Nonlinear Boundary Value Problem of Magnetic Insulation

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Abstract. On the basis of generalization of upper and lower solution method to the singular two point boundary value problems, the existence theorem of solutions for the system, which models a process of magnetic insulation in plasma is proved.

Keywords: upper and lower solution, singular nonlinear system.

1. Introduction.

Investigation of mathematic models of magnetic insulation has been started by P. Degond, N. Ben Abdallah and F. Mehats in 1995 year. In 1996 P. Degond has put to the author of this paper the problem on existence of solutions of limit system (I) and its generalization to the problem with free boundary. The effect of magnetic insulation consists in that the electrons emitted from cathode cannot reach the anode due to the extremely high applied electric and magnetic field; they are reflected by the magnetic forces back to the cathode. Thus there is electronic layer outside of which electromagnetic field is equal to zero [1]. Here two basic regimes are possible: the first, when electrons reach the anode — ”noninsulated” diode and the second one, when electrons rotate back to the cathode — ”insulated” diode. The regime of ”noninsulated” diode is described by the following nonlinear two-point boundary value problem

\[
\begin{align*}
\frac{d^2 \varphi}{dx^2} &= j_x \frac{1 + \varphi(x)}{\sqrt{(1 + \varphi(x))^2 - 1 - a(x)^2}} \triangleq F(\varphi, a); \quad \varphi(0) = 0, \; \varphi(1) = \varphi_L, \\
\frac{d^2 a}{dx^2} &= j_x \frac{a(x)}{\sqrt{(1 + \varphi(x))^2 - 1 - a(x)^2}} \triangleq G(\varphi, a); \quad a(0) = 0, \; a(1) = a_L,
\end{align*}
\]

(I)

where \( j_x > 0, \; x \in [0, 1]; \) \( \varphi \) is the potential of electric field and the potential of magnetic field is \( a \).

Our main goal consists in search of positive solutions of system (I) that is \( \varphi > 0, \; a > 0 \) and their dependences upon parameter \( j_x \). Here there are some interesting questions about solvability of this problem, because the system (I) is singular in zero for \( \varphi = 0 \) and in this connection, we can not say about properties of monotonicity of right parts on the interval \( \varphi \in [0, \infty) \) and, hence, about Lipschitz condition. The problem (I) has no a property of quasimonotonicity in cone. Thus a standard upper and lower solution method, developed
for the systems of semilinear elliptic equations in partially ordered Banach space [3], does
not work. In spite of this fact, we show the existence of lower and upper solutions of problem
(I) without conditions of local Lipschitz continuity and quasimonotonicity using sufficiently
simple techniques. To this purpose, we modify the P.J. McKenna and W. Walter [4] theorem
of existence of lower and upper solutions for arbitrary elliptic systems

\[ \triangle u + f(x, u) = 0 \quad \Omega, \quad u = 0 \text{ on } \partial \Omega, \]

where \( u = (u_1, \ldots, u_n) \), \( f = (f_1, \ldots, f_n) \) are \( n \) - vectors, \( \Omega \) – is an open bounded subset of
\( \mathbb{R}^M \) with smooth boundary \( \partial \Omega \), and \( f(x, u) \) is uniformly Hölder continuous (with exponent
\( \alpha \)) in \( x \) and Lipschitz continuous in \( u \).

In section 2 we will prove theorem 2 and propositions 1, 2 on the existence of semitrivial
solutions of problem (I) by upper and lower solution method. The estimations to the value
of electrostatic potential on the anode \( \varphi_L \) and the current \( j_x \) are obtained. In section 3 we
formulate the principal theorem 4 on the existence of positive solutions of problem (I) and
the estimation to the value of magnetic field on the anode \( a_L \) is given.

We note that system (I) was studied in [2] by a shooting method with \( \beta = a'(0) \) and \( j_x \) as shooting parameters. The strategy is: given the values of \( \beta \) and \( j_x \), solve (I) with the
Cauchy conditions \( \varphi(0) = 0, a(0) = 0, \varphi'(0) = 0, a'(0) = \beta, \) and then adjust the values in
order to fulfill the conditions \( \varphi(1) = \varphi_L \) and \( a(1) = a_L \).

2. Existence of semitrivial solutions of problem (I).

Let us introduce the definition of cone in a Banach space \( X \).

**Definition 1.** Let \( X \) be a Banach space. A nonempty convex closed set \( P \subset X \) is called
a cone, if it satisfies the conditions:

(i) \( x \in P, \lambda \geq 0 \) implies \( \lambda x \in P \);

(ii) \( x \in P, -x \in P \) implies \( x = O \), where \( O \) denotes zero element of \( X \).

\( \leq \) is the order in \( X \) induced by \( P \), i.e. \( x \leq y \) if and only if \( y - x \) is an element of \( P \).

We will denote \([x, y]\) the closed order interval between \( x \) and \( y \), i.e.

\[ [x, y] = \{ z \in X : x \leq z \leq y \}. \quad (1) \]

We will also assume that the cone \( P \) is normal in \( X \), i.e. order intervals are norm bounded.

In \( X \)

\[ X \equiv \{(u, v) : u, v \in C^1(\bar{\Omega}), u = v = 0\} \]
we introduce the norm \( |U|_X = |u|_{C^1} + |v|_{C^1} \), and the norm \( |U|_X = |u|_{\infty} + |v|_{\infty} \) in \( C \), where
\( U = (u, v) \). Here \( P \) is given by

\[ P = \{(u, v) \in X : u \geq 0, v \geq 0 \text{ for all } x \in \Omega\}. \quad (2) \]

So, if \( u \neq 0, v \neq 0 \) belong to \( P \), then \(-u, -v\) does not belong. We will work with classical
spaces on the intervals \( I = [a, b] \), \( \hat{I} = ]a, b[ \), \( I = (a, b) \):
\( C(\bar{I}) \) with norm \( \| u \|_\infty = \max \{ |u(x)| : x \in \bar{I} \} \); 
\( C^1(\bar{I}) = \| u \|_\infty + \| u' \|_\infty \);
\( C_{loc}(I) \), which contains all functions that are locally absolutely continuous in \( I \). We introduce a space \( C_{loc}(I) \) because the problem (I) is singular for \( \varphi = 0 \). The order \( \leq \) in cone \( P \) is understood in the weak sense, i.e. \( y \) is increasing if \( a \leq b \) implies \( y(a) \leq y(b) \) and \( y \) is decreasing if \( a \leq b \) implies \( y(a) \geq y(b) \).

**Theorem 1.** [5] (comparison principle in cone) Let \( y \in C(\bar{I}) \cap C_{loc}(I) \). The function \( f \) is defined on \( I \times \mathbb{R} \). Let \( f(x,y) \) is increasing in \( y \) function, then

\[
\begin{align*}
&v'' - f(x,v) \geq w'' - f(x,w) \text{ a.e. on } I, \\
v(a) \leq w(a), & \quad v(b) \leq w(b)
\end{align*}
\]

implies

\[
v \leq w \text{ on } \bar{I}.
\]

Remark 1. Let \( f(x,y) \) is decreasing, then theorem 1 remains without changes, if both parts of (3) multiply onto \(-1\).

For the convenience of defining an ordering relation in cone \( P \) we make a transformation for the problem (I). Let \( F(\varphi, a) \) and \( G(\varphi, a) \) be defined by (I). Then through the transformation \( \varphi = -u \) the problem (I) is reduced to the form

\[
\begin{align*}
-\frac{d^2u}{dx^2} &= j_x \frac{1 - u}{\sqrt{(1 - u)^2 - 1 - a^2}} \triangleq \tilde{F}(j_x, u, a), & u(0) = 0, & u(1) = \varphi_L, \\
\frac{d^2a}{dx^2} &= j_x \frac{a}{\sqrt{(1 - u)^2 - 1 - a^2}} \triangleq \tilde{G}(j_x, u, a), & a(0) = 0, & a(1) = a_L.
\end{align*}
\]

(II)

We note that all solutions of the problem (I), as well the problem (II), are symmetric with respect to the transformation of sign for the magnetic potential \( a : (\varphi, a) = (\varphi, -a) \) or the same \((u, a) = (u, -a)\). Thus we must search only positive solutions \( \varphi > 0, a > 0 \) in cone \( P \) or only negative ones: \( \varphi < 0, a < 0 \). Thanks to the symmetry of problem it is equivalently and does not yields the extension of the types of sign- defined solutions of the problem (I) (respect.(II)). Once more, we note that introduction of negative electrostatic potential in problem (II) is connected with more convenient relation between order in cone and positiveness of Green function for operator \(-u''\) that we use below.

**Definition 2.** A pair \([\varphi_0, a_0], (\varphi^0, a^0)\) is called

a) sub-super solution of the problem (I) relative to \( P \), if the following conditions are satisfied

\[
\begin{align*}
&\{ (\varphi_0, a_0) \in C_{loc}(I) \cap C(\bar{I}) \times C_{loc}(I) \cap C(\bar{I}), \\
&(\varphi^0, a^0) \in C_{loc}(I) \cap C(\bar{I}) \times C_{loc}(I) \cap C(\bar{I}) \},
\end{align*}
\]

\[
\begin{align*}
\varphi''_0 - j_2 \frac{1 + \varphi_0}{\sqrt{(1 + \varphi_0)^2 - 1 - a^2}} \triangleq F(\varphi_0, a) \leq 0 \text{ in } I,
\end{align*}
\]

\[
\begin{align*}
&\{ (\varphi^0, a^0) \in C_{loc}(I) \cap C(\bar{I}) \times C_{loc}(I) \cap C(\bar{I}),
\end{align*}
\]
\begin{align*}
\left(\varphi^0\right)^{\prime\prime} - j_x \frac{1 + \varphi^0}{\sqrt{(1 + \varphi^0)^2 - 1 - a^2}} \triangleq F(\varphi^0, a) & \geq 0 \text{ in } I \forall a \in [a_0, a^0]; \\
a^{\prime\prime} - j_x \frac{a_0}{\sqrt{(1 + \varphi)^2 - 1 - a_0^2}} \triangleq G(\varphi, a_0) & \leq 0 \text{ in } I,
\end{align*}
(5)

\begin{align*}
\varphi^0 & \leq \varphi^0, \ a_0 \leq a^0 \text{ in } I \\
\end{align*}
(6)

and on the boundary
\begin{align*}
\varphi_0(0) & \leq 0 \leq \varphi^0(0), \ \varphi_0(1) \leq \varphi_L \leq \varphi^0(1), \\
a_0(0) & \leq 0 \leq a^0(0), \ a_0(1) \leq a_L \leq a^0(1);
\end{align*}
(7)

b) sub-sub solution of the problem (I) relative to \( P \), if a condition (4) is satisfied and
\begin{align*}
\varphi^{\prime\prime} - F(j_x, \varphi_0, a_0) & \leq 0 \text{ in } I, \\
a^{\prime\prime} - G(j_x, \varphi_0, a_0) & \leq 0 \text{ in } I
\end{align*}
(9)

and on the boundary
\begin{align*}
\varphi_0(0) & \leq 0, \ \varphi_0(1) \leq \varphi_L, \ a_0(0) \leq 0, \ a_0(1) \leq a_L.
\end{align*}
(10)

Remark 2. In definition 2 the expressions with square roots we take by modulus \(|(1 + \varphi)^2 - 1 - a^2|\).

By analogy with (9), (10) we may introduce the definition of super-super solution in cone.

**Definition 3.** The functions \( \Phi(x, x_{a_i}, j_x) \), \( \Phi_1(x, x_{\varphi_j}, j_x) \) we shall call a *semitrivial* solutions of the problem (I), if \( \Phi(x, x_{a_i}, j_x) \) is a solution of the scalar boundary value problem
\begin{align*}
\varphi^{\prime\prime} = F(j_x, \varphi, x_{a_i}) = j_x \frac{1 + \varphi}{\sqrt{(1 + \varphi)^2 - 1 - (x_{a_i})^2}},
\end{align*}
(III)

\( \varphi(0) = 0, \ \varphi(1) = \varphi_L \),

and \( \Phi_1(x, x_{\varphi_j}, j_x) \) is a solution of the scalar boundary value problem
\begin{align*}
a^{\prime\prime} = G(j_x, x_{\varphi_j}, a) = j_x \frac{a}{\sqrt{(1 + x_{\varphi_j})^2 - 1 - a^2}},
\end{align*}
(IV)
\[ a(0) = 0, \quad a(1) = a_L. \]

Here \( x_{a_i}, i = 1, 2, 3 \) and \( x_{\varphi_j}, j = 1, 2 \) are respectively, the \textit{indicators} of semitrivial solutions \( \Phi(x, x_{a_i}, j), \Phi_1(x, x_{\varphi_j}, j) \) defined by the following way:
\[ x_{a_1} = 0, \text{ if } a(x) = 0; \]
\[ x_{a_2} = a^0, \text{ if } a = a^0 \text{ be upper solution of the problem (IV)}; \]
\[ x_{a_3} = a_0, \text{ if } a = a_0 \text{ be lower solution of the problem (IV)}; \]
\[ x_{\varphi_1} = \varphi^0, \text{ if } \varphi = \varphi^0 \text{ be upper solution of the problem (III)}; \]
\[ x_{\varphi_2} = \varphi_0, \text{ if } \varphi = \varphi_0 \text{ be lower solution of the problem (III)}. \]

From definition 3 we obtain the following types of scalar boundary value problems for semitrivial (in sense of definition 3) solutions of (I) (resp.(II)):
\[ \varphi'' = F(\varphi, 0) = j_x \frac{1 + \varphi}{\sqrt{(1 + \varphi)^2 - 1}}, \quad \varphi(0) = 0, \quad \varphi(1) = \varphi_L. \tag{A_1} \]
\[ \varphi'' = F(\varphi, a^0) = j_x \frac{1 + \varphi}{\sqrt{(1 + \varphi)^2 - 1 - (a^0)^2}}, \quad \varphi(0) = 0, \quad \varphi(1) = \varphi_L. \tag{A_2} \]
\[ \varphi'' = F(\varphi, a_0) = j_x \frac{1 + \varphi}{\sqrt{(1 + \varphi)^2 - 1 - (a_0)^2}}, \quad \varphi(0) = 0, \quad \varphi(1) = \varphi_L. \tag{A_3} \]
\[ a'' = G(\varphi^0, a) = j_x \frac{a}{\sqrt{(1 + \varphi^0)^2 - 1 - a^2}}, \quad a(0) = 0, \quad a(1) = a_L. \tag{A_4} \]
\[ a'' = G(\varphi_0, a) = j_x \frac{a}{\sqrt{(1 + \varphi_0)^2 - 1 - a^2}}, \quad a(0) = 0, \quad a(1) = a_L. \tag{A_5} \]

We shall find the solutions of problems (A_1) \( \cdots \) (A_3) with condition
\[ \varphi_0 < \varphi^0, \]
where \( \varphi_0(x_{a_1}), \varphi^0(x_{a_2}) \) are respectively, lower and upper solutions of problem (A_1). The solution \((\varphi, a)\) of problem (I) should be belong to the interval
\[ \varphi \in \Phi(\varphi, 0) \cap \Phi(\varphi, a^0) \cap \Phi(\varphi, a_0), \]
\[ a \in \Phi_1(\varphi^0, a) \cap \Phi_1(\varphi_0, a). \]
Moreover, the ordering of lower and upper solutions of problems (A_1) \( \cdots \) (A_3) is satisfied
\[ \varphi_0(x_{a_1}) < \varphi_0(x_{a_2}) < \varphi_0(x_{a_3}) < \varphi^0(x_{a_2}) < \varphi^0(x_{a_1}). \]

We shall seek the solution of problems (A_4) \( \cdots \) (A_5) with condition
\[ a_0 < a^0, \]
in this case the following ordering of lower and upper solutions of problems (A_4) \( \cdots \) (A_5)
\[ a_0(\varphi_1) < a_0(\varphi_2) < a^0(\varphi_2) < a^0(\varphi_1). \]
is satisfied.

We go over to the direct study of the problem (III), which includes a cases \((A_1) - (A_3)\). Let us consider the boundary value problem (III) with

\[
F(x, \varphi) : (0, 1] \times (0, \infty) \to (0, \infty).
\]

In condition \((B_1)\) for \(F(x, \varphi)\) we dropped index \(a_i\), considering a general case of nonlinear dependence \(F\) of \(x\).

We shall assume that \(F\) is Caratheodory function, i.e.

\[
F(\cdot, s) \text{ is measurable for all } s \in R, \tag{B_2}
\]

\[
F(x, \cdot) \text{ is continuous a.e. for } x \in]0, 1[, \tag{B_3}
\]

and the following conditions hold

\[
\int_0^1 s(1 - s)F ds < \infty. \tag{B_4}
\]

\[
\partial F/\partial \varphi > 0, \text{ i.e. } F \text{ is increasing in } \varphi. \tag{B_5}
\]

There are \(\gamma(x) \in L^1([0, 1])\) and \(\alpha \in R, 0 < \alpha < 1\) such that

\[
|F(x, s)| \leq \gamma(x)(1 + |s|^{-\alpha}), \forall (x, s) \in ]0, 1[ \times R. \tag{B_6}
\]

We are interested in a positive classical solution of equation (III), i.e. \(\varphi > 0\) in \(P\) for \(x \in]0, 1[\) and \(\varphi \in C([0, 1]) \cap C^2([0, 1]).\) The problem (III) is singular, therefore, condition \((B_1)\) is not fulfilled on the interval \(\varphi \in (0, \infty)\) and in this connection, the well-known theorems [3] on existence of lower and upper solution in cone \(P\) does not work. It follows from theorem 1, since \(F\) in (III) is increasing in \(\varphi\), then \(\varphi < w\) for \(x \in]0, 1[\), where \(\varphi\) and \(w\) satisfy to the differential inequality (3).

**Theorem 2.** Assume conditions \((B_2) - (B_6)\). Then there exists a positive solution \(\varphi \in C([0, 1]) \cap C^2([0, 1])\) to the boundary value problem (III).

Proof. Let \(\varphi > 0\) is a solution of problem (III). By theorem 1 \(\varphi < w\) for \(x \in]0, 1[\). Take \(\epsilon > 0\) and consider equation

\[
\varphi''_\epsilon = j \frac{1 + \varphi_\epsilon + \epsilon}{\sqrt{(1 + \varphi_\epsilon + \epsilon)^2 - 1 - (x_0)^2}} \triangleq F_\epsilon(j, \varphi_\epsilon + \epsilon, x_0). \tag{11}
\]

\[
\varphi_\epsilon(0) = 0, \quad \varphi_\epsilon(1) = \varphi_L.
\]

Let \(w\) and \(\varphi\) are upper and lower solutions of equation (11) (below, in proposition 1 is shown that such solutions really exist). Hence the theorem on monotone iterations [6] gives an existence of classical solution \(\varphi_\epsilon\) of equation (11), which satisfies \(w > \varphi_\epsilon > \varphi\) for \(x \in]0, 1[\).
and is bounded in $C$. Thus $F_{\epsilon}(j_x, \varphi_{\epsilon} + \epsilon, x_{a_{\epsilon}})$ is bounded and there exists uniform limit $\lim_{\epsilon \to 0} \varphi_{\epsilon} = \varphi$. It follows from the last, if $0 < \eta < \frac{1}{2}$, then $\lim_{\epsilon \to 0} F_{\epsilon}(j_x, \varphi_{\epsilon} + \epsilon, x_{a_{\epsilon}}) = F(j_x, \varphi, x_{a_{\epsilon}})$ uniformly on $[\eta, 1-\eta]$ and $\varphi > 0$ for $x \in [\eta, 1-\eta]$.

Since $\varphi_{\epsilon}$ converges uniformly on $[0, 1]$, this implies the existence of uniform limit $\lim_{\epsilon \to 0} \varphi_{\epsilon}'(\eta)$. Therefore there exists $\lim_{\epsilon \to 0} \varphi_{\epsilon}''(x)$ on compact subsets $(0, 1)$ and $\{\varphi_{\epsilon}'\}$ uniformly converges on $(0, 1)$ to the differentiable function $\varphi'$ on $[\eta, 1-\eta]$. It follows from the last, $\varphi$ is twice differentiable on $[\eta, 1-\eta]$, $\varphi''(x) = F(j_x, \varphi, x_{a_{\epsilon}}), x \in [\eta, 1-\eta]$ and $u \in C([0, 1]) \cap C^2([0, 1])$ is a positive solution of problem (III).

Remark 3. Delicate moment in proof of theorem 2 is connected with finding of a lower $\varphi$ and an upper solution for perturbated problem (11). As a lower solution we can take solution of equation $(A_1)$ (semitrivial solution $\varphi$), then an upper solution will be, for example, maximal solution of equation $(A_1)$.

Application of monotone iteration techniques to the equation (III) gives an existence of maximal solution $\bar{\varphi}(x, j_x)$ such that

$$\varphi(x, x_j) \leq \bar{\varphi}(x, x_j) < w(x) \quad x \in [0, 1].$$

(12)

**Proposition 1.** Let $0 < c \leq j_x \leq j_x^{max}$. Then equation $(A_1)$

$$\varphi'' = F(j_x, \varphi, 0) = j_x \frac{1 + \varphi}{\sqrt{\varphi(2 + \varphi)}},$$

$$\varphi(0) = 0, \quad \varphi(1) = \varphi_L$$

has a lower positive solution

$$u_0 = \delta^2 x^{4/3},$$

(13)

if

$$4\delta^3 \geq 9j_x^{max}(1 + \delta^2)/\sqrt{2 + \delta^2}$$

(14)

and an upper positive solution

$$u^0 = \alpha + \beta x \quad (\alpha, \beta > 0),$$

(15)

here

$$\varphi_L \geq \delta^2,$$

(16)

where $\delta$ is defined from (14).

Remark 4. Square root is taking as $\sqrt{|\varphi(2 + \varphi)|}$ in the case of negative solutions. Here $u^0 = -\epsilon x$ is an upper solution, and $u_0 = -2 + \epsilon$ is a lower solution ($0 < \epsilon < 1$). Hence, equation $(A_1)$ has negative solution only for $0 < \varphi_L < -2$ because $F(x, -2) = -\infty$.

It follows from (14), (16) that a value of current is limited by the value of electrostatic potential on the anode $\varphi_L$

$$j_x \leq j_x^{max} \leq F(\varphi_L).$$

(17)
Analysis of lower and upper solutions (13), (15) exhibits that for \( \delta^2 = \varphi_L > 2 \) and \( \alpha = \beta \leq 1 \) interval in \( x \) between lower and upper solutions is decreased, and for the large values of potential \( \varphi_L \) diode makes on regime \( \varphi_L x^{4/3} \).

**Proposition 2.** Let \( 0 < c \leq j_x \leq j_x^{\text{max}} \). Then equation \((A_4)\)

\[
a'' = G(j_x, \varphi^0, a) = j_x \frac{a}{\sqrt{(1 + \varphi^0)^2 - 1 - a^2}}, \quad a(0) = 0, \quad a(1) = a_L
\]

with a lower solution \( a_0 = 0 \) and an upper solution \( a^0 = u^0 > 0 \), conditions (14), (16) has an unique solution \( a(x, j_x, c) \), which is positive, moreover

\[
0 \leq a_L \leq \sqrt{\varphi^0 (2 + \varphi)}.
\]

Proof. The positive solution of problem \((A_4)\) is concave and be found as a solution of initial problem with \( a(0) = 0, \quad a'(0) = c \), where \( c \) is shooting parameter. The solution \( a = a(x, j_x, c) \) is unique and strongly decreasing in \( c \) because right part of differential equation is decreasing in \( a \). The least nonnegative solution is \( f(x, j_x, 0) = 0 \) and for \( 0 \leq a_L \leq \sqrt{\varphi_L^0 (2 + \varphi_L^0)} \) exists only one solution and no positive solutions for other values \( a_L \).

**Remark 5.** The problem \((A_5)\) is considered by analogy with problem \((A_4)\), change of an upper solution \( a^0 = u^0 \) to a lower \( a^0 = u_0 \) one and \( 0 \leq a_L \leq \sqrt{\varphi_0L (2 + \varphi_0L)} \).

Following to the definition 2 and propositions 1, 2, solutions of problems (III), (IV) we can write in the form (fig.1):

![Figure 1: location of lower \((\varphi_0, a_0)\) and upper \((\varphi^0, a^0)\) solutions](image)

lower-lower \((\varphi_0, a_0)\):

\[
\varphi_0 = u_0 = \delta^2 x^{4/3}, \quad a_0 = 0, \quad \varphi_L \geq \delta^2;
\]

upper-lower \((\varphi^0, a_0)\):

\[
\varphi^0 = u^0 = \alpha + \beta x, \quad a_0 = 0, \quad \delta^2 \leq \varphi_L \leq C, \quad C = \max\{\alpha, \beta\};
\]
lower-upper \((\varphi_0, a^0)\):

\[
\varphi_0 = u_0 = \delta^2 x^{4/3}, \quad a^0 = u^0, \quad \varphi_L \geq \delta^2, \quad a_L \leq \sqrt{(u_0(2 + u_0)};
\]

upper-upper \((\varphi^0, a^0)\):

\[
\varphi^0 = u^0 = \alpha + \beta x, \quad a^0 = u^0, \quad \varphi_L \leq \mathcal{C}, \quad a_L \leq a^0 \leq u^0.
\]

3. Existence of solutions of system (I).

In the previous section we demonstrated the existence of semitrivial solutions of system (I). Here we show existence of solutions for the complete system (I), using the following McKenna-Walter theorem.

**Theorem 3.** [4] Assume conditions \((B_1) - (B_6)\). We assume that there exists the ordered pair \((u, \bar{u})\) — lower and upper solutions, i.e.

\[
\bar{u}, \check{u} \in C_{\text{loc}}((0, 1)]^2 \cap C([0, 1])^2, \quad \bar{u} \leq \check{u} \mid [0, 1]
\]

\[
u(0) \leq 0 \leq \check{u}(0), \quad \check{u}(1) \leq u_L \leq \bar{u}(1); \quad u_L \overset{\Delta}{=} (\varphi_L, a_L),
\]

\[
\forall x \in [0, 1], \quad \forall z \in \mathbb{R}^2,
\]

\[
\bar{u}(x) \leq z \leq \check{u}(x), \quad z_k = u_k(x);
\]

\[
-\bar{u}''(x) \geq h_k(x, z) \tag{19}
\]

and

\[
\forall x \in [0, 1], \quad \forall z \in \mathbb{R}^2,
\]

\[
\bar{u}(x) \leq z \leq \bar{u}(x), \quad z_k = \bar{u}_k(x);
\]

\[
-\bar{u}''(x) \leq h_k(x, z) \tag{20}
\]

for all \(k \in \{1, 2\}\). Then there exists a solution \(u \in C^2((0, 1)]^2 \cap C([0, 1])^2\) of the problem

\[
-u'' = h(\cdot, u(\cdot)) \mid [0, 1]
\]

\[u(0) = 0, \quad u(1) = u_L.\]

For keeping of ordering of lower and upper solutions in theorem 3 (in cone \(P\)) we write differential inequalities (19), (20) in the following form

\[
\forall z \in [v(x), w(x)], \quad z_1 = w_1(x): \quad
\]

\[
\pm w''_1(x) \overset{(\geq)}{=} \pm F_1(w_1(x), z_2)
\]

\[
\forall z \in [v(x), w(x)], \quad z_1 = v_1(x): \quad
\]

\[
\pm v''_1(x) \overset{(\leq)}{=} \pm F_1(v_1(x), z_2)
\]
∀z ∈ [v(x), w(x)]; z_2 = w_2(x)

±w''(x) \leq ±F_2(z_1, w_2)

∀z ∈ [v(x), w(x)]; z_2 = v_2(x)

±v''(x) \geq ±F_2(z_1, v_2).

Remark 6. Change of signs with (+) to (−) in differential inequalities is connected with adjustment of signs and ordering (≤) of lower (upper) solutions of system (I) in definition 2 and lower (upper) solutions in theorem 3.

From the last relations we obtain

\[
\begin{align*}
\begin{cases}
  w''(x) = F_1(w_1(x), 0) \leq F_1(w_1, z_2) \\
  v''(x) \geq \sup_{z_2} F_1(v_1(x), z_2)
\end{cases}, \quad (21)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  w''(x) \leq F_2(z_1, w_2) \\
  v''(x) \geq \sup_{z_1} F_2(z_1, v_2)
\end{cases}. \quad (22)
\end{align*}
\]

From inequality \( v''(x) \geq \sup_{z_1} F_2(z_1, v_2) \) we get estimations to the value of magnetic field on the anode \( a_L \)

\[
a_L \leq \frac{j_L}{2} \leq \frac{j_L^{\max}}{2} \leq \frac{\mathcal{F}(\varphi_L)}{2} \quad (23)
\]

taking account of (17) and \( \Theta_L > 0 \). Under realization of estimation (23) diode works in ”noninsulated” regime, moreover the value \( a_L \) is limited by value of electrostatic potential on the anode \( \varphi_L \) with critical value \( \varphi_L = 2 \). In increasing of magnetic potential \( a_L \) diode transfers in ”isolated” regime that leads to more complicated problem with free boundary.

Thus we have the following main result of this paper.

**Theorem 4.** Assume conditions \((B_3), (B_6)\) and inequalities \((14), (17), (23)\). Then the problem (I) possesses a positive solution \( P \) such that

\[
\begin{align*}
\begin{cases}
  \varphi''_0 \geq j_x F(\varphi_0, z_2), \; z_2 \in [0, \varphi^0] \\
  (\varphi^0)'' \leq j_x F(\varphi^0, z_2), \; z_2 \in [0, \varphi^0] 
\end{cases},
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  a''_0 \geq G(j_x, z_1, a_0), \; z_1 \in [\varphi_0, \varphi^0] \\
  (a^0)'' \leq G(j_x, z_1, a^0), \; z_1 \in [\varphi_0, \varphi^0]
\end{cases},
\end{align*}
\]

where \( \varphi_0 = \delta^2 x^{4/3} \) is a lower solution of problem \((A_1)\), \( \varphi^0 = \alpha + \beta x \) \((\alpha, \beta > 0)\) is an upper solution of problem \((A_1)\) with condition \( \varphi_L \geq \delta^2 \); \( a_0 = 0 \) is a lower solution of problem \((A_4)\) with condition \( 0 \leq a_L \leq \sqrt{\varphi^0(2 + \varphi^0)} \).

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