Mixing with descendant fields in perturbed minimal CFT models

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ABSTRACT: We extend the analysis of the RG trajectory connecting successive minimal CFT models $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$ for $p \gg 1$, performed by A. Zamolodchikov, to the fields $\varphi_{n,n^\pm 3}$. This required a close investigation of mixing with the descendant fields at the level 2. In particular we identify those specific linear combinations of UV fields which flow to the IR fields $\varphi_{n,3,n}$ and $\varphi_{n,-3,n}$. We report also the results of the calculation of the same mixing coefficients through the recent RG domain wall approach by Gaiotto. These results are in complete agreement with the leading order perturbation theory.
Introduction

In his work [1] A. Zamolodchikov has investigated two dimensional QFTs (denoted as $\mathcal{M}_{p,p-1}$) which are perturbations of the minimal models $\mathcal{M}_p$ by the relevant primary field $\varphi_{1,3}$. In large $p$ limit he has shown that this theory corresponds to the RG trajectory connecting two successive minimal models $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$. In other words $\mathcal{M}_{p,p-1}$ interpolates between the theories $\mathcal{M}_p$ (in UV limit) and $\mathcal{M}_{p-1}$ (in IR limit).

Zamolodchikov has investigated in details the renormalization of the fields $\varphi_{n,n}$, $\varphi_{n,n\pm1}$, $\varphi_{n,n\pm2}$ and computed their matrices of anomalous dimensions. Next to leading order calculations of the same matrices of anomalous dimensions have been carried out in the recent paper [5]. Note also that the $\mathcal{N}=1$ super-symmetric analogue of this RG flow is analysed in [6].

In this paper we extend Zamolodchikov’s analysis to the fields $\varphi_{n,n\pm3}$. The relative complexity of our case is due to strong mixing with the second level descendants of $\varphi_{n,n\pm1}$. The general formula of Zamolodchikov for the matrix of anomalous dimensions is designed for the case of primary fields, that is why we will adjust his formula to make it applicable also for the descendant fields. Effectively we deal with the mixing of 10 different fields, hence we need to calculate an $10 \times 10$ matrix of anomalous dimensions. We find that under RG the UV fields $\varphi_{n,n\pm3}$ flow to the definite combinations of the IR fields $\varphi_{n\pm3,n}$ and the second level descendants of $\varphi_{n\pm1,n}$ and explicitly calculate the mixing coefficients. We report also the result of calculation of the same mixing coefficients obtained through the recent RG domain wall approach by Gaiotto [7], which in large $p$ limit completely agrees with our leading order perturbation theory result.

The paper is organized as follows.

In section 1 we briefly recall Zamolodchikov’s method of leading order calculation of the anomalous dimensions in perturbed conformal theories. We describe here how to adjust his method for the cases with descendant fields.

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1 Another remarkable aspect of the $\varphi_{1,3}$ perturbation, namely its integrability [4], will not be discussed in this paper.
Section 2 is devoted to the calculation of the three point functions of up to second level descendant fields which we use in the next section for the calculation of the effective structure and normalization constants.

In section 3 we present our calculation of the \(10 \times 10\) matrix of anomalous dimensions.

In section 4 we report the results of the calculation of mixing coefficients obtained through the RG domain wall approach suggested by Gaiotto [7] and get a complete agreement with the leading order perturbation theory results.

In Appendix we collect few relevant facts about minimal models of the two dimensional CFT.

1. Zamolodchikov’s theory and its adjustment

We will briefly present here the leading order perturbation theory developed by A. Zamolodchikov to investigate the renormalization of fields in a conformal field theory perturbed by relevant operators. Denote the action density by \( \mathcal{H}(x, g^i) \), where \( g^i, i = 1, 2, \ldots, n \) are the (renormalized) coupling constants. It is assumed that \( g_i = 0 \) corresponds to a CFT and the primary spinless fields \( \Phi_i \equiv \partial \mathcal{H}/\partial g^i \) are the perturbing operators which are conventionally normalized as

\[
\langle \Phi_i(x)\Phi_j(0) \rangle|_{x^2=1} = \delta_{i,j} + O(g^2) \quad (1.1)
\]

In the case when the dimensions \( \Delta_i \) of these fields satisfy the conditions \( 0 < \epsilon_i \equiv 1 - \Delta_i \ll 1 \) and \( g \lesssim \epsilon \), A. Zamolodchikov has derived a simple expression for the matrix of the anomalous dimensions

\[
\gamma^j_i(g) = \Delta_i \delta^j_i + C^j_{ik} g^k + O(g^2), \quad (1.2)
\]

where \( C^j_{ik} \) are related to the structure constant \( C_{ijk} \equiv \langle \Phi_i(1)\Phi_j(0)\Phi_k(\infty) \rangle|_{g=0} \)

\[
C^j_{ik} = \pi C_{ijk} \quad (1.3)
\]

Suppose we have a set of primary fields with close to each other dimensions. Suppose further that no other field (primary or descendant) of the dimension approximately equal to those of the set can be generated by means of the OPE of the fields from our set with the perturbing fields. Then for the matrix of anomalous dimensions the same formula \((1.2)\) can be used, with the indices \( i \) and \( j \) running over all the fields of our set. Unfortunately such closed with respect to OPE sets of primary fields are rare. In most cases the closed in above mentioned sense sets will include besides primaries also descendant fields.

The key ingredient of Zamolodchikov’s derivation \((1.2)\) is the evaluation of the first order perturbative integral

\[
\int d^2y \varphi_i(x, \bar{x}) \varphi_j(0, 0) \varphi_k(y, \bar{y}) = C_{ijk}(x^2)^{\Delta_k-\Delta_i-\Delta_j} \\
\times \int d^2y (x-y)^{\Delta_j-\Delta_i-\Delta_k} y^{\Delta_j-\Delta_i-\Delta_k} (\bar{x}-\bar{y})^{\Delta_j-\Delta_i-\Delta_k} \bar{y}^{\Delta_j-\Delta_i-\Delta_k} = (x^2)^{1-\Delta_i-\Delta_j-\Delta_k} C_{ijk} I^{k}_{ij} \quad (1.4)
\]
where in first equality the standard expression for the three point functions of primary fields is used and

\[
I_{ij}^k = \pi \frac{\Gamma(\Delta_i - \Delta_j - \Delta_k + 1) \Gamma(\Delta_j - \Delta_i - \Delta_k + 1) \Gamma(2\Delta_k - 1)}{\Gamma(\Delta_i + \Delta_k - \Delta_j) \Gamma(\Delta_j + \Delta_k - \Delta_i) \Gamma(2 - 2\Delta_k)}
\] (1.5)

It is important to note that the form of \( x \) dependence of the integral (1.4) is completely determined from the dimensional analysis and should remain the same also in the case when \( \varphi_i \) or \( \varphi_j \) (or both) are descendants. Only the numerical coefficient in that case would be different. This observation immediately suggests the following strategy for the calculation of the matrix of anomalous dimensions in the case when descendant fields are present:

- normalize all the (possibly descendant) fields as in eq. (1.1)
- calculate the integrals of the three point functions
- separate factors of the form (1.5) from overall numerical coefficients and denote the remaining parts as \( \tilde{C}_{ijk} \)

then the matrix of the anomalous dimensions would be

\[
\gamma^j_i(g) = \Delta_i \delta^j_i + \pi \tilde{C}_{ijk} g^k + O(g^2),
\] (1.6)

Let us come back to the main case of our interest, namely to the theory with a single coupling constant denoted as \( M_{p,p-1} \). We are interested in the behavior of the spinless fields \( \Phi_{n,m}(x,g) \) (\( g \) is the coupling constant) in \( M_{p,p-1} \). At \( g = 0 \), \( \Phi_{n,m}(x,0) = \varphi_{n,m}(x) \) by definition (\( \varphi_{n,m}(x) \) are the primaries of \( M_p \)). Let \( n - m = l > 0 \), \( n, m \ll p \). The structure of OPE with the perturbing field \( \varphi_{1,3} \) schematically is

\[
\varphi_{n,m}\varphi_{1,3} = [\varphi_{n,m}] + [\varphi_{n,m+2}] + [\varphi_{n,m-2}]
\] (1.7)

where the square brackets stand for the corresponding conformal family. The cases when \( l = 0, 1, 2 \) are analysed in [L]. As it is easy to see from (1.7), the field \( \varphi_{n,m} \) by itself alone constitutes a closed in already discussed sense set. Hence, it does not mix any other field and the matrix of anomalous dimensions is one dimensional. When \( l = 1 \), the closed set consists of two primaries \( \{\varphi_{n,n+1}, \varphi_{n,n-1}\} \). Already for \( l = 2 \) we have mixing with a descendant (in this case with a derivative of a primary) and the set is \( \{\varphi_{n,n+2}, \partial \bar{\partial} \varphi_{n,n}, \varphi_{n,n-2}\} \). The next case which is the main subject of this paper includes 10 different fields. Besides the primaries \( \varphi_{n,n+3} \), also second level descendants of the fields \( \varphi_{n,n+1} \), altogether 8 descendants should be included to get a closed set.

The RG trajectory of \( M_{p,p-1} \) has two fixed points at \( g = 0 \) and at \( g = g_* \)

\[
2\pi g_* = \sqrt{3} \epsilon + O(\epsilon^2)
\] (1.8)

where \( \epsilon = 2/(p + 1) \) is a small parameter. The fixed points at \( g = 0 \) and \( g = g_* \) are described by the minimal models \( M_p \) and \( M_{p-1} \) respectively. The fields \( \Phi_\alpha(x,g_*) \) (\( \alpha \) is an index numbering the fields of a closed set) should be identified with some combination
of fields from $\mathcal{M}_{p-1}$. To specify this map it is necessary to calculate the respective matrix of anomalous dimensions. The eigenvalues of this matrix at $g = g_*$ are the dimensions of those fields from $\mathcal{M}_{p-1}$ in term of which the fields $\Phi_\alpha(x, g_*)$ can be expanded. On the other hand the components of an eigenvector show, which combination of the fields $\Phi_\alpha(x, g_*)$ gives us the specific field from the IR theory $\mathcal{M}_{p-1}$ whose dimension is equal to the respective eigenvalue.

2. Calculation of three point functions

Let us begin with the investigation of the fields $\Phi_{n,n+3}(x, g)$ which at vanishing coupling constant coincide with the primaries $\varphi_{n,n+3}$ of the theory $\mathcal{M}_p$. As we discussed at the end of the previous section we should include into consideration also the second level descendants of the fields $\varphi_{n,n+1}$

$$L_a \bar{L}_b \varphi_{n,n+1}, \quad a, b \in \{1, 2\} \quad (2.1)$$

where,

$$L_1 \equiv L_{-1}^2$$

$$L_2 \equiv L_{-1}^2 - \frac{2\Delta(2\Delta + 1)}{3} L_{-2} \quad (2.2)$$

Here $L_n$ are Virasoro generators and $\Delta$ is the dimension of the primary field on which this generators act. The form of the operator $L_2$ is chosen so that the fields $L_2 \varphi$ are quasi primaries. For the resulting properly normalized 10 fields we make the following assignment

$$\Phi_1(x, 0) \equiv N_1^{-\frac{1}{2}} \varphi_{n,n+3}(x), \quad \Phi_2(x, 0) \equiv N_2^{-\frac{1}{2}} L_1 \bar{L}_1 \varphi_{n,n+1}(x),$$

$$\Phi_3(x, 0) \equiv N_3^{-\frac{1}{2}} L_1 \bar{L}_2 \varphi_{n,n+1}(x), \quad \Phi_4(x, 0) \equiv N_4^{-\frac{1}{2}} L_2 \bar{L}_1 \varphi_{n,n+1}(x),$$

$$\Phi_5(x, 0) \equiv N_5^{-\frac{1}{2}} L_2 \bar{L}_2 \varphi_{n,n+1}(x), \quad \Phi_6(x, 0) \equiv N_6^{-\frac{1}{2}} L_1 \bar{L}_1 \varphi_{n,n-1}(x),$$

$$\Phi_7(x, 0) \equiv N_7^{-\frac{1}{2}} L_1 \bar{L}_2 \varphi_{n,n-1}(x), \quad \Phi_8(x, 0) \equiv N_8^{-\frac{1}{2}} L_2 \bar{L}_2 \varphi_{n,n-1}(x),$$

$$\Phi_9(x, 0) \equiv N_9^{-\frac{1}{2}} L_2 \bar{L}_2 \varphi_{n,n-3}(x), \quad \Phi_{10}(x, 0) \equiv N_{10}^{-\frac{1}{2}} \varphi_{n,n-3}(x) \quad (2.3)$$

where the constants $N_1, \ldots, N_{10}$ are determined from the normalization condition. As seen from (2.3) our fields are either primaries, quasi primaries or their derivatives. As we saw in section 1 to calculate the matrix of anomalous dimensions we need all three point functions of the form $\langle \Phi_i \Phi_j \Phi \rangle|_{g=0}$. It is sufficient to calculate three point correlation functions of the primary and quasi primary fields, since we can always take out the derivatives from the correlation functions. The basic correlation functions that should be calculated are of the form $\langle L_{-2} \varphi_1 \varphi_2 \varphi_3 \rangle$ and $\langle L_{-2} \varphi_1 L_{-2} \varphi_2 \varphi_3 \rangle$ where $\varphi_1, \varphi_2, \varphi_3$ are primary fields. All three point functions of our interest can be easily derived from the correlators of this kind simply taking derivatives and using the holomorphic anti-holomorphic factorization property. Our problem boils down to the calculation of the correlation functions $\langle T \varphi_1 \varphi_2 \varphi_3 \rangle$ and
\langle TT\varphi_1\varphi_2\varphi_3 \rangle$ where $T$ is the Energy-Momentum tensor. Such correlators can be computed using conformal Ward identities \cite{2}. The results are

\begin{equation}
\langle T(\xi)\varphi_1\varphi_2\varphi_3 \rangle = \sum_{i=1}^{3} \left( \frac{\Delta_i}{(\xi - x_i)^2} + \frac{1}{\xi - x_i} \right) \partial \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) \rangle
\end{equation}

and

\begin{equation}
\langle T(\xi_1)T(\xi_2)\varphi_1\varphi_2\varphi_3 \rangle = \frac{c}{2(\xi_1 - \xi_2)^4} \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) \rangle
\end{equation}

where $c$ is the Virasoro central charge. Taking into account that

\begin{equation}
L_n = \oint \frac{d\xi}{2\pi i} \xi^{n+1} T(\xi)
\end{equation}

we get

\begin{equation}
\langle L_{-2}\varphi_1\varphi_2\varphi_3 \rangle = \oint_{c(x_1)} \frac{d\xi}{2\pi i} \frac{1}{\xi - x_1} \langle T(\xi)\varphi_1\varphi_2\varphi_3 \rangle
\end{equation}

and

\begin{equation}
\langle L_{-2}\varphi_1 L_{-2}\varphi_3 \rangle = \oint_{c(x_1)} \frac{d\xi_1}{2\pi i} \oint_{c(x_2)} \frac{d\xi_2}{2\pi i} \frac{1}{\xi_1 - x_1} \frac{1}{\xi_2 - x_2} \langle T(\xi_1)T(\xi_2)\varphi_1\varphi_2\varphi_3 \rangle
\end{equation}

where $c(x_1)$, $c(x_2)$ are small contours surrounding respectively the points $x_1$ and $x_2$ in anti-clockwise direction. Using \cite{2.2}, \cite{2.4}, \cite{2.5}, \cite{2.7} and \cite{2.8} we obtain that

\begin{equation}
\langle \mathcal{L}_2 \varphi_1(x_1, \bar{x}_1)\varphi_2(x_2, \bar{x}_2)\varphi_3(x_3, \bar{x}_3) \rangle = \sum_{i=1}^{3} \left( \frac{\Delta_i}{(\bar{x} - x_i)^2} + \frac{1}{\bar{x} - x_i} \right) \partial \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) \rangle
\end{equation}

and

\begin{equation}
\langle \mathcal{L}_2 \varphi_1(x_1, \bar{x}_1)\mathcal{L}_2 \varphi_2(x_2, \bar{x}_2)\varphi_3(x_3, \bar{x}_3) \rangle = \sum_{i=1}^{3} \left( \frac{\Delta_i}{(\bar{x} - x_i)^2} + \frac{1}{\bar{x} - x_i} \right) \partial \langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3) \rangle
\end{equation}
where \( C_{123} \) is the structure constant related to the three-point function
\[
\langle \varphi_1(x_1, \bar{x}_1) \varphi_2(x_2, \bar{x}_2) \varphi_3(x_3, \bar{x}_3) \rangle.
\]
Integrals over three point functions can be calculated using the formula
\[
\int d^2y y^{\alpha_1-1} \bar{y}^{\alpha_2-1}(1-y)^{\beta_1-1}(1-\bar{y})^{\beta_2-1} = \pi \frac{\Gamma(\alpha_1)\Gamma(\beta_1)\Gamma(1-\alpha_2-\beta_2)}{\Gamma(1-\alpha_2)\Gamma(1-\beta_2)\Gamma(\alpha_1+\beta_1)}
\]
where the numbers \( \alpha_1 - \alpha_2 \) and \( \beta_1 - \beta_2 \) are supposed to be integers. Then we calculate the constants \( \tilde{C}_{ijk} \) following the line described in section II.

3. Calculation of the matrix of anomalous dimensions

First let us recall that our expression (1.6) is derived with the assumption that the fields satisfy the normalization condition (1.1). Orthogonality of the fields (2.3) follows from the fact that they are primaries, quasi primaries or their derivatives. The normalization of the quasi primary fields can be easily fixed through the replacement \( \Delta_1 = \Delta_2, \Delta_3 = 0 \) in (2.10) which reduces the three point functions to two point ones. For the normalization constants \( N_i \) we get the following values
\[
N_1 = N_{10} = 1
\]
\[
N_2 = 4\Delta^2_{(n,n+1)} (2\Delta_{(n,n+1)} + 1)^2 (2\Delta_{(n,n+1)} + 2)^2 (2\Delta_{(n,n+1)} + 3)^2
\]
\[
N_3 = N_4 = \frac{4}{9} \Delta_{(n,n+1)} (2\Delta_{(n,n+1)} + 1)^2 (2\Delta_{(n,n+1)} + 2)^2 (2\Delta_{(n,n+1)} + 3)
\]
\[
\times \left( 2c\Delta_{(n,n+1)} + 16\Delta^2_{(n,n+1)} - 10\Delta_{(n,n+1)} + c \right)
\]
\[
N_5 = \frac{4}{81} (2\Delta_{(n,n+1)} + 1)^2 (2c\Delta_{(n,n+1)} + 16\Delta^2_{(n,n+1)} - 10\Delta_{(n,n+1)} + c)^2
\]
\[
N_6 = 4\Delta^2_{(n,n-1)} (2\Delta_{(n,n-1)} + 1)^2 (2\Delta_{(n,n-1)} + 2)^2 (2\Delta_{(n,n-1)} + 3)^2
\]
\[
N_7 = N_8 = \frac{4}{9} \Delta_{(n,n-1)} (2\Delta_{(n,n-1)} + 1)^2 (2\Delta_{(n,n-1)} + 2)^2 (2\Delta_{(n,n-1)} + 3)
\]
\[
\times \left( 2c\Delta_{(n,n-1)} + 16\Delta^2_{(n,n-1)} - 10\Delta_{(n,n-1)} + c \right)
\]
\[
N_9 = \frac{4}{81} (2\Delta_{(n,n-1)} + 1)^2 (2c\Delta_{(n,n-1)} + 16\Delta^2_{(n,n-1)} - 10\Delta_{(n,n-1)} + c)^2
\]
where the central charge \( c \) and the dimensions \( \Delta_{(n,m)} \) are given by (A.1) and (A.2). Now we have all necessary ingredients to start the calculation of the matrix of anomalous dimensions given by the expression (1.6). Remind that \( M_{p,p-1} \) has only one coupling constant denoted as \( g \). The perturbing field we simply denote by \( \varphi_{13} \equiv \Phi \) and its dimension as \( \Delta_{(1,3)} \equiv \Delta \). Accordingly we will suppress the index \( k \) in quantities \( \tilde{C}_{ijk} \) and \( I_{ijk} \). Specializing the discussion of the section II into our case we see that the constants \( \tilde{C}_{ij} \) should be derived from the equality
\[
\int d^2y(\Phi_i(x)\Phi_j(0)\Phi(y))|_{g=0} = (x^2)^{1-\Delta_i-\Delta_j-\Delta}\tilde{C}_{ij}I_{ij}
\]
where \( I_{ij} \) is given by the eq. (3.3) with \( \Delta_k \) replaced by \( \Delta \).
As an example let us demonstrate the calculation of \( \tilde{C}_{47} \). We need the correlation function

\[
\langle L_2 \tilde{L}_1 \varphi_{n,n+1}(x_1, \bar{x}_1) L_1 \tilde{L}_2 \varphi_{n,n-1}(x_2, \bar{x}_2) \varphi_{1,3}(y, \bar{y}) \rangle
= \frac{\partial^2}{\partial \bar{x}_1 \partial x_2} \frac{\partial^2}{\partial \bar{y} \partial y} \langle L_2 \varphi_{n,n+1}(x_1, \bar{x}_1) \tilde{L}_2 \varphi_{n,n-1}(x_2, \bar{x}_2) \varphi_{1,3}(y, \bar{y}) \rangle
\]

(3.3)

Using the factorization property and the eq. (2.9) we get

\[
\langle L_2 \varphi_{n,n+1}(x_1, \bar{x}_1) \tilde{L}_2 \varphi_{n,n-1}(x_2, \bar{x}_2) \varphi_{1,3}(y, \bar{y}) \rangle
= a_1 a_2 C_{(n,n+1)(n,n-1)(1,3)} x_1 x_2 \frac{\Delta_{(1,3)} - \Delta_{(n,n-1)} - \Delta_{(n,n+1)} - 2\Delta_{(n,n+1)} - 2\Delta_{(n,n+1)} - 2\Delta_{(n,n+1)}}{x_1 x_2}
\times (x_1 - y) \Delta_{(n,n-1)} - \Delta_{(n,n+1)} - 2\Delta_{(1,3)} (x_2 - y) \Delta_{(n,n+1)} + 2\Delta_{(n,n+1)} - \Delta_{(n,n+1)} - 2\Delta_{(1,3)}
\]

(3.4)

where

\[
a_{1,2} = -\frac{1}{3} \left( -\Delta_{(n,n+1)} + \Delta_{(n,n+1)}^2 + \Delta_{(n,n+1)} + 2\Delta_{(n,n+1)} \Delta_{(n,n+1)} - 3\Delta_{(n,n+1)} + \Delta_{(1,3)} + 2\Delta_{(n,n+1)} \Delta_{(1,3)} + 6\Delta_{(n,n+1)} \Delta_{(1,3)} - 3\Delta_{(1,3)}^2 \right)
\]

(3.5)

while the structure constant (see [A.3])

\[
C_{(n,n+1)(n,n-1)(1,3)} = \frac{(n^2 - 1)^{1/2}}{\sqrt{3n}} + O(\epsilon)
\]

(3.6)

Performing the integration over \( y \) with the help of the eq. (2.11) we finally get

\[
\tilde{C}_{47} = N_4 \frac{1}{2} N_7 \frac{3}{2} a_1 a_2 C_{(n,n+1)(n,n-1)(1,3)} (-\Delta_{(1,3)} - \Delta_{(n,n-1)} - \Delta_{(n,n+1)} - 1)^2 (-\Delta_{(1,3)} - \Delta_{(n,n-1)} - \Delta_{(n,n+1)} - 2)^2
\]

(3.7)

where

\[
s = \Delta_{(n,n+1)} - \Delta_{(n,n-1)} - \Delta_{(1,3)} + 1
\]

\[
t = \Delta_{(n,n-1)} - \Delta_{(n,n+1)} - \Delta_{(1,3)} + 1
\]

(3.8)

It remains to insert the values of the quantities \( N_4, N_7, C_{(n,n+1)(n,n-1)(1,3)}, a_1, a_2, s, t \) and the respective dimensions. So we get

\[
\tilde{C}_{47} = \frac{35 (n^2 - 4)^{3/2} \epsilon^2}{144 \sqrt{3n}} + O(\epsilon^3)
\]

(3.9)

It is obvious from (1.1) that only the terms of order \( O(\epsilon^0) \) in \( \tilde{C}_{ijk} \) should be kept to get the matrix element \( \tilde{\gamma}_{ij} \) with the required accuracy. So, for \( \gamma_{4,7} \) we get zero.
In a similar manner one can calculate all other matrix elements. Here are the nonzero matrix elements $\gamma_{ij}$ with index $i \leq j$ (remind that this matrix is symmetric)

\[
\begin{align*}
\gamma_{1,1} &= \frac{9}{4} + \left(-\frac{9}{8} - \frac{3n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (4 + n)} \sqrt{\frac{n+3}{n+1}}; \\
\gamma_{2,2} &= \frac{9}{4} + \left(-\frac{9}{8} - \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (2 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{3,3} &= \frac{9}{4} + \left(-\frac{9}{8} - \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (4 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{4,4} &= \frac{9}{4} + \left(-\frac{9}{8} - \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (6 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{5,5} &= \frac{9}{4} + \left(-\frac{9}{8} - \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (8 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{6,6} &= \frac{9}{4} + \left(-\frac{9}{8} + \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (2 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{7,7} &= \frac{9}{4} + \left(-\frac{9}{8} + \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (2 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{8,8} &= \frac{9}{4} + \left(-\frac{9}{8} + \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (2 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{9,9} &= \frac{9}{4} + \left(-\frac{9}{8} + \frac{n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (2 + n)} \sqrt{\frac{n+3}{2n+1}}; \\
\gamma_{10,10} &= \frac{9}{4} + \left(-\frac{9}{8} + \frac{3n}{4}\right) \epsilon + 2\pi g \sqrt{\frac{3}{4} (4 + n)} \sqrt{\frac{n+3}{2n+1}}.
\end{align*}
\]

(3.10)

At the point $g = g_*$ (see eq. (3.8)) where the conformal invariance is recovered the eigenvalues of this matrix are

\[
\begin{align*}
\tilde{\Delta}_1 &= \frac{3}{4} + \left(\frac{3n}{4}\right) \epsilon \\
\tilde{\Delta}_2 &= \tilde{\Delta}_3 = \tilde{\Delta}_4 = \tilde{\Delta}_5 = \frac{9}{4} + \left(\frac{1}{8} + \frac{n}{4}\right) \epsilon \\
\tilde{\Delta}_6 &= \tilde{\Delta}_7 = \tilde{\Delta}_8 = \tilde{\Delta}_9 = \frac{9}{4} + \left(\frac{1}{8} - \frac{n}{4}\right) \epsilon \\
\tilde{\Delta}_{10} &= \frac{9}{4} + \left(\frac{9}{8} - \frac{3n}{4}\right) \epsilon
\end{align*}
\]

(3.11)

It is not difficult to see that the first eigenvalue corresponds to $\varphi_{(n-1)}^{(p-1)}$, next four eigenvalues correspond to the descendants $\mathcal{L}_a \mathcal{L}_b \varphi_{(n+3,n)}^{(p-1)}$, further come the four descendants $\mathcal{L}_a \mathcal{L}_b \varphi_{(n-1,n)}^{(p-1)}$ and the last eigenvalue corresponds to $\varphi_{(n-3,n)}^{(p-1)}$. Since the first and the last eigenvalues are non-degenerated, the corresponding eigenvectors are fixed uniquely (we assume standard unite normalization)

\[
\begin{align*}
\varphi_{(n+3,n)}^{(p-1)}(x) &= \frac{6}{n(n+1)(n+2)} \Phi_1(x, g_*) + \frac{6\sqrt{\frac{n+3}{n+2}}}{n(n+2)} \Phi_5(x, g_*) \\
&+ \frac{3\sqrt{(n+3)(n-1)}}{n(n+1)} \Phi_9(x, g_*) + \frac{\sqrt{n^2 - 9}}{n} \Phi_{10}(x, g_*) \\
\varphi_{(n-3,n)}^{(p-1)}(x) &= \frac{\sqrt{n^2 - 9}}{n} \Phi_1(x, g_*) - \frac{3\sqrt{(n-3)(n+1)}}{(n-1)n} \Phi_5(x, g_*) \\
&+ \frac{6\sqrt{\frac{n-3}{n-1}}}{(n-2)n} \Phi_9(x, g_*) - \frac{6}{n(n-1)(n-2)} \Phi_{10}(x, g_*)
\end{align*}
\]

(3.12)

4. Comparison with RG domain wall approach

Denote our 10 properly normalized fields of the ultraviolet theory as $\Phi_\alpha(x,0) \equiv \varphi_\alpha(x)$. Then the one-point functions of the product theory with Gaiotto’s boundary condition are
of the form

\[ \langle \varphi_{n+3, n} \varphi_{\alpha} | R G \rangle = M_\alpha^{(+) \frac{\sqrt{\varphi_{p-2} \varphi_{p}}} {S_{p-1}^{(+) \frac{1}{1, n}}} } \]  \tag{4.1} \]

where

\[ n_1 = n + 3; \quad n_2 = n_3 = n_4 = n_5 = n + 1; \quad n_6 = n_7 = n_8 = n_9 = n - 1; \quad n_{10} = n - 3 \]  \tag{4.2} \]

The modular matrix of the SU(2)_k WZW model is given by

\[ S_{m, n}^{(k)} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{\pi m n}{k + 2} \right) \]  \tag{4.3} \]

and the coefficients \( M_\alpha^{(+)} \) have been computed following the algorithm described in \[7\].

The computation is rather lengthy, e.g. to calculate the coefficient \( M_1 \) one should consider 42 dimensional space of level 9/2 fields in the theory \( SM_{k+1} \times M_3 \) (\( SM_{k+1} \) is the \( N = 1 \) super-conformal series [8, 10, 11, 11], \( k \equiv p - 1 \), \( M_3 \) is the Ising model). Here we present only the final results, leaving details of computations for a later publication.

\[ M_1^{(+)} = \left( -6 k^2 - 28 k - 33 \right) n^5 - \left( k + 4 \right) \left( 10 k^2 + 40 k + 39 \right) n^4 \]
\[ + 2 \left( k + 3 \right) \left( 3 k^3 + 20 k^2 + 44 k + 35 \right) n^3 - 2 \left( 7 k^5 + 83 k^4 + 400 k^3 + 978 k^2 + 1206 k + 588 \right) n^2 \]
\[ - \left( k + 3 \right) \left( 12 k^5 + 166 k^4 + 863 k^3 + 2135 k^2 + 2493 k + 1095 \right) n \]
\[ -3 \left( k + 1 \right) \left( k + 3 \right)^2 \left( 2 k + 3 \right) \left( 3 k + 5 \right) \]
\[ \times \left( k^2 \left( n + 1 \right) \left( k + 3 \right) \right)^{\frac{1}{2}} \]
\[ \times \left( k^2 + k \left( 5 n + 26 \right) + k \left( 17 n + 46 \right) - \left( n - 3 \right) \left( n + 1 \right) \right) \]
\[ \left( 2 k^2 - 2 k n + 2 \right) \left( 2 k^2 - 2 k n + 2 \right) \]
\[ \times \left( k + 3 \right)^2 \left( k + 4 \right) \left( 2 k + 3 \right) \left( 3 k + 5 \right) \left( 6 k^2 + 27 k + 26 \right) \]
\[ + \left( k + 3 \right) \left( 96 k^5 + 1077 k^4 + 4708 k^3 + 9995 k^2 + 10220 k + 3996 \right) n \]
\[ + \left( -51 k^5 - 665 k^4 - 3338 k^3 - 8123 k^2 - 9653 k - 4542 \right) n^2 \]
\[ + \left( 26 k^3 + 183 k^2 + 410 k + 312 \right) n^4 + \left( -60 k^4 - 525 k^3 - 1699 k^2 - 2406 k - 1254 \right) n^3 \]
\[ + 2 \left( 2 k^2 + 2 k - 3 \right) n^5 + \left( -7 k - 18 \right) n^6 \]
\[ \times \left( k + 2 \right) \left( n + 1 \right) \left( 2 k - n + 3 \right) \left( 2 k + n + 3 \right) \left( 3 k^2 - 2 k n + 2 \right) \left( n - 4 \right) n + 15 \]
\[ \]
Here we present several formulae concerning the unitary minimal series of 2d CFT. We thank Rubik Poghossian for introducing us to the subject.

Acknowledgement

We thank Rubik Poghossian for introducing us to the subject.

A. Minimal models of the 2d CFT

Here we present several formulae concerning the unitary minimal series of 2d CFT $\mathcal{M}_p$. The central charge is given by

$$c = 1 - \frac{6}{p(p+1)} \quad (A.1)$$

The primary fields are denoted by $\varphi_{n,m}$, $n = 1, 2, \ldots, p - 1$, $m = 1, 2, \ldots, p$ and the corresponding dimensions are [12]

$$\Delta_{(n,m)} = \frac{(n-m)^2}{4} + \frac{n^2-1}{4p} - \frac{m^2-1}{4(p+1)} \quad (A.2)$$

Notice also that the identification $\varphi_{n,m} \equiv \varphi_{p-n,p+1-m}$ holds. The structure constants of the operator algebra have been computed in [13]. Here we present a little bit simpler expression borrowed from [14]

$$C_{(n_1,m_1)(n_2,m_2)(n_3,m_3)} = \rho^{4s+2t-2s-1} \times \frac{\gamma(p-1)\gamma(m_1-n_1 \rho^{-1})\gamma(m_2-n_2 \rho^{-1})\gamma(-m_3+n_3 \rho^{-1})}{\gamma(1-\rho^{-1})\gamma(-n_1+m_1 \rho)\gamma(-n_2+m_2 \rho)\gamma(n_3-m_3 \rho)}$$

$$\times \prod_{i=1}^{s} \prod_{j=1}^{t} ((i-j \rho)(i+n_3-(j+m_3 \rho))(i-n_1-(j-m_1 \rho))(i-n_2-(j-m_2 \rho))^{-2}$$

$$\times \prod_{i=1}^{s} \gamma(i \rho^{-1})\gamma(-m_3+(i+n_3 \rho^{-1})\gamma(m_1+(i-n_1 \rho^{-1})\gamma(m_2+(i-n_2 \rho^{-1})$$

$$\times \prod_{j=1}^{t} \gamma(j \rho)\gamma(-n_3+(j+m_3 \rho)\gamma(n_1+(j-m_1 \rho)\gamma(n_2+(j-m_2 \rho)$$ \quad (A.3)

where

$$\rho = \frac{p}{p+1}; \quad \gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)}; \quad s = \frac{n_1+n_2-n_3-1}{2}; \quad t = \frac{m_1+m_2-m_3-1}{2}$$

\[\]
References

[1] A. Zamolodchikov, Renormalization Group and Perturbation Theory Near Fixed Points in Two-Dimensional Field Theory, Sov.J.Nucl.Phys. 46 (1987) 1090.

[2] A. Belavin, A. Polyakov and A. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl.Phys. B241 (1984). 333–380.

[3] D. Friedan, Z. Qiu and S. Shenker, Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions, Phys.Rev.Lett. v. 52 (1984) 1575–1578.

[4] A. Zamolodchikov, Higher Order Integrals of Motion in Two-Dimensional Models of the Field Theory with a Broken Conformal Symmetry, JETP Lett. 46 (1987) 160-164.

[5] R. Poghossian, Two Dimensional Renormalization Group Flows in Next to Leading Order, arXiv:1303.3015 [hep-th].

[6] R. Poghossian, Study of the Vicinities of Superconformal Fixed Points in Two-dimensional Field Theory, Sov.J.Nucl.Phys. 48 (1988) 763.

[7] D. Gaiotto, Domain Walls for Two-Dimensional Renormalization Group Flows, arXiv:1203.1052 [hep-th].

[8] H. Eichenherr, Minimal operator algebras in superconformal quantum field theory Phys.Lett. B151 (1985) 26–30.

[9] M. Bershadski, V. Knizhnik, M. Teitelman, Superconformal symmetry in two dimensions Phys.Lett. B151 (1985) 31–36.

[10] D. Friedan, Z. Qiu, S. Shenker, Superconformal invariance in two dimensions and the tricritical Ising model Phys.Lett. B151 (1985) 37–43,

[11] A. Zamolodchikov, R. Poghossian, Operator algebra in two-dimensional superconformal field theory Sov.J.Nucl.Phys. 47 (1988) 929–936,

[12] V. G. Kac, Highest weight representations of infinite dimensional Lie algebras, Proc. Internat. Congress Mathematicians (Helsinki, 1978).

[13] Vl. Dotsenko, V. Fateev, Operator algebra of two-dimensional conformal theories with central charge $C \leq 1$, Phys.Lett. B154 (1985) 291–295.

[14] R. Poghossian, Fields with spin in the minimal models $M(p)$ $(c < 1)$ of two-dimensional conformal field theory, preprint YERPHI-1198-75-89. [KEK library link]