TREES IN RENORMING THEORY

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INTRODUCTION

Renorming theory is that branch of functional analysis which investigates problems of the form: for which Banach spaces \( X \) does there exist a norm on \( X \), equivalent to the given norm, with some good geometrical property of smoothness or strict convexity? The hope is to give answers in terms of familiar linear topological properties such as reflexivity, separability, separability of the dual space \( X^* \) and so on. The philosophy is well summed up by the title of Section VII.4 of Day’s book [2], “Isomorphisms to improve the norm”. An up-to-date account of the theory is given in the recent and authoritative text of Deville, Godefroy and Zizler [3].

In the next section we shall give full definitions of all the smoothness and convexity conditions to be considered. For the moment, let us recall that a norm \( \| \cdot \| \) on a Banach space \( X \) is said to be Fréchet-smooth (F) if the function \( x \mapsto \| x \| \) is Fréchet-differentiable except at 0. We say that \( \| \cdot \| \) is locally uniformly convex (LUR) if, whenever \( x \) and \( x_n \) \((n \in \mathbb{N})\) are elements of \( X \) such that \( \| x_n \| \to \| x \| \) and \( \| x + x_n \| \to 2\| x \| \) as \( n \to \infty \), we necessarily have \( \| x - x_n \| \to 0 \). One reason for being interested in norms with this property is that if the dual norm \( \| \cdot \|^* \) on \( X^* \) is LUR then the original norm \( \| \cdot \| \) on \( X \) is (F).

A class of Banach spaces which has turned out to be of special importance is the class of strong differentiability spaces, now called Asplund spaces. A Banach space \( X \) is said to be an Asplund space if every continuous convex real-valued function, defined on a convex open subset of \( X \), is Fréchet-differentiable at all points of a dense \( G_δ \) subset of its domain. Work of Stegall, Namioka and Phelps established remarkable equivalent characterizations of this class of space: \( X \) is Asplund if and only if the dual space \( X^* \) has the Radon–Nikodym Property and if and only if every separable subspace of \( X \) has separable dual. In particular, a space \( C_0(L) \), with \( L \) locally compact, is Asplund if and only if \( L \) is scattered. Preiss [16] recently established dense differentiability for Lipschitz functions on Asplund spaces, and Asplund spaces have also proved to be of interest in Optimization Theory.

In the case of a separable space \( X \), there is a very tight connection between renorming and the Asplund Property. Indeed, in this special case the following are all equivalent: separability of the dual space \( X^* \); the existence on \( X \) of an equivalent norm with LUR dual norm; the existence on \( X \) of an equivalent norm which is (F). The question whether there are implications between Fréchet renormability and the Asplund Property for general Banach spaces was posed in [2]. One implication was established by Ekeland and Lebourg [5] in the course of their work on perturbed optimization problems: if \( X \) has an equivalent Fréchet-differentiable norm then \( X \)
is Asplund. That the converse does not hold was the main result of the author’s paper [8].

The example presented in [8] was a Banach space $C_0(L)$ with $L$ a locally compact scattered space of a special kind, namely a tree. From the point of view of renorming theory this Banach space was very bad indeed, admitting neither a strictly convex renorming nor a Gâteaux-differentiable renorming. In order to achieve this degree of bad behaviour, the tree had to be chosen to be very large and the author suggested at the end of [8] that it might be interesting to study renormings of the spaces of continuous functions on trees with a more subtle structure. The present paper is the result of that study. The aim has been, wherever possible, to establish necessary and sufficient conditions, expressed in terms of the combinatorial structure of a tree $\Upsilon$, for the existence of equivalent good norms on the space $C_0(\Upsilon)$. We are then able to give counterexamples to a number of open questions about renormings by considering suitably chosen trees.

As has been found in Logic and General Topology (see for instance [19]), trees are very agreeable objects to work with, offering a diversity of behaviour within a structure that is sufficiently simple to admit precise analysis. Thus we are able to offer fairly satisfactory necessary and sufficient conditions on a tree $\Upsilon$ for the existence of equivalent LUR or strictly convex norms on $C_0(\Upsilon)$ and for norms with the Kadec Property. In particular, we show that for a finitely branching tree $\Upsilon$ the space $C_0(\Upsilon)$ admits a Kadec renorming. Since some finitely branching trees fail the condition for strictly convex renormability, we obtain an example of a Banach space that is Kadec renormable but not strictly convexifiable. Consideration of specially tailored examples enables us to answer the “three-space problem” for strictly convex renorming: there exists a Banach space $X$ with a closed subspace $Y$ such that both $Y$ and the quotient $X/Y$ admit strictly convex norms, while $X$ does not. We also solve a problem about the the property of mid-point locally uniform convexity (MLUR), showing that this does not imply LUR renormability.

In the case of smoothness properties of renormings, we give a necessary and sufficient condition for the existence of a Fréchet-smooth renorming, which turns out (rather surprisingly) to be the same as the condition for LUR renormability. We also show that when $C_0(\Upsilon)$ admits a Fréchet smooth norm it even admits a $C^\infty$ norm. By considering another specially tailored tree we obtain solution to the quotient problem for Fréchet differentiable renorming: there is a Banach space $X$ with a Fréchet differentiable norm and a closed subspace $Y$ such that the quotient space $X/Y$ admits no Fréchet differentiable renorming. We have few results about Gâteaux differentiability, though we do give an example of a space $X$ (as always, of the form $C_0(\Upsilon)$) which is not strictly convexifiable even though it admits an equivalent norm with strictly convex dual norm. This seems to be the first example of a space with a Gâteaux smooth norm which is not strictly convexifiable.

Finally we include some results that are not about renorming theory in the strict sense. The theorem of Ekeland and Lebourg already cited in this introduction was in fact proved with hypothesis weaker than the existence of a Fréchet smooth norm. They needed only a non-trivial Fréchet differentiable function of bounded support, what is usually referred to as a “bump function”. A number of more recent results have been established subject to hypotheses of this type, including the interesting variational results of Deville, Godefroy and Zizler [4]. Using a non-linear version of our technique for the construction of Fréchet differentiable norms, as presented in [13], we show that $C_0(\Upsilon)$ admits a $C^\infty$ bump function for every tree $\Upsilon$, so that
there certainly exist spaces that admit bump functions but no good norms. Finally, using another result from \([13]\), we show that \(C_0(T)\) always admits \(C^\infty\) partitions of unity.

Much of the early work on this paper was done during the author’s stay at the Équipe d’Analyse, Université Paris VI in the academic year 1991–2. Thanks are due to other members of that team, and especially to Gilles Godefroy, for providing a friendly and constantly stimulating working environment. A number of preliminary versions of this work have been circulated since 1991 and a simple account of the counterexamples to the three-space problem and the quotient problem was given in the seminar paper \([12]\). These examples, and the author’s early \((C^1)\) bump function result are presented or cited in \([3]\).

1. Some renorming techniques

The authors of \([3]\) record as their Fact II.2.3 the inequality

\[
\phi(x)^2 + \phi(y)^2 - 2\phi(\frac{1}{2}(x + y))^2 \geq \frac{1}{2}(\phi(x) - \phi(y))^2,
\]

valid when \(\phi\) is a non-negative convex function. It follows that the convergence to 0 of \(\phi(x)^2 + \phi(x_n)^2 - 2\phi(\frac{1}{2}(x + x_n))^2\) is equivalent to the convergence of both \(\phi(f_n)\) and \(\phi(\frac{1}{2}(f_n + f))\) to the limit \(\phi(f)\). This permits a useful equivalent formulation of the definition of local uniform convexity: \(\| \cdot \|\) is LUR if and only if the convergence to 0 of \(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\) implies the norm convergence of \(x_n\) to \(x\). If, at the end of the last sentence, we replace norm with weak we have the definition of a weakly locally uniformly convex (wLUR) norm. A different weakening of LUR leads to the property known as midpoint locally uniform convexity (MLUR): a norm is MLUR if the convergence to zero of \(\|x + h_n\|^2 + \|x - h_n\|^2 - 2\|x\|^2\) (or, equivalently, the convergence to the limit \(\|x\|\) of both \(\|x + h_n\|\) and \(\|x - h_n\|\)) implies the norm convergence of \(h_n\) to 0. Of course, both wLUR and LUR imply strict convexity, a norm being strictly convex if \(\|x\| = \|y\| = \frac{1}{2}\|x + y\|\) implies \(x = y\).

While, as we shall see, it is not strictly speaking a “convexity” condition, it is convenient to mention here the Kadec property: we say that a norm \(\| \cdot \|\) is Kadec if the weak and norm topologies coincide on the unit sphere \(\text{sph} X = \{ x \in X : \|x\| = 1 \}\). For Banach spaces \(X\) not containing \(\ell_1\) (and in particular for spaces \(X = C_0(L)\), with \(L\) scattered) we have the following sequential characterization: \(X\) has the Kadec Property if and only if weakly convergence implies norm convergence for sequences in the unit sphere. It is not hard to show that a LUR norm is Kadec. A striking theorem of Troyanski \([20\text{ or } 3, \text{ IV.3.6}]\) states that if a Banach space admits both a Kadec renorming and a strictly convex renorming then it admits a LUR renorming.

The Kadec Property is closely related to two other notions, in which the linear structure of a Banach space is not involved. One of these is \(\sigma\)-fragmentability, a property introduced by Jayne, Namioka and Rogers \([14]\). We say that the weak topology of a Banach space \(X\) is \(\sigma\)-fragmented by the norm (or, more briefly, that \(X\) is \(\sigma\)-fragmentable) if, for every \(\epsilon > 0\), there is a countable covering \((X_n)_{n \in \mathbb{N}}\) of \(X\) such that, for each \(n\) and each non-empty subset \(Y\) of \(X_n\), there is a non-empty subset of \(Y\) which is relatively open in the weak topology and has norm diameter smaller than \(\epsilon\). Recent work of Namioka and Pol shows that \(\sigma\)-fragmentability of a Banach space \(X\) is characterized by a purely topological property of \(X\) in the
weak topology. It is shown in [14] that a Banach space with a Kadec norm is \( \sigma \)-fragmentable, and it is conjectured that any \( \sigma \)-fragmentable Banach space admits a Kadec renorming. At the end of Section 6 we verify this conjecture for spaces \( C_0(T) \). We shall say that a locally compact space \( L \) has the Namioka Property \( N^* \) if, for every Baire space \( B \) and every function \( \psi : B \to C_0(L) \) which is continuous into the topology \( \tau_p \) of pointwise convergence on \( L \), the set of points of continuity of \( \psi \) into the norm topology is dense in \( B \). If \( L \) is a locally compact scattered space and \( C_0(L) \) is \( \sigma \)-fragmentable then \( L \) has the Namioka Property. (In general, the Namioka Property is implied by \( \sigma \)-fragmentability of the pointwise topology \( \tau_p \), rather than of the weak topology of \( C_0(L) \).)

We now pass from definitions to renorming techniques. The next lemma contains an idea that is at the heart of many LUR renorming proofs. In the following form it may be found in [3].

**Lemma 1.1.** Let \( X, \| \cdot \|_0 \) be a Banach space, let \( I \) be a set and let \( (\phi_i)_{i \in I} \) and \( (\psi_i)_{i \in I} \) be families of non-negative convex functions on \( X \) which are uniformly bounded on bounded subsets of \( X \). For \( f \in X, m \in \mathbb{N} \) and \( i \in I \) define

\[
\phi(f) = \sup_{i \in I} \phi_i(f) \\
\theta_{i,m}(f) = \phi_i(f)^2 + 2^{-m} \psi_i(f)^2 \\
\theta_m(f) = \sup_{i \in I} \theta_{i,m}(f) \\
\theta(f) = \|f\|_0^2 + \sum_{m=1}^{\infty} 2^{-m} \theta_m(f).
\]

If \( f \) and \( f_n \) in \( X \) are such that \( \theta(f) + \theta(f_n) - 2\theta(\frac{1}{2} (f + f_n)) \to 0 \) as \( n \to \infty \), then there exists a sequence \( (i_n) \) in \( I \) such that :

1. \( \phi_{i_n}(f) \), \( \phi_{i_n}(f_n) \) and \( \phi(f_n) \) all converge to \( \phi(f) \) as \( n \to \infty \);
2. \( \psi_{i_n}(f_n)^2 + \psi_{i_n}(f)^2 - 2\psi_{i_n}(\frac{1}{2} (f_n + f))^2 \) tends to 0 as \( n \to \infty \).

Moreover, there is a norm \( \| \cdot \| \) on \( X \), equivalent to \( \| \cdot \|_0 \), such that the above conclusion holds whenever \( 2\|f\|^2 + 2\|f_n\|^2 - \|f + f_n\|^2 \to 0 \).

In Section 6 we shall use a variant of this technique, appropriate to the construction of Kadec norms, rather than LUR norms. The proof is rather easier than that of 1.1.

**Proposition 1.2.** Let \( X \) be a topological space, let \( I \) be a set and let \( \phi_i, \psi_i : X \to \mathbb{R} \) \((i \in I)\) be uniformly bounded families of lower semicontinuous functions. For each \( f \in X \) define

\[
\phi(f) = \sup_{i \in I} \phi_i(f) \\
\theta_m(f) = \sup_{i \in I} [\phi_i(f) + 2^{-m} \psi_i(f)] \\
\theta(f) = \sum_{m=1}^{\infty} 2^{-m} \theta_m(f).
\]

If \( (f_n) \) is a sequence which converges to \( f \) in \( X \) and is such that \( \theta(f_n) \to \theta(f) \) then there exists a sequence \( (i_n) \) in \( I \) such that \( \phi_{i_n}(f) \) and \( \phi_{i_n}(f_n) \) and \( \phi(f_n) \) all converge
as \( n \to \infty \) to the limit \( \phi(f) \) whilst \( \psi_{i_n}(f_n) - \psi_{i_n}(f) \) tends to 0. If \( I \) is equipped with a topology under which it is sequentially compact then we may suppose \((i_n)\) to be a convergent sequence in this topology.

**Proof.** For each \( m \) the function \( \theta_m \) is l.s.c. on \( X \). Since \( f_n \to f \) in \( X \) and \( \theta(f_n) \to \theta(f) \) it must therefore be that \( \theta_m(f_n) \to \theta_m(f) \) for all \( m \). Choose, for each \( m \), an element \( j_m \) of \( I \) and a natural number \( n_m \) such that

\[
\phi_{j_m}(f) + 2^{-m} \psi_{j_m}(f) > \sup_{p \geq n_m} \theta_m(f_p) - 2^{-2m}.
\]

By lower semicontinuity we may suppose \( n_m \) to be chosen so that \( \phi_{j_m}(f_p) > \phi_{j_m}(f) - 2^{-2m} \) and \( \psi_{j_m}(f_p) > \psi_{j_m}(f) - 2^{-m} \) whenever \( p \geq n_m \). We may also assume that \( n_{m+1} > n_m \).

We now let \((m(k))_{k \in \mathbb{N}}\) be any subsequence of the natural numbers, chosen, if we wish, so that \((j_m(k))\) converges with respect to a given sequentially compact topology on \( I \). Define \( i_p = j_{m(k_p)} \) where \( k_p = \max\{k \in \mathbb{N} : n_{m(k)} \leq p\} \). With these definitions, we certainly have \( p \geq n_{m(k_p)} \) so that

\[
\phi_{i_p}(f) + 2^{-m(k_p)} \psi_{i_p}(f) > \sup_{q \geq n_{m(k_p)}} \theta_{m(k_p)}(f_q) - 2^{-2m(k_p)}
\]

\[
\geq \phi_{i_p}(f_p) + 2^{-m(k_p)} \psi_{i_p}(f_p) - 2^{-2m(k_p)}.
\]

Since we also have \( \phi_{i_p}(f_p) > \phi_{i_p}(f) - 2^{-2m(k_p)} \) and \( \psi_{i_p}(f_p) > \psi_{i_p}(f) - 2^{-m(k_p)} \), it must be that

\[
|\phi_{i_p}(f) - \phi_{i_p}(f_p)| < 2^{-2m(k_p)} + 1
\]

\[
|\psi_{i_p}(f) - \psi_{i_p}(f_p)| < 2^{-m(k_p)} + 1.
\]

Thus \( \phi_{i_p}(f) - \phi_{i_p}(f_p) \) and \( \psi_{i_p}(f) - \psi_{i_p}(f_p) \) tend to 0 as \( p \to \infty \). We also have the inequalities

\[
\phi_{i_p}(f) + 2^{-m(k_p)} \sup_i \|\psi_i\|_\infty \geq \phi_{i_p}(f) + 2^{-m(k_p)} \psi_{i_p}(f)
\]

\[
\geq \sup_{q \geq n_{m(k_p)}} \theta_{m(k_p)}(f_q) - 2^{-2m(k_p)}
\]

\[
\geq \limsup_{q \to \infty} \phi(f_q) - 2^{-2m(k_p)}
\]

\[
\geq \liminf_{q \to \infty} \phi(f_q) - 2^{-2m(k_p)}
\]

\[
\geq \phi(f) - 2^{-2m(k_p)}
\]

\[
\geq \phi_{i_p}(f) - 2^{-2m(k_p)},
\]

which yield the convergence of \( \phi_{i_p}(f) \) and \( \phi(f_p) \) to the limit \( \phi(f) \). \( \square \)

Apart from the above Proposition, the main ingredient in Section 6 will be the use of recursion to define norms. There is nothing particularly new in this approach, but it may make Section 6 clearer if we give here some more elementary examples of recursively defined norms. We start with a construction of a LUR renorming of \( C[0,\Omega] \), where \( \Omega \) is an ordinal. The same approach may be used in greater generality to give an alternative construction of LUR norms on certain spaces having projectional resolutions of the identity [3, Proposition VII.1.6]. It should be emphasized that the existence of such norms is well known and that following proposition is included here for the (possible) interest of the method, rather than any originality of the conclusion.
Proposition 1.3. Let $\Omega$ be an ordinal. There is a unique real-valued function $\Phi$ with domain $C[0, \Omega] \times \{ (\alpha, \gamma) : 0 \leq \alpha \leq \gamma \leq \Omega \}$, satisfying the inequality $0 \leq \Phi(f, \alpha, \gamma) \leq \|f\|_{\infty}$ and the identities

$$
\Phi(f, \alpha, \alpha) = |f(\alpha)|
$$

$$
16\Phi(f, \alpha, \gamma)^2 = 4\|f+ [\alpha, \gamma]\|_{\infty}^2 + f(\alpha)^2 + \text{osc} (f+ [\alpha, \gamma])^2
$$

$$+
\sum_{m=1}^{\infty} 2^{-m} \sup_{\alpha \leq \beta < \gamma} \left( (f(\beta + 1) - f(\beta))^2 + 2^{-m}\Phi(f, \alpha, \beta)^2 + 2^{-m}\Phi(f, \beta + 1, \gamma)^2 \right)
$$

for all $f \in C[0, \Omega]$ and $\alpha < \gamma \leq \Omega$. If we define $\|f\|_{\text{ord}} = \Phi(f, 0, \Omega)$, then $\|f\|_{\text{ord}}$ is a locally uniformly convex norm on $C[0, \Omega]$ satisfying $\frac{1}{2}\|f\|_{\infty} \leq \|f\|_{\text{ord}} \leq \|f\|_{\infty}$.

**Proof.** As with many results on recursion, it is convenient to give a proof using a fixed-point theorem. Let $\mathcal{D}$ be the set of all triples $(f, \alpha, \gamma)$ with $f \in C[0, \Omega]$ and $0 \leq \alpha \leq \gamma \leq \Omega$; let $\mathcal{X}$ be the space of all functions from $\mathcal{D}$ into the real interval $[0, 1]$, which are positively homogeneous in $f$. Equip $\mathcal{X}$ with the metric

$$
d(\Phi, \Psi) = \sup \{ |\Phi(f, \alpha, \gamma) - \Psi(f, \alpha, \gamma)| : 0 \leq \alpha \leq \gamma \leq \Omega, \|f\|_{\infty} \leq 1 \}.
$$

Define $T : \mathcal{X} \to \mathcal{X}$ by

$$
(T\Psi)(f, \alpha, \alpha) = |f(\alpha)|
$$

$$
16(T\Psi)(f, \alpha, \gamma)^2 = 4\|f+ [\alpha, \gamma]\|_{\infty}^2 + f(\alpha)^2 + \text{osc} (f+ [\alpha, \gamma])^2
$$

$$+
\sum_{m=1}^{\infty} 2^{-m} \sup_{\alpha \leq \beta < \gamma} \left( (f(\beta + 1) - f(\beta))^2 + 2^{-m}\Psi(f, \alpha, \beta)^2 + 2^{-m}\Psi(f, \beta + 1, \gamma)^2 \right)
$$

The mapping $T$ is a strict contraction on the complete metric space $\mathcal{X}$ and so has a unique fixed point $\Phi$. Evidently, $\|f\|_{\text{ord}} = \Phi(f, 0, \Omega)$ defines a norm on $C[0, \Omega]$ satisfying $\frac{1}{2}\|f\|_{\infty} \leq \|f\|_{\text{ord}} \leq \|f\|_{\infty}$. We have to show that this norm is LUR. Let $f$ be a given element of $C[0, \Omega]$ with $\|f\|_{\infty} \leq 1$ and let $\epsilon$ be a positive real number.

We shall need an easy result about continuous functions on ordinals.

**Fact.** There exists a positive real number $\eta$ such that if $0 \leq \alpha < \beta \leq \Omega$ and $|f(\beta) - f(\beta + 1)| < \eta$ for all $0 \leq \beta \leq \gamma$ then $|f(\alpha) - f(\beta)| < \epsilon$ for all $0 \leq \beta \leq \gamma$.

If $|f(\beta) - f(\beta + 1)| < \frac{\epsilon}{2}$ for all $\beta \in [0, \Omega]$ there is nothing to prove. Otherwise, we start a recursion, choosing $\beta_1$ to be the greatest ordinal with $|f(\beta_1) - f(\Omega)| \geq \frac{\epsilon}{2}$. Subsequently, if $\Omega > \beta_1 > \cdots > \beta_j$ have been defined, then either we have $|f(\beta) - f(\beta_j)| < \frac{1}{2}\epsilon$ for all $0 \leq \beta < \beta_j$, in which case we stop, or else we take $\beta_{j+1}$ to be the greatest ordinal with $0 \leq \beta_{j+1} < \beta_j$ and $|f(\beta_{j+1}) - f(\beta_j)| \geq \frac{\epsilon}{2}$. By well-ordering, the process does stop at some stage, after the definition of $\beta_k$, say. We observe that the oscillation of $f$ on each of the intervals $[0, \beta_k]$, $(\beta_k, \beta_{k-1}]$, \ldots \ldots $(\beta_1, \Omega]$ is smaller than $\epsilon$ and that $f(\beta_j) \neq f(\beta_j + 1)$ for each $j$. We set $\eta = \min\{|f(\beta_j) - f(\beta_j + 1)| : 1 \leq j \leq k\}$. If $\alpha < \gamma$ are such that the interval $[\alpha, \gamma]$ contains no $\beta$ with $|f(\beta) - f(\beta + 1)| \geq \eta$, then $[\alpha, \gamma]$ must be contained in one of the intervals $[0, \beta_k]$, $(\beta_k, \beta_{k-1}]$, \ldots $(\beta, \Omega]$. Thus the oscillation of $f$ on $[\alpha, \gamma]$ is smaller than $\epsilon$, as required.

We now define $m(\alpha, \gamma)$ to be the (finite!) number of $\beta$ such that $0 \leq \beta < \gamma$ and $|f(\beta) - f(\beta + 1)| \geq \eta$. We shall use induction on $m(\alpha, \gamma)$ to establish the following assertion, which will obviously complete the proof of 1.3.
Claim. If \( f_n \in C[0, \Omega] \) are such that \( 2\Phi(f_n, \alpha, \gamma)^2 + 2\Phi(f, \alpha, \gamma)^2 - \Phi(f + f_n, \alpha, \gamma)^2 \to 0 \) as \( n \to \infty \) then \( \limsup \| (f - f_n) \| [\alpha, \gamma]\|_\infty < 4\epsilon. \)

If \( m(\alpha, \gamma) = 0 \) then, by the way in which we chose \( \eta \), the oscillation of \( f \) on \([\alpha, \gamma]\) is smaller than \( \epsilon \), so that \( \text{osc} (f \upharpoonright [\alpha, \gamma]) < \epsilon \). From our hypothesis about \( f_n \) and convexity arguments, we see that \( f_n(\alpha) \) tends to \( f(\alpha) \) and \( \text{osc} (f_n \upharpoonright [\alpha, \gamma]) \) tends to \( \text{osc} (f \upharpoonright [\alpha, \gamma]) \) as \( n \to \infty \). Thus, for all large enough \( n \) we have \( |f(\alpha) - f_n(\alpha)| < \epsilon \) and \( \text{osc} (f_n \upharpoonright [\alpha, \gamma]) < 2\epsilon \), giving \( \| (f - f_n) \| [\alpha, \gamma]\|_\infty < 4\epsilon. \)

We now assume inductively that if \( \beta \) and \( \delta \) are such that \( m(\beta, \delta) < m(\alpha, \gamma) \) and \((g_n)\) is a sequence such that \( 2\Phi(g_n, \beta, \delta)^2 + 2\Phi(f, \beta, \delta)^2 - \Phi(f + g_n, \beta, \delta)^2 \to 0 \), then \( \limsup \| (f - g_n) \| [\beta, \delta]\|_\infty < 4\epsilon. \) If the assertion we are trying to prove is false, then we may, by passing to a subsequence, assume that \( \| (f - f_n) \| [\alpha, \gamma]\|_\infty \geq 4\epsilon \) for all \( n \). We are finally ready to apply Proposition 1.1 with

\[
\phi_\beta(f) = |f(\beta + 1) - f(\beta)|, \\
\psi_\beta(f) = (\Phi(f, \alpha, \beta)^2 + \Phi(f, \beta + 1, \gamma)^2)^{\frac{1}{2}}.
\]

There exists a sequence \((\beta_n)\) such that the conclusions of 1.1 hold, in particular,

\[
|f(\beta_n + 1) - f(\beta_n)| \to \sup_{\alpha \leq \beta \leq \gamma} |f(\beta + 1) - f(\beta)| \\
2\Phi(f_n, \alpha, \beta_n)^2 + 2\Phi(f, \alpha, \beta_n)^2 - \Phi(f + f_n, \alpha, \beta_n)^2 \to 0 \\
2\Phi(f_n, \beta_n, \delta)^2 + 2\Phi(f, \beta_n, \delta)^2 - \Phi(f + f_n, \beta_n, \delta)^2 \to 0.
\]

For all large enough \( n \), it must be that \( \beta_n \) is in the finite set of \( \beta \) with \( |f(\beta) - f(\beta + 1)| \geq \eta \) and, for a suitable subsequence, \( \beta_{n_k} \) will take the same value \( \beta \). Since we have \( m(\alpha, \beta) < m(\alpha, \gamma) \) and \( m(\beta, \gamma) < m(\alpha, \gamma) \) our inductive hypothesis is applicable, giving

\[
\limsup \| (f - f_{n_k} \upharpoonright [\alpha, \beta]\|_\infty < 4\epsilon \\
\limsup \| (f - f_{n_k} \upharpoonright [\beta, \gamma]\|_\infty < 4\epsilon,
\]

whence

\[
\limsup \| (f - f_{n_k} \upharpoonright [\alpha, \gamma]\|_\infty < 4\epsilon,
\]

which completes the proof. \( \square \)

It may amuse the reader to note that the same approach could be used to establish the local uniform convexity of Day’s norm on \( c_0(\Gamma) \). We recall that we define \( \| \cdot \|_{\text{Day}} \) by

\[
\| f \|_{\text{Day}}^2 = \sup \sum_{n=1}^\infty 2^{-n} f(\gamma_n)^2,
\]

where the supremum is taken over all sequences \((\gamma_n)\) of distinct elements of \( \Gamma \). One could also regard \( \| f \|_{\text{Day}} \) as being \( \Phi(f; \Gamma) \), where \( \Phi \) is the unique function defined
on the Cartesian product $c_0(\Gamma) \times \mathcal{P}(\Gamma)$ which satisfies the inequality $0 \leq \Phi(f; \Delta) \leq \|f\|_{\infty}$ as well as the identities

$$\Phi(f; \emptyset) = 0$$

$$\Phi(f; \Delta)^2 = \sum_{m=1}^{\infty} 2^{-m} \sup_{t \in \Delta} \left[ \frac{1}{2} f(t)^2 + (\frac{3}{2})^m \Phi(f; \Delta \setminus \{t\})^2 \right]$$

for $f \in c_0(\Gamma)$ and $\emptyset \neq \Delta \subseteq \Gamma$. Simplifying the proof of 1.3, we see that an argument using 1.1 and proceeding by induction on the number of $\gamma$ for which $|f(\gamma)| \geq \epsilon$ enables to show that $\| \cdot \|_{\text{Day}}$ is LUR.

We now pass to questions of smoothness. A real-valued function $\phi$, defined on an open subset $U$ of a Banach space $X$ is said to be Gâteaux-differentiable at $x \in U$ if there exists an element $\eta$ of the dual space $X^*$ such that, for all $h \in X$, $t^{-1}(\phi(x + th) - \phi(x)) \rightarrow \langle \eta, h \rangle$ as $t \rightarrow 0$. It is usual to write $D\phi(x)$ for this element $\eta$. In the case where $\psi$ is a norm, there is a simple criterion: $\| \cdot \|$ is Gâteaux-differentiable at $x$ if and only if there is a unique $\xi \in X^*$ satisfying $\|\xi\|^* = 1$, $\langle \xi, x \rangle = \|x\|$; moreover, $D\| \cdot \|(x)$ is this element $\xi$. For economy of notation, we shall write $x^*$ for $D\| \cdot \|(x)$. A norm is said to be Gateaux-smooth (or simply smooth) if it Gateaux-differentiable at all points except 0.

By definition, a function $\phi : U \rightarrow \mathbb{R}$ is Fréchet-differentiable at $x$ if it is Gâteaux-differentiable and the convergence of $t^{-1}(\phi(x + th) - \phi(x))$ to $\langle D\phi(x), h \rangle$ as $t \rightarrow 0$ is uniform over $h$ in the unit ball of $X$. In the case of a norm, we have the important criterion of Šmulian [17 or 3, I.1.4]: a norm $\| \cdot \|$ is Fréchet-differentiable at $x$ if and only if it is Gâteaux-differentiable at $x$ and we have $\|\xi_n - x^*\| \rightarrow 0$, whenever $(\xi_n)$ is a sequence in ball $X^*$ with $\langle \xi_n, x \rangle \rightarrow \|x\|$.

For the construction of Fréchet-smooth norms our main method will be one based on Talagrand’s construction [18] for the space $C[0, \Omega]$. When modified as in [10] and [13], this approach even yields norms which are infinitely differentiable (except at 0 of course). Given a locally compact space $L$ and a set $M$, we shall say that a bounded linear operator $c_0(L) \rightarrow c_0(L \times M)$ is a Talagrand operator for $L$ if for every non-zero $f$ in $C_0(L)$, there exist $t \in L$ and $u \in M$ with $\|f(t)\| = \|f\|_{\infty}$ and $(Tf)(t, u) \neq 0$.

**Proposition 1.4** (Theorem 1 of [13]). Let $L$ be a set and let $U(L)$ be the subset of the direct sum $\ell_\infty(L) \oplus c_0(L)$ consisting of all pairs $(f, x)$ such that $\|f\|_{\infty}$ and $\|x\|_{\infty}$ are both strictly less than $\|f\| + \frac{1}{2}\|x\|_{\infty}$. The space $\ell_\infty(L) \oplus c_0(L)$ admits an equivalent norm $\| \cdot \|$ with the following properties:

1. $\| \cdot \|$ is a lattice norm, in the sense that $\|(g, y)\| \leq \|(f, x)\|$ whenever $|g| \leq |f|$ and $|y| \leq |x|$;
2. $\| \cdot \|$ is infinitely differentiable on the open set $U(L)$;
3. locally on $U(L)$, $\|(f, x)\|$ depends on only finitely many non-zero coordinates; that is to say, for each $(f^0, x^0) \in U(L)$ there is a finite $N \subseteq L$ and an open neighbourhood $V$ of $(f^0, x^0)$ in $U(L)$, such that for $(f, x) \in V$ the norm $\|(f, x)\|$ is determined by the values of $f_t$ and $x_t$ with $t \in N$ and such that $f_t \neq 0$, $x_t \neq 0$ for all such $(f, x)$ and $t$.

The following Corollary appears in both [10] and [13].
Corollary 1.5. Let $L$ be a locally compact space which admits a Talagrand operator. Then $C_0(L)$ admits a $C^\infty$ renorming.

Proof. Let $T : C_0(L) \to c_0(L \times M)$ be a Talagrand operator, normalized so that $\|Tf\|_\infty \leq \frac{1}{2}\|f\|_\infty$ for all $f$ and let $S : C_0(L) \to \ell_\infty(L \times M)$ be defined by $(Sf)(t,u) = f(t)$. The pair $(Sf,Tf)$ is in $U(L \times M)$ whenever $f$ is a non-zero element of $C_0(L)$ and so the norm on $C_0(L)$ defined by $\|f\| = \|(Sf,Tf)\|$ is infinitely differentiable except at 0. \qed

2. Some preliminaries about trees

By definition, a tree is a partially ordered set $(\Upsilon, \preceq)$ with the property that for every $t \in \Upsilon$ the set $\{s \in \Upsilon : s \preceq t\}$ is well-ordered by $\preceq$. In any tree, we use normal interval notation, so that, for instance, $(s,u) = \{t \in \Upsilon : s < t \preceq u\}$. For convenience of notation, we introduce two “imaginary” elements, not in $\Upsilon$, denoted $0$ and $\infty$, and having the property that $0 < t < \infty$ for all $t \in \Upsilon$. This allows us to extend our interval notation to include expressions like $(0,t]$ and $[t,\infty)$. Note that, by definition, each $(0,t]$ is well-ordered, but that $[t,\infty)$ need not be. For each $t \in \Upsilon$ there is a unique ordinal $r(t)$ with the same order type as $(0,t)$. Fremlin [6] says that a tree is Hausdorff if, whenever $r(t)$ is a limit ordinal and $(0,t') = (0,t)$, we necessarily have $t = t'$. Such a tree may be equipped with a locally compact, and Hausdorff (!), topology which may be characterized as the coarsest for which all intervals $(0,t]$ are open and closed. We shall consider only trees that are Hausdorff.

We shall be studying norms on the space $C_0(\Upsilon)$ of real-valued functions $f$ on $\Upsilon$, which are continuous for the locally compact topology and are such that, for all $\epsilon > 0$ the set $\{t \in \Upsilon : |f(t)| \geq \epsilon\}$ is compact for that topology. It will occasionally be convenient to give meaning to the expressions $f(0)$ and $f(\infty)$ when $f \in C_0(\Upsilon)$, taking both of them to equal 0. It is useful to note that the space $C_0(\Upsilon)$ is the closed linear span in $\ell_\infty(\Upsilon)$ of the indicator functions $1_{(0,t]} (t \in \Upsilon)$. Many results about general elements of $C_0(\Upsilon)$ may be established easily using uniform approximation by linear combinations of these indicators. Here are two examples.

Lemma 2.1. Let $\Upsilon$ be a tree, let $f$ be in $C_0(\Upsilon)$ and let $\delta$ be a positive real number. There exists $k \in \mathbb{N}$ such that if $H$ is a subset of $\Upsilon$ whose elements are pairwise incomparable then $|f(t)| \geq \delta$ for at most $k$ elements $t$ of $H$.

Lemma 2.2. Let $\Upsilon$ be a tree, let $f$ be in $C_0(\Upsilon)$ and let $\delta$ be a positive real number. For all but finitely many $s \in \Upsilon$, there exists $t \in s^+$ such that $|f(t) - f(s)| < \delta$, while $|f(u)| < \delta$ for all $v \in [s,\infty) \setminus [t,\infty)$.

For any tree $\Upsilon$ and any $t \in \Upsilon$ we write $t^+$ for the set of all immediate successors of $t$ in $\Upsilon$, that is to say that $t^+$ contains those elements $u$ with the property that $s < u$ if $s \preceq t$. By convention, we write $0^+$ for the set of minimal elements of $\Upsilon$ and define $\Upsilon^+ = \bigcup_{t \in \Upsilon} t^+$. The elements of $\Upsilon^+$ are exactly those $t$ for which $r(t)$ is 0 or a limit ordinal; they may also be characterized as the isolated points of $\Upsilon$ for its locally compact topology. For $u \in \Upsilon^+$ we write $u^-$ for the unique $t \in \Upsilon \cup \{0\}$ such that $u \in t^+$. We say that $\Upsilon$ is (in-) finitely branching if $t^+$ is (in-) finite for all $t \in \Upsilon$. For each $t \in \Upsilon$ we say that $\Upsilon$ is a dyadic tree if each $t^+$ contains two elements. A subset $\Gamma$ of a tree $\Upsilon$ is said to be ever-branching if for every $t \in \Gamma$ the intersection $\Gamma \cap [t,\infty)$ is not totally ordered.

The height of a tree $\Upsilon$ is defined to be $ht(\Upsilon) = \sup\{r(t) + 1 : t \in \Upsilon\}$; a branch of $\Upsilon$ is a maximal totally ordered subset; we say that $\Upsilon$ is a full tree of height $\alpha$
if every branch has order-type \( \alpha \). The tree considered in [8] was a full uncountably branching tree of height \( \omega_1 \).

We say that two elements \( s, t \) of a tree \( \Upsilon \) are \textit{incomparable} if neither \( s \preceq t \) nor \( t \preceq s \) holds. A subset of \( \Upsilon \) whose elements are pairwise incomparable is called an \textit{antichain}. For a subset \( S \) of \( \Upsilon \) we define \( \min S \) to be the set of elements of \( S \) that are minimal for the tree order \( \preceq \). If \( s \in S \) then \( (0, s] \) contains a member of \( \min S \) (because \( (0, s] \) is well-ordered). We write \( \max S \) for the set of maximal elements of a subset \( S \) of \( \Upsilon \). For a subset \( S \) of \( \Upsilon \) we define \( \min S \) to be the set of elements of \( S \) that are minimal for the tree order \( \preceq \). If \( s \in S \) then \( (0, s] \) contains a member of \( \min S \) (because \( (0, s] \) is well-ordered). We write \( \max S \) for the set of maximal elements of a subset \( S \) of \( \Upsilon \). In general, non-empty subsets of \( \Upsilon \) need not have maximal elements, but if \( S \) is a \textit{compact} subset of \( \Upsilon \) then \( \max S \) is finite and \( S \subseteq \bigcup_{s \in \max S} (0, s] \) (since the sets \( (0, s] \) \( s \in S \) form an open cover of \( S \)). Of course, for any \( S \subseteq \Upsilon \), the sets \( \min S \) and \( \max S \) are antichains.

There are other interesting topologies with which a tree may be equipped, as well as the locally compact topology we have been looking at so far. Logicians are often concerned with what is sometimes called the \textit{“forcing topology”}, whose basic open sets are of the form \( [t, \infty) \). This satisfies the \( T_0 \) separation axiom, but (except in the trivial case where \( \Upsilon \) is an antichain) not the \( T_1 \) axiom. We shall be interested in a variant of this topology, which will here be called the reverse topology, and which we define to be the coarsest topology for which all the subsets \( [t, \infty) \) are open and closed. A base of neighbourhoods of \( t \) for the reverse topology consists of the sets

\[
[t, \infty) \setminus \bigcup_{u \in F} [u, \infty),
\]

with \( F \) a finite subset of \( t^+ \). Evidently \( t \) is an isolated point for this topology if and only if \( t^+ \) is finite, so that \( \Upsilon \) is reverse-discrete if and only if \( \Upsilon \) is finitely branching. The following observation goes some way towards explaining why the reverse topology is of interest.

**Proposition 2.3.** For any tree \( \Upsilon \), the map \( u \mapsto 1_{(0, u]} \) is a homeomorphic embedding from \( \Upsilon \) in the reverse topology into \( C_0(\Upsilon) \) in the topology \( \tau_p \) of pointwise convergence.

**Proof.** Restricted to the set of indicator functions, the topology \( \tau_p \) may be characterized as the coarsest for which all sets \( \{ f : f(t) = 1 \} \) \( t \in \Upsilon \) are open and closed. Since \( 1_u(t) = 1 \) if and only if \( u \in [t, \infty) \), we see that this corresponds exactly to our definition of the reverse topology. \( \square \)

**Corollary 2.4.** If \( \Upsilon \) is an infinitely branching tree which is a Baire space for the reverse topology then \( \Upsilon \) does not have the Namioka property.

**Proof.** We have already noted that an infinitely branching tree has no reverse-isolated points, so that \( u \mapsto 1_{(0, u]} \) has no points of continuity from the reverse topology into the norm topology of \( C_0(\Upsilon) \). On the other hand, this map is continuous into \( \tau_p \) and, by hypothesis, \( \Upsilon \) is reverse-Baire. \( \square \)

The easy argument at the start of [8] shows that a full tree of height \( \omega_1 \) is reverse Baire. However, examples exist [19 or 12] of infinitely-branching reverse-Baire trees in which every branch is countable. It is shown in [12] that infinitely branching Baire trees have all the bad properties of the tree considered in [8]. Of course, a consequence of this is that if we want to study the fine structure of trees in renorming theory we have to look at trees which are not Baire for the reverse topology.
As already mentioned in the Introduction, we shall obtain necessary and sufficient conditions for the existence on $C_0(\Upsilon)$ of various good renormings. These conditions will all be expressed in terms of increasing real-valued functions on $\Upsilon$. A function $\rho : \Upsilon \to \mathbb{R}$ is said to be increasing if $s \preceq t \implies \rho(s) \leq \rho(t)$ and to be strictly increasing if $s \prec t \implies \rho(s) < \rho(t)$. A tree is said to be $\mathbb{R}$-embeddable if it admits a strictly increasing real-valued function. A tree which admits a strictly increasing rational-valued function may be said to be $\mathbb{Q}$-embeddable, but it is more usual to call such a tree special. An equivalent definition is to say that a tree is special if it is a countable union of antichains. If Baire trees are very bad from the point of view of renorming theory, then special trees are very good. This may be readily deduced from known results.

**Proposition 2.5.** Let $L$ be a locally compact space which is a countable union of closed subsets $L_n \ (n \in \mathbb{N})$. If, for each $n$, $C_0(L_n)$ admits an equivalent norm which is LUR and has LUR dual norm, then the same is true for $C_0(L)$.

**Proof.** For each $n$, let $\| \cdot \|_n$ be an equivalent norm on $C_0(L_n)$, which is LUR, with LUR dual norm. We may suppose that $\| \cdot \|_n \leq \| \cdot \|_\infty$. The $\ell_2$ direct sum $Y = (\bigoplus\{C_0(L_n), \| \cdot \|_n\})_2$, may be equipped with its natural norm $\| \cdot \|_Y$, where

$$\| (g_n) \|_Y^2 = \sum_n \| g_n \|_n^2.$$ 

This norm is LUR and has LUR dual norm. We define a bounded linear operator $T : C_0(L) \to Y$ by $(Tf)_n = f |_{L_n}$ and note that the image of the transpose operator $T^*$ contains each Dirac functional $\delta_t \ (t \in L)$. Thus $T^{**}$ is injective and Theorem VII.2.6 of [3] may be applied to give the required renorming of $C_0(L)$. □

**Corollary 2.6.** If $\Upsilon$ is a special tree then $C_0(\Upsilon)$ admits an equivalent norm which is LUR and has LUR dual norm.

**Proof.** If $A$ is an antichain in $\Upsilon$ then $A$ is closed and discrete in the locally compact topology of $\Upsilon$. Because $A$ is discrete, $C_0(A) = c_0(A)$, a space with a renorming which is LUR and LUR*. Thus, if $\Upsilon$ is expressible as a countable union of antichains, Proposition 2.5 is applicable. □

Although less nice than special trees, $\mathbb{R}$-embeddable trees still have some good properties.

**Proposition 2.7.** If $\Upsilon$ is $\mathbb{R}$-embeddable, there exists a bounded linear injection from $C_0(\Upsilon)$ into $c_0(\Upsilon)$. Hence $C_0(\Upsilon)$ admits an equivalent strictly convex norm.

**Proof.** Let $\rho : \Upsilon \to \mathbb{R}$ be strictly increasing. By replacing $\rho$, if necessary, with $e^\rho/(1 + e^\rho)$, we may suppose that $\rho$ takes values in the real interval $(0, 1)$. We define

$$(Rf)(t) = \begin{cases} (\rho(t) - \rho(t^-))f(t) & \text{if } t \in \Upsilon^+ \\ 0 & \text{otherwise,} \end{cases}$$

and note that $R$ is a bounded linear operator from $C_0(\Upsilon)$ into $\ell_\infty(\Upsilon)$. If $f$ is in the kernel of $R$, then $f(t) = 0$ for all $t \in \Upsilon^+$ because $\rho(t) \neq \rho(t^-)$ for all such $t$. Since $\Upsilon^+$ is dense in $\Upsilon$, we deduce that $f$ is everywhere, and hence that $R$ is injective.
To show that $R$ takes values in $c_0(\mathcal{Y})$ it is enough to show that $R1_{(0,u]}$ is in $c_0(\mathcal{Y})$ for all $u \in \mathcal{Y}$. This may be done by a calculation since

$$\sum_{t \in \mathcal{Y}} \|(Rf)(t)\| = \sum_{t \in (0,u] \cap \mathcal{Y}^+} (\rho(t) - \rho(t^-)) \leq \rho(u) \leq 1,$$

since we are adding up the jumps of the non-negative increasing function $\rho$ along the totally ordered set $(0,u)$. What we have shown is that $R1_{(0,u]}$ is even in $\ell_1(\mathcal{Y})$. \hfill \box

Special trees and $\mathbb{R}$-embeddable trees are well-established in the literature of Logic and Set Theory. In the next section we shall pass on to a definition that is motivated by our applications in renorming.

3. Good points, bad points and $\mu$-functions

Given a tree $\mathcal{Y}$ and an increasing function $\rho : \mathcal{Y} \to \mathbb{R}$, we shall say that an element $t$ of $\mathcal{Y}$ is a good point for $\rho$ if there is a finite subset $F$ of $t^+$ such that $\inf_{u \in t^+ \setminus F} \rho(u) > \rho(t)$. Provided there is no ambiguity about which is the function $\rho$ we shall write $F_t = \{ u \in t^+ : \rho(u) = \rho(t) \}$ and $\delta_t = \inf\{ \rho(u) - \rho(t) : u \in t^+ \setminus F_t \}$. An element of $\mathcal{Y}$ which is not a good point for $\rho$ will be called a bad point for $\rho$.

We can get an idea of the way in which good points and bad points are going to enter into the theory by establishing straightaway some necessary conditions for the existence on $C_0(\mathcal{Y})$ of an equivalent norm with the Kadec property and for the existence of an equivalent strictly convex norm. Whenever we have a renorming $\| \cdot \|$ of $C_0(\mathcal{Y})$, we may define an increasing function $\mu : \mathcal{Y} \to \mathbb{R}$ by

$$\mu(t) = \inf\{ \| f \| : f = 1_{(0,t]} + f' : \text{supp} f' \subseteq (t,\infty) \}. $$

The function $\mu$ defined in the lemma that follows, and variants of it, will be used throughout the present work.

**Lemma 3.1.** Let $\| \cdot \|$ be an equivalent norm on $C_0(\mathcal{Y})$ and let $\mu$ be the increasing function defined by

$$\mu(t) = \inf\{ \| f \| : f = 1_{(0,t]} + f' \text{ with supp } f' \subseteq (t,\infty) \}. $$

If $t$ is a bad point for $\mu$ then $\|1_{(0,t]}\| = \mu(t)$.

**Proof.** It is clear from the definition of $\mu$ that $\|1_{(0,t]}\| \geq \mu(t)$. Badness implies that there is a sequence $(u_n)$ of distinct elements of $t^+$ such that $\mu(u_n) \to \mu(t)$ as $n \to \infty$. Thus there are functions $f_n$ of the form $f_n = 1_{(0,u_n]} + f'_n$, with supp $f'_n \subseteq (u_n,\infty)$, such that $\|f_n\| \to \mu(t)$. Now $(f_n)$ is a norm-bounded sequence and converges pointwise on $\mathcal{Y}$ to $1_{(0,t]}$. Thus this sequence converges weakly to $1_{(0,t]}$, which shows that $\|1_{(0,t]}\| \leq \lim \|f_n\| = \mu(t)$. \hfill \box

**Proposition 3.2.** If $C_0(\mathcal{Y})$ admits an equivalent Kadec norm then the function $\mu$ associated with such a norm has no bad points.

**Proof.** We observe that if $t$ is a bad point then the sequence $(f_n)$ defined in the proof of Lemma 3.1 converges weakly to $1_{(0,t]}$ and satisfies $\|f_n\| \to \|1_{(0,t]}\|$, but does not converge in norm, since $\|f_n - 1_{(0,t]}\|_\infty \geq 1$ for all $n$. Since this contradicts the Kadec property of the norm, $\mu$ can in fact have no bad points. \hfill \box
Proposition 3.3. Let $\Upsilon$ be a tree and suppose that $C_0(\Upsilon)$ admits a strictly convex renorming. Then the associated function $\mu$ has the following property: for every $s \in \Upsilon$ there is at most one bad point $t$ satisfying $t \succeq s$ and $\mu(t) = \mu(s)$. In particular, $\mu$ is strictly increasing on the set of its bad points.

Proof. Suppose that $s$ is in $\Upsilon$ and that $t,u$ are bad points with $s \preceq t, s \preceq u$ and $\mu(s) = \mu(t) = \mu(u) = \alpha$. Then we have

$$\|1_{(0,t]}\| = \alpha, \quad \|1_{(0,u]}\| = \alpha$$

by Lemma 3.1, while

$$\|\frac{1}{2}(1_{(0,t]} + 1_{(0,u]})\| = \|1_{(0,s]} + \frac{1}{2}(1_{(s,t]} + 1_{(s,u]})\| \geq \alpha,$$

by the definition of $\mu(s)$. The strict convexity of the norm implies that we must have $1_{(0,t]} = 1_{(0,u]}$ so that $t = u$. □

The next proposition gives a second property possessed by the $\mu$-function associated with a strictly convex norm. If $\rho : \Upsilon \to \mathbb{R}$ is increasing, we shall say that a point $u$ of $\Upsilon$ is a fan point for $\rho$ if $u$ is a member of an ever-branching subset on which $\rho$ is constant. There is a useful construction that one may carry out in this case. Suppose that $\rho$ takes the constant value $\alpha$ on some ever-branching subset $T$ of $\Upsilon$. Since $T$ is ever-branching, we can choose for any $u$ in $T$, two incomparable elements of $(u,\infty) \cap T$, $u_0$ and $u_1$, say. Similarly we may choose incomparable elements $u_{00}$ and $u_{01}$ of $(u_0,\infty) \cap T$, and it is possible to continue in such a way as to embed a full dyadic tree of height $\omega$ in $[u,\infty) \cap T$, the nodes being labelled $u_{\sigma}$ ($\sigma \in \{0,1\}^n$, $n \in \mathbb{N}$). We define a function $\phi_u \in C_0(\Upsilon)$ by

$$\phi_u = \frac{1}{2}(1_{(u,u_0]} + 1_{(u,u_1]}) + \sum_{n \geq 1, \sigma \in \{0,1\}^n} 2^{-n-1}(1_{(u_\sigma,u_{\sigma0}]} + 1_{(u_\sigma,u_{\sigma1}]})$$

Notice that $\phi_u$ takes the value $\frac{1}{2}$ at $u_0$ and $u_1$, the value $\frac{1}{2}$ at $u_{00}, u_{01}, u_{10}, u_{11}$, and so on. We shall call $\phi_u$ a fan function for $\rho$ at $u$.

Proposition 3.4. If $\Upsilon$ is a tree and $\| \cdot \|$ is a strictly convex renorming of $C_0(\Upsilon)$ then the corresponding function $\mu : \Upsilon \to \mathbb{R}$ is constant on no ever-branching subset of $\Upsilon$.

Proof. Suppose if possible that $\mu$ takes the constant value $\alpha$ on some ever-branching subset $T$. We fix $u$ in $T$ and introduce $u_\sigma$ ($\sigma \in \bigcup_{n \in \mathbb{N}} \{0,1\}^n$) as above. We may choose, for each $n$ and each $\sigma \in \{0,1\}^n$, an element $\psi_\sigma$ of $\Upsilon$ with $\text{supp} \psi_\sigma \subseteq (u_\sigma,\infty)$ and $\|1_{(u_{\sigma0},\sigma}] + \psi_\sigma\| \leq \alpha + 2^{-n}$. Notice that $\|\psi_\sigma\|_\infty \leq \|1_{(u_{\sigma0},\sigma]} + \psi_\sigma\|_\infty$ because $\psi_\sigma$ is supported on $(u_\sigma,\infty)$ and that $\|\psi_\sigma\|_\infty$ is thus at most $M(\alpha + 2^{-n})$.

For each $n$, we set

$$y_n = 2^{-n} \sum_{\sigma \in \{0,1\}^n} \left[1_{(0,u_\sigma]} + \psi_\sigma\right]$$

and note that $\|y_n\| \leq \alpha + 2^{-n}$. Moreover, if $\phi_u$ is the fan function introduced above, $1_{(0,u]} + \phi_u - y_n$ is supported on the union of the disjoint sets $(u_\sigma,\infty)$ ($\sigma \in \{0,1\}^n$) and we have

$$\|1_{(0,u]} + \phi_u - y_n\|_\infty \leq \max_{\sigma \in \{0,1\}^n} \|1_{(u_\sigma,\infty]}\|_\infty + 2^{-n} \|\psi_\sigma\|_\infty$$

$$\leq 2^{-n-1} + 2^{-n} M(\alpha + 2^{-n})$$.
Thus $\mathbf{1}_{[0,u]} + \phi_u$ is the norm limit of $y_n$ and so satisfies $\|\mathbf{1}_{[0,u]} + \phi_u\| = \alpha$.

Now the same argument may be applied with $u$ replaced in turn by $u_0$ and by $u_1$. We find that

$$\|\mathbf{1}_{[0,u_0]} + \phi_{u_0}\| = \|\mathbf{1}_{[0,u_1]} + \phi_{u_1}\| = \alpha,$$

where $\phi_{u_j}$ are defined in the obvious way. This contradicts strict convexity of the norm $\|\cdot\|$ since

$$\mathbf{1}_{[0,u]} + \phi_u = \frac{1}{2}\left(\mathbf{1}_{[0,u_0]} + \phi_{u_0} + \mathbf{1}_{[0,u_1]} + \phi_{u_1}\right).$$

\[ \square \]

The propositions that we have just proved give a good idea of why bad points and fan points are relevant to convexity properties of equivalent norms on $C_0(\Upsilon)$. The connection with smoothness is suggested by the next one. It is convenient to introduce here a more general notion of $\mu$-function. If $t$ is in $\Upsilon$ we shall, as in [11] write $C_t$ for the set of $f \in C_0(\Upsilon)$ such that $\text{supp} f = (0,t]$ and such that $f \upharpoonright (0,t]$ is increasing. For $f \in C_t$ and $u \in [t,\infty)$ we set

$$\mu(f,u) = \inf\{\|f + f(t)\mathbf{1}_{[t,u]} + g\| : g \in C_0(\Upsilon), \text{ supp } g \subseteq (u,\infty)\}.$$ 

Notice that, if $f \in C_t$ and $t \leq u \leq v$ we can rewrite $\mu(f,v)$ as $\mu(g,v)$ with $g = f + f(t)\mathbf{1}_{[t,u]} \in C_u$.

**Proposition 3.5.** Let $\Upsilon$ be a tree, let $t$ be in $\Upsilon$ and let $f$ be in $C_t$. Suppose that there exists a function $\hat{f}$ which attains the infimum in the definition of $\mu(f,t)$, that is to say that $\text{supp} (\hat{f} - f) \subseteq (t,\infty)$ and $\|\hat{f}\| = \mu(f,t)$. If the norm $\|\cdot\|$ is Gateaux-differentiable at $\hat{f}$ (with differential $\hat{f}^*$), then necessarily $(\hat{f},h) = 0$ for all $h$ with $\text{supp } h \subseteq (t,\infty)$. This situation occurs in particular for $\hat{f} = f$ if $t$ is a bad point for $\mu(f,\cdot)$ and for $\hat{f} = f + f(t)\phi_t$ if $t$ is a fan point for this function.

**Proof.** The assumption about $\hat{f}$ is that, for all $h$ with $\text{supp } h \subseteq (t,\infty)$, the function $\lambda \mapsto \|\hat{f} + \lambda h\| : \mathbb{R} \to \mathbb{R}$ has a minimum at $\lambda = 0$. By Gateaux differentiability and elementary calculus, we get $(\hat{f}^*,h) = 0$. The assertions about the cases where $t$ is a bad point or a fan point are proved just as in Propositions 3.1 and 3.4. \[ \square \]

To finish this section we record an easy fact about uniform approximation. We may regard $\mu(f,u)$ as being the distance, calculated with respect to the norm $\|\cdot\|$, from $f + f(t)\mathbf{1}_{[t,u]}$ to the subspace $Z_u = \{g \in C_0(\Upsilon) : \text{ supp } g \subseteq (u,\infty)\}$, or as being the quotient norm $\|f + \mathbf{1}_{[t,u]} + Z_u\|$ in the quotient space $C_0(\Upsilon)/Z_u$. Either way, it is very easy to establish the following lemma.

**Lemma 3.5.** If $(f_n)$ is a sequence in $C_t$ which converges uniformly to $f$ then $\mu(f_n,\cdot)$ converges uniformly on $[t,\infty)$ to the limit $\mu(f,\cdot)$. If $u$ is a bad point for all the functions $\mu(f_n,\cdot)$ then it is a bad point for $\mu(f,\cdot)$.

4. LUR renormings

The aim of this section is to establish a necessary and sufficient condition on the tree $\Upsilon$ for the existence of a locally uniformly convex norm on $C_0(\Upsilon)$ equivalent to the supremum norm $\|\cdot\|_{\infty}$. In fact, this result can be deduced from the theorems in
Sections 5 and 6 about strictly convex renormings and Kadec renormings, together with Troyanski’s theorem. However, the direct proof that we are about to give is significantly shorter and easier to follow than such a roundabout approach, and we hope the reader will excuse the slight redundancy. We also show that the existence of a weakly LUR renorming of $C_0(\Upsilon)$ implies the existence of a LUR renorming. It is an open problem whether wLUR renormability implies LUR renormability for arbitrary Asplund spaces, though this is the case for spaces which have a norm which is both Fréchet-smooth and wLUR [3, Prop. 2.6].

**Theorem 4.1.** Let $\Upsilon$ be a tree. The following are equivalent:

1. there exists a locally uniformly convex renorming of $C_0(\Upsilon)$;
2. there exists a weakly locally uniformly convex renorming of $C_0(\Upsilon)$;
3. there exists an increasing function $\rho: \Upsilon \to \mathbb{R}$ which is constant on no ever-branching subset of $\Upsilon$ and which has no bad points.

We already know from Propositions 3.2 and 3.4 that the function $\mu$ associated with an equivalent LUR norm on $C_0(\Upsilon)$ can have no bad points and can be constant on no ever-branching subset. Thus the implication $(1) \implies (3)$ is certainly true. We now pass to the implication $(3) \implies (1)$.

**Proposition 4.2.** Let $L$ be a locally compact space and let $U_i$ $(i \in I)$ be a family of open and closed subsets of $L$ such that, for each $i$, there is an equivalent LUR norm $\| \cdot \|_i$ on $C_0(U_i)$. For any bounded linear operator $S: C_0(L) \to c_0(I)$, there is an equivalent norm $\| \cdot \|$ on $C_0(\Upsilon)$ which is locally uniformly convex at each $f$ such that the set $\{ t \in L : f(t) \neq 0 \}$ is contained in the union $\bigcup \{ U_i : i \in I \text{ and } (Sf)(i) \neq 0 \}$.

**Proof.** We may assume that $\| \cdot \|_i \leq \| \cdot \|_\infty$ for all $i$. For each non-empty finite subset $F$ of $L$ and each $f \in C_0(L)$, we define

$$\phi(F; f) = \left[ \sum_{i \in F} ((Sf)(i))^2 \right]^{\frac{1}{2}};$$

$$\psi(F; f) = \left[ \| f, 1_L \setminus \bigcup_{i \in F} U_i \|_\infty^2 + \sum_{i \in F} \| f \setminus U_i \|_2^2 \right]^{\frac{1}{2}}.$$ 

Taking $I = I_m = \{ F \subseteq I : \# F = m \}$ in Lemma 1.1, we obtain, for each positive integer $m$, a norm $\| \cdot \|_m$ on $C_0(L)$. We set $\| f \|_m^2 = \sum_{i=1}^{m} 2^{-m} \| f \|_m^2$ and claim that this new norm $\| \cdot \|$ is locally uniformly convex at every $f$ satisfying the condition given above.

Assume without loss of generality that $\| f \|_\infty = 1$ and let $\epsilon$ be a positive real number. There is a finite subset $H_0$ of $I$ such that $(Sf)(t) \neq 0$ for all $i \in H_0$ and $s \in L : |f(s)| \geq \epsilon \subseteq \bigcup_{i \in H_0} U_i$. We may fix a positive real number $\alpha$ such that $H_0$ is contained in the finite set $H = \{ t \in L : |(Sf)(t)| \geq \alpha \}$, noting that, for some $\eta > 0$ we shall have $(Sf)(u)^2 < \alpha^2 - \eta$ whenever $u \in I \setminus H$. Let $m = \# H$ and choose $k \in \mathbb{N}$ so that $k > 4 \eta^{-1}$. Notice that $\phi(H; f) = \sup_{F \subseteq I_m} \phi(F; f)$ and that we have the following “rigidity property”:

$$G \in I_m, \phi(G; f)^2 \geq \phi(H; f)^2 - \eta \implies G = H.$$ 

If $f_n$ are such that $2\| f_n \|_m^2 + 2\| f \|_m^2 - \| f + f_n \|_m^2 \to 0$ then, we certainly have $2\| f_n \|_m^2 + 2\| f \|_m^2 - \| f + f_n \|_m^2 \to 0$ and so, by Lemma 1.1, there exists a sequence
(F_n) in I_m such that (among other things) \( \phi(F_n; f) \to \sup_{F \in I_m} \phi(F; f) \) as \( n \to \infty \). Because of the rigidity that we have just observed, \( F_n \) must equal \( H \) for all large enough \( m \). Thus, by another of the conclusions of Lemma 1.1, we see that
\[
\psi(H; f)^2 + \psi(H; f_n)^2 - 2\psi(H; \frac{1}{2}(f + f_n))^2 \to 0
\]
as \( n \to \infty \), which in turn implies that
\[
\|f_n \upharpoonright U_i\|^2 + \|f \upharpoonright U_i\|^2 - 2\|\frac{1}{2}(f + f_n) \upharpoonright U_i\|^2 \to 0
\]
for all \( i \in H \), and that
\[
\|f \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|^2 + \|f_n \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|^2 - 2\|\frac{1}{2}(f + f_n) \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|^2 \to 0.
\]
The assumed local uniform convexity of the norms \( \| \cdot \|_i \) now gives us uniform convergence of \( f_n \) to \( f \) on the union \( \bigcup_{i \in H} U_i \), while the second of the above limits certainly implies that
\[
\|f_n \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|^2 \to \|f \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|^2.
\]
Since \( H \) contains our original \( H_0 \), we have \( \|f \mathbf{1}_{L \setminus \bigcup_{i \in H} U_i}\|_\infty < \epsilon \) and we thus see that \( \|f - f_n\|_\infty \leq 2\epsilon \) for all large enough \( n \). \( \square \)

We now define an operator \( S \) which will here be used in an application of Proposition 4.2 and which will turn up again in later sections of the paper. Recall from Section 3, that for an element of \( \Upsilon \) which is a good point for \( \rho \) there exist \( \delta_i > 0 \) and a finite (possibly empty) subset \( F_i \subseteq t^+ \) such that
\[
\rho(u) \begin{cases} 
= \rho(t) & (u \in F_i) \\
\geq \rho(t) + \delta_i & (u \in t^+ \setminus F_i). 
\end{cases}
\]
For \( f \in c_0(\Upsilon) \) we define
\[
(Sf)(t) = \begin{cases} 
\frac{\delta_i}{1 + \#F_i} \left[ f(t) - \sum_{u \in F_i} f(u) \right] & \text{if } t \text{ is a good point} \\
0 & \text{if } t \text{ is a bad point}. 
\end{cases}
\]

Although for the immediate purposes of this section we are dealing with a tree equipped with a function \( \rho \) with no bad points it is convenient for later applications to state the following two lemmas in greater generality.

**Lemma 4.3.** Let \( \Upsilon \) be a tree and let \( \rho : \Upsilon \to \mathbb{R}_+ \) be a bounded increasing function. The operator \( S \) defined above is a bounded linear operator from \( c_0(\Upsilon) \) into \( \ell_1(\Upsilon) \).

**Proof.** Certainly \( S \) is a bounded linear operator from \( c_0(\Upsilon) \) into \( \ell_\infty(\Upsilon) \). To show that \( S \) takes values in \( c_0(\Upsilon) \) it is enough to show that \( S \mathbf{1}_{(0, u)} \in c_0(\Upsilon) \) for all \( u \). For \( t \in (0, u] \) let us write \( t' \) for the unique element of \( t^+ \cap (0, u] \); it is clear that \( (S \mathbf{1}_{(0, u)})(t) \) is non-zero if and only if \( t = u \) or \( t < u \) and \( t' \notin F_t \). Proceeding as in the proof of Proposition 2.7 we calculate
\[
\sum_{t \in \Upsilon} |(S \mathbf{1}_{(0, u)})(t)| = \delta_u + \sum_{t < u, t' \notin F_t} \frac{\delta_i}{1 + \#F_t} \\
\leq \delta_u + \sum_{t < u} (\rho(t') - \rho(t)) \\
\leq \delta_u + \rho(u)
\]
deducing that \( S \mathbf{1}_{(0, u)} \) is in \( \ell_1(\Upsilon) \). \( \square \)
Lemma 4.4. Let $\Upsilon$ be a tree, let $\rho: \Upsilon \to \mathbb{R}$ be a bounded increasing function which is continuous for the locally compact topology on $\Upsilon$, let $f$ be in $C_0(\Upsilon)$ and let $s$ be an element of $\Upsilon$ with $f(s) \neq 0$. If the set $[s, \infty) \cap \rho^{-1}(s) = \{ t \in \Upsilon : t \geq s \text{ and } \rho(t) = \rho(s) \}$ has no ever-branching subset and contains only points that are good for $\rho$ then there exists $t$ in this set such that $(Sf)(t) \neq 0$.

Proof. If no such $t$ exists we have $f(t) = \sum_{u \in F_t} f(u)$ whenever $t \geq s$ and $\rho(t) = \rho(s)$. Define a set $T$ by

$$T = \{ t \geq s : \text{there exists } v \geq t \text{ such that } \rho(v) = \rho(s) \text{ and } f(v) \neq 0 \}.$$ 

By hypothesis, $T$ is not ever-branching so that there is some $t_0 \in T$ such that $T \cap [t_0, \infty)$ is totally ordered. Now, for $t \in T \cap [t_0, \infty)$ we have $f(t) = \sum_{u \in F_t} f(u)$ and at most one of the $f(u)$ can be non-zero. Thus $f$ is actually constant on $T \cap [t_0, \infty)$ and this constant value is non-zero because of the way we defined $T$. By the fact that $f \in C_0(\Upsilon)$ and by the assumed continuity of $\rho$ we see that the set $T$, which can be written $T = \{ t \geq t_0 : \rho(t) = \rho(s) \text{ and } f(t) = f(t_0) \}$, is compact. So $T$ has a maximal point, $t$ say. This maximality implies that for any $v \geq t$ either $\rho(v) > \rho(t)$ or $f(v) = 0$; in particular, either $F_t$ is empty or else $f(u) = 0$ for all $u \in F_t$. In any case, $\sum_{u \in F_t} f(u) = 0 \neq f(t)$ and $(Sf)(t) \neq 0$ as claimed. □

We can now establish the implication $(3) \implies (1)$. We may assume that $\rho$ takes values in the real interval $(0, 1)$. We may furthermore suppose that $\rho$ is continuous for the locally compact topology of $\Upsilon$, that is to say that $\rho(t) = \sup_{s \leq t} \rho(s)$ whenever $t$ is a limit point of $\Upsilon$. This involves no loss of generality, since replacing $\rho(t)$ by $\lim_{n \uparrow t} \rho(s)$ at limit points $t$ does not introduce new bad points and does not create new ever-branching sets of constancy of $\rho$. For $t \in \Upsilon$ we define $U_t = (0, t]$, noting that $C_0(U_t)$ has an equivalent LUR norm since $U_t$ is homeomorphic to the interval of ordinals $[0, \tau(t)]$. Lemma 4.3 shows that $S$ takes values in $c_0(\Upsilon)$. Since all points of $\Upsilon$ are good and $\rho$ is constant on no ever-branching subset, Lemma 4.4 tells us that whenever $f(s) \neq 0$ there is some $t$ with $(Sf)(t) \neq 0$ and $s \in U_t$. Thus the hypotheses of Proposition 4.2 are satisfied and a LUR norm may be constructed on $C_0(\Upsilon)$.

We now pass to the implication $(2) \implies (3)$. If we have a norm on $C_0(\Upsilon)$ which is wLUR and not LUR then the associated function $\mu$ cannot be constant on an ever-branching subset (by Proposition 3.4) but it may quite easily have bad points. We show however that it may be modified to give a new function $\rho$ without bad points.

Lemma 4.5. Let $\| \cdot \|$ be an equivalent weakly LUR norm on $C_0(\Upsilon)$. The associated function $\mu : \Upsilon \to \mathbb{R}$ has the following property: whenever $(t_n)$ is an increasing sequence of points which are bad for $\mu$ and $t$ is another bad point with $t \geq \lim_n t_n$, we have $\mu(t) > \lim \mu(t_n)$.

Proof. Set $f_n = 1_{(0, t_n]}$ and $f = 1_{(0, t]}$. By Lemma 3.1 we have $\|f_n\| = \mu(t_n)$ and $\|f\| = \mu(t)$ and by definition of $\mu$ $\|f + f_n\| \geq \mu(t_n)$. If it were the case that $\mu(t) = \lim_n \mu(t_n)$ then we should have $\|f_n\| \to \|f\|$ and $\|f + f_n\| \to \|f\|$, which contradicts the wLUR property, since $f(t) = 1$, and $f_n(t) = 0$ for all $n$. □

We can now complete the proof of the theorem. We already know that the function $\mu$ associated with a wLUR norm on $C_0(\Upsilon)$ is strictly increasing on the set
of its bad points (Proposition 3.3) and that it satisfies the condition established in
the above lemma. Together these two facts tell us that for any point \( t \) which is bad
for \( \mu \) we have
\[
\mu(t) > \sup\{\mu(s) : s \text{ is bad for } \mu \text{ and } s \prec t\}.
\]
We define
\[
\rho(t) = \mu(t) + \sup\{\mu(s) : s \text{ is bad for } \mu \text{ and } s \prec t\},
\]
and note first that a point which is good \( \mu \) must certainly be good for \( \rho \). If \( t \) is bad
for \( \mu \), on the other hand, then for any \( u \in t^+ \) we have
\[
\rho(u) = \mu(u) + \mu(t) > 2\mu(t) \geq \rho(t)
\]
by what we have just observed. Thus \( t \) is good for \( \rho \). The fact that \( \rho \) is constant on
no ever-branching subset follows immediately from the same property for \( \mu \), and so
the proof of Theorem 4.1 is complete.

5. Strictly convex renormings

In this section we establish a necessary and sufficient condition for the existence
on \( C_0(\Upsilon) \) of an equivalent strictly convex norm and we further show that
this condition yields the existence of an equivalent norm that is midpoint-locally
uniformly convex (MLUR). The condition may be regarded as a mild weakening
of \( \mathbb{R} \)-embeddability and the theorem implies the result about \( \mathbb{R} \)-embeddable trees
that we presented as Proposition 2.7.

**Theorem 5.1.** For a tree \( \Upsilon \) the following are equivalent:

1. \( C_0(\Upsilon) \) admits an equivalent strictly convex norm;
2. \( C_0(\Upsilon) \) admits an equivalent MLUR norm;
3. there exists on \( \Upsilon \) an increasing real-valued function \( \rho \) satisfying the two condi-
tions:
   (i) \( \rho \) is constant on no ever-branching subset of \( \Upsilon \);
   (ii) for any \( s \in \Upsilon \) there is at most one bad point \( u \) with \( s \prec u \) and \( \rho(u) = \rho(s) \).
4. there is a bounded linear injection from \( C_0(\Upsilon) \) into some space \( c_0(I) \).

The fact that (1) implies (3) has already been established in Proposi-
tions 1.8 and 2.2, and certainly either of (2) or (4) implies (1). We have to show that (3)
implies (2) and (4). So let \( \rho : \Upsilon \to (0,1) \) be an increasing function satisfying (i)
and (ii).

To establish (4) we define \( R : C_0(\Upsilon) \to c_0(\Upsilon), S : C_0(\Upsilon) \to c_0(\Upsilon) \) by
\[
(Rf)(t) = \begin{cases} 
\rho(t) - \rho(t^-)f(t) & \text{if } t \in \Upsilon^+ \\
0 & \text{otherwise}
\end{cases}
\]
\[
(Sf)(t) = \begin{cases} 
\delta_t(\#F_t)^{-1}(f(t) - \sum_{u \in F_t} f(u)) & \text{if } t \text{ is good}, \\
0 & \text{otherwise}.
\end{cases}
\]
We shall show that the operator \( R \oplus S : C_0(\Upsilon) \to c_0(\Upsilon) \oplus c_0(\Upsilon) \) is injective.

Suppose then that \( f \in C_0(\Upsilon) \) is such that \( Rf = Sf = 0 \). If \( f \neq 0 \) we choose a
minimal element \( t \) of \( \Upsilon \) with \( f(t) \neq 0 \). We notice straightaway that by Lemma 4.4
there must be at least one bad point \( u \) with \( u \prec t \) and \( \rho(u) = \rho(t) \). By continuity of
f and minimality of t, it cannot be that t is a limit element and so t has an immediate predecessor \( s = t^- \); since \( Rf = 0 \) it must be that \( \rho(t) = \rho(s) \). By minimality of t we have \( f(s) = 0 \) and since \( Sf = 0 \) we have \( \sum_{v \in F_s \setminus \{t\}} f(v) = -f(t) \neq 0 \). Let \( t' \) be any element of \( F_s \setminus \{t\} \) with \( f(t') \neq 0 \). By Lemma 4.4 applied this time to \( t' \) there exists at least one bad point \( u' \) with \( u' \gg t' \) and \( \rho(u') = \rho(t') = \rho(s) \). The existence of two bad points \( u, u' \) with \( u, u' > s \) and \( \rho(u) = \rho(u') = \rho(s) \) contradicts property (ii) of the function \( \rho \).

We now set about constructing a MLUR renorming of \( C_0(\mathcal{Y}) \). To get an idea of how one can have such a renorming when no LUR renorming exists one can start with the observation that in the unit sphere of the space \( C[0,1] \), under the supremum norm, there are no points of local uniform convexity, although the constant functions \( \pm 1 \) are points of mid-point local uniform convexity. The following lemma presents a small generalization of this remark. For a bounded function \( f \) defined on a set \( V \), we write \( \text{osc}(f) = \sup_{s.t \in V} |f(s) - f(t)| \) and we define the oscillation norm \( \| \cdot \|_{\text{osc}} \) by

\[
\|f\|_{\text{osc}}^2 = \|f\|_\infty^2 + \text{osc}(f)^2.
\]

**Lemma 5.2.** An element \( g \) of \( \ell_\infty(V) \) is a point of mid-point local uniform convexity of \( \| \cdot \|_{\text{osc}} \) if and only if \( g \) takes at most two values. If \( g \) is “within \( \epsilon \) of being two-valued”, in the sense that there exist real numbers \( a, b \) such that \( g \) takes values in \([a - \epsilon, a + \epsilon] \cup [b - \epsilon, b + \epsilon] \), then \( g \) has the following property:

\[
\|g + h\|_{\text{osc}}^2 + \|g - h\|_{\text{osc}}^2 - 2\|g\|_{\text{osc}}^2 < \epsilon^2 \implies \|h\| < 4\epsilon.
\]

**Proof.** We give a proof only of the second assertion. It is not hard to see that the hypothesis about \( g \pm h \) implies that

\[
\|g + h\|_\infty^2 + \|g - h\|_\infty^2 - 2\|g\|_\infty^2 < \epsilon^2,
\]

\[
\text{osc}(g + h)^2 + \text{osc}(g - h)^2 - 2\text{osc}(g)^2 < \epsilon^2,
\]

and that these inequalities imply

\[
\|g \pm h\|_\infty < \|g\|_\infty + \epsilon, \quad \text{osc}(g \pm h) < \text{osc}(g) + \epsilon.
\]

Now let \( A = \sup g[V] \) and \( B = \inf g[V] \) and assume without loss of generality that \( \|g\|_\infty = A \), \( \text{osc} g = A - B \). By what we have just proved we have

\[
\|g \pm h\|_\infty < A + \epsilon, \quad \text{osc}(g \pm h) < A - B + \epsilon.
\]

If \( s \in V \) is such that \( g(s) \geq A - 2\epsilon \) it is now clear that \( |h(s)| \) can be at most \( 3\epsilon \). It is clear also that if \( s \) is such that \( g(s) \) is sufficiently close to \( A \) then \( |h(s)| < \epsilon \).

We now choose such an \( s \) and consider \( t \) such that \( g(t) \leq B + 2\epsilon \). We have \( |h(t)| = |g(s) - (g(t) + \sigma h(t))| - |g(t) - g(s)| \) for a suitable choice of sign \( \sigma = \pm 1 \). Thus

\[
|h(t)| \leq \epsilon + |(g + \sigma h)(s) - (g + \sigma h)(t)| - |g(s) - g(t)|
\]

\[
\leq \epsilon + \text{osc}(g + \sigma h) - (A - B) + 2\epsilon \leq 4\epsilon.
\]

We have now shown that \( |h| < 3\epsilon \) on the set where \( A - 2\epsilon \leq g \leq A \) and that \( |h| < 4\epsilon \) on the set where \( B \leq g \leq B + 2\epsilon \). The hypothesis that \( g \) is almost two-valued tells us these are the only possibilities and hence that \( \|h\|_\infty \leq 4\epsilon \), as claimed. □

We now present a proposition which is analogous to Proposition 4.2.
Proposition 5.3. Let $L$ be a locally compact space, let $(U_i)_{i \in I}$, $(V_i)_{i \in I}$ be families of open and closed subsets of $L$ and let $T : C_0(Y) \to c_0(I)$ be a bounded linear operator. Assume that, for each $i$, there is an equivalent MLUR norm $\| \cdot \|_i$ on $C_0(U_i)$. Assume further that, for all $f \in C_0(L)$ and all $\epsilon > 0$, the set $\{ t \in L : f(t) \neq 0 \}$ is contained in the union

$$\bigcup \{ U_i : (Tf)(i) \neq 0 \} \cup \bigcup \{ V_i : (Tf)(i) \neq 0 \}
$$

and $f \mid V_i$ is within $\epsilon$ of being two-valued).

Then $C_0(L)$ admits a MLUR renorming.

Proof. For technical reasons it is convenient to introduce a total order on the index set $I$. When $m$ is a positive integer and $F$ is a subset of $I$ with $\# F = m$ there is thus a fixed increasing order $i_0 < i_1 < \cdots < i_{m-1}$ in which to write the elements of $F$. If $\pi \in \{0,1\}^m$ we write $F^\pi$ for the subset $\{i_j : j < m \text{ and } \pi(j) = 1\}$ of $F$.

Now, for each positive integer $m$, each subset $F$ of $I$ with $\# F = m$ and each $\pi \in \{0,1\}^m$, we define

$$\phi(F; f) = \left[ \sum_{i \in F} ( (Tf)(i) )^2 \right]^{\frac{1}{2}} ;$$

$$\psi(\pi, F; f) = \| f 1_{L \setminus (\bigcup_{i \in F} U_i \cup \bigcup_{i \in F^\pi} V_i) } \|_{\infty}^2 + \sum_{i \in F} \| f \mid U_i \|_{i}^2 + \sum_{i \in F^\pi} \| f \mid V_i \|_{\text{osc}}^2 .$$

We now apply the norm construction of Lemma 1.1 obtaining a norm $\| \cdot \|$ such that

$$\| f \| = \| f \|_{\infty}^2 + \sum_{m=1}^{\infty} \alpha 2^{m-2^m} \sum_{\pi \in \{0,1\}^m} \theta(m, \pi; f),$$

where

$$\theta(m, \pi; f) = \sum_{l=1}^{\infty} 2^{-l} \sup \# F = m \left[ \phi(F; f)^2 + 2^{-l} \psi(\pi, F; f)^2 \right].$$

For any $f \in C_0(L)$ and any $\epsilon > 0$ there exists a finite $H_0 \subset I$ such that $\{ t \in L : |f(t)| \geq \epsilon \}$ is contained in the union of the $U_i$ ($i \in H_0$), together with those $V_i$ ($i \in H_0$) for which $f \mid V_i$ is within $\epsilon$ of being two-valued. As in the proof of Proposition 4.2, we find $\alpha > 0$ such that $H_0 \subseteq H = \{ i \in I : |(Tf)(i)| \geq \alpha \}$ and we set $m = \# H$. As a small refinement, we let $\pi \in \{0,1\}^m$ be the function such that $H^\pi$ contains exactly those $i \in H$ such that $f \mid V_i$ is within $\epsilon$ of being two-valued.

Now let $(h_n)$ be a sequence in $C_0(L)$ such that $\| f + h_n \|_{\infty}^2 + \| f - h_n \|_{\infty}^2 - 2 \| f \|_{\infty}^2 \to 0$. For the $m$ and $\pi$ which we fixed above, we have

$$\theta(m, \pi; f_n) + \theta(m, \pi; f) - 2 \theta(m, \pi; \psi(f + f_n)) \to 0$$

as well. If $(F_n)$ is the sequence given by the conclusion of Lemma 1.1 then, as in the proof of Proposition 4.2, we must have $F_n = H$ for all large enough $n$. We also have

$$\psi(H, \pi; f + h_n)^2 + \psi(H, \pi; f + h_n)^2 - 2 \psi(H, \pi; f) \to 0$$
as \( n \to \infty \), which implies in turn that
\[
\| (f + h_n) \|_i^2 + \| (f - h_n) \|_i^2 - 2 \| f \|_i^2 \to 0 \quad (i \in H),
\]
\[
\| (f + h_n) \|_{\text{osc}}^2 + \| (f - h_n) \|_{\text{osc}}^2 - 2 \| f \|_{\text{osc}}^2 \to 0 \quad (i \in H^+),
\]
\[
\| (f \pm h_n) \|_{111} \to \| f \|_{111} \quad (U_{i \in H} \cup U_{i \in H^+} \|_i \|_i) \to \| f \|_{111} \leq \epsilon.
\]

These limits give us, for all large enough \( n \),
\[
| h_n | < \epsilon \text{ on } U_i \ (i \in H) \text{ by the MLUR property of } \| \cdot \|_i,
\]
\[
| h_n | < 4\epsilon \text{ on } V_i \ (i \in H^+) \text{ by Lemma 5.2},
\]
\[
| h_n | < 2\epsilon \text{ on the rest of } L.
\]

We have proved that \( \| h_n \|_{\infty} \to 0 \) and thus that \( \| \cdot \| \) is MLUR. \( \square \)

We are now ready to complete the proof of Theorem 5.1 by showing that, subject to condition (3), we can find \( I, T, U_i, V_i \) to which Proposition 5.3 may be applied.

We take \( T \) to be the operator \( R \oplus S : c_0(\mathcal{Y}) \to c_0(\mathcal{Y}) \oplus c_0(\mathcal{Y}) \) that we have already looked at in this section. For \( t \in \mathcal{Y} \) we define \( a(t) \) to be the minimal element of \( \{ s \in (0, t] : \rho(s) = \rho(t) \} \) and, if there is a bad point \( u \) with \( u \gg a(t) \) and \( \rho(u) = \rho(t) \) we define \( b(t) \) to be this point (which is unique because of condition (ii)). We set \( U_t = (0, t] \cup (a(t), b(t)] \) and \( V_t = [t, \infty) \), noting that since \( U_t \) is homeomorphic to an interval of ordinals there is certainly a LUR renorming of \( C_0(U_t) \).

**Lemma 5.4.** Let \( \mathcal{Y} \) and \( \rho \) satisfy conditions (i) and (ii) and suppose further that \( \rho \) is continuous for the locally compact topology of \( \mathcal{Y} \). Let \( f \) be in \( C_0(\mathcal{Y}) \) and let \( s \) be an element of \( \mathcal{Y} \), with \( f(s) \neq 0 \), which is not in the union \( \bigcup \{ U_t : (Rf)(t) \neq 0 \text{ or } (Sf)(t) \neq 0 \} \). Then the restriction of \( f \) to \( [a(s), \infty) \) is a constant multiple of the indicator function \( 1_{[a(s), b(s)]} \). Moreover, \( a(s) \) is a limit element of \( \mathcal{Y} \) and there is a sequence \( (r_n) \) increasing to \( a(s) \) in \( \mathcal{Y} \) such that \( (Rf)(r_n) \neq 0 \) for all \( n \).

**Proof.** We start by noting that certainly there is no \( t \gg s \) with \( (Sf)(t) \neq 0 \). It follows from Lemma 4.4 that \( [s, \infty) \cap \rho^{-1}(\rho(s)) \) cannot contain only good points, so it must be that \( b(s) \) exists and that \( s \in [a(s), b(s)] \). Now for any \( t \) with \( t \gg a(s) \) and \( \rho(t) = \rho(s) \) we have \( a(t) = a(s) \) and \( b(t) = b(s) \), so that \( s \in U_t \) whence, by hypothesis, \( (Sf)(t) = 0 \). Using Lemma 4.4 again, we can now see that on the set \( [a(s), \infty) \cap \rho^{-1}(\rho(s)) \) our function \( f \) can be nonzero only on \( [a(s), b(s)] \), and since \( (Sf)(t) = 0 \) for all \( t \in [a(s), b(s)] \) it must be that \( f \) is constant on \( [a(s), b(s)] \).

If \( a(s) \) were not a limit element of \( \mathcal{Y} \) we should have \( \rho(a(s)) > \rho(a(s)^-) \) by the minimality in the definition of \( a(s) \) and since \( f(a(s)) = f(s) \neq 0 \) we should have \( (Rf)(a(s)) \neq 0 \) and \( s \in [a(s), b(s)] \subseteq U_{a(s)} \), contrary to assumption. Since \( a(s) \) is a limit point with \( \rho(a(s)) > \rho(r) \) when \( r \ll a(s) \) there is a sequence \( (r_n) \) increasing to \( a(s) \) consisting of successor elements with \( \rho(r_n) > \rho(r_n^-) \). Because \( f \) is continuous it is non-zero on some neighbourhood of \( a(s) \) and so we may assume that \( f(r_n) \neq 0 \) for all \( n \).

Finally we have to show that if \( s' \gg a(s) \) and \( \rho(s') > \rho(s) \) then \( f(s') \neq 0 \). We note that for such an \( s' \) we have \( s \in U_t \) whenever \( s' \in U_t \) so that \( (Sf)(t) = 0 \) whenever \( s' \in U_t \). Hence, if \( f(s') \) is not 0, all of the above analysis is applicable to \( s' \) and in particular \( a(s') \) is a limit element, with \( f(s') \neq 0 \). Also there exists a sequence \( (r_n') \) increasing to \( a(s') \) with \( (Rf)(r_n') \neq 0 \) for all \( n \). But this cannot be, since we have \( s \in (0, r'_n] \subseteq U_{r'_n} \) for suitably large \( n \). \( \square \)
The lemma we have just proved really does all the work for us since, given \( s \) with \( f(s) \neq 0 \) and \( s \notin \bigcup \{ U_t : (Rf)(t) \neq 0 \text{ or } (Sf)(t) \neq 0 \} \), we get a sequence \( (r_n) \) increasing to \( a(s) \) and consisting of points with \( (Rf)(r_n) \neq 0 \). Given \( \epsilon > 0 \) we may find \( n \) such that \( |f(r_n) - f(a(s))| < \epsilon \) for all \( r_n \) and \( s \notin \bigcup \{ U_t : (Rf)(t) \neq 0 \text{ or } (Sf)(t) \neq 0 \} \), the second of these properties resulting from the compactness of the set where \( |f| \geq \epsilon \). Together with what we have already proved about \( f \) restricted to \( [r_n, \infty) \), this shows that on \( V_{r_n} = [r_n, \infty) \) the function \( f \) is indeed within \( \epsilon \) of being two-valued.

We have now shown that the hypotheses of Proposition 5.3 are satisfied and we have thus completed the proof of the theorem.

6. Kadec renormings

**Theorem 6.1.** Let \( \Upsilon \) be a tree. The Banach space \( C_0(\Upsilon) \) admits an equivalent norm with the Kadec Property if and only if there exists an increasing function \( \rho : \Upsilon \to \mathbb{R} \) with no bad points.

We have already seen that if \( C_0(\Upsilon) \) admits a Kadec norm then the associated \( \mu \) function has no bad points. We shall devote this section to the construction of a Kadec renorming, starting from a function \( \rho \) with no bad points. As in the proof of Theorem 4.1, we may suppose that \( \rho \) takes values in \((0,1)\) and is continuous for the locally compact topology of \( \Upsilon \).

For \( r \in \Upsilon \), \( f \in C_0(\Upsilon) \) and \( 1 \leq l \in \mathbb{N} \) we make the following definitions:

\[
\Delta^\pm(f;r) = \inf \{ \|g \pm f\|_\infty : g \in C_0[r, \infty) \text{ and } g \text{ is decreasing} \} ;
\]

\[
A_l(f,r) = \frac{1}{l} \sup \left\{ \sum_{k=1}^{l} |f(s_k)| : \{s_1, s_2, \ldots, s_l\} \text{ is an antichain in } [r, \infty) \right\} .
\]

Notice that \( A_1(f,r) = \|f\|_{[r, \infty)} \). We recall our standard notation for good points:

\[
F_t = \{ u \in t^+ : \rho(u) = \rho(t) \} , \quad \delta_t = \inf \{ \rho(v) - \rho(t) : v \in t^+ \setminus F_t \} .
\]

**Lemma 6.2.** There are uniquely determined functions \( \Phi, \Psi, \Theta, \Omega, \Sigma, \Xi : C_0(\Upsilon) \times (\Upsilon \cup \{0\}) \to \mathbb{R}^+ \) and \( \Xi : C_0(\Upsilon) \times (\Upsilon \cup \{0\}) \times (\Upsilon \cup \{0\}) \to \mathbb{R}^+ \) which satisfy the inequalities

\[
\Phi(f;s), \Psi(f;s), \Theta(f;s), \Omega(f;s), \Sigma(f;s), \Xi(s,t) \leq \|f\|_\infty
\]
for $f \in C_0(\Upsilon)$ and $s \leq t \in \Upsilon$ as well as the identities:

$$7 \Phi(f; s) = \Delta^+(f; s) + \Delta^-(f; s) + \sum_{t=1}^{\infty} 2^{-t} \Lambda_t(f; s)$$

$$+ \Sigma(f; s) + \Psi(f; s) + \Theta(f; s) + \Omega(f; s);$$

$$2 \Sigma(f; s) = \sum_{m,l=1}^{\infty} 2^{-m-l} \sup_{s \leq t} \left[ |f(t)| + 2^{-m} \Xi(f; s, t) + 2^{-l} \Phi(f; t) \right];$$

$$4 \Psi(f; s) = \sum_{m,l=1}^{\infty} 2^{-m-l} \sup_{s \leq t \leq u \leq \infty} \left[ |f(s) - f(t) + f(u)| + 2^{-m} \Xi(f; s, t) + 2^{-l} \Phi(f; t) \right];$$

$$2 \Theta(f; s) = \sum_{m,l=1}^{\infty} 2^{-m-l} \sup_{s \leq t} \frac{1}{1 + \# F_t} \left[ \delta_l(f(t) - \sum_{u \in F_t} f(u)) + \sum_{u \in F_t} \left( 2^{-m} \Xi(f; s, u) + 2^{-l} \Phi(f; u) \right) \right];$$

$$2 \Omega(f; s) = \sum_{m,l=1}^{\infty} 2^{-m-l} \sup_{s \leq t} \left[ (\rho(t) - \rho(t^-)) |f(t)| + 2^{-m} \Xi(f; s, t) + 2^{-l} \Phi(f; t) \right];$$

$$2 \Xi(f; s, t) = \| f \|_{(s,t]} \| \Phi(f,1_{\Upsilon \setminus (0,t,\infty)}; r).$$

As functions of $f$, all of $\Phi, \Sigma, \Psi, \Theta, \Omega, \Xi$ are $\tau_p$-lower semicontinuous seminorms on $C_0(\Upsilon)$. They are continuous for the locally compact topology of $\Upsilon$ as functions of their arguments $s, t$. All of these functions are decreasing in $s$, and $\Xi$ is increasing in $t$.

**Proof.** The existence and uniqueness are immediate consequences of Banach’s fixed-point theorem in a suitable space of functions. The other assertions are straightforward to verify. ∎

It is clear that $\| f \| = \Phi(f; 0)$ defines a norm on $C_0(\Upsilon)$ such that $\frac{1}{2} \| f \| \leq \| f \| \leq \| f \|$. We shall show that $\| \cdot \|$ has the Kadec Property.

**Lemma 6.3.** Let $f_n$ and $f$ in $C_0(\Upsilon)$ and $r \in \Upsilon$ be such that $f_n \to f$ pointwise and $\Phi(f_n; r) \to \Phi(f; r)$. Then $\Sigma(f_n; r), \Psi(f_n; r), \Theta(f_n; r), \Omega(f_n; r), \Delta^+(f_n; r),$ and $\Lambda_l(f_n; r)$ $(l \geq 1)$ converge as $n \to \infty$ to the limits $\Sigma(f; r), \Psi(f; r), \Theta(f; r), \Omega(f; r),$ $\Delta^+(f; r),$ and $\Lambda_l(f; r),$ respectively. In particular, $\| f_n \|_{(r, \infty)} \to \| f \|_{(r, \infty)} \|_{\infty}.$

**Proof.** This follows immediately from the $\tau_p$-lower semicontinuity of the terms in the sum defining $\Phi$. ∎

**Lemma 6.4.** Let $f_n$ and $f$ in $C_0(\Upsilon)$ and $r \in \Upsilon$ be such that $f_n \to f$ pointwise and $\Phi(f_n; r) \to \Phi(f; r)$. Let $\epsilon$ be any real number with $0 < \epsilon \leq \| f \|_{(r, \infty)} \|_{\infty}$. There exists $s \in (r, \infty)$ such that

1. $\| f(s) \|_{[s, \infty)} \|_{\infty} \geq \epsilon$;
2. $f_n \to f$ uniformly on $(r, s]$;
3. $\Phi(f_n, 1_{\Upsilon \setminus (0, s, \infty)}; r) \to \Phi(f, 1_{\Upsilon \setminus (0, s, \infty)}; r)$;
4. $\Phi(f_n; 1_{[r, \infty)}; r) \to \Phi(f; 1_{[r, \infty)}; r)$.

**Proof.** We may assume that $\| f \|_{\infty} < 1$, so that the same is true for any of the functions $\Phi, \Sigma$ and so on, associated with $f$. The set $M$ of maximal elements of
\[ \{ t \in [r, \infty) : |f(t)| \geq \epsilon \} \text{ is finite and we can choose a natural number } m \text{ such that } |f(s)| < \epsilon - 2^{-m} \text{ whenever } s \in [r, \infty) \setminus \bigcup_{t \in M} [r, t]. \]

By Lemma 6.3, \( \Sigma(f_n; r) \to \Sigma(f; r) \) as \( n \to \infty \) and it follows similarly that

\[
\sum_{i=1}^{\infty} 2^{-i} \sup_{r \leq s} \left| f_n(s) \right| + 2^{-m} \Xi(f_n; r, s) + 2^{-i} \Phi(f_n, s) \]

tends to

\[
\sum_{i=1}^{\infty} 2^{-i} \sup_{r \leq s} \left| f(s) \right| + 2^{-m} \Xi(f; r, s) + 2^{-i} \Phi(f, s). \]

Now by applying Lemma 1.2 with \( I = [r, \infty) \), \( \phi_s(g) = |g(s)| + 2^{-m} \Xi(g; r, s) \), \( \psi_s(g) = \Phi(g; s) \), we see that there is a sequence \( (s_n) \) in \([r, \infty)\) such that \( |f(s_n)| + 2^{-m} \Xi(f; r, s_n) \) and \( |f_n(s_n)| + 2^{-m} \Xi(f_n; r, s_n) \) and \( \sup_{t \in [r, \infty)} |f_n(t)| + 2^{-m} \Xi(f_n; r, t) \) all converge as \( n \to \infty \) to the limit sup \( t \in [r, \infty) \) \( |f(t)| + 2^{-m} \Xi(f; r, t) \), and moreover such that \( \Phi(f_n; s_n) - \Phi(f; s_n) \) tends to zero. We may further suppose that \( (s_n) \) converges in the sequentially compact space \( \Upsilon \). Its limit must be a point \( s \) of \( \Upsilon \), since \( f(s_n) \) does not tend to zero as \( n \to \infty \). In fact, \( s \) is a point at which the supremum \( \sup_{t \in [r, \infty)} |f(t)| + 2^{-m} \Xi(f; r, t) \) is attained and hence, by our original choice of \( m \), it must be in \( [r, \infty) \). If \( t \in M \) is such that \( s \preceq t \) then we have \( |f(s)| \geq |f(t)| + \Xi(f; r, t) - \Xi(f; r, s) \geq |f(t)| \), since \( \Xi(f; r, t) \) is increasing. Thus \( |f(s)| \geq \epsilon \).

It is easy to see that (4) holds. Indeed, we have \( \liminf \Phi(f_n; s) \geq \Phi(f; s) \) by \( \tau_p \)-lower semicontinuity. On the other hand, since each \( \Phi(g; t) \) is a continuous decreasing function of \( t \), we have \( \Phi(f; s) \to \Phi(f; s) \) and \( \Phi(f_n; s) \geq \Phi(f_n; s) \). Using these facts and remembering that \( \Phi(f_n; s) - \Phi(f; s) \) tends to zero, we see that \( \limsup \Phi(f_n; s) \leq \limsup \Phi(f_n; s) = \lim \Phi(f_n; s) = \Phi(f; s) \).

We next show that \( \Xi(f_n; r, s) \to \Xi(f; r, s) \) as \( n \to \infty \). By \( \tau_p \)-lower semicontinuity, we have \( \liminf \Xi(f_n; r, s) \geq \Xi(f; r, s) \). On the other hand,

\[
\limsup \Xi(f_n; r, s) \leq \limsup \left\{ \sup_{t \in [r, \infty)} \left( 2^m |f_n(t)| + \Xi(f_n; r, t) \right) - 2^m |f_n(s)| \right\}
= \limsup \left\{ 2^m |f(s_n)| + \Xi(f; r, s_n) \right\} - 2^m |f(s)|
= 2^m |f(s)| + \Xi(f; r, s) - 2^m |f(s)| = \Xi(f; r, s).
\]

The definition of \( \Xi \) and the usual lower semicontinuity argument now gives us the convergence of \( \Phi(f_n; 1_{\Upsilon \setminus (0, \infty)}; r) \) to \( \Phi(f; 1_{\Upsilon \setminus (0, \infty)}; r) \), which is (3), as well as the convergence of \( \|f_n\|_{(r, \infty), \text{ord}} \) to \( \|f\|_{(r, \infty), \text{ord}} \), from which (2) follows because \( \| \cdot \|_{\text{ord}} \) is a Kadec norm. \( \square \)

The theorem will follow from the following proposition, the form of which is chosen to simplify a proof by induction.

**Proposition 6.5.** Let \( \epsilon \) be a positive real number, let \( q \) be in \( \Upsilon \cup \{ 0 \} \) and let \( f \) be an element of \( C_0(\Upsilon) \). If the sequence \( (f_n) \) in \( C_0(\Upsilon) \) tends pointwise to \( f \) and is such that \( \Phi(f_n; q) \to \Phi(f, q) \), then \( \limsup_{n \to \infty} \| (f - f_n) \|_{(q, \infty)} < 2\epsilon \).

We define

\[ m(g; r; \epsilon) = \max \{ \# A : A \text{ is an antichain in } [r, \infty) \text{ and } |g(t)| \geq \epsilon \text{ for all } t \in A \}, \]
and note that if \( m(f; q; \epsilon) = 0 \) then \(|f|_{[q, \infty]} \parallel_{\infty} < \epsilon \). Since \( \Phi(f_n; q) \rightarrow \Phi(f, q) \), we have \( \parallel f_n \parallel_{[q, \infty]} \parallel_{\infty} = \Lambda_1(f_n; q) \rightarrow \Lambda_1(f; q) = \parallel f \parallel_{[q, \infty]} \parallel_{\infty} \), so that \( \parallel f_n \parallel_{[q, \infty]} \parallel_{\infty} \) is smaller than \( \epsilon \) for all sufficiently large \( n \).

We now suppose that \( m(f, q, \epsilon) > 0 \) and assume inductively that our result is true for any pair \( g, r \) such that \( m(g; r; \epsilon) < m(f; q; \epsilon) \).

We write \( K \) for the subset

\[
K = \left\{ r \in [q, \infty) : \parallel f \parallel_{[r, \infty]} \parallel_{\infty} \geq \epsilon \text{ and } \lim\sup_{n \rightarrow \infty} \Phi(f_n, r) = \Phi(f, r) \right\}
\]

and set

\[
H = \{ s \in K : \parallel f(s) \parallel = \parallel f \parallel_{[s, \infty]} \parallel_{\infty} \}.
\]

Evidently \( K \) is non-empty since \( q \in K \). Our first task is to show that \( H \) is non-empty.

**Claim 1.** For every \( r \in K \) there exists \( s \in H \cap [r, \infty) \).

**Proof.** Since \( \Phi(f_n; r) \rightarrow \Phi(f; r) \) we can apply Lemma 6.4 (with the \( \epsilon \) of that lemma equal to \( \parallel f \parallel_{[r, \infty]} \parallel_{\infty} \)) getting an element \( s \) of \([r, \infty)\). We have \( \parallel f(s) \parallel = \parallel f \parallel_{[r, \infty]} \parallel_{\infty} \geq \epsilon \) and \( \Phi(f_n; s) \rightarrow \Phi(f; s) \), by (1) and (4) in that Lemma. Now \( \parallel \Phi(f_n, r) \parallel \rightarrow \parallel \Phi(f, r) \parallel \) tends to the limit \( \Phi(f, 1 \setminus (0, s, \infty); r) \). Since \( \parallel f(s) \parallel \geq \epsilon \) and \( \T \setminus (0, s, \infty) \) contains no points comparable with \( s \) we have \( m(f, 1 \setminus (0, s, \infty); r, \epsilon) < m(f, q, \epsilon) \) so that our inductive hypothesis yields \( \lim\sup \parallel (f - f_n) \parallel_{[r, \infty) \setminus (r, s, \infty)} \parallel_{\infty} < 2\epsilon \). Finally, the assertion (2) of Lemma 6.4 tells us that \( f_n \rightarrow f \) uniformly on \([r, s] \), so that certainly \( \lim\sup \parallel (f - f_n) \parallel_{[r, s]} \parallel_{\infty} < 2\epsilon \). \( \square \)

We may suppose that \( H \) is totally ordered by \( \leq \). Indeed, if \( r, s \in H \) are incomparable, we have \([q, \infty) = (\langle q, \infty \rangle \setminus [r, \infty)) \cup (\langle q, \infty \rangle \setminus [s, \infty)) \). From the way in which we defined \( K \) it follows that \( \lim\sup \parallel (f - f_n) \parallel_{[s, \infty]} \parallel_{\infty} < 2\epsilon \) which is what we want to prove. Since \( \parallel f \parallel_{[r, \infty]} \parallel_{\infty} \geq \epsilon \) for all \( r \in H \), the set \( H \) is relatively compact in \( \T \) and so has a supremum \( s \) in \( \T \), with \( \parallel f(s) \parallel \geq \epsilon \) by continuity of \( f \). We shall next show that \( s \) is in \( H \).

To show that \( \Phi(f_n; s) \rightarrow \Phi(f; s) \) is not hard, since by continuity of \( \Phi(f; \cdot) \) we can find, given \( \eta > 0 \), some \( r \in H \) with \( \Phi(f; r) < \Phi(f; s) + \eta \). Since each \( \Phi(f_n; \cdot) \) is a decreasing function, we therefore have

\[
\lim\sup \Phi(f_n; s) \leq \lim \Phi(f_n; r) = \Phi(f; r) < \Phi(f; s) + \eta.
\]

As in the proof of Lemma 6.4, consider the finite set \( M = \max \{ u \in [s, \infty) : |f(u)| \geq \epsilon \} \) and choose \( m \in \mathbb{N} \) such that \( |f(t)| < \epsilon - 2^{-m} \) whenever \( t \in [s, \infty) \setminus \bigcup_{u \in M} [s, u) \). Given \( \eta > 0 \) we may choose \( r \in H \) such that \( \Phi(f; r) < \Phi(f; s) + \eta \) and \( \parallel f \parallel_{[r, s]} \parallel_{\infty} < \parallel f(s) \parallel + \eta 2^{-m} \). We may also suppose that \( \parallel f \parallel_{[r, \infty) \setminus [r, s, \infty)} \parallel_{\infty} < \epsilon \). Since \( r \in H \) we have \( \Phi(f_n; r) \rightarrow \Psi(f; r) \) and so we may apply Lemma 6.4, obtaining \( s' \in [r, \infty) \) such that

\[
|f(s')| + 2^{-m} \Xi(f; r, s') = \lim_{j \rightarrow \infty} \left( |f_n(s')| + 2^{-m} \Xi(f_n; r, s') \right)
\]

\[
= \lim_{j \rightarrow \infty} \sup_{|s| \in [r, \infty)} \left( |f_n(s)| + 2^{-m} \Xi(f_n; r, s) \right),
\]
and such that, moreover, \( \Phi(f_n; s') \to \Phi(f; s') \). By our choice of \( m \) it must be that \( |f(s')| \geq \epsilon \). As in our proof that \( s' \) is non-empty, we show that \( s' \) is in \( H \), which implies that \( s' \leq s \). By the way in which we chose \( r \), we have \( |f(s')| \leq |f(s)| + \eta \), so that

\[
\limsup_{j \to \infty} \Xi(f_{n_j}; r, s) \leq \limsup_{j \to \infty} \left( 2^m|f_n(s)| + \Xi(f_n; r, s) \right) - 2^m|f(s)|
= 2^m|f(s')| + 2^{-m}\Xi(f; r, s') - 2^m|f(s)|
\leq \Xi(f; r, s') + \eta
\leq \Xi(f; r, s) + \eta
\leq \liminf_{j \to \infty} \Xi(f_n; r, s) + \eta.
\]

Thus \( \Xi(f_n; r, s) \to \Xi(f; r, s) \), which, as we have seen, implies that \( f_n \) converges to \( f \) uniformly on \([r, s]\) and that \( \|f_n|_{[r_1, \infty) \setminus [r_1, s, \infty)\} \|_\infty \) tends to \( \|f|_{[r_1, \infty) \setminus [r_1, s, \infty)\} \|_\infty \), a quantity known to be smaller than \( \epsilon \).

We have now found a maximal element \( s \) of \( H \). To finish the proof we have to show that \( \limsup \| (f - f_n) |_{[s, \infty)} \|_\infty < 2\epsilon \).

**Claim 2.** Assume that one of the following holds:

(a) \( f \) is not monotone on \([s, \infty)\);
(b) there is some \( t \in [s, \infty) \) with \( f(t) \neq \sum_{u \in F} f(u) \);
(c) there is some \( t \in \Upsilon^+ \cap (s, \infty) \) such that \( \rho(t) \neq \rho(t^-) \) and \( f(t) \neq 0 \).

Then there exists \( w \in (r, \infty) \) such that:

1. \( f_n \to f \) uniformly on \([s, w]\);
2. \( \Phi(f_n, 1_{\Upsilon \setminus (0, w, \infty)}; s) \to \Phi(f, 1_{\Upsilon \setminus (0, w, \infty)}; s) \);
3. \( \Phi(f_n; w) \to \Phi(f; w) \).

**Proof.** This is very similar to the proof of Lemma 6.4 except that in the cases (a), (b) and (c), we use the functions \( \Psi, \Theta \) and \( \Omega \), respectively, instead of \( \Sigma \). Consider, for example, case (a), where \( f \) is not monotone on \([s, \infty)\). Since \( |f(s)| = \|f|_{[s, \infty)} \|_\infty \), we see that

\[
\sup_{s \leq t \leq u} |f(s) - f(t) + f(u)| > \|f|_{[s, \infty)} \|_\infty.
\]

For suitably chosen \( m \in \mathbb{N} \) therefore, the supremum of \( |f(s) - f(t) + f(u)| + 2^{-m}\Xi(f; s, t) \) is strictly greater than \( \|f|_{[s, \infty)} \|_\infty + 2^{-m} \). We apply Lemma 1.2 with \( I = [s, \infty) \), \( \phi(t) = \sup_{u \in [t, \infty)} |f(s) - f(t) + f(u)| + 2^{-m}\Xi(f; s, t) \) and \( \psi(t) = \Phi(f; t) \), getting a sequence \( (t_n) \) which converges in \( \Upsilon \cap \{s, \infty\} \). and which satisfies, among other things, \( \phi_{n_0}(f) \to \sup \phi_{n_0}(f) \). The limit cannot be \( \infty \) and cannot be \( s \) since in these cases we should have \( \lim sup \phi_{n_0}(f) \leq \|f|_{[s, \infty)} \|_\infty + 2^{-m} \), which is a contradiction, since \( (t_n) \) satisfies, among other things, \( \phi_{n_0}(f) \to \sup \phi_{n_0}(f) \). We take \( w = t \) and can now establish (1), (2) and (3) as in the proof of Lemma 6.4. \( \square \)

Suppose now that we are in the situation where the above Claim is applicable and \( w \in (s, \infty) \) has properties (1),(2),(3). If \( \|f|_{[w, \infty)} \|_\infty \geq \epsilon \) then \( m(f, 1_{\Upsilon \setminus (0, w, \infty)}; s, \epsilon) \) is smaller than \( m(f, g, \epsilon) \), so that \( \lim sup \| (f - f_n) |_{[s, \infty) \setminus [s, w, \infty)} \|_\infty < 2\epsilon \) by (2) and our inductive hypothesis. Taken together with (1), this tells us that \( w \) is in \( K \), so that \( [w, \infty) \cap H \neq \emptyset \), contradicting the maximality of \( s \) in \( H \).
Suppose now that \( \| f \|_{[s, \infty) \setminus [s, w, \infty]} \| \infty \geq \epsilon \). Our inductive hypothesis, together with (3), gives \( \limsup \| (f - f_n) \|_{[w, \infty]} \| \infty < 2\epsilon \). It is not hard to see that if we apply Lemma 6.4 to the functions \( f_n, 1_{T^* \setminus [0, w, \infty)} \) and \( f, 1_{T^* \setminus [0, w, \infty)} \) then we obtain an element of \( H \cap (s, \infty) \), again contradicting maximality of \( s \).

The only remaining possibility is that both \( \| f \|_{[s, \infty]} \| \infty \) and \( \| f \|_{[s, \infty) \setminus [s, w, \infty]} \| \infty \) are smaller than \( \epsilon \). In this case, the initial "\( m(f, q, \epsilon) = 0 \)" case of the proposition gives us \( \limsup \| f \|_{[s, \infty]} \| \infty < 2\epsilon \) and \( \limsup \| f \|_{[s, \infty) \setminus [s, w, \infty]} \| \infty < 2\epsilon \). Taken together with (1), these give us what we want to prove.

Finally we have to deal with the case where none of the possibilities (a),(b),(c) occurs.

**Claim 3.** If none of (a),(b),(c) holds then:

(1) \( f \) is a monotone function on \([s, \infty] \);
(2) for every \( t \in [s, \infty] \) we have \( f(t) = \sum_{u \in F} f(u) \);
(3) \( \supp f \cap [s, \infty] \subseteq \{ t \in [s, \infty] : \rho(t) = \rho(s) \} \);
(4) for each \( l \geq 1 \) and each \( t \in [s, \infty] \), \( A_i(f, t) = l^{-1} \| f(t) \| \);
(5) for any \( \eta > 0 \) there exists a finite antichain \( A \subseteq [s, \infty] \) such that \( 0 < \| f(t) \| < \eta \) for all \( t \in A \) and such that \( \sum_{t \in A} \| f(t) \| > \| f(s) \| - \eta \).

**Proof.** Assertions (1) and (2) follow immediately from the failure of (a) and (b). If (3) were not true, there would exist \( t \) minimal subject to \( f(t) \neq 0 \) and \( \rho(t) > \rho(s) \); by monotonicity, \( t \) would be minimal subject to \( \rho(t) > \rho(s) \) and so \( t \) would be in \( \mathcal{T}^* \) by the assumed continuity of \( \rho \). Thus (c) would hold, contrary to supposition.

Without loss of generality, we may suppose that \( f(s) > 0 \). The monotonicity now implies that \( f \) is everywhere non-negative on \([s, \infty] \), since by convention \( f(\infty) \) has been taken to be 0. To prove (4) we proceed by induction on \( l \), the case \( l = 1 \) being true by monotonicity. Now let \( t \in [s, \infty] \) and let \( A \subseteq [t, \infty] \) be an antichain of size \( l \) with \( f(t) > 0 \) for all \( w \in A \). Define \( u \) to be the greatest element such that \( u \leq w \) for all \( w \in A \). By (3), \( \rho(w) = \rho(s) \) for all \( w \in A \), and so \( A = \bigcup_{v \in F_u} A \cap [v, \infty] \). By choice of \( u \) we have \( \#(A \cap [v, \infty] \infty < l \) for all \( v \), whence \( \sum_{w \in A \cap [v, \infty]} f(w) \leq f(v) \) by inductive hypothesis. Thus

\[
\sum_{w \in A} f(w) = \sum_{v \in F_u} \sum_{w \in A \cap [u, \infty)} f(w) \leq \sum_{v \in F_u} f(v) = f(u) \leq f(t)
\]

by (2) and monotonicity.

To prove (5), we consider the infinite antichain \( B = \min\{ v \in [s, \infty] : 0 < f(v) < \eta \} \); provided we can prove that \( f(t) = \sum_{v \in B} f(v) \), it is clear that a suitable finite subset of \( B \) will do for \( A \). Define \( g(t) = \sum_{v \in B \cap [t, \infty]} f(v) \) and consider \( g \) as a function on the set \( T = \{ t \in [s, \infty] : f(t) \geq \eta \} \). Notice that \( g \) is continuous and that for all \( t \in T \) we have \( g(t) = \sum_{u \in u^+ \cap T} f(t) + \sum_{u \in u^+ \cap T} g(t) \). Suppose if possible that the subset \( U = \{ u \in T : f(u) = g(u) \} \) is empty. The subset \( U \) can have no maximal elements since if \( u \) were such an element we should have

\[
g(u) = \sum_{v \in u^+ \cap T} f(v) + \sum_{v \in u^+ \cap T} g(v) = \sum_{v \in u^+} f(v) = f(u),
\]

by (2), (3) and maximality of \( u \) in \( U \). Thus there exists some \( w \in T \setminus U \) which is a limit point of \( U \). For some \( t \in U \cap [s, w) \) the set \( T \cap [t, w] \) is totally ordered;
Corollary 6.6. There is an Asplund space $X$ which admits a Kadec renorming but no strictly convex renorming.

Proof. If $Y$ is a full dyadic tree of height $\omega_1$ then any increasing function $\rho : Y \to \mathbb{R}$ is constant on some set $[u, \infty)$. Hence, by 3.4, $C_0(Y)$ is not strictly convexifiable. On the other hand this space admits a Kadec renorming, since all points of a finitely branching tree are good, even for a constant function $\rho$. □

As promised earlier, we finish this section with a result about $\sigma$-fragmentability.

Proposition 6.7. For a tree $Y$ the space $C_0(Y)$ admits a Kadec renorming if and only if it is $\sigma$-fragmentable.

Proof. As we have already remarked, one implication is already known. Let us assume therefore that $C_0(Y)$ is $\sigma$-fragmentable. By the results of [9] $Y$ is a countable
union $\bigcup_{n \in \mathbb{N}} \Delta_n$ of subsets $\Delta_n$ that are discrete in the reverse topology. For $t \in \Upsilon$ we define $M_t = \{ m \in \mathbb{N} : [t, \infty) \cap \Delta_m = \emptyset \}$ and set $\rho(t) = \sum_{m \in M_t} 2^{-m}$. Obviously $\rho$ is an increasing function. We shall show that all points $t$ are $\rho$-good. Any $t$ is in $\Delta_n$ for some $n$ and, by reverse-discreteness, there is a reverse-open neighbourhood $U$ of $t$ such that $U \cap \Delta_n = \{ t \}$. We may take $U$ to be of the form $[t, \infty) \setminus \bigcup_{u \in F} [u, \infty)$ for a suitable finite subset $F$ of $t^+$. Evidently, $\rho(v) \geq \rho(t) + 2^{-n}$ for all $v \in t^+ \setminus F$, which shows that $t$ is a good point for $\rho$. \hfill $\square$

7. Strictly convex dual norms

In this section we present a sufficient condition for the existence on $C_0(\Upsilon)$ of an equivalent norm having strictly convex dual norm. The reader should be warned that this condition is probably quite far from being necessary, though we shall see in Section 10 that it is sufficiently general to be satisfied even in some cases where $C_0(\Upsilon)$ has neither a Fréchet-differentiable renorming nor a strictly convex renorming.

**Theorem 7.1.** Suppose that on the tree $\Upsilon$ there is an increasing function $\rho : \Upsilon \to \mathbb{R}$ which is constant on no strictly increasing sequence in $\Upsilon$. Then there is an equivalent norm on $C_0(\Upsilon)$ with strictly convex dual norm.

**Proof.** The dual space of $C_0(\Upsilon)$ may be identified with $\ell_1(\Upsilon)$ and the norm dual to the supremum norm is of course the usual $\ell_1$-norm $\|\xi\|_1 = \sum_{t \in \Upsilon} |\xi(t)|$. We shall construct an equivalent norm on $\ell_1(\Upsilon)$ which is strictly convex and lower semicontinuous for the weak* topology $\sigma(\ell_1(\Upsilon), C_0(\Upsilon))$. Useful weak* lower semicontinuous functions to use as building blocks for our norm are functions of the form $\xi \mapsto \|\xi \upharpoonright V\|_1$ with $V$ an open subset of $\Upsilon$. In particular, we may take $V$ of the form $[s, \infty)$ or $\{ s \}$ when $s$ is a successor element of $\Upsilon$.

Let us write $\Upsilon_0$ for the set of all $t \in \Upsilon^+$ such that $\rho(t) > \rho(t^-)$. Notice that without spoiling the assumed property of $\rho$ we may modify that function so that it takes rational values at all points of $\Upsilon_0$.

For each rational $q$ we note that the wedges $[s, \infty)$, with $s \in \Upsilon_0$ and $\rho(s) = q$, are disjoint, so that the family $(\|\xi \upharpoonright [s, \infty)\|_1)_{s \in \Upsilon_0 \cap \rho^{-1}(q)}$ is in $\ell_1(\Upsilon_0 \cap \rho^{-1}(q)) \subseteq C_0(\Upsilon^+ \cap \rho^{-1}(q))$. If we define

$$
\Phi(q; \xi) = \left( \sum_{m=1}^{\infty} 2^{-m} \sup_{F \subseteq \Upsilon_0 \cap \rho^{-1}(q)} \sum_{s \in F} \|\xi \upharpoonright [s, \infty)\|_1^{2} \right)^{\frac{1}{2}}
$$

the function $\Phi(q; \cdot)$ is weak* lower semicontinuous by our earlier observation. So also is the norm defined (using suitable positive constants $c(q)$) by

$$
\|\xi\|^2 = \|\xi\|_1^2 + \|\xi(s)\|_{\Upsilon_0 \cap \rho^{-1}(q)}^2 \|\xi\|_{\Upsilon^+ \cap \rho^{-1}(q)}^2 + \sum_{q \in \mathbb{Q}} c(q) \Phi(q; \xi)^2.
$$

To see that this norm is strictly convex notice first that if $\xi, \eta \in \ell_1(\Upsilon)$ are such that $\|\xi\| = \|\eta\| = \frac{1}{2} \|\xi + \eta\|$ then $\xi(s) = \eta(s)$ for all $s \in \Upsilon^+$ because of the second term in the definition of $\|\cdot\|$. Next, because of the $\Phi$ terms, we must have

$$
\|\xi \upharpoonright [s, \infty)\|_1 = \|\eta \upharpoonright [s, \infty)\|_1 = \frac{1}{2} \|\xi + \eta \upharpoonright [s, \infty)\|_1.
$$
for every $s \in \Upsilon_0$.

We now consider a limit element $t$ of $\Upsilon$. Since $\rho$ is constant on no strictly increasing sequence in $\Upsilon$, there exist elements $s_n$ of $\Upsilon_0$ such that $t = \lim s_n$. The sets $[s_n, \infty)$ form a decreasing sequence with intersection $[t, \infty)$ and so we can now conclude that

$$\|\xi \upharpoonright [t, \infty)\|_1 = \|\eta \upharpoonright [t, \infty)\|_1 = \frac{1}{2}\|\xi + \eta \upharpoonright [t, \infty)\|_1.$$  

Again using our assumption about $\rho$ we see that the minimal elements of $\{v \in (t, \infty) : \rho(v) > \rho(t)\}$ are all in $\Upsilon_0$ and that $\{u \in (t, \infty) : \rho(u) = \rho(t)\}$ contains only points of $\Upsilon^+$. We have

$$\|\xi \upharpoonright [t, \infty)\|_1 = |\xi(t)| + \sum_{u \in (t, \infty), \, \rho(u) = \rho(t)} |\xi(u)| + \sum_{u \in \min \{v \in (t, \infty) : \rho(v) > \rho(t)\}} \|\xi \upharpoonright [v, \infty)\|_1,$$

with similar expressions for $\eta$ and $\xi + \eta$. From what we have already shown we can now deduce that

$$|\xi(t)| = |\eta(t)| = \frac{1}{2}|\xi(t) + \eta(t)|,$$

which is to say $\xi(t) = \eta(t)$.  \qed

8. Fréchet-differentiable renormings

In this section we shall establish a necessary and sufficient condition for the existence on $C_0(\Upsilon)$ of an equivalent Fréchet-differentiable norm. Rather surprisingly, this condition turns out to be the same as the one obtained in Section 4 for LUR renormings.

**Theorem 8.1.** For a tree $\Upsilon$ the following are equivalent:

1. there is a $C^\infty$ renorming of $C_0(\Upsilon)$;
2. there is a Fréchet-differentiable renorming of $C_0(\Upsilon)$;
3. there is an increasing real-valued function $\rho$ on $\Upsilon$ which has no bad points and is constant on no ever-branching subset;
4. $C_0(\Upsilon)$ admits a Talagrand operator.

It is only the implications (2) $\implies$ (3) and (3) $\implies$ (4) which need proof and we shall deal with the second of these first. We start by making an observation about sets with no ever-branching subset.

**Lemma 8.2.** Let $\Upsilon$ be a tree and let $U \subseteq \Upsilon$ have no ever-branching subset. We may define an ordinal-valued function $i_U$ on $U$ which has the following properties:

1. $i_U$ is decreasing for the tree-order;
2. for any totally-ordered subset $V$ of $U$, $i_U$ takes only finitely many values on $V$;
3. for each $t \in U$ there is at most one $u$ in $t^+ \cap U$ with $i_U(u) = i_U(t)$.

**Proof.** We define $i_U$ in a standard way, starting from a derivation. For a non-empty subset $W$ of $U$ we define

$$W' = W \setminus \{w \in W : W \cap [w, \infty) \text{ is totally ordered}\}.$$
Since $U$ has no ever-branching subset, $W'$ is a proper subset of $W$ whenever $W$ is non-empty. We then define recursively

$$
U^{(0)} = U,
U^{(\beta)} = \bigcap_{\alpha < \beta} (U^{(\alpha)})',
$$

and set $i_U(t) = \beta$ when $t \in U^{(\beta)} \setminus U^{(\beta+1)}$.

It is clear that $i_U$ is decreasing for the tree-order. If $V$ is a totally ordered subset of $U$ then $V$ is well-ordered and a decreasing ordinal-valued function on a well-ordered set can take only finitely many values. Finally, if $t$ is in $U$ and $i_U(t) = \beta$ then $U^{(\beta)} \cap [t, \infty)$ is totally ordered, so that $U^{(\beta)}$ can contain at most one element of $t^+$.

Let us now suppose that $\Upsilon$ is a tree equipped with an increasing real-valued function $\rho$ which has no bad points and which is constant on no ever-branching subset. As usual, we may suppose that $\rho$ takes values in $(0, 1)$ and is continuous for the locally compact topology of $\Upsilon$. We shall say that a pair $(s, u)$ of subset. As usual, we may suppose that $\rho$ and so $(s, u)$ is special pair if either $s = u$ or $\rho(s) = \rho(u)$ and there exists $t \in s'$ with $t < u$ and $i_{\rho^{-1}(\rho(s))}(t) < i_{\rho^{-1}(\rho(s))}(s)$. It follows from assertion (2) of Lemma 8.2 that for any $u \in \Upsilon$ there are only finitely many $s$ such that $(s, u)$ is special. We define $T : C_0(\Upsilon) \to c_0(\Upsilon \times \Upsilon)$ by setting

$$(Tf)(s, u) = \begin{cases} 
\frac{\delta_u}{1 + \#F_u} \left[ f(u) - \sum_{v \in F_u} f(v) \right] & \text{if } (s, u) \text{ is special}, \\
0 & \text{otherwise},
\end{cases}
$$

where, as previously, we write $F_u$ for the finite set $\{v \in u^+ : \rho(v) = \rho(u)\}$.

Clearly $T$ is a bounded linear operator from $C_0(\Upsilon)$ into $\ell_\infty(\Upsilon \times \Upsilon)$ with $\|T\| \leq 1$. To show that $T$ takes values in $c_0(\Upsilon \times \Upsilon)$ we note that when $(Tf)(s, u)$ is not zero it equals $(Sf)(u)$, where $S : C_0(\Upsilon) \to c_0(\Upsilon)$ is the operator defined in Section 4. For any $f \in C_0(\Upsilon)$ and any $\epsilon > 0$, Lemma 4.3 tells us that there are only finitely many $u \in \Upsilon$ with $\|(Sf)(u)\| > \epsilon$. For each of these $u$ there are only finitely many $s$ with $(s, u)$ special and so $|(Tf)(s, u)|$ exceeds $\epsilon$ for only finitely many pairs $(s, u)$. We have now shown that $T$ takes values in $c_0(\Upsilon \times \Upsilon)$.

To show that $T$ is a Talagrand operator we let $f$ be a non-zero element of $C_0(\Upsilon)$ and let $s$ be an element of $\Upsilon$ maximal in the tree order subject to the condition $|f(s)| = \|f\|_\infty$. If $f(s) \neq \sum_{t \in F_s} f(t)$ then we have $(Tf)(s, s) \neq 0$. Suppose then that $f(s) = \sum_{t \in F_s} f(t)$. By assertion (3) of Lemma 8.2, there is at most one element $t_0$ of $F_s$ with $i_{\rho^{-1}(\rho(s))}(t_0) = i_{\rho^{-1}(\rho(s))}(s)$. By maximality, we certainly do not have $f(s) = f(t_0)$ for such a $t_0$, and so there must exist some $t \in F_s$ with $i_{\rho^{-1}(\rho(s))}(t) < i_{\rho^{-1}(\rho(s))}(s)$ and $f(t) \neq 0$. Now by Lemma 4.4 there exists some $u \succ t$ with $\rho(u) = \rho(t) = \rho(s)$ and $(Sf)(u) \neq 0$. For this $u$, the pair $(s, u)$ is special and so $(Tf)(s, u) = (Sf)(u) \neq 0$. This completes the proof that that (3) implies (4) in the theorem.

We now move on to the implication (2) $\implies$ (3). Let us suppose that $\| \cdot \|$ is a Fréchet-smooth norm on $C_0(\Upsilon)$, equivalent to $\| \cdot \|_\infty$. We may assume that $\| \cdot \| \leq \| \cdot \|_\infty \leq M \| \cdot \|$, where $M$ is a positive constant. We shall construct an
increasing sequence of increasing functions \( \rho_n : \mathcal{Y} \rightarrow \mathbb{R} \) and an increasing sequence of subsets \( \Delta_n \) of \( \mathcal{Y} \). To specify precisely the properties possessed by our system, it will be convenient to introduce some further notation. We write \( B_n \) for the set of all \( t \in \mathcal{Y} \) which are bad points or fan points for \( \rho_n \). For \( t \in B_n \) we set
\[
\epsilon^n_t = \begin{cases} 
\rho_n(t) & \text{if } t \text{ is a minimal element of } B_n \\
\rho_n(t) - \sup \{ \rho_n(s) : s \in (0, t) \cap B_n \} & \text{otherwise}
\end{cases}
\]
and let \( \Gamma_n \) be the set of all \( t \in B_n \) with \( \epsilon^n_t > 0 \). We notice that
\[
\rho_n(t) = \sup \{ \rho_n(s) : s \in (0, t) \cap \Gamma_n \}
\]
whenever \( t \in B_n \setminus \Gamma_n \).

For \( t \in \mathcal{Y} \) and \( n \in \mathbb{N} \), we define \( C^n_t \) be the set of functions \( f \in C_0(\mathcal{Y}) \) such that
1. \( \text{supp } f = (0, t] \);
2. \( f \upharpoonright (0, t] \) is increasing;
3. whenever \( r \in (0, t) \) and \( s \) is the unique member of \( t^+ \cap (0, t] \), we have \( f(r) = f(s) \) unless \( r \in \Delta_n \).

The content of condition (3) is that, regarded as increasing functions on \( (0, t] \), the functions in \( C^n_t \) have jumps only at points of \( \Delta_n \).

Finally, for all \( t \), we define
\[
\delta^n_t = \inf_{u \in t^+} \rho_n(u) - \rho_n(t).
\]

The crucial properties of the system to be constructed are:

\begin{enumerate}
\item \( \rho_n \) takes values in the interval \( (0, 2 - 2^{-n}) \);
\item \( \delta^n_t > 0 \) for all \( t \in \Delta_n \);
\item for \( n \geq 1 \), \( \bigcup_{n < m} \Gamma_m \subseteq \Delta_n \);
\item if \( u \in B_n \) and \( g \in C^n_u \) then \( u \) is a bad point or a fan point for the function \( \mu(g, \cdot) \);
\item if \( u \in B_n \setminus \Gamma_n \) and \( g \in C^n_u \) then \( \mu(g, u) = \sup \{ \mu(g, 1_{[0, q]}(t) : t \in \Gamma_n \cap (0, u) \} \).
\end{enumerate}

We now show how to perform the construction so that (a) to (e) hold.

We start by setting \( \rho_0(t) = \mu(t) \) and \( \Delta_0 = \emptyset \). Thus (a), (b) and (c) certainly hold for \( n = 0 \). Applying our definitions, \( B_0 \) is the set of bad points and fan points for the \( \mu \) function. Now, since \( \Delta_0 = \emptyset \), each set \( C^n_t \) contains only scalar multiples of the indicator function \( 1_{[0, q]} \). For such a function \( g = \lambda 1_{[0, q]} \), we have \( \mu(g, u) = |\lambda| \mu(u) = |\lambda| \rho_0(u) \) for all \( u \in [t, \infty) \), and so (d) holds as well. The final assertion (e) follows from the observation we made just after the definition of \( \Gamma_n \).

Now let us suppose that \( \rho_0, \ldots, \rho_n \) and \( \Delta_0, \ldots, \Delta_n \) have been constructed and satisfy (a) to (e). It follows from (b) that \( \Delta_n \cap (0, t) \) is countable for each \( t \in \mathcal{Y} \) (indeed, \( \sum_{s \in \Delta_n \cap (0, t)} \delta^n_s \leq \rho(t) \)). So each set \( C^n_t \) is norm-separable; we fix, for each \( t \), a norm-dense sequence \( \{ f^n_k,t \}_{k \in \mathbb{N}} \) in this set and also fix an enumeration \( (q_t)_{t \in \mathbb{N}} \) of the non-negative rationals. We define
\[
\sigma^n_t(u) = \begin{cases} 
\sum_{k, l \in \mathbb{N}} 2^{-k-l-2} \| f^n_k,t \|_\infty + q_l)^{-1} \mu(f^n_k,t, q_l, u) & \text{if } t < u \\
\text{otherwise,}
\end{cases}
\]
where
\[
\mu(f, \delta, u) = \mu(f + (f(t) + \delta) 1_{(t, u]}(u) \\
= \inf \{ \| f + (f(t) + \delta) 1_{(t, u]} + g \| : g \in C_0(\mathcal{Y}) \text{ and } \text{supp } g \subseteq (u, \infty) \}
\]
is another variation on a familiar theme. We notice that \( \sigma_t(u) \leq 1 \) and that 
\( \sigma_t(u) \geq M^{-1} \) when \( u \in (t, \infty) \).

We now define \( \Delta_{n+1} = \Delta_n \cup \Gamma_n \) and set

\[
\rho_{n+1} = \rho_n + 2^{-n-3} \sum_{t \in \Gamma_n} \epsilon^n_t \sigma^n_t + 2^{-n-3} \sum_{t \in \Delta_n} \delta^n_t \sigma^n_t.
\]

Of course, we need to show that the series defining \( \rho_{n+1} \) converges and that \( \rho_{n+1} \)

\( \text{takes values in } (0, 2 - 2^{-n-1}) \) in order that \( \text{(a) should hold at stage } n+1 \). This is

indeed so since for any \( u \in \Upsilon \) we have

\[
\rho_{n+1}(u) \leq \rho_n(u) + 2^{-n-3} \sum_{t \in \Gamma_n \cap (0,u)} \epsilon^n_t + 2^{-n-3} \sum_{t \in \Delta_n \cap (0,u)} \delta^n_t
\]

\[
\leq (1 + 2 - 2^{-n-2}) \rho_n(u) \leq (1 + 2 - 2^{-n-2})(2 - 2^{-n}) < 2 - 2^{-n}.
\]

To see that \( \text{(b) holds we note that if } t \in \Gamma_n \text{ and } u \in t^+ \text{ then } \sigma^n_t(t) = 0 \) and

\( \sigma^n_t(u) \geq 1/M \) so that \( \delta^{n+1} \geq 2^{n-3}\epsilon^n_t/M > 0 \). Assertion \( \text{(c) holds because of the} \)

way we defined \( \Delta_{n+1} \).

Now let \( u \) be a bad point for \( \rho_{n+1} \). It is clear from the definition that if \( t \in \Delta_n \cap (0,u) \) and \( k, l \) are natural numbers then \( u \) is bad for the function \( \mu(f_k^n, q^l, \cdot) \).

By uniform approximation, we see that the same is true for all functions \( \mu(f, \delta, \cdot) \) with \( f \in C^n_t \) and \( \delta \in \mathbb{R}^+ \). Looking at this another way, we see that \( u \) is bad for each function \( \mu(g, \cdot) \) where \( g \) is a function in \( C^n_u \) of the form \( f + (f(t) + \delta)\mathbf{1}_{(t,u]} \) with \( t \in \Delta_n \cap (0,u) \), \( f \in C^n_t \) and \( \delta \geq 0 \). The set of functions of this form being

norm-dense in \( C^n_u \), we see that \( u \) is bad for all of the functions \( \mu(g, \cdot) \) with \( g \in C^n_u \). A similar argument works for the case where \( u \) is a fan point for \( \rho_{n+1} \), since if \( \rho^{n+1}(v) = \rho_{n+1}(u) \) for all \( v \) in some ever-branching subset \( T \) of \([u, \infty)\) then all the

functions \( \mu(f_k^n, q^l, \cdot) \) are constant on the same set \( T \). In this way we establish \( \text{(d)}. \)

We have already noted that

\[
\rho_{n+1}(u) = \sup \{ \rho_{n+1}(t) : t \in \Gamma_{n+1} \cap (0,u) \}
\]

whenever \( u \in B_{n+1} \setminus \Gamma_{n+1} \). It follows from the definition of \( \rho_{n+1} \) that

\[
\mu(f_k^n, q_t, u) = \sup \{ \mu(f_k^n, q_t, t) : t \in \Gamma_{n+1} \cap (0,u) \}
\]

for all \( t \in \Delta_n \) and all \( k, l \). A uniform approximation argument now leads us to \( \text{(e)}. \)

Having completed the construction of the sequence \( \rho_n \) we define \( \rho(t) \) to be

\[
\lim_{n \to \infty} \rho_n(t).
\]

It is this function that we shall show has no bad points and no fan points. Suppose then that \( u \) is a bad point or a fan point for \( \rho \). By construction, \( u \) is in \( B_n \) for all \( n \), and since \( B_{n+1} \cap \Gamma_n \subseteq B_{n+1} \cap \Delta_{n+1} = \emptyset \) by \( \text{(b)} \) and \( \text{(c)}, \) \( u \) is in none of the sets \( \Gamma_n \).

We shall now construct an increasing sequence \( (t_n) \) in \((0,u)\), with \( t_n \in \Gamma_n \), and

a summable sequence \( (\Delta_n) \) of non-negative real numbers, and shall define \( f_n \in C^n_u \),

\( g_n \in C^n_u \) by

\[
f_n = 1_{(0,t_n]} + \sum_{m=0}^{n-1} \delta_m 1_{(t_m,t_n]},
\]

\[
g_n = 1_{[0,u]} + \sum_{m=0}^{n-1} \delta_m 1_{[t_m,u]}.
\]
Since $u$ and $t_n$ are bad or fan for $\mu(f_n, \cdot)$, we may consider, as in Proposition 3.5, $\mu$-attaining modifications $\hat{f}_n$ and $g_n$ of $f_n$. The construction will be carried out in such a way that the following hold:

(i) for all $n$, $\mu(f_n, t_n) \geq \mu(f_n, u) - 2^{-n}$;
(ii) for each $n$, $\langle \hat{f}_{n+1}^*, 1_{(t_n, t_{n+1})} \rangle \geq 1/4M$.

Before giving the details, let us indicate why we shall end up with a contradiction.

Define the function $g$ by

$$g = 1_{[0, u]} + \sum_{m=0}^{\infty} \delta_m 1_{[t_m, u]}.$$ 

Thus $g$ is the norm limit as $n \to \infty$ of $g_n$. So by (d) and uniform approximation, $u$ is a bad point or a fan point for $\mu(g, \cdot)$. By (i), we have

$$\|g\| = \mu(g, u) = \lim_{n \to \infty} \mu(g_n, u) = \lim_{n \to \infty} \mu(f_n, t_n) = \lim_{n \to \infty} \|\hat{f}_n\|.$$ 

Also $\text{supp}(g - f_n) \subseteq (t_n, \infty)$ and $\text{supp} \hat{f}_n^* \cap (t_n, \infty) = \emptyset$ by Lemma 3.5. Thus

$$\langle \hat{f}_n^*, \hat{g} \rangle = \langle \hat{f}_n^*, f_n \rangle = \|\hat{f}_n\|$$

which tends to $\|\hat{g}\|$ as $n \to \infty$. Hence $\|\hat{g} - \hat{f}_n^*\|^* \to 0$ by Šmulian’s Criterion. But of course this contradicts (ii).

To finish the proof we have to show how to construct $t_n$ and $\delta_n$. We start by choosing an arbitrary $t_0 \in \Gamma_0 \cap (0, u)$. (There are such points since $u$ is in $B_0 \setminus \Gamma_0$.) If $t_0 < t_1 < \cdots < t_n$ and $\delta_0, \ldots, \delta_{n-1}$ have been defined already (so that $f_n \in C_{t_n}$ is determined as above), we choose $\delta_n$ to be the greatest $\delta \geq 0$ such that

$$\mu(f_n, \delta, u) \leq \mu(f_n, 0, u) + \delta/2M.$$ 

To see that this definition makes sense, we note that the set of $\delta$ satisfying this inequality is non-empty (since it certainly contains 0) and bounded above, since we have $\mu(f_n, \delta, u) \geq \delta/M$ for any $\delta$. Now, by construction, $u$ is a bad or fan point for each of the functions $\mu(f_n, \delta, \cdot)$ and so, as in Proposition 3.5, it is the case that for all $\delta \geq 0$

$$\mu(f_n, \delta, u) = \|f_n + (f_n(t_n) + \delta)(1_{(t_n, u]} + \phi_u)\|,$$

where $\phi_u$ is either 0 or a fan function at $u$. Thus $\delta_n$ is maximal subject to

$$\|f_n + (f_n(t_n) + \delta_n)(1_{(t_n, u]} + \phi_u)\| \leq \|f_n + f_n(t_n)(1_{(t_n, u]} + \phi_u)\| + \delta/2M.$$ 

We recall that the function $f_n + (f_n(t_n) + \delta_n)(1_{(t_n, u]} + \phi_u)$ is what we have decided to call $\hat{g}_{n+1}$. Because of the maximality in the definition of $\delta_n$ it must be that $\langle \hat{g}_{n+1}^*, 1_{(t_n, u]} + \phi_u \rangle \geq 1/2M$. 


The function \( g_n \) is in \( C^{n+1}_u \) and \( u \) is in \( B_{n+1}\setminus \Gamma_{n+1} \). So by (e) there is a sequence \((s_k)\) in \( \Gamma_{n+1}\cap (0,u) \) with

\[
\mu(g_n1_{(0,s_k)}, s_k) \to \mu(g_n, u)
\]
as \( n \to \infty \). We shall choose \( t_{n+1} \) to be one of the \( s_k \); evidently, any sufficiently large \( k \) will give us (i). If we write \( h_k = g_n1_{(0,s_k)} \) then \( s_k \), being an element of \( \Gamma_{n+1} \), is a bad point or a fan point for \( \mu(h_k, \cdot) \) so that we can define \( \hat{h}_k \) as usual.

By an argument we used earlier, we have

\[
\langle \hat{h}_k^*, \hat{g}_n \rangle = \langle \hat{h}_k^*, h_k \rangle = \|\hat{h}_k\| = \mu(h_k, s_k) \to \mu(g_n, u) = \hat{g}_n,
\]
so that \( \|\hat{h}_k^* - \hat{g}_n^*\|^* \to 0 \) by \( \check{\text{Smulyan’s Criterion}} \). In particular, for sufficiently large \( k \), we have \( \langle \hat{h}_k^*, 1_{(t_n,u]} \rangle + \phi_u \geq 1/4M \). Since \( \text{supp} \hat{h}_k \cap (s_k, \infty) = \emptyset \) we have, in fact,

\[
\langle \hat{h}_k, 1_{(t_n,s_k]} \rangle \geq 1/4M,
\]

All that remains is to verify that the series \( \sum_{n=0}^\infty \delta_n \) converges. To do this, we notice first that, by the way in which \( \delta_n \) was chosen, we always have

\[
\mu(f_n, \delta_n, u) = \mu(f_n, 0, u) + \delta_n/2M,
\]
or, equivalently,

\[
\mu(g_{n+1}, u) = \mu(g_n, u) + \delta_n/2M.
\]

Thus, for any \( N \geq 1 \),

\[
\sum_{0 \leq n < N} \delta_n = \|g_N\|_\infty \leq M\mu(g_N, u) = M\mu(u) + \frac{1}{2} \sum_{0 \leq n < N} \delta_n,
\]
whence \( \sum_{0 \leq n < N} \delta_n \leq 2M \). This establishes the desired convergence and finishes the proof.

9. Bump functions and partitions of unity

In this section, \( \mathcal{Y} \) will be an arbitrary tree. We shall show how to define a non-trivial \( C^\infty \) function of bounded support (that is to say, a bump function) on \( C_0(\mathcal{Y}) \) and how to obtain \( C^\infty \) partitions of unity on this space. We shall use constructions from the author’s paper [12], from which we quote the following two results. In the first of them, the set \( U(L) \) is the one defined in Section 1 in the context of Talagrand operators.

**Proposition 9.1 (Corollary 3 of [12]).** Let \( X \) be a Banach space, let \( L \) be a set and let \( k \) be a positive integer or \( \infty \). Suppose that there exist continuous mappings \( S : X \to \ell_\infty(L) \), \( T : X \to c_0(L) \) with the following properties:

1. for all \( x \in X \) the pair \((Sx, Tx)\) is in \( U(L) \cup \{0\}\);
2. the coordinates of \( S \) and of \( T \) are all \( C^k \) functions on the sets where they are non-zero;
3. \( \|Sx\|_\infty \to \infty \) as \( \|x\| \to \infty \).

Then \( X \) admits a \( C^k \) bump function.
Proposition 9.2 (Theorem 2 of [12]). Let $X$ be a Banach space, let $L$ be a set and let $k$ be a positive integer or $\infty$. Let $T : X \rightarrow c_0(L)$ be a function such that each coordinate $x \mapsto T(x)_i$ is of class $C^k$ on the set where it is non-zero. For each finite subset $F$ of $L$, let $R_F : X \rightarrow X$ be of class $C^k$ and assume that the following hold:

1. for each $F$, the image $R_F[X]$ admits $C^k$ partitions of unity;
2. $X$ admits a $C^k$ bump function;
3. for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - R_F x\| < \varepsilon$
   if we set $F = \{ \gamma \in L : |(T x)(\gamma)| \geq \delta \}$.

Then $X$ admits $C^k$ partitions of unity.

Theorem 9.3. For any tree $\Upsilon$ there is a $C^\infty$ bump function on $C_0(\Upsilon)$.

Proof. We shall define $L, S, T$ satisfying the conditions of Theorem 9.1. Let $L$ be $\Upsilon \times \mathbb{N}$ and define $S : C_0(\Upsilon) \rightarrow \ell_\infty(L)$ by $(S f)(s, n) = f(s)$. It is obvious that the coordinates of $S$ are $C^\infty$ and that $\|S f\|_\infty \rightarrow \infty$ as $\|f\|_\infty \rightarrow \infty$.

In order to define $T$, we first fix a $C^\infty$ function $\phi : \mathbb{R}^+ \rightarrow [0, 1]$ satisfying

$\phi(x) = \begin{cases} 
0 & \text{if } |x| \leq \frac{1}{2} \\
1 & \text{if } |x| \geq 1,
\end{cases}$

and set $\psi = 1 - \phi$. For $f \in C_0(\Upsilon)$, $s \in \Upsilon$ and $n \in \mathbb{N}$, we define

$$(T f)(s, n) = \begin{cases} 
0 & \text{if } f(s) = 0 \text{ or there exists } t \in s^+ \text{ with } f(s) = f(t) \\
-\phi(2^n f(s)) \prod_{t \in s^+} \psi(2^{-n} f(t)/(f(t) - f(s))) & \text{otherwise.}
\end{cases}$$

We shall show that the hypotheses of Theorem 9.1 are satisfied. First we have to show that $T$ takes values in $c_0(\Upsilon \times \mathbb{N})$. Consider some fixed $n$. By Lemma 2.2, for all but finitely many $s \in \Upsilon$ there exists $t \in s^+$ with $|f(t) - f(s)| < 2^{-2n-2}$. If $s, t$ have this property then either $|f(s)| < 2^{-n-1}$, in which case $\phi(2^n f(s)) = 0$, or $|f(s)| \geq 2^{-n-1}$, in which case we have $|f(t)| \geq 2^{-n-1} - 2^{-2n-2} \geq 2^{-n} - 2^{-2n-2}$ and hence $\psi(2^{-n} f(t)/(f(t) - f(s))) = 0$. Thus there are, for each $n$, only finitely many $s$ with $(T f)(s, n) \neq 0$. Combined with the inequality $|(T f)(s, n)| \leq 2^{-n}$, this shows that $T f$ is in $c_0(\Upsilon)$.

Now let $s$ and $n$ be fixed and consider the coordinate mapping $f \mapsto (T f)(s, n)$. We shall show that every $f_0 \in C_0(\Upsilon)$ has a neighbourhood on which this function is $C^\infty$. If $f_0 \in C_0(\Upsilon)$ is such that $f_0(s) = 0$ or there exists $t \in s^+$ such that $f(t) = f(s)$, then by the calculation in the last paragraph we have $(T f)(s, n) = 0$ whenever $\|f - f_0\|_\infty < 2^{-2n-3}$. So our mapping is certainly infinitely differentiable at $f_0$. Now consider $f_0$ such that $f_0(s) \neq 0$ and $\delta = \inf \{|f(t) - f(s) : t \in s^+| \neq 0$. Let $V$ be the set of all $f$ with $\|f - f_0\|_\infty < \frac{\delta}{2}$ and let $F$ be the (finite) set of all $t \in s^+$ such that $|f_0(t)| \geq \frac{1}{2} \delta$. For $f \in V$ and $t \in s^+ \setminus F$ we have $|f(t)/(f(t) - f(s))| \leq \frac{1}{2}$, whence $\psi(2^{-n} f(t)/(f(t) - f(s))) = 1$. Thus, on $V$, our function $(T \cdot)(s, n)$ is the product of finitely many $C^\infty$ functions and is thus itself $C^\infty$.

To finish the proof we shall show that for every non-zero $f \in C_0(\Upsilon)$ there is some $(s, n) \in L$ with $(T f)(s, n) \neq 0$ and $|(S f)(s, n)| = \|S f\|_\infty$, that is to say $|f(s)| = \|f\|_\infty$. We choose $s$ to be maximal in $\Upsilon$ subject to $|f(s)| = \|f\|_\infty$. Thus $f(s) \neq 0$ and there is no $t \in s^+$ with $f(t) = f(s)$. Let $\delta = \inf \{|f(s) - f(t) : t \in s^+\}$ and choose $n$ so that $2^n \|f\|_\infty \geq 1$ and $2^{-n} \|f\|_\infty \leq \frac{1}{2} \delta$. We then have $\phi(2^n f(s)) = 1$ and $\psi(2^{-n} f(t)/(f(t) - f(s))) = 1$ for all $t \in s^+$, whence $(S f)(s, n) = 2^{-n}$. □
Theorem 9.4. For any tree $\mathcal{T}$ the space $C_0(\mathcal{T})$ admits $C^\infty$ partitions of unity.

Proof. In 9.2 we take $X$ to be $C_0(\mathcal{T})$ and let $L$ and $T$ be as in the proof of Theorem 9.3. We have to define the “reconstruction operators” $R_F$ and establish the hypotheses (1) and (3) of 9.2. For a finite subset $F$ of $L = \mathcal{T} \times \mathbb{N}$ we define

$$R_F f(s) = \begin{cases} f(s) & \text{if there exists } (t, n) \in F \text{ with } s \leq t \\ 0 & \text{otherwise.} \end{cases}$$

The image $R_F[X]$ is just the set of continuous functions supported on the finite union of intervals $\bigcup_{(t, n) \in F} [0, t]$. Now this union is homeomorphic to some closed interval of ordinals $[0, \Omega]$ and it is shown in [12] that the space of continuous functions $C[0, \Omega]$ admits $C^\infty$ partitions of unity. Thus hypothesis (1) holds.

Finally, given a non-zero $f \in C_0(\mathcal{T})$ and $\epsilon > 0$, we note that the set $M$ of maximal elements of $\{ t \in \mathcal{T} : |f(t)| \geq \epsilon \}$ is finite. As in the proof of Theorem 9.3, there exists, for each $t \in M$, a natural number $n_t$ such that $(Sf)(t, n_t) = 2^{-n_t}$. If we set $\delta = 2^{-\max_{t \in M} n_t}$ and $F$ is the set of all $(s, m) \in L$ with $(Tf)(s, m) \geq \delta$, then $|f(u)| < \epsilon$ for all $u$ in $\mathcal{T} \setminus \bigcup_{(s, m) \in F} [0, s]$. Hence $\|f - R_F f\|_\infty < \epsilon$ as required. $\Box$

10. Examples

In this section we consider in detail three particular trees, which provide most of the counterexamples mentioned in the introduction. Our starting point is a $\mathbb{R}$-embeddable, non-$\mathcal{Q}$-embeddable tree which is well-known to set-theorists. We recall that the standard realization of a full countably branching tree of height $\omega_1$ is the set of all functions $t$ with domain some countable ordinal and codomain the set $\omega$ of natural numbers. For the rest of this section, $\Lambda$ will denote a certain subtree of this tree, namely the set of all injections $t$ with domain $\text{dom} t$ a countable ordinal and image im $t$ a co-infinite subset of $\omega$. This tree is $\mathbb{R}$-embeddable since the function $\lambda : t \mapsto \sum_{\alpha \in \text{dom} t} 2^{-t(\alpha)}$ is strictly increasing. We define the $\lambda$-topology $T_\lambda$ on $\Lambda$ by taking basic neighbourhoods of $t$ to be of the form $\{ u \in [t, \omega) : \lambda(u) < \lambda(t) + \epsilon \}$. We start with a key lemma about this topology.

Lemma 10.1. For the topology $T_\lambda$ the tree $\Lambda$ is a Baire space.

Proof. It is convenient to work with a slightly different description of the $\lambda$-topology. For $t \in \Lambda$ and $n \in \omega$ let $[t, \omega)_n = \{ u \in [t, \omega) : t \prec u \text{ and } n \cap \text{dom} u = n \cap \text{dom} t \}$. Then we have

$$[t, \omega)_{p+1} = \{ u \in [t, \omega) : \lambda(u) < \lambda(t) + 2^{-p} \},$$

so that the sets $[t, \omega)_n$ form a base of neighbourhoods of $t$ for the $\lambda$-topology.

We shall now show that $(\Lambda, T_\lambda)$ is a Baire space by showing that it is $\alpha$-favourable [1]. The strategy for the player (a) is as follows: if at stage $n$, player $\beta$ plays $[t_n, \omega)_n$, then $\alpha$ chooses $r_n$ to be the $n$th element of $\omega \setminus \text{im} t_n$ and plays $[t_n, \omega)_{q_n}$ where $q_n = \max \{ p_n, r_n + 1 \}$. At the end of the game, we have a sequence $t_0 \ll t_1 \ll \cdots$ in $\Lambda$ and there is certainly an ordinal $\alpha = \sup_n \text{dom} t_n$ and an injection $t : \alpha \rightarrow \omega$ satisfying $t \upharpoonright \text{dom} t_n = t_n$ for all $n$. The only question is whether this $t$ is in $\Lambda$, that is to say whether $\text{im} t$ is a co-infinite subset of $\omega$. I claim, in fact, that none of the $r_n$ are in the range of $t$: since $\text{im} t = \bigcap_n \text{im} t_n$ it is enough to show that $r_n \notin \text{im} t_m$ for all $m$. If $m \leq n$ we have $\text{im} t_m \subseteq t_n$ and $r_n \neq t_n$, while for $m \geq n$ we have $\text{im} t_m \subseteq [t_n, \omega)_{q_n} = [t_n, \omega)_{r_n} + 1$ so that $\text{im} t_m \cap [0, r_n] = \text{im} t_n \cap [0, r_n]$, which again shows that $r_n \notin \text{im} t_m$. Finally, since we chose $r_n$ to be the $n$th member of $\omega \setminus \text{im} t_n$, and since $\text{im} t_n \supseteq \text{im} t_m$ when $m < n$, all the $r_n$ are distinct. $\Box$
Corollary 10.2. The $\mathbb{R}$-embeddable tree $\Lambda$ does not have the Namioka Property. Hence $C_0(\Lambda)$ has no Kadec norm, and certainly no LUR norm—though by 3.1 it does have a MLUR norm.

Proof. If $t \in \Lambda$ has domain $\alpha$ then the immediate successors of $t$ in $\Lambda$ have domain $\alpha + 1 = \alpha \cup \{\alpha\}$ and are given by $t.n$ where

$$t.n \upharpoonright \alpha = t, \quad (t.n)(\alpha) = n.$$ 

There is one such successor for each $n \in \omega \setminus \text{im } t$. Since $\lambda(t.n) \to \lambda(t)$ as $n$ tends to $\infty$ in the infinite set $\omega \setminus \text{im } t$, we see that $t$ is the $T_\lambda$ limit of the sequence $t.n$ and, in particular that $\Lambda$ has no $T_\lambda$-isolated points. On the other hand, since $T_\lambda$ is finer than the reverse topology, the map $t \mapsto 1_{[0, t]}$ is continuous from $(\Lambda, T_\lambda)$ into $C_0(\Lambda), T_p$. Thus $\Lambda$ does not have the Namioka Property as we have just shown that $(\Lambda, T_\lambda)$ is a Baire space. \qed

Remark 10.3. It follows from the above Corollary, together with our necessary and sufficient condition for Kadec renormability, that every increasing real-valued function of $\Lambda$ has bad points. There is a direct way to see this. Let $\rho : \Lambda \to \mathbb{R}$ be increasing. For any $\alpha \in \mathbb{R}$ the set $\{t \in \Lambda : \rho(t) > \alpha\}$ is a union of wedges $[s, \infty)$, and is hence $T_\lambda$-open. Thus $\rho$ is $T_\lambda$ lower semicontinuous. Since $\Lambda, T_\lambda$ is a Baire space, there are points $t$ at which $\rho$ is $T_\lambda$-continuous. Since, for any $t$, the immediate successors of $t$ form a sequence which converges to $t$ for $T_\lambda$, a point at which $\rho$ is $T_\lambda$-continuous must be bad for $\rho$.

We shall now use the tree $\Lambda$ considered above to answer negatively the “three-space problem” about strictly convex renormings, showing that there exists a Banach space $X$ which has a closed subspace $Y$ with a LUR norm, such that the quotient space $X/Y$ has a strictly convex norm, while $X$ itself has no strictly convex renorming.

Proposition 10.4. There is a tree $\Upsilon$ which contains $\Lambda$ as a closed subset, with $\Upsilon \setminus \Lambda$ discrete, such that $C_0(\Upsilon)$ admits no strictly convex renorming.

Corollary 10.5 (Three-Space Problem). There is a Banach space $X$ and a subspace $Y$ of $X$, isomorphic to $c_0(\mathbb{R})$, such that the quotient $X/Y$ admits a strictly convex renorming, while $X$ does not.

Proof of 10.5. We take $X$ to be $C_0(\Upsilon)$ and $Y$ to be the subspace consisting of all functions $f$ with $f \upharpoonright \Lambda = 0$. The quotient $X/Y$ may be identified with $C_0(\Lambda)$, which admits a strictly convex norm (and even an MLUR norm). The subspace $Y$ itself may be identified with $C_0(\Upsilon \setminus \Lambda)$, which is isomorphic to $c_0(\mathbb{R})$ since $\Upsilon \setminus \Lambda$ is discrete and of cardinality the continuum. \qed

Proof of 10.4. For each $t \in \Lambda$ we partition $t^+$ as the union of infinite sets $A_1(t), \cup A_2(t)$, and augment the tree $\Lambda$ by introducing elements $(t, 1)$ and $(t, 2)$ with the property that $t \prec (t, i) \prec u$ whenever $u \in A_i(t)$. We write $\Upsilon$ for the the resulting tree, which contains $\Lambda$ as a closed subset. We note that $\Upsilon \setminus \Lambda = \Lambda \times \{1, 2\}$ is open and discrete. To show that $C_0(\Upsilon)$ is not strictly convexifiable we shall show that for any increasing function $\phi : \Upsilon \to \mathbb{R}$ there exists $t \in \Lambda$ such that both $(t, 1)$ and $(t, 2)$ are $\phi$-bad, with $\phi(t, 1) = \phi(t, 2) = \phi(t)$.

Given an increasing function $\phi$ on $\Upsilon$ we define $\psi : \Lambda \to \mathbb{R}$ by $\psi(t) = \phi \upharpoonright \Lambda$. We remarked earlier in this section that there exist points at which $\psi$ is $T_\lambda$-continuous.
Let \( t \) be any such point, let \( i \) be 1 or 2 and let \( u_n (n \in \omega) \) be a sequence of distinct points of \( A_i(t) \). Then we have \( t < (t, i) < u_n \), and hence \( \psi(t) \leq \phi(t, i) \leq \psi(u_n) \), for all \( n \). But the sequence \((u_n)\) converges to \( t \) for the topology \( T_\lambda \) so that, by continuity of \( \psi \) at \( t \), we have \( \phi(t, i) = \phi(t) = \lim_n \phi(u_n) \). \( \square \)

A very similar construction will now enable us to answer the quotient problem for Fréchet-differentiable renorming.

**Proposition 10.6.** Let \( \Delta \) be a dyadic tree equipped with an increasing function \( \rho : \Delta \to \mathbb{R}^+ \) having the property that, for all \( t \in \Delta \) there is exactly one element \( u \) of \( t^+ \) such that \( \rho(u) = \rho(t) \). Then \( C_0(\Delta) \) admits a \( C^\infty \) renorming.

**Proof.** Since \( \Delta \) is finitely branching, the function \( \rho \) has no bad points. The assumption that each successor set \( t^+ \) contains only one point \( u \) with \( \rho(u) = \rho(t) \) ensures that \( \rho \) is constant on no ever-branching subset of \( \Delta \). Thus \( C_0(\Delta) \) admits a \( C^\infty \) renorming by Theorem 8.1. \( \square \)

In fact, in the simple case we have just considered, it would have been very easy to write down a Talagrand operator and thus establish the proposition without recourse to the general result 8.1. For each \( t \in \Delta \) let the elements of \( t^+ \) be denoted \( t^* \) and \( \hat{t} \), with \( \rho(\hat{t}) > \rho(t) \) and \( \rho(t^*) = \rho(t) \). We set

\[
(Tf)(t) = (\rho(\hat{t}) - \rho(t))(f(t) - f(t^*)).
\]

It is easy to see that if \( t \) is maximal subject to \( |f(t)| = \|f\|_\infty \) then \( (Tf)(t) \neq 0 \). We show that \( \Phi \) takes values in \( c_0(\Delta) \) by applying the argument used for the operator \( S \) in the Lemma 4.3. (Indeed, when we note that we here have \( F_t = \{t\} \) for all \( t \), we can see that the operator \( T \) is just \( 2S \).)

**Proposition 10.7.** The tree \( \Lambda \) may be embedded as a closed subset of a dyadic tree \( \Delta \) satisfying the hypotheses of Proposition 10.6.

**Proof.** We augment the tree \( \Lambda \) in a way analogous to what we have just done in the context of the three-space problem for strictly convex renorming. We enumerate, for each \( t \in \Lambda \), the elements of \( t^+ \) (in \( \Lambda \)) as \( t_n (n \in \mathbb{N}) \) and introduce a sequence \( t'_n \) of new points in such a way that, in the augmented tree \( \Delta \), \( t \) has exactly two immediate successors, \( t_0 \) and \( t'_0 \), while for each \( n \) the immediate successors of \( t'_n \) are \( t_{n+1} \) and \( t'_{n+1} \). We extend the function \( \lambda \) to \( \Delta \) by setting \( \lambda(t'_n) = \lambda(t) \) for all \( t \) and \( n \) and observe that the hypotheses of Proposition 10.6 are satisfied. \( \square \)

**Corollary 10.8** (Quotient Problem for Fréchet Renormability). The Banach space \( C_0(\Lambda) \) is a quotient of a space with \( C^\infty \) norm, but does not itself admit a Fréchet renorming.

**Proof.** Since \( \Lambda \) is a closed subset of \( \Delta \), \( C_0(\Lambda) \) is a quotient of \( C_0(\Delta) \), a space which admits a \( C^\infty \) renorming. However, \( C_0(\Lambda) \) admits no Fréchet renorming, by Theorem 8.1, since every increasing real-valued function on \( \Lambda \) has bad points. \( \square \)

11. Open Problems

As we remarked in the Introduction, this paper has not offered many results about Gateaux differentiability. Even our sufficient condition for the existence of a strictly convex dual norm on \( C_0(T)^* \), established in Section 7, does not appear to
be a necessary condition. The only case in which the author knows that \( C_0(\mathcal{Y}) \) does not admit a Gateaux smooth norm is when \( \mathcal{Y} \) is a Baire tree [11]. There is a very big gap between this situation and the cases where 7.1 and 8.1 of this paper give positive results. By way of a specific problem, one can ask whether the existence of a strictly convex renorming of \( C_0(\mathcal{Y}) \) implies the existence of a smooth renorming. Significant progress in this area will no doubt depend on the development of new ways to construct Gateaux-smooth norms. To the best of the author’s knowledge, the only approaches available at present use either strict convexity of the dual norm or Talagrand operators (the latter, of course, yielding Fréchet-smoothness).

The other class of problems left open by this paper are those where trees have yielded positive results, rather than counterexamples. In one case it is clear that the behaviour of trees is not representative of general locally compact spaces: we saw in 5.1 and 8.1 that the existence of an equivalent strictly convex or Fréchet-smooth norm on \( C_0(\mathcal{Y}) \) implies the existence of a bounded linear injection from \( C_0(\mathcal{Y}) \) into a space \( c_0(\Gamma) \); the Cieselski–Pol space [3, VII.4.9] shows that \( C(K) \) may admit a LUR renorming and a \( C^\infty \) renorming but no such injection into \( c_0(\Gamma) \). The following are problems where more work is needed, including perhaps the systematic investigation of new classes of examples. In each case, \( L \) denotes a locally compact, scattered space.

**Problem 11.1.** Is there any logical connection between LUR renormability of \( C_0(L) \) and Fréchet-smooth renormability of that space?

**Problem 11.2.** If \( C_0(L) \) admits a Fréchet-smooth renorming, does it necessarily admit a \( C^\infty \) renorming? This question seems to be open even in the case where the \( \beta^{th} \) derived set \( L^{(\beta)} \) is empty, for some countable ordinal \( \beta \) [3, VII.4].

**Problem 11.3.** If \( C_0(L) \) admits a wLUR renorming, does it necessarily admit a LUR renorming?

**Problem 11.4.** If \( C_0(L) \) admits a strictly convex renorming, does it necessarily admit a MLUR renorming?

**Problem 11.5.** If \( C_0(L) \) is \( \sigma \)-fragmentable, does it necessarily admit a Kadec renorming?

Of the above problems, 11.1, 11.3, 11.4 and 11.5 are also open with \( C_0(L) \) replaced by a general Asplund space. To finish, we should perhaps remind the reader that the most important open problems in the area of non-separable renorming theory are those concerning bump functions and partitions of unity. Does every Asplund space admit a \( C^1 \) bump function? Does a space with a \( C^1 \) bump function necessarily admit \( C^1 \) partitions of unity? In the special case of spaces \( C_0(L) \) we may ask whether there always exist \( C^\infty \) bump functions.

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