1/H_i}} + \frac{3}{2} \frac{(1 + 2^{H_1 - H_2})^{1/H_i}}{2u^{1/H_i}} \leq \frac{(1 + 2^{H_1 - H_2})^{1/H_i}}{u^{1/H_i}} \left( \frac{1}{2K_i} + \frac{3}{2} \right).
\]

Further, from definition (9) of \( Z(p) \) we get the following inequality.

\[
Z(p) \leq \left( \frac{1}{\beta_p} \int_0^{\beta_p} \left( \frac{1 + 2^{H_1 - H_2}}{Q} \right)^{\frac{N_1 N_2}{2u^Q}} \mu \right)^{1/\mu}
\]

\[
= \frac{N_1 N_2}{2} \left( 1 + 2^{H_1 - H_2} \right)^{\frac{1}{Q \beta_p^{1/\mu}}} \left( \int_0^{\beta_p} \frac{1}{u^Q} \right)^{1/\mu} = \frac{N_1 N_2}{2p^Q} \left( \frac{1}{1 - Q \mu} \right)^{1/\mu}.
\]

As \( \mu \to 0 \), we have

\[
Z(p) \leq \frac{N_1 N_2}{2} \left( \frac{e}{p} \right)^Q.
\]

Thus, we obtain

\[
I_{[1,2]}(p) \leq N_1 N_2 \left( \frac{e}{p} \right)^Q \exp \left\{ -\frac{\epsilon^2 (1 - p)}{2 \left( 4^{H_1 + H_2} + (1 + 2^{H_1 - H_2})^{2} \frac{p}{1-p} \right)} \right\}
\]

As before, denote \( \eta = \max\{H_1, H_2\} \).

**Corollary 4.6.** Under the conditions of Proposition 4.5 for \( \epsilon > 2 \) we have

\[
I_{[1,2]}(p) \leq N_1 N_2 \exp \{ Q + \frac{1}{2 \left( 4^{H_1 + H_2} + 1 \right)} \} \epsilon^2 Q \exp \left\{ -\frac{3\epsilon^2}{2 \cdot 4^\eta \left( 4^{H_3 + 4^{1-H}} \right)} \right\}.
\]

**Proof.** The corollary follows from (17), if we put \( p = 1/\epsilon^2 \).
Theorem 4.7. Let $T = [1, \infty)^2$, $\rho = \rho_2$ and $X = \{X(t), t = (t_1, t_2) \in \mathbb{R}_+^2\}$ be a centered self-similar Gaussian random field with stationary rectangular increments. Let $\varphi : (0, +\infty)^2 \to (0, +\infty)$ be an increasing function in each coordinate. Suppose that for any $D > 0$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \exp \{-D\varphi(2^n, 2^m)\} < +\infty. \quad (19)$$

Denote

$$C_1 = N_1N_2 \exp\{Q + \frac{1}{2(4H_1H_2 + 1)}\} \text{ and } C_2 = \frac{3}{2 \cdot 4^H (4^H 3 + 4^{1-H})}.$$

If $\varepsilon > \frac{2}{\varphi(1)},$ then

$$Y(\varepsilon) := P \left\{ \sup_{t \in [1, +\infty)^2} \frac{|X(t)|}{t_1^{\frac{1}{2}H_1} t_2^{\frac{1}{2}H_2} \varphi(t)} > \varepsilon \right\} \leq C_1 \varepsilon^{2Q} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi^{2Q}(2^n, 2^m) \exp \{C_2 \varepsilon^2 \varphi^2(2^n, 2^m)\}. \quad (20)$$

Proof. At first, we have the following obvious inequality

$$P \left\{ \sup_{t \in [1, +\infty)^2} \frac{|X(t)|}{t_1^{\frac{1}{2}H_1} t_2^{\frac{1}{2}H_2} \varphi(t)} > \varepsilon \right\} \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P \left\{ \sup_{t_1 \in [2^{n-1}, 2^n]} t_1^{\frac{1}{2}H_1} t_2^{\frac{1}{2}H_2} \varphi(t) > \varepsilon \right\}.$$ 

Then from monotonicity of $\varphi$ we get for all $n, m > 1$:

$$P \left\{ \sup_{t_1 \in [2^{n-1}, 2^n], t_2 \in [2^{m-1}, 2^m]} \frac{|X(t)|}{t_1^{\frac{1}{2}H_1} t_2^{\frac{1}{2}H_2} \varphi(t)} > \varepsilon \right\} \leq P \left\{ \sup_{t_1 \in [2^{n-1}, 2^n]} t_1^{\frac{1}{2}H_1} t_2^{\frac{1}{2}H_2} |X(t)| > \varepsilon \varphi(2^{n-1}, 2^m) \right\}.$$

By self-similarity, we obtain the following equality for all $n, m \geq 1$:

$$P \left\{ \sup_{t_1 \in [2^{n-1}, 2^n], t_2 \in [2^{m-1}, 2^m]} \frac{2^{(1-n)H_1}2^{(1-m)H_2}}{\varphi(2^{n-1}, 2^{m-1})} |X(t)| > \varepsilon \right\} = P \left\{ \sup_{t \in [1, 2]^2} \frac{|X(t)|}{\varphi(2^{n-1}, 2^{m-1})} > \varepsilon \right\}.$$

Thus,

$$Y(\varepsilon) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P \left\{ \sup_{t \in [1, 2]^2} \frac{|X(t)|}{\varphi(2^{n-1}, 2^{m-1})} > \varepsilon \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} P \left\{ \sup_{t \in [1, 2]^2} |X(t)| > \varepsilon \varphi(2^{n-1}, 2^{m-1}) \right\}.$$

It follows from Corollary 4.6 that for $\varepsilon > \frac{2}{\varphi(1)}$ we have

$$Y(\varepsilon) \leq C_1 \varepsilon^{2Q} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi^{2Q}(2^n, 2^m) \exp \{-C_2 \varepsilon^2 \varphi^2(2^n, 2^m)\}. \quad (19)$$

This completes the proof.
Corollary 4.8. If for any $H \in (0,1)^2$ the series
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{2Q}(2^n,2^m)}{\exp \left\{ 2\frac{\varphi^{2}(2^n,2^m)}{\varphi^{2}(1)} \right\}} < +\infty, \]
then for $\varepsilon > \frac{2}{\varphi(1)} \sqrt{\frac{2}{24}} (4H3 + 41^{-H})$,
\[ Y(\varepsilon) \leq C_1 \varepsilon^{2Q} \exp \left\{ -\frac{\varepsilon^2 \varphi^2(1)}{C_2} \right\} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{2Q}(2^n,2^m)}{\exp \left\{ 2\frac{\varphi^{2}(2^n,2^m)}{\varphi^{2}(1)} \right\}} . \tag{21} \]

Proof. Denote
\[ u = \frac{3}{4 \cdot 4^n (4^nH3 + 41^{-H})} \varphi^2(1) \quad \text{and} \quad v_{n,m} = 2\frac{\varphi^{2}(2^n,2^m)}{\varphi^{2}(1)}, \quad n,m \geq 0. \]

It can easily be checked that $u \varepsilon^2 > 2$ and $v_{n,m} > 2, n,m \geq 0$. Recall that for $u \varepsilon^2, v_{n,m} > 2$ we have $u \varepsilon^2 + v_{n,m} \leq u \varepsilon^2 v_{n,m}$. It follows from (20) that for $\varepsilon > \frac{2}{\varphi(1)} \sqrt{\frac{2}{24}} (4H3 + 41^{-H}) > \frac{2}{\varphi(1)}$ we have
\[ Y(\varepsilon) \leq C_1 \varepsilon^{2Q} \exp \left\{ -\frac{3\varepsilon^2}{4 \cdot 4^n (4^nH3 + 41^{-H})} \varphi^2(1) \right\} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{2Q}(2^n,2^m)}{2\frac{\varphi^{2}(2^n,2^m)}{\varphi^{2}(1)}} . \]

The corollary is proved. \hfill \square

We present the example of applying Corollary 4.8.

Example 2. Let $\varphi_1, \varphi_2$ be the positive functions of $R^2_+$ to $R$ such that
\[ \varphi_1(x) = \sqrt{(2 + \delta) \ln \left( \ln(x_1x_2) + e \right)}, x = (x_1, x_2) \in R^2_+ \]
and
\[ \varphi_2(x) = \sqrt{(2 + \delta) \ln(e + \ln_2(x_1)) + \ln(e + \ln_2(x_1))}, x = (x_1, x_2) \in R^2_+. \]

Then
\[ \varphi_1(2^n,2^m) = \sqrt{\ln(n+m+e)}, n,m \in \{0\} \cup N, \]
\[ \varphi_2(2^n,2^m) = \sqrt{\ln(n+e) + \ln(m+e)}, n,m \in \{0\} \cup N, \]
and $\varphi_1(1) = \varphi_2(1) = 1$.

Therefore, from (21) we get
\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi_1^{2Q}(2^n,2^m)}{2\frac{\varphi_1^2(2^n,2^m)}{\varphi_1^2(1)}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi_1^{2Q}(2^n,2^m)}{\exp \left\{ 2\frac{\varphi_1^2(2^n,2^m)}{\varphi_1^2(1)} \right\}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(n+m+e)}{(n+m+e)^{2Q}}. \]

Hence, from Corollary 4.8 we have
\[ P \left\{ \sup_{t \in [1,\infty]^2} \frac{|X(t)|}{H_1 t_1^2 H_2 t_2^2 \varphi_1(t)} > \varepsilon \right\} \leq C_1 \varepsilon^{2Q} \exp \left\{ -\frac{C_2}{2} \frac{\varepsilon^2}{\varphi_1^2(1)} \right\} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\ln(n+m+e)}{(n+m+e)^{2Q}}. \]

Thus, we obtain probability distributions for extremes of a normalized self-similar Gaussian random field with stationary rectangular increments that defined in $[1,\infty)^2$. 

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