NONCONVEX SET INTERSECTION PROBLEMS:
FROM PROJECTION METHODS TO THE NEWTON METHOD
FOR SUPER-REGULAR SETS

C.H. JEFFREY PANG

Abstract. The problem of finding a point in the intersection of closed sets can be solved by the method of alternating projections and its variants. It was shown in earlier papers that for convex sets, the strategy of using quadratic programming (QP) to project onto the intersection of supporting halfspaces generated earlier by the projection process can lead to an algorithm that converges multiple-term superlinearly. The main contributions of this paper are to show that this strategy can be effective for super-regular sets, which are structured nonconvex sets introduced by Lewis, Luke and Malick. Manifolds should be approximated by hyperplanes rather than halfspaces. We prove the linear convergence of this strategy, followed by proving that superlinear and quadratic convergence can be obtained when the problem is similar to the setting of the Newton method. We also show an algorithm that converges at an arbitrarily fast linear rate if halfspaces from older iterations are used to construct the QP.

CONTENTS

1. Introduction 1
2. Preliminaries 4
3. Basic local convergence for super-regular SIP 5
4. Connections with the Newton method 11
5. An algorithm with arbitrary fast linear convergence 14
6. Two step SHQP 19
7. Global strategies 21
8. Conclusion 23
References 23

1. INTRODUCTION

For finitely many closed sets $K_1, \ldots, K_m$ in $\mathbb{R}^n$, the Set Intersection Problem (SIP) is stated as:

\[(\text{SIP}): \text{Find } x \in K := \bigcap_{i=1}^{m} K_i, \text{ where } K \neq \emptyset. \quad (1.1)\]

Date: June 30, 2015.
2010 Mathematics Subject Classification. 90C30, 90C55, 47J25.
Key words and phrases. super-regularity, supporting halfspaces, quadratic programming, alternating projections.
One assumption on the sets $K_i$ is that projecting a point in $\mathbb{R}^n$ onto each $K_i$ is a relatively easy problem.

A popular method of solving the SIP is the Method of Alternating Projections (MAP), where one iteratively projects a point through the sets $K_i$ to find a point in $K$. For more on the background and recent developments of the MAP and its variants, we refer the reader to [BB96, BR09, ER11], as well as [Deu01, Chapter 9] and [BZ05, Subsubsection 4.5.4]. We refer to the references mentioned earlier for a commentary on the applications of the SIP for the convex case (i.e., when all the sets $K_i$ in (1.1) are convex).

1.1. The convex SIP. One problem of the MAP is slow convergence. As discussed in the previously mentioned references, in the presence of a regular intersection property, one can at best expect linear convergence of the MAP. A few acceleration methods were explored. The papers [GPR67, GK89, BDHP03] explored the acceleration of the MAP using a line search in the case where $K_i$ are linear subspaces. See also the papers [HRER11, Pan15a] for newer research for this particular setting.

In [Pan15b], we looked at a different method for the convex SIP (i.e., the SIP (1.1) when the sets $K_i$ are all convex). Each projection generates a halfspace containing the intersection of the sets $K_i$, and one can project onto the intersection of a number of these halfspaces using standard methods in quadratic programming (for example an active set method [GI83] or an interior point method). We call this the SHQP (supporting halfspace and quadratic programming) strategy. This strategy is illustrated in Figure 1.1. We refer to [Pan15b] for more on the history on the SHQP strategy, and we point out a few earlier papers that had some ideas of the SHQP strategy [Pie84, GP98, GP01, BCK06, PM79, MPH81].

![Figure 1.1](image-url)

Figure 1.1. Refer to the diagram on the left. The method of alternating projections on two convex sets $K_1$ and $K_2$ in $\mathbb{R}^2$ with starting iterate $x_0$ arrives at $x_3$ in three iterations. The point $x_4$ is the projection of $x_1$ onto the intersection of halfspaces generated by projecting onto $K_1$ and $K_2$ earlier. One can see that $d(x_4, K_1 \cap K_2) < d(x_3, K_1 \cap K_2)$, illustrating the potential of the SHQP (supporting halfspace and quadratic programming) strategy elaborated in [Pan15b]. The diagram on the right shows that such a heuristic need not be effective for nonconvex sets.

The main result in [Pan15b] is to show the following: For a convex SIP satisfying the linearly regular intersection property (Definition 2.5), we have an algorithm that achieves multiple-term superlinear convergence if enough halfspaces generated from earlier projections are stored to form the quadratic programs to be solved in later iterations. While the proof of this result suggests keeping an impractically huge
The number of halfspaces to guarantee the fast convergence, simple examples like the one in Figure 1.1 suggests that the number of halfspaces that need to be used to obtain the fast convergence can actually be quite small.

1.2. The nonconvex SIP. We quote from [LLM09] on the applications and background of the SIP in the nonconvex case (i.e., when the sets $K_i$ in (1.1) are not known to be convex): An example of a nonconvex set that is easy to project onto is the set of matrices with some fixed rank. The method of alternating projections for nonconvex problems appear in areas such as inverse eigenvalue problems [CC96, Chu95], pole placement [Ors06, YO06], information theory [TDHS05], low-order control design [GB00, GS96, OHM06], and image processing [BCL02, MTW14, WA86]. Previous convergence results on nonconvex alternating projection algorithms have been uncommon, and have either focused on a very special case (see, for example [CC96, LM08]), or have been much weaker than for the convex case [CT90, TDHS05]. For more discussion, see [LM08]. More recent works on the nonconvex SIP include [BLPW13b, BLPW13a, HL13]. See also [ABRS10].

For the nonconvex problem, the projection onto a nonconvex set need not generate a supporting halfspace. It is easy to construct examples such that the halfspace generated by the projection process will not contain any point in the intersection. (See for example the diagram on the right in Figure 1.1.) The notion of super-regularity (See Definition 2.2) was first defined in [LLM09]. They also showed how super-regularity is connected to various other well-known properties in variational analysis. In the presence of super-regularity, they established the linear convergence of the MAP.

1.3. Contributions of this paper. The main contribution of this paper is to make two observations about super-regular sets. The first observation is that once a point is close enough to a super-regular set, the projection onto this set produces a halfspace that locally separates a point from the set. (This observation is used to prove Claim (a) in Theorem 3.8.) With this observation, the SHQP strategy can be carried over to super-regular sets. The second observation is that if one of the sets is a manifold, then we can use a hyperplane to approximate the manifold instead of using a halfspace in the QP subproblem and still obtain convergence of our algorithms. See (2.3).

In Section 3 we show that under typical conditions in the study of alternating projections, an algorithm (Algorithm 3.1) that has a sequence of projection steps and SHQP steps that visits all the sets will converge linearly to a point in the intersection. In Section 4 we show that the SHQP strategy applied to find a point in the intersection of manifolds and super-regular sets with only one unit normal on its boundary points will converge superlinearly. The convergence is quadratic under added conditions. This makes a connection to the Newton method. Lastly, in Section 5 we show that arbitrary fast linear convergence is possible when enough halfspaces from previous iterations are kept to form the quadratic programs to accelerate later iterations.

1.4. Notation. The notation we use are fairly standard. We let $B(x, r)$ be the closed ball with center $x$ and radius $r$, and we denote the projection onto a set $C$ by $P_C(\cdot)$. 
2. Preliminaries

In this section, we recall some definitions in nonsmooth analysis and some basic background material on the theory of alternating projections that will be useful for the rest of the paper.

**Definition 2.1.** (Normal cones and Clarke regularity) For a closed set $C \subset \mathbb{R}^n$, the regular normal cone at $\bar{x}$ is defined as

$$\hat{N}_C(\bar{x}) := \{ y \mid \langle y, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|) \text{ for all } x \in C \}. \tag{2.1}$$

The limiting normal cone at $\bar{x}$ is defined as

$$N_C(\bar{x}) := \{ y \mid \text{there exists } x_i \to \bar{x}, y_i \in \hat{N}_C(x_i) \text{ such that } y_i \to y \}. \tag{2.2}$$

When $\hat{N}_C(\bar{x}) = N_C(\bar{x})$, then $C$ is Clarke regular at $\bar{x}$. If $C$ is Clarke regular at all points, then we simply say that it is Clarke regular.

An important tool for our analysis for the rest of the paper is the following notion of regularity of nonconvex sets.

**Definition 2.2.** [LLM09, Proposition 4.4] (Super-regularity) A closed set $C \subset \mathbb{R}^n$ is super-regular at a point $\bar{x} \in C$ if, for all $\delta > 0$ we can find a neighborhood $V$ of $\bar{x}$ such that

$$\langle z - y, v \rangle \leq \delta \|z - y\| \|v\| \text{ for all } z, y \in C \cap V \text{ and } v \in N_C(y).$$

We say that $C$ is super-regular if it is super-regular at all points.

The discussion in [LLM09] also shows that

1. Super-regularity at a point implies Clarke regularity there [LLM09, Corollary 4.5]. (The converse is not true [LLM09, Example 4.6].)
2. Either amenability at a point or prox-regularity at a point implies super-regularity there [LLM09, Propositions 4.8 and 4.9].

We assume that all the sets involved in this paper are super-regular. In view of property (1), we will not need to distinguish between $\hat{N}_C(\bar{x})$ and $N_C(\bar{x})$ for the rest of the paper.

**Remark 2.3.** (On manifolds) It is clear that if $M$ is a smooth manifold in the usual sense, then $M$ is super-regular. Moreover,

$$\hat{N}_M(\bar{x}) = N_M(\bar{x}) \implies -v \in N_M(x). \tag{2.3}$$

For all $x \in M$, $v \in N_M(x)$ implies $-v \in N_M(x)$. For the rest of our discussions, we shall let a manifold be a super-regular set satisfying (2.3).

The following property relates $d(x, \cap_{i=1}^m K_i)$ to $\max_{1 \leq t \leq m} d(x, K_t)$.

**Definition 2.4.** (Local metric inequality) We say that a collection of closed sets $K_l \subset \mathbb{R}^n$, $l = 1, \ldots, m$ satisfies the local metric inequality at $\bar{x}$ if there is a $\beta > 0$ and a neighborhood $V$ of $\bar{x}$ such that

$$d(x, \cap_{i=1}^m K_i) \leq \beta \max_{1 \leq t \leq m} d(x, K_t) \text{ for all } x \in V. \tag{2.4}$$

A concise summary of further studies on the local metric inequality appears in [Kru06], who in turn referred to [BBL99, Iof00, NT01, NY04] on the topic of local metric inequality and their connection to metric regularity. Definition 2.4 is sufficient for our purposes. The local metric inequality is useful for proving the
linear convergence of alternating projection algorithms [BB93, LLM09]. See [BB96] for a survey.

**Definition 2.5.** (Linearly regular intersection) For closed sets $K_l \subset \mathbb{R}^n$, we say that \{\{K_l\}\} has linearly regular intersection at $x \in K := \cap_{l=1}^{m} K_l$ if the following condition holds:

$$\text{If } \sum_{l=1}^{m} v_l = 0 \text{ for some } v_l \in N_{K_l}(x), \text{ then } v_l = 0 \text{ for all } l \in \{1, \ldots, r\}. \tag{2.5}$$

The linearly regular intersection property appears in [RW98, Theorem 6.42] as a condition for proving that $N_{\cap_{l=1}^{m} K_l}(x) = \sum_{l=1}^{m} N_{K_l}(x)$. As discussed in [Kru06] and related papers, linearly regular intersection is related to the sensitivity analysis of the SIP (1.1). Linearly regular intersection implies the linear convergence of the method of alternating projections. Furthermore, linearly regular intersection implies local metric inequality, but the converse is not true.

The following easy and well known principle is used to prove the Fejér monotonicity of iterates in Theorems 5.2 and 5.3.

**Proposition 2.6.** (Fejér monotonicity) Suppose $C$ is a closed convex set in $\mathbb{R}^n$, with $x \notin C$ and $y \in C$. Then for any $\lambda \in [0, 1]$,

$$\|y - (P_C(x) + \lambda(P_C(x) - x))\| \leq \|y - x\|,$$

and the inequality is strict if $\lambda \in (0, 1)$.

3. BASIC LOCAL CONVERGENCE FOR SUPER-REGULAR SIP

In the absence of additional information on the global structure of a nonconvex SIP, the analysis of convergence must necessarily be local. In this section, we discuss how super-regularity can give a halfspace that locally separates a point from the intersection of the sets. This leads to the local linear convergence of an alternating projection algorithm that incorporates QP steps whenever possible.

We begin with the algorithm that we study for this section.

**Algorithm 3.1.** (Basic algorithm) Let $K_l$ be (not necessarily convex) closed sets in $\mathbb{R}^n$ for $l \in \{1, \ldots, m\}$. From a starting point $x_0 \in \mathbb{R}^n$, this algorithm finds a point in the intersection $K := \cap_{l=1}^{m} K_l$.

01 For iteration $i = 0, 1, \ldots$
02 Set $x_i^0 = x_i$.
03 Find sets $S_1, \ldots, S_m \subset \{1, \ldots, m\}$ such that $\cup_{i=1}^{m} S_i = \{1, \ldots, m\}$.
04 For $j = 1, \ldots, m$
05 Find $x_{i,j,l} \in P_{K_l}(x_i^{j-1})$ for all $l \in S_j$
06 For $l \in S_j$, define halfspace/ hyperplane $H_{i,j,l}$ by

$$H_{i,j,l} := \begin{cases} \{x : \langle x_i^{j-1} - x_{i,j,l}, x - x_{i,j,l} \rangle = 0 \} & \text{if } K_l \text{ is a manifold} \\ \{x : \langle x_i^{j-1} - x_{i,j,l}, x - x_{i,j,l} \rangle \leq 0 \} & \text{otherwise}. \end{cases}$$

07 Define the polyhedron $F_i^j$ by $F_i^j = \cap_{(k,l) \in \tilde{S}_i} H_{i,k,l}$, where
08 $\tilde{S}_i \subset \{1, \ldots, m\} \times \{1, \ldots, m\}$ is such that $\{j\} \times S_j \subset \tilde{S}_i$ and \begin{equation} \tilde{S}_i := \{ (k,l) : l \in S_k, k \in \{1, \ldots, j\}, \text{ and } \} \end{equation} implies $k_1 = k_2$. \tag{3.1a}

09 Find $x_{i,j} \in F_i^j$.
10 Set $x_i^j = x_{i,j}$.

11 If $\|x_i^j - x_i^{j-1}\| < \varepsilon$, halt.
12 Set $i := i + 1$.
13 Go to 01.

**Remark.** The line search on $S_i$ is local with respect to $x_i^j$.
09 Set \( x^i_i = P_{F_i^j}(x^{i-1}_i) \).
10 end for
11 Set \( x_{i+1} = x^m_i \).
12 end

We allow some of the \( S_j \)'s to be empty as long as the condition \( \cup_{i=1}^m S_i = \{1, \ldots, m\} \) is satisfied. When \( S_j = \{j\} \) and \( \tilde{S}_j^i = \{(j, j)\} \) for all \( i, j \), Algorithm 3.1 reduces to the alternating projection algorithm. Algorithm 3.1 has the given design because we believe that by performing QP steps with polyhedra that bound the sets \( K_i \) better, the convergence to a point in \( K \) can be accelerated. Yet, we still retain the flexibility of the size of the QPs so that each step can be performed with a reasonable amount of effort.

Remark 3.2. (Mass projection) Another particular case of Algorithm 3.1 we will study in Section 4 is when \( S_1 = \{1, \ldots, m\} \), \( S_j = \emptyset \) for all \( j \in \{2, \ldots, m\} \), and \( \tilde{S}_j^i = \{j\} \times S_j \) for all \( i, j \in \{1, \ldots, m\} \). In such a case, Algorithm 3.1 is simplified to

\[
x_{i+1} = P_{\cup_{j \in S_i} H_{i,j}}(x_i).
\]

Remark 3.3. (On the polyhedron \( F^j_i \)) The polyhedron \( F^j_i \) is defined by intersecting some of the halfspaces/ hyperplanes \( H_{i,k,l} \). The line \( (3.1b) \) in (3.1) defining \( \tilde{S}_j^i \) ensures that no two of the halfspaces/ hyperplanes \( H_{i,k,l} \) that are intersected to form \( F^j_i \) come from projecting onto the same set. To see why we need \( (3.1b) \), observe that we can draw two tangent lines to a manifold in \( \mathbb{R}^2 \) that do not intersect, which would lead to \( F^j_i = \emptyset \).

Remark 3.4. (Treatment of manifolds) Another feature of this algorithm is that when \( K_i \) is a manifold, the set \( H_{i,j,l} \) is a hyperplane instead. Manifolds are super-regular sets. We take advantage of property (2.3) of manifolds to create a more logical algorithm. The hyperplane is a better approximate of a manifold than a halfspace, and we may expect faster convergence to a point in \( K \) when we use hyperplanes instead. Another advantage of using hyperplanes is that quadratic programming algorithms resolve equality constraints (which are always tight) better than they resolve inequality constraints (where determining whether each constraint is tight at the optimal solution requires some effort).

The lemma below will be useful in studying the convergence of the algorithms throughout this paper.

Lemma 3.5. (Linear convergence conditions) Let \( K \) be a set in \( \mathbb{R}^n \). Suppose an algorithm generates iterates \( \{x_i\} \) such that

1. There exists some \( \rho \in (0, 1) \) such that \( d(x_{i+1}, K) \leq \rho d(x_i, K) \), and
2. There exists a constant \( c > 0 \) such that \( \|x_{i+1} - x_i\| \leq c d(x_i, K) \).

Then the sequence \( \{x_i\} \) converges to a point \( \bar{x} \in K \), and we have, for all \( i \geq 0 \),

1. \( \|x_i - \bar{x}\| \leq \frac{\rho^i}{1 - \rho} d(x_0, K) \),
2. \( \mathcal{B}(x_{i+1}, \frac{c}{1 - \rho} d(x_{i+1}, K)) \subset \mathcal{B}(x_i, \frac{\rho^i}{1 - \rho} d(x_i, K)) \).
Proof. For any \( j \geq 0 \), we have
\[
\|x_{i+j+1} - x_{i+j}\| \leq cd(x_{i+j}, K) \leq c p^j d(x_i, K).
\]
Standard arguments in analysis shows that \( \{x_i\} \) is a Cauchy sequence which converges to a point \( \bar{x} \in K \). Both parts (a) and (b) are straightforward. \( \square \)

The next result shows how such derived halfspaces relate to the original halfspaces.

Lemma 3.6. (Derived supporting halfspaces) Let \( \bar{x} \in \mathbb{R}^n \), and suppose \( H_1, H_2, \ldots, H_k \) are \( k \) halfspaces containing \( \bar{x} \) such that \( d(\bar{x}, \partial H_i) \), the distance from \( \bar{x} \) to the boundary of each halfspace \( H_i \), is at most \( \alpha \). Suppose the normal vectors of each halfspace \( H_i \) is \( v_i \), where \( \|v_i\| = 1 \), and the constant \( \eta \) defined by
\[
\eta := \min \left\{ \left\| \sum_{i=1}^{k} \lambda_i v_i \right\| : \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \text{ for all } i \in \{1, \ldots, k\} \right\}
\]  
(3.2)

is positive. (i.e., \( \eta \neq 0 \).) Let \( F \) be the intersection of these halfspaces. Let \( \bar{H} \) be the halfspace containing \( F \) produced by projecting from a point \( x' \notin F \) onto \( F \). In other words, the halfspace \( \bar{H} \) is defined by
\[
\{ x : \langle x' - P_F(x'), x - P_F(x') \rangle \leq 0 \}.
\]
Then the distance of \( \bar{x} \) from the boundary of \( \bar{H} \) is at most \( \frac{1}{\eta} \alpha \).

As a consequence, suppose \( H_i \) are defined by \( H_i = \{ x : \langle v_i, x \rangle \leq \alpha \} \). Let \( v = \sum_{i=1}^{k} \frac{\lambda_i v_i}{\|\sum_{i=1}^{k} \lambda_i v_i\|} \) for some nonzero vector \( \lambda \in \mathbb{R}^k \) that has nonnegative components, and \( H \) be \( H = \{ x : \langle v, x \rangle \leq \frac{\alpha}{\eta} \} \). Then we have \( \cap_{i=1}^{k} H_i \subset H \subset H_i \).

Proof. We remark that \( \eta \) is the distance of the origin to the convex hull of \( \{v_i\} \). We can eliminate halfspaces if necessary and assume that \( k \geq 1 \), and that \( P_F(x') \) lies on the boundaries of all the halfspaces. The KKT condition tells us that \( x' - P_F(x') \) lies in the conical hull of \( \{v_i\} \). By Caratheodory’s theorem, we can assume that \( k \) is not more than the dimension \( n \). We can also eliminate halfspaces if necessary so that the vectors \( \{v_i\} \) are linearly independent.

Suppose each halfspace \( H_i \) is defined by \( \{ x : \langle v_i, x \rangle \leq b_i \} \), where \( b_i \in \mathbb{R} \). Since \( P_F(x') \) lies on the boundaries of the halfspaces \( H_i \), we have
\[
\langle v_i, P_F(x') \rangle = b_i \text{ for all } i.
\]  
(3.3)

Define the hyperslab \( S_i \) by
\[
S_i := \{ x : \langle v_i, x \rangle \in [b_i - \alpha, b_i] \}.
\]  
(3.4)

Since the distance from \( \bar{x} \) to the boundaries of each halfspace \( H_i \) were assumed to be at most \( \alpha \), the point \( \bar{x} \) is inside all the hyperslabs \( S_i \).

Let \( \bar{v} \) be the vector \( \frac{x' - P_F(x')}{\|x' - P_F(x')\|} \). We now study the problem
\[
\begin{align*}
\min_{x} & \quad \langle \bar{v}, x \rangle \\
\text{s.t.} & \quad x \in S_i \text{ for all } i \in \{1, \ldots, k\}.
\end{align*}
\]  
(3.5)

If the above problem were a maximization problem instead, then an optimizer is \( P_F(x') \). Consider the point \( P_F(x') - \alpha d \), where \( d \) is the direction defined through
\[
\langle v_i, d \rangle = 1 \text{ for all } i, \text{ and } d \in \text{span}(\{v_i\}_{i=1}^{k}).
\]  
(3.6)
Since the vectors \( \{ v_i \}_{i=1}^k \) are linearly independent, such a \( d \) exists, and can be calculated by \( d = QR^{-T}1 \), where \( 1 \) is the vector of all ones, \( QR \) is the QR factorization of \( V \), and \( V \) is the matrix formed by concatenating the vectors \( \{ v_i \}_{i=1}^k \). We can use (3.3) and (3.6) to calculate that
\[
\langle v_i, P_F(x') - \alpha d \rangle = b_i - \alpha \quad \text{for all } i,
\]
so \( P_F(x') - \alpha d \) is on the other boundary of all the hyperslabs \( S_i \). Furthermore, since
\[
N_{\{ v_i \}_{i=1}^k} S_i (P_F(x') - \alpha d) = -N_{\{ v_i \}_{i=1}^k} S_i (P_F(x')), \quad \text{we have} \quad -\bar{v} \in N_{\{ v_i \}_{i=1}^k} S_i (P_F(x') - \alpha d).
\]
Hence \( P_F(x') - \alpha d \) is a minimizer of (3.5).

We proceed to find the optimal value of (3.5). Since \( \bar{v} \) lies in the conical hull of \( \{ v_i \}_{i=1}^k \), \( \bar{v} \) can be written as \( \frac{V\lambda}{\|V\lambda\|} \), where \( \lambda \in \mathbb{R}_+^k \) is a vector with nonnegative elements such that its elements sum to one. We can calculate
\[
\left( \frac{V\lambda}{\|V\lambda\|} \right)^T d = \frac{1}{\|V\lambda\|} \lambda^T V^T QR^{-T} 1
\]
\[
= \frac{1}{\|V\lambda\|} \lambda^T R^T Q^T QR^{-T} 1 = \frac{1}{\|V\lambda\|} \lambda^T 1 = \frac{1}{\|V\lambda\|}.
\]
By the definition of \( \eta \), we have \( \frac{1}{\|V\lambda\|} \geq \frac{1}{\eta} \). This means that the minimum value of (3.5) is at least
\[
\langle \bar{v}, P_F(x') - \alpha d \rangle = \langle \bar{v}, P_F(x') \rangle - \frac{\alpha}{\eta}.
\]
Since \( \bar{x} \in S_i \) for all \( i \in \{1, \ldots, k\} \), we can deduce that \( \bar{x} \) lies in the hyperslab
\[
\{ x : \langle \bar{v}, x \rangle \in [\langle \bar{v}, P_F(x') \rangle - \frac{\alpha}{\eta}, \langle \bar{v}, P_F(x') \rangle] \}.
\]
In other words, \( \bar{x} \) lies in the halfspace \( \{ x : \langle \bar{v}, x \rangle \leq \langle \bar{v}, P_F(x') \rangle \} \), and the distance from \( \bar{x} \) to the boundary of this halfspace is at most \( \frac{1}{\eta} \alpha \), which is the conclusion we seek.

The final paragraph is easily deduced from the main result. \( \square \)

**Remark 3.7.** (The formula \( \eta \)) We remark that the use of the notation \( \eta \) in Lemma 3.6 is consistent with the notation of [Kru06] and related papers, where the relationship of the constants related to the sensitivity analysis of the SIP (1.1) and linearly regular intersection are studied.

We now prove our result on the convergence of Algorithm 3.1.

**Theorem 3.8.** (Local linear convergence of general Algorithm) Suppose \( K_i \), where \( l \in \{1, \ldots, m\} \), are super-regular at \( x^* \in K = \bigcap_{i=1}^m K_i \). Suppose that \( \eta \) defined by
\[
\eta := \min \left\{ \|v_i\| : v_i \in N_{K_i}(x^*), \sum_{i=1}^m \|v_i\| = 1 \right\}
\]
is positive. (i.e., \( \eta \neq 0 \).) This is equivalent to \( \{ K_i \}_{i=1}^m \) having linear regular intersection at \( x^* \), which in turn implies that the local metric inequality holds at \( x^* \). If \( x_0 \) is sufficiently close to \( x^* \), then Algorithm 3.7 converges to a point in \( K \) \( Q \)-linearly (i.e., at a rate bounded above by a geometric sequence).

**Proof.** Since the local metric inequality holds at \( x^* \), let \( \beta \geq 1 \) and \( V \) be a neighborhood of \( x^* \) such that
\[
d(x, K) \leq \beta \max_i d(x, K_i) \quad \text{for all } x \in V.
\]
Let
\[ \rho = \sqrt{1 + \frac{1}{\beta^2 m^2} + \frac{1}{4\beta^4 m^3} - \frac{1}{\beta^2 m^2} + \frac{1}{2\beta^3 m^5} - \frac{1}{16\beta^4 m^6} + \frac{1}{16\beta^4 m^6}} \]
and
\[ c = \sqrt{m \left[ 1 + \frac{1}{4m^3\beta^2} \right] + \frac{1}{16m^6\beta^4}} \]
(3.7a)
and (3.7b)

It is clear to see that if \( m \geq 2 \), then \( \rho < 1 \). Choose \( \delta > 0 \) such that \( \delta \leq \frac{(1-\rho)\eta}{16m^4\beta^2c} \).

Since \( x^* \) is super-regular at all sets \( K_i \), where \( l \in \{1, \ldots, m\} \), we can shrink the neighborhood \( V \) if necessary so that for all \( l \in \{1, \ldots, m\} \), we have
\[ \langle v, z - y \rangle \leq \delta \|v\|\|z - y\| \text{ for all } z, y \in K_l \cap V \text{ and } v \in N_{K_l}(y). \]

By the outer semicontinuity of the normal cones, we can shrink \( V \) if necessary so that for all \( x \in V \), we have
\[ \min \left\{ \left\| \sum_{i=1}^m v_i \right\| : v_i \in N_{K_i}(x), \; x \in K_v, \; \sum_{i=1}^m \|v_i\| = 1 \right\} \geq \frac{\eta}{2}. \]

Suppose \( x_0 \) is close enough to \( x^* \) such that \( B(x_0, \frac{c}{1-\rho}d(x_0, K)) \subset V \). Provided that we prove conditions (1) and (2) in Lemma 3.5, we have the convergence of \( x_i \) to \( x \). The convergence of \( x_i \) to \( x \) would be at the rate suggested in Lemma 3.5(a).

If \( x \in K \cap B(x_i, \frac{c}{1-\rho}d(x_i, K)) \) and \( x_i^{-1}, x_{i,j} \in B(x_i, \frac{c}{1-\rho}d(x_i, K)) \), then
\[ \left\langle \frac{x_i^{-1} - x_{i,j}}{\|x_i^{-1} - x_{i,j}\|}, x - x_{i,j} \right\rangle \leq \delta \|x - x_{i,j}\| \leq \delta \frac{2c}{1-\rho}d(x_i, K) \leq \frac{\eta}{8m^4\beta^2}d(x_i, K). \]

Define the halfspace \( H_{i,j,l}^+ \) by
\[ H_{i,j,l}^+ := \left\{ x : \left\langle \frac{x_i^{-1} - x_{i,j,l}}{\|x_i^{-1} - x_{i,j,l}\|}, x - x_{i,j,l} \right\rangle \leq \frac{\eta}{8m^4\beta^2}d(x_i, K) \right\}. \]

(Note that the halfspace \( H_{i,j,l} \) defined in Algorithm 3.1 is similar to \( H_{i,j,l}^+ \) with the exception that the right hand side of the inequality is zero.) We have \( K_i \cap B(x_i, \frac{c}{1-\rho}d(x_i, K)) \subset H_{i,j,l} \). Note that \( x_i \) is the projection of \( x_i^{-1} \) onto \( F_i \). Define the halfspace \( H_{i,j}^+ \) by
\[ H_{i,j}^+ := \left\{ x : \left\langle \frac{x_i^{-1} - x_i}{\|x_i^{-1} - x_i\|}, x - x_i \right\rangle \leq \frac{1}{4m^4\beta^2}d(x_i, K) \right\}. \]

By Lemma 3.6 we have
\[ K \cap B(x_i, \frac{c}{1-\rho}d(x_i, K)) \subset \cap_{l=1}^m H_{i,k,l}^+ \subset H_{i,j}^+. \]

Note that almost exactly the same arguments works if the set \( K_l \) is a manifold, but we may have to take \( -\frac{x_i^{-1} - x_i}{\|x_i^{-1} - x_i\|} \) as the normal vector of \( H_{i,j,l}^+ \) instead and define \( H_{i,j}^+ \) differently, depending on the multipliers in the KKT condition.
Claim:
(a) If \( \|x_i^j - x_i^{j-1}\| \geq \frac{1}{2m^4\beta^2} d(x_i, K) \), then
\[
d(x_i^j, K)^2 \leq d(x_i^{j-1}, K)^2 - \left[ \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K) \right]^2 + \left[ \frac{1}{4m^4\beta^2} d(x_i, K) \right]^2.
\]
(b) If \( \|x_i^j - x_i^{j-1}\| \leq \frac{1}{2m^4\beta^2} d(x_i, K) \), then \( d(x_i^j, K) \leq d(x_i^{j-1}, K) + \frac{1}{2m^4\beta^2} d(x_i, K) \).

Part (b) is obvious. We now prove part (a). Let \( y \) be any point in \( P_K(x_i^{j-1}) \), and let \( z = P_{H_i^+}(x_i^{j-1}) \).

Note that \( \angle zyx_i^j \geq \pi/2 \), we apply cosine rule to get
\[
d(x_i^j, K)^2 = \|y - x_i^j\|^2 \leq \|y - z\|^2 + \|z - x_i^j\|^2 \leq \|y - x_i^{j-1}\|^2 - \|x_i^j - z\|^2 + \|z - x_i^{j-1}\|^2 = d(x_i^{j-1}, K)^2 - \left[ \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K) \right]^2 + \left[ \frac{1}{4m^4\beta^2} d(x_i, K) \right]^2.
\]

This completes the proof of the claim.

\[\begin{array}{c}
X_i^{j-1} \\
H_i^+ \\
Y
\end{array}\]

Figure 3.1. This figure illustrates the proof in the claim of Theorem 3.8. Note that \( d_1 = \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K) \) and \( d_2 = \frac{1}{4m^4\beta^2} d(x_i, K) \).

It now remains the prove conditions (1) and (2) of Lemma 3.5. By local metric inequality, there is some \( j \in \{1, \ldots, m\} \) such that \( d(x_i, K_j) \geq \frac{1}{m} d(x_i, K) \). Hence there is a distance \( \|x_i^j - x_i^{j-1}\| \) that will be at least \( \frac{1}{m\beta} d(x_i, K) \). Making use of the claim earlier, we have the following estimate of \( d(x_i+1, K) \).

\[
d(x_i+1, K)^2 \leq \left[ d(x_i, K) + \frac{m}{2m^4\beta^2} d(x_i, K) \right]^2 - \left[ \frac{1}{m\beta} d(x_i, K) - \frac{1}{4m^4\beta^2} d(x_i, K) \right]^2 + \left[ \frac{m}{4m^4\beta^2} d(x_i, K) \right]^2 = \left[ 1 + \frac{1}{\beta^2} + \frac{1}{4\beta^4} - \frac{1}{\beta^6} \frac{1}{2} + \frac{1}{16\beta^8} - \frac{1}{16\beta^8} + \frac{1}{16\beta^8} \right] d(x_i, K)^2 = \rho^2 d(x_i, K)^2.
\]
This proves that $d(x_{i+1}, K) \leq \rho d(x_i, K)$. Next,

$$\|x_{i+1} - x_i\| \leq \sum_{j=1}^{m} \|x_i^j - x_i^{j-1}\|$$

$$\leq \frac{1}{4m^4\beta^2} d(x_i, K) + \sum_{j=1}^{m} \max \left\{ \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K), 0 \right\}$$

$$\leq \frac{1}{4m^4\beta^2} d(x_i, K) + \left( \sum_{j=1}^{m} \max \left\{ \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K), 0 \right\} \right)^2.$$  \hfill (*)

By the analysis in \((3.11)\), the fact that $d(x_{i+1}, K)^2 \geq 0$ gives

$$0 \leq d(x_{i+1}, K)^2 \leq \left[ d(x_i, K) + \frac{1}{2m^4\beta^2} d(x_i, K) \right]^2$$

$$+ \left[ \frac{m}{4m^4\beta^2} d(x_i, K) \right]^2 - \sum_{j=1}^{m} \max \left\{ \|x_i^j - x_i^{j-1}\| - \frac{1}{4m^4\beta^2} d(x_i, K), 0 \right\}^2.$$  \hfill (**)

We thus deduce that the term marked (*) in \((3.12)\) is at most

$$\sqrt{m} \left[ \left( 1 + \frac{1}{2m^4\beta^2} \right)^2 + \frac{1}{16m^6\beta^4} d(x_i, K) \right].$$

Thus the constant $c$ in Lemma \((3.5)\) can be taken to be what was given in \((3.13)\). \qed

Remark 3.9. (On the condition $\eta > 0$ in Theorem \((3.8)\). The condition $\eta > 0$ is required in the proof of Theorem \((3.8)\) only when $|S_i| > 1$, when halfspaces are aggregated. So in the case of alternating projections, the weaker condition of local metric inequality is sufficient.

4. Connections with the Newton Method

To find a point in $\{ x \in \mathbb{R}^n : F(x) = 0 \}$ for some smooth $F : \mathbb{R}^n \to \mathbb{R}^m$, the method of choice is to use the Newton method provided that the linear system in the Newton method can be solved quickly enough. Note that the set $\{ x : F(x) = 0 \}$ can be written as the intersection of the manifolds $M_j := \{ x : F_j(x) = 0 \}$ for $j \in \{ 1, \ldots, m \}$, where $F_j : \mathbb{R}^n \to \mathbb{R}$ is the $j$th component of $F(\cdot)$. Note that the manifolds $M_j$ are of codimension 1. This section gives conditions for which the SHQP strategy can converge superlinearly or quadratically when the sets involved satisfy the conditions for fast convergence in the Newton method.

The following result was proved in [Pan15b] for convex sets, but is readily generalized to Clarke regular sets, which we do so now.

Theorem 4.1. (Supporting hyperplane near a point) Suppose $C \subset \mathbb{R}^n$ is Clarke regular, and let $\bar{x} \in C$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any point $x \in B_\delta(\bar{x}) \cap C \setminus \{ \bar{x} \}$ and supporting hyperplane $A$ of $C$ with unit normal $v \in N_C(x)$ at the point $x$, we have

$$\langle v, x - \bar{x} \rangle \leq \epsilon \| x - \bar{x} \|.$$  \hfill (4.1)
Proof. Let \( \delta > 0 \) be small enough so that for any \( x \in [B_\delta(\bar{x}) \cap C] \setminus \{\bar{x}\} \) and unit normal \( v \in N_C(x) \), we can find \( \bar{v} \in N_C(\bar{x}) \) such that \( \|v - \bar{v}\| < \frac{\delta}{2} \) and that \( \langle \bar{v}, x - \bar{x}\rangle \leq \frac{1}{2} \|x - \bar{x}\| \). Then we have
\[
\langle v, x - \bar{x}\rangle = \langle v - \bar{v}, x - \bar{x}\rangle + \langle \bar{v}, x - \bar{x}\rangle \\
\leq \|v - \bar{v}\|\|x - \bar{x}\| + \frac{\delta}{2} \|x - \bar{x}\| \\
\leq \epsilon \|x - \bar{x}\|.
\]
Thus we are done. \( \square \)

We identify a property that will give multiple-term quadratic convergence. Compare this property to that in Theorem 4.1.

**Definition 4.2.** (Second order supporting hyperplane property) Suppose \( C \subset \mathbb{R}^n \) is a closed convex set, and let \( \bar{x} \in C \). We say that \( C \) has the second order supporting hyperplane (SOSH) property at \( \bar{x} \) (or more simply, \( C \) is SOSH at \( \bar{x} \)) if there are \( \delta > 0 \) and \( M > 0 \) such that for any point \( x \in [B_\delta(\bar{x}) \cap C] \setminus \{\bar{x}\} \) and \( v \in N_C(x) \) such that \( \|v\| = 1 \), we have
\[
\langle v, x - \bar{x}\rangle \leq M \|x - \bar{x}\|^2.
\]

It is clear how (4.1) compares with (4.2). The next two results show that SOSH is prevalent in applications.

**Proposition 4.3.** (Smoothness implies SOSH) Suppose function \( f : \mathbb{R}^n \to \mathbb{R} \) is \( C^2 \) at \( \bar{x} \). Then the set \( C = \{x \mid f(x) \leq 0\} \) is SOSH at \( \bar{x} \).

Proof. Consider \( \bar{x}, x \in C \). In order for the problem to be meaningful, we shall only consider the case where \( f(\bar{x}) = 0 \). We also assume that \( f(x) = 0 \) so that \( C \) has a tangent hyperplane at \( x \). An easy calculation gives \( N_C(\bar{x}) = \mathbb{R}_+ \{\nabla f(\bar{x})\} \) and \( N_C(x) = \mathbb{R}_+ \{\nabla f(x)\} \).

Without loss of generality, let \( \bar{x} = 0 \). We have
\[
f(x) = f(0) + \nabla f(0)x + \frac{1}{2} x^T \nabla^2 f(0)x + o(\|x\|^2).
\]

\[
\Rightarrow f(0)x + \frac{1}{2} x^T \nabla^2 f(0)x = o(\|x\|^2).
\]

Since \( f(x) = f(0) = 0 \) and \( |\nabla f(0) - \nabla f(x)|x| = x^T \nabla^2 f(0)x + o(\|x\|^2) \), we have
\[
-\nabla f(x)(x) = [\nabla f(0) - \nabla f(x)]x + \frac{1}{2} x^T \nabla^2 f(0)x + o(\|x\|^2) = O(\|x\|^2).
\]

Therefore, we are done. \( \square \)

**Proposition 4.4.** (SOSH under intersection) Suppose \( K_l \subset \mathbb{R}^n \) are closed sets that are SOSH at \( \bar{x} \) for \( l \in \{1, \ldots, m\} \). Let \( K := \cap_{l=1}^m K_l \), and suppose that \( \{K_l\}_{l=1}^m \) satisfy the linear regular intersection property at \( \bar{x} \). Then \( K \) is SOSH at \( \bar{x} \).

Proof. Since each \( K_l \) is SOSH at \( \bar{x} \), we can find \( \delta > 0 \) and \( M > 0 \) such that for all \( l \in \{1, \ldots, m\} \) and \( x \in K_l \cap B_\delta(\bar{x}) \) and \( v \in N_{K_l}(x) \), we have
\[
\langle v, x - \bar{x}\rangle \leq M \|v\|\|x - \bar{x}\|^2.
\]

Claim 1: We can reduce \( \delta > 0 \) if necessary so that
\[
\sum_{l=1}^m v_l = 0, v_l \in N_{K_l}(x) \text{ and } x \in K \cap B_\delta(\bar{x}) \quad (4.3)
\]

implies \( v_l = 0 \) for all \( l \in \{1, \ldots, m\} \).
Suppose otherwise. Then we can find \( \{x_i\}_{i=1}^{\infty} \in K \) such that \( \lim x_i = \bar{x} \) and for all \( i > 0 \), there exists \( v_{i,t} \in N_{K_t}(x_i) \) such that \( \sum_{t=1}^{m} v_{i,t} = 0 \) but not all \( v_{i,t} = 0 \). We can normalize so that \( \|v_{i,t}\| \leq 1 \), and for each \( i \), \( \max_t \|v_{i,t}\| = 1 \). By taking a subsequence if necessary, we can assume that \( \lim v_{i,t} \), say \( \bar{v}_i \), exists for all \( i \). Not all \( \bar{v}_i \) can be zero, but \( \sum_{t=1}^{m} \bar{v}_i = 0 \). The outer semicontinuity of the normal cone mapping implies that \( \bar{v}_i \in N_{K_t}(\bar{x}) \). This contradicts the linear regular intersection property, which ends the proof of Claim 1.

**Claim 2:** There exists a constant \( M' \) such that whenever \( x \in B_\delta(\bar{x}) \cap K \), \( v_i \in N_{K_i}(x) \) and \( v = \sum_{i=1}^{m} v_i \), then \( \max \|v_i\| \leq M'\|v\| \).

Suppose otherwise. Then for each \( i \), there exists \( x_i \in B_\delta(\bar{x}) \cap K \) and \( \bar{v}_{i,t} \in N_{K_t}(x_i) \) such that \( \bar{v}_i = \sum_{t=1}^{m} \bar{v}_{i,t}, \|\bar{v}_i\| \leq \frac{1}{l} \), and \( \max_t \|\bar{v}_{i,t}\| = 1 \) for all \( i \). As we take limits to infinity, this would imply that (4.30) is violated, a contradiction. This ends the proof of Claim 2.

Since (4.30) is satisfied, this means that \( N_K(x) = \sum_{i=1}^{m} N_{K_t}(x) \) for all \( x \in B_\delta(\bar{x}) \cap K \) by the intersection rule for normal cones in [RW98, Theorem 6.42]. Then each \( v \in N_K(x) \) can be written as a sum of elements in \( N_{K_t}(x) \), say \( v = \sum_{i=1}^{m} v_i \), where \( v_i \in N_{K_t}(x) \), and \( \max \|v_i\| \leq M'\|v\| \). Then

\[
\langle v, x - \bar{x} \rangle = \sum_{i=1}^{m} \langle v_i, x - \bar{x} \rangle \leq M\|\bar{x} - x\|^2 \sum_{i=1}^{m} \|v_i\| \leq M\|\bar{x} - x\|^2 m M'\|v\|.
\]

Thus we are done. \( \square \)

We now make a connection to the Newton method. Consider the mass projection algorithm.

**Theorem 4.5.** (Connection to Newton method) Consider Algorithm (2.2) for the case when \( S_1 = \{1, \ldots, m\} \) and \( S_j = \emptyset \) for all \( j \in \{2, \ldots, m\} \) at all iterations \( t \), and \( S_t^j = \{j\} \times S_j \) for all \( j \in \{1, \ldots, m\} \). See Remark (2.2). Let \( x^* \in K := \cap_{i=1}^{m} K_i \). Suppose the following hold

1. Each set \( K_i \) is super-regular.
2. For each \( l \in \{1, \ldots, m\} \), \( K_l \) is either a manifold, or \( N_{K_l}(x) \) contains at most one point of norm 1 for all \( x \in K_l \) near \( x^* \).
3. The sets \( \{K_i\}_{i=1}^{m} \) has linearly regular intersection at \( x^* \).

Then provided \( x_0 \) is close enough to \( x^* \), the convergence of the iterates \( \{x_i\} \) to some \( \bar{x} \in K \) is superlinear. Furthermore, the convergence is quadratic if all the sets \( K_i \) satisfy the SOSC property.

**Proof.** By Theorem (3.8) the convergence of the iterates \( \{x_i\} \) to \( \bar{x} \) is assured. What remains is to prove that the convergence is actually superlinear, or quadratic under the additional assumption. Without loss of generality, let \( \bar{x} = 0 \). We first prove the superlinear convergence. The proof in Theorem (3.8) assures that there is some \( \beta \geq 1 \) such that \( d(x_i, K) \leq \beta \max_t d(x_i, K_t) \) for all iterates \( x_i \).

Let \( x_i \) be an iterate. Recall that \( x_{1,i} = P_{K_i}(x_i) \). The projection of \( x_i \) onto the polyhedron gives \( x_{i+1} \). Let \( v_{i,j}^+ \) be the unit normal in \( N_{K_j}(x_{i+1,j}) \) in the direction of \( x_{i+1,j} - x_{i+1,1,j} \), and let \( v_{j}^+ \) be the unit normal in \( N_{K_j}(x_{i,1,j}) \) that is close to \( v_{i,j}^+ \).
The proof of Theorem 3.8 uses Lemma 3.5. Hence there are constants \( c \) and \( \rho \in (0, 1) \) such that \( \|x_i\| \leq \frac{c}{1-\rho} d(x_i, K) \) for all \( i \). By local metric inequality, let the index \( j \) be such that \( d(x_{i+1}, K) \leq \beta d(x_{i+1}, K_j) \). We let \( \kappa = \frac{c \beta}{(1-\rho)} \). Then

\[
\langle v^+_{j, i+1} - x_{i+1,1,j} \rangle = \| x_{i+1} - x_{i+1,1,j} \| (4.4)
\]

\[
= d(x_{i+1}, K_j) \geq \frac{1}{\beta} d(x_{i+1}, K) \geq \frac{1}{\kappa} \| x_{i+1} \|.
\]

Consider the neighborhood \( U \) such that if \( x \in U \) and \( v \in N_{K_j}(x) \{0\} \), then \( \| \frac{u}{\|u\|} - \frac{v}{\|v\|} \| \leq \frac{1}{2\kappa} \) for some \( \bar{v} \in N_{K_j}(x) \{0\} \). If \( i \) is large enough, then \( x_i \in U \) and \( x_{i,1,j} \in U \) for all \( j \in \{1, \ldots, m\} \), which leads to

\[
\| v^+_{j, i} - v^+_{j, i+1} \| \leq \| v^+_{j, i} - \bar{v}_j \| + \| v^+_{j, i+1} - \bar{v}_j \| \leq \frac{1}{2\kappa}, (4.5)
\]

where \( \bar{v}_j \) is the appropriate unit vector in \( N_{K_j}(x) \{0\} \). For any \( \delta > 0 \), we can reduce the neighborhood \( U \) if necessary so that by super-regularity,

\[
\langle v^+_{j, i}, 0 - x_{i+1,1,j} \rangle \leq \delta \| x_{i+1,1,j} \|. (4.6)
\]

Claim: \( \| x_{i+1,1,j} \| \leq \frac{\delta}{\sqrt{1-\delta^2}} \| x_{i+1} \| \)

We know that \( x_{i+1} = x_{i+1,1,j} + t v^+_{j, i} \), where \( t = \| x_{i+1} - x_{i+1,1,j} \| > 0 \). By super-regularity, we have \( \cos^{-1} \delta \leq \angle x_{i+1} x_{i+1,1,j} 0 \). Note that \( \sqrt{1-\delta^2} = \sin \cos^{-1} \delta \).

Some simple trigonometry ends the proof of the claim.

Choose \( \delta \) small enough so that \( \delta \frac{\delta}{\sqrt{1-\delta^2}} \leq \frac{1}{4\kappa} \). From (4.6), we have

\[
\langle v^+_{j, i}, 0 - x_{i+1,1,j} \rangle \leq \delta \| x_{i+1,1,j} \| \leq \frac{\delta}{\sqrt{1-\delta^2}} \| x_{i+1} \| \leq \frac{1}{4\kappa} \| x_{i+1} \|. (4.7)
\]

Then combining (4.4), (4.7) and (4.5), we get

\[
\langle v^0_{j, i+1} \rangle = \langle v^+_{j, i+1} - x_{i+1,1,j} \rangle + \langle v^+_{j, i+1,1,j} \rangle + \langle v^0_{j, i+1,1,j} \rangle + \langle v^0_{j, i+1} \rangle \ (4.8)
\]

\[
\geq \frac{1}{\kappa} \| x_{i+1} \| - \frac{1}{4\kappa} \| x_{i+1} \| - \frac{1}{2\kappa} \| x_{i+1} \| = \frac{1}{4\kappa} \| x_{i+1} \|.
\]

Choose any \( \epsilon > 0 \). Theorem 4.1 implies that \( \langle v^0_{j, i+1} \rangle \leq \epsilon \| x_{i+1} \| \) for all \( i \) large enough. We have the following set of inequalities.

\[
\langle v^0_{j, i+1} \rangle \leq \langle v^0_{j, i+1} \rangle \leq \epsilon \| x_{i+1} \| \leq \frac{\epsilon}{\sqrt{1-\delta^2}} \| x_{i+1} \|.
\]

The first inequality comes from the fact that \( x_{i+1} \) has to lie in the halfspaces constructed by the previous projection. If \( K_j \) is a manifold, then the first inequality is in fact an equation. The last inequality is from the highlighted claim above.)

Combining (4.8) and (4.9) gives \( \| x_{i+1} \| \leq \frac{4\epsilon}{\sqrt{1-\delta^2}} \| x_i \| \), which is what we need.

In the case where \( K_j \) has the SOSH property near \( x \), (4.9) can be changed to give \( \langle v^0_{j, i+1} \rangle \leq \frac{M}{1-\delta^2} \| x_i \| \) for some constant \( M \), which gives \( \| x_{i+1} \| \leq \frac{4\epsilon M}{1-\delta^2} \| x_i \| \).

This completes the proof. \( \square \)

5. An algorithm with arbitrary fast linear convergence

In this section, we show the arbitrary fast linear convergence of Algorithm 5.1 for the nonconvex SIP when the sets are super-regular. Motivated by the fast convergent algorithm in [Pan15b], Algorithm 5.1 collects old halfspaces from previous projections to try to accelerate the convergence in later iterations.

We now present an algorithm that can achieve arbitrarily fast linear convergence.
Algorithm 5.1. (Local super-regular SHQP) Let $K_l$ be (not necessarily convex) closed sets in $\mathbb{R}^n$ for $l \in \{1, \ldots, m\}$. From a starting point $x_0 \in \mathbb{R}^n$, this algorithm finds a point in the intersection $K := \bigcap_{l=1}^m K_l$.

**Step 0:** Set $i = 1$, and let $\bar{p}$ be some positive integer.

**Step 1:** Choose $\tilde{j}_i \in \text{arg max}_j \{d(x_{i-1}, K_j)\}$. (i.e., we take only an index which gives the largest distance.)

**Step 2:** Choose some $\tau_i \in [0, 1)$. Define $x_i^{(\tilde{j}_i)} \in \mathbb{R}^n$, $a_i^{(\tilde{j}_i)} \in \mathbb{R}^n$ and $b_i^{(\tilde{j}_i)} \in \mathbb{R}$ by

\[
\begin{align*}
x_i^{(\tilde{j}_i)} & \in P_{K_{\tilde{j}_i}}(x_{i-1}), \\
a_i^{(\tilde{j}_i)} & = x_{i-1} - x_i^{(\tilde{j}_i)}, \\
\text{and } b_i^{(\tilde{j}_i)} & = \langle a_i^{(\tilde{j}_i)}, x_i^{(\tilde{j}_i)} \rangle + \tau_i \langle a_i^{(\tilde{j}_i)}, x_{i-1} - x_i^{(\tilde{j}_i)} \rangle.
\end{align*}
\]

Let $x_i = P_{\tilde{F}_i}(x_{i-1})$, where the set $\tilde{F}_i \subset \mathbb{R}^n$ is defined by

\[
\tilde{F}_i := \{ x | \langle a_i^{(\tilde{j}_i)}, x \rangle \leq b_i^{(\tilde{j}_i)} \text{ for max}(1, i - \bar{p}) \leq l \leq i \}.
\]

**Step 3:** Set $i \leftarrow i + 1$, and go back to step 1.

There are some differences between Algorithm 5.1 and that of Algorithm 5.1. Firstly, in step 1, we take only one index $j$ in $\{1, \ldots, m\}$ that gives the largest distance $d(x_{i-1}, K_j)$. Secondly, the term $\tau_i \langle a_i^{(\tilde{j}_i)}, x_{i-1} - x_i^{(\tilde{j}_i)} \rangle$ is added in (5.1c) to account for the nonconvexity of the set $K_j$.

The parameter $\tau_i$ in Algorithm 5.1 requires tuning to achieve fast convergence. This tuning may not be easy to perform.

**Lemma 5.2.** (Convergence of Algorithm 5.1) Suppose that in Algorithm 5.1, the sets $K_l$ are all super-regular at a point $x^* \in K = \bigcap_{l=1}^m K_l$ for all $l \in \{1, \ldots, m\}$, and the local metric inequality holds, i.e., there is a $\beta > 0$ and a neighborhood $V_1$ of $x^*$ such that

\[
d(x, \bigcap_{l=1}^m K_l) \leq \beta \max_{1 \leq l \leq m} d(x, K_l) \text{ for all } x \in V_1.
\]

Then for any $\tau \in (0, 1)$, we can find a neighborhood $U$ of $x^*$ such that

- For any $x_0 \in U$, Algorithm 5.1 with $\tau_i = \tau$ for all $i$ generates a sequence $\{x_i\}$ that converges to some $\bar{x} \in V_1$ so that
  \[
  \|x_{i+1} - \bar{x}\| \leq \|x_i - \bar{x}\| \text{ for all } i \geq 0,
  \]
  and
  \[
  \|x_i - \bar{x}\| \leq L \max_{l \in \{1, \ldots, m\}} d(x_1, K_l),
  \]

where

\[
\rho := \frac{\sqrt{\beta^2 - (1 - \tau)^2}}{\beta} \text{ and } L := \frac{\beta}{1 - \rho}.
\]

**Proof.** By the super-regularity of the sets $K_l$, for any $\delta > 0$, there exists a neighborhood $V_2$ of $x^*$ such that for any $l \in \{1, \ldots, m\}$, we have

\[
\langle z - y, v \rangle \leq \delta \|z - y\| \|v\| \text{ for all } z, y \in K_l \cap V_2, v \in N_{K_l}(y).
\]

We choose $\delta \geq 0$ to be small enough so that $\delta \leq \frac{\tau(1 - \rho)}{2\beta^2}$. 

Claim: If \( x_{i-1} \) are such that \( \mathbb{B}(x_{i-1}, \frac{1}{1-\rho}d(x_{i-1}, K)) \subset V_1 \cap V_2 \), then \( K \cap \mathbb{B}(x_{i-1}, \frac{1}{1-\rho}d(x_{i-1}, K)) \subset H_i \), where the halfspace \( H_i := \{ x : \langle a_i^{(j)}, x \rangle \leq b_i^{(j)} \} \) is defined by \( [5.4] \).

Proof of Claim: Suppose \( x' \in K \cap \mathbb{B}(x_{i-1}, \frac{1}{1-\rho}d(x_{i-1}, K)) \). Since \( K \cap V_2 \), we have
\[
\langle x_{i-1} - x_i^{(j)}, x' - x_i^{(j)} \rangle \leq \delta \| x_{i-1} - x_i^{(j)} \| \| x' - x_i^{(j)} \|,
\]
where \( x_i^{(j)} \) is the point in \( P_{K_i}(x_{i-1}) \subset K_i \) in \( [5.1] \). Also, \( x' \) was assumed to lie in \( \mathbb{B}(x_{i-1}, \frac{1}{1-\rho}d(x_{i-1}, K)) \). Note that \( d(x_{i-1} - x_i^{(j)}) \leq d(x_{i-1}, K) \). So we have
\[
\| x' - x_i^{(j)} \| \leq \| x' - x_{i-1} \| + \| x_{i-1} - x_i^{(j)} \| \leq \left( \frac{1}{1-\rho} + 1 \right) d(x_{i-1}, K).
\]
Note that \( \frac{1}{1-\rho} + 1 \leq \frac{2}{1-\rho} \). From the above inequality, we have
\[
\langle x_{i-1} - x_i^{(j)}, x' - x_i^{(j)} \rangle \leq \delta \| x_{i-1} - x_i^{(j)} \| \| x' - x_i^{(j)} \| \leq \frac{2\delta}{1-\rho} \| x_{i-1} - x_i^{(j)} \| \leq \frac{2\delta}{1-\rho} d(x_{i-1}, K)^2.
\]
Recall that \( \delta \leq \frac{\tau (1-\rho)}{2\rho^2} \). Local metric inequality gives \( \| x_{i-1} - x_i^{(j)} \| \geq \frac{1}{\beta} d(x_i, K) \), so
\[
\langle x_{i-1} - x_i^{(j)}, x' - x_i^{(j)} \rangle \leq \frac{\tau}{\beta^2} d(x_{i-1}, K)^2 \leq \tau \| x_{i-1} - x_i^{(j)} \| ^2.
\]
The above inequality is precisely \( \langle a_i^{(j)}, x' \rangle \leq b_i^{(j)} \), so \( x' \in H_i \). This ends the proof of the claim.

Suppose \( \mathbb{B}(x_0, \frac{1}{1-\rho}d(x_0, K)) \subset V_1 \cap V_2 \). If the conditions of Lemma \([3.5]\) are satisfied, then we have convergence to some \( \bar{x} \).

We try to prove that \( d(x_{i+1}, K) \leq \rho d(x_i, K) \). Recall that \( x_{i+1} = P_{\tilde{F}_{i+1}}(x_i) \). By making use of the claim above, the previous halfspaces generated all contain \( K \cap \mathbb{B}(x_i, \frac{1}{1-\rho}d(x_i, K)) \), so \( \tilde{F}_{i+1} \) is a polyhedron that contains \( K \cap \mathbb{B}(x_i, \frac{1}{1-\rho}d(x_i, K)) \).

It is clear that \( K \cap \mathbb{B}(x_i, \frac{1}{1-\rho}d(x_i, K)) \neq \emptyset \), so \( \tilde{F}_{i+1} \) is nonempty. It is obvious that \( d(x_i, \tilde{F}_{i+1}) \leq d(x_i, K) \), so \( \| x_i - x_{i+1} \| \leq (1 - \tau) d(x_i, K) \). The distance \( d(x_i, H_{i+1}) \) is at least \( \frac{\beta}{2}d(x_i, K) \), so \( \| x_i - x_{i+1} \| \geq \frac{1}{\beta} d(x_i, K) \). We then have
\[
 d(x_{i+1}, K)^2 \leq d(x_i, K)^2 - \| x_i - x_{i+1} \|^2 \\
\leq d(x_i, K)^2 - \frac{(1-\tau)^2}{\beta^2} d(x_i, K)^2 \\
= \frac{\sqrt{\beta^2 - (1-\tau)^2}}{\beta} d(x_i, K)^2.
\]
We can now apply Lemma \([5.6] \). The conclusion \([5.2] \) comes from the fact that \( \{ x_i \} \), by construction, is obtained by projection onto convex sets that contain \( \bar{x} \) and the theory of Féjér monotonicity. The conclusion \([5.5a] \) is straightforward from Lemma \([5.5a] \) and local metric inequality.

We now prove the theorem on the arbitrary fast multiple-term linear convergence of Algorithm \([5.1] \).

**Theorem 5.3.** (Arbitrary fast linear convergence) Consider the setting of Theorem \([6.2] \). If \( \bar{p} \) in Algorithm \([6.4] \) is finite and sufficiently large, then for all \( \tau \in (0, 0.5) \)
the iterates of Algorithm 5.1 with \( \tau \) assume that \( \{x_i\} \). Moreover,

\[
\limsup_{i \to \infty} \frac{\|x_{i+\bar{p}} - \bar{x}\|}{\|x_i - \bar{x}\|} \leq 8\bar{L}\tau,
\]

where \( \bar{L} = \frac{\beta}{1-p} \) and \( \bar{p} = \frac{\sqrt{2\beta^2 - 1/4}}{\beta} \).

**Proof.** The basic strategy is to prove the inequalities (5.9) and (5.10) like in [Pan15b Theorem 5.12], with a bit more attention put into handling the nonconvexity.

By Lemma 5.2, the convergence of the iterates \( \{x_i\} \) to some \( \bar{x} \in K \) is assured. Without loss of generality, suppose that \( \bar{x} = 0 \). Let \( v_i^* := \frac{x_i - x_{i+1}^{(j+1)}}{\|x_i - x_{i+1}^{(j+1)}\|} \), where \( x_{i+1}^{(j+1)} \) is defined through (5.1a). We shall prove that if \( x_i \) is defined through (5.1a), and \( v_i^* \) and \( v_k^* \) belong to the same ball of radius \( \frac{1}{4} \), covering \( S^{n-1} \). We thus have \( \|v_i^* - v_k^*\| \leq \frac{1}{4} \). (The key in choosing \( \bar{p} \) is to obtain the last inequality.)

We shall prove that if \( i \) is large enough, we have the two inequalities

\[
\langle v_i^*, x_k \rangle \leq 2\tau\|x_j\| \tag{5.9}
\]

and

\[
\frac{1}{4\bar{L}}\|x_k\| \leq \langle v_i^*, x_k \rangle. \tag{5.10}
\]

In view of the Fejér monotonicity condition (5.4), these two inequalities give \( \|x_{i+\bar{p}}\| \leq \|x_k\| \leq 8\bar{L}\tau\|x_i\| \), which gives the conclusion we seek.

We first prove (5.9). Since \( x_k \) lies in \( \tilde{F}_k \), it lies in the halfspace with normal \( v_i^* \) passing through \( (1 - \tau)x_{j+1}^{(j+1)} + \tau x_j \). (Recall that \( x_{j+1}^{(j+1)} \) was defined in (5.1a), and lies in \( P_{K_{ij+1}}(x_j) \). This gives us

\[
\langle v_i^*, x_k \rangle \leq \langle v_i^*, (1 - \tau)x_{j+1}^{(j+1)} + \tau x_j \rangle \tag{5.11}
\]

\[
= (1 - \tau)\langle v_i^* - \bar{v}, x_{j+1}^{(j+1)} \rangle + (1 - \tau)\langle \bar{v}, x_{j+1}^{(j+1)} \rangle + \tau\langle v_i^*, x_j \rangle,
\]

where \( \bar{v} \) is some vector with norm 1 in \( N_{K_{ij+1}}(\bar{x}) \). Since \( \lim_{i \to \infty} x_i = \bar{x} \), we can assume that \( \{x_i\} \) is sufficiently close to \( \bar{x} \) so that:

1. the vector \( \bar{v} \), by the outer semicontinuity of the normal cone mapping \( x \mapsto N_{K_{ij+1}}(x) \), can be chosen to be such that \( \|v_i^* - \bar{v}\| \leq \frac{\tau}{3} \), and

2. by the super-regularity of \( K_{ij+1} \) at \( \bar{x} \), we have \( (1 - \tau)\|x_{j+1}^{(j+1)}\| \leq \frac{\tau}{3}\|x_{j+1}^{(j+1)}\| \).

Note that \( (1 - \tau)x_{j+1}^{(j+1)} + \tau x_j \) is the projection of \( x_{j+1}^{(j+1)} \) onto one of the halfspaces defining \( \tilde{F}_{j+1} \) and that \( \tau < \frac{1}{3} \). From the principle in Proposition 2.6, we have \( \|x_{j+1}^{(j+1)}\| \leq \|x_j\| \). Since \( \|v_i^*\| = 1 \), we have \( \langle v_i^*, x_j \rangle \leq \|x_j\| \). Continuing the arithmetic in (5.11), we have

\[
\langle v_i^*, x_k \rangle \leq (1 - \tau)(\bar{v}, x_{j+1}^{(j+1)}) + (1 - \tau)\langle \bar{v}, x_{j+1}^{(j+1)} \rangle + \tau\langle v_i^*, x_j \rangle
\]

\[
\leq (1 - \tau)\left(\frac{\tau}{3} + \frac{\tau}{3}\right)\|x_{j+1}^{(j+1)}\| + \tau\|x_j\|
\]

\[
\leq \frac{2\tau}{3}\|x_j\| + \tau\|x_j\| < 2\|x_j\|.
\]
This ends the proof of (5.9). Next, we prove (5.10). Recall that \( x_{k+1}^{(j_k+1)} \in P_{K_{j_{k+1}}} (x_k) \) was defined in (5.1a). Note that provided \( \tau < \frac{1}{4} \), the \( \rho = \frac{\sqrt{\beta^2 - (1-\tau)^2}}{\beta} \) in (5.6) is less than \( \bar{\rho} = \frac{\sqrt{\beta^2 - 1/4}}{\beta} \). Hence the \( L = \frac{\beta}{1-\rho} \) in (5.6) is less than \( L = \frac{\beta}{1-\bar{\rho}} \). By using the definition of \( v_k^* = \frac{x_k - x_{k+1}^{(j_k+1)}}{\|x_k - x_{k+1}^{(j_k+1)}\|} \) and (5.5), we have

\[
\langle v_k^*, x_k - x_{k+1}^{(j_k+1)} \rangle = \|x_k - x_{k+1}^{(j_k+1)}\| = d(x_k, K_{j_{k+1}}) \geq \frac{1}{L} \|x_k\| \geq \frac{1}{L} \|x_k\|. 
\tag{5.12}
\]

By the super-regularity of \( K_{j_{k+1}} \) at \( \bar{x} \) and the fact that \( \bar{x} = \lim_{i \to \infty} x_i \), we can assume that \( x_k \) is close enough to \( \bar{x} \) so that

\[
\langle v_k^*, 0 - x_{k+1}^{(j_k+1)} \rangle \leq \frac{1}{4L} \|x_{k+1}^{(j_k+1)}\| \leq \frac{1}{4L} \|x_k\|. \tag{5.13}
\]

(Note that the inequality on the right follows from the same proof of \( \|x_{k+1}^{(j_k+1)}\| \leq \|x_j\| \).) Combining (5.12) and (5.13) as well as \( \|v_j^* - v_k^*\| \leq \frac{1}{2L} \) gives us

\[
\langle v_j^*, x_k \rangle = \langle v_k^*, x_k - x_{k+1}^{(j_k+1)} \rangle + \langle v_k^*, x_{k+1}^{(j_k+1)} \rangle + \langle v_j^* - v_k^*, x_k \rangle \geq \frac{1}{L} \|x_k\| - \frac{1}{4L} \|x_k\| - \frac{1}{2L} \|x_k\| = \frac{1}{4L} \|x_k\|.
\]

This ends the proof of (5.10), which concludes the proof of our result. \( \square \)

The large parameter \( \bar{\rho} \) is an upper bound on when we can find \( v_j^* \) and \( v_k^* \) such that \( \|v_j^* - v_k^*\| \leq \frac{1}{2L} \). We hope that the upper bound needed in a practical implementation would be much smaller than \( \bar{\rho} \).

**Remark 5.4.** (Towards superlinear convergence) The coefficient of 8 in (5.8) can be reduced, but this does not detract us from the point that as \( \tau \searrow 0 \), the right hand side of (5.8) goes to zero. So there is a choice of parameters \( \{\tau_i\}_{i=1}^{\infty} \) that can be chosen at each iteration of Algorithm 5.1 so that superlinear convergence is achieved, even though there doesn’t seem to be a good way of choosing how the parameters \( \tau \) go to zero. If the parameter \( \tau \) goes to zero too fast, the Fejér monotonicity (5.4) of the iterates may not be maintained, which may mean that Lemma 5.2 may not hold, i.e., the iterates \( \{x_i\}_{i=1}^{\infty} \) may not converge. Contrast this to the convex SIP in \( \text{Pan15b} \), where setting \( \tau \equiv 0 \) gives *multiple-term superlinear convergence*

\[
\lim_{i \to \infty} \frac{\|x_{i+1} - \bar{x}\|}{\|x_i - \bar{x}\|} = 0
\]

instead of multiple-term arbitrary linear convergence (5.8). In view of nonconvexity, the observation in Remark 5.3 has to be overcome, so we believe that this arbitrary fast convergence is difficult to improve on in general.

**Remark 5.5.** (Simplification in (5.12)) The inequality \( d(x_k, K_{j_{k+1}}) \geq \frac{1}{L} \|x_k\| \) in (5.12) follows easily from (5.5). But in \( \text{Pan15b} \), some effort was spent to prove the inequality \( \lim \sup_{k \to \infty} \frac{1}{\|x_k\|} d(x_k, K_{j_{k+1}}) \geq \frac{1}{\beta} \). The proof of the multiple-term superlinear convergent algorithm for convex problems in \( \text{Pan15b} \) can thus be shortened considerably.
If some of the sets \( K_i \) are known to be convex sets or affine subspaces, then this information can be taken into account by setting the appropriate \( \tau_i \) to zero when creating the halfspaces defined by (5.1).

6. Two step SHQP

The algorithms in this paper need not guarantee that \( \{d(x_i, K)\} \) is nonincreasing. In this section, we give an example of additional conditions needed for the SHQP to have this property. Consider the following algorithm.

**Algorithm 6.1.** (2-SHQP) Let \( K_1, K_2 \) be two closed sets in \( \mathbb{R}^n \), and \( K = K_1 \cap K_2 \). This algorithm tries to find a point \( x \in K \) using a starting iterate \( x_0 \).

01 Set \( i = 0 \)
02 Loop
03 Set \( x_{i+1} \) to be an element in \( P_{K_1}(x_i) \) and \( i \leftarrow i + 1 \).
04 Set \( x_{i+1} \) to be an element in \( P_{K_2}(x_i) \) and \( i \leftarrow i + 1 \).
05 If \( x_{i+1} - x_{i} \) is not in \( \pi/2 \), then
06 set \( x_{i+1} = P_{K_1}(x_{i+1} - x_{i} \leq 0, \langle x_{i+1} - x_{i} \rangle \leq 0) \).
07 else
08 set \( x_{i+1} = x_i \)
09 end if
10 \( i \leftarrow i + 1 \)
11 end loop

In line 6, \( x_{i+1} \) is the projection of \( x_i \) onto the polyhedron formed by intersecting the last two halfspaces generated by the projection process. See Figure 6.1 for an illustration of the first few iterates \( x_1 \), \( x_2 \) and \( x_3 \) formed by a single iteration of the loop. If the “if” block in lines 5 to 9 is removed, then the algorithm reduces to an alternating projection algorithm. We now analyze the effectiveness of this “if” block.

**Proposition 6.2.** (2-SHQP) Consider Algorithm 6.1. Let \( \delta \in (0, 1) \). Let \( x^* \in K \) and let a neighborhood \( V \) of \( x^* \) be such that

\[
\begin{align*}
(1) \quad &\langle v, y-z \rangle \leq \delta \|v\|\|y-z\| \text{ for all } y, z \in K_1 \cap V, \quad l \in \{1, 2\} \text{ and } v \in N_{K_1}(z), \text{and} \\
(2) \quad &d(x, K) \leq \beta \max_{l \in \{1, 2\}} d(x, K_l) \text{ for all } x \in V.
\end{align*}
\]

Let \( x_1, x_2 \) be successive iterates of Algorithm 6.1. Suppose \( \mathcal{B}(x_1, (\beta+1)\|x_1-x_2\|) \subset \mathcal{V} \). Let \( \theta := \angle x_2 x_1 x_0 < \pi/2 \). If

\[
\delta \beta \cos \theta + (\beta + 1) < \frac{1}{2} \cos \theta, \tag{6.1}
\]

then \( d(x_3, K) < d(x_2, K) \).

Conditions (1) and (2) are consequences of the super-regularity condition and local metric inequality condition respectively, so they will be satisfied when close to \( K \).

**Proof.** Since property (2) and the fact that \( x_1 \in K_1 \) implies that

\[
d(x_2, K) \leq \beta \max_{l \in \{1, 2\}} d(x_2, K_l) = \beta d(x_2, K_1) \leq \beta \|x_1 - x_2\|,
\]

the set \( \mathcal{B}(x_2, \beta \|x_2 - x_1\|) \cap K \) is not empty. Hence

\[
\emptyset \neq \mathcal{B}(x_2, \beta \|x_2 - x_1\|) \subset \mathcal{B}(x_1, (\beta+1)\|x_2 - x_1\|) \subset \mathcal{V}. \tag{6.2}
\]
Let y be any point in \( \mathbb{B}(x_2, \beta \|x_2 - x_1\|) \cap K \). By property (1), we have
\[
(y - x_2, x_1 - x_2) \leq \delta \|y - x_2\| \|x_1 - x_2\| \leq \beta \delta \|x_1 - x_2\|^2.
\] (6.3)
In other words, \( \mathbb{B}(x_2, \beta \|x_2 - x_1\|) \cap K \subset H_2 \), where \( H_2 \) is the halfspace defined by
\[
H_2 := \{x : (x - x_2, x_1 - x_2) \leq \beta \delta \|x_1 - x_2\|^2\}.
\]
Next, from (6.2), we can make use of the argument similar to (6.3) to prove that
\[
\mathbb{B}(x_1, (\beta + 1) \|x_2 - x_1\|) \cap K \subset H_1,
\]
where \( H_1 \) is the halfspace defined by
\[
H_1 := \{x : (x - x_1, x_0 - x_1) \leq (\beta + 1) \delta \|x_0 - x_1\|^2\}.
\]
This implies that
\[
\emptyset \neq \mathbb{B}(x_2, \beta \|x_2 - x_1\|) \cap K \subset H_1 \cap H_2.
\] (6.4)

![Figure 6.1](image-url)

**Figure 6.1.** This figure illustrates the proof of Proposition 6.2. The dotted lines show the boundaries of \( H_1 \) and \( H_2 \).

We refer to Figure 6.1, which shows the two dimensional cross section containing \( x_0, x_1 \) and \( x_2 \). The point \( x_3 \) is also shown in the figure, and is the projection of \( x_2 \) onto \( H_1 \cap H_2 \). We now calculate the minimal value of \( \langle \frac{x_3 - x_2}{\|x_3 - x_2\|}, x \rangle \), where \( x \) ranges over \( H_1 \cap H_2 \). This minimal value can be seen to be \( d_1 - d_2 - d_3 \), where \( d_1, d_2 \) and \( d_3 \) are the distances as indicated in Figure 6.1. These distances can be calculated to be
\[
\begin{align*}
d_1 &= \|x_2 - x_3\| = \|x_1 - x_2\| \cot \theta, \\
d_2 &= \beta \delta \|x_1 - x_2\| \cot \theta, \\
and \quad d_3 &= (\beta + 1) \delta \|x_1 - x_2\| / \sin \theta.
\end{align*}
\]
We can check that (6.1) is equivalent to \( d_2 + d_3 < \frac{1}{4} d_1 \). As long as (6.1) holds, the region \( H_1 \cap H_2 \) lies on the same side as \( x_3 \) of the perpendicular bisector of the points \( x_2 \) and \( x_3 \). Hence all the points in \( H_1 \cap H_2 \) are closer to \( x_3 \) than to \( x_2 \). Since \( H_1 \cap H_2 \) contains all the points in \( P_K(x_2) \) by (6.4), we thus have \( d(x_3, K) < d(x_2, K) \) as needed.

Note that if \( \theta < \pi/2 \) is too close to \( \pi/2 \), then the condition (6.1) can fail. In fact, if \( \theta > \cos^{-1} \delta \), one can check that condition (1) in Proposition 6.2 does not rule out \( x_2 \) being inside \( K_1 \), so there would be no point calculating \( x_3 \). The supporting halfspaces as calculated by the projection process can be too aggressive for super-regular sets. For example, one can draw a manifold in \( \mathbb{R}^2 \) such that the intersection
the manifold and a halfspace generated by the projection process consists of only one point. Two halfspaces of this kind would give an empty intersection with the manifold. Therefore, one has to relax the halfspaces.

We remark that the procedure in (6.3) shows how to construct halfspaces under the super-regularity condition, and can be augmented into Algorithm 5.1 as long as we have a good estimate for $\delta$.

7. Global strategies

In this section, we discuss methods for when local methods of the nonconvex SIP are not appropriate. In Example 7.1 we show that while the theory for the convex SIP suggests that one should not backtrack, backtracking is however suggested for the nonconvex problem, which can lead to the Maratos effect and slows down convergence.

The problem of finding a point in the intersection of a finite number of closed sets $K_l \subset \mathbb{R}^n$, where $l = 1, \ldots, m$, can be equivalently cast as the problem of finding a point that minimizes $f(x)$, where $f(x)$ can be chosen as

$$d(x, \cap_{l=1}^m K_l),$$

$$\sum_{l=1}^m d(x, K_l)^2,$$

$$\max_{l \in \{1, \ldots, m\}} \{d(x, K_l)\},$$

or some other function similar to those presented above. In the event that the intersection $\cap_{l=1}^m K_l$ is nonempty, then any point in $K := \cap_{l=1}^m K_l$ would be a global minimizer of $f(\cdot)$. The function in (7.1a) is the function of choice, but $\cap_{l=1}^m K_l$ can be only be estimated well locally with the techniques in Section 3. Instead of trying to minimize $f(\cdot)$, the problem that really needs to be solved is the one of finding an $x$ in $\{\tilde{x} : f(\tilde{x}) \leq 0\}$. This is a simpler problem which can be solved by a subgradient projection method that is somewhat simpler than the minimization problem. A bundle method [HUL93] [BGLS06] adapted for a nonconvex objective function can be used to solve the nonconvex SIP. (See also [BWWX14] [Pan14] for the principles of a finitely convergent algorithm for this setting. This idea of finite convergence goes back to [PM79] [MPH81] [Fuk82] [PI88] for the convex case and the smooth case.)

A standard procedure in optimization algorithms is the line search procedure. A search direction is calculated, and the next solution is obtained by a line search along this search direction. For the nonconvex SIP, the search direction can be calculated by projecting onto a polyhedron formed by intersecting a number of previously generated halfspaces. There are two ways we can backtrack to obtain decrease in some objective function (in (7.1) or otherwise). Firstly, one can remove halfspaces that describe the polyhedron. It is sensible to remove the older halfspaces since they become less reliable. This has the effect of reducing the distance from the current iterate to the polyhedron, so the search direction is more likely to give decrease. The problem of projecting onto the polyhedron with one halfspace removed can be solved effectively from the old solution using a warmstart quadratic programming algorithm (for example, the active set method of [Gol86]). Secondly, one can use the usual backtracking line search.
We note however that in the pursuit of obtaining decrease in the objective function, we may encounter the Maratos effect (see [NW06, Section 15.5], who in turn cited [Mar78]) which slows convergence.

**Example 7.1.** (Backtracking slows convergence) In this example, we show how the SHQP strategy for a convex SIP converges quickly for a problem, but would be slowed down by backtracking when treated as a nonconvex SIP. Consider the sets $K_1, K_2 \subset \mathbb{R}^3$ where $K_1 = H_1$ and $K_2 = H_2 \cap H_3$, where the halfspaces $H_1, H_2$ and $H_3$ are defined by

\[
H_1 := \{ x \in \mathbb{R}^3 : (0, 1, 0)x \leq 0 \},
\]

\[
H_2 := \{ x \in \mathbb{R}^3 : (1/3, -1, 0)x \leq -2 \},
\]

and $H_3 := \{ x \in \mathbb{R}^3 : (-1, -1, 1)x \leq 0 \}$. Let the point $x_0 = (0, 1, 0)$. The projection of $x_0$ onto $K_1$ and $K_2$ generates the halfspaces $H_1$ and $H_2$ respectively. The projection of $x_0$ onto $H_1 \cap H_2$ is $x_1 := (-6, 0, 0)$. We can calculate that

\[
d(x_0, K_1) = 1, \quad d(x_0, K_2) = 3/\sqrt{10}, \quad d(x_1, K_1) = 0 \quad \text{and} \quad d(x_1, K_2) = 2\sqrt{3}.
\]

(7.2)

The projection of $x_1$ onto $K_2$ generates $H_3$, and once we project $x_1$ onto $H_1 \cap H_2 \cap H_3$, we found a point in $K_1 \cap K_2$. If this SIP were solved as a nonconvex SIP, the values in (7.2) fitted into the objective function (7.1b) or (7.1c) suggests that one has to backtrack in some manner, and this actually slows down the convergence. (See Figure 7.1 for an illustration.)

\[\text{Figure 7.1. This figure illustrates the two dimensional cross section in } \{ x \in \mathbb{R}^3 : x_3 = 0 \} \text{ in the example in Example 7.1. Note that the projection of } x_1 \text{ onto } K_2 \text{ lies outside this cross section.}\]

We recall the method of averaged projections for finding a point in $\cap_{l=1}^{m} K_l$, where $K_l \subset \mathbb{R}^n$ for all $l \in \{1, \ldots, m\}$, is defined by

\[
x_{i+1} = \frac{1}{m} \sum_{l=1}^{m} P_{K_l}(x_i).
\]

(7.3)

It was noticed that this formula corresponds to the method of alternating projections between the two sets in $\mathbb{R}^nm$ defined by

\[
D := \{(x, x, \ldots, x) : x \in \mathbb{R}^n\}
\]

and $\mathbf{K} := K_1 \times K_2 \times \cdots \times K_m$. 
It is easy to see that $f(x_{i+1}) \leq f(x_i)$ if $x_{i+1}$ is defined by (7.3) and $f(\cdot)$ is defined by (7.1b) since $\sqrt{f(x)}$ is the distance of $(x, \cdots, x) \in D$ to $K$. Moreover, if $f(x_{i+1}) = f(x_i)$, then $x_i$ is the minimizer.

In the SHQP strategy for nonconvex problems, we can use backtracking to find the next iterate $x_i$ of the form

$$tP_{\tilde{F}_i}(x_{i-1}) + (1-t)x_{i-1},$$

where $t \in (0, 1]$ and $\tilde{F}_i$ is the polyhedron defined by intersecting previously generated halfspaces like in Algorithm 5.1. We can instead find an iterate of the form

$$tP_{\tilde{F}_i}(x_{i-1}) + (1-t)\frac{1}{m}\sum_{l=1}^{m} P_{K_l}(x_{i-1}).$$

Other heuristics for the nonconvex problem are also possible. For example, if one is certain that the intersection is nonempty, then one can try to avoid points in the balls $\mathbb{B}(x_i, d(x_i, K_l))$ for all $i \geq 0$ and $l \in \{1, \ldots, m\}$. If some of the sets are spectral sets (i.e., the set of symmetric matrices solely described by their eigenvalues), then the results in [LM08] can also be applied.

8. Conclusion

We hope our results make the case that in solving feasibility problems involving super-regular sets, one should use the SHQP procedure as much as possible to accelerate convergence once close enough to the intersection. The size of the QPs to be solved can be kept to be of a manageable size if we combine with projection methods like in Algorithm 3.1.

References

[ABRS10] H. Attouch, J. Bolte, P. Redont, and A. Soubeyran, Proximal alternating minimization and projection methods for nonconvex problems: An approach based on the Kurdyka-Łojasiewicz inequality, Math. Oper. Res. 35 (2010), no. 2, 438–457.

[BB03] H.H. Bauschke and J.M. Borwein, On the convergence of von Neumann’s alternating projection algorithm for two sets, Set-Valued Anal. 1 (1993), 185–212.

[BB06] ———, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (1996), 367–426.

[BBL99] H.H. Bauschke, J.M. Borwein, and W. Li, Strong conical hull intersection property, bounded linear regularity, Jameson’s property (G), and error bounds in convex optimization, Math. Program., Ser. A 86 (1999), no. 1, 135–160.

[BCK06] H.H. Bauschke, P.L. Combettes, and S.G. Kruk, Extrapolation algorithm for affine-convex feasibility problems, Numer. Algorithms 41 (2006), 239–274.

[BCL02] H.H. Bauschke, P.L. Combettes, and D.R. Luke, Phase retrieval, error reduction algorithm, and variants: A view from convex optimization, J. Opt. Soc. Am. 19 (2002), no. 7, 1334–1345.

[BDHP03] H.H. Bauschke, F. Deutsch, H.S. Hundal, and S.-H. Park, Accelerating the convergence of the method of alternating projections, Trans. Amer. Math. Soc. 355 (2003), no. 9, 3433–3461.

[BGLS06] J.F. Bonnans, J.C. Gilbert, C. Lemaréchal, and C.A. Sagastizábal, Numerical optimization: Theoretical and practical aspects, 2 ed., Springer, 2006, Original French edition was published in 1997.

[BLPW13a] H.H. Bauschke, D.R. Luke, H.M. Phan, and X.F. Wang, Restricted normal cones and the method of alternating projections: Applications, Set-valued Var. Anal. 21 (2013), no. 3, 475–501.

[BLPW13b] ———, Restricted normal cones and the method of alternating projections: Theory, Set-valued Var. Anal. 21 (2013), no. 3, 431–473.

[BR09] E.G. Birgin and M. Raydan, Dykstra’s algorithm and robust stopping criteria, Encyclopedia of Optimization (C. A. Floudas and P. M. Pardalos, eds.), Springer, US, 2 ed., 2009, pp. 828–833.
[BWWX14] H.H. Bauschke, Caifang Wang, Xianfu Wang, and Jia Xu, On the finite convergence of a projected cutter method, ArXiv e-prints (2014).
[BZ05] J.M. Borwein and Q.J. Zhu, Techniques of variational analysis, Springer, NY, 2005, CMS Books in Mathematics.
[CC96] X. Chen and M.T. Chu, On the least squares solution of inverse eigenvalue problems, SIAM J. Numer. Anal. 33 (1996), 2417–2430.
[Chu95] M.T. Chu, Constructing a Hermitian matrix from its diagonal entries and eigenvalues, SIAM J. Matrix Anal. 16 (1995), 207–217.
[CT90] P.L. Combettes and H.J. Trussell, Method of successive projections for finding a common point of sets in metric spaces, J. Optim. Theory Appl. 67 (1990), no. 3, 487–507.
[Deu01] F. Deutsch, Best approximation in inner product spaces, Springer, 2001, CMS Books in Mathematics.
[ER11] R. Escalante and M. Raydan, Alternating projection methods, SIAM, 2011.
[Fuk82] M. Fukushima, A finitely convergent algorithm for convex inequalities, IEEE Trans. Automat. Control 27 (1982), no. 5, 1126–1127.
[GB00] K.M. Grigoriadis and E. Beran, Alternating projection algorithm for linear matrix inequalities problems with rank constraints, Advances in Linear Matrix Inequality Methods in Control, SIAM, 2000.
[GI83] D. Goldfarb and A. Idnani, A numerically stable dual method for solving strictly convex quadratic programs, Math. Programming 27 (1983), 1–33.
[GK89] W.B. Gearhart and M. Koshly, Acceleration schemes for the method of alternating projections, J. Comput. Appl. Math. 26 (1989), 235–249.
[Gol86] D. Goldfarb, Efficient primal algorithms for strictly convex quadratic programs, Fourth IFMA Workshop in Numerical Analysis, Guanajuato, Mexico, 1984 (J.P. Hennart, ed.), Springer-Verlag, Berlin, 1986, pp. 11–25.
[GP98] U.M. García-Palomares, A superlinearly convergent projection algorithm for solving the convex inequality problem, Oper. Res. Lett. 22 (1998), 97–103.
[GP01] ________, Superlinear rate of convergence and optimal acceleration schemes in the solution of convex inequality problems, Inherently Parallel Algorithms in Feasibility and Optimization and their Applications (D. Butnariu, Y. Censor, and S. Reich, eds.), Elsevier, 2001, pp. 297–305.
[GPR67] L.G. Gubin, B.T. Polyak, and E.V. Raik, The method of projections for finding the common point of convex sets, USSR Comput. Math. Math. Phys. 7 (1967), no. 6, 1–24.
[GS96] K.M. Grigoriadis and R.E. Skelton, Low-order control design for LMI problems using alternating projection methods, Automatica 32 (1996), 1117–1125.
[HL13] R. Hesse and D.R. Luke, Nonconvex notions of regularity and convergence of fundamental algorithms for feasibility problems, SIAM J. Optim. 23 (2013), no. 4, 2397–2419.
[HRER11] L. M. Hernández-Ramos, R. Escalante, and M. Raydan, Unconstrained optimization techniques for the acceleration of alternating projection methods, Numer. Funct. Anal. Optim. 32 (2011), no. 10, 1041–1066.
[HUL93] J.-B. Hiriart-Urruty and C. Lemaréchal, Convex analysis and minimization algorithms I & II, Springer, 1993, Grundlehren der mathematischen Wissenschaften, Vols 305 & 306.
[Ilo00] A.D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Surveys 55 (2000), no. 3, 501–558.
[Kru06] A.Y. Kruger, About regularity of collections of sets, Set-Valued Anal. 14 (2006), 187–206.
[LLM09] A.S. Lewis, D.R. Luke, and J. Malick, Local linear convergence for alternating and averaged nonconvex projections, Found. Comput. Math. 9 (2009), no. 4, 485–513.
[LM08] A.S. Lewis and J. Malick, Alternating projection on manifolds, Math. Oper. Res. 33 (2008), 216–234.
[Mar78] N. Maratos, Exact penalty function algorithms for finite dimensional and control optimization problems, Ph.D. thesis, University of London, 1978.
[MPH81] D.Q. Mayne, E. Polak, and A.J. Heunis, Solving nonlinear inequalities in a finite number of iterations, J. Optim. Theory Appl. 33 (1981), 207–221.
S. Marchesini, Y.-C. Tu, and H.-T. Wu, *Alternating projection,ptychographic imaging and phase synchronization*.  

H.V. Ngai and M. Théra, *Metric inequality, subdifferential calculus and applications*, Set-Valued Anal. 9 (2001), 187–216.  

J. Nocedal and S.J. Wright, *Numerical optimization*, 2 ed., Springer, 2006.  

K.F. Ng and W.H. Yang, *Regularities and their relations to error bounds*, Math. Program., Ser. A 99 (2004), 521–538.  

R. Orsi, U. Helmke, and J. Moore, *A Newton-like method for solving rank constrained linear matrix inequalities*, Automatica 42 (2006), 1875–1882.  

R. Orsi, *Numerical methods for solving inverse eigenvalue problems for nonnegative matrices*, SIAM J. Matrix Anal. 28 (2006), 190–212.  

C.H.J. Pang, *Finitely convergent algorithm for nonconvex inequality problems*, (preprint) (2014).  

A.R. De Pierro and A.N. Iusem, *A finitely convergent “row-action” method for the convex feasibility problem*, Appl. Math. Optim. 17 (1988), 225–235.  

G. Pierra, *Decomposition through formalization in a product space*, Math. Programming 28 (1984), 96–115.  

E. Polak and D.Q. Mayne, *On the finite solution of nonlinear inequalities*, IEEE Trans. Automat. Control AC-24 (1979), 443–445.  

R.T. Rockafellar and R.J.-B. Wets, *Variational analysis*, Grundlehren der mathematischen Wissenschaften, vol. 317, Springer, Berlin, 1998.  

J.A. Tropp, I.S. Dhillon, R.W. Heath, and T. Strohmer, *Designing structured tight frames via an alternating projection method*, IEEE Trans. Inf. Theory 51 (2005), 188–209.  

C.A. Weber and J.P. Allebach, *Reconstruction of frequency-offset Fourier data by alternating projection on constraint sets*, 24th Allerton Conference Proc. (Urbana-Champaign, IL), 1986, pp. 194–201.  

K. Yang and R. Orsi, *Generalized pole placement via static output feedback: a methodology based on projections*, Automatica 42 (2006), 2143–2150.  

Current address: Department of Mathematics, National University of Singapore, Block S17 08-11, 10 Lower Kent Ridge Road, Singapore 119076  

E-mail address: matpchj@nus.edu.sg