Superradiant instabilities for short-range non-negative potentials on Kerr spacetimes and applications

Georgios Moschidis

August 9, 2016

Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544, United States, gm6@math.princeton.edu

Abstract

In [40], Shlapentokh-Rothman established that the wave equation $\Box_{g_{M,a}}\psi = 0$ on subextremal Kerr spacetimes $(\mathcal{M}_{M,a}, g_{M,a})$, $0 < |a| < M$, does not admit real mode solutions. This is a highly non-trivial result, in view of the phenomenon of superradiance, i.e. the fact that the stationary Killing field $T$ fails to be causal on the horizon $\mathcal{H}$. The analogue of this result fails for long-range perturbations of the wave equation, such as the Klein–Gordon equation $\Box_{g_{M,a}}\psi - \mu^2 \psi = 0$ for $\mu > 0$, as was shown by Shlapentokh-Rothman in [39]. The question naturally arises whether the absence of real modes persists under the addition of an arbitrary short-range non-negative potential $V$ to the wave equation or under changes of the metric $g_{M,a}$ in the far away region of $\mathcal{M}_{M,a}$ (retaining the causality of $T$ there).

In this paper, we answer the above question in the negative in both cases. First, for any $0 < |a| < M$, we establish the existence of real mode solutions $\psi$ to equation $\Box_{g_{M,a}}\psi - V \psi = 0$, for a suitably chosen time-independent real potential $V$ with compact support in space, satisfying the sign condition $V \geq 0$. Exponentially growing modes are also obtained after perturbing the potential $V$. Then, as an application of the above results, we construct a family of spacetimes $(\mathcal{M}_{M,a}, g_{M,a}^{(\text{def})})$ which are compact in space perturbations of $(\mathcal{M}_{M,a}, g_{M,a})$, have the same symmetries as $(\mathcal{M}_{M,a}, g_{M,a})$ and moreover admit real and exponentially growing mode solutions to equation $\Box_{g} \psi = 0$. The nature of our construction forces, however, the spacetimes $(\mathcal{M}_{M,a}, g_{M,a}^{(\text{def})})$ to contain stably trapped null geodesics. We also construct a more complicated family of spacetimes $(\mathcal{M}_{0}, g_{M,a}^{(b)})$ admitting real and exponentially growing mode solutions to the wave equation, on which the trapped set is normally hyperbolic, at the expense of $g_{M,a}^{(b)}$ having conic asymptotics.

The above results are in contrast with the case of stationary asymptotically flat (or conic) spacetimes $(\mathcal{M}, g)$ with a globally timelike Killing field $T$, where the absence of real modes for equation $\Box_{g} \psi - V \psi = 0$ is immediate. On such spacetimes, this fact gives a useful continuity criterion for showing stability for a smooth family of equations $\Box_{g} \psi - V_\lambda \psi = 0$, with $\lambda \in [0,1]$ and $V_0 = 0$: It suffices to bound the resolvent for frequencies in a neighborhood of $\omega = 0$ for all $\lambda \in [0,1]$. We show explicitly that this criterion fails on Kerr spacetime, by constructing a potential $V$ so that for the smooth family of equations $\Box_{g_{M,a}} \psi - \lambda V \psi = 0$, $\lambda \in [0,1]$, a real mode first appears at $\omega = \omega_0 \in \mathbb{R}\backslash\{0\}$ for $\lambda = \lambda_0 \in (0,1]$.

Contents

1 Introduction
1.1 Theorem 1: An instability result for $\Box \psi - V \psi = 0$ on Kerr spacetimes .................
1.2 Theorem 2: An instability result for $\Box \psi = 0$ on short-range deformations of the Kerr metric ........
1.3 An example with normally hyperbolic trapping ......................................................
1.4 Discussion of the role of superradiance as a mechanism of instability .........................
1.5 Local energy decay for the family $\Box \psi - V_\lambda \psi = 0$ via continuity in $\lambda \in [0,1]$ ...........
1.6 Outline of the paper ........................................................................................................
1.7 Acknowledgements .........................................................................................................
2 Conventions on constants and vector field currents

3 Basic properties of the Kerr spacetimes
  3.1 Special coordinate charts and Killing fields
  3.2 Separation of the wave equation on Kerr spacetimes
  3.3 Fourier decomposition and mode solutions
  3.4 The superradiant frequency regime

4 Proof of Theorem 1
  4.1 Detailed statement and sketch of the proof of Theorem 1
  4.2 The main lemmas
  4.3 Proof of Proposition 4.1

5 Proof of Theorem 2

6 Superradiant instabilities for the free wave equation on spacetimes with normally hyperbolic trapping
  6.1 Theorem 3: Statement and remarks on the proof
  6.2 Construction of the auxiliary spacetimes
  6.3 Separation of the wave equation and frequency decomposition
  6.4 An integrated local energy decay estimate in the case $a \ll M$
  6.5 The deformed metric $g_{M,a}^{(a,R_s)}$
  6.6 A generalisation of Theorem 1
  6.7 Proof of Theorem 3 and Proposition 6.1

7 A zero-frequency continuity criterion for decay and its failure in the presence of superradiance
  7.1 The resolvent operator
  7.2 A zero-frequency continuity criterion for decay for the family (7.2) in the non-superradiant case
  7.3 Failure of the zero-frequency continuity criterion in the presence of superradiance

A Proof of Proposition 7.1

B Proof of Proposition 7.2

C A topological lemma

1 Introduction

The celebrated Kerr family of spacetimes $(M, g_{M,a})$, first discovered in 1963 (see [29]), is a 2-parameter family of solutions to the vacuum Einstein equations

$$Ric_{\mu\nu} = 0,$$

parametrised by the mass $M$ and the angular momentum per unit mass $a$. In the Boyer–Lindquist coordinate chart $(t, r, \vartheta, \varphi) : M_{M,a} \to \mathbb{R} \times (r_+, +\infty) \times S^2$, the metric $g_{M,a}$ takes the form

$$g_{M,a} = -(1 - \frac{2Mr}{\varphi^2})dt^2 - \frac{4Mar}{\varphi^2}dtd\varphi + \frac{\varphi^2}{\Delta}dr^2 + \varphi^2d\vartheta^2 + \sin^2 \vartheta \frac{\Pi}{\varphi^2}d\varphi^2,$$
where
\begin{align}
(1.3) \quad \varphi^2 &= r^2 + a^2 \cos^2 \theta, \\
(1.4) \quad \Delta &= (r - r_+) \cdot (r - r_-), \\
(1.5) \quad r_\pm &= M \pm \sqrt{M^2 - a^2}, \\
(1.6) \quad \Pi &= (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta.
\end{align}

The Schwarzschild metric corresponds to (1.2) for \( a = 0 \).

In the so called \textit{subextremal} parameter range \( 0 \leq |a| < M \), the maximal extension \((\tilde{M}_{M,a}, \tilde{g}_{M,a})\) of the Kerr spacetime \((\mathcal{M}_{M,a}, g_{M,a})\), first constructed by Carter in [4], has two asymptotically flat ends, and contains a black hole and a white hole region, which are bounded by a future and a past event horizon \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) respectively. The union \( \mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \) is the event horizon of \((\tilde{M}_{M,a}, \tilde{g}_{M,a})\), while the intersection \( \mathcal{H}^+ \cap \mathcal{H}^- \) (which is non empty) is the so called \textit{bifurcation sphere}. In this extension, \((\mathcal{M}_{M,a}, g_{M,a})\) is identified with the domain of outer communications of one of the two asymptotically flat ends of \((\tilde{M}_{M,a}, \tilde{g}_{M,a})\).

The wave equation
\begin{equation}
\Box_{g_{M,a}} \psi = 0
\end{equation}
on \((\mathcal{M}_{M,a}, g_{M,a})\), for \( 0 \leq |a| < M \), has been extensively studied. A first result relevant to the stability (i.e. boundedness and decay) properties of equation (1.7) in the full subextremal range \( 0 \leq |a| < M \) is the proof by Whiting, in [44], that equation (1.7) does not admit exponentially growing mode solutions, i.e. solutions \( \psi \) of the form
\begin{equation}
\psi(t, r, \theta, \varphi) = e^{-i \omega t} \varphi(r, \theta, \varphi)
\end{equation}
such that \( \text{Im}(\omega) > 0 \), with \( \varphi \) being smooth up to \( \mathcal{H} \setminus \mathcal{H}^- \) and having finite energy on the \( \{ t = \text{const} \} \) slices. This result was extended by Shlapentokh-Rothman in [40], where the non-existence of outgoing real mode solutions of (1.7) on \((\mathcal{M}_{M,a}, g_{M,a})\), \( 0 \leq |a| < M \), was established. Recall that a solution \( \psi \) to (1.7) is called an outgoing real mode solution if it is of the form (1.8) with \( \omega \in \mathbb{R} \setminus \{ 0 \} \), such that \( \psi \) is smooth up to \( \mathcal{H} \setminus \mathcal{H}^- \) and has finite energy flux through a hyperboloidal hypersurface \( \mathcal{S} \) terminating at future null infinity and intersecting \( \mathcal{H}^+ \) transversally (satisfying also \( \mathcal{S} \cap \mathcal{H}^- = \varnothing \)), but has infinite energy flux through the \( \{ t = \text{const} \} \) hypersurfaces; see [40] (or Section 3.3) for more details.

Notice that, in view of the fact that the stationary Killing field \( T = \partial_t \) fails to be causal on \( \mathcal{H}^+ \) when \( a \neq 0 \), the results of [44] [40] are highly non-trivial in this case (unlike the Schwarzschild case \( a = 0 \), where they would simply follow from the energy identity for \( T \); see Section 1.4). The absence of an everywhere causal Killing field on \((\mathcal{M}_{M,a}, g_{M,a})\) when \( a \neq 0 \) gives rise to the phenomenon of \textit{superradiance} for equation (1.7), which we will discuss in more detail in Section 1.4.

The results of [40] were then used in [13], where quantitative decay estimates for solutions \( \psi \) to (1.7) on \((\mathcal{M}_{M,a}, g_{M,a})\) for \( 0 \leq |a| < M \) were obtained. For earlier stability results in the Schwarzschild case (i.e. when \( a = 0 \) and the very slowly rotating Kerr case (i.e. when \( |a| \ll M \)), see [28] [8] [10] [8] [2] [3] and [13] [14] [11] [11] [11] respectively. It should be noted, also, that the techniques employed in [40] are robust enough to yield a mode stability statement for \textit{small short-range} potential perturbations of equation (1.7).

In contrast to the above results, superradiance-related instability phenomena come into the picture in the case of \textit{long-range} perturbations to equation (1.7), showing that the mode stability results of [14] [40] can not be extended to include this case: In [39], Shlapentokh-Rothman constructed exponentially growing mode solutions (as well as real mode solutions) to the Klein–Gordon equation
\begin{equation}
\Box_{g_{M,a}} \psi - \mu^2 \psi = 0
\end{equation}
on \((\mathcal{M}_{M,a}, g_{M,a})\) with \( a, \mu \neq 0 \). This result was anticipated by the heuristics of [10] [45] [17] (see also the numerics of [18] [19] and references therein).

The question naturally arises, therefore, whether the mode stability results of [40] can be extended to include \textit{large short-range} deformations of equation (1.7), in the form of either potential perturbations or metric deformations retaining the causal character of \( T \) at each point. In this paper, we will provide a negative answer to this question in both cases.
Figure 1.1: A hyperboloidal hypersurface $S \subset M_{M,a}$ terminating at $I^+$ and intersecting $H^+$ transversally, such that $S \cap H^- = \emptyset$ (depicted above intersected with the 1+1 dimensional slice $\{\theta = \pi/2, \phi = 0\} \subset M_{M,a}$). An outgoing real mode solution $\hat{\psi}$ to equation (1.7) has finite energy flux through $S$, but infinite energy flux through the hypersurface $\{t = 0\}$.

1.1 Theorem 1: An instability result for $\Box \psi - V \psi = 0$ on Kerr spacetimes

Our first result will show that the mode stability statement of [44, 40] cannot be extended to include the case when an arbitrary term of the form $-V \psi$, with $V \geq 0$ compactly supported in the region $\{r \geq r_0\}$, is added to the wave equation (1.7) on $(M_{M,a}, g_{M,a})$ for any $a$ in the subextremal range $0 < |a| < M$. In particular, we will establish the following:

**Theorem 1** (short version). For any $0 < |a| < M$, any $\omega_R \in \mathbb{R}\setminus\{0\}$, any $r_0 \gg 1$ large in terms of $\omega_R$ and $a$, any $0 < \omega_I \ll 1$ small in terms of $\omega_R$ and $a$, there exists a $V : M_{M,a} \to [0, +\infty)$ compactly supported in the region $\{r \geq r_0\}$ and satisfying $\partial_t V = \partial_{\phi} V = 0$, such that the equation

(1.10)  \[ \Box_{g_{M,a}} \psi - V \psi = 0 \]

admits an outgoing mode solution with frequency parameter $\omega_R + i\omega_I$. In particular, (1.10) admits a real mode solution with $\omega_I = 0$, and an exponentially growing mode solution with $\omega_I > 0$.

For a more detailed statement of Theorem 1, see Section 4. For the definition of an outgoing mode solution, see Section 3.3.

Note, in contrast, that, on a stationary and asymptotically flat spacetime with an everywhere causal Killing field (e.g. Schwarzschild exterior), adding a potential $V$ with the properties described in Theorem 1 to the wave equation does not “destroy” the decay properties of the corresponding solutions. It is due to the phenomenon of superradiance (more precisely, the fact that $H^+ \cap \{g(T,T) > 0\} \neq \emptyset$) that a construction of a real mode solution to equation (1.10) in the case $a \neq 0$ is possible. For a discussion on the role of superradiance as a mechanism of instability, see Section 1.4.
1.2 Theorem 2: An instability result for $\Box \psi = 0$ on short-range deformations of the Kerr metric

As an application of Theorem 1, we will infer the existence of real and exponentially growing modes for the wave equation associated to stationary and compactly supported in space deformations $g_{M,a}^{(de)}$ of the Kerr metric $g_{M,a}$ with the same ergoregion as $g_{M,a}$:

**Theorem 2** (short version). For any $0 < |a| < M$, any $\omega_R \in \mathbb{R}\setminus\{0\}$, any $0 \leq \omega_I \ll 1$ and any $r_0 > 1$, there exists a stationary and axisymmetric Lorentzian metric $g_{M,a}^{(de)}$ on $\mathcal{M}_{M,a}$, coinciding with $g_{M,a}$ outside the region \{$r_0 \leq r \leq r_0 + C$\} (for some $C \gg 1$) and satisfying $g_{M,a}^{(de)}(\partial_t, \partial_t) < 0$ in \{$r_0 \leq r \leq r_0 + C$\}, such that the wave equation

\[
\Box g_{M,a}^{(de)} \psi = 0
\]

completely separates in the Boyer–Lindquist coordinate chart and admits an outgoing mode solution with frequency parameter $\omega_R + i\omega_I$. In particular, \{(1.11)\} admits a real mode solution with $\omega_I = 0$, and an exponentially growing mode solution with $\omega_I > 0$.

For a more detailed statement of Theorem 2 see Section 5.

1.3 An example with normally hyperbolic trapping

The spacetimes $(\mathcal{M}_{M,a}, g_{M,a}^{(de)})$ of Theorem 2 possess the same ergoregion structure as $(\mathcal{M}_{M,a}, g_{M,a})$, but our specific construction forces the structure of the trapped set to be different. In particular, the region $r_0 \leq r \leq r_0 + C$ of $(\mathcal{M}_{M,a}, g_{M,a}^{(de)})$ contains stable trapped null geodesics. As a consequence, the behaviour of high frequency solutions to equations (1.7) and (1.11) is substantially different.

Despite this aspect of the spacetimes $(\mathcal{M}_{M,a}, g_{M,a}^{(de)})$, in general there is no reason for a connection to exist between the structure of the trapped set outside the ergoregion (which manifests itself in the high frequency behaviour of solutions to (1.7) and superradiance-related mode-instabilities (which is a purely fixed frequency phenomenon). In order to better clarify the irrelevance of the structure of trapping to the existence of a real mode solution to (1.11), it would be preferable to have an example of a spacetime $(\mathcal{M}, g)$ possessing a “nice” trapped set and at the same time admitting real or exponentially growing modes. In Section 6 we construct a spacetime $(\mathcal{M}_0, g_{M,a}^{(h)})$ which has the symmetries of the Kerr exterior $(\mathcal{M}_{M,a}, g_{M,a})$, such that the trapped set of $(\mathcal{M}_0, g_{M,a}^{(h)})$ is normally hyperbolic and the wave equation

\[
\Box g_{M,a}^{(h)} \psi = 0
\]

admits an outgoing real mode solution (which in turn yields, after a suitable perturbation, an exponentially growing mode solution). However, our specific construction forces the spacetime $(\mathcal{M}_0, g_{M,a}^{(h)})$ to be asymptotically conic, instead of asymptotically flat like the Kerr exterior $(\mathcal{M}_{M,a}, g_{M,a})$. For more details regarding this technically involved (in comparison to the proof of Theorem 2) construction, see Section 6 and Theorem 5.

1.4 Discussion of the role of superradiance as a mechanism of instability

In this section, we will try to put into some context the phenomenon of superradiance and its relation to instability results concerning the wave equation

\[
\Box \psi = 0
\]

on a general class of spacetimes $(\mathcal{M}, g)$.

Let $(\mathcal{M}, g)$ be a globally hyperbolic, stationary and asymptotically flat spacetime with stationary Killing field $T$, possibly bounded by an event horizon $\mathcal{H}$. For the purposes of this section, and in analogy with the properties

---

1 The discussion in this section also applies to asymptotically conic spacetimes, as the ones discussed in Section 1.3.
of the subextremal Kerr exterior \((\mathcal{M}_{M,a}, g_{M,a})\), the spacetime \((\mathcal{M}, g)\) will be called superradiant if \(T\) fails to be causal everywhere on \(\mathcal{M}\). In the case when \(T\) is everywhere timelike on \(\mathcal{M}\setminus \mathcal{H}\), the spacetime \((\mathcal{M}, g)\) will be called non-superradiant\(^2\). It can be readily shown that a spacetime \((\mathcal{M}, g)\) as above is superradiant if and only if there exist solutions \(\psi\) to (1.13) such that their \(T\)-energy flux through future null infinity \(\mathcal{I}^+\) is greater than their \(T\)-energy flux through a Cauchy hypersurface \(\Sigma\) of \(\mathcal{M}\) (for the definition of the \(T\)-energy flux, see below; for the definition of \(\mathcal{I}^+\) on a general asymptotically flat spacetime, see \([33]\)).

We will now proceed to examine some general mode stability statements for equation (1.13) on non-superradiant spacetimes and then highlight the failure of these statements on superradiant spacetimes. To this end, it will be convenient to distinguish among superradiant spacetimes the ones having a non-empty future event horizon \(\mathcal{H}^+\) satisfying \(\mathcal{H}^+ \cap \{g(T, T) > 0\} \neq \emptyset\) (such as Kerr exterior spacetime \((\mathcal{M}_{M,a}, g_{M,a})\) when \(a \neq 0\)).

### 1.4.1 Mode stability results on non-superradiant spacetimes

Let \((\mathcal{M}, g)\) be a spacetime as above which is non-superradiant, and let \(V : \mathcal{M} \to \mathbb{R}\) be a smooth function, having compact support in space and satisfying \(T(V) = 0\). In this case, equation

\[
\Box_g \psi - V \psi = 0,
\]

satisfies the following mode stability statement for real and non-zero frequencies:

1. a) No outgoing real mode solutions at a non-zero real frequency parameter \(\omega\) exist for equation (1.14).

   b) No \(L^2\) “eigenfunctions” (i.e. solutions \(\psi\) of the form (1.8) such that \(\int_{t=0} |\psi|^2 < +\infty\) at a non-zero real frequency parameter \(\omega\) exist for equation (1.14).

The above statement can be inferred from the fact that the energy flux associated to the vector field \(T\), which is obtained by integrating the divergence-free current

\[
\mathcal{E}_g[\psi] = J^T \psi - V |\psi|^2 g_{\mu\nu} T^\nu
\]

over a chosen causal hypersurface (see Section 2 for our notations on vector field currents), is positive definite both on \(\mathcal{H}^+\) and on future null infinity \(\mathcal{I}^+\). Thus, integrating the identity

\[
\nabla^T \mathcal{E}_g[\psi] = 0
\]

over suitable subregions of \(\mathcal{M}\) (yielding an identity for the \(T\)-energy flux of \(\psi\) on the associated boundary hypersurfaces, the so called \(T\)-energy identity) yields that a real mode solution of (1.14) must have vanishing radiation field on \(\mathcal{I}^+\) and that it must actually be an \(L^2\) eigenfunction for (1.14). However, using a unique continuation argument (similar to the ones appearing in \([36, 34]\), or using the Carleman estimates of \([38, 32]\)), we can infer that any solution \(\psi\) to (1.14) of the form (1.8) such that \(\int_{t=0} |\psi|^2 < +\infty\) vanishes identically on \(\mathcal{M}\); see also \([40]\).

Assuming, moreover, that the potential function \(V\) in (1.14) satisfies the non-negativity condition \(V \geq 0\), one can readily obtain the following results in addition to 1.a–1.b, yielding the full “mode stability” statement for equation (1.14):

2. Equation (1.14) does not have a zero eigenvalue or a zero resonance\(^3\).

3. Equation (1.14) does not admit any exponentially growing mode solutions, since the energy norm for solutions to (1.14) associated to the \(T\)-energy flux is positive definite and conserved.

In particular, these results are in contrast with the situation in Theorem 1.

---

\(^2\)Note that, under the above definitions, there exist spacetimes which are neither superradiant nor non-superradiant: For instance, stationary spacetimes \((\mathcal{M}, g)\) on which \(T\) is everywhere causal and identically null on an open set \(\mathcal{U} \subset \mathcal{M}\), or the spacetimes considered in \([22]\), do not fall in either category.

\(^3\)Equation (1.14) is said to admit a zero eigenvalue if there exists a solution \(\psi\) to (1.14) which is smooth up to \(\mathcal{H}\) (if non-empty) and satisfies \(T \psi = 0\) and \(\|\psi\|_{L^2(\Sigma)} < +\infty\), where \(\Sigma\) is a Cauchy hypersurface of \((\mathcal{M}, g)\). If \(\psi\) satisfies \(T \psi = 0\), \(\|\psi\|_{L^2(\Sigma)} = +\infty\) but \(\|\psi\|_{L^2(\Sigma)} < +\infty\), then \(\psi\) is called a zero resonance.
1.4.2 Superradiant spacetimes: the case $\mathcal{H}^+ \cap \{g(T,T) > 0\} = \emptyset$

Let us now examine the behaviour of solutions to (1.14) in the case $(\mathcal{M}, g)$ has a non-empty ergoregion (i.e. $(\{g(T,T) > 0\}) \neq \emptyset$) with the property that either $\mathcal{H}^+ \neq \emptyset$, or $\mathcal{H}^+ = \emptyset$, and $\mathcal{H}^+ \cap \{g(T,T) > 0\} = \emptyset$ (note that the Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$, $a \neq 0$, does not satisfy this property). The condition $\mathcal{H}^+ \cap \{g(T,T) > 0\} = \emptyset$ implies the positivity of the flux of (1.15) through $\mathcal{H}^+$.

On such a spacetime $(\mathcal{M}, g)$, there exist smooth solutions $\psi$ to equation (1.13) with compactly supported initial data, such that the energy of $\psi$ grows to infinity as time increases; see [23, 31]. While the proof of [31] does not yield the existence of exponentially growing mode solutions to equation (1.13) on $(\mathcal{M}, g)$, it is reasonable to expect that, in general, such a mode exists. Thus, it is in general expected that the mode stability statements 2 and 3 (concerning the non-existence of solutions with a frequency parameter $\omega$ satisfying $\omega = 0$ or $Im(\omega) > 0$) fail in this case, even under the sign condition $V \geq 0$ for the potential term in equation (1.13). However, such a spacetime $(\mathcal{M}, g)$ always satisfies Statement 1.a, i.e. $(\mathcal{M}, g)$ does not admit an outgoing real mode solution to (1.14) in this case. This fact can be inferred as follows: the T-energy identity and the positivity of the flux of (1.15) through $\mathcal{H}^+$ imply that any solution $\psi$ to (1.14) which is of the form (1.15) for some $\omega \in \mathbb{R}\setminus\{0\}$ and satisfies

$$(1.17) \quad \lim_{r \to +\infty} \left( \partial_r \psi - i \omega \psi \right) = 0$$

in each asymptotically flat end of $(\mathcal{M}, g)$, has necessarily vanishing T-energy flux through future null infinity $\mathcal{I}^+$. Therefore, since $\omega \neq 0$, it can be readily verified that, for any Cauchy hypersurface $\Sigma$ of $(\mathcal{M}, g)$:

$$(1.18) \quad \int_{\Sigma} |\psi|^2 < +\infty. $$

Thus, equation (1.14) in this case does not admit an outgoing real mode solution.

**Remark.** In general, we can not exclude the existence of $L^2$ “eigenfunctions” at a non-zero real frequency parameter $\omega$ for spacetimes $(\mathcal{M}, g)$ as above, i.e. Statement 1.b might not hold. However, the conditions (1.18) and $\omega \in \mathbb{R}\setminus\{0\}$ imply, through a suitable unique continuation argument that can be obtained by adapting the Carleman-type estimates of [32] (or the estimates of Section 6 of [31]), that any $L^2$ “eigenfunction” $\psi$ will be identically 0 in the connected component of $\mathcal{M}\setminus\{g(T,T) > 0\}$ which contains the asymptotically flat region of $\mathcal{M}$. Thus, under a stronger unique continuation assumption for equation (1.14) in a neighborhood of the ergoregion (satisfied, for instance, when both $(\mathcal{M}, g)$ and the potential $V$ are analytic), the statement 1.b can also be established.

1.4.3 Superradiant spacetimes: the case $\mathcal{H}^+ \cap \{g(T,T) > 0\} \neq \emptyset$

Let us finally examine the case when $(\mathcal{M}, g)$ has a non-empty ergoregion and a non-empty future event horizon, with the property $\mathcal{H}^+ \cap \{g(T,T) > 0\} \neq \emptyset$ (such as the Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ when $a \neq 0$). In this case, the energy identity of $T$ no longer yields a positive energy flux through $\mathcal{H}^+$, and, thus, the vanishing of the flux of a real mode solution $\psi$ through $\mathcal{I}^+$ can not be inferred as before. Therefore, on such a spacetime, the aforementioned argument leading to the non-existence of real mode solutions $\psi$ to equation (1.14) no longer applies, and the real mode stability statement 1.a might, in general, fail (in addition to the statements 1.b, 2 and 3). In particular, Theorems 1 and 2 provide examples of spacetimes $(\mathcal{M}, g)$ and potentials $V$ such that the mode stability statement 1.a for equation (1.14) fails.

In view of the aforementioned discussion, the proof of [40], that equation (1.7) on $(\mathcal{M}_{M,a}, g_{M,a})$ (with $0 < |a| < M$) does not admit real mode solutions, is highly non-trivial, and relies on the specific algebraic structure of $(\mathcal{M}_{M,a}, g_{M,a})$. According to Theorem 1, this structure is “destroyed” by adding a compactly supported non-negative potential term $V\psi$ to equation (1.7), a modification that would be completely harmless in the non-superradiant case (preserving the mode stability properties of (1.13) in the case where $T$ is everywhere causal on $\mathcal{M}$, as explained in Section 1.4.1).

---

\[4\] We should note that it is not at all clear if there exists a spacetime $(\mathcal{M}, g)$ with a non-empty ergoregion and no event horizon, such that $(\mathcal{M}, g)$ does not admit an exponentially growing mode. A positive answer to this question would be particularly interesting, as it would show that mode stability results for equation (1.7) on superradiant spacetimes can coexist with instability results in physical space.
1.5 Local energy decay for the family $\Box_g \psi - V_\lambda \psi = 0$ via continuity in $\lambda \in [0, 1]$

As a final example of the differences between superradiant and non-superradiant spacetimes concerning the behaviour of solutions to equation (1.14), we will state below a simple continuity criterion for integrated local energy decay for a family of wave equations with potential on non-superradiant spacetimes, which utterly fails in the presence of superradiance.

As we showed in Section [1.4.1] on a globally hyperbolic, stationary and asymptotically flat spacetime $(\mathcal{M}, g)$, possibly bounded by an event horizon $\mathcal{H}$ and possessing a Killing field $T$ which is everywhere timelike on $\mathcal{M}\setminus\mathcal{H}$, the mode stability statements 1.a and 1.b of Section [1.4.1] hold. This fact gives rise to the following zero-frequency continuity criterion for integrated local energy decay for the family of equations (1.19)

$$\Box_g \psi - V_\lambda \psi = 0,$$

$\lambda \in [0, 1]$, with $V_\lambda : \mathcal{M} \to \mathbb{R}$ having compact support in space, satisfying $T(V_\lambda) = 0$ and depending smoothly on $\lambda$:

**Zero-frequency continuity criterion for integrated local energy decay:** Provided the family (1.19) satisfies an integrated local energy decay estimate (possibly with loss of derivatives) when $\lambda = 0$, and the resolvent of $\Box - V_\lambda$ can be bounded for frequencies $\omega$ in a neighborhood of 0 for all $\lambda \in [0, 1]$, then (1.19) satisfies a similar integrated local energy decay estimate for all $\lambda \in [0, 1]$.

See Section 7 for a more detailed statement of the above criterion, as well as for a definition of the resolvent operator in this setting.

Let us now examine whether the above criterion can be extended to the case when $(\mathcal{M}, g)$ is superradiant. As we did in Sections [1.4.2] and [1.4.3] we have to differentiate between the cases $\{g(T, T) > 0\} \cap \mathcal{H}^+ = \emptyset$ and $\{g(T, T) > 0\} \cap \mathcal{H}^+ \neq \emptyset$. In view of [23] (see also our forthcoming [31]), in the case when $\{g(T, T) > 0\} \neq \emptyset$ and $\mathcal{H}^+ = \emptyset$ (or at least $\{g(T, T) > 0\} \cap \mathcal{H}^+ = \emptyset$), there exist solutions $\psi$ to equation (1.13) which are not square integrable in time. Thus, in order to study conditions yielding an integrated local energy decay estimate for families of equations of the form (1.19) on superradiant spacetimes, it is necessary to restrict to spacetimes $(\mathcal{M}, g)$ with $\mathcal{H}^+ \neq \emptyset$ satisfying $\{g(T, T) > 0\} \cap \mathcal{H}^+ \neq \emptyset$, such as the subextremal Kerr spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ with $a \neq 0$.

In view of Theorem 1, the aforementioned zero-frequency continuity criterion fails in the case of $(\mathcal{M}_{M,a}, g_{M,a})$ with $a \neq 0$. In particular, there exists a suitable potential $V$ on $\mathcal{M}_{M,a}$ so that, for the family

$$(1.20) \quad \Box_{g_{M,a}} \psi - \lambda V \psi = 0$$

with $\lambda \in [0, 1]$, a real mode solution at some non-zero frequency $\omega_0$ first appears for $\lambda = \lambda_0 \in (0, 1)$, while the resolvent operator associated to (7.18) remains bounded for frequencies close to 0 for all $\lambda \in [0, 1]$. Note that, when $\lambda$ is close to 0, the robustness of the method of [10] implies that (1.20) does not admit any real mode solutions. See Section 7 for more details.

1.6 Outline of the paper

This paper is organised as follows:

In Section 2, we will introduce the conventions on the notations of constants and vector fields that will be used throughout the paper.

In Section 3, we will review the basic properties of the subextremal Kerr spacetimes $(\mathcal{M}_{M,a}, g_{M,a})$, including the separability of the wave operator $\Box_{g_{M,a}}$, and we will introduce the notion of an outgoing mode solution of the wave equation.

In Section 4, we will provide a more detailed statement and the proof of Theorem 1. Similarly, a more detailed statement and the proof of Theorem 2 will be presented in Section 5.

In Section 6, we will construct a class of asymptotically conic spacetimes with normally hyperbolic trapping, admitting outgoing real or exponentially growing mode solutions to the wave equation.

Finally, in Section 7, we will present a zero-frequency continuity criterion for decay for the family of equations (1.19) on non-superradiant spacetimes, and we will then examine its failure in the presence of superradiance. To this end, we will also present a definition of the resolvent operator for the wave equation on a general class of spacetimes. We should note that Section 7 can be read independently of Sections 5 and 6.
1.7 Acknowledgements

I would like to express my gratitude to my advisor Mihalis Dafermos for providing me with comments, ideas and assistance while this paper was being written. I would also like to thank Igor Rodnianski for many insightful comments and suggestions. Finally, I would like to thank Yakov Shlapentokh-Rothman and Stefanos Aretakis for many helpful conversations.

2 Conventions on constants and vector field currents

We will adopt the same conventions for denoting constants, volume forms and vector field currents as was done in [32] [33].

In particular, capital letters (e.g. $C$) will be used to denote “large” constants, typically appearing on the right hand side of inequalities, while lower case letters (e.g. $c$) will be used to denote “small” constants. The same characters will be frequently used to denote different constants. The dependence of these constants on various unfixed parameters will be usually made explicit through the use of appropriate subscripts.

The notation $f_1 \lesssim f_2$ for two real functions $f_1, f_2$ will be used to imply that there exists some $C > 0$, such that $f_1 \leq C f_2$. We will also write $f_1 \sim f_2$ when we can bound $f_1 \leq f_2$ and $f_2 \leq f_1$, while $f_1 \ll f_2$ will be equivalent to the statement that $\frac{|f_1|}{|f_2|}$ can be bounded by some sufficiently small (depending on the context) constant $c > 0$. In each of the aforementioned cases, the dependence of the implicit constants on any unfixed parameters will be clear from the context. For any function $f : \mathcal{A} \to [0, +\infty)$ on some set $\mathcal{A}$, $\{ f \gg 1 \}$ will denote the subset $\{ f \geq C \}$ of $\mathcal{A}$ for some constant $C \gg 1$.

The natural volume form $dg$ associated to the metric $g$ of a Lorentzian manifold $(\mathcal{M}^{d+1}, g)$ is defined, in any local coordinate chart $(x^0, x^1, x^2, \ldots x^d)$, as:

$$dg = \sqrt{-\det(g)} dx^0 \cdots dx^d.$$ 

We will often omit the notation for $dg$ in the expression of integrals over measurable subsets of $\mathcal{M}$. The same rule will apply when integrating over any spacelike hypersurface $\mathcal{S}$ of $(\mathcal{M}, g)$ using the natural volume form of its induced (Riemannian) metric.

We will also frequently use the language of currents and vector field multipliers for the wave equation: On any Lorentzian manifold $(\mathcal{M}, g)$, associated to the wave operator

$$\square_g = \frac{1}{\sqrt{-\det(g)}} \partial_\alpha \left( \sqrt{-\det(g)} \cdot g^{\alpha \beta} \partial_\beta \right)$$

is a $(0, 2)$-tensor called the energy momentum tensor $Q$. For any smooth function $\psi : \mathcal{M} \to \mathbb{C}$, the energy momentum tensor $Q$ evaluated at $\psi$ is given by the expression

$$Q_{\alpha \bar{\beta}}(\psi) = \frac{1}{2} \left( \partial_\alpha \psi \cdot \partial_\bar{\beta} \bar{\psi} + \partial_\bar{\beta} \bar{\psi} \cdot \partial_\alpha \psi \right) - \frac{1}{2} \left( \partial^\gamma \psi \cdot \partial_\gamma \bar{\psi} \right) g_{\alpha \bar{\beta}}.$$ 

For any continuous and piecewise $C^1$ vector field $X$ on $\mathcal{M}$, the following associated currents can be defined almost everywhere on $\mathcal{M}$:

$$J^X_\mu(\psi) = Q_{\mu \nu}(\psi) X^\nu,$$

$$K^X(\psi) = Q_{\mu \nu}(\psi) \nabla^\mu X^\nu.$$ 

The following divergence identity then holds almost everywhere on $\mathcal{M}$:

$$\nabla^\mu J^X_\mu(\psi) = K^X(\psi) + \text{Re} \left\{ \left( \square_g \psi \right) \cdot X \bar{\psi} \right\}.$$
3 Basic properties of the Kerr spacetimes

In this Section, we will provide an overview of the basic properties of the subextremal Kerr spacetimes that will be used throughout the rest of the paper.

3.1 Special coordinate charts and Killing fields

In the present paper, we will work on subextremal Kerr exterior spacetimes \( (\mathcal{M}_{M,a}, g_{M,a}) \), where \( g_{M,a} \) is given in the Boyer–Lindquist \( (t, r, \theta, \varphi) \) chart by the expression (1.2). Keeping the notation introduced in Section 1, we will denote with \( (\widetilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a}) \) the maximally extended Kerr spacetime, while \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) will denote the future and past event horizons associated to one (fixed) asymptotically flat end of \( (\widetilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a}) \). The original Kerr spacetime \( (\mathcal{M}_{M,a}, g_{M,a}) \) will be identified with the domain of outer communications of this asymptotically flat end of \( (\widetilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a}) \).

We will work entirely on \( (\mathcal{M}_{M,a}, g_{M,a}) \), but we will need to consider functions on \( \mathcal{M}_{M,a} \), which are regular up to \( \mathcal{H}^+ \backslash \mathcal{H}^- \). Since the Boyer–Lindquist coordinate chart can not be extended smoothly beyond \( \mathcal{H}^+ \backslash \mathcal{H}^- \) (in view of the fact that the expression (1.2) becomes singular as \( r \to r_+ \)), we will sometimes use the Kerr star coordinate chart \( (t^*, r, \theta, \varphi^*) \) on \( \mathcal{M}_{M,a} \), which is regular up to \( \mathcal{H}^+ \backslash \mathcal{H}^- \). The new \( t^*, \varphi^* \) coordinate variables in this chart are given by the expressions

\[
t^* = t + \int_{2r_+}^{r} \frac{x^2 + a^2}{(x - r_+)(x - r_-)} \, dx
\]

and

\[
\varphi^* = \varphi + \int_{2r_+}^{r} \frac{a}{(x - r_+)(x - r_-)} \, dx.
\]

In Kerr star coordinates \( (t^*, r, \theta, \varphi^*) \), \( \mathcal{H}^+ \backslash \mathcal{H}^- \) corresponds to \( \mathbb{R} \times \{r_+\} \times S^2 \), where \( S^2 \) is parametrised by \( (\theta, \varphi^*) \) in the standard way.

The vector fields \( T = \partial_t \) and \( \Phi = \partial_{\theta} \) (in the Boyer–Lindquist coordinate chart) are Killing fields for \( g_{M,a} \), and extend to smooth Killing fields across \( \mathcal{H}^+ \backslash \mathcal{H}^- \) (and, in fact, on the whole of \( (\widetilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a}) \)). The hypersurface \( \mathcal{H}^+ \backslash \mathcal{H}^- \) is a non-degenerate Killing horizon of \( (\widetilde{\mathcal{M}}_{M,a}, \tilde{g}_{M,a}) \), with associated Killing field

\[
K = T - \frac{a}{2Mr_+} \Phi.
\]

We will call the vector field \( T \) the stationary Killing field of \( (\mathcal{M}_{M,a}, g_{M,a}) \), while \( \Phi \) will be called the axisymmetric Killing field.

3.2 Separation of the wave equation on Kerr spacetimes

A remarkable feature of the Kerr metric is that the wave equation (1.7) on \( \mathcal{M}_{M,a} \) is separable. The separability of the wave equation on \( (\mathcal{M}_{M,a}, g_{M,a}) \) was discovered by Carter in [5] and is a consequence of the fact that in addition to the Killing fields \( T, \Phi \), the Kerr metric also possesses an additional Killing tensor \( K_{a\beta} \) (as explained by Walker and Penrose in [12]).

In particular, for any \( \omega \in \mathbb{C} \) and any \( m \in \mathbb{Z} \), equation (1.7) admits formal solutions of the form

\[
\psi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\theta} S(\theta) R(r),
\]

where \( S(\theta) \) satisfies (for some \( \lambda \in \mathbb{C} \))

\[
-\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) + \left( \frac{m^2}{\sin^2 \theta} - a^2 \omega^2 \cos^2 \theta \right) S = \lambda S
\]

and \( R(r) \) satisfies

\[
\Delta \frac{d}{dr} \left( \frac{dR}{dr} \right) + \left( a^2 m^2 - 4Mar\omega m + (r^2 + a^2)^2 \omega^2 - \Delta(\lambda + a^2 \omega^2) \right) R = 0.
\]
When \( \omega \in \mathbb{R} \), equation (3.3), combined with the boundary condition

\[
S(\vartheta) \text{ is bounded at } \vartheta = 0, \pi, \tag{3.5}
\]
defines a well-posed Sturm–Liouville problem, with a set of eigenfunctions \( \{ S_{\omega ml} \}_{l \geq |m|} \) forming an orthonormal basis of \( L^2(\sin \vartheta d\vartheta) \), with corresponding real eigenvalues \( \{ \lambda_{\omega ml} \}_{l \geq |m|} \). In particular, when \( \omega = 0 \), \( S_{\omega ml} \) are the standard spherical harmonics, and \( \lambda_{\omega ml} = l(l + 1) \). The construction of the pair \( (S_{\omega ml}, \lambda_{\omega ml}) \) can also be performed, via perturbation theory, in the case when \( \omega \) is complex with \( |Im(\omega)| \ll 1 \). See [10, 15] for more details.

Let us define, for convenience, the auxiliary radial function \( r_* = r_*(r) : (r_*, +\infty) \to (-\infty, +\infty) \) by solving

\[
\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}. \tag{3.6}
\]

Having defined \( (S_{\omega ml}, \lambda_{\omega ml}) \) as above for \( (\omega, m, l) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|} \) with \( |Im(\omega)| \ll 1 \), after setting

\[
\lambda = \lambda_{\omega ml} \text{ in equation } (3.4), \tag{3.7}
\]
and \( \omega = \lambda_{\omega ml} \), we infer that

\[
u''_{\omega ml} + (\omega^2 - V_{\omega ml}) \nu_{\omega ml} = 0, \tag{3.8}
\]
where \( ' \) denotes differentiation with respect to \( r_* \) and

\[
V_{\omega ml} = \frac{4Mr_m\omega - a^2m^2 + \Delta(\lambda_{\omega ml} + a^2\omega^2)}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)}(a^2\Delta + 2Mr(r^2 - a^2)). \tag{3.9}
\]

The aforementioned separability is preserved after adding to the wave equation (1.7) a term of the form

\[
d_{g_{m,a}} \psi - \frac{(r^2 + a^2)^2}{(r - r_*)(r - r_-)r^2}V(r)\psi = 0 \tag{3.10}
\]

admits for any \( \omega \in \mathbb{C} \) and any \( m \in \mathbb{Z} \) solutions of the form

\[
\psi(t, r, \vartheta, \phi) = e^{-i\omega t} e^{im\varphi} S(\vartheta) R(r), \tag{3.11}
\]

with \( S \) as before and \( R \) solving

\[
\frac{\Delta}{dr} \left( \frac{dR}{dr} \right) + (a^2m^2 + (r^2 + a^2)\omega^2 - \Delta(\lambda + a^2\omega^2) - (r^2 + a^2)^2V) R = 0. \tag{3.12}
\]

In the case \( |Im(\omega)| \ll 1 \) (and with \( (S_{\omega ml}, \lambda_{\omega ml}), l \geq |m|, \) as before), setting \( u_{\omega ml}(r_*) \equiv (r^2 + a^2)^{1/2}R(r) \) in equation (3.12) for \( \lambda = \lambda_{\omega ml} \), we obtain:

\[
u''_{\omega ml} + (\omega^2 - V_{\omega ml} - V) \nu_{\omega ml} = 0. \tag{3.13}
\]

### 3.3 Fourier decomposition and mode solutions

Any smooth function \( \Psi : M_{M,a} \to \mathbb{C} \) which is square integrable in the \( t \) variable can be represented as

\[
\Psi(t, r, \vartheta, \phi) \equiv \sum_{(m,l) \in \mathbb{Z}^2_{\geq |m|}} \int_{-\infty}^{\infty} e^{-i\omega t} e^{im\varphi} S_{\omega ml}(\vartheta) \Psi_{\omega ml}(r) d\omega
\]

for some \( \Psi_{\omega ml} : (r_*, +\infty) \to \mathbb{C} \) which is square integrable in \( \omega \) and square summable in \( (m,l) \).

For any function \( F \) with compact support in \( M_{M,a} \), any solution \( \psi \) to equation

\[
d_{g_{m,a}} \psi = F \tag{3.14}
\]
which is smooth up to $\mathcal{H}^+ \backslash \mathcal{H}^-$ and is supported on the set $\{t \geq t_F\}$, where
\begin{equation}
    t_F = \inf \{t_0 \in \mathbb{R} | \{t = t_0\} \cap \text{supp}(F) \neq \emptyset\},
\end{equation}
is square integrable in time (according to the polynomial decay estimates established in [15]) and can thus be decomposed as
\begin{equation}
    \phi(t, r, \vartheta, \varphi) = \sum_{(m,l) \in \mathbb{Z} \times \mathbb{Z}_{\geq |m|}} \int_{-\infty}^{\infty} e^{-ict} e^{im\varphi} S_{\omega m l}(\vartheta) R_{\omega m l}(r) \, d\omega,
\end{equation}
with $u_{\omega m l}$ (defined in terms of $R_{\omega m l}$ as in (3.17)) satisfying for almost all $\omega \in \mathbb{R}$:
\begin{equation}
    \begin{cases}
        u''_{\omega m l} + (\omega^2 - V_{\omega m l}) u_{\omega m l} = (r^2 + a^2)^{-3/2} \Delta (\varphi^2 F)_{\omega m l} \\
        u'_{\omega m l} - i\omega u_{\omega m l} \to 0 \text{ as } r_+ \to +\infty \\
        u'_{\omega m l} + i(\omega - \frac{am}{2Mr_+}) u_{\omega m l} \to 0 \text{ as } r_+ \to -\infty.
    \end{cases}
\end{equation}
See [15] for more details.

We will now define the notion of an outgoing mode solution to equation (3.10):

**Definition.** Let $V : (r_+, +\infty) \to \mathbb{C}$ be any smooth and compactly supported function. A solution $\psi$ to equation
\begin{equation}
    \Box_{g_{M,a}} \psi - \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)} \varphi^2 V(r) \psi = 0
\end{equation}
on $(\mathcal{M}_{M,a}, g_{M,a})$ will be called an outgoing mode solution (or simply a mode solution) with parameters $(\omega, m, l) \in (\mathbb{C} \backslash \{0\}) \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$, $0 \leq \text{Im}(\omega) < 1$, if $\psi$ is of the form
\begin{equation}
    \psi(t, r, \vartheta, \varphi) = e^{-ict} e^{im\varphi} S_{\omega m l}(\vartheta) \cdot R_{\omega m l}(r),
\end{equation}
with $u_{\omega m l}$ (defined in terms of $R_{\omega m l}$ as in (3.17)) satisfying
\begin{equation}
    \begin{cases}
        u''_{\omega m l} + (\omega^2 - V_{\omega m l} - V) u_{\omega m l} = 0 \\
        e^{-i\varphi r_+} (u'_{\omega m l} - i\omega u_{\omega m l}) \to 0 \text{ as } r_+ \to +\infty \\
        e^{i(\omega - \frac{am}{2Mr_+}) r_+} (u'_{\omega m l} + i(\omega - \frac{am}{2Mr_+}) u_{\omega m l}) \to 0 \text{ as } r_+ \to -\infty.
    \end{cases}
\end{equation}

**Remark.** The above definition of a mode solution also extends to time frequencies $\omega$ with $\text{Im}(\omega) < 0$, provided the boundary conditions at $r_\pm = \pm \infty$ in (3.20) are replaced by the equivalent (in the $\text{Im}(\omega) \geq 0$ case) conditions $u_{\omega m l} \sim e^{i\varphi r_+}$ and $u_{\omega m l} \sim e^{-i(\omega - \frac{am}{2Mr_+}) r_+}$, as $r_+ \to +\infty, -\infty$ respectively. Such solutions to the wave equation (1.7) are known in the physics literature as quasinormal modes, and have been studied extensively (see e.g. [6]). For a definition of quasi-normal modes on more general spacetimes, see [20] [21] [13]. Notice, also, that it is straightforward to extend the above definition of an outgoing mode solution to any metric $g$ on $\mathcal{M}_{M,a}$, for which $\Box_g$ completely separates in the Boyer–Lindquist coordinate chart (such as the metrics constructed in Sections 5 [6].)

The notion of an outgoing mode solution can be introduced for more general stationary and asymptotically flat spacetimes: Let $(\mathcal{M}, g)$ be a smooth and globally hyperbolic spacetime, which is stationary (with stationary Killing field $T$), asymptotically flat (with the asymptotics described by Assumption 1 in Section 2.1.1 of [32]) and possibly bounded by a future event horizon $\mathcal{H}^+$ and a past event horizon $\mathcal{H}^-$ (for the relevant definitions, see Section 2.1.1 of [32]). Let $t : \mathcal{M} \to \mathbb{R}$ satisfy $T(t) = 1$, with $\{t = 0\}$ being a Cauchy hypersurface of $(\mathcal{M}, g)$, and let $N$ be a globally timelike vector field on $(\mathcal{M}, g)$ satisfying $[T, N] = 0$ and $N = T$ in the asymptotically flat region of $(\mathcal{M}, g)$. Finally, let $\mathcal{S} \subset \mathcal{M}$ be a smooth, inextendible spacelike hyperboloidal hypersurface satisfying $\mathcal{S} \cap \mathcal{H}^- = \emptyset$, intersecting $\mathcal{H}^+$ transversally (in the case $\mathcal{H}^+ \neq \emptyset$) and terminating at future null infinity $\mathcal{I}^+$ (see Section 3.1 of [33]). We can then introduce the following definition:
Definition. A smooth solution $\psi$ to equation
\begin{equation}
\Box_g \psi - V \psi = 0,
\end{equation}
for some smooth $V : \mathcal{M} \to \mathbb{C}$ satisfying $T(V) = 0$, is called an outgoing mode solution with frequency parameter $\omega \in \mathbb{C}\backslash\{0\}$, $\text{Im}(\omega) \geq 0$, if $\psi$ has the following properties:

1. The function $\psi$ is of the form
\begin{equation}
\psi = e^{-i\omega t} \psi_\omega
\end{equation}
with $T(\psi_\omega) = 0$.

2. We have
\begin{equation}
\int_S J^N_\mu(\psi)n^\mu_S < +\infty,
\end{equation}
where $n_S$ is the future directed unit normal to $S$.

3. In the case $\omega \in \mathbb{R}\backslash\{0\}$, we have
\begin{equation}
\int_{\{t=0\}} J^N_\mu(\psi)n^\mu = +\infty,
\end{equation}
where $n$ is the future directed unit normal to $\{t = 0\}$.

We should remark that (3.23) is a condition on both the regularity of $\psi$ near $\mathcal{H}^+\backslash\mathcal{H}^-$ and the decay properties of the first derivatives of $\psi$ near $I^+$.

Notice that, specialising the latter definition to the case of Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$, a mode solution $\psi$ with parameters $(\omega, m, l)$ on $\mathcal{M}_{M,a}$, according to our former definition, is automatically an outgoing mode solution with frequency parameter $\omega$, according to the latter definition. See also [40] for the relation between the outgoing radiation condition at $r = +\infty$ and condition (3.23).

3.4 The superradiant frequency regime

The superradiant frequency regime of $(\mathcal{M}_{M,a}, g_{M,a})$, consists of those frequency triads $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}$ for which the limits
\begin{equation}
F_\pm[u_{\omega ml}] = \lim_{r_\ast \to \pm \infty} \pm \text{Im}(\omega u_{\omega ml}^{-1} \cdot u'_{\omega ml})
\end{equation}
for any non-zero function $u_{\omega ml}$ satisfying (3.17) have opposite sign. In view of the boundary conditions of (3.17) at $r_\ast = \pm \infty$, it readily follows that the superradiant frequency regime has the following form:
\begin{equation}
\mathbb{A}^{(a,M)}_{\text{supp}} = \left\{ (\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|} \mid \omega \left( \omega - \frac{am}{2M_{r_\ast}} \right) < 0 \right\}.
\end{equation}
For a more detailed discussion about the structure of the superradiant frequency regime (3.26), as well as its relation to energy estimates for solutions to (1.7), see [40] and references therein.

4 Proof of Theorem [1]

In this Section, we will provide a more detailed statement and the proof of Theorem [1]
4.1 Detailed statement and sketch of the proof of Theorem 1

A more detailed statement of Theorem 1 is the following:

**Theorem 1** (detailed version). For any $0 < |a| < M$ and any frequency triad $(\omega_R, m, l) \in (\mathbb{R}\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}_{\pm m}$ in the superradiant regime (3.26), there exist constants $C_{\omega_R ml} > r_+$ and $C^{(0)}_{\omega_R ml} > 0$ depending only on $\omega_R, m, l$ (as well as the Kerr parameters $M, a$) such that for any $r_0 > C_{\omega_R ml}$ and any $\omega \geq 0$ sufficiently small in terms of $(\omega_R, m, l, r_0)$, there exists a smooth function $V: (r_+, \infty) \rightarrow [0, 2\omega_R^2]$ supported on $\{r_0 \leq r \leq r_0 + C^{(0)}_{\omega_R ml}\}$ such that the equation

\[(4.1) \quad \Box_g \psi - \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)\rho^2} V(r) \cdot \psi = 0\]

on Kerr spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ admits an outgoing mode solution with parameters $(\omega_R + i\omega_I, m, l)$.

The proof of Theorem 1 will be mainly based on the following stronger proposition, which immediately yields Theorem 1 as a special case:

**Proposition 4.1.** For any $0 < |a| < M$, any frequency triad $(\omega_R, m, l) \in (\mathbb{R}\backslash\{0\}) \times \mathbb{Z} \times \mathbb{Z}_{\pm m}$ in the superradiant regime (3.26), any smooth function $\mathcal{G}: \mathbb{R}^4 \rightarrow \mathbb{C}$ such that $\mathcal{G}[\omega_R, 0, \epsilon, \rho] \equiv 0$ and $\mathcal{G}[\omega_R, r, \epsilon, \rho] \equiv 0$, there exist constants $C_{\omega_R ml} > r_+$ and $C^{(0)}_{\omega_R ml} > 0$, depending on $\omega_R, m, l, M, a, r_0$, such that, for any $r_0 > C_{\omega_R ml}$, any $0 < \epsilon_1 \leq 1$ and any $0 \leq \omega_I < 1$ sufficiently small in terms of $\omega_R, m, l, M, a, r_0, \epsilon_1$ and $\mathcal{G}$, there exists a smooth real valued function $V: (r_+, \infty) \rightarrow [0, +\infty)$ supported on $\{r_0 \leq r \leq r_0 + C^{(0)}_{\omega_R ml}\}$ and satisfying

\[(4.2) \quad V(r) + \frac{4Mram\omega_R - a^2m^2 + \Lambda(\omega_{\omega_R ml} + a^2\omega_R^2)}{(r^2 + a^2)^2} \leq (1 + \epsilon_1)\omega_R^2,
\]

such that equation

\[(4.3) \quad u'' + \left((\omega_R + i\omega_I)^2 - V(\omega_R + i\omega_I)ml(r_*) + \mathcal{G}(\omega_R, \omega_I, V(r(r_*)), r_*) - V(r(r_*))\right)u = 0,
\]

where $V_{\omega ml}(r_*) = V_{\omega ml}(r(r_*))$ is given by (3.9), admits a solution $u$ satisfying the following boundary conditions at $r_+ = \pm\infty$:

\[(4.4) \quad \lim_{r_+ \rightarrow \infty} \left(e^{-i(\omega_R + i\omega_I)r_+} \left(u' - i(\omega_R + i\omega_I)u\right)\right) = 0,
\]

\[(4.5) \quad \lim_{r_+ \rightarrow -\infty} \left(e^{i(\omega_R + i\omega_I - \frac{am}{2Mr_+})r_+} \left(u' + i(\omega_R + i\omega_I - \frac{am}{2Mr_+})u\right)\right) = 0.
\]

Notice that, in view of the separation of the wave equation on $(\mathcal{M}_{M,a}, g_{M,a})$ and the definition of a mode solution (see Sections 3.2, 3.3 respectively), Theorem 1 is obtained after setting $\mathcal{G} \equiv 0$ and $\epsilon_1 = 1$ in the statement of Proposition 4.1. Choosing the function $\mathcal{G}$ and the parameter $\epsilon_1$ in a different way will be useful in the proof of Theorem 2.

The proof of Proposition 4.1 will occupy Sections 4.2 and 4.3. We will now proceed to sketch the main ideas of the proof.

**Sketch of the proof of Proposition 4.1.** The proof of Proposition 4.1 will be mainly based on the following straightforward observation, which we state here in the form of a lemma:

**Lemma 4.1.** For any function $w: \mathbb{R} \rightarrow \mathbb{C}$ satisfying an equation of the form

\[(4.6) \quad \frac{d^2w}{dx^2} + \Omega \cdot w = 0
\]

for some real function $\Omega: \mathbb{R} \rightarrow \mathbb{R}$, the quantity $\text{Im}\left(\frac{d\bar{w}}{dx}, \bar{w}\right)$ is constant in $x$.  

14
Notice that in the case of the separated equation \( (3.13) \), the conserved quantity \( \text{Im}(\frac{du\partial_{\omega m l}}{dx^2} \cdot \bar{u}_{\omega m l}) \) is proportional to the frequency separated \( T \)-energy current (see, e.g. [19]).

As a first step towards establishing Proposition \( 4.1 \), we will show that for any \( \omega \in \mathbb{R}\setminus\{0\} \) and any two solutions \( u_1, u_2 : [a, b] \rightarrow \mathbb{C} \) of

\[
\frac{d^2 u}{dx^2} + \omega^2 u = 0
\]
satisfying

\[
\text{Im}(u_1 \bar{u}_1) = \text{Im}(u_2 \bar{u}_2),
\]
assuming also that \( b - a \) is large enough, there exists a piecewise constant potential \( V : [a, b] \rightarrow [0, \omega^2] \) which is identically 0 in a neighborhood of \( x = a, b \), such that the modified equation

\[
\frac{d^2 u}{dx^2} + (\omega^2 - V) u = 0
\]

admits a solution \( u \) such that \( u = u_1 \) in a neighborhood of \( x = a \) and \( u = e^{i\vartheta_2} u_2 \) in a neighborhood of \( x = b \), for some suitable \( \vartheta_2 \in [0, 2\pi) \). See Lemma \( 4.13 \) for more details. Note that, in view of Lemma \( 4.1 \), the condition \( (4.8) \) is necessary for the existence of such a function \( V \).

The proof of Lemma \( 4.13 \) will be based on the observation that the functions \( u_1, u_2 : [a, b] \rightarrow \mathbb{C} \) trace out two ellipses \( C_1, C_2 \) in the complex plane, having the same orientation in view of \( (4.8) \). This fact will be used to show that, for a suitable value of \( \vartheta_2 \) and a well chosen closed interval \( [x_1, x_2] \subset (a, b) \), the curve \( u : [a, b] \rightarrow \mathbb{C} \) defined so that

1. \( u = u_1 \) on \( [a, x_1] \),
2. \( u = e^{i\vartheta_2} u_2 \) on \( [x_2, b] \) and
3. \( u = \varepsilon_{\vartheta_2} \) on \( [x_1, x_2] \), where \( \varepsilon_{\vartheta_2} : [x_1, x_2] \rightarrow \mathbb{C} \) is a parametrization of one of the common tangent lines of the ellipse \( C_1 \) and the rotated ellipse \( e^{i\vartheta_2} C_2 \)

is a \( C^1 \) and piecewise \( C^2 \) function from \( [a, b] \) to \( \mathbb{C} \). In particular, \( u \) will satisfy an ode of the form \( (4.9) \).

The second step in the proof of Proposition \( 4.1 \) will consist of showing the following: Even if equation \( (4.7) \) is perturbed by introducing a small imaginary part for \( \omega \) (as well as other small complex valued terms), and the function \( V \) is in addition required to be smooth, one can still obtain a suitable potential \( V \) and a smooth solution \( u \) of the (perturbed version of) equation \( (4.9) \), such that \( u \) induces any chosen initial data on \( a \) and \( b \) which are close enough to the initial data induced by \( u_1 \) and \( \lambda u_2 \), respectively, for some suitable \( \lambda \in \mathbb{C}\setminus\{0\} \). See Lemma \( 4.3 \) for more details.

Finally, the proof of Proposition \( 4.1 \) will be completed in Section \( 4.3 \) roughly along the following lines: We will first choose the functions \( u_1 \) and \( u_2 \) (appearing, for instance, in the statement of Lemma \( 4.3 \)) so that they satisfy the boundary conditions \( (4.4) \) and \( (4.5) \), respectively, for \( \omega_I = 0 \), normalised so that \( (4.8) \) also holds (this is possible only in the case when the frequency triad \( (\omega_R, m, l) \) lies in the superradiant regime \( (3.26) \)). Then, the proof will follow by applying Lemma \( 4.3 \) for equation \( (4.3) \) on the interval \( \{ r_0 \leq r \leq r_0 + C^{(0)}_{\omega_R m l} \} \), for a well chosen perturbation of the initial data induced by \( u_1, u_2 \) at \( r = r_0 \), \( r = r_0 + C^{(0)}_{\omega_R m l} \) respectively.

### 4.2 The main lemmas

In this section, we will state and prove some lemmas concerning the behaviour of solutions to the ordinary differential equation \( (4.7) \) with equal constants of motion. These lemmas lie at the heart of the proof of Proposition \( 4.1 \) and some of the technical details involved in their proof will be needed in the constructions of Section \( 6 \).
Lemma 4.2. Let \( \omega, L_0 \in \mathbb{R} \setminus \{0\} \) and \( C_0 > 0 \), and let \( a < b \) be two real numbers such that \( b - a > \frac{2\pi}{\omega} + \frac{\pi C_0^2}{|L_0|} + 2 \). Let \( u_1, u_2 : [a, b] \to \mathbb{C} \) be two solutions of the ordinary differential equation

\[
d\frac{d^2 u}{dx^2} + \omega^2 u = 0
\]

with equal and non-zero constant of motion (see Lemma 4.1)

\[
\operatorname{Im}(\frac{du_1}{dx} \cdot \bar{u}_1) = \operatorname{Im}(\frac{du_2}{dx} \cdot \bar{u}_2) = L_0 \neq 0,
\]

satisfying also

\[
\sup_{[a, b]} (|u_1| + |u_2|) \leq C_0,
\]

such that \( \frac{u_1}{u_2} \) is not a constant function. Then for any \( x_0 \in [a, b] \) such that \( [x_0, x_0 + \frac{2\pi}{\omega} + \frac{\pi C_0^2}{|L_0|}] \subset (a, b) \), there exists a \( C^1 \) and piecewise \( C^2 \) function \( \tilde{u} : [a, b] \to \mathbb{C} \) satisfying for some suitable \( \theta_2 \in [0, 2\pi) \) and \( x_1, x_2 \in [x_0, x_0 + \frac{2\pi}{\omega} + \frac{\pi C_0^2}{|L_0|}] \) with \( x_1 < x_2 \):

\[
\begin{cases}
\frac{d^2 \tilde{u}}{dx^2} + (\omega^2 - \tilde{V}(x)) \tilde{u} = 0 \\
\tilde{u} = u_1 \text{ on } [a, x_1] \\
\tilde{u} = e^{i\theta_2} u_2 \text{ on } [x_2, b],
\end{cases}
\]

where the piecewise constant function \( \tilde{V} : \mathbb{R} \to [0, \omega^2] \) is defined in terms of \( x_1, x_2 \) as:

\[
\tilde{V}(x) = \begin{cases}
0, & x \in \mathbb{R} \setminus [x_1, x_2] \\
\omega^2, & x \in [x_1, x_2].
\end{cases}
\]

Proof. For \( j = 1, 2 \), we can decompose

\[
u_j(x) = A_j e^{i\alpha x} + B_j e^{-i\beta x}
\]

for some \( A_j, B_j \in \mathbb{C} \). Since \( b - a > \frac{2\pi}{\omega} \) and the constant of motion \( \operatorname{Im}(\frac{du_j}{dx} \cdot \bar{u}_j) \) for \( u_j \) is non-zero, the image of \( u_j \) is an ellipse \( C_j \) in the plane of the complex numbers, with its center at the origin. In view of the equality (4.11), we have

\[
|A_j|^2 - |B_j|^2 = |A_2|^2 - |B_2|^2 = \omega^{-1}L_0.
\]

Thus, in view of (4.16) and the fact that \( u_1/u_2 \) was assumed to not be identically constant, we infer that the semi-major and semi-minor axes of \( C_1, C_2 \) have different length, i.e.

\[
|A_1| + |B_1| \neq |A_2| + |B_2| \quad \text{and} \quad |A_1| - |B_1| \neq |A_2| - |B_2|.
\]

Therefore, for any \( \theta \in [0, 2\pi) \), the ellipses \( C_1 \) and \( e^{i\theta} C_2 \) (where \( e^{i\theta} C_2 \) is the rotation of \( C_2 \) by a \( \theta \)-angle around the origin) intersect at four distinct points which vary smoothly with \( \theta \). Moreover, the orientation induced on \( C_1, C_2 \) by their parametrization through \( u_j : [a, b] \to \mathbb{C} \) is the same, and, in particular, it is clockwise if \( L_0 < 0 \) and counter-clockwise if \( L_0 > 0 \).

In view of (4.16) and (4.17), for any \( \theta \in [0, 2\pi) \) there exist four straight lines \( \varepsilon^{(j)}_\theta, j = 1, \ldots, 4 \), which lie in the exterior of \( C_1 \) and \( e^{i\theta} C_2 \) and are tangent to both, and these lines depend smoothly on \( \theta \), satisfying also \( \varepsilon^{(j)}_\theta = \varepsilon^{(j)}_\phi \) (notice that the smooth dependence of \( \varepsilon^{(j)}_\theta \) on \( \theta \) is a consequence of (4.17)). We will parametrize these lines as \( \varepsilon^{(j)}_\theta : \mathbb{R} \to \mathbb{C} \), with

\[
\varepsilon^{(j)}_\theta(x) = a^{(j)}_\theta x + b^{(j)}_\theta
\]
by the polar rays through the points $P_\lambda$ and $\varepsilon$ of (4.19).

Figure 4.1: The two ellipses $C_1$ and $e^{i\theta}C_2$, with their four common tangents $\varepsilon_\theta^{(j)}$ and the associated tangent points $P_{k,\theta}^{(j)}$.

for some $p_0^{(j)}, b_0^{(j)} \in \mathbb{C}$ (that will be fixed more precisely shortly), depending smoothly on $\theta$.

Let $P_{1,\theta}^{(j)}$ and $P_{2,0}^{(j)}$ be the points where $\varepsilon_\theta^{(j)}$ meets $C_1$ and $e^{i\theta}C_2$ respectively. Without loss of generality, we will assume that $|A_1| + |B_1| > |A_2| + |B_2|$ in (4.17). Then, it follows that, if $\varepsilon_{\text{max}}^{(0)}$ is the straight line in $\mathbb{C}$ defined by the major semi-axis of $C_1$, no one of the points $P_{k,\theta}^{(j)}$ ($k = 1, 2, j = 1, \ldots, 4$) belongs to $\varepsilon^{(0)}$ for any $\theta \in [0, 2\pi]$. Notice also that $P_{k,\theta}^{(j)}$ vary smoothly with $\theta$ in view of (4.17) and (4.16) (and $P_{k,0}^{(j)} = P_{k,2\pi}^{(j)}$). Thus, for any $k = 1, 2$ and $j = 1, \ldots, 4$, the point $P_{k,\theta}^{(j)}$ remains on the same side of $\varepsilon^{(0)}$ for all $\theta \in [0, 2\pi]$.

Let $x_{1,\theta}^{(j)}, x_{2,\theta}^{(j)} \in \mathbb{R}$ be the unique points for which $\varepsilon_0^{(j)}(x_{1,\theta}^{(j)}) = P_{1,\theta}^{(j)}$ and $\varepsilon_0^{(j)}(x_{2,\theta}^{(j)}) = P_{2,\theta}^{(j)}$, $j = 1, \ldots, 4$. Let us also define $y_{1,\theta}^{(j)}$ to be the unique point in $[x_0, x_0 + 2\pi]$ such that $u_1(y_{1,\theta}^{(j)}) = P_{1,\theta}^{(j)}$. Since $\varepsilon_0^{(j)}$ is tangent to $C_1$ at $P_{1,\theta}^{(j)}$, we have

$$\varepsilon_0^{(j)}(x_{1,\theta}^{(j)}) = u_1(y_{1,\theta}^{(j)})$$

and

$$\frac{dx_0^{(j)}}{dx}(x_{1,\theta}^{(j)}) = \lambda_{1,\theta}^{(j)} \frac{du_1}{dx}(y_{1,\theta}^{(j)})$$

for some $\lambda_{1,\theta}^{(j)} \in \mathbb{R}\backslash\{0\}$. We will uniquely fix the the parametrization (4.18) of $\varepsilon_0^{(j)}$ by requiring that $x_{1,\theta}^{(j)} = y_{1,\theta}^{(j)}$ and $\lambda_{1,\theta}^{(j)} = 1$.

Notice that for two values of $j$ (say $j = 1, 3$), we have $x_{1,\theta}^{(j)} < x_{2,\theta}^{(j)}$ for all $\theta \in [0, 2\pi]$, i.e. in this parametrization of $\varepsilon_0^{(j)}$, $P_{1,\theta}^{(j)}$ lies before $P_{2,\theta}^{(j)}$. From now on, we will work only with the family of lines $\varepsilon_0^{(1)}$ and the family of points $P_{1,\theta}^{(1)}$ and $P_{2,\theta}^{(1)}$. In view of (4.11), (4.12), and the fact that $\lambda_{1,\theta}^{(1)} = 1$ in (4.20), as well as the fact that the angle formed by the polar rays through the points $P_{1,\theta}^{(1)}$ and $P_{2,\theta}^{(1)}$, is at most $\pi$, we can bound for any $\theta \in [0, 2\pi]$:  

$$x_{2,\theta}^{(1)} - x_{1,\theta}^{(1)} \leq \pi C_0^2 / |x_0|.$$ 

\(^6\)In case $|A_1| + |B_1| > |A_2| + |B_2|$, the same analysis goes through with the roles of $u_1$ and $u_2$ interchanged.
Thus, \( x_{2,2}^{(1)} \in [x_0, x_0 + \frac{2\pi}{\omega} + \frac{\pi C_0^2}{|L_0|}) \subset (a, b) \).

We have shown that the map \( h_2 : [0, 2\pi] \to \mathbb{C} \setminus \{0\} \),

\[
(4.22) \quad h_2(\theta) = P_{2,\theta}^{(1)}
\]
is a smooth map satisfying \( h_2(0) = h_2(2\pi) \), and moreover \( h_2([0, 2\pi]) \) does not intersect \( \varepsilon_0 \). Defining, now, the map \( h_2 : [0, 2\pi] \to \mathbb{C} \setminus \{0\} \),

\[
(4.23) \quad \tilde{h}_2(\theta) = e^{i\theta} u_2(x_{2,\theta}^{(1)}),
\]
we readily verify that \( \tilde{h}_2 \) is also smooth, satisfying \( \tilde{h}_2(0) = \tilde{h}_2(2\pi) \), and furthermore the curve \( \theta \mapsto \tilde{h}_2(\theta) \) has winding number (around the origin) equal to 1. Therefore, there exists some \( \vartheta_2 \in [0, 2\pi) \) and some \( \lambda > 0 \) such that

\[
(4.24) \quad h_2(\vartheta_2) = \lambda \tilde{h}_2(\vartheta_2).
\]

Since \( h_2(\vartheta_2), \tilde{h}_2(\vartheta_2) \in e^{i\vartheta_2} \mathcal{C}_2 \) and each ray from the origin intersects the ellipse \( e^{i\vartheta_2} \mathcal{C}_2 \) only once, we thus infer that \( \lambda = 1 \), i.e.

\[
(4.25) \quad h_2(\vartheta_2) = \tilde{h}_2(\vartheta_2).
\]

In view of (4.22), (4.23) and (4.25), we have:

\[
(4.26) \quad \varepsilon^{(1)}_{\vartheta_2}(x_{2,\vartheta_2}^{(1)}) = e^{i\vartheta_2} u_2(x_{2,\vartheta_2}^{(1)}).
\]

Since \( \varepsilon^{(1)}_{\vartheta_2} \) is tangent to \( e^{i\vartheta_2} \mathcal{C}_2 \) at \( P_{2,\vartheta_2}^{(1)} \), we also have for some \( \lambda_2 \in \mathbb{R} \setminus \{0\} \):

\[
(4.27) \quad \frac{d\varepsilon^{(1)}_{\vartheta_2}}{dx}(x_{2,\vartheta_2}^{(1)}) = \lambda_2 e^{i\vartheta_2} \frac{du_2}{dx}(x_{2,\vartheta_2}^{(1)}).
\]

Thus, the equalities (4.11), (4.27) and (4.20) (in view of the fact that \( \varepsilon \) is linear) readily implies that \( \lambda_2 = \lambda_1 = 1 \).

All in all, after setting (for notational simplicity) \( x_k = x_k^{(1)} \) for \( k = 1, 2 \) and \( \varepsilon = \varepsilon^{(1)}_{\vartheta_2} \), the equalities (4.19), (4.20), (4.26) and (4.27) yield

\[
(4.28) \quad \varepsilon(x_1) = u_1(x_1), \quad \varepsilon(x_2) = e^{i\vartheta_2} u_2(x_2)
\]
and

\[
(4.29) \quad \frac{d\varepsilon}{dx}(x_1) = \frac{du_1}{dx}(x_1), \quad \frac{d\varepsilon}{dx}(x_2) = e^{i\vartheta_2} \frac{du_2}{dx}(x_2)
\]
with \( \vartheta_2 \in [0, 2\pi) \) and

\[
(4.30) \quad x_0 \leq x_1 < x_2 < x_0 + \frac{2\pi}{\omega} + \frac{\pi C_0^2}{|L_0|}.
\]

If we define the function \( \bar{u} : [a, b] \to \mathbb{C} \) as the unique \( C^1 \) and piecewise \( C^2 \) solution of the initial value problem

\[
(4.31) \quad \begin{cases}
\frac{d^2 \bar{u}}{dx^2} + (\omega^2 - \bar{V}(x)) \bar{u} = 0 \\
\bar{u}(a) = u_1(a) \\
\frac{d\bar{u}}{dx}(a) = \frac{du_1}{dx}(a),
\end{cases}
\]
where \( \bar{V} \) is defined by (4.14) for the chosen values of \( x_1, x_2 \), then \( \bar{u} \) satisfies the following properties (readily inferred in view of (4.28), (4.29) and the fact that the function \( \varepsilon(x) \) satisfies \( \frac{d^2 \varepsilon}{dx^2} = 0 \)):
\[ \tilde{u}(x) = u_1(x) \text{ for } x \in [a, x_1], \]
\[ \tilde{u}(x) = \varepsilon(x) \text{ for } x \in [x_1, x_2], \]
\[ \tilde{u}(x) = e^{ib_2}u_2(x) \text{ for } x \in [x_2, b]. \]

Thus, the proof of the lemma is complete. \( \square \)

The following lemma will allow us to mollify the piecewise constant potential \( \tilde{V} \) of Lemma 4.2 as well as extend Lemma 4.2 in order to include smooth perturbations of equation 4.10.

**Lemma 4.3.** Let \( \omega, L_0 \in \mathbb{R}\setminus\{0\}, C_0 > 0, a < b, x_0 \in (a, b) \) and \( u_1, u_2 : [a, b] \to \mathbb{C} \) be as in the statement of Lemma 4.2 and let \( Z : \mathbb{R}^3 \times [a, b] \to \mathbb{C} \) be a smooth function such that \( Z(\omega, 0, v, x) = 0 \) for all \((v, x) \in \mathbb{R} \times [a, b]\). We will use the following notation for the absolute value of the Wronskian of \( u_1, u_2 \) (which is constant in \( x \in [a, b] \)):

\[
(4.32) \quad W[u_1, u_2] = \left| \frac{du_1}{dx} - u_1 \frac{du_2}{dx} \right|.
\]

Then, for any \( 0 < \varepsilon_1 \leq 1 \), there exists a \( \delta_0 \) sufficiently small in terms of \( \omega, C_0, L_0, \varepsilon_1, Z \) and \( W[u_1, u_2] \), where \( Z = \sup_{|\omega| \leq 1, |v| \leq 2\varepsilon_0} \sum_{j=0}^2 \int_a^b |\partial_{\omega_j} Z(\omega, \omega_1, v, x)| \, dx, \)

such that for any \( \omega \in (-\delta_0, \delta_0) \) and any initial data sets \((u_a^{(0)}, u_a^{(1)}), (u_b^{(0)}, u_b^{(1)}) \in \mathbb{C}^2 \) satisfying

\[
(4.33) \quad |u_a^{(0)} - u_1(a)| + |u_a^{(1)} - \frac{du_a}{dx}(a)| + |u_b^{(0)} - u_2(b)| + |u_b^{(1)} - \frac{du_b}{dx}(b)| < \delta_0,
\]

there exists a smooth function \( V : \mathbb{R} \to [0, (1 + \varepsilon_1)\omega^2] \) supported in \([ x_0 - \delta_0, x_0 + \delta_0 + \frac{2\pi}{\omega_0} + \frac{\varepsilon_1^2}{4\omega_0} ] \) and a smooth solution \( u \) to equation

\[
(4.34) \quad \frac{d^2 u}{dx^2} + (\omega^2 + Z(\omega, \omega_1, V(x), x) - V(x)) u = 0
\]

such that \( (u(a), \frac{du}{dx}(a)) = (u_a^{(0)}, u_a^{(1)}) \) and \( (u(b), \frac{du}{dx}(b)) = (\lambda u_b^{(0)}, \lambda u_b^{(1)}) \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \).

**Remark on the proof of Lemma 4.3.** Setting \( Z = 0 \), \( (u_a^{(0)}, u_a^{(1)}) = (u_1(a), \frac{du_1}{dx}(a)) \), \( (u_b^{(0)}, u_b^{(1)}) = (u_1(b), \frac{du_1}{dx}(b)) \) and \( \varepsilon_1 = 1 \) in the statement of Lemma 4.3 does not obscure any of the main difficulties associated to the proof of the lemma, but the notation is substantially simplified. Thus, it might be advisable for the reader to adopt this simplifying assumption on \( Z, (u_a^{(0)}, u_a^{(1)}), (u_b^{(0)}, u_b^{(1)}) \) and \( \varepsilon_1 \) at first reading.

Under this simplification, the proof of Lemma 4.3 proceeds by showing that, after suitably perturbing and then smoothing out the potential \( \tilde{V} : \mathbb{R} \to [0, \omega^2] \) of Lemma 4.2, obtaining a new potential \( V : \mathbb{R} \to [0, 2\omega^2] \), the two solutions of equation (4.35) coinciding with \( \tilde{u} \) (of Lemma 4.2) around \( x = a \) and \( x = b \), respectively, have vanishing Wronskian on \([a, b]\) (and hence differ only by a constant multiple).

The aforementioned perturbation of \( \tilde{V} \) is achieved through small variations \( \bar{x}_1 \) of the point \( x_1 \) in the definition of (4.14), as well as an addition of a term of the form \( \omega^2 \chi_{[c-x_1, c-x_1+\eta]} \) for some small \( \eta \in \mathbb{R} \) (where \( \chi_{[c,d]} \) equals the characteristic function of \([c, d]\) if \( c \leq d \), and minus the characteristic function of \([d, c]\) if \( c > d \)). Denoting with \( \tilde{V}_{\bar{x}_1, \eta} \) the perturbed potential, and with \( u(\bar{x}_1, \eta) \) the Wronskian of the two solutions of

\[
(4.36) \quad \frac{d^2 u}{dx^2} + (\omega^2 - \tilde{V}_{\bar{x}_1, \eta}) u = 0
\]

coinciding with \( \tilde{u} \) near \( x = a \) and \( x = b \), respectively, it is shown that the image of \( u \) (as a function of the parameters \( \bar{x}_1, \eta \)) contains an open neighborhood of \( 0 \in \mathbb{C}^2 \). This fact is then shown to imply that \( 0 \) belongs to the image of the associated Wronskian for equation (4.36) after a suitable mollification of the rough potential \( \tilde{V}_{\bar{x}_1, \eta} \), yielding the proof of Lemma 4.3 after a suitable choice of the parameters \( \bar{x}_1, \eta \). The rough dependence of the mollified potential on the mollifying parameter poses the main difficulty in the last step.

\footnote{That would not be always true had we chosen to perturb \( \tilde{V} \) only by varying the two points \( x_1, x_2 \) in (4.14).}
Proof. Let $\tilde{V}: \mathbb{R} \to [0, \omega^2]$, $\varnothing_2$ and $\tilde{u}: [a, b] \to \mathbb{C}$ be as in the statement of Lemma \ref{lemma}. Let $\eta_0 > 0$ be sufficiently small in terms of $\omega, x_1, a, b, C_0, L_0$ and $\mathcal{W}[u_1, u_2]$, and let us use the notation

\begin{equation}
\mathcal{D}_{\eta_0} = (x_1 - \eta_0, x_1 + \eta_0) \times (-\eta_0, \eta_0).
\end{equation}

For any $(\bar{x}_1, \eta) \in \mathcal{D}_{\eta_0}$, we define the function $\tilde{V}_{\bar{x}_1, \eta}: \mathbb{R} \to [0, 2\omega^2]$ as follows:

\begin{equation}
\tilde{V}_{\bar{x}_1, \eta}(x) = \begin{cases} 
0, & x \in (-\infty, \bar{x}_1) \cup (x_2, +\infty), \\
\omega^2, & x \in [\bar{x}_1, \frac{x_1 + x_2}{2} + \eta] \cup [\frac{x_1 + x_2}{2}, \bar{x}_2], \\
(1 - \varepsilon_1)\omega^2, & x \in (\frac{x_1 + x_2}{2} + \eta, \frac{x_1 + x_2}{2}), 
\end{cases}
\end{equation}

when $\eta \geq 0$, and

\begin{equation}
\tilde{V}_{\bar{x}_1, \eta}(x) = \begin{cases} 
0, & x \in (-\infty, \bar{x}_1) \cup (x_2, +\infty), \\
\omega^2, & x \in [\bar{x}_1, \frac{x_1 + x_2}{2} + \eta] \cup [\frac{x_1 + x_2}{2}, \bar{x}_2], \\
(1 + \varepsilon_1)\omega^2, & x \in (\frac{x_1 + x_2}{2} + \eta, \frac{x_1 + x_2}{2}), 
\end{cases}
\end{equation}

when $\eta \leq 0$. Notice that $\tilde{V}_{\bar{x}_1, 0} \equiv \tilde{V}$.

We will also define the functions $\tilde{u}_{\bar{x}_1, \eta}^{(a)}, \tilde{u}_{\bar{x}_1, \eta}^{(b)}: [a, b] \to \mathbb{C}$ as solutions to the following initial value problems:

\begin{equation}
\begin{cases} 
\frac{d^2\tilde{u}_{\bar{x}_1, \eta}^{(a)}}{dx^2} + (\omega^2 - \tilde{V}_{\bar{x}_1, \eta})\tilde{u}_{\bar{x}_1, \eta}^{(a)} = 0 \\
\tilde{u}_{\bar{x}_1, \eta}^{(a)}(a) = \tilde{u}(a) \\
\frac{d\tilde{u}_{\bar{x}_1, \eta}^{(a)}}{dx}(a) = \frac{\partial u}{\partial x}(a) 
\end{cases}
\end{equation}

and

\begin{equation}
\begin{cases} 
\frac{d^2\tilde{u}_{\bar{x}_1, \eta}^{(b)}}{dx^2} + (\omega^2 - \tilde{V}_{\bar{x}_1, \eta})\tilde{u}_{\bar{x}_1, \eta}^{(b)} = 0 \\
\tilde{u}_{\bar{x}_1, \eta}^{(b)}(b) = \tilde{u}(b) \\
\frac{d\tilde{u}_{\bar{x}_1, \eta}^{(b)}}{dx}(b) = \frac{\partial u}{\partial x}(b). 
\end{cases}
\end{equation}

Notice that

\begin{equation}
\tilde{u}_{\bar{x}_1, 0}^{(a)} = \tilde{u}_{\bar{x}_1, 0}^{(b)} = \tilde{u}.
\end{equation}

Furthermore, the map $(\bar{x}_1, \eta) \to (\tilde{u}_{\bar{x}_1, \eta}^{(a)}, \tilde{u}_{\bar{x}_1, \eta}^{(b)})$ on $\mathcal{D}_{\eta_0}$ is $C^0$ in the $(C^1([a, b]) \cap C^2([a, b \setminus I_{\eta_0}]))^2$ topology and $C^1$ in the $(C^1([a, b \setminus I_{\eta_0}])^2$ topology, where

\begin{equation}
I_{\eta_0} = (x_1 - \eta_0, x_1 + \eta_0) \cup (\frac{x_1 + x_2}{2} - \eta_0, \frac{x_1 + x_2}{2} + \eta_0) \cup (x_2 - \eta_0, x_2 + \eta_0) \subset [a, b].
\end{equation}

Let us define the function $w: \mathcal{D}_{\eta_0} \to \mathbb{C}$ as the Wronskian of the pair $\tilde{u}_{\bar{x}_1, \eta}^{(a)}, \tilde{u}_{\bar{x}_1, \eta}^{(b)}$ for any $x_3 \in [a, b]$:

\begin{equation}
w(\bar{x}_1, \eta) = \left. \left( \frac{d\tilde{u}_{\bar{x}_1, \eta}^{(a)}}{dx} \tilde{u}_{\bar{x}_1, \eta}^{(b)} - \tilde{u}_{\bar{x}_1, \eta}^{(a)} \frac{d\tilde{u}_{\bar{x}_1, \eta}^{(b)}}{dx} \right) \right|_{x=x_3}.
\end{equation}

Notice that the value of the right hand side of (4.44) is independent of the choice of $x_3 \in [a, b]$. In view of the fact that the map $(\bar{x}_1, \eta) \to (\tilde{u}_{\bar{x}_1, \eta}^{(a)}, \tilde{u}_{\bar{x}_1, \eta}^{(b)})$ is $C^1$ in the $(C^1([a, b \setminus I_{\eta_0}])^2$ topology, we deduce that $w \in C^1(\mathcal{D}_{\eta_0})$. Furthermore,

\begin{equation}
w(\bar{x}_1, 0) = 0
\end{equation}
in view of (4.42).

For any \((\bar{x}_1, \eta) \in D_{\nu_0}\), with \(\eta > 0\), the functions \(\tilde{u}_{\bar{x}_1, \eta}^{(a)}\) are of the form (with \(\gamma \in \{a, b\}) ::

\[
\tilde{u}_{\bar{x}_1, \eta}^{(\gamma)}(x) = \begin{cases} 
A^{(1, \gamma)}(\bar{x}_1, \eta) e^{i \omega x} + B^{(1, \gamma)}(\bar{x}_1, \eta) e^{-i \omega x}, & x \in [a, \bar{x}_1] \\
A^{(2, \gamma)}(\bar{x}_1, \eta) x + B^{(2, \gamma)}(\bar{x}_1, \eta), & x \in [\bar{x}_1, \frac{x_1 + x_2}{2}] \\
A^{(3, \gamma)}(\bar{x}_1, \eta) e^{i \omega x} + B^{(3, \gamma)}(\bar{x}_1, \eta) e^{-i \omega x}, & x \in [\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} + \eta] \\
A^{(4, \gamma)}(\bar{x}_1, \eta) x + B^{(4, \gamma)}(\bar{x}_1, \eta), & x \in [\frac{x_1 + x_2}{2} + \eta, x_2] \\
A^{(5, \gamma)}(\bar{x}_1, \eta) e^{i \omega x} + B^{(5, \gamma)}(\bar{x}_1, \eta) e^{-i \omega x}, & x \in [x_2, b],
\end{cases}
\]

(4.46)

where the constants \(A^{(j, \gamma)} \in \mathbb{C}, j \in \{1, \ldots, 5\}, \gamma \in \{a, b\}\) are uniquely determined by the initial conditions of (4.40), (4.41) and the requirement that the functions \(\tilde{u}_{\bar{x}_1, \eta}^{(\gamma)}\) are \(C^1\). Thus, using the identity (4.42), we can readily calculate that

\[
\partial_{x_1} u_1^{\bar{x}_1, \eta}(x_1, 0) = -e^{2}(\tilde{u}(x_1)) \eta^{2},
\]

(4.47)

\[
\partial_{x_1} u_1^{\bar{x}_1, \eta}(x_1, 0) = -\eta^{2}(\tilde{u}(x_1) + \tilde{u}(x_1 + x_2)) / 2.
\]

(4.48)

Since \(\tilde{u}(x_1) = u_1(x_1)\) and \(\tilde{u}(x_2) = e^{i \theta_2} u_2(x_2)\) differ by a polar angle of magnitude less than \(\pi\), and \(\tilde{u}(\frac{x_1 + x_2}{2}) = u_1(x_1) + e^{i \theta_2} u_2(x_2)\) is the middle-point of the line segment defined by \(\tilde{u}(x_1)\) and \(\tilde{u}(x_2)\) (because \(\tilde{u}\) is linear on \([x_1, x_2]\)),

we deduce that \(\tilde{u}(x_1)\) and \(\tilde{u}(\frac{x_1 + x_2}{2})\) differ by a polar angle smaller than \(\frac{\pi}{2}\). Hence, the linear span of \((\tilde{u}(x_1))^2\) and \((\tilde{u}(\frac{x_1 + x_2}{2}))^2\) (viewed as vectors in \(\mathbb{C} \cong \mathbb{R}^2\)) is the whole plane, and thus, in view of (4.47)–(4.48), we deduce that the derivative map \(Dw: TD_{\nu_0} \to TC\) is invertible at \((\bar{x}_1, \eta) = (x_1, 0)\), satisfying in particular the lower bound:

\[
|Dw|(\bar{x}_1, \eta) = (x_1, 0) \geq c\eta \omega^2 |Im(u_1(x_1) \cdot e^{-i \theta_2} u_2(x_2))| > 0.
\]

(4.49)

Notice also that, in view of the fact that \(u_1, u_2: [a, b] \to \mathbb{C}\) are expressed as (4.15) satisfying (4.16), and \(x_1, x_2\) define a common tangent of \(u_1\) and \(e^{i \theta_2} u_2\), from (4.49) we can estimate (for an absolute constant \(c > 0\))::

\[
|Dw|(\bar{x}_1, \eta) = (x_1, 0) \geq c \epsilon_1 \omega \mathcal{W}[u_1, u_2].
\]

(4.50)

Let \(K: \mathbb{R} \to [0, +\infty)\) be a smooth function supported in \([-1, 1]\) such that \(\int_{\mathbb{R}} K(x) dx = 1\). For any \((\bar{x}_1, \eta) \in D_{\nu_0}\) and \(\delta > 0\), let us define the function \(V^{(\delta)}_{\bar{x}_1, \eta}: \mathbb{R} \to [0, 1 + \epsilon_1 \omega^2]\) as the convolution:

\[
V^{(\delta)}_{\bar{x}_1, \eta}(x) = \int_{-\infty}^{+\infty} \hat{V}_{\bar{x}_1, \eta}(y - x) \cdot \delta^{-1} K(\delta^{-1} y) dy.
\]

(4.51)

Notice that for any \(\delta > 0\), \(V^{(\delta)}_{\bar{x}_1, \eta}\) is a smooth function supported in \([x_1 - \delta, x_2 + \delta]\) and, as \(\delta \to 0\), the function \(V^{(\delta)}_{\bar{x}_1, \eta}\) converges to \(\hat{V}_{\bar{x}_1, \eta}\) pointwise everywhere except at the points \(\hat{\mathcal{A}} = \{\bar{x}_1, x_2, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} + \eta\}\).

We will set \(V^{(0)}_{\bar{x}_1, \eta} = \hat{V}_{\bar{x}_1, \eta}\). Notice that, since \(\hat{V}_{\bar{x}_1, \eta}\) is piecewise constant, has bounded support and is uniformly bounded by \(2 \omega^2\), \(|V^{(\delta)}_{\bar{x}_1, \eta} - V^{(0)}_{\bar{x}_1, \eta}|\) is bounded by \(4 \epsilon_2^2\) and supported in the set \(x: dist(x, \hat{\mathcal{A}}) \leq \delta\) for any \(0 \leq \delta \leq 1\). In particular, for any \(0 \leq \delta_2, \delta_3 \leq 1\), we can bound (provided \(\epsilon_0\) is smaller than some absolute constant \(\epsilon_0\)):

\[
\sup_{(\bar{x}_1, \eta) \in D_{\nu_0}} \int_{-\infty}^{+\infty} |V^{(\delta_2)}_{\bar{x}_1, \eta}(x) - V^{(\delta_3)}_{\bar{x}_1, \eta}(x)| dx \leq C \omega^2 |\delta_3 - \delta_2|.
\]

(4.52)

Let \(\delta_0, \delta_1 > 0\) be small constants (their magnitude will be specified in more detail later). Let us define the set \(B^{(a, b)}_{\nu_0, \nu_2} \subset C^4\) as the set of all \((u^{(a)}_0, u^{(1)}_0, u^{(0)}_0, u^{(1)}_0) \in \mathbb{C}^4\) satisfying (4.34). For any \((\delta, \omega) \in (0, \delta_1) \times (0, \delta_0), (u^{(0)}_0, u^{(1)}_0, u^{(0)}_0, u^{(1)}_0) \in B^{(a, b)}_{\nu_0, \nu_2}\) and \((\bar{x}_1, \eta) \in D_{\nu_0}\), we define the functions \(u^{(a)}(a, b) : [a, b] \to \mathbb{C}\) as solutions to the following initial value problems:

\[
\begin{cases}
\frac{d^2 u^{(a)}}{dx^2} + \omega^2 + Z(\omega, \omega, V^{(\delta)}_{\bar{x}_1, \eta}(x), x) - V^{(\delta)}_{\bar{x}_1, \eta}(x)) u^{(a)} = 0 \\
u^{(a)}(a) = u^{(a)}(0) \\
\frac{d u^{(a)}}{dx} (a) = u^{(a)}(1)
\end{cases}
\]

(4.53)
and

\[
\begin{align*}
\left\{ \frac{dxu^{(b)}}{dx} + (\omega^2 + Z(\omega, \omega, V(\beta)(x), x) - V(\beta)(x))u^{(b)} = 0 \\
u^{(b)}(b) = u_b^{(0)} \\
\frac{du^{(a)}}{dx}(b) = u_b^{(1)}
\right. 
\end{align*}
\] (4.54)

From now on, we will use the shorthand notation \( \mathcal{U} = (u^{(a)}, u^{(b)}) \) and \( \mathcal{U}_0 = (u_1(a), \frac{du}{dx}(a), u_2(b), \frac{du}{dx}(b)) \).

We will also set

\[
\mathcal{U}(\delta, \omega, \mathcal{U}, \bar{x}, \eta) = (u^{(a)}, u^{(b)}).
\] (4.55)

Notice that, in view of the fact that \( Z(\omega, 0, \cdot) = 0 \), we have \( \mathcal{U}(0, 0, \mathcal{U}_0, \bar{x}, \eta) = (u^{(a)}, u^{(b)}) \). Furthermore, by differentiating (4.53) and (4.54) with respect to \( \omega, \mathcal{U}, \bar{x}, \eta \), we readily infer that the map \( \mathcal{U} : [0, \delta_1] \times (-\delta_0, \delta_0) \times B^{(u_1, u_2)} \times D_{\mathcal{U}_0} \rightarrow \left( C^1([a, b]) \cap C^2([a, b]) \right)^2 \) (defined by (4.55)) has the following regularity properties:

1. \( \mathcal{U} \in C^0 \left( [0, \delta_1] \times (-\delta_0, \delta_0) \times B^{(u_1, u_2)} \times D_{\mathcal{U}_0} \rightarrow \left( C^1([a, b]) \cap C^2([a, b]) \right)^2 \right) \).

2. \( \mathcal{U} \in C^0 \left( [0, \delta_1] \rightarrow C^2 ( -\delta_0, \delta_0) \times B^{(u_1, u_2)} \rightarrow C^1 ( D_{\mathcal{U}_0} \rightarrow \left( C^2([a, b]) \right)^2 \right) \).

3. \( \mathcal{U} \in C^0 \left( [0, \delta_1] \rightarrow C^2 ( -\delta_0, \delta_0) \times B^{(u_1, u_2)} \rightarrow C^0 ( D_{\mathcal{U}_0} \rightarrow \left( C^2([a, b]) \right)^2 \right) \).

In the above, \( C^k(A \rightarrow \mathcal{Y}) \) (or \( C^{k,a}(A \rightarrow \mathcal{Y}) \)) denotes the space of \( C^k \) (or \( C^{k,a} \), respectively) functions defined on the manifold \( A \) and taking values in the Banach space \( \mathcal{Y} \). Note that \( \delta \) is a consequence of (4.52).

Thus, extending (4.44) on the whole of \( \mathcal{D}_{\delta, \delta_0, \mathcal{U}_0} = [0, \delta_1] \times (-\delta_0, \delta_0) \times B^{(u_1, u_2)} \times D_{\mathcal{U}_0} \) as the Wronskian of the associated pair \( u^{(a)}, u^{(b)} \):

\[
w(\tilde{\delta}, \omega, \mathcal{U}, \bar{x}, \eta) = \left( \frac{du^{(a)}}{dx} - u^{(a)} \frac{du^{(b)}}{dx} \right)_{x=\bar{x}},
\] (4.56)

we infer that

\[
w \in C^0 \left( [0, \delta_1] \rightarrow C^2 ( -\delta_0, \delta_0) \times B^{(u_1, u_2)} \rightarrow C^1 ( D_{\mathcal{U}_0} \rightarrow \left( C^2([a, b]) \right)^2 \right) \right) \cap C^0 \left( [0, \delta_1] \rightarrow C^2 ( -\delta_0, \delta_0) \times B^{(u_1, u_2)} \rightarrow C^0 ( D_{\mathcal{U}_0} \rightarrow \left( C^2([a, b]) \right)^2 \right) \right),
\] (4.57)

satisfying the following bounds in view of (4.53), (4.54) and (4.52) (provided \( \delta_0, \delta_1, \eta_0 \) are smaller than some absolute constant \( c_0 > 0 \):

\[
\sum_{j_1+j_2=0}^2 \sum_{j_1+j_2=0}^1 |\partial_{\omega}^{j_1} D_{\mathcal{U}}^{j_2} \partial_{\tilde{x}}^{j_3} \partial_{w}^{i_4} w| \leq C(\omega, \mathcal{Z}, C_0, L_0)
\] (4.58)

and

\[
\sup_{\delta_2, \delta_3 \in (0, \delta_1)} \sum_{j_1+j_2=0}^2 \frac{|\partial_{\omega}^{j_1} D_{\mathcal{U}}^{j_2} w(\delta_2, \cdot) - \partial_{\omega}^{j_1} D_{\mathcal{U}}^{j_2} w(\delta_3, \cdot)|}{|\delta_2 - \delta_3|} \leq C(\omega, \mathcal{Z}, C_0, L_0),
\] (4.59)

where \( C(\omega, \mathcal{Z}, C_0, L_0) > 0 \) depends only on \( \omega, \mathcal{Z}, C_0, L_0 \).

In view of (4.49), (4.58) and the fact that \( w(0, 0, \mathcal{U}_0, x_1, 0) = 0 \), we infer that, provided \( \eta_0, \delta_0 \) are sufficiently small in terms of \( \omega, \mathcal{Z}, C_0, L_0, \varepsilon_1 \) and \( \mathcal{W}[u_1, u_2] \), then for any \( (\omega, \mathcal{U}) \in (-\delta_0, \delta_0) \times B^{(u_1, u_2)} \) the map

\[
w(0, \omega, \mathcal{U}, \cdot) : D_{\mathcal{U}_0} \rightarrow \mathbb{C}
\] (4.60)
is a diffeomorphism onto an open neighborhood of \( 0 \in \mathbb{C} \), with \( w(\{0\} \times \{\omega_1\} \times \{\mathcal{W}\} \times \mathcal{D}_{\delta_1/2}) \) containing a disk of radius at least \( c = c(\omega, Z, C_0, L_0, W[u_1, u_2]) \). Therefore, in view of (4.59), provided \( \delta_1 \) is sufficiently small in terms of \( \omega, Z, C_0, L_0, W[u_1, u_2], \varepsilon_1, \eta_0 \) and \( \delta_0 \), we have for all \( (\delta, \omega_1, \mathcal{W}) \in [0, \delta_1) \times (-\delta_0, \delta_0) \times \mathcal{B}_{\delta_0}^{(u_1, u_2)} \):

\[
(4.61) \quad 0 \notin w(\{\delta\} \times \{\omega_1\} \times \{\mathcal{W}\} \times \mathcal{D}_{\delta_0} \setminus \mathcal{D}_{\delta_0/2}).
\]

Thus, (4.59), (4.61) and Lemma C.1 of the Appendix imply that:

\[
(4.62) \quad 0 \notin w(\{\delta\} \times \{\omega_1\} \times \{\mathcal{W}\} \times \mathcal{D}_{\delta_0}) \quad \text{for all} \quad (\delta, \omega_1, \mathcal{W}) \in [0, \delta_1) \times (-\delta_0, \delta_0) \times \mathcal{B}_{\delta_0}^{(u_1, u_2)}.
\]

The relation (4.62) implies (in view of the definition (4.56) of the Wronskian) that for any \( (\delta, \omega_1, \mathcal{W}) \in [0, \delta_1) \times (-\delta_0, \delta_0) \times \mathcal{B}_{\delta_0}^{(u_1, u_2)} \), there exist some \( (\hat{x}_1, \hat{\eta}) \in \mathcal{D}_{\delta_0} \) and a \( \lambda \in \mathbb{C}\{0\} \) so that the pair \( (u^{(a)}, u^{(b)}) \) associated to \( (\delta, \omega_1, \mathcal{W}, \hat{x}_1, \hat{\eta}) \) satisfies on \([a, b] \):

\[
(4.63) \quad u^{(a)} = \lambda u^{(b)}.
\]

We finally construct the required function \( u \) solving equation (4.35) as follows: Fix a \( 0 < \hat{\delta} < \delta_1 \), and define \( (\hat{x}_1, \hat{\eta}) \in \mathcal{D}_{\delta_0} \) and \( \lambda \in \mathbb{C}\{0\} \) in terms of \((\delta, \omega_1, \mathcal{W})\) as above, so that (4.63) holds. Then, setting \( V = V^{(\hat{\delta})}_{\hat{x}_1, \hat{\eta}} \) and \( u = u^{(a)} \), we deduce that \( u \) satisfies equation (4.35), and moreover \( (u(a), \frac{du}{dx}(a)) = (u^{(0)}, u^{(1)}(a)) \) and \( (u(b), \frac{du}{dx}(b)) = (\lambda u^{(0)}, \lambda u^{(1)}) \). Thus, the proof of the lemma is complete.

In Section 4 we will also need the following refinement of Lemma 4.3 providing an estimate on the change of the potential \( V \) of Lemma 4.3 under smooth variations of equation (4.35) and the associated initial data:

**Lemma 4.4.** Let \( \omega, L_0 \in \mathbb{R}\{0\}, C_0 > 0, a < b, x_0, u_1, u_2 : [a, b] \to \mathbb{C} \) be as in the statement of Lemma 4.3. Let \( Z^{(s)} : \mathbb{R}^3 \times [a, b] \to \mathbb{C} \) be a family of functions depending smoothly on \( s \in [0, 1] \), satisfying for all \( s \in [0, 1] \)

\[
(4.35) \quad Z^{(s)}(\omega, x_0, 0, 0) = 0.
\]

Let \( Z^{(s)} : \mathbb{R}^3 \times [a, b] \to \mathbb{C} \) be a family of functions depending smoothly on \( s \in [0, 1] \), satisfying for all \( s \in [0, 1] \)

\[
(4.35) \quad Z^{(s)}(\omega, x_0, 0, 0) = 0.
\]

Let \( \delta_0 > 0 \) be sufficiently small in terms of \( \omega, C_0, L_0, \max_{s \in [0,1]} Z^{(s)} \) and \( W[u_1, u_2] \), where

\[
Z^{(s)} = \sup_{|\omega| \leq 2\omega_0} \sup_{|\omega| \leq 1} \sum_{j=0}^2 \int_a^b \left| \frac{1}{\delta_0} \frac{\partial}{\partial \omega_j} Z^{(s)}(\omega, \omega_1, v, x) \right| dx.
\]

Then, for any \( s \in [0, 1] \), any \( \omega_1 \in (-\delta_0, \delta_0) \) and any family of initial data sets \( \mathcal{W}^{(s)} = (u^{(0)}, u^{(1)}, u^{(0)}, u^{(1)}) \in C^4 \)

\[
(4.65) \quad \left| u^{(0)}_a(a) - u_1(a) \right| + \left| u^{(1)}_a(a) - \frac{du}{dx}(a) \right| + \left| u^{(0)}_b(b) - u_2(b) \right| + \left| u^{(1)}_b(b) - \frac{du}{dx}(b) \right| < \delta_0,
\]

there exists a family of smooth functions \( V^{(s)} : \mathbb{R} \to [0, 2\omega_0] \), \( s \in [0, 1] \), supported in \([x_0 - \delta_0, x_0 + \delta_0 + \frac{2\pi + \pi C^2}{\omega_0}] \), satisfying for all \( 0 \leq s_1 \leq s_2 \leq 1 \)

\[
(4.66) \quad \int_a^b \left| V^{(s_1)}(x) - V^{(s_2)}(x) \right| dx \leq C(\omega, C_0, L_0, \max_{s \in [0,1]} Z^{(s)}), W[u_1, u_2] \int_{s_1}^{s_2} \left( \frac{d}{ds} \mathcal{W}^{(s)} \right) + \sup_{|\omega|\leq 1} \int_a^b \left| \frac{\partial}{\partial \omega_j} Z^{(s)}(\omega, \omega_1, v, x) \right| dx \right) ds,
\]

and a smooth family of solutions \( u^{(s)} \) to equation

\[
(4.67) \quad \frac{d^2 u^{(s)}}{dx^2} + (\omega^2 + Z^{(s)}(\omega, \omega_1, V^{(s)}(x), x) - V^{(s)}(x)) u^{(s)} = 0
\]

such that \( (u^{(s)}(a), \frac{du^{(s)}}{dx}(a)) = (u^{(0)}, u^{(1)}) \) and \( (u^{(s)}(b), \frac{du^{(s)}}{dx}(b)) = (\lambda^{(s)} u^{(0)}, \lambda^{(s)} u^{(1)}) \) for some \( \lambda^{(s)} \in \mathbb{C}\{0\} \).

**Proof.** Let \( \delta_1, \eta_0 \) be sufficiently small in terms of \( \omega, C_0, L_0, \max_{s \in [0,1]} Z^{(s)} \) and \( W[u_1, u_2] \). We will use the same notations and conventions as in the proof of Lemma 4.3.
Let $V^{(0)} : \mathbb{R} \to [0, 2\omega^2]$ be the potential function associated to the pair of functions $(u_1, u_2)$ and the initial data tetrad $(u_a^{(0)}, u_a^{(1, 0)}, u_b^{(0)}, u_b^{(1, 0)})$ as in Lemma 4.3, yielding a smooth solution $u^{(0)}$ to (4.67) for $s = 0$. In particular, following the proof of Lemma 4.3, $V^{(0)}$ is of the form $V^{(0)}_t(\alpha(0))$ (defined according to (4.51)) for some $\delta \in (0, \delta_1)$ and some $(\bar{x}_1(0), \eta(0)) \in D_{\bar{u}_0}$ such that

$$w(\delta, \omega, \bar{\mathcal{Z}}^{(0)}, \bar{x}_1(0), \eta(0)) = 0,$$

where $w$ is defined as (4.56), with $(u^{(a)}, u^{(b)})$ solving (4.53) and (4.54) with $\mathcal{Z}^{(0)}$ replacing $\mathcal{Z}$.

In view of the estimates (4.49), (4.58) and (4.59) (as well as (4.65)), provided $\delta_1, \delta_0, \eta_0$ are sufficiently small in terms of $\omega, C_0, L_0$, $\max_{s \in [0, 1]} \mathcal{Z}(s)$ and $\mathcal{W}[u_1, u_2]$, there exists a pair $(\bar{x}_1(s), \eta(s)) \in D_{\bar{u}_0}$ for any $s \in [0, 1]$ such that

$$w(\delta, \omega, \mathcal{Z}(s), \bar{x}_1(s), \eta(s)) = 0.$$

In particular, an application of the implicit function theorem (in view again of (4.49), (4.58) and (4.59)) implies that

$$\left| \frac{d}{ds} \mathcal{Z}(s) \right| + \left| \frac{d}{ds} \eta(s) \right| \leq C(\omega, C_0, L_0, \max_{s \in [0, 1]} \mathcal{Z}(s), \mathcal{W}[u_1, u_2]) \cdot \left( \left| \frac{d}{ds} \mathcal{Z}(s) \right| + \sup_{|v| \leq 2a^2, |x| \leq 1} \sum_{j=0}^{2} \int_{0}^{1} |\partial_s \partial_{\omega}^j \mathcal{Z}(s)(\omega, \omega, v, x)| dx \right).$$

Setting $V(s) = V^{(5)}_{\bar{x}_1(s)}(\eta(s))$ for $s \in [0, 1]$, the existence of a smooth solution $u^{(s)}$ to (4.67) satisfying the assumptions of the lemma follows, in view of (4.69), as in the end of the proof of Lemma 4.3. Furthermore, in view of (4.38), (4.39), (4.51) and (4.70), inequality (4.66) follows readily.

### 4.3 Proof of Proposition 4.1

Let $u_{inf}$ be the unique (up to multiplication by a complex constant) solution of the ordinary differential equation

$$u'' + (\omega_R^2 - V_{\omega_R ml})u = 0,$$

satisfying the outgoing condition

$$u'_{inf} - i\omega_R u_{inf} \to 0$$
as $r_* \to +\infty$, and let $u_{hor}$ be the solution of

$$u_{hor} + i(\omega_R - \frac{am}{2Mr_*})u_{hor} \to 0$$
as $r_* \to -\infty$. Notice that in view of the form of equation (4.71) and the conditions (4.72) and (4.73), the following limits are well defined in $\mathbb{C}$:

$$\lim_{r_* \to +\infty} (e^{-i\omega_R r_*} u_{inf}(r_*)) \equiv u_{inf}(+\infty)$$

$$\lim_{r_* \to -\infty} (e^{i(\omega_R - \frac{am}{2Mr_*}) r_*} u_{hor}(r_*)) \equiv u_{hor}(-\infty)$$

for some $u_{inf}(+\infty), u_{hor}(-\infty) \in \mathbb{C} \backslash \{0\}$.

The quantity $\text{Im}(u' \bar{u})$ is constant as a function of $r_*$ for both $u_{inf}$ and $u_{hor}$, since they both satisfy (4.71) and $\omega_R, V_{\omega_R ml} \in \mathbb{R}$ (see the remark below Proposition 4.1). Thus, from (4.72), (4.73), (4.74) and (4.75) we deduce that

$$\text{Im}(u'_{inf} \bar{u}_{inf}) = |u_{inf}(+\infty)|^2 \omega_R$$

and

$$\text{Im}(u'_{hor} \bar{u}_{hor}) = |u_{hor}(-\infty)|^2 \left( \frac{am}{2Mr_*} - \omega_R \right).$$
Since \((\omega_R, m, l)\) lies in the superradiant regime \([3.20]\), the quantities \([4.76]\) and \([4.77]\) are of the same sign. Thus, there exists a \(a_3 \in \mathbb{R}\setminus\{0\}\), such that by rescaling \(u_{inf} \rightarrow a_3 u_{inf}\) we have
\[
(4.78) \quad \text{Im}(u'_{inf} \cdot \bar{u}_{inf}) = \text{Im}(u'_{hor} \cdot \bar{u}_{hor}).
\]
From now on, we will assume that \(u_{inf}\) has been rescaled like this.

For any \(\omega_t \geq 0\), we will also define the functions \(u_{inf}^{(\omega_t)}, u_{hor}^{(\omega_t)} : \mathbb{R} \rightarrow \mathbb{C}\) as the unique solutions of equation
\[
(4.79) \quad u'' + \left((\omega_R + i\omega_t)^2 - V(\omega_R + i\omega_t)ml\right)u = 0
\]
satisfying
\[
(4.80) \quad \lim_{r_* \rightarrow +\infty} (e^{-i(\omega_R + i\omega_t)r_*} u^{(\omega_t)}(r_*) \right) = u_{inf}(+\infty)
\]
and
\[
(4.81) \quad \lim_{r_* \rightarrow -\infty} (e^{i(\omega_R + i\omega_t)r_*} u^{(\omega_t)}(r_*) \right) = u_{hor}(-\infty)
\]
respectively.

We will assume that \(C_{\omega R m l}\) has been chosen large in terms of \(\omega_R, m, l, M, a\), so that in the region \(r \geq C_{\omega R m l}\) we can bound
\[
(4.82) \quad 0 \leq V_{\omega_R m l} \leq \frac{\omega_R^2}{2}.
\]
This is possible since \(V_{\omega_R m l} \rightarrow 0\) as \(r_* \rightarrow +\infty\) and \(V_{\omega_R m l} \geq 0\) for \(r\) sufficiently large in terms of \(\omega_R, m, l, M, a\). For a \(C(0)_{\omega R m l} > 0\) sufficiently large in terms of \(\omega_R, m, l, M, a\), we will fix \(\chi : \mathbb{R} \rightarrow [0, 1]\) to be a smooth cut-off function such that \(\chi \equiv 0\) on \((-\infty, r_* (r_0)] \cup [r_* (r_0) + C(0)_{\omega_R m l}, +\infty)\) and \(\chi \equiv 1\) on \([r_* (r_0) + 1, r_* (r_0) + C(0)_{\omega_R m l} - 1]\).

The proof of Proposition \([4.1]\) will follow by constructing, for any \(0 \leq \omega_t < \infty\) (sufficiently small in terms of \(\omega_R, m, l, M, a, \varepsilon_1\) and \(G\)), a smooth function \(V_f : \mathbb{R} \rightarrow [0, (\frac{1}{\mathcal{T}} + \varepsilon_1)\omega_R]\) supported on \([r_* (r_0) + 2, r_* (r_0) + C(0)_{\omega_R m l} - 2]\) such that, using the ansatz
\[
(4.83) \quad V = -\chi(V_{\omega_R m l}(r_*) - \frac{3\omega_R^2}{4}) + V_f(r_*),
\]
the equation
\[
(4.84) \quad u'' + \left((\omega_R + i\omega_t)^2 - V_{\omega_R m l} + G(\omega_R, \omega_t, V(r_*), r_*) - V(r_*)\right)u = 0
\]
admits a solution \(u\) which satisfies \(u \equiv u_{inf}^{(\omega_t)}\) for \(r_* \leq r_* (r_0)\) and \(u \equiv \lambda u_{inf}^{(\omega_t)}\) for \(r_* \geq r_* (r_0) + C(0)_{\omega_R m l}\) and some \(\lambda \in \mathbb{C}\setminus\{0\}\).

To this end, we will make use of Lemma \([4.3]\).

The ordinary differential equation
\[
(4.85) \quad u'' + \left(\omega_R^2 - V_{\omega_R m l} + \chi(V_{\omega_R m l} - \frac{3\omega_R^2}{4})\right)u = 0
\]
admits two unique solutions \(u_1, u_2 : \mathbb{R} \rightarrow \mathbb{C}\) satisfying \(u_1 \equiv u_{hor}\) for \(r_* \leq r_* (r_0)\) and \(u_2 \equiv u_{inf}\) for \(r_* \geq r_* (r_0) + C(0)_{\omega_R m l}\).

For \(r_* \in [r_* (r_0) + 1, r_* (r_0) + C(0)_{\omega_R m l} - 1]\), \(u_1\) and \(u_2\) satisfy
\[
(4.86) \quad u'' + \left(\frac{\omega_R^2}{2}\right) u = 0.
\]
By perturbing \(\chi\) on the interval \([r_* (r_0) + C(0)_{\omega_R m l} - 1 \leq r_* \leq r_* (r_0) + C(0)_{\omega_R m l}\), if necessary, we will assume without loss of generality that \(\frac{u_1}{u_2}\) is not constant on \(\mathbb{R}\) (and thus also on any open interval of \(\mathbb{R}\)), so that \(\mathcal{W}[u_1, u_2]\) satisfies the lower bound:
\[
(4.86) \quad \mathcal{W}[u_1, u_2] = |u'_1 u_2 - u'_2 u_1| \geq c_{\omega_R m l r_0} > 0.
\]
\(^8\)which is allowed since \(u_{inf}\) was only defined up to multiplication by a non zero complex constant
Notice that, by comparing (4.84) to equation (4.87)
\[ u'' + \left( \omega_R^2 - \chi_R(r_*)V_{\omega_R ml}(r_*) \right) u = 0, \]
for some fixed \( R_* \in \mathbb{R} \), where \( \chi_R : \mathbb{R} \to +\infty \) is a step function satisfying \( \chi_R \equiv 0 \) for \( r_* \leq R_* \) and \( \chi_R \equiv 1 \) for \( r_* > R_* \), we can bound, in view of (4.83) and the boundary conditions (4.74) and (4.75) (see also [7]):
\[
\sup_{r_* \in \mathbb{R}} |u_1(r_*)| \leq C(\omega_R, m)|u_{h; 0}(\infty)| + C \int_{R_*}^{\infty} |V_{\omega_R ml} + \chi(\frac{\omega_R^2}{4} - V_{\omega_R ml})| dr_* + \\
+ C \int_{-\infty}^{R_*} |V_{\omega_R ml} - V_{\omega_R ml}(r_*) + \chi(\frac{\omega_R^2}{4} - V_{\omega_R ml})| dr_*
\]
and
\[
\sup_{r_* \in \mathbb{R}} |u_2(r_*)| \leq C(\omega_R, m)|u_{inf}(\infty)| + C \int_{R_*}^{\infty} |V_{\omega_R ml} + \chi(\frac{\omega_R^2}{4} - V_{\omega_R ml})| dr_* + \\
+ C \int_{-\infty}^{R_*} |V_{\omega_R ml} - V_{\omega_R ml}(r_*) + \chi(\frac{\omega_R^2}{4} - V_{\omega_R ml})| dr_*
\]
Thus, choosing the constant \( C_{\omega_R ml}^{(0)} \) sufficiently large in terms of \( \omega_R, m, l, M, a \) we can bound (recall (4.78)):
\[
\frac{\sup_{R} \left( |u_1| + |u_2| \right)^2}{I_m(u_1', u_1)} \ll C_{\omega_R ml}^{(0)}
\]
(notice that the left hand side of (4.90) is invariant under multiplication of \( u_1, u_2 \) with the same non-zero constant).

We will also define the functions \( u_1^{(\omega_1)}, u_2^{(\omega_2)} : \mathbb{R} \to \mathbb{C} \) as the unique solutions of equation
\[
u'' + \left( (\omega_R + i\omega_I)^2 + G(\omega_R, \omega_I, -\chi(V_{\omega_R ml}(r_*) - \frac{3\omega_R^2}{4}), r_*) - V(\omega_R ml + \chi(\frac{3\omega_R^2}{4} - V_{\omega_R ml})) \right) u = 0
\]
satisfying \( u_1^{(\omega_1)} \equiv u_{h; 0} \) for \( r_* \leq r_0(r_0) \) and \( u_2^{(\omega_2)} \equiv u_{inf} \) for \( r_* \geq r_0(r_0) + C_{\omega_R ml}^{(0)} \). Notice that \( u_1^{(0)} = u_1 \) and \( u_2^{(0)} = u_2 \).

Setting
\[
\mathcal{Z}(\omega_R, \omega_I, v, r_*) \equiv 2i\omega_R\omega_I - \omega_R^2 - V(\omega_R ml + \chi(\frac{3\omega_R^2}{4} - V_{\omega_R ml})) + V_{\omega_R ml}(r_*) + G(\omega_R, \omega_I, v - \chi(V_{\omega_R ml}(r_*) - \frac{3\omega_R^2}{4}), r_*),
\]
equation (4.83) restricted on \([r_* (r_0) + 1, r_* (r_0) + C_{\omega_R ml}^{(0)} - 1]\) becomes:
\[
u'' + \left( (\frac{\omega_R^2}{2})^2 + \mathcal{Z}(\omega_R, \omega_I, V_f(r_*), r_*) - V_f(r_*) \right) u = 0.
\]
Thus, the existence of a function \( V_f : \mathbb{R} \to [0, \frac{\omega_R^2}{2}] \) supported on \([r_* (r_0) + 2, r_* (r_0) + C_{\omega_R ml}^{(0)} - 2]\) such that the equation (4.93) admits a solution \( u \) coinciding with \( u_1^{(\omega_1)} \) on \([r_* (r_0) + 1, r_* (r_0) + 2]\) and with \( \lambda u_2^{(\omega_2)} \) on \([r_* (r_0) + C_{\omega_R ml}^{(0)} - 2, r_* (r_0) + C_{\omega_R ml}^{(0)} - 1]\) for some \( \lambda \in \mathbb{C} \setminus \{0\} \) is guaranteed by Lemma 4.3 (setting \( \frac{\omega_R^2}{2} \) in place of \( \omega \) there, as well as \( \epsilon = 1 \), in view of (4.88), (4.89), (4.90) and (4.86). Thus, the proof can be concluded by extending \( u \) on the whole of \( \mathbb{R} \) under the requirement that it coincides with \( u_1^{(\omega_1)} \) for \( r_* \leq r_0(r_0) + 1 \) and with \( \lambda u_2^{(\omega_2)} \) for \( r_* \geq r_0(r_0) + C_{\omega_R ml}^{(0)} - 1 \).

5 Proof of Theorem 2

In this section, we will provide a more detailed statement and the proof of Theorem 2.

In particular, a more detailed statement of Theorem 2 is the following:
**Theorem 2** (detailed version). For any $0 < |a| < M$ and any frequency triad $(\omega_0, m_0, l_0) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{Z} \times \mathbb{Z}[m]$ in the superradiant regime $(3.26)$, there exist constants $C_{\omega_0 m_0 l_0} > r_0$ and $C^{(0)}_{\omega_0 m_0 l_0} > 0$ depending only on $\omega_0 m_0 l_0, a, M$, such that for any $r_0 > C_{\omega_0 m_0 l_0}$ and any $\omega_I \geq 0$ sufficiently small in terms of $\omega_0 m_0 l_0, r_0$, there exists a Lorentzian metric $g^{(def)}_{M,a}$ on $\mathcal{M}_{M,a}$ such that:

1. The vector fields $T, \Phi$ are Killing vector fields for $g^{(def)}_{M,a}$.
2. $g^{(def)}_{M,a} \equiv g_{M,a}$ for $\{r \leq r_0\}$ and $\{r \geq r_0 + C^{(0)}_{\omega_0 m_0 l_0}\}$.
3. $g^{(def)}_{M,a}(T, T) < 0$ for $r \geq r_0$.
4. The wave equation
   \[
   (5.1) \quad \Box_{g_{M,a}} \psi = 0
   \]
   completely separates in the Boyer–Lindquist coordinate chart.
5. Equation $(5.1)$ admits an outgoing mode solution with parameters $(\omega_0 + i\omega_I, m_0, l_0)$.

**Proof.** Let $C_{\omega_0 m_0 l_0}$ and $C^{(0)}_{\omega_0 m_0 l_0}$ be as in the proof of Theorem 1.

For a function $h : (2M, +\infty) \to (0, +\infty)$ satisfying $h = 1$ for $r \not\in [r_0, r_0 + C^{(0)}_{\omega_0 m_0 l_0}]$ (to be defined later), let us introduce the following metric on $\mathcal{M}_{M,a}$, expressed in the Boyer–Lindquist coordinate chart:

\[
(5.2) \quad g^{(def)}_{M,a} = -h(r)(1 - \frac{2Mr}{\varphi^2})dt^2 - h(r)\frac{4Mar^2}{\varphi^2}dt\varphi + h(r)\frac{\varphi^2}{\Delta}dr^2 + h(r)\varphi^2d\vartheta^2 + h(r)\sin^2\vartheta\frac{\Pi}{\varphi^2}d\varphi^2.
\]

Notice that $(5.2)$ satisfies the following properties:

1. The vector fields $\partial_t, \partial_\varphi$ are Killing vector fields for $g^{(def)}_{M,a}$.
2. We have $g^{(def)}_{M,a} = g_{M,a}$ for $r \not\in [r_0, r_0 + C^{(0)}_{\omega_0 m_0 l_0}]$, since $h = 1$ there.
3. We have $g^{(def)}_{M,a}(\partial_t, \partial_t) < 0$ in the region $\{r \geq r_0\}$ (provided $C_{\omega_0 m_0 l_0}$ is sufficiently large).
4. The wave equation
   \[
   (5.3) \quad \Box_{g^{(def)}_{M,a}} \psi = 0
   \]
   on $(\mathcal{M}_{M,a}, g^{(def)}_{M,a})$ completely separates in the $(t, r, \vartheta, \varphi)$ coordinate chart.

The latter property is deduced as follows: The wave operator on $(\mathcal{M}_{M,a}, g^{(def)}_{M,a})$ has the following form:

\[
(5.4) \quad h(r)\varphi^2\Box_{g^{(def)}_{M,a}} \psi = h^{-2}(r)\varphi^2\partial_t(\Delta \partial_t \psi) - (\frac{(r^2 + a^2)^2}{\Delta} + a^2 \sin^2 \vartheta)\partial_r^2 \psi + (\frac{1}{\sin^2 \vartheta} - \frac{a^2}{\Delta})\partial_\varphi^2 \psi - \frac{4Mar}{\Delta} \partial_t \partial_\varphi \psi - \frac{1}{\sin \vartheta} \partial_\vartheta(\sin \vartheta \partial_\varphi \psi).
\]

Therefore, for any $(\omega, m) \in \mathbb{C} \times \mathbb{Z}$, equation $(5.3)$ admits solutions of the form $(5.2)$, with $S(\theta)$ satisfying $(5.3)$ and $R(r)$ satisfying

\[
(5.5) \quad h^{-2}(r)\Delta \frac{d}{dr} \left(\Delta \frac{dR}{dr}\right) + (a^2 m^2 - 4Mar\omega m + (r^2 + a^2)\omega^2 - \Delta(\lambda + a^2 \omega^2))R = 0.
\]
In particular, for any \((\omega, m, l) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{Z}_{2|m|}\) with \(|Im(\omega)|\) sufficiently small, equation (5.3) admits solutions of the form
\[
\psi(t, r, \vartheta, \varphi) = e^{-i\omega t} e^{im\varphi} S_{\omega ml}(\vartheta) R_{\omega ml}(r),
\]
with \(S_{\omega ml}\) defined as in Section 3.2 and \(u_{\omega ml}\) (defined in terms of \(R\) by (3.7)) satisfying
\[
u''_{\omega ml} + \left(h^2\omega^2 - V_{\omega ml; h}\right) u_{\omega ml} = 0,
\]
where \(''\) denotes differentiation with respect to the variable \(r\) (defined by (3.6)), and \(V_{\omega ml; h}\) is defined as:
\[
V_{\omega ml; h}(r) = h^2(r) \frac{4M r a m_m - a^2 m^2 + \Delta(\lambda_{\omega ml} + a^2\omega^2)}{(r^2 + a^2)^2} + \frac{\Delta}{(r^2 + a^2)}(a^2\Delta + 2Mr(\omega^2 - a^2)).
\]

We will now show that (5.3) admits an outgoing mode solution with parameters \((\omega_0 + i\omega, m_0, l_0)\), provided \(0 \leq \omega \ll 1\) and the function \(h\) is chosen appropriately. Let us set
\[
\varepsilon_1 = \frac{1}{2} \inf_{r \in [r_0, r_0 + C^{(0)}_{\omega_0 m_0 l_0}]} \left(\frac{\Delta}{(r^2 + a^2)^2}(a^2\Delta + 2Mr(\omega^2 - a^2))\right) > 0
\]
and
\[
G[\omega_0, \omega, l, r, \omega] = \frac{Im\left((\omega_0 + i\omega)^2 - \frac{4Mr a m_m}{(r^2 + a^2)^2} + \Delta(\lambda_{\omega ml} + a^2(\omega_0 + i\omega)^2)\right)}{Re\left((\omega_0 + i\omega)^2 - \frac{4Mr a m_m}{(r^2 + a^2)^2} + \Delta(\lambda_{\omega ml} + a^2(\omega_0 + i\omega)^2)\right)}.
\]

Let \(V(r) : \mathbb{R} \to [0, +\infty)\) be the potential function provided by Proposition 4.1 for the function \(G\) above and the parameters \((\omega_0, m_0, l_0), r_0, \varepsilon_1\) and \(\omega_0\). Finally, let us then define
\[
h^2(r) = 1 - \left(Re\left((\omega_0 + i\omega)^2 - \frac{4Mr a m_m}{(r^2 + a^2)^2} + \Delta(\lambda_{\omega ml} + a^2(\omega_0 + i\omega)^2)\right)\right)^{-1} V(r).
\]

Notice that, in view of (3.9), (4.2) and (5.9), provided \(\omega_0\) is sufficiently small in terms of \(\omega_0, m_0, l_0, M, a, r_0, \varepsilon_1\), the relation (5.11) indeed guarantees that
\[
h^2 > 0,
\]
and hence \(h\) is a smooth positive function. Since \(V(r)\) is supported in \([r_0, r_0 + C^{(0)}_{\omega_0 m_0 l_0}]\), from (5.11) we have
\[
h \equiv 1 \text{ outside } [r_0, r_0 + C^{(0)}_{\omega_0 m_0 l_0}].
\]

Notice also that (5.10) and (5.11) yield:
\[
h^2(\omega_0 + i\omega)^2 - V_{\omega_0 + i\omega, m_0 l_0; h}(r) = (\omega_0 + i\omega)^2 - V_{\omega_0 + i\omega, m_0 l_0}(r) + G[\omega_0, \omega, V(r), r] - V(r)
\]
(where \(V_{\omega ml}\) is defined by (3.9)).

In view of (5.14), equation (5.7) for the frequency triad \((\omega_0 + i\omega, m_0, l_0)\) becomes
\[
u'' + \left((\omega_0 + i\omega)^2 - V_{\omega_0 + i\omega, m_0 l_0}(r_*) + G[\omega_0, \omega, V(r_*), r_*)\right) u = 0.
\]

In view of Proposition 4.1 (and our choice of \(V\)), provided \(\omega_0\) is sufficiently small in terms of \(\omega_0, m_0, l_0, M, a, r_0, \varepsilon_1\), equation (5.15) admits a solution \(u\) satisfying at \(r_* = \pm \infty\) the boundary conditions (3.20). Thus, we finally deduce that:

5. With \(h\) defined as above, equation (5.3) admits an outgoing real mode solution with parameters \((\omega_0, m_0, l_0)\).
6 Superradiant instabilities for the free wave equation on spacetimes with normally hyperbolic trapping

According to our discussion in Section 1.3 in order to better clarify the absence of any connection between superradiance related mode instabilities and the structure of trapping outside the ergoregion, it would be preferable to replace the spacetimes \((\mathcal{M}_{M,a}, g_{M,a}^{(def)})\) of Theorem 2 with a family of spacetimes admitting real or exponentially growing mode solutions to (1.13) and at the same time having a “nice” trapped set. In this section, we will construct such a family of spacetimes \((\mathcal{M}_0, g_{M,a}^{(h)})\). We note already that the nature of our construction forces the spacetimes \((\mathcal{M}_0, g_{M,a}^{(h)})\) to be asymptotically flat instead of asymptotically flat (see the remarks below Theorem 3).

6.1 Theorem 3: Statement and remarks on the proof

The main result of this section will be the following:

**Theorem 3.** For any \(\omega_R \in \mathbb{R}\setminus\{0\}\) and \(0 \leq \omega_I < 1\) (small in terms of \(\omega_R\)), there exists a smooth family of globally hyperbolic Lorentzian metrics \(g_{(h)}^{(b)}\), \(M > 0, a > 0\), on the ambient “Schwarzschild exterior” manifold \(\mathcal{M}_0 = \mathbb{R} \times (2M, +\infty) \times S^2\), such that for any \(M > 0, R_b > 2M, k \in \mathbb{N}\) and \(\epsilon > 0\), there exists an \(a_0 \in (0, M)\) so that for \(0 < a \leq a_0\) the following properties are satisfied:

1. In the \((t, r, \theta, \phi)\) coordinate chart on \(\mathcal{M}_0\), the vector fields \(T = \partial_t\) and \(\Phi = \partial_\phi\) are Killing vector fields for \(g_{M,a}^{(h)}\).
2. The spacetime \((\mathcal{M}_0, g_{M,a}^{(h)})\) is asymptotically conic.
3. The metric \(g_{M,a}^{(h)}\) can be smoothly extended up to \(\mathcal{H} = \{r = 2M\}\) in the “Schwarzschild star” coordinate chart on \(\mathcal{M}_0\). In this extension, \(\mathcal{H}\) is the event horizon associated to the asymptotically conic end of \((\mathcal{M}_0, g_{M,a}^{(h)})\). Furthermore, \(\mathcal{H}\) is a Killing horizon with positive surface gravity.
4. On the closed subset \(\{2M \leq r \leq R_b\}\) of \(\mathcal{M}_0 \cup \mathcal{H}^*\), the metrics \(g_{M,a}^{(h)}\) and \(g_{M,a}\) are \(\epsilon\)-close in the \(C^k\) norm defined with respect to the “Schwarzschild star” coordinate chart on \(\{2M \leq r \leq R_b\}\). Furthermore, the ergoregion of \((\mathcal{M}_0, g_{M,a}^{(h)})\) is non-empty, contained in the set \(\{r \leq 2M + \epsilon\}\), and
\[
\sup_{\mathcal{M}_{M,0}} g_{M,a}^{(h)}(T, T) < \epsilon.
\]
5. The wave equation
\[
\Box g_{M,a}^{(h)} \psi = 0
\]
completely separates in the \((t, r, \theta, \phi)\) coordinate chart.
6. The wave equation (6.2) admits an outgoing mode solution with frequency parameter \(\omega_R + i\omega_I\).
7. The trapped set of \((\mathcal{M}_0, g_{M,a}^{(h)})\) is normally hyperbolic.

The proof of Theorem 3 will be presented in Section 6.7.

The metric \(g_{M,a}^{(h)}\) in Theorem 3 is constructed as an asymptotically conic perturbation of an asymptotically flat metric \(g_{M,a}^{(l)}\) on \(\mathcal{M}_0\) with normally hyperbolic trapped set - see Sections 6.2-6.5. The metric \(g_{M,a}^{(l)}\) bears many algebraic and geometric similarities with the Kerr metric \(g_{M,a}\), but it does not solve the vacuum Einstein equations.

The main difference between the spacetimes \((\mathcal{M}_{M,a}, g_{M,a}^{(def)})\) of Theorem 2 and \((\mathcal{M}_0, g_{M,a}^{(h)})\) of Theorem 3 lies exactly in the structure of the trapped set: The spacetime \((\mathcal{M}_{M,a}, g_{M,a}^{(def)})\) contains stable trapped null geodesics, while the trapped set of \((\mathcal{M}_0, g_{M,a}^{(h)})\) is normally hyperbolic. The normal hyperbolicity of the trapped set of \((\mathcal{M}_0, g_{M,a}^{(h)})\) can actually be deduced from the high frequency integrated local energy decay statement of Proposition 6.1.

At this point, we should remark the following:
Let $\hat{g}$ be the construction of the auxiliary metric $g_{M,a}^{(1)}$ as an intermediate step for establishing Theorem 3, was motivated as follows: Attempting to modify the Kerr metric $g_{M,a}$ in the region $\{r \gg 1\}$, so that the wave equation $\Box g_{M,a} \psi = 0$ for the new metric $\hat{g}_{M,a}$ admits an outgoing mode solution while at the same time remaining completely separable, one encounters the following obstacle to controlling the structure of the trapped set: The separability of $\Box \hat{g}_{M,a}$, combined with the requirement that $\hat{g}_{M,a} = g_{M,a}$ in the region $\{r \lesssim 1\}$ and the fact that the angular separation variable $\Lambda$ for $\Box g_{M,a}$ depends on the time separation variable $\omega$ (see Section 3.2 for the separation of the wave equation on Kerr spacetimes), imply a rigid relation between $\hat{g}_{M,a}(\partial_{\tau}, \partial_{\varphi})$ and $\hat{g}_{M,a}(\partial_{\varphi}, \partial_{\varphi})$, which leaves almost no room for deforming $g_{M,a}$ without introducing stably trapped null geodesics. In view of these difficulties, we introduced a novel metric $g_{M,a}^{(1)}$ on $M_0$, which has many algebraic and analytic similarities with the slowly rotating Kerr metric (such as the separability of the wave operator and the polynomial decay of solutions to the wave equation), but for which the angular separation variable $\Lambda$ does not depend on $\omega$.

2. The conic asymptotics of the metric $g_{M,a}^{(h)}$ in Theorem 3 are a technical necessity imposed by the methods used in the proof, which, in view of the conditions imposed on the structure of the trapped set, enforces a monotonicity condition on the angular components of $g_{M,a}^{(h)}$. It would be of particular interest to examine whether $g_{M,a}^{(h)}$ can be replaced by an asymptotically flat metric with similar properties. However, since the $r^n$-weighted estimates of $\Box \hat{g}_{M,a}$ also apply in the asymptotically conic case, the conic asymptotics of $g_{M,a}^{(h)}$ pose no additional difficulties (compared to the asymptotically flat case) in the study of the decay properties of solutions to (6.2).

As we remarked before, the normal hyperbolicity of the trapped set of the spacetimes $(M_0, g_{M,a})$ of Theorem 3 can be viewed as a consequence of the statement that $(M_0, g_{M,a}^{(h)})$ satisfies a high frequency integrated local energy decay estimate:

**Proposition 6.1.** Let $(M_0, g_{M,a}^{(h)})$ be the spacetimes of Theorem 3. Let $\hat{t} : M_0 \cup \mathcal{H}^+ \to \mathbb{R}$ be a smooth function with spacelike level sets intersecting $\mathcal{H}^+$, such that $T(\hat{t}) = 1$ and $\hat{t} \equiv t$ for $\{r \geq 3M\}$. Then, for any $\epsilon > 0$, there exists a (small) parameter $a_0 > 0$ and (large) parameters $2M \ll R_\epsilon \ll R_\epsilon$, with $R_\epsilon$ independent of $\epsilon$, so that the following statement holds for any $0 \leq a \leq a_0$: For any solution $\psi$ to the inhomogeneous wave equation

$$\Box g_{M,a}^{(h)} \psi = F$$

on $(M_0, g_{M,a}^{(h)})$ which is smooth up to $\mathcal{H}^+$, we can bound for any $\tau_1 \leq \tau_2$:

$$\int_{\{\tau_1 \leq \tau \leq \tau_2\}} \left((1 - \frac{2M}{r})r^{-2}\partial_{\tau}' \partial_{\varphi}' \right)^2 + \chi_{r \geq 3M}(r) \cdot r^{-2} J_\mu^N(\psi) N^\mu + r^{-4} \partial_{\varphi}' \partial_{\varphi}' \right)^2 \, dg_{M,a}^{(h)} + \int_{\{\tilde{t} = \tau_2\}} J_\mu^N(\psi) \bar{n}^\mu \leq C_{R_\epsilon, R_\epsilon} \left( \int_{\{\tilde{t} = \tau_1\}} J_\mu^N(\psi) \bar{n}^\mu + Z_{\tau_1, \tau_2}[F, \psi; R_\epsilon] + \int_{\{R_\epsilon \leq \tau_\epsilon \leq R_\epsilon\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} \min \{ |T \psi|^2, |\psi|^2 \} \, dg_{M,a}^{(h)} \right),$$

$$\int_{\{\tau_1 \leq \tau \leq \tau_2\}} \left(r^{-2} J_\mu^N(\psi) N^\mu + r^{-4} \partial_{\varphi}' \partial_{\varphi}' \right) \, dg_{M,a}^{(h)} + \sum_{j = 0}^{1} \int_{\{\tilde{t} = \tau_2\}} J_\mu^N(T^j \psi) \bar{n}^\mu \leq C_{R_\epsilon, R_\epsilon} \left( \sum_{j = 0}^{1} \int_{\{\tilde{t} = \tau_1\}} J_\mu^N(T^j \psi) \bar{n}^\mu + \sum_{j = 0}^{1} Z_{\tau_1, \tau_2}[T^j F, T^j \psi; R_\epsilon] + \int_{\{R_\epsilon \leq \tau_\epsilon \leq R_\epsilon\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} \min \{ |T \psi|^2, |\psi|^2 \} \, dg_{M,a}^{(h)} \right).$$

In the above, the vector field $N$ is everywhere timelike on $M_0 \cup \mathcal{H}^+$ and satisfies $[T, N] = 0$ and $N = T$ for $r \geq R_2 \gg 1$, $Z_{\tau_1, \tau_2}[F, \psi; R_\epsilon]$ is defined as

$$Z_{\tau_1, \tau_2}[F, \psi; R_\epsilon] \equiv \int_{\{r \leq R_\epsilon\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} |F|^2 + \int_{\{r \geq R_\epsilon\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} |F| \cdot (|\partial_{\varphi}' \partial_{\varphi}'| + r^{-1} |\psi|) + \int_{\{\tau_1 \leq \tau \leq \tau_2\}} |F| \cdot |T \psi|,$$

This is simply the condition $h' \geq 0$ appearing in the proof of Lemma 6.3.
is everywhere negative on $[0, 1]$ vanishes on $[3M - c, 3M + c]$ and is identically 1 outside $(3M - 2c, 3M + 2c)$, and the constant $C_{R_+, R_-}$ depends only on the precise choice of $R_+$ and $R_-$.

The proof of Proposition 6.1 will be presented in Section 6.7 as a part of the proof of Theorem 3.

The estimates (6.4) and (6.5) can be viewed as integrated local energy decay estimates (with loss of derivatives) for solutions $\psi$ to (6.3) in the case when the time frequency of $\phi$ is very high or very low (so that the last term in the right hand side of (6.4) and (6.5) can be absorbed into the left hand side). These high frequency properties of (6.4) for solutions $V$ for suitably large values), the vector fields is stationary and axisymmetric, with stationary Killing field $\Phi = \partial_\theta$ (both coordinate vector fields defined in the fixed $(t, r, \theta, \varphi)$ coordinate chart). When $a = 0$, (6.6) is simply the Schwarzschild exterior metric.

When $a > 0$, the Killing field $T$ becomes spacelike in the region $\{r < r_1(\theta)\}$, where $r_1(\theta)$ is the positive solution of

$$1 - \frac{2M}{r_1} - \frac{a^2 M^2 \sin^2 \theta}{r_1^3} = 0$$

(notice that $r_1(\theta) > 2M$ for $a > 0$ and $\theta \neq 0, \pi$). However, the span of the Killing fields $T, \Phi$ contains everywhere on $\mathcal{M}_0$ a timelike direction. This follows from the fact that the determinant

$$\mathcal{D} = \det \begin{pmatrix} g_{M,a}^{(1)}(T, T) & g_{M,a}^{(1)}(T, \Phi) \\ g_{M,a}^{(1)}(\Phi, T) & g_{M,a}^{(1)}(\Phi, \Phi) \end{pmatrix} = -(1 - \frac{2M}{r})^2 \sin^2 \theta$$

is everywhere negative on $\mathcal{M}_0$, except on the axis $\theta = 0, \pi$, where $\Phi$ vanishes and $T$ is timelike.

The spacetime $(\mathcal{M}_0, g_{M,a}^{(1)})$ does not contain any black hole or white hole region, i.e. the domain of outer communications of the asymptotically flat region $\{r \gg 1\}$ of $\mathcal{M}_0$ is the whole $\mathcal{M}_0$. This can be inferred as follows: Let us fix a vector field $V$ in the span of $\{T, \Phi\}$, such that $V$ is everywhere on $\mathcal{M}_0$ future pointing and timelike, $V \equiv T$ in the asymptotic region $\{r \gg 1\}$ and $[T, V] = [\Phi, V] = 0$. Then for some $h : (2M, +\infty) \to (1, +\infty)$ (taking suitably large values), the vector fields

$$V_1 = \partial_r + h(r) V$$

6.2 Construction of the auxiliary spacetimes $(\mathcal{M}_0, g_{M,a}^{(1)})$

For any $M > 0$ and $a \geq 0$, we define the following metric on $\mathcal{M}_0 = \mathbb{R} \times (2M, +\infty) \times S^2$ in the $(t, r, \theta, \varphi)$ coordinate system (where $t, r$ are the projections onto the first two factors of $\mathcal{M}_0$ and $(\theta, \varphi)$ are the usual polar coordinates on $S^2$):

$$g_{M,a}^{(1)} = -\left(1 - \frac{2M}{r} - \frac{a^2 M^2 \sin^2 \theta}{r^4} \right) dt^2 - \frac{2M a \sin^2 \theta}{r} dt d\varphi + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$

Notice that $(\mathcal{M}_0, g_{M,a}^{(1)})$ is a smooth, globally hyperbolic and asymptotically flat spacetime. Furthermore, $(\mathcal{M}_0, g_{M,a}^{(1)})$ is stationary and axisymmetric, with stationary Killing field $T = \partial_t$ and axisymmetric Killing field $\Phi = \partial_\theta$ (both coordinate vector fields defined in the fixed $(t, r, \theta, \varphi)$ coordinate chart). When $a = 0$, (6.6) is simply the Schwarzschild exterior metric.
and

\begin{equation}
V_2 = \partial_r - h(r)V
\end{equation}

are future directed and past directed timelike vector fields respectively. Furthermore, \(V_1 r = V_2 r = 1\), and thus, starting from any point \(p \in M_0\), the flow of \(V_1, V_2\) reaches the asymptotically flat region \(\{ r \gg 1 \}\) in finite time. Thus, any point \(p \in M_0\) communicates with the asymptotically flat region through both future and past directed causal curves.

In the limit \(r \to 2M\), the expression (6.6) for the metric breaks down, however, the spacetime is future incomplete, with incomplete null geodesics approaching \(r = 2M\). It turns out that the spacetime can be smoothly extended to the future “beyond” \(r = 2M\), and this follows immediately after the following change of coordinates: By introducing the new coordinate functions

\begin{align}
t^* &= t + \tilde{t}(r) \\
\varphi^* &= \varphi + \tilde{\varphi}(r),
\end{align}

where

\begin{align}
\frac{d\tilde{t}}{dr} &= \left(1 - \frac{2M}{r}\right)^{-1}, \\
\frac{d\tilde{\varphi}}{dr} &= \left(1 - \frac{2M}{r}\right)^{-1} aMr^{-3},
\end{align}

the expression for \(g^{(1)}_{M,a}\) in the \((t^*, r, \vartheta, \varphi^*)\) coordinate chart on \(M_0\) becomes:

\begin{equation}
g^{(1)}_{M,a} = -\left(1 - \frac{2M}{r} - \frac{a^2 M^2 \sin^2 \vartheta}{r^4}\right)(dt^*)^2 - \frac{2Ma \sin^2 \vartheta}{r} dt^* d\varphi^* + 2dt^* dr + r^2 (d\vartheta^2 + \sin^2 \vartheta (d\varphi^*)^2).
\end{equation}

Thus, \(g^{(1)}_{M,a}\) can be smoothly extended beyond \(r = 2M\). In such a smooth extension \((\tilde{M}_0, \tilde{g}^{(1)}_{M,a})\) of \((M_0, g^{(1)}_{M,a})\), where \(\tilde{M}_0 = \mathbb{R} \times (2M - \delta, +\infty) \times S^2\) for some \(0 < \delta < 2M\), \(M_0\) has a non-empty boundary \(\mathcal{H}^+\), on which \(r\) extends continuously (with \(r|_{\mathcal{H}^+} = 2M\)). It can be readily verified (in the \((t^*, r, \vartheta, \varphi^*)\) coordinate chart) that \(\mathcal{H}^+\) is actually a smooth, null hypersurface, and thus \(g^{(1)}_{M,a}\) extends uniquely (independently of the particular choice of the extension \((\tilde{M}_0, \tilde{g}^{(1)}_{M,a})\)) as a smooth Lorentzian metric on the manifold with boundary

\begin{equation}
\overline{M}_0 = M_0 \cup \mathcal{H}^+
\end{equation}

(on which the \((t^*, r, \vartheta, \varphi^*)\) coordinate chart is smooth). Furthermore, the Killing fields \(T, \Phi\) extend smoothly on \(\mathcal{H}^+\). Notice that \(T\) is spacelike on \(\mathcal{H}^+\) (except on the points \(\vartheta = 0, \pi\) of \(\mathcal{H}^+\), where \(T\) is null).

It can be readily verified that in any such extension \((\tilde{M}_0, \tilde{g}^{(1)}_{M,a})\) of \((M_0, g^{(1)}_{M,a})\), the domain of outer communications of \((\tilde{M}_0, \tilde{g}^{(1)}_{M,a})\) is precisely \((\mathcal{M}_0, g^{(1)}_{M,a})\), and \(\mathcal{H}^+\) is the future event horizon. Furthermore, \(\mathcal{H}^+\) is also a Killing horizon: Introducing the vector field

\begin{equation}
K = T + \frac{a}{8M^2} \Phi,
\end{equation}

we notice that \(K\) is a Killing vector field for \(g^{(1)}_{M,a}\) (as a linear combination of \(T, \Phi\) with constant coefficients), and furthermore \(g^{(1)}_{M,a}(K, K) = 0\) on \(r = 2M\).

\textbf{Remark.} Let us note that the existence of a Killing field parallel to the null generator of \(\mathcal{H}^+\) does not follow in this case by Hawking’s theorem (see [25]), since \((\mathcal{M}_0, g^{(1)}_{M,a})\) does not satisfy the null energy condition (in general, one would expect the null generator of \(\mathcal{H}^+\) to be a \(\vartheta\)-dependent linear combination of \(T, \Phi\)).
Since the vector field $K$ satisfies
\begin{equation}
\tag{6.18}
g_{M,a}^{(1)}(K, K) = -\left(1 - \frac{2M}{r}\right)\left[1 - a^2 \sin^2 \vartheta \left(1 - \frac{2M}{r}\right)^2 (r^2 + 2Mr + 4M^2)^2 \right],
\end{equation}
and thus $\partial_r g_{M,a}^{(1)}(K, K)|_{\mathcal{H}^+} > 0$, in the $(t^*, r, \vartheta, \varphi^*)$ coordinate chart, the Killing horizon $\mathcal{H}^+$ is a non-degenerate horizon with positive surface gravity.

### 6.3 Separation of the wave equation and frequency decomposition on ($\mathcal{M}_0, g_{M,a}^{(1)}$)

The wave operator $\Box_{g_{M,a}^{(1)}}$ on ($\mathcal{M}_0, g_{M,a}^{(1)}$) takes the form:
\begin{equation}
\tag{6.19}
\Box_{g_{M,a}^{(1)}} \psi = \partial_t^2 \psi + 2\partial_t \left(1 - \frac{2M}{r}\right) \partial_t \psi - r^{-1} \partial_r (r^2 \partial_r \psi) + r^{-2} \left(\sin \vartheta \right)^{-1} \partial_\vartheta \left(\sin \vartheta \partial_\vartheta \psi\right) - 2M \partial_r \vartheta \partial_r \psi + \left(\left(1 - \frac{2M}{r}\right)^2 (\sin \vartheta)^{-2} - a^2 M^2 r^{-4}\right) r^{-2} \partial_\vartheta^2 \psi.
\end{equation}

Therefore, the wave equation $\Box_{g_{M,a}^{(1)}} \psi = 0$ separates on ($\mathcal{M}_0, g_{M,a}^{(1)}$), i.e. it admits solutions of the form
\begin{equation}
\tag{6.20}
\psi(t, r, \vartheta, \varphi) = e^{-i\omega t} e^{im\varphi} S_{ml}(\vartheta) \cdot R_{am}(r)
\end{equation}
for $(\omega, m, l) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{Z}_{[m]}$, where $e^{im\varphi} S_{ml}(\vartheta)$ are the usual spherical harmonics on $S^2$, with $\{S_{ml}\}_{(m, l) \in \mathbb{Z} \times \mathbb{Z}_{[m]}}$ being the usual set of eigenfunctions of the Sturm–Liouville problem
\begin{equation}
\tag{6.21}
\begin{cases}
-\sin \vartheta \partial_\vartheta \left(\sin \vartheta \partial_\vartheta \right) S_{ml} + \frac{m^2}{\sin^2 \vartheta} S_{ml} = l(l + 1) S_{ml} \\
S_{ml}(\vartheta) \text{ is bounded at } \vartheta = 0, \pi,
\end{cases}
\end{equation}
and $R_{am}(r)$ satisfies the ordinary differential equation
\begin{equation}
\tag{6.22}
r^{-2} \left(1 - \frac{2M}{r}\right) \frac{d}{dr} \left(r^2 \left(1 - \frac{2M}{r}\right) \frac{dR_{am}(r)}{dr}\right) + \left\{\omega^2 - 2M a r^{-3} \omega m + a^2 M^2 r^{-6} - \left(1 - \frac{2M}{r}\right)^2 r^{-2} (l + 1)\right\} R_{am}(r) = 0.
\end{equation}

Using on $\mathcal{M}_0$ the auxiliary radial function $r_+: (2M, +\infty) \to (-\infty, +\infty)$ defined by
\begin{equation}
\tag{6.23}
\frac{dr_+}{dr} \equiv \left(1 - \frac{2M}{r}\right)^{-1}
\end{equation}
and setting
\begin{equation}
\tag{6.24}
u_{am} = r_+ R_{am},
\end{equation}
equation (6.22) becomes:
\begin{equation}
\tag{6.25}
u_{am}'' + \left(\omega^2 - V_{am}\right) \nu_{am} = 0,
\end{equation}
where $'$ denotes differentiation with respect to $r_+$, and
\begin{equation}
\tag{6.26}
V_{am}(r) = \frac{(1 - \frac{2M}{r})(l + 1) + 2aMr^{-1}\omega m - a^2 M^2 r^{-4} m^2}{r^2} + \left(1 - \frac{2M}{r}\right)^2 \frac{2M}{r^3}.
\end{equation}

Any smooth function $\Psi : \mathcal{M}_0 \to \mathbb{C}$ which is square integrable in the $t$ variable can be uniquely decomposed as
\[\Psi(t, r, \vartheta, \varphi) \equiv \sum_{(m, l) \in \mathbb{Z} \times \mathbb{Z}_{[m]}} \int_{-\infty}^{\infty} e^{-i\omega t} e^{im\varphi} S_{ml}(\vartheta) \Psi_{am}(r) \, d\omega\]

33
for some $\Psi_{aml} : (r_*, +\infty) \to \mathbb{C}$. With $\mathcal{M}_0$ defined as in (6.16), if $\psi : \mathcal{M}_0 \to \mathbb{C}$ is a smooth function which is supported on a set of the form \( \{ t \geq t_0 \} \) for some $t_0 \in \mathbb{R}$ and is square integrable in the $t^*$ variable, satisfying

\[
\Box g_{M,a}^{(1)} \psi = F
\]

for some smooth function $F : \mathcal{M}_0 \to \mathbb{C}$ which is square integrable in $t^*$, then $u_{aml}(r) \pm r\Psi_{aml}(r)$ satisfies for all $(m,l) \in \mathbb{Z} \times \mathbb{Z}_{z[m]}$ and almost all $\omega \in \mathbb{R}$:

\[
\begin{align*}
\left\{ \begin{array}{ll}
& u_{aml}'' + \left( \omega^2 - V_{aml} \right) u_{aml} = (r - 2M) \cdot F_{aml} \\
& u_{aml}' - i\omega u_{aml} \to 0 \text{ as } r_* \to +\infty \\
& u_{aml}' + i (\omega - \frac{am}{8M^2}) u_{aml} \to 0 \text{ as } r_* \to -\infty.
\end{array} \right.
\end{align*}
\]

The derivation of this ordinary differential equation for the Fourier transform of $\psi$ follows in exactly the same way as for subextremal Kerr spacetimes (see [15] for more details). The last boundary condition of (6.28) is derived from the fact that $\psi$ is smooth on $\mathcal{H}^+$, and $\partial_r$ extends smoothly on $\mathcal{H}^+$ as $\partial_r|_{\mathcal{H}^+} = K|_{\mathcal{H}^+}$.

Let us also remark that for any smooth $V : (2M, +\infty) \to \mathbb{C}$ of compact support, equation

\[
\Box g_{M,a}^{(1)} \psi - V(r) \psi = F
\]

(with $\psi, F$ having the same asymptotic behaviour as before) also separates, leading to the following variant of (6.28) for $u_{aml}(r) \pm r\Psi_{aml}(r)$ for all $(m,l) \in \mathbb{Z} \times \mathbb{Z}_{z[m]}$ and almost all $\omega \in \mathbb{R}$:

\[
\begin{align*}
\left\{ \begin{array}{ll}
& u_{aml}'' + \left( \omega^2 - V_{aml} - \left(1 - \frac{2M}{r}\right) \cdot V \right) u_{aml} = (r - 2M) \cdot F_{aml} \\
& u_{aml}' - i\omega u_{aml} \to 0 \text{ as } r_* \to +\infty \\
& u_{aml}' + i (\omega - \frac{am}{8M^2}) u_{aml} \to 0 \text{ as } r_* \to -\infty.
\end{array} \right.
\end{align*}
\]

The superradiant frequency regime of $(\mathcal{M}_0, g_{M,a}^{(1)})$, defined as for the Kerr exterior spacetime $(\mathcal{M}_{M,a}, g_{M,a})$ (see Section 3.4), consists of those frequency triads $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{z[m]}$ for which the limits

\[
\mathcal{F}_a[u_{aml}] = \lim_{r_+ \to +\infty} \pm \text{Im}(\omega u_{aml}^{-1} \cdot u_{aml}')
\]

for any non-zero function $u_{aml}$ satisfying (6.28) have opposite sign. In view of the boundary conditions of (6.28), it readily follows that the superradiant frequency regime has the following form

\[
\mathcal{A}^{(a,M)}_{sup} = \left\{ (\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{z[m]} \mid \omega \left( \omega - \frac{am}{8M^2} \right) < 0 \right\}
\]

The metric $g_{M,a}^{(1)}$ on $\mathcal{M}_0$ approaches the Schwarzschild exterior metric $g_{M,0}$ smoothly as $a \to 0$. Based on this fact, the following lemma can be readily inferred (the proof of which will be omitted):

**Lemma 6.1.** There exists some (large) $C_0 > 1$ such that the following statement holds: For any $\delta_0 > 0$, there exists some $0 < a_0 \ll M$, such that for any $0 < a < a_0$ the potential $V_{aml}$ has the following properties:

1. For any $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{z[m]}$, we have

\[
|V_{aml}(r) - V_{ml}^{(Sch)}(r)| \leq \delta_0 r^{-2} \left( \omega^2 + r^2 m^2 \right),
\]

where $V_{ml}^{(Sch)}$ is the corresponding potential for the Schwarzschild metric $g_{M,0}$:

\[
V_{ml}^{(Sch)}(r) = \frac{(1 - \frac{2M}{r})(l+1)}{r^2} + \frac{(1 - \frac{2M}{r})}{r^3} \cdot 2M.
\]

2. For any $(\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{z[m]} \setminus \{0\}$ such that $l \geq \max\{ C_0^{-1} |\omega|, C_0 \}$, there exists some $r_{aml} \in (3M - \delta_0, 3M + \delta_0)$ such that $\frac{dV_{aml}}{dr}$ has only a simple root at $r = r_{aml}$, and moreover

\[
\left( 1 - \frac{r}{r_{aml}} \right) \cdot \frac{dV_{aml}}{dr} \geq C_0^{-1} (l^2 + \omega^2) r^{-3}
\]
6.4 An integrated local energy decay estimate for \((M_0, g_{M,a}^{(1)})\) in the case \(a \ll M\)

Using Lemma 6.1, we will establish the following integrated local energy decay estimate in the case \(a \ll M\), along the lines of the corresponding proof in the case of a very slowly rotating Kerr spacetimes in [14]:

**Proposition 6.2.** With the notation as in Section 6.2, there exists a (small) \(a_0, \delta_0 > 0\) and (large) \(C > 0, R_0 > 2M\), such that for any \(0 < a < a_0\) and any smooth solution \(\psi\) of (6.27) on \(\mathcal{M}_0\) supported on a set of the form \(\{t \geq t_0\}\) for some \(t_0 \in \mathbb{R}\) with the property that \(\psi, F\) are square integrable in the \(t^*\) variable, the following integrated local energy decay estimates hold:

\[
\int_{\mathcal{M}_0} \left( (1 - \frac{2M}{r}) r^{-2} |\partial_r \psi|^2 + \chi_{r \leq 3M}(r) \cdot r^{-2} J^N_{\mu}(\psi) N^\mu + r^{-4} |\psi|^2 \right) \leq C \cdot Z[F, \psi; R_0],
\]

\[(6.40)\]

\[
\int_{\mathcal{M}_0} \left( r^{-2} J^N_{\mu}(\psi) N^\mu + r^{-4} |\psi|^2 \right) \leq C \cdot \left( Z[F, \psi; R_0] + Z[T F, T \psi; R_0] \right),
\]

where

\[(6.37)\]

\[
Z[F, \psi; R_0] \equiv \int_{\{r < R_0\}} |F|^2 + \int_{\{r \geq R_0\}} F \cdot \left( r^{-1} + O(r^{-2}) \right) \partial_r (r \bar{\psi}) + \int_{\{r \geq R_0\}} F \cdot O(r^{-2}) \bar{\psi} + \int_{\mathcal{M}_0} F \cdot T \bar{\psi},
\]

\[(6.38)\]

\(N\) is a \(T\)-invariant timelike vector field on \(\mathcal{M}_0\) such that \(N \equiv T\) in the region \(\{r \gg 1\}\), and the cut-off function \(\chi_{r \leq 3M} : (2M, +\infty) \to [0, 1]\) vanishes in \([3M - \delta_0, 3M + \delta_0]\) and is identically 1 on \((2M, 3M - \delta_0) \cup [3M + 2\delta_0, +\infty)\).

**Remark.** Note that for \(a \ll M\), the uniform energy boundedness results of [13] apply on \((\mathcal{M}_0, g_{M,a}^{(1)})\).

**Proof.** Let us assume first that the following estimates hold for the frequency separated equation (6.28) for any \((\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\geq |l|}\), any \(R_1^* \gg 1\) and any \(\varepsilon_1 > 0\) sufficiently small:

\[
\int_{-R_1^*}^{R_1^*} r^{-2} |u_{\omega m l}|^2 + r^{-2} (\omega^2 + l^2 + r^2) |u_{\omega m l}|^2 \, dr_\ast \leq \]

\[
\leq C R_1^* \delta_0 m^2 |u_{\omega m l}(\infty)|^2 + C \int_{R_1^*}^{\varepsilon_1 R_1^*} (\varepsilon_1 r^{-2} + r^{-3}) \omega^2 |u_{\omega m l}|^2 +
\]

\[
+ C R_1^* \int_{-\infty}^{+\infty} Re \left\{ (r - 2M) F_{\omega m l} \cdot \left( f_{\omega m l} \tilde{u}_{\omega m l} + (r^{-1} h_{\omega m l} + i \omega) \tilde{u}_{\omega m l} \right) \right\} \, dr_\ast,
\]

for \(|\omega| \gg l\) or \(|\omega| \ll l\), and

\[
\int_{-R_1^*}^{R_1^*} r^{-2} |u_{\omega m l}|^2 + r^{-2} \left\{ (1 - \frac{r_{\omega m l}}{r})^2 (\omega^2 + l^2) + r^{-2} \right\} |u_{\omega m l}|^2 \, dr_\ast \leq \]

\[
\leq C R_1^* \delta_0 m^2 |u_{\omega m l}(\infty)|^2 + C \int_{R_1^*}^{\varepsilon_1 R_1^*} (\varepsilon_1 r^{-2} + r^{-3}) \omega^2 |u_{\omega m l}|^2 +
\]

\[
+ C R_1^* \int_{-\infty}^{+\infty} Re \left\{ (r - 2M) F_{\omega m l} \cdot \left( f_{\omega m l} \tilde{u}_{\omega m l} + (r^{-1} h_{\omega m l} + i \omega) \tilde{u}_{\omega m l} \right) \right\} \, dr_\ast,
\]

for \(|\omega| \sim l\), where the functions \(f_{\omega m l}, h_{\omega m l}\) depend on the precise choice of \(\omega, m, l, R_1^*, \varepsilon_1\), and are bounded independently of \(\omega, m, l\).

**Remark.** Notice that the constants in front of the second terms of the right hand sides of (6.39), (6.40) are independent of \(R_1^*\).

Then, combining (6.39) and (6.40) with the red shift type estimates of Section 7 of [14] (see also [11]) associated to the \(K\) vector field of \(\mathcal{M}_0\) and the general \(\partial_r\)-Morawetz type inequalities of [33] for the region \(\{r \geq R_0\}\), one obtains both the integrated local energy decay estimates (6.36) and (6.37) (see e.g. [11] [14]).
We will now proceed to establish (6.39) and (6.40) for all \((\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_{\leq |l|}\). We will first deal with the very low frequency regime \(|\omega| \ll 1\). Fixing a small \(\varepsilon_0 > 0\) (depending only on the geometry of the family \((M_0, g_{M_0}^{(1)})\) for \(a > 0\)), inequalities (6.36) and (6.37) for solutions \(\psi\) to (6.27) with frequency support contained in \(\{|\omega| \leq \varepsilon_0\}\) follow readily by repeating the proof of Proposition 6.1 of [32] for equation (6.27). Thus, it remains to obtain (6.39) and (6.40) in the case \(|\omega| \geq \varepsilon_0\).

In the case \(|\omega| \gg l\) or \(\varepsilon_0 \leq |\omega| \ll l\), provided \(a_0\) is sufficiently small, in view of (6.33) (and the boundary conditions of (6.28) for \(u_{am\ell}\) at \(r_* = \pm \infty\)), it follows immediately from the corresponding frequency separated estimates in [11] for Schwarzschild and very slowly rotating Kerr spacetimes that

\[
\int_{-R_1^*}^{R_1^*} (r^{-2}|u_{am\ell}'|^2 + r^{-2}(\omega^2 + l^2 + r^2)|u_{am\ell}|^2) \, dr_* \leq C R_1^* \{(\omega^2 + \delta_0 m^2)|u_{am\ell}(-\infty)|^2 - \omega^2 |u_{am\ell}(+\infty)|^2\} + C R_1^* \int_{-\infty}^{+\infty} Re\{(r - 2M) F_{am\ell} \cdot (f_{am\ell} u_{am\ell} + r^{-1} h_{am\ell} u_{am\ell})\} \, dr_* + C \int_{R_1^*}^{\varepsilon_1 R_1^*} \left(\varepsilon_1 r^{-2} + r^3\right) \omega^2 |u_{am\ell}|^2,
\]

for suitable functions \(f_{am\ell}, h_{am\ell}\) with the properties described above. Therefore, inequality (6.39) follows from (6.41), in view of the the microlocal \(T\)-energy identity

\[
\omega^2 |u_{am\ell}(+\infty)|^2 - \omega(\omega - \frac{am}{8M^2}) |u_{am\ell}(-\infty)|^2 = \int_{-\infty}^{+\infty} Im \{(r - 2M) F_{am\ell} \cdot \omega \bar{u}_{am\ell}\} \, dr_*,
\]

where

\[
|u_{am\ell}(\pm \infty)|^2 \leq \lim_{r_* \to \pm \infty} |u_{am\ell}(r_*)|^2.
\]

Finally, in the frequency regime \(|\omega| \sim l\) (where trapping takes place), inequality (6.40) follows in a similar way as for the very slowly rotating Kerr spacetimes in Section 5 of [14]. Let us introduce a smooth function \(f_{am\ell} : \mathbb{R} \to [-1, 1]\) such that \(f_{am\ell}(r_*) : (2M, +\infty) \to [-1, 1]\) extends smoothly up to \(r = 2M\), satisfying the following properties:

1. \(f_{am\ell}' \geq 0\) and \(f_{am\ell}' \geq c R_1^* > 0\) on \([-R_1^*, R_1^*]\).
2. \(f_{am\ell} < 0\) for \(r < r_{am\ell}\) and \(f_{am\ell} > 0\) for \(r > r_{am\ell}\).
3. \(-f_{am\ell} V_{am\ell}' - \frac{1}{2} f_{am\ell}'' \geq 0\) and \(-f_{am\ell} V' - \frac{1}{2} f_{am\ell}'' \geq c R_1^* > 0\) on \([-R_1^*, R_1^*]\).
4. \(f_{am\ell}\) is independent of \((\omega, m, l)\) for \(r_* \geq r_*(R_0)\).

Such a function clearly exists, in view of the property (6.35) of \(V_{am\ell}\) (see also the related construction in [15]). After multiplying (6.28) with \(2f_{am\ell} u' + f_{am\ell}'' \bar{u}\) and integrating by parts (using also the boundary conditions of (6.28) for \(u_{am\ell}\) at \(r_* = \pm \infty\)), we obtain:

\[
\int_{-\infty}^{+\infty} \left(2 f_{am\ell}' |u'|^2 - (f_{am\ell}' V_{am\ell}' + f_{am\ell}'' |u|^2)\right) \, dr_* \leq -\int_{-\infty}^{+\infty} Re\{(r - 2M) F \cdot (2f_{am\ell} u' + f_{am\ell}'' \bar{u})\} \, dr_* + 2 f_{am\ell} \cdot \left(\omega - \frac{am}{8M^2}\right)^2 |u_{am\ell}(-\infty)|^2 - \omega^2 |u_{am\ell}(+\infty)|^2\right).
\]

Thus, from (6.42), (6.44) and the properties of \(f_{am\ell}\), we obtain (6.40) in the case \(|\omega| \sim l\). Therefore, the proof of Proposition 6.2 is complete. \(\Box\)
Remark. Notice that, while the integrated local energy decay estimates of Proposition 6.2 were established for solutions \( \psi \) of (6.27) on \( \mathcal{M}_0 \) which are square integrable in \( t^* \), Proposition 6.2 was used as a black box (combined with the fact that the uniform energy boundedness results of [13] hold on \( (\mathcal{M}_0, g_{M,a}^{(1)}) \)), also implies that an estimate of the form (7.4) (with \( k = 1 \)) holds for solutions \( \psi \) of (6.27) which are not necessarily square integrable in \( t^* \) (arguing as in the proof of Proposition 7.2 see Section B of the Appendix). In particular, equation (6.27) for \( F = 0 \) on \( (\mathcal{M}_0, g_{M,a}^{(1)}) \) (for a sufficiently small) does not admit real outgoing mode solutions

Furthermore, fixing \( t: \mathcal{M}_0 \to \mathbb{R} \) to be a smooth function satisfying \( T(t) = 1 \), with level sets which are spacelike hyperboloids terminating at \( \mathcal{I}^+ \) and intersecting \( \mathcal{H}^+ \), the results of [33] imply (in view of (6.37)) that smooth solutions \( \psi \) of the wave equation on \( (\mathcal{M}_0, g_{M,a}^{(1)}) \) (for a sufficiently small) with suitably decaying initial data on a Cauchy hypersurface of \( \mathcal{M}_0 \) decay at a \( t^{-\frac{3}{2}} \) rate.

6.5 The deformed metric \( g_{M,a}^{(h,R_0)} \)

For any \( M > 0 \), \( a > 0 \), any positive constant \( R_* > 2M \) and any smooth function \( h: (2M, +\infty) \to [1, +\infty) \) \( \) such that \( h \equiv 1 \) for \( r \leq R_* \) and \( h = O(1) \) as \( r \to +\infty \), we introduce the following metric on \( \mathcal{M}_0 \):

(6.45) \[
\begin{align*}
g_{M,a}^{(h,R_0)}(r,t) = & -\left(1 - \frac{2M}{r} - a^2 M^2 \sin^2 \theta \right) dt^2 - 2M a \cdot h(r) \sin \theta \frac{dr}{r} - \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + \frac{h(r)}{2} \left(\theta^2 + \sin^2 \theta d\varphi^2\right).
\end{align*}
\]

Since \( h \equiv 1 \) on \( \{ r \leq R_* \} \), we have \( g_{M,a}^{(1)} = g_{M,a}^{(h,R_0)} \) on \( \{ 2M < r < R_* \} \), and thus \( g_{M,a}^{(h,R_0)} \) also extends as a smooth metric on \( \mathcal{M}_0 \).

Notice that

(6.46) \[
g_{M,a}^{(1)}(T,T) = g_{M,a}^{(h,R_0)}(T,T)
\]
everywhere on \( \mathcal{M}_0 \), and thus \( (\mathcal{M}_0, g_{M,a}^{(h,R_0)}) \) has the same ergoregion as \( (\mathcal{M}_0, g_{M,a}^{(1)}) \).

The wave operator associated to \( g_{M,a}^{(h,R_0)} \) takes the form:

(6.47) \[
\square g_{M,a}^{(h,R_0)} \psi = -h^{-2} \partial_r \left( h^2 \partial_r \right) \psi + r^{-2} h^{-2} \sin \theta \partial_\theta \left( \sin \theta \partial_\theta \psi \right) + \frac{-\partial_t^2 \psi - 2Mar^{-2} \partial_r \psi + \left(1 - \frac{2M}{r} \right)^{-2} \partial_r^2 \psi}{1 - \frac{2M}{r}}.
\]

Therefore, the wave equation

(6.48) \[
\square g_{M,a}^{(h,R_0)} \psi = 0
\]
separates (as in the case of \( g_{M,a}^{(1)} \)): For any \( (\omega, m, l) \in \mathbb{C} \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|} \), (6.48) admits solutions of the form (6.20), with \( S_{ml}(\theta) \) satisfying (6.21), and \( R_{am\ell}(r) \) satisfying the following ordinary differential equation (compare with (6.25)):

(6.49) \[
u_{am\ell}^\prime + \left( \omega^2 - V_{am\ell,h}^{(h)} \right) u_{am\ell} = 0,
\]

where \( \prime \) denotes differentiation with respect to the \( r_* \) variable (defined by (6.23)), \( u_{am\ell} \) is defined in terms of \( R_{am\ell} \) as

(6.50) \[
u_{am\ell}(r) \equiv h(r) \cdot r R_{am\ell}(r)
\]
and \( V_{am\ell,h}^{(h)} \) is defined as:

(6.51) \[
V_{am\ell,h}^{(h)}(r) \equiv V_{am\ell}(r) + \frac{h''(r)}{h(r)} + 2 \left( 1 - \frac{2M}{r} \right) \frac{h'(r)}{r \cdot h(r)}.
\]
and
\[
V_{\omega ml}(r) = \left(1 - \frac{2M}{r}\right)(h(r))^{-2}(1 + 2aMr^{-1}(h(r))^{-1}\omega m - a^2M^2r^{-4}(h(r))^{-2}m^2 + \left(1 - \frac{2M}{r}\right)\frac{2M}{r^3}.
\]

Similarly, the inhomogeneous wave equation
\[
\square_{\gamma_\omega(r)} \varphi = F
\]
separates as:
\[
u''_{\omega ml} + (\omega^2 - V_{\omega ml}(h))u_{\omega ml} = h \cdot (r - 2M)F_{\omega ml}
\]

In addition to equation (6.49), in the next sections we will also study the behaviour of solutions to the following simplified version of (6.49):
\[
u''_{\omega ml} + (\omega^2 - V_{\omega ml}(h))u_{\omega ml} = 0.
\]

6.6 A generalisation of Theorem 1

Let us fix a large parameter \(R_\infty \gg 1\), and let us introduce the following class of real functions:
\[
\mathcal{B}_{R_\infty} = \{h: \mathbb{R} \rightarrow [1, +\infty) \text{ continuous, such that } h \equiv 1 \text{ on } (-\infty, R_\infty]\}.
\]

In this Section, we will study for any \((\omega, m, l) \in (\mathbb{C}\{0\}) \times \mathbb{Z} \times \mathbb{Z}_{\geq 0}\) with \(Im(\omega) \geq 0\) and any function \(h \in \mathcal{B}_{R_\infty}\) the ordinary differential equation (6.55). Recall that the two variables \(r_*, r\) (which will both be used as arguments for the functions considered in this section) are related by (6.23). We will also set \(\omega = \omega_R + i\omega_I\).

Let us define \(u_{in f}^{(h)}, u_{hor}^{(h)}: \mathbb{R} \rightarrow \mathbb{C}\) to be the unique solutions of (6.55) satisfying the following asymptotic boundary conditions at \(r_* = \pm \infty\) respectively:
\[
\lim_{r_* \rightarrow +\infty} (e^{r_* - r_{in f}}u_{in f}^{(h)}(r_*)) = u_{in f}(+\infty)
\]
\[
\lim_{r_* \rightarrow -\infty} (e^{r_* - r_{hor}}u_{hor}^{(h)}(r_*)) = u_{hor}(-\infty)
\]

for some \(u_{in f}(+\infty), u_{hor}(-\infty) \in \mathbb{C}\{0\}\) (fixed independently of the particular choice of the function \(h\)). Then, the classical theory of ordinary differential equations (see e.g. Chapter 3, Section 3, and Chapter 4, Section 1, of [7]) yields the following stability result for equation (6.55) for any two functions \(h_1, h_2 \in \mathcal{B}_{R_\infty}\) (using also the fact that \(h \geq 1\) for any \(h \in \mathcal{B}_{R_\infty}\)):
\[
e^{\omega_I|r_*|(u_{hor}^{(h_1)} - u_{hor}^{(h_2)}) + (u_{in f}^{(h_1)})' - (u_{in f}^{(h_2)})'(r_*)}\leq C_{\omega ml}\|u_{hor}(-\infty)\| \int_{R_\infty}^{\max\{r_*, R_\infty\}} |V_{\omega ml}(r(x)) - V_{\omega ml}(r(x))| dx,
\]
\[
e^{\omega_I|r_*|(u_{in f}^{(h_1)} - u_{in f}^{(h_2)}) + (u_{in f}^{(h_1)})' - (u_{in f}^{(h_2)})'(r_*)}\leq C_{\omega ml}\|u_{in f}(+\infty)\| \int_{R_\infty}^{\max\{r_*, R_\infty\}} |V_{\omega ml}(r(x)) - V_{\omega ml}(r(x))| dx,
\]
\[
C_{\omega ml}^{-1} \min\{|u_{hor}(-\infty)|, |u_{in f}(+\infty)|\} \leq e^{\omega_I|r_*|(u_{hor}^{(h_1)} + (u_{hor}^{(h_1)})' + (u_{in f}^{(h_1)})' + (u_{in f}^{(h_1)})') \leq C_{\omega ml}\|u_{hor}(-\infty)\| + |u_{in f}(+\infty)|
\]

By repeating the proof of Theorem 1 for equation (6.55) in place of (3.8), and with the boundary conditions (6.57) and (6.58) in place of (4.74) and (4.75), and using Lemma 4.4 in place of 4.3, we readily deduce the following result (the details of the proof are exactly the same and hence will be omitted):
Lemma 6.2. For any function \( h \in B_{R_\infty} \) and any frequency triad \((\omega_R, m, l) \in \mathcal{S}^{(a, M)}_{\text{up}}\), there exist constants \( C_{\omega_R m l, h}, C_{\omega_R m l}^{(0)} \gg 1 \) such that for any \( r_0 \geq C_{\omega_R m l, h} \) and any \( 0 \leq \omega_I \ll 1 \) sufficiently small in terms of \( \omega_R, m, l, r_0 \) and \( R_\infty \), there exists a function \( V^{(h, r_\infty)} : \mathbb{R} \to [0, 2\omega_I^2] \) supported in \([r_0, r_0 + C_{\omega_R m l, h}^{(0)}]\) so that equation

\[
(u^{(h)})'' + \left((\omega_R + i\omega_I)^2 - V^{(h)}_{(\omega_R + i\omega_I)ml}\right)u^{(h)} = 0
\]

admits a non-zero solution \( u^{(h)} \) for which the limits \( \lim_{r_* \to +\infty} (e^{-i(\omega_R + i\omega_I)r_*}u^{(h)}(r_*)) \) and \( \lim_{r_* \to -\infty} (e^{i(\omega_R + i\omega_I)r_*}) \) exist in \( \mathcal{C}\{0\} \).

Furthermore, for any \( h_1, h_2 \in B_{R_\infty} \) and \( \omega_I \) as above, we can bound (provided \( R_\infty \) has been fixed sufficiently large in terms of \( \omega_R, m, l \)):

\[
\int_{r_0}^{r_0 + C_{\omega_R m l}^{(0)}} |V^{(h_1, r_\infty)}(r_*) - V^{(h_2, r_\infty)}(r_*)| \, dr_* \leq C_{\omega_R m l} \int_{R_\infty}^{+\infty} \left| V^{(h_1)}_{\omega_R m l}(r(r_*)) - V^{(h_2)}_{\omega_R m l}(r(r_*)) \right| \, dr_*.
\]

Remark. The estimate (6.63) follows from the estimate (4.66) (for the family \( u^{(s)} \) \( s \leq u^{(h_1) + (1-s)h_2} \)) of Lemma 4.4 in view also of the bounds (6.59)–(6.61).

By inspecting the proof of Theorem 1 we readily infer, in view of the estimates (6.59) and (6.60) (as well as the fact that \( h \geq 1 \) for any \( h \in B_{R_\infty} \)), that the constants \( C_{\omega_R m l, h}, C_{\omega_R m l}^{(0)} \) in the statement of Lemma 6.2 can be chosen independently of \( h \in B_{R_\infty} \) (we will thus use the notation \( C_{\omega_R m l, h}, C_{\omega_R m l}^{(0)} \) from now on).

Finally, let us also note that \( V^{(1, r_\infty)} \) is not identically 0 (since that would imply that the wave equation on the spacetime \((M_0, g_0)\) admits an outgoing real or exponentially growing mode solution, violating the statement of Proposition 6.2). In particular, the proof of Theorem 1 yields the following lower bound for \( V^{(1, r_\infty)} \) (uniform in \( r_0, \omega_I \)) holds:

\[
\int_{r_0}^{r_0 + C_{\omega_R m l}^{(0)}} V^{(1, r_\infty)}(r_*) \, dr_* \geq C_{\omega_R m l} > 0.
\]

Moreover, since any \( h \in B_{R_\infty} \) satisfies \( h \geq 1 \), from (6.52) we deduce that

\[
\lim_{R_\infty \to +\infty} \sup_{R_\infty} \int_{R_\infty}^{+\infty} \left| V^{(h)}_{(\omega_R + i\omega_I)ml}(r(r_*)) \right| \, dr_* = 0
\]

uniformly in \( h \in B_{R_\infty} \) (for fixed \( \omega_R, m, l \) and any \( \omega_I \) sufficiently small in terms of \( \omega_R, m, l \)). Thus, (6.63), (6.64) and (6.65) imply that, provided \( R_\infty \) is sufficiently large in terms of \( \omega_R, m, l \), there exists some \( c_{\omega_R m l} > 0 \) depending only on \( \omega_R, m, l, r_0 \) and \( R_\infty \) such that for any \( h \in B_{R_\infty} \), any \( r_0 > C_{\omega_R m l} \) and any \( \omega_I \) is sufficiently small in terms of \( \omega_R, m, l, r_0 \) and \( R_\infty \):

\[
\int_{r_0}^{r_0 + C_{\omega_R m l}^{(0)}} V^{(h, r_\infty)}(r_*) \, dr_* \geq \int_{r_0}^{r_0 + C_{\omega_R m l}^{(0)}} V^{(1, r_\infty)}(r_*) \, dr_* - C_{\omega_R m l} \int_{R_\infty}^{+\infty} \left( V^{(h)}_{(\omega_R + i\omega_I)ml}(r(r_*)) + V^{(1)}_{(\omega_R + i\omega_I)ml}(r(r_*)) \right) \, dr_* > c_{\omega_R m l} > 0.
\]

6.7 Proof of Theorem 3 and Proposition 6.1

We will now proceed to establish Theorem 3 and Proposition 6.1. In view of (6.45), (6.47), (6.49) and the corresponding properties of the spacetimes \((M_0, g^{(1)}_{M, a})\), we immediately obtain the following properties of \((M_0, g^{(h, R_2)}_{M, a})\) for any \( M > 0, a > 0 \) and \( R > 1 \):

1. In the \((t, r, \vartheta, \varphi)\) coordinate chart on \( M_0 \), the vector fields \( T = \partial_t \) and \( \Phi = \partial_\varphi \) are Killing vector fields for \( g^{(h, R_2)}_{M, a} \).
2. The spacetime \(( \mathcal{M}_0, g_{M,a}^{(h,R_s)})\) is asymptotically conic.

3. The metric \(g_{M,a}^{(h,R_s)}\) can be smoothly extended up to \(\mathcal{H} = \{ r = 2M \}\) in the “Schwarzschild star” coordinate chart on \(\mathcal{M}_0\). In this extension, \(\mathcal{H}\) is the event horizon associated to the asymptotically conic end of \(( \mathcal{M}_0, g_{M,a}^{(h,R_s)})\). Furthermore, \(\mathcal{H}\) is a Killing horizon with positive surface gravity.

4. For any \(\varepsilon > 0\), \(k \in \mathbb{N}\), there exists an \(a_0 > 0\) so that if \(a \leq a_0\), the metrics \(g_{M,a}^{(h,R_s)}\) and \(g_{M,a}\) are \(\varepsilon\)-close in the \(C^k\) norm defined with respect to the “Schwarzschild star” coordinate chart on the closed subset \(\{2M \leq r \leq R_0\}\) of \(\mathcal{M}_0 \cup \mathcal{H}^+\). The ergoregion of \((\mathcal{M}_0, g_{M,a}^{(h)})\) is non-empty, contained in the set \(\{ r \leq 2M + \varepsilon \}\), and

\[
\sup_{\mathcal{M}_{M,0}} g_{M,a}^{(h)}(T,T) < \varepsilon.
\]

5. The wave equation \((6.68)\) completely separates in the \((t, r, \vartheta, \phi)\) coordinate chart. Furthermore, given \((\omega_R, m, l) \in \mathcal{Q}_{sup}^{(a,M)}\), provided \(R_s\) is sufficiently large in terms of \(\omega_R, m, l, a, M\), and, additionally, \(\omega_I\) is sufficiently small in terms of \(\omega_R, m_0, l_0, a, M, R_s\) and \(a \ll M\), the function \(h\) can be chosen appropriately so that the following properties also hold on \((\mathcal{M}_0, g_{M,a}^{(h,R_s)})\) (thus completing the proof of Theorem 3 and Proposition 6.1):

6. The wave equation \((6.48)\) admits an outgoing mode solution with frequency parameter \(\omega_R + \iota \omega_I\).

7. The trapped set of \((\mathcal{M}_0, g_{M,a}^{(h)})\) is normally hyperbolic. Furthermore, any solution \(\psi\) to the inhomogeneous wave equation \((6.53)\) on \((\mathcal{M}_0, g_{M,a}^{(h)})\) which is smooth up to \(\mathcal{H}^+\) satisfies, for any \(\tau_1 \leq \tau_2\), the estimates \((6.4)\) and \((6.5)\).

The statements \([6]\) and \([7]\) above follow directly as corollaries of Propositions \([6.3]\) and \([6.4]\) respectively:

**Proposition 6.3.** For any \(a > 0\), \(M > 0\) and any superradiant frequency triad \((\omega_R, m, l) \in \mathcal{Q}_{sup}^{(a,M)}\), there exists an \(R_s > 2M\) large in terms of \(\omega_R, m, l, a, M\), so that for any \(0 \leq \omega_I \ll 1\) small in terms of \(\omega_R, m, l, R_s\), there exists a smooth and increasing function \(h: \mathbb{R} \to [1, +\infty)\) satisfying \(h \equiv 1\) on \((-\infty, R_s]\) and \(h(r_s) = C_1 + O(r_s^{r_1})\) as \(r_s \to +\infty\), for which the wave equation \((6.48)\) on \((\mathcal{M}_0, g_{M,a}^{(h,R_s)})\) admits an outgoing mode solution with frequency parameter \(\omega_R + \iota \omega_I\).

**Proof.** In view of Lemma 6.2 and the form \((6.51)\) of the potential for \(g_{M,a}^{(h,R_s)}\), in order to construct the function \(h\) so that \((6.48)\) has a mode solution with frequency parameter \(\omega_R + \iota \omega_I\), it suffices to solve the following initial value problem on \(\mathbb{R}\):

\[
\begin{cases}
  h''(r_s) + 2r^{-1}(1 - \frac{2M}{r})h'(r_s) = V^{(h,R_s)}(r_s) \cdot h(r_s), \\
  h(R_s) = 1, \quad h'(R_s) = 0,
\end{cases}
\]

where \(V^{(h,R_s)}: \mathbb{R} \to [0, 2\omega_R^2]\) is the function provided by Lemma 6.2 for \(R_s\) in place of \(r_0\).

**Remark.** Notice that the relation \((6.68)\) is not an ordinary differential equation, since \(V^{(h,R_s)}\) is a non-local (and non-linear) operator acting on the function \(h\). Notice also that, since \(V^{(h,R_s)}\) is supported in \(\{r_s \geq R_s\}\), a solution \(h\) to \((6.68)\) (if it exists) will be identically equal to 1 on \((-\infty, R_s]\).

We will solve \((6.68)\) using a Picard-type iteration scheme, assuming \(R_s\) has been fixed sufficiently large. Let \(h_n: \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}\) be defined by the recursive relation

\[
\begin{cases}
  h_n''(r_s) + 2r^{-1}(1 - \frac{2M}{r})h_n'(r_s) - V^{(h_{n-1},R_s)}(r_s) \cdot h_n(r_s) = 0, \\
  h_n|_{r = R_s} = 1, \quad h_n'|_{r = R_s} = 0,
\end{cases}
\]
with \( h_0 \equiv 1 \). In view of (6.23), equation (6.69) can be rewritten as

\[
(6.70) \quad \begin{cases} 
   r^{-2}(r^2 h'_n) = V^{(h_{n-1}, R_+)} h_n, \\
   h_n|_{r=R_-} = 1, \ h'_n|_{r=R_-} = 0. 
\end{cases}
\]

In order for (6.70) to be well defined, we first need to establish that for all \( n \in \mathbb{N} \), \( h_n \) is a smooth function belonging to the class \( B_{R_-} \), i.e. (6.56). To this end, it suffices to show that \( h_n(r_*) \geq 1 \) for all \( r_* \geq R_+ \). This follows readily by induction: Assuming that \( h_{n-1} \) is smooth and belongs to the class \( B_{R_-} \) (this statement is true for \( h_0 \)), the function \( V^{(h_{n-1}, R_+)} \) is well defined by Lemma 6.2, and thus a unique smooth solution \( h_n : \mathbb{R} \to \mathbb{R} \) of the (inhomogeneous) linear ordinary differential equation (6.70) exists. Since \( V^{(h_{n-1}, R_+)} \geq 0 \), integrating (6.70) yields

\[
(6.71) \quad h'_n \geq 0,
\]

and hence \( h_n(r_*) \geq h_n(R_+) = 1 \) for all \( r_* \geq R_+ \). Since \( V^{(h_{n-1}, R_+)} \) is supported in \([R_+, +\infty)\), \( h_n \) is identically 1 on \((-\infty, R_-]\), in view of the conditions \( h_n|_{r=R_-} = 1, \ h'_n|_{r=R_-} = 0 \), and thus \( h_n \in B_{R_-} \).

By integrating equation (6.70), we obtain the following implicit formula for \( h_n \) for any \( r_* \geq R_+ \):

\[
(6.72) \quad h_n(r_*) = 1 + \int_{R_-}^{r_*} \left( \frac{1}{r^2(\varphi)} \right) \left( \int_{R_-}^{\varphi} r^2(\sigma)V^{(h_{n-1}, R_+)}(\sigma) h_n(\sigma) \ d\sigma \right) d\varphi = 1 + \int_{R_-}^{r_*} r^2(\sigma) \left( \int_{\sigma}^{r_*} r^{-2}(\varphi) d\varphi \right) V^{(h_{n-1}, R_+)}(\sigma) h_n(\sigma) d\sigma.
\]

Let us set

\[
(6.73) \quad R^{(1)}_+ = R_+ + C^{(0)}_{\omega R ml},
\]

where \( C^{(0)}_{\omega R ml} \) is the constant in the statement of Lemma 6.2 for \( h \equiv 1 \) (see the remarks below Lemma 6.2). Since \( V^{(h_{n-1}, R_+)} \) is supported in the interval \([R_+, R^{(1)}_+]\) and \( 0 \leq V^{(h_{n-1}, R_+)} \leq 2\omega R \), an application of Grönwall’s inequality on (6.72) readily yields that there exists some \( C^{(1)}_{\omega R ml} \) depending only on \( \omega, m, l \) (and \( a, M \)), so that for all \( n \in \mathbb{N} \):

\[
(6.74) \quad \max_{r_* \in [R_+, R^{(1)}_+]} \left( |h'_n(r_*)| + |h_n(r_*)| \right) \leq C^{(1)}_{\omega R ml}.
\]

Furthermore, in view of (6.72) and (6.66) (as well as the fact that \( h_n \geq 1 \)), we obtain the following lower bound for \( h'_n(R^{(1)}_+) \) for some \( C^{(1)}_{\omega R ml} > 0 \) depending only on \( \omega, m, l \) (and \( a, M \)):

\[
(6.75) \quad h'_n(R^{(1)}_+) \geq C^{(1)}_{\omega R ml} > 0.
\]

Since \( V^{(h_{n-1}, R_+)} \equiv 0 \) for \( r_* \geq R^{(1)}_+ \), we obtain the following expression for any \( r_* > R^{(1)}_+ \) by integrating (6.70) over \([R^{(1)}_+, r_*]\):

\[
(6.76) \quad h_n(r_*) = h_n(R^{(1)}_+) + \left( R^{(1)}_+ \right)^2 \cdot h'_n(R^{(1)}_+) \int_{R^{(1)}_+}^{r_*} r^{-2}(\varphi) d\varphi.
\]

Thus, in view of (6.74), (6.75), (6.76) and (6.23) (as well as the fact that \( h_n \geq 1 \)), we obtain for any \( r_* \geq R^{(1)}_+ \):

\[
(6.77) \quad 1 + C^{(1)}_{\omega R ml} R^{(1)}_+ \left( \frac{r_* - R^{(1)}_+}{r_*} \right) \leq h_n(r_*) \leq C^{(1)}_{\omega R ml} R^{(1)}_+.
\]

The expression (6.72) yields the following formula for the differences \( h_n - h_{n-1} \) for any \( r_* \geq R_+ \) and any \( n \geq 2 \) (recall that \( h_n - h_{n-1} \equiv 0 \) for \( r_* \leq R_- \)):

\[
(6.78) \quad h_n(r_*) - h_{n-1}(r_*) = \int_{R_-}^{r_*} r^2(\sigma) \left( \int_{\sigma}^{r_*} r^{-2}(\varphi) d\varphi \right) V^{(h_{n-1}, R_+)}(\sigma) (h_n(\sigma) - h_{n-1}(\sigma)) d\sigma + \int_{R_-}^{r_*} r^2(\sigma) \left( \int_{\sigma}^{r_*} r^{-2}(\varphi) d\varphi \right) V^{(h_{n-1}, R_+)}(\sigma) - V^{(h_{n-2}, R_+)}(\sigma) h_{n-1}(\sigma) d\sigma.
\]
Notice that for any \( r_+ \in [R_+, R_+^{(1)}] \) and any \( \sigma \in [R_+, r_+] \), we can bound, provided \( R_+ \gg 1 \) (in view of (6.23)):

\[
(6.79) \quad r^2(\sigma) \left( \int_\sigma^{r_+} r^{-2}(\hat{\varphi}) \, d\hat{\varphi} \right) \leq C(R_+^{(1)} - R_+) \leq C_{\omega_R ml}.
\]

In view of (6.74), (6.79) and the fact that \( V^{(h_+, R_+)} \) is supported in \( [R_+, R_+^{(1)}] \) and satisfies \( 0 \leq V^{(h_+, R_+)} \leq 2\omega_R^2 \) for any \( k \in \mathbb{N} \), from (6.78) we obtain the following estimates for any \( r_+ \in [R_+, R_+^{(1)}] \):

\[
(6.80) \quad |h_n(r_+) - h_{n-1}(r_+)| \leq C_{\omega_R ml} \int_{R_+}^{r_+} |h_n(\sigma) - h_{n-1}(\sigma)| \, d\sigma + C_{\omega_R ml} \int_{R_+}^{r_+} |V^{(h_{n-1}, R_+)}(\sigma) - V^{(h_{n-2}, R_+)}(\sigma)| \, d\sigma
\]

and

\[
(6.81) \quad |h_n'(r_+) - h_{n-1}'(r_+)| \leq C_{\omega_R ml} \int_{R_+}^{r_+} |h_n(\sigma) - h_{n-1}(\sigma)| \, d\sigma + C_{\omega_R ml} \int_{R_+}^{r_+} |V^{(h_{n-1}, R_+)}(\sigma) - V^{(h_{n-2}, R_+)}(\sigma)| \, d\sigma.
\]

Thus, an application of Gronwall’s inequality yields:

\[
(6.82) \quad \sup_{r_+ \in [R_+, R_+^{(1)}]} \left( |h_n(r_+) - h_{n-1}(r_+)| + |h_n'(r_+) - h_{n-1}'(r_+)| \right) \leq C_{\omega_R ml} \int_{R_+}^{R_+^{(1)}} |V^{(h_{n-1}, R_+)}(\sigma) - V^{(h_{n-2}, R_+)}(\sigma)| \, d\sigma.
\]

Moreover, the following bound is a consequence of (6.76) and the fact that \( R_+^{(1)} \leq 2R_+ \) (provided \( R_+ \) is sufficiently large in terms of \( \omega_R, m, l, M, a \)):

\[
(6.83) \quad \sup_{r_+ \in [R_+, R_+^{(1)}]} \left( |h_n(r_+) - h_{n-1}(r_+)| + |h_n'(r_+) - h_{n-1}'(r_+)| \right) \leq C_{\omega_R ml} R_+ \sup_{r_+ \in [R_+, R_+^{(1)}]} \left( |h_n(r_+) - h_{n-1}(r_+)| + |h_n'(r_+) - h_{n-1}'(r_+)| \right).
\]

Fix some \( \varepsilon_{\omega_R ml} > 0 \) small in terms of \( \omega_R, m, l, M, a \). Provided \( R_+ \) is sufficiently large in terms of \( \omega_R, m, l, M, a \) and the specific value of \( \varepsilon_{\omega_R ml} \), we will establish the following bound for the right hand side of (6.82) for any \( n \in \mathbb{N} \):

\[
(6.84) \quad \int_{R_+}^{R_+^{(1)}} |V^{(h_+, R_+)}(r_+) - V^{(h_{n-1}, R_+)}(r_+)| \, dr_+ \leq \varepsilon_{\omega_R ml} \sup_{r_+ \in [R_+, R_+^{(1)}]} \left( |h_n(r_+) - h_{n-1}(r_+)| + |h_n'(r_+) - h_{n-1}'(r_+)| \right).
\]

In view of (6.69) and (6.52) (using also the fact that \( h_n \equiv h_{n-1} \) for \( r_+ \leq R_+ \)), we can estimate (recall that \( h_n \geq 1 \) for all \( n \in \mathbb{N} \)):

\[
(6.85) \quad \int_{R_+}^{R_+^{(1)}} |V^{(h_+, R_+)}(r_+) - V^{(h_{n-1}, R_+)}(r_+)| \, dr_+ \leq C_{\omega_R ml} \int_{R_+}^{R_+^{(1)}} \left( |h_n^2(r_+) - h_{n-1}^2(r_+)| + r_+^{-1} |h_n^2(r_+) - h_{n-1}^2(r_+)| \right) r_+^{-2} \, dr_+ \\
\leq C_{\omega_R ml} \int_{R_+}^{R_+^{(1)}} \frac{|h_n(r_+) - h_{n-1}(r_+)|}{h_n(r_+)h_{n-1}(r_+)} r_+^{-2} \, dr_+.
\]

Notice that we can trivially bound

\[
(6.86) \quad \int_{R_+}^{R_+^{(1)}} \frac{|h_n(r_+) - h_{n-1}(r_+)|}{h_n(r_+)h_{n-1}(r_+)} r_+^{-2} \, dr_+ \leq C_{\omega_R ml} R_+^{-2} \sup_{r_+ \in [R_+, R_+^{(1)}]} |h_n(r_+) - h_{n-1}(r_+)|.
\]

Furthermore, in the region \( r_+ \geq R_+^{(1)} \), in view of the expression (6.76) we can bound

\[
(6.87) \quad |h_n(r_+) - h_{n-1}(r_+)| \leq C \left( 1 + R_+^{(1)} \left( \frac{r_+ - R_+^{(1)}}{r_+} \right) \right) \left( |h_n(R_+^{(1)}) - h_{n-1}(R_+^{(1)})| + |h_n'(R_+^{(1)}) - h_{n-1}'(R_+^{(1)})| \right).
\]
Thus, \((6.87)\) and the left hand side of \((6.77)\) imply that

\[
\int_{R^{(1)}_+} \sup_{r_\ast \in \{R_+, R^{(1)}_+\}} \left( \left| h_n(r_\ast) - h_{n-1}(r_\ast) \right| + \left| h'_n(r_\ast) - h'_{n-1}(r_\ast) \right| \right) d r_\ast \leq C_{\omega Rml} \left( \int_{R^{(1)}_+} r_\ast^{-2} d r_\ast \right) \sup_{r_\ast \in \{R_+, R^{(1)}_+\}} \left( \left| h_n(r_\ast) - h_{n-1}(r_\ast) \right| + \left| h'_n(r_\ast) - h'_{n-1}(r_\ast) \right| \right).
\]

Adding \((6.86)\) and \((6.88)\) and using \((6.85)\), we deduce:

\[
\int_{R^{(1)}_+} \left| V(h_n, R_\ast)(r_\ast) - V(h_{n-1}, R_\ast)(r_\ast) \right| d r_\ast \leq C_{\omega Rml} R^{(1)}_+^{-1} \sup_{r_\ast \in \{R_+, R^{(1)}_+\}} \left( \left| h_n(r_\ast) - h_{n-1}(r_\ast) \right| + \left| h'_n(r_\ast) - h'_{n-1}(r_\ast) \right| \right),
\]

and thus \((6.84)\) follows, provided \(R_+\) has been fixed sufficiently large in terms of \(\omega_R, m, l, M, a\) and the specific value of \(\varepsilon_{\omega Rml}\).

We will now proceed to show (using \((6.84)\)) that the sequence \(\{h_n\}_{n\in\mathbb{N}}\) converges in \(C^1(\mathbb{R})\) to a solution \(h\) of \((6.68)\), with the desired properties. Provided \(\varepsilon_{\omega Rml}\) was chosen sufficiently small in terms of \(\omega_R, m, l, M, a\), from \((6.82)\) and \((6.84)\) we infer that for some \(\theta < 1\) and any \(n \geq 2\):

\[
\sup_{r_\ast \in \{R_+, R^{(1)}_+\}} \left( \left| h_n(r_\ast) - h_{n-1}(r_\ast) \right| + \left| h'_n(r_\ast) - h'_{n-1}(r_\ast) \right| \right) < \theta \sup_{r_\ast \in \{R_+, R^{(1)}_+\}} \left( \left| h_{n-1}(r_\ast) - h_{n-2}(r_\ast) \right| + \left| h'_{n-1}(r_\ast) - h'_{n-2}(r_\ast) \right| \right).
\]

Then \((6.90)\) implies that \(\{h_n|_{\{R_+, R^{(1)}_+\}}\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(C^1([R_+, R^{(1)}_+])\), and thus \((6.83)\) (in view of the fact that \(h_n \equiv 1\) for \(r_\ast \leq R_+\)) yields that \(\{h_n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(C^1(\mathbb{R})\), converging to a function \(h \in C^1(\mathbb{R})\). Notice that we immediately obtain that \(h \in B_{R_\infty}\), since \(B_{R_\infty}\) is a closed subset of \(C^1(\mathbb{R})\). Furthermore, in view of \((6.63)\) and the fact that \(h_n\) converges to \(h\) in the \(C^1\) norm, we obtain that

\[
\lim_{n \to +\infty} \sup_{r_\ast \geq R_+} \int_{\mathbb{R}} \left| V(h_n, R_\ast) - V(h, R_\ast) \right| d \sigma = 0,
\]

and thus, from the expression \((6.72)\) we obtain for any \(r_\ast \geq R_+\):

\[
h(r_\ast) = 1 + \int_{R_+}^{r_\ast} \left( \frac{1}{r^2(\rho)} \int_{R_+}^{\rho} r^2(\sigma) V(h, R_\ast)(\sigma) h(\sigma) d \sigma \right) d \rho.
\]

Thus, \(h\) is a smooth solution of equation \((6.68)\), satisfying \(h \equiv 1\) for \(r_\ast \leq R_+\) and

\[
h(r_\ast) = C_1 + C_2 \int_{R_+}^{r_\ast} r^{-2} d r_\ast
\]

for \(r_\ast \geq R^{(1)}_+\) and some constants \(C_1, C_2\) depending on \(\omega_R, m, l, M, a\) (in view of \((6.76)\)). Furthermore, \(h' \geq 0\) on all of \(\mathbb{R}\), in view of \((6.71)\). \(\blacksquare\)

The following lemma will be used in the proof of Proposition \(6.3\):

**Lemma 6.3.** Let \(h, R_+\) be as in the statement of Proposition \(6.3\). Provided that \(a\) has been fixed sufficiently small in terms of \(M\) (so that Proposition \(6.2\) applies on \(\mathcal{M}_0^{(1)}\)) and \(R_+\) has been fixed sufficiently large in terms of the geometry of \(\mathcal{M}_0^{(1)}\), there exists a constant \(1 < R_- < R_+,\) independent of \(a, R_+\) and large in terms of the geometry of the undeformed spacetime \(\mathcal{M}_0^{(1)}\), such that the following integrated local energy decay estimates hold for solutions \(\psi\) to the inhomogeneous wave equation \((6.53)\) on \(\mathcal{M}_0^{(h, R_\ast)}\) with \(\psi, F\) as in the statement of Proposition \(6.2\):

\[
\int_{\mathcal{M}_0} \left( \chi_{\text{Im3}}(r) \cdot r^{-2} J^N_\mu(\psi) N^\mu + (1 - \chi_{\text{Im3}}(r)) \left| \partial_r \psi \right|^2 + r^{-4} \left| \psi \right|^2 \right) d g_{M,a}^{(h, R_\ast)} \leq C_{R_-} Z[F, \psi; R_-] + C_{R_- R^{(1)}_+} \int_{R_- \leq r_\ast \leq R^{(1)}_+} \min \{ |T\psi|^2, |\psi|^2 \} d g_{M,a}^{(h, R_\ast)},
\]

(6.94)
\[ \int_{\mathcal{M}_0} \left( r^{-2} J^N \psi N^\mu + r^{-3} |\psi|^2 \right) d\mathcal{M}_{M,a} \leq \]
\[ \leq C_{R} Z[F, \psi, R_+] + C_{R} Z[TF, T\psi, R_-] + C_{R_+R_+^{(1)}} \int_{\{r \leq r_+^{(1)}\}} \min \left\{ |T\psi|^2, |\psi|^2 \right\} d\mathcal{M}_{M,a}, \]

where \( Z[F, \psi, R_-] \) is defined as in the statement of Proposition 6.2 and \( R_+^{(1)} \) is defined in terms of \( R_+, h \) as in the proof of Proposition 6.3. The constants \( C_{R_-} \) and \( C_{R_+R_+^{(1)}} \) in the right hand side of (6.94) and (6.95) depend only on the precise choice of \( R_- \) and \( R_+^{(1)} \) respectively, as well as the geometry of the spacetime \( (\mathcal{M}_0, g_{M,a}^{(1)}) \).

Proof. Let \( R_+^{(1)} > 0 \) be any constant sufficiently large in terms of the geometry of \( (\mathcal{M}_0, g_{M,a}^{(1)}) \) as in the statement of Proposition 6.2 assuming without loss of generality that \( R_+ \) is sufficiently large in terms of the geometry of \( (\mathcal{M}_0, g_{M,a}^{(1)}) \) so that \( R_+ \gg (R_+^{(1)})^2 \). Let also \( R_+^{(1)} \) be defined as in the proof of Proposition 6.3.

For any \( (\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_2 \parallel l \parallel \), the frequency separated equation (6.54) is the same as (6.28) in the region \( \{r_+ \leq (R_+^{(1)})^2 \} \) (since \( h \equiv 1 \) there). Furthermore, choosing \( \varepsilon_1 = (R_+^{(1)})^{-1} \) in the proof of Proposition 6.2 if \( f_{\omega ml} \) and \( h_{\omega ml} \) are the seed functions appearing in the proof of Proposition 6.2, then

\[ f_{\omega ml}' = h_{\omega ml} = 0 \]
in the region \( \{r_+ \geq (R_+^{(1)})^2 \} \). Therefore, by repeating the proof of Proposition 6.2 for \( \varepsilon_1 = (R_+^{(1)})^{-1} \), using exactly the same functions \( f_{\omega ml} \) and \( h_{\omega ml} \) as in that proof, we readily obtain the following estimates for the frequency separated equation (6.54) for any \( (\omega, m, l) \in \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}_2 \parallel l \parallel \) (assuming that the limits \( \lim_{r_+ \to \infty} \left( e^{(\omega - \frac{m}{2\pi^2} \parallel l \parallel) R_+ \omega} u_{\omega ml} \right) \) exist):

\[ \int_{-R_+^{(1)}}^{R_+^{(1)}} \left( r^2 |u_{\omega ml}'|^2 + r^{-2} (\omega^2 + l^2 + r - 2) |u_{\omega ml}|^2 \right) dr_+ \leq \]
\[ \leq C_{R_+} \delta_0 m^2 |u_{\omega ml}|(-\infty)|^2 + C \int_{R_+^{(1)}}^{R_+^{(1)}} \left( (R_+^{(1)})^{-1} r^{-2} + r^{-3} \right) \omega_2 |u_{\omega ml}|^2 + C_{R_+R_+^{(1)}} \int_{R_+}^{R_+^{(1)}} |u_{\omega ml}|^2 + \]
\[ + C_{R_+} \int_{-\infty}^{+\infty} \text{Re} \left\{ (r - 2M) h F_{\omega ml} \cdot f_{\omega ml} u_{\omega ml}' + (r^{-1} h_{\omega ml} + i\omega) \bar{u}_{\omega ml} \right\} dr_+, \]

for \( |\omega| \gg l \) or \( |\omega| \ll l \), and

\[ \int_{-R_+^{(1)}}^{R_+^{(1)}} (r^{2} |u_{\omega ml}|^2 + r^{-2} \left( \left\{ 1 - \frac{r_{\omega ml}}{r} \right\} (\omega^2 + l^2) + r - 2 \right) |u_{\omega ml}|^2) \right) dr_+ \leq \]
\[ \leq C_{R_+} \delta_0 m^2 |u_{\omega ml}|(-\infty)|^2 + C \int_{R_+^{(1)}}^{R_+^{(1)}} \left( (R_+^{(1)})^{-1} r^{-2} + r^{-3} \right) \omega_2 |u_{\omega ml}|^2 + C_{R_+R_+^{(1)}} \int_{R_+}^{R_+^{(1)}} |u_{\omega ml}|^2 + \]
\[ + C_{R_+} \int_{-\infty}^{+\infty} \text{Re} \left\{ (r - 2M) h F_{\omega ml} \cdot f_{\omega ml} u_{\omega ml}' + (r^{-1} h_{\omega ml} + i\omega) \bar{u}_{\omega ml} \right\} dr_+, \]

for \( |\omega| \sim l \). Notice that in obtaining (6.97) and (6.98), we used the following one sided bound for the derivative of \( V_{\omega ml;h}^{(b)} \) in the region \( \{r_+ \geq (R_+^{(1)})^2 \} \) (following from the non-negativity condition \( h' \geq 0 \)), provided \( R_+^{(1)} \) is sufficiently large in terms of the geometry of \( (\mathcal{M}_0, g_{M,a}^{(1)}) \):

\[ \left( V_{\omega ml;h}^{(b)} \right)' \geq \begin{cases} 0, & r_+ \in \left[ (R_+^{(1)})^2, R_+ \right] \cup \left[ R_+^{(1)}, +\infty \right) \\ -C_{R_+R_+^{(1)}}, & r_+ \in \left[ R_+, R_+^{(1)} \right]. \end{cases} \]

Observe that the right hand side of (6.99) is independent of the frequency parameters \( (\omega, m, l) \).
From (6.97) and (6.98) we obtain after summing in \(m, l\), integrating in \(\omega\), and using the red shift type estimates of Section 7 of [14], in the region \(\{r_* \leq -R_1^*\}:

\[
\int_{\{r_* \leq -R_1^*\}} \left(\chi_{r=3M}^3(r) \cdot r^{-2} J_\mu^N(\psi) N^\mu + (1 - \chi_{r=3M}^3(r)) |\partial_r \psi|^2 + r^{-4} |\partial_t \psi|^2\right) \, dg_{\pi, a}^{(1)} \leq \nabla
\]

\[
\leq C_{R_1^*} \left(\left. \left| \frac{\hat{Z}[F, \psi; R_1^*] + C \int_{(R_1^* \leq r, \leq (R_1^*)^2]} (R_1^*)^{-1} r^{-2} + r^{-3}) |T| |\psi|^2 \, dg_{\pi, a}^{(1)} + \right. \right. + C_{R_1^*, (R_1^*)} \int_{(R_1^* \leq r, \leq (R_1^*)^2]} |\psi|^2 \, dg_{M, a}^{(h, R_1^*)} \right) 
\]  

(6.100) and

\[
\int_{\{r_* \leq -R_1^*\}} \left(\frac{r^{-2} J_\mu^N(\psi) N^\mu + r^{-4} |\partial_t \psi|^2}{\psi} \right) \, dg_{\pi, a}^{(1)} \leq \nabla
\]

\[
\leq C_{R_1^*} \left(\left. \left| \frac{\hat{Z}[F, \psi; R_1^*] + Z[T F, T \psi; R_1^*] + C \int_{(R_1^* \leq r, \leq (R_1^*)^2]} (R_1^*)^{-1} r^{-2} + r^{-3}) |T| |\psi|^2 \, dg_{\pi, a}^{(1)} + \right. \right. + C_{R_1^*, (R_1^*)} \int_{(R_1^* \leq r, \leq (R_1^*)^2]} |\psi|^2 \, dg_{M, a}^{(h, R_1^*)} \right) 
\]

(6.101) and integrating by parts over \(r\), multiplying (6.102) with \(\hat{\pi} \psi / \psi\), and let us set

\[
(\psi = \psi \cdot \hat{\pi}, \psi)
\]

(6.102)\(h \cdot (r - 2M) F = -\partial_t^2 \psi + \partial_r^2 \psi + \left(1 - \frac{2M}{r}\right) h^{-2} - \frac{2M}{r} \Delta_{\nabla^2} \psi + \frac{2aM r^{-3} h^{-1} \partial_t \partial_r \psi}{r} - a^2 M^2 r^{-6} \partial_r \psi + \left(-\left(1 - \frac{2M}{r}\right) \frac{2M}{r} \frac{h''(r)}{h(r)} + 2\left(1 - \frac{2M}{r}\right) \frac{h'(r)}{r \cdot h(r)} \right) \psi,
\]

where

\[
(6.103)
\]

\[
\psi \equiv \psi \cdot \hat{\pi}, \psi.
\]

Let \(\chi_1 : \mathbb{R} \to [0, 1]\) be a smooth function satisfying \(\chi \equiv 0\) on \((-\infty, \frac{1}{2}\) and \(\chi \equiv 1\) on \([1, +\infty)\), and let us set \(\chi_{R_1^*}(r_* \equiv \chi_1 \left(\frac{r_*}{R_1^*}\right)\). Multiplying (6.102) with

\[
\chi_{R_1^*}(r_* \equiv \chi_1 \left(\frac{r_*}{R_1^*}\right) \cdot \frac{r^n}{1 + r^n} \partial_r \hat{\pi}
\]

(6.104) for some \(0 < \tau_1 < 1\) and integrating by parts over \(M_0\), we obtain, provided \(R_1^*\) is sufficiently large in terms of \(a, M\) (using also the fact that \(h \geq 1\):

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} \frac{1}{2} \chi_{R_1^*}(r_\ast) \left\{ \left(\eta r^{-1 - \eta} + O_{\eta}(r^{-2 - \eta})\right) \left(|\partial_r \psi|^2 + (1 + O_{\eta}(r^{-1})|\partial_t \psi|^2 + (6M r^{-4} + O_{\eta}(r^{-5}))|\psi|^2 + + 2\left(h''(r) - h^{-2} - 2O_{\eta}(r^{-4})\left(|\partial_\theta \psi|^2 + \frac{1 + O(r^{-1})}{\sin^2\theta} |\partial_r \psi|^2\right)\right) \right\} \sin \theta \, d\phi \, d\theta \, dr_\ast \, dt = \nabla
\]

\[
= -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} \left\{ \chi_{R_1^*}(r_\ast) \cdot \frac{r^n}{1 + r^n} Re \left\{ \hat{F} \cdot \partial_r \hat{\psi} \right\} + \chi_{R_1^*}(r_\ast) \left( O(1)|\partial_\theta \psi|^2 + O(r^{-2})|\psi|^2 \right) + + \frac{1}{2} \left( h''(r) - 2\left(1 - \frac{2M}{r}\right) \frac{h'(r)}{r \cdot h(r)} \right) |\psi|^2 \right\} \sin \theta \, d\phi \, d\theta \, dr_\ast \, dt,
\]

where

\[
|\partial_\theta \psi|^2 \equiv |\partial_\theta \psi|^2 + |\partial_\theta \psi|^2 + |\partial_\theta \psi|^2 + \sin^{-2} \theta |\partial_\theta \psi|^2.
\]

(6.106)
Thus, in view of the fact that \( h' \geq 0 \) on \( \mathbb{R} \) and
\[
\frac{h''(r)}{h(r)} + 2\left(1 - \frac{2M}{r}\right)\frac{h'(r)}{r \cdot h(r)} = 0
\]
for \( r \notin [R_*, R_*^{(1)}] \), from (6.105) we obtain (provided \( R_*^{1} \) is sufficiently large in terms of \( \eta \)):
\[
(6.107) \quad \int_{(r \geq R_*)} r^{-1-\eta} |\nabla \psi|^2 \, dg_{M,a}^{(h,R_*)} \leq C_r(\int_{\frac{1}{2} R_*^{1}} r^{-1} |\partial_r \psi|^2 + r^{-3} |\psi|^2) \, dg_{M,a}^{(1)} + \int_{(R_* \leq r \leq R_*^{(1)})} |\psi|^2 \, dg_{M,a}^{(h,R_*)} + Z[F, \psi; R_*].
\]

From (6.100), (6.101) and (6.107), we readily obtain:
\[
\int_{\mathcal{M}_0} \left( r^{-2} J_\mu^n (\psi) N^\mu + (1 - \chi_{r \equiv 3M}(r)) |\partial_r \psi|^2 + r^{-4} |\psi|^2 \right) \, dg_{M,a}^{(h,R_*)} \leq \]
\[
\leq C_{R_*^{1}} Z[F, \psi; R_*] + C_{R_*^{1}} \int_{(\frac{1}{2} R_*^{1} \leq r \leq R_*^{(1)})} |\psi|^2 \, dg_{M,a}^{(h,R_*)}
\]
and
\[
\int_{\mathcal{M}_0} \left( r^{-2} J_\mu^n (\psi) N^\mu + r^{-4} |\psi|^2 \right) \, dg_{M,a}^{(h,R_*)} \leq \]
\[
\leq C_{R_*^{1}} (Z[F, \psi; R_*] + Z[T F, T \psi; R_*]) + C_{R_*^{1}} \int_{(\frac{1}{2} R_*^{1} \leq r \leq R_*^{(1)})} |\psi|^2 \, dg_{M,a}^{(h,R_*)}.
\]

Therefore, inequalities (6.94) and (6.95) readily follow by combining (6.100), (6.101), (6.108) and (6.109).

The statement will now follow as a consequence of the following proposition:

**Proposition 6.4.** Let \( h, R_*, R_+ \) be as in the statement of Lemma 6.3. Provided that \( a \) has been fixed sufficiently small in terms of \( M \), the integrated local energy decay estimates (6.6) and (6.5) hold for any \( \tau_1 \leq \tau_2 \) and any solution \( \psi \) to the inhomogeneous wave equation (6.53) on \( (\mathcal{M}_0, \bar{g}_{M,a}^{(h,R_*)}) \) which is smooth up to \( \mathcal{H}^+ \). In particular, the trapped set of \( (\mathcal{M}_0, \bar{g}_{M,a}^{(h)}) \) is normally hyperbolic.

**Proof.** Let \( \chi_{\text{hor}} : (2M, +\infty) \to [0,1] \) be a smooth function such that \( \chi_{\text{hor}} \equiv 1 \) on \( (2M, \frac{9M}{4}) \) and \( \chi_{\text{hor}} \equiv 0 \) on \( (\frac{10M}{4}, +\infty) \), and let us introduce the vector field
\[
(6.110) \quad \tilde{K} \equiv T + \chi_{\text{hor}} \cdot \frac{a}{8M^2} \Phi.
\]
Notice that \( \tilde{K} \equiv K \) for \( r \leq \frac{9M}{4} \) and \( \tilde{K} \equiv T \) for \( r \geq \frac{10M}{4} \), and furthermore \( \tilde{K} \) is everywhere future directed and timelike on \( \mathcal{M}_{M,0} \) (but merely null on \( \mathcal{H}^+ \)), provided \( a \) is sufficiently small. In particular, \( \tilde{K} \) is a Killing vector field on \( \{ r \leq \frac{9M}{4} \} \cup \{ r \geq \frac{10M}{4} \} \), while on \( \{ \frac{9M}{4} \leq r \leq \frac{10M}{4} \} \) we can bound for any fixed and \( T \)-invariant reference Riemannian metric \( g_{Rm} \) on \( \mathcal{M}_0 \) (provided \( a \) is sufficiently small):
\[
(6.111) \quad |(\tilde{K})_{\pi_{(h,R_*)}^{(g_{Rm})}}|_{g_{Rm}} \leq C_{g_{Rm}} a,
\]
where \( (\tilde{K})_{\pi_{(h,R_*)}^{(g_{Rm})}} \) is the deformation tensor of \( \tilde{K} \) with respect to \( g_{M,a}^{(h,R_*)} \). In view of the energy identity for the vector field \( \tilde{K} \) (and the fact that \( \tilde{K} \) is everywhere causal), the bound (6.111) implies that for any \( \tau_1 \leq \tau_2 \) we can bound (omitting the volume form notation for simplicity)
\[
(6.112) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} \int_{\{ \tau = \tau \}} J_\mu (\psi) \bar{n}^\mu \leq \int_{\{ \tau = \tau_1 \}} J_\mu (\psi) \bar{n}^\mu + C \int_{\{ \frac{9M}{4} \leq r \leq \frac{10M}{4} \}} J_\mu (\psi) \tilde{K}^\mu + C \int_{\{ \tau_1 \leq \tau \leq \tau_2 \}} |\tilde{K} \psi| F.
\]
Let $N$ be a $T$-invariant, smooth and everywhere timelike on $\mathcal{M}_0 \cup \mathcal{H}^+$, such that $N \equiv T$ on $\{r \geq \frac{10M}{3}\}$, and furthermore satisfying on $\{r \leq 2M + c\}$ for some small $c > 0$ depending only on the geometry of $(\mathcal{M}_0, g^{(1)}_{M,a})$ (and independent of $a$, provided $a$ is sufficiently small):

\[(6.113) \quad K^N(\psi) \geq cJ^N_\mu(\psi)N^\mu.\]

Such a vector field $N$ can always be constructed, in view of the fact that the surface gravity of $\mathcal{H}^+$ is positive; see Section 7 of [14]. Notice also that $N$ can be chosen so that in the region $\{2M + c \leq r \leq \frac{10M}{3}\}$ the following bound holds:

\[(6.114) \quad |K^N(\psi)| \leq C|J^N_\mu(\psi)N^\mu|\]

for some constant $C$ independent of $a$ (provided $a$ is sufficiently small), while in the region $\{r \geq \frac{10M}{3}\}$, we have $K^N = K^T = 0$. Thus, the energy identity for $N$ yields for any $\tau_1 \leq \tau_2$:

\[(6.115) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} \int_{\{t \equiv \tau\}} J^N_\mu(\psi)\tilde{n}^\mu + c \int_{\{\tau_1 \leq \tau \leq \tau_2\} \cap \{r \leq 2M + c\}} J^N_\mu(\psi)N^\mu \leq \int_{\{t \equiv \tau_1\}} J^N_\mu(\psi)\tilde{n}^\mu + C \int_{\{2M + c \leq r \leq \frac{10M}{3}\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} J^N_\mu(\psi)\tilde{n}^\mu + C \int_{\{\tau_1 \leq \tau \leq \tau_2\}} |N\psi| |F|.

Using (6.112) (integrated in $t$) to control the second term of the right hand side of (6.115), we readily obtain for any $\tau_1 \leq \tau_2$:

\[(6.116) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} E_N[\psi](\tau) + c \int_{\tau_1}^{\tau_2} E_N[\psi](\tau) \, d\tau \leq E_N[\psi](\tau_1) + C \cdot (\tau_2 - \tau_1)E_N[\psi](\tau_1) + Ca \int_{\tau_1}^{\tau_2} \mathcal{G}[\psi](\tau_1; \tau) \, d\tau + C\mathcal{F}[F, \tau_2, \tau_1],\]

where

\[\mathcal{G}[\psi](\tau_1; \tau) \equiv \int_{\{\frac{10M}{3} \leq r \leq \frac{10M}{3}\} \cap \{\tau_1 \leq \tau \}} J^K_\mu(\psi)\tilde{K}^\mu,\]

\[\mathcal{F}[F, \tau_2, \tau_1] \equiv \int_{\{\tau_1 \leq \tau \leq \tau_2\}} |N\psi| |F| + \int_{\tau_1}^{\tau_2} \left( \int_{\{\tau_1 \leq \tau \leq \tau_2\}} |\tilde{K}\psi| |F| \right) \, d\tau.

From (6.116) and Gronwall’s inequality, we thus infer for any $\tau_1 \leq \tau_2$:

\[(6.117) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} \int_{\{t \equiv \tau\}} J^N_\mu(\psi)\tilde{n}^\mu \leq C \int_{\{t \equiv \tau_1\}} J^N_\mu(\psi)\tilde{n}^\mu + Ca \int_{\{\frac{10M}{3} \leq r \leq \frac{10M}{3}\} \cap \{\tau_1 \leq \tau \leq \tau_2\}} J^K_\mu(\psi)\tilde{K}^\mu + C \int_{\{\tau_1 \leq \tau \leq \tau_2\}} (|\tilde{K}\psi| + |N\psi|) \cdot |F|.

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function, such that $\chi \equiv 1$ on $(-\infty, -1]$ and $\chi \equiv 0$ on $[0, +\infty)$, and let us define for any $\tau_1 \leq \tau_2$ the function $\chi_{\tau_1, \tau_2} : \mathcal{M}_M, 0 \to [0, 1]$:

\[(6.118) \quad \chi_{\tau_1, \tau_2} \equiv \chi(\bar{\tau} - \tau_2)\chi(\tau_1 - \bar{\tau}).\]

Notice that $\chi_{\tau_1, \tau_2} \psi$ satisfies

\[(6.119) \quad \Box_{g^{(h,r_2)}_{M,a}} (\chi_{\tau_1, \tau_2} \psi) = \chi_{\tau_1, \tau_2} F + 2\nabla^\mu \chi_{\tau_1, \tau_2} \nabla_\mu \psi + \left( \Box_{g^{(h,r_2)}_{M,a}} \chi_{\tau_1, \tau_2} \right) \cdot \psi,\]

and both $\chi_{\tau_1, \tau_2} \psi$ and $\Box_{g^{(h,r_2)}_{M,a}} (\chi_{\tau_1, \tau_2} \psi)$ have compact support in the $t^*$ variable. Hence, applying Lemma 6.3 for
\( \chi_{\tau_1, \tau_2} \psi \) in place of \( \phi \), we readily obtain
\[
\int_{\tau_1 \leq \tau \leq \tau_2} \chi_{r=3M(r)} \cdot r^{-2} J^N_{\mu}(\psi) N^\mu + r^{-4} |\phi|^2 \leq
\]
\[
(6.120) \quad \leq C_{R_\tau} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu + C_{R_\tau} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu +
\]
\[
+C_{R_\tau R^{(1)}} \int_{\tau_1 \leq \tau \leq \tau_2} \min \{ |T\psi|^2, |\phi|^2 \}.
\]

Remark. Notice that, in obtaining (6.120) from (6.94) and (6.119), the following estimate is used:
\[
(6.121) \quad \int_0^\tau \nabla^\mu \chi_{\tau_1, \tau_2} \psi + \left( \nabla_{g_{M,a}} \chi_{\tau_1, \tau_2} \right) \cdot \psi - \chi_{\tau_1, \tau_2} \psi \leq C \int_{\tau_1} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu + C_{R_\tau \tau_2} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu +
\]
\[
+C_{R_\tau R^{(1)}} \int_{\tau_1 \leq \tau \leq \tau_2} \min \{ |T\psi|^2, |\phi|^2 \}.
\]

Inequality (6.121) is inferred using integrations by parts in \( \partial \tau \) and \( T \), combined with local-in-time energy estimates for \( \phi \), in the spirit of [11, 15].

From (6.117) and (6.120) we obtain:
\[
(6.122) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} \int_{(\tau_1 \leq \tau)} J^N_{\mu}(\psi) \bar{n}^\mu \leq C \int_{\tau_1} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu + C_{R_\tau a} \int_{\tau_1} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu +
\]
\[
+C_{R_\tau R^{(1)}} \int_{\tau_1 \leq \tau \leq \tau_2} \min \{ |T\psi|^2, |\phi|^2 \}.
\]

Recall that the constant \( R_\tau \) can be chosen independently of \( a \) (provided \( a \) is sufficiently small). Therefore, for \( a \) sufficiently small, the third term of the right hand side of (6.122) can be absobed into the left hand side, yielding:
\[
(6.123) \quad \sup_{\tau_1 \leq \tau \leq \tau_2} \int_{(\tau_1 \leq \tau)} J^N_{\mu}(\psi) \bar{n}^\mu \leq C_{R_\tau} \int_{\tau_1} \int_{\tau_1 \leq \tau \leq \tau_2} J^N_{\mu}(\psi) \bar{n}^\mu + C_{R_\tau R^{(1)}} \int_{\tau_1 \leq \tau \leq \tau_2} \min \{ |T\psi|^2, |\phi|^2 \} +
\]
\[
+C_{R_\tau} \int_{\tau_1 \leq \tau \leq \tau_2} \min \{ |T\psi|^2, |\phi|^2 \}.
\]

From (6.120) and (6.123), we hence obtain the estimate (6.4). Inequality (6.6) then follows by applying the same procedure for \( T\psi \) in place of \( \phi \), and using (6.95) instead of (6.94).

The normal hyperbolicity of the trapped set of \( (\mathcal{M}_0, g_{M,a}) \) can be readily inferred by a simple computation (using the fact that \( h' \geq 0 \)).

\[ \Box_g \psi = 0 \]

7 \ A zero-frequency continuity criterion for decay and its failure in the presence of superradiance

In Section 1.4 we discussed some differences between superradiant and non-superradiant spacetimes concerning the behaviour of solutions to equation (1.14). In the context of this discussion, we will present a specific example of such a difference: As a consequence of Theorem 1.4, we will show that a simple zero-frequency continuity criterion for extending an integrated local energy decay estimate for equation
\[
(7.1) \quad \Box_g \psi = 0
\]
on a non-superradiant spacetime \( (\mathcal{M}, g) \) to the family of equations
\[
(7.2) \quad \Box_g \psi - V_{\lambda} \psi = 0,
\]
where \( V_{\lambda} \) is time independent and depends smoothly on \( \lambda \in [0, 1] \) (with \( V_0 = 0 \)), utterly fails in the superradiant case.
The outline of this section is as follows: In Section 7.1, we will introduce the definition of the resolvent operator associated to the family (7.2) on a general class of stationary and asymptotically flat spacetimes and derive some of its main properties in the case when equation (7.1) satisfies a suitable integrated local energy decay estimate. In Section 7.2, we will state a zero-frequency continuity criterion for integrated local energy decay for the family (7.2), valid on non-superradiant spacetimes. Finally, in Section 7.3, we will show how the analogue of this continuity criterion fails in the presence of superradiance.

7.1 The resolvent operator

Before stating the aforementioned continuity criterion, we will introduce the notion of the resolvent operator for the family (7.2) on a general stationary and asymptotically flat spacetime, and we will state the main properties satisfied by the resolvent operator under the assumption that an integrated local energy decay estimate holds for (7.2) when \( \lambda = 0 \).

Figure 7.1: In the case when \((M, g)\) is the subextremal Kerr exterior spacetime \((M_{M,a}, g_{M,a})\), the intersection of the hypersurfaces \(\tilde{\Sigma}, \Sigma\) and \(S\) with the \(1+1\) dimensional slice \((\theta, \phi) = (\pi/2, 0)\) is schematically as depicted above.

Let \((M^{d+1}, g), d \geq 3\), be a smooth, globally hyperbolic, stationary and asymptotically flat spacetime\(^{10}\) possibly bounded by an event horizon \(\mathcal{H}\) (see \[32\] for the relevant definition). Let \(T\) be the stationary Killing field of \((M, g)\), normalised so that it is future directed in the asymptotically flat region of \((M, g)\), and let \(\Sigma\) be a Cauchy hypersurface of \((M, g)\). We will assume without loss of generality that \(\Sigma\) can be chosen so that \(T\) is everywhere transversal to \(\Sigma\). Let also \(\Sigma\) be a smooth, spacelike hypersurface of \(\mathcal{M}\), which is not necessarily a Cauchy hypersurface of \(\mathcal{M}\), such that \(\Sigma\) differs from \(\tilde{\Sigma}\) only near \(\mathcal{H}\) (and coincides everywhere with \(\Sigma\) in the case \(\mathcal{H} = \emptyset\)) and satisfies \(\mathcal{H} \cap \Sigma \subset I^+(\Sigma)\setminus\tilde{\Sigma}\). For any set \(A \subset \mathcal{M}\), we will denote with \(J_\tau(A)\) the image of \(A\) under the flow of \(T\) for time \(\tau\). We will also fix a \(T\)-invariant globally timelike vector field \(N\) on \((\mathcal{M}, g)\), such that \(N \equiv T\) in the asymptotically flat region of \((\mathcal{M}, g)\).

Let \(t : \mathcal{M} \to \mathbb{R}\) be defined by the condition \(t|_{\Sigma} = 0\) and \(T(t) = 1\) (hence, \(\{t = \tau\} = J_\tau(\Sigma)\)). Notice that in the case of Kerr spacetime \((\mathcal{M}_{M,a}, g_{M,a})\), this is a different function than the Kerr \(t\) coordinate (see Figure 4.1). Let \(V_\lambda : \mathcal{M} \to \mathbb{R}\) be a family of smooth \(T\)-invariant functions supported in the same set \(\cup_{\tau \in \mathbb{R}} J_\tau(K)\), with \(K \subset \Sigma\setminus\mathcal{H}\).

\(^{10}\)The results of this section also apply on asymptotically conic spacetimes without any change.

49
compact, such that $V_t$ depends continuously (in the $C^\infty$ topology) on the parameter $\lambda \in [0, 1]$ and $V_0 = 0$. Finally, let us fix a spacelike hyperboloidal hypersurface $S$ in the future of $\Sigma$, terminating at future null infinity $I^+$ as in Section 3.1 of [33], with $S$ intersecting $\mathcal{H}$ transversally and satisfying $\mathcal{H} \cap S \subset I^+(\Sigma)$ in the case $\mathcal{H} \neq \emptyset$. We will define the function $\tau : \mathcal{M} \to \mathbb{R}$ associated to $S$ by solving $T(\tau) = 1$, $\tau|_S = 0$.

We will assume that for any smooth function $F : \mathcal{M} \to \mathbb{R}$ which is compactly supported when restricted to the $\{t = \text{const}\}$ hypersurfaces, the (unique) smooth solution $\psi$ to

$$
\begin{aligned}
\Box_g \psi &= F, \\
(\psi, T\psi)|_{t=0} &= (0, 0),
\end{aligned}
$$

(7.3)

on $\{t \geq 0\}$ satisfies the following integrated local energy decay estimate (for any $0 < \eta < \frac{1}{2}$) for some $k \in \mathbb{N}$ and any $t_f \geq 0$:

$$
\int_{\{0 \leq t \leq t_f\}} (1 + r)^{-1-2\eta} (J_\mu(\psi) N^\mu + (1 + r)^{-2} |\psi|^2) \, dg \leq \sum_{j=0}^k \int_{\{0 \leq t \leq t_f\}} (1 + r)^{1+2\eta} |T^j F|^2 \, dg.
$$

(7.4)

The integrated local energy decay estimate (7.4) allows us to define the free “resolvent” operator $R(\Box_g; \omega)$ for any $\omega \in \mathbb{C}$ with $\text{Im}(\omega) \geq 0$:

**Proposition 7.1.** Let $(\mathcal{M}, g)$ be a spacetime as above, so that any smooth solution $\psi$ to (7.3) satisfies the integrated local energy decay estimate (7.4). For any $\omega \in \mathbb{C}$ with $\text{Im}(\omega) \geq 0$ and any function $F \in L^2_{cp}(\Sigma)$, there exists a unique function $\varphi \in H^1_{loc}(\Sigma)$, such that the functions $e^{-i\omega t} F$ and $e^{-i\omega t} \varphi$ on $\cup_{t \in \mathbb{R}} J(\Sigma) \simeq \mathbb{R} \times \Sigma$ (where $\cup_{t \in \mathbb{R}} J(\Sigma) \subset \mathcal{M}$ coincides with $\mathcal{M}$ in the case $\mathcal{H} \neq \emptyset$) satisfy

$$
\begin{aligned}
\Box_g(e^{-i\omega t} \varphi) &= e^{-i\omega t} F, \\
\int_S J_\mu N^\mu(e^{-i\omega t} \varphi) n^\mu_S &= 0, \quad \text{and} \quad \lim_{r \to +\infty}(e^{-i\omega t} \varphi)|_{S^0} = 0,
\end{aligned}
$$

(7.5)

where $n^\mu_S$ is the future directed unit normal to the hypersurface $S$. The operator $R(\Box_g; \omega) : L^2_{cp}(\Sigma) \to H^1_{loc}(\Sigma)$ defined by

$$
R(\Box_g; \omega) F \doteq \varphi
$$

(7.6)

is uniformly bounded on $\{\omega \in \mathbb{C} \mid \text{Im}(\omega) \geq 0\}$ with respect to the operator norm

$$
\|R(\Box_g; \omega)\|_{L^2_{cp}(\Sigma)} = \sup_{F \in C_0^\infty(\Sigma)} \frac{\| (1 + r)^{-\frac{1}{2} - \eta} \nabla_{g_E} R(\Box_g; \omega) F \|_{L^2(\Sigma)} + \| (1 + r)^{-\frac{1}{2} - \eta} (|\omega| + (1 + r)^{-1}) R(\Box_g; \omega) F \|_{L^2(\Sigma)}}{(1 + |\omega|^k) \| (1 + r)^{\frac{1}{2} + n} F \|_{L^2(\Sigma)}}
$$

(7.7)

for any $0 < \eta < \frac{1}{2}$ (where $k$ is the same number appearing in the right hand side of (7.4)). Furthermore, $R(\Box_g; \omega)$ is Hölder continuous in $\omega$ for $\text{Im}(\omega) \geq 0$ with respect to the norm

$$
\|R(\Box_g; \omega)\|_{L^2_{cp}(\Sigma)} = \sup_{F \in C_0^\infty(\Sigma)} \frac{\| (1 + r)^{-\frac{1}{2} - \eta} \nabla_{g_E} R(\Box_g; \omega) F \|_{L^2(\Sigma)} + \| (1 + r)^{-\frac{1}{2} - \eta} (|\omega| + (1 + r)^{-1}) R(\Box_g; \omega) F \|_{L^2(\Sigma)}}{(1 + r)^{\frac{1}{2} + n} F \|_{L^2(\Sigma)}}
$$

(7.8)

for any $0 < \eta < \frac{1}{2}$ and any $0 < \eta_0 < \frac{1}{2} - \eta$ and, for any $F, \varphi_0 \in L^2_{cp}(\Sigma)$, the inner product $\langle \varphi_0, R(\Box_g; \omega) F \rangle_{L^2(\Sigma)}$ is a holomorphic function of $\omega$ for $\text{Im}(\omega) > 0$.

**Remark.** In the case of a product Lorentzian metric $g = -dt^2 + g_E$, our definition of the free resolvent operator $R(\Box_g; \omega)$ coincides with the more standard definition of the free resolvent operator $R(\Delta_{g_E}; \omega)$ associated to the time independent operator $\Delta_{g_E}$, appearing for instance in [30].

---

11Here, $L^2_{cp}(\Sigma)$ denotes the subspace of $L^2(\Sigma)$ spanned by functions of compact support.
For the proof of Proposition 7.1 see Section A of the Appendix.

We will now turn to the family of operators (7.2). It can be readily established, through a simple application of Gronwall’s inequality, that there exists a $C_0 > 0$ depending on the geometry of $(\mathcal{M}, g)$ and the family $V_\lambda$, such that, for any $\lambda \in [0, 1]$ and any solution $\psi : \{t \geq 0\} \to \mathbb{C}$ to

\[
(\square_g - V_\lambda)\psi = 0, \\
(\psi, T\psi)|_{t=0} = (\psi_0, \psi_1),
\]

where the initial data $(\psi_0, \psi_1) : \Sigma \to \mathbb{C}^2$ are smooth and compactly supported, we can estimate for any $\tau \geq 0$

\[
(7.10) \quad \int_{(t=\tau)} J^N_\mu(\tau) n^\mu \leq C_0 e^{C_0\tau} \int_{(t=0)} J^N_\mu(\psi) n^\mu.
\]

The bound (7.10), combined with an application of the Fourier–Laplace transformation in the $t$ variable on (7.9), readily yields that, for any $\mathcal{F} \in L^2_{cp}(\Sigma)$ and any $\omega \in \mathbb{C}$ with $Im(\omega) > C_0$, there exists a unique $\varphi \in H^1_{loc}(\Sigma)$ solving

\[
(7.11) \quad \begin{cases}
(\square_g - V_\lambda)(e^{-i\omega t}\varphi) = e^{-i\omega t}\mathcal{F}, \\
\int_{\Sigma} J^N_\mu(e^{-i\omega t}\varphi)n^\mu_\Sigma < +\infty.
\end{cases}
\]

Thus, we can introduce the following definition:

**Definition.** For any $\omega \in \mathbb{C}$ with $Im(\omega) > C_0$ and any $\lambda \in [0, 1]$, we define the operator $R(\square_g - V_\lambda) : L^2_{cp}(\Sigma) \to H^1_{loc}(\Sigma)$ so that, for any $\mathcal{F} \in L^2_{cp}(\Sigma)$, $\varphi = R(\square_g - V_\lambda)\mathcal{F}$ is the unique solution of (7.11).

**Remark.** It can be readily shown that $R(\square_g - V_\lambda)$ is holomorphic in $\omega$ for $Im(\omega) > 0$.

We will now show the following:

**Lemma 7.1.** Let $(\mathcal{M}, g)$ be as in proposition 7.1. For any $\lambda \in [0, 1]$, the operator $R(\square_g - V_\lambda) : L^2_{cp} \to H^1_{loc}$, defined by the relation (7.11), can be extended as a meromorphic operator-valued function of $\omega$ on the whole of $\{\omega \in \mathbb{C} : Im(\omega) > 0\}$, bounded up to $\omega \in \mathbb{R}$ with respect to the norm (7.7), except, possibly, at $\omega = 0$ and at all the values $\omega \in \mathbb{R}\{0\}$ for which equation (7.2) admits an outgoing mode solution with frequency parameter $\omega$ (see the definition at the end of Section 3.3). Furthermore, the poles of $R(\square_g - V_\lambda)$ for $Im(\omega) > 0$ depend continuously on $\lambda$, except at the values of $\lambda$ where they reach the real axis.

**Proof.** As a consequence of Rellich’s embedding theorem, in view also of the boundedness of $R(\square_g) : L^2_{cp} \to L^2_{cp}$ and the compact support in space of $V_\lambda$, the operator $R(\square_g) \circ V_\lambda : H^1_{\omega} \rightarrow H^1_{\omega}$ is bounded and compact when $Im(\omega) \geq 0$. Furthermore, it also follows (in view of Lemma 7.1) that $R(\square_g) \circ V_\lambda$ is holomorphic in $\omega$ when $Im(\omega) > 0$.

Therefore, the Fredholm alternative implies that the operator $(1 - R(\square_g) \circ V_\lambda)^{-1}$ is a meromorphic function of $\omega$ for $Im(\omega) > 0$, bounded up to $\omega \in \mathbb{R}$ with respect to the operator norm $\| \cdot \|_{H^1_{\omega} \rightarrow H^1_{\omega}}$, except at those frequencies $\omega \in \mathbb{R}$ for which

\[
(7.13) \quad \begin{cases}
(\square_g e^{-i\omega t}\varphi) = 0, \\
\int_{\Sigma} J^N_\mu(e^{-i\omega t}\varphi)n^\mu_\Sigma < +\infty \text{ and } \lim_{\tau \to +\infty}(e^{-i\omega t}\varphi)|_S = 0
\end{cases}
\]

admits a non-trivial solution $\varphi \in H^1_{loc}(\Sigma)$. Furthermore, in view of the continuous dependence of $V_\lambda$, the poles of $(1 - R(\square_g) \circ V_\lambda)^{-1}$ for $Im(\omega) > 0$ depend continuously on $\lambda$ (except, of course, at the values of $\lambda$ where they reach the real axis); see [26].

51
In the region $\text{Im}(\omega) > C_0$ where $R(\Box_g - V_\lambda; \omega)$ was defined, the following relation holds:

$$
R(\Box_g - V_\lambda; \omega) = \left( \text{Id} - R(\Box_g; \omega) \circ V_\lambda \right)^{-1} \circ R(\Box_g; \omega). \tag{7.14}
$$

Thus, using the relation (7.14), the operator $R(\Box_g - V_\lambda; \omega)$ can be extended as a meromorphic operator-valued function of $\omega$ on $\{ \omega \in \mathbb{C} : \text{Im}(\omega) > 0 \}$ with the required properties. Furthermore, on $\{ \omega \in \mathbb{C} : \text{Im}(\omega) > 0 \} \setminus \mathcal{P}_\lambda$, where $\mathcal{P}_\lambda \subset \{ \omega \in \mathbb{C} : \text{Im}(\omega) > 0 \}$ are the poles of $R(\Box_g - V_\lambda; \omega)$, the relation (7.14) implies that, for any $\mathcal{F} \in L^{2}_{\omega}(\Sigma)$, the function $\varphi = R(\Box_g - V_\lambda; \omega)\mathcal{F}$ satisfies

$$
\int_{\Sigma} J^N_n(e^{-i\omega t} R(\Box_g; \omega)\mathcal{G})n^{\omega}_{\mu} < +\infty. \tag{7.15}
$$

and the fact that, for any $\mathcal{G} \in L^{2}_{\omega}(\Sigma)$, we have

$$
\int_{\Sigma} J^N_n(e^{-i\omega t} R(\Box_g; \omega)\mathcal{G})n^{\omega}_{\mu} < +\infty. \tag{7.16}
$$

7.2 A zero-frequency continuity criterion for decay for the family (7.2) in the non-superradiant case

Having introduced the resolvent family $R(\Box_g - V_\lambda; \omega)$ and stated its basic properties stemming from the integrated local energy decay estimate (7.4), we can now proceed to examine sufficient zero-frequency conditions on the family $V_\lambda$, i.e. conditions related to the boundedness of $R(\Box_g - V_\lambda; \omega)$ near $\omega = 0$ so that (7.4) also holds for equation (7.17). We will first consider the case when $(\mathcal{M}, g)$ is the product Lorentzian manifold $(\mathbb{R} \times \Sigma, -dt^2 + g_\Sigma)$, where $(\Sigma, g_\Sigma)$ is a complete asymptotically flat Riemannian manifold, with $-dt^2 + g_\Sigma$ having the asymptotics described by Assumption 1 of [32]. In that case, the following result can be readily established:

**Proposition 7.2.** Let $(\mathcal{M}, g) = (\mathbb{R} \times \Sigma, -dt^2 + g_\Sigma)$, with $-dt^2 + g_\Sigma$ having the asymptotics described by Assumption 1 of [32], and let $V_\lambda : \mathcal{M} \to \mathbb{R}$, $\lambda \in [0, 1]$, be as described above. Assume that there exists some (small) $\epsilon > 0$, so that the near-zero frequency bound

$$
\sup_{|\omega| \leq \epsilon} \| R(\Box_g - V_\lambda; \omega) \|_{L,\eta} < +\infty \tag{7.17}
$$

holds for all $\lambda \in [0, 1]$. Then, for any $\lambda \in [0, 1]$ and any smooth $F : \mathcal{M} \to \mathbb{R}$ which has compact support when restricted on the $\{ t = \text{const} \}$ hypersurfaces, the solution $\psi$ to

$$
\Box_g \psi - V_\lambda \psi = F, \quad (\psi, T\psi)|_{\omega = 0} = (0, 0), \tag{7.18}
$$

satisfies the integrated local energy decay estimate (7.4).

**Remark.** The proof of Proposition 7.2 consists of showing that the spectrum of the operator $\Delta_{g_\Sigma} - V_\lambda$ does not obtain a discrete component as $\lambda$ varies in $[0, 1]$. Note that, in case the potentials $V_\lambda$ are assumed to be small, Proposition 7.2 holds without assuming (7.16), even under substantially weaker assumptions on the regularity and decay properties of the potentials $V_\lambda$, see e.g. [27, 37].

For the proof of Proposition 7.2 see Section B of the Appendix. Let us remark that Proposition 7.2 also applies on stationary spacetimes $(\mathcal{M}, g)$, possibly bounded by an event horizon $\mathcal{H}$, where the stationary vector field $T$ is everywhere timelike on $\mathcal{M} \setminus \mathcal{H}$ and $\mathcal{H}$ has positive surface gravity (so that the red-shift estimates of Section 7 of [14] apply).

7.3 Failure of the zero-frequency continuity criterion in the presence of superradiance

Let us now examine the case of a stationary spacetime $(\mathcal{M}, g)$ with a non-empty ergoregion, i.e. $\{ g(T, T) > 0 \} \neq \emptyset$. There is no point in considering the case when either $\mathcal{H}^+ = \emptyset$, or $\mathcal{H}^+ \neq \emptyset$ but $\mathcal{H}^+ \cap \{ g(T, T) > 0 \} = \emptyset$, since, according to [23] (see also our forthcoming [31]), the integrated local energy decay estimate (7.4) does not hold in this case. Therefore, we will only consider spacetimes $(\mathcal{M}, g)$ for which $\mathcal{H}^+ \neq \emptyset$ and $\{ g(T, T) > 0 \} \cap \mathcal{H}^+ \neq \emptyset$. 

52
On such a spacetime \((M, g)\), we will show that the zero-frequency condition of Proposition 7.2 is no longer sufficient to guarantee an integrated local energy decay estimate of the form (7.4) for the family (7.17), \(\lambda \in [0, 1]\). In particular, as a corollary of Theorem 1 we will establish the following:

**Corollary 7.1.** On Kerr exterior spacetime \((M_{M,a}, g_{M,a})\) with \(0 < |a| < M\), for any fixed \((\omega_R, m, l) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{Z} \times \mathbb{Z}_{\geq |m|}\) in the superradiant regime (3.26) and \(r_0 > 1\) as in the statement of Theorem 1, defining \(V : [r_+, +\infty) \rightarrow [0, +\infty)\) as the potential function of Theorem 1 for \(\omega_I = 0\), the family of equations

\[
\square_{g_{M,a}} \psi - \lambda \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)\rho^2} V(r) \psi = 0,
\]

\(\lambda \in [0, 1]\), has the following properties:

1. There exists a \(\lambda_0 \in (0, 1]\) such that (7.18) satisfies an integrated local energy decay estimate for all \(\lambda \in [0, \lambda_0]\).

2. There exists an \(\epsilon > 0\) so that for all \(\lambda \in [0, 1]\) we can bound:

\[
\sup_{|\omega| \leq \epsilon} \|R(\square_{g_{M,a}} - \lambda \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)\rho^2} V(r); \omega)\|_{L^q, n} < +\infty.
\]

3. There exists a \(\lambda_1 \in (0, 1]\) such that (7.18) does not admit an outgoing real or exponentially growing mode solution for \(\lambda < \lambda_1\), but admits a real mode solution for \(\lambda = \lambda_1\). As a consequence, no integrated local energy decay estimate of the form (7.4) holds for (7.18) when \(\lambda = \lambda_1\).

Therefore, the analogue of Proposition 7.2 does not hold on \((M_{M,a}, g_{M,a})\), \(a \neq 0\).

**Proof.** The robustness of the estimates of [15] guarantee that (7.18) satisfies an integrated local energy decay estimate when \(\lambda \in [0, \lambda_0]\) with \(\lambda_0 < 1\), while the near-zero frequency estimates of Section 8.7 of [15] apply to (7.18) for all \(\lambda \in [0, 1]\) (without any change in the proof), in view of the sign condition \(V \geq 0\), yielding (7.19). It thus remains to show the existence of a value \(\lambda_1 \in (0, 1]\) such that (7.18) does not admit an outgoing real or exponentially growing mode solution for \(\lambda < \lambda_1\), but admits a real mode solution for \(\lambda = \lambda_1\). We will also show that this implies that no integrated local energy decay estimate holds for (7.18) when \(\lambda = \lambda_1\).

Let us set for any \(\lambda \in [0, 1]\):

\[
V_\lambda \doteq \lambda \frac{(r^2 + a^2)^2}{(r - r_+)(r - r_-)\rho^2} V(r).
\]

Let \(\lambda_1\) be defined as the infimum of all \(\lambda \in (0, 1]\) for which equation (7.18) admits an outgoing real or exponentially growing mode solution. In view of our aforementioned remarks, it is necessary that \(\lambda_1 > 0\). The estimates of [15] imply that, when \(Re(\omega) >> 1\), the operators \(Id - R(\square_{g_{M,a}}; \omega) \circ V_\lambda\) are invertible on the space \(H^1_0 + r^r(\Sigma)\) uniformly in \(\lambda\) (see the proof of Lemma 7.1 for the relevant notation). Thus, the identity (7.14) (see also the remarks in the proof of Lemma 7.1 on the compactness of \(R(\square_{g}; \omega) \circ V_\lambda\), and dependence of the poles of \(R(\square_{g} - V_\lambda; \omega)\) on \(\lambda\)) implies that, when \(\lambda = \lambda_1\), there exists at least one \(\omega \in \mathbb{R}\) such that the operator \(Id - R(\square_{g_{M,a}}; \omega) \circ V_\lambda\) has non trivial kernel, i.e. equation (7.18) admits an outgoing real mode solution.

For \(\lambda = \lambda_1\) and \(\omega \in \mathbb{R}\) as above, let \(\varphi : M_{M,a} \rightarrow \mathbb{C}\) be a non-trivial \(T\)-invariant function solving

\[
\begin{cases}
(\square_{g_{M,a}} - V_\lambda)(e^{i\omega t} \varphi) = 0, \\
\int_S n_\mu \left( e^{i\omega t} \varphi \right) n^\mu < +\infty \text{ and } \lim_{r \rightarrow +\infty} \left( e^{i\omega t} \varphi \right) |_S = 0,
\end{cases}
\]

where \(S \subset M_{M,a}\) is a spacelike hyperboloidal hypersurface terminating at \(I^+\) and intersecting \(\mathcal{H}^+\backslash\mathcal{H}^-\) transversally, defined as in Section (7.1). We will show that (7.21) implies that no estimate of the form (7.4) can hold for equation (7.18) when \(\lambda = \lambda_1\). Assume, for the sake of contradiction, that (7.4) holds for equation (7.18). Then, the uniqueness part of the proof of Lemma 7.1 (see Section A of the Appendix) applies, without any change, yielding that any function \(\varphi\) satisfying (7.21) must be identically 0; thus, we obtain a contradiction. \(\square\)
The fact that Proposition 7.2 fails to apply on \((\mathcal{M}_{M,a}, g_{M,a})\) highlights that, generally, there is no “cheap” way of controlling the resolvent operator \(R(\Box_g - V; \cdot)\) for frequencies in a neighborhood of the real axis on superradiant spacetimes \((\mathcal{M}, g)\), as is the case on spacetimes with an everywhere timelike Killing field \(T\). Therefore, controlling the behaviour of the resolvent operator \(R(\Box_g - V; \cdot)\) in the near zero frequency regime \(\{|\omega| \ll 1\}\) is not enough to exclude the emergence of “spectral” instabilities for the family \((7.17)\) elsewhere on the real axis as \(\lambda\) varies in \([0, 1]\).

**A Proof of Proposition 7.1**

The proof of Proposition 7.1 will consist of several steps. We will first show that for any \(\mathcal{F} \in L^2_{\text{cp}}(\Sigma)\) and any \(\omega \in \mathbb{C}\) with \(\text{Im}(\omega) \geq 0\), there exists a solution \(\psi \in H^1_{\text{loc}}(\Sigma)\) to (7.5), which is also unique, establishing, thus, that the free resolvent operator \(R(\Box_g; \cdot)\) is well defined. In this step, we will also obtain some useful estimates for the operator \(R(\Box_g; \cdot)\). Then, we will proceed to establish the Hölder continuity and holomorphicity properties of the operator \(R(\Box_g; \cdot)\).

1. Existence

We will first show that, for any \(\mathcal{F} \in L^2_{\text{cp}}(\Sigma)\) and any \(\omega \in \mathbb{C}\) with \(\text{Im}(\omega) \geq 0\), there exists a solution \(\psi \in H^1_{\text{loc}}(\Sigma)\) to (7.5). Let \(\chi : \mathbb{R} \to [0, 1]\) be a smooth function such that \(\chi(t) = 0\) for \(t \leq 0\) and \(\chi(t) = 1\) for \(t \geq 1\), and let us define the function \(\hat{\psi} : \{t \geq 0\} \to \mathbb{C}\) as the unique \(H^1_{\text{loc}}\) solution of the initial value problem

\[
\begin{aligned}
\Box_g \hat{\psi} & = \chi(t) e^{-i\omega t} \mathcal{F}, \\
(\hat{\psi}, T\hat{\psi})|_{t=0} & = (0, 0).
\end{aligned}
\]

(A.1)

In view of the integrated local energy decay estimate (7.4), we can bound for any \(t_f \geq 2\) and any \(0 < \eta < \frac{1}{2}\) after multiplying both sides of (7.4) with \((\int_1^{t_f} e^{2Im(\omega)t} dt)^{-1}\):

\[
(A.2) \quad \left(\int_1^{t_f} e^{2Im(\omega)t} dt\right)^{-1} \int_{\{0 \leq t \leq t_f\}} (1 + r)^{-1-2\eta}(J^N(\psi) N^\mu + (1 + r)^{-2}|\psi|^2) d\Sigma \leq C_{\eta}(1 + |\omega|^k) \int_\Sigma (1 + r)^{1+2\eta}|\mathcal{F}|^2 d\Sigma,
\]

where \(C_{\eta}\) depends on \(\eta\) and \(\sum_{j=0}^k \sup_{t \in [0,1]} |\frac{d^j}{dt^j}(t)|\).

Commuting (A.1) with \(T + i\omega\) and applying (7.4) for the commuted equation (after multiplying, again, both sides of (7.4) with \((\int_1^{t_f} e^{Im(\omega)t} dt)^{-1}\) ), we obtain:

\[
(A.3) \quad \left(\int_1^{t_f} e^{2Im(\omega)t} dt\right)^{-1} \int_{\{0 \leq t \leq t_f\}} (1 + r)^{-1-2\eta}(J^N_{\mu}(T\psi + i\omega\psi)) N^\mu + (1 + r)^{-2}|T\psi + i\omega\psi|^2 d\Sigma \leq \]

\[
\leq C_{\eta}\left(\int_1^{t_f} e^{2Im(\omega)t} dt\right)^{-1} \sum_{j=0}^k \int_{\{0 \leq t \leq t_f\}} (1 + r)^{1+2\eta}|T_j(T + i\omega)(\chi(t)e^{-i\omega t} \mathcal{F})|^2 d\Sigma \leq \]

\[
\leq C_{\eta}\left(\int_1^{t_f} e^{2Im(\omega)t} dt\right)^{-1} \int_\Sigma (1 + r)^{1+2\eta}|\mathcal{F}|^2 d\Sigma,
\]

where we used the fact that \((T + i\omega)(e^{-i\omega t} \mathcal{F}) = 0\) and \(T\chi\) is supported in \(\{0 \leq t \leq 1\}\).

Using the \(r^p\)-weighted energy estimate of Theorem 5.1 of [33] for \(p = 1\), \(\tau_1 = 0\) and \(\tau_2 = t_f\), in conjunction with the integrated local energy decay estimate (7.4), we readily obtain

\[
(A.4) \quad \int_0^{t_f} \left(\int_{\{t = \tau\}} J^N(\psi) n^\mu|_{\Sigma}\right) d\tau \leq \]

\[
\leq C_2(1 + |\omega|^k) \int_{\{0 \leq t \leq t_f\}} (1 + r)^3|e^{-i\omega t} \mathcal{F}|^2 d\Sigma,
\]

54
where \( n_S \) is the future directed unit normal to the \( \{ t = \text{const} \} \) hypersurfaces. Notice that, while \( e^{-\text{mast}} \) is unbounded on the \( \{ t = \text{const} \} \) hypersurfaces when \( Im(\omega) > 0 \), the right hand side of (A.5) is finite, since \( \mathcal{F} \) was assumed to have compact support on \( \Sigma \). In particular, (A.4) implies that, for any \( t_f \geq 2 \):

\[
(A.5) \quad \left( \int_1^{t_f} e^{2Im(\omega)t} dt \right)^{-1} \int_0^{t_f} \left( \int_{\{t=\tau\}} J^N_{\mu}(\psi) n^\mu_S \right) d\tau \leq C_{\chi, \omega, \mathcal{F}}
\]

where \( C_{\chi, \omega, \mathcal{F}} \) depends on \( \chi, \omega, \mathcal{F} \). Similarly, using Theorem 5.3 of [33] for \( p = 2\eta, t_1 = 0 \) and \( t_2 = t_f \), combined with (7.4), we can estimate:

\[
(A.6) \quad \left( \int_1^{t_f} e^{2Im(\omega)t} dt \right)^{-1} \int_{\{0 \leq \tau \leq t_f\}} (1 + r)^{-1+2\eta} \left( \left( \nabla_\mu \bar{T}^\mu \psi \right)^2 + (1 + r)^{-2} |\dot{\psi}|^2 \right) d\gamma \leq \left( \int_1^{t_f} e^{2Im(\omega)t} dt \right)^{-1} \int_{\{0 \leq \tau \leq t_f\}} (1 + r)^{-1+2\eta} |\mathcal{F}|^2 d\Sigma.
\]

Adding (A.2) and (A.6), we thus obtain:

\[
(A.7) \quad \left( \int_1^{t_f} e^{2Im(\omega)t} dt \right)^{-1} \int_{\{0 \leq \tau \leq t_f\}} (1 + r)^{-1+2\eta} \left( \left( \nabla_\mu \bar{T}^\mu \psi \right)^2 + (1 + r)^{-4\eta} J^N_{\mu}(\psi) N^\mu + (1 + r)^{-2} |\dot{\psi}|^2 \right) d\gamma \leq C_{\tau, \chi}(1 + |\omega|^k) \int_\Sigma (1 + r)^{1+2\eta} |\mathcal{F}|^2 d\Sigma.
\]

For any integer \( j \in \{0, \ldots, [t_f]-1\} \), let us define the non-negative quantities

\[
(A.8) \quad f_j = \frac{\int_{\{j \leq \tau \leq j+1\}} (1 + r)^{-1+2\eta} \left( \left( \nabla_\mu \bar{T}^\mu \psi \right)^2 + (1 + r)^{-4\eta} J^N_{\mu}(\psi) N^\mu + (1 + r)^{-2} |\dot{\psi}|^2 \right) d\gamma}{C_{\tau, \chi}(1 + |\omega|^k) \int_\Sigma (1 + r)^{1+2\eta} |\mathcal{F}|^2 d\Sigma} + \frac{\int_{\{j \leq \tau \leq j+1\}} (1 + r)^{-1+2\eta} \left( J^N_{\mu}(T \psi + i \omega \bar{\psi}) N^\mu + (1 + r)^{-2} |\dot{T} \psi + i \omega \dot{\psi}|^2 \right) d\gamma}{C_{\tau, \chi}(t_f, t_f) e^{2Im(\omega)t}} + \frac{\int_{\{j \leq \tau \leq j+1\}} (1 + r)^{1+2\eta} |\mathcal{F}|^2 d\Sigma}{C_{\tau, \chi}(t_f, t_f) e^{2Im(\omega)t}},
\]

and

\[
(A.9) \quad g_j = \int_j^{j+1} e^{2Im(\omega)t} dt,
\]

where the denominators of the terms in the right hand side of (A.8) are precisely the right hand sides of inequalities (A.7), (A.3) and (A.5) respectively. Then, the inequalities (A.7), (A.3) and (A.5) imply that, for any \( t_f \geq 2 \):

\[
(A.10) \quad \left( \sum_{j=1}^{[t_f]-1} g_j \right)^{-1} \sum_{j=1}^{[t_f]-1} f_j \leq 3.
\]

An application of the pigeonhole principle on the relation (A.10) yields that there exists a \( j_0 \in \{1, \ldots, [t_f]-1\} \) such that

\[
(A.11) \quad \frac{f_{j_0}}{g_{j_0}} \leq 3.
\]

Therefore, we deduce that, for any \( t_f \geq 2 \), there exists a \( t_{in} = t_{in}(t_f) \in [1, t_f - 1] \) such that

\[
(A.12) \quad \left( \int_{t_{in}}^{t_{in}+1} e^{2Im(\omega)t} dt \right)^{-1} \int_{\{t_{in} \leq \tau \leq t_{in}+1\}} \int_{\{0 \leq \tau \leq t_f\}} (1 + r)^{-1+2\eta} \left( \left( \nabla_\mu \bar{T}^\mu \psi \right)^2 + (1 + r)^{-4\eta} J^N_{\mu}(\psi) N^\mu + (1 + r)^{-2} |\dot{\psi}|^2 \right) d\gamma \leq C_{\tau, \chi}(1 + |\omega|^k) \int_\Sigma (1 + r)^{1+2\eta} |\mathcal{F}|^2 d\Sigma;
\]

55
In view of (A.19) and the fact that (A.21)

Since the functions (A.13) and (A.14)

We also define the sequence of functions

noting that (A.1) (and the fact that (A.17) and (A.18)) implies that, as \( n \to \infty \), we can bound for any \( 0 \leq t \leq 1 \):

The bound (A.18) implies that, as \( n \to +\infty \), there exists a subsequence of \( \{ \psi_n \}_{n \in \mathbb{N}} \) (assuming, without loss of generality, that this subsequence is in fact the whole sequence \( \{ \psi_n \}_{n \in \mathbb{N}} \)) converging weakly in \( H^1_{\text{loc}}(\{0 \leq t \leq 1\}) \) to a function \( \hat{\psi} \) satisfying the bound:

Since the functions \( \psi_n \) satisfy (A.17) and \( \psi_n \) converge weakly in \( H^1_{\text{loc}} \) to \( \hat{\psi} \), we readily infer that \( \hat{\psi} \) also satisfies (A.17). In view of (A.19) and the fact that \( t_{f,n} \to +\infty \) as \( n \to +\infty \), we infer that

(A.22)

\[ T\hat{\psi} + i\omega \hat{\psi} = 0, \]
i.e. $\hat{\psi}$ is of the form

(A.23) \[ \hat{\psi} = e^{-iat} \varphi \]

for some $\varphi \in H_{1oc}^1(\Sigma)$. We will extend $\hat{\psi}$ on the whole of $\cup_{t \in \mathbb{R}} \mathcal{J}_t(\Sigma)$ by the relation (A.23).

Since the functions $\psi_n$ satisfy (A.17) and $\{\psi_n\}_{n \in \mathbb{N}}$ converges weakly in $H_{1oc}^1(\{0 \leq t \leq 1\})$ to $\hat{\psi}$, we readily infer that $\hat{\psi}$ also satisfies (A.17) on $\{0 \leq t \leq 1\}$. Since $\hat{\psi}$ was extended on the whole of $\cup_{t \in \mathbb{R}} \mathcal{J}_t(\Sigma)$ by the relation (A.23) and the metric $g$ is $T$-invariant, we infer that $\hat{\psi}$ satisfies (A.17) on the whole of $\cup_{t \in \mathbb{R}} \mathcal{J}_t(\Sigma)$. Thus, the function $\varphi$ satisfies (7.5), with the condition that $e^{-iat} \varphi$ having finite energy flux through $\mathcal{S}$ and $\lim_{t \to +\infty} (e^{-iat} \varphi)|_{\mathcal{S}}$ being a direct consequence of (A.20). Furthermore, $\varphi$ satisfies

(A.24) \[ \int_{\Sigma} (1 + r)^{-1-2r} \left( |\nabla g \varphi|^2_{g_{\Sigma}} + (|\varphi|^2 + (1 + r)^{-2})|\varphi|^2 \right) d\mathcal{S} \leq C_{\varphi} (1 + |\varphi|^4) \int_{\Sigma} (1 + r)^{1+2r} |\mathcal{F}|^2 d\mathcal{S} \]

and

(A.25) \[ \int_{\Sigma} (1 + r)^{-1+2r} \left( |\nabla g \varphi|^2_{g_{\Sigma}} + (|\varphi|^2 + (1 + r)^{-2})|\varphi|^2 \right) d\mathcal{S} \leq C_{\varphi} (1 + |\varphi|^4) \int_{\Sigma} (1 + r)^{1+2r} |\mathcal{F}|^2 d\mathcal{S} \]

in view of (A.21) (and the fact that $T$ is everywhere transversal to $\Sigma$).

2. Uniqueness

Having established the existence of a solution $\varphi \in H_{1oc}^1(\Sigma)$ to (7.5), we will now proceed to show that this solution is unique. In particular, we will show that, for any $\omega \in \mathbb{C}$ with $\text{Im}(\omega) \geq 0$, if $\varphi \in H_{1oc}^1(\Sigma)$ satisfies

(A.26) \[
\begin{align*}
\square_g (e^{-iat} \varphi) &= 0, \\
\int_{\mathcal{S}} \mathcal{J}^N (e^{-iat} \varphi) n_{\mathcal{S}}^k < +\infty \text{ and } \lim_{t \to +\infty} (e^{-iat} \varphi)|_{\mathcal{S}} = 0,
\end{align*}
\]

then $\varphi \equiv 0$.

For any $0 < a \leq 1$, we will introduce the auxiliary function

(A.27) \[ \varphi_a = \chi(t) e^{at} e^{-iat} \varphi. \]

In view of (A.26), as well as the fact that $\chi \equiv 0$ for $t \leq 0$, $\varphi_a$ satisfies

(A.28) \[
\begin{cases}
\square_g \varphi_a = \mathcal{A}_{\text{cut}} + \mathcal{A}_{\text{exp}}, \\
(\varphi_a, T\varphi_a)|_{t=0} = (0,0),
\end{cases}
\]

where

(A.29) \[ \mathcal{A}_{\text{cut}} \doteq 2\chi'(t) \nabla g t \nabla \mu (e^{at} e^{-iat} \varphi) + \square_g (\chi(t)) e^{at} e^{-iat} \varphi \]

and

(A.30) \[ \mathcal{A}_{\text{exp}} \doteq 2ae^{at} \nabla g t \nabla \mu (e^{-iat} \varphi) + (a^2 \nabla g t \nabla \mu t + a \square_g t) e^{at} e^{-iat} \varphi. \]

In view of the fact that

(A.31) \[ \bar{t} \leq t - \frac{1}{2} r - C, \]
we can estimate for any \( \tau \geq 0 \) and any \( \beta > 0 \):

\[
\sup_{t=\tau} (1+r)^2 e^{2\alpha t} \leq C_\beta a^{-\beta} e^{2\alpha \tau},
\]

where \( C_\beta > 0 \) depends only on \( \beta \). In view of the flat asymptotics of \( g \) and the fact that \( \mathcal{F} \) is compactly supported, the conditions \( \text{Im}(\omega) \geq 0 \) and \( \sum_{j=1}^2 \int_S j_\mu^N (e^{i\omega t} \dot{\varphi}) n^S \) imply that there exists a \( \beta > 0 \) such that

\[
\int_\Sigma \left( (1+r)^{-\beta} |\nabla_\Sigma \varphi|^2 + (1+r)^{-\beta-2} |\varphi|^2 \right) < +\infty
\]

(note that, in the case \( \text{Im}(\omega) > 0 \), \( \varphi \) in fact belongs to the space \( H^1(\Sigma) \); the bound [A.33] becomes non-trivial only when \( \text{Im}(\omega) = 0 \)). Therefore, [A.29], [A.32] and [A.33], combined with the fact that \( \chi \equiv 0 \) for \( t \geq 1 \), imply that, for any \( \tau \geq 0 \):

\[
\sum_{j=1}^k \int_{\{t=\tau\}} (1+r)^2 |T_j A_{\text{cut}}|^2 \, dg_S \leq \begin{cases} C_{\chi, \varphi} a^{-\beta-2}, & 0 \leq \tau \leq 1, \\ 0, & \tau \geq 1, \end{cases}
\]

where \( C_{\chi, \varphi} > 0 \) depends on \( \chi, \omega, \varphi \).

The finiteness of the \( J^N \)-energy flux of \( e^{-i\alpha t} \varphi \) through \( S \), combined with a Hardy type inequality, implies that, for any \( \tau \geq 0 \):

\[
\int_{\{t=\tau\}} \left( |\nabla_\mu \vec{T}_j^\mu (e^{-i\alpha t} \varphi)|^2 + (1+r)^{-2} |e^{-i\alpha t} \varphi|^2 \right) \, dg_S \leq C e^{2\text{Im}(\omega)\tau} \int_S J_\mu^N (e^{-i\alpha t} \varphi) n_\mu^S < +\infty.
\]

Since \( \text{Im}(\omega) > 0 \) and \( S \subset J^+(\Sigma) \), from [A.35] we can deduce the following estimate on the slice \( \{t=\tau\} \) for any \( \tau \geq 0 \):

\[
\int_{\{t=\tau\}} \left( |\nabla_\mu \vec{T}_j^\mu (e^{-i\alpha t} \varphi)|^2 + (1+r)^{-2} |e^{-i\alpha t} \varphi|^2 \right) \, dg_S \leq C e^{2\text{Im}(\omega)\tau} \int_S J_\mu^N (e^{-i\alpha t} \varphi) n_\mu^S < +\infty.
\]

Therefore, on \( \{t \geq 0\} \), from [A.30] and [A.36], as well as the fact that

\[
|\Box_g \varphi| \leq C(1+r)^{-1}
\]

(following from the flat asymptotics of (\( \mathcal{M}, g \)) and the definition of the hyperboloidal hypersurface \( S \), see also Section 3.1 of [33]), we infer that, for any \( \tau \geq 0 \) and any \( 0 < \eta < \frac{1}{2} \):

\[
\sum_{j=0}^k \int_{\{t=\tau\}} (1+r)^{1+2\eta} |T_j A_{\text{exp}}|^2 \, dg_S \leq C_{\chi, \varphi} a^{1-2\eta} e^{2(\alpha+1\text{Im}(\omega))\tau}.
\]

Applying the integrated local energy decay estimate [7.4] for [A.28], we obtain for any \( t_f \geq 0 \) in view of [A.34] and [A.38]:

\[
\int_{0 \leq t \leq t_f} (1+r)^{-1-2\eta} \left( J_\mu^N (\varphi_\alpha) N^\mu + (1+r)^{-2} |\varphi_\alpha|^2 \right) \, dg_S \leq C_{\chi, \varphi} (a^{1-2\eta} e^{2(\alpha+1\text{Im}(\omega))t})^t_0.
\]

Thus, in view of the relation [A.27], [A.39] yields for any \( 0 < a \leq 1 \), any \( t_f \geq 0 \) and any \( 0 < \eta < \frac{1}{2} \):

\[
\int_\Sigma (1+r)^{-3-2\eta} |\varphi|^2 \, dg_S \leq C_{\chi, \varphi} \left( e^{a(\alpha+1\text{Im}(\omega))t} \right)^t_0 + a^{1-2\eta}.
\]

Choosing \( t_f = a^{-1} \) in [A.40] and letting \( a \to 0 \), we thus infer that \( \varphi \equiv 0 \). Hence, we have established the uniqueness for solutions to (7.5), and, thus, the resolvent operator \( R(\Box_g; \omega) \) given by (7.6) is well defined, with \( R(\Box_g; \omega) \) being uniformly bounded on \( \{\omega \in \mathbb{C} | \text{Im}(\omega) \geq 0\} \) with respect to the norm (7.7) in view of [A.24]. Furthermore, for any \( \mathcal{F} \in L^2_{\chi, \varphi}(\Sigma) \), \( \varphi = R(\Box_g; \omega)\mathcal{F} \) satisfies the bound [A.25].

58
3. Hölder continuity

We will now show that the operator $R(\omega; \psi)$ is Hölder continuous as a function of $\omega \in \mathbb{C}$ with $\text{Im}(\omega) \geq 0,$ with respect to the norm (7.8). For any given $F \in L^2_{\text{loc}}(\Sigma)$ and any $\omega_1, \omega_2$, let us set $\varphi_1 = R(\omega_1; \psi)$ and $\varphi_2 = R(\omega_2; \psi)$.

We will show that, for any $0 < \eta < \frac{1}{2}$, $0 < \tau_0 < \frac{1}{2} - \eta$ (assuming without loss of generality that $|\omega_1 - \omega_2| < 1$):

\[
\int_{\Sigma} (1 + r)^{-1-2\eta}(|\nabla_{g_{\Sigma}} (\varphi_1 - \varphi_2)|^2 + (|\omega_1|^2 + (1 + r)^{-2})|\varphi_1 - \varphi_2|^2) \, dg_{\Sigma} \leq \leq C_{\eta_0 \lambda}\int_{\Sigma} (1 + r)^{1+2\tau} e^{2|\text{Im}(\omega_1)| (1 + |\omega_1|^k)} (1 + |\omega_1|^k) \int_{\Sigma} (1 + r)^{1+2\tau_0} |F|^2 \, dg_{\Sigma}.
\]

For any $0 < a \leq 1$, we define

\[
\varphi_a = \chi(t)e^{a(t - \tau_0)}(e^{-i\omega_1 t} - e^{-i\omega_2 t}),
\]

noticing that $\varphi_a$ satisfies

\[
\int_{\Sigma} (1 + r)^{-1-2\eta}(|\nabla_{g_{\Sigma}} (\varphi_1 - \varphi_2)|^2 + (|\omega_1|^2 + (1 + r)^{-2})|\varphi_1 - \varphi_2|^2) \, dg_{\Sigma} \leq C_{\eta_0 \lambda}\int_{\Sigma} (1 + r)^{1+2\tau} e^{2|\text{Im}(\omega_1)| (1 + |\omega_1|^k)} (1 + |\omega_1|^k) \int_{\Sigma} (1 + r)^{1+2\tau_0} |F|^2 \, dg_{\Sigma}.
\]

and

\[
\int_{\Sigma} (1 + r)^{-1-2\eta}(|\nabla_{g_{\Sigma}} (\varphi_1 - \varphi_2)|^2 + (|\omega_1|^2 + (1 + r)^{-2})|\varphi_1 - \varphi_2|^2) \, dg_{\Sigma} \leq C_{\eta_0 \lambda}\int_{\Sigma} (1 + r)^{1+2\tau} e^{2|\text{Im}(\omega_1)| (1 + |\omega_1|^k)} (1 + |\omega_1|^k) \int_{\Sigma} (1 + r)^{1+2\tau_0} |F|^2 \, dg_{\Sigma}.
\]

The relation (A.42) implies that

\[
\varphi_a = \chi(t)e^{a(t - \tau_0)}(e^{-i\omega_1 t} - e^{-i\omega_2 t})\varphi_2.
\]
Thus, (A.48) and (A.49) yield

\[(A.50)\]

\[
\int_{\{1 \leq t \leq T\}} (1 + r)^{-1-2\eta} e^{2\alpha t} \left( J^N_{\mu} (e^{-i\omega_1 t} (\varphi_1 - \varphi_2)) N^{\mu} + (1 + r)^{-2} |e^{-i\omega_1 t} (\varphi_1 - \varphi_2)|^2 \right) dt \leq 
\]

\[
\leq C_N \int_{\{0 \leq t \leq T\}} (1 + r)^{-1-2\eta} e^{2\alpha t} \left( J^N_{\mu} ((e^{-i\omega_1 t} - e^{-i\omega_2 t}) \varphi_2) N^{\mu} + (1 + r)^{-2} (e^{-i\omega_1 t} - e^{-i\omega_2 t}) \varphi_2^2 \right) dt + 
\]

\[
+ C_{\varphi_2} \left\{ \int_{0 \leq t \leq T} \chi(t)(1 + r)^{1+2\eta} e^{2\alpha t} \left( |\omega_1 e^{-i\omega_1 t} - \omega_2 e^{i\omega_2 t}| \right)^2 dt 
\]

\[
+ \sum_{j=1}^{2} \left( a^{2\eta_0} \int_{0}^{\eta} e^{2(\alpha + \tau \omega_1)(\tau+1)} dt \right) (1 + |\omega_j|^k) \int_{\Omega} (1 + r)^{1+2\eta_0} |F|^2 d\gamma \right\}.
\]

From (A.50), we readily obtain using the bound (A.24) for \( \varphi_2 \) for the first term in the right hand side:

\[(A.51)\]

\[
(1 + r)^{-1-2\eta} \left( \sum_{j=1}^{2} \left( a^{2\eta_0} \int_{0}^{\eta} e^{2(\alpha + \tau \omega_1)(\tau+1)} dt \right) (1 + |\omega_j|^k) \right) \int_{\Omega} (1 + r)^{1+2\eta_0} |F|^2 d\gamma \leq 
\]

and thus:

\[(A.52)\]

\[
(1 + r)^{-1-2\eta} \left( \sum_{j=1}^{2} \left( a^{2\eta_0} \int_{0}^{\eta} e^{2(\alpha + \tau \omega_1)(\tau+1)} dt \right) (1 + |\omega_j|^k) \right) \int_{\Omega} (1 + r)^{1+2\eta_0} |F|^2 d\gamma \leq 
\]

Therefore, choosing \( t_f = a^{-1-\eta_0} \) and \( a = |\omega_1 - \omega_2|^{-1-4\eta_0} \), assuming without loss of generality that \( |\omega_1 - \omega_2| < 1 \), (A.52) readily yields the required Hörder continuity estimate:

\[(A.53)\]

\[
(1 + r)^{-1-2\eta} \left( \sum_{j=1}^{2} \left( a^{2\eta_0} \int_{0}^{\eta} e^{2(\alpha + \tau \omega_1)(\tau+1)} dt \right) (1 + |\omega_j|^k) \right) \int_{\Omega} (1 + r)^{1+2\eta_0} |F|^2 d\gamma \leq 
\]

and thus:

\[4. \text{ Holomorphicity}\]

Finally, we will show that for any \( 0 < \eta < \frac{1}{2} \) and any \( F, \varphi_0 \in L^2_{cp}(\Sigma) \), the inner product \( \langle \varphi_0, R(\varphi_0, \omega) F \rangle_{L^2(\Sigma)} \) is a holomorphic function of \( \omega \) when \( \text{Im} (\omega) > 0 \). This can be readily established using the classical Morera’s theorem,
since, as we showed, \( \langle \varphi_0, R(\Box_g; \omega) \mathcal{F} \rangle_{L^2(\gamma)} \) is continuous in \( \omega \), and the right hand side of (7.5) vanishes upon complex integration over any piecewise smooth closed loop \( \gamma \in \{ \omega \in \mathbb{C} : \text{Im}(\omega) > 0 \} \) (and, hence, the same arguments leading to the uniqueness of solutions to (7.5) show that \( \int_\gamma R(\Box_g; \omega) \mathcal{F} \, d\omega \) also vanishes).

**B Proof of Proposition 7.2**

In order to establish the integrated local energy decay estimate (7.4) for equation (7.17), we will first show that the condition (7.16) implies that, for all \( \lambda \in [0, 1] \), \( R(\Box_g - V_{\lambda}; \cdot) \) has no poles in the half plane \( \{ \omega : \text{Im}(\omega) > 0 \} \), and the following bound holds on the strip \( \{0 \leq \text{Im}(\omega) \leq \frac{\pi}{2}\} \):

(B.1) \[
\sup_{0 \leq \text{Im}(\omega) \leq \frac{\pi}{2}} \|R(\Box_g - V_{\lambda}; \omega)\|_{L^\infty} < +\infty.
\]

The effective limiting absorption principles established in [38] imply that, provided \( \varepsilon > 0 \) is sufficiently small depending on the precise choice of the family \( V_{\lambda} \), the following non-zero real frequency bound holds for all \( \lambda \in [0, 1] \):

(B.2) \[
\sup_{\{0 \leq \text{Im}(\omega) \leq \frac{\pi}{2}\} \cap \{0 \leq |\omega| \leq \varepsilon\}} \|R(\Box_g - V_{\lambda}; \omega)\|_{L^\infty} < +\infty.
\]

Thus, (7.16) and (B.2) yield (B.1) for all \( \lambda \in [0, 1] \). The non-existence of poles for \( R(\Box_g - V_{\lambda}; \cdot) \) in the half plane \( \{ \omega : \text{Im}(\omega) > 0 \} \) follows readily from the following facts:

1. The poles of \( R(\Box_g - V_{\lambda}; \cdot) \) in \( \{ \omega : \text{Im}(\omega) > 0 \} \) vary continuously with \( \lambda \), except when reaching the real axis (see Lemma 7.1).
2. \( R(\Box_g - V_0; \cdot) = R(\Box_g; \cdot) \) has no poles in \( \{ \omega : \text{Im}(\omega) > 0 \} \).
3. The bound (B.1) guarantees that no poles of \( R(\Box_g - V_{\lambda}; \cdot) \) exist in \( \{0 \leq \text{Im}(\omega) \leq \varepsilon\} \) for all \( \lambda \in [0, 1] \).
4. There exists some \( C \gg 1 \) depending on the family \( V_0 \) so that \( R(\Box_g - V_{\lambda}; \cdot) \) has no poles in the region \( \{\text{Im}(\omega) > C\} \).

This follows from the fact that the all poles of \( R(\Box_g - V_{\lambda}; \cdot) \) in \( \{\omega \in \mathbb{C} : \text{Im}(\omega) > 0\} \) must lie on the imaginary semi-axis \( \{\omega = ia, a > 0\} \) (in view of the fact that \( \Delta g_\varepsilon - V_0 \) is essentially self-adjoint), combined with the fact that \( R(\Box_g - V_{\lambda}; \cdot) \) is holomorphic in the region \( \text{Im}(\omega) > 1 \) (see the definition and the remark below (7.11)).

We will now proceed to establish the integrated local energy decay estimate (7.4). The bound (B.1) readily implies (after an application of the Fourier transform in the \( t \)-variable) the following bound for any smooth \( \tilde{\psi} : \mathcal{M} \rightarrow \mathbb{C} \) such that the restriction of \( \tilde{\psi} \) on the \( \{t = \text{const}\} \) hypersurfaces is compactly supported and both \( \tilde{\psi} \) and its first derivatives are square integrable in \( t \):

(B.3) \[
\int_{\mathcal{M}} (1 + r)^{-1 - 2\varepsilon}(J^T_{\mu} (\tilde{\psi}) T^\nu + r_+^{-2} |\tilde{\psi}|^2) \leq \lambda \sum_{j=0}^k \int_{\mathcal{M}} (1 + r)^{1 + 2\varepsilon} |T^j(\Box_g \tilde{\psi})|^2.
\]

Furthermore, the bound (B.1) combined with the absence of poles for \( R(\Box_g - V_0; \cdot) \) in the upper half plane imply that for all \( \lambda \in [0, 1] \):

(B.4) \[
\inf_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Sigma (|\nabla g_\varepsilon \varphi|^2 + V_0 |\varphi|^2) \, dg_{\Omega}}{\int_{\text{supp}(V_0)} |\varphi|^2 \, dg_{\Sigma}} > 0.
\]

The lower bound (B.4), in turn, implies that the \( T \)-energy flux (associated to the problem (7.17), \( \lambda \in [0, 1] \)) of any smooth and suitably decaying function \( \psi : \mathcal{M} \rightarrow \mathbb{R} \), is positive definite on the \( \{t = \text{const}\} \) hypersurfaces, i.e. for any \( s \in \mathbb{R} \):

(B.5) \[
\int_{\{t = s\}} (J^T_{\mu} (\psi) \tilde{n}^\mu + r_+^{-2} |\tilde{\psi}|^2) \leq \lambda \int_{\{t = s\}} (J^T_{\mu} (\psi) \tilde{n}^\mu + V_0 |\tilde{\psi}|^2).
\]
Remark. The lower bound \([B.4]\) can be obtained as follows: Assume that \((B.4)\) fails to hold for some \(\lambda \in [0,1]\), then there exists a smooth and compactly supported function \(V_{\delta,2}: \Sigma \to (-\infty,0]\) with \(V_{\delta,2} = -1\) on \(\text{supp}(V_\delta)\), such that for all \(\delta > 0:\)

\[
\inf_{\varphi \in C_0^\infty(\Sigma)} \frac{\int_{\Sigma} (|\nabla g_\xi \varphi|^2 + |V_\delta + \delta V_{\delta,2}|^2 |\varphi|^2) \, dg_{\Sigma}}{\int_{\text{supp}(V_\delta)} |\varphi|^2 \, dg_{\Sigma}} \leq -\delta < 0.
\]

Thus, in view of the compactness of the support of \(V_\delta + \delta V_{\delta,2}\), a standard minimization argument (see e.g. \([35]\)) yields that for any \(\delta > 0\), there exists a \(\lambda_\delta > 0\) and an \(L^2\) solution \(\varphi_\delta\) to the eigenvalue problem:

\[
\Delta_{g_\xi} \varphi_\delta - (V_\delta + \delta V_{\delta,2}) \varphi_\delta = \lambda_\delta^2 \varphi_\delta,
\]

i.e. \(R(\square_g - V_\delta - \delta V_{\delta,2}; \cdot)\) has a pole at \(\omega = \partial_\delta\) for any \(\delta > 0\). However, for \(\delta > 0\) sufficiently small, the estimate \((B.1)\) also holds for \(R(\square_g - V_\delta - \delta V_{\delta,2}; \cdot)\) in place of \(R(\square_g - V_\delta; \cdot)\) \(^{12}\) and thus our previous analysis establishing the absence of resonances in the upper half plane for the family \(R(\square_g - V_\delta; \cdot)\) (with parameter \(\lambda\)) also applies for the family \(R(\square_g - V_\delta - \delta V_{\delta,2}; \cdot)\) (with parameter \(\delta\)), yielding a contradiction.

Let, now, \(\psi\) be a smooth solution to \((7.17)\) for some \(\lambda \in [0,1]\) and some smooth function \(F: \mathcal{M} \to \mathbb{C}\), such that \(\text{supp}(F) \subset \{t \geq 0\}\) (so that \(\psi \equiv 0\) for \(t \leq 0\)). We will assume without loss of generality that \(F\) is compactly supported in \(\mathcal{M}\), and we will show that

\[
\int_{(t \geq 0)} (1 + r)^{-1-2\nu}(J^N_\mu(\psi))N^\mu + (1 + r)^{-2} |\varphi|^2 \, dg \leq \epsilon \sum_{j=0}^k \int_{(t \geq 0)} (1 + r)^{1+2\nu} |T^j F|^2 \, dg.
\]

The estimate \((7.4)\) for any \(t_f \geq 0\) and any function \(F\) which does not necessarily have compact support in the \(t\) variable (but with \(\text{supp}(F) \cap \{t = \tau\}\) being compact for any \(\tau \geq 0\)) can be obtained from \((B.3)\) as follows: Fixing a smooth function \(\chi_1: \mathbb{R} \to [0,1]\) such that \(\chi_1 \equiv 1\) on \((-\infty,-1]\) and \(\chi_1 \equiv 0\) on \([0,\infty)\), let us define for any \(t_f \geq 0\) the function \(\psi_{t_f}: \mathcal{M} \to \mathbb{C}\) by solving

\[
\begin{cases}
(\square_g - V_\delta)\psi_{t_f} = \chi_1(t-t_f)F, \\
(\psi_{t_f}, T\psi_{t_f})|_{t=0} = (0,0).
\end{cases}
\]

Note that \(\psi \equiv \psi_{t_f}\) on \(\{0 \leq t \leq t_f - 1\}\) in view of \((7.17)\). Since \(\chi_1(t-t_f)F\) has compact support in \(\mathcal{M}\), an application of \((B.8)\) for \(\psi_{t_f}\) yields:

\[
\int_{(t \geq 0)} (1 + r)^{-1-2\nu}(J^N_\mu(\psi_{t_f}))N^\mu + (1 + r)^{-2} |\varphi_{t_f}|^2 \, dg \leq \epsilon \sum_{j=0}^k \int_{(0 \leq t \leq t_f)} (1 + r)^{1+2\nu} |T^j F|^2 \, dg.
\]

Furthermore, the domain of dependence property for \((7.17)\) combined with local-in-time energy estimates and a Cauchy–Schwarz inequality readily yield:

\[
\int_{(t_f-1 \leq t \leq t_f)} (1 + r)^{-1-2\nu}(J^N_\mu(\psi))N^\mu + (1 + r)^{-2} |\varphi|^2 \, dg \lesssim \\
\leq \int_{(t_f-2 \leq t \leq t_f-1)} (1 + r)^{-1-2\nu}(J^N_\mu(\psi))N^\mu + (1 + r)^{-2} |\varphi|^2 \, dg + \\
+ \int_{(t_f-1 \leq t \leq t_f)} (1 + r)^{1+2\nu} |F|^2 \, dg.
\]

Therefore, since \(\psi = \psi_{t_f}\) on \(\{0 \leq t \leq t_f - 1\}\), \((7.4)\) can be readily obtained from \((B.10)\) and \((B.11)\).

\(^{12}\)In view of the fact that \(R(\square_g - V_\delta - \delta V_{\delta,2}) = (1 - R(\square_g - V_\delta) \circ \partial_\delta V_{\delta,2})^{-1} R(\square_g - V_\delta)\)
We will now proceed to establish (B.8) when $F$ is compactly supported in $\mathcal{M}$. Let $\chi : \mathbb{R} \to (0, 1]$ be a smooth function such that $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = e^{-x}$ for $x \geq 1$, and let us define for any $\delta > 0$ the function $\chi_{\delta} : \mathcal{M} \to (0, 1]$ by the relation:

\begin{equation}
\chi_{\delta} = \chi(\delta \cdot) \tag{B.12}
\end{equation}

Commuting \((7.17)\) with $T^j$, $j \leq \lfloor \frac{d-1}{2} \rfloor$, the energy flux identity (for any $\tau > 0$)

\begin{equation}
\int_{\{\tilde{t} = \tau\}} (J^T_{\mu}(T^j \phi) \bar{n}^\mu + V_\phi |T^j \phi|^2) = -2 \int_{\{0 \leq \tilde{t} \leq \tau\}} T^{j+1} \phi \cdot T^j F \tag{B.13}
\end{equation}

combined with elliptic estimates (using equation \((7.17)\)) and the Sobolev embedding theorem, implies that $\phi$ is uniformly bounded on $\mathcal{M}$. Therefore, the function $\chi_{\delta} \phi$ is square integrable in the $t$ variable. In view of this fact and the relation

\begin{equation}
\Box_g (\chi_{\delta} \phi) - V_\phi \chi_{\delta} \phi = \chi_{\delta} F + 2 \theta^\mu \chi_{\delta} \cdot \partial_\mu \phi + \Box_g \chi_{\delta} \cdot \phi \tag{B.14}
\end{equation}

from \((B.3)\) (for $\chi_{\delta} \phi$ in place of $\tilde{\phi}$ there) and \((B.12)\) we obtain \(^{13}\)

\begin{equation}
\int_{\mathcal{M}} \left( 1 + r \right)^{-1-2\xi} \left( J^T_{\mu}(\chi_{\delta} \phi) T^\mu + (1 + r)^{-2} |\chi_{\delta} \phi|^2 \right) \leq \lambda_{\gamma, \delta} \tag{B.15}
\end{equation}

where, for some fixed $R \gg 1$ and any $0 < p \leq 2$:

\begin{equation}
\mathcal{E}^{(p)}[\phi](\tau) \leq \int_{\{\tilde{t} = \tau\} \cap \{r \leq R\}} \left( r^p |\partial_r \phi|^2 + r^{p-2} |\phi|^2 \right), \tag{B.16}
\end{equation}

\begin{equation}
\mathcal{E}_{cn}[\phi](\tau) \leq \int_{\{\tilde{t} = \tau\} \cap \{r \leq R\}} \left( J^T_{\mu}(\phi) \bar{n}^\mu + (1 + r)^{-2} |\phi|^2 \right), \tag{B.17}
\end{equation}

the $\partial_r$-derivative in \((B.16)\) being considered with respect to the (polar) coordinate chart $(\tilde{t}, r, \sigma)$ in the region $\{r \geq R\}$.

In order to obtain \((B.8)\) for $\phi$ from \((B.15)\), it suffices to show that the second term of the right hand side of \((B.15)\) converges to 0 as $\delta \to 0$. The $r^p$-weighted estimates of Section 5 of [33], for $p = 2$, yield for any $\tau > 0$ (provided $R$ is sufficiently large):

\begin{equation}
\mathcal{E}^{(2)}[\phi](\tau) + \int_{-\infty}^{\tau} \mathcal{E}^{(1)}[\phi](s) \, ds \leq \int_{\mathcal{M}} (1 + r)^{3} |F|^2 + \int_{\{\tilde{t} \leq \tau\} \cap \{r \leq R\}} \left( J^T_{\mu}(\phi) T^\mu + |\phi|^2 \right), \tag{B.18}
\end{equation}

In view of the $T$-energy flux identity \((B.13)\) and the lower bound \((B.5)\), we obtain from \((B.18)\):

\begin{equation}
\mathcal{E}^{(2)}[\phi](\tau) + \int_{-\infty}^{\tau} \mathcal{E}^{(1)}[\phi](s) \, ds \leq C(F)(1 + \tau), \tag{B.19}
\end{equation}

where $C(F) > 0$ depends on the precise choice of $F$. From \((B.19)\) and the compact support of $F$ (yielding $\mathcal{E}^{(p)}[\phi](\tau) = 0$ for $\tau \ll 1$) we deduce that for any $\tau > 0$

\begin{equation}
\int_{-\infty}^{\tau} \mathcal{E}^{(2)}[\phi](s) \, ds + \tau \int_{-\infty}^{\tau} \mathcal{E}^{(1)}[\phi](s) \, ds \leq C(F)(1 + \tau)^2, \tag{B.20}
\end{equation}

\(^{13}\)Notice the bound $|\Box_g \chi_{\delta}| \leq \delta^2 (1 + r)^{-1} \min \{e^{-\delta \cdot}, 1\}$, which follows from the fact that $\Box_g \chi_{\delta} = -2 \partial_t \partial_r \chi_{\delta} + \partial_r^2 \chi_{\delta} + (d-1) r^{-1} \partial_r \chi_{\delta} + r^{-2} \Delta_{g_{\mu \nu}, \delta} \chi_{\delta} + O(r^{-1}) \{\partial^2 \chi_{\delta}, \partial_r \chi_{\delta}\}$ in the polar coordinate chart $(\tilde{t}, r, \sigma)$ in the region $r \gg 1$.\[^{13}\]
and thus a standard interpolation argument yields the following qualitative bound for $\psi$:

(B.21) \[ \int_{-\infty}^{\infty} \mathcal{E}^{(1+2\gamma)}[\psi](s) \leq C(F)(1+\tau)^{1+2\gamma}. \]

Therefore, for any integer $0 \leq j \leq k$, setting for simplicity

(B.22) \[ f_j(\tau) \equiv \mathcal{E}^{(1+2\gamma)}[T^j\psi](s) + \mathcal{E}_{en}[T^j\psi], \]

in view of (B.21) and (B.13), we can bound for any integer $0 \leq j \leq k$, any $\delta > 0$ and any $s_0 \gg 1$:

(B.23) \[ \int_0^{+\infty} \delta^2 e^{-\delta s} f_j(s) \, ds \leq \int_0^{s_0} \delta^2 f_j(s) \, ds + \int_{s_0}^{+\infty} \delta^2 e^{-\delta s} f_j(s) \, ds \leq C(F)
\left( \delta^2 s_0^{1+2\gamma} + \int_{s_0}^{+\infty} e^{-\delta s} \delta^2 s^2 \, ds \right) \leq C(F) \left( \delta^2 s_0^{1+2\gamma} + e^{-\delta s_0} (\delta s_0^2 + \delta^{-1}) \right). \]

Choosing $s_0 = \delta^{-1} \frac{2\gamma(1+2\gamma)}{1+\gamma}$ in (B.23), we obtain as $\delta \to 0$ (since $2\gamma < 1$)

(B.24) \[ \lim_{\delta \to 0} \int_0^{+\infty} \delta^2 e^{-\delta s} f_j(s) \, ds = 0. \]

Thus, letting $\delta \to 0$, (B.24) yields the desired integrated local energy decay estimate (B.8). \hfill \square

\section*{C \ A topological lemma}

We will establish the following lemma on the image of a continuous family of maps from the unit ball to itself:

\textbf{Lemma C.1.} Let $\mathcal{F} : [0,1] \times B_1^n \to B_1^n$ be a continuous map, where $B_1^n$ is the open ball of radius $\rho$ in $\mathbb{R}^n$. Assume also that $\mathcal{F}([0] \times \cdot) : B_1^n \to B_1^n$ is a homeomorphism onto an open neighborhood of $0_{\mathbb{R}^n}$ and that for any $t \in [0,1]$ we have

(C.1) \[ \mathcal{F}([t] \times (B_1^n \setminus B_{1/2}^n)) \subset B_1^n \setminus \{0\}. \]

Then for any $t \in [0,1]$:

(C.2) \[ 0_{\mathbb{R}^n} \in \mathcal{F}([t] \times B_1^n). \]

\textbf{Proof.} Because of (C.1), for any $t \in [0,1]$ the map $\mathcal{F}([t] \times \cdot) : B_1^n \to B_1^n$ induces a well defined group homomorphism

(C.3) \[ \mathcal{F}_{\text{hom}}(t) : H_n(B_1^n, B_1^n \setminus B_{1/2}^n) \to H_n(B_1^n, B_1^n \setminus \{0_{\mathbb{R}^n}\}), \]

where $H_n(A, B)$ is the $n$-th reduced homology group of $A$ relative to $B \subset A$ (see [23]). In this case, $H_n(B_1^n, B_1^n \setminus B_{1/2}^n) \cong \mathbb{Z} \cong H_n(B_1^n, B_1^n \setminus \{0_{\mathbb{R}^n}\})$.

Because $\mathcal{F}$ is continuous, the map (C.3) is continuous in $t$ and, hence, since its domain and range are discrete, it is constant in $t$. Because $\mathcal{F}([0] \times \cdot) : B_1^n \to B_1^n$ is a homeomorphism onto an open neighborhood of $0_{\mathbb{R}^n}$, $\mathcal{F}_{\text{hom}}(0)$ is non-trivial, and thus (C.3) is also non-trivial for any $t \in [0,1]$. This implies that (C.2) holds, since if $0_{\mathbb{R}^n} \notin \mathcal{F}([t] \times B_1^n)$ then $\mathcal{F}_{\text{hom}}(t)$ is identically 0. Thus, the proof of the Lemma is complete. \hfill \square
References

[1] L. Andersson and P. Blue. Hidden symmetries and decay for the wave equation on the Kerr spacetime. *Annals of Mathematics*, 182(3):787–853, 2015.

[2] P. Blue and A. Soffer. Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates. *Advances in Differential Equations*, 8(3):595–614, 2003.

[3] P. Blue and J. Sterbenz. Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Communications in Mathematical Physics*, 268(2):481–504, 2006.

[4] B. Carter. Global structure of the Kerr family of gravitational fields. *Physical Review*, 174(5):1559, 1968.

[5] B. Carter. Hamilton-Jacobi and Schrödinger separable solutions of Einstein’s equations. *Communications in Mathematical Physics*, 10(4):280–310, 1968.

[6] S. Chandrasekhar. *Selected Papers, Volume 6: The Mathematical Theory of Black Holes and of Colliding Plane Waves*, volume 6. University of Chicago Press, 1991.

[7] W. A. Coppel. *Stability and asymptotic behavior of differential equations*. Heath, 1965.

[8] M. Dafermos and I. Rodnianski. A proof of Price’s law for the collapse of a self-gravitating scalar field. *Inventiones mathematicae*, 162(2):381–457, 2005.

[9] M. Dafermos and I. Rodnianski. A note on energy currents and decay for the wave equation on a Schwarzschild background. *arXiv preprint arXiv:0710.0171*, 2007.

[10] M. Dafermos and I. Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Communications on Pure and Applied Mathematics*, 62(7):859–919, 2009.

[11] M. Dafermos and I. Rodnianski. Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a|\ll M$ or axisymmetry. *arXiv preprint arXiv:1010.5132*, 2010.

[12] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In *XVIIth International Congress on Mathematical Physics*, pages 421–432, 2010.

[13] M. Dafermos and I. Rodnianski. A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. *Inventiones mathematicae*, 185(3):467–559, 2011.

[14] M. Dafermos and I. Rodnianski. Lectures on black holes and linear waves. In *Evolution equations*, *Clay Mathematics Proceedings*, volume 17, pages 97–205, 2013.

[15] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman. Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a|<M$. *Annals of Mathematics*, 183(3):787–913, 2016.

[16] Th. Damour, N. Deruelle, and R. Ruffini. On quantum resonances in stationary geometries. *Lettere al Nuovo Cimento*, 15(8):257–262, 1976.

[17] S. Detweiler. Klein–Gordon equation and rotating black holes. *Physical Review D*, 22(10):2323, 1980.

[18] S. Dolan. Instability of the massive Klein–Gordon field on the Kerr spacetime. *Physical Review D*, 76(8):084001, 2007.

[19] S. Dolan. Superradiant instabilities of rotating black holes in the time domain. *Physical Review D*, 87(12):124026, 2013.

[20] S. Dyatlov. Quasi-normal modes and exponential energy decay for the Kerr-de Sitter black hole. *Communications in Mathematical Physics*, 306(1):119–163, 2011.
[21] S. Dyatlov. Asymptotics of linear waves and resonances with applications to black holes. *Communications in Mathematical Physics*, 335(3):1445–1485, 2015.

[22] F. C. Eperon, H. S. Reall, and J. E. Santos. Instability of supersymmetric microstate geometries. arXiv preprint arXiv:1607.06828, 2016.

[23] J. L. Friedman. Ergosphere instability. *Communications in Mathematical Physics*, 63(3):243–255, 1978.

[24] A. Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[25] S. W. Hawking. Black holes in general relativity. *Communications in Mathematical Physics*, 25(2):152–166, 1972.

[26] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag Berlin Heidelberg, 1966.

[27] T. Kato. Wave operators and similarity for some non-selfadjoint operators. In *Contributions to Functional Analysis*, pages 258–279. Springer, 1966.

[28] B. S. Kay and R. M. Wald. Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Classical and Quantum Gravity*, 4(4):893, 1987.

[29] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical review letters*, 11(5):237, 1963.

[30] R. B. Melrose. *Geometric scattering theory*, volume 1. Cambridge University Press, 1995.

[31] G. Moschidis. A proof of Friedman’s ergosphere instability for scalar waves. preprint.

[32] G. Moschidis. Logarithmic local energy decay for scalar waves on a general class of asymptotically flat spacetimes. *Annals of PDE*, 2:5, 2016. doi:10.1007/s40818-016-0010-8.

[33] G. Moschidis. The $r^p$-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications. *Annals of PDE*, 2:6, 2016. doi:10.1007/s40818-016-0011-7.

[34] F. Odeh. Note on differential operators with a purely continuous spectrum. *Proceedings of the American Mathematical Society*, pages 363–366, 1965.

[35] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: Vol. 1: Functional Analysis*. Academic press, 1972.

[36] F. Rellich. Über das asymptotische Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 53:57–65, 1943.

[37] I. Rodnianski and W. Schlag. Time decay for solutions of schrödinger equations with rough and time-dependent potentials. *Inventiones mathematicae*, 155(3):451–513, 2004.

[38] I. Rodnianski and T. Tao. Effective limiting absorption principles, and applications. *Communications in Mathematical Physics*, 333:1–95, 2015.

[39] Y. Shlapentokh-Rothman. Exponentially growing finite energy solutions for the Klein-Gordon equation on sub-extremal Kerr spacetimes. *Communications in Mathematical Physics*, 329(3):859–891, 2013.

[40] Y. Shlapentokh-Rothman. Quantitative mode stability for the wave equation on the Kerr spacetime. *Annales Henri Poincaré*, pages 1–57, 2013.

[41] D. Tataru and M. Tohaneanu. A local energy estimate on Kerr black hole backgrounds. *International Mathematics Research Notices*, 2011(2):248–292, 2011.

[42] M. Walker and R. Penrose. On quadratic first integrals of the geodesic equations for type $\{22\}$ spacetimes. *Communications in Mathematical Physics*, 18(4):265–274, 1970.
[43] C. Warnick. On quasinormal modes of asymptotically anti-de Sitter black holes. *Communications in Mathematical Physics*, 333(2):959–1035, 2015.

[44] B. F. Whiting. Mode stability of the Kerr black hole. *Journal of Mathematical Physics*, 30(6):1301–1305, 1989.

[45] T. Zouros and D. Eardley. Instabilities of massive scalar perturbations of a rotating black hole. *Annals of physics*, 118(1):139–155, 1979.