We prove that static, spherically symmetric, asymptotically flat, regular solutions of the Einstein-Yang-Mills equations are unstable for arbitrary gauge groups, at least for the “generic” case. This conclusion is derived without explicit knowledge of the possible equilibrium solutions.
1 Introduction

In several recent papers [1, 2, 3, 4] we have studied important aspects of the Einstein-Yang-Mills (EYM) system for arbitrary gauge groups. In particular, we investigated the classification and properties of spherically symmetric EYM solitons (magnetic structure, Chern-Simons numbers) and a generalization of the Birkhoff theorem for the non-Abelian case. We also worked out the generalization of the first law of black hole physics (Bardeen-Carter-Hawking formula), allowing for additional Higgs and dilaton fields [5, 6]. For other studies of these and related topics we refer to [7, 8, 9].

In the present paper we prove that static, spherically symmetric, asymptotically flat, regular solutions of the EYM equations for any gauge group are unstable, at least in the “generic” case (defined below). In a recent letter [10] we have already sketched how we arrived at this result. Here, we present full details of the proof and discuss also some further mathematical issues involved.

On physical grounds, this instability was expected, because we had shown earlier that the Bartnik-McKinnon solutions [11] for the gauge group SU(2), as well as the related black hole solutions [2, 3, 4] are unstable [13, 16, 17, 18]. A mathematical proof of this expectation presents, however, quite a challenge, since one can not rely on any knowledge of the possible solutions (apart from regularity and boundary conditions).

Our strategy is based on the study of the pulsation equations describing linear radial perturbations of the equilibrium solutions and involves the following main steps: First, we show that the frequency spectrum of a class of radial perturbations is determined by a coupled system of radial “Schrödinger equations”. Eigenstates with negative eigenvalues correspond to exponentially growing modes. Using the variational principle for the ground state it is then proven that there always exist unstable modes (at least for “generic” solitons).

There is, unfortunately, no direct way to apply our method to black holes, because of problems related to the boundary conditions at the horizon. We have, however, recently used a similar procedure to establish also the instability of gravitating regular sphaleron solutions of the SU(2) Einstein-Yang-Mills-Higgs system with a SU(2) Higgs doublet [19], which have been constructed numerically in [20].

The paper is organized as follows: In Sec. 2 we recall some basic facts
and equations of our previous work [2, 4] which will be needed in the present analysis. In Sec. 3 we then derive the linearized perturbation equations for solitons (and black holes) and bring them into a convenient partially decoupled form. The resulting eigenvalue problem is discussed in Sec. 4, and in Sec. 5 we show the existence of unstable modes.

2 Spherically symmetric EYM fields

We begin with a convenient description of gauge fields with spherical symmetry (for derivations see [2]). Let us fix a maximal torus $T$ of the gauge group $G$ with the corresponding integral lattice $I$ ($= \text{kernel of the exponential map restricted to the Lie algebra } LT \text{ of the torus } T$). In addition, we choose a basis $S$ of the root system $R$ of real roots. The corresponding fundamental Weyl chamber

\begin{equation}
K(S) = \{ H \in LT \mid \alpha(H) > 0 \text{ for all } \alpha \in S \}
\end{equation}

plays an important role in what follows. We have shown in [2] that to a given spherically symmetric gauge field configuration, there belongs in a natural way a canonical element $H_\lambda \in I \cap K(S)$ which characterizes the corresponding principal bundle $P(M,G)$ over the spacetime manifold $M$ admitting an SU(2) action. If the solution is regular at the origin, $H_\lambda$ lies in a small finite subset of $I \cap K(S)$, which we have described in [4]. In much of our discussion we exclude (for technical reasons) the possibility that $H_\lambda$ lies on the boundary of the fundamental Weyl chamber. The term generic always refers to fields for which the classifying element $H_\lambda$ is contained in the open Weyl chamber $K(S)$.

The SU(2) action on $P(M,G)$ by bundle automorphisms induces an action on the base manifold $M$. An SU(2)-invariant connection in $P(M,G)$ defines a connection in each subbundle over a single orbit of this induced action, which by Wangs theorem is described by a linear map $\Lambda: \text{LSU}(2) \to LG$, depending smoothly on the orbit and satisfying

\begin{equation}
\Lambda_1 = [\Lambda_2, \Lambda_3], \quad \Lambda_2 = [\Lambda_3, \Lambda_1], \quad \Lambda_3 = -H_\lambda/4\pi.
\end{equation}

Here, $\Lambda_k := \Lambda(\tau_k)$ and $2i\tau_k$ are the Pauli matrices. These equations imply
that $\Lambda_+ := \Lambda_1 + i\Lambda_2$ lies in the following direct sum of root spaces $L_\alpha$ of $LG_C$:

$$
\Lambda_+ \in \bigoplus_{\alpha \in \Sigma} L_\alpha ,
$$

(3)

$$
\Sigma := \{ \alpha \in R_+ \mid \alpha(H_\lambda) = 2 \} .
$$

$R_+$ denotes the set of positive roots in $R$ (relative to the basis S). In the
generic case $\Sigma$ turns out to be a basis of a root system contained in $R$. (This
is proven in Appendix A of Ref. [4].)

The $LG$-valued functions on the orbit space determine part of the con-
nnection on $P(M,G)$. Before giving a parametrization of the YM fields in a
convenient gauge, we fix our conventions in parametrizing the Lorentz metric
g on $M$ which is, of course, assumed to be invariant under the induced SU(2)
action. We use standard Schwarzschild-like coordinates and set

$$
g = -NS^2 dt^2 + N^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) ,
$$

(4)

where the metric functions $N$ and $S$ depend only on $r$ and $t$. We use also
the usual mass fraction $m(r, t)$, defined by $N =: 1 - 2m/r$.

In Ref. [2] it is shown that there exists always a (local) gauge such that
the gauge potential $A$ takes the form

$$
A = \tilde{A} + \hat{A} ,
$$

(5)

with

$$
\tilde{A} = \Lambda_2 d\vartheta + (\Lambda_3 \cos \vartheta - \Lambda_1 \sin \vartheta) d\phi
$$

(6)

and

$$
\hat{A} = NS A dt + B dr ,
$$

(7)

where $A$ and $B$ commute with $H_\lambda$ (i.e. with $\Lambda_3$). If $H_\lambda$ is generic its central-
izer is the infinitesimal torus $LT$. Hence, $A$ and $B$ are $LT$-valued and $\tilde{A}$ is thus Abelian.

For the example of the gauge group SU(2), $H_\lambda$ is an integer multiple of
$4\pi \tau_3$: $H_\lambda = 4\pi k \tau_3$ with $k \in Z$, and the only solutions of (3) are $\Lambda_1 = \Lambda_2 = 0,$
$\Lambda_3 = k \tau_3$, or

$$
\Lambda_1 = w \tau_1 + \tilde{w} \tau_2 , \quad \Lambda_2 = \mp \tilde{w} \tau_1 \pm w \tau_2 , \quad \Lambda_3 = \pm \tau_3 .
$$

(8)

We introduce some further notation which is used below. A suitably
normalized Ad($G$)-invariant scalar product on $LG$ will be denoted by $\langle \cdot , \cdot \rangle$. 

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We use the same symbol for the hermetian extension to $LG_C$ (antilinear in the first argument), and $| \cdot |$ means the corresponding norm. Note that the original $\text{Ad}(G)$-invariance extends on $LG_C$ to
\[
\langle X, [Z,Y] \rangle + \langle [c(Z), X], Y \rangle = 0,
\]
where $c$ is the conjugation in $LG_C$.

If we insert the parametrizations (4)-(7) into the EYM equations, we obtain a system of partial differential equations for the metric functions $N, S$ and the YM amplitudes $\Lambda_{\pm}, A, B$. For our instability proof it will suffice to write them in the temporal gauge $A = 0$. Specializing the results of [2] (and using slightly different notation) they read as follows:

The Einstein equations give two constraint equations for the $r$ derivative (denoted by a dash) and the $t$ derivative (denoted by a dot) of $m$
\[
m' = \frac{\kappa}{2} \left\{ NG + r^2 p_\theta \right\}, \quad \dot{m} = \frac{\kappa}{2} NH,
\]
($\kappa := 8\pi G$), and the ($rr$)-equation reduces to
\[
\frac{S'}{S} = \frac{\kappa}{r} G,
\]
where
\[
G = \frac{1}{2} \left\{ (NS)^{-2} |\dot{\Lambda}_+|^2 + |\Lambda'_+ + [B, \Lambda_+]|^2 \right\},
\]
\[
H = \text{Re} \langle \dot{\Lambda}_+, \Lambda'_+ + [B, \Lambda_+] \rangle,
\]
\[
p_\theta = \frac{1}{2r^4} \left\{ |\tilde{F}|^2 + |\hat{F}|^2 \right\}
\]
with
\[
\tilde{F} = \frac{r^2}{S} \dot{B}, \quad \hat{F} = \frac{i}{2} [\Lambda_+, \Lambda_-] - \Lambda_3.
\]
The YM equations decompose into
\[
\frac{2}{NS} \left( \frac{r^2}{S} \dot{B} \right)' + \left[ \Lambda_+, \Lambda'_- + [B, \Lambda_-] \right] + \left[ \Lambda_-, \Lambda'_+ + [B, \Lambda_+] \right] = 0,
\]
The last equation is the Gauss constraint. For the generic case the term proportional to $[\mathcal{B}, \dot{\mathcal{B}}]$ in (18) vanishes.

For static solutions all time derivatives disappear and the basic equations simplify considerably. (For the Bartnik-McKinnon solutions $\Lambda$ is of the form (8) with $\tilde{w} = 0$, $\Lambda_3 = \tau_3$ and $\tilde{A} = 0$ in (5).)

### 3 Perturbation equations

In this section we study time-dependent perturbations of a given static, asymptotically flat solution of the coupled EYM equations (10), (11), (16)-(18). Regular solutions are of purely magnetic type ($\mathcal{A} = 0$ in (5)) with vanishing YM charge. Unfortunately, this is not yet rigorously proven under satisfactory weak fall-off conditions, but there is strong evidence for this (see [4, 21] for partial results.) The perturbation equations which we shall derive hold also for black holes, if these are assumed to be of purely magnetic type.

From now on the symbols $\Lambda_{\pm}$, $N$, $S$, etc. refer to the equilibrium solution and time-dependent perturbations are denoted by $\delta \Lambda_{\pm}$, $\delta \mathcal{B}$, etc.. All basic equations are linearized about the equilibrium solution.

First, we linearize the right hand sides of the Einstein equations (14) and (17). Since $\mathcal{B}$ and $\dot{\Lambda}_{\pm}$ vanish for the equilibrium solution, the first order variation of the source $G$ is

$$
\frac{1}{S}\left(\frac{1}{NS} \dot{\Lambda}_+ \right)' - \frac{1}{S} \left(\left(NS \{ \Lambda'_+ + [\mathcal{B}, \Lambda_+] \} \right) \right)' - N \left\{ [\mathcal{B}, \Lambda'_+] + [\mathcal{B}, [\mathcal{B}, \Lambda_+]] \right\} + \frac{i}{r^2} [\hat{\mathcal{F}}, \Lambda_+] = 0 ,
$$

(17)

$$
2 \left(\frac{r^2}{S} \dot{\mathcal{B}} \right)' + 2 \frac{r^2}{S} [\mathcal{B}, \dot{\mathcal{B}}] + \frac{1}{NS} \left\{ [\Lambda_+, \dot{\Lambda}_-] + [\Lambda_-, \dot{\Lambda}_+] \right\} = 0 .
$$

(18)

Here, the last term vanishes, because the properties of the scalar product mentioned earlier (notably (9)) imply that

$$
-2 \text{Re} \left\langle \Lambda'_+, [\Lambda_+, \delta \mathcal{B}] \right\rangle = \left\langle [\Lambda_+, \Lambda'_-] + [\Lambda_-, \Lambda'_+] , \delta \mathcal{B} \right\rangle ,
$$

(20)
and the YM equation (16) for the equilibrium solution shows that

\[ [\Lambda_+, \Lambda'_-] + [\Lambda_-, \Lambda'_+] = 0. \tag{21} \]

Thus

\[ \delta G = \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle. \tag{22} \]

The only first order variation for \( p_\theta \) comes from \( \delta |\hat{\mathcal{F}}|^2 = 2 \langle \hat{\mathcal{F}}, \delta \hat{\mathcal{F}} \rangle \), with (see eq. (15))

\[ \delta \hat{\mathcal{F}} = \frac{i}{2} [\Lambda_+, \delta \Lambda_+] - \frac{i}{2} [\Lambda_-, \delta \Lambda_+], \tag{23} \]

giving

\[ \delta p_\theta = \frac{1}{r^4} \Re \langle i [\hat{\mathcal{F}}, \Lambda_+] \rangle, \tag{24} \]

Now we can work out the variation of the first Einstein equation in (10). With (22), (24) and (11) for the equilibrium solution, we find

\[ \delta m' = - \frac{S'}{S} \delta m + \frac{\kappa}{2} \left\{ N \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle + \Re \langle \frac{i}{r^2} [\hat{\mathcal{F}}, \Lambda_+] \rangle \right\}. \tag{25} \]

For the commutator in the last term we use the unperturbed YM equation (17), i.e.

\[ \frac{i}{r^2} [\hat{\mathcal{F}}, \Lambda_+] = N \frac{S'}{S} \Lambda'_+ + N' \Lambda'_+ + N \Lambda''_+, \tag{26} \]

whence

\[ \delta m' = - \frac{S'}{S} \delta m + \frac{S'}{S} \left\{ \frac{\kappa}{2} N \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle \right\} + \left\{ \frac{\kappa}{2} N \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle \right\} \tag{27} \]

or

\[ (\delta m S)' = \left\{ \frac{\kappa}{2} N S \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle \right\} \tag{28} \]

Therefore, \( \delta m \) must be of the form

\[ \delta m = \frac{\kappa}{2} N \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle + \frac{f(t)}{S}, \tag{29} \]

where \( f(t) \) is a function of \( t \) alone. This function is determined by considering the variation of the second Einstein equation in (10), which reads

\[ \delta \dot{m} = \frac{\kappa}{2} N \Re \langle \Lambda'_+, \delta \Lambda'_+ \rangle. \tag{30} \]
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Thus we have also
\[
\delta m = \frac{\kappa}{2} N \text{Re} \left\langle \Lambda'_+, \delta \Lambda_+ \right\rangle + g(r),
\]  
(31)

with a function \(g(r)\) of \(r\) alone. By comparing (29) and (31) we arrive at the remarkably simple result
\[
\delta m = \frac{\kappa}{2} N \text{Re} \left\langle \Lambda'_+, \delta \Lambda_+ \right\rangle,
\]  
(32)

which generalizes an observation already made in [15].

The variation of the Einstein equation (11) is immediately obtained with (22)
\[
\delta \left( \frac{S'}{S} \right) = \frac{\kappa}{r} N \text{Re} \left\langle \Lambda'_+, \delta \Lambda'_+ \right\rangle.
\]  
(33)

Before also linerizing the YM equations we introduce a decomposition of \(\Lambda_+\) and \(\delta \Lambda_+\) into “real” and “imaginary” parts. For this we introduce (for \(\alpha \in \Sigma\)) a basis \(e_\alpha\) of the root spaces \(L_\alpha\) in the direct sum (3) with respect to which we expand the unperturbed \(\Lambda_+\) as well as its perturbation \(\delta \Lambda_+\),
\[
\Lambda_+ = \sum_{\alpha \in \Sigma} w^\alpha e_\alpha, \quad \delta \Lambda_+ = \sum_{\alpha \in \Sigma} \delta w^\alpha e_\alpha.
\]  
(34)

Then we have
\[
\delta \Lambda_\pm = \delta X_\pm \pm i \delta Y_\pm
\]  
(35)

with
\[
\delta X_+ = \sum_{\alpha \in \Sigma} \text{Re} (\delta w^\alpha) e_\alpha, \quad \delta X_- = \sum_{\alpha \in \Sigma} \text{Im} (\delta w^\alpha) e_\alpha
\]  
(36)

and the corresponding expansion for \(\delta X_-\) and \(\delta Y_-\) with \(e_\alpha\) replaced by \(c(e_\alpha)\) \(\in \ L_{-\alpha}\), because \(\delta \Lambda_- = c(\delta \Lambda_+)\) and thus
\[
\delta X_- = c(\delta X_+), \quad \delta Y_- = c(\delta Y_+).
\]  
(37)

We shall call \(\delta X_\pm, \delta Y_\pm\) the real and imaginary parts of the perturbations \(\delta \Lambda_\pm\). It was shown in [3] that the unperturbed \(\Lambda_+\) can be chosen to have only a real part.
This decomposition will lead to a significant decoupling of the perturbation equations. Note in particular, that the variations $\delta m$ and $\delta p_\theta$ in (24) and (32) depend only on the real part $\delta X_+^-$:

\[
\delta m = \frac{\kappa}{2}N\langle \Lambda'_+, \delta X_+ \rangle, \tag{38}
\]

\[
\delta p_\theta = \frac{1}{r^4}\langle i[\hat{F}; \Lambda_+], \delta X_+ \rangle. \tag{39}
\]

We consider now the first variation of the YM equation (17). Its decomposition into real and imaginary parts yields

\[
-\frac{1}{NS^2} \delta \ddot{X}_+ = -N\delta X_+'' - \frac{(NS)'}{S}\delta X_+ - i\frac{1}{r^2}[\Lambda_+, \delta \hat{F}] + i\frac{1}{r^2}[\hat{F}, \delta \Lambda_+] - \delta N \Lambda_+'' - \delta \left( \frac{(NS)'}{S} \right) \Lambda_+ \tag{40}
\]

and

\[
-\frac{1}{NS^2} \delta \ddot{Y}_+ = -N\left\{ \delta Y_+'' + i[\Lambda_+, \delta \mathcal{B}]' + i[\Lambda'_+, \delta \mathcal{B}] \right\} \tag{41}
\]

\[
= \frac{(NS)'}{S}\left\{ \delta Y_+'' + i[\Lambda_+, \delta \mathcal{B}] \right\} + i\frac{1}{r^2}[\hat{F}, \delta Y_+].
\]

The third term on the right hand side of (40) is indeed real and can be written, using (22), as

\[
i\frac{1}{r^2}[\Lambda_+, \delta \hat{F}] = \frac{1}{r^2} \text{ad} (\Lambda_+) \text{ad} (\Lambda_-) \delta X_+. \tag{42}
\]

Equation (41) can be simplified further. From (38) and the equilibrium equation (26) we deduce

\[
-\delta N \Lambda_+'' = \frac{2}{r} \delta m \Lambda_+''
\]

\[
= \kappa N \text{Re} \langle \Lambda'_+, \delta X_+ \rangle \Lambda_+''
\]

\[
= \kappa \text{Re} \langle \Lambda'_+, \delta X_+ \rangle \left\{ -\frac{(NS)'}{S} \Lambda_+ + \frac{i}{r^2}[\hat{F}, \Lambda_+] \right\},
\]
and the Einstein equations (10), (11) give
\[ -\delta \frac{(NS)'}{S} = -\frac{2}{r^2}\delta m + \kappa r \delta \rho \theta . \] (43)

If we use here (38) and (39) we see that the last two terms in (40) can be expressed as follows:
\[ -\delta N\Lambda'' - \delta \frac{(NS)'}{S} = \frac{1}{NS^2} \left\{ -(p_\ast \Lambda_+) \frac{\kappa}{r} \left( \frac{(NS)'}{NS} + \frac{1}{r} \right) \langle p_\ast \Lambda_+, \delta X_+ \rangle + (p_\ast \Lambda_+) \frac{\kappa S}{r^3} \langle [\hat{F}, \Lambda_+], \delta X_+ \rangle + [\hat{F}, \Lambda_+] \frac{\kappa S}{r^3} \langle p_\ast \Lambda_+, \delta X_+ \rangle \right\} . \] (44)

where we have introduced the differential operator
\[ p_\ast = -i NS \frac{\partial}{\partial r} . \] (45)

Inserting these expressions into (40) gives the following pulsation equation for the real amplitude \( \delta X_+ \) of the YM field
\[ \delta \ddot{X}_+ + U_{XX} \delta X_+ = 0 , \] (46)

where the operator \( U_{XX} \) is given by
\[ U_{XX} = p_\ast^2 + \frac{NS^2}{r^2} \text{ad}(i\hat{F}) - \frac{1}{NR^2} \text{ad}(NS\Lambda_+) \text{ad}(NS\Lambda_-) \]
\[ - (p_\ast \Lambda_+) \frac{\kappa}{r} \left( \frac{(NS)'}{NS} + \frac{1}{r} \right) \langle p_\ast \Lambda_+, \cdot \rangle \]
\[ + (p_\ast \Lambda_+) \frac{\kappa S}{r^3} \langle [\hat{F}, \Lambda_+], \cdot \rangle + [\hat{F}, \Lambda_+] \frac{\kappa S}{r^3} \langle p_\ast \Lambda_+, \cdot \rangle . \] (47)

It is remarkable that the perturbations \( \delta Y_\pm \) and \( \delta B \) do not appear in (46) and that the back reaction of gravitation on \( \delta X_+ \) can be described by an effective potential (last three terms in (47)).

Equation (41) can easily be brought into the form
\[ \delta \ddot{Y}_+ + U_{YY} \delta Y_+ + U_{YB} \delta B = 0 , \] (48)
where

\[ U_{YY} = p_*^2 + \frac{NS^2}{r^2}\text{ad}(i\tilde{F}), \]  
(49)

\[ U_{YB} = p_*\text{ad}(NS\Lambda_+) + \text{ad}(NS p_*\Lambda_+). \]  
(50)

We have thus achieved a partial decoupling, because neither \( \delta X_+ \), nor the metric perturbations appear in (48).

We proceed with the linearization of the YM equation (16). The variation of the last two terms is

\[ \left[ \Lambda_+ - \Lambda_-, \delta B \right] + \left[ \Lambda_+, \delta \Lambda_- \right] - \left[ \Lambda_-', \delta \Lambda_+ \right] + \text{conjugate}, \]  
(51)

which leads (with \( \delta \Lambda_\pm = \delta X_\pm \pm i\delta Y_\pm \)) to

\[ -\left[ \Lambda_+, \left[ \Lambda_-, \delta B \right] \right] + i\left[ \Lambda_+, \delta Y_- \right] + i\left[ \Lambda_-', \delta Y_+ \right] \]

\[ + \left\{ \left[ \Lambda_+, \delta X_- \right] - \left[ \Lambda_-', \delta X_+ \right] \right\} + \text{conjugate}. \]

Here, the terms in the first curly bracket are in \( LT \), while those in the second are in \( iLT \). The latter are compensated by their conjugates and we find

\[ Nr^2\delta \tilde{B}_+ + U_{BB} \delta B_+ + U_{BY} \delta Y_+ = 0, \]  
(52)

with

\[ U_{BB} = -\text{ad}(NS\Lambda_+)\text{ad}(NS\Lambda_-), \]  
(53)

\[ U_{BY} = -\text{ad}(NS\Lambda_-)p_* + \text{ad}(NS p_*\Lambda_-) \]  
(54)

At this point we collect the results obtained so far as follows: Let

\[ \delta \Psi = \begin{pmatrix} \delta Y_+ \\ \delta B \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & Nr^2 \end{pmatrix}, \]  
(55)

then (18) and (52) can be written as a \( 2 \times 2 \) matrix equation

\[ T\ddot{\delta \Psi} + U\delta \Psi = 0, \]  
(56)
with

\[ U = \begin{pmatrix} U_{YY} & U_{YG} \\ U_{BY} & U_{BG} \end{pmatrix}. \quad (57) \]

The operators in this matrix are given in eqs. \((19), (50), (53)\) and \((54)\).

The perturbation equations \((46)\) and \((56)\) do not include the Gauss constraint \((18)\), whose linearization is easily found to be

\[
\frac{\partial}{\partial t} \left\{ p_\ast \left( \frac{r^2}{S} \delta B \right) - [\Lambda_+, \delta Y_-] \right\} = 0.
\quad (58)
\]

The role of this constraint will be discussed later.

In concluding this section we emphasize once more, that the perturbation equations hold also for black holes, if these are assumed to be of purely magnetic type. In our further discussion we will, however, consider only perturbations of uncharged regular solutions.

### 4 The eigenvalue problem

It is natural to introduce the following scalar product for \(LG_C\)-valued functions on \(R_+\):

\[
\langle \phi | \psi \rangle = \int_0^\infty \langle \phi, \psi \rangle \frac{dr}{NS}, \quad (59)
\]

because the operators \(U_{XX}, U\) and \(T\) are symmetric with respect to this scalar product on a dense domain of \(L^2\)-functions. This can easily be seen by using

\[
\langle \phi | p_\ast \psi \rangle = \langle p_\ast \phi | \psi \rangle \quad (60)
\]

for smooth functions which vanish at the origin, and

\[
\langle \phi | \text{ad}(Z) \psi \rangle = -\langle \text{ad}(c(Z)) \phi | \psi \rangle \quad (61)
\]

for all \(LG_C\)-valued functions \(\phi, \psi, Z\) in \(L^2\) (see \((3)\)).

We specialize now to harmonic perturbations

\[
\delta X_+(r, t) = \xi(r) e^{-i\omega t}, \quad \delta \Psi(r, t) = \delta \Phi(r) e^{-i\omega t}, \quad (62)
\]

whose frequencies satisfy the eigenvalue equations

\[
U_{XX} \xi = \omega^2 \xi, \quad (63)
\]
and
\[ U \delta \Phi = \omega^2 T \delta \Phi. \] (64)

It should be remarked at this point that (47) and (63) reduce for the Bartnik-McKinnon solution [11] to the eigenvalue problem derived in Ref. [15], where it was shown that this has exactly one unstable mode. (A similar instability for the “colored” black hole was found in [16].)

Let us now turn to the role of the linearized Gauss constraint (58) in conjunction with the eigenvalue problem (64). We show first that a variation \( \delta \Phi \) is orthogonal with respect to the scalar product defined by (65),
\[ \langle \cdot | \cdot \rangle_T = \langle \cdot | T | \cdot \rangle, \] (65)
to all gauge variations
\[ \delta \Phi_{\text{gauge}} = \begin{pmatrix} i [\chi, \Lambda+] \\ \chi' \end{pmatrix}, \] (66)
if and only if the curly bracket in (58) vanishes. Note that these gauge variations arise if (5) is subjected to the gauge transformation \( g = \exp(\epsilon \chi) \), because (6) and (7) show that this induces the infinitesimal transformation
\[ \Lambda_+ \rightarrow \Lambda_+ - [\chi, \Lambda_+], \quad B \rightarrow B + \chi'. \] (67)

To prove the statement just made, we compute
\[ \langle \delta \Phi | T | \delta \Phi_{\text{gauge}} \rangle = \int_0^\infty \left\{ \langle \delta Y_+, -i [\Lambda_+, \chi] \rangle + \langle \delta B, N r^2 \chi' \rangle \right\} \frac{dr}{NS}, \]
\[ = -\int_0^\infty \left\langle \left( \frac{r^2}{S} \delta B \right)' + \frac{i}{NS} [\Lambda_-, \delta Y_+] \right\rangle \chi \, dr. \] (68)

Here we have used (3) and made a partial integration, dropping a boundary term. This is allowed if \( \chi \) is regular at the origin and vanishes sufficiently fast at infinity. Since \( \chi \) is otherwise arbitrary and \( i [\Lambda_-, \delta Y_+] = -i [\Lambda_+, \delta Y_-] \), eq. (68) implies our claim.

In the next section we will show that the eigenvalue equation (64) has at least one mode \( \delta \Phi \) with \( \omega^2 < 0 \). Such a mode is orthogonal with respect to the scalar product (63) to any zero mode of (64) and thus in particular
to $\delta \Phi_{\text{gauge}}$ in (66). (This follows since different eigenmodes of (64) are orthogonal with respect to the scalar product (65), because $U$ and $T$ are symmetric with respect to (59).) Hence, we can conclude that the Gauss constraint is automatically satisfied.

From (64) we obtain

$$\omega^2 = \frac{\langle \delta \Phi | U | \delta \Phi \rangle}{\langle \delta \Phi | T | \delta \Phi \rangle}, \quad (69)$$

and for the frequency $\omega_0$ of the fundamental mode we have the minimum principle

$$\omega_0^2 = \min_{\delta \Phi} \frac{\langle \delta \Phi | U | \delta \Phi \rangle}{\langle \delta \Phi | T | \delta \Phi \rangle}. \quad (70)$$

We do not discuss here the precise mathematical nature of the eigenvalue problem (domains of definition, essential selfadjointness, etc.), because the functional analytic aspects are very similar to other well-studied eigenvalue problems.

5 Instability of generic EYM solitons

We are now ready to establish the main point of this paper:

For a given regular solution with $\Lambda_+ = \sum_{\alpha \in \Sigma} w^\alpha e_\alpha$ we shall construct a one-parameter family of field configurations $\Lambda_{(\chi)}^+, B_{(\chi)}$ such that the variational expressions (70) for $\delta \Lambda_\pm = (d \Lambda_{(\chi)}^+ / d \chi)_{\chi=0}$, $\delta B = (d B_{(\chi)} / d \chi)_{\chi=0}$ is negative. This family is chosen of the following form:

$$\Lambda_{(\chi)}^+ = \text{Ad}(\exp(-\chi Z)) \left\{ \Lambda_+ \cos(\chi) + i \Lambda_+ (\infty) \sin(\chi) \right\}, \quad (71)$$

$$B_{(\chi)} = \chi Z', \quad (72)$$

where $Z$ is an LT-valued function of $r$ with the properties

$$\lim_{r \to 0, \infty} [Z, \Lambda_+] = i \Lambda_+ (\infty), \quad \text{supp} \ Z' \subseteq [1 - \epsilon, 1 + \epsilon] \quad (73)$$

for an $\epsilon > 0$. The existence of such a function can be seen as follows: Let $\{h_\alpha\}_{\alpha \in \Sigma}$ be the dual basis of $2\pi \Sigma$ and put

$$Z = \sum_{\alpha \in \Sigma} Z^\alpha h_\alpha, \quad Z^\alpha = \begin{cases} w^\alpha(\infty)/w^\alpha(0) & \text{for } r < 1 - \epsilon, \\ 1 & \text{for } r > 1 + \epsilon. \end{cases} \quad (74)$$
It is easy to verify that both conditions in (73) are satisfied. (In Appendix A of ref. [4] we have shown that \( w^\alpha(0) \neq 0 \) for all \( \alpha \in \Sigma \).)

We note a few properties of the family (71), (72). The equilibrium solution is clearly obtained for \( \chi = 0 \). Applying a gauge transformation with \( g = \exp(-\chi Z) \) we obtain

\[
\Lambda_{(\chi)+} \to \Lambda_+ \cos(\chi) + i \Lambda_+(\infty) \sin(\chi), \quad B_{(\chi)} \to 0.
\]

The first variations of (71) and (72) are

\[
\delta \Lambda_+ = -[Z, \Lambda+] + i \Lambda_+(\infty), \quad \delta B = Z',
\]

and these satisfy by construction the desired boundary conditions

\[
\lim_{r \to 0, \infty} \delta \Lambda_+ = 0, \quad \lim_{r \to 0, \infty} \delta B = 0.
\]

(\( \delta B \) vanishes even outside \( [1 - \epsilon, 1 + \epsilon] \).) Finally, \( \delta \Lambda_+ \) has only an imaginary component

\[
\delta Y_+ = -i \delta \Lambda_+ = i [Z, \Lambda+] + \Lambda_+(\infty)
\]

and thus by (70)

\[
\omega_0^2 \leq \frac{\langle \delta \Phi | U | \delta \Phi \rangle}{\langle \delta \Phi | T | \delta \Phi \rangle}.
\]

with \( \delta \Phi = (\delta Y_+, \delta B) \) given by (78) and the second eq. in (74).

This judicious choice of trial functions fulfills our goal: The denominator in (79) is finite and the numerator turns out to be strictly negative!

The first of these two points is simple. Since \( \delta B \) in (76) has compact support, we have to check only whether

\[
\int_0^\infty \langle \delta Y_+, \delta Y_+ \rangle \frac{dr}{NS} < \infty.
\]

By construction, \( i \delta Y_+ = \Lambda_+(r) - \Lambda_+(\infty) \) for \( r > 1 + \epsilon \). Since \( N \) and \( S \) both approach 1 at infinity, the integral is finite if \( \Lambda_+(r) - \Lambda_+(\infty) \) is assumed to converge to zero faster than \( r^{-1/2} \).

The calculation of the numerator in (79) is somewhat tedious. Considerable simplifications occur by separating a gauge mode in \( \delta \Phi \):

\[
\delta \Phi = \delta \Phi_{\text{gauge}} + \begin{pmatrix} \Lambda_+(\infty) \\ 0 \end{pmatrix}, \quad \delta \Phi_{\text{gauge}} = \begin{pmatrix} i [Z, \Lambda_] \\ Z' \end{pmatrix}.
\]
Clearly $U\delta\Phi_{\text{gauge}} = 0$, and thus (49) and (54) give

$$U\delta\Phi = \begin{pmatrix} U_{YY}\Lambda_+\left(\infty\right) \\ U_{BY}\Lambda_+\left(\infty\right) \end{pmatrix} = \frac{NS^2}{r^2} \begin{pmatrix} i[\hat{F},\Lambda_+\left(\infty\right)] \\ -inr^2[\Lambda'_-,\Lambda_+\left(\infty\right)] \end{pmatrix}. \quad (82)$$

From (81) and (82) we obtain

$$\langle \delta\Phi | U | \delta\Phi \rangle = \int \langle i[Z,\Lambda_+], i[\hat{F},\Lambda_+\left(\infty\right)] \rangle \frac{S}{r^2} \, dr$$

$$+ \int \langle \Lambda_+\left(\infty\right), i[\hat{F},\Lambda_+\left(\infty\right)] \rangle \frac{S}{r^2} \, dr$$

$$+ \int \langle Z', -i[\Lambda'_-,\Lambda_+\left(\infty\right)] \rangle NS \, dr. \quad (83)$$

Let us denote the integrands of the three terms by $J_1$, $J_2$ and $J_3$. We find immediately

$$J_1 = \frac{S}{r^2} \langle \hat{F}, [\Lambda_+\left(\infty\right), [Z,\Lambda_-]] \rangle, \quad (84)$$

$$J_2 = 2 \frac{S}{r^2} \langle \hat{F}, \Lambda_3 \rangle. \quad (85)$$

In the second equation we have used (15) and the vanishing of the YM charge, implying that $\lim_{r \to \infty} \Lambda(r)$ is a homomorphism from $LSU(2)$ to $LG$, whence

$$i[\Lambda_+(\infty),\Lambda_-(\infty)] = 2\Lambda_3. \quad (86)$$

In a next step we show that the first and the last term in (83) compensate each other. For this we rewrite the third term, performing a partial integration and making use of the equilibrium equation (26), as follows

$$\int J_3 \, dr = i \int \langle NS\Lambda'_+, [Z,\Lambda_+(\infty)]' \rangle \, dr$$

$$= i \langle NS\Lambda'_+, [Z,\Lambda_+(\infty)] \rangle \bigg|_0^\infty$$

$$- \int \langle \hat{F}, [\Lambda_-, [Z,\Lambda_+(\infty)]] \rangle \frac{S}{r^2} \, dr.$$

$$\quad (87)$$
The boundary term vanishes and the double commutator in the last term is equal to \([\Lambda_+(\infty), [Z, \Lambda_-]]\), as is seen by using the Jacobi identity and the fact that \([\Lambda_+(\infty), \Lambda_-]\) is in \(iLT\). Comparing this result with (84) shows that there remains indeed only the second term in (83). Thus from (85) we obtain the intermediate result

\[
\langle \delta \Phi \mid U \mid \delta \Phi \rangle = 2 \int \langle \hat{F}, \Lambda_3 \rangle \frac{S}{r^2} dr .
\]  

(88)

Finally, we show that the last expression can be transformed into a form with a definite sign:

\[
2 \int \langle \hat{F}, \Lambda_3 \rangle \frac{S}{r^2} dr = - \int \left\{ NS|\Lambda'_+|^2 + 2 \frac{S}{r^2} |\hat{F}|^2 \right\} dr .
\]  

(89)

In order to see this we perform a partial integration of the first term on the right and use again the YM equation (87):

\[
\int \langle \Lambda'_+, NS\Lambda'_+ \rangle \, dr = \langle \Lambda_+, NS\Lambda'_+ \rangle \bigg|_0^\infty - \int \langle \Lambda_+, i [\hat{F}, \Lambda_+] \rangle S \frac{1}{r^2} \, dr .
\]  

(90)

The boundary term vanishes and the integral combines with the last term in (89) to the left hand side, because we have (see (13))

\[
2|\hat{F}|^2 = \langle \hat{F}, i [\Lambda_+, \Lambda_-] - \Lambda_3 \rangle = \langle \Lambda_+, i [\hat{F}, \Lambda_+] \rangle - \langle \hat{F}, \Lambda_3 \rangle .
\]  

(91)

All together, we have established the crucial result

\[
\langle \delta \Phi \mid U \mid \delta \Phi \rangle = - \int \left\{ NS|\Lambda'_+|^2 + 2 \frac{S}{r^2} |\hat{F}|^2 \right\} dr .
\]  

(92)

This expression is finite and strictly negative. Hence we have shown that \(\omega_0\) is indeed negative, and thus there exist unstable modes of (64). These fulfill, we recall, automatically the linearized Gauss constraint (58).

One can show that the expression (92) is also equal to the second variation of the Schwarzschild mass for the one-parameter family (69), (72). (This is the way we arrived originally at the variation (72)). For a systematic discussion of the relation between variational principles for the spectra of radial pulsations and second variations of the total mass, we refer to [22].
In summary, we have proven the instability of all generic, regular equilibrium solutions. More precisely, we have established the following

**Theorem.** A static, spherically symmetric, asymptotically flat, regular solution of the EYM eqs. (10), (11), (16)-(18) is unstable if the following three conditions are satisfied:

(i) The solution is generic (i.e. the classifying element $H_\lambda = -4\pi \Lambda_3$ lies in the open fundamental Weyl chamber).

(ii) The (magnetic) YM charge vanishes (i.e. $\lim_{r\to\infty} \Lambda(r)$ is a homomorphism from $LSU(2)$ to $LG$).

(iii) Asymptotically $\Lambda_+(r) - \Lambda_+(\infty) \sim r^{-\alpha}$ with $\alpha > 1/2$.

We emphasize again the strong evidence, that the assumptions of the theorem together with condition (i) already imply condition (ii). Also, we would like to stress that we were able to draw this conclusion assuming only weak asymptotic conditions for the solitons. In particular, the fall-off conditon (iii) of the theorem is mild and is certainly fulfilled for the Bartnik-McKinnon solutions, as was shown rigorously in [23]. The same is true for the solutions which have been found numerically by H.P. Künzle for the group $SU(3)$ [21]. (For both types the exponent $\alpha$ is equal to one.)

**Acknowledgments**

We would like to thank Markus Heusler for discussions and comments and to Michael Volkov for suggestive remarks on stability problems. This work was supported by the Swiss National Science Foundation.

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